Counting mapping class group orbits under shearing coordinates

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Abstract
Let $S_{g,n}$ be an oriented surface of genus $g$ with $n$ punctures, where $2g - 2 + n > 0$ and $n > 0$. Any ideal triangulation of $S_{g,n}$ induces a global parametrization of the Teichmüller space $T_{g,n}$ called the shearing coordinates. We study the asymptotics of the number of the mapping class group orbits with respect to the standard Euclidean norm of the shearing coordinates. The result is based on the works of Mirzakhani.

Keywords Hyperbolic surfaces · Mapping class group · Orbits counting · Shearing coordinates · Teichmüller theory · Weil–Petersson volume

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Contents
1 Introduction ............................................... 2
2 Preliminaries ............................................... 3
  2.1 Teichmüller space and Fenchel-Nielsen coordinates ........................................... 3
  2.2 Asymptotically piecewise linear functions ...................................................... 4
3 The shearing coordinates ......................................... 5
  3.1 Shear between adjacent triangles .............................................................. 5
  3.2 The shearing coordinates ............................................................ 6
  3.3 Relation between shear and length ...................................................... 7
4 Mirzakhani’s counting result and the bounding condition ........................ 9
  4.1 Restatement of the bounding condition ...................................................... 9
  4.2 Relation between length and shear ...................................................... 10
  4.3 The norm of shearing coordinates is bounding ........................................... 14
5 Weil–Petersson volume under shearing coordinates ........................... 14
  5.1 Weil–Petersson volume form .............................................................. 15
  5.2 Proof of Theorem 1.1 ........................................ 17
References .................................................. 18

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1 Introduction

There are plenty of results about counting closed geodesics on hyperbolic surfaces. One of the most significant results is Mirzakhani’s count of simple closed geodesics. Let $X$ be a complete hyperbolic metric on $S_{g,n}$ and let $\gamma$ be a simple closed curve. Denote the hyperbolic length of the geodesic representation of $\gamma$ on $X$ by $\ell_\gamma (X)$. Let

$$s_X(L,\gamma) := \# \{ \beta \in \text{Mod}_{g,n} \cdot \gamma \mid \ell_\beta (X) \leq L \}$$

be the number of simple closed curves in the mapping class group orbit of $\gamma$ with hyperbolic length at most $L$. Mirzakhani [9] proved that, as $L \to +\infty$,

$$s_X(L,\gamma) \sim \frac{n_\gamma \cdot B(X)}{b_{g,n}} L^{6g-6+2n}.$$

In the above formula, the coefficient $n_\gamma$ is determined by the topological type of the curve, and $B(X)$ is an integrable function on the moduli space, endowed with the Weil–Petersson volume form. The integration of $B(X)$ defines the constant $b_{g,n}$. The above result is extended to arbitrary closed curves or multi-curves by Mirzakhani [11], see also [2]. There is also a different proof from another viewpoint, see [5].

There are two well-known parametrizations of the Teichmüller space related to the Weil–Petersson symplectic form. One is the Fenchel–Nielsen coordinates, which is defined by choosing a pants decomposition. The other is the shearing coordinates associated to an ideal triangulation (or, in general, a maximal geodesic lamination) [15]. As shown by Mirzakhani [11], by counting the mapping class group orbit of a fixed hyperbolic surface in Teichmüller space, one can understand the distribution of lengths and twists of curves in a random pants decomposition. A similar question is how a random ideal triangulation of a hyperbolic surface looks like.

In this paper, we count the number of the mapping class group orbits in the shearing coordinates. The main result is the following:

**Theorem 1.1** Let $\Delta$ be a given ideal triangulation of $S_{g,n}$, where $2g - 2 + n > 0$ and $n > 0$. Let $Sh_\Delta : T_{g,n} \to \mathbb{R}^{6g-6+3n}$ be the associated shearing coordinates of the Teichmüller space. Then for any $X \in T_{g,n}$, as $L \to +\infty$, we have:

$$\# \{ \phi \in \text{Mod}_{g,n} \mid \| Sh_\Delta (\phi \cdot X) \| \leq L \} \sim \frac{n_\Delta \cdot B(X)}{b_{g,n}} L^{6g-6+2n}.$$

Moreover, the coefficient $n_\Delta$ is determined by the topological type of $\Delta$ and can be expressed as

$$n_\Delta = \mu_{wp} \{ Y \in T_{g,n} \mid \| Sh_\Delta (Y) \| \leq 1 \}.$$

Here $\| \cdot \|$ is the standard Euclidean norm of $\mathbb{R}^{6g-6+3n}$, and $\mu_{wp}$ denotes the Weil–Petersson volume form on the Teichmüller space. Note that for any mapping class $\phi \in \text{Mod}_{g,n}$, $Sh_\Delta (\phi \cdot X) = Sh_{\phi^{-1} \Delta} (X)$. Thus (1.1) counts the number of mapping class group orbit of a given ideal triangulations.

To prove Theorem 1.1, we use the following result of Mirzakhani [11, Theorem 1.1], which is later generalized by Arana-Herrera [1, Theorem 5.5].
Theorem 1.2 [1,11] Let $F: T_{g,n} \to \mathbb{R}_+$ be a positive, continuous, proper function that is asymptotically piecewise linear and bounding with respect to the Fenchel–Nielsen coordinates. Then
\[
\lim_{L \to +\infty} \frac{\# \{ \phi \in \text{Mod}_{g,n} \mid F(\phi \cdot X) \leq L \}}{L^{6g-6+2n}} = \frac{B(X) \cdot r(F)}{b_{g,n}},
\]
where
\[
r(F) := \lim_{L \to +\infty} \frac{\mu_{wp} \{ Y \in T_{g,n} \mid F(Y) \leq L \}}{L^{6g-6+2n}}.
\]

See Sects. 2.2 and 4.1 for the definitions of asymptotically piecewise linear and bounding functions, respectively. The most important examples are hyperbolic lengths of closed curves. To apply Theorem 1.2, we show that the shearing coordinates satisfy the following properties:

(C1) The shear on each edge of $\Delta$ is asymptotically piecewise linear with respect to the Fenchel-Nielsen coordinates. We observe that each shear can be described by an asymptotically piecewise linear function of the hyperbolic lengths of some closed curves. The proof is presented in Sect. 3.

