A MATRIX REALIGNMENT METHOD FOR RECOGNIZING ENTANGLEMENT

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Motivated by the Kronecker product approximation technique, we have developed a very simple method to assess the inseparability of bipartite quantum systems, which is based on a realigned matrix constructed from the density matrix. For any separable state, the sum of the singular values of the matrix should be less than or equal to 1. This condition provides a very simple, computable necessary criterion for separability, and shows powerful ability to identify most bound entangled states discussed in the literature. As a byproduct of the criterion, we give an estimate for the degree of entanglement of the quantum state.

Keywords: separability, density matrix, bipartite quantum system

1. Introduction
Quantum entangled states have recently emerged as basic resources in the rapidly expanding field of quantum information processing, with remarkable applications such as quantum teleportation, cryptography, dense coding and parallel computation. However, two fundamental questions have only been partially answered: how do we know if a given quantum state is entangled, and how entangled is it still after interacting with a noisy environment?

From a practical point of view, the state of a composite quantum system is called "unentangled" or "separable" if it can be prepared in a "local" or "classical" way. A separable bipartite system can be expressed as an ensemble realization of pure product states $|\psi_i\rangle_A |\phi_i\rangle_B$ occurring with a certain probability $p_i$:

$$\rho_{AB} = \sum_i p_i \rho^A_i \otimes \rho^B_i, \quad (1)$$

where $\rho^A_i = |\psi_i\rangle_A \langle \psi_i|$, $\rho^B_i = |\phi_i\rangle_B \langle \phi_i|$, $\sum_i p_i = 1$, and $|\psi_i\rangle_A$, $|\phi_i\rangle_B$ are normalized pure states of subsystems $A$ and $B$, respectively. If no convex linear combination exists for a given $\rho_{AB}$, then the state is called "entangled".

There have been considerable efforts in recent years to analyze the separability and quantitative character of quantum entanglement. For a pure state $\rho_{AB}$, it is separable iff $\rho_{AB} \equiv \rho_A \otimes \rho_B$ where $\rho_A$ is the reduced density matrix defined as $\rho_A = Tr_B \rho_{AB}$ and $\rho_B = Tr_A \rho_{AB}$. However, for a generic mixed state $\rho_{AB}$, finding a decomposition as in Eq. (1) or proving that it does not exist is a non-trivial task (see [5] and references therein). The first important breakthrough was made by Peres who proposed that partial transposition...
with respect to one subsystem of the density matrix for a separable state is positive, i.e., has non-negative eigenvalues \([6]\). Now known as the \(PPT\) criterion, this was shown by Horodecki \textit{et al} to be sufficient for bipartite systems of \(2 \times 2\) and \(2 \times 3\) \([7]\). In the same paper, a necessary and sufficient condition for separability was found by establishing a close connection between positive map theory and separability. Soon after, Wootters succeeded in computing the “\textit{entanglement of formation}” \([8]\) and thus obtained a separability criterion for \(2 \times 2\) mixtures \([9]\).

The “\textit{reduction criterion}” proposed independently in \([10]\) and \([11]\) gives another necessary criterion which is equivalent to the \(PPT\) criterion for \(2 \times n\) composite systems but is generally weaker than the \(PPT\) criterion. Pittenger \textit{et al} gave also a sufficient criterion for separability connected with the Fourier representations of density matrices \([12]\). Later, Nielsen \textit{et al} \([13]\) presented another necessary criterion called the \textit{majorization criterion}: the decreasingly ordered vector of the eigenvalues for \(\rho_{AB}\) is majorized by that of \(\rho_A\) or \(\rho_B\) alone for a separable state. A new method of constructing \textit{entanglement witnesses} for detecting entanglement was given in \([7]\) and \([14, 15]\). There are also some necessary and sufficient criteria of separability for low rank cases of the density matrix, as shown in \([16, 17]\). In addition, it was shown in \([18]\) and \([19]\) that a necessary and sufficient separability criterion is also equivalent to certain sets of equations.

However, despite these advances, a practical computable criterion for generic bipartite systems is mainly limited to the \(PPT\), reduction and majorization criteria, as well as a recent extension of the \(PPT\) criterion based on semidefinite programs \([20]\). The \(PPT\) criterion has been considered a strong one up to now, but in general it is still not sufficient for higher dimensions of greater than or equivalent to \(3\). Several counterexamples of \textit{bound entangled states} with \(PPT\) properties were provided in \([21]\), and their entanglement does not seem to be “useful” for distillation \([22]\). These states are “weakly” inseparable and it is very hard to establish with certainty their inseparability \([11]\).

