EXISTENCE OF SOLUTION OF A MICROWAVE HEATING MODEL AND ASSOCIATED OPTIMAL FREQUENCY CONTROL PROBLEMS

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Abstract. Microwave heating has been widely used in various fields during recent years. However, it also has a common problem of uneven heating. In this paper, optimal frequency control problem for microwave heating process is considered. The cost function is defined such that the temperature profile at the final stage has a relative uniform distribution in the field. The controlled system is a coupled by Maxwell equations with nonlinear heating equation. The existence of a weak solution for coupled system is proved. The weak continuity of the solution operator is also shown. Moreover, the existence of a global minimizer of the optimal frequency control problems is proved.

1. Introduction. Currently, the applications of microwave heating technology are very broad and great economic value. However, the major drawback associated with microwave heating is the existence of nonuniform temperature distribution and permittivity variations, which will lead to some problems, such as overheating and thermal runaway in the heated media; see, e.g., [12, 13].

Microwave heating is highly sensitive to frequency. The electric field evolves with frequency, forming various hot spots at different positions. In practical application, how to choose microwave frequency is closely related to the matter of heating. The experiment shows that the high frequency microwave osmosis heating material is shallow, but the heating time is fast. The slow frequency microwave osmosis heating material is deep, but the heating time is slow. Therefore, microwave frequency plays an important role in microwave heating technology. Many researchers studied that frequency-selected improved heating uniformity and heating efficiency by solving

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Maxwell equations coupled with heat equation numerically or by experiment. One can refer to [4,14,15] and therein.

In this paper, the uniform temperature distribution of heated stuff attained through optimal control theory is studied. An optimal frequencies control modeling is proposed to improve the uniform heating performance to refrain from hot spot. We formulate this model as an optimal control problem in which the underlying dynamics is governed by Maxwell’s equations coupled with nonlinear heat conduction with a source generated by microwaves. The goal of the control is to reach a relatively uniform heat profile at a specific time by controlling the microwave frequency.

Optimal control associated with partial differential equations or systems as underlying states have been studied by many researchers. There are several classical books, see e.g., [1,3,11], and more recent ones, see e.g., [7–9,16]. For optimal control problems of microwave heating, Li, Tang and Yin [10] considered a similar problem where the control is chosen from the heat source. Wei and Yin [17,18] considered a boundary control problem where the external electric field is chosen as the control function.

In this paper, we study how to choose desired frequency as we like by designing various semiconductor devices. Control appears in both coefficients and right side of Maxwell’s equations and heat equation. It poses numerous mathematical challenges to analysis of microwave heating models due to the coupled Maxwell equations and nonlinear heat equation. An approach has been considered in the paper [18]. But we deduced more precise model and admit more general coefficients condition which is not necessary satisfy assumption H(2.3) in [18]. Here, the heat equation is strongly nonlinear, that is, quasi-linear equation. We use the monotone operator theory to prove the existence of a solution. In addition, one of the difficulties in the present paper is the nonlinear heat source often belongs to $L^1$ for this type of coupled systems. This causes a serious problem when one needs to prove the existence of an optimal control. By applying some recent estimates obtained in [19–21] and techniques for elliptic-parabolic equations in [2,6], we are able to obtain a better estimate for the nonlinear heat source (see Lemma 3.7 below). With this new estimate, we can overcome the difficulties to establish the existence result.

This paper is organized as follows. In Section 2, we use a unified approach to derive the mathematical model for microwave and induction heating. The optimal control problem is formulated, in which the control function is the microwave frequency. In Section 3, we prove that the underlying system has a weak solution for any frequency in a suitable admissible set. Moreover, some important estimates are derived in this section. In Section 4, we prove the weak sequential continuity of the solution map and deduce the existence of a global minimizer of associated optimal control problems as an application.

2. The formulation of an optimal control problem. Suppose that a targeted substance, say, food-like material, is placed in a microwave processor cavity, denoted by $\Omega$ with $C^1$-boundary $S = \partial \Omega$. Let $E(x,t)$ and $H(x,t)$ denote the electric and magnetic fields at $x \in \Omega$ and time $t$. Hereafter, a bold letter represents a vector function in $\mathbb{R}^3$. From electromagnetic theory, Maxwells equations in $\Omega$ can be expressed by

$$
\begin{align*}
\varepsilon E_t + \sigma E &= \nabla \times H, \\
\mu H_t + \nabla \times E &= 0, \\
\nabla \cdot H &= 0.
\end{align*}
$$

(2.1)
where Ohms law $J = \sigma E$ is used; $\varepsilon$, $\mu$, and $\sigma$ are the electric permittivity, magnetic permeability, and electric conductivity, respectively.

We note that $\nabla \times H = 0$ holds automatically as long as an initial field $H(x, 0)$ satisfies the same condition. Due to the high frequency of the microwaves, the scale of the time variable in electromagnetic fields is very different from that for heat conduction. It is common in practice to assume that the electric and magnetic fields are time harmonic, with frequency depend on $x$. With this assumption, Maxwell's equations can be reduced to a Helmholtz type of system. Indeed, let

$$
\begin{align*}
\mathbf{E}(x, t) &= \hat{\mathbf{E}}(x)e^{i\omega(x)t} , \\
\mathbf{H}(x, t) &= \hat{\mathbf{H}}(x)e^{i\omega(x)t},
\end{align*}
$$

where $i$ denotes the unit complex number.

