Clock Synchronization, Dirac Observables and Gauge Variables in Canonical Gravity and the Objectivity of Spacetime.

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Abstract. This is a review of the chrono-geometrical structure of special and general relativity with a special emphasis on the role of non-inertial frames and of the conventions for the synchronization of distant clocks. ADM canonical metric and tetrad gravity are analyzed in a class of space-times suitable to incorporate particle physics by using Dirac theory of constraints, which allows to arrive at a separation of the genuine degrees of freedom of the gravitational field, the Dirac observables describing generalized tidal effects, from its gauge variables, describing generalized inertial effects. A background-independent formulation (the rest-frame instant form of tetrad gravity) emerges, since the chosen boundary conditions at spatial infinity imply the existence of an asymptotic flat metric. By switching off the Newton constant in presence of matter this description deparametrizes to the rest-frame instant form for such matter in the framework of parametrized Minkowski theories. The problem of the objectivity of the space-time point-events, implied by Einstein’s Hole Argument, is analyzed.

1. The Chrono-Geometrical Structure of Special Relativity
In special relativity the chrono-geometrical structure of Minkowski space-time is non dynamical. It replaces the notions of absolute time and instantaneous Euclidean 3-space of the Galilei space-time underlying Newtonian physics. The Galilei relativity principle is replaced by the relativistic one. In both cases ideal inertial observers with their associated inertial frames, employing Cartesian coordinates, and connected by a kinematical group (either Galilei or Poincare’) of transformations are privileged. The light postulates state that the two-way (or round trip) velocity of light \( c \) (only one clock is needed in its definition) is constant and isotropic.

The Lorentz signature of Minkowski 4-metric tensor implies that every time-like observer can identify the light-cone (the conformal structure, i.e. the locus of the trajectories of light rays) in each point of the world-line. However there is no notion of an instantaneous 3-space, of a spatial distance and of a one-way velocity of light between two observers (the problem of the synchronization of distant clocks). Since the relativity principle privileges inertial observers and Cartesian coordinates \( x^\mu = (x^\alpha = ct; \vec{x}) \) with the time axis centered on them (inertial frames), the \( x^\alpha = const. \) hyper-planes of inertial frames are usually taken as Euclidean instantaneous 3-spaces, on which all the clocks are synchronized. Indeed they can be selected with Einstein’s convention for the synchronization of distant clocks to the clock of an inertial observer. This inertial observer \( A \) sends a ray of light at \( x^\alpha_i \) to a second accelerated observer \( B \), who reflects

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it towards A. The reflected ray is reabsorbed by the inertial observer at $x^\gamma_0$. The convention states that the clock of B at the reflection point must be synchronized with the clock of A when it signs $\frac{1}{2} (x^\sigma_0 + x^\sigma_1)$. This convention selects the $x^\sigma = \text{const.}$ hyper-planes of inertial frames as simultaneity 3-spaces and implies that with this synchronization the two-way and one-way velocities of light coincide and the spatial distance between two simultaneous point is the (3-geodesic) Euclidean distance.

However, real observers are never inertial and for them Einstein’s convention for the synchronization of clocks is not able to identify globally defined simultaneity 3-surfaces, which could also be used as Cauchy surfaces for Maxwell equations. The 1+3 point of view tries to solve this problem starting from the local properties of an accelerated observer, whose world-line is assumed to be the time axis of some frame. Since only the observer 4-velocity is given, this only allows to identify the tangent plane of the vectors orthogonal to this 4-velocity in each point of the world-line. Then, both in special and general relativity, this tangent plane is identified with an instantaneous 3-space and 3-geodesic Fermi coordinates are defined on it and used to define a notion of spatial distance. However this construction leads to coordinate singularities, because the tangent planes in different points of the world-line will intersect each other at distances from the world-line of the order of the (linear and rotational) acceleration radii of the observer. Another type of coordinate singularity arises in all the proposed uniformly rotating coordinate systems: if $\omega$ is the constant angular velocity, then at a distance $r$ from the rotation axis such that $\omega r = c$, the $g_{\theta \theta}$ component of the induced 4-metric vanishes. This is the so-called horizon problem for the rotating disk: the time-like 4-velocity of an observer sitting on a point of the disk becomes light-like in this coordinate system when $\omega r = c$.

See Ref.[1] for a review of these topics and for the locality hypothesis [standard clocks and rods do not feel acceleration and at each instant the detectors of the instantaneously comoving inertial observer give the correct data].

This state of affairs and the need of predictability (a well-posed Cauchy problem for field theory) lead to the necessity of abandoning the 1+3 point of view and to shift to the 3+1 one. In this point of view, besides the world-line of an arbitrary time-like observer, it is given a 3+1 splitting of Minkowski space-time, namely a foliation of it whose leaves are space-like hyper-surfaces. Each leaf is both a Cauchy surface for the description of physical systems and an instantaneous (in general Riemannian) 3-space, namely a notion of simultaneity implied by a clock synchronization convention different from Einstein’s one. Even if it is unphysical to give initial data on a non-compact space-like hyper-surface, this is the only way to be able to use the existence and uniqueness theorem for the solutions of partial differential equations. The extra structure of the 3+1 splitting of Minkowski space-time allows to enlarge its atlas of 4-coordinate systems with the definition of Lorentz-scalar observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$, $A = \tau, r$. Here $\tau$ is the proper time of the accelerated observer or any monotonically increasing function of it, and is used to label the simultaneity leaves $\Sigma_{\tau}$ of the foliation. On each leaf $\Sigma_{\tau}$ the point of intersection with the world-line of the accelerated observer is taken as the origin of curvilinear 3-coordinates $\sigma^r$, which can be assumed to be globally defined since each $\Sigma_{\tau}$ is diffeomorphic to $R^3$. To the coordinate transformation $x^\mu \mapsto \sigma^A$ ($x^\mu$ are the standard Cartesian coordinates) is associated an inverse transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$, where the functions $z^\mu(\tau, \sigma^r)$ describe the embedding of the simultaneity surfaces $\Sigma_{\tau}$ into Minkowski space-time. The 3+1 splitting leads to the following induced 4-metric (a functional of the embedding): $4g_{AB}(\tau, \sigma^r) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} = 4g_{AB}[z(\sigma)]$, where $4g_{\mu\nu} = \epsilon (+ - -)$ with $\epsilon = \pm 1$ according to particle physics or general relativity convention respectively. The quantities $\frac{\partial z^\mu}{\partial \sigma^A}$ are cotetrad fields on Minkowski space-time.

