Proof of a Conjecture on the Genus Two Free Energy Associated to the $A_n$ Singularity

Yulong Fu, Si-Qi Liu, Youjin Zhang, Chunhui Zhou
Department of Mathematical Sciences, Tsinghua University
Beijing 100084, P. R. China

Abstract

In a recent paper [8], it is proved that the genus two free energy of an arbitrary semisimple Frobenius manifold can be represented as a sum of contributions associated with dual graphs of certain stable algebraic curves of genus two plus the so called genus two G-function, and for a certain class of Frobenius manifolds it is conjectured that the associated genus two G-function vanishes. In this paper, we prove this conjecture for the Frobenius manifolds associated with simple singularities of type A.

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1 Introduction

The notion of Frobenius is a geometrical characterization of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations of associativity that arise in the study of 2D topological field theory (TFT)[1, 2, 4, 20]. For a 2D TFT with \( n \) primary fields, the generating function of its correlators, called the free energy, has the genus expansion

\[
F(t) = \sum_{g \geq 0} e^{2g-2} F_g(t),
\]

(1.1)

where for any \( g \in \mathbb{Z}_{\geq 0} \) the function \( F_g(t) \) is called the genus \( g \) free energy, it is a function defined on the large phase space of the 2D TFT with coordinates

\[
t = (t^\alpha p), \quad \alpha = 1, \ldots, n, \quad p = 0, 1, 2, \ldots
\]

The restriction of the genus zero free energy \( F_0(t) \) yields a function

\[
F(v^1, \ldots, v^n) = F_0(t)|_{t^\alpha p=0(p>0),\ v^\alpha v^\alpha>0}
\]

of \( n \) variables \( v^1, \ldots, v^n \) which satisfies the WDVV equations of associativity. In terms of the corresponding Frobenius manifold, the function \( F(v) \) is called the potential, and the variables \( v^1, \ldots, v^n \) are called the flat coordinates of the Frobenius manifold w.r.t. the flat metric defined by

\[
\eta_{\alpha\beta} = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\gamma \partial v^\beta}, \quad \alpha, \beta = 1, \ldots, n.
\]

(1.2)

One of the important subjects of study in the theory of Frobenius manifold is to reconstruct the full genera free energy \( F(t) \) in terms of the geometric structure of the Frobenius manifold. The reconstruction of the genus zero free energy \( F_0(t) \) is achieved with the help of a particular solution

\[
v(t) = (v^1(t), \ldots, v^n(t))
\]

(1.3)

of an integrable hierarchy of the form

\[
\frac{\partial v^\alpha}{\partial t^\beta q} = K^\alpha_{\beta q} (v^\gamma) v^\gamma x, \quad \alpha, \beta = 1, \ldots, n, \quad q \geq 0.
\]

Here and in what follows summation w.r.t. repeated upper and lower greek indices is assumed. This integrable hierarchy is defined on the loop space of the Frobenius manifold, and is called the Principal Hierarchy in [7]. With the identification of the spatial variable \( x \) with the time variable \( t^{1,0} \), this particular solution of the integrable hierarchy has the following relation with the genus zero free energy \( F_0(t) \):

\[
v^\alpha(t) = \eta^\alpha\gamma \frac{\partial^2 F_0(t)}{\partial t^{1,0} \partial t^{\gamma,0}}, \quad \alpha = 1, \ldots, n.
\]
Here \((\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}\).

For the reconstruction of the higher genera free energies for any semisimple Frobenius manifold, an algorithm is given in [7] by solving recursively the so called loop equation of the Frobenius manifold. The genus \(g \geq 1\) free energy \(F_g(t)\) can be represented in the form

\[
F_g(t) = \tilde{F}_g \left( v(t), v_x(t), \ldots, v^{(3g-2)}(t) \right),
\]

where \(v(t)\) is the particular solution of the Principal Hierarchy, and

\[
v^{(k)}(t) = (\partial_x^k v^1(t), \ldots, \partial_x^k v^n(t)).
\]

In particular, the genus one free energy has the expression \([3, 6, 13]\)

\[
F_1(t) = \frac{1}{24} \log \det(\mathcal{C}_{\alpha\beta\gamma}(v(t))v^\gamma_x(t)) + G(v(t)),
\]

where

\[
c_{\alpha\beta\gamma}(v) = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\gamma}
\]

and \(G(v)\) is called the genus one \(G\)-function of the semisimple Frobenius manifold. The function \(G(v)\) is defined by the isomonodromic tau function \(\tau_I\) and the Jacobian \(J\) of the transformation between the flat coordinates \(v^1, \ldots, v^n\) and the canonical coordinates \(u_1, \ldots, u_n\) (see their meaning below) of the Frobenius manifold via the formula

\[
G(v) = \log \frac{\tau_I(v)}{J^{1/24}(v)}.
\]

Note that in the expression (1.5) of the genus one free energy \(F_1\), the first term in the r.h.s. of the formula can be represented by the genus zero three point correlation functions since we have

\[
c_{\alpha\beta\gamma}(v(t))v^\gamma_x(t) = \frac{\partial^3 F_0(t)}{\partial u^\alpha \partial v^\beta \partial v^\gamma}. \quad (1.6)
\]

Thus the genus one free function can be represented as the summation of two parts, the first part can be represented explicitly by using the genus zero correlation functions, and the second part is given by the \(G\)-function defined on the Frobenius manifold. In general one do not know the explicit expression of the \(G\)-function given in terms of the flat coordinates. However, it is proved in [14] [19] that for a semisimple Frobenius manifold that is associated to a simple singularity or, equivalently, to the a Coxeter group of ADE type, the function \(G(v)\) vanishes.

Similar to the above expression of the genus one free energy, it is shown in a recent paper [8] that the genus two free energy can also be represented as a summation of two parts: the first part is given in an explicit way by some genus zero correlation functions, and the second part is given by the so called genus two \(G\)-function.
Theorem 1.1 ([8]) Let $M$ be a semisimple Frobenius manifold of dimension $n$. Then the genus two free energy has the expression

$$F_2 = \sum_{p=1}^{16} c_p Q_p + G^{(2)}(u, u_x, u_{xx}). \quad (1.7)$$

Here each term $Q_p$ corresponds to a dual graph of a stable curve of arithmetic genus two and can be represented by some genus zero correlation functions; $c_1, \ldots, c_{16}$ are some constants. The function $G^{(2)}(u, u_x, u_{xx})$ is called the genus two $G$-function of the Frobenius manifold, and has an explicit expression (A.1) represented in terms of the canonical coordinates $u_1, \ldots, u_n$ of the Frobenius manifold.

In [8] it is also conjecture that for a certain class of Frobenius manifolds the genus two G-functions equal to zero.

Conjecture 1.2 If $M$ is a Frobenius manifold associated to an ADE singularity or an extended affine Weyl groups of ADE type, then

$$G^{(2)}(u, u_x, u_{xx}) = 0. \quad (1.8)$$

The construction of the two classes of Frobenius manifold structures mentioned in the above conjecture can be found in [4, 5, 17]. They can also be interpreted in terms of cohomological field theory and Gromov-Witten invariants of $\mathbb{P}^1$-orbifolds, see [10, 11, 12, 15, 16, 18] and references therein.

The purpose of the present paper is to prove the following theorem.

Theorem 1.3 For the class of Frobenius manifolds obtained from the simple singularities of type A, the above conjecture holds true.

The paper is organized as follows. In Section 2 we represent the rotation and Lamé coefficients of the Frobenius manifolds associated to the simple singularities of type A in terms of their superpotentials. In Section 3 we prove some identities that will be used in the subsequent sections. In Section 4 we prove the vanishing of the coefficients $G^{(2)}_i(u)$ and $G^{(2)}_{ij}(u)$ that appear in the expression of the genus two G-function. In Section 4–6 we finish the proof of Theorem 1.3. Section 7 is an conclusion of the paper.

2 The Rotation and Lamé Coefficients

Let us recall the definition of the rotation coefficients and Lamé coefficients of a semisimple Frobenius manifold $(M^n, \cdot, \cdot, e, E)$. Near each point of the Frobenius manifold there is a system of local coordinates $u_1, \ldots, u_n$ given by the roots of the characteristic polynomial of the operator of multiplication by the Euler vector $E$. They are called the canonical coordinates of the
Frobenius manifold. In these coordinates the multiplication table defined on the tangent space of $M$ is given by

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}, \quad i, j = 1, \ldots, n.$$ 

In the canonical coordinates the unity vector field $e$ and the Euler vector field $E$ have the expressions

$$e = \sum_{i=1}^{n} \frac{\partial}{\partial u_i}, \quad E = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i},$$

and the flat metric $\langle , \rangle$ of the Frobenius manifold takes the diagonal form

$$\sum_{i=1}^{n} \eta_{ii}(u) du_i^2.$$

The Lamé coefficients $h_i$ and the rotation coefficients $\gamma_{ij}$ of the above diagonal metric are defined by

$$h_i = h_i(u) = \sqrt{\eta_{ii}}, \quad i = 1, \ldots, n$$

and

$$\gamma_{ij} = \gamma_{ji} = \frac{1}{h_i} \frac{\partial h_j}{\partial u_i} \quad \text{for} \quad i \neq j, \quad \gamma_{ii} = 0$$

for some choice of the signs of the square roots. They satisfy the following equations:

$$\frac{\partial h_i}{\partial u_k} = \gamma_{ik} h_k \quad \text{for} \quad i \neq k, \quad \frac{\partial h_i}{\partial u_i} = -\sum_{k=1}^{n} \gamma_{ik} h_k, \quad (2.1)$$