In this paper we focus on an inseparability test for a generic bipartite quantum system of arbitrary dimensions. The basic procedure is the same as that of the cross norm criterion presented in \([23]\) but is mathematically much more straightforward. Motivated by the Kronecker product approximation of the density matrix \([24, 25]\) we have derived a directly computational method to recognize entangled states based on a realigned matrix constructed from the density matrix. An estimate is also given for depicting the “\textit{distance}” from a quantum state to the maximally entangled state and to the maximally mixed state. Several typical examples of bound entangled states are shown to be recognized by this criterion in Section 3. A brief summary and discussion are given in the last section.

2. A necessary separability criterion based on a matrix realignment method

In this section we will present an inseparability criterion to recognize entangled states based on simple matrix analysis. Some of its characteristics are shown in several propositions. At the same time, we give a possible measure for the degree of entanglement as a by-product. The main tools used are a technique developed by Loan and Pitsianis \([24, 25]\) for the Kronecker product approximation of a given matrix, and some results from matrix analysis (see Chapters 3 and 4 of \([26]\) for a more extensive background). We shall first review some of the notation
and results required in this paper.

**Definition:** For each $m \times n$ matrix $A = [a_{ij}]$, where $a_{ij}$ is the matrix entry of $A$, we define the vector $\text{vec}(A)$ as

$$\text{vec}(A) = [a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}]^T.$$ 

Let $Z$ be an $m \times m$ block matrix with block size $n \times n$. We define a realigned matrix $\tilde{Z}$ of size $m^2 \times n^2$ that contains the same elements as $Z$ but in different positions as

$$\tilde{Z} = \begin{bmatrix}
\text{vec}(Z_{1,1})^T \\
\vdots \\
\text{vec}(Z_{m,1})^T \\
\vdots \\
\text{vec}(Z_{1,m})^T \\
\vdots \\
\text{vec}(Z_{m,m})^T
\end{bmatrix}, \quad (2)$$

so that the singular value decomposition for $\tilde{Z}$ is

$$\tilde{Z} = U \Sigma V^\dagger = \sum_{i=1}^{q} \sigma_i u_i v_i^\dagger, \quad (3)$$

where $U = [u_1 u_2 \cdots u_m] \in \mathbb{C}^{m \times m^2}$ and $V = [v_1 v_2 \cdots v_n] \in \mathbb{C}^{n \times n^2}$ are unitary, $\Sigma$ is a diagonal matrix with elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q \geq 0$ and $q = \min(m^2, n^2)$. In fact, the number of nonzero singular values $\sigma_i$ is the rank $r$ of matrix $\tilde{Z}$, and $\sigma_i$ are exactly the nonnegative square roots of the eigenvalues of $\tilde{Z} \tilde{Z}^\dagger$ or $\tilde{Z}^\dagger \tilde{Z}$ [26]. Based on the above constructions, Loan and Pitsianis obtained the following representation for $Z$

$$Z = \sum_{i=1}^{r} (X_i \otimes Y_i), \quad (4)$$

with $\text{vec}(X_i) = \sqrt{\sigma_i} u_i$ and $\text{vec}(Y_i) = \sqrt{\sigma_i} v_i^\ast$ [24, 25].

For any given density matrix $\rho_{AB}$ we can associate a realigned version $\tilde{\rho}_{AB}$ according to the transformation of Eq. 2. For example, a $2 \times 2$ bipartite density matrix $\rho$ can be transformed as:

$$\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix} \rightarrow \tilde{\rho} = \begin{pmatrix}
\rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\
\rho_{31} & \rho_{41} & \rho_{32} & \rho_{42} \\
\rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\
\rho_{33} & \rho_{34} & \rho_{43} & \rho_{44}
\end{pmatrix}. \quad (5)$$

For any separable system $\rho_{AB}$, we arrive at the following main theorem:

**Theorem:** If an $m \times n$ bipartite density matrix $\rho_{AB}$ is separable, then for the $m^2 \times n^2$ matrix

$$\tilde{\rho} = \begin{pmatrix}
\rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\
\rho_{31} & \rho_{41} & \rho_{32} & \rho_{42} \\
\rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\
\rho_{33} & \rho_{34} & \rho_{43} & \rho_{44}
\end{pmatrix}$$

we can express $\tilde{\rho}$ as a sum of rank-one matrices:

$$\tilde{\rho} = \sum_{i=1}^{r} (X_i \otimes Y_i)$$
the Ky Fan norm $N(\tilde{\rho}_{AB}) \equiv \sum_{i=1}^{q} \sigma_i(\tilde{\rho}_{AB})$, which is the sum of all the singular values of $\tilde{\rho}_{AB}$, is $\leq 1$, or equivalently $\log(N(\tilde{\rho}_{AB})) \leq 0$, where $q = \min(m^2, n^2)$.

