UNCONDITIONALITY OF ORTHOGONAL SPLINE SYSTEMS IN $H^1$

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Abstract. We give a simple geometric characterization of knot sequences for which the corresponding orthonormal spline system of arbitrary order $k$ is an unconditional basis in the atomic Hardy space $H^1[0,1].$

1. Introduction.

This paper belongs to a series of papers studying properties of orthonormal spline systems with arbitrary knots. The detailed study of such systems started in 1960's with Z. Ciesielski's papers [2, 3] on properties of the Franklin system, which is an orthonormal system consisting of continuous piecewise linear functions with dyadic knots. Next, the results by J. Domsta (1972), cf. [11], made it possible to extend such study to orthonormal spline systems of higher order – and higher smoothness – with dyadic knots. These systems occurred to be bases or unconditional bases in several function spaces like $L^p[0,1]$, $1 \leq p < \infty$, $C[0,1]$, $H^p[0,1]$, $0 < p \leq 1$, Sobolev spaces $W^{p,k}[0,1]$, they give characterizations of BMO and VMO spaces, and various spaces of smooth functions (Hölder functions, Zygmund class, Besov spaces). One should mention here names such as Z. Ciesielski, J. Domsta, S.V. Bochkarev, P. Wojtaszczyk, S.-Y. A. Chang, P. Sjölin, J.-O. Strömberg (for more detailed references see e.g. [13], [15], [16]). Nowadays, results of this kind are known for wavelets.

The extension of these results to orthonormal spline systems with arbitrary knots has begun with the case of piecewise linear systems, i.e. general Franklin systems, or orthonormal spline systems of order 2. This was possible due to precise estimates of the inverse to the Gram matrix of piecewise linear $B$-spline bases with arbitrary knots, as presented in [19]. First results in this direction were obtained in [5] and [13]. We would like to mention here two results by G.G. Gevorkyan and A. Kamont. First, each general Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$, cf. [14], Second, there is a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$, cf. [15]. We note that in both of these results, an essential tool for their proof is the association of a so called characteristic interval to each general Franklin function $f_n$. 

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The case of splines of higher order is much more difficult. Let us mention that the basic result – the existence of a uniform bound for $L^\infty$-norms of orthogonal projections on spline spaces of order $k$ with arbitrary order (i.e. a bound depending on the order $k$, but independent of the sequence of knots) – was a long-standing problem known as C. de Boor’s conjecture (1973), cf. [8]. The case of $k = 2$ was settled even earlier by Z. Ciesielki [2], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981), cf. [7, 9], but the positive answer in the general case was given by A. Yu. Shadrin [22] in 2001. A much simplified and shorter proof of this theorem was recently obtained by M. v. Golitschek (2014), cf. [24]. An immediate consequence of A.Yu. Shadrin’s result is that if a sequence of knots is dense in $[0, 1]$, then the corresponding orthonormal spline system of order $k$ is a basis in $L^p[0, 1], 1 \leq p < \infty$ and $C[0, 1]$. Moreover, Z. Ciesielki [4] obtained several consequences of Shadrin’s result, one of them being some estimate for the inverse to the $B$-spline Gram matrix. Using this estimate, G.G. Gevorkyan and A. Kamont [16] extended a part of their result from [15] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order $k$ is a basis in $H^1[0, 1]$. Further extension required more precise estimates for the inverse of $B$-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A.Yu. Shadrin [21]. Using these estimates, M. Passenbrunner [20] proved that for each sequence of knots, the corresponding orthonormal spline system of order $k$ is an unconditional basis in $L^p[0, 1], 1 < p < \infty$. The main result of the present paper is to give a characterization of those knot sequences for which the corresponding orthonormal spline system of order $k$ is an unconditional basis in $H^1[0, 1]$.

The paper is organized as follows. In Section 2 we give necessary definitions and we formulate the main result of this paper – Theorem 2.4. In Sections 3 and 4 we recall or prove several facts needed for our results. In particular, in Section 4 we recall precise pointwise estimates for orthonormal spline systems with arbitrary knots, the associated characteristic intervals and some combinatoric facts for characteristic intervals. Then Section 5 contains some auxiliary results, and the proof of Theorem 2.4 is done in Section 6.

The results contained in this paper were obtained independently by the two teams G. Gevorkyan, K. Keryan and A. Kamont, M. Passenbrunner at the same time, so we have decided to work out a joint paper.

2. Definitions and the main result.

Let $k \geq 2$ be an integer. In this work, we are concerned with orthonormal spline systems of order $k$ with arbitrary partitions. We let $\mathcal{T} = (t_n)_{n=2}^\infty$ be a dense sequence of points in the open unit interval such that each point occurs at most $k$ times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called $k$-admissible. For $n$ in the range $-k + 2 \leq n \leq 1$, let $S_n^{(k)}$ be the space of polynomials of order $n + k - 1$ (or degree $n + k - 2$) on the
interval \([0, 1]\) and \((f_n^{(k)})_{n=-k+2}^{1}\) be the collection of orthonormal polynomials in \(L^2 \equiv L^2[0, 1]\) such that the degree of \(f_n^{(k)}\) is \(n + k - 2\). For \(n \geq 2\), let \(T_n\) be the ordered sequence of points consisting of the grid points \((t_j)_{j=0}^{n}\) counting multiplicities and where the knots 0 and 1 have multiplicity \(k\), i.e., \(T_n\) is of the form

\[
T_n = \{0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \leq \cdots \leq \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1\}.
\]

In that case, we also define \(S_n^{(k)}\) to be the space of polynomial splines of order \(k\) with grid points \(T_n\). For each \(n \geq 2\), the space \(S_n^{(k)}\) has codimension 1 in \(S_n^{(k)}\) and, therefore, there exists a function \(f_n^{(k)} \in S_n^{(k)}\) that is orthonormal to the space \(S_{n-1}^{(k)}\). Observe that this function \(f_n^{(k)}\) is unique up to sign.

**Definition 2.1.** The system of functions \((f_n^{(k)})_{n=-k+2}^{\infty}\) is called orthonormal spline system of order \(k\) corresponding to the sequence \((t_n)_{n=0}^{\infty}\).

We will frequently omit the parameter \(k\) and write \(f_n\) and \(S_n\) instead of \(f_n^{(k)}\) and \(S_n^{(k)}\), respectively.

Let us note that the case \(k = 2\) corresponds to orthonormal systems of piecewise linear functions, i.e. general Franklin systems.

We are interested in characterizing sequences of knots \(T\) such that the system \((f_n^{(k)})_{n=-k+2}^{\infty}\) is an unconditional basis in \(H^1 = H^1[0, 1]\). By \(H^1 = H^1[0, 1]\) we mean the atomic Hardy space on \([0, 1]\), cf \([6]\). A function \(a : [0, 1] \to \mathbb{R}\) is called an atom, if either \(a \equiv 1\) or there exists an interval \(\Gamma\) such that the following conditions are satisfied:

(i) \(\text{supp } a \subset \Gamma\),
(ii) \(|a|_\infty \leq |\Gamma|^{-1}\),
(iii) \(\int_0^1 a(x) \, dx = \int_\Gamma a(x) \, dx = 0\).

Then, by definition, \(H^1\) consists of all functions \(f\) that have the representation

\[
f = \sum_{n=1}^{\infty} c_n a_n
\]

for some atoms \((a_n)_{n=1}^{\infty}\) and real scalars \((c_n)_{n=1}^{\infty}\) such that \(\sum_{n=1}^{\infty} |c_n| < \infty\). The space \(H^1\) becomes a Banach space under the norm

\[
||f||_{H^1} := \inf \sum_{n=1}^{\infty} |c_n|,
\]

where \(\inf\) is taken over all atomic representations \(\sum c_n a_n\) of \(f\).

To formulate our result, we need to introduce some regularity conditions for a sequence \(T\).

For \(n \geq 2, \ell \leq k\) and \(i\) in the range \(k - \ell + 1 \leq i \leq n + k - 1\), we define \(D_{n, i}^{(\ell)}\) to be the interval \([\tau_{n,i}, \tau_{n,i+\ell}]\).

**Definition 2.2.** Let \(\ell \leq k\) and \((t_n)_{n=0}^{\infty}\) be an \(\ell\)-admissible (and therefore \(k\)-admissible) point sequence. Then, this sequence is called \(\ell\)-regular with
parameter $\gamma \geq 1$ if

$$\frac{|D_{n,i}^{(\ell)}|}{\gamma} \leq |D_{n,i+1}^{(\ell)}| \leq \gamma|D_{n,i}^{(\ell)}|, \quad n \geq 2, \quad k - \ell + 1 \leq i \leq n + k - 2.$$ 

So, in other words, $(t_n)$ is $\ell$-regular, if there is a uniform finite bound $\gamma \geq 1$, such that for all $n$, the ratios of the lengths of neighboring supports of B-spline functions (cf. Section 3.2) of order $\ell$ in the grid $T_n$ are bounded by $\gamma$.

The following characterization for $(f_n^{(k)})$ to be a basis in $H^1$ is the main result of [16]:

**Theorem 2.3** ([16]). Let $k \geq 1$ and let $(t_n)$ be a $k$-admissible sequence of knots in $[0,1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order $k$. Then, $(f_n^{(k)})$ is a basis in $H^1$ if and only if $(t_n)$ is $k$-regular with some parameter $\gamma \geq 1$

In this paper, we prove the characterization for $(f_n^{(k)})$ to be an unconditional basis in $H^1$. The main result of our paper is the following:

**Theorem 2.4.** Let $(t_n)$ be a $k$-admissible sequence of points. Then, the corresponding orthonormal spline system $(f_n^{(k)})$ is an unconditional basis in $H^1$ if and only if $(t_n)$ satisfies the $(k - 1)$-regularity condition with some parameter $\gamma \geq 1$.

Let us note that in case $k = 2$, i.e. for general Franklin systems, both Theorems 2.3 and 2.4 were obtained by G. G. Gevorkyan and A. Kamont in [15]. (In the terminology of the current paper, the condition of strong regularity from [15] is now 1-regularity, and the condition of strong regularity for pairs from [15] is now 2-regularity.)

The proof of Theorem 2.4 follows the same general scheme as the proof of Theorem 2.2 in [15]. In Section 5 we introduce four conditions (A) – (D) for series with respect to orthonormal spline systems of order $k$ corresponding to a $k$-admissible sequence of points. Then we study relations between these conditions under various regularity assumptions on the underlying sequence of points. Having done this, we proceed with the proof of Theorem 2.4 in Section 6.

3. Preliminaries

The parameter $k \geq 2$ will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$, such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all $t$, where $t$ denotes all implicit and explicit dependencies that the expressions $A$ and $B$ might have. If the constants $c_1, c_2$ depend on an additional parameter $p$, we write this as $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim, \gtrsim, \lesssim_p, \gtrsim_p$. For a subset $E$ of the real line, we denote by $|E|$ the Lebesgue measure of $E$ and by $1_E$ the characteristic function of $E$. If $f : \Omega \to \mathbb{R}$ is a real valued function and $\lambda$ is a real parameter, we write the level set of all points at which $f$ is greater than $\lambda$ as $[f > \lambda] := \{\omega \in \Omega : f(\omega) > \lambda\}$. 
3.1. Properties of regular sequences of points. The following Lemma describes geometric decay of intervals in regular sequences (recall the notation $D_{n,i}^{(t)} = [\tau_{n,i}, \tau_{n,i+1}]$):

**Lemma 3.1.** Let $(t_n)$ be a $k$-admissible sequence of points that satisfies the $\ell$-regularity condition for some $1 \leq \ell \leq k$ with parameter $\gamma$ and let $D_{n_1,i_1}^{(t)} \supset \cdots \supset D_{n_{2\ell},i_{2\ell}}^{(t)}$ be a strictly decreasing sequence of sets defined above. Then,

$$|D_{n_{2\ell},i_{2\ell}}^{(t)}| \leq \frac{\gamma^\ell}{1 + \gamma^\ell} |D_{n_1,i_1}^{(t)}|.$$

**Proof.** We set $V_j := D_{n_j,i_j}^{(t)}$ for $1 \leq j \leq 2\ell$. Then, by definition, $V_1$ contains $\ell + 1$ grid points from $T_{n_1}$ and it contains at least $3\ell$ grid points of the grid $T_{n_{2\ell}}$. As a consequence, there exists an interval $D_{n_{2\ell},m}^{(t)}$ for some index $m$ that satisfies

$$\text{int}(D_{n_{2\ell},m}^{(t)} \cap V_{2\ell}) = \emptyset, \quad D_{n_{2\ell},m}^{(t)} \subset V_1, \quad \text{dist}(D_{n_{2\ell},m}^{(t)}, V_{2\ell}) = 0.$$

The $\ell$-regularity of $(t_n)$ now implies

$$|V_{2\ell}| \leq \gamma^\ell |D_{n_{2\ell},m}^{(t)}| \leq \gamma^\ell (|V_1| - |V_{2\ell}|),$$

i.e., $|V_{2\ell}| \leq \frac{\gamma^\ell}{1 + \gamma^\ell} |V_1|$, which proves the assertion of the lemma. \qed

3.2. Properties of B-spline functions. We define the functions $(N_{n,i}^{(k)})_{i=1}^{n+k-1}$ to be the collection of B-spline functions of order $k$ corresponding to the partition $T_n$. Those functions are normalized in such a way that they form a partition of unity, i.e., $\sum_{i=1}^{n+k-1} N_{n,i}^{(k)}(x) = 1$ for all $x \in [0,1]$. Associated to this basis, there exists a biorthogonal basis of $S_n$, which is denoted by $(N_{n,i}^{(k)*})_{i=1}^{n+k-1}$. If the setting of the parameters $k$ and $n$ is clear from the context, we also denote those functions by $(N_i)_{i=1}^{n+k-1}$ and $(N_i^*)_{i=1}^{n+k-1}$, respectively.

