Projective discrete modules over profinite groups

Alexandru Chirvasitu and Ryo Kanda

Abstract

We show that the category of discrete modules over an infinite profinite group has no non-zero projective objects and does not satisfy Ab4*. We also prove the same types of results in a generalized setting using a ring with linear topology.

Key words: profinite group, discrete module, projective object, abelian category, Grothendieck category, linear topology

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Introduction

The starting point for the present note is the pervasive phenomenon whereby abelian categories encountered “in nature”, housing various (co)homology theories, tend to have enough injectives but much more rarely have enough projectives.

This is familiar, for instance, for various flavors of sheaves:

• According to [Har77, Exercise III.6.2] the categories of modules, coherent modules and quasi-coherent modules on projective lines over infinite fields (with the Zariski topology) do not have enough projectives.

• For a more general class of schemes, it is shown in [Kan18, Theorem 1.1] that the category of quasi-coherent sheaves on a non-affine divisorial noetherian scheme (e.g. a non-affine quasi-projective schemes over a commutative noetherian ring) does not have enough projectives.

• On a slightly different note, locally connected Hausdorff spaces with no isolated points admit no non-zero projective sheaves of abelian groups [Bre97, p. 30, Exercise 4].
Here, we examine the category of discrete modules over a profinite group $G$, used in defining
the cohomology groups $H^i(G, -)$ (see, e.g., [Ser02, §2] for background and Section 1 below for
definitions).

Our main results show that if $G$ is infinite then the category of discrete modules

• admits no non-zero projective object (Theorem 2.1) and

• does not satisfy Grothendieck’s Ab4* condition (Proposition 2.2), that is, products are not
  exact.

Either of the two properties implies that the category does not have enough projectives.

We also consider the analogous question over fields (rather than the integers), resulting in a
characterization of those profinite groups for which the category of discrete modules has enough (or
equivalently, non-zero) projectives in characteristic $p$ (Theorem 3.1): they are exactly those whose
Sylow $p$-subgroups in the sense of [Ser02, §1.4] are finite. This will also provide a characterization
of projectives in the said category (Corollary 3.3).

A natural question is whether the techniques used in the proofs of the main results can be
applied to more general Grothendieck categories. The discrete modules over a profinite group form
a prelocalizing subcategory of the category of all modules, so one reasonable attempt is to generalize
them to a prelocalizing subcategory of the category of modules over a ring. It is known that such
subcategories bijectively correspond to linear topologies of the ring (see [Ste75, Proposition VI.4.2]).
We will express a necessary condition in terms of a linear topology, and prove a generalized results
for those prelocalizing subcategories satisfying the condition (Theorem 4.2).

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1 Preliminaries

Let $G$ be a profinite group. We denote by $(\mathcal{N}, \leq)$ the poset of open normal subgroups of $G$, ordered
by inclusion.

**Definition 1.1** A discrete $G$-module is a (left) $G$-module $M$ with the property that the action
$G \times M \to M$ is continuous when $M$ is equipped with the discrete topology.

We write $\text{d}_G\text{Mod}$ for the category of discrete $G$-modules, $\text{GMod}$ for the category of all $G$-modules
(without topology). Note that $\text{GMod}$ is canonically equivalent to the category of left modules over
the group algebra $\mathbb{Z}[G]$.

**Remark 1.2** Discrete modules can be characterized as those $G$-modules $M$ with the property that

$$M = \lim_{\longrightarrow} M^H$$

where $M^H$ is the submodule of $M$ consisting of those elements $x$ that are fixed by every element
of $H$ (see [Wil98, Proposition 6.1.2]). The direct limit can be replaced by a sum or a union.

Since $\text{d}_G\text{Mod}$ is a full subcategory of the Grothendieck category $\text{GMod}$ closed under subobjects,
quotient objects, and coproducts (such a subcategory is called prelocalizing, weakly closed, or a
hereditary pretorsion class), it is also a Grothendieck category. The forgetful functor \( \text{Mod}_G \to \text{Mod}_G \) has a right adjoint that sends a \( G \)-module \( M \) to its largest discrete submodule \( \lim_{\to H} M^H \).