(C2) The Euclidean norm of the shearing coordinates is bounding with respect to the Fenchel-Nielsen coordinates. The proof is presented in Sect. 4. We first give an equivalent definition of the bounding condition. Then we express the length functions in terms of the shearing coordinates, again in an asymptotically linear way.

To compute the coefficient $r(F)$ in Theorem 1.2 when $F$ is the shearing norm, we show that the Weil–Petersson volume form is equal to Euclidean volume form under the shearing coordinates, up to a scaling constant. This is done in §5.

2 Preliminaries

2.1 Teichmüller space and Fenchel-Nielsen coordinates

We recall some basic notions from the theory of Teichmüller spaces. For more details, see [7,8].

Given a topological surface $S_{g,n}$, its Teichmüller space $T_{g,n}$ is the space of all complete hyperbolic metrics up to isotopy. More precisely, a point in $T_{g,n}$ is an equivalence class of pairs $(f, \Sigma)$, where $f$ is an orientation-preserving homeomorphism from $S_{g,n}$ to a complete hyperbolic surface $\Sigma$. Two pairs $(f_1, \Sigma_1)$ and $(f_2, \Sigma_2)$ are equivalent if and only if $f_2 \circ f_1^{-1}$ is homotopic to an isometry from $\Sigma_1$ to $\Sigma_2$.

Let $\gamma$ be a closed curve on $S_{g,n}$, which is neither homotopic to a point nor to a puncture. Given $X \in T_{g,n}$ represented by a pair $(f, \Sigma)$, the curve $f(\gamma)$ is freely homotopic to a unique closed geodesic on $\Sigma$. The hyperbolic length of $f(\gamma)$ on $\Sigma$ depends on the equivalence class of the pair. Thus it defines a function on $T_{g,n}$ called the length function of $\gamma$, denoted by $\ell_\gamma$.

The hyperbolic length can also be expressed by the trace of a matrix. If $X \in T_{g,n}$ corresponds to a Fuchsian representation $\rho_X: \pi_1(S_{g,n}) \to \text{PSL}(2, \mathbb{R})$, then for any closed curve $\gamma \in \pi_1(S_{g,n})$, we have
\[
|\text{tr}(\rho_X(\gamma))| = \cosh \left( \frac{\ell_\gamma(X)}{2} \right).
\]
Let $\text{Mod}_{g,n}$ be the mapping class group of $S_{g,n}$, i.e. the group of isotopy classes of orientation-preserving self-homeomorphisms leaving each puncture fixed. Then $\text{Mod}_{g,n}$ acts on $T_{g,n}$ by changing the markings. If $X \in T_{g,n}$ is represented by $(f, \Sigma)$ and $\phi \in \text{Mod}_{g,n}$, then $\phi \cdot X \in T_{g,n}$ is represented by $(f \circ \phi^{-1}, \Sigma)$. In particular, for a closed curve $\gamma$, we have $\ell_{\gamma}(\phi \cdot X) = \ell_{\phi^{-1} \cdot \gamma}(X)$. The action of $\text{Mod}_{g,n}$ on $T_{g,n}$ is properly discontinuous, thus the orbit of any point is discrete.

Recall that a pair of pants is a topological surface homeomorphic to $S_{0,3}$. A pants decomposition of $S_{g,n}$ is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Let $\mathcal{P} = \{\alpha_i\}_{i=1}^{3g-3+n}$ be a pants decomposition of $S_{g,n}$. The Fenchel-Nielsen coordinates adapted to $\mathcal{P}$ consist of the length functions of $\alpha_i$ and the twist parameters $\tau_{\alpha_i}$ along $\alpha_i$. For the precise definition of twist parameter, see [4, §3.3]. We have a homeomorphism

$$\text{FN}_\mathcal{P} : T_{g,n} \longrightarrow \mathbb{R}^{3g-3+n} \times \mathbb{R}^{3g-3+n} \quad X \longmapsto \left( l_{\alpha_i}(X), \tau_{\alpha_i}(X) \right).$$

The Fenchel–Nielsen coordinates induce a canonical symplectic 2-form on $T_{g,n}$, which is called the Weil-Petersson symplectic form. A remarkable fact due to Wolpert [16] is that the Weil-Petersson symplectic form does not depend on the choice of the pants decomposition. Thus it gives a volume form $\mu_{wp}$ on $T_{g,n}$ which is invariant under the action of the mapping class group.

### 2.2 Asymptotically piecewise linear functions

We introduce the notion of asymptotically piecewise linear function, following [11, §4].

A closed cone $C \subset \mathbb{R}^k$ is a noncompact closed region bounded by finitely many hyperplanes, that is,

$$C = \bigcap_{i=1}^{m} \{ x \in \mathbb{R}^k \mid R_i(x) \geq 0 \},$$

where each $R_i(x) = r_{i,1}^1 x_1 + \cdots + r_{i,k}^k x_k$ is a linear function.

We say that $x$ tends to infinity in $C$, denoted by $x \to C_\infty$, if $x$ stays within the closed cone $C$ and

$$\min_{i=1 \ldots m} \{ R_i(x) \} \to +\infty.$$

Geometrically, this means that $x$ stays asymptotically away from the boundary of $C$.

Let $F : C \to \mathbb{R}$ be a function on a closed cone. We say that $F$ is asymptotically linear if there exists a linear function $L : C \to \mathbb{R}$ and some real number $c$ such that

$$\lim_{x \to C_\infty} \left( F(x) - L(x) \right) = c.$$

For this we write $F \sim L$ in $C$. Note that $F \sim L$ in $C$ if and only if for any $\varepsilon > 0$, there exists $A > 0$ such that

$$\left| F(x) - (L(x) + c) \right| < \varepsilon$$

for any $x \in C$ satisfying $\min \{ R_i(x) \} > A$. 

