Classical information capacities of some single qubit quantum noisy channels

Liang Xian-Ting

1Department of Physics and Institute of Mathematics, Huaibei College, Huainan, Huan 418008, China
2Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, China

Abstract

By using the Holevo-Schumacher-Westmoreland (HSW) theorem and through solving eigenvalues of states out from the quantum noisy channels directly, or with the help of the Bloch sphere representation, or Stokes parametrization representation, we investigate the classical information capacities of some well-known quantum noisy channels.

PACS numbers: 03.67.H

Introduction

Calculating the information capacities of quantum noisy channels is an important task for studying quantum communication [1]. It has attracted much interest and many methods and results are proposed and obtained [2]. This interest was mainly stimulated by interplay between quantum communication theory and quantum information ideas related to more recent development in quantum computing and quantum communication. Unlike classical channels, which are adequately characterized by a single capacity, a quantum channel has several distinct capacities. They include (1) classical capacity $C_{cp}$, for transmitting classical information by encoding it with quantum product states and decoding it with individual measurement [3] [4]; (2) classical capacity $C_{pe}$, for transmitting classical information by encoding it with quantum product states and decoding it with collective measurement [5] [6] [7]. Similarly, there are another two capacities denoted by $C_{cp}$ and $C_{ce}$: (3) entanglement-assisted classical capacity $C_{ea}$, which describes the capacity of transmitting intact quantum states by the help of prior entanglement between the sender and the receiver; (4) quantum capacity $Q$, a supremum of coherent information which is the correspondence of mutual information in classical information theory [8] [9]; and (5) classical assisted quantum capacity $Q_c$ [10]. In general, it is difficult to calculate these capacities by their definitions. Scientists discovered some clear expressions which may simplify the calculation but there are still involving some technical problems. So for calculating the capacities distinctly some special methods are developed in last years. By using some special methods we have investigated the entanglement-assisted classical information capacities of some single qubit quantum noisy channels [11]. In this paper our investigation focus on classical information capacities $C_{pe}$ for some well-known quantum noisy channels. In subsection 2.1 we will investigate the capacities of depolarizing and erasure quantum noisy channels. The capacities of these two kinds of channels can be calculated by solving eigenvalues of output states directly. In subsection 2.2 we will calculate the capacities of several well-known quantum noisy channels with the help of Bloch sphere representation of qubit quantum state. In subsection 2.3 we use the Stokes parametrization representation of qubit quantum state to investigate the capacities of amplitude damping channel and splaying channel. A brief conclusion will close this paper in the last section.

I. CALCULATING THE CAPACITIES OF QUANTUM NOISY CHANNELS

Interactions with the environment are the fundamental source of noise in both classical and quantum systems. It is often not easy to find exact models for the environment or the system-environment interaction. However, some quantum noisy models, for example, depolarizing channel, phase damping channel, two-Pauli channel, amplitude damping channel etc. [12] can attain a high degree of accuracy in modeling of noise in circuits of quantum computation and quantum communication [13]. So calculating the capacities of quantum noisy channels is a significative work. We start our research with reviewing some concepts as follows.

* Operator sum representation: Every completely positive trace-preserving map $\varepsilon$ can be regarded as a channel which can be represented (non-uniquely) in the Kraus form

$$\rho' = \varepsilon(\rho) = \sum E_k^\dagger \rho E_k$$

which is also called operator sum representation. Here, $E_k$ is the Kraus operators and $\rho$ is the density matrix of input state and $\rho'$ is the density matrix of output state.

* Unital map: [15] If $\varepsilon$ map the identity operator to itself i.e.

$$\varepsilon (I) = I,$$

this map is a unital map.

* Bloch sphere representation: A density matrix of a qubit quantum pure state can be expressed as

$$\rho = \frac{1}{2} \left( \begin{array}{cc} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 - \cos \theta \end{array} \right).$$
This expression is called Bloch sphere representation of qubit quantum state.

\* Holevo-Schumacher-Westmoreland (HSW) theorem: The classical information capacity (encoding with pure product states and decoding with collective measurement) of quantum channel is given by

\[ C_{pe} = \sup_{\{\rho_j, \rho_j\}} \left( S(\sum_j p_j \rho_j) - \sum_j p_j S(\rho_j) \right). \]  

(4)

Here, \( S(\tau) = -\tau \log_2(\tau) \) (where and throughout this paper the logarithms are taken to base two) denotes the von Neumann entropy, \( p_j \) is the probability of state \( \rho_j \) in ensemble \( \{\rho_j, \rho_j\} \).

