Exact Safety Verification of Interval Hybrid Systems
Based on Symbolic-Numeric Computation*

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Abstract

In this paper, we address the problem of safety verification of interval hybrid systems
in which the coefficients are intervals instead of explicit numbers. A hybrid symbolic-
numeric method, based on SOS relaxation and interval arithmetic certification, is pro-
posed to generate exact inequality invariants for safety verification of interval hybrid
systems. As an application, an approach is provided to verify safety properties of non-
polynomial hybrid systems. Experiments on the benchmark hybrid systems are given
to illustrate the efficiency of our method.

1. Introduction

As a tool of modelling cyber-physical systems, hybrid systems are dynamical systems gov-
erned by interacting discrete and continuous dynamics. The continuous dynamics of a hybrid
system is specified by differential equations, and for discrete transitions, the hybrid system
changes state instantaneously and possibly discontinuously. Among the most important re-
search issues in formal analysis of hybrid systems are safety, i.e., deciding whether a given
property holds in all the reachable states, and its dual problem reachability, i.e., deciding if
there exists a trajectory starting from the initial set that reaches a state satisfying the given
property. Due to the infinite number of possible states in state spaces, safety verification
and reachability analysis of hybrid systems presents a challenge. For general (exact) hybrid
systems, some well-established techniques \cite{26,7,16,21,32,22,35,34} based on invariant
generation have been proposed for safety verification of the systems. However, when apply-
ning these techniques, one can not avoid numerical errors or may suffer from high complexity.

\*This material is supported in part by the National Natural Science Foundation of China under Grants
91118007, 61021004(Yang,Wu), and the Fundamental Research Funds for the Central Universities under Grant
78210043(Yang,Wu).
To take advantage of the efficiency of numerical computation and the error-free property of symbolic computation, we proposed in [36] a hybrid symbolic-numeric method via exact sums-of-squares (SOS) representation to construct differential invariants for continuous dynamic systems, and generalized in [15, 37] the idea for safety verification of polynomial hybrid systems.

A common assumption made on hybrid systems is that the coefficients of the involved equations are specific values. In practice, however, due to the increasing complexity of modern systems, some disturbance and modeling errors may be contained in the system description, and, in addition, there may be noisy and inexact data involved in the realistic problem. All these factors may contribute to inexactness of the data used to describe the hybrid systems. To take this uncertainty into account, it would be more reasonable and appropriate to use intervals rather than concrete but inexact data to represent the hybrid systems. This motivates us to introduce the notion of interval polynomial hybrid systems, by which we mean the differential equations in hybrid systems are represented as polynomials with interval coefficients.

In this paper, we consider safety verification of interval polynomial hybrid systems, i.e., deciding whether none of trajectories of an interval hybrid system starting from the initial set can enter some unsafe regions in the state spaces. In [37] we applied a symbolic-numeric computation method, based on bilinear matrix inequality (BMI) solving and exact SOS polynomials representations, to deal with exact safety verification for polynomial hybrid systems. In this paper, we extend the techniques in [37] to generate exact invariants for verifying interval hybrid systems. The idea lies in applying interval arithmetic to verify positive semidefiniteness of interval matrices and existence of solutions to interval polynomial equations. As an application, we apply the above approach to verify safety of non-polynomial hybrid systems by relaxing continuous dynamics of non-polynomial forms to those of interval polynomial forms, and then studying safety of the latter system whose set of trajectories contains that of the original non-polynomial system.

The contributions of our paper are as follows. First, an approach is proposed to verify safety property of an interval hybrid system, therefore, safety property is guaranteed for an arbitrary hybrid system within the given interval system. Moreover, our approach can generate exact invariants instead of approximate ones, overcoming the unsoundness of verification caused by numerical errors [20]. And in comparison with some symbolic approaches based on qualifier elimination, our approach is more efficient and practical, because parametric polynomial optimization problem based on SOS relaxation can be solved in polynomial time theoretically. Second, a key problem we consider in safety verification is that of determining nonnegativity of interval multivariate polynomials, which is a fundamental problem in real algebraic geometry. Thirdly, for a non-polynomial function, we propose a rigorous polynomial approximation method to compute its approximate polynomial with polynomial lower and upper bounds of the interpolation error. Compared with the classical Taylor approximation, the polynomial bounds we give is much sharper.

The rest of the paper is organized as follows. In Section 2, we introduce some notions related to interval hybrid systems. Section 3 is devoted to determining nonnegativity of interval multivariate polynomials. In Section 4, two techniques which combine SOS relaxation with interval arithmetic are proposed to generate invariants of interval hybrid systems with small and large radii, respectively. As an application, safety verification of non-polynomial
2. Interval Hybrid Systems and Safety Verification

Let us first review some notions of general hybrid systems [9, 32].

**Definition 1 (Hybrid System).** A hybrid system is a tuple \( H : \langle V, L, \Theta, D, \Psi, \ell_0 \rangle \) with

- \( V = \{ x_1, ..., x_n \} \), a set of real-valued system variables;
- \( L \), a finite set of locations;
- \( \ell_0 \in L \), the initial location;
- \( \mathcal{T} \), a set of transitions. Each transition \( \tau : (\ell, \ell', g_\tau, \rho_\tau) \in \mathcal{T} \) consists of a prelocation \( \ell \in L \), a postlocation \( \ell' \in L \), the guard condition \( g_\tau \) over \( V \), and an assertion \( \rho_\tau \) over \( V \cup V' \) representing the next-state relation, where \( V' = \{ x'_1, ..., x'_n \} \) denotes the next-state variables;
- \( \Theta \), an assertion specifying the initial condition;
- \( D \), a map that associates each location \( \ell \in L \) to a differential rule (a.k.a. a vector field) \( D(\ell) \), an autonomous system \( \dot{x}_i = f_{\ell,i}(V) \) for each \( x_i \in V \), written briefly as \( \dot{x} = f_{\ell}(x) \);
- \( \Psi \), a map that sends \( \ell \in L \) to a location invariant \( \Psi(\ell) \), an assertion over \( V \).

In reality, due to measuring errors or disturbance, the data involved in the systems may be inaccurate. It is then reasonable to consider hybrid systems in which some data are given as interval estimates rather than specific values, the so-called *interval hybrid systems*. Similar to Definition 1, an interval hybrid system \( IH \) is defined to be a tuple

\[ \langle V, L, \mathcal{T}, \Theta, [D], \Psi, \ell_0 \rangle, \]

where \( V, L, \mathcal{T}, \Theta, \Psi, \ell_0 \) are the same as in Definition 1, while \([D]\) represents a map sending each location \( \ell \in L \) to an interval differential rule \([D(\ell)]\) of the form

\[ \dot{x}_i = [f]_{\ell,i}(x) \quad i = 1, ..., n, \]

by which we mean \([f]_{\ell,i}(x)\) is a real function with interval coefficients; for brevity, we write \([D(\ell)]\) as \( \dot{x} = [f](x) \); For more details on interval arithmetic, please refer to Appendix A.

A hybrid system \( H : \langle V, L, \mathcal{T}, \Theta, D, \Psi, \ell_0 \rangle \) is said to be *within* an interval hybrid system \( IH : \langle V, L, \mathcal{T}, \Theta, [D], \Psi, \ell_0 \rangle \) if \( f_{\ell,i}(x) \in [f]_{\ell,i}(x) \) for each \( \ell \in L \) and \( i = 1, ..., n \), or written briefly as \( D(\ell) \in [D](\ell) \).

In this paper, we will mainly study safety verification of interval hybrid systems. Recall that a hybrid system is said to be *safe* if none of the trajectories starting from any state in the initial set can evolve to an unsafe region. Similarly, given a prespecified unsafe region \( X_u \subset \mathbb{R}^n \), an interval system \( IH : \langle V, L, \mathcal{T}, \Theta, [D], \Psi, \ell_0 \rangle \) is said to be *safe* if every hybrid
system within IH is safe. This is to say, none of the trajectories of interval hybrid system IH starting from any state in the initial set can evolve to \(X_u\), or, equivalently, any state in \(X_u\) is not reachable.

Recall that an invariant of a hybrid system \(H\) is an over-approximation of all the reachable states of the system \(H\). Since generating invariants of arbitrary form for hybrid systems is computationally hard, the usual technique is to compute inductive invariants. It is shown in [37] that safety verification of general hybrid systems can be reduced to finding inductive invariants (a.k.a. barrier certificates in [22]) of hybrid systems, as described in the following theorem.

