NON-EISENSTEIN COHOMOLOGY OF
LOCALLY SYMMETRIC SPACES FOR GL$_2$
OVER A CM FIELD

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Abstract

Let $F$ be a CM field, let $p$ be a prime number. The goal of this paper is to show, under mild conditions, that the modulo $p$ cohomology of the locally symmetric spaces $X$ for GL$_2(F)$ with level prime to $p$ is concentrated in degrees belonging to the Borel-Wallach range $[q_0, q_0 + \ell_0]$ after localizing at a "strongly non-Eisenstein" maximal ideal of the Hecke algebra. From this result, we deduce expected consequences on the structure of the first and last cohomology groups as modules over the Hecke algebra.

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1 Introduction

Let $G/\mathbb{Q}$ be a connected reductive group and let $X = G_{\infty}/K_{\infty}$ be the symmetric space for $G_{\infty} = G(\mathbb{R})$. For any neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we can consider the locally symmetric space $X_K = G(\mathbb{Q})\backslash (G(\mathbb{A}_f)/K \times X)$.

Let us define the integers

$$l_0 := rk(G_{\infty}) - rk(K_{\infty}) \text{ and } q_0 := \frac{1}{2}(\dim_{\mathbb{R}} X - l_0)$$

A well-known conjecture (see for instance [CG18, Section 9, Conjecture B (4) (a)]) predicts that the cohomology of the locally symmetric space $X_K$ with any $\mathbb{F}_p$-local system vanishes after localizing at a non-Eisenstein maximal ideal of the Hecke algebra of $G$, outside of the range $[q_0, q_0 + l_0]$. It was motivated by its known analogue for $\mathbb{Q}_p$-local systems (cf. [BW80] Theorem III.5.1).

When $F$ is a CM field with degree $2d$ and $G = \text{Res}_{F/\mathbb{Q}}GL_2$, the dimension of $X$ is equal to $4d - 1$, $q_0 = d$ and $l_0 = 2d - 1$.

In this paper, we assume that the CM field $F$ is the compositum of a totally real field $F^+$ and an imaginary quadratic field $F_0$. Let $p$ be a prime. The spherical Hecke

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algebra $T = T^S$ (defined in Section 2) is generated by $T_{v,1}, T_{v,2}$ for all finite places $v \notin S$.

Let $\mathcal{O}$ be the valuation ring of a $p$-adic field, let $\varpi$ be a uniformizing parameter in $\mathcal{O}$; Given an algebraic representation $\mathcal{V}$ of $G$ over $\mathcal{O}$, the Hecke algebra $T$ acts on the Betti cohomology groups $H^i(X_K, \mathcal{V})$, resp. $H^i(X_K, \mathcal{V}/\varpi^m\mathcal{V})$. Let $m \subset T$ be a maximal ideal in the support of $H^i(X_K, \mathcal{V})$.

By a theorem of Scholze [Sch15, Theorem I.3], there exists a sufficiently big finite extension $\kappa$ of $\mathbb{F}_p$, and a continuous semisimple Galois representation

$$\bar{\rho}_m : G_F \to \text{GL}_2(\kappa)$$

which is characterized, up to conjugation, as follows: for every $v \notin S$, $\bar{\rho}_m$ is unramified at $v$ and the characteristic polynomial of $\bar{\rho}_m(\text{Frob}_v)$ is equal to $X^2 - T_v,1X + T_v,2N(v)(mod \, m)$.

Let us say that $\bar{\rho}_m$ has very large image if there exists a subfield $\kappa' \subset \kappa$ such that $\text{SL}_2(\kappa') \subset \text{Im}(\bar{\rho}_m) \subset \kappa' \times \text{GL}_2(\kappa')$.

We write $\mathcal{V} = \mathcal{V}_\lambda$ for a representation of highest weight $\lambda = (m_\tau, n_\tau)_\tau$ for all embeddings $\tau$ of $F$. Let $c$ be the complex conjugation. We assume that $\lambda = (m_\tau, n_\tau)_\tau$ is pure, that is:

$m_\tau + n_{\tau c} = n_\tau + m_{\tau c}$ for all $\tau$'s.

A cuspidal representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$ is called of cohomological weight $\lambda = (m_\tau, n_\tau)_\tau$ if it occurs in $H^i(X_K, \mathcal{V}_\lambda \otimes \mathbb{C})$. The purity weight $w = w(\lambda)$ of $\lambda$ is defined as $w = n_\tau + m_{\tau c} = n_\tau + m_{\tau c}$ for any $\tau$.

Our main result is

**Theorem 1.1.** Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ with level $K$, cohomological weight $\lambda$ and purity weight $w$. Assume that $p$ splits in $F_0$, $K_p$ is hyperspecial and $\pi$ is unramified outside of $S$. Let $m \subset T^S$ be the maximal ideal of the Hecke algebra associated to $(\pi, p)$.

Assume that $w$ is even, $n_\tau \equiv n_{\tau c} (mod \, 2)$ for all $\tau \in \text{Hom}(F, E)$, $p > 2d^2(d + 1)(-2w + 4)$ and that $\bar{\rho}_m$ has very large image. Then, for any $m \geq 1$,

$$H^i(X_K, \mathcal{V}_\lambda/\varpi^m\mathcal{V})_m = 0 \quad \text{for all } i \notin [q_0, q_0 + l_0].$$

Note that the assumption of very large image implies that $\bar{\rho}_m$ is absolutely irreducible, in other words, that the maximal ideal $m$ is non-Eisenstein. To recall that the image of Galois is very large, we call our maximal ideal strongly non-Eisenstein. Note also that we allow the weight $\lambda$ to be non-regular so that the cuspidal representation $\pi$ can be a base change from a subfield $F_1$ of $F$ of a cuspidal representation $\pi_1$, provided the corresponding maximal ideal $m_1$ is strongly non-Eisenstein.
Remark 1.2. By Poincaré duality, it suffices to prove the vanishing for $i < q_0 = d$ (see Proposition 8.2 in the text).

Remark 1.3. There are many vanishing results after localizing at non-Eisenstein maximal ideal for $l_0 = 0$, such as: Mokrane-Tilouine [MT02] for $G = GSp_{2g}$, Dimitrov [D03] $G = \text{Res}_{F/Q}\text{GL}_2$ where $F$ is a totally real field, Emerton-Gee [EG15] for $G = U(2,1)$, Caraiani-Scholze [CS19] for compact unitary Shimura varieties and $G = \text{Res}_{F/Q}U(n,n)$ where $F$ is a totally real field.

Recall that $V_\lambda$ denotes the local system in $\mathcal{O}$-modules associated to the representation $V_\lambda$ of highest weight $\lambda$.

Corollary 1.4. Let $\pi$ be as above, with cohomological weight $\lambda = (m_\tau,n_\tau)$ and purity weight $w$. Let $p$ and $m$ as above.

Assume that $w$ is even, $n_\tau = n_{\tau c} \pmod{2}$ for all $\tau \in \text{Hom}(F,E)$, $p > 2d^2(d+1)(-2w+4)$ and that $\overline{\rho}_m$ has very large image. Then, for any $m \geq 1$,

$$H^i(X_K,V_\lambda)_m = 0 \quad \text{for all } i \notin [q_0,q_0 + l_0].$$

This corollary follows easily (see Corollary 8.11) from the main theorem. Note that $H^{q_0}(X_K,V_\lambda)_m$ is torsion-free, but it is not the case in general for $i > q_0$ (and in particular for $i = q_0 + l_0$).

Finally, recall that Calegari and Geraghty introduced Conjectures A (see [CG18, Section 5.3]) and B (see [CG18, Section 9.1]) about the existence of Galois representations over localized Hecke algebras $T^S$ and $T^S_Q$ satisfying local global compatibilities at all primes, which we denote by (LGC). Furthermore they assume in Conjecture B that $H^i(X_K,V_\lambda/\varpi)_m = 0$, for $i \notin [q_0,q_0 + l_0]$. This is our Main Theorem. Moreover, if $p$ splits completely in $F$, the existence of the Galois representations defined over $T^S$ and $T^S_Q$ as desired (except (LGC)) has been established in [CGH+20, Theorem 6.1.4]. Note that Calegari-Geraghty established (LGC) at primes in $Q$ by [CG18, Lemma 9.6] for $n = 2$ and that results towards (LGC) at other primes are proven in [ACC+18]. We deduce from [CG18, Proposition 6.4] and from our main theorem the result:

Corollary 1.5. Under the assumptions of Theorem 8.9, if we assume (LGC), then $H^{q_0+\ell_0}(X_K,V_\lambda)_m$ is free of rank one over $T^S$ and $H^{q_0}(X_K,V_\lambda)_m \cong \text{Hom}(T^S,\mathcal{O})$ as $T^S$ module.

See comments before Theorem 8.12 for more details on the status of Conjecture B of [CG18].
Let us give a sketch of proof of the Main Theorem. First, all considerations until the end of Section 6 are valid for $G_n = \text{Res}_{F/\mathbb{Q}}GL_n$ for $n$ arbitrary. From Section 7 on, we specialize $n$ to be 2. We hope to generalize our result to larger values of $n$ later.

Let $X_K$ be the locally symmetric space of $G_n$ with level $K$ and $\tilde{X}_K$ be the locally symmetric space of $\tilde{G}_n := \text{Res}_{F/\mathbb{Q}}U(n,n)$ with level $\tilde{K}$. Note that $\tilde{X}_K$ is a PEL Shimura variety defined over $F_0$. We fix a non-Eisenstein maximal ideal $\mathfrak{m} \subset T^S$. We can define a maximal ideal of Hecke algebra of $\tilde{G}_n$, $\tilde{\mathfrak{m}} = S^{-1}(\mathfrak{m}) \subset T^S$ where $S : T^S \to T^S$ is the Satake transform recalled in Section 2.

By choosing $\tilde{K}$ and $\tilde{\lambda}$ adapted to the choice of $K$ and $\lambda$, we can define by Theorem 4.2.1 of [ACC+18] a Hecke-equivariant embedding $H^i(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}} \hookrightarrow H^i(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)_{\tilde{\mathfrak{m}}}$ as explained in Subsection 5.2.

We shall prove for any $n$ that $H^i(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}} = 0$ for $i < d$. Note that $d$ is not equal to $q_0$ for $n > 2$, but happens to be equal to $q_0$ when $n = 2$. For this purpose, it is enough to prove that $H^i(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)_{\tilde{\mathfrak{m}}} = 0$ for $i < d$. By Nakayama’s Lemma (or dévissage), it suffices to show $H^i(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)[\tilde{\mathfrak{m}}] = 0$ for $i < d$.

For this, the first step is Theorem 6.3 it is valid for weights $\tilde{\lambda}$ which are mildly regular (as defined in section 6) and sufficiently small with respect to $p$. This theorem states that for $i < d$, the Galois semisimplification of the boundary cohomology $H^i(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)[\tilde{\mathfrak{m}}]$ viewed as a $G_{F_0}$-representation where $G_{F_0}$ is the absolute Galois group of $F_0$, is a direct sum of characters. In our proof, we use the main result of Pink’s thesis [P92] and Kostant formula [Kos61] for computing the spectral sequence associated to the boundary stratification of the minimal compactification of $\tilde{X}_K$. Then, we use vanishing results of [LS13]. Note that for $n = 2$, the mild regularity is automatic.

By the generalized Eichler-Shimura relations due to Wedhorn (Theorem 7.1 Section 7), the Hecke polynomial $H_{v_0}$ at an unramified place $v_0$ of $F_0$ annihilates $H^i_{\mathfrak{m}}(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)$. Moreover, $H_{v_0}$ modulo $\tilde{\mathfrak{m}}$ coincides with the characteristic polynomial of $T(\text{Frob}_{v_0})$ where $T$ denotes the twisted tensor induction representation

$$T = (\otimes \text{Ind}_{F_0}^{F})((\wedge^n \rho_{\tilde{\mathfrak{m}}}) \otimes \rho_{\psi}^\vee(n(n + 1)/2 - 2n^2))$$

where $\rho_{\tilde{\mathfrak{m}}} \cong \rho_{\mathfrak{m}} \otimes \rho_{\mathfrak{m}}^\vee \otimes \varepsilon^{1-2n}$ and $\rho_{\psi}$ is the Galois representation associated to the central character of $\rho_{\mathfrak{m}}$ ($\varepsilon$ is the cyclotomic character of $F$).

Hence the characteristic polynomial of $T$ annihilates $H^i_{\mathfrak{m}}(\tilde{X}_K, \mathcal{V}_{\tilde{\lambda}}/\varpi)[\tilde{\mathfrak{m}}]$.

From now we assume that $n = 2$ and $\lambda$ satisfies the partial regularity condition (PR) (see subsection 7.2). This won’t restrict our main result because Corollary 8.3 shows that the Main Theorem for dominant weights $\lambda$ satisfying the condition (PR)
implies the Main Theorem for any pure dominant weight \( \lambda = (m_\tau, n_\tau)_{\text{Hom}(F,E)} \) where \( n_\tau \equiv n_{\tau_c} \pmod{2} \) for any \( \tau \in \text{Hom}(F,E) \).

In subsection 7.3, we assume that \( m \) is strongly non-Eisenstein and we study the Galois representation \( \left( \otimes \text{Ind}^{F_0}_F \right)((\wedge^2 \rho^\vee_m)^{-5}) \) restricted to \( G_F \) where \( \tilde{F} \) is the normalization of \( F \) in \( \bar{Q} \). We prove that if \( \gcd(p - 1, w + 1) = 1 \), the restriction of \( (\otimes \text{Ind}^{F_0}_F)((\wedge^2 \rho^\vee_m)^{-5}) \) to \( G_F \) factors through:

\[
H(F_\ell) := \{(M_i)_{i=1}^d \in \prod_{i=1}^d \text{GL}_2(F_\ell) \mid \exists \delta \in \det(G_F), \forall i, \det M_i = \delta\}
\]

In subsection 8.2, we observe that, by ”almost incompatibility” of the Fontaine-Laffaille weights of the twisted tensor induction representation \( T \)'s and those of the boundary cohomology, there are only three characters \( \chi_i, i = 1, 2, 3, \) of \( G_{F_0} \) which can occur simultaneously in the representation \( T \) and in the boundary cohomology groups \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi)[\bar{m}] \) for \( i < d \), viewed as \( G_{F_0} \)-representations. They are given explicitly as

\[
\chi_1 := (\otimes \text{Ind}^{F_0}_F)((\wedge^2 \rho^\vee_m)^{-5}), \quad \chi_2 = (\otimes \text{Ind}^{F_0}_F)((\wedge^2 \rho^\vee_m)(1)) \quad \text{and} \quad \chi_3 := (\otimes \text{Ind}^{F_0}_F)(\varepsilon^{-2}).
\]

Actually, in subsection 3.4, we compute the Fontaine-Laffaille weights of the \( G_{F_0} \)-representations \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi) \). By comparison of these weights to those of the characters \( \chi_1, \chi_2, \chi_3 \), we see that the characters \( \chi_2 \) and \( \chi_3 \) cannot occur in \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi)[\bar{m}] \) for \( i < d \). Therefore the only character which can occur in \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi)[\bar{m}] \) is \( \chi_1 \). Its Fontaine-Laffaille weight \( d \cdot w \) is the minimal Fontaine-Laffaille weight which can occur in \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi)[\bar{m}] \).

In subsection 8.3, we compute the multiplicity of the Fontaine-Laffaille weight \( d \cdot w \) inside \( H^i_d(X_K, \mathcal{V}_\lambda/\varpi)[\bar{m}] \). We find it is equal to the \( \kappa_p \)-dimension of \( H^{i+1}(X_K^{\text{tor}}, \mathcal{W}^{\text{can}}_{\lambda, \varpi})_{\bar{m}} \) (\( \kappa_p = F_\ell \) is the residue field of \( p \subset \mathcal{O}_{F_0} \)). By Serre duality, \( H^{d-i-1}(X_K^{\text{tor}}, \mathcal{W}^{\text{can}}_{-2\rho_{nc-w_0,G,\lambda,p}})_{\bar{m}} \) is dual to \( H^{i+1}(X_K^{\text{tor}}, \mathcal{W}^{\text{can}}_{\lambda, \varpi})_{\bar{m}} \); thus, the \( \kappa_p \)-dimension of this coherent cohomology group is equal to the multiplicity of the Fontaine-Laffaille weight \( d \cdot w \) in \( H^{d-i-1}(X_K, \mathcal{V}_\mu/\varpi)_{\bar{m}} \), where \( \mu := (-n_\tau + 2, -m_\tau + 2, m_\tau - 2, n_\tau - 2)_{\tau \in \mathcal{I}} \). Now, a vanishing result of Caraiani and Scholze [CS19] shows that \( H^{4d-i-1}(X_K, \mathcal{V}_\mu/\varpi)_{\bar{m}} = 0 \). Therefore, \( H^{4d-i-1}(X_K, \mathcal{V}_{\mu, \lambda}^{\text{can}}_{-2\rho_{nc-w_0,G,\lambda,p}})_{\bar{m}} \) is zero.

