ON \textit{q}-DEFORMED LOGISTIC MAPS

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Abstract. We consider the logistic family \(f_a\) and a family of homeomorphisms \(\phi_q\). The \textit{q}-deformed system is given by the composition map \(f_a \circ \phi_q\). We study when this system has non zero fixed points which are LAS and GAS. We also give an alternative approach to study the dynamics of the \textit{q}-deformed system with special emphasis on the so-called Parrondo’s paradox finding parameter values \(a\) for which \(f_a\) is simple while \(f_a \circ \phi_q\) is dynamically complicated. We explore the dynamics when several \textit{q}-deformations are applied.

1. Introduction. The complexity of some mechanic and thermodynamic systems has produced the development of techniques as nonextensive statistical mechanics, which differ from the standard ones [31]. In this framework \textit{q}-deformed models appear (see [11, 18]).

Recall briefly some background. By a discrete system we mean a pair \((X, f_n)\), where \(X\) is a (compact) metric space and \(f_n : X \rightarrow X\) is a sequence of continuous maps. Given \(x \in X\), its orbit under \(f\) is given by the solution of the difference equation
\[
\begin{cases}
x_{n+1} = f_n(x_n), \\
x_0 = x.
\end{cases}
\]
When \(f_n = f\) is constant, the pair \((X, f)\) is a classical discrete dynamical system. When the sequence \(f_n\) is periodic, we have a discrete periodic system. Along this paper, we assume that \(X = I = [0, 1]\).

Roughly speaking a \textit{q}-deformation allows us to modify or introduce a deformation in a discrete dynamical system by using a parameter \(q\) such that, when \(q\) converges to 1, the original discrete dynamical system is obtained. This modified version of the original system could help to study the complexity of the system in the limit. Although different functions have been analyzed using \textit{q}-deformations, see [25] for Hénon map, logistic map and its \textit{q}-deformations received a great interest (see [4, 16] and other references therein). In addition, \textit{q}-deformations have been also studied with fractional systems (see [21, 32]). Among the applications of \textit{q}-deformations we can mention [5, 27].

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We can simplify the analysis of a \( q \)-deformed map if we see the deformation as a periodic system of period two. So, we consider the logistic family \( f_a(x) = ax(1-x) \), \( a \in (0,4] \) and, following [16], the family of homeomorphisms
\[
\phi_q(x) = \frac{x}{1+(1-q)(1-x)},
\]
for any \( x \in I \) and \( q \in (-\infty,2) \). Observe that when \( q \) tends to 1 then \( \phi_q(x) \) tends to \( x \). Additionally, \( \phi_q \) is a strictly increasing homeomorphism. The \( q \)-deformed discrete dynamical system is then given by the map \( F_{a,q} = f_a \circ \phi_q \), which is the two power of the periodic sequence
\[
(\phi_q, f_a, \phi_q, f_a, \ldots).
\]
Shifting it we obtain the one given by the periodic sequence
\[
(f_a, \phi_q, f_a, \phi_q, \ldots),
\]
from which we may obtain the map \( \Phi_{q,a} = \phi_q \circ f_a \). Both maps, \( F_{a,q} \) and \( \Phi_{q,a} \), share most of their dynamical properties (see [8]). So, we can take any of them depending on the problem we want to study.

In this paper we go a step further and wonder about the dynamics of a deformed map when several \( q \)-deformations are applied; in other words, we analyze the behavior of the periodic system
\[
(\phi_{q_1}, \ldots, \phi_{q_k}, f_a, \phi_{q_k}, \ldots, \phi_{q_1}, f_a, \ldots),
\]
for some integer \( k \). The deformed system is defined by the map \( F_{a,q_k,\ldots,q_1} = f_a \circ \phi_{q_k} \circ \ldots \circ \phi_{q_1} \), but we can make use of \( \Phi_{q_k,\ldots,q_1,a} = \phi_{q_k} \circ \ldots \circ \phi_{q_1} \circ f_a \) when it will be more suitable for practical reasons. As we will show, the key point for this general case is that the composition of two \( q \)-deformations is a \( q \)-deformation as well.

We will analyze the local asymptotic stability (LAS) of fixed points of the single and generalized \( q \)-deformations of the logistic family \( f_a \), and study when a LAS fixed point is also globally asymptotically stable (GAS). We point out that this is a natural problem when studying the stability of difference equations (see e.g. [13]).

