Abstract

Quantization procedures play an essential role in microlocal analysis, time-frequency analysis and, of course, in quantum mechanics. Roughly speaking the basic idea, due to Dirac, is to associate to any symbol, or observable, \( a(x, \xi) \) an operator \( \text{Op}(a) \), according to some axioms dictated by physical considerations. This led to the introduction of a variety of quantizations. They all agree when the symbol \( a(x, \xi) = f(x) \) depends only on \( x \) or \( a(x, \xi) = g(\xi) \) depends only on \( \xi \):

\[
\text{Op}(f \otimes 1)u = fu, \quad \text{Op}(1 \otimes g)u = F^{-1}(g Fu)
\]

where \( F \) stands for the Fourier transform. Now, Dirac aimed at finding a quantization satisfying, in addition, the key correspondence

\[
[\text{Op}(a), \text{Op}(b)] = i\text{Op}(\{a, b\})
\]

where \([, ,]\) stands for the commutator and \(\{, ,\}\) for the Poisson brackets, which would represent a tight link between classical and quantum

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mechanics. Unfortunately, the famous Groenewold–van Hove theorem states that such a quantization does not exist, and indeed most quantization rules satisfy this property only approximately.

Now, in this note we show that the above commutator rule in fact holds for the Born-Jordan quantization, at least for symbols of the type $f(x) + g(\xi)$. Moreover we will prove that, remarkably, this property completely characterizes this quantization rule, making it the quantization which best fits the Dirac dream.

1 Introduction

The theory of pseudodifferential operators has had many avatars since its inception in the mid 1960s; it has developed into a major branch of operator theory since the pioneering work of R. Beals, H. Duistermaat, C. Fefferman, L. Hörmander, J. J. Kohn, R. Melrose, L. Nirenberg, M. A. Shubin, M. E. Taylor, and many others. One early precursor, having its origin in quantum mechanics, and which gained its mathematical lettres de noblesse only in 1979 following the work of Hörmander [21], is the theory of Weyl operators. It was observed by Stein [29], §75–7.6, that the Weyl pseudodifferential calculus is uniquely characterized by its symplectic covariance with respect to conjugation with metaplectic operators (see Wong [35] for a detailed proof of this property). In the present paper we consider another class of pseudodifferential operators, which is a relative newcomer in the mathematical literature, and which we are going to characterize in terms of the so-called Dirac correspondence. These operators are the Born–Jordan pseudodifferential operators familiar to mathematicians working in quantization problems and in time-frequency analysis. They can be defined as follows (we will give alternative definitions as we go): assuming here for simplicity that $a \in \mathcal{S}(\mathbb{R}^{2n})$ we define for, $\tau \in \mathbb{R}$, the Shubin operator $\text{Op}_\tau(a)$ by

$$\text{Op}_\tau(a)u(x) = (2\pi)^{-n} \int e^{i(x-y,\xi)}a((1-\tau)x + \tau y)u(y)dyd\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ (the case $\tau = \frac{1}{2}$ corresponding to Weyl operators).

The Born–Jordan operator $\text{Op}_{BJ}(a)$ with symbol $a$ is then, by definition, the average

$$\text{Op}_{BJ}(a) = \int_0^1 \text{Op}_\tau(a)d\tau.$$

Now, for a given quantization rule $a \mapsto \text{Op}(a)$, regarded as a mapping

$$\mathcal{S}'(\mathbb{R}^{2n}) \rightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)),$$
natural desirable properties are the formulas (1) and (2) below:

\[
\text{Op}(f \otimes 1)u = fu,
\quad \text{Op}(1 \otimes f)u = F^{-1}(fFu)
\] (1)

for \(u \in \mathcal{S}(\mathbb{R}^n)\), where \(F\) stands for the Fourier transform in \(\mathbb{R}^n\), and

\[
[\text{Op}(a), \text{Op}(b)] = i \text{Op}\{a, b\}
\] (2)

where \([\cdot, \cdot]\) is the commutator and \(\{a, b\}\) the Poisson bracket associated with the standard symplectic form.

While property (1) is a natural requirement for any honest pseudodifferential calculus, property (2) (which is closely related to the physicists “Dirac correspondence”) is of a slightly more subtle nature. For a better understanding of the importance of this property we have to briefly recall the notion of prequantization (see Gotay [16], Gotay et al. [17], and Tuynman [30] for detailed discussions of the state of the art; also see Berndt [2]; Englis [11] discusses in a short paper the existence of nonlinear quantizations and Abraham Marsden [1] address the question from a more function-theoretical point of view). One requires that if \(a \mapsto \text{Op}(a)\) is a continuous linear mapping associating to a real symbol \(a\) on \(\mathbb{R}^{2n}\) a symmetric operator \(\text{Op}(a)\) defined on some dense subspace of \(L^2(\mathbb{R}^n)\) then this mapping should satisfy, in addition to some other axioms, the condition (2). It turns out that it is in principle impossible to achieve this goal; this impossibility is the famous result of Groenewold [18], later completed by van Hove [31, 32], which is a “no-go” result. It says (in its strong form) that one cannot quantize the Poisson algebra of polynomials in \(\mathbb{R}^n\), beyond those of degree \(\leq 2\) (we briefly discuss this at the end of the paper).

In the present note we will show that:

- **This obstruction can be bypassed if one limits oneself to symbols of the type**
  \[
a(x, \xi) = f(x) + g(\xi)
  \] (3)
  
  and choose \(\text{Op}(a) = \text{OP}_{BJ}(a)\);

- **Conversely, \(\text{OP}_{BJ}\) is the only pseudodifferential quantization satisfying** (2) **at least for symbols of the type (3).**

In (3) the functions \(f, g\) are smooth and are allowed to grow at most polynomially, together with their derivatives. Notice that for the Weyl quantization \(\text{Op}_{1/2}\) the formula (2) holds, in general, only for polynomial symbols of order \(\leq 2\).
Observe that assuming conditions (1) and (2), at least for symbols of the type (3), uniquely forces the values $\text{Op}(a)$ when $a$ is in the linear space spanned of symbols of type (3) and their Poisson brackets. We will show that this space is dense in $S'(\mathbb{R}^{2n})$ so that $\text{Op}(a)$ is then uniquely characterized by these properties; in fact, we have $\text{Op}(a) = \text{Op}_{B}(a)$ for every $a \in S'(\mathbb{R}^{2n})$.

