Novel Results on the Number of Runs of the Burrows-Wheeler-Transform

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Abstract. The Burrows-Wheeler-Transform (BWT), a reversible string transformation, is one of the fundamental components of many current data structures in string processing. It is central in data compression, as well as in efficient query algorithms for sequence data, such as webpages, genomic and other biological sequences, or indeed any textual data. The BWT lends itself well to compression because its number of equal-letter-runs (usually referred to as $r$) is often considerably lower than that of the original string; in particular, it is well suited for strings with many repeated factors. In fact, much attention has been paid to the $r$ parameter as measure of repetitiveness, especially to evaluate the performance in terms of both space and time of compressed indexing data structures.

In this paper, we investigate $\rho(v)$, the ratio of $r$ and of the number of runs of the BWT of the reverse of $v$. Kempa and Kociumaka [FOCS 2020] gave the first non-trivial upper bound as $\rho(v) = O(\log^2(n))$, for any string $v$ of length $n$. However, nothing is known about the tightness of this upper bound. We present infinite families of binary strings for which $\rho(v) = \Theta(\log n)$ holds, thus giving the first non-trivial lower bound on $\rho(n)$, the maximum over all strings of length $n$.

Our results suggest that $r$ is not an ideal measure of the repetitiveness of the string, since the number of repeated factors is invariant between the string and its reverse. We believe that there is a more intricate relationship between the number of runs of the BWT and the string's combinatorial properties.

Keywords: Burrows-Wheeler-Transform, compressed data structures, string indexing, repetitiveness, combinatorics on words
1 Introduction

Since its introduction in 1994 by Michael Burrows and David J. Wheeler, the Burrows-Wheeler Transform (BWT) [6] has played a fundamental role in lossless data compression and string-processing algorithms. The BWT of a word \( w \) can be obtained by concatenating the last characters of the lexicographically-sorted conjugates (that is, rotations) of \( w \). Among its many fundamental properties, this permutation turns out to be invertible and more compressible than the original word \( w \). The latter property follows from the fact that sorting the conjugates of \( w \) has the effect of clustering together repeated factors; as a consequence, characters preceding those repetitions are clustered together in the BWT, and thus repetitions in \( w \) tend to generate long runs of equal characters in its BWT. The more repetitive \( w \), the lower the number \( r \) of such runs. This fact motivated recent research on data structures whose size is bounded as a function of \( r \): the most prominent example in this direction, the \( r \)-index [13], is a fully-compressed index of size \( O(r) \) able to locate factor occurrences in log-logarithmic time each. Other examples of recent algorithms working in runs-bounded space include index construction [14] and data compression in small working space [1, 24, 25].

As it turns out, \( r \) is a member of a much larger family of word-repetitiveness measures that have lately generated much interest in the research community. Examples of those measures include (but are not limited to) the number \( z \) of factors in the LZ77 factorization [21], the number \( b \) of the smallest bidirectional macro scheme [26], and the size \( e \) of the CDAWG [4]. More recently, it was shown that all those compressors are particular cases of a combinatorial object named string attractor [16] whose size \( \gamma \) lower-bounds all measures \( r, z, g, b, \) and \( e \). In turn, in [19] it was shown that \( \gamma \) is lower-bounded by another measure, \( \delta \), which is linked to factor complexity (that is, to the number of distinct factors of each length) and better captures the word’s repetitiveness. On the upper-bound side, the papers [16,19] provided approximation ratios of all measures but \( r \) with respect to \( \gamma \). Finding an upper-bound for \( r \) remained an open problem until the recent work of Kempa and Kociumaka [15], who showed that, for any word of length \( n \), \( r = O(\delta \log^2 n) \) (which in turn implies \( r = O(\gamma \log^2 n) \)). As stated explicitly in [15], this implies the first upper bound on the ratio \( \rho \) between \( r \) and the number of runs in the BWT of the reverse of the word, namely \( \rho = O(\log^2 n) \).

This leaves open the interesting question of whether this bound is tight. In this paper, we give a first answer to this question by exhibiting an infinite family of binary words whose members satisfy \( \rho = \Theta(\log n) \). This contrasts the experimental observation made in [2, 25] that \( \rho \) appears to be constant on real repetitive text collections, and shows that \( r \) is not a strong repetitiveness measure since—unlike \( b, g, \gamma \), and \( \delta \)—it is not invariant under reversal.

