FORMAL GROUPS AND GEOMETRIC QUANTIZATION

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ABSTRACT. The complex projective spaces, considered as prequantized symplectic manifolds, are roughly to the complete symmetric functions as those projective spaces, regarded as complex-oriented manifolds, are to Newton’s power sums.

Introduction

This paper is concerned with the statistical mechanics of a commutative algebra $B_* \subset B_* \otimes \mathbb{R}$ of cobordism classes of geometrically prequantized symplectic manifolds, which is to some extent well-understood [3,16]: $B_*$ is naturally isomorphic to the bordism ring $\text{MU}_* CP_\infty$ of compact complex-oriented manifolds carrying a suitable complex line bundle with connection, and the principal technical result below is an explicit formula [Prop 3.1] for the (injective) Hurewicz homomorphism

$$h : B_* \cong \text{MU}_* CP_\infty \to H_*(CP_\infty; H_*(	ext{MU}, \mathbb{Z})) \cong S_* \otimes \mathbb{Z}[b(n) \mid n \geq 1]$$

which identifies these manifolds in terms of their characteristic numbers $\text{I}$. This formula is easily accessible by modern homotopy theory, though it may not be so familiar; the point is that in our case this homomorphism has an interesting interpretation as a kind of partition function, analogous to the construction which assigns to a Riemannian manifold $(M, g)$, the trace of its heat kernel $\exp(t \Delta_g)$ [1,2,5,8,12,21 ...]. An essentially equivalent corollary is that (as observed by Friedrich and McKay [6,7], Miščenko’s logarithm

$$\log_{\text{MU}}(z) = \sum_{k \geq 1} \frac{CP_{k-1} z^k}{k} \in \text{MU}^2(CP_\infty)$$

for complex cobordism can be regarded as a kind of cumulant generating function (or Helmholtz free energy, or as (the negative of) a kind of Shannon entropy) for $B_*$. 

Date: January 2020.

1991 Mathematics Subject Classification. 53D50, 55N22, 57R17.

$^1$Here $S_*$ is the ring of classical symmetric functions, and $b(n)$ is the $n$th divided power of an element $b$. Further technical definitions will be provided soon.
§I, algebraic and geometric preliminaries

1.1 For our purposes, a geometrically prequantized symplectic manifold \([4]\) will be a \(2n\)-dimensional closed compact smooth manifold \((V, L, \nabla_L, j)\) together with a Hermitian complex line bundle \(L\) with connection \(\nabla_L\) on \(V\), with symplectic (i.e., closed and nondegenerate) connection form \(\omega = j\Omega(\nabla_L)\) (with \([\omega/2\pi i] \in H^2_{\text{dr}}(V)\) equal to the Chern class \(c_1(L)\) of the line bundle \(L : V \to \mathbb{C}P_{\infty}\) (or, equivalently, classified by a map to the Narasimhan-Ramanan space \(BT\) of bundles with Hermitian connection)). There is a contractible space of almost-complex structures \(\{ j \in \text{End}(T_V) | j^2 = -1 \}\) on \(V\), compatible with \(\omega\) in the sense that \(\omega(j-, -)\) is a Riemannian metric, and we assume that such a \(j\) has been chosen. The associated Liouville volume \(\omega^n/n!\) defines a class in \(H^2_{\text{dR}}(V)\) Poincaré dual to the orientation class in \(H_{2n}(V; \mathbb{Z})\). This will usually be summarized as the assertion that \((V, \omega)\) is a (prequantized) symplectic manifold. Relaxing the integrality condition on \(\omega\) defines the real completion \((B \otimes \mathbb{R})_*\) of \(B_*\).

Following VL Ginzburg [8,9 Th. H.10], a cobordism

\[ W : (V_0, \omega_0) \to (V_1, \omega_1) \]

between two such manifolds is a compact \((2n + 1)\)-dimensional manifold \(W\) together with a Hermitian line bundle \(L : W \to BT\) with connection, which restricts to the given line bundle with connection on \(\partial W = V^0_0 \coprod V_1\). We furthermore assume that the kernel of the curvature form \(\omega_L : T_W \to T^*_W\) is a real line bundle on \(W\), trivial on \(\partial W\). This defines the (monoidal) cobordism category of geometrically prequantized manifolds, whose cobordism ring is known:

**Theorem:** The map

\[ B_* \ni [V, L, \nabla_L, j] \to [V, j, L] \in MU_* \mathbb{C}P_{\infty} \]

defines an isomorphism \([3,16]\) of (evenly graded, commutative) torsion-free algebras.

