Is the Standard Model Renormalizable?

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Abstract

In this paper, we study the renormalizability of the standard model in the Landau gauge. On the basis of the Ward-Takahashi identities, we derive exact expressions for the physical masses of $W$ and $Z$ as well as the renormalized coupling constants in the theory. We show that it is impossible to make all these renormalized quantities finite. Thus the quantum theory of the standard model with the divergent amplitudes obeying the Ward-Takahashi identities is not renormalizable.

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1. Introduction

In a recent paper[1], we showed that the standard model is not renormalizable in the alpha gauge, the only possible exception being the Landau gauge which is a special case of the alpha gauge with alpha equal to zero. In this paper, we go on to investigate if the standard model in the Landau gauge is renormalizable.

In the standard model, the number of renormalized or physical parameters exceed that of independent bare parameters. For example, once we give the bare vacuum expectation value of the Higgs field $v_0$ a non-zero value, we generate not just a non-zero renormalized $v$, but also non-zero physical masses of $W$ and $Z$. Since quantities in the quantum theory of the standard model have ultraviolet divergences, it is natural to ask if all three renormalized parameters can be made finite by a choice of one bare parameter. Similarly, the two bare electro-weak coupling constants generate renormalized coupling constants of the charged weak current, the neutral weak current, as well as the electromagnetic current. Can all these renormalized coupling constants be made finite with proper choices of the two bare coupling constants?

For almost three decades, people have agreed that the answer to these questions is yes. The key point is the validity of the Ward-Takahashi identities, which, many people believe, render all renormalized quantities finite. In this paper, we spell out the specific relations satisfied by the renormalized parameters as consequences of the Ward-Takahashi identities.

How do the Ward-Takahashi identities yield relations on the $W$-mass and the $Z$-mass? Quantum electrodynamics offers a hint. In QED, the Ward-Takahashi identity for the photon propagator insures that there are no radiative corrections to the propagator of the longitudinal photon. This means that the 1PI self-energy for the longitudinal photon vanishes identically. Now at $k^2 = 0$, the 1PI self-energy for the transverse photon is equal to that for the longitudinal photon. Consequently, the 1PI self-energy for the transverse photon vanishes at $k^2 = 0$. Thus the inverse of the propagator for the transverse photon vanishes at $k^2 = 0$, signifying the existence of a massless vector meson in QED.

Making use of the equality of the 1PI self-energy at zero momentum of the transverse propagator and that of the longitudinal propagator, we shall show that the Ward-Takahashi...
identities indeed lead to relations among $v$, the renormalized electro-weak coupling constants, and the physical $W$ mass and the physical $Z$ mass. These relations resemble their classical counterparts, with additional factors involving ultraviolet divergent quantities.

We shall show that the Ward-Takahashi identities for vertex functions also lead to specific relations satisfied by the renormalized coupling constants. There are ultraviolet-divergent quantities in these relations.

In the derivation of these relations, we shall need the forms of various propagators expressed in terms of the 1PI self-energy amplitudes. We shall list them in the next section.

2. The propagators in the Landau Gauge

In the Landau gauge, the unphysical Higgs mesons $\phi^0$ and $\phi^\pm$ decouple from the longitudinal vector mesons and hence we have

$$G^{\phi^0\phi^0}(k^2) = \frac{i}{k^2 - \Pi_{\phi^0\phi^0}(k^2)} , \quad (2.1a)$$

and

$$G^{\phi^+\phi^-}(k^2) = \frac{i}{k^2 - \Pi_{\phi^+\phi^-}(k^2)} , \quad (2.1b)$$

where $G^{\phi^0\phi^0}$, for example, is the Fourier transform of $\langle 0| T \phi^0(x) \phi^0(0)|0 \rangle$, with $T$ the time-ordering operator, and $\Pi_{\phi^0\phi^0}$ is the 1PI amplitude of $G^{\phi^0\phi^0}$.

The propagators involving longitudinal vector mesons vanish in the Landau gauge. However, we cannot simply ignore such propagators. This is because, in the Ward-Takahashi identities, a field operator of a longitudinal vector meson is always multiplied by a factor $1/\alpha$, where $\alpha$ is the gauge parameter. Thus, as we take the limit $\alpha \to 0$, terms in these longitudinal propagators which are proportional to $\alpha$ should be retained. Therefore, we shall need the approximate forms of these propagators up to the order of $\alpha$. We have

$$G^{W^+W^-}_{\mu\nu}(k^2) \approx G^{ZZ}_{\mu\nu}(k^2) \approx G^{AA}_{\mu\nu}(k^2) = -i\alpha \frac{k_\mu k_\nu}{(k^2)^2} , \quad (2.2a)$$

and

$$G^{ZA}_{\mu\nu}(k^2) \approx 0 , \quad (2.2b)$$
In (2.2a) and (2.2b), we have chosen the gauge parameter $\alpha$ for $A$, $Z$ and $W$ to be the same, and have neglected terms of the order of $\alpha^2$. The propagators for the transverse components of the vector mesons have not been included in (2.2).

