12C within the Semimicroscopic Algebraic Cluster Model

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Abstract. The Semimicroscopic Algebraic Cluster Model (SACM) is applied to 12C as a system of three α-clusters. The microscopic model space, which observes the Pauli-Exclusion-Principle (PEP), is constructed. It is shown that the 12C nucleus can effectively be treated as a two-cluster system 8Be+α. The experimental spectrum is well reproduced. The geometrical mapping is discussed and it is shown that the ground state must correspond to a triangular structure, which is in agreement with other microscopic calculations. The non-zero B(E2;0+ → 2+1) transition requires a mixing of SU(3) irreducible representations (irreps) whose consequences are discussed. The Hoyle state turns out to contain large shell excitations. The results are compared to another phenomenological model, which assumes a triangular structure and, using simple symmetry arguments, can reproduce the states observed at low energy. This model does not observe the PEP and one objective of our contribution is to verify the extend of importance of the PEP.

Key words. nuclear clusters, algebraic model, Pauli Principle

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1 Introduction

In recent years, the 12C nucleus enjoyed an increased interest, especially concerning the structure of the Hoyle state [4]. Experiments revealed new states [2] and microscopic calculations show a triangular structure with peaks in the density corresponding approximately to the center of α particles. Particularly, the Hoyle state [4] is important in the understanding of the fusion in stars of three α-particles to the 12C nucleus. In [3] a no-core shell model calculation was performed in order to describe this state, requiring many shell excitations for the description of the Hoyle state. Taking into account the cluster structure of 12C in terms of three α particles provides an advantage to reduce significantly the Hilbert space needed to describe 12C. In another model [21], claimed to be a cluster model, a simple geometric structure was assumed, namely a triangle with an α particle at each corner. Though this model works quite well, it ignores the PEP, one of the most important principles in nature. The motivation of our contribution is to compare the model in [2] to the SACM and test to what extent the PEP plays a role in 12C. One attempt was already taken in [5] using the multi-channel symmetry, which relates different clusterizations to the same Hamiltonian. Here, we will discuss the 12C nucleus as a three-α cluster system.

The paper is organized as follows: In section 2 an introduction to the basic concepts of the SACM with three clusters is presented. Also in section 2 the model space is constructed, first for the two-α subsystem 8Be and then the third α particle is added. It is shown that the final model space can also be obtained considering a two-cluster system of 8Be+α. In section 3 some geometrical considerations are presented, where simple symmetry arguments and a geometrical mapping leads to a triangular structure of 12C in its ground state. In Section 4 a Hamiltonian is proposed, being a combination of a dominant SU(3) part with one mixing term. The spectrum is reproduced in the case of the SU(3) limit with the same quality as in [2]. With the mixing it is even better than in [4]. It will be shown that the mixing of SU(3) irreps is crucial and that taking into account the PEP is important. It is also shown that the cluster model permits 1+ states at low energy, which are forbidden in the oblate symmetric top model [2].

2 The SACM and the construction of the model space for 12C

For light nuclei, the SU(3) symmetry is approximately realized. A useful basis is available when the nucleus is di-
vided into several clusters. Within the SACM \[6,7\], each cluster is represented by an \(SU(3)\)shell model irreducible representation (irrep) \((\lambda_k, \mu_k)\), where \(k\) refers to the number of the cluster. The clusters are not excited because this would lead to a double counting of shell model states, as will be illustrated further below.

For a three-cluster system, the model space in constructed in four steps:

1) Multiply the first two cluster irreps with the irrep of their relative motion:

\[
((\lambda_1, \mu_1) \otimes (\lambda_2, \mu_2) \otimes (n_\rho, 0)) = \sum_{\lambda_{12}, \mu_{12}} m_{(\lambda_{12}, \mu_{12})} (\lambda_{12}, \mu_{12}) \quad (1)
\]

