STABILITY CONDITIONS ON AFFINE NOETHERIAN SCHEMES

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Abstract. We show that the existence of locally finite stability conditions on the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on an affine Noetherian scheme $X$ is equivalent to $\dim X = 0$. We also study the space of stability conditions on the category of morphisms $\mathcal{M}_X$ in the derived category of the scheme $X$. Similarly to the case of $\mathcal{D}^b(X)$, the existence of $\text{Stab} \mathcal{M}_X$ is equivalent to $\dim X = 0$. Finally we show that the spaces of stability conditions on $\mathcal{D}^b(X)$ and on $\mathcal{M}_X$ are homotopy equivalent.

1. Introduction

1.1. Stability conditions on affine schemes. Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on an algebraic variety $X$. The space $\text{Stab} \mathcal{D}^b(X)$ of stability conditions on $\mathcal{D}^b(X)$, introduced by Bridgeland \cite{bridgeland}, is an effective mathematical object for the study of algebraic geometry. For instance $\text{Stab} \mathcal{D}^b(X)$ has many applications not only to the derived category $\mathcal{D}^b(X)$ (cf. \cite{boucksom} and \cite{bridgeland}) but also to moduli spaces of sheaves on the variety $X$ (cf. \cite{bayer-macri-toda}, \cite{bridgeland} and \cite{bridgeland2}).

Now the non-emptiness of $\text{Stab} \mathcal{D}^b(X)$ is non-trivial and fundamental. If $X$ is smooth and projective with $\dim X \leq 2$, then $\text{Stab} \mathcal{D}^b(X)$ is non-empty by \cite{boucksom} and \cite{bridgeland}. If $X$ is smooth and projective with $\dim X = 3$, the generalized Bogomolov-Gieseker inequality introduced by Bayer-Macri-Toda \cite{bayer-macri-toda} gives a sufficient condition for the non-emptiness. On the other hand, there seems no study of a necessary condition for the non-emptiness. In this note we show that the non-emptiness of stability conditions on an affine Noetherian scheme leads a property for dimension of the scheme.

Theorem 1.1. Let $X$ be an affine Noetherian scheme of a Noetherian ring $R$. Then $\text{Stab} \mathcal{D}^b(X)$ is non-empty if and only if $\dim X = 0$. Moreover, if $\dim X = 0$, then $\text{Stab} \mathcal{D}^b(X)$ is isomorphic to $\mathbb{C}^n$ where $n$ is the number of the points in $X$.

Based on the theorem above, it might be interesting to study what kind of categorial properties of a triangulated category $\mathcal{D}$ guarantee the existence of (locally finite) stability conditions on $\mathcal{D}$. Roughly, our proof is based on the existence of “many” global functions on the affine scheme $X = \text{Spec} R$. Though we do not understand such an existence in terms of triangulated categories, the following question represents one of the directions.

Question 1. Let $\mathcal{D}$ be a $R$-linear triangulated category over a Noetherian ring $R$. Suppose $\text{Stab} \mathcal{D} \neq \emptyset$. Does $\dim R = 0$ holds?

It might be interesting to study a relation between the existence of stability conditions and $R$-linear structures. To answer the question, a new idea could be needed.
1.2. Stability conditions on morphisms. We further study the space of stability conditions on morphisms in the bounded derived category of an affine Noetherian scheme, which is our second interest. Note that the category of morphisms in a triangulated category is not triangulated in general. Recall that the derived category \( D^b(X) \) of a Noetherian scheme \( X \) is obtained by the homotopy category \( h(D^b(X)) \) of a stable infinity category \( D^b_{\text{coh}}(X) \) of quasi-coherent sheaves with bounded coherent cohomologies. Then the homotopy category \( h(D^b_{\text{coh}}(X)^{\Delta^1}) \) of the infinity category \( D^b_{\text{coh}}(X)^{\Delta^1} \) of morphisms in the infinity category \( D^b_{\text{coh}}(X) \) is triangulated, and hence is a reasonable candidate of the triangulated category of morphisms in \( D^b(X) \).

From now on let us denote by \( M_X \) the category \( h(D^b_{\text{coh}}(X)^{\Delta^1}) \) of morphisms. We note that \( M_X \) is equivalent to the bounded derived category of representations of the \( A_2 \) quiver when \( X \) is the affine scheme of a field by the author \([11, \text{Corollary 6.2}]\).

Basically we are interested in a relation between the spaces of stability conditions on \( D^b(X) \) and on \( M_X \). In our previous paper \([11]\), we showed that there exist natural two homomorphic maps \( d_0^*, d_1^*: \text{Stab} D^b(X) \rightarrow \text{Stab} M_X \) and that these maps are closed embeddings whose images do not intersect each other. Moreover \( \sigma \in \text{Stab} D^b(X) \) is full if and only if \( d_i^* \sigma \) is full where \( i \in \{0, 1\} \). The following naive problem suggested in \([11]\) is based on an expectation of the contractibleness of the space of stability conditions:

**Problem 1.2** ([11 Problem 1.1]). Is \( \text{Stab} M_X \) homotopy equivalent to \( \text{Stab} D^b(X) \)?

One of the natural expectations is that these spaces of stability conditions are both contractible unless they are empty, in particular they are homotopy equivalent to each other. However it seems difficult to prove the homotopy equivalence in general. Moreover, to the best of our knowledge, there are no examples of triangulated categories whose space of stability conditions are non-contractible. So if the answer to Problem 1.2 is negative, we might find an interesting example which is non-contractible space.

The second aim is to give an answer to Problem 1.2 when \( X \) is an affine Noetherian scheme. More precisely we show the following:

**Theorem 1.3** (=Corollary 4.11). Let \( X \) be an affine Noetherian scheme. \( \text{Stab} M_X \) is homotopy equivalent to \( \text{Stab} D^b(X) \).

The proof of Theorem 1.3 might be interesting as follows. We first prove an analogous statement to Corollary 3.9

**Theorem 1.4** (=Corollary 4.2, Theorem 4.8). Let \( X \) be an affine Noetherian scheme. Then \( \text{Stab} M_X \) is non-empty if and only if \( \dim X = 0 \). Moreover, if \( \dim X = 0 \), then \( \text{Stab} M_X \) is isomorphic to \( \mathbb{C}^{2n} \) where \( n \) is the number of points in \( X \).

Thus, if \( \dim X > 0 \) then there is nothing to study the homotopy types of \( \text{Stab} M_X \) and of \( \text{Stab} D^b(X) \). Now suppose \( \dim X = 0 \). Roughly we show that the space of stability conditions is invariant under nilpotent thickening. To be precise, let \( X_0 \) be the set of closed points of \( X \). Then the closed embedding \( i: X_0 \rightarrow X \) induces a faithful exact functor \( i_*: M_{X_0} \rightarrow M_X \). In general an exact functor between triangulated categories does not induce a map between the spaces of stability conditions. However, using the inducing construction due to \([12]\), we show that the functor \( i_* \) implies a continuous map \( i_*^{-1}: \text{Stab} M_X \rightarrow \text{Stab} M_{X_0} \) contravariantly and that the map gives an isomorphism. Using Qiu \([13]\) and Dimitrov-Katzarkov \([10]\), we see that \( \text{Stab} M_{X_0} \) is contractible. Then, combining Theorem 1.1, Theorem 1.3 follows.

