The BRST Charge for the $\hat{D}(2,1;\alpha)$ Non-Linear Quasi-Superconformal Algebra

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Abstract

The quantum BRST charge for the most general, two-dimensional, non-linear, $N=4$ quasi-superconformal algebra $\hat{D}(1,2;\alpha)$, whose linearisation is the so-called ‘large’ $N=4$ superconformal algebra, is constructed. The $\hat{D}(1,2;\alpha)$ algebra has $su(2)_{k^+} \oplus su(2)_{k^-} \oplus \tilde{u}(1)$ Kač-Moody component, and $\alpha = k^-/k^+$. As a pre-requisite to our construction, we check the $\hat{D}(1,2;\alpha)$ Jacobi identities and construct a classical BRST charge. Then, we analyse the quantum BRST charge nilpotency conditions and find the only solution, $k^+ = k^- = -2$. The $\hat{D}(1,2;1)$ algebra is actually isomorphic to the $SO(4)$-based Bershadsky-Knizhnik non-linear quasi-superconformal algebra. We argue about the existence of a new string theory with (i) the non-linearly realised $N=4$ world-sheet supersymmetry, (ii) non-unitary matter in a $\hat{D}(1,2;\alpha)$ representation of $k = -2$ and $c = -6$, and (iii) negative ‘critical dimension’.

1Supported in part by the ‘Deutsche Forschungsgemeinschaft’ and the NATO grant CRG 930789
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Introduction. The critical $N$-extended fermionic string theories with $N \leq 4$ world-sheet supersymmetries are based on the two-dimensional (2d) linear $N$-extended superconformal algebras (SCAs) which are gauged \[1\]. The string world-sheet fields usually form a linear $N$-extended superconformal multiplet coupled to the $N$-extended 2d conformal supergravity fields which are gauge fields of the $N$-extended SCA. The only known $N = 4$ string theory was constructed by gauging the ‘small’ linear $N = 4$ SCA \[1, 2\], and it is of some interest to know how many different $N = 4$ string theories can be constructed at all, despite of their apparently negative ‘critical dimensions’. The $N = 4$ fermionic strings are relevant in the search for the ‘universal string theory’ \[3\], and they are expected to have deep connections with integrable models \[4, 5\].

It has been known for some time that there are two different linear $N = 4$ SCAs which are (affine versions of) finitely-generated Lie superalgebras: the so-called ‘small’ linear $N = 4$ SCA with the $\hat{su}(2)$ Kac-Moody (KM) component \[1\], and the so-called ‘large’ linear $N = 4$ SCA with the $\hat{su}(2) \oplus \hat{su}(2) \oplus \hat{U}(1)$ KM component \[3, 7\]. Unlike the ‘small’ $N = 4$ SCA mentioned above, the ‘large’ $N = 4$ SCA has subcanonical charges, or ‘currents’ of conformal dimension $1/2$. This observation already implies that no supergravity or string theory based on the ‘large’ $N = 4$ SCA exists, because there are no 2d gauge fields which would correspond to the fermionic charges of dimension $1/2$. \[3\]

However, when a number of world-sheet supersymmetries exceeds two, there are, in fact, more opportunities to build up new string theories, by using 2d non-linear quasi-superconformal algebras which are known to exist for an arbitrary $N > 2$. By an $N$-extended quasi-superconformal algebra (QSCA) we mean a graded associative algebra whose contents is restricted to canonical charges of dimension 2, $3/2$ and 1, which (i) contains the Virasoro subalgebra, and (ii) $N$ real supercurrents of conformal dimension $3/2$, whose operator product expansion (OPE) has a stress tensor of dimension 2, (iii) satisfies the Jacobi identity, and (iv) has the usual spin-statistics relation. \[4\] By definition, a QSCA is an ‘almost’ usual SCA, except it may not be a Lie superalgebra but its OPEs have to be closed on quadratic composites of the fundamental set of canonical generators. The QSCAs can, therefore, be considered on equal footing with the $W$ algebras \[8\] without, however, having currents of spin higher than two. Though QSCAs do not belong, in general, to ordinary (finitely-generated) affine Lie superalgebras, but, so to say, to infinitely-generated Lie superalgebras, they are still closely related with finite Lie superalgebras \[10, 11\].

