Signed Shape Tilings of Squares

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Abstract

Let $T$ be a tile made up of finitely many rectangles whose corners have rational coordinates and whose sides are parallel to the coordinate axes. This paper gives necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Q}$-weighted tiles with the same shape as $T$, and necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Z}$-weighted tiles with the same shape as $T$. The main tool we use is a variant of F. W. Barnes's algebraic theory of brick packing, which converts tiling problems into problems in commutative algebra.

1 Introduction

In [3] Dehn proved that an $a \times b$ rectangle $R$ can be tiled by finitely many nonoverlapping squares if and only if $a/b$ is rational. More generally, suppose we allow the squares to have weights from $\mathbb{Z}$. An arrangement of weighted squares is a tiling of $R$ if the sum of the weights of the squares covering a region is 1 inside of $R$ and 0 outside. Dehn’s argument applies in this more general setting, and shows that $R$ has a $\mathbb{Z}$-weighted tiling by squares if and only if $a/b$ is rational. In [4] this result is generalized to give necessary and sufficient conditions for a rectangle $R$ to be tilable by $\mathbb{Z}$-weighted rectangles with particular shapes. In this paper we consider a related question: Given a tile $T$ in the plane made up of finitely many weighted rectangles, is there a weighted tiling of a square by tiles with the same shape as $T$?

We define a rectangle in $\mathbb{R} \times \mathbb{R}$ to be a product $[b_1, b_2) \times [c_1, c_2)$ of half-open intervals, with $b_1 < b_2$ and $c_1 < c_2$. Let $A$ be a commutative ring with unity. An $A$-weighted tile is represented by a finite $A$-linear combination $L = a_1R_1 + \cdots + a_nR_n$ of disjoint rectangles. Associated to each such $L$ there is a function $f_L : \mathbb{R}^2 \to A$ which is supported on $\bigcup R_i$.

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and whose value on $R_i$ is $a_i$. We say that $L_1$ and $L_2$ represent the same tile if $f_{L_1} = f_{L_2}$. An example of a $\mathbb{Z}$-weighted tile is given in Figure 1. We may form the sum $T_1 + T_2$ of two weighted tiles $T_1, T_2$ by superposing them in the natural way. For $a \in A$ the tile $aT$ is formed from $T$ by multiplying all the weights of $T$ by $a$. The set of all $A$-weighted tiles forms an $A$-module under these operations.

Let $U$ be an $A$-weighted tile and let $\{T_\lambda : \lambda \in \Lambda\}$ be a set of $A$-weighted tiles. We say that the set $\{T_\lambda : \lambda \in \Lambda\}$ $A$-tiles $U$ if there are weights $a_1, \ldots, a_n \in A$ and tiles $\tilde{T}_1, \ldots, \tilde{T}_n$, each of which is a translation of some $T_{\lambda_i}$, such that $a_1\tilde{T}_1 + \cdots + a_n\tilde{T}_n = U$. Note that we are allowed to use as many translated copies of each prototile $T_\lambda$ as we need, but we are not allowed to rotate or reflect the prototiles. Given an $A$-weighted tile $T$ and a real number $\rho > 0$ we define $T(\rho)$ to be the image of $T$ under the rescaling $(x, y) \mapsto (\rho x, \rho y)$. We say that an $A$-weighted tile $T'$ has the same shape as $T$ if there exists $\rho > 0$ such that $T'$ is a translation of $T(\rho)$. We say that $T$ $A$-shapetiles $U$ if $\{T(\rho) : \rho > 0\}$ $A$-tiles $U$. If $U'$ has the same shape as $U$ then $T$ $A$-shapetiles $U'$ if and only if $T$ $A$-shapetiles $U$.

In this paper we consider tiles $T$ constructed from rectangles whose corners have rational coordinates. We prove two main results about such tiles. First, we show that if $T$ is a $\mathbb{Q}$-weighted tile whose weighted area is not 0, then $T$ $\mathbb{Q}$-shapetiles a square. Second, if $T$ is a $\mathbb{Z}$-weighted tile we give necessary and sufficient conditions for $T$ to $\mathbb{Z}$-shapetile a square.

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2 Polynomials and tiling

Say that $T$ is a lattice tile if $T$ is an $A$-weighted tile made up of unit squares in $\mathbb{R}^2$ whose corners are in $\mathbb{Z}^2$. We will associate a (generalized) polynomial $f_T$ to each $A$-weighted lattice tile $T$. Our approach is similar to that used by F.W.Barnes in \cite{2}, except
that the polynomials that we construct differ from Barnes’s polynomials by a factor $(X - 1)(Y - 1)$. Including this extra factor will allow us to generalize the construction to non-lattice tiles at the end of the section.

Our polynomials will be elements of the ring

$$A[X^Z, Y^Z] := A[X, Y, X^{-1}, Y^{-1}],$$

which is naturally isomorphic to the group ring of $\mathbb{Z} \times \mathbb{Z}$ with coefficients in $A$. To begin we associate the polynomial $X^iY^j(X - 1)(Y - 1)$ to the unit square $S_{ij}$ with lower left corner $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Given an $A$-weighted lattice tile

$$T = \sum_{i,j} w_{ij}S_{ij},$$

by linearity we associate to $T$ the polynomial

$$f_T(X, Y) = \sum_{i,j} w_{ij}X^iY^j(X - 1)(Y - 1).$$

One consequence of this definition is that translating a tile by a vector $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to multiplying its polynomial by $X^iY^j$. The map $T \mapsto f_T$ gives an isomorphism between the $A$-module of $A$-weighted lattice tiles in the plane and the principal ideal in $A[X^Z, Y^Z]$ generated by $(X - 1)(Y - 1)$.

