APPLYING POLICY ITERATION FOR TRAINING
RECURRENT NEURAL NETWORKS
TECHNICAL REPORT

ISTVÁN SZITA AND ANDRÁS LÓRINCZ

Abstract. Recurrent neural networks are often used for learning time-series data. Based on a few assumptions we model this learning task as a minimization problem of a nonlinear least-squares cost function. The special structure of the cost function allows us to build a connection to reinforcement learning. We exploit this connection and derive a convergent, policy iteration-based algorithm. Furthermore, we argue that RNN training can be fit naturally into the reinforcement learning framework.

Keywords: recurrent neural networks, policy iteration, sequence learning, reinforcement learning

1. Introduction

Recurrent neural networks (RNNs) are attractive tools for the learning of time-series. However, traditional long-term prediction methods either work as iterated 1-step methods (Fig. 1(a)) or by direct learning of the $k$-step-ahead value (Fig. 1(b)). There is vast amount of literature on RNNs, which is reviewed in the accompanying paper\textsuperscript{1} In the former scheme, during the learning phase, prediction errors are computed relative to the previous sequence values, so errors do not accumulate. However, this does not capture well the behavior in the testing phase, when errors cannot be corrected step-by-step, so they accumulate rather quickly. The latter scheme escapes this trap, but it needs a different estimator for each lookahead time $k$, which is far from being economic (although implementations exist, see, e.g.

\textsuperscript{1}This technical report is a supplementary material for the manuscript ‘PI-RANHA: Policy Iteration for Recurrent Artificial Neural Networks with Hidden Activities’ written by I. Szita and A. Lőrincz. The interested reader is kindly referred to the a recent review on this topic (Boné and Crucianu, 2002) and references therein.
We take a novel approach: perform iterated prediction without correction, and formulate an objective function that directly takes all the prediction errors into account (see Fig. 1(c)).

The resulting objective function is a least-squares function of highly nonlinear terms, being intractable for traditional minimization techniques. We proceed by showing that the task can be interpreted as a reinforcement learning problem of a hypothetical agent in an abstract environment. This connection enables the minimization of our objective function using a version of policy iteration.

The outline of the paper is as follows. The theoretical part is made of three sections. First, we define the learning task and derive the learning rules of our algorithm in Section 2. In Section 3 the optimization problem is rewritten into a Policy Iteration Algorithm for Recurrent Artificial (neural) Networks with Hidden Activities (PIRANHA).\footnote{PIRANHA can be viewed as a gradient based approach. The purpose of this technical report is to extend the gradient based description of the accompanying paper to the framework of RL and to provide insight why and how PIRANHA works.} Using this novel form, we give a convergence proof of PIRANHA in Section 4. In Section 5 we argue that considering RNN learning as a RL process is consistent and fits well into the “classical” RL framework and to ongoing recent efforts in RL. The last section summarizes our results.

2. Definitions and Basic Concepts

2.1. The Network Architecture. We consider fully connected recurrent neural networks with $m$ input neurons and $n > m$ hidden neurons. We do not use a separate output layer; the states of the first $m$ hidden neurons are considered as outputs. The state of each neuron

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Comparison of time series learning methods}
\end{figure}

Black dots: the original time series, dotted arrows: predictions, gray arrows: corrections during learning.
is a real number in the interval $[-1, +1]$, because neurons admit activation – squashing – function, which maps onto interval $[-1, +1]$. An example is function $\sigma(z) = \tanh(z)$. The input, output and the hidden state at time $t$ are denoted by $v_t \in [-1, +1]^m$, $\hat{x}_{t+1} \in [-1, +1]^m$ and $u_t \in [-1, +1]^n$, respectively. No explicit bias term is applied, instead a constant 1 might make the $(m+1)^{st}$ component of the input. Let the recurrent, the input and the output weight matrices be denoted by $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m+1}$, and $H \in \mathbb{R}^{m \times n}$, respectively. Let $H$ be a fixed matrix projecting to the first $m$ components of $u_t$. The dynamics of the network is as follows:

$$u_{t+1} = \sigma(Fu_t + Gv_t), \quad (1)$$
$$\hat{x}_{t+1} = Hu_{t+1}, \quad (2)$$

with the initial condition $u_0 = 0$.

2.2. Problem Description. Suppose that a time series $\{x_t \in \mathbb{R}^m \mid t = 1, 2, 3, \ldots\}$ and a sequence length $T$ is given. We have to find weight matrices $F$ and $G$ so that the network is able to replay the first $T$ elements of the sequence in correct temporal order, i.e. for any $1 < t < T$, having input $v_{t'} := x_{t'}$ for $t' \leq t$ and generating $v_{t'} := \hat{x}_{t'}$ for $t < t' \leq T$, the total squared error $\sum_{t'=t}^{T} \|\hat{x}_{t'} - x_{t'}\|^2$ is small.

