ON YOUNG’S CONVOLUTION INEQUALITY FOR HEISENBERG GROUPS

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Abstract. Young’s convolution inequality provides an upper bound for the convolution of functions in terms of $L^p$ norms. It is known that for certain groups, including Heisenberg groups, the optimal constant in this inequality is equal to that for Euclidean space of the same topological dimension, yet no extremizing functions exist. For Heisenberg groups we characterize ordered triples of functions that nearly extremize the inequality.

The analysis relies on a characterization of approximate solutions of a certain class of functional equations. A result of this type is developed for a class of such equations.

1. Introduction

This paper characterizes ordered triples of functions that nearly extremize Young’s convolution inequality for Heisenberg groups. We first review Young’s inequality with sharp constant for Euclidean spaces, then review the corresponding inequality for Heisenberg groups, recalling observations of Klein and Russo [13] and of Beckner [2] concerning the distinction between the Euclidean and Heisenberg settings. For Heisenberg groups we introduce a group of symmetries of the inequality, along with a special class of ordered triples of Gaussian functions. Our main theorem states that an ordered triple of functions nearly extremizes the inequality if and only if it differs by a small amount, in the relevant norm, from the image of one of these special ordered triples of Gaussians under some element of the symmetry group. Our conclusion is of “$o(1)$” type; we do not obtain an explicit upper bound on the difference of norms as a function of the discrepancy from exact extremization.

The proof combines a preexisting characterization of near extremizers of Young’s inequality for Euclidean groups with the structure of Heisenberg groups and with a characterization of approximate solutions of certain functional equations.

1.1. Young’s inequality for Euclidean groups. In its classical form, Young’s convolution inequality for the Euclidean group $\mathbb{R}^m$ states that the convolution $f * g$ of functions $f, g$ satisfies the upper bound

\[ \|f * g\|_{L^r(\mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^m)} \|g\|_{L^q(\mathbb{R}^m)} \]

whenever $p, q, r \in [1, \infty]$ and $r^{-1} = p^{-1} + q^{-1} - 1$. In its sharp form established by Beckner [1] for the case when all three of $p, q, r'$ are less than or equal to 2, and subsequently established independently by Brascamp and Lieb [3] and by Beckner for the full range of exponents, it states that

\[ \|f * g\|_{L^r(\mathbb{R}^m)} \leq C_{p,q}^n \|f\|_{L^p(\mathbb{R}^m)} \|g\|_{L^q(\mathbb{R}^m)} \]

with

\[ C_{p,q} = A_p A_q A_{r'} \quad \text{where} \quad A_s = s^{1/2s}t^{-1/2t} \quad \text{with} \quad t = s' ; \]
here and below \( s' = s/(s - 1) \) denotes the exponent \( s' \) conjugate to \( s \). The factor \( C_{p,q} \) is strictly less than 1 provided that \( p, q, r \in (1, \infty) \), and \( C_{p,q}^n \) is the optimal constant in this inequality for all exponents and all dimensions.  

Write \( p = (p_1, p_2, p_3) \) with \( p_j \in [1, \infty] \), \( f = (f_1, f_2, f_3) \), and \( x = (x_1, x_2, x_3) \) where each \( x_j \in \mathbb{R}^m \). We use the notational convention

\[
(1.4) \quad \|f\|_p = \prod_{j=1}^3 \|f_j\|_{p_j}.
\]

An ordered triple \( p = (p_1, p_2, p_3) \) of exponents is said to be admissible if \( p_j \in [1, \infty] \) and \( \sum_{j=1}^3 p_j^{-1} = 2 \).

Rather than work with the bilinear operation \((f, g) \mapsto f \ast g\), we will work with the trilinear form

\[
(1.5) \quad \mathcal{T}(f) = \mathcal{T}_{\mathbb{R}^m}(f) = \int_{x_1+x_2+x_3=0} \prod_{j=1}^3 f_j(x_j) \, d\lambda_{\mathbb{R}^m}(x)
\]

where \( \lambda_{\mathbb{R}^m} \) is the natural Lebesgue measure on

\[
(1.6) \quad \Lambda_{\mathbb{R}^m} = \{ x \in (\mathbb{R}^m)^3 : x_1 + x_2 + x_3 = 0 \}.
\]

That is,

\[
\lambda_{\mathbb{R}^m}(E) = \int_{\mathbb{R}^m \times \mathbb{R}^m} \mathbf{1}_E(x_1, x_2, -x_1 - x_2) \, dx_1 \, dx_2.
\]

The three variables \( x_1, x_2, x_3 \) may be freely permuted in the discussion of \( \lambda_{\mathbb{R}^m} \).

For \( p \in [1, \infty]^3 \) define the constant

\[
(1.7) \quad A_p = \prod_{j=1}^3 p_j^{1/2p_j} q_j^{-1/2q_j}
\]

where \( q_j \) is the exponent conjugate to \( p_j \), with \( \infty^{\pm 1/\infty} \) interpreted as 1. Then \( A_p \) is strictly less than 1 whenever \( p \) is admissible and each \( p_j \) belongs to the open interval \((1, \infty)\). The inequality of Beckner and Brascamp-Lieb can be restated as

\[
(1.8) \quad \left| \mathcal{T}_{\mathbb{R}^m}(f) \right| \leq A_p \|f\|_p
\]

whenever \( p \) is admissible. The factor \( A_p \) is optimal for all exponents.

By a Gaussian function \( G \) with domain equal to a Euclidean space \( \mathbb{R}^m \) we mean a function

\[
(1.9) \quad G(x) = ce^{-|L(x-a)|^2 + ix \cdot b}
\]

where \( c \in \mathbb{C} \), \( a \in \mathbb{R}^m \), \( b \in \mathbb{R}^m \), and \( L : \mathbb{R}^m \to \mathbb{R}^m \) is an invertible linear endomorphism. A linear imaginary term, \( ix \cdot b \), is allowed in the exponent, but the quadratic part of the exponent is real. In other contexts, the term “Gaussian” may refer to functions that are either more, or less, general.

For the Euclidean group \( \mathbb{R}^m \), extremizing triples \( f \) for Young’s convolution inequality exist for all admissible exponent triples \( p \) with each \( p_j \in (1, \infty) \). All such triples were characterized by Brascamp and Lieb [3]. For each admissible \( f \in L^{p_1} \times L^{p_2} \times L^{p_3} \) there exists \( \gamma(p) = (\gamma_1, \gamma_2, \gamma_3) \in (0, \infty)^3 \) with the following property. Suppose that \( \|f_j\|_{p_j} > 0 \) for each index \( j \). If \( |\mathcal{T}_{\mathbb{R}^m}(f)| = A_p \|f\|_p \) then each function \( f_j \) is a Gaussian function

\[
G_j = c_j e^{-\rho_j |L_j(x-a_j)|^2 + ix \cdot b_j}
\]

Moreover, the ordered triple \((G_1, G_2, G_3)\) is compatible in the sense that \( a_1 + a_2 + a_3 = 0, b_1 = b_2 = b_3, L_1 = L_2 = L_3 \), and \( \rho_i / p_j = \gamma_i / \gamma_j \) for all \( i, j \in \{1, 2, 3\} \). Conversely, if each \( f_j \) is Gaussian and if these functions are compatible
in the sense indicated, then \( |\mathcal{T}_{\mathbb{R}^m}(f)| = A^m_p \|f\|_p \). \( \gamma(p) \) is uniquely specified by \( p \) if one requires that \( \gamma_1 = 1 \).

A yet sharper formulation of Young’s inequality for \( \mathbb{R}^m \) is developed in [4]. If \( \|f_j\|_{p_j} = 1 \) for each index \( j \) and if \( \mathcal{T}(f) \geq A^m_p - \delta \) then \( f \) lies within distance \( \varepsilon(\delta) \) of an extremizing triple of Gaussians, in the sense that \( \|f_j - G_j\|_{p_j} \leq \varepsilon(\delta) \), and \( \varepsilon(\delta) \to 0 \) as \( \delta \to 0 \). For a partial range of admissible exponents \( p \), this is shown [11] to hold with \( \varepsilon(\delta) = C(\delta) \delta^{1/2} \).

1.2. Young’s inequality for Heisenberg groups. Let \( d \in \mathbb{N} \), and identify \( \mathbb{R}^{2d+1} \) with \( \mathbb{R}^{2d} \times \mathbb{R} \). The Heisenberg group \( \mathbb{H}^d \) is \( \mathbb{R}^{2d+1} \) as a set, with the group law

\[
\begin{align*}
    z \cdot z' &= (x, t) \cdot (x', t') = (x + x', t + t' + \sigma(x, x'))
\end{align*}
\]

where \( z = (x, t) \), \( z' = (x', t') \), and \( \sigma : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}^1 \) is the symplectic form

\[
\begin{align*}
    \sigma(x, x') &= \sum_{j=1}^{d} (x_j x'_{j+d} - x_{j+d} x'_j).
\end{align*}
\]

Although we use multiplicative notation for the group law, we denote the the group identity element by \( 0 = (0, 0) \). The Heisenberg multiplicative inverse of \( (x, t) \) is \( (-x, -t) \). There are of course many alternative isomorphic formulations of this group law, some of which are in common use. By a Gaussian function \( G : \mathbb{H}^d \to \mathbb{C} \) we mean a Gaussian function \( G : \mathbb{R}^{2d+1} \to \mathbb{C} \), with respect to the coordinate system for \( \mathbb{H}^d \) introduced above.

\( L^p \) norms on \( \mathbb{H}^d \) are defined with respect to Lebesgue measure on \( \mathbb{R}^{2d+1} \), and will be denoted by \( \| \cdot \|_{L^p} \) and more succinctly by \( \| \cdot \|_p \). Throughout this paper, integrals over \( \mathbb{H}^d \) or subsets of \( \mathbb{H}^d \) measure are understood to be with respect to Lebesgue measure, unless the contrary is explicitly indicated. Convolution is defined to be \( f * g(u) = \int_{\mathbb{H}^d} f(\nu^{-1})g(\nu) \, d\nu \).

This bilinear operation is associative, but not commutative, on the Schwartz space.

We phrase Young’s inequality for \( \mathbb{H}^d \) in terms of the trilinear form

\[
\begin{align*}
    \mathcal{T}_{\mathbb{H}^d}(f) &= \int_{z_1 z_2 z_3 = 0} \prod_{j=1}^{3} f_j(z_j) \, d\lambda(z)
\end{align*}
\]

where \( z_1 z_2 z_3 \) is the threefold \( \mathbb{H}^d \) product and \( \lambda = \lambda_{\mathbb{H}^d} \) is the natural Lebesgue measure on \( \mathbb{H}^d \).

\[
\begin{align*}
    \Lambda_{\mathbb{H}^d} &= \{ z \in (\mathbb{H}^d)^3 : z_1 z_2 z_3 = 0 \}.
\end{align*}
\]

That is,

\[
\begin{align*}
    \lambda(E) &= \int_{\mathbb{H}^d \times \mathbb{H}^d} 1_E(z_1, z_2, z_2^{-1} z_1^{-1}) \, dz_1 \, dz_2
\end{align*}
\]

and the roles of the variables \( z_1, z_2, z_3 \) can be interchanged provided that noncommutativity of the group law is taken properly into account. Recall that the group identity element of \( \mathbb{H}^d \) is denoted by \( 0 \). Just as in the Euclidean case, it is elementary that \( |\mathcal{T}_{\mathbb{H}^d}(f)| \leq \|f\|_p \) whenever \( f_j \in L^{p_j} \) for all \( j \) and \( p \) is admissible.

Klein and Russo [13] and Beckner [2] have observed that the sharper inequality

\[
\begin{align*}
    |\mathcal{T}_{\mathbb{H}^d}(f)| \leq A^{2d+1}_p \|f\|_p
\end{align*}
\]

holds, with the same constant factor on the right-hand side as for Euclidean space of dimension \( 2d + 1 \). Moreover, \( A^{2d+1}_p \) is the optimal constant in this inequality. Beckner has
observed further that there exist no extremizing functions, that is, \(|T_{H^d}(f)|\) is strictly less than \(A_{p}^{2d+1}||f||_{p}\) whenever all three functions have positive norms\(^1\).

The nonexistence of extremizing functions can be viewed differently. For each \(s \in \mathbb{R}\), the set \(H^{2d+1}\) is a group under the operation \(+_s\) defined by

\[(x, t) +_s (x', t') = (x + x', t + t' + s\sigma(x, x')).\]

This group is isomorphic to \(H^d\) if \(s \neq 0\), and to the Euclidean group \(\mathbb{R}^{2d+1}\) for \(s = 0\). Haar measure is Lebesgue measure in these coordinates, for all \(s\). The optimal constant in Young’s convolution inequality is \(A_{p}^{2d+1}\) for every \(s\). A datum \((f, s)\) realizes this optimal constant if and only if \(s = 0\) and \(f\) is a maximizing ordered triple \(G\) for \(H^d\). Theorem 2.2 below, could be reformulated as an assertion that \((f, s)\) nearly realizes the optimal constant only if \((f, s)\) is close to such a datum \((G, 0)\), in an appropriate sense.

In a series of papers [5], [6], [7], [8], [9], [10], [11], [12] we have studied various sharp inequalities for which extremizing functions (respectively ordered tuples of functions or sets) exist and have previously been characterized. We have shown that functions (respectively ordered tuples of functions or sets) that nearly extremize the inequalities are nearly equal, in appropriate norms or other measures of approximation, to extremizing functions (respectively ordered tuples of functions or sets). The present paper characterizes ordered triples of functions that nearly extremize Young’s inequality for Heisenberg groups — despite the nonexistence of exact extremizers.

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2. Definitions and main theorem

Our main result will state that if \(f\) nearly extremizes Young’s inequality for \(H^d\) then there exists an ordered triple \((G_1, G_2, G_3)\) of Gaussians with certain properties, such that \(\|f_j - G_j\|_{p_j}\) is small for each index \(j\). In order to formulate this result precisely, several definitions are required.

