**PROP profile of Poisson geometry**

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“The genetic code appears to be universal; ...”

*Britannica.*

0. Introduction. The first instances of algebraic and topological strongly homotopy, or infinity, structures have been discovered by Stasheff [St] long ago. Since that time infinities have acquired a prominent role in algebraic topology and homological algebra. We argue in this paper that some classical local geometries are of infinity origin, i.e. their smooth formal germs are (homotopy) representations of cofibrant PROPs $\mathcal{P}_\infty$ in spaces concentrated in degree zero; in particular, they admit natural infinity generalizations when one considers homotopy representations of $\mathcal{P}_\infty$ in generic differential graded (dg) spaces. The simplest manifestation of this phenomenon is provided by the Poisson geometry (or even by smooth germs of tensor fields!) and is the main theme of the present paper. Another example is discussed in [Mer2]. The PROPs $\mathcal{P}_\infty$ are minimal resolutions of PROPs $\mathcal{P}$ which are graph spaces built from very few basic elements, *genes*, subject to simple engineering rules. Thus to a local geometric structure one can associate a kind of a code, *genome*, which specifies it uniquely and opens a new window of opportunities of attacking differential geometric problems with the powerful tools of homological algebra. In particular, the genetic code of Poisson geometry discovered in this paper has been used in [Mer3] to give a new short proof of Kontsevich’s [Ko1] deformation quantization theorem.

Formal germs of geometric structures discussed in this paper are pointed in the sense that they vanish at the distinguished point. This is the usual price one pays for working with (di)operads without “zero terms” (as is often done in the literature). As structural equations behind the particular geometries we study in this paper are homogeneous, this restriction poses no problem: say, a generic non-pointed Poisson structure, $\nu$, in $\mathbb{R}^n$ can be identified with the pointed one, $h\nu$, in $\mathbb{R}^{n+1}$, $h$ being the extra coordinate.

We introduce in this paper a dg free dioperad whose generic representations in a graded vector space $V$ can be identified with pointed solutions of the Maurer-Cartan equations in the Lie algebra of polyvector fields on the formal manifold associated with $V$. The cohomology of this dioperad can not be computed directly. Instead one has to rely on some fine mathematics such as Koszulness [GiKa, G] and distributive laws [Mar1, G]. One of the main results of this paper is a proof of Theorem 3.2 which identifies the cohomology of that dg free dioperad with a surprisingly small dioperad, Lie$^1$Bi, of Lie 1-bialgebras, which are almost identical to the dioperad, LieBi, of usual Lie bialgebras except that degree of generating Lie and coLie operations differ by 1 (compare with Gerstenhaber versus Poisson algebras). The dioperad Lie$^1$Bi is proven to be Koszul. We use the resulting geometric interpretation of Lie$^1$Bi$_\infty$ algebras to give their homotopy classification (see Theorem 3.4.5) which is an extension of Kontsevich’s homotopy classification [Ko1] of $\mathbb{L}_\infty$ algebras.

As a side remark we also discuss graph and geometric interpretations of strongly homotopy Lie bialgebras using Koszulness of the latter which was established in [G].

1. Geometry $\Rightarrow$ PROP profile $\Rightarrow$ Geometry$_\infty$. Let $\mathcal{P}$ be an operad, or a dioperad, or even a PROP admitting a minimal dg resolution. Let $\mathcal{P}\text{Alg}$ be the category of finite dimensional dg $\mathcal{P}$-algebras, and $\mathcal{D}(\mathcal{P}\text{Alg})$ the associated derived category (which we understand here as the homotopy category of $\mathcal{P}_\infty$-algebras, $\mathcal{P}_\infty$ being the minimal resolution of $\mathcal{P}$).

For any locally defined geometric structure $\text{Geom}$ (say, Poisson, Riemann, Kähler, etc.) it makes sense talking about the category of formal $\text{Geom}$-manifolds. Its objects are formal pointed manifolds
(non-canonically isomorphic to \((\mathbb{R}^n, 0)\) for some \(n)\) together with a germ of formal \text{Geom}-structure at the distinguished point.

1.1. Definition. The operad/dioperad/PROP \(P\) is called a PROP-profile, or genome, of a geometric structure \text{Geom} if

- the category of formal \text{Geom}-manifolds is equivalent to a full subcategory of the derived category \(D(P\text{Alg})\), and
- there is no sub-(di)operad of \(P\) having the above property.

1.2. Definition. If \(P\) is a PROP-profile of a geometric structure \text{Geom}, then a generic object of \(D(P\text{Alg})\) is called a formal \text{Geom}_\infty-manifold.

Presumably, \text{Geom}_\infty-structure is what one gets from \text{Geom} by means of the extended deformation theory.

Local geometric structures are often non-trivial and complicated creatures — the general solution of the associated defining system of nonlinear differential equations is not available; it is often a very hard job just to show existence of non-trivial solutions. Nevertheless, if such a structure \text{Geom} admits a PROP-profile, \(P = \text{Free}(\mathcal{E})/\text{Ideal}\), then \text{Geom} can be non-ambiguously characterized by its “genetic code”: genes are, by definition, the generators of \(\mathcal{E}\), and the engineering rules are, by definition, the generators of \(\text{Ideal}\). And that code can be surprisingly simple, as examples 1.3-1.5 and the table below illustrate.

1.3. Hertling-Manin’s geometry and the \(G\)-operad. A Gerstenhaber algebra is, by definition, a graded vector space \(V\) together with two linear maps,

\[
\circ : \odot^2 V \rightarrow V, \quad \bullet : \odot^2 V \rightarrow V[1]
\]

satisfying the identities,

- (i) \(a \circ (b \circ c) - (a \circ b) \circ c = 0\) (associativity);
- (ii) \([a \bullet b] \bullet c = [a \bullet [b \bullet c]] + (-1)^{|b||a|+|b|+|a|} [b \bullet [a \bullet c]]\) (Jacobi identity);
- (iii) \([a \circ (b \bullet c)] = a \circ [b \bullet c] + (-1)^{|b||c|+1} [a \bullet c] \circ b\) (Leibniz type identity).

The operad whose algebras are Gerstenhaber algebras is often called the \(G\)-operad. It has a relatively simple structure, \(\text{Free}(E)/\text{Ideal}\), with \(E\) spanned by two corollas,

\[
E = \text{span} \left\{ \circ = \overrightarrow{\circ}, \bullet = \overrightarrow{\bullet} \right\}
\]

and with engineering rules (i)-(iii). The minimal resolution of the \(G\)-operad has been constructed in \[\text{GetJo}\] and is often called a \(G_\infty\)-operad. The derived category of Gerstenhaber algebras is equivalent to the category whose objects are isomorphism classes of minimal \(G_\infty\)-structures on graded vector spaces \(V\). Let \((M, \ast)\) be the formal pointed graded manifold whose tangent space at the distinguished point is isomorphic to a vector space \(V\), and let us choose an arbitrary torsion-free affine connection \(\nabla\) on \(M\).