The sum of the least-square errors for all possible $t'$ values characterizes the replay capability of the network with the network weights given. However, this cost function may have pitfalls for ordinary gradient-based methods, because little changes in weights may considerably influence the output many steps ahead. Sensitivity may become crucial under this condition.

Our key observation to the solution is that the same problem emerges in reinforcement learning (RL). We shall reformulate the learning task to highlight this connection and we shall apply RL algorithms to optimize our problem.

2.3. Constructing an Appropriate Cost Function. The traditional approach would be to search for weight matrices $F$ and $G$ that minimize the reconstruction error

$$\sum_{t=0}^{T} \|\hat{x}_t - x_t\|^2 = \sum_{t=0}^{T} \|Hu_t - x_t\|^2$$

subject to the constraints (1)–(2). It is well known that such recurrent network optimization tasks are hard, because the objective function has many local minima and is often ill-conditioned. Let us investigate some possible reasons: consider the case when all the predicted output
values \( \hat{x}_t \) are correct, except for a single time step \( t_0 \). The hidden state \( u_{t_0} \) for this time step is also bad, and we can modify it only by modifying the weights \( F \) and \( G \). However, this modification is likely to compromise all the other errors, resulting in a total increase in the objective value. The reason for this phenomenon is that the hidden states for different time steps are very strongly coupled by Eq. (1).

The underlying idea of our approach is that if this coupling is relaxed, the resulting error surface may become smoother. Naturally, in the end we have to enforce Eq. (1), but first let us consider a different set of constraints. Suppose that the sequence \( U = (u_0, u_1, u_2, \ldots) \) is an arbitrary state sequence, not necessarily belonging to an RNN. When can we say that matrices \( F, G \) and a state \( u_t \) in the sequence is a good predictor? The one-step prediction from \( u_t \) is

\[
\hat{x}_{t \rightarrow (t+1)} = H \sigma (Fu_t + Gx_t),
\]

for two steps it is

\[
\hat{x}_{t \rightarrow (t+2)} = H \sigma (F \sigma (Fu_t + Gx_t) + G \hat{x}_{t \rightarrow (t+1)}) = H \sigma ((F + GH) \sigma (Fu_t + Gx_t)),
\]

and so on. Let us introduce the notation

\[
s^{(1)}_{t,(F,G)}(u) := \sigma (Fu + Gx_t)
\]

for the effect of \((F, G)\) on a single RNN state \( u \in \mathbb{R}^n \) and

\[
s^{(k)}_{t,(F,G)}(u) := \sigma \left( (F + GH)s^{(k-1)}_{t,(F,G)}(u) \right)
\]

to describe the effect of applying \((F, G)\) \( k \) \((k > 1)\) times on state \( u \). For later use we also define \( s^{(0)} \) as the identity function.

Using these notations, the \( k \)-step prediction error of state \( u_t \) is

\[
\left( s^{(k)}_{t,(F,G)}(u_t) - x_{t+k} \right).
\]

(For the sake of notational simplicity, we assume that the input sequence \( x_t \) is defined for \( t > T \) as well.) The overall cost of state \( u_t \) should be the the sum of the \( k \)-step prediction error norms, for all time instants and for all \( k \)s. To ensure convergence, we use a geometrically decaying weight sequence with decay rate \( \gamma \). So the total cost of the state sequence \( U \) is

\[
J(U, F, G) = \sum_{k=0}^{\infty} \sum_{t=1}^{T} \gamma^k \left\| H \left( s^{(k)}_{t,(F,G)}(u_t) \right) - x_{t+k} \right\|^2
\]

Note that if a set of weights \((F, G)\) minimizes Eq. (3) subject to Eq. (1), it also minimizes Eq. (9), and technical assumptions can ensure
that the converse also holds. Note also that cost function in Eq. (9) has a large number of partial sequences and so, it has an enormous number of adjustable quantities. This gives one great freedom in accomplishing the minimization. However, there is a price to pay: minimization is ill-posed, because the number of parameters is much larger than the number of data points. We restrict the choices to the parameters of the original problem, but in a way which is different from Eq. (3), and which reflects the structure of the sequence replay problem better.

To this end, note that if a particular state $u_t$ is a good predictor at time step $t$ using weights $(F, G)$, then $\sigma(Fu_t + Gx_t)$ is a good predictor at time step $t + 1$, following from the structure of the cost function. Let

$$S_{(F,G)}U := (u'_0, u'_1, \ldots, u'_T), \text{ where}$$

$$u'_0 = 0,$$

$$u'_{t+1} = \sigma(Fu_t + Gx_t) \text{ for } 0 \leq t < T.$$ 

Using this notation the above statement says that if $U$ is a good predictor with $(F, G)$, $S_{(F,G)}U$ is also a good predictor. This argument can be applied iteratively, yielding that $S^k_{(F,G)}U$ is a good predictor for $k > 1$. However, it is easy to see that for $k > T$ $S^k_{(F,G)}U$ is equal to the state sequence generated by (11), and will be denoted by $U_{(F,G)}$:

$$U_{(F,G)} := (u_0, u_1, \ldots, u_T), \text{ where}$$

$$u_0 = 0,$$

$$u_{t+1} = \sigma(Fu_t + Gx_t) \text{ for } 0 \leq t < T.$$ 

This justifies the following improvement scheme:

- fix $U$, $F_0$, $G_0$
- find $(F, G)$ for which $J(S_{(F,G)}U, F_0, G_0)$ is small
- continue with $U := U_{(F,G)}$, $F_0 := F$ and $G_0 := G$

In Section 4 we will formally prove the correctness of the method, but before doing so, we elaborate on some of the details.