**Theorem 1.** [[22], [36]] Let \(H : \langle V, L, T, \Theta, D, \Psi, \ell_0 \rangle\) be a general hybrid system. Suppose that for each location \(\ell \in L\), there exists a function \(\varphi_\ell(x)\) such that

(i) \(\Theta \models \varphi_\ell(x) \geq 0\);

(ii) \(\varphi_\ell(x) \geq 0 \land g(\ell, \ell') \land \rho(\ell, \ell') \models \varphi_\ell(x') \geq 0\), for any transition \((\ell, \ell', g, \rho)\) going out of \(\ell\);

(iii) \(\varphi_\ell(x) \geq 0 \land \Psi(\ell) \models \varphi_\ell(x) > 0\), where \(\partial \varphi_\ell(x)\) denotes the Lie-derivative of \(\varphi_\ell(x)\) along the vector field \(D(\ell)\), i.e., \(\partial \varphi_\ell(x) = \sum_{i=1}^n \frac{\partial \varphi_\ell}{\partial x_i} f_{\ell,i}(x)\).

Then \(\varphi_\ell(x) \geq 0\) is an (inductive) invariant of the hybrid system \(H\) at location \(\ell\). If, moreover,

(iv) \(X_u(\ell) \models \varphi_\ell(x) < 0\) for any \(\ell \in L\),

then the safety of the system \(H\) is guaranteed.

The notion of inductive invariants can be generalized for interval hybrid systems, as defined in the following

**Definition 2 (Inductive invariant).** For an interval hybrid system IH : \(\langle V, L, T, \Theta, [D], \Psi, \ell_0 \rangle\), an inductive assertion map \(I\) of IH is a map that associates with each location \(\ell \in L\) an assertion \(I(\ell)\) that holds initially and is preserved by all discrete transitions and continuous flows of IH. More formally, the map \(I\) satisfies the following requirements:

[Initial] \(\Theta \models I(\ell_0)\).

[Discrete Consecution] For each discrete transition \(\tau : (\ell, \ell', g, \rho)\) starting from a state satisfying \(I(\ell)\), taking \(\tau\) leads to a state satisfying \(I(\ell')\), i.e., \(I(\ell) \land g \land \rho \models I(\ell')\) where \(I(\ell')\) represents the assertion \(I(\ell)\) with the current state variables \(x_i\)’s replaced by the next state variables \(x_i\)’s, respectively.

[Continuous Consecution] For location \(\ell \in L\) and states \(\langle \ell, x_1 \rangle, \langle \ell, x_2 \rangle\) such that \(x_2\) evolves from \(x_1\) according to any differential rule \(D(\ell) \in [D](\ell)\), if \(x_1 \models I(\ell)\) then \(x_2 \models I(\ell)\).

The difference between inductive invariants of interval hybrid systems and those of general hybrid systems lies in that for continuous consecution, any differential rule contained in the interval differential rule must be considered. Then Theorem 1 can be modified for verifying safety of interval hybrid systems, as described in the following.
Theorem 2. Let $IH : \langle V, L, T, \Theta, [D], \Psi, \ell_0 \rangle$ be an interval hybrid system. Suppose that for each $\ell \in L$, there exists a function $\varphi_\ell(x)$ satisfying the conditions (i-ii) in Theorem 1, and

(iii') $\varphi_\ell(x) \geq 0 \wedge \Psi(\ell) = \varphi_\ell(x) > 0$, here $\varphi_\ell(x)$ denotes the Lie-derivative of $\varphi_\ell(x)$ along any differential rule $D(\ell) \in [D(\ell)]$, i.e., $\varphi_\ell(x) = \sum_{i=1}^{n} \frac{\partial \varphi_\ell}{\partial x_i} f_{\ell,i}(x)$, for any $f_{\ell,i}(x) \in [f_{\ell,i}](x)$.

Then $\varphi_\ell(x) \geq 0$ is an (inductive) invariant of the interval hybrid system $IH$ at location $\ell$. If, moreover, the condition (iv) in Theorem 1 is satisfied, then the safety of the system $IH$ is guaranteed.

In our preceding papers [15, 37], a symbolic-numeric method based on SOS relaxation, Gauss-Newton refinement and rational vector recovery techniques is proposed to generate polynomial inequality invariants $\varphi_\ell(x) \geq 0$ at each location $\ell \in L$ for general polynomial hybrid systems. This method can not be applied directly on interval hybrid systems. In the sequel, we will combine BMI solving with interval arithmetic to compute polynomial invariants $\varphi_\ell(x) \geq 0$ which satisfy conditions in Theorem 2. For brevity, we will abuse the notation $\varphi_\ell(x)$ to represent both the polynomial $\varphi_\ell(x)$ and the invariant $\varphi_\ell(x) \geq 0$.

3. Nonnegativity of Interval Polynomials

To determine whether a polynomial inequality $\varphi_\ell(x) \geq 0$ is an invariant of an interval hybrid system, by Theorem 2 (iii') it suffices to decide whether a multivariate polynomial $\varphi_\ell(x)$ with interval coefficients is positive semidefinite. In the sequel, we will call a polynomial with interval coefficients an interval polynomial. Denote by $IR[x]$ the set of interval multivariate polynomials in $x$. The first problem to be investigated is the following

Problem 1. Given an interval polynomial $[\psi](x) \in IR[x]$, verify whether it is positive semidefinite, or the validity of the interval inequality

$$[\psi](x) \geq 0, \forall x \in \mathbb{R}^n.$$ 

It is well known that the problem of testing positive semidefiniteness of real polynomials is NP-hard (when the degree is at least four). As stated in Appendix B, a sufficient condition for a multivariate polynomial to be positive semidefinite is that there exists an SOS polynomial (or rational function) representation. In [10, 11, 19], some symbolic-numeric methods were proposed to determine whether a multivariate polynomial $\psi(x)$ with rational coefficients is positive semidefinite by computing its exact SOS representations, or equivalently, to determine if there exists a symmetric matrix $W \in \mathbb{R}^{k \times k}$ satisfying exactly

$$\psi(x) = m(x)^T \cdot W \cdot m(x) \quad \text{and} \quad W \succeq 0,$$

where $W \succeq 0$ denotes that $W$ is positive semidefinite. These methods cannot be applied directly to verifying positive semidefiniteness of an interval polynomial $[\psi](x) \in IR[x]$, since there are infinitely many polynomials in the interval, and it is impossible to provide certificates of SOS representations for infinitely many polynomials in $[\psi](x)$. For Problem 1, we will only prove existence of SOS representations for polynomials in $[\psi](x)$. This problem can be further distinguished into two cases according to the radii of the coefficient intervals: the
coefficient intervals of $[\psi](\mathbf{x}) \geq 0$ are all smaller (resp. larger) than the given threshold. In the sequel, we will describe how to deal with the former case, and the latter case will be discussed in subsection 4.1.

Let $[W]$ be an interval matrix such that $[W] \succeq 0$, i.e., every matrix within $[W]$ is positive semidefinite. If for any polynomial $\psi(\mathbf{x})$ within $[\psi](\mathbf{x})$, there exists a matrix $W \in [W]$ such that the condition (1) holds exactly, then we have $[\psi](\mathbf{x}) \geq 0$. Thus the first case of Problem 1 can be transformed into the problem of finding an interval matrix $[W] \succeq 0$ for $[\psi](\mathbf{x})$.

Suppose that there exists an approximate SOS decomposition of the mid-point function $\text{mid}\psi(\mathbf{x}) \in [\psi](\mathbf{x})$:

$$\text{mid}\psi(\mathbf{x}) \approx \mathbf{m}(\mathbf{x})^T \cdot \hat{W} \cdot \mathbf{m}(\mathbf{x})$$

(2)

where $\hat{W} \succeq 0$. Having $\hat{W}$, we will consider how to compute an interval matrix $[W] \succeq 0$ of minimal radius, such that $[W]$ contains $\hat{W}$ and for any $\psi(\mathbf{x}) \in [\psi](\mathbf{x})$ there always exists a matrix $W \in [W]$ satisfying the condition (1) exactly. Considering whether the matrix $\hat{W}$ is of full rank, there are two cases to be addressed.

3.1. $\hat{W}$ is of full rank

Suppose that $\hat{W}$ in (2) is of full rank numerically, namely, the minimal eigenvalue of $\hat{W}$ is greater than the given tolerance $\tau > 0$. Let

$$[W] := \hat{W} + [\Delta W]$$

be an interval matrix perturbed from $\hat{W}$ where $[\Delta W] \in \mathbb{IR}^{k \times k}$. If, for any $\psi(\mathbf{x}) \in [\psi](\mathbf{x})$, there exists a matrix $\Delta W \in [\Delta W]$ which satisfies

$$\psi(\mathbf{x}) = \mathbf{m}(\mathbf{x})^T \cdot (\hat{W} + \Delta W) \cdot \mathbf{m}(\mathbf{x})$$

(3)

and $\hat{W} + \Delta W \succeq 0$ exactly, then we have $[\psi](\mathbf{x}) \geq 0$. Since $\hat{W}$ is positive definite and of full rank, according to matrix perturbation theory we have $\hat{W} + [\Delta W] \succeq 0$ as long as the radius of interval matrix $[\Delta W]$ is small enough.