The adjoint property tells us how to compute inverse limits in \( \text{Mod}_G \). Let \( \{M_i\}_{i \in I} \) is an inverse system in \( \text{Mod}_G \) and denote by \( \lim_{\to i} M_i \) the inverse limit taken in \( \text{Mod}_G \) and by \( \lim^{f}_{\to i} M_i \) the one taken in \( \text{Mod}_G \) (which is simply the limit of corresponding abelian groups; the ‘\( f \)’ superscript stands for “full”). Then

\[
\lim_{\to i} M_i = \lim_{\to H \in \mathcal{N}} \left( \lim_{\to i} M_i \right)^H,
\]

which is the largest discrete submodule of \( \lim^{f}_{\to i} M_i \). ♦

We will refer briefly to coalgebras and comodules over fields, for which our background reference is [DNR01, Chapters 1 and 2].

Write \( \mathcal{M}^C \) for the category of right \( C \)-comodules. According to [DNR01, Corollary 2.4.21], \( \mathcal{M}^C \) has enough projectives if and only if every finite-dimensional \( C \)-comodule has a (finite-dimensional again) projective cover. Furthermore, according to the proof of [DNR01, Corollary 2.4.22] every projective object in \( \mathcal{M}^C \) is a direct sum of finite-dimensional projective objects. These observations will be of use later.

2 Main results

**Theorem 2.1** Let \( G \) be a profinite group. The category \( \text{Mod}_G \) has non-zero projective objects if and only if \( G \) is finite.

**Proof** One implication is obvious, so we assume that \( G \) is an infinite profinite group and that \( P \) is a nonzero projective object in \( \text{Mod}_G \). \( P \) can be expressed as a union of its submodules \( P^H \) as in (1), and hence we have a surjection

\[
\bigoplus_{H \in \mathcal{N}} P^H \to P.
\]

In turn, every \( P^H \) is surjected upon by some free \( G/H \)-module \( F_H \), which is a direct sum of copies of \( \mathbb{Z}[G/H] \), and we have an epimorphism

\[
F := \bigoplus_{H \in \mathcal{N}} F_H \to P.
\] (2)

Now let \( H_0 > H_1 > \cdots \) be a strictly descending sequence of groups in \( \mathcal{N} \) (one exists, since \( G \) is assumed infinite). For each non-negative integer \( i \) we construct a surjection \( E_i \to F \) defined by substituting \( \mathbb{Z}[G/(H \cap H_i)] \) for every \( \mathbb{Z}[G/H] \) summand of \( F \) and surjecting

\[
\mathbb{Z}[G/(H \cap H_i)] \to \mathbb{Z}[G/H]
\]

naturally.

We now claim that the projectivity of \( P \) entails a factorization

\[
P \xrightarrow{\lim_{\to i} E_i} F \xrightarrow{id} P
\] (3)

where the limit is taken in the category \( \text{Mod}_G \).
To see this, note first that $P$ splits off as a summand of $F$. Since $E_0$ surjects onto the latter, we further obtain an direct summand embedding $P \to E_0$; now repeat the procedure to lift this to a map $P \to E_1$ fitting into a triangle

$$
P \longrightarrow E_1 \longrightarrow E_0$$

Continuing this recursively will produce (3).

The contradiction will follow if we show that the limit $\lim_{i} E_i$ in (3) vanishes. As explained in Remark 1.2, the limit in $DiG\text{-Mod}$ is obtained as the largest discrete submodule of the limit $\lim_{i}^f E_i$ in $G\text{-Mod}$. We thus have to argue that $\lim_{i}^f E_i$ contains no non-zero elements fixed by every open normal subgroup $H \triangleleft G$. To see this, recall that the connecting morphisms

$$E_{i+1} \to E_i, \quad i \in \mathbb{N}$$

whose filtered limit we are taking are coproducts of copies of the standard epimorphisms

$$\mathbb{Z}[G/(H \cap H_{i+1})] \to \mathbb{Z}[G/(H \cap H_i)]$$

for the summands $\mathbb{Z}[G/H]$ of $F$. To illustrate the claimed vanishing of the limit without irrelevant notational overhead we will consider the simpler limit

$$\lim_{i}^f \mathbb{Z}[G/(H \cap H_i)]$$

along the canonical surjections.

