Wheel-free planar graphs

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August 19, 2014

Abstract

A wheel is a graph formed by a chordless cycle C and a vertex u not in C that has at least three neighbors in C. We prove that every 3-connected planar graph that does not contain a wheel as an induced subgraph is either a line graph or has a clique cutset. We prove that every planar graph that does not contain a wheel as an induced subgraph is 3-colorable.

AMS classification: 05C75

*Partially supported by Agence Nationale de la Recherche under reference ANR 10 JCJC 0204 01.
†Supported by NSF grants DMS-1001091 and IIS-1117631.
‡Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-1265563.
§Partially supported by ANR project Stint under reference ANR-13-BS02-0007. Also Labex Milyon, INRIA, Université de Lyon, Université Lyon 1.
1 Introduction

All graphs in this paper are finite and simple. A graph $G$ contains a graph $F$ if an induced subgraph of $G$ is isomorphic to $F$. A graph $G$ is $F$-free if $G$ does not contain $F$. For a set of graphs $\mathcal{F}$, $G$ is $\mathcal{F}$-free if it is $F$-free for every $F \in \mathcal{F}$. An element of a graph is a vertex or an edge. When $S$ is a set of elements of $G$, we denote by $G \setminus S$ the graph obtained from $G$ by deleting all edges of $S$ and all vertices of $S$.

A wheel is a graph formed by a chordless cycle $C$ and a vertex $u$ not in $C$ that has at least three neighbors in $C$. Such a wheel is denoted by $(u,C)$; $u$ is the center of the wheel and $C$ the rim. Observe that $K_4$ is a wheel (in some papers on the same subject, $K_4$ is not considered as a wheel). Wheels play an important role in the proof of several decomposition theorems. Little is known about wheel-free graphs. The only positive result is due to Chudnovsky (see [1] for a proof). It states that every non-null wheel-free graph contains a vertex whose neighborhood is made of disjoint cliques with no edges between them. No bound is known on the chromatic number of wheel-free graphs. No decomposition theorem is known for wheel-free graphs. However, several classes of wheel-free graphs were shown to have a structural description.

• Say that a graph is unichord-free if it does not contain a cycle with a unique chord as an induced subgraph. The class of $\{K_4, \text{unichord}\}$-free graphs is a subclass of wheel-free graphs (because every wheel contains a $K_4$ or a cycle with a unique chord as an induced subgraph), and unichord-free graphs have a complete structural description, see [10].

• It is easy to see that the class of graphs that do not contain a subdivision of a wheel as an induced subgraph is the class of graphs that do not contain a wheel or a subdivision of $K_4$ as induced subgraphs. Here again, this subclass of wheel-free graphs has a complete structural description, see [7].

• The class of graphs that do not contain a wheel as a subgraph does not have a complete structural description so far. However, in [9] (see also [2]), several structural properties for this class are given.

• A propeller is a graph formed by a chordless cycle $C$ and a vertex $u$ not in $C$ that has at least two neighbors in $C$. So, wheels are just special propellers, and the class of propeller-free graphs is a subclass of wheel-free graphs. In [3], a structural description of propeller-free graphs is given.
Interestingly, every graph that belongs to one of the four classes described above is 3-colorable (this is shown in cited papers). One might conjecture that every wheel-free graph is 3-colorable, but this is false as shown by the graph represented on Figure 1 (it is wheel-free and has chromatic number 4). Also, the four classes have polynomial time recognition algorithms, so one could conjecture that so does the class of wheel-free graphs. But it is proved in [5] that it is NP-hard to recognize them. All this suggest that possibly, no structural description of wheel-free graphs exists.

Figure 1: The Ramsey graph $R(3, 5)$, the unique graph $G$ satisfying $|V(G)| \geq 13$, $\alpha(G) = 4$ and $\omega(G) = 2$.

In this paper, we study planar wheel-free graphs. A clique cutset of a graph $G$ is a clique $K$ such that $G \setminus K$ is disconnected. When the clique has size three, it is referred to as a $K_3$-cutset. When $R$ is a graph, the line graph of $R$ is the graph denoted by $L(R)$ defined as follows: the vertex-set of $L(R)$ is $E(G)$, and two vertices $x$ and $y$ of $L(R)$ are adjacent if they are adjacent edges of $R$. We prove the following theorems.

**Theorem 1.1** If $G$ is a 3-connected wheel-free planar graph, then either $G$ is a line graph or $G$ has a clique cutset.

We now give a complete description of 3-connected wheel-free planar graphs, but we first need some terminology. A graph is basic if it is the line graph of a graph $H$ such that either $H$ is $K_{2,3}$, or $H$ can be obtained from a 3-connected cubic planar graph by subdividing every edge exactly once. We need to name four special graphs: the claw, the diamond, the butterfly and the paw, that are represented in Figure 2. Basic graphs have a simple characterization given below.
Theorem 1.2 Let $G$ be a graph. The following statements are equivalent.

1. $G$ is basic.

2. $G$ is a 3-connected wheel-free planar line graph.

3. $G$ is 3-connected, planar and \{\(K_4\), claw, diamond, butterfly\}-free.

With Theorems 1.1 and 1.2 we may easily prove the complete description of 3-connected wheel-free planar graphs. By the Jordan curve theorem, a simple closed curve $C$ in the plane partitions its complement into a bounded open set and an unbounded open set. They are respectively the interior and the exterior of $C$.

Theorem 1.3 The class $\mathcal{C}$ of 3-connected wheel-free planar graphs is the class of graphs that can be constructed as follows: start with basic graphs and repeatedly glue previously constructed graphs along cliques of size three that are also face boundaries.

Proof — By Theorem 1.2 a basic graph is in $\mathcal{C}$. Also gluing along cliques of size three that are also face boundaries preserves being in $\mathcal{C}$ (in particular, it does not create wheels, because wheels have no clique cutset). It follows that the construction only constructs graphs in $\mathcal{C}$.

