Isomorphic groupoid $C^*$-algebras associated with different Haar systems

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Abstract

We shall consider a locally compact groupoid endowed with a Haar system $\nu$ and having proper orbit space. We shall associated to each appropriated cross section $\sigma : G^{(0)} \rightarrow G^F$ for $d_F : G^F \rightarrow G^{(0)}$ (where $F$ is a Borel subset of $G^{(0)}$ meeting each orbit exactly once) a $C^*$-algebra $M^*_\sigma (G, \nu)$. We shall prove that the $C^*$-algebras $M^*_\sigma (G, \nu_i)$ associated with different Haar systems are $*$-isomorphic.

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1 Introduction

The $C^*$-algebra of a locally compact groupoid was introduced by J. Renault in [8]. The construction extends the case of a group: the space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest $C^*$-norm making its representations continuous. In order to define the convolution on groupoid one needs to assume the existence of a Haar system which is an analogue of Haar measure on a group. Unlike the case for groups, Haar systems need not to be unique. A result of Paul Muhly, Jean Renault and Dana Williams establishes that the $C^*$-algebras of $G$ associated with two Haar systems are strongly Morita equivalent (Theorem 2.8/p. 10 [3]). If the groupoid is transitive they have proved that if $G$ is transitive then the $C^*$-algebra of $G$ is isomorphic to $C^* (H) \otimes K (L^2 (\mu))$, where $H$ is the isotropy group $G^u_\mu$ at any unit $u \in G^{(0)}$, $\mu$ is an essentially unique measure on $G^{(0)}$, $C^* (H)$ denotes the group $C^*$-algebra of $H$, and $K (L^2 (\mu))$ denotes the compact operators on $L^2 (\mu)$ (Theorem 3.1/p. 16 [3]). Therefore the $C^*$-algebras of a transitive groupoid $G$ associated with two Haar systems are $*$-isomorphic.

In [7] Arlan Ramsay and Martin E. Walter have associated to a locally compact groupoid $G$ a $C^*$-algebra denoted $M^* (G, \nu)$. They have considered the universal representation $\omega$ of $C^* (G, \nu)$ -the usual $C^*$-algebra associated to...

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a Haar system $\nu = \{\nu^u, u \in G^{(0)}\}$ (constructed as in [3]). Since every cyclic representation of $C^* (G, \nu)$ is the integrated form of a representation of $G$, it follows that $\omega$ can be also regarded as a representation of $B_c (G)$, the space of compactly supported Borel bounded function on $G$. Arlan Ramsay and Martin E. Walter have used the notation $M^* (G, \nu)$ for the operator norm closure of $\omega (B_c (G))$. Since $\omega$ is an $*$-isomorphism on $C^* (G, \nu)$, we can regarded $C^* (G, \nu)$ as a subalgebra of $M^* (G, \nu)$.

We shall assume that the orbit space of the groupoid $G$ is proper and we shall choose a Borel subset $F$ of $G^{(0)}$ meeting each orbit exactly once and such that $F \cap [K]$ has a compact closure for each compact subset $K$ of $G^{(0)}$. For each appropriated cross section $\sigma : G^{(0)} \rightarrow G^F$ for $d_F : G^F \rightarrow G^{(0)}$, $d_F (x) = d (x)$, we shall construct a $C^*$-algebra $M^*_\sigma (G, \nu)$ which can be viewed as a subalgebra of $M^* (G, \nu)$. If $\nu_1 = \{\nu_1^u, u \in G^{(0)}\}$ and $\nu_2 = \{\nu_2^u, u \in G^{(0)}\}$ are two Haar systems on $G$, we shall prove that the $C^*$-algebras $M^*_\sigma (G, \nu_1)$ and $M^*_\sigma (G, \nu_2)$ are $*$-isomorphic.

For a transitive (or more generally, a locally transitive) groupoid $G$ we shall prove that the $C^*$-algebras $C^* (G, \nu)$, $M^* (G, \nu)$ and $M^*_\sigma (G, \nu)$ coincide.

For a principal proper groupoid $G$, we shall prove that

$$C^* (G, \nu) \subset M^*_\sigma (G, \nu) \subset M^* (G, \nu).$$

Let $\pi : G^{(0)} \rightarrow G^{(0)}/G$ be the quotient map and let $\nu_i = \{\varepsilon_u \times \mu_i^u, u \in G^{(0)}\}$, $i = 1, 2$ be two Haar systems on the principal proper groupoid $G$. We shall also prove that if the Hilbert bundles determined by the systems of measures $\{\mu_i^u\}$ have continuous bases in the sense of Definition [24] then $*$-isomorphism between $M^*_\sigma (G, \nu_1)$ and $M^*_\sigma (G, \nu_2)$ can be restricted to a $*$-isomorphism between $C^* (G, \nu_1)$ and $C^* (G, \nu_2)$.

For establishing notation, we include some definitions that can be found in several places (e.g. [3, 4]). A groupoid is a set $G$ endowed with a product map

$$(x, y) \rightarrow xy : G^{(2)} \rightarrow G$$

where $G^{(2)}$ is a subset of $G \times G$ called the set of composable pairs, and an inverse map

$$x \rightarrow x^{-1} : G \rightarrow G$$

such that the following conditions hold:

1. If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.

2. $(x^{-1})^{-1} = x$ for all $x \in G$.

3. For all $x \in G$, $(x, x^{-1}) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx)x^{-1} = z$.

4. For all $x \in G$, $(x^{-1}, x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps $r$ and $d$ on $G$, defined by the formulae $r (x) = xx^{-1}$ and $d (x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of $G$, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd (x) = r (x)x = x$. 

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Units will usually be denoted by letters as \( u, v, w \) while arbitrary elements will be denoted by \( x, y, z \). It is useful to note that a pair \((x, y)\) lies in \( G^{(2)} \) precisely when \( d(x) = r(y) \), and that the cancellation laws hold (e.g., \( xy = xz \) iff \( y = z \)). The fibres of the range and the source maps are denoted \( G^u = r^{-1}(\{u\}) \) and \( G_v = d^{-1}(\{v\}) \), respectively. More generally, given the subsets \( A, B \subset G^{(0)} \), we define \( G^A = r^{-1}(A) \), \( G_B = d^{-1}(B) \) and \( G^A_B = r^{-1}(A) \cap d^{-1}(B) \). The reduction of \( G \) to \( A \subset G^{(0)} \) is \( G|A \). The equivalence relation \( u \sim v \) iff \( G^u \neq \emptyset \) is an equivalence relation on \( G^{(0)} \). Its equivalence classes are called orbits and the orbit of a unit \( u \) is denoted \( [u] \). A groupoid is called transitive iff it has a single orbit. The quotient space for this equivalence relation is called the orbit space of \( G \) and denoted \( G^{(0)}/G \). We denote by \( \pi : G^{(0)} \to G^{(0)}/G, \pi(u) = \dot{u} \) the quotient map. A subset of \( G^{(0)} \) is said saturated if it contains the orbits of its elements. For any subset \( A \) of \( G^{(0)} \), we denote by \( [A] \) the union of the orbits \([u]\) for all \( u \in A \).

A topological groupoid consists of a groupoid \( G \) and a topology compatible with the groupoid structure. This means that:

1. \( x \to x^{-1} \) \( : G \to G \) is continuous.
2. \( (x, y) \) \( : G^{(2)} \to G \) is continuous where \( G^{(2)} \) has the induced topology from \( G \times G \).

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [6] that measured groupoids may be assume to have locally compact topologies, with no loss in generality.

If \( X \) is a locally compact space, \( C_c(X) \) denotes the space of complex-valuated continuous functions with compact support. The Borel sets of a topological space are taken to be the \( \sigma \)-algebra generated by the open sets. The space of compactly supported bounded Borel function on \( X \) is denoted by \( B_c(X) \).

For a locally compact groupoid \( G \), we denote by

\[ G' = \{ x \in G : r(x) = d(x) \} \]

the isotropy group bundle of \( G \). It is closed in \( G \).

