A Modular Non-Rigid Calabi-Yau Threefold

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Abstract. We construct an algebraic variety by resolving singularities of a quintic Calabi-Yau threefold. The middle cohomology of the threefold is shown to contain a piece coming from a pair of elliptic surfaces. The resulting quotient is a two-dimensional Galois representation. By using the Lefschetz fixed-point theorem in étale cohomology and counting points on the variety over finite fields, this Galois representation is shown to be modular.

1. Introduction

In this paper we investigate the geometry and arithmetic of a Calabi-Yau threefold $X \subset \mathbb{P}^4 \times \mathbb{P}^4$ given as a complete intersection of five hypersurfaces of bidegree $(1,1)$. $X$ is a partial desingularization common to a pair of quintic threefolds $F$ and $G$ in $\mathbb{P}^4$; we will in fact be interested in a big resolution $\tilde{X}$ of $X$. So $\tilde{X}$ is not actually a Calabi-Yau threefold, but it is birational to one.

A conjecture of Fontaine and Mazur predicts that two-dimensional $l$-adic Galois representations coming from geometry should be modular. More precisely, the statement is that a continuous irreducible two-dimensional $l$-adic representation of the absolute Galois group $G_{\mathbb{Q}}$ that is isomorphic to a Tate twist of a subquotient of an étale cohomology group of a variety $X/\mathbb{Q}$ should be modular. This is a higher-dimensional generalization of the Taniyama-Shimura conjecture on the modularity of elliptic curves over $\mathbb{Q}$, as proved by Wiles, Taylor et al.

A rigid Calabi-Yau threefold $X$ defined over $\mathbb{Q}$ has two-dimensional middle cohomology, and is thus expected to be modular. We expect the $L$-series of the Galois action on the $l$-adic cohomology to be, up to factors associated to the primes of bad reduction of $X$, the Mellin transform of a weight 4 modular form. Recently Dieulefait and Manoharmayum proved that rigid Calabi-Yau threefolds that have good reduction at 3 and 7, or at 5 and another suitable prime, are modular. A handful of explicit examples of modular Calabi-Yau threefolds are known. Some examples are given in [25], [31], [16], [32], [33].

Some nonrigid Calabi-Yau threefolds have been shown to be modular by Hulek-Verrill and Schütt. Here the middle cohomology group of $X$ has dimension greater than 2, so one must figure out how to extract a 2-dimensional piece on which $G_{\mathbb{Q}}$ acts.

Hulek-Verrill found threefolds in a toric variety with $h^3 = 4, 6$ and 10; in each case they showed that the semisimplification of $H^3$ was a direct sum $\oplus_i W_i \oplus V$.

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where the $W_i$ came from elliptic surfaces defined over $\mathbb{Q}$ and $V$ was the remaining quotient. The $W_i$ were thus isomorphic to cohomology groups of elliptic curves twisted by $(-1)$, and $V$ was shown to correspond to a modular form of weight 4.

Schütt constructed some fiber products of rational elliptic surfaces; he showed that each of their middle cohomology groups also broke up into a sum of two-dimensional pieces coming from elliptic curves and a leftover two-dimensional piece corresponding to a modular form of weight 4.

Our threefold $\tilde{X}$ is constructed by resolving singularities of a pair of Horrocks-Mumford quintic threefolds $F$ and $G$. The Horrocks-Mumford bundle $HM$ is a stable indecomposable rank 2 vector bundle over $\mathbb{P}^4$, and zero sets of its sections are abelian surfaces. We take zero sets of a pair of sections of the line bundle $\wedge^2(HM) \cong \mathcal{O}(5)$ as our quintic threefolds $F$ and $G$; they are thus pencils of abelian surfaces.

First we take a common partial resolution $X$ of $F$ and $G$ as mentioned at the outset; it is a Calabi-Yau threefold given as a complete intersection in $\mathbb{P}^4 \times \mathbb{P}^4$. We then blow up the singularities of $X$ to obtain $\tilde{X}$. Unfortunately, there exists no model for a small resolution of $X$ over $\mathbb{Q}$.

By studying the geometry of $\tilde{X}$ and by exploiting the Weil conjectures, we are able to show that $h^3(\tilde{X}) = 6$. We show that the semisimplification of the Galois representation $H^3(\tilde{X})$ is a direct sum of a two-dimensional piece $V$ and a four-dimensional piece $W$. The four-dimensional piece $W$ arises from the cohomology of a pair of elliptic surfaces $E_1$ and $E_2$ that are complex conjugates of each other. Thus their union is defined over $\mathbb{Q}$, and as a Galois representation $W$ is induced from a representation of the subgroup $G_\mathbb{Q}(i)$. We then show that the two-dimensional piece $V$ is modular; by using a theorem of Faltings-Serre-Livné [20], we are able to prove this by studying the reduction of $\tilde{X}$ modulo a finite set of primes. In practice this amounts to counting the points on $\tilde{X}$ over $\mathbb{F}_p$, a task which can easily be done by computer.

This paper is organized as follows: In section 2 we review the construction and key properties of the Horrocks-Mumford vector bundle. In section 3 we construct the Horrocks-Mumford quintic threefolds $F$ and $G$ as pencils of abelian surfaces. In section 4 we construct the common resolution $\tilde{X}$ of the threefolds $F$ and $G$ and study its geometry. In section 5 we find the elliptic surfaces $E_1$ and $E_2$ in $\tilde{X}$, count points and apply Livné’s method to show that $\tilde{X}$ is modular.

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2. The Horrocks-Mumford vector bundle

2.1. Construction of the bundle. The Horrocks-Mumford vector bundle $HM$ is a stable, indecomposable rank 2 bundle over the complex projective space $\mathbb{P}^4$. It is essentially the only known bundle satisfying these properties; all other such bundles that are currently known are derived from $HM$ by twisting by powers of the sheaf $\mathcal{O}(1)$ or by taking pullbacks to branched covers of $\mathbb{P}^4$. It was first discovered by Horrocks and Mumford in [14], and has been further studied by many other authors (see for example [5], [6], [15], [25]). In this section we will describe the construction of $HM$ and explain some of its properties that we will use later.

The following exposition of the Horrocks-Mumford bundle is taken from [13]. A monad is a three-term complex
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\[ A \xrightarrow{p} B \xrightarrow{q} C \]
of vector bundles where \( p \) is injective and \( q \) is surjective. The cohomology of the monad

\[ E = \text{ker} \ q / \text{im} \ p \]
is also a vector bundle. To construct the Horrocks-Mumford bundle using a monad we fix a vector space

\[ V \cong \mathbb{C}^5. \]
Denote its standard basis by \( e_i, i \in \mathbb{Z}/5 \). On the projective space \( \mathbb{P}^4 = \mathbb{P}(V) \), we have the Koszul complex

\[
0 \rightarrow \mathcal{O} \xrightarrow{\wedge^s} V \otimes \mathcal{O}(1) \xrightarrow{\wedge^s} \wedge^2 V \otimes \mathcal{O}(2) \\
\wedge^s \mathcal{O}(3) \xrightarrow{\wedge^s} \wedge^4 V \otimes \mathcal{O}(4) \rightarrow \mathcal{O}(5) \rightarrow 0.
\]
Now the quotient \( \mathcal{O}(1) \otimes V / \mathcal{O} \) is isomorphic to the tangent sheaf \( T \), and in the Koszul complex the sheaf of cycles \( \text{im}(\mathcal{O}(i) \otimes \wedge^i V) \subset \mathcal{O}(i+1) \otimes \wedge^{i+1} V \) is isomorphic to \( \wedge^i T \). Thus from the map

\[
\mathcal{O}(2) \otimes \wedge^2 V \xrightarrow{\mathcal{O}(1)} \mathcal{O}(3) \otimes \wedge^3 V
\]
we obtain the sequence of maps

\[
\mathcal{O}(2) \otimes \wedge^2 V \xrightarrow{p_0} \wedge^2 T \xrightarrow{q_0} \mathcal{O}(3) \otimes \wedge^3 V
\]
where the first map is surjective and the second is injective.

Horrocks and Mumford defined the following maps

\[
f^+ : V \rightarrow \wedge^2 V, \quad f^+(\sum v_i e_i) = \sum v_i e_{i+2} \wedge e_{i+3}, \\
f^- : V \rightarrow \wedge^2 V, \quad f^-(\sum v_i e_i) = \sum v_i e_{i+1} \wedge e_{i+4}.
\]
Using these maps one can define

\[
p : V \otimes \mathcal{O}(2) \xrightarrow{(f^+ f^-)^{(2)}} 2 \wedge^2 V \otimes \mathcal{O}(2) \xrightarrow{2p_0} 2 \wedge^2 T \\
q : 2 \wedge^2 T \xrightarrow{2q_0} 2 \wedge^3 V \otimes \mathcal{O}(3) \xrightarrow{(-f^- f^+)^{(3)}} V^* \otimes \mathcal{O}(3).
\]
One easily checks that \( q \circ p = 0 \). Hence we obtain a monad

\[
V \otimes \mathcal{O}(2) \xrightarrow{p} 2 \wedge^2 T \xrightarrow{q} V^* \otimes \mathcal{O}(3).
\]
Its cohomology

\[ HM = \text{ker} \ q / \text{im} \ p \]
is the Horrocks-Mumford bundle. It is a rank 2 bundle, and its total Chern class \( c(HM) \) equals \( c(\wedge^2 T)^2 c(V^* \otimes \mathcal{O}(3))^{-1} c(V \otimes \mathcal{O}(2))^{-1} \). Using the splitting principle, one computes this class to be \( 1 + 5H + 10H^2 \). Therefore, zero sets of sections of \( HM \) are surfaces of degree 10; Horrocks and Mumford showed that the generic zero set is a smooth abelian surface.
2.2. Symmetries of $HM$ and invariant quintics. The study of $HM$ has been greatly expedited by the fact that it admits a large group of discrete symmetries. Consider the Heisenberg group of rank 5, which we denote by $H_5$. We present it as a subgroup of $GL_5(\mathbb{C})$ generated by the matrices

$$
\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & \epsilon & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon^3 \end{pmatrix},
$$

where $\epsilon = e^{\frac{2\pi i}{5}}$ is a primitive fifth root of unity. $H_5$ is a central extension

$$1 \to \mu_5 \to H_5 \to \mathbb{Z}/5 \times \mathbb{Z}/5 \to 1$$

where $\sigma$ is sent to $(1, 0)$ and $\tau$ to $(0, 1)$. Here $\mu_5$ is the multiplicative group of fifth roots of unity.

In fact, the normalizer $N_5$ of $H_5$ in $SL_5(\mathbb{C})$ preserves $HM$. $N_5$ is a semidirect product of $H_5$ with the binary icosahedral group $SL(2, \mathbb{Z}_5)$. We will need the following elements of $N_5$:

$$
\iota = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 & \epsilon & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}.
$$

These matrices act on $N_5/\mu_5 \simeq \mathbb{Z}/5 \times \mathbb{Z}/5$ by conjugation; Horrocks and Mumford showed that the action is unimodular. Their images in $SL_2(\mathbb{Z}_5)$ are

$$
\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.
$$

En route to determining the sections of $HM$, Horrocks and Mumford determined the $N_5/H_5$-module $\Gamma_{H_5}(\mathcal{O}(5))$ of $H$-invariants of $\Gamma(\mathcal{O}(5))$, i.e. Heisenberg-invariant quintics in $\mathbb{P}^4$. It is six-dimensional, spanned by the polynomials

$$
\sum x_i^5, \sum x_i^3 x_{i+1} x_{i+4}, \sum x_i x_{i+1}^2 x_{i+4},
$$

$$
\sum x_i^3 x_{i+2} x_{i+3}, \sum x_i x_{i+2}^2 x_{i+3}^2, x_0 x_1 x_2 x_3 x_4
$$

where the sums are taken over powers of $\sigma$. The base locus of this space of quintics is the set of 25 lines $L_{ij}$, where

$$
L_{00} = \{ x \in \mathbb{P}^4 : x_0 = x_1 + x_4 = x_2 + x_3 = 0 \},
$$

$$
L_{ij} = \sigma^i \tau^j L_{00}.
$$

Since $c(HM) = 1 + 5H + 10H^2$, $c_1(\wedge^2(HM)) = 5H$ and thus $\wedge^2(HM) \cong \mathcal{O}(5)$. Hence if $s_1$ and $s_2$ are sections of $HM$, the zero set of the section $s_1 \wedge s_2$ of $\wedge^2 H M$ is a (singular) quintic Calabi-Yau threefold that has the structure of a pencil of abelian surfaces.

**Proposition 2.1.** For generic sections $s_1$ and $s_2$ of $HM$, the singularities of the resulting threefold are the 100 nodes coming from the intersection of $Z(s_1)$ and $Z(s_2)$.
Proof. For a generic choice of \( s_1, s_2, Z(s_1) \) and \( Z(s_2) \) intersect transversely in 100 points; these points form the base locus of the pencil.

Let \( p \) be a point at which \( Z(s_1) \) and \( Z(s_2) \) intersect transversely. Choose a trivialization of \( HM \) near \( p \), and put \( s_1 = (s_{11}, s_{12}) \) and \( s_2 = (s_{21}, s_{22}) \) relative to this trivialization. Since \( Z(s_1) \) and \( Z(s_2) \) intersect transversely, we may then use \( s_{11}, s_{12}, s_{21} \) and \( s_{22} \) as local coordinates on \( \mathbb{P}^4 \) centered at \( p \). The local equation for the threefold is then

\[
\begin{align*}
\begin{aligned}
s_{11} s_{22} - s_{12} s_{21} &= 0.
\end{aligned}
\end{align*}
\]

Hence \( p \) is a node. \( \square \)

Nodes on threefolds result from the vanishing of an \( S^3 \) cycle on a smooth family of threefolds. One expects that degenerating the \( S^3 \) cycles and then resolving the singularities will cause the Betti number \( h^3 \) to drop. We will be interested in birationally equivalent Calabi-Yau threefolds with low Betti number. Taking one-parameter families of abelian surfaces in \( \mathbb{P}^4 \) gives us a quick way of manufacturing nodal Calabi-Yau threefolds, whose singularities can then be resolved. In [25], Schoen studied the Fermat quintic \( Q \) defined by the equation

\[
\begin{align*}
\begin{aligned}
x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0 x_1 x_2 x_3 x_4 &= 0.
\end{aligned}
\end{align*}
\]

Schoen showed that it was a Horrocks-Mumford quintic with 125 nodes instead of the usual 100, and he proved that the blowup \( Q \) of \( Q \) was rigid and modular. Other nodal Calabi-Yau threefolds whose resolutions are modular were studied in [31].

Remark. Instead of manufacturing nodal Calabi-Yaus in \( \mathbb{P}^4 \), one can also use as the ambient space other Fano fourfolds such as \( \mathbb{P}^3 \times \mathbb{P}^1 \); we then want to consider a rank 2 bundle whose determinant bundle is anticanonical. As above, we can then consider surfaces cut out by sections of the bundle and take pencils of these surfaces to obtain other nodal Calabi-Yau threefolds. In [18] and [19], Lange has proven the existence of abelian surfaces in \( \mathbb{P}^1 \times \mathbb{P}^3 \) and by the Serre construction found the rank 2 bundle \( V \) whose zero sections yield these surfaces.