On the other hand, in [10], we studied the parameter region for which the \( q \)-deformed system is chaotic and checked that the so-called Parrondo’s paradox (see e.g. [9]) appears in this system by finding parameter values of \( a \) and \( q \) such that \( f_a \) has a simple dynamics while the dynamics of \( \Phi_{q,a} \) is complicated. Note that the dynamics of \( \phi_q \) is trivially simple and hence, combining two dynamically simple maps we obtain a complicated dynamic behavior. It is worth to mention that a periodic sequence given by homeomorphisms cannot support the paradox because the composition of homeomorphisms is another homeomorphism. Hence, a periodic sequence given by homeomorphisms and exactly one unimodal map, i.e with two monotone pieces, is the simplest scenario in which Parrondo’s paradox “simple + simple = complex” is possible.

This paper is organized as follows. Next Section is devoted to introduce some basic mathematical background which is necessary to understand the paper properly. Section 3 will be devoted to analyze the stability of fixed points of \( q \)-deformations of the logistic family, characterizing when they are LAS and GAS. In Section 4 we will analyze the complexity of the dynamics of \( q \)-deformations of the logistic family. In particular, we compute the topological entropy for single and several \( q \)-deformations. We check the evolution of Parrondo’s paradox when the number of applied \( q \)-deformations increases.
2. Mathematical background.

2.1. Attractors, Lyapunov exponents and negative Schwarzian derivative.

First, we fix the notation (see e.g. [15]). Let \( I = [0, 1] \) and let \( f : I \to I \) be a continuous interval map. As usual, for \( n \in \mathbb{N} \), \( f^n = f^{n-1} \circ f \), \( f^1 = f \). For \( x \in I \), its orbit under \( f \) is given by the sequence \( \{ f^n(x) \} \). Its \( \omega \)-limit set, \( \omega(x, f) \), is the set of accumulation points of the orbit.

A point \( x \in I \) is periodic with period \( n \) if \( f^n(x) = x \) and \( f^i(x) \neq x \) for \( i < n \). A periodic point of period one is a fixed point. A fixed point \( x \) is locally asymptotically stable (LAS) if there exists a non-trivial interval (positive length) \( J \) such that for each \( y \in J \lim_{n \to \infty} (f^n(y)) = x \). If \( J = (0, 1) \) we say that \( x \) is globally asymptotically stable (GAS). In particular, if \( x \) is LAS (resp. GAS), then \( |f'(x)| \leq 1 \) (see [13, pp. 25]) whenever \( f \) is differentiable. The stability criteria for nonhyperbolic fixed points is more complex and may be found in [13, pp. 28-39]. A periodic point \( x \) of period \( n \) is locally asymptotically stable (LAS) if there exists a non-trivial interval (positive length) \( J \) such that for each \( y \in J \), we have that \( \omega(y, f) = \{ x, f(x), ..., f^{n-1}(x) \} \).

For differentiable maps, if \( x \) is a locally stable periodic point of period \( n \), then \( |(f^n)'(x)| \leq 1 \).

A set \( \Omega \) is called forward invariant under \( f \) if \( f(\Omega) = \Omega \). The basin of attraction of a forward invariant set \( \Omega \), \( B(\Omega) \) is given by

\[
B(\Omega) = \{ x : \omega(x, f) \subset \Omega \}.
\]

Following Milnor [22], a forward invariant set \( \Omega \) is called a (minimal) metric attractor if \( B(\Omega) \) has positive Lebesgue measure and satisfies that if \( \Omega' \) is a forward invariant compact set strictly contained in \( \Omega \), then \( B(\Omega') \) has Lebesgue measure equal to zero. If \( f : I \to I \) is differentiable, a point \( c \) is called a turning point if \( f'(c) = 0 \). If \( f \) is differentiable enough, a turning point \( c \) is nonflat if \( f^n(c) \neq 0 \) for some \( n \geq 2 \). If the map \( f \) is \( C^3 \), its Schwarzian derivative at \( x \) is given by

\[
S(f)(x) = \frac{f'''(x)}{f'(x)^2} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2
\]

for all \( x \in I \) such that \( f'(x) \neq 0 \). The Schwarzian derivative of a composition is obtained as follows (see [28, Theorem 2.1]).

**Theorem 2.1.** Let \( f \) and \( g \) be \( C^3 \)-maps, then

\[
S(f \circ g)(x) = (S(f)(g(x)) \cdot (g'(x))^2 + S(g)(x),
\]

at those points \( x \in I \) for which these expressions are well defined.

Recall that \( f : I \to I \) is unimodal if it attains the maximum at a point \( c \in (0, 1) \) such that \( f \) is strictly increasing on \([0, c] \) and strictly decreasing on \([c, 1]\). Regarding LAS and GAS we have the following results which can be found in [28].

**Theorem 2.2.** If a unimodal map \( f \) is \( C^3 \) with \( S(f)(x) < 0 \) and has a unique fixed point \( x \in (0, 1) \) which is LAS, then it is GAS.

