The Hilbert transform, rearrangements, and logarithmic determinants

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This is an extended version of notes prepared for the talk at the conference “Rajchman-Zygmund-Marcinkiewicz 2000”. They are based on recent papers [13] and [15] (see also [14] and [16]). The authors thank Professor Zelazko for the invitation to participate in the proceedings of this conference.

§1

Let \( g \) be a bounded measurable real-valued function on \( \mathbb{R} \) with a compact support.

We shall use the following notations:

The Hilbert transform of \( g \):
\[
(\mathcal{H}g)(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{t - \xi} dt ,
\]
the prime means that the integral is understood in the principal value sense at the point \( t = \xi \).

The (signed) distribution function of \( g \):
\[
N_g(s) = \begin{cases} 
\text{meas} \{ x : g(x) > s \}, & \text{if } s > 0; \\
-\text{meas} \{ x : g(x) < s \}, & \text{if } s < 0.
\end{cases}
\]

The (signed) decreasing rearrangement of \( g \): \( g_d \) is defined as the distribution function of \( N_g \): \( g_d = N_{N_g} \).

Less formally, the functions \( N_g \) and \( g_d \) can be also defined by the following properties: they are non-negative and non-increasing for \( s > 0 \), non-positive and non-increasing for \( s < 0 \), and
\[
\int_{\mathbb{R}} \Phi(g(t)) \, dt = \int_{\mathbb{R}} \Phi(s) \, dN_g(s) = \int_{\mathbb{R}} \Phi(g_d(t)) \, dt ,
\]

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for any function \( \Phi \) such that at least one of the three integrals is absolutely convergent.

We shall use notation \( A \lesssim B \), when \( A \leq C \cdot B \) for a positive numerical constant \( C \). We shall write \( A \lesssim_{\lambda} B \), if \( C \) in the previous inequality depends on the parameter \( \lambda \) only.

**Theorem 1.1** Let \( g \) be a bounded measurable real-valued function with a compact support. Then

\[
\|Hg_d\|_{L^1} \leq 4 \|Hg\|_{L^1}.
\]

Hereafter, \( L^1 \) always means \( L^1(\mathbb{R}) \).

**Remarks:**

1.3 Estimate (1.2) can be extended to a wider class of functions after an additional regularization of the Hilbert transform \( Hg_d \) (see §3 below).

1.4 Probably, the constant 4 on the RHS is not sharp. However, Davis’ discussion in [3] suggests that (1.2) ceases to hold without this factor on the RHS.

1.5 Theorem 1.1 yields a result of Tsereteli [19] and Davis [3]: if \( g \in \text{Re}H^1 \), then \( g_d \) is also in \( \text{Re}H^1 \), and \( \|Hg_d\|_{L^1} \lesssim \|g\|_{\text{Re}H^1} \), where \( \text{Re}H^1 \) is the real Hardy space on \( \mathbb{R} \).

1.6 Theorem 1.1 can be extended to functions defined on the unit circle \( \mathbb{T} \). Let \( g(t) \) be a bounded function on \( \mathbb{T} \), \( g_d \) be its signed decreasing rearrangement, and \( \tilde{g} \) be a function conjugate to \( g \):

\[
\tilde{g}(t) = \frac{1}{2\pi} \int_{\mathbb{T}} g(\xi) \cot \frac{t - \xi}{2} d\xi.
\]

Then

\[
\|\tilde{g}_d\|_{L^1(\mathbb{T})} \leq 4 \|\tilde{g}\|_{L^1(\mathbb{T})}.
\]

Juxtapose this estimate with Baernstein’s inequality [4]:

\[
\|\tilde{g}\|_{L^1(\mathbb{T})} \leq \|\tilde{g}_s\|_{L^1(\mathbb{T})},
\]

where \( g_s \) is the symmetric decreasing rearrangement of \( g \). In particular, if \( g_s \) has a conjugate in \( L^1 \), then any rearrangement of \( g \) has a conjugate in \( L^1 \), and if some rearrangement of \( g \) has a conjugate in \( L^1 \), then the conjugate of \( g_d \) is in \( L^1 \). We are not aware of a counterpart of Baernstein’s inequality for the Hilbert transform and the \( L^1(\mathbb{R}) \)-norm.
Here, we shall prove Theorem 1.1. WLOG, we assume that
\[
\int_{\mathbb{R}} g(t) \, dt = 0,
\]
otherwise
\[
(\mathcal{H}g)(\xi) = -\frac{1}{\pi \xi} \int_{\mathbb{R}} g(t) \, dt + O(1/\xi^2), \quad \xi \to \infty,
\]
and the $L^1$-norm on the RHS of (1.2) is infinite.