First of all, note that the number of adjustable parameters is the same as in the original method, but the parameters $F$ and $G$ play a different role: they are chosen so that they minimize the multi-step prediction errors, and relation (11) is part of the full optimization problem.\(^3\)

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\(^3\)Several numerical simulations showed the step $U := U_{(F,G)}$ could be substituted by $U := S_{(F,G)}U$ or $U := S^k_{(F,G)}U$ for any $k$ and under such conditions, constraint (11) does not appear at all. However, theoretical analysis is easier for the definition, which includes Eq. (11) and this definition will be used here.
2.4. Computing the Gradient. In this subsection we work out the details of the above scheme.

Let us denote the state sequence at iteration \( i \) by \( U_i \), and the weight matrices by \( F_i \) and \( G_i \), respectively. We would like to find \( F \) and \( G \) so that

\[
\begin{align*}
J'_{F, G} &:= J(S_{F, G} U_i, F_i, G_i) = \\
&= \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \gamma^k \left\| H \left( s_{t+1, (F, G)}(u_t) \right) - x_{t+k} \right\|^2 \\
&:= \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \gamma^k \|e_{t,k}\|^2
\end{align*}
\]

is minimized, that is, we are minimizing the cost by propagating all states \( u_t \) by \( (F, G) \) once, and then by propagating with \( (F_i, G_i) \) further.

In order to do this, we have to compute the gradient of the cost function with respect to the weights. For any \( 1 \leq a, b, c \leq n \) and \( 1 \leq d \leq m + 1 \), the gradients are given by

\[
\begin{align*}
\left( \frac{\partial J'}{\partial F_{ab}} \right)(F_i, G_i) &= \sum_{t=1}^{T} \sum_{k=1}^{\infty} \gamma^k \left( e_{t,k} \right)^\top H[m(k)]_a[u_t]_b \\
\left( \frac{\partial J'}{\partial G_{cd}} \right)(F_i, G_i) &= \sum_{t=1}^{T} \sum_{k=1}^{\infty} \gamma^k \left( e_{t,k} \right)^\top H[m(k)]_c \begin{bmatrix} x_{t+k} \\ 1 \end{bmatrix}_d,
\end{align*}
\]

where

\[
m(k) := \prod_{j=1}^{k} \sigma' \left( [s_{t, (F, G)}(u_t)]_j \right) F_i,
\]

\((.)^\top\) denotes transposition and \([v]_b\) denotes the \( b^{th} \) component of vector \( v \).

If we can ensure that the terms \( m(k) \) remain bounded, then the above sums are convergent, because all terms can be bounded by \( C \cdot \gamma^k \). This means that if \( K \) is a sufficiently large positive number, then the gradients of

\[
\hat{J}'(F, G) := \sum_{t=1}^{T} \sum_{k=1}^{K} \gamma^k \|e_{t,k}\|^2
\]

will be arbitrarily close to (13) and (14), so we can use it as an approximation.
Using the above formulae, we can obtain weight matrices better than \((F_i, G_i)\) by taking a gradient descent step. The description of the algorithm is therefore complete. We have summarized it in Fig. 2. In the next section we will show that the algorithm is can be seen as policy iteration of an RL problem, which justifies its name, Policy Iteration for Recurrent Artificial (neural) Networks with Hidden Activities, or PIRANHA for short.

| input: \(\alpha, K, T; (x_1, \ldots, x_T)\) |
| initialize \(F_0, G_0\); \(i := 0\); |
| for \(i := 1\) to \(\max_{\text{iter}}\) |
| \(U_i := U_{(F_i, G_i)}\); |
| for \(t := 0\) to \(T\) |
| \(m_t(1) := F_i;\) |
| for \(k = 1 \ldots (K - 1)\), |
| \(m_t(k + 1) := m_t(k) \cdot \sigma' \left( [s_t^{(k)}]_{a \cdot u_t} \right) F_i;\) |
| \(e_{t,k} := H \left( s_t^{(k-1)} \cdot s_{t+1}^{(1)} \cdot s_{t+1}^{(1)} \cdot u_t \right) - x_{t+k};\) |
| \([\Delta F]_{ab} := \sum_{t=1}^{T} \sum_{k=1}^{K} \gamma^k \left( e_{t,k} \right)^\top H[m_t(k)]_a[u_t]_b;\) |
| \([\Delta G]_{cd} := \sum_{t=1}^{T} \sum_{k=1}^{K} \gamma^k \left( e_{t,k} \right)^\top H[m_t(k)]_c[x_{t+k}]_d;\) |
| \(F_{i+1} = F_i - \alpha_i \cdot \Delta F;\) |
| \(G_{i+1} = G_i - \alpha_i \cdot \Delta G;\) |
| \(i := i + 1;\) |

