PL DENSITY INVARIANT FOR TYPE II DEGENERATING K3 SURFACES, MODULI COMPACTIFICATION AND HYPER-KÄHLER METRIC

YUJI ODAKA

Abstract. A protagonist here is a new-type invariant for type II degenerations of K3 surfaces, which is explicit piecewise linear convex function from the interval with at most 18 nonlinear points. Forgetting its actual function behavior, it also classifies the type II degenerations into several combinatorial types, depending on the type of root lattices as appeared in classical examples.

From differential geometric viewpoint, the function is obtained as the density function of the limit measure on the collapsing hyper-Kähler metrics to conjectural segments, as in [HSZ19]. On the way, we also reconstruct a moduli compactification of elliptic K3 surfaces by [AB19], [ABE20], [Brun15] in a more elementary manner, and analyze the cusps more explicitly.

We also interpret the glued hyper-Kähler fibration of [HSVZ18] as a special case from our viewpoint, and discuss other cases, and possible relations with Landau–Ginzburg models in the mirror symmetry context.

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§1. Introduction

In this paper, to each type II degeneration of polarized K3 surfaces \( \pi: (\mathcal{X}, \mathcal{L}) \rightarrow \Delta = \{ t \in \mathbb{C} \mid |t| < 1 \} \), we associate an explicit piecewise linear (PL) convex function \( V = V_\pi: [0, 1] \rightarrow \mathbb{R}_{\geq 0}(\cup \infty) \) over the interval, as a new-type invariant, and discuss its geometric meanings from various geometric perspectives. The nondifferential points of \( V \) are at most 18 points, and anyhow the behavior of \( V \) is completely classified. If \( \mathcal{L} \) are assumed to be of relative Hodge (integral) class as in algebraic geometry, the function \( L \) is rational, while if we extend to relative Kähler class on (not necessarily algebraic) \( \mathcal{X} \), then we obtain not necessarily rational bend points.

From differential geometric perspective, this is done by considering the behavior of hyper-Kähler metrics on the fibers \( \mathcal{X}_t = \pi^{-1}(t) \) with the Kähler class in \( \mathbb{R}_{>0}(\cup \infty) \) with diameter bounded rescale, because our function \( V \) is the density function of a limit measure
on the conjectural limit interval as predicted in recent [HSZ19]. As inferred from such background, we can actually define $V$ not only for holomorphic one parameter degeneration but also for more general sequences of type II.

The ends behavior of $V$ is encoded in the root lattices of type D or E, while the open part is reflected in type A lattices. This root lattice-theoretic information has classically appeared and studied at least in lower degree case, for example, in [Fri84], and also in recent [AET19, §3B, 9.10], [LO19, §1], and [ABE20]. Our exploration aims to reveal their hidden meanings.

1.1. History of this work

This paper originally stems out as a part of the series for ongoing joint work with Oshima on collapsing of hyper-Kähler metrics, with recent focus on K3 surfaces to segments, with great inspirations input from [ABE20] and [HSZ19] as well. Our whole framework depends on the one initiated in our previous joint paper [OO21] (its short summary is [OO18]), whose particular focus of the latter part was on type III degenerations and associated collapsing to spheres.

In addition, the recent log KSBA (Koll’ar-Shepherd-Barron-Alexeev) style explicit compactification work of moduli of elliptic K3 surfaces by [AB19], [ABE20], and [Brun15] has much to do with our work. In particular, $V$ implicitly appears in [ABE20] in the form of their integral affine spheres construction, and used in the projective moduli variety construction, much to our surprise then. Next, we give the comparison, partially to give an overview of this paper.

1.2. Comparison and organization

While [ABE20, §7A] implicitly obtained the definition of $V$ in the form of its graphs as integral affine spheres, Oshima [Osh] also had definition of $V$ independently, as a function for the collapsing K3 surfaces to segment. Then, he proves that it is the limit measure of the McLean metric on $\mathbb{P}^1$, from periods calculation along explicit 2-cycles.

Section 3 of this paper provides another algebrogeometric proof of the theorem of Oshima, which is the heart of the paper. Before that, in preparatory Section 2, we give an elementary reproof and analysis of the stable reduction corresponding to [ABE20]. The reproof has virtue for the arguments in Section 3. More precisely speaking, the information of asymptotic behavior of singular fibers analyzed in Section 2, not only the location of limits of discriminants, is crucially used in Section 3.

In Section 3, we also connect our work with [HSVZ18]. More precisely, we interpret [HSVZ18] as a special case when the lattice theoretic label (combinatorial information) of our PL function $V$ is of type $EAE$. Note that there are other interesting cases whose label include type $D$ at the ends and also the label can be longer. Indeed, it can involve at most 20 lattices as the longest possibility, because it readily follows from [ABE20] or our discussions.

Section 2 can and does work over an arbitrary algebraically closed field $K$ of characteristic neither 2 nor 3, unless otherwise stated. The assumption on characteristic is frequently used, especially for the Weierstrass standard form description of elliptic curves and the reducedness of the finite group schemes $\mu_2$ and $\mu_3$ over $K$. On the other hand, Section 3 works over $\mathbb{C}$, because, for instance, discussions involve hyper-Kähler metrics.
§2. Moduli of elliptic K3 surfaces revisited

2.1. Review of [OO21, §7] and analysis of cusps

In the work [OO21, §7] on collapsing of K3 surfaces, the moduli $M_W(\mathbb{C})$ of complex Weierstrass elliptic K3 surfaces played an important role, because it parametrizes real two-dimensional collapses (tropical K3 surfaces) of Kähler K3 surfaces. Still keeping it as one of the motivations, we first make further analysis on $M_W$ in this paper. It also naturally extends to other field $K$. First, we set up or recall the notation.

We set $\mathbb{A}^{22}$, which parametrizes the coefficients of degree 8 polynomial $g_8$ and the coefficients of degree 12 polynomial $g_{12}$.

Recall from [OO21, §7.1] that $\overline{M_W}$ is nothing but the GIT quotient of $\mathbb{A}^{22}_{g_8,g_{12}} \setminus \{0\}$ by the action of GL(2), or in other words, that of

$$\mathbb{P}(2,2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3)$$

by the further action of SL(2). We denote the homogeneous coordinates of the base $\mathbb{P}^1$ as $s_1$ and $s_2$, and set $s := \frac{s_1}{s_2}$.

Recall that [OO21, §7.2.1] shows $\overline{M_W}$ is isomorphic to the Satake–Baily–Borel compactification for an appropriate $O(2,18)$ orthogonal symmetric variety (and also has the structure as a double (anti-)holomorphic) covering over the boundary component $M_{K3}(a)$ of $\overline{M_{K3}}$ in [OO21, §6], which appears in the context of F-theory, for example, as classical F-theory moduli space in [CM05]. See [OO21, §6.1], in particular its last discussion for the proof of Theorem 6.6 of §7.3.7 in [OO21] for the details.

Now, we head toward more explicit understanding of cusps of the compactification. From the uniformization structure $M_W \simeq \Gamma \backslash \mathcal{D}$, with orthogonal symmetric domain $\mathcal{D}$, there is the natural branch divisor $B$ in $M_W$ with the standard coefficients. From [Mum77, Prop. 3.4], it follows that $(\overline{M_W}, \overline{B})$ is the log canonical model, and the three cusps $M_W^{an}$, $M_W^{seg}$, and $M_W^{an} \cap M_W^{seg}$ are the set of all log canonical centers.

An important point to notice is that $\text{Supp}(\overline{B})$ actually contain both of $M_W^{an}$ and $M_W^{seg}$. Indeed, as we see below later, $M_W$ (without the branch divisor) are log terminal around both log canonical centers.

More direct way to see it is as follows. Recall from [OO21, §7.1.5] that the locus $S_b$ corresponding to (b) in [OO21, §7.1.5], that is, the surface in $M_W$ isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$, parametrize Kummer surfaces for the product of elliptic curves $E_1 \times E_2$ and the closure include both $M_W^{an}$ and $M_W^{seg}$. As [OO21, Prop. 7.8] shows, for all such Kummer surfaces, the corresponding Weierstrass models contain four $D_4$-singularities which are ordinary cusps fiberwise, as a birational transform of $(E_1 \times E_2) / (\mathbb{Z} / 2\mathbb{Z})$. The Heegner divisor of $M_W$, which corresponds to their partial smoothings with a single $A_1$-singularity, contains the locus $S_b$ obviously.

2.1.1. Around $M_W^{an}$

Because the locus $M_W^{an} \setminus M_W^{seg}$ locates inside the (strictly) stable locus inside the GIT quotient $\overline{M_W}$ (cf. [OO21, §7.1.1]), it follows that the stabilizer of the $GL(2)$-action on $\mathbb{A}^{22}$ which represents a point inside $M_W^{an}$ is finite. Furthermore, it is generically the Klein four
group, that is, \((\mathbb{Z}/2\mathbb{Z})^\oplus 2\), and becomes larger only at finite points in \(M_{\text{fin}}^W\) (e.g., when the corresponding degree 4 polynomial \(G_4\) is \(s_1s_2(s_1 - s_2)(s_1 + s_2)\) (or \(s^3 - s\) in the way written in [OO21]), so that the corresponding stabilizer group is \((\mathbb{Z}/2\mathbb{Z})^\oplus 3\).

Before our statements, we define the following singularity.

**Definition 2.1.** A canonical Gorenstein threefold singularity whose germ is written as

\[
\vec{0} \in [X^2 = YZW] \subset \mathbb{A}^4
\]

is denoted as \(\mathcal{A}_1^{(3)}\) in this paper. Indeed, each component of the singular locus meeting at \(\vec{0}\),

- \(X = Z = W = 0, Y \neq 0\),
- \(X = Y = W = 0, Z \neq 0\),
- \(X = Y = Z = 0, W \neq 0\),

is transversally two-dimensional \(A_1\)-singularity (\(cA_1\)), hence the name. It is also easy to see that this coincides with the quotient singularity by \((\mathbb{Z}/2\mathbb{Z})^\oplus 2 = K_4\) of \(A_3\) acting by the eigenvalues

\[
(1,1,1,1) \quad \text{by the unit } e \text{ of } K_4,
(1,-1,1,-1) \quad \text{by an element } a \text{ of } K_4,
(-1,-1,1,1) \quad \text{by an element } b \text{ of } K_4,
(-1,1,1,-1) \quad \text{by the element } ab \text{ of } K_4.
\]

**Theorem 2.2.** At general points in \(M_{\text{fin}}^W\), \(M_w\) is formally (hence also analytically if \(K = \mathbb{C}\)) isomorphic to

\[
(\mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)}) \times \mathbb{A}^6,
\]

hence canonical Gorenstein singular in particular.

It is interesting, because, with the branch divisor, it becomes one of strictly log canonical locus.

**Proof.** We use the Luna slice theorem [Luna73] (see also the exposition [Dre04, 5.3]). Take a general point \(p\) in \(M_{\text{fin}}^W\) and its lift \(\bar{p}\) to \(\mathbb{A}^{22}_{g_8,g_{12}}\) as \((P_4^2, P_3^3)\), where \(P_4 \in \mathcal{O}_{P^1}(4)\) is of the form \((s_1^2 - \epsilon^2 s_2^2)(s_2^2 - \epsilon^2 s_1^2)\), so that its stabilizer is \(K_4\) generated by

\[
\begin{align*}
\text{(switch)}: s_1 &\mapsto s_2, & s_2 &\mapsto s_1, \\
(-1)_{s_1}: s_1 &\mapsto -s_1, & s_2 &\mapsto s_2.
\end{align*}
\]

Now, we construct slice at the above point in \(\mathbb{A}^{22}_{g_8,g_{12}}\) with respect to the natural \(\text{SL}(2)\)-action as follows. Consider the following regular parameter system (or holomorphic coordinates at neighborhood) around \((P_4^2, P_3^3) \in \mathbb{A}^{22}_{g_8,g_{12}}\): they are formed by coeff \(P_4\), the coefficients of the polynomial \(P_4\), which is introduced before, and those of

\[
\begin{align*}
R_{\text{fin}}^{\text{fin}} &\in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)), \\
Q_{\text{fin}}^{\text{fin}} &\in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)), \\
R'_{\text{fin}}^{\text{fin}} &\in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)),
\end{align*}
\]
each of which are linear combinations of:

- (for $R^{\text{rfn}}$)

\[
\begin{align*}
& s_1^3 s_2^5 + s_1^5 s_2^3, \\
& s_1^2 s_2^6 + s_1^6 s_2^2,
\end{align*}
\]

- (for $Q^{\text{rfn}}$)

\[
\begin{align*}
& s_1^4 + s_2^4, \\
& s_1^3 s_2 + s_1 s_2^3, \\
& s_1^2 s_2^2,
\end{align*}
\]

- (for $R^{'\text{rfn}}$)

\[
\begin{align*}
& s_1^{10} s_2^2 + s_1^2 s_2^{10}, \\
& s_1^9 s_2^3 + s_1^3 s_2^9, \\
& s_1^8 s_2^4 + s_1^4 s_2^8, \\
& s_1^7 s_2^5 + s_1^5 s_2^7,
\end{align*}
\]

and we consider the points

\[ (g_8 = P_4^2 + R^{\text{rfn}}, g_{12} = P_4^3 + (3P_4)^2 Q^{\text{rfn}} + R^{'\text{rfn}}), \]

for those $R^{\text{rfn}}, Q^{\text{rfn}}, R^{'\text{rfn}}$ which are generated by special ones above. Then, this forms a stab$(\tilde{p})$-invariant étale slice. In addition, the action of stab$(\tilde{p}) \simeq K_4$ whose generators we recall as

\[ (\text{switch})\iota: s_1 \mapsto s_2, \quad s_2 \mapsto s_1, \]

\[ (\text{switch})\iota: s_1 \mapsto -s_1, \quad s_2 \mapsto s_2, \]

acts with eigenvalues $-1$ or $1$ on each basis vector above. Looking at the eigenvalues, the assertion readily follows.

2.1.2. Around $M^{\text{seg}}_W$

Now, take a point $p \in M^{\text{seg}}_W$ and its lift $\tilde{p}$ as $(c_1 s_1^4 s_2^4, c_2 s_1^6 s_2^6)$ for some $c_1, c_2 \in K$, and consider the stabilizer group at the point with respect to the natural $GL(2)$-action, which we denote as stab$(\tilde{p})$. It is simply isomorphic to $G_m(K) \rtimes \mu_2(K)$ which acts as either

\[ \{ s_1 \mapsto c s_1, s_2 \mapsto c^{-1} s_2 \mid c \neq 0 \}, \]

\[ \{ s_1 \mapsto c s_2, s_2 \mapsto c^{-1} s_1 \mid c \neq 0 \}. \]

From the easy calculation of the tangent space to the orbit $GL(2)\tilde{p}$, we can take stab$(\tilde{p})$-invariant étale slice at $\tilde{p}$ as

\[
S(\tilde{p}) := \tilde{p} + \{(n_{0 \leq i \leq 8, i \neq 3,4,5} k \cdot s_1^i s_2^{8-i}, \oplus n_{0 \leq j \leq 12} k \cdot s_1^j s_2^{12-j}) \subset A_{98,912}^{22} \}.
\]
Here, we apply the Luna slice theorem (see [Dre04], [Luna73]) again to see the local structure around $M^\text{seg}_W$. From the above description of the slice $S(\bar{p})$, it is locally
\begin{equation}
(S(\bar{p})//((\mathbb{G}_m(K) \times \mu_2(K))) \equiv ((\mathbb{A}^{18}_K//\mathbb{G}_m(K))/\mu_2(K)) \times K.
\end{equation}
The weights for the $\mathbb{G}_m(K)$-action on $\mathbb{A}^{18}_K$ are twice the following:
\begin{equation}
-4, -3, -2, 2, 3, 4,
\end{equation}
because which correspond to the coefficients of $g_8$, further followed by
\begin{equation}
-6, -5, -4, -3, -2, -1, 2, 3, 4, 5, 6,
\end{equation}
because which correspond to the coefficients of $g_{12}$. Recall that, in general, affine toric variety is characterized as GIT quotient of affine space by a linear action of some algebraic torus [Cox95, §2]. By applying it to our situation conversely, it follows that $\mathbb{A}^{18}_K//\mathbb{G}_m(K)$ is isomorphic to\(^1\) the 17-dimensional affine toric variety $U_\sigma$ corresponding to $S_\sigma = \sigma^\vee \cap M$ defined as follows.

2.1.3. Cone description

If we consider $w: \mathbb{R}^{18}_{\geq 0} \to \mathbb{R}$ the inner product with the above vector $(-4, -3, -2, 2, 3, 4, -6, -5, -4, -3, -2, -1, 2, 3, 4, 5, 6)$, then for $S_\sigma := \mathbb{Z}^{18} \cap w^{-1}(0)$ and $\sigma := S_\sigma^\vee$ in the dual vector space $(\mathbb{R}^{18})^\vee$, the above GIT quotient corresponds to this $\sigma \subset N \otimes \mathbb{R}$.

It is easy to see that this is nothing but the affine cone of self-product of weighted projective space
\begin{equation}
\mathbb{P}^8(1, 2, 2, 3, 4, 4, 5, 6) \times \mathbb{P}^8(1, 2, 2, 3, 4, 4, 5, 6)
\end{equation}
with respect to the $(\mathbb{Q})$-line bundle $\mathcal{O}(1, 1)$. Therefore, germ at any point in $M^\text{seg}_W$ in $\tilde{M}_W$ is isomorphic to the product of smooth curve with the affine cone of $\text{Sym}^2(\mathbb{P}^8(1, 2, 2, 3, 4, 4, 5, 6))$ with respect to the descend of $\mathcal{O}(1, 1)$.

