Answering FO+MOD queries under updates on bounded degree databases*

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Abstract
We investigate the query evaluation problem for fixed queries over fully dynamic databases, where tuples can be inserted or deleted. The task is to design a dynamic algorithm that immediately reports the new result of a fixed query after every database update.

We consider queries in first-order logic (FO) and its extension with modulo-counting quantifiers (FO+MOD), and show that they can be efficiently evaluated under updates, provided that the dynamic database does not exceed a certain degree bound.

In particular, we construct a data structure that allows to answer a Boolean FO+MOD query and to compute the size of the result of a non-Boolean query within constant time after every database update. Furthermore, after every update we are able to immediately enumerate the new query result with constant delay between the output tuples. The time needed to build the data structure is linear in the size of the database.

Our results extend earlier work on the evaluation of first-order queries on static databases of bounded degree and rely on an effective Hanf normal form for FO+MOD recently obtained by Heimberg, Kuske, and Schweikardt (LICS 2016).

1 Introduction
Query evaluation is a fundamental task in databases, and a vast amount of literature is devoted to the complexity of this problem. In this paper we study query evaluation on relational databases in the “dynamic setting”, where the database may be updated by inserting or deleting tuples. In this setting, an evaluation algorithm receives a query \( \varphi \) and an initial database \( D \) and starts with a preprocessing phase that computes a suitable data structure to represent the result of evaluating \( \varphi \) on \( D \). After every database update, the data structure is updated so that it represents the result of evaluating \( \varphi \) on the updated database. The data structure shall be designed in such a way that it quickly provides the query result, preferably in constant time (i.e., independent of the database size). We focus on the following flavours of query evaluation.

- **Testing**: Decide whether a given tuple \( \overline{a} \) is contained in \( \varphi(D) \).
- **Counting**: Compute \( |\varphi(D)| \) (i.e., the number of tuples that belong to \( \varphi(D) \)).
- **Enumeration**: Enumerate \( \varphi(D) \) with a bounded delay between the output tuples.

Here, as usual, \( \varphi(D) \) denotes the \( k \)-ary relation obtained by evaluating a \( k \)-ary query \( \varphi \) on a relational database \( D \). For Boolean queries, all three tasks boil down to

- **Answering**: Decide if \( \varphi(D) \neq \emptyset \).

*This is the full version of the conference contribution [3].
Compared to the dynamic descriptive complexity framework introduced by Patnaik and Immerman [17], which focuses on the expressive power of first-order logic on dynamic databases and has led to a rich body of literature (see [18] for a survey), we are interested in the computational complexity of query evaluation. The query language studied in this paper is $\text{FO} + \text{MOD}$, the extension of first-order logic $\text{FO}$ with modulo-counting quantifiers of the form $\exists^{\text{mod} m} x \psi$, expressing that the number of witnesses $x$ that satisfy $\psi$ is congruent to $i$ modulo $m$. $\text{FO} + \text{MOD}$ can be viewed as a subclass of SQL that properly extends the relational algebra.

Following [2], we say that a query evaluation algorithm is efficient if the update time is either constant or at most polylogarithmic ($\log^c n$) in the size of the database. As a consequence, efficient query evaluation in the dynamic setting is only possible if the static problem (i.e., the setting without database updates) can be solved in linear or pseudo-linear ($n^{1+\varepsilon}$) time. Since this is not always possible, we provide a short overview on known results about first-order query evaluation on static databases and then proceed by discussing our results in the dynamic setting.

**First-order query evaluation on static databases.** The problem of deciding whether a given database $D$ satisfies a $\text{FO}$-sentence $\varphi$ is $\text{AW}[\ast]$-complete (parameterised by $\|\varphi\|$) and it is therefore generally believed that the evaluation problem cannot be solved in time $f(\|\varphi\|)\|D\|^c$ for any computable $f$ and constant $c$ (here, $\|\varphi\|$ and $\|D\|$ denote the size of the query and the database, respectively). For this reason, a long line of research focused on increasing classes of sparse instances ranging from databases of bounded degree [19] (where every domain element occurs only in a constant number of tuples in the database) to classes that are nowhere dense [9]. In particular, Boolean first-order queries can be evaluated on classes of databases of bounded degree in linear time $f(\|\varphi\|)\|D\|$, where the constant factor $f(\|\varphi\|)$ is 3-fold exponential in $\|\varphi\|$ [19] [7]. As a matter of fact, Frick and Grohe [7] showed that the 3-fold exponential blow-up in terms of the query size is unavoidable assuming $\text{FPT} \neq \text{AW}[\ast]$.

Durand and Grandjean [5] and Kazana and Segoufin [11] considered the task of enumerating the result of a $k$-ary first-order query on bounded degree databases and showed that after a linear time preprocessing phase the query result can be enumerated with constant delay. This result was later extended to classes of databases of bounded expansion [12]. Kazana and Segoufin [12] also showed that counting the number of result tuples of a $k$-ary first-order query on databases of bounded expansion (and hence also on databases of bounded degree) can be done in time $f(\|\varphi\|)\|D\|$. In [6] an analogous result was obtained for classes of databases of low degree (i.e., degree at most $\|D\|^{\omega(1)}$) and pseudo-linear time $f(\|\varphi\|)\|D\|^{1+\varepsilon}$; the paper also presented an algorithm for enumerating the query results with constant delay after pseudo-linear time preprocessing.

**Our contribution.** We extend the known linear time algorithms for first-order logic on classes of databases of bounded degree to the more expressive query language $\text{FO} + \text{MOD}$. Moreover, and more importantly, we lift the tractability to the dynamic setting and show that the result of $\text{FO}$ and $\text{FO} + \text{MOD}$-queries can be maintained with constant update time. In particular, we obtain the following results. Let $\varphi$ be a fixed $k$-ary $\text{FO} + \text{MOD}$-query and $d$ a fixed degree bound on the databases under consideration. Given an initial database $D$, we construct in linear time $f(\|\varphi\|, d)\|D\|$ a data structure that can be updated in constant time $f(\|\varphi\|, d)$ when a tuple is inserted into or deleted from a relation of $D$. After each update the data structure allows to

- immediately answer $\varphi$ on $D$ if $\varphi$ is a Boolean query (Theorem 1.1),
- test for a given tuple $\overline{a}$ whether $\overline{a} \in \varphi(D)$ in time $O(k^2)$ (Theorem 6.1),
- immediately output the number of result tuples $|\varphi(D)|$ (Theorem 8.1), and
- enumerate all tuples $(a_1, \ldots, a_k) \in \varphi(D)$ with $O(k^3)$ delay (Theorem 9.4).
For fixed \( d \), the parameter function \( f(\|\varphi\|, d) \) is 3-fold exponential in terms of the query size, which is (by Frick and Grohe [7]) optimal assuming \( \text{FPT} \neq \text{AW}[\bullet] \).

Outline. Our dynamic query evaluation algorithm crucially relies on the locality of \( \text{FO}+\text{MOD} \) and in particular an effective Hanf normal form for \( \text{FO}+\text{MOD} \) on databases of bounded degree recently obtained by Heimberg, Kuske, and Schweikardt [10]. After some basic definitions in Section 2 we briefly state their result in Section 3 and obtain a dynamic algorithm for Boolean \( \text{FO}+\text{MOD} \)-queries in Section 4. After some preparations for non-Boolean queries in Section 5 we present the algorithm for testing in Section 6. In Section 7 we reduce the task of counting and enumerating \( \text{FO}+\text{MOD} \)-queries in the dynamic setting to the problem of counting and enumerating independent sets in graphs of bounded degree. We use this reduction to provide efficient dynamic counting and enumeration algorithms in Section 8 and 9, respectively, and we conclude in Section 10.

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2 Preliminaries

We write \( \mathbb{N} \) for the set of non-negative integers and let \( \mathbb{N}_{\geq 1} := \mathbb{N} \setminus \{0\} \) and \( [n] := \{1, \ldots, n\} \) for all \( n \in \mathbb{N}_{\geq 1} \). By \( 2^M \) we denote the power set of a set \( M \). For a partial function \( f \) we write \( \text{dom}(f) \) and \( \text{codom}(f) \) for the domain and the codomain of \( f \), respectively.

Databases. We fix a countably infinite set \( \text{dom} \), the domain of potential database entries. Elements in \( \text{dom} \) are called constants. A schema is a finite set \( \sigma \) of relation symbols, where each \( R \in \sigma \) is equipped with a fixed arity \( \text{ar}(R) \in \mathbb{N}_{\geq 1} \). Let us fix a schema \( \sigma = \{ R_1, \ldots, R_{|\sigma|} \} \). A database \( D \) of schema \( \sigma \) (\( \sigma \)-db, for short) is, of the form \( D = (R_1^D, \ldots, R_{|\sigma|}^D) \), where each \( R_i^D \) is a finite subset of \( \text{dom}^{\text{ar}(R_i)} \). The active domain \( \text{adom}(D) \) of \( D \) is the smallest subset \( A \) of \( \text{dom} \) such that \( R_i^D \subseteq A^{\text{ar}(R_i)} \) for each \( R_i \) in \( \sigma \).

The Gaifman graph of a \( \sigma \)-db \( D \) is the undirected simple graph \( G^D = (V, E) \) with vertex set \( V := \text{adom}(D) \), and there is an edge between vertices \( u \) and \( v \) whenever \( u \neq v \) and there are \( R \in \sigma \) and \((a_1, \ldots, a_{\text{ar}(R)}) \in R^D \) such that \( u, v \in \{a_1, \ldots, a_{\text{ar}(R)}\} \). A \( \sigma \)-db \( D \) is called connected if its Gaifman graph \( G^D \) is connected; the connected components of \( D \) are the connected components of \( G^D \). The degree of a database \( D \) is the degree of its Gaifman graph \( G^D \), i.e., the maximum number of neighbours of a node of \( G^D \). Throughout this paper we fix a number \( d \in \mathbb{N} \) and restrict attention to databases of degree at most \( d \).

Updates. We allow to update a given database of schema \( \sigma \) by inserting or deleting tuples as follows (note that both types of commands may change the database’s active domain and the database’s degree). A deletion command is of the form \text{delete} \( R(a_1, \ldots, a_r) \) for \( R \in \sigma \), \( r = \text{ar}(R) \), and \( a_1, \ldots, a_r \in \text{dom} \). When applied to a \( \sigma \)-db \( D \), it results in the updated \( \sigma \)-db \( D' \) with \( R'^D = R^D \setminus \{a_1, \ldots, a_r\} \) and \( S'^D = S^D \) for all \( S \in \sigma \setminus \{R\} \).

An insertion command is of the form \text{insert} \( R(a_1, \ldots, a_r) \) for \( R \in \sigma \), \( r = \text{ar}(R) \), and \( a_1, \ldots, a_r \in \text{dom} \). When applied to a \( \sigma \)-db \( D \) in the unrestricted setting, it results in the updated \( \sigma \)-db \( D' \) with \( R'^D = R^D \cup \{a_1, \ldots, a_r\} \) and \( S'^D = S^D \) for all \( S \in \sigma \setminus \{R\} \). In this paper, we restrict attention to databases of degree at most \( d \). Therefore, when applying an insertion command to a \( \sigma \)-db \( D \) of degree \( \leq d \), the command is carried out only if the resulting database \( D' \) still has degree \( \leq d \); otherwise \( D \) remains unchanged and instead of carrying out the insertion command, an error message is returned.
Queris. We fix a countably infinite set \( \var{\text{var}} \) of variables. We consider the extension \( \FO+\MOD \) of first-order logic \( \FO \) with modulo-counting quantifiers. For a fixed schema \( \sigma \), the set \( \FO+\MOD[\sigma] \) is built from atomic formulas of the form \( x_1=x_2 \) and \( R(x_1, \ldots, x_{\arity(R)}) \), for \( R \in \sigma \) and variables \( x_1, x_2, \ldots, x_{\arity(R)} \in \var{\text{var}} \), and is closed under Boolean connectives \( \land, \lor \), existential first-order quantifiers \( \exists x \), and modulo-counting quantifiers \( \exists^{m} x \), for a variable \( x \in \var{\text{var}} \) and integers \( i, m \in \mathbb{N} \) with \( m \geq 2 \) and \( i < m \). The intuitive meaning of a formula of the form \( \exists^{m} x \) is that the number of witnesses \( x \) that satisfy \( \psi \) is congruent \( i \) modulo \( m \). As usual, \( \forall, \land, \lor, \to, \iff \) will be used as abbreviations when constructing formulas. It will be convenient to add the quantifier \( \exists^{m} x \), for \( m \in \mathbb{N} \); a formula of the form \( \exists^{m} x \) expresses that the number of witnesses \( x \) which satisfy \( \psi \) is \( \geq m \). This quantifier is just syntactic sugar and does not increase the expressive power of \( \FO+\MOD \).

The quantifier rank \( qr(\varphi) \) of a \( \FO+\MOD \)-formula \( \varphi \) is the maximum nesting depth of quantifiers that occur in \( \varphi \). By free(\( \varphi \)) we denote the set of all free variables of \( \varphi \), i.e., all variables \( x \) that have at least one occurrence in \( \varphi \) that is not within a quantifier of the form \( \exists x \), \( \exists^{m} x \), or \( \exists^{\mod{n}} x \). A sentence is a formula \( \varphi \) with free(\( \varphi \)) = \( \emptyset \).