FIGURE 2. Pseudo-code of the PIRANHA algorithm

3. INTERPRETING PIRANHA AS POLICY ITERATION

Notice that the cost function \(9\) is formally very similar to the cost function of a reinforcement learning problem. Furthermore, the proposed algorithmic solution is very much like policy iteration, a widely used algorithm for solving RL problems. Although this similarity is only formal, as in our case uncertain transitions are not considered, the policy iteration formalism can be matched perfectly, and this gives us valuable insight how PIRANHA works. Furthermore, the similarity enables us to prove that under appropriate conditions the algorithm is convergent.

Firstly, we give a brief overview of the reinforcement learning framework and policy iteration. Next we reformulate the sequence learning problem as a special case of RL, and point out some important differences concerning traditional RL problems. Finally, using the policy
iteration formalism, we present our main result about the convergence of PIRANHA.

3.1. The Reinforcement Learning Framework and Policy Iteration. RL deals with agents that make decisions, i.e., selects actions in a stochastic environment. The state of the environment is influenced by the decisions of the agent. From the point of view of the agent, the actual state is (fully or partially) observed, actions are selected and state-dependent immediate costs are received. RL aims to minimize the total cumulated cost by finding the optimal decisions (optimal policy) for the agent.

In most cases, RL problems are treated as Markov decision processes, i.e., states are fully observable and rewards and successor states depend only on the current state and action but not on the history. For the sake of simplicity, we also assume that costs are bounded and deterministic, and depend only on the current state. Let us denote the state space and the actions of the agent by $\mathcal{S}$ and $\mathcal{A}$, respectively. The dynamics of the environment is characterized by the transition probability function $P : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$ and the immediate cost function $c : \mathcal{S} \to \mathbb{R}$. A deterministic policy $\pi : \mathcal{S} \to \mathcal{A}$ of the agent is a mapping from states to actions. Future costs to be cumulated may be discounted by some factor $0 \leq \gamma \leq 1$, so the expected cost of a state $s_0$ is

$$J^\pi(s_0) = E(c(s_0) + \gamma c(s_1) + \gamma^2 c(s_2) + \ldots),$$

where $E(.)$ denotes expectation, and for each $t \geq 0$, $a_t = \pi(s_t)$ and the system arrives at $s_{t+1}$ with probability $P(s_t, a_t, s_{t+1})$. The task is to find an optimal policy $\pi^*$, for which $J^{\pi^*}(s) \leq J^\pi(s)$ for any state $s$ and any policy $\pi$.

Policy iteration approaches the optimal policy by a two-phase iteration procedure: at each iteration $i$, the current policy $\pi_i$ is first evaluated, i.e. $J^{\pi_i}(s)$ is computed (or approximated), usually by letting the agent to take many steps using $\pi_i$ and processing the experienced costs. The other phase is called policy improvement. In this phase the policy is modified by using the inequality

$$J^{\pi_i}(s) = c(s) + \gamma \sum_{s'} P(s, \pi(s), s') J^{\pi_i}(s'),$$

and

$$\geq \min_a \left( c(s) + \gamma \sum_{s'} P(s, a, s') J^{\pi_i}(s') \right),$$

which implies that

$$\pi_{i+1}(s) := \arg \min_a \left( c(s) + \gamma \sum_{s'} P(s, a, s') J^{\pi_i}(s') \right)$$
is an improvement over \( \pi_i \).\(^4\) The improvement can be gradual: there is no need to find the minimum, but it is sufficient if a partial and/or approximate step is made in that direction.

If transition probabilities, policies and cost function values are stored in a lookup-table, with separate entries for each different argument, policy iteration converges to an optimal policy under appropriate conditions. This is true even if approximations are used in either step or \( P \) and \( c \) is not known in advance but have to be estimated from the agent-environment interaction. For details about the conditions of the various convergence results, see ([Bertsekas and Tsitsiklis, 1996, Sutton and Barto, 1998]).

In some cases, the construction of the lookup-table is not feasible, e.g., when the state and/or the action spaces are continuous. Then one needs to revert to function approximation methods. It has been shown, however, that policy iteration for function approximators can be divergent even for the simplest cases ([Bertsekas and Tsitsiklis, 1996]) and, in turn, the proof of convergence becomes a central issue.