2.1. The symplectic group. Denote by \(Sp(2d)\) the symplectic group of all invertible linear mappings \(S: \mathbb{R}^{2d} \to \mathbb{R}^{2d}\) satisfying

\[\sigma(Sx, Sx') = \sigma(x, x') \quad \text{for all } x, x' \in \mathbb{R}^{2d},\]

To \(S \in Sp(2d)\) is associated the group automorphism \((x, t) \mapsto (Sx, t)\) of \(H^d\).

Let \(J\) denote the \(2d \times 2d\) matrix

\[J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\]

where \(I\) is the \(d \times d\) identity matrix. Since \(\sigma(x, y) = \langle x, Jy \rangle\) for \(x, y \in \mathbb{R}^{2d}\), the identity \(\sigma(Sx, Sy) \equiv \sigma(x, y)\) that defines \(Sp(2d)\) is equivalent to \(\langle Sx, JSy \rangle \equiv \langle x, Jy \rangle\). Thus \(S \in Sp(2d)\) if and only if \(S^*JS = J\).

\(^1\)Klein and Russo do not explicitly discuss existence of extremizers for Young’s inequality, but do prove a closely related result: There exist no nonzero extremizers for the Heisenberg group analogue of the \(L^p \to L^{p'}\) Hausdorff-Young inequality when the conjugate exponent \(p'\) is an even integer.
2.2. Symmetries. Let $\Psi = (\psi_1^*, \psi_2^*, \psi_3^*)$ be an ordered 3-tuple of invertible linear mappings $\psi_j : L^p(\mathbb{H}^d) \to L^p(\mathbb{H}^d)$. Consider the functional
\begin{equation}
\Phi(f) = |T_{\mathbb{H}^d}(f)| \|f\|_p^{-1},
\end{equation}
defined for all $f$ satisfying $\|f\|_p \neq 0$. Given $p$, we say that $\Psi$ is a symmetry of the inequality (1.13), or of the functional $\Phi$, if $\Phi(\Psi f) = \Phi(f)$ for all $f \in L^{p_1} \times L^{p_2} \times L^{p_3}$ with $\|f\|_p \neq 0$. These 3-tuples form a group under componentwise composition.

Most of the symmetries of $\Phi$ relevant to our considerations are defined in terms of mappings of the underlying space $\mathbb{H}^d$. To any diffeomorphism $\psi$ of $\mathbb{H}^d$ we associate a linear operator on functions $f : \mathbb{H}^d \to \mathbb{C}$, defined by
$$\psi^*(f) = f \circ \psi.$$ We next list four families of ordered triples $(\psi_1, \psi_2, \psi_3)$ of diffeomorphisms of $\mathbb{H}^d$ such that $\Psi = (\psi_1^*, \psi_2^*, \psi_3^*)$ is a symmetry of $\Phi$. The first three of these families are:
\begin{equation}
\begin{cases}
(i) & \psi_j(x, t) = (rx, r^2t) \quad \text{with } r \in \mathbb{R}^+ \\
(ii) & \psi_j(z) = (u_jzw_j) \quad \text{with } w_1 = u_2^{-1}, w_2 = u_3^{-1}, \text{ and } w_3 = u_1^{-1}.
\end{cases}
\end{equation}
The fourth family is defined by
\begin{equation}
\psi_j(x, t) = (x, t + \varphi_j(x))
\end{equation}
where $(\varphi_1, \varphi_2, \varphi_3)$ is an ordered triple of affine mappings from $\mathbb{R}^{2d}$ to $\mathbb{R}^1$ that satisfies $\sum_{k=1}^{3} \varphi_k(x_k) = 0$ whenever $\sum_{k=1}^{3} x_k = 0$. In (i), $r$ is independent of $j$; likewise $S$ is independent of $j$ in (ii). In (ii), $u_jz_jw_j$ is the $\mathbb{H}^d$ group product of these three elements.

A fifth family of symmetries is defined in terms of modulations of functions, rather than diffeomorphisms of the underlying space. For any $u \in \mathbb{R}^{2d}$ define $\Psi = (\psi_1, \psi_2, \psi_3)$ by
\begin{equation}
(\psi_j f)(x, t) = e^{iu \cdot x} f(x, t).
\end{equation}
The exponent $iu \cdot x$ depends only on the coordinate $x$, not on $t$.

Each component of each element of each of these five families is an invertible bounded linear operator on $L^p(\mathbb{H}^d)$ for all $p \in [1, \infty]$. By the composition $\Psi \circ \Psi'$ of two such ordered triples we mean the ordered triple $(\psi_1 \circ \psi'_1, \psi_2 \circ \psi'_2, \psi_3 \circ \psi'_3)$ defined by componentwise composition.

**Lemma 2.1.** Each of the ordered triples of linear operators $\Psi$ listed above is a symmetry of the ratio $\Phi$ for every admissible $p$.

The straightforward verifications are left to the reader. \hfill \Box

**Definition 2.1.** $\mathcal{G}(\mathbb{H}^d)$ denotes the group of all ordered triples $\Psi$ of diffeomorphisms of $\mathbb{H}^d$ that can be expressed as compositions of finitely many symmetries of the inequality (1.14), with each factor being one of the five types introduced above.

2.3. Special ordered triples of Gaussians on $\mathbb{H}^d$.

**Definition 2.2.** Let $d \geq 1$ and $\varepsilon > 0$. A canonical $\varepsilon$–diffuse Gaussian is a function $G : \mathbb{H}^d \to \mathbb{C}$ of the form
$$G(x, t) = e^{-|Lx|^2} e^{-at^2} e^{ibt}$$
where $a > 0$, $b \in \mathbb{R}$, and $L : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is an invertible linear endomorphism, which together satisfy
\begin{equation}
\max(a^{1/2}, a, |b|) \cdot \|L^{-1}\|^2 \leq \varepsilon.
\end{equation}
Recall the ordered triple $\gamma(p)$ introduced above in the discussion of maximizers for Young’s inequality for $\mathbb{R}^m$.

**Definition 2.3.** Let $p$ be admissible. An ordered triple $G = (G_1, G_2, G_3)$ of canonical $\varepsilon$–diffuse Gaussians

$$G_j(x, t) = e^{-|L_j x|^2} e^{-a_j t^2} e^{ib_j t}$$

is said to be $p$–compatible if there exist $L, a, b$ such that $L_j = \gamma_j^{1/2} L$, $a_j = \gamma_j a$, and $b_j = b$ for all $j \in \{1, 2, 3\}$.

**Definition 2.4.** Let $d \geq 1$ and let $\varepsilon > 0$ be small. An ordered triple $G = (G_1, G_2, G_3)$ of Gaussian functions $G_j : \mathbb{H}^d \to \mathbb{C}$ is $\varepsilon$–diffuse and $p$–compatible if there exist $\Psi \in \mathcal{G}(\mathbb{H}^d)$, scalars $c_j \in \mathbb{R}^+$, and a $p$–compatible ordered triple $(\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ of canonical $\varepsilon$–diffuse Gaussian functions such that

$$G_j = c_j \tilde{G}_j$$

for each index $j \in \{1, 2, 3\}$.

### 2.4. Main theorem.

**Theorem 2.2.** For each $d \geq 1$ and each admissible ordered triple $p$ of exponents there exists a function $\delta \mapsto \varepsilon(\delta)$ satisfying $\lim_{\delta \to 0} \varepsilon(\delta) = 0$ with the following property. Let $f \in L^p(\mathbb{H}^d)$ and suppose that $\|f_j\|_{p_j} \neq 0$ for each $j \in \{1, 2, 3\}$. Let $\delta \in (0, 1)$ and suppose that $|T_{\mathbb{H}^d}(f)| \geq (1 - \delta) A_p^{2d+1} \|f\|_p$. Then there exists a $p$–compatible $\varepsilon(\delta)$–diffuse ordered triple of Gaussians $G = (G_1, G_2, G_3)$ such that

$$\|f_j - G_j\|_{p_j} < \varepsilon(\delta) \|f_j\|_{p_j} \text{ for } j \in \{1, 2, 3\}. \quad (2.8)$$

Thus $G_j = c_j \tilde{G}_j$ where $c_j \in \mathbb{C}$, $(\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ is a canonically $\varepsilon(\delta)$–diffuse $p$–compatible ordered triple of Gaussians, and $\Psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{G}(\mathbb{H}^d)$. All five types of elements of $\mathcal{G}(\mathbb{H}^d)$ are encountered in the analysis.

The technique developed here has been adapted to the $ax + b$ group, and an analogue of Theorem 2.2 for that group has been established, by E. Scerbo [15].

### 3. Approximate solutions of functional equations

A principal ingredient of the analysis is a quantitative expression of the unsolvability of a variant of the functional equation

$$\varphi(x) + \psi(y) + \xi(x + y) = 0. \quad (3.1)$$

This variant takes the form

$$\varphi(x) + \psi(y) + \xi(x + y) + \sigma(x, y) = 0 \quad (3.2)$$

where the functions $\varphi, \psi, \xi$ have domains equal to $\mathbb{R}^{2m}$. Its unsolvability is formulated below, in quantitative terms, as Proposition 7.4.

An ad hoc argument that relies on the antisymmetry of $\sigma(x, y)$ will enable us to deduce the information needed concerning $\{\mathcal{Z}_2\}$ from what is already known about approximate solutions of (3.1). This leads naturally to analogous questions about more general functional equations, for which this ad hoc argument may not apply. We therefore digress to present the following general result, which is suggested and motivated by considerations in this paper, but is not actually used in the proofs of the main theorems.

Consider the difference operators

$$\Delta_h f(x) = f(x + h) - f(x), \quad (3.3)$$

where \( x \in \mathbb{R}^d \) and \( + \) denotes the Euclidean group operation. Let \( B \) be an arbitrary ball of positive, finite radius in \( \mathbb{R}^d \) and let \( \tilde{B} \) be a ball of positive, finite radius in \( \mathbb{R}^d \) centered at the origin.

**Theorem 3.1.** For each dimension \( d \geq 1 \), each nonnegative integer \( D \), and each \( \eta > 0 \) there exists a function \( \delta \mapsto \varepsilon(\delta) \) satisfying \( \lim_{\delta \to 0} \varepsilon(\delta) = 0 \) with the following property. Suppose that \( |B| \geq \eta|B| \), \( 0 < \delta \leq 1 \), and \( A \in [0, \infty) \). Let \( \varphi : B + \tilde{B} \to \mathbb{C} \) be Lebesgue measurable. Suppose that there exists a function \( B \times \tilde{B} \ni (x, h) \mapsto P_h(x) \in \mathbb{C} \) such that
\[
|\Delta_h \varphi(x) - P_h(x)| \leq A
\]
for all \((x, h) \in B \times \tilde{B}\) with the exception of a set of measure \( \leq \delta |B| \cdot |\tilde{B}| \). Suppose that
\[
P_h(x) = \sum_{|\alpha| \leq D} a_\alpha(h)x^\alpha
\]
is a polynomial function of \( x \) of degree \( \leq D \) whose coefficients \( a_\alpha \) are Lebesgue measurable functions of \( h \). Then there exists a polynomial \( Q \) of degree at most \( D + 1 \) such that
\[
|\varphi(x) - Q(x)| \leq CA
\]
for all \( x \in B \) outside a set of measure \( \leq \varepsilon(\delta)|B| \). The constant \( C \) and function \( \varepsilon \) depend only on \( d, D, \eta \).

This is proved in [11]. In the simplest case \( D = 0 \), the assumption is that \( |\varphi(x + h) - \varphi(x) - a(h)| \leq A \) for nearly all points of \( B \times \tilde{B} \); one has an approximate version of the fundamental functional equation (3.1). In that special case, Theorem 3.1 is proved in [6].

It is natural to also record a multiplicative analogue the preceding theorem.

**Theorem 3.2.** For each dimension \( d \geq 1 \), each nonnegative integer \( D \), and each \( \eta > 0 \) there exists a function \( \delta \mapsto \varepsilon(\delta) \) satisfying \( \lim_{\delta \to 0} \varepsilon(\delta) = 0 \) with the following property. Suppose that \( |B| \geq \eta|B| \), \( 0 < \delta \leq 1 \), and \( A \in [0, 2] \). Let \( \varphi : B + \tilde{B} \to \mathbb{R} \) be Lebesgue measurable. Suppose that there exists a function \( B \times \tilde{B} \ni (x, h) \mapsto P_h(x) \in \mathbb{R} \) such that
\[
|e^{i(\varphi(x+h)-\varphi(x))}e^{-iP_h(x)} - 1| \leq A
\]
for all \((x, h) \in B \times \tilde{B}\) with the exception of a set of measure \( \leq \delta |B| \cdot |\tilde{B}| \). Suppose that
\[
P_h(x) = \sum_{|\alpha| \leq D} a_\alpha(h)x^\alpha
\]
is a polynomial function of \( x \) of degree \( \leq D \) whose coefficients \( a_\alpha \) are Lebesgue measurable real-valued functions of \( h \). Then there exists a polynomial \( Q \) of degree at most \( D + 1 \) such that
\[
|e^{i\varphi x}e^{-iQ(x)} - 1| \leq CA
\]
for all \( x \in B \) outside a set of measure \( \leq \varepsilon(\delta)|B| \). The constant \( C \) and function \( \varepsilon \) depend only on \( d, D, \eta \).

4. Analogue for twisted convolution

Consider twisted convolution of functions with domains \( \mathbb{R}^{2d} \). The associated trilinear forms are
\[
T_{\mathbb{R}^{2d}, \lambda}(f) = \int_{(\mathbb{R}^{2d})^3} e^{i\lambda \sigma(x_1, x_2)} \prod_{j=1}^{3} f_j(x_j) \, d\lambda_{\mathbb{R}^{2d}}(x)
\]
where \( 0 \neq \lambda \in \mathbb{R} \) is a parameter and \( \mathbf{x} = (x_1, x_2, x_3) \). Since \(|\mathcal{T}_{\mathbb{R}^{2d}, \lambda}(f)| \leq \mathcal{T}_{\mathbb{R}^{2d}}(|f_1|, |f_2|, |f_3|)\), one has

\[
(4.2) \quad |\mathcal{T}_{\mathbb{R}^{2d}, \lambda}(f)| \leq A_{p}^{2d} \prod_{j=1}^{3} \|f_j\|_{p_j}
\]

for admissible \( p \). The constant \( A_{p}^{2d} \) is optimal [13], as one sees by considering ordered triples of Gaussians that extremize Young’s inequality for \( \mathbb{R}^{2d} \) and are concentrated near 0. Again, there exist no extremizing triples [13].

