Abstract. In this paper, we study the atomic structure of Puiseux monoids generated by monotone sequences. To understand this atomicity, it is often useful to know whether the monoid is bounded, in the sense that it has a bounded generating set. We provide necessary and sufficient conditions for atomicity and boundedness to be transferred from a monotone Puiseux monoid to all its submonoids. Finally, we present two special subfamilies of monotone Puiseux monoids and fully classify their atomic structure.

1. Introduction

Puiseux monoids were introduced in [8], where their atomic structure is studied. They are a natural generalization of numerical semigroups; however, while numerical semigroups are always atomic and minimally generated by their finite sets of atoms (irreducible elements), Puiseux monoids exhibit a very complex atomic structure. For instance, there are nontrivial Puiseux monoids having no atoms at all (i.e., being antimatter), whereas others, failing to be atomic, contain infinitely many irreducible elements.

Most of the Puiseux monoids whose sets of atoms have been determined can be “nicely” generated, meaning that they contain generating sets with convenient properties: finite, bounded, strongly bounded, etc. The simplicity of such generating sets allows us to have more control over the Puiseux monoid under study and, as a consequence, to better describe its atomic structure.

In this paper, we will continue the study of the atomic structure of nicely generated Puiseux monoids, focusing now on those generated by a monotone sequence of rationals; we call them monotone Puiseux monoids. Although the atomic behavior of this family will play the fundamental role here, we also study its boundedness. Even though boundedness does not seem to be related to atomicity a priori, by imposing certain boundedness conditions we can control drastically the atomic structure. The following results, whose terminology is recalled in the next section, shed light on this fact.

**Theorem 1.1.** [8, Theorem 5.2] Let $R = \{r_n \mid n \in \mathbb{N}\}$ be a strongly bounded subset of rationals generating the Puiseux monoid $M$. If $d(r_n)$ divides $d(r_{n+1})$, the sequence $\{d(r_n)\}$ is unbounded, and $\cap p\mathbb{Z}$ is finite for all prime $p$, then $M$ is antimatter.

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Theorem 1.2. [8, Theorem 5.8] Let $M$ be a strongly bounded finite Puiseux monoid. Then $M$ is atomic if and only if $M$ is isomorphic to a numerical semigroup.

Since every submonoid of a Puiseux monoid $M$ is again a Puiseux monoid, it is natural to ask whether a property of $M$ is inherited by all its submonoids. We say that a property $P$ is hereditary on $M$ if every submonoid of $M$ satisfies $P$. Additionally, we say that a property $P$ is hereditary on a class $C$ of monoids if it is hereditary on every member of $C$. As part of our study of monotone Puiseux monoids, we will find subfamilies where being atomic or monotone is hereditary.

In Section 2, we establish the nomenclature we will be using throughout this paper. In Section 3, we study the structure of Puiseux monoids that can be generated by increasing sequences. By contrast, Puiseux monoids that can be generated by decreasing sequences are investigated in Section 4. Later, in Section 5, we focus on the study of a special class of decreasing Puiseux monoids, that one whose members are precisely those generated by reciprocals of primes; we show that atomicity is hereditary on this class. Finally, in Section 6, we describe the atomic structure of Puiseux monoids generated by geometric sequences (which are monotone), characterizing, in particular, those that are atomic.

2. Background and Notation

In this section, we fix notation and establish the nomenclature we will use later. To do this, we recall some basic definitions related to commutative semigroups and their sets of atoms. Reference material on commutative semigroups can be found in [9] of Grillet. In addition, the monograph [7] of Geroldinger and Halter-Koch offers extensive background information on atomic monoids and non-unique factorization theory.

The double-struck symbols $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and non-negative integers, respectively. If $r$ is a real number, then we write $\mathbb{Z}_{\geq r}$ instead of $\{z \in \mathbb{Z} \mid z \geq r\}$; with a similar intention, we write $\mathbb{Q}_{\geq r}$ and $\mathbb{Q}_{> r}$. If $r \in \mathbb{Q}_{> 0}$, then the unique $a, b \in \mathbb{N}$ such that $r = a/b$ and $\gcd(a, b) = 1$ are denoted by $n(r)$ and $d(r)$, respectively. For $R \subseteq \mathbb{Q}_{> 0}$, the sets $n(R) = \{n(r) \mid r \in R\}$ and $d(R) = \{d(r) \mid r \in R\}$ are called numerator set and denominator set of $R$, respectively.

In this sequel, the unadorned term monoid always means commutative cancellative monoid. Let $M$ be a monoid. Because every monoid here is assumed to be commutative, we will use additive notation; in particular, “+” denotes the operation of $M$, while 0 denotes the identity element. We use the symbol $M^\times$ to denote the set $M\setminus\{0\}$. For $a, c \in M$, we say that $a$ divides $c$ in $M$ and write $a \mid_M c$ if $c = a + b$ for some $b \in M$. We write $M = \langle S \rangle$ when $M$ is generated by $S$, and say that $M$ is finitely generated if it can be generated by a finite set.

The set of units of $M$ is denoted by $M^\times$. An element $a \in M \setminus M^\times$ is irreducible or an atom if $a = u + v$ implies that either $u \in M^\times$ or $v \in M^\times$. We denote the set of atoms of $M$ by $\mathcal{A}(M)$. Every monoid $M$ in this paper will be reduced, which means that $M^\times$
contains only the zero element. Therefore $\mathcal{A}(M)$ will be contained in each generating set. The monoid $M$ is atomic if $M = \langle A(M) \rangle$. On the other hand, if $\mathcal{A}(M)$ is empty, then we say that $M$ is antimatter. Antimatter domains are defined in the same way as antimatter monoids are; they have been investigated by Coykendall et al. in [4].

A numerical semigroup $N$ is a cofinite submonoid of the additive monoid $\mathbb{N}_0$. Every numerical semigroup has a unique minimal set of generators, which happens to be finite. Additionally, if a numerical semigroup $N$ is minimally generated by positive integers $a_1, \ldots, a_n$, then $\gcd(a_1, \ldots, a_n) = 1$ and $\mathcal{A}(N) = \{a_1, \ldots, a_n\}$. Thus, every numerical semigroup is an atomic monoid containing finitely many atoms. A great introduction to the realm of numerical semigroups can be found in [5] by García-Sánchez and Rosales.

A Puiseux monoid is an additive submonoid of $\mathbb{Q}_{\geq 0}$. Albeit a natural generalization of numerical semigroups, Puiseux monoids are not always atomic. In fact, a Puiseux monoid is atomic if and only if it contains a minimal set of generators, which, in this case, must be unique. In addition, if 0 is not a limit point of a Puiseux monoid $M$, then $M$ is atomic. We will use these two facts throughout this paper without explicit mention. The atomicity of Puiseux monoids was studied in [6], where the reader can find further results related to the atomic structure of these objects. A Puiseux monoid $M$ is said to be bounded if $M$ can be generated by a bounded subset of rational numbers. Besides, we say that $M$ is strongly bounded if $M$ can be generated by a set of rationals $R$ such that $n(R)$ is bounded. The family of strongly bounded Puiseux monoids is strictly contained in that of bounded Puiseux monoids. Additionally, there are Puiseux monoids that are not bounded. The following example illustrates these observations.