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3In conformal field theory, ‘currents’ of dimension $1/2$ are just free fermions \[5\].
4We exclude from our analysis all kinds of twisted (Q)SCAs with unusual relations between spin and statistics (they are, however, relevant for topological field theory and topological strings \[8\]).
The full classification of QSCAs has been done by Fradkin and Linetsky [10]. Their classification is based on the classical results of Kač [11] about finite simple Lie superalgebras. When \( N = 4 \), the only different QSCAs are just \( su(1,1|2) \) and \( \hat{D}(2,1;\alpha) \). The \( su(1,1|2) \) QSCA is, in fact, the ‘small’ \( SU(2) \)-based linear \( N = 4 \) SCA. The non-linear \( \hat{D}(2,1;\alpha) \) QSCA was extracted by Goddard and Schwimmer [12] from the ‘large’ linear \( N = 4 \) SCA by factoring out free fermions and boson. When \( \alpha = 1 \), i.e. \( k^+ = k^- \equiv k \), it reduces to the Bershadsky-Knizhnik \( SO(4) \)-based quasi-superconformal algebra [13, 14]. The \( \hat{D}(2,1;\alpha) \) QSCA has the non-linear \( N = 4 \) supersymmetry but includes only canonical charges, which implies the existence of a new \( N = 4 \) conformal supergravity and a new \( N = 4 \) string theory to be obtained by coupling this supergravity with appropriate 2d matter, along the lines of constructing the \( W \) gravities and \( W \) strings.

The algebra. Let \( J^{a\pm}(z) \) be the internal symmetry currents, where \( a, b, \ldots \) are the adjoint indices of \( SU(2) \), and \( \pm \) distinguishes between the two \( SU(2) \) factors. We label the four-dimensional fundamental (vector) representation space of \( SO(4) \) by indices \( i, j, \ldots \). The self-dual components of the KM currents, \( J^{\pm a}(z) \) can be unified into an antisymmetric tensor \( J^{ij}(z) \) in the adjoint of \( SO(4) \),

\[
J^{ij}(z) = (t^{a-})^{ij} J^{a-}(z) + (t^{a+})^{ij} J^{a+}(z),
\]

where the antisymmetric \( 4 \times 4 \) matrices \( t^{a\pm} \) satisfy the relations

\[
[t^{a\pm}, t^{b\pm}] = -2\varepsilon^{abc} t^{c\pm}, \quad [t^{a+}, t^{a-}] = 0, \quad \{t^{a\pm}, t^{b\pm}\} = -2\delta^{ab}.
\]

These matrices can be explicitly represented as

\[
(t^{a\pm})^{ij} = \varepsilon^{aij} \pm (\delta_i^4 \delta_j^4 - \delta_i^4 \delta_j^4),
\]

and satisfy the identity

\[
\sum_a (t^{a\pm})^{ij} (t^{a\pm})^{kl} = \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk} \pm \varepsilon^{ijkl}.
\]

The OPEs describing the action of \( J^{a\pm}(z) \) read

\[
J^{a\pm}(z) J^{b\pm}(w) \sim \frac{\varepsilon^{abc} J^{c\pm}(w)}{z - w} + \frac{-k^{\pm} \delta^{ab}}{2(z - w)^2},
\]

\[
J^{a\pm}(z) G^i(w) \sim \frac{1}{4} (t^{a\pm})^{ij} G^j(w) \frac{1}{z - w},
\]

where two arbitrary ‘levels’ \( k^{\pm} \) for both independent \( su(2) \) KM components have been introduced.
The general ansatz for the OPE of two fermionic supercurrents can be written as
\[
G^i(z)G^j(w) \sim b_1 \frac{\delta^{ij}}{(z-w)^3} + 2T(w)\delta^{ij} + \frac{1}{2}(b_2 + b_3) \left[ \frac{J^{ij}(w)}{(z-w)^2} + \frac{1}{2} \partial J^{ij}(w) \right] + \frac{1}{2}(b_2 - b_3) \varepsilon^{ijkl} \left[ \frac{J^{kl}(w)}{(z-w)^2} + \frac{1}{2} \partial J^{kl}(w) \right] + \frac{1}{4} b_4 \varepsilon^{ijkl} \varepsilon^{jklm} : \frac{J^{lm}G^{pq}}{z-w} : (w),
\]
or, equivalently,
\[
G^i(z)G^j(w) \sim \frac{b_1\delta^{ij}}{(z-w)^3} + \frac{1}{(z-w)^2} \left[ b_2(t^{a+})^{ij}J^{a+}(w) + b_3(t^{a-})^{ij}J^{a-}(w) \right] + \frac{1}{z-w} \left[ 2T(w)\delta^{ij} + \frac{1}{2} \partial \left\{ b_2(t^{a+})^{ij}J^{a+}(w) + b_3(t^{a-})^{ij}J^{a-}(w) \right\} \right] + \frac{b_4}{z-w} \left\{ t^{a+}J^{a+} - t^{a-}J^{a-} \right\}^{(i} \varepsilon_{k}^{ij} \left( t^{b+}J^{b+} - t^{b-}J^{b-} \right)^{j)k} : (w),
\]
where we have used the fact that
\[
\frac{1}{z-w} \varepsilon^{ijkl} T^{kl}(z) = \left( t^{a+} \right)^{ij} J^{a+}(z) - \left( t^{a-} \right)^{ij} J^{a-}(z),
\]
as a consequence of eq. (1).

