Abstract

Research on nearest-neighbor methods tends to focus somewhat dichotomously either on the statistical or the computational aspects — either on, say, Bayes consistency and rates of convergence or on techniques for speeding up the proximity search. This paper aims at bridging these realms: to reap the advantages of fast evaluation time while maintaining Bayes consistency, and further without sacrificing too much in the risk decay rate. We combine the locality-sensitive hashing (LSH) technique with a novel missing-mass argument to obtain a fast and Bayes-consistent classifier. Our algorithm’s prediction runtime compares favorably against state of the art approximate NN methods, while maintaining Bayes-consistency and attaining rates comparable to minimax. On samples of size \( n \) in \( \mathbb{R}^d \), our pre-processing phase has runtime \( O(dn \log n) \), while the evaluation phase has runtime \( O(d \log n) \) per query point.

1 Introduction

In the sixty or so years since the introduction of the nearest-neighbor paradigm, a large amount of literature has been devoted to analyzing and refining this surprisingly effective classification method. Although the 1-NN classifier is not in general Bayes-consistent (Cover and Hart, 1967), taking a majority vote among the \( k \) nearest neighbors does guarantee Bayes consistency, provided that \( k \) increases appropriately in sample size (Stone, 1977; Devroive and Győrfi, 1985; Zhao, 1987). However, the \( k \)-NN classifier presents issues of its own. A naive implementation involves storing the entire sample, over which a linear-time search is performed when evaluating the hypothesis on test points. For large samples sizes, this approach is prohibitively expensive in terms of storage memory and computational runtime.

Until recently, research on NN-based methods tended to focus somewhat dichotomously either on the statistical or the computational aspects. On the statistical front, the most commonly investigated questions involve Bayes consistency and rates of convergence under various distributional assumptions (Hall et al., 2008; Kpotufe, 2009; Gadat et al., 2016; Chaudhuri and Dasgupta, 2014). An orthogonal body of literature developed a host of techniques for evaluating the hypothesis (or an approximation to it) on test points in runtime considerably better than linear in sample size.

Exact NN search methods suffer from either space or query time that is exponential in the dimension \( d \) (Samet, 2006). To overcome this problem, approximate NN search was proposed. Broadly speaking, these techniques construct a hierarchical net during the offline pre-processing (learning) phase (Krauthgamer and Lee, 2004; Beygelzimer et al., 2006; Gottlieb et al., 2014), or seek to condense the sample down to a smaller yet nearly-faithful subsample (Hart, 1968; Gates, 1972; Ritter et al., 1973; Wilson and Martinez, 2006).
2000; Gottlieb et al., 2018), or perform some sort of dimensionality reduction (Indyk and Motwani, 1998; Charikar, 2002; Datar et al., 2004; Andoni and Indyk, 2008; Gottlieb et al., 2016). The speedup in search time is offset by a degraded classification accuracy, and with rare exceptions (Gottlieb et al., 2014), this tradeoff has not been addressed in the literature.

The aim of this paper is to combine the best of both worlds: to reap the advantages of fast evaluation time while maintaining Bayes consistency, with the risk decaying at a rate not much worse than minimax. We combine the locality-sensitive hashing (LSH) technique of Datar et al. (2004) with a novel missing-mass argument to construct a fast, Bayes-consistent LSH-based classifier.

Our contribution. Our main contribution consists of constructing a fast and Bayes-consistent classifier in $\mathbb{R}^d$. Our algorithm’s prediction runtime compares favorably against state of the art approximate NN methods. An additional advantage our method enjoys over the latter is provable Bayes-consistency — and a convergence rate that is off by a power of 2 from the minimax rate. The concentration inequality for a generalized notion of missing mass developed in the course of our analysis may be of independent interest.

Related work. Following the pioneering work of Cover and Hart (1967), it was shown by Devroye and Gyorfi (1985); Zhao (1987) that the k-NN classifier is strongly Bayes-consistent. Some of the classic results on k-NN risk decay rates were later refined by taking into account the noise margin, i.e., the data distribution around the decision boundary. In particular, Chaudhuri and Dasgupta (2014) obtain minimax rates of the form $O(n^{-\alpha(d+1) \over 2d+\beta})$, where $\alpha$ is a Hölder-like smoothness exponent of the regression function $\eta(x) = \mathbb{P}(Y = 1 | X = x)$ and $\beta$ is a Tsybakov noise exponent. To obtain this rate, they require $k = \Theta(n^{2\beta \over 2d+\beta})$, which slows down the query time by an additional poly$(n)$ factor. A recently proposed alternative approach, based on sample compression and 1-NN classification has been shown to be Bayes-consistent in doubling metric spaces (Kontorovich et al., 2017) — and in fact is universally consistent in all spaces where Bayes consistency is possible (Hanneke et al., 2019).

Various approximate NN techniques have been proposed to speed up the query time. One such result was obtained by Har-Peled et al. (2012), who show that $(r, cr, p_1, p_2)$-sensitive LSH families (see definition below) achieve an approximate NN query time of $O(d^n\rho)$, where $\rho = \log(1/p_1)$. Other approximation methods include fast $\varepsilon$-net constructions (Krauthgamer and Lee, 2004), where query time (after sample compression, as in Gottlieb et al. (2018)) is $O(1/\varepsilon^d)$ but does not depend on $n$. No risk convergence (or even Bayes consistency) analysis is known for any classifier using these methods — absent which, as we argue in the discussion below Table 1 comparisons to our approach are not meaningful.

The recent work of Xue and Kpotufe (2018) proposes aggregating denoised 1-NN predictors over a small number of distributed subsamples. This approach, which requires distributed computing resources, can achieve nearly the accuracy of k-NN while matching the prediction time of 1-NN. Since the present paper does not assume access to parallel processors, this result is incomparable to ours.

Paper outline. The structure of this paper is as follows. Section 2 contains the relevant definitions and notations. Section 3 discusses our main contributions. In section 4 we present the LSH based learner algorithm. Full detailed proofs are deferred to the supplementary material.
2 Preliminaries

Learning model. We work in the standard agnostic learning model [Mohri et al., 2012; Shalev-Shwartz and Ben-David, 2014], whereby the learner receives a sample $S$ consisting of $n$ labeled examples $\{(x_i, y_i)\}_{i=1}^n$ drawn iid from an unknown distribution $D$ over $X \times Y$. In this work we take $X = \{0, 1\}^d$ equipped with an $\ell_p$ metric $|x - x'|^p = \sum_{i=1}^d |x_i - x'_i|^p$; when the subscript $p$ is omitted, its default value is always $p = 2$: $\|\cdot\| \equiv \|\cdot\|_2$. For simplicity of exposition, we take $Y = \{0, 1\}$; the extension to the multiclass case is straightforward.