**Example 2.1** Let $a, b, c, d$ be integers such that $a, b \geq 1$ and let $T$ be the $a \times b$ rectangle whose lower left corner is at $(c, d)$. Then the polynomial associated to $T$ is

$$f_T(X, Y) = \sum_{i=c}^{c+a-1} \sum_{j=d}^{d+b-1} X^iY^j(X - 1)(Y - 1) = X^cY^d(X^a - 1)(Y^b - 1).$$

In section 4 we will need to work with non-lattice tiles. To represent these more general tiles systematically we introduce a new set of building blocks to play the role that the unit squares $S_{ij}$ play in the theory of lattice tiles. For $\alpha, \beta \in \mathbb{R}^2$ let $R_{\alpha\beta}$ denote the oriented rectangle with vertices $(0, 0), (\alpha, 0), (\alpha, \beta), (0, \beta)$. Note that if exactly $k$ of $\alpha, \beta$ are negative then $R_{\alpha\beta}$ is equal to $(-1)^k$ times a translation of $R_{|\alpha|, |\beta|}$. We can express any rectangle in terms of the rectangles $R_{\alpha\beta}$:

**Example 2.2** Let $\alpha, \beta > 0$ and let $R'_{\alpha\beta}$ be the translation of the rectangle $R_{\alpha\beta}$ by the vector $(\sigma, \tau) \in \mathbb{R}^2$. Then $R'_{\alpha\beta} = R_{\alpha+\sigma, \beta+\tau} - R_{\alpha+\sigma, \tau} - R_{\sigma, \beta+\tau} + R_{\sigma, \tau}$. In particular, we have $S_{ij} = R_{i+1, j+1} - R_{i+1, j} - R_{i, j+1} + R_{ij}$.

In fact the following holds:

**Lemma 2.3** Every $A$-weighted tile $T$ can be expressed uniquely as an $A$-linear combination of rectangles $R_{\alpha\beta}$ with $\alpha, \beta \in \mathbb{R}^2$. 

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Proof: By Example 2.2 every rectangle is an $A$-linear combination of the rectangles $R_{\alpha\beta}$. Therefore every $A$-weighted tile is an $A$-linear combination of the $R_{\alpha\beta}$. Suppose

$$c_1R_{\alpha_1\beta_1} + c_2R_{\alpha_2\beta_2} + \cdots + c_nR_{\alpha_n\beta_n} = 0$$

is a linear relation such that the pairs $(\alpha_i, \beta_i)$ are distinct and $c_i \neq 0$ for $1 \leq i \leq n$. Choose $j$ to maximize the distance from the origin to the far corner $(\alpha_j, \beta_j)$ of $R_{\alpha_j\beta_j}$. None of the other rectangles in the sum can overlap the region around $(\alpha_j, \beta_j)$. Since $c_j \neq 0$, this gives a contradiction. Therefore the set $\{R_{\alpha\beta} : \alpha, \beta \in \mathbb{R}^x\}$ is linearly independent over $A$, which implies the uniqueness part of the lemma. \qed

In order to represent arbitrary $A$-weighted tiles algebraically we introduce a generalization of the polynomials $f_T$. Let $A[X^R, Y^R]$ denote the set of “polynomials” with coefficients from $A$ where the exponents of $X$ and $Y$ are allowed to be arbitrary real numbers. The natural operations of addition and multiplication make $A[X^R, Y^R]$ a commutative ring with unity. The ring $A[X^R, Y^R]$ is naturally isomorphic to the group ring of $\mathbb{R} \times \mathbb{R}$ with coefficients in $A$, and contains $A[X^\mathbb{Z}, Y^\mathbb{Z}]$ as a subring.

For $\alpha, \beta \in \mathbb{R}^x$ define $f_{R_{\alpha\beta}} = (X^\alpha - 1)(Y^\beta - 1) \in A[X^R, Y^R]$. By Lemma 2.3 this definition extends linearly to give a well-defined element $f_T \in A[X^R, Y^R]$ associated to any $A$-weighted tile $T$. It follows from Example 2.2 that this definition agrees with that given earlier if $T = S_{ij}$ is a unit lattice square, and hence also if $T$ is any lattice tile. The map $T \mapsto f_T$ gives an isomorphism between the $A$-module of $A$-weighted tiles and an $A$-submodule of $A[X^R, Y^R]$. The next lemma implies that this $A$-submodule is actually an ideal in $A[X^R, Y^R]$.\hfill 

Lemma 2.4 Let $T$ be an $A$-weighted tile and let $T'$ be the translation of $T$ by the vector $(\sigma, \tau) \in \mathbb{R} \times \mathbb{R}$. Then $f_{T'} = X^\sigma Y^\tau f_T$.

Proof: Let $R'_{\alpha\beta}$ be the translation of $R_{\alpha\beta}$ by $(\sigma, \tau)$. Using Example 2.2 we get

$$f_{R'_{\alpha\beta}} = X^\sigma Y^\tau (X^\alpha - 1)(Y^\beta - 1) = X^\sigma Y^\tau f_{R_{\alpha\beta}},$$

so the lemma holds for $T = R_{\alpha\beta}$. Therefore by Lemma 2.3 the lemma holds for all tiles $T$. \qed

The next result gives a further relation between ideals and tiling.

Proposition 2.5 Let $U$ be a tile, let $\{T_{\lambda} : \lambda \in \Lambda\}$ be a collection of tiles, and let $I \subset A[X^R, Y^R]$ be the ideal generated by the set $\{f_{T_{\lambda}} : \lambda \in \Lambda\}$. Then $\{T_{\lambda} : \lambda \in \Lambda\}$ $A$-tiles $U$ if and only if $f_U \in I$.

Proof: We have $f_U \in I$ if and only if

$$f_U(X,Y) = \sum_{i=1}^k a_i X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X,Y)$$

for some $a_i \in A$, $\sigma_i, \tau_i \in \mathbb{R}$, and $\lambda_i \in \Lambda$. Since $X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X,Y)$ is the polynomial associated to the translation of $T_{\lambda_i}$ by the vector $(\sigma_i, \tau_i)$, we have $f_U \in I$ if and only if $U = a_1 \hat{T}_1 + \cdots + a_k \hat{T}_k$, with $\hat{T}_i$ a translation of $T_{\lambda_i}$. Therefore $f_U \in I$ if and only if $\{T_{\lambda} : \lambda \in \Lambda\}$ $A$-tiles $U$. \qed
Corollary 2.6 Let \( \{ T_\lambda : \lambda \in \Lambda \} \) be a collection of lattice tiles, let \( I \) be the ideal in \( A[\mathbb{Z}^2, \mathbb{Z}^2] \) generated by the set \( \{ f_{T_\lambda} : \lambda \in \Lambda \} \), and let \( U \) be a lattice tile such that \( f_U \in I \). Then \( \{ T_\lambda : \lambda \in \Lambda \} \) \( A \)-tiles \( U \).