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik} \gamma_{kj} \quad \text{for} \quad k \neq i, j, \quad \frac{\partial \gamma_{ij}}{\partial u_i} = \sum_{k=1}^{n} (u_j - u_k) \gamma_{ik} \gamma_{kj} - \gamma_{ij} u_i - \gamma_{ij} u_j. \quad (2.2)$$

Now let us consider semisimple Frobenius manifold associated to the singularity $f(z) = z^{n+1}$ of type $A_n$. The miniversal unfolding of the function $f(z)$ is given by

$$\lambda(z, t) = z^{n+1} + t_n z^{n-1} + \cdots + t_1, \quad (z, t) \in \mathbb{C} \times B, \quad (2.3)$$

where $B$ is an open ball in $\mathbb{C}^n$. Denote by $C \subset B$ the caustic and $M = B \setminus C$. Then on $M$ there is a semisimple Frobenius manifold structure given by the flat metric

$$\langle \partial', \partial'' \rangle_t = -\text{Res}_{z=\infty} \frac{(\partial' \lambda(z, t))(\partial'' \lambda(z, t))}{\partial_{\lambda(z, t)}} dz \quad (2.4)$$

and the multiplication

$$\langle \partial' \cdot \partial'', \partial''' \rangle_t = -\text{Res}_{z=\infty} \frac{(\partial' \lambda(z, t))(\partial'' \lambda(z, t))(\partial''' \lambda(z, t))}{\partial_{\lambda(z, t)}} dz \quad (2.5)$$

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for any $\partial^t$, $\partial^\alpha$, $\partial^\beta \in T_tM$. The unity vector field and the Euler vector field are defined by

$$e = \partial \partial t_1, \quad E = \sum_{\alpha=1}^{n} \frac{n+2-\alpha}{n+1} t_\alpha \partial \partial t_\alpha. \quad (2.6)$$

Note that the flat coordinates $v_1, \ldots, v_n$ of the metric $\langle \ , \ \rangle$ can be represented as quasihomogenius polynomials of $t_1, \ldots, t_n$ by the formula

$$v_\alpha = -\frac{n+1}{n+1-\alpha} \text{Res}_{z=\infty} \lambda(z)^{\frac{n+1}{n+1-\alpha}} dz, \quad \alpha = 1, \ldots, n.$$ 

For example, when $n = 1, 2, 3$ the corresponding Frobenius manifolds have the following potentials respectively:

\begin{align*}
F(v) &= \frac{1}{12} (v_1)^3, \quad v_1 = t_1. \\
F(v) &= \frac{1}{6} (v_1)^2 v_2 - \frac{1}{216} (v_2)^4, \quad v_1 = t_1, \ v_2 = t_2. \\
F(v) &= \frac{1}{8} (v_1)^2 v_3 + \frac{1}{8} v_1 (v_2)^2 - \frac{1}{64} (v_2)^2 (v_3)^2 + \frac{1}{3840} (v_3)^5, \\
&\quad v_1 = t_1 - \frac{1}{8} t_3^2, \ v_2 = t_2, \ v_3 = t_3.
\end{align*}

Let $z_1, \ldots, z_n$ be the critical points of $\lambda(z, t)$ satisfying

$$\partial z \lambda(z, t)|_{z=z_i} = 0, \quad i = 1, \ldots, n.$$ 

Then the canonical coordinates of the Frobenius manifold are given by the critical values

$$u_i(t) = \lambda(z_i, t), \quad i = 1, \ldots, n. \quad (2.7)$$

The Lamé coefficients $h_i$ and the rotation coefficients $\gamma_{ij}$ can be represented as follow [3]:

\begin{align*}
h_i &= \frac{1}{\sqrt{\lambda''(z_i, t)}}, \quad \gamma_{ij} = \frac{h_i h_j}{(z_i - z_j)^2}, \quad i, j = 1, \ldots, n. \quad (2.8)
\end{align*}

Here and in what follows we denote $\lambda(z, t)$ by $\lambda(z)$, and derivatives of the function $\lambda(z)$ are taken w.r.t. the variable $z$.

To simplify the expressions that will be given in what follows, we introduce the notations

$$H_i = \frac{1}{2} \sum_{j \neq i} u_{ij} \gamma_{ij}^2, \quad 1 \leq i \leq n \quad (2.9)$$

with $u_{ij} = u_i - u_j$, and

$$z_{ij} = z_i - z_j, \quad C_{ik} = \frac{\lambda^{(k)}(z_i)}{\lambda''(z_i)}, \quad i, k = 1, \ldots, n. \quad (2.10)$$

The following lemma lists some identities that will be used frequently in the proof of Theorem 1.3.
Lemma 2.1 The Lamé coefficients $h_i$ and the rotation coefficients $\gamma_{ij}$ satisfy the following identities:

$$\partial_k h_i = \frac{h_i h_k^2}{z_{ik}} \quad \text{for } k \neq i; \quad \partial_i h_i = h_i^3 \left( \frac{1}{4} C_{i3}^2 - \frac{1}{6} C_{i4} \right); \quad (2.11)$$

$$\partial_k \gamma_{ij} = \frac{h_i h_j h_k^2}{z_{ik}^2 z_{jk}^2} \quad \text{for distinct } i, j, k; \quad (2.12)$$

$$\partial_i \gamma_{ij} = \frac{h_i^3}{z_{ij}^2} \left( \frac{3}{z_{ij}^2} + \frac{C_{i3}}{z_{ij}} + \frac{C_{i3}^2}{4} - \frac{C_{i4}}{6} \right); \quad (2.13)$$

$$H_i = \frac{1}{48} h_i^2 \left( C_{i3}^2 - C_{i4} \right). \quad (2.14)$$

Here $\partial_k = \frac{\partial}{\partial u_k}$.

Proof The lemma can be proved by using the identities (2.1), (2.2), the formulae (2.8) and the residue theorem on the complex plane. Let us show the details of the derivation for the identity (2.12) as follows:

$$\partial_i \gamma_{ij} = \frac{1}{u_{ij}} \left( \sum_{k \neq i,j} u_{jk} \gamma_{ik} \gamma_{kj} - \gamma_{ij} \right) = \frac{h_i h_j}{u_{ij}} \sum_{k \neq i,j} \frac{u_j - u_k}{z_{ik}^2 z_{jk}^2 \lambda'(z_k)} - \frac{\gamma_{ij}}{u_{ij}}$$

$$= \frac{h_i h_j}{u_{ij}} \sum_{k \neq i,j} \text{Res}_{z_k} \lambda(z) - \lambda(z) \left( z - z_k \right)^2 (z - z_j)^2 \lambda'(z) - \frac{\gamma_{ij}}{u_{ij}}$$

$$= \frac{h_i h_j}{u_{ij}} \left( \text{Res}_{z_k = z_i} + \text{Res}_{z_k = z_j} \right) \lambda(z) - \lambda(z) \left( z - z_i \right)^2 (z - z_j)^2 \lambda'(z) - \frac{\gamma_{ij}}{u_{ij}}$$

$$= \frac{h_i^3}{z_{ij}^2} \left( \frac{3}{z_{ij}^2} + \frac{C_{i3}}{z_{ij}} + \frac{C_{i3}^2}{4} - \frac{C_{i4}}{6} \right).$$

Here the last equality is due to a residue formula for $R_6(2, 2)$ given in appendix B. The lemma is proved. \[\Box\]

Remark 2.2 In what follows we will frequently calculate residues of some rational functions, so for the readers convenience we list some useful residue formulae in Appendix B.

3 Some useful identities

In this section, we list some identities that will be used in the subsequent sections to prove the main theorem of the present paper.

