Quantum Resonances of Kicked Rotor and $SU(q)$ group

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The quantum kicked rotor (QKR) map is embedded into a continuous unitary transformation generated by a time-independent quasi-Hamiltonian. In some vicinity of a quantum resonance of order $q$, we relate the problem to the regular motion along a circle in a $(q^2 - 1)$-component inhomogeneous “magnetic” field of a quantum particle with $q$ intrinsic degrees of freedom described by the $SU(q)$ group. This motion is in parallel with the classical phase oscillations near a non-linear resonance.

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For years, Chirikov’s standard quantum map $[\text{1}]$, i.e. a planar quantum rotor driven by very short periodic kicks, served as a cornerstone model for many investigations of onset and manifestations of dynamical chaos in quantum systems. Indeed, the standard map provides a local description for a large class of dynamical systems and it is of quite general interest to understand its properties. The unitary Floquet transformation $U$ which evolves the wave function $\psi(\theta)$ over each kick period is given by:

$$U = U_r \cdot U_k \equiv \exp \left(-\frac{i}{2}T\hat{m}^2\right) \cdot \exp(-ik\cos\theta) \quad (1)$$

and consists of the operator $U_k$ of a kick with strength $k$ and a consecutive free rotation $U_r$ during the time $T$. Here $\hat{m} = -id/d\theta$ and we put $\hbar = 1$.

In the classical limit the motion depends on the single parameter $K = T k$. When this parameter exceeds unity the classical dynamics yields unrestricted diffusive growth of the mean kinetic energy. The quantum dynamics is appreciably richer $[\text{2}]$. In particular the phenomenon of dynamical localization has been discovered $[\text{3,4}]$, namely the quantum motion mimics the classical diffusive behaviour only up to a time $t_R \sim k^2$ after which it enters a stationary oscillatory regime.

However, at present, we are still very far from a satisfactory understanding of this simple model of quantum chaos. The main obstacle for the analytical and numerical investigations are the so called quantum resonances which take place at a fixed value of $k$ for the everywhere dense set of the rational values $\zeta = T/4\pi = p/q$, the integer numbers $p$ and $q$ being mutually prime $[\text{5}]$. In such a regime, the mean kinetic energy increases quadratically for asymptotic large times. Quantum resonances may strongly influence the motion in some vicinities around them. In other words they have certain widths. It is in this feature the main difference between the QKR and disordered quantum systems and the main reason for the failure of analytical tools in particular those which use supersymmetric technique $[\text{6}]$. Therefore any attempt to improve our understanding of quantum chaotic motion must confront with the so far unsolved core problem of the motion in vicinities of resonances.

In this paper we develop a general approach to the problem of the motion in a vicinity of a quantum resonance with arbitrary order $q$. We show that the most important role is played by the resonances with small $q$: for finite regions around them the quantum motion is explicitly shown to be regular and dominates the motion for all $\zeta$ values inside these regions. More precisely, the motion is described, for large though finite times, by a conservative Hamiltonian with one rotational degree of freedom and with a discrete spectrum. On the contrary, when the resonance order is large enough the resonant quadratic growth appears only in the remote time asymptotic and during a long time the motion reveals universal features characteristic of the localized quantum chaos. We demonstrate on a specific example that our approach allows to predict the quantum evolution with great accuracy for long time (see Fig.1) even though the corresponding classical motion is chaotic and exponentially unstable.

Basically, we use the same idea of embedding the map $[\text{4}]$ into a continuous unitary transformation which has been used before $[\text{7}]$ in the study of the classical counterpart of $[\text{4}]$. The Floquet operator is represented in the form $U = \exp(-i\mathcal{H})$, so that the motion appears as a conservative evolution in a continuous time $t$ of a system with one rotational degree of freedom described by the effective time-independent quasi-Hamiltonian $\mathcal{H}$. The latter cannot generally be found in a closed form. Rather, it is formally expressed as an infinite sum of successive commutators. However, when the condition of a quantum resonance is fulfilled it simplifies enormously $[\text{8}]$ and reduces to a pure phase (or local gauge) transformation from the unitary unimodular group $SU(q)$. Moreover, in some vicinities of quantum resonances, the sum of commutators can be truncated thus allowing to obtain new insight on the nature of the motion.

