WEIGHTED CONDITIONAL TYPE OPERATORS BETWEEN
DIFFERENT ORLICZ SPACES

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Abstract. In this note we consider weighted conditional type operators between different Orlicz spaces and generalized conditional type Hölder inequality that we defined in [2]. Then we give some necessary and sufficient conditions for boundedness of weighted conditional type operators. As a consequence we characterize boundedness of weighted conditional type operators and multiplication operators between different $L^p$-spaces. Finally, we give some upper and lower bounds for essential norm of weighted conditional type operators.

1. Introduction and Preliminaries

The continuous convex function $\Phi : \mathbb{R} \to \mathbb{R}$ is called a Young’s function whenever

1. $\Phi(x) = 0$ if and only if $x = 0$.
2. $\Phi(-x) = \Phi(x)$.
3. $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$, $\lim_{x \to -\infty} \Phi(x) = \infty$.

With each Young’s function $\Phi$ one can associate another convex function $\Phi^* : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\Phi^*(y) = \sup \{ x | y - \Phi(x) : x \geq 0 \}, \ y \in \mathbb{R}.$$ 

Then $\Phi^*$ is called the complementary Young’s function of $\Phi$. The following properties also are immediate from the definition of Young functions.

**Proposition 1.1.**[4] Let $\Phi$ be a Young’s function. Then $\Phi$, $\Phi^*$ are strictly increasingly so that their inverses $\Phi^{-1}$, $\Phi^{-1}$ are uniquely defined and

(i) $\Phi(a) + \Phi(b) \leq \Phi(a + b), \ \Phi^{-1}(a) + \Phi^{-1}(b) \geq \Phi^{-1}(a + b), \ a, b \geq 0$,

(ii) $a < \Phi^{-1}(a) \Phi^{-1}(a) \leq 2a, \ a \geq 0$.

A Young’s function $\Phi$ is said to satisfy the $\Delta_2$ condition (globally) if $\Phi(2x) \leq k\Phi(x), \ x \geq x_0 \geq 0$ ($x_0 = 0$) for some constant $k > 0$. Also, $\Phi$ is said to satisfy the $\Delta'(\nabla')$ condition, if $\exists c > 0$ ($b > 0$) such that

$$\Phi(xy) \leq c\Phi(x)\Phi(y), \ x, y \geq x_0 \geq 0$$

$$(\Phi(bxy) \geq \Phi(x)\Phi(y)), \ x, y \geq y_0 \geq 0).$$

If $x_0 = 0(y_0 = 0)$, then these conditions are said to hold globally. If $\Phi \in \Delta'$, then $\Phi \in \Delta_2$.

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Let \( \Phi, \Psi \) be two Young’s functions, then \( \Phi \) is stronger than \( \Psi \), \( \Phi \geq \Psi \) or \( \Psi \leq \Phi \) in symbols, if
\[
\Psi(x) \leq \Phi(ax), \quad x \geq x_0 \geq 0
\]
for some \( a \geq 0 \) and \( x_0 \), if \( x_0 = 0 \) then this condition is said to hold globally. A detailed discussion and verification of these properties may be found in [4].

Let \((\Omega, \Sigma, \mu)\) be a measure space and \( \Phi \) be a Young’s function, then the set of \( \Sigma \)-measurable functions
\[
L^\Phi(\Sigma) = \{ f : \Omega \to \mathbb{C} : \exists k > 0, \int_\Omega \Phi(|f|)d\mu < \infty \}
\]
is a Banach space, with respect to the norm \( N_\Phi(f) = \inf \{ k > 0 : \int_\Omega \Phi(\frac{|f|}{k})d\mu \leq 1 \} \). \((L^\Phi(\Sigma), N_\Phi(.)\) is called Orlicz space. If \( \Phi \in \Delta_2 \), then the dual space of \( L^\Phi(\Sigma) \) is equal to \( L^{\Phi^*}(\Sigma) \). The usual convergence in the orlicz space \( L^\Phi(\Sigma) \) can be introduced in term of the orlicz norm \( N_\Phi(.) \) as \( u_n \to u \) in \( L^\Phi(\Sigma) \) means \( N_\Phi(u_n - u) \to 0 \). Also, a sequence \( \{ u_n \}_{n=1}^{\infty} \) in \( L^\Phi(\Sigma) \) is said to converges in \( \Phi \)-mean to \( u \in L^\Phi(\Sigma) \), if
\[
\lim_{n \to \infty} I_\Phi(u_n - u) = \lim_{n \to \infty} \int_\Omega \Phi(|u_n - u|)d\mu = 0.
\]

For a sub-\( \sigma \)-finite algebra \( A \subseteq \Sigma \), the conditional expectation operator associated with \( A \) is the mapping \( f \to E^Af \), defined for all non-negative, measurable function \( f \) as well as for all \( f \in L^1(\Sigma) \) and \( f \in L^\infty(\Sigma) \), where \( E^Af \), by the Radon-Nikodym theorem, is the unique \( A \)-measurable function satisfying
\[
\int_A f d\mu = \int_A E^Af d\mu, \quad \forall A \in A.
\]
As an operator on \( L^1(\Sigma) \) and \( L^\infty(\Sigma) \), \( E^A \) is idempotent and \( E^A(L^\infty(\Sigma)) = L^\infty(A) \) and \( E^A(L^1(\Sigma)) = L^1(A) \). Thus it can be defined on all interpolation spaces of \( L^1 \) and \( L^\infty \) such as, Orlicz spaces [1]. We say the measurable function \( f \) is conditionable with respect to \( \sigma \)-subalgebra \( A \subseteq \Sigma \) if \( E^A(f) \) is defined. If there is no possibility of confusion, we write \( E(f) \) in place of \( E^A(f) \). This operator will play a major role in our work and we list here some of its useful properties:

- If \( g \) is \( A \)-measurable, then \( E(fg) = E(f)g \).
- \( \varphi(E(f)) \leq E(\varphi(f)) \), where \( \varphi \) is a convex function.
- If \( f \geq 0 \), then \( E(f) \geq 0 \); if \( f > 0 \), then \( E(f) > 0 \).
- For each \( f \geq 0 \), \( S(f) \subseteq S(E(f)) \), where \( S(f) = \{ x \in X : f(x) \neq 0 \} \).