So the essential part is the isomorphism \( i_*^{-1}: \text{Stab} M_X \rightarrow \text{Stab} M_{X_0} \) induced by the faithful functor \( i_* \). Due to non-functoriality of taking the space of stability conditions, our method might be rare and interesting.
2. Preliminaries

2.1. Derived categories of coherent sheaves. Let $X$ be a Noetherian scheme and $D^b(X)$ be the bounded derived category of coherent sheaves on $X$. A global function $r \in H^0(X, \mathcal{O}_X)$ gives an endomorphism $\mu_r : E \to E$ of $E \in D^b(X)$ via multiplication by $r$. We refer to the morphism $\mu_r$ as the multiplication by $r \in H^0(X, \mathcal{O}_X)$. The multiplication $\mu_r$ is just the value of the morphism $\mu$ as algebras

\[
\mu : \Gamma(X, \mathcal{O}_X) \cong \text{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{O}_X) \to \text{Hom}_{D^b(X)}(E, E).
\]

The following condition for an object $E \in D^b(X)$ is crucial for us:

**Definition 2.1.** An object $E$ in $D^b(X)$ has the isomorphic property if $E$ satisfies the following:

(Ism) For any $r \in R$, the morphism $\mu_r : E \to E$ is an isomorphism if $\mu_r$ is non-zero.

Recall that the support $\text{Supp} \ E$ of a complex $E \in D^b(X)$ is the union of the support of the $i$-th cohomology of $E$:

\[
\text{Supp} \ E = \bigcup_{i \in \mathbb{Z}} \text{Supp} \ H^i(E).
\]

Note that $\text{Supp} \ E$ is closed since $E$ is a bounded complex.

2.2. Inducing stability conditions. Let $D$ be a triangulated category. Following the original article [7], the set of locally finite stability conditions on $D$ is denoted by $\text{Stab} \ D$. Recall that a stability condition consists of a pair $\sigma = (Z, \mathcal{P})$ where $Z$ is a group homomorphism from the Grothendieck group of $D$ to $\mathbb{C}$ and $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ is the collection of full sub-abelian categories $\mathcal{P}(\phi)$ of $D$. An object $A \in D$ is said to be $\sigma$-semistable if $A$ is in $\mathcal{P}(\phi)$ and $A$ is non-zero. Moreover the object $A$ is said to be $\sigma$-stable if $A$ is simple in $\mathcal{P}(\phi)$.

An exact functor $F : D \to D'$ between triangulated categories does not induces a map between spaces of stability conditions in general. However, a “good” functor $F : D \to D'$ induces a map $F^{-1}$ from a subset of $\text{Stab} \ D'$ to $\text{Stab} \ D$ due to Macr´ı-Mehrotra-Stellari [12]. Let us briefly recall the construction of $F^{-1}$.

Let $F : D \to D'$ be an exact functor between triangulated categories. Assume that $F$ satisfies the following additional condition

\[
(\text{Ind}) \quad \text{Hom}_D(F(a), F(b)) = 0 \text{ implies } \text{Hom}_D(a, b) = 0 \text{ for any } a, b \in D.
\]

Let $\sigma' = (Z', \mathcal{P}') \in \text{Stab} \ D'$. Define $F^{-1}\sigma'$ by the pair $(Z, \mathcal{P})$ where

\[
Z = Z' \circ F, \quad \mathcal{P}(\phi) = \{x \in D \mid F(x) \in \mathcal{P}'(\phi)\}.
\]

By the definition of $F^{-1}\sigma'$, the pair $F^{-1}\sigma'$ is a stability condition on $D$ if and only if $F^{-1}\sigma'$ has the Harder-Narasimhan property.

**Lemma 2.2 ([12] Lemma 2.9).** Notation is the same as above. The map $F^{-1} : \text{Dom}(F^{-1}) \to \text{Stab} \ D$ is continuous.

**Remark 2.3.** Recall that the universal cover $\widetilde{\text{GL}}_2^+(\mathbb{R})$ of $\text{GL}_2^+(\mathbb{R})$ has the right action to the space of stability conditions. The map $F^{-1}$ is $\widetilde{\text{GL}}_2^+(\mathbb{R})$-equivariant by the definition of $F^{-1}$.
2.3. Semiorthogonal decompositions and stability conditions. Collins–Polishchuck [9] proposed a construction of stability conditions on a triangulated category $\mathcal{D}$ from a semiorthogonal decomposition. A key ingredient of the construction is a reasonable stability condition on a triangulated category.

**Definition 2.4 ([9] pp. 568).** A stability condition $\sigma = (\mathcal{A}, Z)$ on a triangulated category $\mathcal{D}$ is reasonable if $\sigma$ satisfies

$$0 < \inf \{|Z(E)| \in \mathbb{R} \mid E \text{ is semistable in } \sigma\}.$$

**Remark 2.5.** A reasonable stability condition is locally finite by [9, Lemma 1.1]. Unfortunately we do not know whether the converse holds or not. For instance, if $\dim \mathcal{D} = 1$, then any stability condition on $\mathcal{D}$ is reasonable.

Let $\mathcal{D}$ be a triangulated category. Recall that a pair $(\mathcal{D}_1, \mathcal{D}_2)$ of full triangulated subcategories of $\mathcal{D}$ is said to be a semiorthogonal decomposition of $\mathcal{D}$ if the pair satisfies

1. $\text{Hom}_\mathcal{D}(E_2, E_1) = 0$ for any $E_i \in \mathcal{D}_i$ $(i = 1, 2)$, and
2. any object $E \in \mathcal{D}$ is decomposed into a pair of objects $E_i \in \mathcal{D}_i$ $(i = 1, 2)$ by the following distinguished triangle in $\mathcal{D}$:

$$E_2 \to E \to E_1 \to E_2[1].$$

The situation will be denoted by the symbol $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ or simply $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. In addition to the first condition (1) above, if $\text{Hom}_\mathcal{D}(E_1, E_2) = 0$ holds, the semiorthogonal decomposition is said to be orthogonal.

**Proposition 2.6 ([9]).** Let $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. The left adjoint of the inclusion $\mathcal{D}_1 \to \mathcal{D}$ is denoted by $\tau_1$ and the right adjoint of the inclusion $\mathcal{D}_2 \to \mathcal{D}$ is denoted by $\tau_2$. Let $\sigma_i = (Z_i, \mathcal{P}_i)$ be a reasonable stability condition on $\mathcal{D}_i$ for $i \in \{1, 2\}$. Suppose that $\sigma_1$ and $\sigma_2$ satisfy the following conditions

1. $\text{Hom}_\mathcal{D}^{<0}(\mathcal{P}_1(0, 1], \mathcal{P}_2(0, 1]) = 0$ and
2. There is a real number $a \in (0, 1)$ such that $\text{Hom}_\mathcal{D}^{<0}(\mathcal{P}_1(a, a + 1], \mathcal{P}_2(a, a + 1]) = 0$.