Before turning to the consideration of the general
case, we illustrate our approach on the simplest and strongest resonances \( q = 1, 2, 4 \) (see also \([3]\)). For the main resonance \( q = 1 \) the rotation operator is equivalent to identity. The Floquet transformation \([1]\) is then simply \( e^{-i v(\theta)} \), where \( v(\theta) = k \cos \theta \) is the interaction potential. After \( t \) repetitions of this transformation, an initially isotropic wave function gets \( \sim k t \) harmonics. This number is naturally measured by the operator \( \hat{m}(t) \equiv e^{i \hat{m} \cdot \hat{v}(\theta) t} = \hat{m} - v(\theta)t \). This yields the increase law \( E_\kappa(t) = (\Delta \hat{m}^2(t))/2 = r t^2 \) with the resonant growth rate \( r(\kappa) = \langle \hat{v}^2 \rangle = k^2/4 \) for the kinetic energy. The symbol \( \langle ... \rangle \) stands here for the average over the angle \( \theta \).

For the other two resonances \( q = 2, 4 \) the rotation operator \( U_r \) has only two eigenvalues \( (1, -1) \) and \( (1, -i) \) respectively. The corresponding eigenfunctions \( \psi(\pm)(\theta) \) are periodic and contain only even (odd) harmonics for the first (second) eigenvalue in each case. By writing the wave function in the form of a spinor \( \Psi(\theta) \)

\[
\Psi^{T(\text{transposed})}(\theta) = \left( \psi^{(+)}(\theta), \psi^{(-)}(\theta) \right)
\]

the rotation is described by a matrix operator \( U_r = e^{-i \alpha r/\sigma_3} \) with \( \alpha_r = \pi/2 \) for \( q = 2 \) and \( \alpha_r = -\pi/4 \) if \( q = 4 \). The kick matrix has in both cases the same form \( e^{-i \alpha r} \). Here \( \sigma_j \) are 2 \times 2 Pauli matrices. Simple manipulations cast, up to an irrelevant constant phase, the Floquet operator, that acts on the spinor \( \Psi(\theta) \), into the form of a transformation from the \( SU(2) \) group

\[
U^{(\text{res})}_q = \exp(-i w \cdot n \cdot \sigma) \equiv \exp\left(-i \hat{\mathcal{H}}^{(\text{res})}(\theta)\right),
\]

where for \( q = 2, w = \pi/2, \) and \( n = (0, \sin v, \cos v) \) while for \( q = 4, w = \arccos(\cos v/\sqrt{2}) \) and \( n = (1 + \sin^2 v)^{-1/2}(\sin v, -\sin v, \cos v) \). The hermitian operator in the exponent is equivalent to the Hamiltonian of the spin 1/2 in a magnetic field oriented along the vector \( n \).

The unified representation \([3]\) allows us to calculate in closed form the evolution of the angular momentum operator \( \hat{M} = \text{diag}(\hat{m}, \hat{\bar{m}}) \)

\[
\Delta \hat{M}(t) = e^{i w \cdot n \cdot \sigma t} \hat{M} e^{-i w \cdot n \cdot \sigma t} - \hat{M} \]

\[
= -\int_0^t d\tau \left[ w(t') \cdot \sigma + w(t')' \cdot e^{-i w \cdot n \cdot \sigma \tau} \sigma \right]
\]

\[
= -\left( w(t) n + \frac{\sin 2\omega t}{2} n' + \frac{1 - \cos 2\omega t}{2} n' \times n \right) \cdot \sigma.
\]