For fixed distinct indices $i, k \in \{1, 2, \ldots, n\}$, let us denote

$$A_{i,k,p} := \sum_{j \neq k,i} \frac{1}{z_{jp}^2}.$$
In order to simplify the notations we will write $A_{i,k,p}$ by $A_p$ in what follows, keeping in mind that the indices $i, k$ are fixed. By using identities among symmetric polynomials we have

$$
\sum_{j_1 < \ldots < j_p \atop j_1, \ldots, j_p \neq i, k} \frac{1}{z_{kj_1} \cdots z_{kj_p}} = \begin{cases} 
\frac{1}{2}(A_1^2 - A_2), & p = 2; \\
\frac{1}{6}(A_1^3 - 3A_1 A_2 + 2A_3), & p = 3; \\
\frac{1}{24}(A_1^4 - 6A_1^2 A_2 + 8A_1 A_3 + 3A_2^2 - 6A_4), & p = 4; \\
\frac{1}{120}(A_1^5 - 10A_1^3 A_2 + 20A_1^2 A_3 + 15A_1 A_4^2 - 30A_1 A_4 - 20A_2 A_3 + 24A_5), & p = 5; \\
\frac{1}{720}(A_1^6 - 15A_1^4 A_2 + 40A_1^3 A_3 + 45A_1^2 A_4^2 - 90A_1^2 A_4 + 144A_1 A_5 - 120A_1 A_2 A_3 - 15A_2^2 + 90A_2 A_4 + 40A_3^2 - 120A_6), & p = 6.
\end{cases}
$$

(3.1)

These identities enable us to represent the rational functions $z_{ik}^p C_{k,p+2}$ in terms of $A_p$ by using the following relation:

$$
z_{ik}^p C_{k,p+2} = (p + 1)! z_{ik}^p \sum_{j_1 < \ldots < j_p \atop j_1, \ldots, j_p \neq i, k} \frac{1}{z_{kj_1} \cdots z_{kj_p}}
= (p + 1)! \left( z_{ik}^p \sum_{j_1 < \ldots < j_p \atop j_1, \ldots, j_p \neq i, k} \frac{1}{z_{kj_1} \cdots z_{kj_p}} - z_{ik}^{p-1} \sum_{j_1 < \ldots < j_{p-1} \atop j_1, \ldots, j_{p-1} \neq i, k} \frac{1}{z_{kj_1} \cdots z_{kj_{p-1}}} \right).
$$

For example, when $p = 1, 2$ we have the following identities respectively:

$$
z_{ik} C_{k,3} = 2 \left( 1 + A_1 z_{ik} \right),
$$

(3.3)

$$
z_{ik}^2 C_{k,4} = 3 \left( A_1^2 - A_2 \right) z_{ik}^2 - 6A_1 z_{ik}.
$$

(3.4)

We can also represent the rational function $\frac{h_i^2}{h_i^2}$ in terms of $A_p$ due to the relation

$$
\frac{h_i^2}{h_i^2} = - \prod_{j \neq k, i} \frac{z_{ij}}{z_k - z_j} = - \left( \sum_{j_1 < \ldots < j_p \atop j_1, \ldots, j_p \neq k, i} \frac{1}{z_{kj_1} \cdots z_{kj_p}} \right) z_{ik}^p.
$$

(3.5)

By using the above identities, we arrive at the following lemma.

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Lemma 3.1  For any fixed $i \in \{1, 2, \ldots, n\}$ we have the following four identities:

\[
\sum_{l \neq i} \frac{h_l^2}{z_{il}^2} \left( C_{l3}^2 - C_{l4} - \frac{2C_{l3}}{z_{il}} \right) \tag{3.6}
\]

\[
= -\frac{h_i^2}{12} \left( 6C_{i3}^4 - 15C_{i3}^2C_{i4} + 4C_{i4}^2 + 7C_{i3}C_{i5} - 2C_{i6} \right);
\]

\[
\sum_{l \neq i} \frac{h_l^4}{z_{il}^4} \left( \frac{3}{z_{il}^2} - \frac{C_{l3}}{z_{il}} \right) \tag{3.7}
\]

\[
= \frac{h_i^4}{240} \left( 75C_{i3}^4 - 120C_{i3}^2C_{i4} + 20C_{i4}^2 + 30C_{i3}C_{i5} - 4C_{i6} \right);
\]

\[
\sum_{l \neq i} \frac{h_l^2}{z_{il}^2} \left( C_{l3}^2 - C_{l4} - \frac{4C_{l3}}{z_{il}} \right) \tag{3.8}
\]

\[
= -\frac{h_i^2}{240} \left( 135C_{i3}^6 - 525C_{i3}^4C_{i4} + 480C_{i3}^2C_{i4}^2 - 60C_{i4}^4 + 270C_{i3}^3C_{i5} - 300C_{i3}C_{i4}C_{i5} + 30C_{i3}^2C_{i6} - 108C_{i3}^2C_{i6} + 32C_{i3}C_{i7} - 6C_{i8} \right);
\]

\[
\sum_{l \neq i} \frac{h_l^2}{z_{il}^2} \left( C_{l3}^2 - C_{l4} - \frac{4C_{l3}}{z_{il}} \right) \tag{3.9}
\]

\[
= -\frac{h_i^2}{720} \left( 135C_{i3}^6 - 525C_{i3}^4C_{i4} + 480C_{i3}^2C_{i4}^2 - 60C_{i4}^4 + 270C_{i3}^3C_{i5} - 300C_{i3}C_{i4}C_{i5} + 30C_{i3}^2C_{i6} - 108C_{i3}^2C_{i6} + 32C_{i3}C_{i7} - 6C_{i8} \right).
\]

**Proof**  Let us take the first identity as an example to illustrate the proof of the lemma. Regarding $z_i$ as an independent variable, we denote the functions defined by the r.h.s and the l.h.s. of (3.6) by $f_1(z_i)$ and $f_2(z_i)$ respectively. Since both of these rational functions tend to zero when $z_i$ tends to infinity, in order to prove the identity (3.6) we only need to show that these two functions have identical principal parts at each of their poles $z_k (k \neq i)$. Note that the orders of these poles do not exceed 5 and $z_i = z_k$ is not a zero or a pole of the function, so it suffices to show that the Taylor expansions of the functions $z_{ik}^4 f_1(z_i)/h_i^2$ and $z_{ik}^4 f_2(z_i)/h_i^2$ in $z_i - z_k$ coincide up to the terms of order 4.

It follows from the identities (3.1)–(3.5) that

\[
\frac{z_{ik}^4 f_1(z_i)}{h_i^2} = -8 - 2A_1 z_{ik} + (A_1^2 + A_2) z_{ik}^2 + \frac{2}{3} \left( A_1^3 - 3A_1 A_2 - 4A_3 \right) z_{ik}^3 \\
+ \frac{1}{6} \left( A_1^4 - 12A_1^2 A_2 + 3A_2^2 - 4A_1 A_3 + 12A_4 \right) z_{ik}^4 + \mathcal{O}(z_{ik}^5). \tag{3.10}
\]
By using the identities
\[
\sum_{j \neq k, l} \frac{1}{z_{ij}^m} = \sum_{m \geq 0} (-1)^m \frac{(-p)^m}{m!} A_{p+m} z_{ik}^m, \quad p \geq 1
\]
and (3.2) we can also represent the functions \(z_{ik}^p C_{i,p+2}\) in terms of the functions \(A_1, A_2, \ldots\). Thus we can obtain the Taylor expansion of the function \(z_{ik}^p f(z)/h_i^4\) at \(z_k\), and it turns out that its Taylor expansion coincide with (3.10) up to the \(z_{ik}^4 = (z_i - z_k)^4\) term. The lemma is proved. □

4 Vanshing of \(G_i^{(2)}\) and \(G_{ij}^{(2)}\)

Let us fix \(k \neq i\) and regard \(\frac{u_{k,x}}{u_{i,x}} (k \neq i)\) as an independent variable in the expression of \(G_i^{(2)}\) given in Appendix A, then the sum of all its coefficients that appear in the expression of \(G_i^{(2)}\) is given by
\[
T (\frac{u_{k,x}}{u_{i,x}}) := \frac{\partial_k h_i H_i}{60 h_i^4} - \frac{7 \partial_i h_i \partial_k h_i}{5760 h_i^4} - \frac{\gamma_{ik} H_k}{120 h_i h_k} - \sum_l \frac{\gamma_{il} \partial_k h_i}{5760 h_i^2 h_l} - \frac{\gamma_{ik} \partial_k h_k}{1152 h_i h_k} + \frac{\partial_k \gamma_{ik}}{1920 h_i^3} + \sum_{l \neq k, i} \frac{\partial_l \gamma_{il}}{5760 h_i h_l} + \frac{\partial_k \gamma_{ik}}{5760 h_i h_k}
\]
(4.1)

By using the formulae given in (2.8) and in Lemma 2.1 we obtain the identities
\[
\sum_l \frac{\gamma_{il} \partial_k h_i}{5760 h_i^2 h_l} = \frac{h_k^2}{5760 z_{ik}^2} \sum_{l \neq i} \frac{1}{z_{il}^2} = \frac{h_k^2}{5760 z_{ik}^2} \sum_{l \neq i} \text{Res}_{z=z_i} \frac{\lambda''(z)}{(z-z_i)^2 \lambda(z)} = -\frac{h_k^2}{5760 z_{ik}^2} \text{Res}_{z=z_i} \frac{\lambda''(z)}{(z-z_i)^2 \lambda(z)} = \frac{h_k^2}{5760 z_{ik}^2} \left( \frac{C_i^2}{4} - \frac{C_i}{3} \right),
\]
and
\[
\sum_{l \neq k, i} \frac{\partial_k \gamma_{il}}{5760 h_i h_l} = \frac{h_k^2}{5760 z_{ik}^2} \sum_{l \neq i, k} \frac{1}{z_{il}^2} = \frac{h_k^2}{5760 z_{ik}^2} \left( \sum_{l \neq k} \frac{1}{z_{kl}^2} - \frac{1}{z_{ik}^2} \right)
\]
\[
= \frac{h_k^2}{5760 z_{ik}^2} \left( \frac{C_i^2}{4} - \frac{C_i}{3} - \frac{1}{z_{ik}^2} \right).
\]

Here we used the residue formula for \(R_i(2)\) given in Appendix B. In a similar way, by using (2.8) and the residue formula for \(R_5(2, 2)\) given in Appendix
B we obtain the identity

\[ \sum_{k} h_{k}\gamma_{i}^{J} \sum_{k} \frac{1}{z_{ik}^{2} z_{kl}^{2}} = \frac{h_{k}^{2}}{1920} \sum_{k} \text{Res}_{z=z_{i}} \frac{\lambda''(z) (z - z_{i})^{2} (z - z_{k})^{2} \lambda'(z)}{\lambda''(z) (z - z_{i})^{2} (z - z_{k})^{2} \lambda'(z)} \]

\[ = - \frac{h_{k}^{2}}{1920} (\text{Res}_{z=z_{i}} + \text{Res}_{z=z_{k}}) \left( \frac{\lambda''(z)}{(z - z_{i})^{2} (z - z_{k})^{2} \lambda'(z)} \right) \]

\[ = - \frac{h_{k}^{2}}{1920 z_{ik}^{2}} \left( \frac{6}{z_{ik}^{2}} - \frac{C_{i3} - C_{k3}}{z_{ik}} - \frac{C_{i4}^{2}}{4} + \frac{C_{i4}}{3} - \frac{C_{k3}^{2}}{4} + \frac{C_{k4}}{3} \right). \]

Now the vanishing of \( T(\frac{h_{k,x}}{u_{i,x}}) \) easily follows from the above three identities.

