Heat kernel estimates on spaces with varying dimension

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Abstract
Z.-Q. Chen and S. Lou (Ann. Probab. 2019) constructed Brownian motion on a space with varying dimension, in which a 1-dimensional space and a 2-dimensional space are connected at one point, and derived sharp two-sided estimates for its transition density (heat kernel). In this paper, we obtain sharp two-sided heat kernel estimates on spaces with varying dimension, in which two spaces of general dimension are connected at one point. On these spaces, if the dimensions of the two constituent parts are different, the volume doubling property fails with respect to the measure induced by the associated Lebesgue measures. Thus the parabolic Harnack inequalities fail and the heat kernels do not enjoy Aronson type estimates. Our estimates show that the on-diagonal estimates are independent of the dimensions of the two parts of the space for small time, whereas they depend on their transience or recurrence for large time.

Key words Space of varying dimension, Brownian motion, transition density, heat kernel estimates

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1 Introduction
The heat kernel, the fundamental solution of the heat equation, has been studied in many areas, both for mathematical interest and for its importance in applications. The heat kernel is the transition density of Brownian motion, and it is difficult to determine its explicit form except in some special cases, such as on Euclidean spaces. Thus, heat kernel estimates have been studied on various spaces, see for example, [2, 9, 10, 12, 14, 18]. In a remarkable series of result, Grigor’yan [9], Saloff-Coste [18] and Sturm [19, 20] proved that the following are equivalent on a metric measure space: (i) the volume doubling property and scaled Poincaré inequalities, (ii) the parabolic Harnack inequalities, (iii) Aronson type estimates of the heat kernel. These results were extended to the setting of graphs in [5].

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In studies of heat kernel estimates, the volume doubling property is a natural assumption. However, there are many spaces that do not satisfy this property. One such example is a space with varying dimension given as following: for fixed \( \varepsilon > 0 \),

\[
\mathbb{R}^2_+ \cup \mathbb{R}_+ \cup \{a^*\} := \{(x,0) \mid x \in \mathbb{R}^2, |x| > \varepsilon\} \cup \{(0,0,x) \mid x > 0\} \cup \{a^*\}.
\]

Here, we identify \( x \in \mathbb{R}^2 \mid |x| \leq \varepsilon\) and \( 0 \in \mathbb{R} \) with a point \( a^* \). Z.-Q. Chen and S. Lou [3] constructed a stochastic process on \( \mathbb{R}^2_+ \cup \mathbb{R}_+ \cup \{a^*\} \) that they called Brownian motion with varying dimension (BMVD). Note that \( \mathbb{R}^2_+ \cup \mathbb{R}_+ \cup \{a^*\} \) was considered instead of \( \mathbb{R}^2 \cup \mathbb{R}_+ \) because 2-dimensional Brownian motion never hits 0. For BMVD, the following heat kernel estimates were given. To state the result, let \( \rho \) be the shortest path metric derived from the Euclidean metric on the two parts of the space (the precise definition is denoted below) and \(|x|_\rho\) be the distance between \( x \) and \( a^* \) with respect to \( \rho \).

**Theorem 1.1.** ([3, Theorem 1.3, 1.4]) \[I\] Let \( T > 0 \) be fixed. The transition density \( p(t,x,y) \) of BMVD satisfies the following estimates when \( t \in (0,T] \).

(i) For \( x \in \mathbb{R}_+ \) and \( y \in \mathbb{R}^2_+ \cup \mathbb{R}_+ \cup \{a^*\} \),

\[
p(t,x,y) \approx \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t}.
\]

(ii) For \( x, y \in \mathbb{R}^2_+ \cup \{a^*\} \) with \(|x|_\rho \vee |y|_\rho < 1 \),

\[
p(t,x,y) \approx \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t} + \frac{1}{t} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t},
\]

and for \( x, y \in \mathbb{R}^2_+ \cup \{a^*\} \) with \(|x|_\rho \vee |y|_\rho \geq 1 \),

\[
p(t,x,y) \approx \frac{1}{t} e^{-\rho(x,y)^2/t}.
\]

**[II]** The transition density \( p(t,x,y) \) of BMVD satisfies the following estimates for \( t \geq 8 \).

(i) For \( x, y \in \mathbb{R}^2_+ \cup \{a^*\} \),

\[
p(t,x,y) \approx \frac{1}{t} e^{-\rho(x,y)^2/t}.
\]

(ii) For \( x \in \mathbb{R}_+ \) and \( y \in \mathbb{R}^2_+ \cup \{a^*\} \), when \(|y|_\rho \leq 1 \),

\[
p(t,x,y) \approx \frac{1}{t} \left(1 + \frac{|x| \log t}{\sqrt{t}}\right) e^{-\rho(x,y)^2/t},
\]

and when \(|y|_\rho > 1 \),

\[
p(t,x,y) \approx \frac{1}{t} \left(1 + \frac{|x| \log \left(1 + \frac{\sqrt{t}}{|y|_\rho}\right)}{\sqrt{t}}\right) e^{-\rho(x,y)^2/t}.
\]

(iii) For \( x, y \in \mathbb{R}_+ \),

\[
p(t,x,y) \approx \frac{e^{-|x-y|^2/t}}{\sqrt{t}} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|}{\sqrt{t}}\right) + \frac{e^{-((|x|_\rho+|y|_\rho)^2/t}}{t} \left(1 + \frac{(|x| + |y|) \log t}{\sqrt{t}}\right).
\]
Here and throughout this paper, we use the notation $a \wedge b := \min \{ a, b \}$, $a \vee b := \max \{ a, b \}$, and for $(t, x, y) \in A \subset [0, \infty) \times (\mathbb{R}_e^d \cup \mathbb{R}_e^d \cup \{a^*\}) \times (\mathbb{R}_e^d \cup \mathbb{R}_e^d \cup \{a^*\})$ and non-negative functions $f(t, x, y), g(t, x, y), h(t, x, y)$,

$$fe^{-h} \lesssim ge^{-h}$$

($\lesssim$, respectively) means that there exist $C > 0, c_1 > 0, c_2 > 0$, independent of $(t, x, y) \in A$, such that $fe^{-c_1 h} \leq Cge^{-c_2 h}$ for $(t, x, y) \in A$ ($\gtrsim$, respectively). Moreover,

$$fe^{-h} \asymp ge^{-h}$$

means that $fe^{-h} \lesssim ge^{-h}$ and $fe^{-h} \gtrsim ge^{-h}$. In computations, constants $C, c$ may change from line to line.

Concerning other work for heat kernel estimates on spaces with varying dimension, S. Lou deduced such for Brownian motion with drift on $\mathbb{R}^2 \cup \mathbb{R}_+ \cup \{a^* \}$ in [16] and obtained an explicit expression for the heat kernel of distorted Brownian motion on $\mathbb{R}^3 \cup \mathbb{R}_+ \cup \{a^* \}$ in [17].

In this paper, we estimate the heat kernel for Brownian motion on spaces with general varying dimension. To introduce the setting more precisely, let $d \geq d' \geq 1$ and $\varepsilon, \varepsilon' > 0$. We define

$$\mathbb{R}_e^d := \{ x \in \mathbb{R}^d ; |x| > \varepsilon \}, \mathbb{R}_e^d := \{ x \in \mathbb{R}^{d'} ; |x| > \varepsilon' \},$$

where $| \cdot |$ is the Euclidean norm. For simplicity, set $\mathbb{R}_e^1 := \mathbb{R}_+ := (0, \infty)$ for all $\varepsilon$. For $\mathbb{R}^d$ and $\mathbb{R}^d$, we identify $\{ x \in \mathbb{R}^d ; |x| \leq \varepsilon \}$ and $\{ x \in \mathbb{R}^{d'} ; |x| \leq \varepsilon' \}$ with a point $a^*$. We will establish heat kernel estimates for Brownian motion on $\mathbb{R}_e^d \cup \mathbb{R}_e^{d'} \cup \{a^*\}$, where $\mathbb{R}_e^d \cup \mathbb{R}_e^{d'}$ means $\{(x,0,\cdots,0) | x \in \mathbb{R}_e^d \} \cup \{(0,\cdots,0,y) | y \in \mathbb{R}_e^{d'} \}$.

We define a neighborhood of $a^*$ as $\{a^*\} \cup (U_1 \cap \mathbb{R}_e^d) \cup (U_2 \cap \mathbb{R}_e^{d'})$ for some neighborhoods $U_1$ in $\{ x \in \mathbb{R}^d ; |x| \leq \varepsilon \}$ and $U_2$ in $\{ x \in \mathbb{R}^{d'} ; |x| \leq \varepsilon' \}$. Moreover, we consider the topology on $\mathbb{R}_e^d \cup \mathbb{R}_e^{d'} \cup \{a^*\}$ induced by the neighborhoods. We denote the Borel $\sigma$-field by $\mathcal{B} := \mathcal{B}(\mathbb{R}^d \cup \mathbb{R}_e^{d'} \cup \{a^*\})$.

For a constant $p > 0$, we define $m_p(A) := m^{(d)}(A \cap \mathbb{R}_e^d) + p m^{(d')} (A \cap \mathbb{R}_e^{d'})$ for $A \in \mathcal{B}$. Here, $m^{(d)}$ is the Lebesgue measure on $\mathbb{R}^d$. In particular, $m_p(\{a^*\}) = 0$.

We extend the definition of Brownian motion with varying dimension as follows. In Theorem 2.1, we will describe the existence and the uniqueness of a process satisfying the following definition.

**Definition 1.2.** Let $d \geq d' \geq 1$, $\varepsilon, \varepsilon' > 0$ and $p > 0$. **Brownian motion with varying dimension** (BMVD in abbreviation) with parameters $(\varepsilon, \varepsilon', p)$ on $\mathbb{R}_e^d \cup \mathbb{R}_e^{d'} \cup \{a^*\}$ is an $m_p$-symmetric diffusion $X = (\{X_t \}, \{\mathbb{P}_x \})$ on $\mathbb{R}_e^d \cup \mathbb{R}_e^{d'} \cup \{a^*\}$ such that:

(i) its part process on $\mathbb{R}_e^d$ or $\mathbb{R}_e^{d'}$ has the same law as Brownian motion killed upon leaving $\mathbb{R}_e^d$ or $\mathbb{R}_e^{d'}$, respectively,

(ii) it admits no killings on $a^*$.
Throughout the paper, $X = \{X_t\}, \{P_x\}$ denotes BMVD, $E_x$ denotes the expectation corresponding to $P_x$ and $P_t f(x) := E_x (f(X_t))$ for a bounded Borel measurable function $f$. Let $p(t, x, y)$ be the heat kernel with respect to $m_p$ whose existence will be proved in Proposition 2.2. Let $C_c^\infty$ be the set of all smooth functions with compact support and $\sigma_K := \inf \{t > 0 \mid X_t \in K\}$ be the hitting time of $K \in \mathcal{B}$.

Next, we introduce a distance $\rho$ on $\mathbb{R}^d \cup \mathbb{R}_c^d \cup \{a^*\}$, as follows,

$|x|_\rho := \begin{cases} |x| - \varepsilon & \text{for } x \in \mathbb{R}_c^d, \\ |x| - \varepsilon' & \text{for } x \in \mathbb{R}_c^d, \\ 0 & \text{for } x = a^*. \end{cases}$

\[
\rho(x, y) := (|x|_\rho + |y|_\rho) \wedge |x - y| \quad \text{for } x, y \in \mathbb{R}_c^d \cup \mathbb{R}_c^d \cup \{a^*\}.
\]

Here, for $x \in \mathbb{R}_c^d \cup \{a^*\}, y \in \mathbb{R}_c^d \cup \{a^*\}$ or $x \in \mathbb{R}_c^d \cup \{a^*\}, y \in \mathbb{R}_c^d \cup \{a^*\}$, we define $|x - y| := \infty$.

The following theorems are the main results in this paper.

**Theorem 1.3 (Small time estimates).** Let $d \geq d' \geq 1$ and $T \geq 1$ be fixed. The heat kernel $p(t, x, y)$ satisfies the following estimates when $t \in (0, T]$.

1. For $x, y \in \mathbb{R}_c^{d'}$ with $|x|_\rho \vee |y|_\rho \leq 1$,

\[
p(t, x, y) \approx \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/4t} + \frac{1}{t^{d'/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/4t}.
\]

2. For $x, y \in \mathbb{R}_c^{d'}$ with $|x|_\rho \vee |y|_\rho > 1$,

\[
p(t, x, y) \approx \frac{1}{t^{d'/2}} e^{-\rho(x, y)^2/4t}.
\]

3. For $x, y \in \mathbb{R}_c^d$ with $|x|_\rho \vee |y|_\rho \leq 1$,

\[
p(t, x, y) \approx \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/4t} + \frac{1}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/4t}.
\]

4. For $x, y \in \mathbb{R}_c^d$ with $|x|_\rho \vee |y|_\rho > 1$,

\[
p(t, x, y) \approx \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/4t}.
\]

5. For $x \in \mathbb{R}_c^d \cup \{a^*\}, y \in \mathbb{R}_c^{d'} \cup \{a^*\}$,

\[
p(t, x, y) \approx \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/4t}.
\]
Note that when \( d' = 1 \) and \( d = 2 \), the estimates are the same as those in Theorem 1.1 \([I]\). Intuitively, if BMVD hits \( a^* \), or both \( x \) and \( y \) are close to \( a^* \), a 1-dimensional effect appears in the heat kernel. If either \( x \) or \( y \) is far from \( a^* \), the dimension on which BMVD lives affects the heat kernel. We will prove Theorem 1.3 in Section 3.

Concerning large time estimates, we give four theorems depending on the dimensions of the two parts of the space.

**Theorem 1.4 (Large time estimates I).** Let \( d \geq 3 \), \( d' = 1 \) and \( T \) be large. The heat kernel \( p(t, x, y) \) satisfies the following estimates when \( T \leq t \).

(i) For \( x, y \in \mathbb{R}_+ \) with \( |x|_\rho \wedge |y|_\rho > 1 \),

\[
p(t, x, y) \asymp \frac{|x||y|}{\sqrt{t(|x| + \sqrt{t})(|y| + \sqrt{t})}} e^{-\rho(x, y)^2/t}.
\]

(ii) For \( x, y \in \mathbb{R}_d^2 \) with \( |x|_\rho \wedge |y|_\rho > 1 \),

\[
p(t, x, y) \asymp \frac{1}{t^{3/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}.
\]

(iii) For \( x \in \mathbb{R}_+ \cup \{a^*\}, y \in \mathbb{R}_d^2 \cup \{a^*\} \) or \( x, y \in \mathbb{R}_+ \) with \(|y|_\rho \leq 1 \) or \( x, y \in \mathbb{R}_d^2 \) with \(|x|_\rho \leq 1 \),

\[
p(t, x, y) \asymp \left( \frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-\rho(x, y)^2/t}.
\]

Since 1-dimensional Brownian motion is recurrent and \( d \)-dimensional Brownian motion is transient for \( d \geq 3 \), if BMVD starting from a point in \( \mathbb{R}_+ \) enters \( \mathbb{R}_d^2 \) and stays there for a long time, it is likely to escape to infinity. Thus, intuitively \( \mathbb{R}_+ \) affects the heat kernel more than \( \mathbb{R}_d^2 \). We will prove Theorem 1.4 in Section 4 using the projection.

**Theorem 1.5 (Large time estimates II).** Let \( d = d' = 2 \) and \( T \) be large. The heat kernel \( p(t, x, y) \) satisfies the following estimates when \( T \leq t \).

(i) For \( x, y \in \mathbb{R}_e^2 \) or \( x, y \in \mathbb{R}_e^2 \),

\[
p(t, x, y) \asymp \frac{1}{t} e^{-\rho(x, y)^2/t}.
\]

(ii) For \( x \in \mathbb{R}_e^2 \cup \{a^*\} \) and \( y \in \mathbb{R}_e^2 \cup \{a^*\} \),

\[
p(t, x, y) \asymp \frac{1}{t} \left( U_t(x) U_t(y) + \frac{U_t(x) \log |y|}{\log (1 + t|y|)} + \frac{U_t(y) \log |x|}{\log (1 + t|x|)} \right) e^{-\rho(x, y)^2/t}.
\]

Here, \( U_t(x) := \frac{1}{\log(t+|x|)} + \left( 1 - \frac{\log|y|}{\log \sqrt{t}} \right)_+ \).
Theorem 1.6 (Large time estimates III). Let \(d \geq 3, d' = 2\) and \(T\) be large. The heat kernel \(p(t, x, y)\) satisfies the following estimates when \(T \leq t\).

