ERRATUM FOR RICCI-FLAT GRAPHS WITH GIRTH AT LEAST FIVE

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Abstract. This erratum will correct the classification of Theorem 1 in [1] that misses the Triplex graph.

In Theorem 1 of [1], the classification of Ricci-flat graph with girth \( g(G) \geq 5 \) missed one graph – the Triplex graph, as discovered by three authors: Cushing, Kangaslampi, and Liu. Here is the correct theorem.

**Theorem 1.** Suppose that \( G \) is a Ricci-flat graph with girth \( g(G) \geq 5 \). Then \( G \) is one of the following graphs,

1. the infinite path,
2. cycle \( C_n \) with \( n \geq 6 \),
3. the dodecahedral graph,
4. the Petersen graph,
5. the half-dodecahedral graph,
6. the Triplex graph.

![四個圖形](image)

**Figure 1.** The four Ricci-flat graphs with girth 5

This error was caused by an incorrect implicit statement (in [1]) that any 3-regular Ricci-flat graph \( G \) has a surface embedding whose faces are all pentagons. In this erratum, we analyze the case that \( G \) does not have a surface embedding whose faces are all pentagons. We will show that this case leads a unique missing graph — the Triplex graph. An alternative method to correct Theorem 1 in [1] is given in [2].

Recall that Lemma 3 item 2 in [1] states:

**Lemma 1.** For any edge \( xy \) of a graph of girth at least 5, if \( d_x = d_y = 3 \) and \( \kappa(x, y) = 0 \), then \( xy \) belongs to two 5-cycles \( P_1 \) and \( P_2 \) such that \( P_1 \cap P_2 = xy \).

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Since $G$ contains no cycle of length 3 or 4, any $C_5$ contains the edge $xy$ is uniquely determined by a 3-path passing through $xy$. Since $d_x = d_y = 3$, there are four 3-paths of form $x_iyy_j$ for $i, j = 1, 2$. Here $x_1, x_2$ are two neighbors of $x$ other than $y$ and $y_1, y_2$ are two neighbors of $y$ other than $x$. We say two $C_5$'s are opposite to each other at $xy$ if one $C_5$ passes through $x_iyy_j$ and the other one passes through $x_{3-i}yy_{3-j}$. The above lemma says that there is a pair of opposite $C_5$'s sharing the edge $xy$. We say an edge $xy$ is irregular if there are exactly three $C_5$ passing through it.

From this lemma, we have the following corollary.

**Corollary 1.** If $G$ is a 3-regular Ricci-flat graph and contains no irregular edge, then $G$ can be embedded into a surface so that all faces are pentagons.

**Proof.** View $G$ as 1-dimension skeleton and glue pentagons to $G$ recursively. Starting with any $C_5$ and glue a pentagon to it as a face, call the two-dimensional region $M$. If $M$ contains a boundary edge $xy$, by induction, $xy$ is on one face $C_5$. We glue a pentagon face to the opposite $C_5$ at $xy$ to enlarge $M$. Since every edge is not irregular, the process will continue until $M$ has no boundary edge. When this process ends, we get an embedding of $G$ into some surface so that every face is a $C_5$.

We are ready to fix the proof of Theorem 1 in [1].

**Proof of Theorem 1.** Since in the original proof of Theorem 1 in [1], we have taken care of all the cases except that $G$ is 3-regular and contains an irregular edge $xy$. Let us show this case leads to a unique graph — the Triplex graph.

Let $xy$ is an irregular edge. It is contained in three $C_5$'s: $ux_2xyy_2u$, $vx_1xyy_1v$, and $wx_2xyy_1w$. The path $x_1xyy_2$ is not in any $C_5$. Let $w_1$ be the third neighbor of $x_1$, and $w_2$ be the third neighbor of $y_2$. Then $w_1, w_2$ are two distinct vertices, and they cannot be coincident with any vertex on the three $C_5$'s. This is our starting configuration (See Figure 2 with solid lines).

Now consider the edge $xx_1$. Observe that the path $w_1x_1xy$ is not on any $C_5$. Thus, the path $w_1x_1xx_2$ must be extended to a $C_5$. Either $w_1u$ is an edge or $w_1w$ is an edge. Similarly, by considering the edge $yy_2$, either $w_2v$ or $w_2v$ is an edge. These four possible edges
are shown as dashed lines i), ii), iii), and iv) in Figure 2. There are four combinations: i)+iii), i)+iv), ii)+iii), ii)+iv). The combination i)+iii) is impossible since $d_w = 3$. The two cases i)+iv) and ii)+iii) are symmetric. Essentially we have two cases to consider:

Case: i)+iv): Now consider the edge $w_1 x_1$. By Lemma 1, there are a pair of opposite $C_5$ sharing the edge $w_1 x_1$. The unmarked third neighbor of $w_1$ must be $u$. But this creates a $C_4$: $w_1 w x_2 u w_1$. Contradiction!

Case: ii)+iv). Let $w_3$ be the third neighbor of $w$. ($w_3$ is distinct from $w_1$ and $w_2$ since the girth of $G$ is at least 5.) Applying Lemma 1 on the edge $wx_2$, we have a pair of opposite $C_5$’s passing through $wx_2$. This will force $w_3 w_1$ to be an edge. Similarly, by considering $wy_1$, we conclude that $w_3 w_2$ must be an edge. This completes a 3-regular graph. It is easy to check this is the Triplex graph.

□
Figure 4. Unique way to complete into the Triplex graph.

References

[1] Y. Lin, L. Lu, and S.-T. Yau, *Ricci-flat graphs with girth at least five*, Comm. Anal. Geom. 22 (2014), no 4, 671-687.

[2] D. Cushing, R. Kangaslampi, Y. Lin, S. Liu, L. Lu, and S.-T. Yau, *Ricci-flat cubic graphs with girth five*, preprint.

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