We have

$$
\begin{align*}
\mathbf{E}_t(x, t) &= \hat{\mathbf{E}}(x)e^{i\omega(x)t}i\omega(x) = i\omega(x)\mathbf{E}(x, t), \\
\mathbf{H}_t(x, t) &= \hat{\mathbf{H}}(x)e^{i\omega(x)t}i\omega(x) = i\omega(x)\mathbf{H}(x, t).
\end{align*}
$$

Replace (2.2) into the (2.1), and it can be obtained

$$
\begin{align*}
\varepsilon i \omega(x)\mathbf{E}(x, t) + \sigma \mathbf{E}(x, t) &= \nabla \times \mathbf{H}(x, t), \\
\mu i \omega(x)\mathbf{H}(x, t) + \nabla \times \mathbf{E}(x, t) &= 0.
\end{align*}
$$

Therefore,

$$
\nabla \times \left( \frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{E}(x, t) \right) + \xi(x, t) = 0,
$$

where $\gamma(x) = \frac{1}{\mu(x)}$ and

$$
\xi(x, \omega(x)) = -\varepsilon(x)\omega(x) + i\sigma(x) \triangleq -a_1(x, \omega(x)) + ia_2(x, \omega(x)).
$$

To derive the equation of heat conduction, we need to derive the heat density generated by microwaves. There are two types of current: displacement currents $J_d = \varepsilon(i\omega)E$ and eddy currents, $J_e = \sigma E$, by Ohms law. For a dielectric material, the displacement current dominates the current flow, while for a metallic material, the eddy current dominates the flow. We combine the two types of material by using a unified quantity, denoted by $J_{total}$. The total current density can be expressed by

$$
J_{total} = (a_2(x, \omega(x)) + ia_1(x, \omega(x)))E.
$$

For microwave heating and inductive heating, the time-average power dissipated in a material per unit volume is given by (12, Chapter 3)

$$
Q(x, t) = \frac{1}{2}Re[\mathbf{E} \cdot \mathbf{J}_{total}^*] = \frac{1}{2}a_2(x, \omega(x))|\mathbf{E}|^2,
$$

where $\mathbf{J}_{total}^*$ represents the complex conjugate of $\mathbf{J}_{total}$.

Let $u(x, t)$ denote the heated stuff temperature at $x \in \Omega$ and time $t$. With the above local heat source, by using Fouriers law and the conservation of energy, one can easily see that the temperature $u(x, t)$ satisfies a nonlinear heat equation with an internal source generated by microwaves:

$$
\rho cu_t - \nabla [k(x, u)\nabla u] = \frac{1}{2}a_2(x, \omega(x))|\mathbf{E}|^2, \text{ in } Q_T,
$$

where $Q_T = \Omega \times (0, T]$, $\rho$ is the density, $c$ the specific heat, and $k(x, u)$ the heat conductivity.
We sum up the above derivation and normalize certain physical constants to obtain the following mathematical model:

$$
\begin{cases}
\nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times E) + (-a_1(x, \omega(x)) + ia_2(x, \omega(x)))E = 0, & x \in \Omega, \\
u_t - \nabla^2[k(x, u)\nabla u] = \frac{1}{2}a_2(x, \omega(x))|E|^2, & (x, t) \in Q_T, \\
n \times E(x) = n \times G(x), & x \in \partial\Omega, \\
u_n(x, t) = 0, & (x, t) \in S_T, \\
u(x, 0) = u_0, & x \in \Omega,
\end{cases}
\tag{2.8}
$$

where $n$ is the outward unit normal on $S = \partial\Omega$, $u_n = \nabla u \cdot n$ is the normal derivative on $S$, and $G(\cdot)$ is the time-harmonic electric field generated by external optoelectronic devices be given, $\omega(\cdot) \in U_{ad}$ is control variable with the admissible set $U_{ad} = \{\omega : \Omega \to \mathbb{R} \text{ is measurable} : 0 < \omega_0 \leq \omega(x) \leq \omega_1\}$ where $\omega_0$ and $\omega_1$ are two positive constants.

**Optimal control problem (P):** Given a time $T > 0$ and a desired temperature $u_T(\cdot)$ in $\Omega$, we would like to find an optimal frequency control $\omega(\cdot) \in U_{ad}$ such that the cost functional $J(\omega; E, u)$ reaches its minimum, where

$$
J(\omega; E, u) := \int_{\Omega} |u(x, T) - u_T(x)|^2 dx + \frac{\lambda}{2} \int_{\Omega} |\omega(x)|^2 dx,
\tag{2.9}
$$

where the constant $\lambda \geq 0$ is the regularization parameter, the function pair $(E, u)$ is the solution of underlying system (2.8) corresponding to $\omega(\cdot) \in U_{ad}$.

**Remark 2.1.** In the above model derivation, the electric field $E$ is assumed to be time harmonic. However, it still depends on time variable from the heat conduction, since the time scale for electric waves is much faster than the time variable in the heat conduction.

3. **Existence of solution for the underlying system.** Throughout this paper, $\Omega$ is always assumed to be a bounded and simply connected domain in $\mathbb{R}^3$, and the boundary of $\Omega$ is $C^1$-continuous.