(i) For \(x, y \in \mathbb{R}^d\),
\[
p(t, x, y) \approx \frac{1}{t(\log t)^2|x|^{d-2}|y|^{d-2}}e^{-(|x|_\rho+|y|_\rho)^2/t} + \frac{1}{t^{d/2}}e^{-\rho(x,y)^2/t}.
\]

(ii) For \(x, y \in \mathbb{R}^2\),
\[
p(t, x, y) \approx \log (1 + |x|) \log (1 + |y|) e^{-\rho(x,y)^2/t}.
\]

(iii) For \(x \in \mathbb{R}^d \cup \{a^*\}, y \in \mathbb{R}^2 \cup \{a^*\},\)
\[
p(t, x, y) \approx \log (1 + |x|) \log (1 + |y|) e^{-\rho(x,y)^2/t}.
\]

\(p(t, x, y) \approx \frac{1}{(\log (1+|y|))^{d'}} + \left(\frac{1}{2 \log(1+|y|)} - \frac{1}{\log t}\right)_{+}.
\]

For \(d = d' = 2\), BMVD is recurrent and a 2-dimensional effect appears in the large time estimates. For \(d \geq 3, d' = 2\), we have a mixed case of recurrent and transient parts of the space. In this case, \(\mathbb{R}^2\) affects the heat kernel more than \(\mathbb{R}^d\) for a similar reason as in the case of \(d \geq 3, d' = 1\). We will prove Theorem 1.5 and 1.6 in Section 6 using Doob’s \(h\)-transform and the relative Faber-Krahn inequality.

Theorem 1.7 (Large time estimates IV). Let \(d \geq d' \geq 3\) and \(T\) be large. The heat kernel \(p(t, x, y)\) satisfies the following estimates when \(T \leq t\).

(i) For \(x, y \in \mathbb{R}^{d'}\),
\[
p(t, x, y) \approx \frac{1}{t^{d'/2}}e^{-\rho(x,y)^2/t}.
\]

(ii) For \(x, y \in \mathbb{R}^d\) with \(|x|_\rho \lor |y|_\rho \leq 1\),
\[
p(t, x, y) \approx \frac{1}{t^{d'/2}}e^{-\rho(x,y)^2/t}.
\]

For \(x, y \in \mathbb{R}^d\) with \(|x|_\rho \lor |y|_\rho > 1\),
\[
p(t, x, y) \approx \frac{1}{t^{d'/2}|x|^{d-2}|y|^{d-2}}e^{-(|x|_\rho+|y|_\rho)^2/t} + \frac{1}{t^{d/2}}e^{-\rho(x,y)^2/t}.
\]

(iii) For \(x \in \mathbb{R}^d \cup \{a^*\}, x \in \mathbb{R}^{d'} \cup \{a^*\},\)
\[
p(t, x, y) \approx \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}|y|^{d-2}}\right)e^{-\rho(x,y)^2/t}.
\]
For \( d \geq d' \geq 3 \), both Brownian motion on \( \mathbb{R}^d \) and \( \mathbb{R}^{d'} \) are transient. Intuitively, \( d \geq d' \) yields that \( d \)-dimensional Brownian motion escape to infinity faster than \( d' \)-dimensional Brownian motion. Thus, \( \mathbb{R}^d \) affects the large time heat kernel more than \( \mathbb{R}^{d'} \) if BMVD starts near \( a^* \). We will prove Theorem 1.7 in Section 5 by estimating \( p(t, a^*, a^*) \) and using \( P_x(\sigma_{a^*} \in ds) \).

In related works, A. Grigor'yan, L. Saloff-Coste and S. Ishiwata obtained heat kernel estimates for Brownian motion on the connected sum of manifolds \([12, 14]\). To explain their results, we present the following definition.

**Definition 1.8.** Let \( M_1 \) and \( M_2 \) be \( n \)-dimensional manifolds. A connected sum \( M := M_1 \# M_2 \) is a manifold constructed by removing a ball inside each manifold and gluing together these boundary spheres. A non-empty compact set \( K \subset M \) is a central part of \( M \) if the exterior \( M \setminus K \) is a disjoint union of open sets \( E_1 \) and \( E_2 \) such that each \( E_i \) is homeomorphic to \( M_i \setminus K_i \) for some compact \( K_i \subset M_i \).

Let \( S^{d-d'} \) be the \( d-d' \) dimensional unit sphere. For \( d \geq d' \geq 1, \epsilon, \epsilon' > 0 \) and \( p > 0 \), our large time heat kernel estimates for BMVD are, up to the distances with which the results are stated, of the same form as those for Brownian motion on \( \mathbb{R}^d \#(\mathbb{R}^{d'} \times S^{d-d'}) \) given in \([12, 14]\). In fact, in order to prove Theorem 1.4, 1.6, we borrow some techniques from \([12]\).

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## 2 Preliminary

Throughout the paper, we fix \( \epsilon, \epsilon', p > 0 \). In this section, we first prove the existence and the uniqueness of BMVD. We then show the existence and some properties of the heat kernel for BMVD. We also prove the space with varying dimension fails to the volume doubling property and we give some lemmas that will be used in Section 4-6.

**Theorem 2.1.** For \( d \geq d' \geq 1, \epsilon, \epsilon' > 0 \) and \( p > 0 \), BMVD with parameters \((\epsilon, \epsilon', p)\) on \( \mathbb{R}^d \cup \mathbb{R}^{d'} \cup \{a^*\} \) exists and is unique in law. Furthermore, its
associated Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'} \cup \{a^*\}; m_p)\) is given by

\[
\mathcal{F} := \left\{ f \in L^2(\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'} \cup \{a^*\}; m_p) \mid \begin{array}{l}
f|_{\mathbb{R}^d_\varepsilon} \in H^1(\mathbb{R}^d_\varepsilon), f|_{\mathbb{R}^d_{\varepsilon'}} \in H^1(\mathbb{R}^d_{\varepsilon'}) \\
f(x) = f(a^*) \text{ q.e. on } \partial \mathbb{R}^d_\varepsilon \cup \partial \mathbb{R}^d_{\varepsilon'}
\end{array} \right\},
\]

\[
\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^d_\varepsilon} \nabla f \cdot \nabla g \, dm_p + \frac{1}{2} \int_{\mathbb{R}^d_{\varepsilon'}} \nabla f \cdot \nabla g \, dm_p \text{ for } f, g \in \mathcal{F}.
\]

**Proof.** The proof is the same as that of [3, Theorem 2.2]. \(\square\)

**Proposition 2.2.** There exists a heat kernel \(p(t, x, y)\) with respect to \(m_p\), which is continuous for each \(t > 0\). Moreover, for all \(t > 0\), it holds that \(p(t, a^*, a^*) \lesssim t^{-d/2} \vee t^{-d'/2}\).

**Proof.** \(\| \cdot \|_{L^1}\) denotes \(L^1\)-norm with respect to \(m_p\). Since \(\mathbb{R}^d_\varepsilon\) and \(\mathbb{R}^d_{\varepsilon'}\) have smooth boundaries, for all \(f \in \mathcal{F} \cap L^1(\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'} \cup \{a^*\})\), by classical Nash’s inequality, there is \(C > 0\) such that

\[
\|f|_{\mathbb{R}^d_\varepsilon}\|_{L^2}^{1+2/d} \leq C \|f|_{\mathbb{R}^d_\varepsilon}\|_{L^1}^{2/d} \|\nabla f|_{\mathbb{R}^d_\varepsilon}\|_{L^2},
\]

\[
\|f|_{\mathbb{R}^d_{\varepsilon'}}\|_{L^2}^{1+2/d'} \leq C \|f|_{\mathbb{R}^d_{\varepsilon'}}\|_{L^1}^{2/d'} \|\nabla f|_{\mathbb{R}^d_{\varepsilon'}}\|_{L^2}.
\]

Then, for all \(f \in \mathcal{F}\), we have

\[
\|f\|_{L^2}^2 \leq C \left( \mathcal{E}(f, f)^{d/d+2}\|f\|_{L^1}^{4/d+2} + \mathcal{E}(f, f)^{d'/d'+2}\|f\|_{L^1}^{4/d'+2} \right).
\]

By [2, Corollary 2.12], the heat kernel \(p(t, x, y)\) with respect to \(m_p\) exists and the desired inequality holds for a.e. \(x, y\), so it is sufficient to prove the continuity of \(p(t, \cdot, \cdot)\). By Definition 1.2 (i), \(p(t, \cdot, \cdot)\) is continuous on \((\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'}) \times (\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'})\). For fixed \(t, y\), \(p(t, x, y) = \int p(t/2, x, z)p(t/2, z, y)dm_p(z) = p_{t/2}(t/2, \cdot, y)\) is quasi-continuous ([4, Proposition 3.1.9]) and, since \(a^*\) is nonpolar for \(X, p(t, \cdot, \cdot)\) is continuous. By the symmetry, \(p(t, \cdot, \cdot)\) is continuous. \(\square\)

In this paper, for \(x\) and \(r > 0\), we define \(B(x; r) := \{y \in \mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'} \cup \{a^*\} \mid \rho(x, y) < r\}\).

**Proposition 2.3.** For \(d > d' \geq 1\), the volume doubling property fails on \(\mathbb{R}^d_\varepsilon \cup \mathbb{R}^d_{\varepsilon'} \cup \{a^*\}\) for \(m_p\).

**Proof.** For \(r > 0\), we take \(x \in \mathbb{R}^d\) with \(|x| = r + \varepsilon\), see Figure 1, then we have

\[
m_p(B(x; r)) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d \quad \text{and} \quad m_p(B(x; 2r)) \geq \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} (r + \varepsilon)^d + \frac{p r^{d'/2} \left((r + \varepsilon')^{d'} - (r + \varepsilon)^d\right)}{\Gamma(d'/2 + 1)} \geq c (r^d + r^{d'})\]

Now, if there exists \(C > 0\) such that \(m_p(B(x; 2r)) \leq C m_p(B(x; r))\) for all \(x\), then we obtain \(r^d + r^{d'} \leq c r^d\), so \(1 + r^{d-d} \leq c \). \(1 + r^{d-d} \to \infty\) as \(r \to 0\) and this is a contradiction. \(\square\)
Let \( p_{R^d_\varepsilon}(t, x, y) \) be the transition density of the part process of BMVD killed upon exiting \( R^d_\varepsilon \). According to [22], the following proposition holds.

**Proposition 2.4.** Let \( d \geq 2 \). For \( x, y \in R^d_\varepsilon \) and \( t > 0 \), it holds that

\[
p_{R^d_\varepsilon}(t, x, y) \asymp \frac{1}{t^{d/2}} \left( 1 \wedge \frac{|x|}{\sqrt{t} \wedge 1} \right) \left( 1 \wedge \frac{|y|}{\sqrt{t} \wedge 1} \right) e^{-|x-y|^2/t}. \quad (2.1)
\]

Let \( p_{R^d_\varepsilon}(t, x, y) := \int_0^t p(t-s, a^*, y) P_x(\sigma_{a^*} \in ds) \) for \( x, y \in R^d_\varepsilon \). In order to estimate \( p_{R^d_\varepsilon}(t, x, y) \), we prepare some lemmas for \( \sigma_{a^*} \). According to [1, Theorem 3], the following two lemmas hold when \( \varepsilon = 1 \). By the scaling, they hold for every \( \varepsilon > 0 \).

**Lemma 2.5.** For \( d \geq 3 \) and \( x \in R^d_\varepsilon \), it holds that

\[
\mathbb{P}(\sigma_{a^*} \in ds) \asymp \frac{|x|}{|x|} e^{-|x|^2/s} |x|^{s/2} + s^{3/2}|x|^{(d-3)/2} ds.
\]

**Lemma 2.6.** For \( d = 2 \) and \( x \in R^2_\varepsilon \), it holds that

\[
\mathbb{P}(\sigma_{a^*} \in ds) \asymp \frac{|x|}{|x|} \left( 1 + \log \left( 1 + \frac{|x|}{s} \right) \right) \left( 1 + \log \left( 1 + \frac{|x|}{s} \right) \right) e^{-|x|^2/s} ds.
\]

We will use the following elementary estimate.

**Lemma 2.7.** Let \( d \geq 3 \). Then for \( t \geq 1 \) and \( x \in R^d_\varepsilon \), we have

\[
\frac{e^{-|x|^2/t}}{t^{d/2} + t^{3/2}|x|^{(d-3)/2}} \geq \frac{e^{-|x|^2/t}}{t^{d/2}}. \quad (2.2)
\]

**Proof.** When \( |x| \leq \sqrt{t}/2 \), (2.2) follows from \( t^{d/2} + t^{3/2}|x|^{(d-3)/2} \lesssim t^{d/2} + t^{3/2} \).
When $|x| > \sqrt{t}/2$,

\[
\frac{e^{-c|x|^2}}{t^{d/2} + \rho^3/2|x|^{(d-3)/2}} \geq \frac{e^{-(c+1)|x|^2}}{t^{d/2} + \rho^3/2|x|^{(d-3)/2}} \geq \frac{e^{-|x|^2}}{t^{d/2} + t^{(d+3)/4}} \geq \frac{e^{-|x|^2}}{t^{d/2}}.
\]

In the next two lemmas, we obtain the estimates of hitting distribution.

**Lemma 2.8.** Let $d \geq 3$. Then for $x \in \mathbb{R}^d$ and $t > 1$, we have

\[
\mathbb{P}_x(\sigma_{x^*} \leq t) \asymp \frac{1}{|x|^{d-2}} e^{-|x|^2/t}. \tag{2.3}
\]

**Proof.** When $|x|_\rho \geq 1$, (2.3) follows from [11, Theorem 4.4 (1)]. When $|x|_\rho < 1$, $\mathbb{P}_x(\sigma_{x^*} \leq t) \leq 1$ and there is some $C > 0$ with $\mathbb{P}_x(\sigma_{x^*} \leq t) \geq \mathbb{P}_x(\sigma_{x^*} \leq 1) \geq C$, so (2.3) holds. \qed

**Lemma 2.9.** Let $d = 2$. Then for $x \in \mathbb{R}^2$ with $|x|_\rho \geq 1$, we have

(i) $\mathbb{P}_x(\sigma_{x^*} \leq t) \asymp \frac{1}{\log |x|} e^{-|x|^2/t}$ for $0 < t < 2|x|^2$,

(ii) $\mathbb{P}_x(\sigma_{x^*} \leq t) \asymp 1 - \frac{\log |x|}{\log \sqrt{t}}$ for $2|x|^2 \leq t$.

**Proof.** See [11, Theorem 4.11]. \qed

The next lemma gives the relations between $e^{-\rho(x,y)^2/t}$, $e^{-(|x|_\rho + |y|_\rho)^2/t}$ and $e^{-|x-y|^2/t}$ for large time.

**Lemma 2.10.** Let $T > 0$ be fixed and $d \geq 1$. For $T \leq t$ and $x, y \in \mathbb{R}^d$, we have

(i) $e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \asymp e^{-(|x|_\rho + |y|_\rho)^2/t}$ if $|x|_\rho \vee |y|_\rho > 1$,

(ii) $e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \asymp e^{-(|x|_\rho + |y|_\rho)^2/t}$ if $|x|_\rho \vee |y|_\rho \leq 1$,

(iii) $e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \asymp e^{-(|x|_\rho + |y|_\rho)^2/t}$ if $|x|_\rho > 1 > b \geq |y|_\rho$ for some $b$.

**Proof.** (i) When $|x|_\rho \vee |y|_\rho > 1$, we may assume $|x|_\rho > 1$ without loss of generality. If $\rho(x, y) = |x-y|$, there is nothing to prove. If $\rho(x, y) = |x|_\rho + |y|_\rho$, then we have

$\rho(x, y) \leq |x-y| \leq |x| + |y| = |x|_\rho + |y|_\rho + 2\varepsilon \leq (2\varepsilon + 1)(|x|_\rho + |y|_\rho) = (2\varepsilon + 1)\rho(x, y)$.

Hence, the desired estimate holds.

(ii) When $|x|_\rho \vee |y|_\rho \leq 1$, we have

\[
\frac{(|x|_\rho + |y|_\rho)^2}{t} \leq \frac{4}{T} \leq \frac{4}{T} + \frac{|x-y|^2}{t} \leq \frac{4}{T} + \frac{(|x| + |y|)^2}{t} \leq \frac{4}{T} + \frac{2(2\varepsilon)^2}{T} + \frac{2(|x| + |y|_\rho)^2}{t}.
\]
Hence, desired estimate holds.

