ON A THEOREM OF N. KATZ AND BASES IN IRREDUCIBLE REPRESENTATIONS

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Dedicated to the memory of Leon Ehrenpreis

1.

Abstract. N. Katz has shown that any irreducible representation of the Galois group of $\mathbb{F}_q((t))$ has unique extension to a special representation of the Galois group of $k(t)$ unramified outside 0 and $\infty$ and tamelyramified at $\infty$. In this paper we analyze the number of not necessarily special such extensions and relate this question to a description of bases in irreducible representations of multiplicative groups of division algebras.

Let $k = \mathbb{F}_q$, $q = p^r$ be a finite field, $\bar{k}$ the algebraic closure of $k$, $F := k((t))$ and $\bar{F}$ be the algebraic closure of $F$. The restriction on $\bar{k} \subset \bar{F}$ defines a group homomorphism

$$\text{Gal}(\bar{F}/F) \to \text{Gal}(\bar{k}/k) = \hat{\mathbb{Z}}$$

and we define the Weil group of the field $F$ as the preimage $G_0 \subset \text{Gal}(\bar{F}/F)$ of $\mathbb{Z} \subset \hat{\mathbb{Z}}$ under this homomorphism.

We denote by $\mathbb{P}^1$ the projective line over $k$, set $E := k(t)$ and denote by $S$ the set of points of $\mathbb{P}^1$. For any $s \in S$ we denote by $E_s$ the completion of $E$ at $s$. Using the parameter $t$ on $\mathbb{P}^1$ we identify the fields $E_0$ and $E_\infty$ with $F$ and therefore identify $G_0$ with the Weil groups of the fields $E_0$ and $E_\infty$.

Let $\bar{E}$ be the maximal extension of the field $E$ unramified outside 0 and $\infty$ and tamely ramified at $\infty$. We denote by $G \subset \text{Gal}(\bar{E}/E)$ the Weil group corresponding to the extension $\bar{E}/E$. We have the natural imbeddings

$$G_0 \hookrightarrow G, G_\infty \hookrightarrow G$$

well defined up to conjugation. Therefore for any complex representation $\rho$ of $G$ the restrictions to $G_0, G_\infty$ define representations $\rho_0, \rho_\infty$ of the corresponding local groups. The group $G$ has a unique maximal
quotient $\mathcal{G}$ such that the Sylow $p$-subgroup of $\mathcal{G}$ is normal. As shown by N.Katz ([5]) the composition $\mathcal{G}_0 \to \mathcal{G}$ is an isomorphism.

**Remark** A finite-dimensional irreducible representation $\rho_0$ of $\mathcal{G}$ is called *special* if it factors through a representation of the group $\mathcal{G}$. One can restate the theorem of N.Katz by saying that for any irreducible representation $\rho_0$ of $\mathcal{G}_0$ there exists a unique special representation $\rho_{sp}$ of the group $\mathcal{G}$ whose restriction to $\mathcal{G}_0$ is equivariant to $\rho_0$.

Let $D_0$ be a skew-field with center $F$, $\dim_F D_0 = n^2$, $G_0 := D_0^*$ be the multiplicative group of $D_F$ and $\rho_0$ be an $n$-dimensional indecomposable representation of the group $\mathcal{G}_0$.

**Definition 1.1.** a) We denote by $\tilde{\sigma}(\rho_0)$ the irreducible discrete series representation of the group $GL_n(F)$ which corresponds to $\rho_0$ under the local Langlands correspondence (see for example [3]) and by $\sigma(\rho_0)$ the irreducible representation of the group $G_0$ which corresponds to $\tilde{\sigma}(\rho_0)$ as in [1].

b) We denote by $r(\rho_0)$ the formal dimension of the representation $\tilde{\sigma}(\rho_0)$ where the formal dimension is normalized in such a way that the formal dimension of the Steinberg representation is equal to 1. Analogously for any indecomposable representation $\rho_\infty$ of the group $\mathcal{G}_\infty$ we define an integer $r(\rho_\infty)$.

c) We denote by $A(\rho_0)$ the set of equivalence classes of $n$-dimensional irreducible representations $\rho$ of the group $\mathcal{G}$ whose restriction to $\mathcal{G}_0$ is equivalent to $\rho_0$ and the restriction to $\mathcal{G}_\infty$ is indecomposable.

**Theorem 1.2.** For any $n$-dimensional irreducible $\overline{\mathbb{Q}}_l$-representation of the group $\mathcal{G}_0$ the sum $\sum_{\rho \in A(\rho_0)} r(\rho_\infty)$ is equal to $r(\rho_0)$.

