FALTINGS HEIGHTS OF BIG CM CYCLES
AND DERIVATIVES OF $L$-FUNCTIONS

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Abstract. We give a formula for the values of automorphic Green functions on the special rational 0-cycles (big CM points) attached to certain maximal tori in the Shimura varieties associated to rational quadratic spaces of signature $(2d,2)$. Our approach depends on the fact that the Green functions in question are constructed as regularized theta lifts of harmonic weak Maass forms, and it involves the Siegel-Weil formula and the central derivatives of incoherent Eisenstein series for totally real fields. In the case of a weakly holomorphic form, the formula is an explicit combination of quantities obtained from the Fourier coefficients of the central derivative of the incoherent Eisenstein series. In the case of a general harmonic weak Maass form, there is an additional term given by the central derivative of a Rankin-Selberg type convolution.

1. Introduction

In 1985, Gross and Zagier discovered a beautiful factorization formula for singular moduli [GZ]. This has inspired a lot of interesting work, including Dorman’s generalization to odd discriminants [Do], Elkies’s examples on Shimura curves [El] and Lauter’s conjecture on the Igusa $j$-invariants ([GL, Ya2, Ya1]), among others. In his thesis, Schofer [Scho] proved a much more general factorization formula for the ‘small’ CM values of Borcherds modular functions on a Shimura variety of orthogonal type via regularized theta liftings. The proof is very natural and is based on a method introduced in [Ku3]. Two of the authors adapted the same idea to study the ‘small’ CM values of automorphic Green functions and discovered a direct link between the CM value and the central derivative of a certain Rankin-Selberg $L$-function. This direct link is used to give a different proof of the well-known Gross-Zagier formula [BY2]. Here ‘small’ means that the CM cycles are associated to quadratic imaginary quadratic fields. On the other hand, the two authors also extended Gross and Zagier’s factorization formula, using a method close to Gross and Zagier’s original idea, to ‘big’ CM values of some Hilbert modular functions on a Hilbert modular surface. Here ‘big’ means that the CM cycle is associated to a maximal torus of the reductive group giving the Hilbert modular surface.

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A motivating question for this paper is whether this ‘big’ CM value result can also be derived using the regularized theta lifting method in [Scho] and [BY2], which is more natural and simpler. While the small CM cycles are constructed systematically and associated to rational negative two planes in the quadratic space defining the Shimura variety, no big CM cycles are constructed this way. In Section 2, we describe a way to construct big CM cycles in some special Shimura varieties (including Hilbert modular surfaces), and study their Galois conjugates. In Sections 3–5, we extend the CM value result in [BY2] to this situation. In Section 6, we restrict to the special case of Hilbert modular surfaces and give a new proof of the main results in [BY1] and a generalization. Actually, to get the CM cycles in [BY1] from this construction is not straightforward and quite interesting. An arithmetic application is given at the end of Section 6. We now describe this work in more detail.

Let \((V, Q_V)\) be a rational quadratic space of signature \((2d, 2)\) for some positive integer \(d \geq 1\). Let \(G = \text{GSpin}(V)\) and let \(K \subset G(\hat{\mathbb{Q}})\) be a compact open subgroup \(1\). Let \(D\) be the associated Hermitian domain of oriented negative 2-planes in \(V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}\), and let

\[
X_K = G(\mathbb{Q}) \backslash \left( \mathbb{D} \times G(\hat{\mathbb{Q}})/K \right)
\]

be the associated Shimura variety which has a canonical model over \(\mathbb{Q}\). Assume that there is a totally really number field \(F\) of degree \(d + 1\) and a two-dimensional \(F\)-quadratic space \((W, Q_W)\) of signature

\[
\text{sig}(W) = ((0, 2), (2, 0), \ldots, (2, 0))
\]

with respect to the \(d + 1\) embeddings \(\{\sigma_j\}_j\) such that

\[
V = \text{Res}_{F/\mathbb{Q}} W, \quad Q_V(x) = \text{tr}_{F/\mathbb{Q}} Q_W(x).
\]

Then there is an orthogonal direct sum decomposition

\[
V(\mathbb{R}) = \bigoplus_j W_{\sigma_j}, \quad W_{\sigma_j} = W \otimes_{F, \sigma_j} \mathbb{R}.
\]

The negative 2-plane \(W_{\sigma_0}\) gives rise to two points \(z_0^\pm\) in \(\mathbb{D}\). Let \(T\) be the preimage of \(\text{Res}_{F/\mathbb{Q}} \text{SO}(W) \subset \text{SO}(V)\) in \(G\). Then \(T\) is a maximal torus associated to the CM number field \(E = F(\sqrt{-\det W})\), and we obtain a ‘big’ CM cycle in \(X_K\):

\[
Z(W, z_0^\pm) = T(\mathbb{Q}) \backslash \left( \{ z_0^\pm \} \times T(\hat{\mathbb{Q}})/K_T \right),
\]

where \(K_T = T(\hat{\mathbb{Q}}) \cap K\). The CM cycle \(Z(W, z_0^\pm)\) is defined over \(F\), and the formal sum \(Z(W)\) of all its Galois conjugates as a 0-cycle in \(X_K\) is defined over \(\mathbb{Q}\). We refer to Section 2 for details.

Let \(L\) be an even integral lattice in \(V\) such that \(K\) preserves \(L\) and acts trivially on \(L'/L\), where \(L'\) is the dual lattice. Let \(S_L\) be the space of locally constant functions on \(\hat{V} = V \otimes \hat{\mathbb{Q}}\) which are \(L\)-invariant and have support in \(\hat{L'}\), and let \(\rho_L\) be the associated

\[\text{We write } \hat{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \text{ for the finite adèles of } \mathbb{Q}, \text{ where } \hat{\mathbb{Z}} = \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z}.\]
Weil representation’ of $\text{SL}_2(\mathbb{Z})$ on it. For each harmonic weak Maass form $f \in H_{1-d, \rho_L}$, there is a corresponding special divisor $Z(f)$ (determined by the principal part of $f$) and an automorphic Green function $\Phi(\cdot, f)$ which is constructed in [BE] as a regularized theta lift of $f$ (see Section 3). On the other hand, associated to $L$, there is also an incoherent (vector valued and normalized) Hilbert Eisenstein series $E^*(\tau, s, L, 1)$ of parallel weight 1 (see Section 4) such that its diagonal restriction to $\mathbb{Q}$ is a weight $d+1$ non-holomorphic modular form with representation $\rho_L$. Let $E(\tau, L)$ be the ‘holomorphic part’ of $E^*(\tau^\Delta, 0, L, 1)$, where, for $\tau \in \mathbb{H}$, we put $\tau^\Delta = (\tau, \cdots, \tau) \in \mathbb{H}^{d+1}$. Finally define the generalized Rankin-Selberg $L$-function

$$L(s, \xi(f), L) = \langle E^*(\tau^\Delta, s, L, 1), \xi(f) \rangle_{\text{Pet}}$$

to be the Petersson inner product of the pullback of the Eisenstein series and the holomorphic cusp form $\xi(f)$ of weight $d+1$, given by the differential operator $\xi(f) = 2i v^{1-d} \partial f / \partial \bar{\tau}$. In Section 5, we prove the following general formula, which is similar to that in [Sch0] and [BY2] Theorem 5.2.

**Theorem 1.1.** Let the notation be as above. Then

$$\Phi(Z(W, 1/2, f) = \frac{\deg Z(W, z_0^\pm)}{\Lambda(0, \chi_{E/F})} \left( \text{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L) \rangle] - L'(0, \xi(f), L) \right).$$

Here $\chi_{E/F}$ is the quadratic Hecke character of $F$ associated to $E/F$, $f^+$ is the holomorphic part of $f$, and $\text{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L) \rangle]$ is the constant term of

$$\langle f^+(\tau), \mathcal{E}(\tau, L) \rangle = \sum_{\mu \in L'/L} f^+(\tau, \mu) \mathcal{E}(\tau, L, \mu),$$

where $f^+(\tau, \mu)$ is the $\mu$-component of $f^+$, and $\mathcal{E}(\tau, L, \mu)$ is the $\mu$-component of $\mathcal{E}(\tau, L)$.

In the special case that $f$ is weakly holomorphic, $\Phi(\cdot, f)$ is the Petersson norm of a meromorphic modular form $\Psi(\cdot, f)$ on $X_K$ given by the Borcherds lift of $f$. The second summand on the right hand side of (1.3) vanishes and the first summand gives an explicit formula for the evaluation of $\Psi(\cdot, f)$ on the CM cycle $Z(W)$.

Note that the first summand $\text{CT}[\langle f^+(\tau), \mathcal{E}(\tau, L) \rangle]$ is of arithmetic nature and this theorem suggests two interesting conjectures about arithmetic intersection numbers and Faltings heights of big CM cycles, see Conjectures 5.4 and 5.5.

Also note that, in contrast with the situation in [BY2], the function $L(s, \xi(f), L)$ is not a standard Rankin-Selberg integral, since it involves the pullback of a Hilbert modular Eisenstein series. We expect that it is related to a Langlands $L$-function for the group $G$ and hope to pursue this idea in a subsequent paper.

To explain the Hilbert modular surface case in [BY1], let $E$ be a non-biquadratic quartic CM number field with real quadratic subfield $F = \mathbb{Q}(%\sqrt{D})$ with fundamental discriminant
D. Let $\sigma$ be the non-trivial Galois automorphism of $F$. Let
\[ V := \{ A \in M_2(F) : \sigma(A) = A^t \} = \{ A = \begin{pmatrix} u & bv_D \\ 0 & \sigma(u) \end{pmatrix} : u \in F, a, b \in \mathbb{Q} \} \]
and let
\[ L = \{ A = \begin{pmatrix} u & bv_D \\ 0 & \sigma(u) \end{pmatrix} : u \in \mathcal{O}_F, a, b \in \mathbb{Z} \}. \]
Here $A \mapsto A^t$ is the main involution of $M_2(F)$. The group
\[ G(\mathbb{Q}) = \text{GSpin}(V)(\mathbb{Q}) = \{ g \in \text{GL}_2(F) : \det g \in \mathbb{Q}^\times \} \]
acts on $V$ via $g.A = \sigma(g)A g^{-1}$. The Shimura variety $X_K$ is a Hilbert modular surface, and, for suitable choice of $K$, is isomorphic to $\text{SL}_2(\mathcal{O}_F) \backslash \mathbb{H}^2$. Now we describe the CM cycle $\text{CM}(E)$ in \cite{BY1}, the locus of abelian surfaces over $\mathbb{C}$ with CM by $\mathcal{O}_E$, as a formal sum of $Z(W)$’s. For a principally polarized CM abelian surface $A = (A, \kappa, \lambda)$ of CM type $(\mathcal{O}_E, \Sigma)$, let $M = H_1(A, \mathbb{Z})$ with the action of $\mathcal{O}_E$ induced by $\kappa$ and the symplectic form $\lambda$ induced by the polarization. Define the lattice
\[ L(A) = \{ j \in \text{End}(M) : j \circ \kappa(a) = \kappa(\sigma(a)) \circ j, a \in \mathcal{O}_F, j^* = j \} \]
of special endomorphisms of $M$ with $\mathbb{Z}$-quadratic form $Q(j) = j^2$, where $j^*$ is the ‘Rosati’ involution induced by $\lambda$. Let $V(A) = L(A) \otimes \mathbb{Q}$. Then one can show that the rank 4 quadratic lattice $(L(A), Q) \cong (L, \det)$ is independent of the choice of $A$. On the other hand, let $\tilde{E}$ be the reflex field of $(E, \Sigma)$, which is generated by the type norms $N_{\Sigma}(r), r \in E$, defined in \cite{HY}, and let $\tilde{F} = \mathbb{Q}(\sqrt{D})$ be the real quadratic subfield of $\tilde{E}$. It turns out (\cite{HY}, see also Section 9) that $V(A)$ has a natural $\tilde{E}$-vector space structure together with an $\tilde{F}$-valued quadratic form $Q_A$ such that
\[ N_{\Sigma}(r) \cdot j = \kappa(r) \circ j \circ \kappa(\tilde{r}) \]
for any $r \in E$, and
\[ \text{tr}_{\tilde{F}/\mathbb{Q}} Q_A(j) = Q(j), \quad j \in V(A). \]
Let $W(A) = (V(A), Q_A)$ be the resulting 2-dimensional quadratic space over $\tilde{F}$. Then the rational torus associated to $W(A)$ is
\[ T(R) = \{ r \in (R \otimes \mathbb{Q}) E^\times : r\tilde{r} \in R^\times \} \]
and its rational points $T(\mathbb{Q})$ act on $W(A)$ via $r \cdot j = \frac{1}{r\tilde{r}} \kappa(r) \circ j \circ \kappa(\tilde{r})$. However, in general, different $A$’s might give different $\tilde{F}$-quadratic spaces and different incoherent Eisenstein series $E^*(\tilde{r}, s, L(A), 1)$. The CM cycle $\text{CM}(E)$ is a union of such CM cycles, and Theorem 11 gives a formula for the CM value $\Phi(\text{CM}(E), f)$ in terms of several incoherent Eisenstein series. When $D \equiv 1 \mod 4$ is a prime, however, the formula becomes simple and only one incoherent Eisenstein series is involved, as in \cite{BY1} (see Theorem 6.8). We have the following result (Theorem 6.11).
Theorem 1.2. Let $E$ be a CM quartic field with discriminant $D^2 \tilde{D}$ with $D \equiv 1 \mod 4$ prime and $\tilde{D} \equiv 1 \mod 4$ square free and with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$. Let $f \in H_{0, \rho_L}$ as above. Then
\[
\Phi(CM(E), f) = \frac{\deg(CM(E))}{2 \Lambda(0, \chi)} \left( CT[(f^+, \mathcal{E}(\tau, \tilde{\mathcal{L}}))] - L'(0, \xi(f), \tilde{\mathcal{L}}) \right).
\]
Here $\tilde{\mathcal{L}} = O_{\tilde{E}}$ with $\tilde{F}$-quadratic form $\tilde{Q}(r) = -\frac{1}{\sqrt{D}} r \tilde{r}$, and $\Lambda(s, \chi)$ is the complete $L$-function of the quadratic Hecke character $\chi$ of $F$ associated to $E/F$ defined in (4.6).

We expect the factor $\frac{\deg(CM(E))}{2 \Lambda(0, \chi)}$ to be 1 and prove it in some special cases in Section 6. We also give a scalar modular form version of this theorem and arithmetic applications in Section 6. In particular, we have the following result.

Theorem 1.3. Assume that $E$ is a quartic CM number field with absolute discriminant $d_E = D^2 \tilde{D}$ and real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ such that $D \equiv 1 \mod 4$ is prime and $\tilde{D} \equiv 1 \mod 4$ is square-free. Assume further that $O_E = O_F + O_F w + \sqrt{\Delta}$ is free over $O_F$, where $w, \Delta \in O_F$. Let $X$ be a regular toroidal compactification of the moduli stack of principally polarized abelian surfaces with real multiplication by $O_F$, [Ra], [DP]. For any $f \in H_{0, \rho_L}$ with $c^+(0, 0) = 0$, let $\tilde{Z}(f)$ be the closure of $Z(f)$ in $X$ and let $\hat{Z}(f) = (\tilde{Z}(f), \Phi(\cdot, f)) \in \widehat{CH}^1(X)_C$. Let $CM(E)$ be the moduli stack of principally polarized abelian surfaces with CM by $O_E$. Then
\[
\langle \hat{Z}(f), CM(E) \rangle_{Fal} = -\frac{1}{4} L'(0, \xi(f), \tilde{\mathcal{L}}).
\]

The idea of constructing big CM cycles was communicated to one of the authors (T.Y.) a couple of years ago by Eyal Goren in a private conversation. We thank him for sharing his idea. A slightly more general type of CM point is discussed in [Shim, Section 5], and our result (Theorem 1.1) can undoubtedly be extended to that case.

It is interesting to note that the Shimura variety $Sh(G, D)$ attached to $G = GSpin(V)$ is of PEL-type only for small values of $d$ where accidental isomorphisms occur. In these cases, the moduli theoretic interpretation of the 0-cycles defined in Section 2 is slightly subtle. Thus, for example, as shown in Section 6, in the Hilbert modular surface case, the 0-cycle associated to abelian surfaces with CM by a non-biquadratic quartic CM field $E/F$ is a union of the 0-cycles constructed in Section 3 for the reflex field $\tilde{E}/\tilde{F}$.