Let \( G \) be a locally compact second countable groupoid equipped with a Haar system, i.e. a family of positive Radon measures on \( G \), \( \{\nu^u, u \in G^{(0)}\} \), such that

1. For all \( u \in G^{(0)} \), \( \text{supp} (\nu^u) = G^u \).
2. For all \( f \in C_c(G) \),
\[
(\nu^u)(f) = \int f(x) \, d\nu^u(x) \quad : \quad G^{(0)} \to \mathbb{C}
\]
is continuous.
3. For all \( f \in C_c(G) \) and all \( x \in G \),
\[
\int f(y) \, d\nu^{r(x)}(y) = \int f(xy) \, d\nu^{d(x)}(y)
\]

As a consequence of the existence of continuous Haar systems, \( r, d : G \to G^{(0)} \) are open maps ([10]). Therefore, in this paper we shall always assume that \( r : G \to G^{(0)} \) is an open map.
If $\mu$ is a measure on $G^{(0)}$, then the measure $\nu = \int \nu^u d\mu(u)$, defined by

$$
\int f(y) d\nu(y) = \int \left( \int f(y) d\nu^u(y) \right) d\mu(u), \quad f \geq 0 \text{ Borel}
$$

is called the measure on $G$ induced by $\mu$. The image of $\nu$ by the inverse map $x \to x^{-1}$ is denoted $\nu^{-1}$. $\mu$ is said quasi-invariant if its induced measure $\nu$ is equivalent to its inverse $\nu^{-1}$. A measure belongs to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant.

If $\mu$ is a quasi-invariant measure on $G^{(0)}$ and $\nu$ is the measure induced on $G$, then the Radon-Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of $\mu$.

In order to define the $C^*$-algebra of a groupoid the space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest $C^*$-norm making its representations continuous. For $f, g \in C_c(G)$ the convolution is defined by:

$$
f * g(x) = \int f(xy) g(y^{-1}) d\nu^d(x)(y)
$$

and the involution by

$$
f^*(x) = \overline{f(x^{-1})}.
$$

Under these operations, $C_c(G)$ becomes a topological $*$-algebra.

A representation of $C_c(G)$ is a $*$-homomorphism from $C_c(G)$ into $\mathcal{B}(H)$, for some Hilbert space $H$, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $\mathcal{B}(H)$. The full $C^*$-algebra $C^*(G)$ is defined as the completion of the involutive algebra $C_c(G)$ with respect to the full $C^*$-norm

$$
\|f\| = \sup \|L(f)\|
$$

where $L$ runs over all non-degenerate representation of $C_c(G)$ which are continuous for the inductive limit topology.

Every representation $(\mu, G^{(0)} \ast \mathcal{H}, L)$ (see Definition 3.20/p.68 [4]) of $G$ can be integrated into a representation, still denoted by $L$, of $C_c(G)$. The relation between the two representation is:

$$
\langle L(f) \xi_1, \xi_2 \rangle = \int f(x) \langle L(x) \xi_1(d(x)), \xi_2(r(x)) \rangle d\nu^u(x) d\mu(u)
$$

where $f \in C_c(G), \xi_1, \xi_2 \in \int_{G^{(0)}} \mathcal{H}(u) d\mu(u)$.

Conversely, every non-degenerate $*$-representation of $C_c(G)$ is obtained in this fashion (see [8] or [4]).

2 The decomposition of a Haar system over the principal groupoid

First we present some results on the structure of the Haar systems, as developed by J. Renault in Section 1 of [4] and also by A. Ramsay and M.E. Walter in
Section 2 of [7].

In Section 1 of [9] Jean Renault constructs a Borel Haar system for \( G' \). One way to do this is to choose a function \( F_0 \) continuous with conditionally support which is nonnegative and equal to 1 at each \( u \in G^{(0)} \). Then for each \( u \in G^{(0)} \) choose a left Haar measure \( \beta^u \) on \( G_u \) so the integral of \( F_0 \) with respect to \( \beta^u \) is 1.

Renault defines \( \beta^u \) as usual. If \( z \) is another element in \( G_u \), then \( \beta^u(z) = \beta^u(x) \) if \( x \in G_u \). If \( K \) is a compact subset of \( G \), then \( \sup_u \beta^u(K) < \infty \). Renault also defines a 1-cocycle \( \delta \) on \( G \) such that for every \( u \in G^{(0)} \), \( \delta^u \) is the modular function for \( \beta^u \). \( \delta \) and \( \delta^{-1} = 1/\delta \) are bounded on compact sets in \( G \).

Let \( R = (r, d)(G) = \{(r(x), d(x)), x \in G\} \) be the graph of the equivalence relation induced on \( G^{(0)} \). This \( R \) is the image of \( G \) under the homomorphism \( (r, d) \), so it is a \( \sigma \)-compact groupoid. With this apparatus in place, Renault describes a decomposition of the Haar system \( \{\nu^u, u \in G^{(0)}\} \) for \( G \) over the equivalence relation \( R \) (the principal groupoid associated to \( G \)). He proves that there is a unique Borel Haar system \( \alpha \) for \( R \) with the property that

\[
\nu^u = \int \beta^u \alpha^u(s, t) \quad \text{for all } u \in G^{(0)}.
\]

In Section 2 [7] A. Ramsay and M.E. Walter prove that

\[
\sup_u \alpha^u((r, d)(K)) < \infty, \text{ for all compact } K \subset G.
\]

For each \( u \in G^{(0)} \) the measure \( \alpha^u \) is concentrated on \( \{u\} \times [u] \). Therefore there is a measure \( \mu^u \) concentrated on \( [u] \) such that \( \alpha^u = \varepsilon_u \times \mu^u \), where \( \varepsilon_u \) is the unit point mass at \( u \). Since \( \{\alpha^u, u \in G^{(0)}\} \) is a Haar system, we have \( \mu^u = \mu^v \) for all \( (u, v) \in R \), and the function

\[
u^u = \int f(s) \mu^u(s)
\]

is Borel for all \( f \geq 0 \) Borel on \( G^{(0)} \). For each \( u \) the measure \( \mu^u \) is quasi-invariant (Section 2 [7]). Therefore \( \mu^u \) is equivalent to \( d_u(v^u) \) (Lemma 4.5/p. 277 [5]).

If \( \eta \) is a quasi-invariant measure for \( \{\nu^u, u \in G^{(0)}\} \), then \( \eta \) is a quasi-invariant measure for \( \{\alpha^u, u \in G^{(0)}\} \). Also if \( \Delta_R \) is the modular function associated to \( \{\alpha^u, u \in G^{(0)}\} \) and \( \eta \), then \( \Delta = \delta \Delta_R \circ (r, d) \) can serve as the modular function associated to \( \{\nu^u, u \in G^{(0)}\} \) and \( \eta \).

Since \( \mu^u = \mu^v \) for all \( (u, v) \in R \), the system of measures \( \{\mu^u\}_u \) may be indexed on the elements of the orbit space \( G^{(0)}/G \).
Definition 1 We shall call the pair of the system of measures

\[ \{ \beta^u \}_{(u,v) \in R}, \{ \mu^u \}_{u \in G^{(0)}/G} \]

(described above) the decomposition of the Haar system \( \{ \nu^u, u \in G^{(0)} \} \) over the principal groupoid associated to \( G \). Also we shall call \( \delta \) the 1-cocycle associated to the decomposition.

Remark 2 Let us note that the system of measures \( \{ \beta^u \} \) and the 1-cocycle do not depend on the Haar system.

Lemma 3 Let \( G \) be a locally compact second countable groupoid with the bundle map \( r|G' \) of \( G' \) open. Let \( \{ \nu^u, u \in G^{(0)} \} \) be a Haar system on \( G \) and \( (\{ \beta^u \}, \{ \mu^u \}) \) its decomposition over the principal groupoid associated to \( G \). Then for each \( f \in C_c(G) \) the function

\[ x \to \int f(y) \, d\beta^r_{d(x)}(y) \]

is continuous on \( G \).

Proof. By Lemma 1.3/p. 6, for each \( f \in C_c(G) \) the function \( u \to \int f(y) \, d\beta^u_{d(x)}(y) \) is continuous.