Let $D_X$ denote the induced marginal distribution over $X$ and let $\eta$ be the conditional probability over the labels: $\eta(x) = \mathbb{P}(Y = y|X = x)$. This function is said to be $(\alpha, L)$-Hölder if

$$|\eta(x) - \eta(x')| \leq L \|x - x'\|_p^\alpha, \quad x, x' \in X. \quad (1)$$

Based on the training sample $S$, the learner produces a hypothesis $h : X \rightarrow \{0, 1\}$ whose empirical error is defined by $\hat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}[h(x_i) \neq y_i]$ and whose generalization error is defined by $R(h) = \mathbb{P}(h(x) \neq y)$. The Bayes-optimal risk is defined as $R^* = \inf_h R(h)$, where the infimum is over all measurable hypotheses. This infimum is achieved by the Bayes-optimal classifier, $h^*$, given by

$$h^*(x) = \arg\max_{y \in \{0, 1\}} \mathbb{P}(Y = y|X = x).$$

A learning algorithm mapping a sample $S$ of size $n$ to a hypothesis $h_n$ is said to be (weakly) Bayes-consistent if $\lim_{n \to \infty} \mathbb{E}[R(h_n)] = R^*$. (For strong Bayes consistency, the convergence is almost-sure rather than in expectation, but this paper deals with the former notion.)

Locality Sensitive Hashing. Let $H$ be a family of hash functions mapping a metric space $(\mathcal{M}, \rho)$ to some set $U$. The family $H$ is called $(r, cr, p_1, p_2)$-sensitive if for any two points $x, x' \in \mathcal{M}$, using a function $h \in H$ which is drawn from some distribution $\mathbb{P}_H$:

- $\rho(x, x') \leq r \implies \mathbb{P}_H(h(x) = h(x')) \geq p_1,$
- $\rho(x, x') \geq cr \implies \mathbb{P}_H(h(x) = h(x')) \leq p_2.$

In order for a locality-sensitive hash (LSH) family to be useful, it must satisfy inequalities $p_1 > p_2$ and $c > 1$ (Datar et al., 2004).

$k$-missing mass. For a sample $S = (X_1, \ldots, X_n)$ drawn iid from a discrete distribution $P = (p_1, p_2, \ldots)$ over $\mathbb{N}$, the missing mass is the total mass of the atoms not appearing in $S$. Let us define a generalized notion, the $k$-missing mass. For $i, k \in [n]$ and $j \in \mathbb{N}$, define $\psi_{i,j} = \mathbb{1}[X_i = j]$ and $\xi_j^{(k)} = \mathbb{1}[\sum_{i=1}^n \psi_{i,j} < k]$; in words, $\xi_j^{(k)}$ is the indicator for the event that the $j$th atom was observed fewer than $k$ times. The $k$-missing mass is the following random variable:

$$U_n^{(k)} = \sum_{j \in \mathbb{N}} p_j \xi_j^{(k)} \quad (2)$$

(for $k = 1$, this is the usual missing mass).

\[1\] by replacing “majority vote” in Section 4 by the plurality label, as done in Kontorovich et al. (2017).
Table 1: A comparison of the various algorithms’ runtimes (OptiNet is given in Algorithm 1 of Hanneke et al. (2019)). Note that while the query time of $k$-NN may be improved (e.g., to $O(dkn^{1/c^2})$ using an LSH family) and the training time of OptiNet can be improved to $O(C_dn \log n)$ via fast $\varepsilon$-net (Gottlieb et al., 2014), the effect of the approximate NN techniques on Bayes consistency is not understood — much less the effect on the risk decay rates. Indeed, one can trivially speed up any learning algorithm by discarding all but a tiny fraction of the training sample. This will obviously significantly degrade the risk rate, which illustrates that runtime comparisons are only meaningful among techniques with comparable risk rates.

### 3 Main Results

Our first contribution is the construction of a sequence $H_n$ of $(r_n, c r_n, p_1, p_2)$-sensitive families with the following properties:

S1. $p_1^2 > p_2$

S2. $r_n \to 0$ as $n \to \infty$

S3. $r_n = o(\sqrt{n})$.

Following Datar et al. (2004), our construction (given in Section 4.1) is based on $p$-stable distributions.

Using this construction, we design a learning algorithm (Alg. 1) with runtime $O(dn \log n)$, for the pre-processing phase and evaluation (online) runtime $O(d \log n)$. The pre-processing phase and evaluation times are compared to other algorithms in Table 1.

In addition to achieving an exponential speed-up over the state of the art, our algorithm enjoys the property of being Bayes-consistent. The price we pay for the computational speedup is a quadratic slow-down of the convergence rate:

**Theorem 1.** Let $\mathcal{X} = [0, 1]^d$, $\mathcal{Y} = \{0, 1\}$, and $\mathcal{D}$ be a distribution over $\mathcal{X} \times \mathcal{Y}$ for which the conditional probability function, $\eta$, is $(\alpha, L)$-Hölder. Let $f_n$ denote the classifier constructed by Algorithm 1 on a sample $S_n \sim \mathcal{D}^n$. Then the LSH learner is weakly Bayes-consistent: $\lim_{n \to \infty} \mathbb{E}[R(f_n)] = R^*$. Further, $\mathbb{E}[R(f_n)] - R^* = O(n^{-\frac{c^2}{d+2}})$.

**Remark.** Since we rely on the LSH techniques developed by Indyk and Motwani (1998); Datar et al. (2004); Andoni and Indyk (2008), it might appear that we are “beating them at their own game” by achieving an exponential speedup over the state-of-the-art runtimes based on LSH. A more accurate conceptual explanation would be that we are “playing a different game”. Namely, while the latter works focus on the approximate nearest neighbor problem, our goal is rather to efficiently label a test point, without guaranteeing anything about its approximate nearest neighbor in the sample. Instead, we guarantee that with high probability, most of the points in a query point’s hash bucket will be in its close proximity.
search in high-dimensional space. It is based on the definition of LSH family to describe a randomized hashing framework for efficient approximate nearest neighbor.

The term Locality-Sensitive Hashing (LSH) was introduced by Indyk and Motwani (1998). For the Euclidean metric we pick a random projection of $\mathbb{R}^d$ onto a 1-dimensional line and chop the line into segments of length $w$, shifted by a random value $b \in [0, w)$. Formally, $h_{a,b}(x) = \lfloor \frac{ax+b}{w} \rfloor$, where the projection vector $a \in \mathbb{R}^d$ is constructed by picking each coordinate of $a$ from the standard normal $N(0, 1)$ distribution. The choice of $w$ is made according to the sample size. A generalization of this approach to $\ell_p$ norms for any $p \in (0, 2]$ is possible as well: this is done by picking the vector $a$ from so-called

**Open problem.** Is there an NN-based classification algorithm with query evaluation time $O(n)$ that achieves, under the conditions of Theorem 1, the minimax risk rate of $O(n^{-1/2+\epsilon})$?