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1.1 Notations

In this section, we follow rather closely the notations of [ACC+18, Section 2].

Let \( F = F^+ F_0 \) be a CM field which is the compositum of an imaginary quadratic field \( F_0 \) and a totally real field \( F^+ \). Let \( \Delta_F \) be the discriminant of \( F \). We denote by \( c \) the complex conjugation of \( F \). Let \( d = [F^+: \mathbb{Q}] \). For a number field \( K \), let \( K_\mathbb{A} \) be its adele ring, \( K_f \) its finite part and \( K_\infty \) its infinite part so that \( K_\mathbb{A} = K_f \times K_\infty \).

Let \( \Psi_n \) be the matrix with 1’s on the anti-diagonal and 0’s elsewhere and set \( J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix} \).

Let \( \tilde{H} \) resp. \( \tilde{H}_c \) the quasi-split unitary group scheme over \( \mathcal{O}_{F^+} \) whose points on a ring \( R \) are given by

\[
\tilde{H}(R) = \{ g \in GL_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) | ^t g J_n g^c = J_n \}.
\]

resp.

\[
\tilde{H}(R) = \{ (g, c) \in GL_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) \times R^\times | ^t g J_n g^c = c \cdot J_n \}.
\]

If \( \bar{v} \) is a place of \( F^+ \) which splits in \( F \), then a choice of place \( v|\bar{v} \) of \( F \) determines a canonical isomorphism \( \iota_v : \tilde{H}(F^+_v) \cong GL_{2n}(F_v) \). Denote \( \tilde{G} := \text{Res}_{\mathcal{O}_{F^+}/\mathbb{Z}} \tilde{H} \) and \( \tilde{G} := \text{Res}_{\mathcal{O}_{F^+}/\mathcal{O}_F} \tilde{H} \).

We write \( \tilde{T} \subset \tilde{B} \subset \tilde{H} \) for the subgroups consisting of, respectively, the diagonal and the upper-triangular matrices in \( \tilde{H} \). Similarly we write \( H \subset P \subset \tilde{H} \) for the Siegel Levi and parabolic subgroups consisting of matrices which are diagonal, respectively, upper triangular by \( n \times n \)-blocks. We have \( H \cong \text{Res}_{\mathcal{O}_{F^+}/\mathcal{O}_F} \text{GL}_n \). We introduce \( \tilde{B} = \tilde{B} \cap H \) and we remark that \( T = \tilde{T} \cap H = \tilde{T} \). Let us define \( G = \text{Res}_{\mathcal{O}_{F^+}/\mathbb{Z}} H \).

Let \( K \subset G_f \) be a neat compact open subgroup and \( X_K \) be the adelic locally symmetric space of level \( K \):

\[
X_K = G(\mathbb{Q}) \backslash G_h/K \mathbb{Q}_K^\times K_\infty.
\]

Let \( \tilde{K} \subset \tilde{G}_f \) be a neat compact open subgroup of \( \tilde{G}_f \) compatible with \( K \) in the sense that \( G_f \cap \tilde{K} = K \). We also consider the unitary Shimura variety

\[
\tilde{X}_{\tilde{K}} = \tilde{G}(\mathbb{Q}) \backslash \tilde{G}_h/\tilde{K} \mathbb{K}_\infty.
\]
and let $\tilde{K} \subset \tilde{G}_f$ such that $\tilde{K} \cap \tilde{G}_f = \tilde{K}$. We also consider the unitary Shimura variety 

$$\tilde{X}_\tilde{K} = \tilde{G}(\mathbb{Q}) \backslash \tilde{G}_\lambda / \tilde{K} \tilde{K}_\infty.$$ 

The real dimension of $X_K$ is equal to $d \cdot \dim GL_n(\mathbb{C}) - d \cdot \dim U(n) - 1 = 2n^2d - n^2d - 1 = n^2d - 1$ and the real dimension of $\tilde{X}_\tilde{K}$ is equal to $d \cdot \dim U(n, n) - d \cdot \dim(U(n) \times U(n)) = 4n^d - 2n^2d = 2n^d$.

We denote by $S(\tilde{K})$ the set of places $\tilde{v}$ of $F^+$ where $\tilde{K}_{\tilde{v}}$ is not hyperspecial. Let $S'$ be the set of places of $F$ above those of $S(\tilde{K})$.

Let $p$ be a rational prime which splits in $F_0$, say, $(p) = pp^c$. Let $S_p$, resp. $\tilde{S}_p$, be the set of places of $F$, resp. $F^+$ above $p$; We put $\tilde{S} = S(\tilde{K}) \cup \tilde{S}_p$ and $S = S' \cup S_p$. Note that $S$ is stable by complex conjugation. Let $E$ be a finite extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$ large enough to contain the images of all the embeddings of $F$ in $\bar{\mathbb{Q}}_p$. Let $\mathcal{O}$ be its valuation ring, $\varpi$ a uniformizing parameter and $\kappa = \mathcal{O}/(\varpi)$ be its residue field. Note that $\tilde{G}_p = \tilde{G} \times \mathbb{Q}_p = \text{Res}_{F^+/\mathbb{Q}} GL_{2n} \times \mathbb{Q}_p$.

Let $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\Hom(F^+, E)}$ be a dominant weight for $(\tilde{G}_p, \tilde{B}_p, \tilde{T}_p)$. Let $\lambda \in (\mathbb{Z}_+^n)^{\Hom(F, E)}$ be a dominant weight for $(G, B, T)$.

Let $\tilde{I} \subset \text{Hom}(F, E)$ be a subset such that $\text{Hom}(F, E) = \tilde{I} \amalg \tilde{I}^c$. We write $\tilde{\tau}$ for unique element of $I$ which extends $\tau \in \text{Hom}(F^+, E)$ to $F$.

**Definition 1.6.** We say that the dominant weights $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\Hom(F^+, E)}$ and $\lambda \in (\mathbb{Z}_+^n)^{\Hom(F, E)}$ are corresponding if $\tilde{\lambda}_\tilde{\tau} = (-\lambda_{\tilde{\tau}, c,n}, -\lambda_{\tilde{\tau}, c,n-1}, \cdots, -\lambda_{\tilde{\tau}, c,1}, \lambda_{\tilde{\tau}, 1}, \lambda_{\tilde{\tau}, 2}, \cdots, \lambda_{\tilde{\tau}, n})$

Let $\mathcal{V}_\lambda$, resp. $\mathcal{V}_{\tilde{\lambda}}$, be the finite free $\mathcal{O}$-module with algebraic action of $G_p$, resp. $\tilde{G}_p$, of highest weight $\lambda$ resp. $\tilde{\lambda}$. It defines a local system on $X_K$, resp. $\tilde{X}_{\tilde{K}}$, which we still denote $\mathcal{V}_\lambda$, resp. $\mathcal{V}_{\tilde{\lambda}}$. There are a priori several choices for the modules $\mathcal{V}_\lambda$ and $\mathcal{V}_{\tilde{\lambda}}$ (see corollary 1.9 of [PT02]). But since $p$ will be large with respect to $\lambda$, they all coincide.
2 Hecke algebras

In this section we will define Hecke algebras for $H$ and $\tilde{H}$ and important elements in these Hecke algebras which we will use continuously in this note.

Definition 2.1. Let $G$ be a locally profinite group, and let $U \subset G$ be an open compact subgroup. We write $\mathcal{H}(G, U)$ for the set of compactly supported, $U$-biinvariant functions $f : G \to \mathbb{Z}$.

Let $\ell \neq p$ be a rational prime. Let $K$ be a finite extension of $\mathbb{Q}_\ell$ with parameter $\varpi_K$ and residue field $\kappa$ of cardinality $q = \ell^m$. Let $G$ be a quasi-split reductive group scheme over $\mathcal{O}_K$, $S$ is a maximal $\mathcal{O}_K$-split torus of $G$, $T$ is the maximal torus which centralizes $S$, and $B$ is a Borel subgroup containing $T$. Let $N$ be the unipotent radical of $B$. The Weyl group $W(G, T)$ acts on $T$ by conjugation. Let $dn$ be the Haar measure on $\mathcal{N}(K)$ such that $dn(\mathcal{N}(\mathcal{O}_K)) = 1$. It takes values in $\mathbb{Z}[q^{\pm 1}]$. Let $\delta_B : \mathcal{T}(K) \to q^Z$ be the modulus of $B$, given by the formula $t \mapsto |\det_K \text{Ad}_N(t)|_K$ ($|\cdot|_K$ is the absolute value of $K$ normalized by $|\varpi_K|_K = q^{-1}$). The Satake homomorphism

$$N : f \mapsto (t \mapsto \delta_B(t)^{1/2} \int_{n \in \mathcal{N}(K)} f(tn)dn)$$

gives us the isomorphism:

$$N : \mathcal{H}(G(K), \mathcal{G}(\mathcal{O}_K)) \otimes \mathbb{Z}[q^{\pm 1/2}] \to \mathcal{H}(\mathcal{T}(K), \mathcal{T}(\mathcal{O}_K))^W(G, T) \otimes \mathbb{Z}[q^{\pm 1/2}]$$

If there exist a character $\chi_G \in X^*(G)$ such that the character $t \mapsto \delta_B(t)^{1/2} \chi_G(t)^{1/2}$ takes values in $q^Z$, then we get an isomorphism

$$N' : \mathcal{H}(G(K), \mathcal{G}(\mathcal{O}_K)) \otimes \mathbb{Z}[q^{\pm 1}] \to \mathcal{H}(\mathcal{T}(K), \mathcal{T}(\mathcal{O}_K))^W(G, T) \otimes \mathbb{Z}[q^{\pm 1}]$$

given by formula $f \mapsto (t \mapsto |\chi_G(t)|^{1/2} N(f)(t))$.

Let us to consider $G = \tilde{H} \times \mathcal{O}_{F_v^+}$.

If $\bar{v}$ splits in $F$, then there is an canonical isomorphisms $\iota_v : \tilde{H}(F_v^+) \cong \text{GL}_{2n}(F_v)$ and $T(F_v^+) \cong F_v^{2n}$. Therefore, The Weyl group $W(\tilde{H}_v, T_v) = S_{2n}$ and the Hecke algebra $\mathcal{H}(T(F_v^+), T(\mathcal{O}_{F_v^+})) \cong \mathbb{Z}[Y_1^{\pm 1}, Y_2^{\pm 1}, \cdots, Y_{2n}^{\pm 1}]$.

If $\bar{v}$ is inert in $F$, then there is an canonical isomorphisms $\iota_v : \tilde{H}(F_v^+) \cong U(n, n)(F_v)$ and $T(F_v^+) \cong F_v^n$. Therefore, The Weyl group $W(\tilde{H}_v, T_v) = S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ and the Hecke algebra $\mathcal{H}(T(F_v^+), T(\mathcal{O}_{F_v^+})) \cong \mathbb{Z}[X_1^{\pm 1}, X_2^{\pm 1}, \cdots, X_n^{\pm 1}]$. 

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Definition 2.2. Let \( \bar{v} \) be a place of \( F^+ \) which is unramified in \( F \).

1) Suppose that \( \bar{v} \) splits in \( F \). Put \( \chi_G = \det^{2n-1} \), then we have

\[
N' : \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \otimes \mathbb{Z}[q^{\pm 1}_v] \cong \mathbb{Z}[q^{\pm 1}_v][Y_1^{\pm 1}, Y_2^{\pm 1}, ..., Y_{2n}^{\pm 1}]^{S_{2n}}
\]

For each \( i = 1, ..., 2n \), we write \( \bar{T}_{v,i} \) for the element of \( \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \otimes \mathbb{Z}[q^{\pm 1}_v] \) which corresponds under the \( S \)atake transform to the element \( q^{(2n-i)/2}_v e_i(Y_1, ..., Y_{2n}) \) (\( e_i \) is \( i \)-th symmetric polynomial)

2) Suppose that \( \bar{v} \) is inert in \( F \). Since \( q_v = q_v^2 \), the \( S \)atake isomorphism gives a canonical isomorphism

\[
N' : \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \otimes \mathbb{Z}[q^{\pm 1}_v] \cong \mathbb{Z}[q^{\pm 1}_v][X_1^{\pm 1}, X_2^{\pm 1}, ..., X_n^{\pm 1}]^{S_n \times (\mathbb{Z}/2\mathbb{Z})^n}
\]

we write \( \bar{T}_{v,i} \) for the element of \( \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \otimes \mathbb{Z}[q^{\pm 1}_v] \) which corresponds under the \( S \)atake transform to the element \( q^{(2n-i)/2}_v e_i(Y_1, ..., Y_{2n}) \), where \( \{Y_1, Y_2, ..., Y_{2n}\} = \{X_1^{\pm 1}, ..., X_n^{\pm 1}\} \).

For each \( \bar{v} \) of \( F^+ \) we have an algebra homomorphism \( \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \rightarrow \mathcal{H}(H(F^+_\bar{v}), H(\mathcal{O}_{F^+_\bar{v}})) \), we call it \( S_{\bar{v}} = r_G \circ r_P \) unnormalized \( S \)atake transform, where algebra homomorphism \( r_P : \mathcal{H}(\bar{H}(F^+_\bar{v}), \bar{H}(\mathcal{O}_{F^+_\bar{v}})) \rightarrow \mathcal{H}(P(F^+_\bar{v}), P(\mathcal{O}_{F^+_\bar{v}})) \) defines by \( f \mapsto f|_{P(F^+)} \) and algebra homomorphism \( r_G : \mathcal{H}(P(F^+_\bar{v}), P(\mathcal{O}_{F^+_\bar{v}})) \rightarrow \mathcal{H}(H(F^+_\bar{v}), H(\mathcal{O}_{F^+_\bar{v}})) \) defines by \( f \mapsto (x \mapsto \int_{U(F^+)} f(xy)du) \).

Definition 2.3. Let \( \bar{v} \) be a place of \( F^+ \) which is unramified in \( F \).

1) Suppose that \( \bar{v} \) splits in \( F \). The unnormalized \( S \)atake transform corresponds under the \( S \)atake isomorphism to the map

\[
\mathbb{Z}[q^{\pm 1}_v][Y_1^{\pm 1}, Y_2^{\pm 1}, ..., Y_{2n}^{\pm 1}]^{S_{2n}} \rightarrow \mathbb{Z}[q^{\pm 1}_v][W_1^{\pm 1}, W_2^{\pm 1}, ..., W_n^{\pm 1}, Z_1^{\pm 1}, Z_2^{\pm 1}, ..., Z_n^{\pm 1}]^{S_n \times S_n}
\]

where the set \( \{Y_1, ..., Y_{2n}\} \) is in bijection with \( \{q_v^{n/2} Z_1^{-1}, ..., q_v^{n/2} Z_n^{-1}, q_v^{-n/2} W_1, ..., q_v^{-n/2} W_n\} \). For all \( i = 1, ..., n \), let \( T_{v,i} \in \mathcal{H}(H(F_v), H(\mathcal{O}_{F_v})) \) corresponds to \( q_v^{(n-i)/2} e_i(W_1, ..., W_n) \).

2) Suppose instead that \( \bar{v} \) is inert in \( F \). Then the unnormalized \( S \)atake transform corresponds under the \( S \)atake isomorphism to the map

\[
\mathbb{Z}[q^{\pm 1}_v][X_1^{\pm 1}, X_2^{\pm 1}, ..., X_n^{\pm 1}]^{S_n \times (\mathbb{Z}/2\mathbb{Z})^n} \rightarrow \mathbb{Z}[q^{\pm 1}_v][W_1^{\pm 1}, W_2^{\pm 1}, ..., W_n^{\pm 1}]^{S_n}
\]

where the set \( \{X_1, ..., X_n\} \) is in bijection with \( \{q_v^{-n/2} W_1, ..., q_v^{-n/2} W_n\} \). For all \( i = 1, ..., n \), let \( T_{v,i} \in \mathcal{H}(H(F_v), H(\mathcal{O}_{F_v})) \) corresponds to \( q_v^{(n-i)/2} e_i(W_1, ..., W_n) \).
Let $\tilde{T}^S := \otimes_{v \notin S}(\mathcal{H}(\tilde{H}(F_v^+), \tilde{H}(O_{F_v^+}))) \otimes O$, resp. $T^S := \otimes'_{v \notin S}(\mathcal{H}(H(F_v), \tilde{H}(O_{F_v}))) \otimes O$, be base change to $O$ the abstract spherical Hecke algebra outside $S$ for $\tilde{H}$, resp. for $H$. Let $S = \otimes_{v \notin S}S_v$: $\tilde{T}^S \rightarrow T^S$ be the twisted Satake homomorphism.

