Pricing Multivariate European Equity Option Using Gaussians Mixture Distributions and EVT-Based Copulas.

A Preprint

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May 25, 2021

Abstract

In this article, we present an approach which allows to take into account the effect of extreme values in the modeling of financial asset returns and in the valorisation of associated options. Specifically, the marginal distribution of assets returns is modeled by a mixture of two gaussiens distributions. Moreover, we model the joint dependence structure of the returns using a copula function, the extremal one, which is suitable for our financial data, particularly the extreme value copula. Applications are made on the Atos and Dassault Systems actions of the CAC40 index. Monte-Carlo method is used to compute the values of some equity options such as the call on maximum, the call on minimum, the digital option and the spreads option with the basket (Atos, Dassault systems) as underlying.

Keywords: Options, Extremes values, Gaussians mixture; Copulas; Monte-Carlo; European market.

2010 Mathematical Subject Classification: 60E05; 91B24; 91G20; 91G60; 91G70.
2010 JEL classification: C02; F31; G13.

1 Introduction

Since the pioneering work of Black-Scholes [1] and Cox and al. [6] (respectively in the continuous and discrete case), option pricing has become a crucial topic in finance. Indeed, considering an european-type option on an underlying asset with a price $S_t$; strike $K$ and expiration $T$, Black and Scholes have made it possible to determine a formula for the price of such options under certain assumptions, the fundamental of which are the lack of arbitrage opportunity and that on the price $S_t$ of the asset underlying ($S_t$ follows geometric Brownian motion), i.e.

$$dS_t = \mu S_t dt + \sigma S_t W_t,$$

where $\mu$ and $\sigma$ are constant and $W_t$ is a standard geometric Brownian motion.
Thus, the formula of the relative theoretical value \( \frac{C_t}{S_t} \) of a call option is then given by:

\[
C_t^{\text{call}} = N(d_1) - \kappa e^{-r(T-t)} N(d_2),
\]

and the relative theoretical value of a put is given by:

\[
P_t^{\text{put}} = -N(-d_1) + \kappa e^{-r(T-t)} N(-d_2),
\]

where \( N(.) \) is the standard normal distribution; \( d_1 = \frac{(r + \frac{\sigma^2}{2})(T - t) - \ln(k)}{\sigma \sqrt{T - t}} \); \( d_2 = d_1 - \sigma \sqrt{T - t} \) and \( \kappa = \frac{K}{S_t} \) the relative strike and \( r \) the risk-free interest rate.

Options are essential financial products allowing to their holders to hedge against the risk of falls in their investments. This is how we are increasingly seeing the creation of several types of options such as exotic options; multivariate options; etc., with the aim of providing more security. As a result, valuation models are also evolving. Of all the multiple option pricing models, it turns out that each one is primarily based on the dynamics of underlying asset pricing model (for options with only one underlying) or the asset portfolio (for options on multiple assets), when market assumptions are known. In fact, since the assumption of no arbitrage opportunity (NAO) in the markets is the basis of the fundamental results obtained in finance, it’s considered by default\(^1\). The advantage under this NAO assumption is that, associated with that of market completeness, there is a single risk-neutral probability for which the discounted flows are martingales. In the univariate case, one of the most interesting results obtained in this direction on valuation is that of Breeden and al.\(^3\). It states that the second derivative (when it exists and continuous) of the price of a standard option relatively to the strike coincides with the risk-neutral density. Indeed, if \( D_t \) is the price of a european option of an underlying asset with price \( X_t \) having for pay-off \( g(X_T) \), \( T \) the time to expiration and \( r \) the risk-free interest rate; then the risk-neutral density \( f^*(X_T) \) is linked to \( D_t \) by:

\[
D_t = e^{-r(T-t)} \mathbb{E}^* \{ g(X_T) \} = e^{-r(T-t)} \int g(X_T) f^*(X_T) dX_T.
\]

For valuation in the multivariate framework, this risk-neutral formula is a simple generalization. Talponen and al.\(^{19}\) recently gave the multivariate version of the univariate result.

Multivariate options (rainbow; digital; quantos; etc.), which will be the main subject of our study in this paper, constitute the central themes of current research on financial risk coverage. The advantage lies in the fact that they offer better coverage against risks. Indeed, the basic idea is that when the option is a function of several assets, the fall in value of one asset is compensated by the rise of another asset in the portfolio. Thus the association or dependence between assets plays a major role in the pricing of these types of options. To take such an aspect into account in the valuation, the use of the copula is a good alternative.

