ORDER STRUCTURE AND TOPOLOGICAL PROPERTIES OF
THE SET OF MULTIPLE $t$-VALUES

ENDE PAN

Abstract. In this paper, we compute the iterated derived sets of the set of multiple $t$-values under the usual topology of $\mathbb{R}$. Our results imply that the set of multiple $t$-values, ordered by $\geq$, is a well-ordered set. We determine its type of order, which is $\omega^2$, where $\omega$ is the smallest infinite ordinal. There exists a unique bijection from the set of multiple $t$-values to $\mathbb{N}^2$, which reverses the orders. We provide some description of this bijection.

1. Introduction and statement of main results
For a finite sequence $k = (k_1, \ldots, k_d)$ of positive integers with $k_1 > 1$, the multiple zeta value $\zeta(k)$ is defined by the following infinite series
\[
\zeta(k) = \zeta(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}.
\]

In recent years there are some different variations of the multiple zeta values. In this paper we consider the multiple $t$-values, which were first systematically studied by Hoffman in [1]. Here a multiple $t$-value $t(k)$ is defined as
\[
t(k) = t(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{1}{(2m_1 - 1)^{k_1} \cdots (2m_d - 1)^{k_d}}.
\]

(1.1)

Usually $d$ is called the depth, $k_1 + \cdots + k_d$ is called the weight, and $k$ is called an admissible multi-index. By convention we denote the empty index by $\emptyset$, and define $t(\emptyset)$ to be 1.

Previous research on multiple $t$-values revealed that they remarkable parallel to, and contrast with, multiple zeta values. In [5], Zhao obtained some type of sum formula for multiple $t$-values. More precisely for positive integers $d, n$ with $d < n + 1$, let $T(2n, d)$ be the sum of all multiple $t$-values with even arguments whose weights are $2n$ and whose depths are $d$. Then it was proved in [5] that
\[
T(2n, d) = \frac{(-1)^{n-d} 2^{2n}}{4^n (2n)!} \sum_{l=0}^{n-d} \binom{n-l}{d} \binom{2n}{2l} E_{2l},
\]

where $E_j$ is Euler’s number. More general weighted sums of multiple $t$-values at even arguments with polynomial weights were studied by Li and Xu in [4]. In [1] Hoffman gave explicit formulas for the multiple $t$-values with repeated arguments

2010 Mathematics Subject Classification. 11M32, 06A05.
Key words and phrases. multiple $t$-values, order structure, derived set.
analogous to those known for multiple zeta values by quasi shuffle products. For example, the well-known identities
\[
\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n + 1)!}, \quad \zeta(\{4\}_n) = \frac{2^{2n+1} \pi^{4n}}{(4n + 2)!}, \quad \zeta(\{6\}_n) = \frac{6(2\pi)^{6n}}{(6n + 3)!}
\]
(where \(\{k\}_n\) means \(k\) repeats \(n\) times) have multiple \(t\)-values counterparts
\[
t(\{2\}_n) = \frac{\pi^{2n}}{(2n)!2^{2n}}, \quad t(\{4\}_n) = \frac{\pi^{4n}}{(4n)!2^{2n}}, \quad t(\{6\}_n) = \frac{3(\pi)^{6n}}{(6n)!4^n}.
\] (1.2)

Furthermore, Hoffman provided some interesting conjectures. For example he conjectured that the dimension of the rational vector space generated by weight \(n\) multiple \(t\)-values equals to the \(n\)th Fibonacci number.

For multiple zeta values, Kumar has a unconventional idea. In [2], he studied the order structure and the topological properties of the set \(Z\) of all multiple zeta values. Taking the usual order and the usual topology of the set \(\mathbb{R}\) of real numbers, Kumar computed the derived sets of the topological subspace \(Z\) of \(\mathbb{R}\), and showed that the set \(Z\), ordered by \(\geq\), is well-ordered with the order type \(\omega^3\), where \(\omega\) is the smallest infinite ordinal. Inspired by the work of Kumar, Li and the author studied the topological properties of some \(q\)-analogues of multiple zeta values in [3].