Then there exists a unique reasonable stability condition $\mathcal{g}(\sigma_1, \sigma_2)$ on $\mathcal{D}$ glued from $\sigma_1$ and $\sigma_2$ whose heart $\mathcal{A}$ of the t-structure of $\mathcal{g}(\sigma_1, \sigma_2)$ is given by

$$\mathcal{A} = \{E \in \mathcal{D} \mid \tau_i(E) \in \mathcal{P}_i((0, 1]) (i = 1, 2)\}$$

and whose central charge $Z$ is given by $Z(E) = Z_1(\tau_1(E)) + Z_2(\tau_2(E))$.

2.4. A category of morphisms. Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on a Noetherian scheme $X$. The category of morphisms in $\mathcal{D}^b(X)$, introduced by the author, is one of generalizations of the derived category of representations of the $A_2$ quiver. Let us briefly recall the construction.

Let $\mathcal{D}_{\text{coh}}^b(X)$ be the stable infinity category of quasi-coherent sheaves on a Noetherian scheme $X$ with bounded coherent cohomologies. Then the homotopy category $\text{h}(\mathcal{D}_{\text{coh}}^b(X))$ of the infinity category is equivalent to the derived category $\mathcal{D}^b(X)$.

The homotopy category $\text{h}(\mathcal{D}_{\text{coh}}^b(X)^{A_1})$ of the infinity category $\mathcal{D}_{\text{coh}}^b(X)^{A_1}$ of morphisms in the infinity category $\mathcal{D}_{\text{coh}}^b(X)$ is a reasonable candidate of a triangulated category of morphisms in $\mathcal{D}^b(X)$. Thus we refer to $\text{h}(\mathcal{D}_{\text{coh}}^b(X)^{A_1})$ as the category of morphisms in $\mathcal{D}^b(X) = \text{h}(\mathcal{D}_{\text{coh}}^b(X))$. 


**Definition 2.7.** Let $X$ be a Noetherian scheme. The category of morphisms in $X$ is denoted by $M_X$. If $X$ is the affine scheme of a Noetherian ring $R$, we simply write $M_{\text{Spec } R}$ as $M_R$.

Note that a morphism $[f : E \to F]$ in $D^b(X)$ determines an object in $M_X$. There exist pairs of adjoint functors between $D^b(X)$ and $M_X$:

$$D^b(X) \xrightarrow{d_0} M_X \xleftarrow{s} d_1,$$

where $d_0([E \to F]) = F$, $d_1([E \to F]) = E$ and $s(E) = [\text{id}_E : E \to E]$. Moreover $d_1$ has the right adjoint $j_1$ and $d_0$ has the left adjoint $j_s$,

$$j_1 : D^b(X) \to M_X \text{ and } j_s : D^b(X) \to M_X,$$

where $j_1(E) = [E \to 0]$ and $j_s(E) = [0 \to E]$.

**Lemma 2.8.** Let $X$ be a Noetherian scheme and define the subcategories of the triangulated category $M_X$ by

$$\begin{align*}
(M_X)_{/0} & := \{[E \to 0] | E \in \text{h}(D^b_{\text{coh}}(X))\}, \\
(M_X)_{0/} & := \{[0 \to E] | E \in \text{h}(D^b_{\text{coh}}(X))\}, \text{ and} \\
(M_X)_s & := \{[\text{id} : E \to E] | E \in \text{h}(D^b_{\text{coh}}(X))\}.
\end{align*}$$

The triangulated category $M_X$ has three semiorthogonal decompositions:

$$\begin{align*}
(\langle (M_X)_s, (M_X)_{/0} \rangle), \\
(\langle (M_X)_{0/}, (M_X)_s \rangle), \text{ and} \\
(\langle (M_X)_{/0}, (M_X)_{0/} \rangle).
\end{align*}$$

**Proof.** The first two decompositions follow from [11, Lemma 2.14]. Since $j_s$ is the left adjoint of $d_0$ and $j_1$ is the right adjoint of $d_1$, we have canonical morphisms $j_s \circ d_0(f) \to f$ and $f \to j_1 \circ d_1(f)$ for $f \in M_X$. Then the sequence

$$j_s \circ d_0(f) \longrightarrow f \longrightarrow j_1 \circ d_1(f) \longrightarrow j_s \circ d_0(f)[1]$$

gives a distinguished triangle in $M_X$ since the triangulated structure on $M_X$ is defined object-wise.

Note that $(M_X)_{/0}$ (resp. $(M_X)_{0/}$) is the essential image of $j_1$ (resp. $j_s$). The adjunction $d_1 \dashv j_1$ implies

$$\text{Hom}_{M_X}(j_sE, j_lF) \cong \text{Hom}_{D^b(X)}(d_1 \circ j_s(E), F) = \text{Hom}_{D^b(X)}(0, F) = 0.$$

This gives the proof of (2.5). \qed

**Remark 2.9.** Let $M_X = \langle M_1, M_2 \rangle$ be one of semiorthogonal decompositions in Lemma 2.8. Then both components $M_1$ and $M_2$ are equivalent to $D^b(X)$ in any cases and equivalences are respectively given by

$$\begin{align*}
s : D^b(X) \to (M_X)_s \\
j_1 : D^b(X) \to (M_X)_{/0} \\
j_s : D^b(X) \to (M_X)_{0/}.
\end{align*}$$

Throughout this note, we always identify $D^b(X)$ with components of semiorthogonal decompositions of $M_X$.  

Let \((\text{coh } X)^{\Delta^1}\) be the category of morphisms in the abelian category \(\text{coh } X\) of coherent sheaves on \(X\). The category \((\text{coh } X)^{\Delta^1}\) is also abelian, and we obtain the bounded derived category \(D^b((\text{coh } X)^{\Delta^1})\) by the localization of quasi-isomorphisms.

**Proposition 2.10** ([11, Corollary 6.2]). Let \(X\) be a Noetherian scheme. The triangulated category \(D^b((\text{coh } X)^{\Delta^1})\) is equivalent to \(\mathcal{M}_X\). In particular the category \(\mathcal{M}_X\) has a natural bounded \(t\)-structure \((\mathcal{M}_X^{\leq 0}, \mathcal{M}_X^{\geq 1})\) whose heart is equivalent to the abelian category \((\text{coh } X)^{\Delta^1}\).

**Remark 2.11.**

1. We refer to the \(t\)-structure \((\mathcal{M}_X^{\leq 0}, \mathcal{M}_X^{\geq 1})\) as the canonical \(t\)-structure on \(\mathcal{M}_X\).