**The first reduction**: instead of (1.2), we shall prove inequality
\[
||\mathcal{H}N_{g}\||_{L^1} \leq 2||\mathcal{H}g||_{L^1},
\]
then its iteration gives (1.2).

We introduce a (regularized) logarithmic determinant of $g$:
\[
u_g(z) \overset{\text{def}}{=} \int_{\mathbb{R}} K(zg(t)) \, dt, \quad K(z) = \log|1 - z| + \text{Re}(z).
\]
This function is subharmonic in $\mathbb{C}$ and harmonic outside of $\mathbb{R}$.

**List of properties of $\nu_g$**:
Since $g$ is a bounded function with a compact support,
\[(2.3a) \quad \nu_g(z) = O(|z|^2), \quad z \to 0,\]
and by (2.1)
\[(2.3b) \quad \nu_g(z) = \int_{\mathbb{R}} \log|1 - zg(t)| \, dt = O(|z|), \quad z \to \infty.\]

In particular,
\[(2.3c) \quad \int_{\mathbb{R}} \frac{|\nu_g(x)|}{x^2} < \infty.\]

Next,
\[(2.4) \quad \int_{\mathbb{R}} \frac{\nu_g(x)}{x^2} \, dx = 0.\]
This follows from the Poisson representation:

\[ u_g(iy) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{u_g(x)}{x^2 + y^2} \, dy, \quad y > 0. \]

Dividing by \( y \), letting \( y \to 0 \), and using (2.3a), we get (2.4).

Further,

\[ u_g(1/t) = -\pi (\mathcal{H}N_g)(t). \quad (2.5) \]

Indeed, integrating by parts and changing variables, we obtain for real \( x \)'s:

\[
\begin{align*}
 u_g(x) &= \int_{\mathbb{R}} \log |1 - xs| \, dN_g(s) \\
 &= x \int_{\mathbb{R}} \frac{N_g(s)}{1 - xs} \, ds \\
 &= -\pi (\mathcal{H}N_g)(1/x).
\end{align*}
\]

We have done the second reduction: Instead of (2.2), we shall prove inequality

\[ \int_{\mathbb{R}} \frac{u_g(x)}{x^2} \, dx \leq \pi \| \mathcal{H}g \|_{L^1}. \quad (2.6) \]

Then combining (2.4) and (2.6), we get (2.2).

Now, we set

\[ f(t) = g(t) + i(\mathcal{H}g)(t). \]

This function has an analytic continuation into the upper half-plane:

\[ f(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{g(t)}{t - z} \, dt. \]

We define the regularized logarithmic determinant of \( f \) by the equation

\[ u_f(z) = \int_{\mathbb{R}} K(zf(t)) \, dt. \quad (2.7) \]

The positivity of this subharmonic function is central in our argument:

**Lemma 2.8 (cf. [4])**

\[ u_f(z) \geq 0, \quad z \in \mathbb{C}. \]
Proof of Lemma 2.8: It suffices to consider such $z$’s that all solutions of the equation $zf(w) = 1$ are simple and not real. Then

$$u_f(z) = \text{Re} \left\{ \int_{\mathbb{R}} \left[ \log(1 - zf(t)) + zf(t) \right] dt \right\}$$

$$= \text{Re} \left\{ z^2 \int_{\mathbb{R}} \frac{tf(t)f'(t)}{1 - zf(t)} dt \right\}$$

$$= \text{Re} \left\{ 2\pi iz^2 \sum_{\{w: zf(w) = 1\}} \text{Res}_w \left( \frac{zf(\zeta)f'(\zeta)}{1 - zf(\zeta)} \right) \right\}$$

$$= 2\pi \sum_{\{w: zf(w) = 1\}} \text{Im}(w) \geq 0.$$ 

The application of the Cauchy theorem is justified since $f(\zeta) = O(1/\zeta^2)$ when $\zeta \to \infty$, $\text{Im}(\zeta) \geq 0$. Done.

