Shape Preserving Rational Cubic Spline Fractal Interpolation

A. K. B. CHAND AND P. VISWANATHAN
Department of Mathematics
Indian Institute of Technology Madras
Chennai - 600036, India
Email: chand@iitm.ac.in ; amritaviswa@gmail.com
Phone: 91-44-22574629
Fax : 91-44-22574602

Abstract. Fractal interpolation functions (FIFs) developed through iterated function systems (IFSs) prove more versatile than classical interpolants. However, the applications of FIFs in the domain of ‘shape preserving interpolation’ are not fully addressed so far. Among various techniques available in the classical numerical analysis, rational interpolation schemes are well suited for the shape preservation problems and shape modification analysis. In this paper, the capability of FIFs to generalize smooth classical interpolants, and the effectiveness of rational function models in shape preservation are intertingly exploited to provide a new solution to the shape preserving interpolation problem in fractal perspective. As a common platform for these two techniques to work together, we introduce rational cubic spline FIFs involving tension parameters for the first time in literature. Suitable conditions on parameters of the associated IFS are developed so that the rational fractal interpolant inherits fundamental shape properties such as monotonicity, convexity, and positivity present in the given data. With some suitable hypotheses on the original function, the convergence analysis of the $C^1$-rational cubic spline FIF is carried out. Due to the presence of the scaling factors in the rational cubic spline fractal interpolant, our approach generalizes the classical results on the shape preserving rational interpolation by Delbourgo and Gregory [SIAM J. Sci. Stat. Comput., 6 (1985), pp. 967-976]. The effectiveness of the shape preserving interpolation schemes are illustrated with suitably chosen numerical examples and graphs, which support the practical utility of our methods.

KEYWORDS : Fractal, Iterated Function System, Fractal Interpolation, Rational Cubic Spline FIF, Convergence, Shape Preservation, Positivity, Monotonicity, Convexity.

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1. INTRODUCTION

Mandelbrot [30] coined the term fractal to refer to a new class of mathematical objects that does not easily fit into the classical Euclidean geometric settings. Fractal curves have been applied successfully in various problems of natural sciences and engineering [19,27,30,47,48] in the last 30 years. These curves are constructed deterministically [2,3] by using FIFs from suitable choices of IFSs [28]. The description of a FIF through an IFS is based on the Read-Bajraktarević operator, which has its origin in the theory of functional equations [1,37]. Fractal interpolation defined through a functional relation provides a constructive way to describe a prescribed data set, in contrast to the descriptive ways employed in the traditional interpolation techniques. The
functional relation involved in the definition of a FIF gives self-similarity on small scales. Thus, FIFs are generally self-affine in nature, and their Hausdorff-Besicovitch dimensions are non-integers. These features enable FIFs to provide non-smooth approximants suitable for data that define dynamic relations corresponding to real processes. Barnsley and Harrington [4] observed that a FIF can be indefinitely integrated any number of times to yield a hierarchy of smoother functions, and consequently developed differentiability of a FIF. The above mentioned observation has a paramount importance in the construction of smooth fractal interpolants, and initiated a striking relationship between classical splines and fractal functions. By allowing the admissibility of various types of boundary conditions, Chand and Kapoor [6] generalized the above construction of spline FIFs, and developed cubic spline FIFs through moments. Some classical interpolations emerge as a special case of smooth fractal interpolations [7,33], and therefore FIFs provide satisfactory generalization for smooth interpolants. Thus, fractal splines provide a single specification method to obtain a very large class of interpolants with significant differences in their visual properties, which can be effectively utilized in geometric modelling and design environment. Moreover, if an experimental data is approximated by a spline FIF, then the fractal dimension of certain derivative of this FIF can be used as an index for analysis of complexity of the data. In summary, alternative solutions to the approximation and reconstruction problems offered by the fractal methodology are capable of modelling experimental data that are either smooth or non-smooth in nature, and flexibility in the choice of interpolant is its intrinsic characteristic.

The data set obtained from an experiment or a simulation is a sample only, and it may be neither ideal for a physical interpretation of the underlying phenomenon nor for the idea of a scientist/engineer. Various interpolation methods, specifically splines, can be used to construct a mathematical model for the sample data. Standard interpolation techniques indeed explore a few characteristics of the data, say for instance, scalar/vector nature of the data, scattered/ordered distribution of the data, and degree of smoothness. However, they often show undesired oscillations, and violate many additional qualitative characteristics hidden in the given data. To obtain a valid physical interpretation of the underlying process, it is important to develop interpolation schemes that honor such properties present in the data, particularly when the data is produced by some scientific phenomena. Examples of few such prevalent features are positivity, monotonicity, and convexity. Constraining the range of an interpolation function so as to yield a credible visualization of the data by adhering to these type of intrinsic qualitative characteristics is generally called the ‘shape preserving interpolation’. Since an interpolation method which accurately represent physical reality is a demand in design and manufacturing of products such as car bodies, aircrafts and ship hulls, in modelling of paths of particles, in abstract and physical processes, in economics, in social and physical sciences, in the description of geological and medical phenomena, etc., the development of interpolation and approximation techniques for shape preservation and shape control is inevitable. Smooth shape preserving curves and surfaces are extremely important in areas such as scientific visualization, data plot, meteorological monitoring, signal/image processing, computer graphics, CAD, and CAM.

Research on shape preserving interpolation problems has been originated with some existence-type results by Wolibner [49] and Kammerer [29]. These results do not provide any additional
information on the shape preserving polynomial. A constructive approach to the shape preserving interpolation was popularized by Schweikert. As a substitute for cubic polynomials in the $C^2$-cubic spline interpolation, Schweikert [41] introduced tension parameters through solutions of suitable differential equations in order to preserve monotonicity and/or convexity, locally or globally. Based on the Schweikert’s tension splines, several researchers have developed various methods for the evaluation of tension parameters (for instance, see [10, 13, 36, 39, 45]). Main issues connected with hyperbolic tension splines are (i) development of automatic algorithm for choice of free parameters is complicated (ii) it is computationally complex to work with, especially for very large/small values of tension parameters involved in it. Hence, an alternative through a minimization problem involving first derivatives of the original function was discussed by Neilson [34]. Foley [21] pursued further generalization of this minimization problem to gain on shape properties. Schmidt and Heß [42] developed necessary and sufficient conditions for a cubic polynomial to be nonnegative on a given interval, and presented an algorithm for generating nonnegative interpolatory curve for a given nonnegative data. Fritsch and Carlson [23] developed ‘fit and modify’ type cubic spline interpolation algorithm to preserve monotonicity of a given data. Later, Fritsch and Butland [22] proposed a modified technique to simplify the Fritsch-Carlson algorithm. Renka [38] produced an algorithm to choose tension parameters automatically in the $C^1$-exponential spline, and with an iterative procedure, smoothness of this spline was extended to $C^2$. McAllister, Passow and Roulier [32, 35] considered piecewise polynomial interpolation schemes of variable degree to preserve monotonicity and convexity features of the given data, and this idea was further extended by Costantini [11, 12]. Preserving monotonicity/convexity properties with polynomial splines by inserting some additional knots between data points are investigated in [20, 31, 43, 44]. The uniqueness of a spline interpolant for a prescribed data turns out to be a disadvantage as far as shape preservation and shape modification issues are concerned. By introducing suitable shape control parameters to the spline structure, various geometric properties inherent from the initial data can be retained. This motivated construction of various kinds of rational splines with shape parameters. The idea of using rational functions with tension parameter was conceived by Späh [46]. Wide applicability of rational interpolants may be attributed to their ability to accommodate a wider range of shapes than the polynomial family, excellent asymptotic properties, capability to model complicated structures, better interpolation properties, and excellent extrapolating powers. Gregory and Delbourgo [15, 24] have developed a piecewise rational quadratic function, which does not involve any shape parameters, to produce a monotonic interpolant to a given monotonic data. Rational interpolants in [9, 14, 26] preserve convexity property hidden in the given data. The pioneer work of Gregory and Delbourgo [17] through the rational cubic spline with one family of shape parameters stimulated a large amount of research in the direction of monotonicity/convexity preserving rational spline interpolation. Recently, Sarfraz et al. [40] developed a rational spline interpolation scheme that discuss all the three basic shape properties, namely, positivity, monotonicity, and convexity within one mathematical model. Due to the increasing demand of shape modelings in engineering and industrial problems, the shape preserving interpolation still continues to be a core topic of research in the subfield of interpolation.

On contrary to classical spline interpolants, the presence of vertical scaling factors enables FIFs to generate wide range of interpolants from which an experimenter can choose the most
suitable one according to the specificities of the problem or simply at his/her discretion. Despite the generality offered by the fractal interpolation techniques, the possibility of operating methods of fractal interpolation and shape preservation together, is not well explored. On the other hand, from the knowledge gained from the classical shape preserving polynomial interpolation techniques, it is felt that preserving fundamental shape properties via polynomial FIFs would be difficult or impossible. Thus, for an initial exposition, the need for constructing shape preserving rational FIFs is strongly felt. To best of our knowledge, no work has appeared in this direction hitherto. The reason for this may be partially attributed to the hypothesis that introducing and validating the notion of shape preservation in FIFs is cumbersome due to its implicit nature. As a consequence, the first relevant work in the context of the shape preserving rational FIF is being initiated in the reference [8], wherein the rational quadratic FIF is constructed without any shape parameter. But this rational quadratic fractal interpolation scheme is not suitable for convexity preservation problems.

In the present paper, we propose the construction of a rational cubic spline FIF with shape parameters based on the work done in the references [6,17]. The rationale behind selecting such an expression is that the suitable values of shape parameters and scaling factors would drive the rational fractal interpolant towards a piecewise linear interpolant, and thereby ensures the shape of the interpolant to coincide with that of a prescribed data. With this observation as a theoretical support on the existence of shape preserving rational FIFs, subsequently, we discuss on how to constrain the parameters involved in the rational cubic FIF so as to incorporate the fundamental shape properties such as monotonicity, convexity, and positivity that are present in the given data, without sacrificing the smoothness of the fractal interpolant. This provide in practice, a solution for shape preserving interpolation problem in fractal perspective, and an automatic algorithm for selecting free parameters to control and modify the shape. Moreover, the rational spline FIF introduced here possesses the benefit of including all the three important shape properties within a single mathematical model. Construction of such a rational cubic spline FIF has advantage that, for a prescribed data, one can have infinite number of rational cubic spline FIFs depending on the vertical scaling factors and shape parameters, thus giving a large flexibility and diversity in the choice of an interpolant. The flexibility offered by this method may be effectively utilized in the fine tuning of the shape of the interpolant with some optimization techniques. By suitable choice of shape parameters that verify the monotonicity condition, our cubic spline FIF reduces to a lower degree rational spline FIF discussed in [8] wherein proper choice of scaling factors generate monotonic curves for given monotonic data. In addition, the present scheme provides an advancement to [8] by incorporating convexity and positivity properties of the given data. With a special choice of shape parameters satisfying convexity condition, our cubic/ quadratic rational FIF reduces to lower order form, which is the fractal generalization of the classical rational interpolant discussed in [14]. Again, by proper choices of scaling factors and shape parameters, our rational cubic FIF degenerates to the classical rational cubic interpolating function introduced in [17], and the rational quadratic functions discussed in [14, 24]. Therefore, the present paper offers a novel idea of setting a common platform for the two techniques, namely fractal interpolation and shape preserving interpolation to operate together, and in the process collectively generalize the three different classical rational interpolation schemes available in literature. To explicate the effectiveness of the rational cubic spline fractal interpolation scheme,
convergence analysis is carried out using convergence results of rational cubic spline interpolant by Delbourgo and Gregory [17]. By admitting a relatively weaker condition on data generating function \( f \), namely \( f \in C^1 \), the uniform convergence of the classical rational cubic spline can be established. This serves as an addendum to the convergence results by Delbourgo and Gregory, and it is further utilized to establish the uniform convergence of the developed rational cubic spline FIF.

The rest of the paper is organized as follows. In Section 2, some basic results for FIFs and polynomial spline FIFs are given. The construction of a \( C^1 \)-rational cubic spline FIF with one family of shape parameters is enunciated, and its convergence analysis is carried out in Section 3. Section 4 focuses on constraining the shape parameters and scaling factors involved in the \( C^1 \)-rational fractal spline structure so as to preserve monotonicity, convexity, and positivity characteristics of the given interpolation data. In Section 5, the results developed in Sections 3-4 are illustrated by generating certain examples of shape preserving rational cubic spline FIFs. Finally, some concluding remarks and open problems are discussed in Section 6.

2. BASICS OF POLYNOMIAL SPLINE FIF

The understanding of a FIF is originated on the fixed point of an IFS. Based on the observation that the indefinite integral of a FIF is again a fractal function interpolating to a different set of data, the construction of a polynomial spline FIF from given data was discussed in the reference [4], and this construction was generalized in [6]. To shed some light on the polynomial spline FIF, we start with a short introduction.