For every $K \in \mathcal{N}$ the image of the $K$-invariants through

$$\mathbb{Z}[G/(H \cap H_j)] \to \mathbb{Z}[G/(H \cap H_i)], \quad j > i$$

consists of $[K \cap H \cap H_i : K \cap H \cap H_j]$-multiples in the latter free abelian group. Indeed, if we write $N_i = H \cap H_i$ and $N_j = H \cap H_j$, then an element of $\mathbb{Z}[G/N_j]^K$ is of the form

$$\sum_{gN_j \in G/N_j} n_{gN_j} \cdot gN_j$$

such that $n_{gN_j} = n_{g'N_j}$ whenever $gKN_j = g'KN_j$. In particular, the elements of $G/N_j$ in the same coset of $(K \cap N_i)N_j$ have the same coefficient, and they are sent to a single element of $G/N_i$ whose coefficient is a multiple of

$$[(K \cap N_i)N_j : N_j] = [K \cap N_i : K \cap N_j] = [K \cap H \cap H_i : K \cap H \cap H_j].$$

Since $[G : K \cap H] < \infty$, the sequence $\{K \cap H \cap H_j\}^\infty_{i=0}$ is strictly descending. Thus the index $[K \cap H \cap H_i : K \cap H \cap H_j]$ grows indefinitely with $j$ for fixed $i$. It follows that the image of

$$\left(\lim_{i}^f \mathbb{Z}[G/(H \cap H_i)]\right)^K$$

in every $\mathbb{Z}[G/(H \cap H_i)]$ vanishes. In conclusion, as claimed, the maximal discrete submodule

$$d_{DiG\text{-Mod}} \ni \lim_{i}^f \mathbb{Z}[G/(H \cap H_i)] \subseteq \lim_{K}^f \mathbb{Z}[G/(H \cap H_i)] \in DiG\text{-Mod}$$

is trivial. \qed
Theorem 2.1 shows in particular that for infinite $G$ the category of discrete $G$-modules fails to have enough projective modules. That failure is in fact even stronger: recall that for a Grothendieck category, having enough projectives implies

- having non-zero projectives and
- being Ab4*, i.e. having exact products.

The next result proves that the second condition is also violated in the present context:

**Proposition 2.2** Let $G$ be a profinite group. The category $\mathbb{G}_G\text{Mod}$ satisfies Ab4* if and only if $G$ is finite.

**Proof** Again, one implication is obvious, so we assume that $G$ is infinite. We will show that the product of the augmentation morphisms

$$\mathbb{Z}[G/H] \to \mathbb{Z}[G/G] = \mathbb{Z}$$

for $H \in \mathcal{N}$ is not epic. Indeed, for fixed $K \in \mathcal{N}$ the image of $\mathbb{Z}[G/H]^K$ through (4) is contained in the ideal generated by $[K : H]$ whenever $H \leq K$. It follows that the all-1 element of the product $\prod_{H \in \mathcal{N}} \mathbb{Z}$ is not contained in the image of $\prod_{H \in \mathcal{N}} \mathbb{Z}[G/H]$, proving the claim. ■

3 Ground fields

The situation is rather different when working over a field $k$ in place of $\mathbb{Z}$. First, recall the notion of ***supernatural number*** from [Ser02, §1.3]: simply a formal product of the form

$$\prod_{\text{primes } p} p^{n_p}, \quad n_p \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$ 

A profinite group $G$ has an order $|G|$, well-defined as a supernatural number as the least common multiple of all orders $G/H$ for $H \in \mathcal{N}$. Similarly, we can define the index $[G : H]$ as a supernatural number for every closed subgroup $H \leq G$. There is also a concept of ***Sylow $p$-subgroup*** of $G$, i.e. a closed subgroup whose supernatural order is of the form $p^{n_p}$ (meaning it is pro-$p$, i.e. a filtered limit of finite $p$-groups) and whose index in $G$ does not have $p$ as a factor (or is coprime to $p$, in short). We refer to [Ser02, Chapter 1] for details.

Now let $k$ be a ground field of characteristic $p$ (a prime or zero) and write $\mathbb{G}_G\text{Vect}$ for the category of discrete $G$-modules over $k$ (note that we are suppressing $k$ from the notation, for brevity). The main result of the present section is a characterization of those $G$ for which this category admits non-zero projectives.