The last result in this section shows what happens to \( f_T \) when we replace \( T \) by a rescaling.

Lemma 2.7 Let \( T \) be an \( A \)-weighted tile and let \( \rho \) be a positive real number. Then \( f_{T(\rho)} = f_T(X^\rho, Y^\rho) \).

**Proof:** Let \( \alpha, \beta \in \mathbb{R}^\times \). Then \( R_{\alpha,\beta}(\rho) = R_{\rho\alpha,\rho\beta} \) and hence
\[
f_{R_{\alpha,\beta}(\rho)} = (X^{\rho\alpha} - 1)(Y^{\rho\beta} - 1) = f_{R_{\alpha,\beta}}(X^\rho, Y^\rho).
\]
Therefore the lemma holds for \( T = R_{\alpha,\beta} \). It follows from Lemma 2.3 that the lemma holds for all tiles \( T \). \( \square \)

3 Tiling with rational weights

This section is devoted to proving the following theorem:

**Theorem 3.1** Let \( T \) be a \( \mathbb{Q} \)-weighted tile made up of rectangles whose corners all have rational coordinates. Then \( T \) \( \mathbb{Q} \)-shapetiles a square if and only if the weighted area of \( T \) is not zero.

**Proof:** It is clear that if the weighted area of \( T \) is zero then \( T \) cannot shapetile a square with nonzero area. Assume conversely that \( T \) has nonzero weighted area. By rescaling and translation we may assume that \( T \) is a lattice tile in the first quadrant. Let \( T(\mathbb{N}) \) denote the set \( \{ T(k) : k \in \mathbb{N} \} \) of positive integer rescalings of \( T \). To complete the proof of Theorem 3.1 it suffices to prove that \( T(\mathbb{N}) \) \( \mathbb{Q} \)-tiles a square. First we will prove that \( T(\mathbb{N}) \) \( \mathbb{C} \)-tiles a square; from this it will follow easily that \( T(\mathbb{N}) \) \( \mathbb{Q} \)-tiles a square.

Since \( T \) is a lattice tile in the first quadrant, \( f_T \in \mathbb{Q}[X,Y] \) is a polynomial in the ordinary sense. We begin by interpreting the hypothesis that the weighted area of \( T \) is nonzero in terms of \( f_T \).

**Lemma 3.2** There is a polynomial \( f^*_T \in \mathbb{Q}[X,Y] \) such that
\[
f_T(X,Y) = (X - 1)(Y - 1)f^*_T(X,Y).
\]
Moreover, the weighted area of \( T \) is equal to \( f^*_T(1,1) \), and hence \( f^*_T(1,1) \neq 0 \).

**Proof:** Since the polynomial associated to the unit square \( S_{ij} \) is
\[
f_{S_{ij}}(X,Y) = X^iY^j(X - 1)(Y - 1),
\]
the lemma holds for $S_{ij}$. It follows by linearity that the lemma holds for all lattice tiles in the first quadrant. \hfill \Box

Let $I$ denote the ideal in $\mathbb{C}[X^Z, Y^Z]$ generated by $\{f_{T(k)} : k \in \mathbb{N}\}$ and let
\[ g_t(X,Y) = (X^t-1)(Y^t-1) \]
be the polynomial associated to an $l \times l$ square with lower left corner $(0,0)$. To show that $T(\mathbb{N})$ $\mathbb{C}$-tiles a square it suffices by Corollary 2.8 to show that $g_t \in I$ for some positive integer $l$. In order to get information about $I$ we consider the set $V(I) \subset \mathbb{C}^\times \times \mathbb{C}^\times$ of common zeros of the elements of $I$. The set $V(I)$ is essentially the union of the lines $X = 1$ and $Y = 1$ with the “shape variety” of $T(\mathbb{N})$ as defined by Barnes [2, §3].

We wish to determine which points $(\alpha, \beta) \in \mathbb{C}^\times \times \mathbb{C}^\times$ might be in $V(I)$. Let $m$ be the $X$-degree of $f_T$, let $n$ be the $Y$-degree of $f_T$, and define $\Upsilon \subset \mathbb{C}^\times$ by
\[ \Upsilon = \{ \zeta \in \mathbb{C}^\times : \zeta^k = 1 \text{ for some } 1 \leq k \leq 2mn \}. \]

**Lemma 3.3** $V(I) \subset (\mathbb{C}^\times \times \Upsilon) \cup (\Upsilon \times \mathbb{C}^\times)$.

*Proof:* Let $(\alpha, \beta) \in V(I)$, and suppose neither $\alpha$ nor $\beta$ is in $\Upsilon$. By Lemma 2.7 and Lemma 3.2 we have
\[ 0 = f_{T(k)}(\alpha, \beta) = f_T(\alpha^k, \beta^k) = (\alpha^k-1)(\beta^k-1)f_T^*(\alpha^k, \beta^k) \]
for all $k \geq 1$. Since $\alpha$ and $\beta$ aren’t in $\Upsilon$ this implies $f_T^*(\alpha^k, \beta^k) = 0$ for $1 \leq k \leq 2mn$. Therefore by Lemma 3.4 below there exist $c, d \in \mathbb{Z}$ such that $f_T^*(X^c, X^d) = 0$. It follows that $f_T^*(1,1) = 0$, contrary to Lemma 3.2. We conclude that if $(\alpha, \beta) \in V(I)$ then at least one of $\alpha, \beta$ must be in $\Upsilon$. \hfill \Box

**Lemma 3.4** Let $K$ be a field and let $f^* \in K[X,Y]$ be a nonzero polynomial with $X$-degree $m - 1$ and $Y$-degree $n - 1$. Assume there are $\alpha, \beta \in K^\times$ such that
\begin{enumerate}
  \item $\alpha$ and $\beta$ are not $k$th roots of $1$ for any $1 \leq k \leq 2mn$, and 
  \item $f^*(\alpha^k, \beta^k) = 0$ for all $1 \leq k \leq 2mn$.
\end{enumerate}

Then there exist relatively prime integers $c, d$ with $1 \leq c \leq n - 1$ and $1 \leq |d| \leq m - 1$ such that $f^*(X^c, X^d) = 0$.