The following propagators depend on the specific form of the gauge fixing terms we use in the effective Lagrangian. In the main text of this paper, we shall choose the gauge fixing terms to be the ones given by (A.1). Then we have[1]

$$G_{\eta A}^{W+} (k^2) = G_{\mu}^{W+} (k^2) \approx -\frac{i\alpha k^\mu}{M_0} \frac{\Pi_{W+} (k^2)}{k^2 [k^2 - \Pi_{W+} (k^2)]},$$  \hspace{1cm} (2.3a)

and

$$G_{\mu}^{V\phi} (k^2) = -G_{\mu}^{\phi V} (k^2) \approx -\frac{\alpha k^\mu}{M_0} \frac{\Pi_{V\phi} (k^2)}{k^2 [k^2 - \Pi_{V\phi} (k^2)]},$$  \hspace{1cm} (2.3b)

where $V$ is either $A$ or $Z$ and where $M_0$ and $M'_0$ are the bare masses of $W$ and $Z$, respectively.

Some of the ghost propagators in the Landau gauge are given by[1]

$$G_{\eta A}^{\eta A} (k^2) = \frac{i}{k^2} \frac{1 + g_0^2 \cos^2 \theta F (k^2)}{1 + g_0^2 F (k^2)},$$  \hspace{1cm} (2.4a)

and

$$G_{\eta A}^{\xi A} (k^2) = G_{\eta Z}^{\xi Z} (k^2) = -\frac{i}{k^2} \frac{g_0^2 \cos \theta \sin \theta F (k^2)}{1 + g_0^2 F (k^2)},$$  \hspace{1cm} (2.4b)

and

$$G_{\eta Z}^{\xi Z} (k^2) = \frac{i}{k^2} \frac{1 + g_0^2 \sin^2 \theta F (k^2)}{1 + g_0^2 F (k^2)},$$  \hspace{1cm} (2.4c)

where

$$\Gamma_{\eta A [W^+ \xi^-]} - \Gamma_{\eta A [W^- \xi^+]} \equiv k^\nu e_\nu F (k^2).$$  \hspace{1cm} (2.5)

In (2.4), $\eta_A$ and $\xi_A$ are the ghost fields associated with $A$, $\eta_Z$ and $\xi_Z$ are the ghost fields associated with $Z$, $\theta$ is the Weinberg angle, and $g_0$ is the bare weak coupling constant. In (2.5), $\Gamma_{\eta A [W^+ \xi^-]}$ is the 3-point function with the fields $W^+$ and $\xi^-$ joined at the same space-time point and with the propagator of the external $\eta_A$ omitted.

All of the propagators above have singularities at $k^2 = 0$. We shall express these propagators in terms of their wavefunction renormalization constants. We have

$$G_{\phi^0 \phi^0} (k^2) \equiv \frac{iZ_{\phi^0 \phi^0} (k^2)}{k^2},$$  \hspace{1cm} (2.6)

and

$$G_{\eta^i \xi^j} (k^2) \equiv \frac{iZ_{\eta^i \xi^j} (k^2)}{k^2}.$$  \hspace{1cm} (2.7)
We shall also put
\[ G_{\mu}^{W+\phi-}(k^2) \equiv -\frac{i\alpha k_{\mu}M_{0}Z_{W+\phi-}(k^2)}{(k^2)^2}, \]  
(2.8a)

and
\[ G_{\mu}^{V\phi}(k^2) \equiv -\frac{\alpha k_{\mu}M'_{0}Z_{V\phi}(k^2)}{(k^2)^2}, \]  
(2.8b)

where \( V \) is either \( A \) or \( Z \). From (2.1) and (2.6) we have
\[ Z_{\phi\phi}(k^2) = \frac{1}{[1 - \Pi_{\phi\phi}(k^2)/k^2]^{-1}}, \]  
(2.9a)

and
\[ Z_{\phi+\phi-}(k^2) = \frac{1}{[1 - \Pi_{\phi+\phi-}(k^2)/k^2]^{-1}}. \]  
(2.9b)

From (2.3) and (2.8), we get
\[ Z_{W+\phi-}(k^2) = \frac{\Pi_{W+\phi-}(k^2)}{M_{0}^2}Z_{\phi+\phi-}(k^2), \]  
(2.10a)

and
\[ Z_{V\phi}(k^2) = \frac{\Pi_{V\phi}(k^2)}{M'_{0}^2}Z_{\phi\phi}(k^2). \]  
(2.10b)

where \( V \) is either \( A \) or \( Z \). From (2.4), we have
\[ Z_{\eta\xi\lambda}(k^2) - \frac{\sin \theta}{\cos \theta}Z_{\eta\xi\lambda}(k^2) = 1, \]  
(2.11a)

and
\[ Z_{\eta\xi\lambda}(k^2) - \frac{\cos \theta}{\sin \theta}Z_{\eta\xi\lambda}(k^2) = 1. \]  
(2.11b)

Finally, we turn to the propagators for the transverse components of the vector mesons. The vector mesons \( W \) and \( Z \) are massive. Following Peterman, Stuckelberg[2], Gell-Mann and Low[3], we put the propagator for the transverse \( W \) as
\[ -iZ_{W+\phi-}(k^2) \frac{k^2}{k^2 - M_{W}^2}, \]  
(2.12)

where \( M_{W} \) is the physical mass of \( W \). The ratio of \( Z_{W+\phi+}(k^2) \) and \( Z_{W+\phi+}(0) \) is proportional to the effective weak charge[2,3,4].