3. Abelian surfaces in \( \mathbb{P}^4 \)

3.1. Sections of \( HM \). In the previous section we mentioned that an abelian surface in \( \mathbb{P}^4 \) is projectively equivalent to \( Z(s) \) for some section \( s \) of \( HM \). Since \( h^0(HM) = 4 \), \( \mathbb{P}^3 \) is a parameter space of (possibly degenerate) abelian surfaces in \( \mathbb{P}^4 \).

Any vector bundle over \( \mathbb{P}^1 \) splits into a direct sum of line bundles; for most lines in \( \mathbb{P}^4 \), the restriction of \( HM \) is isomorphic to \( O(2) \oplus O(3) \). Lines \( L \) such that \( HM|_L \) is isomorphic to \( O(2 - a) \oplus O(3 + a) \) are called jumping lines of order \( a \). It is well known that the 25 lines \( L_{ij} \) are jumping lines of order 3; the restriction of \( HM \) to these lines is isomorphic to \( O(-1) \oplus O(6) \). Barth, Hulek and Moore ([5]) proved that the restriction map \( \Gamma_{HM} \longrightarrow \Gamma_{HM|_{L_{00}}} \) is injective, and they were able to determine the sections of \( \Gamma_{HM|_{L_{00}}} \).

Proposition 3.1. Let \( \lambda \) and \( \mu \) be the restrictions of the coordinates \( x_1 \) and \( x_2 \) to \( L_{00} \). Then the image of \( \Gamma_{HM} \) in \( \Gamma_{HM|_{L_{00}}} \) is spanned by the sections \( t_0 = x_0^6 + 2\mu^5\lambda, t_1 = \mu^6 - 2\mu\lambda^5, t_2 = 5\lambda^4\mu, t_3 = 5\lambda^2\mu^3 \) of \( O(-1) \oplus O(6) \). \( \square \)

Given a section \( s \) of \( HM \), we can associate to it the vector \( (c_0, c_1, c_2, c_3) \) representing the coordinates of \( s_{L_{00}} \) with respect to the basis \( t_0, t_1, t_2, t_3 \). The coordinates
$c_i$ are then homogeneous coordinates on the moduli space $\mathbb{P}^3$ of abelian surfaces in $\mathbb{P}^4$. We will call these coordinates BHM coordinates.

The polynomial $c_0t_0 + c_1t_1 + c_2t_2 + c_3t_3$ determines the singularities of $Z(s)$:

**Theorem 3.2.** Let $s$ be a section of $HM$, and let $f = c_0t_0 + c_1t_1 + c_2t_2 + c_3t_3$ be its restriction to $L_0$. The degeneracies of $Z(s)$ are determined by the multiplicities of the roots $(\lambda : \mu)$ of $f$:

| Multiplicities of roots | Degeneracy of $X(s)$ |
|-------------------------|----------------------|
| $(1, 1, 1, 1, 1)$       | smooth               |
| $(2, 1, 1, 1, 1)$       | translation scroll of elliptic normal curve |
| $(3, 1, 1, 1)$          | tangent scroll of elliptic normal curve     |
| $(2, 2, 1, 1)$          | union of five quadrics                      |
| $(2, 2, 2)$             | doubled elliptic quintic scroll             |
| $(4, 2)$                | union of five double planes                  |

The automorphisms $\mu, \nu$ and $\delta$ of $\mathbb{P}^4$ induce automorphisms, also denoted $\mu, \nu$ and $\delta$, of the moduli space $\mathbb{P}^3$. In BHM coordinates, they have the following form:

$$
\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} \epsilon^4 \\ \epsilon \\ \epsilon^3 \end{pmatrix}, \quad \delta = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 1 & \eta & -\eta \\ 1 & -1 & -\eta & \eta' \\ \eta' & -\eta & 1 & -1 \\ -\eta & \eta' & -1 & 1 \end{pmatrix},
$$

where $\eta = \epsilon + \epsilon^4$ and $\eta' = \epsilon^2 + \epsilon^3$.

**3.2. A pencil of abelian surfaces.** One can ask if the abelian surfaces corresponding to fixed points of these automorphisms have any interesting properties; this is how we found the threefold $X$. Let us find the fixed points of the automorphism $\mu$; these correspond to the eigenspaces of the matrix

$$
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
$$

We find the eigenspace $V_{-1}$ spanned by the vectors $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$ and the eigenspace $V_1$ spanned by $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ These correspond to lines in the moduli space $\mathbb{P}^3$, also denoted $V_{-1}$ and $V_1$. Our threefold $X$ will be derived from the threefolds swept out by the abelian surfaces in $V_{-1}$ and $V_1$.

As in [6], we will use the Clebsch diagonal cubic

$$X_3 = \{ c \in \mathbb{P}^3 : c_0^3c_3 + c_1^2c_2 - c_0c_2^2 - c_1c_3^2 = 0 \}.$$

$X_3$ is the image of the $SL(2, \mathbb{F}_5)$-equivariant rational map $p : \mathbb{P}^2 \to \mathbb{P}^3$ sending $(y_1 : y_2 : y_3)$ to

$$
x = (y_1y_3^2 - y_2^3 : y_3^2 - y_1y_2 : y_2y_3 - y_2y_3^2 : y_3y_1^2 - y_2y_3^2).
$$

The rational map $p$ is undefined at the points $(1 : 0 : 0)$ and $(1 : \epsilon^k : \epsilon^{-k})$; these six points correspond to the six exceptional divisors when we exhibit $X_3$ as $\mathbb{P}^2$ blown up in six points.

Recall the configuration of 27 lines on the cubic surface: the cubic surface is isomorphic to $\mathbb{P}^2$ blown up in six points $p_1, p_2, \ldots, p_6$. The lines $E_m, m = 1, 2, \ldots, 6$
are the exceptional divisors. The lines $F_{mn}, 1 \leq m < n \leq 6$ are the proper transforms of the lines through $p_m$ and $p_n$. The lines $G_n, n = 1, 2, \ldots, 6$ are the proper transforms of the conics through the five points other than $p_n$.

To locate the six exceptional divisors in $X_3$, temporarily dehomogenize $y$ by setting $y_1 = 1$. Also set $y_2 = 0$ and consider what happens when we let $y_3$ approach 0; we see that $p((1 : 0 : 0))$ approaches the point $(0 : 0 : 1)$. Now set $y_3 = 0$ and consider what happens when $y_2$ approaches 0; $p((1 : 0 : 0))$ approaches the point $(0 : 0 : 1 : 0)$. Hence the line $E_0$ in $X_3$ is spanned by the points $(0 : 0 : 1 : 0)$ and $(0 : 0 : 1 : 0)$. Repeating the same local analysis at the other five points, we find that the lines $E_k, k = 1, 2, \ldots, 5$ are spanned by the points $(-3e^{2k} : -2e^k : 1 : -e^{-2k})$ and $(2e^{-k} : 3e^{-2k} : e^{2k} : -1)$.

**Lemma 3.3.** The line $V_{-1}$ is exactly the line $F_{56}$.

**Proof.** One checks that the line $V_{-1}$ intersects the $E$-lines $E_5, E_6$ and no others.

Let $F$ be the threefold swept out by the abelian surfaces parametrized by $F_{56} = V_{-1}$. Let $G$ be the threefold swept out by $V_1$. We can determine what all the fibres of $F$ are:

**Proposition 3.4.** The singular fibres of $F$ are as follows: there are two fibres of type $(2,2,1,1)$ corresponding to unions of five quadrics and two fibres of type $(2,2,2)$ corresponding to doubled elliptic quintic scrolls.

The singular fibres of $G$ are as follows: there are two fibres of type $(2,2,1,1)$ corresponding to unions of five quadrics and two fibres of type $(3,1,1,1)$ corresponding to tangent scrolls of elliptic normal curves.

**Proof.** For $(a : b) \in \mathbb{P}^1$ and $(\lambda : \mu) \in \mathbb{P}^1$, consider the correspondence in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the condition that $(\lambda : \mu)$ is a root of the polynomial $a(t_0 - t_1) + b(t_2 - t_3)$. This is a $(1, 6)$ correspondence and thus has genus zero. Projection to the first copy of $\mathbb{P}^1$ is a map between rational curves of degree 6. By the Riemann-Hurwitz theorem, there are 10 ramification points (counted with multiplicity).

The line $F_{56}$ intersects $E_5$ and $E_6$ at $(2 : -2 : 4 : 4)$ and $(0 : 0 : 1 : -1)$ respectively. The corresponding sextic polynomials in $\lambda, \mu$ have multiplicities $(2,2,1,1)$, so the corresponding fibres are unions of five quadrics each.

Four branch points have already been accounted for. We found that the line $F_{56}$ intersects the curve $G_6$ in the points $(5 : -5 : 2i + 1 : -2i - 1)$ and $(5 : -5 : -2i + 1 : 2i - 1)$. These points correspond to polynomials of type $(2,2,2)$. They account for the other six ramification points. We have therefore found all the singular fibres of $F$.

For $G$, we look at the correspondence on $\mathbb{P}^1 \times \mathbb{P}^1$ where $(\lambda : \mu)$ is a root of $a(t_0 + t_1) + b(t_2 + t_3)$. This time, the correspondence has two horizontal components because $(1 : i)$ and $(1 : -i)$ are common roots of $t_0 + t_1$ and $t_2 + t_3$. The remaining part of the correspondence is a $(1, 4)$ curve that maps surjectively onto the first copy of $\mathbb{P}^1$, so there are 6 ramification points.

By inspection, the polynomial $a(t_0 + t_1) + b(t_2 + t_3)$ has repeated roots when $(a : b) = (1 : 0), (0 : 1), (1 : \frac{3 + 4i}{5})$ and $(1 : \frac{3 - 4i}{5})$ of type $(2,2,1,1), (2,2,1,1), (3,1,1,1)$ and $(3,1,1,1)$ respectively. Eliminating the common roots $(1 : \pm i)$, the remaining roots have multiplicity $(2,2), (2,2), (2,1,1)$ and $(2,1,1)$ respectively. This accounts for the 6 ramification points.

Once we have the defining equation for $F$, simple calculations will show that the quadric surface $T_0 = \{ x : x_0 = x_1x_4 - x_2x_3 = 0 \}$ and its translates $T_i = \sigma^iT_0$
are contained in $F$; the $T_i$ are $\tau$-invariant. Put $U_0 = \delta T_0$; we then have $U_0 = \{ x : \Sigma_i x_i = \Sigma_{i \neq j} x_i x_j = 0 \}$. The quadric surface $U_0$ and its translates $U_i = \tau^i U_0$ are contained in $F$ as well; the $U_i$ are $\sigma$-invariant. Since the surfaces $\cup_i T_i$ and $\cup_i U_i$ are of degree 10 and invariant under $H_5$, they must be two of the singular fibres in the abelian surface fibration of $F$.

Similarly, one checks that the quadric surface $Q_0 = \{ z : z_0 = z_1 z_4 + z_2 z_3 = 0 \}$ and its translates $Q_i = \sigma^i Q_0$ are in $G$, and that the quadric surface $R_0 = \delta Q_0 = \{ z : \Sigma_i z_i = \Sigma_{i \neq j} z_i z_j = 0 \}$ and its translates $Q_i = \tau^i Q_0$ are in $G$. These two unions of five quadrics are two of the singular fibres in the fibration of $G$.

We can identify the other fibres of $F$; first we need to identify certain elliptic curves in $\mathbb{P}^4$. Aure, Decker, Hulek, Popescu and Ranestad [11] have shown that the set of $H_5$-equivariant elliptic normal curves in $\mathbb{P}^4$ is parametrized by $\mathbb{P}^1$, to the point $(\lambda : \mu)$ corresponds the elliptic normal curve $E_{\lambda(\mu)}$ defined by the set of equations

\begin{equation}
q_{i}^{(\lambda, \mu)}(x) = -\lambda \mu x_i^2 - \mu^2 x_{i+1} x_{i+4} + \lambda^2 x_{i+2} x_{i+3}.
\end{equation}

By inspection, we see that the curves corresponding to $(1 : i)$ and $(1 : -i)$ are in $F$ and $G$. Denote these curves by $E_1$ and $E_2$ respectively. Proposition 4.3 in [11] shows that the elliptic quintic scrolls $Q_1$ and $Q_2$ contain $E_1$ and $E_2$ respectively, where $Q_1$ is defined by the equations

\begin{equation}
x_i^3 + x_i x_{i+1} x_{i+4} + x_i x_{i+2} x_{i+3} - i (x_{i+1} x_{i+3} + x_i x_{i+4} + x_{i+1} x_{i+2} + x_{i+3} x_{i+4}) = 0
\end{equation}

and $Q_2$ is defined by replacing $i$ above with its complex conjugate $-i$. Simple calculations show that $Q_1$ and $Q_2$ are contained in $F$. Being $H_5$-invariant, they must be the elliptic quintic scroll fibers of $F$.

4. The threefold $X$

\subsection{Definition of $X$}

Although we have defined the Horrocks-Mumford bundle only over $\mathbb{C}$, the pencils of abelian surfaces it defines are quintics in $\mathbb{P}^4$, and the quintics we are interested in have integer coefficients and can thus be studied over arbitrary fields $k$.

In particular, we have the Horrocks-Mumford quintics $F$ and $G$ defined by the lines $V_{-1}$ and $V_1$ in $\mathbb{P}^4$. In [21], Manolache was able to determine the equation of the Horrocks-Mumford quintic in terms of the BHM-coordinates of the parametrizing line:

\textbf{Theorem 4.1.} Suppose the Horrocks-Mumford quintic $X$ is determined by the line passing through the points $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$. Then the defining equation for $X$ is

\begin{equation}
25(a_3 b_4 - a_4 b_3)(\Sigma x_0 x_1 x_2 x_3 x_4)
+ 5(a_2 b_3 - a_3 b_2)(\Sigma x_0 x_2^2 x_3^2)
- 5(a_1 b_3 - a_3 b_1)(\Sigma x_0^2 x_2^2 x_3)
+ 5(a_2 b_4 - a_4 b_2)(\Sigma x_0^3 x_1 x_4)
- 5(a_1 b_4 - a_4 b_1)(\Sigma x_0^2 x_2^2 x_4)
+ (a_1 b_2 - a_2 b_1)(\Sigma x_0^5 - x_0 x_1 x_2 x_3 x_4)
\end{equation}

where the sums are taken over cyclic permutations of the indices.
An easy calculation shows that $F$ is defined by the equation

$$
\Sigma_i (x_i^3 x_{i+1} x_{i+4} + x_i^3 x_{i+2} x_{i+3} - x_i x_{i+1}^2 x_{i+4}^2 - x_i^2 x_{i+2}^2 x_{i+3}^2) = 0
$$

and that $G$ is defined by the equation

$$
\Sigma_i (z_i^3 z_{i+1} z_{i+4} - z_i^3 z_{i+2} z_{i+3} - z_i z_{i+1}^2 z_{i+4}^2 + z_i z_{i+2}^2 z_{i+3}^2) = 0.
$$

where the summations are taken over cyclic permutations of the indices. As before, $F$ and $G$ are both invariant under the action of the matrices $\sigma$ and $\tau$.