where \(m_{(\lambda_{12}, \mu_{12})}\) is the multiplicity of the corresponding irrep.\(\rho\) denotes the relative motion between the first two \(\alpha\)-particles, whose relative vector is \(\rho\) and \(n_\rho\) is the number of relative oscillation quanta. Due to the Wildermuth condition \[8\] there is a lower limit for \(n_\rho\). This number is determined by the difference of the oscillation quanta in the parent nucleus and the sum of the oscillation quanta in each cluster. For example, in a two-\(\alpha\) system, each \(\alpha\) particle carries zero quanta, i.e. the sum is zero. The united nucleus \(^8\text{Be}\) has four nucleons in the s-shell and four in the p-shell, thus, it carries 4 quanta. The difference is therefore 4 quanta which is the value for \(n_\rho\). Using a lower number would imply that at least one nucleon has to be put into an already occupied state, which is forbidden by the PEP. Thus, the Wildermuth condition is necessary in order to satisfy the PEP. A further restriction in the list can be applied for a symmetric two-cluster system \[9\].

For a two-\(\alpha\)-cluster system, \((-1)^{n_\rho}\) has to be positive, thus only even \(n_\rho\) are allowed. Because the \(^8\text{Be}\) has to be in its ground state \(SU(3)\) irreps, the \(n_\rho\) is fixed to 4, but we also will discuss what happens when the \(^8\text{Be}\) cluster is excited, in order to strengthen our argument, concerning the double counting.

2) Take the result (1) and determine all the products \((\lambda_\lambda_3, \mu_3) \otimes [(\lambda_{12}, \mu_{12}) \otimes (n_\lambda, 0)]\). The \(n_\lambda\) denotes the number of relative quanta of the third cluster to the center of mass of the first two clusters and \(\lambda\) is its relative vector. The \(n_\lambda\) is determined in the same manner as in 1); Assuming that \(^8\text{Be}\) is in its ground state \(SU(3)\) irrep, adding a further \(\alpha\) cluster, the two-\(\alpha\) subsystem \(^8\text{Be}\) already carries 4 quanta, while the third \(\alpha\) particle carries zero quanta, the sum is 4. The united nucleus \(^{12}\text{C}\) carries 8 quanta, having 8 nucleons in the p-shell. Thus, the Wildermuth condition is \(n_{\lambda 0} = 4\).

3) The third cluster is combined with each \((\lambda_{12}, \mu_{12})\) and the result can be schematically written as a sum, namely

\[
(\lambda_3, \mu_3) \otimes [(\lambda_{12}, \mu_{12}) \otimes (n_\lambda, 0)] = \sum_{(\lambda, \mu)} m_{(\lambda, \mu)} (\lambda, \mu) \quad (2)
\]

where \(m_{(\lambda, \mu)}\) denotes the multiplicity of \((\lambda, \mu)\).

The list of irreps in (2) with (1) still contains in general irreps which are not allowed by the PEP. The SACM provides us with a method to eliminate the non-allowed states:

4) In each step, the shell model space for the corresponding sub-cluster system is constructed. For example, in the two-\(\alpha\) cluster subsystem, construct the shell model space of \(^8\text{Be}\) and determine the overlap with the shell model space of the two-cluster system, which results in the SACM-model space of the 2-cluster subsystem. When the remaining list is coupled with the third cluster and \((n_\lambda, 0)\), the resulting list has to be matched with the shell model space of \(^{12}\text{C}\). Also to mention is that in each step the center of mass has to be subtracted, which is particularly easy within the harmonic oscillator basis, for details please consult \[9\].

The steps 1) to 4) are now applied more specifically to the three \(\alpha\)-cluster system: In the two-\(\alpha\) cluster subsystem we obtained for the minimal number of relative oscillation quanta 4. The product of all irreps can be resumed by

\[
(0, 0) \otimes (0, 0) \otimes (n_\rho = 4, 0) = (4, 0) \quad . \quad (3)
\]

Suppose that the \(^8\text{Be}\) is excited to (6,0). It now carries 2 more quanta, which implies that the minimal number of \(n_\lambda\) quanta so satisfy the PEP is reduced to 2. Calculating the overlap of the space obtained via (2) with the shell model leads to a list of allowed irreps which is contained in the list when the \(^8\text{Be}\) cluster is in the (4,0) irrep. Thus, in order to avoid a double counting the \(^8\text{Be}\) nucleus has to be in its ground state irrep. Exciting the \(^8\text{Be}\) cluster to larger \(n_\rho\) leads to the same observation.