To complete the proof of the theorem, we shall use an argument borrowed from the perturbation theory of compact operators [5]. We use auxiliary functions $f_1 = g + i|\mathcal{H}g|$ and

$$u_1(z) = \int_{\mathbb{R}} \left| \log \left| \frac{1 - zg(t)}{1 - zf_1(t)} \right| \right| dt.$$ 

Then on the real axis

$$u_g(x) = u_1(x) + u_f(x), \quad x \in \mathbb{R},$$

so that $u_g(x) \geq u_1(x)$, or $u_g(x) \leq u_1(x) = -u_1(x)$, since $u_1(x) \leq 0$, $x \in \mathbb{R}$.

Next, we need an elementary inequality: if $w_1, w_2$ are complex numbers such that $\text{Re}(w_1) = \text{Re}(w_2)$ and $|\text{Im}(w_1)| \leq \text{Im}(w_2)$, then for all $z$ in the upper half-plane,

$$\left| \frac{1 - zw_1}{1 - zw_2} \right| < 1.$$ 

Due to this inequality the function $u_1$ is non-positive in the upper half-plane. Since this function is harmonic in the upper half-plane, we obtain

$$\int_{\mathbb{R}} \frac{u_g(x)}{x^2} dx \leq -\int_{\mathbb{R}} \frac{u_1(x)}{x^2} dx$$

$$= -\lim_{y \to 0} \int_{\mathbb{R}} \frac{u_1(x)}{x^2 + y^2} dx$$

$$\leq -\pi \lim_{y \to 0} \frac{u_1(iy)}{y}.$$
\[- \pi \lim_{y \to 0} \frac{1}{y} \int_{\mathbb{R}} \log \left| \frac{1 - iyg(t)}{1 - iyg(t) + y|Hg(t)|} \right| \, dt \]
\[= \pi \int_{\mathbb{R}} |(Hg)(t)| \, dt.\]

This proves (2.6) and therefore the theorem. \(\square\)

§3

Here, we will formulate a fairly complete version of estimate (2.2). The proof given in [15] follows similar lines as above, however is essentially more technical.

Now, we start with a real-valued measure \(d\eta\) of finite variation on \(\mathbb{R}\), and denote by \(g = H\eta\) its Hilbert transform. By \(||\eta||\) we denote the total variation of the measure \(d\eta\) on \(\mathbb{R}\). Let \(R_g = H^{-1}N_g\) be a regularized inverse Hilbert transform of \(N_g\):

\[R_g(t) \overset{def}{=} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|s|>\epsilon} N_g(s) \frac{1}{t-s} \, ds.\]

The integral converges at infinity due to the Kolmogorov weak \(L^1\)-type estimate

\[N_g(s) \lesssim ||\eta||/s, \quad 0 < s < \infty.\]

Existence of the limit when \(\epsilon \to 0\) (and \(t \neq 0\)) follows from the Titchmarsh formula [18] (cf. [15]):

\[
\lim_{s \to 0} sN_g(s) = \frac{\eta(\mathbb{R})}{\pi}.
\]

**Theorem 3.1** Let \(d\eta\) be a real measure supported by \(\mathbb{R}\). Then

\[\int_{\mathbb{R}} R_g^+(t) \, dt \leq ||\eta_{h.c.}||, \quad (3.2)\]

\[\int_{\mathbb{R}} R_g^-(t) \, dt \leq ||\eta|| - |\eta(\mathbb{R})|, \quad (3.3)\]

and

\[\int_{\mathbb{R}} R_g(t) \, dt = |\eta(\mathbb{R})| - ||\eta_{\text{sing}}||. \quad (3.4)\]

**Corollary 3.5** The function \(R_g\) always belongs to \(L^1\) and its \(L^1\)-norm does not exceed \(2||\eta||\).
The classical Boole theorem says that if \( d\eta \) is non-negative and pure singular, then \( N_g(s) = \eta(\mathbb{R})/s \), and therefore \( R_g \) vanishes identically. The next two corollaries can be viewed as quantitative generalizations of this fact:

**Corollary 3.6** If \( d\eta \geq 0 \), then \( R_g(t) \) is non-negative as well, and \( \|R_g\|_{L^1} = \eta_{a.c.}(\mathbb{R}) \).

