Microscopic Calculation of $\Lambda - \alpha$ Folding Potential

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Abstract

We construct a folding potential between the $\alpha$ and $\Lambda$ particles based on underlying nucleon-nucleon and hyperon-nucleon interactions. Starting from a phenomenological $\Lambda$-N potential and a Gaussian form of the $\alpha$-particle wave function we obtain for the built $\alpha$-$\Lambda$ interaction a bound $\Lambda^5$He state with the binding energy (3.10 MeV), which is consistent with recent experimental data $3.12 \pm 0.02$ MeV. When in turn an exact solution of the four-body Faddeev-Yakubovsky equation for the $\alpha$-particle calculated with the CDBon, Nijmegen or Argonne V18 realistic nucleon-nucleon potential is used and the phenomenological Gaussian $\Lambda$-N potential is replaced by the realistic (Nijmegen NSC97f) potential approximated by a rank-1 separable form, then $\Lambda^5$He is overbound. In particular, its binding energy given by the folding potential generated with the $\alpha$ particle wave function based on the CDBon potential is 7.47 MeV. Although the rank-1 separable $\Lambda$-N potential reproduces the exact scattering length and the effective range of the original $\Lambda$-N potential, the overbinding results from the lack of the required repulsive properties in the assumed separable form.

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The strangeness sector of few-body systems still poses many problems despite many years of strong efforts\cite{1}. Hyperon-nucleon (YN) and hyperon-hyperon (YY) forces are to a large extent unknown due to the sparsity of direct YN or indirect YY data. Thus the situation is quite different from the nucleon-nucleon (NN) case, where thousands of available data points strongly constrain the spin-momentum dependencies of nucleon-nucleon force models. In the case of Λ-N or Σ-N (strangeness S=-1 sector) the set of scattering data is very small \cite{2} and is not sufficient to determine well the properties of those forces. On the theoretical side one is still far away from a solution of the underlying theory of the strong interaction, QCD, and therefore effective approaches are used to generate forces. They are either based on meson exchanges, like the ones by the Nijmegen \cite{3}, Jülich\cite{4}, or Ehime\cite{5} groups, or on quark models, like the ones by the Kyoto-Niigata \cite{6}, the Tokyo\cite{7}, the Salamanca \cite{8} or the Beijing\cite{9} groups. Despite all this work, there remains a lot of uncertainty about the properties of the baryon-baryon forces, especially in the strangeness S=-2 sector.

Beside the above-mentioned potentials, there exist also modified versions of the realistic Nijmegen potential, widely used in \cite{10}, which are phase equivalent to the original ones and are parameterized in the Gaussian form. These YN and YY interactions constitute the dynamical input for few-body equations, whose solutions lead to observables, which can be compared to experimental data. The lightest hypernuclei $^3\Lambda\text{H}$, $^4\Lambda\text{H}$ and $^4\Lambda\text{He}$ with strangeness S=-1 have been studied extensively, by evaluating their binding energies and lowest excitation energies by employing various potential models. The results are still ambiguous. The binding energy of the lightest hypernucleus $^3\Lambda\text{H}$ can be satisfactorily reproduced using directly some of the Nijmegen forces \cite{11}.

However, when the Nijmegen YN forces NSC89 \cite{12} and NSC97a-f \cite{13} are employed, no adequate description of $^4\Lambda\text{H}$ and $^4\Lambda\text{He}$ is achieved \cite{14}, whereas more phenomenological forces come closer to the data \cite{15}. Phenomenological central Λ-N potentials overbind $^4\Lambda\text{He}$, which is well known for its anomalously small binding energy. This problem seems to be solved by a recent variational 5-body calculation\cite{15} using forces stemming from the original Nijmegen ones.

Certain heavier hypernuclei can also be viewed as few-body systems assuming their cluster structure in terms of the $\alpha$-particles. Phenomenological Λ – $\alpha$ potentials of a simple Gaussian type were used e.g. to study double-Λ hypernuclei \cite{16}. Recently, we predicted the existence of quasi-bound state of the Σ – Σ – $\alpha$ system employing a phenomenological Σ – $\alpha$ potential \cite{17}. The common drawback of all phenomenological potentials is that they have some parameters to be fit to experimental data. In order to obtain a potential without any unknown parameters, one has to solve a N-body problem driven by baryon-baryon interactions which is, however, very difficult beyond the four-body system.

