DIFFEOMORPHISMS PRESERVING MORSE-BOTT FUNCTIONS

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Abstract. Let \( f : M \to \mathbb{R} \) be a Morse-Bott function on a closed manifold \( M \), so the set \( \Sigma_f \) of its critical points is a closed submanifold whose connected components may have distinct dimensions. Denote by \( \mathcal{S}(f) = \{ h \in \mathcal{D}(M) \mid f \circ h = f \} \) the group of diffeomorphisms of \( M \) preserving \( f \) and let \( \mathcal{D}(\Sigma_f) \) be the group of diffeomorphisms of \( \Sigma_f \). We prove that the “restriction to \( \Sigma_f \)” map \( \rho : \mathcal{S}(f) \to \mathcal{D}(\Sigma_f), \rho(h) = h|_{\Sigma_f} \), is a locally trivial fibration over its image \( \rho(\mathcal{S}(f)) \).

1. Introduction

Homotopy properties of groups of leaf-preserving diffeomorphisms for nonsingular foliations are studied in many papers, see e.g. [1], [2], [3], [4], [5], [6], [7], [8] and references therein. Most of the results concern with the extension of results by M. Herman [9], W. Thurston [10], J. Mather [11], [12], D. B. A. Epstein [13] on proving perfectness of such groups.

However for singular foliations their groups of diffeomorphisms are less studied, e.g. [14], [15], [16], [17].

The present paper is devoted to certain deformational properties of groups of leaf-preserving diffeomorphisms of codimension one foliations with Morse-Bott singularities. These foliations, in particular, foliations by level sets of Morse-Bott functions, play an important role in Hamiltonian dynamics and Poisson geometry, see e.g. [18], [19], [20], [21], [22].

Let \( M \) be a smooth compact manifold and \( \mathcal{F} \) be a Morse-Bott foliation on \( M \) such that every connected components of \( \partial M \) is a leaf of \( \mathcal{F} \), see Definition 4.1. Thus the set \( \Sigma \) of singular leaves of \( \mathcal{F} \) is a disjoint union of finitely many closed submanifolds of \( M \). Let \( \mathcal{D}(M) \) and \( \mathcal{D}(\Sigma) \) be the groups of all smooth diffeomorphisms of \( M \) and \( \Sigma \) respectively, and \( \mathcal{D}(\mathcal{F}) \) be the subgroup of \( \mathcal{D}(M) \) consisting of leaf-preserving diffeomorphisms of \( \mathcal{F} \).

The well-known result by J. Cerf [23] and R. Palais [24] implies that the “restriction to \( \Sigma \)” map \( \rho : \mathcal{D}(M) \to \mathcal{D}(\Sigma), \rho(h) = h|_\Sigma \), is a locally trivial fibration over its image \( \rho(\mathcal{D}(M)) \). Since \( \rho \) is a homomorphism, the latter statement is equivalent to existence of local sections of \( \rho \), i.e. if \( \alpha = h|_\Sigma \in \mathcal{D}(\Sigma) \) for some \( h \in \mathcal{D}(M) \), then there exists an open neighborhood \( V \subset \mathcal{D}(\Sigma) \) of \( \alpha \) and a continuous map \( \xi : V \to \mathcal{D}(M) \) such that \( \xi(\beta)|_\Sigma = \beta \) for all \( \beta \in V \).

However, the authors did not find in the available literature that \( \rho|_{\mathcal{D}(\mathcal{F})} : \mathcal{D}(\mathcal{F}) \to \mathcal{D}(\Sigma) \) is a locally trivial fibration as well, i.e. that one can guarantee that \( \xi(\beta) \) is also leaf preserving.

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The aim of the present paper is to show that this is true for foliations that in a neighborhood of every singular leaf coincide with level sets of some Morse-Bott function, see Theorems 3.3 and 4.2. In particular, it will allow to relate homotopy groups of $D(F)$ and its subgroup $D(F, \Sigma)$, consisting of diffeomorphisms fixed on $\Sigma$, via homotopy groups of $D(\Sigma)$, see Corollary 3.6. This would be useful when the dimension of $\Sigma$ is small.

2. Preliminaries

Throughout the paper the word “smooth” means $C^\infty$, and the spaces of smooth maps are always endowed with the corresponding strong $C^\infty$ topologies and their subspaces with the induces ones.

We will say that a not necessarily surjective map $p : E \to B$ is a locally trivial fibration over its image if the map $p : E \to p(E)$ is a locally trivial fibration. It follows from path lifting property for $p$ that $p(E)$ is a union of certain path components of $B$. Moreover, if $E_x$ is a path component of $E$, then $p(E_x)$ is a path component of $B$ and the restriction map $p : E_x \to p(E_x)$ is a locally trivial fibration as well.

Let $M$ be a smooth manifold. For a subset $X \subset M$ denote by $D(M, X)$ the group of diffeomorphisms of $M$ fixed on $X$, and by $D_{id}(M, X)$ its identity path component. If $F$ is a (possibly singular) foliation on $M$, then we will denote by $D(F, X)$ the group of leaf-preserving diffeomorphisms of $M$ fixed on $X$, and by $D_{id}(F, X)$ its identity path component. If $X$ is empty, then we will omit it from notation. E.g. we write $D(M)$ instead of $D(M, \emptyset)$ and $D(F)$ instead of $D(F, \emptyset)$.

3. Morse-Bott maps

Let $M$ be a smooth compact manifold and $P$ be either the real line or the circle. Notice also that there is a natural right action of the groups of diffeomorphisms $D(M)$ of $M$ on the space $C^\infty(M, P)$ of smooth maps $M \to P$. For $f \in C^\infty(M, P)$ and a subset $X \subset M$ let

$$S(f) = \{ h \in D(M) \mid f \circ h = f \}, \quad S(f, X) = S(f) \cap D(M, X)$$

be the stabilizers of $f$ with respect to the above action of $D(M)$ and the induced action of $D(M, X)$. Let also $S_{id}(f)$ and $S_{id}(f, X)$ be the identity path components of the corresponding stabilizers.

**Definition 3.1.** A smooth map $f : M \to P$ will be called Morse-Bott if it satisfies the following conditions:

1. $f$ takes a constant value at each connected component of $\partial M$ and has no critical points in $\partial M$;

2. the set $\Sigma$ of critical points of $f$ is a union of at most countable locally finite family of mutually disjoint submanifolds $\bigcup C_i$ and for each $x \in C_i$ the Hessian of $f$ at $x$ is non-degenerate in the direction transversal to $C_i$.

If all critical points of a Morse-Bott map $f$ are isolated, then $f$ is called Morse. The well known analogue of Morse lemma, e.g. [25, Theorem 2], claims that condition (2) is equivalent to the following one:
for every critical point $x \in M$ of $f$ there exist an open neighborhood $U$ of $x$ in $M$, an open neighborhood $V$ of $f(x)$ in $P$, two open embeddings $h : U \rightarrow \mathbb{R}^m$ and $\phi : V \rightarrow \mathbb{R}$, and two integers $\lambda, k$ such that $0 \leq \lambda \leq k \leq m$, $h(x) = 0$, $\phi(f(x)) = 0$, $\phi$ preserves orientation, and

$$\phi \circ f \circ h^{-1}(v_1, \ldots, v_m) = -v_1^2 - \cdots - v_\lambda^2 + v_{\lambda+1}^2 + \cdots + v_k^2.$$  

In [16], [26], [27], [28] the second author computed the homotopy types of $S_{\text{id}}(f, X)$ for a large class of smooth maps $f : M \rightarrow P$ (which includes all Morse maps) on all compacts surfaces $M$, where $X$ is a finite (possibly empty) collection of regular components of level sets of $f$ and critical points.

**Theorem 3.2.** [26, Theorem 1.3], [27, Theorem 2.1], [28, Theorem 3.1]. Let $M$ be a compact surface, $f : M \rightarrow P$ be a Morse map, and $X$ be a finite (possibly empty) collection of regular components of level sets of $f$ and critical points. Then $S_{\text{id}}(f, X)$ is contractible if and only if either of the following conditions holds true:

- $M$ is non-orientable;
- $f$ has at least one saddle, i.e. a critical point of index 1;
- $\chi(M) < |X|$, e.g. this hold if $X$ contains a regular component of some level set $f^{-1}(c)$, $c \in P$.

