Quantum memory assisted entropic uncertainty and entanglement dynamics in classical dephasing channels

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We investigate the dynamics of entropic uncertainty relations, tightness, and concurrence in two non-interacting qubits produced in the Werner state and subjected to classical channels. In two contexts, common and independent qubit-noise configurations, a Gaussian Ornstein Uhlenbeck process is used to control the noisy effects of the local external fields. Using parameter optimization, we establish long-term stability in two qubits while reducing the environment’s disorder entropic impact and maintaining entanglement. Using entropic uncertainty relations, tightness and concurrence, we show that in the presence of Ornstein Uhlenbeck noise, two-qubit coherence and entanglement in local fields are both weak and readily lost. Despite this, longer entanglement and lower entropy can be predicted within a limited range of the noise parameter and purity estimator of the two-qubit state. Furthermore, the measured rate of entropy rise outpaces the rate of disentanglement generation. In addition, we also utilized the tightness measure to estimate the uncertainty in the dynamics of bipartite entropic relations.

Keywords: Entropic uncertainty, Entanglement, Tightness, common and independent classical fields, concurrence, OU noise

I. Introduction

The uncertainty principle is a well-known concept in physics. It serves as a reminder that nature is ambiguous and that there is a fundamental limit to our understanding of quantum particles and, as a result, nature’s lowest scales [1]. We can only expect a probability calculation of where things are and how they will act from these scales. The uncertainty principle gives a level of fuzziness to quantum theory, in contrast to Isaac Newton’s clockwork universe, where everything follows clear-cut laws and prediction is straightforward if the initial conditions are known [2]. The uncertainty principle in quantum mechanics is both a fundamental feature and a significant departure from classical physics. Any pair of incompatible observables obeys a specific type of uncertainty relationship, imposing final constraints on measurement accuracy while also laying the theoretical groundwork for future technologies, such as quantum encryption and quantum information [3–6]. The newly empirically validated entropic uncertainty principle has piqued interest in its potential applications from various perspectives. A new sort of Heisenberg relation known as the quantum memory assisted entropic uncertainty relation has just been constructed, according to Renes and Boileau’s concept [7, 8]. The entropic uncertainty relation is used in cryptographic security.
Quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects, notwithstanding their spatial separation, must be explained in terms of one another [15]. As a result, there are correlations between the systems’ observable physical characteristics. Even though quantum physics makes it difficult to forecast which set of measurements will be observed, it is feasible to combine two particles into a single quantum state so that when one is detected as spin-up, the other is always detected as spin-down, and vice versa. Quantum entanglement has been utilized in experiments to establish quantum teleportation [16], and it has potential applications in quantum computing [17], quantum cryptography [18], communications [19], quantum radar [20] and entanglement swapping [21].

The preservation of entanglement and the level of uncertainty in open quantum systems are inextricably linked and a hotly debated topic. In this case, we want to discuss the dynamical map of a Werner type mixed entangled state and the relationship between entanglement and entropic uncertainty and its regulation. This is significant because the dynamics of open quantum systems are crucial for the development of quantum protocols and the inter-transmission of information between two locations [21]. The main source of uncertainty is that quantum systems cannot be completely isolated from their external mediums, which can accommodate a variety of disorders [22–25]. These disorders generate a variety of noises, which, when superimposed on the phase factors of the systems, reduce the efficiency of quantum processes and phenomena [26–28]. As a result, research into such topics can help to reduce the actual causes of quantum mechanical application failure while also improving relative precision and measurement accuracy.