Hence, if we blow up $M^\text{seg}_W$ with the descent of the vertex, we get
\begin{equation}
\text{Sym}^2(\mathbb{P}^8(1, 2, 2, 3, 4, 4, 5, 6))
\end{equation}
as fibers over any point at $M^\text{seg}_W$. We suspect that this corresponds to the variation of two rational elliptic surfaces.

**Remark 2.3.** Looijenga [Looi76] (cf. also Friedman et al. [FMW97, pp. 681–682]) proves the following by use of the Weyl formula for affine root systems (Macdonald). We wonder if one can explain somewhat mysterious coincidence of the appeared exponents and those in (16) and (17), in a more systematic manner.

**Theorem 2.4.** ([BS78], [Looi76]; cf. also Pinkham [Pin77], [FMW97]) For each elliptic curve $E$, and root lattice $Q$ and its dual root lattice $Q^\vee$, $(E \otimes Q^\vee)/W(Q)$ is isomorphic to the weighted projective space of dimension $\text{rk}(Q)$. The weights are, for example,
\begin{equation}
\mathbb{P}(\underbrace{1, 1, 1}_{4}, \underbrace{2, 2, 2, 2, 2, 2, 2, 2, 2}_{l-3})
\end{equation}

\(^1\) This isomorphism is also easy to see directly, in this special case, since the weights of the $\text{stab}(p)(\simeq \mathbb{G}_m(K))$-action involve 1 and the acting algebraic torus is one-dimensional.
for $D$, and

\[(19) \quad \mathbb{P}(1,2,3,4,5,6)\]

for $E$. Note that if $Q$ is of $A,D,E,F,G$ type, then $Q = Q'$ by their self-duality.

2.2. Algebrogeometric compactification after [ABE20] - elementary reconstruction -

2.2.1. Introduction to this section

In this section, we reconstruct and analyze one of the algebrogeometric compactifications of $M_W$ recently studied in [ABE20, especially §§4C and 7], denoted $F^{\text{rc}}$ in [ABE20]. There was also a preceding work [Brun15] before that, and there is also a closely related independent work [AB19, especially §§5 and 9]. In this paper, we call the compactification $\overline{M}_W^{\text{ABE}}$. [ABE20] shows that its normalization $\overline{M}_W^{\text{ABE},\nu}$ is a toroidal compactification, whose corresponding admissible rational polyhedral fan is what they call rational curves fan $\Sigma_{\text{rc}}$ [ABE20, §4C], as introduced as $\mathcal{F}$ in [Brun15, Chap. 12], because the considered boundary on K3 surfaces are weighted sum of rational curves in the polarization, as in [BL00] and [YZ96].

We briefly describe the points of our reconstruction of $\overline{M}_W^{\text{ABE}}$, especially the difference with [ABE20]. Our methods certainly overlap with the discussions in [ABE20], and even some exposition of Section 2.2 also parallel theirs, but the main point of our logic here is to replace some of essential parts of [ABE20] (especially the implicit/indirect stable reductions) by a simple elementary analysis of Weierstrass normal forms, so that the properness follows directly and construction extends even over $\mathbb{Z}[1/6]$. Furthermore, there is an independent nice work by [AB19] which constructs $\overline{M}_W^{\text{ABE}}$ and described the boundary components in [[AB19, Section 9] (of version 3), mainly from the viewpoints of the minimal model program again and twisted stable maps of [AV02].

In turn, our method based on explicit analysis of Weierstrass equations also helps to answer certain differential geometric questions shared with Oshima after the paper [HSZ19] and fruitful discussions with Honda. More precisely, as it is culminated in §3.1.2, we show and determine the very rich nontrivial moduli of all the limit measures of (further) Gromov–Hausdorff collapses from tropical K3 surfaces to an interval. For algebraic geometers, one can say that this gives a new invariant for type II degenerations of K3 surfaces, as a PL function of one real variable.

As another virtue for algebraic perspective of the reconstruction, we also do not rely on the general theory of Kollár–Shepherd–Barron–Alexeev moduli of semi-log-canonical models, which in turn depends on the Minimal Model Program (three-dimensional relative semistable MMP) in this case). Furthermore, from our construction, the presence of fibration structures on each degenerate surface come for free, which [ABE20, §7C] proved by some discussions on periods and deformation theory.

Furthermore, our (re)proof also do not logically use the tropical K3 surfaces or the key PL functions, although we finally aim to clarify the meaning of those tropics appeared in [ABE20] and [Osh]. We expect that this reconstruction also provides convenience for future study of limits of K3 metrics at different rescale.

In this section, we first briefly review the irreducible components of stable degenerations introduced in [ABE20] (see also [AB19, 8.13]) and give alternative description to each.
2.2.2. Preparation

2.2.2.1. Notations and convention

• (Recall) the base $\mathbb{P}^1$ of elliptic K3 surfaces in our concern has homogeneous coordinates $s_1, s_2$ and $s := s_1/s_2$.

• $g_8 = \sum_i a_i s^i \in H^0(\mathbb{P}^1_s, O(8) = O(8[\infty]))$.

• $g_{12} = \sum_i b_i s^i \in H^0(\mathbb{P}^1_s, O(12) = O(12[\infty]))$.

• $\Delta_{24} = \sum_i d_i s^i \in H^0(\mathbb{P}^1_s, O(24) = O(24[\infty]))$.

• $g_4 \in H^0(\mathbb{P}^1_s, O(4) = O(4[\infty]))$.

• $g_6 \in H^0(\mathbb{P}^1_s, O(6) = O(6[\infty]))$.

• Following [FriMrg94, e.g., §1.4], $\mathbb{P}_E$ for a locally free coherent sheaf $\mathcal{E}$ denotes the projective bundle as the covariant projectivization, rather than Grothendieck contravariant notation.

• (Notes added in the revision) 8 months after the first version of this paper on arXiv, the preceded preprint [ABE20], which had been a great inspiration for us, is revised to its version 4 in which the authors change some notation. In particular, the notation of the types of the components of the degenerations from $A, D, E, \tilde{D}, \tilde{E}$ to $A, C, E, \tilde{C}, \tilde{E}$. We use and keep the notation after their previous version as we explicitly and independently describe in §§2.2.2.3, 2.2.2.4, 2.2.2.5, and 2.2.2.6.

2.2.2.2. Degenerate surfaces over the compactified moduli by [ABE20] We briefly recall the degenerate surfaces over the boundary of $\overline{M}^{\text{ABE}}$. We explore and classify the prime divisors later in §2.2.4.

First, we focus on the type III degenerations parametrized on the normalization of $\overline{M}^{\text{ABE}}$, that is, the toroidal compactification $\overline{M}^\text{toroidal,} \Sigma^\text{rc}$ with respect to the rational curves cone $\Sigma^\text{rc}$ (see [ABE20, §4C], [Brun15, §12]), which first parametrizes special Kulikov degenerations up to the flops of the Kulikov type either:

\[
\begin{cases}
XI \cdots IX, \\
XI \cdots IY, \\
YI \cdots IY.
\end{cases}
\]

Each symbol refers to an irreducible components, but they are not all the components. We omitted the subindices (called charge as invariant of the integral affine singularities, in [ABE20], [AET19]), whose sum is 24. When we pass to the ultimate KSBA degeneration, then many of the components are contracted, so that we get a surface of the stable type:

\[
\begin{cases}
\DA \cdots \AD, \\
\DA \cdots \AE, \\
\EA \cdots \AE,
\end{cases}
\]

respectively, because

$X$ turns to $E$ with subindex 3 less, $Y_2Y_{d+2}$ turns to $D$ with (total) subindex 4 less, and $I$ turns to $A$ with subindex 1 less during this contraction process. These $A, D, E$ correspond to the root lattices of the same symbols.
From here, we recall some of the surface components including type II case, and give some different elementary descriptions for our purpose of the reconstruction of $\mathcal{M}_{W}^{\text{ABE}}$.

2.2.2.3. A-type surface  About the A-type surface [ABE20, §7G], we have nothing new to add to [ABE20, §7G], so we simply recall it for readers’ convenience. For the nodal rational curve $C$, that is, the rational curve with only one singularity which is the node, consider $C \times \mathbb{P}^1 \to \mathbb{P}^1$ with marked $k$ fibers over the points which are neither over 0 nor $\infty$. The normalization is $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

2.2.2.4. D-type surface  For any square-free quadric polynomial $P_2$ of $s$, regarded as an element of $H^0(\mathbb{P}^1_s, \mathcal{O}(2)) = \mathcal{O}(2[\infty])$, the fibers of

\[
X^{W}_{3P_2^2, P_3^2} := [y^2z = 4x^3 - 3P_2^2xz^2 + P_3^2z^3] \to \mathbb{P}^1_s
\]

\[
[y^2z = (2x - P_2z)^2(x + P_2z)]
\]

\[
\subset \mathbb{P}_{\mathbb{P}^1} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} \right),
\]

as fibration over $\mathbb{P}_s^1$, are generically (irreducible) nodal rational curves, with at most two cuspidal rational curves over the roots of $P_2$. Recall that our notation for projective bundle is, as §2.2.2.1 declares, the covariant projectivization of a locally free sheaf.

The normalization of this surface is the $\mathbb{P}^1$-fiber bundle with fiber coordinates $[y : x - P_2z]$, which is $\mathbb{P}^1_2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(1))$, the Hirzebruch surface $\mathbb{F}_1$. So, as $P_3$ is square-free, the surface coincides with the underlying fibered surface of $D_k$-surface ($k$ only makes difference of the boundary divisors) which [ABE20, §7G] writes. The fiberwise ordinary cusps are simply pinch points as [ABE20].

As we note in §2.2.2.1, our notation follows version 3 of [ABE20], and version 4 of [ABE20] refers to the above surface as type $C$. Recall that the lattice generated by the root system of $C_n$ is the same as that corresponding to $D_n$, although the lengths of roots are not all same.

2.2.2.5. E-type and \(\tilde{E}\)-type surfaces  For general $g_4, g_6$,

\[
X^{W}_{g_4, g_6} := [y^2z = 4x^3 - g_4xz^2 + g_6z^3] \subset \mathbb{P}_{\mathbb{P}^1} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} \right)
\]

becomes a rational elliptic surface with only ADE singularities (cf. [Kas77], [Mir81]). We specify the $I_{9-k}$ Kodaira-type fiber [Kod63] as the boundary, then we call this type of log surface $E_k$ ($k = 1$ has two types). If $k$ reaches 9, we rather denote $\tilde{E}_9$ which is nothing but the rational elliptic surface minus a smooth elliptic curve fiber.

Here, we allude to the fact that this $E_k$ ($k \leq 8$) surface (resp. $\tilde{E}_9$) is exactly the Landau–Ginzburg model for Del Pezzo surfaces (resp. rational elliptic surface) in the context of mirror symmetry as [AKO06] showed the homological mirror symmetry type statement. Furthermore, the associated lattices coincides with those of Del Pezzo surfaces [Manin, Chap. IV, §25]. See [CJL21] for related work.

2.2.2.6. D-type surface  We discuss $\tilde{D}_{16}$-type surface similarly to the one discussed in §2.2.2.4. For a square-free quartic polynomial $Q_4 \in H^0(\mathbb{P}^1_s, \mathcal{O}(4[\infty]))$, we consider as in
we have algebraic surfaces and simple birational geometry. At one end of the one parameter family, parameter family of fibered surfaces cf., \((31)\). We do so by using only pure algebraic geometry degenerations over the roots of \(G_4\) (see [OO21, §§7.1.1 and 7.1.3] for details). The normalization of this surface is the \(\mathbb{P}^1\)-fiber bundle with fiber coordinates \([y : x - G_4 z]\), which is \(\mathbb{P}^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))\), the Hirzebruch surface of degree 2, that is, \(F_2\).

We remark here that the log KSBA surface parametrized along the same strata as [ABE20, §7F] consists of 18 components, and the middle ruled components are all not open K-polystable in the sense of [Od20a], unless the 16\(\mathbb{P}^1\)'s on the top components all the way flopped down to the bottom components, so that all the middle components become trivial \(\mathbb{P}^1\)-bundle over the elliptic curve.

As we note in §2.2.2.1, our notation follows version 3 of [ABE20], and version 4 of [ABE20] refers to the above surface as type \(\tilde{C}\).

2.2.2.7. Mutations of Y-surfaces Recall from [ABE20] that two types of parts of Kulikov degenerations \((Y_2)_{Y_2}(I_a \cdots)\) and \((Y_2)_{Y_2}(I_a \cdots)\), modulo corner blowups, parametrized at the toroidal compactification \(\overline{\mathcal{M}_W}^{\tor}\), are not distinguished once we contract them to the KSBA models (they become \(D_0\mathbb{A}_2 \cdots\) type) parametrized at \(\overline{\mathcal{M}_W}^{\ABE}\). This is the main reason of non-normality of \(\overline{\mathcal{M}_W}^{\ABE}\), as explained in [ABE20, §7I].

Here, we reinterpret this by elementarily (by explicit equations) constructing a one parameter family of fibered surfaces cf., (31). We do so by using only pure algebraic geometry of algebraic surfaces and simple birational geometry. At one end of the one parameter family, we have \(Y_2(I_a \cdots)\) surface, while at the other end, we see degeneration to \(Y_2'(I_a \cdots)\). The generic fiber is \(Y_3(I_a \cdots)\). This is the transition we should observe at the outer (and left) part of the [ABE20, §7] type Kulikov degeneration.

For that purpose, recall the Hirzebruch surface \(F_1\) and \(F_0 = \mathbb{P}^1 \times \mathbb{P}^1\), and \(\mathbb{P}^1\)-bundles over the common base \(\mathbb{P}^1\) are elementary transforms of each other. Therefore, there is a common noncorner blowup which we write as \(\varphi_S : S \to \mathbb{P}^1\) (this corresponds to \(Y_3\) in [ABE20]) and we denote their centers in \(F_i\) are \(p_i(i = 0, 1)\). We denote the projections as \(\pi_i : F_i \to \mathbb{P}^1\) which satisfies \(\pi_0 \circ \varphi_0 = \pi_1 \circ \varphi_1\).

In general, if we take a general conic in \(\mathbb{P}^2\) and its strict transform \(D_1\) in \(S, F_i(i = 0, 1)\), then the projection to \(\mathbb{P}^1\) has two ramifying points as [ABE20, §7B] writes. It is easy to see that after the automorphism, we can and do assume that \(p_i \in F_i\) is one of two points \(D_1 \cap \pi_i^{-1}(\infty)\) for both \(i\).

Here, we use the construction of [Ohno21, §3.1], which originally aimed to partially establish the CM degree minimization conjecture (cf. [Od20c], [Ohno21]) in the context of K-stability, by the author. One main point is we consider extra direction by introducing \(\mathbb{A}^1\). We consider the blowup of \(\mathbb{P}^1 \times \mathbb{A}^1\) at \((\infty, (t = 0))\) (resp. \((\infty, 1))\), which we denote by \(\beta_i : B_i \to \mathbb{P}^1 \times \mathbb{A}^1\).
Then, take the fiber product with 
\[ \Pi_i = (\pi_i \times id): F_i \times \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1, \]
for \( i = 0 \) (resp. \( i = 1 \)) and further blow up the total space along a smooth closed curve \((\{\infty\} \times (\mathbb{A}^1 \setminus \{\{i\}\}))(\simeq \mathbb{A}^1_K)\). Then, we obtain \(^2\)
\[ \tilde{\Pi}_i: F_i \to \text{Bl}_{(\infty,i)}(\mathbb{P}^1 \times \mathbb{A}^1). \]
We can glue these two for \( i = 0 \), since the blowups of \( F_0 \) at \( p_0 \) and \( F_1 \) at \( p_1 \) coincide, and obtain
\[ \tilde{\Pi}: F \to \text{Bl}_{(\infty,0) \cup (\infty,1)}(\mathbb{P}^1 \times \mathbb{A}^1). \]
We denote the fiber over \( t \) by \( \tilde{F}_t \). Then, \( F_t \) is
\[ (F_0 \to \mathbb{P}^1) \cup (S \to \mathbb{P}^1) = Y_2'I_a \text{ for } t = 0, \]
\[ S \to \mathbb{P}^1 = Y_3 \text{ for } t \neq 0,1, \]
\[ (F_1 \to \mathbb{P}^1) \cup (S \to \mathbb{P}^1) = Y_2I_a \text{ for } t = 1. \]
This interesting family \( \{F_t\}_t \) with two different degenerations at \( t = 0 \) and \( t = 1 \) exactly describes the switch between \( Y_2Y_2 \) and \( Y_2Y_2' \) in the context of \([ABE20]\). Recall from \([ABE20]\) (also see \([Osh]\)) that the corresponding PL functions to each of (32), (33), and (34) start with slopes 8, 7, and 8, respectively.

2.2.2.8. Slight extension of ADE lattices In \([ABE20]\), over \( K = \mathbb{C} \), they used the periods and corresponding Torelli theorems for components of the degeneration of elliptic K3 surfaces after \([Fri15]\) and \([GHK15]\).

The convention of denoting each component by \( A, D, E \) comes from it, but for such description, they indirectly used the following slight extension of the usual ADE lattices; and allows \( D_i \) for \( i = 1, 2, 3 \) and also \( E_i \) for \( i = 1, 2, 3, 4, 5 \). We logically do not need it until §3.2, but for the convenience of readers, we clarify here.

