On the nuclear dependence of the $\mu^- \rightarrow e^-$ conversion branching ratio

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**Abstract**

The variation of the coherent branching ratio $R_{\mu e}$ (ratio of the $\mu^- \rightarrow e^-$ reaction rate divided by the total muon-capture rate) through the periodic table is studied by using exact muon wave functions. It was found that, by using very heavy nuclei (e.g. $^{197}\text{Au}$, the SINDRUM II target) as $\mu^- \rightarrow e^-$ conversion stopping-targets, the above ratio is favored by a factor of about four to five than by using light ones (e.g. $^{48}\text{Ti}$, chosen as PRIME target).

**Key words:** rare muon decays, muon-to-electron conversion, muon capture, lepton flavor violation

**PACS:** 23.40.Bw, 24.10.-i, 13.35.Bv, 12.60.Cn

The present status of the exotic neutrinoless $\mu^- \rightarrow e^-$ conversion in nuclei,

$$\mu^-_0 + (A, Z) \rightarrow e^- + (A, Z)^*,$$

has comprehensively been discussed recently [1–3] both from experimental [4–9] and theoretical [10–15] point of view. Process (1) violates the $L_i$, ($i = \mu, e$) quantum numbers and has been proposed [12] as one of the best probes to test the existence of charged-lepton flavor conservation, among a great number of similar processes predicted by modern gauge and supersymmetric theories.

Several experiments have been designed to explore process (1) and performed at PSI (and earlier at TRIUMF) on $^{48}\text{Ti}$, $^{208}\text{Pb}$ and $^{197}\text{Au}$ targets [1,4,7,8]. They have put so far, only bounds on the branching ratio $R_{\mu e}$. Presently, the published best upper limit is $R_{\mu e}^{\text{Ti}} \leq 6.1 \times 10^{-13}$ [7]. The ongoing SINDRUM II experiment (PSI) is now using $^{197}\text{Au}$ as stopping target and the extracted preliminary limit constitutes an improvement over the previous one set on a heavy target ($^{208}\text{Pb}$ [4]) by two orders of magnitude [7,8]. The planned MECO experiment (Brookhaven) is going to use $^{27}\text{Al}$ target and a very intense pulsing
muon-beam [5,6] to reach a sensitivity of roughly $R_{\mu e}^{Al} \leq 2 \times 10^{-17}$ [5,6] which implies an improvement of the present limits by about three orders of magnitude. It should be mentioned that very recently a new $\mu^- \rightarrow e^-$ conversion experiment on $^{48}$Ti (PRIME) was announced to be performed at KEK [2,9] aiming to push the limit down to $R_{\mu e}^{Ti} \leq 10^{-18}$. In all these experiments, the signature of reaction (1) is a single electron with energy $E_e = m_{\mu} - \epsilon_b - m_e$ (neglecting recoil) where $\epsilon_b$ is the muon binding energy in the 1s orbit and $m_{\mu}$ ($m_e$) the muon (electron) mass.

From the theoretical point of view [12–17], process (1) constitutes a very good interplay between atomic, nuclear, particle and non-standard physics [14,15]. The $\mu^- \rightarrow e^-$ conversion Hamiltonians which result in the context of many extensions of the standard model proposed up to now, in general, give rise to coherent and incoherent processes [16]. In several models, like those for which the isoscalar couplings of the vector and scalar interactions are not very small [3,16], the coherent action of all nucleons in $\mu^- \rightarrow e^-$ leads to enhancement of conversion electrons thereby making it potentially a sensible indicator for lepton flavor violation (LFV) effects [10–13]. In addition, the $g.s. \rightarrow g.s.$ transitions are favored due to Pauli blocking effects which prevent the formation of excited states. The present work is motivated from the necessity to investigate the nuclear physics aspects of process (1). We study the nuclear structure dependence of the coherent branching ratio $R_{\mu e}$ throughout the periodic table by performing exact calculations of the muon-nucleus overlap integrals.