When the third \(\alpha\) particle is added we obtain through the multiplication of (0,0) again (4,0). This irreps has to be multiplied with \((n_\lambda, 0)\), with \(n_\lambda \geq n_{\lambda 0}\), i.e. for \(0\omega\) it is \((4,0)\)Be \(\otimes (4,0)\), for \(1\omega\) it is \((4,0)\)Be \(\otimes (5,0)\), for \(2\omega\) it is \((4,0)\)Be \(\otimes (6,0)\), etc.

In Table (1) the \(SU(3)\) content of up to \(6\omega\) excitations are listed, i.e. it represents the model space of the SACM for \(^{12}\text{C}\) as a three \(\alpha\) cluster system.

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
\(n\omega\) & \((\lambda, \mu)\) \\
\hline
 0 & (0,4) \\
 1 & (3,3) \\
 2 & (2,4), (4,3), (6,2) \\
 3 & (3,4), (5,3), (7,2), (9,1) \\
 4 & (4,4), (6,3), (8,2), (10,1), (12,0) \\
 5 & (5,4), (7,3), (13,0) \\
 6 & (6,4), (8,3), (10,2), (12,1) \\
\hline
\end{tabular}
\caption{Model space of the \(^{12}\text{C}\) nucleus within the SACM for up to \(6\omega\) excitations. Note, that for \(n = 2\), the (4,3) irrep contains a positive parity, spin one state.}
\end{table}

Note, that the model space is microscopic, i.e., it observed the PEP!

From Table(1) one can already infer the spin content of the spectrum, using the rules of J.P. Elliott \[10\]. For even
n, the states have positive parity and for odd n, the states have negative parity. The lowest negative parity states are contained in the (3,3) irrep which has two bands [10], one with $L = 1^−, 2^−, 3^−$ and $4^−$ (it does not contain a $5^−$ state!) and the other band with $3^+, 4^−, 5^−$ and $6^−$. Note, there is only one $5^−$ state at low energy. When the $SU(3)$-breaking term is switched on, these states become mixed with higher lying $SU(3)$ irreps of the same parity. As it will turn out further below, the Hoyle state does not come from a $2\hbar\omega$ excitation but from a $4\hbar\omega$ one. Note also, that there are several $1^+$ states possible, which are forbidden in the oblate symmetric top model of [2]. For example, the $(4,3)$ irrep at $2\hbar\omega$ contains one $1^+$ state.

3 Geometrical considerations

In [11] a geometrical mapping of the SACM Hamiltonian was presented. The potential is defined as the expectation value of this Hamiltonian with respect to a coherent state, which takes into account the Wildermuth condition. The same structure can be used here, namely a direct product of a coherent state for the two-cluster subsystem and one for the $^8\text{Be}+\alpha$ system. The result gives a hint to the ground state configuration of a cluster system.

Due to the PEP two $\alpha$-clusters have a finite minimal distance, i.e., they cannot overlap completely with a vanishing relative distance. The relation obtained [11] can be readily extended to the three-cluster system of $^{12}\text{C}$ and its given by

$$\rho_0 = \sqrt{\frac{\hbar n_\rho}{\mu_{\alpha\alpha} \omega}}, \quad \lambda_0 = \sqrt{\frac{\hbar n_\lambda}{\mu_{\text{Be}\alpha} \omega}},$$

(4)

where $\mu_{\alpha\alpha}$ is the reduced mass of the two-$\alpha$ subsystem and $\mu_{\text{Be}\alpha}$ the reduced mass of the $^8\text{Be}+\alpha$ cluster system. Because the $\alpha$-clusters are indistinguishable (they are symmetric under permutation) the distance $\rho_0$ between any cluster has to be the same. This already indicates that the three-$\alpha$ cluster system has to be ordered in a triangle. When a $SU(3)$ mixing interaction is taken into account, the $n_\rho$ and $n_\lambda$ values increase, thus, the estimation given here yields lower limits, otherwise the consequences about the geometric structure remain.