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Roughly speaking, being asymptotically linear means that, far away from the hyperplanes, the function behaves asymptotically like a linear function. Simple examples of asymptotically linear functions are $\cosh^{-1}(e^x)$ and $\sinh^{-1}(e^x)$. We have
\[
\cosh^{-1}(e^x) \sim x, \quad \sinh^{-1}(e^x) \sim x \text{ in } \mathbb{R}_+.
\]
A function is defined to be \textit{asymptotically piecewise linear}, if one can divide its domain of definition into finitely many closed cones such that the function is asymptotically linear on each cone. Similarly, a vector-valued function $F$ is asymptotically (piecewise) linear if each component is asymptotically (piecewise) linear. Equivalently, there exists a linear transformation $L: \mathbb{R}^k \to \mathbb{R}^l$ and $c \in \mathbb{R}^l$ such that
\[
\|F(x) - (L(x) + c)\| \to 0 \quad \text{as } x \to C_{\infty}.
\]
We will use the abbreviation “A(P)L” for "asymptotically (piecewise) linear" throughout this paper.

The following composition law is easy to prove.

\textbf{Proposition 2.1} (Composition law for APL functions) Let $C$ be a closed cone in $\mathbb{R}^k$ and let $F_i : C \to \mathbb{R}$ be APL functions for $i = 1, \ldots, m$. Let $C'$ be a closed cone in $\mathbb{R}^m$ and let $G : C' \to \mathbb{R}$ be an APL function. Assume that for all $x \in C$, $(F_1(x), \ldots, F_m(x)) \in C'$. Then $H := G(F_1, \ldots, F_m)$ is again an APL function.

Given a family $\mathcal{M}$ of APL function, let $\mathcal{F}$ be the set of functions generated by $\mathcal{M}$, under arithmetic operations, rational multiplication or $N$-th root:
\[
f \pm g, \quad f \cdot g, \quad f, \quad r \cdot f, \quad \sqrt{f}.
\]
Then for each $f \in \mathcal{F}$, the functions $\sinh^{-1}(e^f)$, $\cosh^{-1}(e^f)$ are APL. In this paper, all APL functions we considered are obtained from such construction.

The following basic result is due to Mirzakhani [11, Theorem 4.1]:

\textbf{Proposition 2.2} For any closed curve $\gamma$ on $S_{g,n}$, the hyperbolic length function $\ell_\gamma : T_{g,n} \to \mathbb{R}_+$ is an APL function with respect to any given Fenchel-Nielsen coordinates.

In fact, Mirzakhani [11] shows that the transformation between Fenchel-Nielsen coordinates associated to any two different pants decomposition is APL. Thus the property of being APL with respect to Fenchel-Nielsen coordinates does not depend on the the choice of the pants decomposition.

\section{The shearing coordinates}

The shearing coordinates of $T_{g,n}$ were introduced by Thurston [15]. We will show that, for any ideal triangulation, the shear on each edge is an APL function with respect to the Fenchel-Nielsen coordinates.

\subsection{Shear between adjacent triangles}

Let $T_1, T_2$ be a pair of adjacent ideal geodesic triangles in the hyperbolic plane, with a common edge $c$. Note that all ideal geodesic triangles are isometric. An ideal triangle has a unique inscribed circle tangent to its three edges. Denote the tangent points of the inscribed circles on the common edge $c$ by $p_i (i = 1, 2)$. 

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**Definition 3.1** The shear on the common edge $c$ of two adjacent ideal triangles $T_1, T_2$ is the signed distance from $p_1$ to $p_2$, denoted by $s_c$. Here $c$ is oriented such that $T_1$ is on the left and $T_2$ is on the right.

One can check that interchanging the order of $T_1, T_2$ does not change the shear. See Fig. 1 for the case of a positive shear.

The following formula relates shear with cross-ratio. We adopt the upper half plane model for the hyperbolic plane. The ideal boundary $\partial \mathbb{H}$ is identified with $\mathbb{R} \cup \{\infty\}$. Denote the geodesic with two different end points $x, y \in \partial \mathbb{H}$ by $[x, y]$. And denote the ideal triangle with three different ideal vertices $x, y, z \in \partial \mathbb{H}$ by $[x, y, z]$.

**Proposition 3.2** Let $x_1, x_2, x_3, x_4$ be four distinct points on $\partial \mathbb{H}$, in counterclockwise order. Let $T_1 = [x_1, x_2, x_3]$ and $T_2 = [x_1, x_3, x_4]$, with the common edge $c = [x_1, x_3]$. Then

$$s_c = \ln \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)}.$$  \hspace{1cm} (3.1)

### 3.2 The shearing coordinates

Let $\Delta$ be an ideal triangulation on $S_{g,n}$. For any $X \in T_{g,n}$ represented by $(f, \Sigma)$, $f(\Delta)$ is homotopic to an ideal geodesic triangulation of $\Sigma$. For each edge $c \in \Delta$, the shearing on $f(c)$ under the hyperbolic metric of $\Sigma$ is denoted by $s_c(X)$. (To define $s_c(X)$, we can lift the map $f : S_{g,n} \to \Sigma$ to the universal covers.) Note that $s_c(X)$ is independent of the choice of the representation $(f, \Sigma)$, thus well-defined on $T_{g,n}$.

We can recover the hyperbolic structure by gluing those hyperbolic ideal triangles with the data of triangulation and shears. See Fig. 2 for an example, which shows an ideal triangulation of $S_{1,1}$ in the universal covering space.

**Theorem 3.3** [3, Theorem 3.6] The map $Sh_\Delta : T_{g,n} \to \mathbb{R}^{6g-6+3n}$, defined by

$$Sh_\Delta(X) := (s_c(X))_{c \in \Delta},$$

is a homeomorphism onto its image.

The shear parameters are not all independent. At each puncture, the completeness of the hyperbolic structure induces a linear equation of the shears on the edges emitting from it. In fact, the sum of these shears should be zero. Since there are $n$ punctures, there are $(6g-6+2n)$ independent parameters, which coincides with the dimension of $T_{g,n}$.

