LEAVITT PATH ALGEBRAS WITH BOUNDED INDEX OF NILPOTENCE

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Abstract. In this paper we completely describe graphically Leavitt path algebras with bounded index of nilpotence. We show that the Leavitt path algebra $L_K(E)$ has index of nilpotence at most $n$ if and only if no cycle in the graph $E$ has an exit and there is a fixed positive integer $n$ such that the number of distinct paths that end at any given vertex $v$ (including $v$, but not including the entire cycle $c$ in case $v$ lies on $c$) is less than or equal to $n$. Interestingly, the Leavitt path algebras having bounded index of nilpotence turn out to be precisely those that satisfy a polynomial identity. Furthermore, Leavitt path algebras with bounded index of nilpotence are shown to be directly-finite and to be $\mathbb{Z}$-graded $\Sigma$-V rings. As an application of our results, we answer an open question raised in [10] whether an exchange $\Sigma$-V ring has bounded index of nilpotence.

1. Introduction

The objective of this paper is to characterize Leavitt path algebras with bounded index of nilpotence. A ring $R$ is said to have bounded index of nilpotence if there is a positive integer $n$ such that $x^n = 0$ for all nilpotent elements $x$ in $R$. If $n$ is the least such integer then $R$ is said to have index of nilpotence $n$. We show that the Leavitt path algebra $L := L_K(E)$ of a directed graph $E$ over a field $K$ has index of nilpotence at most $n$ if and only if no cycle in the graph $E$ has an exit and there is a fixed positive integer $n$ such that the number of distinct paths that end at any given vertex $v$ (including $v$, but not including the cycle $c$ in case $v$ lies on $c$) is less than or equal to $n$. In this case, $L$ becomes a subdirect product of matrix rings $M_t(K)$ or $M_t(K[x,x^{-1}])$ of finite order $t \leq n$. Examples are constructed showing that $L$ need not decompose as a direct sum of these matrix rings $M_t(K)$ or $M_t(K[x,x^{-1}])$, though the decomposition is possible when $E$ is row-finite. We show that a Leavitt path algebra $L$ with bounded index of nilpotence is always directly-finite and that $L$ is a $\mathbb{Z}$-graded $\Sigma$-V ring, that is, each graded simple left/right $L$-module is graded $\Sigma$-injective. Examples show that the converse of these statements do not hold. Interestingly, it turns out that the graphical conditions on $E$ that ensure $L$ has a bounded index of nilpotence are exactly the same graphical conditions on $E$ that were shown in [4] to imply that $L$ satisfies a polynomial identity. When $E$ is a finite graph, these graphical conditions also imply that $L$ has GK-dimension $\leq 1$. Such

2010 Mathematics Subject Classification. 16D50, 16D60.

Key words and phrases. Leavitt path algebras, bounded index of nilpotence, direct-finiteness, simple modules, injective modules.

The work of the second author is partially supported by a grant from Simons Foundation (grant number 426367).
statements illustrate a unique phenomenon in the study of Leavitt path algebras where a single graph property of $E$ often implies different ring-theoretic properties for $L$ and these ring-theoretic properties are usually independent of each other for general rings (see [15] for several illustrations of this phenomenon of Leavitt path algebras). This feature of Leavitt path algebras makes them really useful tools in constructing examples of rings of various desired ring-theoretic properties. Finally, as an application of our results, we answer a question raised in [10], whether an exchange $\Sigma$-$V$ ring has bounded index of nilpotence.

For the general notation, terminology and results in Leavitt path algebras, we refer the reader to [1]. We give below an outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges.

A vertex $v$ is called a sink if it emits no edges and a vertex $v$ is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is a vertex which emits infinitely many edges. For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A path $\mu$ of length $n > 0$ is a finite sequence of edges $\mu = e_1e_2\cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n - 1$. In this case $\mu^* = e_n^*\cdots e_2^*e_1^*$ is the corresponding ghost path. A vertex is considered a path of length 0.