We will need the following well known formula for the derivative of a linear combination of B-spline functions: if $g = \sum_{j=1}^{n+k-1} a_j N_{n,j}^{(k)}$, then

$$g' = (k - 1) \sum_{j=2}^{n+k-1} (a_j - a_{j-1}) \frac{N_{n,j}^{(k-1)}}{|D_{n,j}^{(k-1)}|}.$$

(3.1)

We now recall an elementary property of polynomials.

**Proposition 3.2.** Let $0 < \rho < 1$. Let $I$ be an interval and $A \subset I$ be a subset of $I$ with $|A| \geq \rho |I|$. Then, for every polynomial $Q$ of order $k$ on $I$,

$$\max_{t \in I} |Q(t)| \lesssim_{\rho,k} \sup_{t \in A} |Q(t)| \quad \text{and} \quad \int_I |Q(t)| \, dt \lesssim_{\rho,k} \int_A |Q(t)| \, dt.$$

We continue with recalling a few important results for B-splines $(N_i)$ and their dual functions $(N_i^*)$.
Proposition 3.3. Let $1 \leq p \leq \infty$ and $g = \sum_{j=1}^{n+k-1} a_j N_j$, where the collection $(N_i)_{i=1}^{n+k-1}$ are the B-splines of order $k$ corresponding to the partition $\mathcal{T}_n$. Then,

$$\text{(3.2)} \quad |a_j| \lesssim_k |J_j|^{-1/p} \|g\|_{L^p(J_j)}, \quad 1 \leq j \leq n + k - 1,$$

where $J_j$ is a subinterval $[\tau_{n,i}, \tau_{n,i+1}]$ of $[\tau_{n,j}, \tau_{n,j+k}]$ of maximal length. Additionally,

$$\text{(3.3)} \quad \|g\|_p \sim_k \left( \sum_{j=1}^{n+k-1} |a_j|^p |D_{n,j}^{(k)}| \right)^{1/p} = \|(a_j |D_{n,j}^{(k)}|^{1/p})_{j=1}^{n+k-1}\|_{\ell_p}.$$

Moreover, if $h = \sum_{j=1}^{n+k-1} b_j N_j^*$,

$$\text{(3.4)} \quad \|h\|_p \lesssim_k \left( \sum_{j=1}^{n+k-1} |b_j|^p |D_{n,j}^{(k)}|^{-1/p} \right)^{1/p} = \|(b_j |D_{n,j}^{(k)}|^{-1/p})_{j=1}^{n+k-1}\|_{\ell_p}.$$

The two inequalities (3.2) and (3.3) are Lemma 4.1 and Lemma 4.2 in [10] Chapter 5, respectively. Inequality (3.4) is a consequence of Shadrin’s theorem [22], that the orthogonal projection operator onto $S_n^{(k)}$ is bounded on $L^\infty$ independently of $n$ and $\mathcal{T}_n$. For a deduction of (3.4) from this result, see [4] Property P.7.

The next thing to consider are estimates for the inverse $(b_{ij})_{i,j=1}^{n+k-1}$ of the Gram matrix $(\langle N_i, N_j \rangle_{i,j=1}^{n+k-1})$. Later, we will need one special property of this matrix, which is that $(b_{ij})_{i,j=1}^{n+k-1}$ is checkerboard, i.e.,

$$\text{(3.5)} \quad (-1)^{i+j} b_{ij} \geq 0 \quad \text{for all } i, j.$$

This is a simple consequence of the total positivity of the Gram matrix $(\langle N_i, N_j \rangle_{i,j=1}^{n+k-1})$, cf. [7, 18]. Moreover, we need the following lower estimate for $b_{i,i}$:

$$\text{(3.6)} \quad |D_{n,i}^{(k)}|^{-1} \lesssim_k b_{i,i}.$$

This estimate is a consequence of the total positivity of the $B$-spline Gram matrix, the $L^2$-stability of $B$-splines and the following Lemma 3.4

Lemma 3.4 (20). Let $C = (c_{ij})_{i,j=1}^{n}$ be a symmetric positive definite matrix. Then, for $(d_{ij})_{i,j=1}^{n} = C^{-1}$ we have

$$c_{ii}^{-1} \leq d_{ii}, \quad 1 \leq i \leq n.$$

3.3. Some results for orthonormal spline systems. We recall now two results concerning orthonormal spline series, which we will need in the sequel.

Theorem 3.5 (21). Let $(f_n)_{n=-k+2}^{\infty}$ be the orthonormal spline system of order $k$ corresponding to an arbitrary $k$-admissible point sequence $(t_n)_{n=0}^{\infty}$. Then, for an arbitrary $f \in L^1 \equiv L^1[0, 1]$, the series $\sum_{n=-k+2}^{\infty} \langle f, f_n \rangle f_n$ converges to $f$ almost everywhere.
Let \( f \in L^p \equiv L^p[0,1] \) for some \( 1 \leq p < \infty \). Since the orthonormal spline system \((f_n)_{n \geq -k+2}\) is a basis in \( L^p \), we can write \( f = \sum_{n=-k+2}^{\infty} a_n f_n \). Based on this expansion, we define the square function \( Pf := \left( \sum_{n=-k+2}^{\infty} |a_n f_n|^2 \right)^{1/2} \) and the maximal function \( Sf := \sup_{n} \left| \sum_{m \leq n} a_n f_n \right| \). Moreover, given a measurable function \( g \), we denote by \( Mg \) the Hardy-Littlewood maximal function of \( g \) defined as

\[
Mg(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)| \, dt,
\]

where the supremum is taken over all intervals \( I \) containing the point \( x \). The connection between the maximal function \( Sf \) and the Hardy-Littlewood maximal function is given by the following result:

**Theorem 3.6** ([21]). If \( f \in L^1 \), we have

\[
Sf(t) \lesssim_k Mg(t), \quad t \in [0,1].
\]

4. **Properties of orthogonal spline functions and characteristic intervals**

4.1. **Estimates for \( f_n \).** This section treats the calculation and estimation of one explicit orthonormal spline function \( f_n^{(k)} \) for fixed \( k \in \mathbb{N} \) and \( n \geq 2 \) induced by the \( k \)-admissible sequence \((t_n)_{n=0}^{\infty}\). Most of the presented results are taken from [20].

Here, we change our notation slightly. We fix the parameter \( n \) and let \( i_0 \) be an index with \( k + 1 \leq i_0 \leq n + k - 1 \) such that \( \mathcal{T}_{n-1} \) equals \( \mathcal{T}_n \) with the point \( \tau_{i_0} \) removed. In the points of the partition \( \mathcal{T}_n \), we omit the parameter \( n \) and \( \mathcal{T}_n \) is thus given by

\[
\mathcal{T}_n = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \leq \cdots \leq \tau_{i_0} \leq \cdots \leq \tau_{n+k-1} < \tau_{n+k} = \cdots = \tau_{n+2k-1} = 1).
\]

We denote by \((N_i : 1 \leq i \leq n+k-1)\) the B-spline functions corresponding to \( \mathcal{T}_n \).

An (unnormalized) orthogonal spline function \( g \in \mathcal{S}_n^{(k)} \) that is orthogonal to \( \mathcal{S}_{n-1}^{(k)} \), as calculated in [20], is given by

\[
(4.1) \quad g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=i_0-k}^{i_0} \sum_{\ell=1}^{n+k-1} \alpha_j b_{j\ell} N_{\ell},
\]

where \((b_{j\ell})_{j,\ell=1}^{n+k-1}\) is the inverse of the Gram matrix \((\langle N_j, N_\ell \rangle)_{j,\ell=1}^{n+k-1}\) and the sequence \((\alpha_j)\) is given by

\[
(4.2) \quad \alpha_j = (-1)^{j-i_0+k} \left( \prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell} \right) \left( \prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell} \right), \quad i_0 - k \leq j \leq i_0.
\]
We remark that the sequence \((\alpha_j)\) alternates in sign and since the matrix 
\[
(b_{j\ell})_{j,\ell=1}^{n+k-1}
\]
is checkerboard, we see that the B-spline coefficients of \(g\), namely 
\[
w_{\ell} := \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}, \quad 1 \leq \ell \leq n + k - 1,
\]
satisfy 
\[
\left| \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell} \right| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j\ell}|, \quad 1 \leq j \leq n + k - 1.
\]

In the following Definition 4.1, we assign to each orthonormal spline function 
a characteristic interval that is a grid point interval \([\tau_j, \tau_{j+k}]\) and lies 
in the proximity of the newly inserted point \(\tau_{i_0}\). The choice of this interval 
is crucial for proving important properties of the system 
\[
(f_n^{(k)})_{n=-\infty}^{\infty}
\]
which has its origins in [14], where it is proved that general Franklin systems are unconditional bases in \(L_p\), \(1 < p < \infty\).

**Definition 4.1.** Let \(T_n, T_{n-1}\) be as above and \(\tau_{i_0}\) be the new point in \(T_n\) that 
is not present in \(T_{n-1}\). We define the characteristic interval \(J_n\) corresponding 

to the pair \((T_n, T_{n-1})\) as follows.

1. Let 
\[
\Lambda^{(0)} := \{i_0 - k \leq j \leq i_0 : |[\tau_j, \tau_{j+k}]| \leq 2 \min_{i_0-k \leq \ell \leq i_0} |[\tau_{\ell}, \tau_{\ell+k}]|\}
\]
be the set of all indices \(j\) for which the corresponding support of the 
B-spline function \(N_j\) is approximately minimal. Observe that \(\Lambda^{(0)}\) is 
nonempty.

2. Define 
\[
\Lambda^{(1)} := \{j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell \in \Lambda^{(0)}} |\alpha_{\ell}|\}.
\]

For an arbitrary, but fixed index \(j^{(0)} \in \Lambda^{(1)}\), set 
\[J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]\].

3. The interval \(J^{(0)}\) can now be written as the union of \(k\) grid intervals 
\[
J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \quad \text{with } j^{(0)} \text{ as above.}
\]

We define the characteristic interval \(J_n\) to be one of the above \(k\) intervals that has maximal length.

A few clarifying comments to these definitions are in order. Roughly speaking, 
we first take the B-spline support \([\tau_j, \tau_{j+k}]\) intersecting the new point 
\(\tau_{i_0}\) with minimal length and then we choose as \(J_n\) the largest grid point 
interval in \([\tau_j, \tau_{j+k}]\). This definition guarantees the concentration of \(f_n\) at 
\(J_n\) in terms of the \(L_p\)-norm (cf. Lemma 4.3) and the exponential decay of \(f_n\) 
away from \(J_n\) (cf. Lemma 4.4), which are crucial for further investigations.

An important ingredient in the proof of Lemma 4.3 is Proposition 3.3, being 
the reason why we choose the largest grid point interval as \(J_n\). Further important properties of the collection \((J_n)\) of characteristic intervals are
that they form a nested family of sets and for a subsequence of decreasing characteristic intervals, their lengths decay geometrically (cf. Lemma 4.3).

Next we remark that the constant 2 in step (1) of Definition 4.1 could also be an arbitrary number $C > 1$, but $C = 1$ is not allowed. This is in contrast to the definition of characteristic intervals in [14] for piecewise linear orthogonal functions ($k = 2$), where precisely $C = 1$ is chosen, step (2) of Definition 4.1 is omitted and $j^{(0)}$ is an arbitrary index in the set $\Lambda^{(0)}$.

