Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics: II. Noether’s theorems and Casimirs

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Abstract
Conservation laws in ideal gas dynamics and magnetohydrodynamics (MHD) associated with fluid relabeling symmetries are derived using Noether’s first and second theorems. Lie dragged invariants are discussed in terms of the MHD Casimirs. A nonlocal conservation law for fluid helicity applicable for a non-barotropic fluid involving Clebsch variables is derived using Noether’s theorem, in conjunction with a fluid relabeling symmetry and a gauge transformation. A nonlocal cross helicity conservation law involving Clebsch potentials, and the MHD energy conservation law are derived by the same method. An Euler–Poincaré variational approach is also used to derive conservation laws associated with fluid relabeling symmetries using Noether’s second theorem.

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1. Introduction
In a recent paper, Webb et al (2014a) (herein referred to as paper I) used Lie dragging techniques (e.g. Tur and Yanovsky 1993) and Hamiltonian methods using Clebsch variables to investigate advected invariants and helicities in ideal fluid mechanics and magnetohydrodynamics (MHD).
The main aim of the present paper is to derive some of the conservation laws of paper I, by using Noether’s theorems and gauge transformations, and to relate the invariants obtained by the Lie dragging approach to fluid relabeling symmetries and the Casimirs of ideal MHD and gas dynamics associated with non-canonical Poisson brackets. A conference paper by Webb et al (2014a) also studies Lie dragging techniques and advected invariants in MHD and fluid dynamics.

In paper I, we derived the helicity conservation law in fluid dynamics and the cross helicity conservation law in MHD. In the simplest case of a barotropic equation of state for the gas in which the gas pressure \( p = p(\rho) \) depends only on the gas density one obtains local conservation laws for helicity in fluids and cross helicity in MHD (i.e. the conserved densities and fluxes depend only on the density \( \rho \), the magnetic induction \( B \), the fluid velocity \( \mathbf{u} \) and the entropy \( S \)). For the case of cross helicity a local conservation law also holds for a non-barotropic equation of state with \( p = p(\rho, S) \) provided the magnetic field induction \( B \) lies in the constant entropy surfaces \( S = \text{const.} \) (i.e. \( \mathbf{B} \cdot \nabla S = 0 \)). For the general case of a non-barotropic equation of state, generalized nonlocal conservation laws for helicity and cross helicity were obtained by using Clebsch potentials. One of the main aims of the present paper is to show how the nonlocal helicity and cross helicity conservation laws arise from fluid relabeling symmetries, gauge transformations and Noether’s theorem.

The basic MHD model of paper I is described in section 2.

Section 3 gives a short synopsis of Clebsch variables and Lagrangian and Hamiltonian formulations of ideal fluid mechanics and MHD. Section 3 also gives an overview of the MHD Casimirs, i.e. functionals \( C \) that have zero Poisson bracket with \( \{C, K\} = 0 \) for functionals \( K \) dependent on the physical variables. There is an overlap in the Casimir functionals and the class of functionals that are Lie dragged by the flow.

In section 4, conservation laws for both barotropic \( (p = p(\rho)) \) and non-barotropic equations of state \( p = p(\rho, S) \) obtained in paper I are described.

Section 5 discusses Lagrangian MHD and fluid dynamics as developed by Newcomb (1962).

The Lagrangian approach is used in section 6, to write down the invariance condition for the action under fluid relabeling symmetries and gauge transformations (e.g. Padhye and Morrison 1996a, 1996b). We derive the Eulerian version of the invariance condition including the effects of gauge transformations, and use Noether’s theorem to derive the nonlocal helicity and cross helicity conservation laws and also the Eulerian energy conservation equation, using fluid relabeling symmetries.

Section 7 uses the Euler–Poincaré approach to study the MHD equations (e.g. Holm et al 1998, Cotter and Holm 2013). It shows the important role of the Lagrangian map, which corresponds to the Lie group of transformations between the Lagrangian fluid labels and the Eulerian position of the fluid element. The Euler–Poincaré equation for the MHD system, using Eulerian variations is equivalent to the Eulerian MHD momentum equation. The Euler–Poincaré approach is used to develop Noether’s second theorem and the generalized Bianchi identity for representative fluid relabeling symmetries. The connection of this approach to the more classical approach to Noether’s theorem of section 6 is described. Section 8 concludes with a summary and discussion.

2. The model

The basic MHD equations used in the model are the same as in paper I. The physical quantities \( (\rho, \mathbf{u}^T, p, S, \mathbf{B}^T)^T \) denote the density \( \rho \), fluid velocity \( \mathbf{u} \), gas pressure \( p \), entropy \( S \) and
magnetic field induction $B$ respectively. The equations consist of the mass continuity equation, the MHD momentum equation written in conservation form using the Maxwell and fluid stress energy tensors and the momentum flux $\rho u$, the entropy advection equation, Faraday’s induction equation in the MHD limit, the first law of thermodynamics and Gauss’s law $\nabla \cdot B = 0$. Faraday’s equation, from paper I, can be written in the form:

$$\left( \frac{\partial}{\partial t} + L_u \right) B \cdot dS \equiv \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + u \nabla \cdot B \right) \cdot dS = 0. \quad (2.1)$$

Thus, Faraday’s equation corresponds to a conservation law in which the magnetic flux $B \cdot dS$ is Lie dragged with the flow, where $L_u = u \cdot \nabla$ is the Lie derivative (tangent vector) vector field $u$ representing the fluid velocity. The first law of thermodynamics for an ideal gas:

$$dQ = T \, dS = dU + p \, dV, \quad (2.2)$$

is used where $U$ is the internal energy of the gas per unit mass and $V = 1/\rho$ is the specific volume of the gas. The internal energy of the gas per unit volume is $\varepsilon(\rho, S) = \rho U$. In terms of $\varepsilon(\rho, S)$ the first law of thermodynamics gives the equations:

$$\rho T = \varepsilon_S, \quad h = \varepsilon_\rho, \quad p = \rho \varepsilon_\rho - \varepsilon, \quad (2.3)$$

$$\frac{1}{\rho} \nabla p = TVS - \nabla h, \quad (2.4)$$

where $h$ is the gas enthalpy.

We also require that Gauss’s law $\nabla \cdot B = 0$ is satisfied. However, in the Hamiltonian formulation of MHD, setting $\nabla \cdot B = 0$ can give rise to problems in ensuring that the Jacobi identity is satisfied for all functionals of the physical variables (e.g. Morrison and Greene 1980, 1982, Holm and Kupershmidt 1983a, 1983b, Chandre et al 2013).

### 3. Hamiltonian approach

In this section we discuss the Hamiltonian approach to MHD and gas dynamics. In section 3.1 we give a brief description of a constrained variational principle for MHD using Lagrange multipliers to enforce the constraints of mass conservation; the entropy advection equation; Faraday’s equation and the so-called Lin constraint describing in part, the vorticity of the flow (i.e. Kelvin’s theorem). This leads to Hamilton’s canonical equations in terms of Clebsch potentials (Zakharov and Kuznetsov 1997). In section 3.2 we transform the canonical Poisson bracket obtained from the Clebsch variable approach to a non-canonical Poisson bracket written in terms of Eulerian physical variables (see e.g. Morrison and Greene 1980, 1982, Morrison 1982, Holm and Kupershmidt 1983a, 1983b). In section 3.3 we obtain the MHD Casimir equations using the non-canonical variables $\Psi = (M, A, \rho, \sigma)$ where $M = \rho u$ is the MHD momentum flux, $\sigma = \rho S$ and $A$ is the magnetic vector potential in which the gauge is chosen so that the 1-form $\alpha = A \cdot dx$ is an invariant advected with the flow.

#### 3.1. Clebsch variables and Hamilton’s equations

Consider the MHD action (modified by constraints):

$$J = \int d^3x \, dt, \quad (3.1)$$
where

\[
L = \left\{ \frac{1}{2} \rho u^2 - \epsilon(\rho, S) - \frac{B^2}{2\mu_0} \right\} + \phi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \\
+ \beta \left( \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S \right) + \lambda \left( \frac{\partial \mu}{\partial t} + \mathbf{u} \cdot \nabla \mu \right) \\
+ \Gamma \cdot \left( \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) \right).
\] (3.2)

The Lagrangian in curly brackets equals the kinetic minus the potential energy (internal thermodynamic energy plus magnetic energy). The Lagrange multipliers \( \phi, \beta, \lambda \) and \( \Gamma \) ensure that the mass, entropy, Lin constraint, Faraday equations are satisfied. We do not enforce \( \nabla \cdot \mathbf{B} = 0 \), since we are interested in the effect of \( \nabla \cdot \mathbf{B} \neq 0 \) (which is useful for numerical MHD where \( \nabla \cdot \mathbf{B} \neq 0 \)) (see section 2, and paper I for further discussion of this issue).

Stationary point conditions for the action are \( \delta J = 0, \delta J/\delta \mathbf{u} = 0 \) gives the Clebsch representation for \( \mathbf{u} \):

\[
\mathbf{u} = \nabla \phi - \frac{\beta}{\rho} \nabla S - \frac{\lambda}{\rho} \nabla \mu + \mathbf{u}_M
\] (3.3)

where

\[
\mathbf{u}_M = -\left( \nabla \times \Gamma \right) \times \mathbf{B} - \Gamma \nabla \cdot \mathbf{B}.
\] (3.4)

is magnetic contribution to \( \mathbf{u} \). Setting \( \delta J/\delta \phi, \delta J/\delta \beta, \delta J/\delta \lambda, \delta J/\delta \Gamma \) consecutively equal to zero gives the mass, entropy advection, Lin constraint, and Faraday (magnetic flux conservation) constraint equations. Setting \( \delta J/\delta \rho, \delta J/\delta S, \delta J/\delta \mu, \delta J/\delta \beta \) equal to zero gives evolution equations for the Clebsch potentials \( \phi, \beta, \lambda \) and \( \Gamma \) (see Webb et al 2014a, paper I for details).

The Hamiltonian functional for the system is given by:

\[
\mathcal{H} = \int H \, d^3x \quad \text{where} \quad H = \frac{1}{2} \rho u^2 + \epsilon(\rho, S) + \frac{B^2}{2\mu_0}.
\] (3.5)

One can show that the evolution equations for \( (\rho, \phi, \mathbf{B}, \Gamma, S, \beta, \mu, \lambda) \) satisfy Hamilton’s equations for functionals \( F \):

\[
\dot{F} = \{ F, H \} \quad \text{where} \quad \dot{F} = \frac{\partial F}{\partial t},
\] (3.6)

and the canonical Poisson bracket is defined by the equation:

\[
\{ F, G \} = \int d^3x \left( \frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta S} \frac{\delta G}{\delta \beta} - \frac{\delta F}{\delta \beta} \frac{\delta G}{\delta S} + \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta \lambda} - \frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \mu} \right).
\] (3.7)

Note that \( \{ \rho, \phi \}, \{ S, \beta \}, \{ \mu, \lambda \} \) are canonically conjugate variables (see paper I). The canonical Poisson bracket (3.7) satisfies the linearity, skew symmetry and Jacobi identity necessary for a Hamiltonian system (i.e. the Poisson bracket defines a Lie algebra).

### 3.2. Non-Canonical Poisson Brackets

Morrison and Greene (1980, 1982) introduced non-canonical Poisson brackets for MHD. Morrison and Greene (1980) gave the non-canonical Poisson bracket for the case \( \nabla \cdot \mathbf{B} = 0 \). Morrison and Greene (1982) gave the form of the Poisson bracket in the more general case where \( \nabla \cdot \mathbf{B} \neq 0 \). A detailed discussion of the MHD Poisson bracket and the Jacobi identity is given in Morrison (1982). Holm and Kupershmidt (1983) point out that their
Poisson bracket has the form expected for a semi-direct product Lie algebra, for which the Jacobi identity is automatically satisfied. Chandre et al. (2013) use Dirac’s theory of constraints to derive properties of the Poisson bracket for the $\nabla \cdot B = 0$ case.

Introduce the new variables:

$$ M = \rho u, \quad \sigma = \rho S. $$

(3.8)

Noting that

$$ M = \rho u = \rho \nabla \phi - \beta \nabla S - \lambda \nabla \mu + B \cdot (\nabla \Gamma)^T - B \cdot \nabla \Gamma - \Gamma (\nabla \cdot B) $$

(3.9)

and transforming the canonical Poisson bracket (3.7) from the old variables $(\rho, \phi, S, \beta, B, \Gamma)$ to the new variables $(\rho, \sigma, B, M)$ we obtain the Morrison and Greene (1982) non-canonical Poisson bracket:

$$\{F, G\} = - \int d^3 x \left\{ \rho \left[ \frac{\delta F}{\delta M} \cdot \nabla \left( \frac{\delta G}{\delta \rho} \right) - \frac{\delta G}{\delta M} \cdot \nabla \left( \frac{\delta F}{\delta \rho} \right) \right] \right\}
+ \sigma \left[ \frac{\delta F}{\delta M} \cdot \nabla \left( \frac{\delta G}{\delta \sigma} \right) - \frac{\delta G}{\delta M} \cdot \nabla \left( \frac{\delta F}{\delta \sigma} \right) \right]
+ M \left[ \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right]
+ B \left[ \left( \nabla \frac{\delta F}{\delta M} \cdot \frac{\delta G}{\delta B} \right) - M \frac{\delta G}{\delta B} \cdot \nabla \frac{\delta F}{\delta B} \right].$$

(3.10)

The bracket (3.10) has the Lie–Poisson form and satisfies the Jacobi identity for all functionals $F$ and $G$ of the physical variables, and in general applies both for $\nabla \cdot B \neq 0$ and $\nabla \cdot B = 0$.

