Evidence for a null entropy of extremal black holes

Shahar Hod

The Racah Institute for Physics, The Hebrew University, Jerusalem 91904, Israel

(November 2, 2021)

Abstract

We present some arguments in support of a zero entropy for extremal black holes. These rely on a combination of both quantum, thermodynamic, and statistical physics arguments. This result may shed some light on the nature of these extreme objects. In addition, we show that within a quantum framework the capture of a particle by an initially extremal black hole always results with a final nonextremal black hole.

I. INTRODUCTION

Extremal black holes have an important and controversial status in black-hole physics. It had been traditionally believed that an extremal black hole is the limiting case of its nonextremal counterpart; when the inner Cauchy horizon and the outer event horizon coincide, the nonextremal black hole becomes an extremal one [1]. However, this traditional point of view has been recently challenged by Hawking et al. [2], who based their arguments on the qualitative differences between the topologies of extremal and nonextremal black holes, differences which raise doubts about limiting arguments.

While it is well established that a nonextremal black hole bears an entropy which is proportional to its surface area $S_{BH} = A/4\hbar$ [3,4] (we use gravitational units in which $G = c = 1$), there is no general agreement on the entropy of extremal black holes. Based on the different topologies of extremal and nonextremal Reissner-Nordström black holes,
Hawking et al. [2] and Teitelboim [3] argued that extremal black holes have zero entropy even though their event horizon has nonzero area. For further reading see e.g., [6–9] and references therein.

In this paper we examine the consistency of the Bekenstein-Hawking area-entropy relation $S_{BH} = A/4\hbar$ with the properties of extremal black holes. To this end, we shall use a combination of both quantum, thermodynamic, and statistical physics arguments: In Sec. II we construct a gedanken experiment in which the (quantum) generalized second law of thermodynamics is shown to be incompatible with the area-entropy relation as applied to extremal black holes. In Sec. III we show that the standard quantization of angular momentum and electric charge in nature, when applied to extremal black holes, are incompatible with the area-entropy relation. The same argument leads to a zero-entropy conjecture for extremal black holes.

II. GEDANKEN EXPERIMENTS WITH EXTREMAL BLACK HOLES

We consider a neutral object which is lowered towards an extremal Kerr-Newman black hole. We challenge the validity of accepted physical laws in the most ‘dangerous’ situation, i.e., when the energy delivered to the black hole is as small as possible. We therefore bring the object as close to the horizon as possible, and then drop it in. The descent of the body, if sufficiently slow, is known to be an adiabatic process which causes no change in the black-hole horizon area [10–12].

To zeroth order in particle-hole interaction the energy (energy-at-infinity) $E^{(0)}$ of the object in the black-hole spacetime is given by Carter’s integrals (constants of motions). As first shown by Christodoulou [14] (see also [15]), $E^{(0)}(r = r_+) = \Omega^{(0)}L_z$ at the point of capture, where $\Omega^{(0)} = a/(r_+^2 + a^2)$ is the angular velocity of the black hole, $L_z$ is the conserved angular momentum of the particle, and $r_+ = M$ is the location of the black-hole horizon.

One should also consider first-order interactions between the black hole and the object’s
angular momentum: As the particle spirals into the black hole (in the case \( L_z^2 \neq 0 \)) it interacts with the black hole, so the horizon generators start to rotate, such that at the point of assimilation the black-hole angular velocity \( \Omega \) has changed from \( \Omega^{(0)} \) to \( \Omega^{(0)} + \Omega^{(1)}_c \). The corresponding first-order energy correction is \( \mathcal{E}^{(1)} = \Omega^{(1)}_c L_z \). On dimensional analysis we expect \( \Omega^{(1)}_c \) to be of the order of \( O(L_z/M^3) \). In fact, Will \[16\] has performed a perturbation analysis for the problem of a ring of particles rotating around a slowly rotating (neutral) black hole, and found \( \Omega^{(1)}_c = L_z/4M^3 \). As would be expected from a perturbative approach, \( \Omega^{(1)}_c \) is proportional to \( L_z \). To our best knowledge, no exact calculation of \( \Omega^{(1)}_c \) has been performed for generic Kerr-Newman black holes. We therefore write \( \mathcal{E}^{(1)} = \omega L_z^2 \), and obtain

\[
\mathcal{E} = \mathcal{E}^{(0)} + \mathcal{E}^{(1)} = \frac{aL_z}{M^2 + a^2} + \omega L_z^2 ,
\]

for the particle’s energy at the point of capture.