The 3 matrices \( \sigma_a, a = 1, 2, 3 \) are generators of the adjoin representation of the \( SU(2) \) group while the vectors \( n, n' \) and \( n' \times n \) define an orthogonal basis in this space. At a given angle, the evolution \([3]\) consists of two different contributions. The linearly growing part appears because of inhomogeneity of the magnetic field while the periodic part is due to the spin rotation in this field. The corresponding kinetic energy evolution reads

\[
E_k(t) = r t^2 + \chi(t)
\]

where

\[
r(q; k) = \frac{1}{2} \langle (w')^2 \rangle, \quad \chi(q; k; t) = \frac{1}{2} (w^2 \sin^2 w t).
\]

When \( q = 2 \), the quantity \( w = \pi/2 \) is independent of the angle \( \theta \) so that the quadratically growing term disappears and the function \( \chi(t) \) is purely periodic

\[
E_k(t) = \frac{1}{4} k_2^2 \sin^2(\frac{1}{2} \pi t).
\]

Only the spin rotation contributes and the energy jumps between values 0 and \( k^2/4 \) when the time \( t \) runs over integers. On the other hand, for \( q = 4 \) both contributions exist. The function \( \chi(t) \) fluctuates with time approaching as \( \text{const}/\sqrt{t} \) the finite positive value

\[
\chi \approx \frac{k_2^2}{2 \pi} \int_0^\pi d\theta \sin^2 \theta \left[ 1 + \sin^2 v \right]^{-2}.
\]

Let us now detune slightly from the exact resonance, \( T = T^{(\text{res})} + \kappa \). The Floquet operator looks then as \( U_{\kappa,q}(\kappa) = \exp\left(-\frac{i \kappa M^2}{4}\right) U^{(\text{res})}_q = \exp\left(-\frac{i \kappa}{4} \hat{\mathcal{H}}(\kappa)\right) \). Representing the quasi-Hamiltonian as

\[
\hat{H} = \kappa \hat{\mathcal{H}}^{(\text{res})}(\theta) + \kappa^2 Q(\kappa),
\]

we come to the condition

\[
\exp\left(-\frac{i \kappa M^2}{4}\right) = \exp\left(-\kappa \int_0^t d\tau e^{-i \hat{\mathcal{H}}^{(\text{res})}\tau} Q(\kappa) e^{i \hat{\mathcal{H}}^{(\text{res})}\tau}\right).
\]

The symbol \( T^* \) indicates the antichronological ordering. One can formally solve this equation by expanding the operator \( Q(\kappa) \) over the small detuning \( \kappa \). The effective quasi-Hamiltonian resulting from such a procedure appears in the form of the series

\[
\hat{H} = \kappa \hat{\mathcal{H}}^{(\text{res})}(\theta) + \frac{1}{2} \left\{ J, F_1(\theta) \right\}_+ + \frac{1}{2} J F_2(\theta) J + ...
\]

which is, in particular, an expansion in powers of the angular momentum \( J = \kappa M \) in which the zero-order term generates the resonant Floquet transformation. All operators in \([1]\) are one or two dimensional matrices depending on the order \( q = 1 \) or 2, 4. Being a factor in front of the angle derivative, the detuning \( \kappa \) plays in \([1]\) the role of the dimensionless Planck’s constant. Keeping only the terms written in eq. \([1]\) explicitly, the quasi-Hamiltonian \( \hat{H} \), when \( q = 2, 4 \), is formally equivalent to the Hamiltonian of a “particle” with the spin 1/2 which moves along a circle in an inhomogeneous magnetic field. The term linear in the angular momentum \( J \) mimics a sort of spin-orbital interaction. Due to the periodic boundary condition, the eigenvalue spectrum \( \{ \epsilon \} \) of \( \hat{H} \) is discrete so that the detuning restricts the growth
of the energy to a finite maximal value. This justifies the expansion over the angular momentum.