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2. The Shimura variety and its special points

As in Section 1, let $F$ be a totally real number field of degree $d+1$ over $\mathbb{Q}$ with embeddings $\{\sigma_j\}_{j=0}^d$ into $\mathbb{R}$. Let $W, (\ , \ )_W$ be a quadratic space over $F$ of dimension 2 with signature $\text{sig}(W) = ((0,2), (2,0), \ldots, (2,0))$.

Let $V = \text{Res}_{F/\mathbb{Q}}W$ be the underlying rational vector space with bilinear form $(x,y)_V = \text{tr}_{F/\mathbb{Q}}(x,y)_W$. There is an orthogonal direct sum

\begin{equation}
V \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_j W_{\sigma_j}
\end{equation}

of real quadratic spaces where $W_{\sigma_j} = W \otimes_{F,\sigma_j} \mathbb{R}$, and $\text{sig}(V) = (2d,2)$. Let $G = \text{GSpin}(V)$.

Then there is a homomorphism

\begin{equation}
\text{Res}_{F/\mathbb{Q}} \text{GSpin}(W) \rightarrow G
\end{equation}

of algebraic groups over $\mathbb{Q}$ which, on real points, gives the homomorphism

\begin{equation}
\text{Res}_{F/\mathbb{Q}} \text{GSpin}(W)(\mathbb{R}) = \prod_j \text{GSpin}(W_{\sigma_j}) \rightarrow \text{GSpin}(V \otimes_{\mathbb{Q}} \mathbb{R}) = G(\mathbb{R}),
\end{equation}

associated to the decomposition (2.1).

**Lemma 2.1.** Let $T$ be the inverse image in $G$ of the subgroup $\text{Res}_{F/\mathbb{Q}} \text{SO}(W)$ of $\text{SO}(V)$. Then $T$ is a maximal torus of $G$ and is the image of the homomorphism (2.2).

Note that there is thus an exact sequence

\begin{equation}
1 \rightarrow \mathbb{G}_m \rightarrow T \rightarrow \text{Res}_{F/\mathbb{Q}} \text{SO}(W) \rightarrow 1
\end{equation}

of algebraic groups over $\mathbb{Q}$, where $\mathbb{G}_m$ is the kernel of the homomorphism $\text{GSpin}(V) \rightarrow \text{SO}(V)$.

A more explicit description of $T$ can be given as follows. The even part $C^0_F(W) = E$ of the Clifford algebra of $W$ over $F$ is a CM field of degree $2d + 2$ over $\mathbb{Q}$. The odd part of the Clifford algebra $C^1_F(W) = W = Ew_0$ is a one dimensional vector space over $E$ with quadratic form $Q_W(aw_0) = \alpha N_{E/F}(a)$, where $a = Q_W(w_0) \in E^\times$ is an element with $\sigma_0(\alpha) < 0$ and $\sigma_j(\alpha) > 0$ for $j \geq 1$. Then, on rational points, we have

\[
\begin{array}{cccc}
\text{Res}_{F/\mathbb{Q}} \text{GSpin}(W)(\mathbb{Q}) & \rightarrow & T(\mathbb{Q}) & \rightarrow & \text{Res}_{F/\mathbb{Q}} \text{SO}(W)(\mathbb{Q}) \\
\| & & \| & & \| \\
E^\times & \rightarrow & E^\times/F^1 & \rightarrow & E^\times/F^X
\end{array}
\]

where $E^\times/F^X \simeq E^1$, via $\beta \mapsto \beta/\bar{\beta}$ is the kernel of $N_{E/F}$, and $F^1$ is the kernel of $N_{F/\mathbb{Q}}$.

Fixing an identification $S = \text{Res}_{C/\mathbb{R}} \mathbb{G}_m \simeq \text{GSpin}(W_{\sigma_0})$, we obtain a homomorphism $h_0 : S \rightarrow G_\mathbb{R}$ of algebraic groups over $\mathbb{R}$ corresponding to the inclusion in the first factor.
Let $\mathbb{D}$ be the $G(\mathbb{R})$-conjugacy class of $h_0$. Let $\{e_0, f_0\}$ be a standard basis of $W_0 \subset V \otimes_{\mathbb{Q}} \mathbb{R}$. Then it is easy to check
\[ gh_0g^{-1} \mapsto \mathbb{R}ge_0 + \mathbb{R}gf_0 \]
gives a bijection between $\mathbb{D}$ and the set of oriented negative 2-planes in $V \otimes_{\mathbb{Q}} \mathbb{R}$. We will not distinguish between the two interpretations of $\mathbb{D}$. Note that the choice of orientation determined by $\{e_0, f_0\}$ is equivalent to the choice of an extension of $\sigma_0$ to an embedding of $E$ into $\mathbb{C}$, which we also denote by $\sigma_0$.

Let $K$ be a compact open subgroup of $G(\mathbb{Q})$, where $\hat{F}$ stands for the finite adeles of a number field $F$. Let $X_K = \text{Sh}(G, h_0)_K$ be the canonical model of the Shimura variety over $\mathbb{Q}$ with
\[ X_K(\mathbb{C}) = G(\mathbb{Q}) \setminus (\mathbb{D} \times G(\hat{\mathbb{Q}})/K). \]
By construction, the homomorphism $h_0$ factors through $T_\mathbb{R}$ and is fixed by conjugation by $T(\mathbb{R})$, so we have, for any $g \in G(\hat{\mathbb{Q}})$, a special 0-cycle in $X_K$ according to [Mi, Page 325]
\[ Z(T, h_0, g)_K = T(\mathbb{Q}) \setminus (\{h_0\} \times T(\hat{\mathbb{Q}})/K_T^g) \to X_K, \quad [h_0, t] \mapsto [h_0, tg] \]
where $K_T^g = T(\hat{\mathbb{Q}}) \cap gKg^{-1}$. Note that $K_T^g$ depends only on the image of $g$ in $\text{SO}(V)(\hat{\mathbb{Q}})$. We will usually drop the subscript $K$ and identify $Z(T, h_0, g)$ with its image in $X_K$, but every point in $Z(T, h_0, g)$ is counted with multiplicity $\frac{2}{w_{K,T,g}}$ and $w_{K,T,g} = #(T(\mathbb{Q}) \cap gKg^{-1})$. In particular, for a function $f$ on $X_K$, we have
\[ f(Z(T, h_0, g)) = \frac{2}{w_{K,T,g}} \sum_{t \in T(\mathbb{Q}) \setminus T(\hat{\mathbb{A}})/K_T^g} f(h_0, tg). \]
When $g = 1$, we will further abbreviate notation and write $Z(T, h_0)$ for $Z(T, h_0, 1)$.

The 0-cycle $Z(T, h_0)$ is defined over $\sigma_0(E)$, the reflex field of $(T, h_0)$. We next describe its Galois conjugates $\tau(Z(T, h_0))$ for $\tau \in \text{Aut}(\mathbb{C}/\mathbb{Q})$.

For $j \in \{0, \ldots, d\}$, let $W(j)$ be the unique (up to isomorphism) quadratic space over $F$ such that $W(j) \otimes_F F_v$ and $W \otimes_F F_v$ are isometric for all finite place $v$ of $F$, and such that
\[ \text{sig}(W(j)) = ((2, 0), \ldots, (2, 0), (0, 2), (2, 0) \ldots, (2, 0)). \]
Note that, although the quadratic spaces $W = W(0)$ and $W(j)$ over $F$ are not isomorphic for $j \neq 0$, there is an isomorphism $C^0_F(W(j)) \simeq C^0_F(W) = E$ of their even Clifford algebras. Let $V(j) = \text{Res}_{F/Q}(W(j))$ with bilinear form $(x, y)_{V(j)} = \text{tr}_{F/Q}(x, y)_{W(j)}$. The signature of $V(j)$ is $(2d, 2)$ and the quadratic spaces $V(j)$ and $V$ are isomorphic. We fix an isomorphism
\[ V(j) \xrightarrow{\sim} V \]
and hereafter identity $V(j)$ with $V$. Let $T(j)$ be the preimage of $\text{Res}_{F/Q} \text{SO}(W(j)) \subset \text{SO}(V)$ in $G$ and let $h_0(j) : \mathbb{S} \to G_{\mathbb{R}}$ be the homomorphism defined, as above, by an identification of $\mathbb{S}$ with $\text{GSpin}(W(j) \otimes_{F,\sigma_j} \mathbb{R})$. For $g \in G(\hat{\mathbb{Q}})$, the analogue of the construction above yields a special 0-cycle $Z(T(j), h_0(j), g)$ on $X_K$ defined over $\sigma_j(E)$.

We fix an $\hat{F}$-linear isometry

\begin{equation}
\mu_j : W(j)(\hat{F}) \xrightarrow{\sim} W(\hat{F}).
\end{equation}

Noting that there are canonical identifications $W(j)(\hat{F}) = V(j)(\hat{\mathbb{Q}})$ and $W(\hat{F}) = V(\hat{\mathbb{Q}})$, and using the fixed identification of $V$ and $V(j)$, there is a unique element $g_{j,0} \in \text{O}(V)(\hat{\mathbb{Q}})$ such that the diagram

\begin{equation}
\begin{array}{ccc}
W(j)(\hat{F}) & \xrightarrow{\mu_j} & W(\hat{F}) \\
\| & & \| \\
V(\hat{\mathbb{Q}}) & \xrightarrow{g_{j,0}^{-1}} & V(\hat{\mathbb{Q}})
\end{array}
\end{equation}

Modifying the isometry $\mu_j$ by an element of $\text{O}(W)(\hat{F})$, if necessary, we can assume that $g_{j,0} \in \text{SO}(V)(\hat{\mathbb{Q}})$. For any element $g_j \in G(\hat{\mathbb{Q}})$ with image $g_{j,0}$ in $\text{SO}(V)(\hat{\mathbb{Q}})$, the finite adele points of the tori $T(j)$ and $T$ are related, as subgroups of $G(\hat{\mathbb{Q}})$, by

\begin{equation}
T(j)(\hat{\mathbb{Q}}) = g_j T(\hat{\mathbb{Q}}) g_j^{-1},
\end{equation}

and hence

\begin{equation}
K_{T(j)}^{g_j} = g_j K_T g_j^{-1}.
\end{equation}

These relations depend only on the image $g_{j,0}$ of $g_j$.

The reciprocity laws for the action of $\text{Aut}(\mathbb{C})$ on special points of Shimura varieties [MS.I], [MS.II], [Mi], yields the following result.

**Lemma 2.2.** Let the notation be as above and let $\tau \in \text{Aut}(\mathbb{C}/\mathbb{Q})$.

1. If $\tau = \sigma_j \circ \sigma_0^{-1}$ on $\sigma_0(E)$, then there is a preimage $g_j$ of $g_{j,0}$, unique up to an element of $\mathbb{Q}^\times$, such that

\[ \tau(Z(T, h_0)) = Z(T(j), h_0(j), g_j). \]

2. If $\tau = \rho$ is complex conjugation, then

\[ \tau(Z(T, h_0)) = Z(T, h_0^\sigma). \]

Here $h_0^\sigma$ is the map from $\mathbb{S}$ to $G_{\mathbb{R}}$ induced by $\mathbb{S} \to \text{GSpin}(W_{\sigma_0}), z \mapsto \bar{z}$.

We will write

\[ Z(T(j), h_0^\pm(j), g_j) = Z(T(j), h_0^+(j), g_j) + Z(T(j), h_0^-(j), g_j). \]
We will also write $z_0^\pm(j) \in \mathbb{D}$ for the oriented negative two planes in $V(\mathbb{R})$ associated $h_0^\pm(j)$. Let

\begin{equation}
Z(W) = \sum_{j=0}^{d} Z(T(j), z_0^\pm(j), g_j) \in Z^{2d}(X_K)
\end{equation}

Then $Z(W)$ is a 0-cycle defined over $\mathbb{Q}$.

### 3. Special divisors and automorphic Green functions

In this section, we briefly review the special divisors defined in [Ku2] and their ‘automorphic’ Green functions defined by the first author and Funke using regularized theta liftings [Br2], [BF]. We prove that these special cycles do not intersect with the special cycles defined in Section 2.

Let $x \in V(\mathbb{Q})$ be a vector of positive norm. We write $V_x$ for the orthogonal complement of $x$ in $V$ and $G_x$ for the stabilizer of $x$ in $G$. So $G_x \cong \text{GSpin}(V_x)$. The sub-Grassmannian

\begin{equation}
\mathbb{D}_x = \{ z \in \mathbb{D}; \ z \perp x \}
\end{equation}

defines an analytic divisor of $\mathbb{D}$. For $g \in G(\mathbb{Q})$ we consider the natural map

\begin{equation}
G_x(\mathbb{Q}) \backslash \mathbb{D}_x \times G_x(\mathbb{Q}) / (G_x(\mathbb{Q}) \cap gKg^{-1}) \rightarrow X_K, \ (z, g_1) \mapsto (z, g_1g).
\end{equation}

Its image defines a divisor $Z(x, g)$ on $X_K$, which is rational over $\mathbb{Q}$. For $m \in \mathbb{Q}_{>0}$ and $\varphi \in S(V(\mathbb{Q}))^K$, if there is an $x_0 \in V(\mathbb{Q})$ with $Q(x_0) = m$, we define the weighted cycle

\begin{equation}
Z(m, \varphi) = \sum_{g \in G_{x_0}(\mathbb{Q}) \backslash G(\mathbb{Q}) / K} \varphi(g^{-1}x_0)Z(x_0, g).
\end{equation}

It is a divisor on $X_K$ with complex coefficients. Note that, since $\varphi$ has compact support in $V(\mathbb{Q})$ and the orbits of $K$ on the compact set $G(\mathbb{Q}) \cdot x_0 \cap \text{supp}(\varphi)$ are open, the sum is finite. If there is no $x_0 \in V(\mathbb{Q})$ such that $Q(x_0) = m$, we set $Z(m, \varphi) = 0$.

**Proposition 3.1.** Let the notation be as above. Then $Z(m, \varphi)$ and $Z(T(j), h_0^\pm(j), g_j)$ do not intersect in $X_K$.

**Proof.** It suffices to show that $Z(x, g_1) \cap Z(T, h_0, g_2)$ is empty for every $x \in V(\mathbb{Q})$ with $Q(x) > 0$ and $g_1, g_2 \in G(\mathbb{Q})$. Suppose $P = [z, hg_1] = [z_0, tg_2]$ is in the intersection, where $z_0 = Re_0 + Re_0$ is the negative two plane associated to $h_0$, and $z$ is a negative two-plane in $V(\mathbb{R})$ which is orthogonal to $x$. Then there are $\gamma \in G(\mathbb{Q})$ and $k \in K$ such that

\[(\gamma)_\infty z = z_0, \quad \hat{\gamma}hg_1k = tg_2.\]

Here $\hat{\gamma}$ is the image of $\gamma$ in $G(\mathbb{Q})$. Let $y = \gamma x \in V(\mathbb{Q})$. Then $x \perp z$ implies that $y \perp z_0$, i.e., $(\sigma_0(y), e_0) = (\sigma_0(y), f_0) = 0$. This implies that $\sigma_0(y) = 0$ and thus $y = 0$, a contradiction. \hfill \Box
Let $L$ be an even integral lattice in $V$, i.e., $Q(x) = \frac{1}{2}(x, x) \in \mathbb{Z}$ for $x \in L$, and let

$$L' = \{ y \in V : (x, y) \in \mathbb{Z}, \text{ for } x \in L \} \supset L$$

be its dual. For $\mu \in L'/L$, we write $\varphi_\mu = \text{char}(\mu + \hat{L}) \in S(V(\hat{Q}))$ and $Z(m, \mu) = Z(m, \varphi_\mu)$, where $\hat{L} = L \otimes \hat{\mathbb{Z}}$. Associated to the reductive dual pair $(\text{SL}_2, O(V))$ there is a Weil representation $\omega = \omega_\psi$ of $\text{SL}_2(A)$ on the Schwartz space $S(V(A))$, where $\psi$ is the ‘canonical’ unramified additive character of $\mathbb{Q} \setminus A$ with $\psi_\infty(x) = e(x)$. Since the subspace $S_L = \bigoplus C\varphi_\mu \subset S(V(\hat{Q}))$ is preserved by the action of $\text{SL}_2(\hat{\mathbb{Z}})$, there is a representation $\rho_L$ of $\Gamma = \text{SL}_2(\mathbb{Z})$ on this space defined by the formula

$$\rho_L(\gamma)\varphi = \bar{\omega}(\hat{\gamma})\varphi$$

where $\hat{\gamma} \in \text{SL}_2(\hat{\mathbb{Z}})$ is the image of $\gamma$. This representation is given explicitly by Borcherds as

$$\rho_L(T)(\varphi_\mu) = e(Q(\mu^2)) \varphi_\mu, \quad (3.4)$$

$$\rho_L(S)(\varphi_\mu) = \frac{e((2 - n)/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(- (\mu, \nu)) \varphi_\nu, \quad (3.5)$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, see e.g. [Bo1], [Ku3], [Br2]. Note that the complex conjugate $\bar{\rho}_L$ is thus the restriction of $\omega$ to the subgroup $\text{SL}_2(\mathbb{Z}) \subset \text{SL}_2(\hat{\mathbb{Z}})$.