With this choice a structure of \(G_\infty\)-algebra on a graded vector space \(V\) can be suitable described as

- a degree 1 smooth vector field \(\partial\) on \(M\) satisfying the integrability condition \([\partial, \partial] = 0\) and vanishing at the distinguished point \(*\); (if \(\partial\) has zero at \(*\) of second order, then the \(G_\infty\)-structure is called minimal);

\[\text{Any operad/dioperad/etc. can be represented as a quotient of the free operad/dioperad/etc., } \text{Free}(\mathcal{E}) \text{ generated by a collection of } \Sigma_m\text{-left/}\Sigma_m\text{-right modules } \mathcal{E} = \{\mathcal{E}(m, n)\}_{m,n \geq 1}, \text{ by an } \text{Ideal}. \text{ Often there exists a canonical, “common factors canceled out”, representation like this.}\]
| Genome $\mathcal{P}$ | generic representation of $\mathcal{P}_\infty$ in $\mathbb{R}^n$ | generic representation of $\mathcal{P}_\infty$ in a graded vector space $V$ |
|--------------------------------|-------------------------------------------------|-------------------------------------------------|
| $\mathcal{P}$ is the $G$-operad | smooth formal Hertling-Manin structure in $\mathbb{R}^n$ [HeMa] | smooth formal Hertling-Manin$_\infty$ structure in $\hat{V}$ [Mer1] |
| Genes: $\begin{array}{c} \searrow \swarrow \\ \downarrow \uparrow \end{array}$, $\begin{array}{c} \nearrow \nwarrow \\ \downarrow \uparrow \end{array}$ | Engineering rules: $\begin{array}{c} \searrow \swarrow \\ \downarrow \uparrow \end{array} - \begin{array}{c} \searrow \swarrow \\ \uparrow \downarrow \end{array} = 0$ | $\begin{array}{c} \searrow \swarrow \\ \downarrow \uparrow \end{array} + \begin{array}{c} \searrow \swarrow \\ \uparrow \downarrow \end{array} = 0$ |
| | | $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \\ \downarrow \downarrow \uparrow \uparrow \end{array} - \begin{array}{c} \downarrow \downarrow \swarrow \searrow \\ \uparrow \uparrow \downarrow \downarrow \end{array} = 0$ |
| $\mathcal{P}$ is the dioperad TF | smooth formal section of $\otimes^2 T_{\mathbb{R}^n}$ (variants: of $\wedge^2 T_{\mathbb{R}^n}$ or of $\odot^2 T_{\mathbb{R}^n}$) vanishing at 0 | structure, $(\hat{V}, \bar{\delta} \in T_{\hat{V}^*})$, of a smooth dg manifold together with a smooth section $\phi$ of $\otimes^2 T_{\hat{V}^*}$ (variants: of $\wedge^2 T_{\hat{V}^*}$ or of $\odot^2 T_{\hat{V}^*}$) vanishing at 0 and satisfying $Lie_\bar{\delta}\phi = 0$. |
| Genes: $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array}$, $\begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array}$ | Rules: $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} + \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} = 0$ | $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} + \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} + \begin{array}{c} \downarrow \downarrow \downarrow \uparrow \uparrow \end{array} = 0$ |
| | | $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} + \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} + \begin{array}{c} \downarrow \downarrow \downarrow \uparrow \uparrow \end{array} + \begin{array}{c} \downarrow \downarrow \downarrow \uparrow \uparrow \end{array} = 0$ |
| $\mathcal{P}$ is the dioperad Lie$^2$Bi | smooth formal Poisson structure in $\mathbb{R}^n$ vanishing at 0 | structure, $(\hat{V} \oplus \hat{V}^* [1], \bar{\delta})$, of a smooth dg manifold together with an odd symplectic form $\omega_{odd}$ on $\hat{V} \oplus \hat{V}^* [1]$ such that the homological vector field $\bar{\delta}$ is hamiltonian and vanishes on $0 \oplus \hat{V}^* [1]$ |
| Genes: $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array}$, $\begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array}$ | Rules: $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} - \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} = 0$ | $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} + \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} = 0$ |
| | | $\begin{array}{c} \downarrow \downarrow \swarrow \searrow \end{array} + \begin{array}{c} \downarrow \downarrow \nwarrow \nearrow \end{array} + \begin{array}{c} \downarrow \downarrow \downarrow \uparrow \uparrow \end{array} + \begin{array}{c} \downarrow \downarrow \downarrow \uparrow \uparrow \end{array} = 0$ |

**Notations:** For a graded vector space $V$, $\hat{V}$ stands for the formal graded manifold (non-canonically) isomorphic to the formal neighbourhood of 0 in $V$, and $T_{\hat{V}}$ stands for the tangent bundle on $\hat{V}$.
• a collection of homogeneous tensors,

\[ \{ \mu_{n_1, \ldots, n_k} : T_M^{\otimes n_1} \otimes T_M^{\otimes n_2} \otimes \cdots \otimes T_M^{\otimes n_k} \to T_M[k + 1 - n_1 - \ldots - n_k] \}_{n_i \geq 1, n_i + k \geq 2} \]

satisfying an infinite tower of quadratic algebraic and differential equations. The first two floors of this tower read as follows: the data \( \{ \mu_1 \} \) (with \( \mu_1 := Lie_0 \)) makes the tangent sheaf \( T_M \) into a sheaf of \( C_\infty \) algebras\(^2\) satisfying an “integrability” condition,

\[ [\mu_\bullet, \mu_\bullet]_{G_\infty} = Lie_3 \mu_\bullet \]

for a certain bi-differential operator \([\cdot, \cdot]_{G_\infty}\) whose leading term is just the usual vector field bracket of values of \( \mu_\bullet \). It is also required that each tensor \( \mu_\bullet, \ldots, \mu_k : T_M^{\otimes k} \to T_M \) vanishes if the input contains at least one pure shuffle product,

\[ (v_1 \otimes \cdots \otimes v_k) \star (v_{k+1} \otimes \cdots \otimes v_n) := \sum_{\text{Shuffles } \sigma \text{ of type } (k,n)} (-1)^{Koszul(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad v_i \in T_M. \]

A change of the connection \( \nabla \) alters the tensors \( \mu_{\bullet_1, \ldots, \bullet_k} \), \( k \geq 2 \), but leaves the homotopy class of the \( G_\infty \)-structure on \( V \) invariant.

If the vector space \( V \) is concentrated in degree 0, i.e. \( V \simeq \mathbb{R}^n \), then a \( G_\infty \)-structure on \( V \) reduces just to a single tensor field \( \mu_2 : T_M^{\otimes 2} \to T_M \) which makes the tangent sheaf into a sheaf of commutative associative algebras, and satisfies the differential equations,

\[ [\mu_2, \mu_2]_{G_\infty} = 0. \]

The explicit form for the bracket \([\cdot, \cdot]_{G_\infty}\) can be read off from the \( G_\infty \) operad structural equations rather straightforwardly (see [Mer1] for details),

\[ [\mu_2, \mu_2]_{G_\infty}(X, Y, Z, W) = [\mu_2(X, Y), \mu_2(Z, W)] - \mu_2([\mu_2(X, Y), Z], W) - \mu_2(Z, [\mu_2(X, Y), W]) - \mu_2(X, [\mu_2(Z, W)]) - \mu_2([\mu_2(X, Z), \mu_2(Y, W)]) + \mu_2(X, \mu_2(Z, W)) + \mu_2(\mu_2([X, Z], \mu_2(Y, W))) + \mu_2([X, \mu_2(Z, W)], \mu_2(Y, W)) + \mu_2([X, Z], \mu_2(Y, W)) + \mu_2([X, \mu_2(Z, W)], \mu_2(Y, W)). \]

The resulting geometric structure is precisely the one discovered earlier by Hertling and Manin [HeMa] in their quest for a weaker notion of Frobenius manifold; they call it an \( F \)-manifold structure on \( V \).

Hertling-Manin’s geometric structures arise naturally in the theory of singularities [He] and the deformation theory [Mer1].

1.4. Germs of tensor fields.

A TF bialgebra is, by definition, a graded vector space \( V \) together with two linear maps,

\[ \delta \equiv \bigcup_a : V \otimes^2 V \rightarrow V[1], \quad [\bullet] \equiv \bigcup_a : V \otimes^2 V \rightarrow \mathbb{R}[a \otimes b] \]

satisfying the identities,

(i) \([a \bullet b] \bullet c] = [a \bullet [b \bullet c]] + (-1)^{|b||a|+|b||a|}[b \bullet [a \bullet c]] \) (Jacobi identity);

(ii) \(\delta[a \bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 + (-1)^{|a||b|+|a||b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a]) \) (Leibniz type identity).

\(^2\)\( C_\infty \) stands for the minimal resolution of the operad of commutative associative algebras.
There are obvious versions of the above notion with \( \delta \) taking values in \( \wedge^2 V \) and \( \odot^2 V \), i.e. with the gene realizing either the trivial or sign representations of \( \Sigma_2 \).

