Nonlinear Hodge equations in vector bundles

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Abstract

A gauge-invariant form of the nonlinear Hodge equations is studied. 1991 MSC: 58E15 (Classical field theory)

1 Introduction

Under suitably harsh assumptions, many natural phenomena can be expressed as solutions of the linear Hodge-Kodaira equations, in which stationary fields appear as harmonic forms. If the drastic physical assumptions of the linear theory are relaxed, then at first glance the Hodge-de Rham interpretation appears to crumble before a bewildering variety of nonlinear variational theories. Nonlinear Hodge theory, introduced nearly 30 years ago by L. M. Sibner and R. J. Sibner [SS1, SS2], can be viewed as an extension of the unified geometric interpretation achieved for linear field equations to the quasilinear case. (Specifically, a nonlinear "mass density" term is introduced into the linear Hodge-Kodaira equations for differential forms on a Riemannian manifold. If this mass density term is constant, then the nonlinear Hodge equations reduce to the linear Hodge-Kodaira equations.)

Although many of the results of nonlinear Hodge theory extend to differential forms of arbitrary degree (see, e.g., [Si1] and [SS4]), 1-forms occupy a special place in that a 1-form which is closed under exterior differentiation can be interpreted as the gradient of a 0-form (or, in physical terms, as the field of a scalar potential). This interpretation is exploited in [SS1]-[SS3]. A 2-form which is closed under covariant exterior differentiation can, in certain
circumstances, be interpreted as the curvature of a connection 1-form (or, in physical terms, as the field of a vector potential). This interpretation leads to enriched geometry, as the relevant bundle need no longer be the cotangent bundle of a manifold, as in conventional nonlinear Hodge theory, but can be a bundle with curvature.

This is the motivation for generalizing aspects of nonlinear Hodge theory to sections of a vector bundle having nonabelian structure group. This extension adds extra nonlinearities and nontrivial gauge invariance to the equations of [SS1, SS2]. From a variational point of view, its aim is to draw certain nonquadratic energies of Yang-Mills type into the family of variational integrals amenable to a nonlinear Hodge-de Rham interpretation. From a geometric point of view, the generalized equations obtained bear the same relation to harmonic curvature on a bundle that the nonlinear Hodge equations bear to harmonic forms on a manifold, leading to a quasilinear generalization of harmonic curvature. From a purely analytic point of view, the extra nonlinearities of the bundle-valued equations yield new insights into the form-valued equations. For example, these ideas lead to a weakening of the conventional "irrotationality" assumption for stationary nonlinear Hodge flow on a manifold [O4].

We introduced the main features of this generalized nonlinear Hodge theory in two previous papers [O2], [O3]. Here we consider a variety of technical points, some of which were ignored in earlier work, others only sketched, and still others treated inadequately. As a consequence we obtain revised proofs of some of the results in [O2], [O3].

We note that energy functionals which are nonquadratic in the bundle curvature have already appeared in the physics literature in the context of individual models - for example, the well known model introduced by Tchrakian for higher-dimensional gauge theories [T].

In the sequel we denote by $C$ generic positive constants which generally depend on dimension and which may change in value from line to line.

## 2 A nonlinear Hodge theory for 2-forms

Let $M$ be a finite, oriented, $n$-dimensional Riemannian manifold and $X$ a vector bundle over $M$ having compact structure group $G \subset SO(m)$. Let $A \in \Gamma (M, adX \otimes T^*M)$ be a connection 1-form with curvature 2-form $F_A$, 

where

\[ F_A = dA + \frac{1}{2} [A, A] = dA + A \wedge A \]

and \([,\) is the Lie bracket of the Lie algebra \(\mathfrak{g}\) associated to \(G\). Geometrically, \(\mathfrak{g}\) is the fiber of the adjoint bundle \(ad X\). Here \(d : \Lambda^p \to \Lambda^{p+1}\) is the flat exterior derivative and \(\wedge\), the wedge product on differential forms. Sections of the automorphism bundle \(Aut X\) are called gauge transformations. These act tensorially on \(F_A\) but affinely on \(A\), a fact which leads to certain analytic difficulties. For details of this geometric construction see, e.g., [MM].

We consider a stored energy functional of the form

\[
E = \frac{1}{2} \int_M \left( \int_0^Q \rho(s) ds \right) dM,
\]

(1)

where \(Q = |F_A|^2 = \langle F_A, F_A \rangle\) is an inner product on the fibers of the bundle \(ad X \otimes \Lambda^2 (T^* M)\) (the inner product on \(ad X\) being induced by the normalized trace inner product on \(SO(m)\) and that on \(\Lambda^2 (T^* M)\), by the exterior product \(* \langle F_A \wedge *F_A \rangle\), where \(* : \Lambda^p \to \Lambda^{n-p}\) is the Hodge involution); \(\rho : \mathbb{R}^+ \to \mathbb{R}^+\) is a \(C^1\) function satisfying

\[
K^{-1}(Q + k)^q \leq \rho(Q) + 2Q \rho'(Q) \leq K(Q + k)^q
\]

for some positive constant \(K\) and nonnegative constants \(k, q\).

The functional (1) is a generalization of the nonlinear Hodge energy introduced in [SS2] for \(X = T^* M\) and \(Q = |\omega|^2\), where \(\omega \in \Gamma (M, \Lambda^p (T^* M))\). (See also [U1], page 221.) Critical points of (1) with respect to an admissible cohomology class of closed \(p\)-forms satisfy the nonlinear Hodge equations

\[
\delta (\rho(Q) \omega) = 0,
\]

(3)

\[
d\omega = 0,
\]

(4)

which were introduced and extensively studied by L. M. and R. J. Sibner [SS1]-[SS4]. Choosing, for \(X = T^* M, p = 1, \) and \(\omega = d\varphi, \rho(Q)\) to be

\[
\rho(Q) = \left(1 - \frac{\gamma - 1}{2} Q \right)^{1/(\gamma-1)},
\]

where \(\gamma > 1\) is a constant, provides (1) with an interpretation as the energy functional for the stationary, polytropic flow of a compressible fluid having
adiabatic constant \( \gamma \). The scalar \( \varphi \) is a possibly multivalued potential for the velocity field \( \omega \). The multivalued nature of \( \varphi \) describes circulation of the flow, e.g., about an obstacle with handles. In this case inequality (2) with \( q = 0 \) is a condition for subsonic flow. \( Q_{\text{crit}} = 2/(\gamma + 1) \) is the squared speed at the transition from subsonic to supersonic flow. The Euler-Lagrange equations for variations of (1) with \( p = 1 \) yield the continuity equations for the flow in Eulerian coordinates [SS1]. Analogous interpretations can be given to topics in elasticity and thermodynamics, including nonrigid-body rotation and capillarity. Applications to magnetic materials and minimal surfaces are given in [O3] and [SS2], respectively.