Demanding associativity of the combinations \(TGG, JGG \) and \(GGG \) determines the parameters \(b_1, b_2, b_3, b_4 \), and, hence, all of the QSCA 3- and 4-point ‘structure constants’, viz.
\[
b_1 = \frac{4k^+k^-}{k^+ + k^- + 2}, \quad b_4 = \frac{-2}{k^+ + k^- + 2}, \quad b_2 = \frac{-4k^-}{k^+ + k^- + 2}, \quad b_3 = \frac{-4k^+}{k^+ + k^- + 2},
\]
as well as the central charge,
\[
c = \frac{6(k^+ + 1)(k^- + 1)}{k^+ + k^- + 2} - 3,
\]
in agreement with ref. [12].

We define \(\alpha\)-parameter of this \(\hat{D}(1, 2; \alpha)\) QSCA as a ratio of its two KM ‘levels’,
\[
\alpha \equiv \frac{k^-}{k^+},
\]
which measures the relative asymmetry between the two \(su(2)\) KM algebras in the whole algebra. When \(\alpha = 1\), i.e. \(k^- = k^+ = k\), the \(\hat{D}(1, 2; 1)\) QSCA is just the \(SO(4)\) Bershadsky-Knizhnik QSCA [13, 14] with the central charge \(c = 3k\).

In the vector notation, the \(\hat{D}(1, 2; \alpha)\) QSCA non-trivial OPEs take the form
\[
T^{ij}(z)G^{k}(w) \sim \frac{1}{z-w} \left[ \delta^{ik}G^{j}(w) - \delta^{ij}G^{k}(w) \right],
\]
and
\[
2T(w)\delta^{ij} + \frac{1}{2} \partial \left\{ b_2(t^{a+})^{ij}J^{a+}(w) + b_3(t^{a-})^{ij}J^{a-}(w) \right\} + \frac{b_4}{z-w} \left\{ t^{a+}J^{a+} - t^{a-}J^{a-} \right\}^{(i} \varepsilon_{k}^{ij} \left( t^{b+}J^{b+} - t^{b-}J^{b-} \right)^{j)k} : (w),
\]
where we have used the fact that
\[
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In the vector notation, the \(\hat{D}(1, 2; \alpha)\) QSCA non-trivial OPEs take the form
\[
T^{ij}(z)G^{k}(w) \sim \frac{1}{z-w} \left[ \delta^{ik}G^{j}(w) - \delta^{ij}G^{k}(w) \right],
\]
\[ J^{ij}(z)J^{kl}(w) \sim \frac{1}{z-w} \left[ \delta^{ik}J^{jl}(w) - \delta^{ik}J^{jl}(w) + \delta^{il}J^{jk}(w) - \delta^{il}J^{jk}(w) \right] - \frac{1}{2}(k^+ + k^-) \frac{\delta_{ij} \delta^{kl} - \delta^{il} \delta^{jk}}{(z-w)^2} - \frac{1}{2}(k^+ - k^-) \frac{\varepsilon^{ijkl}}{(z-w)^2} , \]

\[ G^i(z)G^j(w) \sim \frac{4k^+ k^-}{(k^+ + k^- + 2)(z-w)^3} \frac{\delta_{ij}}{z-w} + \frac{2T(w)\delta_{ij}}{(z-w)^2} - \frac{k^+ + k^-}{k^+ + k^- + 2} \left[ \frac{2J^{ij}(w)}{(z-w)^2} + \frac{\partial J^{ij}(w)}{z-w} \right] - \frac{\delta_{ij}}{2(k^+ + k^- + 2)} \frac{\varepsilon^{ijkl}}{(z-w)} . \]

In terms of the Fourier modes of the currents \( \mathcal{O}_\lambda \) of dimension \( \lambda \), defined by \( \mathcal{O}_n = \oint (dz/2\pi i) z^{n+\lambda-1} \mathcal{O}_\lambda(z) \), one finds instead

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} , \\
[L_m, G^i_r] &= (\frac{m-r}{m})G^i_{m+r} , \quad [L_m, J^{a^\pm}_{n}] = -nJ^{a^\pm}_{m+n} , \\
[J^{a^\pm}_{m}, G^i_r] &= \frac{1}{2} (t^{a^\pm})^{ij} G^j_{m+r} , \quad [J^{a^\pm}_{m}, J^{b^\pm}_{n}] = \varepsilon^{abc} J^{c^\pm}_{m+n} - \frac{1}{2} k^\pm m \delta^{ab} \delta_{m+n} , \\
\{G^i_r, G^j_s\} &= \frac{2k^+ k^-}{k^+ + k^- + 2} (r^2 - \frac{1}{2}) \delta^{ij} + 2\delta^{ij} L_{m+n} \\
&+ \frac{2}{k^+ + k^- + 2} \left( s - r \right) \left[ k^- (t^{a^+})^{ij} J^{a^+}_{r+s} + k^+ (t^{a^-})^{ij} J^{a^-}_{r+s} \right] \\
&- \frac{2}{k^+ + k^- + 2} \left( t^{a^+} J^{a^+} - t^{a^-} J^{a^-} \right)^{(i)}_k \left( t^{b^+} J^{b^+} - t^{b^-} J^{b^-} \right)^{(j)}_k .
\end{align*}
\]