\* If a map is unital, then the classical information capacity \( C_{pe} \) can be obtained with orthogonal input [15].

By using above definitions, theorem and proposition we now investigate the capacities \( C_{pe} \) for some well-known quantum noisy channels.

### A. By solving the eigenvalues of output states directly

In this subsection we will investigate the classical information capacities \( C_{pe} \) of depolarizing channel and the erasure channel through solving eigenvalues of output states directly.

- **Depolarizing channel:** At first, we investigate the memoryless depolarizing channel (memoryless is assumed throughout this paper for all channels). Depolarizing channel is an important type of quantum noisy channels. It models a decohering qubit that has particularly nice symmetry properties. There are many practical quantum processes corresponding to this model [2]. Its Kraus operators are

\[ E_0^d = \sqrt{1-\eta} I, E_1^d = \sqrt{\frac{\eta}{3}} \sigma_1, E_2^d = \sqrt{\frac{\eta}{3}} \sigma_2, E_3^d = \sqrt{\frac{\eta}{3}} \sigma_3. \]  

(5)

\( \sigma_i \) (\( i = 1, 2, 3 \)) denote the Pauli matrices; \( I \) is the identity matrix in the Hilbert space of \( C^{2\times 2} \); \( \eta \) is the error occurring probability of quantum state passing through a depolarizing channel. So when a quantum state \( \rho \) is transmitted through this channel the state becomes

\[ \rho' = \epsilon^d(\rho) = (1-\eta)\rho + \frac{\eta}{3}(\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3), \]  

(6)

where \( \epsilon^d \) denotes a map of depolarizing channel (we use the first or the first two or three letters of the names of the channels as the superscripts to differentiate the map \( \epsilon \), Kraus operators \( E_i \) and so on of different channels); we set \( \rho \) is the input pure state and \( \rho' \) is the output state (in general, it is mixed state) of input state \( \rho \). This map \( \epsilon^d \) is a unital map. From the following relationship

\[ 2I = \rho + \sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3 \]  

(7)

we can obtain the two output states \( \rho_1' \) and \( \rho_2' \) of a pair of input encoding states \( \rho_1 \) and \( \rho_2 \) as

\[ \rho_{1,2}' = \left( 1 - \frac{4\eta}{3} \right) \rho_{1,2} + \frac{2\eta}{3} I. \]  

(8)

\( \rho_1' \) has the eigenvalues \( \alpha_1^d = 1 - 2\eta/3, \alpha_2^d = 2\eta/3, \) similarly \( \rho_2' \) has the eigenvalues \( \beta_1^d = 1 - 2\eta/3, \beta_2^d = 2\eta/3 \) (in the following we always denote the eigenvalues of states \( \rho_1' \) with \( \alpha_i \) and \( \rho_2' \) with \( \beta_i \) \( i = 1, 2 \)).

So \( S(\epsilon^d(\rho)) = H(2\eta/3) \) which does not depend on \( \rho_j \) at all. Here, \( H(\tau) \) denotes the binary entropy i.e. \( H(\tau) = -\tau \log_2(\tau) - (1-\tau) \log_2(1-\tau) \). Thus, \( C_{pe} \) can be achieved by maximizing the entropy of mixed state \( \rho^d = \sum_j p_j \epsilon^d(p_j) \) which may be done through a pair orthogonal input states (we denote the mixed density matrix \( \sum_j p_j \epsilon(p_j) \) always by \( \rho \) in this paper). For example, we calculate it by simply choosing the \( |0\rangle \) and \( |1\rangle \) of single qubit, so we immediately obtain a eigenvalues of mixed state \( \rho = \sum_j p_j \epsilon(p_j) \) by \( \gamma_j \). Thus, we can obtain its capacity as

\[ C_{pe}^d = 1 - H \left( \frac{2\eta}{3} \right). \]  

(9)

- **Erasure channel:** The capacity of another quantum noisy channel, erasure channel can also be investigated as same as depolarizing channel [16]. When a quantum state is transmitted through this channel the undisturbed probability is \( 1-\eta \). In case of an error, the quantum state is replaced by \( |\xi\rangle \) that is orthogonal to all quantum states of the system. In another words, the error make the state out of its original Hilbert space i.e. the information is erased with probability \( \eta \). So when a quantum state \( \rho \in \mathcal{H} = C^{2\times 2} \) pass through this channel the state becomes

\[ \rho' = \epsilon^e(\rho) = (1-\eta)\rho + \eta |\xi\rangle \langle \xi|, |\xi\rangle \notin \mathcal{H}. \]  

(10)

The capacity can be easily calculated. It is \( 1-\eta \) [17,18].