The expression for $R_{\mu e}$, to leading order in the non-relativistic reduction, for the coherent process has been written in the form [16]

$$R_{\mu e} \equiv \frac{\Gamma_{\mu e}(A, Z)}{\Gamma_{\mu c}(A, Z)} = \frac{G^2_F}{2\pi} Q \frac{p_e E_e |\mathcal{M}_{V,S}^{(0)}|^2}{\Gamma_{\mu c}},$$

(2)

where $\Gamma_{\mu e}$ stands for the $\mu^- \rightarrow e^-$ conversion rate and $\Gamma_{\mu c}$ for the total rate of the ordinary muon-capture, $\mu^- + (A, Z) \rightarrow \nu_{\mu} + (A, Z - 1)$ [18,19]. The factor $G^2_F/2$ corresponds to non-photonic mechanisms and for photonic ones it should be replaced by the ratio $(4\pi\alpha)^2/q^4$, where $\alpha$ is the fine structure constant. In Eq. (2), $p_e$ denotes the outgoing-electron momentum connected to the excitation energy $E_x$ of the daughter nucleus ($E_x = E_f - E_{gs}$) through the relation

$$p_e \approx q = m_{\mu} - \epsilon_b - E_x.$$

(3)

$q = |q|$ is the magnitude of the momentum transfer (we neglect the electron mass). The quantity $Q$ of Eq. (2) depends very weakly on the nuclear structure and, in principle, it contains scalar (S), vector (V), axial-vector (A), pseudo-
scalar (P), and tensor (T) coupling terms [16]. Especially for photonic diagrams, which are the main concern of this work, Q is rather nuclear-structure independent. Hence, the main nuclear physics aspects of \( R_{\mu e} \) are accumulated in the last fraction of Eq. (2).

The matrix elements \( M_\alpha^{(\tau)} \), \( \alpha = V, S, A, P, T \), which enter the expression of \( R_{\mu e} \) are defined by

\[
M_\alpha^{(\tau)} = \langle f | \sum_{j=1}^{A} \Theta_\alpha^{(j)}(e^{-i \mathbf{q} \cdot \mathbf{r}_j}) \Phi_\mu(\mathbf{r}_j) | i \rangle, \tag{4}
\]

\( \tau = 0 \) for isoscalar and \( \tau = 1 \) for isovector operators, respectively, where \( |i\rangle \) the initial and \( |f\rangle \) the final nuclear state. \( \Phi_\mu(\mathbf{r}_j) \) represents the muon wave function evaluated at the position of the \( j^{th} \) target-nucleon. The functions \( \Theta_\alpha^{(j)}(j) \) contain the spin-isospin dependence of the \( \mu^- \rightarrow e^- \) operator [3]. For the coherent process (|\( f \rangle = |i\rangle \), in the case of scalar and vector interactions, \( M_{V, S}^{(\tau)} \) are written in terms of the ground-state proton, neutron densities as

\[
M_{V, S}^{(\tau)}(q) = \int [\rho_p(\mathbf{r}) \pm \rho_n(\mathbf{r})] e^{-i \mathbf{q} \cdot \mathbf{r}} \Phi_\mu(\mathbf{r}) d^3 \mathbf{r} = F_p(q) \pm F_n(q), \tag{5}
\]

the (+) sign corresponds to \( \tau = 0 \) and the (-) to \( \tau = 1 \) channel, where

\[
F_{p,n}(q) = \int \rho_{p,n}(\mathbf{r}) e^{-i \mathbf{q} \cdot \mathbf{r}} \Phi_\mu(\mathbf{r}) d^3 \mathbf{r}. \tag{6}
\]

The proton (neutron) density \( \rho_p (\rho_n) \) is normalized to the atomic number \( Z \) (neutron number \( N \)) of the nucleus in question. For our purposes here, the required densities \( \rho_p \) are taken from experiment [20]. For photonic mechanisms only protons of the target-nucleus contribute and hence, \( M_{V, S}^{(0)} = M_{V, S}^{(1)} = F_p(q) \).

For light and medium nuclei (see discussion of the results below) \( M_\alpha^{(\tau)} \) can be reliably evaluated in a straightforward way by factorizing outside the integrals of Eq. (4) a suitably averaged muon wave function \( \langle \Phi_{1s}^{1s} \rangle \). Under these conditions Eq. (4) is approximated by

\[
\overline{M}_\alpha^{(\tau)} = \langle \Phi_{1s}^{1s} \rangle \langle f | \sum_{j=1}^{A} \Theta_\alpha^{(j)}(e^{-i \mathbf{q} \cdot \mathbf{r}_j}) | i \rangle \equiv \langle \Phi_{1s}^{1s} \rangle M_\alpha^{(\tau)}, \tag{7}
\]

where \( M_\alpha^{(\tau)} \) involve the pure nuclear physics aspects of the \( \mu^- \rightarrow e^- \) process. Equation (7) for photonic diagrams, is written as \( \overline{M}_{V, S}^{(0,1)} = F_p(q) = \langle \Phi_{1s}^{1s} \rangle Z F_p(q) \). For the mean muon wave function \( \langle \Phi_{1s} \rangle \), a simplified expression
(see e.g. Ref. [18] and Eqs. (22), (23) of Ref. [17]) was used in muon capture studies by many authors [18]. In previous estimations of the branching ratio \( R_{\mu e} \), the same expression for \( \langle \Phi^{1\mu} \rangle \) was adopted [13] for both the numerator and the denominator of Eq. (2) in order to reduce the uncertainties inserted via the use of Eq. (7) in \( \mu^- \rightarrow e^- \) and \( \mu^- \rightarrow \nu_\mu \) processes.