In this paper, we study the order structure and the topological properties of multiple \(t\)-values. Let \(T\) be the set of all multiple \(t\)-values. Under the usual topology, we determine the sequence \((T^{(n)})_{n \geq 0}\) of the derived sets of the topological subspace \(T\) of \(\mathbb{R}\). Here \(T^{(0)} = T\) and for any \(n \in \mathbb{N}\), \(T^{(n)}\) is the set of accumulation points of \(T^{(n-1)}\). As usually, we denote \(T^{(1)}\) by \(T'\) and \(T^{(2)}\) by \(T''\). To state our result about \(T^{(n)}\), we introduce the concept of the tail of multiple \(t\)-values. For an admissible multi-index \(k = (k_1, \ldots, k_d)\) and a positive odd integer \(n\), we define
\[
t(k)_n = t(k_1, \ldots, k_d)_n = \sum_{m_1 > \cdots > m_d > n} \frac{1}{(2m_1 - 1)^{k_1} \cdots (2m_d - 1)^{k_d}}.
\]

We set \(t(\emptyset)_n = 1\). Then under the usual topology of \(\mathbb{R}\), the derived sets of \(T\) are described as in the following theorem.

**Theorem 1.1.** We have
\[
T' = \{t(k) \mid k \text{ is admissible}\} \cup \{0, 1\}
\]
and \(T^{(n)} = \{0\}\) for any positive integer \(n \geq 2\).

For the order structure of the set \(T\), we have the following two results.

**Theorem 1.2.** The set \(T\), ordered by \(\geq\), is a well-ordered set with \(t(2)\) as the maximum element.

**Theorem 1.3.** The type of the order of \((T, \geq)\) is \(\omega^2\), where \(\omega\) is the smallest infinite ordinal.

The paper is organized as follows. In Section 2, we study the derived sets of \(T\) and give a proof of Theorem 1.1. In Section 3, we recall the definition of well-ordered sets and give a proof of Theorem 1.2. In Section 4, we recall some properties of ordinals, and then give a proof of Theorem 1.3. By Theorem 1.3 there is a unique bijection \(\Phi: (T, \geq) \to (\mathbb{N}^2, \leq)\), which reverses the orders. We give some description of this bijection at the end of Section 4.
2. Proof of Theorem \[1.1\]

In this section, we give a proof of Theorem \[1.1\].

We recall a formula.

Lemma 2.1 \([1]\). Let \(G = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\) be Catalan’s constant, then we have

\[
\sum_{j=1}^{\infty} t(2, \{1\}_j-1) = 2G.
\]

To prove Theorem \[1.1\] we have to know the behaviour of the convergent sequences in the space \(T\). Recall that a sequence \((a(n))_{n \in \mathbb{N}}\) is injective if for any \(n \neq m\), it holds \(a(n) \neq a(m)\).

Theorem 2.2. Let \((k(n))_{n \in \mathbb{N}} = ((k_1(n), \ldots, k_d(n)))_{n \in \mathbb{N}}\) be an injective sequence of admissible multi-indices. If 0 is not the accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\), then there exist an infinite subset \(D\) of \(\mathbb{N}\) and a positive integer \(d\) such that

(i) \(d(n) = d\) for all \(n \in D\),
(ii) and if \(d > 1\), there are positive integers \(k_1, \ldots, k_{d-1}\), with \(k_i(n) = k_i\) for all \(n \in D\) and \(1 \leq i < d\).

Proof. If the sequence \((d(n))_{n \in \mathbb{N}}\) is unbounded, there exists a subsequence \((n_k)_{k \in \mathbb{N}}\) of \(\mathbb{N}\) such that \(\lim_{k \to \infty} d(n_k) = \infty\). Then Lemma \[2.1\] implies that

\[
\lim_{k \to \infty} t(2, \{1\}_{d(n_k)-1}) = 0.
\]

From the fact

\[
0 < t(k(n_k)) \leq t(2, \{1\}_{d(n_k)-1}),
\]

we find that 0 is an accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\), a contradiction. Hence there exist an infinite subset \(D_1\) of \(\mathbb{N}\) and a positive integer \(d\), such that \(d(n) = d\) for all \(n \in D_1\). If \(d = 1\), we take \(D = D_1\), and complete the proof of the theorem.

