Strange-metal behavior in a Fermi liquid with strange scatterers

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A variety of low-temperature, normal-state properties of optimally and overdoped cuprate superconductors, including the DC and optical transport responses, are sufficiently anomalous that they might seem to be inconsistent with any quasiparticle description. However, we show by explicit construction that the most salient phenomena can be accounted for in a system with localized “strange scatterers” embedded into a Fermi liquid with a conventional quasiparticle description. Such scatterers could originate from “two-level systems” associated with an electronic glassy state with short-range charge-order correlations.

INTRODUCTION

A variety of the observed properties of optimally and overdoped cuprates, both at temperatures in some range above $T_c$ and, with superconductivity suppressed by a high magnetic field, in the low-temperature ($T \to 0$) limit, appear to be inconsistent with any quasiparticle description. One of the most prominent of these properties is a $T$-linear term in the resistivity [1–3],

$$\rho(T) = \rho_0 + AT + \ldots,$$

with $A > 0$ (and $\ldots$ signifying terms proportional to higher powers of $T$). In principle, such a linear-in-$T$ dependence of the resistivity can result from electron-phonon scattering at temperatures larger than the Debye frequency, $\omega_D$ [4]. However, $T$-linear resistivity is also observed at temperatures significantly lower than $\omega_D$. The coefficient $A$ is nonzero (i.e., the resistivity is linear in $T$ at the lowest temperatures) over a range of doping, typically extending from optimal doping to the vicinity of the end of the superconducting dome [2, 5], as shown schematically in Fig. 1.

Moreover, seminal angle-resolved photoemission (ARPES) data [6–11] suggested that the imaginary part of the electron self-energy—at least near optimal doping—is linear in $\omega$ for $\omega > T$, a behavior that is not expected in Fermi liquid theory. In addition, the frequency-dependent conductivity exhibits an unusually large spectral weight at frequencies much larger than the scattering rate extracted from the DC resistivity [12, 13]. Stimulated by these observations, a variety of interesting “exotic” proposals have been explored theoretically, based on the proposition that no quasiparticle-based theory can account for the anomalous aspects of the observed phenomena.

On the other hand, many observed properties imply or strongly suggest the existence of quasiparticles, including quantum oscillations that are well accounted for by the Lifshitz-Kosevitch formula [14–17], the fact that the Wiedemann-Franz law is obeyed at sufficiently low temperature [18–20], and magnetotransport that can be accounted for on the basis of Boltzmann transport theory in great detail [21–23]. Furthermore, at least in somewhat overdoped cuprates ($p > p^*$), in the same doping range where the resistivity is linear in $T$ at low temperature, ARPES measures a Fermi surface that is strikingly similar to that obtained from density functional theory [24–26]. Moreover, with increased resolution, it has become clear that the self-energy measured in ARPES is much more structured than was initially appreciated, exhibiting conventional features of strong electron-phonon cou-
pling. At the very least, this makes inferences concerning the nonexistence of quasiparticles more complicated [27–30], particularly at relevant energy scales—viz. at frequencies of order the temperatures relevant to the strange metal—which are often too small to resolve in ARPES.

Here, we explore the possibility that the salient features of the cuprate strange-metal phenomenology can be understood from a conventional Fermi liquid perspective. We construct a model that consists of a fluid of weakly-interacting quasiparticles interacting with a set of localized “strange scatterers.” In our model, the strange scatterers are represented by a set of random, spatially localized two-level systems (TLSs), which we treat in the same manner as in the theory of TLSs in structural glasses [31, 32]. In the context of the cuprates, we imagine that these TLSs originate from an electronic glass, presumably associated with the existence of local charge-density-wave (CDW) or nematic glass order [33–35]. Evidence of short-range CDW and nematic order in the cuprates is manifold [36–50].

It is important to stress that we are dealing exclusively with the physics of the regime identified as “strange metal” in Fig. 1, where (with superconductivity suppressed) the resistivity is linear in temperature, \( h/e^2 \) per Cu-O plane. Our analysis does not extend to the “bad metal” regime.

**THE MODEL**

We consider the following Hamiltonian:

\[
H = H_e + H_{\text{TLS}},
\]

(2)

where \( H_e \) is the Hamiltonian of free fermions on a lattice with an appropriate simple band dispersion \( \varepsilon(\vec{k}) \), and

\[
H_{\text{TLS}} = \sum_j \hat{h}_j \cdot \vec{\tau}_j + \sum_j V_j [1 + \vec{g}_j \cdot \vec{\tau}_j] (\hat{n}_j - \bar{n}_j),
\]

(3)

where the sums are over the (assumed random) positions of TLSs with concentration \( c \), \( \hat{n}_j - \bar{n}_j \) is the deviation from the average fermion density at site \( j \), and \( \vec{\tau}_j \) with \( a = x, y, z \) are pseudo-spin-1/2 Pauli operators representing the TLS degree of freedom at site \( j \). The operators \( \vec{\tau}_j \) and \( \tau_j^a \) are even under time reversal and \( \tau_j^0 \) is odd. It is important to stress that the two states with \( \tau_j^z = \pm 1 \) represent two distinct configurations of a localized electronic mode and are not related by time-reversal symmetry. Here \( \vec{h}_j \), \( V_j \) and \( \vec{g}_j \) are random variables, whose distributions and physical significance will be discussed below and in Appendix A. Time-reversal symmetry requires \( \vec{h}_j^0 = \vec{g}_j^0 = 0 \). We will use the freedom to define a local \( z \)-axis at each site to set \( \vec{h}_j = h_j \hat{z} \) with \( h_j \geq 0 \).

