A PROJECTIVE $C^*$-ALGEBRA RELATED TO $K$-THEORY

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Abstract. The $C^*$-algebra $\mathcal{C}$ is the smallest of the $C^*$-algebras $qA$ introduced by Cuntz \cite{1} in the context of $KK$-theory. An important property of $\mathcal{C}$ is the natural isomorphism

$$K_0(A) \cong \lim \begin{array}{c} q \setminus \mathcal{B} \setminus \mathcal{V} \\ \end{array} [\mathcal{C}, M_n(A)].$$

Our main result concerns the exponential (boundary) map from $K_0$ of a quotient $B$ to $K_1$ of an ideal $I$. We show if a $K_0$ element is realized in $\text{hom}(\mathcal{C}, B)$ then its boundary is realized as a unitary in $\tilde{I}$. The picture we obtain of the exponential map is based on a projective $C^*$-algebra $\mathcal{P}$ that is universal for a set of relations slightly weaker than the relations that define $\mathcal{C}$. A new, shorter proof of the semiprojectivity of $\mathcal{C}$ is described. Smoothing questions related the relations for $\mathcal{C}$ are addressed.

1. Introduction

The simplest nonzero projective $C^*$-algebra is $C_0(0, 1]$. A quotient of this is $\mathcal{C}$, the simplest nonzero semiprojective $C^*$-algebra. The first is universal for the relation $0 \leq x \leq 1$ and the second for $p^* = p^2 = p$. When lifting a projection from a quotient, one must either settle for a lift that is only a positive element or confront some $K$-theoretical obstruction to finding a lift that is a projection. We consider noncommutative analogs of these two $C^*$-algebras.

We use $\hat{A}$ to denote the unitization of $A$, where a unit $1$ is to be added even in $1_A$ exists. For elements $h$, $x$ and $k$ of $A$, we use the notation

$$(1) \quad T(h, x, k) = \begin{bmatrix} 1 - h & x^* \\ x & k \end{bmatrix} \in M_2(\hat{A}).$$

We will show that there is a $C^*$-algebra $\mathcal{P}$ with generators $h$, $k$ and $x$ that are universal for the relations

$$hk = 0,$$

$$0 \leq T(h, x, k) \leq 1.$$ 

Moreover, $\mathcal{P}$ is projective. This does not appear to be a familiar $C^*$-algebra, but it has a familiar quotient. The relations

$$hk = 0,$$

$$T(h, x, k)^* = T(h, x, k)^2 = T(h, x, k)$$

have as their universal $C^*$-algebra the semiprojective $C^*$-algebra

$$\mathcal{C} = \{ f \in C_0((0, 1], M_2) \mid f(1) \text{ is diagonal} \}.$$  

Key words and phrases. $C^*$-algebras, semiprojectivity, $K$-theory, boundary map, projectivity, lifting.
A complicated proof of the semiprojectivity of $q/\mathbb{BV}$, was given in [2]. Subsequent proofs found with Eilers and Pederson in [3] and [4] worked in the context of noncommutative CW-complexes. Those proofs did not utilize the fact that $q/\mathbb{BV}$ is similar to the noncommutative Grassmannian $G^{nc}_{2}$, c.f. [5]. The proof here uses this connection.

The importance of $q/\mathbb{BV}$ to $K$-theory is illustrated by the isomorphism

$$K_{0}(A) \cong [q/\mathbb{BV}, A \otimes \mathcal{K}] \cong \lim_{\rightarrow} [q/\mathbb{BV}, M_{n}(A)].$$

For example, see [1] and [6].

Our main result concerns the exponential (boundary) map from $K_{0}$ of a quotient $B$ to $K_{1}$ of an ideal $I$. If we look at $K_{0}$ as

$$K_{0}(D) \cong \lim_{\rightarrow} [q/\mathbb{BV}, M_{n}(D)],$$

then given

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

we show that a $K_{0}$ element realized in $\text{hom}(q/\mathbb{BV}, B)$ has boundary in $K_{1}(I)$ that can be realized as a unitary in $\tilde{I}$.