At first glance, it might seem natural to carry over the same definition to arbitrary spline orders $k$, but at some point in the proof of Theorem 4.2 we estimate $\alpha_{j^{(0)}}$ by the constant $(C - 1)$ from below, which has to be strictly greater than zero in order to establish (4.5). Since Theorem 4.2 is also used in the proofs of both Lemma 4.3 and Lemma 4.4, this is the reason for a different definition of characteristic intervals here than in [14], in particular for step (2) of Definition 4.1.

**Theorem 4.2** ([20]). With the above definition (4.3) of $w_\ell$ for $1 \leq \ell \leq n + k - 1$ and the index $j^{(0)}$ given in Definition 4.1,

\begin{equation}
|w_{j^{(0)}}| \lesssim_k b_{j^{(0)}, j^{(0)}}.
\end{equation}

**Lemma 4.3** ([20]). Let $\mathcal{T}_n, \mathcal{T}_{n-1}$ be as above and $g$ be the function given in (1.1). Then, $f_n = g/\|g\|_2$ is the $L^2$-normalized orthogonal spline function corresponding to $(\mathcal{T}_n, \mathcal{T}_{n-1})$ and

$$\|f_n\|_{L^p(J_n)} \sim_k \|f_n\|_p \sim_k |J_n|^{1/p-1/2} \sim_k |J_n|^{1/2}\|g\|_p, \quad 1 \leq p \leq \infty,$$

where $J_n$ is the characteristic interval associated to $(\mathcal{T}_n, \mathcal{T}_{n-1})$.

By $d_n(x)$ we denote the number of points in $\mathcal{T}_n$ between $x$ and $J_n$ counting endpoints of $J_n$. Correspondingly, for an interval $V \subset [0, 1]$, by $d_n(V)$ we denote the number of points in $\mathcal{T}_n$ between $V$ and $J_n$ counting endpoints of both $J_n$ and $V$.

**Lemma 4.4** ([20]). Let $\mathcal{T}_n, \mathcal{T}_{n-1}$ be as above, $g = \sum_{j=1}^{n+k-1} w_j N_j$ be the function in (4.1) with $(w_j)_j^{n+k-1}$ as in (4.3) and $f_n = g/\|g\|_2$. Then, there exists a constant $0 < q < 1$ that depends only on $k$ such that

\begin{equation}
|w_j| \lesssim_k \frac{q^{d_n(x)}}{|J_n| + \text{dist}(\text{supp} N_j, J_n) + |D_{n,j}^k|} \quad \text{for all } 1 \leq j \leq n + k - 1.
\end{equation}

Moreover, if $x < \inf J_n$, we have

\begin{equation}
\|f_n\|_{L^p(0,x)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \text{dist}(x, J_n))^{1-1/p}}, \quad 1 \leq p \leq \infty.
\end{equation}

Similarly, for $x > \sup J_n$,

\begin{equation}
\|f_n\|_{L^p(x,1)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \text{dist}(x, J_n))^{1-1/p}}, \quad 1 \leq p \leq \infty.
\end{equation}
4.2. Combinatorics of characteristic intervals. Next, we recall a combinatorial result about the relative positions of different characteristic intervals:

**Lemma 4.5 ([20])**. Let \( x, y \in (t_n)_{n=0}^\infty \) such that \( x < y \). Then there exists a constant \( F_k \) only depending on \( k \) such that

\[
N_0 := \text{card}\{ n : J_n \subseteq [x, y], |J_n| \geq |[x, y]|/2 \} \leq F_k,
\]

where \( \text{card} \) denotes the cardinality of the set \( E \).

Similarly to [14] and [15], we need the following estimate involving characteristic intervals and orthonormal spline functions:

**Lemma 4.6.** Let \((t_n)\) be a \( k\)-admissible point sequence in \([0, 1]\) and let \((f_n)_{n \geq -k+2}\) be the corresponding orthonormal spline system of order \( k \). Then, for each interval \( V = [\alpha, \beta] \subset (0, 1) \),

\[
\sum_{n : J_n \subseteq V} |J_n|^{1/2} \int_V |f_n(t)| \, dt \lesssim_k |V|.
\]

Once we know the estimates for orthonormal spline functions as in Lemma 4.4 and the basic combinatorial result for their characteristic intervals, i.e. Lemma 4.5, this result follows by the same line of arguments that was used in the proof of Lemma 4.6 in [14], so we skip its proof.

Instead of Lemma 3.4 of [15], we will use the following:

**Lemma 4.7.** Let \((t_n)_{n=0}^\infty\) be a \( k\)-admissible knot sequence that satisfies the \((k-1)\)-regularity condition and let \( \Delta = D_{m,i}^{(k-1)} \) for some indices \( m \) and \( i \). For \( \ell \geq 0 \), let

\[
N(\Delta) := \{ n : \text{card}(\Delta \cap T_n) = k, J_n \subseteq \Delta \},
\]

\[
M(\Delta, \ell) := \{ n : d_n(\Delta) = \ell, \text{card}(\Delta \cap T_n) \geq k, |J_n \cap \Delta| = 0 \},
\]

where in both definitions we count the points in \( \Delta \cap T_n \) including multiplicities. Then,

\[
(4.9) \quad \frac{1}{|\Delta|} \sum_{n \in N(\Delta)} |J_n| \lesssim_k 1 \quad \text{and} \quad \sum_{n \in M(\Delta, \ell)} \frac{|J_n|}{\text{dist}(J_n, \Delta) + |\Delta|} \lesssim_{k, \gamma} (\ell + 1)^2.
\]

**Proof.** For every \( n \in N(\Delta) \), there are only the \( k - 1 \) possibilities \( D_{m,i}^{(1)}, \ldots, D_{m,i+k-2}^{(1)} \) for \( J_n \) and by Lemma 4.5 each interval \( D_{m,j}^{(1)}, j = i, \ldots, i + k - 2 \) occurs at most \( F_k \) times as a characteristic interval. This implies the first inequality in (4.9).

We now prove the second inequality in (4.9). To begin with, assume that each \( J_n, n \in M(\Delta, \ell) \) lies to the right of \( \Delta \), since the other case is covered by similar methods. The argument is split in two parts depending on the value of the parameter \( \ell \), beginning with \( \ell \leq k \). In that case, for \( n \in M(\Delta, \ell) \), let \( J_n^{1/2} \) be the unique interval determined by the conditions

\[
\sup J_n^{1/2} = \sup J_n, \quad |J_n^{1/2}| = |J_n|/2.
\]
Since \( d_n(\Delta) = \ell \) is constant, we group the occurring intervals \( J_n \) into packets, where all intervals in one packet have the same left endpoint and maximal intervals from different packets are disjoint (up to possibly one point). By Lemma \[4.3\], each point \( t \in [0,1] \) belongs to at most \( F_k \) intervals \( J_n^{1/2} \).

The \((k-1)\)-regularity and the fact that \( \ell \leq k \) now imply \( |J_n| \lesssim_{k,\gamma} |\Delta| \) and \( \text{dist}(\Delta, J_n) \lesssim_{k,\gamma} |\Delta| \) for \( n \in M(\Delta, \ell) \) and thus, every interval \( J_n \) for \( n \in M(\Delta, \ell) \) is a subset of a fixed interval whose length is comparable to \( |\Delta| \). So, putting these things together,

\[
\sum_{n \in M(\Delta, \ell)} \frac{|J_n|}{\text{dist}(J_n, \Delta) + |\Delta|} \leq \frac{1}{|\Delta|} \sum_{n \in M(\Delta, \ell)} |J_n| = \frac{2}{|\Delta|} \sum_{n \in M(\Delta, \ell)} \int_{J_n^{1/2}} dx \lesssim_{k,\gamma} 1,
\]

which completes the case of \( \ell \leq k \).

Next, assume \( \ell \geq k+1 \) and define \( (L_j)_{j=1}^\infty \) as the strictly decreasing sequence of all sets \( L \) that satisfy

\[
L = D^{(k-1)}_{n,i} \quad \text{and} \quad \sup L = \sup \Delta
\]

for some index \( n \) and \( i \). Moreover, set

\[
M_j(\Delta, \ell) := \{ n \in M(\Delta, \ell) : \text{card}(L_j \cap T_n) = k \},
\]
i.e., \( L_j \) is a union of \( k-1 \) grid point intervals in the grid \( T_n \). Then, since \( |\Delta| + \text{dist}(J_n, \Delta) \gtrsim_{\gamma} |\Delta| + \text{dist}(t, \Delta) \) for \( t \in J_n^{1/2} \) by \((k-1)\)-regularity,

\[
\sum_{n \in M_j(\Delta, \ell)} \frac{|J_n|}{\text{dist}(J_n, \Delta) + |\Delta|} \lesssim_{k,\gamma} \sum_{n \in M_j(\Delta, \ell)} \int_{J_n^{1/2}} \frac{1}{\text{dist}(t, \Delta) + |\Delta|} \, dt.
\]

If \( n \in M_j(\Delta, \ell) \) we get, again due to \((k-1)\)-regularity,

\[
\inf J_n^{1/2} \geq \inf J_n \geq \gamma^{-k} |L_j| + \sup \Delta,
\]

and

\[
\sup J_n^{1/2} \leq \inf J_n + |J_n| \lesssim_{k,\gamma} C_k \gamma^\ell |L_j| + \sup \Delta
\]

for some constant \( C_k \) only depending on \( k \). Combining this with Lemma \[4.3\] which implies that each point \( t \) belongs to at most \( F_k \) intervals \( J_n^{1/2} \),

\[
\sum_{n \in M_j(\Delta, \ell)} \int_{J_n^{1/2}} \frac{1}{\text{dist}(t, \Delta) + |\Delta|} \, dt \lesssim \int_{J_n^{1/2}} \frac{1}{\gamma^{-k}|L_j| + |\Delta|} \, \frac{1}{s} \, ds.
\]

Next we will show that the above intervals of integration can intersect at most for roughly \( \ell \) indices \( j \). Let \( j_2 \geq j_1 \), so that \( L_{j_1} \supset L_{j_2} \) and write \( j_2 = j_1 + 2kr + t \) with \( t \leq 2k-1 \). Then, by Lemma \[3.1\]

\[
C_k \gamma^\ell |L_{j_2}| \leq C_k \gamma^\ell |L_{j_1 + 2kr}| \leq C_k \gamma^\ell \eta^r |L_{j_1}|,
\]

where \( \eta = \gamma^{k-1}/(1 + \gamma^{k-1}) < 1 \). If now \( r \geq C_k \gamma^\ell \) for some suitable constant \( C_{k,\gamma} \) depending only on \( k \) and \( \gamma \), we have

\[
C_k \gamma^\ell |L_{j_2}| \leq \gamma^{-k} |L_{j_1}|.
\]
Thus, each point $s$ in the integral in (4.10) for some $j$ belongs to at most $C_{k, \gamma \ell}$ intervals $[\gamma^{-k}|L_j| + |\Delta|, C_{k, \gamma \ell}|L_j| + |\Delta|]$ where $j$ is varying. So we conclude by summing over $j$

$$
\sum_{n \in M(\Delta, \ell)} \frac{|J_n|}{\text{dist}(J_n, \Delta) + |\Delta|} \leq C_{k, \gamma \ell} \int_{|\Delta|}^{(1+C_{k, \gamma \ell})|\Delta|} \frac{1}{s} \, ds \leq C_{k, \gamma \ell}^2.
$$

This completes the analysis of the case $\ell \geq k + 1$ and thus, the proof of the lemma is finished. $\square$

5. Four conditions on spline series and their relations

Let $(t_n)$ be a $k$-admissible sequence of knots with the corresponding orthonormal spline system $(f_n)_{n \geq -k+2}$. For a sequence $(a_n)_{n \geq -k+2}$ of coefficients, let

$$
P := \left( \sum_{n=-k+2}^{\infty} a_n^2 f_n^2 \right)^{1/2} \quad \text{and} \quad S := \max_{m \geq -k+2} \left| \sum_{n=-k+2}^{m} a_n f_n \right|.
$$

If $f \in L^1$, we denote by $Pf$ and $Sf$ the functions $P$ and $S$ corresponding to the coefficient sequence $a_n = \langle f, f_n \rangle$, respectively. Consider the following conditions:

(A) $P \in L^1$,
(B) The series $\sum_{n=-k+2}^{\infty} a_n f_n$ converges unconditionally in $L^1$,
(C) $S \in L^1$,
(D) There exists a function $f \in H^1$ such that $a_n = \langle f, f_n \rangle$.