In order to extend the variational problem to sections of a vector bundle, we form an admissible class of connections by choosing a smooth base connection \( D \) in the space of connections compatible with \( G \) and considering the class of connections \( D + A \), where \( A \) is a section of \( adX \otimes T^*M \) which lies in the largest Sobolev space for which the energy \( E \) is finite; details are given in [U3] for the case \( \rho \equiv 1 \). We take variations by computing \( (d/dt)(F_{D+tA}) \) at the origin of \( t \). Using the fact that for any smooth section \( \sigma \) we have

\[
F_{D+tA}(\sigma) = \left( F + tDA + t^2 A \wedge A \right)(\sigma),
\]
we obtain

\[
\delta E = \frac{1}{2} \int_M \rho(Q) \delta Q \, dM = \frac{1}{2} \int_M \rho(Q) \frac{d}{dt} \bigg|_{t=0} |F + tDA + t^2 A \wedge A|^2 \, dM. \tag{5}
\]

Notice that \( D = d + [A, \cdot] \). Letting \( t = 0 \), the right-hand side of (5) can be written

\[
\int_M \rho(Q) \langle DA, F_A \rangle \, dM = \int_M \langle DA, \rho(Q)F_A \rangle \, dM. \tag{6}
\]

We assume that either \( \partial M = 0 \) or, if not, that \( F_A \) satisfies a "Neumann" boundary condition of the form

\[
i^*(\ast F) = 0 \tag{7}
\]
on \( \partial M \), where \( i^* \) is the pull-back under inclusion of the boundary of \( M \) in \( M \). This is equivalent in local coordinates to prescribing zero boundary data for \( F \) in a direction normal to \( \partial M \); see, e.g., [Ma] for details in the case \( \rho \equiv 1 \).
Set $\delta E = 0$ equal to zero. Then (5) and (6) imply

$$0 = \int_M \langle DA, \rho(Q)F_A \rangle \ dM =$$

$$\int_M d(A \wedge * \rho(Q)F_A) + \int_M \langle A, D^* (\rho(Q)F_A) \rangle \ dM$$

$$= \int_{\partial M} A_\theta \wedge (\rho(Q)F_A)_N + \int_M \langle A, D^* (\rho(Q)F_A) \rangle \ dM,$$

where $D^*$ denotes the formal adjoint of the exterior covariant derivative $D$; $\theta$ denotes tangential component on the boundary and $N$, the normal component there. Condition (7) implies Euler-Lagrange equations of the form

$$D^* (\rho(Q)F) = 0. \quad (8)$$

Because $F$ is a curvature 2-form, it satisfies an additional condition

$$DF = 0. \quad (9)$$

(This is the second Bianchi identity.) This paper is concerned with analytic properties of the system (8), (9).

If we write these equations as a system of equations for Lie-algebra-valued forms, they can be written

$$\delta (\rho(Q)F_A) = - * [A, * \rho(Q)F_A], \quad (10)$$

$$dF_A = - [A, F_A]. \quad (11)$$

Here $\delta : \Lambda^p \to \Lambda^{p-1}$ is the adjoint of the exterior derivative $d$. If $G$ is abelian, then the Lie brackets in (10), (11) vanish and eqs. (8), (9) reduce to the system

$$\delta (\rho(Q(F)) F) = dF = 0,$$

which are the nonlinear Hodge equations for the 2-form $F$ in a local trivialization of $X$. If in addition $\rho \equiv 1$, then we obtain the Hodge-Kodaira equations for 2-forms. If $G$ is nonabelian and $\rho \equiv 1$, then eqs. (8) reduce to the Yang-Mills equations, the equations for the classical limit of quantum fields.

Equations (10) have certain formal similarities to the continuity equation for a velocity field, having components $v^\alpha$, of a stationary, polytropic, compressible fluid on a Riemannian manifold $M$ possessing a $C^1$ metric tensor $g_{\alpha\beta}$ and affine connection $\Gamma^\gamma_{\alpha\beta}$, where $\alpha, \beta = 1, \ldots, n$, that is,

$$\partial_\alpha (\rho(Q)v^\alpha) + \rho(Q)v^\alpha \Gamma^\gamma_{\alpha\beta} = 0. \quad (12)$$
Here $\partial_{\alpha} = \partial/\partial x^{\alpha}$ for $x = (x^1, ..., x^n) \in M$ and $Q = g^{\alpha\beta}v_{\alpha}v_{\beta}$. If the flow is parallel on $M$, then

$$\partial_{\beta}v^{\alpha} + v^{\gamma}\Gamma^{\alpha}_{\gamma\beta} = 0,$$

(13)

which is to say that the covariant derivative of $v$ vanishes with respect to the connection $\Gamma^{\alpha}_{\gamma\beta}$. Equation (13) is thus the geometric analogue for parallel 1-tensors on a manifold of the Bianchi identity (9) for curvature 2-forms (on a vector bundle). If $g \equiv \det (g^{\alpha\beta})$ is a $C^1$ function of $x$, then we can write

$$\Gamma^{\beta}_{\alpha\beta} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\alpha}} \sqrt{g}.$$

On a Riemannian manifold the operator $\delta$ explicitly involves the metric:

$$\delta_M (\vartheta) = - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\alpha}} (\sqrt{g} \vartheta_\alpha).$$

Applying the product rule to $\delta_M (\vartheta)$ with $\vartheta = \rho(Q)v$, we conclude that on a differentiable Riemannian manifold eq. (3) is exactly dual to (12), as the former equation results from replacing the tangent bundle in the latter equation by the cotangent bundle. (This relation between (3) and (12) was introduced in [SS1].) But eq. (4) asserts that the flow is irrotational: there is no circulation about any curve homologous to zero. In euclidean space every parallel flow is irrotational, but this is not true in general (the simplest example being flow along a great circle of a sphere), so on a manifold condition (4) differs somewhat from condition (13) — c.f. [O4].

In the case of abelian $G$, for given $\lambda \in \overline{Ker d}$ an admissible class is defined by the set of $\omega \in \overline{Ker d}$ for which $\omega - \lambda \in \overline{Im d}$. This condition prescribes a cohomology class of admissible forms, leading to a complete existence theory for 1-forms [SS2]. This theory extends to $p$-forms on a compact Riemannian manifold [Si1], but appears to fail for nonabelian $G$.

Another analytic difficulty is that, in distinction to the conventional Yang-Mills equations, eqs. (8), (9) cannot be written as a diagonal elliptic system even in a good gauge. Two obvious consequences are that the technique used in [Ma] to solve boundary-value problems for the 4-dimensional Yang-Mills equations will not work for (8), (9) and that Hölder continuity for solutions of (8), (9) does not automatically imply any higher regularity.
3 The smoothness of solutions

In this section we derive regularity and gauge-improvement results for solutions of the system (8), (9).

The following is a revision of Theorem 1.1 of [O2]. (The unnecessary restriction that $Q$ be positive is removed; an estimate for nonlinear Hodge fields [O4] is used to obtain a bound on the bundle curvature; the role of the exponential gauge in the proof is clarified; a gauge-invariant Campanato estimate is constructed.)

**Theorem 1** Let the pair $(A, F_A)$ weakly satisfy eqs. (8), (9) in a bounded, open, type-A domain $\Omega \subset \mathbb{R}^n$. Let $\rho$ satisfy condition (2) with $q = 0$. Suppose that $F_A \in L^s(\Omega)$ for some $s > n/2$. Then $A$ is equivalent via a continuous gauge transformation to a connection $\tilde{A}$ such that $F_{\tilde{A}}$ is Hölder continuous in $\Omega$.