A detailed discussion and verification of most of these properties may be found in [3]. We recall that an \( A \)-atom of the measure \( \mu \) is an element \( A \in A \) with \( \mu(A) > 0 \) such that for each \( F \in A \), if \( F \subseteq A \), then either \( \mu(F) = 0 \) or \( \mu(F) = \mu(A) \). A measure space \((\Omega, \Sigma, \mu)\) with no atoms is called a non-atomic measure space. It is well-known fact that every \( \sigma \)-finite measure space \((\Omega, \Sigma, \mu)\) can be partitioned uniquely as \( \Omega = (\bigcup_{n \in \mathbb{N}} C_n) \cup B \), where \( \{ C_n \}_{n \in \mathbb{N}} \) is a countable collection of pairwise disjoint \( \Sigma \)-atoms and \( B \), being disjoint from each \( C_n \), is non-atomic [5].

Let \( f \in L^\Phi(\Sigma) \). It is not difficult to see that \( \Phi(E(f)) \leq E(\Phi(f)) \) and so by some elementary computations we get that \( N_\Phi(E(f)) \leq N_\Phi(f) \) i.e, \( E \) is a contraction on the Orlicz spaces. As we defined in [2], we say that the pair \((E, \Phi)\) satisfies the generalized conditional-type Hölder-inequality (or briefly GCH-inequality) if there exists some positive constant \( C \) such that for all \( f \in L^\Phi(\Omega, \Sigma, \mu) \) and \( g \in
In section 3 we find some upper and lower bounds for weighted conditional type operators between different Orlicz spaces by considering the pair \((E, \Phi)\) that satisfy GCH-inequality in [2]. The results of the section 2 generalizes some results of [2] and [5].

In section 3 we find some upper and lower bounds for weighted conditional type operators on Orlicz spaces.

2. Bounded weighted conditional type operators

First we give a definition of weighted conditional type operator.

**Definition 2.1.** Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and let \(A\) be a \(\sigma\)-subalgebra of \(\Sigma\) such that \((\Omega, A, A)\) is also \(\sigma\)-finite. Let \(E\) be the corresponding conditional expectation operator relative to \(A\). If \(u \in L^0(\Sigma)\) (the spaces of \(\Sigma\)-measurable functions on \(\Omega\)) such that \(uf\) is conditionable and \(E(uf) \in L^\Psi(\Sigma)\) for all \(f \in D \subseteq L^\Phi(\Sigma)\), where \(D\) is a linear subspace, then the corresponding weighted conditional type operator (or WCT operator) is the linear transformation \(R_u : D \rightarrow L^\Psi(\Sigma)\) defined by \(f \rightarrow E(uf)\).

In the first theorem we give some necessary conditions for boundedness of \(R_u : L^\Phi(\Sigma) \rightarrow L^\Psi(\Sigma)\), when \(\Phi \preceq \Psi\) and some sufficient conditions, when \(\Phi \succeq \Psi\).

**Theorem 2.2.** Let WCT operator \(R_u : D \subseteq L^\Phi(\Sigma) \rightarrow L^\Psi(\Sigma)\) be well defined, then the followings hold.

(a) Let \(\mu(\Omega) < \infty\) (\(\mu(\Omega) = \infty\)) and \(\Phi \preceq \Psi\) (globally). Then
(i) If \(R_u\) is bounded from \(L^\Phi(\Sigma)\) into \(L^\Psi(\Sigma)\), then \(E(u) \in L^\infty(A)\).
(ii) If \(\Psi \in \Delta^d(\text{globally})\) and \(R_u\) is bounded from \(L^\Phi(\Sigma)\) into \(L^\Psi(\Sigma)\), then \(\Psi^{-1}(E(\Psi^*(u))) \in L^\infty(A)\).

(b) Let \(\mu(\Omega) < \infty\) (\(\mu(\Omega) = \infty\)) and \(\Psi \preceq \Phi\) (globally). Moreover, if \((E, \Psi)\) satisfies the GCH-inequality and \(\Psi^{-1}(E(\Psi^*(u))) \in L^\infty(A)\), then \(R_u\) is bounded.

In this case, \(\|R_u\| \leq C\|\Psi^{-1}(E(\Psi^*(u)))\|_\infty\), where the constant \(C\) comes from GCH-inequality.

**Proof.** (a)-(i) Suppose that \(E(u) \notin L^\infty(A)\). If we set \(E_n = \{w \in \Omega : |E(u(w))| > n\}\), for all \(n \in \mathbb{N}\), then \(E_n \in A\) and \(\mu(E_n) > 0\). Since \((\Omega, A, \mu)\) has the finite subset property, we can assume that \(0 < \mu(E_n) < \infty\), for all \(n \in \mathbb{N}\).

By definition of \(E_n\) we have
\[
R_u(\chi_{E_n}) = E(u \chi_{E_n}) = E(u) \chi_{E_n} > n \chi_{E_n}.
\]
Since \(\Phi \preceq \Psi\) and the Orlicz's norm is monotone, thus there exists a positive constant \(c\) such that
\[
\|R_u(\chi_{E_n})\|_\Psi \geq \frac{1}{c}\|R_u(\chi_{E_n})\|_\Phi > \frac{1}{c}\|n \chi_{E_n}\|_\Phi = \frac{n}{c}\|\chi_{E_n}\|_\Phi.
\]
This implies that \(R_u\) isn't bounded. Therefore \(E(u)\) should be essentially bounded.
(a)-(ii) If $\Psi^{-1}(E(\Psi^*(u))) \notin L^\infty(A)$, then $\mu(E_n) > 0$, where 

$$E_n = \{w \in \Omega : \Psi^{-1}(E(\Psi^*(u)))(w) > n\}$$

and so $E_n \in A$. Since $\Psi \in \Delta'$, then $\Psi^* \in \nabla'$, i.e., $\exists b > 0$ such that $\Psi^*(by) \geq \Psi^*(x)\Psi^*(y)$, $x, y \geq 0$.