**Remark:** Data of the sort described above can be used to define various (almost Kähler, Spin\(^c\) Dirac, twisted signature . . . ) elliptic differential operators, which behave nicely with respect to products and cobordisms; the work [12] of Liu and Xu, for example, was one of the main motivations for this paper. Symplectomorphism groups are generally infinite-dimensional, however, so it seems unreasonable to expect a close analog of the rich spectral theory of the classical Laplace-Beltrami operator in this context. But even if we may not be able to hear the shape of a prequantized manifold, we can at least hear its cobordism class.

1.2.1 It will useful to have some examples. We will regard the complex projective spaces \(\mathbb{C}P_n = (\mathbb{C}^\times - 0)/\mathbb{C}^\times\) as symplectic manifolds \(\mathbb{C}P_n(\omega)\)
when endowed with the Fubini-Study symplectic form

$$\omega = \frac{i}{2} \partial \bar{\partial} \log |z|^2 \in \Omega^2(\mathbb{C}P_n),$$

with canonical line bundle pulled back along the standard inclusion

$$i_n : \mathbb{C}P_n \rightarrow \bigcup_{n \geq 0} \mathbb{C}P_n := \mathbb{C}P_\infty \cong BT.$$  

More generally, \( \mathbb{C}P_n(m\omega) \) will be defined by the line bundle \( L \otimes m \) with symplectic form \( m\omega \), \( m \in \mathbb{Z}^\times > 0 \). Let \( b_{\text{MU}} = [i_n : \mathbb{C}P_n \rightarrow \mathbb{C}P_\infty] \in \text{MU}_n \mathbb{C}P_\infty \) and define the generating function

$$b_{\text{MU}}(z) = 1 + \sum_{n \geq 1} b_{n} z^n,$$

where \( b_{n} = b_{n}(1)/n! \) is a divided power.

Steenrod’s cycle class homomorphism

$$h : MU_* X \rightarrow H_*(X, \mathbb{Z})$$

sends a bordism class \( \kappa = [x : M \rightarrow X] \in MU_*(X) \) to \( x_*[M] \), where \( [M] \in H_{2n}(M; \mathbb{Z}) \) is the orientation class. Thus

$$h(b_{n}^{\text{MU}}) = i_n^*[\mathbb{C}P_n] \in H_{2n}(\mathbb{C}P_\infty; \mathbb{Z}).$$

The composition

$$H_*(M; \mathbb{Z}) \xrightarrow{[\omega^n]} H_*(M; \mathbb{R}) \xrightarrow{\mu^{-1}(\cdot)^{-1}} H^*_{\text{dR}}(M) \xrightarrow{\iota_*} H^*_{\text{dR}}(M)$$

sends \( [M] \in H_{2n}(M; \mathbb{Z}) \) to the Riemannian (hence, in our case, Liouville) volume

$$([\frac{\omega^n}{n!}])[M] = \text{vol}(M, \omega);$$

thus the cycle map sends \( \kappa \) to Weyl’s leading term in the asymptotic expansion [21] for the trace of the heat kernel \( \exp(t\Delta_g) \) of \( (M, \omega) \).

### §II Characteristic numbers

#### 2.1

Ravenel and Wilson [19] describe \( \text{MU}_* \mathbb{C}P_\infty \) as follows:

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2 We will try to be careful with gradings, which can play a rather subtle role. If \( A_* \) is a \( \mathbb{Z} \)-graded module, then \( A^* = A_{-\cdot} \). We will use graded book-keeping indeterminates, \( e.g., z_0, z_1, \ldots \) of cohomological degree +2.
Theorem As an algebra,
\[ \text{MU}_* \mathbb{C}P_\infty \cong \text{MU}_*[b_n^{\text{MU}} \mid n \geq 1]/(b_0^{\text{MU}}(z_0) \cdot b_1^{\text{MU}}(z_1) = b^{\text{MU}}(z_0 + \text{MU} z_1)) \, . \]

It is also a cocommutative Hopf \( \text{MU}_* \)-algebra, with coproduct \( \Delta b^{\text{MU}}(z) = (b^{\text{MU}} \otimes 1)(z) \otimes_{\text{MU}} (1 \otimes b^{\text{MU}})(z) \).

