QCD Effective Couplings in Minkowskian and Euclidean Domains

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Abstract. We argue for essential upgrading of the defining equations (9.5) and (9.6) in Section 9.2. "The QCD coupling ... " of PDG review and their use for data analysis in the light of recent development of the QCD theory. Our claim is twofold. First, instead of universal expression (9.5) for \( \bar{\alpha}_s \), one should use various ghost-free couplings \( \alpha_E(Q^2) \), \( \alpha_M(s) \) ... specific for a given physical representation. Second, instead of power expansion (9.6) for observable, we recommend to use nonpower functional ones over particular functional sets \( \{ \alpha_k(Q^2) \} \), \( \{ \alpha_k(s) \} \) ... related by suitable integral transformations. We remind that use of this modified prescription results in a better correspondence of reanalyzed low energy data with the high energy ones.

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1. PREAMBLE

The main message consists of two statements:

A: Instead of common effective QCD coupling \( \bar{\alpha}_s \), (with its ghost defect) as, e.g., it is implicitly mentioned by eq.(9.5) of PDG review [1], one should use (at least) two different ghost-free forms for QCD effective coupling \( \alpha_E(Q^2) \) in the Euclidean and \( \alpha_M(s) \) in the Minkowskian (and, possibly, some others) pictures;

B: The RG-invariant perturbative expansions for observables, like eq.(9.6) in PDG,

\[
O(\xi) = o_1 \bar{\alpha}_s(\xi) + o_2 \bar{\alpha}_s^2(\xi) + o_3 \bar{\alpha}_s^3(\xi) + ..., 
\]

– in powers of the same \( \bar{\alpha}_s \) in different pictures, Euclidean (\( \xi = Q^2 \)) or Minkowskian (\( \xi = s \)) are neither based theoretically, nor adequate practically to low-energy QCD. Instead, one should use diverse nonpower functional expansions

\[
d(Q^2) = \sum_{i \geq 1} d_i \alpha_i(Q^2), \quad r(s) = \sum_{i \geq 1} d_i \mathcal{A}_i(s),
\]

(each particular one for a given representation) over nonpower sets of ghost-free functions like \( \{ \alpha_k(Q^2) \} \) in Euclidean and \( \{ \alpha_k(s) \} \) in Minkowskian, mutually related by suitable integral transformations.

Below we demonstrate that a reasonable revising of the above mentioned PDG Eqs. essentially modifies the results of the analysis of some low energy data like GLM and Bjorken sum-rules, \( \tau \)-lepton and Ypsilon decays and \( e^+ e^- \) inclusive cross-sections (Sections 9.3., 9.4. and 9.6 in PDG).
As a result, new overall fit for Euclidean data in terms of $\alpha_E(Q^2)$ and Minkowskian data in $\alpha_M(s)$ results in (see our recent review [2]) $\bar{\alpha}_s(M^2_Z) = 0.123$ with an essentially smaller $\chi^2$ than the commonly accepted one.

2. THE APT ESSENCE AND STRUCTURE

2a. Minkowskian And Euclidean Couplings $\alpha_M$ and $\alpha_E$

RG defined invariant coupling $\bar{\alpha}(Q^2)$ is a real function of space-like argument $Q^2$. It effectively sums up UV logs into an expression with ghost. In the 1-loop QCD case

$$\alpha_s^{(1)}(Q^2) = \frac{\alpha_s}{1 + \beta_0 \alpha_s \ln(Q^2/\mu^2)} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}$$

Instead, in the APT scheme[2], we deal with differing ghost-free couplings

\begin{align*}
\text{Minkowskian: } & \alpha_M^{(1)}(s) = \frac{1}{\beta_0} \arccos \frac{L}{\sqrt{L^2 + \pi^2}} \bigg|_{L>0} = \frac{1}{\beta_0} \arctan \frac{\pi}{L}; \quad \text{and} \\
\text{Euclidean: } & \alpha_E^{(1)}(Q^2) = \frac{1}{\beta_0} \left[ 1 - \frac{\Lambda^2}{Q^2 - \Lambda^2} \right]; \quad \ell = \ln \frac{Q^2}{\Lambda^2}, \quad L = \ln \frac{s}{\Lambda^2}.
\end{align*}

\hspace{1cm}

\text{FIGURE 1. Comparison of usual QCD coupling $\alpha_s$ with Euclidean $\alpha_{an} = \alpha_E$ and Minkowskian one $\bar{\alpha} = \alpha_M$ in a few GeV region.}

On Fig.1 one can see the comparison of $\bar{\alpha}_s$ with $\alpha_E$ and $\alpha_M$ in the 1-2 GeV region. Transition to the “$s$ picture” performed first by contour integration by Radyushkin[3], Krasnikov and Pivovarov [4], (see also [5])