In the present work, we discuss the dynamics of entanglement and entropic uncertainty relation in a two-qubit entangled Werner type state under the influence of classical fields. To limit our problem, we consider Ornstein Uhlenbeck (OU) noise generation in the classical fields and the main reason for the disentanglement of the two qubits and the relative degree of entropic uncertainty. In microscopic view, the OU noise is caused by the Brownian motion of the particles, which can be found nearly in every quantum mechanical process [28]. This makes our more significant because of its widespread presence and noisy actions in such operations. We prefer the classical context of environments rather than the non-classical ones because the local former provides more degrees of freedom to examine the dynamical maps of quantum systems [22–24]. Two types of two-qubit spin squeezing models were used to investigate thermal quantum correlations and the entropic uncertainty relation in the presence of quantum memory [29]. For two atoms and the relative dynamics of the entanglement and uncertainty was found to be greatly dependent upon the temperature parameters [30]. For different three levels systems, the dynamics of entropic uncertainty reveal that the corresponding entanglement losses and rise in entropy and uncertainty is regulated by the coupling strengths of the random telegraphs noise [31]. The demonstration of the evolution of entropic uncertainty in the multi-measurement process has shown that the Markovianity and non-Markovianity of the fields can be traced back to the degree of uncertainty and noise [32]. A new type of long-range reaction was used to achieve long-distance entanglement in the spin system [33, 34]. In a similar case, the authors in [36] investigated the dynamics of entropic
uncertainty in three qubits and they found that the classical depolarizing noise and environments enhance entropy. Besides, they found that designing the system-environment coupling between three qubits and environments can also lead to enhanced entanglement preservation and lower entropy. Thus, the above literature concludes that entropic uncertainty and entanglement are controlled by the different variables and fields, which must be thoroughly investigated for practical implementation of the quantum protocols.

We assume two kinds of system-environment coupling: common qubit-noise (CQN) and independent qubit-noise (IQN) configurations. Both qubits will be coupled to a common environment characterized by an OU noise source in the CQN environment. The two qubits are coupled with two independent local environments in the IQN configuration case. This will help to conclude the entropy and disentanglement level for the increasing number of environments. Entanglement has already been shown to degrade differently in different types and number of environments [22–28].

This paper is organized as: In the Sec.II, we give the details of the physical model, estimators of entropy and entanglement and OU noise application. The explicit results and discussion are written in Sec.III. In Sec.IV, we summarize our results in few remarks.

II. Suggested model and dynamics

Our model comprises two non-interacting qubits initially prepared in Werner type entangled state coupled to a classical environment. OU noise is considered to be the primary cause of dephasing and entropic increase in classical environments. We examine CQN and IQN configurations, which are two different designs of system-environment coupling approaches. In CQN configuration, the dynamical map of the two qubits is studied under the influence of a single OU noise source. In IQN configuration, the system is considered evolving under the influence of two independent OU noise sources. The Hamiltonian, which characterizes the current model, is written as:

\[ H(t) = H_1(t) \otimes I_2 + I_1 \otimes H_2(t), \]  

where \( H_n(t) = \kappa I + \lambda \chi_n(t) \sigma_z \) with \( n \in \{1, 2\} \). \( \kappa \) represents the energy of the \( n \)th qubit, \( I \) and \( \sigma_z \) are the identity and Pauli matrices of depolarizing classical channels while \( \lambda \) is the coupling constant, regulating the strength of linking between the qubits and classical environments. The terms \( \chi_n(t) \) represent the stochastic parameter of the fields and control the flipping of the qubits between \( \pm 1 \).

For the time-evolution of the two qubits in classical fields, we employ the time unitary operation by:

\[ U(t) = \exp[-iH(t)]. \]  

Time evolved state of the two qubits, when prepared in the initial state \( \rho_{\text{in}} \) is obtained using:

\[ \rho(t) = U(t)\rho_{\text{in}}U^\dagger(t). \]  

We assume that the OU process, a stochastic mathematical process with applications in both physical sciences and finance, impacts the classical fields to account for noise. This term in physics
FIG. 1. Shows the schematic diagram of two qubits $Q_i$ and $Q_j$ connected with square-like boxes means classical environments of two types: common qubit-noise (left) and independent qubit-noise configuration (right). OUNS denotes the Ornstein Uhlenbeck noisy sources. The reddish-green lines represent the connection between the classical channels and noisy sources characterized by the stochastic parameter $\Delta(t)$. The glow around the qubits shows the entropic action of the environments while the wavy lines above the qubits represent the dynamics and the relative coupling strengths $\lambda$ while its diminishing size shows the resultant dephasing effects.