A path $\mu = e_1\cdots e_n$ in $E$ is closed if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. A closed path $\mu$ as above is called simple provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \cdots, n$. The closed path $\mu$ is called a cycle if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. An exit for a path $\mu = e_1\cdots e_n$ is an edge $e$ such that $s(e) = s(e_1)$ for some $i$ and $e \neq e_i$.

An infinite rational path $p$ is an infinite path of the form $p = x_1x_2\cdots x_n\cdots$ where there is an $m \geq 1$ such that $x_k = g$, a fixed closed path for all $k \geq m$ and that $x_k$ is an edge $e_k$ if $k < m$. Thus $p$ will be of the form $p = e_1e_2\cdots e_m g g g g g \cdots$ where $g$ is a closed path and the $e_i$ are edges. An infinite path which is not rational is called an irrational path.

A graph $E$ is said to satisfy Condition $(K)$, if every vertex $v$ on a closed path $c$ is also the base of a another closed path $c'$ different from $c$. A graph $E$ is said to satisfy Condition $(L)$ if every cycle has an exit.

If there is a path from vertex $u$ to a vertex $v$, we write $u \geq v$. A subset $D$ of vertices is said to be downward directed if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset $H$ of $E^0$ is called hereditary if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is saturated if, for any regular vertex $v$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

Given an arbitrary graph $E$ and a field $K$, the Leavitt path algebra $L_K(E)$ is defined to be the $K$-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The “CK-1 relations”) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$. 

(4) (The “CK-2 relations”) For every regular vertex \( v \in E^0 \),
\[
v = \sum_{e \in E^1, s(e) = v} ee^*.
\]

Every Leavitt path algebra \( L_K(E) \) is a \( \mathbb{Z} \)-graded algebra, namely, \( L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n \)
induced by defining, for all \( v \in E^0 \) and \( e \in E^1 \), \( \deg(v) = 0 \), \( \deg(e) = 1 \), \( \deg(e^*) = -1 \). Here the \( L_n \) are abelian subgroups satisfying \( L_m L_n \subseteq L_{m+n} \) for all \( m, n \in \mathbb{Z} \). Further, for each \( n \in \mathbb{Z} \), the homogeneous component \( L_n \) is given by
\[
L_n = \{ \sum k_i \alpha_i \beta_i^* \in L : |\alpha_i| - |\beta_i| = n \}.
\]

Elements of \( L_n \) are called homogeneous elements. An ideal \( I \) of \( L_K(E) \) is said to be a graded ideal if \( I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n) \). If \( A, B \) are graded modules over a graded ring \( R \), we write \( A \cong_{gr} B \) if \( A \) and \( B \) are graded isomorphic and we write \( A \oplus_{gr} B \) to denote a graded direct sum. We will also be using the usual grading of a matrix of finite order. For this and for the various properties of graded rings and graded modules, we refer to [6], [8] and [12].

A breaking vertex of a hereditary saturated subset \( H \) is an infinite emitter \( w \in E^0 \setminus H \) with the property that \( 0 < |s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty \). The set of all breaking vertices of \( H \) is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{s(e) = v, r(e) \notin H} ee^* \). Given a hereditary saturated subset \( H \) and a subset \( S \subseteq B_H \), \( (H, S) \) is called an admissible pair. Given an admissible pair \( (H, S) \), the ideal generated by \( H \cup \{ v^H : v \in S \} \) is denoted by \( I(H, S) \). It was shown in [16] that the graded ideals of \( L_K(E) \) are precisely the ideals of the form \( I(H, S) \) for some admissible pair \( (H, S) \). Moreover, \( L_K(E)/I(H, S) \cong L_K(E/\{(H, S)) \). Here \( E/\{(H, S)) \) is a Quotient graph of \( E \) where \( (E/\{(H, S)) = (E^0/H) \cup \{ e' : e \in E^1 \text{ with } r(e) \in B_H \} \) and \( (E/\{(H, S)) = \{ e \in E^1 : s(e) \notin H \} \cup \{ e' : e \in E^1 \text{ with } r(e) \in B_H \} \) and \( r, s \) are extended to \( (E/\{(H, S)) \) by setting \( s(e') = s(e) \) and \( r(e') = r(e) \). It is known (see [14]) that if \( P \) is a prime ideal of \( L \) with \( P \cap E^0 = H \), then \( E^0/H \) is downward directed.