Consider now the complete intersection threefold $X$ in $\mathbb{P}^4(x) \times \mathbb{P}^4(z)$ given by the matrix equation

$$
M(x)z = 0,
$$

where

$$
M(x) = \begin{pmatrix}
-x_3 & x_1 & x_4 & -x_2 \\
-x_3 & -x_4 & x_2 & x_0 \\
x_1 & -x_4 & -x_0 & x_3 \\
-x_2 & x_0 & x_3 & -x_1
\end{pmatrix}.
$$

Note that this is equivalent to the matrix equation

$$
L(z)x = 0,
$$

where

$$
L(z) = \begin{pmatrix}
z_2 & -z_4 & -z_1 & z_3 \\
z_4 & z_3 & -z_0 & -z_2 \\
-z_3 & -z_0 & z_4 & -z_1 \\
z_1 & -z_3 & -z_0 & z_2
\end{pmatrix}.
$$

Note also that $\det M(x)$ and $\det L(z)$ give us the equations for $F$ and $G$ respectively (up to a factor of 2). Hence the projections $\pi_1$ and $\pi_2$ of $X$ onto each factor give us $F$ and $G$. In [23], Moore first considered the matrices

$$
M(x, y) = \begin{pmatrix}
x_0 y_0 & x_3 y_2 & x_1 y_4 & x_4 y_1 & x_2 y_3 \\
x_3 y_3 & x_1 y_0 & x_4 y_2 & x_2 y_4 & x_0 y_1 \\
x_1 y_1 & x_4 y_3 & x_2 y_0 & x_0 y_2 & x_3 y_4 \\
x_4 y_4 & x_2 y_1 & x_0 y_3 & x_3 y_0 & x_1 y_2 \\
x_2 y_2 & x_0 y_4 & x_3 y_1 & x_1 y_3 & x_4 y_0
\end{pmatrix},
$$

$$
L(z, y) = \begin{pmatrix}
z_0 y_0 & z_2 y_4 & z_4 y_3 & z_1 y_2 & z_3 y_1 \\
z_4 y_1 & z_1 y_0 & z_3 y_4 & z_0 y_3 & z_2 y_2 \\
z_3 y_2 & z_0 y_1 & z_2 y_0 & z_4 y_4 & z_1 y_3 \\
z_2 y_3 & z_4 y_2 & z_1 y_1 & z_3 y_0 & z_0 y_4 \\
z_1 y_4 & z_3 y_3 & z_0 y_2 & z_2 y_1 & z_4 y_0
\end{pmatrix}.
$$

For a generic choice of $y \in \mathbb{P}^4$, the threefold determined by $\det M(x, y) = 0$ and the threefold determined by $\det L(z, y) = 0$ are both Horrocks-Mumford quintics with
the expected 100 nodes. For our threefolds $F$ and $G$, we have taken $y = (0 : 1 : -1 : -1 : 1)$.

4.2. Singularities of $X$. We need to know the singularities of $X$:

**Proposition 4.2.** Over a field of characteristic not equal to 2 or 5, $X$ has 60 singular points. The 60 singular points of $X$ are all ordinary double points (nodes).

**Remark.** In characteristic 0, Gross and Popescu [11] have studied the two-parameter family of threefolds $X_y$ given by performing the construction above for $y$ a generic point of the plane

$$\mathbb{P}^2 = \{ y : y_1 - y_4 = y_2 - y_3 = 0 \}.$$  

Thus $X_y$ is a common partial resolution of the quintic threefolds $F_y$ and $G_y$. Our threefold $X$ is thus a special member of this family. Over $\mathbb{C}$, Gross and Popescu proved the following statements for a generic choice of $y$:

1. $F_y$ is singular along the union of two elliptic curves $D_{1,y}$ and $D_{2,y}$. These curves are the base curves of the elliptic quintic scrolls $Q_{1,y}$ and $Q_{2,y}$ appearing in the abelian surface fibration of $F_y$, and they intersect in the 25 points comprising the $H_5$ orbit of $y$. These 25 points comprise the base locus of the pencil of abelian surfaces.

2. $G_y$ is singular along a disjoint union of two elliptic curves $E_{1,y}$ and $E_{2,y}$.

3. $X_y$ is singular along 50 nodes, 2 of which lie over each point of the Heisenberg orbit of $y$ when projecting via the map $\pi_1$. We call these 50 nodes regular nodes.

Essentially our threefold $X$ is a special member of the family $X_y$ where $y = (0 : 1 : -1 : -1 : 1)$. Over $\mathbb{C}$, it is easy to check that $X$ has the 50 expected nodes and 10 others. Over fields of arbitrary characteristic, we resort to brute-force computation. A basic trick we use is to compute Gröbner bases of ideals over $\mathbb{Z}$ in order to obtain results valid in fields of unknown characteristic.

We will begin the proof of the proposition after we dispense with some preliminary results. If $x$ is a point in $\mathbb{P}^4(x)$, we say that $x$ is a rank $n$ point if rank $M(x) = n$. Similarly, if $z$ is a point in $\mathbb{P}^4(z)$, we say that $z$ is a rank $n$ point if rank $L(z) = n$.

**Lemma 4.3.** *If $x$ is a point of $F$, then $x$ is a rank 4 point or a rank 3 point.*

**Proof.** If $x$ is a point of $F$, then det $M(x) = 0$. Hence the rank of det $M(x)$ is at most 4.

The 3 by 3 matrices

$$\begin{pmatrix} -x_3 & x_1 \\ -x_4 & x_2 \\ x_1 & -x_3 \\ -x_2 & x_4 \\ x_2 & -x_1 \\ x_0 & -x_2 \\ -x_0 & x_3 \\ x_3 & -x_1 \\ -x_1 & x_4 \\ x_4 & -x_0 \\ -x_2 & x_3 \\ x_0 & -x_1 \\ -x_1 & x_4 \\ x_3 & -x_0 \\ x_4 & x_2 \\ -x_0 & x_3 \\ x_2 & -x_1 \\ x_1 & x_3 \\ -x_4 & x_2 \\ x_1 & -x_2 \\ -x_3 & x_0 \\ x_0 & -x_1 \\ -x_3 & x_4 \\ x_2 & x_1 \\ -x_2 & x_3 \\ x_4 & -x_0 \\ -x_1 & x_4 \\ x_3 & -x_0 \\ x_0 & x_2 \\ -x_4 & x_1 \\ x_2 & x_3 \\ -x_3 & x_0 \\ x_4 & x_2 \\ x_1 & x_3 \\ -x_4 & x_0 \\ x_0 & x_2 \\ -x_1 & x_4 \\ x_3 & -x_0 \\ x_1 & x_2 \\ -x_3 & x_1 \\ x_4 & x_3 \\ -x_0 & x_2 \\ x_4 & x_0 \\ -x_2 & x_1 \\ x_3 & -x_0 \\ x_0 & x_4 \\ -x_1 & x_2 \\ x_2 & x_0 \\ -x_3 & x_4 \\ x_3 & -x_0 \\ x_4 & x_2 \\ x_1 & x_0 \\ -x_4 & x_3 \\ x_2 & x_0 \\ -x_1 & x_4 \\ x_0 & x_3 \\ -x_2 & x_1 \\ x_1 & x_2 \\ -x_3 & x_4 \\ x_3 & -x_0 \\ x_4 & x_2 \end{pmatrix}$$
are all 3 by 3 minors of \( M(x) \). Their determinants give us \( \pm 2x_i x_j x_k \) for all subsets \( \{i, j, k\} \) of \( \{0, 1, 2, 3, 4\} \). Suppose these determinants are all zero. Then three of the \( x_i \) must be zero.

Without loss of generality, we can assume either that \( x_0, x_1, x_2 = 0 \) or that \( x_0, x_2, x_4 = 0 \). Suppose the former holds. One sees then that the matrix

\[
\begin{pmatrix}
-x_3 & x_4 \\
-x_4 & -x_3 \\
-x_4 & x_0 \\
x_0 & x_3
\end{pmatrix}
\]

has at least three linearly independent columns. A similar result holds of \( x_0, x_2, x_4 \) are zero. Hence the rank of \( M(x) \) is at least 3 for all \( x \).

\[\square\]

**Lemma 4.4.** If \( z \) is a point of \( G \), then \( z \) is a rank 4 point or a rank 3 point.

**Proof.** Again, if \( z \) is a point of \( G \), then \( \det L(z) = 0 \). Hence the rank of \( \det L(z) \) is at most 4.

The polynomials \( \pm(z_1 z_{i+1}^2 + z_{i+2}^2 z_{i+3}) \) and \( \pm(z_1^2 z_{i+3} + z_{i+2}^2 z_{i+4}) \) are all determinants of 3 by 3 minors of \( L(z) \). Suppose that these minors are all zero. Without loss of generality, assume \( z_0 = 1 \).

Since \( z_0 z_1^2 + z_2^2 z_3 = 0 \), we have \( z_1^2 = -z_2^2 z_3 \). From \( z_1 z_2^2 + z_1 z_2 = 0 \), we get \( z_4 = -z_2^2 z_2 = z_2^2 z_3 \). From \( z_2 z_3^2 + z_2 z_0 = 0 \), we get \( z_2 z_3(1 + z_2^2) = 0 \).

On the other hand, we also have \( z_2^2 z_2 + z_2^2 z_3 = 0 \). From this we see that \( z_2^2 z_3 - z_2^2 z_3 = 0, \) or \( (z_2^2 - 1)z_2^2 z_3 = 0 \). Thus we must have either \( z_2 = 0 \) or \( z_3 = 0 \). In either case, we have \( z = (1: 0 : 0 : 0 : 0) \), which is a rank 4 point. Hence there are no points of rank lower than 3.

\[\square\]

**Lemma 4.5.** Points of \( F \) or \( G \) of rank less than 4 are singular points.

**Proof.** More generally, let \( A \) be an \( n \) by \( n \) matrix whose \( ij \) entry is the indeterminate \( x_{ij} \). Using the Laplace expansion formula, we see that \( \frac{\partial \det(A)}{\partial x_{ij}} \) is equal to \( (-1)^{i+j} \det A_{ij} \), where \( A_{ij} \) is the \( ij \) minor of \( A \).

Now suppose that the \( x_{ij} \) are functions of some other indeterminates \( y_k \). By the chain rule, \( \frac{\partial \det(A)}{\partial y_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} \det A_{ij} \frac{\partial x_{ij}}{\partial y_k} \). If \( y \) is a point of rank less than \( n \), then the determinants of all the \( (n-1) \) by \( (n-1) \) minors are zero, and hence \( \frac{\partial \det(A)}{\partial y_k} \) is zero for all \( k \).

\[\square\]

**Lemma 4.6.** The projections \( \pi_1 : X \to F \) and \( \pi_2 : X \to G \) are isomorphisms outside the rank 3 loci of \( F \) and \( G \) respectively.

**Proof.** If \( x \) is a rank 4 point of \( \mathbb{P}^4(x) \), this means that the kernel of \( M(x) \) is 1-dimensional. Hence the kernel of \( M(x) \) defines a unique point \( z \) in \( \mathbb{P}^4(z) \). We thus have a regular map \( f_1 : F - F_4 \to \pi^{-1}(F - F_4) \).

Now given a point \( (x, z) \) in \( X \) such that \( x \) is a rank 4 point, the coordinates of \( z \) are given by the determinants of \( 4 \) by \( 4 \) minors of \( M(x) \). Hence the map \( \pi_1 : \pi^{-1}(F - F_4) \to (F - F_4) \) is also regular, proving the result for \( F \). A similar result holds for \( G \).

\[\square\]
Lemma 4.7. Suppose the characteristic of the base field is not 2. Then the rank 3 locus of $G$ is the union $E$ of $E_1$ and $E_2$, where $E_1$ is cut out by the polynomials $iz_i^2 + zi_{i+1}z_{i+4} + zi_{i+2}z_{i+3}$ and $E_2$ is cut out by the polynomials $-iz_i^2 + zi_{i+1}z_{i+4} + zi_{i+2}z_{i+3}$.

Proof. This is proven by using Macaulay2. One shows that the radicals of the ideals defining the two varieties have identical Gröbner bases, up to constant factors of 2 and 5.

Question. Is there a geometric explanation for this result?

Lemma 4.8. If the characteristic of the base field is not 2 or 5, $E_1$ and $E_2$ are elliptic normal curves.

Proof. This was proven by Fisher in Chapter 4 of [7]. Assuming the characteristic of the base field is not 5, Fisher constructs the universal curve $X(5) \subset X(n) \times \mathbb{P}^4$ as the closure of the scheme defined by the 4 by 4 Pfaffians of the 5 by 5 matrix

$$
\begin{pmatrix}
-a_1x_1 & -a_2x_2 & a_2x_3 & a_1x_4 \\
-a_1x_1 & -a_1x_3 & -a_2x_4 & a_2x_0 \\
a_2x_2 & a_1x_3 & -a_1x_0 & -a_2x_1 \\
a_2x_3 & a_2x_4 & a_1x_0 & -a_1x_2 \\
a_1x_4 & -a_2x_0 & a_2x_1 & a_1x_2
\end{pmatrix}
$$

where $a = (0 : a_1 : a_2 : -a_2 : -a_1)$. One checks that $E_1$ and $E_2$ are the curves obtained when $a = (0 : 1 : -i : i : -1)$ and $(0 : 1 : i : -i : 1)$ respectively. By considering the $SL_2(\mathbb{Z}/5\mathbb{Z})$ action on $X(5)$, Fisher shows that the fibers are smooth elliptic normal curves when $a_1a_2(a_1^{10} - 11a_1^5a_2^5 - a_2^{10}) \neq 0$, in analogy with the case where the base field is $\mathbb{C}$.

Note that $S_1 = \pi_2^{-1}(E_1)$ and $S_2 = \pi_2^{-1}(E_2)$ are elliptic ruled surfaces in $X$.

We can now classify the singularities of $X$.

Proof of Proposition 4.12. A point $(x, z)$ of $X$ is a singular point if and only if the kernel of the matrix $(L(z) \ M(x))$ has dimension at least 6, i.e. if the rank is at most 4. This is equivalent to saying that the rank of the transposed matrix

$$
\begin{pmatrix}
L^T(z) \\
M(x)
\end{pmatrix}
$$

is at most 4, which is equivalent to saying that the kernel of $L^T(z)$ and the kernel of $M(x)$ have nontrivial intersection.

Case 1. $x$ is a rank 4 point.

Since $(x, z)$ is a point of $X$, $z$ is in the kernel of $M(x)$. Therefore $z$ must span the space ker $L^T(z)$ in $M(x)$. Hence $L^T(z)z = 0$. Simple algebra shows that the only singular points $(x, z)$ in this case are $((1 : 0 : 0 : 0 : 0), (1 : 0 : 0 : 0 : 0))$ and $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ and their orbits under $(\sigma, \sigma)$ and $(\tau^3, \tau)$. We will call these nodes in $X \sigma = \text{nodes}$ and $\tau = \text{nodes}$ respectively. We will also use the same nomenclature for the images of these nodes in $F$ and $G$.

Case 2. $x$ is a rank 3 point.

We will suppose that $z$ must then be a rank 3 point. Using Macaulay2, we show that if $(x, z)$ is a singular point of $X$ and $x$ is a rank 3 point of $F$, then some $x_i$ or some $z_i$ is zero. (The code can be found in the Appendix.)
Suppose that some \( x_i \) is zero. Without loss of generality, suppose \( x_0 \) is zero. The polynomials \((x_1x_2 + x_4^2)^2, (x_2x_4 + x_3^2)^2, (x_1x_3 + x_2^2)^2, (x_1x_4 + x_3^2)^2 \) and \((x_2x_3 - x_1x_4)^2\) are all \( 4 \times 4 \) minors of \( M(x) \) when \( x_0 = 0 \).