The dioperad whose algebras are TF bialgebras is denoted by TF. This quadratic dioperad is Koszul so that one can construct its minimal resolution using the results of [G, GiKa, Mar1]. It turns out that the structure of \( \text{TF}_\infty \)-algebra on a graded vector space \( V \) is the same as a pair of collections of linear maps,

\[
\{ \mu_n : \odot^n V \to V[1] \}_{n \geq 1},
\]
and

\[
\{ \phi_n : \odot^n V \to V \otimes V \}_{n \geq 1},
\]
satisfying a system of quadratic equations which are best described using a geometric language. Let \( M \) be the formal graded manifold associated to \( V \). If \( \{ e_{\alpha}, \alpha = 1, 2, \ldots \} \) is a homogeneous basis of \( V \), then the associated dual basis \( t^\alpha \), \( |t^\alpha| = -|e_\alpha| \), defines a coordinate system on \( M \). The collection of tensors \( \{ \mu_n \}_{n \geq 1} \) can be assembled into a germ, \( \partial \in T_M \), of a degree 1 smooth vector field,

\[
\partial := \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n t^{\alpha_1} \cdots t^{\alpha_n} \mu_{\alpha_1 \cdots \alpha_n} \frac{\partial}{\partial t^{\beta}} \frac{\partial}{\partial t^{\beta}}
\]
where

\[
\epsilon = \sum_{k=1}^{n} |e_{\alpha_k}| (1 + \sum_{i=1}^{k} |e_{\alpha_i}|)
\]
the numbers \( \mu_{\alpha_1 \cdots \alpha_n} \) are defined by

\[
\mu_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) = \sum \mu_{\alpha_1 \cdots \alpha_n} \beta e_{\beta},
\]
and we assume here and throughout the paper summation over repeated small Greek indices.

Another collection of linear maps, \( \{ \phi_n \} \), can be assembled into a smooth germ, \( \phi \in \otimes^2 T_M \), of a degree zero contravariant tensor field on \( M \),

\[
\phi := \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n t^{\alpha_1} \cdots t^{\alpha_n} \phi_{\alpha_1 \cdots \alpha_n} \beta_1 \beta_2 \frac{\partial}{\partial t^{\beta_1}} \otimes \frac{\partial}{\partial t^{\beta_2}}
\]
where

\[
\epsilon = |e_{\beta_2}|(|e_{\beta_1}| + 1) + \sum_{k=1}^{n} \sum_{i=1}^{k} |e_{\alpha_k}||e_{\alpha_i}|
\]
and the numbers \( \mu_{\alpha_1 \cdots \alpha_n} \beta_1 \beta_2 \) are defined by

\[
\mu_n(e_{\alpha_1}, \ldots, e_{\alpha_n}) = \sum \mu_{\alpha_1 \cdots \alpha_n} \beta_1 \beta_2 e_{\beta_1} \otimes e_{\beta_2}.
\]

1.4.1. Proposition. The collections of tensors,

\[
\{ \mu_n : \odot^n V \to V[1] \}_{n \geq 1} \quad \text{and} \quad \{ \phi_n : \odot^n V \to V \otimes V \}_{n \geq 1},
\]
define a structure of \( \text{TF}_\infty \)-algebra on \( V \) if and only if the associated smooth vector field \( \partial \) and the contravariant tensor field \( \phi \) satisfy the equations,

\[
[\partial, \partial] = 0
\]
and

\[
\text{Lie}_\partial \phi = 0,
\]
where \([ , ]\) stands for the usual bracket of vector fields and \( \text{Lie}_\partial \) for the Lie derivative along \( \partial \).
If \( V \) is finite dimensional and concentrated in degree zero, then a TF\(_{\infty}\)-structure in \( V \) is just a germ of a smooth rank 2 contravariant tensor on \( V \) vanishing at 0.

1.5. Poisson geometry and the dioperad of Lie 1-bialgebras.

A \textit{Lie 1-bialgebra} is, by definition, a graded vector space \( V \) together with two linear maps,

\[
\delta : V \rightarrow \wedge^2 V, \quad [\bullet] : \wedge^2 V \rightarrow V[1]
\]

satisfying the identities,

(i) \((\delta \otimes \text{Id})\delta a + \tau(\delta \otimes \text{Id})\delta a + \tau^2(\delta \otimes \text{Id})\delta a = 0\), where \( \tau \) is the cyclic permutation (123) represented naturally on \( V \otimes V \otimes V \) (co-Jacobi identity);

(ii) \([a \bullet b] \bullet c = [a \bullet [b \bullet c]] + (-1)^{|a||b|+|a|}[a \bullet [b \bullet c]]\) (Jacobi identity);

(iii) \(\delta[a \bullet b] = \sum a_1 \wedge [a_2 \bullet b] - (-1)^{|a_1||a_2|}a_2 \wedge [a_1 \bullet b] + [a \bullet a_1] \wedge b_2 - (-1)^{|b_1||a_2|}a \bullet [b_2 \bullet b_1] \wedge b_1\) (Leibniz type identity).

The dioperad whose algebras are Lie 1-bialgebras is denoted by \( \text{Lie}^1\text{Bi} \). The superscript 1 in the notation is used to emphasize that the two basic operations

\[
\delta = \bigwedge, \quad [\bullet] = \bigcirc
\]

have homogeneities differed by 1.

Similarly one can introduce the notion of \textit{Lie n-bialgebras}: coLie algebra structure on \( V \) plus Lie algebra structure on \( V[−n] \) plus an obvious Leibniz type identity. Homotopy theory of Lie n-bialgebras splits into two stories, one for \( n \) even, and one for \( n \) odd. The even case (more precisely, the case \( n = 0 \)) has been studied by Gan [G]. In this paper we study the odd case, more precisely, the case \( n = 1 \).

The dioperad \( \text{Lie}^1\text{Bi}_\infty \) is Koszul. Hence one can use the machinery of [G, GiKa, Mar1] to construct its minimal resolution, the dioperad \( \text{Lie}^1\text{Bi}_\infty \). The structure of a \( \text{Lie}^1\text{Bi}_\infty \) algebra on a graded vector space \( V \) is a collection linear maps,

\[
\{\mu_{m,n} : \wedge^m V[2-m]\}_{m \geq 1, n \geq 1}
\]

satisfying a system of quadratic equations which can be described as follows. Let \( M \) be the formal graded manifold associated to \( V \). If \( \{e_\alpha, \alpha = 1, 2, \ldots\} \) is a homogeneous basis of \( V \), then the associated dual basis \( t^\alpha \), \( |t^\alpha| = -|e_\alpha| \), defines a coordinate system on \( M \). For a fixed \( m \) the collection of tensors \( \{\mu_{m,n}\}_{n \geq 1} \) can be assembled into a germ, \( \Gamma_m \in \wedge^m T_M \), of a smooth polyvector field (vanishing at \( 0 \in M \)),

\[
\Gamma_m := \sum_{n=1}^\infty \frac{1}{m!n!}(-1)^t \alpha_1 \cdots \alpha_n \mu_{\alpha_1 \ldots \alpha_n} \beta_1 \cdots \beta_m \frac{\partial}{\partial t^{\beta_1}} \wedge \cdots \wedge \frac{\partial}{\partial t^{\beta_m}}
\]

where

\[
\epsilon = \sum_{k=1}^n |e_{\alpha_k}|(2 - m + \sum_{i=1}^k |e_{\alpha_i}|) + \sum_{k=1}^n (|e_{\beta_k}| + 1) \sum_{i=k+1}^n |e_{\beta_i}|
\]

and the numbers \( \mu_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_m} \) are defined by

\[
\mu_{m,n}(e_{\alpha_1}, \ldots, e_{\alpha_n}) = \sum \mu_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_m} e_{\beta_1} \wedge \cdots \wedge e_{\beta_m}.
\]

1.5.1. Proposition. \textit{A collection of tensors,} \( \{\mu_{m,n} : \wedge^m V[2-m]\}_{m \geq 1, n \geq 1} \), \textit{defines a structure of \( \text{Lie}^1\text{Bi}_\infty \)-algebra on} \( V \) \textit{if and only if the associated smooth polyvector field,}

\[
\Gamma := \sum_{m \geq 1} \Gamma_m \in \wedge^* T_M,
\]
satisfies the equation

$$[\Gamma, \Gamma] = 0,$$

where $[,]$ stands for the Schouten bracket of polyvector fields.