Also, $\Phi, \Psi \in \Delta_2$. Thus $(L^\Phi)^* = L^{\Psi^*}$ and $(L^\Psi)^* = L^{\Psi^*}$ and so $T^* = M_u : L^{\Psi^*}(A) \to L^{\Phi^*}(\Sigma)$, is also bounded. Hence for each $k > 0$ we have

$$\int_\Omega \Psi^*\left(\frac{k\chi_{E_n}}{N_{\Phi^*}(\chi_{E_n})}\right) d\mu = \int_\Omega \Psi^*\left(\frac{k\chi_{E_n}}{\mu(E_n)}\right) d\mu$$

$$\geq \int_{E_n} \Psi^*(u)\Psi^*\left(\frac{ck\Psi^{-1}(\frac{1}{\mu(E_n)})}{b}\right) d\mu$$

$$\geq \left(\int_{E_n} E(\Psi^*(u)) d\mu\right) \Psi^*\left(\frac{ck}{b^2}\right)\Psi^*\left(\frac{1}{\mu(E_n)}\right)$$

$$\geq \Psi^*(n)\mu(E_n) \frac{1}{\mu(E_n)} \Psi^*\left(\frac{ck}{b^2}\right)$$

$$= \Psi^*(n)\Psi^*\left(\frac{ck}{b^2}\right).$$

Thus

$$\int_\Omega \Psi^*\left(\frac{k\chi_{E_n}}{N_{\Phi^*}(\chi_{E_n})}\right) d\mu = \int_\Omega \Psi^*(kM_u(f_n)) d\mu \geq \Psi^*(n)\Psi^*\left(\frac{k}{b^2}\right) \to \infty$$

as $n \to \infty$, where $f_n = \frac{\chi_{E_n}}{N_{\Phi^*}(\chi_{E_n})}$. Thus $N_{\Phi^*}(M_u(f_n)) \to \infty$, as $n \to \infty$. Since $N_{\Phi^*}(M_u(f_n)) \leq N_{\Phi^*}(M_u(f_n))$, then $N_{\Phi^*}(M_u(f_n)) \to \infty$, as $n \to \infty$. This is a contradiction, since $M_u$ is bounded.

(b) Put $M = ||\Psi^{-1}(E(\Psi^*(u)))||_{\infty}$. For $f \in L^\Phi(\Sigma)$ we have

$$\int_\Omega \Psi\left(\frac{E(u)}{CMN_{\Phi^*}(f)}\right) d\mu = \int_\Omega \Psi\left(\frac{E(u)}{CM}\frac{1}{N_{\Phi^*}(f)}\right) d\mu$$

$$\leq \int_\Omega \Psi\left(\frac{C\Psi^{-1}(E(\Psi(\frac{1}{N_{\Phi^*}(f)})))\Psi^{-1}(E(\Psi^*(u)))}{CM}\right) d\mu$$

$$\leq \int_\Omega \Psi\left(\Psi^{-1}(E(\Psi(\frac{f}{N_{\Phi^*}(f)})))\right) d\mu$$

$$= \int_\Omega \Psi\left(\frac{f}{N_{\Phi^*}(f)}\right) d\mu.$$

Now for the case that $\mu(\Omega) = \infty$ and $\Psi \leq \Phi$ globally, easily we get that

$$\int_\Omega \Psi\left(\frac{f}{N_{\Phi^*}(f)}\right) d\mu \leq \int_\Omega \Phi\left(\frac{f}{N_{\Phi^*}(f)}\right) d\mu \leq 1.$$
then we have

\[
\int_{\Omega} \Psi\left(\frac{f}{N_{\Phi}(f)}\right) d\mu \leq \Psi(T)\mu(\Omega) + \int_{\Omega \setminus E} \Phi\left(\frac{f}{N_{\Phi}(f)}\right) d\mu \\
\leq N + \int_{\Omega} \Phi\left(\frac{f}{N_{\Phi}(f)}\right) d\mu \\
\leq N + 1.
\]

So \( N_{\Psi}(R_u(f)) \leq CM N_{\Phi}(f) \), \( N_{\Psi}(R_u(f)) \leq CM (N + 1) N_{\Phi}(f) \), respectively for infinite and finite cases. Thus \( R_u \) is bounded in both cases and \( \|R_u\| \leq C \|\Psi^* - 1(E(\Psi^*(u)))\|_{\infty} \), \( \|R_u\| \leq C(N + 1) \|\Psi^* - 1(E(\Psi^*(u)))\|_{\infty} \), respectively for infinite and finite cases.

**Theorem 2.3.** Let WCT operator \( R_u : D \subseteq L^\Phi(\Sigma) \to L^\Psi(\Sigma) \) be well defined. Then the followings hold.

(a) Let \( \Phi^* \circ \Psi^{-1} \geq \Theta \) globally for some Young’s function \( \Theta \). If \( \Phi^* \in \Delta' \) (globally), \( \Theta \in \nabla' \) (globally) and

(i) \( E(\Phi^*(\bar{u})) = 0 \) on \( B \),

(ii) \( \sup_{n \in \mathbb{N}} \frac{E(\Phi^*(\bar{u}))(A_n)}{\Phi^*(\Psi^{-1}(1/\mu(A_n)))} < \infty \),

then \( R_u \) is bounded. In another case, if \( \Phi^*, \Psi^{-1} \in \Delta' \) (globally),

\[
\sup_{n \in \mathbb{N}} \frac{E(\Phi^*(\bar{u}))(A_n)}{\Phi^*(\Psi^{-1}(1/\mu(A_n)))} < \infty
\]

and (i) holds, then \( R_u \) is bounded.

(b) If \( \Phi^* \circ \Psi^{-1} \) is a Young’s function, \( \Phi^* \in \nabla' \) globally and \( R_u \) is bounded from \( L^\Phi(\Sigma) \) into \( L^\Psi(\Sigma) \), then

(i) \( E(\Phi^*(\bar{u})) = 0 \) on \( B \),

(ii) \( \sup_{n \in \mathbb{N}} \frac{E(\Phi^*(\bar{u}))(A_n)}{\Phi^*(\Psi^{-1}(1/\mu(A_n)))} < \infty \).