**Theorem 4.1.** For each \( d \geq 1 \) and each admissible ordered triple \( p \) of exponents there exists a function \( \delta \mapsto \varepsilon(\delta) \) satisfying \( \lim_{\delta \to 0} \varepsilon(\delta) = 0 \) with the following property. Let \( f \in L^p(\mathbb{R}^{2d}) \) and suppose that \( \|f_j\|_{p_j} \neq 0 \) for each \( j \in \{1, 2, 3\} \). Let \( \delta \in (0, 1) \) and suppose that \( |\mathcal{T}_{\mathbb{R}^{2d}, \lambda}(f)| \geq (1 - \delta)A_{p}^{2d}\|f\|_{p} \). Then there exist \( S \in \text{Sp}(2d) \) and a \( p \)-compatible ordered triple of Gaussians \( G = (G_1, G_2, G_3) \) such that \( G_j = G_j \circ S \) satisfy

\[
(4.3) \quad \|f_j - G_j^2\|_{p_j} < \varepsilon(\delta)\|f_j\|_{p_j} \quad \text{for} \quad j \in \{1, 2, 3\}
\]

and \( G_j \) take the form

\[
(4.4) \quad G_j(x) = c_{j}e^{-\gamma_{j}(p)|L(x-a_{j})|^{2}}e^{ix \cdot v}
\]

where \( v \in \mathbb{R}^{2d} \), \( 0 \neq c_{j} \in \mathbb{C} \), \( a_1 + a_2 + a_3 = 0 \), and

\[
(4.5) \quad |\lambda| \cdot \|L^{-1}\|^{2} \leq \varepsilon(\delta).
\]

The proof of this theorem follows that of Theorem 2.2 with some simplifications. Details are left to the reader.

5. **Nonexistence of extremizers and value of the optimal constant**

We begin by reviewing proofs that the optimal constant in Young’s inequality for \( \mathbb{H}^d \) equals the optimal constant for Euclidean space of dimension \( 2d + 1 \), and that extremizing triples do not exist. To show that the constant for \( \mathbb{H}^d \) is at least as large as for \( \mathbb{R}^{2d+1} \), let \( \varepsilon > 0 \) be small, and consider the ordered triple of functions \( f_\varepsilon = (f_{j,\varepsilon} : 1 \leq j \leq 3) \) with \( f_{j,\varepsilon}(x,t) = e^{-\gamma_{j}|x|^2}e^{-\varepsilon\gamma_{j}t^2} \) and \( \gamma(p) = (\gamma_1, \gamma_2, \gamma_3) \). For each \( \varepsilon > 0 \), \( f_\varepsilon \) extremizes Young’s inequality for \( \mathbb{R}^{2d+1} \). One finds by a simple change of variables \( t = \varepsilon^{-1/2}s \) that

\[
(5.1) \quad \frac{\mathcal{T}_{\mathbb{H}^d}(f_\varepsilon)}{\mathcal{T}_{\mathbb{R}^{2d+1}}(f_\varepsilon)} \to 1 \quad \text{as} \quad \varepsilon \to 0.
\]

To prove the reverse implication, let \( f_j \in L^{p_j}(\mathbb{H}^d) \) be nonzero nonnegative functions which are otherwise arbitrary. Define

\[
(5.2) \quad \begin{cases} 
F_j(x) = \|f_j(x, \cdot)\|_{L^{p_j}}(\mathbb{R}) \\
 f_{j,x}(t) = f_j(x, t)/F_j(x) \quad \text{if} \quad F_j(x) \neq 0,
\end{cases}
\]

with instead \( f_{j,x}(t) \equiv 0 \) if \( F_j(x) = 0 \). Write \( \mathbf{x} = (x_1, x_2, x_3) \). Then

\[
(5.3) \quad \mathcal{T}_{\mathbb{H}^d}(f) = \int_{\mathbb{H}^d} \prod_{j=1}^{3} F_j(x_j) \mathcal{T}_{\mathbb{R}^{1}}(f_{1,x_1}, f_{2,x_2}, f_{3,x_3}) d\lambda_{\mathbb{R}^{2d}}(\mathbf{x})
\]

where

\[
(5.4) \quad f_{3,x}^\dagger(s) = f_{3,x_3}(s + \sigma(x_1, x_2)).
\]
Straightforward calculation gives $f_{3.x_3}(s+\sigma(x_1,x_2)+\sigma(x_1+x_2,x_3))$ as the natural definition of $f_{3,x}(s)$, but outside of a $\lambda_{\mathbb{R}^{2d}}$-null set this simplifies to $f_{3,x}(s+\sigma(x_1,x_2))$ since $x_1+x_2+x_3=0 \implies \sigma(x_1+x_2,x_3) = \sigma(x_1+x_2,-x_1-x_2) = 0$.

Therefore

$$|T_{\mathbb{R}}(f_{1,x_1},f_{2,x_2},f_{3,x_3})| \leq A_p \prod_{j=1}^{3} \|f_{j,x_j}\|_{p_j} \leq A_p$$

with equality only if $\prod_{j=1}^{3} F_j(x_j) \neq 0$ and $(f_{1,x_1},f_{2,x_2},f_{3,x_3})$ is an extremizing triple for Young’s inequality for $\mathbb{R}$. Inserting this into (5.5) gives

$$|T_{\mathbb{H}}(f)\| \leq A_p \int_{x_1+x_2+x_3=0} \prod_{j=1}^{3} F_j(x_j) d\lambda_{\mathbb{H}^{2d}}(x)$$

$$= A_p \mathcal{T}_{\mathbb{H}^{2d}}(F_1,F_2,F_3) \leq A_p A_p^{2d} \prod_{j=1}^{3} \|F_j\|_{L^{p_j}(\mathbb{H}^{2d})} = A_p^{2d+1} \|f\|_p.$$ 

This proves that the optimal constant for $\mathbb{H}^{2d}$ cannot exceed the optimal constant for $\mathbb{R}^{2d+1}$.

This analysis implicitly proves that extremizers do not exist for $\mathbb{H}^d$. For arbitrary non-negative $f_j \in L^p(\mathbb{H}^d)$ with positive norms, we have shown that equality holds only if both

(i) for $\lambda$–almost every $x \in (\mathbb{R}^{2d})^3$, $(f_{1,x_1},f_{2,x_2},f_{3,x_3})$ is an extremizing triple for Young’s inequality for $\mathbb{R}$ and

(ii) $(F_1,F_2,F_3)$ is an extremizing triple for Young’s inequality for $\mathbb{R}^{2d}$.

By the characterization of equality in Young’s inequality for $\mathbb{R}^{2d}$, each $F_j$ must be a Gaussian; in particular, $F_j$ is nonzero almost everywhere. Likewise, $f_{j,y}$ must be a Gaussian for almost every $y \in \mathbb{R}^{2d}$ for each index $j \in \{1,2,3\}$. Moreover, $(f_{1,x_1},f_{2,x_2},f_{3,x_3})$ must be $p$–compatible. Expressing

$$f_{j,y}(s) = c_j(y)e^{-\gamma_j(y)(s-a_j(y))^2+ib_j(y)s},$$

compatibility forces the functional equation

$$a_1(y_1) + a_2(y_2) + a_3(-y_1-y_2) + \sigma(y_1,y_2) = 0$$

for almost every $(y_1,y_2) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$.

**Lemma 5.1.** There exists no ordered triple of measurable functions $a_j : \mathbb{R}^{2d} \to \mathbb{C}$ that satisfies the functional equation (5.5) for almost every $(y_1,y_2) \in (\mathbb{R}^{2d})^2$.

**Proof of Lemma 5.1.** Write (5.5) with the roles of $y_1,y_2$ interchanged, and add the result to (5.5). Since $\sigma$ is antisymmetric, its contributions cancel, leaving

$$a_1(x_1) + a_2(x_2) + a_3(-x_1-x_2) = 0$$

for almost every $(x_1,x_2) \in (\mathbb{R}^{2d})^3$, where $a = \frac{1}{2}a_1 + \frac{1}{2}a_2$. As is well known, any measurable solutions of this functional equation must agree almost everywhere with affine functions. Thus $a_3$ is affine.

Inserting this conclusion into (5.5), we conclude that there exist functions $\tilde{a}_j$, which differ from $a_j$ by affine functions, such that $\tilde{a}_1(x_1) + \tilde{a}_2(x_2) + \sigma(x_1,x_2) = 0$ almost everywhere. By freezing almost any value of $x_2$ one finds that $\tilde{a}_1$ agrees almost everywhere with an affine function. The same reasoning applies to $\tilde{a}_2$. But the original equation (5.5) cannot hold with all three functions $a_j$ affine, since $\sigma$ is not affine. 

□
This paper establishes a more quantitative form of Lemma 5.1 and reduces Theorem 2.2 to this result by elaborating on the reasoning shown above. Klein and Russo [13] have shown how the same type of reasoning as that shown above can be applied to certain semidirect product Lie groups. Much of the quantitative analysis below extends straightforwardly to more general semidirect products. However, each semidirect product leads to its own analogue of the variant (5.5) of the classical functional equation (3.1). In this paper we analyze only one such variant, leaving a general investigation for future work. Forthcoming work of E. Scerbo [15] will adapt this analysis to the $ax + b$ group.

**Remark 5.1.** There is no solution $(a_1, a_2, a_3)$ of (5.5) in the sense of distributions. This remark does not subsume Lemma 5.1 since the lack of any assumption in that lemma that the functions $a_j$ are locally integrable prevents their being interpreted as distributions.

To show this, write $y_j = (y_{j,k})_{1 \leq k \leq 2d}$. Applying $\frac{\partial^2}{\partial y_{1,m} \partial y_{1,n}}$ gives

$$\frac{\partial^2 a_1}{\partial y_{1,m} \partial y_{1,n}}(y_1) + \frac{\partial^2 a_3}{\partial y_{1,m} \partial y_{1,n}}(y_1 + y_2) \equiv 0,$$

whence $\frac{\partial^2 a_3}{\partial y_{1,m} \partial y_{1,n}}(y_1 + y_2)$ is independent of $y_2$ as a distribution. Therefore $a_3$, and hence $a_1$, are quadratic polynomials. The same applies to $a_2$.

Now consider any $k \in \{1, 2, \ldots, d\}$ and apply $\frac{\partial^2}{\partial y_{1,k} \partial y_{2,k+d}} + \frac{\partial^2}{\partial y_{2,k} \partial y_{1,k+d}}$ to both sides of (5.5). This differential monomial annihilates $\sigma(y_1, y_2)$. It results that $\frac{\partial^2}{\partial y_{1,m} \partial y_{1,n}} a_3 \equiv 0$. By applying $\frac{\partial^2}{\partial y_{1,m} \partial y_{2,n}}$ for other pairs $m, n$ one obtains $\frac{\partial^2}{\partial y_{1,m} \partial y_{n}} a_3 \equiv 0$ for all $m, n$. Thus $a_3$ is an affine function.

Once this is known, apply to $\frac{\partial^2}{\partial y_{1,m} \partial y_{1,n}}$ to conclude that $a_1$ is affine. In the same way, $a_2$ is affine. (5.5) now expresses $\sigma(y_1, y_2)$ as a sum of three affine functions, contradicting the definition of $\sigma$.

6. SUFFICIENCY

**Proposition 6.1.** Let $d \geq 1$, and let $p$ be admissible. For each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ satisfying $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$ with the following property. For any $p$–compatible $\varepsilon$–diffuse ordered triple $G = (G_1, G_2, G_3)$ of Gaussian functions,

$$T_{H^d}(G) \geq (1 - \eta(\varepsilon)) A_p^{2d+1} \prod_{j=1}^{3} \|G_j\|_{p_j}.$$

More generally, it follows immediately from the triangle inequality that if $G$ is $p$–compatible and $\varepsilon$–diffuse, and if $\|f_j - G_j\|_{p_j} \leq \varepsilon \|f_j\|_{p_j}$ for all $j \in \{1, 2, 3\}$ then

$$|T_{H^d}(f)| \geq (1 - \eta(\varepsilon)) A_p^{2d+1} \prod_{j=1}^{3} \|f_j\|_{p_j},$$

where the function $\eta$ is modified but is still $o_{\varepsilon}(1)$.

The following notation will be used throughout the analysis, here and below.