**Example 2.1.** Let $P$ denote the set of primes. Consider the bounded Puiseux monoid

$$M = \langle A \rangle,$$

where $A = \left\{ \frac{p - 1}{p} \mid p \in P \right\}$.

Since the set of atoms of $M$ is precisely $A$, one has that $M$ cannot be strongly bounded. On the other hand, let

$$M' = \langle A' \rangle,$$

where $A' = \left\{ \frac{p^2 - 1}{p} \mid p \in P \right\}$.

In this case, the set of atoms of $M'$ is $A'$. Because $A'$ is an unbounded set, it follows that $M'$ is not a bounded Puiseux monoid.

### 3. Increasing Puiseux Monoids

We are in a position now to begin our study of the atomic structure of Puiseux monoids generated by monotone sequences.

**Definition 3.1.** A Puiseux monoid $M$ is said to be increasing (resp., decreasing) if it can be generated by an increasing (resp., decreasing) sequence. A Puiseux monoid is monotone if it is either increasing or decreasing.
Not every Puiseux monoid is monotone, as the next example illustrates.

**Example 3.2.** Let \( p_1, p_2, \ldots \) be an increasing enumeration of the set of prime numbers. Consider the Puiseux monoid \( M = \langle A \cup B \rangle \), where

\[
A = \left\{ \frac{1}{p_{2n}} \mid n \in \mathbb{N} \right\} \quad \text{and} \quad B = \left\{ \frac{p_{2n-1} - 1}{p_{2n-1}} \mid n \in \mathbb{N} \right\}.
\]

It follows immediately that both \( A \) and \( B \) belong to \( A(M) \). So \( M \) is atomic, and \( A(M) = A \cup B \). Every generating set of \( M \) must contain \( A \cup B \) and so will have at least two limit points, namely, 0 and 1. Since every monotone sequence of rationals can have at most one limit point in the real line, we conclude that \( M \) is not monotone.

The following proposition describes the atomic structure of the family of increasing Puiseux monoids.

**Proposition 3.3.** Every increasing Puiseux monoid is atomic. Moreover, if \( \{r_n\} \) is an increasing sequence of positive rationals generating a Puiseux monoid \( M \), then

\[
A(M) = \{ r_n \mid r_n \notin \langle r_1, \ldots, r_{n-1} \rangle \}.
\]

**Proof.** The fact that \( M \) is atomic follows from observing that \( r_1 \) is a lower bound for \( M^\bullet \) and so 0 is not a limit point of \( M \). To prove the second statement, set

\[
A = \{ r_n \mid r_n \notin \langle r_1, \ldots, r_{n-1} \rangle \},
\]

and rename the elements of \( A \) in a strictly increasing sequence (possibly finite), namely, \( \{a_n\} \). Note that \( M = \langle A \rangle \) and \( a_n \notin \langle a_1, \ldots, a_{n-1} \rangle \) for any \( n \in \mathbb{N} \). Since \( a_1 \) is the smallest nonzero element of \( M \), we have that \( a_1 \in A(M) \). Suppose now that \( n \) is a natural such that \( 2 \leq n \leq |A| \). Because \( \{a_n\} \) is a strictly increasing sequence and \( a_n \notin \langle a_1, \ldots, a_{n-1} \rangle \), one finds that \( a_n \) cannot be written as a sum of elements in \( M \) in a non-trivial manner. Hence \( a_n \) is an atom for every \( n \in \mathbb{N} \) and, therefore, we can conclude that \( A(M) = A \). \( \square \)

Now we use Proposition 3.3 to show that every Puiseux monoid that is not isomorphic to a numerical semigroup has an atomic submonoid with infinitely many atoms. Let us first prove the next lemma.

**Lemma 3.4.** Let \( M \) be a nontrivial Puiseux monoid. Then \( d(M^\bullet) \) is finite if and only if \( M \) is isomorphic to a numerical semigroup.

**Proof.** Suppose first that \( d(M^\bullet) \) is finite. Since \( M \) is not trivial, \( M^\bullet \) is not empty. Take \( a \in \mathbb{N} \) to be the least common multiple of \( d(M^\bullet) \). Since \( aM \) is a submonoid of \( \mathbb{N}_0 \), it is isomorphic to a numerical semigroup. Furthermore, the map \( \varphi : M \to aM \) defined by \( \varphi(x) = ax \) is a monoid isomorphism. Thus, \( M \) is isomorphic to a numerical semigroup. Conversely, suppose that \( M \) is isomorphic to a numerical semigroup. Because every numerical semigroup is finitely generated, so is \( M \). Hence \( d(M^\bullet) \) is finite. \( \square \)
Proposition 3.5. If $M$ is a nontrivial Puiseux monoid, then it satisfies exactly one of the following conditions:

1. $M$ is isomorphic to a numerical semigroup;
2. $M$ contains an atomic submonoid with infinitely many atoms.

Proof. Suppose that $M$ is not isomorphic to any numerical semigroup. Take $r_1 \in M^*$. By Lemma 3.4 the set $d(M^*)$ is not finite. Therefore $d((r_1)^*)$ is strictly contained in $d(M^*)$. Take $r'_2 \in M^*$ such that $d(r'_2) \notin d((r_1)^*)$. Let $r_2$ be the sum of $m_2$ copies of $r'_2$, where $m_2$ is a natural number so that $\gcd(m_2, d(r'_2)) = 1$ and $m_2r'_2 > r_1$. Setting $r_2 = m_2r'_2$, we notice that $r_2 > r_1$ and $r_2 \notin \langle r_1 \rangle$. Now suppose that $r_1, \ldots, r_n \in M$ have been already chosen so that $r_{i+1} > r_i$ and $r_{i+1} \notin \langle r_1, \ldots, r_i \rangle$ for $i = 1, \ldots, n - 1$. Once again, be using Lemma 3.4 we can guarantee that $d(M^*) \setminus d((r_1, \ldots, r_n)^*)$ is not empty. Take $r'_{n+1} \in M^*$ such that $d(r'_{n+1}) \notin d((r_1, \ldots, r_n)^*)$, and choose $m_{n+1} \in \mathbb{N}$ so that $\gcd(m_{n+1}, d(r'_{n+1})) = 1$ and $m_{n+1}r'_{n+1} > r_n$. Taking $r_{n+1} = m_{n+1}r'_{n+1}$, one finds that $r_{n+1} > r_n$ and $r_{n+1} \notin \langle r_1, \ldots, r_n \rangle$. Using the method just described, we obtain an infinite sequence $\{r_n\}$ of elements in $M$ satisfying that $r_{n+1} > r_n$ and $r_{n+1} \notin \langle r_1, \ldots, r_n \rangle$ for every $n \in \mathbb{N}$. By Proposition 3.3 the submonoid $N = \{r_n \mid n \in \mathbb{N}\}$ is atomic and $\mathcal{A}(N) = \{r_n \mid n \in \mathbb{N}\}$. Hence $M$ has an atomic submonoid with infinitely many atoms, namely, $N$.