**Proof.** (a) If we prove that the operator \( M_{\bar{u}} : L^\Psi(\mathcal{A}) \to L^\Phi(\Sigma) \) is bounded, then we conclude that \( R_u = (M_{\bar{u}})^* \) from \( L^\Phi(\Sigma) \) into \( L^\Psi(\Sigma) \) is bounded. So we prove the operator \( M_{\bar{u}} \) is bounded under given conditions. Since \( \Phi^* \in \Delta' \) and \( \Theta \in \nabla' \) globally, then there exist \( b, b' > 0 \) such that the following computations holds. Put \( M = \sup_{n \in \mathbb{N}} \frac{E(\Phi^*(\bar{u}))(A_n)}{\Phi^*(\Psi^{-1}(1/\mu(A_n)))} \). Therefore, for every \( f \in L^\Psi(\mathcal{A}) \) we get
that

\[
\int_{\Omega} \Phi^*(\frac{\bar{u} f}{N_{\Psi^*}(f)}) d\mu \leq b \int_{\Omega} E(\Phi^*(\bar{u})) \Phi^*(\frac{f}{N_{\Psi^*}(f)}) d\mu
\]

\[
= b \sum_{n=1}^{\infty} E(\Phi^*(\bar{u}))(A_n) \Phi^*(\frac{f}{N_{\Psi^*}(f)})(A_n) \mu(A_n)
\]

\[
\leq b \sum_{n=1}^{\infty} E(\Phi^*(\bar{u}))(A_n) \Theta \Psi^*(\frac{f}{N_{\Psi^*}(f)})(A_n)
\]

\[
\leq Mb \sum_{n=1}^{\infty} \Phi^* \Psi^{*-1}(\mu(A_n)) \Theta \Psi^*(\frac{f}{N_{\Psi^*}(f)})(A_n)
\]

\[
\leq Mb \sum_{n=1}^{\infty} \Theta(\frac{\mu(A_n)}{b}) \Theta \Psi^*(\frac{f}{N_{\Psi^*}(f)})(A_n)
\]

\[
\leq Mb \Theta(\frac{1}{b}) \int_{\Omega} \Psi^*(\frac{f}{N_{\Psi^*}(f)}) d\mu
\]

\[
\leq Mb \Theta(\frac{1}{b}).
\]

This implies that \( \int_{\Omega} \Phi^*(\frac{\bar{u} f}{N_{\Psi^*}(f)}) d\mu \leq 1 \) and so \( \|M_{ul}\| \leq (Mb \Theta(\frac{1}{b}) + 1) \). Therefore the WCT operator \( R_u \) is bounded. For the other case also by the same way we get that \( R_u \) is bounded.

(b) Suppose that \( R_u = M_{ul} \) from \( L^\Phi(\Sigma) \) into \( L^\Psi(\Sigma) \) is bounded, then the multiplication operator \( M_{ul} \) from \( L^\Phi^*(A) \) into \( L^\Phi^*(A) \) is bounded. First, we show that \( E(\Phi^*(\bar{u})) = 0 \) on \( B \). Suppose on the contrary. Thus we can find some \( \delta > 0 \) such that \( \mu(\{w \in B : E(\Phi^*(\bar{u}))(w) > \delta\}) > 0 \). Take \( F = \{w \in B : E(\Phi^*(\bar{u}))(w) > \delta\} \). Since \( F \subseteq B \) is a \( A \)-measurable set and \( A \) is \( \sigma \)-finite, then for each \( n \in N \), there exists \( F_n \subseteq F \) with \( F_n \in A \) such that \( \mu(F_n) = \frac{\delta}{2^n} \). Define \( f_n = \frac{x F_n}{N_{\Psi^*}^*(x F_n)} \). It is clear that \( f_n \in L^\Phi^*(A) \) and \( N_{\Psi^*}(f_n) = 1 \). Hence for each \( k > 0 \) we have

\[
\int_{\Omega} \Phi^*(\frac{k \bar{u} \chi_{F_n}}{N_{\Psi^*}^*(\chi_{F_n})}) d\mu = \int_{\Omega} \Phi^*(k \bar{u} \chi_{F_n} \Psi^{*-1}(\frac{1}{\mu(F_n)})) d\mu
\]

\[
= \int_{\Omega} \Phi^*(k \bar{u} \Psi^{*-1}(\frac{1}{\mu(F_n)})) \chi_{F_n} d\mu
\]

\[
\geq \int_{F_n} \Phi^*(\bar{u}) \Phi^*(\frac{k \Psi^{*-1}(\frac{1}{\mu(F_n)})}{b}) d\mu
\]

\[
\geq (\int_{F_n} E(\Phi^*(\bar{u}))) d\mu \Phi^*(\frac{k}{b^2} \Phi^*(\Psi^{*-1}(\frac{1}{\mu(F_n)})))
\]

\[
\geq \delta \Phi^*(\frac{k}{b^2} \Phi^*(\Psi^{*-1}(\frac{1}{\mu(F_n)}))) \mu(F_n)
\]

\[
= \delta \Phi^*(\frac{k}{b^2} \Phi^*(\Psi^{*-1}(\frac{1}{\mu(F_n)}))).
\]
Since \(\Phi^* \circ \Psi^{-1}\) is a Young’s function, then \(\frac{\Phi^*(\Psi^{-1}(\eta/A_n))}{\eta/M_a} \to \infty\) when \(n \to \infty\).