Conversely, let $G$ be a graph in $\mathcal{C}$. We prove by induction on $|V(G)|$ that $G$ can be constructed as we claim. If $G$ is a line graph, then Theorem 1.2 implies $G$ is basic. So by Theorem 1.1 we may assume that $G$ has a clique cutset. Since $G$ is 3-connected, this clique must be a triangle $K$ whose edges form a closed curve in the plane since $G$ is planar. So, $G$ is obtained by gluing along $K$ the two induced subgraphs of $G$ that are drawn respectively on the closure of the interior and on the closure of the exterior of $K$. These two graphs are easily checked to be 3-connected because $G$ is 3-connected. It follows by induction that $G$ can be constructed from previously constructed graphs by gluing along a triangle that is also a face boundary. \(\square\)

A consequence of our description is the following.
Theorem 1.4  Every wheel-free planar graph is 3-colorable.

We have no conjecture (and no theorem) about the structure of wheel-free planar graphs in general (possibly not 3-connected). In Figure 3 three wheel-free planar graphs of connectivity 2 are represented. It can be checked that they belong to none of the four classes described above (each of them contains a cycle with a unique chord, an induced subdivision of $K_4$, a wheel as a subgraph and a propeller). So we do not understand them. We leave the description of the most general wheel-free planar graph as an open question.

Section 2 gives the proof of Theorem 1.1 and in fact of a slight generalization that we need in Section 4. In Section 3 we prove Theorem 1.2. Theorem 1.4 is proved in Section 4.

Notation, definitions and preliminaries

We use notation and classical results from [4]. Let $G$ be a graph, $X \subseteq V(G)$ and $u \in V(G)$. We denote by $G[X]$ the subgraph of $G$ induced on $X$, by $N(u)$ the set of neighbors of $u$, and by $N(X)$ the set of vertices of $V(G) \setminus X$ adjacent to at least one vertex of $X$; and we define $N_X(u) = N(u) \cap X$. We sometimes write $G \setminus u$ instead of $G \setminus \{u\}$. When $e$ is an edge of $G$, we denote by $G/e$ the graph obtained from $G$ by contracting $e$.

A path $P$ is a graph with $k \geq 1$ vertices that can be numbered $p_1, \ldots, p_k$, and with $k - 1$ edges $p_ip_{i+1}$ for $1 \leq i < k$. The vertices $p_1$ and $p_k$ are the end-vertices of $P$, and $\{p_2, \ldots, p_{k-1}\}$ is the interior of $P$. We also say that $P$ is a $p_1p_k$-path. If $P,Q$ are paths, disjoint except that they have one end-vertex $v$ in common, then their union is a path and we often denote it by $P-vQ$. If $a,b$ are vertices of a path $P$, we denote the subpath of $P$ with end-vertices $a,b$ by $a-P-b$.

A cycle $C$ is a graph with $k \geq 3$ vertices that can be numbered $p_1, \ldots, p_k$, and with $k$ edges $p_ip_{i+1}$ for $1 \leq i \leq k$ (where $p_{k+1} = p_1$).
Let $Q$ be a path or a cycle in a graph $G$. The length of $Q$ is the number of its edges. An edge $e = xy$ of $G$ is a chord of $Q$ if $x, y \in V(Q)$, but $xy$ is not an edge of $Q$. A chord is short if its ends are joined by a two-edge path in $Q$.

We need the following.

**Theorem 1.5** (Harary and Holzmann [6]) A graph is the line graph of a triangle-free graph if and only if it is \{diamond, claw\}-free.

## 2 Almost 3-connected wheel-free planar graphs

A graph $G$ is almost 3-connected if either it is 3-connected or it can be obtained from a 3-connected graph by subdividing one edge exactly once. For a 2-connected graph drawn in the plane, the boundary of every face is a cycle. We need the following consequence.

**Theorem 2.1** Let $G$ be an almost 3-connected graph drawn in the plane, and let $x$ be a vertex of $G$ such that all its neighbors have degree at least three. Let $R$ be the face of $G \setminus \{x\}$ in which $x$ is drawn. Then the boundary of $R$ is a cycle $C$, and $C$ goes through every vertex of $N(x)$.

In this section, we prove the theorem below, which clearly implies Theorem 1.1. We prove the stronger statement below because we need it in the proof of Theorem 1.4.

**Theorem 2.2** If $G$ is an almost 3-connected wheel-free planar graph with no clique cutset, then $G$ is a line graph.

**Proof** — The proof is by contradiction, so suppose that $G$ is an almost 3-connected wheel-free planar graph that has no clique cutset and that is not a line graph.

1. Let $\{a, b, c\}$ be a clique of size three in $G$, and let $P$ be a chordless path of $G \setminus \{b, c\}$ with one end $a$. Then at least one of $b, c$ has no neighbor in $V(P) \setminus \{a\}$.

Suppose $b, c$ both have neighbors in $V(P) \setminus \{a\}$, and $P'$ be the minimal subpath of $P$ such that $a \in V(P')$, and both $b$ and $c$ have neighbors in $V(P') \setminus \{a\}$. We may assume that $P'$ is from $a$ to $x$, $x$ is adjacent to $b$, and $b$ has no neighbor in $V(P') \setminus \{a, x\}$. Then $a-P'-x-b-a$ is an induced cycle, say $C$. Now since $c$ is adjacent to $a$ and $b$, and has a neighbor in $V(P') \setminus \{a\}$, it follows that $(c, C)$ is a wheel, a contradiction. This proves (1).
(2) G is diamond-free.

Suppose that \( \{a, x, b, y\} \) induces a diamond of \( G \), and \( xy \notin E(G) \). Since \( \{a, b\} \) is not a cutset of \( G \), there exists a chordless \( xy \)-path \( P \) in \( G \setminus \{a, b\} \), contrary to (1). This proves (2).

A vertex \( e \) of \( G \) is a corner if \( e \) has degree two, and there exist four vertices \( a, b, c, d \) such that \( E(G[\{a, b, c, d, e\}]) = \{ab, ac, bc, cd, de, eb\} \).

(3) No vertex of \( G \) is a corner.

Suppose that \( e \in V(G) \) is a corner and let \( a, b, c, d \) be four vertices as in the definition. Since \( \{b, c\} \) is not a cutset of \( G \), there exists a chordless \( ad \)-path \( P \) in \( G \setminus \{b, c\} \). But now the path \( a-P-d-e \) contradicts (1). This proves (3).

(4) G contains a claw.