If \( x_1 \) is also zero, we quickly see that all \( x_i \) are zero, a contradiction. So suppose \( x_1 = 1 \). We quickly see that \( x_4 = \epsilon^k, x_2 = -\epsilon^{2k} \) and \( x_3 = -\epsilon^{4k} \). Solving for \( z \) gives us the singular points

\[(x, z) = (0 : 1 : -\epsilon^k : -\epsilon^{2k} : \epsilon^{3k}, 0 : 1 : \pm i\epsilon^{2k} : \mp i\epsilon^{4k} : -\epsilon^k),\]

whose orbits under \((\sigma, \sigma)\) will be the 50 regular nodes of \( X \).

Now suppose instead that some \( z_i \) is zero. Without loss of generality, we may assume \( z_0 = 0 \). If we assume that \( z \) is a rank 3 point of \( G \), then \( z \) is a point of \( E = E_1 \cup E_2 \) with \( z_0 = 0 \). We have equations defining \( E \), and using these equations we recover the 50 points above.

If \( z \) is a rank 4 point, then \( z \) must itself be a rank 4 node of \( G \), since the map \( \pi_2 : X \rightarrow G \) is an isomorphism on the rank 4 locus. Again using Macaulay2, we can find generators of an ideal that defines the set of singular points on \( G \) with \( z_0 = 0 \). We eventually find one of two things: either \( z \) is a \( \sigma \)-node, all of whose corresponding \( x \) are not rank 3 points, or \( z \) is of the form \((0 : 1 : \pm i\epsilon^k : \mp i\epsilon^{2k} : -\epsilon^{3k})\), which are not rank 4 points of \( G \). So we have found no new nodes.

We have assumed throughout that our nodes have \( x_0 = 0 \) or \( z_0 = 0 \); taking the orbits of \((x, z)\) under \((\sigma, \sigma)\), we get all possible values of \( z \) for which there is a node \((x, z)\) with \( x \) and \( z \) rank 3 points.

There are 60 singular points on \( X \): the \( \sigma \)-nodes; the \( \tau \)-nodes; and the 50 regular nodes given by \((0 : 1 : -1 : -1 : 1), (0 : 1 : \pm i : -\pm i : -1)\) and their orbits under \((\sigma, \sigma)\) and \((\tau^3, \tau)\).

To prove that the 60 singular points are indeed nodes, we need to construct local coordinates around each singular point.

\[((1 : 0 : 0 : 0 : 0), (1 : 0 : 0 : 0 : 0))\): Dehomogenize by fixing \( x_0 = 1 \) and \( z_0 = 1 \). \( X \) is then defined by the five equations

\[
\begin{align*}
-x_3z_1 + x_1z_2 + x_4z_3 - x_2z_4 &= 0, \\
-x_3 - x_4z_2 + x_2z_3 + z_4 &= 0, \\
x_1 - x_4z_1 - z_3 + x_3z_4 &= 0, \\
x_4 + x_2z_1 - z_2 - x_1z_4 &= 0, \\
x_2 + z_1 + x_3z_2 - x_1z_3 &= 0.
\end{align*}
\]

(15)

Let \( I \) be the ideal generated by these five polynomials.

Notice that \( z_4 = x_3 \) plus terms of higher order after we complete the coordinate ring at the ideal \((x_1, \ldots, x_4, z_1, \ldots, z_4)\). Similarly \( z_3 = x_1 + \text{h.o.t.} \), \( z_2 = x_4 + \text{h.o.t.} \) and \( z_1 = x_2 + \text{h.o.t.} \).

Replacing the \( z \)'s in this manner, we see that \( X \) is defined by the single equation

\[
-x_3x_2 + x_1x_4 + x_4x_1 - x_2x_3 + \text{h.o.t.} = 0
\]

in the variables \( x_1, x_2, x_3, x_4 \). Let \( J \) be the ideal generated by this formal power series. We see that the completed local ring \( \overline{k}[[x_1, \ldots, x_4, z_1, \ldots, z_4]]/I \) is isomorphic to the completed local ring \( \overline{k}[[x_1, \ldots, x_4]]/J \).

The second-order piece of equation (16) reads...
\[ x^T B x = 0 \] where \[ B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \]. The determinant of \( B \) is 1. Therefore, we can find a matrix \( C \) such that \( B = C^T C \). Put \( y = Cx \); we now have an isomorphism

\[ \mathbb{K}[x_1, \ldots, x_4]/J \rightarrow \mathbb{K}[y_1, \ldots, y_4]/L \]

where \( L \) is generated by some formal power series of the form \( y_1^2 + y_2^2 + y_3^2 + y_4^2 + h.o.t. \) and \( Cx \rightarrow y \).

Finally, we show that there is an isomorphism

\[ \mathbb{K}[y_1 \ldots y_4]/L \rightarrow \mathbb{K}[t_1, \ldots, t_4]/M \]

where \( M \) is generated by \( t_1^2 + t_2^2 + t_3^2 + t_4^2 \). Put \( t_i = y_i + P_{i2}(y) + P_{i3}(y) + \ldots \) where \( P_{ik} \) is a monomial of order \( k \) in the \( y_i's \). \( L \) is generated by \( y_1^2 + y_2^2 + y_3^2 + y_4^2 + L_3(y) + \ldots \). We can find \( P_{i2} \) such that \( \sum_2 2y_iP_{i2} = L_3(y) \), and we can solve for higher \( P_{ik} \) recursively. The resulting expressions for the \( t_i \) define the isomorphism. At last we see that our singular point is a node.

\(((1 : 1 : 1 : 1), (1 : 1 : 1 : 1))\): Dehomogenize by setting \( x_0 = 1 \) and \( z_0 = 1 \). Now put \( x_i = u_i + 1 \) and \( y_i = w_i + 1 \) for \( i = 1, 2, 3, 4 \). The defining equations for \( X \) are now

\[(17)\]

\[
\begin{align*}
    u_1 - u_2 - u_3 + u_4 - w_1 + w_2 + w_3 - w_4 + (-u_3w_1 + u_1w_2 + u_4w_3 - u_2w_4) &= 0, \\
    u_2 - u_3 - u_4 - w_2 + w_3 + w_4 + (u_4w_2 + u_2w_3) &= 0, \\
    u_1 - u_3 - u_4 - w_1 - w_3 + w_4 + (u_4w_1 + u_3w_4) &= 0, \\
    -u_1 + u_2 + u_4 + w_1 - w_2 - w_4 + (u_2w_1 - u_1w_4) &= 0, \\
    -u_1 - u_2 + u_3 + w_1 + w_2 - w_3 + (u_3w_2 - u_1w_3) &= 0.
\end{align*}
\]

Adding them up, we get

\[(18)\]

\[
\begin{align*}
    u_1w_2 - u_1w_3 - u_1w_4 + u_2w_1 + u_2w_3 - u_2w_4 \\
    - u_3w_1 + u_3w_2 + u_3w_4 - u_4w_1 - u_4w_2 + u_4w_3 &= 0.
\end{align*}
\]

Again we can eliminate the \( w_i \); we then have

\[(19)\]

\[
\begin{align*}
    u_1(u_2 - u_3 - u_4) + u_2(u_1 + u_3 - u_4) \\
    + u_3(-u_1 + u_2 + u_4) + u_4(-u_1 - u_2 - u_3) + h.o.t. &= 0.
\end{align*}
\]

after passing to the completion of the coordinate ring.

The second-order piece of \( B \) can be written as

\[
2u^T Bu = 0
\]

with \( B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \). The determinant of \( B \) is 5. Therefore, \(((1 : 1 : 1 : 1), (1 : 1 : 1 : 1))\) is a node.
\((1 : -1 : -1 : 1 : 0), (1 : i : -i : -1 : 0))\): Again dehomogenize by setting \(x_0 = z_0 = 1\). Put \(x = u + (-1, -1, 1, 0), z = w + (i, -1, 1, 0)\). In the \((u, w)\) coordinates, the defining equations for \(X\) are

\[-iu_1 - iu_3 - u_4 - w_1 - w_2 + w_4 + (u_1 w_2 - u_2 w_4 - u_3 w_1 + u_4 w_3) = 0,\]
\[-u_2 - u_3 + iu_4 - w_3 + w_4 + (u_2 w_3 - u_4 w_2) = 0,\]
\[u_1 - iu_4 - w_3 + w_4 + (u_3 w_4 - u_4 w_1) = 0,\]
\[iu_2 + u_4 - w_1 - w_2 + w_4 + (-u_3 w_4 + u_2 w_1) = 0,\]
\[u_1 - u_2 - u_3 + w_1 + w_2 + w_3 + (-u_1 w_3 + u_3 w_2) = 0.\]  

(20)

From these equations, we can obtain the equation

\[-u_1 w_2 + u_2 w_4 + u_3 w_1 - u_4 w_3 + iu_2 w_3 \]
\[-iu_4 w_2 - iu_3 w_4 + iu_4 w_1 - u_1 w_4 + u_2 w_1 = 0.\]  

(21)

This time we want to eliminate \(u_1, u_2, u_3\) and \(w_1\). We have

\[u_1 = iu_4 + w_3 - w_4 + h.o.t.,\]
\[u_2 = iu_4 + iw_3 + \frac{2 - i}{5}w_4 + h.o.t.,\]
\[u_3 = (-1 - i)w_3 + \frac{3 + i}{5}w_4 + h.o.t.,\]
\[w_1 = -w_2 - w_3 + \frac{6 + 2i}{5}w_4 + h.o.t.\]  

(22)

We then get the equation

\[-4iu_4 w_2 - (2 + 2i)u_4 w_3 \]
\[-\frac{4 - 12i}{5}u_4 w_4 - (4 - 2i)w_3 w_4 + \frac{14 - 2i}{5}w_2^2 + h.o.t. = 0.\]  

(23)

The second-order part of this equation can be written in matrix form:

\[v^T B v = 0\]  

(24)

with \(v = (u_4, w_2, w_3, w_4)\) and

\[B = \begin{pmatrix} -2i & -1 - i & \frac{-2 + 6i}{5} \\ -1 - i & -2 + i & \frac{14 - 2i}{5} \\ \frac{-2 + 6i}{5} & \frac{14 - 2i}{5} & -2 + i \end{pmatrix}.\]

The determinant of this matrix is \(-12 + 16i\). So this point is a node. By symmetry, all of our 60 singular points are nodes.

Note that the surface \(S_1\) contains the 25 nodes

\((\sigma^i \tau^3 j, \sigma^i \tau^j)((0 : 1 : -1 : -1 : 1), (0 : 1 : -i : -1 : -1))\)

and that \(S_2\) contains the 25 nodes

\((\sigma^i \tau^3 j, \sigma^i \tau^j)((0 : 1 : -1 : -1 : 1), (0 : 1 : i : -1 : -1)).\)

Furthermore, the surfaces \(\pi_1^{-1} T_i\) contain the 5 \(\sigma\)-nodes and the surfaces \(\pi_1^{-1} U_i\) contain the 5 \(\tau\)-nodes. Hence we can blow up these surfaces in some order to obtain
a projective small resolution $\tilde{X}$ of $X$ over $\mathbb{Q}(\epsilon)$. However, we cannot obtain a model for any small resolution over $\mathbb{Q}$ because the determinant of the local quadratic equation at the node $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$, which is not a square in $\mathbb{Q}$.

Assume from now on that the characteristic of our base field is different from 2 and 5. Since the singularities of $X$ are 60 nodes, we can blow up the nodes to get a smooth threefold $\tilde{X}$. (Note that the equations defining the union of the 60 points are defined over $\mathbb{Z}$, so the blowup is defined over $\mathbb{Z}$.) Each exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and we will need to know when the rulings on each $\mathbb{P}^1 \times \mathbb{P}^1$ are defined over $\mathbb{F}_p$:

**Proposition 4.9.** If $p \equiv 1(\text{mod } 20)$, then all 60 nodes are defined over $\mathbb{F}_p$. If $p \equiv 11(\text{mod } 20)$, then only the 10 $\sigma$-nodes and $\tau$-nodes are defined over $\mathbb{F}_p$. If $p \equiv 9, 13, 17(\text{mod } 20)$, then the $\sigma$-nodes $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ and 10 of the regular nodes are defined over $\mathbb{F}_p$. Otherwise only the $\sigma$-nodes and $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ are defined over $\mathbb{F}_p$.

**Proof.** Whether or not our nodes are defined over $\mathbb{F}_p$ depends on the values of $p$ modulo 4 and 5. If $p \equiv 1(\text{mod } 4)$ and $p \equiv 1(\text{mod } 5)$, then both $i$ and $\epsilon$ are in $\mathbb{F}_p$, and hence all 60 nodes are defined over $\mathbb{F}_p$. If $p \equiv 3(\text{mod } 4)$ and $p \equiv 1(\text{mod } 5)$, then the $\sigma$-nodes and $\tau$-nodes are all defined over $\mathbb{F}_p$. If $p \equiv 1(\text{mod } 4)$ and $p \equiv 2, 3, 4(\text{mod } 5)$, then the $\sigma$-nodes, $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ and the nodes $(0 : 1 : -1 : -1 : 1), (0 : 1 : \pm i : \mp i : -1))$ and their orbits under $(\sigma, \tau)$ are defined over $\mathbb{F}_p$. Finally, if $p \equiv 3(\text{mod } 4)$ and $p \equiv 2, 3, 4(\text{mod } 5)$ then only the $\sigma$-nodes and $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ are defined over $\mathbb{F}_p$. Using the Chinese Remainder Theorem gives us the result. 

**Proposition 4.10.** If $p \equiv 1(\text{mod } 20)$, all 60 nodes have blowups whose rulings are defined over $\mathbb{F}_p$. If $p \equiv 11(\text{mod } 20)$, then the $\sigma$-nodes and $\tau$-nodes have rulings defined over $\mathbb{F}_p$. If $p \equiv 9(\text{mod } 20)$, then the $\sigma$-nodes, $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ and the 10 regular nodes have rulings defined over $\mathbb{F}_p$. If $p \equiv 13, 17(\text{mod } 20)$, then the $\sigma$-nodes and the 10 regular nodes have rulings defined over $\mathbb{F}_p$. If $p \equiv 19(\text{mod } 20)$, then the $\sigma$-nodes and $((1 : 1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1))$ have rulings defined over $\mathbb{F}_p$. Otherwise only the $\sigma$-nodes have rulings defined over $\mathbb{F}_p$.

**Proof.** Given a node defined over $\mathbb{F}_p$, the rulings of the exceptional divisor are defined over $\mathbb{F}_p$ if and only if the determinant of the symmetric matrix defining the node is a square in $\mathbb{F}_p$. We calculated these determinants to be 1, 5 and $-12 + 16i$ (up to square factors). Now 1 is always a square, and $-12 + 16i$ is a square in $\mathbb{F}_p$ if $i$ is defined over $\mathbb{F}_p$. So the $\sigma$-nodes and 50 regular nodes have rulings defined over $\mathbb{F}_p$ as long as the nodes themselves are defined over $\mathbb{F}_p$. Now 5 is a square in $\mathbb{F}_p$ if and only if $p \equiv \pm 1(\text{mod } 5)$. Using the Chinese Remainder Theorem again gives us our result. 

