Effect of noncommutative Chern-Simons term on the magnetic moment of planar fermions

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We study the effect of noncommutative Chern-Simons term on fundamental fermions. In particular the one-loop contribution to the magnetic moment to order $\theta$ is calculated.

I. INTRODUCTION

Noncommutative (NC) field theories are defined on a space-time characterized by the commutation relation: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$; here the noncommutative parameter $\theta^{\mu\nu}$ is treated as a constant. Quantum field theories defined on such a space-time show many interesting features that are absent in ordinary field theories and have recently been vigorously pursued [1]. Though the idea itself is old [2], the recent impetus in this field can be traced to its emergence in string/M(atrix) theory [3, 4]. Keeping the string theoretical considerations aside, NC field theories themselves pose interesting challenges and features. The most discussed properties happen to be the loss of unitarity if one includes time also to be noncommuting [5]. Here it must be pointed out that, the reported violation of unitarity can be overcome if the quantization conditions are defined in a proper manner [6]. Furthermore, contrary to the initial motivation for NC theories [2], namely regulating the ultraviolet divergences; there appear new divergences compared to the corresponding commutative theory. This is due to the coupling of the ultraviolet sector to the infrared, dubbed UV/IR mixing [7].

In the present work we are concerned with NC Chern-Simons (CS) term coupled to planar fermions in the fundamental representation. It is well known that a CS term can be added to the Maxwell lagrangian in 2+1 dimensions which can provide mass to the gauge field independent of Higgs mechanism [8, 9]. Apart from this CS field theories have found applications in various branches of mathematics and physics, notably in knot theory [10] and as an effective description of quantum Hall effect (QHE) [11]. The particular use of NCCS theory in the description of QHE, as shown in [12] and further pursued in [13] is quite interesting.

Independent of its usage in the description of QHE, a number of formal perturbative studies have been conducted [14, 16, 17]. Here we calculate the MM of fermions coupled to the NCCS gauge field. The corresponding calculation in the commutative non-Abelian and the Abelian cases was carried out in [19] and [20, 21] respectively.

The manuscript is organized as follows. In the next section we set up the NCCS action and state

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the Feynman rules. In section III the vertex diagram is evaluated. Our conclusions and discussion of the results are presented in section IV.

II. THE NONCOMMUTATIVE ACTION AND FEYNMAN RULES

The NC action for a pure CS theory in Minkowski space reads

\[ S_{\text{CS}} = \frac{M}{2} \int d^3x \epsilon^{\mu\nu\rho} \left[ A_\mu \star \partial_\nu A_\rho + \frac{2ie}{3} A_\mu \star A_\nu \star A_\rho \right]. \]

(1)

Here \( M \) is the CS coefficient and \( e \) the coupling constant. Note that due to the noncommuting nature of the fields there emerges a three gauge boson interaction term, akin to the commutative non-Abelian theory. Let us consider the three photon interaction term. Since we deal with only spatial noncommutativity one of the star product can be dropped. The rest of the fields being bilinear gives usual product therefore three gauge boson vertex can be dropped \[15\]. This has to be contrasted with the NCCS actions based on Seiberg-Witten map that also leads to the extinction of the three photon vertex \[16\]. This equivalence of the pure NCCS theory has been shown to persist at the quantum level as well \[17\]. To proceed further we add a gauge fixing term

\[ S_{gf} = -\frac{1}{2\xi} \int d^3x (\partial_\mu A^\mu \star \partial_\nu A^\nu). \]

(2)

The Dirac action gives the matter content of the theory

\[ S_{\text{Dirac}} = \int d^3x \bar{\psi} \star (i\gamma^\mu D_\mu - m)\psi, \]

(3)

where \( D_\mu \psi = \partial_\mu \psi - ieA_\mu \star \psi \). The gauge \( (G_{\mu\nu}(p)) \) and Fermionic \( (S(p)) \) propagators are

\[ iG_{\mu\nu}(p) = -\frac{1}{M} \epsilon_{\mu\nu\rho} \frac{p^\rho}{p^2 + i\epsilon}, \]

(4)

and

\[ iS(p) = \frac{i(p + m)}{p^2 - m^2 + i\epsilon}, \]

(5)

respectively. The gauge propagator is defined in the Landau gauge \( (\xi = 0) \) since it was shown that other gauges can introduce spurious infrared divergences \[18\]. The interaction vertices are depicted in Figs. (1) and (2). In what follows the \( i\epsilon \) term in the propagators is to be understood.