Finally, note that conditions (1) and (2) exclude each other; this is because a submonoid of a numerical semigroup is either trivial or isomorphic to a numerical semigroup and so it must contain only finitely many atoms.

Now we split the family of increasing Puiseux monoids into two fundamental subfamilies. We will see that these two subfamilies have different behavior. We say that a sequence of rationals is strongly increasing if it increases to infinity. On the other hand, a bounded increasing sequence of rationals is called weakly increasing.

Definition 3.6. A Puiseux monoid is said to be strongly (resp., weakly) increasing if it can be generated by a strongly (resp., weakly) increasing sequence.

Proposition 3.7. Every increasing Puiseux monoid is either strongly increasing or weakly increasing. A Puiseux monoid is both strongly and weakly increasing if and only if it is isomorphic to a numerical semigroup.

Proof. The first statement follows straightforwardly. For the second statement, suppose that $M$ is a Puiseux monoid that is both strongly and weakly increasing. By Proposition 3.3 the monoid $M$ is atomic, and its set of atoms can be listed increasingly. Let $\{a_n\}$ be an increasing sequence with underlying set $\mathcal{A}(M)$. Suppose, by way of contradiction, that $\mathcal{A}(M)$ is not finite. Since $M$ is strongly increasing, $\{a_n\}$ must be unbounded. However, the fact that $M$ is weakly decreasing forces $\{a_n\}$ to be bounded, rising a contradiction. Hence $\mathcal{A}(M)$ is finite, which implies that $M$ is isomorphic to a numerical semigroup.
To prove the converse implication, take $M$ to be a Puiseux monoid isomorphic to a numerical semigroup. So $M$ is finitely generated, namely, $M = \langle r_1, \ldots, r_n \rangle$ for some $n \in \mathbb{N}$ and $r_1 < \cdots < r_n$. The sequence $\{a_n\}$ defined by $a_k = r_k$ if $k \leq n$ and $a_k = kr_n$ if $k > n$ is an unbounded increasing sequence generating $M$. Similarly, the sequence $\{b_n\}$ defined by $b_k = r_k$ if $k \leq n$ and $b_k = r_n$ if $k > n$ is a bounded increasing sequence generating $M$. Consequently, $M$ is both strongly and weakly increasing. □

We will show that being strongly increasing is hereditary on the class of strongly increasing Puiseux monoids. First, we verify the following lemma.

**Lemma 3.8.** Let $R$ be an infinite subset of $\mathbb{Q}_{\geq 0}$. If $R$ does not have any limit points, then it is the underlying set of a strongly increasing sequence.

**Proof.** For every $r \in R$ and every subset $S$ of $R$, the interval $[0, r]$ must contain only finitely many elements of $S$; otherwise there would be a limit point of $S$ in $[0, r]$. Therefore every nonempty subset of $R$ has a minimum element. So the sequence $\{r_n\}$ recurrently defined by $r_1 = \min R$ and $r_n = \min R \setminus \{r_1, \ldots, r_{n-1}\}$ is strictly increasing and has $R$ as its underlying set. Since $R$ is infinite and contains no limit points, the increasing sequence $\{r_n\}$ must be unbounded. Hence $R$ is the underlying set of the strongly increasing sequence $\{r_n\}$. □

**Theorem 3.9.** A nontrivial Puiseux monoid $M$ is strongly increasing if and only if every submonoid of $M$ is increasing.

**Proof.** If $M$ is finitely generated, then it is isomorphic to a numerical semigroup, and the statement of the theorem follows immediately. So we will assume for the rest of this proof that $M$ is not finitely generated. Suppose that $M$ is strongly increasing. Let us start by verifying that $M$ does not have any real limit points. By Proposition 3.3 the monoid $M$ is atomic. As $M$ is atomic and non-finitely generated, $|\mathcal{A}(M)| = \infty$. Let $\{a_n\}$ be an increasing sequence with underlying set $\mathcal{A}(M)$. Since $M$ is strongly increasing and $\mathcal{A}(M)$ is an infinite subset contained in every generating set of $M$, the sequence $\{a_n\}$ is unbounded. Therefore, for every $r \in \mathbb{R}$, the interval $[0, r]$ contains only finitely many elements of $\{a_n\}$, say $a_1, \ldots, a_k$ for $k \in \mathbb{N}$. Since $\langle a_1, \ldots, a_k \rangle \cap [0, r]$ is a finite set, it follows that $M \cap [0, r]$ is finite as well. Because $|[0, r] \cap M| < \infty$ for all $r \in \mathbb{R}$, it follows that $M$ does not have any limit points in $\mathbb{R}$.

Now suppose that $N$ is a nontrivial submonoid of $M$. Notice that, being a subset of $M$, the monoid $N$ cannot have any limit points in $\mathbb{R}$. Thus, by Lemma 3.8 the set $N$ is the underlying set of a strongly increasing sequence of rationals. Hence $N$ is a strongly increasing Puiseux monoid, and the direct implication follows.

For the converse implication, suppose that $M$ is not strongly increasing. We will check that, in this case, $M$ contains a submonoid that is not increasing. If $M$ is not increasing, then $M$ is a submonoid of itself that is not increasing. Suppose, therefore, that $M$ is increasing. By Proposition 3.3, the monoid $M$ is atomic, and we can list its atoms increasingly. Let $\{a_n\}$ be an increasing sequence with underlying set $\mathcal{A}(M)$.
Because $M$ is not strongly increasing, there exists a positive real $\ell$ that is the limit of the sequence $\{a_n\}$. Since $\ell$ is a limit point of $M$, which is closed under addition, it follows that $2\ell$ and $3\ell$ are both limit points of $M$. Let $\{b_n\}$ and $\{c_n\}$ be sequences in $M$ having infinite underlying sets such that $\lim b_n = 2\ell$ and $\lim c_n = 3\ell$. Furthermore, assume that for each $n \in \mathbb{N}$,

\begin{equation}
|b_n - 2\ell| < \frac{\ell}{4} \quad \text{and} \quad |c_n - 3\ell| < \frac{\ell}{4}.
\end{equation}

Take $N$ to be the submonoid of $M$ generated by the set $A = \{b_n, c_n \mid n \in \mathbb{N}\}$. Let us verify that $N$ is atomic with $A(N) = A$. Note that $A$ is bounded from above by $3\ell + \ell/4$. On the other hand, by using the inequalities (3.1) we get

$$\max\{b_n + b_m, c_n + c_m, b_n + c_m\} > 3\ell + \frac{\ell}{4}$$

for every $n, m \in \mathbb{N}$. This implies that every element of $A$ is an atom of $N$ and so $A(N) = A$. Since every increasing rational sequence has at most one limit point in $\mathbb{R}$, the set $A$ cannot be the underlying set of an increasing rational sequence. As every generating set of $N$ contains $A$, we conclude that $N$ is not an increasing Puiseux monoid. \qed

As a direct consequence of Theorem 3.9, one obtains the following corollary.