An assignment for \( \varphi \) in a \( \sigma \)-db \( D \) is a partial mapping \( \alpha \) from \( \var{\text{var}} \) to \( \text{dom}(D) \), where free(\( \varphi \)) \subseteq \text{dom}(\( \alpha \)). We write \( (D, \alpha) \models \varphi \) to indicate that \( \varphi \) is satisfied when evaluated in \( D \) with respect to active domain semantics while interpreting every free occurrence of a variable \( x \) with the constant \( \alpha(x) \). Recall from \cite{H} that “active domain semantics” means that quantifiers are evaluated with respect to the database’s active domain. In particular, \( (D, \alpha) \models \exists x \psi \) iff there exists an \( a \in \text{dom}(D) \) such that \( (D, \alpha \hat{x} a) \models \psi \), where \( \alpha \hat{x} a \) is the assignment \( \alpha' \) with \( \alpha'(x) = a \) and \( \alpha'(y) = \alpha(y) \) for all \( y \in \text{dom}(\alpha) \setminus \{x\} \). Accordingly, \( (D, \alpha) \models \exists^{m} x \psi \) iff \( \{|a \in \text{dom}(D) : (D, \alpha \hat{x} a) \models \psi\| \geq m \} \), and \( (D, \alpha) \models \exists^{\mod{n}} x \psi \) iff \( \{|a \in \text{dom}(D) : (D, \alpha \hat{x} a) \models \psi\| \equiv i \mod m \} \).

A \( k \)-ary \( \FO+\MOD \) query of schema \( \sigma \) is of the form \( \varphi(x_1, \ldots, x_k) \) where \( k \in \mathbb{N} \), \( \varphi \in \FO+\MOD[\sigma] \), and free(\( \varphi \)) \subseteq \{x_1, \ldots, x_k\} \). We will often assume that the tuple \( (x_1, \ldots, x_k) \) is clear from the context and simply write \( \varphi \) instead of \( \varphi(x_1, \ldots, x_k) \) and \( (D, (a_1, \ldots, a_k)) \models \varphi \) instead of \( (D, \alpha_1 \ldots \alpha_k) \models \varphi \), where \( \alpha_1 \ldots \alpha_k \) denotes the assignment \( \alpha \) with \( \alpha(x_i) = a_i \) for all \( i \in [k] \). When evaluated in a \( \sigma \)-db \( D \), the \( k \)-ary query \( \varphi(x_1, \ldots, x_k) \) yields the \( k \)-ary relation

\[
\varphi(D) := \{ (a_1, \ldots, a_k) \in \text{dom}(D)^k : (D, \alpha_1 \ldots \alpha_k) \models \varphi \}.
\]

\( \text{Boolean} \) queries are \( k \)-ary queries with \( k = 0 \). As usual, for Boolean queries we will write \( \varphi(D) = \text{no} \) instead of \( \varphi(D) = \emptyset \), and \( \varphi(D) = \text{yes} \) instead of \( \varphi(D) \neq \emptyset \); and we write \( D \models \varphi \) to indicate that \( (D, \alpha) \models \varphi \) for any assignment \( \alpha \).

Sizes and Cardinals. The size \( |\sigma| \) of a schema \( \sigma \) is the sum of the arities of its relation symbols. The size \( |\varphi| \) of an \( \FO+\MOD \) query \( \varphi \) of schema \( \sigma \) is the length of \( \varphi \) when viewed as a word over the alphabet \( \sigma \cup \var{\text{var}} \cup \mathbb{N} \cup \{\land, \lor, \exists, \mod, \geq, ()\} \). For a \( k \)-ary query \( \varphi(x_1, \ldots, x_k) \) and a \( \sigma \)-db \( D \), the cardinality of the query result is the number \( |\varphi(D)| \) of tuples in \( \varphi(D) \). The cardinality \( |D| \) of a \( \sigma \)-db \( D \) is defined as the number of tuples stored in \( D \), i.e.,

\[
|D| := \sum_{R \in \sigma} |R|^D.
\]

The size \( |D| \) of \( D \) is defined as \( |\sigma| + |\text{dom}(D)| + \sum_{R \in \sigma} |\text{ar}(R)| |R|^D \) and corresponds to the size of a reasonable encoding of \( D \).

Dynamic Algorithms for Query Evaluation. We adopt the framework for dynamic algorithms for query evaluation of \cite{P}; the next paragraphs are taken almost verbatim from \cite{P}. Following \cite{H}, we use Random Access Machines (RAMs) with \( O(\log n) \) word-size and a uniform cost measure to analyse our algorithms. We will assume that the RAM’s memory is initialised to 0. In particular, if an algorithm uses an array, we will assume that all array entries are initialised to 0, and this initialisation comes at no cost (in real-world computers this can be achieved by using the lazy array initialisation technique, cf. e.g. \cite{IE}). A further assumption is that for every fixed dimension \( k \in \mathbb{N} \), we have available an unbounded number of \( k \)-ary arrays \( A \) such that
for given \((n_1, \ldots, n_k) \in \mathbb{N}^k\) the entry \(A[n_1, \ldots, n_k]\) at position \((n_1, \ldots, n_k)\) can be accessed in constant time. For our purposes it will be convenient to assume that \(\text{dom} = \mathbb{N}_{\geq 1}\).

Our algorithms will take as input a \(k\)-ary \(\text{FO}+\text{MOD}\)-query \(\varphi(x_1, \ldots, x_k)\), a parameter \(d\), and a \(\sigma\)-db \(D_0\) of degree \(\leq d\). For all query evaluation problems considered in this paper, we aim at routines \text{preprocess} and \text{update} which achieve the following.

Upon input of \(\varphi(x_1, \ldots, x_k)\) and \(D_0\), \text{preprocess} builds a data structure \(D\) which represents \(D_0\) (and which is designed in such a way that it supports the evaluation of \(\varphi\) on \(D_0\)). Upon input of a command \text{update} \(R(a_1, \ldots, a_p)\) (with \(\text{update} \in \{\text{insert}, \text{delete}\}\)), calling \text{update} modifies the data structure \(D\) such that it represents the updated database \(D\). The \text{preprocessing time} \(t_p\) is the time used for performing \text{preprocess}; the \text{update time} \(t_u\) is the time used for performing an \text{update}. In this paper, \(t_u\) will be independent of the size of the current database \(D\). By \text{init} we denote the particular case of the routine \text{preprocess} upon input of a query \(\varphi(x_1, \ldots, x_k)\) and the empty database \(D_\emptyset\) (where \(R^{D_\emptyset} = \emptyset\) for all \(R \in \sigma\)). The \text{initialisation time} \(t_i\) is the time used for performing \text{init}. In all dynamic algorithms presented in this paper, the \text{preprocess} routine for input of \(\varphi(x_1, \ldots, x_k)\) and \(D_0\) will carry out the \text{init} routine for \(\varphi(x_1, \ldots, x_k)\) and then perform a sequence of \(|D_0|\) \text{update} operations to insert all the tuples of \(D_0\) into the data structure. Consequently, \(t_p = t_i + |D_0| \cdot t_u\).

In the following, \(D\) will always denote the database that is currently represented by the data structure \(D\).

To solve the \textit{enumeration problem under updates}, apart from the routines \text{preprocess} and \text{update}, we aim at a routine \text{enumerate} such that calling \text{enumerate} invokes an enumeration of all tuples (without repetition) that belong to the query result \(\varphi(D)\). The \textit{delay} \(t_d\) is the maximum time used during a call of \text{enumerate}

- until the output of the first tuple (or the end-of-enumeration message \text{EOE}, if \(\varphi(D) = \emptyset\)),
- between the output of two consecutive tuples, and
- between the output of the last tuple and the end-of-enumeration message \text{EOE}.

To \text{test} if a given tuple belongs to the query result, instead of \text{enumerate} we aim at a routine \text{test} which upon input of a tuple \(\pi \in \text{dom}^k\) checks whether \(\pi \in \varphi(D)\). The \textit{testing time} \(t_t\) is the time used for performing a \text{test}. To solve the \textit{counting problem under updates}, instead of \text{enumerate} or \text{test} we aim at a routine \text{count} which outputs the cardinality \(|\varphi(D)|\) of the query result. The \textit{counting time} \(t_c\) is the time used for performing a \text{count}. To answer a \textit{Boolean} query under updates, instead of \text{enumerate}, \text{test}, or \text{count} we aim at a routine \text{answer} that produces the answer \text{yes} or \text{no} of \(\varphi\) on \(D\). The \textit{answer time} \(t_a\) is the time used for performing \text{answer}. Whenever speaking of a \textit{dynamic algorithm}, we mean an algorithm that has routines \text{preprocess} and \text{update} and, depending on the problem at hand, at least one of the routines \text{answer}, \text{test}, \text{count}, and \text{enumerate}.

Throughout the paper, we often adopt the view of \textit{data complexity} and suppress factors that may depend on the query \(\varphi\) or the degree bound \(d\), but not on the database \(D\). E.g., “linear preprocessing time” means \(t_p \leq f(\varphi, d) \cdot |D_0|\) and “constant update time” means \(t_u \leq f(\varphi, d)\), for a function \(f\) with codomain \(\mathbb{N}\). When writing \(\text{poly}(n)\) we mean \(n^{O(1)}\).

### 3 Hanf Normal Form for \text{FO}+\text{MOD}

Our algorithms for evaluating \text{FO}+\text{MOD} queries rely on a decomposition of \text{FO}+\text{MOD} queries into \textit{Hanf normal form}. To describe this normal form, we need some more notation.

\footnotetext{While this can be accomplished easily in the RAM-model, for an implementation on real-world computers one would probably have to resort to replacing our use of arrays by using suitably designed hash functions.}
Two formulas \( \varphi \) and \( \psi \) of schema \( \sigma \) are called \( d \)-equivalent (in symbols: \( \varphi \equiv_d \psi \)) if for all \( \sigma \)-dbs \( D \) of degree \( \leq d \) and all assignments \( \alpha \) for \( \varphi \) and \( \psi \) in \( D \) we have \( (D, \alpha) \models \varphi \iff (D, \alpha) \models \psi \).

For a \( \sigma \)-db \( D \) and a set \( A \subseteq \text{dom}(D) \) we write \( D[A] \) to denote the restriction of \( D \) to the domain \( A \), i.e., \( R^{D[A]} = \{ \pi \in R^D : \pi \in A^{\text{ar}(R)} \} \), for all \( R \in \sigma \). For two \( \sigma \)-dbs \( D \) and \( D' \) and two \( k \)-tuples \( a = (a_1, \ldots, a_k) \) and \( a' = (a'_1, \ldots, a'_k) \) of elements in \( \text{dom}(D) \) and \( \text{dom}(D') \), resp., we write \( (D, a) \equiv (D', a') \) to indicate that there is an isomorphism \( \pi \) from \( D \) to \( D' \) that maps \( a_i \) to \( a'_i \) for all \( i \in [k] \).

The distance \( \text{dist}^D(a, b) \) between two elements \( a, b \in \text{dom}(D) \) is the minimal length (i.e., the number of edges) of a path from \( a \) to \( b \) in \( D' \)'s Gaifman graph \( G^D \) (if no such path exists, we let \( \text{dist}^D(a, b) = \infty \); note that \( \text{dist}^D(a, a) = 0 \)). For \( r \geq 0 \) and \( a \in \text{dom}(D) \), the \( r \)-ball around \( a \) in \( D \) is the set \( N_r^D(a) = \{ b \in \text{dom}(D) : \text{dist}^D(a, b) \leq r \} \). For a \( \sigma \)-db \( D \) and a tuple \( \bar{a} = (a_1, \ldots, a_k) \) we let \( N_r^D(\bar{a}) := \bigcup_{i \in [k]} N_r^D(a_i) \). The \( r \)-neighbourhood around \( \bar{a} \) in \( D \) is defined as the \( \sigma \)-db \( N_r^D(\bar{a}) := D[N_r^D(\bar{a})] \).

For \( r \geq 0 \) and \( k \geq 1 \), a type \( \tau \) (over \( \sigma \)) with \( k \) centres and radius \( r \) (for short: \( r \)-type with \( k \) centres) is of the form \( (T, \bar{t}) \), where \( T \) is a \( \sigma \)-db, \( \bar{t} \in \text{dom}(T)^k \), and \( \text{dom}(T) = N_r^T(\bar{t}) \). The elements in \( \bar{t} \) are the centres of \( \tau \). For a tuple \( \bar{a} \in \text{dom}(D)^k \), the \( r \)-type of \( \tau \) in \( D \) is defined as the \( r \)-type with \( k \) centres \( (N_r^D(\bar{a}), \bar{a}) \).

For a given \( r \)-type with \( k \) centres \( \tau = (T, \bar{t}) \) it is straightforward to construct a first-order formula \( \text{sph}_r(\bar{x}) \) (depending on \( r \) and \( \tau \)) with \( k \) free variables \( \bar{x} = (x_1, \ldots, x_k) \) which expresses that the \( r \)-type of \( \bar{x} \) is isomorphic to \( \tau \), i.e., for every \( \sigma \)-db \( D \) and all \( \bar{a} = (a_1, \ldots, a_k) \in \text{dom}(D)^k \) we have \( (D, \bar{a}) \models \text{sph}_r(\bar{x}) \iff (N_r^D(\bar{a}), \bar{a}) \equiv (T, \bar{t}) \). The formula \( \text{sph}_r(\bar{x}) \) is called a sphere-formula (over \( \sigma \) and \( \tau \)); the numbers \( r \) and \( k \) are called locality radius and arity, resp., of the sphere-formula.