(iii) When \(|x|_\rho \geq 1 > b \geq |y|_\rho\), we have

\[
\frac{(|x|_\rho + |y|_\rho)^2}{t} - \frac{2|x - y|^2}{t} = \frac{(|x| + |y| - 2\varepsilon)^2 - 2|x - y|^2}{t} \\
\leq \frac{(|x - y| + |y| - 2\varepsilon)^2 - 2|x - y|^2}{t} = \frac{(2|y|_\rho + |x - y|)^2 - 2|x - y|^2}{t}
\]

\[
\leq \frac{2(2|y|_\rho)^2 + 2|x - y|^2 - 2|x - y|^2}{t} = \frac{8|y|_\rho^2}{t} \leq \frac{8b^2}{T}.
\]

Hence, we have \(e^{-|x-y|^2/t} \leq e^{-(|x|_\rho + |y|_\rho)^2/2t} e^{4b^2/T}\) and by (i), the desired estimate holds.

\[\square\]

3 Small time estimate

In this section, we prove Theorem 1.3 in the same way as [3, section 4]. First, we define

\[u(x) := \begin{cases} -|x|_\rho & : x \in \mathbb{R}^d_{\varepsilon'} \\
|x|_\rho & : x \in \mathbb{R}^d_{\varepsilon}
\end{cases}
\]

and \(Y_t := u(X_t)\). Then \(u \in \mathcal{F}^{\text{loc}}\), where \(\mathcal{F}^{\text{loc}}\) denotes the local Dirichlet space of \((\mathcal{E}, \mathcal{F})\). We will prove that the heat kernel for \(Y\) enjoys 1-dimensional Gaussian estimates. Combining this with the fact that \(p_{\mathbb{R}^d_{\varepsilon}}(t, x, y)\) (resp. \(p(t, x, y)\)) depends only on \(|x|_\rho\) and \(|y|_\rho\) for \(x, y \in \mathbb{R}^d_{\varepsilon}\) (resp. \(x \in \mathbb{R}^d_{\varepsilon}, y \in \mathbb{R}^d_{\varepsilon'}\)), we prove Theorem 1.3.

We first derive the stochastic differential equation that \(Y\) satisfies, and then use it to obtain Gaussian heat kernel estimates of \(Y\).

**Proposition 3.1.**

\[dY_t = dB_t + \frac{(d-1)1_{\{Y_t > 0\}}}{2(Y_t + \varepsilon)} dt + \frac{p(d' - 1)1_{\{Y_t < 0\}}}{2(Y_t - \varepsilon')} dt + \frac{\partial p_{\mathbb{R}^d_{\varepsilon}} - p\partial p_{\mathbb{R}^d_{\varepsilon'}}}{\partial p_{\mathbb{R}^d_{\varepsilon}} + p\partial p_{\mathbb{R}^d_{\varepsilon'}}} d\hat{L}^0_t(Y),
\]

where \(|\cdot|\) is the Lebesgue measure, \(B\) is one-dimensional Brownian motion and \(\hat{L}^0(Y)\) is symmetric semimartingale local time of \(Y\) at 0.

**Proof.** By the Fukushima decomposition ([4, Chapter 4]), there exist local martingale additive functional \(M^{[u]}\) and continuous additive functional locally having zero energy \(N^{[u]}\) such that \(Y_t - Y_0 = M^{[u]}_t + N^{[u]}_t\), \(\mathbb{P}_x\)-a.s. for q.e.
\[ x \in \mathbb{R}_+^d \cup \mathbb{R}_-^d \cup \{a^*\}. \] For any \( \psi \in C_c^\infty(\mathbb{R}_+^d \cup \mathbb{R}_-^d \cup \{a^*\}) \), it holds that

\[
\mathcal{E}(u, \psi) = \frac{1}{2} \int_{\mathbb{R}_d^d} \nabla |x| \cdot \nabla \psi dx + \frac{p}{2} \int_{\partial \mathbb{R}_d^d} \nabla(-|x|) \cdot \nabla \psi dx \\
= -\frac{1}{2} \int_{\mathbb{R}_d^d} d - \frac{1}{|x|} \psi dx + \frac{p}{2} \int_{\partial \mathbb{R}_d^d} \psi(0) \frac{\partial |x|}{\partial n} \sigma(dx) \\
+ \frac{p}{2} \int_{\mathbb{R}_d^d} d - \frac{1}{|x|} \psi dx - \frac{p}{2} \int_{\partial \mathbb{R}_d^d} \psi(0) \frac{\partial |x|}{\partial n} \sigma(dx) \\
= -\frac{1}{2} \int_{\mathbb{R}_d^d} d - \frac{1}{|x|} \psi dx + \frac{p}{2} \int_{\mathbb{R}_d^d} d - \frac{1}{|x|} \psi dx - \frac{1}{2} (|\partial \mathbb{R}_d^d| - p |\partial \mathbb{R}_d^d|) \psi(0) \\
= -\int_{\mathbb{R}_d^d \cup (\mathbb{R}_d^d \cup \{a^*\})} \psi(x) \nu(dx),
\]

where \( n \) is the outward normal vector of the surface \( \partial \mathbb{R}_d^d \cup \partial \mathbb{R}_d^d \), \( \sigma \) is the surface measure on \( \partial \mathbb{R}_d^d \cup \partial \mathbb{R}_d^d \), and

\[
\nu(dx) := \frac{d - 1}{2|x|} 1_{\mathbb{R}_d^d}(x) dt - \frac{p(d - 1)}{2|x|} 1_{\mathbb{R}_d^d}(x) dt + \frac{(|\partial \mathbb{R}_d^d| - p |\partial \mathbb{R}_d^d|)}{2} \delta_{\{a^*\}}.
\]

By [6, Theorem 5.5.5], it holds that

\[
dN_t[a] = \frac{(d - 1) 1\{X_t \in \mathbb{R}_d^d\}}{2(u(X_t) + \varepsilon)} dt - \frac{p(d - 1) 1\{X_t \in \mathbb{R}_d^d\}}{2(-u(X_t) + \varepsilon)} dt + (|\partial \mathbb{R}_d^d| - p |\partial \mathbb{R}_d^d|) dL_t^0(X).
\]

Here, \( L_t^0(X) \) is positive continuous additive functional of \( X \) whose Revuz measure is \( \frac{1}{2} \delta_{\{a^*\}} \).

Let \( u_n := (\varepsilon v \cup u) \wedge n \), then \( u_n \in \mathcal{F} \). By [4, Theorem 4.3.11] and strongly locality of (\( \mathcal{E}, \mathcal{F} \)), for any \( \varphi \in \mathcal{F} \cap C_c(\mathbb{R}_d^d \cup \mathbb{R}_d^d \cup \{a^*\}) \), we have \( \int \varphi d\mu_{(u_n)} = 2\mathcal{E}(u_n \varphi, u_n) - \mathcal{E}(u_n^2, \varphi) = \int \varphi \nabla u_n^2 dm_p \). Here, \( d\mu_{(u_n)} \) is the Revuz measure corresponding to \( \langle M^{(u_n)} \rangle \). By [6, Theorem 5.5.2], we obtain \( d\mu_{(u_n)} = |\nabla u_n|^2 dm_p = 1\{|x|_p \leq n\} dm_p \).

It yields \( d\mu_{(u)} = dm_p \). By [4, Theorem 4.1.8], \( \langle M^{(u)} \rangle \) is \( t \) and \( B_t := M_t^{(u)} \) is one-dimensional Brownian motion. Thus it holds that

\[
dY_t = dB_t + \frac{(d - 1) 1\{Y_t > 0\}}{2(Y_t + \varepsilon)} dt + \frac{p(d - 1) 1\{Y_t < 0\}}{2(Y_t - \varepsilon')} dt + (|\partial \mathbb{R}_d^d| - p |\partial \mathbb{R}_d^d|) dL_t^0(X). \tag{3.3}
\]

Next, we show \( dL_t^0(Y) = (|\partial \mathbb{R}_d^d| + p |\partial \mathbb{R}_d^d|) dL_t^0(X) \).

Let \( v(x) := |x|_p \), so \( |Y_t| = v(X_t) \) holds. Then, by the similar computation as above, for one-dimensional Brownian motion \( \hat{B} \), we have

\[
d|Y_t| = d\hat{B}_t + \frac{(d - 1) 1\{Y_t > 0\}}{2(Y_t + \varepsilon)} dt - \frac{p(d - 1) 1\{Y_t < 0\}}{2(Y_t - \varepsilon')} dt + (|\partial \mathbb{R}_d^d| + p |\partial \mathbb{R}_d^d|) dL_t^0(X).
\]
While, by Tanaka’s formula and (3.3), we have
\[
\begin{align*}
\mathrm{d}|Y_t| &= \text{sign}(Y_t)\mathrm{d}Y_t + dL_t^0(Y) \\
&= \text{sign}(Y_t)\mathrm{d}B_t + \frac{(d-1)1_{\{Y_t>0\}}}{2(Y_t+\varepsilon)}\mathrm{d}t - \frac{(d-1)1_{\{Y_t<0\}}}{2(Y_t-\varepsilon')}\mathrm{d}t \\
&\quad -((\partial\mathcal{R}_t^d - p(\partial\mathcal{R}_t^d'))dL_t^0(X) + dL_t^0(Y),
\end{align*}
\]\nwhere \(\text{sign}(x) := 1_{\{x>0\}} - 1_{\{x<0\}}\). By the uniqueness of the decomposition of a continuous semi-martingale to a continuous local martingale and a continuous bounded variation process, we have
\[
\begin{align*}
\mathrm{d}L_t^0(Y) &= \frac{dL_t^0(Y) + dL_t^0(\bar{Y})}{2} = (|\partial\mathcal{R}_t^d| + p|\partial\mathcal{R}_t^d'|)dL_t^0(X)\quad (3.5)
\end{align*}
\]
By (3.3) and (3.5), the desired SDE follows.

**Proposition 3.2.** \(Y\) has a jointly continuous density function \(p(Y)(t, x, y)\) with respect to the Lebesgue measure on \(\mathbb{R}\). Furthermore, for any \(T \geq 1\), \(p(Y)(t, x, y) \propto \sqrt{t}e^{-|x-y|^2/t}\) for \((t, x, y) \in (0, T) \times \mathbb{R} \times \mathbb{R}\).

**Proof.** This follows from the proof of [3, Proposition 4.4].

In the following propositions, we prove Theorem 1.3.

**Proposition 3.3** (Theorem 1.3(iii)). Fix \(T \geq 1\). Then it holds that
\[
p(t, x, y) = \frac{1}{\sqrt{t}}e^{-p(x, y)^2/t} \quad \text{for } t \in (0, T], x \in \mathbb{R}_x^d \cup \{a^*\}, y \in \mathbb{R}_y^d \cup \{a^*\}.
\]

**Proof.** Since BMVD hits \(a^*\), we have
\[
p(t, x, y) = \int_0^t p(t-s, a^*, x)\mathbb{P}_y(\sigma_{a^*} \in ds)
\]
\[
= \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*)\mathbb{P}_x(\sigma_{a^*} \in dw)\mathbb{P}_y(\sigma_{a^*} \in ds).
\]
Thus \((x, y) \mapsto p(t, x, y)\) depends only on \(|x|_\rho\) and \(|y|_\rho\). For \(a > b > 0\), we have
\[
\int_a^b p(Y)(t, -|x|_\rho, |y|_\rho)d|y|_\rho = \mathbb{P}_{-|x|_\rho}(a \leq Y_t \leq b) = \mathbb{P}_x(X_t \in \mathbb{R}_x^d, a \leq |X_t|_\rho \leq b)
\]
\[
= \int_{\{y \in \mathbb{R}_y^d, a \leq |y|_\rho \leq b\}} p(t, x, y)m_\rho(dy)
\]
\[
= \int_a^b \partial B(0; |y|_\rho + \varepsilon)p(t, x, y)d|y|_\rho.
\]

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Thus, we have $p_Y(t, -|x|_{\rho_0}, |y|_{\rho_0}) = |\partial B(0; |y|)|p(t, x, y) \leq |y|^{d-1}p(t, x, y)$.

By Proposition 3.2, it holds that

$$p(t, x, y) \leq \frac{1}{|y|^{d-1}} \frac{1}{\sqrt{t}} e^{-\frac{(|x|_{\rho_0}-|y|_{\rho_0})^2}{2t}} = \frac{1}{|y|^{d-1}} \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t}. \tag{3.7}$$

Since $\varepsilon \leq |y|$ and (3.7), we have

$$p(t, x, y) \leq \frac{1}{|y|^{d-1}} \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t} \leq \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t}.$$

Moreover, if $|y|_{\rho_0} \leq 1$ we have $p(t, x, y) \geq \frac{1}{\sqrt{t}} e^{-\rho(x,y)^2/t}$ and if $|y|_{\rho_0} > 1$ we have

$$p(t, x, y) \geq \frac{1}{|y|^{d-1}} \frac{1}{\sqrt{t}} \left( \frac{t}{\sqrt{T}} \right)^{(d-1)/2} e^{-\rho(x,y)^2/t} \geq \frac{1}{\sqrt{t}} e^{-(c+1)\rho(x,y)^2/t}.$$

\[\square\]

**Proposition 3.4** (Theorem 1.3(ii)). Fix $T \geq 1$, then for all $t \leq T$, $x, y \in \mathbb{R}^d$, it holds that

$$p(t, x, y) \leq \frac{e^{-\rho(x,y)^2/t}}{\sqrt{t}} + \frac{e^{-|x-y|^2/t}}{t^{d/2}} \left( 1 \wedge \frac{|x|_{\rho_0}}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|_{\rho_0}}{\sqrt{t}} \right) \text{ if } |x|_{\rho_0} \vee |y|_{\rho_0} \leq 1,$$

$$p(t, x, y) \leq \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} \text{ if } |x|_{\rho_0} \vee |y|_{\rho_0} > 1.$$

**Proof.** When $d = 1$, the statement holds from Proposition 3.2, so we assume $d \geq 2$.

For $x, y \in \mathbb{R}^d$, it holds that $p(t, x, y) = p_{R^d}(t, x, y) + p_{R^d}(t, x, y)$. Since $p_{R^d}(t, x, y)$ depends only on $|x|_{\rho_0}$ and $|y|_{\rho_0}$, for $0 \leq a < b$, we have

$$\mathbb{P}_x(\sigma_{a^*} < t, X_t \in \mathbb{R}^d, a \leq |X_t|_{\rho_0} \leq b) = \int_{(a \leq |y|_{\rho_0} \leq b)} \mathbb{P}_{R^d}(t, x, y)m_p(dy) \tag{3.8} \geq \int_{a}^{b} (|y|_{\rho_0} + \varepsilon)^{d-1} \mathbb{P}_{R^d}(t, x, y) dy_{|y|_{\rho_0}}.$$

The left hand side of (3.8) is equal to

$$\mathbb{P}^{(Y)}_{|x|_{\rho_0}} (\sigma_0 < t, Y_t > 0, a \leq Y_t \leq b) = \int_{a}^{b} \int_{0}^{t} p_Y(t-s, 0, |y|_{\rho_0}) \mathbb{P}^{(Y)}_{|x|_{\rho_0}} (\sigma_0 \in ds) dy_{|y|_{\rho_0}}.$$

Here, $\mathbb{P}^{(Y)}$ is a probability measure with respect to $Y$. Thus, by using Proposi-
By (3.2), it follows that

\[
(|y|_\rho + \varepsilon)^{d-1}\mathcal{P}_{R^d}(t, x, y) \asymp \int_0^t p^{(Y)}(t - s, 0, |y|_\rho)\mathcal{P}^{(Y)}_{|x|_\rho}(\sigma_0 \in ds)|y|_\rho \\
\asymp \int_0^t p^{(Y)}(t - s, 0, |y|_\rho)\mathcal{P}^{(Y)}_{|x|_\rho}(\sigma_0 \in ds)|y|_\rho \\
= p^{(Y)}(t, -|x|_\rho) |y|_\rho \\
\asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.9}
\]

Case 1: \(|x|_\rho \vee |y|_\rho \leq 1\): Since \(\varepsilon \leq |y|_\rho + \varepsilon \leq 1 + \varepsilon\), we have by (3.9),

\[
\mathcal{P}_{R^d}(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.10}
\]

If \(\rho(x, y) \asymp |x|_\rho + |y|_\rho\), we obtain \(\mathcal{P}_{R^d}(t, x, y) \asymp \frac{1}{\sqrt{e}} e^{-\rho(x, y)^2/t}\).

If \(\rho(x, y) = |x - y|\) and \(|x|_\rho \wedge |y|_\rho \leq \sqrt{t}\), we may assume \(|x|_\rho \leq \sqrt{t}\) without loss of generality. Then, it holds that

\[
\rho(x, y) \leq |x|_\rho + |y|_\rho \leq \sqrt{t} + |y|_\rho \leq \sqrt{t} + |x - y| - \varepsilon \\
= \sqrt{t} + |x|_\rho + |x - y| \leq 2\sqrt{t} + |x - y|. \tag{3.11}
\]

By (3.11), it holds that \(e^{-\rho(x, y)^2/t} \geq e^{-(|x|_\rho + |y|_\rho)^2/t} \geq e^{-2(2\sqrt{t})^2/t} e^{-\rho(x, y)^2/t}\). Thus, by (3.10), we have \(\mathcal{P}_{R^d}(t, x, y) \asymp \frac{1}{\sqrt{e}} e^{-\rho(x, y)^2/t}\).