3.2.1. Advected A formulation. Consider the MHD variational principle using the magnetic vector potential $A$ instead of using $B$ (e.g. Holm and Kupershmidt 1983a, 1983b) The condition that the magnetic flux $B \cdot ds$ is Lie dragged with the flow (i.e. Faraday’s equation) as a constraint equation, is satisfied if the magnetic vector potential 1-form $\alpha = \mathbf{A} \cdot dx$ is Lie dragged by the flow, where $\mathbf{B} = \nabla \times \mathbf{A}$. The condition that $\mathbf{A} \cdot dx$ is Lie dragged with the flow implies:

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{u} \cdot \mathbf{A}) = 0$$

(3.11)

(sic paper I).

We use the variational principle $\delta \mathcal{A} = 0$ where the action $\mathcal{A}$ is given by:

$$\mathcal{A} = \int_{\mathcal{V}} d^3 x \int dt \left\{ \frac{1}{2} \rho |u|^2 - \epsilon (\rho, S) - \frac{1}{2 \mu} |\nabla \times \mathbf{A}|^2 \right\}
+ \phi \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] + \beta \left[ \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S \right] + \lambda \left[ \frac{\partial \mu}{\partial t} + \mathbf{u} \cdot \nabla \mu \right]
+ \gamma \left[ \frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{u} \cdot \mathbf{A}) \right].$$

(3.12)

By setting the variational derivative $\delta \mathcal{A}/\delta \mathbf{u} = 0$ gives the Clebsch variable expansion:

$$\mathbf{u} = \nabla \phi - \frac{\beta}{\rho} \nabla S - \frac{\lambda}{\rho} \nabla \mu - \frac{\gamma (\nabla \times \mathbf{A})}{\rho} + \frac{\nabla \cdot \mathbf{A}}{\rho},$$

(3.13)

for the fluid velocity $\mathbf{u}$.  

5
In terms of the non-canonical variables \((\mathbf{M}, \mathbf{A}, \rho, \sigma)\) where \(\sigma = \rho S\) we obtain the non-canonical Poisson bracket:

\[
\{F, G\} = -\int d^3x \left[ F_M \cdot \nabla(G_M) - G_M \cdot \nabla(F_M) \right] \cdot \mathbf{M} + \rho \left[ F_M \cdot \nabla(G_\rho) - G_M \cdot \nabla(F_\rho) \right] + \sigma \left[ F_M \cdot \nabla(G_\sigma) - G_M \cdot \nabla(F_\sigma) \right] + \nabla \times \mathbf{A} \cdot \left[ G_A \times F_M - F_A \times G_M \right],
\]

(3.14)

where \(F_M \equiv \delta F/\delta \mathbf{M}\) and similarly for the other variational derivatives in (3.14). The non-canonical bracket (3.14) was obtained by Holm and Kupershmidt (1983a, 1983b). It is a skew symmetric bracket and satisfies the Jacobi identity. Holm and Kupershmidt (1983a, 1983b) show that bracket (3.14) corresponds to a semi-direct product Lie algebra.

### 3.3. The MHD Casimirs

The Casimirs are defined as functionals that have zero Poisson bracket with any functional \(K\) defined on the phase space. The condition for a Casimir is:

\[
\{C, K\} = 0,
\]

(3.15)

for arbitrary functionals \(K\). The Casimirs reveal the underlying symmetries of the phase space, implying dependence among the variables used to describe the systems. The reduced Hamiltonian dynamics, taking into account the Casimir constants of motion (note \(C_t = 0\)) takes place on the symplectic leaves foliating the phase space (e.g. Marsden and Ratiu 1994, Morrison 1998, Holm et al 1998, Hameiri 2003, 2004).

To obtain the Casimir determining equations, we introduce the vector:

\[
\zeta = (K_M, K_A, K_\rho, K_\sigma) = (\xi, \chi, \lambda, \nu),
\]

(3.16)

where \(K_M \equiv \delta K/\delta \mathbf{M}\), and similarly for the other variational derivatives in (3.16). The MHD Poisson bracket \([C, K]\) can be written in the form:

\[
[C, K] = \int \frac{\delta C}{\delta \psi^a} A^{ab} \frac{\delta K}{\delta \psi^b} d^3x = \int \frac{\delta C}{\delta \psi^a} A^{ab} \zeta_b d^3x = -\int \xi_a A^{ab} \frac{\delta C}{\delta \psi^b} d^3x,
\]

(3.17)

where \(\psi = (\mathbf{M}, \mathbf{A}, \rho, \sigma)\). The matrix differential operator in (3.17) is skew-symmetric, i.e. \([C, K] = -[K, C]\). From (3.17) it follows that for arbitrary \(\zeta_b = \delta K/\delta \psi^b\), the Casimirs must satisfy the equations:

\[
A^{ab} \frac{\delta C}{\delta \psi^b} = 0.
\]

(3.18)

#### 3.3.1. Casimir equations for advected \(A\)

Using the notation (3.16), the gas dynamic terms in the bracket (3.14) are given by:

\[
\begin{align*}
F_M \cdot \nabla(G_M) - G_M \cdot \nabla(F_M) & = [(C_M \cdot \nabla) \xi - \xi \cdot \nabla(C_M)] \cdot \mathbf{M}, \\
\rho(F_M \cdot \nabla G_\rho - G_M \cdot \nabla F_\rho) & = \rho(C_M \cdot \nabla \lambda - \lambda \cdot \nabla C_\rho), \\
\sigma(F_M \cdot \nabla G_\sigma - G_M \cdot \nabla F_\sigma) & = \sigma(C_M \cdot \nabla \nu - \nu \cdot \nabla C_\sigma).
\end{align*}
\]

(3.19)

where \(G \equiv K\) and \(F \equiv C\). Similarly, the magnetic vector potential terms in the Poisson bracket (3.14) are:

\[
\begin{align*}
(A \cdot F_M) \nabla \cdot G_A - (A \cdot G_M) \nabla \cdot F_A & = (A \cdot C_M) \nabla \cdot \chi - (A \cdot \xi) \nabla \cdot C_A, \\
B \cdot [G_A \times F_M - F_A \times G_M] & = \chi \cdot (C_M \times B) - \xi \cdot (B \times C_A).
\end{align*}
\]

(3.20)

In (3.19)–(3.20) \(B = \nabla \times A\) and we make the identifications \(F = C\) and \(G = K\).
Substituting (3.19)–(3.20) in the Poisson bracket (3.14) and integrating the derivative terms by parts, and dropping the surface terms gives:

\[
[C, K] = \int \left\{ -\xi \cdot \left[ (\nabla \cdot C_M) M + (C_M \cdot \nabla) M + M \cdot (\nabla C_M)^T \right] \right. \\
\left. - [\lambda \nabla \cdot (\rho C_M) + \rho \xi \cdot \nabla C_s] - [\nu \nabla \cdot (\sigma C_M) + \sigma \xi \cdot \nabla C_s] \right. \\
\left. - [\chi \cdot \nabla (A \cdot C_M) + (\xi \cdot A) \nabla \cdot C_A] + \chi C_M \times B - \xi \cdot (B \times C_A) \right\} d^3x = 0.
\]
(3.21)

Setting the coefficients of \( \lambda \) and \( \nu \) equal to zero in (3.21) gives the equations:

\[
\nabla \cdot (\rho C_M) = 0, \quad \nabla \cdot (\sigma C_M) = 0
\]
(3.22)

which are analogous to the steady state mass continuity equation and entropy conservation equation with advection velocity

\[
\hat{v}^x = C_M.
\]
(3.23)

Setting the coefficient of \( \chi \) equal to zero in (3.21) gives the equation:

\[
-C_M \times (\nabla \times A) + \nabla (A \cdot C_M) = 0,
\]
(3.24)

associated with Lie dragging the magnetic vector potential 1-form \( \alpha = A \cdot dx \) with velocity \( \hat{v}^x = C_M \). Noting that \( M = \rho u \) and setting the coefficient of \( \xi \) equal to zero in (3.21) we obtain the equation:

\[
M^I \nabla C_{M^I} + \rho C_M \cdot \nabla (M/\rho) + \rho \nabla C_\rho + \sigma \nabla C_\sigma + A(\nabla \cdot C_A) + B \times C_A = 0.
\]
(3.25)

By noting that for \( B = \nabla \times A \), that

\[
C_A = \nabla \times C_B, \quad \nabla \cdot C_A = 0,
\]
(3.26)

(3.25) reduces to:

\[
M^I \nabla C_{M^I} + \rho C_M \cdot \nabla (M/\rho) + \rho \nabla C_\rho + \sigma \nabla C_\sigma + B \times (\nabla \times C_B) = 0.
\]
(3.27)

Note that this latter result depends on Gauss’s law \( \nabla \cdot B = 0 \) for which \( B = \nabla \times A \).

Padhye and Morrison (1996a, 1996b) give the Casimir solutions:

\[
C(\rho, S, A) = \int_{\nu} \rho G \left( S, \frac{A \cdot B}{\rho}, \frac{B \cdot \nabla S}{\rho}, \frac{B \cdot \nabla (B \cdot S)}{\rho}, \frac{B \cdot \nabla \left( \frac{A \cdot B}{\rho} \right)}{\rho}, \ldots \right) \, d^3x.
\]
(3.28)

It is clear that this family of Casimirs has \( C_M = 0 \) and hence the gauge dependent condition (3.24) does not affect the solution of the Casimir determining equations (3.22) and (3.24).

The Casimir (3.28) can be related to Lie dragged scalars, 1-forms, 2-forms, 3-forms and vector fields (e.g. Webb et al 2014a, paper 1). Let

\[
b = \frac{B}{\rho}, \quad \alpha = \frac{\nabla S}{\rho}, \quad \nu = \nabla \cdot dx,
\]
(3.29)

\[
\beta = d\alpha = B \cdot dS, \quad I = S, \quad \omega = \rho d^3x.
\]

Here \( b \) is a Lie dragged vector field; \( \alpha \) and \( \nu \) are 1-forms that are Lie dragged with the fluid; \( \beta = B \cdot dS \) is the Lie dragged magnetic flux 2-form; \( \omega = \rho d^3x \) is a Lie dragged 3-form and \( I = S \) is an invariant scalar or 0-form that is advected with the fluid (Moiseev et al 1982, Tur and Yanovsky 1993, Webb et al 2014a). Thus

\[
b \cdot \alpha = \frac{B}{\rho} \cdot \nabla \cdot \left( \frac{\nabla S}{\rho} \right) = \frac{\nabla S}{\rho},
\]
(3.30)
and $S$ are invariant, Lie dragged scalars or 0-forms, where the symbol $\iota$ denotes the contraction operator in the algebra of exterior differential forms. Note that the Casimir (3.28) is made up of invariant Lie dragged forms, and hence the Casimir (3.28) is a Lie dragged invariant.

The Casimir equations (3.22)–(3.27) obtained by using the Holm and Kupershmidt (1983a, 1983b) bracket (3.14) are essentially the same as for the Morrison and Greene bracket (see e.g. Padhye and Morrison 1996a, 1996b). Our main aim here is to show that there is a connection between the advected, Lie dragged invariants of the MHD system (e.g. Moiseev et al 1982, Tur and Yanovsky 1993, Webb et al 2014a, paper I), and the solutions of the Casimir equations. Padhye and Morrison (1996a, 1996b) investigate in more detail how the fluid relabeling symmetries are related to the Casimirs.

4. Helicity conservation laws

In this section we outline the helicity conservation laws obtained in paper I.

4.1. Fluid helicities

In ideal fluid mechanics the helicity transport equation has the form:

$$\frac{\partial h_f}{\partial t} + \nabla \cdot \left[ u h_f + \left( h - \frac{1}{2} |u|^2 \right) \omega \right] = \omega T \nabla S + u \cdot (\nabla T \times \nabla S),$$  \hspace{1cm} (4.1)

where $\omega = \nabla \times u$ is the fluid helicity and $h_f = u \cdot \omega$ is the fluid helicity density. For a barotropic gas with $p = p(\rho)$ (4.1) implies the helicity conservation law:

$$\frac{\partial h_f}{\partial t} + \nabla \cdot \left[ u h_f + \left( h - \frac{1}{2} |u|^2 \right) \omega \right] = 0.$$  \hspace{1cm} (4.2)

The generalization of the helicity conservation law (4.2) for the case of a non-barotropic equation of state for the gas (i.e. $p = p(\rho, S)$) is given below (cf proposition 6.1 paper I).

Helicity conservation laws in fluid dynamics are discussed by Moffatt (1969) and Arnold and Khesin (1998).

