ON 3-GAUGE TRANSFORMATIONS, 3-CURVATURES AND
Gray-CATEGORIES

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ABSTRACT. In the 3-gauge theory, a 3-connection is given by a 1-form $A$ valued in the Lie algebra $\mathfrak{g}$, a 2-form $B$ valued in the Lie algebra $\mathfrak{h}$ and a 3-form $C$ valued in the Lie algebra $\mathfrak{l}$, where $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ constitutes a differential 2-crossed module. We give the 3-gauge transformations from a 3-connection to another, and show the transformation formulae of the 1-curvature 2-form, the 2-curvature 3-form and the 3-curvature 4-form. The gauge configurations can be interpreted as smooth Gray-functors between two Gray 3-groupoids: the path 3-groupoid $\mathcal{P}_3(X)$ and the 3-gauge group $\mathcal{G}^G$ associated to the 2-crossed module $\mathcal{L}$, whose differential is $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$. The derivatives of Gray-functors are 3-connections, and the derivatives of lax-natural transformations between two such Gray-functors are 3-gauge transformations. We give the 3-dimensional holonomy, the lattice version of the 3-curvature, whose derivative gives the 3-curvature 4-form. The covariance of 3-curvatures easily follows from this construction. This Gray-categorical construction explains why 3-gauge transformations and 3-curvatures have the given forms. The interchanging 3-arrows are responsible for the appearance of terms concerning the Peiffer commutator $\{,\}$.

1. Introduction

String theory and $M$-theory involve various higher gauge fields, such as the $B$-field in string theory and the $C$-field in 11-dimensional $M$-theory. They are locally given by differential form fields of higher degree and are globally modeled by higher bundles with connections (higher gerbes with connections, higher differential characters) (cf. [1] [2] and references therein). In general, the extended $n$-dimensional relativistic objects appearing in string theory are usually coupled to background fields, which can naturally be the $n$-categorical version of fiber bundles with connections.

For nonabelian bundle gerbes [3] or more generally principal 2-bundles [4], there exists a framework of differential geometry: 2-connections and 2-curvatures (cf. [3] [5] [6] and references therein). There are lattice and differential formulations of the 2-gauge theory [7] [8], which can be applied to some $M$-brane models [9], BF theory [10] [11] and non-Abelian self-dual tensor field theories [12], etc.. The next step is to develop the 3-gauge theory, 3-connections and 3-curvatures for 3-bundles or bundles 2-gerbes [13] [14] [15] [16]. The 3-form gauge potentials have already appeared in physics (cf. [17] [18] and references therein).

Recall that the lattice gauge theory can be formulated in language of categories. Let $(V, E)$ be a directed graph, given by a set $V$ of vertices and a set $E$ of edges. Let $C^{V,E}$ be the associated category: the vertices as objects and the edges as arrows. Then configurations of lattice gauge theory are the functors from the category $C^{V,E}$ to the gauge group $\mathcal{G}^G$, the groupoid associated

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to the Lie group $G$ with one object. A gauge transformation is a natural transformation from one functor to another.

This construction was generalized to the 2-lattice gauge theory by Pfeiffer in [8]. Consider a *simplicial 2-complex* $(V; E; F)$ consisting of sets $V$, $E$ as above and a set $F$ of faces. There is an associated small 2-category $\mathcal{C}^{V; E; F}$: the vertices as objects, the edges as arrows and the faces as 2-arrows. The Lie group is replaced by a crossed module $\mathcal{M} = (\alpha : H \to G, \triangleright)$, from which we can construct a strict Lie 2-group $\mathcal{G}_\mathcal{M}$. Then the configurations of the 2-lattice gauge theory are 2-functors from the generalized lattice $\mathcal{C}^{V; E; F}$ to the 2-gauge group $\mathcal{G}_\mathcal{M}$, i.e., the edges are coloured by group elements of $G$ and the faces are coloured by elements of $G \times H$. A gauge transformation is a pseudonatural transformation from one 2-functor to another. If we take the length of lattice tending to zero, we get the differential 2-gauge theory [7].

To define the 3-lattice gauge theory, we need to replace a crossed module by a 2-crossed module $\mathcal{L}$, which is given by a complex of Lie groups:

\begin{equation}
L \xrightarrow{\delta} H \xrightarrow{\alpha} G,
\end{equation}

with smooth left actions $\triangleright$ of $G$ on $L$ and $H$ by automorphisms and a $G$-equivariant smooth function (the Peiffer lifting) $\{,\} : H \times H \to L$, and to construct the associated Gray 3-groupoid $\mathcal{G}_\mathcal{L}$. A *simplicial 3-complex* $(V; E; F; T)$ consists of sets $V$, $E$, $F$ as above and a set $T$ of tetrahedrons. There is an associated small tricategory $\mathcal{C}^{V; E; F; T}$: objects, 1-arrows and 2-arrows as above, and the tetrahedrons as 3-arrows. The configurations of our 3-lattice gauge theory will be functors from the generalized lattice $\mathcal{C}^{V; E; F; T}$ to the 3-gauge group $\mathcal{G}_\mathcal{L}$. Namely, the edges and faces are coloured as before and the tetrahedrons are coloured by elements of $G \times H \times L$.

A gauge transformation is a lax-natural transformation from one functor to another.

There is another similar, but more mathematical approach to this construction. There exists a bijection between connections and functors (play the role of holonomies) [19]:

$$\Lambda^1(X, g) \cong \{\text{smooth functors } \mathcal{P}_1(X) \to \mathcal{G}^G\},$$

where $\mathcal{P}_1(X)$ is the path groupoid of the manifold $X$, and $\Lambda^k(X, g)$ is the set of $g$-valued $k$-forms on $X$. This is generalized to the 2-gauge theory by Schreiber and Waldorf [20]: there exists a bijection between 2-connections and 2-functors (play the role of 2-dimensional holonomies):

$$\left\{\text{smooth 2-functors } \mathcal{P}_2(X) \to \mathcal{G}^\mathcal{M}\right\} \cong \{A \in \Lambda^1(X, g), B \in \Lambda^1(X, h); dA + A \wedge A = \alpha(B)\},$$

where $\mathcal{P}_2(X)$ is the path 2-groupoid of a manifold $X$ by adding 2-arrows to the path groupoid $\mathcal{P}_1(X)$. See also [21] [22] [23] for 1- and 2-dimensional holonomies.

To construct the 3-gauge theory, we will consider the path 3-groupoid $\mathcal{P}_3(X)$ by adding 3-arrows to the path 2-groupoid $\mathcal{P}_2(X)$, and smooth Gray-functors (play the role of 3-dimensional holonomies) from $\mathcal{P}_3(X)$ to $\mathcal{G}^\mathcal{L}$ [24]. Locally, a 3-connection on an open set $U$ of $\mathbb{R}^n$ is a triple $(A, B, C)$ with $A \in \Lambda^1(U, g), B \in \Lambda^2(U, h)$ and $C \in \Lambda^3(U, l)$. A 3-gauge transformation from a 3-connection $(A, B, C)$ to another one $(A', B', C')$ is given by

\begin{align}
A' &= \text{Ad}_{g^{-1}} A + g^{-1} dg + \alpha(\varphi) \\
B' &= g^{-1} \triangleright B + d\varphi + A' \wedge^\triangleright \varphi - \varphi \wedge \varphi - \delta(\psi) \\
C' &= g^{-1} \triangleright C - d\psi - A' \wedge^\triangleright \psi + \varphi \wedge^\triangleright \psi - B' \wedge^\{1\} \varphi - \varphi \wedge^\{1\} (g^{-1} \triangleright B),
\end{align}
for some $g \in \Lambda^0(U, G)$, $\varphi \in \Lambda^1(U, \mathfrak{h})$, $\psi \in \Lambda^2(U, I)$. Here $I \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\alpha} g$ is a differential 2-crossed module with smooth left actions $\triangleright$ of $g$ on $\mathfrak{h}$ and $I$ by automorphisms and a left action $\triangleright'$ of $\mathfrak{h}$ on $I$ (cf. §2 for notations).

The 1-curvature 2-form, 2-curvature 3-form and 3-curvature 4-form are defined as

\begin{align}
\Omega_1 & := dA + A \wedge A, \\
\Omega_2 & := dB + A \wedge^\triangleright B, \\
\Omega_3 & := dC + A \wedge^\triangleright C + B \wedge^\{\} B,
\end{align}

respectively. Under the 3-gauge transformation (1.2), these curvatures transform as follows:

\begin{align}
\Omega_1' & = g^{-1} \triangleright \Omega_1 + \alpha(B') - \alpha(g^{-1} \triangleright B), \\
\Omega_2' & = g^{-1} \triangleright \Omega_2 + [\Omega_1' - \alpha(B')] \wedge^\triangleright \varphi + \delta(C') - \delta(g^{-1} \triangleright C), \\
\Omega_3' & = g^{-1} \triangleright \Omega_3 - [\Omega_2' - \delta(C')] \wedge^\{\} \varphi + \varphi \wedge^\{\} [g^{-1} \triangleright (\Omega_2 - \delta(C))] - [\Omega_1' - \alpha(B')] \wedge^\{\} \psi.
\end{align}

We define the fake 1-curvature to be $F_1 = \Omega_1 - \alpha(B)$ and the fake 2-curvature to be $F_2 = \Omega_2 - \delta(C)$. Then the 3-curvature 4-form is covariant under the gauge transformations (1.2) if the fake 1- and fake 2-curvatures vanish.

In section 3, we give an elementary proof of the transformation formulae (1.3) of curvatures. This proof, having nothing to do with Gray-categories, is based on some properties of actions of $\triangleright$ and $\{\} \triangleright$ on Lie algebra valued differential forms, which are established in section 2.

Gray-categories are semi-strict tricategories. In a Gray-category there are two possible ways of composing two 2-arrows horizontally

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow \delta & & \downarrow \gamma \\
C & \xrightarrow{g} & C
\end{array}
\]

$((\gamma \#_0 g) \#_1 (f' \#_0 \delta))$ and $(f \#_0 \delta) \#_1 (\gamma \#_0 g')$, cf. §2.13 in [25] and references therein for the pasting theorem for 2-categories. This is an essential difference between 2-categories and Gray-categories. For the 2-crossed module $\mathcal{L}$ in (1.1), $H \xrightarrow{\alpha} G$ is no longer a crossed module in general. The Peiffer lifting $\{\}, \{\} : H \times H \to L$ measures its failure to be a crossed module. The interchanging 3-arrows in $G^\mathcal{L}$ are given by $\{\}$, and are responsible for the appearance of terms $\{\}$ in our formulae of gauge transformations and curvatures.

In section 4, we recall the definitions of a Gray-category, the Gray 3-groupoid constructed from the 2-crossed module $\mathcal{L}$ and lax-natural transformations between two Gray-functors.

In section 5, for a given lax-natural transformation between two Gray-functors $F$ and $\tilde{F}$ from the path 3-groupoid $P_3(X)$ to the 3-gauge group $G^\mathcal{L}$, the naturality of the lax-natural transformation gives us an equation with 3-parameters. We write down explicitly each side of the equation as the composition of several 3-arrows in the Gray 3-groupoid $G^\mathcal{L}$. Then take the derivatives with respect to the parameters at the origin to get the gauge transformation formula for the $C$ field in (1.2). The same is done for the $A$ and $B$ fields. In this construction, we must have $F_1 = F_2 = 0$. 

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Similarly in section 6, we consider a 4-path $\Theta$, whose boundary $\partial \Theta$ can be viewed as the composition of several 3-paths. For a Gray-functor $F$ from the path 3-groupoid $P_3(X)$ to the 3-gauge group $G^X$, $F(\partial \Theta)$ is its 3-dimensional holonomy, the lattice version of the 3-curvature. We write it explicitly as a composition of several 3-arrows in the 3-gauge group $G^X$, and take the derivatives with respect to the parameters at the origin to get the expression of the 3-curvature in (1.3). The covariance of the 3-holonomies under 3-gauge transformations is easily found from this construction.

The correct definition of the 3-curvature on a manifold $X$ was first appeared in [24] as the 2-curvature of a 2-connection on the loop space of $X$. The authors of [24] used it to define the 3-dimensional holonomy, but did not discuss the 3-gauge transformations. When this paper was almost finished, we found the preprint [18], where the authors studied the Penrose-Ward transformation between solutions to self-dual 3-gauge fields on the flat 6-dimensional space $M$ and $M$-trivial holomorphic principal 3-bundles over the twistor space $\mathbb{CP}^6$ (they consider the supersymmetric version). By writing down gauge transformations of relatively flat 3-gauge fields on the correspondence space, they found the gauge transformations of 3-gauge fields and transformation formulae of curvatures on the 6-dimensional space-time $M$. Then they claimed the gauge transformations of 3-gauge fields and the transformation formulae of curvatures in any dimension (cf. (5.17), (5.22) in [18]). The authors informed me that their method, based on the equivalence of the Čech and Dolbeault pictures and solving the Riemann-Hilbert problem, actually works in any dimension and the proofs of (5.17) and (5.22) in [18] is similar to the 6-dimensional case. In this paper we give a detailed proof of the gauge transformations of curvatures. Moreover, with the help of the Gray 3-groupoid constructed from a 2-crossed module, we see why gauge transformations and 3-curvatures are given by (1.2) and (1.3), respectively.

In this paper, we only consider the local 3-gauge theory. See [18] for a discussion of 3-connections on principal 3-bundles and their transformations under coordinate transformations.

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2. 2-crossed modules and Lie algebra valued differential forms

2.1. 2-crossed modules and differential 2-crossed modules. A pre-crossed module $G = (\alpha : H \to G, \triangleright)$ of Lie groups is given by a homomorphism $\alpha : H \to G$ between Lie groups, together with a smooth left action $\triangleright$ of $G$ on $H$ by automorphisms such that $\alpha(g \triangleright e) = g\alpha(e)g^{-1}$, for each $g \in G$ and $e \in H$. The Peiffer commutators in a pre-crossed module are defined as

$$\langle e, f \rangle = efe^{-1}(\alpha(e) \triangleright f^{-1}),$$

for any $e, f \in H$. A pre-crossed module is said to be a crossed module if all of its Peiffer commutators are trivial, i.e.

$$\alpha(e) \triangleright f = efe^{-1}.$$

A 2-crossed module of Lie groups is given by a complex (1.1) of Lie groups together with smooth left actions $\triangleright$ of $G$ on $L$ and $H$ by automorphisms (and on $G$ by conjugations), i.e.,

$$g \triangleright (e_1e_2) = g \triangleright e_1 \cdot g \triangleright e_2, \quad (g_1g_2) \triangleright e = g_1 \triangleright (g_2 \triangleright e),$$

(2.1)
for any \( g, g_1, g_2 \in G, e, e_1, e_2 \in H \) or \( L \), and a \( G \)-equivariant smooth function \( \{,\} : H \times H \to L \), the Peiffer lifting, such that

\[
(2.2) \quad g \triangleright \{e, f\} = \{g \triangleright e, g \triangleright f\},
\]

for any \( g \in G \) and \( e, f \in H \). They satisfy:

1. \( L \xrightarrow{\delta} H \xrightarrow{\triangleright} G \) is a complex of \( G \)-modules, namely, \( \delta \) and \( \alpha \) are \( G \)-equivariant and \( \alpha \circ \delta \) maps \( L \) to \( 1_G \), the identity of \( G \);
2. For each \( e, f \in H \), we have \( \delta\{e, f\} = \langle e, f \rangle \);
3. For each \( l, k \in L \), we have \( [l, k] = \{\delta(l), \delta(k)\} \), where \( [l, k] = lkl^{-1}k^{-1} \);
4. For each \( e, f, g \in H \), we have \( \{e, fg\} = \{e, f\}\{\delta(e), \delta(f)\} \alpha(e) \triangleright \{e, f\} \);
5. For \( e, f \in H \), we have \( \delta\{e, f\} = \{e, f\}\{\langle e, f \rangle\}^{-1} \alpha(e) \triangleright f \);
6. For each \( e \in H \) and \( l \in L \), we have \( \delta(l, e)\{e, \delta(l)\} = l(\alpha(e) \triangleright l^{-1}) \).