**Theorem 2.3.** Let \( f : I \to I \) be a \( C^3 \) interval map with \( S(f)(x) < 0 \) for each \( x \in [0, 1] \) such that \( f'(x) \neq 0 \). Then each LAS periodic orbit attracts at least one critical point or boundary point.

An interval \( J \subset I \) is called wandering if the intervals \( \{ J, f(J), ..., f^n(J), ... \} \) are pairwise disjoint and \( J \) is not contained in the basin of a LAS periodic orbit of \( f \). The following result can be found in [12, Chapter II. Theorem 6.3].
Theorem 2.4 ([15],[12]). Let $f : I \to I$ be a $C^3$ unimodal map with negative Schwarzian derivative and such that $f''(c) \neq 0$ at the unique critical point $c$ of $f$. Then $f$ has no wandering intervals.

Following [14, Corollary 1], we can state that every metric attractor of a $C^3$ unimodal map $f$ is either

A1 a periodic orbit, or
A2 a finite union of pairwise disjoint subintervals intervals $I_1,...,I_k$ such that $f^k(I_j) = I_j$ and $f^k|_{I_j}$ has a dense orbit for $j = 1,2,...,k$, or
A3 a Cantor set,
and there is at most one metric attractor of type A2 or A3 which must attract the orbit of the turning point. If, in addition, the map $f$ has negative Schwarzian derivative the number of attractors is at most two, and just one if 0 is not LAS.

For $x \in J$, its Lyapunov exponent (see e.g. [12]) is defined by,

$$\lambda(f,x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log |f'(f^i(x))|.$$ 

If we compute it at the turning point, we observe that when the Lyapunov exponent is negative, the map $f$ has an attractor of type A1 and therefore, the dynamics of the map $f$ seems to be simple. Thus, positive Lyapunov exponents are necessary to display observable chaos.

2.2. Topological entropy and Li-Yorke chaos. A continuous map $f : I \to I$ is called piecewise monotone if there is a finite collection of subintervals of $I$ whose union is $I$, with $f$ monotone on each of these intervals. If, in addition, the map $f$ is strictly monotone on each of these intervals, then it is said to be strictly piecewise monotone. A maximal interval on which $f$ is monotone is called a lap. The number of pieces of monotonicity of $f$ is called the lap number, denoted by $c(f)$.

The topological entropy $h(f)$ of a continuous map on a compact topological space was introduced in [1], but for piecewise monotone interval maps can be computed as

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log c(f^n),$$

by [24, Theorem 1].

A subset $S \subset I$ is called scrambled if for any pair of points $x,y \in S$ such that $x \neq y$ the following conditions hold

$$0 = \liminf_{n \to \infty} |f^n(x) - f^n(y)| < \limsup_{n \to \infty} |f^n(x) - f^n(y)| = \delta > 0.$$

The map $f$ is said to be chaotic in the sense of Li and Yorke if $I$ contains an uncountable scrambled set (see [19]). It is known that a map with positive topological entropy is chaotic in the sense of Li and Yorke [6]. On the other hand, there exist maps with zero topological entropy and chaotic in the sense of Li and Yorke (see [29]), but that is impossible if $f$ does not have wandering intervals (see [3]).

The question is whether it is possible to compute the topological entropy in a practical way, giving a simple and useful characterization of chaos. Although different algorithms have been proposed in order to compute topological entropy with prescribed accuracy, we will use the one proposed by Block et al. [7]. It allows us to compute the topological entropy of unimodal maps by comparing with maps of the family $T$ of strictly piecewise continuous maps with constant slope $\pm s$, maximum at $1/2$, and such that the topological entropy is known and equal
to log s. The comparison is made by means of the images of turning points of f
and the members of T. It reminds the bisection method (see [10] for a schematic
algorithm description).

3. LAS and GAS in q-deformations.

3.1. Single q-deformations. In this section we consider the q-deformed logistic
family when a single q-deformation is applied. Some results of this section were
partially studied in [16] and [10], although with a slightly different approach focussed
on local stability and chaos. Recall that \( f_a, a \in (0, 4], \) denotes the logistic family,
\( \phi_q, q \in (-\infty, 2), \) denotes the q-deformation and \( F_{a,q} = f_a \circ \phi_q. \) It is easy to check
that \( F_{a,q} \) fulfills the following properties:

P1. The point \( \phi^{-1}_q(1/2) \) is the unique turning point of \( F_{a,q}, \) and therefore this
map is unimodal.