**4.3. Topology of $\tilde{X}$.** We need to compute the topological invariants of $\tilde{X}$. First let $X'$ be a smooth deformation of $X$; that is, let $X'$ be a smooth threefold obtained by intersecting $\mathbb{P}^4 \times \mathbb{P}^4$ with five divisors of type $(1, 1)$. Topologically, $\tilde{X}$ is obtained from $X'$ by contracting 60 copies of $S^3$ and replacing them with 60 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, we have

$$\chi((\tilde{X})) = \chi(X') + 60\chi(\mathbb{P}^1 \times \mathbb{P}^1) = \chi(X') + 240.$$
For $0 \leq i \leq 5$, let $X^i$ denote the intersection of $\mathbb{P}^4 \times \mathbb{P}^4$ with $i$ generic divisors of type $(1,1)$. So we have $\mathbb{P}^4 \times \mathbb{P}^4 = X^0$ and $X' = X^5$. We have the exact sequences

$$
\begin{align*}
0 \rightarrow & \ T(X^1) \rightarrow \ T(\mathbb{P}^4 \times \mathbb{P}^4)|_{X^1} \rightarrow \ N|_{X^1/\mathbb{P}^4 \times \mathbb{P}^4} \rightarrow 0 \\
0 \rightarrow & \ T(X^i) \rightarrow \ T(X^{i-1})|_{X^i} \rightarrow \ N|_{X^i/X^{i-1}} \rightarrow 0 \\
0 \rightarrow & \ TX' \rightarrow \ T(X^4)|_{X'} \rightarrow \ N|_{X'/X^4} \rightarrow 0
\end{align*}
$$

(25)

Let $X$ and $Y$ denote the hyperplane classes on the two copies of $\mathbb{P}^4$. Then the Chern class of the third term in each sequence is equal to $(1 - X - Y)$. We thus have

$$
c(X') = c(\mathbb{P}^4 \times \mathbb{P}^4) \cdot (1 - X - Y)^5 .
$$

(26)

Taking the terms of order three, we have

$$
c_3(X') = (10X^3 + 50X^2Y + 50XY^2 + 10Y^3) - 5(X + Y)(10X^2 + 25XY + 10Y^2) + 10(X + Y)^2(5X + 5Y) - 10(X + Y)^3.
$$

(27)

The Euler characteristic of $X'$ is

$$
\chi(X') = \int_{\mathbb{P}^4 \times \mathbb{P}^4} c_3(N|_{X'/\mathbb{P}^4 \times \mathbb{P}^4}) \cdot c_3(X')
$$

(28)

Now only the terms of degree 8 in the characteristic class contribute to the integral, and the orientation class of $\mathbb{P}^4 \times \mathbb{P}^4$ is $X^4Y^4$. The integral is thus equal to the coefficient of the $X^4Y^4$ term in $c_3(X')(1 + X + Y)^5$, which we compute to be $-100$. We thus have $\chi(X') = -100$ and $\chi(\tilde{X}) = 140$.

We can also compute most of the Hodge numbers of $\tilde{X}$. Since $\tilde{X}$ is obtained from $X'$ by the surgery procedure explained above, we have $h^{0,0} = h^{3,3} = 1, h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = 0$, and $h^{3,0} = h^{0,3} = 1$. The only unknown Hodge numbers are $h^{1,1} = h^{2,2}$ and $h^{1,2} = h^{2,1}$, and we know that $2h^{1,1} - h^{2,1} = 140$.

**Remark.** In section 1, we remarked that Lange [19] had found a rank 2 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^3$ whose zero sections yielded abelian surfaces. The Chern class of this bundle is $1 + (2H_1 + 4H_3) + (8H_1H_3 + 6H_3^2)$, where $H_1$ and $H_3$ denote the pullbacks to $\mathbb{P}^1 \times \mathbb{P}^3$ of the hyperplane class on $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively. Since $c_1(\mathbb{P}^1 \times \mathbb{P}^3) = (2H_1 + 4H_3)$, a pencil of such abelian surfaces would yield a Calabi-Yau threefold. A generic pencil of such abelian surfaces would have a base locus consisting of $2 \cdot 8 \cdot 6 = 96$ nodes. One computes the Euler characteristic of the resulting threefold to be 176. Since $176 < 4 \cdot 96$, it is within the realm of possibility that resolving the singularities would yield a threefold with $h^{1,2}$ small or zero.

**5. Proof that $X$ is modular**

**5.1. Modular forms.** This brief review of modular forms is taken from [3]. Recall that the special linear group $SL_2(\mathbb{Z})$ acts on points $z$ in the upper half plane $\mathcal{H}$: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma z = \frac{az + b}{cz + d}$.

Let $\Gamma(N)$ denote the subgroup of matrices $\gamma$ in $SL_2(\mathbb{Z})$ such that
\[ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}. \]

Call a subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \) a congruence subgroup if it contains \( \Gamma(N) \) for some \( N \). The level of \( N \) is the smallest \( N \) such that \( \Gamma \) contains \( \Gamma(N) \). The most important congruence subgroups are

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \equiv (1, 0, *, 1) \pmod{N} \right\},
\]

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \equiv (*, *, 0, *) \pmod{N} \right\}.
\]

**Definition 5.1.** A modular function \( f \) of weight \( 2k \) and level \( N \) is a holomorphic function on the upper half plane \( \mathcal{H} \) such that \( f(\gamma(z)) = (cz + d)^{-2k}f(z) \) for all \( \gamma \) in some congruence subgroup \( \Gamma \) of level \( N \).

**Definition 5.2.** A modular form \( f \) is a modular function satisfying the following property: for all \( \gamma \) in \( SL_2(\mathbb{Z}) \), the function \( (cz + d)^{-2k}f(\gamma \tau) \) has a Puiseux series expansion \( \sum_{n \geq 0} a_n q^{\frac{1}{n}} \) in fractional powers of \( q = e^{2\pi i \tau} \). We call this series the Fourier expansion of \( f \) at the cusp \( \gamma^{-1}(i\infty) \), since the limit \( q \to 0 \) corresponds to the limit \( z \to i\infty \). A cusp form is a modular form such that the Fourier expansion at each cusp has vanishing constant term.

We will focus on congruence subgroups \( \Gamma \) lying between \( \Gamma_1(N) \) and \( \Gamma_0(N) \). In this case, the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) is in \( \Gamma \). If \( f \) is a modular form with respect to \( \Gamma \), we must then have \( f(\gamma z) = f(z) \) for all \( z \) in \( \mathcal{H} \). Since \( f \) is periodic with period 1, its expansion at the cusp at infinity can be written as a power series in \( q = e^{2\pi i \tau} \).

**5.2. Galois theory.** This quick review of Galois theory is taken from [29]. Recall the \( p \)-adic absolute value on \( \mathbb{Q} \) given by \( |x|_p = p^{-e} \) if \( x \) can be expressed as \( p^e \frac{a}{b} \) with \( a \) and \( b \) both coprime to \( p \). The completion of the field \( \mathbb{Q} \) under this absolute value gives us the field of \( p \)-adic numbers \( \mathbb{Q}_p \), and \( | \cdot |_p \) extends uniquely to \( \mathbb{Q}_p \).

Under this metric, \( \mathbb{Q}_p \) is a locally compact, totally disconnected field. Let \( G_{\mathbb{Q}_p} \) denote the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \); it is identified with the group of continuous automorphisms of \( \mathbb{Q}_p \). \( G_{\mathbb{Q}_p} \) is a topological group; a basis at the identity is given by the collection of subgroups of finite index.

Given an embedding of \( \overline{\mathbb{Q}}_p \) into \( \overline{\mathbb{Q}}_p \), we obtain a closed embedding of \( G_{\mathbb{Q}_p} \) into \( G_{\mathbb{Q}} \); this embedding varies by conjugation as the embedding varies.

Let \( \mathcal{O}_{\mathbb{Q}_p} \) denote the ring of integers of \( \overline{\mathbb{Q}}_p \); it is the ring of elements with absolute value less than or equal to 1. It is a local ring with maximal ideal \( \mathfrak{m}_{\mathbb{Q}_p} \), which is the ideal of elements with absolute value strictly less than 1. The residue field \( \mathcal{O}_{\mathbb{Q}_p}/\mathfrak{m}_{\mathbb{Q}_p} \) is an algebraic closure of \( \mathbb{F}_p^{\infty} \), which we denote by \( \mathbb{F}_p \). We thus obtain a continuous map \( G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \), which is surjective. Its kernel \( I_{\mathbb{Q}_p} \) is called the inertia subgroup of \( G_{\mathbb{Q}_p} \). The group \( G_{\mathbb{F}_p} \) is procyclic, being the inverse limit \( \lim_{\leftarrow} G(\mathbb{F}_p^e/\mathbb{F}_p) \). The Frobenius element \( \text{Frob}_p \) defined by

\[
\text{Frob}_p(x) = x^p
\]
generates a dense subgroup of \( G_{\mathbb{F}_p} \).
We will be looking at representations $\rho : G_\mathbb{Q} \to GL_d(K)$ coming from algebraic geometry in the next section. Given an embedding $\mathbb{Q} \to \mathbb{Q}_p$, we have $I_{Q_p} \subset G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$. We say that $\rho$ is unramified at a prime $p$ if it is trivial on the inertia group $I_p$. In this case, $\rho(Frob_p)$ is well-defined given the choice of embedding $\mathbb{Q} \to \mathbb{Q}_p$, and $tr \rho(Frob_p)$ is well-defined independent of the embedding.

5.3. Modularity of an algebraic variety. Let $X$ be an projective algebraic variety defined over $\mathbb{Z}$. We can then consider the reduction of $X$ modulo a prime $p$, i.e. the scheme $X \times_{\mathbb{Z}} \mathbb{F}_p$.

Fixing once and for all an embedding $\mathbb{Q} \subset \mathbb{C}$, the Galois group $G_{\mathbb{Q}}$ acts on $X$ and thus induces an action on the cohomology groups

$$H^i(X(\mathbb{C}), \mathbb{Q}_l) \cong H^i_{et}(X \times_{\mathbb{Z}} \mathbb{Q}_l),$$

where the left-hand side is the cohomology of $X$ in the analytic topology.

Given a prime $p \neq l$ and an embedding $\mathbb{Q} \subset \mathbb{Q}_p$, we have the decomposition subgroup $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$. In addition, we have the surjective map $G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}$. It is well-known that if $p$ is a prime of good reduction for $X$, then the representation $\rho : G_{\mathbb{Q}} \to GL(H^i(X \times_{\mathbb{Z}} \mathbb{Q}_l))$ is unramified at $p$.

If $p$ is a prime of good reduction for $X$, then we can assign a well-defined value to $\rho(Frob_p)$ by choosing a lifting of $Frob_p$ from $G_{\mathbb{F}_p}$ to $G_{\mathbb{Q}_p}$. By abuse of notation, such a lifting will also be denoted by $Frob_p$. Furthermore, by passing from $X \times_{\mathbb{Z}} \mathbb{Q}_p$ to the reduction $X \times_{\mathbb{Z}} \mathbb{F}_p$, we can consider the action of $Frob_p$ on the cohomology of $X \times_{\mathbb{Z}} \mathbb{F}_p$.

We say that an $n$-dimensional algebraic variety $X$ is modular if for some subquotient $V$ of $H^i(X, \mathbb{Q}_l)$, the numbers $tr Frob_p(V)$ are equal to the coefficients of a cusp form $f$ for all but finitely many primes $p$.

En route to proving Fermat’s Last Theorem, Wiles and Taylor [30] proved that for any elliptic curve $E$ with semistable reduction at 3 and 5, the numbers $tr Frob_p(H^1(X))$ are the coefficients of a modular form; Breuil, Conrad, Diamond and Taylor [2] later proved the statement for all elliptic curves $E$.

5.4. Lefschetz, Weil and counting points. In étale cohomology, we have the Lefschetz theorem [9]:

**Theorem 5.1.** If $f$ is an automorphism of the variety $X$, then the number of fixed points of $f$ is given by the following formula:

$$Fix(f, X) = \sum_{i=0}^{2n} (-1)^i tr f^*(H^i(X)).$$

In the case $X$ is defined over $\mathbb{F}_p$ and $f = Frob_p$, $Fix(f, X)$ is simply the number of points of $X$ over $\mathbb{F}_p$. We see that the number of points of $X$ over $\mathbb{F}_p$ is related to the action of $Frob_p$ on the étale cohomology of $X$.

**Proposition 5.2.** For $p$ congruent to 1 modulo 20, the semisimplification of the action of the Frobenius map $Frob_p$ on $H^2(X \times_{\mathbb{Z}} \mathbb{F}_p, \mathbb{Q}_l)$ is multiplication by $p$.

**Proof.** We prove this statement in two steps.

**Step 1.** The Frobenius map $Frob_p$ acts on $H^2(X \times_{\mathbb{Z}} \mathbb{F}_p, \mathbb{Q}_l)$ by multiplication by $p$.
Consider the embedding \( i : \mathbb{P}^4 \times \mathbb{P}^4 \to \mathbb{P}^{24} \) that sends \((x, z)\) to \(y\) where \(y_{5i+j} = x_iz_j\). Recall that \(X\) is a section of \(\mathbb{P}^4 \times \mathbb{P}^4\) by five divisors of type \((1, 1)\); correspondingly \(i(X)\) is a section of \(i(\mathbb{P}^4 \times \mathbb{P}^4)\) by five hyperplanes; as in section 3 let \(X^i\) denote successive sections of \(\mathbb{P}^4 \times \mathbb{P}^4\) by these hyperplanes, with the exception that here \(X^1\) is \(X\), instead of a deformation of \(X\). Moreover, by Bertini’s theorem these sections can be chosen such that \(X^{i-1} - X^i\) is smooth for all \(i\); for \(p = 101\) a specific choice of hyperplane sections is given in a Macaulay2 program in the Appendix.

By the Lefschetz theorem in étale cohomology,

\[
i^* : H^2(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Q}_l) \to H^2(X \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l)
\]

is an isomorphism that preserves the Frobenius action. (The proof of the Lefschetz theorem depends only on the fact that the \(i(X^{i-1} - X^i)\) are smooth and affine; see [9], p. 106.)

Since all of \(H^2(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Q}_l)\) can be represented by divisors defined over \(\mathbb{F}_{101}\), the Frobenius map acts by multiplication by \(p\) on \(H^2(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Q}_l)\). Thus the Frobenius map acts likewise on \(H^2(X \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l)\). Note that Step 1 is valid for any prime \(p\), not just those congruent to 1 modulo 20.

**Step 2.** The semisimplification of the action of the Frobenius map \(\text{Frob}_p\) on \(H^2(\tilde{X} \times_\mathbb{Q} \mathbb{F}_p, \mathbb{Q}_l)\) is multiplication by \(p\).