In particular, if $V$ is concentrated in degree zero, then the only non-zero summand in $\Gamma$ is $\Gamma_2 \in \wedge^2 T_M$. Hence a $\text{Lie}^3\text{Bi}_\infty$-algebra structure on $\mathbb{R}^n$ is nothing but a germ of a smooth Poisson structure on $\mathbb{R}^n$ vanishing at 0.

1.6. On the content of the rest. Section 2 is a reminder on PROPs, dioperads and Koszulness [G, GiKa, Mar1]. In Sections 3 and 4 we prove Koszulness of the dioperads $\text{Lie}^3\text{Bi}$ and $\text{TF}_\infty$, apply the machinery reviewed in Section 2 to give explicit graph descriptions of their minimal resolutions, $\text{Lie}^3\text{Bi}_\infty$ and $\text{TF}_\infty$, prove Propositions 1.4.1 and 1.5.1 and introduce and study the notion of $\text{Lie}^3\text{Bi}_\infty$ morphisms. Section 5 is a comment on a geometric description of algebras over the dioperad of strongly homotopy Lie bialgebras, and their strongly homotopy maps.

2. PROPs and dioperads [G]. Let $S_f$ be the groupoid of finite sets. It is equivalent to the category whose objects are natural numbers, $\{m\}_{m \geq 1}$, and morphisms are the permutation groups $\{\Sigma_m\}_{m \geq 1}$.

A PROP $P$ in the category, $\text{dgVec}$, of differential graded (shortly, dg) vector spaces is a functor $P : S_f \times S_f^{op} \to \text{dgVec}$ together with natural transformations,

$$\circ_{A,B,C} : \mathcal{P}(A, B) \otimes \mathcal{P}(B, C) \to \mathcal{P}(A, C),$$

$$\otimes_{A,B,C,D} : \mathcal{P}(A, B) \otimes \mathcal{P}(C, D) \to \mathcal{P}(A \otimes B, C \otimes D)$$

and the distinguished elements $\text{Id}_A \in \mathcal{P}(A, A)$ and $s_{A,B} \in \mathcal{P}(A \otimes B, B \otimes A)$ satisfying a system of axioms [A] which just mimic the obvious properties of the following natural transformation,

$$\mathcal{E}_V : (m, n) \to \text{Hom}(V^\otimes n, V^\otimes m),$$

canonically associated with an arbitrary dg space $V$. The latter fundamental example is called the endomorphism PROP of $V$.

Given a collection of dg $(\Sigma_m, \Sigma_n)$-bimodules, $E = \{E(m, n)\}_{m,n \geq 1}$, one can construct the associated free PROP, $\text{Free}(E)$, by decorating vertices of all possible directed graphs with a flow by the elements of $E$ and then taking the colimit over the graph automorphism group. The composition operation $\circ$ corresponds then to gluing output legs of one graph to the input legs of another graph, and the tensor product $\otimes$ to the disjoint union of graphs. Even for a small finite dimensional collection $E$ the resulting free PROP can be a monstrous infinite dimensional object. The notion of dioperad was introduced by Gan [G] as a way to avoid that free PROP “divergence”. In the above setup, a free dioperad on $E$ is built on graphs of genus zero, i.e. on trees.

More precisely, a dioperad $P$ consists of data:

(i) a collection of dg $(\Sigma_m, \Sigma_n)$-bimodules, $\{P(m, n)\}_{m \geq 1, n \geq 1}$;

(ii) for each $m_1, n_1, m_2, n_2 \geq 1$, $i \in \{1, 2, \ldots, m_1\}$ and $j \in \{1, \ldots, n_1\}$ a linear map

$$i^{\circ j} : P(m_1, n_1) \otimes P(m_2, n_2) \to P(m_1 + m_2 - 1, n_1 + n_2 - 1),$$

(iii) a morphism $e : k \to P(1, 1)$ such that the compositions

$$k \otimes P(m, n) \xrightarrow{e \otimes \text{Id}} P(1, 1) \otimes P(m, n) \xrightarrow{\text{Id} \otimes e} P(m, n)$$

and

$$P(m, n) \otimes k \xrightarrow{\text{Id} \otimes e} P(m, n) \otimes P(1, 1) \xrightarrow{\text{Id} \otimes e} P(m, n)$$

are the canonical isomorphisms for all $m, n \geq 1$, $1 \leq i \leq m$ and $1 \leq j \leq n$. 

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These data satisfy associativity and equivariance conditions [G] which can be read off from the example of the endomorphism dioperad $\mathcal{End}_V$ with $\mathcal{End}_V(m,n) = \text{Hom}(V^\otimes n, V^\otimes m)$, $e : 1 \to \text{Id} \in \text{Hom}(V,V)$, and the compositions given by

$$i \circ j : \mathcal{P}(m_1, n_1) \otimes \mathcal{P}(m_2, n_2) \to \mathcal{P}(m_1 + m_2 - 1, n_1 + n_2 - 1)$$

$$f \otimes g \to (\text{Id} \otimes \ldots \otimes f \otimes \ldots \otimes \text{Id})\sigma(\text{Id} \otimes \ldots \otimes g \otimes \ldots \otimes \text{Id}),$$

where $f$ (resp. $g$) is at the $j$th (resp. $i$th) place, and $\sigma$ is the permutation of the set $I = \{1, 2, \ldots, n_1 + m_2 - 1\}$ swapping the subintervals, $I_1 \leftrightarrow I_2$ and $I_4 \leftrightarrow I_5$, of the unique order preserving decomposition, $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 \sqcup I_5$, of $I$ into the disjoint union of five intervals of lengths $|I_1| = i - 1$, $|I_2| = j - 1$, $|I_3| = 1$, $|I_4| = m_2 - j$ and $|I_5| = n_1 - i$.

If $\mathcal{P}$ is a dioperad, then the collection of $(\Sigma_m, \Sigma_n)$ bimodules,

$$\mathcal{P}^{op}(m,n) := (\mathcal{P}(n,m), \text{transposed actions of } \Sigma_m \text{ and } \Sigma_n),$$

is naturally a dioperad as well.

If $\mathcal{P}$ is a dioperad with $\mathcal{P}(m,n)$ vanishing for all $m,n$ except for $(m = 1, n \geq 1)$, then $\mathcal{P}$ is called an operad.

A morphism of dioperads, $F : \mathcal{P} \to \mathcal{Q}$, is a collection of equivariant linear maps, $F(m,n) : \mathcal{P}(m,n) \to \mathcal{Q}(m,n)$, preserving all the structures. If $\mathcal{P}$ is a dioperad, then a $\mathcal{P}$-algebra is a dg vector space $V$ together with a morphism, $F : \mathcal{P} \to \mathcal{End}_V$, of dioperads.

We shall consider below only dioperads $\mathcal{P}$ with $\mathcal{P}(m,n)$ being finite dimensional vector spaces (over a field $k$ of characteristic zero) for all $m,n$.

The endomorphism dioperad of the vector space $k[-p]$, $p \in \mathbb{Z}$, is denoted by $\langle p \rangle$. Thus $\langle p \rangle(m,n)$ is $\text{sgn}^\otimes_p \otimes \text{sgn}^\otimes_{-p}[p(n - m)]$ where $\text{sgn}_m$ stands for the one dimensional sign representation of $\Sigma_m$. Representations of the dioperad $\mathcal{P}(p) : = \mathcal{P} \otimes \langle p \rangle$ in a vector space $V$ are the same as representations of the dioperad $\mathcal{P}$ in $V[p]$.