The main non-trivial assumption we will make is that \( N(\hat{h}) \), the probability density for \( |\hat{h}_j| \), is non-vanishing as \( |\hat{h}| \to 0 \) and remains substantial over an energy window of width \( W \) that is large compared to the relevant temperatures. Here, we are motivated by the success of the analogous assumption made in Refs. [31, 51] to explain the experimental data in dielectric structural glasses. (The existence of TLSs with a non-vanishing density of states at low energies has been confirmed in numerous experiments in structural glasses [32, 52]?)

In order to make analytical progress, we analyze the case where couplings \( |V_j| \) are assumed to be small so they can be treated perturbatively. At the end of the paper, where we make qualitative contact with experiments in the cuprates, we will discuss extrapolating the results to intermediate values of \( |V_j| \).

The physical picture underlying this model is based on an analogy with the theory of structural glasses [31, 51]. For a given TLS, \( \tau^z = \pm 1 \) labels a pair of electronic states localized at two nearby local potential minima of a complicated multi-electron potential landscape. The potential energy along a one-dimensional cut through this landscape—corresponding to the path of the semiclassical trajectory that dominates tunneling processes—is illustrated schematically in Fig. 2. In a “natural” basis for the TLS, the energy difference between the states in each well corresponds to the parameter \( h^2 = (E_1 - E_2)/2 \), while \( h^z = t \) is the tunneling matrix element connecting the two states. (After rotation to the basis that locally diagonalizes the first term in Eq. (3), \( \vec{h} \to \hat{z} (\tau^z/2) \sqrt{(E_1 - E_2)^2 + 4t^2} \). Small amplitude fluctuations of the local density, \( n \), away from \( \bar{n} \), modify both these energies, as well as the average energy \( E = (E_1 + E_2)/2 \), by an amount proportional to \((\bar{n} - \bar{n})\). We have parameterized these changes by an overall magnitude, \( V \), and the dimensionless vector \( \vec{g} \) that encodes the relative changes in these quantities.

![FIG. 2. Schematic illustration of a two-level system. The potential along a collective coordinate of several electrons has two minima. The states localized near these minima, whose wavefunctions as a function of the collective coordinate are illustrated in orange and blue, are labelled by \( \tau_j = \pm 1 \). \( 2h_j^z \) is the energy difference between the two states, and \( h_j^z \) is the tunneling matrix element between them.](image-url)
RESULTS

We consider two distinct mechanisms corresponding to different limiting cases of our model that can produce a linear-in-$T$ scattering rate. Each mechanism involves different microscopic assumptions. The first applies in the limit in which the TLSs are essentially non-dynamical and hence the scattering of electrons from them is quasi-elastic. In this limit, $g_j^2$ is the dominant coupling in Eq. (3). In the second, the impurities have significant quantum dynamics such that the $T$-linear part of the scattering rate is predominantly inelastic (i.e. $g_j^2$ is the dominant coupling). Within our perturbative analysis, interpolating between these two limits is straightforward, and one may consider a situation where both the elastic and the inelastic contributions to the $T$-linear scattering rate are significant.

A. Linear-in-temperature resistivity

One can readily compute the scattering rate of the itinerant electrons perturbatively in $V_j$. (See Appendix A for details.) The imaginary part of the electron self-energy is given by

$$\Sigma''(\omega) = \Sigma''_{el}(\omega) + \Sigma''_{in}(\omega),$$

where $el$ and $in$ refer to the elastic and inelastic contributions, respectively. The elastic contribution is

$$\Sigma''_{el}(\omega) = -\pi e V_j^2 \nu(E_F) \left\{ (1 - g_j^2)^2 + 2g_j^2 \ln(2)N(0)T + \mathcal{O}(T^3) \right\},$$

where $\nu(E_F)$ is the density of electronic states at the Fermi level, and $(\bar{\gamma})$ denotes appropriate disorder averages (see Appendix A); in particular $\bar{g}_j^2$ in the $T$-linear term is the average of $g_j^2$ over the TLSs with $h_j$ near 0. (For simplicity, we have assumed that $V_j$ is independent of $h_j$ and $\bar{g}_j$, so that the averages factorize.) Note that $\Sigma_{el}$ is independent of frequency, reflecting its elastic nature. The real part of the self-energy, $\Sigma'_{el}$, results in an unimportant shift of the chemical potential and will be neglected.

The inelastic contribution to the imaginary part of the self-energy is

$$\Sigma''_{in}(\omega) = -\pi e V_j^2 \nu(E_F) \left\{ \left(\frac{g_j}{2T}\right)^2 \left[ N(0) \frac{\omega}{\gamma} \coth \left( \frac{\omega}{2T} \right) \right] + \mathcal{O}(T^3) \right\},$$

where $(\bar{g}_j^2)^2$ is the average computed for TLSs with $h_j \approx \omega/2$. In this case, $\Sigma_{in}$ is frequency dependent. In fact, it exhibits the same $\omega/T$ scaling as in the “marginal Fermi liquid” [53], but from a very different physical origin—unrelated to any quantum criticality. The real part of $\Sigma_{in}$ can be obtained by a Kramers-Kronig transformation and has an $\omega \log(\omega)$ behavior at low frequency, leading to a logarithmic enhancement of the effective mass and a suppression of the quasiparticle weight at low energies.

We now analyze the two limiting cases discussed above:

- **Case 1: Predominantly elastic scattering.** In this case, the TLSs are slow on electronic time scales and the scattering of electrons from them is quasi-elastic. The $T$-linear dependence of the scattering rate originates from the difference in the scattering cross section from TLSs in the ground and excited states, and the $T$-linear dependence of the concentration of TLSs in their excited state. In order to explain the slope of the measured $T$-linear resistivity, $A > 0$, we must assume that on average, the excited states of the TLSs have a larger cross section than the ground states, $\bar{g}^2_j > 0$.