An added value of the proof we present lies in a surprising insight into the exact structure of the BWT matrix of the words we study: right-extensions of Fibonacci words. This insight allows us to further extend the method to a much larger family of words, giving the number of runs of the BWT for both the word and its reverse, for right-extensions of all standard words. As it turns out, the
words we obtain from Fibonacci words are maximal with respect to ρ within this class. At the same time, we have verified experimentally that these words are not maximal among all words of the same length. This leaves a gap on the maximum on ρ, taken over all words of length n, between our lower bound Ω(log n) and the upper bound of O(log² n) of [15].

As a matter of fact, the reverse of the Fibonacci extensions allow us to prove an even more surprising result: a single character extension can increase the upper bound of ρ by a multiplicative factor Θ(log n). This result is the equivalent of the “one-bit catastrophe” exhibited by Lagarde and Perifel [20] for Lempel-Ziv ’78: using these compression schemes, the compression ratio of the word can change dramatically if just one bit is prepended to the input.

2 Basics

Let Σ = {a, b}, with a < b. A binary word (or string) w is a finite sequence of characters (or letters) from Σ. We denote the i-th character of w by w[i] and index words from 1. We denote by |w| the length of w, and by |w|a resp. |w|b the number of characters a resp. b in w. The empty word ε is the unique word of length 0. The set of words over Σ is denoted Σ*. We write wrev = w[n] · · · w[1] for the reverse of a word w of length n. The word w is a conjugate of the word w if w' = w[i] · · · w[n]w[1] · · · w[i − 1] =: conji(w) for some i = 1, ..., n (also called the i-th rotation of w).

If w = uxa, for some words u, x, v ∈ Σ*, then u is called a prefix, v a suffix, and x a factor of w. A prefix (suffix, factor) u of w is called proper if u ̸= w. A word u is a circular factor of w if it is the prefix of some conjugate of w. A circular factor u is called left-special if both au and bu occur as circular factors. For an integer k ≥ 1, uk = u · · · u is the k-th power of u. A word w is called primitive if w = uk implies k = 1. A word w is primitive if and only if it has exactly |w| distinct conjugates.

For two words v, w, the longest common prefix lcp(v, w) is defined as the maximum length word u such that u is a prefix both of v and of w. The lexicographic order on Σ* is defined by: v < w if either v is a proper prefix of w, or va is a prefix of v and vb is a prefix of w, where u = lcp(v, w). A Lyndon word is a primitive word which is lexicographically smaller than all of its conjugates. To simplify the discussion, we will assume on that w is primitive (but everything can be extended also to non-primitive words).

The Burrows-Wheeler-Transform (BWT) [6] of a word w of length n is a permutation of the characters of w, defined as the sequence of final characters of the lexicographically ordered set of conjugates of w. More precisely, let the BW-array be an array of size n defined as: BW[i] = k if conjk(w) is the i-th conjugate of w in lexicographic order.5 Then bwt(w)[i] = w[BW[i] − 1], where we set w[0] = w[n]. Another way to visualize the BWT is via an (n × n)-matrix containing the lexicographically sorted conjugates of w: the BWT of w equals

5 Note that this is in general not the same as the suffix array SA, since here we have the conjugates and not the suffixes.
the last column of this matrix, read from top to bottom, see Fig. 1. By definition, \( \text{bwt}(w) = \text{bwt}(w') \) if and only if \( w \) and \( w' \) are conjugates.

For a word \( w \), let \( \text{runs}(w) \) denote the number of maximal equal-letter runs of \( w \), and \( r(w) = \text{runs}(\text{bwt}(w)) \). We are now ready for our main definition:

**Definition 1.** Let \( w \in \{a, b\}^* \). We define the runs-ratio \( \rho(w) \) as

\[
\rho(w) = \max \left( \frac{\text{runs}(\text{bwt}(w))}{\text{runs}(\text{bwt}(w^{rev}))}, \frac{\text{runs}(\text{bwt}(w^{rev}))}{\text{runs}(\text{bwt}(w))} \right),
\]

and \( \rho(n) = \max \{ \rho(w) : \|w\| = n \} \).

Note that \( \rho(w) \geq 1 \) holds by definition. Since \( r(w) = r(w^{rev}) \) for all \( w \) with \( \|w\| \leq 6 \), we have \( \rho(n) = 1 \) for \( n < 7 \). In Table 1, we give the values of \( \rho(n) \) for \( n = 7, \ldots, 30 \) (computed with a computer program):

| \( n \) | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|-------|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \( \rho(n) \) | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 | 1.5 |

Table 1. The values of \( \rho(n) \) for \( n = 7, \ldots, 30 \).