Recall that a smooth function $f : \mathbb{H} \to S_L$ is called a harmonic weak Maass form (of weight $k$ with respect to $\Gamma$ and $\rho_L$) if it satisfies:

(i) $f \mid_{k, \rho_L} \gamma = f$ for all $\gamma \in \Gamma$; i.e.,

$$f(\gamma \tau) = (c\tau + d)^k \rho_L(\gamma) f(\tau).$$

(ii) there is a $S_L$-valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n \leq 0} c^+(n, \mu) q^n \varphi_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$;

(iii) $\Delta_k f = 0$, where

$$\Delta_k := -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

is the usual weight $k$ hyperbolic Laplace operator (see [BE]).

The Fourier polynomial $P_f$ is called the principal part of $f$. We denote the vector space of these harmonic weak Maass forms by $H_{k, \rho_L}$. Any weakly holomorphic modular form
is a harmonic weak Maass form. The Fourier expansion of any \( f \in H_{k,\rho_L} \) gives a unique decomposition \( f = f^+ + f^- \), where

\[
(3.6a) \quad f^+(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Q} \atop n \gg -\infty} c^+(n, \mu) q^n \varphi_\mu,
\]

\[
(3.6b) \quad f^-(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Q} \atop n < 0} c^-(n, \mu) \Gamma(1 - k, 4\pi |n| v) q^n \varphi_\mu,
\]

and, for \( a > 0 \), \( \Gamma(s, a) = \int_a^{\infty} e^{-t} t^{s-1} dt \) is the incomplete \( \Gamma \)-function. We refer to \( f^+ \) as the holomorphic part and to \( f^- \) as the non-holomorphic part of \( f \).

Recall that there is an antilinear differential operator \( \xi = \xi_k : H_{k,\rho_L} \rightarrow S_{2-k,\bar{\rho}_L} \), defined by

\[
(3.7) \quad f(\tau) \mapsto \xi(f)(\tau) := 2iv^k \frac{\partial}{\partial \bar{\tau}} f(\tau).
\]

By [BF, Corollary 3.8], one has the exact sequence

\[
(3.8) \quad 0 \rightarrow M^1_{k,\rho_L} \rightarrow H_{k,\rho_L} \xrightarrow{\xi} S_{2-k,\bar{\rho}_L} \rightarrow 0.
\]

Let \( f \in H_{1-d,\bar{\rho}_L} \) be a harmonic weak Maass form of weight \( 1 - d \) with representation \( \bar{\rho}_L \) for \( \Gamma \), and denote its Fourier expansion as in (3.6). Let \( S^\vee_L \) be the dual space of \( S_L \)—the space of linear functionals on \( S_L \), and let \( \{ \varphi_\mu \} \) be the dual basis in \( S^\vee_L \) of the basis \( \{ \varphi_\mu \} \) of \( S_L \). Recall that the Siegel theta function

\[ \theta_L(\tau, z, g) = \sum_\mu \theta(\tau, z, g, \varphi_\mu) \varphi_\mu^\vee \]

is an \( S^\vee_L \)-valued holomorphic modular form of weight \( d - 1 \) for \( \Gamma \) and \( \rho_L \) defined as follows (see [BY2, Section 2] or [Ku3] for details). For \( z \in \mathbb{D} \), one has decomposition

\[ V(\mathbb{R}) = z \oplus z^\perp, \quad x = x_z + x_z^\perp. \]

Let \( (x, x)_z = -(x_z, x_z) + (x_z^\perp, x_z^\perp) \) and define the associated Gaussian by

\[
(3.9) \quad \varphi_\infty(x, z) = e^{-\pi (x, x)_z}.
\]

Then, for \( \tau \in \mathbb{H}, [z, g] \in X_K \), and \( \varphi \in S(V(\hat{\mathbb{Q}})) \), the theta function is given by

\[ \theta(\tau, z, g, \varphi) = v^{\frac{1}{2}(1-d)} \sum_{x \in V(Q)} \omega(g_\tau') \varphi_\infty(x, z) \varphi(g^{-1}x), \quad g_\tau' = \left( v^{\frac{1}{2}}, \frac{uv^{-\frac{1}{2}}}{v^{-\frac{1}{2}}} \right) \in \text{SL}_2(\mathbb{R}). \]

Here \( g \) acts on \( V \) via its image in \( \text{SO}(V) \).

We consider the regularized theta integral

\[
(3.10) \quad \Phi(z, g, f) = \int_{\mathcal{F}} \langle f(\tau), \theta_L(\tau, z, g) \rangle d\mu(\tau)
\]
for $z \in \mathbb{D}$ and $g \in G(\mathbb{Q})$, where $\mathcal{F}$ is the standard domain for $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. The integral is regularized as in [Bo1], [BF], that is, $\Phi(z, g, f)$ is defined as the constant term in the Laurent expansion at $s = 0$ of the function

$$
\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \theta_L(\tau, z, g) \rangle v^{-s} d\mu(\tau).
$$

Here $\mathcal{F}_T = \{ \tau \in \mathbb{H}; |u| \leq 1/2, |\tau| \geq 1, \text{and } v \leq T \}$ denotes the truncated fundamental domain and the integrand

$$
\langle f(\tau), \theta_L(\tau, z, g) \rangle = \sum_{\mu \in L'/L} f_{\mu}(\tau) \theta(\tau, z, g, \varphi_\mu)
$$

is the pairing of $f$ with the Siegel theta function, viewed as a linear functional on the space $S_L$.

The following theorem summarizes some properties of the function $\Phi(z, g, f)$ in the setup of the present paper (see [Br2], [BF]).

**Theorem 3.2.** The function $\Phi(z, g, f)$ is smooth on $X_K \backslash Z(f)$, where

$$
Z(f) = \sum_{\mu \in L'/L} \sum_{m > 0} c^+(-m, \mu) Z(m, \mu).
$$

It has a logarithmic singularity along the divisor $-2Z(f)$. The $(1,1)$-form $dd^c \Phi(z, g, f)$ can be continued to a smooth form on all of $X_K$. We have the Green current equation

$$
dd^c[\Phi(z, g, f)] + \delta_Z = [dd^c \Phi(z, g, f)],
$$

where $\delta_Z$ denotes the Dirac current of a divisor $Z$. Moreover, if $\Delta_z$ denotes the invariant Laplace operator on $\mathbb{D}$, normalized as in [Br2], we have

$$
\Delta_z \Phi(z, g, f) = \frac{n}{4} \cdot c^+(0, 0).
$$

In particular, the theorem implies that $\Phi(z, g, f)$ a Green function for the divisor $Z(f)$ in the sense of Arakelov geometry in the normalization of [SABK]. (If the constant term $c^+(0, 0)$ of $f$ does not vanish, one actually has to work with the generalization of Arakelov geometry given in [BKK].) Moreover, we see that $\Phi(z, g, f)$ is harmonic when $c^+(0, 0) = 0$. Therefore, it is called the automorphic Green function associated with $Z(f)$. Notice also that $Z(f)$ has coefficients in $\mathbb{Q}(f)$, the field generated by the $c(-m, \mu), m > 0$.

4. CM values of Siegel theta functions and Eisenstein series

Recall that, for each $j$, we have fixed an isomorphism $V \simeq V(j) = \text{Res}_{F/\mathbb{Q}} W(j)$ of rational quadratic spaces, and hence an identification

$$
S(V(\mathbb{A}_Q)) = S(W(j)(\mathbb{A}_F)), \quad \varphi \mapsto \varphi_{F,j}
$$
of the corresponding Schwartz spaces. For example, if \( \varphi_F = \bigotimes_w \varphi_{F,w} \in S(W(j)(\mathbb{A}_F)) \), with \( w \) running over the places of \( F \), then the corresponding \( \varphi \in S(V(\mathbb{A}_Q)) \) is also factorizable, with local component \( \varphi_v = \bigotimes_w \varphi_{F,w} \) in the space

\[
S(\text{Res}_{F/Q} W(j)(\mathbb{Q}_v)) = S\left( \bigoplus_w W(j)(F_w) \right) = \bigotimes_w S(W(j)(F_w)).
\]

These identifications are compatible with the Weil representations of \( \text{SL}_2(\mathbb{A}_Q) \) and \( \text{SL}_2(\mathbb{A}_F) \) for our fixed additive character \( \psi \) of \( \mathbb{A}_Q \) and the character \( \psi_F = \psi \circ \text{tr}_{F/Q} \) of \( \mathbb{A}_F \), i.e.,

\[
\omega_{V,\psi}(g')\varphi = \omega_{W(j),\psi_F}(g')\varphi_{F,j},
\]

where, on the right side, we view \( g' \in \text{SL}_2(\mathbb{A}_Q) \) as an element of \( \text{SL}_2(\mathbb{A}_F) \). We write \( \varphi_F \) for \( \varphi_{F,0} \). Moreover, we will frequently abuse notation and write \( \varphi \) for \( \varphi_F \) and identify \( S(W(\mathbb{A}_F)) \) with \( S(V(\mathbb{A})) \). Note that the Weil representations \( \omega_{W(j),\psi_F} \) of \( \text{SL}_2(\mathbb{A}_F) \), which are now all realized on \( S(V(\mathbb{A}_Q)) \), via (4.1), do not coincide in general. The point is that the group \( \text{SL}_2(\hat{F}) \) in the dual pair \( (\text{SL}_2(\hat{F}), \text{Res}_{F/Q} \text{O}(W(j))) \) arises as the commutant in the ambient symplectic group of the subgroup \( \text{Res}_{F/Q} \text{O}(W(j)) \subset \text{O}(V) \), i.e., by a seesaw construction, and these subgroups do not coincide.

Recall that, for each \( j \), we have fixed an isometry \( \mu_j : W(j)(\hat{F}) \xrightarrow{\sim} W(\hat{F}) \), and an element \( g_{j,0} \in \text{SO}(V)(\hat{Q}) \) so that the diagram (2.10) commutes.

**Lemma 4.1.** (i) For any \( \varphi \in S(V(\hat{Q})) \), recall that we identify \( \varphi_{F,0} = \varphi_F \) with \( \phi \) via \( S(W(\hat{F})) \cong S(V(\hat{Q})) \). Then

\[
\mu_j^*(\varphi) = (\omega(g_{j,0})\varphi)_{F,j}.
\]

(ii) The map \( \mu_j^* : S(W(\hat{F})) \rightarrow S(W(j)(\hat{F})) \) intertwines the Weil representations \( \omega_{W,\psi_F} \) and \( \omega_{W(j),\psi_F} \) of \( \text{SL}_2(\hat{Q}) \) on these spaces.

(iii) For \( g' \in \text{SL}_2(\hat{Q}) \), and \( \varphi \in S(V(\hat{Q})) \),

\[
\omega_{W(j),\psi_F}(g')\omega(g_{j,0})\varphi = \omega(g_{j,0})\omega_{W,\psi_F}(g')\varphi.
\]

Here in part (iii), we are working in the fixed space \( S(V(\hat{Q})) \) with natural linear action of \( g \in \text{SO}(V)(\hat{Q}) \), \( \omega(g)\varphi(x) = \varphi(g^{-1}x) \), and the various Weil representation actions of \( \text{SL}_2(\hat{F}) \), as described above.

For \( z \in \mathbb{D} \), the Gaussian \( \varphi_\infty(\cdot, z) \in S(V(\mathbb{R})) \) is defined by (3.9). The points \( z_0^\pm(j) \in \mathbb{D} \) are the fixed points of \( T(j)(\mathbb{R}) \), and

\[
\varphi_\infty(\cdot, z_0^\pm(j)) = \bigotimes_i \varphi_\infty(W(j)_{\sigma_i}),
\]

in

\[
S(V(\mathbb{R})) = S(\text{Res}_{F/Q}(W(j))(\mathbb{R})) = \bigotimes_i S(W(j)_{\sigma_i}),
\]

where \( W(j)_{\sigma_i} = W(j) \otimes_{F,\sigma_i} \mathbb{R} \), and

\[
\varphi_\infty(W(j)_{\sigma_i})(x) = e^{-\pi\|x\|^2}W(j)_{\sigma_i}(x).
\]
is the Gaussian for the definite space $W(j)_{\sigma_i}$. Note that $\varphi_{\infty, W(j)_{\sigma_i}}$ is $\text{SO}(W_{j, \sigma_i})$-invariant, and is an eigenfunction of $\text{SO}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R})$ with respect to the Weil representation $\omega_{W_{j, \sigma_i}}$ of ‘weight’ $+1$ for $i \neq j$ and $-1$ for $i = j$.

For a $K$-invariant Schwartz function $\varphi \in S(V(\hat{Q}))^K$ and $\tau \in \mathbb{H}$, the theta function

$$
\theta(\tau, z, g, \varphi) = \sum_{x \in V(\hat{Q})} \omega_{V}(g') \varphi_{\infty}(x, z) \varphi(g^{-1}x)
$$

is an automorphic function of $[z, g] \in X_K$, where $z \in \mathbb{D}$ and $g \in G(\hat{Q})$. By the preceding discussion, the pullback of this function to $Z(T(j), \mathbb{Z}^+_0(j), g_j)$ coincides with the pullback of the Hilbert theta function associated to the quadratic space $W(j)$,

$$
\theta(\tau, t, (\omega(g_{j,0})\varphi)_{F,j}) = v_j N(\vec{v})^{-\frac{1}{2}} \sum_{x \in W(j)(F)} \omega_{W(j)}(g'_x') \varphi_{\infty, W(j)}(x) (\omega(g_{j,0})\varphi)_{F,j}(t^{-1}x),
$$

via the diagonal embedding of $\mathbb{H}$ into $\mathbb{H}^{d+1}$. Here $\vec{\tau} \in \mathbb{H}^{d+1}$, with components $\tau_r = u_r + iv_r$, $N(\vec{v}) = \prod_r v_r$, and $g'_x' \in \text{SL}_2(\mathbb{R})^{d+1}$ with component $g'_{r_x}$ in the $r$th slot. This theta function has weight

$$
1(j) := (1, \ldots, -1, \ldots, 1),
$$

with $-1$ in the $j$th slot.

Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of $F$ associated to $E/F$, and let $I(s, \chi) = \otimes_v I(s, \chi_v)$ be the representation of $\text{SL}_2(\mathbb{A}_F)$ induced from the character $\chi | \cdot^s$ of the standard Borel subgroup. We write $\Phi^k_{\sigma_i}$ for the unique eigenfunction of $\text{SO}_2(\mathbb{R}) \subset \text{SL}_2(F \otimes_{F, \sigma_i} \mathbb{R})$ in $I(s, \chi_{\sigma_i})$ of weight $k$ with $\Phi^k_{\sigma_i}(1, s) = 1$. We define sections in $I_{\infty}(s, \chi_{\infty}) = \otimes_i I(s, \chi_{\sigma_i})$ by

$$
\Phi^1_{\infty}(s) = \otimes_i \Phi^1_{\sigma_i}(s),
$$

and

$$
\Phi^{1(j)}_{\infty}(s) = \Phi^{-1}_{\sigma_j}(s) \otimes (\otimes_{i \neq j} \Phi^1_{\sigma_i}(s)).
$$

For each $j$, there is an $\text{SL}_2(\hat{F})$-equivariant map

$$
\lambda_j : S(W(j)(\hat{F})) \rightarrow I_F(0, \chi_f), \quad \varphi \mapsto \lambda_j(\varphi)(g) = \omega_{W(j), \psi_F}(g)\varphi(0).
$$

By (ii) of Lemma 4.1 these maps for various $j$’s are related as follows.