The lattice \( D_i \) for \( i < 4 \) is constructed in the same way as those with \( i \geq 4 \). Simply,
\[ D_i := \{(x_1, \ldots, x_i) \in \mathbb{Z}^i \mid \sum_j x_j \in 2\mathbb{Z}\}. \]

In our context, with respect to the fundamental domain, these D-type lattices are naturally realized in \( \Lambda_{\text{seg}} \) as
\[ \langle \alpha_1 - \alpha_3 \rangle \text{ for } i = 1, \]
\[ \langle \alpha_1, \alpha_3 \rangle \text{ for } i = 2, \]
\[ \langle \alpha_1, \alpha_3, \alpha_4 \rangle \text{ for } i = 3. \]

On the other hand, the following inductive construction of \( E_i \) (from \( i = 1 \)) is essentially due to Manin \([Manin]\).

\(^2\) The author also used this construction in a joint work with R. Thomas on K-stability in 2013.
We construct a little extended lattice $E'_i$ for $i = 1, 2, \ldots$ with $E_i \subset E'_i$ which has corank 1 and orthogonal to $K_i$. (Geometrically, it is fairly simple, that is, $E'_i = H^2(S_{9-i},\mathbb{Z})$ where $S_{9-i}$ stands for Del Pezzo surface of degree $9-i$ and $c_1(S_{9-i})^\perp = E_i$.) Here is more elementary construction (through blowup):

\begin{align*}
E'_i &:= \mathbb{Z}l(l^2 = 1), \quad -K_1 = 3l. \\
E'_{i+1} &:= E'_i \oplus \mathbb{Z}e_i(e_i^2 = -1) \quad K_{i+1} = K_i + e_i.
\end{align*}

Allowing the above type D lattices and E lattices with lower indices, we call these such A,D,E lattices and their direct sum as slightly generalized root lattice. See [LO19, §1] for related discussions.

2.2.3. Reconstruction of [ABE20]

In our logic for the reconstruction of the compactification of [ABE20], first we readily construct the desired moduli stack $\overline{M}_{W,\text{ABE}}$, and then we show the desired properties especially the properness as well as the presence of projective coarse moduli spaces $\overline{M}_{W,\text{ABE}}$ ($F^c$ in [ABE20]) later.

Our discussion uses the degenerations of the elliptic K3 surfaces parametrized by $\overline{M}_{W,\text{ABE}}$ simply as a set(!) and denote them by $(X,R) \in \overline{M}_{W,\text{ABE}}$. First, we fix large enough positive integers $m$ and $d$, so that for any $(X,R = s + m \sum f_i) \in \overline{M}_{W,\text{ABE}}$, $R$ is ample and $dR$ is very ample without high cohomology. Obviously, $\chi(X,\mathcal{O}_X(dR))$ does not depend on $(X,R)$s. Then, we take the corresponding Hilbert scheme $H'$. Naturally, $G := \text{SL}(H^0(X,dR))$ acts on $H$.

We take a subset $H$ of $H'$ parametrizing the surfaces $X$ parametrized by $\overline{M}_{W,\text{ABE}}$ embedded by $dR$. Since the subset is characterized as those $\mathcal{O}_P(1)|_X = \mathcal{O}_X(dR)$ (closed condition) as well as the reduced semi-log-canonical-Gorenstein properties of $X$ (open condition), $H$ is a locally closed subset of $H'$.

Then, we put reduced scheme structure on $H$ and set

\begin{equation}
\overline{M}_{W,\text{ABE}} := [H/G],
\end{equation}

the quotient (a priori only Artin) stack. It is straightforward to see that, by expanding the meaning, the corresponding moduli contravariant functor (35) represents is the following which we denote by the same symbol $\overline{M}_{W,\text{ABE}}$: For a scheme $T$ with all residue characteristics other than 2,3,

\begin{equation}
\overline{M}_{W,\text{ABE}}(T) := \{(\mathcal{X},\mathcal{R}) \rightarrow \mathcal{P} \rightarrow T^{\text{red}}\}/\sim,
\end{equation}

where $T^{\text{red}}$ is the reduced subscheme of $T$, that is, the same topological space as $T$ with the structure sheaf replaced by the reduced quotient, $\mathcal{P} \rightarrow T^{\text{red}}$ is a proper morphism whose geometric fibers are all either $\mathbb{P}^1$ or the chain of $\mathbb{P}^1$s, $\mathcal{X} \rightarrow \mathcal{P}$ is a flat proper family of surfaces, $\mathcal{R}$ is its relative Cartier divisor such that for all $t \in T$, the fiber $(\mathcal{X}_t,\mathcal{R}_t) \rightarrow \mathcal{P}_t$ is a fiber of universal family over $H$, and $\sim$ refers to the natural isomorphism as an equivalence relation. See also [PS21, Defs. 4.1 and 4.3 (also 1.1 and 1.2)] (cf. also [ASD73, §1]) for the description of open subfunctor, in which $\mathcal{R}$ is a redundant data.
Now, we prove that this is actually a proper Deligne–Mumford stack (i.e., stable reduction type statements) case by case, so that we reprove the following in an elementary way. (Of course, we do not mean to be short arguments, by the word elementary.)

**Theorem 2.5.** (cf. [ABE20]) The moduli algebraic stacks (constructed above) \( \mathcal{M}_W \subset \overline{\mathcal{M}}_{W}^{ABE} \) of elliptic K3 surfaces and their degenerations over Spec(\( \mathbb{Z}[1/6] \)) (the former is an open substack of the latter) both admit the coarse moduli varieties \( \mathcal{M}_W \subset \overline{\mathcal{M}}_{W}^{ABE} \) (the former is an open subvariety of the latter) such that \( \overline{\mathcal{M}}_{W}^{ABE} \) is projective.

The reason we work away from characteristics 2 and 3 are that our following arguments rely on the (uniform) description of equation of the Weierstrass models (cf. [Mir81], also [Sil85, Appendix A], [PS21]) and their degenerations.

Elementary direct reproof. The existence of coarse moduli spaces as algebraic spaces follows from [KeMo97], since the inertia groups of the moduli stack are nothing but the automorphism of log canonical model \((X,\epsilon R)\) which is finite (cf. [Iit82, Chap. 11], [Amb05, Prop. 4.6]). The projectivity follows from the ampleness of the determinant of direct image sheaves of pluri-log-canonical bundles (see [Fjn18], [KP17]).

Therefore, to reprove Theorem 2.5, it remains to show the following key claim from the valuative criterion of properness relative to Spec(\( \mathbb{Z}[(t)] \)) (e.g., [LM00, §7]). In particular, the uniqueness part shows that the reconstructed compactifications in this section and [ABE20] are identical.

**Theorem 2.6.** (stable reduction; cf. [ABE20]) For any field \( K \) of characteristic different from 2 and 3, and any \((X, R) \to \mathbb{P}^1_s\) parametrized in \( \overline{\mathcal{M}}_{W}^{ABE}(K((t)))\), \((X, R) \to \mathbb{P}^1_s\) has a unique (explicit) model \((\mathcal{X}, \mathcal{R}) \to \mathcal{B}\) over \( K[[t]]\) in \( \overline{\mathcal{M}}_{W}^{ABE}(K[[t]])\).

We fix further notations before giving the details of the proof.

2.2.2.9. Some further notations

- \( K \) denotes the field we take in Theorem 2.6, whose characteristic is coprime to 6. Recall that we use \( s \) for the corresponding coordinate, virtually valued in \( K \).
- Since we only wish to prove properness of the above quotient algebraic stack, we can and do assume that the field \( K \) is actually algebraically closed, just for simpler exposition.
- We denote the obvious trivial model \( \mathbb{P}^1_s \times \text{Spec}(K[[t]])\) of \( \mathbb{P}^1_s \times \text{Spec}(K((t)))\) as \( \mathcal{B}_{\text{triv}}\). We make birational transforms of this \( \mathcal{B}_{\text{triv}}\) to other model \( \mathcal{B} \).
- Discriminant locus of \((X, R) \to \mathbb{P}^1_s\) in \( \mathcal{M}_{W}(K((t)))\) as \( D \subset \mathcal{B} \). The fibers over its reduction \( D \cap (t = 0) \subset \mathcal{B} \) are called really singular in [ABE20] which we follow to use. We call their underlying closed points in the base as real discriminant (points).

**Proof of Theorem 2.6.** The uniqueness part follows from a standard birational geometric result, that is, the general uniqueness of relative log canonical model, not relying on the existence of difficult birational models nor the minimal model program. As this argument is well known to experts, we only sketch the proof for the normal case. Non-normal case is similarly proved after normalization. Suppose we have a priori two different models \([\pi_i: (\mathcal{X}_i, \mathcal{R}_i) \to \mathcal{P}_i \to S] \in \overline{\mathcal{M}}_{W}^{ABE}(S)\) for \( i = 1, 2 \) over a DVR \( S \) with the same generic normal fiber, and we take a bigger normal model \( \tilde{\pi}: \tilde{\mathcal{X}} \to S \) which dominates both \( \mathcal{X}_i \to S \) by morphisms \( f_i: \tilde{\mathcal{X}} \to \)
\( \mathcal{X}_i \) as \( S \)-schemes. We denote the exceptional divisors of \( f_i \) with all coefficients 1 as \( E_i \).

Then, since \((\mathcal{X}_i, \mathcal{R}_i)(i = 1, 2)\) are both relative log canonical models over \( S \), it follows that \( \text{Proj}_S(\pi_s + \pi_m \geq 0 \mathcal{O}(mK_{\mathcal{X}/S} + (f_i)_s \mathcal{R}_i + E_i)) \) is isomorphic to \( \mathcal{X}_1 \) as \( S \)-scheme for both \( i = 1, 2 \).

Hence, \( \mathcal{X}_1 \cong \mathcal{X}_2 \), so that \( \mathcal{R}_i \) and \( \mathcal{P}_i \) also coincide for \( i = 1 \) and \( i = 2 \).

Hence, we focus on the explicit construction of the desired stable reduction to each punctured family lying on \( \mathcal{M}_W \). By lifting to \( \mathbb{A}^{22} \), reduce to the following four cases: Cases 1–4.

**Case 1.** (Type III degenerations from \( \mathcal{M}_W \)) This case amounts to show the following claim.

**Claim 2.7.** (Maximally degenerating stable reduction) Given any \( g_{s}(s) \) in \( \Gamma(H^0(\mathbb{P}^1_s, \mathcal{O}(8))) \) \( \otimes \mathbb{K}[[t]] \) (resp. \( g_{12}(s) \) in \( \Gamma(H^0(\mathbb{P}^1_s, \mathcal{O}(12))) \otimes \mathbb{K}[[t]] \)) such that

\[
\begin{align*}
X^W_{g_s,g_{12}|t \neq 0} & := \{ y^2 z = 4x^3 - g_s(t)xyz^2 + g_{12}(t)z^3 \} \\
(36) & \subset \mathbb{P}^2_{s} (\mathcal{O}_{\mathbb{P}^2_{s}}(4) \oplus \mathcal{O}_{\mathbb{P}^2_{s}}(6) \oplus \mathcal{O}_{\mathbb{P}^2_{s}}),
\end{align*}
\]

as in \([OO21, \S 7.1] \) is an elliptic \( K3 \) surface parametrized in \( \mathcal{M}_W(K((t))) \), that is, only with \( ADE \) singularities and \( g_s|_{t = 0} = 3s^4, g_{12}|_{t = 0} = s^6 \) (i.e., converging to \( \mathcal{M}^{\text{nm,seg}}_W \) in the Satake–Baily–Borel compactification (cf. \([OO21, \S 7]\) ), the corresponding \( X \to \mathbb{P}^1_s \) (resp. \( \mathbb{P}^1_s \times \text{Spec}(K((t))) \) over \( \mathbb{K}[[t]] \), so that \( X|_{t = 0} = \mathcal{B}|_{t = 0} \) is (the only possible) one of those parametrized in \( \mathcal{M}^{\text{ABE}}_W \).

**Step 1.** (End surfaces) To prove Claim 2.7, first we take finitely ramified base change from \( K[[t]] \) to \( K[[t^{1/d}]] \) for some \( d \in \mathbb{Z}_{> 0} \), so that we can and do assume that the roots of \( g_s, g_{12} : g_3 := g_3^s - 27g_2^2 \) are Laurent (not only Puiseux), that is, there are \( \xi_i \in K((t))(i = 1, \ldots, 8) \), \( \eta_i \in K((t))(i = 1, \ldots, 12) \), and \( \chi_i \in K((t))(i = 1, \ldots, 24) \) in the descending order of the valuations \( v_t(-\) along coordinates \( s \) with respect to \( t \) (or additive inverse of the valuation of \( s^{-1} \)).

We first set

\[
\begin{align*}
(38) e(0) & := \min\{v_t(\xi_1), \ldots, v_t(\xi_4), v_t(\eta_1), \ldots, v_t(\eta_6)\}, \\
(39) e(\infty) & := \min\left\{v_t\left(\frac{1}{\xi_5}\right), \ldots, v_t\left(\frac{1}{\xi_8}\right), v_t\left(\frac{1}{\eta_7}\right), \ldots, v_t\left(\frac{1}{\eta_{12}}\right)\right\},
\end{align*}
\]

and after an appropriate elementary transform of the trivially extended \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1_s \times_K K[[t]] \) (we fix this ambiguity below soon), further blow it up to \( \mathcal{B}_1 \to \mathcal{B}_{\text{triv}} \) by the coherent ideal sheaf

\[
(40) \langle s, t^{e(0)} \rangle \cdot \langle s', t^{e(\infty)} \rangle \cdot \mathcal{O}_{\mathcal{B}_{\text{triv}}},
\]

Here, \( s' := \frac{s_2}{s_1} \) and it is also a local uniformizer at \([s_1 : s_2] = [1 : 0] \) (\( \infty \) -point) in the base \( \mathbb{P}^1_s \). Then, the special fiber of \( \mathcal{B}_1 \) over \( t = 0 \) is

\[
(41) \mathbb{P}^1_s \bigcup \mathbb{P}^1_s \bigcup \mathbb{P}^1_s' \bigcup \mathbb{P}^1_{s''},
\]

where the two ends are exceptional curves.

Accordingly, we can naturally degenerate the ambient space \( \mathbb{P}^2_{s} (\mathcal{O}_{\mathbb{P}^2_{s}}(4) \oplus \mathcal{O}_{\mathbb{P}^2_{s}}(6) \oplus \mathcal{O}_{\mathbb{P}^2_{s}}) \) over \( K((t)) \) to over \( K[[t]] \), so that the special fiber over \( t = 0 \) is a connected union of the
following three irreducible components:

i. \( \mathbb{P}^1 \) (\( \mathcal{O}(2) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \)) over \( \mathbb{P}^1 \setminus \{0\} \),

ii. trivial \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^1 \) (i.e., \( \mathbb{P}^2 \times \mathbb{P}^1 \)), and

iii. \( \mathbb{P}^2 \) (\( \mathcal{O}(2) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) \)) over \( \mathbb{P}^1 \setminus \{\infty\} \).

Inside the first component (i), the closure of \( X \) (limit component) appears as

\[
X^W_{\gamma_1^6} := [y^2z = 4x^3 - g_4^s|_{t=0}xz^2 + g_6^s|_{t=0}z^3],
\]

where \( g_4^s = c_4 \prod_{i=1}^4 (s - \xi_i) \) and \( g_6^s = c_6 \prod_{i=1}^6 (s - \eta_i) \), with replaced roots \( \xi \)s and \( \eta \)s. Recall that construction of the model \( B \) above had an ambiguity modulo elementary transform with respect to \( t = 0 \), but we fix it by assuming \( (c_4, c_6) \in K^2 \setminus \bar{0} \). From the construction, \( g_4^s \) and \( g_6^s \) are strictly degrees 4 and 6, respectively, with coefficients 3 and 1, respectively, and \( \Delta_{12} := (g_4^s)^3 - 27(g_6^s)^2 \) has degree at most 11. This means that the component \( X^W_{\gamma_1^6} \) has singular fiber over \( \infty \), which corresponds to the fact that the degeneration is of type III.

Furthermore, from the definition of \( e(0) \), not all of \( \xi \)s and \( \eta \)s vanish. Similarly, in the last component (iii), the closure of \( X \) (limit component) appears as

\[
X^W_{h_4^6} := [y^2z = 4x^3 - h_4^s|_{t=0}xz^2 + h_6^s|_{t=0}z^3],
\]

where \( h_4^s = \prod_{i=5}^{8} (s - \xi_i) \) and \( h_6^s = \prod_{i=7}^{12} (s - \eta_i) \), again with newly replaced roots \( \xi \)s and \( \eta \)s. From the construction, due to \cite[Lem. 1]{Kas77}, if Weierstrass surfaces are generically smooth, they automatically only have ADE singularities (at nonzero finite base coordinates).

When \( K = \mathbb{C} \), in comparison with our asymptotic analysis of McLean’s real Monge–Ampère metrics in \cite[§7.3.3]{OO21}, these end surfaces are where the term (denominator of the second term in \cite[Lem. 7.16]{OO21})

\[
(\log(|g_8|^3 + 27|g_{12}|^2))
\]

becomes dominant. On the other hand, the following step is relevant to expand the divergence of the \( \log(|\Delta_{24}|) \) term.

**Step 2.** (Separating middle \( \chi \)s) In the next step, we consider toric model \( B \) with respect to some combinatorial data coming from the Newton polygon, as the method used classically by \cite{AN99}, \cite{Don02}, and \cite{Mum72b} as follows. We consider the Newton polygon \( \text{Newt}(\Delta_{24}) \) of \( \Delta_{24} \), that is, the convex hull of

\[
\{(i, v_t(d_i)) \mid 0 \leq i \leq 24\} + \mathbb{R}_{\geq 0}(0, 1).
\]

We regard it as a graph of PL convex function \( \varphi_\Delta : [0, 24] \to \mathbb{R} \cup \{\infty\} \). Then, we modify this as follows (this process aims at including the previous step when we consider the toric models).