The importance of these results is double. First, the theory Born–Jordan operators has recently gained considerable interest under the impetus of mathematicians working in harmonic analysis [3, 4, 8, 13] and mathematical physicists [14, 15, 24, 25]. Secondly, as we anticipated, it is intimately related to a mathematical question harking back to the work of Groenewold [18] and van Hove [31, 32] on quantization; we will discuss this at the end of the paper.

This work is structured as follows: we review in Section 2 the basic properties of Born and Jordan’s pseudodifferential calculus we will need to prove our main results (Theorems 4 and 6) in Section 3. In Section 4 we discuss our results from the point of view of quantization.

Notation

We identify $\mathbb{R}^n$ with its dual $(\mathbb{R}^n)^*$ and $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$; if $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ we sometimes write $z = (x, \xi)$. The Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $S(\mathbb{R}^n)$ and its dual (the tempered distributions) by $S'(\mathbb{R}^n)$. We denote by $\delta(x', \xi')$ the Dirac distribution centered at $(x', \xi')$, and by $\mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$ the space of all continuous linear operators from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ (the continuity being understood in the weak sense). The Euclidean scalar product of $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ will be written $\langle x, \xi \rangle$.

For $x = (x_1, \ldots, x_n)$ we write $D = (D_{x_1}, \ldots, D_{x_n})$ where $D_{x_j} = -i\partial_{x_j}$ and $\langle x, D \rangle = x_1 D_{x_1} + \cdots + x_n D_{x_n}$. The Fourier transform $\hat{u}$ of $u \in S(\mathbb{R}^n)$ is the function $\hat{u} \in S(\mathbb{R}^n)$ defined by

$$
\hat{u}(\xi) = \mathcal{F}u(\xi) = (2\pi)^{-n/2} \int e^{-i\langle \xi, x \rangle} u(x) \, dx
$$

where $dx = dx_1 \cdots dx_n$ is the usual Lebesgue measure on $\mathbb{R}^n$. The standard symplectic form on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is defined by $\sigma = d\xi_1 \wedge dx_1 + \cdots + d\xi_n \wedge dx_n$; in coordinates

$$
\sigma(x, \xi; y, \eta) = \langle \xi, y \rangle - \langle \eta, x \rangle.
$$

The corresponding symplectic group is denoted by $\text{Sp}(n)$. 
2 Preliminary Material: Review

2.1 The exponential of a linear review

To make the definitions above rigorous, we have to give a precise sense to the exponential operator \( e^{i(\langle x', x \rangle + \langle \xi', D \rangle)} \) and its variants. This can be done without any recourse to operator functional calculus. Consider the Schrödinger equation

\[
- \frac{i}{\partial t} v = \left( \langle x_0, x \rangle + \langle \xi_0, D \rangle \right) v
\]  

(4)

with initial datum \( u_0 = v(\cdot, 0) \) in \( S(\mathbb{R}^n) \). Its time-one solution \( u_1 = v(\cdot, 1) \) is given by

\[
u_1(x) = e^{i(\langle x_0, x \rangle + \frac{1}{2} \langle x_0, \xi_0 \rangle)} u_0(x + \xi_0);
\]

writing formally the solution \( u \) of (4) as \( e^{i(\langle x_0, x \rangle + \langle \xi_0, D \rangle)t} u_0 \) justifies the notation

\[
e^{i(\langle x_0, x \rangle + \langle \xi_0, D \rangle)} u_0 = e^{i(\langle x_0, x \rangle + \frac{1}{2} \langle x_0, \xi_0 \rangle)} u_0(x + \xi_0).
\]

We will write \( M(x_0, \xi_0) = e^{i(\langle x_0, x \rangle + \langle \xi_0, D \rangle)} \); thus:

\[
M(x_0, \xi_0) u(x) = e^{i(\langle x_0, x \rangle + \frac{1}{2} \langle x_0, \xi_0 \rangle)} u(x + \xi_0).  
\]

(5)

Setting \( x_0 = 0 \) we have in particular \( e^{i\langle \xi_0, D \rangle} u(x) = u(x + \xi_0) \) for \( u \in S(\mathbb{R}^n) \).

One verifies by a direct calculation using the definitions above that the “Baker–Campbell–Hausdorff formulas”

\[
M(x_0, \xi_0) = e^{-\frac{1}{2} \langle \xi_0, x_0 \rangle} e^{i\langle \xi_0, D \rangle} e^{i\langle x_0, x \rangle} e^{-\frac{1}{2} \langle x_0, \xi_0 \rangle} e^{i\langle x_0, D \rangle} e^{i\langle x_0, x \rangle} e^{i\langle \xi_0, D \rangle}  
\]

(6)

hold. Notice that the operator \( M(\xi_0, -x_0) \) is the Heisenberg operator

\[
T(x_0, \xi_0) = e^{is(x_0, \xi_0; x, D)}
\]

[12][33], that is, \( T(tx_0, t\xi_0) \) is the propagator of the Schrödinger equation \(-i\partial_t v = \sigma(x_0, \xi_0; x, D)v\). The operators \( M(x_0, \xi_0) \) and \( T(x_0, \xi_0) \) extend to continuous operators \( S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n) \) whose restrictions to \( L^2(\mathbb{R}^n) \) are unitary, and we have \( M(x_0, \xi_0)^* = M(-x_0, -\xi_0), T(x_0, \xi_0)^* = T(-x_0, -\xi_0) \).