**Lemma 4.2.** For $\varphi \in S(V(\hat{Q}))$, one has

$$
\lambda_j(\mu^*_j(\varphi_F)) = \lambda_0(\varphi_F)
$$

Let $\Phi(\varphi) \in I_f(s, \chi_f)$ be the unique standard section with $\Phi(\varphi)(0) = \lambda_0(\varphi) = \lambda_j(\mu^*_j(\varphi))$. For $\varphi \in S(W(\hat{F})) = S(V(\hat{Q}))$ and $\vec{\tau} = (\tau_0, \ldots, \tau_d) \in \mathbb{H}^{d+1}$ with $\tau_r = u_r + iv_r$, we define the Hilbert-Eisenstein series

$$
E(\vec{\tau}, s, \varphi, 1) = N(\vec{v})^{-\frac{1}{2}} E(g'_x, s, \Phi^1_{\infty} \otimes \Phi(\varphi))
$$
(4.5) \[ E(\tau, s, \varphi, 1(j)) = v_j N(\vec{v})^{-\frac{1}{2}} E(g^j, s, \Phi^1(\vec{v}) \otimes \Phi_\varphi). \]

Here \( N(\vec{v}) = \prod_r v_r \). Note that, \( \Phi^1(\vec{v}) \otimes \mu^*_j(\varphi) \in S(W(j)(A_F)) \) and \( E(\tau, s, \varphi, 1) \) is an incoherent Eisenstein series of parallel weight \( 1 \) (independent of \( j \)). The two Eisenstein series are related as follows by an observation of [Ku3, (2.17)], [BY2, Lemma 2.3], Lemma 4.3.

\[ \bar{\partial}_j = \frac{\partial}{\partial \sigma_j} d\tau_{\sigma_j}. \] Then
\[-2 \bar{\partial}_j \left( E'(\tau, 0, \varphi, 1(j)) d\tau_{\sigma_j} \right) = E(\tau, 0, \varphi, 1(j)) d\mu(\tau_{\sigma_j}). \]

In this paper, we normalize the Haar measure \( dh \) on \( SO(W(j))(A_F) \) so that \( \text{vol}(SO(W(j))(F) \backslash SO(W(j))(A_F)) = 2 \), and write \( dh = dh_\infty dh_f \) where \( dh_\infty = \prod_i dh_\infty_i \) with \( \text{vol}(SO(W(j)_i), dh_\infty_i) = 1 \). For the convenience of the reader, we first recall [Scho, Lemma 2.13].

Lemma 4.4. For any function \( f \) on \( Z(T(j), z_0(j), g_j) = T(j)(\mathbb{Q}) \backslash \{ z_0(j) \} \times T(j)(\hat{\mathbb{Q}}) / K_{T(j)}^{g_j} \), the weighted sum \( (2.6) \) of the values of \( f \) over this discrete finite set is given by
\[ f(Z(T(j), z_0(j), g_j)) = \frac{1}{2} \deg Z(T, z_0) \int_{SO(W(j))(F) \backslash SO(W(j))(F)} f(z_0(j), t) dt. \]

Here \( \deg Z(T, z_0) = \frac{4}{\text{vol}(K_T)} \) is independent of \( j \).

Proof. By [Scho, Lemma 2.13], the formula holds with \( \deg Z(T, z_0) \) replaced by the quantity \( 2 / \text{vol}(K_{T(j)}^{g_j}) \). So it suffices to check \( \text{vol}(K_{T(j)}^{g_j}) = \text{vol}(K_T) \) is independent of \( j \). But this is immediate by (2.11) and (2.12). \qed

Proposition 4.5. With the notation as above,
\[ \theta(\tau, Z(T(j), z_0(j), g_j), \varphi) = C \cdot E(\tau^j, 0, \varphi, 1(j)) \]
where \[ C = \frac{1}{2} \deg(Z(T, z_0)). \]
Proof. Since $\text{vol}(SO(W(j))_{\sigma_i}) = 1$, one has by Lemma 4.4 that
\[
\theta(\tau, Z(T(j), z_0(j), g_j)) = \frac{1}{2} \deg Z(T, z_0) \int_{SO(W(j)/(F) \setminus SO(W(j))(A_F))} \theta(\tau^\Delta, t, (\omega(g_j,0)\varphi)_{F,j}) \, dt,
\]
where the theta function in the integral is given by (4.3). Now the proposition follows from the Siegel-Weil formula.
\[\square\]

For $\chi = \chi_{E/F}$ as above, let
\[
\Lambda(s, \chi) = A^2 \pi^{-\frac{1}{m}} \Gamma(\frac{s+1}{2}) \Lambda^d s + 1 L(s, \chi), \quad A = N_{F/Q}(\partial_F d_{E/F})
\]
be the complete $L$-function of $\chi$. It is a holomorphic function of $s$ with functional equation
\[
\Lambda(s, \chi) = \Lambda(1-s, \chi)
\]
and
\[
\Lambda(1, \chi) = \Lambda(0, \chi) = L(0, \chi) = 2^{d-\delta} \frac{h(E)}{w(E) h(F)} \in \mathbb{Q}^x,
\]
where $2^\delta = |\mathcal{O}_E^x : \mu(E)\mathcal{O}_F^x|$ is 1 or 2. Let
\[
E^*(\vec{\tau}, s, \varphi, 1) = \Lambda(s+1, \chi) E(\vec{\tau}, s, \varphi, 1)
\]
be the normalized incoherent Eisenstein series.

**Proposition 4.6.** Let $\varphi = \varphi_F \in S(V(\hat{Q})) = S(W(\hat{F}))$. For a totally positive element $t \in F_+^\times$, let $a(t, \varphi)$ be the $t$-th Fourier coefficient of $E^{*,t}(\vec{\tau}, 0, \varphi, 1)$ and write the constant term of $E^{*,t}(\vec{\tau}, 0, \varphi, 1)$ as
\[
\varphi(0) \left( \Lambda(0, \chi) \log N(\vec{v}) + a_0(\varphi) \right).
\]
Let
\[
\mathcal{E}(\tau, \varphi) = \varphi(0) a_0(\varphi) + \sum_{n \in \mathbb{Q}_{>0}} a_n(\varphi) q^n
\]
where
\[
a_n(\varphi) = \sum_{t \in F_+^\times, \text{tr}_{F/Q} t = n} a(t, \varphi).
\]
Then, writing $\tau^\Delta$ for the diagonal image of $\tau \in \mathbb{H}$ in $\mathbb{H}^{d+1}$,
\[
E^{*,t}(\tau^\Delta, 0, \varphi, 1) - \mathcal{E}(\tau, \varphi) - \varphi(0) \Lambda(0, \chi) (d+1) \log v
\]
is of exponentially decay as $v$ goes to infinity. Moreover, for $n > 0$
\[
a_n(\varphi) = \sum_p a_{n,p}(\varphi) \log p
\]
with $a_{n,p}(\varphi) \in \mathbb{Q}(\varphi)$, the subfield of $\mathbb{C}$ generated by the values $\varphi(x), x \in V(\hat{Q})$. 
Proof. Let \( \mathcal{C} = \otimes_i \mathcal{C}_i \) be the incoherent collection of local quadratic \( F_v \)-spaces with \( \hat{\mathcal{C}} = \hat{W} \) for finite adeles and \( \mathcal{C}_\infty \) is totally positive definite. Then
\[
\Phi_{\varphi}(0) \otimes \Phi_\infty^1(0) = \lambda(\varphi \otimes \varphi_\infty, \mathcal{C})
\]
for \( \varphi \otimes \varphi_\infty, \mathcal{C} \in S(\mathcal{C}) = S(\hat{W}) \otimes S(\mathcal{C}_\infty) \), where \( \varphi_\infty, \mathcal{C} = \otimes_i \varphi_\infty, \mathcal{C}_i \) is the product of the Gaussians for the positive definite binary quadratic spaces \( \mathcal{C}_i \). Thus \( E(\tau, s, \varphi, 1) \) is an incoherent Eisenstein series according to [Ku1] and \( E^*(\tau, 0, \varphi, 1) = 0 \). By linearity, we may assume that the function \( \varphi = \otimes_v \varphi_v \in S(W(\bar{F})) \) is factorizable, the Fourier expansion can be written as
\[
E^*(\tau, s, \varphi, 1) = E_0^*(\tau, s, \varphi, 1) + \sum_{\nu \in F^\times} E_\nu^*(\tau, s, \varphi, 1)
\]
with
\[
E_\nu^*(\tau, s, \varphi, 1) = A^\frac{s}{2} \prod_{p < \infty} W_{t_p}^*(1, s, \varphi_p) \prod_{i=0}^d W_{\sigma_{i}(t), \sigma_{i}}^*(\tau_i, s, \Phi_{\sigma_{i}}^1)
\]
and
\[
E_0^*(\tau, s, \varphi, 1) = \varphi(0) \Lambda(s + 1, \lambda) N(\tilde{v})^{\frac{s}{2}} + A^\frac{s}{2} \prod_{p < \infty} W_{0, p}^*(1, s, \varphi_p) \prod_{i=0}^d W_{0, \sigma_{i}}^*(\tau_i, s, \Phi_{\sigma_{i}}^1).
\]
Here, for \( g' \in \text{SL}_2(F_p) \),
\[
W_{t_p}^*(g', s, \varphi_p) = L_p(s + 1, \chi_v) W_{t_p}(g', s, \varphi_p)
\]
and
\[
W_{\sigma_{i}(t), \sigma_{i}}^*(\tau_i, s, \Phi_{\sigma_{i}}^1) = \pi^{-\frac{s+2}{2}} \Gamma\left(\frac{s+2}{2}\right) v_i^{-\frac{1}{2}} W_{\sigma_{i}(t), \sigma_{i}}^*(g_{\tau_i}', s, \Phi_{\sigma_{i}}^1)
\]
are the normalized local Whittaker functions, which are computed in [KRY] and [Ya3] in special cases. In particular, [KRY] Proposition 2.6] (see also [Ya3] Proposition 1.4) asserts that
\[
W_{\sigma_{i}(t), \sigma_{i}}^*(\tau_i, 0, \Phi_{\sigma_{i}}^1) = 2 \gamma(\mathcal{C}_i) e(\sigma_i(t)\tau_i), \quad \text{if } \sigma_i(t) > 0,
\]
\[
W_{0, \sigma_{i}}^*(\tau_i, s, \Phi_{\sigma_{i}}^1) = \gamma(\mathcal{C}_i) v_i^{-s/2} \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right),
\]
\[
W_{\sigma_{i}(t), \sigma_{i}}^*(\tau_i, 0, \Phi_{\sigma_{i}}^1) = 0, \quad \text{if } \sigma_i(t) < 0.
\]
Here \( \gamma(\mathcal{C}_i) \) is the local Weil index, an \( 8 \)-root of unity. Moreover, in the last case,
\[
W_{\sigma_{i}(t), \sigma_{i}}^*(\tau_i, 0, \Phi_{\sigma_{i}}^1) = \gamma(\mathcal{C}_i) e(\sigma_i(t)\tau_i) \beta_1(4\pi |\sigma_i(t)| v_i)
\]
is of exponentially decay when \( v_i \) goes to infinity. Here
\[
\beta_1(x) = \int_1^\infty e^{-xt} t^{-1} dt, \quad x > 0.
\]

\footnote{The extra ‘−’ in the formula is due to the fact that we use \( w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) here for the local Whittaker function instead of \( w^{-1} \) in [KRY].}
On the other hand, when everything is unramified at a finite prime \( p \), i.e., \( E/\mathbb{Q} \) is unramified at primes over \( p \), \( \alpha \in \mathcal{O}_{F_{p}^{\infty}} \), and \( \varphi_{p} = \text{char}(\mathcal{O}_{E_{p}}) \), one has ([Ya3 Proposition 1.1]) for \( t \neq 0 \)

\[
\gamma(C_{p})^{-1} W_{t,v}^{*}(1, s, \varphi_{p}) = \begin{cases} 
0 & \text{if } t \notin \mathcal{O}_{p}, \\
\text{ord}_{p} t + 1 & \text{if } p \text{ split in } E/F, t \in \mathcal{O}_{p}, \\
\frac{1}{2}(1 + (-1)^{\text{ord}_{p} t}) & \text{if } p \text{ inert in } E/F, t \in \mathcal{O}_{p}.
\end{cases}
\]

In general, \( \gamma(C_{p})^{-1} W_{t,v}^{*}(1, s, \varphi_{p}) \) is a polynomial of \( N(p)^{-s} \) with coefficients in \( \mathbb{Q}(\varphi_{p}) \) ([KY]). For \( t \neq 0 \), let \( D(t) = D(t, C) \) be the ‘Diff’ set of places \( p \) of \( F \) (including infinite places) such that \( C_{p} \) does not represent \( t \), as defined in [Ku1]. Then \( D(t) \) is a finite set of odd order, and for every \( p \in D(t) \), the local Whittaker function at \( v \) vanishes at \( s = 0 \). So \( E_{t}^{*'}(\vec{\tau}, 0, \varphi) = 0 \) unless \( D(t) \) has exactly one element. Assuming this and restricting \( \vec{\tau} \) to the diagonal \( \tau^{\Delta} = (\tau, \cdots, \tau) \) with \( \tau = u + \sqrt{-1}v \in \mathbb{H} \), there are two subcases.

When \( D(t) = \{ \sigma_{i} \} \) for some \( i \), the above formulae shows that

\[
E_{t}^{*'}(\tau^{\Delta}, 0, \varphi, 1) = W_{\sigma_{(t)}, \sigma_{i}}^{*'}(\tau, 0, \Phi_{\sigma_{1}}^{1}) \prod_{p \neq \sigma_{i}} W_{t,p}^{*'}(\cdot, 0, \cdot)
\]

is of exponential decay when \( v = \text{Im}(\tau) \to \infty \).

When \( D(t) = \{ p \} \) for some finite prime \( p \), \( t \in F_{p}^{\infty} \) is totally positive,

\[
E_{t}^{*'}(\tau^{\Delta}, 0, \varphi, 1) = a(t, \varphi) q^{\text{tr}_{F/Q} t}, \quad q = e(\tau)
\]

for some \( a_{t}(\varphi) \in \mathbb{Q}(\varphi) \log p \), where \( p \) is the prime below \( p \). Here we have used the fact that

\[
\prod_{p < \infty} \gamma(C_{p}) \prod_{i=1}^{d+1} \gamma(C_{\sigma_{i}}) = -1.
\]

Finally, for the constant term, one has (see e.g., [Ya3 Section 1] or [KY])

\[
E_{0}^{*'}(\vec{\tau}, s, \varphi, 1) = \varphi(0) \left( N(\vec{v})^{\frac{1}{2}} \Lambda(s + 1, \chi) + N(\vec{v})^{-\frac{1}{2}} \Lambda(1 - s, \chi)M_{\varphi}(s) \right)
\]

where \( M_{\varphi}(s) \) is a product of finitely many polynomials in \( N(p)^{-s} \) for finitely many ‘bad’ \( p \), and \( M_{\varphi}(0) = -1 \). Recalling that \( E_{0}^{*}(\tau^{\Delta}, 0, \varphi, 1) = 0 \), this gives for \( \tau \in \mathbb{H} \)

\[
E_{0}^{*'}(\tau^{\Delta}, 0, \varphi, 1) = \varphi(0) \left( \Lambda(1, \chi)(d + 1) \log v + 2\Lambda'(1, \chi) + \Lambda(1, \chi)M_{\varphi}'(0) \right).
\]

The constant term of \( E_{t}^{*'}(\tau^{\Delta}, 0, \varphi, 1) \) as a (non-holomorphic) elliptic modular form is

\[
E_{0}^{*'}(\tau^{\Delta}, 0, \varphi, 1) + \sum_{0 \neq t \in F \setminus F/Q \text{ t}=0} E_{t}^{*'}(\tau^{\Delta}, 0, \varphi, 1),
\]

where the last sum is of exponential decay when \( v = \text{Im}(\tau) \to \infty \). This proves the proposition. \( \square \)
5. The main formula

Let \( L \) be an even integral lattice in \( V \), and let \( K \subset G(\hat{\mathbb{Q}}) \) be a compact open subgroup which fixes \( L \) and acts trivially on \( L'/L \). We also assume that \( K \) satisfies the condition
\[
K \cap \mathbb{G}_m(\hat{\mathbb{Q}}) = \hat{\mathbb{Z}}^x,
\]
where \( \mathbb{G}_m \) is the kernel of the homomorphism \( \text{GSpin}(V) \to \text{SO}(V) \). Let \( f \in H_{1-d,\bar{\rho}_L} \) be a harmonic weak Maass form and let \( \Phi(z, h, f) \) be the corresponding ‘automorphic’ Green function for the divisor \( Z(f) \) defined in (3.13).