In the final section we look further into methods for perturbing approximate representations of the relations for $q/\mathbb{BV}$ into true representations, but this time restricting ourselves to using only $C^{\infty}$-functional calculus.

**Lemma 1.1.** The $C^{*}$-algebra

$$q\mathbb{C} = \{ f \in C_{0}((0, 1], \mathbb{M}_{2}) \mid f(1) \text{ is diagonal} \}$$

is universal in the category of all $C^{*}$-algebras for generators $h, k$ and $x$ with relations

$$h^{*}h + x^{*}x = h,$$

$$k^{*}k + xx^{*} = k,$$

$$kx = xh,$$

$$hk = 0.$$

The concrete generators may be taken to be

$$h_{0} = t \otimes e_{11}, \quad k_{0} = t \otimes e_{22}, \quad x_{0} = \sqrt{t - t^{2}} \otimes e_{21}.$$

**Proof.** This is almost identical to Proposition 2.1 in [2]. To see these are equivalent, notice first that the top two relations imply $h$ and $k$ are positive. Since $x^{*}x$ is positive, the relation $x^{*}x = h - h^{2}$ implies $h \leq 1$. It also implies $\|x\| \leq \frac{1}{2}$. Similarly $k \leq 1$. \qed

**Lemma 1.2.** The $C^{*}$-algebra $q\mathbb{C}$ is universal in the category of all $C^{*}$-algebras for generators $h, k, x$ and relations

$$hk = 0,$$

(3) $$T(h, x, k)^{2} = T(h, x, k) = T(h, x, k).$$

**Proof.** Since

$$T(h, x, k) = \begin{bmatrix} 1 - h & x^{*} \\ x & k \end{bmatrix}$$

and

$$T(h, x, k)^{2} = \begin{bmatrix} 1 - 2h + h^{2} + x^{*}x & x^{*}h + x^{*}k \\ x - xh + kx & k^{2} + xx^{*} \end{bmatrix},$$

then given

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

we show that a $K_{0}$ element realized in $\text{hom}(q/\mathbb{BV}, B)$ has boundary in $K_{1}(I)$ that can be realized as a unitary in $\tilde{I}$. In the final section we look further into methods for perturbing approximate representations of the relations for $q/\mathbb{BV}$ into true representations, but this time restricting ourselves to using only $C^{\infty}$-functional calculus.

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$$k^{*}k + xx^{*} = k,$$

$$kx = xh,$$

$$hk = 0.$$
2. Internal Matrix Structures in $C^*$-Algebras

Lemma 2.1. Suppose $A$ is a $C^*$-algebra and $X_{11}, X_{21}, X_{12},$ and $X_{22}$ are closed linear subspaces of $A$. Suppose $X_{ij} = X_{ji}$ and $X_{ij}X_{jk} \subseteq X_{ik}$ and $X_{11}X_{22} = 0$.

(1) The subset
\[ \hat{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \]

is a $C^*$-subalgebra of $M_2(A)$.

(2) The sum $X_{11} + X_{21} + X_{12} + X_{22}$

is a linear direct sum and is a $C^*$-subalgebra of $A$, isomorphic to $\hat{X}$.

(3) There is a homotopy $\theta_t$ of injective $*$-homomorphisms
\[ \theta_t : X_{11} + X_{21} + X_{12} + X_{22} \to M_2(A) \]

so that
\[ \theta_0(x_{11} + x_{21} + x_{12} + x_{22}) = \begin{bmatrix} x_{11} + x_{21} + x_{12} + x_{22} & 0 \\ 0 & 0 \end{bmatrix} \]

and
\[ \theta_1(x_{11} + x_{21} + x_{12} + x_{22}) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \]

Proof. An element $x_{ij}$ of $X_{ij}$ factors as $x_{ij} = x_{ii}y_{jj}$ with $y$ in $A$ and $x_{jj} = |x_{ij}|^2$ in $X_{jj}$ and $x_{ii} = |x_{ij}|^2$ in $X_{ii}$. From here, it is easy to show that $X_{ij}X_{kl} = 0$ if $j \neq k$ and that $X_{ij} \cap X_{kl} = 0$ when $i \neq k$ or $j \neq l$.