The propagator for the transverse \( Z \) and the transverse \( A \) takes a little more manipulation. This is because the transverse \( Z \) mixes with the transverse \( A \). The mixing matrix of
propagators is equal to the inverse of the matrix
\[
\begin{bmatrix}
i(k^2 - \Pi_{AA}^T) & -i\Pi_{AZ}^T \\
-i\Pi_{AZ}^T & i(k^2 - M_0'^2 - \Pi_{ZZ}^T)
\end{bmatrix}
\] (2.13)

where $\Pi_{ZZ}^T$, for example, is the 1PI self-energy amplitude for the transverse $Z$. Let us diagonalize the matrix in (2.13). By (4.1b) below, the determinant of this matrix vanishes at $k^2 = 0$. Thus so does one of its eigenvalues at $k^2 = 0$, with the corresponding eigenvector representing the physical and massless photon which we shall denote as $A'$. The other eigenvector represents the physical $Z$ meson, which we shall denote as $Z'$. The particles $A'$ and $Z'$ are related to $A$ and $Z$ by an angle of rotation $\Theta(k^2)$. It is straightforward to find
\[
\cot \Theta = \frac{M_0'^2 + \Pi_{ZZ}(0)}{\Pi_{AZ}(0)},
\] (2.14)

where $\Theta = \Theta(0)$. Let the propagator for the transverse $Z'$ be represented as
\[
-i\frac{Z_{Z'Z'}^T(k^2)}{k^2 - M_Z^2},
\] (2.15)

where $M_Z$ is the physical mass of $Z$. The propagator for the transverse $A'$ will be represented as
\[
-i\frac{Z_{A'A'}^T(k^2)}{k^2}.
\] (2.16)

We have
\[
Z_{Z'Z'}^T(0) = \frac{M_Z^2}{M_0'^2 + \Pi_{ZZ}(0) + \Pi_{AA}(0)},
\] (2.17)

and
\[
Z_{A'A'}^T(0) = (1 - a)^{-1},
\] (2.18)

where
\[
a \equiv \lim_{k^2 \to 0} \frac{\Pi_{AA}^T(k^2)[M_0'^2 + \Pi_{ZZ}^T(k^2)] - [\Pi_{AZ}^T(k^2)]^2}{k^2[M_0'^2 + \Pi_{ZZ}(0) + \Pi_{AA}(0)].
\]

3. The W Mass and the Vacuum Field Value

In the Landau gauge, the ghosts directly interact only with the gauge vector mesons. According to the Feynman rules, the ghost-ghost-$V$ vertex factor is proportional to $k^\mu$,
where $V$ is a gauge vector meson, $\mu$ is the polarization of $V$ and $k$ is the momentum of the incoming ghost. Since $\mu$ is always transverse, $k^\mu$ is equal to $p^\mu$, where $p$ is the momentum of the outgoing ghost. Consequently, this vertex factor vanishes if either the momentum of the incoming ghost or that of the outgoing ghost vanish. This is a feature which greatly simplifies some of the Ward-Takahashi identities at zero momenta discussed below.

There are three Ward-Takahashi identities associated with the longitudinal $W$. We have made use of two of them in a preceding paper and get the following relation among the 1PI self-energy amplitudes:

$$
(1 + \Pi_{W^+W^-}(k^2) - \Pi_{\phi^+\phi^-}(k^2)) = (1 + \Pi_{W^+\phi^-}(k^2))^{-2},
$$

where $\Pi_{W^+W^-}$, for example, is the 1PI self-energy amplitude for the longitudinal $W$. We shall now explore the consequences of the third identity.

The bare mass of $W$ is equal to $\frac{1}{2}g_0 v_0$. We may choose $v_0$ to be either the classical vacuum expectation value of the Higgs field at which the Higgs potential is minimum, or the quantum vacuum expectation value of the Higgs field. The gauge fixing terms and the ghost terms are different with these two different choices. Thus the Green functions are different if $v_0$ is chosen differently. However, the physical scattering amplitudes are the same if physical quantities are gauge invariant.

We shall, in this paper, choose $v_0$ to be the quantum vacuum expectation value of the Higgs field. With this choice, the third Ward-Takahashi identities associated with the longitudinal $W$ is, at $k^2 = 0$,

$$
Z_{W^+\phi^-}(0) + Z_{\phi^+\phi^-}(0) = Z_{\eta^+\xi^-}(0).
$$

We mention that, for $k^2$ not equal to zero, there is a ghost-ghost-Higgs vertex function appearing in this identity. This vertex function vanishes at $k^2 = 0$ and does not appear in (3.2), for the reason we mentioned above.

Substituting (2.10a) into (3.2), and making use of (3.1), we reduce (3.2) into

$$
\left(1 + \frac{\Pi_{W^+W^-}(0)}{M_0^2}\right)Z_{\phi^+\phi^-}(0) = Z_{\eta^+\xi^-}(0).
$$

We will cast (3.3) into a more useful form. We note that the propagator for the transverse
W may be expressed by its 1PI amplitude as:

\[ -i \frac{k^2 - M_0^2 - \Pi_{W^+W^-}(k^2)}{M_0^2 + \Pi_{W^+W^-}(0)}. \]  

(3.4)

where \( \Pi^T_{W^+W^-} \) is the 1PI self-energy amplitude for the transverse \( W \). From (2.12) and (3.4), we get

\[ Z^T_{W^+W^-}(0) = \frac{M_W^2}{M_0^2 + \Pi^T_{W^+W^-}(0)}. \]  

(3.5)

Making use of the fact that \( \Pi^T_{W^+W^-}(0) \) and \( \Pi_{W^+W^-}(0) \) are equal, we reduce (3.3) into

\[ M_W = \frac{1}{2} g_0 v_0 Z^{++}_{\eta^+\xi^-}(0) \sqrt{Z^T_{W^+W^-}(0) / Z^{++}_{\phi^+\phi^-}(0)}. \]  

(3.6)

Finally, \( v \), the renormalized vacuum expectation value of the Higgs field, is equal to \( v_0 \) divided by \( \sqrt{Z_H(0)} \), where \( Z_H(0) \) is the wavefunction renormalization constant for the physical Higgs field \( H \). Thus we have

\[ v = \frac{v_0}{\sqrt{Z_H(0)}}. \]  

(3.7)

Hence (3.6) becomes

\[ \frac{M_W}{v} = \frac{1}{2} g_0 Z^{++}_{\eta^+\xi^-}(0) \sqrt{Z^T_{W^+W^-}(0) / Z^{++}_{\phi^+\phi^-}(0)}. \]  

(3.8)

If the standard model is renormalizable, the right-side of (3.8) must be finite.