An admissible 3+1 splitting of Minkowski space-time must have the embeddings $z^\mu(\tau, \sigma^r)$ of the space-like leaves $\Sigma_{\tau}$ of the associated foliation satisfying the Møller conditions on
the coordinate transformation \[2\]: \( \epsilon^4 g_{\tau\tau}(\sigma) > 0, \epsilon^4 g_{rr}(\sigma) < 0, \left| \begin{array}{cc} 4 g_{\tau\tau}(\sigma) & 4 g_{rr}(\sigma) \\ 4 g_{\tau\sigma}(\sigma) & 4 g_{ss}(\sigma) \end{array} \right| > 0, \epsilon \det[4 g_{rs}(\sigma)] < 0. \) Moreover, the requirement that the foliation be well defined at spatial infinity may be satisfied by asking that each simultaneity surface \( \Sigma_\tau \) tends to a space-like hyperplane there, namely we must have \( z^\mu(\tau, \sigma^r) \rightarrow x^\mu(0) + \epsilon A^\mu(0) \sigma^r \) for some set of orthonormal asymptotic tetrads \( \epsilon_{\mu}^A \).

As a consequence, any admissible 3+1 splitting leads to the definition of a \textit{non-inertial frame centered on the given time-like observer} and coordinatized with Lorentz-scalar observer-dependent radar 4-coordinates. While inertial frames centered on inertial observers are connected by the transformations of the Poincare' group, the non-inertial ones are connected by passive frame-preserving diffeomorphism: \( \tau \mapsto \tau'(\tau, \sigma^r), \sigma^r \mapsto \sigma'^r(\sigma^r) \). It turns out that Møller conditions forbid uniformly rotating non-inertial frames: only differentially rotating ones are allowed (the ones used by astrophysicists in the modern description of rotating stars). In Ref.[1] there is a detailed discussion of this topic and there is the simplest example of 3+1 splittings whose leaves are space-like hyper-planes carrying admissible differentially rotating 3-coordinates. Moreover, it is shown that to each admissible 3+1 splitting are associated two congruences of time-like observers (the natural ones for the given notion of simultaneity): i) the Eulerian observers, whose unit 4-velocity field is the field of unit normals to the simultaneity surfaces \( \Sigma_\tau \); ii) the observers whose unit 4-velocity field is proportional to the evolution vector field of components \( \partial z^\mu(\tau, \sigma^r)/\partial \tau \); in general this congruence is non-surface forming having a non-vanishing vorticity (like the congruence associated to a rotating disk).

The next problem is how to describe physical systems in non-inertial frames and how to connect different conventions for clock synchronization. The answer is given by \textit{parametrized Minkowski theories} (see Ref.[3]). Given any isolated system (particles, strings, fields, fluids) admitting a Lagrangian description, one makes the coupling of the system to an external gravitational field and then replaces the 4-metric \( 4 g_{\mu\nu}(x) \) with the induced metric \( 4 g_{AB}[z(\tau, \sigma^r)] \) associated to an arbitrary admissible 3+1 splitting. The Lagrangian now depends not only on the matter configurational variables but also on the embedding variables \( z^\mu(\tau, \sigma^r) \). Since the action principle turns out to be invariant under frame-preserving diffeomorphisms, at the Hamiltonian level there are four first-class constraints. As a consequence, Dirac’s theory of constraints implies that the configuration variables \( z^\mu(\tau, \sigma^r) \) are arbitrary \textit{gauge variables}. Therefore, all the admissible 3+1 splittings, namely all the admissible conventions for clock synchronization, and all the admissible non-inertial frames centered on time-like observers are \textit{gauge equivalent}. By adding four gauge-fixing constraints \( \chi^\mu(\tau, \sigma^r) = z^\mu(\tau, \sigma^r) - z^\mu_M(\tau, \sigma^r) \approx 0 \) (\( z^\mu_M(\tau, \sigma^r) \) being an admissible embedding), satisfying an orbit condition, we identify the description of the system in the associated inertial frame centered on a given time-like observer. The resulting effective Hamiltonian for the \( \tau \)-evolution turns out to contain the potentials of the \textit{relativistic inertial forces} present in the given non-inertial frame. Since a non-inertial frame means the use of its radar coordinates, we see that already in special relativity \textit{non-inertial Hamiltonians are coordinate-dependent quantities} like the notion of energy density in general relativity. Therefore, the gauge variables \( z^\mu(\tau, \sigma^r) \) describe the \textit{spatio-temporal appearances} of the phenomena in non-inertial frames, which, in turn, are associated to extended physical laboratories using a metrology for their measurements compatible with the notion of simultaneity of the non-inertial frame (think to the description of the Earth given by GPS).