Set

\[
i_{\epsilon(0)} := \max\{i \mid \varphi_\Delta(i) - \varphi_\Delta(i + 1) \geq e(0)\},
\]

\[
i_{\epsilon(\infty)} := \min\{i \mid \varphi_\Delta(i + 1) - \varphi_\Delta(i) \geq e(\infty)\},
\]
where $e(0)$ and $e(\infty)$ as (38) and (39). We modify $\varphi_{\Delta}$ to $\overline{\varphi}_{\Delta}: [0, 24] \to \mathbb{R} \cup \{\infty\}$ defined as follows:

\begin{equation}
\varphi_{\Delta}(i) := \begin{cases} 
\varphi_{\Delta}(i_{e(0)}) - e(0)(i_{e(0)} - i) & \text{(if } 0 \leq i \leq i_{e(0)}) , \\
\varphi_{\Delta}(i) & \text{(if } i_{e(0)} \leq i \leq i_{e(\infty)}) , \\
\varphi_{\Delta}(i_{e(\infty)}) + e(\infty)(i - i_{e(\infty)}) & \text{(if } i_{e(\infty)} \leq i \leq 24). 
\end{cases}
\end{equation}

Then, consider the toric model (test configuration of $\mathbb{P}^1$) $B$ over $\mathbb{A}^1$ (hence also over $K[[t]]$), corresponding to $\overline{\varphi}_{\Delta}$, that is, for some $c \geq 0$, the moment polytope of (the natural compactification of) $B$ becomes $P_{\Delta,c}$.

In particular, the normal fan of the graph of $\overline{\varphi}_{\Delta}$ gives $B$ by usual toric construction. We fix and take the natural $c$ such that the obtained $B$ has the same end components as $B_1$ in Step 1, that is, the end components of $B_{|t=0}$ are the bases $\mathbb{P}^1_{\{t=0\}}$ and $\mathbb{P}^1_{\{\infty\}}$ of the ends at (63). Indeed, it is possible by our modification (73) of the PL function.

Furthermore, as desired, every other component of $B_{|t=0}$ has at least one point of $D_0(= \bar{D} \cap (t = 0))$. Here, recall that $D$ denotes the discriminant locus defined after Theorem 2.6 whose closure is denoted as $\bar{D}$. This ensures the ampleness of the boundary $R$ in the corresponding irreducible components of the Weierstrass (reducible) fibered surface.

**Step 3. (About end surfaces again)** If the end surface $X^W_{g_4', g_6'} \to \mathbb{P}^1_*$ is generically smooth, it is nothing but a rational elliptic surface, that is, type $E_k$ in [ABE20]. In that case, because of the construction, $\deg \Delta_{12}' = 12$ ($\Delta_{12}'$ does not vanish at $\infty$), so that the fiber over $\frac{s}{t^{16}} = \infty$ cannot be singular.

On the other hand, if the end surface $X^W_{g_4', g_6'}$ has singular general fibers, it means that there is $P_2 \in H^0(\mathcal{O}(2[\infty]))$ such that

\begin{equation}
g_4' = 3P_2^2, g_6' = P_2^3.
\end{equation}

$\deg(P_2)$ cannot be less than 2 from the construction. If this $P_2$ is square-free, then from our discussion in §2.2.2, we get the surface $\mathbb{D}$ type and end the step here. If $P_2$ is not square-free, we continue to next step.

**Step 4. (Modifying almost $D$-type end)** Depending on formulation, this process may be included in Step 1, but, nevertheless, we separated it to make the steps clearer. From here, we treat the left end surfaces in the original sense of Step 1, that is, those map to $s = 0$, that is, defined by $g_4'$ and $g_6'$. (For the right end surface which maps to $s = \infty$, the completely similar arguments work by symmetry, so we avoid repetition of the details of the arguments.)

We continue from the previous step, so suppose $P_2$ is not square-free. Nevertheless, since our generic fiber at $t \neq 0$, $X^W_{g_4, g_{12}}$, was originally at worst ADE, among those (a priori at total 10) roots of $g_4$ or $g_6$, that is, $\xi_i(1 \leq i \leq 4), \eta_j(1 \leq j \leq 6)$, at least two of them do not coincide as elements of $K[[t]]$ (before substitution $t = 0$). Suppose that they are \( \{p, q\} \subset \{\xi_i(1 \leq i \leq 4), \eta_j(1 \leq j \leq 6)\} \) with respect to the new coordinates after Step 1. Write the local uniformizer at $p(0) = q(0)$ for the component $\mathbb{P}^1_{\{t=0\}}$, as $s_{p,q}$. 

We make Puiseux expansions of $p, q$, and set $e_{p, q} := v_t(p - q)$, where $v_t$ denotes the $t$-adic (additive) valuation. Then, do blowup of $B$ (which was the outcome of processes until the previous step) along $(s_{p, q}, t^{e_{p, q}}) \mathcal{O}_B$ whose cosupport is in $\mathbb{P}^1_{\mathbb{Q}(t)} \times \{t = 0\}$, and blow down the surface without $D \cap \{t = 0\}$ if necessary, so that we obtain the situation with square-free $P_2$. Note that by this last step, the resulting model $B$ may not be toric, while toroidal, with respect to the original coordinates (since $p(0) = q(0)$ may not be zero).

**Case 2.** (Type II degenerations) These cases are essentially done in [CM05, §3] via deformation theory and more Hodge-theoretic viewpoints, while the degenerations are slightly modified in [ABE20] (see also [Fri84], [Kon85] including nonelliptic case).

Here, we again recover them by our elementary method using the Weierstrass form as below.

**Subcase.** (to $\tilde{D}_{16}$) This case essentially follows from the GIT picture in [OO21, §7] by applying the GIT stable reduction. Recall that the Satake–Baily–Borel compactification $\overline{M}_{W}^{\text{SBB}}$ coincides with the GIT compactification with respect to the Weierstrass expression [OO21, §7.2.1]. As [OO21, §7.1] shows, the locus $M_{W}^{\text{nn}}$ is in the strictly stable locus, which parametrizes the semi-log-canonical surface of the form (24), which is nothing but $\tilde{D}_{16}$-type in [ABE20].

If we have $(g_8, g_{12}) \in H^0(\mathcal{O}(8)) \times H^0(\mathcal{O}(12))$ over the base $K[[t]]$, with reduction sits in the stable locus mapping down to $M_{W}^{\text{nn}}$, then the GIT stable reduction proves that after finite base change if necessary, if we apply an element of $\text{SL}(2)$ in the coefficient $K((t))$, we get reduction with special fiber of the surface of type (24). This completes the required process.

**Remark 2.8.** By comparing with toroidal compactification, recall that type II locus does not depend on the choice of admissible rational polyhedral decompositions (cf., e.g., [Fri84]). Furthermore, the preimage of $M_{W}^{\text{nn}}$ in it which we write as $M_{W}^{\text{nn}, \text{tor}}$ is a $\text{Aut}(\tilde{D}_{16})$-quotient of the 16th self-fiberproduct of the (coarse moduli of) universal elliptic curve over $\overline{M}_{W}^{\text{nn}} \cong \mathbb{A}^1_j$. ($j$ stands for the $j$-invariant of $E$.) There is a very clear geometric meaning to this phenomenon—by [CM05, §3] and [ABE20, 7.20, 7.22, and 7.44], the 16 real discriminants are arbitrary (for each fixed $E$), which give the difference of these $M_{W}^{\text{nn}}$ and $M_{W}^{\text{nn}, \text{tor}}$.

Note that the parametrized degeneration is slightly different between that in [CM05, §3] and [ABE20] (i.e., the former has two components one of which is those parametrized in [ABE20]—$\tilde{D}_{16}$-surface), but this is unessential difference. Indeed, the relation is by a simple birational transform (at the total space level) as explained in [OO21, §7.1.3].

**Subcase.** (to $\tilde{E}_8 \tilde{E}_6$) We treat the case of degenerating from $M_W$ to $M_{W}^{\text{seg}} \subset M_{W}^{\text{SBB}}$, which we recall to be the $\tilde{E}_8 \tilde{E}_6$-type 1-cusp (see also its GIT interpretation in [OO21, §7]).

Take $(X, R) \to B$ in $\mathcal{M}_W(K((t)))$ which degenerates to $M_{W}^{\text{seg}}$ at the closed point. From [OO21, §7], it follows that we can lift these data to $(g_8, g_{12}) \in H^0(\mathcal{O}(8)) \times H^0(\mathcal{O}(12))$ with coefficients in $K[[t]]$, so that its reduction is $(cs^4, s^6)$ for $c \neq 3$.

Then, we can exploit the same procedure as Step 1 of Case 1, to replace the reduction as the reducible fibered surface

$$\begin{align*}
(X_1 \cup X_2) \to \mathbb{P}^1 \cup \mathbb{P}^1,
\end{align*}$$

(51)
where $X_1$ (resp. $X_2$) is a hypersurface of the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ over the first $\mathbb{P}^1$ (resp. the second $\mathbb{P}^1$), defined by

$$[y^2 z = 4x^3 - g_d' x z^2 + g_0' z^3],$$

$$[y^2 z = 4x^3 - h_d' x z^2 + h_0' z^3],$$

respectively. Then, from our assumption that $c \neq 3$, it follows that the double locus $X_1 \cap X_2$ is smooth elliptic curve fiber, hence this is of $E_8 E_6$-type surface as desired. We have 12 real discriminant points in each base.

**Case 3.** (Further degenerations from type III degenerations) Below, we study the occurring degeneration componentwise. We proceed as follows. In the notations below, we promise that:

1. $\sum l_i = l$.
2. All the subindices are nonnegative.
3. We call the images of really singular fibers (cf. notations below Theorem 2.6) on any of possibly singular $[(X, R) \to B(\simeq \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1)] \in \overline{\mathcal{M}}_W^{\text{ABE}}(K)$ or $\overline{\mathcal{M}}_W^{\text{ABE}}(K((t)))$ as $\chi_{1, \ldots, \chi_{24}}$ (which extends the original meaning in the realm of $M_W$) and continue to call them real discriminant points.
4. Furthermore, before each discussion below, we lift these data $[(X, R) \to B \simeq \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1]$ by fixing gauge, that is, the isomorphism of every rational component with $\mathbb{P}^1$, so that their nodal points have coordinate 0 or $\infty$.

**Subcase.** ($\mathbb{A}_{l-1}$ to $\mathbb{A}_{l_1-1} \mathbb{A}_{l_2-1} \cdots \mathbb{A}_{l_m-1}$) We now concentrate on the base of the component of $A$-type in the degenerated

$$[(X, R) \to \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1] \in \overline{\mathcal{M}}_W^{\text{ABE}}(K((t))),$$

which we denote as $X_A \to \mathbb{P}^1$ here, with coordinate $s_A$. The real discriminant points $\chi_{a+1}, \ldots, \chi_{a+l}$ can be seen as formal Puiseux series, that is, elements of $K((t))$. Note that any of $\chi_{a+i}$ is not 0 nor $\infty$ (as element of $\mathbb{P}^1(K((t)))$. Hence, after finite base change, we can suppose that they all lie in $K((t))$ and we write $\Delta_A(s_A) := \prod_{1 \leq i \leq l}(s_A - \chi_{a+i})$.

Similarly to Step 2 of Case 1, we take Newton polygon Newt($\Delta_A$), its supporting function $\varphi_A$, and the toric degeneration model $\mathcal{B}_A$ over $\mathbb{A}^1$ (hence also over $K[[t]]$) whose corresponding fan is the normal fan of the graph of $\varphi_A$. Or in other words, the natural compactification has moment polytope

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq l, -c \leq y \leq -\varphi_A(x)\}$$

for a constant $c \gg 0$. This is one component of our desired $\mathcal{B}$, that is, the closure of $\mathbb{P}^1_{s_A}$. Then, accordingly, we degenerate the ambient space $\mathbb{P}^2 \times \mathbb{P}^1_{s_A} \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}^{\otimes 3})$ to still trivial $\mathbb{P}^2_{s_A}$-bundle over $\mathcal{B}$, so that we obtain the (semi-log-canonical) union of $\alpha$-type log surfaces as the closure of $X_A$ inside the ambient model $\mathbb{B} \times \mathbb{P}^2$.

**Subcase.** ($\mathbb{D}_{k+l}$ to $\mathbb{D}_k \mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) Next, we consider the base of the component of $D$-type in the degenerated

$$[(X, R) \to \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1] \in \overline{\mathcal{M}}_W^{\text{ABE}}(K((t))),$$
which we denote as \( X_D \to \mathbb{P}^1 \) here, with coordinate \( s_D \). We can and do suppose that the only double curve in \( X_D \), which is the intersection of next surface component, has coordinates \( s_D = \infty \).

Recall from §2.2.2.4 that we have an explicit Weierstrass-type equation for the \( \mathbb{D} \)-type surface (20) in terms of a quadratic polynomial \( P_2(s_D) \) whose coefficients live in \( K((t)) \). By quadratic base change, if necessary, we can further suppose that its two roots are also in \( K((t)) \). Then, by multiplying appropriate powers \( t^{2c}, t^{3c} \) of \( t \) to \( g'_2 \) and \( g''_3 \) which do not change the isomorphism class of original \( X_D \to \mathbb{P}^1 \) (over \( t \neq 0 \)), we can and do assume that coefficients of both lie in \( K[[t]] \) and do not vanish at \( t = 0 \) generically (with respect to \( s_D \)).

If some of real discriminants \( \chi_i \) in the base of \( X_D \) (including two roots of \( P_2 \)) converges to \( \infty \), whose fiber is in the double locus of the surface, then we do weighted blowup of the model and take closure of \( X \) to get desired model of type \( \mathbb{D}A_{\cdots}A \).

**Subcase.** \( (E_{k+1} \to E_k A_{l_{i-1}} \cdots A_{l_m-1}) \) Next, we consider the base of the component of \( E \)-type in the degenerated

\[
((X, R) \to \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1) \in \overline{\mathcal{M}}_{W}^{\text{ABE}}(K((t))),
\]

which we denote as \( \pi_E : X_E \to \mathbb{P}^1_{s_E} \) here, with coordinate \( s_E \). We can and do suppose that the only double curve in \( X_E \), which is the intersection of next surface component, has coordinates \( s_E = \infty \). We consider stable reduction of generic fiber, thus, over \( K(s_E) \), which is from elliptic curve to either elliptic curve or (irreducible) nodal rational curve over whole \( K(s_E)[[t]] \). Correspondingly, we realize this model by multiplying \( t^{2c} \) (resp. \( t^{3c} \))

\[
g_4 \in H^0(\mathbb{P}^1_{s_E}, \mathcal{O}(4)) \quad \text{(resp.} \ g_6 \in H^0(\mathbb{P}^1_{s_E}, \mathcal{O}(6))\text{)}
\]

with appropriate \( c \) (we fix this normalization from now on), so that \( g_4, g_6 \) both become nonzero at \( t = 0 \).

In this subcase, we focus when the generic fiber at \( t = 0 \) is smooth, that is, elliptic curve, which we suppose from now on, and leave the nodal reduction case to the next subcase.

Suppose that the real discriminant points \( \chi_{c+1}, \ldots, \chi_{c+k+l+3} \) below \( X_E \) also all sit in \( K((t)) \) after finite base change if necessary. Then, in a similar manner as before, with respect to the variable \( s_E' := s_E^{-1} \), we set

\[
P_E(s_E') := \prod_{i=1}^{3+k+l} (s_E' - \chi_{c+i}^{-1}),
\]

consider its Newton polygon \( \text{Newt}(P_E) \), then corresponding toric blow-up model \( B_E \to \mathbb{P}^1_{s_E} \times \text{Spec}(K[[t]]) \) with cosupport at \( t = 0, s_E = \infty \). Then, generalizing the stable reduction over \( K(s_E)[[t]] \), we extend ambient space \( \mathbb{P}^1_{s_E} (\mathcal{O}_{s_E} \oplus \mathcal{O}_{s_E}(2) \oplus \mathcal{O}_{s_E}(3)) \) of \( X_E \) to that of \( B \), so that its restriction to \( \mathbb{P}^1_{s_E} \) is \( \mathbb{P}(\mathcal{O}_{s_E} \oplus \mathcal{O}_{s_E}(2) \oplus \mathcal{O}_{s_E}(3)) \) which includes the \( t \)-direction stable
reduction of the generic fiber of $X_E$, and trivial $\mathbb{P}^1$-bundle over the rest of the components of $\mathcal{B}_{|t=0}$. Then, it is easy to see that the closure inside the ambient model over $K[[t]]$ gives reduction to the surface of type $\mathbb{E}_A \cdots A$ in [ABE20].

**Subcase.** ($\mathbb{E}_{k+l}$ to $D_{k-1}A_{l-1} \cdots A_{l,m-1}$) Similarly to the previous subcase, we next treat the case when $t$-direction stable reduction of the generic fiber of $X_E$ becomes nodal (i.e., $j = \infty$). This assumption means

$$
\Delta_{12} = (g'_6)^3 - 27 (g'_6)^2 = 0,
$$

hence we can write $g'_6 = 3P_2^2, g'_6 = P_2^3$. Since we normalized our $g'_i$ to give the $t$-direction stable reduction of the generic fiber, $P_2|_{t=0} \neq 0$ as a polynomial.

If the roots of $P_2|_{t=0}$ remain finite and distinct, then we only need to do toric modifications of the base model $\mathbb{P}^1_{s_R} \times \text{Spec} K[[t]]$ at cosupport $\infty \times 0$ (closed point). As it is completely similar to the just previous subcase, using Newton polygon of the polynomial of $s'_E$ with roots $\chi_i^{-1}$ converging to 0, we omit details.

