The Master Field for Rainbow Diagrams
and
Free Non-Commutative Random Variables

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Abstract

The master field for a subclass of planar diagrams, so called rainbow diagrams, for higher dimensional large N theories is considered. An explicit representation for the master field in terms of noncommutative random variables in the modified interaction representation in the Boltzmannian Fock space is given. A natural interaction in the Boltzmannian Fock space is formulated by means of a rational function of the interaction Lagrangian instead of the ordinary exponential function in the standard Fock space.

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1 Introduction

The problem of summation of all planar diagrams in higher dimensional space-time is still out of reach. Its solution is closely related with problem of finding the leading asymptotics in matrix models for large N and may have important applications to the hadron dynamics [1, 2, 3]. Summation of planar diagrams has been performed only in low dimensional space-time [4].

One can write a closed system of equations for invariant correlation functions in the large N limit, so called the planar Schwinger-Dyson equations, for arbitrary dimension of space-time. In the early 80-s it was suggested [3] that there exists the master field $\Phi(x)$ such that the correlation functions for this field $\Phi(x)$ are equal to the large N limit of invariant correlation functions for matrix models,

$$\lim_{N \to \infty} \frac{1}{N^{1+n/2}} < \text{tr} (M(x_n)...M(x_1)) >= <0|\Phi(x_n)...\Phi(x_1)|0> \quad (1.1)$$

It was suggested that $\Phi(x)$ satisfies the equation [5]

$$[i \frac{\delta S[\Phi]}{\delta \Phi(x)} + 2\Pi(x)]|0> = 0 \quad (1.2)$$

where $S$ is an action and $\Pi$ and $\Phi$ are the subjects of the relation [3, 6]

$$[\Pi(x), \Phi(y)] = i\delta^{(D)}(x-y)|0> <0| \quad (1.3)$$

An operator realization of this algebra proposed in [5] has used the knowledge of all correlation functions. This was considered as an evident drawback of such approach. In that time it was also discussed the problem of finding a generating functional reproducing the planar Schwinger-Dyson equations or equations (1.2), (1.3). It was pointed out that the generating functional cannot depend on one commutative source [7]. It was proposed to use some auxiliary fermionic fields to reproduce the planar Schwinger-Dyson equations [8]. For gauge theories the suitable generating functional is nothing but a functional on paths, i.e. the Wilson loops, that satisfies the Makeenko-Migdal equation [9]. The stochastic equation for large N master fields was proposed in [10].

Recently it has been a reveal of an interest to the problem of constructing the master field for planar graphs. One of origins for this are mathematical works [11, 12, 13] devoted to non-commutative probability, for a review see [14, 15]. Singer has advocated that the essential difficulty of the large number of degrees of freedom in higher dimensional large N matrix models is dealt with finding master fields which live in “large” operator algebras such as the type $II_1$ factor associated with free group. In the recent paper by Gopakumar and Gross [16] the basic concepts of non-commutative probability have been reviewed and applied to the large N limit of matrix models. They stress that if one can solve a matrix model then one can write an explicit expression for the master field as an operator in a well defined Hilbert space. Douglas also proposed to use ideas of non-commutative probability to the large N stochastic approach [17]. The explicit construction of the master fields for several low-dimensional models including $QCD_2$ has been given [16]-[19].

However an effective (i.e. without the knowledge of correlation fonctions) operator realization for the master field for all planar diagrams in higher dimensional space-time is still unknown. Therefore it is worth to try to find an effective operator realization of
the master field for some subset of the planar diagrams. The goal of this letter is to construct an explicit operator realization of the master field for rainbow graphs. Rainbow graphs form a subset of planar graphs. It is turn out that rainbow correlation functions may be obtained by average of the fields with Boltzman statistics. The construction does not require as input correlation functions. More exactly we show that to get a closed set of equations for correlations functions for a model with an interaction in the Boltzmannian Fock space one has to deal with a modified interaction representation. This new interaction representation involves not the ordinary exponential function of the interaction but a rational function and will be given by the formula

$$< \phi(x_m)\ldots\phi(x_1) \frac{1}{1 - g \int dy V_{int}(\phi(y))} >$$ (1.4)

The paper is organized as follows. In Sect.2 we present the Schwinger-Dyson equations for rainbow diagrams. Sect.3 contains a necessary information about the Boltzmannian Fock space. We argue also that to get a closed set of the Schwinger-Dyson type of equation we have to deal with the interaction representation in the Boltzmannian Fock space in the form (1.4). We also show that the corresponding Schwinger-Dyson equations reproduce the Schwinger-Dyson equations for rainbow diagrams.