Inertial frames centered on inertial observers are a special case of gauge fixing in parametrized Minkowski theories. For each configuration of an isolated system there is an special 3+1 splitting associated to it: the foliation with space-like hyper-planes orthogonal to the conserved time-like 4-momentum of the isolated system. This identifies an intrinsic inertial frame, the \textit{rest-frame}, centered on a suitable inertial observer (the Fokker-Pryce center of inertia of the isolated system)
and allows to define the Wigner-covariant rest-frame instant form of dynamics for every isolated system. Let us remark that in parametrized Minkowski theories a relativistic particle with world-line $x^\mu_i(\tau)$ is described only by the 3-coordinates $\sigma^r = \eta^r_i(\tau)$ defined by $x^\mu_i(\tau) = z^\mu(\tau, \eta^r_i(\tau))$ and by the conjugate canonical momenta $\kappa^r_i(\tau)$. The usual 4-momentum $p^\mu_i(\tau)$ is a derived quantity satisfying the mass-shell constraint $\epsilon_{\mu\nu}^{ij} p^\mu_i(\tau) p^\nu_j(\tau) = m^2_i$. Therefore, we have a different description for positive- and negative-energy particles. All the particles on an admissible surface $\Sigma_\tau$ are simultaneous by construction: this eliminates the problem of relative times, which for a long time has been an obstruction to the theory of relativistic bound states and to relativistic statistical mechanics.

This framework made possible to develop a coherent formalism for all the aspects of relativistic kinematics both for N particle systems and continuous bodies and fields: i) the classification of the intrinsic notions of collective variables (canonical non-covariant center of mass; covariant non-canonical Fokker-Pryce center of inertia; non-covariant non-canonical Møller center of energy); ii) canonical bases of center-of-mass and relative variables; iii) canonical spin bases and dynamical body-frames for the rotational kinematics of deformable systems; iv) multipolar expansions for isolated and open systems; v) the relativistic theory of orbits; vi) the Møller radius (a classical unit of length identifying the region of non-covariance of the canonical center of mass of a spinning system around the covariant Fokker-Pryce center of inertia; it is an effect induced by the Lorentz signature of the 4-metric; it could be used as a physical ultraviolet cutoff in quantization). See Ref. [4] for a comprehensive review and the references to the main related papers.

Let us also remark that, differently from Fermi coordinates (a purely theoretical construction), radar 4-coordinates can be operationally defined. As shown in Ref.[1], given four functions satisfying certain restrictions induced by the Møller conditions, the on-board computer of a spacecraft may establish a grid of radar 4-coordinates in its future.

In Ref.[5] there is the quantization of relativistic scalar and spinning particles in a class of non-inertial frames, whose simultaneity surfaces $\Sigma_\tau$ are space-like hyper-planes with arbitrary admissible linear acceleration and carrying arbitrary admissible differentially rotating 3-coordinates. It is based on a multi-temporal quantization scheme for systems with first-class constraints, in which only the particle degrees of freedom $\eta^r_i(\tau), \kappa^r_i(\tau)$ are quantized. The gauge variables, describing the appearances (inertial effects) of the motion in non-inertial frames, are treated as c-numbers (like the time in the Schrödinger equation with a time-dependent Hamiltonian) and the physical scalar product does not depend on them. The previously quoted relativistic kinematics has made possible to separate the center of mass and to verify that the spectra of relativistic bound states in non-inertial frames are only modified by inertial effects, being obtained from the inertial ones by means of a time-dependent unitary transformation. The non-relativistic limit allows to recover the few existing attempts of quantization in non-inertial frames as particular cases.

The main open problem is the quantization of the scalar Klein-Gordon field in non-inertial frames, due to the Torre and Varadarajan no-go theorem [6], according to which in general the evolution from an initial space-like hyper-surface to a final one is not unitary in the Tomonaga-Schwinger formulation of quantum field theory. From the 3+1 point of view there is evolution only among the leaves of an admissible foliation and the possible way out from the theorem lies in the determination of all the admissible 3+1 splittings of Minkowski space-time satisfying the following requirements: i) existence of an instantaneous Fock space on each simultaneity surface $\Sigma_\tau$ (i.e. the $\Sigma_\tau$’s must admit a generalized Fourier transform); ii) unitary equivalence of the Fock spaces on $\Sigma_{\tau_1}$ and $\Sigma_{\tau_2}$ belonging to the same foliation (the associated Bogoliubov transformation must be Hilbert-Schmidt), so that the non-inertial Hamiltonian is a Hermitean
operator; iii) unitary gauge equivalence of the 3+1 splittings with the Hilbert-Schmidt property. The overcoming of the no-go theorem would help also in quantum field theory in curved space-times and in condensed matter (here the non-unitarity implies non-Hermitian Hamiltonians and negative energies).

2. The Chrono-Geometrical Structure of General Relativity

In the years 1913-16 Einstein developed general relativity relying on the equivalence principle (equality of inertial and gravitational masses of bodies in free fall). This led to the geometrization of the gravitational interaction and to the replacement of Minkowski space-time with a pseudo-Riemannian 4-manifold $M^4$ with non vanishing curvature Riemann tensor. The principle of general covariance, at the basis of the tensorial nature of Einstein’s equations, has the two following consequences: i) the invariance of the Hilbert action under passive diffeomorphisms (the coordinate transformations in $M^4$), so that the second Noether theorem implies the existence of first-class constraints at the Hamiltonian level; ii) the mapping of solutions of Einstein’s equations among themselves under the action of active diffeomorphisms of $M^4$ extended to the tensors over $M^4$ (dynamical symmetries of Einstein’s equations). The basic field of metric gravity is the 4-metric tensor with components $g_{\mu\nu}(x)$ in an arbitrary coordinate system of $M^4$. The peculiarity of gravity is that the 4-metric field, differently from the fields of electromagnetic, weak and strong interactions and from the matter fields, has a double role: i) it is the mediator of the gravitational interaction (in analogy to all the other gauge fields); ii) it determines the chrono-geometric structure of the space-time $M^4$ in a dynamical way through the line element $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. As a consequence, the gravitational field teaches relativistic causality to all the other fields: for instance it tells to classical rays of light and to quantum photons and gluons which are the allowed trajectories for massless particles in each point of $M^4$.