A Hanf-sentence (over \( \sigma \)) is a sentence of the form \( \exists x \text{sph}_r(x) \) or \( \exists \sigma \text{sph}_r(x) \), where \( \tau \) is an \( r \)-type (over \( \sigma \)) with 1 centre, for some \( r \geq 0 \). The number \( r \) is called locality radius of the Hanf-sentence. A formula in Hanf normal form (over \( \sigma \)) is a Boolean combination of sphere-formulas and Hanf-sentences (over \( \sigma \)). The locality radius of a formula \( \psi \) in Hanf normal form is the maximum of the locality radii of the Hanf-sentences and the sphere-formulas in \( \psi \). The formula is called \( d \)-bounded if all types \( \tau \) that occur in sphere-formulas or Hanf-sentences of \( \psi \) are \( d \)-bounded, i.e., \( T \) is of degree \( \leq d \), where \( \tau = (T, \bar{t}) \). Our query evaluation algorithms for \( \text{FO} + \text{MOD} \) rely on the following result by Heimberg, Kuske, and Schweikardt [10].

**Theorem 3.1 ([10]).** There is an algorithm which receives as input a degree bound \( d \in \mathbb{N} \) and a \( \text{FO} + \text{MOD}[\sigma] \)-formula \( \varphi \), and constructs a \( d \)-equivalent formula \( \psi \) in Hanf normal form (over \( \sigma \)) with the same free variables as \( \varphi \). For any \( d \geq 2 \), the formula \( \psi \) is \( d \)-bounded and has locality radius \( \leq 4^{\text{ar}(\varphi)} \), and the algorithm's runtime is \( 2^{O(\|\varphi\| + 1)} \).

The first step of all our query evaluation algorithms is to use Theorem 3.1 to transform a given query \( \varphi(\bar{x}) \) into a \( d \)-equivalent query \( \psi(\bar{x}) \) in Hanf normal form. The following lemma summarises easy facts that are useful for evaluating the sphere-formulas that occur in \( \psi \).

**Lemma 3.2.** Let \( d \geq 2 \) and let \( D \) be a \( \sigma \)-db of degree \( \leq d \). Let \( r \geq 0 \), \( k \geq 1 \), and \( \bar{a} = (a_1, \ldots, a_k) \in \text{dom}(D) \).

(a) \( |N_r^D(\bar{a})| \leq k \sum_{i=0}^{r} d^i \leq kd^{r+1} \).

(b) Given \( D \) and \( \bar{a} \), the \( r \)-neighbourhood \( N_r^D(\bar{a}) \) can be computed in time \( (kd^{r+1})^{O(\|\sigma\|)} \).

(c) \( N_r^D(a_1, a_2) \) is connected if and only if \( \text{dist}^D(a_1, a_2) \leq 2r + 1 \).
(d) If $N_r^D(\pi)$ is connected, then $N_r^D(\pi) \subseteq N_\infty^D(a, a_i)$, for all $i \in [k]$.

(e) Let $D'$ be a $\sigma$-db of degree $\leq d$ and let $b = (b_1, \ldots, b_k) \in \text{dom}(D')$.

It can be tested in time $(kd^r+1)^{O(|\sigma|+kd^r+1)} < 2^{O(|\sigma|k^2d^{2r+2})}$ whether $(N_r^D(\pi), \pi) \cong (N_r^{D'}(b), b)$.

Proof. Parts (a)–(d) are straightforward. Concerning Part (e), a brute-force approach is to loop through all mappings from $N_r^D(\pi)$ to $N_r^{D'}(b)$ that map $a_i$ to $b_i$ for every $i \in [k]$ and check whether this mapping is an isomorphism. Each such check can be accomplished in time $n^{O(|\sigma|)}$ for $n := kd^r+1$, and the number of mappings that have to be checked is $\leq n^n$. Thus, the isomorphism test is accomplished in time $n^{O(n+|\sigma|)} = (kd^r+1)^{O(|\sigma|+kd^r+1)}$. □

The time bound stated in part (e) of Lemma 3.2 is obtained by a brute-force approach. When using Luks’ polynomial time isomorphism test for bounded degree graphs [15], the time bound of Lemma 3.2 can be improved to $(kd^r+1)^{\text{poly}(d|\sigma|)}$. However, the asymptotic overall runtime of our algorithms for evaluating FO+MOD-queries won’t improve when using Luks algorithm instead of the brute-force isomorphism test of Lemma 3.2.

4 Answering Boolean FO+MOD Queries Under Updates

In [7], Frick and Grohe showed that in the static setting (i.e., without database updates), Boolean FO-queries $\varphi$ can be answered on databases $D$ of degree $\leq d$ in time $2^{d^{O(|\varphi|)}}|D|$. Our first main theorem extends their result to FO+MOD-queries and the dynamic setting.

Theorem 4.1. There is a dynamic algorithm that receives a schema $\sigma$, a degree bound $d \geq 2$, a Boolean FO+MOD[$\sigma$]-query $\varphi$, and a $\sigma$-db $D_0$ of degree $\leq d$, and computes within $t_p = f(\varphi, d) \cdot ||D_0||$ preprocessing time a data structure that can be updated in time $t_u = f(\varphi, d)$ and allows to return the query result $\varphi(D)$ with answer time $t_a = O(1)$. The function $f(\varphi, d)$ is of the form $2^{d^{O(|\varphi|)}}$.

If $\varphi$ is a $d$-bounded Hanf-sentence of locality radius $r$, then $f(\varphi, d) = 2^{O(|\sigma|d^{2r+2})}$, and the initialisation time is $t_i = O(1\varphi||)$.\[\]

Proof. W.l.o.g. we assume that all the symbols of $\sigma$ occur in $\varphi$ (otherwise, we remove from $\sigma$ all symbols that do not occur in $\varphi$). In the preprocessing routine, we first use Theorem 3.1 to transform $\varphi$ into a $d$-equivalent sentence $\psi$ in Hanf normal form; this takes time $2^{d^{O(|\varphi|)}}$. The sentence $\psi$ is a Boolean combination of $d$-bounded Hanf-sentences (over $\sigma$) of locality radius at most $r := 4^{|\sigma|}(\varphi)$. Let $\rho_1, \ldots, \rho_s$ be the list of all types that occur in $\psi$. Thus, every Hanf-sentence in $\psi$ is of the form $\exists^{\geq k}x \text{sph}_{\rho_j}(x)$ or $\exists^{\text{mod } m}x \text{sph}_{\rho_j}(x)$ for some $j \in [s]$ and $k, i, m \in \mathbb{N}$ with $k \geq 1$, $m \geq 2$, and $i < m$. For each $j \in [s]$ let $r_j$ be the radius of $\text{sph}_{\rho_j}(x)$. Thus, $\rho_j$ is an $r_j$-type with 1 centre (over $\sigma$).

For each $j \in [s]$ our data structure will store the number $A[j]$ of all elements $a \in \text{dom}(D)$ whose $r_j$-type is isomorphic to $\rho_j$, i.e., $(N_r^D(a), a) \cong \rho_j$. The initialisation for the empty database $D_{\emptyset}$ lets $A[j] = 0$ for all $j \in [s]$. In addition to the array $A$, our data structure stores a Boolean value $\text{Ans}$ where $\text{Ans} = \varphi(D)$ is the answer of the Boolean query $\varphi$ on the current database $D$. This way, the query can be answered in time $O(1)$ by simply outputting $\text{Ans}$. The initialisation for the empty database $D_{\emptyset}$ computes $\text{Ans}$ as follows. Every Hanf-sentence of the form $\exists^{\geq k}x \text{sph}_{\rho_j}(x)$ in $\psi$ is replaced by the Boolean constant $\text{true}$. Every Hanf-sentence of the form $\exists^{\text{mod } m}x \text{sph}_{\rho_j}(x)$ is replaced by $\text{true}$ if $i = 0$ and by $\text{false}$ otherwise. The resulting formula, a Boolean combination of the Boolean constants $\text{true}$ and $\text{false}$, then is evaluated, and we let $\text{Ans}$ be the obtained result. The entire initialisation takes time at most $t_i = f(\varphi, d) = 2^{d^{O(|\varphi|)}}$. If $\varphi$ is a Hanf-sentence, we even have $t_i = O(||\varphi||)$.\[\]
To update our data structure upon a command update \( R(a_1, \ldots, a_k) \), for \( k = \text{ar}(R) \) and update \( \in \{\text{insert, delete}\} \), we proceed as follows. The idea is to remove from the data structure the information on all the database elements whose \( r_j \)-neighbourhood (for some \( j \in [s] \)) is affected by the update, and then to recompute the information concerning all these elements on the updated database.

Let \( D_{\text{old}} \) be the database before the update is received and let \( D_{\text{new}} \) be the database after the update has been performed. We consider each \( j \in [s] \). All elements whose \( r_j \)-neighbourhood might have changed, belong to the set \( U_j := N^{D_{\text{old}}}_{r_j}(\mathcal{A}) \), where \( D' := D_{\text{new}} \) if the update command is insert \( R(\mathcal{A}) \), and \( D' := D_{\text{old}} \) if the update command is delete \( R(\mathcal{A}) \).

To remove the old information from \( \mathcal{A}[j] \), we compute for each \( a \in U_j \) the neighbourhood \( T_a := N^{D_{\text{old}}}_{r_j}(a) \), check whether \( (T_a, a) \cong \rho_j \), and if so, decrement the value \( \mathcal{A}[j] \).

To recompute the new information for \( \mathcal{A}[j] \), we compute for all \( a \in U_j \) the neighbourhood \( T'_a := N^{D_{\text{new}}}_{r_j}(a) \), check whether \( (T'_a, a) \cong \rho_j \), and if so, increment the value \( \mathcal{A}[j] \).

Using Lemma 3.2 we obtain for each \( j \in [s] \) that \( |U_j| \leq kd^{r_j+1} \). For each \( a \in U_j \), the neighbourhoods \( T_a \) and \( T'_a \) can be computed in time \( (d^{r_j+1})^{|\sigma|} \), and testing for isomorphism with \( \rho_j \) can be done in time \( (d^{r_j+1})^{|\sigma|+d^{r_j+1}} \). Thus, the update of \( \mathcal{A}[j] \) is done in time \( k \cdot (d^{r_j+1})^{|\sigma|+d^{r_j+1}} \leq 2^{|\mathcal{A}[j]|} \) (note that \( k \leq \|[\sigma]\| \leq |\varphi| \) and \( r_j \leq 4^{|\varphi|} \leq 2^{|\mathcal{A}[j]|} \)).

After having updated \( \mathcal{A}[j] \) for each \( j \in [s] \), we recompute the query answer \( \text{Ans} \) as follows. Every Hanf-sentence of the form \( \exists x \text{ sph}_{\rho_j}(x) \) in \( \psi \) is replaced by the Boolean constant \( \text{true} \) if \( \mathcal{A}[j] \geq k \), and by the Boolean constant \( \text{false} \) otherwise. Every Hanf-sentence of the form \( \exists x \mod m \text{ sph}_{\rho_j}(x) \) is replaced by \( \text{true} \) if \( \mathcal{A}[j] \equiv i \mod m \), and by \( \text{false} \) otherwise. The resulting formula, a Boolean combination of the Boolean constants \( \text{true} \) and \( \text{false} \), is then evaluated, and we let \( \text{Ans} \) be the obtained result. Thus, recomputing \( \text{Ans} \) takes time \( \text{poly}(|\psi|) \).

In summary, the entire update time is \( t_a = f(\varphi, d) = 2^{2^{|\mathcal{A}[j]|}} \). In case that \( \varphi \) is a \( d \)-bounded Hanf-sentence of locality radius \( r \), we even have \( t_a = k \cdot (d^{r+1})^{|\sigma|+d^{r+1}} \leq 2^{|\mathcal{A}[j]|} \). This completes the proof of Theorem 4.1.

In \cite{FG06}, Frick andGrohe obtained a matching lower bound for answering Boolean FO-queries of schema \( \sigma = \{E\} \) on databases of degree at most \( d := 3 \) in the static setting. They used the (reasonable) complexity theoretic assumption \( \text{FPT} \neq \text{AW}[\ast] \) and showed that if this assumption is correct, then there is no algorithm that answers Boolean FO-queries \( \varphi \) on \( \sigma \)-dbs \( D \) of degree \( \leq 3 \) in time \( 2^{2^{2^{|\mathcal{A}[j]|}}} \cdot \text{poly}(|D|) \) in the static setting (see Theorem 2 in \cite{FG06}). As a consequence, the same lower bound holds in the dynamic setting and shows that in Theorem 4.1 the 3-fold exponential dependency on the query size \( |\varphi| \) cannot be substantially lowered (unless \( \text{FPT} = \text{AW}[\ast] \)):

**Corollary 4.2.** Let \( \sigma := \{E\} \) and let \( d := 3 \). If \( \text{FPT} \neq \text{AW}[\ast] \), then there is no dynamic algorithm that receives a Boolean FO[\sigma]-query \( \varphi \) and a \( \sigma \)-db \( D_0 \), and computes within \( t_p \leq f(\varphi) \cdot \text{poly}(|D_0|) \) preprocessing time a data structure that can be updated in time \( t_u \leq f(\varphi) \) and allows to return the query result \( \varphi(D) \) with answer time \( t_a \leq f(\varphi) \), for a function \( f \) with \( f(\varphi) = 2^{2^{2^{|\mathcal{A}[j]|}}} \).