Assume that \(d > 1\). If \((k_1(n))_{n \in D_1}\) is unbounded, there exists a subsequence \(n_k\) of \(D_1\) with the property

\[
\lim_{k \to \infty} k_1(n_k) = \infty.
\]

Since

\[
t(k(n_k)) \leq t(k_1(n_k), \{1\}_{d-1}),
\]

and

\[
\lim_{k \to \infty} t(k_1(n_k), \{1\}_{d-1}) = 0,
\]

a contradiction. Hence there exist an infinite subset \(D_2\) of \(D_1\) and a positive integer \(k_1\) such that \(k_1(n) = k_1\) for all \(n \in D_2\). Now assume that for \(1 \leq j < d - 1\), we have found an infinite subset \(D_j\) of \(\mathbb{N}\) and positive integers \(k_1, \ldots, k_j\), such that \(k_i(n) = k_i\) for \(i = 1, 2, \ldots, j\) and all \(n \in D_j\). Then using the inequality

\[
t(k_1, \ldots, k_j, k_{j+1}(n), \ldots, k_d(n)) < t(k_1, \ldots, k_j) t(k_{j+1}(n), \ldots, k_d(n))
\]

with \(k_{j+1}(n) > 1\), we conclude that there are an infinite subset \(D_{j+1}\) of \(D_j\) and a positive integer \(k_{j+1}\) such that \(k_{j+1}(n) = k_{j+1}\) for all \(n \in D_{j+1}\). Finally we take \(D = D_{d-1}\) and finish the proof.

We can restate Theorem \[2.2\] in the following way.
Corollary 2.3. Let \((k(n))_{n \in \mathbb{N}}\) be an injective sequence of admissible multi-indices such that 0 is not an accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\). Then \((k(n))_{n \in \mathbb{N}}\) has a subsequence of the following type

\[(k, \varphi(n) + 2)_{n \in \mathbb{N}}, \quad (2.1)\]

where \(k\) is a fixed admissible multi-index or an empty index and \((\varphi(n))_{n \in \mathbb{N}}\) is a strictly increasing sequence in \(\mathbb{N}\).

Therefore we have the following corollaries.

Corollary 2.4. Each injective sequence \((k(n))_{n \in \mathbb{N}}\) of admissible multi-indices has a subsequence \((k(n_k))_{k \in \mathbb{N}}\) such that the sequence \((t(k(n_k)))_{k \in \mathbb{N}}\) is strictly decreasing.

Proof. If 0 is an accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\), we obviously get the result. Otherwise, the result follows from Corollary 2.3. \(\square\)

Corollary 2.5. For any real number \(\alpha\), there exist only finitely many admissible multi-indices \(k\) for which \(t(k(n)) = \alpha\).

Proof. We immediately get the result from Corollary 2.4. \(\square\)

Now we come to prove Theorem 1.1

Proof of Theorem 1.1 We first compute \(T'\). For an admissible multi-index \(k = (k_1, \ldots, k_d)\), we take \(k(n) = (k_1, \ldots, k_d, n + 2)\). Then since

\[
\lim_{n \to \infty} t(k(n)) = t(k_1, \ldots, k_d),
\]

we get \(t(k_1) \in T'\). Similarly, taking \(k(n) = (2, \{1\}_{n-1})\) and \(k(n) = (n + 1)\) for \(n \in \mathbb{N}\) respectively, we find 0 and 1 belong to \(T'\).

Conversely, for any nonzero \(\alpha \in T'\), there is a sequence \((k(n))_{n \in \mathbb{N}}\) such that

\[
\lim_{n \to \infty} t(k(n)) = \alpha.
\]

By Corollary 2.3 without loss of generality we may assume that \((k(n))_{n \in \mathbb{N}}\) is one of the following types

1. \((\varphi(n) + 2)_{n \in \mathbb{N}}\);
2. \((k, \varphi(n) + 2)_{n \in \mathbb{N}}\),

where \(k = (k_1, \ldots, k_d)\) is a fixed admissible multi-index and \((\varphi(n))_{n \in \mathbb{N}}\) is a strictly increasing sequence in \(\mathbb{N}\). In the case of type (1), since

\[
\lim_{n \to \infty} t(\varphi(n) + 2) = \lim_{n \to \infty} \left( \frac{1}{(2m-1)\varphi(n) + 2} \right)
= \sum_{m > 0} \lim_{n \to \infty} \frac{1}{(2m-1)\varphi(n) + 2}
= \frac{1}{2m-1} \varphi(n) + 2
= 1,
\]

we get \(\alpha = 1\). And in the case of type (2), since

\[
\lim_{n \to \infty} t(k_1, \ldots, k_d, \varphi(n) + 2)
= \lim_{n \to \infty} \left( \frac{1}{(2m_1-1)k_1 \cdots (2m_d-1)k_d(2m_{d+1}-1)\varphi(n) + 2} \right)
= \sum_{m_1 > \cdots > m_d > m_{d+1} > 0} \lim_{n \to \infty} \frac{1}{(2m_1-1)k_1 \cdots (2m_d-1)k_d(2m_{d+1}-1)\varphi(n) + 2}
\]
\[
\sum_{m_1 > \cdots > m_d > 1} \frac{1}{(2m_1 - 1)^{k_1} \cdots (2m_d - 1)^{k_d}} = t(k_1, \ldots, k_d),
\]

we have \(\alpha = t(k_1, \ldots, k_d)\). Hence the result about \(T'\) is proved.