**Corollary 3.7** If \( d\eta \) is pure singular, then \( R_g(t) \) is non-positive and \( \|R_g\|_{L^1} = \|\eta\| - |\eta(\mathbb{R})| \).

For other recent results obtained with the help of the logarithmic determinant we refer to [8], [14] and [16].

§ 4

In § 2 we used the subharmonic function technique for proving a theorem about the Hilbert transform. The idea of logarithmic determinants also provides us with a connection which works in the opposite direction: starting with a known result about the Hilbert transform, one arrives at a plausible conjecture about a non-negative subharmonic function in \( \mathbb{C} \) represented by a canonical integral of genus one. For illustration, we consider a well known inequality

\[
(4.1) \quad m_f(\lambda) \lesssim \frac{1}{\lambda^2} \int_0^\lambda s m_g(s) ds + \frac{1}{\lambda} \int_\lambda^\infty m_g(s) ds, \quad 0 < \lambda < \infty,
\]

where \( f = g + iHg \), \( g \) is a test function on \( \mathbb{R} \), \( m_f(\lambda) = \text{meas}\{ |f| \geq \lambda \} \), and \( m_g(\lambda) = \text{meas}\{ |g| \geq \lambda \} = N_g(\lambda) - N_g(-\lambda) \). Inequality (4.1) contains as special cases Kolmogorov’s weak \( L^1 \)-type inequality \( \lambda m_f(\lambda) \lesssim \|g\|_{L^1} \), and M. Riesz’ inequality \( \|f\|_{L^p} \lesssim \|g\|_{L^p} \), \( 1 < p \leq 2 \). Inequality (4.1) can be justly attributed to Marcinkiewicz. He formulated his general interpolation theorem for sub-linear operators in [21], the proof was supplied by Zygmund in [22] with reference to Marcinkiewicz’ letter. Its main ingredient is a decomposition \( g = g\chi_{|g|<\lambda} + g\chi_{|g|\geq\lambda} \), where \( \chi_E \) is a characteristic function of a set \( E \). This decomposition immediately proves (4.1), see [4, Section V.C.2].

Define a logarithmic determinant \( u_f \) of genus one as in (2.7), and denote by \( d\mu_f \) its Riesz measure (i.e. \( 1/(2\pi) \) times the distributional Laplacian \( \Delta u_f \)). For each Borelian subset \( E \subset \mathbb{C} \), \( \mu_f(E) = \text{meas}(f^{-1}E^*) \), where \( E^* = \{ z : z^{-1} \in E \} \), and \( f^{-1}E^* \) is the full preimage of \( E \) under \( f \). Now, we can express the RHS and the LHS of inequality (4.1) in the terms of \( \mu_f \).

7
First, observe that the counting function of $\mu_f$ equals

$$
\mu_f(r) \overset{\text{def}}{=} \mu_f\{|z| \leq r\} = \text{meas}\{|f(t)| \geq r^{-1}\} = m_f(r^{-1}).
$$

In order to write down $m_g$ in terms of $\mu_f$, we introduce the Levin-Tsuji counting function (cf. [20], [6]):

$$
n_f(r) = \mu_f\{|z - ir/2| \leq r/2\} + \mu_f\{|z + ir/2| \leq r/2\}
= \mu_f\{|\text{Im}(z^{-1})| \geq r^{-1}\} = \text{meas}\{|g| \geq r^{-1}\} = m_g(r^{-1}).
$$

Now, we can rewrite (4.1) in the form:

$$
(4.2) \quad \mu_f(r) \lesssim r \int_0^r \frac{n_f(t)}{t^2} dt + r^2 \int_r^\infty \frac{n_f(t)}{t^3} dt, \quad 0 < r < \infty.
$$

We shall show that (4.2) persists for any non-negative in $\mathbb{C}$ subharmonic function represented by a canonical integral of genus one. In this case the operator $g \mapsto \mathcal{H}g$ disappears, and the Marcinkiewicz argument seems to be unapplicable anymore.

Let

$$
(4.3) \quad u(z) = \int_{\mathbb{C}} K(z/\zeta) \, d\mu(\zeta),
$$

where $d\mu$ is a non-negative locally finite measure on $\mathbb{C}$ such that

$$
(4.4) \quad \int_{\mathbb{C}} \min\left(\frac{1}{|\zeta|}, \frac{1}{|\zeta|^2}\right) \, d\mu(\zeta) < \infty.
$$

Subharmonic functions represented in this form are called canonical integrals of genus one.