Let \( x \in G \) and \( (x_i) \) be a sequence in \( G \) converging to \( x \). Let \( f \in C_c(G) \) and let \( g \) be a continuous extension on \( G \) of \( y \to f(xy) \, : \, G^d(x) \to \mathbb{C} \). Let \( K \) be the compact set

\[ \left( \{ x, x_i, i = 1, 2, .. \}^{-1} \operatorname{supp}(f) \cup \operatorname{supp}(g) \right) \cap r^{-1}(\{ d(x), d(x_i), i = 1, 2, .. \}). \]

We have

\[ \left| \int f(y) \, d\beta^r_{d(x)}(y) - \int f(y) \, d\beta^r_{d(x_i)}(y) \right| \]

\[ = \left| \int f(xy) \, d\beta^d_{d(x)}(y) - \int f(xy) \, d\beta^d_{d(x_i)}(y) \right| \]

\[ = \left| \int g(y) \, d\beta^d_{d(x)}(y) - \int f(xy) \, d\beta^d_{d(x_i)}(y) \right| \]

\[ \leq \int g(y) \, d\beta^d_{d(x)}(y) - \int f(xy) \, d\beta^d_{d(x_i)}(y) + \]

\[ + \left| \int g(y) \, d\beta^d_{d(x_i)}(y) - \int f(xy) \, d\beta^d_{d(x_i)}(y) \right| \]

\[ \leq \int g(y) \, d\beta^d_{d(x)}(y) - \int g(y) \, d\beta^d_{d(x_i)}(y) + \]

\[ + \sup_{y \in G^d(x_i)} |g(y) - f(xy)| \beta^d_{d(x_i)}(K) \]

A compactness argument shows that \( \sup_{y \in G^d(x_i)} |g(y) - f(xy)| \) converges to 0. Also \( \left| \int g(y) \, d\beta^d_{d(x)}(y) - \int g(y) \, d\beta^d_{d(x_i)}(y) \right| \) converges to 0 because the function \( u \to \int f(y) \, d\beta^u_{d(x)}(y) \) is continuous. Hence

\[ \left| \int f(y) \, d\beta^r_{d(x)}(y) - \int f(y) \, d\beta^r_{d(x_i)}(y) \right| \]

converges to 0. \( \blacksquare \)
**Definition 4** A locally compact groupoid $G$ is proper if the map $(r, d): G \to G(0) \times G(0)$ is proper (i.e. the inverse of each compact subset of $G(0) \times G(0)$ is compact). (Definition 2.1.9/p. 37 [1]).

Throughout this paper we shall assume that $G$ is a second countable locally compact groupoid for which the orbit space is Hausdorff and the map

$$(r, d): G \to R, \ (r, d) (x) = (r (x), d (x))$$

is open, where $R$ is endowed with the product topology induced from $G(0) \times G(0)$. Therefore $R$ will be a locally compact groupoid. The fact that $R$ is a closed subset of $G(0) \times G(0)$ and that it is endowed with the product topology is equivalent to the fact $R$ is a proper groupoid.

Throughout this paper by a groupoid with proper orbit space we shall mean a groupoid $G$ for which the orbit space is Hausdorff and the map

$$(r, d): G \to R, \ (r, d) (x) = (r (x), d (x))$$

is open, where $R$ is endowed with the product topology induced from $G(0) \times G(0)$.

**Proposition 5** Let $G$ be a second countable locally compact groupoid with proper orbit space. Let $\{\nu^u, u \in G(0)\}$ be a Haar system on $G$ and $\{\beta^u_v, \{\mu^u\}\}$ its decomposition over the principal groupoid associated to $G$. Then for each $g \in C_c (G(0))$, the map

$$u \to \int g (v) d\mu^u (v)$$

is continuous.

**Proof.** Let $g \in C_c (G(0))$ and $u_0 \in G(0)$. Let $K_1$ be a compact neighborhood of $u_0$ and $K_2$ be the support of $g$. Since $G$ is locally compact and $(r, d)$ is open from $G$ to $(r, d) (G)$, there is a compact subset $K$ of $G$ such that $(r, d) (K)$ contains $(K_1 \times K_2) \cap (r, d) (G)$. Let $F_1 \in C_c (G)$ be a nonnegative function equal to 1 on a compact neighborhood $U$ of $K$. Let $F_2 \in C_c (G)$ be a function which extends to $G$ the function $x \to F_1 (x) / \int F_1 (y) d\beta^u_v (y), \ x \in U$. We have $\int F_2 (y) d\beta^u_v (y) = 1$ for all $(u, v) \in (r, d) (K)$. Since for all $u \in K_1$,

$$\int g (v) d\mu^u (v) = \int g (v) \int F_2 (y) d\beta^u_v (y) d\mu^u (v) = \int g (d (y)) F_2 (y) d\nu^u (y),$$

it follows that $u \to \int g (v) d\mu^u (v)$ is continuous at $u_0$. 

**Remark 6** Let $G$ be a locally compact second countable groupoid with proper orbit space. Let $\{\nu^u, u \in G(0)\}$ be a Haar system on $G$ and $\{\beta^u_v, \{\mu^u\}\}$ be its decomposition over the associated principal groupoid. If $\mu$ is a quasi-invariant probability measure for the Haar system, then $\mu_1 = \int \mu^u d\mu (u)$ is a Radon

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Let us assume that the range map \( r \).
Conversely, if \( \mu \) follows that there is a Borel set \( \nu \), be the quotient map. Since the quotient space is proper, \( G \) and since \( \mu \) Radon quasi-invariant measure is equivalent to a Radon measure of the form 
\[
\int \mu^u d\tilde{\mu}(u),
\]
where \( \tilde{\mu} \) is a probability measure on the orbit space \( G/G^{(0)} \).

3 A \( C^* \)-algebra associated to a locally compact groupoid with proper orbit space

Let \( G \) be a locally compact second countable groupoid with proper orbit space. Let 
\[
\pi : G^{(0)} \rightarrow G^{(0)}/G
\]
be the quotient map. Since the quotient space is proper, \( G^{(0)}/G \) is Hausdorff. Let us assume that the range map \( r \) is open. As a consequence, the map \( \pi \) is open. Applying Lemma 1.1 \([2]\) to the locally compact second countable spaces \( G^{(0)} \) and \( G^{(0)}/G \) and to the continuous open surjection \( \pi : G^{(0)} \rightarrow G^{(0)}/G \), it follows that there is a Borel set \( F \) in \( G^{(0)} \) such that:

1. \( F \) contains exactly one element in each orbit \([u] = \pi^{-1}(\pi(u))\).
2. For each compact subset \( K \) of \( G^{(0)} \), \( F \cap [K] = F \cap \pi^{-1}(\pi(K)) \) has a compact closure.

For each unit \( u \) let us define \( e(u) = F \cap [u] \) (\( e(u) \) is the unique element in the orbit of \( u \) contained in \( F \)). For each Borel subset \( B \) of \( G^{(0)} \), \( \pi \) is continuous and one-to-one on \( B \cap F \) and hence \( \pi(B \cap F) \) is Borel in \( G^{(0)}/G \). Therefore the map \( e : G^{(0)} \rightarrow G^{(0)} \) is Borel (for each Borel subset \( B \) of \( G^{(0)} \), \( e^{-1}(B) = [B \cap F] = \pi^{-1}(\pi(B \cap F)) \) is Borel in \( G^{(0)} \)). Also for each compact subset \( K \) of \( G^{(0)} \), \( e(K) \) has a compact closure because \( e(K) \subset F \cap [K] \).

Since the orbit space \( G^{(0)}/G \) is proper the map 
\[
(r, d) : G \rightarrow R, (r, d)(x) = (r(x), d(x))
\]
is open and \( R \) is closed in \( G^{(0)} \times G^{(0)} \). Applying Lemma 1.1 \([2]\) to the locally compact second countable spaces \( G \) and \( R \) and to the continuous open surjection \( (r, d) : G \rightarrow R \), it follows that there is a regular cross section \( \sigma_0 : R \rightarrow G \). This means that \( \sigma_0 \) is Borel, \( (r, d)(\sigma_0(u, v)) = (u, v) \) for all \( (u, v) \in R \), and \( \sigma_0(K) \) is relatively compact in \( G \) for each compact subset \( K \) of \( R \).

Let us define \( \sigma : G^{(0)} \rightarrow G^{F} \) by \( \sigma(u) = \sigma_0(e(u), u) \) for all \( u \). It is easy to note that \( \sigma \) is a cross section for \( d : G^{F} \rightarrow G^{(0)} \) and \( \sigma(K) \) is relatively compact in \( G \) for all compact \( K \subset G^{(0)} \).