This is closely related to work \([11]\) of Katz. Here
\[ z_0 + \text{MU} z_1 = \exp_{\text{MU}}(\log_{\text{MU}}(z_0) + \log_{\text{MU}}(z_1)) \]
(with \( \exp_{\text{MU}} \) the formal inverse of \( \log_{\text{MU}} \)) is Quillen’s formal group law for complex cobordism; recall that \( \text{MU}_* \mathbb{C}P_\infty = \text{MU}_* \) is polynomial over \( \mathbb{Z} \) with one generator of each even degree \([15,18]\), and that (following work of Thom)

\[ \text{it is generated over } \mathbb{Q} \text{ (but not } \mathbb{Z} \text{) by the classes } [\mathbb{C}P_n]. \]

2.2 A generator \( b_{(1)} : S^2 = \mathbb{C}P_1 \to \mathbb{C}P_\infty \) of \( \pi_2 \mathbb{C}P_\infty \) defines a homotopy associative map from the free topological monoid generated by the two-sphere, to \( \mathbb{C}P_\infty \), for example with the Segre product. Work of IM James \([20]\) shows this map to be stably equivalent to a map from \( \Omega S^3 \to \mathbb{C}P_\infty \).

In fact a level one projective representation of the loop group \( \text{LSU}(2) \) on a separable Hilbert space \( \mathcal{H} \) defines a continuous homomorphism
\[ \Omega \text{SU}(2) \to \text{PGL}_\mathbb{C}(\mathcal{H}) \simeq \mathbb{C}P_\infty \]
[17] inducing a homomorphism
\[ \text{MU}_* \Omega \text{SU}(2) \cong \text{MU}_*[b] \to \text{MU}_*[b_n^{\text{MU}} \mid n \geq 1] \cong \text{MU}_* \mathbb{C}P_\infty \]
of Hopf algebras, taking the class of the adjoint \( b : S^2 \to \Omega S^3 \) of the identity map \( S^1 \wedge S^2 \to S^3 \) to \( b_{(1)} \). Similarly, the embedding
\[ \text{MU}_* \mathbb{C}P_\infty = \text{MU}_*[c] \to \text{MU}_* \Omega \text{SU}(2) \cong \text{MU}_*[c_n \mid n \geq 1] \]
(of the formal group on the left into the completed ring of divided powers on the right) represents an analog of an exponential map for a formal Lie group. From now on we will identify \( b_{(1)} \) and \( b \).

2.3 The Hurewicz homomorphism
\[ \h : \text{MU}_* X \cong \pi_*(X \wedge \text{MU}) \to H_*(X \wedge \text{MU}; \mathbb{Z}) \cong H_*(X; \mathbb{Z}) \otimes_{\text{S}_*} \]
(where \( H_*(\text{MU}; \mathbb{Z}) \cong H_*(\text{BU}; \mathbb{Z}) \cong \text{S}_* \) is regarded as the classical algebra \( \mathbb{Z}[h_i \mid i \geq 1] \) of complete symmetric functions) can be identified, when \( H_*(X; \mathbb{Z}) \) is torsion-free, with the map which sends \( \nu = [x : M \to X] \) to
\[ \h(\nu) \in \text{Hom}(H^*(\text{BU}), H_*(X)) \cong H_*(X) \otimes H_*(\text{BU}) \]
defined by
\[ \h(\nu)(\alpha) = x_* D_M \nu^*(\alpha) ; \]
where $\nu : M \to BU$ classifies the stable normal bundle of $M$, and $D_M : H^*(M; \mathbb{Z}) \to H_{2n-4}(M; \mathbb{Z})$ is the Poincaré duality map$^3$.

Following Thom and Milnor, the Hurewicz homomorphism $MU_*(pt) \to H_*(BU; \mathbb{Z})$ is injective, and Quillen’s work implies that the image of the formal group law on $MU^\ast CP_\infty$ is isomorphic to the additive group over $H^*(CP_\infty; H_*(BU))$; see [14] for an elegant account. It follows in particular that the characteristic number class $h(CP_{k-1})$ is divisible by $k$.

Composition with the morphism $[1 : MU \to HZ] \in H^0(MU; \mathbb{Z})$ of spectra yields Steenrod’s cycle map

$$h : MU_\ast CP_\infty \xrightarrow{h} H_\ast CP_\infty \otimes S_\ast \xrightarrow{\varepsilon} H_\ast CP_\infty$$

($\varepsilon(h_i) = 0$, $i > 0$) sending $b_n^{MU} \mapsto i_n \cdot D_{CP_n}(c_1) = b_{(n)}$; more generally, we can think of $h$ as sending $(V, \omega)$ to its Liouville volume.

