Elliptic Curves and Algebraic Geometry Approach in Gravity Theory II. Parametrization of a Multivariable Cubic Algebraic Equation

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Abstract

In a previous paper, the general approach for treatment of algebraic equations of different order in gravity theory was exposed, based on the important distinction between covariant and contravariant metric tensor components.

In the present second part of the paper it has been shown that a multivariable cubic algebraic equation can also be parametrized by means of complicated, irrational and non-elliptic functions, depending on the elliptic Weierstrass function and its derivative. As a model example, the proposed before cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian has been investigated.

This is quite different from the standard algebraic geometry approach, where only the parametrization of two-dimensional cubic algebraic equations have been considered. Also, the possible applications in modern cosmological theories has been commented.

1 INTRODUCTION

Recently there has been an increasing interest in finding solutions of the Einstein’s equations in terms of elliptic and theta functions. In [1] a solution of the Einstein’s equations with a perfect fluid energy-momentum tensor was found for the special case of the inhomogeneous Szafrań - Szekeres cosmological model [2, 3,4,5 ] with a metric

$$ds^2 = dt^2 - e^{2\alpha(t,r,y,z)}dr^2 - e^{2\beta(t,r,y,z)}(dy^2 + dz^2) \quad (1.1)$$

Solutions expressed through ultraelliptic functions for the case of an relativistic gravitational field of a rigidly rotating disk of dust have been found in [6, 7, 8]. Further, in [1] important cosmological characteristics for observational cosmology such as the Hubble’s constant $H(t) = \frac{R(t)}{R''(t)}$ and the deceleration parameter $q = -\frac{R'(t)R(t)}{R''(t)}$ have been expressed in terms of the Jacobi’s theta function and of the Weierstrass elliptic function respectively. Thus solutions in terms of elliptic functions may turn out to be an important ingredient in understanding the s. c. ”accelerating cosmological models”.

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Since in [1] a solution has been found only for one component of the Einstein’s equations and also for one additional, limiting case - the FLRW (Friedman - Lemaître - Robertson - Walker) metric, it is important to know whether there are other cosmological metrics, for which solutions in terms of elliptic functions exist.

In [9, 10], instead of searching new solutions for special cases, a more general approach has been proposed. Concretely, it has been shown that the Einstein’s vacuum equations and the gravitational Lagrangian can be represented in the form of multivariable cubic algebraic equations. In [9] this was performed for a specific choice of the contravariant tensor components in the form $\tilde{g}^{ij} = dX^i dX^j$, where $X^i = X^i(x_1, x_2, \ldots, x_n)$ are some generalized coordinates, depending on the initial coordinates $x_1, x_2, \ldots, x$. In [10], this was performed for a general contravariant metric tensor $\tilde{g}^{ij}$. In both papers, it has been assumed that the contravariant metric components differ from the inverse one (i.e. $g_{ij}\tilde{g}^{jk} = \delta^k_i$) and consequently, a distinction should be made between covariant and contravariant metric tensor components. This is the essence of the affine geometry approach, the essence of which have been clarified in the previous paper ([10], Part I).

Most importantly, the derived cubic equations in all cases clearly suggest that in respect to a chosen variable, they can be brought to the parametrizable form of the cubic algebraic equation [11, 12, 13]

$$y^2 = 4x^3 - g_2 x - g_3 .$$

Then, in accordance with the standard algebraic geometry prescriptions, one can set up

$$x = \rho(z) ; \quad y = \rho'(z) ,$$

The main purpose of this second part of the paper will be to demonstrate how this parametrization can be extended to all the variables $dX^i$ in the cubic algebraic equation and thus a non-plane, multivariable cubic algebraic equation will also be possible to be parametrized, but in terms of more complicated functions. This is necessary to be performed in order to find how the contravariant metric components $\tilde{g}^{ij} = dX^i dX^j$ are parametrized - the importance of this will become evident in the next (third) part of this paper.