The Grossmann–Royer \[19][27\] reflection operator \( R(x_0, \xi_0) \) is defined by

\[
R(x_0, \xi_0) = T(x_0, \xi_0) R(0, 0) T(x_0, \xi_0)^*  
\]

(7)

where \( R(0, 0) u(x) = u(-x) \). Explicitly, it is the unitary operator given by

\[
R(x_0, \xi_0) u(x) = e^{2i(\langle \xi_0, x-x_0 \rangle)} u(2x_0 - x).
\]

(8)
2.2 Weyl operators

Let \( A \) be a continuous linear operator \( \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \). Using Schwartz’s kernel theorem ([22], Theorem 5.2.1) one shows that there exists a distribution \( K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n) \) such that \( \langle Au, v \rangle = \langle K, v \otimes u \rangle \) for all \( u, v \in \mathcal{S}(\mathbb{R}^n) \); turning to integral notation, the operator \( A \) is thus formally given by

\[
Au(x) = \int K(x, y)u(y)dy.
\] (9)

The Weyl symbol \( a \) of \( A \) is the tempered distribution on \( \mathbb{R}^{2n} \) given by the Fourier transform

\[
a(x, \xi) = (2\pi)^{n/2} \int e^{-i(y, \xi)} K(x + \frac{1}{2}y, x - \frac{1}{2}y) dy;
\] (10)

the action of the operator \( A = \text{Op}_W(a) \) on \( u \in \mathcal{S}(\mathbb{R}^n) \) is thus given by the formula

\[
Au(x) = (2\pi)^{-n} \int e^{i(x-y, \xi)} a(\frac{1}{2}(x + y), \xi)u(y) dyd\xi
\] (11)

(the integral being interpreted in the distributional sense for \( a \in \mathcal{S}'(\mathbb{R}^{2n}) \)). Performing the change of variables \((y, \xi) \mapsto (2x - x', \xi')\) formula (11) can be rewritten in terms of the Grossmann–Royer operator \( \mathcal{R} \) as

\[
Au = \pi^{-n} \int a(x', \xi')R(x', \xi')u dyd\xi.
\] (12)

Finally, applying the Parseval formula to the integral in (12), we get

\[
Au(x) = (2\pi)^{-n} \int \hat{a}(x', \xi')M(x', \xi')u(x)dyd\xi
\] (13)

(see [14], §6.3.2).

2.3 Shubin’s \( \tau \)-operators

Let \( a \in \mathcal{S}(\mathbb{R}^{2n}) \) and \( \tau \in \mathbb{R} \); replacing the definition (10) of the Weyl symbol with

\[
a_{\tau}(x, \xi) = (2\pi)^{n/2} \int e^{-i(y, \xi)} K(x + \tau y, x - (1 - \tau)y) dy
\] (14)

we get the \( \tau \)-pseudodifferential operator (Shubin [28]) \( A_{\tau} = \text{Op}_{\tau}(a) \):

\[
A_{\tau}u(x) = (2\pi)^{-n} \int e^{i(x-y, \xi)} a((1 - \tau)x + \tau y, \xi)u(y) dyd\xi;
\] (15)
the case $\tau = \frac{1}{2}$ yields the Weyl operator $A = \text{Op}_W(a)$. Equivalently, $A_\tau$ is the operator with Schwartz kernel

$$K_\tau(x,y) = (2\pi)^{-n} \int e^{i(x-y,\xi)} a((1-\tau)x + \tau y, \xi) \, d\xi. \quad (16)$$

The operator $A_\tau$ is a continuous linear mapping $S(S^\infty) \rightarrow S'(\mathbb{R}^n)$; conversely, for every $\tau \in \mathbb{R}$, every such operator $A$ is a Shubin $\tau$-operator, the $\tau$-symbol $a_\tau$ of $A$ being the distribution on $\mathbb{R}^{2n}$ defined by the Fourier transform where $K$ is the Schwartz kernel of $A$. One shows ([14], §9.2.1 and 9.3.1) that, as in the case of Weyl operators, the operator $A_\tau$ can be written

$$A_\tau u = \pi^{-n} \int a(x',\xi') R_\tau(x',\xi')u \, dx' d\xi' \quad (17)$$

where $R_\tau$ is given, for $\tau \neq \frac{1}{2}$, by

$$R_\tau(x',\xi')u = (\Theta_\tau * R(x',\xi'))u \quad (18)$$

$$\Theta_\tau(x,\xi) = \frac{2^n}{2\tau - 1} \exp\left(\frac{2i}{2\tau - 1} \langle x,\xi \rangle \right) \quad (19)$$

and $\Theta_{1/2}(x,\xi) = \delta(x,\xi)$. The Fourier decomposition of $A_\tau$ is then given by

$$A_\tau u(x) = (2\pi)^{-n} \int \tilde{a}(x',\xi') M_\tau(x',\xi') u(x) \, dx' d\xi' \quad (20)$$

where, by definition,

$$M_\tau(x,\xi) = e^{\frac{4}{2\tau - 1}(x,\xi)} M(x,\xi). \quad (21)$$

It is convenient for our purposes to introduce the Shubin symbol classes $\Gamma^m_\rho(\mathbb{R}^{2n})$ (Shubin, [28], §23). By definition $a \in \Gamma^m_\rho(\mathbb{R}^{2n})$ ($m \in \mathbb{R}, 0 < \rho \leq 1$) if $a \in C^\infty(\mathbb{R}^{2n})$ and if for every $\alpha \in \mathbb{N}^{2n}$ there exists $C_\alpha \geq 0$ such that

$$|D^{\alpha}_{(x,\xi)} a(x,\xi)| \leq C_\alpha (1 + |x| + |\xi|)^{m-\rho|\alpha|}.$$ 

Every polynomial $a$ of degree $m$ in the variables $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ belongs to $\Gamma^m_{1/2}(\mathbb{R}^{2n})$. Using standard estimates it is easy to check that if $a \in \Gamma^m_\rho(\mathbb{R}^{2n})$ then $A_\tau = \text{Op}_\tau(a)$ maps continuously $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$.
2.4 Born–Jordan operators