Therefore \(\int_{\Omega} \Phi^*\left(\frac{k \chi A_n}{N\Phi^*(\chi A_n)}\right)d\mu \to \infty\) when \(n \to \infty\) for each \(k > 0\) and so \(N_{\Phi^*}(\bar{u}f_n) \to \infty\) when \(n \to \infty\). This is a contradiction. It remains to prove (ii). Let \(f_n = \chi A_n\), then we have

\[
1 \geq \int_{\Omega} \Phi^*\left(\frac{\bar{u}f_n}{N\Phi^*(\bar{u}f_n)}\right)d\mu \\
\geq \int_{\Omega} \Phi^*\left(\frac{\bar{u}\chi A_n}{\|M_a\|}\right)d\mu \\
\geq \int_{A_n} \Phi^*(\bar{u})\Phi^*\left(\frac{\Psi^{-1}(\eta/\mu(A_n))}{b\|M_a\|}\right)d\mu \\
= \int_{A_n} E(\Phi^*(\bar{u}))\Phi^*\left(\frac{\Psi^{-1}(\eta/\mu(A_n))}{b\|M_a\|}\right)d\mu \\
= E(\Phi^*(\bar{u}))(A_n)\Phi^*\left(\frac{1}{\mu(A_n)}\right)\mu(A_n)\Phi^*\left(\frac{1}{b^2\|M_a\|}\right).
\]

Hence we get that

\[
\sup_n E(\Phi^*(\bar{u}))(A_n)\Phi^*\left(\frac{1}{\mu(A_n)}\right)\mu(A_n) \leq \frac{1}{\Phi^*\left(\frac{1}{b^2\|M_a\|}\right)} < \infty.
\]

This completes the proof.

In the next proposition we give another necessary condition for boundedness of \(R_u\). I think it’s better that others.

**Proposition 2.4.** Let WCT operator \(R_u : D \subseteq L^\Phi(\Sigma) \to L^\Psi(\Sigma)\) be well defined. And let \(\Phi, \Psi \in \triangle'\) and \(\Psi \circ \Phi^{-1}\) be a Young’s function. The WCT operator \(R_u\) into \(L^\Psi(\Sigma)\) is bounded, if the following conditions hold;

(i) \(E(\Phi^*(\bar{u})) = 0\) on \(B\),

(ii) \(\sup_{n \in \mathbb{N}} \frac{\Phi^*\left(\frac{E(\Phi^*(\bar{u}))(A_n))\mu(A_n)}{\Psi^*(\frac{1}{\mu(A_n)})}\right) < \infty\).

In another case, if \(\Psi \circ \Phi^{-1}\) isn’t a Young’s function, but \(\Psi \circ \Phi^{-1} \preceq \Theta\) for some Young’s function \(\Theta\). Then the operator WCT operator \(R_u\) into \(L^\Psi(\Sigma)\) is bounded, if the conditions (i) and (ii) hold.
Proof. Let \( f \in L^\Phi(\Omega) \) such that \( N_\Phi(f) \leq 1 \). Then we have

\[
\int_\Omega \Psi\left( \frac{E(uf)}{N_\Phi(f)} \right) d\mu \\
= \sum_{n=1}^{\infty} \Psi\left( \Phi^{-1}\left( E\left( \frac{f}{N_\Phi(f)} \right) \right)(A_n) \right) \left( \Phi^{-1}\left( E\left( \Phi^*(u) \right) \right)(A_n) \right) \mu(A_n)
\]

\[
\leq M \sum_{n=1}^{\infty} \Psi \circ \Phi^{-1}\left( E\left( \frac{f}{N_\Phi(f)} \right) \right)(A_n) \mu(A_n)
\]

\[
\leq M \sum_{n=1}^{\infty} \Psi \circ \Phi^{-1}\left( E\left( \frac{f}{N_\Phi(f)} \right) \right)(A_n)
\]

\[
\leq M \Psi \circ \Phi^{-1}\left( \int_\Omega E\left( \frac{f}{N_\Phi(f)} \right) d\mu \right)
\]

\[
= M \Psi \circ \Phi^{-1}\left( \int_\Omega \frac{f}{N_\Phi(f)} d\mu \right)
\]

\[
\leq M \Psi \circ \Phi^{-1}(1).
\]

Hence

\[
\int_\Omega \Psi\left( \frac{E(uf)}{M \Psi \circ \Phi^{-1}(1) + 1} \right) d\mu \leq 1.
\]

Consequently we get that

\[
N_\Psi(E(uf)) \leq (M \Psi \circ \Phi^{-1}(1) + 1) N_\Phi(f).
\]

Thus the operator \( R_u \) is bounded. If \( \Psi \circ \Phi^{-1} \lesssim \Theta \) for some Young’s function \( \Theta \), by the same method we get that

\[
N_\Psi(E(uf)) \leq (M \Theta(1) + 1) N_\Phi(f).
\]

So this also states that the operator \( R_u \) is bounded.

Remark 2.5. If \((\Omega, \Sigma, \mu)\) is a non-atomic measure space, then under assumptions of Theorem 2.3, there is not any non-zero bounded operator of the form \( R_u \) from \( L^\Phi(\Sigma) \) into \( L^\Psi(\Sigma) \).

Put \( \Phi(x) = \frac{x^p}{p} \) for \( x \geq 0 \), where \( 1 < p < \infty \). It is clear that \( \Phi \) is a Young’s function and \( \Phi^*(x) = \frac{x^{p'}}{p'} \), where \( 1 < p' < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). If \( \Phi(x) = \frac{x^q}{q} \) for \( x \geq 0 \), where \( 1 < p < q < \infty \). Then \( \Phi^* \circ \Psi^{-1}(x) = \frac{x^{q'}}{p' q'} \). Since \( \frac{q'}{q} > 1 \), then \( \Phi^* \circ \Psi^{-1} \) is a Young’s function. These observations and Theorems 2.3, 2.4 give us the next Remark.

Remark 2.6. Let \( R_u : D \subseteq L^p(\Sigma) \to L^q(\Sigma) \) be well defined. Then the operator \( R_u \) from \( L^p(\Sigma) \) into \( L^q(\Sigma) \), where \( 1 < p < q < \infty \), is bounded if and only if the followings hold:

(i) \( E(|u|^p) = 0 \) on \( B \).

(ii) \( \sup_{n \geq 0} \frac{E(|u|^p)(A_n)}{\mu(A_n)^{p-1}} < \infty \).

In addition, we get that there is not any non-zero bounded operator of the form
$R_u$ from $L^p$ into $L^q$, $(1 < p < q < \infty)$ when the underlying measure space is non-atomic.