Our analysis is facilitated by a bound on the $k$-missing mass of possible independent interest:

**Theorem 2.** Let $U_n(k)$ be the missing mass variable defined in 2. For $\varepsilon > 0$, $n \in \mathbb{N}$ and $1 \leq k \leq n$, we have

(a) $\mathbb{E}[U_n(k)] < 1.6\|P\|_0 k/n$, where $\|P\|_0 = \sum_{j \in \mathbb{N}} \mathbb{I}[p_j > 0]$ is the support size of $P$;

(b) $\mathbb{P}(U_n(k) > \mathbb{E}[U_n(k)] + \varepsilon) \leq 2 \exp\left(-0.09n\varepsilon^2/k\right)$.

**Remark.** Lemma 16.6 in Shalev-Shwartz and Ben-David (2014) claims the bound $\mathbb{E}[U_n(k)] \leq 2\|P\|_0 k/n$ for $k \geq 2$. The proof is an exercise, but a sketch is provided. Since we provide a complete proof (via a different method), with a better constant and without restricting the range of $k$, we decided to include part (a) above. The concentration result in (b) is, to our knowledge, novel.

4 LSH based Learner

Our LSH-based learner (presented formally in Alg. 1) operates as follows. Given a sample $S_n$ of size $n$, we set the radius parameter $r_n$, and pick $m_n = O(\log n)$ functions $\{h_i\}$ from an LSH family $\mathcal{H}$, and define $g_n(x) = (h_1(x), \ldots, h_{m_n}(x))$. Using $g_n$ we then construct the hash table $T$, which contains the training set $S_n$, and each bucket is labeled according to the majority vote among the labels of the $x_i$’s falling into the bucket. Technically, this is done by taking a single pair, which agrees with the majority vote, $(x_i, y_i)$, from the bucket, and inserting it into a new table $T'$, using the same hash function $g_n$. The LSH learner runs in $O(dn \log n)$, and its output is a classifier defined by a (table, hash function) pair.

We denote by $|T|$ the size of the table, namely, the number of buckets in $T$. We use $|T(k)|$ to denote the number of elements in the bucket whose key is $k$. The number of buckets can be reduced, by retaining only the nonempty buckets using (standard) hashing of the values $g_n(x)$. However, in this work we use single hashing.

To label a test point $x$, we need to access the label in $T'(g_n(x))$. This can be done in time $O(d \log n)$ (see Algorithm 2).

4.1 LSH family

The term Locality-Sensitive Hashing (LSH) was introduced by Indyk and Motwani (1998) to describe a randomized hashing framework for efficient approximate nearest neighbor search in high-dimensional space. It is based on the definition of LSH family $\mathcal{H}$, a family of hash functions mapping similar input items to the same hash code with higher probability than dissimilar items. Our LSH learner is using the following family, proposed by Datar et al. (2004). For the Euclidean metric we pick a random projection of $\mathbb{R}^d$ onto a 1-dimensional line and chop the line into segments of length $w$, shifted by a random value $b \in [0, w)$. Formally, $h_{a,b}(x) = \lfloor \frac{ax+b}{w} \rfloor$, where the projection vector $a \in \mathbb{R}^d$ is constructed by picking each coordinate of $a$ from the standard normal $N(0, 1)$ distribution. The choice of $w$ is made according to the sample size. A generalization of this approach to $\ell_p$ norms for any $p \in (0, 2]$ is possible as well: this is done by picking the vector $a$ from so-called
Algorithm 1 LSH based learner

Require:
Sample $S_n = \{(x_i, y_i)\}_{i=1}^n$

Ensure:
LSH based classifier

1: set $m_n = \lfloor \frac{\log n}{2 \log \frac{1}{p_1}} \rfloor$
2: pick $m_n$ functions from $\mathcal{H}_n$ where $\mathcal{H}_n$ is as in Section 4.1
3: Initialize empty hash tables $T, T'$
4: set $g_n = (h_1, \ldots, h_{m_n})$
5: for $i = 1 \rightarrow n$ do
6:  add $(x_i, y_i)$ to $T(g_n(x_i))$
7: end for
8: for bucket $j$ in $T$ do
9:  if $\sum_{(x_i, y_i) \in T(j)} y_i > \frac{|T(j)|}{2}$ then
10:     find $(x', y') \in T(j)$ s.t. $y' = 1$
11:     add $(x', y')$ to $T'(g_n(x'))$
12:  else
13:     find $(x', y') \in T(j)$ s.t. $y' = 0$
14:     add $(x', y')$ to $T'(g_n(x'))$
15: end if
16: end for
17: return $(T', g_n)$

$p$-stable distribution. We compute the probability that two vectors $v_1, v_2 \in \mathbb{R}^d$ collide under a hash function drawn from this family. For the two vectors, let $z = \|v_1 - v_2\|_p$ and let $P(z)$ denote the probability that $v_1, v_2$ collide for a hash function chosen from the family $\mathcal{H}$ described above. For a random vector $\alpha$ whose entries are drawn from a $p$-stable distribution, $\alpha v_1 - \alpha v_2$ is distributed as $zX$ where $X$ is a random variable drawn from a $p$-stable distribution. We get a collision if both $|\alpha v_1 - \alpha v_2| < w$ and a divider does not fall between $\alpha v_1$ and $\alpha v_2$. It is easy to see that

$$\mathbb{P}(h(v_1) = h(v_2)) = P(z) = \int_0^z \phi_p(t)(1 - t z \frac{1}{w}) dt,$$

where $\phi_p$ is the density of the absolute value of the $p$-stable distribution. Notice that for a fixed $w$, this probability depends only on the distance $z$, and it is monotonically decreasing in $z$. Finally, given a sample $S$ of size $n$, we set

$$w = \left( \frac{1.6d^{(d+2)/2}}{n \frac{d+2}{2}} \right)^{\frac{1}{d+1}}.$$

Choosing $r_n = w$, we get

$$p_1 = P(r_n) = \int_0^1 f(t)(1 - t) dt,$$

$$p_2 = P(cr_n) = \int_0^{\frac{1}{c}} f(t)(1 - ct) dt.$$
Algorithm 2 LSH based classifier \( f_{T', g_n} \)

Require:
- hash table \( T' \)
- hash function \( g_n \)
- test point \( x \in X \)

1: if \( T'(g_n(x)) \) is not empty then
2: \((x', y') \leftarrow \text{retrieve element from } T'(g_n(x))\)
3: return \( y' \)
4: else
5: return default label 0
6: end if

For example, for the Euclidean norm, we have
\[
\phi_p(t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}
\]
and \( c = 3 \), which induces a \((r_n, 3r_n, p_1, p_2)\)-sensitive family with
\[
p_1 = P(r_n) \approx 0.367691, \quad p_2 = P(3r_n) \approx 0.131758.
\]

More generally, our Bayes consistency results hold for the LSH learner whenever the \((r_n, cr_n, p_1, p_2)\)-sensitive family \( H_n \) satisfies the properties S1-S3.