We introduce standard words next, following [11]. Given an infinite sequence of integers \( (d_0, d_1, d_2, \ldots) \), with \( d_0 \geq 0, d_i > 0 \) for all \( i > 0 \), called a directive sequence, define a sequence of words \( (s_i)_{i \geq 0} \) of increasing length as follows:

\[
s_0 = b, \quad s_1 = a, \quad s_{i+1} = s_i^{d_i-1} s_{i-1}, \quad \text{for } i \geq 1.
\]

The index \( i \) is referred to as the order of \( s_i \). The best known example is the sequence of Fibonacci words, which are given by the directive sequence \( (1, 1, 1, \ldots) \), and of which the first few elements are as follows:

\[
s_0 = b, \quad s_1 = a, \quad s_2 = ab, \quad s_3 = aba, \quad s_4 = ababa, \quad s_5 = abababa, \quad s_6 = abababaababa,
\]

\[
s_7 = abababaababaababaabaababaabaababaabaababa, \ldots
\]

Note that \( \|s_i\| = F_i \), where \( F_i \) is the Fibonacci sequence, defined by \( F_0 = F_1 = 1 \) and \( F_{i+1} = F_i + F_{i-1} \). Moreover, \( \|s_i\|_{a} = F_{i-1} \) and \( \|s_i\|_{b} = F_{i-2} \), for \( i \geq 2 \).

Standard words are used for the construction of infinite Sturmian words, in the sense that every characteristic Sturmian word is the limit of a sequence of standard words (cf. Chapter 2 of [22]). These words have many interesting combinatorial properties and appear as extreme case in a great range of contexts [7,8,10,12,18]. A fundamental result in connection with the BWT is the following:

\[
\text{bwt}(w) = b^q a^p \text{ with } \gcd(q, p) = 1 \text{ if and only if } w \text{ is a standard word} [23].
\]

### 3 Fibonacci-plus words have \( \rho = \Theta(\log n) \)

Since for a standard word \( s \), \( s^{rev} \) is a conjugate, we have \( \rho(s) = 1 \) for all standard words \( s \). We will show in this section that adding just one character at the end of the word suffices to increase \( \rho \) from 1 to logarithmic in the length of the word.
Definition 2. A word $v$ is called Fibonacci-plus if it is either of the form $sb$, where $s$ is a Fibonacci word of even order $2k$, $k \geq 2$, or of the form $sa$, where $s$ is a Fibonacci word of odd order $2k + 1$, $k \geq 2$. In the first case, $v$ is of even order, otherwise of odd order.

The aim of this section is to prove the following theorem:

**Theorem 1.** Let $v$ be a Fibonacci-plus word, and let $|v| = n$. Then $\rho(s) = \Theta(\log n)$.

We will prove the theorem by showing that, for a Fibonacci-plus word $v$, $r(v) = 4$ (Prop. 2) and $r(v^\text{rev})$ is linear in the order of the word itself (Prop. 3). The statement will then follow by an argument on the length of $v$.

Fibonacci words have very well-known structural and combinatorial properties [9], some of them can be deduced from more general properties that hold true for all standard words (see [3, 5, 11, 12]). In the next proposition we summarize some of these properties, which will be useful in the following.

**Proposition 1 (Some known properties of the Fibonacci words).** Let $s_i$ be the Fibonacci word of order $i \geq 0$. The following properties hold:

1. for all $k \geq 1$, $s_{2k} = x_{2k}ab$ and $s_{2k+1} = x_{2k+1}ba$, where $x_{2k}$ and $x_{2k+1}$ are palindromes ($x_{2k} = \varepsilon$).
2. for all $k \geq 2$,
   - $s_{2k} = x_{2k-1}bax_{2k-2}ab = x_{2k-2}aba x_{2k-1}ab$
   - $s_{2k+1} = x_{2k}abx_{2k-1}ba = x_{2k-1}bax_{2k}ba$.
3. for all $i \geq 2$, $axb$ is a Lyndon word.
4. for all circular factors $y, z$ of $s_i$ with $|y| = |z|$, and for each $c \in \Sigma$, one has that $|y|_c - |z|_c \leq 1$ (Balancedness Property).

**Example 1.** Let us consider $s_8 = abababaaababaababaabaababaaba$ the Fibonacci word of order 8 and length $F_8 = 34$.

One can verify that the prefix $x_8 = abababaaababaababaababaaba$ is a palindrome. Moreover $x_8 = x_7bax_6 = x_6aba x_7$, where $x_7 = abababaaababaaba$ and $x_6 = ababababaab$.

