We derive and generalize the RR twisted tadpole cancellation conditions necessary to obtain consistent D = 4, \( \mathbb{Z}_N \) orbifold compactifications of Type IIB string theory. At least two different types of branes (or antibranes with opposite RR charges) are introduced into the construction. The matter spectra and their contribution to the non-abelian gauge anomalies are computed. Their relation with the tadpole cancellation conditions is also reviewed. The presence of tachyons is a common feature for some of the non-supersymmetric systems of branes.

I. INTRODUCTION

A crucial aspect to consider in the construction of consistent string theories and their compactified versions is whether the theory is free of ultraviolet (UV) divergences. In the perturbative heterotic superstring theory, the absence of UV divergences is guaranteed by the modular invariance of the torus amplitude (the one-loop oriented closed string vacuum amplitude) [1]. This is not the case for orbifold or orientifold compactifications of Type I and Type II superstring theories, where open and unoriented closed string sectors (corresponding to the cylinder, Möbius strip and Klein-bottle amplitudes respectively) are also present in the theory. For these amplitudes there is no modular group and UV divergences may remain present in the theory. This problem arises in the orbifold and orientifold constructions of Type I and Type II string theories in the presence of D-branes, where open and unoriented closed strings appear besides oriented closed strings. Fortunately, we can still construct consistent string theories of this type if the theory is free of Ramond-Ramond (RR) tadpole divergences. The absence of tadpoles is a necessary constraint to guarantee the consistency of the equations of motion of the RR form potentials. In superstring theory, we encounter two different sources of tadpole divergences: Neveu-Schwarz-Neveu-Schwarz (NSNS) tadpoles and RR tadpoles. The presence of NSNS tadpoles represents a shift of the vacuum state in perturbation theory and they can be removed by expanding the background fields (dilaton and metric) around the corrected solutions to the equations of motion. The existence of RR tadpole divergences indicates the presence of unbalanced RR charges in the theory which would couple to the RR form potentials producing inconsistent equations of motion for those potentials [1]. To obtain a consistent superstring theory we need, at least, to impose the vanishing of the RR tadpoles even if in some (non-supersymmetric) cases, the theory is not free of NSNS tadpole divergences. In supersymmetric models, both contributions to the tadpoles cancel each other. Even then, a separate cancellation of the RR tadpoles is required to ensure the consistency of the theory.

Before the “second string revolution”, the heterotic string theory was thought to be the only candidate in which the unification of General Relativity and the Standard Model (SM) could be realized. With the discovery of D-branes [2], all five perturbative descriptions of superstring theory were found to be related to one another and each of them is conjectured to be a different perturbative limit of the same non-perturbative theory (M- theory). As a result, the previous emphasis on the heterotic string became less pronounced. Type I and Type II string theories are now strong candidates for the unification of gravity and gauge interactions within the string theory framework, if suitable types and numbers of D-branes are included. So far, several types of orbifold and orientifold models based on compactifications of Type I and Type II string theories have been successfully constructed in D=4 and D=6 dimensions, with or without space-time supersymmetry [3] - [11]. An alternative approach to model building based on Type II superstring theory is the so-called “bottom-up” approach [4]. It searches for local configurations of D-branes at a \( R^6/\mathbb{Z}_N \) orbifold singularity, in such a way that the world-volume theory is similar to that of the Standard Model, before embedding this local theory (embedding of the orbifold point group only) in a global orbifold theory (embedding of the lattice as well). This approach is appealing because some properties of the model, such as the number of generations and the gauge group of the theory depend only on the local configuration of the D-branes at the orbifold singularities. Simple configurations of Type IIB superstring models with world-volume theories close to the Standard Model have been obtained by placing a stack of D3 branes on top of orbifold singularities with matter fields localized in the 4-dimensional world-volume. The presence of additional branes beyond D3 (e.g. D7) is required.
by the tadpole cancellation conditions which at the same time determine how and when the open strings should be added.

Dp-branes are p-dimensional hyperplanes describing the dynamics of the endpoints of open strings. These endpoints carry gauge quantum numbers and allow gauge interactions and chiral matter to exist within the D-brane worldvolume while gravity remains present in the bulk. D-branes are carriers of RR charges [2]. These charges can be calculated by looking at the RR tadpole. If the Dp-brane has a non-zero RR charge, then it acts as a source term in the equations of motion for the $A_{p+1}$ form field. According to the Gauss law, all the field lines in the compact space must end on an opposite charge. The way to cancel the charge contribution from one brane in a compact space is by adding other Dp-branes in such a way that all the tadpoles are cancelled. In orientifold constructions, the planes that are left invariant by the worldsheet symmetry element of the orientifold group (orientifold planes), also carry RR charges. The one-loop closed string amplitude (Klein-bottle) contains a RR tadpole and D-branes carrying opposite RR charges are needed to neutralize those of the orientifold planes. Therefore, in the orientifold framework of Type IIB string theories, the presence of Dp-branes and with them the open string sector of the theory is required for consistency reasons. However, it is also possible to consider the presence of open strings from the start in orbifold compactifications of Type IIB superstring theories as long as we keep the theory free of RR tadpoles and thus consistent. This is the idea behind the “bottom-up” approach.

In the orientifold theory, the one-loop vacuum amplitudes include the torus, the Klein-bottle, the Möbius strip and the cylinder but as mentioned before, only the last three act as sources for massless tadpoles. The Klein-bottle corresponds to the contribution of unoriented closed strings to the one-loop vacuum amplitude. Alternatively, this amplitude can be pictured as a tree-level closed string amplitude where the closed strings propagate between two orientifold planes. The cylinder (or annulus) amplitude corresponds to the contribution of open strings to the one-loop vacuum amplitude, or equivalently, to the closed string tree-amplitude where the closed strings propagate between two D-branes. The Möbius strip corresponds to the tree-level closed string amplitude where the closed strings propagate between one brane and one orientifold plane. In the bottom-up approach, the absence of orientifold planes leaves the cylinder worldsheet as the only source for tadpole divergences. The different tadpole contributions can be classified according to their volume dependence. According to whether we are computing the tadpoles from the untwisted or the twisted sectors of the orbifold, different volume dependences arise. In the “bottom-up” approach, only cylinder amplitudes contribute to the computation of the tadpoles. In orientifold models, there are additional contributions coming from the Klein-bottle and Möbius strip amplitudes.

In this paper we present a detailed derivation of the twisted tadpole cancellation conditions necessary for obtaining a consistent $D = 4$ Type IIB superstring theory within the “bottom-up” approach, compactified on a 6-dimensional orbifold in the presence of different sets of D-branes. We also compute the contribution from the chiral matter to the anomalies and analyze whether anomalously cancellation is guaranteed when the theory is free of tadpole divergences [3] - [19]. The plan of the paper is as follows. In section 2, we give a generic introduction to the open string mode expansions, Hilbert space and partition functions for $D = 4$ Type IIB orbifolds in the presence of branes. Section 3 is devoted to the study of orbifolds of Type IIB superstring theory in the presence of sets of D9 and D5 branes. We evaluate the chiral fermion spectra of the open string sector and derive its contribution to the gauge anomalies. Section 4 lists equivalent results when in the theory there are present sets of D3-D7, D3-D9, D3-D5, D9-D7 and D5-D7 branes respectively, some of which break supersymmetry. Section 5 discusses the conclusions and some final remarks. An appendix with the properties of the Jacobi theta functions that appear in the computation of the partition functions is given at the end to help understanding the calculations.

II. $D=4$ ORBIFOLDS WITH D-BRANES

In the construction of the four dimensional theory ($\mu = 0,1,2,3$), six of the spatial dimensions of the original 10-dimensional theory ($i = 4,5,6,7,8,9$) are compactified on a $T^6/\mathbb{Z}_N$ orbifold. The six-dimensional orbifold is obtained from a 6-dimensional torus $T^6$ with $\mathbb{Z}_N$ discrete symmetry by dividing out this discrete symmetry. This $\mathbb{Z}_N$ invariant $T^6$ can be realized as a root lattice of a rank-6 Lie algebra, on which the elements of the orbifold group act crystallographically. For ADE semisimple Lie algebras, if we choose the simple roots $\alpha_i$ ($i = 4,5,\ldots,9$) as the basis vectors of the compact subspace, $X = \sum_{i=4}^{9} X_i \alpha_i$, the torus can be defined as:

$$X_i \equiv X_i + 2\pi R_i ,$$  \hspace{1cm} (2.1)

where $i = 4,5,\ldots,9$ and $R_i$ is the $i$-th component of the vector $\vec{R} \equiv \sum_{i=4}^{9} R_i \vec{\alpha}_i$, which belongs to the 6-dimensional lattice $\Lambda = \{ \sum_{i=4}^{9} n_i \alpha_i | n_i \in \mathbb{Z} \}$. We denote the elements of the $\mathbb{Z}_N$ abelian point group as:

$$\mathbb{Z}_N = \{ 1, \theta, \theta^2, \ldots, \theta^{N-1} \} .$$  \hspace{1cm} (2.2)
The orbifold fixed points are determined by the following condition:

\[ X_f \equiv \theta X_f + \sum_{i=4}^{9} n_i \alpha_i , \quad (n_i \in \mathbb{Z}). \]  

(2.3)

Generally, the point group elements act non-diagonally with respect to the simple roots \( \alpha_i \) \( (i = 4, 5, \ldots, 9) \). Alternatively, we can choose a more convenient basis \( e_j \) \( (j = 3, 4, 5) \) for the orbifold basis vectors,

\[ e_{\pm i} \cdot e_{\pm j} = 0, \quad e_{\pm i} \cdot e_{\mp j} = \delta_{ij} \quad (i, j = 1, 2, \cdots, 5) \]  

(2.4)

where now the basis vectors are eigenvectors of the abelian point group elements and the orbifold action is diagonal with respect to the new basis. The simple roots \( \alpha_i \) \( (i = 4, 5, \ldots, 9) \) are linear combinations of the complex basis vectors. With respect to these new basis vectors, the lattice \( \Lambda \) and its dual \( \Lambda^* \) are identical. The orbifold action on an arbitrary coordinate \( X = \sum_{j=1}^{3} (X_j e_j + X_{-j} e_j) \), is now given by:

\[ \theta^k X_{\pm j} = \left\{ \begin{array}{ll}
X_{\pm j} & (j = 1, 2) \\
\exp(\pm 2 \pi i k v_j) X_{\pm j} & (j = 3, 4, 5) 
\end{array} \right. \]  

(2.5)

where \( v_j \) \( (j = 3, 4, 5) \) defines the orbifold twist vector. Type IIB superstring theory without the presence of D-branes is a closed string theory. Under the orbifold compactification, modular invariance of the one-loop scattering amplitudes requires the theory to contain twisted closed string states in addition to the toroidally untwisted closed string states. In the \( k \)-th twisted closed string sector, the worldsheet obeys the following monodromies:

\[ X_{\pm j}(\sigma + 2 \pi, \tau) = \left\{ \begin{array}{ll}
X_{\pm j}(\sigma, \tau) & (j = 1, 2) \\
\exp(\pm 2 \pi i k v_j) X_{\pm j}(\sigma, \tau) & (j = 3, 4, 5) 
\end{array} \right. \]  

(2.6)

where \( k = 0, 1, 2, \ldots, N - 1 \) and \( k = 0 \) refers to the untwisted closed string sector. Superconformal symmetry of the worldsheet implies the following transformation rules under the orbifold action for the fermionic partners:

\[ \psi_{\pm j}(\sigma + 2 \pi + \tau) = \left\{ \begin{array}{ll}
\psi_{\pm j}(\sigma + \tau) & (j = 1, 2) \\
\exp(\pm i 2 \pi k v_j) \psi_{\pm j}(\sigma + \tau) & (j = 3, 4, 5) 
\end{array} \right. \]  

(2.7)

with similar monodromies for the right-moving \( \tilde{\psi}_{\pm j} \) fermions. Based upon the above monodromy properties, we can easily obtain the mode expansions, Hamiltonian (in SCFT), as well as the modular invariant partition functions of the closed string sector. For a more detailed discussion about modular invariance of the closed string sector we refer the reader to [22] and references therein.

Orbifold models of Type IIB superstring theory cannot describe gauge interactions unless open strings are also included. As we have already mentioned in Section I, this is equivalent to adding D-branes into the theory. The type and number of Dp-branes that can be added into the theory is strongly constrained by the twisted RR tadpole cancellation conditions. To our knowledge, there are two fundamental ways of adding Dp-branes into orbifold constructions of Type IIB string theory. The first type of orbifold construction involves the introduction of at least two different types of D-branes. Take a D9-D5 brane system as example. D9-branes embed the full 10-dimensional spacetime and their configuration is automatically symmetric under the \( \mathbb{Z}_N \) orbifold action. However D5-branes, which wrap the three non-compact spatial real dimensions and only one of the compactified complex dimensions, must be located at the orbifold singularities so that they exactly embed one of the compactified complex planes \( (i = 3, 4, 5) \). Different D5-branes are either parallel or perpendicular to each other in the compactified sub-target-space. In the second type of orbifold construction [27] - [32], only one type of Dp-branes is allowed (i.e. D5-branes). These D5-branes, which generally are extended along some “root vectors” of the Coxeter lattice, intersect each other at nontrivial angles so that the total \( T^6 \) configuration preserves the desired \( \mathbb{Z}_N \) symmetry. Our work is restricted to orbifold theories of the first type.