If at least one of the roots of $P_2|_{t=0}$ diverges, then we do toric blowup at $\infty \times 0 \in \mathbb{P}^1_{s_R} \times \text{Spec} K[[t]]$, so that $\mathcal{B}_{|t=0}$ becomes union of $\mathbb{P}^1_s$ with one or two exceptional divisors at each of which the diverging real discriminant converges. Furthermore, if the roots of $P_2$ converge to same points $q$ in $\mathbb{P}^1_{s_R}$, we do weighted blowup of the base model surface at the point $q$, so that the roots converge to different points in the same component which we (still) denote as $\mathbb{P}^1_s$. After that, we contract all irreducible components (curves) of $t = 0$ which do not contain any real discriminant. Then, again similarly, we take ambient space whose restriction to $\mathbb{P}^1_s \times \{t = 0\}$ (resp. other components) is $\mathbb{P}_{t_1}(O_{t_1} \oplus O_{t_1}(2) \oplus O_{t_1}(3))$ (resp. trivial $\mathbb{P}^2$-bundle).

After all these procedures, we obtain the model of reduction type $\mathbb{D}_A \cdots A$.

**Case 4.** (From type II to type III) Now, we deal with the case when the corresponding morphism from $\text{Spec} K[[t]] \to \widetilde{M}_W^{\mathbb{E}_8}$, where the target space refers to the Satake–Baily–Borel compactification, maps generic point inside 1-cusp ($M_W^{\text{seg}}$ and $M_W^{\text{bn}}$ in the [OO21, §7] notation), and maps the closed point to 0-cusp $M_W^{\text{bn,seg}}$. We assume this below and call it ($\ast_{II,III}$).

**Subcase.** ($\mathbb{E}_{9,7}E_8$ to $E_9 \cdots A_{l-1} \cdots A_{l,m-1}$) First, we treat the case when the generic point of $\text{Spec} K[[t]]$ maps to $M_W^{\text{seg}}$. (Other case when the generic point of $\text{Spec} K[[t]]$ maps to $M_W^{\text{bn}}$ is treated in the Subcase after next.) We write the component of $E_9$-surface [ABE20], that is, rational elliptic surface with double locus a single smooth fiber, as $X_E \to \mathcal{B}_E \simeq \mathbb{P}^1$ as local notation. We suppose the double locus fibers over $\infty$.

In case the reduction $t = 0$ gives divergence of some real discriminants in the base $\mathcal{B}_E$ to $\infty$, then we again do the toric blowups of the model completely similarly as in previous steps via Newton polygon technique, so that the real discriminant points only converge finite in the strict transform of $\mathcal{B}_E$ and smooth points in $\mathcal{B}_{|t=0}$ in general. Then, again in the similar manner, we obtain model of polarization whose restriction to $\mathcal{B}_E$ is $O_{t_1} \oplus O_{t_1}(2) \oplus O_{t_1}(3)$ while trivial $O^{\mathbb{E}_8}$ otherwise, projectify it, and take closure of $X_E$ inside.

If such model is generically smooth over the strict transform of $\mathcal{B}_E$ (otherwise, proceed to the next subcase). Then, by the assumption ($\ast_{II,III}$), it follows that the fiber over $\infty$ becomes nodal at $t = 0$ (otherwise, it remains to be in type II locus, that is, 1-cusps of $\overline{M}_W^{\mathbb{E}_8}$). Hence, the reduction for $t = 0$ is the desired fibred surface of type $\mathbb{E}_A \cdots E$. 
Subcase. \((\mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8 \mathbb{P}_8) \to \mathbb{D}_{8-1} \mathbb{A}_{1-1} \cdots \mathbb{A}_{m-1})\) If the obtained model of \((X_E, R) \to B_E\) in the last step is not generically smooth over the strict transform of \(B_E\), then the corresponding elements of \(H^0(\mathcal{O}_{\mathbb{P}_1}(4))\) (resp. \(H^0(\mathcal{O}_{\mathbb{P}_2}(6))\)) which we still prefer to write \(g_\nu^\sigma, g_\nu^\tau\) are of the form \((3P_2^\sigma, P_3^\nu)\) with some \(P_2 \in H^0(\mathcal{O}_{\mathbb{P}_2}(2))\). If \(P_2\) vanishes at \(\infty\), that is, degree at most 1 as a polynomial, then it means that one of the roots of \(P_2\), which is also a real discriminant point, diverges (or converges) to \(\infty\). We do toric blowup of the model of \(B_E\) at this stage by the Newton polygon of the polynomial whose roots are diverging real discriminants, as in the previous steps. The process avoids the divergence of real discriminants \(\infty\) while procuring further rational components in the reduction of base \(B|_{t=0}\). If \(P_2\) is not square-free, we do the same process as Step 4 of Case 1. Then, we contract all irreducible components of \(t = 0\) which do not contain any real discriminants.

Then finally, similarly, we create the model of \(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(3)\) at \(t \neq 0\) as before, its projectivization, and take the closure of \(X_E\) inside, which is our desired model. In this manner, we obtain further degeneration to surface of type \(\mathbb{D}_{1-1} = \mathbb{A}_{1-1}\).

Subcase. \((\mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1 \mathbb{P}_1) \to \mathbb{D}_{1-1} \mathbb{A}_{1-1} \cdots \mathbb{A}_{m-1} \mathbb{D}_b\) with \(a + b + l = 16\) Now, we treat the case when the generic point of \(\text{Spec} K[[t]]\) maps to \(M_{\mathbb{P}_1}^{\text{reg}}\), while the closed point maps to \(M_{\mathbb{P}_1}^{\text{reg}} \cap M_{\mathbb{P}_1}^\text{int}\), that is, degenerations of \(\mathbb{D}_{16}\)-type surfaces to type III surfaces.

We lift the \(K((t))\)-rational point at \(M_{\mathbb{P}_1}^{\text{reg}}\) to \((g_8 = 3G_4^2, g_{12} = G_4^3)\) with \(G_4 \in H^0(\mathcal{O}(4))\) with coefficient \(K((t))\). By multiplying appropriate integer power of \(t\), we can first assume that \(G_4\) has all coefficients in \(K[[t]]\). We also set the solutions of \(G_4\) as \(\sigma_1, \sigma_2, \tau_1, \tau_2\), which we can and do assume to be in \(K((t))\) after finite base change of \(K[[t]]\) if necessary. We suppose \(\sigma_i|_{t=0} = 0, \tau_i|_{t=0} = \infty\).

Similarly to Step 1 of Case 1, we set

\[
(60) \quad f(0) := \min\{\text{val}_t(\sigma_1), \text{val}_t(\sigma_2)\},
\]

\[
(61) \quad f(\infty) := \min\{\text{val}_t(\tau_1^{-1}), \text{val}_t(\tau_2^{-1})\},
\]

and consider the blowup \(B_1 \to B_{\text{triv}}\) by

\[
(62) \quad (s, t f(0)) \cdot (s^{-1}, t f(\infty)).
\]

Then, the special fiber of \(B_1\) over \(t = 0\) is

\[
(63) \quad \mathbb{P}^1_{t f(0)} \cup \mathbb{P}^1_{s} \cup \mathbb{P}^1_{s^{-1} t f(\infty)},
\]

where the two ends are exceptional curves.

Then, as in Step 1 of Case 1, the first component contains the limit of \(\sigma_i|_{t \neq 0}\), and the last component contains the limit of \(\tau_i|_{t \neq 0}\) both different from the nodal points.

Then, similarly to the above,

we degenerate \(\mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(3)\) on the original base \(\mathbb{P}^1\) to the whole model, so that its reduction restricts to:

1. \(\mathbb{P}^1_{t f(0)}(\mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(3) \oplus \mathcal{O}_{\mathbb{P}_1})\) over \(\mathbb{P}^1_{t f(0)}\),
2. trivial \(\mathbb{P}^2\)-bundle over \(\mathbb{P}^1_s\) (i.e., \(\mathbb{P}^2 \times \mathbb{P}^1_s\), and
3. \(\mathbb{P}^1_{t f(\infty)}(\mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(3) \oplus \mathcal{O}_{\mathbb{P}_1})\) over \(\mathbb{P}^1_{t f(\infty)}\).

Then, our first step is to take closure of original \(X\) inside the projectivization of the above \(\mathbb{P}^2\)-bundle on the rational chain.
After this, we do the same procedures as Step 2, Step 3, and then Step 4 of Case 1. Then, we obtain the desired reduction to $DA\cdots AD$-type surface.

By here, we complete the case-by-case reproof of stable reduction type Theorem 2.6. Therefore, the completion of the proof of Theorem 2.5 also follows the above (re)proof of Theorem 2.6 (recall the beginning of our proof).

The identification of the normalization of $M_{W}^{ABE}$ with the toroidal compactification in [ABE20, §7] follows from the fact that the relative locations of the real discriminants in the broken base chain of $\mathbb{P}^1$s are encoded as $(\mathbb{G}_m \otimes \Lambda_i)$. This may also follows again from further analysis in addition to the above, but since this point overlaps more closely with the arguments in [ABE20], we do not pursue this here. See [ABE20, the proof of Prop. 7.45].

Instead, we do some more explicit description.

**Corollary 2.9.** (Of our reproof of Theorem 2.6) The boundary strata of $M_{W}^{ABE}$ which parametrizes degenerated surfaces of the following stable types

\[
\begin{aligned}
&\text{EAE,} \\
&\text{ED,} \\
&\text{E}_A \cdots \text{AD}_k (\text{with } k \geq 9),
\end{aligned}
\]

are not in the closure of two boundary prime divisors of type II.

**Proof.** The first two strata are both 17-dimension by the easy computation, while the type II boundary components are also both 17-dimension, hence the proof follows. For the case of last stratum, the proof follows from our stable reduction arguments (or from the observation below).

We observe that, in our situation at least, if a surface component which corresponds to the lattice of $\Lambda$-type degenerates to those of type $\Lambda_1, \ldots, \Lambda_m$, $\Lambda_1 \oplus \cdots \oplus \Lambda_m$ is a sublattice of $\Lambda$. This is partially explained in [ABE20] and also related to Proposition 3.12 to be explained.

**Remark 2.10.** Recall from [DHT17, §4.1] combined with [CD07, §3.3], the interesting observation that one aspect of the classical Shioda–Inose structure construction to $I1_{1,7}$-lattice polarized (higher Picard rank) K3 surface can be explained by an interesting Jacobian fibration which corresponds to the strata $M_{W}^{np}$. The correspondence is explained via a part of Dolgachev–Nikulin mirror symmetry [Dol96, especially 7.11], that is, the fiber of such Jacobian fibration plus the elliptic fiber of the element of $M_{W}$ provides type II degeneration from $M_{W}$ to $M_{W}^{np}$. This remark is not essentially new.

**2.2.4. Boundary strata of small codimensions**

We classify boundary divisors and boundary strata of codimension 2 of the compactification $M_{W}^{ABE}$. As prime divisors, there are at total of 54 of those as follows:

i. $E_{k_1}A_{k_2}E_{k_3}$ where $k_1 + k_2 + k_3 = 17, 0 \leq k_1 \leq 8, 0 \leq k_2 \leq 17, 0 \leq k_3 \leq 8$. At total, we have 45 boundary prime divisors of this type. The moduli is the product of Weyl group quotient of at total 17-dimensional algebraic tori (divided by left–right involution if $k_1 = k_3$).

ii. $E_{k}D_{17-k}$ where $0 \leq k \leq 8$. Nine of these are boundary prime divisors.
The classifications of 16-dimensional boundary strata are as follows:

i. \(E_{k_1} A_{k_2} A_{k_3} E_{16-k_1-k_2-k_3}\) type with each \(k_i \geq 0\).

ii. \(E_{k_1} A_{k_2} D_{16-k_1-k_2}\) with nonnegative index. By [ABE20, §7I], the normalizations \(\mathcal{M}_W \rightarrow \mathcal{M}_W^{\text{ABE}}\) are nontrivial at the nine irreducible components of those with \(k_1 + k_2 = 16, 0 \leq k_1 \leq 8\).

iii. \(D D\) type. Again, by [ABE20, §7I], the normalizations are nonisomorphic at the one component for \(D_0 D_{16}\).

Hence, the normalizations of \(\mathcal{M}_W^{\text{ABE}}\) are nonisomorphic at \(9 + 1 = 10\) irreducible components of 16-dimension (which is the biggest dimension), and the preimage becomes \(18 + 2 = 20\) components.

§3. Application to type II degeneration of K3 surfaces

3.1. Limit measure along type II degeneration

3.1.1. Limit points

While Section 2 focuses on the elliptic K3 surfaces, their degenerations, and moduli compactification, in Section 3, we apply it to study more general K3 surfaces’ degeneration of type II over \(\mathbb{C}\). The main point is that, as in [OO21], the elliptic K3 structure appears around boundary as special Lagrangian fibration after suitable hyper-Kähler rotation, as expected in the context of the mirror symmetry and shown in [OO21, §4]. If we follow the setup of [OO21, §6], we first observe the following.

**Lemma 3.1.** If we naturally send \(\mathcal{F}_{2d} \ni (X, L)\) into \(\mathcal{M}_{K3}\) by adding \(c_1(L)\) as additional period, type II cusps map to the strata \(\mathcal{M}_{K3}(d)\) (see [OO21, §6]) of the Satake compactification of adjoint type \(\mathcal{M}_{K3}^{\text{Sat,adj}}\).

We refine the statements in Proposition 3.11 which shows the limit existence in a yet another Satake compactification \(\mathcal{M}_{K3}^{\text{Sat}}\) among those nonadjoint types, which especially dominates the above compactification of adjoint type and dilates the zero-dimensional locus \(\mathcal{M}_{K3}(d)\) to 17-dimension.

**Proof.** As it is well known, for type II degeneration, with some fixed marking, \([\text{Re}\Omega_X], [\text{Im}\Omega_X]\) converge to isotropic plane, while obviously \([\omega_X]\) remains the same class. Comparing with §6.2 of [OO21], we obtain the proof. \(\square\)

Note that the locus \(\mathcal{M}_{K3}(d)\) is nothing but the only 0-cusp of the Satake–Baily–Borel compactification of \(\mathcal{M}_{K3}(a)\), which is identified with the moduli of Weierstrass elliptic K3 surfaces modulo the involution (see [OO21, §7]). This is the key point to convert general problem on type II degeneration into type III degeneration of elliptic K3 surfaces. In other words, roughly we divide the diverging isotropic plane into a line plus a line. Note that this conversion is not easy to explicitly describe in general, but we refer to related discussion in §3.4, which partially aims to see the conversion more explicitly.

3.1.2. Limit measure determination via Satake compactification

We now explicitly determine measured Gromov–Hausdorff limits [Fuk87a] of tropical K3 surfaces in the sense of [OO21, §4], so that we can justify the desired PL function invariant \(V\). That is, we study the collapse of two-dimensional spheres \(S^2\) with the McLean metrics.
to unit intervals, through the algebrogeometric compactifications [ABE20] and its study in Section 2.2 of the asymptotic behavior of singular fibers. This is an application of the above stable reduction theorem after [ABE20], providing one way of understanding of measured Gromov–Hausdorff limits’ classification (cf. [Osh] for another way).

We recall that Satake compactification of adjoint representation type coincides with certain generalization of Morgan–Shalen-type compactification [OO21, Th. 2.1]. This is the viewpoint we take in this section.

For our purpose, we introduce the geometric realization map in a non-archimedean manner, which we write as $\Phi(a)$, as follows. This is essentially found by [ABE20, §4] and Oshima [Osh] independently in somewhat different forms. The synchronization of the two works was rather surprising (at least to me), since their original aims were totally different, and also the tools are different: the latter was in more Hodge-theoretic context using a yet another Satake compactification as we define and briefly show below (see [Osh] for details).

No clear reason of the miraculous coincidence has been found yet, while our works mean to take a first step.

3.1.3. Via a yet another Satake compactification

As a preparation of precise statements, while more details are in [Osh], we consider the irreducible representation $\tau$ of $SO_0(3,19)$ whose highest root is only orthogonal to the leftmost one in the Dynkin diagram of [OO21, §6.1]. Then, as [Osh] provides more details, the corresponding Satake compactification (see [Sat60a], [Sat60b]) $M_{K3}^{Sat,\tau}$ has (real) 17-dimensional strata $M_{K3}(d)$ which is

$$O(\Lambda_{seg})/O^+(\Lambda_{seg})/\mathbb{R}_{>0},$$

divided by the involution induced by complex conjugation. Here, $\Lambda_{seg} := p^\perp/p \simeq U \oplus E_8(-1)^{\oplus 2}$ with isotropic plane $p \subset \Lambda_{K3} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 3}$, and

$$C^+(\Lambda_{seg}) := \{ x \in \Lambda_{seg} \otimes \mathbb{R} \mid x^2 > 0 \},$$

hence isomorphic to the 17-dimensional real open unit ball. Its fundamental domain is provided by the classical Vinberg’s method (see [ABE20], [Osh]). Here, we follow [ABE20, 4C] and denote as $P \simeq M_{K3}(d)^\tau$ which is a subdivided Coxeter chamber (modulo the natural involution). $P$ is of the form: $P := \{ x \in C^+(\Lambda_{seg}) \mid (x, \alpha_i) > 0 \}$ for in $\Lambda_{seg}$.

By [OO21, 2.1], we can replace the Satake compactification by certain Morgan–Shalen compactification, which is more useful in this paper.