**Proposition 2.** Let $v$ be a Fibonacci-plus word. Then $r(v) = 4$. In particular,

1. if $v = s_{2k}b$, then $\text{bwt}(v) = bF_{2k-2}F_{2k-1}^{-1}ba$, and
2. if $v = s_{2k+1}a$, then $\text{bwt}(v) = babF_{2k-1}^{-1}F_{2k}$.

**Proof.** We give the proof for even order only. The proof for odd order is analogous.

Let us write $v = sb$, with $s = s_{2k}$. Since Fibonacci words are standard words, it follows that $\text{bwt}(s) = bF_{2k-2}F_{2k-1}^{-1}$ (see Sec. 2). Since $s$ is of even order, it can be written as $s = xab$ for a palindrome $x$ (Prop. 1, part 1); moreover, it follows from the specific form of $x$ (Prop. 1, part 2) that both $xab$ and $xba$ are conjugates. It is further known that the two conjugates $xab$ and $xba$ are at
Let Proposition 3.

Example 1. If \( v = s_2b \) for some \( k \geq 1 \), then BW \((v^rev) = bF_{2k}aF_{2k-1}aF_{2k}b \ldots bF_{2k-2}aF_{2k-1}\).

Our proof is based on a detailed analysis of the structure of the BW matrix of \( v^rev \). We will divide the BW matrix, and thus the BW, into three parts,
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| BW array | rotations of $v = \text{rev}$ BWT$((v))$ | BW array | rotations of $v^{rev} = \text{rev} BWT((v^{rev}))$ |
|----------|-----------------------------------------|----------|------------------------------------------|
| 1 21     | aabaababaabaabbaababaabaabbaababaababaabab b | 1 3      | aabaababaabaababaabaababaababaababaabab b |
| 2 8      | aabaababaabbaabaababaababaabbaababaababaabab b | 2 11     | aabaababaabaababaabaababaababaababaabab b |
| 3 29     | aabaababaabaabbaababaababaabbaababaababaabab b | 3 24     | aabaababaabaababaabaababaababaababaabab b |
| 4 16     | aabaababaabbaabbaababaababaabbaababaababaabab b | 4 6      | aabaababaabbaabbaababaababaababaabab b |
| 5 3      | aabaababaabbaabbaababaababaabbaababaababaabab b | 5 19     | aabaababaabbaabbaababaababaababaabab b |
| 6 24     | aabaababaabbaabbaababaababaabbaababaababaabab b | 6 14     | aabaababaabbaabbaababaababaababaabab b |
| 7 11     | aabaababaabbaabbaababaababaabbaababaababaabab b | 7 27     | aabaababaabbaabbaababaababaababaabab b |
| 8 32     | aabaababaabbaabbaababaababaabbaababaababaabab b | 8 32     | aabaababaabbaabbaababaababaababaabab b |
| 9 19     | aabaababaabbaabbaababaababaabbaababaababaabab b | 9 9      | aabaababaabbaabbaababaababaababaabab b |
| 10 6     | aabaababaabbaabbaababaababaabbaababaababaabab b | 10 22    | aabaababaabbaabbaababaababaababaabab b |
| 11 27    | aabaababaabbaabbaababaababaabbaababaababaabab b | 11 4     | aabaababaabbaabbaababaababaababaabab b |
| 12 14    | aabaababaabbaabbaababaababaabbaababaababaabab b | 12 17    | aabaababaabbaabbaababaababaababaabab b |
| 13 1     | aabaababaabbaabbaababaababaabbaababaababaabab b | 13 12    | aabaababaabbaabbaababaababaababaabab b |
| 14 22    | aabaababaabbaabbaababaababaabbaababaababaabab b | 14 25    | aabaababaabbaabbaababaababaababaabab b |
| 15 9     | aabaababaabbaabbaababaababaabbaababaababaabab b | 15 30    | aabaababaabbaabbaababaababaababaabab b |
| 16 30    | aabaababaabbaabbaababaababaabbaababaababaabab b | 16 7     | aabaababaabbaabbaababaababaababaabab b |
| 17 17    | aabaababaabbaabbaababaababaabbaababaababaabab b | 17 20    | aabaababaabbaabbaababaababaababaabab b |
| 18 4     | aabaababaabbaabbaababaababaabbaababaababaabab b | 18 15    | aabaababaabbaabbaababaababaababaabab b |
| 19 25    | aabaababaabbaabbaababaababaabbaababaababaabab b | 19 28    | aabaababaabbaabbaababaababaababaabab b |
| 20 12    | aabaababaabbaabbaababaababaabbaababaababaabab b | 20 33    | aabaababaabbaabbaababaababaababaabab b |
| 21 33    | aabaababaabbaabbaababaababaabbaababaababaabab b | 21 35    | aabaababaabbaabbaababaababaababaabab b |
| 22 22    | aabaababaabbaabbaababaababaabbaababaababaabab b | 22 2     | aabaababaabbaabbaababaababaababaabab b |
| 23 7     | aabaababaabbaabbaababaababaabbaababaababaabab b | 23 10    | aabaababaabbaabbaababaababaababaabab b |
| 24 28    | aabaababaabbaabbaababaababaabbaababaababaabab b | 24 23    | aabaababaabbaabbaababaababaababaabab b |
| 25 15    | aabaababaabbaabbaababaababaabbaababaababaabab b | 25 5     | aabaababaabbaabbaababaababaababaabab b |
| 26 2     | aabaababaabbaabbaababaababaabbaababaababaabab b | 26 18    | aabaababaabbaabbaababaababaababaabab b |
| 27 23    | aabaababaabbaabbaababaababaabbaababaababaabab b | 27 13    | aabaababaabbaabbaababaababaababaabab b |
| 28 10    | aabaababaabbaabbaababaababaabbaababaababaabab b | 28 26    | aabaababaabbaabbaababaababaababaabab b |
| 29 31    | aabaababaabbaabbaababaababaabbaababaababaabab b | 29 31    | aabaababaabbaabbaababaababaababaabab b |
| 30 18    | aabaababaabbaabbaababaababaabbaababaababaabab b | 30 8     | aabaababaabbaabbaababaababaababaabab b |
| 31 5     | aabaababaabbaabbaababaababaabbaababaababaabab b | 31 21    | aabaababaabbaabbaababaababaababaabab b |
| 32 26    | aabaababaabbaabbaababaababaabbaababaababaabab b | 32 16    | aabaababaabbaabbaababaababaababaabab b |
| 33 13    | aabaababaabbaabbaababaababaabbaababaababaabab b | 33 29    | aabaababaabbaabbaababaababaababaabab b |
| 34 35    | aabaababaabbaabbaababaabbaabbaababaababaabab b | 34 34    | aabaababaabbaabbaababaababaababaabab b |
| 35 34    | aabaababaabbaabbaababaabbaabbaababaababaabab b | 35 1     | aabaababaabbaabbaababaababaababaabab b |