**Proposition 4.1.** The generalized helicity conservation law in ideal fluid mechanics can be written in the form:

$$\frac{\partial}{\partial t} \left[ \Omega \cdot (u + r \nabla S) \right] + \nabla \cdot \left( u [\Omega \cdot (u + r \nabla S)] + \Omega \left( h - \frac{1}{2} |u|^2 \right) \right] = 0.$$  \hspace{1cm} (4.3)

The nonlocal conservation law (4.3) depends on the Clebsch variable formulation of ideal fluid mechanics in which the fluid velocity $u$ is given by the equation:

$$u = \nabla \phi - r \nabla S - \tilde{\lambda} \nabla \mu,$$  \hspace{1cm} (4.4)

where $\phi$, $r$, $S$, $\tilde{\lambda}$, and $\mu$ satisfy the equations:

$$\frac{d\phi}{dr} = \frac{1}{2} |u|^2 - h, \quad \frac{dr}{dr} = -T,$nabla \cdot \left( u \cdot \nabla \right)$$  \hspace{1cm} (4.5)

and $\frac{d}{dr} = \partial / \partial t + u \cdot \nabla$ is the Lagrangian time derivative following the flow. In (4.3) the generalized vorticity $\Omega$ is defined by the equations:

$$w = u - \nabla \phi + r \nabla S \equiv -\tilde{\lambda} \nabla \mu,$$  \hspace{1cm} (4.6)

$$\Omega = \nabla \times w = \omega + \nabla r \times \nabla S,$$  \hspace{1cm} (4.7)

where $\omega = \nabla \times u$ is the fluid vorticity. The one-form $\alpha = w \cdot dx$ and the two-form $\beta = dw = \Omega \cdot dS$ are advected invariants (see paper I). For the barotropic gas case the helicity conservation law (4.3) reduces to (4.2).
Remark. The conservation laws (4.3) is a nonlocal conservation law that involves the nonlocal potentials:

\[ r(x, t) = -\int_0^t T(x_0, t') \, dt' + r_0(x_0), \]

\[ \phi(x, t) = \int_0^t \left( \frac{1}{2} |u|^2 - h \right) (x_0, t') \, dt' + \phi_0(x_0), \]

where \( x = f(x_0, t) \) and \( x_0 = f^{-1}(x, t) \) are the Lagrangian map and the inverse Lagrangian map. The temperature \( T(x, t) = T_0(x_0, t) \) and \( r_0(x_0) \) and \( \phi_0(x_0) \) are ‘integration constants’.

Proposition 4.2 (Ertel’s theorem). Ertel’s theorem in ideal fluid mechanics states that the potential vorticity \( q = \omega \cdot \nabla S/\rho \) is a scalar invariant advected with the flow, i.e.,

\[ \frac{d}{dt} \left( \frac{\omega \cdot \nabla S}{\rho} \right) = 0, \]

where \( \omega = \nabla \times u \) is the fluid vorticity. In paper I it was shown that there is a higher order invariant, the Hollman invariant \( I_h \) (Hollmann (1964)), involving \( I_e \) (see paper I for details).

4.2. MHD helicities

We first discuss the magnetic helicity conservation law, followed by a discussion of cross helicity. A more complete discussion is given in paper I.

4.2.1. Magnetic helicity. For ideal MHD, the magnetic helicity density \( h_m = A \cdot B \) satisfies the conservation law:

\[ \frac{\partial h_m}{\partial t} + \nabla \cdot \left[ u \cdot h_m + B(\phi_E - A \cdot u) \right] = 0, \]

where

\[ E = -\nabla \phi_E - \frac{\partial A}{\partial t} = -u \times B, \quad B = \nabla \times A. \]

If \( \tilde{A} = A + \nabla \Lambda \) where \( \Lambda \) is the gauge potential for \( A \) such that

\[ \Lambda = \int (\phi_E - A \cdot u) \, dt', \]

where the integration in (4.13) is with respect to the Lagrangian time variable \( t' \), then the magnetic helicity conservation law (4.11) reduces to the advection equation:

\[ \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0, \]

where \( \tilde{h} = \tilde{A} \cdot B \) is the magnetic helicity density in this special gauge.

4.2.2. Cross helicity. The cross helicity transport equation from paper I, can be written in the form:

\[ \frac{\partial}{\partial t}(u \cdot B) + \nabla \cdot \left[ (u \cdot B)u + \left( h - \frac{1}{2} |u|^2 \right) B \right] = TB \cdot \nabla S, \]

where \( h_C = u \cdot B \) is the cross helicity density. If \( B \cdot \nabla S = 0 \) the helicity transport equation reduces to the cross helicity conservation law.
Proposition 4.3. The generalized cross helicity conservation law in MHD can be written in the form:
\[
\frac{\partial}{\partial t}[B \cdot (u + r \nabla S)] + \nabla \cdot \left\{ u[B \cdot (u + r \nabla S)] + \left( h - \frac{1}{2} |u|^2 \right) B \right\} = 0, \tag{4.16}
\]
where
\[
u = \nabla \phi - r \nabla S - \tilde{\lambda} \nabla \mu - \frac{(\nabla \times \Gamma_1)}{\rho} \times B - \Gamma \frac{\nabla \cdot B}{\rho}, \tag{4.17}
\]
is the Clebsch variable representation for the fluid velocity \( u \), and \( r(x,t) \) is the Lagrangian temperature integral (4.8) moving with the flow.

In the special cases of either (i) \( B \cdot \nabla S = 0 \) or (ii) the case of a barotropic gas with \( p = p(\rho) \), the conservation law (4.16) reduces to the usual cross helicity conservation law:
\[
\frac{\partial}{\partial t}(u \cdot B) + \nabla \cdot \left[ u(u \cdot B) + \left( h - \frac{1}{2} |u|^2 \right) B \right] = 0. \tag{4.18}
\]

In general the cross helicity conservation equation (4.16) is a nonlocal conservation law, in which the variable \( r(x,t) \) is a nonlocal potential given by (4.8).

Detailed proofs of the above helicity and cross helicity conservation laws were provided in paper I. In paper I, the concept of topological charge was discussed in relation to advected invariants of the ideal fluid and MHD equations (see also Kamchatnov (1982) and Semenov et al (2002)). The physical application of magnetic helicity in solar, space and fusion plasmas is discussed by Woltjer (1958), Moffatt (1978), Moffatt and Ricca (1992), Berger and Field (1984), Finn and Antonsen (1985, 1988), Berger and Ruzmaikin (2000), Bieber et al (1987), Low (2006), Longcope and Malunushenko (2008), Yahalom and Lynden-Bell (2008), Yahalom (2013) and Webb et al (2010a, 2010b, 2011). Tur and Janovsky (1993) and Webb et al (2014a, 2014b) discuss the Godbillon Vey invariant which applies for MHD flows with zero magnetic helicity, i.e. \( \tilde{A} \cdot \nabla \times \tilde{A} = 0 \), where \( \alpha = \tilde{A} \cdot dx \) is Lie dragged with the flow and \( B = \nabla \times \tilde{A} \). Kats (2003) obtains the MHD version of the Ertel invariant. Kuznetsov and Ruban (2000) describe the Hamiltonian dynamics of vortex and magnetic field lines using a mixed Eulerian and Lagrangian description.

5. The Lagrangian map

5.1. Lagrangian MHD

The Lagrangian map: \( x = X(x_0, t) \) is obtained by integrating the fluid velocity equation \( dx/dt = u(x, t) \), subject to the initial condition \( x = x_0 \) at time \( t = 0 \). This approach to MHD was initially developed by Newcomb (1962). In Lagrangian MHD, the mass continuity equation and entropy advection equation are replaced by the equivalent algebraic equations:
\[
\rho = \rho_0(x_0) \frac{J}{J_0}, \quad S = S_0(x_0), \tag{5.1}
\]
where
\[
J = \det(x_{ij}) \quad \text{and} \quad x_{ij} = \frac{\partial x'(x_0, t)}{\partial x'_{ij}}. \tag{5.2}
\]
Similarly, Faraday’s equation (2.3) has the formal solution for the magnetic field induction \( B \) of the form:
\[
B^i = x_{ij} B^j_0 \frac{J^i_0}{J}, \quad \nabla_0 \cdot B_0 = 0. \tag{5.3}
\]
The solution (5.3) for $B^i$ is equivalent to the frozen field theorem in MHD (e.g. Parker 1979), and the initial condition $\nabla_0 \cdot B_0 = 0$ is imposed in order to ensure that Gauss’s law $\nabla \cdot B = 0$ is satisfied.

The Lagrangian map $x = X(x_0, t)$ and its inverse $x_0 = X_0(x, t)$ is discussed in detail in Newcomb (1962), Webb et al (2005b), Webb and Zank (2007) and others. One can show that the Lagrange labels $x_0$ are advected with the flow.

The action for the MHD system is:

$$A = \int \int \mathcal{L} \, d^3x \, dt \equiv \int \int \mathcal{L}_0 \, d^3x_0 \, dt,$$

where

$$\mathcal{L} = \frac{1}{2} \rho |u|^2 - \epsilon(\rho, S) - \frac{B^2}{2\mu} - \rho \Phi, \quad \mathcal{L}_0 = \mathcal{L} J,$$

are the Eulerian ($\mathcal{L}$) and Lagrangian ($\mathcal{L}_0$) Lagrange densities respectively. Using (5.1)–(5.3), and (5.5) we obtain:

$$\mathcal{L}_0 = \frac{1}{2} \rho_0 |x|^2 - J E \left( \frac{\rho_0}{J}, S \right) - \frac{x_i x_0 B_j' B_0^j}{2\mu J} - \rho_0 \Phi,$$

for $\mathcal{L}_0$. Note that in $\mathcal{L}_0 = \mathcal{L}_0(x_0, t; x, x_0, t)$, $x_0$ and $t$ are the independent variables, and $x$ and its derivatives with respect to $x_0$ and $t$ are dependent variables.

The Hamiltonian description of MHD using the Lagrangian map is described by Newcomb (1962) (see also Padhye and Morrison 1996, Webb et al 2005b, Webb and Zank 2007).

### 6. Symmetries and Noether’s theorem in MHD

In this section we discuss Noether’s first theorem in MHD (e.g. Padhye 1998, Webb et al 2005b). We consider the Lagrangian action (5.4), namely

$$A = \int \int \mathcal{L}_0 \, d^3x_0 \, dt,$$

where the Lagrangian density $\mathcal{L}_0$ is given by (5.6).

#### 6.1. Noether’s theorem

**Proposition 6.1 (Noether’s theorem).** If the action (6.1) is invariant to $O(\epsilon)$ under the infinitesimal Lie transformations:

$$x^i = x^i + \epsilon V^x^i, \quad x_0' = x_0 + \epsilon V^x_0, \quad t' = t + \epsilon V^t,$$

and under the divergence transformation:

$$\mathcal{L}_0' = \mathcal{L}_0 + \epsilon D_x \Lambda_0^x + O(\epsilon^2),$$

(here $D_0 \equiv \partial / \partial t$ and $D_i \equiv \partial / \partial x^i_0$ are the total derivative operators in the jet-space consisting of the derivatives of $x^i(x_0, t)$ and physical quantities that depend on $x_0$ and $t$) then the MHD system admits the Lagrangian conservation law:

$$\frac{\partial I_0}{\partial t} + \frac{\partial I_j}{\partial x_0^j} = 0,$$

where

$$I_0 = \rho_0 u^i \dot{V}^x_i + V^x I_0^x + \Lambda_0^0,$$
\[ I' = \hat{V}^x \left[ \left( p + \frac{B^2}{2\mu} \right) \delta^{k} - \frac{B^{k}B^{l}}{\mu} \right] A_{lj} + V^x_0 \mathcal{L}^0 + \Lambda^j_0. \]  

(6.6)

In (6.5)–(6.6)

\[ \hat{V}^x(x_0, t) = V^x(x_0, t) - \left( V^t \frac{\partial}{\partial t} + V^x_0 \frac{\partial}{\partial x^0_0} \right) x^k(x_0, t), \]

(6.7)

is the canonical or evolutionary Lie symmetry transformation generator corresponding to the Lie transformation (6.2) (i.e. \( x'^k = x^k + \epsilon \hat{V}^k, t' = t, x'^j_0 = x^j_0 \)).

Proof of the above form of Noether’s theorem for MHD is given in Webb et al (2005b) and in Webb and Zank (2007). General proofs of Noether’s first theorem are given in Bluman and Kumei (1989) and Olver (1993).

Remark. The action (6.1) is invariant to \( O(\epsilon) \) under the divergence transformation of the form (6.2)–(6.3) provided:

\[ \tilde{X}_L^0 + L^0 \left( D_t V_t + D_{x^0_0} V^x_0 \right) + D_t \Lambda^0_0 + D_{x^0_0} \Lambda^j_0 = 0, \]

(6.8)

where

\[ \tilde{X} = V^k \frac{\partial}{\partial x^k} + V^x_0 \frac{\partial}{\partial x^0_0} + V^x \frac{\partial}{\partial x_0} + V^{x_0} \frac{\partial}{\partial x^0_0} + \cdots, \]

(6.9)

is the extended Lie transformation operator. Here \( \tilde{X} \) gives the transformation rules for the derivatives of \( x^k(x_0, t) \) under Lie transformation (6.2). From Ibragimov (1985):

\[ \check{X} = \tilde{X} + V^u D_u, \]

(6.10)

\[ \check{X} = \hat{V}^x \frac{\partial}{\partial x^k} + D_0 (\hat{V}^x) \frac{\partial}{\partial x_0^k} + D_0 D_{\beta} (\hat{V}^x) \frac{\partial}{\partial x_{\alpha \beta}^k} + \cdots, \]

(6.11)

where \( D_0 = \partial/\partial t \) and \( D_i = \partial/\partial x^0_0 \) denote total partial derivatives with respect to \( t \) and \( x^0_0 \) \((1 \leq i \leq 3)\), \( V^0 \equiv V^t \) and \( V^i \equiv V^{x^i_0} \) respectively. \( \check{X} \) is the extended Lie symmetry operator for the canonical Lie transformation \( x'^k = x^k + \epsilon \hat{V}^k, t' = t \) and \( x'^j_0 = x^j_0 \).

Proposition 6.2. The Lagrangian conservation law (6.4) can be written as an Eulerian conservation law of the form (Padhye 1998):

\[ \frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \]

(6.12)

where

\[ F^0 = \frac{I^0}{J}, \quad F^j = \frac{u^j I^0 + x^j_0 I^k}{J}, \quad (j = 1, 2, 3), \]

(6.13)

are the conserved density \( F^0 \) and flux components \( F^j \).