Otherwise, by (2) and Theorem 1.5, \( G \) is a line graph, a contradiction. This proves (4).

The rest of the proof is in two steps. We first prove the existence of a special cutset, called an “I-cutset” (defined below). Then we use the I-cutset to obtain a contradiction.

Let \( \{u, x, y\} \) be a cutset of size three of \( G \). A component of \( G \setminus \{u, x, y\} \) is said to be degenerate if it has only one vertex, or it has exactly two vertices \( a, b \) and \( G[\{u, x, y, a, b\}] \) has the following edge-set: \( \{xy, ax, ay, ab, bu\} \), and nondegenerate otherwise.

A cutset \( \{u, x, y\} \) of size three of \( G \) is an I-cutset if \( G[\{u, x, y\}] \) has at least one edge and \( G \setminus \{u, x, y\} \) has at least two connected components that are non-degenerate.

(5) G admits an I-cutset.

Fix a drawing of \( G \) in the plane. By (4), \( G \) contains a claw. Let \( u \) be the center of a claw. Let \( u_1', u_2, \ldots, u_k \) \((k \geq 3)\) be the neighbors of \( u \), in cyclic order around \( u \), where \( u_2, \ldots, u_k \) have degree at least three. If \( u_1' \) has degree two, let \( u_1 \) be its neighbor different from \( u \), and otherwise let \( u_1 = u_1' \).

Deleting \( u \), and also deleting \( u_1' \) if \( u_1' \) has degree two, yields a 2-connected graph, drawn in the plane, and therefore, the face \( R \) of this drawing in which \( u \) is drawn is bounded by a cycle \( C \). By Theorem 2.1, \( u_1, u_2, \ldots, u_k \) all belong to \( C \), and are in order in \( C \). For \( i = 1, \ldots, k \), let \( S_i \) be the unique \( u_i u_{i+1} \)-path included in \( C \) that contains none of \( u_i, \ldots, u_k \) except \( u_i \) and \( u_{i+1} \) (subscripts are taken modulo \( k \)).
Assume that \( xy \) is a chord of \( C \). Vertices \( x \) and \( y \) edge-wise partition \( C \) into two \( xy \)-paths, say \( P' \) and \( P'' \). Since \( R \) is a face of \( G \setminus \{u\} \) or of \( G \setminus \{u, u'_1\} \), it follows that \( \{u, x, y\} \) is a cutset of \( G \) that separates the interior of \( P' \) from the interior of \( P'' \). If \( xy \) is not a short chord, then both these interiors contain at least two vertices and therefore \( \{u, x, y\} \) is an \( I \)-cutset of \( G \). So we may assume that \( xy \) is short. If \( x, y \) both belong to \( S_i \) for some \( i \), then \( \{x, y\} \) is a clique-cutset of \( G \), a contradiction. Thus we may assume that for every chord \( xy \) of \( C \), there exists \( i \in \{1, \ldots, k\} \) such that \( x \in S_{i-1}, y \in S_i \) and both \( xu_i \) and \( yu_i \) are edges.

**Claim.** \( k = 3 \) and \( u'_1 \) has degree two.

To prove the claim, assume by way of contradiction that \( u \) has at least three neighbors of degree at least 3. Since \( G \) is wheel-free, \( C \) must have chords. Let \( xy \) be a chord, and choose \( i \in \{1, \ldots, k\} \) such that \( xu_i \) and \( yu_i \) are edges of \( C \). Suppose first that we cannot choose \( xy \) and \( i \) such that \( u_i \) is adjacent to \( u \). Consequently \( i = 1 \), and \( u'_1 \) has degree two; moreover, the cycle obtained from \( C \) by replacing the edges \( xu_1 \) and \( u_1y \) by \( xy \) is induced. Since in this case \( k \geq 4 \), it follows that \( u \) has at least three neighbors in this cycle and so \( G \) contains a wheel, a contradiction.

We can therefore choose \( xy \) and \( i \) such that \( u_i \) is adjacent to \( u \). It follows that \( u_{i+1}, u_{i-1} \) are not consecutive in \( C \), since \( u \) is the center of a claw. We claim that there are no edges between \( S_i \setminus \{u_i\} \) and \( S_{i-1} \setminus \{u_i\} \), except \( xy \). For suppose such an edge exists, say \( ab \). Since \( u_{i+1}, u_{i-1} \) are not consecutive in \( C \), it follows that \( ab \) is a chord of \( C \). Since every chord of \( C \) is short, it follows that \( \{a, b\} = \{u_{i+1}, u_{i-1}\} \) and that \( u_{i+1}u_{i+2} \) and \( u_{i-2}u_{i-1} \) are both edges of \( C \). But now, \( G[u, u_{i+1}, u_{i-1}, u_{i+2}] \) is a wheel or, in case one of \( i - 1, i + 1 \) or \( i + 2 \) equals 1 and \( u'_1 \) has degree 2, \( G[u, u_{i+1}, u_{i-1}, u_{i+2}, u'_1] \) is a wheel.

Hence there are no edges between \( S_i \setminus \{u_i\} \) and \( S_{i-1} \setminus \{u_i\} \) except \( xy \) and thus,

\[
u - u_{i-1} - S_{i-1} - x - y - S_i - u_{i+1} - u
\]

or, in the case where \( i - 1 = 1 \) and \( u'_1 \) exists,

\[
u - u'_1 - u_1 - S_1 - x - y - S_1 - u_3 - u
\]

is an induced cycle containing three neighbors of \( u_i \), a contradiction. This proves the claim.

Observe that the claim implies that every center of a claw in \( G \) has degree three and is adjacent to \( u'_1 \) since \( G \) has at most one vertex of degree two.
Let $x, y$ be the neighbors of $u_2$ in $S_1, S_2$ respectively. Note that possibly $x = u_1$. Observe that, since $u$ is the center of a claw, $y \neq u_3$. Since every center of a claw is adjacent to $u_1'$, it follows that $u_2$ is not the center of a claw and thus $xy$ is an edge. Now,

$$x-y-S_2-u_3-u_1'-u_1-S_1-x$$

must admit a chord, for otherwise $u_2$ is the center of a wheel of $G$. Hence $u_1u_3$ is an edge. Let $z$ be the neighbor of $u_3$ in $S_2$. Since $u_3$ is not the center of claw, $u_1z$ is an edge and thus $u_1'$ is a corner, a contradiction to (3). This proves (5).