Define

\[
e \triangleright' l = l(\delta(l)^{-1}, e),
\]

where \( l \in L \) and \( e \in H \). It is known that \( \triangleright' \) is a left action of \( H \) on \( L \) by automorphisms. This together with the homomorphism \( \delta : L \to H \) defines a crossed module \([24]\). In particular, for any \( h \in H \),

\[
(2.3) \quad h \triangleright' 1_L = \{1_H, h\} = \{h, 1_H\} = 1_L
\]

(Lemma 1.4 in \([24]\)), where \( 1_H \) and \( 1_L \) are the identity of \( H \) and \( L \), respectively.

A differential crossed module is given by a homomorphism of Lie algebras

\[
\mathfrak{e} \xrightarrow{\partial} \mathfrak{g},
\]

together with the smooth left action \( \triangleright \) of \( \mathfrak{g} \) on \( \mathfrak{e} \) by automorphisms (and on \( \mathfrak{g} \) by the adjoint representation), such that

1. For each \( x \in \mathfrak{g} \) and each \( u, v \in \mathfrak{e} \), we have \( x \triangleright [u, v] = [x \triangleright u, v] + [u, x \triangleright v] \);
2. For each \( x, y \in \mathfrak{g} \) and each \( u \in \mathfrak{e} \), we have \( [x, y] \triangleright u = x \triangleright (y \triangleright u) - y \triangleright (x \triangleright u) \);
3. For each \( x \in \mathfrak{g} \) and each \( u \in \mathfrak{e} \), we have \( \partial(x \triangleright u) = [x, \partial(u)] \);
4. For each \( u, v \in \mathfrak{e} \), we have \( \partial(u) \triangleright v = [u, v] \).

A differential 2-crossed module is given by a complex of Lie algebras

\[
\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g},
\]

together with smooth left actions \( \triangleright \) of \( \mathfrak{g} \) on \( \mathfrak{h} \) and \( \mathfrak{l} \) by automorphisms (and on \( \mathfrak{g} \) by the adjoint representation), and a \( \mathfrak{g} \)-equivariant smooth function \( \{,\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l} \), the Peiffer lifting, such that

1. \( \mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g} \) is a complex of \( \mathfrak{g} \)-modules;
2. For each \( u, v \in \mathfrak{h} \), we have \( \delta\{u, v\} = \langle u, v \rangle \), where \( \langle u, v \rangle = [u, v] - \alpha(u) \triangleright v \);
3. For each \( x, y \in \mathfrak{l} \), we have \( [x, y] = \{\delta(x), \delta(y)\} \);
4. For each \( u, v, w \in \mathfrak{h} \), we have \( \{u, v, w\} = \alpha(u) \triangleright \{v, w\} = \{u, [v, w]\} - \alpha(v) \triangleright \{u, w\} - \{v, [u, w]\} \);
5. For each \( u, v, w \in \mathfrak{h} \), we have \( \delta\{u, v, w\} = \{\delta(u), \delta(v), \delta(w)\} \);
6. For each \( x \in \mathfrak{l} \) and \( v \in \mathfrak{h} \), we have \( \{\delta(x), v\} = -\alpha(v) \triangleright x \).
Proposition 2.1. (cf., e.g., Lemma 1.9 in [24]) \( v \triangleright x = -\{\delta(x), v\} \), for \( v \in \mathfrak{h} \) and \( x \in \mathfrak{l} \), defines a left action of \( \mathfrak{h} \) on \( \mathfrak{l} \), which together with the homomorphism \( \delta : \mathfrak{l} \to \mathfrak{h} \) defines a differential crossed module.

2.2. Lie algebra valued differential forms. Given a Lie algebra \( \mathfrak{t} \), we denote by \( \Lambda^k(U, \mathfrak{t}) \) the vector space of \( \mathfrak{t} \)-valued differential \( k \)-forms on \( U \). For \( K \in \Lambda^r(U, \mathfrak{t}) \), we can write \( K = \sum_a K^a X_a \) for some scalar differential \( k \)-forms \( K^a \) and elements \( X_a \in \mathfrak{t} \) (here \( \{X_a\} \) need not to be a basis). We will choose \( \mathfrak{t} \) to be \( \mathfrak{g} \) or \( \mathfrak{h} \) or \( \mathfrak{l} \). Here we assume \( \mathfrak{t} \) to be a matrix Lie algebra. Thus we have \([X, X'] = XX' - X'X \) for \( X, X' \in \mathfrak{t} \).

For \( K = \sum_a K^a X_a \in \Lambda^r(U, \mathfrak{t}) \), \( M = \sum_b M^b X_b \in \Lambda^l(U, \mathfrak{t}) \), define
\[
K \wedge M := \sum_{a,b} K^a \wedge M^b X_a X_b, \quad K \wedge [\cdot] M := \sum_{a,b} K^a \wedge M^b [X_a, X_b],
\]
and define
\[
dK = \sum dK^a X_a.
\]
For \( \Psi_j = \sum_b \Psi_j^b Y_b \in \Lambda^k(U, \mathfrak{h}) \), \( j = 1, 2 \), where \( Y_b \in \mathfrak{h} \), define
\[
(2.4) \quad K \wedge^\Psi \Psi_j := \sum_{a,b} K^a \wedge \Psi_j^b X_a \triangleright Y_b, \quad \Psi_1 \wedge^{\{\cdot\}} \Psi_2 := \sum_{a,b} \Psi_1^a \wedge \Psi_2^b \{Y_a, Y_b\},
\]
where \( K \) is valued in \( \mathfrak{g} \). In a similar way define \( \Psi_1 \wedge^{(\cdot)} \Psi_2, K \wedge^\Psi \psi \) and \( \Psi_j \wedge^\Psi \psi \) for \( \psi \in \Lambda^s(U, \mathfrak{l}) \). It is easy to see that the definitions above are independent of the choice of the expressions \( K = \sum K^a X_a \), etc., by linearity. Here we use the notations in [24]. In [18], \( \Psi_1 \wedge^{\{\cdot\}} \Psi_2 \) is written as \( \{\Psi_1, \Psi_2\} \).

The following proposition gives properties for Lie algebra valued differential forms corresponding to identities in the definition of a differential 2-crossed module.

Proposition 2.2. For \( \Psi \in \Lambda^k(U, \mathfrak{h}), \Psi' \in \Lambda^k(U, \mathfrak{h}), \Phi \in \Lambda^1(U, \mathfrak{h}) \) and \( \psi \in \Lambda^s(U, \mathfrak{l}) \), we have
\begin{enumerate}
\item \( \alpha(\Psi') \wedge^\Phi \Psi = \Psi' \wedge^{\{\cdot\}} \Phi - \Psi' \wedge^{(\cdot)} \Psi \),
\item \(-\alpha(\Psi) \wedge^{\{\cdot\}} \psi = \Psi \wedge^{\{\cdot\}} \delta(\psi) + (-1)^{ks} \delta(\psi) \wedge^{\{\cdot\}} \Psi\),
\item \( \delta(\psi) \wedge^{\{\cdot\}} \Psi = (-1)^{ks} \psi \wedge^{\{\cdot\}} \psi\),
\item \( \Psi \wedge^{\{\cdot\}} (\Phi \wedge \Phi) = (\Psi \wedge^{\{\cdot\}} \Phi) \wedge^{\{\cdot\}} \Phi\),
\item \( \Phi \wedge^{\{\cdot\}} (\Psi \wedge^{\{\cdot\}} \Phi) = \Phi \wedge^{\{\cdot\}} (\Phi \wedge^{\{\cdot\}} \Psi) + \alpha(\Phi) \wedge^{\{\cdot\}} (\Phi \wedge^{\{\cdot\}} \Psi)\).
\end{enumerate}

Proof. (1) Write \( \Psi = \sum_a \Psi^a Y_a, \Psi' = \sum_b (\Psi')^b Y_b \in \Lambda^k(U, \mathfrak{h}) \). Then we have
\[
\alpha(\Psi') \wedge^\Phi \Psi = \sum_{a,b} (\Psi')^b \wedge \Psi^a \alpha(Y_b) \triangleright Y_a = \sum_{a,b} (\Psi')^b \wedge \Psi^a ([Y_b, Y_a] - \langle Y_b, Y_a \rangle)
\]
\[
= \Psi' \wedge^{\{\cdot\}} \Psi - \Psi' \wedge^{(\cdot)} \Psi
\]
by using 2. in the definition of a differential 2-crossed module.

(2) Write \( \psi = \sum_c \psi^c Z_c \in \Lambda^k(U, \mathfrak{l}) \). Then,
\[
-\alpha(\Psi) \wedge^\Phi \psi = -\sum_{a,c} \Psi^a \wedge \psi^c \alpha(Y_a) \triangleright Z_c = \sum_{a,c} \Psi^a \wedge \psi^c ([Y_a, \delta(Z_c)] + \{\delta(Z_c), Y_a\})
\]
\[
= \Psi \wedge^{\{\cdot\}} \delta(\psi) + (-1)^{ks} \delta(\psi) \wedge^{\{\cdot\}} \Psi
\]
by 6. in the definition of a differential 2-crossed module.
(3) Let \( \Psi \) and \( \psi \) be as above.
\[
\delta(\psi) \wedge \{,\} \Psi = \sum_{a,c} \psi^c \wedge \Psi^a \{\delta(Z_c), Y_a\} = -\sum_{a,c} \psi^c \wedge \Psi^a Y_a \triangleright' Z_c \\
= -(-1)^{ks} \sum_{a,c} \Psi^a \wedge \psi^c Y_a \triangleright' Z_c = -(-1)^{ks} \Psi \wedge \triangleright' \psi
\]
by the definition of \( \triangleright' \) in Proposition 2.1.

(4) Write \( \Phi = \sum_b \Phi^b Y_b \in \Lambda^k(U, h) \). Then,
\[
\Psi \wedge \{,\} (\Phi \wedge \Phi) = \sum_{a,b,c} \Psi^a \wedge \Phi^b \wedge \Phi^c \{Y_a, Y_b Y_c\} = \sum_{a,b < c} \Psi^a \wedge \Phi^b \wedge \Phi^c \{Y_a, [Y_b, Y_c]\}
\]
\[
= \sum_{a,b < c} \Psi^a \wedge \Phi^b \wedge \Phi^c (\{\delta[Y_a, Y_b], Y_c\} - \{\delta[Y_a, Y_c], Y_b\})
\]
\[
= \sum_{a,b,c} \Psi^a \wedge \Phi^b \wedge \Phi^c \{Y_a, Y_b, Y_c\} = (\Psi \wedge \{,\} \Phi) \wedge \{,\} \Phi,
\]
by 5. in the definition of a differential 2-crossed module. Here \( \Phi^b \wedge \Phi^c = -\Phi^c \wedge \Phi^b \) since \( \Phi^{*a} \)'s are 1-forms.

(5) Let \( \Psi \) and \( \Phi \) be as above. Then,
\[
(\Phi \wedge \Phi) \wedge \{,\} \Psi = \sum_{a,b,c} \Phi^b \wedge \Phi^c \wedge \Psi^a \{Y_b Y_c, Y_a\} = \sum_{a,b < c} \Phi^b \wedge \Phi^c \wedge \Psi^a \{[Y_b, Y_c], Y_a\}
\]
\[
= \sum_{a,b < c} \Phi^b \wedge \Phi^c \wedge \Psi^a ([Y_b, [Y_c, Y_a]] - [Y_c, [Y_b, Y_a]]) + \alpha(Y_b) \triangleright \{Y_c, Y_a\} - \alpha(Y_c) \triangleright \{Y_b, Y_a\})
\]
\[
= \sum_{a,b,c} \Phi^b \wedge \Phi^c \wedge \Psi^a ([Y_b, [Y_c, Y_a]] + \alpha(Y_b) \triangleright \{Y_c, Y_a\})
\]
\[
= \Phi \wedge \{,\} (\Phi \wedge [,] \Psi) + \alpha(\Phi) \wedge \triangleright\Phi \wedge \{,\} \Phi,
\]
by 4. in the definition of a differential 2-crossed module.

\[\square\]

**Proposition 2.3.** (1) For \( K \in \Lambda^t(U, g) \), \( M \in \Lambda^s(U, g) \) and \( \psi \in \Lambda^s(U, l) \),
\[
\delta(K \wedge \triangleright^0 \psi) = K \wedge \triangleright^0 \delta(\psi),
\]
(2.5)
\[K \wedge \triangleright^o M = K \wedge M - (-1)^{ks} M \wedge K.\]

(2) For \( K \in \Lambda^t(U, g) \), \( \Psi_j \in \Lambda^{k_j}(U, h) \), \( j = 1, 2 \), and \( \gamma \in \Lambda^t(U, \mathfrak{k}) \) (\( \mathfrak{k} = g, h, l \)),
\[
d(K \wedge \triangleright^0 \gamma) = dK \wedge \triangleright^0 \gamma + (-1)^t K \wedge \triangleright^0 d\gamma,
\]
\[
d(\Psi_1 \wedge \{,\} \Psi_2) = (d\Psi_1) \wedge \{,\} \Psi_2 + (-1)^{k_1} \Psi_1 \wedge \{,\} d\Psi_2,
\]
\[
K \wedge \triangleright^o (\Psi_1 \wedge \{,\} \Psi_2) = (K \wedge \triangleright^0 \Psi_1) \wedge \{,\} \Psi_2 + (-1)^{k_1} \Psi_1 \wedge \{,\} (K \wedge \triangleright^0 \Psi_2),
\]
\[
K \wedge \triangleright^o (\Psi_1 \wedge \Psi_2) = (K \wedge \triangleright^0 \Psi_1) \wedge \Psi_2 + (-1)^{k_1} \Psi_1 \wedge (K \wedge \triangleright^0 \Psi_2).
\]

(3) For \( K, M \in \Lambda^1(U, g) \), \( \psi \in \Lambda^2(U, l) \),
\[K \wedge \triangleright^0 (M \wedge \triangleright^0 \psi) + M \wedge \triangleright^0 (K \wedge \triangleright^0 \psi) = (K \wedge M + M \wedge K) \wedge \triangleright^0 \psi.
\]