Replacing \( \sigma \) by 
\[
v \mapsto \sigma(e(v))^{-1} \sigma(v)
\]
we may assume that \( \sigma (e(v)) = e(v) \) for all \( v \). Let us define \( q : G \to G^E_F \) by

\[
q(x) = \sigma(r(x)) x \sigma(d(x))^{-1}, \quad x \in G.
\]

Let \( \nu = \{ \nu^u : u \in G^{(0)} \} \) be a Haar system on \( G \) and let \( \{ \beta^u \}, \{ \mu^u \} \) be its decompositions over the principal groupoid. Let \( \delta \) be the 1-cocycle associated to the decomposition.

Let us denote by \( \mathcal{B}_\sigma (G) \) the linear span of the functions of the form

\[
x \to g_1(r(x)) g(q(x)) g_2(d(x))
\]

where \( g_1, g_2 \) are compactly supported Borel bounded function on \( G^{(0)} \) and \( g \) is a Borel bounded function on \( G^E_F \) such that if \( S \) is the support of \( g \), then the closure of \( \sigma(K_1)^{-1} S \sigma(K_2) \) is compact in \( G \) for all compact subsets \( K_1, K_2 \) of \( G^{(0)} \). \( \mathcal{B}_\sigma (G) \) is a subspace of \( \mathcal{B}_c (G) \), the space of compactly supported Borel bounded function on \( G \).

If \( f_1, f_2 \in \mathcal{B}_\sigma (G) \) are defined by

\[
f_1(x) = g_1(r(x)) g(q(x)) g_2(d(x)) \\
f_2(x) = h_1(r(x)) h(q(x)) h_2(d(x))
\]

then

\[
f_1 * f_2(x) = g * h(q(x)) g_1(r(x)) h_2(d(x)) \left\langle g_2, h_1 \right\rangle_{\pi(r(x))} \\
f_1^*(x) = g_2(r(x)) g(q(x))^{-1} g_1(d(x))
\]

Thus \( \mathcal{B}_\sigma (G) \) is closed under convolution and involution.

Let \( \omega \) be the universal representation of \( C^*(G, \nu) \) the usual \( C^* \)-algebra associated to a Haar system \( \nu = \{ \nu^u, u \in G^{(0)} \} \) (constructed as in [S]). Since every cyclic representation of \( C^*(G, \nu) \) is the integrated form of a representation of \( G \), it follows that \( \omega \) can be also regarded as a representation of \( \mathcal{B}_c (G) \), the space of compactly supported Borel bounded function on \( G \). Arlan Ramsay and Martin E. Walter have used the notation \( M^*(G, \nu) \) for the operator norm closure of \( \omega(\mathcal{B}_c (G)) \). Since \( \omega \) is an *-isomorphism on \( C^*(G, \nu) \), we can regarded \( C^*(G, \nu) \) as a subalgebra of \( M^*(G, \nu) \).

**Definition 7** We denote by \( M^*_\sigma (G, \nu) \) the operator norm closure of \( \omega(\mathcal{B}_\sigma (G)) \).

**Lemma 8** Let \( \{ \mu_1^u \}_u \) and \( \{ \mu_2^u \}_u \) be two systems of measures on \( G^{(0)} \) satisfying:

1. \( \text{supp} (\mu_i^u) = [u] \) for all \( u, i = 1, 2 \)
2. For all compactly supported Borel bounded function \( f \) on \( G^{(0)} \) the function

\[
u \to \int f(v) \mu_1^{\pi(u)}(v)
\]

is bounded and Borel.
Then there is a family \( \{ U_\bar{u} \}_\bar{u} \) of unitary operators with the following properties:

1. \( U_\bar{u} : L^2 (\mu_1^\delta) \to L^2 (\mu_2^\delta) \) is a unitary operator for each \( \bar{u} \in G^{(0)}/G \).

2. For all Borel bounded function \( f \) on \( G^{(0)} \),

\[
u \mapsto U_{\pi(\nu)} (f)
\]

is a bounded Borel function with compact support.

3. For all Borel bounded function \( f \) on \( G^{(0)} \),

\[
U_{\pi(\nu)} (f) = \overline{U_{\pi(\nu)} (f)}
\]

**Proof.** Using the same argument as in [6] (p. 323) we can construct a sequence \( f_1, f_2, \ldots \) of real valued Borel bounded function on \( G^{(0)} \) such that \( \dim (L^2 (\mu_1^\delta)) = \infty \) if and only if \( \| f_n \|_2 \) for \( n = 1, 2, \ldots \) and then \( \{ f_1, f_2, \ldots \} \) gives an ortonormate basis of \( L^2 (\mu_1^\delta) \), while \( \dim (L^2 (\mu_2^\delta)) = k < \infty \) if and only if \( \| f_n \|_2 = 1 \) for \( n \leq k \), and \( \| f_n \|_2 = 0 \) for \( n > k \) and then \( \{ f_1, f_2, \ldots, f_k \} \) gives an ortonormate basis of \( L^2 (\mu_2^\delta) \). Let \( g_1, g_2, \ldots \) be a sequence with the same properties as \( f_1, f_2, \ldots \) corresponding to \( \{ \mu_2^\delta \}_\bar{u} \). Let us define \( U_\bar{u} : L^2 (\mu_1^\delta) \to L^2 (\mu_2^\delta) \) by

\[
U_\bar{u} (f_n) = g_n \text{ for all } n
\]

Then the family \( \{ U_\bar{u} \}_\bar{u} \) has the required properties. ■

**Theorem 9** Let \( G \) be a locally compact second countable groupoid with proper orbit space. Let \( \{ \nu_i^u, u \in G^{(0)} \} \), \( i = 1, 2 \) be two Haar systems on \( G \). Let \( F \) be a Borel subset of \( G^{(0)} \) containing only one element \( e (u) \) in each orbit \([u]\). Let \( \sigma : G^{(0)} \to G^F \) be a cross section for \( d : G^F \to G^{(0)} \) with \( \sigma (e (u)) = e (v) \) for all \( v \in G^{(0)} \) and \( \sigma (K) \) relatively compact in \( G \) for all compact sets \( K \subset G^{(0)} \).

Then the \( C^* \)-algebras \( M^*_\nu (G, \nu_1) \) and \( M^*_\nu (G, \nu_2) \) are \(*\)-isomorphic.

**Proof.** Let \( \{ (\beta_i^u), \{ \mu_i^\delta \} \} \) be the decompositions of the Haar systems over the principal groupoid. Let \( \delta \) be the 1-cocycle associated to the decompositions, \( i = 1, 2 \).

We shall denote by \( \langle \cdot, \cdot \rangle_{i, \bar{u}} \) the inner product of \( \{ L^2 (G^{(0)}, \delta (\cdot) \mu_i^\delta) \} \), \( i = 1, 2 \).

Let us define \( q : G \to G^F_\nu \) by

\[
q (x) = \sigma (r (x)) x \sigma (r (d (x)))^{-1} \text{, } x \in G.
\]

We shall define a \(*\)-homomorphism \( \Phi \) from \( \mathcal{B}_\nu (G) \) to \( \mathcal{B}_\nu (G) \). It suffices to define \( \Phi \) on the set of function on \( G \) of the form

\[
x \to g_1 (r (x)) g (q (x)) g_2 (d (x))
\]

10
Let \( \{ U_\tilde{u} \} \) be the family of unitary operators with the properties stated in Lemma \( \mathbb{N} \) associated to the systems of measures \( \{ \delta ( \sigma ( \cdot ) ) \mu^\tilde{u}_1 \} \), \( i = 1, 2. \)

Let us define \( \Phi \) by

\[
\Phi ( f ) = \left( x \to U_{\pi(\tau(x))} (g_1) (r (x)) \right) g (q (x)) U_{\pi(d(x))} (g_2) (d (x))
\]

where \( f \) is defined by

\[
f (x) = g_1 (r (x)) g (q (x)) g_2 (d (x))
\]

If \( f_1 \) and \( f_2 \) are defined by

\[
\begin{align*}
    f_1 (x) &= g_1 (r (x)) g (q (x)) g_2 (d (x)) \\
    f_2 (x) &= h_1 (r (x)) h (q (x)) h_2 (d (x))
\end{align*}
\]

then

\[
f_1 \ast f_2 (x) = g \ast h (q (x)) g_1 (r (x)) h_2 (d (x)) \langle g_2, \overline{h_1} \rangle_{1, \pi(r(x))}
\]

and consequently

\[
\Phi ( f_1 \ast f_2 ) = g \ast h (q (x)) U_{\pi(\tau(x))} (g_1) (r (x)) U_{\pi(r(x))} (h_2) (d (x)) \langle g_2, \overline{h_1} \rangle_{1, \pi(r(x))}
\]

\[
= \Phi ( f_1 ) \ast \Phi ( f_2 ).
\]