In all other cases $S_{\text{id}}(f, X)$ is homotopy equivalent to the circle.

Suppose $M$ is orientable. Let $\omega$ be any volume form on $M$ and $S(f, \omega)$ be the stabilizer of $f$ with respect to the right action of the group of $\omega$-preserving diffeomorphisms. Then its identity path component $S_{\text{id}}(f, \omega)$ is an abelian group and the inclusion $S_{\text{id}}(f, \omega) \subset S_{\text{id}}(f)$ is a homotopy equivalence. □

The technique of those papers is essentially two-dimensional and based on the observation that for orientable surfaces $S_{\text{id}}(f, X)$ is a subgroup of the group diffemorphisms preserving orbits of the Hamiltonian flow of $f$. Let us mention that the papers [26], [29], [30], [31], [32], [33], [34], [35], [36], [37] are devoted to computations of the homotopy types of orbits of Morse maps on compact surfaces with respect to the above actions.

The present paper is the first of a series of subsequent papers in which we plan to investigate the homotopy type of $S_{\text{id}}(f, X)$ for Morse-Bott maps $f : M \rightarrow P$ on manifolds of arbitrary dimensions. The initial step is to reduce the problem to diffeomorphisms fixed on the set of critical points.

**Theorem 3.3.** Let $f : M \rightarrow P$ be a Morse-Bott map of a smooth compact manifold $M$, so the set $\Sigma_f$ of critical points of $f$ is a disjoint union of smooth mutually disjoint closed submanifolds $C_1, \ldots, C_k$. Let also $X \subset M \setminus \Sigma_f$ be a closed (possibly empty) subset. Then the maps

$$\rho : S(f, X) \rightarrow D(\Sigma_f), \quad \rho(h) = h|_{\Sigma_f};$$

$$\rho_0 : S_{\text{id}}(f, X) \rightarrow D_{\text{id}}(\Sigma_f) \equiv \prod_{i=1}^k D_{\text{id}}(C_i), \quad \rho_0(h) = (h|_{C_1}, \ldots, h|_{C_k}),$$

are locally trivial fibrations over their images and the map $\rho_0$ is surjective.

This theorem can be regarded as a variant of the result by J. Cerf [23] and Palais [24] on local triviality of restrictions to critical submanifolds of Morse-Bott function $f$ for $f$-preserving diffeomorphisms.
In fact Theorem 3.3 holds for more general classes of maps, see Theorem 9.2. On the other hand it says nothing for Morse maps, i.e. when each $C_i$ is a point.

**Remark 3.4.** Evidently,
\[
\ker(\rho) = \mathcal{S}_{id}(f, X \cup \Sigma_f), \quad \ker(\rho_0) = \mathcal{D}(M, \Sigma_f) \cap \mathcal{S}_{id}(f, X),
\]
and these groups are the fibres of the corresponding fibrations (3.1) and (3.2). Notice that $\ker(\rho_0)$ differs from $\mathcal{S}_{id}(f, X \cup \Sigma_f)$ as one may expect at first glance. In fact we have only the inclusion
\[
\mathcal{S}_{id}(f, X \cup \Sigma_f) \subset \mathcal{D}(M, \Sigma_f) \cap \mathcal{S}_{id}(f, X).
\]

There are two standard applications of Theorem 3.3.

**Corollary 3.5** (Ambient isotopy extension for critical submanifolds). Every smooth isotopy $F : \Sigma_f \times [0, 1] \to \Sigma_f$ with $F_0 = \text{id}_\Sigma_f$ extends to an isotopy $F : M \times [0, 1] \to M$ such that $F_t = \text{id}_M$ and $F_t \in \mathcal{S}(f, X)$ for all $t \in [0, 1]$.

**Corollary 3.6.** There is a long exact sequence of homotopy groups:
\[
\cdots \to \prod_{i=1}^{k} \pi_{q+1} \mathcal{D}_{id}(C_i) \to \pi_q \mathcal{S}_{id}(f, X \cup \Sigma_f) \to \pi_q \mathcal{S}_{id}(f, X) \to \cdots \to \prod_{i=1}^{k} \pi_1 \mathcal{D}_{id}(C_i) \to \pi_0 \mathcal{S}(f, X \cup \Sigma_f) \to \pi_0 \mathcal{S}(f, X) \to \pi_0 \rho(\mathcal{S}(f, X)) \to 1,
\]
where the (omitted) base points are the corresponding identity maps.

**Proof.** This is a usual sequence of homotopy groups
\[
\cdots \to \pi_q(\mathcal{S}(f, X \cup \Sigma_f), \text{id}_M) \to \pi_q(\mathcal{S}(f, X), \text{id}_M) \to \pi_q(\mathcal{D}(\Sigma_f), \text{id}_{\Sigma_f}) \to \cdots
\]
of the fibration $\rho : \mathcal{S}(f, X) \to \rho(\mathcal{S}(f, X)) \subset \mathcal{D}(\Sigma_f)$. We just replaced for $q \geq 1$ each group with its identity path component and took to account that $\mathcal{D}_{id}(\Sigma_f)$ is naturally isomorphic with $\prod_{i=1}^{k} \mathcal{D}_{id}(C_i)$. \qed

3.7. **Homotopy types of diffeomorphisms groups.** The following table contains a description of homotopy types of $\mathcal{D}_{id}$ for all connected closed manifolds of dimensions 1 and 2, see e.g. [38], [39], [40]. We present this information just for the convenience of the reader. This would allow to relate homotopy groups of $\mathcal{S}_{id}(f, X)$ and $\mathcal{S}_{id}(f, X \cup \Sigma_f)$ via Corollary 3.6.

| $C$ | dim $\mathcal{D}_{id}(C)$ |
|-----|-------------------------|
| $S^1$ | 1 | $S^1$ |
| $S^2$, $\mathbb{R}P^2$ | 2 | $\text{SO}(3)$ |
| $S^1 \times S^1$ | $S^1 \times S^1$ |
| $D^2$, $S^1 \times I$, Klein bottle | $S^1$ |
| all other cases | point |

In particular, this allows to get some information about the homotopy type of $\mathcal{S}_{id}(f)$ for Morse-Bott maps from 3-manifolds as their critical submanifolds have dimensions $\leq 2$.

The homotopy types of $\mathcal{D}(C)$ for 3-manifolds are studied not so well. Recall that the Smale conjecture holds for a Riemannian 3-manifold $C$ whenever the natural inclusion
Isom(\(C\)) \(\subset \mathcal{D}(\mathcal{C})\) of its isometry group into the groups of diffeomorphisms is a homotopy equivalence. This conjecture was verified e.g. for all closed hyperbolic 3-manifolds (D. Gabai [41]), the unit 3-sphere \(S^3\) with the standard metric (A. Hatcher [42]), and more generally for many classes of elliptic 3-manifolds, e.g. [43]. Chapter 1 of the latter book [43] also contains a comprehensive historical description and the current state of Smale conjecture.

Let us also mention two other results:

- \(\mathcal{D}(S^1 \times S^2)\) is homotopy equivalent to \(O(2) \times O(3) \times \Omega O(3)\), where \(\Omega O(3)\) is a loop space of \(O(3)\), A. Hatcher [44];

- if \(C\) is a closed orientable Haken manifold that does not contain embedded projective plane \(\mathbb{R}P^2\), then \(\mathcal{D}_{\text{id}}(C)\) is homotopy equivalent to one of the following spaces: a point, \(S^1\), \(S^1 \times S^1 \times S^1\), A. Hatcher [45].

In dimensions \(\geq 4\) the situation is very unclear and mostly concerns with computations of the mapping class group \(\pi_q \mathcal{D}(C)\) and finding nontrivial elements in homotopy groups \(\pi_q \mathcal{D}_{\text{id}}(C)\), e.g. [46], [47], [48], [49].