\hspace{1cm}

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\begin{itemize}
\item In this figure taken from our previous papers, a bit different notation $\alpha_{an} = \alpha_E$, $\bar{\alpha} = \alpha_M$ is used. Here $\bar{\alpha}_{appr} = \bar{\alpha}_s - \frac{\pi^2 \beta_0^2}{2\alpha_s^3}$. All the curves are given in the 2-loop approximation for $\Lambda = 350$ GeV.
\end{itemize}
\[
\bar{\alpha}_s(Q^2) \rightarrow \frac{i}{2\pi} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} \bar{\alpha}_s(-z) = \bar{\alpha}_M(s) \equiv \text{Re} \left[ \bar{\alpha}_s \right](s) \quad (3)
\]
results in a ghost-free expression with \(\pi^2\) terms summed.

Reverse transformation \(\mathbb{D} = [\mathbb{R}]^{-1} \quad [6, 7]\) yields a ghost-free expression in the \(Q^2\) picture with subtracted singularity; see below eqs.(7) and (8).

2b. Minkowskian: \(\pi^2\) Summation

Summation of \(\pi^2\)-terms by contour integration (3) for the 1-loop case results in
\[
\bar{\alpha}^{(1)}_s(Q^2) = \frac{1}{\beta_0 L} \to \alpha^{(1)}_M(s) = \frac{1}{\pi \beta_0} \arccos \frac{L}{\sqrt{L^2 + \pi^2}} = \mathcal{A}^{(1)}_1(s),
\]
\[
\left[ \mathcal{A}^{(1)}_1(s) \right]_{L>0} = \frac{1}{\pi \beta_0} \arctan \frac{\pi}{L}; \quad L = \ln \frac{s}{\Lambda^2}. \quad (5)
\]
This expression was first obtained by Radyushkin[3] in the form (5). Later on, Jones and Solovtsov [8] considered the region \(Q^2 \leq \Lambda^2\) and proposed treating expression (4) as a ghost-free Minkowskian effective coupling.

At the same time, the procedure (3) transforms square and cube of \(\bar{\alpha}^{(1)}_s\) into ghost-free forms[4]
\[
\mathcal{A}^{(2)}_1 = \frac{1}{\beta_0^2 [ \pi^2 + L^2 ]}, \quad \mathcal{A}^{(3)}_1 = \frac{L}{\beta_0^3 [ \pi^2 + L^2 ]^2},
\]
which are not powers of \(\alpha^{(1)}_M(s)\). They are rather connected with (4) by the iterative differential relation
\[
\mathcal{A}_{k+1}(s) = -\frac{1}{k \beta_0} \frac{d \mathcal{A}_k(s)}{d \ln s}. \quad (6)
\]

2c. Euclidean: Källen-Lehmann Analyticity

APT uses imperative of the \(Q^2\) analyticity[9] in the form of the Källen–Lehmann spectral representation 3. Being applied to the QCD one-loop case, it gives
\[
\bar{\alpha}^{(1)}_s = \frac{1}{\beta_0 \ell} \Rightarrow \mathcal{A} \left[ \bar{\alpha}^{(1)}_s \right] = \alpha^{(1)}_E(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ell} - \frac{\Lambda^2}{Q^2 - \Lambda^2} \right] = \mathcal{A}^{(1)}_1(Q^2). \quad (7)
\]
For coupling \(\bar{\alpha}^{(1)}_s\) squared
\[
\mathcal{A} \left[ \frac{1}{\ell^2} \right] = \frac{1}{\ln^2(Q^2/\Lambda^2)} + \frac{Q^2 \Lambda^2}{(Q^2 - \Lambda^2)^2} = \beta_0^2 \mathcal{A}^{(2)}_2(Q^2) \neq \left( \beta_0 \mathcal{A}^{(1)}_1(Q^2) \right)^2. \quad (8)
\]

\(^2\) For its explicit form see below eq.(10)
\(^3\) In the form of the first of Eqs.(9) For detail see Refs.[6, 7].
The Minkowskian and Euclidean ghost-free functions are related [10, 2] by $D$ and $R$ transformations:

$$A_k(Q^2) = D[A_k]$$

$$A_k(s) = R[A_k]$$

Accordingly,

$$D[R(s)] = \sum_k d_k A_k(s) \Rightarrow D(Q^2) = \sum_k d_k A_k(Q^2).$$

2d. Sketch Of The Global APT Algorithm

The most convenient form of the APT formalism uses a spectral density $\rho(\sigma) = Im\bar{\alpha}_s(-\sigma)$ taken from the perturbative input

$$\mathcal{A}_k = \frac{1}{\pi} \int_0^\infty \frac{\rho_k(\sigma) d\sigma}{\sigma + Q^2}, \quad \mathcal{A}_k = \frac{1}{\pi} \int_{s}^{\infty} \frac{d\sigma}{\sigma} \rho_k(\sigma). \quad (9)$$

