A STUDY ON NUMERICAL METHODOLOGIES IN SOLVING FLUID FLOW AND HEAT TRANSFER PROBLEMS

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Abstract. Numerical methods are described as techniques by which several mathematical problems are formulated, because they may be easily solved with arithmetic operations. These methodologies have a great impact on the current development of finite element theory and other areas. We have given a short study of numerical methodologies applied in fluid flow and heat and mass transfer problems in mechanical engineering which includes finite difference method, Finite element method, Boundary value problems (general), Lattice Boltzmann's methods, Crank-Nicolson scheme methods, boundary integral method, Runge-Kutta method, Taylor series method and so on. We have discussed some phenomena taking place in fluids such as surface tension, coning, water scattering, Stokes law, gravity-capillary, and unsteady free-surface flows, swirling, and so on. We have also analyzed boundary value problems on boundary problems, eigenvalue problems and found a numerical way to solve these problems. We have presented different numerical methods applied to different fundamental modeling approaches in heat transfer and the performance of the mechanisms (modes) vary concerning the methods applied. The paper is dedicated to demonstrating how the methods are beneficial in solving real-life heat transfer problems in engineering applications. Results of the parameters like thermal conductivity, energy flux, entropy, temperature, etc. have been compared with the existing methods.

Keywords: Heat and Mass Transfer, Fluid Mechanics, Numerical Methods, Boundary Value Problems

1. Introduction

Heat transfer is the flow of thermal energy operated by thermal non-equilibrium, commonly found as a thermal flux defined as the heat flow per unit time at a control surface. It is classified into several mechanisms, such as thermal convection, thermal conduction, thermal radiation, and energy transfer by phase exchanges. Transfer of heat through a wall, heating the free end of a spoon dipped in hot fluid, boiling water in a pan, cooling the hot food in a spoon by blowing air are real-life examples of the given mechanisms. Applications of heat transfer, especially in mechanical engineering, are vast. Major heat transfer applications in mechanical engineering are air conditioners and refrigerators, design of power plant equipment, IC Engines and air compressors, brake cooling in cars, and bearing cooling in wind power plants.

The world around us is filled with fascinating materials/matter. Generally, the matter is classified into three categories called solids, liquids, and gases. We have a common name for liquids and gases known as fluids. A fluid is a substance that has no specific shape and can flow. Fluid mechanics is a branch of mechanical engineering that deals with fluid and its properties. Fluid mechanics is further classified into fluid statics and fluid dynamics. The fluid statics deals with fluids at rest- e.g., water in a closed container, whereas fluid dynamics deals with fluids at motion or flow- e.g., airfall over an airplane wing. This branch of mechanical engineering is essential because it helps us understand liquids and gases' behavior under various external forces, atmospheric conditions, etc. It makes us analyze fluid properties and choose the proper fluid for the required application. There are some essential properties of fluids to be known before examining a particular liquid, including temperature, viscosity, density, pressure, specific volume, specific gravity, and specific weight. Many intense types of research are going worldwide to improve the fluid properties and use them for proper applications. Some applications of fluid
mechanics are Biological applications- blood flow, blood pressure, artificial hearts, Refrigerators, air conditioners- refrigerants, Metrology- weather forecasting, IC engines- coolant, and so on.

Generally, a boundary value problem (BVP) is a system of ordinary differential equations (ODE) having solutions and derivative values specified at more than one point. The boundary conditions are conditions set for the behavior of the answers to a group of differential equations present at the boundary of its domain. Boundary integral equations can be considered a tool for analyzing BVPs for partial differential equations (PDE). Examples of boundary integral equations possibly used are acoustic scattering, elastic theory, waves, potential theory, computation of radio wave attenuation, heat conduction, etc.

This present study concentrates a survey on numerical methods by focusing more on mechanical engineering concepts such as surface tension effects, unsteady free-surface flows, Stoke’s law, coning, gravity capillary free surface flows, Swirling, and water scattering mechanisms. Many scientists and researchers have worked on this vital topic and came out with innovative ideas and applied these methods for various applications.