Though \( \hat{D}(1,2;\alpha) \) is a non-linear QSCA, it can be turned into a linear SCA by adding some ‘auxiliary’ fields, namely, four free fermions \( \psi^i(z) \) of dimension 1/2, and a free bosonic current \( U(z) \) of dimension 1, defining a \( \widehat{U(1)} \) KM algebra \[12\]. The new fields have canonical OPEs,

\[
\psi^i(z)\psi^j(w) \sim \frac{-\delta^{ij}}{z-w} , \quad U(z)U(w) \sim \frac{-1}{(z-w)^2} .
\]

The fermionic fields \( \psi^i(z) \) transform in a (2,2) representation of \( SU(2) \otimes SU(2) \),

\[
J^{a^\pm}(z)\psi^j(w) \sim \frac{1}{2} (t^{a^\pm})^{ij} \psi^j(w) ,
\]

whereas the singlet \( U(1) \)-current \( U(z) \) can be thought of as derivative of a free scalar boson, \( U(z) = i\partial\phi(z) \).
Let us now define the new currents [12]

\[ T_{\text{tot}} = T - \frac{1}{2} : U^2 : - \frac{1}{2} : \partial \psi^i \psi^i : \]

\[ G^i_{\text{tot}} = G^i - U \psi^i + \frac{1}{3 \sqrt{2(k^+ + k^- + 2)}} \epsilon^{ijkl} \psi^j \psi^k \psi^l \]

\[ - \sqrt{\frac{2}{k^+ + k^- + 2}} \psi^j \left[ (t^{a+})^j i J^{a+} - (t^{a-})^j i J^{a-} \right], \]

\[ J^{a\pm}_{\text{tot}} = J^{a\pm} + \frac{1}{4} (t^{a+})^j i \psi^j \psi^j, \]

in terms of the initial \( \hat{D}(1, 2; \alpha) \) QSCA currents \( T, G^i \) and \( J^{a\pm} \). Then the following set of affine generators

\[ \{ T_{\text{tot}}, \ G^i_{\text{tot}}, \ J^{a\pm}_{\text{tot}}, \ \psi^i, \ U \} \]

has closed OPEs among themselves, defining a linear ‘large’ \( N = 4 \) SCA with the \( su(2) \oplus su(2) \oplus u(1) \) KM component! Explicitly, the non-trivial OPEs of this ‘large’ \( N = 4 \) SCA are given by (cf. refs. [3, 7])

\[ T_{\text{tot}}(z) T_{\text{tot}}(w) \sim \frac{1}{2} (c + 3) \frac{1}{(z - w)^4} + \frac{2T_{\text{tot}}(w)}{(z - w)^2} + \frac{\partial T_{\text{tot}}(w)}{z - w}, \]

\[ T_{\text{tot}}(z) O(w) \sim \frac{h O(w)}{(z - w)^2} + \frac{\partial O(w)}{z - w}, \]

\[ J^{a\pm}_{\text{tot}}(z) J^{a\pm}_{\text{tot}}(w) \sim \frac{\epsilon^{abc} J^{c\pm}_{\text{tot}}(w)}{z - w} - \frac{(k^\pm + 1) \delta^{ab}}{2(z - w)^2}, \]

\[ J^{a\pm}_{\text{tot}}(z) G^i_{\text{tot}}(w) \sim \frac{1}{2} (t^{a\pm})^j i G^j_{\text{tot}}(w) \pm \sqrt{\frac{k^\pm + 1}{2(k^+ + k^- + 2)}} \frac{(t^{a\pm})^j i \psi^j}{(z - w)^2}, \]

\[ G^i_{\text{tot}}(z) G^j_{\text{tot}}(w) \sim \frac{2(c + 3) \delta^{ij}}{(z - w)^3} + \frac{2T_{\text{tot}}(w) \delta^{ij}}{z - w} - \frac{2}{k^+ + k^- + 2} \left[ \frac{2}{(z - w)^2} + \frac{1}{z - w} \right] \]

\[ \times \left[ (k^- + 1)(t^{a+})^j i J^{a+}_{\text{tot}}(w) + (k^+ + 1)(t^{a-})^j i J^{a-}_{\text{tot}}(w) \right], \]

\[ \psi^i(z) G^j_{\text{tot}}(w) \sim \frac{1}{z - w} \sqrt{\frac{2}{k^+ + k^- + 2}} \left[ (t^{a+})^j i J^{a+}_{\text{tot}}(w) - (t^{a-})^j i J^{a-}_{\text{tot}}(w) \right] + \frac{U(w) \delta^{ij}}{z - w}, \]

\[ U(z) G^i_{\text{tot}}(w) \sim \frac{\psi^i(w)}{(z - w)^2}, \]

where \( O \) stands for the generators \( G_{\text{tot}}, J_{\text{tot}} \) and \( \psi \) of dimension 3/2, 1 and 1/2, respectively, and the \( \hat{D}(1, 2; \alpha) \) QSCA central charge \( c \) is given by eq. (9).\(^5\)