2. If \(X\) is \(\text{Spec } k\) of a field \(k\) then \((\text{coh } X)^{\Delta^1}\) is nothing but the abelian category of finite dimensional representations of the \(A_2\) quiver.

### 3. Stability conditions on affine schemes

We study the space of stability condition on the derived category of an affine Noetherian scheme.

**Lemma 3.1.** Let \(R\) be a Noetherian domain with \(\dim R > 0\). Suppose that an \(R\)-module \(M\) satisfies the following condition

- The morphism \(\mu_r: M \to M\) is an isomorphism for any \(r \in R \setminus \{0\}\).

Then \(M\) is zero.

**Proof.** Suppose to the contrary that \(M \neq 0\). The assumption implies \(\text{ann}(M) = (0)\). Thus the support of \(M\) is \(\text{Spec } R = X\).

One can choose \(r \in R = H^0(X, \mathcal{O}_X)\) such that \(r\) is not unit in \(R\) since \(\dim X > 0\). Since the morphism \(\mu_r: M \to M\) is an isomorphism, we have \(M \otimes R/(r) = 0\). Thus \(\text{Supp } M\) is a proper closed subset of \(X\) and this gives a contradiction. Hence \(M\) is zero. \(\Box\)

**Lemma 3.2.** Let \(R\) be a Noetherian ring. Suppose that an \(R\)-module \(M\) satisfies the condition \((\text{Ism})\). Then the following holds.

1. \(\text{ann}(M)\) is a prime ideal.
2. If \(M\) is non-zero, then \(\text{ann}(M) = \text{ann}(m)\) for any \(m \in M \setminus \{0\}\). In particular \(\text{ann}(M)\) is the unique associated prime of \(M\).
3. If \(M\) is non-zero, then \(\dim \text{Supp}(M) = 0\).

**Proof.** Suppose that \(ab \in \text{ann}(M)\) and \(a \notin \text{ann}(M)\). Then \(\mu_{ab} = \mu_a \mu_b\) is the zero morphism. Since \(\mu_a\) is an isomorphism by the condition \((\text{Ism})\), \(\mu_b\) has to be zero and \(b\) is in \(\text{ann}(M)\)

Clearly we have \(\text{ann}(M) \subset \text{ann}(m)\) for any \(m \in M \setminus \{0\}\). Let \(r\) be in \(\text{ann}(m)\). Then \(\mu_r\) is not isomorphism. The condition \((\text{Ism})\) implies that \(\mu_r\) is zero. Thus we see \(\text{ann}(M) = \text{ann}(m)\). The last part of the second assertion is obvious.

To complete the proof, we show the assertion \((3)\). Let \(p\) be the prime ideal \(\text{ann}(M)\). If \(p\) is not maximal, \(M\) satisfies the condition \((\text{Ism})\) as \(R/p\)-modules and we have \(\dim R/p > 0\). Then Lemma 3.1 implies \(M = 0\). Hence \(p\) has to be maximal and we see \(\dim \text{Supp}(M) = \dim \text{Supp}(R/p) = 0\). \(\Box\)

**Lemma 3.3.** Let \(R\) be a Noetherian ring with \(\dim R > 0\). Suppose that an object \(E \in D^b(\text{Spec } R)\) satisfies the condition \((\text{Ism})\). If \(E\) is non-zero then \(\dim \text{Supp}(E) = 0\).
Proof. Let $\mu_i^*: H^i(E) \to H^i(E)$ be the $i$-th cohomology of the morphism $\mu_r: E \to E$. Note that $\mu_i^*$ is also the multiplication by $r$ on $H^i(E)$. Since $E$ satisfies the condition (Ism), so does $H^i(E)$ if $H^i(E) \neq 0$. Thus Lemma 3.2 implies $\dim \text{Supp} H^i(E) = 0$ and we have the desired assertion. \hfill \Box

**Theorem 3.4.** Let $X$ be an affine Noetherian scheme with $\dim X > 0$. Then the set $\text{Stab} \mathcal{D}^b(X)$ is empty.

**Proof.** We denote by $R$ the coordinated ring $H^0(X, \mathcal{O}_X)$ of the affine scheme $X$. Suppose to the contrary that there exists a locally finite stability condition $\sigma \in \text{Stab} \mathcal{D}^b(X)$. Since $\sigma$ is locally finite, there exists a $\sigma$-stable object $A \in \mathcal{D}^b(X)$.

Note that any nonzero endomorphism $\varphi: A \to A$ is an isomorphism in $\mathcal{D}^b(X)$. Thus $A$ satisfies the condition (Ism) via the morphism \( \mathcal{H}^{\leq 0} \). Hence we see $\dim \text{Supp}(A) = 0$ for any $\sigma$-stable object by Lemma 3.3.

Taking the Harder-Narasimhan filtration and a Jordan-Hölder filtration, the structure sheaf $\mathcal{O}_X$ is given by a successive extension of finite $\sigma$-stable objects \( \{A_i\}_{i=1}^n \). Thus we have $X = \bigcup_{i=1}^n \text{Supp} A_i$ and this gives a contradiction since $\dim X \neq 0$. \hfill \Box

Thus if $\dim X > 0$ then there is nothing to study $\text{Stab} \mathcal{D}^b(X)$. Next goal is to describe $\text{Stab} \mathcal{D}^b(X)$ for the case $\dim X = 0$.

**Lemma 3.5.** Let $\mathcal{D}$ be a triangulated category, and $A$ the heart of a bounded $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$. The truncation functors with respect to the $t$-structure are respectively denoted by $\tau_{\leq 0}: \mathcal{D} \to \mathcal{D}^{\leq 0}$ and $\tau_{\geq 1}: \mathcal{D} \to \mathcal{D}^{\geq 1}$. The cohomology of $E \in \mathcal{D}$ with respect to the $t$-structure is denoted by $H^i(E)$.

Suppose an object $E \in \mathcal{D}$ satisfies

$$\tau_{\geq m_1+1}E = 0 \text{ and } \tau_{\leq m_2-1}E = 0.$$

Then $\text{Hom}_{\mathcal{D}}(E, E[m_2 - m_1]) \cong \text{Hom}_{\mathcal{D}}(H^{m_1}(E), H^{m_2}(E))$.