If $\mathcal{P}$ is a dioperad, then $\Lambda \mathcal{P} := \{\text{sgn}_m \otimes \mathcal{P}(m,n)[2 - m - n] \otimes \text{sgn}_n\}$ and $\Lambda^{-1} \mathcal{P} := \{\text{sgn}_m \otimes \mathcal{P}(m,n)[m + n - 2] \otimes \text{sgn}_n\}$ are also dioperads.

### 2.1. Cobar dual.

If $T$ is a directed (i.e. provided with a flow which we always assume in our pictures to go from the bottom to the top) tree, we denote by

- $\text{Vert}(T)$ the set of all vertices,
- $\text{edge}(T)$ the set of internal edges; $\text{det}(T) := \wedge |\text{edge}(T)| \text{span}_k(\text{edge}(T))$;
- $\text{Edge}(T)$ is the set of all edges, i.e.
  $$\text{Edge}(T) := \text{edge}(T) \sqcup \{\text{input legs (leaves)}\} \sqcup \{\text{output legs (roots)}\};$$
- $\text{Det}(T) := \wedge |\text{Edge}(T)| \text{span}_k(\text{Edge}(T))$;
- $\text{Out}(v)$ (resp. $\text{In}(v)$) the set of outgoing (resp. incoming) edges at a vertex $v \in \text{Vert}(V)$.

An $(m,n)$-tree is a tree $T$ with $n$ input legs labeled by the set $[n] = \{1, \ldots, n\}$ and $m$ output legs labeled by the set $[m] = \{1, \ldots, m\}$. A tree $T$ is called trivalent if $|\text{Out}(v) \sqcup \text{In}(v)| = 3$ for all $v \in \text{Vert}(T)$.

Let $E = \{E(m,n)\}_{m,n \geq 1}$ be a collection of finite dimensional $(\Sigma_m, \Sigma_n)$ bimodules with $E_{1,1} = 0$. For a pair of finite sets, $I, J \in \text{Objects}(\mathcal{S}_t)$, with $|I| = m$ and $|J| = n$, one defines

$$E(I,J) := \text{Hom}_{\Sigma_t}([m], I) \otimes_{\Sigma_m} E(m,n) \otimes_{\Sigma_n} \text{Hom}_{\Sigma_t}(J, [n]).$$

The free dioperad, $\text{Free}(E)$, generated by $E$ is defined by

$$\text{Free}(E)(m,n) := \bigoplus_{(m,n)-\text{trees } T} E(T),$$

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where

$$E(T) := \bigotimes_{v \in \text{Vert}(T)} E(\text{Out}(v), \text{In}(v)),$$

and the compositions \(i \circ_j\) are given by grafting the \(j\)th root of one tree into \(i\)th leaf of another tree, and then taking the “unordered” tensor product \([\text{MSS}]\) over the set of vertices of the resulting tree.

Let \(\mathcal{P} = \{\mathcal{P}(m,n)\}_{m,n \geq 1}\) be a collection of graded \((\Sigma_m, \Sigma_n)\) bimodules. We denote by \(\bar{\mathcal{P}}\) the collection \(\{\mathcal{P}(m,n)\}_{m,n \geq 1}\) given by \(\mathcal{P}(m,n) := \mathcal{P}(m,n)\) for \(m + n \geq 3\) and \(\mathcal{P}(1,1) = 0\). The collection of dual vector spaces, \(\mathcal{P}^* = \{\mathcal{P}(m,n)^*\}_{m,n \geq 1}\), is naturally a collection of \((\Sigma_m, \Sigma_n)\)-bimodules with the transposed actions. We also set \(\mathcal{P}^\vee = \{\mathcal{P}(m,n)^\vee := \text{sgn}_m \otimes \mathcal{P}(m,n)^* \otimes \text{sgn}_n\}\).

Let \(\mathcal{P}\) be a graded dioperad with zero differential. The \textit{cobar dual} of \(\mathcal{P}\) is the dg dioperad \(\mathcal{D}\mathcal{P}\) defined by

(i) as a dioperad of graded vector spaces, \(\mathcal{D}\mathcal{P} = \Lambda^{-1} \text{Free}(\mathcal{P}^*[-1]) = \text{Free}(\Lambda^{-1}\mathcal{P}^*[-1])\);

(ii) as a complex, \(\mathcal{D}\mathcal{P}(m,n) = \sum_{i=0}^{m+n-3} \mathcal{D}\mathcal{P}^{-i}(m,n)\) with the differential given by dualizations of the compositions \(\mathcal{P}^\bullet\), and edge contractions \([\text{G, GiKa}]\),

\[
\begin{align*}
\mathcal{D}\mathcal{P}^{3-m-n}(m,n) &\overset{d}{\longrightarrow} \mathcal{D}\mathcal{P}_{4-m-n}(m,n) \\
\mathcal{P}^\vee(m,n) &\overset{d}{\longrightarrow} \bigoplus_{|\text{Det}(T)|=1} \mathcal{P}^* \otimes \text{Det}(T) \\
\mathcal{D}\mathcal{P}^{3-m-n}(m,n) &\overset{d}{\longrightarrow} \ldots \overset{d}{\longrightarrow} \mathcal{D}\mathcal{P}^0(m,n)
\end{align*}
\]

where the sums are taken over \((m,n)\)-trees.

\textbf{2.2. Remark.} The vector space \(\mathcal{D}\mathcal{P}\) is bigraded: one grading comes from the grading of \(\mathcal{P}\) as a vector space and another one from trees as in (ii) just above. The differential preserves the first grading and increases by 1 the second one. The \(\mathbb{Z}\)-grading of \(\mathcal{D}\mathcal{P}\) is always understood to be the associated total grading. In particular, \(\deg_{\mathcal{D}\mathcal{P}}\mathcal{P}^\vee(m,n) = \deg_{\text{Vect}}(\mathcal{P}^\vee(m,n)[m + n - 3])\).

\textbf{2.3. Koszul dioperads.} A \textit{quadratic dioperad} is a dioperad \(\mathcal{P}\) of the form

\[
\mathcal{P} = \frac{\text{Free}(E)}{\text{Ideal} < R >},
\]

where \(E = \{E(m,n)\}\) is a collection of finite dimensional \((\Sigma_m, \Sigma_n)\)-bimodules with \(E(m,n) = 0\) for \((m,n) \neq (1,2), (2,1)\), and the \textit{Ideal} in \(\text{Free}(E)\) is generated by a collection, \(R\), of three sub-bimodules \(R(1,2) \subset \text{Free}(E)(1,2), R(2,1) \subset \text{Free}(E)(2,1)\) and \(R(2,2) \subset \text{Free}(E)(2,2)\). The \textit{quadratic dual} dioperad, \(\mathcal{P}^!\), is then defined by

\[
\mathcal{P}^! = \frac{\text{Free}(E^!)}{\text{Ideal} < R^! >},
\]

where \(R^!\) is the collection of the three sub-bimodules \(R^!(i,j) \subset \text{Free}(E^!)(i,j)\) which are annihilators of \(R(i,j), (i,j) = (1,2), (2,2), (2,1)\).

Clearly, \(\mathcal{D}\mathcal{P}^0 = \text{Free}(E^!)\) so that there is a natural epimorphism

\[
\mathcal{D}\mathcal{P}^0 \longrightarrow \mathcal{P}^!.
\]

Its kernel is precisely \(\text{Im} d(\mathcal{D}\mathcal{P}^{-1})\). Hence \(H^i(\mathcal{D}\mathcal{P}) = \mathcal{P}^!\). The quadratic operad \(\mathcal{P}\) is called \textit{Koszul} if the above morphism is a quasi-isomorphism, i.e. \(H^i(\mathcal{D}\mathcal{P}) = 0\) for all \(i < 0\). In that case the operad \(\mathcal{D}\mathcal{P}^!\) provides us with a \textit{minimal resolution} of the operad \(\mathcal{P}\) and is often denoted by \(\mathcal{P}_\infty\). Algebras over \(\mathcal{P}_\infty\) are often called \textit{strong homotopy} \(\mathcal{P}\)-algebras; their most important property is that they can be transferred via quasi-isomorphisms of complexes \([\text{Mar2}]\).