Now we compute \(T''\). Note that for any admissible multi-index \(k\), we have \(t(k) > t(k)_1\). Hence we get \(0 \in T''\) from \(0 \in T'\). Conversely, assume that \(0\) is not the accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\). Then \(0\) is also not the accumulation point of \((t(k(n)))_{n \in \mathbb{N}}\). Hence from Corollary 2.3 we may assume that \((k(n))_{n \in \mathbb{N}}\) is of type (1) or of type (2). Then from the facts

\[
\lim_{n \to \infty} t(\varphi(n) + 2)_1 = \lim_{n \to \infty} \sum_{m > 1} \frac{1}{(2m - 1)^{\varphi(n)+2}} = \sum_{m > 1} \lim_{n \to \infty} \frac{1}{(2m - 1)^{\varphi(n)+2}} = 0,
\]

and

\[
\lim_{n \to \infty} t(k_1, \ldots, k_d, \varphi(n) + 2)_1 = \lim_{n \to \infty} \sum_{m_1 > \cdots > m_d > m_{d+1} > 1} \frac{1}{(2m_1 - 1)^{k_1} \cdots (2m_d - 1)^{k_d} (2m_{d+1} - 1)^{\varphi(n)+2}} = \sum_{m_1 > \cdots > m_d > m_{d+1} > 1} \lim_{n \to \infty} \frac{1}{(2m_1 - 1)^{k_1} \cdots (2m_d - 1)^{k_d} (2m_{d+1} - 1)^{\varphi(n)+2}} = 0,
\]

we have \(T'' = \{0\}\).

Finally, it is obvious that \(T^{(n)} = \{0\}\) for any \(n \geq 2\). □

Similar as Corollary 2.3 we have the following result.

**Corollary 2.6.** For any real number \(\beta\), there exist only finitely many admissible multi-indices \(k\) for which \(t(k)_1 = \beta\).

3. **Proof of Theorem 1.2**

In this section, we give a proof of Theorem 1.2. We recall the definition of a well-ordered set. A partial ordered set \((X, \geq)\) is called well-ordered, if the order \(\geq\) is totally ordered, and any nonempty subset of \(X\) has a maximal element.

**Proof of Theorem 1.2.** As a subset of \(\mathbb{R}\), \(T\) is obviously totally ordered. Now we prove that any nonempty subset \(X\) of \(T\) has a maximal element. If \(X\) is a finite set, it is easy to find the maximal element of \(X\) as \(X\) is totally ordered. Now we investigate another case, \(X\) is an infinite set. Assume that there is no maximal element of \(X\). Then for any \(\alpha_1 \in X\), there is a \(\alpha_2 \in X\), such that \(\alpha_1 < \alpha_2\). Similarly there exists \(\alpha_3 \in X\) with \(\alpha_2 < \alpha_3\). Hence there is an infinite strictly increasing sequence \(\alpha_1 < \alpha_2 < \alpha_3 < \cdots\) in \(X\), which contradicts Corollary 2.3.
We next prove that \( t(2) \) is the maximal element of \( T \). Since \( t(2) = \frac{\pi^2}{8} \), from Lemma 2.1 we obtain

\[
\sum_{j=1}^{\infty} t(2, \{1\}_j) = 2G - \frac{\pi^2}{8} < 1.
\]

Hence we find

\[ t(2) > t(2, \{1\}_j) \]

for all \( j \in \mathbb{N} \). Finally with the inequality \( t(k_1, \ldots, k_{j+1}) \leq t(2, \{1\}_j) \), we complete our proof. \( \square \)

**Corollary 3.1.** \( T' \) is well-ordered with \( t(\emptyset)_1 \) as the maximum element and \( t(2)_1 \) as the second largest element.