Thus by using the identities \( 2.11 - 2.14 \) we can represent the function \( G^{(2)}_{i} \) in the form

\[ G^{(2)}_{i} = \frac{h_{i}^{2}}{5760} \left( \frac{3}{16} C_{i3}^{4} + \frac{1}{3} C_{i3}^{2} C_{i4} - \frac{11}{36} C_{i3}^{2} \right) \]

\[ + \sum_{k \neq i} \left( \frac{7h_{k}^{2}}{576z_{ik}^{3}} \left( C_{k3}^{2} - C_{k4} - \frac{2C_{k3}}{z_{ik}} - \frac{h_{k}^{4}}{384h_{i}^{2} z_{ik}^{2}} \left( \frac{3}{z_{ik}^{2}} - \frac{C_{k3}}{z_{ik}} \right) \right) \right) \]

\[ + \sum_{k \neq i} \left( \frac{h_{i}^{2} - 11h_{k}^{2}}{576z_{ik}^{4}} + \frac{h_{i}^{2} C_{i3}^{2}}{1920z_{ik}^{3}} + \frac{h_{i}^{2}}{5760z_{ik}^{4}} \left( \frac{C_{i4}^{2}}{2} - \frac{2C_{i4}}{3} + \frac{C_{k4}}{3} \right) \right). \]

To simplify the above expression of \( G^{(2)}_{i} \), we need to use the following identities:

\[ \sum_{k \neq i} \frac{1}{z_{ik}^{p}} = \sum_{k \neq i} \text{Res}_{z=z_{k}} \frac{\lambda''(z)}{(z - z_{i})^{p} \lambda'(z)} = - \text{Res}_{z=z_{i}} \frac{\lambda''(z)}{(z - z_{i})^{p} \lambda'(z)}, \quad (4.2) \]

\[ \sum_{k \neq i} \frac{h_{k}^{2}}{z_{ik}^{p}} = \sum_{k \neq i} \text{Res}_{z=z_{k}} \frac{1}{(z - z_{i})^{p} \lambda'(z)} = - \text{Res}_{z=z_{i}} \frac{1}{(z - z_{i})^{p} \lambda'(z)}, \quad (4.3) \]

\[ \sum_{k \neq i} \frac{C_{kq}}{z_{ik}^{p}} = \sum_{k \neq i} \text{Res}_{z=z_{k}} \frac{\lambda^{(q)}(z)}{(z - z_{i})^{p} \lambda'(z)} = - \text{Res}_{z=z_{i}} \frac{\lambda^{(q)}(z)}{(z - z_{i})^{p} \lambda'(z)}. \quad (4.4) \]

Together with the residue formulae for \( R_{1}(p), R_{2}(p) \) and \( R_{4}(p, q) \) that are given in Appendix B, the above identities enable us to simplify the expression of \( G^{(2)}_{i} \) further to obtain

\[ G^{(2)}_{i} = f_{1}(z_{i}) - f_{2}(z_{i}), \]

where

\[ f_{1}(z_{i}) = \sum_{i \neq i} \left( \frac{7h_{i}^{2}}{5760 z_{il}^{4}} \left( C_{i3}^{2} - C_{i4} - \frac{2C_{i3}}{z_{il}} \right) - \frac{h_{i}^{4}}{384h_{i}^{2} z_{il}^{2}} \left( \frac{3}{z_{il}^{2}} - \frac{C_{i3}}{z_{il}} \right) \right), \]
From the identities (3.6), (3.7) it follows that \( f_1(z) = f_2(z) \). Thus we proved that \( G_i^{(2)} \) equals zero.

Now let us proceed to prove the vanishing of \( G_i^{(2)} \). Since \( \gamma_{ii} = 0 \), we only need to show that \( G_i^{(2)} \) is equal to zero for \( i \neq j \). By using the formulae given in (2.8) and Lemma 2.1 we obtain

\[
G_i^{(2)} = \frac{h_i^2 h_j^2}{5760 z_{ij}} \left( \frac{6}{z_{ij}^3} - \frac{C_{i3} - C_{j3}}{z_{ij}} - \frac{1}{2} C_{j3} + \frac{2}{3} C_{j4} \right) + \frac{h_i^2 h_j^2}{2880 z_{ij}^3} \sum_{k \neq i,j} \frac{1}{z_{ik} z_{jk}}. \tag{4.5}
\]

Then the vanishing of \( G_i^{(2)} \) follows from the following residue formula:

\[
\sum_{k \neq i,j} \frac{1}{z_{ik} z_{jk}} = - \sum_{k \neq i,j} \text{Res}_{z=z_k} \frac{\lambda''(z)}{(z-z_i)(z-z_j)^2 \lambda'(z)} \left( \frac{z_{ik}}{z_{ik}^2} + \frac{z_{jk}}{z_{jk}^2} \right) + \frac{C_{i3} - C_{j3}}{2 z_{ij}^2} + \frac{C_{j3}^2}{4} - \frac{C_{j4}^3}{3}.
\]

Thus we have proved that in the expression (A.1) of the function \( G^{(2)} \), the first two terms on the r.h.s. do not give any contribution. In the next two sections, we are to show that the sum of the remaining two terms also equals to zero.

5 The skew symmetry property of \( P_{ij}^{(2)} \) for \( i \neq j \)

In this section we are to prove that the sum

\[
\frac{1}{2} \sum_{i \neq j} P_{ij}^{(2)} (u) u_x^i u_x^j
\]

has no contribution to the function \( G^{(2)} \), i.e. we need to show that

\[
P_{ij}^{(2)} = - P_{ji}^{(2)}, \quad i \neq j.
\]

By using (2.8) and (2.10) we easily obtain the following identity:

\[
\sum_{k,l} \frac{h_i h_j \gamma_{ij} \gamma_{kl}}{h_k h_l} \left( \frac{\gamma_{kl}}{h_l} - \frac{\gamma_{jk}}{2h_j} \right) = \sum_{k \neq i,j} \frac{h_i^2 h_j^2}{z_{ik} z_{jk}} \left( \frac{C_{k3}^2}{4} - \frac{C_{k4}}{3} - \frac{C_{j3}^2}{8} + \frac{C_{j4}}{6} \right) - \sum_{k \neq j} \frac{h_i^2 h_j^2}{z_{ij}^2 z_{jk}} \left( \frac{C_{k3}^2}{4} - \frac{C_{k4}}{3} \right).
\]
Then by using Lemma 2.1 we can simplify $P^{(2)}_{ij}$ to arrive at the following expression of it:

\[
\frac{h^2_i}{z^2_{ij}} \left( \frac{41}{480z^2_{ij}} \right) + \frac{41C_{i3}}{1440z^3_{ij}} + \frac{41(-3C^2_{i3} + 2C_{i4} + 2C_{j4})}{17280z^2_{ij}} + \frac{C_{i3}(-3C^2_{j3} + 4C_{j4})}{17280z_{ij}}
\]

\[
+ \frac{C^2_{j3}(9C^2_{j3} - 30C_{j4}) - 16C_{j4}(C_{i4} - C_{j4}) + C_{i3}(-9C^2_{j3} + 24C_{j4})}{207360}
\]

\[
+ \sum_{k \neq j} \left( \frac{h^2_{i}h^2_{j}C_{k3}}{360z^2_{ij}z^2_{ik}} + \frac{h^2_{j}h^2_{k}(C^2_{i3} - 2C_{i4})}{4320z^2_{ij}z^2_{ik}} - \frac{h^2_{i}h^2_{j}}{240z^2_{ij}z^2_{ik}} - \frac{h^2_{i}h^2_{k}}{720z^2_{ij}z^2_{ik}} \right)
\]

\[
+ \sum_{k \neq i, j} \left( \frac{11h^2_{i}h^2_{j}}{2880z^2_{ij}z^2_{ik}z^2_{jk}} + \frac{h^2_{j}h^2_{k}(C^2_{i3} - 2C_{i4})}{480z^2_{ik}z^2_{jk}} - \frac{h^2_{i}h^2_{j}}{160z^2_{ij}z^2_{ik}} + \frac{h^2_{j}h^2_{k}}{96z^2_{ik}z^2_{jk}} \right)
\]

\[
\sum_{k \neq i} \left( \frac{2h^2_{i}h^2_{k}C_{k3}}{2880z^3_{ik}} \right) - \sum_{k \neq j} \left( \frac{h^2_{i}h^2_{k}(C^2_{i3} - C_{k4})}{z^2_{jk}} \right)
\]