### 3.8. Structure of the paper.

In §4 we define foliations generated by Morse-Bott functions and formulate an analogue of Theorem 3.3 for leaf-preserving diffeomorphisms for such foliations, see Theorem 4.2. In §5 we recall a notion of an Ehresmann fibration in a fibre bundle and some of their properties: convexity and homotopy lifting property, Lemmas 5.1 and 5.2. §6 is devoted to the proof of a well-known “local triviality for restrictions to a submanifold”, Theorem 6.2. Our proof is a modification of E. Lima [50] and uses Ehresmann connections, see Lemma 6.2.3. In §7 we show that in a tubular neighborhood of a critical submanifold of a Morse-Bott map \(f\) there exists an Ehresmann connection “preserving \(f\)”, in the sense that the differential \(df\) annihilates horizontal bundle of that connection. Finally in §8 we prove Theorem 8.2 extending Theorem 4.2, and in §9 deduce from it Theorem 9.2 which extends Theorem 3.3.

### 4. Foliations with Morse-Bott singularities

**Definition 4.1.** cf. [19]. Let \(\mathcal{F}\) be a partition of \(M\). We say that \(\mathcal{F}\) is a foliation with Morse-Bott singularities if there exists a locally finite family of mutually disjoint smooth connected submanifolds \(\Sigma = \{C_i\}_{i \in \Lambda}\) such that

(a) the restriction of \(\mathcal{F}\) to \(M \setminus \bigcup_{i \in \Lambda} C_i\) is a smooth codimension one foliation and every connected component of \(\partial M\) is a leaf of \(\mathcal{F}\);

(b) every \(C_i\) is an leaf of \(\mathcal{F}\), \(\dim C_i < \dim M\), and \(C_i \subset \text{Int} M\);

(c) for each \(i \in \Lambda\) and \(x \in C_i\) there exist two numbers \(\lambda \leq k \in \{0, \ldots, m\}\), an open neighborhood \(U\) of \(x\) in \(M\), and an open embedding \(\phi : U \to \mathbb{R}^m\) such that \(\phi(x) = 0\), and for every leaf \(\omega\) of \(\mathcal{F}\) the image \(\phi(\omega \cap U)\) is a connected component of some level set of the function \(f : \phi(U) \to \mathbb{R}\) defined by

\[
f(v_1, \ldots, v_m) = -v_1^2 - \cdots - v_{\lambda - 1}^2 + v_{\lambda + 1}^2 + \cdots + v_k^2
\]

for \((v_1, \ldots, v_m) \in \phi(U)\).

We will say that \(\mathcal{F}\) is generated by Morse-Bott functions if it satisfies conditions (a), (b), and the following, more stronger than (c), condition (c‘):
(c’) for every \( C_i \) there exist an open neighborhood \( N_i \) and a Morse-Bott function \( f : N_i \to \mathbb{R} \) such that \( C_i = f^{-1}(0) \), this set is the set of critical points of \( f \), and every connected component of \( f^{-1}(t) \setminus C_i \), \( t \in \mathbb{R} \), is contained in some leaf of \( \mathcal{F} \).

It is proved in [19, Theorem B] that if \( \mathcal{F} \) is a foliation with Morse-Bott singularities, and a singular leaf \( C_i \) has a finite holonomy group (e.g. finite fundamental group), then condition \((c’)\) holds for \( C_i \).

**Theorem 4.2.** Let \( \mathcal{F} \) be a foliation generated by Morse-Bott functions on a compact manifold \( M \). Then for every closed (possibly empty) subset \( X \subset M \setminus \Sigma \) the “restriction to \( \Sigma \)” homomorphisms

\[
\rho : \mathcal{D}(\mathcal{F}, X) \to \mathcal{D}(\Sigma), \quad \rho(h) = h|_{\Sigma},
\]

\[
\rho : \mathcal{D}_{id}(\mathcal{F}, X) \to \prod_{i=1}^{k} \mathcal{D}_{id}(C_i), \quad \rho(h) = (h|_{C_1}, \ldots, h|_{C_k}),
\]

are locally trivial fibrations over their images and \( \rho_0 \) is surjective.

This theorem is a particular case of Theorem 8.2 which also implies Theorems 3.3 and 9.2.

5. Ehresmann connections

Let \( p : E \to C \) be a smooth fibre bundle over a smooth connected manifold \( C \). For \( x \in C \) we will denote by \( E_x = p^{-1}(x) \) the fibre over \( x \). Let

\[
VE = \bigcup_{(x,v) \in E} T_{(x,v)}E_x
\]

be the union of all tangent planes to the fibres. The projection \( q : VE \to E \) is called the \emph{vertical subbundle} of \( TE \).

More generally, for an open subset \( Y \subset E \) let

\[
VY = q^{-1}(Y) = TY \cap VE
\]

be the union of tangent spaces to fibres at points of \( Y \). Then a \emph{local Ehresmann connection over} \( Y \) is a smooth vector bundle projection \( \nabla : TY \to VY \) over the identity map of \( Y \) i.e. it makes commutative the following diagram:

A local Ehresmann connection over \( E \) is called an \emph{Ehresmann connection}.

The kernel \( \ker(\nabla) \) of \( \nabla \) is the subbundle of \( TY \) called \emph{horizontal} and usually denoted\(^1\) by \( HY \). Then we have a direct sum splitting \( TY = HY \oplus VY \).

\(^1\)The notation \( HY \) does not include \( \nabla \) itself, however throughout the paper this will not lead to confusion and a concrete connection \( \nabla \) will be clear from the the context.
If \( N \) is a smooth manifold, then a smooth map \( \phi : N \to Y \) is called \emph{horizontal} with respect to \( \nabla \), or simply \( \nabla \)-\emph{horizontal}, if the image of the tangent bundle \( TN \) under the tangent map \( T\phi : TN \to TY \) is contained in the horizontal subbundle \( HY \), i.e.

\[
T\phi(TN) \subset HY.
\]

It is well known, e.g. [51, Theorem 9.8], that \( \nabla \) induces the so-called \emph{parallel transport}. Namely, if \( \gamma : [0,1] \to C \) is a smooth path and \( z \in E_{\gamma(0)} \cap Y \), then there exists an \( \varepsilon \in (0,1] \) and a unique smooth path \( \tilde{\gamma}_z : [0,\varepsilon] \to Y \) such that

(i) \( \tilde{\gamma}_z(0) = z \);
(ii) \( p \circ \tilde{\gamma}_z = \gamma : [0,\varepsilon] \to C \);
(iii) \( \tilde{\gamma}_z \) is \( \nabla \)-horizontal.

This path \( \tilde{\gamma}_z \) is called a \emph{horizontal lift} of \( \gamma \) at \( z \). Moreover, \( \tilde{\gamma}_z \) smoothly depends on \( z \) and \( \gamma \) in the sense described in Lemma 5.2 below. If \( Y = E \) and one can choose \( \varepsilon = 1 \) for any such path \( \gamma \) and \( z \in E_{\gamma(0)} \), then the connection \( \nabla \) is called \emph{complete}.

Let \( Z \subset Y \) be an open subset and \( \alpha = \{A_i\}_{i \in A} \) be an open cover of \( C \). We will say that a local Ehresmann connection \( \nabla : TY \to VY \) over \( Y \) is \emph{complete with respect to} \( (Z,\alpha) \) if for each smooth path \( \gamma : [0,1] \to C \) whose image is contained in some element of \( A_i \) of \( \alpha \) and each \( z \in E_{\gamma(0)} \cap Z \) there exists a horizontal lift \( \tilde{\gamma}_z : [0,1] \to Y \) satisfying the above conditions (i)-(iii).

Thus a complete Ehresmann connection is a local Ehresmann connection over \( E \) being complete with respect to the pair \( (E,\{C\}) \).

**Lemma 5.1.** cf. [52, Exercise 12(iv)]. The set of all Ehresmann connections \( TY \to VY \) over an open subset \( Y \) of \( E \) is convex in the linear space of all bundle morphisms of \( TY \to TE \) over the identity inclusion \( Y \subset E \).