**Remarks.** i) In the following proof we take the $L^n$-norm of $A$ to be small on a sufficiently small ball (in the sense of an $n$-disc). In a part of the proof sketched in an appendix to this paper we additionally require the $L^{n/2}$-norm of $F$ to be small on a small ball. Neither assumption need be stated explicitly in Theorem 1. If $F \in L^s$ for $s > n/2$, the small-$L^{n/2}$-norm assumption for $F$ follows from standard arguments on $\mathbb{R}^n$ (and from Lemma 3.4 of [U3] on a Riemannian manifold). The corresponding assumption for $A$ follows from Theorem 1.3 ii) of [U3] and the Sobolev Theorem.

ii) We exploit the boundedness of $\Omega$ to study eqs. (8), (9) in a small $n$-disc $B$ and employ a covering argument at the end. This allows us to trivialize $X$ locally and understand the notion of weak solution in the sense of [Si1], eq. (1.2b). For abelian $G$, a weak solution of (8), (9) is any curvature 2-form $F_A$ for which $\rho(Q)F_A$ is orthogonal in $L^2$ to the space of d-closed 2-forms $d\zeta \in L^2(B)$ such that $\zeta \in \Lambda^1$ has vanishing tangential data on $\partial B$. For nonabelian $G$, an obvious extension of (1.2b) to inhomogeneous equations allows us to define a weak solution of (8), (9) by the equation

$$\int_B \langle d\zeta, \rho(Q)F_A \rangle * 1 = -\int_B \langle \zeta, *[A, \rho(Q)F_A] \rangle * 1,$$

where $F$ is a curvature 2-form. Our general understanding of weak solutions to gauge-invariant systems is derived from [U3].

iii) For a definition of type-A domain see, e.g., [Gi]. As an example, any Lipschitz domain is type-A.
iv) It is easy to show the existence of weak solutions to (8), (9) by topological arguments, provided that $\rho$ is chosen so that the energy functional is Palais-Smale. An example is given in Corollary 1.2 of [O2].

v) Theorem 1 cannot be improved (for $q = 0$) without improving the existing regularity theory [U3] for the case $\rho = 1$.

The proof of Theorem 1 strongly uses the properties of the exponential gauge in a euclidean $n$-disc $B$ centered at the origin of coordinates in $\mathbb{R}^n$, namely, that in such a gauge

$$A(0) = 0$$

and $\forall x \in B$

$$|A(x)| \leq \frac{1}{2} |x| \cdot \sup_{|y| \leq |x|} |F(y)|$$

(see [U2], Sec. 2).

We also require the following mean-value formula of L. M. Sibner, originally stated for differential forms on a Riemannian manifold, which extends immediately to the case of Lie-algebra-valued sections:

**Lemma 2** (L. M. Sibner[Si1], Lemma 1.1). Let

$$G_J^I(x, \omega) = \sqrt{g} g^{IJ} \rho(Q(\omega)) \omega_I,$$

where $g^{ij}$ is the metric tensor on a compact Riemannian manifold $M; x \in M; g = \det (g^{ij}); I, J$ are multi-indices; $\omega \in \Gamma (M, \Lambda^p(T^*M))$. Let $\rho$ satisfy condition (2) with $q = 0$. Then for $\xi, \eta \in M$ and $\mu, \tau \in \Gamma (M, \Lambda^p(T^*M))$,

$$G_J^I(\xi, \mu) - G_J^I(\eta, \tau) = \alpha^{IJ} (\mu_I - \tau_I) + \beta_J^I \left( \xi^i - \eta^i \right),$$

where $\alpha^{IJ}$ is a positive-definite matrix and

$$|\beta_J^I| \leq C (|\mu(x)| + |\tau(x)|).$$

Finally, we need an *a priori* estimate for smooth solutions:

**Lemma 3** Let the pair $(A, F_A)$ smoothly satisfy eqs. (8), (9) and condition (2) on an open, bounded domain $\Omega \subset \mathbb{R}^n$. Then the scalar $Q = |F|^2$ satisfies the inequality

$$L(Q) + C(Q + k)^q \left( |\nabla A| + |A|^2 \right) Q \geq 0, \quad (15)$$

where $L$ is a divergence-form operator which is uniformly elliptic for $k > 0$. 

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Proof of Lemma 3. The proof is nearly identical to the proof of [O4], Theorem 1, taking (in the notation of [O4]) \( u = A \) and \( \omega = F \). However, eq. (10) contains an inhomogeneous term on the right that is absent in eq. (1) of [O4]. This inhomogeneous term arises as the second term on the right in the equation

\[
\langle F, \Delta (\rho(Q)F) \rangle = -\langle F, \delta d (\rho(Q)F) \rangle + \langle F, d \star [A, \star \rho(Q)F] \rangle 
\]

(c.f. [O4], eq. 14). This term can be estimated by Young’s inequality:

\[
|F| |d \star [A, \star \rho(Q)F]| \leq C (|\nabla A| |\rho(Q) + |F||A| |\nabla (\rho F)|) 
\]

\[
\leq C (Q + k)^q \left( |\nabla A| Q + |\nabla F|^2 + C(\varepsilon) |A|^2 Q \right) 
\]

(c.f. [O4], (18)). The remainder of the derivation of (15) is exactly analogous to the derivation of inequality (7) in [O4], provided the wedge product of differential forms on the right in eq. (2) of [O4] is replaced by a Lie bracket of Lie-algebra-valued sections.

Concerning the ellipticity of the operator \( \tilde{L} \), define a function \( h(Q) \) such that

\[
h'(Q) = \frac{1}{2} \rho(Q) + Q \rho'(Q) 
\]

and an operator \( \tilde{L} \) such that

\[
\tilde{L} (h(Q)) = \sum_{k,j} \partial_k \left\{ \left[ \delta_{kj} - \frac{\rho'(Q)}{h'(Q)} \langle \sigma_k F, \sigma_j F \rangle \right] \partial_k h(Q) \right\}. 
\]

Here \( \delta_{kj} \) is the kronecker delta and \( \sigma_k \) is an antisymmetrization operator equivalent to the operator \( A_i(1) \) in eq. (1.9) of [U1]. Then

\[
\tilde{L} (h(Q)) = \sum_{k,j} \partial_k \left\{ \left[ \frac{1}{2} \rho(Q) + Q \rho'(Q) - \rho'(Q) \langle \sigma_k F, \sigma_j F \rangle \right] \partial_k Q \right\} \equiv L(Q). 
\]

The ellipticity of \( \tilde{L} \) under condition (2) with \( k > 0 \), established on p. 233 of [U1], implies the ellipticity of \( L \) under the same hypothesis. This completes the proof of Lemma 3. □

Proof of Theorem 1. Because \( F_A \in L^s(\Omega) \) for some \( s > n/2 \) there is a continuous gauge transformation in a small disc \( B \subset \subset \Omega \) to a Hodge gauge in which the following conditions are satisfied ([U3], Theorem 2.1):

a) \( \delta A = 0; \)
b) \( x \cdot A = 0 \) on \( \partial B \);

c) \( \| A \|_{1,n/2} \leq C \| F \|_{n/2} \);

d) \( \| A \|_{1,s} \leq C \| F \|_s \) for \( n/2 < s < n \).

Here \( \| \cdot \|_{p,q} \) is the \( H^{p,q} \)-norm and \( \| \cdot \|_p \) is the \( L^p \)-norm on \( B \). Condition a) and the Neumann condition b) allow us to apply the Gaffney-Gårding inequality \([Ga]\)

\[
|\nabla A|^2 \leq C \left( |dA|^2 + \| \delta A \|^2 + |A|^2 \right)
\]

in the form

\[
|\nabla A|^2 \leq C |dA|^2
\]  \hspace{1cm} (16)

(see the proof of \([U3]\), Lemma 2.5), provided that we choose \( B \) so that \( \| A \|_n \) is small.