**Theorem 3.1** For a profinite group $G$ and a field $k$ of characteristic $p$ the following conditions are equivalent:

(a) The category $\mathbb{G}_G\text{Vect}$ has a non-zero projective object.

(b) The category $\mathbb{G}_G\text{Vect}$ has enough projective objects.

(c) The characteristic $p$ of $k$ has finite exponent in the supernatural number $|G|$.

**Remark 3.2** The condition in (c) is by convention assumed to hold vacuously when the characteristic is zero.
Before settling into the proof proper we make the preliminary observation that the category \( \mathcal{d}_G \text{Vect} \) is nothing but the category \( \mathcal{M}^C \) of comodules over the \( k \)-coalgebra \( C = k(G) \) of continuous \( k \)-valued functions on \( G \) (where \( k \) is equipped with the discrete topology).

**Proof of Theorem 3.1** That (a) follows from (b) is obvious, so we focus on proving (c) ⇒ (b) and (a) ⇒ (c).

(a) ⇒ (c). As noted at the end of Section 1, the existence of a non-zero projective entails the existence of a finite-dimensional one \( (P, \text{say}) \). \( P \) will then be projective over some group algebra \( k[G/H] \) for \( H \in \mathcal{N} \), and we can furthermore assume that it is a summand of \( k[G/H] \) (because it can be written as a direct sum of indecomposable projectives, which are summands of \( k[G/H] \)).

Now, if (c) were false then we could find a subgroup \( K \in \mathcal{N} \) of \( H \) with \( p \) dividing \( |H : K| \). Now consider the projection
\[
k[G/K] \to k[G/H] \to P.
\]
Splitting it would provide a summand \( \cong P \) of \( k[G/K] \) acted upon trivially by \( H/K \). But then, since \( p \) divides the order of this latter group, the image of this summand through (5) must vanish. This gives a contradiction and proves the desired implication.

(c) ⇒ (b). By [DNR01, Corollary 2.4.21] it suffices to show that every finite-dimensional discrete module \( M \) admits a surjection by a projective. We can regard \( M \) as a module over \( G/H \) for some \( H \in \mathcal{N} \), and suppose \( H \) is small enough to ensure that \( |G/H| \) and \( |G| \) have the same \( p \)-exponent.

Naturally, \( M \) is a quotient of a finite-dimensional projective \( P \) over \( G/H \). It remains to argue that \( P \) is still projective over \( G \), which will be the goal for the remainder of the proof.

Since \( P \) is \( G/H \)-projective, it must be projective over a Sylow \( p \)-subgroup \( S \subseteq G/H \). Our choice of \( H \) (such that \( [G : H] \) is divisible by the same exact power of \( p \) as \( |G| \)) means that there is a Sylow \( p \)-subgroup \( S \) of \( G \) mapping isomorphically over \( S \). We thus know that the restriction of \( P \) to \( S \) is projective.

Projectivity over \( G \) means showing that all higher cohomology
\[
\text{Ext}^i(P, -) \cong H^i(G, - \otimes P^*), \quad i \geq 1
\]
in the category \( \mathcal{d}_G \text{Vect} \) vanishes. We already know that it vanishes upon restricting via
\[
\text{res} : H^i(G, - \otimes P^*) \to H^i(S, - \otimes P^*),
\]
and the conclusion follows from the fact that this restriction morphism is one-to-one: this is more or less [Ser02, §2.4, Corollary to Proposition 9]. Although the latter result refers to cohomology over \( \mathbb{Z} \), the techniques apply essentially verbatim over \( k \).

The above reasoning applies unequivocally in positive characteristic, but requires interpretation in characteristic zero. In that case the \( p \) referred to throughout will be 0, Sylow subgroups will be trivial, etc. The validity of the proof will not be affected if these obvious modifications are made, using the fact that in characteristic zero the higher cohomology of a profinite group is
\[
H^i(G, -) = \lim_{\substack{\longrightarrow \ \mathcal{N}}} H^i(G/H, -) = \lim_{\substack{\longrightarrow \ \mathcal{N}}} 0 = 0,
\]
i.e. the coalgebra \( k(G) \) is cosemisimple.
We note in passing that as a byproduct of the proof of Theorem 3.1 we obtain the following characterization of projectives in $d_G \text{Vect}$:

**Corollary 3.3** Let $G$ be a profinite group and $k$ a field of characteristic $p$. The following statements hold:

(a) For every open normal subgroup $H$ with $p \nmid |H|$, a $G/H$-module is projective if and only if it is projective over $G$.