Let \( \tilde{\eta} \) be a probability measure on \( G^{(0)}/G \) and \( \eta_i = \int \mu^\tilde{u}_i d\tilde{\eta}(\tilde{u}), i = 1, 2. \) Let \( L_1 \) be the integrated form of a representation \( (L, \mathcal{H} \ast G^{(0)}, \eta_1) \) and \( L_2 \) be the integrated form of \( (L, \mathcal{H} \ast G^{(0)}, \eta_2) \). Let \( B \) be the Borel function defined by:

\[
B (u) = L (\sigma (u))
\]

and \( W : \int_{G^{(0)}}^\oplus \mathcal{H} (u) d\eta_1 (u) \to \int_{G^{(0)}}^\oplus \mathcal{H} (e (u)) d\eta_1 (u) \) be defined by

\[
W (\zeta) = (u \to B (u) (\zeta (u)))
\]

Since every element of \( L^2 (G^{(0)}, \delta (\sigma (\cdot)) \mu^\tilde{w}, \mathcal{H} (e (w))) \) is a limit of linear combinations of elements \( u \to a (u) \xi \) with \( a \in L^2 (G^{(0)}, \delta (\sigma (\cdot)) \mu^\tilde{w}) \) and \( \xi \in \mathcal{H} (e (w)) \), we can define a unitary operator

\[
V_w : L^2 (G^{(0)}, \delta (\sigma (\cdot)) \mu^\tilde{w}, \mathcal{H} (e (w))) \to L^2 (G^{(0)}, \delta (\sigma (\cdot)) \mu^\tilde{w}, \mathcal{H} (e (w)))
\]

by

\[
V_w (u \to a (u) \xi) = U_w (a) \xi
\]

Let \( V : \int_{G^{(0)}}^\oplus \mathcal{H} (e (u)) d\eta_1 (u) \to \int_{G^{(0)}}^\oplus \mathcal{H} (e (u)) d\eta_2 (u) \) be defined by

\[
V (\zeta) = (u \to V_\zeta (\zeta (u)))
\]

If \( \zeta_1, \zeta_2 \in \int_{G^{(0)}}^\oplus \mathcal{H} (e (u)) d\eta_1 (u) \) and \( f \) is of the form

\[
f (x) = g_1 (r (x)) g (q (x)) g_2 (d (x)),
\]
we have

$$\langle W L_1 (f) W^* \zeta_1, \zeta_2 \rangle$$

$$= \int \int g (x) \delta (x) ^{\frac{1}{2}} \langle L (x) A_1 (\dot{w}), B_1 (\dot{w}) \rangle d \beta^{(u)} (x) d \tilde{\eta} (\dot{w})$$

where

$$A_1 (\dot{w}) = \int g_2 (v) \zeta_1 (v) \delta (\sigma (v)) ^{\frac{1}{2}} d \mu^{(v)} (v)$$

$$B_1 (\dot{w}) = \int g_1 (u) \zeta_2 (u) \delta (\sigma (u)) ^{\frac{1}{2}} d \mu^{(u)} (u)$$

Moreover, if $f$ is of the form $f (x) = g_1 (r (x)) g (q (x)) g_2 (d (x))$ and $\zeta_1, \zeta_2 \in \int_{G (0)} H (e (u)) d \eta_2 (u)$, then

$$\langle VWL_1 (f) W^* V^* \zeta_1, \zeta_2 \rangle$$

$$= \int \int g (x) \delta (x) ^{\frac{1}{2}} \langle L (x) A_2 (\dot{w}), B_2 (\dot{w}) \rangle d \beta^{(u)} (x) d \tilde{\eta} (\dot{w})$$

$$= \langle WL_2 (\Phi (f)) W^* \zeta_1, \zeta_2 \rangle$$

where

$$A_2 (\dot{w}) = \int g_2 (v) V^* \zeta_1 (v) \delta (\sigma (v)) ^{\frac{1}{2}} d \mu^{(v)} (v)$$

$$= \int U_\zeta (g_2) (v) \zeta_1 (v) \delta (\sigma (v)) ^{\frac{1}{2}} d \mu^{(v)} (v)$$

$$B_2 (\dot{w}) = \int g_1 (v) V^* \zeta_2 (v) \delta (\sigma (v)) ^{\frac{1}{2}} d \mu^{(v)} (v)$$

$$= \int U_\zeta (g_1) (u) \zeta_2 (u) \delta (\sigma (u)) ^{\frac{1}{2}} d \mu^{(u)} (u)$$

Therefore $\|L_1 (f)\| = \|L_2 (\Phi (f))\|$. Consequently we can extend $\Phi$ to a *-homomorphism between the $M^*_n (G, \nu_1)$ and $M^*_n (G, \nu_2)$. It is not hard to see that $\Phi$ is in fact a *-isomorphism:

$$\Phi^{-1} (f) = \left( x \to U_{\pi (r (x))}^* (g_1) (r (x)) g (q (x)) U_{\pi (d (x))}^* (g_2) (d (x)) \right)$$

for each $f$ of the form

$$f (x) = g_1 (r (x)) g (q (x)) g_2 (d (x)).$$
4 The case of locally transitive groupoids

A locally compact transitive groupoid $G$ is a groupoid for which all orbits $[u]$ are open in $G^{(0)}$. We shall prove that if $G$ is a locally compact second countable locally transitive groupoid endowed with a Haar system $\nu$, then

$$C^* (G, \nu) = M^* (G, \nu) = M^*_\sigma (G, \nu)$$

for any regular cross section $\sigma$.

**Notation 10** Let $\{ \nu^u, u \in G^{(0)} \}$ be a fixed Haar system on $G$. Let $\mu$ be a quasi-invariant measure, $\Delta$ its modular function, $\nu_1$ be the measure induced by $\mu$ on $G$ and $\nu_0 = \Delta^{-\frac{1}{2}} \nu_1$. Let us denote by $II_\mu (G)$ the set

$$\{ f \in L^1 (G, \nu_0) : \| f \|_{II, \mu} < \infty \},$$

where $\| f \|_{II, \mu}$ is defined by

$$\| f \|_{II, \mu} = \sup \left\{ \int |f (x) j (d (x)) k (r (x))| d\nu_0 (x), \int |j|^2 d\mu = \int |k|^2 d\mu = 1 \right\}.$$

If $\mu_1$ and $\mu_2$ are two equivalent quasi-invariant measures, then $\| f \|_{II, \mu_1} = \| f \|_{II, \mu_2}$, because $\| f \|_{II, \mu} = \| II_\mu (|f|) \|$ for each quasi-invariant measure $\mu$, where $II_\mu$ is the one dimensional trivial representation on $\mu$.

Define $\| f \|_I$ to be

$$\sup \{ \| f \|_{II, \mu} : \mu \text{ quasi-invariant Radon measure on } G^{(0)} \}$$

The supremum can be taken over the classes of quasi-invariant measure.

If $\| \|_I$ is the full $C^*$-norm on $C_c (G)$, then

$$\| f \| \leq \| f \|_I \text{ for all } f.$$ (see [7])

**Lemma 11** Let $G$ be a locally compact second countable groupoid with proper orbit space. Let $\{ \nu^u, u \in G^{(0)} \}$ be a Haar system on $G$, $\{ \beta^u_v \}, \{ \mu^u \}$ its decomposition over the principal groupoid associated to $G$ and $\delta$ the associated 1-cocycle. If $f$ is a universally measurable function on $G$, then

$$\| f \|_I \leq \sup_\nu \left( \int \int \left( \int |f (x)| \delta (x)^{-\frac{1}{2}} d\beta^u_v (x) \right)^2 d\mu^u (v) d\mu^w (u) \right)^{\frac{1}{2}}.$$
Proof. Each Radon quasi-invariant measure is equivalent with a Radon measure of the form $\int \mu^\beta d\tilde{\mu}(\dot{u})$, where $\tilde{\mu}$ is a probability measure on the orbit space $G/G^{(0)}$. Therefore for the computation of $\|f\|_{I1}$ it is enough to consider only the quasi-invariant measures of the form $\mu = \int \mu^\beta d\tilde{\mu}(\dot{u})$, where $\tilde{\mu}$ is a probability measure on $G^{(0)}/G$. It is easy to see that the modular function of $\int \mu^\beta d\tilde{\mu}(\dot{u})$ is $\Delta = \delta$.