(6) Let $\{u, x, y\}$ be an $I$-cutset of $G$ where $xy$ is an edge and let $C$ be a nondegenerate connected component of $G \setminus \{u, x, y\}$. Then there exist $v \in \{x, y\}$ and a path $P$ of $G[C \cup \{u, x, y\}]$ from $u$ to $v$, such that the vertex of $\{x, y\} \setminus \{v\}$ has no neighbor in $V(P) \setminus \{v\}$.

Since $G$ does not admit a clique cutset, it follows that $u$ is non-adjacent to at least one of $x, y$. If $u$ is adjacent to exactly one vertex among $x$ and $y$, then the claim holds. So we may assume that $u$ is adjacent to neither $x$ nor $y$.

Since $G$ is $\{\text{diamond, } K_4\}$-free, at most one vertex of $G$ is adjacent to both $x$ and $y$. Let $a$ be such a vertex, if it exists. Let $K = \{x, y, a\}$ if $a$ exists, and let $K = \{x, y\}$ otherwise.

Since $K$ is not a clique cutset in $G$, we deduce that $u$ has a neighbor in every component of $C \setminus K$. Suppose first that there is a component $C'$ of $C \setminus K$ containing a neighbor of one of $x, y$. Let $P$ be a path with interior in $C'$, one of whose ends is $u$, and the other one is in $\{x, y\}$, and subject to that as short as possible. Then only one of $x, y$ has a neighbor in $V(P) \setminus \{x, y\}$, and (6) holds. So we may assume that no such component $C'$ exists, and thus neither of $x, y$ has neighbors in $V(C) \setminus K$.

Let $L = \{a, u\}$ if $a$ exists, and otherwise let $L = \{u\}$. Then $L$ is a cutset in $G$ separating $C \setminus L$ from $x, y$. Since $G$ is almost 3-connected, it follows that $L = \{a, u\}$, and $C \setminus L$ consists of a unique vertex of degree two, so $C$ is degenerate, a contradiction. This proves (6).

For every $I$-cutset $\{u, x, y\}$, some nondegenerate component $C_1$ of $G \setminus \{u, x, y\}$ has no vertex with degree two in $G$; choose an $I$-cutset $\{u, x, y\}$ and $C_1$ such that $|V(C_1)|$ is minimum. We refer to this property as the minimality of $C_1$. Put $G_1 = G[C_1 \cup \{u, x, y\}]$, and $G_2 = G \setminus C_1$. Assume
without loss of generality that \(xy\) is an edge, and let \(C_2 \neq C_1\) be another nondegenerate component.

From (6) and the symmetry between \(x, y\), we may assume without loss of generality that there is a chordless path \(Q\) of \(G_2\) from \(u\) to \(x\) such that \(y\) has no neighbor in \(V(Q) \setminus \{x\}\), and in particular \(u, y\) are non-adjacent. Also, since \(u, y\) both have neighbors in \(C_2\), there is a chordless path \(R\) of \(G_2\) between \(u, y\) not containing \(x\). Since \(u, y\) both have neighbors in \(C_1\), there is a chordless path \(P\) of \(G_1\) between \(u, y\) not containing \(x\). Consequently the union of \(P, Q\) and the edge \(xy\) is a cycle \(S\). Let \(D\) be the disc bounded by \(S\).

Suppose that some edge of \(G_1\) incident with \(x\) is in the interior of \(D\), and some other such edge is in the exterior of \(D\). By adding these two edges to an appropriate path within \(G[C_1]\), we obtain a cycle \(S_0\) drawn in the plane, such that the path formed by the union of \(xy\) and \(P\) crosses it exactly once; and so one of \(y, u\) is in the interior of the disc bounded by \(S_0\), and the other in the exterior. But this is impossible, because \(y, u\) are also joined by the path \(R\), which is included in \(G_2\) and thus is disjoint from \(V(S_0)\). We deduce that we may arrange the drawing so that every edge of \(G_1\) incident with \(x\) belongs to the interior of \(D\). In addition we may arrange that the edge \(xy\) is incident with the infinite face.

Subject to this condition (and from now on with the drawing fixed), let us choose \(P\) so that \(D\) is minimum. Since \(u, x, y \in V(S)\), every component of \(G \setminus V(S)\) has vertex set either a subset of \(C_1\) or disjoint from \(C_1\). Suppose that some vertex \(c\) of \(C_1\) is drawn in the interior of \(D\), and let \(K\) be the component of \(G \setminus V(S)\) containing it. From the choice of \(P\), it follows that there do not exist two non-consecutive vertices of \(P\) both with neighbors in \(K\) and, since \(|N(K)| \geq 3\) (because \(G\) is almost 3-connected and all vertices in \(C_1\) are of degree at least three), and \(N(K) \subseteq V(P) \cup \{x\}\), we deduce that \(|N(K)| = 3\), and \(N(K) = \{x, a, b\}\) say, where \(a, b\) are consecutive vertices of \(P\). From the minimality of \(C_1\), \(\{x, a, b\}\) is not an I-cutset, and thus \(K\) is degenerate. Hence, since \(K\) has no vertex of degree 2, \(|V(K)| = 1\), i.e. \(V(K) = \{e\}\). Therefore \(c\) has degree three, with neighbors \(x, a, b\). But then \(c\) has three neighbors in \(S\), and so \(G\) contains a wheel, a contradiction.