\(^5\)Unlike ref. [3], we put forward the underlying QSCA structure in our notation. It is advantageous to express a given algebra in terms of the smaller number of fundamental charges, whenever it is possible.
Having restricted ourselves to the (Neveu-Schwarz–type, for definiteness) Fourier modes \((L_{tot})_{\pm 1,0}, (G_{tot})_{\pm 1/2,0}, (J_{a_{tot}})^{\pm 0})\), we get a finite-dimensional Lie superalgebra which is isomorphic to the simple Lie superalgebra \(D(1,2;\alpha)\) from the Kač list \([11]\). This explains the reason why we use almost the same (with hat) notation for our affine (infinite-dimensional) QSCA \(\hat{D}(1,2;\alpha)\) defined by eqs. (11) or (12). Note that the finite Lie superalgebra of the ‘large’ \(N = 4\) SCA in eq. (17), defining a ‘linearised’ version of the \(\hat{D}(1,2;\alpha)\) QSCA in eq. (11), is not simple, but contains a \(U(1)\) piece, in addition to the finite-dimensinal \(D(1,2;\alpha)\) subalgebra. The finite-dimensional simple Lie superalgebras \(D(2,1;\alpha)\) at various \(\alpha\) values are not, in general, isomorphic to each other (except of the isomorphism under \(\alpha \to \alpha^{-1}\), interchanging the two \(su(2)\) factors) \([11]\). This is enough to argue about the non-equivalence (for different \(\alpha\)) of the \(\hat{D}(1,2;\alpha)\) QSCAs, which are their affine generalisations.

It is also worthy to notice that the KM ‘levels’ and the central charge of the ‘large’ \(N = 4\) SCA and those of the underlying \(\hat{D}(1,2;\alpha)\) QSCA are different according to eq. (16), namely

\[
k_{\text{large}}^\pm = k^\pm + 1, \quad c_{\text{large}} = c + 3,
\]

which is quite obvious because of the new fields introduced. The exceptional ‘small’ \(N = 4\) SCA with the \(\widehat{su}(2)\) KM component \([7]\) follows from eq. (17) in the limit \(\alpha \to \infty\) or \(\alpha \to 0\), where either \(k^- \to \infty\) or \(k^+ \to \infty\), respectively, and the \(\widehat{su}(2) \oplus \widehat{u}(1)\) KM component decouples from the rest of the algebra. Taking the limit results in the central charge

\[
c_{\text{small}} = 6k,
\]

where \(k\) is an arbitrary ‘level’ of the remaining \(su(2)\) KM component. For an arbitrary \(\alpha\), the ‘large’ \(N = 4\) SCA contains two ‘small’ \(N = 4\) SCAs \([7]\).

The BRST charge. Despite of the apparent non-linearity of the \(\hat{D}(1,2;\alpha)\) QSCA, its quantum BRST charge should be in correspondence with its classical BRST charge, up to renormalisation. The classical BRST charge having the vanishing Poisson bracket with itself can, in fact, be constructed for any algebra of first-class constraints \([13]\). This provides us with a good ansatz for the quantum BRST charge we are looking for. The similar procedure was applied to obtain the quantum BRST charge for the non-linear quantum \(W_3\) algebra \([16]\), and later generalised to any quadratically non-linear \(W\)-type algebra in ref. \([17]\). The nilpotency conditions always require the total (matter + ghosts) central charge to vanish, but also lead to some more constraints on the QSCA parameters, whose consistency is not guaranteed. This is because the constraints imposed by the BRST charge nilpotency condition may be in
conflict with the constraints dictated by the QSCA Jacobi identities.

The BRST quantisation of the $\hat{D}(1, 2; \alpha)$ QSCA requires the following ghosts to be introduced:

- the conformal ghosts $(b, c)$, an anticommuting pair of world-sheet free fermions of conformal dimensions $(2, -1)$, respectively;
- the $N$-extended superconformal ghosts $(\beta^i, \gamma^i)$ of conformal dimensions $(3/2, -1/2)$, respectively, in the fundamental representation of $SO(4) \cong SU(2) \otimes SU(2)$;
- the two pairs of $SU(2)$ ghosts $(\tilde{b}^a, \tilde{c}^a)$ of conformal dimensions $(1, 0)$, respectively, each one in the adjoint representation of $SU(2)$.