Let $j, k \in L^2(G^{(0)}, \mu)$ with $\int |j|^2 d\mu = \int |k|^2 d\mu = 1$. We have

$$\int \int \int \int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x) |j(v)| |k(u)| \mu^\beta(v) \mu^\beta(u) d\tilde{\mu}(\dot{u})$$

$$\leq \int \left(\int \int (\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x))^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}} d\tilde{\mu}(\dot{u})$$

$$\cdot \left(\int \int |j(v)|^2 |k(u)|^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}} \mu^\beta(v) \mu^\beta(u)$$

$$= \sup_{\dot{u}} \left(\int \int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x)\right)^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}} d\tilde{\mu}(\dot{u})$$

$$\cdot \left(\int \int |j(v)|^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}} d\tilde{\mu}(\dot{u})$$

$$\leq \sup_{\dot{u}} \left(\int \int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x)\right)^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}}$$

Consequently,

$$\|f\|_{I1} \leq \sup_{\dot{u}} \left(\int \int \left(\int |f(x)| \delta(x)^{-\frac{1}{2}} d\beta_v^u(x)\right)^2 \mu^\beta(v) \mu^\beta(u) \right)^{\frac{1}{2}}$$

\[\blacksquare\]

If $G$ is locally transitive, each orbit $[u]$ is open in $G^{(0)}$. Each measure $\mu^\beta$ is supported on $[u]$. Since $([u])$ is a partition of $G^{(0)}$ into open sets, it follows that there is a unique Radon measure $m$ on $G^{(0)}$ such that the restriction of $m$ at $C_c([u])$ is $\mu^\beta$ for each $[u]$.

**Corollary 12** Let $G$ be a locally compact second countable locally transitive groupoid endowed with a Haar system $\nu = \{\nu^u, u \in G^{(0)}\}$. Let $f$ be a universally measurable function such that $\|f\|_{I1} < \infty$.

1. If $(f_n)_n$ is a uniformly bounded sequence of universally measurable functions supported on a compact set, and if $(f_n)_n$ converges pointwise to $f$, then $(f_n)_n$ converges to $f$ in the norm of $C^*(G, \nu)$.

2. If $(f_n)_n$ is an increasing sequence of universally measurable nonnegative functions on $G$ that converges pointwise to $f$, then $(f_n)_n$ converges to $f$ in the norm of $C^*(G, \nu)$.
Proof. Let \( \{ \beta_u^v \}, \{ \mu^u \} \) be the decomposition of the Haar system over the principal groupoid associated to \( G \) and \( \delta \) the associated 1-cocycle. Let \( m \) be the unique measure such that restriction of \( m \) at \( C_c([u]) \) is \( \mu^u \) for each \( [u] \). Let \( (f_n)_n \) be a sequence of universally measurable functions supported on a compact set \( K \). Let us write

\[
M = \sup_{u,v} \beta_u^v (K^{-1}).
\]

and let us assume that \( (f_n)_n \) converges pointwise to \( f \). According to Lemma 11, we have

\[
\| f - f_n \|_{II} \leq \sup_{u,v} \left( \int \int \left( |f(x) - f_n(x)|^{2} dg(x) \right)^{2} d\mu^v (v) d\mu^u (u) \right)^{\frac{1}{2}}.
\]

Hence

\[
\| f - f_n \|_{II} \leq M \left( \int \int \left( |f(x) - f_n(x)|^{2} dg(x) \right) dm (v) dm (u) \right)^{\frac{1}{2}}.
\]

If \( \| \| \) denotes the \( C^* \)-norm, then

\[
\lim_n \| f - f_n \| \leq \lim_n \| f - f_n \|_{II} = 0,
\]

because

\[
\int \int \left( |f(x) - f_n(x)|^{2} dg(x) \right) dm (v) dm (u)
\]

converges to zero, by the Dominated Convergence Theorem.

Let \( (f_n)_n \) be an increasing sequence of universally measurable nonnegative functions that converges pointwise to \( f \). Since

\[
\| f - f_n \|_{II} \leq \sup_{u,v} \left( \int \int \left( |f(x) - f_n(x)|^{2} dg(x) \right)^{2} d\mu^v (v) d\mu^u (u) \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int \int \left( |f(x) - f_n(x)|^{2} dg(x) \right)^{2} dm (v) dm (u) \right)^{\frac{1}{2}}
\]

it follows that

\[
\lim_n \| f - f_n \|_{II} = 0.
\]

\[\blacksquare\]

Proposition 13 Let \( G \) be a locally compact second countable locally transitive groupoid endowed with a Haar system \( \nu = \{ \nu^u, u \in G^{(0)} \} \). Then any function in \( B_c(G) \), the space of compactly supported Borel bounded function on \( G \), can be viewed as an element of \( C^* (G, \nu) \).
Proof. Let \( \{\beta_u^v, \mu^u\} \) be the decomposition of the Haar system over the principal groupoid associated to \( G \), and \( \delta \) the associated 1-cocycle. Let \( m \) be the unique measure such that restriction of \( m \) at \( C_c([u]) \) is \( \mu^u \) for each \([u]\). Let \( m \) be a dominant for the family \( \{\mu^u\} \). Let \( \nu_1 \) be the measure on \( G \) define by

\[
\int f(x) \, d\nu_1(x) = \left( \int \left( \int f(x) \, d\beta_u^v(x) \right) \, dm(v) \, dm(u) \right)
\]

for all Borel nonnegative function \( f \). If \( f \in B_c(G) \), then \( f \) is the limit in \( L^2(G, \nu_1) \) of a sequence, \((f_n)_n\), in \( C_c(G) \) that is supported on some compact set \( K \) supporting \( f \). If we write

\[
M = \sup_{u,v} \beta_u^v(K^{-1}),
\]
we have

\[
\|f - f_n\|_{L^2} \leq \sup_{u,v} M \left( \int \left( \int |f(x) - f_n(x)|^2 \, d\beta_u^v(x) \right) \, dm(v) \, dm(u) \right)^{\frac{1}{2}}
\]

\[
\leq M \left( \int \left( \int |f(x) - f_n(x)|^2 \, d\beta_u^v(x) \right) \, dm(v) \, dm(u) \right)^{\frac{1}{2}}.
\]

If \( \|\cdot\| \) denotes the \( C^* \)-norm, then

\[
\lim_n \|f - f_n\| \leq \lim_n \|f - f_n\|_{L^2} = 0.
\]
Thus \( f \) can be viewed as an element in \( C^*(G, \nu) \). □

**Proposition 14** If \( G \) is a locally compact second countable locally transitive groupoid endowed with a Haar system \( \{\nu^u, u \in G^{(0)}\} \) with bounded decomposition, then

\[
C^*(G, \nu) = M^*(G, \nu).
\]

**Proof.** It follows from the Proposition 13. □

**Remark 15** Let \( G \) be locally compact locally transitive groupoid. Let \( F \) be a subset of \( G^{(0)} \) containing only one element \( e(u) \) in each orbit \([u]\). It is easy to see that \( F \) is a closed subset of \( G \) and that \( F \) is a discrete space. Let \( \sigma : G^{(0)} \to G^F \) be a regular cross section of \( d_F \). Let us endow \( \bigcup_{[u]} [u] \times G_{e(u)}^{c(u)} \times [u] \) with the topology induced from \( G^{(0)} \times G^F_e \times G^{(0)} \). The topology of \( \bigcup_{[u]} [u] \times G_{e(u)}^{c(u)} \times [u] \) is locally compact because \( \bigcup_{[u]} [u] \times G_{e(u)}^{c(u)} \times [u] \) is a closed subset of the locally compact space \( G^{(0)} \times G^F_e \times G^{(0)} \). With the operations:

\[
(u, x, v)(v, y, w) = (u, xy, w) \quad (u, x, v)^{-1} = (v, x^{-1}, u)
\]
\[ \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \] becomes a groupoid. Let us define \( \phi : G \rightarrow \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \) by
\[
\phi (x) = \left( r(x), \sigma (r(x)) x \sigma (d(x))^{-1}, d(x) \right)
\]
and note that \( \phi \) is a Borel isomorphism which carries compact sets to relatively compact sets.