Let $D(O)$ be the derived category of complexes of $O$-modules. By section 2.1.2 of [ACC+18], the complex $R\Gamma(X_K, \mathcal{V}_\lambda)$ has $\tilde{T}^S$-module structure, resp. $T^S$-module structure. We denote by $\tilde{T}^S(K, \lambda)$, resp. $T^S(K, \lambda)$, the image of $\tilde{T}^S$, resp. $T^S$, in $\text{End}_{D(O)}(R\Gamma(X_K, \mathcal{V}_\lambda))$, resp. in $\text{End}_{D(O)}(R\Gamma(\tilde{X}_K, \mathcal{V}_\lambda))$.

## 3 Shimura varieties

Let $V = \mathcal{O}_E^n$ be equipped with the skew-hermitian form

$$\langle (x_1, \ldots, x_{2n}), (y_1, \ldots, y_{2n}) \rangle = \sum_{i=1}^{n}(x_i y_{2n+1-i} - x_{2n+1-i} y_i)$$

consider the associated alternating form

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{Q} : (x, y) := \text{tr}_{\mathbb{Q}/\mathbb{Z}}\langle x, y \rangle$$

We define $\tilde{G}$, resp. $\tilde{G}$, as the group scheme representing the functor sending any ring $R$ to

$$\tilde{G}(R) = \{g \in \text{GL}_{O_F}(V)(R)| (gv, gw) = (v, w), \forall v, w \in V\}$$

resp.

$$\tilde{G}(R) = \{(g, c) \in \text{GL}_{O_F}(V)(R) \times R^2| (gv, gw) = c(v, w), \forall v, w \in V\}$$

Since all places above $p$ in $F^+$ split in $F$, there is a direct factor submodule $L \subset V \otimes \mathbb{Z}_p$ of rank $n$ such that $V \otimes \mathbb{Z}_p \cong L \oplus L^\vee(1)$ (this isomorphism is compatible with the alternating form).

Let us denote by $(\cdot, \cdot)_\text{can}: (L \oplus L^\vee(1)) \times (L \oplus L^\vee(1)) \rightarrow \mathbb{Z}_p$ the alternating form

$$((x_1, f_1), (x_2, f_2))_\text{can} = f_2(x_1) - f_1(x_2).$$

### 3.1 Integral model of the Shimura variety

**Definition 3.1.** Let $S$ be a scheme over $\mathbb{Z}[1/\Delta_F]$. An abelian variety with $\tilde{G}$-structure over $T$ is a triple $(A, \iota, \lambda)_T$ where $A$ is an abelian scheme of dimension $[F : \mathbb{Q}]n$ over $T$, $\iota: O_F \rightarrow \text{End}(A)$ is an $O_F$-action such that $\text{Lie}A$ is free of rank $n$ over $O_F \otimes \mathbb{Z}O_T$, and $\lambda: A \rightarrow A^\vee$ is a principal polarization on $A$ whose Rosati involution is compatible with complex conjugation on $O_F$ via $\iota$. 
We fix a neat compact open subgroup $\tilde{K}$ of $\tilde{G}_f$. The functor

$$T \mapsto \{(A, t, \lambda)_T\text{ abelian variety with } \tilde{G} - \text{structure and level } \tilde{K}\}$$

is representable by a scheme $\mathfrak{X}_f^{\tilde{K}}$ which is smooth over $\mathbb{Z}[\frac{1}{S}]$ where $S$ is the set of rational primes $\ell \neq p$ such that there is a place in $S$ above of $\ell$.

**Theorem 3.2.** There is a natural isomorphism of manifolds

$$\tilde{X}_f^{\tilde{K}} \cong \mathfrak{X}_K(\mathbb{C})$$

and also the map $\tilde{G} \hookrightarrow \tilde{G}$ induces a natural map $\tilde{X}_f^{\tilde{K}} \hookrightarrow \tilde{X}_f^{\tilde{K}}$, which is an open and closed immersion.

*Proof.* See [CS19, Lemma 2.1.1 and Proposition 2.1.6].

### 3.2 Automorphic vector bundle

In the sequel, we write $\mathfrak{X}_K$ for the base change of $\mathfrak{X}_f^{\tilde{K}}$ to $\mathbb{Z}_p$.

Let $A \to \mathfrak{X}_f^{\tilde{K}}$ be the universal abelian variety for $\mathfrak{X}_f^{\tilde{K}}$ and let $H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}})$ be the relative de Rham cohomology and $H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) := \text{Hom}_{\mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}}}(H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}), \mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}})$.

We have the canonical pairing $\langle \cdot, \cdot \rangle_\lambda : H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \times H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \to \mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}}(1)$ defined as the composite of $(\text{Id} \times \lambda)_* \text{ followed by the perfect pairing } H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \times H^1_{dR}(A^\vee/\mathfrak{X}_f^{\tilde{K}}) \to \mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}}(1)$ defined by $c_1(\mathcal{L}) \in H^2_{dR}(A \times_{\mathfrak{X}_f^{\tilde{K}}} A^\vee/\mathfrak{X}_f^{\tilde{K}})$ the first Chern class of the Poincaré line bundle $\mathcal{L}$ over $A \times_{\mathfrak{X}_f^{\tilde{K}}} A^\vee$.

$$H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \times H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \xrightarrow{(\text{Id} \times \lambda)_*} H^1_{dR}(A/\mathfrak{X}_f^{\tilde{K}}) \times H^1_{dR}(A^\vee/\mathfrak{X}_f^{\tilde{K}}) \xrightarrow{c_1(\mathcal{L})} \mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}}(1)$$

Let $\mathfrak{X}_f^{(1)}$ be the first infinitesimal neighborhood of the diagonal image of $\mathfrak{X}_f^{\tilde{K}}$ in $\mathfrak{X}_f^{\tilde{K}} \times_{\mathbb{Z}_p} \mathfrak{X}_f^{\tilde{K}}$, and let $pr_1; pr_2 : \mathfrak{X}_f^{(1)} \to \mathfrak{X}_f^{\tilde{K}}$ be the two projections. Then we have by definition the canonical morphism $\mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}} \to \mathcal{P}^{1}_{\mathfrak{X}_f^{\tilde{K}}/\mathbb{Z}_p} := pr_1_*\text{pr}_2^*(\mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}})$, where $\mathcal{P}^{1}_{\mathfrak{X}_f^{\tilde{K}}/\mathbb{Z}_p}$ is the sheaf of principal parts of order 1. The isomorphism $s : \mathfrak{X}_f^{(1)} \to \mathfrak{X}_f^{(1)}$ over $\mathfrak{X}_f^{\tilde{K}}$ swapping the two components of the fiber product then defines an automorphism $s^*$ of $\mathcal{P}^{1}_{\mathfrak{X}_f^{\tilde{K}}/\mathbb{Z}_p}$. The kernel of the structural morphism $\text{str}^* : \mathcal{P}^{1}_{\mathfrak{X}_f^{\tilde{K}}/\mathbb{Z}_p} \to \mathcal{O}_{\mathfrak{X}_f^{\tilde{K}}}$, canonically isomorphic to $\Omega^1_{\mathfrak{X}_f^{\tilde{K}}/\mathbb{Z}_p}$ by definition, is spanned by the image of $s^* - \text{Id}^*$. An important property of the relative de Rham cohomology of any smooth morphism like $A \to \mathfrak{X}_f^{\tilde{K}}$ is that, for any two smooth lifts $\tilde{A}_1 \to \mathfrak{X}_f^{(1)}$ and $\tilde{A}_2 \to \mathfrak{X}_f^{(1)}$ of
Remark 3.3. This Gauss-Manin connection is same with the usual Gauss-Manin connection on the relative de Rham cohomology $H^1_{dR}(A/\mathfrak{x}_K)$. If we consider $A_1 = pr_1^* A$ and $A_2 = pr_2^* A$, then we obtain a canonical isomorphism

$$pr_1^* H^1_{dR}(A/\mathfrak{x}_K) \cong H^1_{dR}(pr_1^* A/\mathfrak{x}_K^{(1)}) \cong H^1_{dR}(pr_2^* A/\mathfrak{x}_K^{(1)}) \cong pr_2^* H^1_{dR}(A/\mathfrak{x}_K)$$

which we denote by $Id^*$ by abuse of notation. On the other hand, the pullback by the swapping automorphism $s : \mathfrak{x}_K^{(1)} \to \mathfrak{x}_K^{(1)}$ defines another canonical isomorphism

$$s^* : pr_1^* H^1_{dR}(A/\mathfrak{x}_K) \cong H^1_{dR}(pr_1^* A/\mathfrak{x}_K^{(1)}) \cong H^1_{dR}(pr_2^* A/\mathfrak{x}_K^{(1)}) \cong pr_2^* H^1_{dR}(A/\mathfrak{x}_K)$$

The Gauss-Manin connection $\nabla : H^1_{dR}(A/\mathfrak{x}_K) \to H^1_{dR}(A/\mathfrak{x}_K) \otimes \mathcal{O}_{\mathfrak{x}_K/\mathbb{Z}_p}$ on $H^1_{dR}(A/\mathfrak{x}_K)$ is the composition

$$H^1_{dR}(A/\mathfrak{x}_K) \xrightarrow{pr_1} H^1_{dR}(pr_1^* A/\mathfrak{x}_K^{(1)}) \xrightarrow{s^* - Id^*} H^1_{dR}(A/\mathfrak{x}_K) \otimes \mathcal{O}_{\mathfrak{x}_K/\mathbb{Z}_p}$$

**Remark 3.3.** This Gauss-Manin connection is same with the usual Gauss-Manin connection on the relative de Rham cohomology $H^1_{dR}(A/\mathfrak{x}_K)$ (cf. [KO68]).

**Definition 3.4.** The principal $\tilde{G}$-bundle over $\mathfrak{x}_K$ is the $\tilde{G}$-torsor

$$\mathcal{E}_G := \text{Isom}_{\mathcal{O}_F \otimes \mathcal{O}_{\mathfrak{x}_K}} [(H^1_{dR}(A/\mathfrak{x}_K(1)), \langle \cdot, \cdot \rangle_\lambda, \mathcal{O}_{\mathfrak{x}_K}), \langle (L \oplus L^\vee(1)) \otimes \mathbb{Z}_p \mathcal{O}_{\mathfrak{x}_K}, \langle \cdot, \cdot \rangle_{\text{can}}, \mathcal{O}_{\mathfrak{x}_K}(1) \rangle]$$

the sheaf of isomorphisms of $\mathcal{O}_{\mathfrak{x}_K}$-sheaves of symplectic $\mathcal{O}_F$-modules.

We can define $\mathcal{E}_P$ (a $P$-torsor) which is all isomorphism such above by adding condition $\text{Lie}_{A^\vee/\mathfrak{x}_K}$ maps to $L^\vee(1) \otimes \mathbb{Z}_p \mathcal{O}_{\mathfrak{x}_K}$ and $\mathcal{E}_G$ (is a $G$-torsor) is all $\mathcal{O}_F \otimes \mathbb{Z} \mathcal{O}_{\mathfrak{x}_K}$-isomorphism between $\text{Lie}_{A^\vee/\mathfrak{x}_K}$ and $L^\vee(1) \otimes \mathbb{Z}_p \mathcal{O}_{\mathfrak{x}_K}$.

**Definition 3.5.** Let $R$ be any $\mathbb{Z}_p$-algebra and $W \in \text{Rep}_R(\tilde{G}_p)$, we define

$$\mathcal{E}_G(W) := (\mathcal{E}_G \otimes_{\mathbb{Z}_p} R) \times_{\tilde{G}_p \otimes \mathbb{Z}_p} W$$

Moreover, for any $W \in \text{Rep}_R(G_p)$, we define

$$\mathcal{E}_G(W) := (\mathcal{E}_G \otimes_{\mathbb{Z}_p} R) \times_{G_p \otimes \mathbb{Z}_p} W$$

which are coherent sheaf on $\mathfrak{x}_K \otimes_{\mathbb{Z}_p} R$. 

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If \( W \in \text{Rep}_R(G_p) \) be free \( R \)-module, we say \( \delta_G(W) \) is an automorphic vector bundle.

**Remark 3.6.** By [LS12, Proposition 3.7], for any \( \tilde{\lambda} \in (\mathbb{Z}^2_+)^{\text{Hom}}(F^+, E) \), there is an idempotent element \( \varepsilon_{\tilde{\lambda}} \) in \( \mathbb{Z}_p[(V \times \mathbb{Z}_p)^m \times S_m] \) and an integer \( t_{\tilde{\lambda}} \) such that

\[
\delta_G(V_{\tilde{\lambda}}) \cong (\varepsilon_{\tilde{\lambda}}) \ast H^i_{dR}(\mathcal{A}^m/\mathcal{X}_{\tilde{\lambda}})(t_{\tilde{\lambda}})
\]

where \( m = |\tilde{\lambda}| \).

### 3.3 Toroidal compactifications of the Shimura variety

We follow the definitions and notations of K.W. Lan’s thesis [LAN13].

To the Shimura variety \( \mathcal{X}_{\tilde{K}} \), one can attach a family of toroidal compactifications. Each is attached to a collection \( \Sigma \) of combinatorial data adapted to \( \tilde{K} \). The corresponding toroidal compactification is denoted \( \mathcal{X}_{\tilde{K}, \Sigma} \).

Let \( \mathcal{P} \) be the set of maximal \( \mathbb{Q} \)-rational parabolic subgroups of \( \tilde{G} \). Let \( P = MU \in \mathcal{P} \).

The combinatorial data \( \Sigma = \cup \Sigma_R \) adapted to the neat level \( \tilde{K} \) is a collection of fans, \( \Sigma_R \), one for each rational boundary component \( R \). Each \( R \) corresponds to its stabilizer \( P_R \), which is the center of the unipotent radical of \( P_R \).

By choosing suitable \( \Sigma \), these compactifications are smooth. Let \( D \) be the boundary of \( \mathcal{X}_{\tilde{K}} \) in \( \mathcal{X}_{\tilde{K}, \Sigma} \). It is a relative Cartier divisor with normal crossings over \( \mathbb{Z}[S^{-1}] \).

The universal abelian scheme \( A \to \mathcal{X}_{\tilde{K}} \) extends to a semi-abelian scheme \( A^{ext} \to \mathcal{X}_{\tilde{K}, \Sigma}^{tor} \), the polarization \( \lambda : A \to A^\vee \) extends to a prime-to-\( p \) isogeny \( \lambda^{ext} : A^{ext} \to (A^{ext})^\vee \) between semi-abelian schemes where \( (A^{ext})^\vee \) is the dual of the semi-abelian scheme \( A^{ext} \).

By proposition of 1.4.3 of [LAN18], we can define a locally free \( \mathcal{O}_{\mathcal{X}^{tor}_{\tilde{K}}} \)-sheaf \( H^1_{dR}(A/\mathcal{X}_{\tilde{K}})^\text{can} \) extending the \( \mathcal{O}_{\mathcal{X}_{\tilde{K}}} \)-sheaf \( H^1_{dR}(A/\mathcal{X}_{\tilde{K}}) \) and such that it satisfies conditions including the existence of an alternating form \( \langle \cdot, \cdot \rangle_\lambda \) extending the Poincaré pairing, and endowed with a connection with logarithmic singularities extending the Gauss-Manin connection \( \nabla \) on \( H^1_{dR}(A/\mathcal{X}_{\tilde{K}}) \). Therefore we can define \( \delta_G^{can} \) and \( \delta_G^{can} \) as last subsection.

For any \( \mathbb{Z}_p \)-algebra \( R \) and \( W \in \text{Rep}_R(G_p) \), there are two extension for \( \delta_G(W) \) to \( \mathcal{X}_{\tilde{K}} \) which is compatible with changing level and refining of by cone decomposition.
The first one is the canonical extension of $\mathcal{E}_G(W)$:

$$
\mathcal{E}_G(W)^{\text{can}} = (\mathcal{E}_G^{\text{can}} \otimes_{\mathbb{Z}_p} R) \times_{G_0 \otimes_{\mathbb{Z}_p} R} W
$$

and the second is the sub-canonical extension of $\mathcal{E}_G$:

$$
\mathcal{E}_G(W)^{\text{sub}} = \mathcal{E}_G(W)^{\text{can}} \otimes_{\mathcal{O}_{X_K}} I_D
$$

where $I_D$ is the $\mathcal{O}_{X_K}$-ideal defining the relative Cartier divisor $D$.

**Remark 3.7.** We have an isomorphism

$$
\mathcal{E}_G(V_{\lambda})^{\text{can(sub)}} \cong (\varepsilon_{\lambda})_* H^m_{dR}(A^m/\mathfrak{X}_{\tilde{K}})^{\text{can(sub)}}(t_{\lambda})
$$

where $m = |\tilde{\lambda}|$.