It is clear that $\hat{X}$ is a $C^*$-subalgebra of $M_2(A)$. Let $w_t$ be a partial isometry in $M_2$ with $|w_t| = e_{11}$ for all $t$ and $w_0 = e_{11}$ and $w_1 = e_{21}$. Define
\[ \psi_t : \hat{X} \to A \otimes M_2 \]

by
\[ \psi_t \left( \sum x_{ij} \otimes e_{ij} \right) = \sum x_{ij} \otimes f^{(t)}_{ij} \]

where
\[ f^{(t)}_{11} = w_t^* w_t, \quad f^{(t)}_{12} = w_t^* \]
\[ f^{(t)}_{21} = w_t, \quad f^{(t)}_{22} = w_t w_t^*. \]

The fact that $X_{ij}X_{kl} = 0$ if $j \neq k$ implies that each $\psi_t$ is a $*$-homomorphism.

The image of $\psi_0$ is
\[ (X_{11} + X_{21} + X_{12} + X_{22}) \otimes e_{11} \]

and so we see that the direct sum of the $X_{ij}$ is a $C^*$-subalgebra of $A$.

Now suppose
\[ \psi_t \left( \sum x_{ij} \otimes e_{ij} \right) = 0. \]

Then for all $r$ and all $s$ we have
\[ 0 = (x_{rs}^* \otimes f^{(t)}_{1r}) \left( \psi_t \left( \sum_{ij} x_{ij} \otimes e_{ij} \right) \right) (x_{rs}^* \otimes f^{(t)}_{s1}) = x_{rs}^* x_{rs} x_{rs}^* \otimes e_{11} \]

which implies $x_{rs} = 0$. Therefore $\psi_t$ is injective.
If we let $\gamma$ denote the obvious isomorphism
$$
\gamma : X_{11} + X_{21} + X_{12} + X_{22} \to (X_{11} + X_{21} + X_{12} + X_{22}) \otimes e_{11}
$$
and $\iota_t$ the inclusion of $\psi_t(\hat{X})$ into $\mathcal{M}_2(A)$ then
$$
\theta_t = \iota_t \circ \psi_t \circ \psi_0^{-1} \circ \gamma
$$
is the desired path of injective $*$-homomorphisms. \hfill $\square$

**Lemma 2.2.** Under the hypotheses of Lemma 2.1, the subset
$$
\left[ \begin{array}{cc} \mathbb{C}1 + X_{11} & X_{12} \\ X_{21} & \mathbb{C}1 + X_{22} \end{array} \right]
$$
is a $C^*$-subalgebra of $\mathcal{M}_2(\hat{A})$, and
$$
\rho \left( \begin{array}{cc} \alpha \mathbb{1} + x_{11} & x_{12} \\ x_{21} & \alpha \mathbb{1} + x_{22} \end{array} \right) = \alpha \oplus \beta
$$
determines a surjection onto $\mathbb{C} \oplus \mathbb{C}$.

**Proof.** This is follows easily from Lemma 2.1. \hfill $\square$

**Lemma 2.3.** Suppose $I$ is an ideal in the $C^*$-algebra $A$ and $h$ and $k$ in $A$ are positive elements. Then
$$
I \cap kAh = kIh
$$

**Proof.** The special case where $h = k$ is routine, and the general case follows via a 2-by-2 matrix trick. \hfill $\square$

### 3. The Exponential Map in $K$-Theory

We chose $b$ as the canonical generator of $K_0(q\mathbb{C}) = \mathbb{Z}$, where $b$ is formed as the class of the projection
$$
P_0 = T(h_0, x_0, k_0)
$$
minus the class of $[1]$. (See (1).)