Thus no vertex in \(C_1\) is drawn in the interior of \(D\). So, since all edges of \(G_1\) incident with \(x\) belong to the interior of \(D\), every neighbor of \(x\) in \(C_1\) belongs to \(P\). Since \(G\) is almost 3-connected and all vertices of \(C_1\) are of degree at least 3, \(x\) has at least one neighbor in \(C_1\). Since \(P \cup R\) is a chordless cycle, it follows that \(x\) has at most two neighbors in \(P\) (counting \(y\)), and so only one neighbor in \(C_1\). Let \(x_1\) be the unique neighbor of \(x\) in \(C_1\).
Since $|V(C_1)| \geq 2$, there is a vertex $x_2$ different from $x_1$ in $C_1$, and since $G$ is almost 3-connected, there are two paths of $G$, from $x_2$ to $u, y$ respectively, vertex-disjoint except for $x_2$, and not containing $x_1$. Consequently both these paths are paths of $G_1$, and so there is a path of $G_1$ between $u, y$, containing neither of $x, x_1$. We may therefore choose a chordless path $P'$ of $G_1$ between $u, y$, containing neither of $x, x_1$. It follows that the union of $P', Q$ and the edge $xy$ is a chordless cycle $S'$ say, bounding a disc $D'$ say; choose $P'$ such that $D'$ is minimal. Since $x_1$ is in $P$ and $xy$ is incident with the infinite face, it follows that $x_1$ is in the interior of $D'$.

Let $Z$ be the set of vertices in $C_1 \setminus \{x_1\}$ that are drawn in the interior of $D'$. We claim that every vertex in $Z$ has degree three, and is adjacent to $x_1$ and to two consecutive vertices of $P'$. For let $c \in Z$, and let $K$ be the component of $G \setminus (V(S') \cup \{x_1\})$ that contains $c$. From the choice of $P'$, no two non-consecutive vertices of $P'$ have neighbors in $K$, and so as before, $N(K) = \{a, b, x_1\}$, where $a, b$ are consecutive vertices of $P'$, and $|V(K)| = 1$. It follows that every vertex in $Z$ has degree three and is adjacent to $x_1$ and to two consecutive vertices of $P'$.

Let $x_1$ have $t$ neighbors in $P'$. Thus $x_1$ has at least $t + 1$ neighbors in the chordless cycle $S'$, and consequently $t \leq 1$ since $G$ does not contain a wheel. The degree of $x_1$ equals $|Z| + t + 1$, and since $x_1$ has degree at least three and $t \leq 1$, we deduce that $Z = \emptyset$, and either $t = 1$, or $t = 0$ and $|Z| > 1$. Choose $z \in Z$, and let $z$ be adjacent to $a, b, x_1$, where $u, a, b, y$ are in order in $P'$.

We claim that $x_1$ is adjacent to neither $u$ nor $y$. For suppose $x_1$ is adjacent to neither $u$ nor $y$. Since $x_1$ is the unique neighbor of $x$ in $C_1$, $\{u, x_1, y\}$ is a cutset of $G$ separating $C_1 \setminus \{x_1\}$ from the rest of the graph. So, since it is not an $I$-cutset and since all vertices in $C_1$ have degree at least 3, $|C_1 \setminus \{x_1\}| = 1$ and thus $C_1 \setminus \{x_1\} = \{z\}$. Since $z$ has degree at least 3, $z$ is adjacent to $u, y$ and $x_1$; and, since $z \notin P'$, $P' = uy$. Hence $G[\{u, y, z, x_1\}]$ is a diamond, a contradiction to \[2\] or else $x_1$ is adjacent to both $u$ and $y$ and $G[\{u, y, z, x_1\}]$ is a wheel, a contradiction. So $x_1$ is adjacent to neither $u$ nor $y$.

If $x_1$ has a neighbor in $P' \setminus \{u, y\}$ (a unique neighbor because $t \leq 1$) between $u$ and $a$, say $v$, then $z$ has three neighbors in the chordless cycle formed by the union of $x_1v$, the subpath of $P'$ between $v$ and $y$, and the edges $yx$ and $xx_1$. On the other hand, if $x_1$ has a neighbor in $P'$ between $b$ and $y$, say $v$, then $x_1$ has three neighbors in the chordless cycle formed by the union of $x_1v$, the subpath of $P'$ between $v$ and $u$, the path $Q$ and the edge $xx_1$. Thus $x_1$ has no neighbor in $P'$, and so $t = 0$ and $|Z| \geq 2$. Let $z' \in Z \setminus \{z\}$, adjacent to $x_1, a', b'$ say, where $a', b'$ are consecutive vertices of $P'$, and $u, a', b', y$ are in order on $P'$. From planarity, $\{a, b\} \neq \{a', b'\}$.
and so we may assume that $u, a, a', y$ are in order on $P'$. But then $z'$ has three neighbors in the chordless cycle formed by the path $y-x_1-x-b$ and the subpath of $P'$ between $b$ and $y$, a contradiction. 

3 A characterization of basic graphs

We need the following.

Theorem 3.1 (Sedlaček [11]) If $H$ is a graph of maximum degree at most three, then $L(H)$ is planar if and only if $H$ is planar.

We now prove the following implications between the three statements of Theorem 1.2.

(1 $\Rightarrow$ 2). Suppose that $G$ is a basic graph. From the definition, $G$ is a line graph of a planar graph $R$ of maximum degree at most 3. Moreover, it is easy to see that $G$ is 3-connected. By Theorem 3.1, $G$ is planar. It remains to check that is wheel-free. If $R = K_{2,3}$, then $G$ is obviously wheel-free. Otherwise, $R$ is obtained from a 3-connected cubic planar graph by subdividing every edge exactly once. Suppose for a contradiction that $(u,C)$ is a wheel in $G$. Since $G = L(R)$, $u$ is an edge of $R$, and we set $u = xy$ where $x$ has degree 3 and $y$ has degree 2. Let $x'$ be the other neighbor of $y$ (so, $x'$ has degree 3 in $R$). In $R$, there are two edges $e$ and $f$ different from $xy$ and incident to $x$. And there are two edges $e'$ and $f'$ different from $x'y$ and incident to $x'$. Since $u$ (seen as a vertex of $G$) has degree 3, the cycle $C$ of $G$ must go through $e, f$ and $yx'$ (also seen as vertices of $G$). But to go in and out from the vertex $yx'$ of $G$, the only way is through $e'$ and $f'$ that are adjacent. It follows that $C$ has a chord, a contradiction.

(2 $\Rightarrow$ 3). Suppose that $G$ is a 3-connected wheel-free planar line graph, say $G = L(R)$. Since $G$ is a line graph, it is claw-free. Since $G$ is wheel-free, it is $K_4$-free.