- **Case 2: Predominantly inelastic scattering.** In this case, the TLSs are dynamical. The inelastic scattering comes from the $x$ component of $\vec{g}_j$. This contribution gives a $T$-linear scattering rate with $A > 0$ without any strong assumptions on the distribution of the $g_j^2$’s. The $T$-linear dependence of the scattering rate originates from the linear growth in phase space available for inelastic scattering with increasing temperature.

Since our model has $\vec{k}$-independent scattering, the transport scattering rate is simply related to the single-particle rate. As a result, the resistivity $\rho(T) = -2\Sigma''(\omega = 0, T)/[\pi^2 v(E_F) v_F^2]$ is of the form of Eq. (1), with a slope

$$A = \frac{2\pi}{e^2 v_F} c V_j^2 N(0) \left[ 2\bar{g}_j^2 \ln(2) + (\bar{g}_j^2)^2 \right].$$

(The fact that inelastic scattering of electrons from TLSs can lead to linear-in-$T$ resistivity was pointed out in the context of metallic glasses in Ref. [54].)

B. Anomalous contribution to the optical conductivity

Within our perturbative approach, the optical conductivity can be written as a sum of a contribution from the itinerant electrons (which we will identify as the “Drude” component, specified by a subscript $D$) and a contribution from the absorption of radiation by the TLSs:

$$\sigma(\omega) = \sigma_D(\omega) + \sigma_{TLS}(\omega),$$

where the Drude part of the conductivity is written as

$$\sigma_D(\omega) = e^2 \nu(E_F) v_F^2 \text{Re} \left[ \frac{1}{i\omega + \Gamma(\omega)} \right].$$

In the simple case where the itinerant electron dispersion is circularly symmetric, $\Gamma(\omega) = 2\Sigma(\omega)$. 
In addition to their effect on the Drude part of the conductivity through the electron self-energy, the TLSs also have a direct contribution to the conductivity, $\sigma_{\text{TLS}}$. Since the TLSs are localized, they make no contribution to the DC conductivity. They do, however, contribute dissipation at finite frequency. To compute $\sigma_{\text{TLS}}$, we consider the direct (dipolar) coupling of the TLSs to an external electric field, which is generically of the form

$$H \rightarrow H + \sum_j \vec{E}(\vec{r}_j) \cdot \left[ \vec{d}_j \cdot \vec{\tau}_j^x + \vec{\epsilon}_j \cdot \vec{\tau}_j^z \right], \quad (10)$$

where $\vec{d}_j$ and $\vec{\epsilon}_j$ are random vectors determined by the size and shape of the TLS. Only the first term causes transitions. A term that couples the electric field to $\tau_j^y$ is forbidden by time-reversal symmetry.

Then, it is straightforward to compute the linear response to $\vec{E}$ as

$$\sigma_{\text{TLS}}(\omega, T) = 2\pi e |\vec{d}_j|^2 N(\omega/2) \omega \tanh(\omega/T). \quad (11)$$

At high frequencies, $\omega \gg T$, $\sigma_{\text{TLS}} \propto \omega$, whereas at $\omega \ll T$, $\sigma_{\text{TLS}} \propto \omega^2/T$. In either case, $\sigma_{\text{TLS}}$ vanishes as $\omega \rightarrow 0$. The width of $\sigma_{\text{TLS}}$ is set by $W$.

**DISCUSSION**

**Issues and additional considerations**

The model presented here is extremely simple, and to some extent reverse engineered to give the desired results. We now consider the various hidden assumptions and subtleties in the two limiting cases, and some of their ramifications.

**Issues for Case 1: non-dynamical TLSs.** It is generically the case that the scattering amplitude in the excited state of a TLS is different than that in its ground state ($g_j^2 \neq 0$). The crucial assumption we made in Case 1 is that $g_j^2 > 0$, i.e., the scattering cross section when the TLS is in its excited state is (on average) larger than the cross section in the ground state. It is not unnatural to suppose that there is a tendency for the excited state to be the stronger scatterer. However, for the important TLSs with small $h_j$, one might expect that the difference in the scattering strengths would be proportionately small on average. This would, however, result in a non-linear $T$ dependence of the scattering rate.

The assumption that $\vec{h}_j$ and $\vec{g}_j$ are parallel encodes the assumption that the TLSs are non-dynamical. Physically, this is reasonable if the TLSs are “heavy”, i.e., if the two states differ by a collective rearrangement of several electrons, or if they are coupled to lattice deformations, such that the tunneling rate between the two states is slow on itinerant electron time scales. However, by the same token, the matrix element induced by an external electric field between the two states is also small, suppressing the contribution of the TLSs to the optical conductivity.

The TLSs may still contribute significantly to the optical conductivity if the tunneling barrier between the two states is uniformly distributed [31]. Since the tunneling rate between the two states depends exponentially on the barrier, this implies that before the rotation performed on Eq. (3) (that orients $\vec{h}_j$ along $\hat{z}$), the probability distribution of $h_j^2$ is proportional to $1/h_j^2$ over a range of $h_j^2$, down to a minimal value corresponding to the maximum tunneling barrier. Since the coupling $\vec{d}_j$ in Eq. (10) is also associated with tunneling between the two levels, we expect its magnitude to be correlated with that of $h_j^2$.

As a result, there are many TLSs with large barriers and negligible tunneling rates that dominate the low-energy properties (such as the specific heat and the DC resistivity), whereas the higher-energy TLSs with smaller tunneling barriers (and hence also substantial values of $|\vec{d}_j|$) control the response at higher frequencies. (Needless to say, the corresponding expression for the optical response would be somewhat modified in this case, in a way that can be represented by an effectively $\omega$-dependent value of $|\vec{d}_j|^2$.)