Specially, if $A = \Sigma$, then $E = I$ and so the multiplication operator $M_u$ from $L^p(\Sigma)$ into $L^q(\Sigma)$ is bounded if and only if

(i) $u = 0$ on $B$.
(ii) $\sup_{n \geq 0} \frac{u(A_n)}{\mu(A_n)^{\frac{q'}{p'}}} < \infty$.

Here we recall a fundamental Lemma, which is as an easy exercise.

Lemma 2.7. Let $\Phi_i$, $i = 1, 2, 3$, be Young's functions for which

$$\Phi_3(xy) \leq \Phi_1(x) + \Phi_2(y), \quad x \geq 0, y \geq 0.$$ 

If $f_i \in L^{\Phi_i}(\Sigma)$, $i = 1, 2$, where $(\Omega, \Sigma, \mu)$ is any measure space, then

$$N_{\Phi_3}(f_1 f_2) \leq 2N_{\Phi_1}(f_1)N_{\Phi_2}(f_2).$$ 

Theorem 2.8. Let $\Phi$ and $\Psi$ be Young's functions an $R_u : D \subseteq L^\Phi(\Sigma) \to L^\Psi(\Sigma)$ be well defined. Then the followings hold:

(i) Suppose that there exists a Young's function $\Theta$ such that

$$\Psi(xy) \leq \Phi(x) + \Theta(y), \quad x \geq 0, y \geq 0$$ 

and $(E, \Phi)$ satisfies GCH-inequality. In this case if $\Phi^{-1}(E(\Phi^*(u))) \in L^\Theta(A)$, then the WCT operator $R_u$ from $L^\Phi(\Sigma)$ into $L^\Psi(\Sigma)$ is bounded.

(ii) Let $\Theta = \Psi \circ \Phi^{-1}$ be a Young’s function, $\Theta \in \triangle_2$ and $\Phi^* \in \triangle_2$. In this case if WCT operator $R_u$ is bounded from $L^\Phi(\Sigma)$ into $L^\Psi(\Sigma)$, then $E(\Phi^*(\bar{u}))$ is bounded. Consequently $\Phi^{-1}(E(\Phi^*(\bar{u}))) \in L^{\Phi^* \circ \Phi^{-1}}(A)$.

Proof. (i) Let $f \in L^\Phi(\Sigma)$ such that $N_{\Phi}(f) \leq 1$. This means that

$$\int_{\Omega} \Phi(\Phi^{-1}(E(\Phi(f))))d\mu = \int_{\Omega} \Phi(f)d\mu \leq 1,$$

hence $N_{\Phi}(\Phi^{-1}(E(\Phi(f)))) \leq 1$. By using GCH-inequality we have

$$N_{\Phi}(E(uf)) \leq N_{\Phi}(\Phi^{-1}(E(\Phi(f))))N_{\Theta}(\Phi^{-1}(E(\Phi^*(u))))$$

$$\leq N_{\Theta}(\Phi^{-1}(E(\Phi^*(u)))).$$ 

Thus for all $f \in L^\Phi(\Sigma)$ we have

$$N_{\Phi}(E(uf)) \leq N_{\Phi}(f)N_{\Theta}(\Phi^{-1}E(\Phi^*(u))).$$

And so the operator $R_u$ is bounded.

(ii) Suppose that $R_u$ is bounded. So the adjoint operator $M_{\bar{u}} = (R_u)^* : L^{\Psi^*}(A) \to L^{\Phi^*}(\Sigma)$ is also bounded. For $f \in L^\Phi(A)$ we have $\Phi^{-1}(f) \in L^{\Psi^*}(A)$. 

Consequently we get that
\[
\int_{\Omega} E(\Phi^*(\bar{u})) f \, d\mu = \int_{\Omega} \Phi^*(\bar{u}) \Phi^*(\Phi^*-1(f)) \, d\mu \\
\leq b \int_{\Omega} \Phi^*(\bar{u}) \Phi^*-1(f) \, d\mu \\
= b \int_{\Omega} \Phi^*(M_u(\Phi^*-1(f))) \, d\mu < \infty.
\]
Therefore \( \int_{\Omega} E(\Phi^*(\bar{u})) f \, d\mu < \infty \) for all \( f \in L^q(\mathcal{A}) \). This implies that \( E(\Phi^*(\bar{u})) \in L^{\Theta^*}(\mathcal{A}) \). This completes the proof.

**Remark 2.9.** Let \( R_u : D \subseteq L^p(\Sigma) \rightarrow L^q(\Sigma) \) be well defined. Then the operator \( R_u \) from \( L^p(\Sigma) \) into \( L^q(\Sigma) \), where \( 1 < q < p < \infty \), is bounded if and only if \( (E(|u|^r))^{\frac{1}{r}} \in L^r(\mathcal{A}) \), where \( r = \frac{p}{p-q} \).

Specially, if \( \mathcal{A} = \Sigma \), then \( E = I \) and so the multiplication operator \( M_u \) from \( L^p(\Sigma) \) into \( L^q(\Sigma) \) is bounded if and only if \( u \in L^r(\Sigma) \).

**Example 2.10.** Let \( \Omega = [-1, 1], d\mu = \frac{1}{2} \, dw \) and \( \mathcal{A} = \langle \{(-a, a) : 0 \leq a \leq 1\} \rangle \) (\( \sigma \)-algebra generated by symmetric intervals). Then
\[
E^A(f)(w) = \frac{f(w) + f(-w)}{2}, \quad w \in \Omega,
\]
where \( E^A(f) \) is defined. Thus \( E^A(|f|) \geq \frac{|f|}{2} \). Hence \( |f| \leq 2E(|f|) \). Let \( \Phi(w) = e^{w^p} - w^p - 1 \) and \( \Psi(w) = \frac{w^p}{p} \) be Young’s functions, where \( p > 1 \). For each \( f \in L^p(\Omega, \Sigma, \mu) \) we have \( \Phi(|f|) \leq 2E(\Phi(|f|)) \). This implies that
\[
E(\Phi(|f|)) \leq 4\Phi^{-1}(E(\Phi(|f|))) \Psi^{-1}(E(\Psi(|g|))).
\]
If \( u \) is a non-zero continuous function on \( \Omega \), then for Young’s function \( \Theta(w) = (1 + w^p)\log(1 + w^p) - w^p \) we have
\[
\Psi(xy) \leq \Phi(x) + \Theta(y), \quad -1 \leq x, y \leq 1.
\]
So by Theorem 2.8 the WCT operator \( R_u \) is bounded from \( L^p \) into \( L^q \). But it is not bounded from \( L^p \) into \( L^p \), because of Theorem 2.3.