It will be demonstrated in section 2 how an embedded system of cubic algebraic equations can be obtained, and each equation from this sequence will be for the algebraic subvariety of the solutions - i.e. the first equation will be for the algebraic variety of $n$ variables, the second one - for $(n - 1)$ variables, the last one will be only for the $dX^1$ variable. In section 3 it has been proved that yet the situation is different from the standard case - the obtained functions for the differentials $dX^1$, $dX^2$, $dX^3$ are complicated irrational functions of the Weierstrass function $\rho(z)$, its derivative $\rho'(z)$, the (covariant) metric tensor and affine connection components, which depend on the global coordinates $X^1$, $X^2$, $X^3$. Therefore, it might seem that due to the dependence on the global coordinates one cannot assert that the functions $dX^1$, $dX^2$, $dX^3$ are "uniformization functions", since they should depend only on the complex coordinate $z$.

In the next (third) part of this paper it shall be proved that yet the uniformization functions for the cubic equation can be found since the generalized coordinates $X^1$, $X^2$, $X^3$ can be expressed as solutions
of a first-order system of nonlinear differential equations and thus the dependence of the differentials on $X^1, X^2, X^3$ is eliminated.

Throughout the whole paper it has been assumed that the second differentials of the generalized coordinates are zero $d^2 X^i = 0$. This may seem to be a serious restriction, but it was necessary to be imposed in order to be able to construct the algorithm for finding the solutions of the cubic equation. The important point is another - since the main tool of the proposed approach is the linear-fractional transformation, will this approach be applicable also for the more complicated cases of 1. the cubic algebraic equation of reparametrization invariance, but with some modifications and technical difficulties imposed in order to be able to construct the algorithm for finding the solutions of the cubic equation. In the subsequent parts of this paper, it will be shown that the approach continues to work, although with some modifications and technical difficulties.

2 EMBEDDED SEQUENCE OF ALGEBRAIC EQUATIONS AND FINDING THE SOLUTIONS OF THE CUBIC ALGEBRAIC EQUATION

The purpose of the present subsection will be to describe the method for finding the solution (i.e., the algebraic variety of the differentials $dX^i$) of the cubic algebraic equation (1.4) (in the limit $d^2 X^k = 0$). The applied method has been proposed first in [9] but here it will be developed further and applied with respect to a sequence of algebraic equations with algebraic varieties, which are embedded into the initial one. This means that if at first the algorithm is applied with respect to the three-dimensional cubic algebraic equation (1.4) and a solution for $dX^3$ (depending on the Weierstrass function and its derivative is found), then the same algorithm will be applied with respect to the two-dimensional cubic algebraic equation with variables $dX^1$ and $dX^2$, and finally to the one-dimensional cubic algebraic equation of the variable $dX^1$ only.

The basic and very simple idea about parametrization of a cubic algebraic equation with the Weierstrass function [11-13] can be presented as follows: Let us define the lattice $\Lambda = \{ m \omega_1 + n \omega_2 \mid m, n \in \mathbb{Z}; \omega_1, \omega_2 \in C, \text{Im } \frac{\omega_1}{\omega_2} > 0 \}$ and the mapping $f: C/\Lambda \to CP^2$, which maps the factorized (along the points of the lattice $\Lambda$) part of the points on the complex plane into the two-dimensional complex projective space $CP^2$. This means that each point $z$ on the complex plane is mapped onto the point $(x, y) = (\rho(z), \rho'(z))$, where $x$ and $y$ belong to the affine curve (1.2). In other words, the functions $x = \rho(z)$ and $y = \rho'(z)$ are uniformization functions for the cubic curve, and it can be proved [13] that the only cubic algebraic curve (but with number coefficients!) which is parametrized by the uniformization functions $x = \rho(z)$ and $y = \rho'(z)$ is the above mentioned affine curve.

In the case of the cubic equation (1.4), the aim will be again to bring the equation to the form (1.2) and afterwards to make equal each of the coefficient functions to the (numerical) coefficients in (1.2).