**Issues for Case 2: dynamical TLSs.** In this case, no special assumption needs to be made on the distribution of $\vec{g}_j$ in order for the (inelastic) scattering rate to grow linearly with $T$. However, since the TLSs have quantum dynamics induced by the electrons (parameterized by $g_j^2$), it is natural to assume that they also have intrinsic quantum dynamics: before moving to a locally aligned basis, $\vec{h}_j$ is determined by two roughly independently distributed random variables, $h_j^z$ and $h_j^x$. Then, it is not easy to explain why the density of states $N(|\vec{h}_j|)$ is constant as $|\vec{h}_j| \rightarrow 0$.

**Issues for both models.** In order for the scattering from TLSs to be large enough to account for the slope of $\rho(T)$ observed experimentally near optimal doping in the cuprates, the TLSs must be relatively dense and their coupling to the itinerant electrons must be sufficiently strong. (We elaborate on the required coupling strength below). In other words, the dimensionless coupling strength $D \equiv N(0) |\nabla|^2 \nu(E_F)$ cannot be small. This implies that effects beyond our perturbative treatment are important. Specifically, induced interactions (essentially, RKKY interactions) will be generated between TLSs so that they cannot be treated independently. For finite $D$, integrating out the itinerant electrons leaves us a version of the random field and random bond Ising model. Moreover, in Case 2, the TLSs become broadened due to their coupling to the itinerant electrons. However, since this is itself a glassy system, it may be possible to reinterpret the two-level systems that we introduced phenomenologically as being renormalized entities, where $N(h)$ effectively incorporates the effects of the induced interactions.
In addition, the form we have chosen for the coupling between the itinerant electrons and the TLSs in Eq. (3) is not the most generic form allowed by symmetry. A more general form would include a variety of “charge-Kondo” couplings, $H_{TLS} \rightarrow \sum_j \left[ h_j \tau_j^+ + V^j \hat{O}_j^+ + \sum_a (\bar{V}_{ja}^2 \cdot \tau_a^+ \hat{O}_j^+ \right]$, where $\hat{O}_j^+$ are (in the most general case, three) non-commuting fermion bilinear operators of some appropriate sort. (Interestingly, the quantum critical character of the two-channel Kondo problem has previously been invoked as a possible source of strange-metal behavior in the cuprates [55].)

**Experimental considerations**

Clearly, our model is too simple to be directly applied to experiments in the cuprates. Nevertheless, it offers a physical picture that naturally captures many salient experimental features. We now discuss in more detail various experimental considerations regarding the applicability of this physical picture, as well as constraints imposed by experiments. (See Appendix B for further discussion.)

**$T$-linear resistivity.** Under the assumptions stated above, both Case 1 and Case 2 produce $T$-linear resistivity in the $T \to 0$ limit and therefore can account for strange metal behaviour at low temperature, without invoking a non-quasiparticle picture. Case 1 even produces a strange metal in the $T \to 0$ limit with purely elastic scattering.

One can then ask what resistivity these models produce at higher temperatures. This motivates a closer examination of the experimental data. While tradition holds that cuprates exhibit $T$-linear resistivity from low temperature up to the melting point at a certain critical doping, the experimental reality is more complicated. For example, Nd-LSCO at $p = 0.24$ and Bi2212 at $p = 0.23$—both perfectly $T$-linear at low temperature—show relatively strong upturns in their resistivity above $T \approx 100K$ [56]. The same is true in the electron-doped cuprates [57]. LSCO, which has the largest range of $T$-linear resistivity, is only truly linear up to around 200K, and then changes slope [58, 59].

These different behaviors of $\rho(T)$ at high temperatures defy the simple picture of linear resistivity with a single slope over a broad temperature range. Instead, we emphasize that the common feature on which we have focused is a $T$-linear resistivity at low temperature with a doping-dependent slope $A$, from optimal doping to the vicinity of the overdoped end of the superconducting dome. It is this feature that can be explained within the TLS model. The $T$-linear resistivity at higher temperatures—in the “bad metal” regime [60] where it exceeds the Mott-Ioffe-Regel limit—presumably cannot be readily understood in the context of Boltzmann transport theory. Since we are effectively invoking Boltzmann transport in our picture of the resistivity in the $T \to 0$ limit, we are implicitly assuming that there is a crossover of some sort separating the two regimes.

Even restricting attention to low temperatures, there can appear to be an impressive dynamical range of $T$-linear resistivity in cases where the extrapolated $T \to 0$ resistivity, $\tilde{\rho}(0)$, is particularly small. For instance, this occurs in some samples of near-optimally doped LSCO, and has been widely interpreted as evidence that the behavior is produced by a quantum critical point of some sort. However, the extrapolation is problematic. In the first place, in these same samples, when superconductivity is suppressed with a magnetic field, the actual $T = 0$ resistivity is always comparable to (or even larger than) the zero field resistivity at $T_c$ [61]. Moreover, although fully-oxygenated YBCO has $T$-linear resistivity from 200K down to $T_c = 88K$, it has a substantially negative extrapolated $\tilde{\rho}(0)$ [62]. If $\tilde{\rho}(0)$ is negative for one range of parameters and positive for another, there can occur conditions in which it is close to zero without this signifying anything special. (As an aside, the negative value of $\tilde{\rho}(0)$ in fully-oxygenated YBCO implies that its extrapolated resistivity cannot remain $T$-linear to $T \to 0$; the fact that this is the “cleanest” cuprate is suggestive that a certain level of elastic disorder may be required for low temperature $T$-linear resistivity. [63])

**Optical conductivity.** There has long been an unresolved issue of interpretation of optical conductivity in the cuprates: either in terms of an “extended Drude” model (i.e. from itinerant electrons only, with an effective scattering rate $\Gamma(\omega)$ that is a strongly increasing function of $\omega$), or in terms of a more conventional Drude component plus a localized “mid-infrared” component. TLSs, both in Case 1 and in Case 2, naturally produce a two-component response—one from the itinerant electrons interacting with the TLSs, and one from the TLSs themselves.