In this paper we derive a folding potential between the $\alpha$ and Λ particles without any additional parameters. Our paper is organized as follows. In the next section we explain how to introduce the folding potential in question. In order to achieve this goal we use the $\alpha$-particle wave function based on realistic NN forces, e.g., a meson theoretical CD-Bonn \cite{18}, Nijmegen \cite{19} and Argonne \cite{20} potentials and we describe the Λ-N interaction by a phenomenological Gaussian form \cite{21} and meson theoretical models, e.g., Chiral \cite{22}, Jülich \cite{23}, Nijmegen \cite{13,24} and Ehime\cite{25}. In order to facilitate calculations, these meson theoretical Λ-N potentials are modified into convenient separable approximations. In Sec. III we calculate binding energies of $^5\Lambda\text{He}$ employing many versions of the folding potential.
Finally, in Sec. IV we discuss and summarize these results.

II. METHOD OF CALCULATION

The construction of the $\Lambda - \alpha$ folding potential is performed by using the $\alpha$-particle wave function from the solution of the Faddeev-Yakubovsky equations based on realistic NN forces. The $\Lambda$-N interaction is first taken in the form of a phenomenological Gaussian potential [21]. Later we will replace the Gaussian potential by a more realistic one. The folding potential $V_{fold}$ is defined by evaluating matrix elements of the inner realistic potential $V_{inner}$ between products of two-cluster wave functions $\psi_\alpha \psi_\Lambda$:

$$V_{fold} = \langle \psi_\alpha \psi_\Lambda | V_{inner} | \psi_\alpha \psi_\Lambda \rangle.$$  \hfill (2.1)

The schematic diagram of the $\Lambda - \alpha$ potential is shown in Fig. 1. For the case at hand, the wave functions $\psi_\alpha$ and $\psi_\Lambda$ correspond to the $\alpha$ and $\Lambda$ particles. The $\alpha$-particle wave function is obtained by solving the Faddeev-Yakubovsky equations [26] in momentum space in the partial wave basis. The natural Jacobi momenta for the (1234)$\Lambda$ partition [26] are

$$\vec{u}_1 = \frac{1}{2}(\vec{k}_1 - \vec{k}_2), \quad \vec{u}_2 = \frac{2}{3} \left\{ \vec{k}_3 - \frac{1}{2}(\vec{k}_1 + \vec{k}_2) \right\},$$

$$\vec{u}_3 = \frac{3}{4} \left\{ \vec{k}_4 - \frac{1}{3}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \right\},$$

$$\vec{u}_\Lambda = \frac{4m_N\vec{k}_\Lambda - m_\Lambda(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)}{4m_N + m_\Lambda}.$$ \hfill (2.2)

where $\vec{k}_i, i = 1, \ldots, 4$ are the individual nucleon momenta, $\vec{k}_\Lambda$ is the momentum of the $\Lambda$ hyperon; $m_N$ and $m_\Lambda$ are the masses of the nucleon and the $\Lambda$ particle. The corresponding relative orbital angular momenta will be denoted by $l_i, i = 1, 2, 3, \Lambda$ and the total spin, total

FIG. 1: A schematic representation of the $\alpha$-$\Lambda$ folding potential.
angular momenta and total isospins in the various subsystems by \( s_i, j_i \) and \( t_i \), respectively (see Fig. 2). The 4N-Λ basis states are introduced via

\[
|u_1u_2u_3u_Λa⟩ := |u_1u_2u_3u_Λ\left[l_2((l_1s_1)j_1\frac{1}{2})s_2\right]j_2,(j_2\frac{1}{2})j_3,(l_3j_3)j_α,(l_Λ\frac{1}{2})j_Λ,J,(t_1\frac{1}{2})t_2(t_2\frac{1}{2})T⟩, (2.4)
\]

where the brackets indicate self-explanatory consecutive couplings to the total five-baryon angular momentum \( J \) and total isospin \( T \) with the corresponding magnetic quantum numbers (not shown for brevity). The quantum numbers for channels \( a \) and \( b \) are listed in Tab. I. The natural Jacobi momenta for the fragmentation \((123)(4Λ)\) are

\[
\vec{v}_Λ = \frac{1}{m_N + m_Λ}(m_Λ\vec{k}_4 - m_N\vec{k}_Λ), \quad \vec{v}_3 = \frac{1}{4m_N + m_Λ}\{3m_N(\vec{k}_4 + \vec{k}_Λ) - (m_Λ + m_N)(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\}. (2.5)
\]