### B. By help with Bloch sphere representation

The classical information capacities of some other quantum noisy channels may not be calculated as easier as above two channels. For example, the capacities of phase damping channel, two Pauli channel, bit flip channel etc. can not be calculated by solving the eigenvalues of the output states directly. In this case, we find that it may be convenient by using Bloch sphere representation of qubit quantum state. Analytically investigating the classical information capacities by using the Bloch sphere representations of qubit states is also restricted in some quantum noisy channels which are expressed by unital maps, but by using this method and with the help of numerical work we can widely investigate the classical
information capacities $C_{pe}$ almost for all of the quantum noisy channels. So this method is very powerful.

Now let us calculate the classical information capacity $C_{pe}$ of phase damping channel by use of Bloch sphere representation.

- Phase damping channel: It has the Kraus operators as

\[ E_0^p = \sqrt{1 - \eta I}, \quad E_1^p = \sqrt{\eta} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad E_2^p = \sqrt{\eta} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \]

(11)

By using Eqs. (11) and (11) we can obtain the density matrix after a state $\rho$ transmitting through this channel as

\[ \rho' = \varepsilon^p(\rho) = (1 - \mu) \rho + \mu \sigma_3 \rho \sigma_3, \]

(12)

where $\mu = \eta/2$. It can be easily seen that this channel is unital we can obtain its capacity by input a pair orthogonal states. By using the Bloch presentation we have

\[ \rho'_j = \varepsilon^p(\rho_j) = \frac{1}{2} \left( \begin{array}{cc} (1 + \cos \theta_j) & e^{-i\epsilon_j} (1 - 2\mu) \sin \theta_j \\ e^{i\epsilon_j} (1 - 2\mu) \sin \theta_j & (1 - \cos \theta_j) \end{array} \right), \]

(13)

where $j = 1, 2$. From above equation we known that when the prior probabilities $p_1 = p_2 = \frac{1}{2}$, $S(\rho') = S\left( \sum_j p_j \varepsilon^p(\rho_j) \right)$ take its maximum value 1bit at $\theta_2 - \theta_1 = \pi$ (orthogonal). Because the eigenvalues of $\varepsilon^p(\rho_1)$ at $\theta_1 = 0$, are

\[ \alpha_1^p = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\mu (1 - \mu) \sin^2 \theta_1} = 1, \]

\[ \alpha_2^p = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\mu (1 - \mu) \sin^2 \theta_1} = 0. \]

Similarly, the eigenvalues of $\varepsilon^p(\rho_1)$ at $\theta_1 = \pi$, are

\[ \beta_1^p = 1, \quad \beta_2^p = 0. \]

(15)

So the classical information capacity $C_{pe}$ of the phase damping channel is 1bit which corresponds to the input encoding states $|0\rangle$ and $|1\rangle$.

Similarly, we can calculate the classical information capacities of bit flip, bit-phase flip, and phase flip channels.

- Bit flip channel: The Kraus operators of bit flip channel are

\[ E_0^{bf} = \sqrt{1 - \eta I}, \quad E_1^{bf} = \sqrt{\eta} \sigma_1. \]

(16)

- Bit-phase flip channel: It has Kraus operators as

\[ E_0^{bpf} = \sqrt{1 - \eta I}, \quad E_1^{bpf} = \sqrt{\eta} \sigma_2. \]

(17)

- Phase flip channel: Its Kraus operators are

\[ E_0^{pf} = \sqrt{1 - \eta I}, \quad E_1^{pf} = \sqrt{\eta} \sigma_3. \]

(18)

The capacities of these channels are all 1bit which correspond to the input encoding states $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, and $\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$ (bit flip channel), $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, and $\frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle)$ (bit-phase flip channel), $|0\rangle$, and $|1\rangle$ (phase flip channel). These results can also be intuitively seen from the evolution of Bloch spheres in these quantum channels, because their Bloch spheres have the same form as Bloch sphere in phase damping channel except for their directions [2].