Let \( A \) be an infinite set and \( R \), a ring. Then \( M_A(R) \) denotes the ring of \( A \times A \) matrices in which all except at most finitely many entries are non-zero.

2. Results

In this section, we characterize Leavitt path algebras having bounded index of nilpotence. We begin with the following useful proposition some part of which might be implicit in earlier works on Leavitt path algebras.

**Proposition 2.1.** Let \( E \) be an arbitrary graph and let \( L := L_K(E) \).

(a) Let \( v \) be a vertex in \( E \) which does not lie on a closed path. If, for some \( n \geq 1 \), there are \( n \) distinct paths \( p_1, \ldots, p_n \) in \( E \) that end at \( v \), then the set
\[
T_n = \{ \sum_{i=1}^{n} k_{ij} p_i^j : k_{ij} \in K \}
\]is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \).
(b) Let \( v \) be a vertex in \( E \) lying on a cycle \( c \) and let \( f \) be an exit for \( c \) at \( v \). Then, for every integer \( n \geq 1 \), the subset 
\[
S_n = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} c_j f f^*(c^*)^j : k_{ij} \in K \right\}
\]
is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \).

(c) Let \( v \) be a vertex lying on a cycle \( c \) without exits in \( E \). If, for some \( n \geq 1 \), there are \( n \) distinct paths \( p_1, \ldots, p_n \) in \( E \) that end at \( v \) and do not go through the entire cycle \( c \), then again the set \( T_n = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* : k_{ij} \in K \right\} \) is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \).

Proof. (a) First observe that \( p_j^* p_k \neq 0 \) if and only if \( p_j = p_k \). Because, if \( p_j^* p_k \neq 0 \), then either \( p_j = p_k p' \) or \( p_k = p_j q' \) for some paths \( p', q' \). Since \( r(p_j) = r(p_k) = v \), we get \( s(p') = v = r(p') \) and \( s(q') = v = r(q') \). Since \( v \) does not lie on a closed path, we conclude that \( p' = v = q' \). So \( p_j = p_k \). Conversely, if \( p_j = p_k \), then clearly \( p_j^* p_k = p_j^* p_j = v \neq 0 \). For all \( i, j \), let \( \varepsilon_{ij} = p_i p_j^* \). Clearly, \( (\varepsilon_{ij})^2 = \varepsilon_{ii} \) and \( \varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{il} \) or 0 according as \( j = k \) or not. Thus the \( \varepsilon_{ij} \) form a set of matrix units and it is readily seen that the set \( T_n = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* : k_{ij} \in K \right\} \) is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \).

(b) Suppose \( c \) is a cycle in \( E \) with an exit \( f \) at a vertex \( v \). Consider the set 
\[
\{ \varepsilon_{ij} = c_i f f^*(c^*)^j : 1 \leq i, j \leq n \}.
\]
Clearly, the \( \varepsilon_{ij} \) form a set of matrix units as \( (\varepsilon_{ii})^2 = \varepsilon_{ii} \) and \( \varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{il} \) or 0 according as \( j = k \) or not. It is then easy to check that the set \( S_n = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} c_i f f^*(c^*)^j : k_{ij} \in K \right\} \) is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \).