The corresponding discrete quantum numbers will be denoted by Greek letters, see the right panel of Fig. 2. The basis states are

\[
|u_1u_2v_3v_Λb⟩ := |u_1u_2v_3v_Λ\left[l_2((l_1s_1)j_1\frac{1}{2})s_2\right]j_2, (λ_ΛΣ_Λ)τ_Λ, (λ_3j_3)τ_3, (τ_Λτ_3)J,(t_1\frac{1}{2})t_2(t_2\frac{1}{2})T⟩, (2.6)
\]

where the brackets indicate again the sequences of couplings of angular momenta and isospins. The Jacobi momenta in these two sets are related via

\[
\vec{u}_3 = \vec{v}_3 + \frac{3}{4}\vec{u}_Λ, \quad \vec{v}_Λ = -\vec{u}_Λ - \frac{m_Λ}{m_N + m_Λ}\vec{v}_3. (2.7)
\]

In order to calculate the folding potential of Eq. (2.1) we first prepare the \( α \)-particle wave function \( ψ_α \) and the YN interaction \( V_{YN} \) as

\[
ψ_α(u_1, u_2, u_3, a) = ⟨u_1u_2u_3a|ψ_α⟩, \quad V_{YN}(v_Λ, v_Λ') = ⟨v_Λ|V_{YN}|v_Λ'⟩. (2.8)
\]
TABLE I: Partial-wave quantum numbers for channels $a$ and $b$ corresponding to Fig. 2.

|   | $l_1$ | $s_1$ | $j_1$ | $s_2$ | $l_2$ | $j_2$ | $j_o$ | $l_\Lambda$ | $j_\Lambda$ |
|---|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|
| $a$ | 1 0 0 0 1/2 0 1/2 0 0 0 0 1/2 | 2 0 1 1 1/2 0 1/2 0 0 0 0 1/2 | 3 2 1 1 1/2 0 1/2 0 0 0 0 1/2 | 4 0 1 1 3/2 2 1/2 0 0 0 0 1/2 | 5 2 1 1 3/2 2 1/2 0 0 0 0 1/2 |
| $b$ | 1 0 0 0 1/2 0 1/2 0 0 0 0 1/2 | 2 0 1 1 1/2 0 1/2 0 0 0 0 1/2 | 3 2 1 1 1/2 0 1/2 0 0 0 0 1/2 | 4 0 1 1 3/2 2 1/2 0 0 0 0 1/2 | 5 2 1 1 3/2 2 1/2 0 0 0 0 1/2 |

Then Eq. (2.1) turns into

$$V_{fold}(u_\Lambda, u'_\Lambda) = 4 \sum_{a,a'} \int d\bar{u}_1 d\bar{u}_2 d\bar{u}_3 d\bar{u}_\Lambda \int d\bar{u}'_1 d\bar{u}'_2 d\bar{u}'_3 d\bar{u}'_\Lambda \psi_\alpha(u_1 u_2 u_3 a) \psi_\Lambda(u_\Lambda)$$

$$\times \langle u_1 u_2 u_3 u_\Lambda | V_{YN} | u_1' u_2' u_3 u_\Lambda' a' \rangle \psi_\alpha(u_1' u_2' u_3' a') \psi_\Lambda(u_\Lambda')$$

$$= 4 \sum_{a,a',b'} \int_0^{\infty} v_3^2 dv_3 \int_{-1}^{1} dx \int_{-1}^{1} dx' \int_{u_3^2}^{d_{\bar{u}_3}^2} \int_{u_3^2}^{d_{\bar{u}'_3}^2} K_\alpha(u_3 u_3' a a') G_{a,b}(u_\Lambda, v_3, x) V_{YN}(v_\Lambda, v'_\Lambda) G_{b',a'}(v_3, v_3', x'),$$  \hspace{1cm} (2.9)

where

$$K_\alpha(u_3, u_3', a, a') = \int d\bar{u}_1 d\bar{u}_2 \psi_\alpha(u_1 u_2 u_3 a) \psi_\alpha(u_1 u_2 u_3 a')$$ \hspace{1cm} (2.10)

and $\psi_\Lambda$ is taken is as a plane wave. The geometrical functions $G_{a,b}(u_\Lambda, v_3, x)$ and $G_{b',a'}(v_3, u_\Lambda, x')$ have been introduced in Refs. [27, 28] and for the sake of the reader are displayed in Appendix A.
III. NUMERICAL RESULTS