Now we investigate two-Pauli channel by using this method.

- Two-Pauli channel: The Kraus operators of two-Pauli channel are

\[ E_0^t = \sqrt{1 - \eta I}, E_1^t = \sqrt{\eta} \sigma_1, E_2^t = \sqrt{\eta} \sigma_2. \]

(19)

From Eqs.(1) and (19) we have

\[ \rho'_j = \varepsilon^t(\rho_j) = \frac{1}{2} \left( \begin{array}{cc} 1 + (1 - 2\mu) \cos \theta_j & e^{-i\epsilon_j} (1 - \eta) \sin \theta_j \\ e^{i\epsilon_j} (1 - \eta) \sin \theta_j & 1 - (1 - 2\mu) \cos \theta_j \end{array} \right), \]

(20)

From above equation we known $S(\rho') = S\left( \sum_j p_j \varepsilon^t(\rho_j) \right)$ takes its maximum value 1bit at $\theta_1 - \theta_2 = \pi$ when the prior probabilities $p_1 = p_2 = \frac{1}{2}$. The eigenvalues of $\varepsilon^t(\rho_1)$ are

\[ \alpha_{1,2}^t = \frac{1 \pm \sqrt{1 - 4\eta (1 - \eta) + \eta (2 - 3\eta) \sin^2 \theta_1}}{2}, \]

(21)

Similarly, the eigenvalues of $\varepsilon^t(\rho_2)$ are

\[ \beta_{1,2}^t = \frac{1 \mp \sqrt{1 - 4\eta (1 - \eta) + \eta (2 - 3\eta) \sin^2 \theta_2}}{2}. \]

(22)

So, when $0 < \eta < \frac{2}{3}$, set $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = -\frac{\pi}{4}$, $\alpha_{1,2}^t = \frac{1}{2} \pm \frac{1}{2} (1 - \eta)$, $\beta_{1,2}^t = \frac{1}{2} \mp \frac{1}{2} (1 - \eta)$ and when $\frac{2}{3} \leq \eta < 1$, $\alpha_{1,2}^t = \frac{1}{2} \pm \frac{1}{2} (2\eta - 1)$, $\beta_{1,2}^t = \frac{1}{2} \mp \frac{1}{2} (2\eta - 1)$ which correspond to the minimum value of von Neumann entropies of $\varepsilon^t(\rho_j)$. So the capacity of this channel is

\[ C_{pe}^t = \left\{ \begin{array}{ll} 1 - H\left(\frac{1}{2} \eta\right), & 0 < \eta < \frac{2}{3}, \\ 1 - H\left(\eta\right), & \frac{2}{3} \leq \eta < 1. \end{array} \right. \]

(23)

It has been shown that the Bloch sphere representation is a powerful tool for analyzing the classical information capacities of quantum noisy channels.

C. By help with Stokes parametrization representation

As mention above, analytically solving the capacities with Bloch sphere representation is not perfect effective
for non-unital quantum channels. At this rate, Stokes parametrization representation \[15\] may help us. In this subsection we will use Stokes parametrization representation to investigate the capacities of amplitude damping channel \[19\] and “spraying” channel \[20\]. Before our furthermore research we review several concepts that will be used in the following.

* The identity and Pauli matrices form a basis for $C^2 \times 2$ so that any quantum state $\rho$ of qubit can be written as \[15\]

$$\rho = \frac{1}{2} \left( I + \vec{w} \cdot \vec{\sigma} \right),$$

where $\vec{w}$ is real and $\vec{w} \in C^3$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ ($\tau$ denotes the transpose of matrix).

* Any quantum state $\rho$ pass through a quantum noisy channel $\varepsilon$ becomes \[15\]

$$\rho^\prime = \varepsilon \left( \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma}) \right) = \frac{1}{2} \left( I + (\vec{t} + T \vec{w}) \cdot \vec{\sigma} \right),$$

where

$$T = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}.$$ (26)

The unital maps correspond to $\vec{t} = 0$.