**Definition 6.1.** For any invertible linear endomorphism $L$ of $\mathbb{R}^{2d}$,

$$\sigma_L(x, y) = \sigma(L^{-1}x, L^{-1}y)$$

for $x, y \in \mathbb{R}^{2d}$. 
Proof of Proposition 6.1. Since the action of $\Theta^1(\mathbb{R}^d)$ preserves the ratio $|T_{\mathbb{R}^d}(f)|/\prod_{j=1}^3 |f_j|_{p_j}$, it suffices to prove this for $p$-compatible ordered triples of canonical $\varepsilon$-diffuse Gaussians. Thus we may assume that

$$G_j(x, t) = e^{-\gamma_j |Lx|^2} e^{-\gamma_j a^2 t^2} e^{ibt}$$

where $L$ is an invertible linear endomorphism of $\mathbb{R}^{2d}$, $a > 0$, $b \in \mathbb{R}$, and $\max(a^{1/2}, |b|)\|L^{-1}\|^2 \leq \varepsilon$. In this situation,

$$T_{\mathbb{R}^d}(G) = \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} e^{-\gamma_1 |Lx_1|^2 - \gamma_2 |Lx_2|^2 - \gamma_3 |L(x_1 + x_2)|^2} \cdot \int_{\mathbb{R} \times \mathbb{R}} e^{-\gamma_1 a^2 t_1^2 - \gamma_2 a^2 t_2^2 - \gamma_3 a(t_1 + t_2 + \sigma(x_1, x_2))^2} e^{i(bt_1 + bt_2 - b(t_1 + t_2 + \sigma(x_1, x_2)))} dt_1 dt_2 dx_1 dx_2.$$

Cancelling where possible and substituting $Lx_j = y_j$ gives $|\det(L)|^{-1} \cdot I$ where

$$I = \int_{\mathbb{R}^{4d}} e^{-\gamma_1 |y_1|^2 - \gamma_2 |y_2|^2 - \gamma_3 |y_1 + y_2|^2} e^{-ib\sigma_L(y_1, y_2)} \cdot \int_{\mathbb{R}^2} e^{-\gamma_1 a^2 t_1^2 - \gamma_2 a^2 t_2^2 - \gamma_3 a(t_1 + t_2)^2} dt_1 dt_2 dy_1 dy_2.$$

Define

$$J = \int_{\mathbb{R}^{4d}} e^{-\gamma_1 |y_1|^2 - \gamma_2 |y_2|^2 - \gamma_3 |y_1 + y_2|^2} \int_{\mathbb{R}^2} e^{-\gamma_1 a^2 t_1^2 - \gamma_2 a^2 t_2^2 - \gamma_3 a(t_1 + t_2)^2} dt_1 dt_2 dy_1 dy_2.$$

$G$ is an extremizing ordered triple for Young’s inequality with exponents $p$ for $\mathbb{R}^{d+1}$, with the same coordinates $(x, t)$. Thus $J = |\det(L)|^2 A_p^{d+1} \prod_{j=1}^3 |G_j|_{p_j}$. Thus it suffices to prove that

$$|I| \geq (1 - o_\varepsilon(1))J.$$

An application of Young’s inequality for $\mathbb{R}^1$ to the inner integral, followed by an application Young’s inequality for $\mathbb{R}^{2d}$ to the remaining outer integral, also reveals that $|I| \leq |\det(L)|^2 A_p^{d+1} \prod_{j=1}^3 |G_j|_{p_j}$.

Let $\varepsilon \mapsto \rho(\varepsilon)$ be a function that tends to $\infty$ slowly as $\varepsilon \to 0$. The same reasoning shows that if the integrand in the integral defining $I$ is replaced by its absolute value, then the contribution of the region $\mathcal{R} = \{(y_1, y_2) \in \mathbb{R}^{4d} : |(y_1, y_2)| > \rho(\varepsilon)\}$ to the integral is $o_\varepsilon(1)$. Since $|b|\|L^{-1}\|^2 \leq \varepsilon$ by hypothesis,

$$|b\sigma_L(y_1, y_2)| \leq |b|\|L^{-1}\|^2 |\rho(\varepsilon)|^2 \leq \varepsilon^{1/2}$$

uniformly for all $(y_1, y_2) \in \mathbb{R}^{4d} \setminus \mathcal{R}$ provided that $\rho(\varepsilon)$ is chosen to satisfy $\rho(\varepsilon) \leq \varepsilon^{-1/4}$. Therefore $|e^{-ib\sigma_L(y_1, y_2)} - 1| = O(\varepsilon^{1/2})$ uniformly for all $y \in \mathbb{R}^{4d} \setminus \mathcal{R}$. Therefore

$$I = \int_{\mathbb{R}^{4d}} e^{-\gamma_1 |y_1|^2 - \gamma_2 |y_2|^2 - \gamma_3 |y_1 + y_2|^2} \int_{\mathbb{R}^2} e^{-\gamma_1 a^2 t_1^2 - \gamma_2 a^2 t_2^2 - \gamma_3 a(t_1 + t_2)^2} dt_1 dt_2 dy_1 dy_2$$

plus $o_\varepsilon(1)$.

Define $\mathcal{R}' = \{(t_1, t_2) \in \mathbb{R}^2 : |(t_1, t_2)| \leq \rho(\varepsilon)\}$. By the same reasoning, to complete the proof it suffices to have

$$e^{-\gamma_3 a^2(t_1 + t_2)\sigma_L(y_1, y_2)} e^{-\gamma_3 a \sigma_L(y_1, y_2)^2} = 1 + o_\varepsilon(1).$$
uniformly for all \((y_1, t_1, y_2, t_2)\) such that \((t_1, t_2) \in \mathbb{R}^2 \setminus \mathbb{R}'\) and \((y_1, y_2) \in \mathbb{R}^d \setminus \mathbb{R}\). This holds because

\[
|a(t_1 + t_2)\sigma_L(y_1, y_2)| \leq a\rho(\varepsilon)\|L^{-1}\|^2 \rho(\varepsilon)^2
\]

\[
|a\sigma_L(y_1, y_2)^2| \leq a\|L^{-1}\|^4 \rho(\varepsilon)^4,
\]

while it is given that \((a^{1/2} + a)\|L^{-1}\|^2 \leq \varepsilon). \hfill \square

7. Two Ingredients

In order to prove Theorem 2.2, we will make the steps of the reasoning in §3 quantitative. The following result from [6], the analogue for \(\mathbb{R}^m\) of our main result for \(\mathbb{H}^d\), will be the first of two main ingredients in the analysis.

**Theorem 7.1.** For each admissible \(\mathbf{p} \in (1, \infty)^3\) and each \(m \in \mathbb{N}\) there exist \(\gamma(\mathbf{p}) = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^+\) and a function \(\delta \mapsto \varepsilon(\delta)\) satisfying \(\lim_{\delta \to 0^+} \varepsilon(\delta) = 0\) with the following property. If \(0 \neq f_j \in L^p_j(\mathbb{R}^m)\) and if \(f = (f_j)_{1 \leq j \leq 3}\) satisfies \(|T_{\gamma m}(f)| \geq (1 - \delta)A_p^m |f|_p\) then there exists an ordered triple of Gaussian functions of the form

\[
G_j(x) = c_j e^{-\gamma_j |L(x) - a_j|^2 + ix \cdot b}
\]

where \(0 \neq c_j \in \mathbb{C}\), \(a_j, b \in \mathbb{R}^m\), \(\sum_{j=1}^3 a_j = 0\), and \(L : \mathbb{R}^m \to \mathbb{R}^m\) is a linear automorphism, such that

\[
|f_j - G_j|_{L^p_j} \leq \varepsilon(\delta) |f|_{L^p_j}
\]

for each \(j \in \{1, 2, 3\}\).

The ordered triple \(\gamma(\mathbf{p})\) is independent of \(m\) but is not uniquely determined in this statement; \((t\gamma_1, t\gamma_2, t\gamma_3)\) works equally well for any \(t \in \mathbb{R}^+\) since a common factor can be absorbed into \(L, a_j\). But \(\gamma(\mathbf{p})\) is uniquely determined with the normalization \(\gamma_1(\mathbf{p}) = 1\), which we enforce henceforth.

The second ingredient is a quantitative expression of the unsolvability of a functional equation. In the discussion that follows, \(\mathbb{B}\) always denotes a ball of finite, positive radius centered at the origin in \(\mathbb{R}^d\). \(\mathbb{B}^*\) denotes the ball centered at 0 whose radius is twice that of \(\mathbb{B}\). Sets of Lebesgue measure zero are negligible for all considerations that follow, so we do not distinguish between open and closed balls. The Cartesian product \(\mathbb{B} \times \mathbb{B}\) is denoted by \(\mathbb{B}^2\). The following two lemmas are established in [6].

**Lemma 7.2.** [6] For each \(d \in \mathbb{N}\) there exist \(\delta_0 > 0\) and a function \(t \mapsto \varepsilon(t)\) satisfying \(\lim_{t \to 0^+} \varepsilon(t) = 0\) such that the following conclusion holds. Let \(A \in [0, \infty)\) and \(\delta \in (0, \delta_0]\). Let \(\varphi, \psi : \mathbb{B} \to \mathbb{C}\) and \(\xi : \mathbb{B}^* \to \mathbb{C}\) be Lebesgue measurable. Suppose that

\[
|\varphi(x) + \psi(y) + \xi(x + y)| \leq A
\]

for all \((x, y) \in \mathbb{B}^2\) outside a set of measure \(\leq \delta |B|^2\). Then there exists an affine function \(h\) such that

\[
|\varphi(x) - h(x)| \leq CA
\]

for all \(x \in \mathbb{B}\) outside a set of measure \(\varepsilon(\delta) |\mathbb{B}|\). The constant \(C\) and function \(\varepsilon\) depend only on \(d\).

In particular, the constants in the conclusions do not depend on \(\mathbb{B}\). The following multiplicative variant of Lemma 7.2 is also proved in [6].
Lemma 7.3. For each dimension $d \geq 1$ there exists a constant $K < \infty$ with the following property. Let $B \subset \mathbb{R}^d$ be a ball with positive radius, and let $\eta \in (0, \frac{1}{2})$. For $j \in \{1, 2, 3\}$ let $f_j : 2B \to \mathbb{C}$ be Lebesgue measurable functions that vanish only on sets of Lebesgue measure zero. Suppose that

$$\{ (x, y) \in B^2 : |f_1(x)f_2(y)f_3(x + y)^{-1} - 1| > \delta |B|^2 \}.$$  

Then for each index $j$ there exists a real-linear function $L_j : \mathbb{R}^d \to \mathbb{C}$ such that

$$\{ x \in B : |f_j(x)e^{-L_j(x)} - 1| > K\eta^{1/2} \} \leq K\delta |B|.$$

The next result is concerned with a Heisenberg variant of Lemma 7.2.

Proof of Proposition 7.4. It is given that $\sigma_L(x, y) \leq CA$ for all $x \in \mathbb{B}$ outside a set of measure $o_1(1)|\mathbb{B}|$. Recall that $\sigma_L(x, y) = \sigma(L^{-1}x, L^{-1}y)$. By $\|T\|$ we mean in (7.7) the usual norm $\sup_{T \neq x \in \mathbb{R}^d} \|T(x)/|x|\|$. The main conclusion is that (7.6) cannot hold, unless $L$ satisfies $\inf_{S \in \text{Sp}(2d)} \|SL^{-1}\| = O(|\mathbb{B}|^{-1/2}A^{1/2})$. Moreover, if (7.6) does hold, then $|\sigma_L(x, y)| \leq CA$ for all $(x, y) \in \mathbb{B}^2$; consequently this term can be dropped from (7.6) to yield $|a_1(x) + a_2(y) + a_3(x + y)| \leq CA$. The conclusion (7.8) follows from this by Lemma 7.2.

Proof of Proposition 7.4. It is given that $|a_1(x) + a_2(y) + a_3(x + y) + \sigma_L(x, y)| \leq A$ for all $(x, y) \in \mathbb{B}^2$ outside a set of measure $\leq \eta |\mathbb{B}|^2$. By interchanging the roles of $x, y$, adding the resulting inequality to this one, and invoking the antisymmetry of $\sigma$, we conclude that $|\hat{a}(x) + \hat{a}(y) + a_3(x + y)| \leq A$ for all $(x, y) \in \mathbb{B}$ outside a set of measure $\leq C\eta|\mathbb{B}|$, where $2\hat{a} = a_1 + a_2$. By Lemma 7.2 this implies that there exists an affine function $\psi_3$ such that $|\psi_3(x) - \psi_3(y)|$ for all $x \in \mathbb{B}$ outside a set of measure $\leq C\eta|\mathbb{B}|$.

$\psi_3(x + y)$ can be expressed as an affine function of $x$ plus an affine function of $y$; these functions can be incorporated into $a_1(x), a_2(y)$, respectively. Combining this information with the hypotheses therefore gives

$$\{ x \in \mathbb{B} : |a_1^+(x) + a_3^+(y) + \sigma_L(x, y)| \leq CA$$
for nearly all \((x, y) \in \mathbb{B} \times \mathbb{B}\), where \(a_j^* - a_j\) is affine. Taking first differences with first to \(x\) gives
\[
(7.10) \quad |\Delta_h a_j^*(x) + \sigma(Lh, Ly)| \leq CA
\]
for nearly all \(x, h, y \in \mathbb{B}\) such that \(x, h, x + h, y \in \mathbb{B}\). By specializing to a typical value of \(y\), one finds that there exists a function \(h \mapsto c(h)\) such that \(|\Delta_h a_j^*(x) - c(h)| \leq CA\) for nearly all \(x, h \in \mathbb{B}\) such that \(x + h \in \mathbb{B}\). Therefore by Lemma 7.2 there exists an affine function \(\psi\) such that \(|a_j^* - \psi| \leq CA\) for nearly all points of \(\mathbb{B}\). Since \(a_1 - a_j^*\) is affine, the same conclusion holds for \(a_1\). Interchanging the roles of the variables \(x, y\) in this argument produces the same conclusion for \(a_2\).

Combining these results for all \(a_j\) with the original hypothesis, we conclude that there exists an affine function \(\psi\) of \((x, y)\) such that \(|\psi(x, y) - \sigma(Lx, Ly)| \leq CA\) for nearly every \((x, y) \in \mathbb{B}\). The same must then hold for every \((x, y) \in \mathbb{B}^* \times \mathbb{B}^*\), since \(\psi, \sigma_L\) are polynomials. By applying \(\partial^2/\partial x_i \partial x_j\) for arbitrary indices \(i, j\) and exploiting the affine character of \(\psi\) together with the homogeneous quadratic nature of \(\sigma(Lx, Ly)\) we conclude that \(|\sigma(Lx, Ly)| \leq CA\) for all \((x, y) \in \mathbb{B}\). According to Lemma 10.1 this implies the existence of \(S \in \text{Sp}(2d)\) such that \(\|SL^{-1}\| \leq CA^{1/2}\).

\section{8. Proof of Theorem 2.22 for nonnegative functions}

Let \(p\) be an admissible ordered triple of exponents in \((1, \infty)^3\), and let \(\delta > 0\) be small. Let \(f_j \in L^{p_j}(\mathbb{H}^d)\) for \(j \in \{1, 2, 3\}\) satisfy \(\|f_j\|_{p_j} = 1\), as we may suppose without loss of generality. Set \(f = (f_1, f_2, f_3)\). Assume that each \(f_j \geq 0\), and suppose that
\[
\mathcal{T}_{\mathbb{H}^d}(f) \geq (1 - \delta)A_p^{2d+1}\|f\|_p = (1 - \delta)A_p^{2d+1}.
\]
Let \(\gamma = \gamma(p) = (\gamma_1, \gamma_2, \gamma_3)\) with \(\gamma_1 = 1\).