More generally, let \( \mathcal{L} = \bigcup_{z \in Y} \mathcal{L}_z \) be a distribution on \( Y \), i.e. a correspondence associating to each \( z \in Y \) some linear subspace \( \mathcal{L}_z \) of the tangent space \( T_zY \), (the dimensions of \( \mathcal{L}_z \) may vary with \( z \), and we do not assume any kind of continuity of the correspondence \( z \mapsto \mathcal{L}_z \)). Denote by \( \Gamma_Y(\mathcal{L}) \) the set of all local Ehresmann connections over \( Y \) whose horizontal bundles are contained in \( \mathcal{L} \). Then \( \Gamma_Y(\mathcal{L}) \) is convex as well. \( \square \)

Using a partition of unity technique one easily deduces from Lemma 5.1 that every smooth fibre bundle \( p : E \to C \) always admits an Ehresmann connection. However, as it is mentioned in [53] the subspace of \emph{complete} Ehresmann connections is not necessarily convex. Hence gluing complete Ehresmann connections defined in local charts gives an Ehresmann connection which is not necessarily complete. Nevertheless, it is shown in [53] that every smooth fibre bundle admits a complete Ehresmann connection. For our purposes it is enough to use only local Ehresmann connections being complete with respect to some pair \( (Z,\alpha) \), see the last paragraph of the proof of Theorem 6.2, and we will not use the result of [53].

**Lemma 5.2** (Homotopy lifting property). cf. [52, Exercise 13(i)]. Let \( \nabla : TY \to VY \) be a local Ehresmann connection complete with respect to an open set \( V \subset Y \) and an open cover \( \alpha = \{A_i\}_{i \in A} \) of \( C \). Let also \( W \) be a smooth manifold and \( F : W \times [0,1] \to C \) be a smooth homotopy. Suppose there exists a smooth map \( \mathbf{F}_0 : W \times 0 \to C \) such that \( p \circ \mathbf{F}_0 = F_0 \).

Then there exists an open neighborhood \( G_0 \) of \( W \times 0 \) in \( W \times [0,1] \) and a unique smooth map \( \mathbf{F} : G \to Y \) having the following properties:
(a) $F(w, 0) = F_0(w)$ for all $w \in G$;
(b) $p \circ F = F|_G$;
(c) if $w \times [0, \varepsilon] \subset G$ for some $w \in G$ and $\varepsilon > 0$, then the path $F_w : [0, \varepsilon] \to E$ defined by $F_w(t) = F(w, t)$ is $\nabla$-horizontal.

In particular, we have the following commutative diagram:

$$
\begin{array}{ccc}
W \times 0 & \xrightarrow{F_0} & Z \xleftarrow{F} Y \xleftarrow{G} E \\
\downarrow & & \downarrow \\
G & \xleftarrow{W \times [0, 1]} & C
\end{array}
$$

If $\nabla$ is a complete Ehresmann connection then one can take $G = W \times [0, 1]$.

Proof. Conditions (a)-(c) uniquely determine the map $F$. Indeed, let $z = (x, v) \in W$ and $\gamma_z : [0, 1] \to C$ be given by $\gamma_z(t) = F(z, t)$. Let $I_z \subset [0, 1]$ be a maximal subinterval such that $\gamma_z(I_z)$ is contained in some element $A_\alpha$ of the restriction to the initial point $z$, and also that $G'$ contains an open neighborhood $G$ of $W \times 0$. Then the restriction $F|_G : G \to Y$ by $F(z, t) = \gamma_z(t)$.

Notice that $\gamma_z$ is a solution of a certain ODE whose coefficients are smooth functions of derivatives of $\gamma$. This implies existence and uniqueness of the lift $\gamma_z$, its smooth dependence on the initial point $z$, and also that $G'$ contains an open neighborhood of $W \times 0$. Then the restriction $F|_G : G \to Y$ is the required map. We leave the details for the reader. \qed

6. Local triviality of restriction maps to singular leaves

Let $p : E \to C$ be a smooth $k$-dimensional vector bundle over a smooth manifold $C$. We will identify $C$ with its image under the zero section.

Let $\mathcal{E}(E; C)$ be the subset of $C^\infty(E, E)$ consisting maps $h : E \to E$ such that $h(C) = C$. For an open $V \subset E$ denote by $\mathcal{E}(E, E \setminus V; C)$ the subset of $\mathcal{E}(E; C)$ consisting of maps fixed on $E \setminus V$. Let also $\mathcal{D}(E)$ be the group of $C^\infty$ diffeomorphisms of $E$ and

$$
\mathcal{D}(E; C) = \mathcal{E}(E; C) \cap \mathcal{D}(E), \quad \mathcal{D}(E, E \setminus V; C) = \mathcal{E}(E, E \setminus V; C) \cap \mathcal{D}(E).
$$

We endow all these spaces with the corresponding $C^\infty$ topologies.

Notice that there is a natural “restriction to $C$ homomorphism”:

$$
\rho : \mathcal{D}(E; C) \to \mathcal{D}(C), \quad \rho(h) = h|_C.
$$

It is well known that $\rho$ is a locally trivial fibration, [24], [23], [50]. We present now a slight modification of the proof from [50] which uses local Ehresmann connections. This will allow us to establish also a local triviality of restrictions to singular leaves for certain groups of foliated diffeomorphisms, see Theorem 8.2.

Remark 6.1. Let $p : E \to C$ be a smooth fibre bundle. Chapter 3 of [43] extends Cerf-Palais result about local triviality for embeddings “in a vertical direction”, that is for submanifolds $X \subset E$ being unions of fibres and diffeomorphisms preserving fibres of vertical subbundle.
Theorem 6.2. cf. [24], [23], [50] Let $C$ be a closed manifold, $p : E \to C$ be a smooth vector bundle over, and $V$ be an open neighborhood of $C$ with compact closure. Then for every subgroup $A \subset \mathcal{D}(E; C)$ containing $\mathcal{D}(E, E \setminus V; C)$ the “restriction to $C$ homomorphism”:

$$\rho : A \to \mathcal{D}(C), \quad \rho(h) = h|_C$$

is a locally trivial fibration. This map is not necessarily onto though its image consists of entire path components of $\mathcal{D}(C)$. If the identity path component $\mathcal{D}_{id}(C)$ of $\mathcal{D}(C)$ is contractible, then $A$ is homeomorphic with $\ker(\rho) \times \rho(A)$.

Proof. We need the following three lemmas. The first one is well known, but for completeness we present a short proof.

Lemma 6.2.1. cf. [24, Theorem A] Let $\rho : A \to B$ be a continuous homomorphisms of topological groups with units $e_A$ and $e_B$ respectively. Suppose there exists an open neighborhood $U$ of $e_B$ in $B$ and a continuous map $\xi : U \to A$ such that

(a) $\xi(e_B) = e_A$;
(b) $\rho \circ \xi(g) = g$ for all $g \in U$.

Then the induced surjective homomorphism $\rho : A \to \rho(A)$ is locally trivial fibration with fiber $\ker(\rho)$.

Moreover, the image of $\rho$ consists of entire path components of $B$. In particular, if $A_e$ and $B_e$ are the path components of the units in $A$ and $B$ respectively, then $\rho(A_e) = B_e$.

If $\xi(U \cap B_e) \subset A_e$, then the restriction map $\rho|_{A_e} : A_e \to B_e$ is a locally trivial fibration with fibre $\ker(\rho) \cap A_e$.