We obtain $\rho_0 \approx \sqrt{\frac{1}{2}}$ fm and $\lambda_0 \approx 2 \sqrt{\frac{1}{2}}$ fm, which is in a very nice agreement with the microscopic cluster model of [12]. We took into account that $\mu_{\alpha\alpha} = \frac{m_\alpha}{2}$ and $\mu_{\text{Be}\alpha} = \frac{2}{3} m_\alpha$ (neglecting energy binding effects), with $m_\alpha$ being the mass of the $\alpha$ particle, we obtain for the ratio of the distances

$$\frac{\lambda_0}{\rho_0} = \sqrt{\frac{3}{2}},$$

(5)

which simply confirms the triangular structure (Pythagoras)-

This shows that with a few elementary considerations, without recurring to ad hoc assumptions and/or complicated calculations, the ground state geometrical configuration of $^{12}\text{C}$ is obtained!

4 The Hamiltonian and results

As a Hamiltonian we propose a combination of a pure $SU(3)$-part and a symmetry breaking term, which is a generator of $O(4)$. The Hamiltonian is also chosen to have for the $SU(3)$ limit the same number of parameter used in [2], where the cut-off $N$ is used as one of the parameters. The choice does not exclude the use of a more general Hamiltonian. As in [2] the total number of quanta is $N = n_\rho + n_\lambda + n_\sigma$, where the $\sigma$-bosons are introduced as a trick to obtain a cut-off.

The model Hamiltonian proposed is

$$H = \hbar \omega n_\sigma - \chi C_2(\lambda, \mu) + t_2 (C_2(\lambda, \mu))^2 + t_4 (C_4(\lambda, \mu)) + \left( a + a_L (-1)^L + a_{Lnp} \Delta n_\pi \right) L^2 + b K^2$$

$$+ b_1 \left[ (\sigma^1)^2 - (\pi^1 \cdot \pi^1) \right] \cdot [h.c.].$$

(6)

The first term is just the harmonic oscillator field and $\hbar \omega$ is fixed via $45 \times A^{-1/3} + 25 \times A^{-2/3}$ [14], where its value is 14.89 for $^{12}\text{C}$. The second term is related to the quadrupole quadrupole interaction [10], with $C_2(\lambda, \mu)$ being the second order Casimir operator. The third and fourth terms allow for corrections in the relative ordering of $SU(3)$ irreps, where $C_4$ is the third order Casimir operator. The fifth term allows to describe changes in the moment of inertia for states with higher shell excitations and when the spin parity changes. The last term in the second line lifts the degeneracy in angular momentum for states within the same $SU(3)$ irrep. Up to here, the Hamiltonian is within the $SU(3)$ limit and permits analytic results, substituting the operators by their corresponding eigenvalues. In the last line, the term mixes $SU(3)$ irreps, it is a generator of a $O(4)$ group. The pure $SU(3)$ part has 7 free parameters, the same as in [2] (including $N$), and will be compared to the oblate top model. For allowing the mixing a further parameter ($b_1$) is added, i.e., in the final calculations there are 8 parameters.

We will apply 3 different sample calculations: i) A pure $SU(3)$ Hamiltonian, which permits the best adjustment to the spectrum, also obtained in [2]. ii) The $B(E2; 0^+_1 \rightarrow 2^+_1)$ is adjusted to about a tenth of the experimental value, which is 8 Wu [14], as done in [4]. iii) The transition value is adjusted to the experimental value.

The $B(E2)$ value mentioned is essential, because it signals an important mixture between $SU(3)$-bands. It will also make it difficult to adjust the spectrum and grouping states into bands, which is easier when a simple algebraic model is used.

The use of $N$ as a parameter in [2] is questionable, because the $N$ is a cut-off parameter. In the oblate top
model the $N$ is related to the anharmonicities of the potential. This interpretation is often used and due to a misleading understanding of the geometrical mapping, as we already investigated in [15,10,17].