We first consider how to construct an interval matrix $[\Delta W]$ with small radius, which satisfies the condition (3). Comparing the coefficients of terms on both sides of (3) gives rise to the following underdetermined linear system with the entries of $\Delta W$ as unknowns $w$:

$$A \cdot w = [v],$$

where $A \in \mathbb{IR}^{s \times r}$ with $s \in \mathbb{Z}^+$ and $r = \frac{k(k+1)}{2}$, $w \in \mathbb{IR}^r$ is a vector composed of columnwise entries of the symmetric matrix $\Delta W$, and $[v] \in \mathbb{IR}^s$ is the coefficient vector of the interval polynomial $[\psi](\mathbf{x}) - \mathbf{m}(\mathbf{x})^T \cdot \hat{W} \cdot \mathbf{m}(\mathbf{x})$. Our goal is to compute a minimal 2–norm interval vector $w$ satisfying $A \cdot w = [v]$. The above problem is then transformed into the following interval least squares problem:

$$\Sigma = \min\{\|w\|_2 : A \cdot w = v \text{ for some } v \in [v]\}.$$
Using the method [31] for solving interval linear systems, we can obtain a solution \([\mathbf{w}'] \in \mathbb{IR}^r\) of \(\Sigma\) and therefore the associated solution \([\Delta W]\) of (3) of minimal radius. Then the remaining task is to verify whether the interval matrix \(\hat{W} + [\Delta W]\) is positive semidefinite. The following theorem provides such a computational criterion.

**Theorem 3.** [27, Theorem 4] Let \([\mathbf{W}]\) be a symmetric interval matrix and \([\mathbf{W}] = [\hat{W} - \Delta W, \hat{W} + \Delta W]\) be its midpoint-radius form. Suppose that \(\rho(\Delta W)\) is the spectral radius of \(\Delta W\) and \(\lambda_{\min}(\hat{W})\) is the minimum eigenvalue of \(\hat{W}\). If \(\rho(\Delta W) \leq \lambda_{\min}(\hat{W})\), then \([\mathbf{W}]\) is positive semidefinite. Moreover, if \(\rho(\Delta W) < \lambda_{\min}(\hat{W})\) then \([\mathbf{W}]\) is positive definite.

We give an example to illustrate the above method.

**Example 1.** Verify \([\psi](\mathbf{x}) \geq 0\) where

\[
[\psi](\mathbf{x}) = 0.9574 - 1.9362x_1 - 0.3404x_2 + [1.1852, 1.2593]x_1^2 \\
- [0.4237, 0.4576]x_1 x_2 + [1.125, 1.2083]x_2^2.
\]

For the mid-point function \(\text{mid}\psi(\mathbf{x})\), we compute its approximate Gram matrix representation \(\text{mid}\psi(\mathbf{x}) \approx \mathbf{m}(\mathbf{x})^T \cdot \hat{W} \cdot \mathbf{m}(\mathbf{x})\) where

\[
\mathbf{m}(\mathbf{x}) = \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}, \quad \hat{W} = \begin{pmatrix} 0.9574 & -0.9681 & -0.1702 \\ -0.9681 & 1.2222 & -0.2203 \\ -0.1702 & -0.2203 & 1.1667 \end{pmatrix}.
\]

It is easy to check that \(\hat{W}\) is of full rank. By solving an associated interval linear system, we obtain the symmetric interval matrix \([\mathbf{W}]\) as follows:

\[
\begin{pmatrix}
[0.9574, 0.9575] & [-0.9681, -0.9680] & [-0.1703, -0.1702] \\
[-0.9681, -0.9680] & [1.1851, 1.2593] & [-0.2289, -0.2118] \\
[-0.1703, -0.1702] & [-0.2289, -0.2118] & [1.1388, 1.1945]
\end{pmatrix}.
\]

For the midpoint-radius form of \([\mathbf{W}]\), we obtain \(0.0422 = \rho(\Delta W) < \lambda_{\min}(\hat{W}) = 0.0461\). According to Theorem 3, \([\mathbf{W}]\) is positive definitive, which proves \([\psi](\mathbf{x}) \geq 0\). □

### 3.2. \(\hat{W}\) is singular

When the matrix \(\hat{W}\) is singular or near to a singular matrix, the perturbed matrix of \(\hat{W}\) may not be positive semidefinite. Therefore, the method in subsection 3.1 does not apply to the case where \(\hat{W}\) is numerically singular.

By expanding the quadratic representation, the equation (2) can be rewritten as

\[
\text{mid}\psi(\mathbf{x}) \approx \sum_{i=1}^l \left( \sum_{\alpha} \hat{q}_{i,\alpha} \mathbf{x}^{\alpha} \right)^2,
\]
where \( l \) is the rank of \( \hat{W} \). Next we will verify, for each \( \psi(x) \in [\psi](x) \), there exist \( q_{i,\alpha} \in \mathbb{R} \) such that
\[
\psi(x) = \sum_{i=1}^{k} \left( \sum_{\alpha} q_{i,\alpha} x^\alpha \right)^2
\] (4)
holds exactly. Let \( q \) be a vector composed of all the \( q_{i,\alpha} \). Comparing the terms of both sides of (4) gives rise to a nonlinear system of the form
\[
F(q) - [v] = 0,
\] (5)
where \( F : \mathbb{R}^r \to \mathbb{R}^s \) with \( r \) the size of \( q \), and \([v] \in \mathbb{I}\mathbb{R}^s \) is an interval vector consisting of coefficients in \([\psi](x)\). Note that \( F(0) = 0 \). Hence, the problem of determining \([\psi](x) \geq 0\) is equivalent to that of verifying existence of real roots of the underdetermined interval nonlinear system (5). The latter problem can be solved in two ways: one is based on existence of real roots for particular interval square nonlinear systems, and the other for particular interval underdetermined nonlinear systems. The details of these two methods are given in Appendix C.

**Remark 1.** If we find a verified real solution to system (5), then \( \psi(x) \geq 0 \) for each \( \psi(x) \in [\psi](x) \). However, the opposite is not true, i.e., even if \([\psi](x) \geq 0\) it is not guaranteed that the above methods can prove existence of real roots of (5).

### 4. Safety Verification of Interval Hybrid Systems

In this section, we study how to verify safe properties of an interval hybrid system. Two techniques will be used depending on the radii of the occurred intervals in the given interval hybrid system. If the radii of the intervals are all larger than a given threshold, we transform the interval hybrid system into an uncertain hybrid system by replacing the intervals with some uncertainties and then generalize the method in [15, 37], which is based on SOS relaxation and rational vector recovery, to compute exact invariants of the uncertain hybrid system. If the radii of the involved intervals are all less than the given threshold, we will apply the interval verification approach in Section 3. For the more general case, when the interval hybrid system contains both intervals of radii smaller than and those of radii larger than the given threshold, the above two techniques will be combined. For simplification, we will only consider the two special cases respectively in subsections 4.1 and 4.2.

#### 4.1. Safety Verification of Interval Hybrid Systems With Large Radii

Let \( \textbf{IH} : \langle V, L, T, \Theta, [D], \Psi, \ell_0 \rangle \) be an interval hybrid system. Suppose that the radii of the intervals in the interval differential rules \([D] \) are all greater than a given threshold \( \epsilon \), say \( \epsilon = 0.1 \). Then some new parameters \( u_1, \ldots, u_t \) will be introduced to replace the interval coefficients, to convert \( \textbf{IH} \) into an uncertain hybrid system \( \textbf{H}_u \) with \( u = (u_1, \ldots, u_t) \), for which Theorem 1 can be extended to handle safety verification.