Proposition 6.3. The Lagrangian conservation law (6.4) with conserved density \( I^0 \) of (6.5), and flux \( I^j \) of (6.6), is equivalent to the Eulerian conservation law:

\[ \frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \]

(6.14)

where

\[ F^0 = \rho u^k \hat{V}^x(x_0, t) + V^k \mathcal{L} + \Lambda^0_0, \]

(6.15)

\[ F^j = \hat{V}^x(x_0, t) (T^{jk} - \mathcal{L} \delta^{jk}) + V^k \mathcal{L} + \Lambda^j_0, \]

(6.16)
\[ T^{jk} = \rho u^j u^k + \left( p + \frac{B^2}{2\mu} \right) \delta^{jk} - \frac{B^j B^k}{\mu}, \]  
(6.17)

\[ \Lambda^0 = \frac{\Lambda_0^0}{J}, \quad \Lambda^j = \frac{u^j \Lambda_0^0 + x_j \Lambda_0^i}{J}. \]  
(6.18)

In (6.14)–(6.18) \( T^{jk} \) is the Eulerian momentum flux tensor (the spatial components of the stress energy tensor) and \( \tilde{F} \) is the canonical symmetry generator (6.6).

**Remark.** For a pure fluid relabeling symmetry \( V^x = V^l = 0 \), and proposition 6.3 gives:

\[ F^0 = \tilde{V}^x \cdot (\rho u) + \Lambda^0, \]  
(6.19)

\[ F = \tilde{V}^x \cdot \left[ \rho u \otimes u + \left( \varepsilon + p + \rho \Phi + \frac{B^2}{\mu_0} - \frac{1}{2} \rho \mu^2 \right) I - \frac{B \otimes B}{\mu_0} \right] + \Lambda, \]  
(6.20)

for the conserved density \( F^0 \) and flux \( F \) where \( \Lambda = (0, \Lambda^1, \Lambda^2, \Lambda^3) \).

**Remark.** Padhye and Morrison (1996a, 1996b) and Padhye (1998) used proposition 6.2 to convert Lagrangian conservation laws to Eulerian conservation laws. Webb et al (2005b) derived Lagrangian and Eulerian conservation laws using propositions 6.1 and 6.3, and studied fully nonlinear MHD waves in a non-uniform and time dependent background flow. Linear waves in a non-uniform background were studied in Webb et al (2005a), extending similar work by Dewar (1970) for WKB waves.

### 6.2. Fluid relabeling symmetries

Consider infinitesimal Lie transformations of the form (6.2)–(6.3), with

\[ V^l = 0, \quad V^x = 0, \quad V^{x_0} \neq 0, \]  
(6.21)

which leave the action (6.1) invariant. The extended Lie transformation operator \( \tilde{X} \) for the case (6.21) has generators:

\[ \tilde{V}^x = -V^{x_0} \cdot \nabla_0 x, \quad V^x = -D_l (V^{x_0}) \cdot \nabla_0 x, \]

\[ V^{x_0} = -\nabla_0 (V^{x_0}) \cdot \nabla_0 x. \]  
(6.22)

The condition (6.8) for a divergence symmetry of the action reduces to:

\[
\begin{align*}
\nabla_0 \cdot (\rho_0 V^{x_0}) \left( \frac{1}{2} |u|^2 - \Phi (x) - \frac{\varepsilon + p}{\rho} \right) & - \frac{3 \varepsilon (\rho, S)}{\rho} V^{x_0} \cdot \nabla_0 S \\
& - D_l (\rho_0 V^{x_0}) \cdot \nabla_0 x \cdot u - \frac{1}{\mu J} (\nabla_0 x) \cdot (\nabla_0 x)^T : (V^{x_0} \cdot \nabla_0 B_0) B_0 \\
& + B_0 B_0 \nabla_0 \cdot V^{x_0} - (B_0 \cdot \nabla_0 V^{x_0}) B_0 = -\partial \Lambda_\alpha^0 / \partial x_\alpha,
\end{align*}
\]

(6.23)

where \( x_0^\alpha = (t, x_0, y_0, z_0) \) is the spatial 4-vector in Lagrange label space. Simple solutions of (6.23) with \( \Lambda_0^\alpha = 0 \) (\( \alpha = 0, 1, 2, 3 \)) are obtained by setting:

\[ \nabla_0 \cdot (\rho_0 V^{x_0}) = 0, \quad V^{x_0} \cdot \nabla_0 S = 0, \quad D_l (\rho_0 V^{x_0}) = 0, \]

\[ \nabla_0 \times (V^{x_0} \times B_0) = 0, \quad \nabla_0 \cdot B_0 = 0, \quad \Lambda_0^\alpha = 0, \]  
(6.24)

where \( \alpha = 0, 1, 2, 3 \). Equations (6.24) are Lie determining equations for the fluid relabeling symmetries obtained by Padhye (1998) and Webb et al (2005b). However, (6.24) do not give the most general solutions for the fluid relabeling symmetries. To obtain other possible solutions of (6.23) it is useful to convert the fluid relabeling divergence symmetry condition to its Eulerian form given below.
Proposition 6.4. The condition (6.23) for a divergence symmetry of the action converted to Eulerian form is:
\[
\nabla \cdot (\rho \dot{V}^x) \left( h + \Phi(x) - \frac{1}{2} |u|^2 \right) + \rho T \dot{V}^x \cdot \nabla S + \rho u \cdot \left( \frac{d\dot{V}^x}{dt} - \dot{V}^x \cdot \nabla u \right) + \frac{B}{\mu_0} \cdot [-\nabla \times (\dot{V}^x \times B)] + \dot{V}^x \nabla \cdot B = -\nabla_a \Lambda^a, \tag{6.25}
\]
where
\[
\nabla_a \Lambda^a = \frac{\partial \Lambda^0}{\partial t} + \frac{\partial \Lambda^i}{\partial x^i}. \tag{6.26}
\]
is the four divergence of the four-dimensional vector \( \Lambda = (\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3) \). The 4-vector \( \Lambda \) is related the Lagrange label space vector \( \Lambda^0_0 \) by the transformations:
\[
\Lambda^a = \Lambda^0_0 B_{\beta a} \equiv \Lambda^0_0 x_{a\beta}, \tag{6.27}
\]
where \( x_{a\beta} = \partial x^a / \partial x^\beta \), \( J = \det(s_{ij}) \) and \( B_{a\beta} = \text{cofac}(\partial s^a_0 / \partial x^\beta) \) (the transformations (6.27) are the same as those in (6.18); note that \( \alpha, \beta \) have values 0, 1, 2, 3).

**Proof.** The proof follows by using (6.1)–(6.7) and the transformations (6.22) relating \( \dot{V}^x, \dot{V}^x, \) and \( \dot{V}^x_0 \) to \( V^x_0 \). \( \square \)

The symmetry conditions (6.25) and Noether’s theorem (proposition 6.3) applied to the fluid relabeling symmetries, and including the gauge potentials \( \Lambda^a (a = 0, 1, 2, 3) \) are used below to derive the nonlocal fluid helicity conservation law (4.3) and the nonlocal cross helicity conservation law (4.16).

Proposition 6.5. The fluid helicity conservation law (4.3), i.e.,
\[
\frac{\partial}{\partial t} \left[ \Omega \cdot (u + r \nabla S) \right] + \nabla \cdot \left\{ u \left[ \Omega \cdot (u + r \nabla S) \right] + \Omega \left( h - \frac{1}{2} |u|^2 \right) \right\} = 0, \tag{6.28}
\]
may be obtained by applying Noether’s theorem (proposition 6.3) in which the fluid relabeling symmetry generator \( \dot{V}^x \) is given by the equations:
\[
\dot{V}^x = \frac{\Omega}{\rho}, \quad \Omega = \nabla \times w, \quad V' = V^x = 0, \quad V^x_0 = -\left( \nabla_\phi \lambda x \right) / \rho_0, \tag{6.29}
\]
and the gauge potentials \( \Lambda^a (a = 0, 1, 2, 3) \) are given by:
\[
\Lambda^0 = r(\Omega \cdot \nabla S), \quad \Lambda^j = r(\Omega \cdot \nabla S) u^j. \tag{6.30}
\]

**Proof.** Because \( \rho \dot{V}^x = \Omega = \nabla \times w \) it follows that \( \nabla \cdot (\rho \dot{V}^x) = 0 \) in (6.25). Also the one form \( \alpha = w \cdot dx \) is Lie dragged with the flow and \( \dot{V}^x = \nabla \times w / \rho = \Omega / \rho \) is an invariant advected vector field, i.e., it satisfies the equation:
\[
\frac{d\dot{V}^x}{dt} - \dot{V}^x \cdot \nabla u \equiv \frac{\partial \dot{V}^x}{\partial t} + [u, \dot{V}^x] = 0. \tag{6.31}
\]
The left hand side of (6.25) reduces to:
\[
\rho T \dot{V}^x \cdot \nabla S = \mathcal{T} \Omega \cdot \nabla S. \tag{6.32}
\]
The gauge potential divergence term on the right hand side of (6.25) reduces to
\[-\nabla \Lambda^\alpha = - \left( \frac{\partial \Lambda^\alpha}{\partial t} + \nabla \cdot \Lambda \right) = - \left( \frac{\partial}{\partial t} \left( r \Omega \cdot \nabla S \right) + \nabla \cdot \left( u (r \Omega \cdot \nabla S) \right) \right) \]
\[= - \left[ \rho \frac{d}{dt} \left( \frac{\Omega \cdot \nabla S}{\rho} \right) + \frac{\Omega \cdot \nabla S}{\rho} \frac{dr}{dt} \right] = - \Omega \cdot \nabla \frac{dr}{dt} = T \Omega \cdot \nabla S, \quad (6.33)\]
which is the same as the left hand side (6.32). Thus the condition (6.25) for a divergence, relabeling symmetry of the action is satisfied. Using (6.29) and (6.30) in the Noether’s theorem (proposition 6.3) gives the nonlocal fluid helicity conservation law (6.28). □

**Proposition 6.6.** The nonlocal cross helicity conservation law (4.16):
\[\frac{d}{dt} \left[ B \cdot (u + r \nabla S) \right] + \nabla \cdot \left[ u [B \cdot (u + r \nabla S)] + \left( h + \frac{1}{2} |u|^2 \right) B \right] = 0, \quad (6.34)\]
is obtained by using Noether’s theorem (proposition 6.3) in which the fluid relabeling symmetry generator \( \hat{V}^x \) has the form:
\[\hat{V}^x = \frac{B}{\rho} \equiv b, \quad V^x = V' = 0, \quad V^x_0 = - \frac{B_0}{\rho_0}, \quad (6.35)\]
and the gauge potentials \( \Lambda^\alpha (\alpha = 0, 1, 2, 3) \) are:
\[\Lambda^0 = r B \cdot \nabla S, \quad \Lambda^i = u^i r B \cdot \nabla S. \quad (6.36)\]

**Proof.** The vector field \( \hat{V}^x = b = B/\rho \) is Lie dragged with the flow, and satisfies the equation:
\[\frac{db}{dt} - b \cdot \nabla u = \frac{\partial b}{\partial t} + [u, b] = 0. \quad (6.37)\]
Also note that \( \nabla \cdot (\rho \hat{V}^x) = \nabla \cdot B = 0 \) (Gauss’s law). Thus, the left hand side of (6.25) reduces to:
\[\rho T \hat{V}^x \cdot \nabla S = T B \cdot \nabla S. \quad (6.38)\]
The divergence term on the right hand side of (6.25) reduces to:
\[-\nabla \Lambda^\alpha = - \left( \frac{\partial}{\partial t} \left( r B \cdot \nabla S \right) + \nabla \cdot \left( u (r B \cdot \nabla S) \right) \right) \equiv - (B \cdot \nabla S) \frac{dr}{dt} = T B \cdot \nabla S. \quad (6.39)\]
Using (6.37)–(6.39) in (6.25) shows that the Lie invariance condition (6.25) is satisfied. Use of (6.35)–(6.36) in Noether’s theorem (proposition 6.3) gives the nonlocal cross helicity conservation law (4.16) or (6.34). □

**Proposition 6.7.** The divergence symmetry condition (6.25) has solutions:
\[\hat{V}^x = u, \quad \Lambda^0 = - \left( \frac{1}{2} \rho |u|^2 - \varepsilon (\rho, S) - \rho \Phi (x) - \frac{B^2}{2 \mu_0} \right) - \rho f(x_0), \quad (6.40)\]
where \( f(x_0) \) is an arbitrary function of \( x_0 \). The gauge potential \( \Lambda^0 = - L - \rho f(x_0) \) where \( L \) is the Eulerian MHD Lagrangian density, including an external gravitational potential \( \Phi (x) \). The conservation laws associated with the solutions (6.40) are the MHD energy conservation equation:
\[\frac{d}{dt} \left( \frac{1}{2} \rho |u|^2 + \varepsilon (\rho, S) + \rho \Phi (x) + \frac{B^2}{2 \mu_0} \right) + \nabla \cdot \left[ \rho u \left( \frac{1}{2} |u|^2 + h + \Phi \right) + \frac{E \times B}{\mu_0} \right] = 0, \quad (6.41)\]
and the conservation law:
\[
\frac{\partial}{\partial t} [\rho f(x_0)] + \nabla \cdot [\rho u f(x_0)] = 0 \quad \text{or} \quad \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) f(x_0) = 0. \tag{6.42}
\]

**Remark.** The MHD energy conservation equation (6.41) is usually associated with the time translation symmetry of the action, for which \( V^i = 1, V^x = 0, V^\psi = 0 \) (\( \psi \) is any of the MHD physical variables \( \rho, u, B \) and \( S \), and \( \Lambda^\alpha = 0 \) \( \alpha = 0, 1, 2, 3 \)). The result (6.41) shows that the energy conservation law (6.41) also arises as a gauge symmetry of the action associated with the fluid relabeling symmetry.