The valuation of multivariate options by copulas is in full development. The copula gives the advantage of joining the marginal and the dependence structure. This is the case for many works on the valuation of options with copulas, the emphasis is first on marginal risk-neutral densities and then on the joint risk-neutral density (risk-neutral copula). For example, we can cite the work of, Cherubuni and al.\(^4\); Cherubuni and al.\(^5\); Rosenberg and al.\(^11\); Salmon and al.\(^13\); Slavchev and al.\(^17\). However, all this work did not take into account the effects of extreme values in the marginal, which is not without effect on valuation (risk of over-valuation or under-valuation). However, there are other copula modeling approaches based on volatility dynamics as in Goorbergh and al.\(^20\).

In this present study, we propose a valuation method for multivariate options allowing to take into account the effects of extreme values in the marginal and the joint structure on the basis of the works of Edier et al.\(^7\) and the use of extreme values copulas.

In the rest of this work, the first section we give the results obtained by Idier et al.\(^7\), which will be

\(^1\)There are markets on which the arbitration assumption is considered.
necessary and some essential notions on copula. In the second section, we expose the methodology used for leading properly the application of the demarche. Then, the obtained results of different estimations and simulations are presented, with their analysis and interpretations. The last section present a conclusion and discussion.

2 Preliminaries

2.1 Results of an approach of modeling financial assets

It is proved that the empirical distribution of financial assets returns has thicker tails than that of the Gaussian distribution. This indicates the presence of extreme values. This fact shows also that the normal distribution does not make it possible to model rigorously the returns of financial assets because it does not take into account the extremes. This is the case with the method proposed by Black-Scholes.

To take into account the effects of extreme values, Idier et al. [7] proposed, as an alternative to the normal distribution, to model the distribution of the rates of return of the underlying asset of a univariate option, under NAO assumption, by a mixture of Gaussian distribution in the continuous framework\(^3\). They justified their choice by the fact that a mixture of Gaussian distributions makes it possible to approximate all the distributions usually used (Gaussian; alpha-stable; Student; hyperbolic; etc.); also that it has certain theoretical properties allowing easy handling in the framework a theoretical model for valuing asset price; that it is easy to simulate and can reproduce various sets (mean, variance, skewness and kurtosis) observed in the data.

Under the assumption that the historical distribution of the returns of the underlying asset \(X_{t+1} = \ln\left(\frac{S_{t+1}}{S_t}\right)\) where \(S_t\) is the price at time \(t\) of the underlying asset, is a mixture of 2 Gaussian distributions. Its density is given by:

\[
f(x) = \sum_{i=1}^{2} p_i n(x, \mu_i, \sigma_i^2),
\]

where

\[
n(x, \mu_i, \sigma_i^2) = \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left\{ -\frac{(x - \mu_i)^2}{2 \sigma_i^2} \right\},
\]

is the density of a Gaussian distribution with mean \(\mu_i\) and standard deviation \(\sigma_i\); \(0 < p_i < 1\) and \(\sum_{i=1}^{2} p_i = 1\).

Moreover, the stochastic discount factor is characterized by an affine exponential form, i.e.

\[
M_{t;+1} = \exp\{\alpha t X_t + \beta t\}.
\]

They establish, under these assumptions, that the risk-neutral distribution is also a Gaussian mixture and that its density \(f^*\) is defined by:

\[
f^*(x) = \sum_{i=1}^{2} v_i n(x, \mu_i + \alpha \sigma_i^2, \sigma_i^2),
\]

where

\[
v_i = \left( p_i \exp\{\mu_i \alpha + \frac{\alpha^2 \sigma_i^2}{2}\} \right) / \left( \sum_{i=1}^{2} p_i \exp\{\mu_i \alpha + \frac{\alpha^2 \sigma_i^2}{2}\} \right) \text{ with } 0 < v_i < 1, \sum_{i=1}^{2} v_i = 1.
\]

\(^3\)Their method is a generalization of the method in the discrete case of Bertholon, Monfort and Pegoraro(2006); Pegoraro(2006)
Thus, they derive the relative theoretical price of an European call with a one-period maturity \((T = t + 1)\) and a relative strike \(k\):

\[
c_t(T, k) = \sum_{i=1}^{2} v_i \gamma_i c_{bs}(\sigma_i^2, \frac{k}{\gamma_i}),
\]

(7)

where \(c_{bs}(., .)\) is the Black-Scholes one period \((T = t + 1)\) formula for the relative price of a call and \(\gamma_i = \exp\{\mu_i + \alpha \cdot \sigma_i^2 - r + \rho_i^2\}\), for \(J = 2\).