If \(\rho(x, y) = |x - y|\) and \(|x|_\rho \wedge |y|_\rho > \sqrt{t}\), by (3.10) and (2.1), we have

\[
p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-|x - y|^2/t} \lesssim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d/2}} e^{-|x - y|^2/t},
\]

\[
p(t, x, y) \gtrsim \frac{1}{\sqrt{t}} e^{-|x - y|^2/t} \gtrsim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d/2}} e^{-|x - y|^2/t}.
\]

Case 2: \(|x|_\rho \vee |y|_\rho > 1\): Without loss of generality, we may assume \(|y|_\rho > 1\). By (3.9), it holds that

\[
\mathcal{P}_{R^d}(t, x, y) \asymp \frac{1}{(|y|_\rho + \varepsilon)^{d-1}} \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
\asymp \frac{1}{2(\varepsilon + 1)(|x|_\rho + |y|_\rho)^{d-1}} \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
\asymp \frac{1}{(|x|_\rho + |y|_\rho)^{d-1}} \left( \frac{|x|_\rho + |y|_\rho^2}{t} \right)^{(d-1)/2} e^{-(|x|_\rho + |y|_\rho^2)/t} \\
\asymp \frac{1}{t^{d/2}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.12}
\]

\[
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\]
By (2.1) and (3.9), we obtain
\[
p(t, x, y) \asymp \frac{1}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t} + \frac{e^{-|x|_\rho|y|_\rho}}{\sqrt{t}(|y|_\rho + \varepsilon)^{d-1}}
\]
\[
\leq \frac{1}{t^{d/2}} \left(e^{-|x-y|^2/t} + e^{-|x|_\rho|y|_\rho}/\sqrt{t}(|y|_\rho + \varepsilon)^{d-1}\right).
\]
(3.13)

If $|x|_\rho \wedge |y|_\rho \leq \sqrt{t}$, then we have $p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}$ in the same way as Case 1.

If $|x|_\rho \wedge |y|_\rho > \sqrt{t}$, then we have $p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}$ since $\rho(x, y) = |x - y| \wedge (|x|_\rho + |y|_\rho)$.

This completes the proof.

\[\square\]

**Proposition 3.5** (Theorem 1.3(i)). Fix $T \geq 1$, then for all $t \in (0, T], x, y \in \mathbb{R}^d$, it holds that
\[
p(t, x, y) \asymp \frac{e^{-\rho(x,y)^2/t}}{\sqrt{t}} + \frac{e^{-|x-y|^2/t}}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) \text{ if } |x|_\rho \vee |y|_\rho \leq 1.
\]
\[
p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} \text{ if } |x|_\rho \vee |y|_\rho > 1.
\]

**Proof.** The proof is the same as that of Proposition 3.4. \[\square\]

This completes the proof of Theorem 1.3.

4 Large time estimate ($d' = 1$)

In this section, we prove Theorem 1.4. Let $d' = 1$. When $d = 1$, $\mathbb{R} \cup \mathbb{R} \cup \{a^\ast\}$ can be identified with $\mathbb{R}$. In this case, BMVD is one-dimensional Brownian motion, so there is nothing to prove. When $d = 2$, it was proved by [3]. Hence we consider the case of $d \geq 3$. In order to obtain sharp estimates, we consider the projection. Let $\varepsilon > 0$ and $S^{d-1}_\varepsilon := \{x \in \mathbb{R}^d : |x| = \varepsilon\}$. We will prove by projecting $(\mathbb{R} \times S^{d-1}_\varepsilon) \# \mathbb{R}^d$ to $\mathbb{R} \cup \mathbb{R}^d \cup \{a^\ast\}$.

The following theorem is a special case of [12, Corollary 6.13].

**Theorem 4.1.** Let $K$ be central part of $M := (\mathbb{R} \times S^{d-1}_\varepsilon) \# \mathbb{R}^d$. Let $E_1 := (M \setminus K) \cap (\mathbb{R} \times S^{d-1}_\varepsilon)$, $E_2 := (M \setminus K) \cap \mathbb{R}^d$, and $E_0 \subset M$ be a precompact open set having smooth boundary and containing $K$. Then heat kernel $\tilde{p}(t, x, y)$ of standard Browian motion $X$ on $M$ satisfies the following estimates for $1 \leq t$.

(i) For $x, y \in E_1$,
\[
\tilde{p}(t, x, y) \asymp \frac{|x|_e |y|_e}{\sqrt{t}(|x|_e + \sqrt{t})(|y|_e + \sqrt{t})} e^{-d(x,y)^2/t}.
\]
(ii) For $x, y \in E_2$,
\[
\hat{p}(t, x, y) \asymp \frac{1}{t^{d/2}|x|^{d-2} |y|^{d-2}} e^{-(|x|_e + |y|_e)^2/2} + \frac{1}{t^{d/2}} e^{-d(x, y)^2/2}.
\]

(iii) For $x \in E_0 \cup E_1, y \in E_0 \cup E_2$,
\[
\hat{p}(t, x, y) \asymp \left( \frac{1}{t^{d/2}} + \frac{|x|_e}{t^{d/2}|y|^{d-2}} \right) e^{-\rho(x, y)^2/2}.
\]

Here, $d$ is a geodesic distance, and $|x|_e := \sup_{z \in K} d(x, z) + 1 + d(x, K)$.

![Diagram](image.png)

Figure 2: $M := (\mathbb{R}_+ \times S^{d-1}_\varepsilon)^\# \mathbb{R}^d$

From now on, we fix $K := ([0] \times \{ x \in \mathbb{R}^d \mid |x| < 1 + \varepsilon \}) \cup ([0, 1) \times S^{d-1}_\varepsilon)$. Then it holds that $M = (\mathbb{R}_+ \times S^{d-1}_\varepsilon) \cup \mathbb{R}_+^d \cup ([0] \times S^{d-1}_\varepsilon)$.

We define $\tilde{m}_p(A) := m(d)(A \cap \mathbb{R}^d) + p \cdot m^{(1,d-1)}(A \cap (\mathbb{R}_+ \times S^{d-1}_\varepsilon))$ for a Borel set $A \subset M$. Here, $m(d)$ and $m^{(1,d-1)}$ are the Lebesgue measures on $\mathbb{R}^d$ and $\mathbb{R}_+ \times S^{d-1}_\varepsilon$, respectively. Then $\tilde{m}_p$-symmetric Brownian motion $\{\hat{X}_t\}$ on $M$ is a time-changed process of standard Brownian motion $\{X_t\}$ on $M$ by a positive continuous additive functional having the Revuz measure $\tilde{m}_p$. To be precise, we have $\hat{X}_t = \hat{X}_t^\tau$, where $A_t := \int_0^t (1_{\mathbb{R}^d} + p1_{\mathbb{R}_+ \times S^{d-1}_\varepsilon})(\hat{X}_s) ds$ and $\tau_t := \{ s > 0 \mid A_s > t \}$. Let $\tilde{p}(t, x, y)$ (resp. $\hat{p}(t, x, y)$) be the heat kernel of $\{\hat{X}_t\}$ (resp. $\{X_t\}$). Since $(1 \land p)t \leq A_t \leq (1 \lor p)t$ and $\frac{1}{1+p} \leq \tau_t \leq \frac{1}{1-p}$, we have $\tilde{p}(t, x, y) \asymp \hat{p}(t, x, y)$. Thus $\hat{p}(t, x, y)$ satisfies the same estimates as Theorem 4.1.

We consider the projection of $\hat{X}$. We define
\[
\hat{X}_t := \begin{cases} 
\hat{X}_t & : \hat{X}_t \in \mathbb{R}^d, \\
\hat{X}_t^{(1)} & : \hat{X}_t \in \mathbb{R}_+ \times S^{d-1}_\varepsilon.
\end{cases}
\]

Here $\hat{X}_t^{(1)} \in \mathbb{R}_+$ is the first element of $\hat{X}_t$.

**Theorem 4.2.** $\hat{X}$ on $\mathbb{R}^d \cup \mathbb{R}_+ \cup \{a^*\}$ satisfies Definition 1.2 (i) and (ii).
**Remark 4.3.** $\tilde{X}$ is not the Markov process. Indeed, once $\tilde{X}$ hits $a^*$, it remember where it came from.

In order to prove Theorem 4.2, we need the following lemma. Note that the Dirichlet form $(\tilde{E}, \tilde{F})$ on $L^2(M; \tilde{m}_p)$ associated with $\tilde{X}$ is $\tilde{F} = H^1(M)$ and $\tilde{E}(f, g) = \frac{1}{2} \int_M \nabla f \cdot \nabla g \, d\tilde{m}_p$ for $f, g \in \tilde{F}$.

**Lemma 4.4.** For $\tilde{m}_p$-symmetric Brownian motion $Z$ on $\mathbb{R}_+ \times S^{d-1}_\varepsilon$, $Z^{(1)}$ is $pm^{(1)}$-symmetric Brownian motion on $\mathbb{R}_+$. Here $Z^{(1)}_t$ is the first element of $Z_t$ and $m^{(1)}$ is the Lebesgue measure on $\mathbb{R}$.

**Proof.** $(\mathcal{E}^Z, \mathcal{F}^Z)$ denotes the Dirichlet form on $L^2(\mathbb{R}_+ \times S^{d-1}_\varepsilon; \tilde{m}_p)$ associated with $Z$, $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ denotes the Dirichlet form on $L^2(\mathbb{R}_+; pm^{(1)})$ associated with $Z^{(1)}$, $R^{(1)}_1 f(x) := \int_0^\infty e^{-t} \mathbb{E}_x(f(Z^{(1)}_t)) \, dt$ denotes a 1-order resolvent for $f \in L^2(\mathbb{R}_+; pm^{(1)})$ and $(\mathcal{E}^*, \mathcal{F}^*)$ denotes the Dirichlet form on $L^2(\mathbb{R}_+; pm^{(1)})$ associated with $pm^{(1)}$-symmetric Brownian motion on $\mathbb{R}_+$.

Then it is sufficient to prove $\mathcal{E}^{(1)}_1 (R^{(1)}_1 f, g) = (f, g)_{pm^{(1)}}$ for $f \in L^2(\mathbb{R}_+; pm^{(1)})$, $g \in \mathcal{F}^*$, where $(\cdot, \cdot)_{pm^{(1)}}$ is $L^2$-inner product of $pm^{(1)}$.

For $f \in L^2(\mathbb{R}_+; pm^{(1)})$, we define $\hat{f} : \mathbb{R}_+ \times S^{d-1}_\varepsilon \to \mathbb{R}$ as $\hat{f}(x_1, x_2) := f(x_1)$. Since $\|\hat{f}\|_{L^2(pm^{(1)})} = \|f\|_{L^2(pm^{(1)} \times \sigma(S^{d-1}_\varepsilon))} < \infty$ for the Lebesgue measure $\sigma$ on $S^{d-1}_\varepsilon$, it holds that $\hat{f} \in L^2(\mathbb{R}_+ \times S^{d-1}_\varepsilon; \tilde{m}_p)$ and for all fixed $x_2 \in S^{d-1}_\varepsilon$,

$$R^{(1)}_1 f(x_1) = \int_0^\infty e^{-t} \mathbb{E}_{x_1}(f(Z^{(1)}_t)) \, dt = \int_0^\infty e^{-t} \mathbb{E}_{(x_1, x_2)}(\hat{f}(Z_t)) \, dt.$$  

Thus we have $R^{(1)}_1 f = R^Z_1 \hat{f} \in \mathcal{F}^Z$, where $R^Z_1 \hat{f}$ is the 1-order resolvent for $\hat{f}$. 

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Figure 3: Projection $M$ to $\mathbb{R}^d_+ \cup \mathbb{R} \cup \{a^*\}$ and $\tilde{X}$ to $\tilde{X}$
Similarly, we can get \( \hat{g} \in \mathcal{F}^Z \) from \( g \in \mathcal{F}^* \). Then we have

\[
\mathcal{E}_1^*(R_1^{(1)} f, g) = \frac{\sigma}{2} \int_{\mathbb{R}^+} \frac{dR_1^{(1)} f}{dx_1} \frac{dg}{dx_1} dx_1 + (R_1^{(1)} f, g)_{pm(1)}
\]

\[
= \left( \frac{1}{\sigma(S^d_{d-1})} \right) \mathcal{F}(\tilde{R}_1^{Z} \hat{f}, \hat{g}) \cdot \frac{1}{\sigma(S^d_{d-1})}
\]

\[
= (f, g)_{pm(1)}.
\]

\( \square \)

**Proof of Theorem 4.2.** (i) By definition, \( \hat{X}^{R_+}_d \), the part process of \( \hat{X} \) on \( \mathbb{R}^d_+ \), is \( \hat{X}^{R_+}_d \). Thus, associated Dirichlet form \( (\mathcal{E}_1^{R_+}_d, \mathcal{F}_1^{R_+}_d) \) on \( L^2(\mathbb{R}^d_+) \) can be written

\[
\mathcal{F}_1^{R_+}_d = \hat{f}^{R_+}_d = \{ f \in H^1(M) \mid f = 0 \text{ in } (\mathbb{R}^d_+)^c \}
\]

\[
= \{ f : (\mathbb{R}^d_+ \times S^d_{d-1}) \cup \mathbb{R}^d_+ \rightarrow \mathbb{R} \mid f = f1_{\mathbb{R}^d_+} \in H^1(\mathbb{R}^d_+) \},
\]

\[
\mathcal{E}_1^{R_+}_d(f, g) = \frac{1}{2} \int_{\mathbb{R}^d_+} \nabla f \cdot \nabla g \, dx \quad \text{for } f, g \in \mathcal{F}_1^{R_+}_d.
\]

Therefore \( \hat{X}^{R_+}_d \) is an absorbing Brownian motion on \( \mathbb{R}^d_+ \). For \( \hat{X}^{R_+}_d \), the part process of \( \hat{X} \) on \( \mathbb{R}_+ \), we have \( \hat{X}^{R_+} = (\hat{X}^{(1)}_t)_{\mathbb{R}_+} \). Hence, by Lemma 4.4, this is absorbing Brownian motion on \( \mathbb{R}_+ \).

(ii) Since \( \hat{X} \) is conservative, \( \hat{X} \) admits no killings on \( a^* \).