**Remark.** The conservation law (6.42) states that an arbitrary function \( f(x_0) \) of the Lagrange labels \( x_0 \) is advected with the flow. Non-trivial examples of this conservation law are obtained for:
\[
f_1(x_0) = \frac{B \cdot \nabla S}{\rho} = \frac{B_0(x_0) \cdot \nabla_0 S(x_0)}{\rho_0(x_0)}, \quad f_2(x_0) = \frac{A \cdot B}{\rho} = \frac{A_0(x_0) \cdot B_0(x_0)}{\rho_0(x_0)}, \tag{6.43}
\]
where \( A \) is chosen so that \( A \cdot dx = A_0(x_0) \cdot dx_0 \) is advected with the flow.

**Proof.** To obtain the solutions (6.40) of the Lie determining equations (6.25) for a divergence symmetry of the action, we note that with \( \vec{V} = u \), (6.25) reduces to:
\[
\nabla \cdot (\rho u) \left( h + \Phi(x) - \frac{1}{2} |u|^2 \right) + \rho T u \cdot \nabla S + \rho u \cdot \frac{\partial u}{\partial t} + \frac{B}{\mu_0} \cdot \left[ -\nabla \times (u \times B) + u(\nabla \cdot B) \right] = -\nabla_u \Lambda^\alpha. \tag{6.44}
\]

Next we use the identities:
\[
T_1 = \rho u \cdot \frac{\partial u}{\partial t} = \frac{1}{2} |u|^2 \nabla \cdot (\rho u) = \frac{\partial}{\partial t} \left( \frac{1}{2} \rho |u|^2 \right) - \frac{1}{2} |u|^2 \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right],
\]
\[
T_2 = \nabla \cdot (\rho u) h + \rho T u \cdot \nabla S = \nabla \cdot (\rho u h) - u \cdot \nabla p = -\frac{\partial \varepsilon}{\partial t},
\]
\[
T_3 = \nabla \cdot (\rho u) \Phi(x) = -\frac{\partial \rho}{\partial t} \Phi = -\frac{\partial}{\partial t} [\rho \Phi(x)],
\]
\[
T_4 = \frac{B}{\mu_0} \cdot \left[ -\nabla \times (u \times B) + u(\nabla \cdot B) \right] = -\frac{\partial}{\partial t} \left( \frac{B^2}{2 \mu_0} \right). \tag{6.45}
\]

In (6.45) use of the mass continuity equation (2.1) gives \( T_1 = \partial((1/2)\rho |u|^2)/\partial t \). The term \( T_2 \) in (6.45) reduces to \(-\partial \varepsilon/\partial t \), where we have used the internal energy evolution equation for the gas:
\[
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\rho u h) = u \cdot \nabla p, \tag{6.46}
\]
where \( \varepsilon = \varepsilon(\rho, S) \). The expression \( T_4 \) in (6.45), using Faraday’s equation reduces to \(-\partial (B^2/2\mu_0)/\partial t \). This result is Poynting’s theorem.

Using the results (6.45), (6.44) reduces to:
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |u|^2 - \varepsilon(\rho, S) - \rho \Phi(x) - \frac{B^2}{2 \mu_0} \right) = -\left( \frac{\partial \Lambda^0}{\partial t} + \frac{\partial \Lambda^i}{\partial x^i} \right). \tag{6.47}
\]

Equation (6.47) has solutions of the form (6.40).

The total energy conservation law (6.41) and the Lagrangian advection conservation law (6.42) now follow by using the symmetry results (6.40) in Noether’s theorem (proposition 6.3). From (6.15)–(6.16) we find:
\[
F^0 = \left( \frac{1}{2} \rho |u|^2 + \varepsilon + \rho \Phi + \frac{B^2}{2 \mu_0} \right) + \rho f(x_0), \tag{6.48}
\]
\[ F = \left[ \rho u \left( \frac{1}{2} \rho |u|^2 + h + \Phi \right) + \frac{E \times B}{\mu_0} \right] + \rho u f(x_0), \]  

(6.49)

where \( F = (F^1, F^2, F^3) \) is the spatial flux and \( E = -(u \times B) \) is the electric field. The MHD energy conservation law (6.41) is obtained by setting \( f(x_0) = 0 \) in (6.48)–(6.49) and using (6.48)–(6.49) for \( F^0 \) and \( F \) in (6.14). Similarly, the conservation law (6.42) for \( f(x_0) \) is obtained by using (6.14). This completes the proof.

### 7. Euler–Poincaré equation approach

Our analysis in this section is based in part, on the analysis of Holm et al (1998) and Cotter and Holm (2013). In action principles in MHD and gas dynamics, it is useful to use both Lagrangian and Eulerian variations. The Euler–Poincaré approach uses Eulerian variations in which \( x \) is held constant. In section 7.1 we derive the MHD Euler–Poincaré equation or Eulerian momentum equation for MHD (see also Holm et al 1998 for a similar approach). In section 7.2, we give an analysis of Noether’s second theorem for MHD and fluid relabeling symmetries which is similar to the analysis of Cotter and Holm (2013). The results from Noether’s second theorem using the Euler–Poincaré approach overlap with the more classical physics approach in section 6. However, there are some subtle issues in Noether’s second theorem that arise in this section, which were not addressed in section 6.

The solution of \( dx/dt = u(x, t) \) with \( x = x_0 \) at \( t = 0 \) is written as \( x = gx_0 = X(x_0, t) \). The inverse map \( x_0 = g^{-1}x \) defines \( x_0 = x_0(x, t) \). The Lagrange label \( x_0 \) is advected with the flow:

\[ \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) x_0 = \left( \frac{\partial}{\partial t} + L_u \right) x_0 = 0. \]  

(7.1)

Write \( x = gx_0, x_0 = g^{-1}x \). Let \( h = g^{-1} \). We find:

\[ \dot{x}_0 = \dot{h}x = -g^{-1}gg^{-1}x = -g^{-1}g x_0. \]  

(7.2)

We identify

\[ \xi = L_u = u \cdot \nabla \equiv g^{-1} \dot{g}, \]  

(7.3)

with the fluid velocity \( u \). Note \( \xi = g^{-1} \dot{g} \) is left invariant vector field. Similarly, for a geometrical object a Lie dragged with the flow:

\[ \left( \frac{\partial}{\partial t} + L_u \right) a = 0. \]  

(7.4)

Let \( a_0 = ga \) then \( \alpha = g^{-1}a_0 \) and

\[ \delta a = \delta(g^{-1})a_0 = -g^{-1}\delta gg^{-1}a_0 = -g^{-1}g\delta a = -\mathcal{L}_\eta(a). \]  

(7.5)

We write

\[ \eta = g^{-1} \delta g. \]  

(7.6)

as the vector field associated with the variations. Note \( \eta \) is a left invariant vector field (i.e. \( (hg)^{-1}\delta(hg) = g^{-1} \delta g \), assuming that \( \delta h = 0 \).

To compute \( \delta \xi \) where \( \xi = g^{-1} \dot{g} \) we note:

\[ \delta \xi = \delta g^{-1} \dot{g} + g^{-1} \delta \dot{g} = -(g^{-1} \delta gg^{-1}) \dot{g} + g^{-1} \delta \dot{g}, \]  

(7.7)

which gives:

\[ \delta \xi = -\eta \xi + g^{-1} \delta \dot{g}. \]  

(7.8)

\[17\]
Similarly, for \( \eta = g^{-1}\delta g \) we find
\[
\dot{\eta} = (g^{-1}) \delta g + g^{-1}\dot{\delta}g = -g^{-1}\dot{g}g^{-1}\delta g + g^{-1}\dot{\delta}g.
\] (7.9)
which gives:
\[
\dot{\eta} = -\xi \dot{\eta} + g^{-1}\delta \dot{g}.
\] (7.10)
Subtract (7.10) from (7.8) gives:
\[
\delta \xi = \dot{\eta} + \xi \dot{\eta} - \eta \dot{\xi} \equiv \dot{\eta} + [\xi, \eta]_L,
\] (7.11)
where \([\xi, \eta]_L = ad_\xi(\eta)_L\) is the left Lie bracket. The right Lie bracket \([\xi, \eta]_R = -[\xi, \eta]_L\).

### 7.1. The Euler–Poincaré equation

Consider the variational principle (Holm et al. 1998, Cotter and Holm 2013) in which the action:
\[
J = \int \ell(u, a) \, d^3x \, dt,
\] (7.12)
is stationary, i.e.
\[
\delta J = \int \left( \frac{\delta \ell}{\delta u} \cdot \delta u + \frac{\delta \ell}{\delta a} \eta \right) \, d^3x \, dt \equiv \int \left( \frac{\delta \ell}{\delta u} \, \delta u \right) + \frac{\delta \ell}{\delta a} \, \eta \, dt = 0.
\] (7.13)
However from (7.11) with \( \xi = u \), and (7.5),
\[
\delta u = \dot{\eta} + [u, \eta], \quad \delta a = -L_\eta(a).
\] (7.14)
Thus
\[
\delta J = \int \left\{ \left( \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) \right) - \left( \eta \frac{\partial}{\partial a} \left( \frac{\delta \ell}{\delta a} \right) \right) \right\} + \left( \frac{\delta \ell}{\delta a} \, \eta \right) \, dt.
\] (7.15)
Integrate (7.15) by parts, and use \( ad_\eta(\eta) = [u, \eta] \) to obtain:
\[
\delta J = \int \left\{ \left( \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) \right) - \left( \eta \frac{\partial}{\partial a} \left( \frac{\delta \ell}{\delta a} \right) \right) + \frac{\delta \ell}{\delta a} \, ad_\eta(\eta) - \frac{\delta \ell}{\delta a} \, L_\eta(a) \right\} \, dt.
\] (7.16)
for \( \delta J \).

In the further analysis of (7.16) it is useful to introduce the diamond operator. The diamond operator \( \diamond \) allows one to take the adjoint of the \( \langle \delta \ell/\delta u, ad_\eta(\eta) \rangle \) term in (7.16) and thereby isolate its \( \eta \) component, by using the formula
\[
\left\{ \frac{\delta \ell}{\delta a} \, \diamond a, \eta \right\} = -\left\{ \frac{\delta \ell}{\delta a}, L_\eta(a) \right\}.
\] (7.17)
A more formal definition of the diamond operator is given below.

**Definition** The diamond operator \( \diamond \) is minus the dual of the Lie derivative, with respect to the pairing induced by the variational derivative \( p = \delta \ell/\delta q \), namely:
\[
\langle p \diamond q, \xi \rangle = \langle p, \, L_\xi(q) \rangle.
\] (7.18)

Using (7.17) and the definition of \( ad_\eta^\ast \):
\[
\left\{ ad_\eta^\ast \left( \frac{\delta \ell}{\delta u} \right), \eta \right\} = \left\{ \frac{\delta \ell}{\delta u}, ad_\eta(\eta) \right\},
\] (7.19)
in (7.16) where \( \diamond \) is the diamond operator (this involves integration by parts, and dropping surface terms). We obtain:
\[
\delta J = \int \left\{ \eta \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) + ad_\eta^\ast \left( \frac{\delta \ell}{\delta u} \right) + \frac{\delta \ell}{\delta a} \, \diamond a \right\} \, dt + \left\{ \frac{\delta \ell}{\delta u}, \eta \right\}_{\partial t}.
\] (7.20)
Assuming the surface term vanishes in (7.20), and \( \eta \) is arbitrary, then \( \delta J = 0 \) implies the Euler–Poincaré equation:

\[
\frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) + ad_u^* \left( \frac{\delta \ell}{\delta u} \right)_R = \frac{\delta \ell}{\delta a} \circ a, \tag{7.21}
\]

where

\[
ad_u^* \left( \frac{\delta \ell}{\delta u} \right)_R = -ad_u^* \left( \frac{\delta \ell}{\delta u} \right)_L. \tag{7.22}
\]

Here, (7.21) is the Euler–Poincaré equation for the variational principle \( \delta J = 0 \) (Holm et al. 1998). In (7.21), \( d/dt = \partial / \partial t \) keeping \( x \) constant. Below, we show that:

\[
ad_u^* \left( \frac{\delta \ell}{\delta u} \right)_R = \nabla \cdot \left( u \otimes \frac{\delta \ell}{\delta u} \right) + (\nabla u)^T \cdot \left( \frac{\delta \ell}{\delta u} \right). \tag{7.23}
\]

**Proof.** To prove (7.23) let \( m = \delta \ell / \delta u \). We obtain:

\[
\{ \eta, ad_u^*(m)_R \} = \langle ad_u(\eta), m \rangle = \langle -[u, \eta]_L, m \rangle
\]

\[
= -\int \left( (\nabla \cdot u - \nabla \cdot \eta) \nabla \right) \cdot m \, dx \, d^3x
\]

\[
= -\int \nabla \cdot \left( (u \otimes \eta - \eta \otimes u) \right) \cdot m \, dx \, d^3x
\]

\[
= \int \nabla \cdot \left( (u \otimes m) + (\nabla u)^T \cdot m \right) \, dx \, d^3x,
\]

where we dropped the surface term. This proves (7.23). \( \square \)

It can be shown that:

\[
\mathcal{L}_u(m \cdot dx \otimes dV) = (\nabla \cdot (u \otimes m) + (\nabla u)^T \cdot m) \cdot dx \otimes dV. \tag{7.25}
\]

For MHD the Lagrange density \( \ell \) is given by:

\[
\ell = \frac{1}{2} \rho u^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu_0}. \tag{7.26}
\]

We now determine the different terms in the Euler–Poincaré equation (7.21).