As the quadrupole transition operator we use the one given in [18], which is a symplectic generator, including connections to multiple $2\hbar \omega$ shell excitations. In an algebraic model it is customary to use the algebraic part of this operator, which does not connect shells, which is valid when inter-shell excitations are not considered. The physical quadrupole operator [18,10] is given by

$$Q_{2m}^{2\text{phys}} = Q_a^o + \frac{\sqrt{6}}{2} (B_{2m}^\dagger + B_{2m})$$

$$B_{2m}^\dagger = (\pi^\dagger \cdot \pi^\dagger), \quad B_{2m} = (\pi \cdot \pi) . \quad (7)$$

The $B_{2m}^\dagger$ operator transforms as a $(2,0)$ SU(3) irrep, while $B_{2m}$ as its conjugate.

As we will see, in order to get a large transition value between states in different bands, a strong mixing of the SU(3) basis is required, which will lead to a less favorable agreement in the spectrum. This is a characteristic feature when simple models are used, as we do here too. In order to obtain a better agreement, more interaction terms have to be included, which will increase the number of parameters, from which we will refrain here for the sake of comparison. We also want to show that sometimes simple models are not enough!

Though, for light nuclei the SU(3) symmetry is well realized, there are strong indications in the electro-magnetic transitions that the SU(3) irreps mix significantly, due to the large deformation of $^{12}$C. In the SU(3) limit, the second excited $0^+$ and $2^+$ states belong to the irrep $(12,0)$, a $4\hbar \omega$ excitation, the first $2^+$ to $(0,4)$ and the third $0^+$-band head to the $(6,2)$ irrep at two excitation quanta, all in agreement with the no-core shell model calculation in [3]. With increasing $B(E2; 0^+_2 \rightarrow 2^+_1)$ the mixing of SU(3) gets larger. We adjusted the spectrum and electromagnetic transitions, assuming the three different values for $B(E2; 0^+_2 \rightarrow 2^+_1)$, discussed further above. These three cases are listed in Table 2 and compared to the experimental values, all in Weisskopf units. The E3-transition value indicates a large octupole component and it is repro-

| $B(EL; J_i \rightarrow J_f)$ | EXP. | $SU(3)$ | case-1 | case-8 |
|-----------------------------|------|---------|--------|--------|
| $B(E2; 2^+_1 \rightarrow 0^+_1)$ | 4.65 | 4.65 | 5.18 | 3.41 |
| $B(E2; 0^+_2 \rightarrow 2^+_1)$ | 8.0 | 0.0 | 0.97 | 8.33 |
| $B(E3; 3^+_1 \rightarrow 0^+_1)$ | 12.0 | 6.32 | 12.23 | 8.64 |

Table 2. List of $B(EL)$-transition values, measured and obtained in three different model calculations: in the first column information is listed on the type of the electro-magnetic transition, the second column lists the corresponding experimental value, the third column assumes exact $SU(3)$ symmetry and in the last two columns the experimental value of $B(E2; 2^+_1 \rightarrow 0^+_1)$ is adjusted to different values (case-1: 1WU, case-8: 8WU), where an increased value corresponds to a larger SU(3) mixing. The E3- transition values are not adjusted.

$$t_1 = t_2 \cdot \frac{\omega}{\hbar}$$

$\omega$ is the angular frequency and $\hbar$ the reduced Planck constant.

$$\lambda = \frac{1}{\hbar} \cdot \frac{\pi}{\lambda}$$

$\lambda$ is the wavelength of light.

$$\kappa = \frac{\mu}{\hbar} \cdot \frac{\pi}{\lambda}$$

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Table 6. SU(3) content of some low lying states with positive parity, given as in percent, for the case of $B(E2; 0^+ \rightarrow 2^+_1) = 8 \text{ WU}$. The numbers are only approximate and not all irreps are shown. The first group refers to the ground state band and the second one to the Hoyle state. Clearly seen is the similar structure of the $0^+_1$ and $2^+_1$ state, some similarity of the $0^+_2$ to the $2^+_2$ state and the similarity vanishes for the $4^+_1$ states.