Let $M(r, u) = \max_{|z| \leq r} u(z)$. A standard estimate of the kernel

$$
K(z) \lesssim \frac{|z|^2}{1 + |z|}, \quad z \in \mathbb{C},
$$

yields Borel’s estimate (cf. [8, Chapter II])

$$
M(r, u) \lesssim r \int_0^r \frac{\mu(t)}{t^2} dt + r^2 \int_r^\infty \frac{\mu(t)}{t^3} dt.
$$

In particular,

$$
M(r, u) = \begin{cases} 
\text{o}(r), & r \to 0 \\
\text{o}(r^2), & r \to \infty.
\end{cases}
$$
**Theorem 4.5** Let \( u(z) \geq 0 \) be a canonical integral (4.3) of genus one, then

\[
M(r, u) \lesssim r \int_0^r \frac{n(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{n(t)}{t^3} \, dt.
\]

The RHS of (4.6) does not depend on the bound for the integral (4.4), this makes the result not so obvious. By Jensen’s formula, \( \mu(r) \leq M(\varepsilon r, u) \), so that \( \mu(r) \lesssim \) the RHS of (4.6). As a corollary we immediately obtain (4.2) and the Marcinkiewicz estimate (4.1).

§5

Here we sketch the proof of Theorem 4.5.

We shall need two auxiliary lemmas. The first one is a version of the Levin integral formula without remainder term (cf. [10, Section IV.2], [6, Chapter 1]). The proof can be found in [13].

**Lemma 5.1** Let \( v \) be a subharmonic function in \( \mathbb{C} \) such that

\[
\int_0^{2\pi} |v(re^{i\theta})| |\sin \theta| \, d\theta = o(r), \quad r \to 0,
\]

and

\[
\int_0^{2\pi} \frac{n(t) + v^-(t) + v^-(t)}{t^2} \, dt < \infty.
\]

Then

\[
\frac{1}{2\pi} \int_0^{2\pi} v(Re^{i\theta}|\sin \theta|) \frac{d\theta}{R \sin^2 \theta} = \int_0^R \frac{n(t)}{t^2} \, dt, \quad 0 < R < \infty,
\]

where \( n(t) \) is the Levin-Tsuji counting function, and the integral on the LHS is absolutely convergent.

The next lemma was proved in a slightly different setting in [11, §2], see also [8, Lemma 5.2, Chapter 6].

**Lemma 5.5** Let \( v(z) \) be a subharmonic function in \( \mathbb{C} \) satisfying conditions (5.2) and (5.3) of the previous lemma, let

\[
T(r, v) = \frac{1}{2\pi} \int_0^{2\pi} v^+(re^{i\theta}) \, d\theta
\]

be its Nevanlinna characteristic function, and let

\[
\mathcal{T}(r, v) = \frac{1}{2\pi} \int_0^{2\pi} v^+(re^{i\theta}|\sin \theta|) \frac{d\theta}{r \sin^2 \theta}
\]
be its Tsuji characteristic function. Then

\[ T(r,v) \leq \frac{\mathcal{I}(r,v)}{r^2}, \quad 0 < R < \infty. \]

For the reader’s convenience, we recall the proof. Consider the integral

\[ I(R) = \frac{1}{2\pi} \int \int_{\Omega_R} \frac{v^+(re^{i\theta})}{r^3} dr d\theta, \]

where \( \Omega_R = \{ z = re^{i\theta} : r > R \} \). Introducing a new variable \( \rho = r/|\sin \theta| \) instead of \( r \), we get

\[ I(R) = \int_{\Omega_R} \frac{v^+(\rho e^{i\theta})}{\rho^2 \sin \theta} \frac{d\theta}{2\pi} \int_0^{2\pi} v^+(\rho |\sin \theta| e^{i\theta}) \frac{d\theta}{\rho^2 \sin \theta} = \int_{\Omega_R} \frac{T(\rho, v)}{\rho^2} d\rho. \]

Now, consider another integral

\[ J(R) = \frac{1}{2\pi} \int \int_{K_R} \frac{v^+(re^{i\theta})}{r^3} dr d\theta, \]

where \( K_R = \{ z : |z| > R \} \). Since \( K_R \subset \Omega_R \), we have \( J(R) \leq I(R) \). Taking into account that

\[ J(R) = \int_{K_R} \frac{dr}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} v^+(re^{i\theta}) d\theta \right\} = \int_{K_R} \frac{T(r,v)}{r^3} dr \]

we obtain (5.6).