In order to provide a more clear description of the developed method, let us divide it into several steps.

Step 1. The initial cubic algebraic equation (1.4) is presented as a cubic equation with respect to the variable $dX^3$ only

$$A_3(dX^3)^3 + B_3(dX^3)^2 + C_3(dX^3) + G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 , \quad (2.1)$$

where naturally the coefficient functions $A_3, B_3, C_3$ and $G^{(2)}$ depend on the variables $dX^1$ and $dX^2$ of the algebraic subvariety and on the metric tensor $g_{ij}$, the Christoffel connection $\Gamma_{ij}^k$, and the Ricci tensor $R_{ik}$:

$$A_3 \equiv 2p \Gamma_{33}^3 g_{3r} ; \quad B_3 \equiv 6p \Gamma_{33}^r g_{3r} dX^\alpha - R_{33} \quad , \quad (2.2)$$

$$C_3 \equiv -2R_{33} dX^\alpha + 2p (\Gamma_{33}^r g_{3r} + 2 \Gamma_{33}^r g_{3r}) dX^\alpha dX^\beta \quad . \quad (2.3)$$

The Greek indices $\alpha, \beta$ take values $\alpha, \beta = 1, 2$ while the indice $r$ takes values $r = 1, 2, 3$. 

Step 2. A linear-fractional transformation

\[ dX^3 = \frac{a_3(z)\tilde{dX}^3 + b_3(z)}{c_3(z)dX^3 + d_3(z)} \]  \hspace{1cm} (2.4)

is performed with the purpose of setting up to zero the coefficient functions in front of the highest (third) degree of \( \tilde{dX}^3 \). This will be achieved if \( G(2)(dX^2, dX^3; g_{ij}, \Gamma_{ij}^k, R_{ik}) = -\frac{a_3Q}{c_3} \), where

\[ Q \equiv A_3a_3^2 + C_3c_3^2 + B_3a_3c_3 + 2c_3d_3C_3 \]  \hspace{1cm} (2.5)

which gives a cubic algebraic equation with respect to the two-dimensional algebraic variety of the variables \( dX^3 \) and \( dX^2 \):

\[ p\Gamma_{\alpha\beta}^r dX^\alpha dX^\beta + K_{\alpha\beta}^{(1)} dX^\alpha dX^\beta + K_{\alpha}^{(2)} dX^\alpha + 2p \left( \frac{a_3}{c_3} \right)^3 \Gamma_{\alpha\beta}^r g_{3r} = 0 \]  \hspace{1cm} (2.6)

and \( K_{\alpha\beta}^{(1)} \) and \( K_{\alpha}^{(2)} \) are the corresponding quantities [9]

\[ K_{\alpha\beta}^{(1)} = -R_{\alpha\beta} + 2p \frac{a_3}{c_3}(1 + 2 \frac{d_3}{c_3})(2\Gamma_{\alpha\beta}^r g_{3r} + \Gamma_{\alpha r}^r g_{3r}) \]  \hspace{1cm} (2.7)

and

\[ K_{\alpha}^{(2)} = 2 \frac{a_3}{c_3} \left[ 3p \frac{a_3}{c_3} \Gamma_{\alpha3}^r g_{3r} - (1 + 2 \frac{d_3}{c_3})R_{\alpha3} \right] . \]  \hspace{1cm} (2.8)

Note that since the linear fractional transformation (with another coefficient functions) will again be applied with respect to another cubic equations, everywhere in (2.4 - 2.7) the coefficient functions \( a_3(z), b_3(z), c_3(z) \) and \( d_3(z) \) bear the indice "3", to distinguish them from the indices in the other linear-fractional trangformations. In terms of the new variable \( n_3 = \tilde{dX}^3 \) the original cubic equation (1.4) acquires the form [9]

\[ \tilde{n}^2 = \mathcal{P}_1(\tilde{n}) m^3 + \mathcal{P}_2(\tilde{n}) m^2 + \mathcal{P}_3(\tilde{n}) m + \mathcal{P}_4(\tilde{n}) , \]  \hspace{1cm} (2.9)