## 2 Rainbow Diagrams

Let us consider the correlation functions of the form

$$< \mathcal{V}(x_n, \ldots x_1) >= \frac{1}{N^{1+n/2}} < \text{tr} (M(x_n)\ldots M(x_1)) >$$ (2.1)

$< \cdot >$ means

$$< \mathcal{O}(M) > = \frac{1}{Z} \int \mathcal{O}(M) \exp\{-S[M]\} dM,$$ (2.2)

where

$$S[M] = \int dx \left\{ \frac{1}{2} \text{tr} (M(-\Delta + m^2)M) + \frac{g}{4N} \text{tr} M^4(x) \right\}$$ (2.3)

Here $M(x)$ is $N \times N$ matrix function. We assume all necessary regularizations. For our purpose it is essential that the regularization is such that the free propagator is

$$< M_{ij}(x)M_{i'j'}(y) >^{(0)} = \delta_{ii'}\delta_{jj'}D(x-y),$$ (2.4)
(− ∆ + m^2) \cdot D(x − y) = \delta^{(D)}(x − y). \hfill (2.5)

We shall consider the external lines corresponding to global invariant Green functions as the lines corresponding to generalized vertex. The rainbow diagrams for \( < V(x_n, ... x_1) > \) are defined as a part of planar non-vacuum diagrams which are topologically equivalent to the graphs with all vertexes lying on some straight line on the the right of generalized vertex and all propagators lying in the half plane. The rainbow diagramms are illustrated for the \( M^3 \)- interaction on Fig.1, where vertexies are drawn by solid double lines and all contraction (propagators) by double dash lines; in that follows we will use also solid lines for propagators.

To write down the rainbow Schwinger-Dyson equations one has to select rainbow diagrams from the both hand sides of the planar Schwinger-Dyson equations. In the large N limit the planar Schwinger-Dyson equations have the form

\[
(− ∆ + m^2) \cdot G_n(x_n, ... x_1) = g G_{n+2}(x_n, ... x_{l+1}, x_l, x_{l+1}, x_{l+1}, x_{l-1}, ... x_1) \\
+ \sum_{i < l} \delta(x_l - x_i) G_{l-i-1}(x_{i-1}, ... x_{i+1}) G_{n+i-1}(x_n, ... x_{l+1}, x_{i-1}, ... x_1) \\
+ \sum_{l < i} \delta(x_l - x_i) G_{i-l-1}(x_{l-1}, ... x_{i+1}) G_{n+l-i-1}(x_n, ... x_{l+1}, x_{l-1}, ... x_1),
\]

where

\[
G_n(x_n, ... x_1) = \lim_{N \to \infty} \frac{1}{N^{1+n/2}} \langle \mathrm{tr} (M(x_n) ... M(x_1)) \rangle 
\]  

The planar Schwinger-Dyson equations (2.6) are written for the case of quartic interaction (2.3) and they are symbolically presented on Fig.2. Now let us consider a modification of the right hand side of (2.6) for correlation functions correspon ding to the rainbow diagrams

\[
\lim_{N \to \infty} \frac{1}{N^{1+n/2}} < \mathrm{tr} (M(x_n) ... M(x_1)) >_{rb} = W_{n+2}(x_{n+2}, ... x_1) \hfill (2.8)
\]
Figure 3: One term in the Schwinger-Dyson equation for rainbow graphs