## 5 Technical Lemmas on Types and Spheres Useful for Handling Non-Boolean Queries

For our algorithms for evaluating non-Boolean queries it will be convenient to work with a fixed list of representatives of \( d \)-bounded \( r \)-types, provided by the following straightforward lemma.

**Lemma 5.1.** There is an algorithm which upon input of a schema \( \sigma \), a degree bound \( d \geq 2 \), a radius \( r \geq 0 \), and a number \( k \geq 1 \), computes a list \( L^\sigma_0(r/k) = \tau_1, \ldots, \tau_\ell \) (for a suitable \( \ell \geq 1 \)) of \( d \)-bounded \( r \)-types with \( k \) centres (over \( \sigma \)), such that for every \( d \)-bounded \( r \)-type \( \tau \) with \( k \) centres
As a first step, we consider each sphere-formula \( \tau \) given the syntax-tree of \( \psi \). The overall time for constructing \( \tau \) can be computed in time \( 2^{O(kd^3)}^{O(|\psi|)} \). Furthermore, upon input of a \( d \)-bounded \( r \)-type \( \tau \) with \( k \) centres (over \( \sigma \)), the particular \( i \in [\ell] \) with \( \tau \cong \tau_i \) can be computed in time \( 2^{O(kd^3)}^{O(|\psi|)} \).

Throughout the remainder of this paper, \( \mathcal{L}^r_d(k) \) will always denote the list provided by Lemma 5.1 The following lemma will be useful for evaluating Boolean combinations of sphere-formulas.

**Lemma 5.2.** Let \( \sigma \) be a schema, let \( r \geq 0 \), \( k \geq 1 \), \( d \geq 2 \), and let \( \mathcal{L}^r_d(k) = \tau_1, \ldots, \tau_\ell \).

Let \( \tau = (x_1, \ldots, x_k) \) be a list of \( k \) pairwise distinct variables. For every Boolean combination \( \psi(\tau) \) of \( d \)-bounded sphere-formulas of radius at most \( r \) (over \( \sigma \)), there is an \( I \subseteq [\ell] \) such that \( \psi(\tau) \equiv_d \bigvee_{i \in I} \text{sph}_{\tau_i}(\tau) \).

Furthermore, given \( \psi(\tau) \), the set \( I \) can be computed in time \( \text{poly}(|\psi|) \cdot 2^{O(kd^3)}^{O(|\psi|)} \).

**Proof.** As a first step, we consider each sphere-formula \( \zeta \) that occurs in \( \psi \) and replace it by a \( d \)-equivalent disjunction of sphere-formulas \( \text{sph}_{\tau_i}(\tau) \) with \( \tau_i \in \mathcal{L}^r_d(k) \). If \( \zeta \) has arity \( k' \leq k \) and radius \( r' \leq r \) and is of the form \( \text{sph}_{\tau_i}(\tau) \) with \( \tau = \tau_j \) for \( 1 \leq i < \cdots < k' \leq k \) and \( \rho = (S, \bar{\sigma}) \) with \( S = s_1, \ldots, s_k \), then we replace \( \zeta \) by the formula \( \zeta' := \bigvee_{j \in I} \text{sph}_{\tau_j}(\tau) \), where \( J \) consists of all those \( j \in [\ell] \) for which \( (T, \bar{t}) = \tau_j \) with \( \bar{t} = t_1, \ldots, t_k \) and for which \( \bar{t} : = t_{\rho_1}, \ldots, t_{\rho_k} \), we have \( (S, \bar{\sigma}) \cong (T[N^k_j(\bar{t})], \bar{t}) \). It is straightforward to see that \( \zeta' \) and \( \zeta \) are \( d \)-equivalent.

Let \( \psi_i \) be the formula obtained from \( \psi \) by replacing each \( \zeta \) by \( \zeta' \). By the Lemmas 5.1 and 3.2, \( \psi_i \) can be constructed in time \( O(|\psi| \cdot 2^{O(kd^3)})^{O(|\psi|)} \).

Note that \( \psi_1 \) is a Boolean combination of formulas \( \text{sph}_{\tau_1}(\tau) \) for \( j \in [\ell] \).

In the second step, we repeatedly use de Morgan’s law to push all \( \neg \)-symbols in \( \psi_1 \) directly in front of sphere-formulas. Afterwards, we replace every subformula of the form \( \neg \text{sph}_{\tau_i}(\tau) \) by the \( d \)-equivalent formula \( \bigwedge_{i \in I \setminus \{j\}} \text{sph}_{\tau_i}(\tau) \). Let \( \psi_2 \) be the formula obtained from \( \psi_1 \) by these transformations. Constructing \( \psi_2 \) from \( \psi_1 \) takes time at most \( O(|\psi_1|) \cdot 2^{O(kd^3)}^{O(|\psi|)} = O(|\psi|) \cdot 2^{O(kd^3)}^{O(|\psi|)} \).

In the third step, we eliminate all the \( \land \)-symbols in \( \psi_2 \). By the definition of the sphere-formulas \( \tau_1, \ldots, \tau_\ell \) we have

\[
\text{sph}_{\tau_i}(\tau) \wedge \text{sph}_{\tau_{i'}}(\tau) \equiv_d \begin{cases} 
\text{sph}_{\tau_i}(\tau) & \text{if } i = i' \\
\bot & \text{if } i \neq i'
\end{cases}
\]

(1)

where \( \bot \) is an unsatisfiable formula. Thus, by the distributive law we obtain for all \( m \geq 1 \) and \( I_1, \ldots, I_m \subseteq [\ell] \) that

\[
\bigwedge_{j \in [m]} \left( \bigvee_{i \in I_j} \text{sph}_{\tau_i}(\tau) \right) \equiv_d \bigvee_{i_1 \in I_1} \cdots \bigvee_{i_m \in I_m} \left( \text{sph}_{\tau_1}(\tau) \wedge \cdots \wedge \text{sph}_{\tau_m}(\tau) \right) \equiv_d \bigvee_{i \in I} \text{sph}_{\tau_i}(\tau)
\]

for \( I := I_1 \cap \cdots \cap I_m \). We repeatedly use this equivalence during a bottom-up traversal of the syntax-tree of \( \psi_2 \) to eliminate all the \( \land \)-symbols in \( \psi_2 \). The resulting formula \( \psi_3 \) is obtained in time polynomial in the size of \( \psi_2 \). Furthermore, \( \psi_3 \) is of the desired form \( \bigvee_{i \in I} \text{sph}_{\tau_i}(\tau) \) for an \( I \subseteq [\ell] \). The overall time for constructing \( \psi_3 \) and \( I \) is \( \text{poly}(|\psi|) \cdot 2^{O(kd^3)}^{O(|\psi|)} \). This completes the proof of Lemma 5.2.

For evaluating a Boolean combination \( \psi(\tau) \) of sphere-formulas and Hanf-sentences on a given \( \sigma \)-db \( D \), an obvious approach is to first consider every Hanf-sentence \( \chi \) that occurs in \( \psi \), to check if \( D \vDash \chi \), and replace every occurrence of \( \chi \) in \( \psi \) with \true\ (resp., \false\) if \( D \vDash \chi \) (resp., \( D \not\vDash \chi \)). The resulting formula \( \psi'(\tau) \) is then transformed into a disjunction \( \psi'(\tau) := \bigvee_{i \in I} \text{sph}_{\tau_i}(\tau) \) by Lemma 5.2 and the query result \( \psi(D) = \psi'(D) \) is obtained as the union of the query results \( \text{sph}_{\tau_i}(D) \) for all \( i \in I \).
While this works well in the static setting (i.e., without database updates), in the dynamic setting we have to take care of the fact that database updates might change the status of a Hanf-sentence $\chi$ in $\psi$, i.e., an update operation might turn a database $D$ with $D \models \chi$ into a database $D'$ with $D' \not\models \chi$ (and vice versa). Consequently, the formula $\psi''(\tau)$ that is equivalent to $\psi(\tau)$ on $D$ might be inequivalent to $\psi(\tau)$ on $D'$.

To handle the dynamic setting correctly, at the end of each update step we will use the following lemma (the lemma’s proof is an easy consequence of Lemma 5.2).

**Lemma 5.3.** Let $\sigma$ be a schema. Let $s \geq 0$ and let $\chi_1, \ldots, \chi_s$ be FO-MOD[\sigma]-sentences. Let $r \geq 0$, $k \geq 1$, $d \geq 2$, and let $L^{\sigma,d}_r(k) = \tau_1, \ldots, \tau_t$. Let $\tau = (x_1, \ldots, x_k)$ be a list of $k$ pairwise distinct variables. For every Boolean combination $\psi$ of a tuple with the corresponding parameters $c$ where $p \neq c$ will be useful to decompose $\psi$ into its connected components as follows. Let $\tau = (T, \overline{t})$ with $\overline{t} = (t_1, \ldots, t_k)$. Consider the Gaifman graph $G^T$ of $T$ and let $C_1, \ldots, C_c$ be the vertex sets of the connected components of $G^T$. For each connected component $C_j$ of $G^T$, let $\overline{t}_j$ be the subsequence of $\overline{t}$ consisting of all elements of $\overline{t}$ that belong to $C_j$, and let $k_j$ be the length of $\overline{t}_j$. Since $(T, \overline{t})$ is an $r$-type with $k$ centres, we have $T = N^T_r(\overline{t})$, and thus $c \leq k$ and $k_j \geq 1$ for all $j \in [c]$. To avoid ambiguity, we make sure that the list $C_1, \ldots, C_c$ is sorted in such a way that for all $j < j'$ we have $i < i'$ for the smallest $i$ with $t_i \in C_j$ and the smallest $i'$ with $t_{i'} \in C_{j'}$.

For each $C_j$ consider the $r$-type with $k_j$ centres $\rho_j = (T[C_j], \overline{t}_j)$. Let $\nu_j$ be the unique integer such that $\rho_j$ is isomorphic to the $\nu_j$-th element in the list $L^{\sigma,d}_r(k_j)$, and let $\tau_{j,\nu_j}$ be the $\nu_j$-th element in this list.

It is straightforward to see that the formula sph$_c(\tau)$ is $d$-equivalent to the formula

$$\text{conn-sph}_c(\tau) := \bigwedge_{j \in [c]} \text{sph}_{\tau_{j,\nu_j}}(\tau_j) \land \bigwedge_{j \neq j'} \neg \text{dist}^{|k_j, k_{j'}|}_{\leq 2r+1}(\tau_j, \tau_{j'}),$$

where $\tau_j$ is the subsequence of $\tau$ obtained from $\tau$ in the same way as $\overline{t}_j$ is obtained from $\overline{t}$, and $\text{dist}^{|k_j, k_{j'}|}_{\leq 2r+1}(\tau_j, \tau_{j'})$ is a formula of schema $\sigma$ which expresses that for some variable $y$ in $\tau_j$ and some variable $y'$ in $\tau_{j'}$ the distance between $y$ and $y'$ is $\leq 2r+1$. I.e., for $\tau = (a_1, \ldots, a_{k_j})$ and $\overline{b} = (b_1, \ldots, b_{k_{j'}})$ we have $(\tau, \overline{b}) \in \text{dist}^{|k_j, k_{j'}|}_{\leq 2r+1}(D) \iff \text{dist}^D(\tau, \overline{b}) \leq 2r+1,$

where $\text{dist}^D(\tau, \overline{b}) \leq 2r+1$ means that $\text{dist}^D(a_i, b_{i'}) \leq 2r+1$ for some $i \in [k_j]$ and $i' \in [k_{j'}]$.

Using the Lemmas 3.2 and 5.1 the following lemma is straightforward.

**Lemma 5.4.** There is an algorithm which upon input of a schema $\sigma$, numbers $r \geq 0$, $k \geq 1$, and $d \geq 2$, and an $r$-type $\tau$ with $k$ centres (over $\sigma$) computes the formula conn-sph$_c(\tau)$, along with the corresponding parameters $c$ and $k_j, \nu_j, \tau_j, \nu_j$ for all $j \in [c]$.

The algorithm’s runtime is $2^{(kd^{r+1})O(|\sigma|)}$.

We define the signature of $\tau$ to be the tuple $\text{sgn}(\tau)$ built from the parameters $c$ and $(k_j, \nu_j, \mu \in [k] : x_\mu$ belongs to $\tau_j)$ obtained from the above lemma. The signature $\text{sgn}^D(\tau)$ of a tuple $\tau$ in a database $D$ (w.r.t. radius $r$) is defined as $\text{sgn}(\rho)$ for $\rho := (N^r_\sigma(\overline{a}), \overline{a})$. Note that $\tau \in \text{sph}_c(D) \iff \text{sgn}^D(\overline{a}) = \text{sgn}(\tau)$. 

10
6 Testing Non-Boolean FO+MOD Queries Under Updates

This section is devoted to the proof of the following theorem.