**Definition 3.2.** (Geometric realization and measure density function) We consider the quotient of

$$\mathcal{M}_W^{MSBJ} \simeq \mathcal{M}_W^{Sat,adj},$$

where the right-hand side denotes the Satake compactification with respect to the adjoint representation of $SO_0(3,19)$ (the isomorphism is proved at [OO21, 2.1] as a general theory) by $O(\Lambda_{seg})/O^+(\Lambda_{seg})$, acting as the complex conjugate involution. Then, we obtain compactifications of $M_{K3}(a)$ in [OO21], respectively, which we denote as

$$M_{K3}(a)^{MSBJ} \simeq M_{K3}(a)^{Sat,adj}.$$
Their common boundaries are hence stratified as follows:

\( (68) \quad \mathcal{M}_{K3}(a) \sqcup \mathcal{M}_{K3}(d)^T \sqcup \{2 \text{ points } p_{\text{seg}} \text{ and } p_{\text{nn}} \}. \)

Note that this space \((68)\), away from the two points \(p_{\text{seg}}\) and \(p_{\text{nn}}\), can be also regarded as a subset of \(\partial \mathcal{M}_{K3}^\text{Sat,\tau} \). From the left-hand side interpretation of \((67)\), \(p_{\text{seg}}\) (resp. \(p_{\text{nn}}\)) corresponds to the prime divisor of toroidal compactifications over the 1-cusp \(M_{W}^\text{nn}\) (resp. \(M_{W}^\text{seg}\)) as [CM05].

Now, we define geometric realization map \(\tilde{\Phi}\) from the above space \((68)\) away from \(p_{\text{nn}}\) to

\[ \{(X,d,\nu) \mid (X,d) \text{ is a compact metric space with diameter one} \]
\[ \text{and } \nu \text{ is a Radon measure}\}/\sim, \]

as follows. Here, \(\sim\) denotes the natural equivalence relation defined as positive constant multiplication of \(\nu\), as follows.

i. \((S^2 \text{ case})\) For \(x \in \mathcal{M}_{K3}(a)\), we define \(\tilde{\Phi}(x)\) as the tropical K3 surface \(\Phi(x)\) as [OO21, §6] with its Monge-Ampere measure (equivalent to the volume form), as the (a priori) additional data.

ii. (Unit interval case) In the following cases, we take \(\Phi(x)\) as a unit interval with certain measure.

(a) (cf. [ABE20, §7A], [Osh]) Recall that any \(x \in \mathcal{M}_{K3}(d)^T\) is represented by some \(l \in P \simeq O(\Lambda_{\text{seg}}) \setminus C^+(\Lambda_{\text{seg}})/\mathbb{R}_{>0}\), which is neither \(p_{\text{seg}}\) nor \(p_{\text{nn}}\). Hence we write \(x\) as \(x(l)\). To such \(l\), [ABE20, §7A] associates a PL function \(V(l)\) from \([0,1]\) for some construction of polygon using it. We set \(\tilde{\Phi}(x(l)) := [0,1]\) with the density function \(V(l)\). While we do not reproduce the details of the functions \(V(l)\) due to [ABE20, §7A] and leave it to [ABE20, §7A] and [Osh], we describe its basic structures and information to some extent at the beginning of §3.1.4.

(b) (A special point \(p_{\text{seg}}\); cf. [Osh]) We set \(\tilde{\Phi}(p_{\text{seg}}) := ([0,1],d,\nu)\) with standard metric \(d\) and \(\nu \equiv 0\).

As [Osh] discusses in more detail, we expect \(\tilde{\Phi}\) is continuous in the sense of the measured Gromov–Hausdorff convergence as a refinement of [OO21, §6.2, Conj. 6.2]. Indeed, the above (i) reflects the collapsing of the Ricci-flat K3 surfaces to tropical K3 surfaces (metrized \(S^2s\)) around \(\mathcal{M}_{K3}(a)\) as proved in [OO21, Th. 6.9], and the above (ba) and (bb) reflect the expectation of the limit measures for the collapsing of the Ricci-flat K3 surfaces to segments.

As one of our main claims in this paper, we give its partial confirmation, depending on our Section 3.

**Theorem 3.3.** (cf. also [Osh] for another proof) The geometric realization map \(\tilde{\Phi}\) is continuous with respect to the measured Gromov–Hausdorff topology in the sense of [Fuk87a].

As we mentioned, [Osh] gives a very different proof of Theorem 3.3, with particular differences notably at the Steps 3 and 4. In particular, his proof uses Satake topology and asymptotic analysis of periods along particular 2-cycles, whereas our proof uses rather the formulation via the Morgan–Shalen-type compactification and Section 3 of this paper.
PL FUNCTION ASSOCIATED TO TYPE II DEGENERATING K3 SURFACES

Proof. First, we fix a notation and make a setup: we take a sequence of \((g_8,g_{12})\) with subindex \(i\) whose corresponding points in \(M_W\) converge to a point in \(M_W^{seg}\), the union of a 1-cusp and the 0-cusp, in the Satake–Baily–Borel compactification \(\overline{M_W}\). Recall that we show that it is isomorphic to the GIT quotient compactification of \(M_W\) with respect to the Weierstrass model description in [OOS21, Th. 7.9]. Taking \((c_1s^4,c_2s^6)\) as a GIT polystable representative of the limit point in \(M_W^{seg}\), by the Luna slice étale theorem at stacky level (cf. [Dre04], [Luna73]), for instance, we can and do assume that our sequence of \((g_8,g_{12})\) converges to it. For later use, for each \(i\), we consider the roots of \(g_8\) (resp. \(g_{12}, \Delta_{24}\)) and denote as \(\{\xi_j\}_{j=1,...,8}\) (resp. \(\{\eta_j\}_{j=1,...,12}\), \(\{\chi_j\}_{j=1,...,24}\)) in ascending order of the absolute values. The natural analogues of \(e(0)\) (38) and \(e(\infty)\) (39) in our stable reduction arguments, that is, sequence version, are

\[
\epsilon := \max\{|\xi_j|,|\eta_j| | 2 \leq j \leq 4, 1 \leq j' \leq 6\},
\]

(69)

\[
\epsilon' := \max\{||\xi_j|^{-1},|\eta_j|^{-1}| | 5 \leq j \leq 7, 7 \leq j' \leq 12\}.
\]

(70)

Step 1. First, this Step 1 focuses on the case when the sequence of \([(g_8,g_{12})]\) converges to a point in the 1-cusp, \(M_W^{seg}\setminus M_W^n\).

In this case, [OOS21, §7.3.2] shows that the corresponding sequence of McLean metrics converges to infinitely long open surface which is asymptotically cylindrical at two ends 0 and \(\infty\), as minimal noncollapsing pointed Gromov–Hausdorff limit.

In this case, for large enough \(i\), that is, with the McLean metric close enough to the above asymptotically cylindrical surface, [OOS21, §7.3.7, notably Lem. 7.26] implies the following: after rescale with fixed diameters, in particular with bounded above distance of \(s = 0\) and \(s = \infty\), the corresponding renormalized \(\rho(r)\) in [OOS21, §7.3.7, notably Lem. 7.26] uniformly converges to 0 (after making \(r\) bounded by rescale) so that even the full measure of the (rescaled) McLean metric also tends to 0 for \(i \to \infty\).

Hence we obtain desired convergence to the interval with 0 measure, as metric measure space, in the sense of, for example, [Fuk87a].

Step 2. This Step 2 provides the first step analysis of the maximally degenerate case when \(c_1 = 3c_2 = 3\), that is, when the sequence of \((g_8,g_{12})\) converges to \((3s^4,s^6)\). The arguments below are borrowed from [Osh], which we follow and leave for the proof. (Our later steps are different from [Osh], with more algebrogeometric or non-archimedean perspectives.) We thank Oshima for the permission to write also here. For each \(i\), we define a cutoff function on \(\mathbb{R}_{>0}\) as

\[
\varphi(r) := \begin{cases} 
-1 & r < \epsilon, \\
\frac{\log r}{\log \epsilon} & \epsilon \leq r \leq \epsilon'^{-1}, \\
1 & r > \epsilon'^{-1}.
\end{cases}
\]

Here, for each \(j\), suppose \(\lim_{i \to \infty} \varphi(|\chi_j|) = x_j\) (the appearances of two indices \(i,j\) are not typo, as \(j\) is fixed here while \(\chi_j\) depends on \(i\)), which is negative for \(j \leq k\) and nonnegative otherwise. In addition, we may assume that \(\frac{\log|D_i|}{\log \epsilon_i} \to d \in [0, +\infty]\), where \(D_i\) denotes the top coefficient of \(\Delta_{24}\). Then, [Osh] determines the limit measure on the interval by using the approximate description of the McLean metric [OOS21, §7.3.3, notably Lem. 7.16]. The limit measure can be described as (up to positive constants’ multiplication) \(V\)
on $[-1,1]$ by

$$V(w) = 12w + d - \sum_{j=1}^{k} \max\{w,x_j\} - \sum_{j=k+1}^{24} \max\{0,w-x_j\},$$

and as in [HSZ19], metric $d$ and measure $\nu$ on the interval $[-1,1]$ as

$$d = V(w)^{1/2}dw, \quad \nu = V(w)dw, \quad \text{if} \ V \neq 0, +\infty,$n

$$d = dw, \quad \nu = dw \quad \text{if} \ V \equiv 0 \ (\text{or} \ V \equiv +\infty).$$

**Lemma 3.4. ([Osh]; compare with [HSZ19])** For the given and fixed sequence of $(g_8,g_{12})$, consider the sequence of the underlying base $\mathbb{P}^1$ with the McLean metric of the Weierstrass elliptic $K3$ surfaces which we rescale to make their diameters $2$. Then, the sequence converges to the above $([-1,1],d,\nu)$ as the metric measure space, up to the multiple of $\nu$.

For convenience, we sketch the proof of [Osh] which depends on [OO21, §§7.3.1 and 7.3.3], but we leave the details to [Osh]. The key claim used in the proof is [OO21, Lem. 7.16], which is a good approximation of the McLean metric $\mu(g_8,g_{12})ds \otimes ds$ on the base $\mathbb{P}^1$ of Weierstrass elliptic $K3$ surfaces. We denote the metrize base $\mathbb{P}^1$ as $B_{g_8,g_{12}}$, but it refers to $\Phi(x)$ in the case of (i). The key term in the approximation of $\mu$ [OO21, Lem. 7.16] is $\log |\Delta_{24}(:=g_8^3-27g_{12}^2)|$, which diverges at the discriminant locus of $B_{g_8,g_{12}}$. Hence, we can approximate the McLean metric $\mu(g_8,g_{12})ds \otimes ds$ by some $\mathbb{C}(1)(:=\{z \in \mathbb{C} \mid |z| = 1\})$-invariant simpler metric $\mu'(g_8,g_{12})ds \otimes ds$ where $\mu'(g_8,g_{12})$ is defined in terms of $\log |\chi_j|s$ and $\log |s|$, away from $s = 0, \chi_j (j = 1, \ldots, 24), \infty$. Then, the limit measure of $(B_{g_8,g_{12}},\mu(g_8,g_{12})ds \otimes ds)$ with respect to the sequence $(g_8,g_{12}) \to (3s^4,s^6)$ is calculable by replacing $\mu$ by simpler $\mu'$, so that we obtain Lemma 3.4. We leave the detailed calculations and estimates to [Osh].

**Step 3.** We consider the normalized compact moduli $\overline{M}_W^{\text{ABE,} \nu}$ and its stacky refinement $\overline{M}_W^{\text{ABE,} \nu}$ (a proper Deligne–Mumford algebraic stack) which comes from the construction of $\overline{M}_W^{\text{ABE}}$ in §2.2.3, that is, by the log KSBA moduli interpretation after [ABE20].

Take an étale chart $\mathcal{U}$ of the stack $\overline{M}_W^{\text{ABE,} \nu}$ which contains the preimage of the $0$-cusp of $\overline{M}_W$. We denote the preimage of the open part $M_W$ as $\mathcal{U} \subset \mathcal{U}$. Denote the corresponding coarse moduli as $\mathcal{M} \subset \mathcal{U}$. Now, we apply the Morgan–Shalen compactification as [Odk18, Appendix] to $\mathcal{M} \subset \mathcal{U}$ and denote it simply as $\mathcal{M}^{\text{MSB}}$.

As preparation, now we define the following modified Newton polygon of the discriminant $\Delta_{24}$ for a sequence of $(g_8,g_{12})$ with respect to $i = 1,2,\ldots$ converging to $(3s^4,s^6)$. For $\Delta_{24}(s) = \sum_{j=1}^{24} d_js^j$, we set

$$\text{Newt}(\Delta_{24}) := \{(j,-\log |d_j|) \mid 0 \leq j \leq 24\} + \mathbb{R}_{\geq 0}(0,1)$$

as an analogue of (45) and modify it by using $\epsilon, \epsilon'$ of (69), a sequence analogue of $\epsilon(0), \epsilon(\infty)$ (during the proof of Claim 2.7), as follows: first, we regard the above Newt$(\Delta_{24})$ as a graph of PL convex function $\varphi^C_\Delta: [0,24] \to \mathbb{R} \cup \{\infty\}$, and modification is defined below. We set

---

3 This convention is for the compatibility with [Osh].
similarly as before

\[
i_\epsilon := \max \{ i \mid \varphi_\Delta(i) - \varphi_\Delta(i + 1) \geq \epsilon \},
\]

\[
i_\epsilon' := \min \{ i \mid \varphi_\Delta(i + 1) - \varphi_\Delta(i) \geq \epsilon' \}.
\]

Again as before, we modify \( \varphi_\Delta^c \) to \( \varphi_\Delta^c : [0, 24] \to \mathbb{R} \cup \{ \infty \} \) as

\[
\varphi_\Delta^c(i) := \begin{cases} 
\varphi_\Delta^c(i_\epsilon) - \epsilon(i_\epsilon - i) & \text{if } 0 \leq i \leq i_\epsilon,
\varphi_\Delta^c(i) & \text{if } i_\epsilon \leq i \leq i_\epsilon',
\varphi_\Delta^c(i_\epsilon') + \epsilon'(i - i_\epsilon') & \text{if } i_\epsilon' \leq i \leq 24.
\end{cases}
\]

We are actually only concerned about it modulo positive constant multiplication, but anyhow denote the graph of \( \varphi_\Delta^c \) as Newt'(\( \Delta_{24} \)). Note that, from the definition using the (archimedean) logarithm, the nondifferentiable points in the domain are not necessarily integers. For instance, along any holomorphic punctured family of \((g_8, g_{12})\) converging to \((3s^4, s^6)\), the obtained limit of the above Newt'(\( \Delta_{24} \)) modulo rescale (fixing the height) becomes our \( P_{\Delta,0} \) in (49), the epigraph of \( \varphi_\Delta \) in (73). We can and do assume that our sequence sits in a neighborhood \( U'' \) of \(((3,0);(1,0))\) in \( \mathbb{A}^{22} = \mathbb{A}^9 \times \mathbb{A}^{13} \) describing the coefficients of \( g_8 \)s and \( g_{12} \)s for each \( i \). We consider the rational map from \( U'' \) to some (arbitrarily fixed) toroidal compactification \( \overline{M_W}^{AMRT,\{\Sigma\}} \) and replace \( U'' \) by its blowup to make it a morphism. We denote the preimage of the boundary as \( D'' \subset U'' \), and set \( U''' := U'' \setminus D'' \).

Now, we apply the functorality of MSBJ construction [Odk18, Appendix §§A.2 and A.15] (more precisely, the analytic extension in [Od20b]), and we obtain a continuous map \( \overline{U'''^{MSBJ}}(U'') \to \overline{M_W}^{MSBJ}(\overline{M_W}^{AMRT,\{\Sigma\}}) \).

From Step 2, the limit of Newt'(\( \Delta_{24} \)) for \( i \to \infty \) decides the measured Gromov–Hausdorff limit of McLean metric sequence (3.4), which is metrically the interval. Thus, the case-by-case proof of Claim 2.7 during that of Theorem 2.6, combined with the above discussion, readily implies that:

**Claim 3.5.** The measured Gromov–Hausdorff limit of McLean metric sequence (3.4) is determined by the limit point inside the Morgan–Shalen-type compactification \( \overline{U'''^{MSBJ}}(U'') \) (if exists).

**Step 4.** If we consider the set of points of the boundary \( \partial \overline{U'''^{MSBJ}}(U'') \), whose (given integral) affine coordinates valued in \( \mathbb{Q} \), it is obviously dense. On the other hand, recall from Step 3 that there is a natural continuous map \( \overline{U^{MSBJ}}(U'') \to \overline{M_W}^{MSBJ}(\overline{M_W}^{AMRT,\{\Sigma\}}) \).

Hence, it is enough to show the following claim.

**Claim 3.6.** For any point \( p \in \partial \overline{U^{MSBJ}}(U'') \) with rational affine coordinates, if we describe its image in \( \overline{M_W}^{MSBJ}(\overline{M_W}^{AMRT,\{\Sigma\}}) \) as \( \overline{l} := \mathbb{R}l \) with \( (0 \neq) l = l(p) \in C^+(\Lambda_{\text{seg}}) \cap \Lambda_{\text{seg}} \otimes \mathbb{Q} \) (we also denote \( \overline{l} = \overline{l(p)} \)), the limit measure density function \( V([HSZ19]; \text{Step 2}) \) for some sequence in \( \overline{M_W} \) converging to \( p \) coincides with \( \overline{\Phi}(|\overline{l}|(p)) \).