§III Conclusion and final remarks

3.1 To state the result below we need some notation for partitions $\pi = 1^{r_1}2^{r_2} \ldots$ of

$$n = |\pi| = \sum_{k \geq 1} kr_k .$$

We write $r_\ast$ for the vector $(r_1, r_2, \ldots)$ of repetitions in $\pi$ and $r = r(\pi) = \sum_{k \geq 1} r_k$ for their sum, and

$$\binom{r}{r_\ast} = \frac{r(\pi)!}{\prod r_k!}$$

for the associated multinomial function. For example, $r_\ast = n, 0, \ldots \Rightarrow r = n, r_\ast = 0, \ldots, 0, 1 \Rightarrow r = 1$.

Proposition

$$b^{MU}(z) = \exp(b \log_{MU}(z)) \in ((MU \otimes \mathbb{Q})_\ast CP_\infty)[[z]]$$

and hence

$$h(b_n^{MU}) = \sum_{|\pi| = n} \binom{r}{r_\ast} \prod_{k \geq 1} (k^{-1} h(CP_{k-1}))^{r_k} b(r) \in S_\ast \otimes \mathbb{Z}[b_\ast] .$$

$^3$From here on, integral (co)homology coefficients will often be omitted. The Chern classes of the tangent bundle (up to a sign) equal the classes defined by complete symmetric functions of roots of the normal bundle $\nu$. Chern-Weil theory presents these global invariants in terms of local curvature forms.
Proof Let \( \kappa(z) = \log b^{\mu}(z) = bz + \ldots \), where \( \log(1 - x) = -\sum_{k \geq 1} \frac{z^k}{k} \) : then \( \kappa(z_0 + \mu z_1) = \kappa(z_0) + \kappa(z_1) \). If now \( z_i = \exp_{\mu}(w_i) = w_i + \ldots \), \( i = 0, 1 \), then \( \exp_{\mu}(w_0 + w_1) = z_0 + \mu z_1 \), so \( \kappa \circ \exp_{\mu}(w_0 + w_1) = \kappa \circ \exp_{\mu}(w_0) + \kappa \circ \exp_{\mu}(w_1) \) and hence \( \kappa \circ \exp_{\mu}(w) = bw \), i.e. \( b^{\mu}(\exp_{\mu}(w)) = \exp(bw) \), so

\[
b^{\mu}(z) = \exp(b \log_{\mu}(z)) = \exp(b \sum_{k \geq 1} \frac{\mu^{k-1}}{k} z^k) ;
\]

but it is elementary [13 I §2.14] that this maps to

\[
\sum_{n \geq 1, |n| = n} \binom{r}{r_\ast} \prod_{i \geq 1} \frac{(k^{-1} \mu(CP_{k-1}))^{r_k}}{r_k!} \cdot (\sum r_k)! \cdot \frac{b^{k} \mu}{(\sum r_k)!} \cdot z^n \in (S_\ast(b \ast))(z) .
\]

This can be reformulated as a

Corollary If \( t \in \mathbb{R}_+ \) then

\[
CP_n(t\omega) = t^n \cdot \text{vol}(CP_n, \omega) + \cdots + t \cdot n^{-1} \text{vol}(CP_1, \omega) \in (B \otimes \mathbb{R})_*,
\]

(analogous to a heat kernel expansion). \( \square \)

3.2 Such relations are familiar from the theory of symmetric functions. If

\[
E(z) = \prod_{i \geq 1} (1 + x_i t) = \sum_{k \geq 0} e_k t^k
\]

\[
H(z) = \prod_{i \geq 1} (1 - x_i t)^{-1} = \sum_{k \geq 0} h_k t^k
\]

then \( E(z)H(-z)^{-1} = 1 \), while

\[
H'(t)/H(t) = \sum_{k \geq 1} p_k t^k
\]

with \( p_k = \sum_{i \geq 1} x_i^k \), so

\[
H(t) = \exp(\sum_{k \geq 1} \frac{p_k}{k} t^k) .
\]

This suggests a formal analogy in which the symplectic cobordism class of \( CP_k(\omega) \) is to the complete symmetric function \( h_k \) as \( bCP_{k-1} \) is to the power sum \( p_k \), with \( b \) playing the role of the inverse temperature \( \beta = 1/kT \) in statistical mechanics.
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