**Corollary 3.10.** Being atomic, increasing, and strongly increasing are hereditary properties on the class of strongly increasing Puiseux monoids.

### 4. Decreasing Puiseux Monoids

Now that we have explored the structure of increasing Puiseux monoids, we will focus on the study of their decreasing counterpart. If a Puiseux monoid is decreasing, then it is obviously bounded. On the other hand, there are bounded Puiseux monoids that are not even monotone; see Example 3.2. However, every strongly bounded Puiseux monoid is decreasing, as we will show in Proposition 4.5.

In the previous section, we proved that being increasing (or strongly increasing) is hereditary on the class comprising all strongly increasing Puiseux monoids. By contrast, the next proposition will show that being decreasing is almost never hereditary, meaning that being decreasing is hereditary only on those Puiseux monoids that are isomorphic to numerical semigroups. First, let us prove the next lemma.

**Lemma 4.1.** If $M$ is a nontrivial decreasing Puiseux monoid, then exactly one of the following conditions holds:

1. $M$ is isomorphic to a numerical semigroup;
2. $M$ contains infinitely many limit points in $\mathbb{R}$.
Proof. Suppose that \( M \) is not isomorphic to a numerical semigroup. Since \( M \) is not trivial, it fails to be finitely generated. Therefore it can be generated by a strictly decreasing sequence \( \{a_n\} \). The sequence \( \{a_n\} \) must converge to a non-negative real number \( \ell \). Since \( \{ka_n\} \subseteq M \) converges to \( k\ell \) for every \( k \in \mathbb{N} \), if \( \ell \neq 0 \), then every element of the infinite set \( \{k\ell \mid k \in \mathbb{N}\} \) is a limit point of \( M \). On the other hand, if \( \ell = 0 \), then every term of the sequence \( \{ka_n\} \) is a limit point of \( M \); this is because for every fixed \( k \in \mathbb{N} \) the sequence \( \{a_k + a_n\} \subseteq M \) converges to \( a_k \). Hence \( M \) has infinitely many limit points in \( \mathbb{R} \).

Now let us verify that at most one of the above two conditions can hold. For this, assume that \( M \) is isomorphic to a numerical semigroup. So \( M \) is finitely generated, namely, \( M = \langle r_1, \ldots, r_n \rangle \), where \( n \in \mathbb{N} \) and \( r_i \in \mathbb{Q}_{>0} \) for \( i = 1, \ldots, n \). For every \( r \in \mathbb{R} \) the interval \([0, r]\) contains only finitely many elements of \( M \). Since \( M \cap [0, r] \) is finite for all \( r \in \mathbb{R} \), it follows that \( M \) cannot have any limit points in the real line. \( \square \)

**Proposition 4.2.** Let \( M \) be a nontrivial decreasing Puiseux monoid. Then exactly one of the following conditions holds:

1. \( M \) is isomorphic to a numerical semigroup;
2. \( M \) contains a submonoid that is not decreasing.

**Proof.** Suppose that \( M \) is not isomorphic to a numerical semigroup. Let us construct a submonoid of \( M \) that fails to be decreasing. Lemma 4.1 implies that \( M \) has a nonzero limit point \( \ell \). Since \( M \) is closed under addition, \( 2\ell \) and \( 3\ell \) are both limit points of \( M \).

An argument as the one given in the proof of Theorem 3.9 will guarantee the existence of sequences \( \{a_n\} \) and \( \{b_n\} \) in \( M \) having infinite underlying sets such that \( \{a_n\} \) converges to \( 2\ell \), \( \{b_n\} \) converges to \( 3\ell \), and the submonoid \( N = \langle a_n, b_n \mid n \in \mathbb{N} \rangle \) of \( M \) is atomic with \( \mathcal{A}(M) = \{a_n, b_n \mid n \in \mathbb{N}\} \). Since every decreasing sequence of \( \mathbb{Q} \) contains at most one limit point, \( \mathcal{A}(M) \) cannot be the underlying set of a decreasing sequence of rationals. As every generating set of \( N \) must contain \( \mathcal{A}(M) \), we can conclude that \( N \) is not decreasing. Hence at least one of the given conditions must hold.

To see that both conditions cannot hold simultaneously, it suffices to observe that if \( M \) is isomorphic to a numerical semigroup, then every nontrivial submonoid of \( M \) is also isomorphic to a numerical semigroup and, therefore, decreasing.

Similarly, as we did in the case of increasing Puiseux monoids, we will split the family of decreasing Puiseux monoids into two fundamental subfamilies, depending on whether 0 is or is not a limit point. We say that a non-negative sequence of rationals is **strongly decreasing** if it is decreasing and it converges to zero. A non-negative decreasing sequence of rationals converging to a positive real is called **weakly decreasing**.

**Definition 4.3.** A Puiseux monoid is **strongly decreasing** if it can be generated by a strongly decreasing sequence of rational numbers. On the other hand, a Puiseux
monoid is said to be \textit{weakly decreasing} if it can be generated by a weakly decreasing sequence of rationals.

Observe that if a Puiseux monoid $M$ is weakly decreasing, then it has a generating sequence decreasing to a positive real number and, therefore, 0 is not in the closure of $M$. Thus, every weakly decreasing Puiseux monoid is atomic. The next proposition describes those Puiseux monoids that are both strongly and weakly decreasing.

**Proposition 4.4.** A decreasing Puiseux monoid is either strongly or weakly decreasing. A Puiseux monoid is both strongly and weakly decreasing if and only if it is isomorphic to a numerical semigroup.

\textit{Proof.} As in the case of increasing Puiseux monoids, the first statement follows immediately. Now suppose that $M$ is a Puiseux monoid that is both strongly and weakly decreasing. Since $M$ is weakly decreasing, 0 is not a limit point of $M$. Let $\{a_n\}$ be a sequence decreasing to zero such that $M = \langle a_n \mid n \in \mathbb{N} \rangle$. Since $M$ is weakly decreasing, it is nontrivial and so $a_n \neq 0$ for some $n \in \mathbb{N}$. If we had $a_n \neq 0$ for every $n \in \mathbb{N}$, then 0 would be a limit point of $M$. Therefore there exists $n \in \mathbb{N}$ for which $a_n = 0$. Because $\{a_n\}$ is decreasing, only finitely many terms of the sequence $\{a_n\}$ are nonzero. Hence $M$ is isomorphic to a numerical semigroup. As in the increasing case, it is easily seen that every numerical semigroup is both strongly and weakly decreasing. \hfill \Box

We mentioned at the beginning of this section that every strongly bounded Puiseux monoid is decreasing. Indeed, a stronger statement holds.