For $a \in \mathcal{S}^{(R^2n)}$ the Born–Jordan operator $A_{BJ} = \text{Op}_{BJ}(a)$ is the operator with kernel $K_{BJ} = \int_0^1 K_\tau d\tau$ where $K_\tau$ is given by (16); equivalently $A_{BJ} = \int_0^1 A_\tau d\tau$ where $A_\tau = \text{Op}_\tau(a)$. Using formulas (20) and (21) it is straightforward to obtain the Fourier decomposition of $A_{BJ}$:

$$A_{BJ}u(x) = (2\pi)^{-n} \int \hat{a}(x',\xi')M_{BJ}(x',\xi')u(x) \, dx' \, d\xi'$$

with

$$M_{BJ}(x,\xi) = \text{sinc}(\frac{1}{2}(x,\xi))M(x,\xi)$$

where, as usual, $\text{sinc} t = \sin(t)/t$.

The terminology “Born–Jordan operator” comes from the following observation: choose $n = 1$ and assume that $a_{r,s}(x,\xi) = x^r\xi^s$ where $r$ and $s$ are positive integers. Then one has ([14], §9.1.2)

$$\text{Op}_\tau(a_{r,s}) = \sum_{k=0}^r \binom{r}{k} \tau^k (1 - \tau)^{r-k} x^k D^s x^{r-k}.$$ 

Integrating both sides of this equality from 0 to 1 with respect to $\tau$ we get, using the properties of the beta function,

$$\text{Op}_{BJ}(a_{r,s}) = \frac{1}{r+1} \sum_{k=0}^r x^k D^s x^{r-k}$$

which is Born and Jordan’s “quantization rule” [5]. The following remark is important: one proves by induction that

$$[x^{r+1},D_x^{s+1}] = (s+1)i \sum_{j=0}^r x^{r-j} D^s x^j$$

hence formula (24) can be rewritten

$$\text{Op}_{BJ}(a_{r,s}) = \frac{1}{i(r+1)(s+1)}[x^{r+1},D_x^{s+1}]$$

**Remark 1** This identity is remarkable because it shows that Born–Jordan operators with polynomial symbols in the $x,\xi$ variables can be expressed as a sum of commutators (see in the context the paper [26] by Pain) and that Born-Jordan quantization enjoys [2] at least for monomial symbols in dimension 1. In other terms, the operators $\text{Op}_{BJ}(a_{r,s})$ are uniquely determined
by the quantization of monomials depending only on \( x \) or \( \xi \). We refer to Theorem 6 and Remark 8 below for the general case of distribution symbols in arbitrary dimension.

An important observation is the following: the adjoint of \( A_{BJ} = \text{Op}_{BJ}(a) \) with respect to the sesquilinear product

\[
(u|v) = \int u(x)\overline{v(x)}dx
\]
on \( S(\mathbb{R}^n) \) is the operator \( A_{BJ}^* = \text{Op}_{BJ}(\overline{a}) \) (hence \( A_{BJ} \) is self-adjoint when \( a \) is real). This follows from the fact that \( A_{BJ}^* = \text{Op}_{1-\tau}(\overline{\tau}) \) if \( A_{\tau} = \text{Op}_{\tau}(a) \) (see [13, 14]).

While the linear mapping

\[
\text{Op}_{W} : S'(\mathbb{R}^{2n}) \rightarrow \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))
\]

which to every symbol \( a \in S'(\mathbb{R}^{2n}) \) associates the corresponding Weyl operator \( A = \text{Op}_{W}(a) \) is a continuous isomorphism [12, 23, 29, 35], this is not true of the mapping

\[
\text{Op}_{BJ} : S'(\mathbb{R}^{2n}) \rightarrow \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n))
\]
because it is not injective. In fact, set \( m(x, \xi) = e^{i(\langle x_0, x \rangle + \langle \xi_0, \xi \rangle)} \); we have \( \hat{m} = (2\pi)^n\delta(x_0, \xi_0) \) and hence by (22) and (23) we obtain

\[
\text{Op}_{BJ}(m) = \int \delta(x_0, \xi_0) \text{sinc}\left(\frac{1}{2} \langle x, \xi \rangle\right) M(x', \xi') dx'd\xi' = \text{sinc}\left(\frac{1}{2} \langle x_0, \xi_0 \rangle\right) M(x_0, \xi_0).
\] (27)

We thus have \( \text{Op}_{BJ}(m) = 0 \) for all \( (x_0, \xi_0) \) such that \( \langle x_0, \xi_0 \rangle \neq 0 \) and \( \langle x_0, \xi_0 \rangle \in 2\pi\mathbb{Z} \). While the surjectivity of \( \text{Op}_{W} \) and \( \text{Op}_{\tau} \) is obvious using Schwartz’s kernel theorem, the proof of the surjectivity of \( \text{Op}_{BJ} \) is rather tricky. The difficulty comes from the following observation: for every \( A \in \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n)) \) there exists \( a \in S'(\mathbb{R}^{2n}) \) such that \( A = \text{Op}_{W}(a) \) hence the mapping \( \text{Op}_{BJ} \) is surjective if and only if we can find \( b \in S'(\mathbb{R}^{2n}) \) such that \( \text{Op}_{BJ}(b) = \text{Op}_{W}(a) \). Comparison of formulas (13) and (22) shows that \( b \) must be a solution of the equation

\[
\hat{b}(x, \xi) \text{sinc}\left(\frac{1}{2} \langle x, \xi \rangle\right) = \hat{a}(x, \xi);
\]

the determination of \( \hat{b} \) (and hence of \( b \)) thus requires a division by the function \( (x, \xi) \mapsto \text{sinc}\left(\frac{1}{2} \langle x, \xi \rangle\right) \), which has infinitely many zeroes. We have
proven in a recent work [9] with E. Cordero that the solution \( \hat{b} \) actually exists in \( \mathcal{S}'(\mathbb{R}^{2n}) \), but the method is quite tricky and does not allow an explicit expression of \( b \), neither does it allow to produce any qualitative results about the regularity properties of \( b \) in terms of those of \( a \). However, as we have shown in [10], the situation is much more satisfactory when one supposes that the symbol \( a \) belongs to one of the Shubin symbol classes \( \Gamma^m_{\rho}(\mathbb{R}^{2n}) \). One has in this case the following result, which in a sense trivializes Born–Jordan operators:

**Proposition 2** If \( \hat{A}_{BJ} = \text{Op}_{BJ}(a) \) with \( a \in \Gamma^m_{\rho}(\mathbb{R}^{2n}) \) there exists, for every \( \tau \in \mathbb{R} \), a symbol \( a_\tau \) belonging to the same symbol class \( \Gamma^m_{\rho}(\mathbb{R}^{2n}) \) such that \( \hat{A}_{BJ} = \text{Op}_{\tau}(a_\tau) \).

Conversely, for any given symbol \( a_\tau \in \Gamma^m_{\rho}(\mathbb{R}^{2n}) \) there exists a symbol \( a \in \Gamma^m_{\rho}(\mathbb{R}^{2n}) \) such that \( \text{Op}_{BJ}(a) = \text{Op}_{\tau}(a_\tau) + R \) where \( R \) is an operator with integral kernel in \( \mathcal{S}(\mathbb{R}^{2n}) \).

In particular, taking \( \tau = \frac{1}{2} \), the operator \( \hat{A}_{BJ} = \text{Op}_{BJ}(a) \) is a Weyl pseudodifferential operator with symbol in the same Shubin class as \( a \).

### 3 The Characteristic Property

In this section we state and prove the main results of this paper.

#### 3.1 Born–Jordan quantization turns Poisson bracket into commutators

Let \( a, b \in C^\infty(\mathbb{R}^{2n}) \) and \( X_a, X_b \) the corresponding Hamiltonian vector fields: \( i_{X_a}\sigma + da = 0 \) and \( i_{X_b}\sigma + db = 0 \). By definition the Poisson bracket of \( a \) and \( b \) is \( \{a, b\} = i_{X_a}i_{X_b}\sigma \). In coordinates, \( X_a = (\partial_\xi a_x, -\partial_x a) \) and \( X_b = (\partial_\xi b_x, -\partial_x b) \), and

\[
\{a, b\} = \sum_{|\alpha|=1} \partial_\xi^\alpha a_x \partial_\xi^\alpha b - \partial_\xi^\alpha b_x \partial_\xi^\alpha a.
\]

Let us now define a convenient class of functions in \( \mathbb{R}^n \).

**Definition 3** Let \( \mathcal{A}(\mathbb{R}^n) \) be the space of all smooth functions \( f \) on \( \mathbb{R}^n \) such that for every \( \alpha \in \mathbb{N}^n \),

\[
|\partial^\alpha f(x)| \leq C_\alpha (1 + |x|)^{m_\alpha} \quad x \in \mathbb{R}^n
\]

for some constants \( C_\alpha > 0 \) and \( m_\alpha \) depending on \( \alpha \).
The relevance of this class of functions is that if \( f \in \mathcal{A}(\mathbb{R}^n) \) and \( u \in \mathcal{S}(\mathbb{R}^n) \) then \( fu \in \mathcal{S}(\mathbb{R}^n) \).

**Theorem 4** Let \( f \in \mathcal{A}(\mathbb{R}^n) \) and \( g \in \mathcal{A}(\mathbb{R}^n) \); set \( a = f \otimes 1 \) and \( b = 1 \otimes g \). Then the operators \( \text{Op}_{BJ}(a) \), \( \text{Op}_{BJ}(b) \) belong to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \) and we have

\[
[\text{Op}_{BJ}(a), \text{Op}_{BJ}(b)] = i \text{Op}_{BJ}\{a, b\}. \tag{28}
\]

**Proof.** An elementary calculation shows that

\[
\text{Op}_{BJ}(a)u = fu, \quad \text{Op}_{BJ}(b)u = \mathcal{F}^{-1}(g \mathcal{F}u) = (2\pi)^{-n/2} \mathcal{F}^{-1}g \ast u. \tag{29}
\]

Since \( f \in \mathcal{A}(\mathbb{R}^n) \) the mapping \( u \mapsto fu \) is continuous on \( \mathcal{S}(\mathbb{R}^n) \), so that \( \text{Op}_{BJ}(a) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \); similarly since \( g \in \mathcal{A}(\mathbb{R}^n) \) the map \( u \mapsto \mathcal{F}^{-1}g \ast u \in \mathcal{S}(\mathbb{R}^n) \) is continuous on \( \mathcal{S}(\mathbb{R}^n) \) and \( \text{Op}_{BJ}(b) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \) as well.

Now, we have

\[
\text{Op}_{BJ}\{a, b\} = \text{Op}\left( \sum_{|\alpha|=1} \partial_x^\alpha f \otimes \partial_\xi^\alpha g \right).
\]

The composed operators \( \text{Op}_{BJ}(a) \text{Op}_{BJ}(b) \) and \( \text{Op}_{BJ}(b) \text{Op}_{BJ}(a) \) are thus well defined, belong to \( \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \), and are given by

\[
\text{Op}_{BJ}(a) \text{Op}_{BJ}(b)(u) = (2\pi)^{-n/2} (\mathcal{F}^{-1}g \ast u)f
\]

\[
\text{Op}_{BJ}(b) \text{Op}_{BJ}(a)(u) = (2\pi)^{-n/2} \mathcal{F}^{-1}g \ast (fu).
\]

It follows that

\[
[\text{Op}_{BJ}(a), \text{Op}_{BJ}(b)]u = (2\pi)^{-n/2} \left[ (\mathcal{F}^{-1}g \ast u)f - \mathcal{F}^{-1}g \ast (fu) \right]. \tag{30}
\]

Let us now show property (28), that is

\[
[\text{Op}_{BJ}(a), \text{Op}_{BJ}(b)]u = i \text{Op}\left( \sum_{|\alpha|=1} \partial_x^\alpha f \otimes \partial_\xi^\alpha g \right)u \tag{31}
\]

for all \( u \in \mathcal{S}(\mathbb{R}^n) \).