**Example 2.11.** Let \( \Omega = [0, 1], \Sigma \) be the \( \sigma \)-algebra of Lebesgue measurable subset of \( \Omega \) and let \( \mu \) be the Lebesgue measure on \( \Omega \). Fix \( n \in \{2, 3, 4\ldots\} \) and let \( s : [0, 1] \rightarrow [0, 1] \) be defined by \( s(w) = w + \frac{1}{n}(\text{mod } 1) \). Let \( \mathcal{B} = \{ E \in \Sigma : s^{-1}(E) = E \} \). In this case
\[
E^\mathcal{B}(f)(w) = \sum_{j=0}^{n-1} f(s^j(w)),
\]
where \( s^j \) denotes the jth iteration of \( s \). The functions \( f \) in the range of \( E^\mathcal{B} \) are those for which the \( n \) graphs of \( f \) restricted to the intervals \( [\frac{j-1}{n}, \frac{j}{n}], 1 \leq j \leq n \), are all congruent. If \( \Phi(w) = e^{w^2} - 1 \) and \( \Psi(w) = \frac{w^2}{\log(w^2)} \), then for \( \Theta(w) = \Phi^*(w^2) \) we have
\[
\Psi(xy) \leq \Phi(x) + \Theta(y), \quad 0 \leq x, y \leq 1.
\]
Then by Theorem 2.8 $R_u$ is a bounded operator from $L^\Phi$ into $L^\Psi$ for every non-zero continuous function $u$. But it is not bounded from $L^\Psi$ into $L^\Phi$, because of Theorem 2.3.

3. Essential norm

Let $\mathfrak{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathfrak{B}$. For $T \in L(\mathfrak{B})$, the Banach algebra of all bounded linear operators on $\mathfrak{B}$ into itself, the essential norm of $T$ means the distance from $T$ to $\mathcal{K}$ in the operator norm, namely $\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}$. Clearly, $T$ is compact if and only if $\|T\|_e = 0$. Let $X$ and $Y$ be reflexive Banach spaces and $T \in L(X,Y)$. It is easy to see that $\|T\|_e = \|T^*\|_e$. In this section we assume that $a_j = \mu(A_j)$, where $A_j$’s are $\mathcal{A}$-atoms.

In the sequel we present an upper bound for essential norm of $EM_u$ on Orlicz space $L^\Phi(\Sigma)$. For this we first recall some results of [2] for compactness of WCT operator $R_u$.

Theorem 3.1. [2] Let WCT operator $R_u$ be bounded on $L^\Phi(\Sigma)$, then the followings hold.

(a) If $R_u$ is compact, then $N_\varepsilon(E(u)) = \{w \in \Omega : E(u)(w) \geq \varepsilon\}$ consists of finitely many $\mathcal{A}$-atoms, for all $\varepsilon > 0$.

(b) If $R_u$ is compact and $\Phi \in \bigtriangleup'$ (globally), then $N_\varepsilon(\Phi^{-1}(E(\Phi(u))))$ consists of finitely many $\mathcal{A}$-atoms, for all $\varepsilon > 0$, where

$$N_\varepsilon(\Phi^{-1}(E(\Phi(u)))) = \{w \in \Omega : \Phi^{-1}(E(\Phi(u)))(w) \geq \varepsilon\}.$$ 

(c) If $(E, \Phi)$ satisfies the GCH-inequality and $N_\varepsilon(\Phi^{-1}(E(\Phi(u))))$ consists of finitely many $\mathcal{A}$-atoms, for all $\varepsilon > 0$, then $T$ is compact.

Corollary 3.2. [2] Under assumptions of theorem 3.1 we have the followings:

(a) If $(E, \Phi)$ satisfies the GCH-inequality and $\Phi \in \bigtriangleup'$ (globally), then $T$ is compact if and only if $N_\varepsilon(\Phi^{-1}(E(\Phi(u))))$ consists of finitely many $\mathcal{A}$-atoms, for all $\varepsilon > 0$.

(b) If $\Phi^* \prec x$ (globally) and $(E, \Phi)$ satisfies the GCH-inequality, then then $T$ is compact if and only if $N_\varepsilon(\Phi^{-1}(E(\Phi(u))))$ consists of finitely many $\mathcal{A}$-atoms, for all $\varepsilon > 0$.

(c) If $(\Omega, \mathcal{A}, \mu)$ is non-atomic measure space, $(E, \Phi)$ satisfies the GCH-inequality and $\Phi \in \bigtriangleup'$ (globally). Then $R_u$ is a compact operator on $L^\Phi(\Sigma)$ if and only if
\[ R_u = 0. \]

**Theorem 3.3.** Let \( R_u : L^\Phi(\Sigma) \rightarrow L^\Phi(\Sigma) \) is bounded. Then

(a) If \( (E, \Phi) \) satisfies in GCH-inequality and \( \beta_2 = \inf \{ \varepsilon > 0 : N_\varepsilon \text{ consists of finitely many } \mathcal{A}-\text{atoms} \} \), where \( N_\varepsilon = N_\varepsilon(\Phi^* (E(u))) \). Then

\[ \| R_u \| \leq C \beta_2, \]

where \( C \) comes from GCH-inequality.