(c) Let \( v \) be the base of a cycle \( c \) without exits and \( p_1, \ldots, p_n \) be \( n \) distinct paths that end at \( v \) and do not go through the entire cycle \( c \). Using the fact that \( c \) is a cycle without exits and repeating the arguments as in (a), it follows that \( p_j^* p_k \neq 0 \) if and only if \( p_j = p_k \). As before let \( \varepsilon_{ij} = p_i p_j^* \) with \( 1 \leq i, j \leq n \). Clearly, \( (\varepsilon_{ij})^2 = \varepsilon_{ii} \) and \( \varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{il} \) or 0 according as \( j = k \) or not. Thus the \( \varepsilon_{ij} \) form a set of matrix units and it is readily seen that \( T_n = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} p_i p_j^* : k_{ij} \in K \right\} \) is a subring of \( L \) isomorphic to the matrix ring \( M_n(K) \). \( \square \)

We are now ready to describe all the Leavitt path algebras with bounded index of nilpotence. It is interesting to note that the Leavitt path algebras having bounded index of nilpotence are precisely those that satisfy a polynomial identity.

 Recall that an algebra \( A \) over a field \( K \) is said to satisfy a polynomial identity if there exists a non-zero element \( f \) in \( K[x_1, \ldots, x_n] \) such that \( f(a_1, \ldots, a_n) = 0 \) for all \( a_i \) in \( A \). Clearly every commutative ring satisfies a polynomial identity but there are many interesting classes of noncommutative rings too that satisfy a polynomial identity. For instance, the Amitsur-Levitzky theorem (see [13]) states that, for any \( n \geq 1 \), the matrix ring \( M_n(R) \) over a commutative ring \( R \) satisfies a polynomial identity of degree \( 2n \). In [4] it is shown that the Leavitt path algebra \( L_K(E) \) of an
arbitrary graph $E$ over a field $K$ satisfies a polynomial identity if and only if no cycle in $E$ has an exit and there is a fixed positive integer $d$ such that the number of distinct paths that end at any given vertex $v$ (including, but not including the entire cycle $c$ in case $v$ lies on $c$) is less than or equal to $d$. When $E$ is a finite graph, then the Leavitt path algebra $L_K(E)$ satisfying a polynomial identity is known to be equivalent to the Gelfand-Kirillov dimension of $L_K(E)$ being at most one [4].

**Theorem 2.2.** Let $E$ be an arbitrary graph. Then the following properties are equivalent for $L := L_K(E)$:

1. $L$ has index of nilpotence less than or equal to $n$;
2. No cycle in $E$ has an exit and there is a fixed positive integer $n$ such that the number of distinct paths that end at any given vertex $v$ (including, but not including the entire cycle $c$ in case $v$ lies on $c$) is less than or equal to $n$;
3. For any graded prime ideal $P$ of $L$, $L/P \cong M_t(K)$ or $M_t(K[x,x^{-1}])$ where $t \leq n$ with appropriate matrix gradings;
4. $L$ is a graded subdirect product of graded rings $\{A_i : i \in I\}$ where, for each $i$, $A_i \cong M_{t_i}(K)$ or $M_{t_i}(K[x,x^{-1}])$ with appropriate matrix gradings where, for each $i$, $t_i \leq n$, a fixed positive integer.
5. $L$ satisfies a polynomial identity.

**Proof.** Assume (i), that is, assume that the index of nilpotence of $L$ is $\leq n$. We claim that no cycle in $E$ can have an exit. Because, otherwise, by Proposition 2.1 (b), $L$ will contain subrings of matrices of arbitrary finite size and this will give rise to unbounded index of nilpotence for $L$. Thus every vertex $v$ in $E$ either does not lie on a closed path or lies on a cycle without exits. If there are more than $n$ distinct paths ending at $v$, then again, by Proposition 2.1 (a) and (c), $L$ will contain a copy of a matrix ring of order greater than $n$ over $K$ which will imply that the index of nilpotence of $L$ is greater than $n$, a contradiction. This proves (ii).