Proof. 1) First we will check that $\rho$ is a locally trivial fibration. Let $a \in A$ and $b = \rho(a) \in B$. Then by (a) the set $U_b = bU$ is an open neighborhood of $b$ in $B$. Define the following map $\phi : U_b \times \ker(\rho) \to \rho^{-1}(U_b)$ by

$$\phi(g, k) = a \xi(b^{-1}g) k.$$

Then $\phi$ is a homeomorphism making commutative the following diagram:

$$
\begin{array}{ccc}
U_b \times \ker(\rho) & \xrightarrow{\phi} & \rho^{-1}(U_b) \\
\downarrow \rho & & \downarrow \rho^{-1} \\
U_b & &
\end{array}
$$

and its inverse $\phi^{-1} : \rho^{-1}(U_b) \to U_b \times \ker(\rho)$ is given by

$$\phi^{-1}(x) = (\rho(x), \xi(b^{-1}\rho(x))^{-1}a^{-1}x).$$

Indeed, notice that

$$\rho(\phi(g, k)) = \rho(a \xi(b^{-1}g) k) = \rho(a) \rho(\xi(b^{-1}g)) \rho(k) = bb^{-1}g = g.$$

Hence

$$\phi^{-1} \circ \phi(g, k) = \phi^{-1}(a \xi(b^{-1}g) k) = (g, \xi(b^{-1}g)^{-1}a^{-1} \xi(b^{-1}g) k) = (g, k),$$

$$\phi \circ \phi^{-1}(x) = \phi(\rho(x), \xi(b^{-1}\rho(x))^{-1}a^{-1}x) = a \xi(b^{-1}\rho(x)) \xi(b^{-1}\rho(x))^{-1}a^{-1}x = x.$$

Thus $\rho : A \to B$ is a locally trivial fibration.

2) Let us prove that $\rho(A)$ consists of entire path components. Let $b = \rho(a) \in \rho(A)$ and $\gamma : [0, 1] \to B$ be a path such that $\gamma(0) = b$ and $\gamma(1) = b'$. We will show that $b' = \rho(a')$ for
there exists an open neighborhood \( \tilde{\xi} \in Z, \alpha \). Lemma 6.2.2. Let \( \alpha \) be a complete Riemannian metric of \( \nabla \). Then one easily checks that \( \tilde{\xi} : [0, 1] \rightarrow A \) such that \( \tilde{\gamma}(0) = a \) and \( \rho \circ \tilde{\gamma} = \gamma \). Denote \( \alpha' = \tilde{\gamma}(1) \). Then \( \beta' = \gamma(1) = \rho \circ \tilde{\gamma}(1) = \rho(a') \).

3) Finally notice that \( V = U \cap B_{\epsilon} \) is an open neighborhood of \( e_{B} \) in \( B_{\epsilon} \). Hence if \( \xi(U \cap B_{\epsilon}) \subset A_{\epsilon} \), then by 1) the restriction \( \rho|_{A_{\epsilon}} \) is a locally trivial fibration as well. \( \square \)

**Lemma 6.2.3.** Let \( p : E \rightarrow C \) be a smooth vector bundle, and \( \nabla : TY \rightarrow V Y \) be a local Ehresmann connection over an open neighborhood \( Y \subset E \subset C \). Suppose \( TC \subset HY \). Then for each open cover \( \beta \) of \( C \) there exists an open neighborhood \( Z \subset Y \subset C \) and an open cover \( \alpha = \{ A_{i} \} \) such that

- \( \alpha \) is a refinement of \( \beta \);
- the closures of \( A_{i} \) are compact and constitute a locally finite family;
- \( \nabla \) is complete with respect to \( (Z, \alpha) \).

**Proof.** Fix a complete Riemannian metric of \( E \) and for each \( x \in C \) and \( \delta > 0 \) denote by \( B_{\delta}(x) \) the open geodesic ball of radius \( \delta \) in the fibre \( E_{x} \) centered at its origin which is identified with the point \( x \) itself. More generally, for a subset \( X \subset C \) and \( \delta > 0 \) denote \( B_{\delta}(X) = \bigcup_{x \in X} B_{\delta}(x) \).

Let \( \gamma : [0, 1] \rightarrow C \subset E \) be a smooth path and \( z = \gamma(0) \). Since \( z \in C \) and \( TC \subset HY \), it follows that the horizontal lift \( \tilde{\gamma}_{z} \) of \( \gamma \) coincides with \( \gamma \). Hence we get from continuous dependence of solutions of ODE on initial data that one can find at most countable cover \( \alpha = \{ A_{i} \}_{i \in \Lambda} \subset C \) and for each \( i \in \Lambda \) a number \( \delta_{i} > 0 \) such that

- \( \alpha \) is a refinement of \( \beta \);
- the closures of \( A_{i} \) are compact and constitute a locally finite family;
- for every smooth path \( \gamma : [0, 1] \rightarrow A_{i} \) and \( z \in B_{\delta_{i}}(A_{i}) \) there exists a horizontal lift \( \tilde{\gamma}_{z} : [0, 1] \rightarrow Y \).

The second property implies that each \( A_{i} \) intersects only finitely many other elements of \( \alpha \), whence for each \( i \in \Lambda \) the following number is positive:

\[ \eta_{i} = \min \{ \delta_{j} \mid A_{j} \cap A_{i} \neq \emptyset, j \in \Lambda \} > 0. \]

Put

\[ Z = \bigcup_{i \in \Lambda} B_{\eta_{i}}(A_{i}). \]

Then one easily checks that \( \nabla \) is complete with respect to the pair \( (Z, \alpha) \). \( \square \)

**Lemma 6.2.3.** [50] Let \( \nabla : TY \rightarrow V Y \) be a local Ehresmann connection over an open neighborhood \( Y \subset E \subset C \) such that \( TC \subset HY \). Then there exists an open neighborhood \( \mathcal{V} \) of \( \text{id}_{C} \) in \( \mathcal{D}(C) \) and a homotopy \( \xi : \mathcal{V} \times [0, 1] \rightarrow \mathcal{D}(E, E \setminus Y; C) \) such that

- \( \xi_{0}(\mathcal{V}) = \text{id}_{E} \), that is \( \xi_{0}(g)(x, v) = (x, v) \) for all \( g \in \mathcal{V} \);
- for each \( g \in \mathcal{V} \) and \( (x, v) \in Y \) the path \( t \mapsto \xi_{t}(g)(x, v) \) is horizontal;
- \( \xi_{t}(\text{id}_{C}) = \text{id}_{E} \) for all \( t \in [0, 1] \);
- \( \rho \circ \xi_{1} = \text{id}_{\mathcal{V}} \), that is \( \xi_{1}(g)(c) = g \) for all \( g \in \mathcal{V} \).

In particular, \( \xi_{1} \) is a section of the restriction map \( \rho \).

**Proof.** The proof follows the line of [50]. By Lemma 6.2.2 there exists an open neighborhood \( Z \subset Y \subset C \) of \( C \) and a finite (as \( C \) is compact) open cover \( \alpha = \{ A_{i} \}_{i=1,...,a} \subset C \) such that \( \nabla \) is complete with respect to \( (Z, \alpha) \).
Fix an embedding of $C$ into $\mathbb{R}^n$ and let $\pi : T \to C$ be its tubular neighborhood. Then there exists $\tau > 0$ such that for every pair $x, y \in C$ with $|x - y| < \tau$ the segment $I_{x,y}$ between $x$ and $y$ is contained in $T$, and its projection $\pi(I_{x,y})$ is contained in some element $A_i$ of the cover $\alpha$.

Denote by $W$ the set of all $g \in D(C)$ such that $|g(x) - x| < \tau$ for each $x \in C$. Then $W$ is an open neighborhood of $\text{id}_C$ in $D(C)$. We will construct a map

$$\xi : W \times [0, 1] \to \mathcal{E}(E, E \setminus Y; C)$$

satisfying (a)-(d), and then choose a smaller neighborhood $V \subset W$ of $\text{id}_C$ such that

$$\xi(V \times [0, 1]) \subset D(E, E \setminus Y; C).$$

Fix a $C^\infty$ function $\lambda : E \to [0, 1]$ such that $\lambda = 0$ on some neighborhood $N$ of $\overline{E \setminus Z}$ and $\lambda = 0$ near $C$. For each $g \in W$ consider the following map:

$$F^g : E \times [0, 1] \to C, \quad F^g(x, v, t) = \pi\left(x + t\lambda(x, tv)(g(x) - x)\right),$$

where $x \in C$, $v \in E_x$, and $t \in [0, 1]$. Then

$$F^g(x, v, 0) = \pi(x) = x = p(x, v).$$

Define also the map $F^g_0 = \text{id}_E : E \times 0 \to E$ by $F^g_0(x, v) = (x, v)$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
E \times 0 & \xrightarrow{F^g_0 = \text{id}_E} & E \\
\downarrow & & \downarrow p \\
E \times [0, 1] & \xrightarrow{F^g} & C
\end{array}$$

We want to construct a diagonal map $F^g : E \times [0, 1] \to E$ extending $F^g_0$ and satisfying $F^g = p \circ F^g$.