Denote by \([u] = [\underline{u}, \overline{u}] \in \mathbb{I}\mathbb{R}^t \) the interval coefficient vector composed of all the interval coefficients occurred in \([D] \), where \( \underline{u} = (\underline{u}_1, \ldots, \underline{u}_t) \) and \( \overline{u} = (\overline{u}_1, \ldots, \overline{u}_t) \). To remove the
intervals \([u] \in \mathbb{R}^s\) in \(\text{IH}\), we introduce a vector \(u \in \mathbb{R}^s\) of uncertainties with the constraints
\[
\vartheta_i(u) = (u_i - u_i)(\bar{u}_i - u_i) \geq 0, \quad i = 1, \ldots, t.
\]

For the uncertain hybrid system \(H_u\), we predetermine a template \(\varphi(x) = \sum \alpha x^\alpha\) of polynomial invariants with the given degree \(d\), where \(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n\) with \(\sum_{i=1}^n \alpha_i \leq d\), and \(\alpha \in \mathbb{R}\) are parameters. For each location \(\ell \in L\), we write \(\varphi_\ell(x) = c^T_\ell \cdot T(x)\), where \(T(x)\) is the (column) vector of all terms in \(x_1, \ldots, x_n\) with total degree \(\leq d\), and \(c_\ell \in \mathbb{R}^\nu\), with \(\nu = \binom{n+d}{n}\), is the coefficient vector of \(\varphi_\ell(x)\). For clarity, we write \(\varphi_\ell(x)\) as \(\varphi_\ell(x, c_\ell)\). Similar to Theorem 1, the problem of computing the invariants \(\varphi_\ell(x)\) of the uncertain hybrid system \(H_u\) can be translated into the following problem
\[
\begin{align*}
\text{find } c_\ell & \in \mathbb{R}^\nu, \quad \forall \ell \in L \\
\text{s.t.} & \quad \Theta \models \varphi_{\ell_0}(x, c_{\ell_0}) \geq 0, \\
& \quad \varphi_\ell(x, c_\ell) \geq 0 \quad \land \quad g(\ell, \ell') \land \rho(\ell, \ell') \models \varphi_{\ell'}(x', c_{\ell'}) \geq 0, \\
& \quad \varphi_\ell(x, c_\ell) \geq 0 \quad \land \quad \Psi(\ell) \land \vartheta(u) \geq 0 \models \varphi_\ell(x, u, c_\ell) > 0, \\
& \quad X_u(\ell) \models \varphi_\ell(x, c_\ell) < 0,
\end{align*}
\]
where \(\varphi_\ell(x, u, c_\ell) = \sum_{i=1}^n \frac{\partial \varphi_\ell}{\partial x_i} \cdot f_{x,i}(x, u)\). Without loss of generality, we consider a simpler form of (6):
\[
\begin{align*}
\text{find } c & \in \mathbb{R}^\nu \\
\text{s.t.} & \quad \varphi_1(x, c) \geq 0, \\
& \quad \varphi_2(x, c) \geq 0 \models \varphi_2(x, c) \geq 0, \\
& \quad \varphi_3(x, u, c) \geq 0 \models \varphi_4(x, u, c) \geq 0,
\end{align*}
\]
where the coefficients of the polynomials \(\varphi_i\)'s are affine in \(c\), for \(i = 1, \ldots, 5\). By Appendix B, the problem (7) can be further transformed into the following polynomial parametric optimization problem
\[
\begin{align*}
\text{find } c & \in \mathbb{R}^\nu \\
\text{s.t.} & \quad \varphi_1(x, c) = m_1(x)^T \cdot W[1] \cdot m_1(x), \\
& \quad \varphi_2(x, c) = m_2(x)^T \cdot W[2] \cdot m_2(x) + (m_3(x)^T \cdot W[3] \cdot m_3(x)) \cdot \varphi_3(x, c), \\
& \quad \varphi_3(x, u, c) = m_4(x, u)^T \cdot W[4] \cdot m_4(x, u) + (m_5(x, u)^T \cdot W[5] \cdot m_5(x, u)) \cdot \varphi_5(x, u, c), \\
& \quad W[i] \succeq 0, \quad i = 1, \ldots, 5,
\end{align*}
\]
which involves both LMI and BMI constraints. As stated in [37], a Matlab package PENBMI solver [13], which combines the (exterior) penalty and (interior) barrier method with the augmented Lagrangian method, can be applied directly on the BMI program, and alternatively, an iterative method can be applied by fixing \(W[5]\) and \(c\) alternatively, which leads to a sequential convex LMI problem.

Since the SDP solvers in Matlab is running in fixed precision, the above techniques yield numerical vector \(c\) and numerical positive semidefinite matrices \(W[1], \ldots, W[5]\), which satisfy the constraints in (8) \textit{approximately}. We will apply the symbolic-numeric method proposed in [37] to obtain exact solutions to (8). The idea is as follows. We first convert \(W[3]\) and \(W[5]\) to the nearby rational positive semidefinite matrices \(\widehat{W}[3]\) and \(\widehat{W}[5]\), respectively, by nonnegative truncated PLDLTPT-decomposition, in which all the diagonal entries of the corresponding diagonal matrix are preserved to be nonnegative. Then, using modified Newton refinement
and rational vector recovery techniques, we can recover the rational vector \( \tilde{c} \) and the rational positive semidefinite matrices \( \tilde{W}^{[1]}, \tilde{W}^{[2]}, \tilde{W}^{[4]} \) from the numerical \( c, W^{[1]}, W^{[2]}, W^{[4]} \), respectively, such that the constraints in (8) hold exactly. For more details, please refer to [37].

### 4.2. Safety Verification of Interval Hybrid systems with Small Radii

In this subsection, we will consider interval hybrid systems with small radii interval coefficients, namely, the radii of the involved intervals are all smaller than the given threshold \( \epsilon \). For such interval hybrid systems, the method described in subsection 4.1 via introducing uncertainties may suffer from high complexity especially when solving the parametric optimization problem (8). Instead, we will consider how to generate invariants of \( \text{IH} \) by determining nonnegativity of interval polynomials: we first compute candidate invariants with rational coefficients, then employ the interval computation method presented in Section 3 to certify that the candidate invariants satisfy the conditions in Theorem 2 exactly.

Suppose that \( |D|(|\ell|) \) of \( \text{IH} \) is given by \( \dot{x} = [f_\ell](x) \) for \( \ell \in L \). Choosing the midpoints of the interval coefficients of \( [f_\ell](x) \) yields a mid-point vector \( \text{mid}f_\ell(x) \) and an associated general hybrid system \( H \) with the vector field \( \dot{x} = \text{mid}f_\ell(x) \), for \( \ell \in L \). Then the symbolic-numeric technique in [37] can be used to generate invariants of \( H \) as follows. Let us predetermined a polynomial template \( \varphi_\ell(x) \geq 0 \) of invariants of \( H \) with \( \deg \varphi_\ell(x) = d \). By Theorem 1, the problem of computing \( \varphi_\ell(x) \) can be translated into the following problem

\[
\begin{aligned}
\text{find } c_\ell & \in \mathbb{R}^p, \quad \forall \ell \in L \\
\text{s.t.} \Theta & \models \varphi_{t_0}(x, c_{t_0}) \geq 0, \\
\varphi_\ell(x, c_\ell) & \geq 0 \land g(\ell, \ell') \land \rho(\ell, \ell') \models \varphi_{\ell'}(x', c_{\ell'}) \geq 0, \\
\varphi_\ell(x, c_\ell) & \geq 0 \land \Psi(\ell) \models \text{mid}f_\ell(x, c_\ell) > 0, \\
X_u(\ell) & = \varphi_\ell(x, c_\ell) < 0,
\end{aligned}
\]

(9)

where \( \text{mid}f_\ell(x, c_\ell) = \sum_{i=1}^{n} \frac{\partial f_\ell}{\partial x_i} \cdot \text{mid}f_\ell(x, c_\ell) \). By use of BMI solving and modified Newton refinement, we can obtain the refined numerical solutions to (9). With the refined vector \( c_\ell \) for \( \ell \in L \), we then apply rational vector recovery technique to obtain a polynomial \( \varphi_\ell(x, \tilde{c}_\ell) \) with rational coefficients. Clearly, \( \varphi_\ell(x, \tilde{c}_\ell) \) can be seen as a candidate invariant of the interval hybrid system \( \text{IH} \).

In the following, we will determine whether \( \varphi_\ell(x, \tilde{c}_\ell) \) satisfies the conditions of invariants of interval hybrid system \( \text{IH} \) in Theorem 2 exactly, i.e.,

\[
\begin{aligned}
\Theta & \models \varphi_{t_0}(x, \tilde{c}_{t_0}) \geq 0, \\
\varphi_\ell(x, \tilde{c}_\ell) & \geq 0 \land g(\ell, \ell') \land \rho(\ell, \ell') \models \varphi_{\ell'}(x', \tilde{c}_{\ell'}) \geq 0, \\
\varphi_\ell(x, \tilde{c}_\ell) & \geq 0 \land \Psi(\ell) \models [\hat{\varphi}_\ell](x, \tilde{c}_\ell) > 0, \\
X_u(\ell) & = \varphi_\ell(x, \tilde{c}_\ell) < 0,
\end{aligned}
\]

(10)

where \( [\hat{\varphi}_\ell](x, \tilde{c}_\ell) = \sum_{i=1}^{n} \frac{\partial \varphi_\ell}{\partial x_i} \cdot [f_\ell](x) \) is an interval polynomial. Observing in (10), all the constraints except the third one are exact constraints. And the SOS-based method presented in subsection 4.1 can be used to determine satisfiability of the exact constraints. To handle the third constraint in (10), we now consider how to determine satisfiability of polynomial inequalities with interval coefficients. More generally, we consider the following problem

\[
\psi_1(x, \tilde{c}) \geq 0 \models [\psi_2](x, \tilde{c}) \geq 0,
\]

(11)
where \([\psi_2](\mathbf{x}, \mathbf{c}) \in \mathbb{R}[\mathbf{x}]\). Let \(\text{mid}\psi(\mathbf{x}, \mathbf{c}) \in [\psi_2](\mathbf{x}, \mathbf{c})\) be the mid-point function of \([\psi_2](\mathbf{x}, \mathbf{c})\).