Let us remark that in all known formulations particle and nuclear physics are a chapter of the theory of representations of the Poincare’ group in inertial frames in the spatially non-compact Minkowski space-time. As a consequence, if one looks at general relativity from the point of view of particle physics, the main problem to get a unified theory is how to reconcile the Poincare’ group (the kinematical group of the transformations connecting inertial frames) with the diffeomorphism group implying the non-existence of global inertial frames in general relativity (special relativity holds only in a small neighborhood of a body in free fall).

Let us consider the ADM formulation of metric gravity and its extension to tetrad gravity (needed to describe the coupling of gravity to fermions; it is a theory of time-like observers endowed with a tetrad field, whose time-like axis is the unit 4-velocity and whose spatial axes are associated to a choice of three gyroscopes) obtained by replacing the ten configurational variables $g_{\mu\nu}(x)$ with the sixteen cotetrad fields $E_{\mu}(x)$ by means of the decomposition $g_{\mu\nu}(x) = E_{\mu}(x) \eta^{(\alpha)(\beta)} E_{\nu}^{(\beta)}(x) \eta_{(\alpha)}$. Then, after having restricted the model to globally hyperbolic, topologically trivial, spatially non-compact space-times (admitting a global notion of time), let us introduce a 3+1 splitting of the space-time $M^4$ and let choose the world-line of a time-like observer. As in special relativity, let us make a coordinate transformation to observer-dependent radar 4-coordinates, $x^\mu \mapsto \sigma^A = (\tau, \sigma^r)$, adapted to the 3+1 splitting and using the observer world-line as origin of the 3-coordinates. Again the inverse transformation, $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$, defines the embedding of the leaves $\Sigma_\tau$ into $M^4$. These leaves $\Sigma_\tau$ (assumed to be Riemannian 3-manifolds diffeomorphic to $R^3$, so that they admit global 3-coordinates $\sigma^r$ and a unique 3-geodesic joining any pair of points in $\Sigma_\tau$) are both Cauchy surfaces and simultaneity surfaces corresponding to a convention for clock synchronization. For the induced 4-metric we get $g_{AB}(\sigma) = g_{\mu\nu}(x) \partial z^r(\sigma) \partial x^\mu = E_A^{(\alpha)} \eta_{(\alpha)(\beta)} E_B^{(\beta)} = \epsilon \left( \begin{array}{ccc} N^2 - 3 g_{rs} N^r N^s & -3 g_{ru} N^u & -3 g_{rs} \\ -3 g_{ru} N^u & -3 g_{ru} N^u & -3 g_{rs} \end{array} \right) (\sigma)$. Here $E_A^{(\alpha)}(\tau, \sigma^r)$ are adapted
cotetrad fields, $N(\tau, \sigma^r)$ and $N^r(\tau, \sigma^r)$ the lapse and shift functions and $g_{rs}(\tau, \sigma^r)$ the 3-metric on $\Sigma_\tau$ with signature $(+++)$. We see that in general relativity the quantities $z^\mu_A = \partial z^\mu / \partial \sigma^A$ are no more cotetrad fields on $M^4$ differently from what happens in special relativity: now they are only transition functions between coordinate charts, so that the dynamical fields are now the real cotetrad fields $E^{(a)}_A(\tau, \sigma^r)$ and not the embeddings $z^a(\tau, \sigma^r)$.

Let us try to identify a class of space-times and an associated suitable family of admissible 3+1 splittings able to incorporate particle physics and giving a model for the solar system or our galaxy (and hopefully allowing an extension to the cosmological context) with the following further requirements (see Ref. [7]): 1) $M^4$ must be asymptotically flat at spatial infinity and the 4-metric must tend asymptotically at spatial infinity to the Minkowski 4-metric in every coordinate system (this implies that the 4-diffeomorphisms must tend to the identity at spatial infinity). Therefore, in these space-times there is an *asymptotic background 4-metric* and this will allow to avoid the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in the bulk. 2) The boundary conditions on each leaf $\Sigma_\tau$ of the admissible 3+1 splittings must be such to reduce the Spi group of asymptotic symmetries to the ADM Poincaré’ group. This means that *super-translations* (direction-dependent quasi Killing vectors, obstruction to the definition of angular momentum in general relativity) must be absent. This is possible only if the admissible 3+1 splittings have all the leaves $\Sigma_\tau$ tending to Minkowski space-like hyper-planes orthogonal to the ADM 4-momentum at spatial infinity [7]. In turn this implies that every $\Sigma_\tau$ is the rest frame of the instantaneous 3-universe and that there are asymptotic inertial observers to be identified with the *fixed stars* (in a future extension to the cosmological context they could be identified with the privileged observers at rest with respect to the background cosmic radiation). 3) The admissible 3+1 splittings should have the leaves $\Sigma_\tau$ admitting a generalized Fourier transform (namely they should be Lichnerowicz 3-manifolds with involution, so to have the possibility to define instantaneous Fock spaces in a future attempt of quantization). 4) All the fields on $\Sigma_\tau$ should belong to suitable weighted Sobolev spaces, so that $M^4$ has no Killing vectors and Yang-Mills fields on $\Sigma_\tau$ do not present Gribov ambiguities (due to the presence of gauge symmetries and gauge copies) [8].