**Proof.** For the simplicity, we may assume $m_2 = 0$ by shifts, if necessary.

The truncation functors give a distinguished triangle

$$\begin{array}{c}
(\tau_{\geq 1}E)[-1] \\
\tau_{\leq 0}E \\
E \\
\tau_{\geq 1}E
\end{array}$$

Since $E[m_1]$ belongs to $\mathcal{D}^{\leq 0}$, we have $\text{Hom}_{\mathcal{D}}(E[m_1], \tau_{\geq 1}E) = \text{Hom}_{\mathcal{D}}(E[m_1], (\tau_{\geq 1}E)[-1]) = 0$. Thus we see

$$\text{Hom}_{\mathcal{D}}(E[m_1], E) \cong \text{Hom}_{\mathcal{D}}(E[m_1], \tau_{\leq 0}E).$$

There is a distinguished triangle

$$\begin{array}{c}
\tau_{\leq -1}(E[m_1]) \\
E[m_1] \\
\tau_{\geq 0}(E[m_1]) \\
\tau_{\leq -1}(E[m_1])[1]
\end{array}$$

The assumption implies $\tau_{\leq 0}E \in \mathcal{D}^{\geq 0}$. Then the vanishings

$$\text{Hom}_{\mathcal{D}}(\tau_{\leq -1}(E[m_1]), \tau_{\leq 0}E) = \text{Hom}_{\mathcal{D}}(\tau_{\leq -1}(E[m_1])[1], \tau_{\leq 0}E) = 0$$

imply the isomorphism:

$$\text{Hom}_{\mathcal{D}}(E[m_1], \tau_{\leq 0}E) \cong \text{Hom}_{\mathcal{D}}(\tau_{\geq 0}(E[m_1]), \tau_{\leq 0}E).$$

Since $\tau_{\leq 0}E$ (resp. $\tau_{\geq 0}(E[m_1])$) is nothing but $H^0(E)$ (resp. $H^{m_1}(E)$), we obtain the desired assertion. \hfill \Box

**Lemma 3.6.** Let $R$ be a zero-dimensional Noetherian local ring. Then $\text{Stab} \mathcal{D}^b(\text{Spec } R)$ is non-empty.
Proof. Put $X = \text{Spec } R$. Recall that any object in $\text{coh}(X)$ is given by a successive extension of the residue field $R/\mathfrak{m}$. Hence one can define a group homomorphism $Z : K_0(\mathbf{D}^b(X)) \to \mathbb{C}$ by $Z(R/\mathfrak{m}) = -1$. Then the pair $\sigma = (\text{coh}(X), Z)$ has the Harder-Narasimhan property in the sense of [7, Definition 2.3] since any object in $\text{coh}(X)$ is $\sigma$-semistable. Thus $\sigma$ is a stability condition on $\mathbf{D}^b(X)$ by [7, Proposition 5.3]. The locally finiteness is obvious since the abelian category $\text{coh}(X)$ is Artinian and Noetherian. Thus $\text{Stab}^b(\text{Spec } R)$ is not empty.

**Proposition 3.7.** Let $R$ be a zero-dimensional Noetherian local ring. Then $\text{Stab}^b(\text{Spec } R)$ is isomorphic to $\mathbb{C}$.

**Proof.** Let us denote by $\mathfrak{m}$ the maximal ideal of $R$. If once we show that $R/\mathfrak{m}$ is stable for all stability conditions on $\mathbf{D}^b(\text{Spec } R)$, the same argument in [11, Proposition 3.7] implies the desired assertion.

We claim that any stable object $A$ for a stability condition is a sheaf up to shifts. To show the claim set $m_1$ and $m_2$ by $m_1 = \max\{i \in \mathbb{Z} \mid \mathcal{H}^i(A) \neq 0\}$ and $m_2 = \max\{i \in \mathbb{Z} \mid \mathcal{H}^i(A) \neq 0\}$. It is enough to show that $m_1 - m_2 = 0$.

Next we have to show that $A$ is $R/\mathfrak{m}$ up to shifts. Lemma 3.3 implies that the support of the stable object $A \in \mathbf{D}^b(\text{Spec } R)$ is annihilated by the maximal ideal $\mathfrak{m}$. Hence each cohomology of $A$ with respect to the standard $t$-structure is an $R/\mathfrak{m}$-module. Since $A$ is stable we have

$$\text{Hom}(A, A[-m]) = 0,$$

for any positive integer $m$. Since the cohomologies of $A$ are vector spaces over the field $R/\mathfrak{m}$, they are isomorphic to the direct sums of $R/\mathfrak{m}$. Hence $m_2 - m_1$ is zero by Lemma 3.3. Thus $A$ is isomorphic to the direct sum $(R/\mathfrak{m})^{\oplus r}$ up to shifts. Since any non-zero endomorphism is invertible, $A$ is isomorphic to $R/\mathfrak{m}$ up to shifts. □

**Theorem 3.8.** Let $X$ be an affine Noetherian scheme with $\dim X = 0$. Then $\text{Stab}^b(\text{Spec } X)$ is isomorphic to $\mathbb{C}^n$ where $n$ is the number of points in $X$.

**Proof.** Let $R$ be the coordinate ring $\mathcal{H}^0(X, \mathcal{O}_X)$. Recall that $R$ is the finite product of Noetherian local rings $\{R_i\}_{i=1}^n$ with $\dim R_i = 0$. Then the derived category $\mathbf{D}^b(X)$ has the orthogonal decomposition

$$\mathbf{D}^b(X) = \bigoplus_{i=1}^n \mathbf{D}^b(\text{Spec } R_i),$$

and the space $\text{Stab}^b(\text{Spec } X)$ is the finite product of $\{\text{Stab}^b(\text{Spec } R_i)\}_{i=1}^n$ by [10, Proposition 5.2]. Hence $\text{Stab}^b(\text{Spec } R)$ is isomorphic to $\mathbb{C}^n$ by Proposition 3.7. □

**Corollary 3.9.** Let $X$ be an affine Noetherian scheme. The space $\text{Stab}^b(\text{Spec } X)$ is not empty if and only if $\dim X = 0$.

**Proof.** The proof is clear from Theorems 3.4 and 3.8. □

4. Stability conditions on morphisms

Basically we are interested in a relation between $\text{Stab}^b(X)$ and $\text{Stab}^b_{M_X}$. One of motivated problems is Problem 1.2. We first study the non-emptiness of $\text{Stab}^b_{M_X}$ for an affine Noetherian scheme $X$.

**Proposition 4.1.** Let $X$ be an affine Noetherian scheme. Then $\text{Stab}^b_{M_X}$ is not empty if and only if $\dim X = 0$. 

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Proof. Assume dim $X = 0$. Then $\text{Stab} \mathcal{D}^b(X)$ is not empty by Proposition 3.8. So Theorem 1.2] implies that $\text{Stab} \mathcal{M}_X$ is not empty.

Assume that dim $X$ is positive. Take $f \in \mathcal{M}_X$ which is stable with respect to a stability condition on $\mathcal{M}_X$. Note that there is a morphism of algebras via term-wise multiplication:

$$\mu: \text{Hom}_{\mathcal{D}^b(X)}(\mathcal{O}_X, \mathcal{O}_X) \to \text{Hom}_{\mathcal{M}_X}(f, f).$$

In particular $d_i \mu: d_i f \to d_i f$ is also the multiplication. Since $f$ satisfies the condition (Ism), so does $d_i f \in \mathcal{D}^b(X)$ ($i \in \{0, 1\}$). Lemma 3.3 implies that the objects $d_i f$ supported in closed points of $X$. This gives a contradiction by the same reason in Theorem 3.4. \[\square\]

Corollary 4.2. Let $X$ be an affine Noetherian scheme. The following are equivalent.

