Complexity reduction of C-Algorithm

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Abstract

The C-Algorithm introduced in \cite{5} is designed to determine isochronous centers for Liénard-type differential systems, in the general real analytic case. However, it has a large complexity that prevents computations, even in the quartic polynomial case.

The main result of this paper is an efficient algorithmic implementation of C-Algorithm, called ReCA (Reduced C-Algorithm). Moreover, an adapted version of it is proposed in the rational case. It is called RCA (Rational C-Algorithm) and is widely used in \cite{1} and \cite{2} to find many new examples of isochronous centers for the Liénard type equation.

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1. Introduction

The use of symbolic computations and Computer Algebra Systems in the Qualitative investigations of ordinary differential equations becomes a standard way, see for instance \cite{9}. One of the important problems in this field is the characterization and the explicit description of isochronous centers for planar polynomials vector fields. See for example \cite{11}.

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This is a companion paper of [1, 2] which are devoted to the seeking out of isochronous centers for real planar polynomial Liénard type equation. These investigations are based on the C-Algorithm introduced in [5] and used in [6] for the cubic case.

The C-Algorithm in its original form has a large computational cost. In the real analytic case as well as in the particular rational case the careful inspection of the formulas used leads to more efficient algorithms, called ReCA (Reduced C-Algorithm) and RCA (Rational C-Algorithm) respectively. The aim of this note is to give a detailed description of them.

Consider the Liénard type differential equation
\[ \ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \] (1)
where \( f \) and \( g \) are defined in a neighborhood of \( 0 \in \mathbb{R} \), or equivalently its associated two dimensional (planar) system
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -g(x) - f(x)y^2 
\end{align*}
\] (2)
In this paper, we assume that \( g(0) = 0 \), which insures that \( O \) is a critical point, and \( xg(x) > 0 \) for \( x \neq 0 \), which insures that the origin is a center. Moreover, we suppose that \( g'(0) = 1 \), so that system (2) is a perturbation of the linear isochronous center \( \dot{x} = y, \dot{y} = -x \).

In our knowledge equation (1) appears for the first time in M. Sabatini’s paper [12], where sufficient conditions for the isochronicity of the origin \( O \) are given for \( C^1 \) functions \( f \) and \( g \) defined in some neighborhood of \( 0 \).

In the real analytic case, necessary and sufficient conditions for isochronicity are given by A. R. Chouikha in [5], where Theorem 1 is proved. We use the same notations as in [5]:
\[
F(x) := \int_0^x f(s)ds, \quad \phi(x) := \int_0^x e^{F(s)}ds.
\] (3)
As \( xg(x) > 0 \) for \( x \neq 0 \), the function \( \xi \) is well defined by the relation
\[
\frac{1}{2}\xi(x)^2 = \int_0^x g(s)e^{2F(s)}ds
\] (4)
and the condition \( x\xi(x) > 0, \forall x \neq 0 \). Such function \( \xi \) is real analytic in some neighborhood of \( 0 \) and \( \xi'(0) = 1 \).

**Theorem 1** (Chouikha, [5]). Let \( f \) and \( g \) be real analytic functions in a neighborhood of \( 0 \), such that \( g(0) = 0 \) and \( xg(x) > 0 \) for \( x \neq 0 \). Then system (1) has an isochronous center at \( O \) if and only if there exists an odd real analytic function \( h \), called the Urabe function, satisfying the following conditions:
\[
\frac{\xi(x)}{1 + h(\xi(x))} = g(x)e^{F(x)},
\] (5)
\[ \phi(x) = \xi(x) + \int_0^{\xi(x)} h(t) \, dt, \quad (6) \]

where \( F(x), \phi(x) \) and \( \xi(x) \) are defined by \((3)-(4)\).

Taking into account \((4)\), it is easy to see that \((5)\) and \((6)\) are equivalent.

This theorem leads to an algorithmic method, named C-Algorithm, which gives necessary conditions for isochronicity by computing the first coefficients of the power series expansion around 0 of both sides of \((5)\). Then, the sufficiency is insured by establishing explicitly the odd Urabe function. More details about the C-Algorithm can be found in \([5,6,2,1]\).