We close this section with two comments:

(a) The relationship (3.8) holds in the Landau gauge, a special case of the alpha gauge. One may derive its counterpart in a general alpha gauge.

(b) The \( W \) meson is unstable, thus the propagator for the transverse \( W \) does not have a pole at a real value of \( k^2 \). We may define \( M \) to be the value of \( k^2 \) at which the real part of the inverse of the propagator for the transverse \( W \) vanishes. This will provide a subtraction condition for the propagator of the transverse \( W \).

4. The \( Z \) Mass and the Vacuum Field Value

Next we consider the Ward-Takahashi identities associated with the longitudinal \( A \) and the longitudinal \( Z \). There are nine of them in all. We have extracted the consequences from
seven of them. Among others, we have

\[ \left[ 1 + \frac{\Pi_{ZZ}(k^2)}{M_0^2} \right] \left[ 1 - \frac{\Pi_{\phi\phi}(k^2)}{k^2} \right] = \left[ 1 + \frac{\Pi_{\phi}(k^2)}{M_0^2} \right]^2, \]  

(4.1a)

\[ \Pi_{AA}(k^2)[M_0^2 + \Pi_{Z\phi}(k^2)] = |\Pi_{AZ}(k^2)|^2. \]  

(4.1b)

and

\[ \Pi_{AA}(k^2) \left[ 1 - \frac{\Pi_{\phi\phi}(k^2)}{k^2} \right] = \frac{[\Pi_{A\phi}(k^2)]^2}{M_0^2}. \]  

(4.1c)

where \( \Pi_{ZZ} \), for example, is the 1PI self-energy amplitude for the longitudinal \( Z \). The remaining two Ward-Takahashi identities are, at \( k^2 = 0 \),

\[ Z_{A\phi}(0) = Z_{\eta_{AA}}(0), \]  

(4.2a)

and

\[ 1 + \frac{\Pi_{Z\phi}(0)}{M_0^2} Z_{\phi\phi}(0) = Z_{\eta_{Z\phi}}(0). \]  

(4.2b)

From (4.1a) and (4.2b), we get

\[ [1 + \frac{\Pi_{Z\phi}(0)}{M_0^2}] Z_{\phi\phi}(0) = [Z_{\eta\xi_Z}(0)]^2. \]  

(4.3)

Making use of eq. (2.17), we get

\[ \frac{M_Z}{v} = \frac{1}{2} \cos \theta \frac{g_0}{\sqrt{Z_{Z\phi}(0)}} \left[ \frac{Z_{\phi\phi}(0)}{Z_{\phi\phi}(0)} + \frac{\Pi_{AA}(0)}{M_0^2} \right] \sqrt{H(0)}. \]  

(4.4)

The right-side of (4.4) must be finite if the standard model is renormalizable.

5. Identities for Three Point Functions

In this section we study the consequences of the Ward-Takahashi identities on three-point functions in the standard model. We shall begin by summarizing a number of relations among the wavefunction renormalization constants which will be used in this section.

From (4.2a) and (4.2b), we obtain

\[ \frac{Z_{\eta\xi_Z}(0)}{Z_{\eta\xi_Z}(0)} = \tan \Theta, \]  

(5.1)
where we have made use of (2.10b), (2.14) and (4.1). Together with (2.11a), and (2.11b), (5.1) give
\[
Z_{\eta z\xi z}(0) = \frac{\sin \theta \cos \Theta}{\sin(\theta - \Theta)}, \tag{5.2a}
\]
\[
Z_{\eta A\xi z}(0) = \frac{\sin \theta \sin \Theta}{\sin(\theta - \Theta)}, \tag{5.2b}
\]
and
\[
Z_{\eta A\xi A}(0) = \frac{1}{2} \frac{\sin(2\theta) \cos \Theta - \cos(2\theta) \sin \Theta}{\cos \theta \sin(\theta - \Theta)}. \tag{5.2c}
\]
Similarly, we may derive from (2.10b) that
\[
\frac{Z_{A\phi\phi}(0)}{Z_{Z\phi\phi}(0) + Z_{\phi\phi\phi}(0)} = \tan \Theta. \tag{5.3}
\]
It is also straightforward to find that
\[
[Z_{A\phi\phi}(0)]^2 + [Z_{Z\phi\phi}(0) + Z_{\phi\phi\phi}(0)]^2 = \left[1 + \frac{\Pi_{AA}(0) + \Pi_{ZZ}(0)}{M_0^2}\right]Z_{\phi\phi\phi}(0).
\]
Thus, by (2.17), we have
\[
Z_{A\phi\phi}(0) = \frac{M_Z}{M'_0} \sqrt{\frac{Z_{\phi\phi\phi}}{Z_{Z\phi\phi}(0)}} \sin \Theta, \tag{5.4a}
\]
and
\[
Z_{Z\phi\phi}(0) + Z_{\phi\phi\phi}(0) = \frac{M_Z}{M'_0} \sqrt{\frac{Z_{\phi\phi\phi}(0)}{Z_{Z\phi\phi}(0)}} \cos \Theta. \tag{5.4b}
\]
By (4.1), eq.(2.14) can also be written as
\[
\cos \Theta = \sqrt{\frac{M_0^2 + \Pi_{ZZ}(0)}{M'_0^2 + \Pi_{ZZ}(0) + \Pi_{AA}(0)}}, \tag{5.5a}
\]
and
\[
\sin \Theta = \sqrt{\frac{\Pi_{AA}(0)}{M'_0^2 + \Pi_{ZZ}(0) + \Pi_{AA}(0)}}. \tag{5.5b}
\]
We shall now derive the Ward identities for the interaction of a lepton \(l\) with a gauge meson.