2.1. Iterated Function System. Let \( (\mathcal{X}, d_\mathcal{X}) \) be a complete metric space. Let \( \mathcal{H}(\mathcal{X}) := \{ A : A \neq \phi, A \) is compact in \( \mathcal{X} \} \). The Hausdorff distance between two sets \( A \) and \( B \) in \( \mathcal{H}(\mathcal{X}) \) is

\[
h(A, B) := \max\{d(A, B), d(B, A)\},
\]

where \( d(A, B) = \max_{x \in A} \min_{y \in B} d_\mathcal{X}(x, y) \). Since \( (\mathcal{X}, d_\mathcal{X}) \) is a complete metric space, so is \( (\mathcal{H}(\mathcal{X}), h) \) (see for instance [3]). Let \( \{w_i : \mathcal{X} \to \mathcal{X}; i \in J\} \) be a collection of continuous functions on \( \mathcal{X} \), where \( J = \{1, 2, \ldots, N - 1\} \). Then, \( \{\mathcal{X}; w_i, i \in J\} \) is called an IFS. If \( d_\mathcal{X}(w_i(x), w_i(y)) \leq |\alpha_i| d_\mathcal{X}(x, y) \forall x, y \in \mathcal{X}, 0 \leq |\alpha_i| < 1 \), then the above IFS is called hyperbolic. The contractive factor of the IFS is defined to be \( |\alpha|_\infty = \max\{|\alpha_i| : i \in J\} \), where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \). For a hyperbolic IFS \( \{\mathcal{X}; w_i, i \in J\} \), the set valued Hutchinson map \( W : \mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{X}) \) is defined as

\[
W(A) = \bigcup_{i=1}^{N-1} w_i(A).
\]

Since \( W \) is a contraction map with contractive factor \( |\alpha|_\infty \), by Banach Fixed Point Theorem, there exist a unique fixed point \( G \in \mathcal{H}(\mathcal{X}) \) of \( W \), i.e., \( G = W(G) \), and it is generated as \( G = \lim_{n \to \infty} W^n(A) \), where \( G \) is independent of initial set \( A \in \mathcal{H}(\mathcal{X}) \). The fixed point \( G \) is called the attractor or the deterministic fractal associated with the IFS \( \{\mathcal{X}; w_i, i \in J\} \).
2.2. Fractal Interpolation Functions. Suppose that the given interpolation data set is \( \{(x_i, y_i) \in I \times \mathbb{R} : i = 1, 2, \ldots, N\} \), where \( x_1 < x_2 < \cdots < x_N \) is the partition of the finite interval \( I = [x_1, x_N] \). Set, \( K = I \times D \), where \( D \) is a suitable compact set in \( \mathbb{R} \). Let \( L_i : I \to I_i = [x_i, x_{i+1}] \) be the affine map satisfying
\[
L_i(x_1) = x_i, \quad L_i(x_N) = x_{i+1}, \quad i \in J,
\]
and \( F_i : K \to D \) be a continuous function such that
\[
F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \quad |F_i(x, y) - F_i(x, y^*)| \leq |\alpha_i||y - y^*|, \quad i \in J,
\]
where \((x, y), (x, y^*) \in K, 0 \leq |\alpha_i| \leq \kappa < 1\) for all \( i \in J \), and \( \kappa \) is a fixed real constant. Define \( w_i(x, y) = (L_i(x), F_i(x, y)) \) for all \( i \in J \). The definition of a FIF originates from the following proposition:

**Proposition 2.1.** [2] The IFS \( \{K; w_i, i \in J\} \) has a unique attractor \( G \) such that \( G \) is the graph of a continuous function \( f : I \to \mathbb{R} \) which interpolates the data \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \), i.e.,
\[
G = \{(x, f(x)) : x \in I\}
\]
and \( f(x_i) = y_i \) for \( i = 1, 2, \ldots, N \).

The aforementioned function \( f \) is called a FIF corresponding to the IFS \( \{K; w_i, i \in J\} \). For the functional equation of \( f \), we proceed as follows. Let \( \mathcal{F} := \{g : I \to \mathbb{R} \mid g \text{ is continuous}, g(x_1) = y_1 \text{ and } g(x_N) = y_N\} \) be endowed with the uniform metric \( d_{\infty}(g, h) := \max\{|g(x) - h(x)| : x \in I\} \). Define the Read-Bajraktarević operator \( T \) on \( (\mathcal{F}, d_{\infty}) \) by
\[
(Tg)(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x)), \quad x \in I_i, \quad i \in J.
\]

Since the functions \( L_i^{-1}, g, F_i \) are all continuous, obviously \( Tg \) is continuous on \( I_i \). Also, using \( (2.1) \) - \( (2.2) \), it can be verified that \( Tg \) is continuous at each of the internal knot points \( x_2, x_3, \ldots, x_{N-1} \). \( Tg(x_1) = y_1 \) and \( Tg(x_N) = y_N \). Consequently, \( Tg \in \mathcal{F} \). It can be proved that \( T \) is a contraction mapping on the complete metric space \( (\mathcal{F}, d_{\infty}) \). Hence \( T \) possesses a unique fixed point, say \( f \) on \( \mathcal{F} \). It follows from \( (2.3) \) that the FIF \( f \) satisfies the following functional relation:
\[
f(x) = F_i(L_i^{-1}(x), f \circ L_i^{-1}(x)), \quad x \in I_i, \quad i \in J.
\]

The most extensively studied fractal interpolation functions so far are defined by the IFS
\[
\begin{align*}
\alpha_i \in \mathbb{R}, \quad b_i \in \mathbb{R}, \quad a_i \in \mathbb{R}, \quad q_i(x) \in \mathbb{R}, \quad i \in J,
\end{align*}
\]
where \( \alpha_i, \ b_i \) are the scaling parameters satisfying \( |\alpha_i| \leq \kappa < 1 \) and \( q_i : I \to \mathbb{R}, \ i \in J \) are suitable continuous functions verifying \( (2.2) \). When \( q_i(x) \) are affine map on \( I \), the line segments parallel to the \( y- \)axis are mapped to the line segments parallel to \( y- \)axis scaled by the factor \( |\alpha_i| \). For this reason, \( \alpha_i \) is called a vertical scaling factor of the map \( w_i \), and \( \alpha \) is the scaling vector of the IFS. Scaling factors are important parameters involved in the IFS that provide a degree of freedom to the corresponding FIF, and permit us to modify properties such as smoothness, shape, and dimension of the generated fractal curve. Also, scaling factors of a suitably defined IFS facilitates the corresponding FIF to approach to a given continuous function \( \mathcal{F} \). From \( (2.1) \),
\[
a_i = \frac{x_{i+1} - x_i}{x_N - x_1}, \quad b_i = \frac{x_Nx_i - x_1x_{i+1}}{x_N - x_1}, \quad i \in J.
\]
2.3. Polynomial Spine FIFs. The existence of differentiable fractal functions is given by Barnsley and Harrington [4]. For a prescribed data, the polynomial $C_r$-spline FIF is obtained as the fixed point of an IFS, where scaling factors $\alpha_i$ and polynomials $q_i(x)$ involved in (2.5) are chosen suitably. For a given set of data points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}, x_1 < x_2 < \cdots < x_N$, consider $\mathcal{F}^r = \{g \in C^r(I, \mathbb{R}) \mid g(x_1) = y_1 \text{ and } g(x_N) = y_N\}$, where $r$ is some non-negative integer. Let $\rho^*$ be the metric on $C^r(I, \mathbb{R})$ induced by the $C_r$-norm. Then $\mathcal{F}^r$ endowed with subspace metric $\rho^*$ is a complete metric space. Define the Read-Bajraktarević operator $T$ on $(\mathcal{F}^r, \rho^*)$ as in (2.3) with $L_i$ and $F_i$ as in (2.5) and $|\alpha_i| \leq \kappa a_i$ for $i \in J$, where $0 < \kappa < 1$. In particular, when $q_i(x)$ is a cubic polynomial and $r = 2$, we can construct the cubic spline FIF $f$ as the fixed point of this operator $T$ (see [6] for details). In general, cubic spline FIFs do not exhibit the monotonicity property inherent in the data. For example, for the monotonic data given by $\{(0, 0.01), (6, 15), (10, 15), (29, 25), (30, 30)\}$, the following cubic spline FIFs are generated with the boundary conditions $f'(0) = 2$ and $f'(30) = -3$, taking three different sets of scaling factors. Since these three cubic FIFs are not monotonic in nature, it is felt that the present work fills the deficiency by introducing and investigating the rational cubic spline FIFs in the following sections.

![Figure 1](image)

**Figure 1.** Monotonicity is lost in cubic FIFs

3. Construction of $C^1$ Rational Cubic Spline FIF Involving One Family of Shape Parameters

Shape preserving interpolation plays a significant role in computer graphics, CAGD, surface analysis, biology, chemistry, geology, meteorology, and other applications (see for e.g., [18,25]). Rational function models that approach to piecewise linear interpolants with increasing values of shape/tension parameters involved in it, can be easily fit into the realm of shape preservation and shape control. In many respects, a rational interpolation function with preassigned poles (i.e., only the numerator polynomial is determined by applying interpolation conditions and denominator polynomial is fixed) leads to a theory very similar to that of a polynomial interpolation. Motivated by these facts, we construct a $C^1$-rational cubic spline FIF with a preassigned quadratic function involving a shape parameter in its denominator such that it can be used as a shape preserving interpolant in various phenomena in science and engineering.
3.1. Construction. Let \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \) be a given set of data points, where \( x_1 < x_2 < \cdots < x_N \). Suppose \( d_i \) denotes the first derivative values defined at knots \( x_i, i = 1, 2, \ldots, N \). Let \( F^1 := \{g \in C^1(I, \mathbb{R}) : g(x_1) = y_1 \text{ and } g(x_N) = y_N\} \). Let \( \rho^* \) be the metric on \( C^1(I, \mathbb{R}) \) induced by \( C^1\)-norm, \( \|g\| := \|g\|_\infty + \|g'\|_\infty \). Then \( F^1 \) is a closed subset of the complete metric space \((F^1, \rho^*)\), and hence complete. Now define the Read-Bajraktarević operator \( T : F^1 \rightarrow F^1 \) by

\[
(Tg)(L_i(x)) = \alpha_i g(x) + \frac{A_i(1 - \theta)^3 + B_i \theta(1 - \theta)^2 + C_i \theta^2(1 - \theta) + D_i \theta^3}{1 + (r_i - 3)\theta(1 - \theta)}, \quad i \in J,
\]

where \( \theta = \frac{x-x_i}{x_{N+1}-x_i} \), \( A_i, B_i, C_i, D_i \) are suitably chosen constants, and \( L_i(x) = \alpha_i x + b_i \) satisfies (2.1) for all \( i \in J \). We assume that the rationality parameters \( r_i > -1 \), so as to make the denominator strictly positive, and thereby excluding the possibility of occurrence of any singularity. The constants \( A_i, B_i, C_i, \) and \( D_i \) are to be uniquely determined in the process by suitable conditions. For \( Tg \) to be differentiable once and its derivative to be a fractal function generated by a suitable IFS, the scaling factors \( \alpha_i \) are chosen such that \( |\alpha_i| \leq \kappa \alpha_i \) for \( i \in J \), where \( 0 < \kappa < 1 \). Hence, the map \( T \) is a contraction map on \( F^1 \). The fixed point \( S \) of \( T \) is a fractal function satisfying

\[
S(L_i(x)) = \alpha_i S(x) + \frac{A_i(1 - \theta)^3 + B_i \theta(1 - \theta)^2 + C_i \theta^2(1 - \theta) + D_i \theta^3}{1 + (r_i - 3)\theta(1 - \theta)}.
\]

Denoting the rational expression involved in (3.2) by \( q_i(x) \), for suitable choice of coefficients in the rational expression, \( S^{(1)} \) is the fixed point of the operator \( T^* : F \rightarrow F \) defined by

\[
(T^*h)(L_i(x)) = \frac{\alpha_i h(x) + q_i^{(1)}(x)}{a_i}, \quad x \in I_i, i \in J,
\]

where \( F = \{h \in C(I, \mathbb{R}) : h(x_1) = d_1 \text{ and } h(x_N) = d_N\} \) is endowed with the uniform metric. Note that the contractive factor of \( T \) is \( \max\{\frac{|\alpha_i|}{a_i} : i \in J\} \leq \kappa < 1 \). Hence from (3.3), \( S^{(1)} \) satisfies the functional equation

\[
S^{(1)}(L_i(x)) = \frac{\alpha_i S^{(1)}(x) + q_i^{(1)}(x)}{a_i}.
\]

To achieve the continuity and the interpolatory properties for \( S \) and \( S^{(1)} \), the constants \( A_i, B_i, C_i, \) and \( D_i \) appearing in the expression are evaluated based on \( C^1\)-Hermite interpolatory conditions, viz., \( S(x_i) = y_i, S(x_{i+1}) = y_{i+1}, S^{(1)}(x_i) = d_i, \) and \( S^{(1)}(x_{i+1}) = d_{i+1} \). It can be seen that, these conditions are in consistent with hypotheses on \( q_i(x) \) stated in Barnsley-Harrington theorem for existence of a smooth FIF [4]. Set \( h_i = x_{i+1} - x_i \). Substituting \( x = x_1 \) in (3.2), and using (2.1),

\[
S(L_i(x_1)) = \alpha_i S(x_1) + A_i,
\]

\[
\implies y_i = \alpha_i y_1 + A_i,
\]

\[
\implies A_i = y_i - \alpha_i y_1.
\]
Similarly, taking \( x = x_N \) in (3.2), and using (2.1), we have \( D_i = y_{i+1} - \alpha_i y_N \). Again, applying \( x = x_1 \) in the derivative of \( S(L_i(x)) \) (from (3.4)),

\[
S^{(1)}(L_i(x))a_i = \frac{B_i}{x_N - x_1} \frac{-r_iA_i}{x_N - x_1},
\]

\[
\implies d_ia_i(x_N - x_1) = \alpha_i d_1(x_N - x_1) + B_i - r_i(y_i - \alpha_i y_1),
\]

\[
\implies B_i = [r_i y_i + \alpha_i d_1] - \alpha_i[r_i y_1 + d_1(x_N - x_1)].
\]

Similarly, the constant \( C_i \) evaluated by substituting \( x = x_N \) in the derivative of \( S(L_i(x)) \) is

\[
C_i = [r_i y_{i+1} - \alpha_i d_{i+1}] - \alpha_i[r_i y_N - d_N(x_N - x_1)].
\]

Therefore, the desired \( C^1 \)-rational cubic spline FIF is described as

\[
S(L_i(x)) = \alpha_i S(x) + \frac{P_i^*(x)}{Q_i^*(x)}, \quad (3.5)
\]

\[
P_i^*(x) \equiv P_i(\theta) = (y_i - \alpha_i y_1)(1 - \theta)^3 + \{(r_i y_i + \alpha_i d_1) - \alpha_i[r_i y_1 + d_1(x_N - x_1)]\} \theta(1 - \theta)^2
\]

\[
+ \{(r_i y_{i+1} - \alpha_i d_{i+1}) - \alpha_i[r_i y_N - d_N(x_N - x_1)]\} \theta^2(1 - \theta) + (y_{i+1} - \alpha_i y_N) \theta^3,
\]

\[
Q_i^*(x) \equiv Q_i(\theta) = 1 + (r_i - 3)\theta(1 - \theta), \quad \theta = \frac{x-x_1}{x_N-x_1}.
\]

The parameters \( r_i > -1 \) can be effectively utilized for shape modification and shape preservation of the \( C^1 \)-rational cubic spline FIF, and hence referred as shape parameters.