We need the following residue formula to calculate the summations in the above expression of $P^{(2)}_{ij}$:

\[
\sum_{k \neq i, j} \frac{C_{k3}}{z^2_{ik}z^3_{jk}} = \sum_{k \neq i, j} \frac{\lambda''(z)}{(z_i - z)^2(z_j - z)^3\lambda'(z)}
\]

\[
= - \left( \text{Res}_{z = z_i} + \text{Res}_{z = z_j} \right) \frac{\lambda''(z)}{(z_i - z)^2(z_j - z)^3\lambda'(z)}
\]

\[
= \frac{6C_{i3} + 4C_{j3}}{z^5_{ij}} + \frac{3C^2_{i3} - 3C^2_{j3} - 6C_{i4} + 6C_{j4}}{2z^4_{ij}}
\]

\[
+ \frac{3C^2_{i3} + 6C^2_{j3} - 8C_{i3}C_{i4} - 16C_{j3}C_{j4} + 6C_{i5} + 12C_{j5}}{12z^3_{ij}}
\]

\[
- \frac{3C^4_{i3} - 10C^2_{j3}C_{j4} + 4(C_{j4})^2 + 7C_{j3}C_{j5} - 4C_{j6}}{24z^2_{ij}}
\]

Similar identities can be obtained by applying the residue formulae for
Thus we proved the expected equation $P_{ij} = -P_{ji}$ for $i \neq j$.

Here the positive integers $p, q$ satisfy $p + q \geq 3$. From these identities and the ones given in (4.2)–(4.4) it follows that

$$P_{ij}^{(2)} = h_i^2 h_j^2 X + \frac{7}{2880 z_{ij}^2} \left( \sum_{k \neq i} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{ik}^3} - \sum_{k \neq j} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{jk}^3} \right)$$

$$= h_i^2 h_j^2 X + \frac{7}{2880 z_{ij}^2} \left( \sum_{k \neq i} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{ik}^3} - \sum_{k \neq j} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{jk}^3} \right)$$

$$- \frac{7 h_i^2}{2880 z_{ij}^2} \sum_{k \neq j} \frac{h_k^2}{z_{jk}^3} \left( C_{k3} - C_{k4} - \frac{2 C_{k3}}{z_{jk}} \right), \quad (5.2)$$

where $X \in \mathbb{Q}[\frac{1}{z_{ij}}, C_{i3}, C_{i3}, C_{i4}, C_{i4}, C_{i5}, C_{i5}, C_{i6}, C_{i6}]$. By using the identity (5.3) that is given in Lemma 5.1, we can simplify $P_{ij}^{(2)}$ further to get the expression

$$P_{ij}^{(2)} = h_i^2 h_j^2 X + \frac{7}{2880 z_{ij}^2} \left( \sum_{k \neq i} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{ik}^3} - \sum_{k \neq j} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{jk}^3} \right)$$

$$- \frac{7 h_i^2}{2880 z_{ij}^2} \left( \frac{1}{2} C_{j3} + \frac{5}{4} C_{j3} C_{j4} - \frac{1}{3} C_{j4} - \frac{7}{12} C_{j3} C_{j5} + \frac{1}{6} C_{j6} \right)$$

$$= \frac{7}{2880 z_{ij}^2} \left( \sum_{k \neq i} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{ik}^3} - \sum_{k \neq j} \frac{2 h_i^2 h_k^2 C_{k3}}{z_{jk}^3} \right) + Y_{ij} - Y_{ji},$$

where

$$Y_{ij} = \frac{h_i^2 h_j^2}{5760 z_{ij}^2} \left( \frac{-22 C_{i3}^2 + 19 C_{i3}^2 - 104 C_{i4} + 15 C_{i3}^3 - 34 C_{i3} C_{i4} + 21 C_{i5}}{z_{ij}} + \frac{45}{4} C_{i3} - 22 C_{i3} C_{i4} - \frac{1}{6} C_{i3} C_{i4} + 5 C_{i4} + \frac{49}{6} C_{i3} C_{i5} - \frac{23}{10} C_{i6} \right)$$

Thus we proved the expected equation $P_{ij}^{(2)} = -P_{ji}^{(2)}$ for $i \neq j$. 

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6 Proof of Theorem 1.3

In this section let us finish the proof of Theorem 1.3. To this end we are left to prove that

\[ \frac{1}{2} P_{ii}^{(2)} + Q_{ii}^{(2)} = 0, \quad i = 1, \ldots, n. \]  

(6.1)

From Lemma 2.1 it follows that the first term in the above mentioned summation has the expression

\[ \frac{1}{2} P_{ii}^{(2)} = \frac{h_i^4}{480} \sum_{k \neq i} \left( \frac{1}{z_{ik}^6} + \frac{C_{k3}}{2z_{ik}^3} \right). \]  

(6.2)

To prove the vanishing of (6.1), we need to calculate the summations w.r.t. the indices \( k, l \) that appear in the expression of \( Q_{ii}^{(2)} \) defined in Appendix A. Such summations are difficult to calculate when their summands have denominators involving explicitly the difference of canonical coordinates \( u_{ik} = u_i - u_k \). Due to this reason let us first pick up all such terms from \( Q_{ii}^{(2)} \) and denote the sum of them by \( \alpha_1 \), i.e.

\[ \alpha_1 := \sum_{k \neq i} \left( \frac{\gamma_{ik} H_k}{576 u_{ik} h_i h_k} + \frac{\gamma_{ik} h_k H_i}{576 u_{ik} h_i h_k^3} - \frac{\partial_i \gamma_{ik} h_k}{576 u_{ik} h_i h_k} - \frac{\partial_k \gamma_{ik}}{576 u_{ik} h_i h_k} \right) \]

\[ + \sum_{k,l \neq i} \left( \frac{u_k \gamma_{ik} \partial_i \gamma_{kl}}{1152 u_{il} h_i h_l} + \frac{h_k u_{kl} \gamma_{ki} \partial_i \gamma_{il}}{1152 u_{ik} h_i^3} \right). \]  

(6.3)

By using Lemma 2.1 we can rewrite the last two terms in the above expression of \( \alpha_1 \) in the following form:

\[ \sum_{k,l \neq i} \frac{u_k \gamma_{ik} \partial_i \gamma_{kl}}{1152 u_{il} h_i h_l} = \sum_{k,l \neq i} \frac{u_k \gamma_{ik} \partial_k \gamma_{kl}}{1152 u_{ik} h_i h_k} \]

\[ = \sum_{k \neq i} \frac{h_k^2}{1152 u_{ik}} \sum_{l \neq k,i} \frac{u_k h_l^2}{2 z_{il}^2 z_{kl}^2} \left( \frac{C_{k3}^2}{4} - \frac{C_{k4}}{6} + \frac{3}{z_{il}^2} + \frac{C_{k3}}{z_{ik} z_{kl}} \right) \]  

(6.4)

\[ \sum_{k,l \neq i} \frac{h_k u_{kl} \gamma_{ki} \partial_i \gamma_{il}}{1152 u_{ik} h_i^3} \]

\[ = \sum_{k \neq i} \frac{h_k^2}{1152 u_{ik}} \sum_{l \neq k,i} \frac{u_k h_l^2}{2 z_{il}^2 z_{kl}^2} \left( \frac{C_{k3}^2}{4} - \frac{C_{k4}}{6} + \frac{3}{z_{il}^2} + \frac{C_{k3}}{z_{il} z_{ik}} \right). \]  

(6.5)

In the r.h.s. of the above formulae the summation w.r.t. \( l \) can be represented as

\[ \sum_{l \neq k,i} u_l h_l^2 \frac{1}{z_{il}^2 z_{kl}^2} = \sum_{l \neq k,i} \text{Res}_{z=z_l} \left( \frac{\lambda(z_k) - \lambda(z)}{(z_i - z)^2 (z_k - z)^2 \lambda'(z)} \right) \]

\[ = - \left( \text{Res}_{z=z_i} + \text{Res}_{z=z_k} \right) \left( \frac{\lambda(z_k) - \lambda(z)}{(z_i - z)^2 (z_k - z)^2 \lambda'(z)} \right). \]

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Then by substituting the r.h.s. of the residue formula for $R_6(2,2)$ given in Appendix B into (6.4) and (6.5) and by using Lemma 2.1, we can represent the sum $\alpha_1$ in terms of $z_{kl}, C_{kl}, h_k$ and $u_k$. Note that all terms that contain $u_{ik}$ in the denominators are cancelled.

Let us continue to exam summations of other terms in $Q^{(2)}_i$. We denote by $\alpha_2$ the terms in $Q^{(2)}_i - \alpha_1$ given by the double summation w.r.t. to the indices $k, l$. By using Lemma 2.1 we have

$$\alpha_2 = \sum_{k \neq i} \left( \frac{h_i^2 h_k^2}{z_{ik}^4} \sum_{l \neq i} \left( \frac{h_l^2 u_{il}}{40 z_{il}^4} \left( \frac{C_{i3}^2}{4} - \frac{C_{i4}}{6} + \frac{3}{z_{ik}^2} + \frac{C_{i3}}{z_{ik}} \right) \right) \right.