| $L^+_i$ | $0^+_1$ | $2^+_1$ | $3^+_1$ | $4^+_1$ | $5^+_1$ |
|--------|---------|---------|---------|---------|---------|
| $n_\pi = 4 : (0,4)$ | 82 | 82 | 48 | 8 | 3 | 31 |
| $n_\pi = 6 : (2,4)$ | 16 | 16 | 4 | 22 | 41 | 16 |
| $n_\pi = 6 : (4,3)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n_\pi = 8 : (4,4)$ | 1 | 2 | 0 | 22 | 28 | 29 |
| $n_\pi = 8 : (6,2)$ | 0 | 0 | 0 | 5 | 6 | 0 |
| $n_\pi = 8 : (6,3)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n_\pi = 8 : (8,2)$ | 0 | 0 | 0 | 2 | 6 | 0 |
| $n_\pi = 8 : (12,0)$ | 0 | 0 | 0 | 33 | 6 | 0 |
| $n_\pi = 10 : (6,4)$ | 0 | 0 | 39 | 7 | 9 | 19 |
| $n_\pi = 10 : (10,2)$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 7. SU(3) content of some low lying states with negative parity, given as in percent, for the case of $B(E2; 0^+_2 \rightarrow 2^+_1) = 8 \text{ WU}$. The numbers are only approximate and not all irreps are shown. The first group refers to the ground state band and the second one to the Hoyle state. Clearly seen is the similar structure of the $1^-_1$ and $2^-_1$ state, the same for the $3^-_1$ to the $4^-_1$. However, the similarity end for the $5^-_1$ state.

| $L^-_i$ | $1^-_1$ | $2^-_1$ | $3^-_1$ | $4^-_1$ | $5^-_1$ |
|---------|---------|---------|---------|---------|---------|
| $n_\pi = 5 : (3,3)$ | 78 | 74 | 82 | 68 | 14 |
| $n_\pi = 7 : (3,4)$ | 5 | 7 | 2 | 10 | 11 |
| $n_\pi = 7 : (5,3)$ | 15 | 14 | 10 | 12 | 11 |
| $n_\pi = 7 : (7,2)$ | 0 | 0 | 0 | 0 | 0 |
| $n_\pi = 7 : (9,1)$ | 0 | 0 | 0 | 0 | 0 |
| $n_\pi = 9 : (5,4)$ | 1 | 4 | 0 | 9 | 59 |
| $n_\pi = 9 : (7,3)$ | 1 | 1 | 1 | 1 | 4 |

Table 8. Spectroscopic factors for $^8\text{Be} + \alpha \rightarrow ^{12}\text{C}$. The first, third, fifth and seventh rows indicate the state in $^{12}\text{C}$ and the second, fourth, sixth and eighth rows list the spectroscopic factor values. The states are ordered into bands, such that the energy of the $4^+_1$ state has a larger energy than $4^+_2$.

| state | $0^+_1$ | $2^+_1$ | $3^+_1$ | $4^+_1$ | $0^+_2$ |
|-------|---------|---------|---------|---------|---------|
| spec. fact. | 0.0866 | 0.0218 | 0.00594 | 0.0417 |
| state | $2^+_1$ | $4^+_1$ | $1^-_1$ | $2^-_1$ |
| spec. fact. | 0.0191 | 0.00662 | 0.0160 | 0.0481 |
| state | $3^-_1$ | $4^-_1$ | $5^-_1$ | $0^-_2$ |
| spec. fact. | 0.000818 | 0.000909 | 0.00504 | 0.0291 |
| state | $1^-_1$ | $1^-_2$ | $2^-_2$ |
| spec. fact. | 0.0449 | 0.0133 | | 0.0024 | 0.053 |

Fig. 1. Spectrum of $^{12}\text{C}$. In the upper row, the left figure is the experimental spectrum and the right figure depicts the result for pure SU(3). In the second row, the left figure depicts the result when the $B(E2; 0^+_2 \rightarrow 2^+_1)$ transition value is adjusted to 1 WU and the right figure when this value is adjusted to the experimental one, namely 8 WU.