For \( \vec{\tau} \in \mathbb{H}^{d+1} \) and \( \tau \in \mathbb{H} \), define \( S_L \)-valued functions by
\[
E(\vec{\tau}, s, L, 1) = \sum_{\mu \in L'/L} E(\vec{\tau}, s, \varphi_\mu, 1) \varphi_\mu^\vee, \quad \mathcal{E}(\tau, L) = \sum_{\mu \in L'/L} \mathcal{E}(\tau, \varphi_\mu) \varphi_\mu^\vee,
\]
where \( \mathcal{E}(\tau, \varphi) \) is defined in Proposition 4.6 and the normalized incoherent Eisenstein series
\[
E^*(\vec{\tau}, s, L, 1) = \Lambda(s + 1, \chi) E(\vec{\tau}, s, L, 1).
\]
Define the \( L \)-function for an cuspidal modular form \( g = \sum a_\mu \varphi_\mu \in S_{d+1,\bar{\rho}_L} \)
\[
\mathcal{L}(s, g, L) = \langle E^*(\tau^\Delta, s, L, 1), g \rangle_{\text{Pet}} := \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \sum_{\mu} g_\mu(\tau) E^*(\tau^\Delta, s, \varphi_\mu, 1) v^{d+1} \, d\mu(\tau).
\]
It can be viewed as the \( g \)-isotypical component of diagonal restriction of the Hilbert-Eisenstein series \( E(\vec{\tau}, s, L, 1) \).

**Remark 5.1.** The Eisenstein series \( E(\vec{\tau}, s, L, 1) \) depends on the \( F \)-quadratic form on \( L \otimes \mathbb{Q} = W \), not just on the \( \mathbb{Q} \)-quadratic form on \( L \otimes \mathbb{Q} = V \). When we need to emphasize this dependence on the \( F \)-quadratic form, we will write \( L(W) \) rather than \( L \) and
\[
E^*(\vec{\tau}, s, L(W), 1) = E^*(\vec{\tau}, s, L, 1), \quad \mathcal{E}(\tau, L(W)) = \mathcal{E}(\tau, L), \quad \mathcal{L}(s, g, L(W)) = \mathcal{L}(s, g, L).
\]
We also caution that \( L(W) \) might not be an \( \mathcal{O}_F \)-lattice, i.e., it might not be \( \mathcal{O}_F \)-invariant.

Since the Eisenstein series has an analytic continuation and is incoherent, the \( L \)-series \( \mathcal{L}(s, g, L) \) has an analytic continuation and is zero at the central point \( s = 0 \). Now we are ready to state and prove the main formula. Here, if \( \sum_n a_n q^n \) is a power series in \( q \), we write
\[
\text{CT} \left[ \sum_n a_n q^n \right] = a_0
\]
for the constant term.

**Theorem 5.2.** For a harmonic weak Maass form \( f \in H_{1-d,\bar{\rho}_L} \) with components \( f = f^+ + f^- \) as in (3.6) and with other notation as above,
\[
\Phi(Z(W), f) = C(W, K) \left( \text{CT} \left[ \langle f^+(\tau), \mathcal{E}(\tau, L(W)) \rangle \right] + \mathcal{L}'(0, \xi(f), L(W)) \right).
\]
where $\xi(f)$ is the image of $f$ under the anti-holomorphic operator $\xi : H_{1-d,\rho_L} \to S_{d+1,\rho_L}$, cf. [3.7], and
\[
C(W, K) = \frac{\deg(Z(T, z_0^+))}{\Lambda(0, \chi)}.
\]

Proof. The proof basically follows the same argument as in [BY2, Theorem 4.8]. We write $L$ in place of $L(W)$. First, by Lemma 4.3 and Proposition 4.5, we have
\[
\Phi(Z(T(j), z_0(j), g_j), f) = \int_{\mathcal{F}} \langle f(\tau), \theta_L(\tau, T(j), z_0(j), g_j) \rangle d\mu(\tau)
\]
\[
= C \int_{\mathcal{F}} \langle f(\tau), E(\tau^\Delta, 0, L, 1(j)) \rangle d\mu(\tau)
\]
\[
= -2C \int_{\mathcal{F}} \langle f(\tau), \partial_j(E'(\tau^\Delta, 0, L, 1) d\tau) \rangle.
\]

Here $C$ is the constant in Proposition 4.5. So, summing on $j$, and recalling the definition (2.13) of $Z(W)$, we have
\[
\Phi(Z(W), f) = -4C \int_{\mathcal{F}} \langle f(\tau), \sum_j \partial_j(E'(\tau^\Delta, 0, L, 1) d\tau) \rangle
\]
\[
= -4C \int_{\mathcal{F}} \langle f(\tau), \partial(E'(\tau^\Delta, 0, L, 1) d\tau) \rangle
\]
\[
= -4C \int_{\mathcal{F}} d(\langle f(\tau), E'(\tau^\Delta, 0, L, 1) d\tau \rangle) + 4C \int_{\mathcal{F}} \langle \partial f(\tau), E'(\tau^\Delta, 0, L, 1) d\tau \rangle
\]
\[
= -C_0 I_1 + 4C_0 I_2,
\]
where $C_0 = 4C\Lambda(0, \chi)^{-1} = C(W, K)$, and
\[
I_1 = \int_{\mathcal{F}} d(\langle f(\tau), E^{s'}(\tau^\Delta, 0, L, 1) d\tau \rangle),
\]
\[
I_2 = \int_{\mathcal{F}} \langle \partial f(\tau), E^{s'}(\tau^\Delta, 0, L, 1) d\tau \rangle.
\]

Recall that
\[
\partial f(\tau) = -\frac{1}{2i} \nu^{d-1} \xi(f) d\tau.
\]

Thus
\[
\langle \partial f(\tau), E^{s'}(\tau^\Delta, 0, L, 1) d\tau \rangle = -\langle \xi(f), E^{s'}(\tau^\Delta, 0, L, 1) \rangle \nu^{d+1} d\mu(\tau)
\]
is actually integrable over the fundamental domain $\mathcal{F}$, and hence
\[
I_2 = -\int_{\mathcal{F}} \langle \xi(f), E^{s'}(\tau^\Delta, 0, L, 1) \rangle \nu^{d+1} d\mu(\tau) = -\mathcal{L}(0, \xi(f), L).
\]
By the same argument as in [Ku3, Proposition 2.5], [Scho, Proposition 2.19], or [BY2, Lemma 4.6], there is a (unique) constant $A_0$ such that

$$I_1 = \lim_{T \to \infty} \left( \int_{\mathcal{F}_T} d(\langle f(\tau), E^{s'}(\tau^A, 0, L, 1) d\tau \rangle) - A_0 \log T \right) = \lim_{T \to \infty} (I_1(T) - A_0 \log T).$$

By Stokes’ theorem, one has

$$I_1(T) = \int_{\partial \mathcal{F}_T} \langle f(\tau), E^{s'}(\tau^A, 0, L, 1) \rangle d\tau$$

$$= - \int_{iT}^{iT+1} \langle f(\tau), E^{s'}(\tau^A, 0, L, 1) \rangle du$$

$$= - \int_{iT}^{iT+1} \langle f^+(\tau), E^{s'}(\tau^A, 0, L, 1) \rangle du + O(e^{-\epsilon T})$$

for some $\epsilon > 0$ since $f^-$ is of exponential decay and $E^{s'}$ is of moderate growth. Proposition 4.6 asserts that $E^{s'}(\tau^A, 0, L) = E(\tau, L) + \Lambda(0, \chi) (d+1) \log(v) + \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Q}} a(m, \mu, v) q^m$

such that $a(m, \mu, v) q^m$ is of exponentially decay as $v \to \infty$. Thus,

$$-I_1(T) = CT[\langle f^+(\tau), E(\tau, L) \rangle] + \Lambda(0, \chi) (d+1) \log T + \sum_{\mu \in L'/L} \sum_{m+n=0} c^+(m, \mu) a(n, \mu, T).$$

The last sum goes to zero when $T \to \infty$. So we can take $A_0 = (d+1) \Lambda(0, \chi)$, and

$$I_1 = -CT[\langle f^+(\tau), E(\tau, L) \rangle]$$

as claimed. \hfill \Box

**Remark 5.3.** There is a sign error in front of $\mathcal{L}'(\xi(f), U, 0)$ in [BY2, Theorem 4.7] and throughout that paper caused by this error. The $+\mathcal{L}'(\xi(f), U, 0)$ in that theorem should be $-\mathcal{L}'(\xi(f), U, 0)$. Accidentally, in the proof of [BY2, Theorem 7.7], there is another sign error relating the Faltings’ height and the Neron-Tate height. Two wrong signs give the correct formula in [BY2, Theorem 7.7], which somehow prevented the authors from discovering the sign error earlier.

As in [BY2], this theorem raises two interesting conjectures. We very briefly describe them and refer to [BY2, Section 5] for details. Assume that there is a regular scheme $\mathcal{X}_K \to \text{Spec} \mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification $\mathcal{X}_K^c$ of $\mathcal{X}_K$. Let $Z(m, \mu)$ and $Z(W)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(m, \mu)$ and $Z(W)$, respectively. Such extensions can be found in low dimensional cases using a moduli interpretation of $\mathcal{X}_K$. For an $f \in H_{1-g,\rho_L}$, the function $\Phi(\cdot, f)$ is a
Green function for the divisor $Z(f)$. Set $Z(f) = \sum_\mu \sum_{m>0} c^+(-m, \mu) Z(m, \mu)$. Then the pair

$$\hat{Z}(f) = (Z(f), \Phi(\cdot, f))$$

defines an arithmetic divisor in $\widehat{CH}^1(X_K)$. Theorem 5.2 provides a formula for the quantity

$$(5.4) \quad \langle \hat{Z}(f), Z(W) \rangle_\infty = \frac{1}{2} \Phi(Z(W), f) ,$$

and inspires the following 'equivalent' conjectures.

**Conjecture 5.4.** Let $\mu \in L'/L$, and let $m \in Q(\mu) + \mathbb{Z}$ be positive. Then $Z(m, \mu)$ and $Z(W)$ intersect properly, and the arithmetic intersection number $\langle Z(m, \mu), Z(W) \rangle_{\text{fin}}$ is equal to $-\frac{1}{2} C(W, K)$ times the $(m, \mu)$-th Fourier coefficient of $E(\tau, L)$.

**Conjecture 5.5.** For any $f \in H_{1-d, \rho_L}$, one has

$$(5.5) \quad \langle \hat{Z}(f), Z(W) \rangle_{\text{Fal}} = \frac{1}{2} C(W, K) (c^+(0, 0)\kappa(0, 0) - L'(0, \xi(f), L)).$$

Here $\kappa(0, 0)$ is the constant term of $E(\tau, L)$.

6. **Hilbert modular surfaces**

In this section, we study the case of Hilbert modular surfaces. Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D$ with non-trivial Galois automorphism $\sigma$ and different $\partial = \partial_F$. Let

$$V = \{ M = \begin{pmatrix} u & b/\sqrt{D} \\ a/\sqrt{D} \sigma(u) \\ \end{pmatrix} : a, b \in \mathbb{Q}, u \in F \} \simeq F \oplus \mathbb{Q}^2,$$

with quadratic form

$$Q(M) = \det(M) = N_{F/\mathbb{Q}}(u) - ab,$$

of signature $(2, 2)$, and let $L$ be the lattice $O_F \oplus \mathbb{Z}^2$. Let $G$ be the algebraic group over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra $R$,

$$G(R) = \{ g \in \text{GL}_2(F \otimes_\mathbb{Q} R) : \det g \in R^\times \}.$$

Then $G \simeq \text{GSpin}(V)$ and the action of $G$ on $V$ is given by

$$g \cdot A = \sigma(g)Ag^{-1}.$$ 

Let

$$K = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\hat{\mathbb{Q}}) : a, d \in \hat{O}_F, c \in \hat{\partial}^{-1}, b \in \hat{\partial} \}.$$ 

Note that the dual lattice of $L$ is $L' \simeq \partial^{-1} \oplus \mathbb{Z}^2$. Then it is easy to check that $K$ preserves $L$ and acts trivially on $L'/L$. By the strong approximation theorem, one has $G(\hat{\mathbb{Q}}) = G(\mathbb{Q})_+ K$ and so

$$X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathbb{D} \times G(\hat{\mathbb{Q}})/K) \simeq X = \Gamma \backslash \mathbb{H}^2,$$
where
\[ \Gamma = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F) : a, d \in \mathcal{O}_F, c \in \partial^{-1}, b \in \partial \} \]
and
\[ D \simeq (\mathbb{H}^+ \times \mathbb{H}^+)_0. \]
Here the subscript indicates the set of pairs \((z_1, z_2) \in \mathbb{H}^+ \times \mathbb{H}^+\) such that \(\text{Im}(z_1)\text{Im}(z_2) > 0\).

Thus, \(X_K(\mathbb{C})\) is a Hilbert modular surface.

In fact, the canonical model of the Shimura variety \(X_K\) over \(\mathbb{Q}\) is the coarse moduli scheme over \(\mathbb{Q}\) of isomorphic classes of principally polarized abelian surfaces with real multiplication, \(A = (A, \kappa_0, \lambda), \kappa_0 : \mathcal{O}_F \to \text{End}(A)\), [Rä, Section 1.27]. Hence \(X = X_K(\mathbb{C})\) can be naturally identified with the set of isomorphism classes of such objects over \(\mathbb{C}\). This interpretation allows us to define CM 0-cycles as follows.

6.1. CM 0-cycles. Let \(E\) be a non-biquadratic quartic CM field with real quadratic subfield \(F = \mathbb{Q}(\sqrt{D})\) with fundamental discriminant \(D\), and let \(\Sigma = \{\sigma_1, \sigma_2\}\) be a fixed a CM type of \(E\). Let \(\tilde{E}\) be the reflex field of \((E, \Sigma)\), the subfield of \(\mathbb{C}\) generated by the type norms
\[ (6.4) \quad N_{\Sigma}(z) = \sigma_1(z)\sigma_2(z), \quad z \in E. \]
Then \(\tilde{E}\) is also a quartic CM number field with real subfield \(\tilde{F} = \mathbb{Q}(\sqrt{\tilde{D}})\) if the absolute discriminant of \(E\) is \(d_E = D^2\tilde{D}\). Notice that, in general, \(\tilde{D}\) is not the fundamental discriminant of \(\tilde{F}\).