**Choice of Derivatives:** The proposed rational fractal interpolation scheme requires input as the function values and the first derivatives at knot points. This is the case, for instance, in a data arising from the discrete solution of an ordinary differential equation. But in most applications, only data positions are available, and the derivatives \( d_i, \ i = 1, 2, \ldots, N \) are not supplied. Thus, estimates of derivatives are necessary. Methods that associate derivatives with data points involve estimates based on nearby slopes or data differences. Depending on the applications, various schemes based on linear combination (e.g., arithmetic mean method) or multiplicative combination (e.g., geometric mean method) of chord-slopes are developed in literature (see for instance [5][16]). We use the following approximations of the derivatives in this paper apart from the direct computation from the given function. With the notation \( \Delta_i = \frac{y_{i+1} - y_i}{h_i}, \ i \in J \), the three point difference approximation for the arithmetic mean method is given by

\[
d_i = \frac{h_i \Delta_{i-1} + h_{i-1} \Delta_i}{h_i + h_{i-1}}, \quad i = 2, 3, \ldots, N - 1
\]

with end conditions

\[
d_1 = \left(1 + \frac{h_1}{h_2}\right)\Delta_1 - \frac{h_1}{h_2} \Delta_{3,1}, \quad \Delta_{3,1} = \frac{y_3 - y_1}{x_3 - x_1},
\]

\[
d_N = \left(1 + \frac{h_{N-1}}{h_N}\right)\Delta_{N-1} - \frac{h_{N-1}}{h_N} \Delta_{N,N-2}, \quad \Delta_{N,N-2} = \frac{y_N - y_{N-2}}{x_N - x_{N-2}}.
\]

The nonlinear approximation for the geometric mean method is given by

\[
d_i = \Delta_{i-1} \frac{h_i}{h_i + h_{i-1}} \Delta_i \frac{h_{i-1}}{h_i + h_{i-1}}, \quad i = 2, 3, \ldots, N - 1
\]
with end conditions
\[
d_1 = \Delta_1^{(1+h_1)\Delta_3,1} \frac{-h_1}{h_2},
\]
\[
d_N = \Delta_{N-1}^{(1+h_{N-1})\Delta_{N,N-2}} \frac{-h_{N-1}}{h_{N-2}}.
\]

Analysis based on Newton series expansion shows that the above methods provide \(O(h^2)\) approximations, where \(h\) is the mesh norm. Alternatively, one can use the derivatives values obtained as a solution of the nonlinear system governed by the \(C^2\)-spline approximations of Delbourgo and Gregory \([15]\). For a prescribed bounded data set, a suitable approximation method yields bounded derivative values. Since the FIF \(S(x)\) in (3.5) is derived as a solution of the fixed point relation \(Tg = g\), the solution is unique for a given choice of vertical scaling factors, shape parameters, and consequently the above discussion leads to the following existence and uniqueness theorem.

**Theorem 3.1.** If a set of scaling factors of the rational cubic IFS satisfy \(|\alpha_i| \leq \kappa a_i, \ i \in J, \ 0 < \kappa < 1\), and the bounded values for the derivative parameters are estimated by a suitable classical method mentioned above, then the corresponding \(C^1\)-rational cubic spline fractal interpolant (3.5) exists, and is unique for a fixed choice of shape parameters and scaling factors.

**Remark 3.1.** Using the notations
\[
E_i := \frac{\theta(1-\theta)h_i\{(2\theta-1)\Delta_i + (1-\theta)d_i - \theta d_{i+1}\}}{1 + (r_i - 3)\theta(1-\theta)},
\]
\[
F_i := \frac{\theta(1-\theta)\{(2\theta-1)(y_N - y_1) + (1-\theta)(x_N - x_1)d_1 - \theta(x_N - x_1)d_N\}}{1 + (r_i - 3)\theta(1-\theta)}
\]
the rational cubic spline FIF in (3.5) can be written as
\[
S(L_i(x)) = \alpha_i S(x) + \{(1-\theta)y_i + \theta y_{i+1} + E_i\} - \alpha_i\{(1-\theta)y_1 + \theta y_N + F_i\}
\] (3.6)

As the shape parameters \(r_i\) are increased, \(E_i\) and \(F_i\) converges to zero. Thus from (3.6), it follows that as scaling factors \(\alpha_i \to 0\) and shape parameters \(r_i \to \infty\), the rational cubic spline FIF \(S(x)\) converges to a piecewise defined linear interpolant. This illustrates that our rational cubic spline FIF can always be rendered shape preserving for suitable choices of scaling factors and shape parameters to be specified latter in Section 4.

**Remark 3.2.** If \(\alpha_i = 0\) for all \(i \in J\), then \(S(x)\) reduces to the classical rational cubic spline \(s(x)\) discussed in \([17]\). Thus in the case of vanishing scaling factors,
\[
S(L_i(x)) = \frac{y_i(1-\theta)^3 + (r_i y_i + h_id_i)\theta(1-\theta)^2 + (r_i y_{i+1} - h_i d_{i+1})\theta^2(1-\theta) + y_{i+1}\theta^3}{1 + (r_i - 3)\theta(1-\theta)}, \ i \in J.
\] (3.7)

Since \(\frac{L_0^{-1}(x-x_i)}{x_N-x_i} = \frac{x-x_i}{h_i}\), from (3.7), for \(x \in [x_i, x_{i+1}]\), we have
\[
S(x) \equiv s(x) = \frac{y_i(1-\theta)^3 + (r_i y_i + h_id_i)\theta(1-\theta)^2 + (r_i y_{i+1} - h_i d_{i+1})\theta^2(1-\theta) + y_{i+1}\theta^3}{1 + (r_i - 3)\theta(1-\theta)}, \ i \in J.
\] (3.8)
where now $\theta = \frac{x-x_i}{h_i}$ is a localized variable. Hence the classical rational cubic spline is derived as a particular case of our rational cubic spline FIF showing the power of fractal methodology. Also, with $r_i = 3$ and scaling factors verifying $|\alpha_i| \leq \kappa a_i$, $\forall i \in J$, our discussion of the rational cubic spline FIF gives a constructive approach to a $C^1$-cubic Hermite FIF: Again, the case $\alpha_i = 0$ and $r_i = 3, \forall i \in J$, recovers the classical piecewise cubic Hermite interpolant.

**Remark 3.3.** If the data is generated from a constant function $g(x) = c$ for all $x \in I$, then $A_i = D_i = c(1-\alpha_i)$ and $B_i = C_i = cr_i(1-\alpha_i)$. Consequently, the rational cubic spline FIF in this case is $S(L_i(x)) = \alpha_i S(x) + c(1-\alpha_i), i \in J$, which is true if and only if $S(x) = c$. Hence, in particular, the rational cubic fractal interpolant to the zero function is the zero function.

**Remark 3.4.** Let $S^*$ be the rational cubic spline FIF to the data set $\{(x_i, y_i+K), i = 1, 2, \ldots, N\}$, where $K$ is a constant. Evaluation of the constants $A_i, B_i, C_i, D_i$, and simplification of the corresponding rational cubic spline FIF gives

$$S^*(L_i(x)) = \alpha_i S^*(x) + \frac{P_i(\theta)}{Q_i(\theta)} + K - \alpha_i K, \quad i \in J,$$

(3.9)

where $P_i(\theta)$ and $Q_i(\theta)$ are given as in (3.5). But at the grid points, $S^*(x_j) = S(x_j) + K$ for $j = 1, 2, \ldots, N$. Thus, from (3.9), we have

$$S^*(L_i(x_j)) = \alpha_i S^*(x_j) + \frac{P_i(\theta)}{Q_i(\theta)} \bigg|_{x=x_j} + K - \alpha_i K = \alpha_i S(x_j) + \frac{P_i(\theta)}{Q_i(\theta)} \bigg|_{x=x_j} + K = S(L_i(x_j)) + K.$$

Since fractal functions are generated iteratively from a given interpolation data, it follows from the above observation that the rational cubic spline FIF for the data $\{(x_i, y_i+K), i = 1, 2, \ldots, N\}$ is $S(x) + K$ provided all $r_i$ are independent of such translations. Analogous to the classical rational cubic spline by Delbourgo and Gregory, choice of $r_i$ in this paper will be independent of such translations. Therefore, changes in location and scale in the raw data results in a rational spline FIF model being mapped to a rational spline FIF model. Hence, as in the case of classical rational function models, rational cubic spline FIFs also form a closed family in the above sense.

### 3.2. Convergence Analysis of Rational Cubic Spline FIFs

To study the effectiveness of the developed rational cubic spline in approximation, an upper bound of the uniform error between the rational cubic spline FIF $S$ and the original function $f \in C^1[x_1, x_N]$ is deduced with the help of the uniform error bound for the classical rational cubic spline $s$ in the following.

For a prescribed set of data and a suitable choice of $\alpha_i$ satisfying $|\alpha_i| \leq \kappa a_i$, $i \in J$, the rational cubic spline FIF $S \in C^1[x_1, x_N]$ is the fixed point of the Read-Bajraktarević operator $T_\alpha$ defined on the metric space $\mathcal{F}^1 = \{g \in C^1(I, \mathbb{R})|g(x_1) = y_1, g(x_N) = y_N\}$ such that

$$(T_\alpha S)(x) = \alpha_i S(L_i^{-1}(x)) + R_i(\alpha_i, \theta),$$

(3.10)

where $R_i(\alpha_i, \theta) = \frac{P_i(\theta)}{Q_i(\theta)}, \quad \theta = \frac{x_{i+1} - x_i}{x_{i+1} - x_i} = \frac{x-x_i}{h_i}, x \in [x_i, x_{i+1}], i \in J$ with $P_i(\theta)$ and $Q_i(\theta)$ are given as in (3.5). Note that the subscript $\alpha$ is used to emphasize the dependence of the map $T$ on scale vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$, and (3.10) is consistent with the notation in Section 3.1 such that $R_i(\alpha_i, \theta) = q_i(L_i^{-1}(x))$. The coefficients of rational function $R_i$ depend on the scaling factor $\alpha_i$, and hence $R_i$ can be thought of as a function of $\alpha_i$ and $\theta$. Let $S$ be the rational cubic spline FIF, and $s$ be the classical rational cubic interpolant for the function $f$ with respect
to the mesh \( x_1 < x_2 < \cdots < x_N \) interpolating the ordinates \( y_1, y_2, \ldots, y_N \) respectively. The interpolants \( S \) and \( s \) are fixed points of \( T_\alpha \) with \( \alpha \neq 0 \) and \( \alpha = 0 \) respectively. We know from [17] that for \( x \in [x_i, x_{i+1}] \),

\[
\begin{align*}
|f(x) - s(x)| &\leq \frac{h_i}{4c_i} \max \{|y_i^{(1)} - d_i|, |y_i^{(1)} - d_{i+1}|\} + \frac{1}{384c_i} \{h_i^4\|f^{(4)}\|_\infty (1 + \frac{|r_i - 3|}{4}) \\
&+ 4|r_i - 3| (h_i^3\|f^{(3)}\|_\infty + 3h_i^2\|f^{(2)}\|_\infty) \},
\end{align*}
\]

(3.11)

where \( y_i^{(1)} = f^{(1)}(x_i) \) for \( i = 1, 2, \ldots, N \). \( \|\cdot\|_\infty \) denotes the uniform norm on interval of interest, and

\[
c_i = \left\{ \begin{array}{ll}
1 + 4r_i, & -1 < r_i < 3, \\
4, & r_i \geq 3.
\end{array} \right.
\]

By triangle inequality,

\[
\|S - f\|_\infty \leq \|S - s\|_\infty + \|s - f\|_\infty.
\]

(3.13)

From (3.10), with a fixed choice of scale vector \( \alpha \neq 0 \) and for \( x \in [x_i, x_{i+1}] \), we obtain that

\[
|T_\alpha S(x) - T_\alpha s(x)| = |\{\alpha_i S(L_i^{-1}(x)) + R_i(\alpha_i, \theta)\} - \{\alpha_i s(L_i^{-1}(x)) + R_i(\alpha_i, \theta)\}|,
\]

\[
\leq |\alpha_i|\|S - s\|_\infty.
\]

Since the above inequality is true for all \( i \in J \),

\[
\|T_\alpha S - T_\alpha s\|_\infty \leq |\alpha_i|\|S - s\|_\infty.
\]

(3.14)

For \( x \in [x_i, x_{i+1}] \) with a choice of \( \alpha \neq 0 \), using (3.10) and the mean value theorem,

\[
|T_\alpha s(x) - T_0 s(x)| = |\{\alpha_i s(L_i^{-1}(x)) + R_i(\alpha_i, \theta)\} - R_i(0, \theta)|,
\]

\[
\leq |\alpha_i|\|s\|_\infty + |\alpha_i|\left|\frac{\partial R_i}{\partial \alpha_i}\right|,
\]

\[
\leq |\alpha_i|\left(\|s\|_\infty + Z_0\),
\]

where \( Z_0 \) is a positive real number to be specified later such that \( |\frac{\partial R_i}{\partial \alpha_i}| \leq Z_0 \forall i \in J \). Thus,

\[
\|T_\alpha s - T_0 s\|_\infty \leq |\alpha_i|\left(\|s\|_\infty + Z_0\).
\]

(3.15)

In order to find an upper bound of \( \|s\|_\infty \), we introduce the notations \( |y|_\infty = \max\{|y_i| : 1 \leq i \leq N\}, |d|_\infty = \max\{|d_i| : 1 \leq i \leq \}, |r|_\infty = \max\{|r_i| : i \in J\} \) and \( h = \max\{h_i : i \in J\} \). From (3.8), for \( x \in [x_1, x_N] \),

\[
|s(x)| \leq \frac{\max\{|P_i^*(\theta) : i \in J, 0 \leq \theta \leq 1\}}{\min\{|Q_i(\theta) : i \in J, 0 \leq \theta \leq 1\}},
\]

where \( P_i^*(\theta) \) is the numerator in (3.8). Using extremum calculations of polynomials,

\[
|P_i^*(\theta)| \leq |y_i|(1 - \theta)^3 + (|r_i||y_i| + h_i|d_i|)(1 - \theta)^2 + (|r_i||y_{i+1}| + h_i|d_{i+1}|)(1 - \theta) + |y_{i+1}|\theta^3,
\]

\[
\Rightarrow \max_{\theta \in [0, 1]} |P_i^*(\theta)| \leq \max\{|y_i|, |y_{i+1}|\} + \frac{1}{4} (|r_i| \max\{|y_i|, |y_{i+1}|\} + h_i \max\{|d_i|, |d_{i+1}|\}),
\]

\[
\Rightarrow \max_{i \in J, \theta \in [0, 1]} |P_i^*(\theta)| \leq |y|_\infty + \frac{1}{4} (|r|_\infty|y|_\infty + h|d|_\infty),
\]
and $|Q_i(\theta)| = Q_i(\theta) \geq c_i$, where $c_i$ is defined as in (3.12). Therefore,
\[\|s\|_\infty \leq \frac{|y|_\infty + \frac{1}{4}(|r|_\infty |y|_\infty + h|d|_\infty)}{\min\{c_i : i \in J\}}.\] (3.16)

Now, from (3.5) and (3.10), for $x \in [x_i, x_{i+1}]$,
\[\frac{\partial R_i}{\partial \alpha_i} = \tilde{P}_i(\theta),\]
where $\tilde{P}_i(\theta) = -\{y_1(1 - \theta)^3 + (r_i y_1 + (x_N - x_1)d_1)\theta(1 - \theta)^2 + (r_i y_N - d_N(x_N - x_1))\theta^2(1 - \theta) + y_N\theta^3\}$. Again using the similar extremum calculations,
\[\left|\frac{\partial R_i}{\partial \alpha_i}\right| \leq Z_0 = \frac{\max\{|y_1|, |y_N|\}(1 + \frac{1}{4}|r|_\infty) + \frac{1}{4}|I|\max\{|d_1|, |d_N|\}}{\min\{c_i : i \in J\}} \forall i \in J,\] (3.17)
where $|I| = x_N - x_1$. Using (3.14) and (3.15),
\[\|S - s\|_\infty = \|T_0 S - T_0 s\|_\infty \leq \|T_0 S - T_0 s\|_\infty \leq |\alpha|_\infty \|S - s\|_\infty + |\alpha|_\infty (\|s\|_\infty + Z_0)\]
which simplifies to
\[\|S - s\|_\infty \leq \frac{|\alpha|_\infty (\|s\|_\infty + Z_0)}{1 - |\alpha|_\infty}.\] (3.18)

Finally, we proved the following theorem from (3.11), (3.13), and (3.18):

**Theorem 3.2.** Let $S(x)$ and $s(x)$ be respectively the rational cubic spline FIF and the classical rational cubic interpolant for the original function $f \in C^4[x_1, x_N]$ with respect to the interpolation data $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$. Suppose that the rational function $R_i(\alpha_i, \theta)$ involved in the IFS generating the FIF $S(x)$ satisfies $|\frac{\partial R_i}{\partial \alpha_i}| \leq Z_0$ for all $i \in J$, for some real constant $Z_0$.