![Fig. 1. BWT-matrices of the Fibonacci-plus word $v = s_9b$ of length 35 and its reverse, underlined the added $b$.](image)

based on the positions of three specific conjugates of $v^{rev}$, and analyse each of these separately.

Now consider the first few conjugates of $v^{rev}$. Since $v = s_{2k}b = x_{2k}abb$, we have $v^{rev} = bbax_{2k}$, noting that $x_{2k}$ is a palindrome. Thus

\[
\begin{align*}
\text{conj}_1(v^{rev}) &= bbax_{2k}, \\
\text{conj}_2(v^{rev}) &= ba x_{2k} b, \\
\text{conj}_3(v^{rev}) &= ax_{2k} bb, \\
\text{conj}_4(v^{rev}) &= x_{2k} bba.
\end{align*}
\]
Since Fibonacci words have no occurrence of $bb$, the conjugate $\text{conj}_1(v^{rev}) = v^{rev}$ is the last row of the matrix. Moreover, by Prop. 1, $ax_{2k}b$ is a Lyndon word, and therefore $\text{conj}_3(v^{rev})$, having only an extra $b$ at the end, is also Lyndon, and thus can be found in the first row. The relative order of the other two conjugates is also clear, since $x_{2k}$ begins with an $a$, thus we have

$$ax_{2k}bb < x_{2k}bba < bax_{2k}b < bbax_{2k}.$$ 

We will now subdivide the BWT-matrix into three parts, according to the positions of these conjugates, and we will call these top part, middle part, and bottom part. The conjugates $ax_{2k}bb$, $x_{2k}bba$ and $bax_{2k}b$ are the first row of the top part, middle part and bottom part, respectively. We use this to partition the BWT into the three corresponding parts $\text{bwt}(v^{rev})_{\text{top}}$, $\text{bwt}(v^{rev})_{\text{mid}}$, and $\text{bwt}(v^{rev})_{\text{bot}}$. Thus we have

$$\text{bwt}(v^{rev}) = \text{bwt}(v^{rev})_{\text{top}} \cdot \text{bwt}(v^{rev})_{\text{mid}} \cdot \text{bwt}(v^{rev})_{\text{bot}}.$$ 

We will prove the form of the BWT of $v^{rev}$ separately for the three parts. In Fig. 2 we give a visual presentation of the proof.