Define \(F_j : \mathbb{R}^{2d} \to [0, \infty]\) and \(f_{j,x} : \mathbb{R}^1 \to [0, \infty]\) as in (5.2); however, the definition of \(f_{j,x}\) will be modified below, for those \(x\) for which \(f(x, t)\) vanishes for almost every \(t\). Set \(F = (F_1, F_2, F_3)\). For \(x \in (\mathbb{R}^{2d})^3\) define
\[
(8.1) \quad f_{j,x}^\dagger(s) = f_3(x_3, s + \sigma(x_1, x_2));
\]
as in (5) this definition will only be relevant when \(x_3 = -x_1 - x_2\). Define a measure \(\nu_F\) on \((\mathbb{R}^{2d})^3\), supported on \(A_{\mathbb{R}^{2d}}\), by
\[
(8.2) \quad d\nu_F(x) = \prod_{j=1}^3 F_j(x_j) \, d\lambda_{\mathbb{R}^{2d}}(x),
\]
where \(A_{\mathbb{R}^{2d}}\) is the natural 4\(d\)–dimensional Lebesgue measure on \(A_{\mathbb{R}^{2d}}\) introduced above. Since \(\|F_j\|_{p_j} = \|f_j\|_{p_j} = 1\) and \(p\) is admissible, Young’s inequality for \(\mathbb{R}^{2d}\) guarantees that \(\nu_F(\mathbb{R}^{2d} \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}) \leq A_p^{2d}\).

\textbf{Lemma 8.1.} For each \(d \geq 1\) and each admissible ordered triple \(p\) there exists \(C < \infty\) with the following property. Let \(f_j \in L^{p_j}(\mathbb{H}^d)\) be nonnegative and satisfy \(\|f_j\|_{p_j} = 1\) for each \(j \in \{1, 2, 3\}\). Let \(\delta > 0\). If \(\mathcal{T}_{\mathbb{H}^d}(f) \geq (1 - \delta)A_p^{2d+1}\) then
\[
(8.3) \quad \mathcal{T}_{\mathbb{R}^{2d}}(F) \geq (1 - \delta)A_p^{2d}
\]
and there exists a set \(E \subset A_{\mathbb{R}^{2d}}\) satisfying
\[
(8.4) \quad \nu_F(E) \leq C\delta^{1/2}
\]
such that for every \( x \in \Lambda_{\mathbb{R}^{2d}} \setminus E \),
\[
\begin{aligned}
F_j(x_j) &\neq 0 \text{ for each } j \in \{1, 2, 3\}, \\
T_{\mathbb{R}^1}(f_{1,x_1}, f_{2,x_2}, f_{3,x_3}) &\geq (1 - o_\delta(1)) A_p.
\end{aligned}
\] (8.5)

A proof of Lemma 8.2 is implicit in the proof in \[\text{[6]}\] that the optimal constant in Young’s inequality for \( \mathbb{H}^d \) does not exceed the optimal constant for \( \mathbb{R}^{2d+1} \). Details are left to the reader.

According to Theorem \[\text{[7]}\] there exists an ordered triple \( G = (G_1, G_2, G_3) \) of Gaussians \( G_j : \mathbb{R}^{2d} \to \mathbb{C} \) that extremizes Young’s convolution inequality for \( \mathbb{R}^{2d} \), of the form
\[
G_j(x) = c_j |\det(L)|^{1/p_j} e^{-\gamma_j |L(x-a_j)|^2},
\]
where \( \gamma = \gamma(p) \), \( a_1 + a_2 + a_3 = 0 \), \( c_j > 0 \), and \( L \) is an invertible linear endomorphism of \( \mathbb{R}^{2d} \), such that \( \|F_j - G_j\|_{L^{p_j}(\mathbb{R}^{2d})} = o_\delta(1) \). The constants \( c_j \) are determined by requiring that \( \|G_j\|_{p_j} = 1 \), as we may require with no loss of generality since \( \|F_j\|_{p_j} = 1 \). Exponential factors \( e^{ix \cdot b_j} \) appear in the conclusion of Theorem \[\text{[7]}\] but can be dropped; since \( F_j \geq 0 \) by its definition, \( |G_j| \) is at least as accurate an approximation to \( F_j \) in \( L^{p_j} \) norm as is \( G_j \).

Define an ordered triple of diffeomorphisms of \( \mathbb{H}^d \) by
\[
(\psi_1(z_1), \psi_2(z_2), \psi_3(z_3)) = (z_1 u, u^{-1}z_2 v, v^{-1}z_3)
\]
where \( u = (-a_1, 0) \) and \( v = (-a_1 - a_2, 0) \). Then \( v^{-1} = (a_1 + a_2, 0) = (-a_3, 0) \). The triple \( \Psi = (\psi_j)_{1 \leq j \leq 3} \) is an element of \( \mathcal{G}(\mathbb{H}^d) \), so upon replacement of \( f_j \) by \( f_j \circ \psi_j \) all of the assumptions and conclusions above are unaffected, and we gain the simplification
\[
G_j(x) = c_j |\det(L)|^{1/p_j} e^{-\gamma_j |Lx|^2}.
\]

**Lemma 8.2.** Let \( f, L, G_j \) be as above. There exist \( \lambda \in \mathbb{R}^+ \), \( S \in \text{Sp}(2d) \), positive scalars \( c_j \), a set \( E' \subset \Lambda_{\mathbb{R}^{2d}} \), affine mappings \( \varphi_j : \mathbb{R}^{2d} \to \mathbb{R} \), and Lebesgue measurable functions \( h_j : \mathbb{R}^{2d} \to [0, \infty) \) of the form
\[
h_j(x, t) = c_j e^{-\lambda \gamma_j (t - \varphi_j(x))^2}
\]
such that \( h_j(x, t) = h_j(x, t) \) satisfy the following conclusions:
\[
\begin{aligned}
\|h_{j,x}\|_{L^{p_j}(\mathbb{R})} &\equiv 1 \text{ for every } x \in \mathbb{R}^{2d} \tag{8.7} \\
\nu_{E'}(E') &\leq o_\delta(1) \tag{8.8} \\
\|f_{j,x} - h_{j,x}\|_{p_j} &\leq o_\delta(1) \text{ for each } j \in \{1, 2, 3\} \text{ for all } x \in \Lambda_{\mathbb{R}^{2d}} \setminus E' \tag{8.9} \\
\|SL^{-1}\| &\leq o_\delta(1) \lambda^{-1/4} \tag{8.10} \tag{8.11}
\end{aligned}
\]
\( \varphi_1(x_1) + \varphi_2(x_2) + \varphi_3(x_3) \equiv 0 \) whenever \( x \in \Lambda_{\mathbb{R}^{2d}} \).

Here \( F_j \) is associated to \( f_j \) as indicated above, and \( (\gamma_1, \gamma_2, \gamma_3) = \gamma(p) \).

**Proof.** Temporarily make the change of variables \( (x, s) \mapsto (y, t) \) in \( \mathbb{H}^d \), with
\[
y = L(x) \text{ and } t = s.
\] (8.12)

We make this same change of variables for each index \( j \in \{1, 2, 3\} \). The resulting diffeomorphism of \( (\mathbb{H}^d)^3 \) corresponds to an element of \( \mathcal{G}(\mathbb{H}^d) \) if and only if \( L \in \text{Sp}(2d) \), which need not hold. So we will revert to the original coordinates after exploiting these new coordinates.

Set
\[
f_j(y_j, t) = f_j(L^{-1}y_j, t),
\] (8.13)
and of course $\tilde{f}_{j,y}(t) = \tilde{f}_j(y,t)$. In these modified coordinates and for these modified functions, the conclusions of Lemma 8.1 coupled with the approximations $\|F_j - G_j\|_{p_j} = o_\delta(1)$, can be stated as follows. Set

\begin{equation}
G_j(y) = c_j e^{-\gamma_j \|y\|^2}.
\end{equation}

\begin{equation}
d\nu_G(y) = \prod_{j=1}^3 G_j(y) \, d\lambda_{\mathbb{R}^{2d}}(y)
\end{equation}

\begin{equation}
\tilde{f}_{3,y}(s) = \tilde{f}_{3,y}(s + \sigma_L(y_1,y_2))
\end{equation}

Recall the notation $\sigma_L(y_1,y_2) = \sigma(L^{-1}y_1,L^{-1}y_2)$. By Lemma 8.1 since $\sum_{j=1}^3 p - j^{-1} = 2$, there is a set $E \subset \Lambda_{\mathbb{R}^{2d}}$ satisfying $\nu_G(E) = o_\delta(1)$ such that

\begin{equation}
\mathcal{T}_{\mathbb{R}^1}(\tilde{f}_{1,y_1}, \tilde{f}_{2,y_2}, \tilde{f}_{3,y}) \geq (1 - o_\delta(1))A_p \quad \text{for all } y = (y_1, y_2, y_3) \in \Lambda_{\mathbb{R}^{2d}} \setminus E.
\end{equation}

Moreover, $\|f_{j,y} \|_{p_j} = 1$ for each $j \in \{1, 2, 3\}$ whenever $y \in \Lambda_{\mathbb{R}^{2d}} \setminus E$.

Let $\delta \to \rho(\delta)$ be a function that tends to infinity slowly as $\delta \to 0^+$, to be chosen below. This function may also depend on $d, p$ but is independent of $f$. Define $B$ to be the closed ball of radius $\rho(\delta)$ centered at the origin in $\mathbb{R}^{2d}$. The $L^p$ norm of $G_j$ on the complement of $B$ is $o_\delta(1)$ since $\lim_{\delta \to 0} \rho(\delta) = \infty$. $G_j$ is bounded above uniformly in $\delta$, and on $B^*$, and is bounded below by $ce^{-C\rho(\delta)^2}$. Thus by (8.17), under the convention that $y = (y_1, y_2, y_3)$ is regarded as a function $y(y_1, y_2)$ of $(y_1, y_2)$ via the relation $y_3 = -y_1 - y_2$, (8.17) holds for all $(y_1, y_2) \in B \times \mathbb{R}^d$ outside a set of Lebesgue measure $\leq \nu_G(E) c^{-1} e^{C\rho(\delta)^2}$. Choose $\rho(\delta)$ to tend to infinity so slowly that this product is $\leq o_\delta(1)$ and hence, since $|B| \to \infty$ as $\rho \to \infty$, is $\leq o_\delta(1)|B|^2$. This is possible because $\nu_G(E) = o_\delta(1)$ tends to zero at a rate that depends on $\delta, p, d$ but is otherwise independent of $f$ and of the choice of $\rho(\delta)$.

By (8.17) and Theorem 7.1 for each $j \in \{1, 2, 3\}$, for all $y_j \in B$ outside a set whose Lebesgue measure is $o_\delta(1)|B|$, there exists a positive Gaussian function $\mathbb{R}^1 \ni t \mapsto g_{j,y}(t)$ satisfying $\|f_{j,y} - g_{j,y}\|_{p_j} \leq o_\delta(1)$. These functions can be chosen to depend Lebesgue measurably on the parameters $y_j$.

Write $g_{j,y}(t) = c_j \gamma_j e^{-\lambda_j(y)(t-\alpha_j(y))^2}$ where $\lambda_j, c_j, \alpha_j$ are measurable functions with domains $\mathbb{R}^{2d}$; $\lambda_j, c_j$ take values in $(0, \infty)$ and $\alpha_j$ takes values in $\mathbb{R}^1$. For all $y_j \in B$ outside a set of Lebesgue measure $\leq o_\delta(1)|B|$, $\|\tilde{f}_{j,y}\|_{p_j} = 1$. Therefore $(g_{1,y_1}, g_{2,y_2}, g_{3,-y_1-y_2})$ nearly extremizes Young's inequality for $\mathbb{R}^1$, for all $(y_1, y_2) \in B^2$ outside a set of Lebesgue measure $\leq o_\delta(1)|B|^2$.

A first consequence of this near extremality is that

\begin{equation}
\left| \frac{\lambda_i(y_i)}{\lambda_j(y_j)} - \frac{\gamma_i}{\gamma_j} \right| = o_\delta(1)
\end{equation}

for all $(y_1, y_2, y_3) \in B^3$ outside a set of Lebesgue measure $o_\delta(1)|B|^2$ for all indices $i, j \in \{1, 2, 3\}$, where $y_3$ continues to be defined to be $-y_1 - y_2$. Therefore there exists $\lambda \in \mathbb{R}^+$ such that

\begin{equation}
\lambda_j(y) = \lambda \cdot (\gamma_j + o_\delta(1)) \quad \text{for each index } j \in \{1, 2, 3\},
\end{equation}

for all $y \in B$ outside a set of Lebesgue measure $o_\delta(1)|B|$. Thus for each $j \in \{1, 2, 3\}$,

\begin{equation}
|g_{j,y}(t) - c_j e^{-\lambda_j(y)(t-\alpha_j(y))^2}| \leq o_\delta(1)
\end{equation}
in $L^p_j(\mathbb{R}^1)$ norm, for every $y \in \mathbb{B}$ outside a set of Lebesgue measure $o_\delta(1)|\mathbb{B}|$. The coefficients $c_j'$ are now constants, rather than functions of $y \in \mathbb{R}^d$.