**Proof:** Suppose $\rho_{AB}$ has a decomposition of $\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B = \sum_i p_i (U_i^A E_{11}^{(m,m)} U_i^A) \otimes (V_i^B E_{11}^{(n,n)} V_i^B)$ with $0 \leq p_i \leq 1$ satisfying $\sum_i p_i = 1$. Here $E_{ij}^{(k,l)}$ is a $k \times l$ matrix which has entry 1 in position $i,j$ and all other entries as zero. The $U_i^A, B$ are the unitary matrices which diagonalize $\rho_i^A, B$. Applying the properties of Kronecker products:

\[
vec(XYZ) = (Z^T \otimes X)vec(Y), \quad \text{(see } 26\text{)}
\]

\[
Z = X \otimes Y \iff \tilde{Z} = vec(X)vec(Y)^T, \quad \text{(see } 23\text{)}
\]

we have

\[
\rho_i^A \otimes \rho_i^B = vec(\rho_i^A)vec(\rho_i^B)^T
\]

\[
= (U_i^A \otimes U_i^A)vec(E_{11}^{(m,m)})(U_i^B \otimes U_i^B)vec(E_{11}^{(n,n)}))^T
\]

\[
= (U_i^{A*} \otimes U_i^A)vec(E_{11}^{(m,m)})(U_i^{B*} \otimes U_i^B)vec(E_{11}^{(n,n)})^T (U_i^B \otimes U_i^{BT}).
\]

(6)

Since $U_i^A$ and $U_i^B$ are unitary, it is evident that $U_i^{A*} \otimes U_i^A$ and $U_i^{B*} \otimes U_i^B$ are unitary also. Moreover, $vec(E_{11}^{(m,m)})(U_i^B \otimes U_i^{BT})$ has a unique singular value 1. Then $\rho_i^A \otimes \rho_i^B$ has also a unique singular value 1 due to the fact that the Ky Fan norm is unitarily invariant [26]. Therefore, as a convex linear combination of $\rho_i^A \otimes \rho_i^B$, $\tilde{\rho}_{AB}$ should have the Ky Fan norm $N(\tilde{\rho}_{AB}) \leq \sum_i p_i N(\rho_i^A \otimes \rho_i^B) = \sum_i p_i = 1$. Here we have used the inequality $\sum^q_i \sigma_i(A + B) \leq \sum^q_i \sigma_i(A) + \sum^q_i \sigma_i(B)$ where $A$ and $B$ are $k \times l$ matrices and $q = \min(k, l)$ [26].

The cross norm criterion proposed in [23] leads to the same result as our above Theorem, but with the expressions in Dirac bra-ket notation. Here we obtain identical results but from simple matrix analysis. Moreover, our approach has special advantages in its concise and explicit expressions: the two factor spaces $A$ and $B$ just correspond to different rows and columns of $\tilde{\rho}_{AB}$ under the transformation $\rho_{AB} \rightarrow \tilde{\rho}_{AB}$. Also the technology for the proof is much simpler compared with the complicated operator algebra used in [23]. Now we only need to rearrange the entries of $\rho_{AB}$ according to Eq. (2), then compare the sum of the square roots of the eigenvalues for $\rho_{AB}^T \rho_{AB}^T$ (i.e., the Ky Fan norm or trace norm of $\tilde{\rho}_{AB}$) with 1. The separability criterion is strong enough to be sufficient for many lower dimensional systems. It is straightforward to verify sufficiency for the Bell diagonal states [3], Werner states in dimension $d = 2$ [27], and isotropic states in arbitrary dimensions [10], as was done in [23]. So far we have not yet found a close connection between this criterion and the PPT criterion, but from Eq. (2) we have the compatibility relation $N(\tilde{\rho}) = N(\rho^{TA})$, since $\tilde{\rho}$ and $\rho^{TA}$ only differ up to a series of elementary row transformations which are naturally unitary and keep the Ky Fan norm invariant. For pure states, we have the following stronger conclusion:

**Proposition 1:** A bipartite pure state is separable iff $\tilde{\rho}_{AB}$ has a unique singular value 1.

**Proof:** Necessity: Given a bipartite separable pure state we have $\rho_{AB} = \rho^A \otimes \rho^B$. From Eq. (6), $\tilde{\rho}_{AB}$ has a unique singular value 1.
Sufficiency: \( \tilde{\rho}_{AB} \) has a unique singular value \( \sigma_1 = 1 \), so \( \tilde{\rho}_{AB} = \sigma_1 u_1 v_1^\dagger = u_1 v_1^\dagger \). By Eq. (4), we have \( \rho_{AB} = \alpha \otimes \beta \) with \( \text{vec}(\alpha) = u_1 \) and \( \text{vec}(\beta) = v_1 \). Moreover, \( \rho_{AB} = \tilde{\rho}_{AB} \implies \alpha \otimes \beta = \alpha^2 \otimes \beta^2 \). Thus the eigenvalues of \( \alpha \) and \( \beta \) should both be 1 or -1 at the same time because \( \rho_{AB} \) has only one eigenvalue 1. This yields \( \rho^A = \alpha \) and \( \rho^B = \beta \) if \( \alpha \) and \( \beta \) have eigenvalue 1, and \( \rho^A = -\alpha \) and \( \rho^B = -\beta \) if \( \alpha \) and \( \beta \) have eigenvalue -1. Therefore \( \rho_{AB} \) is a pure separable density matrix.

Now we derive a dual criterion based on the Theorem:

**Corollary:** For a separable \( m \times n \) system \( \rho_{AB} \), a permuted version \( \rho_{BA} \) which exchanges the first and second factor spaces of \( \rho_{AB} \) can be defined as

\[
\rho_{BA} = S(n,m) \rho_{AB} S(m,n),
\]

where \( S(m,n) = \sum_{i=1}^m \sum_{j=1}^n E_{ij}^{(m,n)} \otimes (E_{ij}^{(m,n)})^T \). Then for the \( n^2 \times m^2 \) realigned matrix \( \tilde{\rho}_{BA} \), the Ky Fan norm \( N(\tilde{\rho}_{BA}) \) is \( \leq 1 \) or equivalently \( \log(N(\tilde{\rho}_{BA})) \leq 0 \).

**Proof:** For a separable \( m \times n \) system \( \rho_{AB} \), there exists a decomposition: \( \rho_{AB} = \sum_i p_i \rho^A_i \otimes \rho^B_i \).

Using the property \( Y \otimes X = S(n,m)(X \otimes Y)S(m,n) \) where \( X \) is an \( m \times m \) matrix and \( Y \) is an \( n \times n \) one [20], we have \( \rho_{BA} = S(n,m)(\sum_i p_i \rho^A_i \otimes \rho^B_i)S(m,n) = \sum_i p_i \rho^B_i \otimes \rho^A_i \). This density matrix is also separable but associated with an \( n \times m \) system, so we have \( N(\tilde{\rho}_{BA}) \leq 1 \) according to the Theorem.

This criterion is equivalent to the Theorem, due to the property \( Y \otimes X = \text{vec}(Y)\text{vec}(X)^T = (\text{vec}(X)\text{vec}(Y)^T)^T = (X \otimes Y)^T \). Thus we have \( \tilde{\rho}_{AB} = (\tilde{\rho}_{BA})^T \) and further \( N(\tilde{\rho}_{AB}) = N(\tilde{\rho}_{BA}) \). This can also be seen from the symmetry with respect to the two subsystems of the cross norm given in [20].

We expect that \( \log N(\tilde{\rho}_{AB}) \) may depict a possible measure of entanglement for the corresponding system. In fact, for the \( d \)-dimension maximally mixed state \( \rho_m = Id/d^2 \) it is easy to derive \( \tilde{\rho}_m = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle \langle jj| \) and \( \log N(\tilde{\rho}_m) = -\log d \), while for the maximally entangled state \( \rho_c = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle \langle jj| \) we have \( \tilde{\rho}_c = Id/d^2 \) and \( \log N(\tilde{\rho}_c) = \log d \). Noticing the relations \( \rho_m = \tilde{\rho}_c \) and \( \rho_m = \tilde{\rho}_c \), we see that they are dual and \( \log N(\tilde{\rho}) \) is distributed symmetrically about 0 where the separable pure states are located. Hence any bipartite quantum state \( \rho \) can be depicted by such an approximate measure \( \log N(\tilde{\rho}) \) representing its "distance" from \( \rho_m \) and \( \rho_c \). Furthermore we have the following property for \( \log N(\tilde{\rho}) \):

**Proposition 2:** Applying a local unitary transformation leaves \( \log N(\tilde{\rho}_m) \) invariant, i.e.

\[
\log N(\tilde{\rho}_{AB}) = \log N(\tilde{\rho}_{AB}),
\]

where \( \rho_{AB}' = (U \otimes V) \rho_{AB}(U^\dagger \otimes V^\dagger) \) and \( U, V \) are unitary operators acting on the \( A,B \) subsystems, respectively.