**Proof.** Let $\mathbb{A} = \prod_{s \in S} E_s$ the ring of adeles of $E$ and $D$ be a skew-field with center $E$ unramified outside $\{0, \infty\}$, $D_0 := D \otimes_E E_0$ and $D_\infty := D \otimes_E E_\infty$. Then $D_0, D_\infty$ are local skew-fields. Let $\mathcal{G}$ be the multiplicative group of $D$ considered as an the algebraic $E$-group.

It follows from [6] that we can identify the set $A(\rho_0)$ with the set of automorphic representations $\tilde{\pi} = \prod_{s \in S} \tilde{\pi}_s$ of the group $GL_n(\mathbb{A})$ such that the representation $\pi_0$ is equivalent to $\tilde{\sigma}(\rho_0)$ and the representation $\tilde{\pi}_\infty$ is of discrete series. Then it follows from [1] that we can identify the set $A(\rho_0)$ with the set of automorphic representations $\pi = \prod_{s \in S} \pi_s$ of the group $\mathcal{G}(\mathbb{A})$ such that the representation $\pi_0$ is equivalent to $\sigma(\rho_0)$. We will use this identification for the proof of the Theorem 1.2.
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We denote by $N : D_0 \to F$ the reduced norm and define
\[ \mu := \nu \circ N : D_0^* \to \mathbb{Z}, K_0 := \mu^{-1}(0) \]
where $\nu : F^* \to \mathbb{Z}$ is the valuation. Then $K_0 \subset D_0^*$ is a maximal compact subgroup. We define the first congruence subgroup $K_1^0$ by
\[ K_1^0 := \{ k \in K_0 | \mu(k - Id) > 0 \} \]
As is well known $K_1^0$ is a normal subgroup of $D_0^*$ such that $K_0/K_1^0 = F_{q^n}^*$ and $D_0^*/K_1^0 = \mathbb{Z} \ltimes F_{q^n}^*$ where $\mathbb{Z}$ acts on $F_{q^n}^*$ by $(n, x) \to x^{q^n}$.

For any $s \in S - \{0, \infty\}$ we identify the group $G_{E_s}$ with $GL(n, E_s)$ and define $K_s := GL(n, \mathcal{O}_s)$. We write $G_A := D_0^* \times GL_n(A^0)$ where
\[ GL_n(A^\infty) := D_0^* \times \prod_{s \in S - \{0, \infty\}} GL(n, E_s) \]
and define
\[ K^0 := \prod_{s \in S - \{0, \infty\}} K_s \times K_{E^\infty}, K^1 := \prod_{s \in S - \{0, \infty\}} K_s \times K_{E^\infty} \]
where $K_{E^\infty}^1 \subset K_{E^\infty} \subset D_{E^\infty}^*$ is the first congruence subgroup of $G_{E^\infty}$.

For any irreducible representation $\pi$ of the group $G_0$ we denote by $\tilde{\pi}$ the discrete series representation of the group $GL_n(F)$ corresponding to $\pi$ as in [1].

**Lemma 1.3.** a) For any irreducible complex representation $\kappa : D_0^*/K_1^0 \to Aut(W)$ and any character $\chi : K_0/K_1^0 \to \mathbb{C}^*$ we have
\[ \dim(W^x) \leq 1 \]
where $W^x = \{ w \in W | \kappa(k)w = \chi(k)w, k \in K_0 \}$.

b) For any irreducible representation $\pi$ of the group $G_0$ the formal dimension of $\tilde{\pi}$ is equal to the dimension of $\pi$.

**Proof.** Part a) follows from the isomorphism $D_0^*/K_1^0 = \mathbb{Z} \ltimes F_{q^n}^*.$.

Part b) follows from [1]. □

We see that the following equality implies the validity of the Theorem 1.2.

**Claim 1.4.** For any $n$-dimensional irreducible $\mathbb{Q}_l$-representation of the group $G_0$ the sum $\sum_{\pi \in A(\rho_0)} \dim(\pi) = \dim(\sigma(\rho_0))$.

The proof of Claim is based on the following result.

**Proposition 1.5.** The product map $D_0^* \times K^1 \times G_E \to G_A$ is a bijection.
Proof of the Proposition. The surjectivity follows from Lemma 7.4 in [4]. To show the injectivity it is sufficient to check the equality
\[(D_0^* \times K^1) \cap G_E = \{e\}\]
which is obvious. □

We denote by \(\mathbb{C}(G_A/G_E)\) the space of locally constant functions on \(G_A/G_E\) with compact support, by \(\mathbb{C}(G_0)\) the space of locally constant functions on \(G_0\) with compact support and by \(L \subset \mathbb{C}(G_A/G_E)\) the subspace of \(K^1\)-invariant functions. The group \(G_0 \times D_\infty^*/K^1\) acts naturally on \(L\).