(4) For \( \psi \in \Lambda^2(U, l) \) and \( \Psi \in \Lambda^1(U, h) \),
\[
\delta(\psi) \wedge [,] \delta(\psi) = 0, \quad (\Psi \wedge \psi) \wedge [,] \Psi = 0.
\]
(2.6)
(5) (Equivariance) For $K \in \Lambda^*(U, g)$, $\Psi \in \Lambda^*(U, h)$ and $Y \in \Lambda^*(U, t)$ ($t = g, h, 1$),

$$
(g \triangleright \Psi) \wedge \{ \} (g \triangleright \Psi) = g \triangleright (\Psi \wedge \{ \} \Psi),
$$

(2.7)

$$
Ad_gK \wedge^\triangleright (g \triangleright Y) = g \triangleright (K \wedge^\triangleright Y).
$$

Proof.  (1) Write $K = \sum_a K^a X_a, M = \sum_b M^b X_b$ and $\psi = \sum_c \psi^c Z_c$. By $\delta$ being a $g$-homomorphism,

$$
K \wedge^\triangleright \delta(\psi) = \sum_{a,b} K^a \wedge \psi^b X_a \triangleright \delta(Z_b) = \sum_{a,b} K^a \wedge \psi^b \delta(X_a \triangleright Z_b) = \delta(K \wedge^\triangleright \psi).
$$

Since $X_a \triangleright X_b = [X_a, X_b]$ ($\triangleright$ acting on $g$ by the adjoint action), we have

$$
K \wedge^\triangleright M = \sum_{a,b} K^a \wedge M^b X_a \triangleright X_b = \sum_{a,b} K^a \wedge M^b (X_a X_b - X_b X_a) = K \wedge M - (-1)^{ts} M \wedge K.
$$

(2) Write $\Psi_j = \sum_b \Psi^b_j Y_b \in \Lambda^k(U, h)$. Then

$$
d(\Psi_1 \wedge \{ \} \Psi_2) = d\sum_{a,b} \Psi_1^a \wedge \Psi_2^b \{ Y_a, Y_b \} = \sum_{a,b} d\Psi_1^a \wedge \Psi_2^b \{ Y_a, Y_b \} + (-1)^{k_1} \sum_{a,b} \Psi_1^a \wedge d\Psi_2^b \{ Y_a, Y_b \}
$$

$$
= d\Psi_1 \wedge \{ \} \Psi_2 + (-1)^{k_1} \Psi_1 \wedge \{ \} d\Psi_2,
$$

and

$$
K \wedge^\triangleright \left( \Psi_1 \wedge \{ \} \Psi_2 \right) = \sum_{a,b,c} K^c \wedge \Psi_1^a \wedge \Psi_2^b \{ X_a, Y_b \}
$$

$$
= \sum_{a,b,c} K^c \wedge \Psi_1^a \wedge \Psi_2^b (\{ X_a \triangleright Y_b, Y_b \} \{ Y_a, X_c \triangleright Y_b \}),
$$

by $g$-equivariance of $\{ \}$ from (2.2). The proofs of the other identities are similar.

(3) Since $\triangleright$ is a left action of $g$ on $I$ from (2.1), we have

$$
K \wedge^\triangleright (M \wedge^\triangleright \psi) + M \wedge^\triangleright (K \wedge^\triangleright \psi) = \sum_{a,b,c} K^a \wedge M^b \wedge \psi^c (X_a \triangleright (X_b \triangleright Z_c) - X_b \triangleright (X_a \triangleright Z_c))
$$

$$
= \sum_{a,b,c} K^a \wedge M^b \wedge \psi^c [X_a, X_b] \triangleright Z_c.
$$

(4)

$$
\delta(\psi) \wedge \{ \} \delta(\psi) = \sum_{a,b} \psi^a \wedge \psi^b \{ \delta(Z_a), \delta(Z_b) \} = \sum_{a,b} \psi^a \wedge \psi^b [Z_a, Z_b]
$$

by 3. in the definition of a differential 2-crossed module. Here $\psi^a \wedge \psi^b = \psi^b \wedge \psi^a$ since they are 2-forms. It must vanish. And

$$
(\Psi \wedge \Psi) \wedge \{ \} \Psi = \sum_{a,b,c} \Psi^a \wedge \Psi^b \wedge \Psi^c \{ Y_a Y_b, Y_c \} = \sum_{a,b,c} \Psi^a \wedge \Psi^b \wedge \Psi^c (Y_a Y_b Y_c - Y_c Y_a Y_b) = 0,
$$

by $\Psi^a \wedge \Psi^b \wedge \Psi^c = \Psi^c \wedge \Psi^a \wedge \Psi^b$, since $\Psi^a$’s are 1-forms.

(5)

$$
(g \triangleright \Psi) \wedge \{ \} (g \triangleright \Psi) = \sum_{a,b} \Psi^a \wedge \Psi^b \{ g \triangleright Y_a, g \triangleright Y_b \} = \sum_{a,b} \Psi^a \wedge \Psi^b g \triangleright \{ Y_a, Y_b \} = g \triangleright (\Psi \wedge \{ \} \Psi)
$$
by $G$-equivariance of $\{ \cdot \}$ in (2.2). And
\[
Ad_g K \wedge^\triangleright (g \triangleright \Upsilon) = \sum_{a,b} K^a \wedge \Upsilon^b(gX_ag^{-1}) \triangleright (g \triangleright Z_b) = \sum_{a,b} K^a \wedge \Upsilon^b(gX_a) \triangleright Z_b
\]
\[
= \sum_{a,b} K^a \wedge \Upsilon^b g \triangleright (X_a \triangleright Z_b) = g \triangleright (K \wedge^\triangleright \Upsilon),
\]
by $T \triangleright (g \triangleright S) = (Tg) \triangleright S$, which follows from $(g_1g_2) \triangleright S = g_1 \triangleright (g_2 \triangleright S)$. \hfill \Box

**Corollary 2.1.** For $\Psi \in \Lambda^1(U, h)$ and $A \in \Lambda^1(U, g)$, we have

\[
(\alpha(\Psi) \wedge A + A \wedge \alpha(\Psi)) \wedge^\triangleright \Psi = (A \wedge^\triangleright \Psi) \wedge \Psi - \Psi \wedge (A \wedge^\triangleright \Psi) - (A \wedge^\triangleright \Psi) \wedge^{(\cdot)} \Psi.
\]

**Proof.** Write $A = \sum_a A^a X_a$ and $\Psi = \sum_b \Psi^b Y_b$. Then,

\[
(\alpha(\Psi) \wedge A + A \wedge \alpha(\Psi)) \wedge^\triangleright \Psi = \sum_{a,b,c} \Psi^b \wedge A^a \wedge \Psi^c([\alpha(Y_b), X_a] \triangleright Y_c)
\]
\[
= \sum_{a,b,c} \Psi^b \wedge A^a \wedge \Psi^c(\alpha(Y_b) \triangleright (X_a \triangleright Y_c) - X_a \triangleright (\alpha(Y_b) \triangleright Y_c))
\]
\[
= \sum_{a,b,c} \Psi^b \wedge A^a \wedge \Psi^c([Y_b, X_a] \triangleright Y_c) - (Y_b, X_a \triangleright Y_c) - X_a \triangleright [Y_b, Y_c] + X_a \triangleright (Y_b, Y_c)
\]
\[
= \sum_{a,b,c} \Psi^b \wedge A^a \wedge \Psi^c(-[X_a \triangleright Y_b, Y_c] + (X_a \triangleright Y_b, Y_c))
\]
\[
= (A \wedge^\triangleright \Psi) \wedge \Psi - \Psi \wedge (A \wedge^\triangleright \Psi) - (A \wedge^\triangleright \Psi) \wedge^{(\cdot)} \Psi.
\]

Here $X_a \triangleright [Y_b, Y_c] = [X_a \triangleright Y_b, Y_c] + [Y_b, X_a \triangleright Y_c]$ by $\triangleright$ acting on $h$ as automorphisms, and $X_a \triangleright (Y_b, Y_c) = X_a \triangleright \delta\{Y_b, Y_c\} = \delta(X_a \triangleright \{Y_b, Y_c\})$
\[
= \delta(\{X_a \triangleright Y_b, Y_c\} + \{Y_b, X_a \triangleright Y_c\}) = \{X_a \triangleright Y_b, Y_c\} + \{Y_b, X_a \triangleright Y_c\}.
\]
The corollary is proved. \hfill \Box

### 3. Covariance of Curvatures under the 3-Gauge Transformations

#### 3.1. Three kinds of gauge transformations.

There are three kinds of 3-gauge transformations. The 3-gauge transformation of the first kind:

\[
A' = Ad_{g^{-1}} A + g^{-1} dg,
\]
\[
B' = g^{-1} \triangleright B,
\]
\[
C' = g^{-1} \triangleright C,
\]

the 3-gauge transformation of the second kind:

\[
A' = A + \alpha(\varphi),
\]
\[
B' = B + d\varphi + A' \wedge^\triangleright \varphi - \varphi \wedge \varphi,
\]
\[
C' = C - B' \wedge^{(\cdot)} \varphi - \varphi \wedge^{(\cdot)} B,
\]
and the 3-gauge transformation of the third kind:

\[
\begin{align*}
A' &= A, \\
B' &= B - \delta(\psi), \\
C' &= C - d\psi - A' \wedge^\varphi \psi.
\end{align*}
\]

(3.3)

If we write a 3-gauge transformation of the second kind as

\[
\begin{align*}
A'' &= A' + \alpha(\varphi), \\
B'' &= B' + d\varphi + A'' \wedge^\varphi \varphi - \varphi \wedge^\varphi, \\
C'' &= C' - B'' \wedge^\{1\} \varphi - \varphi \wedge^\{1\} B',
\end{align*}
\]

(3.4)

and a 3-gauge transformation of the third kind as

\[
\begin{align*}
\tilde{A} &= A'', \\
\tilde{B} &= B'' - \delta(\psi), \\
\tilde{C} &= C'' - d\psi - \tilde{A} \wedge^\varphi \psi,
\end{align*}
\]

(3.5)

then the composition of (3.1), (3.4) and (3.5) gives

\[
\begin{align*}
\tilde{A} &= A' + \alpha(\varphi) = Adg^{-1}A + g^{-1}dg + \alpha(\varphi), \\
\tilde{B} &= B'' - \delta(\psi) = g^{-1} \triangleright B + d\varphi + \tilde{A} \wedge^\varphi \varphi - \varphi \wedge^\varphi - \delta(\psi),
\end{align*}
\]

and

\[
\begin{align*}
\tilde{C} &= C' - B'' \wedge^\{1\} \varphi - \varphi \wedge^\{1\} B' - d\psi - \tilde{A} \wedge^\varphi \psi \\
&= g^{-1} \triangleright C - \left( \tilde{B} + \delta(\psi) \right) \wedge^\{1\} \varphi - \varphi \wedge^\{1\} (g^{-1} \triangleright B) - d\psi - \tilde{A} \wedge^\varphi \psi \\
&= g^{-1} \triangleright C - d\psi - \tilde{A} \wedge^\varphi \psi - \tilde{B} \wedge^\{1\} \varphi - \varphi \wedge^\{1\} (g^{-1} \triangleright B) + \varphi \wedge^\varphi' \psi
\end{align*}
\]

by using Proposition 2.2 (3). Namely, the composition of (3.1), (3.4) and (3.5) gives the general 3-gauge transformation (1.2).

3.2. Covariance of 1-curvature 2-forms and 2-curvature 3-forms. For the 1-curvature, under the 3-gauge transformation (1.2),

\[
\begin{align*}
dA' + A' \wedge A' &= dg^{-1} \wedge A + g^{-1}dAg - g^{-1}A \wedge dg + dg^{-1} \wedge dg + \alpha(d\varphi) \\
&= (g^{-1}A + g^{-1}dg + \alpha(\varphi)) \wedge (g^{-1}A + g^{-1}dg + \alpha(\varphi)) \\
&= g^{-1}(dA + A \wedge A) + A' \wedge \alpha(\varphi) + A' \wedge \alpha(\varphi) \wedge A' - \alpha(\varphi \wedge \varphi) \\
&= g^{-1} \triangleright \Omega_1 + \alpha(B') - \alpha(g^{-1} \triangleright B),
\end{align*}
\]

(3.6)

by \(\alpha(\delta(\psi)) = 0\). Here \(A' \wedge \alpha(\varphi) + \alpha(\varphi) \wedge A' = A' \wedge^\varphi \alpha(\varphi)\) by the second identity of 2.3 in Proposition 2.3.
For the 2-curvature, under the 3-gauge transformation (1.2), we find that
\[
\begin{align*}
\Omega_2 &= g^{-1} \triangleright \Omega_1 + (g^{-1} \triangleright B + d\varphi) \\
&= g^{-1} \triangleright \Omega_1 + (A' \triangleright \varphi - A' \triangleright d\varphi - d\varphi \wedge \varphi) \\
&- \delta(d\psi) + (g^{-1} \triangleright B + d\varphi + A' \triangleright \varphi - \varphi \wedge \varphi - \delta(\psi)) \\
&= g^{-1} \triangleright (B + A \triangleright \varphi) + (dA' + A' \triangleright A') \triangleright \varphi + \alpha(\varphi) \triangleright (g^{-1} \triangleright B) \\
&- d\varphi \wedge \varphi + \varphi \wedge d\varphi - A' \triangleright \varphi + \alpha(\varphi) \triangleright (g^{-1} \triangleright B) \\
&+ \alpha(\varphi) \triangleright (g^{-1} \triangleright B) - \delta(d\psi + A' \triangleright \psi).
\end{align*}
\] (3.7)

by using Proposition 2.3 (1)-(3) and (5) in the second identity and using Proposition 2.3 (2) in the third identity. Here \(d\varphi \wedge \varphi = d\varphi \wedge \varphi - \varphi \wedge d\varphi\) by definition. \(\Omega_1'\) in (3.6) can be written as
\[
\Omega_1' = g^{-1} \triangleright \Omega_1 + \alpha(d\varphi) + \alpha(\varphi) \wedge A' + A' \wedge \alpha(\varphi) - \alpha(\varphi \wedge \varphi).
\] (3.8)

by using Proposition 2.2 (1) and Proposition 2.3 (4), and
\[
\alpha(\varphi \wedge \varphi) \triangleright \varphi = (\varphi \wedge \varphi) \triangleright \varphi - (\varphi \wedge \varphi) \triangleright \varphi = -(\varphi \wedge \varphi) \triangleright \varphi,
\] (3.9)

by using Corollary 2.1. Substitute (3.8)-(3.10) to \(\Omega_1' \triangleright \varphi\) to get
\[
\Omega_1' \triangleright \varphi = (g^{-1} \triangleright B) \triangleright \Omega_1 \triangleright \varphi + d\varphi \triangleright B \triangleright \varphi - d\varphi \triangleright \varphi \\
+ (A' \triangleright \varphi) \wedge \varphi \wedge \varphi \wedge (A' \triangleright \varphi) - (A' \triangleright \varphi) \triangleright \varphi + (\varphi \wedge \varphi) \triangleright \varphi.
\] (3.11)