**Proposition 4.5.** Every strongly bounded Puiseux monoid is strongly decreasing.

\textit{Proof.} Let $M$ be a strongly bounded Puiseux monoid. Since the trivial monoid is both strongly bounded and strongly decreasing, for this proof we will assume that $M \neq \{0\}$. Let $S \subset \mathbb{Q}_{>0}$ be a generating set of $M$ such that $n(S)$ is bounded. Since $n(S)$ is finite, we can take $m$ to be the least common multiple of the elements of $n(S)$. The map $x \mapsto \frac{1}{m}x$ is an order-preserving isomorphism from $M$ to $M' = \frac{1}{m}M$. Consequently, $M'$ is strongly decreasing if and only if $M$ is strongly decreasing. In addition, $S' = \frac{1}{m}S$ generates $M'$. Since $n(S') = \{1\}$, it follows that $S'$ is the underlying set of a strongly decreasing sequence of rationals. Hence $M'$ is a strongly decreasing Puiseux monoid, which implies that $M$ is strongly decreasing as well. \hfill \Box

Strongly decreasing Puiseux monoids are not always strongly bounded, even if we require them to be finite. For example, if $r \in \mathbb{Q}$ such that $0 < r < 1$ and both $n(r)$ and $d(r)$ are different from 1, then the Puiseux monoid $M_r = \langle r^n \mid n \in \mathbb{N} \rangle$ is atomic and $\mathcal{A}(M_r) = \{r^n \mid n \in \mathbb{N} \}$ (this will be proved in Theorem 6.2). As a result, $M_r$ is finite and strongly decreasing. However, $M_r$ fails to be strongly bounded. On the other hand, not every bounded Puiseux monoid is decreasing, as illustrated in Example 3.2.
Because numerical semigroups are finitely generated, they are both increasing and decreasing Puiseux monoids. We end this section showing that numerical semigroups are the only prototypes of Puiseux monoids that are both increasing and decreasing.

**Proposition 4.6.** A nontrivial Puiseux monoid $M$ is isomorphic to a numerical semigroup if and only if $M$ is both increasing and decreasing.

**Proof.** If $M$ is isomorphic to a numerical semigroup, then it is finitely generated and, consequently, increasing and decreasing.

Conversely, suppose that $M$ is a nontrivial Puiseux monoid that is increasing and decreasing. Proposition 3.3 implies that $M$ is atomic and, moreover, $\mathcal{A}(M)$ is the underlying set of an increasing sequence (because $\mathcal{A}(M) \neq \emptyset$). Suppose, by way of contradiction, that $\mathcal{A}(M)$ is not finite. In this case, $\mathcal{A}(M)$ does not contain a largest element. Since $M$ is decreasing, there exists $D = \{d_n \mid n \in \mathbb{N}\} \subset \mathbb{Q}_{>0}$ such that $d_1 > d_2 > \cdots$ and $M = \langle D \rangle$. Let $m = \min\{n \in \mathbb{N} \mid d_n \in \mathcal{A}(M)\}$, which must exist because $\mathcal{A}(M) \subseteq D$. Since $\mathcal{A}(M)$ is contained in $D$, the minimality of $m$ implies that $d_m$ is the largest element of $\mathcal{A}(M)$, which is a contradiction. Hence $\mathcal{A}(M)$ is finite. Since $M$ is atomic and $\mathcal{A}(M)$ is finite, $M$ is isomorphic to a numerical semigroup. □

## 5. Primary Puiseux Monoids

In this section, we take a step further our search of non-finitely generated atomic Puiseux monoids. We investigate the atomic structure of submonoids of those Puiseux monoids that can be generated by reciprocals of primes. Observe that such Puiseux monoids form a special subclass of strongly decreasing Puiseux monoids.

**Definition 5.1.** A Puiseux monoid $M$ is said to be primary if there exists a set $P$ of primes such that $M = \langle 1/p \mid p \in P \rangle$.

Recall that a property $P$ is hereditary on a class $\mathcal{C}$ of monoids if $P$ is hereditary on each member of the class $\mathcal{C}$. Let $M$ be a Puiseux monoid, and let $N$ be a submonoid of $M$. If $M$ is finitely generated, then $N$ is also finitely generated. Thus, being finitely generated is hereditary on the class of finitely generated Puiseux monoids. As we should expect, not every property of a Puiseux monoid is inherited by its submonoids. For example, being antimatter is not hereditary on the class of antimatter Puiseux monoids; to see this, consider $\mathbb{Q}_{\geq 0}$ and its submonoid $\mathbb{N}_0$. Moreover, as Corollary 5.3 indicates, boundedness and strong boundedness are not hereditary, even on the class of primary Puiseux monoids.

Let $S$ be a set of naturals. If the series $\sum_{s \in S} 1/s$ diverges, $S$ is said to be substantial. If $S$ is not substantial, it is said to be insubstantial (see [3]). For example, it is well known that the set of prime numbers is substantial as it was first noticed by Euler that the series of reciprocal primes is divergent.

**Proposition 5.2.** Let $P$ be a set of primes, and let $M$ be the primary Puiseux monoid $\langle 1/p \mid p \in P \rangle$. If every submonoid of $M$ is bounded, then $P$ is insubstantial.
Proof. Suppose, by way of contradiction, that $P$ is substantial. Then $P$ must contain infinitely many primes. Let $\{p_n\}$ be a strictly increasing enumeration of the elements in $P$. Take $N$ to be the submonoid of $M$ generated by $A = \{a_n \mid n \in \mathbb{N}\}$, where

$$a_n = \sum_{i=1}^{n} \frac{1}{p_i}.$$ 

Since $P$ is substantial, $A$ is unbounded. We will show that $N$ fails to be bounded. For this purpose, we verify that $A(N) = A$, which implies that every generating set of $N$ contains $A$ and, therefore, must be unbounded. Suppose that

$$a_n = a_{n_1} + \cdots + a_{n_{\ell}}$$

for some $\ell, n, n_1, \ldots, n_{\ell} \in \mathbb{N}$ such that $n_1 \leq \cdots \leq n_{\ell}$. Since $\{a_n\}$ is an increasing sequence, $n \geq n_{\ell}$. If $n$ were strictly greater than $n_{\ell}$, after multiplying the equation (5.1) by $m = p_1 \cdots p_n$ and moving every summand but $m/p_n$ to the right-hand side, we would obtain

$$p_1 \cdots p_{n-1} = \sum_{j=1}^{\ell} \sum_{i=1}^{n_j} \frac{m}{p_i} - \sum_{i=1}^{n-1} \frac{m}{p_i},$$

but this cannot happen as every summand in the right-hand side of equation (5.2) is divisible by $p_n$ while $p_n$ does not divide $p_1 \cdots p_{n-1}$. Therefore $n = n_{\ell}$ and so $\ell = 1$. Since $a_n \notin \langle a_1, \ldots, a_{n-1} \rangle$, Proposition 3.3 ensures that $A(N) = A$. Thus, $M$ contains a submonoid that fails to be bounded; but this is a contradiction. Hence the set $P$ is insubstantial.