### 3.4 Local system on the integral model

In this subsection we will define étale local system on $\mathfrak{X}_{\tilde{K}} \times \mathbb{Z}[\frac{1}{\mathfrak{p}}] \bar{\mathbb{Q}}$ and we will compute Fontaine-Laffaille weight of $H^i_{(c)}(\mathfrak{X}_{\tilde{K}}, V_{\lambda}/\varpi)$.

For any positive integer $m$, let $f_m : A^m_{\bar{\mathbb{Q}}} \to \mathfrak{X}_{\tilde{K}} \times \mathbb{Z}[\frac{1}{\mathfrak{p}}] \bar{\mathbb{Q}}$ be base change to $\bar{\mathbb{Q}}$ of the $m$-th fiber product $A^m \to \mathfrak{X}_{\tilde{K}}$ of the universal abelian scheme.

If we fix $m = |\tilde{\lambda}|$,

$$
V_{\lambda, \text{ét}} := (\varepsilon_{\lambda})_* R^m (f_m)_*(\mathcal{O})(-t_{\lambda})
$$

**Proposition 3.8.** For any $\tilde{\lambda} \in (\mathbb{Z}^n_{2m})^{\text{Hom}(F^+, E)}$ there exist an étale local system in $\mathcal{O}$-modules $V_{\lambda, \text{ét}}$ on $\mathfrak{X}_{\tilde{K}} \times \mathbb{Z}[\frac{1}{\mathfrak{p}}] \bar{\mathbb{Q}}$ such that it extend to $\mathfrak{X}_{\tilde{K}}/\mathbb{Z}[\frac{1}{\mathfrak{p}}]$ and:

$$
H^i_{\text{ét},(c)}(\mathfrak{X}_{\tilde{K}} \times \mathbb{Z}[\frac{1}{\mathfrak{p}}] \bar{\mathbb{Q}}, V_{\lambda, \text{ét}}/\varpi^m) \cong H^i_{B,(c)}(\mathfrak{X}_{\tilde{K}}, V_{\lambda}/\varpi^m)
$$

for all positive integer $m$.

**Proof.** See [LS12, Subsection 4.3].

Since $V_{\lambda}$ can be defined over $F_0$, the $\mathcal{O}/\varpi^m$-module $H^i_{B,(c)}(\mathfrak{X}_{\tilde{K}}, V_{\lambda}/\varpi^m)$ carries a $G_{F_0}$-action.

We write $X^*(T)^{+,P} \subset X^*(T)^+$ for the subset of $(B \cap G)$-dominant characters. The set

$$
W^P := \{ w \in W_G | w(X^*(T)^+) \subset X^*(T)^{+,P} \}$$
is the set of representatives with minimal length of the quotient \( W_/G/W_G \). It is called the set of Kostant representatives.

We write \( \ell(w) \) for length of \( w \in W^P \) and \( p_{\check{\lambda}}(w) = [w(\check{\lambda} + \rho) - \rho](H) \) where \( H = (0, 0, \ldots, 0, 1, 1, \ldots, 1) \) (first \( n \) components are 0 and last \( n \) components are 1) and \( \rho \) is half sum of positive root of \( \tilde{G}_p \).

**Definition 3.9.** Let \( \tilde{\lambda} \in (\mathbb{Z}^2)_{\text{Hom}}(F^+, E) \)

1) We say that \( \tilde{\lambda} \) is regular if \( \tilde{\lambda}_{\check{\tau},i} > \tilde{\lambda}_{\check{\tau},i+1} \) for any \( \check{\tau} \in \text{Hom}(F^+, E) \) and \( i \in [1, 2n - 1] \).

2) We set \( |\lambda|_{\text{comp}} := \dim \check{X}_K + 1 + \sum_{\check{\tau},i} |\tilde{\lambda}_{\check{\tau},i}| \). The integer \( |\lambda|_{\text{comp}} \) is called the comparison size of \( \check{\lambda} \).

Let \( \text{Rep}_{\mathbb{Z}_p}(G_{Q_p}) \) be the category of \( G_{Q_p} \)-modules of finite type over \( \mathbb{Z}_p \) and \( MF_{\mathbb{Z}_p}^{[0,p-2]} \) that of finitely generated \( \mathbb{Z}_p \)-modules \( M \) endowed with a filtration \( (\text{Fil}^r M)_r \) such that \( \text{Fil}^r M \) is a direct factor, \( \text{Fil}^0 M = M \) and \( \text{Fil}^{p-1} M = 0 \) together with semi-linear maps \( \phi_r : \text{Fil}^r M \to M \) such that the restriction of \( \phi_r \) to \( \text{Fil}^{r+1} M \) is equal to \( p\phi_{r+1} \) and satisfying the strong divisibility condition: \( M = \sum_{i \in \mathbb{Z}} \phi_i(\text{Fil}^r M) \). Recall that by the theory of Fontaine-Laffaille [FL82], we have a fully faithful functor

\[
T_{cr} : MF_{\mathbb{Z}_p}^{[0,p-1]} \to \text{Rep}_{\mathbb{Z}_p}(G_{Q_p})
\]

A \( p \)-adic representation is called Fontaine-Laffaille if it is in the essential image of \( T_{cr} \).

Let \( \check{X} \) be a smooth and proper scheme over \( \mathbb{Z}_p \) of relative dimension \( g \) and \( D \) a relative divisor with normal crossings of \( X \), we put \( X = \check{X} - D \).

**Theorem 3.10** (Theorem 5.3 of Faltings [F90]). For any \( j \in [0, p - 2] \), we have following isomorphism:

\[
T_{cr}(H^j_{\text{log-cris},(c)}(\check{X}; \mathbb{F}_p)) \cong H^j_{\text{ét},(c)}(X_{\overline{\mathbb{Q}_p}}, \mathbb{F}_p)
\]

This isomorphism is compatible with the action of \( G_{Q_p} \). The isomorphism is functorial in the proper smooth \( \mathbb{Z}_p \)-log scheme \( \check{X} \) and is compatible with the cup product structures and with the formation of the Chern classes of line bundles over \( \check{X} \).

This theorem implies that for any \( \check{\lambda} \in (\mathbb{Z}^2)_{\text{Hom}}(F^+, E) \) such that \( |\check{\lambda}|_{\text{comp}} < p \) we have following isomorphism

\[
T_{cr}(H^j_{\text{log-cris},(c)}(\check{X}_{\text{tor}, K}; \mathcal{V}_{\check{\lambda}/\varpi})) \cong H^j_{\text{ét},(c)}(X_{\overline{\mathbb{Q}_p}}, \mathcal{V}_{\check{\lambda}/\varpi})
\]

(3.2)
Remark 3.11. Since in Faltings theorem the isomorphism is compatible with the formation of chern classes, then the isomorphism 3.2 is compatible with the formation of chern classes. Therefore this isomorphism is compatible with the Hecke action on the both side.

Theorem 3.12 (Lan-Polo [LP18]). Assume that $|\tilde{\lambda}|_{\text{comp}} < p$. Then the set of Fontaine-Laffaille weights of $H^i_{B,(c)}(\tilde{X}_K, V_{\tilde{\lambda}}/\varpi)$ is subset of the set $\{p_\lambda(w) | w \in W^P, \ell(w) \leq i\}$. Moreover, multiplicity of $p_\lambda(w)$ is equal to $\mathbb{F}_p$ dimensional of $H^{i-\ell(w)}(\mathcal{X}_{K,ns}^\text{tor}, \mathcal{E}_G(V_{\lambda_w})^\text{can(sub)})$.

Proof. By theorem 3.10 we have to consider the Hodge filteration of $H^{j}_{\text{log-cris, (c)}}(\mathcal{X}_{K,tor}^\text{K}, V_{\lambda}/\varpi))$. Hence Theorem 5.9 of [LP18] implies this theorem.

Corollary 3.13. Let $\tilde{m} \subset \tilde{T}^S$ be a maximal ideal. Assume that $|\tilde{\lambda}|_{\text{comp}} < p$. Then the set of Fontaine-Laffaille weights of $H^i_{B,(c)}(\tilde{X}_K, V_{\tilde{\lambda}}/\varpi)_{\tilde{m}}$ is subset of the set $\{p_\lambda(w) | w \in W^P, \ell(w) \leq i\}$. Moreover, multiplicity of $p_\lambda(w)$ is equal to $\mathbb{F}_p$ dimensional of $H^{i-\ell(w)}(\mathcal{X}_{K,ns}^\text{tor}, \mathcal{E}_G(V_{\lambda_w})^\text{can(sub)})_{\tilde{m}}$.

Proof. Since the isomorphism 3.2 and theorem 5.9 of [LP18] are compatible with the Hecke action $\tilde{T}^S$, theorem 3.12 implies this corollary.

3.5 Dualizing sheaf of toroidal compactification

Generalizing previous results by Mumford, Harris and Faltings-Chai, K.-W. Lan determined the dualizing sheaf for the arithmetic toroidal compactification of PEL Shimura varieties. The result for $\tilde{G} = \tilde{G}_n$ is as follows. We fix a sufficiently fine fan so that $\mathcal{X}_{K,tor}^\text{K}$ is smooth.

Theorem 3.14. The dualizing sheaf of $\mathcal{X}_{K,tor}^\text{K}$ is isomorphic to $\mathcal{E}_G(V_{-2\rho_{nc}})^{\text{sub}}$ where $\rho_{nc}$ is the half of the sum of all positive roots of $\tilde{G}$ which are not positive roots for $G$.

Proof. See Proposition 2.2.6 of [Har90] for the rational case $\mathcal{X}_{K,tor}^\text{K} \times \mathbb{Q}$ and Theorem 6.4.1.1 (4) of [LAN13] for the integral case.

Corollary 3.15. For any field $\kappa$ such that its characteristic does not belong to $S$ and for any representation $W \in \text{Rep}_\kappa(G_p)$, there is a Serre duality isomorphism

$$R\text{Hom}_\kappa(R\Gamma(\mathcal{X}_{K,ns}^\text{tor}, \mathcal{E}_G(W)^{\text{sub}}), \kappa) \cong R\Gamma(\mathcal{X}_{K,ns}^\text{tor}, \mathcal{E}_G(W^\vee \otimes V_{-2\rho_{nc}})^{\text{can}})).$$

Moreover this isomorphism, is compatible with the Hecke action in the sense the action of $[\tilde{K}g\tilde{K}]$ on the left matches the action of $[\tilde{K}g\tilde{K}]^t = [\tilde{K}g^{-1}\tilde{K}]$ on the right.
Remark 3.16. (1) Note that for using Serre duality we need to have a Cohen-Macaulay scheme. In our case, the compactified Shimura variety $X_{K,\tilde{\kappa}}$ is smooth and therefore is Cohen-Macaulay.

(2) In general case when $K_p$ is not hyperspecial, the compactified Shimura variety $\tilde{X}_K$ is Cohen-Macaulay.

4 Galois representation

Let $\pi$ be a cohomological regular cuspidal automorphic representation of $\text{GL}_n(A_F)$ with level $K$ and cohomological weight $\lambda$ (with pure weight $w \in \mathbb{Z}$). With the notations of the previous section, $\pi$ is unramified outside of $S'$. Assume that $K_p$ is hyperspecial so that $S' \cap S_p = \emptyset$. Let $\mathfrak{m} \subset T^S$ be associated maximal ideal of Hecke algebra. By [HLTT16] we can construct a Galois representation associated to $\mathfrak{m}$, which we denote by $\bar{\rho}_m : G_F \rightarrow \text{GL}_n(\kappa)$, such that for any $v \notin S$, the characteristic polynomial of $\bar{\rho}_m(\text{Frob}_v)$ is equal to:

$$P_v(X) := X^n - T_{v,1}X^{n-1} + \ldots + (-1)^j q_v^{j(j-1)/2} T_{v,j}X^{n-j} + \ldots + (-1)^n q_v^{n(n-1)/2} T_{v,n}, \mod m.$$

We say that $\mathfrak{m}$ is non-Eisenstein if $\bar{\rho}_m$ is absolutely irreducible.

We introduce the Fontaine-Laffaille condition

$$(FL) \quad \text{for any } \tau \in \text{Hom}(F, E) \text{ we have } p - 2n - 1 \geq \lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n}.$$

Let $\varepsilon : G_F \rightarrow \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic charac and $\bar{\varepsilon} : G_F \rightarrow \mathbb{F}_p^\times$ its reduction modulo $p$. From now on we assume $\mathfrak{m}$ non-Eisenstein and $(FL)$.

Theorem 4.1. Assuming the above assumptions and $K_p$ is hyperspecial, then for any $v \in S_p$, the restriction of $\bar{\rho}_m$ to $G_{F_v}$ is Fontaine-Laffaille with $\tau$-Fontaine-Laffaille weight $(\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n})$. Also we have:

$$\bar{\rho}_m|_{I_v} \cong \begin{pmatrix} \delta_1 & * & \cdots & * \\ 0 & \delta_2 & & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_n \end{pmatrix}$$

where $\delta_1, \delta_2, \ldots, \delta_n : I_v \rightarrow \mathbb{F}_p^\times$ are tame characters, whose product equals $\varepsilon^{(w + n(n-1)/2)}$ and whose sum has Fontaine-Laffaille weights $(\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n})_{\tau \in \text{Hom}(F_v, E)}$. 

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See [FL82]. Note that this theorem requires the fact that for every $\tau \in \text{Hom}(F, E)$, the Fontaine-Laffaille weights $\lambda_{\tau, 1} + n - 1, \lambda_{\tau, 2} + n - 2, \ldots, \lambda_{\tau, n}$ are mutually distinct.

**Remark 4.2.** If $\overline{\delta}_i : \mathcal{O}_{F_v} \to \widehat{\mathbb{F}}_p^\times$ is correspondence to $\delta_i$ by local class field theory, then it defines by $x \mapsto \prod_{\tau \in \text{Hom}(F_v, E)} \tau(x)^{\lambda_{\tau, n - i}}$.

## 5 A direct summand of the boundary cohomology

In this section, we mention three important theorems, the first is about the occurrence of the cohomology of $X_K$ in the boundary cohomology of $\tilde{X}_{K, \lambda}$, The last two are cohomology vanishing theorems for $\tilde{X}_{K, \lambda}$.

From this section we denote $\tilde{m} := S^{-1}(m)$

### 5.1 Embedding theorems

In this subsection, we recall a theorem about embedding of cohomology of $X_K$ in the boundary cohomology of $\tilde{X}_{K, \lambda}$. The theorem is theorem 4.2.1 of [ACC+18]. Let $U$ be the unipotent radical of the Siegel parabolic $P$ and $P = H \cdot U$ be a Levi decomposition of the Siegel parabolic $P$ in $H$. Let us say that a compact open subgroup $\tilde{K} \subset \tilde{G}$ is decomposed with respect to the Siegel parabolic $P$ if $P(F_f^+) \cap \tilde{K} = (H(F_f^+) \cap \tilde{K}) \cdot (U(F_f^+) \cap \tilde{K})$. Let $\tilde{K}_P = P(F_f^+) \cap \tilde{K}$ and $\tilde{K}_U = U(F_f^+) \cap \tilde{K}$.

Let $\tilde{X}_{K, \lambda}^{BS}$ be the Borel-Serre compactification of $\tilde{X}_{K, \lambda}$ and $\tilde{X}_{K, \lambda}^P$ be $P$-strata of the Borel-Serre compactification as subsection 2.1 [ACC+18]. Let $\partial \tilde{X}_{K, \lambda} = \tilde{X}_{K, \lambda}^{BS} - \tilde{X}_{K, \lambda}$ be the boundary of $\tilde{X}_{K, \lambda}^{BS}$. The action of the Hecke algebra $\tilde{T}^S$ on $R \Gamma(\tilde{X}_{K, \lambda}, V_{\lambda})$ in the derived category of $\mathcal{O}$-modules $D(\mathcal{O})$ extends canonically to $R \Gamma(\tilde{X}_{K, \lambda}^{BS}, V_{\lambda})$; it preserves $R \Gamma(\partial \tilde{X}_{K, \lambda}, V_{\lambda})$ and $R \Gamma(\tilde{X}_{K, \lambda}^P, V_{\lambda})$ (see [ACC+18] Subsection 2.1).