Comparing the results predicted by the quasi-Hamiltonian with exact numerical simulations of the original quantum map (1), we found that already the approximation (11) describes quite satisfactorily not only the cut-off of the initial resonant growth (when it exists in the resonance regime) and the mean value of the kinetic energy after the growth has been saturated but also delicate features of very irregular quantum fluctuations in the plateau region. The term quadratic in the angular momentum \( J \) turns out to be of the principal importance.

As a typical example we show in Fig.1 the results for the resonance \( q = 4 \). The points are numerical simulations with the exact Floquet operator (1) (only each 4th point is kept in the main part and 500th in the inset) while the solid line corresponds to the evolution described by the time-independent quasi-Hamiltonian (11). The condition under which the influence of the lowest omitted correction is weak gives an estimate of the width \( \Delta \kappa \) of a resonance \([\ref{11}]\). This leads to \( \Delta \kappa \sim 1/k \) for the resonances \( q = 1, 2 \). The resonance \( q = 4 \) is much narrower and \( \Delta \kappa \propto 1/k^2 \). Both estimates are in agreement with the numerical data. Outside these intervals the expansion becomes transparently divergent. In Fig.2 the transient region of the motion near the resonance \( q = 4 \) as it comes from the exact numerical simulations is presented. Two quite different regimes are clearly seen. The crossover from the regular (lower plateau which is well described in terms of the quasi-Hamiltonian) to chaotic (upper line) regimes takes place in a rather narrow interval of the detuning. To explore the regularity domain and the adjacent region, we have fitted the height \( E_{pl} \) of the plateau in Fig.2 as a function of the detuning \( \kappa \), (see Fig.3). The plateau height scales as \( \kappa^{-1} \) in the regularity domain \( \kappa \leq 10^{-4} \), while in the “quantum chaos” region \( \kappa \geq 10^{-3} \), the plateau height is scattered around the expected localization value \( \propto l^2 \). Note that the resonance essentially suppresses the diffusion in the intermediate region as well. Higher corrections to eq. (11) may be relevant in this domain.

Rigorously speaking, an expansion of the kind (11) cannot, in spite of the satisfactory agreement with the exact numerical solution, converge. Indeed, an infinite number of quantum resonances of large order hit the domain \( \Delta \kappa \) of influence of a strong resonance with a small order. In such a resonance of high order \( q \), the Floquet operator is a \( SU(q) \) transformation

\[
U^{(\text{res})}_{p,q} = \exp (-i w \mathbf{n} \cdot \mathbf{\lambda}); \quad n^2 = 1
\]

(12)

which is a plain extension of eq. (3). Here the matrices \( \lambda_a \) are the generators of the \( q \)-dimensional fundamental representation of the group, \( \mathbf{n} \cdot \mathbf{\lambda} \equiv \sum a \eta_a \lambda_a \) and \( \mathbf{n} \) is a unit vector in the \( (q^2 - 1) \)-dimensional adjoint space. The transformation (12) depends on the angle \( \theta \) via the periodic functions \( w(\theta) \) and \( \mathbf{n}(\theta) \) which in turn are expressed in terms of the kick potential \( v(\theta) \). These functions satisfy a system of \( q^2 - 1 \) transcendental equations which, generally, cannot be solved analytically. Nevertheless, some generic information can be obtained even without knowing the explicit solution. Indeed, the evolution of the \( q \times q \) angular momentum matrix is described, similarly to eq. (3), by

\[
\Delta \mathbf{M}(t) = \left[ \mathbf{M}^{(0)} t + \mathbf{M}^{(1)}(t) \right] \cdot \mathbf{\lambda}.
\]

(13)

