Solar System Constraints on \( f(\mathcal{G}) \) Dark Energy

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Abstract. Corrections to solar system gravity are derived for \( f(\mathcal{G}) \) gravity theories, in which a function of the Gauss-Bonnet curvature term is added to the gravitational action. Their effects on Newton’s law, as felt by the planets, and on the frequency shift of signals from the Cassini spacecraft, are both determined. Despite the fact that the Gauss-Bonnet term is quadratic in curvature, the resulting constraints are substantial. It is shown that they practically rule out \( f(\mathcal{G}) \) as a natural explanation for the late-time acceleration of the universe. Possible exceptions are when \( f(\mathcal{G}) \) reduces to something very close to a cosmological constant, or if the form of the function \( f \) is exceptionally fine-tuned.

Keywords: dark energy theory, gravity, string theory and cosmology

1. Introduction

The current accelerated expansion of our universe cannot be explained by conventional general relativity if our universe contains only standard matter and radiation. Some form of additional dark energy, such as a cosmological constant, may be the source of the acceleration, although such models suffer from serious fine-tuning problems. It may be that the acceleration instead comes from corrections to Einstein gravity. Such an approach has the potential to avoid the fine-tuning problems, resulting in a far more credible theory. One candidate for effective dark energy is the quadratic curvature Gauss-Bonnet term

\[
\mathcal{G} = R^2 - 4 R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},
\]

which is a natural addition to the Einstein-Hilbert action \([1]\).

In fact, on its own in four dimensions, the contribution of the Gauss-Bonnet term to the gravitational field equation is trivial. For it to have an effect the theory needs to have extra dimensions, such as in the brane world scenario with one \([2]\) or more \([3]\) additional dimensions, or alternatively the Gauss-Bonnet term can be coupled to a scalar field. Another related possibility is to add a function \( f(\mathcal{G}) \) of it to the gravitational action \([4]\). Such a theory will be the subject of this article.

Its potential to give a more elegant explanation for the universe’s accelerated expansion makes gravity modification very appealing. However it does have one major
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drawback, namely that deviations from general relativity will be felt on all scales, not just cosmological ones. In particular gravity will be altered within our solar system, where high precision tests of general relativity have been performed. As we will show for $f(G)$ gravity, the corrections are generically too large, and allow it to be ruled out as a solution to the dark energy problem (with some severely fine-tuned exceptions).

Within our solar system, gravitational fields are weak. Since the Gauss-Bonnet term is quadratic in curvature, it might be expected that its effects will be sub-dominant, and not significantly constrained. However as we will demonstrate, this reasoning is flawed. In section 2 we will show, after introducing the theory, that if the cosmological contribution of $f(G)$ is to be large enough to act as dark energy, the couplings in $f$ must be extremely large. This will greatly magnify the effects of $G$ in the solar system, and produce corrections to the Newtonian and post-Newtonian potentials, which we will derive in section 3. Strong constraints on these corrections arise from planetary motion and light bending measurements, which are analysed for a more general theory in section 4. In section 5 these are applied to $f(G)$ gravity, and it is shown that the gravitational effects within the solar system are indeed large enough to conflict with the observational data that is available, and thus allow this dark energy candidate to be practically ruled out.

2. Dark energy from $f(G)$ gravity

Working in units with $c = 1$, we will be studying the theory

$$\mathcal{L} = \sqrt{-g} \left[ \frac{R}{2} + f(G) \right].$$

(2)

It is equivalent to coupling a Gauss-Bonnet term to a scalar field with a potential, but without an explicit kinetic term (in contrast to the Gauss-Bonnet theories studied in [5, 6]). In particular, if we take the action

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} [R + \xi(\phi)G - 2V(\phi)]$$

(3)

with

$$V(\phi) = -f(\phi) + \phi f'(\phi), \quad \xi(\phi) = 2f'(\phi),$$

(4)

and then vary it with respect to the scalar field, we obtain $\phi = G$. Substituting this back into the action (3), it reduces to the theory (2).