III. THE ABELIAN VERTEX

The amplitude for the vertex diagram, Fig. (3), in the commutative setting was calculated in Ref. \[20\]. The corresponding vertex amplitude in the NC case is given by

\[ \Gamma_\mu = -\frac{e^2}{M} \epsilon^{\mu\nu\rho\lambda} \int \frac{d^3k}{(2\pi)^3} \epsilon_{\lambda\sigma\rho} \frac{k^\rho}{k^2} \bar{u}(q) \gamma^\sigma (\not{q} + \not{k} + m) \gamma_\mu (\not{p} + \not{k} - m) \gamma^\lambda \frac{u(p)}{[(q + k)^2 - m^2][(p + k)^2 - m^2]} e^{ik \times K}. \]

(6)
\[ \equiv ie\gamma_\mu \exp \left[ \frac{i}{2} \mathbf{p} \times \mathbf{q} \right]. \]

FIG. 1: The fermion-photon vertex.

\[ \equiv 2ieM\epsilon^{\mu\nu\rho} \sin \left[ \frac{(p \times r)}{2} \right]. \]

FIG. 2: The three gauge boson vertex.

FIG. 3: The one-loop vertex diagram.
In the above $\mathcal{K}_\mu = (q - p)_\mu$. Here the gamma matrices are defined as $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3, \gamma^3 = i\sigma_1$, and $g_{\mu\nu}$ is taken to be diag(1,-1,-1) same as that adopted in Ref. [8]. The above integral can be simplified using the identity $\gamma_\mu\gamma_\nu = g_{\mu\nu} - ig_{\mu\rho}\gamma^\rho$ and the mass-shell conditions: $(\not p - m)u(p) = 0$ and $\bar{u}(q)(\not q - m) = 0$. After some lengthy algebra and defining $P_\mu = (q + p)_\mu$ the above amplitude can be cast in the form

$$\Pi_\mu = -\frac{e^2}{M} e^{\not p \times q} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{4i(m\gamma_\mu - q_\mu - P_\mu)}{(2p \cdot k + k^2)(2p \cdot k + k^2)} + \frac{i\gamma_\mu k}{k^2(2p \cdot k + k^2)} + \frac{ik\gamma_\mu}{k^2(2q \cdot k + k^2)} \right. + \left. \frac{4\epsilon_\lambda\sigma\rho q^\rho p^\gamma\gamma_\mu}{k^2(2q \cdot k + k^2)(2p \cdot k + k^2)} \right] e^{ik \times \mathcal{K}}. \tag{7}$$

Terms proportional to $k_\mu$ which were zero in the commutative case, due to symmetry arguments, cannot be dropped because of a momentum dependent phase $exp(ik \times \mathcal{K})$. Writing the above integral as $\Pi_\mu = \Pi_\mu^{(1)} + \Pi_\mu^{(2)} + \Pi_\mu^{(3)} + \Pi_\mu^{(4)}$ for the sake of convenience, and performing the standard noncommutative loop integrations yield

$$\Pi_\mu^{(1)} = -\frac{4ie^2}{M(\sqrt{\pi^3})^3} e^{-\not p \times q} \int_0^1 dx \left\{ 2i(m\gamma_\mu + x\mathcal{K}_\mu + p_\mu - P_\mu) \left[ \frac{\mathcal{K}}{2\Delta_1} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_1) \right. + \left. \tilde{\mathcal{K}}_\mu \left[ \frac{2|\Delta_1|}{|\mathcal{K}|} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_1) \right\}, \tag{8}$$

$$\Pi_\mu^{(2)} = \frac{ie^2\gamma_\mu}{M(2\sqrt{\pi^3})^3} e^{\not p \times q} \int_0^1 dx e^{-ix\not p \times q} \left\{ \tilde{\mathcal{K}}_\mu \left[ \frac{2|\Delta_2|}{|\mathcal{K}|} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_2) \right. + \left. 2ix\not p \left[ \frac{\mathcal{K}}{2\Delta_2} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_2) \right\}, \tag{9}$$

$$\Pi_\mu^{(3)} = \frac{ie^2}{M(2\sqrt{\pi^3})^3} e^{\not p \times q} \int_0^1 dx \left\{ \tilde{\mathcal{K}}_\mu \left[ \frac{2|\Delta_3|}{|\mathcal{K}|} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_3) \right. + \left. 2ix\not p \left[ \frac{\mathcal{K}}{2\Delta_3} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_3) \right\} \gamma_\mu, \tag{10}$$

$$\Pi_\mu^{(4)} = -\frac{e^2\epsilon_\lambda\sigma\rho q^\rho p^\gamma\gamma_\mu}{M(2\sqrt{\pi^3})^3} e^{\not p \times q} \int_0^1 dx \int_0^x dy e^{-i(1-y)p \times q} \tilde{\mathcal{K}}_\mu \left[ \frac{\mathcal{K}}{2|\Delta_4|} \right]^{1/2} K_{1/2}(|\mathcal{K}|\Delta_4). \tag{11}$$