There are experimental reasons to favour a two-component analysis. Firstly, in many underdoped cuprates, the optical conductivity is a non-monotonic function of $\omega$, even above $T_c$ [64]. The conductivity is maximal at $\omega = 0$, drops to a local minimum at $\omega = \omega_{\text{min}} \sim 0.1 \text{ eV}$, and then increases somewhat over a range of $\omega$ before dropping slowly with further increasing $\omega$ up to frequencies of order the charge transfer gap, $\omega \sim 1.5 \text{ eV}$. Such non-monotonic behavior is natural for a two-component response but highly unnatural in a single-component response. Secondly, a large fraction of the optical spectral weight fails to condense even at temperatures far below $T_c$, as would be expected if it reflects absorption by localized modes such as TLSs. The same behavior is also expected in a one-component “dirty” superconductor, but there is ample evidence from the superfluid density, field-dependent heat capacity, ARPES, and theoretical considerations that cuprates are not in the dirty limit. [65, 66].

A specific prediction of our model is that the optical response is spatially inhomogeneous and inherently nonlinear. Optical analogues of NMR methods used to dis-
tangible homogeneous from inhomogeneous broadening of spectral features (e.g., hole burning) could, in principle, unambiguously distinguish our model from homogeneous, non-Fermi-liquid models of strange metals.

**Specific heat.** Another consequence of our model is that the TLSs contribute a $T$-linear term in the specific heat:

$$C_{\text{TLS}}(T) = (\pi^2/12) c N(0) T + \mathcal{O}(T^3).$$

A similar $T$-linear specific heat term motivated the TLS model in insulating glasses [31].

There are other potential $T$-linear contributions to the specific heat, including from itinerant electrons for $T > T_c$, and from nodal quasiparticles in the presence of disorder for $T < T_c$. However, the measured specific heat in the superconducting ($B = 0$) state is an upper bound on any putative density of TLSs, and so the present proposal would be falsified if strange-metal behavior were observed in any cuprate with a vanishing $C/T$. Note that a $T$-linear specific heat has been observed in the superconducting state over a broad doping range in the cuprates [67–69], even far below $T_c$ where one might expect the quasiparticle contribution to be highly suppressed.

We note also that a logarithmic enhancement of the specific heat, $C/T \propto \log(1/T)$, has been reported in the normal state of several overdoped cuprates with $T$-linear resistivity [70, 71]. It is unclear whether this logarithmic behavior persists down to the lowest temperatures over a finite range of doping, or is instead confined to a special family of materials and/or a specific doping value. A $C/T \sim \log(1/T)$ behavior is predicted within our model in the presence of substantial inelastic scattering (as in Case 2 above) due to the logarithmic enhancement of the effective quasiparticle mass.

**Momentum-resolved electron energy loss spectroscopy.** The density-density response function, $\chi(\vec{k}, \omega) \sim k^2 \sigma(\vec{k}, \omega)/\omega$, measurable e.g. in electron energy loss spectroscopy experiments [72], can be related to the $\vec{k}$- and $\omega$-dependent (complex) conductivity; within our model, the contribution of the itinerant electrons to this response has the usual $\vec{k}$ dependence expected of a Fermi liquid, but the contribution of the TLSs is approximately $\vec{k}$-independent. An unexpectedly prominent, largely $\vec{k}$-independent contribution to $\chi$ has, indeed, been recently observed experimentally [72].

**Quasiparticle transport.** Boltzmann transport theory has recently been used to fit angle-dependent magnetoresistance data in the strange metal regime of the cuprate Nd-LSCO [21, 22]. The success of this approach is hard to reconcile with any model of the strange metal that involves sufficiently strong deviations from Fermi-liquid theory to be incompatible with Boltzmann transport. Our model naturally provides a justification for such modelling: at least in the limit of weak coupling, the $T$-linear resistivity only arises because the scatterers are “strange”—the conduction electrons are otherwise conventional.

The validity of a quasiparticle picture has implications for other “anomalous” transport coefficients in the cuprates, including the Hall coefficient and the field dependence of the magnetoresistance [23]. The unusual temperature dependence of the Hall angle has been taken as evidence for separate mechanisms of longitudinal and transverse resistivity [73]. Similarly, the behaviour of the longitudinal magnetoresistance, which crosses over from $B^2$ behaviour in the low-field limit to an extended region of $B$-linear magnetoresistance, has been interpreted in terms of unconventional $B/T$ scaling that goes beyond any conventional quasiparticle description [61, 74].

Both the Hall effect and the magnetoresistance have, however, been shown to be consistent with Boltzmann transport of well-defined quasiparticles once the anisotropy of the Fermi surface and the tendency for forward scattering are taken into account [21, 23, 75–77]. The strong anisotropy of both the Fermi surface and of impurity-induced elastic scattering produces an extended region of $B$-linear magnetoresistance [21, 23], while the isotropic ($k$-independent) $T$-linear scattering rate induced by the TLSs serves to “dilute” the scattering rate anisotropy, producing a crossover in the magnetoresistance to $B^2$ at high temperature. The same Boltzmann calculations also reproduce both the sign and magnitude of the Hall coefficient [21]. The anisotropic impurity scattering used in these calculations indicates a tendency for forward scattering [75]: this could reflect the influence of additional, elastic impurities situated away from the copper oxide planes [76].