- Amplitude damping channel: From the Kraus operators of amplitude damping channel

$$E_0^a = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \eta} \end{pmatrix}, \quad E_1^a = \begin{pmatrix} 0 & \sqrt{\eta} \\ 0 & 0 \end{pmatrix},$$

we can obtain that

$$\varepsilon^a(I) = \begin{pmatrix} 1 + \eta & 0 \\ 0 & 1 - \eta \end{pmatrix},$$

which shows that it is a non-unital channel. From its Kraus operators we can know that the $T$ and $\vec{t}$ correspond to amplitude damping channel are

$$T^a = \begin{pmatrix} \sqrt{1 - \eta} & 0 & 0 \\ 0 & \sqrt{1 - \eta} & 0 \\ 0 & 0 & 1 - \eta \end{pmatrix}, \quad \vec{t}^a = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}.$$ (29)

namely, $\chi_1 = \chi_2 = \sqrt{1 - \eta}$, $\chi_3 = 1 - \eta$, $t_1 = t_2 = 0$, $t_3 = \eta$. From Eq.(25) we can obtain the output state of input state $\rho = \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma})$ as

$$\rho^\prime = \varepsilon^a \left( \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma}) \right) = \frac{1}{2} \left( I + \sqrt{1 - \eta}w_1\sigma_1 + \sqrt{1 - \eta}w_2\sigma_2 + \sigma_3 \right),$$

Due to $\chi_1 = \chi_2 > \chi_3$, and for $0 \leq \eta \leq 1$ in order to calculate the capacity we set $w_1 = 1$, $w_2 = w_3 = 0$. Thus, we have

$$\rho^\prime = \frac{1}{2} \begin{pmatrix} 1 + \eta & \chi_1 \\ \chi_1 & 1 - \eta \end{pmatrix}. \quad (31)$$

Because this channel is non-unital we do not know if orthogonal inputs can give the classical information capacity or not and we also do not know in advance what prior probabilities of input states give the maximum output information. In the following we consider the case of only a pair input encoding states. We suppose one of the input states denoted by $\rho_1 = \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma})$ and the other denoted by $\rho_2 = \frac{1}{2} (I + \vec{w}^\prime \cdot \vec{\sigma})$, thus after the state $\rho_1$ transmitting though this channel we can obtain its output state as $\rho_1^\prime = \varepsilon^a \left( \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma}) \right)$ and state $\rho_2$ transmitting though this channel we can obtain its output state as $\rho_2^\prime = \varepsilon^a \left( \frac{1}{2} (I + \vec{w}^\prime \cdot \vec{\sigma}) \right)$. As the Bloch sphere of the amplitude damping channel is symmetrical about its z axes i.e. the axes of $\theta = 0$ (the meaning of $\theta$ we refer the readers to Eq.(3)) $\vec{w}^\prime$ would be obtained by rotating $\vec{w}$ with $\psi$ angle around the axes $z$, namely, $\vec{w}^\prime = R^a \vec{w}$, where

$$R^a = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (32)$$

So we have

$$\rho_1^\prime = \frac{1}{2} \begin{pmatrix} 1 + \eta & \chi_1 \\ \chi_1 & 1 - \eta \end{pmatrix}, \quad (33)$$

the eigenvalues of which are

$$\alpha_{1,2}^a = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \eta + \eta^2}. \quad (34)$$

$$\rho_2^\prime = \varepsilon^a \left( \frac{1}{2} (I + \vec{w}^\prime \cdot \vec{\sigma}) \right) = \frac{1}{2} \begin{pmatrix} 1 + \eta & \chi_1 \cos \psi + i\chi_2 \sin \psi \\ \chi_1 \cos \psi - i\chi_2 \sin \psi & 1 - \eta \end{pmatrix}. \quad (35)$$

The eigenvalues of $\rho_2^\prime$ are also

$$\beta_{1,2}^a = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \eta + \eta^2}. \quad (36)$$

It is shown that the different output states have the same von Neumann entropy, namely,

$$\sum_j p_j S(\rho_j^\prime) = H\left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \eta + \eta^2} \right), \quad (37)$$

which is neither correlative to prior probabilities nor to the angle $\psi$. So in order to maximize the output information, i.e. obtain the capacity of this channel we only
We set $S(g^s) = S\left(\sum_j p_j e^{\alpha_j} (\rho_j)\right)$. We set $S(g^s)$ gets its maximum when $p_1 = 1 - \tau$ and $p_2 = \tau$, thus,

$$g^s = \sum_{i=1}^{2} p_i \rho_i' = (1 - \tau) \rho_1' + \tau \rho_2' = \frac{1}{2} \left( 1 + \eta \right) \left( \frac{1}{\xi} \right),$$