**Theorem 3.1.** Suppose
$$
\begin{array}{ccc}
0 & \longrightarrow & I^C \\
\longrightarrow & \longrightarrow & \pi \\
& & \longrightarrow & \longrightarrow & A & B & 0
\end{array}
$$
is a short exact sequence of $C^*$-algebras. If $x$ is any element of $K_0(B)$ such that $x = \varphi_*(b)$ for some $*$-homomorphism $\varphi : q\mathbb{C} \to B$, then $\partial(x) = [u]$ in $K_1(I)$ for some unitary $u \in \hat{I}$.

**Proof.** Let
$$
y_0 = \sqrt{t^\frac{1}{2} - t^\frac{3}{2}} \otimes e_{21}
$$
so that $y_0$ is a contraction and
$$
k_0^\frac{1}{2} y_0 h^\frac{1}{2} = x_0.
$$
Orthogonal positive contractions lift to orthogonal positive contractions, so we can find $h$ and $k$ in $A$ with $\pi(h) = \varphi(h_0)$, $\pi(k) = \varphi(k_0)$ and
$$
hh = 0,
0 \leq h \leq 1,
0 \leq k \leq 1.
$$
Now take any \( y \) in \( A \) with \( \pi(y) = \varphi(y_0) \) and let \( x = k^{\frac{1}{2}}y h^{\frac{1}{2}} \) and \( T = T(h, x, k) \).

Then \( \pi(x) = \varphi(x_0) \),

(5) \( \tilde{\pi}^{(2)}(T) = \tilde{\varphi}^{(2)}(P_0) \),

(6) \( T \in \begin{bmatrix} C_1 + hAh & \overline{hAk} \\ \overline{kAh} & C_2 + kAk \end{bmatrix} \),

(7) \( \rho(T) = 1 \oplus 0 \),

and \( T^* = T \).

Let

\[ f(\lambda) = \max(\min(\lambda, 1), 0) \]

and let \( T' = f(T) \). Then equations (5), (6) and (7) hold with \( T' \) replacing \( T \). This means

\[ T' = T(h', x', k') \]

for some \( h', k' \) and \( x' \) in \( A \) that are lifts of \( h, k \) and \( x \), and that

(8) \( h'k' = 0 \),

\( 0 \leq T' \leq 1 \).

This is an interesting lifting result that we will return to below. For now, we turn to the exponential map.

Clearly \( \partial([I]) = 0 \) so we need only compute \( \partial \circ \varphi_*[P_0] \). We have the lifts \( T \) and \( T' \). We prefer to work with \( T' \). A unitary that represents this \( K_1 \) element is \( U' = e^{2\pi i T'} \). Since

\[ \tilde{\pi}^{(2)}(U') = \tilde{\varphi}^{(2)}(e^{2\pi i P_0}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

we know that

\[ U' \in \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} I & I \\ I & I \end{bmatrix} \).

By (5) we know

\[ U' \in \begin{bmatrix} C_1 + hAh & \overline{hAk} \\ \overline{kAh} & C_2 + kAk \end{bmatrix} \].

Putting these facts together we discover

\[ U' \in \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} hAh & \overline{hAk} \\ \overline{kAh} & kAk \end{bmatrix} \subseteq \begin{bmatrix} hAh & \overline{hAk} \\ \overline{kAh} & kAk \end{bmatrix} \sim \].

By Lemma 2.1 there is a path of unitaries in \((M_2(I))\) from

\[ U' = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \]

to

\[ \begin{bmatrix} -I + u_{11} + u_{12} + u_{21} + u_{22} & 0 \\ 0 & I \end{bmatrix} \].