**Remark 1** The existence of the call-put parity relation makes it possible to simplify the task in calculating option price. It is then sufficient to calculate the price of the call to deduce that of the corresponding put (or vice versa) by the relation:

\[
C_t(T, K) + Ke^{-r(T-t)} = P_t(T, K) + S_t.
\]

(8)

2.2 A survey of copulas

In this section we recall the basics notions on copulas. These are the definitions and properties essential for our study. For more details on copulas, see Nelsen [10].

2.2.1 Definitions and properties

The copula is a function allowing to capture the structure of dependence between several random variables.

A function \(C : [0, 1]^d \rightarrow [0, 1]\) is a d-copula if it satisfies the following properties:

i) For all \(u\) in \([0, 1]\), \(C(1, ..., 1, u, 1, ..., 1) = u\);

ii) For all \(u_i\) in \([0, 1]\), \(C(u_1, ..., u_d) = 0\) if at least one of the \(u_i\) is zero;

iii) \(C\) is "grounded" and d-increasing, i.e

\[
\sum_{i_1=1}^{2} ... \sum_{i_d=1}^{2} (-1)^{i_1 + ... + i_d} C(u_{1,i_1}, ..., u_{d,i_d}) \geq 0,
\]

(9)

for all \((u_{1,1}, ..., u_{d,1})\) and \((u_{1,2}, ..., u_{d,2})\) in \([0, 1]^d\) with \(u_{d,1} \leq u_{d,2}\).

The fundamental result on the copula due to Sklar (1969) states that: For a whole multivariate distribution \(F\) with continuous marginal \(F_1, ..., F_d\), then there exists a unique[3] copula \(C : [0, 1]^d \rightarrow [0, 1]\) such that:

\[
F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)).
\]

(10)

Conversely, when \(C\) is a copula and \(F_1, ..., F_d\) are marginal distributions, the function \(F\) defined by (10) is a multivariate distribution of marginal distributions \(F_1, ..., F_d\).

This result makes it possible to deduce several properties of the copula including invariance by any monotonic transformation. Another consequence of Sklar’s theorem is that every copula \(C\) satisfies

\[
max \left( \sum_{i=1}^{d} u_i - d + 1; 0 \right) \leq C(u_1, u_2, ..., u_d) \leq min(u_1, u_2, ..., u_d).
\]

(11)

This relation is the variant in terms of copulas of the Frechet-Hoeffding bounds of a multivariate distribution. The upper bound \(min(u_1, u_2, ..., u_d)\) is the comonotonic copula representing the perfect positive dependence. The lower bound \(max \left( \sum_{i=1}^{d} u_i - d + 1; 0 \right)\) is a copula only for \(d = 2\). In this case it represents the perfect negative dependence.

[3] Uniqueness is not guaranteed when marginal are not continuous.
Remark 2 If \( \bar{F} \) is the multivariate survival distribution of a \( F \) distribution of marginals \( F_i; \ i = 1, \ldots, d \), then the survival copula, denoted by \( \bar{C} \), is defined by:

\[
\bar{F}(x_1, \ldots, x_d) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)).
\]

The survival copula \( \bar{C} \) is related to the copula \( C \), for all \( (u_1, u_2, \ldots, u_d) \in [0,1]^d \), by:

\[
\bar{C}(u_1, u_2, \ldots, u_d) = \sum_{M \subset N} (-1)^m C((1 - u_1)^{1\in M}, (1 - u_2)^{1\in M}, \ldots, (1 - u_d)^{1\in M})
\]

where \( N = \{1, 2, \ldots, d\} \), \( m = |M| \) is the cardinality of \( M \), and \( 1_i \in M \) indicates that \( i \) belongs to \( M \).

It is therefore advisable not to confuse the dual copula with the survival copula.

2.2.2 Sample of Copulas for finance in this study

Archimedian copulas

In the literature, there are several families of copulas and some of which are more suited to financial modeling. Archimedian copulas family includes the models of Clayton, Frank and Gumbel. These copulas have the advantage of capturing the structure of positive or negative dependence between the variables. These types of dependences are characteristics of financial variables, which justifies the use of this copula family. In terms of option pricing, for example, these copulas have been used in Cherubuni and Luciano (4) and in Slavchev and Wilkens (17).