In particular, in \([2,1]\), the authors investigate the practical applicability of the C-Algorithm to the following family of planar polynomial systems, which are perturbations of the linear isochronous center \( \dot{x} = y, \dot{y} = -x \):

\[
\begin{align*}
\dot{x} &= p_0(x) + p_1(x) y \\
\dot{y} &= q_0(x) + q_1(x) y + q_2(x) y^2
\end{align*}
\quad (7)
\]

where \( p_0, p_1, q_0, q_1, q_2 \in \mathbb{R}[x], p_0(0) = p_0'(0) = 0, p_1(0) = 1, q_0(0) = 0, q_0'(0) = -1, q_1(0) = 0. \)

Actually, under some restrictions, such systems are reducible to Liénard type differential equation \((1)\) with rational functions \( f \) and \( g \), and in this case the RCA described in Section \([3]\) is much more efficient than the standard C-Algorithm.

2. The C-Algorithm and the ReCA

In \([2]\), the change of variable \( u = \phi(x) = \xi + \int_0^\xi h(s) \, ds \) is introduced. Let us denote by \( \tilde{g}(u) \) the function of \( u \) represented by both sides of \((5)\)

\[ \tilde{g}(u) = \frac{\xi(x)}{1 + h(\xi(x))} = g(x) e^{F(x)}. \quad (8) \]

The derivative of \( u \) with respect to \( x \) (resp. \( \xi \)) can easily be expressed in terms of \( x \) (resp. \( \xi \)),

\[ \frac{du}{dx} = e^{F(x)} \quad \text{and} \quad \frac{du}{d\xi} = 1 + h(\xi). \]

The C-Algorithm is based on the two sides computations of the derivatives of \( \tilde{g}(u) \) with respect to \( u \) in terms of \( \xi \) and \( x \), which consists in computing the following quantities:

\[
\begin{align*}
\tilde{P}_0(\xi) &= \frac{\xi}{1 + h(\xi)} \\
\tilde{P}_k(\xi) &= \frac{d\tilde{P}_{k-1}(\xi)}{d\xi} \frac{1}{1 + h(\xi)} \\
\tilde{Q}_0(x) &= g(x) e^{F(x)} \\
\tilde{Q}_k(x) &= \frac{d\tilde{Q}_{k-1}(x)}{dx} e^{-F(x)}
\end{align*}
\quad (9)
\]
where \( k \geq 1 \) and evaluating them at 0, that is \( \tilde{P}_k(0) = \tilde{Q}_k(0) \). As
\[
\tilde{P}_k(0) = \frac{d^k \left( \frac{x}{1 + h(x)} \right)}{du^k} \bigg|_{u=0}, \quad \text{and} \quad \tilde{Q}_k(0) = \frac{d^k (g(x)e^{F(x)})}{du^k} \bigg|_{u=0}.
\]
by analyticity, this is equivalent to the equalities \( \tilde{Q}_k(0) = \tilde{P}_k(0), \ k \geq 0 \).

In the next theorem we describe the Reduced C-Algorithm, which takes care of significant multiplicative factors appearing in the formulas. The paper [2] is based on it.

**Theorem 2** (Reduced C-Algorithm - ReCA). Let \( f \) and \( g \) be real analytic functions defined in a neighborhood of 0, such that \( g(0) = 0, g'(0) = 1 \) and \( xg(x) > 0 \) for \( x \neq 0 \). Then system (1) has an isochronous center at \( O \) if and only if, there exists an odd analytic function \( h \) such that for all \( k \geq 0 \),
\[
\tilde{P}_k(0) = \tilde{Q}_k(0)
\]
with the recursive formulas:
\[
\begin{align*}
\tilde{P}_0(\xi) &= \xi \\
\tilde{P}_k(\xi) &= \frac{d\tilde{P}_{k-1}(\xi)}{d\xi}(1 + h(\xi)) - (2k - 1) \tilde{P}_{k-1}(\xi) \frac{dh(\xi)}{d\xi} \\
\tilde{Q}_0(x) &= g(x) \\
\tilde{Q}_k(x) &= \frac{d\tilde{Q}_{k-1}(x)}{dx} - (k - 2)f(x)\tilde{Q}_{k-1}(x)
\end{align*}
\]

Moreover, for all \( M > 0 \), the \( M \) first necessary conditions of isochronicity, \( \tilde{P}_k(0) = \tilde{Q}_k(0) \) for 1 \( \leq k \leq M \) can be obtained by computing the truncated power series expansions around 0 of \( \tilde{P}_k(\xi) \) and \( \tilde{Q}_k(x) \) up to the order \( M - k \).