Estimating the difference quotient of \( F \) as in the proof of Lemma 3.1 of \([O2]\), using properties a)-d), inequality (16), and the fact that exterior operators commute with the difference-quotient operator, we find that \( F \in H^{1,2}(B) \), where \( \tilde{B} \subset \subset B \). For the reader’s convenience this argument is briefly reviewed in an appendix. At this point we would be able to apply Theorem 5.3.1 of \([Mo]\), using Lemma 3 with \( q = 0 \), provided we knew that \( |F|^{\tau} \) was in \( H^{1,2}(B') \) for some \( \tau > 1 \) and some \( B' \subset \tilde{B} \). That this condition is in fact satisfied can be seen by writing inequality (15) in the weak form

\[
\int_B a^{ij} \partial_i u \partial_j \zeta * 1 = 2 \int_B a^{ij} u \frac{\partial u}{\partial x_i} \frac{\partial \zeta}{\partial x_j} d^n x \leq C \left\{ \int_B \left( |\nabla u| + |u|^2 \right) u^2 \zeta d^n x \right\},
\]

where \( u = |F| \); the matrix \( a^{ij} \) satisfies the ellipticity condition \( m_1 |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq m_2 |\xi|^2 \) for positive constants \( m_1 \) and \( m_2 \); \( \zeta \in C_0^\infty(\tilde{B}) \cap \mathbb{R}/\mathbb{R}^+ \).

Choose

\[
\zeta = (u_k + \delta)^{2\tau-2} \eta^2
\]

for \( \eta \in C_0^\infty(\tilde{B}), \eta \geq 0, \delta > 0, \tau > 1 \). The sequence \( \{u_k\} \) is chosen to be increasing and so that \( \lim_{k \to \infty} u_k = u \). We have

\[
\int_B a^{ij} u(u_k + \delta)^{2\tau-3} \partial_i u \partial_j (u_k) \eta^2 * 1 \leq C \int_B u |\nabla u| (u_k + \delta)^{2\tau-2} \eta |\nabla \eta| * 1
\]

\[
+C \int_B \left( |\nabla A| + |A|^2 \right) u^2 (u_k + \delta)^{2\tau-2} \eta^2 * 1 \leq C \left( \int_B (u + \delta)^{2\tau-1} |\nabla u| * 1 + \int_B \left( |\nabla A| + |A|^2 \right) (u + \delta)^{2\tau} * 1 \right). \hspace{1cm} (17)
\]
The extreme right-hand side of inequality (17) can be bounded above by the norms
\[
\| u + \delta \|_{(2\tau - 1)_{s_1}}^{2\tau - 1} \| \nabla u \|_{s_2} + \left( \| \nabla A \| + | A | \right)_{s_3} \left( \int (u + \delta)^{2\tau_{s_4}} \ast 1 \right)^{1/s_4}.
\]
These norms can be made finite for \( \tau \) sufficiently close to 1. For \( n > 6 \), choose \( s_1 = s_2 = 2 \), \( s_3 = n/(n - 4\tau) \), \( s_4 = n/(4\tau) \); then \( s_3 \leq s \) for \( s > n/2 \) if \( \tau \) is close to 1. If \( 2 < n \leq 6 \), choose \( s_1 = s_2 = 2, \ s_3 = n/2\tau, \ s_4 = n/(n - 2\tau) \). The finite \( H^{1,2} \)-norm of \( |F| \) implies \( u \in L^{2n/(n-2)} \) by the Sobolev Theorem and of course \( \| A \|_{1,n/2} \leq \| A \|_{1,n/2} < \infty \). If \( n = 2 \), then the norms are finite for \( s_1 = s_2 = s_3 = s_4 = 2 \) by the Sobolev Theorem.

Using ellipticity, we obtain in place of (17) the estimate
\[
\nu \int_B \eta^2 | \nabla u |^2 \, d^n x \leq C < \infty.
\]
Letting \( \eta = 1 \) on some smaller ball \( B' \) completely contained in \( \widetilde{B} \) and concentric with it allows us to conclude that \( |F|^r \in H^{1,2}(B') \) for some \( r > 1 \). Now we apply Theorem 5.3.1 of [Mo] to conclude that \( |F| \) is bounded in \( B' \).

As gauge transformations act tensorially on \( F \), the curvature remains bounded under continuous gauge transformations. In particular, at the origin of coordinates in an exponential gauge eq. (10) becomes
\[
\delta (\rho_0(Q)F_0) = \delta (\rho_0(Q)dA_0) = 0,
\]
where the subscript indicates that the result of the computation is being evaluated at the origin of \( B' \).

Because \( X \) has been trivialized in \( B' \) we can compare \( F \) to a solution \( d\phi \) of the variational problem associated to the equation
\[
\int_{B'} \langle d(A - \phi), \rho (Q(d\phi)) \rangle \, d\phi \ast 1 = 0. \tag{18}
\]
The 2-form \( d\phi \) exists as a weak \( L^2 \) solution by Proposition 4.3 of [S1]; \( d\phi \) is Hölder continuous by Proposition 4.4 of [S1] (which is derived from [U1]). In particular, the test function \( d(A - \phi) \) is admissible:
\[
\| d(A - \phi) \|_2 \leq \| dA \|_2 + \| d\phi \|_2,
\]
where, in an exponential gauge,
\[
\| dA \|_2 \leq \| dA + \frac{1}{2} [A, A] \|_2 + C \| A \|_2 \leq C \| F \|_2 < \infty.
\]
Combining (18) with (14), we have
\[
\int_{B'} \langle d(A - \varphi), \rho(Q(F)) F_A - \rho(Q(d\varphi)) d\varphi \rangle*1 = \int_{B'} \langle A - \varphi, *[A,*\rho(QF_A)]\rangle*1.
\]
Apply Sibner's mean-value formula (Lemma 2) to the left-hand side of the above identity. Take
\[
\mu = F_A; \tau = d\varphi; \xi = x; \eta = 0.
\]
We obtain the inequality
\[
\int_{B'} |d(A - \varphi)|^2*1 \leq C(\int_{B'} (|F_A| + |d\varphi|)|x|*1 + \int_{B'} |A - \varphi||A||\rho(Q)||F_A|*1 + \int_{B'} |d(A - \varphi)||A|^2*1) \equiv C(i_1 + i_2 + i_3).
\]
We estimate the terms \(i_1, i_2,\) and \(i_3\) individually. If \(R\) is the radius of \(B',\)
\[
i_1 = \int_{B'} (|F_A| + |d\varphi|)|x|*1 \leq C (\|F_A\|_\infty + \|d\varphi\|_\infty) \int_0^R |x|^n d|x| = CR^{n+1}.
\]
In an exponential gauge we have additionally, for \(R < 1,\)
\[
i_2 = \int_{B'} |A - \varphi||A||\rho(Q)||F_A|*1 \leq C(\rho) \left(\int_{B'} |A|^2|F_A|*1 + \int_{B'} |\varphi||A||F_A|*1\right) \leq C (\rho, \|F_A\|_\infty, |\varphi|_{C^{0,\gamma}}) \left[\int_0^R (|x|^{n+1} + |x|^n) d|x|\right] \leq CR^{n+1},
\]
where \(\gamma\) is the Hölder exponent of \(\varphi.\) Young's inequality implies that there is a small positive number \(\varepsilon\) for which
\[
i_3 = \int_{B'} |d(A - \varphi)||A|^2*1 \leq \varepsilon \int_{B'} |d(A - \varphi)|^2*1 + C(\varepsilon) \int_{B'} |A|^4*1 \leq \varepsilon \int_{B'} |d(A - \varphi)|^2*1 + C(\varepsilon, \|F_A\|_\infty) \int_0^R |x|^{n+3} d|x|
Substituting inequalities (20)-(22) into inequality (19) and absorbing small terms on the left, we obtain

\[
\int_{B'} |d(A - \varphi)|^2 * 1 \leq CR^{n+1}.
\]  

(23)

Now we use the fact that mean value minimizes variance over all location parameters. This allows us to replace (23) by the inequality

\[
\int_{B'} |dA - (dA)_{R,0}|^2 * 1 \leq CR^{n+1},
\]

where \((f)_{r,\sigma}\) denotes the mean value of \(f\) in an \(n\)-disc of radius \(r\) centered at the point \(\sigma \in \mathbb{R}^n\). Thus \(dA\) is Hölder continuous on \(B'\), with Hölder exponent \(1/2\), by Campanato’s Theorem (c.f. [Gi], Ch.3).