Varying the action (2) with respect to $g_{\mu\nu}$, we obtain the gravitational field equations

$$G_{\mu\nu} + 8(R_{\mu\nu\sigma} + 2R_{\rho[\nu}g_{\sigma]\rho} - 2R_{\mu[\nu}g_{\sigma]\rho} + Rg_{\mu[\nu}g_{\sigma]\rho])\nabla^\rho \nabla^\sigma f'(G) + [Gf'(G) - f(G)]g_{\mu\nu} = 8\pi G_0 T_{\mu\nu}.$$  

(5)

These can also be obtained from (3). The constant $G_0$ in the above equation is the gravitational coupling of matter, which may or may not be the same as the gravitational coupling $G$ that we perceive on Earth.
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In a cosmological background $(ds^2 = -dt^2 + a(t)^2 dx_j^2)$, we find $G = 24H^2(\dot{H} + H^2)$, with $H = \dot{a}/a$. The Friedmann equation can be written as

$$1 = \frac{8\pi G_0}{3H^2} \rho_{\text{mat}} + \Omega_G,$$

where the Gauss-Bonnet density fraction is

$$\Omega_G = -8H \partial_t f'(G) + \frac{G f'(G) - f(G)}{3H^2}.$$  \hspace{1cm} (7)

We will focus on the case $f(G) = C G^n$. The expression (7) then reduces to

$$\Omega_G = \frac{C(n-1)}{3} [-12(1+3w)]^n \left[1 - 12n \frac{(1+w)}{1+3w}\right] H^{2(2n-1)},$$

where $w = -1 - 2\dot{H}/(3H^2)$ is the effective equation of state for the universe. An important point to note is that if $\Omega_G \sim 0.7$, as is required if $f(G)$ is to give a sufficient contribution to the dark energy density, then $C \sim H_0^{2-4n}$. This is extremely large (for $n > 1/2$), and so we see that the Gauss-Bonnet term must be very strongly coupled if it is to have any chance of explaining the accelerated expansion of our universe.

Note that if a solution of the above theory is solve the dark energy problem, $\Omega_G \sim 0.7$ is necessary, but not sufficient. We also need the solution to actually produce enough acceleration (so $w \approx -1$), and for the cosmological evolution of our universe to reach it (after passing though a period of matter domination, exactly like the one that occurred in our universe). In this work we will mainly be concerned with the magnitude of $\Omega_G$, which will be enough to rule out most possible $f(G)$ dark energy models.

3. Solar system gravity

Within the solar system we can use the approximate Post-Newtonian metric

$$ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j + O(\epsilon^{3/2}).$$

with $\Phi, \Psi \sim \epsilon$ and $\dot{\Phi}, \dot{\Psi} \sim \epsilon^{3/2}$. The gravitational fields of the sun and planets are relatively weak and slowly varying, and $\epsilon \lesssim 10^{-5}$. This allows us to study gravity with a perturbative expansion in $\epsilon$. Note that the metric (9) is motivated by the properties of our solar system, and not by the choice of gravitational theory.

Making no assumptions about the size of $f$ or its derivatives, we find that to leading order in $\epsilon$, the gravitational field equations are

$$\Delta \Phi = 4\pi G_0 \rho_{\text{mat}} + f(G) - \mathcal{G} f'(G) - 4\mathcal{D}(f'(G), \Phi + \Psi) + \epsilon^2 O(1, \mathcal{G} f'', \mathcal{G}^2 f''') + \epsilon^2 O(f, \mathcal{G} f')$$

$$\Delta \Psi = 4\pi G_0 \rho_{\text{mat}} + f(G) - \mathcal{G} f'(G) - 4\mathcal{D}(f'(G), \Psi) + \epsilon^2 O(1, \mathcal{G} f'', \mathcal{G}^2 f''') + \epsilon^2 O(f, \mathcal{G} f')$$

and $\mathcal{G} = 8\mathcal{D}(\Phi, \Psi) + O(\epsilon^3)$. We have introduced the operators

$$\Delta F = \sum_i F_{,ii}, \quad \mathcal{D}(X, Y) = \sum_{i,j} X_{,ij} Y_{,ij} - \Delta X \Delta Y,$$  \hspace{1cm} (12)
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which for functions with only \( r \) dependence reduce to

\[
\Delta F = \frac{1}{r^2} \partial_r (r^2 \partial_r F), \quad \mathcal{D}(X, Y) = -\frac{2}{r^2} \partial_r (r \partial_r X \partial_r Y).
\] (13)

In principle, the higher order (in \( \epsilon \)) terms of the above expansion could give significant contributions if their coefficients are very large. However, for \( f = C G^n \), the corresponding coefficients are all of comparable size, and the higher terms can be dropped. Furthermore, when \( f = C G^n \), we see that the \( f(G) \) and \( G f'(G) \) terms are higher order in \( \epsilon \) than the \( \mathcal{D}(\cdots) \) terms. For the rest of this section we will assume this form for the theory, and so can drop those two terms from the field equations. In this work we will be mainly interested in \( n > 0 \), since theories with negative \( n \) have been studied elsewhere in the literature. In particular, it has been shown while they satisfy solar system constraints [8], they are unable to solve the dark energy problem [9]. We will discuss these points in more detail in section 5.