The reparametrisation ghosts

\[
b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-2}, \quad c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n+1},
\]

have the following OPE and anticommutation relations:

\[
b(z) c(w) \sim \frac{1}{z-w}, \quad \{c_m, b_n\} = \delta_{m+n,0}.
\]

The superconformal ghosts

\[
\beta^i(z) = \sum_{r \in \mathbb{Z}(+1/2)} \beta_r^i z^{-r-3/2}, \quad \gamma^i(z) = \sum_{r \in \mathbb{Z}(+1/2)} \gamma_r^i z^{-r+1/2},
\]

satisfy

\[
\beta^i(z) \gamma^j(w) \sim -\frac{\delta_{ij}}{z-w}, \quad [\gamma_r^i, \beta_s^j] = \delta_{r+s,0}.
\]

An integer or half-integer moding of these generators corresponds to the usual distinction between the Ramond- and Neveu-Schwarz–type sectors.

Finally, the fermionic $SU(2) \otimes SU(2)$ internal symmetry ghosts

\[
\tilde{b}^{a\pm}(z) = \sum_{n \in \mathbb{Z}} \tilde{b}_n^{a\pm} z^{-n-1}, \quad \tilde{c}^{a\pm}(z) = \sum_{n \in \mathbb{Z}} \tilde{c}_n^{a\pm} z^{-n},
\]

have

\[
\tilde{b}^{a\pm}(z) \tilde{c}^{a\pm}(w) \sim \frac{\delta_{ab}}{z-w}, \quad \{c_m^{a\pm}, \tilde{b}_n^{b\pm}\} = \delta_{ab} \delta_{m+n,0}.
\]

In general, given a set of bosonic generators $B_i$ and fermionic generators $F_\alpha$, satisfying a graded non-linear associative algebra,

\[
\{B_i, B_j\}_{\text{P.B.}} = f_{ijk} B_k,
\]

\[
\{B_i, F_\alpha\}_{\text{P.B.}} = f_{i\alpha \beta} F_\beta,
\]

\[
\{F_\alpha, F_\beta\}_{\text{P.B.}} = f_{\alpha \beta} B_i + \Lambda_{\alpha \beta} B_i B_j,
\]

8
in terms of the graded Poisson (or Dirac) brackets, with some 3-point and 4-point ‘structure constants’, \( f_{ij}^k \), \( f_{\alpha \beta} \), \( f_{\alpha \beta}^i \) and \( \Lambda_{\alpha \beta} \), respectively, whose values are supposed to be determined by solving the Jacobi identities, the classical BRST charge \( Q \), satisfying the classical ‘master equation’ \( \{ Q , Q \}_{\text{PB}} = 0 \), is given by \[ Q = c_n B_n + \gamma^a F_a + \frac{1}{2} f_{ij} k_i k_j c^j + f_{\alpha \beta} \beta_{\gamma}^a c^j - \frac{1}{2} f_{\alpha \beta} b_{ij} \gamma^a \gamma^b \gamma^c . \] (27)

Eq. (27) may serve as the starting point in a construction of a quantum BRST charge \( Q_{\text{BRST}} \) associated with a quantum non-linear superalgebra. Since we are actually interested in quantum QSCAs, we can assume that all operators are currents, with a holomorphic dependence on \( z \) (or, equivalently, with an additional affine index), and replace the (graded) Poisson brackets by (anti)commutators. In addition, in quantum theory, one must take into account central extensions and the normal ordering needed for defining products of bosonic generators. Although no general procedure seems to exist, which would explain how to fully ‘renormalise’ the naively quantized (i.e. normally-ordered) charge \( Q \) to a quantum-mechanical operator \( Q_{\text{BRST}} \), the answer is known for a particular class of quantum algebras of the \( W \)-type. Similarly to the quantum \( W_3 \) algebra case considered in ref. [10], the only non-trivial modification in quantum theory essentially amounts to a *multiplicative* renormalisation of the structure constants \( f_{\alpha \beta} \). In our case of the \( \hat{D}(1, 2; \alpha) \) QSCA, this gives the following *ansatz*:

\[
Q_{\text{BRST}} = c_n L_n + \gamma^a F_a + \tilde{c}_{\alpha \beta} F_{\alpha \beta} - \frac{1}{2} (m - n) c_{m n} c_{m n} + n c_{m n} \tilde{c}_{m n}^a \tilde{c}_{m n}^a \\
+ \left( \frac{m}{2} - r \right) c_{m n} \beta_{m n}^i \tilde{\gamma}_{i m n} - b_{r s} \gamma_{r s}^i \gamma_{r s}^i - \frac{1}{2} [c_{m n}^a (t^a)^{ij} \beta_{m n}^i \tilde{\gamma}_{j m n}^i + \eta_2 b_{2 r} (r - s) \tilde{b}_{r s}^{a r} (t^{a r})^{ij} \gamma_{r s}^i \gamma_{r s}^j \\
+ \eta_3 b_{3 r} (r - s) \tilde{b}_{r s}^{a r} (t^{a r})^{ij} \gamma_{r s}^i \gamma_{r s}^j \\
- \frac{1}{2} e^{abc} c_{m n} \beta_{m n}^i \tilde{\gamma}_{a c m n} - \frac{1}{2} e^{abc} c_{m n} \beta_{m n}^i \tilde{\gamma}_{d c m n} - \frac{1}{2} d_b \Lambda_{a A} \Lambda_{a A} J_{r s}^{a B} \tilde{b}_{r s}^{a B} \tilde{b}_{r s}^{d D} \tilde{\gamma}_{r s}^{i} \gamma_{r s}^j \\
- \frac{1}{2} b_{d} \Lambda_{a A} \Lambda_{a A} J_{r s}^{a B} \tilde{b}_{r s}^{a B} \tilde{b}_{r s}^{d D} \tilde{\gamma}_{r s}^{i} \gamma_{r s}^j \, ,
\] (28)