Let \(\rho_0\) be an indecomposable representation of the group \(G_0\). We denote by \((\sigma(\rho_0), V(\rho_0))\) the corresponding representation of the group \(G_0\) and identify the set \(A(\rho_0)\) with the set of automorphic representations \(\pi_a = \prod_{s \in S} \pi_s^a\) of the group \(G(A)\) such that the representation \(\pi_0^a\) is equivalent to \(\sigma(\rho_0)\) and the representation \(\pi_\infty^a\) is trivial on \(K^1\). Let
\[\mathcal{H} := \prod_{s \in S - \{0, \infty\}} \mathcal{H}_s\]
where \(\mathcal{H}_s\) is the spherical Hecke algebra for \(G(F_s) = GL(n, F_s)\). By construction, the commutative algebra \(\mathcal{H}\) acts on the \(D_0^* \times D_\infty^*/K^1\)-module \(L\). For any \(a \in A(\rho_0)\) we define
\[L_a := \text{Hom}_{G_0}(\pi_a^a, \mathbb{C}(G_A/G_E)) = \text{Hom}_{G_0 \times \mathcal{H}}(\sigma(\rho_0), L) \subset \text{Hom}_{G_0}(\sigma(\rho_0), L)\]

Lemma 1.6. a) The restriction \(r : L \to \mathbb{C}(G_0)\) is an isomorphism of \(G_0\)-modules where \(G_0\) acts on \(\mathbb{C}(D_0^*)\) by left translation.

b) \(\text{Hom}_{G_0}(\sigma(\rho_0), L) = V^\vee\) where \(V^\vee\) is the dual space to \(V(\rho_0)\).

c) \(V^\vee = \bigoplus L_a, a \in A(\rho_0)\) where the algebra \(\mathcal{H}\) acts on \(L_a, a \in A(\rho_0)\) by a character \(\chi_a : \mathcal{H} \to \bar{\mathbb{Q}}_l^*, \chi_a \neq \chi_a'\) for \(a \neq a'\) and the representations \(\pi_\infty^a\) of the group \(D_\infty^*/K^1\) on \(M_a\) are irreducible.

d) The representations \(\pi_\infty^a\) are associated with the restriction \(\rho(a)_\infty\) by the local Langlands correspondence.

Proof. The Lemma follows immediately from the Proposition and the strong multiplicity one theorem ([7] and [1]). □

This Lemma implies the validity of Claim and therefore of Theorem 1.2. Indeed we have
\[\dim(V) = \dim(V^\vee) = \sum_{a \in A(\rho_0)} \dim(L_a) = \sum_{a \in A(\rho_0)} \dim(\pi_\infty^a) = \sum_{a \in A(\rho_0)} r(\rho(a)_\infty)\] □
One can ask whether one can extend Theorem 1.2 to the case of other groups. More precisely, let $G$ be a split reductive group with a connected center and $^L G$ be the Langlands dual group. Consider a homomorphism $\rho_0 : \mathcal{G}_0 \to ^L G$ such that the connected component of the centralizer $Z_\rho := Z_{^L G} (\text{Im} (\rho))$ is unipotent. Let $[Z_\rho]$ be the group of connected components of the centralizer $Z_\rho$. Conjecturally, one can associate with $\rho_0$ an $^L G$-packet of irreducible representations $\pi_\rho (\tau)$ of the group $G_0 := G(F)$ parameterized by irreducible representations $\tau$ of $[Z_\rho]$ and there exists an integer $r(\rho_0)$ such that the formal dimension of $\pi_\rho (\tau)$ is equal to $r(\rho) dim(\tau)$.

We denote by $A^G(\rho_0)$ the set of conjugacy classes of homomorphisms $\rho : \mathcal{G} \to ^L G$ whose restriction on $\mathcal{G}_0$ is conjugate to $\rho_0$ and such that the connected component of the centralizer of the restriction on $\mathcal{G}_\infty$ is unipotent.

**Question.** Is it true that $r(\rho_0) = \sum_{a \in A(\rho_0)} r(\rho_\infty)$ where $r(\rho_\infty)$ is defined in the same way as $r(\rho_0)$?

2.

Let $G$ be a reductive group over a local field. As is well known one can realize the spherical Hecke algebra $\mathcal{H}$ of $G$ geometrically, that is as the Grothendick group of the monoidal category of perverse sheaves on the affine Grassmanian. Analogously in the case when $G$ be a reductive group over a global field of positive characteristic the unramified geometric Langlands conjecture predicts the existence of a geometric realization of the corresponding space of automorphic functions.

Let $\mathcal{C}$ be a smooth absolutely irreducible $\mathbb{F}_q$-curve, $q = p^m$, $S$ be the set of geometric points of $\mathcal{C}$, $\Gamma := \pi_1(\mathcal{C})$. For any $s \in S$ we denote by $F_{r,s} \subset \Gamma$ the conjugacy class of the Frobenius at $s$.