First let us consider results for the $\Lambda - \alpha$ potential obtained with some phenomenological $\Lambda$-$N$ force. There are many phenomenological models of the $\Lambda$-$N$ interaction, e.g., a hard-core one represented by exponential functions \[29–31\] or a multi-Gaussian type \[32, 33\] inspired by the YNG Potential \[34\]:

\[ V_{\Lambda N}(r) = \sum_{i=1}^{3} w_i e^{-(r/\beta_i)^2}. \] (3.1)

The spin-dependent phenomenological $\Lambda$-$N$ interaction is parameterized by Hiyama et al.,\[21\] as

\[ V_{\Lambda N}(r) = V_{\Lambda N}^0 (1 + \eta \sigma_{\Lambda} \cdot \sigma_N) e^{-(r/\beta)^2}, \] (3.2)

with $V_{\Lambda N}^0 = -38.19$ MeV, $\beta = 1.034$ fm and $\eta = -0.1$. The $\Lambda$-$N$ potential for the spin triplet is shown in Fig. 3 and the quantum numbers for channels $a$ and $b$ are listed in Tab. I for the total $J^\pi = 1/2$ and the total isospin $T=0$, assuming the positive parity and restricting to $j_\alpha = 0$, $l_1 = 2$, $j_1 = 1$. The phenomenological potentials of Eqs. (3.1)–(3.2) are often converted into the $\Lambda$–$\alpha$ potential with the use of the Resonating Group Method (RGM) Technique \[16, 33, 35\], which is based on the $(0s)^4$ shell-model Gaussian wave function. Specifically, the integral kernel $K$ of Eq. (2.10) originating from the Gaussian $\alpha$ particle S-wave function is given as

\[ K(u_3, u'_3) = 4\pi \left( \frac{2}{3\Omega \pi} \right)^{3/2} \exp\left\{ -\frac{(u_3^2 + u'_3^2)}{3\Omega} \right\}, \] (3.3)

where the width parameter $\Omega$ is a common shell model mode \[16, 35\] taken to be 0.275 fm$^{-2}$.

We obtained a $\Lambda$-$\alpha$ folding potential (shown in Fig 4) for which the $^5\Lambda$He binding energy is -3.10 MeV. The calculated binding energy compares well with the data (-3.12 $\pm$ 0.02 MeV) \[36\]. This consistence is to be expected, since the YN potential of Eq. (3.2) is adjusted to the experimental $^5\Lambda$He binding energy when the RGM technique is employed.

Next we replace the simple Gaussian wave function of the $\alpha$ particle by the wave function based on the realistic NN forces: the CD-Bonn\[18\], Nijmegen \[19\], and Argonne V18 \[20\] potentials. Our experience with the $\alpha$ particle wave functions obtained with these potentials indicates that the S-wave contribution is essential to provide the correct binding energy. Thus in the following calculations we could restrict ourselves to only few partial waves and solve the Faddeev-Yakubovsky equation for the $\alpha$ particle in the basis comprising $^1S_0$, $^3S_1$ and $^3D_1$ states.

On the other hand, the $\Lambda$-$N$ potentials are given in the separable form:

\[ V_{\Lambda N}(v_\Lambda, v'_\Lambda) = -\lambda g(v_\Lambda)g(v'_\Lambda), \] (3.4)

where $\lambda$ and $g(p)$ are the coupling constant and the form factor, respectively. In order to check the accuracy of the separable approximation we prepare two kinds of the separable potentials, e.g. the Yamaguchi type (Y) and the Gaussian type (G). The form factors of these potential are given as

\[ g_Y(v_\Lambda) = \frac{1}{v_\Lambda^2 + \beta_Y^2}. \]
The meson theoretical Λ-N potentials are constructed to describe the Λ-N scattering data. However, due to the sparsity of the data parameters of all potential models are not well determined. Therefore, we use the fact that the Λ-N scattering amplitude in the low energy limit can be determined from the well-known effective range expansion which has the

\[ g_G(v_\Lambda) = \exp\{-\beta_G^2 v_\Lambda^2\}. \]  

(3.5)
Form

\[ k \cot \delta = -\frac{1}{a} + \frac{1}{2}rk^2 + \cdots \]  

where \( k \) denotes the scattering momentum in the center-of-mass system and the parameters \( a \) [fm] and \( r \) [fm] are often called the scattering length and the effective range, respectively. The phase shift \( \delta \) of each partial wave is linked to the scattering amplitude.