*Proof:* Define an $mn \times mn$ matrix $M$ whose columns are indexed by pairs $(i,j)$ with $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$ by letting the $k$th entry in the $(i,j)$ column of $M$ be $\alpha^k \beta^j k$. Since $f^*(\alpha^k, \beta^k) = 0$ for $1 \leq k \leq mn$, the coefficients of $f^*$ give a nontrivial element of the nullspace of $M$. Since $M$ is essentially a Vandermonde matrix this implies
\[ 0 = \det(M) = \alpha^{nm(m-1)/2} \beta^{mn(n-1)/2} \cdot \prod_{(i,j)<(i',j')} (\alpha^i \beta^j - \alpha^{i'} \beta^{j'}) \]
for an appropriate ordering of the pairs \((i, j)\). It follows that \(\alpha^i \beta^j = \alpha^{i'} \beta^{j'}\) for some \((i', j') \neq (i, j)\), so \(\alpha^0 = \beta^0\) for some \((c_0, d_0) \neq (0, 0)\) with \(|c_0| \leq n - 1\) and \(|d_0| \leq m - 1\). The first assumption implies that \(c_0 \neq 0\) and \(d_0 \neq 0\), so we may assume without loss of generality that \(c_0 \geq 1\).

Let \(e = \gcd(c_0, d_0)\) and set \(c = c_0/e\) and \(d = d_0/e\). Then since \((\alpha^e)^d = (\beta^e)^c\) with \(\gcd(c, d) = 1\) there is a unique \(\gamma \in K\) such that \(\gamma^c = \alpha^e\) and \(\gamma^d = \beta^e\). Let \(q\) be an integer such that \(1 \leq q \leq 2mn/e\). Then by the second assumption we have

\[
0 = f^*(\alpha^{eq}, \beta^{eq}) = f^*(\gamma^{eq}, \gamma^{dq}),
\]

and so \(f^*(X^c, X^d) \in K[X, X^{-1}]\) has zeros at \(X = \gamma^d\) for \(1 \leq q \leq 2mn/e\). If these zeros are not distinct then for some \(1 \leq r \leq 2mn/e\) we have \(\gamma^r = 1\) and hence \(1 = \gamma^{eq} \alpha^e\), which violates the first assumption. Therefore \(f^*(X^c, X^d)\) has at least \([2mn/e]\) distinct zeros. On the other hand the degree of the rational function \(f^*(X^c, X^d)\) is at most \((m - 1)|c| + (n - 1)|d|\), and since \(|c| = |c_0/e| \leq (n - 1)/e\) and \(|d| = |d_0/e| \leq (m - 1)/e\) we have

\[
(m - 1)|c| + (n - 1)|d| \leq 2(m - 1)(n - 1)/e < [2mn/e].
\]

Therefore \(f^*(X^c, X^d) = 0\). 

Let \(l \geq 1\) and recall that \(g_l(X, Y) = (X^l - 1)(Y^l - 1)\) is the polynomial associated to an \(l \times l\) square with lower left corner \((0, 0)\). The set \(V(g_l) \subset \mathbb{C}^l \times \mathbb{C}^l\) of zeros of \(g_l\) is the union of the lines \(X = \zeta\) and \(Y = \zeta\) as \(\zeta\) ranges over the \(l\)th roots of 1. It follows from Lemma 3.3 that if we choose \(l\) appropriately (say \(l = (2mn)\)) then \(V(g_l) \supset V(I)\). This need not imply that \(g_l\) is in \(I\), but by Hilbert’s Nullstellensatz [4, VII, Th. 14] we do have \(g_k^k \in I\) for some \(k \geq 1\).

To show there exists \(l\) such that \(g_l \in I\) we use the theory of primary decompositions (see, e. g., chapters 4 and 7 of [2]). Let \(A\) be a commutative ring with 1. We say that the ideal \(Q \subset A\) is a primary ideal if whenever \(xy \in Q\) with \(x \notin Q\) there exists \(a \geq 1\) such that \(g^a \in Q\). By the Hilbert basis theorem, \(\mathbb{C}[X^Z, Y^Z]\) is a Noetherian ring [4, Cor. 7.7]. Therefore there are primary ideals \(Q_1, \ldots, Q_r\) in \(\mathbb{C}[X^Z, Y^Z]\) such that \(I = Q_1 \cap \ldots \cap Q_r\) [4, Th. 7.13]. The radical ideal

\[
P_i = \sqrt{Q_i} = \{ f \in \mathbb{C}[X^Z, Y^Z] : f^r \in Q_i \text{ for some } r \geq 1 \}
\]

of the primary ideal \(Q_i\) is automatically prime, and is called the prime associated to \(Q_i\). We may also characterize \(P_i\) as the smallest prime ideal containing \(Q_i\).

Since \(I = Q_1 \cap \ldots \cap Q_r\) we need to show that there exists \(l \geq 1\) such that \(g_l \in Q_i\) for all \(1 \leq i \leq r\). Observe that if \(l | l'\) then \(g_l | g_{l'}\). Therefore it is enough to show that for each \(i\) there is \(l_i\) such that \(g_{l_i} \in Q_i\), since in that case we have \(g_l \in I\) with \(l = \text{lcm}\{l_1, \ldots, l_r\}\). To accomplish this we first restrict the possibilities for the prime ideals \(P_i\).