P2. By Theorem 2.1, and since \( S(\phi_q) = 0, \) the Schwarzian derivative
\( S(f_a \circ \phi_q)(x) = (S(f_a)(\phi_q(x)) \cdot (\phi'_q(x))^2 + S(\phi_q)(x) = (S(f_a)(\phi_q(x)) \cdot (\phi'_q(x))^2 < 0 \)
since \( S(f_a)(x) < 0 \) for all x such that \( f'_a(x) \neq 0. \)

3.1.1. Existence of non zero fixed points. Recall that \( f_a (q = 1) \) has a non zero fixed
point
\( x_0 = \frac{a - 1}{a} \)
provided \( a > 1. \) For \( q \neq 1, \) we can prove that the non zero solutions of the equation
\( (f_a \circ \phi_q)(x) = x \)
are given by
\( x_0^\pm = \frac{(q - 2)(a + 2(q - 1)) \pm \sqrt{(2 - q)a((1 - q)(a - 4) + a)}}{2(1 - q)^2} \)
whenever
\( (2 - q)((1 - q)(a - 4) + a) \geq 0, \)
which is equivalent to
\( q \geq 1 - \frac{a}{4 - a} = \frac{4 - 2a}{4 - a}. \)
For \( a = 4 \) the above denominator vanishes, but note that in this case the discrimi-
nant \( 16(2 - q) > 0 \) for all \( q \in (-\infty, 2). \) In addition, for \( a = 4, \)
\( x_0^+ = \frac{q^2 - q - 2 + 2\sqrt{2 - q}}{(1 - q)^2} \)
Let \( R^+_\mathbf{I} := (-\infty, 2) \times \{4\}. \) Note that \( F_{4,q}([\phi^{-1}_q(1/2), 1]) = I \) and therefore \( x_0^+ > 0 \)
for \( a = 4. \) As \( F_{4,q} \) is unimodal, if \( x_0^+ > 0, \) then it must be contained in \( (0, \phi^{-1}_q(1/2)). \)

On the other hand, when \( a \neq 4, x_0^+ > 0 \) if
\( (2 - a)(1 - q) + 2(1 - q)^2 - a + \sqrt{(2 - q)a((1 - q)(a - 4) + a)} > 0. \)
Note that
\( (2 - a)(1 - q) + 2(1 - q)^2 - a + \sqrt{(2 - q)a((1 - q)(a - 4) + a)} = 0 \)
when \( 2 - q = a \). It is easy to see that, when \( a \in (0, 2] \), the point \( x^+_0 \leq 0 \) when \( a \leq 2 - q < \frac{4}{4-a} \). Then, the region \( R \) where \( x^+_0 > 0 \) is given by

\[
R^+ = R^+_4 \cup \{(q, a) \in (-\infty, 2) \times (0, 2] : 2 - a < q \} \\
\cup \{(q, a) \in (-\infty, 2) \times (2, 4] : q \geq \frac{4 - 2a}{4 - a} \}.
\]

As \( x^-_0 \leq x^+_0 \), we have that \( x^-_0 \) can be positive when \( x^+_0 \) is so. Then \((2 - a)(1 - q) + 2(1 - q)^2 - a - \sqrt{(2 - q)a((1 - q)(a - 4) + a)} > 0\) is the condition for that. When \( a = 4 \), then

\[
x^-_0 = \frac{q^2 - q - 2 - 2\sqrt{2 - q}}{(1 - q)^2}
\]

which is positive provided \( q < -2 \). From this, it is easy to see that the region \( R^- \) where \( x^-_0 > 0 \) is given by

\[
R^- = \{(q, a) \in (-\infty, 0) \times [2, 4] : 2 - a > q \geq \frac{4 - 2a}{4 - a} \} \cup R^+_4,
\]

where \( R^-_4 := (-\infty, -2) \times \{4\} \). Figure 1 shows both regions. Note that \( R^- \subset R^+ \).

3.1.2. Characterizing when 0 is LAS. If \( q = 1 \) it is well-known that 0 is LAS if \( a \leq 1 \). When \( q \neq 1 \), we have that since 0 is a fixed point for both maps \( f_a \) and \( \phi_q \), it is locally asymptotically stable provided

\[
|F'_{a,q}(0)| = |(f_a \circ \phi_q)'(0)| = |f_a'(0)| \cdot |\phi_q'(0)| = \frac{a}{2 - q} < 1,
\]

which is equivalent to

\[
a < 2 - q.
\]

It can be seen in [13, pp. 30-36] that the condition \( F'_{a,q}(0) = 1 \), that is, \( a = 2 - q \), provides LAS in the following cases:

1. When \( F''_{a,q}(0) = -\frac{2aq}{(2-q)^2} < 0 \), that is, when \( q > 0 \).
2. When \( q = 0 \), because \( F''_{a,0}(0) = -\frac{3a}{4} < 0 \).
Note that the point \((q_0, a_0) = (1, 1)\) belongs to the line \(2 - q = a\), and then the set \(\{(1, a) : a \in (0, 1]\}\) is contained in \(L^0\), the set where 0 is LAS, shown in Figure 2(a).