From (7.13), the variation of the action \( \delta J = \delta J_u + \delta J_a \) where:

\[
\delta J_u = \int \frac{\delta \ell}{\delta u} \cdot \delta u \, d^3x \, dr,
\]

\[
\delta J_a = \int \left( \frac{\delta \ell}{\delta \rho} \delta \rho + \frac{\delta \ell}{\delta S} \delta S + \frac{\delta \ell}{\delta B} \cdot \delta B \right) \, d^3x \, dr. \tag{7.27}
\]

From (7.26) we obtain:

\[
\frac{\delta \ell}{\delta \rho} = \frac{1}{2} u^2 - \varepsilon_\rho = \frac{1}{2} u^2 - h, \quad \frac{\delta \ell}{\delta u} = m = \rho u,
\]

\[
\frac{\delta \ell}{\delta S} = -\varepsilon_S = -\rho T, \quad \frac{\delta \ell}{\delta B} = -\frac{B}{\mu_0}, \tag{7.28}
\]

where \( T \) is the temperature and \( h \) is the enthalpy of the gas.

Using the formulae:

\[
\delta (\rho d^3x) = -\mathcal{L}_u (\rho d^3x) = -\nabla \cdot (\rho u) \, d^3x,
\]

\[
\delta S = -\mathcal{L}_u (S) = -u \cdot \nabla S,
\]

\[
\delta (B \cdot dS) = -\mathcal{L}_u (B \cdot dS) = [\nabla \times (u \times B) - u(\nabla \cdot B)] \cdot dS, \tag{7.29}
\]

(7.29)
we obtain:
\[
\delta \rho = -\nabla \cdot (\rho \mathbf{u}), \quad \delta S = -\mathbf{u} \cdot \nabla S,
\]
\[
\delta \mathbf{B} = [\nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u}(\nabla \cdot \mathbf{B})].
\]
(7.30)

Note that \(\delta \rho, \delta S\) and \(\delta \mathbf{B}\) are Eulerian variations in which \(\Delta \mathbf{x} = -x_i \delta x_i^0\) is replaced by \(\mathbf{u}'\), where \(\Delta \mathbf{x}\) is the Lagrangian variation of \(\mathbf{x}\), and \(x_i = \partial x_i^0 / \partial x^j\) (e.g. Webb et al 2005a, 2005b, Newcomb 1962). Using \(\delta \ell / \delta \mathbf{u} = \rho \mathbf{u} = \mathbf{m}\) in (7.23) gives:
\[
ad^\ast \frac{\delta \ell}{\delta \mathbf{u}} = \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \rho \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right),
\]
(7.31)

for the advected term on the left hand side of the Euler–Poincaré equation (7.21).

Next we find the \((\delta \ell / \delta a) \circ a\) term on right hand side of (7.21). We obtain:
\[
\frac{\delta \ell}{\delta a} = \frac{\delta \ell}{\delta \rho} + \frac{\delta \ell}{\delta S} + \frac{\delta \ell}{\delta \mathbf{B}} \cdot \delta \mathbf{B}
\]
\[
= \frac{\delta \ell}{\delta \rho} (-\nabla \cdot (\rho \mathbf{u})) + \frac{\delta \ell}{\delta S} (-\mathbf{u} \cdot \nabla S) + \frac{\delta \ell}{\delta \mathbf{B}} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u} \nabla \cdot \mathbf{B}].
\]
(7.32)

Thus
\[
\frac{\delta \ell}{\delta a} = -\nabla \cdot \left( \rho \frac{\delta \ell}{\delta \rho} \right) + \nabla \cdot \left[ (\mathbf{u} \times \mathbf{B}) \times \frac{\delta \ell}{\delta \mathbf{B}} \right]
\]
\[
+ \mathbf{u} \cdot \left\{ \rho \nabla \left( \frac{\delta \ell}{\delta \rho} \right) - \frac{\delta \ell}{\delta S} \nabla S + \mathbf{B} \times \left( \nabla \times \left( \frac{\delta \ell}{\delta \mathbf{B}} \right) \right) \right\} - \frac{\delta \ell}{\delta \mathbf{B}} \nabla \cdot \mathbf{B}.
\]
(7.33)

From (7.33) we find:
\[
\frac{\delta \ell}{\delta a} \circ a = \rho \nabla \left( \frac{\delta \ell}{\delta \rho} \right) - \frac{\delta \ell}{\delta S} \nabla S + \mathbf{B} \times \left( \nabla \times \left( \frac{\delta \ell}{\delta \mathbf{B}} \right) \right) - \frac{\delta \ell}{\delta \mathbf{B}} \nabla \cdot \mathbf{B}.
\]
(7.34)

Integrate (7.33) over \(\mathbf{d}^3 \mathbf{x}\) over the volume, \(V\), drop surface terms, and set \(\eta \rightarrow \mathbf{u}\) in (7.20) gives the result (7.34) for \(\delta \ell / \delta a \circ a\).

Using the first law of thermodynamics in the form: \(T \nabla S - \nabla h = -\nabla p / \rho\) and the expressions (7.28) for \(\delta \ell / \delta \rho\), \(\delta \ell / \delta S\), \(\delta \ell / \delta \mathbf{B}\) in (7.34) gives:
\[
\frac{\delta \ell}{\delta a} \circ a = \left( -\nabla p + J \times \mathbf{B} + \frac{\mathbf{B}}{\mu_0} \nabla \cdot \mathbf{B} \right) + \rho \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right).
\]
(7.35)

Using \(\mathbf{ad}^\ast (\delta \ell / \delta \mathbf{u}) \circ a\) from (7.31) and \(\delta \ell / \delta a \circ a\) from (7.35) in the Euler–Poincaré equation (7.21) gives the MHD momentum equation in the form:
\[
\frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + J \times \mathbf{B} + \frac{\mathbf{B}}{\mu_0} \nabla \cdot \mathbf{B}.
\]
(7.36)

The momentum equation (7.36) can also be written in the conservative form:
\[
\frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot \left( \rho \mathbf{u} \otimes \mathbf{u} + \left( p + \frac{B^2}{2 \mu_0} \right) \mathbf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right) = 0,
\]
(7.37)

where the magnetic terms involve the Maxwell stress energy tensor. The above derivation of the Euler–Poincaré equation is essentially that of Holm et al (1998). It is also discussed by Cotter and Holm (2013) in their analysis of symmetries and conservation laws associated with advection of physical quantities i.e., the Tur and Yanovsky (1993) conservation laws.
7.2. Noether's second theorem

Consider the application of the above ideas to obtain a version of Noether's second theorem associated with the symmetries \( \eta \). In the derivation of Noether’s theorem, it is useful to keep track of all the surface or divergence terms that arise when integrating by parts. These terms are assumed to vanish in the derivation of the Euler–Poincaré equation (7.36) or (7.37). The variation of the action \( \delta J \) is again given by (7.13), which reduces to the result (7.15), i.e.,

\[
\delta J = \int \left\{ \frac{\delta \ell}{\delta a} \cdot \eta + [u, \eta] \right\} \, dt + \left( \frac{\delta \ell}{\delta a} \cdot \mathcal{L}_\eta (a) \right) \, dt = \delta J_u + \delta J_a,
\]

where \( \delta J_a \) and \( \delta J_u \) are given by (7.27). Using integration by parts, the first term \( \delta J_u \) in (7.38) reduces to:

\[
\delta J_u = - \int \left\{ \eta, \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) + a \frac{\delta u}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) \right\} \, dt + \int \frac{\partial}{\partial t} \left( \eta \cdot \frac{\delta \ell}{\delta u} \right) + \nabla \cdot \left[ \left( \eta \cdot \frac{\delta \ell}{\delta u} \right) u \right] \, d^3x \, dt.
\]

(7.39)

The variations of the \( a \) variables is given by (7.5), i.e., \( \delta a = -\mathcal{L}_\eta (a) \). Thus, we compute the variations \( \delta (\rho \, d^3x) \), \( \delta S \) and \( \delta (B \cdot dS) \) as in (7.29) but with \( u \) replaced by \( \eta \). The net result from (7.30) is:

\[
\delta \rho = - \nabla \cdot (\rho \eta), \quad \delta S = - \eta \cdot \nabla S, \quad \delta B = \left[ \nabla \times (\eta \times B) \right] - \eta (\nabla \cdot B).
\]

(7.40)

Using the results (7.28) and (7.40) we obtain equation (7.32) but with \( u \) replaced by \( \eta \). The net upshot is the result (7.33) but with \( u \) replaced by \( \eta \), i.e.,

\[
\frac{\delta \ell}{\delta a} \delta a = - \nabla \left( \rho \eta \frac{\delta \ell}{\delta \rho} \right) + \nabla \left[ \left( \eta \times B \right) \times \frac{\delta \ell}{\delta B} \right] + \eta \left\{ \rho \nabla \left( \frac{\delta \ell}{\delta \rho} \right) - \frac{\delta \ell}{\delta S} \nabla S \times B + \left( \nabla \times \left( \frac{\delta \ell}{\delta B} \right) \right) - \frac{\delta \ell}{\delta B} \nabla \cdot B \right\}.
\]

(7.41)

Using (7.41) we obtain:

\[
\delta J_a = \int \frac{\delta \ell}{\delta a} \, d^3x \, dt \int \left\{ \eta, \frac{\delta \ell}{\delta a} \circ a \right\} \, dt + \int \nabla \cdot \left( -\rho \eta \frac{\delta \ell}{\delta \rho} + (\eta \times B) \times \frac{\delta \ell}{\delta B} \right) \, d^3x \, dt.
\]

(7.42)

where \( \delta \ell / \delta a \circ a \) is given by (7.34), or the coefficient of \( \eta \) in (7.41). Adding (7.39) and (7.42) for \( \delta J_a \) and \( \delta J_u \) we obtain:

\[
\delta J = \delta J_u + \delta J_a = - \int \left\{ \eta, \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) + a \frac{\delta u}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) \right\} \, dt + \int \left[ \frac{\partial}{\partial t} \left( \eta \cdot \frac{\delta \ell}{\delta u} \right) + \nabla \cdot \left( \eta \cdot \frac{\delta \ell}{\delta u} - \rho \eta \frac{\delta \ell}{\delta \rho} + (\eta \times B) \times \frac{\delta \ell}{\delta B} \right) \right] \, d^3x \, dt.
\]

(7.43)

We require \( \delta J = 0 \) in (7.43) in order for \( \eta \) to be a variational symmetry of the action. Because there are an infinite number of fluid relabeling symmetries \( \eta \) one cannot automatically assume that the Euler–Lagrange equations (7.21) are satisfied. We can write (7.43) in the form:

\[
\delta J = \int \left\{ \eta, E_{[u, a]} (\ell) \right\} \, dt + \int \left( \frac{\partial D}{\partial t} + \nabla \cdot F \right) \, d^3x \, dt,
\]

(7.44)

where

\[
E_{[u, a]} (\ell) = - \left\{ \frac{\partial}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) + a \frac{\delta u}{\partial t} \left( \frac{\delta \ell}{\delta u} \right) \right\} - \frac{\delta \ell}{\delta a} \circ a.
\]

(7.45)
is the Euler operator and

\[ D = \eta \cdot \frac{\delta \ell}{\delta u}, \quad F = \eta \cdot \frac{\delta \ell}{\delta u} - \rho \frac{\delta \ell}{\delta \rho} + (\eta \times B) \times \frac{\delta \ell}{\delta B}, \]  

(7.46)

are the density \( D \) and flux \( F \) surface terms. Further analysis of (7.44) involving integration by parts is necessary before one can arrive at a conservation law for particular Lie symmetries (which involve arbitrary function(s)). In particular, Padhye and Morrison (1996a, 1996b) and Padhye (1998) use this procedure to obtain Ertel’s theorem, from fluid relabeling symmetries.

The variational equation (7.44) can be written in the form:

\[ \delta J = \int \langle \eta, E(\ell) \rangle \, dt + C(t) + \int \int \nabla \cdot F \, d^3x \, dt, \]  

(7.47)

where

\[ C(t) = \int \int \frac{\partial D}{\partial t} \, d^3x \, dt \equiv \left[ \langle \eta, \frac{\delta \ell}{\delta u} \rangle \right]_0, \]

\[ \left\langle \eta, \frac{\delta \ell}{\delta u} \right\rangle = \int_V d^3x \left( \eta \cdot \frac{\delta \ell}{\delta u} \right), \]  

(7.48)

and \( D \) and \( F \) are given by (7.46).

Using the formulae (7.28) for \( \delta \ell/\delta \rho, \delta \ell/\delta u, \delta \ell/\delta S \) and \( \delta \ell/\delta B \) in (7.46) gives:

\[ D = \hat{V}^x \cdot \rho u + \Lambda^0, \]

\[ F = \hat{V}^x \cdot \left( \rho u \otimes u + \left( \varepsilon + p + \frac{B^2}{\mu_0} - \frac{1}{2} \rho |u|^2 \right) I - \frac{B \otimes B}{\mu_0} \right) + \Lambda, \]  

(7.49)

where use the notation:

\[ \hat{V}^x = \eta. \]  

(7.50)

and we have added potentials \( \Lambda^0 \) and \( \Lambda \) in (7.49) to account for the possibility of gauge transformations, which agrees with the density and flux formulas obtained in section 6 in (6.19)–(6.20), for the conserved density and flux in Noether’s theorem for fluid relabeling symmetries and gauge transformations. Here \( \hat{V}^x \) is the canonical symmetry generator for fluid relabeling symmetries, in which \( x = x(x_0, t) \) is the Lagrangian map, in which the \( x' \) are the dependent variables and Lagrange labels \( x_0 \) are the independent variables (e.g. Webb et al 2005b, Webb and Zank 2007). From Ibragimov (1985) and Webb et al (2005b)

\[ \hat{V}^{x'} = V^{x'} - V^{x'} D_{x''} x' \equiv -V^{x'} x'' , \]  

(7.51)

gives the formula for the canonical symmetry generator \( \hat{V}^x \) in terms of the Lagrange label symmetry generator \( V^{x'} \) where \( x'' = \partial x'/\partial x'' \).