In the special case of g.s. \( \rightarrow \) g.s. transitions for spin zero \((J=0)\) light nuclei, \( M_{V,S}^{(r)} \) are determined by the elastic scattering (monopole) nuclear form factors \( F_p, F_n \) [13] and they are simply given by \( M_{V,S}^{(r)}(q) = Z F_p(q) \pm N F_n(q) \). In the general case of nuclei with ground-state spin \( J \neq 0 \) (e.g. \(^{27}\)Al, the MECO target, with \( J = \frac{5}{2} \)), \( M_{V,S}^{(r)} \) contain, in addition to monopole \((L = 0)\) form factors, contributions arising from other multipoles \((L = 2, 4, \ldots)\). However, the latter contributions are not significant [3].

The main goal of the present work, was to study systematically the exact nuclear-structure dependence of \( R_{\mu e}(A,Z) \) by computing the integrals of Eq. (5) for the dominant coherent \( \mu^- \rightarrow e^- \) conversion and investigate the influence on \( R_{\mu e} \) of the following effects: (i) the approximate evaluation of the muon-nucleus overlap integrals [see Eq. (7)] and (ii) the neglect of the muon-binding energy \( \epsilon_b \) in Eq. (3). The latter assumption implies that for all nuclei in the coherent mode we have

\[
q = p_e \approx m_\mu / c = 0.534 \text{ fm}^{-1}, \quad E_e \approx 105.6 \text{ MeV}. \tag{8}
\]

In this work, the rather simple photonic mechanism, where only the target-protons contribute to \( R_{\mu e} \), was examined. The main steps followed in the calculational procedure and the results obtained are briefly discussed below.

In the first step, the variation of the quantity \( Z|F_Z(q)|^2 \) through the periodic table was studied. As is well known, Weinberg and Feinberg [10] using the above approximations have shown that, for the coherent \( \mu^- \rightarrow e^- \) rate it holds

\[
R_{\mu e} \propto Z|F_{NN}(q)|^2. \tag{9}
\]

In Fig. 1(a) the results obtained for Eq. (9) in the following cases are illustrated:

(i) In the first case, experimental form factors \( F_Z(q) = F_{NN}(q) \) [20] were used at the values of \( q \) given by Eq. (8) (dashed-dotted line) and by Eq. (3) (dotted line).

(ii) In the second case, the phenomenological expression for \( F_{NN}(q) \) [see Eq. (16) of Ref. [10]] was used in Eq. (9) (solid line).
The common feature of the three curves in Fig. 1(a) is the fact that they show a maximum in the region of \( A \approx 60 \) (copper region) as had been estimated in Ref. [10]. This behavior was generally adopted by the experimentalists exploring the \( \mu^- \rightarrow e^- \) process [1,2]. As it can be seen, in the region of light nuclei, the three curves nearly coincide, which means that the approximation of Eq. (8) is reasonable in this region, but for medium and heavy nuclei where \( \epsilon_b \) becomes significant, the obtained rates when \( q \) is given by Eq. (3) are larger by about a factor of two than those when \( q \) is given by Eq. (8).

In the second step of the calculations, the variation of the quantity \( R_{\mu e}/Q_{ph} \) [see Eq. (2)], which contains the main nuclear-structure dependence of \( R_{\mu e} \), was studied. Two cases were distinguished:

(i) In the first case, the mean muon wave function was inserted in Eq. (2). This gives

\[
R_{\mu e} \propto 16\pi\alpha^2 \frac{p_e E_e}{q^4} \frac{\langle \Phi_\mu \rangle^2 Z^2 |F_p(q)|^2}{\Gamma_{\mu c}}
\]  

(10)