describes the velocity of a massive Brownian particle under the influence of friction. In many quantum mechanical protocols, the OU method is a static Gaussian–Markov operation with OU noise, and it has been identified as one of the numerous and primary causes of information, coherence, and quantum correlation losses [28]. OU noise has been extensively studied in the case of single qutrit, two-qubit, three qubits, and hybrid qubit-qutrit states and we find that in each case, the degree and behavior of losses are different [26, 37, 38]. Currently, the OU noise is applied to the dynamical map of the two-qubit mixed Werner state, which is created in the state $\rho_0$. To determine the negative consequences of OU noise, we use a zero-mean Gaussian process ($\langle G(t) \rangle = 0$) to describe the classical field $\mathcal{L}(t)$ affecting the system. This is further defined by the auto-correlation function, which is defined:

$$A(g, t - t') = \frac{\exp[-g|t - t'|]}{2}. \tag{4}$$

We connect the classical noise with the environments in the dynamical map of the two-qubit state using the $\beta$-function, which is written as [28]:

$$\beta_{\text{OU}}(t) = \int_0^t \int_0^t A(s - s') ds ds'. \tag{5}$$

We can extract the final $\beta$-function for the OU noise by plugging the auto-correlation function from Eq.(4) into Eq.(5) as:

$$\beta_{\text{OU}}(t) = \frac{1}{g}[gt + \exp[-gt] - 1], \tag{6}$$

The memory characteristic of the classical environment is controlled by $g$, which is the inverse of the autocorrelation time.
A. The entropic uncertainty measure

Consider the following network with two users, Bob and Alice: Bob generates a qubit in the quantum state of his choice and delivers it to Alice, who must choose between the two measurements and broadcast her decision to Bob. By utilizing the measurement he obtained, Bob’s uncertainty can be reduced. The standard deviation uncertainty relation for two observables $A$ and $B$ can be written as [39]:

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|.$$  

Deutsch proposed the entropic uncertainty relation for any pair of observables by describing uncertainty in terms of Shannon entropy rather than standard deviation [40]. Maassen and Uffink devised a tighter entropic uncertainty expression based on Deutsch’s approach [41]:

$$S(A) + S(B) \geq \log_2 \left( \frac{1}{c} \right), \quad (7)$$

where $S(A)$ is the Shannon entropy, which represents the probability distribution when $A$ is measured, and $S(B)$ is the Shannon entropy when $B$ is measured. $c$ denotes the complementary of $A$ and $B$, and $c = \max_{a,b} |\langle \psi | \phi \rangle|^2$ for non-degenerate observables, where $|\psi\rangle$ and $|\phi\rangle$ are the eigenvectors of $A$ and $B$, respectively. The current definition has been updated into a new form known as a quantum memory assisted entropic uncertainty relation [42], which has also been experimentally tested [43] and can be written as:

$$S(A|1) + S(B|2) \geq S(1|2) + \log_2 \left( \frac{1}{c} \right), \quad (8)$$

where $S(1|2) = S(\rho_{12}) - S(\rho_2)$ is the conditional von-Neumann entropy. In Eq.(8), $R(\tau)$ and $L(\tau)$ represents the left and right-hand sides. To find the difference between the two sides, we use the equation:

$$U(\tau) = L(\tau) - R(\tau) \quad (9)$$

as the tightness of the uncertainty relation. Once the first qubit is measured by $A$, the system’s post-measurement state can be stated as [44]:

$$\rho_{A2} = \sum_n (|\psi_n\rangle_1 \langle \psi_n| \otimes I_2) \rho_{12}(|\psi_n\rangle_1 \langle \psi_n| \otimes I_2), \quad (10)$$

Note that we utilize the Werner state form of the two non-interacting entangled qubits:

$$\rho_o = \frac{1 - p}{4} (I_4) + p|\psi\rangle \langle \psi|, \quad (11)$$

where $|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is the two qubit maximally entangled Bell’s state. $p$ denotes the purity factor, controlling the initial purity in the system and ranges between $0 \leq p \leq 1$. 

B. Concurrence

To assess entanglement, we use concurrence for the bipartite state, which ranges from $1 \geq C(t) \geq 0$. The state is entangled at $C(t) = 1$, but at the lowest bound, the state becomes completely separable. For the two-qubit state, the concurrence measurement can be carried out using the following expression [15, 25]:

$$C = \max\{0, \sqrt{\nu_4}, \sqrt{\nu_3}, \sqrt{\nu_2}, \sqrt{\nu_1}\},$$

(12)

where $\nu_i$ are the eigenvalues of the time evolved density matrix $\rho(t)$. The eigenvalues belong to the matrix $\rho(t)\Phi \rho(t)^* \Phi$ and $\Phi = \sigma_y \otimes \sigma_y$.