Proof of Theorem 4.5: Due to Borel’s estimate condition (5.2) is fulfilled. Due to non-negativity of \( u \) and (4.4), condition (5.3) holds as well. Using monotonicity of \( T(r,u) \), Lemma 5.5, and then Lemma 5.1, we obtain

\[ \frac{T(R,u)}{R^2} \leq 2 \int R \frac{T(r,u)}{r^2} dr \]

\[ \leq 2 \int R \frac{\mathcal{I}(r,u)}{r^2} dr \]

\[ = 2 \int R \frac{dr}{r^2} \int_0^r \frac{n(t)}{t^2} dt \]

\[ = \frac{2}{R} \int_0^R \frac{n(t)}{t^2} dt + 2 \int R \frac{n(t)}{t^3} dt. \]
The inequality $M(r, u) \leq 3T(2r, u)$ completes the job. □

§6

Non-negativity of $u(z)$ in $\mathbb{C}$ seems to be too strong assumption, a more natural one is non-negativity of $u(x)$ on $\mathbb{R}$.

**Theorem 6.1** Let $u(z)$ be a canonical integral (4.3) of genus one, and let $u(x) \geq 0$, $x \in \mathbb{R}$. Then

\begin{equation}
(6.2) \quad M(r, u) \lesssim r^2 \left[ \int_r^\infty \sqrt{\frac{n^*(t)}{t^2}} \frac{dt}{t^3} \right]^2,
\end{equation}

where

\begin{equation}
(6.3) \quad n^*(r) = r \int_0^r \frac{n(t)}{t^2} dt + r^2 \int_r^\infty \frac{n(t)}{t^3} \left(1 + \log \frac{t}{r}\right) dt
\end{equation}

The proof of Theorem 6.1 is given in [13]. The method of proof differs from that of Theorem 4.5, and is more technical than one would wish.

Fix an arbitrary $\epsilon > 0$. Then by the Cauchy inequality

\[ \left[ \int_r^\infty \frac{\sqrt{n^*(t)}}{t^2} dt \right]^2 = \left[ \int_r^\infty \frac{\sqrt{1 + \log^{1+\epsilon} \frac{t}{r}} n^*(t)}{t^{3/2}} \frac{dt}{t^{1/2} \sqrt{1 + \log^{1+\epsilon} \frac{t}{r}}} \right]^2 \]

\[ \lesssim \epsilon \int_r^\infty \frac{n^*(t)}{t^3} \left(1 + \log^{1+\epsilon} \frac{t}{r}\right) \frac{dt}{r} \]

\[ \lesssim \epsilon \frac{1}{r} \int_0^r \frac{n(s)}{s^2} ds + \int_r^\infty \frac{n(s)}{s^3} \left(1 + \log^{3+\epsilon} \frac{s}{r}\right) ds. \]

Thus we get

**Corollary 6.4** For each $\epsilon > 0$,

\begin{equation}
(6.5) \quad M(r, u) \lesssim \epsilon r \int_0^r \frac{n(t)}{t^2} dt + r^2 \int_r^\infty \frac{n(t)}{t^3} \left(1 + \log^{3+\epsilon} \frac{t}{r}\right) dt.
\end{equation}

Estimate (6.5) is slightly weaker than (4.6); however, it suffices for deriving inequalities of M. Riesz and Kolmogorov. Using Jensen’s estimate $\mu(r) \leq M(\epsilon r, u)$, we arrive at
Corollary 6.6 The following inequalities hold for canonical integrals of genus one which are non-negative on the real axis:

M. Riesz-type estimate:

\begin{equation}
\int_0^\infty \frac{\mu(r)}{r^{p+1}} \, dr \lesssim_p \int_0^\infty \frac{n(r)}{r^{p+1}} \, dr, \quad 1 < p < 2,
\end{equation}

weak \((p, \infty)\)-type estimate:

\begin{equation}
\sup_{r \in (0, \infty)} \frac{\mu(r)}{r^p} \lesssim_p \sup_{r \in (0, \infty)} \frac{n(r)}{r^p}, \quad 1 < p < 2,
\end{equation}

and Kolmogorov-type estimate:

\begin{equation}
\sup_{r \in (0, \infty)} \frac{\mu(r)}{r} \lesssim \int_0^\infty \frac{n(r)}{r^2} \, dr.
\end{equation}

Remark 6.10 If the integral on the RHS of (6.9) is finite, then \(u(z)\) has positive harmonic majorants in the upper and lower half-planes which can be efficiently estimated near the origin and infinity, see [13, Theorem 3].