\textbf{2.4. Koszulness criterion.} An \((m,n)\)-tree \(T\) is called \textit{reduced} if each vertex has

- either an outgoing root or at least two outgoing internal edges, and/or
- either an incoming leaf or at least two incoming internal edges.
For a collection, \( E = \{E_{m,n}\}_{m,n \geq 1} \), of \((\Sigma_m, \Sigma_n)\)-bimodules define another collection of \((\Sigma_m, \Sigma_n)\)-bimodules as follows,

\[
\text{Free}(E)(m,n) := \bigoplus_{\text{reduced } (m,n)\text{-trees}} \text{Free}(E)(T).
\]

Let \( P \) be a quadratic dioperad, i.e. \( P = \text{Free}(E)/\text{Ideal } <R> \)
for some generators \( E = \{E(1,2), E(2,1)\} \) and relations \( R = \{R(1,3), R(2,2), R(3,1)\} \). With \( P \) one can canonically associate two quadratic operads, \( P_L \) and \( P_R \), such that

\[
P_L = \frac{\text{Free}(E(1,2))}{\text{Ideal } <R(1,3)>}, \quad P_R^{op} = \frac{\text{Free}(E(2,1))}{\text{Ideal } <R(2,1)>}.
\]

Let us denote by \( P_L \circ P_R^{op} \) the collection of \((\Sigma_m, \Sigma_n)\)-bimodules given by

\[
P_L \circ P_R^{op}(m,n) := \begin{cases} 
P_L(1,n) & \text{if } m = 1, n \geq 1; \\
P_R^{op}(m,1) & \text{if } n = 1, m \geq 1; \\
0 & \text{otherwise.}
\end{cases}
\]

2.4.1. Theorem [G, Mar1, MV]. A quadratic dioperad \( P \) is Koszul if the operads \( P_L \) and \( P_R \) are Koszul and

\( P(i,j) = \text{Free}(P_L \circ P_R^{op})(i,j) \)
for \( (i,j) = (1,3), (2,2), (3,1) \). Moreover, in this case \( P(m,n) = \text{Free}(P_L \circ P_R^{op})(m,n) \) for all \( m,n \geq 1 \).

3. A minimal resolution of \( \text{Lie}^1\text{Bi} \). First we present a graph description of the dioperad \( \text{Lie}^1\text{Bi} \); it will pay off when we discuss \( \text{Lie}^1\text{Bi}_\infty \). By definition (see Sect. 1.5), \( \text{Lie}^1\text{Bi} \) is a quadratic dioperad,

\[
\text{Lie}^1\text{Bi} = \frac{\text{Free}(E)}{\text{Ideal } <R>},
\]

where

(i) \( E(2,1) := sgn_2 \otimes 1_1 \) and \( E(1,2) := 1_1 \otimes 1_2[-1] \), where \( 1_n \) stands for the one dimensional trivial representation of \( \Sigma_n \); let \( \delta \in E(2,1) \) and \( [\bullet] \in E(1,2) \) be basis vectors; we can represent both as directed\(^3\) plane corollas,

\[
\delta = \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array}, \quad [\bullet] = \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array},
\]

with the following symmetries,

\[
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} = - \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array}, \quad \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array} = - \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array};
\]

(ii) the relations \( R \) are generated by the following elements,

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} + \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad \in \text{Free}(E)(3,1)
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array} + \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array} + \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow \\
\end{array} \quad \in \text{Free}(E)(1,3)
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad + \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} + \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\end{array} \quad \in \text{Free}(E)(2,2).
\end{array}
\]

Footnote: In all our graphs the direction of edges is chosen to go from the bottom to the top.
3.1. Proposition. Lie¹Bi is Koszul.

Proof. We have Lie¹BiL = Lie ⊗ {1} and Lie¹BiR = Lie, where Lie stands for the operad of Lie algebras and
\[ \{m\} := \{ (m)(n) := sgn_n^\od m [m(n-1)] \}_{n\geq 1} \]
for the endomorphism operad of k[−m]. As Lie is Koszul [GiKa], both the operads Lie¹BiL and Lie¹BiR are Koszul as well. Next, a straightforward analysis of all calculational schemes in Lie¹Bi represented by directed trivalent \((i, j)-trees with i + j = 5\) shows that they generate no new relations so that
\[
\text{Lie}¹Bi(i, j) = \text{Free} (\text{Lie} \otimes \{1\} \circ \text{Lie}^\od) (i, j)
\]
for \((i, j) = (1, 3), (2, 2), (3, 1)\). Hence by Theorem 2.4.1, the dioperad Lie¹Bi is Koszul. \(\square\)

Proposition 1.5.1 is a straightforward corollary of the following

3.2. Theorem. The minimal resolution, Lie¹Biso, of the dioperad Lie¹Bi can be described as follows.

(i) As a dioperad of graded vector spaces, Lie¹Biso = Free(E), where the collection, \(E = \{E(m, n)\}\), of one dimensional \((\sum_m, \sum_n)\)-modules is given by
\[
E(m, n) := \begin{cases} sgn_m \otimes 1_n [m-2] & \text{if } m + n \geq 3; \\ 0 & \text{otherwise.} \end{cases}
\]

(ii) If we represent a basis element of \(E(m, n)\) by the unique (up to a sign) planar \((m, n)\)-corolla,

with skew-symmetric outgoing legs and symmetric ingoing legs, then the differential \(d\) is given on generators by
\[
d = \sum_{I_1 \cup I_2 = (1, \ldots, m)} (-1)\sigma(I_1 \cup I_2) + |I_1||I_2|
\]
where \(\sigma(I_1 \cup I_2)\) is the sign of the shuffle \(I_1 \cup I_2 = (1, \ldots, m)\).

Proof. Claim (i) follows from the fact that Lie¹Bi¹(m, n) = 1_m \otimes sgn_n[1 - n] and Remark 2.2. Claim 2 is a straightforward though tedious graph translation of the initial term,
\[
(Lie¹Bi¹)\uparrow (m, n) \to d \bigoplus_{(m, n) - \text{trans} \geq 1, \left|\text{edge}(T)\right| = 1} (Lie¹Bi¹)^* \otimes \text{Det}(T),
\]
of the definition 2.1 of the differential \(d\) in DLie¹Bi¹. \(\square\)

3.3. A geometric model for Lie¹Biso structures. Let \(V\) be a finite dimensional graded vector space. Then the graded formal manifold, \(\mathcal{M}\), modeled on the infinitesimal neighbourhood of 0 in the vector space \(V \oplus V^*[1]\) has an odd symplectic form \(\omega\) induced from the natural pairing \(V \oplus V^*[1] \to k[1]\). In particular, the graded structure sheaf \(O_{\mathcal{M}}\) on \(\mathcal{M}\) has a degree −1 Poisson bracket, \(\{ \bullet \}\), such that
\[
\{f \bullet g\} = (-1)^{|f||g|+|f|+|g|} \{g \bullet f\}
\]
and the Jacobi identity is satisfied. The odd symplectic manifold \((\mathcal{M}, \omega)\) has two particular Lagrangian submanifolds, \(\mathcal{L} \subset \mathcal{M}\) and II\(\mathcal{L} \subset \mathcal{M}\) associated with, respectively, the subspaces \(0 \oplus V^*[1] \subset V \oplus V^*[1]\) and \(V \oplus 0 \subset V \oplus V^*[1]\).
3.3.1 Proposition. A Lie$^1$Bi$_\infty$ algebra structure in a graded vector space $V$ is the same as a degree two smooth function $\Gamma \in \mathcal{O}_M$ vanishing on $\mathcal{L} \cup \Pi \mathcal{L}$ and satisfying the equation $\{ \Gamma \cdot \Gamma \} = 0$.