### 3.2. Fitting PIRANHA into the RL Framework.

#### 3.2.1. The Agent and its Environment.

Let us define a hypothetical agent that observes the state sequence \( U \), and based on that, chooses a set of weights \((F, G)\), which is used for prediction. Formally this corresponds to \( S = \mathbb{R}^{n \times T} \), so that a state \( U \in S \) of the agent is the whole sequence of hidden states of the RNN: \( U = (u_1, u_2, \ldots, u_T) \), while the action space will be \( A := \mathbb{R}^{n+n \cdot (m+1)} \). The environment of this hypothetical agent is quite simple: if the agent makes an action \((F, G)\) in state \( U \), then the environment transfers it deterministicly to the successor state \( S_{(F,G)}U \).

In a general-case RL problem, a policy of the agent is a mapping \( \pi : S \to A \). However, in our problem, time independent weight matrices \((F \text{ and } G)\) are searched for, therefore we restrict the mapping to constant policies. Such policies \( \pi_{(F,G)} \) execute the fixed action \((F,G)\) regardless of the current state. For a policy \( \pi \equiv (F,G) \) we will also use the notation \( S_\pi(.) \) instead of \( S_{(F,G)}(.) \).

#### 3.2.2. Costs.

Let \( U = (u_1, u_2, \ldots, u_T) \) be an arbitrary state. Its immediate cost \( c(U) \) is defined as the error of reconstructing the input series

\(^4\)Note that for policy improvement, it is not necessary to choose the minimum in the equation.
Thus, by (17), the total discounted cost of $U$ using policy $\pi \equiv (F, G)$ and discount factor $0 \leq \gamma < 1$ is

$$J_\pi(U) := c(U) + \gamma c(S_\pi U) + \gamma^2 c(S_\pi^2 U) + \ldots$$

$$= \sum_{k=0}^{\infty} \sum_{t=1}^{T} \gamma^k \left\| H(s_{i,(F,G)}^{(k)}(u_t)) - x_{i+t} \right\|^2,$$

which is the same cost function as (9). This cost function reverts to the cost function of Eq. 3 (=Eq. 21) for $\gamma = 0$, but then policy iteration reduces to the iterative 1-step method.

3.2.3. Policy Evaluation. At iteration $i > 0$, we have a policy $\pi_i = \pi_{(F_i, G_i)}$ to be evaluated. The evaluation starts by taking many steps using $\pi_i$, which (by (11)) guarantees that the agent is in state $U_{(F_i, G_i)}$. Therefore we have to evaluate the cost function in a single state $U_{(F_i, G_i)}$ only, which – in contrast to traditional policy iteration – can be done directly.

3.2.4. Policy Improvement. Policy improvement will be accomplished only for the single state $U_{(F_i, G_i)}$, because this is the only state where the full cost is known. As a further deviation from the basic policy improvement strategy (20), we take only a small step towards

$$\arg \min_{(F, G)} \left( c(U_{(F_i, G_i)}) + \gamma \hat{J}(S_{(F,G)} U_{(F_i, G_i)}) \right) :=$$

$$= \arg \min_{(F, G)} \hat{J}'(F, G),$$

and improve the policy gradually:

$$\pi_{i+1}(U_{(F_i, G_i)}) = (F_{i+1}, G_{i+1}) := (F_i, G_i) - \alpha_i \nabla \hat{J}'(F_i, G_i),$$

where $\alpha_i$ is the learning rate for iteration $i$, and $\nabla$ denotes the gradient with respect to components of $(F, G)$.

Iterating these two steps completes policy iteration, and as it can be easily seen, it is identical to the PIRANHA algorithm described in Fig. 2.
3.2.5. Differences from Standard Policy Iteration. There are several important differences between standard policy iteration algorithm and PIRANHA. Firstly, in the policy improvement step we modified the policy so that only the cost of a single state, \( U_i \) is considered. However, changing the policy in \( U_i \) changes the costs of every other state as well, and we have no guarantee that the change is really an improvement for all states (in fact, it can be shown that there will be states for which the cost increases). Thus, we cannot apply any of the convergence results of (approximate) policy iteration algorithms, because they all require improvement over the whole state space. Fortunately, \( J_{\pi_i}(U_i) \geq J_{\pi_{i+1}}(U_{i+1}) \) still holds, but we need to prove this by taking into account the specific structure of the cost function.

4. Analysis

4.1. The Finite-\( K \) Approximation. The gradients of (12) are infinite sums, which are approximated by finite sums up to the \( K \)th term. The first question we have to answer is whether this approximation is feasible. The following lemma shows that for sufficiently small discount factor \( \gamma \), the gradients (13) and (14) are convergent, therefore for sufficiently large \( K \), the gradients of (12) will be approximated well by (16).

**Lemma 1.** Suppose that (a) For all \( i \), \( \|F_i\|_\infty \leq 1/\sqrt{\gamma} \), (b) The slope of the sigmoid function is at most 1, i.e. \( \sup_z \sigma'(z) \leq 1 \). Then for any \( \epsilon_0 > 0 \) there exists a \( K \) so that if (16) is used for calculating the gradient, the difference is less than \( \epsilon_0 \).