III. Analytical results

In this section, we present the entanglement and entropic uncertainty dynamics results obtained using Eq.(9) and (12). The system’s time unitary matrix, obtained using Eq.(2), has the following form:

$$U(t) = \begin{pmatrix}
    e^{it(-2\kappa + (\chi_a(t) + \chi_b(t))\lambda)} & 0 & 0 & 0 \\
    0 & e^{-it(2\kappa - (\chi_a(t) + \chi_b(t))\lambda)} & 0 & 0 \\
    0 & 0 & e^{-it(2\kappa + (\chi_a(t) - \chi_b(t))\lambda)} & 0 \\
    0 & 0 & 0 & e^{-it(2\kappa + (\chi_a(t) + \chi_b(t))\lambda)}
\end{pmatrix},$$

(13)

Note that all operations are carried out using the time evolved density matrix given in Eq.(3) which has the following form:

$$\rho_{CQN}(t) = \begin{pmatrix}
    1 + p & 0 & 0 & \frac{1}{2}e^{4it\chi_a(t)\lambda}p \\
    0 & 1 - p & 0 & 0 \\
    0 & 0 & 1 - p & 0 \\
    \frac{1}{2}e^{-4it\chi_a(t)\lambda}p & 0 & 0 & 1 + p
\end{pmatrix}. $$

(14)

A. Entropic uncertainty and entanglement dynamics in CQN configurations

We discuss the dynamics of entropic uncertainty, tightness and entanglement when two non-interacting qubits are both coupled with a common OU noise source in a single local random field. To include the OU noisy effects in the matrix given by Eq.(3), we average for the CQN configuration as follows:

$$\rho_{CQN}(t) = \langle \rho(\phi_1, t) \rangle_{\theta_1},$$

(15)

where, $\phi_1$ is the combined factor of system and environments while, $\theta_1$ is the superimposed noise phase over the system. In Eq.(15), $\phi_1 = n\chi_1(t)$ and we set $\chi_1 = \chi_2$ where $\theta = -\frac{1}{2}n^2\beta(t)$. The
explicit form of the Eq(15) obtained can be put into the following form:

\[
\rho_{CQN}(t) = \begin{pmatrix}
\frac{1+p}{4} & 0 & 0 & e^{-2\beta t} \\
0 & \frac{1-p}{4} & 0 & 0 \\
0 & 0 & \frac{1-p}{4} & 0 \\
\frac{e^{-2\beta t}}{2} & 0 & 0 & \frac{1+p}{4}
\end{pmatrix}
\] (16)

where \(\beta\)-function is given in Eq.(5). The analytical results of the Eq.(9) and Eq.(12) takes the form as:

\[
U(\tau) = \frac{1}{\log[16]} e^{-\frac{1}{2} n^2 \beta} \left( T_1 + e^{\frac{1}{2} n^2 \beta} (T_2 + T_3) \right),
\] (17)

\[
C(t) = -\sqrt{1-p} \frac{1}{2} T_4 + \frac{1}{2} T_5,
\] (18)

where,

\[
T_1 = -4 \text{ArcTanh} \left[ e^{-\frac{1}{2} n^2 \beta} \right] - 2 \log \left[ 1 - 2 e^{-\frac{1}{2} n^2 \beta} + p \right] + 2 \log \left[ 1 + 2 e^{-\frac{1}{2} n^2 \beta} + p \right],
\]

\[
T_2 = -\log[16] - 2 \log \left[ \frac{1}{4} - \frac{1}{4} e^{-\frac{1}{2} n^2 \beta} \right] - 2 \log \left[ 1 + e^{-\frac{1}{2} n^2 \beta} - 2(1+p) \log[1+p] \right],
\]

\[
T_3 = (1+p) \log \left[ 1 - 2 e^{-\frac{1}{2} n^2 \beta} + p \right] + (1+p) \log \left[ 1 + 2 e^{-\frac{1}{2} n^2 \beta} + p \right],
\]

\[
T_4 = \sqrt{e^{-\frac{1}{2} n^2 \beta} \left( -2 + e^{\frac{1}{2} n^2 \beta} + e^{\frac{1}{2} n^2 \beta} + p \right)},
\]

\[
T_5 = \sqrt{e^{-\frac{1}{2} n^2 \beta} \left( 2 + e^{\frac{1}{2} n^2 \beta} + e^{\frac{1}{2} n^2 \beta} + p \right)}.
\]