Then,
\[\|f - S\|_\infty \leq \frac{h}{4c} \max_{i \in J}\{y^{(1)}_i - d_i, |y^{(1)}_{i+1} - d_{i+1}|\} + \frac{1}{384c} \{h^4\|f^{(4)}\|_\infty (1 + \frac{1}{4}\max|r_i - 3|)\]
\[+ 4 \max_{i \in J}|r_i - 3|(h^3\|f^{(3)}\|_\infty + 3h^2\|f^{(2)}\|_\infty)\} + \frac{|\alpha|_\infty (\|s\|_\infty + Z_0)}{1 - |\alpha|_\infty},\] (3.19)

where $c = \min\{c_i : i \in J\}$.

Due to the principle of construction of a smooth FIF, for the rational cubic spline FIF $S$ to be in the class $C^1[x_1, x_N]$, we impose $|\alpha|_\infty \leq \kappa \alpha_i = \frac{\kappa h}{x_N - x_1} \leq \frac{\kappa h}{x_N - x_1}$. Hence, $|\alpha|_\infty \leq \frac{\kappa h}{x_N - x_1}$, and consequently the rational cubic spline FIF converges uniformly to the original function when the norm of the partition tends to zero. The following convergence results are direct consequences of Theorem 3.2 based on the parameters of the rational cubic spline FIF:

**Corollary 3.1.** Let $S(x)$ be the rational FIF with respect to the data points $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ corresponding to the original function $f \in C^4[x_1, x_N]$. Suppose $d_i, i = 1, 2, \ldots, N$, and $r_i, \alpha_i, i \in J$ are chosen accordingly.
Therefore, local error of the interpolation is given by
\[\text{Local Error} = \mathcal{O}(h^2).\]

(ii) If \(y_i^{(1)} - d_i = O(h^2)\), \(y_{i+1}^{(1)} - d_{i+1}, r_i > -1 \text{ and } |\alpha_i| < a_i^2\), then \(\|f - S\|_\infty = O(h^2).\)

Thus, with some mild conditions on vertical scaling factors, the convergence properties of a rational cubic spline FIF is similar to that of its classical counterpart, which should be considered along with the generality offered by the rational fractal splines.

In the next section, we consider how \(r_i, \alpha_i, i \in J\) can be chosen to preserve the monotonic, convex, or positive shape of the prescribed data.

Remark 3.5. It is worth noticing that (3.16) gives an explicit bound, which depends only on the given data and free shape parameters, for the classical rational cubic interpolant \(s\). An upper bound for the uniform norm of the rational cubic spline FIF can now be easily derived from the triangle inequality \(\|S\|_\infty \leq \|S - s\|_\infty + \|s\|_\infty\), and (3.18).

Motivated by Gregory and Delbourgo’s convergence results for the classical rational cubic spline interpolant \(s\), we assumed that the data generating function \(f\) is in class \(C^4[x_1, x_N]\). Now we establish the uniform convergence of classical rational cubic spline \(s\) with a weaker assumption \(f \in C^1[x_1, x_N]\), and use it to deduce the convergence of our rational cubic spline FIF as in the previous case.

Let \(f \in C^1[x_1, x_N]\) be the unknown function that generates the interpolation data \(\{(x_i, y_i) : i = 1, 2, \ldots, N\}\), and \(s\) be the corresponding classical rational cubic spline FIF. Observe that
\[
Q_i(\theta) = 1 + (r_i - 3)\theta(1 - \theta),
\]
\[
= (1 - \theta)^3 + r_i\theta(1 - \theta)^2 + r_i\theta^2(1 - \theta) + \theta^3.
\]

For \(x \in [x_i, x_{i+1}]\),
\[
f(x) - s(x) = f(x) - \frac{P_*^{(r)}(\theta)}{Q_i(\theta)}
\]
\[
= \frac{1}{1 + (r_i - 3)\theta(1 - \theta)}[(1 - \theta)^3(f(x) - y_i) + r_i\theta(1 - \theta)^2(f(x) - y_i) + r_i\theta^2(1 - \theta)
\]
\[
(f(x) - y_{i+1}) + \theta^3(f(x) - y_{i+1}) - h_id_i\theta(1 - \theta)^2 + h_id_{i+1}\theta^2(1 - \theta)].
\]

Therefore, local error of the interpolation is given by
\[
|f(x) - s(x)| \leq \frac{1}{c_i}\left\{ |f(x) - y_i|[(1 - \theta)^3 + |r_i|\theta(1 - \theta)^2] + |f(x) - y_{i+1}|[|r_i|\theta^2(1 - \theta) + \theta^3]
\]
\[
+ h_i[|d_i|\theta(1 - \theta)^2 + |d_{i+1}|\theta^2(1 - \theta)]\right\},
\]
\[
\leq \frac{1}{c_i}\left\{ \left(\frac{1}{4}|r_i| + 1\right)\omega(f; h) + \frac{1}{4}h_i \max\{|d_i|, |d_{i+1}|\}\right\}.
\]
where \( \omega(f; h) := \sup_{|x_1-x_2| \leq h} \{ |f(x_1) - f(x_2)| : x_1, x_2 \in I \} \) is the modulus of continuity of \( f \). Consequently, a uniform error bound for the classical rational cubic spline FIF \( s \) is

\[
\|f - s\|_\infty \leq \frac{1}{4c} h|d|_\infty + \frac{1}{4c} \omega(f; h)(|r|_\infty + 4).
\]

(3.20)

Now from (3.13), (3.18), and (3.20), we have the following theorem:

**Theorem 3.3.** Let \( S(x) \) and \( s(x) \) be respectively the rational cubic spline FIF and the classical rational cubic interpolant for the original function \( f \in C^1[x_1, x_N] \) with respect to the interpolation data \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \). Suppose that the rational function \( R_i(\alpha_i, \theta) \) involved in the IFS generating the FIF \( S(x) \) satisfies \( |\partial R_i / \partial \alpha_i| \leq Z_0 \) for all \( i \in J \) and for some real constant \( Z_0 \). Then,

\[
\|f - S\|_\infty \leq \frac{1}{4c} h|d|_\infty + \frac{1}{4c} \omega(f; h)(|r|_\infty + 4) + \frac{|\alpha|_\infty (\|s\|_\infty + Z_0)}{1 - |\alpha|_\infty},
\]

where \( c = \min\{c_i : i \in J\} \). In particular, \( S \) converges uniformly to \( f \in C^1[x_1, x_N] \) as the mesh norm tends to zero.

### 4. Shape Preserving Rational Fractal Interpolation

We have already observed that by selecting very large values of shape parameters and pretty small values of scaling factors, the rational cubic spline FIF stretches to a piecewise linear interpolant. Hence, in the limiting configuration, shape of the interpolant agrees with that of given data. In practice, the strategy to select appropriate parameters without sacrificing the \( C^1 \) smoothness of the interpolant is very important. In this section, we investigate how large the shape parameters and how small the scaling factors are to be selected in practice so that the corresponding \( C^1 \)-rational cubic spline FIF will inherit fundamental shape properties hidden in the prescribed data. We discuss the shape preservation of four different types of data: (i) monotonic data, (ii) convex data, (iii) monotonic and convex data, and (iv) positive data, with fixed sign of vertical scaling factors.

#### 4.1. Monotonic Data

Without loss of generality, it is assumed that the given set of data is monotonically increasing, i.e., \( y_1 \leq y_2 \leq \cdots \leq y_N \), and consequently \( \Delta_i = \frac{y_{i+1} - y_i}{h_i} \geq 0 \) \( \forall i \in J \). For a monotonic increasing smooth interpolant \( S(x) \), it is necessary that the derivative parameters satisfy \( d_i \geq 0, i = 1, 2, \ldots, N \). From elementary calculus, we know that \( S(x) \in C^1[x_1, x_N] \) is monotonic increasing if and only if \( S^{(1)}(x) \geq 0 \) for all \( x \in [x_1, x_N] \). Calculation of \( S^{(1)}(L_i(x)) \) from (3.5), and further simplification gives for \( x \in [x_1, x_N] \),

\[
S^{(1)}(L_i(x)) = \frac{\alpha_i}{a_i} S^{(1)}(x) + \frac{R_i \theta^4 + S_i \theta^3(1 - \theta) + U_i \theta^2(1 - \theta)^2 + V_i \theta(1 - \theta)^3 + W_i (1 - \theta)^4}{[1 + (r_i - 3)\theta(1 - \theta)]^2},
\]

(4.1)
If it is assumed that 

Again

Similarly

W

of to avoid the intriguing difficulties that may evolve with the process of maintaining positivity

The nonnegativity restriction on scaling factors is not claimed to be indispensable, yet desirable

to maintain positivity of \( S \)

Since the rational cubic spline FIF is defined implicitly and recursively from the given data points, to maintain positivity of \( S^{(1)} \) in successive iterations and to keep the desired data dependent monotonicity condition to be simple enough, we assume all scaling factors as nonnegative.

The nonnegativity restriction on scaling factors is not claimed to be indispensable, yet desirable to avoid the intriguing difficulties that may evolve with the process of maintaining positivity of \( S^{(1)} \) in successive iterations of the FIF. Also, the nonnegative scaling values are sufficient enough for the present purpose of incorporating shape preserving requirement with the rational cubic spline FIF, and generalizing the classical shape preserving rational cubic spline. Thus, for \( i \in J \) and arbitrary node \( x_j \), sufficient conditions for \( S^{(1)}(L(x_j)) \geq 0 \) are

where necessary condition on derivative parameters are assumed.

Now \( R_i \geq 0 \iff \frac{\alpha_i}{h_i}(x_N - x_1)d_N \leq d_{i+1} \). Observe that if \( d_N = 0 \), then \( R_i \geq 0 \) follows from the assumption on derivative parameters. Otherwise, we impose the following condition on the scaling factor:

\[
\alpha_i \leq \frac{d_{i+1}h_i}{d_N(x_N - x_1)}.
\] (4.3)

Similarly \( W_i \geq 0 \), whenever the scaling factor satisfies

\[
\alpha_i \leq \frac{d_ih_i}{d_1(x_N - x_1)}.
\] (4.4)

Again

If it is assumed that

\[
r_i[h_i \Delta_i - \alpha_i(y_N - y_1)] \geq h_i(d_i + d_{i+1}) - \alpha_i(x_N - x_1)(d_1 + d_N),
\] (4.6)

then it follows from (4.3)-(4.4) that

\[
\begin{align*}
r_i[h_i \Delta_i - \alpha_i(y_N - y_1)] &\geq h_id_i - \alpha_id_1(x_N - x_1) \geq 0, \\
r_i[h_i \Delta_i - \alpha_i(y_N - y_1)] &\geq h_id_{i+1} - \alpha_id_N(x_N - x_1) \geq 0.
\end{align*}
\]
These two conditions are sufficient for \( S_i \geq 0 \) and \( V_i \geq 0 \) respectively. Assume that \( h_i \Delta_i - \alpha_i(y_N - y_1) \geq 0 \), i.e.,
\[
\alpha_i \leq \frac{h_i \Delta_i}{y_N - y_1}. \tag{4.7}
\]

From (4.6), we have
\[
r_i^2[h_i \Delta_i - \alpha_i(y_N - y_1)] \geq r_i h_i(d_i + d_{i+1}) - r_i \alpha_i(x_N - x_1)(d_1 + d_N). \tag{4.8}
\]

From (4.8) and \( 3h_i \Delta_i \geq 3\alpha_i(y_N - y_1) \), it is easy to verify that \( U_i \geq 0 \). From (4.3)-(4.4) and (4.7), for a monotonicity preserving rational cubic FIF, it suffices to choose the scaling parameter \( \alpha_i, \ i \in J \) as
\[
0 \leq \alpha_i \leq \min\{\kappa \alpha_i, \frac{d_i h_i}{d_1(x_N - x_1)}, \frac{d_{i+1} h_i}{d_N(x_N - x_1)}, \frac{\Delta_i h_i}{y_N - y_1}\}, \tag{4.9}
\]
where the first term in the right hand side has come from \( C^1 \)-smoothness. After the choice of scale parameter \( \alpha_i \) according to (4.9), the shape parameters \( r_i \) is selected according to (4.6), and these two conditions are sufficient for \( S^{1}(L_i(x_j)) \geq 0 \). As \( x_1, x_N \) is the attractor of the IFS \( \{ \mathbb{R}; L_i(x), i \in J \} \), by the recursive nature of a rational fractal function, \( S^{1}(L_i(x_j)) \geq 0 \) for all \( i \in J \) and for every knot point \( x_j \) imply that \( S^{1}(x) \geq 0 \) for all \( x \in [x_1, x_N] \).

With necessary condition \( d_i \leq 0, \ i \in J \) assumed to hold, an analogous procedure applies for a monotonic decreasing data set. However, we outline the procedure for the sake of completeness.