Then BMI solver and modified Gauss-Newton refinement can yield the numerical positive semidefinite matrices \(W^{[1]}\) and \(W^{[2]}\), which satisfy the following condition approximately

\[
\text{mid}\psi(\mathbf{x}, \mathbf{c}) \approx \mathbf{m}_2(\mathbf{x})^T \cdot W^{[2]} \cdot \mathbf{m}_2(\mathbf{x}) + (\mathbf{m}_1(\mathbf{x})^T \cdot W^{[1]} \cdot \mathbf{m}_1(\mathbf{x})) \cdot \psi_1(\mathbf{x}, \mathbf{c}).
\]  

(12)

Converting \(W^{[1]}\) to a nearby rational positive semidefinite matrix \(\tilde{W}^{[1]}\) by nonnegative truncated PLDL\(^T\)PT-decomposition, the condition (12) becomes

\[
\text{mid}\psi(\mathbf{x}, \mathbf{c}) - (\mathbf{m}_1(\mathbf{x})^T \cdot \tilde{W}^{[1]} \cdot \mathbf{m}_1(\mathbf{x})) \cdot \psi_1(\mathbf{x}, \mathbf{c}) \approx \mathbf{m}_2(\mathbf{x})^T \cdot W^{[2]} \cdot \mathbf{m}_2(\mathbf{x}).
\]  

(13)

Let \([\tilde{\psi}_2](\mathbf{x}, \mathbf{c})\) be an interval polynomial such that

\[
[\tilde{\psi}_2](\mathbf{x}, \mathbf{c}) = [\psi_2](\mathbf{x}, \mathbf{c}) - (\mathbf{m}_1(\mathbf{x})^T \cdot \tilde{W}^{[1]} \cdot \mathbf{m}_1(\mathbf{x})) \cdot \psi_1(\mathbf{x}, \mathbf{c}).
\]

Since \(\tilde{W}^{[1]} \succeq 0\), it suffices to prove satisfiability of (11) when \([\tilde{\psi}_2](\mathbf{x}, \mathbf{c})\) is nonnegative. Remark that (13) is an approximate SOS decomposition of \([\tilde{\psi}_2](\mathbf{x}, \mathbf{c})\). The nonnegativity of \([\tilde{\psi}_2]\) can be verified by computing the corresponding interval matrix \([W_2]\), either using the method in subsection 3.1 if \(W^{[2]}\) is of full rank, or by proving existences of real roots of the interval nonlinear system, as explained in subsection 3.2.

### 4.3. Experiments

In the following, some examples will be given to illustrate our method on safety verification of interval hybrid systems.

**Example 2.** Consider the classical two-dimensional system given in [12, 22], whose coefficients are approximated and described by the following intervals

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-[0.96, 1.04]x_1 + [0.99, 1.01]x_2 \\
-0.32, 0.347]x_1^3 - [0.98, 1.02]x_2
\end{bmatrix}.
\]

We will verify that all trajectories of the system starting from the initial set

\[
\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1.5)^2 + x_2^2 \leq 0.25\}
\]

will never enter the unsafe region

\[
X_u = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + 1)^2 + (x_2 + 1)^2 \leq 0.16\}.
\]

Set the threshold \(\epsilon = 0.1\) Clearly, all the radii of involved intervals are less than this threshold. Applying the method in subsection 4.2, we obtain the following verified invariant with rational coefficients

\[
\tilde{\varphi}(\mathbf{x}) = \frac{151}{99} x_1 + \frac{152}{99} x_2 + \frac{62}{33} x_1 x_2 + \frac{106}{99} x_1^2 + \frac{4}{9} x_2^2,
\]

which guarantees the safety of the original system. \(\square\)
Example 3. Consider a Moore-Greitzer model of a jet engine with stabilizing feedback operating in the no-stall mode [2]. In this model, the origin is translated to a desired no-stall equilibrium. The dynamic system takes the following form:

\[
\begin{align*}
\dot{x}_1 &= [-1.1, -0.9]x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \\
\dot{x}_2 &= [2.98, 3.02]x_1 - x_2.
\end{align*}
\]

The problem is to verify that all trajectories of the system starting from the initial set

\[\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)^2 + x_2^2 \leq 0.04\}\]

will never reach the unsafe set

\[X_u = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + 1.8)^2 + x_2^2 \leq 0.16\}.\]

Set the threshold $\epsilon$ of radii to be 0.1. Then a new uncertainty $u$ is introduced to replace the interval $[-1.1, -0.9]$. Combine the methods in subsections 4.1 and 4.2 to deal with the uncertain interval system, and we obtain the following verified invariant with rational coefficients

\[\tilde{\varphi}(x_1, x_2) = \frac{2231}{328}x_1 + \frac{652}{123}x_2 + \frac{274}{123}x_2^2 - \frac{46}{41}x_1^2 + \frac{10}{41}x_1x_2 + \frac{1649}{984}x_2^2,\]

which guarantees the safety of the original system.

Example 4. Figure 4 gives a predator-prey hybrid system [24] with interval coefficients:

\[f_1(x) = f_2(x) = \begin{bmatrix} -x_1 + [0.99, 1.01]x_1x_2 \\ [0.875, 1.2]x_2 - x_1x_2 \end{bmatrix}.\]

Suppose the system starts in location $\ell_1$ with an initial state in

\[\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 0.8)^2 + (x_2 - 0.2)^2 \leq 0.01\}.\]

We want to verify that the system never reach the states in

\[X_u(\ell_1) = \{(x_1, x_2) \in \mathbb{R}^2 : 0.8 \leq x_1 \leq 0.9 \land 0.8 \leq x_2 \leq 0.9\}.\]
Set the threshold $\epsilon$ of radii to be 0.1. Then a new uncertainty $u$ is introduced to replace the interval $[0.875, 1.2]$. Applying the above method on the resulting uncertain interval hybrid system $IH_u$, we obtain the following verified invariants with rational coefficients:

$$\tilde{\varphi}_1(x_1, x_2) = -\frac{411}{995} + \frac{346}{995} x_1 + \frac{397}{995} x_2 - \frac{49}{199} x_2^2,$$

$$\tilde{\varphi}_2(x_1, x_2) = \frac{556}{995} - \frac{151}{199} x_1 - \frac{986}{995} x_2 + \frac{22}{995} x_2^2,$$

for locations $\ell_1$ and $\ell_2$, respectively, which ensures the safety of the original hybrid system.

□

5. Safety Verification of Non-polynomial Hybrid system

As an application of the method in Section 4 for safety verification for interval hybrid systems, we will consider how to verify safety of non-polynomial hybrid systems.

Let $H : \langle V, L, \mathcal{T}, \Theta, D, \Psi, \ell_0 \rangle$ be a hybrid system where the initial condition $\Theta$, location invariants $\Psi(\ell)$, the guard condition and reset relation in each transition $\tau \in \mathcal{T}$ are semi-algebraic sets, whereas the continuous systems in the differential rules $D(\ell)$, contain some non-polynomial terms in $x$. For such a non-polynomial hybrid system $H$, we will first transform it into an uncertain interval hybrid system $IH$ through polynomial approximation on the non-polynomial terms, such that $H$ is within $IH$. This implies that the safety of $IH$ suffices to guarantee the safety of $H$, and then the method in Section 4 can be applied to the former problem.