Table 1: Examples of Archimedian copulas.

| Family | Archimedian generator | Copula \( C_\theta(u), u = (u_1, \ldots, u_d) \) |
|--------|-----------------------|-----------------------------------------------|
| Clayton | \( \psi_\theta(t) = -\theta -1, \theta > 0 \) | \( C_\theta(u) = \left( \sum_{i=1}^d (u_i)^{-\theta} - d + 1 \right)^{-\frac{1}{\theta}} \) |
| Frank  | \( \psi_\theta(t) = -\frac{1}{\theta} \log (1 - (1 - e^{-\theta})e^{-t}), \theta > 0 \) | \( C_\theta(u) = -\frac{1}{\theta} \log \left( 1 - \prod_{i=1}^d (e^{-u_i\theta} - 1) \right) \) |
| Gumbel | \( \psi_\theta(t) = (-\ln t)^\theta, \text{with } \theta \geq 1 \) | \( C_\theta(u) = \exp \left( -\left( \sum_{i=1}^d (-\log u_i)^\theta \right) \right) \) |

Elliptical copulas

Other types of copulas used in finance are the normal copula and the t-copula. They belong to the family of elliptical copulas which describe the dependence structure of elliptical distributions. The choice of this family is justified by the fact that elliptic distributions have long been used to model random phenomena in many fields. Despite the demonstration of the leptokurtic character of the returns of financial series, of which they have the weakness to rigorously model, their continued use in finance.

The normal copula is also known as the Gaussian copula. Its expression is:

\[
C_\Sigma(u_1, u_2, \ldots, u_d) = \int_{-\infty}^{N^{-1}(u_1)} \cdots \int_{-\infty}^{N^{-1}(u_d)} \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} y^\top \Sigma^{-1} y \right) dy,
\]

where \( N^{-1} \) is the quantile of the standard normal distribution and \( \Sigma \) is the correlation matrix.
The t-copula is the one describing the dependence of t-distributions. It has the advantage of capturing the dependency in the distribution tails more than the normal copula. Its expression is given by:

\[
C_{v, \Sigma}(u_1, u_2, ..., u_d) = \int_{-\infty}^{t^{-1}_v(u_1)} ... \int_{-\infty}^{t^{-1}_v(u_d)} \Gamma\left(\frac{v+d(|\Sigma|^{1/2})}{v/2}\right) \left(1 + \frac{1}{v} y' \Sigma^{-1} y\right)^{-d/2} dy,
\]

where \(t^{-1}_v\) is the quantile of the t-distribution with \(v\) degree of freedom; \(\Gamma\) is the gamma function and \(\Sigma\) the correlation matrix.

### 2.2.3 Estimation-Adequacy test of a copula

The choice of the copula that rigorously describing multivariate statistical data requires estimation and conformity testing. There are several techniques in the literature for estimating copulas belonging to different families: parametric; semi-parametric and non-parametric. For more details on these methods, see Bouyé [2].

#### Estimation by the IFM method

The IFM method (inference functions of margins) is a two-step estimation method of a copula. It was presented by Shih and Louis [14] in the bivariate case and then developed in dimension greater than two by Joe and Xu [8]. It is carried out as follows:

1) the first step consists in finding the estimators \(\hat{\alpha}_i\) of the parameters \(\alpha_i, \ i = 1, ..., d\) for marginal distributions by maximum likelihood:

\[
\hat{\alpha}_i = \arg\max \sum_{j=1}^{N} \ln f_{n}(x^j_i; \alpha_i);
\]

2) Once the marginal have been determined, we estimate the parameter \(\theta\) of the copula that best describes these marginal by the maximum likelihood.

\[
\hat{\theta} = \arg\max \sum_{j=1}^{N} \ln c(F_{1}(x^j_1; \hat{\alpha}_1); ...; F_{d}(x^j_d; \hat{\alpha}_d); \theta).
\]

One of the advantages of this method is that under certain conditions of regularity, the IFM estimator is consistent and asymptotically normal.

Also, in terms of numerical computation time, this method is better than the "direct" maximum likelihood method since it is simpler and faster.

#### Fit test

To confirm whether the chosen parametric copula models the data well, it is necessary to perform a test. The most powerful tests are based on the processes \(\sqrt{n}(\hat{C} - C_{\theta})\), where \(C\) and \(C_{\theta}\) are respectively the empirical copula and the parametric copula.