(b) If \( a_n \rightarrow 0 \) or \( \{ a_n \}_{n \in \mathbb{N}} \) has no convergent subsequence. Let \( \beta_1 = \inf \{ \varepsilon > 0 : N_\varepsilon \text{ consists of finitely many } \mathcal{A}-\text{atoms} \} \), where \( N_\varepsilon = N_\varepsilon(E(u)) \). Then

\[ \| R_u \| \geq \beta_1. \]

**Proof** (a) Let \( \varepsilon > 0 \). Then \( N_{\varepsilon + \beta_2} \) consist of finitely many \( \mathcal{A} \)-atoms. Put \( u_{\varepsilon + \beta_2} = u_{\chi_{N_{\varepsilon + \beta_2}}} \) and \( R_{u_{\varepsilon + \beta_2}} \). So \( R_{u_{\varepsilon + \beta_2}} \) is finite rank and so compact. And for every \( f \in L^\Phi(\Sigma) \) we have

\[
\int_\Omega \Phi(\frac{R_u(f) - R_{\varepsilon + \beta_2}(f)}{C(\varepsilon + \beta_2)N_\Phi(f)}) d\mu = \int_\Omega \Phi(\frac{E(u f)\chi_{\Omega \setminus N_{\varepsilon + \beta_2}}}{C(\varepsilon + \beta_2)N_\Phi(f)}) d\mu
\]

\[
\leq \int_{\Omega \setminus N_{\varepsilon + \beta_2}} \Phi(\frac{C\Phi^{-1}(E(\Phi(\frac{f}{N_\Phi(f)}))))\Phi^* (E(\Phi^* (u))))}{C(\varepsilon + \beta_2))} d\mu
\]

\[
\leq \int_{\Omega \setminus N_{\varepsilon + \beta_2}} \Phi^{-1}(E(\Phi(\frac{\int f N_\Phi(f)}))))) d\mu \leq \int_\Omega E(\Phi(\frac{\int f N_\Phi(f)}) d\mu
\]

\[
= \int_\Omega \Phi(\frac{\int f}{N_\Phi(f)}) d\mu \leq 1.
\]

This implies that

\[ \| R_u \| \leq \| R_u - R_{u_{\varepsilon + \beta_2}} \| \leq C(\beta_2 + \varepsilon). \]

This mean’s that \( \| R_u \| \leq C \beta_2. \)

(b) Let \( 0 < \varepsilon < \beta_1 \). Then by definition, \( N_{\beta_1 - \varepsilon} = N_{\beta_1 - \varepsilon}(E(u)) \) contains infinitely many atoms or a non-atomic subset of positive measure. If \( N_{\beta_1 - \varepsilon} \) consists a non-atomic subset, then we can find a sequence \( \{ B_n \}_{n \in \mathbb{N}} \) such that \( \mu(B_n) < \infty \) and \( \mu(B_n) \rightarrow 0 \). Put \( f_n = \frac{\chi_{B_n}}{N_\Phi(\chi_{B_n})} \), then for every \( A \in \Sigma \) with \( 0 < \mu(A) < \infty \) we have

\[
\int_\Omega f_n \chi_A d\mu = \mu(A \cap B_n) \Phi^{-1}(\frac{1}{\mu(B_n)}) \leq \Phi^{-1}(\frac{1}{\mu(B_n)}) \rightarrow 0.
\]

when \( n \rightarrow \infty \). Also, if \( N_{\beta_1 - \varepsilon} \) consists infinitely many atoms \( \{ A'_n \}_{n \in \mathbb{N}} \). We set

\[ f_n = \frac{\chi_{A'_n}}{N_\Phi(\chi_{A'_n})}. \]

Then for every \( A \in \Sigma \) with \( 0 < \mu(A) < \infty \) we have

\[
\int_\Omega f_n \chi_A d\mu = \mu(A \cap A'_n) \Phi^{-1}(\frac{1}{\mu(A'_n)}).
\]
If \( \{\mu(A_n)\}_{n \in \mathbb{N}} \) has no convergent subsequence, then there exists \( n_0 \) such that for \( n > n_0 \), \( \mu(A \cap A'_n) = 0 \) and if \( \mu(A_n) \to 0 \) then \( \mu(A'_n) \to 0 \). Thus \( \int_{\Omega} f_n \chi_A \, d\mu = \mu(A \cap A'_n) \Phi^{-1}(\frac{1}{\mu(A'_n)}) \to 0 \) in both cases. These imply that \( f_n \to 0 \) weakly. So

\[
\int_{\Omega} \Phi((\beta_1 - \varepsilon)f_n) \, d\mu \leq \int_{\Omega} \Phi(\frac{E(u)f_n}{N_\Phi(R_u(f_n))}) \, d\mu = \int_{\Omega} \Phi(\frac{R_u(f_n)}{N_\Phi(R_u(f_n))}) \, d\mu.
\]

Thus \( N_\Phi(R_u(f_n)) \geq \beta_2 - \varepsilon \).

Also, there exists compact operator \( T \in L(L^\Phi(\Sigma)) \) such that \( \|R_u\|_e \geq \|T - R_u\| - \varepsilon \). Hence \( N_\Phi(Tf_n) \to 0 \) and so there exists \( N > 0 \) such that for each \( n > N \), \( N_\Phi(Tf_n) \leq \varepsilon \). So

\[
\|R_u\|_e \geq \|R_u - T\| - \varepsilon \geq |N_\Phi(R_u(f_n)) - N_\Phi(Tf_n)| \geq \beta_1 - \varepsilon - \varepsilon,
\]

thus we conclude that \( \|R_u\|_e \geq \beta_1 \).

**Corollary 3.4.** Let \( u : \Omega \to \mathbb{C} \) be \( \Sigma \)-measurable and Let \( M_u : L^\Phi(\Sigma) \to L^\Phi(\Sigma) \).

If \( \beta = \beta_1 = \beta_2 = \inf \{\varepsilon > 0 : N_\varepsilon \text{ consists of finitely many atoms} \} \). Then

(a) \( ||M_u||_e \leq \beta \).

(b) Let \( \Phi \in \Delta_2 \) and \( a_n \to 0 \) or \( \{a_n\}_{n \in \mathbb{N}} \) has no convergent subsequence. Then \( \beta = ||M_u||_e \).

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