Let \(\text{CM}^\Sigma(E)\) be the set of isomorphic classes of principally polarized CM abelian surfaces \(A = (A, \kappa, \lambda)\) over \(\mathbb{C}\) of CM-type \((\mathcal{O}_E, \Sigma)\): \(A\) is a CM abelian surface over \(\mathbb{C}\) with an \(\mathcal{O}_E\)-action \(\kappa : \mathcal{O}_E \hookrightarrow \text{End}(A)\) and a principal polarization \(\lambda : A \to A^\vee\) satisfying the further conditions: (i) the Rosati involution induced by \(\lambda\) induces the complex conjugation on \(E\), and (ii) there are two translation invariants, non-zero differentials \(\omega_1\) and \(\omega_2\) on \(A\) over \(\mathbb{C}\) such that
\[ \kappa(r)^*\omega_i = \sigma_i(r)\omega_i, \quad r \in \mathcal{O}_E, i = 1, 2. \]
There is a natural map (of sets of isomorphism classes)
\[ j^\Sigma : \text{CM}^\Sigma(E) \longrightarrow X, \quad A = (A, \kappa, \lambda) \mapsto (A, \kappa|_F, \lambda). \]
We also let
\[ j : \text{CM}(E) = \bigsqcup_\Sigma \text{CM}^\Sigma(E) \longrightarrow X, \]
so that \(\text{CM}(E)\) defines a 0-cycle on \(X\). The main purpose of this section is to use Theorem 5.2 to derive a formula for \(\Phi(\text{CM}(E), f)\) for any \(f \in H_{0,\rho_L}\), which is a generalization of [BY1, Theorem 1.4].
6.2. CM($E$) as an orbit space I. Given $A = (A, \kappa, \lambda) \in \text{CM}^\Sigma(E)$, let $M = H_1(A, \mathbb{Z})$ with the induced $\mathcal{O}_E$-action and the non-degenerate symplectic form

$$\lambda : M \times M \to \mathbb{Z}$$

coming from the polarization of $A$. In particular, $\lambda$ defines a perfect pairing on $M$ and satisfies

$$\lambda(\kappa(r)x, y) = \lambda(x, \kappa(\bar{r})y), \quad r \in \mathcal{O}_E, x, y \in M,$$

so that $(M, \kappa, \lambda)$ is an $\mathcal{O}_F$-polarized CM module in the sense of [HY]. The action $\kappa$ makes $M$ a projective $\mathcal{O}_E$-module of rank one, isomorphic to a fractional ideal $\mathfrak{A} \subset \mathfrak{a} \subset E$. The polarization $\lambda$ induces a polarization $\lambda_\xi$ on $\mathfrak{a}$ given by

$$(6.5) \quad \lambda_\xi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{Z}, \quad \lambda_\xi(x, y) = \text{tr}_{E/\mathbb{Q}} \xi \bar{\xi} xy,$$

where $\xi \in E^\times$ with $\bar{\xi} = -\xi$. A simple calculation shows that $\lambda$ is a principal polarization if and only if

$$(6.6) \quad a := \xi \partial_{E/F} \mathfrak{A} \cap F = \partial^{-1}.$$

Moreover, $A$ is of CM type $\Sigma$ if and only if $\Sigma(\xi) = (\sigma_1(\xi), \sigma_2(\xi)) \in \mathbb{H}^2$, see for example [BY1, Lemma 3.1].

The converse is also true; given $(\mathfrak{A}, \xi)$ satisfying (6.6), there is a unique CM type $\Sigma$ of $E$ such that $\Sigma(\xi) \in \mathbb{H}^2$, and one has

$$A(\mathfrak{A}, \xi) := (A = (\mathfrak{A} \otimes 1)/(E \otimes_{\mathbb{Q}} \mathbb{R}), \kappa, \lambda_\xi) \in \text{CM}^\Sigma(E).$$

Here we identify $E \otimes \mathbb{R}$ with $\mathbb{C}^2$ via the CM type $\Sigma$.

Two such pairs $(\mathfrak{A}, \xi_\mathfrak{A})$ and $(\mathfrak{B}, \xi_\mathfrak{B})$ are equivalent if there is an $r \in E^\times$ such that

$$\mathfrak{B} = r \mathfrak{A}, \quad r \bar{r} \xi_\mathfrak{B} = \xi_\mathfrak{A}.$$

Let $\text{PF}(E)$ be the set of equivalence classes of pairs $(\mathfrak{A}, \xi_\mathfrak{A})$ satisfying (6.6), and let $\text{PF}^\Sigma(E)$ be the subset of $(\mathfrak{A}, \xi_\mathfrak{A})$ with $\Sigma(\xi_\mathfrak{A}) \in \mathbb{H}^2$. Let $C(E) = I(E)/P(E)$ be the generalized ideal class group of $E$, where $I(E)$ is the group of pairs $(\mathfrak{I}, \zeta)$ where $\mathfrak{I}$ is a fractional ideal of $E$ and $\zeta \in F^\times$ with

$$\mathfrak{I} \mathfrak{I} = \zeta \mathcal{O}_E,$$

and $P(E)$ is the subgroup of pairs $(r \mathcal{O}_E, r \bar{r})$ for $r \in E^\times$. The group $C(E)$ acts on $\text{PF}(E)$ by

$$(6.7) \quad (\mathfrak{I}, \zeta) \cdot (\mathfrak{A}, \xi) = (\mathfrak{A}, \zeta^{-1} \xi).$$

The following lemma is easy to check and is left to the reader.

**Lemma 6.1.** Let the notation be as above. Then

1. The map $(\mathfrak{A}, \xi) \mapsto A(\mathfrak{A}, \xi)$ gives a bijection between $\text{PF}(E)$ and $\text{CM}(E)$ and between
The action of the group $C(E)$ on $PF(E)$ defined by \((6.7)\) is simply transitively.

The action of $C(E)$ on $PF(E)$ gives thus a simply transitive action of $C(E)$ on $CM(E)$, which can also be described via the Serre tensor product construction as in \([HY]\).

### 6.3. Special endomorphisms.

To a pair $(\mathfrak{A}, \xi)$ in $PF^E(E)$, we can associate a lattice
\[
(6.8) \quad L(\mathfrak{A}, \xi) = \{ j \in \text{End}(\mathfrak{A}) : j \circ \kappa(a) = \kappa(\sigma(a)) \circ j, \quad j^* = j \}
\]
of special endomorphisms, and let $V(\mathfrak{A}, \xi) = L(\mathfrak{A}, \xi) \otimes \mathbb{Q}$. Here $j^*$ is the adjoint of $j$ with respect to the pairing $\lambda$ defined by \((6.5)\). If $A = \mathfrak{A}(\mathfrak{A}, \xi)$, we will also write $L(A)$ and $V(A)$ for $L(\mathfrak{A}, \xi)$ and $V(\mathfrak{A}, \xi)$ respectively. In the notation of \([HY]\), $L(A)$ is associated to the polarized CM module $M = H_1(A) = \mathfrak{A}$ (not the lattice of special endomorphisms associated to the polarized CM abelian variety defined in \([HY\), Section 2]). The following slight refinement of \([HY\), Proposition 1.2.2\] shows that $Q_A(j) = j^2$ is a integral quadratic form on $L(\mathfrak{A}, \xi)$.

**Lemma 6.2.** Let $(\mathfrak{A}, \xi) \in PF^E(E)$.

1. There are $\alpha, \beta \in \mathfrak{A}$ such that
   \[
   \mathfrak{A} = \mathcal{O}_F \alpha + \partial^{-1} \beta, \quad \xi(\bar{\alpha} \beta - \alpha \bar{\beta}) = 1.
   \]
2. Let $\alpha$ and $\beta$ be as in (1) and identify $\mathfrak{A}$ with $\mathcal{O}_F \oplus \partial^{-1} \subset F^2$ via
   \[
f := f_{\alpha, \beta} : \mathfrak{A} \to \mathcal{O}_F \oplus \partial^{-1}, \quad x\alpha + y\beta \mapsto \begin{pmatrix} x \\ y \end{pmatrix},
   \]
   and define
   \[
   \kappa := \kappa_{\alpha, \beta} : \mathcal{O}_E \to \text{End}_{\mathcal{O}_F}(\mathcal{O}_F \oplus \partial^{-1}) \subset M_2(F),
   \]
   by
   \[
r(\alpha, \beta) \begin{pmatrix} x \\ y \end{pmatrix} = (\alpha, \beta) \kappa(r) \begin{pmatrix} x \\ y \end{pmatrix}.
   \]
Then the polarization $\lambda_\xi$ becomes the standard symplectic form $\lambda_{st}$ on $F^2$ given by
\[
\lambda_{st}((x_1, y_1)^t, (x_2, y_2)^t) = \text{tr}_{F/\mathbb{Q}}(x_1y_2 - x_2y_1).
\]
3. Moreover, define $j_0 \in \text{End}_{\mathbb{Q}}(F^2)$ by $j_0((x, y)^t) = (\sigma(x), \sigma(y))^t$. Then, for $V$ and $L$ given in \((6.7)\), there is a $\mathbb{Q}$-linear isomorphism $V \sim \to V(\mathfrak{A}, \xi)$ given by $v \mapsto v \circ j_0$ and this isomorphism sends $L$ onto $L(\mathfrak{A}, \xi)$.
4. $Q_\mathfrak{A}(j) = j^2$ is a $\mathbb{Q}$-quadratic form on $V(\mathfrak{A}, \xi)$, and the quadratic lattice $(L(\mathfrak{A}, \xi), Q_{\mathfrak{A}, \xi}) \cong (L, Q)$ is independent of $(\mathfrak{A}, \xi)$.

**Proof.** It is easy to check using \((6.6)\) that the map
\[
\mathcal{O}_F \to \text{Hom}_{\mathcal{O}_F}(\Lambda^2_{\mathcal{O}_F} \mathfrak{A}, \partial^{-1}) = \text{Hom}_{\mathbb{Z}}(\Lambda^2_{\mathcal{O}_F} \mathfrak{A}, \mathbb{Z}), \quad a \mapsto \lambda_\alpha
\]
is an isomorphism ([HY, Lemma 3.1]).

(1): One can always write
\[ \mathfrak{A} = \mathcal{O}_F \alpha + \mathfrak{f} \beta \]
for some fractional ideal \( \mathfrak{f} \) of \( F \) and \( \alpha, \beta \in \mathfrak{A} \). Using the above isomorphism and explicit calculation, one gets
\[ \mathcal{O}_F = (\xi(\overline{\alpha \beta} - \alpha \overline{\beta}))^{-1}. \]
So replacing \( \beta \) by \( a \beta \) and \( \mathfrak{f} \) by \( a^{-1} \mathfrak{f} \) if necessary, for some \( a \), we may choose \( \mathfrak{f} = \partial^{-1} \), and
\[ \xi(\overline{\alpha \beta} - \alpha \overline{\beta}) = 1. \]

For (2), a simple calculation gives
\[ \lambda_{st}(f(z_1), f(z_2)) = \lambda_{\xi}(z_1, z_2) \]
for \( z_l = x_l \alpha + y_l \beta \in \mathfrak{A} \). Write \( j = C \circ j_0 \). Then \( j \circ \kappa(a) = \kappa(\sigma(a)) \circ j \) for all \( a \in F \) if and only if \( C \in M_2(F) \). Write \( w = (0 \ 1) \). Then one has for \( z_l = (x_l, y_l)^t \in F^2 \)
\[
\lambda_{st}(j(z_1), z_2) = \text{tr}_{F/Q}(\sigma(x_1), \sigma(y_1))C^t w(x_2, y_2)^t \\
= \text{tr}_{F/Q}((x_1, y_1)\sigma(C)^t(w(\sigma(x_2), \sigma(y_2))^t) \\
= \text{tr}_{F/Q}((x_1, y_1)w\sigma(C)^t(\sigma(x_2), \sigma(y_2))^t) \\
= \bigg( \bigg( \frac{u}{\sqrt{\sigma(u)}} \bigg) \bigg) (\sigma(x), \sigma(y))^t \in \mathcal{O}_F \oplus \mathfrak{A}
\]
for all \((x, y)^t \in \mathcal{O}_F \oplus \mathfrak{A} \), i.e., \( C \in L \).

For (3), one simply checks that \( j = C \circ j_0 \in L(\mathfrak{A}, \xi) \) satisfies \( j^2 = \det C = u\sigma(u) - ab \), which is an integral quadratic form. So \( (L(\mathfrak{A}, \xi), Q_\mathfrak{A}) \cong (L, Q) \) is independent of \( (\mathfrak{A}, \xi) \).

6.4. \( \text{CM}(\tilde{E}) \) as an orbit space II. The following finer structure on \( V(\mathfrak{A}, \xi) \) was discovered in [HY] Section 1, see also [BY1, Section 4].

**Proposition 6.3.** Let \( (\mathfrak{A}, \xi) \in \text{PF}^\Sigma(E) \). Then \( V(\mathfrak{A}, \xi) \) has an \( \tilde{E} \)-vector space structure defined by
\[ N_\Sigma(r) \bullet j = \kappa(r) \circ j \circ \kappa(\tilde{r}), \quad r \in E, \quad j \in V(\mathfrak{A}, \xi). \]
Moreover there is a unique \( \tilde{F} \)-valued quadratic form \( \tilde{Q}_\mathfrak{A} \) on \( V(\mathfrak{A}, \xi) \) such that
\[
Q_\mathfrak{A}(j) = \text{tr}_{\tilde{F}/Q}\tilde{Q}_\mathfrak{A}(j), \\ \langle \tilde{r} \bullet j_1, j_2 \rangle_\mathfrak{A} = \langle j_1, \tilde{r} \bullet j_2 \rangle_\mathfrak{A}
\]
for any \( j \in V(\mathfrak{A}, \xi) \) and \( \tilde{r} \in \tilde{E} \). Here \( \langle , \rangle_\mathfrak{A} \) is the symmetric bilinear \( \tilde{F} \)-form associated to \( \tilde{Q}_\mathfrak{A} \).

**Proof.** This is described in detail in [HY] Section 1. We just replaced the CM field \( E^2 \) in [HY] by \( \tilde{E} \), which is isomorphic to \( E^2 \) via the reflex homomorphism induced by \( a \otimes b \mapsto \sigma_1(a)\sigma_2(b) \).
Let $W(\mathfrak{A}, \xi) = (V(\mathfrak{A}, \xi), \tilde{Q}_\mathfrak{A})$ be the $F$-quadratic space associated to $(\mathfrak{A}, \xi)$, and let $	ilde{L}(\mathfrak{A}, \xi) = L(\mathfrak{A}, \xi)$ but with the $F$-quadratic form $\tilde{Q}_\mathfrak{A}$. Recall that
\[(\tilde{L}(\mathfrak{A}, \xi), \text{tr}_{F/Q} \tilde{Q}_\mathfrak{A}) \cong (L, Q)\]
as quadratic $\mathbb{Z}$-lattices. By Proposition 6.3, there is an $\alpha \in \tilde{E}^\times$ such that $W(\mathfrak{A}, \xi) \cong (\tilde{E}, \alpha z\tilde{E})$, and so $\text{SO}(W(\mathfrak{A}, \xi)) = \tilde{E}^1$. Let $T_E$ be the algebraic group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra $R$,
\[(6.9) \quad T_E(R) = \{ t \in (E \otimes R)^\times : t\bar{t} \in \mathbb{Q}^\times \}.\]
Note that the embedding $\kappa = \kappa_{\alpha, \beta}$ of (2) of Lemma 6.2 identifies $T_E$ with a maximal torus in the group $G$ defined by (6.2). Let $S_{\tilde{E}}$ be the algebraic group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra $R$,
\[(6.10) \quad S_{\tilde{E}}(R) = \{ t \in (\tilde{E} \otimes R)^\times : t\bar{t} = 1 \}.
\]
In particular, $S_{\tilde{E}}(\mathbb{Q}) = \tilde{E}^1$, and $S_{\tilde{E}} = \text{Res}_{F/Q} \text{SO}(W(\mathfrak{A}, \xi))$. Moreover, by [HY, Lemma 1.4.1], the action of $T_E$ on $V(\mathfrak{A}, \xi)$ defined by
\[(6.11) \quad t \cdot j = \frac{1}{tt} \kappa(t) \circ j \circ \kappa(\bar{t})\]
determines an exact sequence
\[(6.12) \quad 1 \rightarrow \mathbb{G}_m \rightarrow T_E \xrightarrow{\nu_E} S_{\tilde{E}} \rightarrow 1, \quad \nu_E(t) = \frac{t \otimes t}{tt}.
\]
So, under the identification of $G$ with $\text{GSpin}(V)$ given above, $T_E$ is identified with the maximal torus $T$ of $\text{GSpin}(V)$ associated to $W(\mathfrak{A}, \xi)$ by the construction of Section 2. Note the shift in notation (!), so that we are now writing $\tilde{E}$ for the field denoted by $E$ in Section 2. By the construction of Section 2, we then have a CM cycle
\[Z(W(\mathfrak{A}, \xi), z_0^+) = T(\mathbb{Q}) \setminus \{ z_0^+ \} \times T(\tilde{\mathbb{Q}})/U_E,\]
where
\[U_E = \{ r \in \tilde{O}_E^\times : r \bar{r} \in \tilde{Z}^\times \} = K \cap T(\tilde{\mathbb{Q}}).\]
Indeed, $U_E \subset K \cap T(\tilde{\mathbb{Q}})$ and $U_E$ is a maximal compact subgroup of $T(\tilde{\mathbb{Q}})$, so $U_E = K \cap T(\tilde{\mathbb{Q}})$. Let $C(T) = T(\mathbb{Q}) \setminus T(\tilde{\mathbb{Q}})/U_E$ be the ‘class group’ of $T$. Define a homomorphism
\[C(T) \rightarrow C(E), \quad [t] \mapsto [((t), \zeta_t)],\]
where $(t) = t\tilde{\mathcal{O}}_E \cap E$ is the ideal of $E$ associated to $t$, and $\zeta_t \in \mathbb{Q}_{>0}$ with
\[\zeta_t \mathbb{Z} = tt\tilde{\mathbb{Z}} \cap \mathbb{Q}.
\]
Via this group homomorphism, $C(T)$ acts on $\text{CM}(E)$.