In absence of matter the Christodoulou and Klainermann [9] space-times are good candidates: they are near Minkowski space-time in a norm sense, avoid singularity theorems by relaxing the requirement of conformal completability (so that it is possible to follow solutions of Einstein’s equations on long times) and admit gravitational radiation at null infinity. Since the simultaneity leaves $\Sigma_\tau$ are the rest frame of the instantaneous 3-universe, at the Hamiltonian level it is possible to define the rest-frame instant form of metric and tetrad gravity [7, 8, 10]. If matters is present, the limit of this description for vanishing Newton constant will produce the rest-frame instant form description of the same matter in the framework of parametrized Minkowski theories and the ADM Poincaré’ generators will tend to the kinematical Poincaré’ generators of special relativity. Therefore we have obtained a model admitting a *deparametrization of general relativity to special relativity*.

In ADM tetrad gravity the 16 cotetrad fields may be replaced by i) 3 boost parameters $\varphi(\tau, \sigma^r)$ (adapting the cotetrad to $\Sigma_\tau$), ii) cotriads $e^{(a)}_r(\tau, \sigma^r)$ on $\Sigma_\tau$, iii) lapse and shift functions $N(\tau, \sigma^r)$, $N_a(\tau, \sigma^r)$. The local invariances of the ADM action imply the existence of 14 first-class constraints (10 primary and 4 secondary): i) $\pi_N(\tau, \sigma^r) \approx 0$ implying the secondary super-hamiltonian constraint $H(\tau, \sigma^r) \approx 0$; ii) $\pi_{N_\alpha}(\tau, \sigma^r) \approx 0$ implying the secondary super-momentum constraints $H_{\alpha}(\tau, \sigma^r) \approx 0$; iii) $\pi_{e^a}(\tau, \sigma^r) \approx 0$; iv) three constraints $M_{\alpha}(\tau, \sigma^r) \approx 0$ generating rotations of the cotriads. As a consequence there are 14 gauge variables describing the *generalized inertial effects* in the non-inertial frame defined by the chosen admissible 3+1 splitting of $M^4$ centered on an arbitrary time-like observer. The remaining independent ”two + two” degrees of freedom are the gauge invariant DO of the gravitational field describing
generalized tidal effects.

In the canonical approach it is possible to make a separation of the gauge variables from the DO by means of a Shanhugadhasan canonical transformation (see Ref.[11]). These transformations define a canonical basis adapted to the existing first-class constraints. Since no-one knows how to solve the super-hamiltonian constraint (except that in the post-Newtonian approximation), the best we can do is to look for a quasi-Shanhugadhasan canonical transformation adapted to the other 13 first-class constraints (the only constraints to be Abelianized are $M_{(a)}(\tau, \sigma^r) \approx 0$ and $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$ [8]:

$$
\begin{array}{cccc}
\varphi^{(a)} & N & N_r & 3\epsilon_{(a)r}^{(a)} \\
\approx 0 & \approx 0 & \approx 0 & 3\pi_{(a)}^{(a)} \\
\end{array}
\quad
\begin{array}{cccc}
\varphi^{(a)} & N & N_{(a)} & \alpha_{(a)} \\
\approx 0 & \approx 0 & \approx 0 & \approx 0 \\
\xi^r & \phi & \pi_\phi & r_\alpha \\
\end{array}
$$

Here, $\alpha_{(a)}(\tau, \sigma^r)$ are three Euler angles and $\xi^r(\tau, \sigma^r)$ are three parameters giving a coordinatization of the action of 3-diffeomorphisms on the cotriads $3\epsilon_{(a)r}(\tau, \sigma^r)$. The configuration variable $\phi(\tau, \sigma^r) = \left(\text{det}^2 g(\tau, \sigma^r)\right)^{1/12}$ is the conformal factor of the 3-metric: it can be shown that it is the unknown in the super-hamiltonian constraint (also named the Lichnerowicz equation). The gauge variables are $N$, $N_{(a)}$, $\varphi_{(a)}$, $\alpha_{(a)}$, $\xi^r$ and $\pi_\phi$, while $r_\alpha$, $\pi_\alpha$, $\bar{a} = 1, 2$, are the DO of the gravitational field (in general they are not tensorial quantities). Even if we do not know the expression of the final variables in terms of the original ones, we note that this a point canonical transformation with known inverse $3\epsilon_{(a)r}(\tau, \sigma^r') = 3\mathcal{R}_{(a)b}(\alpha_{(c)}(\tau, \sigma^u)) \frac{\delta(\xi^u(\tau, \sigma^u))}{\delta(\alpha^u(\tau, \sigma^u))} \mathcal{R}_{(b)a}(\alpha_{(b)}(\tau, \sigma^a))$, as implied by the study of the gauge transformations generated by the first-class constraints $3\epsilon_{(a)b}$ are reduced cotriads, which depend only on the two configurational DO $r_\alpha$. The point nature of the canonical transformation implies that the old cotriad momenta are linear functionals of the new momenta. The kernel connecting the old and new momenta satisfy elliptic partial differential equations implied by i) the canonicity conditions; ii) the super-momentum constraints $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0; iii)$ the rotation constraints $M_{(a)}(\tau, \sigma^r) \approx 0.$

The first-class constraints are the generators of the Hamiltonian gauge transformations, under which the ADM action is quasi-invariant (second Noether theorem). In particular those generated by the super-hamiltonian constraint $\mathcal{H}(\tau, \sigma^r) \approx 0$ transform an admissible 3+1 splitting into another admissible one by realizing a normal deformation of the simultaneity surfaces $\Sigma_\tau$. As a consequence, all the conventions about clock synchronization are gauge equivalent as in special relativity.