Fig. 2. Spectrum of $^{12}\text{C}$ up to 25 MeV, using the same grouping into bands as in Fig. 1. The $B(E2; 0^+_2 \rightarrow 2^+_1)$ transition value is adjusted to the experimental one, namely 8 WU.
increased the weight for the spectrum. To 8 \( [14] \) Weisskopf units and consequently the mixing into bands according to their \( SU_3 \) irreps, which is listed below the bands. Note the additional 1\(^-\)-band head. The lowest 1\(^-\) state is the band head of 13 \( 8 \) \( (10,1) \) and the second one of 13 \( n_\pi = 6 \ (4,3) \) \( K = 1 \).

In Figure 1, the results are resumed, depicting only the up to now confirmed states. In the \( SU(3) \) limit the agreement to experiment is the best, as also in [2], where the \( B(E2; 0^+_2 \rightarrow 2^+_1) \) is not listed. In the left figure in the second row in Figure 1, the experimental \( B(E2; 0^+_2 \rightarrow 2^+_1) \) is assumed to be 1, which increases the mixing and as a consequence the agreement to experiment is less favorable. With respect to the position of states and the splitting within bands, the result is even better than in [1], where the theoretical \( B(E2; 0^+_2 \rightarrow 2^+_1) \) is listed as a tenth of the experimental value. In the right figure in the second row of Figure 1, the same \( B(E2) \) value is adjusted to 8 \( [14] \). Weisskopf units and consequently the mixing increases significantly which results in a less good agreement for the spectrum.

The fitting routine has the possibility to define a weight for a particular transition (also for the energy values). We increased the weight for \( B(E2; 0^+_2 \rightarrow 2^+_1) \) step by step, starting from zero. At first, the transition value stays very small and the spectrum has a very good agreement to experiment, until at one point where the transition value jumps to the large value which one tries to reproduce and at the same time the agreement in the spectrum also jumps to be less favorable, which shows that the particular transition makes it difficult to adjust the spectrum. More interactions are needed for a good agreement, or a more involved model as the no-core shell model calculations in [3].

In Figure 2, different spectra are shown, ordered in the same manner as in [2], though we do not agree with this ordering (see further below). We do it simply for comparison. In the first row left panel the experimental result is repeated, followed by the \( SU(3) \) result. In the second row the left panel refers to the case when the transition value is adjusted to 1 WU, while the right panel corresponds to 8 Wu. Only the \( SU(3) \) case can be compared to the oblate top model. An important feature are the spin-doublets in the ground state band, the Hoyle-band and the 1\(^-\)-band (again, we use the association used in [2]). The 4\(^+\)– 4\(^+\)-doublets in the ground state band and in the Hoyle-band are also obtained in the \( SU(3) \) limit. The doublet structure in the 1\(^-\)-band, however, is only partially reproduced.

One of the most important features is the appearance of 1\(^+\) cluster states, forbidden in the oblate top model. In Figure 3, the spectrum, ordered according to the SACM into bands within the \( SU(3) \) limit, is shown, including more states not shown in the other figures. A distinct difference is that in the \( SU(3) \) limit there is no 5\(^-\) state in the Hoyle-band. Within the SACM a second 1\(^-\)-band appears, nearly degenerate with the first 1\(^-\)-band. While the first 1\(^-\)-band belongs to the \( n_\pi = 5 \ (3,3) \ (\Delta n_\pi = 1) \) irrep of \( SU(3) \), the second one comes from \( n_\pi = 7 \ (9,1) \ (\Delta n_\pi = 3) \), which is significantly different to the first 1\(^-\)-band. Considering the mixing, the two 1\(^-\)-bands are not exactly degenerate but near to each other.

In general, the spectrum supports the triangular structure proposed in [4,2], though the signatures (spin-doublets, the missing 5\(^-\) state and the appearance of 1\(^+\) cluster states) in the spectrum are not the same, once the PEP is taken into account.

When the mixing is included, the spectrum becomes less and less similar to the experimental one, though for the case when the transition value is adjusted to 1 WU, the agreement is still acceptable to the experiment. The problems are related to the moment of inertia in the Hoyle and the 1\(^-\)-band. In the first one it is too large while in the second one it is too low, though they are in much better agreement to experiment than in [1]. Also the position of the band heads gets increasingly more difficult to adjust. Some of the spin-doublets remain, however it is difficult to judge the case with a maximal mixing. The main indication is that the spin-doublets at larger energy will be diluted. Again the missing 5\(^-\) state, the appearance of 1\(^+\) cluster states and an additional 1\(^-\)-band show the main difference to [2].