**Lemma 16** Let \( G \) be locally compact second countable locally transitive groupoid. Let \( F \) be a subset of \( G(0) \) containing only one element \( e(u) \) in each orbit \( [u] \). Let \( \sigma : G(0) \rightarrow G^F \) be a regular cross section of \( d_F \). Then any compactly supported Borel bounded function on \( G \) is pointwise limit of a uniformly bounded sequence \((f_n)_n\) of Borel functions supported on a compact set supporting \( f \), having the property that each \( f_n \) is a linear combination of functions of the form
\[
x \rightarrow g_1 (r(x)) g \left( \sigma (r(x)) x \sigma (d(x))^{-1} \right) g_2 (d(x))
\]
where \( g_1, g_2 \) are compactly supported Borel bounded function on \( G(0) \) and \( g \) is a compactly supported Borel bounded function on \( G^F \).

**Proof.** Let us endow \( \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \) with the topology induced from \( G(0) \times G^F \times G(0) \) as in Remark 15. The topology of \( \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \) is locally compact. Any compactly supported Borel bounded function on \( G(0) \times G^F \times G(0) \) is pointwise limit of uniformly bounded sequences \((f_n)_n\) of Borel functions supported on a compact set, such that each function \( f_n \) is a linear combination of functions of the form
\[
(u, x, v) \rightarrow g_1 (u) g (x) g_2 (v)
\]
where \( g_1, g_2 \) are compactly supported Borel bounded function on \( G(0) \) and \( g \) is a compactly supported Borel bounded function on \( G^F \). Consequently, any compactly supported Borel bounded function on \( \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \) has the same property. Since \( \phi : G \rightarrow \bigcup_{[u]} [u] \times G_{e(u)}^u \times [u] \) defined by
\[
\phi (x) = \left( r(x), \sigma (r(x)) x \sigma (d(x))^{-1}, d(x) \right)
\]
is a Borel isomorphism which carries compact sets to relatively compact sets, it follows that any compactly supported Borel bounded function on \( G \) can be represented as a pointwise limit of a uniformly bounded sequence \((f_n)_n\) of Borel functions supported on a compact set supporting \( f \), having the property that each \( f_n \) is a linear combination of functions of the form
\[
x \rightarrow g_1 (r(x)) g \left( \sigma (r(x)) x \sigma (d(x))^{-1} \right) g_2 (d(x))
\]
Corollary 17 Let $G$ be locally compact second countable locally transitive groupoid. Let $F$ be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to GF$ be a regular cross section of $d_F$. Then the linear span of the functions of the form

$$x \to g_1(r(x)) g \left( \sigma(r(x)) x \sigma(d(x))^{-1} \right) g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c \big(G^{(0)}\big)$ and $g \in \mathcal{B}_c \big(G^F_F\big)$, is dense in the full $C^*$-algebra of $G$.

Proof. Let $f$ be a function on $G$, defined by

$$f(x) = g_1(r(x)) g \left( \sigma(r(x)) x \sigma(d(x))^{-1} \right) g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c \big(G^{(0)}\big)$ and $g \in \mathcal{B}_c \big(G^F_F\big)$. Then $f \in \mathcal{B}_c(G)$, therefore it can be viewed as an element of the $C^*(G, \nu)$ as we note in Proposition 18. Each $f \in \mathcal{B}_c(G)$ (in particular in $C_c(G)$) is the limit (pointwise and consequently in the $C^*$-norm according to Corollary 18) of a uniformly bounded sequence $(f_n)_n$ of Borel functions supported on a compact set supporting $f$, having the property that each $f_n$ is a linear combination of functions of the required form. 

Proposition 18 Let $G$ is a locally compact second countable locally transitive groupoid endowed with a Haar system $\{\nu^u, u \in G^{(0)}\}$. Let $F$ be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to GF$ be a regular cross section of $d_F$. Then

$$C^*(G, \nu) = M^*(G, \nu) = M^*_\sigma(G, \nu).$$

Proof. We have proved that $C^*(G, \nu) = M^*(G, \nu)$. From the preceding corollary, it follows that the linear span of the functions of the form

$$x \to g_1(r(x)) g \left( \sigma(r(x)) x \sigma(d(x))^{-1} \right) g_2(d(x))$$

where $g_1, g_2 \in \mathcal{B}_c \big(G^{(0)}\big)$ and $g \in \mathcal{B}_c \big(G^F_F\big)$ is dense in $C^*(G, \nu)$. But this space is contained in $\mathcal{B}_\sigma(G)$. Therefore $C^*(G, \nu) = M^*(G, \nu) = M^*_\sigma(G, \nu)$. 

5 The case of principal proper groupoids case

Notation 19 Let $G$ be a locally compact second countable groupoid with proper orbit space. Let $F$ be a Borel subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to GF$ be a regular cross section for $d_F : GF \to$
$G^{(0)}, \sigma_{F}(x) = d(x)$ with $\sigma(e(v)) = c(v)$ for all $v \in G^{(0)}$. Let $q : G \rightarrow G_{F}^{\sigma}$ be defined by

$$q(x) = \sigma(r(x)) x \sigma(d(x))^{-1}$$

We shall endow $G_{F}^{\sigma}$ with the quotient topology induced by $q$. We shall denote by $C_{\sigma}(G)$ the linear span of the functions of the form

$$x \rightarrow g_{1}(r(x)) g \left(\sigma(r(x)) x \sigma(d(x))^{-1}\right) g_{2}(d(x))$$

where $g_{1}, g_{2} \in C_{c}(G^{(0)})$ and $g \in C_{c}(G_{F}^{\sigma})$.

**Proposition 20** With the Notation $\mathbb{1}$, if the space of continuous functions with compact support on $G_{F}^{\sigma}$ (with the respect to the quotient topology induced by $q$) separates the points of $G_{F}^{\sigma}$, then $C_{\sigma}(G)$ is dense in $C_{c}(G)$ (for the inductive limit topology). In particular, if the quotient topology induced by $q$ on $G_{F}^{\sigma}$ is a locally compact (Hausdorff) topology, then $C_{\sigma}(G)$ is dense in $C_{c}(G)$.

**Proof.** If the space of continuous functions with compact support on $G_{F}^{\sigma}$ (with the respect to the quotient topology induced by $q$) separates the points of $G_{F}^{\sigma}$, then $C_{\sigma}(G)$ separates the points of $G$. By Stone-Weierstrass Theorem, it follows that $C_{\sigma}(G)$ is dense in $C_{c}(G)$ (for the inductive limit topology).

**Proposition 21** Let $G$ be a locally compact principal groupoid. If $G$ is proper, then the quotient topology induced by $q$ on $G_{F}^{\sigma}$ is a locally compact (Hausdorff) topology. Consequently, $C_{\sigma}(G)$ is dense in $C_{c}(G)$ for the inductive limit topology (we use the Notation $\mathbb{1}$).

**Proof.** Let $\pi : G \rightarrow G^{(0)}/G$ be the canonical projection. Let us note that for a principal groupoid the condition

$$q(x) = q(y)$$

is equivalent with

$$\pi(r(x)) = \pi(r(y)).$$

First we shall prove that the topology on $G_{F}^{\sigma}$ is Hausdorff. Let $(x_{i})_{i}$ and $(y_{i})_{i}$ be two nets with $q(x_{i}) = q(y_{i})$ for every $i$. Let us suppose that $(x_{i})_{i}$ converges to $x$ and $(y_{i})_{i}$ converges to $y$. Then

$$\lim \pi(r(x_{i})) = \lim \pi(r(y_{i})) = \pi(r(x)) = \pi(r(y))$$

Hence $q(x) = q(y)$, and therefore the topology on $G_{F}^{\sigma}$ is Hausdorff. We shall prove that $q$ is open. If $(z_{i})_{i}$ is a net converging to $q(x)$ in $G_{F}^{\sigma}$, then $\pi \circ r(z_{i})$ converges to $\pi \circ r(x)$. Since

$$\pi \circ r : G \rightarrow G^{(0)}/G$$

is an open map, there is a net $(x_{i})_{i}$ converging to $x$, such that $\pi \circ r(x_{i}) = \pi \circ r(z_{i})$, and consequently $q(x_{i}) = q(z_{i}) = z_{i}$. Hence $q$ is an open map and the quotient topology induced by $q$ on $G_{F}^{\sigma}$ is locally compact.
Theorem 22 Let $G$ be a locally compact second countable groupoid with proper orbit space. Let $F$ be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section for $d : G^F \to G$. Let us assume that the quotient topology induced by $q$ on $G^F$ is a locally compact (Hausdorff) topology. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on $G$. Then

$$C^*(G, \nu) \subset M^*_\sigma(G, \nu) \subset M^*(G, \nu).$$

**Proof.** From Proposition 20 $C^*_\sigma(G)$ is dense in $C_c(G)$ for the inductive limit topology and hence is dense in $C^*(G, \nu)$. Since $C^*_\sigma(G) \subset B^*_\sigma(G)$, it follows that $C^*(G, \nu) \subset M^*_\sigma(G, \nu)$.