In order for $(g_{1,y_1}, g_{2,y_2}, g_{3,y_3})$, with $g_{i,y_i}$ of the form (8.20) and $y_3 = y_3(y_1, y_2) = -y_1 - y_2$, to $(1 - o_\delta(1))$–nearly extremize Young’s inequality for $\mathbb{R}^1$ for every $(y_1, y_2) \in \mathbb{B}^2$ outside a set of Lebesgue measure $o_\delta(1)|\mathbb{B}|^2$, it is necessary that

\begin{equation}
\tag{8.21}
\alpha_1(y_1) + \alpha_2(y_2) + \alpha_3(-y_1 - y_2) + \sigma_L(y_1, y_2) \leq \lambda^{-1/2} \cdot o_\delta(1)
\end{equation}

for all $(y_1, y_2) \in \mathbb{B}^2$ outside a set of Lebesgue measure $o_\delta(1)|\mathbb{B}|^2$. By Proposition 7.3, this implies the existence of affine functions $\varphi_j : \mathbb{R}^d \to \mathbb{R}$ satisfying for each $j \in \{1, 2, 3\}$

\begin{equation}
\tag{8.22}
|\alpha_j(y) - \varphi_j(y)| \leq o_\delta(1) \cdot \lambda^{-1/2}
\end{equation}

for all $y \in \mathbb{B}$ outside a set of Lebesgue measure $o_\delta(1)|\mathbb{B}|$, and satisfying

$$\varphi_1(x_1) + \varphi_2(x_2) + \varphi_3(-x_1 - x_2) \equiv 0.$$ 

Moreover, there exists $S \in \text{Sp}(2d)$ such that

\begin{equation}
\tag{8.23}
\|SL^{-1}\| \leq \lambda^{-1/4}o_\delta(1)|\mathbb{B}|^{-1/2d} \leq \lambda^{-1/4}o_\delta(1).
\end{equation}

Equivalently, $L = \tilde{L} \circ S$ where $\tilde{L}$ satisfies a lower bound

\begin{equation}
\tag{8.24}
|\tilde{L}(v)| \geq \lambda^{1/4}q(\delta)^{-1}|v|
\end{equation}

uniformly for all $0 \neq v \in \mathbb{R}^{2d}$, where $\eta(\delta) \to 0$ as $\delta \to 0$. These properties of $L$ will be exploited below.

Define Gaussian functions

\begin{equation}
\tag{8.25}
\tilde{g}_{j,y}(t) = c_j' e^{-\lambda_\gamma_j(t - \varphi_j(y))^2}.
\end{equation}

(8.22) implies that $\|\tilde{f}_{j,y} - \tilde{g}_{j,y}\|_{p_j} \leq o_\delta(1) = o_\delta(1)\|\tilde{f}_{j,y}\|_{p_j}$ for all $y \in \Lambda_{\mathbb{R}^{2d}} \setminus E'$, with $\nu_{\mathcal{T}}(E') = o_\delta(1)$. A consequence, since $\tilde{G}_j \in L^1$, is that

\begin{equation}
\tag{8.26}
\|\tilde{f}_{j,y}(t)F_j(y) - \tilde{g}_{j,y}(t)\tilde{G}_j(y)\|_{L^p_j(\mathbb{R} \times \mathbb{R}, dy \, dt)} \leq o_\delta(1)
\end{equation}

for each $j \in \{1, 2, 3\}$. Therefore

\begin{equation}
\tag{8.27}
\|\tilde{f}_{j,y}(t)F_j(y) - \tilde{g}_{j,y}(t)\tilde{G}_j(y)\|_{L^p_j(\mathbb{R}^{2d} \times \mathbb{R}, dy \, dt)} \leq o_\delta(1).
\end{equation}

Returning to the original coordinates $(x, t)$ for $\mathbb{H}^d$, define

\begin{equation}
\tag{8.28}
\tilde{h}_{j}(x, t) = \tilde{g}_{j,y}(t) = \tilde{g}_{j,L(x)}(t) = c_j' e^{-\lambda_\gamma_j(t - \varphi_j \circ L(x))^2}.
\end{equation}

The next step is to simplify matters by exploiting symmetries. We apply in sequence two elements $\Psi \in \mathcal{S}(\mathbb{H}^d)$. The first is $\Psi = (\psi_1, \psi_2, \psi_3)$, with $\psi_j(x_j, t_j) = t_j - \varphi_j \circ L(x_j)$. The second takes the form $\psi_j(x, t) = (S(x), t)$, where $S$ is as in (8.21). Replace $f_j$ by $f_j \circ \psi_j$ for each of these in turn, continuing to denote by $f_j$ the resulting functions and by $F_j$ the associated functions with domains $\mathbb{R}^{2d}$. Likewise compose $\tilde{h}_j$ with each of these in turn, and denote by $\tilde{h}_{j}^2$ the resulting composed functions. Matters are thereby reduced to the situation in which

\begin{align*}
\tilde{h}_{j,x}^2(t) &= c_j e^{-\lambda_\gamma_j t^2}, \\
F_j(x) &= c_j e^{-\gamma_j |L(x)|^2}, \\
\|f_{j,x} - \tilde{h}_{j,x}^2\|_{p_j} &\leq o_\delta(1) \quad \forall x \in \Lambda_{\mathbb{R}^{2d}} \setminus E''
\end{align*}

where $E'' \subset \Lambda_{\mathbb{R}^{2d}}$ satisfies $\nu_{\mathcal{T}}(E'') \leq o_\delta(1)$ and $\tilde{L}, \lambda$ are related by (8.24).
The next reduction is an automorphic change of variables in $\mathbb{H}^d$ of the form

$$(x,t) \mapsto (\psi(x,t), y) = (\eta(\delta)^{-1} y, \eta(\delta)^{-2} \lambda^{1/2} t),$$

where $\eta(\delta)$ is the function introduced in (9.3). Setting $\psi_j = \psi$ for all three indices $j$ defines an element $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \mathcal{G}(\mathbb{H}^d)$. In these new coordinates, the conclusion is that $\|f_j - f_j^*\|_{p_j} \leq o_6(1)$ where

$$f_j^*(z,r) = c_j e^{-\gamma j |L'z|^2} e^{-\gamma j \epsilon t^2},$$

where $L' : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ is linear and satisfies $|L'z| \geq |z|$ for all $z \in \mathbb{R}^d$, and $\epsilon \leq \epsilon(\delta)$ where $\epsilon(\delta)$ tends to 0 as $\delta \to 0$, and depends also on $p, d$ as well as on $\delta$, but not otherwise on $f$. This completes the analysis of nonnegative near-extremizers $f$. \hfill \Box

9. The complex-valued case

Let $\delta > 0$ be small, and consider an arbitrary complex-valued $f = (f_1, f_2, f_3)$ satisfying $\|f_j\|_{p_j} \neq 0$ for each index $j$, and $|T_{\mathbb{H}^d}(f)| \geq (1 - \delta) A_{p}^{2d+1} \|f\|_p$. Since $T_{\mathbb{H}^d}(|f_1|, |f_2|, |f_3|) \geq |T_{\mathbb{H}^d}(f)|$, we may apply the result proved above for nonnegative near-extremizers to conclude that there exists $\Psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{G}(\mathbb{H}^d)$ such that for each $j \in \{1, 2, 3\}$,

$$\left\{ \begin{array}{l}
\|f_j \circ \psi_j - G_j\|_{p_j} \leq o_6(1) \|f_j \circ \psi_j\|_{p_j} \\
G_j(x,t) = c_j e^{-\gamma j |Lx|^2} e^{-\gamma j \epsilon t^2}
\end{array} \right.$$

where $c_j \asymp 1$, $|Lx| \geq |x|$ for all $x \in \mathbb{R}^{2d}$, and $\epsilon \leq o_6(1)$. By replacing $f_j$ by $f_j \circ \psi_j$ multiplied by an appropriate normalizing constant factor, we may also assume that $\|f_j\|_{p_j} = 1$ and then likewise that $\|G_j\|_{p_j} = 1$.

Write $f_j = e^{i\alpha_j} |f_j|$ where $\alpha_j : \mathbb{H}^d \to \mathbb{R}$ is measurable. We seek to analyze the factors $e^{i\alpha_j}$. Since $\|f_j - e^{i\alpha_j} G_j\|_{p_j} = \|f_j| G_j\|_{p_j} \leq o_6(1)$,

$$|T_{\mathbb{H}^d}(e^{i\alpha_1} G_1, e^{i\alpha_2} G_2, e^{i\alpha_3} G_3)| \geq (1 - o_6(1)) A_{p}^{2d+1}.$$

Thus it suffices to prove that $(e^{i\alpha_j} G_j : 1 \leq j \leq 3)$ satisfies the conclusions of Theorem 2.2.

So we redefine $f_j$ to be $e^{i\alpha_j} G_j$ henceforth.

By multiplying these functions by unimodular constants, we may assume without loss of generality that $T_{\mathbb{H}^d}(f)$ is real and positive. Since then $\text{Re} T_{\mathbb{H}^d}(f_1, f_2, f_3) \geq (1 - o_6(1)) T_{\mathbb{H}^d}(|f_1|, |f_2|, |f_3|)$,

$$(9.1) \quad |\prod_{j=1}^{3} e^{i\alpha_j(z_j)} - 1| = o_6(1) \quad \text{for all } z \in (\mathbb{H}^d)^3 \text{ outside a set satisfying } \nu_G(E) \leq o_6(1)$$

where $d\nu_G(z) = \prod_j G_j(z_j) d\lambda_{\mathbb{H}^d}(z)$. 

Let $\rho = \rho(\delta)$ be a positive quantity that tends to infinity slowly as $\delta \to 0$ and is to be chosen below, and let $\mathbb{B} \subset \mathbb{R}^{2d}$ be the ball of radius 1 centered at 0. By (9.2),

$$(9.2) \quad |e^{i\alpha_1(L^{-1} y_1, t_1)} e^{i\alpha_2(L^{-1} y_2, t_2)} e^{i\alpha_3(-L^{-1} y_1 - L^{-1} y_2, -t_1 - t_2 - \sigma L(y_1, y_2))} - 1| \leq o_6(1)$$

for all $((y_1, t_1), (y_2, t_2)) \in (\mathbb{B} \times [-\rho^{-1/2}, \rho^{-1/2}])^2$ outside a set of Lebesgue measure less than or equal to $o_6(1) \cdot \epsilon^{-1}$ provided that the function $\rho$ is chosen so that $\rho(\delta) \to \infty$ sufficiently slowly as $\delta \to 0$. Therefore according to Lemma 7.3, for each index $j$, $\alpha_j$ takes the form

$$(9.3) \quad e^{i\alpha_j(L^{-1} y, t)} = e^{i(a_j(y)t + b_j(y) + o_6(1))}$$

for $y \in \mathbb{B}$ and $|t| \leq \rho(\delta) \epsilon^{-1/2}$ outside a set of Lebesgue measure $o_6(1) \epsilon^{-1/2}$. The coefficients $a_j, b_j$ are real-valued measurable functions.
We will use informal language “for nearly all $y \in \mathbb{B}$” to indicate a Lebesgue measurable subset $A \subset \mathbb{B}$ satisfying $|A| \leq o_{\delta}(1)|\mathbb{B}|$, where the quantity $o_{\delta}(1)$ depends on $\delta$, and tends to 0 as $\delta \to 0$ while $p, d$ remain fixed. “Nearly all $(y_1, y_2) \in \mathbb{B}^2$” has a corresponding meaning.

Invoking (9.3) together with (9.2) for typical $(t_1, t_2)$ and also for typical $(t'_1, t'_2)$ satisfying $|t_j|, |t'_j| \leq \rho(\delta)\varepsilon^{-1/2}$, considering products of the exponential factors, and setting $u_j = t'_j - t_j$ gives

$$
|e^{iu_1a_1(L^{-1}y_1)}e^{iu_2a_2(L^{-1}y_2)}e^{-i(u_1+u_2)a_3(-L^{-1}y_1-L^{-1}y_2)} - 1| \leq o_{\delta}(1)
$$

for nearly all $(y_1, y_2) \in \mathbb{B}^2$ and nearly all $(u_1, u_2)$ satisfying $|u_j| \leq \frac{1}{\rho(\delta)}\varepsilon^{-1/2}$ outside a set of Lebesgue measure $o_{\delta}(1)\varepsilon^{-1}$. The advantage of (9.4) over (9.2) is that $b_j$ and $\sigma_L$ have been eliminated.

This last inequality can be equivalently written

$$
|e^{iu_1[a_1(L^{-1}y_1)-a_3(-L^{-1}(y_1-y_2))]}e^{iu_2[a_2(L^{-1}y_1)-a_3(-L^{-1}(y_1-y_2))]} - 1| \leq o_{\delta}(1).
$$

By Lemma 12.1 below, (9.5) implies that

$$
|a_1(L^{-1}y_1) - a_3(-L^{-1}(y_1-y_2))| \leq o_{\delta}(1)\varepsilon^{1/2}
$$

for nearly all $(y_1, y_2) \in \mathbb{B}^2$. Note that unlike the functions $\alpha_j$, which are only determined up to addition of arbitrary measurable functions taking values in $2\pi\mathbb{Z}$, the constituent parts $a_j$ can be pinned down as $\mathbb{R}$-valued, rather than $\mathbb{R}/2\pi\mathbb{Z}$-valued, functions.

Therefore there exists a real number $\tilde{a}$ such that $|a_j(L^{-1}y) - \tilde{a}| \leq o_{\delta}(1)\varepsilon^{1/2}$ for nearly all $y \in \mathbb{B}$ for $j = 1, 3$. The same reasoning gives the same conclusion for $j = 2$. Thus for each $j \in \{1, 2, 3\}$,

$$
e^{ia_j(L^{-1}y,t)} = e^{i\tilde{a}t}e^{ib_j(y)} + o_{\delta}(1)
$$

for all $(y, t) \in \mathbb{B} \times [-\rho(\delta)\varepsilon^{-1/2}, \rho(\delta)\varepsilon^{-1/2}]$ outside a set of Lebesgue measure $o_{\delta}(1)\varepsilon^{-1/2}$. Thus

$$
\|e^{ia_j(x,t)}G_j(x,t) - e^{i(\tilde{a}t+b_j(L_j(x)))}G_j(x,t)\|_{L^p(\mathbb{H}^d)} \leq o_{\delta}(1)\|f_j\|_{p_j};
$$

so we may replace $a_j(x,t)$ by $\tilde{a}t + b_j(L_j(x))$.