Recall that \(\pi : \tilde{X} \to X\) is a blowup of 60 ordinary double points. From the Leray spectral sequence for \(\pi\), we obtain an exact sequence

\[
0 \to H^2(X \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l) \to H^2(\tilde{X} \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l) \to \bigoplus_{i=1}^{60} H^2(Q_i, \mathbb{Q}_l),
\]

where the \(Q_i\) are the exceptional divisors. For \(p \equiv 1 \pmod{20}\), the rulings on all the \(Q_i\) are defined over \(\mathbb{F}_p\). Hence the Frobenius map acts by multiplication by \(p\) on the \(Q_i\). From the exact sequence, we see that the semisimplification of the Frobenius map acts by multiplication by \(p\) on \(H^2(\tilde{X} \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l)\). \(\Box\)

We will now concentrate our attention on the prime \(p = 101\). Let us collect the information we have so far about the cohomology of \(\tilde{X} \times_{\mathbb{Q}} \mathbb{F}_{101}\):

1. \(h^0 = h^6 = 1\).
2. \(h^3 = h^5 = 0\).
3. The semisimplification of the \(\text{Frob}_{101}\) action on \(H^2\) is multiplication by 101. By Poincaré duality, the semisimplification of the \(\text{Frob}_{101}\) action on \(H^4\) is multiplication by 101^2.
4. \(2h^2 - h^3 = 138\).
5. \(#X(\mathbb{F}_{101}) = 1 + 101h^2 + 101^2h^2 + 101^3 - \text{tr} \text{Frob}_{101}(H^3(\tilde{X} \times_{\mathbb{F}_p} \mathbb{F}_{101}, \mathbb{Q}_l)).\)
6. For primes \(p\) of good reduction, \(|\text{tr} \text{Frob}_p(H^3(\tilde{X} \times_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l))| \leq h^3p^{\frac{3}{2}} = (138 - 2h^2)p^{\frac{3}{2}}\) by the Weil conjectures.

The number of points in \(\tilde{X}\) is easily computed by the following procedure:

1. For a given prime \(p\), count the number of points in \(G\) using a computer. (The code is in the Appendix.) We start by counting points in \(G\) instead of \(X\) because it is faster, the running time of the program being an exponential function of the number of variables.
2. Add \(p\) times the number of points in \(E\), since each point in \(E\) is replaced by a copy of \(\mathbb{P}^3\) upon passage to \(X\). Note that \(E\) has points defined over \(\mathbb{F}_p\) only when \(p \equiv 1 \pmod{4}\).
(3) Add the number of points arising from the blowup of the nodes.

The \( \sigma \)-nodes are defined for all \( \mathbb{F}_p \), and the rulings over the exceptional divisors exist over all \( \mathbb{F}_p \). Hence each of these five nodes adds \( p^2 + p \) points to the total.

The node \( ((1 : 1 : 1 : 1), (1 : 1 : 1 : 1 : 1)) \) is defined over all \( \mathbb{F}_p \), but the other \( \tau \)-nodes in its orbit are defined only if \( p \equiv 1 \pmod{5} \). The rulings over the exceptional divisors exist over \( \mathbb{F}_p \) only if \( \sqrt{5} \) is defined in \( \mathbb{F}_p \), i.e., if \( p \equiv \pm 1 \pmod{5} \). If the rulings are defined over \( \mathbb{F}_p \), we add \( p^2 + p \) points. Otherwise we add only \( p^2 \) points.

The node \( ((0 : 1 : -1 : 1), (0 : 1 : -i : -1)) \) is defined over \( \mathbb{F}_p \) if \( p \equiv 1 \pmod{4} \), as are the other nodes in its \( \sigma \)-orbit. However, the other nodes in its \( H_5 \)-orbit are defined over \( \mathbb{F}_p \) only if \( i \) and \( \epsilon \) are both defined. If the nodes are defined, then the rulings over the exceptional divisors are as well. Hence we add \( p^2 + p \) times the number of these nodes.

For \( p = 101 \), we obtain \( \# \mathcal{X}(\mathbb{F}_{101}) = 1770940 \). Thus

\[
\text{tr} \text{Frob}_{101}(H^3(\tilde{X})) = 1 + 101h^2 + 101^2h^2 + 101^3 - 1770940,
\]

so we must have

\[
|1 + 101h^2 + 101^2h^2 + 101^3 - 1770940| \leq (138 - 2h^2)101^{\frac{5}{2}}.
\]

Separating the absolute value inequality into two inequalities, we have

\[
\begin{align*}
     h^2(101 + 101^2 + 2 \cdot 101^{\frac{5}{2}}) & \leq 138 \cdot 101^{\frac{5}{2}} - 1 - 101^3 + 1770940, \\
     h^2(101 + 101^2 - 2 \cdot 101^{\frac{5}{2}}) & \geq -138 \cdot 101^{\frac{5}{2}} - 1 - 101^3 + 1770940.
\end{align*}
\]

Since \( h^2 \) must be an integer, the two inequalities force \( h^2 \) to be equal to 72. We then get \( h^3 = 6 \). (This trick for computing \( h^2 \) is due to Werner and van Geemen in [31].)

For \( p \not\equiv 1 \pmod{20} \), we no longer know that the semisimplification of \( \text{Frob}_p \) acts by multiplication by \( p \) on \( H^2 \). However, we know that \( \text{Frob}_p \) acts on \( H^2(\tilde{X}) \) by multiplication by \( p \). We also know that \( \oplus_{i=1}^{60} H^2(Q_i, \mathbb{Q}_i) \) is spanned by algebraic cycles, so the eigenvalues of \( \text{Frob}_p \) acting on this space are all \( p \) times roots of unity. Using the exact sequence (29) again, the eigenvalues of the semisimplification of \( \text{Frob}_p \) acting on \( H^2(\tilde{X}) \) are all \( p \) times roots of unity. By the Weil conjectures, the trace of \( \text{Frob}_p \) is a rational integer. Hence the trace of \( \text{Frob}_p \) must be \( p \) times an integer \( h \).

Suppose the eigenvalues of \( \text{Frob}_p \) acting on \( H^2(\tilde{X}) \) are \( p\zeta_i \), with the \( \zeta_i \) being roots of unity. Choosing a basis of \( H^2(\tilde{X}) \) and a Poincaré dual basis of \( H^4(\tilde{X}) \), the action of \( \text{Frob}_p \) on \( H^2(\tilde{X}) \) can be represented as a matrix. This matrix is similar to a matrix of upper Jordan blocks having diagonal entries \( p\zeta_i \). By Poincaré duality, the action of \( \text{Frob}_p \) on \( H^4(\tilde{X}) \) will be \( p^2 \) times the contragredient of the action on \( H^2(\tilde{X}) \). Thus the matrix of \( \text{Frob}_p \) acting on \( H^4(\tilde{X}) \) will be similar to a matrix of lower triangular blocks having diagonal entries \( p^2 \zeta_i \), with the same multiplicities as in \( H^2(\tilde{X}) \). Hence the trace of \( \text{Frob}_p \) acting on \( H^4 \) is \( hp^2 \).

Furthermore, we have the additional piece of information that \( h^2 = 72 \). So we can use the Weil conjectures again. We now have

\[
|1 + ph + p^2h + p^3 - \#(\tilde{X})| \leq 6p^{\frac{5}{2}}.
\]
This is equivalent to the two inequalities

\begin{align}
(34) \quad & h(p + p^2) \leq #(\tilde{X}) - 1 - p^3 - 6p^2 \\
(35) \quad & h(p + p^2) \geq 1 + p^3 + 6p^2 - #(\tilde{X}).
\end{align}

It turns out that for \( p = 59, 67, 71 \) and \( p \geq 79 \), the inequalities determine \( h \) exactly. Our calculations are listed in Appendix B.

**Conjecture 5.1.** For primes \( p \) of good reduction, the trace of \( \text{Frob}_p \) acting on \( H^3(\tilde{X}) \) is 12p, 20p, 24p or 72p depending on whether \( i \) and \( \epsilon \) are defined over \( \mathbb{F}_p \).

There is a unique cusp form \( f \) of level 5 and weight 4, whose Fourier coefficients \( a_p \) can be found in William Stein’s Modular Forms Database [27]. It is equal to \((\eta(q)\eta(q^p))^4\), where \( \eta \) is the Dedekind function. The first few coefficients of the \( q \)-expansion of \( f \) are as follows:

\begin{align}
(36) \quad & f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 + \ldots
\end{align}

Comparing the values of \( \text{tr} \text{Frob}_p(H^3) \), we notice that for primes \( p \equiv 3 \pmod{4} \), \( a_p = \text{tr} \text{Frob}_p(H^3) \), whereas for primes \( p \equiv 1 \pmod{4} \), \( \text{tr} \text{Frob}_p(H^3) - a_p = p(2p + 2 - #(E)) \).

### 5.5. Proof of the main theorem.

We can finally prove the main theorem of this paper:

**Theorem 5.3.** The third cohomology group \( H^3(\tilde{X}) \) is modular in the following sense: as a Galois representation, its semisimplification is a direct sum of a modular rank 2 motive \( V \) and a 4-dimensional piece \( W \cong H^1(\tilde{S}, \mathcal{O}_l)(-1) \).

**Proof.** First we note that \( \tilde{S} \) and \( \tilde{X} \) are defined over \( \mathbb{Z} \). Since we will only consider the coefficient group \( \mathcal{O}_l \), we can use the proper-smooth base change theorem (see for example [22]) to pass from \( \tilde{X} \) to \( \tilde{X} \times_{\mathbb{Z}} \mathbb{F}_p \) (for \( p \neq 2, 5, l \)) and to \( \tilde{X} \times_{\mathbb{Z}} \mathbb{C} \). In addition, we can pass from étale cohomology on \( \tilde{X} \times_{\mathbb{Z}} \mathbb{C} \) to analytic cohomology by the comparison theorem.

We present the proof in several steps.

**Step 1.** \( H^1(\tilde{S} \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l)(-1) \cong \text{Ind}_{G_{\mathcal{O}_l}(i)} G_{\mathcal{O}_l} H^1((E_1 \times_{\mathbb{Z}} \mathcal{O}_p) \times_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{O}_l)(-1) \). For now, we use the analytic topology. Recall that the map \( \pi : \tilde{S} \to S \) is just the blowup at 50 points. A standard Mayer-Vietoris argument (see for example [10], p. 473) shows that \( H^1(\tilde{S} \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l) \cong H^1(S \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l) \).

The map \( \pi : S \to E \) is the projectivization of the rank 2 bundle \( \ker L \to E \). Using the analytic topology for now, the Leray-Hirsch theorem tells us that \( H^*(S, \mathcal{O}_l) \) is a truncated polynomial ring over \( H^*(E, \mathcal{O}_l) \) generated by the single element \( c_1(\ker L) \), which has dimension 2. Hence \( \pi^* : H^1(E, \mathcal{O}_l) \to H^1(S, \mathcal{O}_l) \) is an isomorphism. This statement also holds in étale cohomology.

Finally, recall that \( E = E_1 \cup E_2 \), where \( E_1 \) and \( E_2 \) are complex conjugates of each other. Thus \( H^1(E \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l) \cong \text{Ind}_{G_{\mathcal{O}_l}(i)} G_{\mathcal{O}_l} H^1((E_1 \times_{\mathbb{Z}} \mathcal{O}_p) \times_{\mathbb{F}_p} \mathbb{F}_p, \mathcal{O}_l) \). Putting everything together completes Step 1.

**Step 2.** \( H^1(\tilde{S} \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l)(-1) \) is a subrepresentation of \( H^3(\tilde{X} \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{O}_l) \).
Note that $H^1(\tilde{S} \times \mathbb{F}_p, \mathbb{Q}_l)(-1)$ is isomorphic to $H^3_S(\tilde{X} \times \mathbb{F}_p, \mathbb{Q}_l)$ (see [22], p. 98). So it is sufficient to show that inclusion induces an injection

$$j^*: H^3_S(\tilde{X} \times \mathbb{F}_p, \mathbb{Q}_l) \to H^3(\tilde{X} \times \mathbb{F}_p, \mathbb{Q}_l).$$

By base change and the comparison theorem, it suffices to prove the same statement in the complex analytic topology.

In the analytic topology, we have the Lefschetz duality diagram (see [24], p. 429)

$$
\begin{array}{ccc}
H^3_S(\tilde{X} \times \mathbb{C}, \mathbb{Q}_l) & \xrightarrow{j^*} & H^3(\tilde{X} \times \mathbb{C}, \mathbb{Q}_l) \\
\downarrow \cong & & \downarrow \cong \\
H_3(\tilde{S} \times \mathbb{C}, \mathbb{Q}_l) & \xrightarrow{j^*} & H_3(\tilde{X} \times \mathbb{C}, \mathbb{Q}_l)
\end{array}
$$

where the vertical arrows are isomorphisms. Hence it is sufficient to show that inclusion induces an injective map $j_*: H_3(\tilde{S} \times \mathbb{C}, \mathbb{Q}_l) \to H_3(\tilde{X} \times \mathbb{C}, \mathbb{Q}_l)$. We do this by computing intersection classes of cycles in $H_3(\tilde{S} \times \mathbb{C}, \mathbb{Q}_l)$.

The 3-cycles in $\tilde{S}$ are of the form $\alpha_i \times \mathbb{P}^1$ and $\beta_j \times \mathbb{P}^1$, where $\alpha_i$ and $\beta_j$ generate $H_1(E_i)$. Note that the $\alpha_i$ and $\beta_j$ can be chosen to miss the exceptional divisors.

We have the fundamental result

$$j_*(\alpha) \cap j_*(\beta) = j_*(PD[\tilde{S}]|_S \cap \alpha \cap \beta)$$

for any homology cycles $\alpha$ and $\beta$.

Note that

$$[\tilde{S}]|_S = (K_{\tilde{S}} - K_{\tilde{X}} + \tilde{S})*|_S = K_S - K_{\tilde{X}}|_{\tilde{S}}.$$

But $X$ has trivial canonical bundle. Hence $K_{\tilde{X}}$ is supported on its exceptional fibers. Therefore the restriction of $K_{\tilde{X}}$ to $\tilde{S}$ is supported on the exceptional fibers of $\tilde{S}$. Let $\gamma_i$ and $\delta_i$ be 1-cycles generating the first homology of $E_i$, and put $\alpha_i = \gamma_i \times C_i$ and $\beta_i = \delta_i \times C_i$, where $C_i$ is a ruling of $S_i$.

Since the cycles $\alpha_i$ and $\beta_j$ can be chosen to miss the exceptional fibers, we have

$$PD[\tilde{S}]|_S \cap \alpha_i \cap \beta_j = PD[K_S|_S \cap \alpha_i \cap \beta_j].$$

The canonical bundle of $\tilde{S}$ is well-known; it is simply $-2(D) + \Sigma E_i$, where $D$ is a horizontal section and the $E_i$ are the exceptional divisors. We also have

$$\gamma_i \cap \gamma_j = \delta_i \cap \delta_j = 0,$$

$$\gamma_1 \cap \delta_2 = \gamma_2 \cap \delta_1 = 0,$$

$$\gamma_i \cap \delta_i = C_i,$$

where $C_i$ is a line belonging to the ruling of $S_i$.

Therefore we have

$$j_*(\alpha_i) \cap j_*(\alpha_j) = j_*(\beta_i) \cap j_*(\beta_j) = 0$$

and

$$j_*(\alpha_i) \cap j_*(\beta_j) = -2\delta_{ij}.$$
which is nonsingular. Hence $H_3(\tilde{S})$ injects into $H_3(\tilde{X})$.

Upon passing to the semisimplification, we now see that as Galois representations,

$$H^3(\tilde{X} \times \mathbb{Z} \mathbb{F}_p, \mathbb{Q}_l) = V \oplus \text{Ind}_{G_{2(\ell)}}^{G_2} H^1(E \times \mathbb{Z} \mathbb{F}_p, \mathbb{Q}_l)(-1),$$

with $V$ some undetermined 2-dimensional piece. This observation was first made by Taylor [28].

**Step 3.** Away from the primes of bad reduction, the traces of $V$ coincide with the coefficients of the unique modular form $f$ of level 5 and weight 4.