\( \square \)

**Proposition 4.5.** Let \( \tilde{p}(t, x, y) \) be a transition density of \( \hat{X} \) with respect to \( m_p \). Then it holds that

\[
\tilde{p}(t, x, y) = \begin{cases} 
\tilde{p}(t, \tilde{x}, y) & : y \in \mathbb{R}^d, \\
\int_{S^d_{d-1}} \tilde{p}(t, \tilde{x}, (y, y_2)) \, dy_2 & : y \in \mathbb{R}^d_+ \cup \{a^*\}.
\end{cases}
\]

\( \text{(4.1)} \)

Here, for \( x \in \mathbb{R}^d_+ \cup \mathbb{R}^d_+ \cap \{a^*\} \) and fixed \( x_2 \in S^d_{d-1} \), define \( (a^*, x_2) := (0, x_2) \) and

\[
\tilde{x} := \tilde{x}(x_2) := \begin{cases} 
x & : x \in \mathbb{R}^d_+, \\
(x, x_2) & : x \in \mathbb{R}^d_+ \cup \{a^*\}.
\end{cases}
\]

\( \text{(4.2)} \)

**Proof.** For any \( f \in C_c(\mathbb{R}^d_+ \cup \mathbb{R}^d_+ \cup \{a^*\} \cap \mathcal{F}) \), we define \( \tilde{f} : M \rightarrow \mathbb{R} \) by

\[
\tilde{f}(\tilde{x}) := \begin{cases} 
f(\tilde{x}) & : \tilde{x} \in \mathbb{R}^d_+, \\
f(\tilde{x}_1) & : \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in (\mathbb{R}_+ \cup \{0\}) \times S^d_{d-1}.
\end{cases}
\]

\[19\]
Now, for \( x \in \mathbb{R}_+ \cup \mathbb{R}_+^d \cup \{a^*\} \), we take \( x_2, x_3^* \in S^{d-1}_z \) and define \( \tilde{x}(x_2), \tilde{x}(x_3^*) \in M \) as (4.2). Then, since \( \tilde{f} \) is independent of \( x_2 \) and \( x_3^* \), it holds that

\[
\mathbb{E}_x(f(\tilde{X}_t)) = \mathbb{E}_{\tilde{x}(x_2)}(\tilde{f}(\tilde{X}_t)) = \mathbb{E}_{\tilde{x}(x_3^*)}(\tilde{f}(\tilde{X}_t)).
\]

Thus we have \( \tilde{p}(t, \tilde{x}(x_2), y) = \tilde{p}(t, \tilde{x}(x_3^*), y) \) for all \( y \), so we simply write \( \tilde{x} \) as \( \tilde{x}(x_2) \). Moreover, we have

\[
\mathbb{E}_x(f(\tilde{X}_t)) = \mathbb{E}_{\tilde{x}}(\tilde{f}(\tilde{X}_t))
\]

(4.3)

\[
= \int_{\mathbb{R}_+^d} \tilde{f}(\tilde{y})\tilde{p}(t, \tilde{x}, \tilde{y})\hat{m}_p(dy) + \int_{\mathbb{R}_+ \times S^{d-1}_z} \tilde{f}(\tilde{y})\tilde{p}(t, \tilde{x}, \tilde{y})\hat{m}_p(dy)
\]

\[
= \int_{\mathbb{R}_+^d} f(y)\tilde{p}(t, \tilde{x}, y)m_p(dy) + \int_{\mathbb{R}_+} f(y)\left(\int_{S^{d-1}_z} \tilde{p}(t, \tilde{x}, (y, y_2))dy_2\right)m_p(dy).
\]

The left hand side of (4.3) is equal to

\[
\int_{\mathbb{R}_+^d} f(y)\tilde{p}(t, x, y)m_p(dy) + \int_{\mathbb{R}_+} f(y)\tilde{p}(t, x, y)m_p(dy).
\]

Thus it yields (4.1).

\[\square\]

**Proof of Theorem 1.4.** Fix large \( T > 0 \). First, note that \( \tilde{X} \) fails the Markov property when it hits the boundary of \( S^{d-1}_z \). If \( \tilde{X} \) hits the boundary of \( S^{d-1}_z \), rotate the next excursion of \( \tilde{X} \) by a random angle uniform over \([0, 2\pi)^d\) and continue this process. Then this new process is the Markov process and BMVD on \( \mathbb{R}_+ \cup \mathbb{R}_+^d \cup \{a^*\} \), so it is sufficient to estimate the heat kernel of the process by the uniqueness of BMVD. For \( f \in C_c(\mathbb{R}_+ \cup \{a^*\}) \) and \( x \in \mathbb{R}_+ \cup \mathbb{R}_+^d \cup \{a^*\} \), we have \( \mathbb{E}_x(f(X_t)) = \mathbb{E}_x(f(\tilde{X}_t)) \), so \( p(t, x, y) = \tilde{p}(t, x, y) \) holds for \( y \in \mathbb{R}_+ \cup \{a^*\} \). Thus, by Proposition 4.5, we have

\[
p(t, x, y) \asymp \int_{S^{d-1}_z} \tilde{p}(t, \tilde{x}, (y, y_2))dy_2 \text{ for } y \in \mathbb{R}_+ \cup \{a^*\}, \tag{4.4}
\]

where \( \tilde{x} \) is given as in (4.2).

We next consider the relation between the distance \( d \) on \( M \) and \( \rho \) on \( \mathbb{R}_+ \cup \mathbb{R}_+^d \cup \{a^*\} \).

(i) (Figure 4, left) For \( x, y \in \mathbb{R}_+ \cup \{a^*\} \), since \( S^{d-1}_z \) is bounded, there exists a constant \( C > 0 \) with \( \rho(x, y) \leq d(\tilde{x}, \tilde{y}) \leq C + \rho(x, y) \). Hence, for \( t \geq T \), it holds that \( e^{-\rho(x, y)^2/t} \asymp e^{-d(\tilde{x}, \tilde{y})^2/t} \) and \( |\tilde{x}| \asymp 1 + d(\tilde{x}, K) = |x| = |x|^\rho \).

(ii) (Figure 4, right) For \( x \in \mathbb{R}_+^d, y \in \mathbb{R}_+ \cup \{a^*\} \), since \( S^{d-1}_z \) is bounded, there exists a constant \( C > 0 \) with \( \rho(x, y) \leq d(x, y) \leq C + \rho(x, y) \). Then for \( t \geq T \), it holds that \( e^{-\rho(x, y)^2/t} \asymp e^{-d(x, y)^2/t} \).

Thus, for \( x \) or \( y \in \mathbb{R}_+ \cup \{a^*\} \), the desired estimates follow from Theorem 4.1 and the boundedness of \( S^{d-1}_z \).
For $x, y \in \mathbb{R}^2$ with $|x|_{\rho} \land |y|_{\rho} > 1$ and $t \geq T$, by the above estimate, Lemma 2.8, Theorem 1.3, Lemma 2.5, Proposition 2.4, Lemma 2.10 and Lemma 2.7, we obtain that

$$p(t, x, y) = \int_0^{t/2} + \int_{t/2}^{t-1} + \int_0^{t} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}^2}(t, x, y)$$

$$\lesssim \left( \frac{e^{-|y|_{\rho}^2/t}}{t^{d/2}} + \frac{e^{-|y|_{\rho}^2/2}}{|t|} \right) e^{-|x|_{\rho}^2/t}$$

$$+ \int_{t/2}^{t-1} + \int_0^{t} p(t-s, a^*, y) e^{-|y|_{\rho}^2/2} ds + \frac{1}{t^{d/2}} e^{-|x-y|_{\rho}^2/t}$$

and

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}^2}(t, x, y)$$

$$\gtrsim \left( \frac{x|y|_{\rho}}{t^{d/2}} + \frac{1}{t^{d/2}} \right) e^{-|x-y|_{\rho}^2/t}.$$
and
\[ p(t, x, y) \geq \int_0^{t/2} p(t - s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \gtrsim \left( \frac{|x|}{t^{3/2} |y|^{d/2}} + \frac{1}{t^{d/2}} \right) e^{-\rho(x,y)^2/t}. \]

This completes the proof of Theorem 1.4. \( \square \)

**Remark 4.6.** In [14], the heat kernel estimate for Brownian motion on \((\mathbb{R}_+ \times S^1)^\# \mathbb{R}^2\) is obtained. Therefore, by the same way as in this section, we can obtain the large time estimate on \(\mathbb{R}_+ \cup \mathbb{R}_2^2 \cup \{a^*\}\). By elementary computations, this estimate is the same as the one appearing in [3].

## 5 Large time estimate \((d' \geq 3)\)

In this section, we will prove Theorem 1.7. We assume \(d \geq d' \geq 3\). Moreover, we may assume \(\varepsilon, \varepsilon' < 1\) without loss of generality.

For \(x, y \in \mathbb{R}^d_\varepsilon\), it holds that
\[ p(t, x, y) = p_d(t, x, y) + \int_0^t \int_0^{t-s} p(t - s - u, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in du) \mathbb{P}_y(\sigma_{a^*} \in ds). \]

For \(x \in \mathbb{R}^d_\varepsilon, y \in \mathbb{R}^d_{\varepsilon'}\), it holds that
\[ p(t, x, y) = \int_0^t \int_0^{t-s} p(t - s - u, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in du) \mathbb{P}_y(\sigma_{a^*} \in ds). \]

So, we consider the estimate of \(p(t, a^*, a^*)\) in order to prove Theorem 1.7.

**Proposition 5.1.** For \(t > 0\), we have
\[ p(t, a^*, a^*) \lesssim \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}. \]

**Proof.** By Proposition 2.2 and the small time estimate (Theorem 1.3), we have
\[ p(t, a^*, a^*) \lesssim \frac{1}{\sqrt{t}} \wedge \left( \frac{1}{t^{d/2}} \wedge \frac{1}{t^{d'/2}} \right) = \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}. \]

\( \square \)

**Proposition 5.2.** For \(t > 0\),
\[ p(t, a^*, a^*) \asymp \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}. \tag{5.1} \]

**Proof.** By Theorem 1.3 and Lemma 2.5, for \(t \geq 2\) and \(x \in \mathbb{R}^d_{\varepsilon'}\) with \(\sqrt{7} \leq |x| \leq 2\sqrt{t}\),
\[ p(t, a^*, x) = \int_0^t p(t - s, a^*, a^*) \mathbb{P}(\sigma_{a^*} \in ds) \geq \int_{t-1}^t p(t - s, a^*, a^*) \mathbb{P}(\sigma_{a^*} \in ds) \gtrsim \int_{t-1}^t (t - s)^{-1/2} \frac{|x|^2}{|x|^2} e^{-\rho(x,y)^2/s} ds, \tag{5.2} \]
Since \( t/2 \leq t - 1 \) and \( \sqrt{2} - \varepsilon \leq |x|_\rho \), we have
\[
p(t, a^*, x) \gtrsim \frac{e^{-|x|_\rho^2/t}}{t^{d/2} + t^{3/2}|x|(d-3)/2} \geq \frac{e^{-(2\sqrt{2})^2/t}}{t^{d/2} + t^{3/2}(2\sqrt{2})(d-3)/2} \gtrsim \frac{1}{t^{d/2}}. \tag{5.3}
\]

By the Markov property and (5.3), we have
\[
p(2t, a^*, a^*) \geq \int_{\{x \in \mathbb{R}^d : \sqrt{7} \leq |x| \leq 2\sqrt{7}\}} p(t, a^*, x)^2 m_p(dx) \]
\[
\gtrsim \int_{\{x \in \mathbb{R}^d : \sqrt{7} \leq |x| \leq 2\sqrt{7}\}} t^{-d} m_p(dx) \]
\[
= \int_{\sqrt{7}}^{2\sqrt{7}} r^{d-1} dr \times pt^{-d} = \frac{1}{t^{d/2}}, \tag{5.4}
\]
where we used polar coordinates \( r := |x| \). (5.4) and the small time estimate (Theorem 1.3) imply \( p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d/2}} \) for \( t > 0 \). Thus (5.1) follows from it and Proposition 5.1.

We will prove Theorem 1.7, by using the on-diagonal estimate at \( a^* \) and hitting probability.

**Proposition 5.3.** Let \( d \geq d' \geq 3 \). Then \( p(t, x, y) \) satisfies the following estimates when \( 1 \leq t \):

(i) For \( x, y \in \mathbb{R}^d \), \( p(t, x, y) \lesssim t^{-d'/2} e^{-\rho(x,y)^2/t} \).

(ii) For \( x, y \in \mathbb{R}^d \) with \( |x|_\rho \vee |y|_\rho \leq 1 \), \( p(t, x, y) \lesssim t^{-d'/2} e^{-\rho(x,y)^2/t} \).
For \( x, y \in \mathbb{R}^d \) with \( |x|_\rho \vee |y|_\rho > 1 \),
\[
p(t, x, y) \lesssim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-(|x|_\rho+|y|_\rho)^2/t} + \frac{1}{t^{d'/2}} e^{-\rho(x,y)^2/t}.
\]

(iii) For \( x \in \mathbb{R}^d \cup \{a^*\}, y \in \mathbb{R}^d \cup \{a^*\} \),
\[
p(t, x, y) \lesssim \left( \frac{1}{t^{d'/2}|y|^{d-2}} + \frac{1}{t^{d'/2}|x|^{d-2}} \right) e^{-\rho(x,y)^2/t}.
\]

**Proof.** We will prove the estimates by comparing with \((\mathbb{R}^d \times S^d_{\varepsilon-d}) \# \mathbb{R}^d\). First, we assume \( \varepsilon \leq \varepsilon' \) (See Figure 5).

Let \( \tilde{p}(t, x, y) \) be the heat kernel of Brownian motion \( \tilde{X} \) on \((\mathbb{R}^d \times S^d_{\varepsilon-d}) \# \mathbb{R}^d\), where \( S^d_{\varepsilon-d} := \{ x \in \mathbb{R}^{d-d+1} : |x| = \varepsilon \} \). According to [12, Example 4.5 and Example 5.5], for \( t > 1 \), \( \tilde{p}(t, x, y) \) has sharp estimates as the right hand side of this proposition up to the difference between \( \rho \) and \( d \), where \( d \) is a geodesic distance on \((\mathbb{R}^d \times S^d_{\varepsilon-d}) \# \mathbb{R}^d\). Furthermore, let \( K := (\overline{B}(0; \varepsilon') \times S^d_{\varepsilon-d}) \cup \overline{B}(0; \varepsilon) \) then, for \( t > 1 \) and \( \tilde{x}, \tilde{y} \in K \), it holds that \( \tilde{p}(t, \tilde{x}, \tilde{y}) \asymp t^{-d'/2} \).
Here $\overline{B}^d$ is a closed ball on $\mathbb{R}^d$. By combining with small time estimates ([12, Theorem 5.10]), for $t > 0$ and $\tilde{x}, \tilde{y} \in K$, we obtain $\tilde{p}(t, \tilde{x}, \tilde{y}) \asymp t^{-d/2} e^{-d(\tilde{x}, \tilde{y})/t} \wedge t^{-d'/2}$.

By proposition 5.2, we have $p(t, a^*, a^*) \asymp t^{-1/2} \wedge t^{-d'/2} \leq t^{-d'/2} \wedge t^{-d'/2}$,

$$P_x(\sigma_{a^*} \in ds) = \tilde{P}_x(\tilde{\sigma}_K \in ds), \quad p_{R^d}(t, x, y) = \tilde{p}_{R^d \setminus K}(t, x, y) \text{ for } x, y \in \mathbb{R}^d$$

$$\bar{P}_x(\sigma_{a^*} \in ds) = \tilde{\bar{P}}_{(x, x_2)}(\tilde{\sigma}_K \in ds) \text{ for } x \in \mathbb{R}^d_e, \ x_2 \in S^{d-d'}_d,$$

where $\tilde{\bar{P}}, \tilde{\sigma}_K$ and $\tilde{p}_{R^d \setminus K}$ are those for the process $\tilde{X}$. Moreover, for $x, y \in \mathbb{R}^d_e$, $x_2 \in S^{d-d'}_d$, by the projection for the part process on $\mathbb{R}^d_e$, which is the same reason as the proof of Proposition 4.5 and continuity of $\tilde{p}$,

$$p_{R^d_e}(t, x, y) = \int_{y_2 \in S^{d-d'}_d} \tilde{p}_{R^d_e \setminus K}(t, (x, x_2), (y, y_2)) \lesssim \max_{y_0 \in S^{d-d'}_d} \tilde{p}_{R^d_e \setminus K}(t, (x, x_2), (y, y_0)).$$

Hence we have

$$p(t, x, y) = p_{R^d_e}(t, x, y) + \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*) P_x(\sigma_{a^*} \in dw) P_y(\sigma_{a^*} \in ds)$$

$$= p_{R^d_e}(t, x, y) + \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*) P_x(\sigma_{a^*} \in dw) P_y(\sigma_{a^*} \in ds)$$

$$\lesssim \tilde{p}_{R^d_e \setminus K}(t, x, y) + \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} \tilde{p}(t-s-w, \tilde{X}_s, \tilde{X}_w) \tilde{P}_x(\tilde{\sigma}_K \in dw) \tilde{P}_y(\tilde{\sigma}_K \in ds)$$

$$= \tilde{p}(t, x, y) \text{ for } x, y \in \mathbb{R}^d.$$
\[ p(t, x, y) = \int_0^t \int_0^{t-s} p(t - s - w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \]
\[ = \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} p(t - s - w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \]
\[ \lesssim \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} \tilde{p}(t - s - w, \tilde{X}_s, \tilde{X}_w) \tilde{\mathbb{P}}_{\tilde{\sigma}_K} \tilde{p}(\tilde{\sigma}_K \in dw) \tilde{\mathbb{P}}_{\tilde{y}}(\tilde{\sigma}_K \in ds) \]
\[ = \tilde{p}(t, x, y) \quad \text{for } x \in \mathbb{R}_d^d, y \in \mathbb{R}_d^{d'}, y_2 \in S^{d-d'}_\varepsilon \text{ and } \tilde{y} := (y, y_2), \]
and
\[ p(t, x, y) = p_{\mathbb{R}_d^{d'}}(t, x, y) + \int_0^t \int_0^{t-s} p(t - s - w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \]
\[ = p_{\mathbb{R}_d^{d'}}(t, x, y) + \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} \tilde{p}(t - s - w, a^*, a^*) \tilde{\mathbb{P}}_{\tilde{\sigma}_K} \tilde{p}(\tilde{\sigma}_K \in dw) \tilde{\mathbb{P}}_{\tilde{y}}(\tilde{\sigma}_K \in ds) \]
\[ \lesssim \tilde{p}_{\mathbb{R}_d^{d'}}(t, \tilde{x}, \tilde{y}) + \tilde{E}_x \tilde{E}_y \int_0^t \int_0^{t-s} \tilde{p}(t - s - w, \tilde{X}_s, \tilde{X}_w) \tilde{\mathbb{P}}_{\tilde{\sigma}_K} \tilde{p}(\tilde{\sigma}_K \in dw) \tilde{\mathbb{P}}_{\tilde{y}}(\tilde{\sigma}_K \in ds) \]
\[ = \tilde{p}(t, \tilde{x}, \tilde{y}), \]
where we denote \( \tilde{x} := (x, x_2), \tilde{y} := (y, y_2) \) for \( x, y \in \mathbb{R}_d^{d'} \) and
\( x_2, y_2 \in S^{d-d'}_\varepsilon \) with
\[ \max_{y_0 \in S^{d-d'}_\varepsilon} \tilde{p}_{\mathbb{R}_d^{d'}}(t, (x, x_2), (y, y_0)) \]
\[ = \tilde{p}_{\mathbb{R}_d^{d'}}(t, (x, x_2), (y, y_2)). \]