We recall some standard Banach spaces. For convenience, a product space $B^n$ is often simply denoted by $B$. Let

$$
\begin{align*}
H(curl, \Omega) &= \{G(\cdot) \in L^2(\Omega) : \nabla \times G \in L^2(\Omega)\}, \\
H_0(curl, \Omega) &= \{G(\cdot) \in L^2(\Omega) : \nabla \times G \in L^2(\Omega), \ n \times G = 0 \text{ on } \partial\Omega\}, \\
H(div, \Omega) &= \{G(\cdot) \in L^2(\Omega) : \nabla \cdot G \in L^2(\Omega)\}.
\end{align*}
$$

$H_0(curl, \Omega)$ and $H(curl, \Omega)$ are Hilbert spaces equipped with inner product

$$
\langle G, F \rangle = \int_{\Omega} [(\nabla \times G) \cdot (\nabla \times F^*) + G \cdot F^*] dx,
$$

where $F^*$ represents the complex conjugate of $F$. $H^1(\Omega) = W^{1,2}(\Omega)$ is the usual Sobolev space.

We impose some basic assumptions which ensure the well-posedness of the underlying system.

**H(1)** Functions $u_0(\cdot)$ and $u_T(\cdot)$ are nonnegative with $u_0(\cdot), u_T(\cdot) \in L^2(\Omega)$.

**H(2)** Let $k : \Omega \times R \to R$ be a given function and $0 < k_0 \leq k(x, u) \leq k_1 \leq 1$ holds for all $x \in \Omega$, $u \in R$ with positive constants $k_0$ and $k_1$. Suppose that:

- $x \mapsto k(x, u)$ is measurable on $\Omega$ for all $u \in R$,
- $u \mapsto k(x, u)$ is uniformly Lipschitz continuous on $R$ for almost all $x \in \Omega$. 

**H(3)** (a) Assume that the functions \( \sigma : \Omega \to R \) is real, positive, measurable, and bounded function, \( 0 < \sigma(x) \leq \sigma_1 \) for all \( x \in \Omega \) and some fixed constant \( \sigma_1 \). The function \( \varepsilon := \varepsilon_1 + i \varepsilon_2 \) is assumed to be a bounded complex function with positive real and image parts. Moreover, \( 0 < \varepsilon_1 \leq \varepsilon(x), \varepsilon_2(x) \leq \varepsilon_u \) for all \( x \in \Omega \) and some constants \( \varepsilon_1 > 0 \) and \( \varepsilon_u > 0 \).

(b) The function \( \gamma = \frac{1}{\varepsilon} := \gamma_1 + i \gamma_2 \) is assumed to be a bounded complex function, and \( \gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0 \) for all \( x \in \Omega \) and some constant \( \gamma_0 > 0 \).

(c) The function \( \mathbf{G} \) is given and defined on \( S \) with an extension such that \( \mathbf{G}(\cdot) \in H(curl, \Omega) \) with extended function \( \overline{\mathbf{G}} \) satisfies \( \| \overline{\mathbf{G}} \|_{H(curl, \Omega)} \leq c_0 \| \mathbf{G} \|_{L^2(S)} \), where \( c_0 \) is a constant that depends only on \( \Omega \).

For convenience, we shall denote the extended function \( \overline{\mathbf{G}}(\cdot) \) by \( \mathbf{G}(\cdot) \) with \( \mathbf{G}(\cdot) \in H(curl, \Omega) \).

It follows assumption \( \textbf{H}(3)(a) \) and the definition that there are positive constants \( a_0 > 0 \) and \( b_0 > 0 \) such that

\[
0 < a_0 \leq a_i(x, \omega) \leq b_0, \quad i = 1, 2. \tag{3.1}
\]

for any \( x \in \Omega \) and \( \omega \in U_{ad} \).

In the following, we will show that the system (2.8) has a weak solution under the hypotheses \( \textbf{H}(1) - \textbf{H}(3) \) for any given \( \omega(\cdot) \in U_{ad} \).

Consider the following system

\[
\begin{cases}
\nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{E}) + \xi(x, \omega(x))\mathbf{E} = 0, & x \in \Omega, \\
\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, & x \in \partial \Omega. \tag{3.2}
\end{cases}
\]

Let \( \overline{\mathbf{E}} = \mathbf{E} - \mathbf{G} \). Then \( \overline{\mathbf{E}} \) satisfies the following linear system

\[
\begin{cases}
\nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times \overline{\mathbf{E}}) + \xi(x, \omega(x))\overline{\mathbf{E}} = \mathbf{F}, & x \in \Omega, \\
\mathbf{n} \times \overline{\mathbf{E}}(x) = 0, & x \in S, \tag{3.3}
\end{cases}
\]

where \( \mathbf{F} = -(\nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{G}) + \xi(x, \omega(x))\mathbf{G}) \).

**Definition 3.1 (weak solution).** We say \( \overline{\mathbf{E}} \in H_0(curl, \Omega) \) is a weak solution of problem (3.3), or \( \mathbf{E} = \mathbf{E} + \mathbf{G} \) is a weak solution of (3.2), if for all \( \Phi \in H(curl, \Omega) \), it satisfies

\[
-\int_{\Omega} (\frac{\gamma(x)}{\omega(x)} \nabla \times \overline{\mathbf{E}}) \cdot (\nabla \times \Phi) dx + \int_{\Omega} \xi(x, \omega(x))\overline{\mathbf{E}} \cdot \Phi dx = (\mathbf{F}, \Phi),
\]

where \( (\mathbf{F}, \Phi) = -\int_{\Omega} (\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{G}) \cdot (\nabla \times \Phi) dx + \int_{\Omega} \xi(x, \omega(x)) \mathbf{G} \cdot \Phi dx \).

**Theorem 3.2.** Under assumption \( \textbf{H}(3) \) the linear system (3.2) has unique weak solution with \( \overline{\mathbf{E}} = \mathbf{E} - \mathbf{G} \in H_0(curl, \Omega) \).

