The stress intensity factor of the crack on the interface between homogeneous material and FGMs

Li Shimin\textsuperscript{1}, Zhao Xinjun\textsuperscript{1}, Li Minjie\textsuperscript{2}

\textsuperscript{1}Beijing Special Vehicle Institute, Beijing 100072, People’s Republic of China

\textsuperscript{2}The Automobile Proving Ground of the General Armament Department, Nanjing 210028, People’s Republic of China

E-mail: lishimin0625@163.com

Abstract. Based on higher-order asymptotic stress fields for the interfacial crack between homogeneous material and functionally graded materials (FGMs), stress intensity factors were calculated by finite element and node collocation methods. The method was verified by comparing the results with numerical solutions obtained in other studies, with varying degrees of homogeneity. The comparison showed that the obtained results agree well with the numerical solutions.

Keywords: Stress intensity factor; Interface crack; Functionally graded materials

1. Introduction

Stress intensity factor (SIF) is the most important parameter in the fracture analysis of linear elasticity. Analytical method, numerical solution and experimental calibration \cite{1} are the common approaches to calculate SIFs. Analytical methods, including Westergaard stress function method, general complex variable function method, integral transform method, series expansion, Green function method, can be used to solve simple problems such as crack in an infinite plate. For complicated problems, analytical solutions are difficult to obtain, and numerical methods must be applied to obtain SIFs. Numerical methods contain include finite difference method (FDM), boundary collocation method (BCM), finite element method (FEM), boundary element method (BEM), etc. A drawback of numerical methods is that they cannot show the influence of structure size and non-homogeneity on the SIFs explicitly. Experimental calibrations include flexibility method, method of lattice, photo-elastic method, laser holography and laser speckle method, moiré interferometry, etc. Experimental calibrations can only measure the crack tip SIFs in special structures and under external loads. Complex crack problems are usually solved by semi-analytical numerical methods, including integral transform method, Williams’ stress function-boundary element method and conformal transformation-boundary element method.

Among all the mentioned methods for calculating SIFs, the integral transform method is the most popular among researchers; it has been used to solve a large number of crack problems by Erdogan \cite{2}, Wu \cite{3}, Dag \cite{4} and Ozturk \cite{5}. As of the other methods, the conformal transformation-boundary element method has been used by Parameswaran \cite{6} to calculate the former six items in the crack tip stress field of mode I, and discuss the non-homogeneous effect on stress constant value line. Using the Westergaard stress function method, Chalivendra \cite{7} determined the oblique crack tip stress field with the elastic modular varying along the gradient in exponent form. The same method has also been used to determine the crack tip field of anisotropy materials \cite{8}. The Williams’ stress function-
boundary element method, however, has rarely been reported for obtaining SIFs, possibly because (1) it is difficult to obtain the Williams’ solution of stress crack tip field for non-homogeneous and composite structures, and (2) the collocation region is affected by the number of items, although the relation in between has not been proved.

Figure 1 presents a structure where the interface parallels to the x-axis, and the gradient direction parallels to the y-axis. Below the interface is a matrix of homogeneous material, while above the interface is Functionally Graded Materials (FGMs). The elastic modulus function of the FGMs is assumed to be $E(y) = E_1 e^{\beta y}$. Measures of the structure are shown in the diagram. To simplify the analysis, the corresponding polar coordinate has been established. The purpose of this study is to calculate the stress intensity factor of this structure.

2. Higher-order asymptotic stress fields of FGMs with homogenous matrix

According to Reference [9], the first six items of the interface crack tip higher-order asymptotic stress fields are as follows, where $-\pi \leq \theta \leq 0$.

\[
\sigma_y = -\frac{3}{4} B_{11} (5\cos^2 - \cos^3) + B_{12} (5\cos^2 - \cos^3) + B_{11} (5\cos^2 - \cos^3) + B_{12} (5\cos^2 - \cos^3)
\]

In the FGMs region ($0 \leq \theta \leq \pi$),

Figure 1. Interfacial crack of FGMs-coating
\[
\sigma_y = r \left[ \frac{3}{4} B_{11} (5\cos \frac{\theta}{2} \cdot \cos \frac{5\theta}{2}) + \frac{1}{4} B_{12} (\sin \frac{\theta}{2} \cdot \sin \frac{5\theta}{2}) \right] + r^2 \left[ \frac{15}{4} B_{31} (3\cos \frac{\theta}{2} + \cos \frac{3\theta}{2}) + \frac{3}{4} B_{32} \right]
\]

\[
\left( \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right) - \frac{3}{16} \beta \left( \sin \frac{7\theta}{2} - 5\sin \frac{3\theta}{2} - 4\sin \frac{\theta}{2} \right) + \frac{1}{16} \beta \left( B_{22} (\cos \frac{7\theta}{2} - 17\cos \frac{3\theta}{2} - 1\cos \frac{\theta}{2}) \right)
\]

\[
\frac{3}{12} \left[ \frac{35}{B_{11}} (3\cos \frac{\theta}{2} + \cos \frac{3\theta}{2}) + \frac{15}{4} B_{52} (\sin \frac{\theta}{2} + \sin \frac{3\theta}{2}) + \frac{3}{16} \beta B_{32} (10\cos \frac{\theta}{2} + 5\cos \frac{3\theta}{2} + \cos \frac{5\theta}{2}) \right]
\]

\[
\frac{15}{16} \beta \left( 2\sin \frac{\theta}{2} + 3\sin \frac{3\theta}{2} + \sin \frac{5\theta}{2} \right) + \frac{1}{128} \beta^2 B_{11} (3(63\mu + 71)\cos \frac{\theta}{2} + 15(7\mu + 5)\cos \frac{3\theta}{2} + 3(9\mu - 1)\cos \frac{5\theta}{2})
\]

\[
3(9\mu - 1)\cos \frac{5\theta}{2} - (\mu - 3)\cos \frac{9\theta}{2} + \frac{1}{128} \beta^2 B_{12} (7(\mu - 7)\sin \frac{\theta}{2} + 7(5\mu - 11)\sin \frac{3\theta}{2} + (5\mu - 29)\sin \frac{5\theta}{2})
\]

\[
+(1 - \frac{\mu}{3})\sin \frac{9\theta}{2} + or^2 \left( B_{61} \cos \frac{3\theta}{2} - 1 \right) + \cdots
\]