Regarding the Hölder continuity of the curvature, we have, using the linearity of the mean-value operator over sums,

\[
\int_{B'} |F - (F)_{R,0}|^2 * 1 = \int_{B'} |dA + A \wedge A - (F)_{R,0}|^2 * 1 \leq
\]

\[
C \left( \int_{B'} |dA - (F)_{R,0}|^2 * 1 + \int_{B'} |A|^4 * 1 \right) \leq
\]

\[
C \left( \int_{B'} |dA - [(dA)_{R,0} + (A \wedge A)_{R,0}]]|^2 * 1 + R^{n+4} \right) \leq
\]

\[
C \left( \int_{B'} |dA - (dA)_{R,0}|^2 * 1 + \int_{B'} |(A \wedge A)_{R,0}|^2 * 1 + R^{n+4} \right) \leq
\]

\[
C \left( R^{n+1} + R^{n+4} + \int_{B'} \frac{1}{|B'|} \int_{B'} A \wedge A * 1 \right) \leq
\]

\[
C \left[ R^{n+1} + \frac{1}{R^{2n}} \int_0^R \left( \|F\|_\infty^2 \int_0^R |x|^{n+1} d|x| \right)^2 |x|^{n-1} d|x| \right] \leq C \left( R^{n+1} + R^{n+4} \right) \leq CR^{n+1}.
\]  

(24)

Thus \(F\) is Hölder continuous in \(B'\) by Campanato’s Theorem.

We would like to finish the proof of Theorem 1 by covering \(\Omega\) with small \(n\)-disks and repeating the above argument in each disk. We would like to do this but cannot yet. The obstacle is our use of the exponential gauge at the
origin of coordinates. We must show that the Campanato estimate (24) is invariant under continuous gauge transformations in a small ball. Precisely, we show that if $F$ satisfies (24) and if a map $\gamma \in \text{Aut}X$ is continuous at each point $x \in B_r(\sigma)$, where $B$ is an $n$-disc of sufficiently small radius $r$ centered at a point $\sigma$ sufficiently close to the origin, we have

$$\|\gamma^{-1}(x)F(x)\gamma(x) - [\gamma^{-1}(x)F(x)\gamma(x)]_{r,\sigma}\|_2 \leq Cr^3$$

for $\beta > 0$, where $\|\cdot\|_2$ is now the $L^2$-norm on $B_r(\sigma)$. Using the continuity of $\gamma$ to approximate the term $[\gamma^{-1}(x)F(x)\gamma(x)]_{r,\sigma}$ by $[\gamma^{-1}(\sigma)F(x)\gamma(\sigma)]_{r,\sigma}$ for small $r$ and also using the fact that $\gamma$ is unitary, we have

$$\|\gamma^{-1}(x)F(x)\gamma(x) - [\gamma^{-1}(x)F(x)\gamma(x)]_{r,\sigma}\|_2 \approx$$

$$\|\gamma^{-1}(x)F(x)\gamma(x) - [\gamma^{-1}(\sigma)F(x)\gamma(\sigma)]_{r,\sigma}\|_2 =$$

$$\|F(x) - \gamma(x)\gamma^{-1}(\sigma) [F(x)]_{r,\sigma} \gamma(\sigma)\gamma^{-1}(x)\|_2 =$$

$$\|F(x)\gamma(x)\gamma^{-1}(\sigma) - \gamma(x)\gamma^{-1}(\sigma) [F(x)]_{r,\sigma}\|_2 \leq$$

$$\|F(x) \left(\gamma(x)\gamma^{-1}(\sigma) - I\right) + \left(I - \gamma(x)\gamma^{-1}(\sigma)\right) [F(x)]_{r,\sigma}\|_2 + \|F(x) - [F(x)]_{r,\sigma}\|_2,$$

where $I$ is the identity transformation. But this is equivalent to

$$\|\gamma^{-1}(x)F(x)\gamma(x) - [\gamma^{-1}(x)F(x)\gamma(x)]_{r,\sigma}\|_2 \leq$$

$$\left\| \left(\gamma(x)\gamma^{-1}(\sigma) - I\right) \left(F(x) - [F(x)]_{r,\sigma}\right) \right\|_2 + Cr^{n+1} \leq Cr^{n+1}, \quad (25)$$

where the inequality on the far right follows, for sufficiently small $r$ and $\sigma$, from (24) and the boundedness of $\gamma(x)\gamma^{-1}(\sigma) - I$. Inequality (25) shows that the Campanato estimate is preserved under continuous gauge transformations in a small $n$-disc centered at a point close to the origin. Thus in applying our covering argument we can gauge transform out of the exponential gauge and "fan out" from the origin, applying Campanato's Theorem in each ball as we go. Because $\Omega$ is a bounded type-$A$ domain, we will eventually cover the entire set. This completes the proof of Theorem 1. $\Box$
The following is a revision of [O3], Theorem 4.2. (Lemma 3 is used as the fundamental elliptic estimate; implied conditions on dimension and conformal weight are stated explicitly; several elliptic inequalities stated in [O3] are proven; an error in the statement of a lemma in [O3] is corrected.)

**Theorem 4** Let the pair \((A, F_A)\) smoothly satisfy eqs. (8), (9) in \(B/\Sigma\), where \(B\) is a small euclidean \(n\)-disc for \(n \geq 6\) and \(\Sigma\) is a Lipschitz manifold of codimension exceeding \(2n/(n-4)\). Let the section \(\rho\) satisfy condition (2) with \(q = 0\). If \(A\) lies in the space \(H^{1,n/2}(B)\) and \(F_A\) lies in \(L^{n/2}(B)\), then there is a continuous gauge transformation \(g\) such that the pair \((g(A), F_{g(A)})\) is a Hölder continuous solution of (8), (9) in \(B\).

**Remarks.**

i) We require \(n \geq 6\) in order to have \(2n/(n-4) \leq n\); this is necessary in order for the word “codimension” to make sense.

ii) Obviously, the hypothesis on \(A\) is gauge-dependent. This is why one would call Theorem 4 a gauge-improvement theorem rather than a true removable singularities theorem.

iii) We take \(\rho\) to be a section of a line bundle having conformal weight \(w\) (which may be zero) in the sense of, e.g., [O1].

The proof of Theorem 4 depends crucially on Theorem 1 and on the following lemma, which extends Lemma 2.1 of [Si2]. (See also Theorem 3.2 of [GS].)