1. The dimension of $X$ is zero,
2. $\text{Stab} \mathcal{D}^b(X)$ is non-empty, and
3. $\text{Stab} \mathcal{M}_X$ is non-empty.

Proof. The proof is clear from Corollary 3.9 and Proposition 4.1. \[\square\]

Remark 4.3. If dim $X$ is positive, then both $\text{Stab} \mathcal{D}^b(X)$ and $\text{Stab} \mathcal{M}_X$ are empty. In particular they are homotopy equivalent to each other.

Now we further study $\text{Stab} \mathcal{M}_X$ when $X$ is the affine scheme of a zero-dimensional Netherian local ring $R$. To simplify notation, we introduce the following:

Definition 4.4. Let $R$ be a zero-dimensional Noetherian local ring with maximal ideal $m$ and let $k$ be the residue field $R/m$. The categories of morphisms in $\mathcal{D}^b(\text{Spec } k)$ is denoted by $\mathcal{M}_0$. We denote by $\mathcal{A}_R$ (resp. $\mathcal{A}_0$) the heart of the standard $t$-structure $(\mathcal{M}_{\leq 0}^R, \mathcal{M}_{\geq 1}^R)$ on $\mathcal{M}_R$ (resp. $(\mathcal{M}_{\leq 0}^0, \mathcal{M}_{\geq 1}^0)$ on $\mathcal{M}_0$).

The main theorem of this section is Theorem 4.8 below. Let $i: \text{Spec } k \to \text{Spec } R$ be the closed embedding. The exact functor $i_*: \text{coh}(\text{Spec } k) \to \text{coh}(\text{Spec } R)$ induces a functor

$$\mathcal{A}_0 \to \mathcal{A}_R$$

which is also exact. Thus we obtain the functor

$$\mathcal{M}_0 \to \mathcal{M}_R.$$ 

By abusing notation, we denote by $i_*$ these induced functors.

Lemma 4.5. The functor $i_*: \mathcal{M}_0 \to \mathcal{M}_R$ is faithful.

Proof. It is enough to show that the natural morphism

$$i_*^p: \text{Hom}_{\mathcal{M}_0}(f, g[p]) \to \text{Hom}_{\mathcal{M}_R}(i_* f, i_* g[p])$$

is injective for any $f$ and $g \in \mathcal{M}_0$, and for any integer $p$. Recall that any object in $\mathcal{M}_0$ is the finite direct sum of shifts of objects in $\mathcal{A}_0$. Hence we can assume both $f$ and $g$ are in $\mathcal{A}_0$ without loss of generality.

Since the global dimension of $\mathcal{A}_0$ is 1, the left hand side in (4.1) vanishes for $p \notin \{0, 1\}$. Now the claim for $p = 0$ is obvious since $i_*^0$ is an isomorphism. In addition $i_*^1$ is injective since $i_*: \mathcal{A}_0 \to \mathcal{A}_R$ is exact and commutes with direct sums. \[\square\]

Lemma 4.6. Let $f$ and $g$ be in $\mathcal{A}_0$. If $\text{Hom}_{\mathcal{M}_R}(i_* f, i_* g[p])$ is zero for $p \in \{0, 1\}$, then $\text{Hom}_{\mathcal{M}_R}(i_* f, i_* g[p]) = 0$ holds for any $p \in \mathbb{Z}$. 

Proof. Recall that any object in $\mathcal{A}_0$ is the direct sum of indecomposable objects in $\mathcal{A}_0$ and any indecomposable object in $\mathcal{A}_0$ is one of the following:

$$s(k) = [\text{id} : k \to k], j_1(k) = [k \to 0], \text{ or } j_\ast(k) = [0 \to k].$$

It is enough to prove the claim when $f$ and $g$ are indecomposable.

Lemma 4.5 implies $\text{Hom}_{M_0}(f, g[p]) = 0$ for $p \in \{0, 1\}$. Then the pair $(i_\ast(f), i_\ast(g))$ has to be one of the following:

$$(i_\ast(f), i_\ast(g)) = (j_\ast(k), j_1(k)), (s(k), j_\ast(k)), \text{ or } (j_1(k), s(k)).$$

Suppose $(i_\ast(f), i_\ast(g)) = (j_\ast(k), j_1(k))$. Recall that $j_\ast$ is the left adjoint of $d_0$. Hence we see

$$\text{Hom}_{M_R}(j_\ast(k), j_1(k)[p]) \cong \text{Hom}_{D^b(\text{Spec } R)}(k, d_0 \circ j_1(k)[p]) = \text{Hom}_{D^b(\text{Spec } R)}(k, 0) = 0.$$ 

Similarly one can prove the claim for $(i_\ast(f), i_\ast(g)) = (s(k), j_\ast(k))$ by using the adjunction $s \dashv d_1$.

Finally suppose that $(i_\ast(f), i_\ast(g)) = (j_1(k), s(k))$. The adjunction $d_0 \dashv s$ implies

$$\text{Hom}_{M_R}(j_1(k), s(k)[p]) \cong \text{Hom}_{D^b(\text{Spec } R)}(d_0 \circ j_1(k), k[p]) = \text{Hom}_{D^b(\text{Spec } R)}(0, k[p]) = 0.$$ 

We have finished the proof. \hfill \Box

**Proposition 4.7.** Let $\sigma$ be a locally finite stability condition on $M_R$. If $f \in M_R$ is $\sigma$-stable, then $f$ is, up to shifts, one of the following:

$$(4.2) \quad s(k) = [\text{id} : k \to k], \quad j_1(k) = [k \to 0], \quad \text{and } j_\ast(k) = [0 \to k].$$

**Proof.** By the argument in Proposition 4.1 each $i$-th cohomology $H^i(f)$ of $f$ with respect to the $t$-structure $(M_0^R, M_1^R) = \mathcal{A}_0$. It is enough to show that $f$ is in $\mathcal{A}_0$ up to shifts. In fact, if $f$ is in $\mathcal{A}_0$ and stable then $f$ is indecomposable by [11, Lemma 3.3]. Since there are only 3 indecomposable objects in $\mathcal{A}_0$ listed in (4.2), we have the desired assertion.

Without loss of generality, assume that $f$ satisfies

$$0 = \min\{j \in \mathbb{Z} \mid H^j(f) \neq 0\}.$$ 

Put $\ell = \max\{j \in \mathbb{Z} \mid H^j(f) \neq 0\}$ and it is enough to show that $\ell = 0$.