Once D-branes are introduced, there will be open string states stretched between them, describing the matter fields and their gauge interactions. The open strings obey various boundary conditions which determine the matter fields and gauge group. In the remaining parts of this section, we will write the mode expansions of the open strings under the different boundary conditions. For concreteness, we work in the light-cone gauge and focus in each of the possible complex dimensions.
A. Open string mode expansions

1. $j$-th noncompact dimension ($j = 1, 2$)

Open strings satisfy Neumann (N) boundary conditions ($\partial_\sigma X^\pm j(\tau, 0) = \partial_\sigma X^\pm j(\tau, \pi) = 0$) in the directions parallel to the brane but Dirichlet (D) boundary conditions ($X^\pm j(\tau, 0) = 0$ and $X^\pm j(\tau, \pi) = Y^\pm j$) in the perpendicular directions. Therefore, the endpoints of the open strings are only free to move along the parallel directions to the brane. We assume that in each of the noncompact dimensions, open strings satisfy Neumann boundary conditions. Its momentum in these directions is also continuous. The mode expansions for the $X^\pm j$ worldsheet coordinates are given by:

$$X^\pm j(\sigma, \tau) = x^\pm j + 2\alpha' p^\pm j \tau + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}}' \frac{1}{n} a_n^\pm j \exp(-in\tau) \cos(n\sigma), \quad (2.8)$$

where $\sum'$ excludes the $n = 0$ contribution. For the open string $\alpha_0^\pm j = \sqrt{2\alpha'} p^\pm j$ and $(\alpha_n^\mu)^* = \alpha_n^\mu$. For the worldsheet fermions, the mode expansions are:

$$\begin{align*}
\psi^{\pm j}(\sigma, \tau) &= \sqrt{\alpha'} \sum_{n \in \mathbb{Z}} d_n^\pm j \exp[-in(\tau + \sigma)] \\
\tilde{\psi}^{\pm j}(\sigma, \tau) &= \sqrt{\alpha'} \sum_{n \in \mathbb{Z}} d_n^\pm j \exp[-in(\tau - \sigma)],
\end{align*} \quad (2.9)$$

in the $R$ sector and

$$\begin{align*}
\psi^{\pm j}(\sigma, \tau) &= \sqrt{\alpha'} \sum_{r \in \mathbb{Z} + 1/2} b_r^\pm j \exp[-ir(\tau + \sigma)] \\
\tilde{\psi}^{\pm j}(\sigma, \tau) &= -\sqrt{\alpha'} \sum_{r \in \mathbb{Z} + 1/2} b_r^\pm j \exp[-ir(\tau - \sigma)],
\end{align*} \quad (2.10)$$

in the $NS$ sector. After quantization, the mode expansion coefficients are interpreted as the creation and annihilation operators acting on the string Hilbert space, and satisfy:

$$\begin{align*}
[x^{\pm j}, p^{\pm j}] &= -i\delta_{j1}\delta_{11} \\
[x^{\pm j}, p^{\mp j}] &= i\delta_{j2}\delta_{22} \\
[\alpha_n^{\pm j}, \alpha_m^{\mp j}] &= n\delta_{j0}\delta_{n+m,0} \\
\{b_r^{\pm j}, b_s^{\mp j}\} &= \delta_{j0}\delta_{r+s,0} \\
\{d_n^{\pm j}, d_m^{\mp j}\} &= \delta_{j0}\delta_{n+m,0}.
\end{align*} \quad (2.11)$$

All other commutators are zero. The contributions to the total Hamiltonian,

$$H = L_0 = H_0 + H_B + H_{NS-R} \quad (2.12)$$

(where $H_{NS-R} = H_{NS} - H_R$) from the dimensions obeying $NN$ boundary conditions, take the following expressions:

$$H_0(NN) = \alpha' \sum_{j=1,2} p^j p_{-j} \quad (2.13)$$

$$H_B(NN) = N_B(NN) - \frac{1}{12} = \sum_{n=1}^{\infty} (\alpha_{-n}^{-j} \alpha_{n}^{j} + \alpha_{-n}^{-j} \alpha_{n}^{j}) - \frac{1}{12} \quad (j = 2). \quad (2.14)$$

In the above expression, we have considered that each integer modded pair of complex bosons contributes with $-\frac{1}{12}$ towards the zero-point energy. In the light-cone gauge, the physical vibrations are those that are transverse to the worldsheet, in $p - 2$ dimensions ($j \neq 1$). By $N_B(NN)$ we mean the contribution to the total bosonic number operator from the dimensions obeying $NN$ boundary conditions. The Ramond sector contribute:

$$H_R(NN) = N^R_N(NN) + \frac{1}{12} = \sum_{n=1}^{\infty} n(d_{-n}^{-j} d_{n}^{j} + d_{-n}^{-j} d_{n}^{j}) + \frac{1}{12} \quad (j = 2) \quad (2.15)$$

where each integer modded pair of complex fermions contributes with $+\frac{1}{12}$ towards the zero-point energy. In the $NS$ sector:
where each half-integer complex pair of worldsheet fermions contributes with $-\frac{1}{24}$ towards the total zero-point energy. The NS sector Hilbert space is constructed from a non-degenerate ground state $|0\rangle_{NS}$, satisfying:

$$\alpha_r^{\pm j} |0\rangle_{NS} = b_{r}^{\pm j} |0\rangle_{NS} = 0 \quad (r \in Z + 1/2, \ r > 0)$$

and contributes with $-\frac{1}{4}$ to the zero-point energy. On the other hand, the R ground states $|s_j\rangle_R$ are massless but degenerate, forming a spacetime spinor:

$$|s_j\rangle_R = |\pm \frac{1}{2}\rangle_R.$$  

The contribution from the NS sector to the total mass spectrum [1] $\alpha' M^2$ is given by:

$$-\frac{1}{8} + N_B(NN) + N_F^{NS}(NN)$$

and from the R sector:

$$N_B(NN) + N_F^R(NN).$$

2. $j$-th compact complex dimension with NN boundaries ($j = 3, 4, 5$)

If the string has NN boundary conditions along some of the compact directions, both ends of the open strings are free to move along these directions. Therefore, there are no winding modes associated to these dimensions because the open string can continuously wrap and unwrap the dimensions. On the other hand, the momentum is quantized along those directions because the open string cannot transfer longitudinal momentum to the D-brane:

$$p_{\pm j} = \frac{n_{\pm j}}{R_{\pm j}}$$

with $n_{\pm j}$ being an integer and $R_{\pm j} = \sum_{i=4}^{9} R_i e^{z_j} \cdot \tilde{a}_i$. When considering toroidal compactification for open strings, the mode expansions for the worldsheet coordinates are:

$$X^{\pm j}(\sigma, \tau) = x^{\pm j} + 2\alpha' \tau p_{\pm j} + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \sqrt{\frac{1}{n}} \alpha_n^{\pm j} \exp(-in\tau) \cos(n\sigma).$$

The worldsheet fermions have exactly the same mode expansions as those in the noncompact complex dimensions (2.9), (2.10). Moreover, both cases share the same commutation relations for the oscillator ladder operators (2.11). The NS $|0\rangle_{NS}$ and R $|s_j\rangle_R$ ground states are also defined by (2.17) and (2.18). Therefore, both cases contribute in the same way to the total mass spectra (2.19), (2.20), bearing in mind that now the momentum is quantized. However, in the current case, the operators $x_{\pm j}$ commute with everything, including themselves. This property is used to construct the correct Hilbert space. Once we include the orbifold projection operator into our theory, the twisting $\theta^k$ changes the boundary conditions with respect to the toroidal compactification. Now, the momentum modes vanish in the presence of the orbifold action unless it acts trivially ($\theta^k = 1$). The twisted sector is trapped at the fixed point and does not feel the shape of the compact dimension.

3. $j$-th compact complex dimension with DD boundaries ($j = 3, 4, 5$)

The brane configuration consists of two Dp-branes (or Dp-Dq branes) with $p < 9$ (or $p, q < 9$), both being perpendicular to the $j$-th complex plane. Strings with DD boundary conditions stretch between the D-branes and in the toroidal compactification they are compatible with the existence of a winding number $\omega$, since they are attached to the D-brane and cannot unwrap from these compact dimensions. Denoting the distance between the branes in this complex direction as $Y^{\pm j}$, we have $X^{\pm j}(\pi, \tau) - X^{\pm j}(0, \tau) = Y^{\pm j} + 2\pi \omega^{\pm j} R^{\pm j}$ (where $\omega^{\pm j} \in \mathbb{Z}$ are the string winding
numbers along these directions) and the mode expansions for an open string winding $\omega^\pm j$ times around each of the $X^{\pm j}$ compact dimensions (assuming without loss of generality that one of the branes contains the origin) read:

\[
X^{\pm j}(\sigma, \tau) = (Y^{\pm j} + 2\pi\omega^{\pm j}R^{\pm j})\sigma/i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \frac{1}{n}a_n^{\pm j}\exp(-in\tau)\sin(n\sigma).
\] (2.23)

As before, once the orbifold projection operator is introduced into our theory, the boundary conditions differ from those of the toroidal compactification. Only for the planes in which $\theta^k$ acts trivially are winding modes allowed in the compact directions. The mode expansions for the worldsheet fermions are:

\[
\begin{align*}
\psi^{\pm j}(\sigma, \tau) &= \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} d_n^{\pm j}\exp[-in(\tau + \sigma)] \\
\tilde{\psi}^{\pm j}(\sigma, \tau) &= -\sqrt{\alpha'}\sum_{n\in\mathbb{Z}} d_n^{\mp j}\exp[-in(\tau - \sigma)],
\end{align*}
\] (2.24)

in the R sector and

\[
\begin{align*}
\psi^{\pm j}(\sigma, \tau) &= \sqrt{\alpha'}\sum_{r \in Z+1/2} b_r^{\pm j}\exp[-ir(\tau + \sigma)] \\
\tilde{\psi}^{\pm j}(\sigma, \tau) &= -\sqrt{\alpha'}\sum_{r \in Z+1/2} b_r^{\mp j}\exp[-ir(\tau - \sigma)],
\end{align*}
\] (2.25)

in the NS sector. The commutation relations (2.11) and the worldsheet vacuum states (2.17) (2.18) are defined in the same way as before. Following the same analysis as before, the different contributions to the Hamiltonian read:

\[
H_0(DD) = \alpha'\sum_j \left(\frac{2\pi\omega^{+j}R^{+j} + Y^{+j}}{2\pi\alpha'}\right) \left(\frac{2\pi\omega^{-j}R^{-j} + Y^{-j}}{2\pi\alpha'}\right),
\] (2.26)

where the term proportional to the square of the distance between the branes is due to the stretching energy of the string.

\[
H_B(DD) = \sum_j \left(N_B(DD) - \frac{1}{12}\right) = \sum_j \left(\sum_{n=1}^{\infty} (\alpha_n^{-j}\alpha_n^{+j} + \alpha_n^{+j}\alpha_n^{-j}) - \frac{1}{12}\right).
\] (2.27)

In the light-cone gauge, the physical vibrations are those that are transverse to the world sheet, in $p - 2$ dimensions ($j \neq 1$).

\[
H_R(DD) = \sum_j \left(N_R(DD) + \frac{1}{12}\right) = \sum_j \left(\sum_{n=1}^{\infty} n(d_n^{-j}d_n^{+j} + d_n^{+j}d_n^{-j}) + \frac{1}{12}\right)
\] (2.28)

\[
H_{NS}(DD) = \sum_j \left(N_{NS}^R(DD) - \frac{1}{24}\right) = \sum_j \left(\sum_{r=1/2}^{\infty} r(b_r^{-j}b_r^{+j} + b_r^{+j}b_r^{-j}) - \frac{1}{24}\right)
\] (2.29)

The contributions to the total mass spectrum $\alpha'M^2$ from each of the complex dimensions are:

\[
\left(\frac{2\pi\omega^{+j}R^{+j} + Y^{+j}}{2\pi\alpha'}\right) \left(\frac{2\pi\omega^{-j}R^{-j} + Y^{-j}}{2\pi\alpha'}\right) - \frac{1}{8} + N_B(DD) + N_{NS}^R(DD),
\] (2.30)

in the NS sector and:

\[
\left(\frac{2\pi\omega^{+j}R^{+j} + Y^{+j}}{2\pi\alpha'}\right) \left(\frac{2\pi\omega^{-j}R^{-j} + Y^{-j}}{2\pi\alpha'}\right) + N_B(DD) + N_{NS}^R(DD),
\] (2.31)

in the R sector.
The worldsheet fermionic partners obey the following mode expansions:

\[ \psi^{\pm j} (\sigma, \tau) = \sqrt{\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{r} \alpha_r^{\pm j} \exp(-ir\tau) \sin(r\sigma). \]  

(2.32)

The worldsheet bosonic degrees of freedom are compatible with neither the presence of quantized momenta nor winding numbers:

\[ \alpha^{\pm j} \]  

(2.33)

\[ \psi^{\pm j} (\sigma, \tau) = \sqrt{\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{r} \alpha_r^{\pm j} \exp[-ir(\tau + \sigma)] \]

\[ \psi^{\pm j} (\sigma, \tau) = -\sqrt{\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{r} \alpha_r^{\pm j} \exp[-ir(\tau - \sigma)]. \]

in the \( R \) sector and

\[ \psi^{\pm j} (\sigma, \tau) = \sqrt{\alpha} \sum_{n \in \mathbb{Z}} b_n^{\pm j} \exp[-in(\tau + \sigma)] \]

\[ \psi^{\pm j} (\sigma, \tau) = -\sqrt{\alpha} \sum_{n \in \mathbb{Z}} b_n^{\pm j} \exp[-in(\tau - \sigma)]. \]

(2.34)

in the \( NS \) sector of the theory. The ladder operators satisfy the following non-trivial commutation relations:

\[ [\alpha_r^{\pm j}, \alpha_s^{\mp \pm}] = \delta^d r \delta_{r+s,0} \]

\[ \{d_r^{\pm j}, d_s^{\mp \pm}\} = \delta^d r \delta_{r+s} \]

\[ \{b_n^{\pm j}, b_m^{\mp \pm}\} = \delta^d r \delta_{n+m,0}. \]

The various contributions to the total Hamiltonian read:

\[ H_0(ND) = 0, \]

(2.35)

\[ H_B(ND) = \sum_j \left( N_B(ND) + \frac{1}{24} \right) = \sum_j \left( \sum_{r=1/2}^{\infty} (\alpha_{r-j}^{-1} \alpha_{r+j} + \alpha_{r-j} \alpha_{r+j}) + \frac{1}{24} \right), \]

(2.36)

\[ H_R(ND) = \sum_j \left( N_R^B(ND) - \frac{1}{24} \right) = \sum_j \left( \sum_{r=1/2}^{\infty} r(d_{r-j}^{-1} d_{r+j} + d_{r-j} d_{r+j}) - \frac{1}{24} \right), \]

(2.37)

\[ H_N^S(ND) = \sum_j \left( N_N^F(ND) + \frac{1}{12} \right) = \sum_j \left( \sum_{n=1}^{\infty} r(b_{n-j}^{-1} b_{n+j} + b_{n-j} b_{n+j}) + \frac{1}{12} \right). \]

(2.38)

The \( NS \) ground state becomes massive (with zero-point energy \( \frac{1}{8} \)) and degenerate. It is therefore described by a spinor in this sub-target-space:

\[ |s^j_N S \rangle \equiv | \pm \frac{1}{2} \rangle_{NS}. \]

(2.39)

The \( R \) ground state (in \( j \)-th complex plane), \( |0\rangle_R \), remains massless but is non-degenerate under DN (ND) boundaries. It obeys,

\[ \alpha_r^{\pm j} |0\rangle_R = b_r^{\pm j} |0\rangle_R = 0 \quad (r \in \mathbb{Z} + 1/2, r > 0). \]