### 7.2.1. Fluid relabeling determining equations.

For fluid relabeling symmetries, Eulerian physical variables do not change (e.g. Webb and Zank 2007). Ad vected quantities \( a \) satisfy:

\[ \delta a = -\mathcal{L}_\eta(a) = 0, \]  

(7.52)

where \( \eta \) is the vector field generator of the relabeling symmetry.

The Eulerian fluid velocity \( u \) does not change under fluid relabeling symmetry. Thus,

\[ \delta u = \eta + [u, \eta] = 0. \]  

(7.53)

Equation (7.53) is condition for the vector field \( \eta \) to be Lie dragged by the fluid, i.e. \( d\eta/dt = 0 \) moving with the flow.
The conditions (7.52) are equivalent in the case of MHD of setting \( \delta \rho, \delta S \) and \( \delta B \) equal to zero. Using the notation \( \dot{V}^x \equiv \eta \), (7.40) reduce to:
\[
\nabla \cdot (\rho \dot{V}^x) = 0, \quad \dot{V}^x \cdot \nabla S = 0, \\
\nabla \times (\dot{V}^x \times B) = 0,
\]
(7.54)
where we used Gauss’s law \( \nabla \cdot B = 0 \). Setting \( \delta u = 0 \) in (7.53) gives the equation:
\[
\frac{d\dot{V}^x}{dt} - \dot{V}^x \cdot \nabla u = 0,
\]
(7.55)
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla \) is the Lagrangian time derivative moving with the flow. The condition (7.55) shows that the vector field \( \dot{V}^x \) is Lie dragged with the flow.

7.2.2. Noether’s 2nd theorem: mass conservation symmetry. Consider the conservation law associated with the mass conservation equation for the case of an ideal, isobaric fluid, with equation of state \( p = p(\rho) \) (see also Cotter and Holm 2013). For Noether’s second theorem the variation of \( J \), \( \delta J \), is given by (7.47), i.e. we require:
\[
\delta J = \int d^3x \int dt \left[ \eta \cdot E(\ell) + \frac{\partial D}{\partial t} + \nabla \cdot F \right] = 0,
\]
(7.56)
where \( E(\ell) \) is the Euler operator given by (7.45). For the fluid relabeling symmetry for mass conservation, the variation \( \delta a \) of \( a = \rho d^3x \) is set equal to zero, i.e.,
\[
\delta a = -\mathcal{L}_\eta(\rho d^3x) = 0.
\]
(7.57)
Using Cartan’s magic formula:
\[
\mathcal{L}_\eta(a) = d(\eta \lrcorner a) + \eta \lrcorner da,
\]
(7.58)
d\( a \) = 0 (as \( a \) is a three-form in 3D-space), and noting \( \eta \lrcorner a = \rho \eta \cdot dS \), we obtain
\[
\mathcal{L}_\eta(\rho d^3x) = d[\rho \eta \cdot dS] = 0.
\]
(7.59)
By the Poincaré Lemma, there exists a 1-form \( \psi \cdot dx \) such that
\[
\eta \lrcorner a = \rho \eta \cdot dS = d(\psi \cdot dx) \equiv \nabla \times \psi \cdot dS.
\]
(7.60)
Since \( \eta \lrcorner a \) is a conserved advected 2-form, then
\[
\eta = \frac{\nabla \times \psi}{\rho} \quad \text{is a conserved (Lie dragged) vector field.}
\]
(7.61)
A simpler derivation of (7.61) is to note that \( \eta \equiv \dot{V}^x \) satisfies the first Lie determining equation in (7.54), i.e. \( \nabla \cdot (\rho \eta) = 0 \).

The first term in (7.56) containing the Euler operator \( E(\ell) \) is:
\[
T_1 = \int d^3x \int dt \eta \cdot E(\ell) = \int d^3x \int dt \frac{\nabla \times \psi}{\rho} \cdot E(\ell)
\]
\[
= \int d^3x \int dt \left\{ \nabla \cdot [\psi \times E(\ell)/\rho] + \psi \cdot \nabla \times \left( E(\ell)/\rho \right) \right\}
\]
\[
= \int d^3x \int dt \psi \cdot \nabla \times \left( E(\ell)/\rho \right),
\]
(7.62)
where the surface term due to \( \nabla \cdot [\psi \times E(\ell)/\rho] \) is assumed to vanish on the boundary \( \partial V \) of the volume \( V \) of integration.

The remaining integrals in \( \delta J \) in (7.56):
\[
T_2 = \int d^3x \int dt \left( \frac{\partial D}{\partial t} + \nabla \cdot F \right) = C(t) + \int d^3x \int dt \nabla \cdot F,
\]
(7.63)
can be reduced to the form:
\[ T_2 = \int d^3x \int dt \left\{ \psi \cdot \left[ \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) \right] + \nabla \cdot W \right\}. \quad (7.64) \]
where
\[ W = \nabla \times \left[ \left( h + \frac{1}{2} |u|^2 \right) \psi - (\nabla \cdot u)u \right], \quad (7.65) \]
and \( \omega = \nabla \times u \) is the vorticity of the fluid. Note that \( \nabla \cdot W = 0 \), because \( W \) may be written in the form of a ‘curl’: \( W = \nabla \times M \). Put another way
\[ \int \nabla \cdot W \, d^3x = \int \nabla \times M \cdot dS = \int_{\partial \Omega} M \cdot d\mathbf{x}, \quad (7.66) \]
which is zero since \( \partial \partial V \) does not exist (i.e. the boundary of a boundary is zero for a simply connected region: e.g. (Misner et al. 1973). Combining (7.62) and (7.64) we obtain:
\[ \delta J = \int d^3x \int dt \left\{ \psi \cdot \left[ \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) + \nabla \times \left( \frac{E(\ell)}{\rho} \right) \right] + \nabla \cdot W \right\}. \quad (7.67) \]
Thus, invoking the du Bois Reymond lemma of the Calculus of variations and noting that \( \nabla \cdot W = 0 \), (7.67) yields the generalized Bianchi identity:
\[ \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) + \nabla \times \left( \frac{E(\ell)}{\rho} \right) \equiv 0. \quad (7.68) \]
Equation (7.68) is the basic result of Noether’s second theorem, which shows that there are differential relations between the Euler–Lagrange variational derivatives \( E_i(\ell) \) (1 \( \leq i \leq 3 \)) in this case. Note that (7.68) does not necessarily imply that the Euler–Lagrange equations \( E_i(\ell) = 0 \) (1 \( \leq i \leq 3 \)) are satisfied. In the case where \( E_i(\ell) = 0 \) (1 \( \leq i \leq 3 \)), (7.68) implies the vorticity conservation law:
\[ \left( \frac{\partial}{\partial t} + L_u \right) (\omega \cdot dS) = \left( \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) + u \nabla \cdot \omega \right) \cdot dS = 0. \quad (7.69) \]
Note that \( \nabla \cdot \omega = 0 \) as \( \omega = \nabla \times u \) is the vorticity. Equation (7.69) shows that the vorticity 2-form \( \omega \cdot dS \) is advected with the flow.

The generalized Bianchi identity could also be derived using the method of Lagrange multipliers for Noether’s second theorem developed by Hydon and Mansfield (2011). The proof of (7.63)–(7.64) is given below.

**Proof.** We use the analysis of Cotter and Holm (2013) to calculate \( C(\ell) \). Using (7.48) and (7.64) \( C(\ell) \) is given by:
\[ C(\ell) = \int_D \left( \frac{\delta \ell}{\delta u} \cdot \eta \right) d^3x = \int \left( \frac{1}{\rho} \frac{\delta \ell}{\delta u} \right) \rho \eta_j d^3x = \int \frac{1}{\rho} \frac{\delta \ell}{\delta u} (\nabla \times \psi) dS_j \] 
\[ = \int \frac{1}{\rho} \frac{\delta \ell}{\delta u} \cdot d\mathbf{x} \wedge d(\psi \cdot d\mathbf{x}). \quad (7.70) \]
From (7.70)
\[ \frac{dC}{dt} = \int \left\{ \frac{\partial}{\partial t} \left( \frac{1}{\rho} \frac{\delta \ell}{\delta u} \cdot d\mathbf{x} \right) \wedge d(\psi \cdot d\mathbf{x}) + \frac{1}{\rho} \frac{\delta \ell}{\delta u} \cdot d\mathbf{x} \wedge \frac{\partial}{\partial t} [d(\psi \cdot d\mathbf{x})] \right\}. \quad (7.71) \]
Write \( dC/dt = t_1 + t_2 \) where \( t_1 \) is first term and \( t_2 \) second term in (7.71). Note that \( a, \eta \) and \( (\eta, \alpha) \), where \( a = \rho \, d^3x \) are advected with the flow. Thus,
\[ \left( \frac{\partial}{\partial t} + L_u \right) (\eta, \alpha) \equiv \left( \frac{\partial}{\partial t} + L_u \right) [d(\psi \cdot d\mathbf{x})] = 0. \quad (7.72) \]
At this point it is useful to introduce the notation:

\[ \alpha = \frac{1}{\rho} \frac{\delta \ell}{\delta u} \cdot dx, \quad \beta = \mathcal{L}_u(\psi \cdot dx), \quad \gamma = \psi \cdot dx. \] (7.73)

Using the results:

\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta, \quad d(\alpha \wedge \gamma) = 0, \]
\[ \mathcal{L}_u(\alpha \wedge \gamma) = \mathcal{L}_u(\alpha \wedge \gamma) + d\alpha \wedge \mathcal{L}_u(\gamma), \]
\[ \mathcal{L}_u(\alpha \wedge \gamma) = u_\alpha d(\alpha \wedge \gamma) + d[u_\alpha (\alpha \wedge \gamma)], \]
\[ \alpha, \ w \wedge \gamma = d\alpha \wedge \gamma - d(\alpha \wedge \gamma), \] (7.74)

we obtain:

\[ \frac{dC}{dt} = \int \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) (d\alpha) \wedge \gamma + d(\alpha \wedge \beta - u_\beta (d\alpha \wedge \gamma) - \alpha \wedge \gamma) \right\}. \] (7.75)

Using (7.75) for \(dC/dt\) in (7.56) for \(\delta J\) gives:

\[ \delta J = \int \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) (d\alpha) \wedge \gamma + d \left( \psi \times \frac{\mathbf{E}(\ell)}{\rho} \cdot d\mathbf{S} + \mathbf{F} \cdot d\mathbf{S} \right) \right. \\
\left. + \alpha \wedge \beta - u_\beta (d\alpha \wedge \gamma) - \alpha \wedge \gamma \right\} + \int d^3x \int dt \ \psi \cdot \nabla \times \left( \mathbf{E}(\ell)/\rho \right). \] (7.76)

Next we note that the surface term:

\[ d(\mathbf{F} \cdot d\mathbf{S} + \alpha \wedge \beta - u_\beta (d\alpha \wedge \gamma) - \alpha \wedge \gamma) = d(\mathbf{W} \cdot d\mathbf{S}) = \nabla \cdot \mathbf{W} d^3x, \] (7.77)

where

\[ \mathbf{W} = \nabla \times \left[ (\mathbf{h} + \frac{1}{2} |\mathbf{u}|^2) \psi - (\psi \cdot \mathbf{u}) \mathbf{u} \right]. \] (7.78)

Note that \(\nabla \cdot \mathbf{W} = 0\). In (7.78) we assumed a barotropic equation of state, with \(p = p(\rho)\), and used the momentum equation:

\[ u_t - \mathbf{u} \times \mathbf{\omega} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) = T \nabla S - \nabla h, \] (7.79)

to determine \(\alpha_t\). Also note that

\[ \int \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) (d\alpha) \wedge \gamma = \int \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) (\mathbf{\omega} \cdot d\mathbf{S}) \wedge (\psi \cdot dx) \]
\[ = \int \psi \cdot \left[ \mathbf{\omega} \cdot \nabla \times (\mathbf{u} \times \mathbf{\omega}) \right] d^3x. \] (7.80)

Substituting (7.77)–(7.80) into (7.76), and assuming the surface term due to \(\psi \times \mathbf{E}(\ell)/\rho\) is zero, we obtain the result (7.67) for \(\delta J\). This completes the proof.