**Lemma 5** (c.f. [O3], Lemma 3.2). Let the \(p\)-form \(u\) smoothly satisfy the inequality

\[- \frac{\partial}{\partial x^j} \left( a^{ij}(u) \frac{\partial Q}{\partial x^i} \right) + b^j \frac{\partial Q}{\partial x^j} - zQ \leq 0\]  

(26)
on \(B/\Sigma\), with \(a^{ij}\) satisfying the ellipticity condition \(m_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq m_2|\xi|^2\), where \(m_1\) and \(m_2\) are positive constants; \(B\) is a small euclidean \(n\)-disc, \(n > 2(p+1)\), of radius \(\tau\), centered at the origin of coordinates in \(\mathbb{R}^n\); \(\Sigma\) is a compact singular set, completely contained in \(B\), of codimension \(\mu\), where \(n \geq \mu > 2n/(n-2p)\); \(Q = *u \wedge *u\). Assume that the \(L^{n/2}\)-norm of \(z\) is sufficiently small on a small ball \(B' \subset B\) and that the functions \(b^i\) are all in \(L^s(B)\) for some \(s > n\). If \(u \in L^{n/p}(B)\), then \(u \in L^s(B)\) for all \(s < \infty\).

**Remark.** I regret that in the course of a prepublication reduction in the length of [O3] I rather garbled the statement of Lemma 3.2. In particular, the lemma concludes that \(u\) is in \(L^\infty(B)\). That conclusion is unwarranted unless
z lies in some higher $L^p$ space than $L^{n/2}$ or is smoothed by the conclusion of the lemma. As it happens, in each of the two applications of Lemma 3.2 in [O3], one of these ameliorating conditions is satisfied and the conclusion of the lemma is in fact justified in the two special cases in which it is used. Lemma 5 gives a corrected statement of [O3], Lemma 3.2. (A correct version was also circulated as a prepublication preprint of [O3].)

**Proof of Lemma 5.** Integrate inequality (26) against a nonnegative test function $\zeta \in C_0^\infty(B)$ which vanishes on $\Sigma$. Precisely, let

$$\zeta = (\eta \psi)^2 \Xi(Q),$$

where $\eta, \psi \geq 0; \psi(x) = 0 \forall x$ in a neighborhood of $\Sigma; \eta \in C_0^\infty(B')$ where $B' \subset B$ is chosen small enough so that the $L^{n/2}$ norm of $z$, and the $L^n$ norm of $b \equiv \sum_j b^j$, are small on $B'; \Xi(Q) = H(Q)H'(Q)$, where $H(Q) = H_\kappa(Q)$ is the following variant of Serrin’s test function [Se1]:

$$H_\kappa(Q) = \left\{ \begin{array}{ll} Q^{[n/(n-2)]^{n/4p}} & \text{for } 0 \leq Q \leq \ell, \\ \frac{\ell}{2} - \frac{2}{\mu - \epsilon} & \text{for } Q \geq \ell. \end{array} \right.$$  

Notice that $H_\kappa(Q)$ is finite $\forall \kappa < \infty$ but that $H_\kappa(Q)$ is singular if $\kappa$ is infinite. Iterate a sequence of elliptic estimates, taking successively $u \in L^{\alpha(\kappa)}(B')$ for $\alpha(\kappa) = [n/(n-2)]^{n/(n-2)} n/p, \kappa = 0, 1, \ldots$. For each $\kappa$ we have

$$\int_{B'} a^{ij}(u) \partial_i Q \cdot 2(\eta \psi) \partial_j (\eta \psi) \Xi(Q) * 1 + \int_{B'} a^{ij}(u) (\eta \psi)^2 \Xi'(Q) \partial_i Q \partial_j Q * 1$$

$$\leq \int_{B'} |z| \eta \psi)^2 \Xi(Q) * 1 + \int_{B'} |b| |\nabla Q| (\eta \psi)^2 \Xi(Q) * 1.$$  

This inequality can be rewritten in the short-hand form

$$I_1 + I_2 \leq I_3 + I_4$$

or more conveniently

$$I_2 \leq I_3 + I_4 + |I_1|,$$  

the integrals of which we estimate individually.

The definitions of $\Xi$ and $H$ imply the inequalities

$$\Xi'(Q) \geq C (H'(Q))^2$$  

(28)
and
\[ Q \Xi \leq \left( \frac{n}{n-2} \right)^\kappa \frac{n}{4p} H^2. \tag{29} \]

(A wish to satisfy (28) is the motivation behind the lower bound on \( \mu \).

Inequality (28) and the ellipticity condition imply that
\begin{align*}
I_2 &= \int_{B'} a_{ij}^*(u) \eta \psi^2 \Xi'(Q) \partial_i Q \partial_j Q * 1 \geq \\
&= C(m_1) \int_{B'} (\eta \psi)^2 (H'(Q))^2 |\nabla Q|^2 * 1 = C \int_{B'} (\eta \psi)^2 |\nabla H|^2 * 1. \tag{30}
\end{align*}

We have by Young’s inequality
\begin{align*}
I_1 &= \int_{B'} a_{ij}^*(u) \partial_i Q \cdot 2 (\eta \psi) \partial_j (\eta \psi) H(Q) H'(Q) * 1 = \\
&= 2 \int_{B'} (a^*_{ij}(u) (\eta \psi) (\partial_i H)) \partial_j (\eta \psi) H * 1 \leq \\
m_2 \left( \varepsilon \int_{B'} (\eta \psi)^2 |\nabla H| * 1 + C(\varepsilon) \int_{B'} (\eta \psi)^2 |\nabla H|^2 * 1 \right) \equiv i_{11} + i_{12}.
\end{align*}

For small \( \varepsilon \), the integral \( i_{11} \) can be subtracted from the lower bound on \( I_2 \) in (30). Using inequality (29) and the Sobolev inequality, we can write
\begin{align*}
I_3 &= \int_{B'} |z(Q(\eta \psi))^2 \Xi(Q) * 1 \leq \left( \frac{n}{n-2} \right)^\kappa \frac{n}{4p} \int_{B'} |z(\eta \psi)^2 H^2 * 1 \leq \\
&= C \| z \|_{n/2} \left( \int_{B'} (\eta \psi H)^{2n/(n-2)} * 1 \right)^{(n-2)/n} \leq C' \| z \|_{n/2} \| \eta \psi H \|_{1,2}^2.
\end{align*}

Expanding the term on the right by the product rule and using Young’s inequality, we have
\begin{align*}
I_3 &\leq C \| z \|_{n/2} \left\{ \int_{B'} \left[ |\nabla (\eta \psi)|^2 + (\eta \psi)^2 \right] H^2 * 1 + \int_{B'} (\eta \psi)^2 |\nabla H|^2 * 1 \right\} \\
&\equiv i_{31} + i_{32}. \tag{31}
\end{align*}

The integral \( i_{32} \) can be subtracted from the lower bound on \( I_2 \) in (30), as \( B' \) has been chosen so that the product of our independent constant \( C \) and the \( L^{n/2} \) norm of \( z \) is small. Young’s inequality yields
\begin{align*}
I_4 &= \int_{B'} |b| |\nabla Q| (\eta \psi)^2 H(Q) H'(Q) * 1 = \int_{B'} |b| (\eta \psi)^2 H |\nabla H| * 1 \leq \\
&\quad \text{...}
\end{align*}
\[
C(\varepsilon) \int_{B'} |b|^2 (\eta \psi)^2 H^2 \ast 1 + \varepsilon \int_{B'} (\eta \psi)^2 |\nabla H|^2 \ast 1.
\]  
(32)

We can write inequality (32) in the short-hand form

\[
I_4 \leq i_{41} + i_{42}.
\]