(2.40)

The total contribution to the mass spectrum \( \alpha'M^2 \) from each of the complex dimensions is:

\[ \frac{1}{8} + N_B(ND) + N_N^S(ND), \]

(2.41)

in the \( NS \) sector and:

\[ N_B(ND) + N_N^R(ND), \]

(2.42)

in the \( R \) sector.
B. Partition Functions

In the orbifold theory under consideration we only need to include the cylinder amplitudes $C$ in order to compute the tadpole cancellation conditions. The general expression for the cylinder amplitudes is given by:

$$C_{pq} = \frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{2t} Tr_{pq}[q^H(1 + (-1)^F)\theta^k]$$

(2.43)

where $q = e^{-2\pi t}$ and $t$ is the cylinder modulus, the proper time in the open string channel. The coefficient $\frac{1}{2N}$ comes from the GSO and $\mathbb{Z}_N$ orbifold projection operators. The subscript "pq" means that the amplitude is evaluated in the 1-loop open string picture in which the open string has one end on a D$p$-brane and the other on a D$q$-brane. For convenience, we rewrite the amplitude as

$$C_{pq} = \frac{1}{4N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} Z_{pq}$$

(2.44)

where the trace $Z_{pq}$ is referred to as the Partition function:

$$Z_{pq} = Tr_{pq}[(1 + (-1)^F)\theta^k e^{-2\pi tH}].$$

(2.45)

The GSO projection operator remains the same for the amplitude of interaction between two anti-branes but should be modified into $\frac{1 - (-1)^F}{2}$ when the interaction is between a D$p$-brane and an anti-D$q$-brane (D$q$). This results from the fact that the interaction between two D-branes and a brane-antibrane pair have the same sign for the NSNS sector but opposite sign for the RR sector [39] [41]. In the light-cone gauge, if an open string obeys $2n$ ($n \leq 4$) DD and NN boundary conditions, it will obey $8 - 2n$ mixed (ND-DN) boundary conditions. According to this, we assumed the following formal expressions for the fermion number operators in the NS:

$$(-1)^F = (-1)^{1 + \sum_{j=2}^{n} \sum_{r>0} (b^+_r b_{r,j} + b^-_r b_{r,-j}) + \sum_{j=n+1}^{n} \sum_{r>0} (b^+_r b_{n,j} + b^-_r b_{n,-j})} \prod_{k=n+1}^{5} \left( b^+_0 b^+_0 - b^-_0 b^-_0 \right)$$

(2.46)

and R sectors,

$$(-1)^F = (-1)^{\sum_{j=2}^{n} \sum_{r>0} (d^+_r d_{r,j} + d^-_r d_{r,-j}) + \sum_{j=n+1}^{n} \sum_{r>0} (d^+_r d_{n,j} + d^-_r d_{n,-j})} \prod_{k=2}^{n} \left( d^+_0 d^+_0 - d^-_0 d^-_0 \right)$$

(2.47)

respectively. According to the above definitions, both operators obey the right anticommutation relations and satisfy unitarity and $(-1)^{2F} = 1$. Furthermore, the NS vacuum is odd under $(-1)^F$ when the number of complex dimensions equals $n = 2$ and in the R sector it allows us to construct Weyl spinors in even-dimensional space-times. The corresponding GSO projection operators $P^{GSO}_F = \frac{1 + (-1)^F}{2}$ are hermitian, as expected.

C. Open string Hilbert space

The full open string Hilbert space can be constructed as the direct product of the sub-Hilbert spaces corresponding to each of the complex degrees of freedom. In the light-cone gauge, we can eliminate the oscillator variables for the $j = 1$ complex plane, reducing to four ($j = 2, 3, 4, 5$) the number of complex degrees of freedom. Let us take as examples for the construction of the full Hilbert space the 99 and 953 sectors, where by 53 we denote a D5-brane embedding the $j = 3$ complex plane in addition to the usual three dimensional non-compact space.

The 99 sector consists of open strings stretched between two, not necessarily different, D9-branes. All complex planes obey NN boundary conditions and as a consequence the NS sector of each of the complex planes is half-integer moded (2.10) and its full ground state (2.17) is tachyonic, has a total mass (2.19) of $\alpha' M^2 = -1/2$ and is non-degenerate (2.17). It obeys:

$$\alpha^{kj}_n |0\rangle_{NS} = b^{kj}_r |0\rangle_{NS} = 0 \quad (j = 2, \ldots, 5; n \in \mathbb{Z}, n > 0 \text{ and } r \in \mathbb{Z} + 1/2, r > 0),$$

(2.48)

with fermion number:
\[ \exp(i\pi F)\langle 0 \rangle_{NS} = -\langle 0 \rangle_{NS} \] (2.49)

where the minus sign corresponds to the contribution from the ghost ground state. This tachyonic state can be removed from the spectrum via the following GSO projection:

\[ \exp(i\pi F)|\text{physical}\rangle_{NS} = |\text{physical}\rangle_{NS}. \] (2.50)

In the R sector, each of the complex planes is integer-moded (2.9) and its ground state massless (2.20) and degenerate (2.18):

\[ |\bar{s}\rangle_R \equiv |s_2, s_3, s_4, s_5\rangle_R \quad (s_a = \pm \frac{1}{2}; a = 2, 3, 4, 5). \] (2.51)

The GSO projection in the Ramond sector can be implemented either by

\[ \sum_a s_a = 0 \pmod{2} \] (2.52)

or

\[ \sum_a s_a = 1 \pmod{2}. \] (2.53)

The relevant Hilbert space is then constructed by acting with the bosonic and fermionic oscillator creation operators on the NS and R ground states. When considering the 99 sector, the sign flip in the GSO projection does not project out the NS sector ground state and tachyonic excitations with \( \alpha'M^2 = -1/2 \) remain in the spectrum, whereas the would-be massless states are projected out. The \( 953 \) sector consists of open strings stretched between one D9-brane and one D53-brane. The complex planes \( j = 2, 3 \) are subject to NN boundary conditions. The rest \( (j = 4, 5) \) obey mixed boundary conditions. As before, the total Hilbert space is constructed by acting with the oscillator creation operators on the NS (2.34), (2.39):

\[ |\bar{s}\rangle \equiv |s_4, s_5\rangle_{NS} \quad (s_a = \pm \frac{1}{2}; a = 4, 5) \] (2.54)

and R (2.9), (2.18):

\[ |\bar{s}\rangle_R \equiv |s_2, s_3\rangle_R \quad (s_a = \pm \frac{1}{2}; a = 2, 3) \] (2.55)

massless ground states. For the \( 953 \) or \( 953 \) sectors, the NS ground states are also massless. The fact that the endpoints of the open strings are distinguishable makes it natural for them to carry extra degrees of freedom in addition to the fields propagating in the bulk. It is allowed by all the symmetries of the theory to add at each endpoint of the string a new but non-dynamical quantum degree of freedom, known as Chan-Paton degrees of freedom. These new non-dynamical degrees of freedom have a major effect on the space-time physics despite obeying trivial worldsheet dynamics. In consistent string theories, these quantum numbers are actually gauge quantum numbers. We may label the open string states by \( (\lambda^{M}_{pq})_{ab} |\Psi, ab > \), where \( \Psi \) refers to the worldsheet degrees of freedom, \( (p, q) \) to the type of brane the string endpoints are attached to \( (pq \) sector) and \( (a, b) \) are the Chan-Paton indices labelling the particular branes of the stack of Dp or Dq branes respectively. The superindex \( M \) varies depending upon the matter being considered: gauge bosons, fermions or matter scalars.

Massive string states have masses of the order of \( M_{String} \), usually far heavier than all the particles of the Standard Model. Thus, only massless string states are interesting from the phenomenological point of view, acquiring small masses through symmetry breaking effects. Massless open string states arise from open strings with zero length (coincident D-branes), otherwise, there would be a contribution to the mass term coming from the tension of the stretched string. If our theory contains \( n \) coincident Dp-branes, each endpoint of the string can be in one of \( n \) states. The set of \( n^2 \) \( (\lambda_{pq}) \) Hermitian matrices form a complete set of states for the two endpoints. They are known as the Chan-Paton matrices (or wavefunctions) and they are generators of \( U(n) \), describing the gauge interactions. The theory of placing D-branes on top of orbifold singularities is obtained by keeping the states invariant under the combined geometrical and Chan-Paton orbifold action. The geometrical orbifold action acts on the worldsheet degrees of freedom while the Chan-Paton action acts on the Chan-Paton degrees of freedom. In general:

\[ \theta^k(\lambda^{M}_{pq})_{ab} |\Psi, ab > = (\gamma_{k,p})_{aa'}(\lambda^{M}_{pq})_{a'b'} |\theta^k\Psi, a'b' > \equiv (\gamma_{k,q})^{1}_{b'b} = \exp(2\pi icM)(\gamma_{k,p})_{aa'}(\lambda^{M}_{pq})_{a'b'} |\Psi, a'b' > \equiv (\gamma_{k,q})^{1}_{b'b} \] (2.56)
where $c_M$ depends on the type of matter we are considering. The projection for the $\lambda^M$ matrices reads:

$$\lambda^M = \exp(2\pi ic_M)(\gamma_{k,p})\lambda^M(\gamma_{k,q}^{-1}).$$  \hfill (2.57)

The gamma matrices $\gamma_{k,p} = \gamma_{k^q,p}$ ($k = 0, 1, \ldots, N - 1$), represent the embedding of the $\mathbb{Z}_N$ orbifold point group actions on the Chan-Paton degrees of freedom. These matrices should form a unitary, not necessarily irreducible, representation of the orbifold group $\mathbb{Z}_N$. Without any loss of generality, they can be defined as:

$$\gamma_{1,p} = \text{diag} \left( I_{n_0^{(p)}}, \alpha I_{n_1^{(p)}}, \ldots, \alpha^j I_{n_j^{(p)}}, \ldots, \alpha^{N-1}I_{n_{N-1}^{(p)}} \right)$$  \hfill (2.58)

where $\alpha = e^{\frac{2\pi i}{N}}$, $I_{n_i}$ is the $n_i \times n_i$ identity matrix and $\sum_i n_i = n$, the total number of D$p$-branes. Similarly, for the D$p$-branes,

$$\gamma_{1,p} = \text{diag} \left( I_{m_0^{(r)}}, \alpha I_{m_1^{(r)}}, \ldots, \alpha^j I_{m_j^{(r)}}, \ldots, \alpha^{N-1}I_{m_{N-1}^{(r)}} \right).$$  \hfill (2.59)

Gauge bosons in consistent interacting theories must always transform in the adjoint representation of the gauge group. For $U(n)$ gauge theories, if the endpoints of the string run over the $n$ and $\bar{n}$ representations of $U(n)$, this is automatically satisfied. Massless gauge bosons correspond to open string states in the NS sector, of the form $\lambda^G_{pq}b^{0,1/2}0, pq >$, with $\mu$ running along the usual non-compact space time coordinates. Their projection (2.57) is then given by:

$$\lambda^G = (\gamma_{k,p})\lambda^G(\gamma_{k,q}^{-1}).$$  \hfill (2.60)

Using the expression for the $\gamma_{1,p}$ matrices (2.58), we get:

$$(\lambda^G_{pq})_{ab} = (\gamma_{k,p})_{aa'}(\lambda^G_{pq})_{a'b'}(\gamma_{k,q}^{-1})_{b'b} = \exp\left(\frac{2\pi ia}{N}\right)\exp\left(-\frac{2\pi ib}{N}\right)(\lambda^G_{pq})_{ab},$$  \hfill (2.61)

where we have used that the $\gamma$ matrices are diagonal. The above projection (2.61) is only satisfied if $a = b$, breaking the original $U(n)$ and $U(m)$ gauge groups into:

$$\bigotimes_{j=0}^{N-1} U(n_j) \text{ and } \bigotimes_{j=0}^{N-1} U(m_j)$$  \hfill (2.62)

respectively. Fermions are described by open string states in the R sector, of the form $\lambda[s_j]_R$, with $s_j = \pm 1/2$, the weights of a spinor representation of SO(8). Before the orbifold projection, we have a $\mathcal{N} = 4$ supersymmetric $U(n)$ gauge theory with four adjoint fermions transforming in the $4$ of $SU(4)$. The orbifold projection is then given by:

$$\lambda^F = \exp(2\pi i a_j \cdot s_j)(\gamma_{k,p})\lambda^F(\gamma_{k,q}^{-1})$$  \hfill (2.63)

where $a_j = a_2, a_3, a_4, a_5$ defines the orbifold action on the fermions with $a_2 + a_3 + a_4 + a_5 = 0 \mod N$. Using (2.58):

$$\lambda^F = \exp(2\pi i (a_j \cdot s_j + \frac{a - b}{N}))\lambda^F$$  \hfill (2.64)

and chiral fermions transform in bifundamental representations $(n_j, \bar{u}_{j+Nn_j}s_j)$. When $a_2 = 0$, the $\mathbb{Z}_N$ orbifold action belongs to $SU(3)$, preserving the supersymmetry of the closed string sector. Massless complex scalars in space-time belong to the NS sector of the theory and are obtained from the states $\lambda^S\Psi^j_{-1/2}0, pq >$, where $j = 3, 4, 5$ labels the three orbifold complex planes. Their projection can be deduced from the orbifold action on the fermions:

$$\lambda^S = \exp(2\pi iv_j \cdot s_j)(\gamma_{k,p})\lambda^S(\gamma_{k,q}^{-1})$$  \hfill (2.65)

where $v_j = (v_3, v_4, v_5)$ is the orbifold twist vector with $v_3 = a_4 + a_5, v_4 = a_3 + a_5, v_5 = a_3 + a_4$. Supersymmetry imposes $v_3 + v_4 + v_5 = 0 \mod N$, so $a_j = -v_j$. For the massless spectrum of open strings stretched between two antibranes or one brane and one antibrane, we obtain analogous results. Tachyons are scalar states in the NS sector of the form $\lambda^t_{pq}0, pq >$ and obey the following projection:

$$\lambda^t = (\gamma_{k,p})\lambda^t(\gamma_{k,q}^{-1}).$$  \hfill (2.66)

In general, the matter content is as shown in the following table:
The corresponding antiparticles transform in the complex conjugate representation. If the \( pq \) sector has a tachyonic ground state, a suitable GSO projection will project this state out of the NS sector and we should only consider the massless scalar fields for this sector. On the other hand, the GSO projection for the \( pq \) sector will have a sign flip which will project out the massless scalars but will leave the tachyons. If the \( pq \) NS ground state is massless, the corresponding massless scalars of the \( pq \) will also be present in the spectrum.