The first such identity is obtained by setting to zero the vacuum expectation value of the BRST variation of
\[
T\eta_A(x)l(y)\bar{l}(z).
\]
We get
\[
\frac{1}{\alpha} < 0 | T \partial_\mu A^\mu(x) l(y) \bar{l}(z) | 0 > = ig_0 < 0 | T \eta_A(x) \left[ \frac{1}{2} L - \sin^2 \theta \sin \theta \xi_A(y) \right] l(y) \bar{l}(z) | 0 > \\
- \frac{ig_0}{\sqrt{2}} < 0 | T \eta_A(x) \xi^- (y) \nu(y) \bar{l}(z) | 0 > \\
- ig_0 < 0 | T \eta_A(x) l(y) \bar{l}(z) \left[ \frac{1}{2} R - \sin^2 \theta \cos \theta \xi(z) + \sin \theta \xi_A(z) \right] | 0 > \\
+ \frac{ig_0}{\sqrt{2}} < 0 | T \eta_A(x) l(y) \bar{\nu}(z) \xi^+(z) | 0 > .
\]

(5.6)

where \( \alpha \) is the gauge parameter, \( \nu \) is the neutrino associated with \( l \), \( L = \frac{1}{2} (1 + \gamma_5) \) and \( R = \frac{1}{2} (1 - \gamma_5) \).

Next we set to zero the vacuum expectation value of the BRST variation of
\[ T \eta_Z(x) l(y) \bar{l}(z) . \]

We get
\[
< 0 | T \left[ \frac{\partial_\mu Z^\mu(z)}{\alpha} + M_0^0 \phi^0(x) \right] l(y) \bar{l}(z) | 0 > = ig_0 < 0 | T \eta_Z(x) \left[ \frac{1}{2} L - \sin^2 \theta \sin \theta \xi_Z(y) + \sin \theta \xi_A(y) \right] l(y) \bar{l}(z) | 0 > \\
- \frac{ig_0}{\sqrt{2}} < 0 | T \eta_Z(x) \xi^- (y) \nu(y) \bar{l}(z) | 0 > \\
- ig_0 < 0 | T \eta_Z(x) l(y) \bar{l}(z) \left[ \frac{1}{2} R - \sin^2 \theta \cos \theta \xi_Z(z) + \sin \theta \xi_A(z) \right] | 0 > \\
+ \frac{ig_0}{\sqrt{2}} < 0 | T \eta_Z(x) l(y) \bar{\nu}(z) \xi^+(z) | 0 > .
\]

(5.7)

We note that the longitudinal \( A \) and the longitudinal \( Z \) mix with the unphysical neutral Higgs meson \( \phi^0 \). Thus the external longitudinal photon in the left-side of (5.6), for example, may propagate into a photon, or a \( \phi^0 \). For this reason, the term on the left-side of (5.6) gives rise to two terms, each of which corresponds to a channel of propagation. Similarly, the term on the left-side of (5.7) also give rise to two terms. We mention that, by (2.2b), the longitudinal \( A \) does not propagate into the longitudinal \( Z \) in the Landau gauge.

We shall take the Fourier transform of (5.6) and (5.7), denoting the momenta of the outgoing lepton, the incoming lepton, and the outgoing gauge meson by \( p' \), \( p \), and \( k \), respectively, with
\[ p = p' + k. \]
We take $\alpha$ to zero to get to the Landau gauge. Then we multiply the resulting expression by $ik^2$ (to get rid of the propagator of the external longitudinal photon) as well as by $S^{-1}_l(p)$ from the right and $S^{-1}_l(p')$ from the left (to eliminate the propagators of the external leptons). We then differentiate the resulting equation with respect to $k\mu$ with $p$ fixed and on-shell, take the limit $k \to 0$ and insert the equation between physical lepton spinor functions. We get

$$-i\Gamma_{\mu l}^\prime(p, p, 0) - M\prime_0 Z_{A\phi^0}(0) \frac{1}{2} L - \sin^2 \theta \frac{1}{Z_l(m^2)} \gamma^\mu \frac{1}{\cos \theta} + e_0 Z_{A\xi^0}(0) \gamma^\mu,$$

and

$$-i\Gamma_{\mu l}^\mu(p, p, 0) - M_0 [Z_{Z\phi^0}(0) + Z_{\phi^0\phi^0}(0)] \frac{1}{2} L - \sin^2 \theta \cos \theta \frac{1}{Z_l(m^2)} \gamma^\mu + e_0 Z_{2\xi^0}(0) \gamma^\mu,$$

where $Z_l(p^2)$ is the wavefunction renormalization constant for the lepton, $p^2 = m^2$, and $m$ is the mass of the lepton. There are 4-point amplitudes in the Ward-Takahashi identities, but they vanish as we set $k = 0$.