The usual, Einstein gravity (\( f \equiv 0 \)) solution is \( \Phi = \Psi = -U \), with

\[
U = 4\pi G_0 \int d^3 x' \rho_{\text{mat}}(\vec{x}', t) \frac{1}{|\vec{x} - \vec{x}'|}.
\] (14)

Treating the sun as a uniform sphere we find that \( U \), and the Gauss-Bonnet term, reduce to

\[
U_{\text{ext}} = \frac{G_0 m_\odot}{r}, \quad \mathcal{G}_{\text{ext}} \approx 48 \frac{(G_0 m_\odot)^2}{r^6}
\] (15)

outside the sun \( (r > R_\odot) \), and

\[
U_{\text{int}} = \frac{G_0 m_\odot}{2R_\odot} \left(3 - \frac{r^2}{R_\odot^2}\right), \quad \mathcal{G}_{\text{int}} \approx -48 \frac{(G_0 m_\odot)^2}{R_\odot^6}
\] (16)

inside it \( (r < R_\odot) \).

A general analysis for non-trivial \( f \) would be difficult. However we know that for any viable theory, the resulting gravitational potentials must be very close to the standard form. Following the approach of [3], we start by assuming that the potentials are very close to the usual \( 1/r \) form. Deviations from this can be treated perturbatively. By bounding their size, we can derive constraints on \( f \). Of course for a wide range of \( f(G) \) theories, \( \Phi \) and \( \Psi \) will be radically different from \( 1/r \), and our approach will not give their approximate form correctly. However, since we already know (by definition) that such theories fail to give the observed gravitational potentials, and there is no need to study them in the first place. Hence our approach will cover all potentially relevant cases.

For the interior of the sun, we see that \( \mathcal{D}(f'(\mathcal{G}_{\text{int}}), U_{\text{int}}) = 0 \). The interior solution (16), up to the addition of a constant, is therefore still valid (recall that we are ignoring the \( f, \mathcal{G} f' \) terms as they are sub-dominant for \( f = C G^n \)).

For the exterior solution we take \( \Phi = -G m_\odot/r + \delta \Phi \) and \( \Psi = -G m_\odot/r + \delta \Psi \). Here \( G \) is the approximate gravitational coupling that we perceive, and need not be equal to the bare coupling \( G_0 \). The perturbations to the usual potentials resulting from the \( f \) dependent \( \mathcal{D}(\cdots) \) operator are given by \( \delta \Phi \) and \( \delta \Psi \). Such an analysis will only be valid
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if $\delta \Phi, \delta \Psi \ll U$, which will indeed be the case for us. The field equations (10) and (11) reduce to

$$\Delta \delta \Phi \approx 8D \left( C_n G^{n-1}, \frac{G m_\odot}{r} \right), \quad \Delta \delta \Psi \approx 4D \left( C_n G^{n-1}, \frac{G m_\odot}{r} \right)$$

(17)

with $G \approx 48(G m_\odot)^2/r^6$. Solving these, and requiring continuity of $\Phi, \Psi$, and their first derivatives, gives the perturbed exterior potentials

$$\Phi = -\frac{m_\odot}{r} \left[ G_1 - \frac{A G}{(1 + s) r^s} \right], \quad \Psi = -\frac{m_\odot}{r} \left[ G_1 - \frac{GB}{(1 + s) r^s} \right] \quad \text{(for } s \neq -1)$$

$$\Phi = -\frac{m_\odot}{r} \left[ G_1 - A r \ln r \right], \quad \Psi = -\frac{m_\odot}{r} \left[ G_1 - GB r \ln r \right] \quad \text{(for } s = -1)$$

(18)

where we have introduced

$$A = 2B = 2n(n - 1) C 48^n (G m_\odot)^{2(n-1)}, \quad G_1 = G_0 + A R_\odot^{-s}, \quad s = 2(3n - 2).$$

(19)

We see that there are finite width effects appearing in the gravitational potential. This is due to the presence of higher than second order derivative operators appearing in the gravitational field equations. This also implies that the usual treatment of the sun, and other objects, as point-like masses is no longer valid.