### III. ORBIFOLD MODELS WITH D9 AND D5 BRANES

#### A. Gauge group and fermion content

In this section we discuss the massless matter content and the consistency conditions for a system in the presence of a number \( n \) of D9-branes, a number \( u^{(i)} \) of D5\(_i\)-branes, a number \( m \) of D9-branes and a number \( w^{(i)} \) of D5\(_i\) branes, within the bottom-up approach. D9-branes embed the full 10-dimensional space-time and therefore the boundary conditions are NN in all directions. However, D5\(_i\)-branes wrap around the usual four-dimensional non-compact space-time and only one of the compact complex planes \((k = 3, 4, 5)\). In this sense, there are three possible types of D5\(_i\)-branes. Between any two D-branes, there is an open string, with boundary conditions summarized as follows \((i \neq l \neq m \neq i)\):

| String sector | \( j = 2 \) | \( j = 1 \) | \( j = 1 \) | \( j = m \) |
|---------------|------------|------------|------------|------------|
| 99            | NN         | NN         | NN         | NN         |
| 95\(_i\)      | NN         | NN         | ND         | NN         |
| 5\(_i\)       | NN         | NN         | DN         | DN         |
| 5\(_i\)\(_i\) | NN         | NN         | DD         | DD         |
| 5\(_i\)\(_i\) | NN         | ND         | DN         | DD         |

A necessary condition for supersymmetry is that there is an equal number of bosonic and fermionic states transforming under the same representation at each mass level. In the presence of antibranes, the system generally has broken supersymmetry. We assume a general embedding for the action of the \( \mathbb{Z}_N \) orbifold point group on the Chan-Paton degrees of freedom:

\[
\begin{align*}
\gamma_{1,9} &= \text{diag} \left( I_{n_0}, \alpha I_{n_1}, \ldots, \alpha^j I_{n_j}, \ldots, \alpha^{N-1} I_{n_{N-1}} \right) \\
\gamma_{1,5r} &= \text{diag} \left( I_{w_0^{(r)}}, \alpha I_{w_1^{(r)}}, \ldots, \alpha^j I_{w_j^{(r)}}, \ldots, \alpha^{N-1} I_{w_{N-1}^{(r)}} \right) \\
\gamma_{1,9} &= \text{diag} \left( I_{m_0}, \alpha I_{m_1}, \ldots, \alpha^j I_{m_j}, \ldots, \alpha^{N-1} I_{m_{N-1}} \right) \\
\gamma_{1,5r} &= \text{diag} \left( I_{w_0^{(r)}}, \alpha I_{w_1^{(r)}}, \ldots, \alpha^j I_{w_j^{(r)}}, \ldots, \alpha^{N-1} I_{w_{N-1}^{(r)}} \right)
\end{align*}
\] (3.1)
with \( \alpha = e^{\frac{2\pi i}{99}} \). For the time being, the non-negative integers \( n_j, u_j^{(r)}, m_j \) and \( w_j^{(r)} \) are kept arbitrary. The NS ground state is tachyonic. After imposing the GSO projection, the two massless gauge bosons and six complex scalars survive the projection. The R sector contains eight fermionic states \( |s_2s_3s_4s_5\rangle_R \), four of which are left handed \( (s_2 = -\frac{1}{2}) \) (2.51). In general, unbroken supersymmetry requires the number of ND complex dimensions to be a multiple of two [20]. Before the orbifold projection and choosing (2.52) as the GSO projection, the left-handed space-time fermions are:

\[
|\psi_1\rangle = \left| -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right>, \\
|\psi_2\rangle = \left| -\frac{1}{2}, -\frac{1}{2}, 1, \frac{1}{2} \right>, \\
|\psi_3\rangle = \left| -\frac{1}{2}, 1, -\frac{1}{2}, \frac{1}{2} \right>, \\
|\psi_4\rangle = \left| -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2} \right>. 
\]

Under the orbifold projection, it follows that

\[
\theta^k|s_2s_3s_4s_5\rangle = \exp[2\pi i k(a_3s_3 + a_4s_4 + a_5s_5)]|s_2s_3s_4s_5\rangle.
\]

Explicitly,

\[
\theta^k|\psi_1\rangle = |\psi_1\rangle, \\
\theta^k|\psi_2\rangle = e^{-2\pi i ka_3}|\psi_2\rangle, \\
\theta^k|\psi_3\rangle = e^{-2\pi i ka_4}|\psi_3\rangle, \\
\theta^k|\psi_4\rangle = e^{-2\pi i ka_5}|\psi_4\rangle
\]

since \( a_3 + a_4 + a_5 = 0 \). Thus, it follows from (3.3) and (2.63) that the left-handed chiral fermion spectrum in the 99 sector is

\[
\sum_{j=0}^{N-1} [n_j, n_{\bar{j}}] + \sum_{r=3}^{5} (n_j, n_{\bar{j} + Nv_r}], 
\]

where the subindices are understood modulo \( N \) and \( n_j = -v_j \) when \( a_2 = 0 \). The corresponding antiparticles, the right-handed fermions, transform in the complex conjugate representation \( \sum_{j=0}^{N-1} [(\bar{n}_j, n_j) + \sum_{r=3}^{5} (\bar{n}_j - Nv_r, n_j)] \). The projections for the open strings in the 99 sector are completely analogous to those in the 99 sector. The 99 sector has an opposite GSO projection and tachyonic states survive in the NS sector. With a similar analysis, the fermion content in the 5f51 sector reads,

\[
\sum_{j=0}^{N-1} [(u_j^{(i)}, \bar{u}_j^{(i)}) + \sum_{r=3}^{5} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)], 
\]

For the 5f51 sectors \( (i \neq l) \), the R fermionic states are of the form \( |s_2s_m\rangle \) before the orbifold projection, where \( m = 3, 4, 5 \) as long as \( m \neq i \neq l \neq m \). We choose \( s_m = -s_2 \) in order to implement the GSO projection (2.52) in the same way as before. The possible left-handed states are then \( |s_2s_m\rangle = \left| -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right> \). Under the orbifold projection,

\[
\theta^k|s_2s_m\rangle = e^{2\pi i k s_m} |s_2s_m\rangle = e^{2\pi i k - \frac{s_m}{2}} |s_2s_m\rangle. 
\]

This equation leads to the following R states,

\[
\sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(l)} \frac{s_m}{2}). 
\]
Table 1: Spectrum in the 95 configuration

Now we are in a position to calculate the contribution $A_{n_{j}}$ from the massless fermions in the world-volume of the D9 brane to the $SU(n_{j})$ gauge anomaly (for simplicity, we will exclude the contribution from the brane-antibrane sectors):

$$A_{n_{j}} = \sum_{r=3}^{5} (n_{j+N_{v_{r}}} - n_{j-N_{v_{r}}}) + \sum_{i=3}^{5} (u_{j-N_{v_{r}}/2}^{(i)} - u_{j+N_{v_{r}}/2}^{(i)})$$

$$= \sum_{r=3}^{5} [(n_{j+N_{v_{r}}} - n_{j-N_{v_{r}}}) - (u_{j+N_{v_{r}}/2}^{(r)} - u_{j-N_{v_{r}}/2}^{(r)})]. \quad (3.8)$$

Using that [4]:

$$n_{j} = \frac{1}{N} \sum_{k=0}^{N-1} \exp(-2\pi ikj/N) Tr \gamma_{k,9}$$

$$u_{j}^{(r)} = \frac{1}{N} \sum_{k=0}^{N-1} \exp(-2\pi ikj/N) Tr \gamma_{k,5_{r}} \quad (3.9)$$

and the mathematical identity

$$\sum_{r=3}^{5} \sin(2\pi kv_{r}) = -4 \prod_{r=3}^{5} \sin(\pi kv_{r}) \quad (3.10)$$

we can rewrite (3.8) as

$$A_{n_{j}} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left( \sum_{r=3}^{5} 2 \sin(\pi kv_{r}) Tr \gamma_{k,9} + \sum_{r=3}^{5} 2 \sin(\pi kv_{r}) Tr \gamma_{k,5_{r}} \right). \quad (3.11)$$
Similarly, the contribution from the chiral matter present at a particular D5\(_i\) brane to the non-abelian \(SU(u_j^{(i)})\) gauge anomaly is:

\[
A_{u_j^{(i)}} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left\{ \prod_{r=3}^{5} 2\sin(\pi kv_r) \right\} Tr\gamma_{k,5_i} + \sum_{l,m \neq i} (2\sin(\pi kv_m) Tr\gamma_{k,5_l} + 2\sin(\pi kv_l) Tr\gamma_{k,5_l}). \tag{3.12}
\]

**B. Partition functions**

**1. 99 sector**

In this section, we will provide a stringy analysis for obtaining the tadpole cancellation conditions of the orbifold models under consideration. First, we need to compute the one-loop cylinder amplitudes of all possible open string sectors of the theory. A general string state is the product of three pieces: a zero mode part (2.13), a part constructed using worldsheet bosonic oscillators (2.14) and a part using (NS or R) worldsheet fermionic oscillators (2.15), (2.16). This allows us to factorize the trace in (2.43) so that:

\[
C_{99} = \frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{2t} \left[ Tr^{(H_0)}(\theta^k q^H_0) Tr^{(B)}(\theta^k q^H_B) \left( Tr^{GSO(\text{NS})}(\theta^k q^{H\text{NS}}) - Tr^{GSO(\text{R})}(\theta^k q^H_R) \right) \right] \\
= \frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{2t} \left( z^{(H_0)}_{99} z^{(B)}_{99} z^{(F)}_{99} \right), \tag{3.13}
\]

where \(q = e^{-2\pi t}\). The minus sign in the \(\text{R}\) sector is due to the space-time statistics and it confirms that the \(\text{R}\) sector leads to space-time fermions while the \(\text{NS}\) sector contains space-time bosons. Quantum states will be labelled by the eigenvalues of the corresponding Hamiltonian operator and computing the trace over all the possible quantum states means integrating over all that is continuous and summing over all that is discrete.

**Zero modes**

The trace over the zero mode contributions factorises into a product of contributions from each space-time dimension. Since in the 99 sector we have NN boundary conditions in all directions, the contribution from the zero modes to the partition function is given by:

i) The trace of \(p^2\) in each of the longitudinal (\(\mu = 0, 1, 2, 3\)) non-compact directions (\(j = 1, 2\) complex dimensions):

\[
\int_{-\infty}^{\infty} \frac{dp_\mu}{2\pi} q^{\alpha^\prime p_\mu} = iV_\mu (8\pi^2 t \alpha')^{-1/2}, \tag{3.14}
\]

for each value of \(\mu\) since \(\theta^k\) acts trivially on these non-compact coordinates. We used \(<p|p'| >= 2\pi \delta(p-p')\) as our state normalization and \(V_\mu\) is the (infinite) volume of the coordinate \(X_\mu\). The \(i\) factor comes from the Wick rotation in the integration of the \(\mu = 0\) component.

ii) When \(\theta^k\) acts trivially on a compactified complex coordinate \(z^j\) obeying NN boundary conditions, there exists quantized momenta in the worldsheet mode expansion (2.22). Thus when \(kv_j = \text{integer}\), we should also consider a sum over the quantized momenta along the compact complex direction \(z^j\) obeying NN boundary conditions. For each of the complex dimensions, this is given by:

\[
(\sum_{n=\infty}^{n=-\infty} q^{\alpha'(\pi^j t)})^2 \rightarrow \frac{V_j}{8\pi^2 \alpha' t} \text{ as } t \rightarrow 0 \tag{3.15}
\]

since it is in the \(t \rightarrow 0\) limit of the open string amplitude where we find the contribution to the tadpoles. We have defined \(V_j = (2\pi R_j)^2\), where \(2\pi R_j\) is the periodicity of the \(z^j\) complex plane. This limit can be easily calculated using Poisson resummation formula \(\sum_{n=\infty}^{n=-\infty} e^{-\pi an^2} = a^{-1/2} \sum_{m=\infty}^{m=-\infty} e^{-\pi m^2/a}\).
iii) In each of the (complex) compactified directions $z^j$ ($j = 3, 4, 5$) in which $\theta^k$ acts non-trivially, the momentum $p_j$ is zero. If any of the complex planes of the compact space ($j = 3, 4, 5$) satisfy NN boundary conditions, the value of the field is free to fluctuate in those directions contributing to the trace as follows:

$$Tr[\theta^k] = \int dz^r <z^r|\theta^k|z^r> = \int dz^r \delta((1 - e^{2\pi i k v_r})z^r) = (2\sin\pi k v_r)^{-2},$$  \hspace{1cm} (3.16)

where we have used $<z^r|z^r> = \delta(z^r - z^r)$ and that $\int dz\delta(az) = \frac{1}{|a|}$.

For the 99 sector (obeying NN boundary conditions in all directions), this contribution gives:

$$Tr[\theta^k] = \int dz^3 dz^4 dz^5 <z^3 z^4 z^5|\theta^k|z^3 z^4 z^5> = \prod_{r=3}^{5} (2\sin\pi k v_r)^{-2}. \hspace{1cm} (3.17)$$

iv) The contribution from the trace of the orbifold projection operator on the Chan-Paton degrees of freedom ($Tr\gamma_{k, 9}$), where $k$ denotes the twisted sector and 9 labels the D9-brane sector. In the 99 sector we would get a contribution from each of the branes giving ($Tr\gamma_{k, 9}$)($Tr\gamma_{k, 9}^{-1}$).