**Proof:** From Eq. (4), we have a decomposition of \( \rho_{AB} = \sum_{i=1}^q \alpha_i \otimes \beta_i \), where \( q = \min(m^2, n^2) \).
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Setting \( \gamma_i = (U \otimes V)(\alpha_i \otimes \beta_i)(U^\dagger \otimes V^\dagger) \), we have

\[
\tilde{\gamma}_i = \text{vec}(U\alpha_i U^\dagger)\text{vec}(V\beta_i V^\dagger)^T
= (U^* \otimes U)(\text{vec}(\alpha_i)\text{vec}(\beta_i))(V^\dagger \otimes V^T).
\]

(9)

Summing all the components \( \tilde{\gamma}_i \), we have

\[
\tilde{\rho}_{AB} = (U^* \otimes U)\tilde{\rho}_{AB}(V^\dagger \otimes V^T).
\]

Since \( U \) and \( V \) are unitary, it is evident that \( U^* \otimes U \) and \( V^\dagger \otimes V^T \) are unitary, hence \( \log N(\tilde{\rho}_{AB}) = \log N(\tilde{\rho}_{AB}) \) since \( N(\tilde{\rho}) \) is a unitary invariant norm. \( \square \)

In addition, like any unitary invariant norm, \( N(\tilde{\rho}) \) is convex [26], but \( \log N(\tilde{\rho}_{AB}) \) is not convex due to the concave logarithmic function. Therefore, this rough “measure” is not exact because the criterion is not a necessary and sufficient one, but it depicts the entanglement to some degree in a subtle way independent of the PPT criterion.

3. Application of the criterion to bound entangled states

In this section we give some typical examples to display the power of our separability criterion to distinguish the bound entangled states from separable states.

**Example 1:** 3 \times 3 bound entangled states constructed from unextendible product bases (UPB)

In [28], Bennett et al introduced a 3 \times 3 inseparable bound entangled state from the following bases:

\[
|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle,
|\psi_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |0\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle,
|\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle),
\]

from which the density matrix can be expressed as

\[
\rho = \frac{1}{4}(I + \sum_{i=0}^{4}|\psi_i\rangle\langle\psi_i|).
\]

(10)

Direct computation gives \( \log N(\tilde{\rho}) \approx 0.121 \), which shows that there is slight entanglement in this state.

When the UPB have the following construction [28]

\[
|\psi_j\rangle = |\vec{v}_j\rangle \otimes |\vec{v}_j \mod 5\rangle, \quad j = 0, \ldots, 4
\]

where the vectors \( \vec{v}_j \) are \( \vec{v}_j = N(\cos \frac{2\pi j}{5}, \sin \frac{2\pi j}{5}, h) \), with \( j = 0, \ldots, 4 \) and \( h = \frac{1}{3} \sqrt{1 + \sqrt{5}} \) and \( N = 2/\sqrt{5 + \sqrt{5}} \), then the PPT state of Eq. (10) gives \( \log N(\tilde{\rho}) \approx 0.134 \), which identifies this bound entangled state.

**Example 2:** Horodecki 3 \times 3 bound entangled state
Horodecki gives a very interesting weakly inseparable state in [21] which cannot be detected by the PPT criterion. The density matrix $\rho$ is real and symmetric:

$$
\rho = \frac{1}{8a+1} \begin{bmatrix}
a & 0 & 0 & 0 & a & 0 & 0 & a \\
a & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & a & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a & \frac{\sqrt{1-a^2}}{2} \\
a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2}
\end{bmatrix},
$$

where $0 < a < 1$. To compare our result with the computation for the entanglement of formation in [29], we consider a mixture of the Horodecki state and the maximally mixed state $\rho_p = p \rho + (1-p) I/9$, and show the function

$$
f = \max(0, \log N(\tilde{\rho}_p))
$$

in Fig. 1.