**Proposition 3.4** The shearing coordinates reduces to a homeomorphism from $T_{g,n}$ to a linear subspace $C_\Delta$ of dimension $(6g - 6 + 2n)$.
Fig. 2 An ideal triangulation of $S_{1,1}$ in $\mathbb{D}$. The ideal quadrilateral $ABCD$ is a fundamental domain. The shear on each edge is labelled as a colored segment.

See [3] for the proofs.

### 3.3 Relation between shear and length

Let $X = (f, \Sigma)$ be a point in $T_{g,n}$, and let $\Gamma$ be a Fuchsian group such that $\Sigma = \mathbb{H}/\Gamma$. In the following, we use the trace formula (2.1) to represent the shear as a function of hyperbolic lengths.

Since each ideal vertex is corresponding to the fixed point of some parabolic element in $\Gamma$, we first recall some basic properties of parabolic elements in $\text{SL}_2(\mathbb{R})$.

**Lemma 3.5** Suppose that $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{SL}_2(\mathbb{R})$ is a parabolic element. Then

1. $|\text{tr}(g)| = |A + D| = 2$;
2. The unique fixed point of $g$ on the ideal boundary is $x = \frac{A - D}{2C}$.

**Proposition 3.6** Suppose that

$$g_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \text{SL}_2(\mathbb{R}), \ i = 1, 2$$
represent two parabolic elements in the Fuchsian group $\Gamma$, with fixed points $x_i \neq \infty$ on the boundary. Then
\[
\text{tr} (g_1 \cdot g_2) = \frac{1}{2} \text{tr} (g_1) \text{tr} (g_2) - C_1 C_2 (x_1 - x_2)^2. 
\] (3.2)

In particular, $C_1 C_2 (x_1 - x_2)^2$ is invariant under conjugation.

**Proof** By Lemma 3.5, one gets the equations for $i = 1, 2$:
\[
\begin{align*}
A_i D_i - B_i C_i &= 1, \\
(A_i + D_i)^2 &= 4, \\
A_i - D_i &= 2x_i C_i.
\end{align*}
\]
Note that $x_i \neq \infty$ implies $C_i \neq 0$. Solve for $B_i, D_i$:
\[
\begin{align*}
D_i &= A_i - 2x_i C_i, \\
B_i &= -x_i^2 C_i.
\end{align*}
\]
Then
\[
\text{tr} (g_1 \cdot g_2) = A_1 A_2 + B_1 C_2 + B_2 C_1 + D_1 D_2 = 2(A_1 - x_1 C_1)(A_2 - x_2 C_2) - C_1 C_2 (x_1 - x_2)^2
\]
\[
= 2 \left( A_1 - \frac{A_1 - D_1}{2} \right) \left( A_2 - \frac{A_2 - D_2}{2} \right) - C_1 C_2 (x_1 - x_2)^2.
\]
\[\square\]

**Remark 3.7** If $g_1$ and $g_2$ are distinct parabolic elements, then either $g_1 g_2$ or $g_1 g_2^{-1}$ is hyperbolic.

**Corollary 3.8** Let $T_1, T_2$ be a pair of adjacent ideal triangles on $S_{g,n}$, with common edge $c$. There exist four closed curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ on $S_{g,n}$ such that $s_c(X)$ is an APL function of the hyperbolic length of $\gamma_i$:
\[
s_c(X) = F_c \left( \ell_{\gamma_1}(X), \ell_{\gamma_2}(X), \ell_{\gamma_3}(X), \ell_{\gamma_4}(X) \right). 
\]
The closed curves and the APL function depend only on the topology of $T_1$ and $T_2$.

**Proof** Denote $\Sigma$ by $\mathbb{H}/\Gamma$ as before. Let $\tilde{T}_1, \tilde{T}_2$ be a pair of adjacent preimage of $T_1, T_2$, with ideal vertices $x_1, x_2, x_3, x_4$ in counterclockwise order and a common edge $[x_1, x_3]$.

Each $x_i$ corresponds to a primitive parabolic element $g_i \in \Gamma$. We may also assume that all $C_i \neq 0$, otherwise we can replace $\Gamma$ by an appropriate conjugation. For $i \neq j$, let $\gamma_{ij}$ be the closed curve corresponding to the invariant geodesic axis of the hyperbolic element $g_i g_j$ or $g_i g_j^{-1}$. Combining formula (2.1) with Proposition 3.6, we have:
\[
-C_i C_j (x_i - x_j)^2 = \pm 2 \cosh \left( \frac{\ell_{\gamma_{ij}}(X)}{2} \right) - \frac{\text{tr}(g_i \cdot g_j)}{2},
\]
where $\text{tr}(g_i) = \pm 2$ are constants.

Since $x_i \neq 0$, we can rewrite formula (3.1) as
\[
s_c(X) = \frac{1}{2} \ln \frac{C_1 C_2 (x_1 - x_2)^2 C_3 C_4 (x_3 - x_4)^2}{C_1 C_4 (x_1 - x_4)^2 C_2 C_3 (x_2 - x_3)^2}.
\]
It is easy to verify that \( s_c(X) \) is APL with respect to the four length functions (using the discussion after Proposition 2.1).

Given a pair of ideal triangles, the choices of \( g_i \)'s and \( \gamma_{ij} \)'s only depend on the fundamental group of the surface. Since \( T_{g,n} \) is simply connected, when the hyperbolic structure on the surface changes continuously in \( T_{g,n} \), all signs appeared in the above function remains the same. As a result, the function is determined by topology. \( \square \)

Given an ideal triangulation \( \Delta \), for each edge \( c \in \Delta \) we can choose four closed curves and a function \( F_c \) as above. By Proposition 2.2, each variable in Corollary 3.8 is an APL function with respect to the Fenchel-Nielsen coordinates. By the composition law in Proposition 2.1, we obtain:

**Theorem 3.9** Given an ideal triangulation \( \Delta \) on the surface \( S_{g,n} \), the shearing coordinates \( Sh_\Delta : T_{g,n} \rightarrow \mathbb{R}^{6g-6+3n} \) is APL with respect to the Fenchel-Nielsen coordinates.