We similarly rewrite the integral inequality (27) in the form

\[
I_2 - (i_{32} + i_{42} + i_{11}) \leq C(i_{12} + i_{31} + i_{41}).
\]  
(33)

Notice that the left-hand side of (33) remains nonnegative for small \(\varepsilon\) when \(I_2\) is replaced by the extreme right-hand integral in (30). Moreover,

\[
i_{41} \leq C \|b\|_n^2 \|\eta \psi H\|_{2(n/(n-2))}^2 \leq C \|b\|_n^2 \|\eta \psi H\|_{1,2}^2,
\]  
(34)

which is analogous to \(I_3\). Because \(b \in L^s\) for \(s > n\), the \(L^n\)-norm of \(b\) is small on \(B'\) if \(B'\) is small. We simultaneously estimate the terms of \(i_{31}\) and \(i_{41}\) which involve the gradient of \(\psi\). There exists ([Se2], c.f. Lemma 2 and p. 73) a sequence of functions \(\xi_\nu\) such that:

a) \(\xi_\nu \in [0, 1] \forall \nu\);

b) \(\xi_\nu \equiv 1\) in a neighborhood of \(\Sigma \forall \nu\);

c) \(\xi_\nu \to 0\) a.e. as \(\nu \to \infty\);

d) \(\nabla \xi_\nu \to 0\) in \(L^{\mu-\varepsilon}\) as \(\nu \to \infty\).

Apply the product rule to the squared \(H^{1,2}\) norms in \(i_{31}\) [inequality (31)] and in \(i_{41}\) [inequality (34)] letting \(\psi = \psi_\nu = 1 - \xi_\nu\). Observing that the cross terms in \((\nabla \eta) \psi\) and \((\nabla \psi) \eta\) can be absorbed into the other terms by applying Young’s inequality, we estimate

\[
\lim_{\nu \to \infty} \int_{B'} \eta^2 |\nabla \psi_\nu|^2 H^2 \ast 1 \leq \lim_{\nu \to \infty} C(\ell) \int_{B'} |\nabla \psi_\nu|^2 \left(\frac{\mu-2+\varepsilon}{\mu-\varepsilon}\right)^{\alpha/(\mu-\varepsilon)} \ast 1
\]

\[
\leq \lim_{\nu \to \infty} C(\ell) \|\nabla \psi_\nu\|_{\mu-\varepsilon}^2 \|u\|_{\alpha(\mu-\varepsilon)/(\mu-\varepsilon)}^{\alpha(\mu-\varepsilon)/(\mu-\varepsilon)} = 0.
\]  
(35)

Having shown that the integral on the left in (35) is zero for every value of \(\ell\), we can now let \(\ell\) tend to infinity. We obtain via Fatou’s Lemma the inequality

\[
\int_{B'} \eta^2 |\nabla \left(\frac{Q^{\alpha(\kappa)/2}}{\eta}\right)|^2 \ast 1 \leq \int_{B'} |\nabla \eta|^2 Q^{\alpha(\kappa)/2} \ast 1.
\]  
(36)

Thus \(Q^{\alpha(\kappa)/4}\) is in \(H^{1,2}\) on some smaller disc on which \(\eta = 1\). But then, because \(u\) is assumed to be smooth away from the singularity and \(\Sigma\) is compact,
\( Q^{\alpha(\kappa)/4} \) must be in \( H^{1.2} \) on the larger disc as well. Now apply the Sobolev inequality to conclude that \( u \) is now in the space \( L^{\alpha(\kappa+1)}(B) \). Because the sequence \( \{n/(n-2)\}^{\kappa} \) obviously diverges, we conclude after a finite number of iterations of this argument that \( Q^c \) is in \( H^{1.2}(B) \) for any positive value of \( c \). A final application of the Sobolev inequality yields the assertion of Lemma 5.\( \square \)

Proof of Theorem 4. Use (15) to apply Lemma 5 with \( u = |F| \), \( z = |\nabla A| + |A|^2 \), and \( b^j \equiv 0 \forall j \). For \( F \in L^{n/2} \) the small-norm assumptions of the lemma follow from conventional scaling arguments in a Hodge gauge (see the comments following Theorem 1.3 of [U3]). Remark iii) following the statement of Theorem 4 implies that eqs. (8), (9) remain unaffected by such changes of scale. We conclude that \( F \) is in \( L^p \) for some \( p > n/2 \).

The continuous gauge transformation guaranteed by Theorem 1.3 of [U3] can be applied up to the boundary of \( \Sigma \), using the methods of [SS5], as \( \Sigma \) is Lipschitz. Because the test functions used in proving Lemma 5 did not require any more smoothness than is implied by the definition of weak solution adopted in Remark ii) following Theorem 1, we can use (36) to show that \( F \) is a weak solution of (8), (9) in all of \( B \). Theorem 1 now implies that \( F \) is Hölder continuous in \( B \).\( \square \)

Remark. A similar argument implies a removable singularities theorem for solutions of eqs. (3), (4). Apply Lemma 5, taking \( z \) to be an upper bound on the sectional curvature of the restriction to \( B \) of the Riemannian manifold \( M \). Thus \( z \) can be chosen to be zero for a sufficiently small singular set. For any singular set \( z \) can be chosen to lie in a higher \( L^p \) space than \( n/2 \), as the singularity is in \( T^*M \) rather than in the metric on \( M \). In this case the arguments of the lemma imply, using Theorem 5.3.1 of [Mo], that the \( p \)-form \( \omega \) (taking \( u = |\omega| \)) is a bounded weak solution of eqs. (3), (4) in \( B \). Observing that the arguments of [S1], Section 4 are local, we can apply them in \( B \) to conclude that \( \omega \) is Hölder continuous in all of \( B \). Details are given in Theorem 3.1 of [O3].

4 Appendix

In this section we sketch the proof of a technical lemma required for the proof of Theorem 1. Details are given in [O2], pp. 387-392.

Lemma 6 ([O2], Lemma 3.1). Under the hypotheses of the theorem, \( F_A \in H^{1.2}(B) \) for sufficiently small \( n \)-disc \( B \).
Proof. It is sufficient to prove Lemma 6 in a Hodge gauge. In eq. (14) replace the admissible test function \( \zeta(x) \) by the admissible test function \( \zeta(x - he_i) \), where \( e_i \) is the \( i \)th basis vector for \( \mathbb{R}^n \), \( i = 1, \ldots, n \), and \( h \) is a positive constant. Then eq. (14) assumes the form

\[
\int_B \langle d\zeta(x - he_i), \rho(Q(x)) F(x) \rangle \, d^n x = - \int_B \langle \zeta(x - he_i), [A(x), *\rho(Q(x)) F(x)] \rangle \, d^n x.
\]

(37)

If we subject both sides of identity (37) to the coordinate transformation \( y = x - he_i \), subtract eq. (14) from the resulting equation and divide through by \( h \), we obtain

\[
\int_B \langle d\zeta(x), \frac{\rho(Q(x + he_i)) F(x + he_i) - \rho(Q(x)) F(x)}{h} \rangle \, d^n x =\]

\[
- \int_B \langle \zeta(x), [\Delta_{i,h} A(x), *\rho(Q(x + he_i)) F(x + he_i)] \rangle \, d^n x - \int_B \langle \zeta(x), [A(x), \{\rho(Q(x + he_i)) F(x + he_i) - (Q(x)) F(x)\}] \rangle \, d^n x,
\]