3.1.3. Characterizing when \(x^\pm_0\) are LAS. Consider the non zero fixed point \(x^+_0\). The LAS condition is given by
\[
|\left(f_a \circ \phi_q\right)^\prime(x_0)| = \left|f_a^\prime(\phi_q(x_0)) \cdot \phi_q^\prime(x_0)\right| < 1,
\]
which reads as
\[
\left|(1 - q)(4 - a) - \frac{(a - 2)\left(\sqrt{a} + \sqrt{(2 - q)(a - 4)(1 - q) + a}\right)}{\sqrt{a}}\right| < 2.
\]
We can check that
\[
(1 - q)(4 - a) - \frac{(a - 2)\left(\sqrt{a} + \sqrt{(2 - q)(a - 4)(1 - q) + a}\right)}{\sqrt{a}} = 2 \quad (4)
\]
if and only if
\[
q = \begin{cases} 
2 - a & \text{if } a \leq 2, \\
\frac{4 - 2a}{4 - a} & \text{if } a \geq 2.
\end{cases}
\]
Note that if \(a \leq 2\) and \(q = 2 - a\), the point \(x^+_0\) does not exist. It can be seen in [13, pp. 28-39] that the equality does not provide LAS. We point out that equation (4) is the condition for the saddle-node bifurcation (see [17]).

Similarly, we have that
\[
(1 - q)(4 - a) - \frac{(a - 2)\left(\sqrt{a} + \sqrt{(2 - q)(a - 4)(1 - q) + a}\right)}{\sqrt{a}} = -2 \quad (5)
\]
if and only if \(a \geq 2\) and
\[
q = \frac{4 + 2a - a^2}{4 - a}.
\]
It can be seen in [13, pp. 28-39] that the equality, jointly with the fact that the map has negative Schwarzian derivative, implies that \(x^+_0\) is LAS. We point out that equation (5) are the conditions for a period doubling bifurcation, where a LAS fixed point becomes unstable and a LAS periodic point of period two arises (see [30]).

Note that in both cases (4) and (5), the value \(a = 4\) vanishes some denominators. In this case,
\[
F^\prime_{a,q}(x^+_0) = -1 - \sqrt{2 - q} < -1,
\]
and thus it is never LAS. Finally, note that \((q_0, a_0) = (1, 3)\) belongs to the curve \(q = \frac{4 + 2a - a^2}{4 - a}\) and so \(\{(1, a) : a \in (0, 3]\}\) is contained in \(L^+\), the set where \(x^+_0\) is LAS, shown in Figure 2(b).

Repeating the same argument for \(x^-_0\), we easily realize that it is never LAS.

Figure 2 shows the parameter region where \(x^+_0\) is LAS.

We summarize the above results as follows.

**Theorem 3.1.** Let \(q \in (-\infty, 2)\) and \(a \in (0, 4]\). Then

1. The fixed point 0 is LAS if \((q, a)\) belongs to the region
   \[
   L^0 = \{(q, a) \in (-\infty, 2) \times (0, 2] : q \leq 2 - a\} \\
   \cup\{(q, a) \in (-\infty, 2) \times (2, 4] : q < 2 - a\}.
   \]
2. The fixed point $x_0^+$ is LAS if $(q,a)$ belongs to the region

$$L^+ = \{(q,a) \in (-\infty, 2) \times (0, 2] : q > 2 - a\}$$

$$\cup \{(q,a) \in (-\infty, 2) \times (2, 4) : \frac{4 - 2a}{4 - a} < q \leq \frac{4 + 2a - a^2}{4 - a}\}.$$ 

3. The fixed point $x_0^-$ is never LAS.

3.1.4. Parameter regions where the fixed points are GAS. A first observation is that any non zero fixed point has infinitely many preimages. So, the parameter region where $x_0^- \geq 0$ cannot be a GAS region for any fixed point unless $x_0^+ = x_0^-$. Similarly, when 0 and $x_0^+$ coexist, 0 is never GAS. Additionally, if 0 and $x_0^+$ are LAS, then $x_0^-$ exists and it is positive. This is clear since if 0 is LAS and $F_{a,q}'(0) \leq 1$, then $x_0^+ > 0$ if and only if $F_{a,q}(c) \geq c$. Then, unless $x_0^+ = x_0^- = c$, $F_{a,q}$ must have a fixed point in $(0,c)$. Taking into account Theorem 3.1, we can state the following result that characterizes when LAS implies GAS.