- \frac{1}{1440 z_{il}^4} \left( \frac{3}{z_{il}^2} - \frac{C_{i3}}{z_{ik}} - \frac{C_{i4}^2}{4} + \frac{C_{i4}}{6} \right)\left( \frac{z_{il}^2}{z_{ik}^2} + \frac{C_{i3}}{z_{ik}} \right) \sum_{l \neq k, i} \frac{h_{lk}^2 u_{lk}}{z_{il}^2}

- \frac{h_i^2}{34560 z_{ik}^2} \left( \frac{36 h_i^2 + 24 h_k^2}{z_{ik}^2} + \frac{C_{i3}}{z_{ik}} (12 h_i^2 + 12 h_k^2) \right)

+ \left. (3C_{i3}^2 - 2C_{i4}) (2h_i^2 + h_k^2) \right) \sum_{l \neq kl} \frac{1}{z_{ik}^2}.

(6.6)

By using the residue formulae for $R_3(4)$ and $R_6(2,2)$ given in Appendix B we have

$$\sum_{l \neq i} \frac{h_l^2 u_{il}}{z_{il}^4} = \sum_{l \neq i} \text{Res}_{z=z_l} \frac{\lambda(z_i) - \lambda(z)}{(z - z_i)^4 \lambda'(z)} = - \text{Res}_{z=z_l} \frac{\lambda(z_i) - \lambda(z)}{(z - z_i)^4 \lambda'(z)}$$

$$= \frac{1}{24} (C_{i3}^2 - C_{i4})$$

$$\sum_{l \neq k, i} \frac{h_{lk}^2 u_{lk}}{z_{ik}^4} = \sum_{l \neq k, i} \text{Res}_{z=z_{kl}} \frac{\lambda(z) - \lambda(z_k)}{(z_l - z)^2 (z_k - z)^2 \lambda'(z)}$$

$$= - \left( \text{Res}_{z=z_k} + \text{Res}_{z=z_l} \right) \frac{\lambda(z) - \lambda(z_k)}{(z_l - z)^2 (z_k - z)^2 \lambda'(z)}$$

$$= - \frac{h_{lk}^2 u_{lk}}{z_{ik}^2} \left( \frac{3}{z_{ik}^2} + \frac{C_{i3}}{z_{ik}} + \frac{C_{i3}^2}{4} - \frac{C_{i4}}{6} \right) \frac{1}{z_{ik}^2}$$

Substituting these expressions of the summation w.r.t. the index $l$ into the r.h.s. of (6.6), we obtained an expression of $\alpha_2$ without summations w.r.t. the index $l$.

Let us denote by $\beta_0$ the summation of the first four terms in the expression of $Q^{(2)}_i$, i.e.

$$\beta_0 = \frac{4H_i^3}{5h_i^4} - \frac{7\partial h_i H_i^2}{10h_i^4} + \frac{7(\partial h_i)^2 H_i}{48h_i^4} - \frac{(\partial h_i)^3}{120h_i^5}.$$
and we denote $\beta_1 = \frac{1}{2} I^{(2)}_{ii} + Q^{(2)}_i - \beta_0$. Then it follows from the result of our calculation for $\alpha_1, \alpha_2$ that

$$
\beta_1 = \sum_{k \neq i} \left( -\frac{h_i^4 h_k^2 u_{ik}}{16 z_{ik}} - \frac{h_i^4 C_{i3} h_k^2 u_{ik}}{24 z_{ik}} - \frac{4 h_i^4 h_k^2}{960 z_{ik}} - \frac{h_i^4}{120 z_{ik}} - \frac{h_i^4 (25 C_{i3} - 10 C_{i4}) h_k^2}{1440 z_{ik}}
\right)
\sum_{k \neq i} \left( -\frac{h_i^4 C_{i3}}{480 z_{ik}} + \frac{h_i^4 C_{i4}}{960 z_{ik}} - \frac{73 h_i^4 C_{i3} h_k^2}{960 z_{ik}} - \frac{h_i^4 (15 C_{i3} - 10 C_{i4}) h_k^2}{4320 z_{ik}} + \frac{h_i^4 (3 C_{i3} - 4 C_{i4})}{11520 z_{ik}}
\right)
+ \frac{h_i^4 C_{i4}}{5760 z_{ik}}
+ \frac{h_i^4 (53 C_{i3}^2 + 38 C_{i4}) h_k^2}{69120 z_{ik}} - \frac{h_i^2 (32 C_{i3} C_{i4} + 5 C_{i5}) h_k^2}{20736 z_{ik}} + \frac{h_i^4 C_{i4} C_{k4}}{17280 z_{ik}} + \frac{h_i^4 (9 C_{i4} - 12 C_{i3} C_{i4} + 4 C_{i4})}{414720 z_{ik}}
+ \frac{h_i^4 (3 C_{i3} - 2 C_{i4}) C_{k4}}{829440 z_{ik}}
- \frac{h_i^2}{4608} \left( \frac{C_{i3}^2}{5} + \frac{C_{i4}}{3} \right)
\sum_{k \neq i} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4}^2 - \frac{2 C_{i3}}{z_{ik}} \right)
+ \frac{h_i^2 C_{i3}}{2304}
\sum_{k \neq i} \frac{h_k^2}{z_{ik}} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4} + \frac{3 C_{i3}^2}{z_{ik}} \right)
+ \frac{17 h_i^2}{11520} \sum_{k \neq i} \frac{h_k^2}{z_{ik}} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4} - \frac{4 C_{i3}^2}{z_{ik}} \right).
$$

By using the identities (12), (13), (14), the identity

$$
\sum_{k \neq i} \frac{h_k^2}{z_{ik}} = \sum_{k \neq i} \text{Res}_{z=z_k} \frac{\lambda(z) - \lambda(z)}{(z_i - z)^p \lambda'(z)} = - \text{Res}_{z=z_i} \frac{\lambda(z_i) - \lambda(z)}{(z_i - z)^p \lambda'(z)}, \quad (6.7)
$$

and the residue formulae for $R_1(p), R_2(p), R_3(p), R_4(p, q)$ we obtain

$$
\frac{1}{2} I^{(2)}_{ii} + Q^{(2)}_i
= h_i^4 Z - \frac{h_i^4}{4608} \left( \frac{C_{i3}^2}{5} + \frac{C_{i4}}{3} \right)
\sum_{k \neq i} \frac{h_k^2}{z_{ik}} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4} - \frac{2 C_{i3}}{z_{ik}} \right)
+ \frac{h_i^2 C_{i3}}{2304}
\sum_{k \neq i} \frac{h_k^2}{z_{ik}} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4} + \frac{3 C_{i3}^2}{z_{ik}} \right)
+ \frac{17 h_i^2}{11520} \sum_{k \neq i} \frac{h_k^2}{z_{ik}} \left( C_{i3}^2 C_{k3} - C_{i4} C_{k4} - \frac{4 C_{i3}^2}{z_{ik}} \right).
$$

Here $Z \in \mathbb{Q}[C_{i3}, C_{i4}, C_{i5}, C_{i6}, C_{i7}, C_{i8}]$. We can simplify the above expression
further to get

\[
\frac{1}{2} P^{(2)}_{ij} + Q^{(2)}_i = \frac{h_i^2 C_{i3}}{2304} \sum_{k \neq i} \frac{h_k^2}{z_{ik}^3} \left( C_{k3}^2 - C_{k4} - 3C_{k3} \right) + \frac{17h_i^2}{11520} \sum_{k \neq i} \frac{h_k^2}{z_{ik}^4} \left( C_{k3}^2 - C_{k4} - 4C_{k3} \right) + \frac{h_i^2}{8294400} (3420C_{i3}^6 - 12525C_{i3}^4C_{i4} + 6390C_{i3}^3C_{i5} + 10260C_{i3}^2C_{i4}^2 \\
- 2496C_{i3}^2C_{i6} - 6000C_{i3}C_{i4}C_{i5} + 694C_{i3}C_{i7} - 1020C_{i4}^3 \\
+ 884C_{i4}C_{i6} + 510C_{i5}^2 - 102C_{i8} ) .
\]

Then the equality (6.1) follows from the identities (3.8), (3.9) given in Lemma 3.1.

Finally, By combining the equality (6.1) with the results of the previous sections on the vanishing of \( G^{(2)}_i \), \( G^{(2)}_{ij} \) and \( P_{ij} + P_{ji} \ (i \neq j) \) we arrive at the result of Theorem 1.3, i.e. \( G^{(2)} = 0 \). The Theorem is proved.

7 Conclusion

We proved in this paper the vanishing of the genus two G-functions for the Frobenius manifolds obtained from the simple singularities of type A by using the formulae of the rotation coefficients and the Lamé coefficients given in [8]. Similar formulae are also given in [8] for the Frobenius manifolds associated to the simple singularities type D, and for the ones associated to the extended affine Weyl groups of type A and type D. We hope that the prove of the Conjecture 1.2 can also be obtained for these classes of Frobenius manifolds by using a similar argument as given in the present paper. We will return to the proof of the conjecture in subsequent publications.