**Proof.** Similar as the proof of Theorem 1.2 to show that \( T' \) is well-ordered, we need the fact that for any injective sequence \( (k(n))_{n \in \mathbb{N}} \) of admissible multi-indices, there is a strictly decreasing infinite subsequence of \( (t(k(n)))_{n \in \mathbb{N}} \). While one can get this fact similar as Corollary 2.4.

We next compute the maximum element and the second largest element of \( T' \).

From the equations

\[ t(2)_1 = t(2) - 1 \]

and

\[ t(2, \{1\}_j)_1 = t(2, \{1\}_j) - t(2, \{1\}_{j-1})_1 \quad (j \geq 1), \]

we obtain

\[
\sum_{j=1}^{\infty} t(2, \{1\}_{j-1})_1 = \sum_{j=1}^{\infty} t(2, \{1\}_{j-1}) - 1 - \sum_{j=1}^{\infty} t(2, \{1\}_{j-1})_1.
\]

So

\[
\sum_{j=1}^{\infty} t(2, \{1\}_{j-1})_1 = \frac{2G - 1}{2} \approx 0.416,
\]

Hence we have

\[ t(\emptyset)_1 = 1 > t(2, \{1\}_{j-1})_1 \geq t(k_1, \ldots, k_j)_1 \]

for all \( j \in \mathbb{N} \) and we find \( t(\emptyset)_1 \) is the maximum element. Similarly, since

\[ t(2)_1 \approx 0.232 > \sum_{j=1}^{\infty} t(2, \{1\}_j)_1 > t(2, \{1\}_j)_1 \geq t(k_1, \ldots, k_{j+1})_1 \]

for all \( j \in \mathbb{N} \), we have \( t(2)_1 \) is the second largest element. \( \square \)

4. **Proof of Theorem 1.3**

In this section, we first give a proof of Theorem 1.3. We recall some lemmas.

**Lemma 4.1 (2).** Any well-ordered subset of \( \mathbb{R} \) is countable, where \( \mathbb{R} \) is the extended real line.

**Lemma 4.2 (2).** Let \( A \) be a well-ordered subset of \( \mathbb{R} \). If \( A \) has order type \( \omega \mu + \nu \), where \( \mu \) is an ordinal and \( \nu \) is a finite ordinal, then the order type of \( A' \) is \( \mu \) if \( \mu \) is finite and \( \mu + 1 \) if \( \mu \) is infinite.
Proof of Theorem 1.3. Recall the division algorithm of ordinals. For ordinals γ, α, β with γ < αβ, there exist unique ordinals α′, β′, such that γ = αβ′ + α′, and α′ < α, β′ < β. Since T′ is countable, we may assume the ordinal of T′ is ωµ + ν, where µ is an ordinal and ν is a finite ordinal. From Lemma 4.2, the ordinal of T″ is µ or µ + 1. But T″ = {0}, which implies that the ordinal of T″ is 1. Hence we get µ = 1. Since 0 is the smallest element of T′, we find ν = 1. Now we assume the ordinal of T is ωµ′ + ν′, where µ′ is an ordinal and ν′ is a finite ordinal. From Lemma 4.2 and the fact that the ordinal of T′ is ω + 1, we have µ′ = ω. Since there is no smallest element in T, we get ν′ = 0. Therefore, the ordinal of T is ω².

Theorem 1.3 guarantees that there exists a unique bijection

$$\Phi : (T', \geq) \rightarrow (\mathbb{N}^2, \leq),$$

reverses the orders. Here \(\mathbb{N}^2\) is endowed with the lexicographical order. At the end of this section, we provide some description of the map \(\Phi\).

Since the ordinal of T′ is ω + 1, there exists a unique bijection

$$\Psi : (T', \geq) \rightarrow (\mathbb{N}, \leq),$$

which reverses the orders. Here \(\mathbb{N} = \mathbb{N} \cup \{\infty\}\) and \(\infty\) is the maximal element of \(\mathbb{N}\). Since 0 is the minimal element of T′, we have \(\Psi(0) = \infty\). And it is easy to compute the image of nonzero elements of T′ under the map \(\Psi\).