2. Numerical methodologies: Heat and Mass transfer

Antonopoulos [1] proposed a solution for problems in heat conduction with their boundary conditions on non-homogeneous composite regions like slabs, cylinders, etc., using the separation of variables method and orthogonal expansion methods for both the surfaces. It has been solved using a method of homogenization where a dependent variable is $O(x, t)$ and is connected to the original variable (temperature $t$) $O(x, t)$ by the variable function $q(x,t)$. While equating these with orthogonal properties of the eigenvalues and differentiating them, the other factors which might change the equations include thermal conductivity, coefficients, and the environmental temperature.

Although authors Ozisik [2], Bulavin [3] have tried to answer the heat conduction problem using boundary value methods in a multilayer structure and Tittle [4] has solved a heat conduction problem using Laplace transform methods. Hence, the author solves the outer boundary problem for the regional layers on both the boundaries (by the outer boundary surface and the surface fluids) by acknowledging the problem as a non-homogeneous problem and solving it by the orthogonal expansion and method of separation of variables.

The heat conduction analysis of 3-D fiber-reinforced materials has been formulated in [5] using the boundary element method. This approach is sophisticated enough to counter high-temperature gradients and allows fibers to interact through the matrix. Using the boundary element method (BEM), modeling holes and inclusions of fiber-reinforced materials seems effective using the boundary element method (BEM).

Henry [6] and Banerjee [7] gave a thorough exploration of 3-D modeling of holes. They discovered that for inclusions and holes being less than 10 percent of the minor extent of the region, Fourier series on three-term expansion had to be done to cover all possible methods of deformation with an at most error of 1%. Meanwhile, an iterative method has been delivered in [8, 9] based on the rapid multipole method for a sizable single-region BEM system of equations, terminated in a condition that the inclusions had to be deformed and conduct heat through them. Similarly, an idea of 3D modeling a composite system based on BEM has been evolved in [10] by an assumption of Poisson’s ratio of the fiber and matrix to be the same.

After several observations and references, the authors in this study came to an ideology of the formulation of BEM, which contains convolution integrals that had to be merged in both spaces and in the time since it is an excellent precise method compared to the other available methods. Also, treatment of these integrals was introduced in this study.

Numerical accuracy of both the single region BEM and multi-region BEM have been compared and evaluated using the preferred analytical method. As for testing the accuracy of a single-part BEM, heat conduction through a 3-D single fiber unit cube was considered. 2D BEM showed an excellent comparison result. While testing the accuracy of multi-region BEM, a heat transfer through multiple 3-D fibers unit cube was considered. As the number of fibers increased, the authors obtained precise, accurate multi-region BEM results.
To exhibit the usefulness and application of this procedure for fiber-reinforced materials, the authors implemented evolved analysis into temporary heat transfer in a turbine blade. The authors observed from the experimental results that the heat at the blade's tip is transferred toward the base quicker than the heat at a similar edge. The result of fibers tends to reduce the temperature faster near the end than the similar blade. Hence this approach was helpful with and without the fiber inclusions.

A new BEM convolution integrals method was established such that the results accurately prove its merit use in real-life problems, especially for tiny diameter fibers alone.

A numerical simulation and mathematical modeling of a thermo-electro magneto-hydrodynamic problem have been presented for an induction heating furnace in [11]. The recent growth in the industries has caused a need for change in the induction furnace with modern and advanced methods. This work was done using the principle of eddy-current induced oscillating magnetic fields around a cylindrical vessel with a coil surrounding it, which produces heat and the material undergoes a phase change inside the vessel.

The need for numerical simulation and modeling in the industries for various induction furnace designs of the system has been explained by analyzing and operating on the system's mode, frequency, and power.

The numerical simulations of the induction furnace by finite element method [12] and hybrid boundary element method [13] have been solved with the given data (the current intensities and magnetic vector). Therefore, the thermal model can be improved by combining convective heat transfer, which guides us to offer the numerical solution of a hydrodynamic problem. This result illustrates the impact of using the convective effects while finding the temperature field.

A new 4-step fragmented finite element method was implemented, examining a conjugate heat transfer between the unsteady and solid viscous flow. The interest in this approach in [14] is due to the consistent, integrated heat transfer combined with the fluid-solid bonding.

Misra and Sarkar [15] have applied the statement of Galerkin to solve momentum, continuity, and energy equations. Malatip et al. [16] progressed in a Streamline upwind Petrov-Galerkin (SUPG) finite element method for investigating conjugate heat transfer problems. AlAmiri [17] studied the uniform state alteration in a fluid-saturated porous cavity in a vertical wall(conducting) using FEM.