**Estimating scattering strength of TLSs.** Since $\rho(T) \propto \Sigma''(T)$, scattering from TLSs contributes both to the $T$-linear and the residual ($T$-independent) scattering, with the ratio of the two—as can be seen in Eqs. (5) and (6)—determined by the values of $\tilde{g}_j$. To characterize the magnitude of the $T$-linear term in the resistivity, it is useful to define the dynamical range

$$\alpha = [\rho(T_{\text{max}}) - \rho_0]/\rho_0,$$

where $\rho_0$ is the residual resistivity, and $T_{\text{max}}$ is the maximal temperature at which our analysis still holds and the resistivity is approximately $T$-linear (corresponding to the “strange metal” regime in Fig. 1). We focus on cases where superconductivity can be suppressed to zero by a combination of a magnetic field and overdoping, such that $\rho(T)$ is linear in $T$ between $T_{\text{max}}$ and the lowest measured temperatures, and $\rho_0$ can be determined directly without having to extrapolate down from $T_c$ to $T \to 0$. Importantly, our analysis relies on the validity of Boltzmann transport theory, and therefore $T_{\text{max}}$ must be smaller than the temperature at which the resistivity approaches the Mott-Ioffe-Regel limit. The precise numerical definition of the Mott-Ioffe-Regel limit is somewhat arbitrary, but for concreteness, we use the criterion $k_F \ell = 2\pi$, where $\ell$ is the mean free path. This translates to a three-dimensional resistivity of $\rho(T_{\text{max}}) \lesssim 250 \mu\Omega \text{cm}$. (Here, we have used the inter-layer
spacing of a typical cuprate, $d \approx 6\text{Å}$, and the fact that the
two-dimensional resistivity is related to the mean free path by
$\rho_{2D} = h / (e^2 k_F l)$.

As an example, we discuss Nd-LSCO at $p = 0.24$,
where upon suppressing superconductivity with a field, the resistivity is linear down to a remarkably low temperature of a few degrees Kelvin [1]. As mentioned above, the $T$-linear regime extends up to about $T \approx 100K$, where there is a noticeable increase of slope. At $T \approx 100K$ the resistivity is about $75\mu\Omega cm$, still below the Mott-Ioffe-Regel limit. Hence, for Nd-LSCO at $p = 0.24$, we identify $T_{\text{max}} \approx 100K$. From the data of Ref. [1], $\rho_0 \approx 25\mu\Omega cm$, giving $\alpha \sim 2$. To obtain similar values of $\alpha$ in our model [see Eqs. (5), (6) and (7)], we need to factor $\mathcal{N}(0)/T_{\text{max}} \sim T_{\text{max}}/W$. This factor is intrinsically small; roughly estimating $W$ from the width of $\sigma(\omega)$, we obtain $T_{\text{max}}/W \sim 0.1$. Using this value, in Case 1 we need $g_{ij}^z \sim 0.75$ (corresponding to the scattering amplitude of electrons from the excited state of a typical TLS being larger than the amplitude in the ground state by a factor of $\sim 7$), where we have taken for simplicity $g_{ij}^z$ to be a constant, independent of $j$. In Case 2 we need $[\langle g_{ij}^z \rangle^2]^{1/2} \sim 5$.

A comparison of the experimental magnitude of $d\rho/dT$ with our model is given in Appendix B.

CONCLUSION

As far as the physics of the itinerant electrons is concerned, the model we have introduced is in no way exceptional. It is only the TLSs that are in any way strange. In limiting Case 1 of our model, although the scattering rate is linear in $T$, the scattering is elastic—or quasi-elastic if dynamical terms are included perturbatively. The energy of the quasiparticles is well defined, and hence the relaxation rate (and with it the equilibration rate) is parametrically smaller than $T$—much the same way as it is for the electron-phonon problem at temperatures above the Debye temperature. To the extent that the notion of a “Planckian” relaxation rate is a statement about equilibration rates, the proposed mechanism is highly sub-Planckian.

In Case 2, the scattering is not elastic, and there is a form of $\omega/T$ scaling that governs the TLS contribution to the electron self-energy and (correspondingly) the Drude piece of the conductivity. As already mentioned, this is not a reflection of quantum critical behavior, but rather is a manifestation of glass physics. Moreover, in the perturbative regime where our analysis is well controlled, the scattering rate is still small (by a prefactor) compared to the Planckian rate, $k_B T / h$. It does not violate any putative Planckian bound, and indeed it more or less saturates it if extrapolated to the largest values of $c$ and $V_j$ for which the perturbative treatment has a chance of being applicable. On the other hand, so long as the rate is small compared to $k_B T / h$ (i.e. everywhere our results are controlled), there are still well-defined quasiparticles; if not in the asymptotic Landau sense, at least in the sense that the excitation energies are large compared to the decay rate. [78]

That such a simple model can account for many of the salient observed properties of the strange metal phase in the cuprates seems to us significant. At least the present results raise questions concerning the necessity of more radical approaches. On the other hand, this does not mean that the strange-metal behavior is unimportant. The fact, reflected in the schematic phase diagram in Fig. 1, that $A$ decreases with increasing $x$ and only vanishes for values of $x$ somewhere near the overdoped edge of the superconducting dome [5], implies an intimate—still to be explored—relation between the strange metal and high temperature superconductivity [79].

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**Appendix A: Details of the model and solution**

After rotation to a local basis for each TLS in which \( \hat{h}_j = h_j \hat{z} \) with \( h_j \geq 0 \), the Hamiltonian of our model is

\[
H = \sum_{h\sigma} \varepsilon(h) c_{h\sigma}^\dagger c_{h\sigma} + \sum_j h_j \hat{r}_j^z + \sum_j V_j (1 + g_j^z \hat{r}_j^z + g_j^x \hat{r}_j^x) (\hat{n}_j - \bar{n}_j). \tag{A1}
\]

Here

\[
\hat{n}_j = \frac{1}{\Omega} \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} \tag{A2}
\]

and \( \Omega \) is the volume of the system. The sums over \( j \) run from 1 to \( N_{\text{TLS}} \), and the overall concentration of TLSs is \( c = N_{\text{TLS}}/\Omega \).