where \( \mathcal{P}_1(\tilde{n}), \mathcal{P}_2(\tilde{n}), \mathcal{P}_3(\tilde{n}) \) and \( \mathcal{P}_4(\tilde{n}) \) are complicated functions of the ratios \( \frac{a_3}{c_3}, \frac{b_3}{d_3} \) and \( A_3, B_3, C_3 \) (but not of the ratio \( \frac{a_3}{c_3} \), which is very important). The variable \( m \) denotes the ratio \( \frac{a_3}{c_3} \) and the variable \( \tilde{n} \) is related to the variable \( n \) through the expression

\[ \tilde{n} = \sqrt{k_3} \sqrt{C_3} \left[ n + L_1^{(3)} \frac{B_3}{C_3} + L_2^{(3)} \right] , \]  \hspace{1cm} (2.10)

where

\[ k_3 \equiv \frac{b_3}{d_3} \frac{c_3}{d_3} (\frac{c_3}{d_3} + 2) \]  \hspace{1cm} (2.11)

\[ L_1^{(3)} \equiv \frac{1}{2} \frac{b_3}{c_3} \frac{c_3}{d_3} + 2 \]  \hspace{1cm} ; \hspace{1cm} L_2^{(3)} \equiv \frac{1}{c_3} + 2 . \]  \hspace{1cm} (2.12)

The subscript "3" in \( L_1^{(3)} \) and \( L_2^{(3)} \) means that the corresponding ratios in the R. H. S. also have the same subscript. Setting up the coefficient functions \( \mathcal{P}_1(\tilde{n}), \mathcal{P}_2(\tilde{n}), \mathcal{P}_3(\tilde{n}) \) equal to the number coefficients \( 4, 0, -g_2, -g_3 \) respectively, one can now parametrize the resulting equation

\[ \tilde{n}^2 = 4m^3 - g_2m - g_3 \]  \hspace{1cm} (2.13)

according to the standard prescription

\[ \tilde{n} = \rho'(z) = \frac{dp}{dz} ; \hspace{1cm} \frac{a_3}{c_3} \equiv m = \rho(z) . \]  \hspace{1cm} (2.14)
Taking this into account, representing the linear-fractional transformation (2.4) as (dividing by \( c_3 \))

\[
dX^3 = \frac{\frac{a}{c_3}dX^3 + \frac{b}{c_3}}{dX^3 + \frac{a}{c_3}} \quad (2.15)
\]

and combining expressions (2.10) for \( \bar{n} \) and (2.15), one can obtain the final formulae for \( dX^3 \) as a solution of the cubic algebraic equation

\[
dX^3 = \frac{\frac{b}{c_3} + \rho(z)\rho'(z) - L^3_B}{\frac{a}{c_3} + \rho'(z) - L^3_C} \quad (2.16)
\]

In order to be more precise, it should be mentioned that the identification of the functions \( \bar{P}_1(\bar{n}) \), \( \bar{P}_2(\bar{n}) \), \( \bar{P}_3(\bar{n}) \) with the number coefficients gives some additional equations [9], which in principle have to be taken into account in the solution for \( dX^3 \). This has been investigated to a certain extent in [9], and will be continued to be investigated. Here in this paper the main objective will be to show the dependence of the solutions on the Weierstrass function and its derivative. Since only the ratios \( \frac{\rho}{\bar{P}} \) enter these additional relations, and not \( \frac{\rho}{\bar{P}} \) (which is related to the Weierstrass function), they do not affect the solution with respect to \( \rho(z) \) and \( \rho'(z) \).

Since \( B_3 \) and \( C_3 \) depend on \( dX^1 \) and \( dX^2 \), the solution (2.16) for \( dX^3 \) shall be called the embedding solution for \( dX^1 \) and \( dX^2 \).