Also, a numerical method with second-order time accuracy has been developed to extend the splitting finite element algorithm. Triangular finite element methods have been deployed for relating the equations. The four-step fractional way with an exact order of the triangular finite element was engaged by the finite element algorithm. Formulations of finite element, fractional four-step, and SUPG methods have been presented.

Overall, the streamline upwind Petrov-Galerkin method (SUPG) and the four-step fragmented finite element were presented for examining the conjugate heat transfer between the unsteady and solid viscous flow. The authors assessed the productivity of this coupled finite element method with several examples and can model and solve both the tangible and fluid regions simultaneously and calculate the temperatures directly with a fluid-solid interface.

The flow and heat transfer through an unsteady stretching sheet implanted in a porous medium based on a theoretical analysis using finite difference method (FDM) in the presence of thermal radiation has been considered and studied in [18]. This numerical approach is powerful enough to solve various kinds of problems in recent research in this field. The accuracy of this method proved to be in outstanding accordance with the other methods.

The extended work has been done in [19] on the linearly changing variable with the distance from a stationary point by the steady 2-D boundary flow on a stretching sheet, resulting in findings of the fluid bordering the stretching sheet in a boundless zone by fluid and heat transfer. Likewise, the result of thermo-capillarity and
magnetic field on a shaky stretching sheet in a thin liquid film have been explored in [20]. Also, the homotopy analysis of a finite thickness in a wobbly stretching sheet has been studied in [21].

After observing a few references, the author launched the finite difference method (FDM) to solve the two non-linear ODEs. The formulation and solution of the problem have been stated and are solved using FDM.

A new implicit finite difference method has been applied to solve ordinary differential equations in a thin film over an unsteady stretching sheet dipped in a porous medium by flow and heat transfer in the effect of the thermal radiation. It is easy to use and proves out to be giving logical calculations, and the general form can be used for differential equations.

The Crank Nicolson type has been used to solve heat conduction problems by a linearly dependent function on temperature. The results obtained in the computation were verified with the values of the analytical section of Jankowska [22]. Formulation of Heat conduction problems, conventional and interval FDM were presented and discussed. Numerical experiments were proposed and confirmed that the required solution belongs to the interim solutions obtained with the temporary method.

Interval FDM was used to solve heat conduction problems and such problems prepared with Pennes' equation. A new method based on finite differences was proposed for the endpoint error approximation for adding the local truncation error in the sequel of the interim solutions.

Solution of heat conduction problems proposed in [23-25] by the Crank-Nicolson type explained the accuracy of the interval solution. Solving initial- BVPs for PDEs, an interval method used in [26] based on a standard finite difference scheme with local truncation error. Moreover, the solution of the one-dimensional heat equation with heat sources was found in [27] using a bio-heat transfer problem and Pennes equation.

The temperature difference in human skin at various layers of the skin subcutaneous tissues of a parabolic heat equation using a finite difference method has been assessed [28]. Also, the thermal stress when compared at different layers in the human skin is very helpful in knowing the human body's behavior for heat conduction.

Khanday and Saxena [29-31] have identified that the variation finite element method could determine the mass and temperature at different layers of the skin concerning environmental temperature. Thron [32] included a new theory that said that the human head did not change its thermal temperature due to the environmental temperature change and showed a rough idea of the thermal conductivity and its levels on the human skin.

The authors solved the parabolic heat equation by taking the boundary conditions and through the finite element method, obtaining the non-linear solution, and solving it numerically. Hence, various distributions with radial distance and the tissues' heat generation and temperature distribution were found.

Boundary-value inverse heat conduction has been discussed in [33] using Laplace transformation and the regularization method used in Fourier transform.

Inverse Laplace transforms are used in [34,35] to complete many of their mathematical problems (Cauchy and 2D problems). Inverse Laplace transforms largely helped to obtain the solution to these complex problems.

The Fourier transform is used in [36, 37] concerning space variables to solve various heat induction problems. The basis of these methods renders an idea for solving the inverse Fourier transforms concerning the time variable.

To increase the stability and obtain a required solution, the author used the regularization method. This method is used in the equation to remove the unstable numerical solution.