When the CQN configuration is considered, Fig.2 analyze the left, right-hand sides of the Eq.(8), tightness and concurrence as well as the noise effects caused by the OU process in the dynamical map of two qubits. When OU noise dephasing effects are present, entanglement decreases and entropy increases in classical environments. In two qubits, the entropy functions \(L(\tau)\) and \(R(\tau)\) are increasing, while the tightness and entanglement functions, \(U(\tau)\) and \(C(\tau)\) are found decreasing. Although the difference between \(L(\tau)\) and \(R(\tau)\) is insignificant, the high rate of entropy increase cannot be omitted. The results of \(U(\tau)\), which show dynamics in a small restricted elevation (as the slope elevation starts at \(1.5 \times 10^{-2}\)), confirm the minimal difference between the \(L(\tau)\) and \(R(\tau)\). As a result, the \(L(\tau)\) and \(R(\tau)\) results are in good agreement with \(U(\tau)\). From the \(C(\tau)\) results, we can see that classical fields with Brownian motion disorders cause entanglement to degrade and entropy to increase. It is simple to deduce that the rate of disentanglement lags the entropy growth by comparing the entropic uncertainty and concurrence dynamics. This means that an increase in entropy causes the disentanglement of the two qubits. Despite this, the noise parameter \(g\) regulates entropic uncertainty and entanglement loss, and as \(g\) rises, entropy rises and entanglement falls. Under the current noise and parameter settings, the system becomes completely separable because of high entropy. We find that the current results differ completely from those described in [45], where previous results showed revivals in \(U(\tau)\), \(R(\tau)\), and \(L(\tau)\). The maximum values of the two sides of uncertainty and tightness do not match, which is important. The qualitative monotone behaviour and dynamical map of the two non-equivalent sides, on the other hand, were similar.
FIG. 2. Dynamics of $R(\tau)$ (a), $L(\tau)$ (b), $U(\tau)$ (c) and $C(\tau)$ (d) in two qubits state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ initially prepared in the state $\rho_0$ given in Eq.(11) common qubit-noise configuration with parameter settings: $g = 0.4$, $p = 1$ and time parameter $\tau = 3$.

1. Entropic uncertainty and concurrence against noise parameter and purity factor in CQN configuration

The left and right hands of the entropic uncertainty relation, tightness and concurrence, are displayed in Fig.3 when the system is linked with a single classical environment. The new results are qualitatively similar to those in Fig.2, although they differ in quantitative terms. Lower g values allowed the $L(\tau)$ and $R(\tau)$ to achieve ultimate saturation heights after the maximum entropic rise however taking a much longer time, which is the main reason for the disparity between the Fig.3 and Fig.2. The difference between the $L(\tau)$ and $R(\tau)$ is minor when compared to the results obtained for $g = 0.4$ in Fig.2 and 3. This shows that entropic uncertainty is primarily caused and controlled by the noise parameter g and that the two are inexorably related. The results of $U(\tau)$ show that $L(\tau) > R(\tau)$ and depict that the gap between the two sides of the uncertainty relation becomes narrow and finally vanishes with time. The entanglement decays monotonically under the influence of OU noise, as seen by the $C(\tau)$ measure. As the values of g increase, the entanglement decreases. The findings of $C(\tau)$ suggest that rising entropy directly affects the degree of entanglement between qubits and that entropic uncertainty increases faster than entanglement diminishes. This suggests that entropic uncertainty is a critical contributor in
entangled quantum systems losing their entanglement. We discovered that for low g values, we could maintain entanglement for a long time, even though the state eventually becomes separable. The current entropic uncertainty results contradict those reported in [45, 46], where the qualitative dynamics of entropic uncertainties, tightness, and entanglement dynamics are vastly different.