The matrices \( \mathbf{M}^{(0)} \) are expressed in terms of the \( q \)-1 zero modes of the matrix \( (\mathbf{n} \cdot \mathbf{\lambda}) \) which acts in the \( (q^2 - 1) \)-dimensional adjoint space, while the quasi-periodic (as long as the angle \( \theta \) is kept fixed) matrix \( \mathbf{M}^{(1)}(t) \) is formed by the \( q(q-1)/2 \) pairs of mutually conjugate modes with finite eigenvalues \( \pm \xi_a \). These two matrices determine respectively the resonant growth rate

\[
r(p, q; k) = \left[ \frac{1}{2} \sum a \eta_a (|M_a^{(0)}|^2) \right]
\]

(14)

and the asymptotic value

\[
\chi_{\infty} = 2 \sum (\xi_a > 0) (\xi_a^{-2} |\mathbf{n}^\ast \cdot \mathbf{\chi}(b)|^2 \sum a \eta_a |\chi_a(b)|^2) \cdot (15)
\]

The quantities \( \eta_a \) in this formulae select the \( (2q-1) \)-dimensional active subspace of the adjoint space, which is reachable for the motion under the isotropic initial conditions. A small detuning from the considered resonance kills the unrestricted growth created by the resonance itself and gives rise to the \( q \times q \) quasi-Hamiltonian matrix of the same structure as in (11). The motion looks like that of a “particle” with \( q \) intrinsic degrees of freedom described by the \( SU(q) \) group, along a circle in a \( (q^2 - 1) \)-component “magnetic” field.

Returning to the convergence problem, one sees that, independently of the number of the corrections taken into account, the quasi-Hamiltonian approach fails to reproduce the unrestricted resonant growth created by the resonance points within the domain of the influence of a strong resonance (i.e. with a small order) around which the expansion is performed. Indeed, the spectrum of the quasi-Hamiltonian is always discrete whereas the resonant growth implies continuous spectrum.

However, as confirmed by numerical data, the growth rate \( r(p, q; k) \) is exponentially small when \( q \) noticeably exceeds the localization length \( \xi \). Therefore, the resonant growth reveals itself only on a very remote time asymptotics. Qualitative arguments presented in (3) connect this fact with the exponentially weak overlap of the neighboring localized parts of the globally delocalized quasienergy eigenfunctions. Exponential effects of such a kind, which are characteristic of the tunneling, are well known to be beyond the reach of perturbation expansions. That is why they cannot be described in the framework of the quasi-Hamiltonian method. The
latter reproduces well only those features of the motion, which are determined by the discrete component of the quasienergy spectrum, in particular, the function $\chi(t)$ which attains its asymptotical value much faster. Therefore inside the width of a strong resonance the time dependence of these functions is dictated for all weak ones by their strongest brother.

On the other hand, if the order is very large, $q \gg l$, and the resonance lies in the region of typical irrationalations being far from all strong resonances, already a very small detuning suffices for killing the quadratic growth with exponentially small rate. At the same time, such a shift does not influence the function $\chi(t)$ which reproduces on a large (though finite) time scale characteristic features of the “localized quantum chaos”.

The behavior becomes more complicated if a number of resonances with comparable and moderate orders are neighbouring and their domains overlap. The expansion near one of them forms a plateau which lasts until the quadratic growth in a next resonance of the same strength reveals itself so that the original expansion fails. However, the expansion near the new resonance cuts off this growth and forms a higher plateau until a next resonance comes to the action. Such a pattern of repeatedly reappearing regimes of resonant growth has been discovered in [10].

In conclusion, the concept of the time-independent quasi-Hamiltonian of a quantum map is proposed. The motion of the quantum kicked rotor in a quantum resonance of order $q$ is exactly described by a continuous transformation from the $SU(q)$ group. The motion in some vicinity of this resonance is proved to be similar to that of a quantum particle with $q$ intrinsic degrees of freedom along a circle in an inhomogeneous $(q^2 - 1)$-component “magnetic” field. The latter is in analogy with the classical phase oscillations near a non-linear resonance. The resonances with sufficiently small $q$ master the motion in finite domains near the resonance points though the widths of this domains rapidly diminish with the order.

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