Suppose for a contradiction that $G$ contains a diamond. It follows that $R$ contains a paw (see Figure 2), say a triangle $xyz$ and vertex $t$ adjacent to $x$ and to none of $y$ or $z$. Since $G$ is 3-connected, the removal of the edge $xt$ in $R$ keeps $R$ connected. It follows that in $R$, there is a path $P$ from $t$ to $y$ or $z$, that does not use the edge $tx$. The edges of $P$, together with the edges $tx, xy, yz$ and $zx$ form a wheel in $G$, a contradiction.

Suppose finally that $G$ contains a butterfly. The vertex of degree 4 in the butterfly is an edge $xy$ in $R$, and both $x$ and $y$ have degree at least 3 (because
of the butterfly), and in fact exactly 3 (because $G$ contains no $K_4$). Since $G$ is 3-connected, the removal of the vertex $xy$ of $G$ makes a 2-connected graph. It follows that in $R$, there exists a cycle through $x$ and $y$ that does not go through the edge $xy$. Hence, the edges of this cycle form the rim of a wheel in $G$ (the center is the vertex $xy$ of $G$). This is a contradiction.

(3 $\Rightarrow$ 1). By Theorem 1.5, $G$ is the line graph of a triangle-free graph $R$. Since $G$ is $K_4$-free, every vertex of $R$ has degree at most 3. In particular, since $G$ is planar, by Theorem 3.1 $R$ must be planar. Also, if $R$ has a cutvertex $x$, at least one pair of edges incident to $x$ form a cutset (of vertices) in $G$, because $G$ has at least four vertices since it is 3-connected. This is a contradiction to the 3-connectivity of $G$. It follows that $R$ is 2-connected.

If two adjacent vertices of $R$ have degree 3, then $G$ contains a diamond or a butterfly, a contradiction. Hence, $R$ is edge-wise partitioned into its branches, where a branch in a graph is a path of length at least 2, whose ends have degree at least 3 and whose internal vertices have degree 2. In fact, every branch of $R$ has length exactly 2 because a branch of length at least 3 would yield a vertex of degree 2 in $G$, a contradiction to its 3-connectivity.

Suppose that there is a pair of vertices $x, y$ of degree 3 in $R$ such that at least two distinct branches $P, Q$ have ends $x$ and $y$. We denote by $e$ (resp. $f$) the edge incident to $x$ (resp. $y$) that does not belong to $P$ or $Q$. Now, $G \setminus \{e, f\}$ is disconnected (contradicting $G$ being 3-connected), unless $e$ and $f$ are the only edges of $R$ that do not belong to $P$ and $Q$. But in this case, $R = K_{2,3}$. So, from here on, we may assume that for all pairs of vertices $x, y$ from $R$, there is at most one branch of $R$ with ends $x$ and $y$.

It follows that by suppressing all vertices of degree 2 of $R$, a cubic graph $R'$ is obtained (suppressing a vertex of degree 2 means contracting one the edges incident to it). Suppose that $R'$ is not 3-connected. This means that $R' \setminus \{x, y\}$ is disconnected where $x$ and $y$ are vertices of $R'$. Since $R'$ is cubic, for at least one component $C_x$ of $R' \setminus \{x, y\}$, $x$ has a unique neighbor $x'$ in $C_x$. Also, $y$ has a unique neighbor $y'$ in some component $C_y$. Now, $xx'$ and $yy'$ are two edges of $R'$ whose removal disconnects $R'$. These two edges are subdivided in $R$, but they still yield two edges whose removal disconnects $R$. This yields two vertices in $G$ whose removal disconnects $G$, a contradiction to $G$ being 3-connected. We proved that $R$ is obtained from a 3-connected cubic graph (namely $R'$) by subdividing once every edge.
4 Coloring wheel-free planar graphs

A coloring of $G$ is a function $\pi : V(G) \to C$ such that no two adjacent vertices receive the same color $c \in C$. If $C = \{1, 2, \ldots, k\}$, we say that $\pi$ is a $k$-coloring of $G$. An edge-coloring of $G$ is a function $\pi : E(G) \to C$ such that no two adjacent edges receive the same color $c \in C$. If $C = \{1, 2, \ldots, k\}$, we say that $\pi$ is a $k$-edge-coloring of $G$. Observe that an edge-coloring of a graph $H$ is also a coloring of $L(H)$.

A graph $R$ is chordless if every cycle in $R$ is chordless. A way to obtain a chordless graph is to take any graph and to subdivide all edges. It follows that basic graphs are in fact line graphs of chordless graphs. This is the property of basic graphs that we rely on in this section.

It is proved in [8] that for all $\Delta \geq 3$ and all chordless graphs $G$ of maximum degree $\Delta$, $G$ is $\Delta$-edge-colorable (for $\Delta = 3$, a simpler proof is given in [7]). Unfortunately, this result is not enough for our purpose and we reprove it for $\Delta = 3$ in a slightly more general form. A graph is almost chordless if at most one of its edges is the chord of a cycle.

**Theorem 4.1** If $G$ is an almost chordless graph with maximum degree three, then $G$ is 3-edge-colorable.

**Proof** — Let $G$ be a counter-example with the minimum number of edges. Let $X \subseteq V(G)$ be the set of vertices of degree three and $Y = V(G) \setminus X$ the set of vertices of degree at most two.

(1) $Y$ is a stable set.

For suppose that there exists an edge $uv$ such that $u$ and $v$ belong to $Y$. From the minimality of $G$ there exists a 3-edge-coloring of $G \setminus uv$. Since $u, v \in Y$, it is easy to extend the 3-edge-coloring of $G \setminus uv$ to a 3-edge-coloring of $G$, a contradiction. This proves (1).

(2) $G$ is 2-connected.

Otherwise $G$ has a cut-vertex $v$, so $V(G) \setminus \{v\}$ partitions into two nonempty sets of vertices $C_1$ and $C_2$ with no edges between them. A 3-edge-coloring of $G$ can be obtained easily from 3-edge-colorings of $G[C_1 \cup \{v\}]$ and $G[C_2 \cup \{v\}]$, a contradiction. This proves (2).

(3) If $e, f$ are disjoint edges of $G$, then $G \setminus \{e, f\}$ is connected.