Let \(q = (2mn)\). We observed above that \(g_q^k \in I\) for some positive integer \(k\). Since \(P_i \supset Q_i \supset I\) this implies that \(g_q^k \in P_i\). Therefore some irreducible factor of

\[
g_q(X, Y)^k = \prod_{\zeta^q = 1} (X - \zeta)^k(Y - \zeta)^k
\]
lies in the prime ideal $P_i$. It follows that $X - \zeta \in P_i$ or $Y - \zeta \in P_i$ for some $\zeta \in \mathbb{C}^\times$ such that $\zeta^q = 1$.

Assume without loss of generality that $X - \zeta \in P_i$. Then $P_i$ contains the prime ideal $(X - \zeta)$ generated by the irreducible polynomial $X - \zeta$. If $P_i \neq (X - \zeta)$ let $h$ be an element of $P_i$ which is not in $(X - \zeta)$. By dividing $X - \zeta$ into $h(X,Y)$ we see that $h(\zeta, Y) \in P_i$. Since $P_i$ is prime and $\mathbb{C}$ is algebraically closed this implies that some linear factor $Y - \alpha$ of $h(\zeta, Y)$ is in $P_i$. Therefore $P_i$ contains the maximal ideal $(X - \zeta, Y - \alpha)$, so in fact $P_i = (X - \zeta, Y - \alpha)$. Moreover, we must have $\alpha \neq 0$ since $Y$ is a unit in $\mathbb{C}[X^2, Y^2]$. It follows that if $X - \zeta \in P_i$ then either $P_i = (X - \zeta)$ or $P_i = (X - \zeta, Y - \alpha)$ for some $\alpha \in \mathbb{C}^\times$.

We will make repeated use of the following elementary fact about primary ideals.

**Lemma 3.5** Let $Q$ be a primary ideal and set $P = \sqrt{Q}$. If $gh \in Q$ with $h \not\in P$ then $g \in Q$.

**Proof:** Since $h \not\in P$ we have $h^a \not\in Q$ for all $a \geq 1$. Therefore by the definition of primary ideal we have $g \in Q$. \hfill \Box

Assume now that $P_i = (X - \zeta)$ with $\zeta^q = 1$. Then $X^q - 1$ has a simple zero at $X = \zeta$. Therefore by Lemma 2.7 and Lemma 3.2 we have

$$f_{T(q)}(X,Y) = f_T(X^q, Y^q) = (X^q - 1)(Y^q - 1) f_T^*(X^q, Y^q) = (X - \zeta) h(X,Y)$$

for some $h \in \mathbb{C}[X, Y]$. Moreover we have $h(\zeta, Y) \neq 0$, since otherwise $0 = f_T^*(\zeta^q, Y^q) = f_T^*(1, Y^q)$, which would imply $f_T^*(1, 1) = 0$, contrary to Lemma 3.2. Therefore $h \not\in P_i = (X - \zeta)$. It follows by Lemma 3.3 that $X - \zeta \in Q_i$, and hence that $g_q \in Q_i$.

Now assume $P_i = (X - \zeta, Y - \alpha)$. If $\alpha$ is an $r$th root of 1 for some $r \geq 1$ then $X^{qr} - 1$ has a simple zero at $X = \zeta$ and $Y^{qr} - 1$ has a simple zero at $Y = \alpha$. As in the previous case this implies

$$f_{T(qr)}(X,Y) = (X^{qr} - 1)(Y^{qr} - 1) f_T^*(X^{qr}, Y^{qr}) = (X - \zeta)(Y - \alpha) h(X,Y)$$

for some $h \in \mathbb{C}[X, Y]$. Since $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1, 1) \neq 0$, we have $h(\zeta, \alpha) \neq 0$, and hence $h \not\in P_i$. Applying Lemma 3.3 we get $(X - \zeta)(Y - \alpha) \in Q_i$, and hence $g_{qr} \in Q_i$. If $\alpha$ is not a root of 1 we may choose $r \geq 1$ so that $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1, \alpha^{qr}) \neq 0$, since $f_T^*(1, 1) \neq 0$ implies that $f_T^*(1, Y)$ has only finitely many zeros. Then $X^{qr} - 1$ has a simple zero at $X = \zeta$ and $Y^{qr} - 1$ is nonzero at $Y = \alpha$. By an argument similar to those used above we have $f_{T(qr)}(X,Y) = (X - \zeta) h(X,Y)$ for some $h \in \mathbb{C}[X, Y]$ such that $h(\zeta, \alpha) \neq 0$. This implies $h \not\in P_i$, so by Lemma 3.3 we get $X - \zeta \in Q_i$, and hence $g_q \in Q_i$.

We’ve shown now that for each $1 \leq i \leq r$ there is $i_t \geq 1$ such that $g_{i_t} \in Q_i$. Therefore we have $g_t \in I$ with $l = \text{lcm}\{l_1, \ldots, l_r\}$. It follows from Corollary 2.6 that $T(\mathbb{N})$ $\mathbb{Q}$-tiles an $l \times l$ square. To prove that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square it is sufficient to prove that $g_t$ is in
the ideal $I_0$ in $\mathbb{Q}[X^Z, Y^Z]$ generated by $T(\mathbb{N})$. Equivalently, we need to show that $g_l$ is in the $\mathbb{Q}$-span of the set

$$\mathcal{E} = \{X^iY^jf_{T(k)} : i, j, k \in \mathbb{Z}, k \geq 1\}.$$ 

We have shown that $g_l$ is in the $\mathbb{C}$-span of $\mathcal{E}$. Since $g_l$ and the elements of $\mathcal{E}$ are all in $\mathbb{Q}[X^Z, Y^Z]$, and

$$\mathbb{C}[X^Z, Y^Z] \cong \mathbb{Q}[X^Z, Y^Z] \otimes_{\mathbb{Q}} \mathbb{C},$$

it follows immediately that $g_l$ is in the $\mathbb{Q}$-span of $\mathcal{E}$. This completes the proof of Theorem 3.1. □

Corollary 3.6 Let $T$ be a $\mathbb{Z}$-weighted tile made up of rectangles whose corners all have rational coordinates. Assume that the weighted area of $T$ is not zero. Then there exists a positive integer $w$ such that $T(\mathbb{N})$ $\mathbb{Z}$-tiles a square with weight $w$.