With the initial nonnegative assumptions on vertical scaling factors, sufficient conditions for \( S^{1}(L_i(x_j)) \leq 0 \) are \( R_i \leq 0, S_i \leq 0, U_i \leq 0, V_i \leq 0, \) and \( W_i \leq 0 \).

\[
R_i \leq 0 \iff d_{i+1} \leq \frac{\alpha_i(x_N - x_1)d_N}{h_i} \quad \text{and} \quad S_i \leq 0 \iff d_i \leq \frac{\alpha_i(x_N - x_1)d_1}{h_i}. \tag{4.10}
\]

From the assumption \( d_i \leq 0, \ i = 1, 2, \ldots, N, \) we obtain condition (4.3) and (4.4) on scaling factors. Again, assume
\[
r_i[h_i \Delta_i - \alpha_i(y_N - y_1)] \leq h_i(d_i + d_{i+1}) - \alpha_i(x_N - x_1)(d_1 + d_N), \tag{4.11}
\]
from which it follows that \( S_i \leq 0, V_i \leq 0 \). Finally, analogous to (4.7), the additional assumption \( h_i \Delta_i - \alpha_i(y_N - y_1) \leq 0 \) implies \( U_i \leq 0 \).

**Remark 4.1.** It is worth mentioning that if \( \Delta_i = 0 \), then in view of (4.9), we take \( \alpha_i = 0 \) for monotonicity of the rational cubic spline FIF. Also in this case, \( d_i = d_{i+1} = 0 \). Consequently, \( S(L_i(x)) = y_i = y_{i+1} \), i.e., to say that, \( S \) reduces to a constant on the interval \([x_i, x_{i+1}]\).

**Remark 4.2.** When all \( \alpha_i = 0 \), the rational cubic FIF reduces to the classical rational cubic spline. In this case, the shape preserving conditions (4.9) are obviously true, and the condition (4.6) reduces to
\[
r_i \geq (d_i + d_{i+1})/\Delta_i. \tag{4.10}
\]

Thus (4.10) is the sufficient condition for monotonicity of the classical rational cubic spline [17, p. 970].

**Remark 4.3.** For a given strict monotonic data, we select \( \alpha_i \) satisfying (4.9) with \( \alpha_i < \frac{h_i \Delta_i}{y_N - y_1} \), and then fix shape parameters according to
\[
r_i = 1 + \frac{h_i(d_i + d_{i+1}) - \alpha_i(x_N - x_1)(d_1 + d_N)}{h_i \Delta_i - \alpha_i(y_N - y_1)}, \tag{4.11}
\]
so that the monotonicity condition (4.6) is satisfied. Then, the rational cubic spline FIF reduces to the rational quadratic FIF described as follows:

\[
S(L_i(x)) = \alpha_i S(x) + \frac{P_i(\theta)}{Q_i(\theta)},
\]

where \( P_i(\theta) = (y_{i+1} - \alpha_i y_N) \Delta_i \theta^2 + \beta_i (y_{i+1} + y_i d_i + y_i d_{i+1} + \alpha_i y_i d_i + \alpha_i y_i d_{i+1}) + \alpha_i^2 (y_N d_i + y_i d_N) \theta (1 - \theta) + (y_i - \alpha_i y_N) \Delta_i (1 - \theta)^2, \)

\( Q_i(\theta) = \Delta_i \theta^2 + \beta_i (\alpha_i (d_i + d_{i+1}) - \alpha_i (d_i + d_N)) \theta (1 - \theta) + \Delta_i (1 - \theta)^2, \)

and

\[
\beta_i = \frac{(y_{i+1} - y_i) - \alpha_i (y_N - y_i)}{\Delta_i (x_N - x_1)}. \]

For \( \Delta_i = 0, \) we choose \( \alpha_i = 0, \) and define \( S(L_i(x)) = y_i = y_{i+1}. \) The above quadratic/quadratic form of the rational spline FIF is studied detailedly in [8]. If all \( \alpha_i = 0 \) in (4.12), then the corresponding rational quadratic FIF reduces to the classical monotonic rational quadratic interpolant studied in [24]. In this degenerated case, the necessary condition \( d_i \geq 0 \) is also sufficient for monotonicity of the interpolant (see [24]).

**Remark 4.4.** It is to be noted that, for the shape parameters specified in (4.11) and \(|\alpha_i| < a_i^4\), we get \( r_i - 3 = O(h_i^2). \) Consequently, from corollary 3.1 optimal \( O(h_i^4) \) bound on the interpolation error can be achieved, provided derivatives are chosen with \( O(h_i^3) \) accuracy. Hence, for a monotonic rational cubic spline FIF with optimal error bound, it is enough to choose

\[
0 \leq \alpha_i \leq \min \left\{ \kappa a_i^4, \frac{d_{i+1} a_i}{d_N}, \frac{d_{i+1} a_i}{d_N}, \kappa^* \frac{\Delta_i h_i}{y_N - y_1} \right\}, 0 < \kappa, \kappa^* < 1, \quad \text{and} \quad r_i \text{ as in (4.11).}
\]

This observation is similar to that in the case of the classical rational cubic spline [17].

The above discussions for monotonicity preserving rational cubic spline FIFs are summarized in the following theorem:

**Theorem 4.1.** For a given monotonic increasing (decreasing) data \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \), let \( S(x) \) be the rational cubic spline FIF described as in (3.5). Assume that the necessary conditions on derivative parameters are satisfied. Then, the following conditions on scaling factors and shape parameters on each subinterval are sufficient for \( S(x) \) to be monotonic on the interpolating interval \( I = [x_1, x_N]. \)

\[
0 \leq \alpha_i \leq \min \left\{ \kappa a_i^4, \frac{d_{i+1} a_i}{d_N}, \frac{d_{i+1} a_i}{d_N}, \kappa^* \frac{\Delta_i h_i}{y_N - y_1} \right\}, 0 < \kappa, \kappa^* < 1, \]

\[
r_i \geq \frac{h_i (d_i + d_{i+1}) - \alpha_i (x_N - x_1)(d_1 + d_N)}{h_i \Delta_i - \alpha_i (y_N - y_1)}. \]

In particular, if \( r_i = 1 + \frac{h_i (d_i + d_{i+1}) - \alpha_i (x_N - x_1)(d_1 + d_N)}{h_i \Delta_i - \alpha_i (y_N - y_1)}, \) then \( S(x) \) reduces to the rational quadratic spline FIF given in (4.12), and in this case the above mentioned conditions on \( \alpha_i \) alone are sufficient for the rational FIF to be monotonic.

The above theorem is used in Section 5 for the construction of visually pleasing and monotonicity preserving rational cubic spline FIFs.
4.2. **Convex Data.** Without loss of generality, it is assumed that a strictly convex set of data is given so that

\[ \Delta_1 < \Delta_2 < \cdots < \Delta_{N-1}. \]  

To avoid the possibility of \( S(x) \) having straight line segments, it is necessary that the derivatives at knot points satisfy

\[ d_1 < \Delta_1 < d_2 < \cdots < d_i < \Delta_i < \cdots < d_N. \]  

Assuming \( S(x) \) to be twice differentiable, \( S(x) \) is convex if and only if \( S''(x) \geq 0 \) \( \forall x \in [x_1, x_N] \). For \( S(x) \) to be twice differentiable, and \( S''(x) \) to be a fractal function, we assume that \( |\alpha_i| \leq \kappa \alpha_i^2 \) for \( i \in J \), where \( 0 < \kappa < 1 \). Calculating \( S''(L_i(x)) \) for \( x \in [x_1, x_N] \) from (4.1), and performing a detailed manipulation, we have

\[
S''(L_i(x)) = \frac{\alpha_i}{a_i^2} S''(x) + \frac{2[A_i^* \theta^3 + B_i^* \theta^2(1-\theta) + C_i^* \theta(1-\theta)^2 + D_i^*(1-\theta)^3]}{h_i[1 + (r_i - 3)\theta(1-\theta)]^3},
\]

where

\[
\begin{align*}
A_i^* &= r_i(d_{i+1} - \Delta_i) + d_i - d_{i+1} - \frac{\alpha_i}{h_i} \left\{ r_i[d_N(x_N - x_1) - (y_N - y_1)] + (x_N - x_1)(d_1 - d_N) \right\}, \\
B_i^* &= 3(d_{i+1} - \Delta_i) - \frac{3\alpha_i}{h_i} [d_N(x_N - x_1) - (y_N - y_1)], \\
C_i^* &= 3(\Delta_i - d_i) - \frac{3\alpha_i}{h_i} [(y_N - y_1) - d_1(x_N - x_1)], \\
D_i^* &= r_i(\Delta_i - d_i) + d_i - d_{i+1} - \frac{\alpha_i}{h_i} \left\{ r_i[(y_N - y_1) - d_1(x_N - x_1)] + (x_N - x_1)(d_1 - d_N) \right\}.
\end{align*}
\]

Hence, if scaling factors are taken to be nonnegative, and necessary conditions on derivative parameters are assumed, then a set of sufficient conditions for \( S''(L_i(x_j)) \geq 0 \) with arbitrary knot point \( x_j \) and \( i \in J \) is given by

\[ A_i^* \geq 0, B_i^* \geq 0, C_i^* \geq 0, \text{ and } D_i^* \geq 0. \]

From the Three Chords Lemma for convex functions, it follows that a convex set of data should necessarily satisfy \( d_1 \leq \frac{y_N - y_1}{x_N - x_1} \leq d_N \), where inequalities remain strict for strict convexity.

Now \( B_i^* \geq 0 \Leftrightarrow d_{i+1} - \Delta_i \geq \frac{\alpha_i}{h_i} [d_N(x_N - x_1) - (y_N - y_1)] \). Observing that if \( d_N(x_N - x_1) - (y_N - y_1) = 0 \), then \( B_i^* \geq 0 \) is obviously satisfied, we obtain the condition on scaling factor as

\[ \alpha_i \leq \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}. \]

Similarly, \( C_i^* \geq 0 \Leftrightarrow \Delta_i - d_i \geq \frac{\alpha_i}{h_i} [(y_N - y_1) - d_1(x_N - x_1)] \) gives

\[ \alpha_i \leq \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)}. \]

Therefore, to obtain \( S''(L_i(x_j)) \geq 0 \) for all \( i \in J \) and knot points \( x_j, j = 1, 2, \ldots, N \), it suffices to have \( A_i^* \geq 0, D_i^* \geq 0, \) and

\[
0 \leq \alpha_i \leq \min \left\{ \kappa \alpha_i^2 \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}, \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)} \right\}.
\]
For convex fractal interpolant, the conditions on shape parameter $r_i$ are found from the assumptions $A_i^* \geq 0$, $D_i^* \geq 0$. Hence, it is sufficient to take

$$r_i \geq \max \left\{ \frac{d_{i+1} - d_i + (\alpha_i/h_i) (x_N - x_i) (d_1 - d_N)}{d_{i+1} - \Delta_i - (\alpha_i/h_i) [d_N (x_N - x_i) - (y_N - y_1)]}, \frac{d_{i+1} - d_i + (\alpha_i/h_i) (x_N - x_i) (d_1 - d_N)}{d_{i+1} - \Delta_i - (\alpha_i/h_i) [(y_N - y_1) - d_i (x_N - x_i)]} \right\}$$

which is equivalent to take

$$r_i \geq 1 + \frac{M_i}{m_i}, \quad (4.17)$$

where

$$M_i = \max \{d_{i+1} - \Delta_i - \frac{\alpha_i}{h_i} [d_N (x_N - x_i) - (y_N - y_1)], \Delta_i - d_i - \frac{\alpha_i}{h_i} [(y_N - y_1) - d_i (x_N - x_i)]\};$$

$$m_i = \min \{d_{i+1} - \Delta_i - \frac{\alpha_i}{h_i} [d_N (x_N - x_i) - (y_N - y_1)], \Delta_i - d_i - \frac{\alpha_i}{h_i} [(y_N - y_1) - d_i (x_N - x_i)]\}.$$

With the above assumptions $A_i^* \geq 0$, $B_i^* \geq 0$, $C_i^* \geq 0$, and $D_i^* \geq 0$, we will show that $S''(x) > 0$ for all $x \in [x_1, x_N]$. From (4.15),

$$S^{(2)}(L_i(x_1)) = \frac{\alpha_i}{a_i^2} S^{(2)}(x_1) + \frac{2 D_i^*}{h_i}, \quad i \in J$$

$$\implies S^{(2)}(x_i) = \frac{\alpha_i}{a_i^2} S^{(2)}(x_1) + \frac{2 D_i^*}{h_i}, \quad i \in J.$$

Taking $i = 1$ in (4.18), we have $S^{(2)}(x_1) = \frac{2 D_1^*}{h_1} \left[ 1 - \frac{\alpha_1}{a_1^2} \right]^{-1}$. But $\alpha_1 < a_1^2$, and hence using $D_1^* \geq 0$, it follows that $S^{(2)}(x_1) \geq 0$. Using this in (4.18), we obtain $S^{(2)}(x_i) \geq 0$ for $i = 2, 3, \ldots, N - 1$. Again from (4.15) with $i = N - 1$ and $x = x_N$, we have $S^{(2)}(x_N) = \frac{2 A_{N-1}^*}{h_{N-1}} \left[ 1 - \frac{\alpha_{N-1}}{a_{N-1}^2} \right]^{-1}$. But $\alpha_{N-1} < a_{N-1}^2$, and hence using $A_{N-1}^* \geq 0$, it is proved that $S^{(2)}(x_N) \geq 0$. As $\alpha_i \geq 0$ for $i = 1, 2, \ldots, N - 1$ and $S^{(2)}(x_i) \geq 0$ for $i = 1, 2, \ldots, N$, using the recursive nature of a fractal function, we conclude that $S^{(2)}(x) \geq 0$ for all $x \in [x_1, x_N]$, whenever $A_i^* \geq 0$, $B_i^* \geq 0$, $C_i^* \geq 0$, $D_i^* \geq 0$. Hence, if scaling factors $\alpha_i$ and shape parameters $r_i$ are chosen according to (4.16) and (4.17) respectively, then the corresponding rational cubic FIF is convex in nature.

**Remark 4.5.** If $\Delta_i = \Delta_{i+1}$, then for a convex fractal interpolant, the above discussion suggests the choice of vertical scaling factor $\alpha_i$ to be zero. Also $d_i = d_{i+1} = \Delta_i$. Thus, in this case the rational cubic FIF becomes $S(L_i(x)) = (1 - \theta)y_i + \theta y_{i+1}$, i.e., to say that $S$ reduces to a straight line on $[x_i, x_{i+1}]$, as would be expected.