**Proof.** First, the computation of the first two derivatives of \( \tilde{g}(u) \) with respect to \( u \) in terms of both \( x \) and \( \xi \) gives
\[
\begin{align*}
\tilde{g}'(u) &= \frac{1 + h(\xi) - \xi \frac{dh(\xi)}{d\xi}}{(1 + h(\xi))^3} = \frac{d}{dx}g(x) + g(x)f(x), \\
\tilde{g}''(u) &= -\frac{(1 + h(\xi)) \xi \frac{d^2h(\xi)}{d\xi^2} + 3 \frac{d^2h(\xi)}{d\xi^2} \left( 1 + h(\xi) - \xi \frac{dh(\xi)}{d\xi} \right)}{(1 + h(\xi))^5} \\
&= \left( \frac{d^2g(x)}{dx^2} + \frac{dg(x)}{dx}f(x) + g(x)\frac{df(x)}{dx} \right) e^{-F(x)}.
\end{align*}
\]

These formulas strongly suggest that the \( k^{th} \) derivative of \( \tilde{g}(u) \) can be written both in terms of \( x \) and \( \xi \) as follows:
\[
\tilde{g}^{(k)}(u) = \frac{\tilde{P}_k(\xi)}{(1 + h(\xi))^{2k+1}} = \tilde{Q}_k(x) e^{(1-k)F(x)},
\]

4
with \( \hat{P}_k \) and \( \hat{Q}_k \) verifying the induction formulas (10).

Then, we assume that this is so up to the order \( n - 1 \). Differentiating the two sides of the equality \( \tilde{g}^{(n-1)}(u) \) with respect to \( u \) in terms of \( x \) and \( \xi \) gives

\[
\tilde{g}^{(n)}(u) = \frac{\hat{P}_n(\xi)}{(1 + h(\xi))^{2n+1}} = \hat{Q}_n(x) e^{(1-n)F(x)},
\]

where \( \hat{P}_n(\xi) \) and \( \hat{Q}_n(x) \) satisfy formulas (10) as expected.

As \( F(0) = 0 \) and \( h(0) = 0 \), necessary and sufficient conditions are given by \( \hat{P}_k(0) = \hat{Q}_k(0), k \geq 0 \).

When our aim is to establish a fixed number \( M \) of necessary conditions, we restrict ourselves to the power series expansion around 0 of \( \hat{P}_k \) and \( \hat{Q}_k \) for \( 1 \leq k \leq M \) up to the order \( M - k \), which is the minimal necessary truncation order. Indeed, by formulas (10), to obtain \( \hat{P}_M(0) \) and \( \hat{Q}_M(0) \) it is sufficient to compute a power series expansion around 0 of \( \hat{P}_M(\xi) \) and \( \hat{Q}_M(x) \) up to order 0 (i.e. constant terms) which require the power series expansion around 0 of \( \hat{P}_{M-1}(\xi) \) and \( \hat{Q}_{M-1}(x) \) up to order 1, and so on.

In the practical use of the described algorithm we are concerned with a finite number \( M \) of necessary conditions. When a candidate for an isochronous center is identified, we try to write down its Urabe function under a closed-form expression, and prove the sufficiency using again Theorem 1.

### 3. The RCA

In this section we restrict ourselves to systems (2) for which \( f \) and \( g \) are rational functions. For this particular case, we describe an easy to handle couple of polynomial recursive formulas which gives the \( k \)th derivatives of each side of (5). Those formulas apply in particular to systems (7) when they are reducible to Liénard type differential equation. The paper [1] is based on it.

We denote \( f(x) = N_f(x) / D_f(x) \) (resp. \( g(x) = N_g(x) / D_g(x) \)), where \( N_f, N_g, D_f \) and \( D_g \) are polynomials such that \( D_f(0) = 1, D_g(0) = 1 \) and \( \text{pgcd}(N_f, D_f) = 1, \text{pgcd}(N_g, D_g) = 1 \).