We will discuss the relations between those four conditions and we will prove the implications indicated in the subsequent picture, where some results need certain regularity conditions imposed on the point sequence $(t_n)$, which is also indicated in the image.

Let us recall that in case of orthonormal spline systems with dyadic knots, the relations (and equivalences) of these conditions have been studied by several authors, also in case $p < 1$, see e.g. \cite{23, 1, 12}. For general Franklin systems corresponding to arbitrary sequences of knots, the relations of these
conditions were discussed in \cite{15} (and earlier in \cite{13}, also for \(p < 1\), but for a restricted class of point sequences). In the sequel, we follow the approach from \cite{15} and we adapt it to the case of spline orthonormal systems of order \(k\).

We begin with the implication \((B) \Rightarrow (A)\), which is a consequence of Khinchin’s inequality:

**Proposition 5.1** ((B) \(\Rightarrow\) (A)). Let \((t_n)\) be a \(k\)-admissible sequence of knots with the corresponding general orthonormal spline system \((f_n)\) and let \((a_n)\) be a sequence of coefficients. If the series \(\sum_{n=-k+2}^{\infty} a_n f_n\) converges unconditionally in \(L^1\), then \(P \in L^1\). Moreover,

\[
\|P\|_1 \leq \sup_{\varepsilon \in \{-1,1\}^2} \left\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \right\|_1.
\]

Next, we investigate the implications \((A) \Rightarrow (B)\) and \((A) \Rightarrow (C)\). Let us note that once we know the estimates and combinatorial results of Sections 3 and 4, the proof is the same as the proof of Proposition 4.3 in \cite{15}, so we just state the result.

**Proposition 5.2** ((A) \(\Rightarrow\) (B) and (A) \(\Rightarrow\) (C)). Let \((t_n)\) be a \(k\)-admissible sequence of knots and let \((a_n)\) be a sequence of coefficients such that \(P \in L^1\). Then, \(S \in L^1\) and \(\sum a_n f_n\) converges unconditionally in \(L^1\); moreover,

\[
\sup_{\varepsilon \in \{-1,1\}^2} \left\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \right\| \lesssim_k \|P\|_1 \quad \text{and} \quad \|S\|_1 \lesssim_k \|P\|_1.
\]

Next we discuss \((D) \Rightarrow (A)\).

**Proposition 5.3** ((D) \(\Rightarrow\) (A)). Let \((t_n)\) be a \(k\)-admissible point sequence that satisfies the \((k-1)\)-regularity condition with parameter \(\gamma\). Then there exists a constant \(C_{k,\gamma}\) depending only on \(k\) and \(\gamma\) such that for each atom \(\phi\),

\[
\|P\phi\|_1 \leq C_{k,\gamma}.
\]

Consequently, if \(f \in H^1\), then

\[
\|Pf\|_1 \leq C_{k,\gamma}\|f\|_{H^1}.
\]

Before we proceed with the proof, let us remark that essentially the same arguments give a direct proof of \((D) \Rightarrow (C)\), under the same assumption of \((k-1)\)-regularity of the sequence of points \((t_n)\), and moreover

\[
\|Sf\|_1 \leq C_{k,\gamma}\|f\|_{H^1}.
\]

We do not present it here, since we have the implications \((D) \Rightarrow (A)\) under the assumption of \((k-1)\)-regularity and \((A) \Rightarrow (C)\) under the assumption of \(k\)-admissibility only. Note that Proposition 6.1 in Section 6 shows that – without the assumption of \((k-1)\)-regularity of the point sequence – the implications \((D) \Rightarrow (A)\) and \((D) \Rightarrow (C)\) need not be true.
Proof of Proposition 5.3 Let \( \phi \) be an atom with \( \int_0^1 \phi(u) \, du = 0 \) and let \( \Gamma = [\alpha, \beta] \) be an interval such that \( \text{supp} \phi \subset \Gamma \) and \( \sup |\phi| \leq |\Gamma|^{-1} \). Define \( n_\Gamma := \max \{ n : \text{card}(T_n \cap \Gamma) \leq k - 1 \} \), where in the maximum, we also count multiplicities of knots. It will be shown that

\[
\| P_1 \phi \|_1, \| P_2 \phi \|_1 \lesssim_{\gamma, k} 1,
\]

where

\[
P_1 \phi = \left( \sum_{n \leq n_\Gamma} a_n^2 f_n^2 \right)^{1/2} \quad \text{and} \quad P_2 \phi = \left( \sum_{n > n_\Gamma} a_n^2 f_n^2 \right)^{1/2}.
\]

First, we consider \( P_1 \) and prove the stronger inequality

\[
\sum_{n \leq n_\Gamma} |a_n| \| f_n \|_1 \lesssim_{k, \gamma} 1,
\]

where \( a_n = \langle \phi, f_n \rangle \). For each parameter \( n \leq n_\Gamma \), we define \( \Gamma_{n, \alpha} \) as the unique closed interval \( D_{n,j}^{(k-1)} \) with minimal index \( j \) such that

\[
\alpha \leq \min D_{n,j}^{(k-1)} + 1.
\]

We note that \( \Gamma_{n, \alpha} \) satisfies

\[
\Gamma_{n_1, \alpha} \supseteq \Gamma_{n_2, \alpha} \quad \text{for } n_1 \leq n_2,
\]

and, by \( (k - 1) \)-regularity,

\[
|\Gamma_{n, \alpha}| \gtrsim_{\gamma, k} |\Gamma|.
\]

Let \( g_n = \sum_{j=1}^{n+k-1} w_j N_{n,j}^{(k)} \) be the unnormalized orthogonal spline function as in (4.1) and the coefficients \( (w_j) \) as in (4.3). For \( \xi \in \Gamma \), we have (cf. (3.1))

\[
|g_n'(\xi)| \lesssim_k \sum_j \frac{|w_j| + |w_{j-1}|}{|D_{n,j}^{(k-1)}|},
\]

where we sum only over those indices \( j \) such that \( \Gamma \cap \text{supp} N_{n,j}^{(k-1)} = \Gamma \cap D_{n,j}^{(k-1)} \neq \emptyset \). By \( (k - 1) \)-regularity, all lengths \( |D_{n,j}^{(k-1)}| \) in this summation range are comparable to \( |\Gamma_{n, \alpha}| \). Moreover, by (4.6),

\[
|w_j| \lesssim_k \frac{q_{d_n(\tau_{n,j})}}{|J_n| + \text{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}|}.
\]

Again by \( (k - 1) \)-regularity, for \( j \) in the summation range of the sum (5.1),

\[
|D_{n,j}^{(k-1)}| \gtrsim_{k, \gamma} |\Gamma_{n, \alpha}|,
\]

\[
\text{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}| \gtrsim_{k, \gamma} \text{dist}(J_n, \Gamma_{n, \alpha}) + |\Gamma_{n, \alpha}|.
\]

Therefore, combining the above inequalities, we estimate the right hand side in (5.1) further and get, with the notation \( \Gamma_n := \Gamma_{n, \alpha} \),

\[
|g_n'(\xi)| \lesssim_{k, \gamma} \frac{1}{|\Gamma_n| |J_n| + \text{dist}(J_n, \Gamma_n) + |\Gamma_n|} \frac{q_{d_n(\Gamma_n)}}{|J_n| + \text{dist}(J_n, \Gamma_n) + |\Gamma_n|}.
\]
As a consequence, for an arbitrary point \( \tau \in \Gamma \),
\[
|a_n| = \left| \int_\Gamma \phi(t)[f_n(t) - f_n(\tau)] \, dt \right| \leq \int_\Gamma \left| \frac{1}{|\Gamma|} \sup_{\xi \in \Gamma} |f_n'(\xi)||t - \tau| \right| \, dt
\]
\[
\lesssim_k |\Gamma| |J_n|^{1/2} \sup_{\xi \in \Gamma} |g_n'(\xi)| \lesssim_{k,\gamma} \frac{|\Gamma| |J_n|^{1/2} q,\gamma_n}{|J_n|} + \text{dist}(J_n, \Delta_1) + |\Delta_1|.
\]

Let \( \Delta_1 \supseteq \cdots \supseteq \Delta_s \) be the collection of all different intervals appearing as \( \Gamma_n \) for \( n \leq n_{\Gamma} \). By Lemma \ref{lemma:3.1}, we have some geometric decay in the measure of \( \Delta_i \). Now fix \( \Delta_i \) and \( \ell \geq 0 \) and consider indices \( n \leq n_{\Gamma} \) such that \( \Gamma_n = \Delta_i \) and \( d_n(\Gamma_n) = \ell \). By the latter display and Lemma \ref{lemma:4.3},
\[
|a_n||f_n|| \lesssim_{k,\gamma} \frac{|\Gamma|}{|\Delta_i|} |J_n|q,\gamma_n + \text{dist}(J_n, \Delta_1) + |\Delta_1|,
\]
and thus, Lemma \ref{lemma:4.7} implies
\[
\sum_{\Gamma_n = \Delta_i, d_n(\Gamma_n) = \ell} |a_n||f_n|| \lesssim_{k,\gamma} (\ell + 1)^2 q,\gamma_n \frac{|\Gamma|}{|\Delta_i|}.
\]

Now, summing over \( \ell \) and then over \( i \) (recall that \( |\Delta_i| \) decays like a geometric progression by Lemma \ref{lemma:3.1} and \( |\Delta_i| \gtrsim_{k,\gamma} |\Gamma| \) since \( n \leq n_{\Gamma} \)) yields
\[
\sum_{n \leq n_{\Gamma}} |a_n||f_n|| \lesssim_{k,\gamma} 1.
\]

This implies the desired inequality \( \|P_1 \phi\|_1 \lesssim_{k,\gamma} 1 \) for the first part of \( P \phi \).

Next, we look at \( P_2 \phi \) and define the set \( V \) as the smallest interval that has grid points in \( T_{n_{\Gamma}+1} \) as endpoints and which contains \( \Gamma \). Moreover, \( \tilde{V} \) is defined to be the smallest interval with gridpoints in \( T_{n_{\Gamma}+1} \) as endpoints and such that \( \tilde{V} \) contains \( k \) grid points in \( T_{n_{\Gamma}+1} \) to the left of \( \Gamma \) and as well \( k \) grid points in \( T_{n_{\Gamma}+1} \) to the right of \( \Gamma \). We observe that due to \( (k - 1) \)-regularity and the fact that \( \Gamma \) contains at least \( k \) gridpoints from \( T_{n_{\Gamma}+1} \),
\[
|V| \sim_{k,\gamma} |\tilde{V}| \sim_{k,\gamma} |\Gamma|,
\]
\[
|(\tilde{V} \setminus V) \cap [0, \inf \Gamma]| \sim_{k,\gamma} |(\tilde{V} \setminus V) \cap [\sup \Gamma, 1]| \sim_{k,\gamma} |\tilde{V}|.
\]

First, we consider the integral of \( P_2 \phi \) over the set \( \tilde{V} \) and obtain by the Cauchy-Schwarz inequality
\[
\int_{\tilde{V}} \left\| P_2 \phi(t) \right\|_2 \leq \left\| \mathbf{1}_{\tilde{V}} \right\|_2 \left\| \phi \right\|_2 \leq \frac{1}{|\tilde{V}|^{1/2}} \lesssim_{k,\gamma} 1.
\]

It remains to estimate \( \int_{\tilde{V}} \left\| P_2 \phi(t) \right\|_2 \). Since for \( n > n_{\Gamma} \), the endpoints of \( \tilde{V} \) are grid points in \( T_n \), for \( J_n \) there are only the possibilities \( J_n \subset \tilde{V}, J_n \) is to the right or \( J_n \) is to the left of \( \tilde{V} \). We begin with considering \( J_n \subset \tilde{V} \), in which case
\[
|a_n| = \left| \int_\Gamma \phi(t)f_n(t) \, dt \right| \leq \frac{\|f_n\|_1}{|\Gamma|} \lesssim_k \frac{|J_n|^{1/2}}{|\Gamma|}.
\]
and therefore, by Lemma 4.6 and (5.3),

$$\sum_{n : J_n \in V, n > n_G} |a_n| \int_{\tilde{V}^c} |f_n(t)| \, dt \lesssim_k \frac{1}{|\Gamma|} \sum_{n : J_n \in \tilde{V}} |J_n|^{1/2} \int_{\tilde{V}^c} |f_n(t)| \, dt \lesssim_k \frac{1}{|\Gamma|} \lesssim_k 1.$$