**Remark 4.6.** When all $\alpha_i = 0$, the condition (4.16) is obviously true and the condition (4.17) reduces to

$$r_i \geq 1 + \frac{M_i^*}{m_i^*} \quad \text{with} \quad (4.19)$$

$$M_i^* = \max \{d_{i+1} - \Delta_i, \Delta_i - d_i\}, \quad m_i^* = \min \{d_{i+1} - \Delta_i, \Delta_i - d_i\}.$$

This is the sufficient condition for convexity in the classical rational cubic spline [17], p. 971.
Remark 4.7. In particular, if we choose
\[ r_i = 1 + \frac{M_i}{m_i} + \frac{m_i}{M_i}, \] (4.20)
with \( \alpha_i \) satisfying \( 0 \leq \alpha_i \leq \min \left\{ \kappa \alpha_i^2, \kappa_* \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}, \kappa^* \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)} \right\}, 0 < \kappa, \kappa^*, \kappa_* < 1, \)
so as to settle the convexity in question, then the rational cubic spline FIF \( S \) in (3.5) reduces to a lower-order form given by
\[ S(L_i(x)) = \alpha_i S(x) + (1 - \theta)(y_i - \alpha_i y_1) + \theta(y_{i+1} - \alpha_i y_N) - \frac{h_i \theta(1 - \theta) G_i H_i}{G_i(1 - \theta) + H_i \theta}, \] (4.21)
where
\[ G_i := (d_{i+1} - \Delta_i) - \frac{\alpha_i}{h_i} [d_N(x_N - x_1) - (y_N - y_1)] \text{ and } \]
\[ H_i := (\Delta_i - d_i) - \frac{\alpha_i}{h_i} [(y_N - y_1) - d_1(x_N - x_1)]. \]

Our particular choice of shape parameters given in (4.20) verifying the convexity condition can be justified as follows. For the shape parameters as in (4.20), and vertical scaling factors satisfying \( |\alpha_i| < \alpha_i^4 \), we have \( r_i - 3 = O(h_i^2) \), and consequently we obtain optimal \( O(h^3) \) bound on interpolation error provided derivatives are estimated with \( O(h^3) \) accuracy.

The main points in the above discussion are extracted in the form of following theorem:

Theorem 4.2. Given a convex data \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \), assume that derivative parameters satisfy the necessary convexity condition expressed in (4.14). Then, the following conditions on scaling factors and shape parameters are sufficient for the corresponding \( C^1 \)-rational cubic spline FIF \( S(x) \) to be convex on \( I = [x_1, x_N] \):

\[ 0 \leq \alpha_i \leq \min \left\{ \kappa \alpha_i^2, \kappa_* \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)}, \kappa^* \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)} \right\}, 0 < \kappa, \kappa^*, \kappa_* < 1, \]
\[ r_i \geq \max \left\{ \frac{d_{i+1} - d_i + (\alpha_i/h_i) (x_N - x_1) (d_1 - d_N)}{d_{i+1} - d_i + (\alpha_i/h_i) (x_N - x_1) (d_1 - d_N)} \right\}, \Delta_i - d_i < d_1(x_N - x_1). \]

The preceding theorem is used in Section 5 for the construction of visually pleasing and convexity preserving rational cubic spline FIFs.

4.3. Convex and Monotonic Data. We now consider the possibility that the data satisfy both the monotonic increasing condition \( y_1 < y_2 < \cdots < y_N \), and the strictly convex condition (4.13). The derivative parameters must then satisfy the following inequalities:
\[ 0 \leq d_1 < \Delta_1 < d_2 < \cdots < \Delta_{i-1} < d_i < \Delta_i < \cdots < d_N. \] (4.22)

We claim that the convex interpolation method described in the previous subsection is suitable for obtaining a convex and monotonic fractal interpolant. To verify this claim, we proceed as
follows. Assume that the sufficient conditions (4.16) on vertical scaling factors for convexity of the rational cubic FIF hold. Rearrangement of these inequalities give
\[ 0 \leq \alpha_i < \frac{h_i(d_{i+1} - \Delta_i)}{d_N(x_N - x_1) - (y_N - y_1)} \implies \Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1) < d_{i+1} - \frac{\alpha_i}{h_i}d_N(x_N - x_1) \]
and
\[ 0 \leq \alpha_i < \frac{h_i(\Delta_i - d_i)}{(y_N - y_1) - d_1(x_N - x_1)} \implies d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1) < \Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1). \]
Combining two inequalities obtained above, we have
\[ d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1) < \Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1) < d_{i+1} - \frac{\alpha_i}{h_i}d_N(x_N - x_1) \quad (4.23) \]
Since \( a_i < 1 \), the condition \( \alpha_i \leq \kappa a_i^2 \) given in (4.16) implies the condition \( \alpha_i \leq \kappa a_i \) in (4.9). Also
\[ \alpha_i \leq \kappa a_i, d_i \geq 0 \forall i \implies \alpha_i d_i \leq \kappa \frac{h_i}{x_N - x_1}d_1 \implies d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1) \geq d_i - \kappa d_1 \quad (4.24) \]
Hence from (4.22) and (4.24), we have \( d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1) \geq 0 \). Consequently, (4.23) yield \( \Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1) \geq 0 \) and \( d_{i+1} - \frac{\alpha_i}{h_i}d_N(x_N - x_1) \geq 0 \). Thus, we get sufficient condition for monotonicity on scale factors \( \alpha_i (i = 1, 2, \ldots, N - 1) \).
Assume that sufficient condition (4.17) on the shape parameters \( r_i \) for convexity of the rational cubic FIF is true. Now we will prove that, this condition implies the condition (4.6) on \( r_i \) for monotonicity of the rational cubic fractal interpolant. Without loss of generality, assume that
\[ M_i = [d_{i+1} - \frac{\alpha_i}{h_i}d_N(x_N - x_1)] - [\Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1)], \]
\[ m_i = [\Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1)] - [d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1)]. \]
Denote \( P_i^* = d_i - \frac{\alpha_i}{h_i}d_1(x_N - x_1), Q_i^* = \Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1), R_i^* = d_{i+1} - \frac{\alpha_i}{h_i}d_N(x_N - x_1) \).
From (4.23), we have \( P_i^* \leq Q_i^* \leq R_i^* \). Again, with these notations
\[ M_i = R_i^* - Q_i^* = \max\{R_i^* - Q_i^*, Q_i^* - P_i^*\} \implies Q_i^* \leq \frac{P_i^* + R_i^*}{2}. \quad (4.25) \]
The sufficient condition (4.6) for monotonicity of a rational cubic FIF can be rearranged as
\[ r_i \geq \frac{d_i + d_{i+1} - \frac{\alpha_i}{h_i}(d_1 + d_N)(x_N - x_1)}{\Delta_i - \frac{\alpha_i}{h_i}(y_N - y_1)} = \frac{P_i^* + R_i^*}{Q_i^*}. \quad (4.26) \]
The sufficient condition (4.17) for convexity of a rational cubic FIF in above notations becomes
\[ r_i \geq 1 + \frac{R_i^* - Q_i^*}{Q_i^* - P_i^*} = \frac{R_i^* - P_i^*}{Q_i^* - P_i^*}. \quad (4.27) \]
Note that (4.27) implies (4.26), if \( \frac{R_i^* - P_i^*}{Q_i^* - P_i^*} \geq \frac{P_i^* + R_i^*}{Q_i^*} \), which is equivalent to the condition described in (4.25). But the condition (4.25) is obviously true in this case due to our assumptions. The proof is similar if we assume that \( M_i = Q_i^* - P_i^* \) and \( m_i = R_i^* - Q_i^* \). Thus, we have proved the sufficient condition for convexity of a rational cubic FIF on shape parameters \( r_i \) gives the
Since conditions described by region parameters determined by \( R \) would be then monotonic as well, which can be demonstrated as follows. Using an elementary rational cubic spline \( \text{FIF} \). In fact, given a monotonic increasing convex data with derivative parameters verifying \( \text{(4.22)} \), any convex rational cubic fractal interpolant corresponding to the data would be then monotonic as well, which can be demonstrated as follows. Using an elementary result from the calculus of fractal function \( [4,6] \), we have

\[
S^{(1)}(L_i(x)) = a_i \int_{x_1}^{x} S^{(2)}(L_i(t)) \, dt + d_i
\]

Convexity of the fractal interpolant \( S \) on \([x_1, x_N]\) imply that \( S^{(2)}(L_i(x)) \geq 0 \) \( \forall i \in J \). Now in view of \( \text{(4.28)} \), monotonicity assumption \( d_i \geq 0 \) imply \( S^{(1)}(L_i(x)) \geq 0 \) \( \forall i \in J \). Since \( I \) is the attractor of the IFS \( \{\mathbb{R}; L_i(x) : i \in J\} \), it follows from the recursive nature of a FIF that \( S^{(1)}(x) \geq 0 \) \( \forall x \in [x_1, x_N] \), yielding monotonicity of \( S \).

4.4. **Positive Data.** For a given set of data points \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \) satisfying the condition \( y_i > 0 \) \( \forall i = 1, 2, \ldots, N \), the problem is to find suitable conditions on free parameters involved in the fractal interpolant \( S(x) \) described in \( \text{(3.5)} \), so as to assure \( S(x) > 0 \) for all \( x \in I = [x_1, x_N] \). As observed earlier, the condition \( r_i > -1 \) guarantees strictly positive denominator \( Q_i(\theta) \). Since \( S \) is defined recursively through an implicit function, to get simple data dependent conditions for positivity, we assume \( \alpha_i \geq 0 \) for all \( i \in J \). With these initial assumptions on shape parameters and scaling factors, it follows from \( \text{(3.5)} \) that positivity of \( S(L_i(x_j)) \) depends on positivity of the cubic polynomial \( P_i(\theta)|_{x=x_j} \). Since the rational FIF is generated iteratively from function values at knot points, and \( I \) is the attractor of the IFS \( \{\mathbb{R}; L_i(x), i \in J\} \), then \( S(L_i(x_j)) > 0 \) for all \( i \in J \) and for all knot points \( x_j \), will ensure positivity of \( S(x) \) on \( I \). Now, according to Schmidt and Heß \( [42] \), \( P_i(\theta)|_{x=x_j} > 0 \) if and only if \( (A_i, B_i, C_i, D_i) \in R_1 \cup R_2 \), where \( A_i, B_i, C_i, D_i \) are defined as in Section 3.1, and

\[
R_1 = \{(\lambda_i, \beta_i, \gamma_i, \delta_i) : \lambda_i > 0, \beta_i > 0, \gamma_i > 0, \delta_i > 0\},
\]

\[
R_2 = \{(\lambda_i, \beta_i, \gamma_i, \delta_i) : \lambda_i > 0, \beta_i > 0, 4\delta_i\beta_i^3 + 4\lambda_i\gamma_i^3 + 27\delta_i^2\gamma_i^2 - 18\lambda_i\beta_i\gamma_i\delta_i - \gamma_i^2\beta_i^2 > 0\}.
\]

Since conditions described by region \( R_2 \) requires a lot of computation, to get a sufficient condition for positivity of \( P_i(\theta) \), we use comparatively efficient and reasonably acceptable choice of parameters determined by \( R_1 \).

\[
A_i > 0 \iff y_i - \alpha_i y_1 > 0 \iff \alpha_i < \frac{y_i}{y_1}.
\]

Similarly

\[
D_i > 0 \iff y_{i+1} - \alpha_i y_N > 0 \iff \alpha_i < \frac{y_{i+1}}{y_N}.
\]

Hence, we constrain the vertical scaling factors according to \( 0 \leq \alpha_i \leq \min\{\kappa \alpha_i, \kappa^* \frac{y_i}{y_1}, \kappa^* \frac{y_{i+1}}{y_N}\} \), where \( 0 < \kappa, \kappa^*, \kappa_* < 1 \). Now,

\[
B_i > 0 \iff [r_i y_i + h_i d_i] - \alpha_i [r_i y_1 + d_1(x_N - x_1)] > 0 \iff r_i > \frac{-h_i d_i + \alpha_i d_1(x_N - x_1)}{y_i - \alpha_i y_1}.
\]
A similar analysis gives that \( C_i > 0 \iff r_i > \frac{h_id_{i+1} - \alpha_id_N(x_N - x_1)}{y_{i+1} - \alpha_iy_N} \). Thus, we have the following theorem:

**Theorem 4.3.** Let \( \{(x_i, y_i) : i = 1, 2, \ldots, N\} \) be a given set of positive data and \( d_i, i = 1, 2, \ldots, N \) be first derivative values at the knots \( x_i \). If for each \( i \in J \), the scaling factors and shape parameters satisfy the following conditions, then the corresponding rational cubic spline FIF \( S(x) \) described in (3.5) is positive on the interpolation interval \( I \).

\[
0 \leq \alpha_i \leq \min\left\{\kappa \alpha_i, \kappa^* \frac{y_i}{y_1}, \kappa_+ \frac{y_{i+1}}{y_N}\right\}, 0 < \kappa, \kappa^*, \kappa_+ < 1,
\]

\[
r_i > \max\{-1, \frac{-h_id_i + \alpha_id_1(x_N - x_1)}{y_i - \alpha_iy_1}, \frac{h_id_{i+1} - \alpha_id_N(x_N - x_1)}{y_{i+1} - \alpha_iy_N}\}.
\]

As a special case of the preceding theorem, it is worth to single out the following remark:

**Remark 4.8.** If we set \( \alpha_i = 0 \forall i \in J \), the positivity condition on \( r_i \) reduces to \( r_i > \max\{-1, -\frac{h_id_i}{y_i}, \frac{h_id_{i+1}}{y_{i+1}}\} \), which provide sufficient condition for positivity of the classical rational cubic spline. This demonstrates that the rational cubic spline by Delbourgo and Gregory can be used to solve positivity preservation problems as well.

5. **Examples and Discussion**

In this section, we consider several numerical examples to support the practical utility of the \( C^1 \)-rational cubic spline fractal interpolation scheme and its particular cases introduced in Sections 3-4. To bring out the nature of generalization achieved through our rational FIF and to compare with its classical counterpart, we base our discussion on examples provided in Delbourgo and Gregory [17]. A large flexibility in the choice of shape preserving \( C^1 \)-rational interpolant offered by the proposed fractal approach is revealed by choosing suitable sets of vertical scaling factors, shape parameters, and also by estimating derivative parameters by various methods. The choices of parameter values rounded off to four decimal places are tabulated for a quick reference.