The Cramer-von Mises statistic is by far the most used because it gives satisfactory results. It is defined by:

\[
\int_{[0;1]^d} n(\hat{C} - C_{\theta})d\hat{C}.
\]
\[ AIC = 2m - \log(l(\theta)) \]  
\[ BIC = m \log(n) - \log(l(\theta)). \]  

where \( l(\theta) \) is the model likelihood for the estimated parameter \( \theta \), \( m \) the number of estimated parameters and \( n \) the data size.

3 Methodology and Application

The price of a multivariate option is a function of the density of the joint distribution. Thus, their valuation requires the determination of the joint risk-neutral density. To do this, it suffices to determine the marginal risk-neutral densities and then to choose the copula that best describes their dependence structure by using Sklar’s theorem. This perspective is possible because the objective copula can be matched with the risk-neutral joint copula, under certain conditions (see Rosenbergh [12]).

3.1 Methodology

Our approach consists firstly in determining the marginal risk-neutral distributions by the procedure used by Idier and al. [7]. This in order to take into account the effect of extreme values in the margins. We will also limit ourselves to the case of a mixture of two Gaussians in this study. Then, we will choose among the families of copulas listed in the section 2.2.2 the one that best suits the study. And finally we will determine the prices of the multivariate options by numerical integration (Monte-Carlo method) by using the formulas provided below for multivariate options considered.

We will be particularly interested by rainbow options (those relating to the maximum or the minimum of several assets, etc.). These kinds of options have been the subject of many studies as in Stulz [18] and Jonshon [9].

Consider \( d \) assets whose price at maturity \( T \) are denoted by \( S_1^T; ...; S_d^T \) and denote by \( X_1^T; ...; X_d^T \), respectively, the returns associated to each asset at instant \( T \) (with for all \( i = 1, ..., d; X_i^T = \log(S_i^T/S_i^t) \)). For a chosen strike price \( K \), we consider the following different types of rainbows: spread option; option on the maximum; option on the minimum and digital option.

3.1.1 Spread Option

Having a pay-off equal to \( \max(S_2^T - S_1^T - K; 0) \), its value is calculated by:

\[
V_{OS}(t) = e^{-r(T-t)} \mathbb{E}^* \left\{ \max(S_2^T - S_1^T - K; 0) \right\} = e^{-r(T-t)} \mathbb{E}^* \left\{ \int_{\mathbb{R}} \mathbb{I}_{\{K+S_1^T \leq x \leq S_2^T\}} dx \right\}
\]

which gives,

\[
V_{OS}(t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \mathbb{P}^* \{ K + S_1^T \leq x \} - \mathbb{P}^* \{ K + S_1^T \leq x \text{ et } S_1^T \leq x \} dx
\]

and finally,"
where

\[ x_i = \frac{x}{S_i}, \quad k_i = \frac{K}{S_i}, \]

and \( P_{t,i} \) is the put price’s, for \( i = 1; 2 \).

### 3.1.2 Call on the maximum

Its pay-off is equal to \( \max \{ \max(S^T_1; \ldots; S^T_d) - K; 0 \} \). Thus, its price at maturity is given by:

\[
V_{CMax}(t) = e^{-r(T-t)} \mathbb{E}^* \{ \max \{ \max(S^T_1; \ldots; S^T_d) - K; 0 \} \} = e^{-r(T-t)} \mathbb{E}^* \left\{ \int_{\mathbb{R}} \mathbb{I}_{k \leq \max(S^T_1; \ldots; S^T_d)} \, dx \right\}
\]

we obtain,

\[
V_{CMax}(t) = e^{-r(T-t)} \int_{K}^{+\infty} 1 - \mathbb{P}^* \{ S^T_1 \leq x; \ldots; S^T_d \leq x \} \, dx
\]

and finally,

\[
V_{CMax}(t) = e^{-r(T-t)} \int_{K}^{+\infty} 1 - C \left( e^{r(T-t)} \frac{\partial P_{t,1}}{\partial k_1}(T; x_1); \ldots; e^{r(T-t)} \frac{\partial P_{t,d}}{\partial k_d}(T; x_d) \right) \, dx
\]

where \( x_i = \log(\frac{x}{S^i_t}) \) and \( P_{t,i} \) is the put price’s, for \( i = 1; 2 \).