The author used the direct and inverse Fourier transforms concerning the time variable to solve the inverse BVP and used the regularization method after the formation of the equation to increase the accuracy, stability, and reduction of errors.
Solution of the heat equation for the exterior region of 3-dimensional domains by applying Boundary Element Method (BEM), a union of Laguerre transforms and integral equations have been studied [38]. On the other hand, the authors used integral equation methods for the non-stationary problems [39,40] and solved them in a direct approach using time boundary integral [41,42] and an alternative approach where a semi-discretization was employed for the non-stationary problem [43-45].

Laguerre transformation has been used in [46] to obtain a series of stationary problems, often in Klein-Gordon equation form. This method overcomes the difficulty of finding the inversion of the Laplace transform compared to the standard numerical methods. Also, the authors use the convergence properties of Laguerre transformation for dealing with error estimates [47].

The BEM is much more cost-efficient and precise, having minor error estimate values. Compared to the commercial numerical approach needing the inversion of Laplace transform, this method is advantageous as it doesn’t require the same.

Similarly, other commercial numerical methods like the finite difference method (FDM) and the finite element method (FEM) are not applicable for solving the study problem as they possess an unbounded solution domain. These methods require some truncation or artificial boundary for solving the problem using FEM or FDM. Meshless methods are also not applicable for solving unbounded 3-D time-dependent problems [48].

A numerical evaluation method for finding the heat transfer of an inclined moving heat source has been studied in [49]. The basis of origin was from the equation presented by Hahn RS for the temperature distribution of an inclined stripe of heat source in the integral form [50,51]. The solution was used for finding the heat evolved at the shear plane in the metal cutting process.

Hahn’s solution has been applied for modeling heat transfer for real-life applications [52]. Trigger and Chao discovered some numerical values for the integrals such that Hahn’s integral key can be expanded in series [53]. Thus the asymptotic formulae were derived using which the series was calculated, and necessary parameters were found. Malkin also used Jaeger’s asymptotic formula in [54] to expand a model for identifying the thermal damage during the surface grinding process.

The numerical evaluation using series yielded the same results as that of numerical integration, and it was found that the former is 29.6 times faster than the latter.

Lattice Boltzmann and finite volume methods were coupled to solve transient conduction and radiation heat transfer problems [55]. The delight in this approach is because of its solvability of the energy equation and acquiring the heat source, respectively.

Mishra and Roy [56] used the lattice Boltzmann method (LBM) and finite volume method (FVM) for solving transient conduction and heat transfer problems to get the radiative information of the source, which were applied to solve the energy equation and radiative transfer equation (RTE).

Formulation of the LBM and FVM with their coupling performance was presented, with results being discussed in this paper. The authors considered both 1-D and 2-D cases, obtaining results following Talukdar and Mishra [57].

Overlapping the solver nodes of LBM and FVM and halfway boundary treatment of the thermal LBM led to avoiding the interpolation operations between these solvers. The authors used the 1-D and 2-D combined conduction and radiation heat transfer to test the potentiality of this method. This method was validated and found to be in good agreement compared with the results and the existing literature studies.

The control volume finite element method (CVFEM) and the Lattice Boltzmann method are used in [58] to solve heat transfer problems in a cylindrical medium. LBM has been used in [59] for conduction and heat transfer problems to stiffen 2D semi-transparent material.
After good observation from the literature survey, the authors concluded that the computational grids were not overlapped and decided to superpose the grids of these solvers by using the same grid systems for LBM and FVM. This implementation led to the one-dimensional planar and two-dimensional square enclosure, releasing the building operations.

The article [60] dealt with the method for solving heat conduction problems of a homogeneous substrate coated by an inhomogeneous material consisting of a circular-shaped non-insulated region. Numerical analyses for the in-homogenous area for various temperatures were taken and solved.

Solution of the heat-conduction problem of an inhomogeneous material using the dual integral equations method and the asymptotic method, which increased the contact between the two surfaces eliminating the contact problem has been highlighted in [61,62]. The problem for the inhomogeneous material was later solved by Krenev L [63] by approximating the kernels of the dual integral equations.

Numerical analysis for a mixed fixed boundary problem with a varying coefficient has been found. The authors then converted the equation into a dual integral equation that is the basis for the boundary conditions. A set of kernels was obtained using a bilateral asymptotic method to double integrals.

After observations done by the author, the analysis of mixed BVP for arbitrarily in-homogeneous solids could be effectively used in temperature prediction for various types of in-homogenous coatings. Further, this method could also be used for layers with monotonic and non-monotonic distributions of the heat-conduction coefficient.