**Proof.** The difference of derivatives with respect to some component \( F_{ab} \) is

\[
(27) \quad \left| \left( \frac{\partial J'}{\partial F_{ab}} \right) (F_i, G_i) - \left( \frac{\partial \hat{J}'}{\partial F_{ab}} \right) (F_i, G_i) \right| =
\]

\[
(28) \quad = \left| \sum_{t=1}^{T} \sum_{k=K+1}^{\infty} \gamma^k (e_{t,k})^\top H[m(k)]_a[u_t]_b \right| \leq \]

\[
(29) \quad \leq \sum_{k=K+1}^{\infty} \sum_{t=1}^{T} \gamma^k \|e_{t,k}\|_\infty \|H\|_\infty \|m(k)\|_\infty \|u_t\|_\infty
\]

But \( u_t \) is an inner state of a neural network, so \( \|u_t\|_\infty \leq 1 \), and similarly, \( \|e_{t,k}\|_\infty \leq 2 \), and \( \|H\|_\infty = 1 \) because it is a projection matrix. Using (13) and Assumption (b), the norm of \( m(k) \) can be bounded from above.
by

\[
\|m(k)\|_\infty \leq \|F_i\|_\infty^k \leq \gamma^{-k/2},
\]

so the difference of the exact and approximate derivatives can be bounded as

\[
\sum_{k=K+1}^{\infty} \sum_{t=1}^{T} \gamma^k \cdot 2 \cdot 1 \cdot \gamma^{-k/2} \cdot 1 = \sum_{k=K+1}^{\infty} 2T \gamma^{k/2} = \frac{2T \gamma^{(K+1)/2}}{1 - \gamma^{1/2}} := C_0 \gamma^{K/2},
\]

which is can be made less than \(\epsilon_0\) for sufficiently large \(K\). One may use (14) to show that the same bound applies for the differences of derivatives with respect to elements of \(G\). \(\square\)

4.2. Convergence proof. We should not that proving the convergence of the algorithm of Table 2 is easy, because it is a gradient descent method, therefore, as it is well known, it converges to a (local) minimum if the step sizes \(\alpha_i\) are sufficiently small.

However, the algorithm offers new possibilities within the framework of reinforcement learning as we shall discuss it later. We give an alternative proof that is somewhat more complicated, but exploits the policy iteration reformulation. This derivation reflects the mechanism and the potentials of the algorithm better.

5. The proof of the Convergence Theorem

First, suppose that \(J\) and \(J'\) can be computed exactly. We proceed by a series of lemmas:

Let \(\pi_i = (F_i, G_i)\) be the policy at iteration \(i\), \(U_{\pi_i} = (u_0, u_1, \ldots, u_T)\), furthermore, let the gradient of \(J'\) be denoted by \(\Delta \pi = (\Delta F, \Delta G)\) as above.

**Lemma 2.** For \(\Delta \pi \neq 0\) there exists a number \(\alpha_0 > 0\) such that for all \(0 < \alpha \leq \alpha_0\),

\[
J^{\pi_i + \alpha \Delta \pi}(U_{\pi_i}) < J^{\pi_i}(U_{\pi_i}).
\]

**Proof.** By definition, \(\Delta \pi\) is the steepest descent direction of \(c(U_{\pi_i}) + J^{\pi_i}(S_{\pi_i} U_{\pi_i})\), and it is nonzero, therefore for sufficiently small \(\beta_0\), for all \(0 < \beta \leq \beta_0\),

\[
c(U_{\pi_i}) + \gamma J^{\pi_i}(S_{\pi_i + \beta \Delta \pi} U_{\pi_i}) <
\]

\[
c(U_{\pi_i}) + \gamma J^{\pi_i}(S_{\pi_i} U_{\pi_i}) = J^{\pi_i}(U_{\pi_i}).
\]
Let
\[ U := \{ U \in \mathbb{R}^{n \times (T+1)} \mid c(U) + \gamma J^{\pi_i}(S_{\pi_i+\alpha \Delta \pi} U) < J^{\pi_i}(U) \text{ for all } \beta \leq \beta_0 \} \]
be the set of states for which \( \pi_i + \beta_0 \Delta \pi \) is an improvement over \( \pi_i \).
This is an open set, because \( c(U) + \gamma J^{\pi_i}(S_{\pi_i+\beta_0 \Delta \pi} U) - J^{\pi_i}(U) \) is a continuous function of \( U \). Furthermore, \( U_{\pi_i} \in U \), so it also contains a ball with some positive radius \( r \) centered on \( U_{\pi_i} \).