**Bosonic partition function**

We compute the trace over the bosonic oscillator states in the basis of the operators $\alpha_n$ and $\alpha_n$. In the light-cone gauge we get:

$$Z^{(B)}_{99} = [q^{-1/12} \prod_{n=1}^{\infty} (1 - q^n)^{-2}] \cdot \prod_{r=3}^{5} q^{-1/12} \times \prod_{n=1}^{\infty} \frac{1}{(1 - q^n e^{2\pi i k v_r})(1 - q^n e^{-2\pi i k v_r})}. \hspace{1cm} (3.18)$$

The first term in brackets represents the contribution from the non-compact dimensions. In particular, the two real ($j = 2$ complex plane) physical dimensions of the light-cone gauge. The rest is the contribution from the three compact complex planes in which the orbifold action takes place. Each integer-modded complex pair of bosons contributes with $-\frac{1}{12}$ towards the zero point energy. Making use of the expressions given in Appendix A, the total bosonic contribution to the 99 sector can be rewritten in terms of the theta functions as follows:

$$Z^{(B)}_{99} = \eta(t) \cdot \prod_{r=3}^{5} \frac{(-2\sin\pi k v_r)}{\left(\frac{2}{\frac{2}{2} + k v_r}\right)(t)}. \hspace{1cm} (3.19)$$

**Fermionic partition function**

We need to consider the contributions from both, the NS and the R sectors of the theory:

$$Z^{(F)}_{99} = Z^{(NS)}_{99} - Z^{(R)}_{99}. \hspace{1cm} (3.20)$$

In the 99 sector, the NS fermionic oscillators are half-integer modded (2.10). The contribution from this sector is given by:

$$Z^{(NS)}_{99} = q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 \cdot \prod_{r=3}^{5} q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{n-1/2} e^{2\pi i k v_r})(1 + q^{n-1/2} e^{-2\pi i k v_r}) \hspace{1cm} (3.21)$$

The two terms correspond to the trace computations with and without the insertion of the $(-1)^F$ GSO operator, respectively. Since the oscillators are fermionic, their square is zero and no more terms can appear. Each half-integer modded worldsheet fermion complex pair contributes with $-\frac{1}{12}$ towards the zero-point energy. The R worldsheet fermions are integer modded (2.9), their contribution to the partition function being:

$$Z^{(R)}_{99} = 2q^{1/12} \prod_{n=1}^{\infty} (1 + q^n)^2 \cdot \prod_{r=3}^{5} q^{1/12} (2\cos\pi k v_r) \prod_{n=1}^{\infty} (1 + q^n e^{2\pi i k v_r})(1 + q^{n-1} e^{-2\pi i k v_r}). \hspace{1cm} (3.22)$$
Each integer modded pair contributes with $\frac{1}{2}$ towards the zero-point energy. The $Tr[(-1)^F]$ vanishes in the $R$ sector as the expansion is integer-moded. Using the formula (A.5) of the appendix, the total contribution from the worldsheet fermions can be rewritten as:

$$Z^{(F)}_{99} = \eta^{-1}(t) \sum_{a,b=0,1/2} \eta_{ab} \frac{a/b}{(t)} \prod_{r=3}^{5} \vartheta \left[ a/b, b + kv_r \right](t),$$

where $\eta_{ab} = (-1)^{2(a+b+2ab)}$.

General expression

Combining all contributions, the final expression for the partition function in the $99$ sector $Z_{99} = Z^{H_0} \cdot Z^{H_0}_{66} \cdot Z^{E}_{55}$ reads:

$$Z_{99} = iV_4(8\pi^2\alpha' t)^{-2}(Tr\gamma_{k,9})(Tr\gamma_{k,9}^{-1}) \prod_{r=3}^{5} (2 \sin \pi kv_r)^{-2} \sum_{a,b=0,1/2} \eta_{ab} \frac{a/b}{\eta(t)} \prod_{r=3}^{5} (-2 \sin \pi kv_r) \vartheta \left[ a/b, b + kv_r \right](t),$$

where a sum over quantized momenta should also be included if $kv_i = integer$. In the one-loop open string picture in which we computed the partition function, the $NS$ open string sector corresponds to taking $a = 0$ ($b = 0, 1/2$) and the $R$ open string sector corresponds to taking $a = 1/2$ and $b = 0$ (since it vanishes for $b = 1/2$) [1]. In the dual picture (see section C), the contribution to the $NSNS$ closed string tadpole divergences is contained in the $Z\left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right](t)$ piece. On the other hand, the $RR$ tadpole divergences are contained within the $Z\left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right](t)$ component of the total partition function. By virtue of the Riemann identities satisfied by the $\vartheta$ functions, the total partition function vanishes when the model is supersymmetric. This result is not expected in other models where supersymmetry is broken. Take as an example a brane-antibrane pair, which breaks all supersymmetries. In this case, the contribution to the $RR$ closed string amplitude picks up a minus sign and as a consequence the total partition function no longer vanishes.

2. $55$ sectors

We denote by $5_1$ a D5-brane that wraps around the four dimensional non-compact space-time ($j = 1, 2$) and one of the compact $j = 3, 4, 5$ complex planes. Thus, there are DD boundary conditions in the $l$-th and $m$-th directions transverse to the $5_1$ branes. Oscillator mode expansions with DD boundary conditions have integer modes but include windings instead of momenta. A general $5_15_1$ brane system satisfies DD boundary conditions in the $m$-th direction perpendicular to both D5-branes and mixed DN(ND) boundary conditions in the other two complex directions if $i \neq l$. If $i = j$, the system would obey NN boundary conditions in the $i = l$ direction and DD boundary conditions in the remaining compact dimensions. As before, the non-compact dimensions satisfy NN boundary conditions.

Zero modes

The contribution from the zero modes to the partition function of a system of $5_15_1$ branes is given by:

$$Z_{5_15_1}^{(H_0)} = iV_4(8\pi^2\alpha' t)^{-2}(2 \sin \pi kv_i)^{-2}(Tr\gamma_{k,5_1})(Tr\gamma_{k,5_1}^{-1}).$$

By comparing this expression with the bosonic partition function of the $99$ system (3.19), we observe that the difference arises from the computation of the trace of the orbifold operator in the complex planes of the compact space. Now, only one of the complex directions, in contrast to the three of the $99$ case, obeys NN boundary conditions. We should also consider a sum over quantized momenta (windings) if $kv_i$ ($kv_l, kv_m$) are integers. For the sum over windings:

$$\sum_{w=\pm \infty} q^{\frac{i}{2}(wR_i)^2} \rightarrow \frac{2\pi^2\alpha'}{V_i t} \quad \text{as} \quad t \to 0.$$
The term that depends on the distance \( Y^2 \) between the D5\(_i\) branes (2.26) is not relevant in the tree channel infrared limit. For the 5\(_i\)5\(_j\) \((i \neq j)\), we get:

\[
Z_{5_i5_j}^{(H_0)} = iV_\alpha(8\pi^2\alpha' t)^{-2}(Tr\gamma_{k,5_i})(Tr\gamma_{k,5_j})^{-1}
\]  

(3.27)

where a sum over windings along the \( z^m \) complex plane should be also included if \( kv_m = \text{integer} \). The contribution from the trace of the orbifold operator \( \theta^k \) is absent because none of the compact dimensions satisfy NN boundary conditions.

**Bosonic partition function**

The contribution from the bosonic oscillator states to the partition function of a system of 5\(_i\)5\(_j\) branes is the same as in the 99 case because the bosonic mode expansions are integer-modded for both NN or DD boundary conditions. For the 5\(_i\)5\(_j\) \((i \neq j)\), the bosonic contribution is:

\[
Z_{5_i5_j}^{(B)} = [q^{-1/12} \prod_{n=1}^{\infty} (1 - q^n)^{-2} ] \cdot [q^{-1/12} \prod_{n=1}^{\infty} (1 - q^n e^{-2\pi ik(v_i + v_j)})^{-1} (1 - q^n e^{2\pi ik(v_i + v_j)})^{-1}]
\]  

(3.28)

We can distinguish the contribution from the compact complex plane with DD boundary conditions \((m)\) with integer-modded expansions (2.23) and the contribution from the compact complex planes with DN(DN) boundary conditions \((i, l \neq m)\) with half-integer modded expansions (2.32). The bosonic contribution coming from the non-compact dimensions remains the same. Each complex pair of half-integer modded bosons contributes with \(-\frac{1}{2\pi i}\) towards the zero-point energy and each complex pair of integer modded bosons contributes with \(-\frac{1}{12}\) to the zero-point energy. In terms of the theta functions:

\[
Z_{5_i5_j}^{(B)} = \frac{\eta(t)}{2\sin \pi k v_i (2\sin \pi k v_i)} \times \prod_{r=3}^{5} \left( -2\sin \pi k v_r \right) \cdot \theta\left[ \frac{1}{2} + k v_i \right] (t) \cdot \theta\left[ \frac{1}{2} + k v_j \right] (t).
\]  

(3.29)

**Fermionic partition function**

The fermionic contribution in the 5\(_i\)5\(_j\) sector is the same as the fermionic contribution in the 99 sector. Although the D-brane configurations are different and obey different boundary conditions, the fact that the fermionic oscillator expansions are integer-modded for the R sector and half-integer modded for the NS sector irrespective of whether the boundary conditions are NN or DD, makes both fermionic partition functions look the same. For the 5\(_i\)5\(_j\) sector, the fermionic partition function can be written as:

\[
Z_{5_i5_j}^{(F)} = [q^{-1/48} \prod_{n=1}^{\infty} (1 + q^n)^{-1/2}]^2 \cdot [q^{-1/24} \prod_{n=1}^{\infty} (1 + q^n e^{2\pi i k(v_i + v_j)}) (1 + q^n e^{-2\pi i k(v_i + v_j)})]
\]  

\[
\cdot \prod_{i,j} [q^{1/12} e^{i \pi k v_i} \prod_{n=1}^{\infty} (1 + q^n e^{2\pi i k v_i}) (1 + q^n e^{-2\pi i k v_i})] - [q^{-1/48} \prod_{n=1}^{\infty} (1 + q^n)^{-1/2}]^2
\]  

\[
\cdot [q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i k(v_i + v_j)}) (1 - q^n e^{-2\pi i k(v_i + v_j)})] \cdot \prod_{i,l} [q^{1/12} e^{i \pi k v_i} \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i k v_i}) (1 - q^n e^{-2\pi i k v_i})]
\]  

\[
- [q^{1/12} \prod_{n=1}^{\infty} (1 + q^n)]^2 \cdot [q^{1/12} e^{-i \pi k(v_i + v_j)} \prod_{n=1}^{\infty} (1 + q^n e^{2\pi i k(v_i + v_j)}) (1 + q^n e^{-2\pi i k v_m})]
\]  

\[
\cdot \prod_{i,l} [q^{-1/24} \prod_{n=1}^{\infty} (1 + q^n e^{-2\pi i k v_i}) (1 + q^n e^{2\pi i k v_i})].
\]  

(3.30)

The first two contributions come from the NS fermionic sector and the last one from the R sector respectively. The oscillator expansions for the NS sector are half-integer modded in the \( m \)-th direction obeying DD boundary conditions.
(2.25) but integer-modded in the other two complex directions $i, l$ obeying mixed DN(ND) boundary conditions (2.34). The $R$ sector has integer-modded expansions in the $m$-th direction (2.24) but half-integer expansions in the remaining two (2.33). Each integer-modded complex fermion contributes with $\frac{1}{12r}$ and each half-integer modded complex fermion with $-\frac{1}{24r}$ to the zero-point energy respectively. In terms of theta functions:

$$Z_{\xi_i, \xi_i}^{(F)} = \eta^{-4}(t) \sum_{a, b = 0, 1/2} \eta_{ab} \theta \left[ \frac{a}{b} \right] (t) \cdot \frac{\theta \left[ \frac{1}{2} - a \right] (t)}{\theta \left[ \frac{a}{b + kv_l} \right] (t)} \cdot \frac{\theta \left[ \frac{1}{2} - a \right] (t)}{\theta \left[ \frac{a}{b + kv_r} \right] (t)} \cdot \prod_{r = 3}^5 \theta \left[ \frac{a}{b + kv_r} \right] (t).$$  \hspace{1cm} (3.31)

**General expression**

Combining all the contributions, the general expression for the $5_i5_i$ brane system is given by:

$$Z_{5,5_i} = iV_4(8\pi^2 \alpha' t)^{-2}(Tr \gamma_{k,5_i})(Tr \gamma_{l,5_i}^{-1}) \cdot (2 \sin \pi kv_i)^{-2} \cdot \sum_{a, b = 0, 1/2} \eta_{ab} \eta^3(t) \frac{\theta \left[ \frac{a}{b} \right] (t)}{\theta \left[ \frac{1}{2} + kv_r \right] (t)} \cdot \prod_{r = 3}^5 \theta \left[ \frac{a}{b + kv_r} \right] (t).$$ \hspace{1cm} (3.32)

For the $5_i5_i$ brane system, the expression for the partition function reads:

$$Z_{5,5_i} = iV_4(8\pi^2 \alpha' t)^{-2}(Tr \gamma_{k,5_i})(Tr \gamma_{l,5_i}^{-1}) \cdot \sum_{a, b = 0, 1/2} \eta_{ab} \eta^3(t) \frac{\theta \left[ \frac{a}{b} \right] (t)}{\theta \left[ \frac{1}{2} + kv_r \right] (t)} \cdot \prod_{r = 3}^5 \theta \left[ \frac{a}{b + kv_r} \right] (t).$$  \hspace{1cm} (3.33)

### 3. $95_i$ and $5_i9$ sectors ($k = 3, 4, 5$)

A $95_i$ brane system obeys DN(ND) boundary conditions in the $l$-th and $m$-th complex coordinates perpendicular to the $5_i$ D-branes and NN boundary conditions in the $i$-th complex direction. As before, $5_i$ branes wrap around the non-compact dimensions and the $i$-th complex plane.

**Zero modes**

The contribution from the zero modes reads:

$$Z_{95_i}^{(H_0)} = iV_4(8\pi^2 \alpha' t)^{-2}(Tr \gamma_{k,9})(Tr \gamma_{l,5_i}^{-1})(2 \sin \pi kv_i)^{-2},$$ \hspace{1cm} (3.34)

where a sum over quantized momenta should be also included if $kv_i$ is integer.