3.1 Bottom part

**Proposition 4.** $\text{bwt}(v^{rev})_{\text{bot}} = ba^{F_{2k} - 2}.$

**Proof.** By definition, the bottom part starts with the conjugate $\text{conj}_2(v) = bax_{2k}b.$ Since $ax_{2k}bb$ is Lyndon (Prop. 1, part 3), it is smaller than all other conjugates, and therefore, $bax_{2k}b$ is smaller than all other conjugates starting with $b$. Thus, the bottom part consists exactly of all conjugates starting with $b$. The number of $b$’s in $v$, and thus in $v^{rev}$ is $F_{2k} - 2 + 1.$ Since $s_{2k}$ has no occurrence of $bb$, every $b$ in $v^{rev}$ except the one in position 2 is preceded by an $a$, thus $bax_{2k}b$ is the only conjugate ending in $b$. This proves the claim.

3.2 Middle part

**Lemma 1.** The left-special circular factors of $v^{rev}$ are exactly the prefixes of $x_{2k-1}b$ and the prefixes of $bax_{2k-2}.$

**Proof.** Let $u$ be a left-special circular factor of $v^{rev} = bbax_{2k}$. From Proposition 1, $v^{rev} = bbax_{2k-1}bax_{2k-2} = bbax_{2k-2}abx_{2k-1}$. Since $bb$ occurs only once, $u$ does not contain $bb$ as factor. Moreover, from combinatorial properties of standard words (see [5]), it is known that for each $0 \leq h \leq F_{2k} - 2$, there is exactly one left-special circular factor of $bax_{2k}$ having length $h$ and it a prefix of $x_{2k}.$ Since $x_{2k-1}ba$ (that is a prefix of $x_{2k}$) occurs exactly once in $v^{rev}$ and $bax_{2k-2}$ has exactly two occurrences (one preceded by $b$ and followed by $a$, the other one preceded by $a$ and followed by $b$), either $u$ is prefix of $x_{2k-1}b$ or it is prefix of $bax_{2k-2}.$
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Fig. 2. A sketch of the BWT-matrix of $v^{rev}$ where $v$ is a Fibonacci-plus word.
Lemma 2. Let $s_{2k}$ be a Fibonacci word of even order. Then, for all $i = 0, \ldots, k-2$, $ax_{2(k-i)}b$ and $ax_{2(k-i)-1}b$ have $F_{2i}$ and $F_{2i+1}$ occurrences, respectively, as circular factors of $s_{2k}$.

Proof. The statement can be proved by induction on $i$. For $i = 0$, the statement follows from the fact that $ax_{2k}b$ and $ax_{2k-1}b$ have just $F_0 = F_1$ occurrence. Let us suppose the statement is true for all $j < i$. Note that $ax_{2(k-i)-2}b$ appears as suffix of $ax_{2(k-i)}b$ and as suffix of $ax_{2(k-i)-1}b$. Moreover, such two occurrences are distinct because $ax_{2(k-i)-1}b$ is not a suffix of $ax_{2(k-i)}b$. This means that, by using the inductive hypothesis, the number of occurrences of $ax_{2(k-i)-2}b$ is $F_{2i} + F_{2i+1} = F_{2i+2}$. Analogously, $ax_{2(k-i)-3}b$ appears as prefix of $ax_{2(k-i)-1}b$ and as prefix of $ax_{2(k-i)-2}b$. Moreover, such two occurrences are distinct because $ax_{2(k-i)-2}b$ is not a prefix of $ax_{2(k-i)-1}b$. This means that the number of occurrences of $ax_{2(k-i)-3}b$ is $F_{2i+1} + F_{2i+2} = F_{2i+3}$.

Proposition 5. $bwt(v^{\text{rev}})_{\text{mid}} = a^{F_0}b_0^{F_2}b_1^{F_3} \ldots a^{F_{k-4}}b_{k-2}.$

Proof. For all $2 \leq i < j$, $x_i$ is a prefix (and also a suffix) of $x_j$. This means that the rotations starting with $x_jbb$ are lexicographically greater than $x_jbb$. Moreover, for $1 \leq i \leq k-2$, $x_{2(k-i)}b$ is not a prefix of $x_{2k-1}b$. Thus, by Lemma 1, $x_{2(k-i)}b$ is not left-special. Therefore, each occurrence of $x_{2(k-i)}b$ is preceded by the same character; this character must be $a$, since otherwise, both $bx_{2(k-i)}b$ and $ax_{2(k-i)}a$ would be factors, contradicting the fact that $s_{2k}$ is balanced (Prop. 1, part 4). Therefore, all occurrences of $x_{2(k-i)}b$ correspond to a run of $a$’s in the BWT. The length of this run is $F_{2i}$ by Lemma 2. The claim follows from the fact that each $x_{2(k-i)-1}bb$ occurs exactly once and it is preceded by $b$.