### 4 Mirzakhani’s counting result and the bounding condition

The original bounding condition in Theorem 1.2 is proposed by Mirzakhani [11]. It is used to reduce the counting problem in the Teichmüller space to a problem in some specific cone-shaped region. Here we adopt the definition in [1].

#### 4.1 Restatement of the bounding condition

Under the Fenchel-Nielsen coordinates \((\ell_i, \tau_i)_{i=1}^{3g-3+n}\) adapted to some pants decomposition \( P \), the Teichmüller space \( T_{g,n} \) admits a partition into countably many convex polytopes of the form

\[
C^m_P := \left\{ Y \in T_{g,n} \mid m_i \cdot \ell_i(Y) \leq \tau_i(Y) \leq (m_i + 1) \cdot \ell_i(Y) \right\}
\]

with \( m := (m_1, \ldots, m_{3g-3+n}) \in \mathbb{Z}^{3g-3+n} \).

**Definition 4.1** A function \( \mathcal{F} : T_{g,n} \rightarrow \mathbb{R}_+ \) is bounding with respect to the Fenchel-Nielsen coordinates \((\ell_i, \tau_i)_{i=1}^{3g-3+n}\), if for every \( Y \in T_{g,n} \) there exists a constant \( C > 0 \) such that for every \( m \in \mathbb{Z}^{3g-3+n} \) and every \( Z \in \text{Mod}_{g,n} \cdot Y \cap C^m_P \cap \mathcal{F}^{-1}((0, L]), \)

\[
\ell_i(Z) \leq C \cdot \frac{L}{\max\{|m_i|, |m_i + 1|\}}.
\]

(4.1)

This means that, when the value of \( \mathcal{F} \) grows, the length of the \( i \)-th pants curve grows at a linear rate, and it is also proportional to the twist component.

**Proposition 4.2** The bounding condition (4.1) is equivalent to the following condition: for every \( Y \in T_{g,n} \),

\[
\sup_{Z \in \text{Mod}_{g,n} \cdot Y} \left\{ \frac{\ell_i(Z) + |\tau_i(Z)|}{\mathcal{F}(Z)} \right\} < +\infty.
\]

(4.2)

**Proof** We write \( \ell_i = \ell_i(Z) \) and \( \tau_i = \tau_i(Z) \) for simplicity. Suppose (4.1) holds. When \( m_i \geq 0 \) and \( Z \in C^m_P \), we have \( 0 \leq m_i \leq \tau_i / \ell_i \leq m_i + 1 \). Then
\[ C \geq (m_i + 1) \frac{\ell_i}{F} \geq \frac{m_i + 2 \ell_i}{2F} = \frac{\ell_i + (m_i + 1)\ell_i}{2F} \geq \frac{\ell_i + \tau_i}{2F}. \]

Thus
\[ \frac{\ell_i + \tau_i}{F} \leq 2C. \]

When \( m_i < 0 \), we have \( 0 \leq -m_i - 1 \leq |\tau_i|/\ell_i \leq -m_i \). Then
\[ C \geq (-m_i) \frac{\ell_i}{F} \geq \frac{-m_i + 1 \ell_i}{2} = \frac{\ell_i - (m_i + 1)\ell_i}{2F} = \frac{\ell_i - |\tau_i|}{2F}. \]

So (4.2) holds.

Now suppose \( F \) satisfies (4.2), with upper bound \( K > 0 \). If \( Z \in \text{Mod}_{g,n} \cdot Y \) with \( 0 \leq m_i \leq \tau_i/\ell_i \leq m_i + 1 \), then
\[ K \geq \frac{\ell_i + \tau_i}{F} \geq \frac{\ell_i + m_i\ell_i}{F}. \]

If \( m_i \leq \tau_i/\ell_i \leq m_i + 1 \leq 0 \), then
\[ K \geq \frac{\ell_i - \tau_i}{F} \geq \frac{\ell_i - (m_i + 1)\ell_i}{F} = \frac{-m_i\ell_i}{F}. \]

Thus (4.1) holds.

Note that each part of the proof utilizes one side of the condition \( m_i \leq \tau_i/\ell_i \leq m_i + 1 \).

\[ \square \]

### 4.2 Relation between length and shear

Our aim here is to prove that \( F(X) = \| Sh_\Delta(X) \| \) satisfies the inequality (4.2). In the following theorem, we have a formula of the length function in terms of shears. This is the key result in this paper.

**Theorem 4.3** Let \( \gamma \) be a non-degenerated closed curve on \( S_{g,n} \), and let \( X \in T_{g,n} \) be represented by \( \mathbb{H}/\Gamma \). Let \( g \in \Gamma \) be a primitive hyperbolic element corresponding to \( \gamma \). Let \((s_1, \ldots, s_K) = Sh_\Delta(X)\) be the shearing coordinates of \( X \) associated to the triangulation \( \Delta \). Then \( |\text{tr}(g)| = 2 \cosh(\ell_\gamma(X)/2) \) is a polynomial of variables \( \{e^{\pm s_i/2}\}_{i=1}^K \) with rational coefficients:
\[ \cosh(\ell_\gamma/2) \in \mathbb{Q}[e^{\pm s_1/2}, \ldots, e^{\pm s_K/2}]. \]

The polynomial only depends on the topological type of \( \Delta \) and \( \gamma \).

**Proof** We will give a precise algorithm to compute the matrix \( g \in \text{SL}_2(\mathbb{R}) \).

Passing to the universal cover, we fix an orientation of \( \gamma \) and choose one intersection of \( \gamma \) and \( \Delta \) as an initial point (here we have identified \( \gamma \) with the axis of \( g \)). Up to a conjugation, we may assume that the initial point of \( \gamma \) is the complex number \( i \) in the upper half plane, contained in the ideal triangle \( T_1 = [-1, 1, \infty] \).

Traveling forward along \( \gamma \), we have a sequence of triangles \((T_1, \ldots, T_{N+1} = gT_1)\). Let \( a_i \) be the common ideal edge of \( T_i, T_{i+1}, i = 1, \ldots, N \). The sequence \((a_i)_{i=1}^N\) depends only on...
the type of $\gamma$ and $\Delta$, thus a topological data. The shear on $a_i$ is denoted by $s_i$. Our algorithm consists of three steps:

1) **The initial data: “Left & Right sequences”**.