By [ACC+18] Theorem 2.4.2, the natural embedding $\tilde{X}_{K, \lambda}^P \hookrightarrow \partial \tilde{X}_{K, \lambda}$ induces an isomorphism:

$$R \Gamma(\tilde{X}_{K, \lambda}^P, V_{\lambda})_{\tilde{m}} \cong R \Gamma(\partial \tilde{X}_{K, \lambda}, V_{\lambda})_{\tilde{m}}$$

**Remark 5.1.** The key ingredients for proving the above isomorphism are

1) $\tilde{\rho}_{\tilde{m}} \cong \tilde{\rho}_m \oplus \tilde{\rho}_m^{c, V} \otimes \varepsilon^{1 - 2n}$ is the direct sum of two $n$-dimensional absolutely irreducible Galois representations,

2) if $R \Gamma(\tilde{X}_K^Q, V_{\lambda})_{\tilde{m}} \neq 0$ for another $\mathbb{Q}$-rational standard parabolic subgroup $Q$ of $\tilde{G}$, then $\tilde{\rho}_{\tilde{m}}$ admits another decomposition as a sum of irreducible Galois representations with respect to $Q$, which is a contradiction.
Denote $\tilde{K}_P = \tilde{K} \cap p(\mathbb{Q}_f)$ and $\tilde{K}_U = \tilde{K} \cap U(\mathbb{Q}_f)$. Arguing in the same way as on [NT16, p. 58], we see that there is an isomorphism

$$R\Gamma(\tilde{X}_K, \mathcal{V}_\lambda) \cong R\Gamma(\tilde{K}_P \times K_S, R\Gamma(\text{Int}^{P_S \times K_S} \chi_G, R\Gamma_{\text{Inf}^{U \times K_S} \chi_G, \mathcal{V}_\lambda}))$$ (5.1)

where $X_G := G(\mathbb{Q}) \setminus X_G \times G(\mathbb{Q}_f)$ and $R\Gamma_{\text{Inf}^{U \times K_S} \chi_G, \mathcal{V}_\lambda}$ is the derived functor of the functor of $\tilde{K}_U,S$-fixed points.

The following theorem is [ACC+18, Theorem 4.2.1] for a special partition $(S_1, S_2)$ of the set $S_p$ of places above $p$; namely, $S_1 = S_p$ and $S_2 = \emptyset$.

**Theorem 5.2.** Let $\tilde{K} \subset \tilde{G}_f$ be a neat compact open subgroup which is decomposed with respect to the parabolic subgroup $P = G \cdot U$, and with the property that for each $\bar{v} \in S_p, \tilde{K}_{U,\bar{v}} = U(\mathcal{O}_{F_v})$. Let $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\text{Hom}(F^+, E)}$ and $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, E)}$ be corresponding dominant weights for $\tilde{G}$ and $G$, respectively. We also assume that $p > n^2$.

Then for any $m \geq 1$, $R\Gamma(X_K, \mathcal{V}_\lambda / \varpi^m)_m$ is a $\tilde{T}^S$-equivariant direct summand of $R\Gamma(\partial \tilde{X}_K, \mathcal{V}_\lambda / \varpi^m)_m$.

**Proof.** By using isomorphism [5.1] We should show that $\mathcal{V}_\lambda$ is direct summand of $R\Gamma_{\text{Inf}^{U_S} \chi_G, \mathcal{V}_\lambda}$ as $K_S$-representation. Note that there is a $K_P$-equivariant embedding $\mathcal{V}_\lambda \to \mathcal{V}_\bar{\lambda}$, which splits after restriction to $K$.

The morphism $\mathcal{V}_\lambda \to R\Gamma_{\text{Inf}^{U_S} \chi_G, \mathcal{V}_\lambda}$ is the composition of the given map $\mathcal{V}_\lambda \to \mathcal{V}_\bar{\lambda}$ with the morphism $\mathcal{V}_\bar{\lambda} \to R\Gamma_{\text{Inf}^{U_S} \chi_G, \mathcal{V}_\lambda}$ whose existence is assured by the universal property of the derived functor.

The $R\Gamma_{\text{Inf}^{U_S} \chi_G, \mathcal{V}_\lambda} \to \mathcal{V}_\lambda$ is the composition of the morphism $R\Gamma_{\text{Inf}^{U_S} \chi_G, \mathcal{V}_\lambda} \to \mathcal{V}_\lambda$ (given by restriction to the trivial subgroup) and the $K$-equivariant splitting $\mathcal{V}_\bar{\lambda} \to \mathcal{V}_\lambda$. This completes the proof.

By the above theorem we can conclude that

$$H^i(X_K, \mathcal{V}_\lambda / \varpi)_m \hookrightarrow H^i(\tilde{X}_K, \mathcal{V}_\bar{\lambda} / \varpi)_m$$ (5.2)

as $\tilde{T}^S$-equivariant direct summand.

### 5.2 Vanishing Theorems for the unitary Shimura variety

The first theorem is the main theorem of [CS19] which states the vanishing of the cohomology of $\tilde{X}_K$ below the middle degree, after localization at certain maximal
ideals \( \mathfrak{m} \) of the Hecke algebra. Let \( L \) be a number field with absolute Galois group \( G_L \) and \( \kappa \) a finite field of characteristic \( p \). Let \( \bar{r} : G_L \to \text{GL}_n(\kappa) \) be a continuous representation.

**Definition 5.3.** 1) We say that a prime \( \ell \neq p \) is decomposed generic for \( \bar{r} \) if \( \ell \) splits completely in \( L \) and for all places \( v|\ell \) of \( L \), \( \bar{r}|_{G_L} \) is unramified and the eigenvalues (with multiplicity) \( \alpha_1, \ldots, \alpha_n \in \bar{\kappa} \) of \( \bar{r}(\text{Frob}_v) \) satisfy \( \alpha_i/\alpha_j \notin \{1, q_v\} \) for all \( i \neq j \).

2) We say that \( \bar{r} \) is decomposed generic if there exists a prime \( \ell \neq p \) which is decomposed generic for \( \bar{r} \).

**Theorem 5.4.** Assume the following conditions

1. \( F^+ \neq \mathbb{Q} \);
2. \( \bar{\rho}_m \) is of length at most 2;
3. \( \bar{\rho}_m \) is decomposed generic,

Then \( H^i(\tilde{X}_K, \mathcal{V}_\lambda/\varpi)\mathfrak{m} = 0 \) for all \( i < d \) and \( H^i_c(\tilde{X}_K, \mathcal{V}_\lambda/\varpi)\mathfrak{m} = 0 \) for all \( i > d \).

The second theorem is the main theorem of [LS13].

**Theorem 5.5.** Assume that \( p > |\lambda|_{\text{comp}} \), that the weight \( \tilde{\lambda} \) is regular and \( \tilde{K}_p \) is hyperspecial. then \( H^i(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) = 0 \) for all \( i < n^2d = \dim \tilde{X}_K \).

Let us consider the long exact sequence of the boundary:

\[ \ldots \to H^i_c(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \to H^i(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \to H^i(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \to \ldots \]

By Theorem 5.4, we can conclude that

\[ H^i_c(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \cong H^i_c(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \mathfrak{m} \]  \hspace{1cm} (5.3)

as \( \tilde{T}^S \)-module for all \( i < n^2d - 1 \).

Also under the assumptions of Theorem 5.5, we can also conclude that

\[ H^i_c(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \cong H^i_c(\tilde{X}_K, \mathcal{V}_{\lambda}/\varpi) \]  \hspace{1cm} (5.4)

as \( \tilde{T}^S \)-module for all \( i < n^2d - 1 \).
6 Boundary cohomology of $\tilde{X}_K$

Our aim in this section is to prove that for any $i < d$, the semi-simplification of $H^i_\partial(\tilde{X}_K, \mathcal{V}_\lambda/\varpi)[m]$ as $G_{F_0}$-module is a direct sum of characters.

Let $j : \tilde{X}_K \hookrightarrow \tilde{X}_K^* \hookrightarrow \partial\tilde{X}_K^*$ where $j$, rep. $i$, is the open immersion, resp. closed immersion of the boundary, into the minimal compactification of $\tilde{X}_K$ over $\mathbb{Z}_{\geq 1}$. It yields an isomorphism between $H^i_\partial(\tilde{X}_K, \mathcal{V}_\lambda/\varpi)$ and $H^i(\partial\tilde{X}_K^*, j^*R_i\mathcal{V}_\lambda/\varpi)$.

We have a spectral sequence:

$$H^i(\partial\tilde{X}_K^*, j^*R^2j_*\mathcal{V}_\lambda/\varpi) \Rightarrow H^{i+j}(\partial\tilde{X}_K^*, j^*R_i\mathcal{V}_\lambda/\varpi)$$

Let $P_a = M_aU_{P_a}$ be the $a$-th standard maximal parabolic subgroup of $\tilde{G}$ for any $a \in [1, n]$ where the Levi subgroup $M_a$ is isomorphic to $G_a \times \text{Res}_{\mathbb{F}/\mathbb{Z}}GL_{n-a}$ and $U_{P_a}$ denotes the unipotent radical of $P_a$. Let $P_{a,h}$ be the inverse image of $\tilde{G}_a$ under the homomorphism $\kappa_a : P_a \rightarrow P_a/U_{P_a} = : M_a$ Denote $M_{a,l} := \text{Res}_{\mathbb{F}/\mathbb{Z}}GL_{n-a}$ and $M_{a,h} := \tilde{G}_a$.

Over $\mathbb{C}$, we have a stratification of the boundary of the minimal compactification of the form $\partial\tilde{X}_K^* = \tilde{X}_{n-1} \sqcup \tilde{X}_{n-2} \sqcup \ldots \sqcup \tilde{X}_0$ where $\tilde{X}_a$ is a disjoint union of $U(a,a)$-Shimura varieties $\tilde{X}_a = \bigsqcup_x \tilde{X}_{a,x}$, where $x$ runs over the finite set $P(\mathbb{Q})P_{a,h}(\mathbb{Q}_f)\backslash \tilde{G}(\mathbb{Q}_f)/\tilde{K}$, and $\bar{x}$ denotes an arbitrary representative of $x$ in $\tilde{G}(\mathbb{Q}_f)$; for later use, we may and do choose $x$ so that its $p$-component $x_p$ is trivial and also we have $\tilde{X}_{a,x} := \tilde{G}_a(\mathbb{Q})\backslash \tilde{G}_a(\mathbb{Q}_\lambda)/\tilde{K}_{a,h}^x$ where $\tilde{K}_{a}^x = x(\tilde{K} \cdot \tilde{K}_{\infty})x^{-1} \cap P_a(\mathbb{Q}_\lambda)$ and $\tilde{K}_{a,h}^x := \tilde{K}_{a}^x \cap \tilde{G}_a(\mathbb{Q}_\lambda)$.

**Remark 6.1.** This stratification extends to the integral structure of Shimura varieties (cf. Theorem 7.2.4.1 of [LAN13]).

This stratification on the minimal compactification gives us a stratification spectral sequence of étale cohomology groups (carrying Galois action of $G_{F_0}$):

$$E_1^{a-1,b} := \bigoplus_x H_{c}^{a+b-1}(\tilde{X}_{a,x}, R^j_\lambda\mathcal{V}_\lambda/\varpi|_{\tilde{X}_{a,x}}) \Rightarrow H^{a+b-1}(\partial\tilde{X}_K^*, j^*R_i\mathcal{V}_\lambda/\varpi) \quad (6.1)$$

This spectral sequence degenerates at $E_2$. There is also another spectral sequence of étale cohomology groups (also carrying Galois action of $G_{F_0}$):

$$E_1^{b,c} = H_c^{b,c}(\tilde{X}_{a,x}, R^j_\lambda\mathcal{V}_\lambda/\varpi|_{\tilde{X}_{a,x}}) \Rightarrow H^{b+c}(\tilde{X}_{a,x}, R^j_\lambda\mathcal{V}_\lambda/\varpi|_{\tilde{X}_{a,x}}) \quad (6.2)$$

This spectral sequence degenerates at $E_2$. By the main result of [P92], the locally constant sheaf $R^b_\lambda j_*\mathcal{V}_\lambda/\varpi|_{\tilde{X}_{a,x}}$ is isomorphic to the locally constant sheaf associated to the $\tilde{K}_{a,h}^x$-module
\[
\bigoplus_{s=0}^{b} H^{b-s}(\Gamma_{M_{a,l}}(x), H^{s}(\Gamma_{U_{P_a}}(x), \mathcal{V}_{k}/\pi))
\]

Where
\[
\Gamma_{M_{a,l}}(x) := M_{a,l}(Q) \cap (\tilde{K}_{a,l}^{x} \times M_{a,l}(Q_{\infty})), \quad \text{for } \tilde{K}_{a,l}^{x} := \kappa_a(\tilde{K}^{x}) \cap M_{a,l}(Q_f)
\]
and
\[
\Gamma_{U_{P_a}}(x) := U_{P_a}(Q) \cap (\tilde{K}^{x} \cap U_{P_a}(Q_{\infty}))
\]

Assume that \(\tilde{\lambda}\) is \(p\)-small. Then by Theorem B of Polo and Tilouine [PT02], we have for each \(s \leq q\) an isomorphism of \(M_a\)-modules:

\[
H^{s}(\Gamma_{U_{P_a}}(x), \mathcal{V}_{k}/\pi) \cong \bigoplus_{w \in W_{P_a}, \ell(w)=s} \mathcal{V}_{M_{a},\tilde{\lambda}_w}/\pi
\]

where \(\tilde{\lambda}_w = w(\tilde{\lambda} + \rho) - \rho\). Therefore as \(\tilde{K}_{a,h}^{x}\)-module, we have

\[
H^{q-s}(\Gamma_{M_{a,l}}(x), H^{s}(\Gamma_{U_{P_a}}(x), \mathcal{V}_{k}/\pi)) \cong \bigoplus_{w \in W_{P_a}, \ell(w)=s} H^{b-s}(\Gamma_{M_{a,l}}(x), \mathcal{V}_{M_{a,l},\tilde{\lambda}_w}/\pi) \otimes \mathcal{V}_{M_{a,h},\tilde{\lambda}_w}/\pi
\]

In particular, the Galois action on the cohomology of \(\tilde{X}_{a,x}\) arises only from the second factors of each summand. Denote \(V_{w,a}^{\tilde{\lambda}} := H^{s}(\Gamma_{M_{a,l}}(x), \mathcal{V}_{M_{a,l},\tilde{\lambda}_w}/\pi)\). Then we can compute the left-hand side of the spectral sequence [6.2]. Namely, we have an isomorphism of \(G_{F_0}\)-modules:

\[
H^{c}(\tilde{X}_{a,x}, R^b j_{!*} \mathcal{V}_{k}/\pi|_{\tilde{X}_{a,x}}) \cong \bigoplus_{w \in W_{P_a}, \ell(w) \leq b} H^{c}(\tilde{X}_{a,x}, \mathcal{V}_{M_{a,h},\tilde{\lambda}_w}/\pi) \otimes V_{w}^{b-\ell(w),a} \tag{6.3}
\]

**Definition 6.2.** We say \(\tilde{\lambda} \in (\mathbb{Z}^{2n}_{+})^{\text{Hom}(F^{+}, E)}\) is **mildly regular** if for any \(\bar{r} \in \text{Hom}(F^{+}, E)\)

\[
(\tilde{\lambda}_{r,2}, \cdots, \tilde{\lambda}_{r,2n-1}) \in \mathbb{Z}^{2n-2}_{+}
\]

is a regular weight.

**Theorem 6.3.** Assume \(\tilde{\lambda}\) is mildly regular and \(|\tilde{\lambda}|_{\text{comp}} < p\). Then any irreducible Galois sub-representation \(W\) of \(H^{b}_{c}(\tilde{X}_{K}, \mathcal{V}_{\tilde{\lambda}}/\pi)\) is one dimensional for all \(j < d\).
Proof. We argue by induction on \( n \). Assume that \( n = 1 \). Since the boundary of the minimal compactification of a \( U(1,1) \)-Shimura variety is only a disjoint union of points, the boundary cohomology is the cohomology of a zero dimensional variety and therefore is a sum of characters as Galois module.

The induction hypothesis being as follows:

Hypothesis: the theorem holds for all integers \( k < n \).

Since the spectral sequences \( 6.1 \) and \( 6.2 \) degenerate at \( E_2 \), its irreducible constituents provide the irreducible constituents of its abutment.

Let \( j \in [0, d-1] \). We want to compute the boundary cohomology \( H^j_\partial(\tilde{X}_K; \mathcal{V}_\lambda/\varpi) \).

By the spectral sequence \( 6.1 \) it is enough to check that the semisimplification of \( H^j_\partial(\tilde{X}_K; \mathcal{V}_\lambda/\varpi|_{\tilde{X}_k,\bar{x}}) \) for the strata \( \tilde{X}_k,\bar{x} \) for \( k < n \) is a direct sum of characters. For this purpose, by the spectral sequence \( 6.2 \) it is enough to show for all \( k < n \) that for every pair \((b, c)\) such that \( j = b + c \), \( b + c < d \), the semisimplification of \( H^c_\partial(\tilde{X}_k,\bar{x}, V_{M_{k,h},\tilde{\lambda}_w}/\varpi) \) for all \( \bar{x} \) and \( w \in W^{F_k} \), is a direct sum of characters.

By \( 6.3 \) it suffices to show that for every \( c < d \) and \( k = 0, \ldots, n-1 \), the Galois module \( H^c_\partial(\tilde{X}_k,\bar{x}, V_{M_{k,h},\tilde{\lambda}_w}/\varpi) \) for all \( \bar{x} \) and \( w \in W^{F_k} \), is a direct sum of characters.