**Step 3.** Let us now consider the two-dimensional cubic equation (2.6). Following the same approach and finding the "reduced" cubic algebraic equation for \( dX^1 \) only, it shall be proved that the solution for \( dX^2 \) is the embedding solution for \( dX^1 \).

For the purpose, let us again write down eq. (2.6) in the form (2.1), singling out the variable \( dX^2 \):

\[
A_2(dX^2)^3 + B_2(dX^2)^2 + C_2(dX^2) + G^{(1)}(dX^1, g_{ij}, \Gamma^k_{ij}, R_{ik}) \equiv 0 \quad (2.17)
\]

where the coefficient functions \( A_2, B_2, C_2 \) and \( G^{(1)} \) are the following:

\[
A_2 \equiv 2p\Gamma^r_{22}g_{2r} \quad ; \quad B_2 \equiv K^{(1)}_{22} + 2p[2\Gamma^r_{12}g_{2r} + \Gamma^r_{22}g_{1r}]dX^1 \quad , (2.18)
\]

\[
C_2 \equiv 2p[\Gamma^r_{11}g_{2r} + 2\Gamma^r_{12}g_{1r}]dX^1 + (K^{(1)}_{12} + K^{(1)}_{21})dX^1 + K^{(2)}_2 \quad , (2.19)
\]

\[
G^{(1)} \equiv 2p\Gamma^r_{11}g_{1r}(dX^1)^3 + K^{(1)}_{11}(dX^1)^2 + K^{(2)}_1dX^1 + 2pp^3(z)\Gamma^r_{33}g_{3r} \quad . (2.20)
\]

Note that the starting equation (2.6) has the same structure of the first terms, if one makes the formal substitution \( -R_{\alpha\beta} \rightarrow K^{(1)}_{\alpha\beta} \) in the second terms, but eq. (2.6) has two more additional terms \( K^{(2)}_1dX^1 + 2pp^3(z)\Gamma^r_{33}g_{3r} \). Therefore, one might guess how the coefficient functions will look like just by taking into account the above substitution and the contributions from the additional terms. Revealing the general structure of the coefficient functions might be particularly useful in higher dimensions, when one would have a "chain" of cubic algebraic equations. Concretely for the three-dimensional case, investigated here, \( C_2 \) in (2.19) can be obtained from \( C_3 \) in (2.3), observing that there will be an additional contribution from the term \( K^{(2)}_1dX^\alpha \) for \( \alpha = 2 \). Also, in writing down the coefficient functions in (2.1) it has been accounted that as a result of the previous parametrization \( \frac{a}{c_3} = \rho(z) \).

Since eq. (2.17) is of the same kind as eq. (2.1), for which we already wrote down the solution, the expression for \( dX^2 \) will be of the same kind as in formulae (2.16), but with the corresponding functions \( A_2, B_2, C_2 \) instead of \( A_3, B_3, C_3 \). Taking into account (2.18 - 2.19), the solution for \( dX^2 \) can be written as follows:

\[
dX^2 = \frac{\frac{1}{\sqrt{k_2}\rho(z)\rho'(z)}(\sqrt{C_2 + h_1(dX^1)^2} + h_2(dX^1) + h_3)}{\frac{1}{\sqrt{k_2}\rho'(z)}(\sqrt{C_2 + l_1(dX^1)^2} + l_2(dX^1) + l_3)} \quad , (2.21)
\]
where \( h_1, h_2, h_3, l_1, l_2, l_3 \) are expressions, depending on \( \frac{b_2}{c_2}, \frac{d_2}{c_2}, \Gamma_{\alpha\beta} \ (r = 1, 2, 3 ; \alpha, \beta = 1, 2) \), \( g_{\alpha\beta}, K^{(1)}_{12}, K^{(1)}_{21} \) and on the Weierstrass function. They will be presented in Appendix A.

The representation of the solution for \( dX^2 \) in the form (2.21) shows that it is an "embedding" solution of \( dX^1 \) in the sense that it depends on this function. Correspondingly, the solution (2.16) for \( dX^3 \) is an "embedding" one for the variables \( dX^1 \) and \( dX^2 \).