Now, let $J_n$ be to the right of $\tilde{V}$. The case of $J_n$ to the left of $\tilde{V}$ is then considered similarly. By (4.7) for $p = \infty$,

$$|a_n| \leq \frac{1}{|\Gamma|} \int_{\tilde{V}} |f_n(t)| \, dt \leq \frac{1}{|\Gamma|} \int_{\tilde{V}} |f_n(t)| \, dt \lesssim_k \frac{q^{d_n(V)} |J_n|^{1/2}}{\text{dist}(\tilde{V}, J_n) + |J_n|}.$$

This inequality, Lemma 4.3 and the fact that $\text{dist}(V, J_n) \gtrsim_{k, \gamma} \text{dist}(V, J_n) + |V|$ (cf. (5.3)) allow us to deduce

$$\sum_{n > n_G} |a_n| \parallel f_n \parallel_1 \lesssim_{k, \gamma} \sum_{n > n_G} \frac{q^{d_n(V)} |J_n|}{\text{dist}(\tilde{V}, J_n) + |\tilde{V}|}.$$  

Note that $V$ can be a union of $k - 1$, $k$ or $k + 1$ intervals from $\mathcal{T}_{n_{k+1}}$; therefore, let $V^+$ be a union of $k - 1$ grid intervals form $\mathcal{T}_{n_{k+1}}$, with right endpoint of $V^+$ coinciding with the right endpoint of $V$. Then, if $J_n$ is to the right of $V$ then $d_n(V) = d_n(V^+)$, $\text{dist}(V, J_n) = \text{dist}(V^+, J_n)$ and – because of $(k - 1)$-regularity of the point sequence $\infty \sim_{k, \gamma} |V^+|$, which implies

$$\sum_{n > n_G} \frac{q^{d_n(V^+)} |J_n|}{\text{dist}(\tilde{V}, J_n) + |\tilde{V}|} \lesssim_{k, \gamma} \sum_{n > n_G} \frac{q^{d_n(V^+)} |J_n|}{\text{dist}(V^+, J_n) + |V^+|}.$$  

Finally, we employ Lemma 4.7 to conclude

$$\sum_{n > n_G} |a_n| \parallel f_n \parallel_1 \lesssim_{k, \gamma} \sum_{\ell = 0}^{\infty} q^\ell \sum_{d_n(V^+) = \ell} \frac{|J_n|}{\text{dist}(V^+, J_n) + |V^+|} \lesssim_{k, \gamma} \sum_{\ell = 0}^{\infty} (\ell + 1)^2 q^\ell \lesssim_k 1.$$

To conclude the proof, note that if $f \in H^1$ and $f = \sum_{m=1}^{\infty} c_m \phi_m$ is an atomic decomposition of $f$, then $\langle f, f_n \rangle = \sum_{m=1}^{\infty} c_m \langle \phi_m, f_n \rangle$, and $Pf(t) \leq \sum_{m=1}^{\infty} |c_m| P \phi_m(t)$.  

Finally, we discuss (C) \Rightarrow (D).

**Proposition 5.4 ((C) \Rightarrow (D)).** Let $(t_n)$ be a $k$-admissible sequence of knots in $[0, 1]$ satisfying the $k$-regularity condition with parameter $\gamma$ and let $(a_n)$ be a sequence of coefficients such that $S = \sup_m \{ \sum_{n \leq m} a_n f_n \} \in L^1$. Then, there exists a function $f \in H^1$ with $a_n = \langle f, f_n \rangle$ for each $n$. Moreover, we have the inequality

$$\|f\|_{H^1} \lesssim_{k, \gamma} \|Sf\|_1.$$
Proof. If $S \in L^1$, then there is a function $f \in L^1$ such that $f = \sum_{n \geq -k+2} a_n f_n$ with convergence in $L^1$. Indeed, this is a consequence of the relative weak compactness of uniformly integrable subsets in $L^1$ and the basis property of $(f_n)$ in $L^1$. Thus, we need only show that $f \in H^1$ and this is done by finding a suitable atomic decomposition of $f$.

We define $E_0 = B_0 = [0,1]$ and, for $r \geq 1$,

$$E_r = [S > 2^r], \quad B_r = [\mathcal{M} 1_{E_r} > c_{k,\gamma}],$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function and $0 < c_{k,\gamma} \leq 1/2$ is a small constant only depending on $k$ and $\gamma$ which is chosen according to a few restrictions that will be given during the proof. We note that

$$\mathcal{M} 1_{E_r} (t) = \sup_{I \ni t} \frac{|I \cap E_r|}{|I|}, \quad t \in [0,1],$$

where the supremum is taken over all intervals containing the point $t$. Since the Hardy-Littlewood maximal function operator $\mathcal{M}$ is of weak type $(1,1)$, we have the inequality $|B_r| \lesssim_{k,\gamma} |E_r|$. Due to the fact that $S \in L^1$, $|E_r| \to 0$ and, as a consequence, $|B_r| \to 0$ as $r \to \infty$. Now, decompose the open set $B_r$ into a countable union of disjoint open intervals

$$B_r = \bigcup_{\kappa} \Gamma_{r,\kappa},$$

where for fixed $r$, no two intervals $\Gamma_{r,\kappa}$ have a common endpoint and the above equality is up to measure $0$ (each open set of real numbers can be decomposed into a countable union of open intervals, but it can happen that two intervals have the same endpoint. In that case, we collect those two intervals to one $\Gamma_{r,\kappa}$). This can be achieved by taking as $\Gamma_{r,\kappa}$ the collection of level sets of positive measure of the function $t \to |[0,t] \cap B^c_r|$.

Moreover, observe that if $\Gamma_{r+1,\xi}$ is one of the intervals in the decomposition of $B_{r+1}$, then there is an interval $\Gamma_{r,\kappa}$ in the decomposition of $B_r$ such that $\Gamma_{r+1,\xi} \subset \Gamma_{r,\kappa}$.

Based on this decomposition, we define the following functions for $r \geq 0$:

$$g_r(t) := \begin{cases} f(t), & t \in B_r^c, \\ \frac{1}{|\Gamma_{r,\kappa}|} \int_{\Gamma_{r,\kappa}} f(t) \, dt, & t \in \Gamma_{r,\kappa}. \end{cases}$$

Observe that $f = g_0 + \sum_{r=0}^{\infty} (g_{r+1} - g_r)$ in $L^1$ and $g_{r+1} - g_r = 0$ on $B_r^c$. As a consequence,

$$\int_{\Gamma_{r,\kappa}} g_{r+1}(t) \, dt = \int_{\Gamma_{r,\kappa} \cap B^c_{r+1}} g_{r+1}(t) \, dt + \int_{\Gamma_{r,\kappa} \cap B_{r+1}} g_{r+1}(t) \, dt$$

$$= \int_{\Gamma_{r,\kappa} \cap B^c_{r+1}} f(t) \, dt + \sum_{\xi : \Gamma_{r+1,\xi} \cap \Gamma_{r,\kappa}} \int_{\Gamma_{r+1,\xi}} f(t) \, dt$$

$$= \int_{\Gamma_{r,\kappa}} f(t) \, dt = \int_{\Gamma_{r,\kappa}} g_r(t) \, dt.$$

The main step of the proof is to show that

$$|g_r(t)| \leq C_{k,\gamma} 2^r, \quad \text{a.e. } t \in [0,1]$$

(5.4)
for some constant $C_{k,\gamma}$ only depending on $k$ and $\gamma$. Once this inequality is proved, we take $\phi_0 \equiv 1$, $\eta_0 = \int_0^1 f(u) \, du$ and

$$\phi_{r,\kappa} := \frac{(g_{r+1} - g_r) \mathbb{1}_{r,\kappa}}{C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|}, \quad \eta_{r,\kappa} = C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|$$

and observe that $f = \eta_0 \phi_0 + \sum_{r,\kappa} \eta_{r,\kappa} \phi_{r,\kappa}$ is the desired atomic decomposition of $f$ since

$$\sum_{r,\kappa} \eta_{r,\kappa} \leq C_{k,\gamma} \sum_{r,\kappa} 2^r |\Gamma_{r,\kappa}| = C_{k,\gamma} \sum_r 2^r |B_r| \lesssim_{k,\gamma} 2^r |E_r| \lesssim \|f\|_1.$$ 

Thus it remains to prove inequality (5.4).

In order to do this, we first assume $t \in B_r^c$. Additionally, assume that $t$ is a point such that the series $\sum_n a_n f_n(t)$ converges to $f(t)$ and that $t$ does not occur in the grid point sequence $(t_n)$. By Theorem 3.5, this holds for a.e. point in $[0, 1]$. We fix the index $m$ and let $V_m$ be the maximal interval where the function $S_m := \sum_{n \leq m} a_n f_n$ is a polynomial of order $k$ that contains the point $t$. Then, $V_m \not\subset B_r$, and since $V_m$ is an interval containing the point $t$,

$$|V_m \cap E_r^c| \geq (1 - c_{k,\gamma}) |V_m| \geq |V_m|/2.$$ 

Since $|S_m| \leq 2^r$ on $E_r^c$, the above display and Proposition 3.2 imply $|S_m| \lesssim_k 2^r$ on $V_m$ and in particular $|S_m(t)| \lesssim_k 2^r$. Now, $S_m(t) \to f(t)$ as $m \to \infty$ by the assumptions on $t$ and thus,

$$|g_r(t)| = |f(t)| \lesssim_k 2^r.$$ 

This concludes the proof of (5.4) in the case of $t \in B_r^c$.

Next, we fix an index $\kappa$ and consider $g_r$ on $\Gamma := [\alpha, \beta] := \Gamma_{r,\kappa}$. Let $n_\Gamma$ be the first index such that there are $k + 1$ points from $T_{n_\Gamma}$ contained in $\Gamma$, i.e., there exists a support $L_{n_\Gamma,i}^{(k)}$ of a B-spline function of order $k$ in the grid $T_{n_\Gamma}$ that is contained in $\Gamma$. Additionally, we define the intervals

$$U_0 := [\tau_{n_\Gamma,i-k}, \tau_{n_\Gamma,i}], \quad W_0 := [\tau_{n_\Gamma,i+k}, \tau_{n_\Gamma,i+2k}].$$

Note that if $\alpha \in T_{n_\Gamma}$, then $\alpha$ is a common endpoint of $U_0$ and $\Gamma$, otherwise $\alpha$ is an interior point of $U_0$. Similarly, if $\beta \in T_{n_\Gamma}$, then $\beta$ is a common endpoint of $W_0$ and $\Gamma$, otherwise $\beta$ is an interior point of $W_0$. By $k$-regularity of the point sequence $(t_n)$, $\max(|U_0|, |W_0|) \lesssim_{k,\gamma} |\Gamma|$. We first estimate the part $S_\Gamma := \sum_{n \leq n_\Gamma} a_n f_n$ and show the inequality $|S_\Gamma| \lesssim_{k,\gamma} 2^r$ on $\Gamma$. Observe that on $\Delta := \overline{U_0} \cup \Gamma \cup W_0$, $S_\Gamma$ can be represented as a linear combination of B-splines $(N_j)$ on the grid $T_{n_\Gamma}$ of the form

$$S_\Gamma(t) = h(t) := \sum_{j=i-2k+1}^{i+2k-1} b_j N_j(t),$$

for some coefficients $(b_j)$. For $j = i - 2k + 1, \ldots, i + 2k - 1$, let $J_{j}$ be a maximal interval of $\supp N_j$ and observe that due to $k$-regularity, $|J_{j}| \sim_{k,\gamma} |\Gamma| \sim_{k,\gamma} |\supp h|$. 
If we assume that \( \max_{j} |S_{\Gamma}| > C_{k}2^{r} \), where \( C_{k} \) is the constant obtained from Proposition 3.2 with \( \rho = 1/2 \), then Proposition 3.2 implies that \( |S_{\Gamma}| > 2^{r} \) on a subset \( I_{j} \) of \( J_{j} \) with measure \( \geq |J_{j}|/2 \). As a consequence,

\[
|\text{supp } h \cap E_{\Gamma}| \geq |J_{j} \cap E_{\Gamma}| \geq |J_{j}|/2 \geq k_{,\gamma} |\text{supp } h|.
\]

We choose the constant \( c_{k,\gamma} \) in the definition of \( B_{r} \) sufficiently small to guarantee that this last inequality implies \( \text{supp } h \subset B_{r} \). This is a contradiction to the choice of \( \Gamma \), which implies that our assumption \( \max_{j} |S_{\Gamma}| > C_{k}2^{r} \) is not true and thus

\[
\max_{j} |S_{\Gamma}| \leq C_{k}2^{r}, \quad j = i - 2k + 1, \ldots, i + 2k - 1.
\]

By local stability of B-splines, i.e., inequality (3.2) in Proposition 3.3, this implies

\[
|b_{j}| \lesssim k 2^{r}, \quad j = i - 2k + 1, \ldots, i + 2k - 1,
\]

and so \( |S_{\Gamma}| \lesssim k 2^{r} \) on \( \Delta \). This means

\[
(5.5) \quad \int_{\Gamma} |S_{\Gamma}| \lesssim k 2^{r}|\Gamma|,
\]

which is inequality (5.4) for the part \( S_{\Gamma} \).