For $m, n \in \mathbb{N}_0$ such that $n > 0$ and $\gcd(m, n) = 1$, Dirichlet’s theorem states that the set $P$ of all primes $p$ satisfying that $p \equiv m \pmod{n}$ is infinite. For a relatively elementary proof of Dirichlet’s theorem, see [11]. Furthermore, it is also known that the set $P$ is substantial; indeed, as indicated in [1, page 156], there exists a constant $A$ for which

$$\sum_{p \in P, p \leq x} \frac{1}{p} = \frac{1}{\varphi(n)} \log \log x + A + O\left(\frac{1}{\log x}\right),$$

where $\varphi$ is the Euler totient function. In particular, the set comprising all primes of the form $4k + 1$ (or $4k + 3$) is substantial. The next corollary follows immediately from Proposition 5.2 and equation (5.3).

Corollary 5.3. Let $m, n \in \mathbb{N}_0$ such that $n > 0$ and $\gcd(m, n) = 1$, and let $P$ be the set of all primes $p$ satisfying $p \equiv m \pmod{n}$. Then the primary Puiseux monoid $M = \langle 1/p \mid p \in P \rangle$ contains an unbounded submonoid.

We say that a monoid $M$ is hereditarily atomic if each submonoid of $M$ is atomic, i.e., being atomic is an hereditary property on $M$. Numerical semigroups and strongly increasing Puiseux monoids are hereditarily atomic. More generally, if $M$ is a Puiseux
monoid not having 0 as a limit point, then no submonoid of \( M \) has 0 as a limit point and, as a consequence, \( M \) is hereditarily atomic. According to Theorem 5.5, every primary Puiseux monoid is hereditarily atomic.

Let \( P \) be a set of primes, and let \( r \in \mathbb{Q}_{>0} \). We denote by \( D_P(r) \) the set of primes \( p \in P \) dividing \( d(r) \). Besides, if \( R \subseteq \mathbb{Q}_{>0} \), then we set \( D_P(R) = \bigcup_{r \in R} D_P(r) \). The following lemma is used in the proof of Theorem 5.5.

**Lemma 5.4.** Let \( P \) be a set of primes and, for \( n \in \mathbb{N} \), let \( r, r_1, \ldots, r_n \) be positive rationals such that \( r = r_1 + \cdots + r_n \). Then \( D_P(r) \subseteq D_P(r_1) \cup \cdots \cup D_P(r_n) \).

**Proof.** Take \( p \in D_P(r) \). Then \( p \) is a prime in \( P \) dividing \( d(r) \). Multiplying the equation \( r = r_1 + \cdots + r_n \) by \( d = d(r)d(r_1)\ldots d(r_n) \), we get

\[
d(r_1)\ldots d(r_n)n(r) = \sum_{i=1}^{n} m_i n(r_i),
\]

where \( m_i = d/d(r_i) \) for every \( i \in \{1, \ldots, n\} \). Since \( p \) divides each summand on the right-hand side of equation (5.4), it must divide \( d(r_i) \) for some \( i \in \{1, \ldots, n\} \). Therefore \( p \in D_P(r_i) \), and the desired set inclusion follows. \( \square \)

**Theorem 5.5.** Every primary Puiseux monoid is hereditarily atomic.

**Proof.** Let \( P \) be a set of primes, and let \( M \) be the primary Puiseux monoid generated by the set \( \{1/p \mid p \in P\} \). If \( P \) is finite, then \( M \) is isomorphic to a numerical semigroup, and so every submonoid of \( M \) is atomic. So we assume that \( P \) contains infinitely many primes. Let \( p_1, p_2, \ldots \) be an increasing enumeration of the elements in \( P \). First, we show that for all \( x \in M^* \) there exist only finitely many \( N \in \mathbb{N} \) such that

\[
x = \sum_{i=1}^{N} \alpha_i \frac{1}{p_{n_i}},
\]

for some \( \alpha_1, \ldots, \alpha_N, n_1, \ldots, n_N \in \mathbb{N} \) with \( n_1 < \cdots < n_N \). Since \( \{1/p \mid p \in P\} \) generates \( M \), there exists at least one natural \( N_0 \) such that equation (5.5) holds. Let us check that if a natural \( N \) satisfies (5.5), then \( N \leq x + N_0 \). Suppose, by way of contradiction, that \( N \) is a natural number greater than \( x + N_0 \) and

\[
x = \sum_{j=1}^{N} \beta_j \frac{1}{p_{m_j}},
\]

for some \( \beta_1, \ldots, \beta_N, m_1, \ldots, m_N \in \mathbb{N} \) with \( m_1 < \cdots < m_N \). Equation (5.6) forces the cardinality of the set \( \{j \in \{1, \ldots, N\} \mid p_{m_j} \text{ divides } \beta_j\} \) to be at most \( \lfloor x \rfloor \). Since \( N > x + N_0 \), there exists \( k \in \{1, \ldots, N\} \) such that \( p_{m_k} \notin \{p_{n_1}, \ldots, p_{n_{N_0}}\} \).
and \( p_{m_k} \nmid \beta_k \). After equaling both right-hand sides of equations (5.6) and (5.7), and multiplying the resulting equality by \( q = p_{n_1} \cdots p_{n_{N_0}} p_{m_1} \cdots p_{m_{N_0}} \), one obtains

\[
(5.7) \quad \beta_k B_k + \sum_{j=1, j \neq k}^N \beta_j B_j - \sum_{i=1}^{N_0} \alpha_i A_i = 0,
\]

where \( A_i = q/p_{n_i} \) for \( i = 1, \ldots, N_0 \) and \( B_j = q/p_{m_j} \) for \( j = 1, \ldots, N \). Note that every summand in (5.7) except the first one is divisible by \( p_{m_k} \). But this is a contradiction and, therefore, every \( N \) satisfying (5.5) is less than or equal to \( x + N_0 \). Hence there are only finitely many \( N \in \mathbb{N} \) satisfying (5.5).