**Step 4.** It remains now to investigate the one-dimensional cubic algebraic equation

\[
A_1(dX^1)^3 + B_1(dX^1)^2 + C_1(dX^1) + G^{(0)}(g_{ij}, \Gamma_{ij}, R_{ik}) \equiv 0 \quad ,
\]

obtained from the two-dimensional cubic algebraic equation (2.17) after applying the linear-fractional transformation

\[
dX^2 = \frac{a_2(z)dX^2 + b_2(z)}{c_2(z)dX^2 + d_2(z)}
\]

and setting up to zero the coefficient function before the highest (third) degree of \( (dX^2)^3 \). Taking into account that as a result of the previous parametrization \( \frac{a_2}{c_2} = \rho(z) \), the coefficient functions \( A_1, B_1, C_1 \) and \( D_1 \) are given in a form, not depending on \( dX^2 \) and \( dX^3 \):

\[
A_1 \equiv 2p_1^{(1)} g_{1r} ,
\]

\[
B_1 \equiv F_3 \rho(z) + K^{(1)}_{11} = 2p(1 + 2 \frac{d_2}{c_2})[2\Gamma_{12} g_{1r} + \Gamma_{11} g_{2r}] \rho(z) + K^{(1)}_{11} ,
\]

\[
C_1 \equiv F_1 \rho^2(z) + F_2 \rho(z) + K^{(2)} = 2p[2\Gamma_{12} g_{2r} + \Gamma_{22} g_{1r}] \rho^2(z) + \frac{1}{2} \frac{d_2}{c_2}(K^{(1)}_{12} + K^{(1)}_{21}) \rho(z) + K^{(2)} ,
\]

\[
G^{(0)} = 2p[\Gamma_{12} g_{2r} + \Gamma_{33} g_{3r}] \rho^3(z) + K^{(1)}_{22} \rho^2(z)
\]

The solution for \( dX^1 \) can again be written in a form (2.16), but with \( \frac{b_1}{c_1}, \frac{d_1}{c_1}, L^{(1)}_1, L^{(1)}_2, k_1 \) and \( B_1, C_1 \) instead of these expressions with the indice "3".

Taking into account formulae (2.24 - 2.27) for \( A_1, B_1 \) and \( C_1 \), the final expression for \( dX^1 \) can be written as

\[
dX^1 = \frac{1}{\sqrt{k_2}} \rho(z) \rho'(z) \sqrt{F_1 \rho^2 + F_2 \rho(z) + K^{(2)}_{12} + f_1 \rho^3 + f_2 \rho^2 + f_3 \rho + f_4}
\]

\[
\frac{1}{\sqrt{k_2}} \rho'(z) \sqrt{F_1 \rho^2(z) + F_2 \rho(z) + K^{(2)}_{12} + g_1 \rho^2(z) + g_2 \rho(z) + g_3}
\]

where \( F_1, F_2, f_1, f_2, f_3, f_4, g_1, g_2 \) and \( g_3 \) are functions (also to be given in Appendix A), depending on \( g_{\alpha\beta}, \Gamma_{\alpha\beta}^{(1)} \ (\alpha, \beta = 1, 2) \) and on the ratios \( \frac{b_2}{c_2}, \frac{d_2}{c_2}, \frac{b_1}{c_1}, \frac{d_1}{c_1} \).