Since \( \hat{\partial_x}f(\xi) = i\xi_j\hat{f}(\xi) \), by (22) and (23) we have

\[
\text{Op}_{BJ}\left( \sum_{|\alpha|=1} \partial_x^\alpha f \otimes \partial_\xi^\alpha g \right)u(x) = (-2)(2\pi)^{-n} I(x) \tag{32}
\]

where

\[
I(x) = \int \hat{f}(x')\hat{g}(\xi') \sin\left( \frac{1}{2}(x', \xi') \right)e^{i((x', x') + \frac{1}{2}(x', \xi'))}u(x + \xi') \, dx' \, d\xi'.
\]
Writing \( \sin t = (e^{it} - e^{-it})/2i \) we have \( I(x) = I_1(x) + I_2(x) \) where

\[
I_1(x) = \frac{1}{2i} \int \hat{f}(x') \hat{g}(\xi') e^{i(x',x)+(x',\xi')} u(x + \xi') dx' d\xi'
\]

\[
I_2(x) = -\frac{1}{2i} \int \hat{f}(x') \hat{g}(\xi') e^{i(x',x)} u(x + \xi') dx' d\xi'.
\]

Performing the change of variables \((x', \xi') \mapsto (x'', \xi'' - x)\) these integrals become

\[
I_1(x) = \frac{1}{2i} \int \hat{f}(x'') \hat{g}(\xi'' - x) e^{i(x'',\xi'')} u(\xi'') dx'' d\xi''
\]

\[
I_2(x) = -\frac{1}{2i} \int \hat{f}(x'') \hat{g}(\xi'' - x) e^{i(x'',x)} u(\xi'') dx'' d\xi''.
\]

Using successively the identity \( \hat{g}(\xi'' - x') = \mathcal{F}^{-1} g(x - \xi'') \), Fubini’s theorem, and the Fourier inversion formula we get the expressions

\[
I_1(x) = \frac{(2\pi)^{n/2}}{2i} \left[ \mathcal{F}^{-1} g * (fu) \right](x)
\]

\[
I_2(x) = -\frac{(2\pi)^{n/2}}{2i} f(x) \left[ \mathcal{F}^{-1} g * u \right](x)
\]

and hence, by (32),

\[
\text{OPBJ} \left( \sum_{|\alpha|=1} \partial_x^\alpha f \otimes \partial_x^\alpha g \right) u = -i(2\pi)^{-n/2} \left[ f(\mathcal{F}^{-1} g * u) - \mathcal{F}^{-1} g * (fu) \right].
\]

Together with (30) this proves the equality (31). \( \square \)

Let us call \( h \in \mathcal{S}'(\mathbb{R}^{2n}) \) a “physical Hamiltonian” if \( h(x, \xi) = f(x) + g(\xi) \) with \( f, g \in \mathcal{A}(\mathbb{R}^n) \). The following consequence of Theorem 3 is straightforward:

**Corollary 5** Let \( h \) and \( k \) be physical Hamiltonians, and set \( H = \text{OPBJ}(h) \), \( K = \text{OPBJ}(k) \). We have

\[
[H, K] = i \text{OPBJ}(\{h, k\}). \tag{33}
\]

**Proof.** Writing \( h(x, \xi) = f(x) + g(\xi) \) and \( k(x, \xi) = d(x) + e(\xi) \) we have, using the linearity of the Poisson bracket and the fact that \( \{f, d\} = \{g, e\} = 0 \),

\[
\{h, k\} = \{f, e\} + \{g, d\}.
\]

Let \( F = \text{OPBJ}(f) \), and so on. In view of the equalities (29) we have \( DF = DF \) and \( GE = EG \) and hence

\[
[H, K] = [F, E] + [G, D]
\]

formula (33) follows since \( [F, E] = i \text{OPBJ}(\{f, e\}) \) and \( [G, D] = i \text{OPBJ}(\{g, d\}) \) in view of (28). \( \square \)

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3.2 The characteristic property of Born-Jordan operators

We now prove a converse of Theorem 4. Consider the space

\[ A_0(\mathbb{R}^n) = \{ e^{i \langle x', \cdot \rangle} : x' \in \mathbb{R}^n \} \]

of purely imaginary exponentials in \( \mathbb{R}^n \).

**Theorem 6** Let \( \text{Op} : \mathcal{S}'(\mathbb{R}^{2n}) \rightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) \) be such that

\[
\text{Op}(f \otimes 1)u = fu, \quad \text{Op}(1 \otimes f)u = \mathcal{F}^{-1}(f \mathcal{F}u)
\]

if \( f \in A_0(\mathbb{R}^n) \), and

\[
[\text{Op}(a), \text{Op}(b)] = i \text{Op}(\{a, b\})
\]

for all \( a = f \otimes 1 \) with \( f \in A_0(\mathbb{R}^n) \) and \( b = 1 \otimes g \) with \( g \in A_0(\mathbb{R}^n) \).

Then

\[ \text{Op}(a) = \text{Op}_{BJ}(a) \]

for all \( a \in \mathcal{S}'(\mathbb{R}^{2n}) \).

We need the following known density result (we report on the short proof for the sake of completeness).