Assume (ii). Let $P = I(H,S)$ be a graded prime ideal of $L$. Our hypothesis implies that no cycle in $E \backslash (H,S)$ has an exit and that $n$ is also the upper bound for the number of distinct paths ending at any vertex in $E \backslash (H,S)$. So $E \backslash (H,S)$ contains no infinite irrational paths. This means that every path ends at a sink or at a cycle without exits. Also, as $I(H,S)$ is a graded prime ideal, Theorem 3.12 of [1] implies that $(E \backslash (H,S))^0$ is downward directed. Consequently, $E \backslash (H,S)$ contains either (a) exactly one sink $w$ or (b) exactly one cycle $c$ without exits based at a vertex $v$.

Now in case (a), there are no more than $n$ distinct paths ending at $w$ and, in case (b), there are no more than $n$ paths which end at $v$ and do not go through the cycle $c$. We then appeal to Corollary 2.6.5 and Lemma 2.7.1 of [1] to conclude that $L/P \cong L_K(E \backslash (H,S)) \cong M_t(K)$ or $M_t(K[x,x^{-1}])$ according as $E \backslash (H,S)$ contains a sink or a cycle without exits. This proves (iii).

Assume (iii). Now, for any graded prime ideal $P$, $L/P \cong M_t(K)$ or $M_t(K[x,x^{-1}])$ with appropriate matrix gradings where $t \leq n$, a fixed positive integer. It is known that the intersection $I$ of all graded prime ideals of $L$ is zero. For the sake of completeness, we shall outline the argument. If $I \neq 0$, being a graded ideal, $I$ the will contain vertices. But, given any vertex $v$, a graded ideal $M$ maximal among graded ideals with respect to $v \notin M$ is a graded prime ideal, because, for any two homogeneous elements $a, b$, if $a \notin M$ and $b \notin M$, then $v \in M + LaL$ and $v \in M + LbL$. Consequently, $v = v^2 \in (M + LaL)(M + LbL) = M + LaLbL$. Since
Example 2.3. Consider the following “infinite clock” graph $E$:

Thus $E^0 = \{v\} \cup \{w_1, w_2, \ldots, w_n, \cdots\}$ where the $w_i$ are all sinks. For each $n \geq 1$, let $e_n$ denote the single edge connecting $v$ to $w_n$. The graph $E$ is acyclic and so every ideal of $L$ is graded [8]. The number of distinct paths ending at any given sink (including the sink) is $\leq 2$. For each $n \geq 1$, $H_v = \{w_i : i \neq n\}$ is a hereditary saturated set, $B_{H_n} = \{v\}$ and $(E \setminus (H_n, B_{H_n}))^0 = \{v, w_n\}$ is downward directed. Also $E \setminus (H_n, B_{H_n})$ trivially satisfies Condition (L). Hence the ideal $P_n$ generated by $H_n \cup \{v - e_n e_n^*\}$ is a graded primitive ideal by Theorem 4.3(iii) of [13] and $L_K(E)/P \cong M_2(K)$. Moreover, every graded primitive (equivalently, prime) ideal $P$ of $L_K(E)$ is equal to $P_n$ for some $n$. By [9] Theorem 4.12, $L_K(E)$ is a graded $\Sigma$-$V$ ring.

But $L_K(E)$ cannot decompose as a direct sum of the matrix rings $M_2(K)$. Because, otherwise, $v$ would lie in a direct sum of finitely many copies of $M_2(K)$. Since the ideal generated by $v$ is $L_K(E)$, $L_K(E)$ will then be a direct sum of finitely many copies of $M_2(K)$. This is impossible since $L_K(E)$ contains an infinite set of orthogonal idempotents $\{e_n e_n^* : n \geq 1\}$.

We can also describe the internal structure of this ring $L_K(E)$. The socle $S$ of $L_K(E)$ is the ideal generated by the sinks $\{w_i : i \geq 1\}$, $S \cong \bigoplus_{\mathbb{N}_0} M_2(K)$ and $L_K(E)/S \cong K$.