**Theorem 6.1.** There is a dynamic algorithm that receives a schema \( \sigma \), a degree bound \( d \geq 2 \), a \( k \)-ary \( \text{FO}+\text{MOD}[\sigma] \)-query \( \varphi(\pi) \) (for some \( k \in \mathbb{N} \)), and a \( \sigma \)-db \( D_0 \) of degree \( \leq d \), and computes \( |D_0| \)-preprocessing time a data structure that can be updated in time \( t_u = f(\varphi, d) \) and allows to test for any input tuple \( \pi \in \text{dom}^k \) whether \( \pi \in \varphi(D) \) within testing time \( t_t = O(k^2) \). The function \( f(\varphi, d) \) is of the form \( 2^{2O(1+d)} \).

For the proof, we use the lemmas provided in Section 5 and the following lemma.

**Lemma 6.2.** There is a dynamic algorithm that receives a schema \( \sigma \), a degree bound \( d \geq 2 \), numbers \( r \geq 0 \) and \( k \geq 1 \), an \( r \)-type \( \tau \) with \( k \) centres (over \( \sigma \)), and a \( \sigma \)-db \( D_0 \) of degree \( \leq d \), and computes \( |D_0| \)-preprocessing time a data structure that can be updated in time \( t_u = 2^{(k d + 1)^O(1+|\tau|)} \) and allows to test for any input tuple \( \pi \in \text{dom}^k \) whether \( \pi \in \text{sp}_{\tau}(D) \) within testing time \( t_t = O(k^2) \).

**Proof.** The preprocessing routine starts by using Lemma 5.4 to compute the formula \( \text{conn-sph}_{\tau}(\pi) \), along with the according parameters \( c \) and \( k \), \( \tau \), \( \pi \), \( \tau \), \( \nu \), \( \nu \) for each \( j \in [c] \). This is done in time \( 2^{(k d + 1)^O(1+|\tau|)} \). We let \( \text{sgn}(\tau) \) be the signature of \( \tau \) (defined directly after Lemma 5.4). Recall that \( \text{conn-sph}_{\tau}(\pi) \equiv_d \text{sp}_{\tau}(\pi) \), and recall from equation 2 the precise definition of the formula \( \text{conn-sph}_{\tau}(\pi) \). Our data structure will store the following information on the database \( D \):

- the set \( \Gamma \) of all tuples \( \bar{b} \in \text{dom}(D)^k \) where \( k' \leq k \) and \( N_\tau^{D}(\bar{b}) \) is connected, and
- for every \( j \in [c] \) and every tuple \( \bar{b} \in \Gamma \) of arity \( k_j \), the unique number \( \nu_\tau \) such that \( \rho_\tau := (N_\tau^{D}(\bar{b}), \nu_\tau) \) is isomorphic to the \( \nu_\tau \)-th element in the list \( L_\tau^{\sigma,c}(k_j) \).

We want to store this information in such a way that for any given tuple \( \bar{b} \in \text{dom}^k \) it can be checked in time \( O(k) \) whether \( \bar{b} \in \Gamma \). To ensure this, we use a \( k' \)-ary array \( \Gamma_{k'} \) that is initialised to 0, and where during update operations the entry \( \Gamma_{k'}(\bar{b}) \) is set to 1 for all \( \bar{b} \in \Gamma \) of arity \( k' \). In a similar way we can ensure that for any given \( j \in [c] \) and any \( \bar{b} \in \Gamma \) of arity \( k_j \), the number \( \nu_\tau \) can be looked up in time \( O(k) \).

The test routine upon input of a tuple \( \pi = (a_1, \ldots, a_k) \) proceeds as follows.

First, we partition \( \pi \) into \( \pi_1, \ldots, \pi_{c'} \) (for \( c' \leq k \)) such that \( C_j := N_\tau^{D}(\pi_j) \) for \( j \in [c'] \) are the connected components of \( N_\tau^{D}(\bar{\pi}) \). As in the definition of the formula \( \text{conn-sph}_{\tau}(\pi) \), we make sure that this list is sorted in such a way that for all \( j < j' \) we have \( i < i' \) for the smallest \( i \) with \( a_i \in C_j \) and the smallest \( i' \) with \( a_i \in C_{j'} \). All of this can be done in time \( O(k^2) \) by first constructing the graph \( H \) with vertex set \( [k] \) and where there is an edge between vertices \( i \) and \( j \) iff the tuple \( (a_i, a_j) \) belongs to \( \Gamma \), and then computing the connected components of \( H \).

Afterwards, for each \( j \in [c'] \) we use time \( O(k) \) to look up the number \( \nu_{\pi_j} \). We then let \( \text{sgn}^D(\bar{\pi}) \) be the tuple built from \( c' \) and \( (\pi_j, \nu_{\pi_j}, \{\mu \in [k] : a_\mu \text{ belongs to } \pi_j\}) \). It is straightforward to see that \( \bar{\pi} \in \text{conn-sph}_{\tau}(D) \) iff \( \text{sgn}^D(\bar{\pi}) = \text{sgn}(\tau) \). Therefore, the test routine checks whether \( \text{sgn}^D(\bar{\pi}) = \text{sgn}(\tau) \) and outputs “yes” if this is the case and “no” otherwise. The entire time used by the test routine is \( t_t = O(k^2) \).

To finish the proof of Lemma 6.2, we have to give further details on the preprocess routine and the update routine. The preprocess routine initialises \( \Gamma \) as the empty set \( \emptyset \) and then performs \( |D_0| \)-update operations to insert all the tuples of \( D_0 \) into the data structure. The update routine proceeds as follows.

Let \( D_{\text{old}} \) be the database before the update is received and let \( D_{\text{new}} \) be the database after the update has been performed. Let the update command be of the form \( \text{update } R(a_1, \ldots, a_{\text{ar}(R)}) \). We let \( r' := r + (\text{ar}(R) - 1)(2r+1) \). All elements whose \( r' \)-neighbourhood might have changed
structures. Afterwards, it recomputes for storing “no” otherwise.

To ensure that this test can be done in time \( t_u = 2^{(k + 1)O(|\sigma|)} \).

Theorem 6.1 is now obtained by combining Theorem 3.1, Lemma 6.2, Theorem 4.1, and Lemma 5.3.

Proof of Theorem 6.1. For \( k = 0 \), the theorem immediately follows from Theorem 4.1.

Consider the case where \( k \geq 1 \).

As in the proof of Theorem 4.1, we assume w.l.o.g. that all the symbols of \( \sigma \) occur in \( \varphi \).

We start the preprocessing routine by using Theorem 3.1 to transform \( \varphi(\bar{\pi}) \) into a \( d \)-equivalent query \( \psi(\bar{\pi}) \) in Hanf normal form; this takes time \( 2^{dO(|\sigma|)} \).

The formula \( \psi \) is a Boolean combination of \( d \)-bounded Hanf-sentences and sphere-formulas (over \( \sigma \)) of locality radius at most \( r := 4^{q_{\varphi}(r)} \), and each sphere-formula is of arity at most \( k \). Let \( \chi_1, \ldots, \chi_s \) be the list of all Hanf-sentences that occur in \( \psi \).

We use Lemma 5.1 to compute the list \( L_{\sigma,d}^\tau(k) = \tau_1, \ldots, \tau_\ell \). In parallel for each \( i \in [\ell] \), we use the algorithm provided by Lemma 6.2 for \( \tau := \tau_i \).

Furthermore, for each \( j \in [s] \), we use the algorithm provided by Theorem 4.1 upon input of the Hanf-sentence \( \varphi := \chi_j \).

In addition to the components used by these dynamic algorithms, our data structure also stores

- the set \( J := \{ j \in [s] : D \models \chi_j \} \),
- the particular set \( I \subseteq [\ell] \) provided by Lemma 5.3 for \( \psi(\bar{\pi}) \) and \( J \), and
- the set \( K := \{ \text{sgn}(\tau_i) : i \in I \} \), where for each type \( \tau \), \( \text{sgn}(\tau) \) is the signature of \( \tau \) defined directly after Lemma 5.4.

The test routine upon input of a tuple \( \bar{\pi} = (a_1, \ldots, a_k) \) proceeds in the same way as in the proof of Lemma 6.2 to compute in time \( O(k^2) \) the signature \( \text{sgn}^D(\bar{\pi}) \) of the tuple \( \bar{\pi} \).

For every \( i \in [\ell] \) we have \( \bar{\pi} \in \text{sp}_\tau(D) \iff \text{sgn}^D(\bar{\pi}) = \text{sgn}(\tau_i) \).

Thus, \( \bar{\pi} \in \varphi(D) \iff \text{sgn}^D(\bar{\pi}) \in K \).

Therefore, the test routine checks whether \( \text{sgn}^D(\bar{\pi}) \in K \) and outputs “yes” if this is the case and “no” otherwise.

To ensure that this test can be done in time \( O(k^2) \), we use an array construction for storing \( K \) (similar to the one for storing \( \Gamma \) in the proof of Lemma 6.2).

The update routine runs in parallel the update routines for all the used dynamic data structures.

Afterwards, it recomputes \( J \) by calling the answer routine for \( \chi_j \) for all \( j \in [s] \).

Then, it uses Lemma 5.3 to recompute \( I \).

The set \( K \) is then recomputed by applying Lemma 5.4 for \( \tau := \tau_i \) for all \( i \in I \).

It is straightforward to see that the overall runtime of the update routine is \( t_u = 2^{O(|\pi|)} \).

This completes the proof of Theorem 6.1.

\[ \square \]

7 Representing Databases by Coloured Graphs

To obtain dynamic algorithms for counting and enumerating query results, it will be convenient to work with a representation of databases by coloured graphs that is similar to the representation used in [6]. For defining this representation, let us consider a fixed \( d \)-bounded \( r \)-type \( \tau \) with \( k \) centres (over a schema \( \sigma \)).

Use Lemma 5.4 to compute the formula \( \text{conn-sph}_a(\bar{\pi}) \) (for \( \bar{\pi} = (x_1, \ldots, x_k) \)) and the according parameters \( c \) and \( k_j, \nu_j, r_j, \tau_j, \nu_j \), and let \( \text{sgn}(\tau) \) be
the signature of \( \tau \). To keep the notation simple, we assume w.l.o.g. that \( \mathbf{x}_1 = x_1, \ldots, x_{k_1}, \mathbf{x}_2 = x_{k_1+1}, \ldots, x_{k_1+k_2} \) etc.

Recall that \( \text{sph}_r(\mathbf{x}) \) is \( d \)-equivalent to the formula

\[
\text{conn-sph}_r(\mathbf{x}) := \bigwedge_{j \in [c]} \text{sph}_{r_j,\nu_j}(\mathbf{x}_j) \land \bigwedge_{j \neq j'} -\text{dist}^r_{\leq 2r+1}(\mathbf{x}_j, \mathbf{x}_{j'}). 
\]

To count or enumerate the results of the formula \( \text{sph}_r(\mathbf{x}) \) we represent the database \( D \) by a \( c \)-coloured graph \( G_D \). Here, a \( c \)-coloured graph \( G \) is a database of the particular schema

\[
\sigma_c := \{ E, C_1, \ldots, C_c \},
\]

where \( E \) is a binary relation symbol and \( C_1, \ldots, C_c \) are unary relation symbols. We define \( G_D \) in such a way that the task of counting or enumerating the results of the query \( \text{sph}_r(\mathbf{x}) \) on the database \( D \) can be reduced to counting or enumerating the results of the query

\[
\varphi_c(z_1, \ldots, z_c) := \bigwedge_{j \in [c]} C_j(z_j) \land \bigwedge_{j \neq j'} -E(z_j, z_{j'})
\]

on the \( c \)-coloured graph \( G_D \). The vertices of \( G_D \) correspond to tuples over \( \text{dom}(D) \) whose \( r \)-neighbourhood is connected; a vertex has colour \( C_j \) if its associated tuple \( \mathbf{x} \) is in \( \text{sph}_{r_j,\nu_j}(D) \); and an edge between two vertices indicates that \( \text{dist}^r(D, \mathbf{x}, \mathbf{y}) \leq 2r+1 \), for their associated tuples \( \mathbf{x} \) and \( \mathbf{y} \). The following lemma allows to translate a dynamic algorithm for counting or enumerating the results of the query \( \varphi_c(z_1, \ldots, z_c) \) on \( c \)-coloured graphs into a dynamic algorithm for counting or enumerating the results of the query \( \text{sph}_r(\mathbf{x}) \) on \( D \).

**Lemma 7.1.** Suppose that the counting problem (the enumeration problem) for \( \varphi_c(\mathbf{x}) \) on \( \sigma_c \)-dbs of degree at most \( d' \) can be solved by a dynamic algorithm with initialisation time \( t_i(c, d') \), update time \( t_u(c, d') \), and counting time \( t_c(c, d') \) (delay \( t_d(c, d') \)). Then for every schema \( \sigma \) and every \( d \geq 2 \) the following holds.

1. Let \( r \geq 0, k \geq 1, \tau \) a \( d \)-bounded \( r \)-type with \( k \) centres, and fix \( d' := d2^{(2r+1)} \) and \( \tilde{t}_x := \max_{c=1}^k t_x(c, d') \) for \( t_x \in \{ t_i, t_u, t_c, t_d \} \). The counting problem (the enumeration problem) for \( \text{sph}_r(\mathbf{x}) \) on \( \sigma \)-dbs of degree at most \( d \) can be solved by a dynamic algorithm with counting time \( \tilde{t}_c \) (delay \( \tilde{O}(\tilde{t}_d k) \)), update time \( \tilde{t}_u \leq t_u d^{O(k^2 r + k |\sigma|)} + 2^{O(|\sigma| k^2 r^2)} \), and initialisation time \( \tilde{t}_i \).