Suppose that $z_0^+ = z_0^+(W(\mathfrak{A}, \xi)) \in \mathbb{D}^+$ is such that the point $[z_0^+, 1] \in X_K(\mathbb{C}) = X$ corresponds to the isomorphism class of $\mathbf{A} = \mathbf{A}(\mathfrak{A}, \xi) \in \text{CM}^+(E)$. Then, for $t \in T(\tilde{\mathbb{Q}})$, $[z_0^+, t]$ corresponds to $t \cdot \mathbf{A}$. The points $z_0^{\pm}$ go to the same point in $X$ since $\text{diag}(1, -1) \in$
G(\mathbb{Q}) \cap K. On the other hand \( \tilde{A} = (A, \tilde{\kappa}, \lambda) \in \text{CM}^\Sigma(E) \) also has the same image in \( X \) as \( A \). So we can view \( Z(W(A, \xi), z_0^+) \) as the \( C(T) \)-orbit of \( A \) in \( \text{CM}^\Sigma(E) \) and \( Z(W(A, \xi), z_0^-) \) as the \( C(T) \)-orbit of \( \tilde{A} \) in \( \text{CM}^\Sigma(E) \). Let \( \Sigma' \) and \( \Sigma'' \) be the other two CM types of \( E \).

**Lemma 6.4.** (1) For any \( t \in C(T) \) and \( (\mathfrak{A}, \xi) \in \text{PF}^\Sigma(E) \), there is an isomorphism

\[
(W(t \bullet (\mathfrak{A}, \xi)), \tilde{Q}_{t \mathfrak{A}}) \cong (W(\mathfrak{A}, \xi), \tilde{Q}_\mathfrak{A})
\]

of quadratic spaces over \( \hat{F} \) inducing an isomorphism \( (\hat{L}(t \bullet (\mathfrak{A}, \xi)), \tilde{Q}_{t \mathfrak{A}}) \cong (\hat{L}(\mathfrak{A}, \xi), \tilde{Q}_\mathfrak{A}) \).

(2) There is a class \( c = (\mathfrak{A}, \xi) \in C(E) \) such that \( c \bullet (\mathfrak{A}, \xi) \) has CM type \( \Sigma'' \) for every \( (\mathfrak{A}, \xi) \in \text{PF}^\Sigma(E) \) and \( (\hat{L}(c \bullet (\mathfrak{A}, \xi)), \tilde{Q}_{c \bullet \mathfrak{A}}) \cong (\hat{L}(\mathfrak{A}, \xi), \tilde{Q}_\mathfrak{A}) \).

**Proof.** (1) It is clear that

\[
\hat{f} : \hat{L}(\mathfrak{A}, \xi) \mapsto \hat{L}(t \bullet (\mathfrak{A}, \xi)) \quad j \mapsto j_t,
\]

is a \( \hat{\mathbb{Z}} \)-linear isomorphism, where \( j_t(m) = j(m) \) for \( m \in \hat{\mathfrak{A}} \). It is also clear that the induced isomorphism \( \hat{f}_\mathfrak{A} : \hat{W}(\mathfrak{A}, \xi) \to \hat{W}(t \bullet \mathfrak{A}, \xi) \) is \( \hat{E} \)-linear. One checks

\[
Q(j_t) = j_t^2 = j^2 = Q(j), \quad \text{for all } j \in W(\mathfrak{A}, \xi).
\]

So, by uniqueness, one has

\[
\tilde{Q}_{t \mathfrak{A}}(j_t) = \tilde{Q}_{\mathfrak{A}}(j), \quad \text{for all } j \in \hat{W}(\mathfrak{A}, \xi).
\]

Thus, \( \hat{f} \) is an \( \hat{F} \)-quadratic isomorphism which sends \( \hat{L}(\mathfrak{A}, \xi) \) onto \( \hat{L}(t \bullet (\mathfrak{A}, \xi)) \). On the other hand, since both \( (\mathfrak{A}, \xi) \) and \( t \bullet (\mathfrak{A}, \xi) \) are of the CM type \( \Sigma \), there is an \( \hat{F}_\Sigma \)-quadratic isomorphism \( f_\infty \) between \( W(\mathfrak{A}, \xi)_\infty \) and \( W(t \bullet (\mathfrak{A}, \xi))_\infty \) by [HY, Proposition 1.3.5]. By the Hasse principle, one has \( W(\mathfrak{A}, \xi) \cong W(t \bullet (\mathfrak{A}, \xi)) \). This proves (1). Claim (2) is [HY, Proposition 1.4.3].

**Corollary 6.5.** For \( A = A(\mathfrak{A}, \xi) \in \text{CM}^\Sigma(E) \), let \( Z(A) = Z(W(A, \xi)) \) be the CM cycle defined in Section 2. Then, as a subset of \( \text{CM}(E) \), \( Z(A) \) is the union of the \( C(T) \)-orbits of \( A, \tilde{A}, c \bullet A, \) and \( c \bullet \tilde{A} \), where \( c \) is a fixed element in \( C(E) \) satisfying the condition in Lemma 6.4.

**Remark 6.6.** By the definition in Section 2, each point in \( Z(A) \) is counted with multiplicity \( \frac{1}{w_E} \), where \( w_E \) is the number of roots of unity in \( E \). Furthermore, since \( z_0^+ \) and \( z_0^- \) go to the same point in \( X \) (resp. \( A \) and \( \tilde{A} \) go to the same point in \( X \)), the image of a point in \( \text{CM}(E) \) in \( X \) is counted with multiplicity \( \frac{1}{w_E} \) in this paper.

**Remark 6.7.** A key point here is that the 0-cycle \( \text{CM}(E) \) associated to the non-biquadratic CM field \( E/F \) via moduli coincides with a union of 0-cycles associated to the quadratic spaces \( W(\mathfrak{A}, \xi) \) for the non-biquadratic CM fields \( \tilde{E}/\tilde{F} \) via the Shimura variety construction of Section 2. Note that the Shimura variety \( \text{Sh}(G, \mathbb{D}) \) for \( G = \text{GSpin}(V) \) is only PEL in
this case due to an accidental isomorphism; this accounts for the duality between the roles of the fields $E/F$ and $\tilde{E}/\tilde{F}$.

Combining Theorem 5.2 with Corollary 6.5, one has the following theorem.

**Theorem 6.8.** Let $f \in H_{0,\bar{\rho}_L}$, and let $E$ be a non-biquadratic CM quartic field with real subfield $F$ and a CM type $\Sigma$. Let

$$c(E) = \frac{4 |C(T)|}{w_E \Lambda(0, \chi)}.$$

(1) For $A \in \text{CM}^E(E)$, we have

$$\Phi(Z(A), f) = c(E) \left( CT[(f^+, \mathcal{E}(\tau, \tilde{L}(A)))] - \mathcal{L}'(0, \xi(f), \tilde{L}(A)) \right).$$

(2) We have

$$\Phi(CM(E), f) = c(E) \sum_{A \in CM(T) \setminus \text{CM}^E(E)} \left( CT[(f^+, \mathcal{E}(\tau, \tilde{L}(A)))] - \mathcal{L}'(0, \xi(f), \tilde{L}(A)) \right).$$

Note that, for $A = A(2\mathfrak{A}, \xi)$, we are writing $\tilde{L}(A)$ for $\tilde{L}(2\mathfrak{A}, \xi)$ and $W(A)$ for $W(2\mathfrak{A}, \xi)$. Notice also that $\tilde{L}(A)$ is $L = L(V)$ with $\tilde{F}$-quadratic form $\tilde{Q}_A$.

**Proof.** By Theorem 5.2 and Corollary 6.5, one has

$$\Phi(Z(A), f) = c_1(E) \left( CT[(f^+, \mathcal{E}(\tau, \tilde{L}(A)))] - \mathcal{L}'(0, \xi(f), \tilde{L}(A)) \right)$$

with

$$c_1(E) = \frac{\deg Z(W(A), z^\pm_0)}{\Lambda(0, \tilde{\chi}).}$$

Here $\tilde{\chi}$ is the quadratic Hecke character of $\tilde{E}$ associated to $\tilde{E}/\tilde{F}$. By the proof of [Ya4, Proposition 3.3], one has

$$\Lambda(s, \chi) = \Lambda(s, \tilde{\chi}).$$

So $c_1(E) = c(E)$ by Remark 6.6. Claim (2) follows from (1) and Corollary 6.5. □

### 6.5. Integral structure.

In this section, we assume that $d_E = D^2 \tilde{D}$ with $D \equiv 1 \mod 4$ prime and $\tilde{D} \equiv 1 \mod 4$ square free, and give a more explicit formula for the CM value $\Phi(CM(E), f)$. Let $\Sigma$ be again a CM type of $E$ and let $\tilde{E}$ be its reflex field. Consider the $\tilde{F}$-quadratic space

$$\tilde{W} = \tilde{E}, \quad \tilde{Q}(z) = -\frac{z \tilde{z}}{\sqrt{\tilde{D}}}$$

with even integral lattice $\tilde{L} = \mathcal{O}_{\tilde{E}}$. 

Proposition 6.9. Let the notation and assumption be as above. Then for any $A \in \text{CM}^2(E)$, there is an $\tilde{F}$-quadratic isomorphism

$$\phi_A : (W(A), \tilde{Q}_A) \cong (\tilde{W}, \tilde{Q})$$

such that $\phi_A(\tilde{L}(A))$ is in the same genus as $\tilde{L}$. In particular $\tilde{L}(A)$ is an $\mathcal{O}_F$-module and all $(\tilde{L}(A), \tilde{Q}_A)$ are in the same genus.

Proof. Let $A = A(\mathfrak{a}, \xi) \in \text{CM}^2(E)$ and let $\alpha$ and $\beta$ be chosen as in Lemma 6.2. Define

$$\phi : W(A) = V \rightarrow \tilde{E}, \quad \phi(A) = \frac{1}{\sqrt{D}}(\sigma_1(\alpha), \sigma_1(\beta))A\sigma_2(\alpha, \sigma_2(\beta))^t, \quad w = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}).$$

We first prove that $\phi$ is an isomorphism of quadratic spaces over $\tilde{F}$ between $(W(A), \tilde{Q}_A)$ and $(\tilde{E}, -\frac{z \bar{z}}{\sqrt{D}N(\mathfrak{a})})$. To verify the claim, let

$$V_1 = \{A \in M_2(F) : \sigma(A) = A^t\} = \{(\begin{smallmatrix} a & b \\ \sigma(u) & a \end{smallmatrix}) : a, b \in \mathbb{Q}, u \in F\}$$

with quadratic form $Q_1(A) = D \det A$, and let

$$\phi_1 : V \rightarrow V_1, \quad A \mapsto \frac{1}{\sqrt{D}}Aw,$$

$$\phi_2 : V_1 \rightarrow E(\Sigma), \quad A \mapsto (\pi_1(\alpha), \pi_1(\beta))A(\pi_2(\alpha), \pi_2(\beta))^t.$$

Then $\phi = \phi_2 \circ \phi_1$. It is easy to check that $\phi_1$ is a $\mathbb{Q}$-isomorphism between $(V, \det)$ and $(V_1, D \det)$. On the other hand, $\phi_2$ is basically the map in [BY1, (4.11)], and is a $\mathbb{Q}$-isomorphism between $(V_1, D \det)$ and $(\tilde{E}, -\text{tr}_{\tilde{F}/\mathbb{Q}} \frac{z \bar{z}}{\sqrt{D}N(\mathfrak{a})})$. So $\phi$ is a $\mathbb{Q}$-quadratic space isomorphism. Next, For $\tilde{r} = N_{\Sigma}(r) \in \tilde{E}$ with $r \in E^\times$, one has

$$\phi(\tilde{r} \bullet A) = \frac{1}{\sqrt{D}}(\sigma_1(\alpha), \sigma_1(\beta))\kappa(r)A\sigma_2(\kappa(r)^t)w(\sigma_2(\alpha), \sigma_2(\beta))^t$$

$$= \frac{1}{\sqrt{D}}(\sigma_1(r\alpha), \sigma_1(r\beta))Aw\sigma_2(\kappa(r)^t)^t(\sigma_2(\alpha), \sigma_2(\beta))^t$$

$$= \frac{1}{\sqrt{D}}(\sigma_1(r\alpha), \sigma_1(r\beta))Aw(\sigma_2(r\alpha), \sigma_2(r\beta))^t$$

$$= \sigma_1(r)\sigma_2(r)\phi(A) = \tilde{r}\phi(A).$$

So $\phi$ is $\tilde{E}$-linear. So $\phi$ is an $\tilde{F}$-quadratic space isomorphism between $(W(A), \tilde{Q}_A)$ and $(\tilde{E}, -\frac{z \bar{z}}{\sqrt{D}N(\mathfrak{a})})$, as claimed.

Second,

$$L^0(\partial_F^{-1}) = \{(\begin{smallmatrix} b \\ \sigma(\lambda) \end{smallmatrix})_a \in V_1 : a \in \frac{1}{D}\mathbb{Z}, b \in \mathbb{Z}, \lambda \in \partial^{-1}\}$$
is a lattice in $(V_1, D \det)$. Then, by [BY1, Proposition 4.7], one has

$$\phi(L(A)) = \phi_2\phi_1(L)$$

$$= \phi_2(L^0(\partial_F^{-1}))$$

$$= N_\Sigma(A).$$

Here $N_\Sigma(A)$ is the type norm of $A$ defined as

$$N_\Sigma(A) = \sigma_1(A)\sigma_2(A)O_M \cap \tilde{E}$$

for any Galois extension $M$ of $\mathbb{Q}$ containing both $E$ and $\tilde{E}$. Thus, $\tilde{L}(A)$ is actually a fractional ideal in $\tilde{E}$, and in particular an $O_{\tilde{F}}$-lattice, and we have

$$(6.18) \quad \phi : \tilde{L}(A) = (L(A), Q_A) \cong (N_\Sigma(A), -\frac{z\bar{z}}{\sqrt{D}N(A)}).$$

Third, we prove that for every $A = A(\mathfrak{a}, \xi) \in CM^\Sigma(E)$, one has for every finite prime $p$ of $\tilde{F}$

$$(6.19) \quad (N_\Sigma(A)_p, -\frac{1}{\sqrt{D}}\frac{z\bar{z}}{N(A)}) \cong (\tilde{L}_p, \tilde{Q}).$$

Notice that ([BY1, Corollary 4.5])

$$(6.20) \quad N_{E/\mathbb{Q}}\partial_{E/\tilde{F}} = D, \quad N_{E/\tilde{F}}(N_\Sigma(A)) = N(\mathfrak{a})O_F.$$
Let
\[ (6.21) \quad E^* (\vec{\tau}, s, \vec{L}, 1) = \sum_{\mu \in \vec{L}' / \vec{L}} E^* (\vec{\tau}, s, \varphi_{\mu}, 1) \varphi_{\mu} \]
be the associated incoherent Eisenstein series, and let \( \mathcal{E}(\tau, \vec{L}) \) be the holomorphic part of \( E^* (\vec{\tau}, \Delta, 0, \vec{L}) \) with \( \tau \in \mathbb{H} \). Note that
\[ \vec{L}' / \vec{L} \simeq \partial_{\vec{E} / \hat{\vec{E}}}^{-1} \mathcal{O}_{\vec{E}} \simeq \mathbb{Z} / D \mathbb{Z}. \]

**Remark 6.10.** In [BY1, Section 6] (where our \( \vec{F} \) is denoted by \( F \)), a slightly different \( \vec{F} \)-quadratic space is used: \( \vec{W}^+ = \vec{E}, \quad \vec{Q}^+(z) = \frac{1}{\sqrt{D}} z \bar{z} \)
with lattice \( \vec{L}^+ \). This corresponds to \( \text{CM}^{\Sigma'}(E) \) where \( \Sigma' \) is a CM type of \( E \) which is not \( \Sigma \) or its complex conjugation. Notice that \( -1 \in N_{\vec{E} / \hat{\vec{E}}} \hat{\mathcal{O}}_{\vec{E}}^x \). So \( \hat{\vec{L}} \) is isomorphic to \( \vec{L}^+ \), and the associated incoherent Eisenstein series are the same:
\[ E^* (\vec{\tau}, s, \vec{L}, 1) = E^* (\vec{\tau}, s, \vec{L}^+, 1). \]

It is interesting to compare this with Lemma 6.4(2).