Assume that the location invariant $\Psi(\ell)$ is a compact set for each location $\ell$. Consider the continuous dynamics of a hybrid system $H$ at location $\ell$:

$$\dot{x}_i = f_i(x) = f_{i0}(x) + \sum_{j=1}^{s} f_{ij}(x)\phi_{ij}(x), \; i = 1, \ldots, n, \quad (14)$$

where $x$ takes values in $\Psi(\ell) \subseteq \mathbb{R}^n$, $f_{ij}(x)$ are polynomials for $j = 0, 1, \ldots, s$, and $\phi_{ij}(x)$ are non-polynomials for $j = 1, \ldots, s$. We will approximate the functions $\phi_{ij}(x)$ with polynomials $g_{ij}(x)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, s$. Let $\mu_{ij}$ be the bound of $|\phi_{ij}(x) - g_{ij}(x)|$ for all $x \in \Psi$, namely,

$$|\phi_{ij}(x) - g_{ij}(x)| \leq \mu_{ij}, \quad \text{for all } x \in \Psi(\ell). \quad (15)$$

Making use of the relation (15) for each location $\ell \in L$, we can construct an interval polynomial hybrid system $IH : \langle V, L, \mathcal{T}, \Theta, [D], \Psi, \ell_0 \rangle$, where the interval differential rule $[D(\ell)]$ given by

$$\dot{x}_i = [f_i](x) = f_{i0}(x) + \sum_{j=1}^{s} f_{ij}(x)(g_{ij}(x) + [-\mu_{ij}, \mu_{ij}]), \; i = 1, \ldots, n, \quad (16)$$

encloses the non-polynomial system (14) in $H$, that is, $f_i(x) \in [f_i](x)$ for all $x \in \Psi(\ell)$.

The key point of the above idea is to compute an approximate polynomial and the associated bound for the given non-polynomial function. For a non-polynomial function $\phi(x)$ with
\( x \in \Psi(\ell) \), we will compute the approximate polynomial \( g(x) \in \mathbb{R}[x] \) with a verified bound \( \mu \in \mathbb{R}_+ \), such that
\[
|\phi(x) - g(x)| < \mu, \forall x \in \Psi(\ell),
\]
and the bound \( \mu \) is as small as possible.

A classic method of polynomial approximation is Taylor expansion. In this paper, to obtain a tighter error bound, multivariate polynomial interpolation[6] is applied to compute an approximate polynomial with the error bound. Furthermore, the technique of oversampling is explored to get better approximate polynomials, i.e., the number of the interpolation points is greater than that of the terms of the target polynomial \( g(x) \). Given the interpolation points, the approximate polynomial \( g(x) \) can be obtained by solving a least squares problem. Specifically, predetermine a polynomial template of \( g(x) \) with a given degree \( r \):
\[
g(x) = c^T \cdot T(x),
\]
where \( T(x) \) is the (column) vector consisting of all terms in \( x_1, \ldots, x_n \) with total degree \( \leq r \), and \( c \in \mathbb{R}^\nu \), with \( \nu = \binom{n+r}{n} \), is the coefficient vector of \( g(x) \). We then construct a mesh \( M \) on \( \Psi(\ell) \) with a small spacing \( s \in \mathbb{R}_+ \), and compute \( y_j = \phi(v_j) \in \mathbb{R} \) for \( 1 \leq j \leq m \) at mesh points \( \{v_1, v_2, \ldots, v_m\} \). Let the coefficient vector \( c \) of \( g(x) \) be unknowns. We can construct a linear system
\[
A \cdot c = y,
\]
where \( A = (T(v_1)^T, T(v_2)^T, \ldots, T(v_m)^T)^T \) is of size \( m \times \nu \) with \( m > \nu \). By solving the above overdetermined system, we obtain \( g(x, c) \) as the approximation of \( \phi(x) \) with \( x \in \Psi(\ell) \). Having \( g(x, c) \), one will compute the verified error bound \( \mu \), namely, \( |\phi(x) - g(x, c)| < \mu, \forall x \in \Psi(\ell) \).

**Lemma 1.** [38, Theorem 3] Let \( K \subset \mathbb{R}^n \) be a convex polyhedron, and \( V_1, V_2, \ldots, V_m \) and \( d \) be the vertices and diameter of \( K \) respectively. Suppose \( \psi : K \to \mathbb{R} \) is a continuous and differential function on \( K \), then for all \( a_1, a_2, \ldots, a_m \in \mathbb{R}_+ \) such that \( a_1 + a_2 + \ldots + a_m = 1 \), we have
\[
|\psi(x) - (a_1 \psi(V_1) + a_2 \psi(V_2) + \ldots + a_m \psi(V_m))| \leq \frac{n}{n+1} \beta d,
\]
where \( \beta = \sup_{x \in K} \| \nabla \psi(x) \| \).

For the error function \( r(x) = \phi(x) - g(x, c) \), we will estimate its bound with \( x \) in the mesh \( M \) by the following theorem.

**Theorem 4.** Suppose that \( s \) and \( \{v_1, v_2, \ldots, v_m\} \) are the mesh spacing and mesh points of \( M \), respectively. Let \( \mu_0 = \max\{r(v_1), r(v_2), \ldots, r(v_m)\} \), and \( \beta' = \sup_{x \in M} \| \nabla r(x) \| \), then for all \( x \in M \),
\[
|r(x)| \leq \frac{n}{n+1} \beta' s + \mu_0.
\]

**Proof.** We know that \( r(x) \) is a continuous and differential function on \( M \). Thus, according to Lemma 1, for all \( a_1, a_2, \ldots, a_m \in \mathbb{R}_+ \) such that \( a_1 + a_2 + \ldots + a_m = 1 \),
\[
|r(x) - (a_1 r(v_1) + a_2 r(v_2) + \ldots + a_m r(v_m))| \leq \frac{n}{n+1} \beta' s.
\]
Figure 2: Approximate $e^x$, $-2 \leq x \leq 2$ by $g(x, \hat{c}) + [-\mu, \mu]$ (solid line: $e^x$, dot line: $g(x, \hat{c}) \pm \mu$).

Then, we have

$$|r(x)| \leq \frac{n}{n+1} \beta' s + |(a_1 r(v_1) + a_2 r(v_2) + ... + a_m r(v_m))| \leq \frac{n}{n+1} \beta' s + \mu_0.$$ 

\[\square\]

**Example 5.** Consider the function $\phi(x) = e^x$ with $\Psi : -2 \leq x \leq 2$. We want to compute a polynomial $g(x)$ and the associated verified error bound $\mu$ such that

$$|\phi(x) - g(x)| < \mu, -2 \leq x \leq 2.$$ 

First, we construct a mesh $M$ on $\Psi$ with the spacing $s = \frac{1}{4}$. For a polynomial of the form $g(x, c) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, it is easy to find an approximate polynomial

$$g(x, \hat{c}) = 0.9173 + 0.9562x + 0.6797x^2 + 0.2117x^3.$$ 

According to Theorem 4, we can also compute the error bound $\mu = 0.2937$. The results are as shown in Figure 2. \[\square\]

Stated as above, once we obtain an interval polynomial hybrid system $\mathbf{IH}$ from $\mathbf{H}$ through polynomial approximation such that $\mathbf{H}$ is within $\mathbf{IH}$, the method in Section 4 can be used to verify safety of $\mathbf{IH}$, which ensures safety of $\mathbf{H}$. The following example is presented to illustrate our method for safety verification of a non-polynomial hybrid system.

**Example 6.** Consider the following two-tanks hybrid system [25] depicted in Figure 3 with

$$f_1(x) = \left[\frac{1 - \sqrt{x_1}}{\sqrt{x_1} - \sqrt{x_2}}\right], \quad f_2(x) = \left[\frac{1 - \sqrt{x_1} - x_2 + 1}{\sqrt{x_1} - x_2 + 1 - \sqrt{x_2}}\right],$$

where $x = (x_1, x_2)$ denotes the liquid levels. In [25], the authors verified that the system starting in location $\ell_1$ with an initial state in

$$\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 5.5)^2 + (x_2 - 0.25)^2 \leq 0.0625\}$$

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never reach the states of

\[ X_u(\ell_1) = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 4.25)^2 + (x_2 - 0.25)^2 \leq 0.0625\}. \]

Here, we enlarge both radii of initial and unsafe regions to 0.49, that is,

\[ \Theta = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 5.5)^2 + (x_2 - 0.25)^2 \leq 0.2401\} \]

and

\[ X_u(\ell_1) = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 4.25)^2 + (x_2 - 0.25)^2 \leq 0.2401\}, \]

and consider again safety verification of the given system. We first compute an interval polynomial system given by \([f_1](\mathbf{x})\) and \([f_2](\mathbf{x})\) to enclose the original system where

\[
[f_1](\mathbf{x}) = \begin{bmatrix}
[0.1658, 0.173] - 0.3377x_1 + 0.0114x_1^2 \\
[0.6465, 0.8615] + 0.3377x_1 - 1.7115x_2 - 0.0114x_1^2 + 0.8241x_2^2
\end{bmatrix}
\]

and

\[
[f_2](\mathbf{x}) = \begin{bmatrix}
- [0.1204, 0.132] - 0.3316x_1 + 0.3319x_2 + 0.0135x_1^2 - 0.0269x_1x_2 + 0.0137x_2^2 \\
[0.6716, 0.6898] + 0.3316x_1 - 0.9576x_2 - 0.0135x_1^2 + 0.0269x_1x_2 + 0.0572x_2^2
\end{bmatrix}.
\]

We obtain the following verified invariants with rational coefficients

\[
\tilde{\varphi}_1(\mathbf{x}) = -\frac{1069}{994}x_1 - \frac{145}{142}x_1 - \frac{367}{497}x_2 + \frac{121}{497}x_1^2 + \frac{160}{497}x_1x_2 + \frac{242}{497}x_2^2,
\]

\[
\tilde{\varphi}_2(\mathbf{x}) = \frac{9621}{994}x_1 + \frac{20}{71}x_1 + \frac{899}{497}x_2 + \frac{989}{994}x_1^2 + \frac{349}{994}x_1x_2 - \frac{1487}{994}x_2^2,
\]

which satisfy the conditions in Theorem 2 exactly. Therefore, the safety of the original hybrid system is verified.