### 3.1.3 Call on the minimum

It admits for pay-off \( \max \{ \min(S^T_1; \ldots; S^T_d) - K; 0 \} \) and its value at maturity is then defined by:

\[
V_{CMin}(t) = e^{-r(T-t)} \mathbb{E}^* \{ \max \{ \min(S^T_1; \ldots; S^T_d) - K; 0 \} \} = e^{-r(T-t)} \mathbb{E}^* \left\{ \int_{\mathbb{R}} \mathbb{I}_{k \geq \min(S^T_1; \ldots; S^T_d)} \, dx \right\}
\]

which is equal to,

\[
V_{CMin}(t) = e^{-r(T-t)} \mathbb{E}^* \left\{ \int_{K}^{+\infty} \mathbb{I}_{\min(S^T_1; \ldots; S^T_d) \geq x} \, dx \right\} = e^{-r(T-t)} \int_{K}^{+\infty} \mathbb{P}^* \{ S^T_1 \geq x; \ldots; S^T_d \geq x \} \, dx
\]

at the end, we obtain

\[
V_{CMin}(t) = e^{-r(T-t)} \int_{K}^{+\infty} \mathbb{P}^* \{ X^T_1 \geq x_1; \ldots; X^T_d \geq x_d \} \, dx
\]

\[
= e^{-r(T-t)} \int_{K}^{+\infty} C \left( -e^{r(T-t)} \frac{\partial C_{t,1}}{\partial k_1}(T; x_1); \ldots; -e^{r(T-t)} \frac{\partial C_{t,d}}{\partial k_d}(T; x_d) \right) \, dx
\]

where \( x_i = \log(\frac{x}{S^i_t}) \) and \( C_{t,i} \) is the call price’s, for \( i = 1; 2 \).
3.1.4 Digital Option

It has for pay-off \( I\{S_T^i \geq K_i; \ldots; S_T^d \geq K_d\} \). Thus, its value at maturity is given by

\[
V_{ODig}(t) = e^{-r(T-t)}E^*\left\{ I\{S_T^i \geq K_i; \ldots; S_T^d \geq K_d\} \right\} = e^{-r(T-t)}E^*\left\{ S_T^i \geq K_i; \ldots; S_T^d \geq K_d\right\}
\]

which give us,

\[
V_{ODig}(t) = e^{-r(T-t)}C\left( -e^{r(T-t)}\frac{\partial C_{t,1}}{\partial k_1}(T;k_1); \ldots; e^{r(T-t)}\frac{\partial C_{t,d}}{\partial k_d}(T;k_d) \right)
\]

where \( k_i = \frac{K_i}{S_t^i} \), \( C_{t,i} \) is the call \( i \) price’s, for \( i = 1; \ldots; d \) and \( C \) the survival copula.

Remark 3 Not to forget that the quantities \(-e^{r(T-t)}\frac{\partial C_{t,d}}{\partial k_d}(T;k_d)\) and \(e^{r(T-t)}\frac{\partial P_{t,d}}{\partial k_d}(T;k)\) are both equal to the risk neutral distribution wich density is given by relation (6) for our study.

3.2 Applications

We will focus on the bivariate options on the pair of Atos and Dassault Systems shares. The data were obtained from Investing.com and relate to the components of the CAC40 index of the Paris stock exchange. The collected data concerns the closing prices for the period from July 01, 2014 to June 30, 2020 (1534 days).

At first, for each of the two assets (Atos; Dassault Systems) the parameters of the two Gaussian regimes constituting the Gaussian mixture are determined (table 2) as well as the proportions of each diet.

| Table 2: Estimated parameters of the Gaussian mixture and their proportion. |
|---|---|---|---|
| Regime 1 | Regime 2 | Gaussian mixture | Empirical distribution |
| Atos | Moyenne | -0.00072328 | 0.0000764489 | 0.00013704 | 0.0001370 |
| | Ecart-type | 0.0603574 | 0.013530408 | 0.02142835 | 0.0214349 |
| | Skewness | 0 | 0 | -0.6131518 | -2.823688 |
| | Kurtosis | 3 | 3 | 15.7 | 42.81632 |
| | Proportion | 0.07845771 | 0.921542310 | – | – |
| Dassault systems | Moyenne | 0.00110506 | -0.00101651 | 0.0007598786 | 0.0007599 |
| | Ecart-type | 0.01014017 | 0.03315738 | 0.01577694 | 0.01630201 |
| | Skewness | 0 | 0 | -0.03745108 | 0.1949504 |
| | Kurtosis | 3 | 3 | 10.00661 | 12.39682 |
| | Proportion | 0.83729906 | 0.16270094 | – | – |