Proof: By Theorem 3.1 we know that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square $R$, so there are rational numbers $a_1, \ldots, a_n$ and tiles $T_1, \ldots, T_n$, each a translation of some $T(k_i) \in T(\mathbb{N})$, such that $R = a_1T_1 + \ldots + a_nT_n$. Let $w \geq 1$ be a common denominator for $a_1, \ldots, a_n$. Then $wR = wa_1T_1 + \ldots + wa_nT_n$, and $wa_i \in \mathbb{Z}$ for $1 \leq i \leq n$. Therefore $T(\mathbb{N})$ $\mathbb{Z}$-tiles $wR$. □

4 Tiling with integer weights

Let $T$ be a $\mathbb{Z}$-weighted lattice tile, and assume that the weighted area of $T$ is not zero. By Corollary 3.6 we know that $T \mathbb{Z}$-shapetiles a square with weight $w$ for some positive integer $w$. We wish to find necessary and sufficient conditions for $T$ to $\mathbb{Z}$-shapetile a square with weight 1. To express these conditions we need a definition. Given $\mu \in \mathbb{Q} \cup \{\infty\}$ we say that two lattice squares $S_{ij}$ and $S_{ij'}$ belong to the same $\mu$-slope class if the line joining their centers has slope $\mu$. The tile $T$ can be decomposed into a sum $T = C_1 + \cdots + C_k$ of lattice tiles such that for each $i$ the unit lattice squares which make up $C_i$ all belong to the same $\mu$-slope class.

Proposition 4.1 Let $T$ be a $\mathbb{Z}$-weighted lattice tile and let $n$ be a positive integer. Let $c$ and $d$ be relatively prime integers and set $\mu = -c/d$. Then the $\mu$-slope classes of $T$ all have weighted area divisible by $n$ if and only if $f_T$ is an element of the ideal $((X^d - Y^c)(X - 1)(Y - 1), n(X - 1)(Y - 1))$ in $\mathbb{Z}[X^Z, Y^Z]$.

Proof: The $\mu$-slope classes of $T$ all have weighted area divisible by $n$ if and only if we can write $T = T_1 + nT_2$, where $T_1$ and $T_2$ are $\mathbb{Z}$-weighted lattice tiles such that the $\mu$-slope classes of $T_1$ all have weighted area zero. Write the decomposition of $T_1$ into its $\mu$-slope classes as $T_1 = C_1 + \cdots + C_k$. Since $\mu = -c/d$ with $c$ and $d$ relatively prime, the lattice squares $S_{ij}$ and $S_{ij'}$ are in the same $\mu$-slope class if and only if $S_{ij'}$ is the translation...
of $S_{ij}$ by $(dr, -cr)$ for some $r \in \mathbb{Z}$. Therefore if $C_i$ is the $\mu$-slope class of $T_1$ containing $S_{ij}$ we have
\[ f_{C_i}(X, Y) = g(X^dY^{-c})X^iY^j(X - 1)(Y - 1) \]
for some $g \in \mathbb{Z}[X^{\mathbb{Z}}]$. Since the weighted area of $C_i$ is zero we see that $0 = f_{C_i}^\mu(1, 1) = g(1)$, which implies $X - 1 \mid g(X)$. It follows that $(X^dY^{-c} - 1)(X - 1)(Y - 1)$ divides $f_{C_i}$ for $1 \leq t \leq k$, and hence also that $(X^dY^{-c} - 1)(X - 1)(Y - 1)$ divides $f_{T_1}$. Conversely, if $(X^dY^{-c} - 1)(X - 1)(Y - 1)$ divides $f_{T_1}$, it is easy to check that the $\mu$-slope classes of $T_1$ all have weighted area zero. It follows that the $\mu$-slope classes of $T$ all have area divisible by $n$ if and only if we can write
\[ f_{T}(X, Y) = (X^dY^{-c} - 1)(X - 1)(Y - 1)h_1(X, Y) + n(X - 1)(Y - 1)h_2(X, Y) \]
for some $h_1, h_2 \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Since $Y^c$ is a unit in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ this is equivalent to $f_T \in ((X^d - Y^c)(X - 1)(Y - 1), n(X - 1)(Y - 1))$. □

**Theorem 4.2** Let $T$ be a $\mathbb{Z}$-weighted lattice tile. Then $T$ $\mathbb{Z}$-shapetiles a square if and only if the two following conditions hold:

1. The weighted area of $T$ is not zero.

2. For every $\mu \in \mathbb{Q}^\times$ the gcd of the weighted areas of the $\mu$-slope classes of $T$ is 1.

**Proof:** Let $T$ be a tile which satisfies conditions 1 and 2. To show that $T$ $\mathbb{Z}$-shapetiles a square it is sufficient by Corollary 3.6 to show that $T(\mathbb{N}) \cup \{wR\} \mathbb{Z}$-tiles a square, where $R$ is an $l \times l$ square and $l, w$ are positive integers. Let $S = S_{00}$ be the unit lattice square with lower left corner $(0, 0)$. If $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles an $a \times a$ square then by rescaling we see that $T(\mathbb{N}) \cup \{wR\}$ $\mathbb{Z}$-tiles an $la \times la$ square. Therefore it is sufficient to show that $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles a square. Let $J$ be the ideal in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $\{f_{T(k)} : k \in \mathbb{N}\} \cup \{w(X - 1)(Y - 1)\}$. By Corollary 2.6 it is sufficient to show that $g_i \in J$ for some $l \geq 1$.