2. The counting problem (the enumeration problem) for \( k \)-ary \( \text{FO+MOD} \)-queries \( \varphi(\mathbf{x}) \) on \( \sigma \)-dbs of degree at most \( d \) can be solved with counting time \( O(1) \) (delay \( O(\tilde{t}_d k) \)), update time \( (\tilde{t}_u + \tilde{t}_c)2^{d^{O(|\tau|)}} \), and initialisation time \( \tilde{t}_i 2^{d^{O(|\tau|)}} \) where \( \tilde{t}_x = \max_{c=1}^k t_x(c, d^{O(|\tau|)}) \) for \( t_x \in \{ t_i, t_u, t_c, t_d \} \).

**Proof.** We prove part [1] by a reduction from \( \text{conn-sph}_r(\mathbf{x}) \) to \( \varphi_c \). We use the notation introduced at the beginning of Section 7 and we let \( \tau_j := \tau_{j,\nu_j} \) for every \( j \in [c] \). For a \( \sigma \)-db \( D \) we let \( G_D \) be the \( \sigma_c \)-db with

\[
C^{G_D}_j := \{ v_\mathbf{x} : \mathbf{x} \in \text{dom}(D)^k_j \text{ with } (N^D_{r_j}(\mathbf{x}), \mathbf{x}) \equiv \tau_j \}, \text{ for all } j \in [c], \text{ and }
\]

\[
E^{G_D} := \{ (v_\mathbf{x}, v_\mathbf{y}) \in V^2 : \text{dist}^D(D, (\mathbf{x}), (\mathbf{y})) \leq 2r+1 \},
\]

where \( V := \bigcup_{j \in [c]} C^{G_D}_j \). We will shortly write \( E \) and \( C_j \) instead of \( E^{G_D} \) and \( C^{G_D}_j \).

Using Lemma 3.2 (and the fact that \( \tau_j \) is connected) we obtain that \( (v_\mathbf{x}, v_\mathbf{y}) \in E \) if \( N^D_r(\mathbf{x}, \mathbf{y}) \) is connected. If \( N^D_r(\mathbf{x}, \mathbf{y}) \) is connected, then \( \mathbf{y} \in (N^D_{r+|\mathbf{y}|-1}(\mathbf{x}))^{\mathbf{x}} \). It follows that the degree of \( G_D \) is bounded by \( d2^{(2r+1)} \). Furthermore, by the definition of \( G_D \) and \( \varphi_c \) we get that
(π₁, . . . , πc) ∈ sphr(D) ⇐⇒ (vπ₁, . . . , vπc) ∈ ϕc(GD), for all tuples π₁, . . . , πc where πj has arity k_j for each j ∈ [c]. As a consequence, |sphr(D)| = |ϕc(GD)|, and we can therefore use the count routine for ϕc on GD to count the number of tuples in sphr(D). Furthermore, for each tuple (vπ₁, . . . , vπc) ∈ ϕc(GD) we can compute (π₁, . . . , πc) in time O(k). Therefore, given an enumerate routine for ϕc(GD) with delay t_d we can produce an enumeration of sphr(D) with delay O(t_dk).

It remains to show how to construct and maintain GD when the database D is updated. As initialisation for the empty database D∅ we just perform the init routine of the dynamic algorithm for ϕ_c(π) on σ_c-dbs of degree at most d'. The update routine of the dynamic algorithm for sphr(π) on σ-dbs of degree at most d is provided by the following claim.

Claim 7.2. If Dnew is obtained from Dold by one update step, then GDnew can be obtained from GDold by dO(k²r+k|σ|) update steps and additional computing time 2dO(|σ|k²d²r²+2).

Proof. Let the update command be of the form update R(a₁, . . . , a_{ar(R)}) with π = (a₁, . . . , a_{ar(R)}). Let r' = r + (k-1)(2r+1). Let D' ∈ {Dold, Dnew} be the database whose relation R contains the tuple π (either before deletion or after insertion). Note that all elements whose r'-neighbourhood might have changed belong to the set U := N^{D'}D (π).

For every j ∈ [c] and every tuple b of arity at most k of elements in U, we check whether the r-type (N^{Dnew}_r(b), b) of b is isomorphic to τ_j. Depending on the outcome of this test, we include or exclude b from the relation C_j. Note that it indeed suffices to consider the tuples b built from elements in U: The r-type of some tuple b is changed by the update command only if N^{D'}D(b) contains some element from π. Furthermore, we only have to consider tuples b whose r-neighbourhood N^{D'}r(b) is connected. Using Lemma 3.2[4], we therefore obtain that each component of b belongs to N^{D'}r(π) = U.

Afterwards, we update the coloured graph’s edge relation E: We consider all tuples b and b' of arity ≤ k built from elements in U, and check whether (1) there is a j ∈ [c] such that b ∈ C_j, (2) there is a j' ∈ [c] such that b' ∈ C_j', and (3) dist^{Dnew}(b, b') ≤ 2r+1. If all three checks return the result “yes”, then we insert the tuple (v(b), v(b')) into E, otherwise we remove it from E.

It remains to analyse the runtime of the described update procedure. By Lemma 3.2 |U| \leq ar(R)d^{r+1} ≤ \|σ\|^{d²(2r+1)} \leq d^{O(kr+k|σ|)} \leq d^{O(kr+|σ|)}. Furthermore, U can be computed in time (ar(R)d^{r+1})^{O(|σ|)} ≤ d^{O(kr|σ|+|σ|^2)}. The number of tuples b that we have to consider is at most |U|^{k+1} \leq d^{O(k²r+k|σ|)}.

For each such b we use Lemma 3.2[4] to check in time 2^{O(|σ|k²d²r²+2)} whether the r-type of b is isomorphic to τ_j, for some j ∈ [c]. In summary, for updating the sets C₁, . . . , Cc we use at most c|U|^{k+1} \leq d^{O(k²r+k|σ|)} calls of the update routine of the dynamic algorithm on coloured graphs, and in addition to that we use computation time at most 2^{O(|σ|k²d²r²+2)}.

By a similar reasoning we obtain that also the edge relation E can be updated by at most d^{O(k²r+k|σ|)} calls of the update routine of the dynamic algorithm on coloured graphs and additional computation time at most 2^{O(|σ|k²d²r²+2)}. For this note that we can use and maintain an additional array that allows us to check, for any a_i and b_j, in constant time whether dist^{D}(a_i, b_j) ≤ 2r+1. This completes the proof of Claim 7.2.

Finally, the preprocess routine of the dynamic algorithm for sphr(π) proceeds in the obvious way by first calling the init routine for D∅ and then performing |D₀| update steps to insert all the tuples of D₀ into the data structure. This completes the proof of part (1) of Lemma 7.1.

We now turn to the proof of part (2) of Lemma 7.1 For k = 0, the result follows immediately from Theorem 3.1. Consider the case where k ≥ 1. W.l.o.g. we assume that all the symbols of σ occur in ϕ (otherwise, we remove from σ all symbols that do not occur in ϕ). We start the preprocessing routine by using Theorem 3.1[1] to transform ϕ(π) into a d-equivalent query ψ(π) in Hanf normal form; this takes time 2\^d^{O(|σ|)}. The formula ψ is a Boolean combination.
of \(d\)-bounded Hanf-sentences and sphere-formulas (over \(\sigma\)) of locality radius at most \(r := 4^{q\tau(r)}\), and each sphere-formula is of arity at most \(k\). Note that for \(d' := d^{2k^2(2r+1)}\) as used in the lemma’s part (1), it holds that \(d' = d^{2\Theta(\ell\log \ell)}\). Let \(\chi_1, \ldots, \chi_s\) be the list of all Hanf-sentences that occur in \(\psi\) (recall that \(s \leq 2d^{O(\ell\log \ell)}\)).

We use Lemma 5.1 to compute the list \(L^\psi_{\sigma,d}(k) = \tau_1, \ldots, \tau_\ell\) (note that \(\ell \leq 2d^{O(\ell\log \ell)}\)). In parallel for each \(i \in [\ell]\), we use the dynamic algorithm for \(\text{sph}_\tau(\mathfrak{F})\) provided from the lemma’s part (1). Furthermore, for each \(j \in [s]\), we use the dynamic algorithm provided by Theorem 4.1 upon input of the Hanf-sentence \(\varphi := \chi_j\). In addition to the components used by these dynamic algorithms, our data structure also stores

- the set \(J := \{j \in [s] : D \models \chi_j\}\),
- the particular set \(I \subseteq [\ell]\) provided by Lemma 5.3 for \(\psi(\mathfrak{F})\) and \(J\), and
- the cardinality \(n = |\varphi(D)|\) of the query result.

The count routine simply outputs the value \(n\) in time \(O(1)\). The enumerate routine runs the enumerate routine on \(\text{sph}_\tau(D)\) for every \(i \in [s]\). Note that this enumerates, without repetition, all tuples in \(\varphi(D)\), because by Lemma 5.3 \(\varphi(D)\) is the union of the sets \(\text{sph}_\tau(D)\) for all \(i \in [s]\), and this is a union of pairwise disjoint sets. The update routine runs in parallel the update routines for all used dynamic data structures. Afterwards, it recomputes \(J\) by calling the answer routine for \(\chi_j\) for all \(j \in [s]\). Then, it uses Lemma 5.3 to recompute \(I\). The number \(n\) is then recomputed by letting \(n = \sum_{i \in [s]} n_i\), where \(n_i\) is the result of the count routine for \(\tau_i\). It is straightforward to verify that the overall runtime of the update routine is bounded by \((\hat{t}_n + \hat{t}_c)2d^{O(\ell\log \ell)}\).

8 Counting Results of FO+MOD Queries Under Updates

This section is devoted to the proof of the following theorem.

Theorem 8.1. There is a dynamic algorithm that receives a schema \(\sigma\), a degree bound \(d \geq 2\), a \(k\)-ary FO+MOD(\(\sigma\))-query \(\varphi(\mathfrak{F})\) (for some \(k \in \mathbb{N}\)), and a \(\sigma\)-db \(D_0\) of degree \(\leq d\), and computes within \(t_p = f(\varphi,d) \cdot |D_0|\) preprocessing time a data structure that can be updated in time \(t_u = f(\varphi,d)\) and allows to return the cardinality \(|\varphi(D)|\) of the query result within time \(O(1)\).

The function \(f(\varphi,d)\) is of the form \(2d^{O(\ell\log \ell)}\).

The theorem follows immediately from Lemma 7.1[2] and the following dynamic counting algorithm for the query \(\varphi_c(\mathfrak{F})\).

Lemma 8.2. There is a dynamic algorithm that receives a number \(c \geq 1\), a degree bound \(d \geq 2\), and a \(\sigma_c\)-db \(G_0\) of degree \(\leq d\), and computes \(|\varphi_c(G)|\) with \(d^{O(c^2)}\) initialisation time, \(O(1)\) counting time, and \(d^{O(c^2)}\) update time.

Proof. Recall that \(\varphi_c(z_1, \ldots, z_c) = \bigwedge_{i \in [c]} C_i(z_i) \land \bigwedge_{j \neq j'} -E(z_j, z_{j'})\). For all \(j, j' \in [c]\) with \(j \neq j'\) consider the formula \(\theta_{j,j'}(z_1, \ldots, z_c) := E(z_j, z_{j'}) \land \bigwedge_{i \in [c]} C_i(z_i)\). Furthermore, let \(\alpha(z_1, \ldots, z_c) := \bigwedge_{i \in [c]} C_i(z_i)\). Clearly, for every \(\sigma_c\)-db \(G\) we have

\[
\alpha(G) = C_1^G \times \cdots \times C_c^G,
\]

\[
\varphi_c(G) = \alpha(G) \setminus \bigcup_{j \neq j'} \theta_{j,j'}(G),
\]

and hence, \(|\varphi_c(G)| = |\alpha(G)| - \left| \bigcup_{j \neq j'} \theta_{j,j'}(G) \right|\).

By the inclusion-exclusion principle we obtain for \(J := \{(j, j') : j, j' \in [c], j \neq j'\}\) that

\[
\left| \bigcup_{j \neq j'} \theta_{j,j'}(G) \right| = \sum_{i \notin K \subseteq J} (-1)^{|K|-1} \left| \bigcap_{(j, j') \in K} \theta_{j,j'}(G) \right| = \sum_{i \notin K \subseteq J} (-1)^{|K|-1} |\varphi_K(G)|
\]
for the formula \( \varphi_K(z_1, \ldots, z_c) := \bigwedge_{i \in [c]} C_i(z_i) \land \bigwedge_{(j, j') \in K} E(z_j, z_{j'}) \).

Our data structure stores the following values:

- \(|C_i^G|\), for each \( i \in [c] \), and \( n_1 := |\alpha(G)| = \prod_{i \in [c]} |C_i^G| \),
- \(|\varphi_K(G)|\), for each \( K \subseteq J \) with \( K \neq \emptyset \), and
- \( n_2 := \sum_{G \neq K \subseteq J} (-1)^{|K|-1} |\varphi_K(G)| \) and \( n_3 := n_1 - n_2 \).