Thus \( \partial \circ \varphi_*(b) = \partial \circ \varphi_*(P_0) \) is represented in \( \tilde{J} \) by the unitary

\[ u = -I + u_{11} + u_{12} + u_{21} + u_{22} \].

□
**Theorem 3.2.** (Theorem 3.9) \(qC\) is semiprojective.

**Proof.** The proof of Theorem 3.1 is easily modified to give a new proof of this result. One needs to assume that \(I\) is the closure of the increasing union of ideals in \(A\). After the lift \(T\) is obtained in \(B/I_1\), one can replace \(I_1\) by \(I_n\) with there now being a hole in the spectrum of \(T\) around \(\frac{1}{2}\). Replacing the role of \(f\) by

\[
 f_\pm(\lambda) = \begin{cases} 0 & \text{if } \lambda < \frac{1}{2} \\ 1 & \text{if } \lambda \geq \frac{1}{2} \end{cases},
\]

and following the same construction, one finds \(T'\) that is a projection. The components of \(T'\) then provide a lift in \(B/I_n\) that is a representation of the generators of \(qC\). \(\square\)

**Corollary 3.3.** There is a universal \(C^*\)-algebra \(P\) for generators \(h, k\) and \(x\) for which

\[
 hk = 0, \\
 0 \leq T(h, x, k) \leq 1.
\]

The surjection \(\theta : P \to qC\) that sends generators to generators is projective.

**Proof.** Once we show \(P\) exists, the proof of the projectivity of \(\theta\) is contained in the proof of Theorem 3.1.

By \([1]\) we need only show that these relations are invariant with respect of inclusions, are natural, are closed under products, and are represented by a list of zero elements. (This last requirement was erroneously missing in \([1]\).) See also \([7]\). Details are left to the reader. \(\square\)

**Theorem 3.4.** The \(C^*\)-algebra \(P\) is projective.

**Proof.** Since \(t^2 \leq t\) in \(C_0((0, 1])\), the matrix \(T = T(h, x, k)\) satisfies \(T^2 \leq T\). From this we deduce \(x^*x \leq h - h^2\). Similarly, \(xx^* \leq k - k^2\). By \([3]\) Lemma 2.2.4 we can factor \(x\) as \(x = k^\frac{1}{2} y h^\frac{1}{2}\) for some \(y\) in \(P\). The rest of the proof is identical to argument between equations (4) and (8). \(\square\)

4. Relations

In this section we briefly examine a class of relations somewhat more complicated than \(\ast\)-polynomials. See \([7, \text{[1]}]\) for different approaches to relations in \(C^*\)-algebras.

Consider sets of relations of the form

\[
 f(p(x_1, \ldots, x_n)) = 0,
\]

either where \(p\) is a self-adjoint \(\ast\)-polynomial in \(n\) noncommuting variables with \(p(0) = 0\) and

\[
 f \in C_0(R \setminus \{0\}),
\]

or where \(p\) is not necessarily self-adjoint, \(p(0) = 0\) and \(f\) is analytic on the plane. The point to restricting to these relations is that

\[
 f(p(x_1, \ldots, x_n))
\]

makes sense, no matter the norm of the \(C^*\)-elements \(x_j\), and so

\[
 \|f(p(x_1, \ldots, x_n))\| \leq \delta
\]

is a common-sense way to define an approximate representation.
Certainly a set $\mathcal{R}$ of relations on $x_1, \ldots, x_n$ of this restricted form is invariant with respect to inclusion, is natural, and each is satisfied when all the indeterminants are set to 0. Therefore, $\mathcal{R}$ will define a universal $C^*$-algebra if and only if it bounded, meaning for all $j$ we have

$$\sup \{ \|\hat{x}_j\| \mid \hat{x}_1, \ldots, \hat{x}_n \text{ is a representation of } \mathcal{R} \} < \infty.$$ 