The next step, we determine the parameters \((\alpha; \beta)\) of the stochastic discount factor defined by the relation (5) thanks to the assumptions of the model in particular that of lack of arbitration opportunity. The results obtained for a risk-free rate \(r = 0.025\) are presented in the table 3.

| Table 3: Stochastic discount factor parameters for data returns with a risk-free rate \(r = 0.025\). |
|---|---|
| Atos | Dassault |
| \(\alpha\) | 36.1209027 | 37.70565 |
| \(\beta\) | -0.3610132 | -0.3500945 |
3.2.1 Copulas fitting results

We present the results of the copula estimates associated with our data. In each case, we will base ourselves on the Cramer-Von Mises statistic and/or the AIC criteria for the choice of the best copula.

In the table 4, we present the estimated parameters (and their Cramer-Von statistics) for bivariate copulas. It then emerges that the three copulas with the best Cramer-Von Mises statistic are Tawn’s copula, Frank’s copula, Gumbel’s copula in that order.

The table 5 gives the AIC and BIC of the parameters estimated for the bivariate copulas chosen. Based on these criteria, the four best candidate copulas for our bivariate data are the normal copula, the Husler-Reiss’s copula, the Galambos’s copula and the Gumbel’s copula.

We can notice that for our data, the performance of each fitted copula differs according to the criterion. It is then difficult to make a particular choice on a copula in such situation on the basis of the two criteria (Cramer-Von Mises statistic vs AIC) combined. Nevertheless, if there is a choice to be made between these two criteria, it would be more judicious to base oneself on the Cramer-Von Mises test.

Table 4: Parameters of bivariate copulas selected and their Cramer-Von Mises test statistics.

|                    | Normale | Clayton | Gumbel | Frank | Tawn | Galambos | Husler-Reiss |
|--------------------|---------|---------|--------|-------|------|----------|-------------|
| Parameter          | 0.2822  | 0.1766  | 1.344  | 2.3166| 0.6868| 0.5995   | 0.9798      |
| Statistique de Test| 1.469   | 3.257   | 0.72485| 0.6136| 0.6136| 0.79162  | 0.84238     |
| p-value            | 0.0005  | 0.0005  | 0.0005 | 0.0005| 0.0005| 0.0005   | 0.0005      |

Table 5: AIC and BIC of the estimated parameters (by the maximum likelihood.) for the selected bivariate copulas.

|                   | Normale | Clayton | Gumbel | Frank | Tawn | Galambos | Husler-Reiss |
|-------------------|---------|---------|--------|-------|------|----------|-------------|
| $\log(1/\theta)$ | -365.9  | -325.1  | 345.4  | 291.8 | 320  | 354.7    | 358         |
| AIC               | -729.8  | -648.2  | -688.8 | -581.6| -638 | -707.4   | -714        |
| BIC               | -724.46 | -642.86 | -683.46| -576.26| -632.66| -702.06  | -708.66     |

3.2.2 Options prices by Monte-Carlo approach

In this section, we give the simulation results of the prices (for one period $T = t + 1$) of all options presented in the section above based on the basket (Atos, Dassault systems). The tables 6 and 7 give, respectively, the prices of the call on maximum and the call of maximum. We fixe the price of each asset of the bivariate basket to 120. The values of their prices are calculated when it is out-of-the money (OTM); at-the-money (ATM) and in-the-money (ITM). For the cases of digital option and the spread option, we fixe, respectively, the price of the basket to $S = (120, 130)$ and $S = (100, 120)$. We give the prices of these options in the table 8 and 9 obtained also for different strikes. The prices are calculated by Monte-carlo method with $N = 10^5$ simulations.

For the case of the call on maximum (table 6), the price obtained by normale is superior than the prices with all others copulas (archimedian and extreme) in all the three situations without only the case when it is at the money with Cayton’s copula. We notice that the prices obtained with the others are approximatily the same (weak discrepancies). The normale copula over-estimate the price compared to the Tawn copula with has the good fitness test.