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A The genus two $G$-function

We recalled the following expression of the genus two $G$-function $G^{(2)}(u, u_x, u_{xx})$ that is given in the appendix of [3]:

\[
G^{(2)}(u, u_x, u_{xx}) = \sum_{i=1}^{n} G^{(2)}(u, u_x) u_{xx}^i + \sum_{i \neq j} G^{(2)}_{ij}(u) \frac{(u_x^j)^3}{u_x^i} + \frac{1}{2} \sum_{i,j} P^{(2)}_{ij}(u) u_x^i u_x^j + \sum_{i=1}^{n} Q^{(2)}_{i}(u) (u_x^i)^2
\]

with coefficients written in terms of the Lamé coefficients $h_i = h_i(u)$ and rotation coefficients $\gamma_{ij} = \gamma_{ij}(u)$ of the semisimple Frobenius manifold. Here

\[
G^{(2)}_i = \frac{\partial_x h_i H_i}{60 u_{i,x} h_i^3} - \frac{3 \partial_x h_i H_i}{40 h_i^4} + \frac{19 (\partial_x h_i)^2}{2880 h_i^4} - \frac{7 \partial_x h_i \partial_x h_i}{5760 u_{i,x} h_i^4}
+ \sum_k \left[ \gamma_{ik} \frac{\partial_x H_i}{120 h_i h_k} + \frac{\partial_x \gamma_{ik}}{120 h_i h_k} \left( \gamma_{ik} \frac{u_{k,x}}{u_{i,x}} \right) \right] - \frac{\gamma_{ik} h_i h_k}{384 h_i^4} \frac{\partial_x h_i}{h_k} + \frac{\partial_x \gamma_{ik}}{h_k} \left( \frac{u_{k,x}}{2880 u_{i,x}} + \frac{7}{2880} \right) + \frac{\gamma_{ik} h_i h_k}{2880 h_i^4}
- \frac{1}{h_i^2} \left( \gamma_{ik} \frac{u_{k,x}}{1152 u_{i,x}} + \frac{19}{720} \right) + \frac{\gamma_{ik}^2}{1440 h_i^2} - \sum_{k,l} \left( \frac{h_i u_{l,k} \gamma_{kl}}{2880 h_k h_i^2} + \frac{u_{k,x} h_{i,k} \gamma_{kl}}{1920 u_{i,x} h_i^2} \right),
\]

\[
G^{(2)}_{ij} = - \frac{\gamma_{ij}^2 H_j}{120 h_j^4} + \frac{\gamma_{ij}^3}{480 h_j^5} - \frac{\gamma_{ij} h_i h_j}{5760} \left( \frac{\partial_x \gamma_{ij}}{h_i^2} + \frac{\partial_x \gamma_{ij}}{h_j^2} \right) + \frac{\gamma_{ij}^2}{5760} \left( \frac{\partial_x h_i}{h_j^3} + \frac{\partial_x h_j}{h_i^3} \right)
+ \sum_k \left( \frac{\gamma_{ik} \gamma_{jk} \gamma_{ij}}{5760 h_k} + \frac{\gamma_{ij}^2}{5760 h_k} \left( \frac{\gamma_{jk}}{h_j} - \frac{\gamma_{ik}}{h_i} \right) \right),
\]

\[
P^{(2)}_{ij} = - \frac{2 \gamma_{ij} H_i H_j}{5 h_i h_j} + \frac{\gamma_{ij} \partial_x h_i H_i}{20 h_i^2 h_j} + \frac{\gamma_{ij} \partial_x h_j H_j}{20 h_i h_j^2} - \frac{19 \gamma_{ij}^2 H_j}{30 h_j^4} - \frac{\gamma_{ij} h_i H_j}{60 h_j^3}
+ \frac{41 \gamma_{ij} H_i^2}{240 h_i h_j} - \frac{41 \gamma_{ij} \partial_x h_i H_i}{1440 h_i h_j^2} - \frac{\partial_x \gamma_{ij} \partial_x h_i}{1440 h_i h_j^3} + \frac{79 \gamma_{ij}^2 \partial_x h_j}{720 h_j^5 h_i^2} - \frac{\gamma_{ij} \partial_x h_i \partial_x h_j}{288 h_j^2} - \frac{\gamma_{ij} h_i \gamma_{ij} h_j}{288 h_j^5}
+ \sum_k \frac{\gamma_{ij} \gamma_{ik} H_j}{60 h_j h_k} - \frac{\gamma_{ij} \gamma_{jk} h_i h_j H_k}{30 h_k^4} - \frac{\gamma_{ij} \gamma_{jk} h_j H_k}{60 h_j^2 h_k^2} - \frac{\gamma_{ij} \gamma_{jk} h_i H_k}{60 h_j h_k^3} - \frac{7 \gamma_{ij} \gamma_{jk} h_i h_j H_k}{60 h_j^2 h_k^2}
- \frac{\gamma_{ij} \gamma_{jk} \partial_x h_i}{720 h_j^3 h_k} - \frac{\gamma_{ij} \gamma_{jk} \partial_x h_j}{240 h_j^3 h_k} - \frac{\gamma_{ij} \gamma_{jk} h_i \partial_x h_j}{1440 h_j^5 h_k^2} + \frac{\gamma_{ij} \gamma_{jk} h_i \partial_x h_j}{720 h_k^4} + \frac{\gamma_{ij} \gamma_{jk} h_i \partial_x h_j}{288 h_k^5}.
\]
Here is the list of most of the residue formulae that we used in this paper.

B Some residue formulae

In these expressions, the summations are taken over indices such that the denominators do not vanish.

\[Q_i^{(2)} = \frac{4H_i^3}{5h_i^2} - \frac{7\partial_i H_i^2}{10h_i^4} + \frac{7(\partial_i h_i)^2 H_i}{48h_i^4} - \frac{(\partial_i h_i)^3}{120h_i^6} + \sum_k \left( \frac{7\gamma_{ik}H_k}{10h_i h_k} - \frac{\gamma_{ik}\partial_i H_i}{120h_i^2 h_k} \right) \]

\[+ \frac{7\partial_i (h_k^{-1\gamma_{ik}}) H_i}{240h_i^5} - \frac{7\gamma_{ik}\partial_i h_i H_k}{80h_i^4 h_k} + \frac{\gamma_{ik}H_k}{576u_i h_i h_k} + \frac{240h_i h_k}{h_i} + \gamma_{ik}h_i H_i \]

\[+ \frac{\gamma_{ik}h_k H_i}{576u_i h_i h_k} - \frac{31\gamma_{ik}^2 H_i}{144h_i^2} + \frac{\gamma_{ik}(\partial_i h_i)^2}{720h_i^3 h_k} + \frac{253\gamma_{ik}^2 \partial_i h_i}{576h_i^4} - \frac{\partial_i\gamma_{ik}\partial_i h_i}{960h_i^2 h_k} - \frac{\gamma_{ik}^2\partial_i h_k}{2880h_i^3} \]

\[- \frac{7\partial_i (h_k^{-1\gamma_{ik}}) \partial_i h_i}{1920h_i^5} - \frac{7\partial_i\gamma_{ik}\partial_i h_k}{576h_i h_k} + \frac{41\partial_i\gamma_{ik}\partial_i h_i h_k}{576h_i^4} + \frac{\partial_i(\gamma_{ik}h_i)\partial_i h_k}{2880h_i^5} \]

\[- \frac{113\gamma_{ik}\partial_i\gamma_{ik}}{576h_i h_k^2} + \frac{3(\partial_i\gamma_{ik} + \partial_i\gamma_{ik}) \gamma_{ik}}{144h_i^2} - \frac{\partial_i\gamma_{ik}h_k}{576u_i h_i h_k} - \frac{\partial_i\gamma_{ik}h_k}{576u_i h_i h_k} - \frac{\gamma_{ik}^2}{240h_i h_k} \]

\[+ \sum_{k,l} \left( \frac{\gamma_{ik}\partial_i (h_l^2\gamma_{il})}{2880h_k h_i h_l} + \frac{\gamma_{ik}\partial_i\gamma_{il}}{2880h_k h_i h_l} + \frac{\gamma_{ik}\partial_i\gamma_{il}}{240h_i h_k} + \frac{\gamma_{ik}\partial_i\gamma_{il}}{2880h_i h_k} + \frac{u_{ik}\gamma_{ik}\partial_i\gamma_{kl}}{1152u_i h_i h_l} \right) \]

\[+ \frac{u_{ik}\gamma_{ik}\gamma_{kl}\partial_i\gamma_{il}}{144h_i^2} + \frac{h_i^2\gamma_{ik}\partial_i\gamma_{il}}{1440h_i^2 h_k} + \frac{u_{ik}\gamma_{ik}\gamma_{kl}\partial_i\gamma_{il}}{1152u_i h_i h_l} + \frac{h_i u_{ik}\gamma_{ik}\partial_i\gamma_{il}}{40h_i^3} \].