(b) The projective objects in $d_G \text{Vect}$ are direct sums of finite-dimensional, indecomposable projectives over $G/H$ for $H$ ranging over the open normal subgroups of $G$ with $p \nmid |H|$.

**Remark 3.4** If the characteristic is zero then the class of subgroups $H$ in Corollary 3.3 is unrestricted, i.e. we range over all of $N$.

Part (a) of Corollary 3.3 has the following partial converse.

**Lemma 3.5** Let $G$ be a profinite group, $k$ a field of characteristic $p$ and $H \leq G$ a closed normal subgroup with $p \nmid |H|$. Then, non-trivial $G/H$-modules cannot be projective over $G$.

**Proof** Let $P$ be a hypothetical non-zero $G/H$-module projective over $G$ and let $T \leq H$ be a (necessarily non-trivial) Sylow $p$-subgroup.

The restriction functor

$$\text{res} : d_G \text{Vect} \to d_T \text{Vect}$$

is left adjoint to the exact functor

$$d_T \text{Vect} \ni M \mapsto \text{Map}_T(G, M) \in d_G \text{Vect},$$

where

- $\text{Map}_T$ denotes continuous $T$-equivariant maps $G \to M$ with $M$ equipped with the discrete topology;
- $T$-equivariant means $f(tg) = tf(g)$ for all $t \in T$ and $g \in G$;
- the $G$-action is given by $g \triangleright f = f(\bullet g)$;

see [RZ10, §6.10]. Since it has an exact right adjoint, it follows that (6) preserves projectivity. In particular, $P$ will be projective over $T$. This, however, is impossible: $T$ is a pro-$p$-group, which Corollary 3.3 (b) ensures cannot have non-zero discrete projectives in characteristic $p$.

We now have the following alternate take on Theorem 2.1:

**Proof of Theorem 2.1 (alternative)** We begin as before, assuming that $G$ is an infinite profinite group and $P$ a non-zero projective object in $d_G \text{Mod}$. We then have the epimorphism (2) onto $P$ from a direct sum $F = \bigoplus_{H \in N} F_H$ of free $G/H$-modules.

By projectivity the epimorphism $P$ splits, realizing $P$ as a summand of $F$. In particular, there is some finite multiset of groups $H_i \in N$ so that

$$\{0\} \neq \left( \bigoplus_i \mathbb{Z}[G/H_i] \right) \cap P \subset F.$$  (7)
Now apply the scalar-extension functor
\[ E_p := \mathbb{F}_p \otimes \mathbb{Z}^{-} : \mathcal{d}_G \text{Mod} \to \mathcal{d}_G \text{Vect} \]
for a finite field \( \mathbb{F}_p \) with \( p \) elements, with \( p \) chosen judiciously (more on this below). The functor preserves projectivity (because it is left adjoint to the exact scalar restriction functor), so \( E_p(P) \) is projective (and clearly non-zero, since it was obtained by scalar-extending a free abelian group).

We now have direct sum decomposition
\[ E_p(P) \oplus \ast \cong E_p(F) = \bigoplus_{H \in \mathcal{N}} \mathbb{F}_p[G/H]^\oplus. \]  
(8)
The summands on the right-most side can be further decomposed as finite direct sums of modules with local endomorphism rings (because they are modules over finite group algebras over \( \mathbb{F}_p \)).

(7) implies its counterpart over \( \mathbb{F}_p \):
\[ \{0\} \neq \left( \bigoplus_i \mathbb{F}_p[G/H_i] \right) \cap (P/pP) \subset F/pF. \]

The proof of [AF92, Theorem 26.5] applied to \( M = F/pF, K = P/pP, \) and \( N = \bigoplus_i \mathbb{F}_p[G/H_i] \) shows that \( P/pP \) has a non-zero summand \( H \) isomorphic to a summand of \( N \). Thus [AF92, Corollary 26.6] implies that \( P/pP \) has a summand, say \( S \), isomorphic to an indecomposable summand of \( \mathbb{F}_p[G/H_i] \) for one of the finitely many \( i \) in (7). Note that \( S \) is projective in \( \mathcal{d}_G \text{Vect} \), being a summand of the projective object \( P/pP \).