Note that \( U_{\pi_i} = S_{\pi_i} U_{\pi_i} \) and the operator \( S_{\pi_i+\beta \Delta \pi} \) is continuous in \( \beta \), so there exists a \( \beta_1 \) so that \( \| U_{\pi_i} - S_{\pi_i+\beta_1 \Delta \pi} U_{\pi_i} \| \leq r \), which means that \( S_{\pi_i+\beta_1 \Delta \pi} U_{\pi_i} \in U \), and in general, for all \( k \) there exists a \( \beta_k \) so that \( S_{\pi_i+\beta_k \Delta \pi} U_{\pi_i} \in U \). Let
\[ \alpha_0 := \min(\beta_0, \beta_1, \ldots, \beta_{T-1}). \]
Then for all \( 0 \leq k < T \) and \( 0 < \alpha < \alpha_0 \)
\[ c(S^k_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) + \gamma J^{\pi_i}(S^k_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) < J^{\pi_i}(S^k_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) \]
by the definition of \( U \). Adding these inequalities with weights \( \gamma^k \) gives
\[ c(U_{\pi_i}) + \gamma c(S_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) + \ldots + \gamma^{T-1} c(S^{T-1}_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) + \gamma^T J^{\pi_i}(S^T_{\pi_i+\alpha \Delta \pi} U_{\pi_i}) < J^{\pi_i}(U_{\pi_i}). \]
The LHS is exactly the cost of taking \( T \) steps using \( \pi_i + \alpha \Delta \pi \), and then using \( \pi_i \). Noting that this is the same as following \( \pi_i + \alpha \Delta \pi \) all the way, so we get
\[ J^{\pi_i+\alpha \Delta \pi}(U_{\pi_i}) < J^{\pi_i}(U_{\pi_i}), \]
which was to be shown. \( \square \)

**Lemma 3.** For \( \Delta \pi \neq 0 \) there exists a number \( \alpha_1 > 0 \) such that for all \( 0 < \alpha \leq \alpha_1 \),
\[ J^{\pi_i+\alpha \Delta \pi}(U_{\pi_i+\alpha \Delta \pi}) < J^{\pi_i}(U_{\pi_i}). \]

**Proof.** The proof proceeds similarly as the proof of Lemma 2. Let
\[ U' := \{ U \in \mathbb{R}^{n \times (T+1)} \mid J^{\pi_i+\alpha \Delta \pi}(U) < J^{\pi_i}(U_{\pi_i}) \text{ for all } 0 \leq \alpha \leq \alpha_0 \} \]
This set is open as well because of the continuity of \( J \), and by Lemma 2, it contains \( U_{\pi_i} \), and thus contains also a ball centered around \( U_{\pi_i} \). This means that for sufficiently small \( \beta_0 \), \( S_{\pi_i+\beta \Delta \pi} U_{\pi_i} \in U' \) for all \( 0 < \beta < \beta_0 \). Setting \( \alpha_1 := \min(\alpha_0, \beta_0) \) proves the statement of the lemma. \( \square \)

These lemmas enable us to prove convergence for the exact cost function case:
Theorem 4. If the learning rates $\alpha_i$ are determined by Lemma 3, PI-RANHA converges to a policy $\bar{\pi}$ that is a (local) minimum of $J^\pi(U_\pi)$.

Proof. As Lemma 3 states, $J^\pi(U_\pi)$ is monotonously decreasing, thus it is necessarily convergent, which also means that the policy series $\pi_i$ converges to some $\pi = \lim_{i \to \infty} \pi_i$. By taking the limit of $\pi_{i+1} = \pi_i + \alpha_i \Delta\pi$ we get by applying lemma 3 to $\bar{\pi}$ that $\bar{\alpha}\Delta\pi = 0$. This means that either $\Delta\pi = 0$ or $\bar{\alpha} = 0$. However, if $\Delta\pi \neq 0$, Lemmas 2 and 3 guarantee the existence of a positive learning rate $\alpha$, so the gradient necessarily vanishes. \qed

Now suppose that we do not compute the exact $J'$ for computing the gradient, but use the approximation

$$J'(F, G) = c(U_i) + \sum_{t=1}^{T} \sum_{k=1}^{K} \gamma^k \|e_{t,k}\|^2$$

instead with some fixed lookahead parameter $K$. The following lemma is a rephrasing of lemma 1, and shows that we can get arbitrarily close to the minimum if $K$ is sufficiently large and the norm of recurrent weights does not grow much beyond 1.

To this end, let us first define the norm of a policy $\pi$: recall that $\pi$ is the concatenation of a $n \times n$ and an $n \times (m+1)$ real-valued matrix. If we regard policies as $n(n+m+1)$ dimensional vectors, then we can define the scalar product $\langle \pi_1, \pi_2 \rangle$ of two policies as normal scalar products in Euclidean space, and the norm of policy $\pi$ as $\|\pi\| = \langle \pi, \pi \rangle^{1/2}$. The norm of matrices and vectors is defined as usual; $\|\cdot\|_\infty$ denotes the maximum-norm.

Theorem 5. Suppose that (a) For all $i$, $\|F_i\|_\infty \leq 1/\sqrt{\gamma}$, (b) The slope of the sigmoid function is at most 1, i.e. $\sup_z \sigma'(z) \leq 1$. Then for any $\epsilon_0 > 0$ there exists a $K$ so that if (43) is used for calculating the gradient $\hat{\pi}_i$, then $\limsup_{i \to \infty} \|\Delta\hat{\pi}_i\| \leq \epsilon_0$.