**Bosonic partition function**

The contribution from the bosonic states to the partition function of a system of $95_i$ branes is given by:

$$Z_{95_i}^{(B)} = \left[ q^{-1/12} \prod_{n = 1}^\infty (1 - q^n)^{-2} \right] \cdot \left[ q^{-1/12} \prod_{n = 1}^\infty (1 - q^n e^{2\pi ikv_i})^{-1} (1 - q^n e^{-2\pi ikv_i})^{-1} \right] \cdot \prod_{l, m}^{1/24} \prod_{n = 1}^\infty \left[ q^{-n/2} e^{2\pi ikv_i} (1 - q^{n-1/2} e^{-2\pi ikv_i})^{-1} \right].$$ \hspace{1cm} (3.35)

The first term in brackets is the contribution from the integer-modded bosonic oscillators in the non-compact dimensions. The second bracket comes from the integer-modded bosonic oscillators in the $i$-th complex compact plane.
obeying NN boundary conditions. The last bracket corresponds to the half-integer modded bosonic oscillators in the 
$t$-th and $m$-th complex directions. Each half-integer modded complex worldsheet boson contributes with $\frac{1}{2\pi}$ towards 
the zero-point energy. In terms of the theta functions:

$$Z_{95_i}^{(B)} = \eta(t) \times \prod_{l,m} \frac{1}{\vartheta \left( \frac{1}{2} + kv_i \right)(t)} \cdot \vartheta \left( \frac{1}{2} - a \right)(t) \cdot \vartheta \left( b + kv_i \right)(t). \quad (3.36)$$

**Fermionic partition function**

The fermionic partition function looks like:

$$Z_{95_i}^{(F)} = \eta^{-4}(t) \sum_{a,b=0,1/2} \eta_{ab} \vartheta \left( \frac{a}{b} \right)(t) \cdot \vartheta \left( \frac{a}{b} + kv_i \right)(t) \cdot \prod_{r=3}^{5} \vartheta \left( \frac{1}{2} - a \right)(t) . \quad (3.37)$$

**General expression**

Combining all the contributions, the full expression for the partition function is given by the expression:

$$Z_{95_i}(\theta^k) = i V_4 (8\pi^2 \alpha')^{-2} (T_{\gamma,9,9}^r) (T_{\gamma,9,5_i}^{-1}) (2 \sin \pi kv_i)^{-2} \sum_{a,b=0,1/2} \vartheta \left( \frac{a}{b} \right)(t) \cdot \vartheta \left( \frac{a}{b} + kv_i \right)(t) \cdot \prod_{r=3}^{5} \vartheta \left( \frac{1}{2} - a \right)(t) . \quad (3.38)$$

The partition function in the 5\text{5i} sector is obtained by exchanging the roles played by the $\gamma_{9,9}$ and $\gamma_{9,5_i}$ gamma functions in the above expression. For the rest of the D-brane systems that we considered, we give the expressions for the partition functions at the end of the paper in appendix B. The method followed for their derivation is the same as the one used so far.

**C. Twisted RR Tadpole Cancellation Conditions**

A RR massless tadpole corresponds to the divergent part of the RR closed string vacuum amplitude. Conformal 
invariance of string theory allows a tree-level closed string amplitude to be pictured alternatively as a one-loop open 
string amplitude. In particular, the cylinder amplitude corresponds to the contribution of open strings to the one-
loop vacuum amplitude or equivalently to the tree-level closed string amplitude where the closed strings propagate 
between two D-branes. Each description reverses the roles of the worldsheet space and time and is more appropiate in 
a different limit of the moduli space. The absence of a modular group for the cylinder worldsheets makes the $t \to 0$ 
cylinder limit quite different from the $\infty$ limit. In the $t \to 0$ limit of the open string amplitude, the radius of the 
circle is very large. In order for the string mode to travel that distance, it must be light and the open string is in the 
IR limit. In the closed string channel, this process can be seen as a short distance effect. On the other hand, 
in the $t \to 0$ limit, the open string is in the UV limit. Now, the string modes do not need to travel long distances in making the loop. However, this is the long distance IR limit of the closed string and the source for the massless 
tadpole. This limit can be easily computed by using the modular transformation properties of the theta functions. In 
the previous section we calculated the cylinder amplitudes in the one-loop open string picture. In this section we will 
factorize the divergent contribution to the RR tadpoles in the tree-channel approximation. The final step requires 
the factorization of the divergences into a sum of perfect squares. It is then that we obtain a very strong tadpole 
cancellation condition for each value $k$ of the twisted sector. Using the Jacobi identities (A.6), (A.7) satisfied by the 
$\vartheta$ functions, we can rewrite the partition functions as:

$$Z_{99} = (1 - 1)i V_4 (8\pi^2 \alpha')^{-2} (T_{\gamma,9,9}^r) (T_{\gamma,9,5_i}^{-1}) (2 \sin \pi kv_v)^{-2} \cdot \vartheta \left( \frac{0}{\frac{1}{2}} \right)(t) \cdot \vartheta \left( \frac{0}{\frac{1}{2} + kv_v} \right)(t) \cdot \prod_{r=3}^{5} \vartheta \left( \frac{1}{2} - a \right)(t) . \quad (3.39)$$
\[ Z_{5,5_{i}} = (1 - 1)iV_{4}(8\pi^2\alpha'^{2})^{-2}(Tr\gamma_{k,5_{i}})(Tr\gamma_{k,5_{i}}^{-1})[2\sin(\pi kv_{l})][2\sin(\pi kv_{l})]^{-2} \cdot \frac{\vartheta \left( \frac{0}{\eta^{2}} \right)}{\eta^{2}(t)} \cdot \prod_{r=3}^{5} \frac{\vartheta \left( \frac{0}{\eta^{2}} + kv_{r} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \] 

(3.40)

\[ Z_{5,5_{i}} = (1 - 1)iV_{4}(8\pi^2\alpha'^{2})^{-2}(Tr\gamma_{k,5_{i}})(Tr\gamma_{k,5_{i}}^{-1}) \cdot \frac{\vartheta \left( \frac{0}{\eta^{2}} \right)}{\eta^{2}(t)} \cdot \prod_{r=3}^{5} \frac{\vartheta \left( \frac{0}{\eta^{2}} + kv_{r} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \] 

(3.41)

\[ Z_{9,9} = (1 - 1)iV_{4}(8\pi^2\alpha'^{2})^{-2}(Tr\gamma_{k,9})(Tr\gamma_{k,9}^{-1}) \cdot \frac{\vartheta \left( \frac{0}{\eta^{2}} \right)}{\eta^{2}(t)} \cdot \prod_{r=3}^{5} \frac{\vartheta \left( \frac{0}{\eta^{2}} + kv_{r} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \cdot \prod_{i=1}^{5} \frac{\vartheta \left( \frac{1}{2} + kv_{l} \right)}{\eta^{2}(t)} \] 

(3.42)

The second term in the (1-1) prefactor corresponds to taking \( a = 0 \) and \( b = \frac{1}{2} \) in the original expressions for the partition functions and represents the contribution from the cylinder amplitudes to the RR couplings in the tree-level closed string picture. The tadpole divergences at \( \frac{1}{2} \to \infty \) in the closed string channel are easily evaluated by taking the \( t \to 0 \) limit in the one-loop cylinder amplitudes of the open string channel. Using the formulas (A.8)-(A.11) given in appendix A, we get as \( t \to 0 \), the following contributions to the RR tadpole divergences:

\[ Z_{99}^{(RR)} \approx -iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{2}{t} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \cdot (Tr\gamma_{k,9})(Tr\gamma_{k,9}^{-1}) \] 

(3.43)

\[ Z_{59}^{(RR)} \approx -iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{2}{t} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \cdot (Tr\gamma_{k,5_{i}})(Tr\gamma_{k,5_{i}}^{-1}) \] 

(3.44)

\[ Z_{59}^{(RR)} \approx -iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{2}{t} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \cdot (Tr\gamma_{k,5_{i}}^{-1})(Tr\gamma_{k,5_{i}}^{-1}) \] 

(3.45)

\[ Z_{99}^{(RR)} \approx -iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{2}{t} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \cdot (Tr\gamma_{k,9})(Tr\gamma_{k,9}^{-1}) \] 

(3.46)

The expression for the contribution to the cylinder amplitude \( C = \sum_{pq} C_{pq} \) in the asymptotic limit of \( t \to 0 \) reads:

\[ C \approx (1 - 1)[-iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{1}{2N} \int_{0}^{\infty} dt \cdot \sum_{k=0}^{N-1} \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \]

\[ \cdot \left( Tr\gamma_{k,9} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]} \right) \cdot \left( Tr\gamma_{k,5_{i}}^{-1} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]} \right) \] 

(3.47)

Using that the \( \gamma_{k,p} \) matrices are both diagonal and unitary,

\[ \gamma_{k,p}^{-1} = \gamma_{k,p}^{*} \quad Tr\gamma_{k,p}^{-1} = Tr\gamma_{k,p}^{*} \] 

(3.48)

(3.47) can alternatively be written as,

\[ C \approx (1 - 1)[-iV_{4}(8\pi^2\alpha'^{2})^{-2} \cdot \frac{1}{2N} \int_{0}^{\infty} dt \cdot \sum_{k=0}^{N-1} \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]}^{-1} \cdot \left( Tr\gamma_{k,9} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]} \right) \cdot \left( Tr\gamma_{k,5_{i}}^{-1} \cdot \prod_{r=3}^{5} \frac{[2\sin(\pi kv_{r})]}{[2\sin(\pi kv_{r})]} \right) \] 

(3.49)

and the RR twisted tadpole cancellation conditions for a system in the presence of different sets of D9-branes and D5-branes are then:
\[
\frac{\text{Tr}\gamma_{k,9}}{\prod_{j=3}^{5}[2\sin(\pi k v_j)]} + \sum_{i=3}^{5} \frac{\text{Tr}\gamma_{k,5_i}}{2\sin(\pi k v_i)} = 0 \quad (k = 1, 2, \cdots, N - 1)
\] (3.50)

and they are required to be satisfied at each fixed point of the six dimensional compact space. If instead of only having Dp-branes, we also allowed the presence of D\(\bar{p}\)-branes, the RR tadpole cancellation conditions should be modified by replacing:

\[
\text{Tr}\gamma_{k,p} \rightarrow \text{Tr}\gamma_{k,\bar{p}}
\] (3.51)

since the brane-antibrane pairs have opposite RR charges. Let us now analyze the relationship between tadpole and anomaly cancellation conditions. For simplicity we assume that our model contains only one type of D5\(i\) branes and that we have compactified on a \(\mathbb{Z}_3\) orbifold. In this scenario, equation (3.50) reads:

\[
\frac{\text{Tr}\gamma_{k,9}}{\prod_{j=3}^{5}[2\sin(\pi k v_j)]} + \frac{\text{Tr}\gamma_{k,5_i}}{2\sin(\pi k v_i)} = 0 \quad (k = 1, 2, \cdots, N - 1)
\] (3.52)

with twisted vector \(v = \frac{1}{3}(1, 1, -2)\). Comparing (3.52) with (3.12), we see that tadpole cancellation guarantees the absence of SU(\(u_{j}^{i}\)) gauge anomalies. On the other hand, to calculate the total contribution to the gauge anomalies from all the chiral matter in the D9 brane, we should include in addition to the chiral matter from the 99 sectors, the remaining chiral matter arising from all the possible 95\(i\) sectors. Since we are considering D5\(i\) branes parallel to the \(z_{i}\) complex plane and we have nine different fixed points with space transverse to the D5\(i\) branes at which we can place them (remember that the \(\mathbb{Z}_3\) orbifold has a total of 27 fixed points), we get:

\[
A_{n_j} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} 2\sin(\pi k v_j) \left\{ \prod_{l,m \neq i} 2\sin(\pi k v_l) [\text{Tr}\gamma_{k,9} + 9\text{Tr}\gamma_{k,5_i}] \right\}
\] (3.53)

which cancels out for the \(\mathbb{Z}_3\) orbifold as a result of the tadpole cancellation. Note that since the tadpole condition (3.52) must be satisfied at all nine fixed points, it ensures that \(\text{Tr}\gamma_{k,5_i}\) is the same at all of them and therefore the factor of nine in the above expression. Thus for a system of D9 and D5\(i\) branes, tadpole cancellation ensures anomaly cancellation (but not vice-versa).

IV. ORBITFOLD MODELS WITH VARIOUS D-BRANES

A. D3-D7 brane orbifold

We now consider a system with a number \(n\) of D3-branes, \(u^{(i)}\) number of D7\(i\)-branes, \(m\) number of D3\(\bar{3}\)-branes and \(w^{(i)}\) number of D7\(\bar{i}\) branes. D3-branes embed the 4-dimensional non-compact Minkowski space-time and sit at some fixed points of the remaining 6-dimensional internal space, while D7-branes occupy an 8-dimensional subspace of the full 10-dimensional spacetime. They wrap the 4-dimensional non-compact spacetime plus two of the complex planes. By D7\(i\) we denote a D7-brane transverse to the \(i\)-th complex plane. As a T-dual version of the D9D5 brane orbifold already discussed, the orbifold containing D3D7 branes also preserves spacetime supersymmetry. We assume the same general embedding for the action of the \(\mathbb{Z}_N\) orbifold point group on the Chan-Paton degrees of freedom (3.1). The spectrum that arises is completely analogous to that in the 95 sector:
Gauge bosons which automatically guarantees the absence of the D3-brane gauge group $SU(n_j)$ anomalies. To verify that (4.3) guarantees the absence of the D7$_i$-brane $SU(u_j^{(i)})$ gauge anomalies, we have to realize that to calculate the contribution to the anomalies from the chiral matter in a particular D7$_i$ brane (for simplicity, let’s assume that we only have one type of D7$_i$ brane), we should not only consider the contribution from the chiral fermions arising in the $\bar{7_i}$ but also

| Sector | Gauge bosons | Tachyonic scalar fields | Massless scalar fields | Fermion (s = - 1/2) |
|--------|--------------|------------------------|-----------------------|---------------------|
| 33     | $N_j=0$      | $U(n_j)$               | Total $N_j$           | $\sum_{j=1}^{5}$   |
| 33     | $j=0$        | $U(m_j)$               | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,7$_i$| $N_j=0$      | $U(u_j^{(i)})$         | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,7$_i$| $j=0$        | $U(w_j^{(i)})$         | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,7$_i$| $N_j=0$      | $U(\bar{u}_j^{(i)})$  | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,7$_i$| $j=0$        | $U(\bar{w}_j^{(i)})$  | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,3   | $N_j=0$      | $U(\bar{u}_j^{(i)})$  | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |
| 7,3   | $j=0$        | $U(\bar{w}_j^{(i)})$  | $\sum_{j=0}^{N_j-1}$  | $\sum_{r=3}^{5}$   |

Table 2: Spectrum in the 37 configuration

and their contribution towards the $SU(n_j)$ and $SU(u_j^{(i)})$ anomalies in the world-volumes of D3 and D7$_i$ branes respectively:

$$A_{n_j} = \frac{2i}{N} \sum_{k=0}^{N_j-1} e^{-2\pi i k j/N} \left( \prod_{r=3}^{5} 2 \sin(\pi k v_r) Tr \gamma_{k,3} + \sum_{r=3}^{5} [2 \sin(\pi k v_r) Tr \gamma_{k,3}] \right)$$  \hspace{1cm} (4.1)

and

$$A_{u_j^{(i)}} = \frac{2i}{N} \sum_{k=0}^{N_j-1} e^{-2\pi i k j/N} \left( \prod_{r=3}^{5} 2 \sin(\pi k v_r) Tr \gamma_{k,3} + \sum_{l,m \neq i} [2 \sin(\pi k v_m) Tr \gamma_{k,3}] + 2 \sin(\pi k v_l) Tr \gamma_{k,3} \right).$$  \hspace{1cm} (4.2)

By using (C.5)-(C.8), the RR twisted tadpole cancellation conditions for a system in the presence of D3 and D7$_i$ branes reads:

$$\prod_{r=3}^{5} 2 \sin(\pi k v_r) Tr \gamma_{k,3} + \sum_{i=3}^{5} [2 \sin(\pi k v_l) Tr \gamma_{k,3}] = 0$$  \hspace{1cm} (4.3)

which automatically guarantees the absence of the D3-brane gauge group $SU(n_j)$ anomalies. To verify that (4.3) guarantees the absence of the D7$_i$-brane $SU(u_j^{(i)})$ gauge anomalies, we have to realize that to calculate the contribution to the anomalies from the chiral matter in a particular D7$_i$ brane (for simplicity, let’s assume that we only have one type of D7$_i$ brane), we should not only consider the contribution from the chiral fermions arising in the $\bar{7_i}$ but also
all the chiral matter arising from all possible 37 sectors, since the D7_i brane embeds all the D3 branes present at the fixed points in the l-th and m-th (l, m ≠ i) complex planes (which are nine for the particular case of the Z_3 orbifold). It is easy to check that this is the case. So again tadpole cancellation ensures non-abelian gauge anomaly cancellation.