Since $A\prime$, not $A$, is the physical photon field, we take the difference of (5.8) multiplied by $\cos \Theta$ and (5.9) multiplied by $\sin \Theta$, getting

$$-i\Gamma_{\mu l}^\mu(p, p, 0) = e_0 Z_{2\xi^0}(0) \cos \Theta - Z_{2\xi^0}(0) \sin \Theta \frac{1}{Z_l(m^2)} \gamma^\mu.$$

The left-side of (5.10) multiplied by $Z_l(m^2)\sqrt{Z^\prime_{A\prime A\prime}}(0)$ is equal to

$$e_0 \sqrt{Z^\prime_{A\prime A\prime}}(0) \cos(\theta - \Theta),$$

where $e$ is the renormalized electric charge. Thus we have

$$e = e_0 \sqrt{Z^\prime_{A\prime A\prime}}(0) \frac{\cos(\theta - \Theta)}{\cos \theta},$$

where (5.2) has been used.

The renormalized charge given by (5.12) is universal, i.e., the same for all leptons[5]. We note that the form in (5.12) differs from its counterpart in QED by the last factor in (5.12). In order that the standard model is renormalizable, the right-side of (5.12) is required to be finite.
Taking the sum of (5.8) multiplied by $\sin \Theta$ and (5.9) multiplied by $\cos \Theta$, we get

$$-i\Gamma^\mu_{lZ'}(p, p, 0) - M_Z \sqrt{\frac{Z_{\phi',\phi'}(0)}{Z'_{Z',Z'}(0)}} \frac{\partial}{\partial k_\mu} \Gamma_{l\phi'}(p - k, p, k)|_{k=0}$$

$$= \frac{\sin \theta}{\sin(\theta - \Theta)} \frac{g_0}{Z_i(m^2)} \frac{1}{2} L - \sin^2 \theta \frac{e_0}{\cos \theta \sin(\theta - \Theta)} Z_i(m^2) \gamma^\mu,$$

where $p^2 = m^2$.

Next we discuss the neutral current for the neutrino. The counterparts of (5.8) and (5.9) are

$$-i\Gamma^\mu_{\nu\phi A}(p, p, 0) - M'_0 Z_{A\phi'}(0) \frac{\partial}{\partial k_\mu} \Gamma_{\nu\phi'}(p - k, p, k)|_{k=0} = -\frac{g_0 Z_{A\xi\xi}(0)}{2 \cos \theta Z_{\nu}(0)} \gamma^\mu,$$  (5.14)

and

$$-i\Gamma^\mu_{\nu\phi Z}(p, p, 0) - M'_0 [Z_{\phi,\phi'}(0) + Z_{\phi',\phi'}(0)] \frac{\partial}{\partial k_\mu} \Gamma_{\nu\phi'}(p - k, p, k)|_{k=0} = -\frac{g_0 Z_{A\xi\xi}(0)}{2 \cos \theta Z_{\nu}(0)} \gamma^\mu,$$  (5.15)

where $p^2 = 0$. Thus the counterpart of (5.10) is

$$\Gamma^\mu_{\nu\phi A'}(p, p, 0) = 0,$$  (5.16)

which says that the renormalized charge of the neutrino is rigorously zero. The counterpart of (5.13) is

$$-i\Gamma^\mu_{\nu\phi Z'}(p, p, 0) - M_Z \sqrt{\frac{Z_{\phi',\phi'}(0)}{Z'_{Z',Z'}(0)}} \frac{\partial}{\partial k_\mu} \Gamma_{\nu\phi'}(p - k, p, k)|_{k=0}$$

$$= -\frac{\tan \theta}{\sin(\theta - \Theta)} g_0 \frac{1}{2 Z_{\nu}(0)} \gamma^\mu,$$  (5.17)

where $p^2 = 0$.

Let us multiply (5.17) by $\sqrt{Z'_{Z',Z'}(0)Z_{\nu}(0)}$. Then the first and the second term on the left-side of the resulting equation are proportional to the renormalized $\nu - \bar{\nu} - Z'$ vertex function and the renormalized $\nu - \bar{\nu} - \phi^0$ vertex function, respectively. If these two renormalized vertex functions are finite, so must be the right-side of the resulting equation. Thus we require that

$$g_Z \equiv \frac{\tan \theta}{\sin(\theta - \Theta)} g_0 \sqrt{Z'_{Z',Z'}(0)},$$  (5.18)

to be finite.
Finally, we consider the charged weak current. We set to zero the vacuum expectation value of the BSRT variation of

\[ T \eta^-(x) \nu(y) \bar{l}(z) . \]

We get

\[
< 0 | T \left( \frac{\partial^{\mu} W^-_{\nu}(x)}{\alpha} - i M_0 \phi^-(x) \right) \nu(y) \bar{l}(z)|0 > \\
= -ig_0 < 0 | T \eta^-(x) \nu(y) \left[ \frac{\xi_Z(y) \nu(y)}{2 \cos \theta} + \frac{\xi^+(y) L l(y)}{\sqrt{2}} \right] \bar{l}(z)|0 > \\
- ig_0 < 0 | T \eta^-(x) \nu(y) \left[ \frac{1}{2} R - \sin^2 \theta \xi_Z(z) + \sin \theta \xi_A(z) \right] \bar{l}(z)|0 > \\
+ \frac{i g_0}{\sqrt{2}} < 0 | T \eta^-(x) \nu(y) \bar{\nu}(z) \xi^+(z)|0 > . \tag{5.19}
\]