In the above inequalities, we used the following estimates in order to treat the effect of \( e^{-d(\tilde{x}, \tilde{y})/t} \) appearing in the estimate of \( \tilde{p}(t, \tilde{x}, \tilde{y}) \) for \( t < 1, \tilde{x}, \tilde{y} \in K \). For \( x, y \in \mathbb{R}_d^d \), we have
\[ \int_{\{0 \leq t-s-w \leq 1, s \geq w\}} p(t - s - w, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \mathbb{P}_x(\sigma_{a^*} \in dw) \]
\[ \lesssim \int_0^t \int_0^{t-w} p(t - s - w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \]
\[ \lesssim \int_0^t \int_0^{t-w} \int_{(w-t-w)/2}^{(w-t-w)/2} \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \]
\[ \leq 2 e^{-|y|^2/t} \mathbb{P}_x(\sigma_{a^*} \leq t) + \int_0^{t/2} \left( \frac{t}{2} - w \right)^{-1/2} e^{-|y|^2/t} \mathbb{P}_x(\sigma_{a^*} \leq t) \]
\[ \lesssim \frac{e^{-|y|^2/t}}{t^{d/2}} + \frac{e^{-|y|^2/t}}{t^{d/2}} \leq \tilde{p}(t, x, y). \]

Thus, by the symmetry, we have
\[ \int_{\{t-s-w \leq 1\}} p(t - s - w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \lesssim \tilde{p}(t, x, y). \]

The same inequalities hold for the cases of \( x \in \mathbb{R}_d^d, y \in \mathbb{R}_d^{d'}, \) and \( x, y \in \mathbb{R}_d^{d'}. \)

By the compactness of \( K \), we can ignore the difference between \( \rho \) and \( d \) and derive upper estimates similarly as in the proof of Theorem 1.4.
If \( \varepsilon > \varepsilon' \), we can prove in the same way as above by exchanging \((\mathbb{R}^d \times S_d^2 - \varepsilon') \# \mathbb{R}^d \) and \( K \) to \((\mathbb{R}^d \times S_d^2 - \varepsilon') \# \mathbb{R}^d \) and \((\mathbb{R}^d \times S_d^2 - \varepsilon') \# \mathbb{R}^d \) and \((\mathbb{R}^d \times S_d^2 - \varepsilon') \# \mathbb{R}^d \), respectively.

**Remark 5.4.** One can prove Proposition 5.3 directly by using the estimates of \( p(t, a^*, a^*) \) and \( \mathbb{P}_x (\sigma_{a^*} \in ds) \).

**Proof of Theorem 1.7.** The upper estimates are already proved in Proposition 5.3, so we consider the lower estimates. In this proof, let \( T > 3 \) be large, and \( t \in [T, \infty) \).

**Step 1** (the estimate of \( p(t, x, a^*) \))

(1) For \( x \in \mathbb{R}^d \), if \( |x|_\rho \geq 1 \), by the Markov property, Theorem 1.3, (5.1), Lemma 2.5, Lemma 2.7 and Lemma 2.8, we have

\[
p(t, x, a^*) \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x (\sigma_{a^*} \in ds) + \int_{t/2}^t p(t-s, a^*, a^*) \mathbb{P} (\sigma_{a^*} \in ds) \\
\geq t^{-d/2} \mathbb{P} (\sigma_{a^*} \leq \frac{t}{2}) + \int_{t/2}^t (t-s)^{-1/2} |x|_{\rho} \mathbb{P} (\sigma_{a^*} \in ds) \\
\geq \left( \frac{1}{t^{d/2} |x|_{\rho}^{d-2}} + \frac{1}{t^{d/2}} \right) e^{-|x|_{\rho}^2 / t}.
\]

For \( x \in \mathbb{R}^d \), if \( |x|_\rho < 1 \), by the Markov property, (5.1) and Lemma 2.8, we have

\[
p(t, x, a^*) \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x (\sigma_{a^*} \in ds) \geq \frac{1}{t^{d/2}} e^{-|x|_{\rho}^2 / t}.
\]

(2) For \( x \in \mathbb{R}^d \), we can prove in the same way as in the case of \( x \in \mathbb{R}^d \). Since the estimate of \( p(t, a^*, a^*) \) depends only on \( d' \), we can derive \( p(t, x, a^*) \geq t^{-d/2} e^{-|x|_{\rho}^2 / t} \) from (1) by changing \( d \) to \( d' \).

**Step 2** (Theorem 1.7(i) and (ii))

(1) For \( x, y \in \mathbb{R}^d \), by (5.1), (2.1), Step 1, Lemma 2.10 and Lemma 2.8, we have

\[
p(t, x, y) = p_{\mathbb{R}^d}(t, x, y) + p_{\mathbb{R}^d}(t, x, y) \\
\quad \geq \frac{1}{t^{d/2}} |x|_{\rho} \mathbb{P}_y (\sigma_{a^*} \in ds) \\
\quad \geq \frac{1}{t^{d/2}} |x|_{\rho} \mathbb{P}_y (\sigma_{a^*} \leq \frac{t}{2}) \\
\quad \geq \frac{1}{t^{d/2}} |x|_{\rho} e^{-\rho(x, y)^2 / t} + p(t, a^*, x) \mathbb{P}_y (\sigma_{a^*} \leq \frac{t}{2}) \\
\quad \geq \frac{1}{t^{d/2}} |x|_{\rho} e^{-\rho(x, y)^2 / t} + p(t, a^*, x) e^{-\rho(x, y)^2 / t}.
\]

(a) If \(|x|_\rho \vee |y|_\rho \leq 1 \), by (5.5) and Lemma 2.10, we have

\[
p(t, x, y) \geq 0 + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2 / t} \geq \frac{1}{t^{d/2}} e^{-\rho(x, y)^2 / t}.
\]
(b) If \( |x|_\rho > 1 \geq |y|_\rho > \frac{1}{2} \), by (5.5), we have
\[
p(t, x, y) \gtrsim \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} + \frac{e^{-|x|_\rho + |y|_\rho)^2/t}}{t^{d/2} |x|^{d-2} |y|^{d-2}} + 0.
\]
(c) If \( |x|_\rho > 1, \frac{1}{2} \geq |y|_\rho \), by (5.5) and Lemma 2.10 (iii), we have
\[
p(t, x, y) \gtrsim \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} + \left( \frac{1}{t^{d/2} |x|^{d-2}} + \frac{1}{t^{d/2} |y|^{d-2}} \right) e^{-|x|_\rho + |y|_\rho)^2/t/t}
\]
\[
\gtrsim 0 + \left( \frac{1}{t^{d/2} |x|^{d-2}} + \frac{1}{t^{d/2} |y|^{d-2}} \right) e^{-|x|_\rho + |y|_\rho)^2/t/t}
\]
\[
\gtrsim \frac{1}{t^{d/2} |x|^{d-2}} e^{-|x|_\rho + |y|_\rho)^2/t/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t/t}.
\]
By the above estimates (a)-(c) and using the symmetry of \( p(t, x, y) \), we obtain the estimates in Theorem 1.7 (ii).

(2) For \( x, y \in \mathbb{R}^d_0 \), we can prove in the same way as in the case of \( x, y \in \mathbb{R}^d_0 \). Since the estimate of \( p(t, a^*, x) \) depends only on \( d' \), and we can derive
\[
p(t, x, y) \gtrsim \frac{e^{-\rho(x,y)^2/t}}{t^{d'/2}}
\]
from (1) by changing \( d \) to \( d' \).

**Step 3 (Theorem 1.7 (iii))**

For \( x \in \mathbb{R}^d_0, y \in \mathbb{R}^{d'_2} \), by Step1, Lemma 2.5, Lemma 2.7 and Lemma 2.8, we obtain
\[
p(t, x, y) \geq \int_0^{t/2} + \int_{t/2}^{t-1} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds)
\]
\[
\gtrsim \frac{e^{-|x|^2/t}}{t^{d/2}} \mathbb{P}_x(\sigma_{a^*} \leq \frac{t}{2}) + \int_{t/2}^{t-1} \frac{e^{-|y|^2/(t-s)}}{t^{d/2} + t^{d/2} + |x|^{d-2} |x|^{d-2} |y|^{d-2}} ds
\]
\[
\gtrsim \frac{1}{t^{d/2} |x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} \frac{e^{-|y|^2/(t-s)}}{t^{d/2} + |x|^{d-2} |x|^{d-2} |y|^{d-2}} ds \geq 0.
\]

(a) If \( |x|_\rho < 1 \), by (5.6) and \( |y| \geq \varepsilon' \), we have
\[
p(t, x, y) \gtrsim \frac{1}{t^{d/2} |x|^{d-2}} e^{-\rho(x,y)^2/t} + 0 \geq \left( \frac{1}{t^{d/2} |x|^{d-2}} + \frac{1}{t^{d/2} |y|^{d-2}} \right) e^{-\rho(x,y)^2/t}.
\]
(b) If \( |x|_\rho \geq 1, |y|_\rho \leq 1 \), by (5.6) and \( 3 < T \leq t \), we have
\[
p(t, x, y) \gtrsim \frac{1}{t^{d/2} |x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} \frac{e^{-1}}{t^{d/2} + t^{d/2} |x|^{d-2} |y|^{d-2}} ds \geq 0.
\]
(c) If $|x|_p \geq 1$, $1 < |x|_p < |y| < \sqrt{t}/2$, by (5.6) and let $\theta := \frac{|y|^2}{t}$, we have

$$p(t, x, y) \gtrsim \frac{1}{td'/2|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} e^{-|y|^2/(t-s)} \frac{ds}{(t-s)^{d'/2}} \frac{|x|_p e^{-|x|_p^2/t}}{|x|}$$

$$\times \frac{1}{td'/2|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{2|y|_p^2/t}^{|y|_p^2} e^{-\theta \rho^{d'/2-2} - 2} \frac{d\theta}{2\rho^{d'/2}} e^{-|y|_p^2/t}$$

$$\times \left(\frac{1}{td'/2|x|^{d-2}} + \frac{1}{td'/2|y|^{d-2}}\right) e^{-\rho(x,y)^2/t}.$$

(d) If $|x|_p \geq 1$, $\sqrt{t}/2 \leq |y|$, by (5.6) and $2t/3 < t - 1$, we have

$$p(t, x, y) \gtrsim \frac{1}{td'/2|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{2t/3} e^{-|y|^2/(t-s)} \frac{ds}{(t-s)^{d'/2}} \frac{1}{td'/2} e^{-|x|_p^2/t}$$

$$\times \frac{1}{td'/2|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{2t/3} t^{-d'/2} \frac{ds}{td'/2} e^{-\rho(x,y)^2/t}$$

$$\times \left(\frac{1}{td'/2|x|^{d-2}} + \frac{1}{td'/2|y|^{d-2}}\right) e^{-\rho(x,y)^2/t}.$$

By (a)-(d), we obtain the assertion of Theorem 1.7 (iii). □

6 Large time estimate ($d' = 2$)

In this section, we will prove Theorem 1.5 and Theorem 1.6. Let $d' = 2$, $d \geq 2$ and without loss of generality, we assume $\varepsilon, \varepsilon' < 1$. For a same reason as in the case of $d' = 3$, we consider the estimate of $p(t, a^*, a^*)$. When $d = d' = 2$, this is easy. When $d \geq 3, d' = 2$, we will obtain the estimate by using Doob’s $h$-transform and the relative Faber-Krahn inequality.

Proposition 6.1. Let $d \geq d' = 2$. Then, for $t > 0$, it holds that

$$t^{-1/2} \wedge t^{-d'/2} \lesssim p(t, a^*, a^*) \lesssim t^{-1/2} \wedge t^{-1}.$$  \hspace{1cm} (6.1)

Proof. The upper estimate follows from Proposition 2.2 and Theorem 1.3. By the Markov property and the Cauchy-Shwarz inequality, for large $M > 0$, we
have

\[ p(t, a^*, a^*) = \int p \left( \frac{t}{2}, a^*, x \right)^2 m_p(dx) \geq \int_{\{ |x| \leq M \sqrt{t} \}} p \left( \frac{t}{2}, a^*, x \right)^2 m_p(dx) \]

\[ \geq m_p \left( \{ |x| \leq M \sqrt{t} \} \right)^{-1} \times \left( \int_{\{ |x| \leq M \sqrt{t} \}} p \left( \frac{t}{2}, a^*, x \right) m_p(dx) \right)^2 \]

\[ \gtrsim \left( \frac{1}{t} \wedge \frac{1}{t^{d/2}} \right) \mathbb{P}_{a^*} \left( |X_t| \leq M \sqrt{t} \right)^2. \quad (6.2) \]

By the proof of [3, Theorem 5.10], there is large \( M > 0 \) such that for all \( t > 0 \), \( \mathbb{P}_{a^*} \left( |X_t| \leq M \sqrt{t} \right) \geq \frac{1}{2} \). Thus the right hand side of (6.2) is equal to \( t^{-1} \wedge t^{-d/2} \) up to a constant multiple. Therefore, by Theorem 1.3 again, we have

\[ p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t}} \wedge \left( \frac{1}{t} \wedge \frac{1}{t^{d/2}} \right) = \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d/2}}. \quad (6.3) \]

\[ \square \]

**Corollary 6.2.** Let \( d = d' = 2 \). Then, for \( t > 0 \), \( p(t, a^*, a^*) \asymp t^{-1/2} \wedge t^{-1} \).

**Proof of Theorem 1.5.** Let \( d = d' = 2 \). We may assume \( \varepsilon \geq \varepsilon' \) without loss of generality. \( \tilde{p}(t, x, y) \) denotes the heat kernel for Brownian motion \( \tilde{X} \) on \( \mathbb{R}^2 \). Then, by [14, Example 2.12], \( \tilde{p}(t, x, y) \) has the estimates of this theorem as a sharp estimate. In particular, it holds that

\[ \tilde{p}(t, x, y) \asymp t^{-1} e^{-d(x, y)/t} \text{ for } t > 0 \text{ and } x, y \in K := \overline{B}^2(0; \varepsilon') \cup \overline{B}^2(0; \varepsilon'), \]

where \( d \) is a geodesic distance on \( \mathbb{R}^2 \). By Corollary 6.2, we have \( p(t, a^*, a^*) \lesssim t^{-1} \). Furthermore, it holds that \( \mathbb{P}_x(\sigma_{a^*} \in ds) = \mathbb{P}_x(\tilde{\sigma}_K \in ds) \) and heat kernels of part processes of \( X \) and \( \tilde{X} \) on \( \mathbb{R}^2 \) are equivalent, where \( \tilde{\mathbb{P}} \) and \( \tilde{\sigma}_K \) are those for \( \tilde{X} \). Thus, by the same way as the proof of Proposition 5.3, it holds that \( p(t, x, y) \lesssim \tilde{p}(t, x, y) \) for \( x, y \in \mathbb{R}^2 \cup \{ a^* \} \), so the upper estimates are proved.

Next, we prove the lower estimates. Let \( T > 0 \) be large and \( t \in [T, \infty) \).