In these expressions, the summations are taken over indices such that the denominators do not vanish.

\[R_i^2 = \frac{C_{i4}^2}{4} + \frac{C_{i4}}{3}; \]

\[R_i^3 = -\frac{1}{8} (C_{i5}^3 - 2C_{i4}C_{i5} + C_{i5}); \]

\[R_i^4 = -\frac{1}{720} (45C_{i4}^3 - 120C_{i4}^2 C_{i5} + 40C_{i4}^2 + 60C_{i4}C_{i5} - 24C_{i5}); \]

\[R_i^5 = -\frac{1}{288} (9C_{i5}^5 - 30C_{i4}^2 C_{i5} + 20C_{i4}C_{i5} + 15C_{i4}^2 C_{i5} - 10C_{i4}C_{i5} - 6C_{i4}C_{i5} + 2C_{i7}); \]

\[\text{Res}_{z = z_i} \frac{n(z)}{z - z_i} \]
\[ R_1(6) = -\frac{1}{60480} (945C_{i3}^6 - 3780C_{i3}^4 C_{i4} + 3780C_{i3}^2 C_{i4}^2 - 560C_{i4}^3 + 1890C_{i3}^3 C_{i5}; \\
-2520C_{i3} C_{i4} C_{i5} + 315C_{i4}^2 - 756C_{i3}^2 C_{i6} + 504C_{i4} C_{i6} + 252C_{i3} C_{i7} - 72C_{i8}) .
\]

Residues of form \( R_2(p) = \text{Res}_{z=z_i} \frac{1}{(z_i-z)p \lambda(z)} \).

\[ R_2(2) = \frac{h_i^2}{12} (3C_{i3}^2 - 2C_{i4}) ; \\
R_2(3) = \frac{h_i^2}{24} (3C_{i3}^3 - 4C_{i3} C_{i4} + C_{i5}) ; \\
R_2(4) = \frac{h_i^2}{720} (45C_{i3}^4 - 90C_{i3}^2 C_{i4} + 20C_{i4}^2 + 30C_{i3} C_{i5} - 6C_{i6}) ; \\
R_2(5) = \frac{h_i^2}{1440} (45C_{i3}^5 - 120C_{i3}^3 C_{i4} + 60C_{i3} C_{i4}^2 + 45C_{i3}^2 C_{i5} - 20C_{i4} C_{i5} \\
-12C_{i3} C_{i6} + 2C_{i7}) ; \\
R_2(6) = \frac{h_i^2}{60480} (945C_{i3}^6 - 3150C_{i3}^4 C_{i4} + 2520C_{i3}^2 C_{i4}^2 - 280C_{i4}^3 + 1260C_{i3}^3 C_{i5} \\
-1260C_{i3} C_{i4} C_{i5} + 105C_{i5}^2 - 378C_{i3}^2 C_{i6} + 168C_{i4} C_{i6} + 84C_{i3} C_{i7} - 12C_{i8}) .
\]

Residues of form \( R_3(p) := \text{Res}_{z=z_i} \frac{\lambda(z)-\lambda(z_i)}{(z_i-z)p \lambda(z)} \).

\[ R_3(4) = -\frac{1}{24} (C_{i3}^2 - C_{i4}) ; \\
R_3(5) = \frac{1}{720} (15C_{i3}^3 - 25C_{i3} C_{i4} + 9C_{i5}) ; \\
R_3(6) = -\frac{1}{1440} (15C_{i3}^4 - 35C_{i3}^2 C_{i4} + 10C_{i4}^2 + 14C_{i3} C_{i5} - 4C_{i6}) ; \\
R_3(7) = -\frac{1}{60480} (315C_{i3}^5 - 945C_{i3}^3 C_{i4} + 560C_{i3} C_{i4}^2 + 399C_{i3}^2 C_{i5} - 231C_{i4} C_{i5} \\
-126C_{i3} C_{i6} + 30C_{i7}) ; \\
R_3(8) = -\frac{1}{120960} (315C_{i3}^6 - 1155C_{i3}^4 C_{i4} + 1050C_{i3}^2 C_{i4}^2 - 140C_{i4}^3 + 504C_{i3}^3 C_{i5} \\
-602C_{i3} C_{i4} C_{i5} + 63C_{i5}^2 - 168C_{i3}^2 C_{i6} + 98C_{i4} C_{i6} + 44C_{i3} C_{i7} - 9C_{i8}) .
\]

Residues of form \( R_4(p, q) := \text{Res}_{z=z_i} \frac{\lambda(q)(z)-\lambda(z)}{(z_i-z)p \lambda(z)} \).

\[ R_4(5, 3) = \frac{1}{1440} (45C_{i3}^6 - 210C_{i3}^4 C_{i4} + 240C_{i3}^2 C_{i4}^2 - 40C_{i4}^3 + 135C_{i3}^3 C_{i5} \\
-200C_{i3} C_{i4} C_{i5} + 30C_{i5}^2 - 72C_{i3}^2 C_{i6} + 52C_{i4} C_{i6} + 32C_{i3} C_{i7} - 12C_{i8}) ; \\
R_4(2, 4) = \frac{1}{12} (3C_{i3}^2 C_{i4} - 2C_{i4}^2 - 6C_{i3} C_{i5} + 6C_{i6}) ; \\
R_4(3, 4) = \frac{1}{24} (3C_{i3}^3 C_{i4} - 4C_{i3} C_{i4}^2 - 6C_{i3}^2 C_{i5} + 5C_{i4} C_{i5} + 6C_{i3} C_{i6} - 4C_{i7}) ; \\
R_4(4, 4) = \frac{1}{720} (45C_{i3}^4 C_{i4} - 90C_{i3}^2 C_{i4}^2 + 20C_{i4}^3 - 90C_{i3}^3 C_{i5} \\
+150C_{i3} C_{i4} C_{i5} - 30C_{i5}^2 + 90C_{i3}^2 C_{i6} - 66C_{i4} C_{i6} - 60C_{i3} C_{i7} + 30C_{i8}) .
\]
Residues of form \( R_5(p, q) := (\text{Res}_{z = z_i} + \text{Res}_{z = z_j}) \left( \frac{\lambda''(z)}{(z_i - z)\lambda'(z)} \right) \).

\[
R_5(2, 2) = \frac{6}{z_{ij}^4} - \frac{C_{i3} - C_{j3}}{z_{ij}^3} - \frac{3C_{i3}^2 + 3C_{i4}^2 - 4C_{i4} - 4C_{j4}}{12z_{ij}^2};
\]

\[
R_5(2, 4) = \frac{15}{z_{ij}^6} - \frac{2C_{i3} - 2C_{j3}}{z_{ij}^5} - \frac{3C_{i3}^2 + 9C_{i4}^2 - 4C_{i4} - 12C_{j4}}{12z_{ij}^4} + \frac{C_{i3}^3 - 2C_{i3}C_{i4} + C_{j3}^3}{4z_{ij}^3};
\]

\[
R_5(4, 2) = \frac{15}{z_{ij}^6} - \frac{2C_{i3} - 2C_{j3}}{z_{ij}^5} - \frac{3C_{i3}^2 + 9C_{i4}^2 - 4C_{i4} - 12C_{j4}}{12z_{ij}^4} - \frac{C_{i3}^3 - 2C_{i3}C_{i4} + C_{i5}}{4z_{ij}^3} - \frac{45C_{j3}^4 - 120C_{j3}^2 C_{j4} + 40(C_{j4})^2 + 60C_{j3} C_{j5} - 24C_{j6}}{720z_{ij}^2};
\]

Residues of form \( R_6(p, q) := (\text{Res}_{z = z_i} + \text{Res}_{z = z_k}) \left( \frac{\lambda''(z) - \lambda''(z)}{(z_i - z)\lambda'(z)} \right) \).

\[
R_6(2, 2) = \frac{1}{2z_{ik}^4} - \frac{h_i^2 u_{ik}}{z_{ik}^3}\left( \frac{3}{z_{ik}^2} + \frac{C_{i3} + C_{i4}^2 + C_{i4}}{4} - \frac{C_{i4}}{6} \right);
\]

\[
R_6(2, 3) = \frac{3}{2z_{ik}^6} - \frac{C_{i3}^3 - C_{i4}^3}{12z_{ik}^5} - \frac{h_i^2 u_{ik}}{z_{ik}^4}\left( \frac{6}{z_{ik}^3} + \frac{3C_{i3} + C_{i4}^2 + C_{i4}}{4} - \frac{C_{i4}}{6} \right);
\]

\[
R_6(2, 4) = \frac{2}{z_{ik}^6} - \frac{C_{i3}^3 - C_{i4}^3}{12z_{ik}^5} - \frac{h_i^2 u_{ik}}{z_{ik}^4}\left( \frac{10}{z_{ik}^3} + \frac{2C_{i3} + C_{i4}^2 + C_{i4}}{4} - \frac{C_{i4}}{6} \right);
\]

\[
R_6(3, 2) = \frac{3}{2z_{ik}^6} - \frac{C_{i3}^3 - C_{i4}^3}{12z_{ik}^5} - \frac{h_i^2 u_{ik}}{z_{ik}^4}\left( \frac{4}{z_{ik}^3} + \frac{3C_{i3} + C_{i4}^2 + C_{i4}}{4} - \frac{C_{i4}}{6} \right);
\]

\[
R_6(4, 2) = \frac{2}{z_{ik}^6} - \frac{C_{i3}^3 - C_{i4}^3}{12z_{ik}^5} - \frac{h_i^2 u_{ik}}{z_{ik}^4}\left( \frac{5}{z_{ik}^3} + \frac{2C_{i3} + C_{i4}^2 + C_{i4}}{4} - \frac{C_{i4}}{6} \right);
\]

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