But the decomposition is possible if the graph is row-finite, as shown in the following theorem.

Theorem 2.4. Let $E$ be a row-finite graph. Then the following properties are equivalent for $L := L_K(E)$:

(i) $L$ has bounded index of nilpotence $\leq n$;

Thus $M$ is a graded prime ideal. But then $v \notin M$ implies $v \notin I$, a contradiction. Thus $\cap \{P : P$ graded prime ideal$\} = 0$. Consequently, $L$ is a graded subdirect product of the graded rings $L/P$ graded isomorphic to $M_t(K)$ or $M_t(K[x, x^{-1}])$ under appropriate matrix gradings, where $t \leq n$, a fixed positive integer. This proves (iv).

Assume (iv). If $t \leq n$, then the matrix rings $M_t(K)$ and $M_t(K[x, x^{-1}])$ will each have index of nilpotence $\leq n$. Consequently, a subdirect product of such rings will also have nilpotence index $\leq n$. This proves (i).

Assume (iv). By the Amitsur-Levitzky theorem [13], both $M_t(K)$ and $M_t(K[x, x^{-1}])$ with $t \leq n$ are polynomial identity rings satisfying a polynomial identity of degree $\leq 2n$. From this, it is clear that the subdirect product $L$ also satisfies a polynomial identity of degree $\leq n$. This proves (v).

The implication (v) $\implies$ (ii) has been established in [4]. \qed

The Leavitt path algebra in Theorem 2.2 need not decompose as a direct sum of matrix rings, as the following example shows.
(ii) There is a fixed positive integer $n$ and a graded isomorphism

$$L \cong_{gr} \bigoplus_{i \in I} M_{n_i}(K) \oplus \bigoplus_{j \in J} M_{n_j}(K[x, x^{-1}])$$

where $I, J$ are arbitrary index sets and, for all $i \in I$ and $j \in J$, $n_i \leq n$ and $n_j \leq n$. In particular, $L$ is graded semi-simple (that is, a direct sum of graded simple left/right ideals).

Proof. Assume (i). By Theorem 2.2, no cycle in $E$ has an exit and the number of distinct paths that end at any vertex $v$ is $\leq n$, with the proviso that if $v$ sits on a cycle $c$, then these paths do not include the entire cycle $c$. If $A$ is the graded ideal generated by all the sinks in $E$ and all the vertices on cycles without exits, then, by Corollary 2.6.5 and Lemma 2.7.1 of [1], $A \cong_{gr} \bigoplus_{i \in I} M_{n_i}(K) \oplus \bigoplus_{j \in J} M_{n_j}(K[x, x^{-1}])$. By giving appropriate matrix gradings, this isomorphism becomes a graded isomorphism. We claim that $L = A$. Let $H \subseteq A$ be the set consisting of all the sinks and all the vertices on cycles in $E$. By hypothesis, every path in $E$ that does not include an entire cycle has length $\leq n$ and ends at a vertex in $H$. So if $u$ is any vertex in $E$, using the fact that all the vertices in $E$ are regular and by a simple induction on the length of the longest path from $u$, we can conclude that $u$ belongs to the saturated closure of $H$. This implies that $L = A \cong_{gr} \bigoplus_{i \in I} M_{n_i}(K) \oplus \bigoplus_{j \in J} M_{n_j}(K[x, x^{-1}])$. This proves (ii).

(ii) $\implies$ (i) follows from the fact that the matrix rings $M_{n_i}(K)$ and $M_{n_j}(K[x, x^{-1}])$ with $n_i, n_j \leq n$ have index of nilpotence $\leq n$. □

One consequence of Theorem 2.2 is the following.

Proposition 2.5. Let $E$ be an arbitrary graph. If $L := L_K(E)$ has bounded index of nilpotence $n$, then $L$ is a graded $\Sigma$-V ring.