To prove Claim 3.6, recall that [ABE20, Th. 1.2] shows that the normalization of the log KSBA compactification of the Weierstrass elliptic K3 surfaces with their (weighted) rational curves cycle type boundaries is the toroidal compactification [AMRT] with respect to the
rational curves cone. As its first step, they construct, for given \(0 \neq l \in C^+(\Lambda_{\text{seg}}) \cap \Lambda_{\text{seg}} \otimes \mathbb{Q}\), a certain Kulikov model \(X_{LR}(l)\) (and its flop \(X'_{LR}(l)\) after a base change). For \(l\), we take such models as the one in Claim 3.6. Furthermore,

one can assume that the image of \(t\) in \(\Delta^*\) converges to \(p\) for \(t \to 0\). Indeed, we can take \(X_{LR}(l)\) to be the pullback of the Kulikov (semistable) model family, constructed in [ABE20], to a generic analytic curve transversally intersecting the open strata of the prime divisor of \(U''\) corresponding to \(p\) (if such divisor does not exist, we simply replace \(U''\) by the blowup satisfying it). Then, [ABE20] showed that its monodromy invariant (cf., e.g., [FriSca86]) is nothing but \(l\) modulo \(O(\Lambda_{\text{seg}})\) in Corollary 7.33 of [ABE20]. It is done using the crucial diffeomorphism from degenerating elliptic K3 surface to a corresponding Symington-type Lagrangian fibration by bare hand [EF19] and then calculating the intersection numbers on the Lagrangian fibration side. Recall from [OO21, Th. 2.8 and Cor. 4.25] that the limit inside the Morgan–Shalen–Boucksom–Jonsson compactification of toroidal compactifications \(\overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}}, \{\Sigma\})\), which is independent of the choice of the admissible decompositions \(\Sigma\), is equivalent to the information of the monodromy on \(U\) of signature \(2,18\).

For each \(X_{LR}(l)\) as above, one can directly see that the limit measure density function by our previous steps combined with the case-by-case explicit proof of Claim 2.7, and coincides with \(\tilde{\Phi}(\bar{l})\) which is determined by the monodromy. Hence, it is determined by the limit inside \(\overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}}, \{\Sigma\})\) by [OO21, Th. 2.8 and Cor. 4.25] and Claim 3.6 for general sequence, and the desired coincidence (Theorem 3.3) finally follows.

3.1.4. Explicit description and examples

Recall that, in particular, \(\tilde{\Phi}(l)\) in case (ba) of Definition 3.2 is as follows (as [ABE20, §7A], [Osh]), which describes all the details from which we borrow. The fundamental polygon \(P\) is divided into \(9 = 3^1 + 1\) maximal chambers, say \(\{P_a'\}_a\), and the points of \([0,1]\) where \((\tilde{\Phi}(l))(0)\) is nondifferentiable can be written as

\[
0 = \frac{q-2}{q_{22}} = \frac{q-1}{q_{22}} = \frac{q_0}{q_{22}} \leq \cdots \leq \frac{q_{19}}{q_{22}} = \frac{q_{20} = q_{21} = q_{22}}{= 1}.
\]

The definitions also imply

\[
\frac{q_1}{q_{22}} = \max \left\{ \left( l, -\frac{1}{3} \beta_L \right), 0 \right\},
\]

with \(\beta_L \in \Lambda_{\text{seg}}\) (see [ABE20, §4C]), and every \(q_j\) is linear at each \(P_a'\) with respect to the description (65).

The values and slopes of the function satisfy

\[
(\tilde{\Phi}(l))(0) = \max \{ (l, \beta_L), 0 \},
\]

\[
\frac{d\tilde{\Phi}(l)(x)}{dx} = 9 - i \quad \text{for any } x \in \left( \frac{q_i}{q_{22}}, \frac{q_{i+1}}{q_{22}} \right),
\]

In particular, \(\tilde{\Phi}(l)\) is convex. Indeed:

- if \((l, \beta_L) \leq 0\), for generic \(l\) under such assumption, \(\tilde{\Phi}(l)(0) = 0\) and the slope of \(\tilde{\Phi}(l)\) starts with \(9\) and decreases by \(1\) at each wall crossing through \(q_j\).
• if \((l, \beta_L) \geq 0\), then for generic \(l\) under such assumption, the slope of \(\tilde{\Phi}(l)\) starts with \(8\) and decreases by \(1\) at each wall crossing through \(q_j\).

In the case \(\tilde{\Phi}(l)(0) = \tilde{\Phi}(l)(1) = 0\) (e.g., §3.3), then note that the barycenter of \(q_i\) is the middle point \(\frac{1}{2}\). The behavior of the function \((\tilde{\Phi}(l))\) around the opposite end \(1\) (denoted by \(R\) in [ABE20]) is completely similar.

**Remark 3.7.** (Relation with [CM05, §5]) For one parameter type III degenerations from \(M_W\) to the locus inside the closure of \(M_W^{\mathrm{un}}\), we expect that the corresponding limit point in \(M_W\) can be explained by the collision of \(18\) blow-up centers \(p_i\) for the stable type II degeneration of those elliptic K3 surfaces introduced in [CM05, §5]. For the combinatorial type of such type III degenerations, recall Corollary 2.9.

**Example 3.8.** (Via Davenport–Stothers triple) Here, we see simple examples of degenerating Weierstrass elliptic K3 surfaces and apply the above to obtain the limit measures of the family of McLean metrized spheres.

In the following two examples, let us denote
\[
g_4(s) := 3(s^4 + 2s),
\]
\[
g_6(s) := s^6 + 3s^2 + \frac{3}{2},
\]
so that
\[
g_4^2 - 27g_6^2 = -27\left(s^3 + \frac{9}{4}\right).
\]
Up to affine transformation, this is known to be the only pair of degree 4, degree 6 polynomials with the degree of \(g_4^2 - 27g_6^2\) is 3. It is an easy example of Davenport–Stothers triple (cf., e.g., [Dav65], [Shi05], [Sto81], [Zan95]).

Our first example is as follows:
\[
g_8(s) := g_4\left(\frac{s}{t}\right)g_4\left(\frac{1}{ts}\right)s^4,
\]
\[
g_{12}(s) := g_6\left(\frac{s}{t}\right)g_6\left(\frac{1}{ts}\right)s^6,
\]
for \(t \to 0\). Then, we see that the density function \(V\) of the limit measure of the tropical K3 surfaces is as follows (modulo rescale):
\[
V(a) = \begin{cases} \frac{a}{1 - a} & 0 \leq a \leq \frac{1}{2}, \\ \frac{1}{2} - a & \frac{1}{2} \leq a \leq 1, \end{cases}
\]
which is directly checkable after our arguments in §2.2.3 and [ABE20, §7A].

**Example 3.9.** (Via Davenport–Stothers triple again) We use the same \(g_4, g_6\) as Example 3.8 but different \(g_8, g_{12}\). Note
\[
\left(s + \frac{1}{s}\right)^{-1} = \frac{s}{s^2 + 1},
\]
\((s + \frac{1}{s})^{-1}\) is near 0 if and only if \(s\) is near 0 or \(\infty\). Thus, \((s + \frac{1}{s})^{-1}\) is near \(\infty\) if and only if \(s\) is near \(\sqrt{-1}\). In this example, we define \(g_8, g_{12}\) as follows:

\[g_8(s) := g_4 \left( \frac{s}{t(s^2 + 1)} \right) \cdot (s^2 + 1)^4,\]
\[g_{12}(s) := g_6 \left( \frac{s}{t(s^2 + 1)} \right) \cdot (s^2 + 1)^6.\]

Then,

\[\Delta_{24}(s) = g_8^3 - 27g_{12}^2 = 0 \in \mathcal{O}_{P^1(24)}|_s\]

if and only if

\[\frac{s}{t(s^2 + 1)} = \chi_i(i = 1, 2, 3)\]

or

\[\frac{s}{s^2 + 1} = \infty\]

with multiplicity 18 if and only if

(84)

\[s + \frac{1}{s} = (t^\chi_i)^{-1}(i = 1, 2, 3)\]

or

(85)

\[s + \frac{1}{s} = 0,\]

where the latter with multiplicity 18. The former (84) happens if and only if

\[s = \frac{1 \pm \sqrt{1 - 4t^2\chi_i^2}}{2},\]

and the latter happens when either \(s = \sqrt{-1}\) with the multiplicity 9 or \(s = -\sqrt{-1}\) with the multiplicity 9 again. Therefore, if we \(t \to 0\), we get \([0, 1]\) with the corresponding \(V\) (modulo rescale) as the same again:

\[V(a) = \begin{cases} 
  a & 0 \leq a \leq \frac{1}{2}; \\
  1 - a & \frac{1}{2} \leq a \leq 1.
\end{cases}\]

In §3.3, we observe that the above two cases are close to the direction of the collapsing of [HSVZ18].

**Example 3.10. (Simplest D-type)** On the other hand, as another simple, our instance of our discussion in the proof of Theorem 2.6, we obtain a different type of \(V\) with \(V(0) = V(1) \neq 0\). Set

\[g_8(s) = 3((s - ta_1)(s - ta_2)(ts - a_3)(ts - a_4))^2,\]
\[g_{12}(s) = ((s - ta_1)(s - ta_2)(ts - a_3)(ts - a_4))^3,\]

for \(a_1 \neq a_2, a_3 \neq a_4\), all lie in \(K\). Then, the Newton polygon of \(\Delta_{24}\) has only two slopes, so that the proof (Cases 1 and 2) of Theorem 2.6 shows that the corresponding \(V\) for \(t \to 0\) is a constant function.
Indeed, this is the simplest prototypical example of D-type degeneration of elliptic K3 surfaces.

From Definition 3.2 of our $\Phi$, and compare with [ABE20, §7A] or [Osh], Theorem 2.6 ensures that $V$ can have much more varieties in general.

### 3.2. Limits along type II degeneration and associated lattices

As claimed in our introduction, we are now ready to give general considerations on limits along $\mathcal{F}_{2d}$ to make sense of the $V$ function for type II degenerations. Suppose that we are in the repeated setup as $(\mathcal{X}, \mathcal{L}) \to \Delta$ in $\mathcal{F}_{2d}$ is a type II polarized degeneration of K3 surfaces, dominated by a Kulikov model $\tilde{\mathcal{X}}$ and the pull back $\tilde{\mathcal{L}}$ of $\mathcal{L}$ to $\tilde{\mathcal{X}}$, and a stable type II degeneration $\mathcal{X}_0 = V_0 \cup V_1$. Then, refining Lemma 3.1, the following holds.

**Proposition 3.11.** For the given $\pi: (\mathcal{X}, \mathcal{L}) \to \Delta$ as above, the naturally associated continuous map $\varphi^\alpha$ from $\Delta \setminus 0 \to \mathcal{M}_{K3}$ continuously extends to a map $\varphi$ from $\Delta$ with $\varphi(0) = c(\mathcal{X}, \mathcal{L})$ in $\mathcal{M}_{K3}(d)_{\tau} \subset \mathcal{M}_{K3}^{Sat, \tau}$. In other words, the limit point inside $\mathcal{M}_{K3}^{Sat, \tau}$ for $t \to 0$ is well defined. In particular, there is the well-defined function $V = V_\pi = V(\mathcal{X}, \mathcal{L}) := \Phi(c(\mathcal{X}, \mathcal{L}))$ on the segment for this $(\mathcal{X}, \mathcal{L})$ as we noted in the beginning of the paper.

**Proof.** The proof is easy as Lemma 3.1, as through a marking $\alpha$ of $H^2(\mathcal{X}_I, Z)$, $\varphi^\alpha(t)$ clearly converges to the image of the Kähler class $\alpha(c_1(\mathcal{L}|_{\mathcal{X}_I}))$ for $t \to 0$. $\square$

We remark that in the collaboration with Oshima, the above limit is expected to describe the limit measure and more generally $\Phi$ to be continuous on whole $\mathcal{M}_{K3} \cup \mathcal{M}_{K3}(d)_{\tau} (\subset \mathcal{M}_{K3}^{Sat, \tau})$ with respect to the measured Gromov–Hausdorff topology, so that the above $V_\pi$ determines the limit measure of the hyper-Kähler metrics on general fibers. [Osh] provides related discussions.

Furthermore, we take a marking $H^2(\mathcal{X}_I, Z) \simeq \Lambda_{K3}$, so that the corresponding isotropic plane is

$$Ze'' \oplus Ze' ,$$

and we denote the image of $c_1(\mathcal{L}|_{\mathcal{X}_I})$ as $v_{2d}$ of norm $2d$. Recall that the canonical isomorphism

$$\langle e'', e' \rangle / \langle e'', e' \rangle \simeq \Lambda_{seg} = HI_{1,17} \cong U \oplus E_8^{\oplus 2} .$$

We write $\langle e'', e' \rangle =: p$. Then, $v_{2d}^\perp \subset p^\perp / p$ is studied classically in, for example, [Fri84], which we denote as $\Lambda_{per}(c) = \Lambda_{per}(c(\mathcal{X}, \mathcal{L}))$.

As a hyper-Kähler rotated side, we take a type III degeneration $\mathcal{X}^{\vee} \to \Delta$ of Weierstrass elliptic K3 surfaces which we suppose to be Kulikov degeneration, that is, $(\mathcal{X}^{\vee}, \mathcal{X}^{\vee}_0)$ is log smooth and is minimal. We put a marking on the smooth fibers, so that the elliptic fiber class is $e''$ and the zero-section class is $f''$. Recall that from [ABE20, §7], an irreducible decomposition of $\mathcal{X}^{\vee}_0$ which we write as $\cup_i V_i^{\vee}$ satisfies each $V_i$ (or its pair) is one of the following forms:

- $XI \cdots IX$,
- $Y_2Y_aI \cdots IX$,
- $Y_2Y_aI \cdots IY_2Y_a$.

It is easy to confirm that after appropriate flops, we can and do assume that the nontoric components (i.e., those with positive charges) all remain at the stable model of [ABE20].
Then, such remaining rational surfaces $V_i$ with normal crossing boundary $\cup_j D_{i,j}$ are encoded as slightly generalized root lattice of type either $DA \cdots AD, DA \cdots AE, EA \cdots AE$ with possibly indices 0s. This is encoded in [ABE20] as $P_{LR}(l)$ (resp. PL function $V$). We denote such lattice as $\Lambda_{ABE}(\mathcal{X}^\vee)$. Note that its rank is generally 0 and at most 17. On the other hand, as a hyper-Kähler rotation of $(\mathcal{X}^\vee, \omega_i)$ with $[\omega_i^\vee] = m_t e'' + f''$ with $m_t \to \infty$, we set $\{(\mathcal{X}_t, \omega_t)\}_t$ of type II for $t \to 0$ (as in [OO21, §4]). We anyhow denote the limit inside the Satake compactification $M_{K3}^{\text{Sat}, \tau}$ formally as $c(\mathcal{X}, \mathcal{L})$. Then, the following holds.

**Proposition 3.12.** In the above setup, the two associated negative definite lattices have canonical inclusion which respects the bilinear forms:

$$\Lambda_{ABE}(\mathcal{X}^\vee) \subset \Lambda_{\text{per}}(c(\mathcal{X}, \mathcal{L})).$$

**Proof.** Recall that the $\Lambda_{ABE}(\mathcal{X}^\vee)$ [ABE20, §§7G and 7H] is the direct sum of the slightly generalized ADE lattices $(\sum_j \mathbb{Z}[D_{i,j}])^\perp \subset H^2(V_i, \mathbb{Z})$. We use the Clemens contraction map $\mathcal{X}_1^\vee \to \mathcal{X}_0^\vee$, and the marking of $\mathcal{X}_1^\vee$, so that we can regard $H^2(\mathcal{X}_0^\vee, \mathbb{Z})$ canonically\(^4\) as a sublattice of $\Lambda_{K3}$.

Any $(\sum_j \mathbb{Z}[D_{i,j}])^\perp \subset H^2(V_i, \mathbb{Z})$ lies in $(1,1)$-part. On the other hand, from the construction of the hyper-Kähler rotation $\mathcal{X}^\vee$, one of its periods (the real part of the cohomology of the holomorphic volume form) converges to $v_{2d}$ as $(2,0)$-part. Hence, they are orthogonal. This completes the proof. \(\square\)

**Example 3.13.** If $2d = 4$, that is, degenerations of quartics, there are certainly examples where the above two lattices $\Lambda_{ABE}(\mathcal{X}^\vee)$ and $\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L})$ do not coincide. For instance, if $v_{2d} = 2e'' + f''$, then

$$\Lambda_{ABE}(\mathcal{X}^\vee) \simeq E_8(-1)^\oplus 2,$$

while

$$\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L}) \simeq \langle -4 \rangle \oplus E_8(-1)^\oplus 2.$$

Furthermore, there is another example with $2d = 4$ such that

$$\Lambda_{ABE}(\mathcal{X}^\vee) \simeq D_8(-1)^\oplus 2,$$

while

$$\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L}) \simeq \langle -4 \rangle \oplus D_8(-1)^\oplus 2.$$

**Remark 3.14.** Similar even negative definite lattices appear also in a slightly different context of Dolgachev–Nikulin mirror symmetry for lattice polarized K3 surfaces [Dol96]. Recall that the Dolgachev–Nikulin mirror (see [DHT17, 4.1], [Dol96, 7.11]) of $\mathcal{F}_{2d}$ says, to each type II degeneration in $\mathcal{F}_{2d}$, there is an associated isotropic element $e(\mathcal{X}, \mathcal{L})$ in $\Lambda_{2d}$ modulo $\tilde{O}(\Lambda_{2d})$.