Finally let us see which is the Dirac Hamiltonian $H_D$ generating the $\tau$-evolution in ADM canonical gravity. In spatially compact space-times without boundary $H_D$ is a linear combination of the primary constraints (each one multiplied by an arbitrary Dirac multiplier, the Hamiltonian version of the undetermined velocities of the configurational approach whose existence is implied by the second Noether theorem) plus the secondary super-hamiltonian and super-momentum constraints multiplied by the lapse and shift functions respectively (consequence of the Legendre transform). As a consequence, $H_D \approx 0$ and in the reduced phase space (quotient of the constraint sub-manifold with respect to the group of gauge transformations) we get a vanishing Hamiltonian. This implies the so-called frozen picture and the problem of how to reintroduce a temporal evolution. Usually one considers the normal (time-like) deformation of $\Sigma_\tau$ induced by the super-hamiltonian constraint as an evolution in a local time variable to be identified (the multi-fingered time point of view with a local either extrinsic or intrinsic time): this is the so-called Wheeler-DeWitt interpretation.

On the contrary, in spatially non-compact space-times the definition of functional derivatives and the existence of a well-posed Hamiltonian action principle (with the possibility of a good
control of the surface terms coming from integration by parts) require the addition of the DeWitt surface term (living on the surface at spatial infinity) to the Hamiltonian. It can be shown [7] that in the rest-frame instant form this term, together with a surface term coming from the Legendre transformation of the ADM action, leads to the Dirac Hamiltonian \( H_D = \hat{E}_{ADM} + \text{constraints} \approx E_{ADM} \). Here \( \hat{E}_{ADM} \) is the strong ADM energy, a surface term analogous to the one defining the electric charge as the flux of the electric field through the surface at spatial infinity in electromagnetism. Since we have \( \hat{E}_{ADM} = E_{ADM} + \text{constraints} \), we see that the non-vanishing part of the Dirac Hamiltonian is the weak ADM energy \( E_{ADM}(\tau, \sigma) \), namely the integral over \( \Sigma_\tau \) of the ADM energy density (in electromagnetism this corresponds to the definition of the electric charge as the volume integral of matter charge density). Therefore there is no frozen picture but a consistent \( \tau \)-evolution.

However, the ADM energy density \( \hat{E}_{ADM}(\tau, \sigma) \) is a coordinate-dependent quantity because it depends on the gauge variables (namely on the inertial effects present in the non-inertial frame): this is the problem of energy in general relativity. Let us remark that in most coordinate systems \( \hat{E}_{ADM}(\tau, \sigma) \) does not agree with the pseudo-energy density defined in terms of the Landau-Lifschiz pseudo-tensor. As a consequence, to get a deterministic evolution for the DO we must fix the gauge completely, that is we have to add 14 gauge-fixing constraints satisfying an orbit condition (so that only one point in each gauge orbit inside the constraint sub-manifold is selected) and to pass to Dirac brackets.

In this way all the gauge variables are fixed to be either numerical functions or well determined functions of the DO. As a consequence, in a completely fixed gauge (i.e. in a non-inertial frame centered on a time-like observer and with its pattern of inertial forces, corresponding to an extended physical laboratory with fixed metrological conventions) the ADM energy density \( \hat{E}_{ADM}(\tau, \sigma) \) becomes a well defined function only of the DO and the Hamilton equations for them with \( E_{ADM} \) as Hamiltonian are a hyperbolic system of partial differential equations for their determination. For each choice of Cauchy data for the DO on a \( \Sigma_\tau \), we obtain a solution of Einstein’s equations in the radar 4-coordinate system associated to the chosen 3+1 splitting of \( M^4 \).

A universe \( M^4 \) (a 4-geometry) is the equivalence class of all the completely fixed gauges with gauge equivalent Cauchy data for the DO on the associated Cauchy and simultaneity surfaces \( \Sigma_\tau \). In each gauge we find the solution for the DO in that gauge (the tidal effects) and then the explicit form of the gauge variables (the inertial effects). Moreover, also the extrinsic curvature of the simultaneity surfaces \( \Sigma_\tau \) is determined. Since the simultaneity surfaces are asymptotically flat, it is possible to determine their embeddings \( z^\mu(\tau, \sigma^r) \) in \( M^4 \). As a consequence, differently from special relativity, the conventions for clock synchronization and the whole chrono-geometrical structure of \( M^4 \) (gravito-magnetism, 3-geodesic spatial distance on \( \Sigma_\tau \), trajectories of light rays in each point of \( M^4 \), one-way velocity of light) are dynamically determined.

Let us remark that, if we look at Minkowski space-time as a special solution of Einstein’s equations with \( r_\theta(\tau, \sigma^r) = \pi_\theta(\tau, \sigma^r) = 0 \) (zero Riemann tensor, no tidal effects, only inertial effects), we find [7] that the dynamically admissible 3+1 splittings (non-inertial frames) must have the simultaneity surfaces \( \Sigma_\tau \) 3-conformally flat, because the conditions \( r_\theta(\tau, \sigma^r) = \pi_\theta(\tau, \sigma^r) = 0 \) imply the vanishing of the Cotton-York tensor of \( \Sigma_\tau \). Instead, in special relativity, considered as an autonomous theory, all the non-inertial frames compatible with the Möller conditions are admissible, namely there is much more freedom in the conventions for clock synchronization.