![Fig. 1. Estimate of degree of entanglement for a Horodecki 3 x 3 bound entangled state. When $a = 0.236$ we have maximal entanglement of the state for $0.9955 < p \leq 1$.](image)

We see that $\log N(\tilde{\rho}_p) > 0$ when $p = 1$ and $0 < a < 1$, which completely identifies these bound entangled states. Furthermore, when $a = 0.236$, $\log N(\tilde{\rho}_p)$ has a maximum of 0.0044 and follows a similar trend to that of the degree of entanglement for $\rho_p$ (with a maximum at $a = 0.225$) given in [29]. It can be seen that we obtain an upper bound $p = 0.9955$ for $\rho_p$ which still has entanglement when $a = 0.236$.

To test the criterion, we programmed a routine to analyze various PPT entangled states described in the literature. On a 600MHz desktop computer we performed a systematic search by checking $10^5$ randomly chosen examples of the seven parameter family of PPT entangled states in [30] within half an hour. The Theorem could detect entanglement in about 21% of
these bound entangled states satisfying $\rho = \rho^T$ and gave a maximum value for $\log N(\tilde{\rho})$ of 0.22, which is a convincing demonstration of the strength of the criterion.

**Remark 1:**

One might expect the Theorem to be equivalent to the PPT criterion for a $2 \times 2$ system. Unfortunately, this is not the case, and certain $2 \times 2$ entangled states cannot be recognized by our criterion. We take the following example from [7] which is a separable state only when $p = \frac{1}{2}$ or $a = 0$ or $a = 1$:

$$
\rho = \begin{pmatrix}
    pa^2 & 0 & 0 & pab \\
    0 & (1-p)a^2 & (1-p)ab & 0 \\
    0 & (1-p)ab & (1-p)b^2 & 0 \\
    pab & 0 & 0 & pb^2
\end{pmatrix}
$$  \hspace{1cm} \text{(13)}

where $a, b > 0$ and $|a|^2 + |b|^2 = 1$. We plot the value of $f = \max(0, \log N(\tilde{\rho}))$ in Fig. 2 with respect to $a$ and $p$. We can see that, apart from when $p = \frac{1}{2}$ or $a = 0$ or 1, there are still some regions where $f = 0$. Thus these $2 \times 2$ entangled states cannot be completely detected by our criterion.

**Remark 2:**

It is interesting to make a comparison between $\log N(\tilde{\rho})$ or $N(\tilde{\rho}) - 1$ with the entanglement of formation $E_f$ [8]. By direct comparison, for the density matrix of Eq. 11 $\log N(\tilde{\rho})$ or $N(\tilde{\rho}) - 1$ is less than the entanglement of formation calculated.
As for the state of a $2 \times 2$ system, we can conveniently calculate the entanglement of formation by Wootter’s formula \[9\]. By straightforward computation, we have

1. For $2 \times 2$ entangled Werner states \[27\]: \( \log N(\bar{\rho}) \geq E_f = N(\bar{\rho}) - 1 \);

2. For the density matrix of Eq. \[13\]: \( N(\bar{\rho}) - 1 \leq E_f \) but there is no definite relation between \( \log N(\bar{\rho}) \) and \( E_f \);

3. For $3 \times 3$ entangled isotropic states \[10\]: \( N(\bar{\rho}) - 1 \geq E_f \) and \( \log N(\bar{\rho}) \geq E_f \).

Thus we find that there is no ordered relationship between \( E_f \) and \( N(\bar{\rho}) - 1 \) or \( \log N(\bar{\rho}) \).

4. Summary and discussion

Summarizing, we have presented a matrix realignment method to assess the separability of the density matrix for a bipartite quantum system in arbitrary dimensions. The criterion provides a necessary condition for separability and is quite easy to compute. It shows remarkable ability to recognize most known bound entangled states in the literature where the PPT test fails. Moreover, it gives a rough estimate for the degree of entanglement. We also show that this criterion is not equivalent to PPT even for $2 \times 2$ quantum states. Moreover, there is no definite relationship between our estimate and the entanglement of formation. However, in combination with the PPT criterion, we can significantly expand our ability to distinguish directly the entanglement and separability of any quantum state in arbitrary dimensions. Our method has recently been developed further by a linear contraction approach that gives a generic separability criterion \[31\]. It is expected that this method will find more applications in the study of multipartite quantum systems and other problems in quantum information theory and quantum computation. We leave the explicit physical meaning of the criterion as an open question that awaits further study.

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