Here we have applied Proposition 2.2 (1) to \(\alpha(d\varphi) \triangleright \varphi\). Now substitute (3.11) into (3.7) to get
\[
\begin{align*}
\Omega_2 &= g^{-1} \triangleright \Omega_1 + (g^{-1} \triangleright B) \triangleright \varphi - \varphi \wedge \varphi \triangleright (g^{-1} \triangleright B) \\
&+ \alpha(\varphi) \triangleright (g^{-1} \triangleright B) - \delta(d\psi + A' \triangleright \psi).
\end{align*}
\] (3.12)

by Proposition 2.2 (1) again, and similarly
\[
\alpha(\varphi) \triangleright (g^{-1} \triangleright B) \wedge \varphi = (g^{-1} \triangleright B) \triangleright \varphi - (g^{-1} \triangleright B) \triangleright \varphi \\
= -\varphi \wedge \varphi \triangleright (g^{-1} \triangleright B) - (g^{-1} \triangleright B) \triangleright \varphi.
\] (3.13)

Now substitute the summation of (3.13) and (3.14) into (3.12) to get
\[
\begin{align*}
\Omega_2 &= g^{-1} \triangleright \Omega_2 + (g^{-1} \triangleright \Omega_1 - \alpha(g^{-1} \triangleright B)) \triangleright \varphi - (g^{-1} \triangleright B + d\varphi + A' \triangleright \varphi - \varphi \wedge \varphi) \triangleright \varphi \\
&- \varphi \wedge \varphi \triangleright (g^{-1} \triangleright B) - \delta(d\psi + A' \triangleright \psi) \\
= g^{-1} \triangleright (\Omega_1' - \alpha(B')) \triangleright \varphi - B' \triangleright \varphi - \delta(\psi) \triangleright \varphi \\
- \varphi \wedge \varphi \triangleright (g^{-1} \triangleright B) - \delta(d\psi + A' \triangleright \psi) \\
= g^{-1} \triangleright (\Omega_1' - \alpha(B')) \triangleright \varphi + \delta(C') - \delta(g^{-1} \triangleright C)
\end{align*}
\]
by \( \delta(\psi) \wedge (\cdot) \varphi = -\delta(\varphi \wedge (\cdot) \psi) \) from applying \( \delta \) to Proposition 2.2 (3) and
\[
\delta(C') - g^{-1} \triangleright \delta(C) = -\delta(d\psi + A' \wedge \varphi) + \delta(\varphi \wedge A' \psi) - B' \wedge (\cdot) \varphi - \varphi \wedge (\cdot) (g^{-1} \triangleright B).
\]

3.3. Covariance of 3-curvature 4-forms. (1) Under the 3-gauge transformation (3.1) of the first kind,
\[
\Omega'_3 = g^{-1} \triangleright dC - g^{-1} dgg^{-1} \wedge (\cdot) C + (Ad_{g^{-1}} A + g^{-1} dg) \wedge A \wedge (g^{-1} \triangleright C)
+ (g^{-1} \triangleright B) \wedge [\Omega' - (g^{-1} \triangleright B)] = g^{-1} \triangleright (dC + A \wedge C) + B \wedge (\cdot) B = g^{-1} \triangleright \Omega_3,
\]
by equivariance (2.7) in Proposition 2.3. We have already seen from the above subsection that under this transformation
\[
\Omega'_2 = g^{-1} \triangleright \Omega_2, \quad \Omega'_1 = g^{-1} \triangleright \Omega_1.
\]

(2) Under the 3-gauge transformation (3.1) of the second kind, we have
\[
\Omega'_3 = dC' + A' \wedge (\cdot) C' + B' \wedge (\cdot) B'
= dC + A' \wedge C - (dB' + A' \wedge B') \wedge (\cdot) \varphi + \varphi \wedge (\cdot) (dB + A' \wedge B)
-B' \wedge (\cdot) (d\varphi + A' \wedge (\cdot) \varphi) - (d\varphi + A' \wedge (\cdot) (d\varphi + A' \wedge (\cdot) \varphi) \wedge (\cdot) B + B' \wedge (\cdot) B'
\]
(3.17)
\[
= \Omega_3 + \alpha(\varphi) \wedge (\cdot) C - \Omega'_2 \wedge (\cdot) \varphi + \varphi \wedge (\cdot) \Omega_2 + \varphi \wedge (\cdot) (\alpha(\varphi) \wedge A' \wedge B)
-B' \wedge (\cdot) (\varphi \wedge \varphi) - (\varphi \wedge \varphi) \wedge (\cdot) B,
\]
by applying Proposition 2.3 (2) and using the following in the last identity, i.e.,
\[
-B' \wedge (\cdot) (d\varphi + A' \wedge (\cdot) \varphi) + B' \wedge (\cdot) B' = B' \wedge (\cdot) B - B' \wedge (\cdot) (\varphi \wedge \varphi),
\]
and
\[
-(d\varphi + A' \wedge (\cdot) \varphi) \wedge (\cdot) B + B' \wedge (\cdot) B = B \wedge (\cdot) B - (\varphi \wedge \varphi) \wedge (\cdot) B.
\]
But
\[
-B' \wedge (\cdot) (\varphi \wedge \varphi) = -(B' \wedge (\cdot) \varphi) \wedge (\cdot) \varphi
\]
by applying Proposition 2.2 (4) and
\[
\alpha(\varphi) \wedge (\cdot) C = -\varphi \wedge (\cdot) \delta(C) + \delta(C) \wedge (\cdot) \varphi
= -\varphi \wedge (\cdot) \delta(C) + \delta(C') \wedge (\cdot) \varphi + [B' \wedge (\cdot) \varphi + \varphi \wedge (\cdot) B] \wedge (\cdot) \varphi
\]
by applying Proposition 2.2 (2). Now substitute the summation of (3.18) and (3.19) into (3.17) to get
\[
\Omega'_3 = \Omega_3 - [\Omega'_2 - \delta(C')] \wedge (\cdot) \varphi + \varphi \wedge (\cdot) [\Omega_2 - \delta(C)] + \mathcal{E}
\]
with
\[
\mathcal{E} = (\varphi \wedge (\cdot) B) \wedge (\cdot) \varphi + \varphi \wedge (\cdot) (\alpha(\varphi) \wedge (\cdot) B) - (\varphi \wedge \varphi) \wedge (\cdot) B = 0.
\]
This vanishing is because we have that
\[
\varphi \wedge (\cdot) [\alpha(\varphi) \wedge (\cdot) B] = \varphi \wedge (\cdot) (\varphi \wedge (\cdot) B) - \varphi \wedge (\cdot) (\varphi \wedge (\cdot) B)
\]
by Proposition 2.2 (1), and
\[
\varphi \wedge (\cdot) B = \varphi \wedge (\cdot) B + \alpha(\varphi) \wedge (\cdot) (\varphi \wedge (\cdot) B)
\]
by Proposition 2.2 (1), and
by Proposition 2.2 (5), and by substituting (3.22) (3.23) into (3.21) we have

\[
E = \delta(\varphi \wedge {}^\uparrow \downarrow B) \wedge {}^\uparrow \downarrow \phi - \varphi \wedge {}^\uparrow \downarrow \delta(\varphi \wedge {}^\uparrow \downarrow B) - \alpha(\varphi) \wedge {}^\uparrow \downarrow (\varphi \wedge {}^\uparrow \downarrow B) = 0
\]

by applying Proposition 2.2 (2) to \( \psi = \varphi \wedge {}^\uparrow \downarrow B \).

(3) Under the 3-gauge transformation (3.5) of the third kind,

\[
\tilde{\Omega}_3 = \delta \tilde{C} + \tilde{A} \wedge {}^\uparrow \downarrow \tilde{C} + \tilde{B} \wedge {}^\uparrow \downarrow \tilde{B} = dC'' + A'' \wedge {}^\uparrow \downarrow C'' + B'' \wedge {}^\uparrow \downarrow B'' + \mathcal{R}
\]

with

\[
\mathcal{R} = -dA'' \wedge {}^\uparrow \downarrow \psi - A'' \wedge {}^\uparrow \downarrow (A'' \wedge {}^\uparrow \downarrow \psi) - B'' \wedge {}^\uparrow \downarrow \delta(\psi) - \delta(\psi) \wedge {}^\uparrow \downarrow \psi
\]

\[
= -dA'' + A'' \wedge {}^\uparrow \downarrow \alpha(B'') \wedge {}^\uparrow \downarrow \psi = -(\Omega''_1 - \alpha(B'')) \wedge {}^\uparrow \downarrow \psi.
\]

since \( \delta(\psi) \wedge {}^\uparrow \downarrow \delta(\psi) = 0 \) by Proposition 2.3 (4), and

\[
-\delta(\psi) \wedge {}^\uparrow \downarrow \psi - \delta(\psi) \wedge {}^\uparrow \downarrow \psi = \alpha(B'') \wedge {}^\uparrow \downarrow \psi,
\]

by Proposition 2.2 (2) and \( A'' \wedge {}^\uparrow \downarrow \psi = (A'' \wedge A'') \wedge {}^\uparrow \downarrow \psi \) by Proposition 2.3 (3). In summary we know that under this transformation,

\[
\tilde{\Omega}_1 - \alpha(\tilde{B}) = \Omega''_1 - \alpha(B''),
\]

\[
\tilde{\Omega}_2 - \delta(\tilde{C}) = \Omega''_2 - \delta(C''),
\]

\[
\tilde{\Omega}_3 = \Omega''_3 - (\Omega_1 - \alpha(\tilde{B})) \wedge {}^\uparrow \downarrow \psi.
\]

The last two identities are already proved in subsection 3.2.

Since any 3-gauge transformation (1.2) can be written as a composition of (3.1), (3.4) and (3.3), identities (3.0), (3.15), (3.16), (3.20) and (3.25) imply the transform formula (1.4) of the 3-curvature for general 3-gauge transformations.

4. The \textbf{Gray-categories} and the \textbf{lax-natural transformations}

4.1. Gray-categories. Gray is a closed symmetric monoidal category with the Gray tensor product [29]. The underlying category is the category of 2-categories and 2-functors between them. A Gray-category is a category enriched over Gray. Any tricategory is triequivalent to a Gray-category [29] [27]. Gray-categories are the semi-strictification of tricategories. It is also well-known that the homotopy category of Gray 3-categories is equivalent to the homotopy category of 3-types (cf., e.g., [28]).

The unpacked version of this definition is as follows. A Gray-category \( \mathcal{C} \) (cf. [30] [31] and references therein) consists of collections \( C_0 \) of objects, \( C_1 \) of 1-arrows, \( C_2 \) of 2-arrows and \( C_3 \) of 3-arrows, together with

- functions \( s_n, t_n : C_i \to C_n \) for all \( 0 \leq n < i \leq 3 \), called \( n \)-source and \( n \)-target,
- functions \( \#_n : C_{n+1} \times T_n \times C_{n+1} \to C_{n+1} \) for all \( 0 \leq n < 3 \), called vertical composition,
- functions \( \#_n : C_i \times T_n \times C_{n+1} \to C_i \) and \( \#_n : C_{n+1} \times T_n \times C_i \to C_i \) for all \( n = 0, 1, n+1 < i \leq 3 \), called whiskering,
- a function \( \#_0 : C_2 \to C_3 \), called the interchanging 3-arrow (horizontal composition),
- a function \( id_n : C_i \to C_{i+1} \) for all \( 0 \leq i \leq 2 \), called identity,

such that

(1) \( \mathcal{C} \) is a 3-skeletal reflexive globular (cf. [32]) set;
(2) for every $C, C' \in C_0$, the collection of elements of $C$ with 0-source $C$ and 0-target $C'$ forms a 2-category $C(C, C')$, with $n$-composition in $C(C, C')$ given by $\#_{n+1}$ and identities given by $id_x$:

(3) for each $g : C' \to C''$ in $C_1$ and every $C, C'' \in C_0$, $g \#_0$ is a 2-functor $C(C', C'') \to C(C', C'')$, and $\#_0 g$ is a 2-functor $C(C, C') \to C(C, C'')$:

(4) for every $C, C', C'' \in C_0$, $id_{C', \#_0}$ is equal to the identity 2-functor $C(C', C'') \to C(C', C'')$, and $\#_0 id_{C'}$ is equal to the identity 2-functor $C(C, C') \to C(C, C')$:

(5) for $\gamma : C \xrightarrow{f} C'$ and $\delta : C' \xrightarrow{g} C''$ in $C_2$, we have the interchanging 3-arrow $\gamma \#_0 \delta$

\[ s_2(\gamma \#_0 \delta) = (\gamma \#_0 g) \#_1 (f' \#_0 \delta) \quad \text{and} \quad t_2(\gamma \#_0 \delta) = (f \#_0 \delta) \#_1 (\gamma \#_0 g'); \]

(6) for $C \xrightarrow{f} C'$ and $\delta : C' \xrightarrow{g} C''$ in $C_2$,

\[ (\gamma \#_1 \gamma') \#_0 \delta = [(\gamma \#_0 g) \#_1 (\gamma' \#_0 \delta)] \#_2 [(\gamma \#_0 \delta) \#_1 (\gamma' \#_0 g')]; \]

and for $\gamma : C \xrightarrow{f} C'$ and $C' \xrightarrow{g} C''$ in $C_2$,

\[ \gamma \#_0 (\delta \#_1 \delta') = [(\gamma \#_0 \delta) \#_1 (f' \#_1 \delta')] \#_2 [(f \#_1 \delta) \#_1 (\gamma \#_0 \delta')]; \]

(7) for $\varphi : C \xrightarrow{f} C'$ in $C_3$ and $\delta : C' \xrightarrow{g} C''$ in $C_2$,

\[ (\gamma \#_0 \delta) \#_2 [(f \#_0 \delta) \#_1 (\varphi \#_0 g')] = [(\varphi \#_0 g) \#_1 (f' \#_0 \delta)] \#_2 (\gamma' \#_0 \delta). \]

(8) for $f : C \to C'$ in $C_1$ and $\delta : C' \xrightarrow{g} C''$ in $C_2$, $id_f \#_0 \delta = id_{f \#_0 \delta}$, and for $\gamma : C \xrightarrow{f} C'$ in $C_2$ and $f : C' \to C''$ in $C_1$, $\gamma \#_0 id_f = id_{\gamma \#_0 f}$.