(38)

where

\[
\Delta_{i,h} u(x) = \frac{u(x + he_i) - u(x)}{h}.
\]

Apply Lemma 2 to each of the terms

\[
\frac{\rho(Q(x + he_i)) F(x + he_i) - \rho(Q(x)) F(x)}{h}
\]

enclosed in braces on each side of (38). Choose \( \xi = x + he_i, \eta = x, \mu = F(x + he_i) \), and \( \tau = F(x) \). We obtain

\[
\int_B \langle d\zeta, \alpha^{ij} \Delta_{i,h} F + \beta_i^j e_i \rangle \, d^n x =
\]

\[
- \int_B \langle \zeta(x), [\Delta_{i,h} A(x), *\rho(Q(x + he_i)) F(x + he_i)] \rangle \, d^n x - \int_B \langle \zeta(x), [A(x), \{\alpha^{ij} \Delta_{i,h} F + \beta_i^j e_i\}] \rangle \, d^n x.
\]

(39)

The function \( \zeta(x) = \eta(x) \Delta_{i,h} A(x) \) is an admissible test function for all \( h \) by [U3], Theorem 1.3 ii), provided \( \eta \in C_0^\infty(B) \). Choose \( \eta(x) \) to be nonnegative

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∀x ∈ B and η(x) ≡ 1 for x ∈ B', where B' is a ball completely contained in B. Using the fact that d commutes with \(\Delta_{i,h}\), we can write (39) in the form

\[
\int_B \eta |\Delta_{i,h} (dA)|^2 d^n x \leq C \left\{ \int_B (|A(x) + he_i| + |A(x)|) |\Delta_{i,h} A| |\Delta_{i,h} (dA)| d^n x + \int_B |\Delta_{i,h} A|^2 (|F(x + he_i)|) d^n x \\
+ \int_B (|A(x)| + 1) |\Delta_{i,h} (dA)| |\Delta_{i,h} A| d^n x + \int_B (|A(x) + 1|) |\Delta_{i,h} A|^2 (|A(x + he_i)| + |A(x)|) d^n x \\
+ \int_B (|F(x + he_i)| + |F(x)|) |\Delta_{i,h} A| d^n x \right\}.
\] (40)

In order to estimate the right-hand side of (40) we use the relation

\[
\int_B |\nabla |\Delta_{i,h} A||^2 d^n x \leq \int_B |\nabla (\Delta_{i,h} A)|^2 d^n x \leq C \left( \int_B |\Delta_{i,h} (dA)|^2 d^n x + \int_B |\Delta_{i,h} A|^2 d^n x \right),
\] (41)

which follows from the Kato and Gaffney-Gårding inequalities, and from the facts that the exterior derivative and its adjoint commutes with \(\Delta_{i,h}\) and \(\delta A = 0\). Now we have

\[
\int_B (|A(x) + he_i| + |A(x)|) |\Delta_{i,h} A| |\Delta_{i,h} (dA)| d^n x \leq \frac{C}{\varepsilon} \left( \int_B |\Delta_{i,h} A||^2 d^n x + \int_B |\Delta_{i,h} A|^2 d^n x \right)
+ \varepsilon \int_B |\Delta_{i,h} A|^2 d^n x.
\]

Using (41) and the fact that A is small in \(L^n\) on B, terms involving \(|\Delta_{i,h} (dA)|^2\) can be subtracted from the left-hand side of (40). The other terms on the right in (40) can be handled similarly, using the fact that F is small in \(L^{n/2}\) on B. Letting h tend to zero, we obtain

\[
\int_{B'} |\nabla (dA)|^2 * 1 < \infty.
\]
The proof of Lemma 6 is completed by applying the Sobolev inequality to \(dA\) and writing
\[
\int_{B'} |\nabla F|^2 \ast 1 \leq C \left( \int_{B'} |\nabla (dA)|^2 \ast 1 + \|A\|_n^2 \|\nabla A\|_{2n/(n-2)}^2 \right). \quad \square
\]

References

[B] L. Bers, Mathematical Aspects of Subsonic and Transonic Gas Dynamics, Wiley, New York, 1958.

[Ga] M. P. Gaffney, The harmonic operator for exterior differential forms, \textit{Proc. N. A. S.} \textbf{37} (1954), 48-50.

[Gi] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton University Press, Princeton, 1983.

[GS] B. Gidas and G. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, \textit{Commun. Pure Appl. Math.} \textbf{34} (1981), 525-598.

[Ma] A. Marini, Dirichlet and Neumann boundary value problems for Yang-Mills connections, Ph.D. dissertation, The University of Chicago, 1990.

[MM] K. B. Marathe and G. Martucci, The Mathematical Foundations of Gauge Theories, North-Holland, Amsterdam, 1992.

[Mo] C. B. Morrey, Jr., Multiple Integrals in the Calculus of Variations, Springer, New York, 1966.

[O1] T. H. Otway, The coupled Yang-Mills-Dirac equations for differential forms, \textit{Pacific J. Math.} \textbf{146}, No. 1 (1990), 103-113.

[O2] T. H. Otway, Yang-Mills fields with nonquadratic energy, \textit{J. Geometry \& Physics} \textbf{19} (1996), 379-398.

[O3] T. H. Otway, Properties of nonlinear Hodge fields, \textit{J. Geometry \& Physics} \textbf{27}(1998), 65-78.

[O4] T. H. Otway, An elliptic inequality for nonlinear Hodge fields, Los Alamos National Laboratory automated electronic archive at \texttt{xxx.lanl.gov}, \texttt{math-ph/9806007} (1998).

[Se1] J. Serrin, Local behavior of solutions of quasilinear equations, \textit{Acta Math.} \textbf{111} (1964), 247-302.

[Se2] J. Serrin, Removable singularities of solutions of elliptic equations, \textit{Archs. Ration. Mech. Analysis} \textbf{17} (1964), 67-78.

[Si1] L. M. Sibner, An existence theorem for a nonregular variational problem, \textit{Manuscripta Math.} \textbf{43} (1983), 45-72.
[Si2] L. M. Sibner, The isolated point singularity problem for the coupled Yang-Mills equations in higher dimensions, *Math. Ann.* 271 (1985), 125-131.

[SS1] L. M. Sibner and R. J. Sibner, A nonlinear Hodge-de Rham theorem, *Acta Math.* 125 (1970), 57-73.

[SS2] L. M. Sibner and R. J. Sibner, Nonlinear Hodge theory: Applications, *Advances in Math.* 31 (1979), 1-15.

[SS3] L. M. Sibner and R. J. Sibner, A maximum principle for compressible flow on a surface, *Proc. Amer. Math. Soc.* 71(1) (1978), 103-108.

[SS4] L. M. Sibner and R. J. Sibner, A subelliptic estimate for a class of invariantly defined elliptic systems, *Pacific J. Math.* 94(2) (1981), 417-421.

[SS5] L. M. Sibner and R. J. Sibner, Classification of singular Sobolev connections by their holonomy, *Commun. Math. Phys.* 144(1992), 337-350.

[T] Tchrakian, D. H., N-dimensional instantons and monopoles, *J. Math. Physics* 21 (1980), 166-169.

[U1] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, *Acta Math.* 138 (1977), 219-240.

[U2] K. Uhlenbeck, Removable singularities in Yang-Mills fields, *Commun. Math. Physics* 83 (1982), 11-30.

[U3] K. Uhlenbeck, Connections with $L^p$ bounds on curvature, *Commun. Math. Physics* 83 (1982), 31-42.