**Proof.** Define a form \( B_{\xi}[\overline{\mathbf{E}}, \Phi] \) as follows

\[
B_{\xi}[\overline{\mathbf{E}}, \Phi] := -\int_{\Omega} (\frac{\gamma(x)}{\omega(x)} \nabla \times \overline{\mathbf{E}}) \cdot (\nabla \times \Phi) dx + \int_{\Omega} \xi(x, \omega(x))\overline{\mathbf{E}} \cdot \Phi dx,
\]

for any \( \overline{\mathbf{E}} \in H_0(curl, \Omega) \) and \( \Phi \in H(curl, \Omega) \). It is easy to see that \( B_{\xi}[\overline{\mathbf{E}}, \Phi] \) is bilinear form.
Next, since $\gamma(x)$ is a bounded complex function and $\omega(x)$ is a bounded and positive, and $a_i(x,\omega(x))$ are bounded functions from (3.1), we readily check
\[
|B_\xi(\tilde{E}, \Phi)| = \left| - \int_\Omega \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{E} \right) \cdot (\nabla \times \Phi) dx + \int_\Omega \xi(x,\omega(x)) \tilde{E} \cdot \Phi dx \right|
\leq \frac{\sqrt{70}}{\omega_0} \int_\Omega |\nabla \times \tilde{E}| \cdot |\nabla \times \Phi| dx + \sqrt{2}b_0 \int_\Omega |\tilde{E}| \cdot |\Phi| dx 
\leq C_1 |\tilde{E}|_{H^0(curl,\Omega)} ^2 |\Phi|_{H^0(curl,\Omega)} ^2,
\]
where $C_1 = \max(\frac{\sqrt{70}}{\omega_0}, \sqrt{2}b_0)$.

Further more, we have
\[
|B_\xi(\tilde{E}, \tilde{E})| = \left| - \int_\Omega \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{E} \right) \cdot (\nabla \times \tilde{E}) dx + \int_\Omega (a_1(x,\omega(x)) - a_2(x,\omega(x))) \tilde{E} \cdot \tilde{E} dx \right|
\geq \sqrt{\left(\frac{70}{\omega_1} \|
abla \times \tilde{E}\|^2_{L^2(\Omega)} - b_0 \|	ilde{E}\|^2_{L^2(\Omega)} \right)^2 + \left(\frac{70}{\omega_1} \|
abla \times \tilde{E}\|^2_{L^2(\Omega)} + a_0 \|	ilde{E}\|^2_{L^2(\Omega)} \right)^2}
\geq \min\left\{\frac{70}{\omega_1}, a_0\right\} \left(\|
abla \times \tilde{E}\|^2_{L^2(\Omega)} + \|	ilde{E}\|^2_{L^2(\Omega)} \right)
\geq C_2 \left(\|
abla \times \tilde{E}\|^2_{L^2(\Omega)} + \|	ilde{E}\|^2_{L^2(\Omega)} \right),
\]
where constant $C_2 = \min\left\{\frac{70}{\omega_1}, a_0\right\}$. Thus
\[
|B_\xi(\tilde{E}, \Phi)| \geq C_2 |\tilde{E}|_{H^0(curl,\Omega)} ^2.
\]

Now fix $\Phi(\cdot) \in L^2(\Omega)$ and set
\[
(\Phi, \Phi) := (\tilde{F}, \Phi)_{L^2(\Omega)}
\]
\[
= - \int_\Omega \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{G} \right) (\nabla \times \Phi) dx + \int_\Omega \xi(x,\omega(x)) \tilde{G} \cdot \Phi dx.
\]
It is easy to show that $(\Phi, \Phi)_{L^2(\Omega)}$ is bounded linear functional on $L^2(\Omega)$. It follows from Lax-Milgram Theorem that there is unique function $\tilde{E} \in H_0(curl,\Omega)$ satisfying
\[
B_\xi(\tilde{E}, \Phi) = (\Phi, \Phi),
\]
for all $\Phi \in H(curl,\Omega)$.

Consider the following problem
\[
\begin{cases}
    u_t - \nabla [k(x,u) \nabla u] = \frac{1}{2} a_2(x,\omega(x)) |\tilde{E}|^2 & (x,t) \in Q_T, \\
    u_n(x,t) = 0 & (x,t) \in S_T, \\
    u(x,0) = u_0(x) & x \in \Omega.
\end{cases}
\]
Let $V = H^1(\Omega)$, $H = L^2(\Omega)$, then $V^* = H^{-1}(\Omega)$ and $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple.
Define space $W[0, T]$ as

$$W[0, T] = \{ u \in L^2(0, T; H^1(\Omega)), u_t \in L^2(0, T; H^{-1}(\Omega)) \},$$

with the norm

$$\|u\|_{W[0, T]}^2 = \|u\|_{L^2(0, T; V)}^2 + \|u_t\|_{L^2(0, T; V')}^2.$$  

Then $W[0, T]$ is a Banach space and the embedding $W[0, T] \hookrightarrow C([0, T]; H)$ is continuous and $W[0, T] \hookrightarrow L^2([0, T]; H)$ is compact ([1], Proposition 23), where $\frac{d}{dt}$ means the generalized derivative of real functions on $[0, T]$.

By applying Sobolev’s embedding with the dimension $N = 3$ ([22], Page 1027) and the solution $E \in H(curl, \Omega)$ of (3.2), we know that the $E \in L^6(\Omega)$. Hence, the function $f(t) = \frac{1}{2}a_2(\cdot, \omega(\cdot))|E(\cdot, t)|^2$ is in $L^2(\Omega)$ and the following formula is well defined.