The Hamiltonian contains five sets of random variables: the TLS energies \( h_j \), coupling constants \( V_j, g_j^z, g_j^x \), and positions \( r_j \). Let \( P(h_j, V_j, g_j^z, g_j^x, r_j) \) denote the joint probability density of these variables. We assume that the properties of each TLS are independent and identically distributed (i.i.d.):

\[
P(h_j, V_j, g_j^z, g_j^x, r_j) = \prod_j p(h_j, V_j, g_j^z, g_j^x, r_j). \tag{A3}
\]

The positions \( r_j \) are taken to be uniformly random over the system. For simplicity, we assume that \( V_j, g_j^z \), and \( g_j^x \) are mutually independent, but we do allow their distributions to depend on \( h_j \):

\[
p(h, V, g^z, g^x, r) = \frac{1}{\Omega} N(h) p_c(V|h) p_c(g^z|h) p_c(g^x|h). \tag{A4}
\]

Here \( 1/\Omega \) is the probability density of \( r, N(h) \) is the probability density of \( h \), and \( p_c(V|h) \) is the conditional probability density of \( V \) given \( h \), etc.

We denote conditional averages as \( \langle \ldots \rangle(h) \) and unconditional averages as \( \langle \ldots \rangle \), e.g.

\[
\bar{V}^2(h) = \int_{-\infty}^{\infty} dV p_c(V|h) V^2, \tag{A5}
\]

\[
\bar{V}^2 = \int_0^\infty dh N(h) \bar{V}^2(h). \tag{A6}
\]
The key assumptions needed to obtain $T$-linear resistivity in our model are that $N(h)$, $\bar{V}^2(h)$, and either $\bar{g}^2(h)$ (in Case 1) or $\langle g^2 \rangle^2(h)$ (in Case 2) approach positive $O(1)$ values as $h \to 0$. We also assume that $dN/dh(0) = 0$; this follows from regularity of the distribution $N(h)$ before the rotation to a local $z$-basis for each TLS. Similarly, it is natural to assume $dV^2/dh(0) = 0$.

1. TLS heat capacity

Since our entire approach is perturbative, the specific heat is dominated by a zeroth order ($V_j = 0$) contribution consisting of a sum of contributions from the itinerant electrons and the TLSs. At a temperature $T = 1/\beta$, the average energy of the TLSs is

$$E_{TLS}(T) = \sum_j h_j \langle \tau_j^z \rangle = -\sum_j h_j \tanh(\beta h_j). \quad (A7)$$

The TLS contribution to the specific heat is therefore

$$C_{TLS} = \frac{1}{\Omega} \frac{dE_{TLS}}{dT} = \frac{1}{\Omega} \sum_j (\beta h_j)^2 \text{sech}^2(\beta h_j). \quad (A8)$$

This expression is clearly self-averaging. In terms of the overall concentration of TLSs, $c$, and the probability density of the TLS energies, $N(h)$,

$$C_{TLS} = c \int_0^\infty dh \; N(h) \; (\beta h)^2 \text{sech}^2(\beta h). \quad (A9)$$

The hyperbolic factor cuts off the integral at $h \sim T$. In the low $T$ limit, we can Taylor expand $N(h) = N(0) + N'(0)h + \ldots$ and integrate term-by-term. Noting that $N'(0) = 0$, we obtain Eq. (12) of the main text.

2. Electron self energy

We compute the electron self energy via standard finite-temperature perturbation theory in $V_j$. The calculation is identical to the textbook treatment of impurity scattering, the only difference being that our impurities (TLSs) have dynamics. To lowest nontrivial (i.e. second) order in $V_j$, after averaging over TLS positions, we obtain

$$\text{Im} \Sigma'^R(k,\omega) = \Sigma''_{cl}(\omega) + \Sigma''_{in}(\omega), \quad (A10)$$

with

$$\Sigma''_{cl}(\omega) = -\pi \nu(\omega) \frac{1}{\Omega} \sum_j V_j^2 \left\{ (1 + g_j^2)^2 p_j + (1 - g_j^2)(1 - p_j) \right\}, \quad (A11)$$

$$\Sigma''_{in}(\omega) = -\frac{\pi}{\Omega} \sum_j V_j^2 \left\{ \nu(\omega + 2h_j)[1 - f(\omega + 2h_j)] p_j + \nu(\omega - 2h_j)f(\omega - 2h_j) p_j + \nu(\omega + 2h_j)f(\omega + 2h_j)(1 - p_j) + \nu(\omega - 2h_j)[1 - f(\omega - 2h_j)](1 - p_j) \right\}. \quad (A12)$$

Here $f(\varepsilon) = (e^{\beta\varepsilon} + 1)^{-1}$ is the Fermi function, $\nu(\varepsilon)$ is the electronic density of states, and

$$p_j = \frac{1}{2} [1 - \tanh(\beta h_j)] \quad (A13)$$

is the probability that the $j$th TLS is in its excited state. The expressions for $-2\Sigma''$ are of course identical to what one would write down using Fermi’s golden rule for the total scattering rate into and out of an electronic state with wavevector $k$ and energy $\omega$.