**Lemma 7** The linear span of \( A_0(\mathbb{R}^n) \) is dense in \( \mathcal{S}'(\mathbb{R}^n) \).

**Proof.** Since the Fourier transform is an isomorphism of \( \mathcal{S}'(\mathbb{R}^n) \), it is sufficient to prove that the linear span of the set of Dirac delta functions \( \delta_{x'} \), \( x' \in \mathbb{R}^n \), is dense in \( \mathcal{S}'(\mathbb{R}^n) \). To this end, observe that \( C_c^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{S}'(\mathbb{R}^n) \); on the other hand, every function \( f \in C_c^\infty(\mathbb{R}^n) \) is the limit in \( \mathcal{S}'(\mathbb{R}^n) \) of the finite sums \( (1/k)^n \sum_{x' \in (1/k)\mathbb{Z}^n} f(x')\delta_{x'} \) as \( k \rightarrow +\infty \), as one sees by approximating the pairing \( \langle \langle f, \phi \rangle \rangle = \int f(x)\phi(x) \, dx \), \( \phi \in \mathcal{S}(\mathbb{R}^n) \), by Riemann sums. \( \blacksquare \)

**Proof of Theorem 6.** By Lemma 7 (applied in dimension \( 2n \)) the exponentials \( e^{i(\langle x', \cdot \rangle + \langle \xi', \cdot \rangle)} \), \( x', \xi' \in \mathbb{R}^n \), span a dense subspace of \( \mathcal{S}'(\mathbb{R}^{2n}) \). Since both the quantizations \( \text{Op} \) and \( \text{Op}_{BJ} \) are linear and continuous on \( \mathcal{S}'(\mathbb{R}^{2n}) \), it is sufficient to prove that they coincide on \( A_0(\mathbb{R}^{2n}) \). By (27) this amounts to prove that

\[
\text{Op}(e^{i(\langle x', \cdot \rangle + \langle \xi', \cdot \rangle)}) = \text{sinc}(\frac{1}{2} \langle x', \xi' \rangle) e^{i(\langle x', x \rangle + \langle \xi', D \rangle)}.
\]
By (34) we have
\[ \text{Op}(e^{i(x',\cdot)} \otimes 1)u(x) = e^{i(x',x)}u(x), \quad \text{Op}(1 \otimes e^{i(\xi',\cdot)})u(x) = e^{i(\xi',D)}u(x) \quad (37) \]
for \( u \in \mathcal{S}(\mathbb{R}^n) \).

Assume now \( \langle x', \xi' \rangle \neq 0 \). The condition (35) implies
\[ [\text{Op}(e^{i(x',\cdot)} \otimes 1), \text{Op}(1 \otimes e^{i(\xi',\cdot)})] = \frac{1}{i} \langle x', \xi' \rangle \text{Op}(e^{i(\langle x',\cdot \rangle + \langle \xi',\cdot \rangle)}), \]
that is
\[ \text{Op}(e^{i(\langle x',\cdot \rangle + \langle \xi',\cdot \rangle)}) = \frac{i}{\langle x', \xi' \rangle} [\text{Op}(e^{i(x',\cdot)} \otimes 1), \text{Op}(1 \otimes e^{i(\xi',\cdot)})]. \]

In view of (37) and (6) we have
\[ \text{Op}(e^{i(x',\cdot)} \otimes 1) \text{Op}(1 \otimes e^{i(\xi',\cdot)}) = e^{-\frac{1}{2n} \langle x', \xi' \rangle} e^{i(\langle x',x \rangle + \langle \xi',D \rangle)} \]
\[ \text{Op}(1 \otimes e^{i(\xi',\cdot)}) \text{Op}(e^{i(x',\cdot)} \otimes 1) = e^{\frac{1}{2n} \langle x', \xi' \rangle} e^{i(\langle x',x \rangle + \langle \xi',D \rangle)} \]
and hence
\[ \text{Op}(e^{i(\langle x',\cdot \rangle + \langle \xi',\cdot \rangle)}) = \frac{i}{\langle x', \xi' \rangle} \left( e^{-\frac{1}{2n} \langle x', \xi' \rangle} - e^{\frac{1}{2n} \langle x', \xi' \rangle} \right) e^{i(\langle x',x \rangle + \langle \xi',D \rangle)} \]
which is (36).

The case \( \langle x', \xi' \rangle = 0 \) follows by continuity, because both sides of (36) are continuous functions of \( x', \xi' \) valued in \( L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) \).

This concludes the proof. \( \blacksquare \)

**Remark 8** As an alternative, since we already proved in Theorem 4 that the Born-Jordan quantization enjoys the properties in the statement of Theorem 6, in the proof of the latter we could limit ourselves to showing that there is at most one quantization \( \text{Op} : \mathcal{S}'(\mathbb{R}^{2n}) \rightarrow L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)) \) satisfying those properties. Now, those conditions force the values \( \text{Op}(a) \) when \( a \) is a symbol of the type
\[ e^{i(\langle x',\cdot \rangle + \langle \xi',\cdot \rangle)}, \quad \langle x', \xi' \rangle \neq 0, \quad (38) \]
because, as we saw, such symbols can be written (up to a multiplicative constant) as the Poisson bracket of symbols in \( \mathcal{A}_0(\mathbb{R}^n) \). On the other hand, symbols of the type (38) span a dense subset of \( \mathcal{S}'(\mathbb{R}^{2n}) \) (this is true by Lemma 7 without the condition “\( \langle x', \xi' \rangle \neq 0 \)”, but also under this additional condition, because those exponential functions are continuous, as functions of \( x', \xi' \), valued in \( \mathcal{S}'(\mathbb{R}^{2n}) \) and the set of \( (x', \xi') \in \mathbb{R}^{2n} \) such that \( \langle x', \xi' \rangle \neq 0 \) is dense in \( \mathbb{R}^{2n} \).
4 Discussion

The original concept of quantization in physics consists in trying to assign to “observables” (= real valued symbols) on \( \mathbb{R}^{2n} \) self-adjoint operators on a Hilbert space (usually \( L^2(\mathbb{R}^{2n}) \)) according to certain rules, dictated by physical considerations. Mathematically speaking, this amounts in constructing a continuous mapping \( \text{Op} \) from some Poisson algebra of functions defined on \( \mathbb{R}^{2n} \) and such that:

(i) The operators \( \text{Op}(x_j) \) and \( \text{Op}(\xi_j) \) are given by \( \text{Op}(x_j)u = x_ju \) and \( \text{Op}(\xi_j)u = D_ju \);

(ii) \( [\text{Op}(a), \text{Op}(b)] = i \text{Op}(\{a, b\}) \) (when \( [\text{Op}(a), \text{Op}(b)] \) exists).