By assumption for each $x \in C$ the path $\pi(I_{x,g(x)})$ is contained in some element $A_i$ of the cover $\alpha$. Since $\nabla$ is complete with respect to the pair $(Z, \alpha)$, we get from Lemma 5.2 that there exists a unique smooth lift $F^g : Z \times [0, 1] \to E$ such that $F^g = p \circ F^g$ and for every $(x, v) \in Z$ the path $t \mapsto F^g(x, v, t)$ is horizontal. Extend $F^g$ to the map $F^g : E \times [0, 1] \to E$ by $F^g(z, t) = z$ for $z \in E \setminus Z$.

We claim that $F^g$ is smooth. It suffices to show that $F^g(z, t) = z$ for $z \in Z \cap N$.

Let $(x, v) \in Z \cap N$. Then $\lambda(x, v) = 0$, whence

$$F^g(x, v, t) = \pi\left(x + \lambda(x, v)(g(x) - x)\right) = \pi(x) = x$$

for all $t \in [0, 1]$. In other words, $t \mapsto F^g(x, v, t)$ is a constant path, whence its lifting $t \mapsto F^g(x, v, t)$ is also constant, that is $F^g(x, v, t) = F^g(x, v, 0) = (x, v)$.

Thus each $F^g_i$ is supported in $Z$, so in particular, we have a well-defined map

$$\xi : W \times [0, 1] \to \mathcal{E}(E, E \setminus Y; C), \quad \xi(g, t)(x, v) = F^g(x, v, t).$$

One easily checks that $\xi$ is continuous. Let us verify properties (a)-(d) for $\xi$. Evidently, (a) and (b) follow from the definition.
(c) We should prove that \( \xi_t(\text{id}_C) = \text{id}_E \) for all \( t \in [0, 1] \). By the construction \( \xi_t(g) \) is fixed on \( E \setminus Z \) for all \( g \in \mathcal{W} \) and \( t \in [0, 1] \). Therefore it suffices to prove that \( \xi_t(\text{id}_C)(x, v) = (x, v) \) for \( (x, v) \in Z \). Notice that

\[
F^{\text{id}_C}(x, v, t) = \pi(x + t\lambda(x, tv)(x - x)) = \pi(x) = x
\]

for all \( t \in [0, 1] \). In other words, the path \( t \mapsto F^{\text{id}_C}(x, v, t) \) is constant, and therefore so is its horizontal lift \( t \mapsto F^{\text{id}_C}(x, v, t) \). Hence

\[
\xi(\text{id}_C)(x, v) = F^{\text{id}_C}(x, v, 1) = F^{\text{id}_C}(x, v, 0) = (x, v).
\]

(d) Recall that we identified \( C \) with the zero section of \( p \). Let \( (x, 0) \in C \subset E, g \in \mathcal{W} \), and \( \gamma_x : [0, 1] \rightarrow C \) be given by \( \gamma_x(t) = F^g(x, 0, t) \). By assumption \( \lambda(x, 0) = 1 \) for all \( x \in C \), whence

\[
\gamma_x(0) = F^g(x, 0, 0) = \pi(x) = x,
\]

\[
\gamma_x(1) = F^g(x, 0, 1) = \pi(x + \lambda(x, 0)(g(x) - x)) = \pi(g(x)) = g(x).
\]

The latter equality holds since \( g(x) \in C \). As \( TM \subseteq HY \), it follows that the horizontal lift of \( \gamma_x \) coincides with \( \gamma_x \) itself. In other words,

\[
F^g(x, 0, t) = (F^g(x, 0, t), 0) \equiv F^g(x, 0, t).
\]

In particular,

\[
\rho \circ \xi(g)(x) = \xi(g)|_C(x) = \xi(g)(x, 0) = F^g(x, 0, 1) = F^g(x, 0, 1) = (g(x), 0) \equiv g(x),
\]

which means that \( \rho \circ \xi(g) = g \).

Now let us show how to choose a neighborhood \( \mathcal{V} \). As \( \nabla \) and therefore \( \nabla \) is compact, the group \( \mathcal{D}(E, E \setminus Y; C) \) is open in \( \mathcal{E}(E, E \setminus Y; C) \). Moreover, due to (c),

\[
\xi(\text{id}_C \times [0, 1]) = \text{id}_E \in \mathcal{D}(E, E \setminus Y; C).
\]

Hence the set \( Q := \xi^{-1}(\mathcal{D}(E, E \setminus Y; C)) \) will be an open neighborhood of \( \text{id}_C \times [0, 1] \) in \( \mathcal{W} \times [0, 1] \). Since \( [0, 1] \) is compact, there exists an open neighborhood \( \mathcal{V} \) of \( \text{id}_C \) in \( \mathcal{W} \) such that \( \mathcal{V} \times [0, 1] \subset Q \). Then the restriction \( \xi|_{\mathcal{V} \times [0, 1]} : \mathcal{V} \times [0, 1] \rightarrow \mathcal{D}(E, E \setminus Y; C) \) will be a required homotopy.

Now we can prove Theorem 6.2. Let \( \nabla \) be any Ehresmann connection on \( E \) such that \( TC \subset HE \). The latter inclusion holds for instance for any linear connection. Then by Lemma 6.2.3 the map (6.1) admits a local section on some neighborhood of \( \text{id}_C \), whence by Lemma 6.2.1 the map \( \rho \) is a locally trivial fibration.

7. Fiberwise Morse lemma

The following lemma shows that the implication \((2) \Rightarrow (2')\) in Definition 3.1 can be regarded as a “fiberwise” analogue of Morse lemma.

**Lemma 7.1.** Let \( U \subset \mathbb{R}^k \) and \( V \subset \mathbb{R}^l \) be open neighborhoods of the corresponding origins and \( f : U \times V \rightarrow \mathbb{R} \) be a smooth function having the following properties:

1. \( f^{-1}(0) = U \times 0 \) and this set coincides with the set of critical points of \( f \);
(2) at each critical point \((u, 0) \in U \times 0\) the Hessian of \(f\) in the \(V\)-direction
\[
H(f, u) = \left( \frac{\partial^2 f}{\partial v_i \partial v_j}(u, 0) \right)_{i,j=1,\ldots,l}
\]

is non-degenerate, where \(v = (v_1, \ldots, v_l)\) are coordinates in \(\mathbb{R}^l\).

Then there exist smaller neighborhoods \(\tilde{U} \subset U\) and \(\tilde{V} \subset V\) of the corresponding origins and a smooth embedding \(h: \tilde{U} \times \tilde{V} \to U \times V\) such that

(a) \(h(u, v) = (u, g(u, v))\), \((u, v) \in \tilde{U} \times \tilde{V}\), for some smooth map \(g: \tilde{U} \times \tilde{V} \to V\), i.e. \(h\) preserves the first coordinate \(u\);
(b) \(f \circ h(u, v_1, \ldots, v_l) = -v_1^2 - \cdots - v_\lambda^2 + v_{\lambda+1}^2 + \cdots + v_l^2\) for some \(\lambda\).

Proof. The principal observation is that in the arguments of the standard proof of Morse lemma all changes of coordinates preserve coordinate \(u\), see e.g. [54, Chapter 2, §6]. This will guarantee condition (a). We leave the details for the reader. \(\square\)

Corollary 7.2. Let \(C\) be a smooth manifold without boundary manifold and \(p: E \to C\) be a smooth \(l\)-dimensional vector bundle. Identify \(C\) with its image under the zero section in \(E\). Let \(W\) be an open neighborhood of \(C\) in \(E\) and \(f: W \to \mathbb{R}\) be a Morse-Bott function such that \(f^{-1}(0) = C\) and this set coincides with the set of all critical points of \(f\). Denote by \(\mathcal{L}\) the distribution of the kernels of the differential \(df\) of \(f\). Then there exists a local Ehresmann connection \(\nabla: TY \to VY\) over some open neighborhood \(Y\) of \(C\) whose horizontal bundle \(HY\) is contained in \(\mathcal{L}\).