Lemma 4.3. For any nonzero \(\beta \in T'\), we have \(\Psi(\beta) = \text{card}(B_\beta)\), where

\[ B_\beta = \{\alpha \in T'| \alpha \geq \beta\}. \]

Therefore, we may index the set \(T' - \{0\}\) by \(\mathbb{N}\) as

\[ T' - \{0\} = \{\beta_1 > \beta_2 > \cdots > \beta_n > \cdots\}. \]

We also set \(\beta_0 = \infty\). For any \(r \in \mathbb{N}\), we set

\[ I_r = \{k|k| \text{ is admissible, } t(k)_1 = \beta_r\}. \]

Which is a finite set by Corollary 2.6. Set

\[ L_r = \{(k, n)|k \in I_r, n \in \mathbb{N}, t(k, n) < \beta_{r-1}\} \]

and

\[ P_r = \{(k, n)|k \in I_r, n \in \mathbb{N}, t(k, n) \geq \beta_{r-1}\}. \]

It is easy to show that \(P_r\) is finite for any \(r \in \mathbb{N}\), and in particular \(P_1 = \emptyset\). For any \(\alpha \in T\), we set

\[ \Phi(\alpha) = (\Phi_L(\alpha), \Phi_R(\alpha)) \in \mathbb{N}^2. \]

Then we have the following lemma.

Lemma 4.4. For any \(l \in L_r\), we have \(\Phi_L(t(l)) = r\).

Proof. We prove it by induction on \(r\). For any \(l = (k, n) \in L_1\), if \(\Phi_L(t(l)) > 1\), then we have

\[ (1, 1) < (1, 2) < \cdots < (1, n) < \cdots < \Phi(t(l)). \]

Applying \(\Phi^{-1}\), we get an injective sequence \((k(n))_{n \in \mathbb{N}}\), such that

\[ t(k(1)) > t(k(2)) > \cdots > t(k(n)) > \cdots > t(l). \]

Therefore there exists an element \(\beta \in T'\) such that

\[ \beta \geq t(l). \]
Hence
\[ \beta_1 = t(k)_1 < t(k, n) = t(l) \leq \beta, \]
which is impossible.

Now assume that \( r > 1 \), and the result holds for any positive integer less than \( r \). Then for any \( l = (k, n) \in L_{r-1} \), we have \( \Phi(t(l)) = \Phi(t(k, n)) = (r-1, \tau(n)) \). Since \( t(k, n) > t(k, n+1) \), we have \( (\tau(n))_{n \in \mathbb{N}} \) is a strictly increasing sequence in \( \mathbb{N} \). For any \( l' \in L_r \), we set \( \Phi(t(l')) = (r', \tau) \). We first prove \( r' \geq r \). If \( r' < r-1 \), then for any \( l \in L_{r-1} \), we have
\[ \Phi(t(l')) < \Phi(t(l)), \]
which implies
\[ t(l') > t(l) > \beta_{r-1}, \]
a contradiction. If \( r' = r-1 \), since \( (\tau(n))_{n \in \mathbb{N}} \) is a strictly increasing sequence in \( \mathbb{N} \), we can find a big enough positive integer \( m \) such that \( \tau(m) > \tau \). Then there exists a multi-index \( l = (k, m) \in L_{r-1} \) such that
\[ \Phi(t(l')) < \Phi(t(l)), \]
which implies
\[ t(l') > t(l) > \beta_{r-1}, \]
a contradiction. We next prove \( r' \leq r \). If \( r' > r \), then
\[ (r, 1) < (r, 2) < \cdots < \Phi(t(l')). \]
Applying \( \Phi^{-1} \), there exists an injective sequence of admissible multi-indices \( (k(n))_{n \in \mathbb{N}} \) such that
\[ t(k(1)) > t(k(2)) > \cdots > t(k(n)) > \cdots > t(l'). \]
Since
\[ \text{card}(\bigcup_{1 \leq i \leq r-1} \mathcal{P}_i) \]

is finite, there exists an accumulation point \( \beta \) of \( (t(k(n)))_{n \in \mathbb{N}} \) such that
\[ \beta \in \mathcal{T}' - \{ \beta_1, \ldots, \beta_{r-1} \} \]
and \( \beta \geq t(l') \), we get a contradiction. \( \square \)

While for elements in \( \mathcal{P}_r \), we have the following lemma.

**Lemma 4.5.** For any \( l \in \mathcal{P}_r \), we have \( \Phi_L(t(l)) = \min\{ m \in \mathbb{N} \mid t(l) > \beta_m \} \).