where two quantum renormalisation parameters \( \eta_2 \) and \( \eta_3 \) have been introduced, and \( \Lambda_{a A} \) denote the \( \hat{D}(1, 2; \alpha) \) QSCA 4-point ‘structure constants’ \( (A = +, -) \),

\[
\Lambda_{a A} \equiv (t^a + J^a(z) - t^{a -} J^a(z))^{(i} \, _k \left( t^{b+} J^{b+}(z) - t^{b-} J^{b-}(z) \right)^{j)k} .
\] (29)

We find always useful to represent a quantum BRST charge as

\[
Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} \, \tilde{j}_{\text{BRST}}(z) ,
\] (30)
where the BRST current \( j_{\text{BRST}}(z) \) is defined \textit{modulo} total derivative. In particular, eq. (28) can be rewritten as

\[
j_{\text{BRST}}(z) = cT + \gamma^i G^i + \bar{c}^{A} J^{A} + bc\partial c - \partial \bar{c}^{A} \partial c^{A} - \frac{1}{2} \partial \gamma^i \partial \beta^i - \frac{3}{2} c \beta^i \partial \gamma^i - b \gamma^i \gamma^i
\]

\[
- \frac{1}{2} \partial \bar{c}^{A} (t^{a A}) \beta^i \gamma^j - \left[ \eta_2 b_2 \bar{b}^{a+} (t^{a+})^{ij} + \eta_3 b_3 \bar{b}^{a-} (t^{a-})^{ij} \right] (\gamma^i \partial \gamma^j - \gamma^j \partial \gamma^i)
\]

\[
- \frac{1}{2} \varepsilon^{abc} \bar{c}^{a+} \bar{c}^{b+} \bar{c}^{c+} \bar{b}^{c+} - \frac{1}{2} \varepsilon^{abc} \bar{c}^{a-} \bar{c}^{b-} \bar{c}^{c-} - \frac{1}{2} b_4 \Lambda_{a AbB}^i \Lambda_{a AbB}^j \gamma^i \gamma^j
\]

\[
- \frac{1}{24} b_4^{ij} \Lambda_{a AbB}^i \Lambda_{c D d D}^{k l} \varepsilon [a c e b B d D] \bar{b}^{a c} \bar{b}^{b B} \bar{b}^{d D} (\bar{b}^{e+} + \bar{b}^{e-}) \gamma^i \gamma^j \gamma^k \gamma^l.
\]

(31)

The central extensions (anomalies) of the ghost-extended QSCA need not form a linear supermultiplet, and they actually do not. Therefore, the vanishing of any anomaly alone does \textit{not} automatically mean the vanishing of the others, unlike in the linear case.

The most tedious part of calculational handwork in computing \( Q_{\text{BRST}}^2 \) can be avoided when using either the Mathematica Package for computing OPEs [18] or some of the general results in ref. [17]. In particular, as was shown in ref. [17], quantum renormalisation of the 3-point structure constants in the quantum BRST charge should be \textit{multiplicative}, whereas the non-linearity 4-point ‘structure constants’ should \textit{not} be renormalised at all — the facts already used in the BRST charge ansatz above. Most importantly, among the contributions to the \( Q_{\text{BRST}}^2 \), only the terms \textit{quadratic} in the ghosts are relevant. Their vanishing imposes the constraints on the central extension coefficients of the QSCA and simultaneously determines the renormalisation parameter \( \eta \). The details can be found in the appendices of ref. [17]. The same conclusion comes as a result of straightforward calculation on computer. Therefore, finding out the nilpotency conditions amounts to calculating only a few terms ‘by hands’, namely, those which are quadratic in the ghosts. This makes the whole calculation as simple as that in ordinary string theories based on linear SCAs [3].