(9) For every $c \in C(C, C')_p$, $c' \in C(C', C'')_q$ and $c'' \in C(C'', C'''_r$ with $p + q + r \leq 2$,

\[ c \#_0 (c' \#_0 c'') = (c \#_0 c') \#_0 c''. \]

Here (5)-(7) are the definition of the interchanging 3-arrow and its functoriality. Following [24], the definition of a Gray-category is a little bit different from the standard one. We write $\alpha \#_{n, \beta}$ instead of $\beta \#_{n, \alpha}$ when $t_n(\alpha) = s_n(\beta)$. A (strict) Gray-functor $F : C \to C'$ between Gray-categories $C$ and $C'$ is given by maps $F_i : C_i \to C'_i$, $i = 0, \ldots, 3$, preserving all compositions, identities, interchanges, sources and targets, strictly. A Gray 3-groupoid is a Gray-category whose $k$-arrows are all equivalences, for all $k = 1, 2, 3$. 

\[ \text{WEI WANG} \]
4.2. The Gray 3-groupoid $G^\mathcal{Z}$ constructed from the 2-crossed module $\mathcal{L}$. Given a 2-crossed module $\mathcal{L}$, we can construct a Gray 3-groupoid $G^\mathcal{Z}$ (cf. §1.2.5 in [24]) with a single object by putting $G_0^\mathcal{Z} = \{ \bullet \}$, $G_1^\mathcal{Z} = G$, $G_2^\mathcal{Z} = G \times H$ and $G_3^\mathcal{Z} = G \times H \times L$. This construction appeared in [33], with different conventions, and also in a slightly different language in [34] [35].

For a 2-arrow $(X,e) \in G_2^\mathcal{Z}$,

$$s_1(X,e) = X, \quad t_1(X,e) = \alpha(e)^{-1}X,$$

and for a 3-arrow $(X,e,l) \in G_3^\mathcal{Z}$,

$$s_1(X,e,l) = X, \quad t_1(X,e,l) = \alpha(e)^{-1}X,$$

$$s_2(X,e,l) = (X,e), \quad t_2(X,e,l) = (X,\delta(l)^{-1}e).$$

The vertical composition of two 2-arrows is defined as

$$(X,e)\#_1(\alpha(e)^{-1}X,f) = (X,ef),$$

and the vertical composition of two 3-arrows is defined as

$$(X,e,l)\#_2(X,\delta(l)^{-1}e,k) = (X,el).$$

The $\#_1$-composition of two 3-arrows is

$$(X,e,l)\#_1(\alpha(e)^{-1}X,f,k) = (X,ef,(e \triangleright k)l),$$

whose 2-source is $(X,ef)$ and the 2-target is $(X,\delta((e \triangleright k)l)^{-1}ef) = (X,\delta(l)^{-1}e \cdot \delta(k)^{-1}f)$.

The interchanging 3-arrow is defined as

$$(X,e)\#_0(Y,f) = (XY,e(\alpha(e)^{-1}X) \triangleright f, e \triangleright \{e^{-1},X \triangleright f\}^{-1}),$$

whose 2-source and 2-target are

$$s_2((X,e)\#_0(Y,f)) = (XY,e(\alpha(e)^{-1}X) \triangleright f),$$

$$t_2((X,e)\#_0(Y,f)) = (XY,(X \triangleright f)e),$$

respectively. This is because $\delta(e \triangleright \{e^{-1},X \triangleright f\}^{-1})^{-1}e(\alpha(e)^{-1}X) \triangleright f = ee^{-1}(X \triangleright f)e(\alpha(e)^{-1}X) \triangleright f^{-1}e^{-1}(\alpha(e)^{-1}X) \triangleright f = (X \triangleright f)e$.

Whiskering by a 1-arrow is defined as

$$X\#_0(Y,e) = (XY,Y \triangleright e), \quad (Y,e)\#_0X = (YX,e),$$

$$X\#_0(Y,e,l) = (XY,Y \triangleright e,X \triangleright l), \quad (Y,e,l)\#_0X = (YX,e,l).$$

The 3-arrow whiskered by a 2-arrow from above is defined as

$$(X,e)\#_1(\alpha(e)^{-1}X,f,k) = (X,ef, e \triangleright k)$$

(i.e., the composition (4.2) of a 3-arrow with a trivial 3-arrow $(X,e,1_L)$) and the one from below is

$$(X,e,l)\#_1(\alpha(e)^{-1}X,f) = (X,ef,l).$$

**Remark 4.1.** By (4.3) and (4.7), in the Gray 3-groupoid $G^\mathcal{Z}$, whiskering from right by a 1-arrow or by a 2-arrow from below is trivial in the sense that the principal element $(e \circ l)$ is unchanged.
4.3. **The lax-natural transformations.** (cf. §5.1 in [30]) Let \( F, \tilde{F} : C \to D \) be \textbf{Gray}-functors between \( \text{Gray} \) categories \( C \) and \( D \). A \textit{lax-natural transformation} \( \Psi : F \to \tilde{F} \) consists of the following data:

- for every object \( C \) of \( C \) a 1-arrow \( \Psi_C : \tilde{F}(C) \to F(C) \) in \( D \),
- for every arrow \( f : C \to C' \) in \( C \) a 2-arrow \( \Psi_f \) in \( D \):
  \[
  \begin{array}{c}
  \tilde{F}(C) \xrightarrow{\tilde{f}} \tilde{F}(C') \\
  \downarrow_{\Psi_C} \quad \downarrow_{\Psi_{C'}} \\
  F(C) \xrightarrow{f} F(C')
  \end{array}
  \]

- for every 2-arrow \( \gamma : C \xrightarrow{f} C' \) in \( C \), we have a 3-arrow \( \Psi_{\gamma} \) in \( D \):
  \[
  \begin{array}{c}
  \tilde{F}(f) \xrightarrow{\tilde{\gamma}} \tilde{F}(f') \\
  \downarrow_{\Psi_C} \quad \downarrow_{\Psi_C} \\
  F(f) \xrightarrow{f} F(f')
  \end{array}
  \]

  satisfying the following conditions:

1. (natrality) for every 3-arrow \( \varphi : C \xrightarrow{\gamma} C' \) in \( C \),

\[
\Psi_{\varphi} \#_2[\tilde{F}(\varphi) \#_0 \Psi_{C'} \#_1 \Psi_{f'}] = [\Psi_{f'} \#_1 (\Psi_{C} \#_0 F(\varphi))] \#_2 \Psi_{\gamma'} : \]

\[
\begin{array}{c}
\tilde{F}(f') \xrightarrow{\tilde{F}(\varphi)} \tilde{F}(f'') \\
\downarrow_{\Psi_C} \quad \downarrow_{\Psi_C} \\
F(f') \xrightarrow{f} F(f'')
\end{array}
\]
(2) (functoriality with respect to 0-composition of 1-arrows) for every \( C \xrightarrow{f} C' \xrightarrow{f'} C'' \) in \( C \),
\[ \Psi_{f \#_0 f'} = [\tilde{F}(f) \#_0 \Psi_{f'}] \#_1 [\Psi_{f \#_0 F(f')}] : \]

\[
\begin{array}{c}
\xymatrix{ 
\tilde{F}(C) 
\ar[rr]^{\tilde{F}(f \#_0 f')} & & \tilde{F}(C'') \\
\ar[rr]_{\Psi_{f}} & & \\
F(C) 
\ar[rr]_{F(f \#_0 f')} & & F(C'') 
}
\end{array}
\]

(3) (functoriality with respect to 1-composition of 2-arrows) for every \( C \xrightarrow{f} C' \xrightarrow{f''} C'' \) in \( C \),
\[ \Psi_{\gamma \#_1 f''} = [\Psi_{\gamma \#_1 (\Psi_{C \#_0 F(\gamma')})}] \#_2 \left[ (\tilde{F}(\gamma) \#_0 \Psi_{C''}) \#_1 \Psi_{\gamma'} \right] : \]

\[
\begin{array}{c}
\xymatrix{ 
\tilde{F}(C) 
\ar[rr]^{\tilde{F}(f) \#_0 [\Psi_{\gamma \#_1 (\Psi_{C \#_0 F(\gamma')})]}} & & \tilde{F}(C'') \\
\ar[rr]_{\Psi_{f}} & & \\
\ar[rr]_{\Psi_{\gamma \#_1 (\Psi_{C \#_0 F(\gamma')})}} & & \\
F(C) 
\ar[rr]_{F(f) \#_0 [\Psi_{\gamma \#_1 (\Psi_{C \#_0 F(\gamma')})]]} & & F(C'') 
}
\end{array}
\]

(4) (functoriality with respect to 0-composition of a 2-arrow with a 1-arrow) for every \( C \xrightarrow{f} C' \xrightarrow{f''} C'' \) in \( C \),
\[ \Psi_{\gamma \#_0 f''} = \left[ (\tilde{F}(f) \#_0 \Psi_{f''}) \right] \#_1 \left( \Psi_{\gamma \#_0 F(f'')} \right) \#_2 \left[ (\tilde{F}(\gamma) \#_0 \Psi_{f''})^{-1} \right] \#_1 \left( \Psi_{f' \#_0 F(f'')} \right) : \]

\[
\begin{array}{c}
\xymatrix{ 
\tilde{F}(C) 
\ar[rr]^{\tilde{F}(f) \#_0 [\Psi_{\gamma \#_0 F(f'')}]} & & \tilde{F}(C'') \\
\ar[rr]_{\Psi_{f}} & & \\
\ar[rr]_{\Psi_{\gamma \#_0 F(f'')} \#_1 \Psi_{f''}} & & \\
\ar[rr]_{\Psi_{\gamma \#_0 F(f'')} \#_1 \Psi_{f''}} & & \\
F(C) 
\ar[rr]_{F(f) \#_0 [\Psi_{\gamma \#_0 F(f'')} \#_1 \Psi_{f''}]} & & F(C'') 
}
\end{array}
\]
J is a groupoid, called the S is denoted by X. The quotient of the set of 1-paths of ∂ γ by the relation of the rank-1 homotopy, is denoted by \( S_1(X) \) for every \( C \xrightarrow{f} C' \) in \( C \),

\[
\Psi_{f\#\alpha'\gamma'} = \left[ \left( F(f) \# 0 \Psi_f \right) \# 1 \left( \Psi_f \# 0 F(\gamma') \right) \right] \# 2 \left[ \left( F(f) \# 0 \Psi_f \right) \# 1 \left( \Psi_f \# 0 F(f'') \right) \right]
\]

(5) (functoriality with respect to 0-composition of a 1-arrow with a 2-arrow) for every \( C \xrightarrow{f} C' \xrightarrow{f''} C'' \) in \( C \),

\[
\Psi_{f\#\alpha'\gamma'} = \left[ \left( F(f) \# 0 \Psi_f \right) \# 1 \left( \Psi_f \# 0 F(\gamma') \right) \right] \# 2 \left[ \left( F(f) \# 0 \Psi_f \right) \# 1 \left( \Psi_f \# 0 F(\gamma'') \right) \right]
\]

6. (functoriality with respect to identities) for every \( C \) in \( C \), \( \Psi_{id_C} = id_{\Psi_{C'}} \), and for every \( f : C \to C' \) in \( C \), \( \Psi_{id_f} = id_{\Psi_f} \).

In the definition of \( \Psi_{\gamma\#0\alpha'\gamma''} \) in (4) and \( \Psi_{f\#\alpha'\gamma'} \) in (5), the interchange of 3-arrows are used to interchange the order of 2-arrows.

5. The 3-Connections and the 3-Gauge Transformations

5.1. 1-path, 2-path and 3-path groupoids. For a positive integer \( n \), an \( n \)-path is a smooth map \( \alpha : [0,1]^n = [0,1] \times [0,1]^{n-1} \to X \), for which there exists an \( \epsilon > 0 \) such that \( \alpha(t_1,\ldots,t_n) = \alpha(0,t_2,\ldots,t_n) \) for \( t_1 \leq \epsilon \), and analogously for any other face of \([0,1]^n\), of any dimension. We will abbreviate property as saying that \( \alpha \) has a product structure close to the boundary of the \( n \)-cube. We also require that \( \alpha(\{0\} \times [0,1]^{n-1}) \) and \( \alpha(\{1\} \times [0,1]^{n-1}) \) both consist of just a single point.

Given an \( n \)-path \( \alpha \) and an \( i \in \{1,\ldots,n\} \), we define \( (n-1) \)-paths \( \partial_i^- \alpha \) and \( \partial_i^+ \alpha \) by restricting it to \([0,1]^{i-1} \times \{0\} \times [0,1]^{n-i} \) and \([0,1]^{i-1} \times \{1\} \times [0,1]^{n-i} \). By definition, \( \partial_i^\pm \alpha \) must be constant \( (n-1) \)-paths. Given two \( n \)-paths \( \alpha \) and \( \beta \) with \( \partial_i^+(\alpha) = \partial_i^- (\beta) \), we have obviously the composition \( \alpha \#_i \beta \) (we only consider \( n = 1,2,3 \) in this paper). The product structure guarantees \( \alpha \#_i \beta \) also to be a \( n \)-path. This is why the condition of product structure is imposed to an \( n \)-path.

For example, a 1-path is a smooth map \( \gamma : [0,1] \to X \) with sitting instants, i.e., there exists \( 0 < \epsilon < \frac{1}{2} \) such that \( \gamma(t) = \gamma(0) \) for \( 0 \leq t < \epsilon \) and \( \gamma(t) = \gamma(1) \) for \( 1 - \epsilon < t \leq 1 \). Two 1-paths \( \gamma_1, \gamma_2 : [0,1] \to X \) are called rank-1 homotopic (cf. [36]) if there exists a 2-path \( \Gamma \) such that (1) \( \partial_0^- (\Gamma) = \gamma_1 \), \( \partial_0^+ (\Gamma) = \gamma_2 \); (2) the differential of \( \Gamma \) at each point of \([0,1]^2\) has at most rank 1.

The quotient of the set of 1-paths of \( X \), by the relation of the rank-1 homotopy, is denoted by \( S_1(X) \). We call the elements of \( S_1(X) \) 1-tracks. The category with objects \( X \) and arrows \( S_1(X) \) is a groupoid, called the (thin) path groupoid \( \mathcal{P}_1(X) \) of \( X \).

The quotient of the set of 2-paths of \( X \), by the relation of the laminated rank-2 homotopy, is denoted by \( S_2^\prime(X) \). We call the elements of \( S_2^\prime(X) \) laminated 2-tracks. A 3-path \( (t_1,t_2,t_3) \to J(t_1,t_2,t_3) \) is called good if the restrictions \( \partial_2^\pm (J) \) each are independent of \( t_3 \). Denote by \( S_3(X) \)
the set of all good 3-paths up to the rank-3 homotopy (with the laminated boundary). (cf. [24] for the laminated rank-2 homotopy and rank-3 homotopy with the laminated boundaries. We will not use these concepts precisely). Vertical and horizontal compositions of laminated 2-tracks, whiskering 2- and 3-tracks by 1-tracks, the interchange 3-tracks, vertical compositions of 3-tracks, etc., are all well defined. Boundaries $\partial_1^\pm$ of good 3-paths are 0-sources and 0-targets $S_0(X) \to X$, boundaries $\partial_2^\pm$ of good 3-paths are 1-sources and 1-targets $S_1(X) \to S_1(X)$, and boundaries $\partial_3^\pm$ of good 3-paths are 2-sources and 2-targets $S_3(X) \to S_3^1(X)$.