Assume $\ell = 1$. Now we claim $\text{Hom}_{M_R}(H^1(f), H^0(f)[1]) = 0$. Recall that there is a distinguished triangle

$$H^0(f) \xrightarrow{\tau_0} f \xrightarrow{\tau_1} H^1(f)[-1] \xrightarrow{} H^0(f)[1]$$

by the truncation for the canonical $t$-structure. If $\text{Hom}_{M_R}(H^1(f), H^0(f)[1]) \neq 0$, then there exists a non-zero morphism $\varphi : H^1(f)[-1] \to H^0(f)$. Then the composite $\tilde{\varphi} = \tau_0 \circ \varphi \circ \tau_1$ is a non-zero endomorphism of $f$ since $f$ is in $M_1^R \cap M_0^R$. Then the $\sigma$-stability of $f$ implies that $\tilde{\varphi}$ is an isomorphism. On the other hand $\varphi \circ \tilde{\varphi}$ is zero by $\text{Hom}_{M_R}(H^0(f), H^1(f)[-1]) = 0$. Thus $\text{Hom}_{M_R}(H^1(f), H^0(f)[1])$ has to be zero if $\ell = 1$.

Recall the spectral sequence given by

$$(4.3) \quad E_2^{pq} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{M_R}(H^i(f), H^{i+q}(f)[p]) \Rightarrow \text{Hom}_{M_R}(f, f[p + q]) = E^{p+q}.$$ 

Since we are assuming $\ell = 1$, we see $E_2^{0, -1} \cong E_\infty^{0, -1}$. By the $\sigma$-stability of $f$, the vanishing $\text{Hom}_{M_R}(f, f[-1]) = 0$ implies that $\text{Hom}_{M_R}(H^1(f), H^0(f))$ is zero. Then Lemma 4.6 implies the vanishings

$$\text{Hom}_{M_R}(H^1(f), H^0(f)[p]) = 0 \quad (\forall p \in \mathbb{Z}).$$

Thus $f$ has to be split. Since $f$ is indecomposable by [11, Lemma 3.3], this gives a contradiction. Hence we see $\ell \neq 1$. \hfill \Box
Next assume \( \ell \geq 2 \). By the spectral sequence (4.3), we see \( E^{0, -\ell}_2 \cong E^{0, -\ell}_\infty \) and \( E^{1, -\ell}_2 \cong E^{1, -\ell}_\infty \). Then the vanishing \( \operatorname{Hom}_{M^R}(f, [n]) = 0 \) for any \( n \in \mathbb{N} \) imply \( E^{0, -\ell}_2 = E^{1, -\ell}_2 = 0 \). By Lemma 4.6, we have \( \operatorname{Hom}_{M^R}(H^i(f), H^0(f)[p]) = 0 \) for any \( p \in \mathbb{Z} \). Thus \( E^{p, -\ell}_2 = 0 \) holds for any integer \( p \). By induction on \( q \), one can see the following

\[
E^{0, q}_2 = 0 \quad \text{for} \quad q < 0 \quad \text{and} \quad E^{1, q}_2 = 0 \quad \text{for} \quad q < -1.
\]

Then the following hold by (4.3):

\[
\begin{aligned}
\operatorname{Hom}_{M^R}(H^\ell(f), H^0(f)[q]) &= 0 \quad \forall q \in \mathbb{Z} \\
\operatorname{Hom}_{M^R}(H^\ell(f), H^j(f)) &= 0 \quad 0 < j < \ell \\
\operatorname{Hom}_{M^R}(H^j(f), H^0(f)) &= 0 \quad 0 < j < \ell.
\end{aligned}
\] (4.5)

Since each cohomology is in the heart \( \mathcal{A}_0 \) and both \( H^0(f) \) and \( H^\ell(f) \) are non-zero, we see \( H^j(f) = 0 \) for \( 0 < j < \ell \). Thus we obtain the distinguished triangle

\[
H^0(f) \longrightarrow f \longrightarrow H^\ell(f)[-\ell] \longrightarrow H^0(f)[1].
\]

The first vanishing in (4.5) implies that \( f \) is the direct sum \( H^0(f) \oplus H^\ell(f)[-\ell] \) and this gives a contradiction. Hence \( \ell \) has to be zero.

\[\square\]

**Theorem 4.8.** Let \( i : \Spec k \to \Spec R \) be the closed embedding. The following hold.

1. \( \operatorname{Dom}(i_*^{-1}) = \operatorname{Stab} M^R \)
2. \( i_*^{-1} : \operatorname{Stab} M^R \to \operatorname{Stab} M_0 \) is an isomorphism as complex manifolds.
3. \( \operatorname{Stab} M^R \) is isomorphic to \( \mathbb{C}^2 \).

**Proof.** We first show \( \operatorname{Dom}(i_*^{-1}) \supset \operatorname{Stab} M^R \). Take \( \sigma \in \operatorname{Stab} M^R \) arbitrary. It is enough to show that \( i_*^{-1}\sigma \) defined by (2.2) has the Harder-Narasimhan property for any \( f \in M_0 \).

Note that an indecomposable object in \( M_0 \) is also, up to shifts, one of the objects in (1.2) since the global dimension of \( \mathcal{A}_0 \) is 1. If \( f \) and \( g \) in \( M_0 \) has the Harder-Narasimhan filtration with respect to \( i_*^{-1}\sigma \), one can construct the Harder-Narasimhan filtration of the direct sum \( f \oplus g \). So it is necessary to show that any indecomposable object \( f \in M_0 \) has the Harder-Narasimhan filtration.

Since \( \sigma \) is locally finite, any object in \( M^R \) is given by a successive extension of finite \( \sigma \)-stable objects. Thus the classes of \( \sigma \)-stable objects generate \( K_0(M^R) \cong K_0(D^b(Spec R))^{\oplus 2} \). Since \( \operatorname{rank} K_0(M^R) = 2 \), two of the objects \( f_1 \) and \( f_2 \) in (1.2) should be \( \sigma \)-stable. If the other object \( g \) in (1.2) is semistable then any indecomposable object in \( M_0 \) has the trivial Harder-Narasimhan filtration for \( i_*^{-1}\sigma \).

We have to discuss three cases of \( g \). Note that there is the following distinguished triangle of the objects in (1.2):

\[
\begin{aligned}
j_* (k) &\longrightarrow s(k) &\longrightarrow j^!(k) &\longrightarrow j_* (k)[1]
\end{aligned}
\] (4.6)

There is no loss of generality in assuming that

\[
(f_1, f_2) = \begin{cases}
(s(k), j^!(k)[1]) & \text{if} \; g = j_* (k) \\
(j_* (k)[1], s(k)) & \text{if} \; g = j^!(k) \\
(j^!(k), j_* (k)) & \text{if} \; g = s(k).
\end{cases}
\] (4.7)

Then we have the distinguished triangle from the triangle (4.6) in each cases:

\[
f_2 \quad \longrightarrow \quad g \quad \longrightarrow \quad f_1 \quad \longrightarrow \quad f_2 [1].
\] (4.8)

Moreover the triplet \((f_1, f_2, g)\) corresponds to three semiorthogonal decomposition of \( M^R \) in Lemma 2.8.
Let $\phi_i$ be the phase of the stable object $f_i$. Since $\text{Hom}_{\text{M}_R}(f_1, f_2[1])$ is non-zero, the stability of $f_1$ and $f_2$ implies $\phi_1 < \phi_2 + 1$. If $\phi_2 > \phi_1$, then the filtration (1.8) gives the Harder-Narasimhan filtration of $g$. If $\phi_2 = \phi_1$, then $g = s(k)$ is semistable by (1.8). If $\phi_2 < \phi_1$, then the inequality $\phi_2 < \phi_1 < \phi_2 + 1$ holds. Without loss of generality we can assume that the heart $\mathcal{P}(0, 1]$ of $\sigma$ is the extension closure generated by $f_1$ and $f_2$ by $\widetilde{\text{GL}}_2^+(\mathbb{R})$-action. Then the non-trivial subobject of $g$ is only $f_2$. Hence $g$ is stable by $\phi_2 < \phi_1$. Thus, if $g = s(k)$, then $\sigma$ is in $\text{Dom}(i_{\ast}^{-1})$ and we have $\text{Dom}(i_{\ast}^{-1}) = \text{Stab}_{\text{M}_R}$.