Assuming $\omega, T \ll E_F$, we can regard the density of states $\nu(\varepsilon)$ as a constant equal to its value at the Fermi energy, $\nu(E_F)$. Then the expressions simplify to

$$\Sigma''_{cl}(\omega) = -\pi \nu(E_F) \frac{1}{\Omega} \sum_j V_j^2 \left\{ (1 - g_j^2)^2 + 2g_j^2[1 - \tanh(\beta h_j)] \right\}, \quad (A14)$$

$$\Sigma''_{in}(\omega) = -\pi \nu(E_F) \frac{1}{\Omega} \sum_j V_j^2 (g_j^2)^2 \frac{1 + \cosh(\beta \omega)}{\cosh(2\beta h_j) + \cosh(\beta \omega)}. \quad (A15)$$

Again, the expressions are clearly self-averaging. In terms of $c$, $N(h)$, and the conditional and unconditional averages
discussed above,
\[
\Sigma'_\nu(\omega) = -\pi c \nu(E_F) \left\{ \frac{V^2}{2} \left( 1 - g_j^2 \right)^2 + 2 \int_0^\infty dh N(h) \frac{V^2(h)}{g^2(h)} \left[ 1 - \tanh(\beta h) \right] \right\},
\]
\[
\Sigma''_\nu(\omega) = -\pi c \nu(E_F) \int_0^\infty dh N(h) \frac{V^2(h)}{g^2(h)} \frac{1 + \cosh(\beta \omega)}{\cosh(2\beta h) + \cosh(\beta \omega)}.
\]

The hyperbolic functions cut off the integrals at \( h \sim \max(\omega, T) \). When these are sufficiently small, we can Taylor expand the various quantities about \( h = 0 \) and integrate term-by-term. Doing so yields Eqs. (5) and (6) of the main text.

**Appendix B: Bad vs. Strange Metal**

In this Appendix, we elaborate on the applicability of our model and its limitations in different regimes. The TLS model has the best chance of being applicable in the strange metal regime of Fig. 1, on the overdoped side and at low temperatures. In this regime, the low-temperature resistivity (measured by suppressing \( T_c \)) is well described by the form \( \rho(T) = \rho_0 + AT + BT^2 \), where \( B \) is weakly doping dependent and \( A \) decreases with increasing doping, vanishing near the end of the superconducting dome [5]. In particular, the resistivity per copper oxide plane in this regime is substantially smaller than \( h/e^2 \), such that a Boltzmann description may apply. As the end of the dome is approached, both \( A \) and the temperature extent of \( T \)-linear resistivity (up to \( T_{\text{max}} \sim A/B \)) become small, and hence a perturbative treatment of the linear-in-\( T \) scattering may be justified. (It is worth noting that a similar situation pertains in the organic superconductors of the TMTSF family, with pressure playing the role of hole doping [5].)

As optimal doping is approached, \( \partial \rho/\partial T \) at low temperature increases, and one has to worry about non-perturbative effects beyond our treatment (as discussed in the main text). Nevertheless, extrapolating our results to this regime, we may estimate the parameters needed to explain the observed \( \partial \rho/\partial T \). At optimal doping, \( \partial \rho/\partial T \) in many cuprates is of the order of 0.5 – 1 \( \mu \Omega \mathrm{cm} / \mathrm{K} \). Using a typical inter-plane distance of 6 Å, this translates into a slope of the 2D resistivity per CuO plane that can be expressed as \( \partial \rho_{2D}/\partial T = h/(e^2 T_0) \), with \( T_0 \approx 1500 – 3000 \mathrm{K} \). Using Eqs. (5), (6), and (7), we obtain:
\[
\frac{1}{T_0} = \frac{cV_j^2}{v_F^2} N(0) \left[ 2g_j^2 \ln(2) + \langle g_j^2 \rangle^2 \right].
\]

(Note that here, distances are measured in units of the lattice spacing, such that \( v_F \) has units of energy.) Assuming that both \( V_j^2/v_F^2 \) and the factor in the square brackets are of order one, and that the bandwidth of the TLS, \( W = 1/N(0) \), is of the order of 0.1 eV \( \sim 1000 \mathrm{K} \) [as is implied by the width of \( \sigma(\omega) \)], we find that the concentration of TLS, \( c \), has to be a sizable fraction of one per unit cell in order to explain the value of \( \partial \rho/\partial T \) near optimal doping. (Note that, in our discussion of the ratio \( \alpha = \Delta \rho/\rho_0 \), we concluded that the term in the square brackets in Eq. B1 needs to be somewhat larger than 1, which would imply that \( c \) can be somewhat smaller.)

Another important question is whether the processes that dominate the resistivity near optimal doping change substantially with increasing \( T \) in the range of \( T \) at which the resistivity crosses the Mott-Ioffe-Regel limit. As discussed in the main text, it is almost never the case that \( \partial \rho/\partial T \) at optimal doping is constant from \( T = 0 \) to the highest temperatures. Rather, there are subtle but noticeable changes in the slope. Further evidence for an intermediate-temperature crossover is provided by the low-frequency optical conductivity, which, in the same general range of temperatures, changes from having a maximum to a minimum at \( \omega = 0 \) as the temperature is raised. These observations are at least consistent with the proposition that the \( T \)-linear resistivity in the high-temperature bad metal regime may have a different origin than the low-\( T \) strange metal regime.

Indeed, as has been noted in several places [60, 82, 83], bad metal behavior has been seen in a high-temperature regime of a wide variety of strongly correlated materials, including not only the cuprates but also the Fe-based superconductors [84], the superconducting nickelates [85], several organic superconductors including alkali-doped \( \text{C}_{60} \) [86], the metallic ferromagnet \( \text{SrRuO}_3 \) [87], the quasi-1D charge-density-wave material TTF-TCNQ [88], and realizations of the Hubbard model in cold atoms systems [89]. Moreover, analytic and numerical investigations of the high temperature transport properties of a large variety of model systems with substantially different microscopic interactions have likewise yielded remarkably similar results, including studies of the Hubbard model [90 – 93], models of quantum critical nematic materials [94 – 96], and large-\( N \) generalizations of the electron-phonon problem [97]. The ubiquity of this behavior argues against its association with any particular low-temperature property of the materials in question.

To summarize, our model assumes that the origin of the \( T \)-linear resistivity in the strange metal and the bad metal regimes are different. We claim that this assumption is not inconsistent with the available data. A detailed understanding of the bad metal regime, and an explanation of the fact that near optimal doping the change in \( \partial \rho/\partial T \) between the two regimes is often relatively small, are interesting challenges beyond the scope of the current work.