In Eq. (1), \( A_{ij} \) and \( B_{ij} \) ( \( i = 1 \ldots 6, \ j = 1, 2 \) ) are unknown constants obtained by the far field boundary conditions. Then, the SIFs of mode I and II can be obtained as

\[
K_I = \lim_{r \to 0} \sqrt{2\pi r} \cdot \sigma_y (r, 0) = 3B_{11} \sqrt{2\pi}, \quad K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \cdot \sigma_{y\theta} (r, 0) = -B_{12} \sqrt{2\pi}
\]

3. Finite element simulation

The composites model was used to simulate the change of the FGMs, and then the model of the FGMs was simplified, as shown in Figure 2. A quadrilateral element was adopted in the lattice division and refined in the field of \( \frac{r}{a} \leq 1 \). In the plane stress condition, a uniform load \( \sigma_0 = 20 \text{ Mpa} \) was applied to the upper face, while the lower face remained fixed. Because of the symmetrical loads, the x-direction displacement on the left boundary was 0. The field \( \frac{r}{a} \leq 1 \) near the crack tip field was divided into two fields, with the quadrilateral elements of \( \frac{a}{50} \) and \( \frac{a}{100} \), respectively. The SIFs were calculated accordingly by finite element method and node collocation method, and the difference
between the calculated SIFs was only 1.2%. Therefore, the quadrilateral element of $\theta/50$ was adopted to meet the precision requirement and save calculation time. The stress results were obtained from the element nodes in the field where $0 \leq \theta \leq \pi$ and $0.1 < r \leq 1.0$, and the node collocation region was $R_{i-1} < r \leq R_i$ ($R_i = 0.1a \times i, i = 2 \cdots 10$).

Based on Eq. (1), the unknown coefficients of the asymptotic stress field were calculated by least square method. According to the definition of SIFs, $K_i = 3\sqrt{2\pi B_{11}}$, $K_i$ was acquired. The normalized SIFs in different node collocation regions, as shown in Figure 3, were compared with the results of Reference [2]. The black triangles in Figure 3 denote the normalized SIFs. The node collocation calculation stops when the difference of solution between the study and the references is greater than 10%. The field adopted in this study was basically within $R/a \leq 0.6$.

Figure 4 shows the comparison between the numerical solutions obtained in this study and those of Reference [2]. In the diagram, $E_i = 4800$ MPa, $\mu = 0.3$ and the measures of the structure are $h_1 = 100$ mm, $h_2 = 1$ mm, $a = 1$ mm, $b = 100$ mm. It can be seen that as the degree of non-homogeneity of the exponent FGMs or transition layer increases, the SIFs on the physical discontinuous line will minish.
The relative errors of the obtained results are listed in Table 1. It can be seen that the errors are all less than 4%, indicating good accuracy of the results. The reasonability and validity of the crack tip field on the physical discontinuous line are thus proved.

### Table 1. Relative error of the results obtained in this study based on those of Reference [2]

| $\beta a$ | Results of Reference [2] | Results in this study | Relative error |
|---------|--------------------------|-----------------------|----------------|
| -2      | 2.087                    | 2.03                  | -2.7311931     |
| -1.5    | 1.936                    | 1.992                 | 2.874175153    |
| -1.25   | 1.866                    | 1.895                 | 1.548264206    |
| -1      | 1.799                    | 1.738                 | -3.390772651   |
| -0.75   | 1.735                    | 1.722                 | -0.756442846   |
| -0.5    | 1.675                    | 1.649                 | -1.545406861   |
| -0.25   | 1.618                    | 1.607                 | -0.679851669   |
| -0.01   | 1.566                    | 1.556                 | -0.66234129    |
| 0.25    | 1.514                    | 1.5                   | -0.924702774   |
| 0.5     | 1.466                    | 1.428                 | -2.581329598   |
| 0.75    | 1.422                    | 1.417                 | -0.321812436   |
| 1       | 1.38                     | 1.362                 | -1.304347826   |
| 1.5     | 1.304                    | 1.292                 | -0.930761041   |
| 2       | 1.237                    | 1.276                 | 3.152789006    |

### 4. Conclusions

By comparing the results obtained in this study with numerical results, the reasonability and validity of the crack tip field on the physical discontinuous line were proved. Meanwhile, this field can effectively describe the whole field stress and the effect of the degree of non-homogeneity on the crack tip stress field. The following conclusions were made.
(1) The first six items of the crack tip stress field on the physical discontinuous line of exponent FGMs/homogeneous materials can exactly depict the crack tip stress in the range of $r/a \leq 0.6$. With this range, the three-dimensional effects of the crack tip can be avoided.

(2) As the degree of non-homogeneity of the exponent FGMs or transition layer increases, the SIFs on the physical discontinuous line will minish.

(3) The results obtained in this study were a full-field asymptotic description. The physical discontinuous line possesses the characters of the general solutions. It can explicitly depict the crack tip stress field in the condition of different structures, external loads and effect of elastic modulus gradient. This method in the study can improve the assessment of the fracture coefficient in fracture experiments, and provides valuable reference for future studies.

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