In the 1-loop case

$$\rho^{(1)}_1 = \frac{1}{\beta_0 [L^2 + \pi^2]}; \quad L_\sigma = \ln \frac{\sigma}{\Lambda^2}; \quad \rho^{(1)}_{k+1}(\sigma) = -\frac{1}{k \beta_0} \frac{d \rho^{(1)}_k(\sigma)}{d L_\sigma}$$

These expressions were generalized for a higher-loop case and for real QCD with transitions across quark thresholds. This global APT was successively used for fitting of various data, e.g. for describing mass spectrum of light mesons [11] and for description of pion formfactor [12]. Logic of the APT scheme is displayed$^4$ in Fig.2.

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$^4$ Here, the distance picture with functions $\alpha_0$ and $\{\mathcal{R}_k\}$ is mentioned. It is related with $Q^2$ picture by Fourier transformation $\mathcal{R}$. For detail, see Ref.[13].
3. THE APT RESUME

3a. Non-Power Ghost-Free Sets \( \{\mathcal{A}_k\}, \{\mathcal{B}_k\} \)

By construction, all APT expansion functions \( \mathcal{A}_k \) and \( \mathcal{B}_k \) (for 2-loop etc., as well) are free of unphysical singularities and at weak-coupling limit tend to powers \( \bar{\alpha}_k^\epsilon \) of common QCD coupling. On Fig.3 we demonstrate the behavior of the first three functions.

Their more detailed properties can be described as follows:

**I.** First ones, new couplings, \( \alpha_E, \alpha_M \):
- ♦ are monotonic and IR finite, \( \alpha_E(0) = \alpha_M(0) = 1/\beta_0 \simeq 1.4 \)
- ♦ in the UV limit \( \sim 1/\ln x \sim \bar{\alpha}_s(x) \).

**II.** All the other functions \( (k \geq 2) \):
- ♤ start from zero \( \mathcal{A}_k(0), \mathcal{B}_k(0) = 0; \)
- ♤ in the UV limit \( \sim 1/(\ln x)^k \sim \bar{\alpha}_s^k(x) \).
- ♦ 2nd ones, \( \mathcal{A}_2, \mathcal{B}_2 \) obey max at \( \sim \Lambda^2 \).
- ♦ Higher ones, \( \mathcal{A}_{k \geq 3}; \mathcal{B}_{k \geq 3} \) oscillate near \( \Lambda^2 \) with \( k - 1 \) zeroes.

![Figure 3](image)

**FIGURE 3.** a – Space-like and time-like APT couplings for 1-, 2- and 3-loop case in a few GeV domain. b – “Distorted mirror symmetry” for global expansion functions. All the solid curves here correspond to exact two–loop solutions \( \mathcal{A}_{2,3} \) and \( \mathcal{B}_{2,3} \), expressed in terms of the Lambert function. They are compared with powers of APT couplings \( \alpha_E \) and \( \alpha_M \) depicted by dotted lines.

The last property results [12, 14] in the reduced renormalization-scheme and higher loop sensitivity and better convergence [15] in the low-energy region, see below Sect.3b.

3b. Non-Power Expansions: Quick Loop Convergence

New effective couplings are related by integral transformations (3) and

\[
\alpha_E(Q^2) = Q^2 \int_0^\infty \frac{\alpha_M(s) ds}{(s+Q^2)^2} \equiv D[\alpha_M](Q^2). \tag{10}
\]
The same transformations induce a nonpower structure
\[ \mathcal{A}_k(s) \rightarrow q_k(Q^2) \equiv \mathbb{D}[\mathcal{A}_k](Q^2) \]
of expansion functions for observables.

Due to this, instead of the “PDG–recommended” universal power-in-\(\bar{\alpha}_s\) expansion,
\[ d_{pu}(Q^2/s) = d_1 \bar{\alpha}_s(Q^2/s) + d_2 \bar{\alpha}_s^2 + d_3 \bar{\alpha}_s^3 \] (11)
one should use non-power expansions
\[ d_{an}(Q^2) = d_1 \alpha_E(Q^2) + d_2 \alpha_2(Q^2) + d_3 \alpha_3(Q^2) + \ldots, \] (12)
\[ r_\pi(s) = d_1 \alpha_M(s) + d_2 \alpha_2(s) + d_3 \alpha_3(s) + \ldots. \] (13)