Clearly the above expressions (18) are different from the usual Einstein gravity result, and are likely to come into conflict with the high precision measurements of gravity in the solar system. By bounding the corrections, we will be able to constrain the parameters in the function $f(G)$, and hence its dark energy contribution.

4. Solar system gravity tests

We will now review two sources of gravitational constraints coming from the solar system. First a test of Newton’s law from planetary motion, and then the frequency shift of light rays, which tests a relativistic effect. The results will apply to any theory giving potentials of the form (18), and not just $f(G) = C G^n$ gravity. We will apply them to the $f(G)$ dark energy models in section 5.

Corrections to the Newtonian potential can be bounded by considering their effect on bodies orbiting the sun. For an expression of the form (18), the gravitational acceleration experienced by a body at distance $r$ from the sun is

$$g_{\text{acc}}(r) = -\frac{d\Phi}{dr} = -\frac{m_\odot}{r^2} \left[ G_1 - A \frac{G}{r^s} \right] \equiv \frac{G m_{\text{eff}}(r)}{r^2}. \quad (20)$$

For a body following an elliptical orbit with semi-major axis $a$, Kepler’s third law gives the period of the orbit as $2\pi \sqrt{a^3/(G m_\odot)}$. Bounds on corrections to the effective mass are then related to the uncertainties in the measurement of $a$ by

$$\frac{\delta m_{\text{eff}}(a_\alpha)}{m_\odot} < 3 \frac{\delta a_\alpha}{a_\alpha}, \quad (21)$$

where the index $\alpha$ runs over all bodies which are orbiting the sun. Values of $\delta a$ for the planets can be found in [10]. The above relation has previously been used to bound
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dark matter in the solar system \cite{11}, the cosmological constant \cite{12}, and another class of Einstein-Gauss-Bonnet gravity models \cite{5}.

As it stands, the above constraint (21) depends on $G_1$, which is related to the unmeasured constant $G_0$. By combining constraints from two bodies, we can eliminate it to obtain

$$|A (a_\alpha^{-s} - a_\beta^{-s})| < 3 \left( \frac{\delta a_\alpha}{a_\alpha} + \frac{\delta a_\beta}{a_\beta} \right),$$

which needs to be satisfied for all choices of $\alpha, \beta$. For $s \geq -1$ the strongest bounds come from the inner planets, firstly because these are the bodies for which we have the best data, and secondly since they are closest to the sun, where the gravity corrections are strongest.

To satisfy the bound (22), either $A$ will need to be very small, $s$ will need to be close to zero (in which case the corrections to the potential will mimic Newtonian gravity), or $s$ will need to be less than $-1$ (in which case the corrections are suppressed at the smaller distances where better data is available).

Further constraints come from signals between man-made spacecraft and the Earth. The sun’s gravitational field produces a time delay in the signals, measurement of which provides an additional test of gravity in the solar system. Furthermore, unlike the planetary constraint (22), this is sensitive to relativistic effects, and so can be used to rule out models which mimic Newtonian gravity.

For a light-ray starting at the Earth, passing close to the sun’s surface, continuing to the spacecraft ($r_e$ from the sun), and then returning by the same route, the time delay in the signal is

$$\Delta t = -2 \int_{a_\oplus}^{r_e} [\Phi(r) + \Psi(r)] dx. \quad (23)$$

The signal’s path is approximated by $r = \sqrt{x^2 + b^2}$, where the impact parameter $b$ is defined as the smallest value of $r$ on the light ray’s path. A small value of $b$ will maximise the above time delay. Particularly good data was obtained for the Cassini spacecraft while making its journey to Saturn. During 2002, the impact parameter fell as low as $b = 1.6 R_\oplus \approx 0.0074$ AU (the value of $r_e$ at this point was 8.43 AU). The actual measurements obtained were not of the time delay, but its frequency shift given by \cite{13}

$$y = \frac{d\Delta t}{dt} \approx \frac{d\Delta t}{db} \frac{db}{dt} = -\frac{4Gm_\odot}{b} \frac{db}{dt} \left[ 2 + (2.1 \pm 2.3) \times 10^{-5} \right]. \quad (24)$$