Proof. The manifold $M$ is isomorphic to the total space of the shifted cotangent bundle, $T^*_M[1]$, of the manifold $M$ of Proposition 1.5.1. Hence smooth functions on $M$ are the same as smooth polyvector fields on $M$, and the Poisson bracket $\{ \cdot \}$ on $M$ is the same as the Schouten bracket on $M$. \hfill $\square$

3.4. Lie$^1$Bi$_\infty$ morphisms. Let $(V, \{ \mu_{m,n} \})$ and $(V', \{ \mu'_{m,n} \})$ be two Lie$^1$Bi$_\infty$ algebras.

3.4.1. Definition. A Lie$^1$Bi$_\infty$ morphism $F : V \to V'$ is, by definition, a symplectomorphism, $F : (\mathcal{M}, \omega) \to (\mathcal{M}', \omega')$ such that $F(\mathcal{L}) \subset \mathcal{L}'$, $F(\Pi \mathcal{L}) \subset \Pi \mathcal{L}'$ and $F^*\Gamma' = \Gamma$.

Thus a Lie$^1$Bi$_\infty$ morphism $F : V \to V'$ is a pair of collections of linear maps,

\[
\{ F_{m,n} : \mathcal{O}^m V \otimes \wedge^n V^* \to V'[\neg n] \}_{m \geq 1, n \geq 0}, \quad \{ \bar{F}_{m,n} : \mathcal{O}^m V \otimes \wedge^n V^* \to V'[1-n] \}_{n \geq 0, m \geq 1}
\]

satisfying the system equations, $F^*(\omega') = \omega$ and $F^*\Gamma' = \Gamma$. In particular, the equation $F^*(\omega') = \omega$ says that the linear maps,

\[
F_{1,0} : (V, \mu_{1,1}) \to (V', \mu'_{1,1}) \quad \text{and} \quad \bar{F}_{0,1} : (V^*, \mu^*_{1,1}) \to (V'^*, \mu'^*_{1,1})
\]

are morphisms of complexes, while the equation $F^*(\omega') = \omega$ says that the composition,

\[
F_{0,1} \circ \bar{F}_{1,0} : V \to V
\]

is the identity map.

3.4.2. Definition. A Lie$^1$Bi$_\infty$-morphism $F : V \to V'$ is called a quasi-isomorphism if the morphisms of complexes

\[
F_{1,0} : (V, \mu_{1,1}) \to (V', \mu'_{1,1}) \quad \text{and} \quad \bar{F}_{0,1} : (V^*, \mu^*_{1,1}) \to (V'^*, \mu'^*_{1,1})
\]

induce isomorphisms in cohomology.

3.4.3. Remark. One might get an impression that the notions introduced above make sense only for finite dimensional Lie$^1$Bi$_\infty$ algebras. However, this is no more than an artifact of the geometric intuition we tried to rely on in our definitions. In fact, everything above (and below) make sense for infinite dimensional representations as well. For example, one can replace ($\ast$) by

\[
\{ F_{m,n} : \mathcal{O}^m V \to \wedge^n V \otimes V'[\neg n] \}_{m \geq 1, n \geq 0}, \quad \{ \bar{F}_{m,n} : \mathcal{O}^m V \otimes V'[n-1] \to \wedge^n V \}_{n \geq 0, m \geq 1},
\]

and reinterpret the equations defining Lie$^1$Bi$_\infty$ morphism accordingly. For example, with this reinterpretation it is the morphism $F_{1,0} \circ \bar{F}_{0,1}$ which is the identity map.

3.4.4. Contractible and minimal Lie$^1$Bi$_\infty$-structures. Let $V$ be a graded vector space and $(\mathcal{M} = V \oplus V^*[1], \omega)$ the associated odd symplectic manifold (as in Sect. 3.3). There is a one-to-one correspondence between differentials, $d : V \to V$, and quadratic degree 2 function, $\Gamma_{quad}$, on $(\mathcal{M}, \omega)$, vanishing on $\mathcal{L} \cup \Pi \mathcal{L}$ and satisfying $[\Gamma_{quad} \cdot \Gamma_{quad}] = 0$. If $H^*(V, d) = 0$, the associated data, $(\mathcal{M}, \omega, \Gamma_{quad})$, is called a contractible Lie$^1$Bi$_\infty$-structure on $V$.

A Lie$^1$Bi$_\infty$ structure, $(\mathcal{M}, \omega, \Gamma)$, on $V$ is called minimal if $\Gamma = 0 \mod P^3$, where $I$ is the ideal of the distinguished point, $\mathcal{L} \cap \Pi \mathcal{L}$, in $\mathcal{M}$. Put another way, the formal power series $\Gamma$ in some (and hence any) coordinate system on $\mathcal{M}$ begins with cubic terms at least.

3.4.5. Theorem (homotopy classification of Lie$^1$Bi$_\infty$-structures, cf. [Ko1, Ko2]). Each Lie$^1$Bi$_\infty$ algebra is isomorphic to the tensor product of a contractible Lie$^1$Bi$_\infty$ algebra and a minimal one.

Proof. Let $(\mathcal{M}, \omega, \Gamma)$ be the geometric equivalent of any given Lie$^1$Bi$_\infty$ algebra. To prove the statement we have to construct coordinates, $(x^a, y^a, z^a, \psi_a, \phi_a, \xi_a)$, on $\mathcal{M}$ such that
(i) \( \omega = \left( \sum_a (dx^a \wedge d\psi_a + dy^a \wedge d\phi_a) + \sum \omega_i dz^\alpha \wedge d\xi_\alpha, \right) \)

(ii) \( \mathcal{L} \) is given by \( x^a = y^a = z^\alpha = 0 \) while \( \Pi \mathcal{L} \) is given by \( \psi_a = \phi_\alpha = \xi_\alpha = 0 \),

(iii) \( \Gamma = \sum \gamma_1^a \psi_\alpha + \Phi(z^\alpha, \xi_\alpha) \) for some formal power series \( \Phi(z^\alpha, \xi_\alpha) \) which begins with cubic terms at least.

For then \( (\mathcal{M}, \omega, \Gamma) \simeq (\mathcal{M}_1, \omega_1, \Gamma_1) \times (\mathcal{M}_2, \omega_2, \Gamma_2) \) with the first factor being a contractible Lie-Bi\( \infty \) structure while the second factor a minimal one.

We shall establish existence of the above coordinates by induction.

As the first step of the induction procedure we choose arbitrary linear coordinates, \( \{t^A\}, A \in \{1, \ldots, \dim V\} \), on \( V \) and the associated dual coordinates, \( \{\chi_A\}, |\chi_A| = -|t^A| + 1 \), on \( V^*[1] \). The odd symplectic form is given in these coordinates as \( \omega = \sum_A dt^A \wedge d\chi_A \), \( \mathcal{L} \) is given by \( t^A = 0 \) while \( \Pi \mathcal{L} \) is given by \( \chi_A = 0 \). Then,

\[
\Gamma = \sum_{A,B} C^A_B t^A \chi_B \mod I^3,
\]

for some constants \( C^A_B \), which are nothing but the coefficients of the differential, \( d : V \to V \), associated to the quadratic bit of \( \Gamma \). As we work over the field of characteristic zero, we can choose a cohomological decomposition of \( V \) with respect to this differential,

\[
V = H(V, d) \oplus B \oplus B[-1],
\]

so that \( d \) vanishes on \( H(V, d) \) and \( B[-1] \) and, on the remaining summand \( B \), it is equal to the natural isomorphism \( B \to B[-1] \). Let \( \{z^\alpha\} \) be some linear coordinates in \( H(V, d) \), \( \{y^a\} \) linear coordinates on \( B \) and \( \{x^a\}, |x^a| = |y^a| - 1 \), the associated (via the natural isomorphism) linear coordinates in \( B[-1] \). Denote by \( (\xi_\alpha, \psi_a, \phi_0) \) the coordinates on \( V^*[1] \) dual to \( (z^\alpha, x^a, y^a) \). In the resulting coordinate system on \( \mathcal{M} \) the conditions (i)-(ii) are satisfied, while the condition (iii) is satisfied modulo \( I^3 \).