Note that \( n_3 = |\varphi_c(G)| \) is the desired size of the query result. Therefore, the **count** routine can answer in time \( O(1) \) by just outputting the number \( n_3 \).

It remains to show how these values can be initialised and updated during updates of \( G \).

The initialisation for the empty graph initialises all the values to 0. In the **update** routine, the values for \(|C_i^G|\) and \( n_1 \) can be updated in a straightforward way (using time \( O(c) \)). For each \( K \subseteq J \), the update of \(|\varphi_K(G)|\) is provided within time \( d^{O(c^2)} \) by the following Claim 8.3.

**Claim 8.3.** For every \( K \subseteq J \), the cardinality \(|\varphi_K(G)|\) of a \( \sigma_c \)-db \( G \) of degree at most \( d \) can be updated within time \( d^{O(c^2)} \) after \( d^{O(c^2)} \cdot |G_0| \) preprocessing time.

**Proof.** Consider the directed graph \( H := (V, K) \) with vertex set \( V := [c] \) and edge set \( K \). Decompose the Gaifman graph of \( H \) into its connected components. Let \( V_1, \ldots, V_s \) be the connected components (for a suitable \( s \leq c \)). For each \( i \in [s] \) let \( H_i := H[V_i] \) be the induced subgraph of \( H \) on \( V_i \). We write \( K_i \) to denote the set of edges of \( H_i \). For every \( i \in [s] \) let \( \ell_i = |V_i| \), and let \( t(i, 1) < t(i, 2) < \cdots < t(i, \ell_i) \) be the ordered list of the vertices in \( V_i \). Consider the query

\[
\varphi_{K_i}(z_{t(i,1)}, \ldots, z_{t(i,\ell_i)}) := \bigwedge_{j \in V_i} C_j(z_j) \land \bigwedge_{(j, j') \in K_i} E(z_j, z_{j'}). \tag{5}
\]

Note that \( \varphi_K \) is the conjunction of the formulas \( \varphi_{K_i} \) for all \( i \in [s] \). Since the variables of the formulas \( \varphi_{K_i} \) for \( i \in [s] \) are pairwise disjoint, we have \( \varphi_K(G) = \varphi_{K_1}(G) \times \cdots \times \varphi_{K_s}(G) \) (modulo permutations of the tuples), and thus \(|\varphi_K(G)| = \prod_{i \in [s]} |\varphi_{K_i}(G)|\).

For each \( i \in [s] \), the value \(|\varphi_{K_i}(G)|\) can be computed as follows. For every \( v \in \text{dom}(G) \) we consider the set \( S^v_i := \{ (w_{t(i,1)}, \ldots, w_{t(i,\ell_i)}) \in \varphi_{K_i}(G) : w_{t(i,1)} = v \} \). Since the Gaifman graph of \( H_i \) is connected and has \( \ell_i \) nodes, it follows that each component of every tuple in \( S^v_i \) is contained in the \( \ell_i \)-neighbourhood of \( v \) in \( G \), and this neighbourhood contains at most \( d^{\ell_i + 1} \) elements. Therefore, \(|S^v_i| \leq (d^{\ell_i + 1})^{\ell_i} \), and using breadth-first search starting from \( v \), the set \( S^v_i \) can be computed in time \( d^{O(c^2)} \). Note that \( \varphi_{K_i}(G) \) is the disjoint union of the sets \( S^v_i \) for all \( v \in \text{dom}(G) \). Therefore, \(|\varphi_{K_i}(G)| = \sum_{v \in \text{dom}(G)} |S^v_i|\).

In our data structure we store for every \( i \in [s] \) and every \( v \in \text{dom}(G) \) the number \( n_{i,v} = |S^v_i| \).

Moreover, for every \( i \in [s] \) we store the sum \( n_i = \sum_{v \in \text{dom}(G)} n_{i,v} = |\varphi_{K_i}(G)| \).

The initialisation for the empty \( \sigma_c \)-db \( G_0 \) sets all these values to 0. Whenever the colour of a vertex of \( G \) is updated or an edge is inserted or deleted, we update all affected numbers accordingly. Note that a number \( n_{i,v} \) changes only if \( v \) is in the \( c \)-neighbourhood around the updated edge or vertex in the graph \( G \). Hence, for at most \( 2d^{\ell_i + 1} \) vertices \( v \), the numbers \( n_{i,v} \) are affected by an update, and each of them can be updated in time \( d^{O(c^2)} \). Moreover, for each \( i \in [s] \), the sum \( n_i \) can be updated in time \( O(d^{\ell_i + 1}) \) by subtracting the old value of \( n_{i,v} \) and adding the new value of \( n_{i,v} \) for each of the at most \( 2d^{\ell_i + 1} \) relevant vertices \( v \). Finally, it takes time \( O(c) \) to compute the updated value \(|\varphi_K(G)| = \prod_{i \in [s]} n_i \). The overall time used to produce the update is \( d^{O(c^2)} \).

Once we have available the updated numbers \(|\varphi_K(G)|\) for all \( K \subseteq J \), the value \( n_3 \) can be computed in time \( O(2^J) \leq 2^{O(c^2)} \). And \( n_3 \) is then obtained in time \( O(1) \). Altogether, performing the **update** routine takes time at most \( d^{O(c^2)} \). The **preprocess** routine initialises all values for the empty graph and then uses \(|G_0| \) update steps to insert all the tuples of \( G_0 \) into the data structure. This completes the proof of Lemma 8.2. 

\[ \square \]
9 Enumerating Results of FO+MOD Queries Under Updates

In this section we prove (and afterwards, improve) the following theorem.

**Theorem 9.1.** There is a dynamic algorithm that receives a schema \( \sigma \), a degree bound \( d \geq 2 \), a \( k \)-ary \( \text{FO}+\text{MOD}[\sigma] \)-query \( \varphi(\tau) \) (for some \( k \in \mathbb{N} \)), and a \( \sigma \)-db \( D_0 \) of degree \( \leq d \), and computes within \( t_p = f(\varphi, \sigma) \cdot |D_0| \) preprocessing time a data structure that can be updated in time \( t_u = f(\varphi, \sigma) \) and allows to enumerate \( \varphi(D) \) with \( d^O(|\varphi|) \) delay.

The function \( f(\varphi, \sigma) \) is of the form \( 2^d \mathcal{O}(\varphi) \).

The theorem follows immediately from Lemma 7.1(2) and the following dynamic enumeration algorithm for the query \( \varphi_c(\mathcal{G}) \).

**Lemma 9.2.** There is a dynamic algorithm that receives a number \( c \geq 1 \), a degree bound \( d \geq 2 \), and a \( \sigma_c \)-db \( \mathcal{G}_0 \) of degree \( \leq d \), and computes within \( t_p = d^\text{poly}(c) \cdot |\mathcal{G}_0| \) preprocessing time a data structure that can be updated in time \( d^\text{poly}(c) \) and allows to enumerate the query result \( \varphi_c(\mathcal{G}) \) with \( \mathcal{O}(c^3d) \) delay.

**Proof.** For a \( \sigma_c \)-db \( \mathcal{G} \) and a vertex \( v \in \text{dom}(\mathcal{G}) \) we let \( N^\mathcal{G}(v) \) be the set of all neighbours of \( v \) in \( \mathcal{G} \). I.e., \( N^\mathcal{G}(v) \) is the set of all \( w \in \text{dom}(\mathcal{G}) \) such that \( (v, w) \) or \( (w, v) \) belongs to \( E^\mathcal{G} \).

The underlying idea of the enumeration procedure is the following greedy strategy. We cycle through all vertices \( u_1 \in C^\mathcal{G}_1, u_2 \in C^\mathcal{G}_2 \setminus N^\mathcal{G}(u_1), u_3 \in C^\mathcal{G}_3 \setminus (N^\mathcal{G}(u_1) \cup N^\mathcal{G}(u_2)), \ldots, u_c \in C^\mathcal{G}_c \setminus \left( \bigcup_{i=1}^{c-1} N^\mathcal{G}(u_i) \right) \) and output \( (u_1, \ldots, u_c) \). This strategy does not yet lead to a constant delay enumeration, as there might be vertex tuples \( (u_1, \ldots, u_i) \) (for \( i < c \)) that do extend to an output tuple \( (u_1, \ldots, u_c) \), but where many possible extensions are checked before this output tuple is encountered. We now show how to overcome this problem and describe an enumeration procedure with \( \mathcal{O}(c^3d) \) delay and update time \( d^\text{poly}(c) \).

Note that for every \( J \subseteq [c] \) we have \( |\bigcup_{j \in J} N^\mathcal{G}(u_j)| \leq cd \). Hence, if a set \( C^\mathcal{G}_i \) contains more than \( cd \) elements, we know that every considered tuple has an extension \( u_i \in C^\mathcal{G}_i \) that is not a neighbour of any vertex in the tuple. Let \( I := \{ i \in [c] : |C^\mathcal{G}_i| \leq cd \} \) be the set of small colour classes in \( \mathcal{G} \) and to simplify the presentation we assume without loss of generality that \( I = \{ 1, \ldots, s \} \). In our data structure we store the current index set \( I \) and the set

\[
S := \left\{ (u_1, \ldots, u_s) \in C^\mathcal{G}_1 \times \cdots \times C^\mathcal{G}_s : (u_j, u_{j'}) \notin E^\mathcal{G}, \text{ for all } j \neq j' \right\}
\]

of tuples on the small colours. Note that a tuple \( (u_1, \ldots, u_s) \in C^\mathcal{G}_1 \times \cdots \times C^\mathcal{G}_s \) extends to an output tuple \( (u_1, \ldots, u_c) \in \varphi_c(\mathcal{G}) \) if and only if it is contained in \( S \). We store the current sizes of all colours and this enables us to keep the set \( I \) of small colours updated. Moreover, as \( |S| \leq (cd)^c \), we can update the set \( S \) in time \( d^\text{poly}(c) \) after every update by a brute-force approach. The enumeration procedure is given in Algorithm 1.

**Algorithm 1** Enumeration procedure with delay \( \mathcal{O}(c^3d) \)

1. **for all** \( (u_1, \ldots, u_s) \in S \) **do** \( \text{ENUM}(u_1, \ldots, u_s) \).
2. **Output** the end-of-enumeration message \( \text{EOE} \).
3. **function** \( \text{ENUM}(u_1, \ldots, u_i) \)
4. **if** \( i = c \) **then** output the tuple \( (u_1, \ldots, u_c) \).
5. **else**
6. **for all** \( u_{i+1} \in C^\mathcal{G}_{i+1} \) **do**
7. **if** \( u_{i+1} \notin \bigcup_{j=1}^{i} N^\mathcal{G}(u_j) \) **then** \( \text{ENUM}(u_1, \ldots, u_i, u_{i+1}) \).

It is straightforward to see that this procedure enumerates \( \varphi_c(\mathcal{G}) \). Let us analyse the delay. Since for all \( i > s \) we have \( |C^\mathcal{G}_i| > cd \), it follows that every call of \( \text{ENUM}(u_1, \ldots, u_i) \) leads to at
least one recursive call of \( \text{ENUM}(u_1, \ldots, u_i, u_{i+1}) \). Furthermore, there are at most \( cd \) iterations of the loop in line 7 that do not lead to a recursive call. As every test in line 8 can be done in time \( O(c) \), it follows that the time spans until the first recursive call, between the calls, and after the last call are bounded by \( O(c^2d) \). As the recursion depth is \( c \), the overall delay between two output tuples is bounded by \( O(c^3d) \). \( \Box \)

By using similar techniques as in [6], we obtain the following improved version of Lemma 9.2, where the delay is independent of the degree bound \( d \).

**Lemma 9.3.** There is a dynamic algorithm that receives a number \( c \geq 1 \), a degree bound \( d \geq 2 \), and a \( \sigma_c \)-db \( G_0 \) of degree \( \leq d \), and computes within \( t_p = d^{O(c)} \cdot |G_0| \) preprocessing time a data structure that can be updated in time \( d^{O(c)} \) and allows to enumerate the query result \( \varphi_c(G) \) with \( O(c^2) \) delay.

Before proving Lemma 9.3, let us first point out that Lemma 9.3 in combination with Lemma 7.1[2] directly improves the delay in Theorem 9.1 from \( O(k) \) to \( O(k^3) \), immediately leading to the following theorem.

**Theorem 9.4.** There is a dynamic algorithm that receives a schema \( \sigma \), a degree bound \( d \geq 2 \), a \( k \)-ary \( \text{FO} + \text{MOD}[\sigma] \)-query \( \varphi(\pi) \) (for some \( k \in \mathbb{N} \)), and a \( \sigma \)-db \( D_0 \) of degree \( \leq d \), and computes within \( t_p = f(\varphi, d) \cdot |D_0| \) preprocessing time a data structure that can be updated in time \( t_u = f(\varphi, d) \) and allows to enumerate \( \varphi(D) \) with \( O(k^3) \) delay.

The function \( f(\varphi, d) \) is of the form \( 2^{d^{O(1)}} \).

The rest of the section is devoted to the proof of Lemma 9.3.