Proof. For the proof we will exploit the fact that gradient descent is convergent with approximating the direction of the steepest descent as long as actual direction of descent and the direction of steepest descent enclose an acute angle, i.e. as long as the scalar product is positive. In our case this means that we have to prove that at each iteration, $\langle \Delta\pi, \Delta\hat{\pi} \rangle > 0$. We begin by bounding $\|\Delta\pi - \Delta\hat{\pi}\|$. 

\[ \begin{align*}
\end{align*} \]
Lemma 1 states that the difference of the exact and approximate derivatives can be bounded by

\[
\left| \frac{\partial J'}{\partial F_{ab}} (F_i, G_i) - \frac{\partial \hat{J}'}{\partial F_{ab}} (F_i, G_i) \right| \leq
\]

\[
\leq \frac{2T\gamma^{(K+1)/2}}{1 - \gamma^{1/2}} := C_0\gamma^{K/2},
\]

where \( C_0 \) is a constant independent of \( K \), and the same holds for the differences of derivatives with respect to elements of \( G \).

Therefore the norm of the gradient error \( \Delta \pi - \Delta \hat{\pi} \) can be bounded from above as

\[
\| \Delta \pi - \Delta \hat{\pi} \| \leq \sqrt{n(n + m + 1)}C_0\gamma^{K/2} := C_1\gamma^{K/2}.
\]

This can be made smaller than any fixed \( \epsilon_0 > 0 \) if \( K > \frac{2\log(\epsilon_0/C_1)}{\log \gamma} \).

Applying this result to the scalar product yields

\[
\langle \Delta \pi, \Delta \hat{\pi} \rangle = \langle \Delta \pi, \Delta \pi \rangle + \langle \Delta \pi, \Delta \hat{\pi} - \Delta \pi \rangle
\]

\[
\geq \| \Delta \pi \|^2 - \| \Delta \pi \| \| \Delta \pi - \Delta \hat{\pi} \|
\]

\[
\geq \| \Delta \pi \| (\| \Delta \pi \| - \epsilon_0),
\]

which is greater than zero if \( \| \Delta \pi \| > \epsilon_0 \), but this holds by the assumption of the theorem. \( \square \)

Remark. Assumption (a) requires that the recurrent weights remain relatively small (the norm of \( F \) does not grow much beyond 1), thus avoiding chaotic behavior. Although this constraint cannot be verified \( a \ priori \), it can be enforced by either choosing small \( \gamma \) values or by applying weight regularization technique.

6. Discussion

6.1. A Hierarchy of Reinforcement Learners? In many reinforcement learning problems, the state space is prohibitively large or continuous, so storing all the state values is infeasible. In most cases this problem is resolved by applying some kind of function approximation. It is well known that neural networks have excellent function approximation capabilities, so it is no wonder that they have been widely applied in various RL problems. These neural networks are all feed-forward ones, which is completely satisfactory for estimating the value function of Markovian problems, as there is no need to keep past memories.

However, the application of recurrent neural networks can also be justified, if instead of value function approximation they are used for
identifying abstract states of the problem: they are able to identify 
spatio-temporal \regions of the state space, not only spatial ones. This 
means that they give a tool for handling non-Markovian, partially ob-
servable problems. Furthermore, their prediction capability provides a 
model of the environment as well.

As a consequence, RNNs with PIRANHA fit very naturally into a 
hierarchical RL scheme: The states of the lower level are the raw in-
put (which is continuous and non-Markovian). The lower-level agent 
uses the reconstruction/prediction error as reward signal and is able to 
learn identifying some region of the input, providing high-level state 
description. This state description can then be used by a traditional RL 
agent on the upper level. Working out the details of this hierarchical 
structure is the topic of ongoing research, see, e.g., ([Szita et al., 2002, 
Lőrinz et al., 2003] [Szita et al., 2003] [Bakker and Schmidhuber, 2004a, 
Bakker and Schmidhuber, 2004b]) and references therein.

6.2. Summary. In this article, a solution to the sequence replay prob-
lem was proposed. The question was to represent time series data so 
that it can be reproduced sequentially from seeing only a small portion 
of it. We formalized the task as the minimization of the cost function 
of a recurrent neural network. An analogy to reinforcement learning 
enabled us to adapt policy iteration method to our problem, yielding a 
novel algorithm that we called PIRANHA. We showed that PIRANHA 
is convergent, and fits naturally into the RL framework, yielding a 
hierarchical architecture.

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Department of Information Systems
Eötvös Loránd University of Sciences
Pázmány Péter sétány 1/C
Budapest, Hungary H-1117,
WWW: nipg.inf.elte.hu
E-mails: szityu@stella.eotvos.elte.hu
andras.lorincz@elte.hu