Assume by induction that we have constructed a coordinate system on \( \mathcal{M} \) in which conditions (i)-(iii) are satisfied \mod \( I^{N+1} \). Then we have,

\[
\Gamma = \sum_a y^a \psi_a + \Phi(z^\alpha, \xi_\alpha) + \Gamma^{N+1}(x, y, z, \psi, \phi, \chi) \mod I^{N+2}.
\]

The equation \( [\Gamma \circ \Gamma] = 0 \mod I^{N+2} \) implies,

\[
\delta \Gamma^{N+1} = 0
\]

where \( \delta \) is the following differential on \( \mathcal{O}_\mathcal{M} \),

\[
\delta : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}, \quad f \mapsto [\sum_a y^a \psi_a \bullet f].
\]

Let \( B \in \mathcal{O}_\mathcal{M} \) be an arbitrary polynomial of degree \( N + 1 \) and with \( |B| = 1 \). It gives rise to a symplectomorphism, \( F : \mathcal{M} \to \mathcal{M} \), given as \( \exp v_B \) with the vector field \( v_B \), defined by

\[
v_B : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}, \quad f \mapsto [B \bullet f].
\]

One has,

\[
F^* \Gamma = \sum_a y^a \psi_a + \Phi(z^\alpha, \xi_\alpha) + \Gamma^{N+1} + \delta B \mod I^{N+2}.
\]
Thus $\Gamma^{N+1}(x, y, z, \psi, \phi, \chi)$ is a $\delta$-cycle defined up to a $\delta$-coboundary. As cohomology of $\delta$ in $\mathcal{O}_M$ is equal to $k[[z^\alpha, \xi_\alpha]]$, one can always find $\Gamma^{N+1}$ such that it is a function of $\{z^\alpha, \xi_\alpha\}$ only. 

3.4.6. Corollary. If $F : V \to V'$ is a $\text{Lie}_1 \text{Bi}_\infty$ quasi-isomorphism, then there exists a $\text{Lie}_1 \text{Bi}_\infty$ quasi-isomorphism $G : V' \to V$ such that on the cohomology level $[F_{1,0}] = [G_{0,1}]^*$ and $[G_{1,0}] = [F_{0,1}]^*$.

Proof is exactly the same as the proof of an analogous statement for $L_\infty$ algebras in $[\text{Ko1}].$

4. Minimal resolution of the operad $\text{TF}$. By definition (see Sect. 1.4), $\text{TF}$ is a quadratic dioperad

$$\text{TF} = \frac{\text{Free}(E)}{\text{Ideal} < R>},$$

where

(i) $E(2, 1) := k[\Sigma_2] \otimes 1_1$ and $E(1, 2) := 1_1 \otimes 1_2[-1]$; we represent two basis vectors of $k[\Sigma_2] \otimes 1_1$ by planar corollas

```
  \[ 1 \quad 2 \]
```

and a basis vector of $E(1, 2)$ by the symmetric corolla,

```
  \[ 1 \quad 2 \quad 2 \quad 1 \]
```

(ii) the relations $R$ are generated by the following elements,

```
\[ + \quad + \quad + \quad \in \text{Free}(E)(1, 3) \]
```

```
\[ - \quad - \quad - \quad - \quad \in \text{Free}(E)(2, 2). \]
```

Proposition 1.4.1 follows immediately from the following

4.1. Proposition. The minimal resolution, $\text{TF}_\infty$, of the dioperad $\text{TF}$ can be described as follows:

(i) As a dioperad of graded vector spaces, $\text{TF}_\infty = \text{Free}(E)$, where the collection, $E = \{E(m,n)\}$, of $(\Sigma_m, \Sigma_n)$-modules is given by

$$E(m,n) := \begin{cases} 
    k[\Sigma_2] \otimes 1_n & \text{if } m = 2, n \geq 2; \\
    1_n[-1] & \text{if } m = 1, n \geq 2; \\
    0 & \text{otherwise}. 
\end{cases}$$

(ii) If we represent two basis elements of $E(2, n)$ by planar $(2, n)$-corollas, and the basis element of $E(1, n)$ by planar $(1, n)$ corolla,

```
  \[ 1 \quad 2 \quad \ldots \quad n-1 \quad n \]
```

and, respectively,

```
  \[ 1 \quad 2 \quad \ldots \quad n-1 \quad n \]
```

with symmetric ingoing legs, then the differential $d$ is given on generators by,

```
d = \sum_{J_1 \cup J_2 = (1, \ldots, n)} \sum_{|J_1| \geq 2, |J_2| \geq 1, J_1 \cap J_2 \neq \emptyset} \chi \cdot J_1 \cdot J_2 
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(ii) If we represent a basis element of $E(m,n)$ by the unique space $(m,n)$-corolla,

then the differential $d$ is given on generators by,

$$d = \sum_{J_1 \cup J_2 = \{1, \ldots, m\}} \sum_{|J_1| \geq 2, |J_2| \geq 0} \left( \begin{array}{c} 1 \\ \vdots \\ n-1 \\ n \end{array} \right) - \sum_{J_1 \cup J_2 = \{1, \ldots, m\}} \sum_{|J_1| \geq 1, |J_2| \geq 1} \left( \begin{array}{c} 1 \\ \vdots \\ n \end{array} \right) \left( \begin{array}{c} 1 \\ \vdots \\ 2 \\ \vdots \\ n \end{array} \right)$$

Proof. Using criterion 2.4 it is easy to see that the dioperad $\mathcal{TF}$ is Koszul. Then the cobar dual $\mathcal{DT}^\mathcal{F}$ provides the required minimal resolution. The rest is a straightforward calculation. □

5. A comment on $\text{LieBi}_\infty$ algebras. It was shown in [G] that the dioperad, LieBi, of (usual) Lie bialgebras is Koszul so that its minimal resolution, $\text{LieBi}_\infty$, can be constructed using the techniques reviewed in Section 2. Here we present its explicit graph description; in fact, we prefer to show $\text{LieBi}_\infty(1)$.

5.1. Proposition. The dioperad $\text{LieBi}_\infty(1)$ can be described as follows.

(i) As a dioperad of graded vector spaces, $\text{LieBi}_\infty(1) = \text{Free}(E)$, where the collection, $E = \{E(m,n)\}$, of one dimensional $(\Sigma_m, \Sigma_n)$-modules is given by

$$E(m,n) := \begin{cases} 1_m \otimes 1_n [2m - 3] & \text{if } m + n \geq 3; \\ 0 & \text{otherwise}. \end{cases}$$

(ii) If we represent a basis element of $E(m,n)$ by the unique space $(m,n)$-corolla,

then the differential $d$ is given on generators by,

Let $V$ be a graded vector space, and let $\mathcal{M}$ be the graded formal manifold isomorphic to the neighbourhood of zero in $V[1] \oplus V^*[1]$. The manifold $\mathcal{M}$ has a natural even symplectic structure $\omega$ induced from the paring $V[1] \otimes V^*[1] \to k[2]$; it also has two particular Lagrangian submanifolds, $\mathcal{L}'$ and $\mathcal{L}''$, modeled on the subspaces $0 \oplus V^*[1] \subset V[1] \oplus V^*[1]$ and, respectively, $V \oplus 0 \subset V[1] \oplus V^*[1]$. The symplectic form induces degree $-2$ Poisson bracket, $\{ \cdot, \cdot \}$, on the structure sheaf, $\mathcal{O}_\mathcal{M}$, of smooth functions on $\mathcal{M}$.

The following result has been independently obtained by Lyubashenko [Lyu].

5.2 Corollary. A $\text{LieBi}_\infty$ algebra structure in a graded vector space $V$ is the same as a degree 3 smooth function $f \in \mathcal{O}_\mathcal{M}$ vanishing on $\mathcal{L}' \cup \mathcal{L}''$ and satisfying the equation $\{ f, f \} = 0$.

$\text{LieBi}_\infty$ morphisms and quasi-isomorphisms are defined exactly as in 3.4.1 and 3.4.2; then an obvious analogue of Theorem 3.4.5 holds true. We omit the details.
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