These effective range expansion parameters are directly connected to the quantities \( \beta_Y \), \( \lambda_Y \), \( \beta_G \) and \( \lambda_G \) from Eq. (3.5) by the following relations.

\[
\beta_Y = \frac{3 + \sqrt{9 - 16\frac{r^2}{a^2}}}{4\beta_Y^2}, \\
\lambda_Y = \frac{\pi \mu (r\beta_Y - 1)}{4\beta_Y^2}, \\
\beta_G = \frac{\sqrt{2a + \sqrt{a(2a - \pi r)}}}{2\sqrt{\pi}}, \\
\lambda_G = \frac{ar}{\sqrt{2}\mu \sqrt{a(2a - \pi r)} - (2a - \pi r)\mu},
\]

where \( \mu \) is the reduced mass of the \( \Lambda-N \) system. In the case of the Yamaguchi type form factors these relations are proved in Ref. [37]. Tables II and III collect the scattering lengths and the effective ranges from several \( \Lambda-N \) potentials. Choosing only the \( ^1S_0 \) and \( ^3S_1 \) partial waves of the \( \Lambda N \) potential we can substantially simplify the folding \( \Lambda - \alpha \) potential from the complicated form given in Eq. (2.9) and arrive at

\[
V_{fold}(u_\Lambda, u'_\Lambda) = \int_0^\infty v_3^2 dv_3 \int_{-1}^1 dx \int_{-1}^1 dx' K_\alpha(u_3, u'_3) \{ \frac{1}{4} V_{YN}^{(1S_0)}(v_\Lambda, v'_\Lambda) + \frac{3}{4} V_{YN}^{(3S_1)}(v_\Lambda, v'_\Lambda) \}. 
\]

One can note that the spin-spin dependence of the \( \sigma_\Lambda \cdot \sigma_N \) Hiyama \( \Lambda N \) potential of Eq. (3.2) disappears due to the weighted (1/4 and 3/4) sum in Eq. (3.8).

In Tab. IV we demonstrate the calculated binding energies of the \( ^5\Lambda\text{He} \) hypernucleus. Each row of the table is prepared for one \( \Lambda-N \) potential. The row containing results based on the full potential from Hiyama et al. [21]) is separated from the other ones by a line to indicate that the predictions in all other rows are obtained with separable approximations employing the Yamaguchi type (Y) or the Gaussian type (G) form factors. Columns tell which realistic NN potential is used to calculate the \( \alpha \) particle wave function, necessary to construct the integral kernel \( K_\alpha \) in Eq.(2.10) or in Eq. (3.3). The column denoted as RGM [33] is an exception because here the \( \alpha \) particle wave function has a simple Gaussian form, \( 2^{-3/4}(\pi\Omega)^{-9/4} \exp\{-u_1^2 + (3/4)u_2^2 + (2/3)u_3^2\}/(2\Omega) \} \) with \( \Omega = 0.275 \text{fm}^{-2} \).

From the comparison of the binding energies for the Gaussian wave function with the full Hiyama Gaussian \( \Lambda N \) potential (-3.10 MeV), the approximate Hiyama potential (type Y) (-2.46 MeV) and the approximate Hiyama potential (type G) (3.07 MeV), we can estimate the accuracy of the separable approximation. The accuracy is not better than approximately 0.7 MeV. Using the realistic \( \alpha \) particle wave functions we obtain clear underbinding. It is most evident for the AV18 potential and predictions based on this NN force differ from the others by up to 1.2 MeV.

Surprisingly, the most realistic input for calculations, namely the realistic \( \Lambda-N \) potential and the \( \alpha \) particle wave functions generated by the realistic NN interactions, leads to rather
TABLE II: Scattering lengths $a$ and effective ranges $r$ in fm for $\Lambda$-neutron potential. The Hiyama, Chiral and Jülich models do not differentiate between the $\Lambda$-neutron and the $\Lambda$-proton channel, which is indicated with a star(*)