The choice of the bivariate options with our underlying is simply for academic interest. In fact, they are not exchanged on the market.
Table 6: Prices of the call on maximum with different strikes.

|                | Normale | Clayton | Gumbel | Frank | Galambos | Tawn       | Husler-Reiss |
|----------------|---------|---------|--------|-------|----------|------------|--------------|
| OTM (K=130)    | 2.7126  | 2.6947  | 2.6642 | 2.6888| 2.6593   | 2.6664     | 2.6617       |
| ATM (K=120)    | 7.328   | 7.417   | 7.0226 | 7.101 | 7.0216   | 7.0264     | 7.0245       |
| ITM (K=110)    | 17.211  | 17.057  | 16.743 | 16.794| 16.674   | 16.660     | 16.673       |

In the case of the call on minimum (table 7), the normale copula present a price which is lower than that obtained by any extreme value copula when it is at-the-money or out-of-the-money. We notice the contrary when it is in-the-money.

Particularly, the normale copula give the highest price that of Tawn’s copula when it is in-the-money with a small discrepancy. And, when it is at-the-money or out-of-the-money the Tawn’s copula produce a price superior to that of normale copula with a fairly large gap than the first situation.

Table 7: Prices of the call on minimum with different strikes K.

|                | Normale | Clayton | Gumbel | Frank | Galambos | Tawn       | Husler-Reiss |
|----------------|---------|---------|--------|-------|----------|------------|--------------|
| OTM (K=130)    | 0.0386  | 0.02188 | 0.05882| 0.03639| 0.06085  | 0.05437    | 0.06224      |
| ATM (K=120)    | 1.181   | 0.995   | 1.361  | 1.268 | 1.356    | 1.369      | 1.353        |
| ITM (K=110)    | 10.628  | 10.183  | 10.559 | 10.451| 10.536   | 10.523     | 10.543       |

For prices of the digital option (table 8), that obtained by normale copula is inferior to the others calculated by extreme copula in the three situations of valuation.

Table 8: Prices of the bivariate digital option (paying one unit of the money) with different strikes K.

|                | Normale | Clayton | Gumbel | Frank | Galambos | Tawn       | Husler-Reiss |
|----------------|---------|---------|--------|-------|----------|------------|--------------|
| OTM            | 0.02283713 | 0.014745475 | 0.03258418 | 0.02359763 | 0.0306132 | 0.03181797 | 0.03329281 |
| ATM            | 0.5208968 | 0.507406  | 0.5273577 | 0.5344094 | 0.5258449 | 0.53006026 | 0.5247992  |
| ITM            | 0.97519641 | 0.9751979 | 0.9751963 | 0.9751963 | 0.9751963 | 0.9751964 | 0.9751963  |

For the spreads option (table 9), we notice that when it is at-the-money the normale copula give the high price and the Tawn copula has the small price. Other wise, when its in-the-money the normale copula give the smallest price. Finally, when the option is out-of-the-money, the normale copula give the second greatest price. Comparatively to the price obtained with Tawn’s copula, the price calculated with normale copula is greatest when the option is at-the-money and out-of-the-money but smallest when it is in-the-money.

Table 9: Prices of the bivariate spread option with different strikes K.

|                | Normale | Clayton | Gumbel | Frank | Galambos | Tawn |
|----------------|---------|---------|--------|-------|----------|------|
| OTM (K=30)     | 0.04626 | 0.05037 | 0.01244| 0.02615| 0.00898  | 0.01909|
| ATM (K=20)     | 2.1429  | 1.379   | 0.9609 | 1.0818 | 1.006    | 0.8878 |
| ITM (K=10)     | 6.5442  | 7.8945  | 7.4151 | 7.8331 | 7.5363   | 7.4094 |

Remark 4 When $X$ and $Y$ are two random variables that modeling the returns of two shares, having an extreme value copula, one can compute the discordance function for more information about the dependence between these variables. For more details on this measure, see Simplice and al. [15].
4 Conclusion and discussions

This paper proposes an approach that allows to take in account the effect of extreme values in the marginal and joint distribution of the underlying for the valorisation of multivariate options. For doing so, at first, each marginal distribution of the returns of any underlying asset is modelled by a mixture of gaussien as in Idier et al. [7] and the dependence structure is modelled by a copula. The choice of the best copula is confirmed by fitting and goodness test fit.

An application is made on the basket (Atos, Dassault systems) of the financial market CAC40 reveal that the Tawn’s copula is the best for modelling the dependence structure of their returns. Thus, the prices of four type of options are calculated by use of Monte-Carlo simulation. The simulations results show that the normale copula over-estimate the prices for the call on maximum and the spread option when they are at-the-money. In the case of digital and call on minimum, this copula under-estimate the prices when the options are at-the-money.

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