(38)

where $\xi = \chi_1 (1 - \tau + \tau \cos \psi - i \sin \psi)$. The eigenvalues of $\rho_2'$ are

$$\gamma^s_{1,2} = \frac{1 \pm \sqrt{\eta^2 + \chi_1^2 (1 - A(1 - \cos \psi))}}{2},$$

(39)

where $A = \tau (1 - \tau), 0 < A < 1$. When we take $\psi = \pi$ and $\tau = \frac{1}{2}$, we obtain the maximum value of $S(g^s) = S\left(\sum_j p_j e^{\alpha_j} (\rho_j)\right) = H(\frac{1 - \eta}{2})$. Thus, the classical information capacity $C_{pe}$ of amplitude channel is

$$C_{pe}^s = H\left(\frac{1 - \eta}{2}\right) - H\left(\frac{1 - \sqrt{1 - \eta + \eta^2}}{2}\right).$$

(40)

In the following we will use this kind method to calculate the classical information capacity of “splaying” channel.

- “splaying” channel: The $T$ and $\tilde{T}$ correspond to this channel are

$$T_s = \left(\begin{array}{ccc} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{array}\right), \quad \tilde{T}_s = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{array}\right).$$

(41)

So the output state of input state $\rho_1 = \frac{1}{2} (I + \hat{w} \cdot \hat{\sigma})$ is

$$\rho_1' = e^s \left(\frac{1}{2} (I + \hat{w} \cdot \hat{\sigma})\right) = \frac{1}{2} \left[ I + \chi_1 w_1 \sigma_1 + (t_3 + \chi_3 w_3) \sigma_3 \right].$$

(42)

Due to $\chi_2 = 0$, $\chi_1 > \chi_3$, in order to calculate the capacity we set $w_1 = 1$, $w_2 = w_3 = 0$. Thus, we have

$$\rho_1' = \left(\begin{array}{ccc} \frac{2}{3} & \frac{1}{2} \sqrt{3} \\ \frac{1}{2} \sqrt{3} & \frac{2}{3} \end{array}\right).$$

(43)

which has eigenvalues

$$\alpha^s_1 = \frac{5}{6}, \quad \alpha^s_2 = \frac{1}{6}. \quad (44)$$

Now we hope to find another state by which and $\rho_1$ construct a ensemble $\{ p_j, \rho_j \}$ that make the splaying channel had maximum output information. Because of $\chi_2 = 0$, we can generally obtain the state $\rho_2$ from $\rho_1$ by rotating $\hat{w}$ with $\psi'$ angle, namely, $\hat{w}' = R^s \hat{w}$, where

$$R^s = \left(\begin{array}{ccc} \cos \psi' & 0 & \sin \psi' \\ 0 & 1 & 0 \\ -\sin \psi' & 0 & \cos \psi' \end{array}\right).$$

(45)

So the output state $\rho_2'$ is

$$\rho_2' = e^s \left(\frac{1}{2} (I + R^s \hat{w} \cdot \hat{\sigma})\right) = \frac{1}{2} \left[ I + \chi_1 \sigma_1 \cos \psi' + (t_3 - \chi_3 \sin \psi') \sigma_3 \right] = \left(\begin{array}{ccc} \frac{5}{3} - \frac{6}{\sqrt{3}} \sin \psi' & \frac{\sqrt{3}}{12} (1 + \cos \psi') \\ \frac{\sqrt{3}}{12} (1 + \cos \psi') & \frac{6}{\sqrt{3}} + \frac{1}{12} \sin \psi' \end{array}\right).$$

(46)

The eigenvalues of $\rho_2'$ are

$$\beta^s_{1,2} = \frac{1}{2} \pm \frac{1}{6} \sqrt{4 - 2 \sin \psi' - 2 \sin^2 \psi'}. \quad (47)$$

When $p_1 = p_2$ we have

$$g^s = \sum_j p_j e^{\alpha_j} (\rho_j) = \left(\begin{array}{ccc} \frac{5}{6} - \frac{2}{12} \sin \psi' - \frac{\sqrt{3}}{6} (1 + \cos \psi') \\ \frac{\sqrt{3}}{12} (1 + \cos \psi') & \frac{2}{6} + \frac{1}{12} \sin \psi' \end{array}\right).$$

(48)