Consider the monotonic and convex data set obtained by sampling the function \( 1/x^2 \) at \( x = -2, -1, -0.3, \) and \(-0.2 \). Figure 2 is a default rational cubic spline FIF given in (3.5) with specific choice of vertical scaling factors and shape parameters as \( \alpha_1 = 0.02, \alpha_2 = 0.2354, \alpha_3 = 0.08; r_1 = 6, r_2 = 10, r_3 = 2. \) The corresponding fractal curve is not monotonic, and exhibits undesired inflection points, thereby illustrating the importance of shape preserving rational cubic fractal interpolation schemes developed in Section 4.

First we implement the convex rational cubic spline interpolation scheme discussed in Section 4.2 on the convex data set defined by \( f(x) = 1/x^2 \) on \([-2, -0.2] \), with the interpolation points at \( x = -2, -1, -0.3, \) and \(-0.2 \). Since arithmetic derivative values (see Section 3.1) are well suited for convex interpolation problems, we use arithmetic estimates of the derivative values at all points including the endpoints (here \( d_i \) is negative, and hence becomes unsuitable for a monotonic interpolation scheme of this increasing data). Different sets of vertical scaling factors (see Table 1) that meet the specification in Theorem 4.2 are chosen, and shape parameters are
calculated according to (4.20) so that the resulting convex curves in Figure 3(a) corresponds to the rational quadratic spline FIFs given in (4.21). The classical rational quadratic spline interpolant in Figure 3(a) corresponds to zero scaling values in each subinterval. It can be observed that the classical rational quadratic interpolant in Figure 3(a) assumes negative values which may be undesirable as the original data generating function is positive on $[-2, -0.2]$. Figure 3(b) corresponds to the rational cubic spline FIFs with same sets of scaling factors as in Figure 3(a), and a common set of shape parameters namely $r_1 = 90$, $r_2 = 26800$, $r_3 = 5$ so that the values of parameters are in accordance with Theorem 4.2. This demonstrates the flexibility gained by the fractal methodology due to the presence of vertical scaling factors. In contrast to the rational quadratic spline FIFs in Figure 3(a), the rational cubic spline FIFs in Figure 3(b) corresponding to all the three sets of scalings preserve positivity of the data. Figure 4(a) is generated by the geometric derivative approximations, and by choosing shape parameters according to (4.20). Contrary to the arithmetic derivative settings, all the rational quadratic spline FIFs generated in Figure 4(a) are positive. In this case, apparent changes in interpolatory curves with changes in scaling factors not seem to be significant. Now the convex rational cubic spline FIF scheme is implemented with same sets of vertical scaling factors as in Figure 4(a), and a common choice of shape parameters $r_1 = 6$, $r_2 = 10$, $r_3 = 3$, and corresponding convexity preserving graphs are displayed in Figure 4(b). Since the data generating function is available, we can calculate the exact derivative values to implement our shape preserving interpolation scheme. Figure 5(a) corresponds to the rational quadratic spline FIFs with parameter values as indicated in Table 1, where as the rational cubic fractal curves in Figure 5(b) are generated with same scaling vectors as that of Figure 5(a), and a common set of shape parameter $r_1 = 4$, $r_2 = 97$, $r_3 = 4$. It appears from the figures that the rational cubic fractal scheme with the geometric derivative settings, and the rational quadratic fractal scheme with the exact derivative settings produce visually pleasing convex rational cubic FIFs. The convex rational cubic fractal interpolants in Figures 4-5 are monotonic as well, which supports the theoretical discussion described in Section 4.3.
Now we implement our monotonic rational interpolation scheme on the same monotonic increasing data set namely \( f(x) = 1/x^2 \) sampled at \( x = -2, -1, -0.3, \) and \(-0.2\). Derivative parameters are estimated by the geometric approximations, and scaling/shape parameters are constrained according to Theorem 4.1 to obtain the rational quadratic spline FIFs in Figure 6(a). Monotonic curves in Figure 6(a) corresponding to certain choices of scaling factors exhibits undesirable inflexion. Visually pleasing rational cubic FIFs have been generated in Figure 6(b) with same set of scaling factors as in Figure 6(a), and with a common choice of shape parameters \( r_1 = 6, r_2 = 10, r_3 = 2 \) satisfying (4.6). Derivative parameters obtained from the \( C^2 \) spline approximations of Delbourgo and Gregory are utilized to generate rational quadratic spline fractal interpolants in Figure 7(a), and rational cubic spline FIFs in Figure 7(b) with a common choice of scaling parameters (see Table 1). The shape parameters are constrained according to Theorem 4.1, and a common choice of shape parameters \( r_1 = 8, r_2 = 10, r_3 = 6 \) is taken in Figure 7(b). Next we employ monotonic quadratic and cubic schemes given in Theorem 4.1 to the same data set but now with exact derivative values to obtain fractal interpolating curves in Figure 8. Choices of the rational cubic IFS parameters corresponding to Figure 8(a) are displayed in Table 1, and a common choice of shape parameters \( r_1 = 6, r_2 = 10, r_3 = 6 \) that matches with the scaling vectors in Figure 8(a) are used to generate Figure 8(b). It can be noticed that the classical interpolants in Figures 6(a), 7(a) and 8(a) obtained as a special case of our rational FIFs are the profiles given in Figure 2 of the reference [17]. Also the monotonic curves in Figures 6-8 are in general non-convex. Thus, for a monotonically increasing convex data with derivative parameters chosen to satisfy required conditions, monotonicity preserving rational fractal scheme may not produce a convex monotonic FIF, whereas the convexity preserving rational fractal scheme will automatically produce a convex monotonic fractal interpolant. In other words, even for a monotonic increasing data with derivative parameters satisfying (4.22), the monotonicity condition (4.9) on scaling factors and the condition (4.6) on shape parameters are not sufficient enough to guarantee corresponding convexity conditions for a rational cubic FIF.

Our next data set consists of points uniformly disposed on half of the unit circle. To be more specific, coordinates of the points are given by \( x_i = -\cos((i-1)\frac{\pi}{12}), y_i = -\sin((i-1)\frac{\pi}{12}), i = 1, 2, 3, \ldots, 13 \). Applying the convex rational cubic FIF scheme with derivative values obtained by the arithmetic setting, \( \alpha_i \) according to (4.16), and \( r_i \) satisfying (4.20), we get the rational quadratic spline FIF generated in Figure 9(a). Figure 9(b) has been generated with the same scalings as in Figure 9(a), and a common choice of shape parameters namely \( r_1 = r_{12} = 4, r_2 = r_{11} = 190, r_3 = r_{10} = 60, r_4 = r_9 = 20, r_5 = r_8 = 20, \) and \( r_6 = r_7 = 18 \). Next we take a monotone and convex set of data consisting of points uniformly spaced at \( 15^\circ \) intervals over quarter circle whose coordinates are given by \( x_i = \sin((i-1)\frac{\pi}{12}), y_i = -\cos((i-1)\frac{\pi}{12}), i = 1, 2, 3, \ldots, 7 \). Now we implement our convex rational cubic spline interpolation scheme, and degenerated quadratic scheme with the geometric derivative values (except \( d_1 = 0 \)), and corresponding rational fractal curves are generated in Figures 10(a)-(b). Parameters corresponding to Figure 10(a) are given in Table 2. Rational cubic FIFs in Figure 10(b) correspond to same scaling vectors as in Figure 10(a), and a common choice of shape parameters, viz., \( r_1 = 25, r_2 = r_3 = 15, r_4 = r_5 = r_6 = 10 \). Note that variations among different rational fractal curves in Figures 9(b) and 10(b) are negligible while variations among different rational curves in Figures 3(b) and 4(b) are prominent even if both data sets use same type of derivative approximations.
Hence, we may conclude that the shape variation among various rational fractal curves is data dependent even if $\alpha_i (i = 1, 2, \ldots, N - 1)$ are small, and $r_i (i = 1, 2, \ldots, N - 1)$ are fixed. Again, we use the semicircular convex non-monotone data but this time with the exact derivative values (except that the infinite gradients at both endpoints replaced with $d_1 = -50$, $d_{13} = 50$) to obtain rational FIFs in Figures 11 (a)-(b). Different convex rational fractal curves in Figure 11 (b) are obtained by choosing scaling factors same as that of Figure 11(a), shape parameters being set to $r_1 = 15$, $r_2 = 10$, $r_3 = r_9 = 8$, $r_4 = r_6 = 4$, $r_5 = 5$, $r_7 = 3$, $r_8 = 6$, $r_9 = 8$, $r_{10} = r_{11} = 11$, and $r_{12} = 18$.

Next, we impose our monotonic rational quadratic and cubic fractal interpolation schemes on quarter circle data with (i) the geometric derivatives (ii) the classical $C^2$-rational spline derivatives of Delbourgo and Gregory and (iii) the exact derivatives. Figure 12(a) is obtained by invoking the monotonic rational fractal interpolation scheme with parameters satisfying (4.9), (4.11) (see Table 2), and the derivatives estimated with the geometric approximation (except $d_1 = 0$). To retain the cubic nature of the fractal interpolant, we apply the scheme with rationality parameters as in (4.6), same scalings as in Figure 12(a), and a common choice of shape parameter values as $r_1 = r_2 = r_3 = r_4 = r_5 = 4$ and $r_6 = 6$. The corresponding rational cubic FIFs are generated in Figure 12(b). A similar treatment with the $C^2$-spline derivatives (boundary conditions $d_1 = 0$, $d_7 = 20$), and parameter values as in Table 2 produce Figure 13(a). Figure 13(b) is generated by applying the cubic rational fractal scheme with scalings as that of Figure 13(a), and a common set of shape parameters as in Figure 12(b). Now with the exact derivative values (except for $d_7 = 25$), our rational quadratic fractal interpolation scheme generates Figure 14(a) (for parameters see Table 2). Vertical scaling factors as that of Figure 14(a) and a common set of shape parameter values that suits with those scalings, generate monotonic rational cubic spline FIFs in Figure 14(b). It can be seen from Figures 12 -14 that the monotonic rational quadratic fractal interpolation scheme has inflections in the curves, especially for non zero scaling factors. The classical monotonic rational quadratic interpolants obtained in Figures 12(a), 13(a) and 14(a) by taking scaling factors as zero in our rational FIF corresponds to Figure 4 of the reference [17]. Hence the shape preserving fractal interpolation schemes in the present paper are more general than that of Delbourgo and Gregory [17].

Finally, we implement our positivity preserving rational cubic spline fractal interpolation scheme on the positive data set generated by the function $f(x) = 1/x^2$, with knot points at $x = -2, -1, -0.3, -0.2$. Derivative parameters are estimated by arithmetic mean method and admissible values of scaling factors are calculated using Theorem 4.3. To bring out the shape modification properties of vertical scaling factors in our positivity preserving fractal interpolation scheme, we take three different sets of $\alpha_i$ that suits with a fixed set of shape parameters (see Table 2), and corresponding IFS codes generate Figure 15(a). Again, implementing positivity preserving cubic spline fractal interpolation scheme with a fixed set of shape parameters, namely, $r_1 = 100$, $r_2 = 25$, $r_3 = 1$, and the same scaling vectors as in Figure 15(a), fractal curves in Figure 15(b) are generated. Thus, curves within Figure 15(a) may be used to study the effect of change in scaling factors $\alpha_i$, and comparison between corresponding curves in Figure 15(a), and Figure 15(b) can be used to illustrate the effect of change of shape parameters $r_i$. Numerical experiments conducted on this positive data reveals that, for the zero scaling in each subinterval (i.e., the classical rational cubic spline by Delbourgo and Gregory), the modification
in shape parameters $r_i$ is not capable to produce perceptible changes in the curve pertaining to the last subinterval (see for example the classical cases in Figure 15(a) and Figure 15(b)). On the other hand, with fixed values of $r_i$ changes in $\alpha_i$ produces considerable shape variations in last subinterval (see Figure 15(b)). This illustrates the advantage of the additional scaling parameters involved in our construction. However, for fixed values of rationality parameters as in Figure 15(a), changes in $\alpha_i$ do not produce considerable changes in the last subinterval. Thus, sensitivity of the curve with perturbation in scaling versus perturbation in shape parameters is decisively depends on the data. Next, we estimate the derivative parameters with the geometric mean method, and consider scaling/shape parameters as in Figure 15(a) to produce positive rational cubic spline FIFs in Figure 16(a). Figure 16(b) is generated with same scaling factors as in Figure 15(a), and a fixed set of shape parameter, $r_1 = 100, r_2 = 25, r_3 = 1$. Similar exercise is done with exact derivative values to produce Figure 17(a), for which the corresponding spline parameters are displayed in Table 2. With scaling factors as in Figure 17(a) and with a common set of shape parameters, $r_1 = 100, r_2 = 25, r_3 = 2$, so as to satisfy the sufficient conditions for positivity preservation, Figure 17(b) is generated. Thus, the shape modification properties of scaling factors, shape parameters, and derivative parameters in the positivity preserving cubic spline fractal interpolation scheme can be illustrated by a comparative study of profiles within and across Figures 15(a)-17(b).

From these numerical experiments, we have the following observations. For a given data set, the classical convex or monotonic rational quadratic scheme available in literature \cite{14,24} can produce different interpolating curves with different choice of derivative parameters. However, if slopes are also given at the grid points (for instance, in plotting solution of ordinary differential equations), the classical shape preserving quadratic scheme just give one interpolating curve. This may be a disadvantage for a designer concerned with the shape modification problem. Due to the presence of scaling factors in the construction of a rational FIF, the difficulty that arise due to the uniqueness is settled. Even for the rational cubic fractal scheme involving shape parameters, the presence of scaling factors offers additional flexibility in the choice of interpolant. In the classical case, visually pleasing good results are produced when all exact derivatives are available, whereas in case of a FIF even estimated values of derivatives may produce relatively pleasing results depending on the values of scaling/shape parameters. Also, the monotone, convex and positive fractal interpolants generated here reveal that our monotonicity or convexity schemes may not preserve positivity property inherent in the data, and vice-versa. For a data set with more than one shape characteristic inherent in it, our scheme can be employed by suitably combining the corresponding conditions, to get desired type of fractal interpolants. It depends on the data to say whether the change in the scaling factors or change in the shape parameters influence more on the shape of the rational cubic spline fractal curve. However, change of scaling factors and shape parameters within the specification limits can undoubtedly produce a large class of shape preserving interpolants, from which user can choose one depending on the requirement of the problem. It seems that, as far as easiness of implementation is concerned the proposed shape preserving rational cubic/quadratic spline fractal interpolation algorithms are on par with their classical counterparts discussed earlier in the literature.
6. CONCLUSION

The construction of rational cubic fractal splines involving shape parameters is initiated for the first time in the present work to provide a new solution to the shape preserving interpolation problem. Due to the presence of vertical scaling factors and shape parameters involved in the definition, the proposed $C^1$-rational cubic spine FIF satisfactorily generalize the classical rational splines in [14, 17, 24], and provides an extra freedom for a user to interactively generate visually pleasant curves as desired. For a data with given or estimated derivative values and for a fixed choice of scaling factors and shape parameters, the existence and uniqueness of a $C^1$-rational cubic spline FIF is proved. By deriving suitable data dependent conditions on vertical scaling factors and shape parameters, the existence of rational cubic spline fractal interpolant solving all the three fundamental shape preservation problems is established. Thus, using an IFS involving rational functions with free shape parameters as a medium, the present paper bridges the gap between the new interpolation methods of fractal theory and the existing shape preserving interpolation schemes of the classical numerical analysis providing benefits of one to the other. The proposed rational cubic FIF carries an additional advantage of discussing all the three important shape aspects within one mathematical model. Uniform convergence of rational cubic spline FIF to the original data generating function $f \in C^4[x_1, x_N]$ is established. The convergence analysis shows that $O(h^r), (r = 1, 2, 3, 4)$, error bounds can be achieved by suitable choices of derivatives, vertical scaling factors, and shape parameters. Thus, the present interpolation method has convergence properties similar to that of the classical counterpart, which should be considered along with the flexibility and diversity offered by the new method. Moreover, with a more general condition on data generating function, the uniform convergence of the classical rational cubic spline and the rational cubic spline FIF are established. In the process of convergence analysis, an explicit upper bound for the $L^\infty$-norm of the cubic spline FIF is naturally emerged thereby establishing the boundedness property of the interpolant.