The numerical effect of this change is demonstrated in the Table 1. There, relative contributions in per cent for usual, PT 3-loop power-in-\(\alpha_s\) expansions (11) are confronted with the APT ones (12) and (13). Besides, they are compared with the experimental error given in the last column in the same (i.e., in \(\alpha_s/\pi\)) units.

| Process       | Energy | PT (11) | APT (12)/(13) | Exp Errors |
|---------------|--------|---------|---------------|------------|
| Bjorken SR    | 1.6 GeV| 55      | 26            | 19         | 80         | 19 | 1 | ±14 |
| GLS SRule     | 1.7 GeV| 65      | 24            | 11         | 75         | 21 | 4 | ±20 |
| Incl \(\tau\)-decay | 1.8 GeV| 55      | 29            | 16         | 88         | 11 | 1 | ±8  |
| \(e^+e^-\rightarrow\text{hadr}\) | 10 GeV| 96      | 8             | -4         | 92         | 7  | .5 | ±27 |
| \(Z_0\rightarrow\text{hadr.}\) | 91 GeV| 99      | 3.7           | -2.3       | 97         | 3.5 | -4 | ±4  |

It follows that APT expansion converges much better than common PT one. Besides, the APT 3-loop term contribution is much less than data errors. Effective suppression of higher-loop terms yields also a reduced scheme [12] and loop dependence.

All these nice features of APT are connected with due account for nonanalyticity with respect to usual expansion parameter, the coupling constant at \(\alpha = 0\).

### 3c. The QFT Nonanalyticity In Coupling

Here, we shortly remind a few general arguments on this non-analyticity.

- **General Dyson [16] argument in QED.** Transition \(\alpha \rightarrow -\alpha\) corresponds to \(e \rightarrow ie\); it destroys Hermiticity of Lagrangian and the S-matrix unitarity. Hence, the origin \(\alpha = 0\) in the complex \(\alpha\) plane can not be a regular point.

- **RG + \(Q^2\)-analyticity arguments.** Combining the \(Q^2\) analyticity for a photon propagator in QED with RG invariance, one could define [17] the type of essential singularity at \(\alpha = 0\) as \(\sim e^{-1/\alpha}\).
• Functional integral reasoning. By the method of functional-integral steepest descent for propagators, it was shown\cite{18} that expansion coefficients $c_n \alpha^n$ at $n \gg 1$ behave like $c_n \sim n! n^m$ which corresponds\cite{19} to the same singularity $\sim e^{-1/\alpha}$.

3d. Analytic approximations for 2-, 3-loop $\mathcal{A}_k$ and $\mathcal{A}_k$

Analytic expressions for 2-,3-loop APT Minkowskian $\mathcal{A}_k$ and Euclidean $\mathcal{A}_k$ couplings involving a special Lambert function $W_{-1}$ are rather cumbersome. Due to this, several analytic approximations for them were devised\cite{20, 15}. In addition, we can mention\cite{21} very simple "1-loop-like" model expressions with "two-loop effective logs" $\ell_2, L_2$

$$\mathcal{A}_1(s) = \frac{1}{\pi \beta_0} \arctan \frac{\pi}{L_2}, \quad \mathcal{A}_2 = \frac{1}{\beta_0^2 [L_2^2 + \pi^2]} \ldots; \quad L_2 = L + b \ln L, \quad b = \frac{\beta_1}{\beta_0},$$

$$\mathcal{A}_1^{\text{appr}} = \frac{1}{\beta_0} \left( \frac{1}{\ell_2} - \frac{1}{\exp(\ell_2) - 1} \right) \ldots; \quad \ell_2 = \ell + b \ln \ell,$$

and modified parameter $\Lambda \rightarrow \Lambda_+ = f(\Lambda)$. Such analytic approximations, typically, could provide us with accuracy at the level of few % quite adequate to practical need.

4. CONCLUSION

1. Numerous non-perturbative data (lattice simulations, Schwinger-Dyson eqs solution) reveal the ghost-free $\bar{\alpha}_s$ behavior in low energy region with finite $\bar{\alpha}_s(0)$ value.

2. The "representation invariance" implies that functional expansions – even in powers of some non-singular $\bar{\alpha}(Q^2/s)$ – are not natural and should be changed for non-power perturbative-inspired expansions; this is essential in a few GeV region.

3. Hence, in this region:
   * the notion of a single universal effective charge $\bar{\alpha}_s$ is not adequate,
   ** to correlate data, one needs two effective couplings $\bar{\alpha}_E(Q^2)$ and $\bar{\alpha}_M(s)$.

4. Instead of expansion (9.6) of PDG, one should use APT expansions eqs.(12) and (13) over sets of nonpower functions $\{A_k(Q^2)\}$ and $\{\mathcal{A}_k(s)\}$.

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