§7

Here we mention several questions related to our results.

7.1 How to distinguish the logarithmic determinants (2.7) of \(f = g + i\mathcal{H}g\) from other canonical integrals (4.3) which are non-negative in \(C\)? In other words, let \(dm_f\) be a distribution measure of \(f\); i.e. a locally-finite non-negative measure in \(C\) defined by \(m_f(E) = \text{meas}\{t \in \mathbb{R} : f(t) \in E\}\) for an arbitrary borelian subset \(E \subset \mathbb{C}\). It should to be interesting to find properties of \(dm_f\) which do not follow only from non-negativity of the subharmonic function \(u_f(z)\). A similar question can be addressed to analytic functions \(f(z)\) of Smirnov’s class in the unit disk.

7.2 Let \(X\) be a rearrangement invariant Banach space of measurable functions on \(\mathbb{R}\). That is, the norm in \(X\) is the same for all rearrangements of \(|g|\), and \(||g_1||_X \leq ||g_2||_X\) provided that \(|g_1| \leq |g_2|\) everywhere. For which spaces does the inequality

\[||\mathcal{H}g_d||_X \leq C_X||\mathcal{H}g||_X\]

hold? This question is interesting only for spaces \(X\) where the Hilbert transform is unbounded; i.e. for spaces which are close in a certain sense either to \(L^1\) or to \(L^\infty\). Some natural restrictions on \(X\) can be assumed: the
linear span of the characteristic functions $\chi_E$ of bounded measurable subsets $E$ is dense in $X$, and $||\chi_E||_X \to 0$, when $\text{meas}(E) \to 0$, see [2, Chapter 3].

**7.3** We do not know how to extend estimate (1.2) (as well as (1.8)) to more general operators like the maximal Hilbert transform, the non-tangential maximal conjugate harmonic function, or Calderon-Zygmund operators. A similar question can be naturally posed for the Riesz transform [17].

**7.4** Does Marcinkiewicz-type inequality (4.6) hold under the assumption that a canonical integral $u$ of genus one is non-negative on $\mathbb{R}$? According to a personal communication from A. Ph. Grishin, the exponent $3 + \epsilon$ can be improved in (6.5). However, his technique also does not allow to get rid at all of the logarithmic factor.

**7.5** Let $u(z)$ be a non-negative subharmonic function in $\mathbb{C}$, $u(0) = 0$. As before, by $\mu(r)$ and $n(r)$ we denote the conventional and the Levin-Tsuji counting functions of the Riesz measure $d\mu$ of $u$. Assume that $\mu(r) = o(r)$, $r \to 0$. This condition is needed to exclude from consideration the function $u(z) = |\text{Im}(z)|$ which is non-negative in $\mathbb{C}$ and harmonic outside of $\mathbb{R}$. Let $\mathcal{M}$, $\mathcal{M}(0) = 0$, $\mathcal{M}(\infty) = \infty$, be a (regularly growing) majorant for $n(r)$. What can be said about the majorant for $\mu(r)$? If $\mathcal{M}(r) = r^p$, $1 < p < \infty$, then we know the answer:

$$\sup_{r \in (0, \infty)} \frac{\mu(r)}{r^p} \leq C_p \sup_{r \in (0, \infty)} \frac{n(r)}{r^p},$$

and

$$\int_0^\infty \frac{\mu(r)}{r^{p+1}} \, dr \leq C_p \int_0^\infty \frac{n(r)}{r^{p+1}} \, dr.$$  

It is more difficult and interesting to study majorants $\mathcal{M}(r)$ which grow faster than any power of $r$ when $r \to \infty$, and decay to zero faster than any power of $r$ when $r \to 0$. The question might be related to the classical Carleman-Levinson-Sjoberg “log log-theorem”, and the progress may lead to new results about the Hilbert transform.

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