5 Proof of Theorem 2(a)

Remark. As shown in Berend and Kontorovich (2012), even for \( k = 1 \), one cannot, in general, obtain estimates on \( \mathbb{E}[U^{(k)}_n] \) independent of the support size — unlike concentration bounds, which are dimension-free.

Proof. Decompose \( U^{(k)}_n = X + Y \), where
\[
X = \sum_{j: k \leq np_j} p_j \xi_j^{(k)}, \quad Y = \sum_{j: k > np_j} p_j \xi_j^{(k)}.
\]
(3)

Then \( \mathbb{E}[U^{(k)}_n] = \mathbb{E}[X] + \mathbb{E}[Y] \) and
\[
\mathbb{E}[\xi_j^{(k)}] = P(\text{Bin}(n, p_j) < k) = \sum_{\ell=0}^{k-1} \binom{n}{\ell} p_j^\ell (1 - p_j)^{n-\ell}.
\]

For \( k \leq np_j \), the multiplicative Chernoff bound \( P(\text{Bin}(n, p) < (1 - \delta)np) \leq \exp(-\delta^2 np/2) \) yields \( \mathbb{E}[\xi_j^{(k)}] \leq \exp \left( -\frac{(np_j - k)^2}{2np_j} \right) \), whence
\[
\mathbb{E}[X] \leq \sum_{j: k \leq np_j} p_j \exp \left( -\frac{(np_j - k)^2}{2np_j} \right).
\]
(4)

We estimate this quantity via the simple strategy of maximizing each summand independently over \( p_j \). To this end, define the function \( F(p) = p \exp \left( -\frac{(np_j - k)^2}{2np} \right) \) over \( p \in [k/n, 1] \) and compute
\[
F'(p) = \exp \left( -\frac{(np_j - k)^2}{2np} \right) \frac{2np}{k^2 + np(2 - np)}.
\]
The latter vanishes at
\[ p \in \{ p_+, p_- \} := \frac{1 \pm \sqrt{1 + k^2}}{n}, \]
of which only \( p_+ \) lies in the permitted range \([k/n, 1]\). Since for \( k \leq n \) we always have \( k^2 < n(n+2) \), it follows that \( F'(1) < 0 \), and hence either \( p_+ \leq 1 \) maximizes \( F \) over \([k/n, 1]\) or else \( p_+ > 1 \) (which happens iff \( k^2 > n(n-2) \)) and \( F \) is maximized at \( p = 1 \).

We shall analyze both cases. For the first case, it is a simple exercise to show that
\[ \frac{(np_+ - k)^2}{2np_+} = \frac{(1 + \sqrt{k^2 + 1} - k)^2}{2(1 + \sqrt{k^2 + 1})} \geq \frac{1}{(1 + \sqrt{2})k} \]
and hence
\[ \frac{nF(p_+)}{k} \leq \frac{(1 + \sqrt{k^2 + 1}) \exp(-[1 + \sqrt{2}]k^{-1})}{k} =: G(k). \]

We claim that \( G \) is monotonically decreasing in \( k \). Indeed, \( k^2 \sqrt{k^2 + 1} + 1 e^{\sqrt{2} - 1} [\sqrt{2} - 1]^{-1} G'(k) = k^2 + 1 + \sqrt{k^2 + 1} - (\sqrt{2} + 1)k(1 + \sqrt{k^2 + 1}) < 0, \)
which follows readily from \( k \leq \sqrt{k^2 + 1} \leq k + \sqrt{2} - 1 \), for \( k \geq 1 \). Thus,
\[ G(k) \leq G(1) = (1 + \sqrt{2}) \exp(-[1 + \sqrt{2}]^{-1}) < 1.595457, \]
whence
\[ F(p_+) < 1.6k/n. \] (5)

For the second case, which requires bounding \( F'(1) \), we claim that
\[ \sup_{n \geq 1} \sup_{k \in [1, n]} \exp \left( -\frac{(n-k)^2}{2n} \right) < 1.56k/n. \] (6)

Indeed, putting \( x = k/n \), we can define \( G(x) = \exp \left( -\frac{n^2(1-x)^2}{2n} \right)/x \) and verify that \( G'(x) < 0 \) on \([1/n, 1]\). Thus, the extreme value of \( \exp(-1/4)/2 \approx 1.56 \) in (6) is achieved at \( n = 2 \) and \( k = 1 \).

It follows from (5) and (6) that
\[ \mathbb{E}[X] \leq 1.6k/n \cdot |\{ j \in \mathbb{N} : p_j \geq k/n \}|. \]

The upper bound on \( \mathbb{E}[Y] \) is trivial:
\[ \mathbb{E}[Y] = \sum_{j: k > np_j} p_j \mathbb{E}[\xi_j^{(k)}] \leq k/n \cdot |\{ j \in \mathbb{N} : p_j > 0 \}|. \]

Combining the estimates on \( \mathbb{E}[X] \) and \( \mathbb{E}[Y] \) concludes the proof. \( \square \)
6 Proof of Theorem 2(b)

We begin by observing that the random variables $\xi^{(k)}_j$, though not independent, are negatively associated, as shown in [McAllester and Ortiz (2003)]. Thus, for the purpose of establishing concentration, one may invoke the standard Bernstein-Chernoff exponential bounding argument verbatim [Dubhashi and Ranjan (1998)]. We shall do so in the sequel without further comment.

We maintain the decomposition $U^{(k)}_n = X + Y$ as in (4) and derive concentration bounds on $X$ and $Y$ separately. A bound for $U^{(k)}_n$ will then follow via

$$\mathbb{P}(U^{(k)}_n \geq \mathbb{E}[U^{(k)}_n] + \varepsilon) \leq \mathbb{P}(X \geq \mathbb{E}[X] + \alpha\varepsilon) + \mathbb{P}(Y \geq \mathbb{E}[Y] + (1 - \alpha)\varepsilon),$$

for any choice of $0 \leq \alpha \leq 1$.