Now, we are in a position to prove that every submonoid of \( M \) is atomic. Let us assume, by way of contradiction, that \( M \) contains a non-atomic submonoid \( M' \). Fix \( z \in M' \setminus \langle A(M') \rangle \). Let \( n \in \mathbb{N} \) such that there exist \( x_1, \ldots, x_n \in M'^\ast \) for which \( z = x_1 + \cdots + x_n \). Set \( D = D_P(x_1) \cup \cdots \cup D_P(x_n) \). Since \( D \) contains only finitely many primes, the set

\[
I = \left\{ n \in \mathbb{N} \mid \exists r_1, \ldots, r_n \in M'^\ast : z = \sum_{i=1}^n r_i \text{ and } \bigcup_{i=1}^n D_P(r_i) \subseteq D \right\}
\]

is finite. Take \( m \) to be the maximum of \( I \), and take \( r_1, \ldots, r_m \in M'^\ast \) such that \( z = r_1 + \cdots + r_m \) and \( D_P(r_1) \cup \cdots \cup D_P(r_m) \subseteq D \). Since \( z \not\in \langle A(M') \rangle \), there is an element of \( \{r_1, \ldots, r_m\} \), say \( r_m \) without loss of generality, that is not an atom of \( M' \). Take \( k \in \mathbb{Z}_{\geq 2} \) and \( r'_1, \ldots, r'_k \in M'^\ast \) so that \( r_m = r'_1 + \cdots + r'_k \). By the maximality of \( m \), there exists \( j \in \{1, \ldots, k\} \) for which \( D_P(r'_j) \) fails to be a subset of \( D \). On the other hand, Lemma 5.4 guarantees that \( D_P(r_m) \subseteq D_P(r'_1) \cup \cdots \cup D_P(r'_k) \). Therefore

\[
|D_P(\{r_1, \ldots, r_m\})| < |D_P(\{r'_1, \ldots, r_{m-1}\} \cup \{r'_1, \ldots, r'_k\})|.
\]

So for every \( N \in \mathbb{N} \) there is a natural \( n \) with \( z = r_1 + \cdots + r_n \) for some \( r_1, \ldots, r_n \in M'^\ast \) such that \( |D_P(\{r_1, \ldots, r_n\})| > N \). Writing each \( r_j \) in \( z = r_1 + \cdots + r_n \) as a sum of elements in \( \{1/p \mid p \in P\} \), we would be able to write \( z \) as in equation (5.5) for infinitely many \( N \in \mathbb{N} \), which is a contradiction. Hence every submonoid of \( M \) is atomic, i.e., \( M \) is hereditarily atomic.

\[\Box\]

6. Multiplicatively Cyclic Puiseux Monoids

We know that finitely generated Puiseux monoids are isomorphic to numerical semigroups. It is natural to wonder which are the simplest families of Puiseux monoids that are not isomorphic to numerical semigroups. Since the members of such families must be infinitely generated, it would be convenient to look into classes of Puiseux monoids infinitely generated by well-behaved sequences of rationals.

Numerical semigroups generated by intervals, arithmetic sequences, and generalized arithmetic sequences have been intensely studied (see [6, 2, 10] and the references...
therein). In particular, the simplicity of arithmetic sequences has facilitated the exploration of the combinatorial and algebraic structure of the numerical semigroups they generate as well as the factorization invariants such numerical semigroups exhibit. We may want to study, in principle, the family of Puiseux monoids generated by arithmetic sequences (which happen to be increasing). However, notice that if a Puiseux monoid $M$ is generated by an arithmetic sequence $\{r+ns\}$, where $r,s \in \mathbb{Q}_{>0}$, then the map $x \mapsto d(r)d(s)x$ defines an isomorphism from $M$ onto the numerical semigroup $\langle d(s)n(r) + nd(r)n(s) \mid n \in \mathbb{N} \rangle$. So Puiseux monoids generated by arithmetic sequences are isomorphic to numerical semigroups.

By contrast, Puiseux monoids generated by geometric sequences are not necessarily finitely generated. For example, if $z \in \mathbb{Z}_{\geq 2}$, the Puiseux monoid $M = \langle 1/z^n \mid n \in \mathbb{N} \rangle$ is antimatter, and so it fails to be finitely generated. We might expect that the controlled behavior of a rational geometric sequence leads us to a better understanding of the atomicity and boundedness of the Puiseux monoid it generates. In this section, we will explore the atomicity and boundedness of those Puiseux monoids that can be generated by geometric sequences.

**Definition 6.1.** For $r \in \mathbb{Q}_{>0}$, the Puiseux monoid generated by the positive powers of $r$ is called **multiplicatively $r$-cyclic** (or just **multiplicatively cyclic**) and is denoted by $M_r$, that is, $M_r = \langle r^n \mid n \in \mathbb{N} \rangle$.

**Remark:** Note that Puiseux monoids of the form $\langle ar^n \mid n \in \mathbb{N} \rangle$ for $a,r \in \mathbb{Q}_{>0}$ are not more general than those we defined as multiplicatively $r$-cyclic; this is because multiplication by $a$ gives an isomorphism of Puiseux monoids.

We recall that an antimatter monoid is a monoid having no atoms. The next theorem describes the sets of atoms of multiplicatively cyclic Puiseux monoids, indicating, in particular, which of these monoids are atomic.

**Theorem 6.2.** For $r \in \mathbb{Q}_{>0}$, let $M_r$ be the multiplicatively $r$-cyclic Puiseux monoid. Then the following statements hold.

- If $d(r) = 1$, then $M_r$ is atomic with $\mathcal{A}(M_r) = \{n(r)\}$.
- If $d(r) > 1$ and $n(r) = 1$, then $M_r$ is antimatter.
- If $d(r) > 1$ and $n(r) > 1$, then $M_r$ is atomic with $\mathcal{A}(M_r) = \{r^n \mid n \in \mathbb{N}\}$.

**Proof.** Set $a = n(r)$ and $b = d(r)$. If $b = 1$, then $M_r = \langle a^n \mid n \in \mathbb{N} \rangle = \langle a \rangle$, which immediately implies that $M_r$ is atomic and $\mathcal{A}(M_r) = \{a\}$. Suppose now that $a = 1$ and $b > 1$. In this case, $M_r = \langle 1/b^n \mid n \in \mathbb{N} \rangle$. Since $1/b^n = b(1/b^{n+1})$ for every $n \in \mathbb{N}$, it follows that $M_r$ is antimatter.

Set $R = \{r^n \mid n \in \mathbb{N}\}$. We proceed to show the last statement, that is, the case where $a > 1$ and $b > 1$. First, we argue the case $a > b$. Since $\{r^n\}$ is an increasing sequence generating $M_r$, by Proposition 3.3, the monoid $M_r$ is atomic. Let us find $\mathcal{A}(M_r)$. Take
\[ n \in \mathbb{N} \text{ such that } n > 1, \text{ and suppose that there exist } k, c_k \in \mathbb{N} \text{ and } c_i \in \mathbb{N}_0 \text{ for every } i = 1, \ldots, k - 1 \text{ satisfying that} \]
\[ \frac{a^n}{b^n} = c_1 \frac{a}{b} + \cdots + c_k \frac{a^k}{b^k}. \]
If \( k < n \), then after multiplying equation (6.1) by \( b^n \), it can be easily seen that every prime divisor of \( b \) must divide \( a \), which is not possible because \( \gcd(a, b) = 1 \). Therefore \( r^m \not\in \langle r, \ldots, r^{n-1} \rangle \) for any \( n \in \mathbb{N} \). By Proposition 5.3, one has \( \mathcal{A}(M_r) = R \).