These rules are often complemented by other conditions, for instance, the “von Neumann” rule

(iii) \( \text{Op}(\phi \circ a) = \phi(\text{Op}(a)) \) where \( \phi \) is a real function for which \( \phi(\text{Op}(a)) \) is defined.

Suppose now that the symbols \( a \) and \( b \) are quadratic polynomials:

\[
a(x, \xi) = \frac{1}{2} \langle M_a z, z \rangle, \quad b(x, \xi) = \frac{1}{2} \langle M_b z, z \rangle, \quad z = (x, \xi), \quad M_a \text{ and } M_b \text{ being symmetric matrices.}
\]

The flows of the corresponding Hamiltonian vector fields \( X_a \) and \( X_b \) are linear hence consist of one-parameter subgroups \( (S^t_a)_{t \in \mathbb{R}} \) and \( (S^t_b)_{t \in \mathbb{R}} \) of the symplectic group \( \text{Sp}(n) \). Using the path-lifting theorem it follows that we can lift, in a unique way \( (S^t_a)_{t \in \mathbb{R}} \) and \( (S^t_b)_{t \in \mathbb{R}} \) to one-parameter subgroups of any of the covering groups \( \text{Sp}_q(n) \) of \( \text{Sp}(n) \). Choosing \( q = 2 \) and identifying \( \text{Sp}_2(n) \) with the metaplectic group \( \text{Mp}(n) \), we obtain two one-parameter subgroups \( (\hat{S}^t_a)_{t \in \mathbb{R}} \) and \( (\hat{S}^t_b)_{t \in \mathbb{R}} \) of unitary operator acting on \( L^2(\mathbb{R}^n) \). It now requires some calculations to show that

\[
i \frac{d}{dt} \hat{S}^t_a = A(x, D) \hat{S}^t_a
\]

where the symmetric operator \( A(x, D) \) is formally given by

\[
A(x, D) = \frac{1}{2} \langle M_a(x, D), (x, D) \rangle = \text{Op}_W(a).
\]

(and likewise for \( \hat{S}^t_b \)); a few more calculations \[\text{[20, 16]}\] then show we have

\[
[\text{Op}_W(a), \text{Op}_W(b)] = i \text{Op}_W(\{a, b\})
\]

which is condition (ii). (The latter is easily extended to non-homogeneous quadratic polynomials). Now, the essence of the Groenewold–Van Hove
no-go result is that there exists no quantization $\text{Op}$ whose restriction to the Poisson algebra of quadratic polynomials coincides with $\text{Op}_W$ and still satisfies (ii).

It suffices in fact to show that (ii) cannot hold for the Poisson algebra of polynomials; the proof then boils down to the following observation. Consider the monomial $x^2\xi^2$; it can be written in two different ways using Poisson brackets, namely

$$x^2\xi^2 = \frac{1}{9}\{x^3, \xi^3\} = \frac{1}{3}\{x^2\xi, x\xi^2\}. \tag{39}$$

Let us assume that such an $\text{Op}$ exists; then we would have

$$\frac{1}{9}\text{Op}(\{x^3, \xi^3\}) = \frac{1}{3}\text{Op}(\{x^2\xi, x\xi^2\})$$

and hence

$$\frac{1}{9i}[\text{Op}(x^3), \text{Op}(\xi^3)] = \frac{1}{3i}[\text{Op}(x^2\xi), \text{Op}(x\xi^2)]. \tag{40}$$

Assuming $\text{Op}(x^3)$ is multiplication by $x^3$ and $\text{Op}(\xi^3) = D^3$ we get, after some calculations

$$\frac{1}{9i}[\text{Op}(x^3), \text{Op}(\xi^3)] = x^2D^2 - 2ixD - \frac{2}{3} \tag{41}$$

and similarly, writing

$$\text{Op}(x^2\xi) = \frac{1}{6i}[\text{Op}(x^3), \text{Op}(\xi^2)], \quad \text{Op}(x\xi^2) = \frac{1}{6i}[\text{Op}(x^2), \text{Op}(\xi^3)]$$

we obtain

$$\frac{1}{3i}[\text{Op}(x^2\xi), \text{Op}(x\xi^2)] = x^2D^2 - 2ixD - \frac{1}{3} \tag{42}$$

hence a contradiction. Now comes the crucial point: the conflict between (II) and (II) disappears when one chooses $\text{Op} = \text{Op}_{BJ}$ and one requires a weaker form of (ii), namely

$$(\text{ii}') \quad [\text{Op}(a), \text{Op}(b)] = i\text{Op}(\{a, b\}) \text{ for all } a = f \oplus g \text{ and } b = h \oplus k.$$  

This weaker rule indeed excludes the condition (II) and in fact we have

$$\text{Op}_{BJ}(x^2\xi^2) = x^2D^2 - 2ixD - \frac{2}{3},$$

which agrees with (III).
Since $\text{Op}_{BJ}(a) = \text{Op}_{WJ}(a)$ for all quadratic symbols (see [13, 14, 15]), Born–Jordan quantization really is the most natural quantization. In view of Corollary 5 we moreover have

$$[H, K] = i \text{Op}_{BJ}([h, k])$$  \hspace{1cm} (43)

for all symbols of the type $h(x, \xi) = f(x) + g(\xi)$ and $k(x, \xi) = d(x) + e(\xi)$; from a physical point of view this means that the Dirac correspondence holds for all Hamiltonians of the traditional type “kinetic energy plus potential”, and this result does not hold for any other quantization; in particular if one replaces $\text{Op}_{BJ}$ with the Weyl correspondence $\text{Op}_{W}$ we have generally

$$[H, K] \neq i \text{Op}_{W}([h, k]).$$  \hspace{1cm} (44)

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