In order to estimate the remaining part, we inductively define two sequences \((u_{s}, U_{s})_{s \geq 0}\) and \((w_{s}, W_{s})_{s \geq 0}\) consisting of integers and intervals. Set \( u_{0} = w_{0} = n_{\Gamma} \) and inductively define \( u_{s+1} \) to be the first index \( n > u_{s} \) such that \( t_{n} \in U_{s} \). Moreover, define \( U_{s+1} \) to be the B-spline support \( D_{u_{s+1},i}^{(k)} \) in the grid \( T_{u_{s+1}} \), where \( i \) is the minimal index such that \( D_{u_{s+1},i}^{(k)} \cap \Gamma \neq \emptyset \). Similarly, we define \( w_{s+1} \) to be the first index \( n > w_{s} \) such that \( t_{n} \in W_{s} \) and \( W_{s+1} \) as the B-spline support \( D_{w_{s+1},i}^{(k)} \) in the grid \( T_{w_{s+1}} \), where \( i \) is the maximal index such that \( D_{w_{s+1},i}^{(k)} \cap \Gamma \neq \emptyset \). It can easily be seen that this construction implies \( U_{s+1} \subset U_{s}, W_{s+1} \subset W_{s} \) and \( \alpha \in U_{s}, \beta \in W_{s} \) for all \( s \geq 0 \), or more precisely: if \( \alpha \in T_{u_{s}} \), then \( \alpha \) is a common endpoint of \( U_{s} \) and \( \Gamma \), otherwise \( \alpha \) is an inner point of \( U_{s} \), and similarly, if \( \beta \in T_{w_{s}} \), then \( \beta \) is a common endpoint of \( W_{s} \) and \( \Gamma \), otherwise \( \beta \) is an inner point of \( W_{s} \).

For a pair of indices \( \ell, m \), let

\[
x_{\ell} := \sum_{\nu=0}^{k-1} N_{u_{\ell},i+\nu} \mathbb{1}_{U_{\ell}}, \quad y_{m} := \sum_{\nu=0}^{k-1} N_{w_{m},j-\nu} \mathbb{1}_{W_{m}},
\]

where \( N_{u_{\ell},i} \) is the B-spline function on the grid \( T_{u_{\ell}} \) with support \( U_{\ell} \) and \( N_{w_{m},j} \) is the B-spline function on the grid \( T_{w_{m}} \) with support \( W_{m} \). The function

\[
\phi_{\ell,m} := x_{\ell} + \mathbb{1}_{\Gamma \setminus (U_{\ell} \cup W_{m})} + y_{m}
\]

is zero on \((U_{\ell} \cup \Gamma \cup W_{m})^c\), one on \( \Gamma \setminus (U_{\ell} \cup W_{m}) \) and a piecewise polynomial function of order \( k \) in between. For \( \ell, m \geq 0 \), consider the following subsets of \( \{n : n > n_{\Gamma}\} \):

\[
L(\ell) := \{n : u_{\ell} < n \leq u_{\ell+1}\}, \quad R(m) := \{n : w_{m} < n \leq w_{m+1}\}.
\]
If \( n \in L(\ell) \cap R(m) \), we clearly have \( \langle f_n, \phi_{\ell,m} \rangle = 0 \) and thus

\[
\int_{\Gamma} f_n(t) \, dt = \int_{\Gamma} f_n(t) \, dt - \int_{0}^{1} f_n(t) \phi_{\ell,m}(t) \, dt = A_\ell(f_n) + B_m(f_n),
\]

where

\[
A_\ell(f_n) := \int_{\Gamma \cap U_\ell} f_n(t) \, dt - \int_{U_\ell} f_n(t) x_\ell(t) \, dt,
\]

\[
B_m(f_n) := \int_{\Gamma \cap W_m} f_n(t) \, dt - \int_{W_m} f_n(t) y_m(t) \, dt.
\]

This implies

\[
|\int_{\Gamma} \sum_{n=\gamma+1}^{\infty} a_n f_n(t) \, dt| = \left| \sum_{\ell,m=0}^{\infty} \sum_{n \in L(\ell) \cap R(m)} a_n (A_\ell(f_n) + B_m(f_n)) \right| 
\leq 2 \sum_{\ell=0}^{\infty} \int_{U_\ell} \left| \sum_{n \in L(\ell)} a_n f_n(t) \right| \, dt + 2 \sum_{m=0}^{\infty} \int_{W_m} \left| \sum_{n \in R(m)} a_n f_n(t) \right| \, dt.
\]

Consider the first sum on the right hand side. On \( U_\ell = D_{u_\ell,i}^{(k)} \), the function \( \sum_{n \in L(\ell)} a_n f_n \) can be represented as a linear combination of B-splines \((N_j)\) on the grid \( T_{u_\ell} \) of the form

\[
\sum_{n \in L(\ell)} a_n f_n = h_\ell := \sum_{j=i-k+1}^{i+k-1} b_j N_j,
\]

for some coefficients \((b_j)\). For \( j = i - k + 1, \ldots, i + k - 1 \), let \( J_j \) be a maximal grid interval of \( \text{supp} N_j \) and observe that due to \( k \)-regularity, \( |J_j| \sim_{k,\gamma} |U_\ell| \sim_{k,\gamma} |\text{supp} h_\ell| \). On \( J_j \), the function \( \sum_{n \in L(\ell)} a_n f_n \) is a polynomial of order \( k \). If we assume \( \max_{J_j} \left| \sum_{n \in L(\ell)} a_n f_n \right| > C_k 2^{r+1} \), where \( C_k \) is the constant obtained from Proposition 3.2 with \( \rho = 1/2 \), then Proposition 3.2 implies that \( \left| \sum_{n \in L(\ell)} a_n f_n \right| > 2^{r+1} \) on a set \( J_j^* \subset J_j \) with \( |J_j^*| = |J_j|/2 \), but this means \( \max \left( \left| \sum_{n \leq u_i} a_n f_n \right|, \left| \sum_{n \leq u_{i+1}} a_n f_n \right| \right) > 2^r \) on the set \( J_j^* \). As a consequence,

\[
|E_\ell \cap \text{supp } h_\ell| \geq |E_\ell \cap J_j| \geq |J_j^*| \geq |J_j|/2 \gtrsim_k |\text{supp } h_\ell|.
\]

We choose the constant \( c_{k,\gamma} \) in the definition of \( B_r \) sufficiently small to guarantee that this last inequality implies \( \text{supp } h_\ell \subset B_r \). This is a contradiction to the choice of \( \Gamma \), which implies that our assumption \( \max_{J_j} \left| \sum_{n \in L(\ell)} a_n f_n \right| > C_k 2^{r} \) is not true and thus

\[
\max_{J_j} \left| \sum_{n \in L(\ell)} a_n f_n \right| \leq C_k 2^r, \quad j = i - k + 1, \ldots, i + k - 1.
\]

By local stability of B-splines, i.e., inequality 3.2 in Proposition 3.3, this implies

\[
|b_j| \gtrsim_k 2^{r}, \quad j = i - k + 1, \ldots, i + k - 1,
\]
and so \( \left| \sum_{n \in L(\ell)} a_n f_n \right| \lesssim_k 2^r \) on \( U_\ell \), which gives

\[
\int_{U_\ell} \left| \sum_{n \in L(\ell)} a_n f_n \right| \lesssim_k 2^r |U_\ell|.
\]

Combining Lemma 3.1, the inclusions \( U_{\ell+1} \subset U_\ell \) and the inequality \( |U_0| \lesssim_{k,\gamma} |\Gamma| \), we see that \( \sum_{\ell=0}^\infty |U_\ell| \lesssim_{k,\gamma} |\Gamma| \). Thus we get

\[
\sum_{\ell=0}^\infty \int_{U_\ell} \left| \sum_{n \in L(\ell)} a_n f_n \right| \lesssim_{k,\gamma} 2^r |\Gamma|.
\]

The second sum on the right hand side of (5.7) is estimated similarly which gives

\[
\sum_{m=0}^\infty \int_{W_m} \left| \sum_{n \in R(m)} a_n f_n \right| \lesssim_{k,\gamma} 2^r |\Gamma|.
\]

Combining these estimates with (5.7) and (5.5), we find

\[
\left| \int_\Gamma f(t) \, dt \right| = \left| \int_\Gamma \sum_n a_n f_n(t) \, dt \right| \lesssim_{k,\gamma} 2^r |\Gamma|,
\]

which implies (5.4) on \( \Gamma \) and therefore, the proof of the proposition is completed. \( \Box \)

6. Proof of the main theorem

For the proof of the necessity part of Theorem 2.4, we will use the following:

**Proposition 6.1.** Let \((t_n)\) be a \( k\)-admissible sequence of knots satisfying the \( k\)-regularity condition with parameter \( \gamma \), but which is not satisfying any \((k - 1)\)-regularity condition. Then,

\[
\sup_n \| \| \sup_n |a_n(\phi) f_n| \|_1 = \infty,
\]

where \( \sup \) is taken over all atoms \( \phi \) and \( a_n(\phi) := \langle \phi, f_n \rangle \).

Proposition 6.1 implies in particular that Proposition 5.3 cannot be extended to arbitrary partitions. For the proof of Proposition 6.1 we need the following technical Lemma 6.2.

**Lemma 6.2.** Let \((t_n)\) be a \( k\)-admissible sequence of knots satisfying the \( k\)-regularity condition with parameter \( \gamma \geq 1 \), but is not satisfying any \((k - 1)\)-regularity condition. Let \( \ell \) be an arbitrary positive integer. Then, for all \( A \geq 2 \), there exists a finite increasing sequence \((n_j)_{j=0}^{\ell - 1}\) of length \( \ell \) such that if \( \tau_{n_j,i_j} \) is the new point in \( T_{n_j} \) that is not present in \( T_{n_j-1} \) and

\[
\Lambda_j := [\tau_{n_j,i_j-k}, \tau_{n_j,i_j-1}), \quad L_j := [\tau_{n_j,i_j-1}, \tau_{n_j,i_j}), \quad R_j := [\tau_{n_j,i_j}, \tau_{n_j,i_j+1}),
\]

we have for all indices \( i, j \) in the range \( 0 \leq i < j \leq \ell - 1 \)

1. \( R_i \cap R_j = \emptyset \),
2. \( \Lambda_i = \Lambda_j \),
3. \( (2\gamma - 1)|L_j| \geq ||[\tau_{n_j,i_j-k-1}, \tau_{n_j,i_j-k}]|| \geq \frac{|L_j|}{2\gamma} \).
\( (4) \ |R_j| \leq (2\gamma - 1)|L_j|, \\
(5) \ |L_j| \leq 2(\gamma + 1)k \cdot |R_j|, \\
(6) \ \min(|L_j|, |R_j|) \geq A|\Lambda_j|. \)

**Proof.** First, we choose a sequence \((n_j)_{j=0}^{\infty}\) so that (1) - (4) hold. Next, we choose \((n_m)_{j=0}^{\gamma-1}\) - a subsequence of \((n_j)_{j=0}^{\infty}\) so that (5) and (6) hold as well.