By the Hilbert basis theorem $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ is a Noetherian ring. Therefore the ideal $J$ has a primary decomposition $J = Q_1 \cap \ldots \cap Q_l$. We need to show that there exists $l \geq 1$ such that $g_i \in Q_i$ for all $i$. As in the proof of Theorem 3.1 it is enough to show that for each $i$ there is $l_i \geq 1$ such that $g_{l_i} \in Q_i$. Let $P_i = \sqrt{Q_i}$ be the prime associated to $Q_i$, and suppose $w \notin P_i$. Then since $w(X - 1)(Y - 1) \in Q_i$, by Lemma 3.3 we see that $(X - 1)(Y - 1) = g_i$ is in $Q_i$. If $w \in P_i$ then since $P_i$ is a prime ideal it follows that $P_i$ contains a prime integer $p$ which divides $w$, and hence that $P_i \cap \mathbb{Z} = p\mathbb{Z}$.

For $f \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ let $\overline{f} \in \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the reduction of $f$ modulo $p$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with $p$ elements. Let $\overline{P_i}$ be the ideal in $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ consisting of the reductions modulo $p$ of the elements of $P_i$. Since $p \in P_i$ the ideal $\overline{P_i}$ is prime. Let $\overline{J} \subset \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the ideal consisting of the reductions modulo $p$ of the elements of $J$. Then $\overline{J}$ is generated by $\{\overline{J}_{T(k)} : k \geq 1\}$. Since $P_i \supset J$, we have $\overline{P_i} \supset \overline{J}$. 

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Let $K$ be an algebraic closure of $\mathbb{F}_p$ and let $V(\mathcal{J}) \subset K^\times \times K^\times$ be the set of common zeros of the elements of $\mathcal{J}$. Let $m$ be the $X$-degree of $\mathcal{J}$, let $n$ be the $Y$-degree of $\mathcal{J}$, and define $\mathcal{Y} \subset K^\times \times K^\times$ by

$$
\mathcal{Y} = \{ \zeta \in K^\times : \zeta^k = 1 \text{ for some } 1 \leq k \leq 2mn \}.
$$

**Lemma 4.3** $V(\mathcal{J}) \subset (K^\times \times \mathcal{Y}) \cup (\mathcal{Y} \times K^\times)$.

**Proof:** Let $(\alpha, \beta) \in V(\mathcal{J})$ and suppose neither $\alpha$ nor $\beta$ is in $\mathcal{Y}$. Then for $1 \leq k \leq 2mn$ we have

$$
0 = \mathcal{J}(\alpha, \beta) = \mathcal{J}(\alpha^k, \beta^k) = (\alpha^k - 1)(\beta^k - 1)\mathcal{H}(\alpha^k, \beta^k).
$$

Since $\alpha$ and $\beta$ aren’t in $\mathcal{Y}$ this implies that $\mathcal{H}(\alpha^k, \beta^k) = 0$ for $1 \leq k \leq 2mn$. Therefore by Lemma 3.4 there are relatively prime integers $c, d$ with $c \geq 1$ and $d \neq 0$ such that $\mathcal{H}(X^c, X^d) = 0$. Let $A$ be the quotient ring $\mathbb{F}_p[X^c, Y^c]/(X^d - Y^c)$, and let $x, y$ denote the images of $X, Y$ in $A$. Then $x$ and $y$ are units in $A$ satisfying $x^d = y^c$ with $\gcd(c, d) = 1$, so there is $z = x^a y^b$ in $A^\times$ such that $x = z^c$ and $y = z^d$. Therefore the image of $\mathcal{J}$ in $A$ is given by $\mathcal{J}(x, y) = \mathcal{J}(z^c, z^d)$, which equals zero since $\mathcal{J}(X^c, X^d) = 0$. It follows that $X^d - Y^c$ divides $\mathcal{J}$, and hence that $f_T^\mu$ is in the ideal $(X^d - Y^c, p)$ in $\mathbb{Z}[X^\mathbb{Z}, Y^\mathbb{Z}]$. Therefore $f_T = (X - 1)(Y - 1)$ is in the ideal $((X^d - Y^c)(X - 1)(Y - 1), p(X - 1)(Y - 1))$ in $\mathbb{Z}[X^\mathbb{Z}, Y^\mathbb{Z}]$. Proposition 1.1 now implies that every $\mu$-slope class of $T$ has area divisible by $p$. This violates condition 2 of the theorem, so we have a contradiction. \hfill $\Box$

Set $q = (2mn)!$ and let $V(\mathcal{J}_q) \subset K^\times \times K^\times$ be the set of zeros of $\mathcal{J}_q$. Since $X^q - 1$ has zeros at all elements of $\mathcal{Y}$, we have $V(\mathcal{J}_q) \supset (K^\times \times \mathcal{Y}) \cup (\mathcal{Y} \times K^\times)$. Therefore Lemma 4.3 implies $V(\mathcal{J}_q) \supset V(\mathcal{J})$. Since $\mathcal{P}_i \supset T$ we have $V(\mathcal{J}) \supset V(\mathcal{P}_i)$, and hence $V(\mathcal{J}_q) \supset V(\mathcal{P}_i)$. As in Section 3 Hilbert’s Nullstellensatz implies that $\mathcal{J}_q \supset P_i$ for some $k \geq 1$. Since $P_i$ is prime and

$$
\mathcal{J}_q(X, Y)^k = (X^q - 1)^k(Y^q - 1)^k
$$

we have either $X^q - 1 \in P_i$ or $Y^q - 1 \in P_i$. It follows that $P_i$ contains one of the ideals $(X^q - 1, p)$ or $(Y^q - 1, p)$. We may assume without loss of generality that $P_i \supset (X^q - 1, p)$.

By [1], Prop. 7.14 we have $Q_i \supset P_i^u$ for some $u \geq 1$. Therefore it is enough to prove that for every $u \geq 1$ there is $l \geq 1$ such that $g_l \in P_i^u$. Let $t$ be a positive integer. Expanding $X^{qt} - 1$ in powers of $X^q - 1$ gives

$$
X^{qt} - 1 = -1 + ((X^q - 1) + 1)^t = \sum_{j=1}^{t} \binom{t}{j} (X^q - 1)^j.
$$
If we choose \( t \) to be divisible by a large power of \( p \) then for small values of \( j \geq 1 \) the binomial coefficient \( {t \choose j} \) is divisible by a large power of \( p \). Thus every term in this expansion is divisible either by a large power of \( p \) or a large power of \( X^q - 1 \). It follows that there exists \( t \geq 1 \) such that \( X^q t - 1 \in (X^q - 1, p)^u \). Since \( P^u_1 \supset (X^q - 1, p)^u \) we get \( g_{qt} \in P^u_1 \), as required.