B. D3-D9 brane orbifold

We now consider a system with a number n of D3-branes, a number u of D9-branes, a number m of D3-branes and a number w of D9 branes. The spectrum is shown in the following table:

| Sector | Gauge bosons | Tachyons | Massless scalar fields | Fermion_1 | Fermion_2 |
|--------|--------------|----------|------------------------|-----------|-----------|
| 33     | N-1 \sum_{i=0}^{u} U(n_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + N \nu_r) | \sum_{r=3}^{N-1} (m_j, \bar{m}_j + N \nu_r) | c. c. | c. c. |
| 33     | N-1 \sum_{i=0}^{u} U(n_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (m_j, \bar{m}_j + N \nu_r) | \sum_{r=3}^{N-1} (m_j, \bar{m}_j + N \nu_r) | c. c. | c. c. |
| 33     | N-1 \sum_{i=0}^{u} U(n_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (m_j, \bar{m}_j + N \nu_r) | \sum_{r=3}^{N-1} (m_j, \bar{m}_j + N \nu_r) | c. c. | c. c. |
| 99     | N-1 \sum_{i=0}^{u} U(u_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (u_j, \bar{u}_j + N \nu_r) | \sum_{r=3}^{N-1} (w_j, \bar{w}_j + N \nu_r) | c. c. | c. c. |
| 99     | N-1 \sum_{i=0}^{u} U(u_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (w_j, \bar{w}_j + N \nu_r) | \sum_{r=3}^{N-1} (w_j, \bar{w}_j + N \nu_r) | c. c. | c. c. |
| 99     | N-1 \sum_{i=0}^{u} U(u_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (w_j, \bar{w}_j + N \nu_r) | \sum_{r=3}^{N-1} (w_j, \bar{w}_j + N \nu_r) | c. c. | c. c. |
| 93     | N-1 \sum_{i=0}^{u} U(n_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + N \nu_r) | \sum_{r=3}^{N-1} (n_j, \bar{n}_j + N \nu_r) | c. c. | c. c. |
| 93     | N-1 \sum_{i=0}^{u} U(u_j) | \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + N \nu_r) | \sum_{r=3}^{N-1} (n_j, \bar{n}_j + N \nu_r) | c. c. | c. c. |

Table 3: Spectrum in the 93 configuration

Note that this type of brane system has tachyons and breaks supersymmetry. The mixed 39 sectors obey NN boundary conditions in the j = 2 complex plane but mixed DN boundary conditions in the remaining complex planes j = 3, 4, 5. The NS ground state is massive, therefore there are no massless bosons present in these sectors. The R sectors consist of only one degenerate component | K >. Taking the fermion number to be F = \frac{1}{2} + \sum_i s_i, the two possible choices for the GSO projections are the following:

\[
\sum_a s_a = \frac{1}{2} \quad (\text{mod } 2)
\]

or

\[
\sum_a s_a = -\frac{1}{2} \quad (\text{mod } 2).
\]

The contributions from the fermions to the SU(n_j) and SU(u_j) anomalies are:

\[
A_{n_j} = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \{ i \prod_{r=3}^{5} 2 \sin(\pi kv_r) [Tr \gamma_{k,3} + Tr \gamma_{k,9}] \}
\]

and

\[
A_{u_j} = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \{ i \prod_{r=3}^{5} 2 \sin(\pi kv_r) [Tr \gamma_{k,9} + Tr \gamma_{k,3}] \}
\]

respectively. Using (C.1) (C.5) and (C.9), the tadpole condition can be written as:

23
\[
\prod_{r=3}^{5} 2 \sin(\pi k v_r) |Tr \gamma_{k,3} - Tr \gamma_{k,9}| = 0. \tag{4.8}
\]

Clearly, the cancellation of tadpoles does not guarantee the absence of \(SU(n_j)\) or \(SU(u_j)\) anomalies.

### C. D3-D5 brane orbifold

We now consider a system with a number \(n\) of D3-branes, a number \(u^{(i)}\) of D5\(_i\)-branes, a number \(m\) of D3-branes and a number \(w^{(i)}\) of D5\(_i\) branes. The spectrum reads:

| Sector | Gauge bosons | Tachyonic fields | Massless scalar fields | Fermion \(s = -1/2\) |
|--------|--------------|------------------|-----------------------|---------------------|
| 33     | \(\boxtimes_{j=0}^{N-1} U(n_j)\) | \(j=0\) \(U(m_j)\) | \(\sum_{j=0}^{N-1} (n_j, m_j, NV_{2})\) | \(\sum_{j=0}^{N-1} [(n_j, m_j, NV_{2})] \) |
| 33     | \(\boxtimes_{j=0}^{N-1} U(m_j)\) | \(j=0\) \(U(n_j)\) | \(\sum_{j=0}^{N-1} (m_j, n_j, NV_{2})\) | \(\sum_{j=0}^{N-1} [(m_j, n_j, NV_{2})] \) |
| 33     | \(\boxtimes_{j=0}^{N-1} U(u^{(i)})\) | \(j=0\) \(U(w^{(i)})\) | \(\sum_{j=0}^{N-1} (u^{(i)}_j, w^{(i)}_j, NV_{2})\) | \(\sum_{j=0}^{N-1} [(u^{(i)}_j, w^{(i)}_j, NV_{2})] \) |
| 33     | \(\boxtimes_{j=0}^{N-1} U(w^{(i)})\) | \(j=0\) \(U(u^{(i)})\) | \(\sum_{j=0}^{N-1} (w^{(i)}_j, u^{(i)}_j, NV_{2})\) | \(\sum_{j=0}^{N-1} [(w^{(i)}_j, u^{(i)}_j, NV_{2})] \) |

\[
A_{n_j} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi i k/j/N} \prod_{r=3}^{5} 2 \sin(\pi k v_r) |Tr \gamma_{k,3} - Tr \gamma_{k,9}| \tag{4.9}
\]

\[
A_{u^{(i)}} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi i k/j/N} \prod_{r=3}^{5} 2 \sin(\pi k v_r) |Tr \gamma_{k,5_i} - Tr \gamma_{k,9_i}| \tag{4.10}
\]

respectively and the tadpole cancellation conditions:

Table 4: Spectrum in the 35 configuration

The contribution from the fermionic states towards the cubic \(SU(n_j)\) and \(SU(u^{(i)})\) anomalies are

The contribution from the fermionic states towards the cubic \(SU(n_j)\) and \(SU(u^{(i)})\) anomalies are
From the above spectra, we can easily compute the contributions to the cubic $SU(n_j)$ and $SU(u_j^{(i)})$ anomalies:

\[
A_{n_j} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left[ \prod_{r=3}^{5} 2 \sin(\pi kv_r) \right] Tr_{\gamma_{k,9}}
\]

\[
A_{u_j^{(i)}} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left[ \prod_{r=3}^{5} 2 \sin(\pi kv_r) \right] Tr_{\gamma_{k,7i}} + \sum_{l,m \neq i} [2 \sin(\pi kv_m)] Tr_{\gamma_{k,7l}}.
\]

where we have used (C.2) (C.3) (C.5) and (C.13) for its computation. Again, in this system, cancellation of tadpoles does not guarantee absence of gauge anomalies. It too is non-supersymmetric.

### D. D9-D7 brane orbifold

We now consider a system with a number $n$ of D9-branes, a number $u^{(i)}$ of D7$_r$-branes, a number $m$ of D9-branes and a number $w^{(i)}$ of D7$_l$ branes. The (non-supersymmetric) spectrum reads:

| Sector $| Gauge bosons | Tachyonic scalar fields | Massless scalar fields | Fermion $(s = 1/2)$ |
|---|---|---|---|---|
| $99$ | $U(n_j)$ | $\sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ |
| $99$ | $U(m_j)$ | $\sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ |
| $99$ | $U(u_j^{(i)})$ | $\sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ |
| $99$ | $U(w_j^{(i)})$ | $\sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ |
| $99$ | $97$ | $\sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ |
| $99$ | $97$ | $\sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (u_j^{(i)}, \bar{u}_j^{(i)} + Nv_r)$ |
| $97$ | $97$ | $\sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (w_j^{(i)}, \bar{w}_j^{(i)} + Nv_r)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (n_j, \bar{n}_j + Nv_r)$ |
| $7i$ | $7i$ | $\sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ | $\sum_{r=3}^{5} \sum_{j=0}^{N-1} (m_j, \bar{m}_j)$ |

**Table 5: Spectrum in the 97 configuration**

From the above spectra, we can easily compute the contributions to the cubic $SU(n_j)$ and $SU(u_j^{(i)})$ anomalies:

\[
A_{n_j} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left[ \prod_{r=3}^{5} 2 \sin(\pi kv_r) \right] Tr_{\gamma_{k,9}}
\]

\[
A_{u_j^{(i)}} = \frac{2i}{N} \sum_{k=0}^{N-1} e^{-2\pi ikj/N} \left[ \prod_{r=3}^{5} 2 \sin(\pi kv_r) \right] Tr_{\gamma_{k,7i}} + \sum_{l,m \neq i} [2 \sin(\pi kv_m)] Tr_{\gamma_{k,7l}}.
\]
The twisted RR tadpole cancellation condition:

\[ \text{Tr}\gamma_{k,9} - \sum_{i=3}^{5} 2\sin(\pi k v_1)\text{Tr}\gamma_{k,7_i} = 0 \]  \hspace{1cm} (4.14)

does not guarantee the absence of either SU\(n_j\) or SU\(u_j^{(i)}\) anomalies.

E. D5-D7 brane orbifold

We now consider a system with a number \(n^{(i)}\) of D5\(_i\)-branes, a number \(u^{(i)}\) of D7\(_i\)-branes, a number \(m^{(i)}\) of D\(\bar{5}\)\(_i\) branes and a number \(u^{(i)}\) of D\(\bar{7}\)\(_i\) branes. The spectrum reads:
The contribution to the $SU(n_j)$ and $SU(u_j^{(i)})$ cubic anomalies is:

$$A(n_j^{(i)}) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i k j/N} \left\{ \int_0^1 2 \sin(\pi k v_i) Tr \gamma_{k,5} + i \sum_{l,m \neq i} [2 \sin(\pi k v_m)] Tr \gamma_{k,5} + Tr \gamma_{k,7} \right\}$$

(4.15)
respectively. Using (C.2) (C.3) (C.6) (C.11) and (C.12), we obtain the following expression for the tadpole cancellation conditions:

\[
\prod_{r=3}^{5} 2 \sin(\pi k v_r) \sum_{i=3}^{5} \frac{\text{Tr} \gamma_{k,5i}}{2 \sin(\pi k v_i)} - \sum_{i=3}^{5} \frac{2 \sin(\pi k v_i)}{2 \sin(\pi k v_i)} \text{Tr} \gamma_{k,7i} = 0, \quad (4.17)
\]

which does not guarantee the absence of \( SU(n_j^{(i)}) \) or \( SU(j^{(i)}) \) gauge anomalies.

F. The most general D-brane orbifold

In the most general case, tadpole cancellation conditions read:

\[
\prod_{j=3}^{5} 2 \sin(\pi k v_j) \{ \text{Tr} \gamma_{k,3} - \sum_{i=3}^{5} \frac{\text{Tr} \gamma_{k,5i}}{2 \sin(\pi k v_i)} \} - \{ \text{Tr} \gamma_{k,9} - \sum_{i=3}^{5} \frac{2 \sin(\pi k v_i)}{2 \sin(\pi k v_i)} \text{Tr} \gamma_{k,7i} \} = 0. \quad (4.18)
\]

V. CONCLUSIONS

For many years the attempts to construct phenomenologically semi-realistic models from String Theory mostly focussed on supersymmetric models [3] - [11] [41], despite the fact that no supersymmetric particles have ever been observed. The reason for this is that the weakly coupled Heterotic String has a string scale close to the Planck scale and supersymmetry is the only known method of evading the hierarchy problem. The discovery of the D-brane world, in which gauge theories may inhabit a lower dimensionality, with gravitational interaction in the (higher-dimensional) bulk, has led to the construction of new, non-supersymmetric models with intermediate [4] - [6] or even TeV [27] - [32] string scales. The former arises, for example, when D-branes wrap the coordinate planes of an orbifold (or orientifold) compactified space, such as we have considered, and hidden-sector anti-D-branes transmit supersymmetry breaking to the (observable sector) D-branes. The latter arises when intersecting D-branes wrap a toroidally compactified space transverse to an orbifold or orientifold. A particularly attractive scenario for model building using the first technique is the “bottom-up” approach [4] - [6] in which some approximation to the (supersymmetric) Standard Model is constructed using D3-branes at an orbifold fixed point. The cancellation of the twisted RR charge at this point requires the introduction of other D-branes and to preserve supersymmetry these should be D7-branes. However it is of interest to consider, as we have done, alternatives which are non-supersymmetric, especially since the string scale is no longer tied to the Planck scale. One immediate objection might be that non-supersymmetric string theories often but not always, as we have shown, possess (scalar) tachyons. However these too have been rehabilitated in recent years. If they are electroweak \( SU(2) \) doublets, they may be interpreted as Higgs bosons [30]. Also singlet tachyons imply the existence of a scalar potential with possibly interesting cosmological consequences [34] - [40]. At a more technical level, it is widely believed that twisted tadpole cancellation implies the cancellation of non-abelian anomalies in the emergent gauge field theory, but hitherto this has only been verified in supersymmetric theories.