It is strongly felt that a large flexibility offered in the choice of shape preserving $C^1$-rational cubic fractal splines by our approach can be exploited in CAGD, CAM, Data visualization environment, and in other industrial applications. Since the classical shape preserving rational quadratic and cubic spline interpolants, which have been successfully employed in many mathematical and engineering applications, emerge as a special case of our rational cubic spline fractal interpolation schemes, it might be possible to use the proposed $C^1$-rational cubic FIFs in areas where the classical interpolations stand less satisfactory.

An interesting extension of the work presented here is as follows. Since the rational FIF $S$ and its derivatives are defined piecewisely and implicitly, to avoid any possible intriguing difficulty that may creep into the process of maintaining positivity of $S^{(r)}, (r = 0, 1, 2)$, in successive iterations so as to make the FIF positive, monotonic, and convex respectively, we imposed non-negative restrictions on scaling factors. By this assumption we could obtain easily calculable data dependent constraints for shape preservation, providing a large number of shape preserving interpolants in addition to the classical rational splines. Our predilection for the nonnegativity assumption on scaling factors is attributable to reasons of convenience and relative generality that it offers, rather than of necessity. We leave the case of admissibility of negative scaling factors unsettled. However, we conjecture that by analysis of the extrema of the nonlinear functions...
involved in the corresponding IFS that generate the rational cubic spline FIF and its derivatives, it may be possible to allow negative values of scaling whilst maintaining the required shape preserving properties of the given data. Theoretically this extension leads to the necessary and sufficient conditions for various shape preservation properties, and practically helps to enlarge the family of shape preserving interpolants.

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Tables and Figures:

**TABLE 1.** Scaling and shape parameters involved in convex/monotonic rational cubic FIFs for \( f(x) = 1/x^2 \).

| Table | Figure | Set 1 | Set 2 | Classical |
|-------|--------|-------|-------|-----------|
|       |        | (a)   | (a)   | (a)       |
| Figure 3 |        | \( \alpha_1 = 0.03, \alpha_2 = 0.02, \alpha_3 = 0.002 \) | \( \alpha_1 = 0.025, \alpha_2 = 0.103, \alpha_3 = 0.002 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 16.1723, r_2 = 23.3564, r_3 = 3.1195 \) | \( r_1 = 5.4, r_2 = 120.12, r_3 = 3.1 \) | \( r_1 = 3, r_2 = 20.3602, r_3 = 3 \) |
| Figure 4 |        | \( \alpha_1 = 0.012, \alpha_2 = 0.151, \alpha_3 = 0.003 \) | \( \alpha_1 = 0.01, \alpha_2 = 0.09, \alpha_3 = 0.002 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 6.0843, r_2 = 7.0372, r_3 = 3.0768 \) | \( r_1 = 3.118, r_2 = 8.7613, r_3 = 3.0314 \) | \( r_1 = 6.1315, r_2 = 9.9826, r_3 = 3 \) |
| Figure 5 |        | \( \alpha_1 = 0.0025, \alpha_2 = 0.098, \alpha_3 = 0.003 \) | \( \alpha_1 = 0.001, \alpha_2 = 0.08, \alpha_3 = 0.002 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 3.7759, r_2 = 96.5243, r_3 = 3.1863 \) | \( r_1 = 3.3105, r_2 = 3.0174, r_3 = 3.222 \) | \( r_1 = 3.9, r_2 = 6.0004, r_3 = 3.2976 \) |
| Figure 6 |        | \( \alpha_1 = 0.0137, \alpha_2 = 0.2354, \alpha_3 = 0 \) | \( \alpha_1 = 0.012, \alpha_2 = 0.115, \alpha_3 = 0.012 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 1.1073, r_2 = 1.6973, r_3 = 1.8374 \) | \( r_1 = 2.2654, r_2 = 6.5696, r_3 = 2.7672 \) | \( r_1 = 6.7443, r_2 = 8.5418, r_3 = 2.9983 \) |
| Figure 7 |        | \( \alpha_1 = 0.0055, \alpha_2 = 0.1136, \alpha_3 = 0.0555 \) | \( \alpha_1 = 0.0025, \alpha_2 = 0.057, \alpha_3 = 0.025 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 1.3906, r_2 = 1.2291, r_3 = 1.5835 \) | \( r_1 = 3.3125, r_2 = 4.1226, r_3 = 2.5856 \) | \( r_1 = 4.6232, r_2 = 6.2263, r_3 = 3.3258 \) |
| Figure 8 |        | \( \alpha_1 = 0.0044, \alpha_2 = 0.1152, \alpha_3 = 0.0555 \) | \( \alpha_1 = 0.004, \alpha_2 = 0.054, \alpha_3 = 0.055 \) | \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) |
|       |        | \( r_1 = 1.3906, r_2 = 2.1861, r_3 = 1.5901 \) | \( r_1 = 1.6885, r_2 = 4.2967, r_3 = 1.0693 \) | \( r_1 = 4, r_2 = 6.2667, r_3 = 3 \) |
| Table 2. Scaling and shape parameters for convex, monotonic, and positive rational cubic FIFs |
|---------------------------------------------------------------|
| **Set 1** | \( \alpha_1 = \alpha_{12} = 0.00025 \), \( \alpha_2 = \alpha_{11} = 0.0023 \), \( \alpha_3 = \alpha_{10} = 0.002 \), \( \alpha_4 = \alpha_9 = \alpha_5 = \alpha_8 = 0.0018 \), \( \alpha_6 = \alpha_7 = 0.0019 \) |
| (a) | \( r_1 = r_{12} = 3 \), \( r_2 = r_{11} = 184.3121 \), \( r_3 = r_{10} = 54.6817 \), \( r_4 = r_9 = 18.8576 \), \( r_5 = r_8 = 18.0819 \), \( r_6 = r_7 = 16.6158 \) |
| **Set 2** | \( \alpha_1 = \alpha_{12} = 0.0001 \), \( \alpha_2 = \alpha_3 = \ldots = \alpha_{11} = 0.001 \) |
| (a) | \( r_1 = r_{12} = 3 \), \( r_2 = r_{11} = 15.7515 \), \( r_3 = r_{10} = 5.9372 \), \( r_4 = r_9 = 4.1719 \), \( r_5 = r_8 = 3.3871 \), \( r_6 = r_7 = 3.0425 \) |
| **Classical** | \( \alpha_1 = \alpha_2 = \ldots = \alpha_{12} = 0 \) |
| (a) | \( r_1 = r_{12} = 3 \), \( r_2 = r_{11} = 10.1062 \), \( r_3 = r_{10} = 4.2722 \), \( r_4 = r_9 = 3.3970 \), \( r_5 = r_8 = 3.1112 \), \( r_6 = r_7 = 3.0107 \) |

**Figure 9**

**Figure 10**

| **Set 1** | \( \alpha_1 = 0.003 \), \( \alpha_2 = 0.004 \), \( \alpha_3 = 0.006 \), \( \alpha_4 = 0.01 \), \( \alpha_5 = 0.009 \), \( \alpha_6 = 0.001 \) |
| (a) | \( r_1 = 20.666, r_2 = 9.4434, r_3 = 10.5699, r_4 = 5.0523, r_5 = 7.0462, r_6 = 3.0011 \) |

**Figure 11**

| **Set 2** | \( \alpha_1 = 0.002, \alpha_2 = 0.004, \alpha_3 = 0.002, \alpha_4 = 0.009, \alpha_5 = 0.0099, \alpha_6 = 0.001 \) |
| (a) | \( r_1 = 3.2189, r_2 = 9.4960, r_3 = 3.0003, r_4 = 3.4819, r_5 = 6.9048, r_6 = 3.0005 \) |

**Figure 12**

| **Set 1** | \( \alpha_1 = 0.00025, \alpha_2 = 0.0006, \alpha_3 = \alpha_{10} = 0.0004, \alpha_4 = \alpha_5 = \alpha_8 = \alpha_9 = 0.0003, \alpha_6 = \alpha_7 = \alpha_{12} = 0.0002, \alpha_{11} = 0.0006 \) |
| (a) | \( r_1 = 14.4149, r_2 = r_{11} = 9.8837, r_3 = r_{10} = 4.7212, r_4 = r_9 = 3.4632, r_5 = r_8 = 3.3386, r_6 = r_7 = 3.0061, r_{12} = 13.865 \) |

**Figure 13**

| **Set 2** | \( \alpha_1 = \alpha_4 = \alpha_7 = \alpha_{10} = 0.0001, \alpha_2 = \alpha_3 = 0.0004, \alpha_5 = \alpha_9 = 0.0003, \alpha_6 = \alpha_8 = 0.0002, \alpha_{11} = 0.0006, \alpha_{12} = 0.00025 \) |
| (a) | \( r_1 = 12.9014, r_2 = 4.5597, r_3 = 4.7203, r_4 = 3.0689, r_5 = 3.3305, r_6 = 3.0062, r_7 = 3.0022, r_8 = 3.0548, r_9 = 3.4522, r_{10} = 3.1786, r_{11} = 9.8777, r_{12} = 9.5436 \) |

**Figure 14**

| **Set 1** | \( \alpha_1 = 0.0004, \alpha_2 = 0.007, \alpha_3 = 0.0102, \alpha_4 = 0.014, \alpha_5 = 0.017, \alpha_6 = 0.034 \) |
| (a) | \( r_1 = 1.0417, r_2 = 1.6905, r_3 = 1.8126, r_4 = 1.8132, r_5 = 1.7844, r_6 = 1.5251 \) |

**Figure 15-17**

| **Set 1** | \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0 \) |
| (a) | \( r_1 = 3.0317, r_2 = 3.0436, r_3 = 3.0443, r_4 = 3.1103, r_5 = 3.1387, r_6 = 4.0836 \) |

| **Set 2** | \( \alpha_1 = 0.0025, \alpha_2 = 0.0055, \alpha_3 = 0.008, \alpha_4 = 0.011, \alpha_5 = 0.0145, \alpha_6 = 0.034 \) |
| (a) | \( r_1 = 1.217, r_2 = 1.7025, r_3 = 1.8389, r_4 = 1.8114, r_5 = 1.8088, r_6 = 1.5772 \) |

| **Classical** | \( \alpha_1 = \alpha_2 = \ldots = \alpha_6 = 0 \) |
| (a) | \( r_1 = 3.034, r_2 = 3.0402, r_3 = 3.0554, r_4 = 3.0962, r_5 = 3.2631, r_6 = 4.7858 \) |
(a) Quadratic scheme \( r_i = 1 + \frac{M_i}{m_i} + \frac{m_i}{M_i} \).

(b) Cubic scheme \( r_1 = 90, r_2 = 26800, r_3 = 5 \).

**Figure 3.** Convex rational spline FIFs for \( 1/x^2 \) with the arithmetic derivative values.

(a) Quadratic scheme \( r_i = 1 + \frac{M_i}{m_i} + \frac{m_i}{M_i} \).

(b) Cubic scheme with \( r_1 = 6, r_2 = 10, r_3 = 3 \).

**Figure 4.** Convex rational spline FIFs for \( 1/x^2 \) with the geometric derivative values.

(a) Quadratic scheme \( r_i = 1 + \frac{M_i}{m_i} + \frac{m_i}{M_i} \).

(b) Cubic scheme \( r_1 = 4, r_2 = 97, r_3 = 4 \).

**Figure 5.** Convex rational spline FIFs for \( 1/x^2 \) with the exact derivative values.
FIGURE 6. Monotonic rational spline FIFs for $1/x^2$ with the geometric derivative values.

FIGURE 7. Monotonic rational spline FIFs for $1/x^2$ with the $C^2$-spline derivative values.

FIGURE 8. Monotonic rational spline FIFs for $1/x^2$ with the exact derivative values.
(a) Quadratic scheme \( r_i = 1 + M_i/m_i + m_i/M_i \).

(b) Cubic scheme

**FIGURE 9.** Convex rational spline FIFs for semi circle data with the arithmetic derivative values.

(a) Quadratic scheme \( r_i = 1 + M_i/m_i + m_i/M_i \).

(b) Cubic scheme

**FIGURE 10.** Convex rational spline FIFs for quarter circle data with the geometric derivative values.

(a) Quadratic scheme \( r_i = 1 + M_i/m_i + m_i/M_i \).

(b) Cubic scheme

**FIGURE 11.** Convex rational spline FIFs for semi circle data with the exact derivative values (except \( d_1 = -50 \), \( d_{13} = 50 \)).
Figure 12. Monotonic rational spline FIFs for quarter circle data with the geometric derivative values (except $d_1 = 0$).

Figure 13. Monotonic rational spline FIFs for quarter circle data with the $C^2$-spline derivative values (with $d_1 = 0, d_7 = 20$).

Figure 14. Monotonic rational spline FIFs for quarter circle data with the exact derivative values (except $d_7 = 25$).
**Figure 15.** Positive rational cubic spline FIFs with the arithmetic derivative values.

**Figure 16.** Positive rational cubic spline FIFs with the geometric derivative values.

**Figure 17.** Positive rational cubic spline FIFs with the exact derivative values.