### 3. A Proof that the Solutions \( dX^1, dX^2 \) and \( dX^3 \) Are Not Elliptic Functions

**Proposition 1** The expressions (2.21) for \( dX^2 \) and (2.28) for \( dX^1 \) do not represent elliptic functions.

Proof: The proof is straightforward and will be based on assuming the contrary. Let us first assume that \( dX^1 \) is an elliptic function. Then from standard theory of elliptic functions [11] it follows that \( dX^1 \) (being an elliptic function by assumption) can be represented as

\[
dX^1 = K_1(\rho) + \rho'(z)K_2(\rho)
\]
where \( K_1(\rho) \) and \( K_2(\rho) \) depend on the Weierstrass function only. For convenience one may denote the expressions outside the square root in the nominator and denominator as

\[
Z_1(\rho) \equiv f_1\rho^3(z) + f_2\rho^2(z) + f_3\rho(z) + f_4 \quad , \quad (3.2)
\]

\[
Z_2(\rho) \equiv g_1\rho^2(z) + g_2\rho(z) + g_3 \quad . \quad (3.3)
\]

Then, setting up equal the expressions (3.1) and (2.28) for \( dX^1 \), one can express the function \( K_2(\rho) \) as

\[
K_2(\rho) = \frac{(1-K_1(\rho))\rho \sqrt{F_1\rho^2 + F_2\rho + K_1^{(2)} + \sqrt{K_1}Z_1(\rho) - Z_2(\rho)K_1(\rho)}}{\rho \sqrt{F_1\rho^2 + F_2\rho + K_1^{(2)} + \sqrt{K_1}Z_2(\rho)}} \quad . \quad (3.4)
\]

The R. H. S. of the above expression depends on the derivative \( \rho' \), while the L.H. S. depends on \( \rho \) only. Therefore the obtained contradiction is due to the initial assumption that \( dX^1 \) is an elliptic function, having the representation (3.1). In quite a similar way, it can be proved that \( dX^2 \) and \( dX^3 \) are not elliptic functions. In fact, this is obvious since they are embedding functions of \( dX^1 \), which is not an elliptic function.

4 DISCUSSION

The most important result in this paper is related to the possibility to find the parametrization functions for a multicomponent cubic algebraic surface, again by consequent application of the linear-fractional transformation. The parametrization functions represent complicated irrational expressions of the Weierstrass function and its derivative. The advantage of applying the linear-fractional transformations (2.4) and (2.23) is that by adjusting their coefficient functions (so that the highest - third degree in the transformation equation will vanish), the following sequence of plane cubic algebraic equations is fulfilled (the analogue of eq.(65) in [9]):

\[
P_1^{(3)}(n_{(3)})m_{(3)}^3 + P_2^{(3)}(n_{(3)})m_{(3)}^2 + P_3^{(3)}(n_{(3)})m_{(3)} + P_4^{(3)} = 0 \quad , \quad (4.1)
\]

\[
P_1^{(2)}(n_{(2)})m_{(2)}^3 + P_2^{(2)}(n_{(2)})m_{(2)}^2 + P_3^{(2)}(n_{(2)})m_{(2)} + P_4^{(2)} = 0 \quad , \quad (4.2)
\]

\[
P_1^{(1)}(n_{(1)})m_{(1)}^3 + P_2^{(1)}(n_{(1)})m_{(1)}^2 + P_3^{(1)}(n_{(1)})m_{(1)} + P_4^{(1)} = 0 \quad , \quad (4.3)
\]

where \( m_{(3)}, m_{(2)}, m_{(1)} \) denote the ratios \( \frac{a_3}{c_3}, \frac{a_2}{c_2}, \frac{a_1}{c_1} \) in the corresponding linear - fractional transformations and \( n_{(3)}, n_{(2)}, n_{(1)} \) are the ”new” variables \( \tilde{d}X^3, \tilde{d}X^2, \tilde{d}X^1 \). The sequence of plane cubic algebraic equations (4.1 - 4.3) should be understood as follows: the first one (4.1) holds if the second one (4.2) is fulfilled; the second one (4.2) holds if the third one (4.3) is fulfilled. Of course, in the case of \( n \) variables (i. e. \( n \) component cubic algebraic equation) the generalization is straightforward. Further, since each one of the above plane cubic curves can be transformed to the algebraic equation \( (i = 1, 2, 3) \)

\[
\tilde{n}_{(i)}^2 = P_{1}^{(i)}(\tilde{n}_{(i)})m_{(i)}^3 + P_{2}^{(i)}(\tilde{n}_{(i)})m_{(i)}^2 + P_{3}^{(i)}(\tilde{n}_{(i)})m_{(i)} + P_{4}^{(i)}(\tilde{n}_{(i)}) \quad (4.4)
\]

and subsequently to its parametrizable form, one obtains the solutions of the initial multicomponent cubic algebraic equation.
5 APPENDIX A: SOME COEFFICIENT FUNCTIONS IN THE FINAL SOLUTIONS FOR $dX^1, dX^2, dX^3$ IN SECTION 2