BVP for determining the asymptotic solutions for the energy transfer problems by keeping the depth of the heat exchanger as an unknown variable was observed in [64].

Due to the increasing cost of building and maintaining geothermal heat exchangers, an idea originated by considering a cartesian model consisting of two parallel channels of finite length that carry the heat transferring fluid. A separable boundary-value problem was found in [65,66] for the Laplace transform of a temperature difference between the channels by applying asymptotic techniques.

For a vertical ground heat exchanger(GHE), modification has been done in [67] Kelvin’s line source model, which consists of the radial distribution of the temperature field. Cylindrical heat source methods have been discussed in [68,69]. The authors studied Galerkin's approach for the accurate radial spectral representation of the physical problem. Tilley [70] implemented another method for a concentric geothermal heat exchanger, and he observed that the axial thermal resistance of the system was inversely proportional to the annular gap thicknesses.

Upon restriction to a 2-D cartesian model, an asymptotic approximation was formulated for the transient system to narrow on the coupling of the U-tube arrangement. Laplace transform was used to identify the BVP in two flow domains using the alternate-direction implicit method [71,72].

The eigenvalue obtained in this BVP gives the depletion rate of the temperature difference of the heat-transfer fluid as a function of depth. As they also depend on the frequency, the heat exchangers can be tuned efficiently to harness energy. Also, we know that the Reynolds number is used to determine the type of flow in practical applications.

The heat transfer change and fluid flow behavior in the nanofluid due to the movement of the moving walls have been studied [73]. Three cases have been discussed, and their effects are recorded and observed. The rate of heat transfer is studied by varying the Reynolds number and Richardson number and observed via isotherms, streamlines, and energy flux vectors.

The study was taken in hand by the authors to enhance the thermal conductivity of the conventional fluids, and to optimize their heat transfer performance in several heat transfer applications, some amount of nano-particles were added to the base fluid. The heat transfer characteristics of nanofluids are dependent on the factors such as size, volume fraction, shape, and properties of thermophysical of both solute and the solvent.
Fluid flow and heat transfer were driven by shear and buoyancy forces through the square or rectangular cavities [74]. Earlier studies were mainly done on the first law of thermodynamics, and other studies were done on entropy generation analysis. Maiga et al. analyzed nanofluid flow through numerical methods. They found that the rate of heat transfer increases with the increase in Reynolds number because of nano-particles [75-78]. The problem of differentially heated 2-sided moving lid-top for a transparent fluid was studied by Oztop and Dagtekin and found that fluid flow and heat transfer properties were dominated by convection currents w.r.t. Richardson number [79].

Entropy generation and energy flux have been discussed in [80,81], a nanofluid-filled enclosure with a double-sided lid-driven cavity with the help of three cases, neglecting the boundary assumptions. It provided a comparative analysis of mixed convection in copper–water-based nanofluid enclosures.

The author of this study is interested in studying the convection onsets of different cross-sectional bodies numerically, converting the thermomechanical fourth-order problem into a second-order eigenvalue problem. Horten & Rogers and Lapwood were the first to solve the onset problem on convection [81-89]. The paper's purpose was to find a unique approach to limit eigenvalue problems of marginal stability for porous cylinders. First, a single Helmholtz equation was derived for two sets of given wall conditions. The next step was deriving a continuous spectrum of the Helmholtz equation with Dirichlet or Neumann conditions at the cross-section contour. Finally, the Helmholtz equation was solved by the authors using the method of images.