Tail bounds for $X$

In this section, we always assume that $n \geq 1$, $p \in [0, 1]$ and $1 \leq k \leq np$. Define the function $q = q(k, n, p) := \exp\left(-\frac{(np - k)^2}{2np}\right)$ and the collection of independent Bernoulli variables $\xi'_j \sim \text{Bet}(q(k, n, p))$, as well as $X' := \sum_{j: k \leq np} p_j \xi'_j$. It follows from (11) that $\mathbb{E}[X] \leq \mathbb{E}[X'] = \sum_{j: k \leq np} p_j q(k, n, p_j)$ and from negative association that

$$\mathbb{P}(X \geq \mathbb{E}[X] + \varepsilon) \leq \mathbb{P}(X' \geq \mathbb{E}[X'] + \varepsilon), \quad \varepsilon > 0. \tag{7}$$

Our strategy for bounding (7) is to bound the moment generating function $\mathbb{E}\exp[\lambda(X' - \mathbb{E}[X'])]$ — to which end, it suffices to bound

$$\mathbb{E}\exp[\lambda \xi'_j - \mathbb{E}[\xi'_j]] = q(k, n, p_j) e^{\lambda p_j (1 - q(k, n, p_j))} + (1 - q) e^{-\lambda p_j q(k, n, p_j)}$$

$$=: \Phi(\lambda, k, n, p_j). \tag{8}$$

Lemma 3. For $\Phi$ as defined in (8),

$$\Phi(\lambda, k, n, p) \leq \exp(C_\Phi \lambda^2 pk/n),$$

where $C_\Phi \leq (2 + \sqrt{3})/4 \log(e - 1) < 1.73$ is a universal constant. \footnote{Numerical simulations suggest that $C_\Phi < 0.61$.}

Armed with Lemma 3, the standard argument yields an estimate on (7):

$$\mathbb{P}(X' \geq \mathbb{E}[X'] + \varepsilon) = \mathbb{P}(\exp(\lambda(X' - \mathbb{E}[X'])) \geq e^{\lambda\varepsilon})$$

$$\leq e^{-\lambda\varepsilon} \prod_{j: k \leq np} \mathbb{E}\exp[\lambda \xi'_j - \mathbb{E}[\xi'_j]]$$

$$= e^{-\lambda\varepsilon} \prod_{j: k \leq np} \Phi(\lambda, k, n, p_j)$$

$$\leq e^{-\lambda\varepsilon} \prod_{j: k \leq np} \exp(C_\Phi \lambda^2 p_j k/n)$$

$$\leq \exp(C_\Phi \lambda^2 k/n - \lambda\varepsilon).$$

Choosing $\lambda = \varepsilon n/2kC_\Phi$ yields

$$\mathbb{P}(X \geq \mathbb{E}[X] + \varepsilon) \leq \exp(-\varepsilon^2 n/4kC_\Phi). \tag{9}$$
Tail bounds for \( Y \)

As done for \( X \) in (7), we invoke negative association to obtain

\[
P(Y \geq \mathbb{E}[Y] + \varepsilon) \leq P(Y' \geq \mathbb{E}[Y'] + \varepsilon), \quad \varepsilon > 0,
\]

where \( Y' = \sum_{j:k>np} p_j \xi'_j \) and the \( \xi'_j \sim \text{Ber}(q_j) \) are independent, and

\[
q_j := \sum_{\ell=0}^{k-1} \left( \begin{array}{c} n \\ \ell \end{array} \right) p_j^\ell (1 - p_j)^{n-\ell}.
\]

In particular, \( \mathbb{E}[Y] = \mathbb{E}[Y'] \).

An application of Hoeffding’s inequality yields

\[
P(Y' \geq \mathbb{E}[Y'] + \varepsilon) \leq \exp\left(-\frac{2\varepsilon^2}{\sum_{j:k>np} p_j^2}\right);
\]

it remains to bound \( \sum_{j:k>np} p_j^2 \).

Since the \( p_j \)'s sum to 1, there can be at most \( 2n/k \) elements in the range \( k/2n \leq p_j \leq k/n \), at most \( 4n/k \) in the range \( k/4n \leq p_j \leq k/2n \), and so forth. Continuing in this fashion, we conclude that

\[
\sum_{j:k>np} p_j^2 \leq \sum_{i=1}^{\infty} \left( \frac{k}{2^n} \right)^2 \frac{2^i n}{k} = \frac{k}{n}.
\]

It follows that

\[
P(Y \geq \mathbb{E}[Y] + \varepsilon) < \exp(-2\varepsilon^2 n/k).
\]

Combining these with (7) concludes the proof.

**Proof of Lemma 3.** Throughout the proof, \( n \geq 1, p \in [0,1], 1 \leq k \leq np \) and \( q = q(k,n,p) \) as defined above.

As in the proof of Lemma 3.5(a) in Berend and Kontorovich (2013), we invoke the Kearns-Saul inequality to obtain

\[
q \exp(\lambda(p - pq)) + (1 - q) \exp(-\lambda pq) \leq \exp[(1 - 2q)\lambda^2 p^2/4\log[(1 - q)/q]].
\]

Thus, to prove the Lemma, it suffices to show that

\[
(1 - 2q)/\log[(1 - q)/q] \leq 4C_\Phi k/np.
\]

Holding \( \mu := np \) fixed, put \( x = k/\mu \) and reparametrize \( q \) as \( y(x) = \exp(-\mu(x-1)^2/2) \); our task is now reduced to proving

\[
\sup_{\mu \geq 1} \sup_{x \in [1/\mu, 1]} \frac{1 - 2y(x)}{x \log[(1 - y(x))/y(x)]} \leq 4C_\Phi.
\]
Note that $x \geq 1/\mu$ implies $y \geq \exp(-(\mu - 1)^2/2\mu)$. Reparametrize again via $z := \log(1/y) \leq (\mu - 1)^2/2\mu$; now proving (13) amounts to showing that

$$F(z) := \frac{1 - 2e^{-z}}{(1 - \sqrt{2z/\mu}) \log(e^z - 1)} \leq 4C_\Phi, \quad \mu \geq 1, z \in [0, (\mu - 1)^2/2\mu].$$

Our proof will not require this, but we note that $F$ is always non-negative; this is clear from the parametrization in (13). To prove (13), we consider the two cases $z < 1$ and $z \geq 1$ below, from which the estimate $C_\Phi \leq (2 + \sqrt{3})/4 \log(e - 1) < 1.73$ readily follows.

**Case I:** $z < 1$. This case will follow from the inequalities

$$\sup_{0 < z < 1} \left| \frac{1 - 2e^{-z}}{\log(e^z - 1)} \right| \leq \frac{1}{2},$$

(14)

and

$$\sup_{\mu \geq 1} \sup_{0 < z < \min\{1, (\mu - 1)^2/2\mu\}} \left| \frac{1}{1 - \sqrt{2z/\mu}} \right| \leq 2 + \sqrt{3} \approx 3.73;$$

(15)

combining them implies a bound of $F(z) \leq 1 + \sqrt{3}/2 \approx 1.87$ over the specified range of $\mu$ and $z$. Both (14) and (13) are straightforward exercises. The former is facilitated by the reparametrization $(1 - 2/t)/\log(t - 1)$ while the latter involves analyzing the two cases $(\mu - 1)^2/2\mu \geq 1$, whose boundary is demarcated by $\mu = 2 + \sqrt{3}$.