For potentials of the form \cite{18}, the frequency shift evaluates to

$$y = -\frac{4m_\odot}{b} \frac{db}{dt} \left( 2G_1 I_0 - [A + B] G I_s \right), \quad (25)$$

where we have defined

$$I_s = \frac{b^2}{2} \int_{a_\oplus}^{r_e} \frac{dx}{(b^2 + x^2)^{(3+s)/2)}, \quad (26)$$

which evaluates to a hypergeometric function. A similar analysis was applied to a different Einstein-Gauss-Bonnet theory in \cite{5}, with $s = 6$ due to the simplifying assumptions imposed on the model.
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For small impact parameter, \( b \ll a_\oplus, r_e \), which is the case for the above Cassini bound (24), the leading order behaviour of the above integral is

\[
\mathcal{I}_s \approx \begin{cases} 
\frac{\sqrt{\pi} \Gamma([2 + s]/2)}{2 \Gamma([3 + s]/2)} b^s & s > -2 \\
b^2 \frac{4 r_e a_\oplus}{b^2} s + \ln \left( \frac{r_e a_\oplus}{b^2} \right) & s = -2 \\
-\frac{b^2}{2(2 + s)} \left[ r_e^{-(s+2)} + a_\oplus^{-(s+2)} \right] & s < -2
\end{cases}
\]  

(27)

In particular \( \mathcal{I}_0 \approx 1 \).

Comparing (24) and (25) gives a bound on \( A + B \), although this is of limited use, since like (20) it depends on the undetermined quantity \( G_1 \). To eliminate that, we combine the constraint with (21), giving

\[
-(2 \times 10^{-6}) - 6 \frac{\delta a_\alpha}{a_\alpha} < 2 a_\alpha^{-s} - (A + B) \mathcal{I}_s < (4.4 \times 10^{-5}) + 6 \frac{\delta a_\alpha}{a_\alpha}.
\]  

(28)

Note that the above constraint will need to be satisfied for all choices of \( \alpha \).

Another well-known solar system gravity test comes from the perihelion precession of Mercury. However we will not consider it here as the linearised analysis of section 3 is insufficient to determine it, and a higher order (in \( \epsilon \)) expansion is needed. As it turns out, the above results will be sufficient to eliminate \( f(\mathcal{G}) \) as a viable dark energy component.

5. Constraining \( f(\mathcal{G}) \) gravity

Combining the expression for the effective dark energy fraction (8) with the planetary motion constraint (22), and taking the current effective equation of state for our universe to be \( w = -1 \), we obtain the bound

\[
|\Omega_\mathcal{G}| < \frac{H_0^{2(2n-1)}}{2^{n+1}n r_g^{2(n-1)}} \left| \frac{a_\alpha^{2(2-3n)}}{a_\beta^{2(2-3n)}} \right| \left( \frac{\delta a_\alpha}{a_\alpha} + \frac{\delta a_\beta}{a_\beta} \right),
\]  

(29)

where \( H_0^{-1} \approx 8.8 \times 10^{14} \) AU is the Hubble distance, and \( r_g = Gm_\odot \approx 9.7 \times 10^{-9} \) AU is the gravitation radius of the sun. For \( n > 1 \) we find the strongest constraint comes from taking \( (\alpha, \beta) \) to be Mercury \( (a \approx 0.39 \) AU, \( \delta a/a \approx 1.8 \times 10^{-12} \)) and the Earth \( (a \approx 1 \) AU, \( \delta a/a \approx 0.98 \times 10^{-12} \)). This implies

\[
|\Omega_\mathcal{G}| \lesssim 1.6 \times 10^{-43},
\]  

(30)

which is clearly far too small to account for the dark energy of our universe. For smaller \( n \), a significant dark energy fraction may be possible. Using all the planets in our solar system, we find \( \Omega_\mathcal{G} \lesssim 0.1 \) unless

\[
\left| n - \frac{2}{3} \right| \lesssim 1.1 \times 10^{-27} \quad \text{or} \quad n \lesssim 0.074.
\]  

(31)

The two numerical values come respectively from the Earth-Mercury and Earth-Uranus combinations (Uranus has \( a \approx 19 \) AU, \( \delta a/a \approx 1.3 \times 10^{-8} \)). Note that for the above
bounds on $\Omega_G$ we assumed the equation of state $w = -1$ for our universe. Other values will give slightly weaker or stronger constraints, but they will be of the same order of magnitude.