3. Numerical methodologies: Fluid Mechanics

Surface tension is nothing but the tendency of the liquid surface to shrink into the minimum surface area. For example, when a needle is carefully placed on water, it floats. Hocking and Forbes [90] have discussed the steady withdrawal of an inviscid fluid of finite depth into a line sink, primarily focusing on surface tension acting on the free surface. The authors used the Boundary-integral-equation method to solve this problem numerically. It is also keenly noted that the flow is based on the Froude number. Tuck and Vanden-Broeck [91] have numerically obtained a 'cusp solution for a line sink in the water of infinite depth (i.e., H → ∞), which was already long thought about the solutions of steady flow corresponding to the critical drawdown value. They found a unique solution, at FS = 12.622. Hocking [92] recently provided strong evidence that this solution is critical for this case of 'infinite' depth. Hocking [93] had determined solutions similar to Tuck and Vanden-Broeck, but a boundary beneath the sink was sloping away without bound. These solutions again occurred at a unique Froude number for each angle. The authors also used a complex potential part and Cauchy’s integral formula to satisfy the no flow beneath the sink was sloping away without bound. These solutions again occurred at a unique Froude number for each angle. The authors also used a complex potential part and Cauchy’s integral formula to satisfy the no flow boundary condition. The results provide an exciting insight into the nature of these withdrawal problems [94,95]. As far as the subcritical region is concerned, it appears that all areas can have solutions of the stagnation-point type, except along a single limiting curve. If the cusp solution on the limiting curve corresponds to critical drawdown values, it would not be possible to find single-layer flow solutions.

There are two types of flows: steady flows and unsteady flows. So, unsteady or non-steady flows are those whose properties depend on time, whereas steady flow is those whose properties do not change with time. Researchers Colicchio and Landrini [96] have considered the Mixed Eulerian-Lagrangian Methods (MEL) for free-surface potential flows, which were solved by using boundary-integral equations (BIEs), as well as about the diffusion and dispersion errors were examined in the discrete linearized problem. Stability analysis of the Runge-Kutta and Taylor-expansion schemes has also been discussed. It was shown that MEL methods based on first-order and second-order explicit Runge-Kutta and Taylor-expansion schemes are unstable whereas Higher-order Runge-Kutta and Taylor-expansion schemes have led to conditionally stable forms. The authors confirmed the theoretical predictions of the errors for two different boundary-element techniques. A high-order panel method based on B-Splines was used to solve for the velocity potential. An Euler-McLaurin summation formula was used to solve the velocity field. Longuet-Higgins and Cokelet [97] have introduced the equation of the body motion approach in a free-floating vessel for recurring problems and, independently, by Faltinsen [98] for floating-body problems. Mostly, the free surface is done in a Lagrangian fashion, and the method is referred to as the Mixed Eulerian-Lagrangian (MEL) formulation. A van Neumann analysis [99,100] was performed for the free-surface equations without considering the effect of spatial discretization, and stability conditions were observed. In [101], Nakos et al. had generalized the spectral analysis and had also shown the impact of spatial discretization based on third-order splines. Following [101], Buchmann [102] had also adopted the same method, and in that, he had discussed...
the stability properties of an algorithm based on three-dimensional B-Spline discretization. In this paper, the matrix method was used to show the stability properties of MEL methods that can be used for the linearized problem when stated in a general way by exploiting the properties of the influence matrices, despite the technique adopted to solve the boundary-integral equations. Based on Runge-Kutta and Taylor-Expansion time-integration schemes, this analysis has been developed up to fourth-order accuracy. They have also discussed regridding and interpolation procedures and how they affect the (linear) theoretical growth rate, often used in non-linear simulations.

Fraenkel and Keady [103] settle a question, which was unsolved, considering the contact angle πβ; and this angle is given together with the wedge angle (or vertex angle) 2πα. This paper also discusses an integral equation of boundary-layer type that permits numerical calculation without extrapolating the limiting solution as α→ 0 and the value β0 corresponding to α = 0. Wagner [104] had formulated an infinite wedge entering into the water, which moves vertically downwards with constant speed. Wagner had also found a similarity transformation that eliminates time from the problem in the absence of viscosity, surface tension, and gravity. This paper concerns the free-boundary problem of an infinite wedge entering into the water and moving vertically downwards with constant velocity and meeting, at time zero, the horizontal free surface of the water as discussed in the first part. It also contains two parts or supplements two others: various steps and estimations and the proof of solutions in detail have been given in [105].

One of the main contributions of this paper was the proof for the set of solutions that had been established in [105], which is in such a way that every wedge angle 2πα in the open interval (0,π) occurs at least once. The supremum πβ- of the contact angle πβ is strictly less than π/4. If it were shown that πβ- were equal to π/4, then, in that case, there would be a sequence ((βn, h_n))<n=1 of solutions for which α(βn, h_n) → −1/4 as n → ∞, and this strongly contradicts the fact that 0 <α< 1/2 for an answer. This clearly explains why the contact angle πβ does not tend to π/2 as α → 0. The boundary-layer equation, which was in effect was used in [105] to construct a limiting solution for β → 0 and α → 1/2, also had a significant part under the assumption that β → 1/4 for a sequence of keys.