The eigenvalues of Eq.(48) are

$$\gamma^s_{1,2} = \frac{6 \pm \sqrt{10 - 4 \sin \psi' - 2 \sin^2 \psi' + 6 \cos \psi'}}{12}. \quad (49)$$

Thus, the capacity can be obtained as

$$C_{pe}^s = \max \left\{ H(A) - \frac{1}{2} \left[ H(B) + H\left(\frac{1}{6}\right) \right] \right\}. \quad (50)$$

where $A = \left( 6 - \sqrt{9 - 4 \sin \psi' + \cos 2 \psi' + 6 \cos \psi'} \right) / 12$, $B = \left( 3 - \sqrt{3 - 2 \sin \psi' + \cos 2 \psi'} \right) / 6$. The numerical work shows that if the input states are orthogonal, namely, $t_3 = \frac{3.14159}{6}$, the information output is $I_{pe}^s = 0.268277bits$ (which is bigger than that obtained in [20] a little bit). This is not the maximum output information. When the input states have angle $\psi' = \frac{3.20359}{6}$ (here, $\psi' = \pi$ is the orthogonal case) we can obtain the maximum output information, namely the capacity, $C_{pe}^s = 0.268673bits$. This is also coincided with the result of [20], qualitatively, but the quantity of capacity is less than that Fuchs’ result a little bit. By our method we also complete the demonstration that the splaying channel’s classical information capacity need not be achievable by orthogonal states.

II. CONCLUSIONS

In this paper we have investigated the classical information capacities $C_{pe}$ for some well-known quantum noisy channels by using different representations of qubit quantum states. It is shown that directly calculating capacity with solving eigenvalues of output states is very convenient but it only adapt to a few of channels. By
using this method we investigate the classical information capacities $C_{pe}$ of depolarizing channel and erasure channel. We use of Bloch sphere representation of qubit quantum states calculating the capacities of phase damping channel, two-Pauli channel and flip channels. It shows that the Bloch sphere representation is convenient for analytically calculating the classical information capacities of some quantum noisy channels expressed by unital maps. We use of the Stokes parametrization representation investigating the classical information capacities of non-unital amplitude damping channel and splaying channel. To the former we have obtained a analytical result which has not been reported in other where to our knowledge, and to the latter our result is coincided with Fuchs’ original result qualitatively.

Acknowledgments

This project partly supported by Scientific Research Fund of Hunan Provincial Education Department under Grand No. 01C036

[1] A. S. Holevo, e-print quant-ph/9809023.
[2] M. A. Nielsen and I. L. Chuang, 2000 Quantum Computation and Quantum Information, Cambridge Press.
[3] A. S. Kholevo, Probl. Inf. Transm. 9 (1973) 177.
[4] C. A. Fuchs, 1996 Ph.D thesis, University of New Mexico; e-print quant-ph/9601020.
[5] A. S. Holevo, IEEE Trans. Inf. Theory 44 (1998) 269; e-print quant-ph/9611023.
[6] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. A 54 (1996) 1869.
[7] B. Schumacher, and M. D. Westmoreland, Phys. Rev. A 56 (1997) 131.
[8] A. S. Holevo, e-print quant-ph/0106073.
[9] H. Baruah, M. A. Nielsen, and B. Schumacher, e-print quant-ph/9702049.
[10] B. Schumacher, and M. A. Nielsen, Phys. Rev. A 54 (1996) 2629.
[11] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. Lett. 83 (1999) 3081.
[12] X. T. Liang, H. Y. Fan, Mod. Phys. Lett. B 16 (2002) 441.
[13] A. Uhlmann, J. Phys. A: math. Gen. 34 (2001) 7047.
[14] X. T. Liang, Mod. Phys. Lett. B 16 (2002) 19.
[15] C. King and M. B. Ruskai, IEEE Trans. Info. Theory 47 (2001) 192; e-print quant-ph/9911073.
[16] M. Grasli, T. Beth, and T. Pellizzari, Phys. Rev. A 56 (1997) 33.
[17] C. Bennett, D. P. Divincenzo, and Smolin, Phys. Rev. Lett. 78 (1997) 3217.
[18] M. Keyl, e-print quant-ph/0202112.
[19] J. Preskill, Quantum Information and Computation, http:// www. theory. caltech. edu/~preskill/ ph229.
[20] C. Fuchs, Phys. Rev. Lett. 79 (1997) 1162.