Suppose there exists two disjoint edges $u_1u_2$ and $v_1v_2$ such that $G \setminus \{u_1u_2, v_1v_2\}$ is not connected; then $G \setminus \{u_1u_2, v_1v_2\}$ partitions into two
nonempty sets of vertices $C_1$ and $C_2$ with no edges between them. By (2) we may assume that $\{u_1,v_1\} \subseteq C_1$ and $\{u_2,v_2\} \subseteq C_2$. For $i = 1, 2$, let $G_i$ be the graph obtained from $G[C_i]$ by adding a vertex $m_i$ adjacent to both $u_i$ and $v_i$. If $G_1$ contains a cycle $C$ with a chord $ab$, then $ab$ is a chord of a cycle of $G$ (this is clear when $C$ does not contain $m_1$, and when $C$ contains $m_1$, the cycle is obtained by replacing $m_1$ by a $u_2v_2$-path included in $C_2$ that exists by (2)). It follows that $G_1$ and symmetrically $G_2$ are almost chordless. Moreover they both clearly have maximum degree at most three and, by (1), both $C_1$ and $C_2$ contain vertices of degree three, so $G_1$ and $G_2$ have fewer edges than $G$. Therefore $G_1$ and $G_2$ admit 3-edge-colorings.

Let $\pi_1$ and $\pi_2$ be 3-edge-colorings of respectively $G_1$ and $G_2$. We may assume without loss of generality that $\pi_1(u_1m_1) = \pi_2(u_2m_2) = 1$ and $\pi_1(v_1m_1) = \pi_2(v_2m_2) = 2$. Now, the following coloring $\pi$ is a 3-edge-coloring of $G$: $\pi(u_1v_1) = 1$, $\pi(u_2v_2) = 2$, $\pi(e) = \pi_1(e)$ if $e \in E(G_1)$ and $\pi(e) = \pi_2(e)$ if $e \in E(G_2)$, a contradiction. This proves (3).

(4) $G[X]$ has at most one edge, and if it has one, it is a chord of a cycle of $G$.

Suppose that $xy$ is an edge of $G[X]$ such that $G \setminus xy$ is not 2-connected. Then, there exists a vertex $w$ such that $G \setminus \{xy, w\}$ is disconnected. Let $C_x$ and $C_y$ be the two components of $G \setminus \{xy, w\}$, where $x \in C_x$ and $y \in C_y$. Since $w$ is of degree at most three, $w$ has a unique neighbor $w'$ in one of $C_x, C_y$, say in $C_x$. If $w' = x$, then $x$ is a cut-vertex of $G$ (because $|C_x| > 1$ since $x$ has degree three), a contradiction to (2). So $w' \neq x$ and hence $xy, ww'$ are disjoint, a contradiction to (3).

Therefore, for every edge $xy$ of $G[X]$, $G \setminus xy$ is 2-connected. So, if such an edge exists, by Menger’s theorem there exists a cycle $C$ going through both $x$ and $y$ in $G \setminus xy$, and thus $xy$ is a chord of $C$. Since $G$ is almost chordless, there is at most one such edge. This proves (4).

If $G$ is chordless, then by (1) and (4), $(X, Y)$ forms a bipartition of $G$, so by a classical theorem of König, $G$ is 3-edge-colorable, a contradiction. So let $xy$ be a chord of a cycle of $G$. Let $x'$ and $x''$ be the two neighbors of $x$ distinct from $y$ and let $y'$ and $y''$ be the two neighbors of $y$ distinct from $x$. By (4), $x'$, $x''$, $y'$ and $y''$ are all of degree 2 and by (1), they induce a stable set. If $\{x', x''\} = \{y', y''\}$, then $G$ is the diamond and thus is 3-edge-colorable. If $\{|\{x', x''\} \cap \{y', y''\}| = 1$, say $x' = y'$ and $x'' \neq y''$, then $xx'', yy''$ are disjoint and their deletion disconnects $G$, a contradiction to (3). Hence $x', x'', y'$ and $y''$ are pairwise distinct.

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Let $x'_1$ (resp. $x''_1$, $y'_1$, $y''_1$) be the unique neighbor of $x'$ (resp. $x''$, $y'_1$, $y''_1$) distinct from $x$ (resp. $y$). Let $G'$ be the graph obtained from $G$ by deleting the edge $xy$ and contracting edges $xx'$, $xx''$, $yy'$ and $yy''$. We note $x$ the vertex resulting from the contraction of $xx'$ and $xx''$, and $y$ the vertex resulting from the contraction of $yy'$ and $yy''$. Since $G'$ has maximum degree at most three and is bipartite by (1) and (4), it follows that $G'$ has a 3-edge-coloring $\pi'$ by König’s theorem.

Assume without loss of generality that $\pi'(xx'_1) = 1$, $\pi'(xx''_1) = 2$, $\pi'(yy'_1) = a$ and $\pi'(yy''_1) = b$ where $\{a, b\} \subseteq \{1, 2, 3\}$. Since $\{a, b\} \cap \{1, 2\} \neq \emptyset$, we may assume without loss of generality that $a = 1$, so $b \neq 1$. Let us now extend this coloring to a 3-edge-coloring $\pi$ of $G$. For any edge $e$ of $G$ such that its extremities are not both in $\{x, y, x', x'', y', y'', x'_1, y'_1, y''_1\}$, set $\pi(e) = \pi'(e)$. Set $\pi(x'x'_1) = 1$, $\pi(x''x''_1) = 2$, $\pi(y'y'_1) = 1$ and $\pi(y'y''_1) = b$. Now we can set $\pi(xx') = 2$, $\pi(xx'') = 1$, $\pi(yy') = 2$, $\pi(yy'') = 1$ and $\pi(xy) = 3$. So $\pi$ is a 3-edge-coloring of $G$. 

Note that in the next proof, we do not use planarity, except when we apply Theorem 2.2.

**Proof of Theorem 1.4**

We argue by induction on $|V(G)|$. Suppose first that $G$ admits a clique cutset $K$. Let $C_1$ be the vertex set of a component of $G \setminus K$ and $C_2 = V(G) \setminus (K \cup C_1)$. By induction $G[C_1 \cup K]$ and $G[C_2 \cup K]$ are both 3-colorable and thus $G$ is 3-colorable. So we may assume that $G$ has no clique cutset. If $G$ has a vertex $u$ of degree two, then we can 3-color $G \setminus \{u\}$ by induction and extend the coloring to a 3-coloring of $G$. So we may assume that every vertex of $G$ has degree at least three.