If $\gamma$ enters $T_i$ through one edge, then it must leave through one of the other two edges. Given the orientation of $X$ and $\gamma$, one can tell whether $\gamma$ leave the triangle through the left edge or the right edge. Thus we can say that $T_{i+1}$ lies on the left or on the right hand side of $T_i$. Define $\varepsilon_i = +1$ if $T_{i+1}$ lies on the left, $\varepsilon_i = -1$ if on the right. See Fig. 3 for illustration and examples.

Then we get a sequence of signatures $(\varepsilon_i)_{i=1}^N$, which tells how $\gamma$ passing through each triangles along the path.

2) **The basic matrices of shear**.

The initial data characterizes how $\gamma$ passing through each triangle. Now we construct the basic matrices of Möbius transformation corresponding to the shear deformation.

If $\gamma$ leaves through the right edge, we may first apply a Möbius transformation $R$ to map $(-1, i, 1)$ into $(\infty, 1 + 2i, 1)$. Then the two triangles $T_1, R(T_1)$ have a common edge $[1, \infty]$ with the same tangent points. If $\gamma$ leaves through the left edge, we map $(-1, i, 1)$ into $(-1, -1 + 2i, \infty)$. The corresponding matrices are

$$R := \frac{1}{2} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad L := \frac{1}{2} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$
To unify them, we may define

\[ P_\varepsilon := \frac{1}{2} \begin{pmatrix} 3 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}, \quad \varepsilon = \pm 1 \]

Then \( R = P_{-\varepsilon}, \ L = P_{\varepsilon} \).

We have mapped the entering edge onto the leaving edge by \( P_\varepsilon \) as above. If \( \gamma \) leaves through the right edge, then shearing along the leaving edge means applying a hyperbolic transformation with fixed point \( \{1, \infty\} \) and signed translation distance \( s \). In complex coordinate, the function is \( z \mapsto e^s(z-1) + 1 \). Similarly, for the left edge case, the complex function is \( z \mapsto e^{-s}(z+1) - 1 \). The unified matrices are

\[ H_\varepsilon(s) := \begin{pmatrix} e^{-\varepsilon s/2} - \sinh(s/2) \\ 0 \end{pmatrix} \]

Some calculation shows that the third vertex of \( (H_\varepsilon(s) \circ P_\varepsilon)(T_1) \) other than \( \pm 1, \infty \) is \(-\varepsilon(1 + 2e^{-\varepsilon s})\).

In conclusion, we take \( V_\varepsilon(s) := H_\varepsilon(s) \circ P_\varepsilon \). It maps \( T_1 \) to the next triangle along \( \gamma \).

(3) The composition diagram.

Let \( g \) be a primitive hyperbolic element, corresponding to the translation along \( \gamma \) for a single period. We now compute the matrix representation for \( g \). It is a composition of basic matrices defined as above.

Let \( (T_i)_{i=1}^N, (a_i)_{i=1}^N, (\varepsilon_i)_{i=1}^N \) as before. Denote by \( a_0 \) the edge at which \( \gamma \) enters \( T_1 \). For any \( i = 1, \ldots, N \), there is a unique isometry \( f_i \) that maps \( T_i \) into \( T_{i+1} \) with the edge \( a_{i-1} \) matching \( a_i \). Then

\[ g = f_N \circ \cdots \circ f_1. \]

Denote \( V_{\varepsilon_i}(s_{\varepsilon_i}) \) by \( V_i, i = 1, \ldots, N \) to simplify notations. Note that \( f_1 = V_1 \).

We have the following diagram:
By going to the bottom right corner diagonally and then going up, one get
\[ g = V_1 \circ V_2 \circ \cdots \circ V_{N-1} \circ V_N \circ V_{N-1}^{-1} \circ \cdots \circ V_2^{-1} \circ V_1^{-1} \circ f_1 \]
\[ = V_1 \circ V_2 \circ \cdots \circ V_{N-1} \circ V_N \]
\[ = H_{\varepsilon_1} (s_1) P_{\varepsilon_1} \cdots H_{\varepsilon_N} (s_N) P_{\varepsilon_N} . \]

It is obvious that each element in the matrix is a polynomial of \( e^{\pm s_i/2} \), with rational coefficients. The polynomials are determined by topology.  

For the twist \( \tau_i \) on each pants curve, we have a similar result.

**Corollary 4.4** With the notations in Definition 4.1 and Theorem 4.3, we have:
\[ \cosh^2 (\tau_i) \in \mathbb{Q}(e^{\pm s_1/2}, \ldots, e^{\pm s_N/2}) . \]
Furthermore, on each mapping class group orbit, the denominator of the associated rational function is bounded away from 0.

**Proof** It is known that the twist \( \tau \) along each pants curve \( \gamma \) can be described in terms of the length functions of some closed curves. See [4, Chapter 3] and [11]. In the following, to simplify notation, the length of a curve is denoted by the same notation as the curve itself.

The following formulae will be used later: \[ \cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1, \quad \sinh 2x = 2 \sinh x \cosh x . \]

There are two types of the curves:

1. The curve \( \gamma \) is contained in a (1,1)-type subsurface. See Fig. 4a.

   We have:
   \[ \begin{cases} 
   \cosh d \sinh^2 (\gamma/2) = \cosh (\delta/2) + \cosh^2 (\gamma/2) \\
   \cosh (\mu/2) = \cosh (d/2) \cosh (\tau/2) 
   \end{cases} . \]

   Here \( \delta \) is the other boundary curve of the pants, \( d \) is a segment perpendicular to \( \gamma \), and \( \mu \) is a simple closed curve. It follows that
   \[ 1 + \cosh \tau = \frac{(\cosh \mu + 1)(\cosh \alpha - 1)}{\cosh (\delta/2) + \cosh \alpha} . \]