**Proof of Lemma 9.3.** Consider Algorithm 1 which enumerates \( \varphi_c(G) \) with \( O(c^2d) \) delay. To enumerate the tuples with only \( O(c^2) \) delay, we replace the loop in lines 7–8 by a precomputed “skip” function that allows to iterate through all elements in \( C^G_{i+1} \setminus \bigcup_{j=1}^i N^G(u_j) \) with \( O(c) \) delay.

For every \( i \in [c] \) we store all elements of \( C^G_i \) in a doubly linked list and let \texttt{void} be an auxiliary element that appears at the end of the list. We let \texttt{first}, be the first element in the list and \texttt{succ}(u) the successor of \( u \in C^G_i \). We denote by \( \leq i \) the linear order induced by this list. We let \( E^G \) be the symmetric closure of \( E^G \), i.e., \( E^G = E^G \cup \{(v, u) : (u, v) \in E^G\} \). For every \( i \in [c] \) we define the function

\[
\text{skip}_i(y, V) := \min \left\{ z \in C^G_i \cup \{\texttt{void}\} : y \leq i z \text{ and for every } v \in V, (v, z) \notin E^G \right\},
\]

which assigns to every \( V \subseteq \text{adom}(G) \) with \( |V| \leq c-1 \), and every \( y \in C^G_i \) the next node that is not adjacent to any vertex in \( V \).

Using these functions, our improved enumeration algorithm is given in Algorithm 2. Below, we show that we can access the values \( \text{skip}_i(y, V) \) in time \( O(c) \). By the same analysis as given in the proof of Lemma 9.2, it then follows that Algorithm 2 enumerates \( \varphi_c(G) \) with \( O(c^2) \) delay.

What remains to show is that we can access the values \( \text{skip}_i(y, V) \) for all \( i, y, V \) in time \( O(c) \) and maintain them with \( d^{O(c)} \) update time. At first sight, this is not clear at all, because the domain of \( \text{skip}_i \) has size \( \Omega(|\text{adom}(G)|^c) \). In what follows, we show that for every \( y \), the number of distinct values that \( \text{skip}_i(y, V) \) can take is bounded by \( d^{O(c)} \), and that we can store them in a look-up table with update time \( d^{O(c)} \).

To illustrate the main idea, let us start with a simple example. We want to enumerate \( \varphi_4 \) on a coloured graph \( H \) with four vertex colours blue, red, yellow, and green (in this order) and analyse the call of \text{ENUM}(b, r, y), which is supposed to enumerate all green nodes \( g_t \) that are not adjacent to any of the nodes \( b, r, \) and \( y \). The relevant part of \( H \) is depicted in Figure 1.

The enumeration procedure starts by considering the first element \( g_1 \) in the list of green vertices, but the first element in the actual output is \( g_3 = \text{skip}_i(g_1, \{b, r, y\}) \). Therefore, we have to skip the irrelevant vertices \( g_1, \ldots, g_4 \).
Algorithm 2 Enumeration procedure with delay $O(c^2)$

1: for all $(u_1, \ldots, u_c) \in S$ do
2: \hspace{1em} ENUM$(u_1, \ldots, u_c)$.
3: Output the end-of-enumeration message EOE.
4: 
5: function ENUM$(u_1, \ldots, u_i)$
6: \hspace{1em} if $i = c$ then
7: \hspace{2em} output the tuple $(u_1, \ldots, u_c)$.
8: \hspace{1em} else
9: \hspace{2em} $y \leftarrow$ skip$_{i+1}$(first$_{i+1}$,$\{u_1, \ldots, u_i\}$)
10: \hspace{2em} while $y \neq$ void do
11: \hspace{3em} ENUM$(u_1, \ldots, u_i, y)$.
12: \hspace{2em} $y \leftarrow$ skip$_{i+1}$(succ$_{i+1}$(y),$\{u_1, \ldots, u_i\}$).

Figure 1: Illustration of the relevant part of graph $\mathcal{H}$

To do this, we want to know the neighbours of the vertices that we skip (b and r in our example) when looking at $g_1$. For this purpose, we define inductively new sorts of edges $E^1_i \subseteq E^2_i \subseteq \cdots$ that connect green vertices $g_i$ with $\tilde{E}$-neighbours of skipped vertices. In our example, we first have to skip $g_1$, because it is $\tilde{E}$-connected to b and we indicate this by letting $E^1_1$ be the set of tuples $(g_i, v) \in \tilde{E}^\mathcal{H}$ (see Figure 2).

Figure 2: $\tilde{E}$-edges and $E^1_1$-edges in our example

After realising that even more vertices ($g_2$ and $g_3$) are excluded by b, the next try would be $g_4$. However, this vertex is excluded by its $\tilde{E}$-neighbour r, so we have to take r into account when computing the skip value for $g_1$ and indicate this by the $E^2_1$-edge $(g_1, r)$ (see Figure 3). This immediately leads to an inductive definition: $E^2_1$ contains all pairs of vertices that are already in $E^1_1$ or connected by a path as shown in Figure 4.

The idea outlined above can be formalised as follows. For $i, j \in [c]$, we define inductively the auxiliary edge sets $E^j_i$:

\[
E^1_i := \{ (y, u) : y \in C^G_i \text{ and } (y, u) \in \tilde{E}^G \} \quad \text{and} \quad E^{j+1}_i := E^j_i \cup \{ (y, u) : \text{there are } v, z \text{ with } (y, v) \in E^j_i, (v, z) \in \tilde{E}^G, (\text{succ}_i(z), u) \in \tilde{E}^G \}
\]

Now we define for every $y \in C^G_i$ the set

\[
S^y_i := \{ u : (y, u) \in E^c_i \}.
\]
Note that $|S^y_i| = O(d^c)$. The following claim states that $S^y_i$ are the only vertices we need to take into account when computing $\text{skip}_i(y, V)$.

**Claim 9.5.** For all $i \leq c$, $y \in C^G_i \cup \{\text{void}\}$, and $V \subseteq \text{adm}(G)$ with $|V| \leq c-1$ it holds that

$$\text{skip}_i(y, V) = \text{skip}_i(y, V \cap S^y_i).$$

**Proof.** The proof is identical to the proof of Claim 1 in [6]. For the reader’s convenience, we include a proof here.

If $y = \text{void}$, the lemma is trivial. Hence assume that $y \neq \text{void}$ and let $z := \text{skip}_i(y, V \cap S^y_i)$. By definition we have $y \leq^i z \leq^i \text{skip}_i(y, V)$ and therefore we have to show $z \geq^i \text{skip}_i(y, V)$, which holds if and only if $(u, z) \notin E^G$ for all $u \in V \setminus S^y_i$. If $z = y$, the claim clearly holds as all $E^G$-neighbours of $y$ are contained in $S^y_i$. Hence we have $z >^i y$ and let $z' >^i y$ be the predecessor of $z$, i.e., $z = \text{succ}_i(z')$. Now assume for contradiction that there is an $u \in V \setminus S^y_i$ such that $(\ast) (u, z) \in E^G$. Note that since $z' >^i z = \text{skip}_i(y, V \cap S^y_i)$, there is a $v \in V \cap S^y_i$ such that $(\ast\ast) (v, z') \in E^G$. In the following we show that $(\ast\ast\ast) (y, v) \in E^G_{i-1}$. Note that this finishes the proof of the claim, as by the definition of $E^G_i$, the statements $(\ast)$, $(\ast\ast)$, and $(\ast\ast\ast)$ imply that $u \in S^y_i$, contradicting the assumption that $u \in V \setminus S^y_i$.

To show that $(y, v) \in E^G_{i-1}$, let

$$V_j := \{ v' \in V : (y, v') \in E^G_i \}$$

for all $j \in [c]$. Note that $V_c = V \cap S^y_i$. Furthermore, if there is a $j < c$ with $V_j = V_{j+1}$, then we have

$$V_j = V_{j+1} = \cdots = V_c = V \cap S^y_i.$$  

(9)

Since $|V| \leq c-1$ and $u \in V \setminus S^y_i$, we have $|V \cap S^y_i| \leq c - 2$. In particular, it holds that $V_{c-1} = V \cap S^y_i$. Since $v \in V \cap S^y_i$, it holds that $v \in V_{c-1}$ and thus $(y, v) \in E^G_{i-1}$.

In our dynamic algorithm we maintain an array that allows random access to the values $\text{skip}_i(y, S')$ for all $y \in C^G_i$ and all $S' \subseteq S^y_i$ of size at most $c-1$. By Claim 9.5 we can then compute $\text{skip}_i(y, V)$ by first computing $S' = V \cap S^y_i$ and then looking up $\text{skip}_i(y, S')$. This can be done in time $O(c)$. The next claim states that we can efficiently maintain the sets $S^y_i$.

**Claim 9.6.** There is a data structure that
1. stores the elements from the sets $S_i^y$ and all subsets $S' \subseteq S_i^y$ of cardinality at most $c-1$,

2. allows to test membership in these sets in time $O(1)$, and

3. can be updated in time $d^{\text{poly}(c)}$ after every update of the form $\text{insert } C_i(v)$, $\text{delete } C_i(v)$, $\text{insert } E(u, v)$, and $\text{delete } E(u, v)$.

Proof. Note that $u \in S_i^y \iff (y, u) \in E_i^y$. We store the edge sets $E_i^j$ for all $i, j \in [c]$ in adjacency lists and additionally maintain arrays to allow constant-time access to the list entries. This allows us to store a list of elements from $S_i^y$ and access the elements in $S_i^y$ in constant time. Moreover, as the size of $S_i^y$ is bounded by $O(d^c)$, the number of subsets $S' \subseteq S_i^y$ of cardinality at most $c-1$ is bounded by $O(d^c)$. Consequently, we can provide constant-time access to all these subsets $S'$.

On every insertion or deletion of an edge in $E_i^G$, as well as every insertion or deletion of a vertex in $C_i^G$, at most $O(d)$ pairs in the relation $E_i^1$ change and the relation can be updated in time $O(d)$. Afterwards we update the edge sets $E_i^j$ according to their inductive definition. To do this efficiently, we use a breadth-first search starting from $u$ and $v$, for every tuple $(u, v)$ that has changed in relation $E_i^1$, up to depth $3c$ to identify the relevant nodes that are affected by the change. By using the adjacency lists, this can be done in time $d^{\text{poly}(c)}$ as the degree of the edge sets is bounded by $d^{\text{poly}(c)}$. We leave the details to the reader. □

In our data structure we store the values $\text{skip}_i(y, S')$ for every $i \in [c], y \in C_i^G$ and for all sets $S' \subseteq S_i^y$ of cardinality at most $c-1$. On every insertion or deletion of an edge, we update the sets $S_i^y$ and their subsets $S'$ of cardinality at most $c-1$ and update affected values of $\text{skip}_i(y, S')$. According to Claim 9.6 this can be done in time $d^{\text{poly}(c)}$.

We do the same on updates of the form $\text{insert } C_i(v)$ and $\text{delete } C_i(v)$, but have to do some additional work, as $v$ might occur in the image of skip-functions. Upon insert $C_i(v)$, we insert $v$ at the beginning of the list $C_i$. This ensures that existing skip values will not be affected. Afterwards, we compute the set $S_i^y$ and the values $\text{skip}_i(v, S')$ for all $S' \subseteq S_i^y$ of cardinality at most $c-1$. Again, this can be done in time $d^{\text{poly}(c)}$.

If we receive the update $\text{delete } C_i(v)$, then we have to recom pute all skip values $\text{skip}_i(y, S')$ that point to $v$. Note that (since $G$ has degree $\leq d$) this is only the case for nodes $y \leq_i^v w$ whose distance from $v$ w.r.t. $\text{succ}_i$ is at most $(c-1)d$. Hence, it suffices to recompute $\text{skip}_i(y, S')$ for at most $(c-1)d$ vertices $y$ and all $S' \subseteq S_i^y$ of cardinality at most $c-1$. This can be done in time $d^{\text{poly}(c)}$. By Claim 9.5 all this suffices to access the value for $\text{skip}_i(y, V)$ in time $O(c)$. This concludes the proof of Lemma 9.3. □

10 Conclusion

Our main results show that in the dynamic setting (i.e., allowing database updates), the results of $k$-ary FO+MOD-queries on bounded degree databases can be tested and counted in constant time and enumerated with constant delay, after linear time preprocessing and with constant update time. Here, “constant time” refers to data complexity and is of size $\text{poly}(k)$ concerning the delay and the time for testing and counting. The time for performing a database update is $3$-fold exponential in the size of the query and the degree bound, and is worst-case optimal.

The starting point of our algorithms is to decompose the given query into a query in Hanf normal form, using a recent result of [10]. This normal form is only available for the setting with a fixed maximum degree bound $d$, i.e., the setting considered in this paper.

Recently, Kuske and Scheweikardt [13] introduced a new kind of Hanf normal form for a variant of first-order logic with counting that contains and extends Libkin’s logic FO(Cnt) [14] and Grohe’s logic FO+C [8]. As an application it is shown in [13] that the present paper’s techniques can be lifted from FO+MOD to first-order logic with counting.
An obvious future task is to investigate to which extent further query evaluation results that are known for the static setting can be lifted to the dynamic setting. More specifically: Are there efficient dynamic algorithms for evaluating (i.e., answering, testing, counting, or enumerating) results of first-order queries on other sparse classes of databases (e.g. planar, bounded treewidth, bounded expansion, nowhere dense) or databases of low degree, lifting the “static” results accumulated in [12] [9] [6] to the dynamic setting?

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