Fig. 4 shows the dynamics of the left hand, right-hand side of the entropic uncertainty, tightness and concurrence in the bipartite entangled state when exposed to local fields and OU noise. The qualitative dynamics of the current result against different values of purity factor differ from those seen in Fig. 3. In two-qubit Werner entangled state, we find that the purity factor significantly affects the initial entanglement and level of entropic uncertainty. As seen, the entropy and relative uncertainty increase proportionally as p decreases, with minimum disorder in the system occurring at \( p = 0.99 \) and maximum disorder at \( p = 0.10 \). When \( p > 0.9 \) and \( p < 0.1 \), the \( U(\tau) \) predicts minor variations at the maximum and minimum bounds, respectively. When compared to various purity factor values, the qualitative dynamics of the current result differ from those shown in Fig. 3. After a finite interval of time, the two-qubit Werner state becomes separable for all ranges of p. Under the OU noise, bipartite entanglement was preserved for longer intervals than in [15, 23, 24, 27–29].

B. Entropic uncertainty and entanglement dynamics in IQN configurations

The dynamics of two qubit entropy, tightness and entanglement under the influence of classical fields characterized by OU noise are discussed in this section. The final density matrix for the IQN
FIG. 4. Dynamics of $R(\tau)$ (a), $L(\tau)$ (b), $U(\tau)$ (c) and $C(\tau)$ (d) against different values of $g$ in two qubits state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ initially prepared in the state $\rho_o$ given in Eq.(11) for common qubit-noise configuration when $g = 10^{-1}$.

configuration is obtained by averaging the time evolved density matrix given in Eq.(3) as:

$$\rho_{IQN}(t) = \langle \langle \rho(\phi_1, \phi_2, t) \rangle_{\theta_1} \rangle_{\theta_2},$$

(19)

where, $\chi_1 \neq \chi_2$. The corresponding numerical form of the density matrix can be put into the following form as:

$$\rho_{IQN}(t) = \begin{pmatrix}
\frac{1+p}{4} & 0 & 0 & \frac{1}{2}e^{-4\beta}p \\
0 & \frac{1-p}{4} & 0 & 0 \\
0 & 0 & \frac{1-p}{4} & 0 \\
\frac{1}{2}e^{-4\beta}p & 0 & 0 & \frac{1+p}{4}
\end{pmatrix}$$

(20)

The presence of diagonal and off-diagonal components in the previous matrix indicates that the state was still entangled and coherent. As a result, time evolution limitations and noise parameter value choices have a role in further restricting entanglement and entropic uncertainty. Next, the analytical expressions obtained for the $U(\tau)$ and $C(\tau)$ can be given as:

$$U(\tau) = \frac{1}{\text{Log}[16]} e^{-4\beta} \left( -2p(U1) + e^{4\beta}(U2 - 2(U3) + U4) \right),$$

(21)

$$C(\tau) = -\sqrt{1-p} - \frac{1}{2}\sqrt{1+p - 2e^{-4\beta}p} + \frac{1}{2}\sqrt{e^{-4\beta}(e^{4\beta} + 2p + e^{4\beta}p)}.$$  

(22)
where,

\[
U_1 = 2 \text{ArcTanh} \left[ e^{-4\beta p} \right] + \log \left[ 1 + p - 2e^{-4\beta p} \right] - \log \left[ 1 + p + 2e^{-4\beta p} \right],
\]

\[
U_2 = -2(1 + p)\log(1 + p) + (1 + p)\log \left[ 1 + p - 2e^{-4\beta p} \right],
\]

\[
U_3 = \log \left[ 1 - e^{-4\beta p} \right] + \log \left[ 1 + e^{-4\beta p} \right],
\]

\[
U_4 = (1 + p)\log \left[ 1 + p + 2e^{-4\beta p} \right].
\]

Fig. 5 investigate the dynamics for the left, right hands sides of the entropic uncertainty relations, tightness and concurrence in two qubits coupled with two independent environments characterized by two OU noise sources. The qualitative dynamics of the entropic uncertainty on the left and right hands appear to be identical. However, the inequality remains quantitatively valid, and the two sides are not equal. It can be validated by looking at the U(\tau) findings, which show that there is a difference between the two sides, but it is lower than that seen in CQN configurations. The dynamical mappings of L(\tau) and R(\tau) remained growing functions of entropy. On the other hand, U(\tau) and C(\tau) remained decreasing functions of tightness and entanglement. The present
dynamical map of entanglement under IQN configurations appears to be more suppressed than \( C(\tau) \) dynamics in CQN configurations. As a result, entanglement is better retained in CQN arrangements than in IQN configurations. The rate of entropy rise and entanglement decrease remained proportionate. Entanglement lags the entropy speed, which is a more logical conclusion. As a result, it may be deduced that relative entropy causes disentanglement in the two qubits. The entropy rises monotonically until it reaches its maximum value and then becomes constant. \( C(\tau) \) has a monotonic qualitative dynamical behavior that matches \( L(\tau), R(\tau), \) and \( U(\tau) \). The new results show a lower degree of uncertainty than the tripartite entropic uncertainty explored for the three-qubit GHZ state [46]. In contrast, the GHZ state remained entangled depending on the type of system-environment link, but the current bipartite Werner state becomes completely separable. \( L(\tau), R(\tau), \) and \( U(\tau) \) all have a monotonic qualitative dynamical behaviour that is compatible with \( C(\tau) \). Different types of quantum systems remained entangled depending on the type of system-environment interaction given in [15, 28, 36–38], but the current bipartite Werner state becomes completely disentangled.