Since the \( p \)-component \( x_p \) of the representative \( x \) is equal to 1 by our choice, the level of the Shimura variety \( \tilde{X}_{k,\bar{x}} \) at \( p \) is maximal hyperspecial.

The local system in \( \mathcal{O} \)-modules \( V_{M_{k,h},\tilde{\lambda}_w} \) corresponds to the representation \( V_{M_{k,h},\tilde{\lambda}_w} \) of \( M_{k,h} \) which is an irreducible representation of highest weight \( \tilde{\lambda}' = \tilde{\lambda}_{w,h} \). Indeed, \( \tilde{\lambda}' \) is dominant since \( w \in W^{F_k} \). The irreducibility follows from the condition \( |\tilde{\lambda}'|_{comp} < p \).

To verify this, recall that the assumption \( |\tilde{\lambda}|_{comp} < p \) guarantees that \( |\tilde{\lambda}'|_{comp} < p \) because:

\[
|\tilde{\lambda}'|_{comp} = dk^2 + |\tilde{\lambda}'| \leq dk^2 + |\tilde{\lambda}_w| \leq dk^2 + |w\rho - \rho| \leq dk^2 + |\tilde{\lambda}|
\]

\[
+d(n-k)(n+k-1) = |\tilde{\lambda}| + dn^2 - d(n-k) \leq |\tilde{\lambda}| + dn^2 = |\tilde{\lambda}|_{comp}
\]

Note moreover that \( \tilde{\lambda}' \) is regular. Indeed if

\[
\tilde{\lambda}_w = w(\tilde{\lambda} + \rho) - \rho = (\tilde{\lambda}_{w,\bar{x},1}, \ldots, \tilde{\lambda}_{w,\bar{x},2n})_{\bar{x}} \in \text{Hom}(F^+,E)
\]

then \( \tilde{\lambda}' \) is equal to

\[
\tilde{\lambda}' = (\tilde{\lambda}_{w,\bar{x},k+1}, \tilde{\lambda}_{w,\bar{x},k+2}, \ldots, \tilde{\lambda}_{w,\bar{x},n-k})_{\bar{x}} \in \text{Hom}(F^+,E)
\]

since \( \tilde{\lambda} \) is mildly regular, \( \tilde{\lambda}' \) is regular.

Therefore, Theorem \( 5.5 \) implies that for any \( r < d \), we have \( H^r_\partial(\tilde{X}_{k,\bar{x}}, V_{M_{k,h},\tilde{\lambda}_w}/\varpi) \cong H^{r-1}_\partial(\tilde{X}_{k,\bar{x}}, V_{M_{k,h},\tilde{\lambda}_w}/\varpi) \). By the induction assumption, since \( k < n \), the semisimplification \( H^r_\partial(\tilde{X}_{k,\bar{x}}, V_{M_{k,h},\tilde{\lambda}_w}/\varpi) \) is a direct sum of some characters. \( \square \)
Remark 6.4. The semisimplification is needed in this theorem because of the spectral sequences 6.1 and 6.2 despite the fact that there is no need of semisimplification in formula 6.3.

7 Eichler-Shimura relations

We first recall a result by Wedhorn [Wed00] establishing Eichler-Shimura relations for the group $\tilde{G}$ relevant to our purpose (He proves the Eichler-Shimura relation in the more general case of PEL Shimura varieties at places of good reduction at which the group is split).

7.1 Tensor induction

In this subsection we recall the notion of tensor induction as defined in Section 1 of Yoshida [Yos94].

Let $\rho_0 : G_F \to \text{GL}(V_0)$ be a representation. We fix a decomposition $G_{F_0} = \prod_{i=1}^{d} g_i G_F$ where $g_1, \ldots, g_d \in G_{F_0}$ are all equivalence classes of $G_{F_0}/G_F$. Denote $V := \bigotimes_{i=1}^{d} V_i$ where all $V_i$ is isomorphic to $V_0$. With this choice of representatives $(g_i)$’s, we define a map $\rho : G_{F_0} \to \text{GL}(V)$ by:

$$\rho(h)(\bigotimes_{i=1}^{d} v_i) = \bigotimes_{i=1}^{d} \rho_0(g_i^{-1} h g_{h,i})(v_{h,i})$$

where $g_{h,i} = g_k$ for some $k = 0, \ldots, d$ such that $h^{-1} g_i \in g_{h,i} G_F$ and $v_{h,i} = v_k$. The map $\rho$ is a group homomorphism. If we choose another set of representatives $(g'_i)$ the resulting homomorphism is equivalent to $\rho$. We denote by $\otimes \text{Ind}^{G_{F_0}}_{G_F} \rho_0$ any representation $\rho$ as above. We call it a tensor induction of $\rho_0$ from $G_F$ to $G_{F_0}$.

7.2 A theorem of Wedhorn

Let $v_0$ be an arbitrary finite place of $F_0$, denote $H_{v_0}(x)$ for the characteristic polynomial of $[(\otimes \text{Ind}^{F_0}_{F})((\Lambda^n \rho_\psi) \otimes \rho_\psi(n(n+1)/2-2n^2)](\text{Frob}_{v_0})$ where $\psi$ is central character of $\rho_\psi$ (this is the Hecke polynomial for $U(n,n)/F^+$, see [N18 A5.7]).

Theorem 7.1 (Wedhorn). For any finite place $v_0$ of $F_0$, the endomorphism $H_{v_0}(\text{Frob}_{v_0})$ annihilates $H_{(c)}(\bar{X_K}, V_{\lambda}/\varpi)[\bar{m}]$.  

Comment: the proof requires that the finite place $v_0$ of $F_0$ is above a rational prime $\ell$ which is totally split in $F$. However, Chebotarev density theorem implies
the result for any finite place \( v_0 \) of \( F_0 \).

Therefore, the long exact sequence \( 5.2 \) implies that \( H_{v_0}(\text{Frob}_{v_0}) \) annihilates \( \overline{\rho}_0 := H^1_{\bar{\sigma}}(\bar{X}_{\bar{K}}, \mathcal{V}_{\bar{X}} )[\bar{m}] \).

From now, we fix \( n = 2 \). We can therefore write \( \lambda = (m_\tau, n_\tau)_{\tau \in \text{Hom}(F, E)} \). We make the following assumption (PR) of purity and partial regularity:

(PR) For all \( \tau \in \text{Hom}(F, E) \), we have \( m_\tau = m_\tau \epsilon \), \( n_\tau = n_\tau \epsilon \) and \( 0 > m_\tau \).

The assumption (PR) implies that the central character \( \psi \) of \( \overline{\rho}_m \) is trivial.

For the remaining of this section, we will study \( \overline{\theta} := (\otimes \text{Ind}_{F}^{F_0})(\wedge^2 \overline{\rho}^\vee_{m}(-5)) \). By the definition of \( \bar{m} \), we have an isomorphism between Galois representations \( \overline{\rho}_m \) isomorphic to \( (\wedge^2 \overline{\rho}^\vee_{m}) \oplus (\wedge^2 \overline{\rho}^\vee_{m})(6) \oplus (\overline{\rho}^\vee_{m} \otimes \overline{\rho}^\vee_{m})(3) \).

Since \( \overline{\rho}^\vee_{m} \) isomorphic to \( (\wedge^2 \overline{\rho}^\vee_{m}) \oplus (\wedge^2 \overline{\rho}^\vee_{m})(6) \oplus (\overline{\rho}^\vee_{m} \otimes \overline{\rho}^\vee_{m})(3) \) (7.1)

\[ \wedge^2 (\overline{\rho}^\vee_{m}) \oplus (\wedge^2 \overline{\rho}^\vee_{m})(6) \oplus \epsilon^3 \oplus (\text{Sym}^2 \overline{\rho}^\vee_{m}) \otimes (\det \overline{\rho}_m)(3) \] (7.1)

### 7.3 Large image conditions

For the rest of this paper we assume that:

(LI \( \overline{\rho}_m \)) there exists a power \( q \) of \( p \) such that \( \text{SL}_2(\mathbb{F}_q) \subset \text{Im}(\overline{\rho}_m) \subset \kappa \times \text{GL}_2(\mathbb{F}_q) \).

**Remark 7.2.** The condition (LI \( \overline{\rho}_m \)) implies that \( \overline{\rho}_m \) is decomposed generic. (for example see [AN20, Lemma 2.3])

Let \( \hat{\bar{F}} \) be normal closure of \( F \) in \( \overline{\mathbb{Q}} \). Denote \( pr : \text{PGL}_2(\mathbb{F}_p) \rightarrow \text{GL}_2(\mathbb{F}_p) \) for the projection map. Let \( D = \det \overline{\rho}_m(\bar{G}_F) \).

**Lemma 7.3.** Assume (LI \( \overline{\rho}_m \)) and \( p > w \). then either

\( \overline{\rho}_m(\bar{G}_F) = \text{GL}_2(\mathbb{F}_q)^D \) := \( \{ A \in \text{GL}_2(\mathbb{F}_q) | \det(A) \in D \} \), or

\( \overline{\rho}_m(\bar{G}_F) = (\mathbb{F}_q^\times \text{GL}_2(\mathbb{F}_q))^D \) := \( \{ A \in \mathbb{F}_q^\times \text{GL}_2(\mathbb{F}_q) | \det(A) \in D \} \)

**Proof.** By (LI \( \overline{\rho}_m \)) the group \( pr(\overline{\rho}_m(\bar{G}_F)) \) is isomorphic to \( \text{PGL}_2(\mathbb{F}_q) \) or \( \text{PSL}_2(\mathbb{F}_q) \). The group \( pr(\overline{\rho}_m(\bar{G}_F)) \) is normal subgroup of \( pr(\overline{\rho}_m(\bar{G}_F)) \). Since \( p > w \) and \( pr(\overline{\rho}_m(I_v)) \subset pr(\overline{\rho}_m(\bar{G}_F)) \) then theorem [1.1] implies that \( pr(\overline{\rho}_m(\bar{G}_F)) \) is non-trivial.

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Since $\text{PSL}_2(\mathbb{F}_q)$ is a simple group of index 2 in the group $\text{PGL}_2(\mathbb{F}_q)$, we have $\text{PSL}_2(\mathbb{F}_q) \subset \text{pr}(\rho_m(G_{\hat{F}}))$. In the next lemma, we show that this implies $\text{SL}_2(\mathbb{F}_q) \subset \bar{\rho}_m(G_{\hat{F}})$. So we are done by the following chain of inclusions:

$$(\kappa^{\times}\text{GL}_2(\mathbb{F}_q))^D \subset \bar{\rho}_m(G_{\hat{F}}) \subset (\text{GL}_2(\mathbb{F}_q))^D$$

Lemma 7.4. Let $H$ be a group of center $Z$ and let $\text{pr} : H \to H/Z$ the canonical projection. Let $P$ and $Q$ be two subgroups of $H$ such that $\text{pr}(Q) \subset \text{pr}(P)$. Assume moreover that $Q$ has no non-trivial abelian quotients. Then $Q \subset P$.

Proof. Since $Q$ has no non-trivial abelian quotients, $Q^{\text{der}}$ is equal to $\{1\}$ or $Q$. If $Q^{\text{der}} = \{1\}$, then $Q$ is an abelian group, contradicting the assumption. So $Q^{\text{der}} = Q$ and we have following a chain of reverse inclusions:

$$P \supset P^{\text{der}} = (PZ)^{\text{der}} = (QZ)^{\text{der}} = Q^{\text{der}} = Q$$

Let $\gamma \in \mathbb{F}_{q^2} - \mathbb{F}_q$ be such that $\gamma^2 \in \mathbb{F}_q$. Then:

$$(\text{GL}_2(\mathbb{F}_q))^D = (\text{GL}_2(\mathbb{F}_q))^D \prod (\gamma \text{GL}_2(\mathbb{F}_q))^D$$

Since $\text{pr}(\text{Ind}^{F_0}_{\bar{\rho}_m}(G_{\hat{F}})) \subset \prod_{\tau \in \text{Hom}(F,E)} \text{PGL}_2(\mathbb{F}_q)$ by using Goursat’s lemma as in Lemma 5.2.2 of [R76] and the facts $\text{PSL}_2(\mathbb{F}_q)$ is a simple group and $\text{Aut}(\text{PSL}_2(\mathbb{F}_q)) = \text{PSL}_2(\mathbb{F}_q) \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, we see that there exists a partition $\text{Hom}(F,E) = \bigsqcup_{i \in I} J^i_F$ and for all $\tau \in J^i_F$ there exist an element $\sigma_{i,\tau} \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ such that

$$\text{pr}(\Phi(\text{SL}_2(\mathbb{F}_q)^I)) \subset \text{pr}(\text{Ind}^{F_0}_{\bar{\rho}_m}(G_{\hat{F}})) \subset \text{pr}(\Phi(\text{GL}_2(\mathbb{F}_q)^I)),$$

Where $\Phi = (\Phi_i)_{i \in I} : \text{GL}_2(\mathbb{F}_q)^I \to \text{GL}_2(\mathbb{F}_q)^{\text{Hom}(F,E)}$ is given by $(M_i)_{i \in I} \mapsto (M_i^{\sigma_{i,\tau}})_{i \in I, \tau \in J^i_F}$ (pr : $\text{GL}_2 \to \text{PGL}_2$ is projection map).

Denote $J_{F,v} := \text{Hom}(F_v, E)$. Let

$$H(\mathbb{F}_q) := \{(M_i)_{i \in I} \in \prod_{i \in I} \text{GL}_2(\mathbb{F}_q) | \exists \delta \in D, \forall i, \det M_i = \delta\}.$$

Lemma 7.5. Assuming $(LI \bar{\rho}_m)$ holds and $p > 2w + 2$, then:

$$\text{Ind}^{F_0}_{\rho_m}(I_p) \subset \Phi(H(\mathbb{F}_q))$$
Proof. Let \( h = |J_{F,v}| \) or \( 2|J_{F,v}| \) be the common tame level of the characters \( \delta_1, \delta_2 \) (which are defined in Theorem 4.1) and \( x_h \) be a generator of \( \mathbb{F}_p^\times \). Let \( \tilde{\delta}_1, \tilde{\delta}_2 : \mathbb{F}_p^\times \to \mathbb{F}_p^\times \) be the characters corresponding to \( \delta_1, \delta_2 : I_v \to \mathbb{F}_p^\times \) by local class field theory. By Lemma 7.3, the all of array of the elements of \( \bar{\rho}_m(I_v) \) are in \( \mathbb{F}_q \supset \gamma \mathbb{F}_q \).

Then \( \tilde{\delta}_1(x_h)^2, \tilde{\delta}_2(x_h)^2 \in \mathbb{F}_q \). Since \( \tilde{\delta}_1(x_h) = \prod_{\tau \in J_{F,v}} \tau(x_h)^{m_{\tau}+1} = x_h^{\sum(m_{\tau}+1)p'} \) or \( \prod_{\tau \in J_{F,v}} \tau(x_h)^{p\tau} = x_h^{\sum n_i p} \) where \( n_i \) resp. \( m_i \) is equal to \( n_{\tau} \), resp. \( m_{\tau} \) for \( \tau(x_h) = x_h^{p^i} \).

If \( q = p^j \) then \( \mathbb{F}_q \cap \mathbb{F}_{p^h} = \mathbb{F}_{p^j} \) (\( \mathbb{F}_q \)), we put \( l' := \gcd(l, h) \) and \( h = l' \cdot k \).

Since \( \tilde{\delta}_1(x_h)^2, \tilde{\delta}_2(x_h)^2 \in \mathbb{F}_q \), we have \( \tilde{\delta}_1(x_h)^2, \tilde{\delta}_2(x_h)^2 \in \mathbb{F}_{p^l'} \), hence \( p^{h-1}2(\sum n_i p') (p' - 1) and p^{h-1}2(\sum (m_i+1)p') (p' - 1) \). Hence \( \sum_{i=0}^{k-1} p^{i'} | \sum_{i=0}^{h-1} 2n_i p^i \) and \( \sum_{i=0}^{k-1} p^{i'} | \sum_{i=0}^{h-1} 2(m_i+1)p^i \). Since \( p > 2w + 2 \), the quotient of these numbers is equal to \( \sum_{i=0}^{l'-1} 2n_i p^i \) resp. \( \sum_{i=0}^{l'-1} 2(m_i+1)p^i \) which are even numbers and implies that \( p^{h-1}2(\sum n_i p') (p' - 1) \) and \( p^{h-1}2(\sum (m_i+1)p') (p' - 1) \), then \( \tilde{\delta}_1(x_h), \tilde{\delta}_2(x_h) \in \mathbb{F}_q \). It implies that \( \bar{\rho}_m(I_v) \) is a subgroup of \( \text{GL}_2(\mathbb{F}_q) \).