For Step 3, we invoke a result of Faltings-Serre-Livné [20] which says essentially that it is enough to check the equality at a suitably chosen finite set of primes:

**Theorem 5.4.** (Faltings-Serre-Livné) Suppose that $\rho_1$ and $\rho_2$ are two 2-adic 2-dimensional Galois representations unramified outside a set of primes $S$. Let $K_S$ be the smallest field containing all quadratic extensions of $\mathbb{Q}$ ramified at primes in $S$, and let $T$ be a set of primes disjoint from $S$. Then if

- $\text{tr} \rho_1 \equiv \text{tr} \rho_2 \equiv 0 \pmod{2}$;
- $\{\text{Frob}_p|_{K_S} : p \in T\}$ is equal to the set $\text{Gal}(K_S/K) - \text{Id}$;
- for all $p \in T$, $\text{tr} \rho_1 \text{Frob}_p = \text{tr} \rho_2 \text{Frob}_p$ and $\det \rho_1 \text{Frob}_p = \det \rho_2 \text{Frob}_p$;

then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications.

We will apply this result in the case $\rho_1 = V$ and $\rho_2$ is the modular Galois representation coming from the cusp form $f$ of level 5 and weight 4. Thus we must do the following:

- Check that $\text{tr} \text{Frob}_p(H^3(\tilde{X} \times \mathbb{Z} \mathbb{F}_p, \mathbb{Q}_l))$ is even for primes $p$ of good reduction.
- Check that the coefficients $a_p$ of the modular form $f$ are even for $p \neq 2, 5$.
- Observe that the determinants of both representations are given by $\chi^3$, where $\chi$ is the cyclotomic character.
- Find a suitable set of primes $T_s$.
- Compute $\text{tr} \text{Frob}_p H^3(\tilde{X} \times \mathbb{Z} \mathbb{F}_p, \mathbb{Q}_l)$ for all $p \in T_s$ and check that these are equal to the corresponding coefficients in the modular form $f$.

**Lemma 5.5.** $\text{tr} \text{Frob}_p H^3(\tilde{X} \times \mathbb{Z} \mathbb{F}_p, \mathbb{Q}_l)$ is even for $p \neq 2, 5$.

**Proof.** Recall that

$$\text{tr} \text{Frob}_p(H^3(\tilde{X})) = p^3 + p(p+1)h + 1 - \# \tilde{X}(\mathbb{F}_p).$$

Thus we need only check that $\# \tilde{X}(\mathbb{F}_p)$ is even.

Recall the automorphism of order 4

$$\mu = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
acting on $F$, and notice that it lifts to an automorphism $(\mu, \mu)$ of $X$. Each point $(x, z)$ of $X$ lies in an orbit of size 1, 2 or 4. The points in orbits of size 2 or 4 obviously contribute an even number of points to the total. If any of these points are nodes, then so are the other points in their orbits. Blowing up each node adds either $p^2$ or $p^2 + 2p$ points to the total. Since there must be an even number of such nodes, blowing up adds another even number of points to the total. So we are left with counting the number of points in $\tilde{X}$ coming from fixed points of $X$.

The fixed points of $\mu$ in $F$ are of the form $(1 : x : x : x)$, $(0 : 1 : 1 : 1)$, $(0 : 1 : -1 : -1)$, and $(0 : 1 : \pm i : \mp i)$; this is a total of $p + 3$ points, which is even. Of these points, the only ones that are singular points in $F$ are $(1 : 1 : 1 : 1)$, $(1 : 0 : 0 : 0)$ and $(1 : \pm \frac{1}{2i} : \pm \frac{1}{2i} : \pm \frac{1}{2i})$. The other $p - 1$ points are smooth points in $F$ and thus lift uniquely to points in $X$ fixed by $(\mu, \mu)$.

The points $(1 : \pm \frac{1}{2i} : \pm \frac{1}{2i} : \frac{1}{2i})$ (if they exist over $\mathbb{F}_p$) are singular; their inverse images under $\pi$ are lines. In these lines, the only fixed points of $(\mu, \mu)$ are $(1 : x : x : x)$, $(0 : 1 : -i : -1)$, $(1 : -\frac{1}{2i} : -\frac{1}{2i} : -\frac{1}{2i})$, $(0 : 1 : i : -i : -1)$, and none of these points are nodes. So we have added another even number of points to the total.

Finally, we look at the contribution coming from the points $(1 : 1 : 1 : 1 : 1)$ and $(1 : 0 : 0 : 0 : 0)$ in $F$. The fixed points of $(\mu, \mu)$ in $X$ lying over these points are $(1 : 1 : 1 : 1)$ and $(1 : 0 : 0 : 0)$.

These points are nodes; blowing them up adds either $p^2$ or $p^2 + 2p$ points for each node. So again we end up adding an even number of points to the total. Hence the number of points in $\tilde{X}$ is even.

\[\text{LEMMA 5.6. The coefficients } a_p \text{ of } f \text{ are all even for } p \neq 2, 5.\]

\[\text{PROOF. This proof is essentially Proposition 4.10 in } [20]. f \text{ is a modular form of level } 5. \text{ The corresponding Galois representation } \rho : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Q}_2) \text{ is unramified away from } 2 \text{ and } 5. \text{ Let } L \text{ be the extension of } \mathbb{Q} \text{ cut out by ker } \overline{\rho} \text{, where } \overline{\rho} \text{ denotes the reduction of } \rho \text{ modulo } 2. \text{ If the trace of } \rho \text{ were not congruent to } 0 \text{ (mod } 2) \text{, then the image of } \overline{\rho} : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/2) \text{ would contain one of the elements } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \text{ These matrices are of order } 3. \text{ Therefore, if } L \text{ is the extension of } \mathbb{Q} \text{ cut out by ker } \overline{\rho}, L \text{ is a } C_3 \text{ or } S_3 \text{ extension of } \mathbb{Q} \text{ unramified away from } 2 \text{ and } 5. \text{ The only such field is the splitting field of } g(x) = x^3 - x^2 + 2x + 2 \text{ (see for instance } [17]) \text{, which has Galois group } S_3. \text{ Since } g \text{ is irreducible modulo } 3, \text{ the Frobenius element } \text{Frob}_3 \text{ has order } 3 \text{ in } S_3 = \text{Gal}(\mathbb{Q}/\mathbb{Q}) \cong \text{GL}_2(\mathbb{Z}/2). \text{ Hence the trace of } \rho(\text{Frob}_3) = 1 \text{ in } \mathbb{Z}/2. \text{ But the Fourier coefficient } a_3 \text{ of } f \text{ is } 2, \text{ a contradiction.}\]

\[\text{LEMMA 5.7. Let } T \text{ be the set } \{67, 71, 101, 103, 113, 131, 157\}. \text{ Then as } p \text{ runs over this set, } \text{Frob}_p \text{ runs over all the non-identity members of Gal}(\mathbb{Q}_S/\mathbb{Q}).\]

\[\text{PROOF. The field } \mathbb{Q}_S \text{ is the compositum of all quadratic extensions of } \mathbb{Q} \text{ unramified outside } \{2, 5\}, \text{ which is } \mathbb{Q}(i, \sqrt{2}, \sqrt{5}). \text{ Now Gal}(\mathbb{Q}_S/\mathbb{Q}) \text{ is isomorphic to } (\mathbb{Z}/2)^3, \text{ and the coordinates of } \text{Frob}_p \text{ are just the values of } \left(\frac{p}{7}, \frac{1}{7}, \frac{1}{7}\right). \text{ These are controlled by the behavior of } p \text{ modulo } 40, \text{ as illustrated in Table } [23]. \text{ The set of primes } T \text{ gives us representatives of every element of Gal}(\mathbb{Q}_S/\mathbb{Q}) \text{ other than the identity.}\]
Table 5.1. Images of Frobenius elements in \( \text{Gal}(\mathbb{Q}_s/\mathbb{Q}) \).
Appendix A. C++ and Macaulay2 code

In this appendix we put the C++ and Macaulay2 programs we used for our computations.

```c++
#include<iostream.h> // Count points on G over Fp
int psols;

long long G(long long z0, long long z1, long long z2, long long z3, long long z4)
// the equation for the determinantal quintic
{
    return z0 * z0 * z0 * z1 * z4 + z1 * z1 * z1 * z2 * z0
         + z2 * z2 * z2 * z3 * z1 + z3 * z3 * z3 * z4 * z2 + z4 * z4 * z4 * z0 * z3
         + z0 * z2 * z2 * z3 * z3 + z1 * z3 * z3 * z4 * z4 + z2 * z4 * z4 * z0 * z0
         + z3 * z0 * z0 * z1 * z1 + z4 * z1 * z1 * z2 * z2 - (z0 * z1 * z1 * z4 * z4
         + z1 * z2 * z2 * z0 * z0 + z2 * z3 * z3 * z1 * z1 + z3 * z4 * z4 * z2 * z2
         + z4 * z0 * z0 * z3 * z3 + z0 * z0 * z0 * z2 * z3 + z1 * z1 * z1 * z3 * z4
         + z2 * z2 * z2 * z4 * z0 + z3 * z3 * z3 * z0 * z1 + z4 * z4 * z4 * z1 * z2);
}

void counter(long long z0, long long z1, long long z2, long long z3, long long z4, int p) // count points on G
{
    if (G(z0, z1, z2, z3, z4) % p == 0)
        ++psols;
}

int main()
{
    long long j0, j1, j2, j3, j4;
    int p;
    cout << "Enter a prime p: ";
    cin >> p;
    j0 = 1; // Affine open set with j0 != 0
    for (j1 = 0; j1 < p; ++j1) {
        for (j2 = 0; j2 < p; ++j2) {
            for (j3 = 0; j3 < p; ++j3) {
                for (j4 = 0; j4 < p; ++j4) {
                    counter(j0, j1, j2, j3, j4, p);
                }
            }
        }
    }
    j0 = 0;
    j1 = 1; // Affine A^3 with j0 = 0, j1 != 0
    for (j2 = 0; j2 < p; ++j2) {
        for (j3 = 0; j3 < p; ++j3) {
            for (j4 = 0; j4 < p; ++j4) {
                counter(j0, j1, j2, j3, j4, p);
            }
        }
    }
    j1 = 0;
    j2 = 1; // Affine A^2 with j0 = j1 = 0, j2 != 0
    for (j3 = 0; j3 < p; ++j3) {
        for (j4 = 0; j4 < p; ++j4) {
            counter(j0, j1, j2, j3, j4, p);
        }
    }
    j2 = 0;
    j3 = 1; // Affine A^1 with j0 = j1 = j2 = 0, j3 != 0
    for (j4 = 0; j4 < p; ++j4) {
        counter(j0, j1, j2, j3, j4, p);
    }
    cout << "The number of solutions mod p is " << psols << "\n";
    return 0;
}
```

Program A.1. Counting points on $G$. 

---

# Program A.1. Counting points on $G$. 

---
```c
#include <iostream.h> // Count points on E modulo a prime p
int psols;

long long E0(long long z0, long long z1, long long z2, long long z3, long long z4, long long y)
{
    return (-y * z0 * z0 - z1 * z4 + y * y * z2 * z3);
}

long long E1(long long z0, long long z1, long long z2, long long z3, long long z4, long long y)
{
    return (-y * z1 * z1 - z2 * z0 + y * y * z3 * z4);
}

long long E2(long long z0, long long z1, long long z2, long long z3, long long z4, long long y)
{
    return (-y * z2 * z2 - z3 * z1 + y * y * z4 * z0);
}

long long E3(long long z0, long long z1, long long z2, long long z3, long long z4, long long y)
{
    return (-y * z3 * z3 - z4 * z2 + y * y * z0 * z1);
}

long long E4(long long z0, long long z1, long long z2, long long z3, long long z4, long long y)
{
    return (-y * z4 * z4 - z0 * z3 + y * y * z1 * z2);
}

void counter(long long z0, long long z1, long long z2, long long z3, long long z4, long long y, int p)
// count points
{
    if (E0(z0, z1, z2, z3, z4, y) % p == 0) {
        if (E1(z0, z1, z2, z3, z4, y) % p == 0) {
            if (E2(z0, z1, z2, z3, z4, y) % p == 0) {
                if (E3(z0, z1, z2, z3, z4, y) % p == 0) {
                    if (E4(z0, z1, z2, z3, z4, y) % p == 0) {
                        ++psols;
                    }
                }
            }
        }
    }
}

int main()
{
    long long j0, j1, j2, j3, j4, y;
    int p;
    cout << "Enter a parameter y: "; // usually y is a square root of -1
    cin >> y;
    cout << "Enter a prime p: ";
    cin >> p;
    j0 = 1; // Affine open set with j0 != 0
    for (j1 = 0; j1 < p; ++j1) {
        for (j2 = 0; j2 < p; ++j2) {
            for (j3 = 0; j3 < p; ++j3) {
                for (j4 = 0; j4 < p; ++j4) {
                    counter(j0, j1, j2, j3, j4, y, p);
                }
            }
        }
    }
    j1 = 0; // Affine A^3 with j0 = 0, j1 != 0
    for (j2 = 0; j2 < p; ++j2) {
        for (j3 = 0; j3 < p; ++j3) {
            for (j4 = 0; j4 < p; ++j4) {
                counter(j0, j1, j2, j3, j4, y, p);
            }
        }
    }
    j2 = 1; // Affine A^2 with j0 = j1 = 0, j2 != 0
    for (j3 = 0; j3 < p; ++j3) {
        for (j4 = 0; j4 < p; ++j4) {
            counter(j0, j1, j2, j3, j4, y, p);
        }
    }
    j3 = 0; // Affine A^1 with j0 = j1 = j2 = 0, j3 != 0
    for (j4 = 0; j4 < p; ++j4) {
        counter(j0, j1, j2, j3, j4, y, p);
    }
    counter(j0, j1, 0, 0, 1, y, p);
    cout << "The number of solutions to Ey mod p is 
" << psols << "
";
    return 0;
}
```

Program A.2. Counting points on \( E_1 \) and \( E_2 \).
\[ F = \mathbb{Z}; \]
\[ R = F[x_0, x_1, x_2, x_3, x_4, z_0, z_1, z_2, z_3, z_4]; \]
\[ f_0 = 0 * 0 - x_3 * z_1 + x_1 * z_2 + x_4 * z_3 - x_2 * z_4; \]
\[ f_1 = x_3 * x_0 - 0 * 0 - x_4 * z_2 + x_2 * z_3 + x_0 * z_4; \]
\[ f_2 = x_4 * x_0 + x_2 * z_1 - 0 * 0 + 0 * 0 - x_1 * z_2; \]
\[ f_3 = x_2 * x_0 + x_0 * z_1 + x_3 * z_2 - x_1 * z_4; \]
\[ L = \text{matrix}\{(0, z_2, -z_4, -z_1, z_3), (z_4, 0, z_3, -z_0, -z_2), (-z_3, z_0, 0, z_4, -z_1), (-z_2, -z_4, z_1, 0, z_0), (x_1, -z_3, -z_0, z_2, 0)\}; \]
\[ G = \text{minors}(4, L); \]
\[ M = \text{matrix}\{(0, -x_3, x_1, x_4, -x_2), (-x_3, 0, -x_4, x_2, x_0), (x_1, -x_4, 0, -x_0, x_3), (x_4, x_2, -x_0, 0, -x_1), (-x_2, x_0, x_3, -x_1, 0)\}; \]
\[ F = \text{minors}(4, M); \]
\[ S_0 = \text{diff} f_0; \]
\[ S_1 = S_0 | | \text{diff} f_1; \]
\[ S_2 = S_1 | | \text{diff} f_2; \]
\[ S_3 = S_2 | | \text{diff} f_3; \]
\[ S = S_3 | | \text{diff} f_4; \]
\[ X = \text{ideal}(f_0, f_1, f_2, f_3, f_4); \]
\[ \text{sing} = \text{minors}(5, S); \]
\[ \text{slocus} = X + \text{sing}; \]
\[ \text{slocus2} = \text{slocus} + G + F; \]
\[ k = 100 * x_0 * x_1 * x_2 * x_3 * x_4 * z_0 * z_1 * z_2 * z_3 * z_4; \]
\[ \text{--now check that k is contained in slocus2} \]