There are modifications in the term representing the interaction and also in the Schwinger terms. Indeed, let us consider all possible contractions of a given point $x_m$ with a vertex $v$ of the rainbow diagrams. We have to distinguish the cases of odd and even $m$. For the even $m = 2l$ on the left of this vertex $v$ (see Fig. 3a) one has subgraphs corresponding to rainbow diagrams of the correlation function (Fig. 3b)

$$< (M(x_{2l})M(x_{2l-1})...M(x_1)(V_{int})^{k_1})_{rj} >$$

or

$$< (M^3(x_{2l})M(x_{2l-1})...M(x_1)(V_{int})^{k_1})_{rj} >$$

By using

$$< (M(x_i) ... M(x_k))_{jj'} > = \frac{\delta_{jj'}}{N} < \text{tr} (M(x_i) ... M(x_k)) >$$

these terms reproduce $W_{2l}^{k_1}(x_{2l}, x_{2l-1}...x_1)$ and $W_{2l+2}^{k_1}(x_{2l}, x_{2l}, x_{2l}, x_{2l-1}...x_1)$, respectively. The rest of the diagram Fig. 3a corresponds to $W_{n-2l+2}^{k_2}(x_n, ...x_{2l+1}, x_{2l}, x_{2l})$, $k_2 = k - k_1 - 1$.

One has also to modify contributions from the Schwinger terms, since only one correlator (in our case the correlator corresponding to $G_{n-l+i-1}$) can contain the interaction. Finally we get the following system of equations

$$(-\triangle + m^2)_{x_{2l}} W_n(x_n, ...x_1) = g(W_{2l}(x_{2l}, x_{2l-1}, ...x_1)W_{n-2l+2}(x_n, ...x_{2l+1}, x_{2l}, x_{2l})$$
Let us consider an algebra generated by operators $A(p)$ and $A^+(p)$ satisfying the relations
\[ A(p)A^+(q) = \delta^{(D)}(p - q). \]  
(3.1)

One can realized this algebra in a space which is an analogue of the usual Fock space $[3, 10]$. This space is generated by the vacuum vector $|0\rangle$, $A(p)|0\rangle = 0$ and $n$-particle states of $n$ non-identical particles,

\[ |p_1, \ldots, p_n\rangle = A^+(p_1)\ldots A^+(p_n)|0\rangle \]  
(3.2)

There is no symmetization or antisymmetrization as in the Bose or Fermi cases. We shall call this Fock space the Boltzmannian Fock space (it is also called the free Fock space). One defines

\[ \phi(x) = \phi^+(x) + \phi^-(x) = \frac{1}{(2\pi)^{D/2}} \int \frac{d^Dp}{\sqrt{p^2 + m^2}} (A^+(p)e^{ipx} + A(p)e^{-ipx}) \]  
(3.3)

and therefore

\[ <0|\phi(x)\phi(y)|0\rangle = D(x - y) = \frac{1}{(2\pi)^D} \int \frac{d^Dp}{p^2 + m^2} e^{ip(x-y)} \]  
(3.4)
To calculate the n-point correlation function one has to apply a Boltzmannian Fock space analog of the ordinary Wick theorem. The specific feature of the Wick theorem in this case is that for a given diagram one has not additional symmetry factors related with that an annihilation operator can be contracted with any creation operator on the right. In the Boltzmannian Fock space an annihilation operator can been contracted only with a nearest creation operator on the right. Therefore one sees immediately from the Fig 4b that the correlation function
\[
\langle 0 | \phi(x_{2m}) \ldots \phi(x_{1}) | 0 \rangle = F^{(0)}_{2m}(x_{2m}, \ldots x_{1})
\]
(3.5)
satisfies to the same equations as \( W^{0}_{2m} \), and therefore
\[
F^{(0)}_{2m}(x_{2m}, \ldots x_{1}) = W^{0}_{2m}(x_{2m}, \ldots x_{1}),
\]
i.e.
\[
\lim_{N \to \infty} \frac{1}{N^{1+m}} \langle 0 | \left( M(x_{2m}) \ldots M(x_{1}) \right) | 0 \rangle = \langle 0 | \phi(x_{2m}) \ldots \phi(x_{1}) | 0 \rangle
\]
(3.6)

Let us make a few comments about an operator realization of the algebra (1.3) with \( \pi \) satisfying the requirement
\[
\pi(x)|0\rangle = \frac{i}{2}(-\Delta + m^2)x\phi(x)|0\rangle.
\]
(3.7)