**Theorem 3** (Rational C-Algorithm - RCA). There exists a positive integer \( M_0 \), such that for any \( M \geq M_0 \) the following assertions are equivalents:

1. the origin \( O \) of system (2) is an isochronous center;
2. there exists a real analytic odd function \( h \) satisfying \( P_k(0) = Q_k(0) \).
for all 0 ≤ k ≤ M, where

\[
P_0(\xi) = \xi \\
P_k(\xi) = \left( \frac{dP_{k-1}(\xi)}{d\xi} \right) \left( 1 + h(\xi) \right) - (2k - 1) P_{k-1}(\xi) \frac{dh(\xi)}{d\xi} \\
Q_0(x) = N_g(x) \\
Q_k(x) = Q_{k-1}(x) D_g(x) \left( (1 - k) \frac{D_f(x)}{dx} + (2 - k) N_f(x) \right) \\
- k Q_{k-1}(x) \frac{dD_g(x)}{dx} D_f(x) + \frac{dQ_{k-1}(x)}{dx} D_g(x) D_f(x)
\]

Moreover, as we only need the values of the \( P_k \) and \( Q_k \) at 0, it is sufficient to compute the power series expansions of \( P_k(x) \) and \( Q_k(x) \) at order \( M - k \), i.e. to truncate the polynomials \( P_k \) and \( Q_k \) up to degree \( M - k \).

Proof. As in the proof of the previous theorem, those formulas are found by induction on \( k \):

\[
\hat{g}^{(k)}(u) = \frac{P_k(\xi)}{(1 + h(\xi))^{2k+1}} = \frac{Q_k(x)}{D_f(x)^k D_g(x)^{k+1}} e^{(1-k)F(x)}.
\]

It remains to prove that there exists a finite \( M_0 \) such that the \( M_0 \) first conditions are sufficient. This comes from the Hilbert Basis Theorem, and more precisely the Ascending Chain Condition (see [7]) applied to the ascending chain of ideals \( I_j = \langle P_k(0) - Q_k(0), \; 0 \leq k \leq j \rangle \). Then there exists an \( M_0 \geq 0 \) such that \( I_{M_0} = I_{M_0+1} = \cdots = I_\infty \).

4. Efficiency of the RCA

The original C-Algorithm which is based on (9) will be denoted by \( A_0 \). In this section we study the efficiency of the algorithms resulting from Theorems 2 and 3, that will be denoted by \( A_2 \), \( A_3 \), \( A_4 \) and \( A_5 \):

- \( A_1 \) is a truncated C-algorithm. It is based on the formulas (9), for which we apply the truncation procedure using power series.
- \( A_2 \) is the algorithm based on the formulas (10) of Theorem 2 where the truncation procedure is applied, that is ReCA.
- \( A_3 \) is the algorithm based on the formulas (10) of Theorem 2 without truncations.
- \( A_4 \) is the algorithm based on the formulas (11) of Theorem 3 where the truncation procedure is applied, that is RCA.
- \( A_5 \) is the algorithm based on the formulas (11) of Theorem 3 without truncations.
To compare the efficiency of the above 6 algorithms we will apply them to the quartic system
\[
\begin{align*}
\dot{x} &= -y + a_{1,1} xy + a_{2,1} x^2 y + a_{3,1} x^3 y \\
\dot{y} &= x + b_{2,0} x^2 + b_{3,0} x^3 + b_{0,2} y^2 + b_{1,2} x y^2 + b_{2,2} x^2 y^2 + b_{4,0} x^4
\end{align*}
\] (13)

that is system (15) from [2] or system (3.1) from [1]. By standard reduction this system is reducible to Liénard type equation (1) with
\[
f(x) = \frac{b_{0,2} + b_{1,2} x + b_{2,2} x^2 + a_{1,1} + 2 a_{2,1} x + 3 a_{3,1} x^2}{1 - a_{1,1} x - a_{2,1} x^2 - a_{3,1} x^3}
\]
and
\[
g(x) = (1 - a_{1,1} x - a_{2,1} x^2 - a_{3,1} x^3) \left( x + b_{2,0} x^2 + b_{3,0} x^3 + b_{4,0} x^4 \right).
\]

There are 18 unknowns, 9 for the \(a_{i,j}\) and \(b_{i,j}\) and 9 for the coefficients of the power series expansion of \(h\) up to order 17 (remember that \(h\) is odd). Then, it is reasonable to compute the conditions (11) at least up to order \(M = 19\). Since the depth of the isochronous center still an open problem for system (13), then investigations need a higher number of necessary conditions. In our comparative study we ask for the first 30 necessary conditions by each of the presented algorithms.

| Order of derivation | C-Algorithm \(A_0\) | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) | \(A_5\) |
|---------------------|-------------------|-------|-------|-------|-------|-------|
| 10                  | 230.8             | 0.52  | 0.0   | 6.7   | 0.0   | 5.7   |
| 15                  | 5920.2            | 19.1  | 0.6   | 523.4 | 0.4   | 520.0 |
| 20                  | 1168.6            | 4.6   |       |       |       |       |
| 30                  | 168.7             |       |       |       |       | 84.9  |

Table 1: CPU time in seconds on Pentium 2,4 GHz with 4 Gb of memory

The superiority of ReCA and RCA is obvious as well as the role of the truncation. The absence of values means that in that case the computations failed by lack of memory.