**Proof.** We denote by \( m \) the minimal number such that \( \beta_m < t(l) \). Then we have \( \beta_m \geq t(l) \). We first prove \( \Phi_L(t(l)) \geq m \). If \( m = 1 \), the conclusion is obvious. If \( m > 1 \), by Lemma 4.4 there exists an infinite sequence
\[ (t(l_{m-1}(n)))_{n \in \mathbb{N}} \]
such that for any positive integer \( n \)
\[ t(l_{m-1}(n)) > t(l_{m-1}(n+1)) \geq t(l), \]
where \( l_{m-1} \in L_{m-1} \). Therefore we obtain
\[ (m-1, \zeta(1)) < (m-1, \zeta(2)) < \cdots < (m-1, \zeta(n)) < \cdots < \Phi(t(l)), \]
where \( (\zeta(n))_{n \in \mathbb{N}} \) is a strictly increasing sequence in \( \mathbb{N} \). Hence \( \Phi_L(t(l)) > m-1. \)
We next prove $\Phi_L(t(1)) < m + 1$. If $\Phi_L(t(1)) \geq m + 1$, then
$$(m, 1) < (m, 2) < \cdots < \Phi(t(1)),$$
which implies that for any $l_m \in L_m$, it holds $\Phi(t(l_m)) > \Phi(t(1))$. Then we have $\beta_m \geq t(1)$, which leads to a contradiction.

From Lemma 4.4 and Lemma 4.5, we find that for any $\alpha \in T$ and $r \in N$, we have $\Phi_L(\alpha) = r \Leftrightarrow \beta_r - 1 \geq \alpha > \beta_r$. In other words we can order the subset $T_r = \{ \alpha \in T | \beta_r - 1 \geq \alpha > \beta_r \}$ by $N$ as $(\alpha_r, 1) < \alpha_r, 2 < \cdots < \Phi(t(l_m))$, which implies that for any $l_m \in L_m$, it holds $\Phi(t(l_m)) > \Phi(t(1))$. Then we have $\beta_m \geq t(1)$, which leads to a contradiction. □

**Theorem 4.6.** For any $\alpha \in T$, we have the following equality:
$$\Phi(\alpha) = (\min\{i \in N| \alpha > \beta_i\}, \min\{j \in N| \alpha \geq \alpha_{i,j}\}).$$

**Proof.** We get this claim from the facts that there exists a unique bijection from $(T_i, \geq)$ to $(N, \leq)$ which reverses the orders, and there is a unique bijection $((i, 1), (i, 2), \cdots ) \leq$ to $(N, \leq)$ which preserves the orders. □

By the above description of the bijection $\Phi : (T, \geq) \rightarrow (N^2, \leq)$ and the supplementary definition $t(1) = \infty$, we compute some approximate values and give the following ordering rule of $T \cup T'$:
$$t(1) > t(2) > t(3) > \cdots > t(\emptyset) > t(2, 1) > t(2, 2) > \cdots > t(2) > t(2, 1, 1) > t(2, 1, 2) > \cdots > t(2, 1) > t(3, 1) > t(3, 2) > \cdots > t(3) > \cdots .$$
Hence here we conjecture $\bigcup_{r=1}^{\infty} T_r = \emptyset$ and there are no two different admissible multi-indices $k$ and $l$, such that $t(l) = t(k)$. Moreover combining with Lemma 4.3 and Theorem 4.6 we have the following conjecture.

**Conjecture.** For any multi-index $l = (k, n)$, we have
$$\Phi(t(k, n)) = (m, n),$$
where $k$ is admissible or empty and $m = \min\{i \in N| t(k)_1 > \beta_i\}$.

**References**

[1] M. E. Hoffman, An odd variant of multiple zeta values, *Comm. Number Theory Phys.* 13(2019), 529-567.
[2] K. S. Kumar, Order structure and topological properties of the set of multiple zeta values, *Int. Math. Res. Not.* 2016 (5) (2016), 1541-1562.
[3] Z. Li and E. Pan, Topological properties of $q$-analogues of multiple zeta values, *Int. J. Number Theory*, accepted.
[4] Z. Li and C. Xu, Weighted sum formulas of multiple $t$-values with even arguments, arXiv: 1908.03200.
[5] J. Zhao, Sum formula of multiple Hurwitz-zeta values, *Forum Math.* 2015 (3) (2015), 929-936.

**College of Teacher Education, QuZhou University, No. 78 Jiuhua Road, QuZhou, PR China**

**Email address:** 1710385@tongji.edu.cn