Here and in what follows, a tilde over a momentum indicates that it is contacted with the NC parameter $\theta$ i.e., $\tilde{\mathcal{K}}_\mu = \theta^{\mu\nu}\mathcal{K}_\nu$. Furthermore we have abbreviated $\Delta_2^2 = (x\mathcal{K} + p)^2$, $\Delta_2^2 = m^2x^2$, $\Delta_3^2 = m^2x^2$ and $\Delta_4^2 = m^2(1 - y)^2 - (x - y)(1 - x)\mathcal{K}^2$. In obtaining the above integrals we have used

$$\int_0^\infty dx x^{\nu-1} e^{-\gamma x - \beta / x} = 2(\beta / \gamma)^{\nu/2} K_\nu[2\sqrt{\beta \gamma}], \tag{12}$$
where $K_\nu$ is the modified Bessel function of the second kind. The parametric integrals can be solved elegantly by going over to the rest frame of the electron which implies $p \times q = 0$. Using

$$K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}}$$

and retaining terms to only first order in $\theta$ we get

$$\Gamma^{(1)} \mu = -\frac{ie^2}{2\pi M} \int_0^1 dx \left\{ i[m \gamma_\mu + x \kappa_\mu + p_\mu - \mathcal{P}_\mu] \left[ \frac{1}{|\Delta_1|} - |\tilde{\kappa}| \right] + \tilde{\kappa}_\mu \left[ \frac{1}{|\tilde{\kappa}|} - |\Delta_1| \right] \right\}$$

$$\Gamma^{(2)} \mu = \frac{ie^2 \gamma_\mu}{8\pi M} \int_0^1 dx \left\{ \tilde{\kappa}_\mu \left[ \frac{1}{|\tilde{\kappa}|} - |\Delta_2| \right] + i x \tilde{p} \left[ \frac{1}{|\Delta_2|} - |\tilde{\kappa}| \right] \right\}$$

$$\Gamma^{(3)} \mu = \frac{ie^2}{8\pi M} \int_0^1 dx \left\{ \tilde{\kappa}_\mu \left[ \frac{1}{|\tilde{\kappa}|} - |\Delta_3| \right] + i x \tilde{g} \left[ \frac{1}{|\Delta_3|} - |\tilde{\kappa}| \right] \right\} \gamma_\mu$$

$$\Gamma^{(4)} \mu = -\frac{e^2 \epsilon_{\lambda \sigma \rho} q^\sigma p^\lambda \tilde{K}^{\mu} \gamma_\mu}{4\pi M} \int_0^1 dx \int_0^x dy \left\{ \frac{1}{|\Delta_4|} - \frac{|\tilde{\kappa}|}{2} \right\}. \tag{14}$$

Solving the integrals in the low momentum transfer limit i.e., $\kappa^2 = 0$ and making use of the three dimensional analogue of Gordon’s decomposition

$$\gamma_\mu = \frac{1}{2m} [\mathcal{P}_\mu + i \epsilon_{\mu \nu \lambda} \kappa^{\nu} \gamma^{\lambda}], \tag{15}$$

the various contributions to the one-loop vertex can be written as

$$\Gamma^{(1)} \mu = \frac{ie^2}{4\pi M} \left[ \frac{\epsilon_{\mu \nu \lambda} \kappa^{\nu} \gamma^{\lambda}}{m} - \epsilon_{\mu \nu \lambda} \kappa^{\nu} - \frac{2}{|\tilde{\kappa}|} \left( \frac{1}{|\tilde{\kappa}|} - m \right) \right]$$

$$\Gamma^{(2+3)} \mu = \frac{ie^2}{4\pi M} \left[ i \gamma_\mu - \frac{i \gamma_\mu m |\tilde{\kappa}|}{2} + \tilde{\kappa}_\mu \left( \frac{1}{|\tilde{\kappa}|} - m \right) \right]$$

$$\Gamma^{(4)} \mu = -\frac{e^2}{4\pi M} \gamma_\mu \epsilon_{\lambda \sigma \rho} q^\sigma p^\lambda \tilde{K}^{\mu} \left[ \frac{1}{m} - \frac{|\tilde{\kappa}|}{2} \right]. \tag{16}$$

### IV. CONCLUSIONS

Now that the total contribution for the vertex has been obtained let us study the result. The first terms of $\Gamma^{(1)}$ and $\Gamma^{(2+3)}$ are the usual commutative MM and vertex contributions respectively. The second terms of $\Gamma^{(1)}$ and $\Gamma^{(2+3)}$ give the NC modulated corrections to the MM and the vertex. As can be seen from the signs of these terms they are opposite to that of the commutative contributions. Hence we expect that the second term in $\Gamma^{(1)}$ will reduce the total MM. But then of interest are the third terms of $\Gamma^{(1)}$ and $\Gamma^{(2+3)}$ which happen to be a pure $\theta$ induced contributions to the MM. Such terms have been in the case ordinary QED as well [22]. These terms as can be seen, the one with $1/|\tilde{\kappa}|$ as the coefficient leads to a reduction in the MM whereas that arising from $m$ leads to the increase. The final contribution $\Gamma^{(2+3)}$ the second term is higher order in $\theta$ and hence need not be considered here. The first term is higher order in the derivatives and will not contribute to the MM.
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