As before, we take the limit \( \alpha \) going to zero and take the Fourier transform of eq.(5.19), with the momentum of the outgoing neutrino denoted by \( (p - k) \) and that of the incoming lepton denoted by \( p \). We multiply the Fourier transform of (5.19) by \( i k^2 \) as well as by \( S^{-1}_\nu(p - k) \) from the left and \( S^{-1}_l(p) \) from the right, differentiate with respect to \( k \) with \( p \) fixed, and set \( k \) to zero. Since the masses of the neutrino and the lepton are different, we cannot make both of these particles to be on the mass-shell. We shall choose to have the electron on the mass-shell, i.e., we choose \( p^2 = m^2 \). We apply the resulting expression on the lepton spinor function, setting \( \not{\!p} \) operating on the lepton spinor function to equal to \( m \). We get

\[
-i \Gamma_{\nu W^-}(p, p, 0) + i M_W \sqrt{Z_{W^- + W^-}(0)} \frac{\partial}{\partial k_\mu} \Gamma_{\nu \phi^-}(p - k, p, k)|_{k=0} \\
= -\frac{g_0}{\sqrt{2}} Z_{\eta^- \xi^+}(0) Z_{\nu}(m^2) \gamma^\mu L , \tag{5.20}
\]

where \( p^2 = m^2 \), and where (3.5) has been used.

Let us multiply equation (5.20) by \( \sqrt{Z_{W^- + W^-}(0) Z_{\nu}(m^2) Z_{\nu}(m^2)} \), then the first term and the second term in the left-side of the resulting equation are proportional to the renormalized \( \nu - l - W \) vertex the renormalized \( \nu - \bar{l} - \phi^- \) vertex, respectively. If these two renormalized vertices are finite, so must be the right-side of the resulting equation. Thus

\[
g_W \equiv g_0 Z_{\eta^+ \xi^-}(0) \sqrt{Z_{\nu}(m^2) Z_{W^- + W^-}(0)} , \tag{5.21}
\]

is required to be finite.
6. Discussion

As theoretical physicists all know, the forms of the Ward-Takahashi identities in QED are relatively simple. People also know that these identities have profound consequences, two of them being the vanishing of the photon mass and the universality of the electric charge. All of these consequences are easily extracted from the Ward-Takahashi identities in QED.

In contrast, the Ward-Takahashi identities in Yang-Mills theories are notoriously complex in form. There are numerous terms in these identities due to the existence of interacting ghosts. Such complexities mar the rigorous implications of these identities which are more difficult to explore.

As an example, consider using these identities in the standard model in the Feynman gauge and investigate the question of the universality of the electric charge. The ratio of $Z_{\bar{e}eA}$ and $Z_e$ obtained from these identities is not identically unity—the value of this ratio in QED. Instead, it is equal to a sum of amplitudes. It is possible to calculate this sum perturbatively and showed that, to the one-loop order, it is independent of the lepton mass. But to prove this true to all orders on the basis of these identities in the Feynman gauge appears difficult.

But these identities do give simple and exact consequences. For example, it is easy to prove that the mass of the photon is strictly zero with the Ward-Takahashi identities. It is also possible to prove charge universality from these identities if one uses the unitary gauge[5].

Additional consequences from these identities are found if one uses the Landau gauge. This is because these identities at $k^2 = 0$ simplify in the Landau gauge. One finds relationships between the $W$-mass and the $Z$-mass with the renormalized vacuum expectation value of the Higgs field:

$$\frac{M_W}{\frac{v}{2} g_W} = \sqrt{\frac{Z_H(0) Z_\phi(m^2)}{Z_{\phi^+ \phi^-}(0) Z_l(m^2)}}$$  \hspace{1cm} (6.1)

and

$$\frac{M_Z}{v g_Z} = \frac{\sin(\theta - \Theta)}{\sin \theta} \sqrt{\frac{Z_{\eta \xi Z}(0)}{Z_{\phi^0 \phi^0}(0)} + \frac{\Pi_{AA}(0)}{M_0^2}} Z_H(0).$$ \hspace{1cm} (6.2)

These are non-perturbative and quantum mechanical expressions for the masses of $W$ and $Z$ generated by spontaneous symmetry breaking. If one neglects quantum corrections and
set the right-sides of (6.1) and (6.2) to unity, they are reduced to the well-known classical
formulae for the \(W\) mass and the \(Z\) mass.

One also finds the following three renormalized electro-weak coupling constants:

\[ e \equiv e_0 \sqrt{Z_{\Lambda',\Lambda'}(0)} \frac{\cos(\theta - \Theta)}{\cos \theta}, \quad (6.3) \]

\[ g_z \equiv \frac{g_0}{2} \sqrt{Z_{\Lambda',\Lambda'}(0)} \frac{\tan \theta}{\sin(\theta - \Theta)}, \quad (6.4) \]

and

\[ g_w \equiv g_0 \sqrt{Z_{\Lambda',\Lambda'}(0)} \frac{Z_l(m^2)}{Z_{\nu}(m^2)} Z_{\eta}(0). \quad (6.5) \]

Equation (6.3) gives the electric charge \(e\) in the standard model. Equation (6.4) gives the
coupling constant \(g_z\) for the neutral weak current. And (6.5) gives the coupling constant \(g_w\)
for the charged weak current.