Fraenkel [106] examined the existence theory and used it to derive approximations from ensuring the flow for burnt wedges( wedge angles close to π). The integral equation to which the problem is reduced, which resembles that used in [107] for numerical calculation, agrees with the solutions only if 0 <β< 1/4. This investigation was noted and discussed in [108, 109]. Supercritical withdrawing or coning is the extraction process of two layers of fluids with different densities separated by an interface in a porous media. This phenomenon was investigated by G.C.Hoking and H.Zhang [110] using a coupled integral equation. Coupled integration equations and boundary integration methods were used to frame and solve the equations. Muskat and Wyckoff [111] analyzed the coning phenomenon through analytical methods. Bear and Dagan [112] took the unbounded medium as a reference and investigated critical and single flow phenomena. Critical flow is the flow that occurs when flow velocity equals wave velocity. The analogous problem of supercritical withdrawal in two-layer surface water bodies was taken in [113-115], where an integral-equation approach is applied to find accurate numerical solutions. Yu et al. and Hendersen et al. [116,117] simulated isothermal, nanophasic, and incompressible flow in a supercritical case using finite difference methods.

Stoke’s law is fundamental in fluid mechanics that help in understanding the settlement of particles in water. The law states that the force that opposes the sphere in a viscous medium is directly proportional to velocity, sphere radius, and fluid viscosity. A boundary integral method has been developed in [118] to determine the motion of two spherical liquid drops that obeys stoke’s law in the limit of zero capillarity number. The problem was framed as a system of integral equations of zeroth, first and second Fourier coefficients of perpendicular components of interfacial velocity and traction jump. The angle used as a reference here is the meridional angle. The problem is handled by a high accuracy boundary element method. Numerical solutions have been arrived at [119, 120] for the motions of 2 suspended spherical bodies nearly in contact with each other in an infinite linear flow. Zincenko and Davis [121-123] experimented on numerical methods of drops interception by boundary layer concepts.

Oseen equations play a vital role in describing viscous and incompressible fluid flow at a small Reynolds number in fluid dynamics. We know that Reynolds number is defined as a dimensionless quantity in fluid mechanics used to determine the flow pattern. Research has been done on the viscous, laminar, separated flow downstream of a
sudden expansion in a pipe [124]. Here the flow is done by an Oseen-type equation, but the nonlinearity in the swirl is retained. Here the exact solutions are obtained for a High-Reynolds-number limit. An arbitrary Reynolds number is obtained using Eigen function-expansion procedure with a swirl, which further leads to a non-standard eigenvalue problem. The author has also discussed the effect of the pressure gradients on the velocity profiles. Ramakrishnan and Shankar [125] have discussed that model equations similar to the ones used here have results that satisfy those of the N–S equations at low Re. This shows qualitative features identical to those of the N–S equations for which Re tends to infinity. It was also observed that an increase in the swirl amplitude squeezes the corner recirculation region. These observations also agree with that of Abujelala and Lilley [126]. The formation of a central recirculation bubble, especially when the swirl amplitude is sufficiently high, is one of the most exciting features of these swirling flows, similar to that of the vortex breakdown in case the swirl exceeds the critical value, as in [127-131]. This paper formulates a model that works at all Reynolds numbers and goes smoothly to the limiting form. So, here all the arbitrary values suitable for the Reynolds number's general case will be taken. The eigenvalues of the plane case, which were given in [125], acts as good starting values to compute the present ones in this paper. Even now, they are refined by a Newton–Raphson procedure as before.

The authors used the principle of the argument to find the total number of eigenvalues within a particular region in the right half of the complex plane to ensure no eigenvalues are missed. Moffatt [132] has discussed, in general, that there will be an infinite number of corner eddies of very small or diminishing size and intensity. Since most of the experimental results deal with turbulent flows, it’s challenging to find anything more than qualitative similarity as its mixing properties are different compared to those of laminar flows. One of the most critical limitations of this model would be its restriction to laminar flows. Like all turbulent flows at sufficiently high Reynolds numbers, even this model is turbulent at high Reynolds numbers. This presumably approximates the possible laminar solutions of the Navier–Stokes equations, which can at best reproduce only the qualitative features of natural flows. It should also be noted that the quasi-linearization of the convective terms concerning the Oseen equations makes the model inaccurate in near-field close to the sudden expansion. This also sheds light on the complex phenomena in understanding the importance of swirl.