**Definition 3.3.** We say $u \in W[0, T]$ is a weak solution of (3.4) if

$$\frac{d}{dt} \int_{\Omega} u(x, t)v(x)dx + \int_{\Omega} k(x, u)\nabla u(x, t)\nabla v(x)dx = \frac{1}{2} \int_{\Omega} a_2(x, \omega(x))|E|^2 v(x)dx$$

for all $v \in V$. Then the equation can be written in the form

$$\frac{d}{dt}(u(t), v)_H + a(u(t), v) = (f(t), v)_H \quad \text{on } [0, T],$$

for all $v \in V$.

Now, we introduce the operator $A(t) : V \rightarrow V^*$ and the functional $b(t) \in V^*$ by

$$\langle Aw, v \rangle_v = a(w, v)$$

$$\langle b(t), v \rangle_v = (f(t), v)_H.$$  

Moreover, we set $X = L^2(0, T; V)$ and hence $X^* = L^2(0, T; V^*)$. Define $(Au)(t) = Au(t)$ for all $t \in [0, T]$. Then the original problem is equivalent to the following operator equation

$$\begin{cases} u'' + A(t)u = b(t), \\ u(0) = u_0, \\ u \in X, u_t \in X^*, \end{cases}$$

for all $v \in V$, where $u_0 \in H_0$ and $b \in X^*$ are given.

**Lemma 3.4.** Under the assumptions (H1), (H2) and (H3), then the operator $A$ satisfies the following properties

(1) For each $t \in [0, T]$, the operator $A(t) : V \rightarrow V^*$ is monotone and hemicontinuous.

(2) For each $t \in [0, T]$, the operator $A(t)$ is coercive, i.e. there exist constants $C_1 > 0$ and $C_2 > 0$, such that

$$\langle A(t)v, v \rangle_V \geq C_1\|v\|_V^2 - C_2$$

for all $v \in V$, $t \in [0, T]$.  

(3) Growth condition
\[ \|A(t)v\|_{V'} \leq C_3(t) + C_4\|v\|_V \text{ for all } v \in V, \ t \in [0, T]. \]

**Proof.** (1). Since
\[ < A(u + \lambda v), w > = a(u + \lambda v, w) = \int \Omega k(x, u + \lambda v) \nabla (u + \lambda v) \cdot \nabla w \, dx. \]
It follows from assumption \((H1)\) that \(\lambda \mapsto < A(u + \lambda v), w >\) is continuous on \([0, 1]\) for all \(u, v, w \in V\). Hence \(A\) is hemicontinuous.

\[ \langle Au - Av, u - v \rangle_V = \int \Omega [k(x, u)\nabla u - k(x, v)\nabla v](\nabla u - \nabla v) \, dx \]
\[ = \int \Omega [k(x, u)|\nabla u|^2 - 2k(x, u)k(x, v)\nabla u \nabla v + k(x, v)|\nabla v|^2] \, dx. \]

**Case 1.** \(|\nabla v| = 0\) then
\[ \langle Au - Av, u - v \rangle_V = \int \Omega k(x, u)|\nabla u|^2 \, dx \geq k_0 \int \Omega |\nabla u|^2 \, dx \geq 0. \]

**Case 2.** \(|\nabla v| \neq 0\) then
\[ k(x, u)|\nabla u|^2 - 2k(x, u)k(x, v)\nabla u \nabla v + k(x, v)|\nabla v|^2 \]
\[ = |\nabla v|^2 \left\{ k(x, u)\left|\frac{\nabla u}{|\nabla v|}\right|^2 - 2k(x, u)k(x, v)\frac{|\nabla u|}{|\nabla v|} \cos \theta + k(x, v) \right\}, \]
where \(\theta\) is included angle between \(\nabla u\) and \(\nabla v\).

Let \(y = |\nabla u|^2\) and
\[ P(y) = k(x, u)y^2 - 2k(x, u)k(x, v)y\cos \theta + k(x, v). \]
Since \(k(x, u) \leq k_1 \leq 1\), we have
\[ (-2k(x, u)k(x, v)\cos \theta)^2 - 4k(x, u)k(x, v) \leq 0. \]
Hence \(P(y) \geq 0\) for any \(y \in R\). We claim
\[ \langle Au - Av, u - v \rangle_V \geq 0, \]
for any \(u, v \in V\). That is, \(A\) is monotone.

(2). By the inequality Sobolev ([5], Page 519), we know that
\[ \|\nabla v\|_{L^2(\Omega)}^2 \geq C\|v\|_{L^2(\Omega)}^2, \]
where \(C = \frac{1}{\Gamma(\frac{3}{2})} \Gamma(1+\frac{3}{2})^2 = \frac{1}{\Gamma(\frac{3}{2})^2} \Gamma(1+\frac{3}{2})^2\). It follows from assumption \((H2)\) and inequality above that
\[ \langle Av, v \rangle_V = \int \Omega k(x, v)\nabla v \cdot \nabla vdx \geq k_0\|\nabla v\|_{L^2(\Omega)}^2 \geq \frac{1}{2}k_0\|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{2}k_0 C\|v\|_{L^2(\Omega)}^2 \geq C_1\|v\|_V^2 \]
for all \(v \in V\), where \(C_1 = \min(\frac{1}{2}k_0, \frac{1}{2}k_0 C) > 0\).