3.3 Top part

Lemma 3. Let $i$ be such that $\text{con}(v^{\text{rev}}) < x_{2k}bba$. Then the last character of $\text{con}(v^{\text{rev}})$ is $b$.

Proof. Let $u = \text{lcp}(\text{con}_i(v^{\text{rev}}), x_{2k}bba)$. Then $u$ is a proper prefix of $x_{2k-1}$. This is because there are only two occurrences of $x_{2k-1}$, one followed by $ba$, this is the prefix of $x_{2k}bba$, and the other followed by $bb$, thus greater than $x_{2k}bba$. Therefore, $u' = ua$ is a prefix of $\text{con}_i(v^{\text{rev}})$ but not of $x_{2k-1}$, and thus by Lemma 1 it is not left-special. Now assume that $\text{con}_i(v^{\text{rev}})$ ends with $a$. Then $aua$ is a factor of $v^{\text{rev}}$, and since $u$ does not contain $bb$, it is thus also a factor of $s_{2k}^{rev}$. On the other hand, $ub$ is left-special, since it is a prefix of $x_{2k-1}b$ (Lemma 1), therefore both $ubu$ and $aua$ are factors of $v^{\text{rev}}$, and again, of $s_{2k}^{rev}$. This implies that both $auv^{\text{rev}}$ and $bu^{\text{rev}}b$ are factors of $s_{2k}$. This is a contradiction, since $s_{2k}$ is balanced (Prop. 1, part 4).

Proposition 6. $\text{bwt}(v^{\text{rev}})_{\text{top}} = b^{F_{2k-1+k+1}}$.

Proof. By Lemma 3, $\text{bwt}(v^{\text{rev}})_{\text{top}}$ consists of $b$’s only. The number of $b$’s of $v$ is $F_{2k-2} + 1$, of which we have accounted for $k$ (since 1 is contained in $\text{bwt}(v^{\text{rev}})_{\text{bot}}$ and $k - 1$ in $\text{bwt}(v^{\text{rev}})_{\text{mid}}$), there remaining exactly $F_{2k-2} - k + 1$ $b$’s.
3.4 Putting it all together

Proof. of Prop. 2: The claim for even-order Fibonacci-plus words follows from Propositions 4, 5, and 6. The claim for odd-order Fibonacci-plus words can be proved in an analogous manner.

Proof. of Thm. 1: From Propositions 2 and 3, we have that $\rho(v) = 2k/4 = k/2$. On the other hand, $n = |v| = F_{2k} + 1$, thus by the properties of the Fibonacci numbers, $2k = \Theta(\log n)$, implying that $\rho(v) = k/2 = \Theta(\log n)$.

4 Standard-plus words have $\rho = \mathcal{O}(\log n)$

In this section we consider other infinite families of finite words, defined from standard words. Here we assume that $d_0 \geq 1$, otherwise we could consider the word obtained by exchanging $a$’s and $b$’s and the results still hold true.

Definition 3. A word $v$ is called standard-plus if it is either of the form $sb$, where $s$ is a standard word of even order $2k$, $k \geq 2$, or of the form $sa$, where $s$ is a standard word of odd order $2k + 1$, $k \geq 2$. In the first case, $v$ is of even order, otherwise of odd order.

We show that, when a standard-plus word $v = s_{2k}b$ is considered, the exact asymptotic growth of $\rho$ depends on the directive sequence of the word $s_{2k}$. Here we give the proof of the result for standard-plus words of even order, however an analogous statement can also be proved for standard-plus words of odd order.

Proposition 7. Let $v = s_{2k}b$ be a standard-plus word of even order. Then $r(v) = 4$.

The proof of Proposition 7 is analogous to that of Proposition 2.

Proposition 8. Let $v = s_{2k}b$ be a standard-plus word of even order $2k$, where $s_{2k}$ is the standard word obtained by using the directive sequence $(d_0, d_1, \ldots, d_{2k-2})$ of length $2k - 1$, where $d_0 \geq 1$. If $d_0 = 1$, then $r(v^{rev}) = 2k$. Otherwise, $r(v^{rev}) = 2k + 2$.