**Theorem 5.1.** (Theorem 2.4 in [24]) Let $X$ be a smooth manifold. The sets of 1-tracks, laminated 2-tracks and 3-tracks can be arranged into a Gray 3-groupoid $P_3(X) = (X, S_1(X), S_2(X), S_3(X))$. In particular, $P_2(X) = (X, S_1(X), S_2^1(X))$ is automatically a 2-groupoid.

### 5.2. Vanishing of fake 1- and 2-curvatures

Given a 3-connection $(A, B, C)$, we can construct 1-, 2- and 3-dimensional holonomies, which constitute a smooth Gray-functor from the 3-groupoid $P_3(X)$ to the Gray 3-groupoid $G^X$ (cf. [24]). Conversely, let us derive a 3-connection $(A, B, C)$ as derivatives of a smooth Gray-functor from the 3-groupoid $P_3(X)$ to the Gray 3-groupoid $G^X$. Its fake 1- and 2-curvatures vanish in this case.

For $(x_1, x_2) \in \mathbb{R}^2$, choose a 2-path $\hat{\Sigma}_{x_1, x_2}$ in $S_2^1(\mathbb{R}^2)$ to be

$$
\begin{array}{c}
(0, 0) \quad \gamma_1 \\
(0, x_2) \quad \gamma_2 \\
(\gamma_1 \circ \gamma_2) \\
(\gamma_1 \circ \gamma_2)(x_1, x_2)
\end{array}
$$

(5.1)

In the 2-path $\hat{\Sigma}_{x_1, x_2}$, the wavy line is the 1-source and the dotted line is the 1-target. $\hat{\Sigma}_{x_1, x_2}$ can be constructed by dilation from one fixed 2-path $\hat{\Sigma}_{1, 1}$. So it is a smooth family of 2-paths $\Sigma$ in $S_2^1(\mathbb{R}^2)$. It is important to see that the wavy and the dotted lines are smooth 1-path by the product structure, although their images in $\mathbb{R}^2$ are not smooth.

For fixed $x \in X$ and tangential vectors $v_1, v_2 \in T_x X$, choose a smooth mapping $\Gamma : \mathbb{R}^2 \to X$ such that $\Gamma(0) = x$ and

$$
v_j = \frac{\partial \Gamma}{\partial x_j}(0, 0), \quad j = 1, 2.
$$

(5.2)

Then $\Sigma_{x_1, x_2} := \Gamma \circ \hat{\Sigma}_{x_1, x_2}$ is a 2-path in $S_2^1(X)$. We use notations $\gamma_{x_1}^{x_2}$ for the 1-path of $\Sigma_{x_1, x_2}$ corresponding to line $[0, x_1] \times \{x_2\}$ in $\hat{\Sigma}_{x_1, x_2}$, and $\gamma_{x_1}^{x_2}$ for the 1-path corresponding to line $[0, x_1] \times \{0\}$.

A smooth Gray-functor $F : P_3(X) \to G^X$ is given by smooth mappings $F_0 : S_0(X) \to \{\bullet\}$, $F_1 : S_1(X) \to G$, $F_2 : S_2^1(X) \to G \times H$ and $F_3 : S_3(X) \to G \times H \times L$. Denote by $\pi_H : G \times H \to H$ the projection. Then

$$
F_1(\gamma_{x_1}^{x_2})F_1(\gamma_{x_1}^{x_2}) = \alpha(\pi_H \circ F_2(\Sigma_{x_1, x_2}))^{-1}F_1(\gamma_{x_1}^{x_2})F_1(\gamma_{x_1}^{x_2}).
$$

(5.3)

Define

$$
A_x(v_j) = \frac{\partial F_1(\gamma_{x_1}^{x_2})}{\partial x_j} \bigg|_{x_1 = 0}, \quad B_x(\gamma_1, \gamma_2) = \frac{\partial^2 \pi_H \circ F_2(\Sigma_{x_1, x_2})}{\partial x_1 \partial x_2} \bigg|_{x_1 = 0, x_2 = 0}.
$$

(5.4)
\( j = 1, 2. \) We claim that \( B_x \) is a 2-form, i.e., \( B_x(v_1, v_2) = -B_x(v_2, v_1) \) (cf. Lemma 3.7 in [20]).

Set \( \Gamma(s, t) := \Gamma(t, s) \). Note that \( \Sigma_{x_1, x_2} = \Gamma \circ \Sigma_{x_1, x_2} = \Gamma \circ \Sigma^{-1}_{x_1, x_2} = \Sigma^{-1}_{x_1, x_2} \), where \( \Sigma^{-1}_{x_1, x_2} \) is the 2-arrow inverse to the 2-arrow \( \Sigma_{x_1, x_2} \) under vertical composition. Since the 2-functor \( F \) sends the inverse 2-arrow to the inverse group element, we have \( \pi_H \circ F_2(\Sigma_{x_1, x_2}) = \pi_H \circ F_2(\Sigma_{x_1, x_2})^{-1} \).

Hence, by taking derivatives, we get \( \partial \pi \circ \frac{\partial x}{\partial x_j} \) by using the inverse 2-arrow to the inverse group element, we have \( \pi_H \circ F_2(\Sigma_{x_1, x_2}) = \pi_H \circ F_2(\Sigma_{x_1, x_2})^{-1} \).

Take derivatives \( \frac{\partial x}{\partial x_j} \) at \( (0, 0) \) on both sides of (5.3) to get

\[
A_x(v_2)A_x(v_1) + v_2A_x(v_1) = -\alpha(B_x(v_1, v_2)) + A_x(v_1)A_x(v_2) + v_1A_x(v_2),
\]

by using

\[
(5.5) \quad \left. \frac{\partial \pi_H \circ F_2(\Sigma_{x_1, x_2})}{\partial x_j} \right|_{(0, 0)} = 0,
\]

which follows from \( \pi_H \circ F_2(\Sigma_{x_1, x_2}) = \pi_H \circ F_2(\Sigma_{x_1, x_2})^{-1} \). Thus, \( \mathcal{F}_1 = dA + A \wedge A - \alpha(B) = 0. \)

Now for fixed \( (x_1, x_2, x_3) \in \mathbb{R}^3 \), choose a good 3-path \( \Omega_{x_1,x_2,x_3} \) in \( \mathcal{S}_3(\mathbb{R}^3) \) to be

\[
(0, 0, 0) \quad \Omega_{x_1,x_2,x_3} \quad (x_1, x_2, 0) \quad (x_1, 0, 0) \quad (0, x_2, 0) \quad (0, x_2, x_3) \quad (0, 0, x_3) \quad (0, x_2, x_3)
\]

In the 3-path \( \Omega_{x_1,x_2,x_3} \), the wavy line is its 1-source and the dotted line is its 1-target. \( \Omega_{x_1,x_2,x_3} \) can be constructed by dilation from one fixed good 3-path \( \Omega_{1,1,1} \). So it is a smooth family of good 3-paths \( \hat{\Omega} \) in \( \mathcal{S}_3(\mathbb{R}^3) \).

For fixed \( x \in X \) and tangential vectors \( v_1, v_2, v_3 \in T_xX \), choose a smooth mapping \( \Gamma : \mathbb{R}^3 \to X \) such that \( \Gamma(0, 0, 0) = x \) and

\[
v_j = \frac{\partial \Gamma}{\partial x_j}(0, 0, 0), \quad j = 1, 2, 3.
\]

Then \( \Omega_{x_1,x_2,x_3} := \Gamma \circ \hat{\Omega}_{x_1,x_2,x_3} \) is a good 3-path in \( \mathcal{S}_3(X) \).

We use notations \( \gamma^{x_1,x_j,x_k} \) for the 1-path of \( \Omega_{x_1,x_2,x_3} \) corresponding to line \( [0, x_1] \times \{x_j\} \times \{x_k\} \) in \( \hat{\Omega}_{x_1,x_2,x_3} \), \( \gamma^{x_1} \) for the 1-path corresponding to line \( [0, x_1] \times \{0\} \times \{0\} \), etc. Similarly, we denote by \( \Sigma^{x_1,x_j,x_k} \) the 2-path of \( \Omega_{x_1,x_2,x_3} \) corresponding to the 2-cell \( [0, x_1] \times [0, x_j] \times [0, x_k] \) in \( \hat{\Omega}_{x_1,x_2,x_3} \), and by \( \Sigma^{x_1,x_j} \) the 2-path of \( \Omega_{x_1,x_2,x_3} \) corresponding to the 2-cell \( [0, x_1] \times [0, x_j] \times \{0\} \), etc.
\( \Omega_{x_1,x_2,x_3} \) is a good 3-path in \( S_3(X) \) with the 2-source \( \Sigma_- \) and the 2-target \( \Sigma_+ \) as follows:

More precisely, we have a 3-arrow \((\ast, \Sigma_-, \Omega_{x_1,x_2,x_3})\) in the 3-groupoid \( P_3(X) \) with

\[
\Sigma_- := [\gamma^1_0 \#_0 \Sigma_{x_2,x_3,x_1}] \#_1 [\Sigma_{x_1,x_3} \#_0 \gamma^{x_2,x_1,x_3}] \#_1 [\gamma^3_0 \#_0 \Sigma_{x_1,x_2}] \\
\Rightarrow \Sigma_+ := [\Sigma_{x_1,x_2} \#_0 \gamma^{x_3,x_1,x_2}] \#_1 [\gamma^{x_2} \#_0 \Sigma_{x_1,x_3}] \#_1 [\Sigma_{x_2,x_3} \#_0 \gamma^{x_1,x_2,x_3}].
\]

The 2-source \( \Sigma_- \) and the 2-target \( \Sigma_+ \) are compositions of 3 whiskered 2-arrows, respectively, as follows:

Then \( \hat{F}_2(\Sigma_+) = \delta(\hat{F}_3(\Omega_{x_1,x_2,x_3}))^{-1}\hat{F}_2(\Sigma_-) \), i.e.,

\[
\hat{F}_3(\Sigma_{x_1,x_2}) \cdot F_1(\gamma^{x_2}) \triangleright \hat{F}_2(\Sigma_{x_1,x_3,x_2}) \cdot \hat{F}_2(\Sigma_{x_1,x_3}) \\
= \delta(\hat{F}_3(\Omega_{x_1,x_2,x_3}))^{-1} \cdot F_1(\gamma^{x_1}) \triangleright \hat{F}_2(\Sigma_{x_2,x_3,x_1}) \cdot \hat{F}_2(\Sigma_{x_2,x_3}) \cdot F_1(\gamma^{x_3}) \triangleright \hat{F}_2(\Sigma_{x_1,x_2,x_3}),
\]
where \( \hat{F}_2 = \pi_H \circ F_2, \hat{F}_3 = \pi_L \circ F_3 \) and \( \pi_L : G \times H \times L \to L \) is the projection. Here we use Remark 4.4 that in the Gray 3-groupoid \( \mathcal{G}_L \), whiskering from right by a 1-arrow is trivial. Set

\[
(5.11) \quad C(v_1, v_2, v_3) := \frac{\partial^3 \hat{F}_3(\Omega_{x_1, x_2, x_3})}{\partial x_1 \partial x_2 \partial x_3} \bigg|_{x_1 = x_2 = x_3 = 0}.
\]

\( C \) is a 3-form, in the same way as \( B \) is a 2-form. Take derivatives \( \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \) at \((0, 0, 0)\) on both sides of \((5.10)\), noting \((5.5)\), to get

\[
A(v_2) \triangleright B(v_1, v_3) + v_2 B(v_1, v_3) = -\delta(C(v_1, v_2, v_3)) + A(v_1) \triangleright B(v_2, v_3) + v_1 B(v_2, v_3) + A(v_3) \triangleright B(v_1, v_2) + v_3 B(v_1, v_2).
\]

Thus, \( \mathcal{F}_2 = db + A \wedge^p B - \delta(C) = 0. \)

### 5.3. The gauge transformations.

**Proposition 5.1.** Suppose \((A, B, C)\) and \((\hat{A}, \hat{B}, \hat{C})\) are 3-connections constructed from smooth Gray-functors \( F \) and \( \hat{F} : \mathcal{P}_3(X) \to \mathcal{G}_L \), respectively, in the above subsection, and there exists a lax-natural transformations \( \Psi : F \to \hat{F} \). Then there exist some \( g \in \Lambda^0(X, G), \varphi \in \Lambda^1(X, \mathfrak{h}), \psi \in \Lambda^1(X, \mathfrak{l}) \) such that

\[
\hat{A} = Ad_g A + g dg^{-1} + \alpha(\varphi);
\]

\[
\hat{B} = g \triangleright B + d\varphi + \hat{A} \wedge^p \varphi - \varphi \wedge \varphi - \delta(\psi);
\]

\[
\hat{C} = g \triangleright C - d\psi - \hat{A} \wedge^p \psi + \varphi \wedge^p \psi - \hat{B} \wedge^p \psi - \varphi \wedge^p (g \triangleright B).
\]

This is exactly the 3-gauge transformation in \((1.2)\) with \( g \) replaced by \( g^{-1} \). The transformation formula for the \( A \) field is easy. Let \( \Omega_{x_1, x_2, x_3} \) be the 3-path in \( S_3(X) \) in \((5.6)\) and \( (5.7) \). Set

\[
f^{x,j;i*} := F_1(\gamma^{x,j;i}), \quad \hat{f}^{x,j;i*} := \hat{F}_1(\gamma^{x,j;i}),
\]

\[
F^{x_1,x_2; x_j *} := \pi_H \circ F_2 \left( \Sigma_{x_1,x_2; x_j} \right), \quad \hat{F}^{x_1,x_2; x_j *} := \pi_H \circ \hat{F}_2 \left( \Sigma_{x_1,x_2; x_j} \right),
\]

\[
h^{x,j;i*} := \pi_H \circ \Psi^{x,j;i*}, \quad k^{x, x_j; x_l} := \pi_L \circ \Psi^{x, x_j; x_l},
\]

for \( *= \) empty or \( x_k, \) etc.. For a \( G \)-valued function \( g \) on \( X \), we denote

\[
g^{x_j} = g(\Gamma(x_j, 0, 0)), \quad g^{x_j, x_k} = g(\Gamma(x_j, x_k, 0)) \quad \text{and} \quad g^{x_j, x_k, x_l} = g(\Gamma(x_j, x_k, x_l)),
\]

up to the order of coordinates. Note that

\[
(5.13) \quad \left. f^{x,j;i*} \right|_{x_j = 0} = 1_G, \quad \left. h^{x,j;i*} \right|_{x_j = 0} = 1_L, \quad \left. \frac{\partial k^{x_1,x_j; x_i*}}{\partial x_s} \right|_{x_1 = x_j = 0} = 0, \quad \left. \frac{\partial F^{x_1,x_j; x_i*}}{\partial x_s} \right|_{x_1 = x_j = 0} = 0,
\]

for \( s = i \) or \( j \). The last identity comes from \( F^{x_1,x_j; x_i*} = 1_H \) if \( x_i = 0 \) or \( x_j = 0 \).
By definition, the lax-natural transformation $\Psi$ defines a 3-arrow $\Psi_{\Sigma_{x_1,x_2}}$:

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & f_{x_1} & \ar[ld]_{g_{x_1}} & f_{x_1:x_2} & g_{x_1} & \ar[ld]_{g_{x_1:x_2}} & f_{x_2} & \ar[ld]_{g_{x_2}} & \ar[d]_{f_{x_2}} & f_{x_2:x_1} & \ar[l]_{g_{x_2:x_1}} & \ar[d]_{f_{x_2:x_1}} & f_{x_1} & \ar[l]_{g_{x_1}} & \ar[u]_{g_{x_1:x_2}} & f_{x_2} & \ar[l]_{g_{x_2}} & \ar[u]_{g_{x_2:x_1}} & f_{x_2:x_1} & \ar[l]_{g_{x_2:x_1}} & \ar[u]_{g_{x_2:x_1}} & f_{x_2:x_1}}
\end{array}
\end{array}
\]

such that

\[(5.14) \quad \delta(k^{x_1,x_2})^{-1} \cdot \tilde{f}_{x_1} \triangleright h^{x_2:x_1} \cdot h^{x_1} \cdot g^0 \triangleright F^{x_1,x_2} = \tilde{F}^{x_1,x_2} \cdot \tilde{f}_{x_2} \triangleright h^{x_1:x_2} \cdot h^{x_2}.
\]

Let $A(v)$ and $B(v_1,v_2)$ as before and let

\[
\varphi(v_1) := \frac{\partial h^{x_1}}{\partial x_1} \bigg|_{x_1=0}, \quad \varphi(v_2) := \frac{\partial h^{x_2}}{\partial x_2} \bigg|_{x_2=0}, \quad \psi(v_1,v_2) := \frac{\partial^2 k^{x_1,x_2}}{\partial x_1 \partial x_2} \bigg|_{x_1=x_2=0}.
\]

We can take derivatives $\frac{\partial^2}{\partial x_1 \partial x_2}$ at $(0,0)$ on both sides of (5.14) and use (5.13) to get

\[
-\delta(\psi(v_1,v_2)) + \tilde{A}(v_1) \triangleright \varphi(v_2) + v_1 \varphi(v_2) + \varphi(v_2)\varphi(v_1) + g(x) \triangleright B(v_1,v_2) = \tilde{B}(v_1,v_2) + \tilde{A}(v_2) \triangleright \varphi(v_1) + v_2 \varphi(v_1) + \varphi(v_1)\varphi(v_2).
\]

This is exactly the transformations formula for the $B$ field in (5.12).

5.4. The gauge transformations of the $C$ field: the $\Psi_{\Sigma_-}$ part. $F(\Omega_{x_1,x_2,x_3})$ is a 3-arrow in $G^Z$, whose 2-source $F_2(\Sigma_-)$ and 2-target $F_2(\Sigma_+)$ (cf. (5.7) for $\Sigma_-$ and $\Sigma_+$) are as follows:

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & f_{x_1} & \ar[ld]_{g_{x_1}} & f_{x_1:x_3} & g_{x_1} & \ar[ld]_{g_{x_1:x_2}} & f_{x_2} & \ar[ld]_{g_{x_2}} & \ar[d]_{f_{x_2}} & f_{x_2:x_1} & \ar[l]_{g_{x_2:x_1}} & \ar[d]_{f_{x_2:x_1}} & f_{x_1} & \ar[l]_{g_{x_1}} & \ar[u]_{g_{x_1:x_2}} & f_{x_2} & \ar[l]_{g_{x_2}} & \ar[u]_{g_{x_2:x_1}} & f_{x_2:x_1} & \ar[l]_{g_{x_2:x_1}} & \ar[u]_{g_{x_2:x_1}} & f_{x_2:x_1}}
\end{array}
\end{array}
\]

Similarly for 3-arrow $\tilde{F}(\Omega_{x_1,x_2,x_3})$ in $G^Z$. The naturality (1.9)-(1.8) of the lax-natural transformation $\Psi : F \to \tilde{F}$ implies

\[(5.15) \quad F^*_3(\Omega_{x_1,x_2,x_3}) \cdot \Psi_{\Sigma_+} = \Psi_{\Sigma_-} \cdot F^*_3(\Omega_{x_1,x_2,x_3}).\]
where $F^*(\Omega_{x_1,x_2,x_3})$ is a suitable whiskering of $F(\Omega_{x_1,x_2,x_3})$. $\Psi_{\Sigma_-}$ is the following 3-arrow.

See also the figure $\Psi_{\Sigma_-}$ later. Write 3-arrow $\Psi_{\Sigma_{x_1,x_j;x_k}} = (\ast, \ast, k^{x_1,x_j;x_k})$ in $\mathcal{G}^Z$.

Since the 2-arrow $\Sigma_-$ is the composition of three 2-arrows in (5.8), by the functoriality (4) of the lax-natural transformation, the 3-arrow $\Psi_{\Sigma_-}$ is the whiskered composition of $\Psi_{\gamma^3 \ast 0} \Sigma_{x_2,x_3;x_1}$, $\Psi_{\gamma^3 \ast 0} \Sigma_{x_1,x_2;x_3}$ and $\Psi_{\gamma^3 \ast 0} \Sigma_{x_1,x_2;x_3}$. Let us write it down explicitly as composition of 3-arrows $(g_*, \Sigma_1, K_1), \ldots, (g_*, \Sigma_6, K_6)$ in $\mathcal{G}^Z$, where

\[(5.16)\] $g_* = \tilde{f}^{x_1} f^{x_2;1} f^{x_3;1,x_2,x_1} g^{x_1,x_2,x_3}$, corresponding to the wavy path in the above figures.

The first 2-arrow is $(g_*, \Sigma_0)$, interchanging 2-arrows (3) and (4) in the figure $\Sigma_1$, with

\[(5.17)\] $K_1 = K_0 \triangleright' \left( h^{x_1} \triangleright' \left( h^{x_1} \triangleright \left( f^{x_1} \triangleright f^{x_2;x_3;1} \triangleright f^{x_2;x_3;1} \triangleright f^{x_2;x_3;1} \right)^{-1} \right) \right)$, whose 2-target is the 2-arrow $(g_*, \Sigma_2)$, where $K_0 = \tilde{f}^{x_1} f^{x_2;x_3;1} h^{x_1,x_2,x_2} h^{x_2;x_1}$ is the whiskering corresponding to the composition of the whiskered (1) and (2) by definition (4.16) of the 2-whiskering. The interchanging 3-arrow has this form by the definition (4.3). We will not write $\Sigma_j$'s down explicitly, since we will not use them.

The second 3-arrow is $(g_*, \Sigma_2, K_2)$ with $K_2 = \tilde{f}^{x_1} k^{x_2;x_3;1}$, whose 2-target is $(g_*, \Sigma_3)$. 

\[\]
The third 3-arrow is \((g_\ast, \Sigma_3, K_3)\) with
\[
K_3 = \left[ \tilde{f}^x_1 \triangleright \left( \tilde{F}^{x_2, x_3; x_1}_1 \cdot \tilde{f}^{x_2; x_1}_2 \triangleright h^{x_2; x_1, x_3} \right) \right] \triangleright' k^{x_1, x_3}.
\]
whose 2-target is \((g_\ast, \Sigma_4)\), where the part before \(\triangleright'\) is the whiskering corresponding to the composition of 2-arrows (1) and (2) in the figure \(\Sigma_3\). The fourth 3-arrow is \((g_\ast, \Sigma_4, K_4)\), interchanging 2-arrows (3) and (2) in the figure \(\Sigma_4\), with
\[
K_4 = \left( \tilde{f}^x_1 \triangleright \tilde{F}^{x_2, x_3; x_1}_1 \right) \triangleright' \left[ \tilde{F}^{x_2, x_3}_1 \triangleright' \left\{ \left( \tilde{F}^{x_1, x_3}_1 \right)^{-1}, \left( \tilde{f}^{x_1, x_3}_1 \right) \triangleright h^{x_2; x_1, x_3} \right\} \right],
\]
whose 2-target is \((g_\ast, \Sigma_5)\).

The fifth 3-arrow is \((g_\ast, \Sigma_5, K_5)\), interchanging 2-arrows (5) and (6) in the figure \(\Sigma_4\), with
\[
K_5 = K^0_5 \triangleright' \left[ h^{x_3} \triangleright' \left\{ \left( h^{x_3} \right)^{-1}, \left( \tilde{f}^{x_3}_3 \right) \triangleright F^{x_2; x_3; x_1} \right\} \right]^{-1},
\]
whose 2-target is \((g_\ast, \Sigma_6)\), where \(K^0_5 = \tilde{f}^x_1 \triangleright \tilde{F}^{x_2, x_3; x_1}_1 \cdot \tilde{F}^{x_1, x_3}_1 \cdot \tilde{f}^{x_3}_3 \triangleright \left[ \tilde{f}^{x_1; x_3}_1 \triangleright h^{x_2; x_1, x_3}, h^{x_1, x_3} \right]\) is the whiskering corresponding to the composition of 2-arrows (1), (3), (2) and (4) in the figure \(\Sigma_4\).

The sixth 3-arrow is \((g_\ast, \Sigma_6, K_6)\) with
\[
K_6 = \left[ \tilde{f}^x_1 \triangleright \tilde{F}^{x_2, x_3; x_1}_1 \cdot \tilde{F}^{x_1, x_3}_1 \right] \triangleright' \left( \tilde{f}^{x_3}_3 \triangleright k^{x_1, x_2; x_3} \right).
\]
whose 2-target is \((g_\ast, \Sigma_\tau)\). The last 3-arrow is \((g_\ast, \Sigma_\tau, K_\tau) = \tilde{F}_3(\Omega_{x_1,x_2,x_3})\) with \(K_\tau = \tilde{k}_{x_1,x_2,x_3}\), whose 2-target is \((g_\ast, \Sigma_8)\). Now \(\Psi_{\Sigma}\) in \(\eqref{4.13}\) is \((\ast, \ast, K_1 K_2 \cdots K_6)\) by the functoriality of \(\Psi\) in the definition in \(\S 4.3\), while the RHS of \(\eqref{5.15}\) is \((\ast, \ast, K_1 K_2 \cdots K_7)\).

\[\frac{\partial^3 K_1 K_2 \cdots K_7}{\partial x_1 \partial x_2 \partial x_3}(0,0,0) = \sum_{j=1}^{7} \frac{\partial^3 K_j}{\partial x_1 \partial x_2 \partial x_3}(0,0,0).\]

By \(\eqref{2.3}, \eqref{5.13}\) and \(\eqref{5.18}\), we have

\[\frac{\partial^3 K_1}{\partial x_1 \partial x_2 \partial x_3}(0,0,0) = \left\{ \frac{\partial h_{x_1}}{\partial x_1}, g^0 \triangleright \frac{\partial^2 F_{x_2,x_3;x_1}}{\partial x_2 \partial x_3}(0,0,0) \right\}(0,0,0) = \{\varphi(v_1), g^0 \triangleright B(v_2, v_3)\},\]
for $K_1$ in (5.17), and similarly for the other $K_j$. Consequently, the derivative \( \frac{\partial^3 K_1 K_2 \cdots K_7}{\partial x_1 \partial x_2 \partial x_3} \) at \((0, 0, 0)\) gives

(5.19)

\[
\{ \varphi(v_1), g^0 \triangleright B(v_2, v_3) \} + v_1 \psi(v_2, v_3) + \tilde{A}(v_1) \triangleright \psi(v_2, v_3) + \varphi(v_2) \triangleright' \psi(v_1, v_3) - \{ \tilde{B}(v_1, v_3), \varphi(v_2) \} + \{ \varphi(v_3), g^0 \triangleright B(v_1, v_2) \} + v_3 \psi(v_1, v_2) + \tilde{A}(v_3) \triangleright \psi(v_1, v_2) + \tilde{C}(v_1, v_2, v_3).
\]

5.5. The gauge transformations of the $C$ field: the $\Psi_{\Sigma^+}$ part. Now consider $\Psi_{\Sigma^+}$ in (5.13).

The first 3-arrow is $(g_*, \hat{\Sigma}_1, \hat{K}_1)$ with

\[
\hat{K}_1 = \left[ \left( \tilde{f}^x_1 \tilde{f}^{x_2; x_1} \right) \triangleright h^{x_1; x_1, x_2} \right] \triangleright' k^{x_1, x_2},
\]

whose 2-target is $(g_*, \hat{\Sigma}_2)$. The second 3-arrow is $(g_*, \hat{\Sigma}_2, \hat{K}_2)$, interchanging 2-arrows (2) and (1) in the figure $\hat{\Sigma}_2$, with

\[
\hat{K}_2 = \tilde{F}^{x_1, x_2} \triangleright' \left\{ \left( \tilde{F}^{x_1, x_2} \right)^{-1}, \left( \tilde{f}^{x_1 \tilde{f}^{x_2; x_1}} \right) \triangleright h^{x_3; x_1, x_2} \right\},
\]

whose 2-target is $(g_*, \hat{\Sigma}_3)$. The third 3-arrow is $(g_*, \hat{\Sigma}_3, \hat{K}_3)$, interchanging 2-arrows (4) and (5) in the figure $\hat{\Sigma}_2$, with

\[
\hat{K}_3 = \left[ \tilde{F}^{x_1, x_2} \cdot \tilde{f}^{x_2} \triangleright \left( \tilde{f}^{x_1; x_2} \triangleright h^{x_3; x_1, x_2} \cdot h^{x_1; x_2} \right) \right] \triangleright' \left[ \tilde{h}^{x_2} \triangleright' \left\{ \left( \tilde{h}^{x_2} \right)^{-1}, \left( \tilde{f}^{x_2 \tilde{g}^{x_3}} \right) \triangleright F^{x_1, x_3; x_2} \right\}^{-1} \right],
\]

whose 2-target is $(g_*, \hat{\Sigma}_4)$, where the first bracket is the whiskering corresponding to the composition of 2-arrows (2), (1) and (3) in the figure $\hat{\Sigma}_2$. 
The fourth 3-arrow is \((g_*, \hat{\Sigma}_4, \hat{K}_4)\) with \(\hat{K}_4 = \tilde{F}^{x_1,x_2} \triangleright f^{x_2} \triangleright k^{x_1,x_3;x_4}\), whose 2-target is \((g_*, \hat{\Sigma}_5)\). The fifth 3-arrow is \((g_*, \hat{\Sigma}_5, \hat{K}_5)\) with
\[
\hat{K}_5 = \left[ \tilde{F}^{x_1,x_2} \cdot f^{x_2} \triangleright \left( \tilde{F}^{x_1,x_3;x_2} \cdot \tilde{F}^{x_3;x_2} \triangleright h^{x_1;x_2,x_3} \right) \right] \triangleright' k^{x_2,x_3},
\]
whose 2-target is \((g_*, \hat{\Sigma}_6)\), where the part before \(\triangleright'\) is the whiskering corresponding to the composition of 2-arrows (1), (2) and (3) in the figure \(\hat{\Sigma}_5\). The last 3-arrow is \((g_*, \Sigma_6, \hat{K}_6)\), interchanging 2-arrows (4) and (3) in the figure \(\hat{\Sigma}_5\), with
\[
\hat{K}_6 = \left[ \tilde{F}^{x_1,x_2} \cdot f^{x_2} \triangleright \tilde{F}^{x_1,x_3;x_2} \right] \triangleright' \left( \left( \tilde{F}^{x_2,x_3} \right)^{-1}, \left( \tilde{F}^{x_2,x_3} \right) \triangleright h^{x_1;x_2,x_3} \right),
\]
whose 2-target is \((g_*, \hat{\Sigma}_7) = (g_*, \Sigma_8)\) in the above subsection.