We will also need to use relations of the form

$$(10) \quad g(q(f_1(p_1(x_1, \ldots, x_n)), \ldots, f_m(p_m(x_1, \ldots, x_n)))) = 0$$

where the $f_k, p_k$ and $g, q$ are pairs of continuous functions and $*$-polynomials subscribing to the above rule. In particular this will allow us the relation

$$\|q(f_1(p_1(x_1, \ldots, x_n)), \ldots, f_m(p_m(x_1, \ldots, x_n)))\| \leq C.$$ 

For any $n$-tuple of elements in a $C^*$-algebra $A$ we define $r(x_1, \ldots, x_n)$, again in $A$, by

$$r(x_1, \ldots, x_n) = f(q(f_1(p_1(x_1, \ldots, x_n)), \ldots, f_m(p_m(x_1, \ldots, x_n)))).$$ 

If $x_1, \ldots, x_n$ are is a sub-$C^*$-algebra, then so is $r(x_1, \ldots, x_n)$. Thus we are justified in the notation $r$ instead of the more pedantic $r_A$. Also $r$ is natural. It is still the case that the universal $C^*$-algebra exists if and only if the set of relations is bounded.

**Lemma 4.1.** Suppose

$$r_k(x_1, \ldots, x_n) = 0$$

for $k = 1, \ldots, K$ form a bounded set of relations of the form (10). Suppose

$$s(x_1, \ldots, x_n) = 0$$

is a relation of the form (10) that holds true in

$$\mathcal{U} = C^*(x_1, \ldots, x_n \mid r_k(x_1, \ldots, x_n) = 0 \quad (\forall k)).$$

Then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $y_1, \ldots, y_n$ in a $C^*$-algebra $A$ satisfy

$$\|r_k(y_1, \ldots, y_n)\| \leq \delta \quad (\forall k)$$

then

$$\|s(y_1, \ldots, y_n)\| \leq \epsilon.$$ 

**Proof.** This follows from standard arguments involving the quotient of an infinite direct product by an infinite direct sum. \qed

5. **Smoothing Relations**

We now modify the techniques from Section 3 for a smooth version of semiprojectivity for $q\mathbb{C}$. The result is slightly weaker than [2, Theorem 1.10], but comes with a more reasonable proof. The result involves maps from the generators of $q\mathbb{C}$ to a dense $*$-subalgebra $A_\infty$ of a $C^*$-algebra $A$. The additional hypothesis is that $M_2(A_\infty)$, and not just $A_\infty$, is closed under $C^\infty$ functional calculus on self-adjoint elements. This additional assumption may be no difficulty in examples. The smooth algebras of Blackadar and Cuntz are closed under passing to matrix algebra ([9, Proposition 6.7]).
Lemma 5.1. If \( p^* = p \) is an element of a \( C^* \)-algebra \( A \) and
\[
\|p^2 - p\| = \eta < \frac{1}{4}
\]
then, with \( f_{\frac{1}{2}} \) as in (2), \( f_{\frac{1}{2}}(p) \) is a projection in \( A \) and
\[
\left\| f_{\frac{1}{2}}(p) - p \right\| \leq \eta.
\]
Proof. This is well-known. \( \square \)