| Model ΛN potential                | $a\,(^1S_0)$ | $r\,(^1S_0)$ | $a\,(^3S_1)$ | $r\,(^3S_1)$ |
|-----------------------------------|--------------|--------------|--------------|--------------|
| Hiyama et al. [21]               | -1.28*       | 2.33*        | -0.67*       | 3.08*        |
| Chiral ($\Lambda = 600$) [22, 37]| -2.91*       | 2.78*        | -1.54*       | 2.74*        |
| Jülich04 [23, 37]                | -2.56*       | 2.75*        | -1.66*       | 2.93*        |
| Nimegen ESC16                    | -1.96        | 3.65         | -1.84        | 3.33         |
| Nijmegen NSC97e [13, 37]         | -2.24        | 3.24         | -1.83        | 3.14         |
| Nijmegen NSC97f [13, 37]         | -2.68        | 3.07         | -1.67        | 3.34         |
| Nijmegen NSC89 [12]              | -2.86        | 2.91         | -1.24        | 3.33         |
| Nimegen HC-D model [24, 37]      | -2.03        | 3.66         | -1.84        | 3.32         |
| Ehime set 2 [25]                 | -2.65*       | 3.24*        | -1.80*       | 3.71*        |
| Ehime set A [25]                 | -2.76*       | 3.19*        | -2.064*      | 3.46*        |
| Ehime set B [25]                 | -2.71*       | 3.21*        | -1.95*       | 3.56*        |

TABLE III: Scattering lengths $a$ and effective ranges $r$ in fm for $\Lambda$-proton potential.

| Model ΛN potential                | $a\,(^1S_0)$ | $r\,(^1S_0)$ | $a\,(^3S_1)$ | $r\,(^3S_1)$ |
|-----------------------------------|--------------|--------------|--------------|--------------|
| Nimegen ESC16                     | -1.88        | 3.58         | -1.86        | 3.37         |
| Nijmegen NSC97e [13, 37]          | -2.10        | 3.19         | -1.86        | 3.19         |
| Nijmegen NSC97f [13, 37]          | -2.51        | 3.03         | -1.75        | 3.32         |
| Nijmegen NSC89 [12]               | -2.73        | 2.87         | -1.48        | 3.04         |
| Nimegen HC-D model [24, 37]       | -2.06        | 3.78         | -1.77        | 3.18         |

strong overbinding of the $^5\Lambda$He hypernucleus and moves the predictions away from the data. These numbers are listed below the second horizontal line in Tab. [IV]

IV. SUMMARY AND OUTLOOK

We have prepared many versions of the $\Lambda$-α folding potential which required both α particle wave functions and Λ-N potentials. To this end we considered not only the simple Gaussian form of the α particle wave function but also wave functions generated as rigorous solutions of the four-nucleon Faddeev-Yakubovsky equation with several realistic NN potentials. For the Λ-N potential we have taken the full and approximated phenomenological Hiyama model [33] but also the meson theoretical ones. In order to facilitate our calculations we prepared and utilized simplified separable versions of the Λ-N realistic potentials, taking care to realize exactly their crucial features, the scattering length and the effective range.

First we employed the Hiyama phenomenological Λ-N potential together with the Gaussian α particle wave function and obtained the binding energy of $^5\Lambda$He $-3.10$ MeV, which is in agreement with the experimental data ($-3.12 \pm 0.02$ MeV).

All the ΛN potentials used in this paper reproduce both the scattering length and the ef-
**TABLE IV:** The binding energies of $^5\Lambda$He using the model $\alpha\Lambda$ potentials. The notations (Y) and (G) are corresponding to the Yamaguchi separable form and Gaussian one, respectively. Unit is in MeV.