First of all note that the operator algebra (1.3) is not an unique algebra which follows from the free planar Schwinger-Dyson equation. Indeed, one gets the same equation from the operator relation
\[
(-\Delta + m^2)x\phi(x) = \pi(x), \quad [\pi(x), \phi(y)] = -i\delta^D(x \cdot y) |0\rangle < 0 | + K(x, y)
\]
(3.8)
with \( K(x, y) \) being the subject of relations
\[
\langle 0 | K(y, x_{1})\phi(x_{2}) \ldots \phi(x_{n}) | 0 \rangle > + \langle 0 | \phi(x_{1})K(y, x_{2}) \ldots \phi(x_{n}) | 0 \rangle > + \ldots
\]
(3.9)

The simplest solution of (3.8) and (3.9) is given by
\[
\pi(y) = \frac{i}{2}(-\Delta + m^2)y[\phi^{+}(y)]|0\rangle < 0 | - |0 > < 0 | \phi^{-}(y)],
\]
(3.10)
\[ K(y, x) = \frac{i}{2}(-\Delta + m^2)y[\phi^+(x)\phi^+(y)]0 \rangle < 0| - |0 > < 0|\phi^-(y)\phi^-(x) \] (3.11)
+ \phi^+(y)|0 > < 0|\phi^-(x) + \phi^+(x)|0 > < 0|\phi^-(y)\]

This gives a hint to write a following operator realization of the commutation relations (1.3) with \( \pi \) satisfying the requirement (3.7)

\[ \pi(y) = \frac{i}{2}(-\Delta + m^2)y\{\phi^+(y)|0 > < 0| - |0 > < 0|\phi^-(y) + \]

\[ \sum_{n=1}^{\infty} \int dz_1(-\Delta + m^2)z_1...\int dz_n(-\Delta + m^2)z_n[\phi^+(z_1)...\phi^+(z_n)\phi^+(y)|0 > < 0|\phi^-(z_n)...\phi^-(z_1) \]

\[ - \phi^+(z_1)...\phi^+(z_n)|0 > < 0|\phi^-(y)\phi^+(z_n)...\phi^+(z_1)]\}

For completeness let us present a known solution of equation (2.4) for \( D = 0 \) and \( g = 0 \), i.e. equations

\[ \langle \Phi^{2m} \rangle = \sum_{l=0}^{m-1} \langle \Phi^{2l} \rangle \langle \Phi^{2m-2l-2} \rangle \] (3.13)

where \( \Phi = a^+ + a, aa^+ = 1 \). Let us denote \( c_n = \langle \Phi^{2n} \rangle, \quad c_0 = 1 \). Consider the generating function

\[ Z(g) = c_0 + c_1 g + ... + c_n g^n + ... = \langle \frac{1}{1 - g\Phi^2} \rangle \] (3.14)

One has

\[ Z(g)^2 = c_0^2 + (c_0 c_1 + c_1 c_0) g + ... + (c_0 c_n + ... + c_n c_0) g^n + ... \]

From (3.13) one gets

\[ Z(g)^2 = c_1 + c_2 g + ... + c_{n+1} g^n + ... \]

Therefore \( Z(g) \) satisfies the equation

\[ gZ(g)^2 = Z(g) - 1. \]

One has to take the following solution of this equation

\[ Z(g) = \frac{1 - \sqrt{1 - 4g}}{2g} = 1 + C_2^1 g + ... + \frac{1}{n+1} C_n^m g^n + ... , \]

from which we get

\[ c_n = \langle \Phi^{2n} \rangle = \frac{1}{n+1} C_n^m = \frac{2n!}{n!(n+1)!} \]

### 3.2 Interacting Theory

We want to derive the Schwinger-Dyson equations for theory with interaction in the Boltzmannian Fock space. To find the form of interaction let us consider the following correlation functions

\[ F_m^{(k)}(x_m, ... x_1) = \langle 0|\phi(x_m)...\phi(x_1)(\int dy_1 : (\phi(y_1))^4 :) ... (\int dy_k : (\phi(y_k))^4 :)|0 > ' \] (3.15)

where \( \phi(x) \) is the free field (1.3) and ' means that we do not take into account the diagrams with vacuum subgraphs. We draw all operators \( \phi \) on the straight line. The operators \( \phi(x_i) \)
Figure 5: Some diagrams contributed to the Schwinger-Dyson equations in the Boltzmannian Fock space

corresponding to the external lines are represented by the circles and the operators \( \phi(y_i) \) corresponding to the interaction vertices are represented by the filled circles on Fig. 5.