The above approach can be easily extended to the case of uncertain non-polynomial hybrid systems, by which we mean the continuous dynamics at each location \(\ell\) are given by uncertain non-polynomial systems of the form

\[
\dot{x}_i = f_i(\mathbf{x}, \theta) = f_{i0}(\mathbf{x}, \theta) + \sum_{j=1}^{s} f_{ij}(\mathbf{x}, \theta)\phi_{ij}(\mathbf{x}), \quad \text{for } 1 \leq i \leq n,
\]

where \(\theta \in \Phi \subseteq \mathbb{R}^t\) is a vector of uncertainty. The following example demonstrates how to apply the above approach to verify safety of an uncertain non-polynomial system.
Example 7. Consider an uncertain non-polynomial system given in [5]:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 + \frac{1}{2}(e_1^2 - 1), \\
\dot{x}_2 &= -x_1 - x_2 + \theta x_1 x_2 + x_1 \cos x_1,
\end{align*}
\]

for \(-2 \leq x_1, x_2 \leq 2\) and \(0.98 \leq \theta \leq 1.2\). This system starts with an initial state in

\[\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 0.25\},\]

and we want to verify that the system never reach the states of

\[X_u = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + 1.5)^2 + (x_2 + 1.5)^2 \leq 0.16\}.
\]

To prove the safety of this non-polynomial system, we first compute interval polynomials to approximate the non-polynomial terms \(e^{x_1}\) and \(\cos x_1\) occurred in this system. Based on the above techniques, we obtain the following interval polynomial system

\[
\begin{align*}
\dot{x}_1 &= [-0.1882, 0.1055] - 0.5219 x_1 + x_2 + 0.33985 x_1^2 + 0.10585 x_1^3, \\
\dot{x}_2 &= [-0.2067, 0.0875] x_1 - x_2 + \theta x_1 x_2 - 0.3594 x_1^3,
\end{align*}
\]

which enclosures the original system. For the above interval hybrid system, we obtain the following verified invariant with rational coefficients

\[
\bar{\varphi}(x) = \frac{343}{32} x_1 + \frac{31}{6} x_2 + \frac{25}{48} x_1^2 - \frac{49}{32} x_1^3 - \frac{17}{48} x_1 x_2 - \frac{55}{32} x_2^2,
\]

which guarantees the safety of the original system.

6. Conclusion

In this paper, a hybrid symbolic-numeric method, based on SOS relaxation and interval arithmetic certification, is proposed to generate exact inequality invariants for safety verification of interval hybrid systems. As an application, one approach is provided to verify safety property of non-polynomial hybrid systems. More precisely, we apply a rigorous polynomial approximation method to compute an interval polynomial system, which contains the non-polynomial system, and then compute the exact invariant of the corresponding interval polynomial system to verify the safety property of the original system. Experiments on the benchmark hybrid systems illustrate the efficiency of our algorithm.

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Appendix

A. Interval Arithmetic

Interval arithmetic [1] has been designed for automatically tackling roundoff errors of numerical computations. In this subsection, some notions about interval arithmetic are presented.

Denote the closed intervals by $[a], [b]$, etc. By convention, the left and right endpoints of a closed interval $[a]$ are represented by $\underline{a}$ and $\overline{a}$, respectively, i.e.,

$$[a] = \{ x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a} \}$$

with $\underline{a} = \inf[a]$ and $\overline{a} = \sup[a]$. Any real number $a$ can also be regarded as an interval by identifying $a$ with the “point interval” $[a]$ with $\underline{a} = \overline{a} = a$. Such point intervals are also called degenerate intervals. Let

$$\text{mid}([a]) := \frac{1}{2}(\underline{a} + \overline{a}) \quad \text{and} \quad \text{rad}[a] := \frac{1}{2}(\overline{a} - \underline{a})$$

be the midpoint and radius of the closed interval $[a]$, respectively. Clearly, an interval can also be represented by its midpoint and radius. The set of all intervals over $\mathbb{R}$ is denoted by $\mathbb{R}$. The arithmetic operations $+, -, \cdot, \div$ can be extended from $\mathbb{R}$ to $\mathbb{R}$ in the usual set theoretic sense, and the bounds of the resulting intervals can be computed from the bounds of the operands, see [1] for details.

By $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ we denote the sets of real $n$-dimensional vectors and $m \times n$ matrices over $\mathbb{R}$, respectively. Elements of $\mathbb{R}^n$ are called interval vectors and denoted by $[a], [b]$ and etc, and elements of $\mathbb{R}^{m \times n}$ are called interval matrices and denoted by $[A], [B]$ and etc.
Remark that interval vectors (resp. interval matrices) are sets of vectors (resp. matrices). For interval vectors and matrices, the notions of midpoints and radius, and the arithmetic operations are defined componentwise.

By an interval linear system, we mean a system of the form

$$[A]x = [b],$$

(20)

where $[A] \in \mathbb{IR}^{n \times n}$, $[b] \in \mathbb{IR}^n$ and $x = (x_1, \ldots, x_n)^T$ is a column vector of $n$ unknowns. The set

$$\Sigma = \{ x \in \mathbb{R}^n : Ax = b \text{ for some } A \in [A], b \in [b] \}$$

is called the solution set of the interval system (20). Many efficient algorithms are available in [8, 28, 29, 31] for obtaining guaranteed inclusions [31] for the solution set $\Sigma$.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function. Replacing the real vector $x$ by an intervector $[x] \in \mathbb{IR}^n$ we thus obtain an interval extension $[f]$ of $f$. By the inclusion property of interval arithmetic, the range of $f$ over an interval is contained in its interval extension, i.e. $\{ f(x) : x \in [x] \} \subseteq [f([x])]$. To determine existence of solutions to the nonlinear system $f(x) = 0$, we will use the Krawczyk operators [14] based on Browder fixed points, which is defined as follows. Assume that $[x] \in \mathbb{IR}^n$ is an interval set satisfying $\hat{x} \in [x]$, and $C \in \mathbb{R}^{n \times n}$. The Krawczyk operator is defined as follows

$$K(\hat{x}, [x], f) = \hat{x} - Cf(\hat{x}) + (I - C[f']([x]))([x] - \hat{x}).$$

In practical computation, $C$ is usually chosen to be near the inverse of the Jacobian $f'(\hat{x})$.

Theorem 5. [31] Under the above assumptions, if

$$K(\hat{x}, [x], f) \subset int([x]),$$

where $int([x])$ is the topological interior of $[x]$, then there exists a unique $x^* \in K(\hat{x}, [x], f)$ such that $f(x^*) = 0$.

INTLAB is a MATLAB toolbox [30], which consists of interval calculations, and interval arithmetic for vectors and matrices. Many interval operations in this paper are implemented in MATLAB that uses the INTLAB package supporting rigorous real interval standard functions and interval least squares problem.

B. Sum of Squares Relaxation

We give a brief review on SOS optimization. More details will be found in [18]. Recall that a sufficient condition for determining $\psi(x) \in \mathbb{R}[x]$ to be positive semidefinite is that there exists an SOS decomposition of $\psi(x)$:

$$\psi(x) = \sum_{i=1}^s f_i^2(x), \quad \text{with } f_i(x) \in \mathbb{R}[x],$$

(21)

or, equivalently, $\psi(x)$ can be represented in the Gram matrix form

$$\psi(x) = m(x)^T \cdot W \cdot m(x),$$

(21)
where $W$ is a real symmetric and positive semidefinite matrix over $\mathbb{R}$, and $m(x)$ is a vector of all monomials in $\mathbb{R}[x]$ with degree $\leq \frac{1}{2}\deg r(x)$. Therefore the SOS program (21) can be further converted into the following Semidefinite programming (SDP) problem

$$\inf_W \text{Trace}(W)$$

$$\text{s. t. } \psi(x) = m(x)^T \cdot W \cdot m(x)$$

$$W \succeq 0, W^T = W,$$

(22)

where $\text{Trace}(W)$ acts as a dummy objective function that is commonly used in SDP for optimization problem with no objective functions. Many Matlab packages of SDP solvers, such as SOSTOOLS [23], YALMIP [17], and SeDuMi [33], are available to solve the SDP problem (22) efficiently.