From the arguments in [OO21, 4.14, 4.18, and 6.10], in an open neighborhood of 0-cusp, $e(\mathcal{X}, \mathcal{L})$ induces elliptic fibrations. Then, we expect that the direct sum of ADE lattices, which represent the Kodaira type of reducible degenerations of fibers, coincides

\[^4\] Modulo the monodromy, but the classes in our actual concern are all monodromy invariant, and further, if one fixes a continuous path connecting 0 and 1 in $\Delta$, then it becomes canonical.
with $\Lambda_{\text{ABE}}(X^\vee)$. Indeed, in every $2d \leq 4$ case, they coincide by the calculation of [Dol96, §7].

We conclude the section by making an easy but important remark.

**Proposition 3.15. (Denseness of algebraic limits)** Note that for each $d \geq 1$, we can consider $\overline{\mathcal{F}_{2d}} \to \overline{\mathcal{M}_{K3}^{\text{Sat},\tau}}$ (see Lemma 3.1, also §3.2). If we consider the union of such limits:

$$\bigcup_{d \in \mathbb{Z} > 0} (\partial \mathcal{F}_{2d} \cap \mathcal{M}_{K3}(d)^\tau),$$

then this countable set is dense inside the whole 17-dimensional strata $\mathcal{M}_{K3}(d)^\tau$.

**Proof.** This easily follows, since $\mathcal{M}_{K3}(d)^\tau$ is the quotient of

$$\{\lambda \in \Lambda_{\text{seg}} \otimes \mathbb{R} \mid \lambda^2 > 0\},$$

while $\Lambda_{\text{seg}}$ is an even integral lattice.

This implies the following straightforwardly.

**Corollary 3.16. (Possible PL invariants for type II degenerations)** Possible PL invariants for type II degenerations of polarized K3 surfaces run over a dense subset of which appears in [ABE20, §7A] and [Osh].

This result in particular gives negative answer to the first question of [HSZ19, §2.6] on the behavior of $V$.

### 3.3. [HSVZ18] glued metric and type II limits of algebraic K3 surfaces

The recent work of Hein et al. [HSVZ18] gives construction of compact K3 surfaces at the level of hyper-Kähler structures, by glueings of Tian–Yau metrics and Taub-NUT-type metrics, which maps and collapses to an interval.

In this section, we reveal how [HSVZ18] fits into our picture, therefore giving more structures. As a result, [HSVZ18] roughly corresponds to the following two aspects simultaneously:

**Aspect 1.** the special stable type $\mathfrak{E} \mathfrak{A} \mathfrak{E}$ in [ABE20] (cf. also our §§2.2 and 3.1.2),

**Aspect 2.** also the pushforward of two Lagrangian fibrations on the limiting K3 surfaces.

#### 3.3.1. Review of [HSVZ18] construction

First, we recall their construction here (while we leave full details to [HSVZ18]). They construct compact hyper-Kähler manifolds (hence homeomorphic to the K3 surfaces) by glueing, which maps to an interval, from the following set of data:

- two arbitrary Del Pezzo surfaces $X_1$ with the degrees $d_1 := (-K_{X_1})^2$ and $d_2 := (-K_{X_2})^2$,
- choice of their (isomorphic) smooth anticanonical divisors $D_i \subset X_i (i = 1, 2)$ with an isomorphism $D_1 \simeq D_2$,
- Tian–Yau metrics [TY90] on $X_i \setminus D_i$ (note $\chi(X_i \setminus D_i) = 12 - d_i$) which is cohomologically zero in $H^2(X_i \setminus D_i, \mathbb{R})$,
- a transition region $N$ whose general fibers over the interval are $(T^2 \times \mathbb{R})$ away from $(d_1 + d_2)$-points in the base,
• a hyper-Kähler metric on \( N \) constructed by the Gibbons–Hawking ansatz,
• (parameter specifying the attaching parameter for the \( S^1 \)-rotation), and
• the \textit{collapsing parameter} \( \beta \in (0, 1) \).

As for the Tian–Yau metric of the above situation, they analyzed its asymptotic at the boundary \( D_i \) to identify with ALH (or ALG\(^*\) suggested by [CC21]) with exactly quadratic curvature decay and the noninteger volume growth \( \sim r^{\frac{2}{3}} \) (cf. also [Hein12, Th. 1.5(iii), \( I_b \)-case]), where \( r \) denotes the distance from some arbitrary base point.

From the above data, [Hein12, Th. 1.5(iii), \( I_b \)-case] glues the Tian–Yau hyper-Kähler metrics on \( (X_i \setminus D_i) \) and some Gibbons–Hawking metrics with several (multi-)Taub-NUT asymptotics on \( N \), which collapses to the interval \([0, 1]\) when \( \beta \to 0 \) (also see earlier expectation by Kobayashi [Kob90b, p. 223]), which we here write \( S_\beta \) with its hyper-Kähler metric \( g_\beta \). Furthermore, they provide a continuous map \( F_\beta : S_\beta \to [0, 1] \) which satisfies:

i. The fibers over ends \( F_\beta^{-1}(0) \) and \( F_\beta^{-1}(1) \) are closure of open locus in the Tian–Yau spaces \( X_i \setminus D_i \).
ii. For \( \beta \to 0 \), \( (S_\beta, g_\beta) \) converges in the Gromov–Hausdorff sense to the unit interval with natural affine structure (induced from the behavior of harmonic functions on \( S_\beta \)).\(^5\)
iii. The limit measure on the interval is written as \( \sqrt{V(x)} dx \) with a convex PL function on \([0, 1]\) with \( V(0) = V(1) = 0 \), where \( dx \) stands for the affine structure above.

Remark 3.17. With respect to this affine structure \( dx \), assuming that the Gromov–Hausdorff limit of rescaled metrics with fixed diameters is identified with the dual graph, the natural affine structure with respect to the latter perspective is \( V(x) dx \) (see [BJ17]).

Recall that [TY90] first constructed the hermitian metric \( h \) on the normal bundle for \( D_i \subset X_i \) whose curvature form is Ricci-flat, then solved the complex Monge–Amperé equation with the reference metric of Calabi-ansatz type via \( h \).

As [CFG92], [Fuk87b], and [Fuk89] show, the fibers are infranilmanifolds, indeed simply Heisenberg nilmanifolds (cf. also [HSZ19, §2.2]). In particular, they also confirmed that their hyper-Kähler manifolds are parametrized by 57-dimensional data (plus rescaling data), that is, at least containing some open subset of \( \mathcal{M}_{K3} \). Note that \( V(0) = V(1) = 0 \) condition of the above (iii) infers, as [Osh] proves, it only gives a neighborhood of the strata corresponding to \( E(A)E \)-type sub cone. Hence the direction which involves D-type is missing.

Then, after [HSVZ18], more recent work of Honda et al. [HSZ19] proved similar PL structure for all possible \textit{limit measure} on the Gromov–Hausdorff limit when it is one-dimensional (interval). In §2.6 of [HSZ19], they raise some questions regarding the function \( V \) to which we answer:

• First question in [HSZ19] asks if \( V(p) = 0 \) at the boundary point \( p \) in the case when \( V \) is not constant. This is far from true, from the presence of D-type region combined with Theorem 3.3.
• The second question in [HSZ19], in the situation of [HSVZ18], asks if \( V \) is singular at the \( d_1 + d_2 \) points in the interval. The answer is yes from our conclusion.

\(^5\) We observe that in general this affine structure is not the same as the one induced from non-archimedean structure as used in [BJ17].
• Their third question is about the ratio of slopes. As our analysis so far, the slopes can be normalized to $0, \pm 1, \ldots, \pm 9$, and the ratios are rational as expected.

3.3.2. Our interpretation of [HSVZ18]

Now, we discuss Aspects 1 and 2 of the beginning of §3.3.

3.3.3. For Aspect 1—Landau–Ginzburg model

Recall that [CJL21, Th. 6.4] relates the above Tian–Yau metrics and those of $\frac{4}{3}$-order volume growth gravitational instanton on rational elliptic surfaces [Hein12] by hyper-Kähler rotations (cf. [CJL21, 6.9], [HSVZ18, 2.5]). We expect that our viewpoint may help to clarify relation with the Landau–Ginzburg models [EHX97], as we partially give observation here.

As first instance, we observe the following. For type II degeneration with one component isomorphic to $\mathbb{P}^2$, consider its underlying $\mathbb{R}^2$ below our degenerate elliptic K3 surface of $X_3/E_0$-type (see [ABE20, 7.4] and §§2.2 and 2.2.2). Then, it is the limit of the affine structures of Gross–Siebert program type at [CPS, Exam. 2.4] (see also [LLL20, §3.1]), which has three $I_1$-type singularities of affine structure. Indeed, if three of them collide via moving worms [KS06], it becomes the abovementioned $X_3/E_0$-type singularity of affine structures. [LLL20] also identified it with the affine structure coming from special Lagrangian fibration of a complement of cubic curve in $\mathbb{P}^2$ constructed in [CJL21]. See the details at [CJL21], [CPS], and [LLL20].

Furthermore, [ABE20] with the arguments in this paper provides further evidence to a variant of Doran–Harder–Thompson expectation (see [DHT17], [Dol96]) for K3 surfaces, where mirror is replaced or specialized to be hyper-Kähler rotation, with slight refinement by putting $A$-type surfaces between. In particular, this picture applies for general type II degenerations, with possibly many irreducible components, hence not necessarily Tjurin degeneration in the sense of [DHT17].

Indeed, recall that in [ABE20, §7] moduli compactification and our reconstruction in (2.5), the main role was played by the behavior of singular fibers. Such fact together with our interpretation of $M_{IW}$ as limits of hyper-Kähler rotated K3 surfaces may naturally invoke the homological mirror symmetry type phenomenon after [Sei01], that the Lefschetz vanishing cycles around the degenerations of the elliptic curves reflect the B-model pictures of the degeneration of K3 surfaces. We hope to have further understanding of it in our context in more systematic way in future.

3.3.4. For Aspect 2—relation with two Lagrangian fibrations

We take a sequence of $(g_8, g_{12}) \in H^0(\mathbb{P}_1^4, \mathcal{O}(8)) \times H^0(\mathbb{P}_1^4, \mathcal{O}(12))$ converging to $(3s^4, s^6)$ and the associated Weierstrass K3 surface

$$\pi'' : X := [y^2 z = 4x^3 - g_8(s)xz^2 + g_{12}(s)z^3]$$

$$\subset \mathbb{P}_1^4(O_{\mathbb{P}_1^4}(4) \oplus O_{\mathbb{P}_1^4}(6) \oplus O_{\mathbb{P}_1^4})$$

$$\to B \simeq \mathbb{P}_1^3$$

converging to $\lambda \in \mathcal{M}_{K3}(d)$ in the Satake compactification $\overline{\mathcal{M}}_{K3}^{Sat, \tau}$. For $i \gg 0$, we have two Lagrangian fibrations:

i. As we showed in [OO21, §4], for fixed $m \gg 0$, we obtain a hyper-Kähler rotation $X_m^\vee$ of $X$ which is canonically diffeomorphic to $X$ (so that we can keep the corresponding
marking to original $\varphi$ for $X$) whose holomorphic form $\Omega_m^\vee$ has cohomology class as

$$(86) \quad [\Omega_m^\vee] = |\log \epsilon|^{-1} \text{Re}\Omega + \sqrt{-1/2m} c(f'' + me'').$$

Here, $\epsilon$ is as (69) and $c_i$ is uniquely determined positive constant which automatically converges to 1 for $i \to \infty$. By the same argument as in [OO21, §4], we obtain a fibration structure $\pi': X_m^\vee \to \mathbb{P}_s = B_m^\vee$ defined by the pencil $|e'|$ with the fiber class $e'$. Note that this is a special Lagrangian fibration with respect to the original complex structure, as in [OO21, §4].

i. Original $\pi''$: $X \to B \simeq \mathbb{P}^1_s$, the Weierstrass elliptic fibration structure. The fiber class is $e''$ and is determined as $|e''|$. This is Lagrangian fibration with respect to the holomorphic volume form $\Omega$.

As [HSVZ18] confirms, its glued K3 surfaces form a subset of $\mathcal{M}_{K3}$ which includes an open subset $U_{HSVZ}$ whose closure is in the EAE region of $\mathcal{M}_{K3}(d)^{Sat,\tau}$. We can and do assume that $U_{HSVZ}$ is close to the boundary enough, so that any point has the special Lagrangian fibration $\pi'$ in (i).

**Conjecture 3.18.** For any glued fibration of K3 surface to the segment as in [HSVZ18] so that $p = (F_\beta: X \to [0,1]) \in U_{HSVZ}$, $F_\beta$ factors through both $\pi'$ and $\pi''$. There is a 1-homology class, which we denote $e' \cap e''$, such that

- $e' \cap e''$ is primitive in both $H_1(e',\mathbb{Z})$ and $H_1(e'',\mathbb{Z})$.
- $e' \cap e''$ is monodromy invariant with respect to both $\pi'$ and $\pi''$.

The above conjecture would clarify an interpretation of the nilmanifold (Heisenberg manifold) fiber of [HSVZ18] as $S^1$-bundle over an elliptic curve.

**Remark 3.19.** It would be interesting to see if the conjectural map $B_m^\vee$ coincides with a moment map for a $\mathbb{C}^*$-action on it with the McLean metric and the limit measure is comparable to its Duistermaat–Heckman measure.

**Remark 3.20.** The domain wall crossing [HSVZ18, Th. 1.5] (also treated in type II superstring theory before according to [HSVZ18, Rem. 1.6]) is now reflected as the formation of the singularity of affine structure of $I_w$-type.

### 3.4. Root lattice type and type II degenerations

Suppose we have a type II polarized degeneration of K3 surfaces $\pi: (\mathcal{X}, \mathcal{L}) \to \Delta$. As an example case, suppose the end component of $\mathcal{X}_0$ is $\mathbb{F}_1$. Consider the ample cone of the $\mathbb{F}_1$, which gives the simplest classical instance of 2-ray game (cf. [Take89] for higher dimensional work) of Fano variety: Denote the natural projection $\varphi: \mathbb{F}_1 \to \mathbb{P}^2$, $\psi: \mathbb{F}_1 \to \mathbb{P}^1$, and $H$ the hyperplane in $\mathbb{P}^2$ passing through the center of $\varphi p$, $E$ the exceptional curve, and set the strict transform of $H$ as $H'$, so that $\varphi^*H = H' + E$, as local notation. Then, as is well known and easy, the ample cone is

$$\text{Amp} (\mathbb{F}_1) = \mathbb{R}_{>0}[\varphi^*H] + \mathbb{R}_{>0}[\pi^*\mathcal{O}_{\mathbb{P}^1}(1)],$$

so that each extremal ray corresponds to $\varphi$ and $\pi$.

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6 Recall that in our first sections, the symbol $\pi$ was used as a one parameter degeneration of K3 surfaces.
Our point here is that if we consider MMP of $F_1$ with scaling in $|L|$, then depending on the terminal objects (either $P^1$ or $P^2$), we have a subdivision of the cone:

$$\text{Amp}(F_1) = (\mathbb{R}_{\geq 0}[\varphi^*H] + \mathbb{R}_{\geq 0}[-K_{F_1}])$$

$$+ (\mathbb{R}_{\geq 0}[-K_{F_1}] + \mathbb{R}_{\geq 0}[\psi^*O_{P^1}(1)]).$$

We denote the first cone as $C_1$ and the second as $C_2$. Then, we observe the following: if type II Kulikov degeneration with nef (but generically ample [She83a]) polarization $L$ has end component $V \simeq F_1$, then

- $[L|_V] \in C_1$ if and only if the corresponding PL function has $E$-type singularity at the end, and
- $[L|_V] \in C_2$ if and only if the corresponding PL function has $D$-type singularity at the end.

Now, we conjecture the following.

**Conjecture 3.21. (D vs. E conjecture)** We consider type II polarized degeneration of K3 surfaces $(X, L) \to \Delta$ in $F_{2d}$. Take a simultaneous resolution after base change to make it Kulikov model $\tilde{X}$. We denote the pull back of $L$ to $\tilde{X}$ as $\tilde{L}$ and $X_0 = V_0 \cup V_1$, the stable type II degeneration (see [Fri84], [Kon85]).

Suppose that if we run the MMP with scaling in $\tilde{L}|_{V_i}$ to $V_i$, it ends with ruled surface structure (resp. birational contraction). Then, our hyper-Kähler rotation of $(X_t, L_t)$ limits to $D$-type end of interval (resp. $E$-type end of interval).

**Example 3.22.** Indeed, it at least matches to the four cases of degree 2 examples (see [Fri84, 5.2]; cf. also [AET19], [Sha80]).

**Remark 3.23.** (Strong open K-polystable degenerations on $M_{\text{W}}^m$) For $X := F_2$, $D$ an elliptic bi-section for the ruling, then $X^o := X \setminus D$, for certain range of ample $L$, $(X^o, L^o := L|_{X^o})$ is strongly open K-polystable [Od20a], as in the arguments of [Od20a]. Indeed, [AP06] applied to the crepant contraction to the quadric cone $X \to \mathbb{P}(1,1,2)$ implies that. This appears as $M_{\text{W}}^m$ in [OO21, §7]. We expect that these $D$-type degenerating families bubble off different ALH gravitational instantons along minimal noncollapsing rescaling in the sense of [Od20a, §6].

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7 (Added notes in revision) We note here again that in version 4 of [ABE20], the authors change the notation of type $D$ to type $C$ (see §2.2.2.4).

8 As in Example 3.22, $\tilde{L}|_{V_i}$ can be only semiample and big, as the pullback of ample line bundle on some crepant contraction. Nevertheless, the MMP with scaling still makes sense. If not preferred, one can pass to the crepant contraction induced by $L|_{V_i}$ and discuss on it.
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Yuji Odaka
Department of Mathematics
Kyoto University
Kyoto, Japan
yodaka@math.kyoto-u.ac.jp