Program A.3. Proving that nodes on \( X \) have \( x_0 = 0 \) or \( z_0 = 0 \).
Let $G$ be the quintic threefold as in our thesis. $E_1 \cup E_2$ is a union of two elliptic curves.
We show that over any field of characteristic not equal to 2,
the set of points in $E_1 \cup E_2$ is precisely the set of rank 3 points of $G$.
$I$ is the ideal of the singular locus of $G$.

```plaintext
F = ZZ;
R = F[z0,z1,z2,z3,z4];
L = matrix({{0,-z4,z3,z2,-z1},{-z2,0,-z0,z4,z3},{z4,-z3,0,-z1,z0},{z1,z0,-z4,0,-z2},{-z3,z2,z1,-z0,0}});
I = minors(4,L);
```

-- the ideal of $E_1$ is generated by the elements $I r_i + s_i$, $i = 0,1,2,3,4$.
-- the ideal of $E_2$ is generated by the elements $-I r_i + s_i$, $i = 0,1,2,3,4$.

```plaintext
r0 = z0^2;
r1 = z1^2;
r2 = z2^2;
r3 = z3^2;
r4 = z4^2;
s0 = (z1*z4 + z2*z3);
s1 = (z2*z0 + z3*z4);
s2 = (z3*z1 + z4*z0);
s3 = (z4*z2 + z0*z1);
s4 = (z0*z3 + z1*z2);
```

-- if the characteristic of our base field is not 2, then the ideal of $E_1 \cap E_2$ is generated by the elements
-- $p_i = (I r_i + s_i)(-I r_i + s_i) = (r_i^2 + s_i^2)$
-- and the elements $(I r_i + s_i)(-I r_j + s_j) = r_i r_j + s_i s_j + I(ri sj - si rj)$.
-- But since the characteristic is not 2, the generators $r_i r_j + s_i s_j + I(ri sj - si rj)$
-- can be replaced by $r_i r_j + s_i s_j$ and $r_i s_j - s_i r_j$.

```plaintext
p0 = r0^2 + s0^2;
p1 = r1^2 + s1^2;
p2 = r2^2 + s2^2;
p3 = r3^2 + s3^2;
p4 = r4^2 + s4^2;
qu1 = r0 + s1 + s0 + s1;
qu0 = r0 + s1 + s1 + s0;
qu2 = r0 + r2 + s0 + s2;
qu3 = r0 + s3 + s0 + s3;
qu4 = r0 + s4 + s0 + s4;
qu0 = r0 + s4 + s4 + s0;
qu1 = r1 + r2 + s1 + s2;
qu2 = r1 + s2 + r2 + s1;
qu3 = r1 + s3 + s1 + s3;
qu4 = -r1 + s3 + s3 + s1;
qu1 = -r1 + s4 + s1 + s4;
qu2 = -r1 + s4 + s4 + s1;
qu3 = r2 + s3 + s2 + s3;
qu4 = -r2 + s3 + s3 + s2;
qu0 = r2 + s4 + s2 + s4;
qu1 = r3 + s4 + s4 + s2;
qu2 = r3 + s4 + s4 + s3;
qu3 = -r3 + s4 + s3 + s3;
qu4 = -r3 + s4 + s4 + s3;
E = ideal(p0,p1,p2,p3,p4,q01,q02,q03,q04,q12,q21,q31,q32,q33,q34,q41,q42,q43,q44);
```

-- $E$ is the ideal of the rank 3 locus of $G$.

```plaintext
Erad = radical E;
Irad = radical I;
Egens = transpose gens gb Erad;
Igens = transpose gens gb Irad;
```

-- one checks that the elements of $E_{gens}$ and the elements of $I_{gens}$ are identical, up to factors of 2.
-- Hence over a base field of characteristic not equal to 2, $E_{gens}$ and $I_{gens}$ cut out the same points.
-- The singular locus of $G$ thus consists of $E$ and some extra rank 4 points.

**Program A.4.** Proof that $E_1 \cup E_2$ is the rank 3 locus of $G$. 
-- We want to find the singular points of G.
-- with z0 = 0
F = ZZ;
R = F[z0, z2, z3, z1];
L = matrix{{0, z2, -z4, -z1, z3}, {z4, 0, z3, -z0, -z2}, {-z3, z0, 0, z4, -z1}, {-z2, -z4, z1, 0, z0}, {z1, -z3, -z0, z2, 0}};
-- here G is actually 2 times what it should be but that's ok
-- if we are not in characteristic 2
G = det L;
H = diff(G);
-- the ideal slocus defines the singular locus of G
slocus = ideal H + ideal (G);
slocusgens = transpose gens gb slocus;
-- the ideal slocus0 defines the set of singular points of G
-- with z0 = 0
slocusz0 = slocus + ideal(z0);
slocusz0gens = transpose gens gb slocusz0;
-- checking the Grobner basis slocusz0gens, we find that either some
-- other z_i is 0, in which case x is a sigma-node, or
-- z1 * z4 + z2 * z3 = 0. In this case,
slocus2z0 = slocusz0 + ideal(z1 * z4 + z2 * z3);
slocus2z0gens = transpose gens gb slocus2z0;
-- checking the Grobner basis slocus2z0gens,
-- and setting z1 = 1, one concludes that
-- z0 = 0, z1 = 1, z2 = z3^3, z4 = -z3^4,
-- and z3^10 = -1.

Program A.5. Finding nodes of G that satisfy z0 = 0.

-- consider the segre variety P4 x P4 in P24. our threefold X is a section of P4 by five
-- hyperplanes. we show that one can choose five hyperplanes such that no singularities appear until
-- the final slice. we can then apply the Lefschetz theorem to compute Hodge numbers of X.
F = ZZ/101;
R = F[x0, x1, x2, x3, z0, z1, z2, z3];
x4 = 1;
z4 = 1;
f0 = -x3 * z1 + x1 * z2 + x4 * z3 - x2 * z4;
f1 = -x3 * x0 + x4 * x2 + x2 * z3 + x1 * x4;
f2 = z1 * x0 - x4 * z1 - x0 * z3 + x3 * x4;
f3 = x4 * x0 + x2 * z1 - x0 * z2 - x1 * x4;
f4 = x2 * x0 - x0 * x1 + x3 * x2 - x1 * x3;
x0 = ideal(f0+3*f1+7*f2+2*f3+10*f4);
x1 = ideal(f0+5*f1+27*f2+53*f3+18*f4);
x2 = ideal(f0+54*f1+33*f2+42*f3+20*f4);
x3 = ideal(f0+9*f2+38*f2+19*f3+64*f4);
x4 = ideal(f0);
xj = diff (f0+3*f1+7*f2+2*f3+10*f4);
xj = diff (f0+5*f1+27*f2+53*f3+18*f4);
j2 = diff (f0+54*f1+33*f2+42*f3+20*f4);
j3 = diff (f0+9*f2+38*f2+19*f3+64*f4);
j4 = diff f0;
sing0 = minors(1,xj);
sing1 = minors(2,xj);
sing2 = minors(3,xj);
sing3 = minors(4,xj);
sing4 = minors(5,xj);
slocus0 = x0 + sing0;
slocus1 = x1 + sing1;
slocus2 = x2 + sing2;
slocus3 = x3 + sing3;
slocus4 = x4 + sing4;
-- check that dim slocus0 = dim slocus1 = dim slocus2 = dim slocus 3 = -1
-- and dim slocus4 = 0

Program A.6. Proof that \( \mathbb{P}^4 \times \mathbb{P}^4 \) can be successively sliced to obtain \( X \) in such a manner that singularities only appear at the end.
Appendix B. Calculating traces of $\text{Frob}_p$

| $p$  | 59  | 67  | 71  | 79  | 83  | 89  |
|------|-----|-----|-----|-----|-----|-----|
| $\#G(\mathbb{F}_p)$ | 225766 | 327706 | 407910 | 529886 | 613006 | 751756 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 | 1 | 5 | 1 | 1 | 1 |
| Other nodes defined over $\mathbb{F}_p$ | 0 | 0 | 0 | 0 | 0 | 10 |
| Points on $E_1 \cup E_2$ | 0 | 0 | 0 | 0 | 0 | 180 |
| $i$ in $\mathbb{F}_p$ | 0 | 0 | 0 | 0 | 0 | 1 |
| $\sqrt{5}$ in $\mathbb{F}_p$ | 1 | 0 | 1 | 1 | 0 | 1 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 | 5 | 1 | 1 | 1 | 1 |
| Other nodes defined over $\mathbb{F}_p$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\#X(\mathbb{F}_p)$ | 247360 | 355310 | 459740 | 568280 | 655170 | 897360 |
| $p^3 + 1 - \#X(\mathbb{F}_p)$ | -41980 | -54546 | -101828 | -75240 | -83382 | -192390 |
| $p^2 + p$ | 3540 | 4556 | 5112 | 6320 | 6972 | 8010 |
| $h$ | 12 | 12 | 20 | 12 | 12 | 24 |
| $tr \text{Frob}_p$ on $H^3$ | 500 | 126 | 412 | 600 | 282 | -150 |
| $6p^3$ | 2719.2 | 3290.6 | 3589.6 | 4213.1 | 4537.0 | 5037.8 |
| $a_p$ | 500 | 126 | 412 | 600 | 282 | -150 |
| $tr \text{Frob}_p - a_p$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $(tr \text{Frob}_p - a_p)/p$ | 0 | 0 | 0 | 0 | 0 | 0 |

2p + 2 − $\#(E_1 \cup E_2)(\mathbb{F}_p)$ = 0

Table B.1. Counting points on $X$ and calculating traces of $\text{Frob}_p$, cont.

| $p$  | 97  | 101 | 103 | 107 | 109 | 113 |
|------|-----|-----|-----|-----|-----|-----|
| $\#G(\mathbb{F}_p)$ | 967966 | 1126560 | 1157186 | 1295146 | 1365776 | 1517046 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 | 5 | 1 | 1 | 1 | 1 |
| Other nodes defined over $\mathbb{F}_p$ | 10 | 50 | 0 | 0 | 10 | 10 |
| Points on $E_1 \cup E_2$ | 170 | 200 | 0 | 0 | 220 | 230 |
| $i$ in $\mathbb{F}_p$ | 1 | 1 | 0 | 0 | 1 | 1 |
| $\sqrt{5}$ in $\mathbb{F}_p$ | 0 | 1 | 0 | 0 | 1 | 0 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 | 5 | 1 | 1 | 1 | 1 |
| Other nodes defined over $\mathbb{F}_p$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\#X(\mathbb{F}_p)$ | 1137910 | 1770940 | 1221870 | 1364910 | 1583340 | 1750730 |
| $p^3 + 1 - \#X(\mathbb{F}_p)$ | -225236 | -740638 | -129142 | -139866 | -288310 | -307832 |
| $p^2 + p$ | 9506 | 10302 | 10712 | 11556 | 11990 | 12882 |
| $h$ | 24 | 72 | 12 | 12 | 24 | 24 |
| $tr \text{Frob}_p$ on $H^3$ | 2908 | 1106 | -598 | -1194 | -550 | 1336 |
| $6p^3$ | 5732.1 | 6090.3 | 3589.6 | 6272.1 | 6620.9 | 6828.0 |
| $a_p$ | 386 | 702 | -598 | -1194 | -550 | 1562 |
| $tr \text{Frob}_p - a_p$ | 2522 | 404 | 0 | 0 | 0 | -226 |
| $(tr \text{Frob}_p - a_p)/p$ | 26 | 4 | 0 | 0 | 0 | -2 |

Table B.2. Counting points on $X$ and calculating traces of $\text{Frob}_p$, cont.
Table B.3. Counting points on $X$ and calculating traces of $\text{Frob}_p$, cont.

| $p$ | 127 | 131 | 137 | 139 | 149 | 151 |
|-----|-----|-----|-----|-----|-----|-----|
| $\#G(\mathbb{F}_p)$ | 2143566 | 2421910 | 2685206 | 2802246 | 3437616 | 3669110 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 | 5 | 1 | 1 | 1 | 5 |
| Other nodes defined over $\mathbb{F}_p$ | 0 | 0 | 10 | 0 | 10 | 0 |
| Points on $E_1 \cup E_2$ | 0 | 0 | 290 | 0 | 300 | 0 |
| $i$ in $\mathbb{F}_p$? | 0 | 0 | 1 | 0 | 1 | 0 |
| $\sqrt{5}$ in $\mathbb{F}_p$? | 0 | 1 | 0 | 1 | 1 | 1 |
| $\epsilon$ in $\mathbb{F}_p$? | 0 | 1 | 0 | 0 | 0 | 1 |
| $\#X(\mathbb{F}_p)$ | 2241610 | 2596140 | 3029350 | 2919840 | 3842300 | 3900140 |
| $p^3 + 1 - \#X(\mathbb{F}_p)$ | -193226 | -348048 | -534350 | -457188 |
| $p^2 + p$ | 16256 | 17292 | 18906 | 19460 | 22350 | 22952 |
| $h_1$ | 12 | 20 | 24 | 12 | 24 | 20 |
| $\text{tr Frob}_p$ on $H^3$ | 1846 | -2208 | -4252 | -700 | 2050 | 1852 |
| $6p^2$ | 8587.4 | 8996.2 | 9621.3 | 9832.8 | 10912.7 | 11313.2 |
| $\text{tr Frob}_p - a_p$ | 0 | 0 | -1918 | 0 | 0 | 0 |
| $(\text{tr Frob}_p - a_p)/p$ | 0 | 0 | -14 | 0 | 0 | 0 |

Table B.4. Counting points on $X$ and calculating traces of $\text{Frob}_p$, cont.

| $p$ | 157 |
|-----|-----|
| $\#G(\mathbb{F}_p)$ | 4019026 |
| $\sigma$-nodes defined over $\mathbb{F}_p$ | 5 |
| $\tau$-nodes defined over $\mathbb{F}_p$ | 1 |
| Other nodes defined over $\mathbb{F}_p$ | 10 |
| Points on $E_1 \cup E_2$ | 350 |
| $i$ in $\mathbb{F}_p$? | 1 |
| $\sqrt{5}$ in $\mathbb{F}_p$? | 0 |
| $\epsilon$ in $\mathbb{F}_p$? | 0 |
| $\#X(\mathbb{F}_p)$ | 4473070 |
| $p^3 + 1 - \#X(\mathbb{F}_p)$ | -603176 |
| $p^2 + p$ | 24806 |
| $h_1$ | 24 |
| $\text{tr Frob}_p$ on $H^3$ | -7832 |
| $6p^2$ | 11803.3 |
| $a_p$ | -2494 |
| $(\text{tr Frob}_p - a_p)/p$ | -34 |
| $2p + 2 - \#(E_1 \cup E_2)(\mathbb{F}_p)$ | -34 |
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