**Theorem 3.2.** Let $q \in (-\infty, 2)$ and $a \in (0, 4]$. Then

1. The fixed point 0 is GAS if $(q,a)$ belongs to the region

$$G^0 = \{(q,a) \in (-\infty, 2) \times (0, 2] : q \leq 2 - a\}$$

$$\cup \{(q,a) \in (-\infty, 2) \times (2, 4] : \frac{4 - 2a}{4 - a} > q\}.$$ 

2. The fixed point $x_0^+$ is GAS in the sense that it attracts all orbits in $(0,1)$ if $(q,a)$ belongs to the region

$$G^+ = \{(q,a) \in (-\infty, 2) \times (0, 2] : q > 2 - a\}$$

$$\cup \{(q,a) \in (-\infty, 2) \times (2, 2 + \sqrt{2}] : 2 - a < q \leq \frac{4 + 2a - a^2}{4 - a}\}.$$ 

**Proof.** Just use (P2) and Theorems 2.2, 3.1 and 2.3. □

Figure 3 shows the regions where 0 and $x_0$ are GAS.

### 3.2. Generalized $q$-deformations.

In this section we consider the $q$-deformed logistic family when several $q$-deformations are applied. The next results are the key for a complete study of these maps as they allow us to reduce this case to the single $q$-deformation case.
Proposition 1. Let $q_i \in (-\infty, 2)$ for all $i = 1, \ldots, k$. Then,

$$
\phi_{q_k} \circ \cdots \circ \phi_{q_1} = \phi_{2-\prod_{i=1}^{k}(2-q_i)}.
$$

Proof. Let $\alpha_i = 1 - q_i$ for $1 \leq i \leq k$. So, $\phi_{q_i}$ is rewritten as

$$
\varphi_{\alpha_i}(x) := \frac{x}{1 + \alpha_i - \alpha_i x}.
$$

Let $k = 2$. Then, it is straightforward to check that

$$
(\varphi_{\alpha_2} \circ \varphi_{\alpha_1})(x) = \frac{x}{1 + [(1 + \alpha_1)(1 + \alpha_2) - 1] - [(1 + \alpha_1)(1 + \alpha_2) - 1] x}
$$

$$
= \varphi_{(1+\alpha_1)(1+\alpha_2)-1}(x).
$$

Next, we assume that the result is true for $k - 1$, and prove it for $k$. Then, notice that

$$
\left(1 + \prod_{i=1}^{k-1}(1+\alpha_i) - 1\right)(1 + \alpha_k) - 1 = \prod_{i=1}^{k}(1 + \alpha_i) - 1.
$$

Then

$$
(\varphi_{\alpha_k} \circ \cdots \circ \varphi_{\alpha_1})(x) = (\varphi_{\alpha_k} \circ (\varphi_{\alpha_{k-1}} \circ \cdots \circ \varphi_{\alpha_1}))(x) = \varphi_{\prod_{i=1}^{k}(1+\alpha_i)-1}(x).
$$

Rewriting the expressions with $q_i$ instead of $\alpha_i$, the proof concludes. \qed

Corollary 1. Let $q \in (-\infty, 2)$. Then, for all $x \in [0, 1]$ we have that

$$
\phi^k_q = \phi_{2-(2-q)^k}.
$$

The above results help us to prove the following results, where LAS and GAS are characterized. First, we start by characterizing when the map $f_a \circ \phi_{q_k} \circ \cdots \circ \phi_{q_1}$ has non zero fixed points. They are given by

$$
x_0^\pm = \frac{(2-a)(1-\Theta) + 2(1-\Theta)^2 - a \pm \sqrt{(2-\Theta)a((1-\Theta)(a-4)+a)}}{2(1-\Theta)^2},
$$

where

$$
\Theta = \Theta(q_1, \ldots, q_k) := 2 - \prod_{i=1}^{k}(2-q_i).
$$
Then, we adopt the following notation. For a set \( A \subset (-\infty, 2) \times (0, 4] \) we define for \( q_1, \ldots, q_k \in (-\infty, 2) \) the set \( A(q_1, \ldots, q_k) := \{(q_1, \ldots, q_k, a) \in (-\infty, 2)^k \times (0, 4] : (\Theta(q_1, \ldots, q_k, a), a) \in A \} \). Then, the fixed points \( x_0^+ \) and \( x_0^- \) exist and are greater than zero in the regions \( R^+(q_1, \ldots, q_k) \) and \( R^-(q_1, \ldots, q_k) \), respectively. When \( q_1 = \ldots = q_k = q \), we set

\[
\Theta(q, k) := \Theta(q, \ldots, q) = 2 - (2 - q)^k
\]

and \( A(q, k) := A(q, \ldots, q) \). Then, the regions where the non zero fixed point exist are \( R^+(q, k) \) and \( R^-(q, k) \). In addition, we can prove the following results which characterize LAS and GAS of fixed points.