Inserting this into (9.4) gives

$$
e^{ib_1(L^{-1}y_1)}e^{ib_2(L^{-1}y_2)}e^{ib_3(-L^{-1}y_1-L^{-1}y_2)}e^{-i\tilde{a}\sigma_L(y_1, y_2)} - 1 \leq o_{\delta}(1)
$$

for nearly all $(y_1, y_2) \in \mathbb{B}^2$. From the antisymmetry of $\sigma_L$ it follows that

$$
e^{ib_1+b_2(L^{-1}y_1)}e^{ib_1+b_2(L^{-1}y_2)}e^{ib_3(-L^{-1}y_1-L^{-1}y_2)} - 1 \leq o_{\delta}(1)
$$

for nearly all $(y_1, y_2) \in \mathbb{B}^2$; this can be deduced by interchanging $y_1$ with $y_2$ and considering the product of the two resulting left-hand sides of (9.9).

According to Lemma 7.3 the functions $e^{ib_1+\sigma L^{-1}}$ and $e^{ib_1+\sigma L^{-1}}$ nearly agree with exponentials of imaginary affine functions, at nearly all points of $\mathbb{B}$. Since

$$
e^{ib_1(L^{-1}y_1)}e^{ib_2(L^{-1}y_2)}e^{ib_3(-L^{-1}y_1-L^{-1}y_2)}e^{-i\tilde{a}\sigma_L(y_1, y_2)} - 1 \leq o_{\delta}(1)
$$

for nearly all $(y_1, y_2) \in \mathbb{B}^2$ by (9.9), it follows by invoking this information for $b_3$ that

$$e^{ib_1(L^{-1}y_1)}e^{ib_2(L^{-1}y_2)}e^{-i\tilde{a}\sigma_L(y_1, y_2)}
$$
is nearly equal to the exponential of an imaginary affine function of \((y_1, y_2)\), at nearly all points of \(\mathbb{B}^2\).

Next consider the ratio
\[
\frac{e^{i2b_1(L^{-1}y_1)} e^{i2b_2(L^{-1}(u+y_2))} e^{-i2\alpha_L(y_1,u+y_2)}}{e^{i2b_1(L^{-1}y_1)} e^{i2b_2(L^{-1}y_2)} e^{-i2\alpha_L(y_1,y_2)}} = e^{i2b_2(L^{-1}(u+y_2))} e^{-i2b_2(L^{-1}(y_2))} e^{-i2\alpha_L(y_1,u)}.
\]

From the conclusion of the preceding paragraph one can deduce that the right-hand side of (9.12) nearly coincides with the exponential of an imaginary affine function of \(u\) alone, at nearly all points \((y_1, y_2, u)\) with \(y_1 \in \mathbb{B}\) and \(y_2, u \in \frac{1}{2}\mathbb{B}\). On the right-hand side, only the last exponential factor depends on \(y_1\), so by regarding this quantity as a function of \(y_1\) we conclude that \(|\tilde{a}| \cdot |\sigma_L(v, u)| \leq \sigma_\delta(1)\) for nearly all \((v, u) \in (\frac{1}{2}\mathbb{B})^2\). Therefore
\[
|\tilde{a}| \cdot \sup_{|x|, |y| \leq 1} |\sigma_L(x, y)| \leq \sigma_\delta(1).
\]

Therefore by Lemma 10.1 below, there exists \(S \in \text{Sp}(2d)\) such that \(|\tilde{a}| \cdot ||S^{-1}L||^2 \leq \sigma_\delta(1)\).

Combining this with (9.9) yields
\[
\left| e^{ib_1(L^{-1}y_1)} e^{ib_2(L^{-1}y_2)} e^{ib_3(-L^{-1}y_1-L^{-1}y_2)} - 1 \right| \leq \sigma_\delta(1)
\]
for nearly all \((y_1, y_2) \in \mathbb{B}^2\). By Lemma 10.3 for each \(j \in \{1, 2, 3\}\) there exists an affine function \(L_j : \mathbb{R}^{2d} \rightarrow \mathbb{R}\) such that
\[
\left| e^{ib_j(L^{-1}y)} - e^{iL_j(y)} \right| \leq \sigma_\delta(1)
\]
for nearly all \(y \in \mathbb{B}\). Thus
\[
e^{i\alpha_j(L^{-1}y, t)} = e^{i\tilde{a}t} e^{iL_j(y)} + \sigma_\delta(1)
\]
for \((y, t) \in \mathbb{B} \times \mathbb{R}\) satisfying \(|t| \leq \rho(\delta)\varepsilon^{-1/2}\) outside a set of Lebesgue measure \(\leq \sigma_\delta(1)\varepsilon^{-1/2}\), where \(\tilde{a}\) satisfies (9.13).

This concludes the proof of Theorem 2.2 in the general complex-valued case. \(\square\)

10. Some matrix algebra

**Lemma 10.1.** For any invertible linear endomorphism \(L : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}\),
\[
||L^*JL||^{1/2} = \inf_{S \in \text{Sp}(2d)} ||S^{-1}L||.
\]

**Proof.** That \(||L^*JL|| \leq \inf_{S \in \text{Sp}(2d)} ||S^{-1}L||^2\) is immediate. For any \(L\) and any \(S \in \text{Sp}(2d)\),
\[
||L^*JL|| = ||(S^{-1}L)^* S^* JS(S^{-1}L)|| = ||S^{-1}L^* J(S^{-1}L)|| \leq ||S^{-1}L|| ||J|| ||S^{-1}L|| = ||S^{-1}L||^2.
\]

To establish the reverse inequality, note that since \(L^*JL\) is a nonsingular antisymmetric real matrix, its eigenvalues are imaginary, and come in conjugate pairs; if \(i\lambda\) is an eigenvalue then \(\lambda \neq 0\) and \(-i\lambda\) is also an eigenvalue, and the eigenspace associated to \(-i\lambda\) has the same dimension as the eigenspace associated to \(i\lambda\); coordinatewise complex conjugation interchanges these two eigenspaces. Therefore \(L^*JL\) can be written in the form \(O_1^* KO_1\) where \(O_1 \in O(2d)\) and \(K\) takes the form
\[
K = \begin{pmatrix}
0 & t_1 & 0 & 0 & \cdots & 0 \\
-t_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & t_2 & \cdots & 0 \\
0 & 0 & -t_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]
with $2 \times 2$ blocks $\begin{pmatrix} 0 & t_j \\ -t_j & 0 \end{pmatrix}$ along the diagonal, where $t_j \in \mathbb{R}^+$ and the eigenvalues are $\pm it_j$. Now $t_j \leq \|L^*JL\|$. Defining

$$T = \begin{pmatrix} t_1^{1/2} & 0 & 0 & 0 & \cdots \\ 0 & t_1^{1/2} & 0 & 0 & \cdots \\ 0 & 0 & t_2^{1/2} & 0 & \cdots \\ 0 & 0 & 0 & t_2^{1/2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

(10.3)

gives

$$K = T^* \tilde{J}T$$

(10.4)

where

$$\tilde{J} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

(10.5)

with $2 \times 2$ blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal. Now $\tilde{J} = O_2^*JO_2$ for an appropriate permutation matrix $O_2 \in O(2d)$ and thus we have

$$L^*JL = M^*JM$$

(10.6)

where $M = O_2TO_1$. Equivalently,

$$(LM^{-1})^*J(LM^{-1}) = J,$$

(10.7)

so $LM^{-1} \in \text{Sp}(2d)$. That is, $L = SM$ where $S \in \text{Sp}(2d)$. Equivalently, $M = S^{-1}L$ satisfies

$$\|M\| = \|O_2TO_1\| \leq \|O_2\||\|T\||\|O_1\| = \|T\| = \|L^*JL\|^{1/2},$$

as required. □

11. Integration of difference relations

In this section we establish Theorem 3.1 which is motivated by considerations that have arisen in this paper, but on which the main theorems do not rely. This is done in the hope that it will prove useful in other problems. We continue to use the expressions “nearly every” and “nearly all points” in the same sense as in §9.

Before embarking on the core of the proof of Theorem 3.1 we introduce several simplifications. Firstly, it suffices to prove this in the case in which $B$ is centered at 0, for the hypotheses and conclusions are invariant under translation. Second, it suffices to prove this for the ball $B$ centered at 0 of radius 1. For if the result holds for some ball centered at 0, then it holds uniformly for all such balls, because the hypotheses and conclusions

**Lemma 11.1.** Let $d, m \in \mathbb{N}$. Let $q(x,y) = \sum_{0 \leq |\alpha| \leq m} a_\alpha(y)x^\alpha$ where $a_\alpha$ are Lebesgue measurable functions. Suppose that $|q(x,y)| \leq 1$ for nearly every $(x,y) \in B \times \tilde{B}$. Then for any multi-index $\beta$ satisfying $0 \leq |\beta| \leq m$, $|a_\beta(y)| \leq C$ where $C < \infty$ depends only on $m, d$. 

Before embarking on the core of the proof of Theorem 3.1 we introduce several simplifications. Firstly, it suffices to prove this in the case in which $B$ is centered at 0, for the hypotheses and conclusions are invariant under translation. Second, it suffices to prove this for the ball $B$ centered at 0 of radius 1. For if the result holds for some ball centered at 0, then it holds uniformly for all such balls, because the hypotheses and conclusions
are invariant under dilations. Thirdly, it suffices to prove the theorem for \( A = 1 \), since hypotheses and conclusions are invariant under multiplication of \( \varphi \) by positive scalars, and the case \( A = 0 \) follows from the case \( A > 0 \) with uniform bounds by a straightforward limiting argument. Fourthly, assuming \( \mathbb{B} \) to be centered at the origin, it suffices to prove that there exists \( \rho > 0 \), depending only on \( d, D \), such that the conclusion holds for all \( x \in \rho \mathbb{B} = \{ \rho y : y \in \mathbb{B} \} \) outside a set of measure \( \varepsilon \rho^d |\mathbb{B}| \). Indeed, the full conclusion for \( \mathbb{B} \) itself then follows by combining this weaker conclusion with a Whitney decomposition of \( \mathbb{B} \), as in [6]. One arranges that each Whitney cube \( Q_k \) is contained in a ball \( B_k \) of comparable diameter, such that the ball \( B_k^* \) concentric with \( B_k \) with radius enlarged by a factor of \( \rho^{-1} \) is contained in \( \mathbb{B} \). Invoking the weaker result in its translation and dilation invariant form gives an approximation by an affine function on \( B_k \), provided that \( |B_k|/|\mathbb{B}| \) is not too small as a function of \( \delta \). These affine functions patch together on most of \( \mathbb{B} \) to yield a single globally defined affine function, up to a suitably small additive error. The same reasoning reduces the case of small parameters \( \eta \) to \( \eta = 1 \).

The proof of the theorem will involve multiple steps in which \( \mathbb{B} \) is replaced by a ball \( \rho \mathbb{B} \) where \( \rho' > 0 \) depends only on \( d, D \). The final constant \( \rho \) is the product of all these factors \( \rho' \). We will simplify notation by allowing the value of \( \rho \) to change from one step to the next, so that each of these factors \( \rho' \), and products of successive factors, are denoted by \( \rho \).

The fifth simplification is one of language. Various conclusions will hold for all \( x \in \rho \mathbb{B} \) except for a set of measure at most \( \tau \rho^d |\mathbb{B}| \) where \( \tau > 0 \) depends only on \( d, D, \delta \) and \( \tau \to 0 \) as \( \delta \to 0 \). In this circumstance we will not specify a function \( \delta \mapsto \tau(\delta) \), but will simply write that the conclusions in question hold for nearly all \( x \in \rho \mathbb{B} \) in the same sense we will write “for nearly all \( (x, y) \in \rho \mathbb{B} \times \rho \mathbb{B} \)” and so on.

In the proof we write \( O(1) \) for a quantity that is bounded above by some constant depending only on \( D, \eta \). The value of this quantity is permitted to change from one occurrence to the next.

We will argue by induction on the degree \( D \). The key to this induction is the observation that Theorem 3.1 implies an additional conclusion.

**Corollary 11.2.** Let \( D \) be a nonnegative integer. Under the hypotheses of Theorem 3.1, for each multi-index satisfying \( |\alpha| = D \), there exists an affine function \( \xi_\alpha \) such that the coefficients \( a_\alpha \) in 3.8 satisfy
\[
|a_\alpha(h) - \xi_\alpha(h)| \leq C A \quad \text{for nearly all } h \in \rho \mathbb{B}.
\]

**Proof.** To prove this, assuming Theorem 2.2 for the given degree \( D \), let \( Q \) be a polynomial of degree \( \leq D + 1 \) that satisfies the conclusion 3.9. Then assuming as we may that \( \mathbb{B} \) is centered at 0 and has radius 1, \( \| \Delta_h Q(x) - \Delta_h \varphi(x) \| \leq C A \) for nearly all \( (x, h) \in (\rho \mathbb{B})^2 \). Expand \( \Delta_h Q(x) = \sum_{|\alpha| \leq D} \tilde{a}_\alpha(h) x^\alpha \) where \( \tilde{a}_\alpha \) are polynomials of degrees \( \leq D + 1 - |\alpha| \). In particular, \( \tilde{a}_\alpha \) is affine when \( |\alpha| = D \).

Consider \( \Delta_h Q - \Delta_h \varphi \). Substituting for \( \Delta_h \varphi \) the expression \( \sum_{|\alpha| \leq D} a_\alpha(h) x^\alpha + O(A) \) given in the hypothesis yields
\[
| \sum_{|\alpha| \leq D} (a_\alpha(h) - \tilde{a}_\alpha(h)) x^\alpha | \leq C A
\]
for nearly all \( (x, h) \in (\rho \mathbb{B})^2 \). Invoking Lemma 11.1 gives \( |a_\alpha(x) - \tilde{a}_\alpha(x)| \leq C A \) for nearly all \( x \in \rho \mathbb{B} \), which is the desired additional conclusion for \( |\alpha| = D \). \( \square \)

**Proof of Theorem 3.1.** We proceed by induction on \( D \). Since the proof of Corollary 11.2 for degree \( D \) relied on Theorem 2.2 for that same degree, in the induction it is only permissible to invoke Corollary 11.2 for smaller degrees.
The base case \( D = 0 \) is a corollary of Lemma 7.2. Indeed, it is given that \(|\varphi(x + h) - \varphi(x) - p(h)| \leq A\) for nearly all points \((x, h)\) with \(x \in B\) and \(h \in \mathbb{B}\), where \(p(h)\) is a polynomial of degree zero in \(x\) that depends on \(h\); that is, \(p(h)\) depends only on \(h\). If \(\mathbb{B}\) were equal to \(\mathbb{B}^*\) then this would be a direct application of Lemma 7.2. The general case is proved by combining this special case with a Whitney decomposition of \(B\), as in the analysis in [9].