Now we can specialize \( p \): choose it so as to ensure that it divides the (supernatural) order of \( H = \bigcap_i H_i \) (this is possible, since the latter group has finite index in the infinite profinite group \( G \)). The projectivity of \( S \) over \( G \) contradicts Lemma 3.5, finishing the proof.  

\[ \square \]

4 Ring-theoretic interpretation

Theorem 2.1 and Proposition 2.2 can be shown in a more general setting of a ring with a filter of ideals.

Let \( R \) be a ring. We denote by \( \mathcal{R} \text{Mod} \) the category of (left) \( R \)-modules.

Let \( \mathcal{I} \) be a downward filtered set of (two-sided) ideals of \( R \), that is, \( \mathcal{I} \) is a non-empty set of ideals and for any \( I_1, I_2 \in \mathcal{I} \), there exists \( J \in \mathcal{I} \) such that \( J \subseteq I_1 \) and \( J \subseteq I_2 \). For ease of comparison with the previous section, we use the notation
\[ \mathcal{R}_{\mathcal{I}} \text{Mod} := \{ M \in \mathcal{R} \text{Mod} \mid M = \lim_{I \in \mathcal{I}} \mathcal{R}M_I \} \]
where
\[ M_I := \{ x \in M \mid Ix = 0 \}. \]
The forgetful functor \( \mathcal{R}_{\mathcal{I}} \text{Mod} \to \mathcal{R} \text{Mod} \) admits a right adjoint given by \( M \mapsto \lim_I M_I \). Products in \( \mathcal{R}_{\mathcal{I}} \text{Mod} \) are described in a similar way to Remark 1.2.

In view of [Ste75, Proposition VI.4.2], \( \mathcal{R}_{\mathcal{I}} \text{Mod} \) is the prelocalizing subcategory of \( \mathcal{R} \text{Mod} \) corresponding to a left linear topology of \( R \) that admits a fundamental system of neighborhoods of \( 0 \in R \) consisting of two-sided ideals.
Remark 4.1 For a profinite group $G$, we take the group algebra $R := \mathbb{Z}[G]$ without topology. The poset $\mathcal{N}$ of open normal subgroups of $G$ defines the downward filtered set of ideals
\[ \mathcal{I} := \{ I_H \mid H \in \mathcal{N} \} \]
where $I_H$ is the kernel of the canonical surjection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$. For each $M \in G\text{Mod} = R\text{Mod}$, the largest submodule of $M$ belonging to $G/H\text{Mod}$ is $M^H$ as well as $M_{I_H}$. By the characterization (1) in Remark 1.2, we have $\mathfrak{g}_{G\text{Mod}} = \mathfrak{g}_{R\text{Mod}}$. ♦

For two ideals $I, J \subseteq R$, define the ideal
\[ (J : I) := \{ r \in R \mid Ir \subseteq J \}. \]
Note that $(R/J)_I = (J : I)/J$.

Theorem 4.2 Let $R$ be a ring and $\mathcal{I}$ a downward filtered set of ideals of $R$. Suppose that for each $I \in \mathcal{I}$,
\[ \bigcap_J ((J : I) + I) = I \tag{9} \]
where $J$ runs over all ideals in $\mathcal{I}$ with $J \subseteq I$. Then the following hold:
(a) $\mathfrak{g}_{R\text{Mod}}$ has no non-zero projective objects.
(b) $\mathfrak{g}_{R\text{Mod}}$ does not satisfy Ab4*.

The proof of these are parallel to the proofs of Theorem 2.1 and Proposition 2.2 but needs some modifications. We first rephrase the condition (9).

Lemma 4.3 The hypothesis of Theorem 4.2 is equivalent to the following condition: For any $I, K \in \mathcal{I}$ with $I \subseteq K$ and $0 \neq x \in (R/I)_K$, there exists $J \in \mathcal{I}$ with $J \subseteq I$ such that $x$ does not belong to the image of the canonical morphism
\[ (R/J)_K \to (R/I)_K. \]

Proof For ideals $J \subseteq I \subseteq K$, the commutative diagram
\[ \begin{array}{ccc}
(R/J)_K & \to & (R/I)_K \\
\downarrow & & \downarrow \\
(R/J)_I & \to & (R/I)_I
\end{array} \]
shows that any element of $(R/I)_I$ not in the image of $(R/J)_I \to (R/I)_I$ does not belong to the image of $(R/J)_K \to (R/I)_K$. Thus the condition is equivalent to that with $K = I$.