Assume now that $G$ is 3-connected. By Theorem 2.2 there exists a chordless graph $H$ of maximum degree three such that $G = L(H)$. Hence, by Theorem 4.1, $H$ is 3-edge-colorable and thus $G$ is 3-colorable. So we may assume that the connectivity of $G$ is two.

Let $\{a, b\} \subseteq V(G)$ be such that $G \setminus \{a, b\}$ is disconnected. We choose $\{a, b\}$ to minimize the smallest order of a component of $G \setminus \{a, b\}$, and let $C$ be the vertex set of this component. If $|C| = 1$, then the vertex in $C$ is of degree two in $G$, a contradiction. So $|C| \geq 2$. Let $G'_C$ be the graph obtained from $G[C \cup \{a, b\}]$ by adding the edge $ab$ (that did not exist since $G$ has no clique cutset). Let us prove that $G'_C$ is 3-connected. Since $|C| \geq 2$ and $G'_C$ therefore has at least four vertices, we may assume by contradiction that $G'_C$ admits a 2-cutset $\{x, y\}$. Let $C_1, \ldots, C_k$ ($k \geq 2$) be the vertex sets of the components of $G'_C \setminus \{x, y\}$. Since $ab$ is an edge of $G'_C$, $a$ and $b$ are
included in \( G'_C[C_i \cup \{x, y\}] \) for some \( i \leq k \), say \( i = 2 \). Hence \( \{x, y\} \) is a cutset of \( G \) and \( C_1 \) is a component of \( G \setminus \{x, y\} \) that is a proper subset of \( C \), a contradiction to the minimality of \( C \). So \( G'_C \) is 3-connected. (But it might not be wheel-free.)

Let \( G_C \) be the graph obtained from \( G'_C \) by subdividing \( ab \) once, and let \( m \) be the vertex of degree two of \( G_C \). Since \( G'_C \) is 3-connected, \( G_C \) is almost 3-connected. Suppose that \( G_C \) admits a wheel \((u, R)\). Since \( G \) is wheel-free, \( m \) must be a vertex of \((u, R)\). Since \( m \) is of degree two, \( m \) is in \( R \), and so \( a-m-b \) is a subpath of \( R \). Since \( G \) is 2-connected, there exists a chordless \( ab \)-path \( P \) in \( G \setminus C \). Hence by replacing \( a-m-b \) by \( P \), we obtain a wheel in \( G \), a contradiction. Therefore \( G_C \) is an almost 3-connected wheel-free planar graph.

By Theorem 2.2 there exists a chordless graph \( H \) of maximum degree three such that \( L(H) = G_C \). We are now going to prove there exist two ways to 3-edge-color \( H \), one giving the same color to \( a \) and \( b \) (that are edges of \( H \)), and the other giving distinct colors to \( a \) and \( b \). This implies that there exist two ways to 3-color \( G[C \cup \{a, b\}] \), one giving the same color to \( a \) and \( b \) and the other giving distinct colors to \( a \) and \( b \). Since by the inductive hypothesis there exists a 3-coloring of \( G \setminus C \), it follows that this 3-coloring can be extended to a 3-coloring of \( G \).

We first prove that there exists a 3-edge-coloring \( \pi \) of \( H \) such that \( \pi(a) \neq \pi(b) \). Observe that both ends of \( m \) are of degree two in \( H \). Hence, \( H/m \) is also a chordless graph with maximum degree at most three. Therefore there exists a 3-edge-coloring \( \pi \) of \( H/m \) and clearly \( \pi \) satisfies \( \pi(a) \neq \pi(b) \). It is easy to extend \( \pi \) to a 3-edge-coloring of \( H \) by giving a color distinct from \( \pi(a) \) and \( \pi(b) \) to \( m \).

Let us now prove that there is a 3-edge-coloring of \( H \) such that \( \pi(a) = \pi(b) \). Let \( m = m_am_b \), \( a = m_aa_1 \) and \( b = m_bb_1 \). We claim that \( a_1b_1 \) is not an edge of \( H \). For if \( a_1b_1 \) is an edge of \( H \), then there exists a vertex \( x \) in \( G_C \) adjacent to both \( a \) and \( b \). Since \( G_C \) is almost 3-connected and \( m \) is the only vertex of degree 2 in \( G_C \), \( G_C \setminus \{x, m\} \) is connected, and thus there exists a path \( P \) between \( a \) and \( b \) avoiding \( x \) and \( m \). Since \( a \) and \( b \) are not adjacent, \( P \) is of length at least 2. Naming \( u \) the vertex of \( P \) adjacent to \( a \), \( u \) is adjacent to \( x \), otherwise \( G_C[\{a, x, u, m\}] \) is a claw of \( G_C \), contradicting the fact that \( G_C \) is a line graph. Hence \( x \) has at least three neighbors in the chordless cycle formed by the path \( P \) and the edges \( am \) and \( bm \), a contradiction to the fact that \( G_C \) is wheel-free. So \( a_1b_1 \) is not an edge of \( H \).

Let \( H' \) be the graph obtained from \( H \) by deleting the vertices \( m_a \) and \( m_b \) and adding the edge \( a_1b_1 \). If an edge \( xy \) distinct from \( a_1b_1 \) is the chord of a cycle \( Q \), then since it is not a chord in \( H \), \( Q \) must contain \( a_1b_1 \). Then
by replacing $a_1b_1$ by $a_1\cdot m_a\cdot m_b\cdot b_1$, we deduce that $xy$ is also the chord of a cycle in $H$, a contradiction. Hence $H'$ is almost chordless and thus, by Theorem 4.1 $H'$ admits a 3-edge-coloring $\pi'$. Assume that $\pi'(a_1b_1) = 1$. Then setting $\pi(a_1m_a) = \pi(b_1m_b) = 1$ and $\pi(m_am_b) = 2$, we obtain a 3-edge-coloring of $H$ satisfying $\pi(a_1m_a) = \pi(b_1m_b)$. This completes the proof of Theorem 1.4. \hfill $\square$

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