1. **Entropic uncertainty and concurrence against noise parameter and purity factor in IQN configuration**

![Graphs showing dynamics of R(τ), L(τ), U(τ), and C(τ) against different values of g in two qubits state](image)

FIG. 6. Dynamics of \( R(\tau) \) (a), \( L(\tau) \) (b), \( U(\tau) \) (c) and \( C(\tau) \) (d) against different values of \( g \) in two qubits state, \( |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) initially prepared in the state \( \rho_o \) given in Eq.(11) for independent qubit-noise configuration when \( p = 1 \).

When the system is linked to two independent classical environments, the left and right hands of the entropic uncertainty relationship, as well as tightness and concurrence, are presented in Fig.6. The new findings are like those in Fig.2, although there are some quantitative differences.
As can be seen from $R(\tau)$ and $L(\tau)$, the present case’s entropic uncertainty growing rate is slower than the CQN configuration scenario. The dynamical maps of the tripartite states under classical noises, on the other hand, demonstrate that the entropic uncertainty rise is much smaller [46]. At lower g values, both the $L(\tau)$ and $R(\tau)$ curves reached the ultimate saturation heights. We found the two sides deviating from each other in the curves with an insignificant narrow gap. Compared to [45], the present difference between the two sides is smaller. Even though no entropic revivals were observed in our dynamic setup, we noticed that the occurrence of revivals in the dynamical maps is due to the suppression of entropy, as shown in [49]. Although the preservation intervals are longer in this case, they are consistent with the results obtained for the two-qubit Bell’s state dynamics when subjected to classical conditions with static noise [25]. Apart from that, as seen in Fig.3, entanglement decreases and entropy increases faster for larger values of g. We noticed that the values of the entropic uncertainty relation are lower for smaller g values. As can be seen, the highest elevation of the $U(\tau)$ is $4 \times 10^{-5}$, which is a tiny extent.

![Dynamics of $U(\tau)$ (a), $R(\tau)$ (b), $L(\tau)$ (c) and $C(\tau)$ (d) against different values of p in two qubits state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ initially prepared in the state $\rho_o$ given in Eq.(11) for independent qubit-noise configuration when $g = 10^{-1}$.](image_url)

In Fig.8, we illustrate the dynamics of the two sides of entropy relations, tightness and concurrence in bipartite Werner state coupled to local random fields with OU noise against various purity factor values. The current findings can be traced back to different values of p in Fig.4, and it can be seen that as p decreases, the initial entropy increases. Compared to the CQN set-up in the relevant values of p, the entropic increase in the current case is negligible. We discovered the lowest entropy in the two qubits at $p = 0.99$ and the maximum at $p = 0.1$ in the purity factor limits. $R(\tau) > L(\tau)$, confirming earlier observations. The measurements of $U(\tau)$, $R(\tau)$, and $L(\tau)$ agree,
demonstrating that the initial uncertainty is lowest for $p = 0.99$ and grows for all other values. When the difference between $R(\tau)$ and $L(\tau)$ approaches zero, the tightness curves eventually reach a minimal saturation level. The entanglement preservation duration appears to be influenced by the initially encoded entanglement. Due to the decoherence and entropic nature of the classical environments, the entanglement quickly fades as the initially encoded entanglement lowers. The current dynamical maps in IQN configuration offered less entropy and a longer entanglement retention time than CQN configuration, showing that it is a good resource for practical quantum information processing. This contradicts the findings of the tripartite non-local correlations which remained more robust and preserved in the presence of a common noise source as compared to the bipartite entangled state given in [24, 28, 36, 51].