Since \( p \) is unramified in \( F \) then for any \( y \in \mathbb{F}_p \) we know \( y \) acts on \( \text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m) \) by \( (\bar{\rho}_m(g_i^{-1}yg_i))_i \). So for prove of lemma, it is enough to check that the determinant condition and it is true because \( \bar{\rho}_m(I_p) \) is subgroup of \( \text{GL}_2(\mathbb{F}_q) \) and theorem 4.1.

Lemma 7.6. Assuming that \( (L^I \bar{\rho}_m) \) holds and \( p > 2w + 2 \), then

\[ H(\mathbb{F}_q) \subset \text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m(G_F)) \]

Proof. By 7.3 we have:

\[ \text{pr}(\Phi(SL_2(\mathbb{F}_q)^I)) \subset \text{pr}(\text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m(G_F))) \]

Therefore lemma 7.4 implies that \( (\Phi(SL_2(\mathbb{F}_q)^I)) \subset (\text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m(G_F))) \).

Then the lemma deduces by the fact \( \Phi(H(\mathbb{F}_q)) = \Phi((SL_2(\mathbb{F}_q)^I) \cdot \text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m(I_v))) \).

Denote the fixed field of \( (\text{Ind}_{\mathbb{F}_p}^{\mathbb{F}_q} (\bar{\rho}_m))^{-1}(\Phi(H(\mathbb{F}_q))) \) by \( F' \).

Lemma 7.7. Assume that \( (L^I \bar{\rho}_m) \) holds and \( \gcd(p - 1, w + 1) = 1 \), then the restrictions to \( G_{F'} \) of the \( G_{F_0} \)-representations \( \bar{\theta} \) and \( \bar{\rho}_m^{**} \) factor through \( H(\mathbb{F}_q) \).

Proof. Note that the condition \( \gcd(p - 1, w + 1) = 1 \) implies that the mod \( p \) cyclotomic character factors by \( \text{Im}(\bar{\rho}_m) \). Then, the following diagram implies that the restriction of \( \bar{\theta} \) to \( F' \) factors by \( H(\mathbb{F}_q) \).
Also, by theorem 7.1, \( \text{char} \hat{\theta}(g) = 0 \) for any \( g \in G_{F'} \). By theorem 6.3, the semi-simplification of \( \hat{\rho}_\partial \) is a direct sum of characters, so \( \hat{\theta}(g) = 1 \) implies \( \hat{\rho}_\partial^{ss}(g) = 1 \). Hence, \( \text{Ker} \hat{\theta} \subset \text{Ker} \hat{\rho}_\partial^{ss} \); therefore, \( \hat{\rho}_\partial^{ss} \) factors by \( H(F_\mathbb{F}) \).

\[ \square \]

8 Proof of the Main Theorem

The aim of this section is to prove our main results on the cohomology of locally symmetric space of \( \text{GL}_2(F) \) after localisation at a strongly non-Eisenstein maximal ideal.

8.1 Reduction from an arbitrary weight to the (PR) case

Let \( \lambda = (m_\tau, n_\tau)_{\tau \in \text{Hom}(F,E)} \in (\mathbb{Z}_2^+)_{\text{Hom}(F,E)} \) be an arbitrary dominant weight (so \( m_\tau \geq n_\tau \) for all \( \tau \)'s).

If \( \psi : G_F \rightarrow \mathcal{O}_F^\times \) is a continuous character, we define an automorphism \( f_\psi \) of the Hecke algebra \( T^S \) (defined in section 2) by the formula

\[ f_\psi([K^S gK^S]) = \psi(\text{Art}_F(\text{det}(g)))^{-1}[K^S gK^S] \]

for all \( g \in G_f \) such that \( g_S = 1 \). If \( \mathfrak{m} \subset T^S \) is a maximal ideal, then we define \( \mathfrak{m}(\psi) = f_\psi(\mathfrak{m}) \).

**Proposition 8.1.** Let \( \psi : G_F \rightarrow \mathcal{O}_F^\times \) be a continuous character satisfying the following conditions:

1. For each \( v \in S_p \), \( \psi \) is unramified at \( v \).
2. There is \( m'_\tau = (m'_\tau)_{\tau} \in \mathbb{Z}_{\text{Hom}(F,E)}^{\text{Hom}(F,E)} \) such that for each place \( v \in S_p \), and for each \( k \in \mathcal{O}_{F_v}^\times \), we have

\[ \psi(\text{Art}_{F_v}(k)) = \prod_{\tau \in \text{Hom}(F_v,E)} \tau(k)^{-m'_\tau} \]

Let \( \mu \in (\mathbb{Z}_2^+)_{\text{Hom}(F,E)} \) be the dominant weight defined by the formula \( \mu_\tau = (m'_\tau, m'_\tau) \) for each \( \tau \in \text{Hom}(F,E) \). Then for any \( \lambda \in (\mathbb{Z}_2^+)_{\text{Hom}(F,E)} \) there is an isomorphism

\[ R\Gamma(X_K, V_{\lambda})_m \cong R\Gamma(X_K, V_{\lambda + \mu})_m(\psi) \]
in $D(\mathcal{O})$ which is equivariant for the action of $T^S$ when $T^S$ acts in the usual way on the left-hand side and acts by $f_\psi$ on the right-hand side.

Proof. See Proposition 2.2.14 and Corollary of 2.2.15 of [ACC+18].

There is an involution $\iota : T^S \rightarrow T^S$, resp. $\tilde{\iota} : \tilde{T}^S \rightarrow \tilde{T}^S$, which sends a double coset $[K^S g K^S] \mapsto [K^S g^{-1} K^S]$, resp. $[\tilde{K}^S g K^S] \mapsto [\tilde{K}^S g^{-1} K^S]$. If $m \subset T^S$, resp. $\tilde{m} \subset \tilde{T}^S$, is a maximal ideal with residue field a finite extension of $\kappa$, then we define $m^\vee = \iota(m)$, resp. $\tilde{m}^\vee = \iota(\tilde{m})$.

Proposition 8.2. Let $\lambda = (m_\tau, n_\tau)_{\tau} \in (\mathbb{Z}^2_+)_{\Hom(F,E)}$, resp $\tilde{\lambda} = (\tilde{\lambda}_\tau, 1, \cdots, \tilde{\lambda}_{\tau,4})_{\tau} \in (\mathbb{Z}^4_+)_{\Hom(F^+,E)}$. Denote $\lambda^\vee := (\tau, n_\tau, -m_\tau)_{\tau}$, resp. $\tilde{\lambda}^\vee := (\tau, -\tilde{\lambda}_{\tau,4}, \cdots, -\tilde{\lambda}_{\tau,1})_{\tau}$. Then there is an isomorphism

$$\text{RHom}_\mathcal{O}(R\Gamma(X_K, \mathcal{V}_\lambda)_m, \mathcal{O}) \cong R\Gamma_c(X_K, \mathcal{V}_{\lambda^\vee})_{m^\vee}[4d - 1]$$

resp.

$$\text{RHom}_\mathcal{O}(R\Gamma(\tilde{X}_K, \mathcal{V}_{\lambda^\vee})_{\tilde{m}}, \mathcal{O}) \cong R\Gamma_c(\tilde{X}_K, \mathcal{V}_{\lambda^\vee})_{\tilde{m}^\vee}[8d]$$

in $D(\mathcal{O})$ which is equivariant for the action of $T^S$, resp. $\tilde{T}^S$, when $T^S$, resp. $\tilde{T}^S$, acts by $\iota$, resp $\tilde{\iota}$, on the left-hand side and in its usual way on the right-hand side.

Proof. See Proposition 3.7 of [NT16].

Corollary 8.3. Let $\lambda = (m_\tau, n_\tau)_{\tau} \in (\mathbb{Z}^2_+)_{\Hom(F,E)}$ be a pure dominant weight such that $n_\tau \equiv n_{\tau_c} \pmod{2}$ and such that for any $\tau \in \Hom(F,E)$, $n_\tau$ and $m_\tau$ have the same sign. Then there exists a pair $(\mu, \psi)$ where

- $\mu = (k_\tau, k_\tau)_{\tau \in \Hom(F,E)}$ is a diagonal weight such that $\lambda + \mu$ or $(\lambda + \mu)^\vee$ satisfies the condition (PR),

- and $\psi : G_F \rightarrow \mathcal{O}^\times$ is a continuous character

which satisfies conditions (1) and (2) of Proposition 8.1 for $m' = (k_\tau)_{\tau \in \Hom(F,E)}$.

Proof. Let $k_\tau = \frac{n_{\tau_c} - n_\tau}{2}$ for any $\tau \in \Hom(F,E)$ and $\mu' = (k_\tau, k_\tau)_{\tau \in \Hom(F,E)}$. Hence $\lambda + \mu' = (a_\tau, b_\tau)_{\tau \in \Hom(F,E)}$ is a pure dominant weight with the same purity weight and such that $a_\tau = a_{\tau_c}, b_\tau = b_{\tau_c}$; moreover, all these integers have the same sign.

If $a_\tau \leq 0$, put $\mu = (k_\tau - 1, k_\tau - 1)_{\tau \in \Hom(F,E)}$, then the weight $\lambda^{(PR)} = \lambda + \mu$ satisfies the condition (PR).

If $a_\tau \geq 0$, put $\mu = (k_\tau + 1, k_\tau + 1)_{\tau \in \Hom(F,E)}$, then the weight $\lambda^{(PR)} = (\lambda + \mu)^\vee$ satisfies the condition (PR).

The existence of $\psi$ follows from Lemma 2.2 of [HSBT10].
Remark 8.4. Actually, the proof also shows that one can choose \( \mu \) such that for \( \chi^{(PR)} \) defined as in the proof above, we have \(|w(\chi^{(PR)})| = |w(\lambda)| + 2\), where \( w(\mu) \) denotes the purity weight of \( \mu \).

8.2 Direct summand factors of \( \bar{\rho}_\theta \)

Under the assumptions of Lemma \([7.7]\), we can see \( \bar{\theta} \) as an \( H(\mathbb{F}_q) \)-representation. First, we note that by \([7.1]\) we have a decomposition

\[
\wedge^2(\bar{\rho}_m^\vee(-5)) \cong \det^{-1}(-5) \oplus \det(1) \oplus \varepsilon_0(-2) \oplus (\text{Sym}^2)^\vee \otimes \det(-2)
\]

as \( \text{Im}(\bar{\rho}_m) \)-representations. It follows from this that any irreducible subrepresentation of the restriction of the representation \( \bar{\theta} = (\text{Ind}_{\rho}^{\bar{\rho}})^\wedge (\bar{\rho}_m^\vee(-5)) \) to \( G_{F'} \) is a direct summand of \( \otimes_{\tau \in I} \phi_{\tau} \), where the representation \( \phi_{\tau} \) is either a character or is isomorphic to the twist of \( \text{Sym}^2 \) by a character as \( H(\mathbb{F}_q) \)-representation.

Lemma 8.5. If \( p > d^2(d+1)(-2w+4) \), \( \gcd(p-1,w+1) = 1 \) and \( \chi \) is a character which is direct summand of \( \bar{\rho}_\theta \). Then \( \chi \) is a direct summand of \( \bar{\theta} \) as \( G_{F'} \)-representation.

Proof. Since the restriction \( \chi_{F'} \) of \( \chi \) to \( G_{F'} \) factors by \( H(\mathbb{F}_q) \) and since \( \text{SL}_2(\mathbb{F}_q) \) is a simple group, the character \( \chi_{F'} \) factors into a character of the quotient \( \mathcal{D} \) of \( H(\mathbb{F}_q) \). Therefore, there is an integer \( k \) such that \( \chi_{F'} \) is isomorphic to \( \det^k \) as an \( H(\mathbb{F}_q) \)-representation. So \( \chi_{F'}((M_i)_{i \in I}) = a^k \), where \( M_i = \text{diag}(a,1) \) for \( a \in \mathbb{F}_p \) and any \( i \in I \). We assume that \( \chi \) is not a direct summand of \( \bar{\theta} \).

Since \( \text{char}_{\bar{\theta}(g)}(\bar{\rho}_\theta(g)) = 0 \) for any \( g \in G_{F'} \) and \( \chi \) is a character which is direct summand of \( \bar{\rho}_\theta \), we see that \( \text{char}_{\bar{\theta}(g)}(\chi(g)) = 0 \) for any \( g \in G_{F'} \), hence \( \chi(g) \) is an eigenvalue of \( \bar{\theta}(g) \).

If we put \( M'_i = \text{diag}(ab,b^{-1}) \) for any \( i \in I \), then \( \chi((M'_i)_{i \in I}) = a^k \) is an eigenvalue of \( \bar{\theta}((M'_i)_{i \in I}) \).

The condition \( \gcd(p-1,w+1) = 1 \) implies that there is an element \( t \in \mathbb{Z}/(p-1)\mathbb{Z} \) such that \( \varepsilon \cong \det^t \).

Note that \( a^k \) is an eigenvalue of a matrix of the form \( \otimes_{\tau \in I} \phi_{\tau}((M_i)_{i \in I}) \) where the representation \( \phi_{\tau} \) is either \( \det^{-1} \otimes \varepsilon^{-5} \) or \( \det \otimes \varepsilon \) or \( \varepsilon^{-2} \) or \( (\text{Sym}^2)^\vee \otimes \det \otimes \varepsilon^{-2} \). Hence, using \( \varepsilon \cong \det^t \), we see that the eigenvalues of \( \phi_{\tau}((M_i)_{i \in I}) \) are respectively \( a^{-1} \cdot a^{-5t}, a \cdot a^t, a^{-2t} \) or they belong to \( \{ a^{-1} \cdot a^{-2t} \cdot b^{-2}, a \cdot a^{-2t} \cdot b^2, a^{-2t} \} \). Hence \( a^k \) is of the form \( a^j b^l \) where \(-2d \geq l \geq 2d\) is a non-zero even number. There are only \((-2w+4)d\) possible values for \( j \), because \( (w+1).\{-1-5t,1+t,-2t,-1-2t,1-2t,-2t\} = \{-w+4,w,2,-w+1,w+3,2\} \) (note that \( t(w+1) = -1 \) mod \( p-1 \)).
Since the equation $a^k = a^l x^l \pmod{p}$ has at most $l$ roots and $p > d^2(d+1)(-2w+4)$, there must exist an element $b \in F_p$ such that $a^k$ does not belong to the set of eigenvalues of $\bar{\ell}((M_i^r)_{i \in I})$. Contradiction. Thus, $\chi$ is a direct summand of $\bar{\ell}$ as $G_{F_0}$-representation.

\[\Box\]

**Remark 8.6.** The equation $a^k = a^l x^l \pmod{p}$ has no root or it has exactly $\gcd(p-1, l)$ roots. Therefore, the bound $d^2(d+1)(-2w+4)$ can be improved as

$$2d \cdot \left( \sum_{i=1}^{d} \gcd(p-1, i) \right)(-2w+4)$$

which unfortunately depends on $p$.

By the above Lemma, the character $\chi$ is equal to $\bar{\psi}$, which is a direct summand of $\bar{\ell}$ as $G_{F'}$-representation, so by the Frobenius reciprocity theorem, we have a nonzero $G_{F'}$-equivariant map $\text{Ind}_{F'}^{F_0} \psi \to \chi$.

By Mackey theorem, there is a one-dimensional subquotient in the $\text{Ind}_{F'}^{F_0} \psi$ if and only if for any $\tau \in \text{Gal}(F'/F_0)$ we have $\psi \cong \psi^\tau$. This implies that $\chi$ is isomorphic to $(\otimes \text{Ind}_{F'}^{F_0})((\wedge^2 \rho_m^\vee)(-5))$ or $(\otimes \text{Ind}_{F'}^{F_0})((\wedge^2 \rho_m^\vee)(1))$ or $(\otimes \text{Ind}_{F'}^{F_0})(\varepsilon^0 \cdot 2)$ as $G_{F'}$-representation. Thus by Lemma 7.5 they have the same Fontaine-Laffaille weights.

Therefore the Fontaine-Laffaille weight of $\chi$ is equal to $d(-w+4)$ or $d(w+1)$ or $2d$.

### 8.3 Proof of the main Theorem

**Lemma 8.7.** $\chi$ can not have Fontaine-Laffaille weight $d(-w+4)$ or $2d$.

**Proof.** The character $\chi$ occurs as a subrepresentation of $H^1_{\partial}(X_K, \mathcal{V}_{\lambda}/\varpi)$; by Formula (5.4), $\chi$ is also a subrepresentation of $H^1_{\epsilon}((X_K, \mathcal{V}_{\lambda}/\varpi))$.

By Theorem 3.12, Fontaine-Laffaille weights of $H^1_{\epsilon}((X_K, \mathcal{V}_{\lambda}/\varpi))$ form a subset of $\{p(\bar{w}) | \bar{w} \in W^P, \ell(\bar{w}) \leq i + 1 \}$ where $p(\bar{w}) = -\bar{w}(\rho + \lambda)(H) + p(H)$ ($H = \text{diag}(0, 0, \ldots, 0, -1, -1, \ldots, -1)$).