The SOS programs have many applications, for example, in determining the nonnegativity of a multivariate polynomial over a semialgebraic set. Consider the problem of verifying whether the implication

$$\bigwedge_{i=1}^m (p_i(x) \geq 0) \implies q(x) \geq 0$$

(23)

holds, where $p_i(x) \in \mathbb{R}[x]$ for $1 \leq i \leq m$ and $q(x) \in \mathbb{R}[x]$. According to Stengle’s Positivstellensatz, Schmüdgen’s Positivstellensatz or Putinar’s Positivstellensatz [3], if there exist SOS polynomials $\sigma_i \in \mathbb{R}[x]$ for $i = 0, ..., m$, such that

$$q(x) = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x)p_i(x),$$

then the assertion (23) holds. Therefore, the existence of SOS representations provides a sufficient condition for determining the nonnegativity of $q(x)$ over $\{x \in \mathbb{R}^n : \bigwedge_{i=1}^m p_i(x) \geq 0\}$.

C. Existence of Real Roots for Underdetermined Interval Nonlinear Systems

Consider a nonlinear system

$$F(q) - [v] = 0.$$  

(24)

where $F : \mathbb{R}^r \rightarrow \mathbb{R}^s$ a continuously differentiable function with $r > s$, $r = \text{Dim}(q)$, and $[v] \in \mathbb{IR}^s$. To determine the existence of solution to system (24), we present two methods as follows. The idea of the first method is to transform the underdetermined interval system into the corresponding interval square nonlinear system by fixing some variables as constants, and then generalize Theorem 5 to verify the existence of real roots for this square nonlinear system, while the second method is to generalize the method in [4] to solve the interval underdetermined system (5).

Firstly, suppose that $\hat{q}$ is an approximate solution of (24). Here we assume that the Jacobian matrix $F'(q)$ at $\hat{q}$ is of full row rank. Column pivoting $QR$-decomposition for $F'(q)$ is applied to choose an index set $B = \{k_1, k_2, ..., k_s\}$ such that $F'_B(\hat{q}) \in \mathbb{R}^{s \times s}$ is nonsingular, that is,

$$F'(q) P^T = Q [ R | * ] \text{ with } P \in \mathbb{R}^{r \times r}, Q, R \in \mathbb{R}^{s \times s},$$

22
where $P$ is a permutation matrix, $Q$ is orthogonal and $R$ is upper triangular. The permutation $P$ arises from a greedy strategy to obtain maximum diagonal elements in $R$. Then, the set $B$ can be taken as those components which are permuted to the first $s$ positions by $P$. Thus, $\mathbf{q}$ can be separated into two parts $\mathbf{q} = (\mathbf{q}_B, \mathbf{q}_N)$, where $N = \{1, 2, ..., r\}/B$. Similar to the partition of $\mathbf{q}$, we have $\hat{\mathbf{q}} = (\hat{\mathbf{q}}_B, \hat{\mathbf{q}}_N)$. By use of the evaluations $\mathbf{q}_N = \hat{\mathbf{q}}_N$, (24) becomes the following interval square system

$$G(\mathbf{q}_B) - [\tilde{\mathbf{v}}] = 0,$$

where $G(\mathbf{q}_B) = F(\mathbf{q}_B, \hat{\mathbf{q}}_N) - \mathbf{c}$, and $[\tilde{\mathbf{v}}] = \mathbf{c} + [\mathbf{v}]$, and $\mathbf{c}$ is the constant vector of $F(\mathbf{q}_B, \hat{\mathbf{q}}_N)$, i.e., $\mathbf{c} = F(0, \hat{\mathbf{q}}_N)$.

Observing in (25), the interval coefficients only occur in the constant vector $[\tilde{\mathbf{v}}]$, and $G(\mathbf{q}_B)$ is a real function from $\mathbb{R}^s$ to $\mathbb{R}^s$, meaning that the Jacobian matrix of (25) is the same as one exact square system $G(\mathbf{q}_B) - \mathbf{v} = 0$, where $\mathbf{v}$ is a vector chosen randomly. Taking advantage of this property, it is easy to generalize Theorem 5 to verify the existence of real roots for (25).

**Theorem 6.** Consider the system (25). Let $[\mathbf{q}_B] \in \mathbb{I} \mathbb{R}^s$ be such that $\hat{\mathbf{q}}_B \in [\mathbf{q}_B]$, and $C \in \mathbb{R}^{s \times s}$. If

$$K(\hat{\mathbf{q}}_B, [\mathbf{q}_B], G - [\tilde{\mathbf{v}}]) = \hat{\mathbf{q}}_B - C(G(\mathbf{q}_B) - [\tilde{\mathbf{v}}]) + (I - CG'(G(\mathbf{q}_B)))([\mathbf{q}_B] - \hat{\mathbf{q}}_B) \subset \text{int}([\mathbf{q}_B]),$$

then there is a unique root $\mathbf{q}_B^*$ of (25) in $[\mathbf{q}_B]$ for each $\mathbf{v} \in [\tilde{\mathbf{v}}]$.

**Proof.** If (26) holds, then we have, for each $\mathbf{v} \in [\tilde{\mathbf{v}}]$,

$$K(\hat{\mathbf{q}}_B, [\mathbf{q}_B], G - \mathbf{v}) = \hat{\mathbf{q}}_B - C(G(\mathbf{q}_B) - \mathbf{v}) + (I - CG'(G(\mathbf{q}_B)))([\mathbf{q}_B] - \hat{\mathbf{q}}_B) \subset \text{int}([\mathbf{q}_B]).$$

According to Theorem 5, for $\mathbf{v}$ there exists a unique root $\mathbf{q}_B^*$ of (25) in $[\mathbf{q}_B]$. Hence, for each $\mathbf{v} \in [\tilde{\mathbf{v}}]$, there exists a unique root for the system (25) if (26) holds. □

Alternatively, we also can apply another method provided in [4], to determine the existence of real roots for the underdetermined system (5) directly. The only difference is that we need deal with a special interval underdetermined system while they worked on an exact one. For the same reason as in the above discussion, it is easy to generalize their method in [4] to deal with our problem.

Suppose the Jacobian $F'(\mathbf{q})$ is of full row rank. Following [4], we apply the column pivoting $QR$-decomposition to choose an index set $B = \{k_1, k_2, ..., k_s\}$ such that $F'_B(\mathbf{q}) \in \mathbb{R}^{s \times s}$ is nonsingular. Then, define the function $H : \mathbb{R}^r \rightarrow \mathbb{R}^r$ by

$$\begin{cases}
H_B(\mathbf{q}) = \mathbf{q}_B - F'_B(\mathbf{q})^{-1}(F(\mathbf{q}) - \mathbf{v}), \\
H_N(\mathbf{q}) = \mathbf{q}_N - \alpha(\mathbf{q}_N - \hat{\mathbf{q}}_N),
\end{cases}$$

where $N = \{1, 2, ..., r\}/B$ and $\alpha \in (0, 1)$ is a constant. Obviously, if $\mathbf{q}_B^* \in \mathbb{R}^r$ is a fixed point of $H$, that is $H(\mathbf{q}_B^*) = \mathbf{q}_B^*$, then we have $F(\mathbf{q}_B^*) - \mathbf{v} = 0$ with $\mathbf{q}_N^* = \hat{\mathbf{q}}_N$. Choose two nonnegative numbers $r_1$ and $r_2$, we define the convex set

$$[\mathbf{q}] = \{\mathbf{q} \in \mathbb{R}^r : \|\mathbf{q}_B - \hat{\mathbf{q}}_B\| \leq r_1, \|\mathbf{q}_N - \hat{\mathbf{q}}_N\| \leq r_2\}.$$ 

Now, we can use the following theorem to determine the existence of solutions to the system (5).
Theorem 7. Consider the system (5). Suppose the Jacobian $F'(\hat{q})$ has full row rank, and that
\[
\|F_B'(q) - F_B'(\hat{q})\| \leq K\|q - \hat{q}\| \text{ for } q \in [q].
\]
There is a solution $q^*$ of (5) in $[q]$ for each $v \in [v]$ if
\[
\max_{v \in [v]} \|F_B'(\hat{q})^{-1}F(\hat{q}) - v\| + \|F_B'(\hat{q})^{-1}\| \left(\frac{1}{2}K(r_1 + r_2)r_1 + \max_{q \in [q]} \|F_N'(q)\| r_2\right) \leq r_1.
\]
Remark that Theorem 6 is a special case of Theorem 7 by setting $r_2 = 0$. Compared with Theorem 7, the condition (26) in Theorem 6 is easy to verify in practice.