The functions $h_1, h_2, h_3$ (depending on the Weierstrass function $\rho(z)$) and the functions $l_1, l_2, l_3$ (not depending on $\rho(z)$) in the expression (2.23) for the solution $dX^2$ of the cubic algebraic equation are

$$h_1 \equiv 2p \left( \frac{b_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) \left( 2\Gamma_{12} g_{1r} + \Gamma_{11} g_{2r} \right) , \quad (A1)$$

$$h_2 \equiv \left( \frac{b_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) \left( K_{12}^{(1)} + K_{21}^{(1)} \right) - 2p L_1^{(2)}(\rho(z)) \left( 2\Gamma_{12} g_{2r} + \Gamma_{22} g_{1r} \right) , \quad (A2)$$

$$h_3 \equiv \left( \frac{b_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) K_2^{(2)} - L_1^{(2)}(\rho(z)) K_{22}^{(1)} , \quad (A3)$$

$$l_1 \equiv 2p \left( \frac{d_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) \left( 2\Gamma_{12} g_{1r} + \Gamma_{11} g_{2r} \right) , \quad (A4)$$

$$l_2 \equiv \left( \frac{d_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) \left( K_{12}^{(1)} + K_{21}^{(1)} \right) - 2p L_1^{(2)}(\rho(z)) \left( 2\Gamma_{12} g_{2r} + \Gamma_{22} g_{1r} \right) , \quad (A5)$$

$$l_3 \equiv \left( \frac{d_2}{c_2} - 2 \frac{b_2}{d_2} L_1^{(2)}(\rho(z)) \right) K_2^{(2)} - L_1^{(2)}(\rho(z)) K_{22}^{(1)} . \quad (A6)$$

The functions $F_1, F_2, F_3, f_1, f_2, f_3, f_4, g_1, g_2, g_3$ in the solution for $dX^1$ are the following

$$F_1 \equiv 2p \left( 2\Gamma_{12} g_{2r} + \Gamma_{22} g_{1r} \right) ; \quad F_2 \equiv \left( 1 + 2 \frac{d_2}{c_2} \right) \left( K_{12}^{(1)} + K_{21}^{(1)} \right) , \quad (A7)$$

$$F_3 \equiv 2p \left( 1 + 2 \frac{d_2}{c_2} \right) \left( 2\Gamma_{12} g_{1r} + \Gamma_{11} g_{2r} \right) , \quad (A8)$$

$$f_1 \equiv -2 \frac{b_1}{d_1} L_1^{(1)}(F_1) ; \quad f_3 \equiv \frac{b_1}{c_1} F_2 - L_1^{(1)} , \quad (A9)$$

$$f_2 \equiv \frac{b_1}{c_1} F_1 - L_1^{(1)} F_3 - 2 \frac{b_2}{d_1} L_1^{(1)} F_2 , \quad (A10)$$

$$\tilde{g}_1 \equiv \left( \frac{d_1}{c_1} - 2 \frac{b_1}{d_1} \right) F_1 , \quad (A11)$$

$$\tilde{g}_2 \equiv \left( \frac{d_1}{c_1} - 2 \frac{b_1}{d_1} \right) F_2 - L_1^{(1)} F_3 , \quad (A12)$$

$$\tilde{g}_3 \equiv \left( \frac{d_1}{c_1} - 2 \frac{b_1}{d_1} \right) K_1^{(2)} - L_1^{(1)} K_{11}^{(1)} . \quad (A13)$$
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