**Theorem 1.2** is almost clear; we restate it as follows.

**Theorem 6.11.** Assume \( d_E = D^2 \tilde{D} \) with \( D \equiv 1 \mod 4 \) prime and \( \tilde{D} \equiv 1 \mod 4 \) square free. Let \( \vec{L} = \mathcal{O}_{\vec{E}} \) with quadratic form \( \vec{Q}(z) = -\frac{1}{\sqrt{D}} z \bar{z} \). Then
\[ \Phi(\text{CM}(E), f) = c'(E) \left( \text{CT}[\{ f^+(\tau), \mathcal{E}(\tau, \vec{L}) \}] - \mathcal{L}'(0, \xi(f), \vec{L}) \right). \]
where
\[ c'(E) = \frac{\text{deg}(\text{CM}(E))}{2 \Lambda(0, \chi)}. \]

In particular, when \( \tilde{D} \) is also prime, \( c'(E) = 1 \).

**Proof.** By Proposition 6.9 one has
\[ \mathcal{E}(\tau, \vec{L}(A)) = \mathcal{E}(\tau, \vec{L}), \quad \mathcal{L}(s, \xi(f), \vec{L}(A)) = \mathcal{L}(s, \xi(f), \vec{L}) \]
for all \( A \in \text{CM}^{\Sigma}(E) \). So Theorem 6.8 gives
\[ \Phi(\text{CM}(E), f) = c(E)|C(T) \backslash \text{CM}^{\Sigma}(E)| \left( \text{CT}[\{ f^+(\tau), \mathcal{E}(\tau, \vec{L}) \}] - \mathcal{L}'(0, \xi(f), \vec{L}) \right). \]
Finally,

\[ c(E) |C(T)\setminus \text{CM}^E(E)| = \frac{4}{w_E} \frac{|\text{CM}^E(E)|}{\Lambda(0, \chi)} \]
\[ = \frac{1}{w_E} \frac{|\text{CM}(E)|}{\Lambda(0, \chi)} \]
\[ = \frac{\deg \text{CM}(E)}{2\Lambda(0, \chi)} \]

as claimed. The last claim, \( c'(E) = 1 \), follows from this and \([\text{BY1}, (9.2)]\), since our points here have multiplicity \( 2/w_E \) and our CM\((E)\) is twice the CM cycle there. \( \square \)

6.6. Scalar modular forms. In this subsection, we again assume that \( d_E = D^2\tilde{D} \) with \( D \equiv 1 \mod 4 \) prime and \( \tilde{D} \equiv 1 \mod 4 \) square-free, and translate Theorem 6.11 into the usual language of scalar modular forms, finally compare it with \([\text{BY1}, \text{Theorem } 1.4]\) in the special case considered there. Under the identification

\[ X \cong \text{SL}_2(\mathcal{O}_F)\setminus \mathbb{H}^2, \]

the Hirzebruch-Zagier divisor \( T_n \) defined in \([\text{BY1}]\) is related to the special divisor \( Z(m, \mu) \) via

\[ T_n = \frac{1}{2} \begin{cases} 
Z\left( \frac{\mu}{D}, 0 \right) & \text{if } D|n, \\
Z\left( \frac{\mu}{D}, \mu \right) + Z\left( \frac{\mu}{D}, -\mu \right) & \text{if } D \nmid n.
\end{cases} \tag{6.22} \]

Here, in the second case, \( \mu \in \tilde{L}'/\tilde{L} \) is determined by the condition that \( Q(\mu) \equiv \frac{\mu}{D} \mod 1 \). Let \( k \) be an even integer, and let \( A_{k,\rho} \) be the space of real analytic modular forms of weight \( k \) with representation \( \rho \), where \( \rho = \rho_L \) or \( \bar{\rho}_L \). Let \( A^+_k(D, (\frac{\mu}{D})) \) be the space of real analytic modular forms \( f_{sc}(\tau) = \sum_n a(n, v)q^n \) of weight \( k \) for the group \( \Gamma_0(D) \) with character \( (\frac{\mu}{D}) \) such that \( a(n, v) = 0 \) whenever \( (\frac{D}{n}) = -1 \). Here we use \( f_{sc} \) to denote a scalar valued modular form to distinguish it from vector valued modular forms in this paper. Then the following lemma is proved in \([\text{BB}]\).

**Lemma 6.12.** There is an isomorphism of vector spaces \( A_{k,\rho} \to A^+_k(D, (\frac{\mu}{D})) \),

\[ f = \sum_{\mu \in \tilde{L}'/\tilde{L}} f_\mu \varphi_\mu \mapsto f_{sc} = D^{\frac{k-1}{2}} f_0 |W_D. \]

The inverse map is given by

\[ f_{sc} \mapsto f = \frac{1}{2} D^{\frac{k-1}{2}} \sum_{\gamma \in \Gamma_0(D) \setminus \text{SL}_2(\mathbb{Z})} (f_{sc} |W_D |\gamma) \rho(\gamma)^{-1} \varphi_0, \]
where \( W_D = (D^{-1}) \) denotes the Fricke involution. Moreover, if \( f_{sc}(\tau) = \sum_n a(n, v)q^n \), then \( f \) has the Fourier expansion
\[
f = \frac{1}{2} \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Z}} \tilde{a}(n, v) q^{n/D} \phi_{\mu},
\]
where \( \tilde{a}(n, v) = a(n, v) \) if \( n \not\equiv 0 \pmod{D} \), and \( \tilde{a}(n, v) = 2a(n, v) \) if \( n \equiv 0 \pmod{D} \).

In particular, the constant term of \( f_{sc} \) agrees with the constant term of \( f \) in the \( \varphi_0 \) component. The isomorphisms of Lemma 6.12 take harmonic weak Maass forms to harmonic weak Maass forms, (weakly) holomorphic modular forms to (weakly) holomorphic modular forms, and cusp forms to cusp forms.

Let
\[
E_{sc}^*(\tau, s) = \frac{1}{\sqrt{D}} E^*(\tau^\Delta, s, \phi_0, 1)|W_D
\]
be the scalar image of \( E^*(\tau^\Delta, s, \tilde{L}, 1) \), and let \( E_{sc}(\tau) \) be the holomorphic part of
\[
\tilde{f}(\tau) = \frac{d}{ds} E_{sc}^*(\tau, s)|_{s=0}.
\]
Then \( \tilde{f}(\tau) \) is the function defined in [BY1, (7.2)]. By [BY1 Theorem 7.2], we have the following lemma.

**Lemma 6.13.** Let the notation be as above. Then
\[
E_{sc}(\tau) = -2N'(0, \chi) - 4 \sum_{m \in \mathbb{Z}_{>0}} b_m q^m
\]
where
\[
b_m = \sum_{t = \frac{n + m\sqrt{D}}{2D} \in \mathcal{O}_{E/F}} B_t
\]
with
\[
B_t = (\text{ord}_l + 1)\rho(td_{E/F}1^{-1}) \log N(l)
\]
for some (and any) prime ideal of \( F \) with \( \tilde{\chi}_l(t) = -1 \). Here \( \tilde{\chi} \) is the quadratic Hecke character of \( \tilde{F} \) associated to \( E/F \). Finally
\[
\rho(a) = |\{ \mathfrak{a} \subset \mathcal{O}_{\tilde{E}} : N_{E/F} \mathfrak{a} = a \}|.
\]

Now let \( f_{sc} = f_{sc}^+ + f_{sc}^- \in H_0^+(D, (\frac{D}{-})) \) be a harmonic weak Maass form with holomorphic part
\[
f_{sc}^+(\tau) = \sum_{n \gg -\infty} c^+(n)q^n,
\]
and let
\[ \tilde{c}^+(n) = \begin{cases} 2c^+(n) & \text{if } D|n, \\ c^+(n) & \text{if } D \nmid n. \end{cases} \]

Let \( f \in H_{0,\rho_L} \) be the associated vector valued harmonic weak Maass form. Define
\[
T(f_{sc}) = \sum_{n>0} \tilde{c}^+(-n)T_n, \quad \Phi(z, f_{sc}) := \Phi(z, f).
\]

Then one sees that \( T(f_{sc}) = Z(f) \) by (6.22). Define the Rankin-Selberg \( L \)-series
\[
L_{sc}(s, \xi(f_{sc}), \tilde{L}) = \langle E^*_{sc}(\tau, s), \xi(f_{sc}) \rangle_{\text{Pet}}.
\]

Then a straightforward calculation gives

**Lemma 6.14.** \( (1) \)
\[
L(s, \xi(f), \tilde{L}) = \frac{1}{2} D(D + 1)L_{sc}(s, \xi(f_{sc}), \tilde{L}).
\]

\( (2) \)
\[
CT[(f^+, \xi(\tau, \tilde{L}))] = -2c^+(0)\Lambda'(0, \chi) - 2 \sum_{n>0} \tilde{c}^+(-n)b_n.
\]

Combining this with Theorem 6.11, we obtain:

**Corollary 6.15.** Let \( F = \mathbb{Q}(\sqrt{D}) \) with \( D \equiv 1 \mod 4 \) prime, and let \( E \) be a CM non-biquadratic field with absolute discriminant \( d_E = D^2 \tilde{D} \) where \( \tilde{D} \equiv 1 \mod 4 \) is square free. If \( f_{sc} \in H^+_0(D, (\frac{D}{\cdot})) \), then
\[
\Phi(\text{CM}(E), f_{sc}) = -2c'(E) \left[ \sum_{n>0} \tilde{c}^+(-n)b_n + c^+(0)\Lambda'(0, \chi) + \frac{D(D + 1)}{4} L_{sc}'(0, \xi(f_{sc}), \tilde{L}) \right].
\]

Now we assume that \( f_{sc} = \sum c^+(n)q^n \in H^+_0(D, (\frac{D}{\cdot})) \) is weakly holomorphic, i.e., \( \xi(f_{sc}) = 0 \), and that \( \tilde{c}^+(n) \in \mathbb{Z} \) for \( n < 0 \). Then there is a (up to a constant of modulus 1 unique) meromorphic Hilbert modular form \( \Psi(z, f_{sc}) \) of weight \( c^+(0) \) with a Borcherds product expansion whose divisor is given by
\[
\text{div}(\Psi) = T(f_{sc}),
\]
see [BB, Theorem 9]. Moreover, by construction it satisfies
\[
- \log \| \Psi(z, f_{sc}) \|_{\text{Pet}}^2 = \Phi(z, f_{sc}),
\]
where
\[
\| \Psi(z_1, z_2, f_{sc}) \|_{\text{Pet}}^2 = |\Psi(z_1, z_2, f_{sc})|^2 (4\pi e^{-y_1 y_2}) c^+(0)
\]
is the Petersson metric (normalized in a way which is convenient for our purposes), and \( \gamma = -\Gamma'(1) \) is Euler’s constant.
Corollary 6.16. Let the notation be as in Corollary 6.15 and assume that \( f_{sc} \) is weakly homomorphic. Then
\[
\log \| \Psi(\text{CM}(E), f_{sc}) \|_{\text{Pet}} = c'(E) \sum_{n>0} \tilde{c}^+(-n)b_n + c'(E)c^+(0)\Lambda'(0, \chi).
\]

When \( \tilde{D} \) is also prime, we have \( c'(E) = 1 \). Then this corollary coincides with [BY1, Theorem 1.4], since the CM points in this paper are counted with multiplicity \( \frac{2}{w_E} \), and our CM cycle is twice the CM cycle there as a set with multiplicities.

Combining this corollary with the [Ya1, Theorem 1.2], one has the following theorem, which verifies a special case of Conjecture 5.5.

Theorem 6.17. Assume that \( E \) is a quartic CM number field with absolute discriminant \( d_E = D^2\tilde{D} \) and real quadratic subfield \( F = \mathbb{Q}(\sqrt{D}) \) such that \( D \equiv 1 \pmod{4} \) is prime and \( \tilde{D} \equiv 1 \pmod{4} \) is square-free. Assume further that \( \mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F \frac{w + \sqrt{\Delta}}{2} \) is free over \( \mathcal{O}_F \), where \( w, \Delta \in \mathcal{O}_F \). Then \( c'(E) = 1 \). Moreover, let \( \mathcal{X} \) be a regular toroidal compactification of the moduli stack of principally polarized abelian surfaces with real multiplication by \( \mathcal{O}_F \), [Ra], [DP], and let \( \mathcal{T}_n \) be the closure of \( T_n \) in \( \mathcal{X} \). Let \( \mathcal{CM}(E) \) be the moduli stack of the principally polarized abelian surfaces with CM by \( \mathcal{O}_E \). Then for any \( f_{sc} \in H_0^+(D, (\frac{D}{2})) \), one has
\[
\langle \hat{T}(f_{sc}), \mathcal{CM}(E) \rangle_{\text{Fal}} = \frac{1}{2} c^+(0)\Lambda'(0, \chi) - \frac{D(D+1)}{8}\mathcal{L}'_{sc}(0, \xi(f_{sc}), \tilde{L}).
\]

Here
\[
\hat{T}(f_{sc}) = (T(f_{sc}), \Phi(z, f_{sc})) = \left( \sum_{n>0} \tilde{c}^+(-n)\mathcal{T}_n, \Phi(z, f_{sc}) \right) \in \hat{CH}^1(\mathcal{X})_\mathbb{C}.
\]

Proof. (sketch) First notice that \( \mathcal{CM}(E) \) does not meet with the boundary of \( \mathcal{X} \) and that \( \mathcal{CM}(E) \) intersects with \( \mathcal{T}(f_{sc}) \) properly. Notice also that \( \mathcal{CM}(E)(\mathbb{C}) = \frac{1}{2}\mathcal{CM}(E) \) since each point in \( \mathcal{CM}(E)(\mathbb{C}) \) is counted with multiplicity \( \frac{1}{w_E} \) (each point \( A \) in \( \mathcal{X} \) is counted with multiplicity \( \frac{1}{\text{Aut}(A)} \)).

First, take \( f_{sc} \) to be non-trivial weakly holomorphic with \( c^+(0) = 0 \) so that \( \hat{T}(f_{sc}) = 0 \) in \( \hat{CH}^1(\mathcal{X})_\mathbb{C} \). So
\[
0 = \langle \hat{T}(f_{sc}), \mathcal{CM}(E) \rangle_{\text{Fal}} = \langle T(f_{sc}), \mathcal{CM}(E) \rangle_{\text{fin}} - \frac{1}{2} \log |\Psi(\text{CM}(E), f_{sc})|,
\]
and consequently
\[
\langle T(f_{sc}), \mathcal{CM}(E) \rangle_{\text{fin}} = \frac{1}{2} c'(E) \sum_{n>0} \tilde{c}^+(-n)b_n.
\]
by Corollary 6.16. On the other hand, [Ya1, Theorem 1.2] (which uses Corollary 6.16) asserts that

\[(6.26) \quad \langle T(f_{sc}), CM(E) \rangle_{\text{fin}} = \frac{1}{2} \sum_{n>0} \tilde{c}^+(−n)b_n.\]

So \(c'(E) = 1\).

Now for a general \(f_{sc}\), one has by definition, (6.26) and Corollary 6.15 that

\[
\langle \hat{T}(f_{sc}), CM(E) \rangle_{\text{Fal}} = \langle T(f_{sc}), CM(E) \rangle_{\text{fin}} - \frac{1}{4} \Phi(CM(E), f_{sc})
\]

\[= -\frac{1}{2} \tilde{c}^+(0)A'(0, \chi) - \frac{D(D + 1)}{8} \mathcal{L}'_{sc}(0, \xi(f_{sc}), \tilde{L}).\]

This concludes the proof of the theorem. \( \square \)

We remark that (6.26) verifies a special case of Conjecture 5.4 and that this corollary is a generalization of [Ya1, Theorem 1.3]. A slight refinement of the main result in HY together with Theorem 6.8 should settle Conjectures 5.4 and 5.5 completely for Hilbert modular surfaces. It would also settle the Colmez conjecture ([Co], [Ya4]) for CM abelian surfaces.

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