**Theorem 3.3.** Let \( (q_1, \ldots, q_k) \in (-\infty, 2)^k \) and \( a \in (0, 4] \). Then:
1. The fixed point 0 is LAS if \( (q_1, \ldots, q_k, a) \) belongs to the region \( L^0(q_1, \ldots, q_k) \).
2. The fixed point \( x_0^+ \) is LAS if \( (q_1, \ldots, q_k, a) \) belongs to the region \( L^+(q_1, \ldots, q_k) \).
3. The fixed point \( x_0^- \) is never LAS.

**Proof.** Just apply Proposition 1 and Theorem 3.1 with \( \Theta \).

**Theorem 3.4.** Let \( (q_1, \ldots, q_k) \in (1, \infty)^k \) and \( a \in (0, 4] \). Then
1. The fixed point 0 is GAS if \( (q_1, \ldots, q_k, a) \) belongs to the region \( G^0(q_1, \ldots, q_k) \).
2. The fixed point \( x_0^+ \) is GAS in the sense that it attracts all orbits in \( (0, 1) \) if \( (q_1, \ldots, q_k, a) \) belongs to the region \( G^+(q_1, \ldots, q_k) \).

**Proof.** Just apply Proposition 1 and Theorem 3.2 with \( \Theta \).

When \( q_1 = \ldots = q_k = q \), the regions of Theorems 3.3 and 3.4 are simply \( L^0(q, k) \), \( L^+(q, k) \), \( G^0(q, k) \) and \( G^+(q, k) \). Figure 4 depicts the variation of the sets where the fixed points are GAS when \( k = 2 \) and 5. Note that the equality \( \Theta(q, k) = g(a) \), for some real function \( g \), can be rewritten as \( q = 2 - [2 - g(a)]^{1/k} \). Since

\[
\lim_{k \to \infty} 2 - [2 - g(a)]^{1/k} = 1
\]

we see that \( G^0(q, k) \) tends to \((-\infty, 1] \times (0, 4] \) and \( G^+(q, k) \) tends to \([1, 2] \times (0, 2] \cup \{(1, a) : 1 \leq a \leq 3\} \) when \( k \) increases to \( \infty \).

We show in Figure 5 some bifurcation diagrams where a \( q \)-deformation is applied twice. The phenomenon of bistability appears for suitable \( q \). This was already pointed out in [4] and [10] for single \( q \)-deformations. We point out that this phenomenon is known for population models with the so-called Allee effect given by unimodal maps (see [20] or [26]). The bifurcation diagrams also show the evolution to a chaotic dynamics which was analyzed previously in [10]. We will study the parameter values with a complicated dynamical behavior in the next section.

4. **Complex dynamics of the \( q \)-deformation of the logistic family.** As we stated before, the map \( \Phi_{q,a} = \phi_q \circ f_a \) shares most of its dynamic properties with \( F_{a,q} \) (see [8]). In particular, they have the same topological entropy and their Lyapunov exponents at the turning point have the same sign. Moreover, it is easy to check that \( \Phi_{q,a} \) satisfies the following properties:

**P1.** It follows from the chain rule and the fact that \( \phi_q \) is an homeomorphism that 1/2 is the unique turning point of \( \Phi_{q,a} \), and therefore this map is unimodal.

**P2.** By the same reason, we easily see that 1/2 is non-flat because

\[
\Phi_{q,a}''(1/2) = \Phi_{q,a}'(f_a(1/2))f_a''(1/2) \neq 0.
\]
Figure 4. For $k = 2$, (a) The region $G^0(q, 2)$ where the fixed point 0 is GAS. (b) The region $G^+(q, 2)$ where the fixed point $x_0^+$ is GAS. For $k = 5$, (c) The region $G^0(q, 5)$ where the fixed point 0 is GAS. (d) The region $G^+(q, 5)$ where the fixed point $x_0^+$ is GAS.

P3. By Theorem 2.1, and since $S(\phi_q) = 0$, the Schwarzian derivative

$$S(\Phi_{q,a}) = S(f_a) = -\frac{6}{(1 - 2x)^2} < 0.$$  

Taking the above properties into account, in [10], we computed the topological entropy with accuracy $10^{-4}$. The results are shown in Figure 6, jointly with the estimations of the Lyapunov exponents. The fact that all the maps have the same turning point simplifies somehow the computation of topological entropy. It was found that there is a set of parameter values $a$ such that $f_a$ has zero topological entropy ($a < 3.5699...$) and hence a simple dynamics, while the topological entropy $h(\Phi_{q,a}) > 0$, implying a complicated dynamical behavior. Additionally, it was shown that this complicated dynamical behavior is observable because the Lyapunov exponents are positive for a large subset of the parameters set with positive topological entropy. If we observe Figure 6, we realize that the complexity can increase when $q$ is close to 0.85. However, we have to point out that for parameter values far enough from the interval $(0, 1)$, the application of a $q$-deformation decreases the complexity of $\Phi_{q,a}$.