5. Examples and comments

Let us recall that the isochronicity problem for planar cubic systems (linear center perturbed by cubic nonlinearity) is still open. This fact is due to the huge number of parameters (14 parameters). In the same time, several recent works have proven the power of the algorithmic methods in the characterization of isochronous centers. For instance we quote the normal forms approach established by V. G. Romanovsky [11] and used in several of his coauthored
papers \[4, 8, 9\]. Particularly, this method has proven its performance in the study of time-reversible isochronous centers. Indeed, the paper [4] contains the complete set of time reversible isochronous centers of linear center perturbed by cubic nonlinearity. Hence the cubic isochronicity problem is still open only in the case of non time-reversible systems.

Using Algorithm RCA, we succeeded to establish several new cubic and quartic isochronous centers [1, 2]. Among others, we found in [1] three families of new non time-reversible cubic isochronous systems:

\[
\dot{x} = -y - 2 b_{2,0} x y + x^2 + 2 b_{2,0} x^3 \\
\dot{y} = x - 4 b_{2,0} y^2 - 2 x y + b_{2,0} x^2 + 4 b_{2,0} x^2 y + 2 x^3
\]

(14)

\[
\dot{x} = -y \pm 2 \sqrt{2} x y + x^2 \mp 2 \sqrt{2} x^3 \\
\dot{y} = x \pm 8 \sqrt{2} y^2 - 2 x y \mp 3 \sqrt{2} x^2 \mp 12 \sqrt{2} x^2 y + 10 x^3
\]

(15)

\[
\dot{x} = -y - \frac{1}{2} b_{2,0} x y + x^2 + \frac{1}{2} b_{2,0} x^3 \\
\dot{y} = x - b_{2,0} y^2 - 2 x y + b_{2,0} x^2 + b_{2,0} x y + \left( 2 + \frac{1}{4} b_{2,0} \right) x^3
\]

(16)

In [8] time-reversible isochronous centers of homogeneous quartic perturbation of the linear center are completely established. In [1], using RCA we found a large list of new non time-reversible quartic isochronous centers. We also found the following family of systems, for which we conjecture it has an isochronous center at 0:

\[
\dot{x} = -y + \left( \frac{3}{8} - 2 b_{2,2} \right) x^2 y + \left( \frac{1}{16} + b_{2,2} \right) x^3 y + x y \\
\dot{y} = x - \frac{3 x^2}{4} + \frac{y^2}{4} + \frac{3 x^3}{8} - 2 b_{2,2} x y^2 + b_{2,2} x^2 y^2 - \frac{x^4}{16}
\]

(17)

We were able to prove the isochronicity of the system in few particular cases, for instance for \( b_{2,2} \in \{-\frac{1}{16}, 0, \frac{1}{16}\} \), by computing explicitly the Urabe function:

\[
h_{\{b_{2,2}(\xi) = \frac{1}{16}\}} = \frac{\sqrt{2} \sqrt{2 \xi^2 + 32 \left( \xi^2 + 12 \right) \left( \xi^2 + 4 \right)}}{2 \left( \xi^2 + 4 \right) \left( \xi^2 + 16 \right)},
\]

\[
h_{\{b_{2,2}(\xi) = -\frac{1}{16}\}} = \frac{\sqrt{2} \left( 2 L \left( \frac{\xi^2}{4} \right) + 8 \sqrt{\frac{\xi^2}{4} L \left( \frac{\xi^2}{4} \right) + 3} \right) \left( \frac{\xi^2}{4} \right)}{2 \xi \left( L \left( \frac{\xi^2}{4} \right) + 4 \right) \left( L \left( \frac{\xi^2}{4} \right) + 1 \right)},
\]

where \( L = LambertW \) is the Lambert function (see [10]),

\[
h_{\{b_{2,2}(\xi) = 0\}} = \frac{\sqrt{2} \sqrt{-4 + \xi^2 + 2 \sqrt{4 + 2 \xi^2} \xi \xi^2 + 2 \sqrt{4 + 2 \xi^2} + 2}}{(2 + \xi^2) \left( \sqrt{4 + 2 \xi^2} + 6 \right)}.
\]
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