(3).
\[ \langle Av, v \rangle_V = \int \Omega k(x, v)|\nabla v|^2 dx \leq k_1\|v\|_{V'}^2. \]
Hence,
\[ \|Av\|_{V'} \leq k_1\|v\|_V, \]
for all \( v \in V \).

Now we have checked that all assumptions of Theorem 30.A in [22] hold. By Theorem 30.A of [22], we obtain the existence theorem as follows.

**Theorem 3.5.** Under the assumptions \( H(1), H(2), \) and \( H(3) \), the system (3.7) has unique weak solution \( u \in W[0,T] \) for any given \( E \in H(\text{curl}, \Omega) \).

From what has been discussed above, we obtain the existence of solutions of the coupled system (2.8).

**Theorem 3.6.** The assumptions \( H(1) - H(3) \), the coupled system (2.8) has unique weak solution \( (E, u) \) with \( E - G \in H_0(\text{curl}, \Omega) \) and \( u \in W[0,T] \).

In order to prove the existence of a minimum for the cost functional \( J(\omega; E, u) \), the following estimates ([17], Lemma 3.1.) for solutions of problem (2.8) are essential for the proof of the main result.

**Lemma 3.7.** Under the assumptions \( H(1) - H(3) \), there are positive constants \( C_0, C_1, C_2 \) and \( C_3 \) which depend on known data such that

\[
\int_{\Omega} |E|^2 \, dx \leq C_0, \quad (3.8)
\]

\[
\int_{\Omega} |\nabla \times E|^2 \, dx + \int_{\Omega} |E|^6 \, dx \leq C_1, \quad (3.9)
\]

\[
\int_{\Omega} \left( |\nabla \cdot (a_1 E)|^2 + |\nabla (a_2 E)|^2 \right) \, dx \leq C_2, \quad (3.10)
\]

\[
\max_{t \in [0,T]} \int_{\Omega} |u|^2 \, dx + \int_{0}^{T} \int_{\Omega} |\nabla u|^2 \, dx \, dt \leq C_3. \quad (3.11)
\]

### 4. Existence of an optimal control

In this section, the aim is to prove the existence of a global minimizer by way of weak continuity of the control-to-state mapping. With Theorem 3.6 at hand we can define the control-to-state mapping \( P: \omega \mapsto (E, u) \) for the microwave heating model (2.8).

**Proposition 1.** Under the assumptions \( H(1) - H(3) \) and provided \( \omega(\cdot) \in L^p(\Omega) \) holds, the control-to-state mapping \( P: \omega \mapsto (E, u) \) from \( L^p(\Omega) \) into \( H(\text{curl}, \Omega) \times W[0,T] \) is weakly sequentially continuous for any \( p > 1 \).

**Proof.** Let us consider sequences \( \{\omega_n\} \subset L^p(\Omega) \):

\[
\omega_m \rightharpoonup \omega^* \text{ weakly in } L^p(\Omega), \quad (4.1)
\]

as \( m \to +\infty \). Define \( (E_m, u_m) := P(\omega_m) \), we have to show that

\[
(E_m, u_m) = P(\omega_m) \rightharpoonup P(\omega^*) =: (E^*, u^*).
\]

The definition of mapping \( P \) implies,

\[
\begin{cases}
\int_{\Omega} \frac{1}{\epsilon_n} (\nabla \times E_m) \cdot (\nabla \times \Phi) + \left( -a_1(x, \omega_m) + i a_2(x, \omega_m) \right) E_m \cdot \Phi \, dx = 0, \\
n \times E_m = n \times G, \quad x \in \partial\Omega, \\
\frac{d}{dt} \int_{\Omega} u_m \, dx + \int_{\Omega} k(x, u_m) \nabla u_m \nabla v \, dx = \frac{1}{2} \int_{\Omega} a_2(x, \omega_m) |E_m|^2 v \, dx, \\
u_m(x, 0) = u_0(x), \quad x \in \Omega,
\end{cases} \quad (4.2)
\]

for \( m = 1, 2, \ldots \), where \( \frac{d}{dt} \) denotes the generalized derivative on \([0,T]\).
It follows from the Lemma 3.7 that \( \{E_m, u_m\} \) is bounded in reflexive space \( H(curl, \Omega) \times W[0, T] \). So, there exists a subsequence of \( \{E_m, u_m\} \) such that
\[
E_m \to E^* \text{ weakly in } H(curl, \Omega), \quad (4.3)
\]
\[
u_m \to \nu^* \text{ weakly in } W[0, T], \quad (4.4)
\]
\[
u_m(\cdot, t) \to \nu^*(\cdot, t) \text{ weakly in } L^2(\Omega) \text{ for any } t \in [0, T]. \quad (4.5)
\]
Moreover, the compactness embedding \( H(curl, \Omega) \hookrightarrow L^2(\Omega) \) and \( W[0, T] \hookrightarrow L^2(0, T; H) \) imply that
\[
E_m \to E^* \text{ strongly in } L^2(\Omega), \quad (4.6)
\]
\[
u_m \to \nu^* \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (4.7)
\]
The definitions of the spaces \( H(curl, \Omega) \) and \( W[0, T] \) imply that
\[
\nabla \times E_m \to \nabla \times E^* \text{ weakly in } L^2(\Omega),
\]
\[
\nabla u_m \to \nabla \nu^* \text{ weakly in } L^2(0, T; L^2(\Omega)),
\]
as \( m \to +\infty \).