The assimilation of the object results with a change \( \Delta M = \mathcal{E} \) in the black-hole mass, and a change \( \Delta J = L_z \) in its angular momentum. With the plausible assumption of cosmic censorship \[17\] one may argue from Hawking’s area theorem \[18\] (\( \Delta A \geq 0 \)) to find

\[
\omega \geq \frac{M}{2(M^2 + a^2)^2} .
\]

For the analysis to be self-consistent, the black-hole condition \( M^2 - a^2 - Q^2 \geq 0 \) should be satisfied after the assimilation of the object. This requires

\[
\omega \geq \frac{M(M^2 - 3a^2)}{2(M^2 + a^2)^3} ,
\]

which is always a weaker condition than the condition Eq. (2). We therefore conclude, that provided cosmic censorship is respected, the final black hole (in the case \( L_z^2 \neq 0 \)) is not extremal [the two expressions Eq. (2) and Eq. (3) coincide for the unique case \( a = 0 \) (first considered in \[19\]), in which case second-order interactions between the black hole and the object’s angular momentum should be considered].

The increase in black-hole surface area due to the assimilation of the object is of the order of \( O(|L_z|) \). Evidently, this can be minimized for \( L_z = 0 \), in which case one finds \( \Delta A = 0 \). This result is consistent with Hawking’s (classical) area theorem \[18\].
We next examine the gedanken experiment from the point of view of a *quantum* theory of gravity. A complete quantum theory of gravity is, of course, beyond our present reach. We do have, however, an important fingerprint of the elusive theory. The Bekenstein-Hawking area-entropy relation $S_{BH} = A/(4\bar{\hbar}G/c^3)^{1/2}$ involves the universal constants $\bar{\hbar}$ and $G$ of quantum theory and gravitation, respectively. It therefore allows a glance into the realm of quantum gravity. The concept of black-hole entropy is intimately related to the generalized second law (GSL) of thermodynamics [3,20] “The sum of the black-hole entropy and the common (ordinary) entropy in the black-hole exterior never decreases”.

We realize that the result $\Delta S_{BH} = 0$ (this is a direct consequence of the result $\Delta A = 0$ provided the relation $S_{BH} = A/4\bar{\hbar}$ holds true for extremal black holes) does not respect the GSL; the object’s entropy disappears with no obvious physical mechanism to compensate for its loss. We recall, however, that the essence of a *quantum* theory is the Heisenberg quantum uncertainty principle. It implies that $\delta L_z^2$ cannot vanish identically for a fairly localized object, since according to the uncertainty principle this would give rise to a large uncertainty in its canonically conjugate variable, the azimuthal angle $\phi$. [Recall for example, that the stationary states of the hydrogen atom, which are also eigenstates of the $L_z$ operator (and hence have a definite value of $L_z$, and a vanishing $\delta L_z^2$), have a probability distribution which is $\phi$-independent, i.e., the corresponding wave function is completely unlocalized in the $\phi$ direction, whereas the semiclassical analysis requires the descending object to be fairly localized.] Specifically, we have the uncertainty relation

$$\sqrt{\delta L_z^2} \geq \frac{\hbar}{2\delta \phi} \gtrsim h \ , \eqno (4)$$

where $\delta \phi \ll 1$ if the particle is in the equatorial plane, and $\delta \phi$ can be made of the order of unity if the particle is near the black-hole poles (note that the particle cannot be localized at the pole itself, since this would violate the uncertainty relation between the angle variable $\theta$, and the corresponding canonically conjugate angular momentum).

Taking cognizance of Eq. (4) we obtain $(\Delta A)_{min} = O(\hbar)$ within the quantum framework. According to the standard area-entropy relation, the corresponding increase in black-hole
entropy is of the order of $O(1)$. However, this increase in black-hole entropy cannot guarantee the GSL’s validity; the disappeared entropy (the object’s entropy) is generically larger than $O(1)$. Thus, assuming the validity of the area-entropy relation for extremal black holes one finds $\Delta S_{\text{tot}} = \Delta S_{BH} - S_{\text{object}} < 0$ in our gedanken experiment, in contradiction with the GSL [21]. This motivates the conjecture that the area-entropy relation is not applicable for extremal black holes.