The 0-th 3-arrow is \((g_*, \hat{\Sigma}_0, \hat{K}_0)\) with
\[
\hat{K}_0 = f_3^{x_1,x_2,x_3} \left[ \left( \tilde{F}^{x_2,x_1} \triangleright h^{x_1;x_1,x_2} \cdot h^{x_2;x_1} \right) \triangleright h^{x_1} \right] \triangleright' \left[ g^0 \triangleright k^{x_1,x_2,x_3} \right],
\]
where the part before \(\triangleright'\) is the whiskering corresponding to the composition of 2-arrows (1), (2) and (3) in the figure \(\hat{\Sigma}_1\). Now \(\Psi_{\Sigma_4}\) in (5.15) is \(K_1K_2 \cdots K_6\) and the LHS of (5.15) gives \(K_0K_1 \cdots K_6\). The derivative \(\partial^\alpha \tilde{K}_0 \tilde{K}_1 \cdots \tilde{K}_6\) at \((0,0,0)\) gives
\[
g^0 \triangleright C(v_1, v_2, v_3) + \varphi(v_3) \triangleright' \psi(v_1, v_2) - \left\{ \tilde{B}(v_1, v_2, \varphi(v_3)) + \{v_2, g^0 \triangleright B(v_1, v_3)\} \right\} + v_2 \psi(v_1, v_3) + \tilde{A}(v_2) \triangleright \varphi(v_1, v_3) + \varphi(v_1) \triangleright' \psi(v_2, v_3) - \{\tilde{B}(v_2, v_3, \varphi(v_1))\}.
\]

The derivatives of both sides of (5.15) give (5.19) = (5.20), which is exactly the gauge transformation formula (5.12) for the C-field.

6. The 3-holonomies and the 3-curvatures

For \((x_1, x_2, x_3, x_4) \in \mathbb{R}^4\), consider a smooth family of 4-paths \(\tilde{\Theta}_{x_1,x_2,x_3,x_4} : [0,1]^4 \to [0,x_1] \times \cdots \times [0,x_4] \subset \mathbb{R}^4\), and \(\theta_{x_1,x_2,x_3,x_4} : [0,1]^4 \to [0,x_1] \times \cdots \times [0,x_4] \subset \mathbb{R}^4\), where \(\Gamma\) is a mapping \(\mathbb{R}^4 \to U \subset \mathbb{R}^n\) such that \(\frac{\partial \psi}{\partial x_j}(0,0,0,0) = v_j\) for fixed \(v_j \in T_x U, j = 1, \ldots, 4\).

As before, we use notations \(\gamma_{x_1,x_2,x_3,x_4}\) for 1-paths, \(\Sigma_{x_1,x_2,x_3,x_4}\) for 2-paths and \(\Omega_{x_1,x_2,x_3,x_4}\) for 3-paths. Under the action of a Gray-functor \(F\), we get 1-arrows \(f_{x_1,x_2,x_3,x_4}\), 2-arrows \((*, F^{x_1,x_2,x_3,x_4})\) and 3-arrows \((*, *, k^{x_1,x_2,x_3,x_4})\).

The boundary \(\partial \Theta_{x_1,x_2,x_3,x_4}\) of the 4-path \(\Theta_{x_1,x_2,x_3,x_4}\) is a closed 3-path, which is the composition of two 3-paths corresponding to \(\Sigma_{x_1,x_2,x_3,x_4} \cup \Omega_{x_1,x_2,x_3,x_4}\) and \(\Omega_{x_1,x_2,x_3} \cup \Sigma_{x_1,x_2,x_3} \cup \Sigma_{x_1,x_2,x_3}\), respectively, where \(\Sigma_{x_1,x_2,x_3,x_4}\) are the 2-source and 2-target in \([6.14]\) of the 3-path \(\Omega_{x_1,x_2,x_3,x_4}\). Each of these two 3-paths is a composition of several 3-paths. Then \(F(\partial \Theta_{x_1,x_2,x_3,x_4})\) is the 3-dimensional holonomy, the lattice version of 3-curvature. Let us write it down explicitly.

6.1. 3-arrows corresponding to \(\Sigma_{x_1,x_2,x_3,x_4} \cup \Omega_{x_1,x_2,x_3,x_4}\). The first 3-arrow is \((g_*, \Sigma_1, K_1)\), interchanging 2-arrows (3) and (4) in the figure \(\Sigma_1\), with
\[
K_1 = K_1^0 \triangleright f^x \left( f^{x_1,x_4} \left( f^{x_1,x_4} \right)^{-1}, f^{x_1,x_4} \right) \triangleright F^{x_2,x_3;x_4},
\]
whose 2-target is \((g_*, \Sigma_2)\), where \(K_1^0 = f^x \triangleright (f^{x_2,x_1} \triangleright F^{x_3,x_4;x_1,x_2}) \cdot F^{x_2,x_4} \triangleright f^{x_1,x_4}\) is the whiskering corresponding to the composition of 2-arrows (1) and (2) in the figure \(\Sigma_1\).
is \((g_*, \Sigma_2, K_2)\) with \(K_2 = f^{x_1} \triangleright k^{x_2:x_3:x_4:x_1},\) whose 2-target is \((g_*, \Sigma_3)\).

\[
\text{The sixth 3-arrow is } (g_*, \Sigma_6, K_6) \text{ with } K_6 = [f^{x_1} \triangleright F^{x_2:x_3:x_1}, F^{x_1:x_3}] \triangleright' [f^{x_3} \triangleright k^{x_1:x_2:x_4}].
\]
whose 2-target is \((g_\epsilon, \Sigma_7)\).
The last 3-arrow is \((g_*, \Sigma_7, K_7)\) with \(K_7 = \pi_L \circ F_3(\Omega_{x_1,x_2,x_3}) = k^{x_1,x_2,x_3}\), whose 2-target is \((g_*, \Sigma_8)\).

The derivative \(\frac{\partial^4 K_5 K_2 - K_5}{(x_1,x_2,x_3)}\) at \((0,0,0)\) gives

\[
\begin{align*}
\{B(v_1,v_4), B(v_2,v_3)\} + v_1 C(v_2,v_3,v_4) + A(v_1) &\triangleright C(v_1,v_2,v_4) - \{B(v_1,v_3), B(v_2,v_4)\} \\
+ \{B(v_3,v_4), B(v_1,v_2)\} + v_2 C(v_1,v_2,v_4) + A(v_3) &\triangleright C(v_1,v_2,v_4).
\end{align*}
\]

(6.1)

6.2. 3-arrows corresponding to \(\Omega_{x_1,x_2,x_3} \cup \Sigma_+ \times [0,x_4]\).

The first 3-arrow is \((g_*, \Sigma_1, \hat{K}_1)\) with

\[
\hat{K}_1 = [(f^{x_1} f^{x_2})] \triangleright F^{x_1,x_4; x_1,x_2} \triangleright k^{x_1,x_2,x_4},
\]

whose 2-target is \((g_*, \Sigma_2)\). The second 3-arrow is \((g_*, \Sigma_2, \hat{K}_2)\), interchanging 2-arrows (2) and (1) in the figure \(\Sigma_2\), with

\[
\hat{K}_2 = F^{x_1,x_2} \triangleright \left\{ (f^{x_1,x_2})^{-1}, (f^{x_1} f^{x_2}) \triangleright F^{x_3,x_4; x_1,x_2} \right\},
\]

whose 2-target is \((g_*, \Sigma_3)\). The third 3-arrow is \((g_*, \Sigma_3, \hat{K}_3)\), interchanging 2-arrows (4) and (5) in the figure \(\Sigma_2\), with

\[
\hat{K}_3 = \hat{K}_3^0 \triangleright \left\{ (f^{x_2,x_4})^{-1}, (f^{x_2} f^{x_4}) \triangleright F^{x_1,x_3; x_2,x_4} \right\}^{-1},
\]
whose 2-target is \((g_*, \hat{\Sigma}_4)\), where \(\hat{K}_3^0 = F^{x_1,x_2} \cdot (f^{x_2} f^{x_1; x_2}) \triangleright F^{x_3,x_4; x_1,x_2} \cdot f^{x_2} \triangleright F^{x_1,x_4; x_2}\) is the whiskering corresponding to the composition of 2-arrows (2), (1) and (3) in the figure \(\hat{\Sigma}_2\).

The fourth 3-arrow is \((g_*, \hat{\Sigma}_4, \hat{K}_4)\) with

\[
\hat{K}_4 = F^{x_1,x_2} \triangleright' [f^{x_2} \triangleright k^{x_1,x_3}; x_4; x_2],
\]

whose 2-target is \((g_*, \hat{\Sigma}_5)\). The fifth 3-arrow is \((g_*, \hat{\Sigma}_5, \hat{K}_5)\) with

\[
\hat{K}_5 = [F^{x_1,x_2} \cdot f^{x_2} \triangleright (F^{x_1,x_3}; x_2 \cdot f^{x_3} x_2 \triangleright F^{x_1,x_4}; x_2; x_3)] \triangleright' k^{x_2,x_3; x_4},
\]

whose 2-target is \((g_*, \hat{\Sigma}_6)\), where the part before \(\triangleright'\) is the whiskering corresponding to the composition of 2-arrows (1), (2) and (3) in the figure \(\hat{\Sigma}_5\).

The last 3-arrow is \((g_*, \hat{\Sigma}_6, \hat{K}_6)\), interchanging 2-arrows (4) and (3) in the figure \(\hat{\Sigma}_6\), with

\[
\hat{K}_6 = [F^{x_1,x_2} \cdot f^{x_2} \triangleright F^{x_1,x_3}; x_2] \triangleright' [F^{x_2,x_3} \triangleright' \{F^{x_2,x_3} \triangleright' (F^{x_2,x_3} \cdot f^{x_2} f^{x_3}; x_2 \triangleright F^{x_1,x_4}; x_2; x_3)]\,.
\]

whose 2-target is \((g_*, \hat{\Sigma}_8)\) in the last subsection. The 0-th 3-arrow is

\[
\hat{K}_0 = \pi_L \circ F_3(\Omega_{x_1,x_2}, x_3; x_4) = [f^{x_1} \triangleright (f^{x_2}; x_1 \triangleright F^{x_3,x_4}; x_1, x_2 \cdot F^{x_2,x_4}; x_1) \cdot F^{x_1,x_4}] \triangleright' (F^{x_1,x_4}; x_1 \triangleright k^{x_1,x_2}; x_3; x_4),
\]

where the part before \(\triangleright'\) is the whiskering corresponding to the composition of 2-arrows (1), (2) and (3) in the figure \(\hat{\Sigma}_1\).

The derivative \(\frac{\partial^3 \hat{K}_6 \circ \hat{K}_4 \circ \hat{K}_3 \circ \hat{K}_2}{\partial x_1 \partial x_2 \partial x_3 \partial x_4}\) at \((0, 0, 0, 0)\) gives

\[
\begin{align*}
v_3 C(v_1, v_2, v_3) + A(v_4) &\triangleright C(v_1, v_2, v_3) - \{B(v_1, v_2), B(v_3, v_4)\} + \{B(v_2, v_4), B(v_1, v_3)\} \\
+ v_2 C(v_1, v_3, v_4) + A(v_2) &\triangleright C(v_1, v_3, v_4) - \{B(v_2, v_3), B(v_1, v_4)\}.
\end{align*}
\]
6.3. The covariance of the 3-dimensional holonomies. Denote \( K_8 = \hat{K}_0^{-1}, \ldots, K_{14} = \hat{K}_0^{-1}. \) The 3-dimensional holonomy is defined as \( \mathcal{H}_F = K_1 K_2 \cdots K_{14}. \) Then

\[
\frac{\partial^4 \mathcal{H}_F}{\partial x_1 \partial x_2 \partial x_3 \partial x_4}(0, 0, 0, 0) = (6.1) - (6.2) = \Omega_3.
\]

In our construction above, \( \partial \Theta \) is the composition of 14 3-arrows, say \( \vartheta_1, \ldots, \vartheta_{14}, \) with the 2-target of \( \vartheta_j \) coinciding with the 2-source of \( \vartheta_{j+1} \)'s, i.e.,

\[
t_2(\vartheta_1) = s_2(\vartheta_2), \ldots, t_2(\vartheta_{14}) = s_2(\vartheta_1).
\]

Denote \( \tilde{F}(\vartheta_j) = (\ast, \ast, \tilde{K}_j) \) and \( F(\vartheta_j) = (\ast, \ast, K_j). \) The 0- and 1-source of \( \vartheta_j \) are independent of \( j. \) Denote \( f := s_1(\vartheta_j), c := s_0(\vartheta_j). \) The naturality (4.9)-(4.8) of the lax-natural transformation \( \Psi : F \rightarrow \tilde{F} \) implies

\[
(6.4) \quad \Psi_{s_2(\vartheta_j)} \cdot \tilde{F}_3(\vartheta_j) = \Psi_f \gg' [\Psi_c \gg K_j] \cdot \Psi_{t_2(\vartheta_j)}.
\]

Thus, \( \Psi_f \gg' [\Psi_c \gg K_j] = \Psi_{s_2(\vartheta_j)} \cdot \tilde{K}_j \cdot \Psi_{t_2(\vartheta_j)}, \) which implies that

\[
\Psi_f \gg' [\Psi_c \gg (K_1 K_2 \cdots K_{14})] = \Psi_{s_2(\vartheta_j)} \tilde{K}_1 \tilde{K}_2 \cdots \tilde{K}_{14} \Psi_{s_2(\vartheta_j)}^{-1},
\]

by (6.3) and both \( \gg \) and \( \gg' \) being automorphisms. This is the covariance of the 3-dimensional holonomy under lattice 3-gauge transformations.

7. Discussion

The 3-dimensional holonomy is 3-gauge invariant. We can use the construction in section 5 and 6 to give the construction of a non-Abelian 3-form lattice gauge theory. It is interesting to give a lattice 3-BF theory (cf. [8] [10] for the 2-gauge case), a combinatorial construction of the
topological higher gauge theory as a state sum model. These models are expected to be trivially renormalizable, i.e., independent of the chosen triangulation. Then they will give topological invariants of manifolds.

In the standard lattice gauge theory, the most general gauge invariant expressions are spin networks, the generalization of Wilson loops that includes branchings of the lines with intertwiners of the gauge group at the branching points. The most general 2-gauge invariant expressions will be given by coloured branched surfaces, i.e., by some sort of spin foams. It is quite interesting to consider the most general 3-gauge invariant expressions.

Barrett-Crane-Yetter state sum model of quantum gravity \[37\] is equivalent to a lattice 2-gauge theory. How about the 3-version counterpart of quantum gravity?

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