Proof. (Sketch) Similar to what happens with Fibonacci’s words (see Prop. 1), it is known that $s_{2k} = Cb$, where $C$ is a palindrome, the conjugate $aCb$ is a Lyndon word (see [3, 12]). Then $v^{rev} = bbaC$ and, in order to lexicographically sort the conjugates of $v^{rev}$, we can consider its Lyndon rotation $aCbb$. One can verify that $C \in \{a^{d_0}b, a^{d_0+1}b\}^*$. It is possible to see that $bwt(v^{rev})$ ends with $ba1^{s_{2k}}$, since $baCb$ is the smallest rotation starting with $b$. Moreover, since $t = b(a^{d_0}b)^{d_1}b$ is a suffix of $aCbb$, all rotations of $v^{rev}$ starting with the first occurrence of $a$ in each run $a^{d_0}$ in $t$ determine $d_1$ consecutive $b$’s in $bwt(v^{rev})$. If $d_0 = 1$ such rotations are followed by the rotation $baCb$, otherwise several rotations preceded by $a$ (including the rotations starting with the other $a$’s of $t$) are in between. So, if $d_0 = 1$, the last run of $b$’s has length $d_1 + 1$, otherwise the last two runs of $b$’s have length $d_1$ and 1, respectively.
Finally, when $d_i$ (with odd $i$) is used to generate standard words, a set of consecutive rotations starting with $(a^{d_0}b)^{d_1}a^{d_3+1}b$ and preceded by $b$ is produced. This means that the other runs of $b$’s have length $d_3, d_5, \ldots, d_{2k-3}, |s_{2k}b - (d_1 + d_3 + \ldots + d_{2k-3})$.

**Example 3.** Let us consider the standard-plus word $v$ of even order constructed by using the directive sequence $(2, 3, 1, 2, 1)$. One can verify that

$$v = aabaabaabaabaabaabaabaabaabaabaaabb.$$

Moreover, $\text{bwt}(v^{\text{rev}}) = b^{10}ab^2a^3b^3a^{15}ba^{15}$ and $\text{bwt}(v) = b^{15}a^{33}ba$.

**Theorem 2.** Let $v$ be a standard-plus word of even order $n$. Then $\rho(v) = \mathcal{O}(\log n)$.

**Proof.** By definition, $v = s_{2k}b$ where $s_{2k}$ is a standard word of order $n = 2k$ for some positive $k$. Since $|s_{2k}| \geq F_{2k}$, by Prop. 7 and 8, $\rho(v) \leq \frac{k+1}{2} \in \mathcal{O}(\log n)$.

The following proposition states that among all standard-plus words, Fibonacci-plus words are maximal w.r.t. $\rho$.

**Proposition 9.** Let $v$ be a Fibonacci-plus word, and $v'$ a standard-plus word s.t. $|v| = |v'|$. Then $\rho(v) \geq \rho(v')$.

**Proof.** Follows directly from Prop. 7 and 8, and from the fact that Fibonacci words have the longest directive sequence among all standard words of the same length.

### 5 Conclusion and Outlook

In this paper, we presented the first non-trivial lower bound on the maximum runs-ratio $\rho(n)$ of a word of length $n$. This shows for the first time that the widely used parameter $r$, the number of runs of the BWT of a word, is not an ideal measure of the repetitiveness of the word. Moreover, it proves that for BWT-based compression a parallel result holds to the “one-bit catastrophe” recently shown for LZ78-compression [20].

Several open questions remain. We saw in the previous section that Fibonacci-plus words are maximal among the class of standard-plus words with respect to the runs-ratio $\rho$. However, they stay strictly below $\rho(n)$, the maximum among all words of length $n$, even for lengths up to $n = 30$. In Table 2, we report the values of $\rho(n)$ and compare them to the maximum reached by standard-plus words. Note that this is a Fibonacci-plus word only for $n = 9, 14, 22$.

It is possible to construct binary words of arbitrary length and greater runs-ratio $\rho$ than any standard-plus word of the same length. However, we currently do not know the asymptotic growth of the $\rho$ value for such words. Therefore, the question of closing the gap for $\rho(n)$ between our lower bound $\Omega(\log n)$ and the upper bound $\mathcal{O}(\log^2 n)$ remains open.
Table 2. The values of $\rho(n)$ for $n = 9, \ldots, 30$, and the maximum value of $\rho(n)$ among all standard-plus words of length $n$.

It would be interesting to explore the question also for larger alphabets. Our preliminary experimental results on ternary alphabets indicate that the increase in $\rho$ happens at smaller lengths than for the binary case. This suggests that the effect we showed in this paper, of a divergence between the string’s repetitiveness and $\rho$, may be even more pronounced in real-life applications.

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