C. Purity factor, degree of entropic uncertainty relations and entanglement

The dynamics of the right and left hands sides of the uncertainty relations, as well as tightness and concurrence as functions of the purity factor of the two qubit entangled Werner type state, are discussed in this section. The $R(p)$ and $L(p)$ are both maximum at $p = 0$ and minimum at $p = 1$, according to the current results. This means that entropy reaches its maximum when the two-qubit Werner state becomes completely separable. However, in the CQN configuration, both the $L(p)$ and $R(p)$ entropies are higher than in the IQN setup. The difference between the $L(p)$ and $R(p)$ in the range $0.9 \leq p \leq 0.15$ is smaller than the top and lower limits of $p$. The results of $U(p)$ and $C(p)$ are discussed in this section. 

FIG. 8. Dynamics of $R(p)$ (a), $L(p)$ (b), $U(p)$ (c) and $C(p)$ (d) within full range $p$ in two qubits state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ initially prepared in the state $\rho_0$ given in Eq.(11) for common (non-dashed) independent qubit-noise configuration (dashed) when $\tau = 1$. 

$p = 1$, according to the current results. This means that entropy reaches its maximum when the two-qubit Werner state becomes completely separable. However, in the CQN configuration, both the $L(p)$ and $R(p)$ entropies are higher than in the IQN setup. The difference between the $L(p)$ and $R(p)$ in the range $0.9 \leq p \leq 0.15$ is smaller than the top and lower limits of p. The results of $U(p)$ and $C(p)$ are discussed in this section.
depict similar results as that of $L(p)$ and $R(p)$. At $p = 0$, the state of the $C(p)$ becomes completely separable, and at $p = 1$, it becomes maximally entangled. The variance in entanglement is reduced for $p > 0.9$ and $p < 0.15$, as expected from the entropic uncertainty relations. Compared to the CQN configuration, the entanglement in the IQN configuration appears to be better preserved. As a result, the IQN configuration may be used to simulate quantum information processing protocols realistically.

IV. Conclusion

We study the dynamics of entropy, tightness and entanglement in two qubits coupled to classical fields described by the Ornstein-Uhlenbeck process. The two qubits are created initially in an entangled Werner state regulated by a purity factor. In addition, we consider two different system-environment coupling schemes, namely, common qubit-noise and independent qubit-noise configurations. We use the right and left-hand sides of the uncertainty relations, tightness, and concurrence to study the entropy, relevant uncertainty, and entanglement of the two non-interacting qubits.

Entropy increased due to the noisy action of the classical fields in the dynamical map of two qubits. The discrepancy between the left and right sides of the entropic uncertainty relations is controlled through parameter optimization. The difference between the two sides develops when the noise parameter $g$ is raised. In the case of purity factor $p$, some unusual behaviour has been observed. In the ranges of $0.9 \geq p \geq 0.15$, the entropic uncertainty gap between the two sides is larger, but it becomes exceptionally small at $p > 0.9$ and $p < 0.15$. Our findings show that the left-hand side of the entropic uncertainty relation is more effective than the right-hand side, which is consistent with the findings given in [45]. In the case of concurrence, entanglement decreases as the entropy between the qubits increases and it remains a crucial cause for the disentanglement of the two qubits. The entanglement preservation intervals are controlled by the noise parameters, which decrease proportionately as $g$ rises and increase as $p$ grows. We noticed that the preservation intervals are controlled jointly by the noise parameter $g$ and purity factor of the state $p$. However, the roles of both the parameters in preserving entanglement and inclining entropy are found the opposite.

Finally, under any parameter optimisation values, the two-qubit state eventually reaches separability, with no ultimate solution to avoid the corresponding disentanglement and decoherence. It’s worth mentioning, however, that the phase factors of current systems can be leveraged to accomplish longer entanglement preservation, particularly when $g = 10^{-3}$ is employed. In contrast to the bipartite and tripartite states reported in [12, 15, 23, 25, 45], entanglement in the current study has been maintained for extended periods. The entropy and uncertainty relations can also be depicted in a similar way. Moreover, the entropic uncertainty relations and concurrence all showed
a monotonic behaviour with no revivals.

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