A first application of this formalism [12] has been the determination of post-Minkowskian background-independent gravitational waves in a completely fixed non-harmonic 3-orthogonal
gauge with diagonal 3-metric. It can be shown that the requirements \( r_\bar{a}(\tau, \sigma^r) \ll 1 \), \( \pi_\bar{a}(\tau, \sigma^r) \ll 1 \) lead to a weak field approximation based on a Hamiltonian linearization scheme: i) linearize the Lichnerowicz equation, determine the conformal factor of the 3-metric and then the lapse and shift functions; ii) find \( E_{ADM} \) in this gauge and disregard all the terms more than quadratic in the DO; iii) solve the Hamilton equations for the DO. In this way we get a solution of linearized Einstein’s equations, in which the configurational DO \( r_\bar{a}(\tau, \sigma^r) \) play the role of the two polarizations of the gravitational wave and we can evaluate the embedding \( z^\mu(\tau, \sigma^r) \) of the simultaneity surfaces of this gauge explicitly.

3. The Objectivity of Space-Time and Open Problems

In 1914 Einstein [13], during his researches for developing general relativity, faced the problem arising from the fact that the requirement of general covariance would involve a threat to the physical objectivity of the points of space-time \( M^4 \), which in classical field theories are usually assumed to have a well defined individuality. He formulated the Hole Argument, according to which to each active diffeomorphisms interchanging space-time points is associated a different solution: determinism can be reobtained only abandoning the physical objectivity of space-time points. Einstein avoided the problem with the pragmatic point-coincidence argument: the only real world-occurrences are the (coordinate-independent) space-time coincidences (like the intersection of two world-lines). However, the problem was reopened by Stachel [14] and then by Earman and Norton [15] and this opened a rich philosophical debate that is still alive today. Stachel suggested that a physical individuation of the point-events of \( M^4 \) could be done only by using four individuating fields depending on the 4-metric on \( M^4 \), namely that a tensor field on \( M^4 \) is needed to identify the points of \( M^4 \).

By using Bergmann-Komar [16] passive reinterpretation of active diffeomorphisms, it can be shown [17] that their role in the Hole Argument is equivalent to Hamiltonian gauge transformations restricted to solutions of Einstein’s equations. Moreover, the intrinsic pseudo-coordinates of Bergmann and Komar [18], four scalar functions \( F^A[w_\lambda], \lambda = 1, \ldots, 4 \), of the four eigenvalues \( w_\lambda \) of the Weyl tensor, can be used as individuating fields. As shown in Ref.[17] the individuation of point-events can be done by considering an arbitrary admissible 3+1 splitting of \( M^4 \) with a given time-like observer and the associated radar 4-coordinates \( \sigma^A \) and by imposing the gauge fixings \( \chi^A(\tau, \sigma^r) = \sigma^A - F^A[w_\lambda] \approx 0 \). After having fixed the other gauge freedoms of tetrad gravity, we arrive at a completely fixed gauge in which, after the transition to Dirac brackets, we get \( \sigma^A \equiv F^A[r_\bar{a}(\sigma), \pi_\bar{a}(\sigma)] \), namely that the radar 4-coordinates of a point in \( M^4_{3+1} \) the copy of \( M^4 \) coordinatized with the chosen non-inertial frame, are determined off-shell by the four DO of that gauge: in other words the individuating fields are the genuine tidal effects of the gravitational field.

Some consequences of this identification of the point-events of \( M^4 \) are: 1) The space-time \( M^4 \) and the gravitational field are essentially the same entity. The presence of matter modifies the solutions of Einstein equations, i.e. \( M^4 \), but does not play any role in this identification. Instead matter is fundamental for establishing a (still lacking) dynamical theory of measurement not using test objects. 2) The reduced phase space of this model of general relativity is the space of abstract DO (pure tidal effects without inertial effects), which can be thought as four fields on an abstract space-time \( M^4 = \{\text{equivalence class of all the admissible non – inertial frames } M^4_{3+1} \text{ containing the associated inertial effects}\} \). 3) Each radar 4-coordinate system of an admissible non-inertial frame \( M^4_{3+1} \) has an associated non-commutative structure, determined by the Dirac brackets of the functions \( \hat{F}^A[r_\bar{a}(\sigma), \pi_\bar{a}(\sigma)] \) determining the gauge. 4) Conjecture: there should exist privileged Shanmugadhasan canonical bases of phase space, in which the DO (the tidal effects) are also Bergmann observables [19], namely coordinate-independent scalar tidal effects.
As a final remark, let us note that these results on the identification of point-events are model dependent. In spatially compact space-times without boundary, the DO are constants of the motion due to the frozen picture. As a consequence, the gauge fixings $\chi^A(\tau, \sigma^r) \approx 0$ (in particular $\chi^\tau$) cannot be used to rebuild the temporal dimension: probably only the instantaneous 3-space of a 3+1 splitting can be individuated in this way.

One of the main open problems now is to find the Hamiltonian formulation of the Newman-Penrose formalism, in particular of the 10 Weyl scalars. We must look for the Bergmann observables (the scalar tidal effects) and try to understand which inertial effects may have a coordinate-independent form and which are intrinsically coordinate-dependent like the ADM energy density. We must also look for the existence of a closed Poisson algebra of scalars and for Shanmugadhasan canonical bases incorporating the Bergmann observables, to be used to find new expressions for the super-hamiltonian and super-momentum constraints, hopefully easier to be solved. Moreover, if we can find all the admissible 3+1 splittings of Minkowski space-time which avoid the Torre-Varadarajan no-go theorem, then they can be adapted to tetrad gravity to try to see whether it is possible to arrive at a multi-temporal background- and coordinate-independent quantization of the gravitational field, in which only the Bergmann observables (the scalar tidal effects) are quantized.