With respect to the association of bands, we do not agree to the interpretation given in [2]. As already mentioned, in the \( SU(3) \) limit the content of the positive parity states are not the same as for the negative parity states. Thus, they cannot be ordered into the same band, a consequence of the PEP. The content of the states in terms of the \( SU(3) \) basis is listed in Tables 1, 5 for the adjustment to 1 WU and in Tables 6, 7 for 8 Wu, all in percent. They do not add up all to 100 percent, due to rounding effects and not listing some irreps. The association into bands can be guessed, though for the higher lying states the mixing is important and deviates substantially from the internal structure of the band heads. This problem is
known for a long time [21], making it difficult to associate a given state to a band.

In conclusion, several properties and differences are observed:

- The spin doublets predicted in [2] are only partially reproduced. The doublets at higher energies disappear in the calculation including the PEP.
- There is only one 5− state below 25 MeV. The second 4− state is shifted to larger energies, compared to [2].
- In [2] it is claimed that there is no 1+ cluster state possible. However, the SACM permits these states and some of them are shown in Figure 2. Thus, observed $T = 0$ states may very well be cluster states!
- Within the SACM there is an extra 1−-band, which is in the SU(3) limit degenerate to the first one.
- The association into bands is criticized. While in the oblate top model it is allowed, as it corresponds also to a molecule with three atoms where the PEP is of no importance, in a nuclear cluster system this association is forbidden.

Spectroscopic factors for 8Be+α were also calculated. For details of the algebraic spectroscopic factor operator please consult [21]. In [21] is is demonstrated that the agreement to microscopic exact shell model calculations [22] is extremely good. Though, the reaction 8Be+α → 12C is difficult or not possible to measure, for completeness the values for some spectroscopic factors are listed in Table 8. Due to a possible interest of experimental physicists, which look for the 1+ states below 25 MeV, also the spectroscopic factors of two 1+ states are tabulated and of the first two states of the second 1−-band.

5 Conclusions

We applied the Semimicroscopic Algebraic Cluster Model [6,7] to 12C, as a three-α cluster system. It was shown that it suffices to treat 12C as a 3Be+α two cluster system.

Some geometrical considerations were applied and it was shown that in its ground state 12C must have a triangular structure. The main reason for the finite distance of two α particles is due the minimal number of oscillation quanta, required by the Wildermuth Condition, i.e. due to the observation of the Pauli Exclusion Principle.

Three calculations were performed, using for comparison a simplified version of the SACM: The first calculation was restricted to the SU(3) limit, while in the next two calculations the experimental $B(E2; 0^+_2 \rightarrow 2^+_1)$ transition was taken as 1 WU and 8 WU, respectively, where the last value corresponds to the experimentally observed one. It was shown that this non-zero transition value indicates a strong mixing of bands, due to the large deformation of the 12C nucleus, making it difficult to group the states into bands. The overall agreement to the experiment is very good up to moderately good.

It was shown that the PEP has the following effects:

- There are additional 1+ cluster states, forbidden in the oblate top model.
- There is an extra 1−-band in the spectrum.
- The experimental $B(E2)$ and $B(E3)$ transitions can be described within the SACM.
- The low lying 5− state, predicted in [2] is not present when the PEP is observed.
- The association into bands, as done in [2], is criticized.

We also mentioned some critics on the use of the cut-off value $N$ as a parameter.

There are ways to improve the agreement. One is to use the multi-channel symmetry [9], which reduces the number of free parameters and other possibility is to apply no-core shell model calculations as in [9]. The 12C is not an easy nucleus to describe and sometimes one has to do more than using a simple model.

To resume, The Pauli Exclusion Principle is very important and cannot simply be ignored, even if the results indicate that the model “works”. The PEP is one of the fundamental principles of nature and cannot be set aside!

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