Finally, suppose \( a < b \). Take \( n \in \mathbb{N} \), and write
\[ \frac{a^n}{b^n} = c_n \frac{a^n}{b^n} + \cdots + c_{n+k} \frac{a^{n+k}}{b^{n+k}}, \]
where \( k \in \mathbb{N}_0 \) and \( c_i \in \mathbb{N}_0 \) for every \( i = n, \ldots, n + k \). Notice that \( c_n = 0 \). In this case, \( k \geq 1 \). Let \( p \) be a prime dividing \( a \), and let \( \alpha \) be the maximum power of \( p \) dividing \( a \). Applying the \( p \)-adic valuation function to equation (6.2), one obtains
\[ p^{\alpha n} = v_p \left( \frac{a^n}{b^n} \right) = v_p \left( \sum_{i=1}^{k} c_{n+i} \frac{a^{n+i}}{b^{n+i}} \right) \geq \min_{1 \leq i \leq k} \left\{ v_p \left( c_{n+i} \frac{a^{n+i}}{b^{n+i}} \right) \right\} \geq p^{\alpha(n+m)}, \]
where \( m = \min \{ i \in \{1, \ldots, k\} \mid c_{n+i} \neq 0 \} \). Inequality (6.3) rises a contradiction because \( m \geq 1 \). Therefore \( c_n = 1 \), and so \( c_{n+i} = 0 \) for every \( i \geq 1 \). Since \( (a/b)^n \) cannot be expressed in a nontrivial way as a sum of elements in \( R \), one finds that \( (a/b)^n \) is an atom. Hence \( R \) is the set of atoms of \( M_r \) and, as a result, \( M_r \) is atomic.

With notation as in Theorem 6.2 if \( n(r) = 1 \) or \( d(r) = 1 \), then the multiplicatively cyclic Puiseux monoid \( M_r \) is strongly bounded. If \( n(r), d(r) > 1 \) and \( r < 1 \), then \( M_r \) is bounded. However \( M_r \) cannot be strongly bounded because every generating set of \( M_r \) must contain the set \( R = \{ r^n \mid n \in \mathbb{N} \} \), which is not strongly bounded. By a similar argument, \( M_r \) is not bounded when \( n(r), d(r) > 1 \) and \( r > 1 \).

**Corollary 6.3.** For \( r \in \mathbb{Q}_{>0} \), let \( M_r \) be the multiplicatively \( r \)-cyclic Puiseux monoid. Then the following statements hold.

- If \( n(r) = 1 \) or \( d(r) = 1 \), then \( M_r \) is strongly bounded.
- If \( n(r), d(r) > 1 \) and \( r < 1 \), then \( M_r \) is bounded but not strongly bounded.
- If \( n(r), d(r) > 1 \) and \( r > 1 \), then \( M_r \) is not bounded.

As illustrated by Corollary 5.3, being bounded (or strongly bounded) is not hereditary on the class of primary Puiseux monoids. Additionally, boundedness (resp., strongly boundedness) is not hereditary on the class of bounded (resp., strongly bounded) multiplicatively cyclic Puiseux monoids.
Example 6.4. Let $M$ be the multiplicatively $(1/2)$-cyclic Puiseux monoid, that is, $M = \langle 1/2^n \mid n \in \mathbb{N} \rangle$. It is strongly bounded, and yet its submonoid
\begin{equation}
N = \left\langle \sum_{i=1}^{n} \frac{1}{2^i} \right\vert n \in \mathbb{N} \right\rangle = \left\langle \frac{2^n - 1}{2^n} \right\vert n \in \mathbb{N} \right\rangle
\end{equation}
is not strongly bounded; to see this, it is enough to verify that $\mathcal{A}(N) = S$, where $S$ is the generating set defining $N$ in (6.4). Note that the sum of any two elements of the generating set $S$ is at least one, while every element of $S$ is less than one. Therefore each element of $S$ must be an atom of $N$, and so $\mathcal{A}(N) = S$.

We conclude this paper showing that boundedness (resp., strong boundedness) is almost never hereditary on the class of bounded (resp., strongly bounded) multiplicatively cyclic Puiseux monoids.

Proposition 6.5. For $r \in \mathbb{Q}_{>0}$, let $M_r$ be the multiplicatively $r$-cyclic Puiseux monoid. Then every submonoid of $M$ is bounded (or strongly bounded) if and only if $M_r$ is isomorphic to a numerical semigroup.

Proof. Let $a$ and $b$ denote $n(r)$ and $d(r)$, respectively. To prove the direct implication, suppose, by way of contradiction, that $M_r$ is not isomorphic to a numerical semigroup. In this case, $b > 1$. Consider the submonoid $N = \langle s_1, s_2, \ldots \rangle$ of $M$, where
\[ s_n = \frac{(nb^n + 1)a^n}{b^n} \]
for every natural $n$. Proving the forward implication amounts to verifying that $N$ is not bounded and, as a consequence, not strongly bounded. First, let us check that $\mathcal{A}(N) = \{ s_n \mid n \in \mathbb{N} \}$. Note that
\[ s_{n+1} = \frac{(n+1)b^{n+1} + 1)a^{n+1}}{b^{n+1}} > \frac{(nb^{n+1} + b)a^n}{b^{n+1}} = s_n \]
for each $n \in \mathbb{N}$, and so $\{ s_n \}$ is an increasing sequence. Moreover, it is easy to see that $s_n > n$ for every $n$. Thus, $\{ s_n \}$ is unbounded. Suppose that there exist $k, \alpha_k \in \mathbb{N}$ and $\alpha_i \in \mathbb{N}_0$ for every $i = 1, \ldots, k - 1$ such that $s_n = \alpha_1 s_1 + \cdots + \alpha_k s_k$. Since $\{ s_n \}$ is increasing and $\alpha_k > 0$, we have $k \leq n$. Let $p$ be a prime divisor of $b$, and let $m = v_p(b)$. The fact that $p \mid (nb^n + 1)a^n$ for every natural $n$ implies $v_p(s_n) = -mn$. Therefore
\[ -mn = v_p(s_n) \geq \min_{1 \leq i \leq k} \{ v_p(\alpha_i s_i) \} = \min_{1 \leq i \leq k} \{ v_p(s_i) \} = -mk, \]
which implies that $k \geq n$. Thus, $k = n$ and then $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n = 1$. So $s_n \notin \langle s_1, \ldots, s_{n-1} \rangle$ for every $n \in \mathbb{N}$ and, by Proposition 3.3, $\mathcal{A}(N) = \{ s_n \mid n \in \mathbb{N} \}$. Since $\mathcal{A}(N)$ is unbounded, $N$ cannot be a bounded Puiseux monoid.

On the other hand, if $M_r$ is isomorphic to a numerical semigroup, then it is finitely generated and, hence, bounded and strongly bounded. This gives us the converse implication. \qed
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