Although the first of the exceptions (31) mimics the correct Newtonian limit of gravity, it fails to produce the correct relativistic effects. Using the Cassini constraint (28), we find that for the parameters (19),

$$-\frac{2 \times 10^{-6}}{6} - \frac{\delta a_\alpha}{a_\alpha} < \frac{2^n n r_{\text{g}}^{2n-2}}{H_0^{4n-2}} (2a_\alpha^{4-6n} - 3I_{6n-4}) \Omega_G < \frac{4.4 \times 10^{-5}}{6} + \frac{\delta a_\alpha}{a_\alpha},$$

where we have again used (8) with $w = -1$. Taking $\alpha$ to be Mercury, we find that $|\Omega_G| \lesssim 1.2 \times 10^{-20}$ for $n \geq 0.66$, which includes the first range in (31), thus ruling it out.

It should be pointed out that the errors $\delta a_\alpha$ appearing in [10] are in fact the statistical errors in the values of $a_\alpha$ coming from a least-squares fit of observations. The real errors may well be an order or two of magnitude higher, and would give correspondingly weaker bounds. However the above bounds are so strong that even this would not be enough to save the theory.

The second exception (31) to the bound (29) includes the parameter range $n < 0$. Such models have been studied elsewhere, for example [8, 9, 14]. We see that (for $n < 1/2$) the correction to the Newtonian potential for a general mass $M$ is

$$\delta \Phi \propto n \left( \frac{r^3}{GM} \right)^{1-2n},$$

which actually decreases for larger masses. Obviously, the above expression is not valid at very large distances, where $\delta \Phi$ will cease to be a small perturbation. However for couplings of the size needed to solve the dark energy problem, $C \sim H_0^{2-4n}$, the above $\delta \Phi$ will remain small within the solar system.

We see that for a heavy object such as the sun the corrections to $\Phi$ and $\Psi$ are suppressed, which is why (for low enough $n$) they do not conflict with our solar system constraints. On the other hand, we see that the above expression becomes large for small masses, implying that the weak field approximation we have been using breaks down. This suggests that the theory will predict non-Newtonian gravity for table-top gravity experiments, and so will disagree with observation. However it must not be forgotten that laboratory experiments are performed on the Earth, which has a large gravitational field of its own. The above expression (33) no longer applies since it is now the Earth which gives the dominant contribution to the Gauss-Bonnet term $G$. The resulting bounds on the theory are then far weaker, and do not rule out dark energy models with $n < 0$ [8, 14].

Although $f(G)$ gravity with inverse powers of $G$ can give a large enough $\Omega_G$, and avoids conflict with solar system gravity tests, this is not enough to solve the dark energy problem. During its early evolution, our universe passed through a matter dominated, decelerating phase (with $G = 24H_0^2(\ddot{a}/a) < 0$), before entering the current accelerated phase (with $G > 0$). The Gauss-Bonnet term must therefore have passed through zero at some point. We see from the cosmological field equations (6), (7) that if $f(0)$ is not
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bounded, $\mathcal{G} = 0$ corresponds to curvature singularity, and is unreachable (for finite $H$). Since no transition to accelerated expansion occurs, inverse powers of $\mathcal{G}$ do not give viable dark energy models [9]. However, they may still be relevant to the dark matter problem [14].

So far we have only considered a special subset of the modified gravity theory (2) where the function $f(\mathcal{G})$ is some power of the Gauss-Bonnet term. A more general theory can be expanded as a power series

$$f(\mathcal{G}) = \sum_n C_n \mathcal{G}^n,$$

(34)

where the $n$ are not necessarily integers. Now each term gives a correction to the Newtonian potential $\Phi$ with a different $r$ dependence, and so barring extreme fine-tuning of the $C_n$, we can constrain each of them separately. Similarly for $\Psi$. Hence, using the above results, we find that for each of their contributions to the dark energy, $\Omega^{(n)}_\mathcal{G} \ll 1$. The only exception is $0 \leq n \lesssim 0.074$, which is very similar to a cosmological constant.