**Case II:** $z \geq 1$. This case is facilitated by the trivial estimate

$$\sup_{t \geq 1} \frac{t}{\log(e^t - 1)} \leq 1/\log(e - 1) < 1.85.$$

(16)

Indeed, since $|1 - 2e^{-z}| \leq 1$, it follows from (16) that

$$F(z) \leq G(z) := \frac{1.85}{z(1 - \sqrt{2z/\mu})}$$

over the specified range of $\mu$ and $z$, which is $z \in [1, (\mu - 1)^2/2\mu]$ and $\mu \geq 2 + \sqrt{3}$ (since for smaller $\mu$, the range of $z$ is empty). Now the function $G(z, \mu) := z(1 - \sqrt{2z/\mu})$ is unimodal in $z$ for fixed $\mu$, vanishing at $z = 0$ and at $z = \mu/2$, and achieving a positive maximum value strictly inbetween. As the actual range of $z$ is $1 \leq z \leq (\mu - 1)^2/2\mu < \mu/2$ (the latter inequality holds for all $\mu \geq 1$), to analyze the minimum of $G(\cdot, \mu)$, we need only consider the extreme feasible values $z_1 = 1$ and $z_2 = (\mu - 1)^2/2\mu$. A straightforward computation yields

$$\sup_{\mu \geq 2 + \sqrt{3}} \frac{1}{G(z_1, \mu)} = \sup_{\mu \geq 2 + \sqrt{3}} \frac{1}{1 - \sqrt{2}/\mu} = \frac{1}{1 - \sqrt{4 - 2\sqrt{3}}} = 2 + \sqrt{3}$$

and

$$\sup_{\mu \geq 2 + \sqrt{3}} \frac{1}{G(z_2, \mu)} = \sup_{\mu \geq 2 + \sqrt{3}} \frac{2\mu^2}{(\mu - 1)^2} = 2 + \sqrt{3}.$$

Combining these implies a bound of $F(z) \leq (2 + \sqrt{3})/\log(e - 1) < 6.9$ over the specified range of $\mu$ and $z$. \qed
7 Proof of Theorem 1

Our proof closely follows the argument in Devroye et al. (1996, Theorem 6.1).

Given a test point \( x \in [0,1]^d \) drawn from \( \mathcal{D}_X \), and \( g_n(x) = j \). We would like to know how many sample points are in the bucket \( T(j) \), and what is the ratio of the near (i.e. at distance at most \( < cr_n \)) and distant (i.e. at distance at least \( \geq cr_n \)) points in the bucket. To deal with these questions, we first set some notations. Given a test point \( x \sim \mathcal{D}_X \) and a hash function \( g_n \), we denote by \( A(x) \) the set of points from \( S \) in the same bucket with \( x \), and \( N(x) \) is the size of that bucket. Formally,

\[
A(x) = \{ x_i \in S_n | g_n(x_i) = g_n(x) \} \\
N(x) = \sum_{i=1}^{n} \mathbb{1}[x_i \in A(x)].
\]

In addition, for \( r > 0 \) we denote by \( A_{\text{close}}(x) \) the set of near points from \( S \) in the same bucket with \( x \),

\[
A_{\text{close}}(x) = \{ x_i \in S_n | g_n(x_i) = g_n(x), \| x - x_i \| < cr_n \}
\]

and \( A_{\text{far}}(x) \) is the complementary \( A(x) \setminus A_{\text{close}}(x) \). Finally, we define \( N_{\text{close}}(x) \) and \( N_{\text{far}}(x) \) as the cardinality of the sets \( A_{\text{close}}(x) \) and \( A_{\text{far}}(x) \). Equipped with the preceding notations, we are now ready to prove the Theorem 1.

Define \( \hat{\eta}_n(x) = \frac{1}{N(x)} \sum_{i:x_i \in A(x)} y_i \) and \( \eta^*(x) = \mathbb{E}[\eta(x') | x' \in A_{\text{close}}(x)] \). By Devroye et al. (1996, Theorem 2.2), we have

\[
\mathbb{E}[R(f_{g_n,T^r})] - R^* \leq 2\mathbb{E}[|\hat{\eta}_n(x) - \eta(x)|].
\]

By the triangle inequality,

\[
\mathbb{E}[|\hat{\eta}_n(x) - \eta(x)|] \leq \mathbb{E}[|\hat{\eta}_n(x) - \eta^*(x)|] + \mathbb{E}[|\eta^*(x) - \eta(x)|].
\]

By conditioning on the variables \( \mathbb{1}[x_i \in A(x)] \), \( \mathbb{1}[x_i \in A_{\text{close}}(x)] \), it is easy to see that \( \sum_{i:x_i \in A_{\text{close}}(x)} y_i \) is distributed as \( \text{Bin}(N_{\text{close}}(x), \eta^*(x)) \), a binomial random variable with parameters \( N_{\text{close}}(x), \eta^*(x) \). Thus,

\[
\mathbb{E}\left[|\hat{\eta}_n(x) - \eta^*(x)| \mid \mathbb{1}[x_i \in A(x)], \mathbb{1}[x_i \in A_{\text{close}}(x)]\right] \\
\leq \mathbb{E}\left[\frac{1}{N(x)} \sum_{i:x_i \in A(x)} y_i - \eta^*(x) \mid \mathbb{1}[x_i \in A(x)], \mathbb{1}[x_i \in A_{\text{close}}(x)] \right] \\
+ \mathbb{1}[N(x) = 0] \\
\leq \mathbb{E}\left[\frac{1}{N(x)} \sum_{i:x_i \in A_{\text{close}}(x)} y_i - \eta^*(x) \mid \mathbb{1}[x_i \in A(x)], \mathbb{1}[x_i \in A_{\text{close}}(x)] \right] \\
+ \frac{N_{\text{far}}(x)}{N(x)} + \mathbb{1}[N(x) = 0] \\
= \mathbb{E}\left[\frac{\text{Bin}(N_{\text{close}}(x), \eta^*(x)) - N(x)\eta^*(x)}{N(x)} \mid \mathbb{1}[x_i \in A(x)], \mathbb{1}[x_i \in A_{\text{close}}(x)] \right] \\
+ \frac{N_{\text{far}}(x)}{N(x)} + \mathbb{1}[N(x) = 0] \\
= (\ast) + (\ast\ast) + (\ast\ast\ast).
\]
By Cauchy-Schwarz we have

\[(*) \leq \left( \frac{1}{N(x)^2} \mathbb{E} \left[ \text{Bin}(N_{\text{close}}(x), \eta^*(x)) - N(x)\eta^*(x) \right] \right)^{\frac{1}{2}} \]

\[= \left( \frac{1}{N(x)^2} \left( \mathbb{E} \left[ \text{Bin}(N_{\text{close}}(x), \eta^*(x))^2 \right] - 2N_{\text{close}}(x)N(x)\eta^*(x) + N(x)^2\eta^*(x)^2 \right) \right)^{\frac{1}{2}} \]

\[= \left( \frac{1}{N(x)^2} \left( N_{\text{close}}(x)\eta^*(x)(1 - \eta^*(x)) + \eta^*(x)^2(N(x) - N_{\text{close}}(x)^2) \right) \right)^{\frac{1}{2}}. \]