On Fig. 5 we draw all possible contractions of a given external line with a given vertex. As we have mentioned above there is no here additional factors related with symmetry of graphs. Therefore one has not here the standard factor \( 1/k! \) in the \( k \)-th order of perturbation theory. This remark leads to an important observation that to get a set of equations for correlation functions in the Boltzmannian Fock space we have to consider instead of the usual exponential factor \( \exp\{V_{int}\} \) the rational function

\[
\{1 - V_{int}\}^{-1},
\]

(compare with \( Z(g) \) (3.14)). Therefore we introduce the following correlation functions

\[
F_m(x_m, ...x_1) = \sum_{k=0} g^k F_m^{(k)}(x_m, ...x_1) = <0 | \phi(x_m) ... \phi(x_1) \frac{1}{1 - V_{int}} | 0 >'.
\]

(3.17)

On Fig. 5 \( V_{int} = g \int dy_1(\phi(y_1))^4 \).

Examining all possible contractions of the point \( x_{2l} \) we see that the first two graphs on Fig. 5 reproduce the first two sums on the right hand side of (2.12). In the similar way one sees that for odd point the graphs reproduce the first two sum in the the right hand side of (2.13).

Performing the normal ordering in the expression \( \phi(x_m) ... \phi(x_1) \) we get contributions corresponding to the Schwingers terms in the correlation function \( (-\triangle + m^2)_{x_{2l}} F_m(x_m, ...x_1) \).

This consideration proves that \( F_m \) satisfies to the following equations

\[
(-\triangle + m^2)_{x_{2l}} F_n(x_n, ...x_1) = g(F_{2l}(x_{2l}, x_{2l-1}, ...x_1) F_{n-2l+2}(x_n, ... x_{2l+1}, x_{2l}, x_{2l})
\]

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\[ F_{2l+2}(x_{2l}, x_{2l}, x_{2l-1}, \ldots x_1) F_{n-2l}(x_n, \ldots x_{2l+1}) \]
\[ + \sum_{i<2l} \delta(x_{2l} - x_i) F_{n-2l+i-3}(x_n, \ldots x_{2l+1}, x_{i-1}, \ldots x_1) F_{2l-i+1}^{(0)}(x_{2l-1}, \ldots x_{i+1}) \]
\[ + \sum_{2l<i} \delta(x_{2l} - x_i) F_{n+2l-i-3}(x_n, \ldots x_{i+1}, x_{2l-1}, \ldots x_1) F_{i-2l+1}^{(0)}(x_{i+1}, \ldots x_{2l+1}); \] (3.18)

and the similar equations for the \( x_{2l+1} \).

Comparing (3.18) with (2.12) we see that \( F_m \) satisfies to the equations for the rainbow diagrams. Therefore we get

\[ \lim_{N \to \infty} \frac{1}{N^{1+m/2}} < \text{tr} \left( M(x_m) \ldots M(x_1) \right) \exp \left\{ \frac{g}{4N} \int dy \text{tr} \left( M(y) \right)^4 \right\} >_t^{F_{rb}} = \] (3.19)

\[ < 0 | \phi(x_m) \ldots \phi(x_1) \frac{1}{1 - g \int dy \phi(y)^4} | 0 >_{BF} \]

Symbol \(< . >^F\) denotes the vacuum expectation value in the ordinary Euclidean bosonic Fock space and \(< . >^B_{BF}\) denotes nonvacuum diagrams in the Boltzmannian Fock space.

In conclusion, a model of quantum field theory with interaction in the Boltzmannian Fock space has been considered. We have used the new interaction representation with a rational function of the interaction Lagrangian instead of the exponential function in the standard interaction representation. The Schwinger-Dyson equations were derived and it was shown that the perturbation expansion for the model corresponds to the summation of the rainbow diagrams. The quantum field with this interaction can be interpreted as the master field for the rainbow diagrams in the large \( N \) limit matrix model. The construction of the master field is effective in the sense that it is purely algebraic and doesn’t require the knowledge of correlation functions of the theory with the interaction. Another aspects of quantum field theory in the Boltzmannian Fock space including the Minkowskian formulation are considered in [20].

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