The sequence \((t_n)\) does not satisfy any \((k-1)\)-regularity condition. As a consequence, for all \(C_0\) there exists an index \(n_0\) and an index \(i_0\) such that

\begin{equation}
(6.1) \quad C_0|D_{n_0,i_0-k}^{(k-1)}| \leq |D_{n_0,i_0-k+1}^{(k-1)}| \quad \text{or} \quad |D_{n_0,i_0-k}^{(k-1)}| \geq C_0|D_{n_0,i_0-k+1}^{(k-1)}|.
\end{equation}

We choose \(C_0\) sufficiently large such that with \(C_j := C_{j-1}/\gamma - 1\) for \(j \geq 1\), the condition \(C_k \geq 2\gamma\) is true. We will give an additional restriction for \(C_0\) at the end of the proof. Without loss of generality, we can assume that the first inequality in (6.1) holds. Taking \(\Lambda_0 = [\tau_{n_0,i_0-k}, \tau_{n_0,i_0-1}]\) and \(L_0 = [\tau_{n_0,i_0-1}, \tau_{n_0,i_0}], R_0 = [\tau_{n_0,i_0}, \tau_{n_0,i_0+1}],\) we have

\begin{equation}
(6.2) \quad |[\tau_{n_0,i_0-k+1}, \tau_{n_0,i_0}]| \geq C_0|\Lambda_0|.
\end{equation}

A direct consequence of (6.2) is

\begin{equation}
(6.3) \quad |L_0| \geq (C_0 - 1)|\Lambda_0|.
\end{equation}

By \(k\)-regularity we have

\[|D_{n_0,i_0-k-1}^{(k)}| \geq \frac{|D_{n_0,i_0-k}^{(k)}|}{\gamma} = \frac{|\Lambda_0| + |L_0|}{\gamma},\]

which implies

\begin{align}
|[\tau_{n_0,i_0-k-1}, \tau_{n_0,i_0-k}]| &= |D_{n_0,i_0-k-1}^{(k)}| - |\Lambda_0| \\
&\geq \frac{|\Lambda_0| + |L_0|}{\gamma} - |\Lambda_0| \\
&\geq \frac{|L_0|}{2\gamma} + \frac{|\Lambda_0|}{\gamma} + C_0 - \frac{1}{2\gamma}|\Lambda_0| - |\Lambda_0| \\
&= \frac{|L_0|}{2\gamma} + (\frac{C_0 + 1}{2\gamma} - 1)|\Lambda_0| \geq \frac{|L_0|}{2\gamma},
\end{align}

i.e., the right hand side inequality of (3) for \(j = 0\). To get the upper estimate, note that by \(k\)-regularity

\[|\Lambda_0| + |[\tau_{n_0,i_0-k-1}, \tau_{n_0,i_0-k}]| \leq \gamma(|\Lambda_0| + |L_0|),\]

hence by (6.3)

\begin{equation}
(6.5) \quad |[\tau_{n_0,i_0-k-1}, \tau_{n_0,i_0-k}]| \leq \gamma|L_0| + (\gamma - 1)|\Lambda_0| \leq (2\gamma - 1)|L_0|.
\end{equation}

This and the previous calculation give (3) for \(j = 0\). Therefore, the construction below can be continued either to the right or to the left of \(\Lambda_0\). We continue our construction to the right of \(\Lambda_0\).

We continue by induction. Having defined \(n_j, \Lambda_j, L_j\) and \(R_j\), we take

\[n_{j+1} := \min\{n > n_j : t_n \in \Lambda_j \cup L_j\}, \quad j \geq 0.\]
Unconditionality of Orthogonal Spline Systems in $H^1$

By definition of $R_j$ and $n_{j+1}$, property (1) is satisfied for all $j \geq 0$. We identify $t_{n_{j+1}} = \tau_{n_{j+1},i_{j+1}}$. Thus, by construction, $t_{n_j} = \tau_{n_j,i_j}$ is a common endpoint of $L_j$ and $R_j$ for $j \geq 1$.

In order to prove (2), we will show by induction that

(6.6) $|\tau_{n_j,i_j+k+1}, \tau_{n_j,i_j}| \geq C_j|\Lambda_j|$ and $\Lambda_{j+1} = \Lambda_j$

for all $j = 0, \ldots, k\ell$. We remark that the equation $\Lambda_{j+1} = \Lambda_j$ is equivalent to the condition $\tau_{n_{j+1},i_{j+1}} \in L_j$.

The first inequality of (6.6) for $j = 0$ is exactly (6.2). If the second identity in (6.6) were not satisfied for $j = 0$, i.e., $\tau_{n_1,i_1} \in \Lambda_0$, we would have by $k$-regularity of the point sequence $(t_n)$, applied to the partition $\mathcal{T}_n$,

$$|\Lambda_0| \geq \frac{1}{\gamma}|L_0|,$$

which contradicts (6.3) for our choice of $C_0$. This means we have $\Lambda_1 = \Lambda_0$, and so, (6.6) is true for $j = 0$. Next, assume that (6.6) is satisfied for $j - 1$, where $j$ is in the range $1 \leq j \leq k\ell - 1$. By $k$-regularity, applied to the partition $\mathcal{T}_{n_j}$, and employing repeatedly (6.6) for $j - 1$,

$$|\Lambda_j| + |L_j| = |\Lambda_j \cup L_j| \geq \frac{1}{\gamma}(\tau_{n_j,i_j+1} - \tau_{n_j,i_j-k+1})$$

$$= \frac{1}{\gamma}(\tau_{n_{j-1},i_{j-1}} - \tau_{n_{j-1},i_{j-1}-k+1})$$

$$\geq \frac{C_{j-1}}{\gamma}|\Lambda_{j-1}| = \frac{C_{j-1}}{\gamma}|\Lambda_j|.$$

This means, by the recursive definition of $C_j$,

(6.7) $|L_j| \geq C_j|\Lambda_j|$, 

and in particular the first identity in (6.6) is true for $j$. If the second identity in (6.6) were not satisfied for $j$, i.e., $\tau_{n_{j+1},i_{j+1}} \in \Lambda_j$, we would have by $k$-regularity of the point sequence $(t_n)$, applied to the partition $\mathcal{T}_{n_{j+1}}$,

$$|\Lambda_j| \geq \frac{1}{\gamma}|L_j|,$$

which contradicts (6.7) and our choice of $C_0$. This proves (6.6) for $j$ and thus, property (2) is true for all $j = 0, \ldots, k\ell$.

Moreover, choosing $C_0$ sufficiently large – that is, such that $C_{k\ell} \geq 2(\gamma + 1)kA$, (6.7) implies

(6.8) $|L_j| \geq 2(\gamma + 1)kA|\Lambda_j|$, 

which is a part of (6).

The lower estimate in property (3) is proved by repeating the argument giving (6.4) and using (6.7) instead of (6.4). Moreover, the upper estimate also uses the same arguments as the proof of (6.5), but now we have to use (6.7) as well.

Next, we look at property (4). By $k$-regularity and (6.7), as $C_j > 1$, we have

$$|R_j| + |L_j| \leq \gamma(|L_j| + |\Lambda_j|) \leq 2\gamma|L_j|,$$
which is exactly property (1).

We prove property (5) by choosing a suitable subsequence of \((n_j)_{j=0}^{k\ell}\) and begin with assuming the contrary to (5) for \(k\) consecutive indices, i.e., for an index \(s\),

\[
|R_{s+r}| < \alpha |L_{s+r}| \leq \alpha |L_s|, \quad r = 1, \ldots, k,
\]

where \(\alpha := (2(\gamma + 1)k)^{-1}\). We have \(L_j = L_{j+1} \cup R_{j+1}\) for \(0 \leq j \leq k\ell - 1\). Thus, on the one hand,

\[
|L_s \setminus L_{s+k}| = \sum_{r=1}^{k} |R_{s+r}| \leq \alpha k |L_s|
\]

by (6.9); on the other hand, by \(k\)-regularity of the partition \(\mathcal{T}_{n_s+k}\),

\[
|L_s \setminus L_{s+k}| \geq \frac{1}{\gamma} |L_{s+k}| = \frac{1}{\gamma} \left( |L_s| - \sum_{r=1}^{k} |R_{s+r}| \right) \geq \frac{1 - \alpha k}{\gamma} |L_s|.
\]

Now, (6.10) contradicts (6.11) for our choice of \(\alpha\). We thus have proved that for \(k\) consecutive indices \(s + 1, \ldots, s + k\), there is at least one index \(s + r, 1 \leq r \leq k\), such that (5) is satisfied for \(s + r\). As a consequence we can extract a sequence of length \(\ell\) from \((n_j)_{j=0}^{k\ell}\) satisfying (5). Without restriction, this subsequence is called \((n_j)_{j=0}^{\ell-1}\) again.

Property (6) for \(R_j\) is now a simple consequence of (6.8), property (5) and the choice of the subsequence \((n_j)_{j=0}^{\ell-1}\). Therefore, the proof of the lemma is completed. \(\square\)

Now, we are ready to proceed with the proof of Proposition 6.1.

**Proof of Proposition 6.1.** Let \(\ell\) be an arbitrary positive integer and \(A \geq 2\) a number that will be chosen later. Then, Lemma 6.2 gives us a sequence \((n_j)_{j=0}^{\ell-1}\) such that all conditions in Lemma 6.2 are satisfied. We assume that \(|\Lambda_0| > 0\). Let \(\tau := \tau_{n_0, i_0 - 1}, x := \tau - 2|\Lambda_0|\) and \(y := \tau + 2|\Lambda_0|\). Then we define the atom \(\phi\) by

\[
\phi \equiv \frac{1}{4|\Lambda_0|}(1_{[x, \tau]} - 1_{[\tau, y]})
\]

and let \(j\) be an arbitrary integer in the range \(0 \leq j \leq \ell - 1\). By partial integration, the expression \(a_{n_j}(\phi) = \langle \phi, f_{n_j} \rangle\) can be written as

\[
4|\Lambda_0|a_{n_j}(\phi) = \int_{x}^{\tau} f_{n_j}(t) \, dt - \int_{\tau}^{y} f_{n_j}(t) \, dt
\]

\[
= \int_{x}^{\tau} f_{n_j}(t) - f_{n_j}(\tau) \, dt - \int_{\tau}^{y} f_{n_j}(t) - f_{n_j}(\tau) \, dt
\]

\[
= \int_{x}^{\tau} (x - t) f'_{n_j}(t) \, dt - \int_{\tau}^{y} (y - t) f'_{n_j}(t) \, dt.
\]

In order to estimate \(|a_{n_j}(\phi)|\) from below, we estimate the absolute values of \(I_1 := \int_{x}^{\tau} (x - t) f'_{n_j}(t) \, dt\) from below and of \(I_2 := \int_{\tau}^{y} (y - t) f'_{n_j}(t) \, dt\) from above and begin with \(I_2\).
Consider the function \( g_{n_j} \), which is connected to \( f_{n_j} \) via \( f_{n_j} = g_{n_j} / \| g_{n_j} \|_2 \) and \( \| g_{n_j} \|_2 \sim_k |J_{n_j}|^{-1/2} \) (cf. (4.11) and Lemma 4.3). In the notation of Lemma 6.2, \( g_{n_j} \) is obtained by inserting the point \( t_{n_j} = \tau_{n_j, i} \) to \( T_{n_j - 1} \), and it is a common endpoint of intervals \( L_i \) and \( R_i \). By construction of the characteristic interval \( J_{n_j} \), Lemma 6.2 properties (4) – (6), and the \( k \)-regularity of the point sequence \((t_n)\), we have

\[
(6.12) \quad |J_{n_j}| \sim_{k, \gamma} |L_j| \sim_{k, \gamma} |R_j|.
\]

By Lemma 6.2, property (6), we have \([\tau, y] \subset L_j\), and therefore on \([\tau, y]\), the derivative of the function \( g_{n_j} \) has the representation (cf. (3.1))

\[
g'_{n_j}(u) = (k - 1) \sum_{i = i_j - k + 1}^{i_j - 1} \xi_i N_{n_j, i}^{(k-1)}(u), \quad u \in [\tau, y],
\]

where \( \xi_i = (w_i - w_{i-1})/|D_{n_j, i}^{(k-1)}| \) and the coefficients \( w_i \) are given by (4.3) associated to the partition \( T_{n_j} \). For \( i = i_j - k + 1, \ldots, i_j - 1 \) we have \( L_j \subset D_{n_j, i}^{(k-1)} \), which in combination with the \( k \)-regularity of the point sequence \((t_n)\) and Lemma 6.2 property (6) implies

\[
(6.13) \quad |J_{n_j}| \sim_k |L_j| \sim_{k, \gamma} |D_{n_j, i}^{(k-1)}|, \quad i = i_j - k + 1, \ldots, i_j - 1.
\]

Moreover, by Lemma 4.4

\[
|w_i| \lesssim_k \frac{1}{|J_{n_j}|}, \quad 1 \leq i \leq n_j + k - 1.
\]

Therefore

\[
|f'_{n_j}(t)| \sim_k |J_{n_j}|^{1/2} |g'_{n_j}(t)| \lesssim_{k, \gamma} |L_j|^{-3/2} \quad \text{for } t \in [\tau, y].
\]

Consequently, putting the above facts together,

\[
(6.14) \quad |I_2| \lesssim_{k, \gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}.
\]

We continue with the estimate of \( I_1 \). By properties (3) and (6) of Lemma 6.2 (with \( A \geq 2\gamma \)), we have \([x, \tau] \subset [\tau_{n_j, i_j - k - 1}, \tau_{n_j, i_j - 1}]\) and, therefore, on \([x, \tau]\), \( g'_{n_j} \) has the representation (cf. (3.1))

\[
g'_{n_j}(u) = (k - 1) \sum_{i = i_j - 2k + 1}^{i_j - 2} \xi_i N_{n_j, i}^{(k-1)}(u), \quad u \in [x, \tau].
\]

We split the integral \( I_1 \) as a sum \( I_1 = I_{1,1} + I_{1,2} \) corresponding to the indices \( i \neq i_j - k \) and \( i = i_j - k \) in the above representation of \( g_{n_j} \) on \([x, \tau]\).