1. (a) For \( x \in \mathbb{R}^2 \) with \( |x| \leq 1 \), by Corollary 6.2, we have

\[ p(t, x, a^*) \geq \int_0^{t/2} p(t - s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \asymp \frac{e^{-|x|^2/4t}}{t}. \]

(b) For \( x \in \mathbb{R}^2 \) with \( 1 < |x| \leq \sqrt{t}/2 \), by Lemma 2.6, Lemma 2.9, and Corollary
6.2, we have

\[
p(t, x, a^*) \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) + \int_{t/2}^{t-1} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \\
\geq \frac{1}{t} \left( 1 - \frac{1}{\log \sqrt{t/2}} \right) + \int_{t/2}^{t-1} \frac{1}{t-s} ds \mathbb{P}_x(\sigma_{a^*} \in dt) \\
\geq \frac{1}{t} \left( 1 - \frac{1}{\log \sqrt{t/2}} \right) e^{-|x|^2/t} + \frac{\log |x|}{t \log t} e^{-|x|^2/t} \\
\geq \frac{1}{t} e^{-|x|^2/t}.
\]

(c) For \( x \in \mathbb{R}_{\rho}^2 \) with \( \sqrt{7}/2 < |x| \), by Lemma 2.6, and Corollary 6.2, we have

\[
p(t, x, a^*) \geq \int_0^{t-1} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) = \int_0^{t-1} (t-s)^{-1} ds \mathbb{P}_x(\sigma_{a^*} \in dt) \\
\geq \log t \left( 1 + \log (1 + t) \right) \frac{e^{-|x|^2/t}}{t} \geq \frac{1}{t} e^{-|x|^2/t}.
\]

(2) For \( x, y \in \mathbb{R}_{\rho}^2 \), by (2.1), Lemma 2.10 and Proposition 6.2, it holds that

\[
p(t, x, y) \geq p_{\mathbb{R}^2}(t, x, y) + \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \\
\geq \frac{(1 \wedge |x|\rho)(1 \wedge |y|\rho)}{t} e^{-\rho(x,y)^2/t} + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq t/2). \tag{6.4}
\]

(a) If \( \frac{1}{2} \leq |x|\rho, 1 \leq |y|\rho \), by (6.4), we have

\[
p(t, x, y) \geq \frac{1}{t} e^{-\rho(x,y)^2/t} + \frac{1}{t} e^{-\rho(x,y)^2/t}.
\]

(b) If \( |x|\rho \leq \frac{1}{2}, 1 \leq |y|\rho \), by (6.4) and Lemma 2.10 (iii), we have

\[
p(t, x, y) \geq 0 + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \geq \frac{1}{t} e^{-(|x|\rho + |y|\rho)^2/t} \geq \frac{1}{t} e^{-\rho(x,y)^2/t}.
\]

(c) If \( |x|\rho \leq 1, |y|\rho \leq 1 \), by (6.4) and Lemma 2.10 (ii), we have

\[
p(t, x, y) \geq 0 + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \geq \frac{1}{t} e^{-(|x|\rho + |y|\rho)^2/t} \geq \frac{1}{t} e^{-\rho(x,y)^2/t}.
\]

(3) For \( x \in \mathbb{R}_{\rho}^2, y \in \mathbb{R}_{\rho}^2 \), it holds that

\[
p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds). \tag{6.5}
\]

(a) If \( |x| \wedge |y| \leq \sqrt{7}/2 \), by (6.5) and Lemma 2.9, we have

\[
p(t, x, y) \geq \frac{1}{t} \left( 1 - \frac{\log |y|}{\log \sqrt{t/2}} \right) \geq \frac{1}{t} \log \frac{t}{\log t} - \log |y|.
\]
By the symmetry, we have
\[
p(t, x, y) \gtrsim \frac{\log \frac{t}{2} - \log |x| - \log |y|}{t \log t} \\
\gtrsim \frac{U_t(x)}{t} \left( \frac{U_t(y) + \log |y|}{\log (t|y|)} \right) + \frac{U_t(y)}{t} \left( \frac{U_t(x) + \log |x|}{\log (t|x|)} \right) \\
\approx \frac{e^{-\rho(x, y)^2/t}}{t} \left( U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log (1 + t|y|)} + \frac{U_t(y) \log |x|}{\log (1 + t|x|)} \right).
\]

(b) If \(|x| \wedge |y| \geq \sqrt{t}/2\), by (6.5) and Lemma 2.9, we have
\[
p(t, x, y) \gtrsim \frac{1}{t} \left( 1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) \approx \frac{1}{t \log |y|} e^{-\rho(x, y)^2/t}. 
\]

By the symmetry, we have
\[
p(t, x, y) \gtrsim \frac{1}{t} \left( \frac{1}{\log |x|} + \frac{1}{\log |y|} \right) e^{-\rho(x, y)^2/t} \\
\approx \frac{e^{-\rho(x, y)^2/t}}{t} \left( U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log (1 + t|y|)} + \frac{U_t(y) \log |x|}{\log (1 + t|x|)} \right).
\]

(c) If \(|y| \leq \sqrt{t}/2 \leq |x|\), by (6.5) and Lemma 2.9, we have
\[
p(t, x, y) \gtrsim \frac{1}{t} \left( 1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) e^{-\rho(x, y)^2/t}. 
\]

Moreover, it holds that
\[
p(t, x, y) \geq \int_{t/2}^{2t/3} p(t - s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t \log t} e^{-\rho(x, y)^2/t}. 
\]
Thus we obtain
\[
p(t, x, y) \gtrsim \frac{1 + \log \left( \frac{1}{\log \sqrt{t/2}} \right)}{t \log t} e^{-\rho(x, y)^2/t} \\
\gtrsim \frac{e^{-\rho(x, y)^2/t}}{t} \left( U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log (1 + t|y|)} + \frac{U_t(y) \log |x|}{\log (1 + t|x|)} \right).
\]

By the symmetry, the all cases have been proved.\qed

Next, we prove Theorem 1.6.

**Proposition 6.3.** Let \(d \geq 3, \ d' = 2\). Then, for \(t > 0\), we have
\[
p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t} \left( t + 1 \right)} \frac{1}{(d' + 1)^2}.
\]
Proof. For $t > 3$ and $x \in \mathbb{R}^d_+$, with $\sqrt{t} \leq |x| \leq 2\sqrt{t}$, by Theorem 1.3, Lemma 2.6 and Proposition 6.1, we have
\[
p(t, a^*, x) \geq \int_{t-1}^t p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds)\]
\[
\geq \int_{t-1}^t (t-s)^{-1/2}ds \left( \frac{e^{-|x|^2/t}}{1 + \log |x|} \right) \frac{1 + \log t}{1 + \log (1 + |x|)(1 + \log (t + |x|))} \geq \frac{1}{(t + 1) \log (t + 1)}.
\]

Thus, by the Markov property, we have
\[
p(2t, a^*, a^*) \geq \int_{\{|x| \leq 2\sqrt{t} \}} p(t, a^*, x)^2 m_p(dx) \geq \int_{\{|x| \leq 2\sqrt{t} \}} \frac{1}{(t + 1)^2 (\log (t + 1))^2} m_p(dx) \geq \frac{1}{(t + 1) (\log (t + 1))^2},
\]
where we used polar coordinates. Combining this with Theorem 1.3, the desired estimate is proved. \(\square\)

Next, we prove that the estimate in Proposition 6.3 is sharp by using Doob’s $h$-transform. First, we construct a harmonic function, which is comparable to $1$ on $\mathbb{R}^d_+$ and $1 + \log |x|_\rho$ on $\mathbb{R}^d_\rho$. In the following proposition, we use some ideas from [21, Theorem 2.6].

**Proposition 6.4.** Let $d \geq 3$. Then, there exists a positive harmonic function $h$ on $\mathbb{R}^d_+ \cup \mathbb{R}^d_\rho \cup \{a^*\}$ such that $h \geq 1$ on $\mathbb{R}^d_+$ and $h(x) \approx 1 + \log |x|_\rho$ for $x \in \mathbb{R}^d_\rho$.

We take $\eta \in C^\infty(\mathbb{R}^d_+ \cup \mathbb{R}^d_\rho \cup \{a^*\})$ satisfying $\eta = 1$ on $\{|x|_\rho > 2R\}$ and $\eta = 0$ on $K$. Let
\[
h(x) := (\eta f)(x) + \int_{\mathbb{R}^d_+ \cup \mathbb{R}^d_\rho \cup \{a^*\}} G(x, y) \Delta(\eta f)(y) dy,
\]
where $G(x, y) := \int_0^\infty p(t, x, y)dt$. Since $\mathbb{R}^d_+ \cup \mathbb{R}^2_+ \cup \{a^*\}$ is non-parabolic, we have $G(x, y) < \infty$. It holds that $\Delta (\eta f) \in C^\infty(\mathbb{R}^d_+ \cup \mathbb{R}^2_+ \cup \{a^*\})$ since $(\eta f)(x) = f(x)$ and $\Delta (\eta f)(x) = 0$ for $x$ with $|x|_\rho > 2R$. Hence, for all $x$, we have $\Delta h(x) = 0$, so $h$ is a harmonic function.

For $x$ with $|x|_\rho > 2R$,

$$|f(x) - h(x)| = \left| \int_{\{|y|_\rho \leq 2R\}} G(x, y)\Delta (\eta f)(y)dy \right| \leq C \sup_{\{|y|_\rho \leq 2R\}} G(x, y) \times |\{|y|_\rho \leq 2R\}| \times \sup \Delta (\eta f) \leq C \sup_{\{|y|_\rho \leq 2R\}} G(x, y).$$

By using the elliptic Harnack inequality on $\mathbb{R}^d_+ \cup \{a^*\}$ and $\mathbb{R}^2_+ \cup \{a^*\}$ (see for example [7, Theorem 13.10]), it holds that $|f(x) - h(x)| \leq C G(x, a^*)$ for $x$ with $|x|_\rho > 2R$.

Let fix $x_1 \in \mathbb{R}^d_+$ and $x_2 \in \mathbb{R}^2_+$ with $|x_1|_\rho = |x_2|_\rho = 4R$. For $x \in \mathbb{R}^d_+$ with $|x|_\rho > 4R$, by Lemma 2.5, we have $\mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim \mathbb{P}_x(\sigma_{a^*} \in ds)$. Furthermore, for $x \in \mathbb{R}^2$, with $|x|_\rho > 4R$, by Lemma 2.6 and for large $R$, we have $\mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim \mathbb{P}_x(\sigma_{a^*} \in ds)$. Thus, we have

$$p(t, x, a^*) = \int_0^t p(t-s, a^*, a^*)\mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim p(t, x_1, a^*) + p(t, x_2, a^*)$$

and $G(x, a^*) \lesssim G(x_1, a^*) + G(x_2, a^*) < \infty$. Then, $|f - h|$ is bounded on $\{|x|_\rho > 4R\}$ and, by the continuity of $G$, $|f - h|$ is bounded. □

Let $h$ be a positive harmonic function constructed as above. Define

$$H : L^2(\mathbb{R}^d_+ \cup \mathbb{R}^2_+ \cup \{a^*\}; h^2m_p) \ni f \mapsto fh \in L^2(\mathbb{R}^d_+ \cup \mathbb{R}^2_+ \cup \{a^*\}; m_p),$$

$$\mathcal{E}^h(f,f) := \mathcal{E}(fh,fh) = \int |\nabla(fh)|^2 dm_p$$

for $f \in \mathcal{F}^h := H^{-1} \mathcal{F}$. Then, $H^{-1} \circ P_t \circ H$ admits a transition density $p^h(t, x, y)$ with respect to $dm_p := h^2dm_p$ and $p^h(t, x, y)h(x)h(y) = p(t, x, y)$ holds ([15, Lemma 5.6]). Since $h$ is harmonic, by [15, Proposition 5.7], we have

$$\mathcal{E}^h(f,f) = \int |\nabla fh|^2 dm_p$$

for $f \in \mathcal{F}^h$.

The next lemma follows from [10, Lemma 4.8].

**Lemma 6.5.** Let $q(t, x, y)$ be the transition density function with respect to $m_p$ on $\mathbb{R}^d_+ \cup \{a^*\}$ and $q^h(t, x, y)$ be the $h$-transform of $q(t, x, y)$. Then it holds that $q^h(t, x, x) \lesssim m_p \{\{y \in \mathbb{R}^2_+ \cup \{a^*\}|\rho(x, y) \leq \sqrt{t}\}\}^{-1}$ for $t > 0, x \in \mathbb{R}^2_+ \cup \{a^*\}$.

In order to get the sharp estimate of $p(t, a^*, a^*)$, we imitate the technique of the relative Faber-Krahn inequality appearing [13].
**Lemma 6.6.** For some constant $c>0$, $\alpha_2>0$ and any ball $B:=B(x_0;R)$, let

$$\Lambda_1(B,v) := \frac{c}{R^2} \left( \frac{m^h_p(B)}{v} \right)^{2/d},$$

$$\Lambda_2(B,v) := \frac{c}{R^2} \left( \frac{m^h_p(B)}{v} \right)^{\alpha_2}. $$

Then, for any ball $B_1 \subset \mathbb{R}^d \setminus \{a^*\}$ and a non-empty open subset $\Omega \subset B_1$, we have

$$\inf_{f \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla f|^2 dm^h_p}{\int_{\Omega} |f|^2 dm^h_p} \geq \Lambda_1(B_1, m^h_p(\Omega))$$

and, for any ball $B_2 \subset \mathbb{R}^d \setminus \{a^*\}$ and any open subset $\Omega \subset B_2$, we have

$$\inf_{f \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla f|^2 dm^h_p}{\int_{\Omega} |f|^2 dm^h_p} \geq \Lambda_1(B_2, m^h_p(\Omega)).$$

These inequalities are called the relative Faber-Krahn inequality.

**Proof.** From [13, Proposition 4.2], for a complete weighted manifold, the relative Faber-Krahn inequality holds if the diagonal upper estimate of heat kernel holds. $h \succ 1$ and $q(t, x, x) \leq t^{-d/2}$ on $\mathbb{R}^d \setminus \{a^*\}$, where $q$ is a heat kernel with respect to $m^h_p$ on $\mathbb{R}^d \setminus \{a^*\}$, so the first inequality holds. The second inequality follows from Lemma 6.5. □

**Theorem 6.7.** Let $\alpha := \alpha_2 \wedge \frac{2}{d}$. For $B := B(x_0;R) \subset \mathbb{R}^d \setminus \{a^*\}$ and any open subset $\Omega \subset B$, it holds that

$$\inf_{f \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla f|^2 dm^h_p}{\int_{\Omega} |f|^2 dm^h_p} \geq \Lambda(B, m^h_p(\Omega)), \text{ where } \Lambda(B, v) := \frac{c}{R^2} \left( \frac{F(B)}{v} \right)^{\alpha},$$

$$F(B) := \begin{cases} m^h_p(B) & : B \subset \mathbb{R}^d_i \text{ for } i = 2, d, \\ m^h_p(\{x \in \mathbb{R}^2_i \cup \{a^*\} \mid \rho(x, y_2) \leq R\}) & : a^* \in B \text{ and large } R. \end{cases}$$

Here, $\varepsilon_2 := \varepsilon', \varepsilon_d := \varepsilon$ and $y_2 \in \mathbb{R}^2_i \cup \{a^*\}$ with $|x_0| = |y_2|_\rho$.

**Proof.** When $B \subset \mathbb{R}^d_i$ for $i = 2, d$, the estimate holds by Lemma 6.6. When $a^* \in B := B(x_0;R) \subset \mathbb{R}^d_i \cup \{a^*\}$ for large $R > 0$, for any open subset $\Omega \subset B$ and $f \in C^\infty_0(\Omega) \setminus \{0\}$, it holds that $f|_{\mathbb{R}^d_i \cup \{a^*\}} \in C^\infty_0(\Omega \cap (\mathbb{R}^d_i \cup \{a^*\})) \setminus \{0\}$ and $f|_{\mathbb{R}^d_i \cup \{a^*\}} \in C^\infty_0(\Omega \cap (\mathbb{R}^2_i \cup \{a^*\})) \setminus \{0\}$. For $i = 2, d$, fix $y_i \in \mathbb{R}^i_i \cup \{a^*\}$ satisfying $|y_i|_\rho = |x_0|_\rho$ and $B^i := \{x \in \mathbb{R}^i_i \cup \{a^*\} \mid \rho(x, y_i) \leq 3R + 2\varepsilon + 2\varepsilon'\}$. (1) For $x_0 \in \mathbb{R}^2_i$, we have (see Figure 6)

$$B \cap (\mathbb{R}^2_i \cup \{a^*\}) \subset B(y_2; |x_0|_\rho + 2\varepsilon' + |x_0|_\rho + R) \cap (\mathbb{R}^2_i \cup \{a^*\}) \subset B^2,$$

$$B \cap (\mathbb{R}^d_i \cup \{a^*\}) \subset B(y_d; 2\varepsilon + 2R) \cap (\mathbb{R}^d_i \cup \{a^*\}) \subset B^d \text{ if } R - |x_0|_\rho \leq |y_d|_\rho,$$

$$B \cap (\mathbb{R}^d_i \cup \{a^*\}) \subset B(y_d; 2R) \cap (\mathbb{R}^d_i \cup \{a^*\}) \subset B^d \text{ if } R - |x_0|_\rho > |y_d|_\rho.$$ (2) For $x_0 \in \mathbb{R}^2_i$, by the same way as (1), it holds that $B \cap (\mathbb{R}^d_i \cup \{a^*\}) \subset B^d$ and $B \cap (\mathbb{R}^d \cup \{a^*\}) \subset B^2$. 34
Hence, for $i = 2$ or $d$ satisfying $\int_{B_i} |f|^2 dm^h_p \geq \frac{1}{2} \int_{\Omega} |f|^2 dm^h_p$, by Lemma 6.6 and $m^h_p(B^i) \geq m^h_p(\Omega \cap \mathbb{R}^d_{\varepsilon})$, we have
\[
\int_{\mathbb{R}^d} |\nabla f|^2 dm^h_p \geq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla f|^2 dm^h_p
\]
\[
\geq \Lambda_i \left( B^i, m^h_p(\Omega \cap \mathbb{R}^d_{\varepsilon}) \right) \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 dm^h_p
\]
\[
\geq \frac{c}{(3R + 2\varepsilon + 2\varepsilon')^2} \left( \frac{m^h_p(B^i)}{m^h_p(\Omega \cap \mathbb{R}^d_{\varepsilon})} \right)^\alpha \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 dm^h_p.
\] (6.6)

Hence, the proof is finished if $i = 2$. If $i = d$, we have
\[
m^h_p(B^d) \geq m_p(B(y_d; 3R) \cap \mathbb{R}^d_{\varepsilon}) \geq m_p(B(y_d'; R) \cap \mathbb{R}^d_{\varepsilon}) \approx R^d
\]
\[
\geq (1 + \log (5R))^2 (8R)^2
\]
\[
\geq \int_{B(y_d'; 4R) \cap \mathbb{R}^d_{\varepsilon}} (1 + \log (|y_d| + 3R + 2\varepsilon + 2\varepsilon'))^2
\]
\[
\geq m^h_p(B^2) \geq m^h_p((B(y_d; R) \cap \mathbb{R}^d_{\varepsilon}),
\] (6.7)

where $y_d'$ is the point with $|y_d'| = 2R$ on the line joining $a^*$ and $y_d$. Therefore, by (6.6) and (6.7), the desired inequality holds.