Proof. If $L$ has bounded index of nilpotence $n$, then for any graded primitive ideal $P$ of $L$ (since it is also graded prime), we have, by Theorem 2.2(iii), $L/P \cong_{gr} M_t(K)$ or $M_t(K[x, x^{-1}])$ with appropriate matrix gradings, where $t \leq n$. By [9, Theorem 4.12], we then conclude that $L$ is a graded $\Sigma$-V ring. □

The converse of the above result does not hold, as can be seen in the two examples below.

Example 2.6. Let $E$ be the “inverse infinite clock” graph consisting of a sink $w$ and countably infinite edges $\{e_n : n \geq 1\}$ with $r(e_n) = w$ and $s(e_n) = w_n$ for all $n$.

Then $L_K(E) \cong M_\infty(K)$, the infinite $\omega \times \omega$ matrix with finitely many non-zero entries. Now, under appropriate matrix grading, $M_\infty(K)$ is graded semisimple (that is,
a graded direct sum of graded simple modules) and so all graded left/right $M_\infty(K)$-modules are graded injective and hence $L$ is a graded $\Sigma$-V ring. But $L$ does not have bounded index of nilpotence by Theorem 2.2(ii).

Example 2.7. Consider the following graph $F$ consisting of two cycles $g$ and $c$ connected by an edge:

$$
\begin{array}{c}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
\uparrow & g & \downarrow & \uparrow & c & \downarrow \\
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet
\end{array}
$$

Now $F$ is downward directed, $c$ is a cycle without exits and the various powers of the cycle $g$ give rise to infinitely many distinct paths that end at the base $v$ of the cycle $c$. Hence $L_K(F) \cong M_\infty(K[x,x^{-1}])$ (by Lemma 2.7.1 of [1]) and is graded semisimple. Hence each graded simple module over $L_K(F)$ is graded $\Sigma$-injective. But $L_K(F)$ does not have bounded index of nilpotence by Theorem 2.2(ii).

In the monograph [10], the following open question (6.33: Problem 3, Chapter 6) was raised:

Question 2.8. Does every exchange right $\Sigma$-V ring have bounded index of nilpotence?

We answer this question in the negative in the following remark.

Remark 2.9. Consider the graph $E$ of Example 2.6. As noted there, $L_K(E) \cong M_\infty(K)$ which is semisimple and hence is a $\Sigma$-V ring. Since $E$ is acyclic, $L_K(E)$ is von Neumann regular [2] and hence is an exchange ring. But $L_K(E)$ does not have bounded index of nilpotence by Theorem 2.2(ii).

Remark 2.10. It was shown in [9] that a Leavitt path algebra $L_K(E)$ is directly-finite (equivalently, graded directly-finite with respect to vertices) if and only if no cycle in $E$ has an exit. In view of Theorem 2.2, it is clear that a Leavitt path algebra of bounded index is always directly-finite. But, for arbitrary rings with bounded index of nilpotence, this is not the case. Let $S = \prod R_k$, where each $R_k \cong \mathbb{Z}(p^n)$, the ring of integers modulo a fixed integer $n \geq 2$. Now $(pS)^n = 0$ and if $a \in S$ is nilpotent, then $a \in pS$ and $a^n = 0$. Consequently, $S$ has bounded index of nilpotence. In fact, the index of nilpotence of $S$ is $n$. But $S$ is not directly-finite, since $S \cong \prod_{k \geq 2} R_k \cong \prod_{k \geq 3} R_k \cong \cdots$

Conversely, a directly-finite Leavitt path algebra need not have bounded index of nilpotence. If $E$ is the graph consisting of an infinite line segment

$$
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \cdots
$$

then clearly $L_K(E)$ is directly-finite, but, by Theorem 2.2 (ii), $L_K(E) \cong M_\infty(K)$ does not have bounded index of nilpotence.
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