The results obtained in this way, are presented in Fig. 1(b) and correspond to the two choices of coherent momentum transfer discussed before: (a) by using the values of Eq. (8) (solid line) and (b) by using the values of \( q \) given by Eq. (3) \((E_x = 0 \text{ for coherent mode})\) (dashed line). In choice (a) we see that \( R_{\mu e}(A, Z) \) presents a maximum at the region of \( A \approx 130 \) which is not in accordance with the estimation of Ref. [10]. This is due to the fact that in Ref. [10] the gross nuclear dependence of \( \Gamma_{\mu c} \) was considered as linear in \( Z \) which is equivalent to a constant Primakoff function \( f_{GP}(A, Z) \). It is well known, however, that \( f_{GP}(A, Z) \) is strongly dependent on the mass excess (see e.g. Ref. [13]). In this work, in order to minimize such uncertainties, experimental data for the total muon capture rates \( \Gamma_{\mu c} \) were used [19]. From the results of choice (b) we see that \( R_{\mu e}(A, Z) \) shows a slow increase up to the heaviest nuclei.

(ii) In the second case, the explicit muon-nucleus overlap integral of Eq. (4) was used (numerical integration) in Eq. (2). Then, the variation of \( R_{\mu e}(A, Z) \) relies on the expression

\[
R_{\mu e} \propto 16\pi\alpha^2 \frac{p_e E_e}{q^4} \frac{|F_p|^2}{\Gamma_{\mu c}}
\]  

(11)

This requires the use of the exact muon wave function calculated as in Ref. [21]. Here the values of \( q \) are provided by Eq. (3). The results are represented by stars (*) in Fig. 1(b). We see that, the exact evaluation of the muon-nucleus overlap integrals of Eq. (5), shows linear increase of \( R_{\mu e} \) as function of \( A \) (or \( Z \) [21]). For heavy and very heavy nuclei (region of \( ^{197}Au \) and \( ^{208}Pb \)), }
the use of the exact muon wave function in Eq. (11), gives much larger rates than the approximation of the averaged muon wave function Eq. (9). The comparison is much worse if $\epsilon_b$ is neglected in the kinematics. One must notice that, the presence of the factor $q^{-4}$ in Eq. (11) (photonic mechanism), inserts an additional $(A,Z)$ dependence on the branching ratio $R_{\mu e}$ which had been previously overlooked.

In conclusion, the gross variation of $R_{\mu e}$ obtained by the exact results, shows a linear rise with $A$ which can be attributed to the coherent effect. Due to this behaviour of $R_{\mu e}(A, Z)$, very heavy nuclei are favored by a factor of about five to be used as $\mu^- \rightarrow e^-$ conversion targets. It is worth mentioning that, the above factor may, in some cases, be partly compensated by other experimental advantages of some specific muon-stopping targets [5]. We also remark that in the present work we neglected relativistic atomic effects related to $\mu^- \rightarrow e^-$ process [14] which might influence a bit (depending on the nucleus) the results for $R_{\mu e}$.

In summary, we have investigated the dependence of the $\mu^- \rightarrow e^-$ conversion branching ratios on the nuclear parameters $A$ and $Z$ throughout the periodic table. This exotic process is an interesting and important one to be studied, since stringent bounds for the charged-lepton flavor violating parameters already exist and significant improvements over these limits are feasible in the not-too-distant future from the SINDRUM II, MECO and PRIME experiments. We performed direct calculations of the muon-nucleus overlap integrals (for photonic mechanisms) and found that $R_{\mu e}(A, Z)$ keeps increasing up to the very heavy nuclei. This shows that, by using such isotopes (e.g. $^{197}$Au, present SINDRUM II target) as targets in $\mu^- \rightarrow e^-$ conversion experiments, $R_{\mu e}$ is favored by a factor of about four to five than by using light isotopes. Finally, the present exact results were exploited to test the validity of previous approximations made on the study of the $\mu^- e$ conversion and shed more light in this situation.

The author wishes to thank Dr. Y. Kuno for financial support and Dr. Andries van der Schaaf for fruitful discussions on SINDRUM II experiments.

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FIGURE CAPTION

Fig. 1. Variation of the $\mu^- \to e^-$ conversion branching ratio, $R_{\mu e}$, through the periodic table assuming that the nuclear-structure dependence of $R_{\mu e}$ is described: (a) by Eq. (9) [10] and (b) by Eqs. (10) and (11). For details see the text.
Fig. 1. Variation of the $\mu^{-}\rightarrow e^{-}$ conversion branching ratio, $R_{\mu e}$, through the periodic table assuming that the nuclear-structure dependence of $R_{\mu e}$ is described: (a) by Eq. (9) [10] and (b) by Eqs. (10) and (11). For details see the text.