Corollary 23 Let $G$ be a locally compact second countable principal proper groupoid. Let $F$ be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section for $d : G^F \to G$. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on $G$. Then

$$C^*(G, \nu) \subset M^*_\sigma(G, \nu) \subset M^*(G, \nu).$$

**Proof.** Applying Proposition 21, we obtain that the quotient topology induced by $q$ on $G^F$ is a locally compact (Hausdorff) topology. Therefore $G$ satisfies the hypothesis of Theorem 22.

Definition 24 Let $\{\mu^u\}_{\hat{u}}$ be a system of measures on $G^{(0)}$ satisfying:

1. $\text{supp} (\mu^u) = [u]$ for all $\hat{u}$.

2. For all compactly supported continuous functions $f$ on $G^{(0)}$ the function

$$u \mapsto \int f(v) \mu^{\pi(u)}(v)$$

is continuous.

We shall say that the Hilbert bundle determined by the system of measures $\{\mu^u\}_{\hat{u}}$ has a continuous basis if there is sequence $f_1, f_2, \ldots$ of real valued continuous functions on $G^{(0)}$ such that $\dim (L^2(\mu^u)) = \infty$ if and only if $\|f_n\|_2$ for $n = 1, 2, \ldots$ and then $\{f_1, f_2, \ldots\}$ gives an orthonormate basis of $L^2(\mu^u)$, while $\dim (L^2(\mu^u)) = k < \infty$ if and only if $\|f_n\|_2 = 1$ for $n \leq k$, and $\|f_n\|_2 = 0$ for $n > k$ and then $\{f_1, f_2, \ldots f_k\}$ gives an orthonormate basis of $L^2(\mu^u)$.

Remark 25 Let $\{\mu_1^u\}_{\hat{u}}$ and $\{\mu_2^u\}_{\hat{u}}$ be two systems of measures on $G^{(0)}$ satisfying:

1. $\text{supp} (\mu_i^u) = [u]$ for all $\hat{u}$, $i = 1, 2$. 

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2. For all compactly supported continuous functions \( f \) on \( G^{(0)} \) the function 

\[
    u \mapsto \int f(v) \mu^{\pi(u)}(v)
\]

is continuous

Let us assume that the Hilbert bundles determined by the systems of measures \( \{\mu^{\hat{u}}_1\}_{\hat{u}} \) have continuous bases. Let \( f_1, f_2, \ldots \) be a continuous basis for Hilbert bundle determined by \( \{\mu^{\hat{u}}_1\}_{\hat{u}} \) and let \( g_1, g_2, \ldots \) be a continuous basis for Hilbert bundle determined by \( \{\mu^{\hat{u}}_2\}_{\hat{u}} \). Let us define a unitary operator \( U_{\hat{u}} : L^2(\mu^{\hat{u}}_1) \to L^2(\mu^{\hat{u}}_2) \) by

\[
    U_{\hat{u}}(f_n) = g_n \quad \text{for all } n
\]

Then the family \( \{U_{\hat{u}}\}_{\hat{u}} \) has the following properties:

1. For all Borel bounded function \( f \) on \( G^{(0)} \),

\[
    u \mapsto U_{\pi(u)}(f)
\]

is a bounded Borel function with compact support.

2. For all Borel bounded function \( f \) on \( G^{(0)} \),

\[
    U_{\pi(u)}(f) = \overline{U_{\pi(u)}(f)}
\]

3. For all compactly supported continuous functions \( f \) on \( G^{(0)} \) there is a sequence \( (h_n)_n \) of compactly supported continuous functions on \( G^{(0)} \) such that

\[
    \sup_{\hat{u}} \int |U_{\hat{u}}(f) - h_n|^2 d\mu^{\hat{u}}_2 \to 0 \ (n \to \infty)
\]

Indeed, we can define

\[
    h_n(v) = \sum_{k=1}^n g_k(v) \int f(u) f_k(u) \mu^{\pi(v)}(u).
\]

**Remark 26** Let \( G \) be a locally compact second countable groupoid with proper orbit space. Let \( F \) be a Borel subset of \( G^{(0)} \) containing only one element \( e(u) \) in each orbit \([u]\). Let us assume that \( F \cap [K] \) has a compact closure for each compact subset \( K \) of \( G^{(0)} \), and let \( \sigma : G^{(0)} \to GF \) be a regular cross section for \( d_F : GF \to G^{(0)} \). Let us endow \( GF \) with the quotient topology induced by \( q : G \to GF \)

\[
    q(x) = \sigma(r(x)) x \sigma(d(x))^{-1}, \ x \in G
\]

If \( g \in C_c \left(G^{(0)}_F\right) \) and \( g_1, g_2 \) are two functions on \( G^{(0)} \) with the property that there
is two sequences \((h^1_n)_n\) and \((h^2_n)_n\) of compactly supported continuous functions on \(G^{(0)}\) such that

\[
\sup_u \int |g_i - h^i_n|^2 \, d\mu^u_2 \to 0 \quad (n \to \infty)
\]

for \(i = 1, 2\), then

\[
x \xrightarrow{\mathcal{L}} g_1(r(x)) \ g \left( \sigma(r(x)) \ x \sigma(d(x))^{-1} \right) g_2(d(x))
\]

can be viewed as an element of \(C^* \ (G, \nu)\). Indeed, it is easy to see that

\[
\|f - (h^1_n \circ r) \ (g \circ q) \ (h^2_n \circ d)\|_{L^1} \to 0 \quad (n \to \infty).
\]

**Proposition 27** Let \(G\) be a locally compact second countable principal proper groupoid. Let \(\nu_i = \{\nu^u_i, u \in G^{(0)}\}, i = 1, 2\) be two Haar system on \(G\) and \((\{\beta^u_i\}, \{\mu^q_i\})\) the corresponding decompositions over the principal groupoid. If the Hilbert bundles determined by the systems of measures \(\{\mu^q_i\}_u\) have continuous bases, then the \(C^*\)-algebras \(C^* \ (G, \nu_1)\) and \(C^* \ (G, \nu_2)\) are \(*\)-isomorphic.

**Proof.** We use the Notation [19]. From Proposition 20 \(C^*_\sigma(G)\) is dense in \(C_c(G)\) for the inductive limit topology and hence is dense in \(C^* \ (G, \nu_1)\). We shall define a \(*\)-homomorphism \(\Phi\) from \(C^*_\sigma(G)\) to \(C^* \ (G, \nu_2)\). It suffices to define \(\Phi\) on the set of function on \(G\) of the form

\[
x \rightarrow g_1(r(x)) \ g(q(x)) \ g_2(d(x))
\]

where \(g_1, g_2 \in C_c(G^{(0)})\) and \(g \in C_c(G^F_F)\). Let \(\{U^u_i\}_u\) be the family of unitary operators with the properties stated in Remark 24 associated to the systems of measures \(\{\mu^q_i\}_u, i = 1, 2\).

Let us define \(\Phi\) by

\[
\Phi(f) = (x \to U_{\pi(r(x))} (g_1(r(x)) \ g(q(x)) \ U_{\pi(d(x))} (g_2)(d(x))))
\]

where \(f\) is defined by

\[
f(x) = g_1(r(x)) \ g(q(x)) \ g_2(d(x))
\]

with \(g_1, g_2 \in C_c(G^{(0)})\) and \(g \in C_c(G^F_F)\).

We noted in Remark 24 that the functions of the form \(\Phi(f)\) can be viewed as elements of \(C^* \ (G, \nu_2)\). With the same argument as in the proof of Theorem 4 it follows that \(\Phi\) can be extended to \(*\)-isomorphism between \(C^* \ (G, \nu_1)\) and \(C^* \ (G, \nu_2)\).
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