Another exciting area in mechanical engineering is software simulations. Subgrid-scale turbulence models have been tested in [133] in a square duct for incompressible flow turbulence. They used a numerical simulation database to capture this flow pattern. Germno et al. [134] developed the first SGS dynamic model called the DSM model. Sarveti and Banerji [135] created the second one and named it as DTM model. There were some strengths and flaws with these models. For example, the DSM model over-predicts the subgrid-scale dissipation on average, whereas the DTM model showed the relevant result of SGS dissipation. The current authors assessed both models in large-eddy simulations of stream-wise corner flows. For DSM, they used a Fourier cut-off filter and a modified Gaussian filter, whereas, in DTM, they used only the modified gaussian filter. Generally, the SGS filtering should be done without homogenous directions in a mixture of complex flows. Their idea was to implement the concept of the SGS filter with all three directions using the present method. They got inspired by authors Salvetti and Banerjee [135], Zang [136], and Najjar and Tafti [137], who worked in the same area. There was a problem of second-order communication errors when filtering in wall-normal directions. Vasilyev et al. [138] had developed solutions to this problem which these authors followed.

This review provides the knowledge of finding the different numerical methods by FEM, FDM, FVM, and BVM. Also, these methods help us to understand their advantages and disadvantages as depicted in Table 1.

| Methods                | Advantages                                                                 | Disadvantages                                        |
|------------------------|----------------------------------------------------------------------------|-------------------------------------------------------|
| Finite Element Method  | ● Suitable for Symmetrical and sparse matrices.                             | ● Can’t be done for infinite problem cases, also domain meshing is needed. |
| (FEM)                  | ● Integration of simple functions can be easily made.                      | ● Its computation is a time-consuming process.         |

Table 1. Advantages and Disadvantages of Numerical Methods.
### Finite Difference Method (FDM)
- Simplest method among FEM, FVM, BEM to implement.
- Doesn’t require any numerical integration.
- Very fine grids are required to solve problems.
- Requires domain meshing and is time-consuming.

### Finite Volume method (FVM)
- Ability of adaptive mesh, and can be utilized for unstructured grids.
- Appropriate for turbulence.
- Especially while solving non-conservative laws, this method can be considered less efficient.
- False diffusion and is biased towards edges.

### Boundary Element Method (BEM)
- Here it is suitable for infinite problems and the computation process is less time-consuming compared to other methods.
- Discretization of boundary
- Integral relations can be complicated.
- Non-symmetric matrices.

From literature studies, out of FEM, FDM, FVM, and BEM, we observed that the Finite Difference Method (FDM) is considered to be more suitable for solving fluid flow and heat transfer problems due to its simplicity, efficiency, and less computational time. FDM is chiefly easy to acquire higher-order schemes on regular grids since regular grids are useful for very-large-scale simulations on supercomputers often used in, as mentioned before, meteorological, seismological, and astrophysical simulations.

### 4. Conclusion

Much automated software had gained attention on numerical simulation modeling in heat transfer and fluid mechanics. This review has given an idea to simulate transient and steady-state models with boundary conditions of fluids as input and can be separated as layers based on density. In the current study, different numerical methods have been applied to different fundamental modeling approaches in heat and mass transfer and fluid mechanics applications. Also, it is dedicated to demonstrating how methods such as FEM, FDM, FVM, and BEM are beneficial in solving the real-life heat and mass transfer problems and fluid mechanics in engineering problems. Comparison of various numerical methods has been done to observe their validity and ease of solution for further engineering applications.

### 5. Future Scope

Many researchers are held in the field of advanced elastic solutions. In the future, to study more on the stability of fully localized solitary waves and their solutions. Various studies are on generalizing the algorithm and calculating 3D gravity–capillary flows based on time-dependent models. There are three main areas where developments have to be made for the LES model to be implemented in industrial flows. They are devising an appropriate method for the filtration in homogeneous and non-homogeneous directions, increasing the Re and wall resolution. However, the research is extending to designing software for creating and analyzing various heat transfer devices on their parameters such as material, dimensions, fittings, space positioning, and heat flow rate for effective and efficient output.
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