7.2.3. Cross helicity. To obtain the cross helicity conservation law (4.18) using Noether’s theorem, it is necessary to obtain the appropriate solution of (7.52)–(7.55) for the fluid relabeling symmetries. The condition that the mass 3-form \(\alpha = \rho d^3x\) is a fluid relabeling symmetry using Cartan’s magic formula, and noting \(d\alpha = 0\) requires that:

\[ \mathcal{L}_u(\rho d^3x) = d(\eta \wedge \rho d^3x) = d(\rho \eta \cdot d\mathbf{S}) = \nabla \cdot (\rho \eta) d^3x = 0. \] (7.81)

The entropy variation \(\delta S = -\eta \cdot \nabla S = 0\), and the magnetic field variation \(\delta \mathbf{B} = \nabla \times (\eta \times \mathbf{B}) = 0\) and the fluid velocity variation \(\delta \mathbf{u} = \dot{\eta} + [\mathbf{u}, \eta] = 0\) are all satisfied by the choice:

\[ \eta = \dot{V} = \zeta (x_0) \mathbf{b} \quad \text{where} \quad \mathbf{b} = \frac{\mathbf{B}}{\rho} \quad \text{and} \quad \mathbf{B} \cdot \nabla S = 0. \] (7.82)
Note that \( b = B / \rho \) is an invariant vector field that is Lie dragged with the flow (see (6.37)). From (7.49) the surface term \( D \) in the variational principle (7.44) is given by:
\[
D = \rho u \cdot \eta + \Lambda^0 = \rho u \cdot \zeta(x_0) b + \Lambda^0 \equiv \zeta(x_0) u \cdot B + \Lambda^0.
\]  
(7.83)
Similarly, the flux \( F \) surface term in (7.49) is given by:
\[
F = \zeta(x_0) \frac{B}{\rho} \left[ \rho u \otimes u + \rho \left( h - \frac{1}{2}|u|^2 \right) I + \frac{B^2}{\mu_0} I - \frac{B \otimes B}{\mu_0} \right] + A
\]
\[
= \zeta(x_0) \left[ |u - \frac{1}{2}|u|^2 \right] B + A.
\]  
(7.84)
In (7.83) and (7.84) we have added the gauge potential terms \( \Lambda^0 \) and \( A \). This allows one to make a link to the variational approach of section 6 that includes the effects of gauge transformations in the variational principle and in Noether’s theorem. In section 6, the generalized cross helicity conservation law (6.34) was obtained by setting \( \zeta(x_0) = 1, \Lambda^0 = r B \cdot \nabla S \) and \( A = u B \cdot \nabla S \) where \( dr/dt = -T \) (see equations (6.36)). In the variational principle (7.44) \( \delta J \) reduces to:
\[
\delta J = \int d^3x \int dt \left\{ \zeta(x_0) \frac{B}{\rho} \cdot E(\ell) + \frac{\partial}{\partial t} (\zeta(x_0) u \cdot B) + \nabla \cdot \left[ \zeta(x_0) \left( (u \cdot B)u + \left( h - \frac{1}{2}|u|^2 \right) B \right] + \nabla \cdot \left[ u h_c + \left( h - \frac{1}{2}|u|^2 \right) B \right] \right\} + R,
\]  
(7.85)
where \( h_c = u \cdot B \) is the cross helicity, and
\[
R = h_c \left( \frac{\partial \zeta}{\partial t} + u \cdot \nabla \zeta \right) + \left( h - \frac{1}{2}|u|^2 \right) B \cdot \nabla \zeta + \nabla \cdot \left[ u h_c + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 0.
\]  
(7.86)
In the case \( B \cdot \nabla S(x_0) = B \cdot \nabla \zeta(x_0) = 0, \) and \( \Lambda^\alpha = 0 (\alpha = 0, 1, 2, 3) \), the remainder term in (7.85) and (7.86) \( R = 0 \). The net upshot from (7.85) is the generalized Bianchi identity:
\[
B \cdot E(\ell) + \frac{\partial h_c}{\partial t} + \nabla \cdot \left[ u h_c + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 0.
\]  
(7.87)
Thus, if the Euler–Lagrange equations \( E(\ell) = 0 \) are satisfied, then (7.87) reduces to the cross helicity conservation equation (4.18), i.e.
\[
\frac{\partial h_c}{\partial t} + \nabla \cdot \left[ u h_c + \left( h - \frac{1}{2}|u|^2 \right) B \right] = 0.
\]  
(7.88)
The only constraint on (7.88) is that we require \( B \cdot \nabla S = 0 \). If \( B \cdot n = 0 \) on the boundary \( \partial V_m \) of the volume \( V_m \) of interest, then the integral form of the (7.88) reduces to \( dH_c / dt = 0 \) (see section 3 for further discussion).

### 7.2.4. Helicity in fluids.

In a barotropic, ideal fluid in which the pressure \( p = p(\rho) \) is independent of the entropy \( S \), the helicity density:
\[
h_f = u \cdot \omega \quad \text{where} \quad \omega = \nabla \times u,
\]  
(7.89)
satisfies the conservation law:
\[
\frac{\partial h_f}{\partial t} + \nabla \cdot \left[ u h_f + \left( h - \frac{1}{2}|u|^2 \right) \omega \right] = 0. 
\]  
(7.90)
This conservation law is the analogue of the cross helicity conservation law (7.88) where \( B \rightarrow \omega \) and \( h_c \rightarrow h_f \).
The Lie symmetry associated with the helicity (kinetic helicity) conservation equation (7.90) is:

\[ \eta \equiv \dot{\psi} = \frac{\xi(x_0)\omega}{\rho} \quad \text{where} \quad \omega \cdot \nabla \xi(x_0) = 0. \quad (7.91) \]

One can verify that the solution (7.91) satisfies the fluid relabeling Lie determining equations (7.53)–(7.55) with \( B = 0 \). In particular (7.55) reduces to the vorticity equation:

\[ \frac{d}{dt} \left( \frac{\omega}{\rho} \right) = \frac{\omega \cdot \nabla u}{\rho} \quad \text{or} \quad \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) = 0, \quad (7.92) \]

which applies for a barotropic equation of state with \( p = p(\rho) \). The derivation of the helicity conservation law (7.90) using Noether’s theorem is analogous to the derivation of the cross helicity conservation law (7.88) except that \( B \rightarrow \omega \) and \( h_c \rightarrow h_f \).

7.2.5. Potential vorticity and Ertel’s theorem

Proposition 7.2. Ertel’s theorem states that in ideal compressible fluid mechanics, that the potential vorticity \( q = \omega \cdot \nabla S/\rho \) where \( \omega = \nabla \times u \) is the fluid vorticity, is advected with the flow, i.e. \( dq/\partial t = 0 \) where \( dq/\partial t = \partial /\partial t + u \cdot \nabla \) is the Lagrangian time derivative following the flow.

The Lie determining equations (7.54)–(7.55) admit the symmetry:

\[ \eta \equiv \dot{\psi} = \frac{\nabla \times (\Phi \nabla S)}{\rho} = \frac{\nabla \times \psi}{\rho}, \quad \text{where} \quad \psi = \Phi \nabla S, \quad (7.93) \]

and \( \Phi = \Phi(x_0) \) depends only on the Lagrange labels \( x_0 \), i.e. \( \Phi \) is a 0-form Lie dragged by the flow:

\[ \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + u \cdot \nabla \Phi = 0. \quad (7.94) \]

Note that

\[ \eta \cdot \rho \frac{d^3x}{dt} = \rho \eta \cdot dS = \nabla \times \psi \cdot dS = d(\psi \cdot dx) = d(\Phi dS). \quad (7.95) \]

The condition (7.55) implies \( \dot{\psi} \equiv \eta \) is a Lie dragged vector field which satisfies (7.53). Similarly, the 1-form \( \alpha = \psi \cdot dx \) is Lie dragged with the flow, i.e. \( \psi \) satisfies the equation:

\[ \frac{\partial \psi}{\partial t} + u \times (\nabla \times \psi) + \nabla (u \cdot \psi) = 0. \quad (7.96) \]

Using \( \psi = \Phi \nabla S, (7.96) \) reduces to:

\[ \Phi \nabla \left( \frac{dS}{dt} \right) + \nabla S \frac{d\Phi}{dt} = 0. \quad (7.97) \]

Equation (7.55) is equivalent to the curl of (7.97). Since \( dS/\partial t = 0, (7.97) \) implies \( d\Phi/\partial t = 0. \) Note that \( \psi \cdot dx = \Phi \nabla S \cdot dx \) are Lie dragged 1-forms and hence \( \Phi \) is necessarily an advected invariant 0-form or function.

Proof. Ertel’s theorem

To derive Ertel’s theorem from Noether’s theorem, we require \( \delta J = 0 \) in (7.56). From (7.76):

\[ \delta J = \int dt \int_{\mathcal{V}} \left[ \left( \frac{\partial}{\partial t} + L_u \right) (d\alpha) \wedge \gamma + d(W \cdot dS) \right] + \int dt \int_{\mathcal{V}} d^3x \psi \cdot \nabla \times (E(\ell)/\rho), \quad (7.98) \]

where \( W \) is given by (7.78). Note that \( W \) is a solenoidal vector field, i.e. \( \nabla \cdot W = 0 \). In (7.98) \( \psi = \Phi \nabla S \) and \( d\Phi/\partial t = 0. \) We introduce the notation:

\[ I = \int_{\mathcal{V}} \left( \frac{\partial}{\partial t} + L_u \right) (d\alpha) \wedge \gamma, \quad (7.99) \]
for the first integral in (7.98), where \( \alpha, \beta \) and \( \gamma \) are the differential 1-forms given in (7.73). From (7.99) and (7.73) we obtain:

\[
I = \int_V \frac{d}{dt} \left[ \nabla \times \left( \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \right) \cdot dS \right] \wedge \Phi \nabla S \cdot d\mathbf{x} \\
= \int_V \frac{d}{dt} \left( \omega \cdot dS \right) \wedge \Phi \nabla S \cdot d\mathbf{x} \\
= \int_V \frac{d}{dt} \left( \omega \cdot dS \wedge \Phi \nabla S \cdot d\mathbf{x} \right).
\]  
(7.100)

In (7.100) we use the fact that \( \Phi \) is a 0-form and \( \nabla \cdot d\mathbf{x} \) is a 1-form, which are Lie dragged with the flow. The integral \( I \) in (7.100) can be further reduced to:

\[
I = \int_V \frac{d}{dt} \left( \frac{\omega \cdot \nabla S}{\rho} \Phi \rho \right) \, d^3x.
\]  
(7.101)

Note that \( d/dt (\Phi \rho d^3x) = 0 \) as \( \rho d^3x \) is an invariant 3-form and \( \Phi \) is an invariant 0-form.

Using (7.101) in (7.98) gives:

\[
\delta J = \int_V \int d^3x \left\{ \Phi \left[ \frac{d}{dt} \left( \frac{\omega \cdot \nabla S}{\rho} \right) + \nabla S \cdot \nabla \times \left( \frac{\mathbf{E}(\ell)}{\rho} \right) \right] + \nabla \cdot \mathbf{W} \right\}.
\]  
(7.102)

Because \( \nabla \cdot \mathbf{W} = 0 \), and using the du-Bois Reymond lemma in (7.102), we obtain the generalized Bianchi identity:

\[
\rho \frac{d}{dt} \left( \frac{\omega \cdot \nabla S}{\rho} \right) + \nabla S \cdot \nabla \times \left( \frac{\mathbf{E}(\ell)}{\rho} \right) = 0.
\]  
(7.103)

If the Euler–Lagrange equations \( \mathbf{E}(\ell) = 0 \) are satisfied, then (7.103) implies Ertel’s theorem:

\[
\frac{d}{dt} \left( \frac{\omega \cdot \nabla S}{\rho} \right) = 0.
\]  
(7.104)

This completes the proof. \( \square \)

8. Concluding remarks

In this paper we have used variants of Noether’s theorems to derive nonlocal conservation laws for helicity and cross helicity in ideal fluid dynamics and in MHD. These two conservation laws were derived in Webb et al (2014a). Other conservation laws for advected invariants of MHD and ideal gas dynamics were obtained by using Lie dragging techniques (Webb et al 2014a, Tur and Janovsky 1993). If the gas is isobaric (i.e. the gas pressure \( p = p(\rho) \)), the helicity and cross helicity conservation laws are local conservation laws that depend only on the local variables \((\rho, \mathbf{u}, S, \mathbf{B})\). Also if \( p = p(\rho, S) \) and \( \mathbf{B} \cdot \nabla S = 0 \), the cross helicity conservation law is a local conservation law. For the general case of a non-isobaric gas with \( p = p(\rho, S) \), nonlocal conservation laws were obtained that depend on the non-local Clebsch potentials.

The connection between advected invariants and the Casimir invariants was investigated in section 3. Padhye and Morrison (1996a, 1996b) using the canonical Lie bracket for Lagrangian MHD, used the fluid relabeling symmetry equations to derive the determining equations for the Casimirs.

The nonlocal helicity and cross helicity conservation laws were derived in the present paper by using Clebsch variables in Noether’s theorem and by exploiting fluid relabeling symmetries and gauge symmetries of the action. The energy conservation law in MHD was
also derived by using a fluid relabeling symmetry of the action and including a non-zero gauge potential in the action.

An alternative derivation of the helicity conservation laws was carried out in section 7 where the Euler–Poincaré formulation of Noether’s first theorem and Noether’s second theorem was developed similar to that of Cotter and Holm (2013) (see Holm et al. 1998) for a general account of the Euler–Poincaré equations and semi-direct product Lie algebras applied to Hamiltonian systems). Noether’s second theorem plays an important role in cases where the variational principle admits an infinite class of symmetries. In this case the conservation laws involve so-called Bianchi identities, since the Euler–Lagrange equations are not necessarily independent (e.g. Hydon and Mansfield 2011, Padhye and Morrison 1996a, 1996b). This approach uses Eulerian variations of the action. The use of Lie symmetries for differential equations and Noether’s theorems are described in standard texts (e.g. Olver 1993, Ibragimov 1985, Bluman and Kumei 1989, Bluman et al. 2010). The helicity and cross helicity conservation laws for barotropic and non-barotropic equations of state for the gas, were derived using Noether’s theorems coupled with fluid relabeling symmetries and gauge transformations. One surprising result, was the derivation of the energy conservation equation for MHD by using a fluid relabeling symmetry and a gauge transformation for the action.

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