Assume conversely that \( T \mathbb{Z} \)-shapetiles a square. Then the weighted area of \( T \) is clearly not equal to zero, so condition 1 of Theorem 4.2 is satisfied. We need to show that for every \( \mu \in \mathbb{Q}^* \) the gcd of the weighted areas of the \( \mu \)-slope classes of \( T \) is equal to 1. If we knew that the scale factors and the coordinates of the translation vectors used in shapetiling the square were all in \( \mathbb{Z} \), or even in \( \mathbb{Q} \), we could prove this using polynomials in \( \mathbb{Z}[X^\mathbb{R}, Y^\mathbb{R}] \). Since we have no right to make this assumption, we need to work in the ring \( \mathbb{Z}[X^\mathbb{R}, Y^\mathbb{R}] \).

We may assume that the square which is shapetiled by \( T \) is \( S = S_{00}, \) the unit square with lower left corner \( (0, 0) \). We have then \( S = a_1T_1 + \cdots + a_kT_k \), where \( a_i \in \mathbb{Z} \) and each \( T_i \) is a translation of some \( T(\rho_i) \). Let \( p \) be prime and suppose that for some \( \mu \in \mathbb{Q}^* \) the areas of the \( \mu \)-slope classes of \( T \) are all divisible by \( p \). Let \( c, d \) be integers such that \( \gcd(c, d) = 1 \) and \( \mu = -c/d \). Let \( \overline{f}_T \in \mathbb{F}_p[X^\mathbb{R}, Y^\mathbb{R}] \) be the reduction of \( f_T \) modulo \( p \), and for \( 1 \leq i \leq n \) let \( \overline{f}_{T_i} \in \mathbb{F}_p[X^\mathbb{R}, Y^\mathbb{R}] \) be the reduction of \( f_{T_i} \). Then by Proposition 4.1 we see that \( (X^d - Y^c)(X - 1)(Y - 1) \) divides \( \overline{f}_T \) (in \( \mathbb{F}_p[X^\mathbb{R}, Y^\mathbb{R}] \), and hence also in \( \mathbb{F}_p[X^\mathbb{R}, Y^\mathbb{R}] \)). Therefore by Lemma 2.7 and Lemma 2.4 we see that \( \overline{f}_{T_i} \) is divisible by

\[
(X^{\rho_i d} - Y^{\rho_i c})(X^{\rho_i} - 1)(Y^{\rho_i} - 1).
\]

Define a ring homomorphism \( \Psi : \mathbb{F}_p[X^\mathbb{R}, Y^\mathbb{R}] \rightarrow \mathbb{F}_p[X^\mathbb{R}] \) by setting \( \Psi(f) = f(X^c, X^d) \). Since \( \Psi(X^{\rho_i d} - Y^{\rho_i c}) = 0 \), the divisibility relation from the preceding paragraph implies that \( \Psi(\overline{f}_{T_i}) = 0 \) for \( 1 \leq i \leq n \). On the other hand, since \( \overline{f}_S = \overline{f}_1 = (X - 1)(Y - 1) \), we have

\[
\Psi(\overline{f}_S) = X^{c+d} - X^c - X^d + 1,
\]

which is nonzero since \( c \) and \( d \) are nonzero. Since \( S = a_1T_1 + \cdots + a_kT_k \) we have \( \overline{f}_S = \overline{a}_1\overline{f}_{T_1} + \cdots + \overline{a}_k\overline{f}_{T_k} \) with \( \overline{a}_i \in \mathbb{F}_p \), which gives a contradiction. Therefore the areas of the \( \mu \)-slope classes of \( T \) can’t all be divisible by \( p \), so condition 2 is satisfied. This completes the proof of Theorem 4.2. \( \square \)

**Example 4.4** Let \( T \) be the lattice tile pictured in Figure 3a. Since \( T \) has area \( 4 \neq 0 \), it follows from Theorem 3.1 that \( T \mathbb{Q} \)-shapetiles a square. But since the nonempty \( 1 \)-slope classes of \( T \) both have area 2, Theorem 4.2 implies that \( T \) does not \( \mathbb{Z} \)-shapetile a square.

**Example 4.5** Let \( a, b, c, d \) be positive integers with \( a > c \) and \( b > d \). We construct a lattice tile \( T \) by removing a \( c \times d \) rectangle from the upper right corner of an \( a \times b \) rectangle, as in Figure 3b. The area of \( T \) is \( ab - cd > 0 \), so the first condition of Theorem 4.2 is satisfied. If \( \mu > 0 \) there is a \( \mu \)-slope class of \( T \) consisting of just the
upper left corner square, while if \( \mu < 0 \) there is a \( \mu \)-slope class of \( T \) consisting of just the lower left corner square. In either case \( T \) has a \( \mu \)-slope class whose area is 1. Therefore the second condition of Theorem 4.2 is also satisfied, so \( T \) \( \mathbb{Z} \)-shapetiles a square.

**Example 4.6** The simplest case of Example 4.5 occurs when \( a = b = 2 \) and \( c = d = 1 \). In this case we have \( f_T(X, Y) = (1 + X + Y)(X - 1)(Y - 1) \). A straightforward calculation shows that

\[
XY g_3(X, Y) = (X^3Y^3 - X^2Y^2 - X^4 - X^4Y - X^4Y^2 - Y^4 - XY^4 - X^2Y^4) f_T(X, Y) \\
+ (XY - 1) f_{T(2)}(X, Y) + f_{T(3)}(X, Y).
\]

This gives the \( \mathbb{Z} \)-tiling of a 3 \( \times \) 3 square with lower left corner \((1, 1)\) depicted in Figure 3. The left side of Figure 3 has tiles with weight 1 and the right side has tiles with weight \(-1\). The total weights of the tiles covering each region are indicated.

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