Theorem 5.2. For every \( \epsilon > 0 \), there is a \( \delta > 0 \) so that if \( A_\infty \) is a dense \(*\)-subalgebra of a \( C^* \)-algebra \( A \) for which both \( A_\infty \) and \( M_2(A_\infty) \) are closed under \( C^\infty \) functional calculus on self-adjoint elements, then for any \( h, k \) and \( x \) in \( A_\infty \) for which
\[
\|h^*h + x^*x - h\| \leq \delta, \\
\|k^*k + xx^* - k\| \leq \delta, \\
\|kx - xh\| \leq \delta, \\
\|hk\| \leq \delta,
\]
there exist \( \overline{h}, \overline{k} \) and \( \overline{x} \) in \( A_\infty \) so that
\[
\overline{h}^*\overline{h} + \overline{x}^*\overline{x} - \overline{h} = 0, \\
\overline{k}^*\overline{k} + \overline{x}^*\overline{x} - \overline{k} = 0, \\
\overline{k}\overline{x} - \overline{x}\overline{k} = 0, \\
\overline{h}\overline{k} = 0,
\]
and
\[
\|\overline{h} - h\| \leq \epsilon, \quad \|\overline{k} - k\| \leq \epsilon, \quad \|\overline{x} - x\| \leq \epsilon.
\]
Proof. Let \( \epsilon \) be given, with \( 0 < \epsilon < \frac{1}{4} \). Choose \( \theta > 0 \) so that
\[
\|h' - h''\| \leq \theta, \quad \|k' - k''\| \leq \theta, \quad \|x' - x''\| \leq \theta,
\]
\[
\|h'\| \leq 2, \quad \|k'\| \leq 2, \quad \|x'\| \leq 2,
\]
implies
\[
\|(h'^*h' + x'^*x' - h') - (h''^*h'' + x''^*x'' - h'')\| \leq \frac{\epsilon}{8},
\]
\[
\|(k'^*k' + x'x'^* - k') - (k''^*k'' + x''x''^* - k'')\| \leq \frac{\epsilon}{8},
\]
\[
\|(k'x' - x'h') - (k''x'' - x''h'')\| \leq \frac{\epsilon}{8},
\]
Choose \( g_+ \) some real-valued \( C^\infty \) function on \( \mathbb{R} \) for which
\[
t \leq 0 \implies g_+(t) = 0,
\]
\[
t \geq 0 \implies t - \frac{\theta}{2} \leq g_+(t) \leq t,
\]
and let \( g_-(t) = g_+(-t) \). Choose \( q_+ \) some real-valued \( C^\infty \) functions on \( \mathbb{R} \) for which
\[
t \leq 0 \implies q_+(t) = 0,
\]
\[
t \geq 0 \implies \sqrt{t - t^2} - \frac{\theta}{2} \leq (q_+(t))^2 \sqrt{t - t^2} \leq \sqrt{t - t^2},
\]
and let \( q_-(t) = q_+(-t) \).
Inside \( q \mathbb{C} \), let us have
\[
g_+ \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) = g_+ (t) \otimes e_{11},
\]
and
\[
g_- \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) = g_+ (t) \otimes e_{22}
\]
and
\[
q_- \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) x_0 q_+ \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right)
= (g_+ (t))^2 \sqrt{t - t^2} \otimes e_{21}.
\]
Therefore
\[
\left\| g_+ \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) - h_0 \right\| \leq \frac{\theta}{2},
\]
\[
\left\| g_- \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) - k_0 \right\| \leq \frac{\theta}{2},
\]
\[
\left\| q_- \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) x_0 q_+ \left( \frac{1}{2} \left( h_0 + h^*_0 - k_0 - k^*_0 \right) \right) - x_0 \right\| \leq \frac{\theta}{2}.
\]
Of course, we also know
\[
\| h_0 \| \leq 1, \quad \| k_0 \| \leq 1, \quad \| x_0 \| \leq \frac{1}{2}.
\]
Lemma 4.1 tells us there is a \( \delta > 0 \) so that if \( h, k \) and \( x \) are in a C*-algebra \( A \) with
\[
\| h^* h + x^* x - h \| \leq \delta,
\]
\[
\| k^* k + x x^* - k \| \leq \delta,
\]
\[
\| k x - x h \| \leq \delta,
\]
\[
\| h k \| \leq \delta
\]
then
\[
\left\| g_+ \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right) - h \right\| \leq \theta,
\]
\[
\left\| g_- \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right) - k \right\| \leq \theta,
\]
\[
\left\| q_- \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right) x q_+ \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right) - x \right\| \leq \theta,
\]
\[
\| h \| \leq 2, \quad \| k \| \leq 2, \quad \| x \| \leq 2.
\]
If necessary, replace \( \delta \) with a smaller number to ensure \( \delta < \frac{\theta}{2} \).
Let
\[
\tilde{h} = f_+ \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right),
\]
\[
\tilde{k} = f_- \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right),
\]
\[
h_2 = g_+ \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right),
\]
\[
k_2 = g_- \left( \frac{1}{2} \left( h + h^* - k - k^* \right) \right),
\]
and
\[ x_2 = q_+ \left( \frac{1}{2}(h + h^* - k - k^*) \right) xq_- \left( \frac{1}{2}(h + h^* - k - k^*) \right). \]