In this paper, we have studied the construction of Type IIB orbifold models in four dimensions in the presence of different types of parallel branes and antibranes. We computed the open string spectrum, its contribution to the gauge anomalies and the twisted RR tadpole cancellation conditions in each case. We obtained several supersymmetric and non-supersymmetric configurations of D-branes. For the supersymmetric systems (37 and 95), we verified that tadpole cancellation conditions guarantees the absence of all non-abelian gauge anomalies [4] [19] [24] [25]. For the non-supersymmetric systems, the presence of tachyonic excitations is a common feature (although for some particular cases tachyons get projected out of the spectrum) and the cancellation of the twisted tadpoles does not guarantee the cancellation of the non-abelian gauge anomalies. As a result, additional constraints coming from the cancellation of the non-abelian anomalies should be imposed in order to obtain a consistent theory. The bottom up construction of standard-like models using non-supersymmetric configurations of D-branes is being studied elsewhere [43].
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APPENDIX A

A. Theta $\vartheta$ functions

In this appendix we write the properties of the theta $\vartheta$ functions used in the computation of the partition functions. The theta $\vartheta$ function with characteristics $a$ and $b$ is given by:

$$\vartheta\left[\begin{array}{l} a \\ b \end{array}\right](t) = \sum_{n} q^{\frac{1}{2}(n+a)^2} e^{2i\pi(n+a)b} \quad (A.1)$$

where the variable $q$ is defined as $q = e^{-2\pi t}$. The Dedekind $\eta$ function is:

$$\eta(t) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (A.2)$$

The modular transformation properties are given by:

$$\vartheta\left[\begin{array}{l} a \\ b \end{array}\right](t) = e^{2i\pi ab} t^{-\frac{1}{2}} \vartheta\left[\begin{array}{l} b \\ a \end{array}\right](1/t) \quad (A.3)$$

$$\eta(t) = t^{-\frac{1}{2}} \eta(1/t). \quad (A.4)$$

A fundamental expression for the ratio of these two functions in product form is given by:

$$\frac{\vartheta\left[\begin{array}{l} a \\ b \end{array}\right]}{\eta(t)} = \left[ e^{2i\pi ab} q^{\frac{1}{2}(a^2-b^2)} \right] \prod_{n=1}^{\infty} (1 + q^{n+a-\frac{1}{2}} e^{2\pi ib})(1 + q^{n-a-\frac{1}{2}} e^{-2\pi ib}). \quad (A.5)$$

The $\vartheta$ functions satisfy several abstruse Riemann identities, e.g.,

$$\sum_{a,b} \eta_{a,b} \vartheta\left[\begin{array}{l} a \\ b \end{array}\right] \prod_{r=3}^{5} \vartheta\left[\begin{array}{l} a \\ b + v_r \end{array}\right] = 0 \quad (A.6)$$

$$\sum_{a,b} \eta_{a,b} \vartheta\left[\begin{array}{l} a \\ b \end{array}\right] \vartheta\left[\begin{array}{l} a \\ b + v_5 \end{array}\right] \prod_{r=3}^{4} \vartheta\left[\begin{array}{l} a + \frac{1}{2} \\ b + v_r \end{array}\right] = 0 \quad (A.7)$$

where $\eta_{a,b} = (-1)^{2(a+b+2ab)}$ and $v_3 + v_4 + v_5 = 0$. It is helpful to analyze the various limits that appear in the calculation of the partition functions and the tadpole cancellation conditions:

$$\lim_{b \to 0} \vartheta\left[\begin{array}{l} \frac{1}{2} \\ b \end{array}\right] = \frac{1}{\eta^{2n}} \quad (A.8)$$

$$\lim_{t \to 0} \vartheta\left[\begin{array}{l} 0 \\ \frac{1}{2} \end{array}\right](t) = 2t \quad (A.9)$$
if the components of the twist vector satisfy the constraint $v_i$ that

$$\theta \left[ \begin{array}{c} a \\ b \\ 0 \end{array} \right]_{l,m} (t) = \left\{ \begin{array}{l} (-1)^{[kv_l]} + 1 \\
(-1)^{-[kv_l]} \
\end{array} \right. \text{ if } v_j > 0 \nonumber$$

$$\theta \left[ \begin{array}{c} a \\ b \\ 0 \end{array} \right]_{l,m} (t) = \left\{ \begin{array}{l} \text{if } v_j < 0 \\
\text{if } v_j > 0 \
\end{array} \right. \text{ if } v_j < 0 \nonumber$$

where $[kv_l]$ denotes the integer part of $kv_l$. It can also be verified for all supersymmetric orbifold groups $Z_N$ listed in [4] that

$$( -1 )^{[kv_3]+[kv_4]+[kv_5]} \prod_{r=3}^{5} ( 2 \sin \pi kv_r ) = - \prod_{r=3}^{5} | 2 \sin \pi kv_r | \nonumber$$

if the components of the twist vector satisfy the constraint $v_3 + v_4 + v_5 = 0$.

**APPENDIX B**

B. Partition functions

The other partition functions relevant for the computation of the tadpole cancellation conditions are:

$$Z_{33}(\theta^k) = i V_4 (8 \pi^2 \alpha' t)^2 (Tr \gamma_{k,3})(Tr \gamma_{k,3}^{-1}) \cdot \sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \\ 0 \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

where we have considered that the D3-brane world-volume embeds the full non-compact space-time. All compact complex dimensions obey DD boundary conditions. Like always, the non-compact dimensions obey NN boundary conditions.

$$Z_{7,7_i}(\theta^k) = i V_4 (8 \pi^2 \alpha' t)^2 (Tr \gamma_{k,7_i})(Tr \gamma_{k,7_i}^{-1}) \cdot \prod_{j=3}^{5} (2 \sin \pi kv_j) \cdot \sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

By D7$_l$ we denote a D7-brane transverse to the $z_l$ complex plane. Therefore, in the 7$_i$7$_i$ system there are NN boundary conditions in the $l$-th and $m$-th complex directions and DD boundary conditions in the $i$-th complex plane.

$$Z_{37_i}(\theta^k) = i V_4 (8 \pi^2 \alpha' t)^2 (Tr \gamma_{k,3})(Tr \gamma_{k,3}^{-1}) \cdot \prod_{j=3}^{5} (2 \sin \pi kv_j) \cdot \sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

The 37$_l$ system obeys DD boundary conditions in the $i$-th complex plane and mixed DN boundary conditions in the other two complex directions.

$$Z_{7,7_i}(\theta^k) = i V_4 (8 \pi^2 \alpha' t)^2 (Tr \gamma_{k,7_i})(Tr \gamma_{k,7_i}^{-1}) \cdot (2 \sin \pi kv_i)(2 \sin \pi kv_i) \cdot \prod_{r=3}^{5} (2 \sin \pi kv_r)^{-1}$$

$$\sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

$$\sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

$$\sum_{a,b=0,1/2} \frac{\theta \left[ \begin{array}{c} a \\ b \end{array} \right]_{l,m} (t)}{\eta^3(t)} \cdot \prod_{r=3}^{5} \frac{(-2 \sin \pi kv_r) \theta \left[ \begin{array}{c} a \\ b + kv_r \end{array} \right]_{l,m} (t)}{\theta \left[ \begin{array}{c} 1/2 \\ 1/2 + kv_r \end{array} \right]_{l,m} (t)} \nonumber$$

30
where \( i \neq l \neq m \neq i \). This brane system obeys mixed ND(ND) boundary conditions in the \( i \)-th and \( l \)-th complex directions respectively and NN boundary conditions in the \( m \)-th complex plane.

\[
Z_{39}(\theta^k) = iV_4(8\pi^2\alpha't)^2(T\gamma_{k,3})(T\gamma_{k,9}^{-1}) \sum_{a,b=0,1/2} \psi_a \psi_b \frac{1}{\eta_i(t)} \cdot \prod_{r=3}^5 \psi_l \frac{1/2-a}{b+kv_r} \frac{1/2-kv_r}{0}.
\] (B.5)

This system obeys DN boundary conditions in all complex directions.

\[
Z_{7i,5i}(\theta^k) = iV_4(8\pi^2\alpha't)^2(T\gamma_{k,7i})(T\gamma_{k,5i}^{-1}) \sum_{a,b=0,1/2} \psi_a \psi_b \frac{1}{\eta_i(t)} \cdot \prod_{r=3}^5 \psi_l \frac{1/2-a}{b+kv_i} \frac{1/2-kv_i}{0} \cdot \prod_{l,m=0,1/2} (-2\sin \pi kv_l) \psi_l \frac{1/2-a}{b+kv_l} \frac{1/2-kv_l}{0}.
\] (B.6)

where the \( 97_i \) system obeys NN boundary conditions in the \( l \)-th and \( m \)-th complex planes and mixed ND boundary conditions in the \( i \)-th complex plane perpendicular to the \( D7_i \) brane.

\[
Z_{7i,5i}(\theta^k) = iV_4(8\pi^2\alpha't)^2(T\gamma_{k,7i})(T\gamma_{k,5i}^{-1}) \sum_{a,b=0,1/2} \psi_a \psi_b \frac{1}{\eta_i(t)} \cdot \prod_{r=3}^5 \psi_l \frac{1/2-a}{b+kv_i} \frac{1/2-kv_i}{0} \cdot \prod_{l,m=0,1/2} (-2\sin \pi kv_l) \psi_l \frac{1/2-a}{b+kv_l} \frac{1/2-kv_l}{0}.
\] (B.7)

This system obeys mixed DN(ND) boundary conditions in all the complex planes.

\[
Z_{7i,5i}(\theta^k) = iV_4(8\pi^2\alpha't)^2(T\gamma_{k,7i})(T\gamma_{k,5i}^{-1}) \sum_{a,b=0,1/2} \psi_a \psi_b \frac{1}{\eta_i(t)} \cdot \prod_{r=3}^5 \psi_l \frac{1/2-a}{b+kv_i} \frac{1/2-kv_i}{0} \cdot \prod_{l,m=0,1/2} (-2\sin \pi kv_l) \psi_l \frac{1/2-a}{b+kv_l} \frac{1/2-kv_l}{0}.
\] (B.8)

The \( 7i5i \) system obeys DD boundary conditions in the \( i \)-th complex plane, NN boundary conditions in the \( l \)-th direction and mixed boundary conditions in the \( m \)-th complex plane perpendicular to both the \( 7_i \) and \( 5_i \) branes.

\[
Z_{35i}(\theta^k) = iV_4(8\pi^2\alpha't)^2(T\gamma_{k,3})(T\gamma_{k,5i}^{-1}) \sum_{a,b=0,1/2} \psi_a \psi_b \frac{1}{\eta_i(t)} \cdot \prod_{r=3}^5 \psi_l \frac{1/2-a}{b+kv_r} \frac{1/2-kv_r}{0} \cdot \prod_{l,m=0,1/2} (-2\sin \pi kv_l) \psi_l \frac{1/2-a}{b+kv_l} \frac{1/2-kv_l}{0}.
\] (B.9)

where the system obeys mixed DN boundary conditions in the \( i \)-th complex plane and DD boundary conditions in the other two complex directions.

### APPENDIX C

C. Asymptotic behaviours

We chose \( v_1 > 0, v_2 > 0 \) but \( v_3 < 0 \) in order to guarantee \( \pm v_1 \pm v_2 \pm v_3 = 0 \). Using expressions (A.9) and (A.10), the asymptotic behaviour of the RR \( Z_{99}(\theta^k) \) amplitudes is given by:

\[
Z_{99}^{RR} \cong -iV_4(8\pi^2\alpha't)^2 \cdot \frac{1}{t} \cdot \prod_{r=3}^5 (2\sin \pi kv_r) \cdot (T\gamma_{k,9})(T\gamma_{k,9}^{-1})(-1)^{|kv_1|+|kv_2|+|kv_3|+1}
\] (C.1)
\[
Z_{5,5,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{\prod_{r=3}^{5} (2 \sin \pi k v_r)}{(2 \sin \pi k v_1)^2} \cdot (Tr \gamma_{k,5,5})(Tr \gamma_{k,5,5}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.2)

\[
Z_{5,5,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{\prod_{r=3}^{5} (2 \sin \pi k v_r)}{(2 \sin \pi k v_1)(2 \sin \pi k v_5)} \cdot (Tr \gamma_{k,5,5})(Tr \gamma_{k,5,5}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.3)

\[
Z_{95,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{1}{(2 \sin \pi k v_1)} \cdot (Tr \gamma_{k,9})(Tr \gamma_{k,5,5}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.4)

\[
Z_{33}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \left( \prod_{r=3}^{5} (2 \sin \pi k v_r) \right) \cdot (Tr \gamma_{k,3})(Tr \gamma_{k,3}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.5)

\[
Z_{7,7,1}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{(2 \sin \pi k v_1)^2}{\prod_{r=3}^{5} (2 \sin \pi k v_r)} \cdot (Tr \gamma_{k,7,1})(Tr \gamma_{k,7,1}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.6)

\[
Z_{7,7,1}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \left( \prod_{r=3}^{5} (2 \sin \pi k v_r) \right) \cdot (Tr \gamma_{k,7,1})(Tr \gamma_{k,7,1}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.7)

\[
Z_{7,7,1}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{(2 \sin \pi k v_1)(2 \sin \pi k v_7)}{\prod_{r=3}^{5} (2 \sin \pi k v_r)} \cdot (Tr \gamma_{k,7,1})(Tr \gamma_{k,7,1}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.8)

\[
Z_{7,7,1}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot (2 \sin \pi k v_1) \cdot (Tr \gamma_{k,7,1})(Tr \gamma_{k,7,1}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.9)

\[
Z_{97,1}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{1}{\prod_{r=3}^{5} (2 \sin \pi k v_r)} \cdot (Tr \gamma_{k,9})(Tr \gamma_{k,7,1}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.10)

\[
Z_{7,5,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot (Tr \gamma_{k,7,5})(Tr \gamma_{k,7,5}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]}
\] (C.11)

\[
Z_{7,5,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{(2 \sin \pi k v_1)(2 \sin \pi k v_7)}{(2 \sin \pi k v_5)} \cdot (Tr \gamma_{k,7,5})(Tr \gamma_{k,7,5}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]+1}
\] (C.12)

\[
Z_{35,5}^{(RR)} \cong -i V_4 (8\pi^2 \alpha')^{-2} \cdot \frac{2}{t} \cdot \frac{\prod_{r=3}^{5} (2 \sin \pi k v_r)}{(2 \sin \pi k v_5)} \cdot (Tr \gamma_{k,3})(Tr \gamma_{k,3}^{-1})(-1)^{[kv_1]+[kv_2]+[-kv_3]}
\] (C.13)

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