Hence,

\[(*) \leq \sqrt{\frac{N_{\text{close}}(x)}{4N(x)^2}} + \left( \frac{N_{\text{far}}(x)}{N(x)} \right)^2 \leq \sqrt{\frac{1}{4N(x)}} + \left( \frac{N_{\text{far}}(x)}{N(x)} \right)^2. \]

Hence,

\[\mathbb{E} \left[ |\hat{\eta}_n(x) - \eta^*(x)| \right] \leq \mathbb{E} \left[ \sqrt{\frac{1}{4N(x)}} + \left( \frac{N_{\text{far}}(x)}{N(x)} \right)^2 + \frac{N_{\text{far}}(x)}{N(x)} + \mathbb{1} [N(x) = 0] \right]. \]

Taking expectations,

\[\mathbb{E} [|\hat{\eta}_n(x) - \eta^*(x)|] \leq \mathbb{E} \left[ \sqrt{\frac{1}{4N(x)}} + \left( \frac{N_{\text{far}}(x)}{N(x)} \right)^2 + \frac{N_{\text{far}}(x)}{N(x)} \right] + \mathbb{P}(N(x) = 0) \]

\[\leq \left( \sqrt{2} + 2 \right) \left( \mathbb{P}(N(x) < M) + \mathbb{P}(N_{\text{far}}(x) > \delta N(x)) \right) + \sqrt{\frac{1}{4M}} + \delta^2 + \delta. \]

For the second term, \( \mathbb{E} [|\eta^*(x) - \eta(x)|] \) we use the smoothness assumption on \( \eta \). Since \( \eta^*(x) = \mathbb{E} [\eta(x') | \|x - x'\| \leq cr] \) then

\[\eta(x) - L(cr_n)^\alpha \geq \eta^*(x) \leq \eta(x) + L(cr_n)^\alpha. \]

Hence,

\[\mathbb{E} [|\eta^*(x) - \eta(x)|] \leq L(cr_n)^\alpha. \]

Now, by applying Lemmas 4, 5 and setting \( \delta = \sqrt{\frac{1}{n^{s-\frac{1}{2}}} M = \frac{4^{s-\frac{1}{2}}}{4} } \) we get

\[\mathbb{E} [R(\mathbf{f}_{T^*, g_n})] - R^* \leq 4 \left( 2 \exp(-\frac{1}{8}n^{s-\frac{1}{2}} + 4 \exp(-0.09n^{\frac{1}{2}+\epsilon}) + 2 \left( \frac{1.6}{n^{1-s} \sqrt{d}} \right)^{\frac{1}{4+s}} + \sqrt{\frac{4}{n^{\frac{1}{2}}} + 2^{-\frac{2s-1}{2}}} \right) \]

\[+ \sqrt{\frac{1}{4M}} + \delta^2 + \delta + L(cr_n)^\alpha. \]

Finally, we set \( s = \frac{d+5}{2d+6} \) and for \( d \geq 3 \), we get by straightforward calculation,

\[\mathbb{E} [R(\mathbf{f}_{T^*, g_n})] - R^* \leq 48 \exp(-0.09n^{\frac{1}{2}+\epsilon}) + \frac{73Lc^\alpha \sqrt{d^{\frac{d+2}{4}}} \frac{n}{2^{d+\epsilon}}}{n^{2d+\epsilon}}, \]

which completes the proof.
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\[ A \text{ Proof of Lemma 4} \]

The following lemma states that with high probability, the ratio \( \frac{N_{\text{far}}(x)}{N(x)} \to 0 \) as \( n \) approaches \( \infty \). Throughout this section, \( B(x, r) \) denotes the closed Euclidean \( r \)-ball about \( x \).

**Lemma 4.** Let \( x \sim \mathcal{D}_X \). Then, for all \( \delta > 0, \frac{1}{2} < s < 1 \), the hash table \( T \) calculated by Algorithm 1 satisfies:

\[
\mathbb{P}(N_{\text{far}}(x) > \delta N(x)) \leq 2 \exp(-0.09n^{\frac{1-s}{2}}) + \left( \frac{1.6}{n^{1-s} \sqrt{d}} \right)^{\frac{1}{s+1}} + \frac{1}{n^{\frac{1-s}{2}}} + \exp(-\frac{1}{8}n^{s-\frac{1}{2}}) + 2^{-\frac{8}{3}n^{-\frac{1}{2}}},
\]

where the probability is over \( S_n, x \sim \mathcal{D}^{n+1} \) and the choice of the function \( g_n \).

**Proof.** Fix \( \delta > 0, \varepsilon = \frac{\delta}{\sqrt{d}}, \) and let \( C_1, \ldots, C_t \) be a partition of \([0,1]^d\) into \( t = \left( \frac{1}{\varepsilon} \right)^d \) boxes of length \( \varepsilon \). Notice that for any \( x, x' \) in the same box, we have \( |x - x'| \leq \sqrt{d} \varepsilon \).

Put \( k = n^s \) and define the random variable \( L_{\varepsilon,k}(S_n) = \sum_{i:C_i \cap S_n < k} \mathbb{P}(C_i) \), and note that it is precisely the \( k \)-missing mass (defined in (2)) associated with the distribution \( P = (\mathbb{P}(C_1), \ldots, \mathbb{P}(C_t)) \). By Theorem 2(a), we have \( \mathbb{E}[L_{\varepsilon,k}(S_n)] \leq \frac{1.6kd}{n} \). By the law of total probability,

\[
\mathbb{P}(N_{\text{far}}(x) > \delta N(x)) \leq \mathbb{P}
\left(
L_{\varepsilon,m}(S_n) > \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma
\right) + \mathbb{P}(N_{\text{far}}(x) > \delta N(x) \bigg| L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma).
\]

For the first term in (17), we apply Theorem 2(b):

\[
\mathbb{P}(L_{\varepsilon,m}(S_n) > \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma) \leq \mathbb{P}(L_{\varepsilon,m}(S_n) > \mathbb{E}[L_{\varepsilon,m}(S_n)] + \gamma) \leq 2 \exp\left(-0.09n^{1-s} \gamma^2\right).
\]

For the second term in (17), we have

\[
\mathbb{P}(N_{\text{far}}(x) > \delta N(x) \bigg| L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma)
\leq \mathbb{P}
\left(
|B(x, r_n) \cap S_n| < n^s \bigg| L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma
\right)
\]

\[
+ \mathbb{P}(N_{\text{far}}(x) > \delta N(x), \ |B(x, r_n) \cap S_n| \geq n^s \bigg| L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma)
\]

\[
= (\ast) + (**).
\]