Since $(R/J)_I = (J : I)/J$ and $(R/I)_I = R/I$, the image of $(R/J)_I \to (R/I)_I$ is $((J : I) + I)/I$. Thus the condition is equivalent to
\[ \bigcap_J \frac{(J : I) + I}{I} = 0, \]
which is equivalent to (9). ■
Remark 4.4 The process of proving the vanishing of the limit $\lim_{i} E_i$ in Theorem 2.1 shows in particular that for any $N_i, N_j, K \in \mathcal{N}$ with $N_j \leq N_i$, the image of $\mathbb{Z}[G/N_j]^K \to \mathbb{Z}[G/N_i]^K$ consists of $[K \cap N_i : K \cap N_j]$-multiples and the index grows indefinitely when $N_j$ gets smaller. With the terminology of Remark 4.1, this implies that for every $0 \neq x \in R/I_{N_i}$, there exists some $j$ such that $x$ is not contained in the image of $(R/I_{N_i})_K \to (R/I_{N_i})_K$. Thus the condition in Lemma 4.3 is satisfied and hence the hypothesis of Theorem 4.2 holds.

Proof of Theorem 4.2 (a) Assume that $\mathcal{O}_R$Mod has a nonzero projective object $P$. Let $0 \neq x \in P$ and take $K \in \mathcal{I}$ such that $x \in P_K$. Similarly to the proof of Theorem 2.1, there is an epimorphism $F = \bigoplus_{I \in \mathcal{I}} F(I) \to P$ where $F(I)$ is a free $R/I$-module. We can assume that $F(I_j) \neq 0$ only if $I_j \subseteq K$.

Since $P$ is projective, the epimorphism $F \to P$ splits. We fix a section $P \to F$ and let $\tilde{x} \in F$ be the image of $x$ by the section. There are finitely many summands $R/I_1, \ldots, R/I_n$ of $F$ such that

$$\tilde{x} = \sum_j \tilde{x}_j \in \bigoplus_{j=1}^n R/I_j$$

(10)

where $0 \neq \tilde{x} \in R_{I_j}$. Since $F(I_j) \neq 0$, we have $I_j \subseteq K$ for all $j$.

Applying the condition in Lemma 4.3, we obtain $J_1, \ldots, J_n \in \mathcal{I}$ with $J_j \subseteq I_j$ such that $\tilde{x}_j$ does not belong to the image of

$$(R/J_j)_K \to (R/I_j)_K.$$ 

We construct a module $E$ from $F$ by substituting $R/J_j$ for $R/I_j$. The section $P \to F$ lifts along the canonical epimorphism $E \to F$, and we obtain a commutative diagram

$$P_K \xrightarrow{E_K} F_K.$$ 

However, $\tilde{x}$, the image of $x$ along $P_K \to F_K$, does not belong to the image of $E_K \to F_K$. This is a contradiction.

(b) We fix $I \in \mathcal{I}$ until the end of the proof. We claim that the product of the morphisms $R/J \to R/I$ where $J \in \mathcal{I}$ with $J \subseteq I$, is not an epimorphism. For each $K \in \mathcal{I}$ with $K \subseteq I$, the condition in Lemma 4.3 implies that there is $J \in \mathcal{I}$ with $J \subseteq I$ such that the image of

$$(R/J)_K \to (R/I)_K$$

does not contain $1 \in R/I = (R/I)_K$. The product $\prod_J (R/J)$ in $\mathcal{O}_R$Mod is the direct limit of $(\prod_J (R/J))_K$, where $\prod_J$ denotes the product in $\mathcal{O}_R$Mod and $K$ runs over all $K \in \mathcal{I}$ with $K \subseteq I$. Thus the all-1 element of $\prod_J (R/I)$ does not belong to the image of $\prod_J (R/J)$. ■

References

[AF92] Frank W. Anderson and Kent R. Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992. MR 1245487

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