Another important open problem is to find a refined Shanmugadhasan canonical transformation allowing the addition of any kind of matter to the rest-frame instant form of tetrad gravity. This would allow to study the weak-field approximation to the two-body problem in a post-Minkowskian background-independent way by using a Grassmann regularization of the self-energies. The solution of the Lichnerowicz equation would allow to find the expression of the relativistic Newton and gravito-magnetic action-at-a-distance potentials between the two bodies (sources, among other effects, of the Newtonian tidal effects) and the coupling of the particles to the DO of the gravitational field (the genuine tidal effects) in various radar coordinate systems: it would amount to a re-summation of the $1/c$ expansions of the Post-Newtonian approximation. Also the relativistic version of the quadrupole formula for the emission of gravitational waves from the binary system could be obtained and some understanding of how is distributed the gravitational energy in different coordinate systems could be obtained. It would also be possible to study the deviations induced by Einstein’s theory from the Keplerian standards for problems like the radiation curves of galaxies, whose Keplerian interpretation implies the existence of dark matter.

With more general types of matter (relativistic fluids, electromagnetic field) it should be possible to develop Hamiltonian numerical gravity based on the Shanmugadhasan canonical basis and to study post-Minkowskian approximations based on power expansions in Newton constant. Moreover one should look for strong-field approximations to be used in the gravitational collapse of a ball of fluid.

For an extended version of the talk see Ref.[20]

References
[1] Alba, D. and Lusanna, L. Simultaneity, Radar 4-Coordinates and the 3+1 Point of View about Accelerated Observers in Special Relativity (2003) (gr-qc/0311058); Generalized Radar 4-Coordinates and Equal-Time Cauchy Surfaces for Arbitrary Accelerated Observers (2005), submitted to General Relativity and Gravitation (gr-qc/0501090).
[2] Møller, C. The Theory of Relativity (Oxford University Press, Oxford, 1957).
[3] Lusanna, L. The N- and 1-Time Classical Description of N-Body Relativistic Kinematics and the Electromagnetic Interaction, Int. J. Mod. Phys. A12, 645-722 (1997); The Chronogeometrical Structure of Special and General Relativity: towards a Background-Independent Description of the Gravitational Field and Elementary Particles (2004), invited contribution to the book Progress in General Relativity and Quantum Cosmology (Nova Science) (gr-qc/0404122).
[4] Alba, D., Lusanna, L. and Pauri, M. New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems (2005), invited contribution for the book Atomic and Molecular Clusters: New Research (Nova Science) (hep-th/0505005).

[5] Alba, D. and Lusanna, L. Quantum Mechanics in Non-Inertial Frames with a Multi-Temporal Quantization Scheme: I) Relativistic Particles (hep-th/0502060); II) Non-Relativistic Particles (hep-th/0504060), to appear in Int.J.Mod.Phys..

[6] Torre, C.G. and Varadarajan, M. Functional Evolution of Free Quantum Fields, Clas. Quantum Grav. 16, 2651-2668 (1999).

[7] Lusanna, L. The Rest-Frame Instant Form of Metric Gravity, Gen.Rel.Grav. 33, 1579-1696 (2001) (gr-qc/0101048).

[8] De Pietri, R., Lusanna, L., Martucci, L. and Russo, S. Dirac’s Observables for the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge, Gen.Rel.Grav. 34, 877-1033 (2002) (gr-qc/0105084).

[9] Christodoulou, D., and Klainerman, S. The Global Nonlinear Stability of the Minkowski Space. (Princeton University Press, Princeton, 1993).

[10] Lusanna, L. and Russo, S. A New Parametrization for Tetrad Gravity, Gen.Rel.Grav. 34, 189-242 (2002) (gr-qc/0102074).

[11] Lusanna, L. The Shanmugadhasan Canonical Transformation, Function Groups and the Second Noether Theorem, Inter.J.Mod.Phys. A8, 4193-4233 (1993).

[12] Agresti, J., De Pietri, R., Lusanna, L. and Martucci, L. Hamiltonian Linearization of the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge: a Radiation Gauge for Background-Independent Gravitational Waves in a Post-Minkowskian Einstein Space-Time, Gen.Rel.Grav. 36, 1055-1134 (2004) (gr-qc/0302084).

[13] Einstein, A. Die formale Grundlage der allgemeinen Relativitätstheorie, in Preuss. Akad. der Wiss. Sitz., pp. 1030–1085 (1914).

[14] Stachel, J. Einstein’s Search for General Covariance, 1912–1915. Ninth International Conference on General Relativity and Gravitation, Jena (1980), ed. E.Schmutzer (Cambridge Univ.Press, Cambridge, 1983).

[15] Earman, J. and Norton, J. What Price Spacetime Substantivalism? The Hole Story, British Journal for the Philosophy of Science 38, 515–525 (1987).

[16] Bergmann, P.G. and Komar, A. The Coordinate Group Symmetries of General Relativity, Int.J.Theor.Phys. 5, 15-28 (1972).

[17] Lusanna, L. and Pauri, M. General Covariance and the Objectivity of Space-Time Point-Events, talk at the Oxford Conference on Spacetime Theory (2004), to appear in History and Philosophy of Modern Physics (gr-qc/0503069); The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity. I: Dynamical Synchronization and Generalized Inertial Effects; II: Dirac versus Bergmann Observables and the Objectivity of Space-Time, to appear in Gen.Rel.Grav. (gr-qc/0403081 and 0407007).

[18] Bergmann, P.G. and Komar, A. Poisson Brackets between Locally Defined Observables in General Relativity, Phys.Rev.Lett. 4, 432-433 (1960).

[19] Bergmann, P.G. Observables in General Relativity, Rev.Mod.Phys. 33, 510-514 (1961).

[20] Lusanna, L. General Covariance and its Implications for Einstein’s Space-Times (gr-qc/0510024).