2. The curve \( \gamma \) is contained in a (0,4)-type subsurface. See Fig. 4b.

   We have:
   \[ \begin{cases} 
   \cosh \frac{\mu}{2} = \cosh d \sinh \frac{\beta}{2} \sinh \frac{\beta'}{2} - \cosh \frac{\beta}{2} \cosh \frac{\beta'}{2} \\
   \cosh d = \cosh \tau \sinh h \sinh h' + \cosh h \cosh h' \\
   \cosh h' \sinh \frac{\nu}{2} \sinh \frac{\nu'}{2} = \cosh \frac{\delta'}{2} + \cosh \frac{\gamma}{2} \cosh \frac{\delta'}{2} \cosh \frac{\gamma}{2} \sinh \frac{\nu'}{2} \cosh \frac{\nu'}{2} 
   \end{cases} . \]

   Here \( \beta, \delta, \beta', \delta' \) are the other boundary curves of the pants, and \( d, h, h' \) are the common perpendicular segments, of certain topological type, from \( \beta \) to \( \beta' \), \( \beta \) to \( \gamma \) and \( \beta' \) to \( \gamma \), respectively. \( \mu \) is a simple closed curve. We have
   \[ \cosh \tau = \frac{\cosh d - \cosh h \cosh h'}{\sinh h \sinh h'} . \]
We have shown that $\cosh^2 \tau$ is a rational function of desired form. Note that in a certain mapping class group orbit, the length of pants curves have a lower bound. In both of the above two cases, the denominator is a function of length of pants curves, which is bounded from below.

**Remark 4.5** The results and proofs in this section also hold if $\Delta$ is a maximal lamination with only isolated and closed leaves.

According to [3], for each closed leaf, there is a linear equation about its length and the shear. And the shear on closed leaves are basically the same as twist. These facts ensure the APL property in both directions.

### 4.3 The norm of shear coordinates is bounding

**Proposition 4.6** The norm $\| Sh_{\Delta} \|$ is bounding with respect to the Fenchel-Nielsen coordinates.

**Proof** Let $\| s \|$ be the Euclidean norm of a vector $s$. By Theorem 4.3, for each $i$ there are positive rational numbers $M_i, A_i, N_i, B_i$ such that

$$\cosh(\ell_i/2) \leq M_i e^{A_i \| s \|}, \quad \cosh^2(\tau_i) \leq N_i e^{B_i \| s \|}$$

on a particular mapping class group orbit. Thus $\ell_i, |\tau_i|$ are bounded by linear functions of $\| s \|$, with positive leading coefficients. By Proposition 4.2, $\| Sh_{\Delta} \|$ is bounding with respect to the Fenchel-Nielsen coordinates. \hfill $\square$

### 5 Weil–Petersson volume under shearing coordinates

The last task is to show that under the shearing coordinates, the Weil–Petersson volume form is the Euclidean volume form, up to a scaling constant. To see this, we use the cataclysm coordinates of Thurston [15].
5.1 Weil–Petersson volume form

There is a close relation between the Thurston symplectic form on the space of measured foliations and the Weil–Petersson symplectic form, via the shearing coordinates \[ 14\]. In the special case of ideal triangulation, the relation is rather simple \[ 12\]. Let us describe it in the following.

A measured foliation on a surface is a foliation with singularities together with a transverse measure, which is invariant under homotopic moving along the leaves of foliation. Two measured foliations are equivalent if one can be transformed to the other by isotopies moves and Whitehead moves, which allow to break down or combine the singularities. Usually, a measured foliation refers to an equivalence class. The space of all equivalence classes of measured foliations on a topological surface is denoted by \( \mathcal{MF} \). We refer to \[ 6\] for more details on measured foliations.

Here we shall consider a slightly different kind of measured foliations on surfaces with punctures. A measured foliation is said to be trivial around the punctures if each puncture has a neighborhood on which the induced foliation is a cylinder, foliated by homotopic closed leaves, and any transverse segment converging to a puncture has infinite total measure. Let \( \mathcal{MF}_0 \) be the space of all equivalence classes of measured foliations which are trivial around the punctures.

The space \( \mathcal{MF}_0 \) has a piecewise linear structure. It admits a 2-form called the Thurston symplectic form, and this symplectic form induces a natural volume form.

Given an ideal triangulation \( \Delta \) and a hyperbolic surface \( X \in \mathcal{T}_{g,n} \), there is a partial foliation on \( X \) whose leaves are segments of horocycles centred at the ideal vertices. The non-foliated region in each ideal triangle is a small triangle bounded by three horocycle segments of length 1, meeting tangentially at the tangent points of the inscribed circle. By collapsing each small non-foliated triangle to a 3-pronged singularity, we obtain a foliation. See Fig. 5. We can endow it with a transverse measure such that the measure of any geodesic arc contained in an edge of \( \Delta \) equals to its hyperbolic length. The completeness of the hyperbolic metric guarantees that the leaves near each puncture must be closed. Thus we obtain a foliation \( \mathcal{F}_\Delta(X) \) called horocyclic measured foliation, which is trivial around the punctures.

Let \( \mathcal{MF}_0(\Delta) \) be the open subset of \( \mathcal{MF}_0 \) containing all measured foliations which are transverse to \( \Delta \). By \[ 15\], Proposition 9.4] and \[ 12\], the map

\[
\mathcal{F}_\Delta : \mathcal{T}_{g,n} \longrightarrow \mathcal{MF}_0(\Delta)
\]

\[
X \mapsto \mathcal{F}_\Delta(X)
\]
An ideal triangulation of $S_{1,2}$ with the shear data.

The horocyclic foliation, with colored singular leaves.

The partial measured foliation. Its boundary must consist of closed singular leaves.

**Fig. 6** From the shear data to the partial measured foliation is a homeomorphism. We also refer to [13] for the proof and more results about horocyclic measured foliations.