First notice that \( \tilde{h} \) and \( \tilde{k} \) are orthogonal positive elements of \( A \). Since
\[ q_+ \left( \frac{1}{2}(h + h^* - k - k^*) \right) \]
is in the \( C^* \)-algebra generated by \( \tilde{h} \), and
\[ q_- \left( \frac{1}{2}(h + h^* - k - k^*) \right) \]
is in the \( C^* \)-algebra generated by \( \tilde{k} \), we have \( x_2 \in \tilde{k}A \tilde{h} \). Similarly, \( h_2 \in kAh \) and \( k_2 \in kAk \). Next, observe that \( h_2, k_2 \) and \( x_2 \) are in \( A_\infty \), with \( h_2 \) and \( k_2 \) self-adjoint and
\[ \|h_2 - h\|, \|k_2 - k\|, \|x_2 - x\| \leq \theta. \]

Therefore
\[ \| (h_2^* h_2 + x_2^* x_2 - h_2) - (h^* h + x^* x - h) \| \leq \frac{\epsilon}{8}, \]
\[ \| (k_2 k_2^* + x_2 x_2^* - k_2) - (k k^* + x x^* - k) \| \leq \frac{\epsilon}{8}, \]
\[ \| (k_2 x_2 - x_2 h_2) - (k x - x h) \| \leq \frac{\epsilon}{8} \]
and so
\[ \| h_2^2 + x_2^* x_2 - h_2 \| \leq \delta + \frac{\epsilon}{8} \leq \frac{\epsilon}{4}, \]
\[ \| k_2^2 + x_2 x_2^* - k_2 \| \leq \delta + \frac{\epsilon}{8} \leq \frac{\epsilon}{4}, \]
\[ \| k_2 x_2 - x_2 h_2 \| \leq \delta + \frac{\epsilon}{8} \leq \frac{\epsilon}{4}. \]

Let
\[ T_2 = T(h_2, x_2, k_2) \in \begin{bmatrix} C^1 + \overline{hAh} & \overline{hAh} \\ \overline{kAh} & \overline{kAk} \end{bmatrix}. \]

With \( \rho \) as in Lemma 2.2 \( \rho(T_2) = 1 \oplus 0 \). Since
\[ \|T_2^2 - T_2\| = \begin{bmatrix} -h_2 + h_2^2 + x_2^* x_2 & x_2 k_2 - h_2 x_2^* \\ k_2 x_2 - x_2 h_2 & -k_2 + k_2^2 + x x_2 \end{bmatrix} \]
we have
\[ \|T_2^2 - T_2\| \leq \frac{\epsilon}{2}. \]

Let \( P = f_\frac{1}{2}(T_2) \) and define \( \overline{h}, \overline{k} \) and \( \overline{x} \) via \( T(\overline{h}, \overline{x}, \overline{k}) = P \). As in the proof of Theorem 3.1 we see that \( x_3, k_3 \) and \( x_3 \) satisfy the relations for \( qC \). Since \( f_\frac{1}{2} \) is smooth on intervals containing the spectrum of \( T_2 \), these are elements of \( A_\infty \). \( \Box \)
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