The 2-ghost terms in the \( Q_{\text{BRST}}^2 \) arise from single contractions of the first three linear (in the ghosts) terms of \( Q_{\text{BRST}} \) with themselves and with the next cubic terms of eq. (28), and from double contractions of the latter among themselves. They result in the pole contributions to \( j_{\text{BRST}}(z)j_{\text{BRST}}(w) \), proportional to \((z - w)^{-n}\) with \( n = 1, 2, 3, 4 \). All the residues have to vanish modulo total derivative. We find

\[\text{\footnotesize The total derivative can be fixed by requring the } j_{\text{BRST}}(z) \text{ to transform as a primary field.}\]
• from the terms \(c(z)c(w)/(z - w)^4\):

\[
c_{\text{tot}} \equiv c + c_{\text{gh}} = \left[ \frac{6(k^+ + 1)(k^- + 1)}{k^+ + k^- + 2} - 3 \right] + 6 = 0 ,
\]

(32a)

where the central charge \(c\) is given by eq. (9) and \(c_{\text{gh}} = +6;\)

• from the terms \(\gamma^i(z)\gamma^i(w)/(z - w)^3\):

\[
s_{\text{tot}} \equiv b_1 + (b_1)_{\text{gh}} = b_1 + \frac{3}{2}b_4(k^+ + k^-) - 6(\eta_2b_2 + \eta_3b_3) + 2 = 0 ,
\]

(32b)

where the parameters \(b_1, b_2\) and \(b_4\) are given by eq. (8);

• from the terms \(\tilde{c}^{a\pm}(z)\tilde{c}^{a\pm}(w)/(z - w)^2\):

\[
k_{\text{tot}}^{\pm} \equiv k^{\pm} + 2 = 0 ,
\]

(32c)

• from the terms \(J^{a\pm}(t^{a\pm})^{ij}\gamma^i\partial\gamma^j/(z - w)\):

\[-2\eta_2 b_2 - 2b_4 = -2\eta_3 b_3 - 2b_4 = 0 .
\]

(32d)

Eq. (32a) just means the vanishing total central charge, where the value of \(c_{\text{gh}}\) is dictated by the standard formula of conformal field theory 

\[
c_{\text{gh}} = 2\sum_\lambda n_\lambda (-1)^{2\lambda+1} \left( 6\lambda^2 - 6\lambda + 1 \right)
\]

\[
= 1 \times (-26) + 4 \times (+11) + 6 \times (-2) = +6 ,
\]

(33)

\(\lambda\) is conformal dimension and \(n_\lambda\) is a number of the conjugated ghost pairs: \(\lambda = 2, 3/2, 1\) and \(n_\lambda = 1, 4, 6\), respectively. Eq. (32b) can be interpreted as the vanishing total supersymmetric anomaly. Since the supersymmetry is non-linearly realised, this anomaly does not have to vanish as a consequence of the other equations (32), but, fortunately, it does in our case. Finally, eqs. (32c,d) determine \(k^{\pm}\) and \(\eta_{2,3}\).

The only consistent solution to eq. (32) is

\[
k \equiv k^+ = k^- = -2 .
\]

(34)

This means that the BRST quantisation of the non-linear \(\hat{D}(1, 2; \alpha)\) QSCA can only be consistent if both its \(su(2)\) KM components enter symmetrically, i.e. when this quantum non-linear algebra is actually the \(SO(4)\)-based Bershadsky-Knizhnik QSCA of \(k = -2\) and \(c = 3k = -6\). This is to be compared with the known fact that the quantum BRST charge for the ‘small’ \(N = 4\) SCA, whose all central terms are related and proportional to central charge, is only nilpotent when \(c = -12\).
A connection between the non-linear $SO(4)$-based Bershadsky-Knizhnik QSCA and the ‘small’ linear $SU(2)$-based SCA exists via the linearisation of the former into the ‘large’ linear $SU(2) \otimes SU(2) \otimes U(1)$-based SCA and taking the limit either $k^+ \to 0$ or $k^- \to 0$. Since (i) there is no nilpotent QSCA BRST charge for the case of $k^+ \neq k^-$, and (ii) it does not make sense to gauge and BRST quantise all the generators of the ‘large’ $N = 4$ linear SCA, there seems to be no direct connection between the corresponding BRST charges.

**Conclusion.** In our letter we constructed the quantum BRST charge for the quantum $\hat{D}(2,1;\alpha)$ QSCA. It is only nilpotent if $k^+ = k^- = -2$, when the $\hat{D}(2,1;\alpha)$ QSCA is isomorphic to the $SO(4)$-based Bershadsky-Knizhnik QSCA.

Gauging the local symmetries of the $SO(4)$-based Bershadsky-Knizhnik QSCA results in the positive total ghost central charge contribution, $c_{\text{gh}} = 6$. When adding the matter $(\psi^i, \phi)$ to linearise this algebra, one adds $+3$ to the total central charge. In addition, the anomaly-free solution requires $k = -2 < 0$. Therefore, there is no way to build an anomaly-free string theory when using only unitary representations. With a non-unitary representation of the $SO(4)$-based QSCA of $k = -2$, one can get the desired anomaly-free matter contribution, $c_m = -6$. Unfortunately, a space-time interpretation and a physical significance of the construction, if any, then become obscure. Despite of all this, we believe that it is worthy to know how many string models, consistent from the mathematical point of view, can be constructed. Requiring the existence of a nilpotent quantum BRST operator, one can construct only two of them having $N = 4$ supersymmetry, either linearly or non-linearly realised.

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