Proof. Let \(\lambda\) be the index of \(C\) and \(q: \mathbb{R}^l \to \mathbb{R}\) be given by
\[
q(v_1, \ldots, v_l) = -v_1^2 - \cdots - v_\lambda^2 + v_{\lambda+1}^2 + \cdots + v_l^2.
\]
For each \(x \in C\) fix a trivialization \(\phi_x: U_x \times \mathbb{R}^l \to p^{-1}(U_x)\) over some open neighborhood \(U_x\) of \(x\) with compact closure \(\overline{U_x}\). Then there exists an open neighborhood \(V_x\) of the origin \(0 \in \mathbb{R}^l\) such that \(\phi_x(U_x \times V_x) \subset W \cap p^{-1}(U_x)\). Hence by Lemma 7.1 there exist an open neighborhood \(\tilde{U}_x \subset U_x\) of \(x\), an open neighborhood \(\tilde{V}_x \subset V_x\) of the origin, and a smooth fiberwise (i.e. preserving first coordinate) embedding \(h_x: \tilde{U}_x \times \tilde{V}_x \to U_x \times V_x\) such that the following diagram is commutative:

\[
\begin{array}{ccccccc}
\tilde{V}_x & \xrightarrow{q} & & & & & \\
\tilde{U}_x \times \tilde{V}_x & \xrightarrow{h_x} & U_x \times V_x & \xrightarrow{\phi_x} & W \cap p^{-1}(U_x) & \xrightarrow{f} & \mathbb{R} \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_1 & \downarrow \text{pr}_1 & \downarrow p & \\
U_x & \xrightarrow{p} & & & & & \end{array}
\]

where \(\text{pr}_i\) is the projection to the \(i\)-th multiple, \(i = 1, 2\).

Put \(E_x = \tilde{U}_x \times \tilde{V}_x\) and consider the trivial fibre bundle \(\text{pr}_1: E_x = \tilde{U}_x \times \tilde{V}_x \to \tilde{U}_x\). Then at each point \((u, v) \in E_x\) its tangent space splits into the direct sum
\[
T_{(u,v)}E_x = (T_u\tilde{U}_x \times 0) \oplus (0 \times T_v\tilde{V}_x).
\]
Evidently, its vertical bundle is \( VE_x = \bigcup_{(u,v) \in E_x} 0 \times T_v \hat{V}_x \) and we have a natural Ehresmann connection

\[
\hat{\nabla}_x : TE_x \to VE_x, \quad \hat{\nabla}_x(u,v,a,b) = (u,v,0,b),
\]

\( a \in T_u \hat{U}_x, \ b \in T_v \hat{V}_x, \) with horizontal bundle \( HE_x = \bigcup_{(u,v) \in E_x} T_u \hat{U}_x \times 0. \)

Since \( q \circ pr_2 = f \circ \phi_x \circ h_x \) does not depend on \( u \), it follows that \( HE_x \) is contained in the distribution of kernels of the differential \( df \).

Then the embedding \( \phi_x \circ h_x \) transfers \( \hat{\nabla}_x \) to a local Ehresmann connection \( \nabla_x \) on \( E \) over \( \hat{U}_x \) whose horizontal bundle is contained in the distribution \( L \) of kernels of the differential \( df \).

Now using partition of unity technique one can glue all these local Ehresmann connections \( \{ \nabla_x \} \) into a local Ehresmann connection over some open neighbourhood \( Y \) of \( C \) contained in \( \bigcup_{x \in C} \phi_x \circ h_x (\hat{U}_x \times \hat{V}_x) \). Since local Ehresmann connections constitute a convex set, it follows from Lemma 5.1 that the horizontal bundle of \( \nabla \) is also contained in \( L \). □

### 8. Leaf-preserving diffeomorphisms of singular foliations

**Definition 8.1.** Let \( M \) be a smooth manifold and \( F \) be a partition of \( M \). We say that \( F \) is a **singular foliation of class** \( \mathcal{Z} \) if there exists a locally finite family of mutually disjoint connected smooth submanifolds \( \Sigma = \{C_i\}_{i \in \Lambda} \) such that

1. the restriction of \( F \) to \( M \setminus \bigcup_{i \in \Lambda} C_i \) is a smooth codimension one foliation, and every connected component of \( \partial M \) is a leaf of \( F \);
2. for every connected component \( C \) of \( \Sigma \) the following statements hold:
   (a) \( C \) is an element of \( F \), \( \dim C < \dim M \), and \( C \subset \text{Int} M \);
   (b) there exist a smooth vector bundle \( p : E \to C \), an open neighborhood \( N \) of \( C \) in \( M \), and a smooth open embedding \( \phi : N \to E \) such that \( p \circ \phi(x) = x \) for all \( x \in C \), (so \( N \) is a regular neighborhood of \( C \) and we have the induced foliation \( \phi^* F \) on \( \phi(N) \) and the distribution \( T\phi^* F \) consisting of tangent spaces to the leaves of \( \phi^* F \));
   (c) for each \( x \in C \) there exists a local Ehresmann connection \( \nabla_x \) over some open neighborhood \( Y_x \) of \( \phi(x) \) in \( E \) such that its horizontal bundle is contained in the distribution \( T\phi^* F \).

We will regard \( \Sigma \) as a submanifold whose connected components may have distinct dimensions. Denote by \( \mathcal{D}(\Sigma) \) the group of diffeomorphisms of \( \Sigma \). Then its identity path component \( \mathcal{D}_{id}(\Sigma) \) is canonically isomorphic with \( \prod_{i=1}^k \mathcal{D}_{id}(C_i) \).

Corollary 7.2 implies that every foliation which is generated by Morse-Bott functions, see Definition 4.1, belongs to class \( \mathcal{Z} \). Therefore Theorem 4.2 is a particular case of the following Theorem 8.2:

**Theorem 8.2.** Let \( F \) be a foliation of class \( \mathcal{Z} \) on a compact manifold \( M \), e.g. a foliation generated by Morse-Bott functions. Then for every closed subset \( X \subset M \setminus \Sigma \) the
homomorphisms

\[(8.1) \quad \rho : \mathcal{D}(\mathcal{F}, X) \to \mathcal{D}(\Sigma), \quad \rho(h) = h|_{\Sigma},\]

\[(8.2) \quad \rho_0 : \mathcal{D}_{id}(\mathcal{F}, X) \to \mathcal{D}_{id}(\Sigma) \equiv \prod_{i=1}^{k} \mathcal{D}_{id}(C_i), \quad \rho_0(h) = (h|_{C_1}, \ldots, h|_{C_k}),\]

are locally trivial fibrations over their images and the map \(\rho_0\) is surjective.

**Proof.** Notice that \(\mathcal{D}_{id}(\Sigma)\) is open in \(\mathcal{D}(\Sigma)\). Therefore it suffices to find an open neighborhood \(\mathcal{V}\) of \(id_{\Sigma}\) in \(\mathcal{D}_{id}(\Sigma)\) and a continuous map \(\xi : \mathcal{V} \to \mathcal{D}_{id}(\mathcal{F}, X)\) such that

\[(8.3) \quad \xi(id_{\Sigma}) = id_M, \quad \rho_0 \circ \xi = id_{\mathcal{V}}.\]

Then by Lemma 6.2.1 the maps (8.1) and (8.2) will be locally trivial fibrations over their images.

Let \(N_i\) be an open regular neighborhood of \(C_i\) guaranteed by Definition 8.1, i.e. there exists a vector bundle \(p_i : E_i \to C_i\) and an open embedding \(\phi_i : N_i \to E_i\) having the corresponding properties. One can assume that \(\overline{N_i} \cap \overline{N_j} = \emptyset\) for \(i \neq j\).