| AN potential | NN potential for $\alpha$ particle | RGM | CD-Bonn | Nijm93 | Nijm I | Nijm II | AV18 |
|--------------|----------------------------------|-----|---------|--------|--------|--------|------|
| Hiyama et al. [21] |                                 | -3.10 | -2.99  | -2.18  | -2.54  | -1.98  | -1.95 |
| Hiyama (Y)   |                                 | -2.46 | -2.36  | -1.62  | -1.94  | -1.44  | -1.42 |
| Hiyama (G)   |                                 | -3.07 | -2.89  | -2.12  | -2.46  | -1.92  | -1.90 |
| Chiral ($\Lambda = 600$) [22, 37] (Y) |             | -8.44 | -8.26  | -6.64  | -7.36  | -6.21  | -6.16 |
| Chiral ($\Lambda = 600$) (G) |               | -9.26 | -8.83  | -7.42  | -8.08  | -7.03  | -6.99 |
| Jülich04 [23, 37] (Y) |                  | -8.47 | -8.26  | -6.68  | -7.39  | -6.26  | -6.21 |
| Jülich04 (G) |                     | -9.08 | -8.62  | -7.29  | -7.92  | -6.93  | -6.89 |
| Nimegen ESC16 (Y) |                        | -7.57 | -7.27  | -5.98  | -6.57  | -5.65  | -5.61 |
| Nimegen ESC16 (G) |                       | -7.38 | -6.85  | -5.93  | -6.39  | -5.68  | -5.67 |
| Nijmegen NSC97e [13, 37] (Y) |                   | -8.22 | -7.94  | -6.51  | -7.16  | -6.13  | -6.09 |
| Nijmegen NSC97e (G) |                     | -8.32 | -7.80  | -6.71  | -7.24  | -6.41  | -6.39 |
| Nijmegen NSC97f [13, 37] (Y) |                 | -7.98 | -7.70  | -6.32  | -6.95  | -5.95  | -5.91 |
| Nijmegen NSC97f (G) |                    | -8.00 | -7.47  | -6.44  | -6.95  | -6.16  | -6.14 |
| Nijmegen NSC89 [12] (Y) |                   | -7.12 | -6.88  | -5.54  | -6.15  | -5.19  | -5.15 |
| Nijmegen NSC89 (G) |                    | -7.48 | -7.03  | -5.93  | -6.46  | -5.64  | -5.61 |
| Nimegen HC-D model [24, 37] (Y) |             | -7.62 | -7.32  | -6.02  | -6.62  | -5.68  | -5.64 |
| Nimegen HC-D model (G) |               | -7.48 | -6.96  | -6.01  | -6.48  | -5.75  | -5.74 |
| Ehime set 2 [25] (Y) |                  | -7.65 | -7.32  | -6.08  | -6.66  | -5.76  | -5.73 |
| Ehime set 2 [25] (G) |                  | -7.22 | -6.66  | -5.83  | -6.25  | -5.60  | -5.60 |
| Ehime set A [25] (Y) |                  | -8.79 | -8.45  | -7.05  | -7.70  | -6.68  | -6.64 |
| Ehime set A [25] (G) |                  | -8.45 | -7.85  | -6.89  | -7.37  | -6.62  | -6.61 |
| Ehime set B [25] (Y) |                 | -8.31 | -7.97  | -6.64  | -7.26  | -6.29  | -6.26 |
| Ehime set B [25] (G) |                 | -7.93 | -7.35  | -6.44  | -6.90  | -6.19  | -6.18 |

It has come as a surprise that for the realistic AN potentials we get clear overbinding and results are quite different from the data. The differences range from 2 MeV to 6 MeV. One of the reason of the failure in the description of the data may be the fact that our rank-1 separable approximation is still unsatisfactory, since it realizes only the attractive part of the original potential. Presumably, a higher-rank approximation is necessary to account also for effective range in the $^1S_0$ and $^3S_1$ states. These features are shown in Tabs. III and IV. Because there is no possibility to compare these parameters with the data, these numbers are to some extent arbitrary. Preserving these features rank-1 separable potentials are prepared as given in Eqs. (3.4)–(3.7). The separable approximation of the Gaussian type reproduces very well the prediction based on the original phenomenological potential, yielding the binding energy -3.07 MeV, which is very close to the original -3.10 MeV.

By using the separable approximations both of Yamaguchi and Gaussian type, we have obtained may further results for the $^5\Lambda$He binding energy, which are displayed in Tab. IV. It has come as a surprise that for the realistic AN potentials we get clear overbinding and results are quite different from the data. The differences range from 2 MeV to 6 MeV. One of the reason of the failure in the description of the data may be the fact that our rank-1 separable approximation is still unsatisfactory, since it realizes only the attractive part of the original potential. Presumably, a higher-rank approximation is necessary to account also
for the repulsive properties of the original potential. In near future we plan to introduce a high-rank separable form of the realistic $\Lambda$-N potential or to use directly the original, unabbreviated force in our calculations.