Since \( r_n = \sqrt{d} \varepsilon \), we have \( \{|B(x, r_n) \cap S_n| < n^s\} \implies \{|C(x) \cap S_n| < n^s\} \), where \( C(x) \) is the \( \varepsilon \)-length box containing \( x \). Thus,

\[
(\ast) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma.
\]
We are left to bound the second term in (18).

\[
\begin{align*}
(\ast\ast) \leq & \mathbb{P}\left( N_{close}(x) < \frac{1}{2} n^{s-\frac{1}{2}} \middle| L_{x,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma, \ |B(x, r_n) \cap S_n| > n^s \right) \\
+ & \mathbb{P}\left( N_{far}(x) > \delta N(x), N_{close}(x) \geq \frac{1}{2} n^{s-\frac{1}{2}} \middle| L_{x,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}} + \gamma, \ |B(x, r_n) \cap S_n| > n^s \right) \\
= & \ (\ast\ast) + (\ast\ast\ast).
\end{align*}
\]

Since the algorithm set \( m_n = \left\lfloor \frac{\log n}{2 \log p_1} \right\rfloor \), we have

\[
\mathbb{E}\left[ N_{close}(x) \middle| |B(x, r) \cap S_n| > n^s \right] \geq p_1^{m_n} n^s \\
\geq & p_1^{\frac{\log n}{2 \log p_1}} n^s \\
\geq & \left(2 \log p_1\right)^{\frac{1}{2} \log p_1} \log n \ n^s \\
\geq & n^{s-\frac{1}{2}}.
\]

Let \( Z \sim \text{Bin}(n^s, p_1^{m_n}) \). We have \( \mathbb{E}\left[ N_{close}(x) \middle| |B(x, r) \cap S_n| > n^s \right] \geq \mathbb{E}[Z] = n^{s-\frac{1}{2}} \). In addition, for each \( x' \in A_{close}(x) \) we have \( \mathbb{P}(g_n(x) = g_n(x')) \geq p_1^{m_n} \), and thus, invoking the Chernoff bound,

\[
(\ast\ast\ast) \leq \mathbb{P}(Z < \frac{1}{2} n^{s-\frac{1}{2}}) \\
= \mathbb{P}(Z < \frac{1}{2} \mathbb{E}[Z]) \\
\leq \exp\left( -\frac{1}{8} \mathbb{E}[Z] \right) \\
\leq \exp\left( -\frac{1}{8} n^{s-\frac{1}{2}} \right).
\]

The last term we have to bound is the second term in (19). Notice that

\[
\{N_{far}(x) > \delta N(x), N_{close}(x) \geq \frac{1}{2} n^{s-\frac{1}{2}}\} \implies \{N_{far}(x) > \frac{\delta}{2} n^{s-\frac{1}{2}}\}.
\]

In addition, since \( p_1^2 > p_2 \), we have

\[
\mathbb{E}[N_{far}(x)] \leq p_2^{m_n} n \leq p_1^{2m_n} n \leq p_1^{\frac{\log n}{2 \log p_1}} n = p_1^{-2} = O(1).
\]

Since for each \( x' \in A_{far}(x) \) we have \( \mathbb{P}(g_n(x) = g_n(x')) \leq p_2^{m_n} \), if we let \( Z \sim \text{Bin}(n, p_2^{m_n}) \) then, by Chernoff’s bound,

\[
(\ast\ast\ast\ast) \leq \mathbb{P}(Z > \frac{\delta}{2} n^{s-\frac{1}{2}}) \leq 2 \frac{2^{\frac{1}{2} \log n^{s-\frac{1}{2}}}}{2^{\frac{1}{2} \log n^{s-\frac{1}{2}}}}.
\]

For \( s > \frac{1}{2} \) and large enough \( n \) s.t. \( 2e \mathbb{E}[N_{far}(x)] \leq 2e \mathbb{E}[Z] \leq 2e \leq \frac{\delta}{2} n^{s-\frac{1}{2}} \).

Finally, setting \( \gamma = \sqrt{\frac{1}{n^{s-2}}} \), \( r_n = \left(\frac{1.6 \sqrt{\frac{1}{n^{1-s}}}}{n^{1-s}}\right)^{\frac{1}{2+1}} \) we conclude our proof.
B Proof of Lemma 5

Here we show that with high probability, the variable $N(x) \to \infty$. Namely, the number of sample points at each bucket is increasing as $n$ goes to $\infty$.

**Lemma 5.** Let $x \sim \mathcal{D}_X$ be a test point. Then, for all $M > 0$, $\frac{1}{2} < s < 1$ the hash table calculated by Algorithm 1 satisfies:

$$
\mathbb{P}(N(x) < M) \leq \exp(-\frac{n^{s-\frac{1}{2}}}{2} + M) + 2 \exp(-0.09n^{\frac{1-s}{2}}) + \left(\frac{1.6}{n^{1-s}\sqrt{d}}\right)^{\frac{d}{d+1}} + \sqrt{\frac{1}{n^{\frac{1}{2}}}}.
$$

Where, again, the probability is over $S_n, x \sim \mathcal{D}^{n+1}$ and the choice of the function $g_n$.

**Proof.** Fix $M > 0$. Similar to Lemma 4, we have

$$
\mathbb{P}(N(x) < M) \leq \mathbb{P}(N(x) < M \mid L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}}, |B(x, r_n) \cap S_n| > n^s) + 2 \exp(-0.09n^{\frac{1-s}{2}}) + \frac{1.6\sqrt{d}}{r_n n^{1-s}} + \sqrt{\frac{1}{n^{\frac{1}{2}}}}.
$$

We only have to bound the first term in (20). Observe that

$$
\{N(x) < M\} \implies \{N_{\text{close}}(x) < M\}
$$

and that $\mathbb{E}[N_{\text{close}}(x) \mid |B(x, r_n \cap S_n)| > n^s] \geq \mathbb{E}[Z] = p^{m_n^s} = n^{s-\frac{1}{2}}$. Now for $Z \sim \text{Bin}(n^s, p^{m_n^s})$, if we let $\xi = 1 - \frac{M}{\mathbb{E}[Z]}$, then by Chernoff’s bound we have,

$$
\mathbb{P}(N_{\text{close}}(x) < M \mid L_{\varepsilon,m}(S_n) \leq \frac{1.6}{\varepsilon^d n^{1-s}}, |B(x, r_n) \cap S_n| > n^s) \leq \mathbb{P}(Z < (1 - \xi)\mathbb{E}[Z]) \leq \exp(-\frac{\xi^2}{2}\mathbb{E}[Z]) \leq \exp(-\frac{n^{s-\frac{1}{2}}}{2} + M).
$$

\qed