Note that \([\tau_{n_j, i_j - k - 1}, \tau_{n_j, i_j - k}] \subset D_{n_j, i}^{(k-1)} \) for \( i_j - 2k + 1 \leq i < i_j - k \) and \( L_j \subset D_{n_j, i}^{(k-1)} \) for \( i_j - k < i \leq i_j - 2 \). Therefore, by properties (3) and (6) of Lemma 6.2 and the \( k \)-regularity of the sequence of knots we have

\[
|D_{n_j, i}^{(k-1)}| \sim_{k, \gamma} |L_j| \quad \text{for } i_j - 2k + 1 \leq i \leq i_j - 2, \quad i \neq i_j - k.
\]
So, by arguments analogous to the proof of (6.14) we get (6.15)

\[ |I_{1,1}| \sim_k |J_{n_j}|^{1/2} \int_0^\tau (t - x)^{i_j - 2} \sum_{i = i_j - 2k + 1}^{i_j - 2} \xi_i N_{n_j,i}^{(k - 1)}(t) \, dt \lesssim_{k,\gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}. \]

Moreover, for \( i = i_j - k \), we have \( D_{n_j,i_j - k}^{(k - 1)} = \Lambda_0 \), so we get

\[ |I_{1,2}| \sim_k |J_{n_j}|^{1/2} \int_0^\tau (t - x) \xi_{i_j - k} N_{n_j,i_j - k}^{(k - 1)}(t) \, dt \geq |\xi_{i_j - k}| |J_{n_j}|^{1/2} |\Lambda_0| \int_0^\tau N_{n_j,i_j - k}^{(k - 1)}(t) \, dt \]

(6.16)

\[ = |\xi_{i_j - k}| |\Lambda_0| |J_{n_j}|^{1/2} \left| \frac{D_{n_j,i_j - k}^{(k - 1)}}{k - 1} \right| = |\xi_{i_j - k}| |J_{n_j}|^{1/2} |\Lambda_0|^2. \]

due to the fact that \( t - x \geq |\Lambda_0| \) for \( t \in \supp N_{n_j,i_j - k}^{(k - 1)} \). Since the sequence \( w_j \) is checkerboard, cf. (4.4),

\[ |\xi_{i_j - k}| = |w_{i_j - k} - |w_{i_j - k - 1}| |D_{n_j,i_j - k}^{(k - 1)}| \geq |w_{i_j - k}| |D_{n_j,i_j - k}^{(k - 1)}|. \]

By definition of \( w_{i_j - k} \),

\[ |w_{i_j - k}| \geq |\alpha_{i_j - k}| |b_{i_j - k,i_j - k}|, \]

where \( \alpha_{i_j - k} \) is the factor from formula (4.2) and \( b_{i_j - k,i_j - k} \) is an entry of the inverse of the B-spline Gram matrix, both corresponding to the partition \( T_{n_j} \). Formulas (4.2) and (6.12) imply that \( \alpha_{i_j - k} \) is bounded from below by a positive constant that is only depending on \( k \) and \( \gamma \). Moreover, \( |b_{i_j - k,i_j - k}| \geq \| N_{n_j,i_j - k}^{(k - 1)} \|^{-2} \sim_k |D_{n_j,i_j - k}^{(k - 1)}|^{-1} \), cf. (3.6). Note that \( D_{n_j,i_j - k}^{(k - 1)} = \Lambda_0 \cup L_j \), so \( |D_{n_j,i_j - k}^{(k - 1)}| \sim_{k,\gamma} |L_j| \). Thus, \( |\xi_{i_j - k}| \gtrsim_{k,\gamma} |\Lambda_0|^{-1} |L_j|^{-1} \). Inserting the above calculations in (6.16), we find

(6.17)

\[ |I_{1,2}| \gtrsim_{k,\gamma} |J_{n_j}|^{1/2} |\Lambda_0| |L_j|^{-1/2}. \]

We now impose conditions on the constant \( A \geq 2\gamma \) from the beginning of the proof and property (6) in Lemma 6.2. It follows by (6.17), (6.15) and (6.14) that there are \( C_{k,\gamma} > 0 \) and \( c_{k,\gamma} > 0 \), depending only on \( k \) and \( \gamma \) such that

\[
4|\Lambda_0|^2 a_{n_j}(\phi) \geq |I_{1,2}| - |I_{1,1}| - |I_2| \geq C_{k,\gamma} |\Lambda_0| |L_j|^{-1/2} - c_{k,\gamma} |\Lambda_0|^2 |L_j|^{-3/2} = |\Lambda_0| |L_j|^{-1/2} (C_{k,\gamma} - c_{k,\gamma} |\Lambda_0| |L_j|^{-1}) .
\]

\footnote{Formulas (4.2) is applied with \( T_0 = T_{n_j} \) and corresponding to \( \tau_{n_0} = \tau_{n_j,i_j} \). Then \( |\tau_{n_0} - 1|, |\tau_{n_0}| = L_j \) and \( |\tau_{n_0}, \tau_{n_0} + 1| = R_j \). Because of \( k \)-regularity and \( |\Lambda_0 \cup L_j| \sim_{k,\gamma} |L_j| \), each denominator in (4.2) is \( \sim_{k,\gamma} |L_j| \). Each nominator in (4.2) is bigger than either \( L_j \) or \( R_j \), so by (6.12) and \( k \)-regularity it is \( \sim_{k,\gamma} |L_j| \) as well.}
By property (6) in Lemma 6.2 we have $|\Lambda_0||L_j|^{-1} \leq 1/A$. Choosing $A$ sufficiently large to guarantee

$$C_{k,\gamma} - \frac{c_{k,\gamma}}{A} \geq \frac{C_{k,\gamma}}{2},$$

we get a constant $m_{k,\gamma}$, depending only on $k$ and $\gamma$ such that

$$(6.18) \quad m_{k,\gamma} |L_j|^{-1/2} \leq |a_{n_j}(\phi)|, \quad j = 0, \ldots, \ell - 1.$$ 

Next, we estimate $\int_{R_j} |g_{n_j}(t)| \, dt$ from below. First, employ Proposition 3.3, property (6) of Lemma 6.2 and the $k$-regularity of the point sequence $(t_n)$ to get

$$\int_{R_j} |g_{n_j}(t)| \, dt \gtrsim_{k,\gamma} |R_j||w_{ij}|,$$

where $w_{ij}$ corresponds to the partition $T_{n_j}$. By definition of $w_{ij}$,

$$\int_{R_j} |g_{n_j}(t)| \, dt \gtrsim_{k,\gamma} |R_j||\alpha_{ij}||b_{ij,i_j}|.$$ 

By arguments similar as above, $|\alpha_{ij}|$ is bounded from below by a constant only depending on $k$ and $\gamma$, and $|b_{ij,i_j}| \gtrsim_k |D_{n_j,i_j}^{(k)}|^{-1}$. Since by $k$-regularity, $|R_j| \sim_{k,\gamma} |D_{n_j,i_j}^{(k)}|$, we finally get

$$\int_{R_j} |g_{n_j}(t)| \, dt \gtrsim_{k,\gamma} 1,$$

which means for $f_{n_j}$ that

$$\int_{R_j} |f_{n_j}(t)| \, dt \gtrsim_{k,\gamma} |J_{n_j}|^{1/2} \gtrsim_{k,\gamma} |L_j|^{1/2}.$$ 

Combining this last estimate with (6.18) and (1) of Lemma 6.2

$$\int_0^1 \sup_n |a_n(\phi)f_n(t)| \, dt \geq \sum_{j=1}^{\ell} \int_{R_j} |a_{n_j}(\phi)f_{n_j}(t)| \, dt \gtrsim_{k,\gamma} \ell.$$

This construction applies to every positive integer $\ell$, proving the assertion of the proposition for $|\Lambda_0| > 0$.

The case $|\Lambda_0| = 0$ proceeds similarly, with the difference that the atom $\phi$ is defined as centered at the point $\tau_{n_0,i_0-1}$ and the length of the support is sufficiently small, depending on $\ell$ and $|L_0|$. □

With Proposition 6.1 and the results of Section 5, the proof of Theorem 2.4 follows now the same line of arguments as the proof of Theorem 2.2 in [15], but we present it here for the sake of the completeness.

**Proof of Theorem 2.4.** We start by proving the unconditional basis property of $(f_n) = (f_n^{(k)})$ assuming the $(k - 1)$-regularity of $(t_n)$. If $(t_n)$ is $(k - 1)$-regular, it is not difficult to check that it is also $k$-regular. As a consequence, Theorem 2.3 implies that $(f_n)$ is a basis in $H^1$. Let $f \in H^1$ with $f = \sum a_n f_n$.
\[ H \]

\[
\sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \lesssim_{k, \gamma} S \left( \sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \right) \lesssim_k \left\| P \left( \sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \right) \right\|_1
\]

\[
= \left\| P \left( \sum_{n=m_1}^{m_2} a_n f_n \right) \right\|_1 \lesssim_{k, \gamma} \sum_{n=m_1}^{m_2} a_n f_n \|_{H^1}
\]

where we used Proposition 5.4, Proposition 5.2 and Proposition 5.3, respectively (cf. also the picture on page 12). So, since \( \sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \) converges in \( H^1 \), so does \( f_\varepsilon := \sum \varepsilon_n a_n f_n \) and the same calculation as above shows

\[
\| f_\varepsilon \|_{H^1} \lesssim_{k, \gamma} \| f \|_{H^1}.
\]

This implies that \( (f_n) \) is an unconditional basis in \( H^1 \).

We now prove the converse, i.e., that \( (f_n) \) being an unconditional basis in \( H^1 \) implies the \( (k-1) \)-regularity condition. First, if \( (t_n) \) does not satisfy the \( k \)-regularity condition, \( (f_n) \) is not a basis in \( H^1 \) by Theorem 2.3. Thus, it remains to consider the case when \( (t_n) \) satisfies the \( k \)-regularity condition, but not the \( (k-1) \)-regularity condition. By Theorem 2.3 again, \( (f_n) \) is then a basis in \( H^1 \). Suppose that \( (f_n) \) is an unconditional basis in \( H^1 \). Then, for \( f = \sum a_n f_n \) and \( \varepsilon \in \{-1,1\}^\mathbb{Z} \), the function \( f_\varepsilon := \sum \varepsilon_n a_n f_n \) is also in \( H^1 \).

Since \( \| \cdot \|_1 \leq \| \cdot \|_{H^1} \), the series \( \sum a_n f_n \) also converges unconditionally in \( L^1 \), and thus Proposition 5.1 (i.e., Khinchin’s inequality) implies

\[
\| P f \|_1 \lesssim \sup_{\varepsilon} \| f_\varepsilon \|_1 \leq \sup_{\varepsilon} \| f_\varepsilon \|_{H^1} \lesssim \| f \|_{H^1},
\]

which is impossible due to Proposition 6.1, even for atoms. This concludes the proof of Theorem 2.4.

As an immediate consequence of Theorem 2.4, a fifth equivalent condition to (A)-(D) is the unconditional convergence of \( \sum a_n f_n \) in \( H^1 \):

**Corollary 6.3.** Let \( (t_n) \) be a \( k \)-admissible and \( (k-1) \)-regular sequence of points, with \( (f_n) \) – the corresponding orthonormal spline system of order \( k \). Let \( (a_n) \) be a sequence of coefficients. Then conditions (A) – (D) from Section 5 are equivalent.

Moreover, they are equivalent to the following

(E) The series \( \sum a_n f_n \) converges unconditionally in \( H^1 \).

In addition, for \( f \in H^1 \), \( f = \sum a_n f_n \) we have

\[
\| f \|_{H^1} \sim \| S f \|_1 \sim \| P f \|_1 \sim \sup_{\varepsilon \in \{-1,1\}^\mathbb{Z}} \| \sum \varepsilon_n a_n f_n \|_1,
\]

with the implied constants depending only on \( k \) and the parameter of \( (k-1) \)-regularity of the sequence \( (t_n) \).
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