Let us compute
\[
\left| \int_\Omega \frac{\gamma}{\omega_m} (\nabla \times E_m) \cdot (\nabla \times \Phi) dx - \int_\Omega \frac{\gamma}{\omega^*} (\nabla \times E^*) \cdot (\nabla \times \Phi) dx \right|
\]
\[
= \left| \int_\Omega \left[ \frac{\gamma}{\omega_m} \nabla \times (E_m - E^*) \cdot (\nabla \times \Phi) + \left( \frac{1}{\omega_m} - \frac{1}{\omega^*} \right) \gamma (\nabla \times E^*) \cdot (\nabla \times \Phi) \right] dx \right|
\]
\[
\leq \int_\Omega \left| \frac{\gamma}{\omega_m} \nabla \times (E_m - E^*) \cdot (\nabla \times \Phi) dx + \int_\Omega \left| \frac{\gamma}{\omega_m} - \frac{\gamma}{\omega^*} \right| (\nabla \times E^*) \cdot (\nabla \times \Phi) dx \right|
\]
where \( \omega^0 \) is the upper bound and positive of \( \omega(x) \). Using (4.8) and (4.1) for any \( t \in [0, T] \), we have
\[
\int_\Omega \frac{\gamma(x)}{\omega_m(x)} (\nabla \times E_m(x)) \cdot (\nabla \times \Phi(x)) dx \to \int_\Omega \frac{\gamma(x)}{\omega^*(x)} (\nabla \times E^*(x)) \cdot (\nabla \times \Phi(x)) dx \quad (4.10)
\]
as \( m \to +\infty \). Because \( a_1(x, \omega_m(x)) \) and \( a_2(x, \omega_m(x)) \) are linear with respect to the \( \omega \)-variable from defined (2.5), the \( \omega_m \to \omega^* \) weakly in \( L^2(\Omega) \) implies that
\[
\int_\Omega a_i(x, \omega_m(x)) \cdot \Phi dx \to \int_\Omega a_i(x, \omega^*(x)) \cdot \Phi dx,
\]
as \( m \to +\infty \) for any \( \Phi \in L^2(\Omega) \) and \( i = 1, 2 \). Hence, it follows from (4.7) that
\[
\int_\Omega [-a_1(x, \omega_m) + ia_2(x, \omega_m)] E_m \cdot \Phi dx \to \int_\Omega [-a_1(x, \omega^*) + ia_2(x, \omega^*)] E^* \cdot \Phi dx.
\]
(4.12)
Let \( m \to +\infty \) in (4.2) and using (4.8) that
\[
\int_\Omega \frac{\gamma}{\omega^*} (\nabla \times E^*) \cdot (\nabla \times \Phi) + (-a_1(x, \omega^*) + ia_2(x, \omega^*)) E^* \cdot \Phi dx = 0,
\]
(4.13)
\[
n \times E^*(x) = n \times G(x), \ x \in \Omega.
\]
(4.14)
That is, \( E^* \) is a solution of (3.2) corresponding to \( \omega^* \).

It follows from Lemma 3.7 that \( \{E_m\} \) is bounded in \( L^6(\Omega) \), then there exists a subsequence of \( \{E_m\} \), again denoted by \( \{E_m\} \), such that
\[
E_m \to E^* \text{ weakly in } L^6(\Omega).
\]
The continuous embedding \( L^6(\Omega) \hookrightarrow L^4(\Omega) \) implies that

\[
\int_\Omega |E_m(x)|^2 \to |E^*(x)|^2 |v(x)| dx \to 0 \text{ for any } v \in H^1(\Omega).
\]

From \( 0 < a_0 \leq a_2(x, w) \leq b_0 \) in (3.1) that

\[
\left| \int a_2(x, \omega_m)(|E_m|^2 - |E^*|^2)v dx \right| \leq b_0 \int_\Omega \left| (|E_m|^2 - |E^*|^2) v \right| dx \to 0.
\]

Therefore,

\[
\int_\Omega a_2(x, \omega_m(x))|E_m|^2 v dx \to \int_\Omega a_2(x, \omega^*(x))|E^*|^2 v dx \quad (4.15)
\]

as \( m \to \infty \). From Majorized convergence theorem and (4.15), for all \( \varphi \in C_0^\infty(0, T) \) we have

\[
\lim_{m \to +\infty} \int_0^T \int_\Omega a_2(x, \omega_m(x))|E_m|^2 \varphi(t) dx dt = \int_0^T \int_\Omega a_2(x, \omega^*(x))|E^*|^2 \varphi(t) dx dt.
\]

Next, for all \( \varphi \in C_0^\infty(0, T) \)

\[
\left| \int_0^T \int_\Omega k(x, u_m)\nabla u_m \nabla v(x) \varphi(t) dx dt - \int_0^T \int_\Omega k(x, u^*)\nabla u^* \nabla v(x) \varphi(t) dx dt \right|
\leq \int_0^T \int_\Omega k(x, u_m)|\nabla u_m - \nabla u^*|\nabla v(x) \varphi(t) dx dt
\]

\[
+ \int_0^T \int_\Omega |k(x, u_m) - k(x, u^*)|\nabla u^* \nabla v(x) \varphi(t) dx dt.
\]

(4.17)