In the proof for the inductive step, we operate under the following convention: For \(|\alpha| \leq D - 2\), \(b_\alpha, \tilde{b}_\alpha, c_\alpha\) denote Lebesgue measurable functions, with appropriate domains. An equation involving such functions is to be interpreted as an existence statement; the assertion is that there exist measurable functions such that the equation holds in the indicated domain. These are permitted to change from one occurrence of each symbol to the next. However, this convention is not in force for \(|\alpha| \geq D - 1\); for such indices, the functions \(b_\alpha\) do not change after they are first introduced.

Assume without loss of generality that \(A = 1\). For the inductive step, let \(D \geq 1\), and let \(\varphi, P\) satisfy the hypothesis with \(A = 1\). For \(x, s, t \in \rho \mathbb{B}\) consider
\[
\Delta_s \Delta_t \varphi(x) = \Delta_t \Delta_s \varphi(x)
= \sum_{|\alpha| \leq D} a_\alpha(s) ((x + t)\alpha - x\alpha) + O(1)
= \sum_{|\alpha| = D - 1} (b_\alpha(s) \cdot t + \tilde{b}_\alpha(s)) x^\alpha + \sum_{|\alpha| \leq D - 2} \tilde{b}_\alpha(s, t) x^\alpha + O(1)
\]
for nearly all \((x, s, t) \in (\rho \mathbb{B})^3\) where \(s \mapsto b_\alpha(s)\) are \(\mathbb{R}^d\)-valued measurable functions, and \(s \mapsto \tilde{b}_\alpha(s)\) is real-valued and measurable.

The terms \(\tilde{b}_\alpha(s)\) are bothersome, because differences ought to vanish when \(t = 0\). They can be eliminated by introducing an extra parameter \(t' \in \rho \mathbb{B}\) and considering the resulting approximate functional equation
\begin{equation}
\Delta_s (\Delta_t \varphi(x) - \Delta_t \varphi'(x)) = \sum_{|\alpha| = D - 1} b_\alpha(s) \cdot (t - t') x^\alpha + \sum_{|\alpha| \leq D - 2} \tilde{b}_\alpha(s, t, t') x^\alpha + O(1),
\end{equation}
which holds for nearly all \((x, s, t, t') \in (\rho \mathbb{B})^4\). Now
\[
\Delta_t \varphi(x) - \Delta_t \varphi(x) = \varphi(x + t) - \varphi(x + t') = \Delta_{t-t'} \varphi(x + t').
\]
Therefore substituting \(x = y - t'\) and then \(\tau = t - t'\), and specializing (11.2) to a typical value of \(t'\), gives
\begin{equation}
\Delta_s \Delta_t \varphi(y) = \sum_{|\alpha| = D - 1} b_\alpha(s) \cdot \tau y^\alpha + \sum_{|\alpha| \leq D - 2} c_\alpha(s, \tau) y^\alpha + O(1)
\end{equation}
for nearly all \((y, s, \tau) \in (\rho \mathbb{B})^3\), where the coefficients \(c_\alpha\) are measurable functions.

Specialize to a typical \(\tau \in \rho \mathbb{B}\). With \(\psi = \Delta_{\tau} \varphi\), this conclusion becomes
\[
\Delta_s \psi(y) = \sum_{|\alpha| = D - 1} b_\alpha(s) \cdot \tau y^\alpha + \sum_{|\alpha| \leq D - 2} c_\alpha(s, \tau) y^\alpha + O(1)
\]
for nearly all \((y, s, \tau) \in (\rho \mathbb{B})^3\). Therefore by induction on the degree \(D\) and Corollary 11.2, for each multi-index of degree \(|\alpha| = D - 1\), there exists an \(\mathbb{R}^d\)-valued affine function that agrees to within \(O(1)\) at nearly every point of \(\rho \mathbb{B}\) with \(b_\alpha\). That is, there exist \(\tilde{u}_\alpha \in \mathbb{R}^d \otimes \mathbb{R}^d\) and \(\tilde{\alpha} \in \mathbb{R}^d\) such that
\begin{equation}
|b_\alpha(s) - (\tilde{u}_\alpha \cdot s + \tilde{\alpha})| = O(1)
\end{equation}
for nearly all \(s \in \rho \mathbb{B}\).
Proof. Apply $\Delta_t$ to both sides of (11.6) to obtain
\[
\Delta_t\Delta_s \varphi = \Delta_t \sum_{|\alpha| = D} \sum_{j=1}^d u_\alpha j s_j x^\alpha + \Delta_t \sum_{|\alpha| \leq D-1} b_\alpha(s) x^\alpha + O(1)
\]
\[
= \sum_{|\alpha| = D} \sum_{j=1}^d u_\alpha j s_j \sum_{i=1}^d a_i x^{\alpha - e_i t_i} + \sum_{|\alpha| \leq D-2} b_\alpha(s, t) x^\alpha + O(1)
\]
This is obtained by writing $\Delta_s \Delta_t \varphi = \Delta_t \Delta_s \varphi$, substituting the right-hand side of (11.8) for $\Delta_s \varphi$, applying $\Delta_t$, expanding $(x + \tau)^\alpha$, and invoking Lemma 11.1 to reach a conclusion for the first order Taylor expansion with respect to $\tau$.

It follows that for each multi-index satisfying $|\beta| = D$, $a_\beta$ is approximately affine in the sense that
\[
|a_\beta(s) - (u_\beta \cdot s + v_\beta)| = O(1) \text{ for nearly all } s \in \rho B
\]
for certain $u_\beta \in \mathbb{R}^d \otimes \mathbb{R}^d$ and $v_\beta \in \mathbb{R}^d$. Insert this conclusion into the hypotheses (3.4), (3.8) to obtain
\[
\Delta_s \varphi(x) = \sum_{|\alpha| = D} (u_\alpha \cdot s + v_\alpha) x^\alpha + \sum_{|\alpha| \leq D-1} a_\alpha(s) x^\alpha + O(1)
\]
for nearly all $(x, s) \in (\rho B)^2$. Once again, there are bothersome terms, $v_\alpha x^\alpha$. Once again, these can be removed; consider $\Delta_s \varphi - \Delta_s' \varphi$ and argue as was done for a parallel situation above to establish (11.3). One concludes that
\[
\Delta_s \varphi(x) = \sum_{|\alpha| = D} u_\alpha \cdot s x^\alpha + \sum_{|\alpha| \leq D-1} b_\alpha(s) x^\alpha + O(1)
\]
for nearly all $(x, s) \in (\rho B)^2$, for certain measurable coefficients $b_\alpha$.

We will show below, in Lemma 11.3, that there exists a homogeneous polynomial $q$ of degree $\leq D + 1$ satisfying
\[
\Delta_s q(x) \equiv \sum_{|\alpha| = D} u_\alpha \cdot s x^\alpha + \sum_{|\alpha| \leq D-1} c_\alpha(s) x^\alpha + O(1)
\]
for all $(x, s) \in (\rho B)^2$ and for some (polynomial) coefficient functions $c_\alpha$, with the same $u_\alpha$ as in (11.6). Granting this for the present, set $\psi = \varphi - q$. Then
\[
\Delta_s \psi(x) = \sum_{|\alpha| \leq D-1} c_\alpha(s) x^\alpha + O(1)
\]
for nearly all $(x, s) \in (\rho B)^2$, where $c_\alpha$ are measurable functions. This is the original hypothesis, with $B$ replaced by $\rho B$, $\varphi$ replaced by $\psi$, and $D$ replaced by $D - 1$. Therefore it suffices to apply the induction hypothesis to conclude that $\psi$, and hence $\varphi = \psi + q$, have the required form. This completes the proof of Theorem 2.2 modulo the proof of the next lemma.

\begin{lemma}
There exists a polynomial $q$ of degree $\leq D + 1$ that satisfies (11.7).
\end{lemma}

Proof. Apply $\Delta_t$ to both sides of (11.6) to obtain
\[
\Delta_t \Delta_s \varphi(x) = \Delta_t \sum_{|\alpha| = D} \sum_{j=1}^d u_\alpha j s_j x^\alpha + \Delta_t \sum_{|\alpha| \leq D-1} b_\alpha(s) x^\alpha + O(1)
\]
\[
= \sum_{|\alpha| = D} \sum_{j=1}^d u_\alpha j s_j \sum_{i=1}^d a_i x^{\alpha - e_i t_i} + \sum_{|\alpha| \leq D-2} b_\alpha(s, t) x^\alpha + O(1)
\]
for nearly all \((x, s, t) \in (\mathbb{R}^d)^3\) where \(b_\alpha\) are measurable functions. Since \(\Delta_t \Delta_s \phi = \Delta_s \Delta_t \phi\), we may write the corresponding formula for \(\Delta_s \Delta_t \phi\), equate it to the one derived above, and apply Lemma 11.1 to deduce that for each \(i, j \in \{1, 2, \ldots, d\}\),

\[
\sum_{|\alpha| = D} u_{\alpha, j} x^{\alpha - e_i} = \sum_{|\alpha| = D} u_{\alpha, i} x^{\alpha - e_j} + O(1) \tag{11.9}
\]

for all \(x \in \rho \mathbb{B}\). Equivalently, for each multi-index \(\beta\) satisfying \(|\beta| = D - 1\),

\[
u_{\beta + e_i, j}(\beta_i + 1) = \nu_{\beta + e_j, i}(\beta_j + 1) + O(1)
\]

for each \(i, j\).

On the other hand, a homogeneous polynomial \(Q\) of degree \(D + 1\) satisfies the exact relation \(\Delta_s Q(x) = \sum_{|\alpha| = D} \tilde{u}_{\alpha, j} s_j x^\alpha + R(x, s)\) for some \(R\), where \(x \mapsto R(x, s)\) is a a polynomial of degree \(\leq D - 1\) for each \(s\), if and only if \(\partial Q(x)/\partial x_j = \sum_{|\alpha| = D} \tilde{u}_{\alpha, j} x^\alpha\) for each \(j \in \{1, 2, \ldots, d\}\). This system of equations is solvable for \(Q\) if and only if

\[
\sum_{|\alpha| = D} \tilde{u}_{\alpha, j} x^{\alpha - e_i} = \sum_{|\alpha| = D} \tilde{u}_{\alpha, i} x^{\alpha - e_j}
\]

for all \(i \neq j \in \{1, 2, \ldots, d\}\). Equivalently, for each multi-index \(\beta\) satisfying \(|\beta| = D - 1\),

\[
\tilde{u}_{\beta + e_i, j}(\beta_i + 1) = \tilde{u}_{\beta + e_j, i}(\beta_j + 1)
\]

for each \(i, j\).

The tuple \((u_{\alpha, k} : |\alpha| = D\text{ and } 1 \leq k \leq d)\) satisfies the system of approximate equations (11.10). By elementary linear algebra, there exists a tuple \((\tilde{u}_{\alpha, k})\) with \(|\tilde{u}_{\alpha, k} - u_{\alpha, k}| = O(1)\) for all \(\alpha, k\) that satisfies the corresponding system of exact equations (11.12). This system of equations implies the existence of a homogeneous polynomial \(q\) of degree \(D + 1\) that satisfies \(\partial q(x)/\partial x_j = \sum_{|\alpha| = D} \tilde{u}_{\alpha, j} x^\alpha\) for each \(j \in \{1, 2, \ldots, d\}\). Therefore \(\Delta_s q(x) = \sum_{|\alpha| = D} \tilde{u}_{\alpha, j} s_j x^\alpha + R(x, s)\) where \(R\) is as above.

The proof of Theorem 3.2 is very similar to that of Theorem 8.1. Details are left to the reader.

12. A FINAL LEMMA

The form of the conclusion of the next lemma contrasts with that of Lemma 7.2. In Lemma 7.3, the logarithms of the factors in the hypothesis are only nearly determined up to arbitrary additive corrections in \(2\pi i \mathbb{Z}\). In Lemma 12.1, no such arbitrary additive corrections arise.

**Lemma 12.1.** There exist \(A < \infty\) and \(\delta > 0\) with the following property. Let \(v_j \in \mathbb{R}\) for \(j = 1, 2\). Let \(\eta > 0\). Suppose that

\[
|e^{i(v_1 v_1 - v_2 v_2)} - 1| \leq \eta
\]

for all \((u_1, u_2) \in [0, 1]^2\) outside a set of Lebesgue measure \(\delta\). Then \(|v_1| + |v_2| \leq C\eta\).

**Proof.** There exists \(v_2 \in [0, 1]\) such that

\[
|e^{i u_1 v_1} - e^{i u_2 v_2}| \leq \eta
\]

for all \(u_1 \in [0, 1]\) outside a set \(E\) of measure \(\leq \delta\). We may assume without loss of generality that \(\eta\) is small and that \(v_1 \neq 0\). Let \(A\) be a large constant to be chosen below. If \(|v_1| \geq A\eta\) then there must exist an interval \(I \subset [0, 1]\) of length comparable to \(A|v_1|^{-1} \eta \leq 1\) such that \(|E \cap I| \leq \delta |I|\). The mapping \(I \ni t \mapsto e^{itv_1}\) maps \(I\) in a measure-preserving manner, up to
universal constant factors, to an arc of the unit circle of length comparable to $A\eta$. Because $|E \cap I| \leq \delta |I|$, the image of $I \setminus E$ has diameter comparable to $A\eta$. This contradicts (12.1).

Therefore $|v_1| \leq A\eta$. The same reasoning applies to $v_2$. □

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