**Proposition 6.8.** Let $d \geq 3$ and $d' = 2$. Then, it holds that for $t > 0$,
\[
p(t, a^*, a^*) \approx \frac{1}{\sqrt{t}} \wedge \frac{1}{(t + 1) (\log (t + 1))^2}.
\]

**Proof.** The lower estimate is given in Proposition 6.3, so we prove the upper estimate.
By the same proof of [8, Theorem 5.2], for large \( T > 0, t \in [T, \infty) \) and \( x, y \in \mathbb{R}^d_x \cup \mathbb{R}^d_y \cup \{a^*\} \), it holds that

\[
p^h(t, x, y) \lesssim (t \wedge R^2)^{-1/\alpha} \left( \frac{F(B(x; R))}{R^2} \frac{F(B(y; R))}{R^2} \right)^{-1/2\alpha}.
\] (6.8)

Indeed, in [8], Grigor’yan proved (6.8) for \( t > 0 \) on a smooth connected non-compact complete Riemannian manifold. In the proof, it is used that \( |\nabla \rho| \leq 1 \), where \( \rho \) is a Riemannian distance. In our setting, we consider the space attached by two manifolds on which \( |\nabla \rho| \leq 1 \) still holds. Hence (6.8) holds by the proof of [8, Theorem 5.2].

We take \( R := \sqrt{t} \) and large \( t \), by Theorem 6.7, we have

\[
p^h(t, a^*, a^*) \lesssim t^{-1/\alpha} \left( \frac{m^h_p((x \in \mathbb{R}^d_x \cup \{a^*\}; |x|_\rho \leq \sqrt{t}))}{t} \right)^{-1/\alpha} = m^h_p((x \in \mathbb{R}^d_x \cup \{a^*\}; |x|_\rho \leq \sqrt{t}))^{-1}.
\] (6.9)

Let \( \tilde{B} := B((3\sqrt{t}/2, 0) \cap \mathbb{R}^d_x \cup \mathbb{R}^d_y \cup \{a^*\} \), then we obtain

\[
m^h_p((x \in \mathbb{R}^d_x \cup \{a^*\}; |x|_\rho \leq \sqrt{t})) \geq m^h_p(\tilde{B}) \geq \int_{|x-(3\sqrt{t}/2)| \leq \sqrt{t}/2} (1 + \log |x|_\rho)^2 dx \geq (1 + \log \sqrt{t}/2)^2 |\tilde{B}| \gtrsim (t + 1)(\log (t + 1))^2.
\] (6.10)

By (6.9), (6.10), \( p^h(t, a^*, a^*) = p(t, a^*, a^*)h(a^*)^2 \) and Theorem 1.3, the upper estimate holds.

**Proof of Theorem 1.6.** By comparing with the heat kernel on \( \mathbb{R}^d \times \mathbb{R}^d \) (12), the upper estimates can be proved in the same way as the proof of Proposition 5.3. We prove the lower estimates. Let \( T > 0 \) be large and \( t \in [T, \infty) \).

**Step 1** (the estimate of \( p(t, x, a^*) \))

1. For \( x \in \mathbb{R}^d_x \cup \{a^*\} \), by Proposition 6.8, Theorem 1.3, Lemma 2.8, Lemma 2.5, and Lemma 2.7, we have

\[
p(t, x, a^*) \geq \int_0^{t/2} p(t-s, x, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \geq \frac{1}{t(\log t)^2} \frac{1}{|x|^2} e^{-|x|^2/t} + \frac{1}{t(\log t)^2} e^{-|x|^2/t}.
\]

2. For \( y \in \mathbb{R}^d_y \cup \{a^*\} \) with \(|y| < \sqrt{t}/2 \), by Proposition 6.8, Theorem 1.3, Lemma 2.9 and Lemma 2.6, we have

\[
p(t, y, a^*) \geq \int_0^{t/2} + \int_{t/2}^t p(t-s, y, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \geq \frac{1}{t(\log t)^2} \left( 1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) + \frac{|y|}{t(\log t)^2} e^{-|y|^2/t}.
\]
(3) For $y \in \mathbb{R}^2_{\rho} \cup \{a^*\}$ with $|y| \geq \sqrt{t}/2$, by Theorem 1.3 and Lemma 2.9, we have

$$p(t, y, a^*) \geq \int_{t-2}^t p(t-s, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds)$$

$$\geq \frac{1 + \log |y|}{(1 + \log (1+t/|y|))(1 + \log (t + |y|))} \frac{(|y| + t)^{1/2}}{t^{3/2}} e^{-|y|^2/t}$$

$$\geq \frac{1}{t (\log t)^2} e^{-|y|^2/t} \geq \frac{1}{t (\log t)^2} e^{-|y|^2/t}.$$  

Since $H_\varepsilon(y) \leq (\log (1 + \varepsilon^{-2}) + (2 \log (1 + \varepsilon'))^{-1} \times 1$, these estimates are sharp.

**Step 2** (the proof of Theorem 1.6 (i))

(1) For $x, y \in \mathbb{R}^2_{\rho}$ with $1 \leq |x|_\rho \wedge |y|_\rho$, by (2.1), Step 1 and Lemma 2.8, we have

$$p(t, x, y) \geq \int_0^{t/2} \int_{t-1}^t p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}^2}(t, x, y)$$

$$\geq \frac{1}{t (\log t)^2} e^{-|x|^2/2} \mathbb{P}_y(\sigma_{a^*} \leq t/2) + (|x|_\rho \wedge 1)(|y|_\rho \wedge 1) \frac{1}{t^{3/2}} e^{-\rho(x,y)^2/t}$$

$$\geq \frac{1}{t (\log t)^2} e^{-|x|^2/2} + \frac{1}{t^{3/2}} e^{-\rho(x,y)^2/t}. $$

(2) For $x, y \in \mathbb{R}^2_{\rho}$ with $|x|_\rho < 1$, by Step 1, Lemma 2.5, (2.1), Lemma 2.8, Lemma 2.7 and Lemma 2.10 (iii), we have

$$p(t, x, y) \geq \int_0^{t/2} \int_{t-1}^t p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}^2}(t, x, y)$$

$$\geq \frac{\mathbb{P}_y(\sigma_{a^*} \leq t/2)}{t (\log t)^2} e^{-|x|^2/2} + \frac{1}{t^{3/2}} e^{-\rho(x,y)^2/t}$$

$$\geq \frac{e^{-|x|^2/2}}{t (\log t)^2} e^{-|x|^2/2} + \frac{1}{t^{3/2}} e^{-\rho(x,y)^2/t}$$

$$\geq \frac{1}{t (\log t)^2} e^{-|x|^2/2} + \frac{1}{t^{3/2}} e^{-\rho(x,y)^2/t}.$$  

**Step 3** (the proof of Theorem 1.6 (ii))

(1) For $x, y \in \mathbb{R}^2_{\rho}$ with $|x|_\rho \leq 1, |y|_\rho \leq \sqrt{t}/2$, by Theorem 1.3, Lemma 2.9, (2.1),
and (5.3), we have

\[ p(t, x, y) \geq \int_{t-2}^{t-1} p(t-s, s, a^*) \mathbb{P}_s(\sigma_{a^*} \in ds) + p_{\mathbb{R}^2_s}(t, x, y) \]

\[ \approx \int_{t-2}^{t-1} e^{-|x|^2/(t-s)} ds \log (1 + |y|) \left( \frac{|x| |y| \wedge 1}{t} \right) e^{-\rho(x,y)^2/t} \]

\[ \times \frac{\log (1 + |x|)}{t} \left( \frac{|x| |y| \wedge 1}{t} \right) e^{-\rho(x,y)^2/t} \]

\[ \times \frac{\log (1 + |x|)}{t} \left( \frac{|x| |y| \wedge 1}{t} \right) e^{-\rho(x,y)^2/t} \]

\[ \times \frac{\log (1 + |x|)}{t \log (1 + |y|)} e^{-\rho(x,y)^2/t}. \]

(2) For \( x, y \in \mathbb{R}^2 \), with \( 1 \leq |x| \leq \sqrt{t}/2, |y| \leq \sqrt{t}/2 \), by (2.1), we have

\[ p(t, x, y) \geq p_{\mathbb{R}^2_s}(t, x, y) \geq \frac{1}{t} e^{-\rho(x,y)^2/t} \geq \frac{\log (1 + |x|)}{t (\log t)^2} e^{-\rho(x,y)^2/t} \]

\[ \approx \frac{\log (1 + |x|)}{t (\log (1 + |x|))(\log (1 + |y|))} e^{-\rho(x,y)^2/t}. \]

(3) For \( x, y \in \mathbb{R}^2 \), with \( |x| \leq 1, \sqrt{t}/2 \leq |y| \), by Theorem 1.3, Lemma 2.6 and Lemma 2.10 (iii), we have

\[ p(t, x, y) \geq \int_{t-2}^{t-1} p(t-s, s, a^*) \mathbb{P}_s(\sigma_{a^*} \in ds) \]

\[ \approx \frac{1}{t} \log t e^{-\rho(x,y)^2/t} \geq \frac{\log (1 + |x|)}{t \log t} e^{-\rho(x,y)^2/t} \]

\[ \times \frac{\log (1 + |x|) \log (1 + |y|)}{t (\log t)^2} e^{-\rho(x,y)^2/t}. \]

(4) For \( x, y \in \mathbb{R}^2 \), with \( 1 \leq |x| \leq \sqrt{t}/2, \sqrt{t}/2 < |y| \), by (2.1), we have

\[ p(t, x, y) \geq p_{\mathbb{R}^2_s}(t, x, y) \geq \frac{1}{t} e^{-\rho(x,y)^2/t} \geq \frac{\log (1 + |x|)}{t \log t} e^{-\rho(x,y)^2/t} \]

\[ \times \frac{\log (1 + |x|) \log (1 + |y|)}{t (\log (1 + t |x|))(\log (1 + |y|))} e^{-\rho(x,y)^2/t}. \]

(5) For \( x, y \in \mathbb{R}^2 \), with \( \sqrt{t}/2 \leq |x| \wedge |y| \), by (2.1), we have

\[ p(t, x, y) \geq p_{\mathbb{R}^2_s}(t, x, y) \geq \frac{1}{t} e^{-\rho(x,y)^2/t} \geq \frac{\log (1 + |x|) \log (1 + |y|)}{t (\log (1 + t |x|))(\log (1 + |y|))} e^{-\rho(x,y)^2/t}. \]
Step 4 (the proof of Theorem 1.6 (iii))

(1) For $x \in \mathbb{R}^d_+, y \in \mathbb{R}^d_+$, with $|x|_{\rho} < 1$, by Theorem 1.3 and Lemma 2.6, we have

$$p(t,x,y) \geq \int_{t-2}^{t-1} p(t-s,a^*,x)\mathbb{P}_y(\sigma_{a^*} \in ds)$$

$$\geq \frac{1}{t \log t} \frac{1 + \log |y|}{1 + \log (t + |y|)} e^{-\rho(x,y)^2/t}$$

$$\geq \frac{1}{t \log t} e^{-\rho(x,y)^2/t}$$

$$\geq \left( \frac{1}{t \log t} \right)^{d/2} e^{-\rho(x,y)^2/t}. $$

(2) For $x \in \mathbb{R}^d_+, y \in \mathbb{R}^d_+$, with $1 \leq |x|_{\rho} < |x| \leq \sqrt{t}/2$, $|y|_{\rho} \leq 1$, by Step 1, Theorem 1.3, Lemma 2.8, Lemma 2.5 and $H_1(y) \lessgtr 1$, we have

$$p(t,x,y) \geq \int_{0}^{t/2} p(t-s,a^*,y)\mathbb{P}_x(\sigma_{a^*} \in ds)$$

$$\geq \frac{1}{t \log t} e^{-\rho(x,y)^2/t} + \int_{t-2}^{t-1} e^{-\rho(x,y)^2/t}$$

$$\geq \left( \frac{1}{t \log t} \right)^{d/2} e^{-\rho(x,y)^2/t}. $$

(3) For $x \in \mathbb{R}^d_+, y \in \mathbb{R}^d_+$, with $1 \leq |x|_{\rho} < |x| \leq \sqrt{t}/2$, $1 \leq |y|_{\rho} < |y| \leq \sqrt{t}/2$, by Step 1 and Lemma 2.8, we have

$$p(t,x,y) \geq \int_{0}^{t/2} p(t-s,a^*,y)\mathbb{P}_x(\sigma_{a^*} \in ds) \geq \frac{1}{t \log t} e^{-\rho(x,y)^2/t}. $$

Furthermore, by Step 1 and Lemma 2.9, we have

$$p(t,x,y) \geq \int_{0}^{t/2} p(t-s,a^*,x)\mathbb{P}_y(\sigma_{a^*} \in ds) \geq \frac{e^{-\rho(x,y)^2/t}}{t \log t} \left( 1 - \frac{\log |y|}{\log \sqrt{t}/2} \right)$$

$$\geq \frac{H_1(y)}{t \log t} e^{-\rho(x,y)^2/t}. $$

(4) For $x \in \mathbb{R}^d_+, y \in \mathbb{R}^d_+$, with $1 \leq |x|_{\rho} < |x| \leq \sqrt{t}/2$, $\sqrt{t}/2 \leq |y|$, by Step 1 and Lemma 2.8, we have

$$p(t,x,y) \geq \int_{0}^{t/2} p(t-s,a^*,y)\mathbb{P}_x(\sigma_{a^*} \in ds) \geq \frac{1}{t \log t} e^{-\rho(x,y)^2/t}. $$

Furthermore, by Step 1, Lemma 2.9 and $H_1(y) = (\log (1 + |y|))^{-2}$,

$$p(t,x,y) \geq \int_{0}^{t/2} p(t-s,a^*,x)\mathbb{P}_y(\sigma_{a^*} \in ds) \geq \frac{e^{-\rho(x,y)^2/t}}{t \log |y|} \geq \frac{H_1(y)}{t \log |y|} e^{-\rho(x,y)^2/t}. $$

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(5) For $x \in \mathbb{R}^d_+$, $y \in \mathbb{R}^d_+$ with $\sqrt{t}/2 \leq |x|$, by Step1, Lemma 2.8, we have

$$p(t, x, y) \geq \int_0^{t/2} p(t - s, a^*, y)\mathbb{P}_x(\sigma_{a^*} \in ds) \asymp \frac{1}{t((\log t)^2|x|^{d-2})} e^{-\rho(x, y)^2/t},$$

$$p(t, x, y) \geq \int_0^{t/2} p(t - s, a^*, x)\mathbb{P}_y(\sigma_{a^*} \in ds) \asymp \frac{e^{-\rho(x, y)^2/t}}{t^{d/2}} \mathbb{P}_y(\sigma_{a^*} \leq t/2). \quad (6.11)$$

By the same way as in Step 4 (3),(4), the right hand side of (6.11) is larger than $H_t(y)e^{-\rho(x, y)^2/t}/t^{d/2}$ up to a constant multiple.

By the symmetry, we have proved all the cases and complete the proof of Theorem 1.6.

\textbf{Remark 6.9.} We already proved Theorem 1.4 for the case of $d' = 1, d \geq 3$ in Section 4. Since it is the mixed case of transient and recurrent, it can also be proved by the same way as in this section.

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