Let $E$ be the field of rational functions on $\mathcal{C}$. For any $s \in S$ we denote by $E_s$ the completion of $E$ at $s$ and we denote by $\mathbb{A}$ be the ring of adeles of $E$. Fix a prime number $l \neq p$.

Let $\mathcal{G}$ be a split reductive group, and $\mathcal{K} := \prod_{s \in S} G(\mathcal{O}_s) \subset G(\mathbb{A})$ be the standard maximal compact subgroup. An irreducible representation $(\pi, V) = \otimes_{s \in S} (\pi_s, V_s)$ of $G(\mathbb{A})$ is unramified if $V^\mathcal{K} \neq \{0\}$. In this case $\text{dim}(V^{\mathcal{K}}) = 1$. So for any unramified representation $(\pi, V)$ of the group $G(\mathbb{A})$ there is a special spherical vector $v_{sp} \in V$ defined up to a multiplication by a scalar.
Let $L_G$ be the Langlands dual group and $\rho$ a homomorphism from $\Gamma$ to $L_G(\overline{Q}_l)$ such that for any $s \in S$ the conjugacy class $\gamma_s := \rho(Fr_s) \subset L_G(\overline{Q}_l)$ is semisimple. In such a case we can define unramified representations $(\pi_{\gamma_s}, V_s)$ of local groups $G(E_s)$ and the representation $(\pi(\rho), V_\rho) = \otimes_s (\pi_{\gamma_s}, V_s)$ of the adelic group $G(\mathbb{A})$. According to the unramified geometric Langlands conjecture the homomorphism $\rho$ defines [at least in the case when $\rho$ is tempered] an imbedding $i_\rho : V_\rho \to \overline{Q}_l(K\backslash G(\mathbb{A})/G(E))$ and a function $f_\rho := i_\rho(v_{sp})$ which is defined up to a multiplication by a scalar.

We can identify the set $K\backslash G(\mathbb{A})/G(E)$ with the set of $\mathbb{F}_q$-points of the stack $\mathcal{B}_G$ of principal $G$-bundles on $\mathcal{C}$ and the unramified geometric Langlands correspondence predicts the existence of a perverse Weil sheaf $F(\rho)$ on $\mathcal{B}_G$ such that the function $f_\rho$ is given by the trace of the Frobenius automorphisms on stalks of $F(\rho)$. (See [2])

If one considers ramified automorphic representations $(\pi, V) = \otimes_{s \in S} (\pi_s, V_s)$ of $G(\mathbb{A})$ then there is no natural way to choose a special vector in $V$. So on the "geometric" side one expects not an object $F(\rho)$ but an abelian category $\mathcal{C}(\rho)$ which is a product of local categories $\mathcal{C}(\rho_s)$ such that the Grothendick K-group of the category $[\mathcal{C}(\rho_s)]$ coincides with the subspace $V_{s0}$ of the minimal $K$-type vectors of the space $V_s$ of the local representation. Such geometric realization of the space $V_{s0}$ would define a special basis of vector spaces $V_{s0}$ which would be a non-archimedian analog of Lusztig’s canonical basis. Here we consider only the case of an anisotropic group when the minimal $K$-type subspace $V_{s0}$ coincides with the space $V_s$ of the representation of $G$. Moreover we will only discuss a slightly weaker data of a projective basis where a projective basis in a finite-dimensional vector space $T$ is a decomposition of the space $T$ in a direct sum of one-dimensional subspaces. So one could look for a special basis of vector spaces $V_s$ which would be a non-archimedian analog of the Lusztig’s canonical basis.

Let as before $F := k((t)), D_0$ be a skew-field with center $F, dim_0 D_0 = n^2, G_0$ be the multiplicative group of $D_0$ and $\sigma : G_0 \to Aut(V)$ a complex irreducible continuous representation of the group $G_0$.

**Theorem 2.1.** For any irreducible representation $\tau : D_F^* \to Aut(T)$ of the group $D_F^*$ there exists a "natural" projective basis $= \oplus_a T_a$ of $T$.

**Remark 2.2.** The construction is global. In particular I don’t know how to define a projective basis in the case when $F$ is a local field of
characteristic zero. It would be very interesting to find a local construction of a projective basis.

**The construction.** As follows from Lemma 1.6 c) we have a decomposition $V^\vee = \sum_{a \in A(\rho_0)} M_a$ where the group $D_{\infty}/K_1^1$ acts irreducibly on $M_a$. Therefore the group $\mathbb{F}_{q^n}^* = K_{\infty}/K_3^1$ acts on $M_a$ and we have a decomposition of $M_a$ into the sum of eigenspaces for the action of the group $\mathbb{F}_{q^n}^*$. As follows from Lemma 3 a) these eigenspaces are one-dimensional.

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