Since $\text{Dom}(i_{\ast}^{-1}) = \text{Stab}_{\text{M}_R}$, the map $i_{\ast}^{-1}: \text{Stab}_{\text{M}_R} \to \text{Stab}_{\text{M}_0}$ is not only continuous but also holomorphic. Thus it is enough to show that $i_{\ast}^{-1}$ is bijective since the spaces are complex manifolds.

For the subjectivity, let $\sigma_0 = (Z_0, \mathcal{P}_0)$ be in $\text{Stab}_{\text{M}_0}$. Then two of the objects listed in (4.6) has to be $\sigma_0$-stable. Hence there are three possibilities of two objects $(f_1, f_2)$. Without loss of generality we can assume that the pair $(f_1, f_2)$ is (4.7). Note that these pairs generate semiorthogonal decompositions of not only of $\text{M}_0$ but also of $\text{M}_R$ listed in (2.3), (2.4) and (2.5) respectively.

Then, in any cases, the pair satisfies
\[
\begin{align*}
\text{Hom}_{\text{M}_R}(f_2, f_1[p]) &= 0 \quad (\forall p) \\
\text{Hom}_{\text{M}_R}(f_1, f_2[p]) &= 0 \quad (p \leq 0) \\
\text{Hom}_{\text{M}_R}(f_1, f_2[1]) &\neq 0.
\end{align*}
\]

Let $n$ be the minimal integer which is greater than or equal to $\phi_2 - \phi_1$. Using the identification (2.4), define stability conditions $\sigma_i = (Z_i, \mathcal{P}_i)$ on $D^b(\text{Spec } R)$ by
\[
\begin{align*}
\mathcal{P}_1(0, 1) &= \text{coh}(\text{Spec } R), Z_1(k) := Z_0(f_1), \text{ and} \\
\mathcal{P}_2(0, 1) &= \text{coh}(\text{Spec } R)[-n], Z_2(k) := Z_0(f_2).
\end{align*}
\]

Note that both $\sigma_1$ and $\sigma_2$ are reasonable by Remark 2.5. Since the set of phases of semistable objects for $\sigma_i$ is discrete, the second condition in Lemma 2.6 is automatic. Then the gluing stability condition $\sigma := \text{gl}(\sigma_1, \sigma_2)$ with respect to the corresponding semiorthogonal decomposition on $\text{M}_R$ satisfies $i_{\ast}^{-1}\sigma = \sigma_0$. Hence $i_{\ast}^{-1}$ is surjective.

For the injectivity, let $\tau_1$ and $\tau_2$ be stability conditions on $\text{M}_R$. If $i_{\ast}^{-1}(\tau_1) = i_{\ast}^{-1}(\tau_2) =: \tau_0$, then there exist two indecomposable objects $f_1$ and $f_2$ in $\text{M}_0$ which are stable in $\tau_0$ and whose phases are in the interval $(0, 1]$. Since the heart of $\tau_i$ is the extension closure of $f_1$ and $f_2$, we see $\tau_1 = \tau_2$.

Finally the third assertion follows from 13. In fact, $\text{Stab}_{\text{M}_0}$ is isomorphic to $\mathbb{C}^2$ by 13. Thus the isomorphism $i_{\ast}^{-1}: \text{Stab}_{\text{M}_R} \cong \text{Stab}_{\text{M}_0}$ implies the assertion. □

**Remark 4.9.** The argument above gives an alternative proof of Proposition 3.7 as follows.

Since the global dimension of $k$ is zero, the functor $i_{\ast}: D^b(\text{Spec } k) \to D^b(\text{Spec } R)$ is faithful. By the same argument in the proof of Theorem 4.8 we see $\text{Stab}_{\text{D}^b(\text{Spec } R)} = \text{Dom}(i_{\ast}^{-1})$. Thus we obtain a holomorphic map
\[
i_{\ast}^{-1}: \text{Stab}_{\text{D}^b(\text{Spec } R)} \to \text{Stab}_{\text{D}^b(\text{Spec } k)}.
\]

Then one can easily see that $i_{\ast}^{-1}$ is surjective and injective. Thus $i_{\ast}^{-1}$ gives an isomorphism between $\text{Stab}_{\text{D}^b(\text{Spec } R)}$ and $\text{Stab}_{\text{D}^b(\text{Spec } k)}$.

**Corollary 4.10.** Let $R$ be a zero-dimensional Noetherian ring. The spaces $\text{Stab}_{\text{M}_R}$ is isomorphic to $\mathbb{C}^{2n}$ where $n$ is the number of points of Spec $R$.

**Proof.** By the assumption, $R$ is the direct product $\prod_{i=1}^{n} R_i$ of zero-dimensional Noetherian local ring $\{R_i\}_{i=1}^{n}$. Then $M_R$ has the orthogonal decomposition $M_R = \bigoplus M_{R_i}$ and
\( \text{Stab} \, M_R \) is isomorphic to the product \( \prod_{i=1}^{n} \text{Stab} \, M_{R_i} \) by [10, Proposition 5.2]. Since each \( \text{Stab} \, M_{R_i} \) is isomorphic to \( \mathbb{C}^2 \), we have the desired assertion. \qedhere

**Corollary 4.11.** Let \( R \) be an Noetherian ring. The spaces \( \text{Stab} \, M_R \) and \( \text{Stab} \, D^b(\text{Spec} \, R) \) are homotopy equivalent.

**Proof.** Suppose \( \dim R > 0 \). By Corollary 4.2, both are empty sets.

Suppose \( \dim R = 0 \). Let \( n \) be the number of points in \( \text{Spec} \, R \). Then \( \text{Stab} \, D^b(\text{Spec} \, R) \) is isomorphic to \( \mathbb{C}^n \) by Theorem 3.8. In particular \( \text{Stab} \, D^b(\text{Spec} \, R) \) is contractible. By Corollary 4.10, \( \text{Stab} \, M_R \) is isomorphic to \( \mathbb{C}^{2n} \). Hence \( \text{Stab} \, M_R \) is also contractible and we have finished the proof. \qedhere

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