In addition, by multiplying (5.13) with \(Z_{\Lambda}(m^2) \sqrt{Z_{\Lambda',\Lambda'}(0)}\), we get

\[ -iZ_l(m^2) \sqrt{Z_{\Lambda',\Lambda'}(0)} \Gamma_{\mu}^{\mu}(p, p, 0) - M_Z Z_l(m^2) \sqrt{Z_{\phi,\phi}(0)} \frac{\partial}{\partial k_{\mu}} \Gamma_{\mu}^{\phi}(p - k, p, k)|_{k=0} \]

\[ = g_z \gamma^\mu [L - 2 \sin^2(\theta - \Theta)], \quad (6.6) \]

where \(p^2 = m^2\). We note that both terms in the left-side of (6.6) are renormalized amplitudes.
Thus the right-side of (6.6) must be ultraviolet finite if the standard model is renormalizable.

While the Ward-Takahashi identities in the standard model do lead to relationships
among various renormalized quantities, they by no means imply that all of these quantities
can be chosen ultraviolet finite. This is because there are ultraviolet divergent quantities in
these relations. For the renormalized quantities in the above equations to be finite, the right
sides of these equations must be finite. In particular, if \(g_z\) and the right-side of (6.6) are both
finite, \(\theta - \Theta\) must be ultraviolet finite for the standard model to be renormalizable.

From (6.3) and (6.4), we get

\[ \frac{g_z}{e} \sin[2(\theta - \Theta)] = \sqrt{\frac{Z_{\Lambda',\Lambda'}(0)}{Z_{\Lambda',\Lambda'}(0)}}. \quad (6.7) \]

Thus the right-side of (6.7) must be ultraviolet finite for the standard model to be renor-
malizable. But the right-side of (6.7) is not ultraviolet finite. Indeed, this ratio has been
calculated up to one loop and is known to be ultraviolet divergent[5].

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In conclusion, a quantum gauge field theory is not completely predictive without a prescription of how the ultraviolet divergences are handled. As is well-known, the prescription in QED is that the divergent amplitudes obey the Ward-Takahashi identities. This leads to predictions in spectacular agreements with experiments. If one uses the same prescription for the quantum theory of the standard model, one also finds exact results such as charge universality, the vanishing of the electric charge of the neutrino, and the vanishing of the photon mass. This is borne out by experiments to an extremely high degree, as the sum of the electron charge and the proton charge is less than $10^{-21} e$, the neutrino charge is less than $10^{-13} e$, and the photon mass is less than $6 \times 10^{-16} eV$. On the other hand, this prescription in the standard model leads to a non-renormalizable theory, as not all renormalized quantities can be chosen finite. We believe that the foundation of the quantum theory of the standard model remains to be laid.
Appendix

In the main text of this paper, the gauge fixing terms of the Lagrangian are
\[-\frac{1}{\alpha} (\partial^\mu W_\mu^+ + i \alpha M_0 \phi^+) (\partial^\nu W_\nu^- - i \alpha M_0 \phi^-) - \frac{1}{2\alpha} (\partial_\mu Z^\mu + \alpha M_0 \phi^0)^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2. \quad (A.1)\]

In this Appendix, we discuss briefly how the formulae are modified if the gauge fixing terms are chosen to be, instead,
\[-\frac{1}{\alpha} (\partial^\mu W_\mu^+ + i \alpha M_0 \phi^+) (\partial^\nu W_\nu^- - i \alpha M_0 \phi^-) - \frac{1}{2\alpha} (\partial_\mu Z^\mu)^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2. \quad (A.2)\]

With the gauge fixing terms given by (A2), all the propagators in the limit \(\alpha \to 0\) are of the same form as the ones in Sec. 2 except the ones given by (2.3a) and (2.3b), which are replaced by
\[G_{W \phi}^W (k) = G_{W \phi}^W (k) \approx -\frac{i \alpha k^\mu}{M_0 (k^2)^2} \left(1 - \frac{\Pi_{W \phi} (k^2)}{k^2}ight), \quad (A.3a)\]
and
\[G_{Z \phi}^Z = -G_{Z \phi}^Z \approx -\frac{\alpha k^\mu}{M_0' k^2} \left(1 - \frac{\Pi_{Z \phi} (k^2)}{k^2}ight). \quad (A.3b)\]

Thus all wavefunction renormalization constants remain the same as before except the ones given by (2.10a) and (2.10b), which are replaced by
\[Z_{W \phi} (k^2) = (1 + \frac{\Pi_{W \phi} (k^2)}{M_0^2}) Z_{W \phi} (k^2), \quad (A.4a)\]
and
\[Z_{Z \phi} (k^2) = (1 + \frac{\Pi_{Z \phi} (k^2)}{M_0'^2}) Z_{Z \phi} (k^2). \quad (A.4b)\]

The relations among the 1PI self-energy amplitudes given by (3.1) and (4.1) remain valid.

The Ward-Takahashi identity (3.2) is replaced by
\[Z_{W \phi} (0) = Z_{W \phi} (0). \quad (A.5)\]
while the Ward identities (4.2a) and (4.2b) remain the same as before. Because of these changes, some of the intermediate formulae are now different. For example, we have, instead of (5.3),
\[\frac{Z_{A \phi} (0)}{Z_{Z \phi} (0)} = \tan \Theta. \quad (A.6)\]
However, the final formulae (6.1)–(6.6) stay the same.
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