Finally, we note that our gravity corrections have turned out to be tiny, at least for $f(\mathcal{G}) = C\mathcal{G}^n$ theories with positive $n$, and which also satisfy the solar system constraints. Hence the leading order corrections $\delta \Phi$ and $\delta \Psi$ used in section 3 are much smaller than $Gm_\odot/r$, and the perturbative analysis used there is indeed self-consistent. In the case of $n = 0$, the leading order non-zero corrections to $\Phi$ and $\Psi$ are not covered by our analysis. This case is the same as a cosmological constant, and has been covered elsewhere in the literature (e.g. [12]). It is worth pointing out that a very wide range of $f(\mathcal{G})$ theories do not have $\Phi \approx \Psi \approx -Gm_\odot/r$ as an approximate solution, and so the analysis of this paper does not apply to them. However, since the gravitational field of the sun in these theories has no resemblance to what we have observed in the solar system, they are already ruled out, and there is no need to apply this paper’s analysis.

6. Conclusions

We have seen that a careful analysis of solar system gravity provides a powerful probe of higher curvature gravity theories, and can be used rule out modified gravity models which are intended to solve the dark energy problem. Specifically, strong constraints on Newton’s law are obtained from the motion of the planets. These are sufficient to rule out a large fraction of $f(\mathcal{G})$ dark energy models. One class of model that survives these constraints are those with inverse powers of the Gauss-Bonnet term $\mathcal{G}$, although their cosmological evolution is not consistent with our universe.

The remaining $f(\mathcal{G})$ models which survive the Newtonian constraints either mimic Newtonian gravity, or are very close to a cosmological constant. Further strong constraints come from the measurement of the frequency shift of signals from the Cassini spacecraft. These rule out the models which mimic Newtonian gravity. Hence $f(\mathcal{G})$ is practically ruled out, as it is only viable when it is virtually identical to a cosmological constant.
In principle, with enough fine-tuning, a model could be constructed where the deviations from general relativity cancel for each of the planets (and for the Cassini bound). The non-standard gravity effects could then still be large enough to account for the dark energy. However for this to work the corrections must cancel for every choice of position \( r \), and mass, for which there is a planet, moon, asteroid, laboratory experiment, time delay or frequency shift measurement. Even then, there is no guarantee that the non-standard cosmological evolution will be accelerating, be free from ghosts \([9]\), and satisfy all other cosmological tests, such as producing correct growth of density perturbations, which is also a problem for \( f(G) \) gravity \([15]\). The problem of ghosts could even arise within the solar system, giving another way to constrain the theory there. The presence of ghosts can be determined by analysing perturbations of the solar system metric. However we will not pursue this for our \( f = CG^n \) dark energy \( (n > 0) \) solutions, since they have already been ruled out. Furthermore, we did not actually obtain a solution to the field equations within the solar system (expect when \( \Omega_G \) is negligible), so we do not have a background solution whose perturbations are worth analysing.

Following the example of \( f(R) \) gravity \([16]\), an \( f(G) \) that reduces to constant for large \( G \) and zero at small \( G \) may give a viable dark energy model while satisfying solar system constraints. In fact any \( f(G) \) whose form gives acceptable corrections for solar system curvatures (e.g. a inverse power of \( G \)), but produces acceleration at cosmological scales could be viable. However such a theory will require even more fine-tuning than a conventional cosmological constant, so it is debatable whether this is an improvement.

Our strong constraints on \( f(G) \) dark energy rely on the fact that we can directly link its local and cosmological properties. This is possible because the behaviour of the theory is fully determined by the form of the metric (which is known at both scales) and the constant parameters which determine \( f \). As an alternative type of theory we could have extended the action \([3]\), to include additional kinetic terms for the scalar \( \phi \), as in \([5, 6]\). These can all have \( \phi \)-dependent couplings, and can include higher order terms as well as the usual \((\partial_\mu \phi)^2\). The resulting theory has the same number of degrees of freedom as the one studied in this paper, suggesting the same analysis can be applied. However, the value of the field \( \phi \) is no longer directly determined by the metric: \( \phi = G \). Instead it is related to \( g_{\mu\nu} \) by a differential equation, whose solution can include integration constants which are determined by cosmological rather than local scales. In particular, the value of \( \phi \) at the edge of the solar system is (roughly) zero for \( f(G) \) gravity, and arbitrary for the more general theories. Hence in the latter case connecting the strong local constraints and the cosmological evolution will require further analysis of the theory’s behaviour on intermediate scales. The theories studied in \([5,6]\) therefore still have the potential to solve the dark energy problem, although of course further work is required. Another, more obvious advantage of the above scalar-tensor theories, is that they contain more coupling functions than the single one in \( f(G) \). With a greater range of theories to choose from, there is more hope of finding one which satisfies both local and cosmological constraints.
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