The only representation of $d(-w+4)$ as $p(\bar{w})$ is $p(\bar{w}_0)$, where $\bar{w}_0 \in W^P$ is the longest length element. Since $\ell(\bar{w}_0) = 4d$ and $i < d$, then $d(-w+4)$ is not Fontaine-Laffaille weight of $H^1_{\epsilon}((X_K, \mathcal{V}_{\lambda}/\varpi))$ and also it is not Fontaine-Laffaille weight of $\chi$.

For any $\bar{w} = \otimes_{\tau \in I} \bar{w}_\tau \in W^P$ we have $p(\bar{w}) = \sum_{\tau \in I} p(\bar{w}_\tau)$ and $\ell(\bar{w}) = \sum_{\tau \in I} \ell(\bar{w}_\tau)$. The set $W^P_\tau$ has six elements and every element corresponds to a subset of $\{1, 2, 3, 4\}$.
with two elements.

| Subset of \{1,2,3,4\} | \(\ell(\tilde{w})\) | \(p(\tilde{w})\) |
|------------------------|----------------|-----------------|
| \{1,2\}               | 0              | \(w\)          |
| \{1,3\}               | 1              | \(-m_\tau + n_\tau + 1\) |
| \{1,4\}               | 2              | 2               |
| \{2,3\}               | 2              | 2               |
| \{2,4\}               | 3              | \(-n_\tau + m_\tau + 3\) |
| \{3,4\}               | 4              | \(-w + 4\)     |

We assume that there is \(\tilde{w} \in W^p\) such that \(p(\tilde{w}) = 2d\) and \(\ell(\tilde{w}) \leq d\). Since \(p(\{1,2\}) + p(\{3,4\}) = p(\{1,3\}) + p(\{2,4\})\) and \(\ell(\{1,2\}) + \ell(\{3,4\}) = \ell(\{1,3\}) + \ell(\{2,4\})\), we can assume that there is no \(\bar{\tau} \in \text{Hom}(F^+, E)\) such that \(w_\tau\) corresponds to \(\{3,4\}\) or there is no \(\bar{\tau} \in \text{Hom}(F^+, E)\) such that \(w_\tau\) corresponds to \(\{1,2\}\). The second case is impossible because \(\sum_{\bar{\tau} \in J_{F^+}} \ell(w_\bar{\tau}) \leq d\) and \(-m_\tau + n_\tau + 1 \neq 2\).

The inequality \(\sum_{\bar{\tau} \in \text{Hom}(F^+, E)} \ell(\tilde{w}_\bar{\tau}) \leq d\) implies that the cardinality of \(\{\bar{\tau} \in J_{F^+} | \tilde{w}_\bar{\tau} \leftrightarrow \{1,2\}\}\) is larger than the cardinality of \(\{\bar{\tau} \in J_{F^+} | \tilde{w}_\bar{\tau} \leftrightarrow \{1,2\}, \{1,3\}\}\).

But \(w + 2, w - n_\tau + m_\tau + 3, w - w + 4\) all is less than or equal to 4, then \(p(\tilde{w}) = \sum_{\bar{\tau} \in \text{Hom}(F^+, E)} p(\tilde{w}_\bar{\tau}) \leq 2d\) and we have equality only when there is not \(\bar{\tau} \in \text{Hom}(F^+, E)\) such that it corresponds to \(\{1,2\}\), but it is impossible as above. Therefore \(2d\) is not Fontaine-Laffaille weight of \(H^{i+1}_c(X_\bar{K}, V_\lambda/\varpi)\) and also it is not Fontaine-Laffaille weight of \(\chi\).

\(\square\)

Note that for every finite generated torsion \(T^S\)-module, resp, \(\hat{T}^S\)-module \(V\), we have \(V[\bar{m}] = 0\), resp, \(V[\bar{m}] = 0\) if and only if \(V_\bar{m} = 0\), resp, \(V_\bar{m} = 0\).

**Lemma 8.8.** If \(i < d\), then \(d \cdot w\) is not a Fontaine-Laffaille weight of \(H^{i+1}_c(X_\bar{K}, V_\lambda/\varpi)_{\bar{m}}\).

**Proof.** The integer \(d \cdot w\) is equal to \(p(\text{Id})\) where \(\text{Id}\) is the trivial element of \(W^p\). By Corollary 3.13 the multiplicity of the weight \(d \cdot w\) in \(H^{i+1}_c(X_\bar{K}, V_\lambda/\varpi)_{\bar{m}}\) is equal to the \(\kappa_p\)-dimension of \(H^{i+1}(\tilde{X}^\text{tor}_{K,p}, W^\text{sub}_{\lambda,p})_{\bar{m}}\), where \(\tilde{X}^\text{tor}_{K,p}\) is a smooth toroidal compactification of \(X^\text{tor}_{K,p} \times \kappa_p\) and \(W^\text{sub(can)}_{\lambda,p}\) is the sub-canonical (canonical) extension of \(W_{\lambda,p} : = \mathcal{E}_G(V_{\lambda,\bar{m}})\).

By Serre duality as Proposition 3.15

\[
H^{i+1}(\tilde{X}^\text{tor}_{K,p}, W^\text{sub}_{\lambda,p})_{\bar{m}} \cong H^{4d-i-1}(\tilde{X}^\text{tor}_{K,p}, W^\text{can}_{2\rho_{nc} - w_{0,G} \lambda,\bar{m}})_{\bar{m}},
\]

as \(\kappa_p\) vector space, where \(2\rho_{nc} := ((2,2),(-2,-2))_{\bar{I}}\) and where \(w_{0,G} \in W_G\) is the longest length element. Then \(-2\rho_{nc} - w_{0,G} \lambda = ((m_\tau - 2, n_\tau - 2),(-n_\tau + \ldots

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2, −mτ + 2)ζ ∈ I. Since the multiplicity of d(w − 4) as Fontaine-Laffaille weight in $H^{d-i-1}(\tilde{X}_K, \mathcal{V}_{\bar{\mu}/\varpi})_{\bar{m}^\vee}$ is equal to $\kappa_p$-dimension of $H^{d-i-1}(\tilde{X}_{K,\varphi}, \mathcal{W}^{\text{can}}_{-2pnc-uw_0,\mathcal{G}\lambda,p})_{\bar{m}^\vee}$ where $\bar{\mu} := (-n_\tau + 2, -m_\tau + 2, m_\tau - 2, n_\tau - 2)\bar{\tau} \in \bar{I}$, because $|\bar{\mu}|_{\text{comp}} = |\bar{\lambda}|_{\text{comp}} + 4d < p$. Then by Theorem 5.4, we have $H^{d-i-1}(\tilde{X}_{K,\varphi}, \mathcal{V}_{\bar{\mu}/\varpi})_{\bar{m}^\vee} = 0$ and therefore by corollary 3.13 we have $H^{d-i-1}(\tilde{X}_{\text{tor} K,\varphi}, \mathcal{W}^{\text{can}}_{-2pnc-uw_0,\mathcal{G}\lambda,p})_{\bar{m}^\vee} = 0$.

Let us define another condition $(p\text{-small})$ $p \geq 2d^2(d + 1)(−2w + 4)$ and $w$ is even.

**Theorem 8.9.** Let π be a regular, cuspidal and cohomological automorphic representation of $GL_2(\mathbb{A}_F)$ with level $K$ and weight $\lambda$ such that $K_p$ is hyperspecial. Assume π is unramified outside of $S$ and $\lambda$ satisfies conditions $(p\text{-small})$ and assumption of Corollary 8.3. Let $m \subset T^S$ be the associated maximal ideal of Hecke algebra. Assume that $(LI \rho_m)$ holds. then $H^i(X_K, \mathcal{V}_\lambda/\varpi)_m$ is zero for all $i / \in \lbrack d, 3d - 1 \rbrack$.

**Proof.** First of all by Proposition 8.2, it suffices to prove vanishing of the cohomology for only $i < d$ and the Corollary 8.3 implies that we can assume that $\lambda$ satisfies the condition (PR).

The $(p\text{-small})$ condition guaranties that there is a twist of $\pi$ with pure weight $w'$ such that satisfies $p \geq d^2(d + 1)(−2w' + 4)$ and $\gcd(w' + 1, p - 1) = 1$. Therefore by theorem 8.1, we can assume that $p \geq d^2(d + 1)(−2w + 4)$ and $\gcd(w + 1, p - 1) = 1$. So lemmas 8.5, 8.7 and 8.8 implies that $H^i_{\mathfrak{p}}(\tilde{X}_K, \mathcal{V}_\lambda/\varpi)_\mathfrak{m} = 0$ for all $i < d$. Hence theorem 5.2 implies that $H^i(X_K, \mathcal{V}_\lambda/\varpi)_m = 0$ for all $i < d$. 

**Remark 8.10.** As Remark 8.6 we can improve the bound to

$$4d \cdot \left( \sum_{i=1}^{d} \gcd(p - 1, i) \right)(2|w| + 4)$$

**Corollary 8.11.** Under the assumptions of Theorem 8.9, we have $H^d(X_K, \mathcal{V}_\lambda)_m$ is torsion-free.

**Proof.** It follows from the following long exact sequence:

$$\cdots \to H^{i-1}(X_K, \mathcal{V}_\lambda/\varpi)_m \to H^i(X_K, \mathcal{V}_\lambda)_m \xrightarrow{\times \varpi} H^i(X_K, \mathcal{V}_\lambda)_m \to H^i(X_K, \mathcal{V}_\lambda/\varpi)_m.$$ 

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Recall that Calegari and Geraghty introduced Conjectures A (see [CG18, Section 5.3]) and B (see [CG18, Section 9.1]) about the existence of Galois representations over localized Hecke algebras $T_S$ and $T_Q^S$ with characteristic polynomials given by Hecke polynomials at primes outside $p$ and the level, and satisfying local global compatibilities, which we denote by (LGC), at primes in the level or above $p$. Furthermore they assume in Conjecture B that $H^i(X_K, V_\lambda/\mathcal{V})_m = 0$, for $i \notin [q_0, q_0 + \ell_0]$. This is our Main Theorem.

If $p$ splits completely in $F$, the existence of the Galois representations over $T_S$ and $T_Q^S$ with the expected characteristic polynomial at primes outside the level and $p$ has been established in [CGH+20, Theorem 6.1.4]. Moreover (LGC) holds at primes in $Q$ by [CG18, Lemma 9.6] since $n = 2$. It holds at $p$ in the Fontaine-Laffaille case up to an nilpotent ideal by [ACC+18, Theorem 4.5.1].

Let us assume that the level group $K$ is of Iwahori type $K = K_0(n)$ and that $\bar{\rho}_m$ is minimal at primes dividing $n$ (see [CG18, Definition 3.1 (4)]). Recall that (LGC) holds at primes dividing $n$ modulo a nilpotent ideal by [ACC+18, Theorem 3.1.1].

**Corollary 8.12.** Under the assumptions of Theorem 8.9, if we assume $\bar{\rho}_m$ minimal and that (LGC) holds at primes dividing $n$, then $H^{3d-1}(X_K, V_\lambda)_m$ is free of rank one over $T_S$ and $H^d(X_K, V_\lambda)_m$ is isomorphic to $\text{Hom}(T_S, \mathcal{O})$ as $T_S$-module.

**Proof.** By the assumptions and our Main Theorem, Conjecture B holds. Hence the result follows from [CG18, Theorems 6.3 and 6.4]. The rank is one because it is so if we localize the cohomology at each automorphic representation occuring in $T_S$ by calculations of Borel-Wallach, Chapter 3. The second part follows by Poincaré duality applied to $H^{3d-1}(X_K, V_\lambda)_m^{\vee}$. 

**References**

[ACC+18] Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne, Potential automorphy over CM fields.

[AN20] Patrick B. Allen and James Newton, Monodromy for some rank two Galois representations over CM fields, Documenta Mathematica 25 (2020), 2487-2506.

[BW80] Armand Borel and Nolan R. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups, Annals of Mathematics Studies, vol. 94, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.
[CG18] F. Calegari, D. Geraghty, Modularity Lifting beyond the Taylor-Wiles Method, Inventiones mathematicae 211(3), 1-137

[CGH+20] Ana Caraiani, Daniel R. Gulotta, Chi-Yun Hsu, Christian Johansson, Lucia Mocz, Emanuel Reinecke, and Sheng-Chi Shih, Shimura varieties at level $\Gamma(p^{\infty})$ and Galois representations, Compositio Math. 156 (2020), no. 6, 1152–1230.

[CS19] Ana Caraiani and Peter Scholze, On the generic part of the cohomology of noncompact unitary Shimura varieties, arXiv:1909.01898 [math.NT]

[D05] Mladen Dimitrov, Galois representations modulo $p$ and cohomology of Hilbert modular varieties, Annales scientifiques de l’École Normale Supérieure Ser. 4, 38 (2005), no. 4, 505-551.

[EG15] Matthew Emerton and Toby Gee, $p$-adic Hodge theoretic properties of étale cohomology with mod $p$ coefficients, and the cohomology of Shimura varieties, Algebra and Number Theory 9 (2015), no. 5, 1035–1088.

[F90] Gerd Faltings, Crystalline cohomology and $p$-adic Galois representations, in Algebraic Analysis, ed. J.-I. Igusa, Proc JAMI inaugural Conference, the Johns Hopkins Univ. Press, 1990.

[FL82] Jean-Marc Fontaine, Guy Laffaille, Construction de représentations $p$-adiques, Ann. Sci. E.N.S., 15 (1982), 547608.

[Har90] Michael Harris, Automorphic forms of $\overline{\partial}$-cohomology type as coherent cohomology classes, J. Differential Geom. 32 (1990), no. 1, 1–63

[HLTT16] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne, On the rigid cohomology of certain Shimura varieties, Res. Math. Sci. 3 (2016), 3:37.

[HSBT10] Michael Harris, Nick Shepherd-Barron, and Richard Taylor, A family of Calabi-Yau varieties and potential automorphy, Ann. of Math. (2) 171 (2010), no. 2, 779–813

[KO68] Nicholas M. Katz, Tadao Oda, On the differentiation of De Rham cohomology classes with respect to parameters, J. Math. Kyoto Univ. 8 (1968) 199-213.

[Kos61] Bertram Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. (2) 74 (1961), 329-387.
[LAN13] Kai-Wen Lan, *Arithmetic compactification of PEL-type Shimura varieties*, London Mathematical Society Monographs, vol. 36, Princeton University Press, Princeton, 2013

[LAN18] Kai-Wen Lan, Compactifications of PEL-type Shimura varieties and Kuga families with ordinary loci, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2018).

[LP18] K.-W. Lan and P. Polo, Dual BGG complexes for automorphic bundles, Math. Res. Lett. 25 (2018), no. 1, pp. 85-141.

[LS12] Kai-Wen Lan, Junecue Suh, Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties, Duke Math. J. 161 (2012), no. 6, 1113-1170.

[LS13] Kai-Wen Lan, Junecue Suh, Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties, Adv. Math. 242 (2013), pp. 228-286.

[N18] Jan Nekovář, Eichler-Shimura relations and semisimplicity of étale cohomology of quaternionic Shimura varieties, Ann. Sci. École Norm. Sup. (4) 51 (2018), 1179-1252.

[NT16] James Newton and Jack Thorne, Torsion Galois representations over CM fields and Hecke algebras in the derived category, Forum Math. Sigma 4 (2016), 88.

[MT02] Abdellah Mokrane and Jacques Tilouine. “Cohomology of Siegel varieties with p-adic integral coefficients and applications”, in Cohomology of Siegel varieties, Astérisque 280 (2002), pp. 1–95.

[P92] Richard Pink, On l-adic sheaves on Shimura varieties and their higher images in the Baily-Borel compactification, Math. Ann. 292 (1992), 197-240.

[PT02] Patrick Polo, Jacques Tilouine, Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_p$ for representations with $p$-small weights Astérisque, tome 280 (2002), p. 97-135.

[R76] Kenneth A. Ribet, Galois Action on Division Points of Abelian Varieties with Real Multiplications, American Journal of Mathematics, Vol. 98, No. 3 (1976), 751-804.

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[Sch15] Peter Scholze, On torsion in the cohomology of locally symmetric varieties, Ann. of Math. (2) 182 (2015), no. 3, 945–1066

[S72] Jean-Pierre Serre, Propriétés galoisiennes des points d’ordre fini des courbes elliptiques, Invent. Math. 15 (1972) 259-331.

[Wed00] Torsten Wedhorn. Congruence relations on some Shimura varieties. J. Reine Angew. Math., 524:43-71, 2000

[Yos94] Hiroyuki Yoshida, On the zeta functions of Shimura varieties and periods of Hilbert modular forms, Duke Math. J. 74 (1994) 121-191.

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