**Lemma 8.2.1.** For each \(i = 1, \ldots, k\) there exist an open neighborhood \(\mathcal{V}_i\) of \(id_{C_i}\) in \(\mathcal{D}_{id}(C_i)\) and a continuous map \(\xi_i : \mathcal{V}_i \to \mathcal{D}_{id}(\mathcal{F}, M \setminus N_i) \subset \mathcal{D}_{id}(\mathcal{F}, X)\) such that

(i) \(\xi_i(id_{C_i}) = id_M;\)

(ii) \(\xi_i(g)(C_i) = C_i\) and \(\xi_i(g)|_{C_i} = g\) for all \(g \in \mathcal{V}_i\).

**Proof.** Similarly to the last paragraph of the proof of Corollary 7.2, one can use partition of unity technique to glue local Ehresmann connections \(\{\nabla_x\}_{x \in C_i}\) and get a local Ehresmann connection \(\nabla\) over some open neighborhood \(Y \subset \phi_i(N_i)\) of \(C_i\) in \(E_i\). Since local Ehresmann connections constitute a convex set, and the horizontal bundle of each \(\nabla_x\) is contained in the tangent distribution \(T\phi_i^*\mathcal{F}\) to the foliation \(\phi_i^*\mathcal{F}\) on \(\phi_i(N_i)\), it follows from Lemma 5.1 that the horizontal bundle of \(\nabla\) is also contained in \(T\phi_i^*\mathcal{F}\).

Then by Lemma 6.2.3 there exist an open path-connected neighborhood \(\mathcal{V}_i\) of \(id_{C_i}\) and a homotopy \(\sigma : \mathcal{V}_i \times [0, 1] \to \mathcal{D}(E_i, E_i \setminus Y; C_i)\) having properties (a)-(d) of that lemma.

In particular, by properties (a) and (b) for every \(g \in \mathcal{V}_i, t \in [0, 1]\), and \((x,v) \in Y\) the path \(\gamma_{(x,v)} : t \mapsto \sigma(g,t)(x,v)\) starts at \((x,v)\) and is horizontal, i.e. its tangent vectors are contained in the horizontal bundle of the connection \(\nabla\) and therefore in the distribution \(T\phi_i^*\mathcal{F}\). Hence every path \(\gamma_{(x,v)}\) is contained in some leaf of the foliation \(\phi_i^*\mathcal{F}\).

In other words, \(\sigma_t(g)\) is fixed outside \(Y\) and preserves leaves of \(\phi_i^*\mathcal{F}\). Hence one can define another homotopy \(\eta : \mathcal{V} \times [0, 1] \to \mathcal{D}(M, M \setminus N_i) \subset \mathcal{D}(M, X)\) by

\[\eta_t(g)(x) = \begin{cases} \phi_i^{-1} \circ \sigma_t(g) \circ \phi_i(x), & x \in N_i, \\ x, & x \in M \setminus N_i. \end{cases}\]

Then \(\eta_t(g)\) preserves leaves of \(\mathcal{F}\), so \(\eta(\mathcal{V}_i \times [0, 1]) \subset \mathcal{D}(\mathcal{F}, X)\). Moreover, as \(\mathcal{V}_i\) is path-connected, we get that \(\eta(\mathcal{V}_i \times [0, 1]) \subset \mathcal{D}_{id}(\mathcal{F}, X)\). One easily check that the mapping \(\xi_i = \eta_1 : \mathcal{V} \to \mathcal{D}_{id}(\mathcal{F}, X)\) satisfies conditions (i) and (ii).
Define the following two maps
\[ \tilde{\xi} : \mathcal{V}_1 \times \cdots \times \mathcal{V}_k \to \mathcal{D}_{\text{id}}(\mathcal{F}, X), \quad \tilde{\xi}(g_1, \ldots, g_k) = \xi_1(g_1) \circ \cdots \circ \xi_k(g_k), \]
\[ \psi : \mathcal{V}_1 \times \cdots \times \mathcal{V}_k \to \mathcal{D}_{\text{id}}(\Sigma), \quad \psi(g_1, \ldots, g_k)(x) = \begin{cases} g_1(x), & x \in C_1, \\ \cdots \\ g_k(x), & x \in C_k, \end{cases} \]
for \( g_i \in \mathcal{V}_i \). Then it is easy to verify that
- \( \psi \) is an open embedding,
- its image \( \mathcal{V} := \psi(\mathcal{V}_1 \times \cdots \times \mathcal{V}_k) \) is a neighborhood of \( \text{id}_\Sigma \) in \( \mathcal{D}_{\text{id}}(\Sigma) \),
- \( \tilde{\xi}(\text{id}_{C_1}, \ldots, \text{id}_{C_k}) = \text{id}_M \),
- \( \rho \circ \tilde{\xi} = \psi \).

Hence the map \( \xi = \tilde{\xi} \circ \psi^{-1} : \mathcal{V} \to \mathcal{D}_{\text{id}}(\mathcal{F}, X) \) satisfies (8.3). Theorem 8.2 is completed. \( \square \)

9. Stabilizers of smooth maps \( M \to P \)

Let \( M \) be a smooth compact manifold, \( P \) be either the real line or the circle \( S^1 \), and \( f : M \to P \) be a smooth map. Then one can define a partition \( \mathcal{F}_f \) of \( M \) whose elements are path components of level sets \( f^{-1}(c), c \in P \).

Lemma 9.1. For every \( f \in \mathcal{C}^\infty(M, P) \) and every subset \( X \subset M \) we have that
\[ \mathcal{D}(\mathcal{F}_f) \subset \mathcal{S}(f), \quad \mathcal{D}(\mathcal{F}_f, X) \subset \mathcal{S}(f, X), \quad \mathcal{D}_{\text{id}}(\mathcal{F}_f, X) \subset \mathcal{S}_{\text{id}}(f, X). \]

Proof. By definition \( f \) takes constant values at elements of \( \mathcal{F}_f \), whence \( \mathcal{D}(\mathcal{F}_f) \subset \mathcal{S}(f) \). Therefore \( \mathcal{D}(\mathcal{F}_f, X) \equiv \mathcal{D}(\mathcal{F}_f) \cap \mathcal{D}(M, X) \subset \mathcal{S}(f) \cap \mathcal{D}(M, X) \equiv \mathcal{S}(f, X) \). In particular, \( \mathcal{D}_{\text{id}}(\mathcal{F}_f, X) \subset \mathcal{S}_{\text{id}}(f, X) \). \( \square \)

Notice that Corollary 7.2 implies that for every Morse-Bott map \( f : M \to P \) its foliation \( \mathcal{F}_f \) belongs to class \( \mathcal{Z} \). Therefore Theorem 3.3 is a particular case of the following Theorem 9.2.

Theorem 9.2. Let \( M \) be a smooth compact manifold and \( f : M \to P \) be a smooth map whose foliation \( \mathcal{F}_f \) belongs to class \( \mathcal{Z} \) in the sense of Definition 8.1, so the set \( \Sigma \) of critical points of \( f \) is a disjoint union of smooth mutually disjoint closed submanifolds \( C_1, \ldots, C_k \), and \( X \subset M \setminus \Sigma \) be a closed (possibly empty) subset. Then the maps
\[ \rho : \mathcal{S}(f, X) \to \mathcal{D}(\Sigma), \quad \rho(h) = h|_\Sigma, \]
\[ \rho_0 : \mathcal{S}_{\text{id}}(f, X) \to \mathcal{D}_{\text{id}}(\Sigma) \equiv \prod_{i=1}^k \mathcal{D}_{\text{id}}(C_i), \quad \rho_0(h) = (h|_{C_1}, \ldots, h|_{C_k}), \]
are locally trivial fibrations over their images, and the map \( \rho_0 \) is surjective.

Proof. By Theorem 8.2 there exists an open neighborhood \( \mathcal{V} \) of \( \text{id}_\Sigma \) in \( \mathcal{D}_{\text{id}}(\Sigma) \) and a continuous map \( \xi : \mathcal{V} \to \mathcal{D}_{\text{id}}(\mathcal{F}_f, X) \subset \mathcal{S}_{\text{id}}(f, X) \) such that \( \xi(\text{id}_\Sigma) = \text{id}_M \) and \( \rho_0 \circ \xi = \text{id}_\mathcal{V} \). Then by Lemma 6.2.1 the maps \( \rho \) and \( \rho_0 \) are also locally trivial fibrations over their images. \( \square \)
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