**Proposition 5.1** [12, Corollary 4.2] *The homeomorphism $\mathcal{F}_\Delta$ pulls back the Thurston’s symplectic form on $\mathcal{MF}_0(\Delta)$ to the Weil–Petersson form on $\mathcal{T}_{g,n}$.***

The space $\mathcal{MF}_0(\Delta)$ has a good parametrization. By deleting all the closed leaves of $\mathcal{F}_\Delta(X)$ parallel to the punctures, we obtain a partial measured foliation. Denote the transverse measure of this partial foliation on each edge $c \in \Delta$ by $m_c(X)$, which is the hyperbolic length of a finite geodesic segment. See Fig. 6 for an example of this process on $S_{1,2}$. Then we can embed $\mathcal{MF}_0(\Delta)$ into $\mathbb{R}^{6g-6+3n}_{\geq 0}$ as an Euclidean polyhedron cone:

$$\mathcal{MF}_0(\Delta) \rightarrow \mathbb{R}^{6g-6+3n}_{\geq 0}$$

$$\mathcal{F}_\Delta(X) \mapsto (m_{c_i}(X))_{i=1}^N$$
Fig. 7 The measured foliation in a pair of adjacent ideal triangles

Note that the Thurston’s symplectic form has a simple expression in the above parametrization. And the induced volume form is just a constant multiple of Euclidean volume form on each face of the polyhedron.

Let $C_{\Delta}$ be the image of the shearing coordinates, as a subspace of $\mathbb{R}^{6g-6+3n}$. By the above construction, we can regard the composition map $S_{\Delta} \circ F_{\Delta}^{-1}$ as a map between Euclidean spaces. The following is equivalent to [15, Proposition 9.1].

Proposition 5.2 The coordinate transformation $S_{\Delta} \circ F_{\Delta}^{-1}$ from $\mathcal{MF}_0(\Delta)$ to $C_{\Delta}$ is determined by

$$s_e(X) = \frac{1}{2} \left( m_a(X) + m_c(X) - m_b(X) - m_d(X) \right)$$

for each edge $e \in \Delta$. Here $a, b, e$ and $c, d, e$ are the edges of two adjacent ideal triangles, with $a, b, c, d$ in counterclockwise order.

Proof The formula follows immediately from the definition of shear and the construction of the partial measured foliation. See Fig. 7. \qed

Now that the coordinate transformation is linear in local charts, with constant coefficients, and the Thurston volume form on $\mathcal{MF}_0(\Delta)$ is a constant multiple of Euclidean volume form, the transformation must push forward the Thurston volume form on $\mathcal{MF}_0(\Delta)$ to another constant multiple of Euclidean volume form on $C_{\Delta}$. Together with Proposition 5.1, we obtain the following.

Theorem 5.3 The Weil–Petersson volume form under the shearing coordinates is a constant scaling of the Euclidean volume form on $C_{\Delta}$.

5.2 Proof of Theorem 1.1

Proof It is obvious that the shear norm $\|S_{\Delta}\|$ is proper in $T_{g,n}$. Applying Theorem 3.9 and Proposition 4.6 to Theorem 1.2, we have

$$\lim_{L \to +\infty} \frac{\# \left\{ \phi \in \text{Mod}_{g,n} \mid \|S_{\Delta}(\phi \cdot X)\| \leq L \right\}}{L^{6g-6+2n}} = \frac{n_{\Delta} \cdot B(X)}{b_{g,n}},$$

where

$$n_{\Delta} = \lim_{L \to +\infty} \frac{\mu_{wp} \left\{ Y \in T_{g,n} \mid \|S_{\Delta}(Y)\| \leq L \right\}}{L^{6g-6+2n}}.$$
is obviously homogenous. Thus the coefficient $n_\Delta$ is equal to the volume of the unit ball. This finishes the proof.

\[ \square \]

References

1. Arana-Herrera, F.: Counting hyperbolic multi-geodesics with respect to the lengths of individual components. arXiv e-prints arXiv:2002.10906 (2020)
2. Arana-Herrera, F.: Equidistribution of families of expanding horospheres on moduli spaces of hyperbolic surfaces. Geom. Dedicata 210, 65–102 (2021)
3. Bestvina, M., Bromberg, K., Fujiwara, K., Souto, J.: Shearing coordinates and convexity of length functions on Teichmüller space. Am. J. Math. 135(6), 1449–1476 (2013)
4. Buser, P.: Geometry and spectra of compact Riemann surfaces. Birkhauser Boston, Boston (1992) (Reprinted in 2010)
5. Erlandsson, V., Souto, J.: Mirzakhani’s curve counting(research announcement. arXiv e-prints (2019), arXiv:1904.05091
6. Fathi, A., Laudenbach, F., Poenaru, V., et al.: Travaux de Thurston sur les surfaces. Astérisque, vol. 66–67, Société Mathématique de France, Paris (1979)
7. Farb, B., Margalit, D.: A primer on mapping class groups. Princeton mathematical series, vol. 49. Princeton University Press, Princeton (2012)
8. Hubbard, J.H.: Teichmüller theory and applications to geometry, topology, and dynamics: vol 1 Teichmüller theory. Matrix Editions, Ithaca (2006)
9. Mirzakhani, M.: Growth of the number of simple closed geodesics on hyperbolic surfaces. Ann. Math. II 168(1), 97–125 (2008)
10. Mirzakhani, M.: Ergodic theory of the earthquake flow. Int. Math. Res. Not. IMRN, no. 3, Art. ID rnm116, 39 (2008)
11. Mirzakhani, M.: Counting mapping class group orbits on hyperbolic surfaces. arXiv e-prints (2016). arXiv:1601.03342
12. Papadopoulos, A., Penner, R.C.: The Weil-Petersson symplectic structure at Thurston’s boundary. Trans. Am. Math. Soc. 335(2), 891–904 (1993)
13. Papadopoulos, A., Théret, G.: On Teichmüller’s metric and Thurston’s asymmetric metric on Teichmüller space. In: Papadopoulos, A. (ed.) Handbook of Teichmüller theory, vol. 1, pp. 447–467. European Math. Soc. (2007)
14. Sözen, Y., Bonahon, F.: The weil-Petersson and Thurston symplectic forms. Duke Math. J. 108(3), 581–597 (2001)
15. Thurston, W.P.: Minimal stretch maps between hyperbolic surfaces. arXiv e-prints (1998) https://arxiv.org/abs/math/9801039
16. Wolpert, S.A.: The Fenchel-Nielsen deformation. Ann. Math. II 115(3), 501–528 (1982)