Since \( 0 < k_0 \leq k(x, u) \leq k_1 \) and \( k(x, u) \) is Lipschitz continuous with respect to \( u \) from \( H(2) \), the sequence \( u_m \to u^* \) strongly in \( L^2(0, T; L^2(\Omega)) \) and \( \nabla u_m \to \nabla u^* \) weakly in \( L^2(0, T; L^2(\Omega)) \) in (4.17), one can easily to see that

\[
\left| \int_0^T \int_\Omega k(x, u_m) \nabla u_m - \nabla u^* | \nabla v(x) \varphi(t) dx dt \right|
\leq k_1 \left| \int_0^T \int_\Omega |(\nabla u_m - \nabla u^*) \cdot \nabla v(x) \varphi(t) dx dt \right|
\]

\[
+ L \left| \int_0^T \int_\Omega u_m - u^* | \nabla u^* \nabla v(x) \varphi(t) dx dt \right|
\to 0.
\]

(4.18)

And it follows from \( u_m \to u^* \) strongly in \( L^2(0, T; L^2(\Omega)) \) that

\[
- \int_0^T \int_\Omega u_m(x, t)v(x) \varphi'(t) dx dt \to - \int_0^T \int_\Omega u^*(x, t)v(x) \varphi'(t) dx dt.
\]

(4.19)

for all \( v \in H^1(\Omega) \) and \( \varphi \in C_0^\infty[0, T] \).
Combining the (4.2), (4.16), (4.18) and (4.19) implies that
\[
- \int_0^T \int_\Omega u^*(x,t)v(x)\varphi'(t)dxdt + \int_0^T \int_\Omega k(x,u^*)\nabla u^* \nabla v(x)\varphi(t)dxdt \\
= \frac{1}{2} \int_0^T \int_\Omega a_2(x,\omega^*) |\mathbf{E}^*|^2 v(x)\varphi(t)dxdt.
\]
By performing integration by parts and taking limit for all \(\Psi \in C^\infty(0,T)\) and \(g \in H'(\Omega)\), we have
\[
\int_\Omega u_m(x,T)\Psi(T)g(x)dx - \int_\Omega u_m(x,0)\Psi(0)g(x)dx \\
= \int_0^T \int_\Omega ((u_m)_t(x,t)\Psi(t)g(x) + \Psi'(t)u_m(x,t)g(x)) dxdt
\]
Taking \(m \to \infty\) and from (4.21), noting \(u_m(x,0) = u_0(x)\), we get
\[
\int_\Omega u^*(x,T)\Psi(T)g(x)dx - \int_\Omega u_0(x)\Psi(0)g(x)dx \\
= \int_\Omega u^*(x,T)\Psi(T)g(x)dx - \int_\Omega u^*(x,0)\Psi(0)g(x)dx.
\]
Taking \(\Psi(0) = 1\) in (4.22), we have
\[
\int_\Omega (u_0(x) - u^0(x,0)) \Psi(0)g(x)dx = 0
\]
Since \(H'(\Omega)\) is dense in \(L^2(\Omega)\), this implies
\[
u^*(0) = u_0 \text{ in } L^2(\Omega).
\]
Therefore, \((\mathbf{E}^*, u^*)\) is a weak solution of (2.8) corresponding to \(\omega^*\). That is, \(P(\omega^*) = (\mathbf{E}^*, u^*)\). In the step above we have shown that \((\mathbf{E}_m, u_m) \to (\mathbf{E}^*, u^*)\) for a subsequence. With the arguments above we can prove that every subsequence has a subsequence converging to the same \((\mathbf{E}^*, u^*)\). Therefore, the entire sequence \((\mathbf{E}_m, u_m)\) converges to \((\mathbf{E}^*, u^*)\) weakly. \(\square\)

**Theorem 4.1.** (Existence of an Optimal Control). Under the assumptions \(H(1), H(2), \text{ and } H(3)\), there exists at least one global minimizer \((\omega^*, \mathbf{E}^*, u^*)\) of the problem (P) such that
\[
\omega^* \in U_{ad}, \ \mathbf{E}^* \in H(curl, \Omega), \ u^* \in W[0,T].
\]
*Proof.* The proof follows standard arguments so we can be brief. First we use the control to state map \(P : \omega \to (\mathbf{E}, u)\) to define the reduced functional \(J(\omega) := J(\omega; \mathbf{E}, u)\). The objective functional \(J(\omega)\) is bounded from (2.9), we get the existence of an infimum \(z\),
\[
z := \inf_{\omega \in U_{ad}} J(\omega) \in \mathbb{R}.
\]
Let \(\{\omega_m\}_{m \in \mathbb{N}} \subset U_{ad}\) be a minimizing sequence with \(\lim_{m \to +\infty} J(\omega_m) = z\).

The definition of admissible control set \(U_{ad}\) implies that \(\omega_m \in L^p(\Omega)\) for any \(p \in (1, +\infty)\) and
\[
\|\omega_m\|_{L^p(\Omega)} \leq C
\]
for some constant \(C \geq 0\). Since \(L^p(\Omega)\) is reflexive space, there exists a subsequence of \(\{\omega_m\}\), again denoted by \(\{\omega_m\}\) for simplicity, respectively, such that
\[
\omega_m \to \omega^* \text{ weakly in } L^p(\Omega),
\]
where \( \omega^* \in U_{ad} \) from the Mazur lemma.

The functional \( J \) is weakly sequentially lower semi-continuous in (2.9). Because \( P \) is weakly sequentially continuous from Proposition 4.1, we get

\[
z = \lim_{m \to +\infty} J(\omega_m; P(\omega_m)) = \lim_{m \to +\infty} J(\omega_m) \geq J(\omega^*) \geq z.
\]

Therefore \( \omega^* \), also \( (\omega^*, E^*, u^*) \) is a global minimizer.

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