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Appendix A

The geometrical functions $G_{a,b}(u_\Lambda, v_3, x)$ and $G_{b',a'}(v_3, u'_\Lambda, x')$ were introduced in Refs. [27, 28] and read:

\[
G_{a,b}(u_\Lambda, v_3, x) = \sum_k P_k(x) \sum_{L_1+L_2=l_3} \sum_{L_1'+L_2'=\lambda_\Lambda} u_\Lambda^{L_2+L_2'} v_3^{L_1+L_1'} g_{a,b}^{k,L_1,L_2,L_1',L_2'},
\]

\[
G_{b',a'}(v_3, u'_\Lambda, x') = \sum_k P_k(x') \sum_{L_1+L_2=l_3} \sum_{L_1'+L_2'=\lambda_\Lambda} v_3^{L_2+L_2'} u'_\Lambda^{L_1+L_1'} g_{b,a'}^{k,L_1,L_2,L_1',L_2'}, \tag{4.1}
\]

where the purely geometrical quantities $g_{a,b}^{k,L_1,L_2,L_1',L_2'}$ and $g_{b,a'}^{k,L_1,L_2,L_1',L_2'}$ are given by

\[
g_{a,b}^{k,L_1,L_2,L_1',L_2'} = \frac{1}{2} \delta_{i,j_3} \delta_{s_1,s_1'} \delta_{j_3,j_3'} \delta_{j_2,j_2'} \delta_{j_1,j_1'} \sqrt{\tilde{t}_3 \prod_{a,b} \lambda_\Lambda \lambda_{\Lambda'} \lambda_{\Lambda''} \lambda_{\Lambda'''} \lambda_{\Lambda''''} (-1)^{\lambda_\Lambda+\sigma_\Lambda}}
\]

\[
\times \sum_{LS} \hat{L} \hat{S} \left\{ \frac{j_2}{2} \frac{j_3}{2} \frac{j_3'}{2} \frac{j_1}{2} \frac{\sigma_\Lambda}{\Lambda} \right\} \left\{ \begin{array}{c} l_3 \ j_3 \ j_3' \\ \lambda \ 2 \ j_2 \ j_1 \end{array} \right\} \left\{ \begin{array}{c} \lambda_\Lambda \ \sigma_\Lambda \ \tau_\Lambda \\ \lambda_3 \ \lambda_2 \ \lambda_1 \end{array} \right\} \hat{k} \left( \frac{3}{4} \right) \left( \frac{m_\Lambda}{m_N+m_\Lambda} \right)^{L_2} \tag{4.2}
\]

and

\[
g_{b,a'}^{k,L_1,L_2,L_1',L_2'} = \frac{1}{2} \delta_{i,j_3} \delta_{s_1,s_1'} \delta_{j_3,j_3'} \delta_{j_2,j_2'} \delta_{j_1,j_1'} \sqrt{\tilde{t}_3 \prod_{a,b} \lambda_\Lambda \lambda_{\Lambda'} \lambda_{\Lambda''} \lambda_{\Lambda'''} \lambda_{\Lambda''''} (-1)^{\lambda_\Lambda+\sigma_\Lambda}}
\]

\[
\times \sum_{LS} \hat{L} \hat{S} \left\{ \frac{1}{2} \frac{\sigma_\Lambda}{\Lambda} \right\} \left\{ \begin{array}{c} \lambda_\Lambda \ \Sigma_\Lambda \ \tau_\Lambda \\ \lambda_3 \ \lambda_2 \ \lambda_1 \end{array} \right\} \left\{ \begin{array}{c} l_3 \ j_3 \ j_3' \\ \lambda \ 2 \ j_2 \ j_1 \end{array} \right\} \hat{k} \left( \frac{3}{4} \right) \left( \frac{m_\Lambda}{m_N+m_\Lambda} \right)^{L_2} \tag{4.2}
\]

and

\[
\times \sum_{LS} \hat{L} \hat{S} \left\{ \frac{1}{2} \frac{\sigma_\Lambda}{\Lambda} \right\} \left\{ \begin{array}{c} l_3 \ j_3 \ j_3' \\ \lambda \ 2 \ j_2 \ j_1 \end{array} \right\} \left\{ \begin{array}{c} \lambda_\Lambda \ \lambda_\Lambda' \ \lambda_\Lambda'' \ \lambda_\Lambda''' \ \lambda_\Lambda'''' \\ \lambda_3 \ \lambda_2 \ \lambda_1 \end{array} \right\} \hat{k} \left( \frac{3}{4} \right) \left( \frac{m_\Lambda}{m_N+m_\Lambda} \right)^{L_2} \tag{4.2}
\]
\[
\times \left\{ \frac{f}{L_1} \frac{L}{f' L'_{2}} \frac{k}{k'f'} \right\} C(kL_1f'; 00)C(kL'_2f'; 00)
\] (4.3)

with \( \hat{x} = \sqrt{2x + 1} \).

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