Next, we analyze how the situation changes when several $q$-deformations are applied to $f_a$. Fix $k \in \mathbb{N}$ and $q_1, ..., q_k \in (-\infty, 2)$. Consider the map $\Phi_{q_k, ..., q_1, a} = \phi_{q_k} \circ ... \circ \phi_{q_1} \circ f_a$. By Proposition 1, it is easy that this map satisfies the properties (P1)-(P3). Therefore, we may apply the algorithm for computing its topological
We fix $q = 0$. Bifurcation diagrams when the $q$-deformation $\phi_q$ is applied twice. We compute 10000 points of each orbit, with initial conditions 0.75 (black) and 0.05 (green), and draw the last 200. Black color overwrites green when the attractor is the same, covering completely when 0 is GAS. The parameter $a$ ranges $(0, 4]$ with step size 0.005. The dashed red line represents the unstable fixed point $x_0^{-}$ when it exists, which acts as a separatrix between the basins of attraction of the two attractors. In (b) we depict the region where the existence of two different attractors is possible when we apply the same $q$-deformation twice.

Figure 5.

First, we assume that $q_1 = ... = q_k = q$, and check what happens when $k$ increases. In Figure 7 we show the computation on topological entropy with accuracy $10^{-4}$ for several values of $k$. It can be seen that the region where the dynamic Parrondo’s paradox “combination of simple maps gives a chaotic map” reduces as $k$ increases (compare Figure 7 with Figure 6). Figure 8 shows bifurcation diagrams. In them we observe that a zoom is necessary to observe the chaotic behavior predicted in Figure 7.

We consider different parameters $q_i's$. Here we have two problems. The first one is that the number of computations increases. The second is that we have difficulties in presenting the results in a suitable way, for instance when three different parameters are involved. So, what we have done is to fix some parameter values $a$ and two parameters $q_1$ and $q_2$. Figure 9 shows the results when three and four $q$-deformations are applied and $a = 3.5697$, which makes $f_a$ dynamically simple. This constitutes another example of Parrondo’s paradox. We point out that Lyapunov exponents can be also positive, implying that the topological chaos is also observable.

Finally, we must remark that, although the application of several $q$-deformations seems to decrease the complexity of the corresponding $q$-deformed logistic map, this process is not capable of removing chaos for all the parameters $a$. For $a = 4$, we have that $\Phi_{q_1, ..., q_k, 4}([0, 1/2]) = \Phi_{q_1, ..., q_k, 4}([1/2, 1]) = I$, and therefore $\Phi_{q_1, ..., q_k, 4}$ has a 2-horseshoe. Consequently, the topological entropy of $\Phi_{q_1, ..., q_k, 4}$ is log 2 (see [2, Chapter 4]). As the topological entropy is continuous for unimodal maps with positive entropy, there must be a neighborhood of $\Phi_{q_1, ..., q_k, 4}$ in which the maps $\Phi_{q_1, ..., q_k, a}$ display positive entropy as well. Of course, this neighborhood becomes smaller when $k$ increases.
Figure 6. With accuracy $10^{-4}$, topological entropy of the $q$-deformed logistic map for $a \in [3.5, 4]$ and $q \in [-25, 2]$ (a) and associated level curves (b). Note that the darker regions are, the smaller the topological entropy is. (c) and (d) show the same computations for $a \in [3.5, 4]$ and $q \in [0.7, 1.2]$. (e) and (f) depict the region where the Lyapunov exponents are positive, and hence chaos is physically observable.

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Figure 7. Level curves of the topological entropy of the $q$-deformed logistic maps (a) $\Phi_{q,q,a}$, (b) $\Phi_{q,q,q,a}$, (c) $\Phi_{q,q,q,q,a}$ and (d) $\Phi_{q,q,q,q,q,a}$ for $a \in [3.5, 4]$ and $q \in [0.7, 1.2]$. Note that the darker regions are, the smaller the topological entropy is.

Figure 8. For $q = 0.95$: (a)-(b) Bifurcation diagrams of the $q$-deformed logistic map $\Phi_{q,q,a}$. In (a) we cannot see any chaotic behavior, but a zoom in (b) shows that it exists. The parameter $a$ ranges from 3.569 to 3.57 with step size $10^{-6}$. (c)-(d) The same for $\Phi_{q,q,q,a}$. 

Figure 9. For $a = 3.5697$ and $q_1, q_2 \in [0.7, 1.2]$, Region of positive Lyapunov exponents of the $q$-deformed logistic maps (a) $\Phi_{q_2, q_1, q_1, a}$ and (b) $\Phi_{q_2, q_1, q_2, q_2, a}$. (c) and (d) level curves of the topological entropy of these maps with accuracy $10^{-4}$.

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