The important point to be emphasized is, that within the quantum theory the final black hole (obtained after the capture of the object) is not extremal any more [see the discussion after Eqs. (2) and (3)]. The final black-hole entropy is therefore given by the standard Bekenstein-Hawking relation $S_{BH}^{\text{fin}} = A^{\text{fin}}/4\hbar$. Assuming that the initial black-hole entropy is zero (for an extremal black hole), one obtains $\Delta S_{\text{tot}} = \Delta S_{BH} - S_{\text{object}} = A^{\text{fin}}/4\hbar - S_{\text{object}} > 0$ (for black holes much larger than the test object, as required by the semiclassical analysis), in agreement with the GSL.

The argument presented in this section supports the idea that the area-entropy relation is not valid for extremal black holes. On the other hand, the zero-entropy conjecture for extremal black holes was shown to be compatible with the GSL. [Of course, this is not to say that a zero entropy is the only resolution; an entropy of the form (say) $S = \ln(A/\hbar)$ for extremal black holes would also be compatible with the GSL. However, the gedanken experiment reveals that the standard proportionality between black-hole surface area and entropy, if applied to extremal black holes would contradict the GSL.] In the next section we give further evidence in support of the conjecture that extremal black holes have zero entropy.

**III. ANGULAR-MOMENTUM AND CHARGE QUANTIZATION VS. THE AREA-ENTROPY RELATION**

The quantization of extremal black holes was discussed by Mazur [26] and Bekenstein (see, e.g., [27]): The extremal Kerr black hole is defined by the relation $M^4 = J^2$, which
implies $A = 8\pi M^2$. One enforces the quantization by replacing the total angular-momentum by the well-known eigenvalues of the corresponding quantum operator, namely $J^2 \to j(j + 1)\hbar^2$, where $j$ is a non-negative integer or half-integer. One therefore obtains

$$A_j = 8\pi \sqrt{j(j + 1)}\hbar,$$  \hspace{1cm} (5)

for the area eigenvalues of the extremal Kerr black hole.

In the spirit of Boltzmann-Einstein formula in statistical physics one relates $\exp(S_{BH})$ to the number of microstates of the black hole that correspond to a particular external macrostate. Thus, the thermodynamic relation $S_{BH} = A/(4\hbar)$ between black-hole surface area and entropy implies that the degeneracy corresponding to the $j$th area level is

$$g_j = \exp[2\pi \sqrt{j(j + 1)}].$$  \hspace{1cm} (6)

This quantity is, however, not an integer. We therefore conclude that the area-entropy thermodynamic relation, if applied to extremal black holes, is not compatible with a combination of quantum and standard statistical physics arguments (namely, the Boltzmann-Einstein formula). [This state of affairs should be contrasted with the corresponding situation for non extremal black holes [27,28], where area quantization, statistical physics arguments, and the Bekenstein-Hawking thermodynamic relation all agree !]

The corresponding discussion for the extreme Reissner-Nordström black hole is very similar to the one presented for extremal Kerr black holes. The extreme Reissner-Nordström black hole is defined by the relation $|Q| = M$, which implies $A = 4\pi Q^2$ for the black-hole surface area. The quantization of its area eigenvalues was discussed by Mazur [26] and Bekenstein (see, e.g., [27]); one enforces the quantization by replacing $Q \to qe$, where $q$ is an integer and $e$ is the elementary charge. One thus obtains

$$A_q = 4\pi \alpha q^2 \hbar,$$  \hspace{1cm} (7)

for the area eigenvalues of the extremal Reissner-Nordström black hole, where $\alpha = e^2/\hbar$ is the fine-structure constant. The area spectrum Eq. (7), together with the thermodynamic
relation $S_{BH} = A/(4\hbar)$, imply that the degeneracy corresponding to the $q$th area eigenstate is

$$g_q = \exp(\pi \alpha q^2),$$

which is, again, not an integer. We therefore recover our previous conclusion, that for extremal black holes quantum and statistical physics arguments are not compatible with the area-entropy thermodynamic relation.

Similar analysis reveals the fact, that for an extreme Kerr-Newman black hole (a synthesis of the former two extreme black holes) with or without a magnetic monopole, the area-entropy thermodynamic relation is inconsistent with quantum and statistical physics arguments. This supports the idea that extremal black holes do not comply with the standard area-entropy relation. Moreover, taking cognizance of Eqs. (5) and (7) one finds that the entropy of extremal black holes should equal a constant (the logarithm of an integer) in order to be compatible with the standard statistical physics interpretation of entropy.

We should comment, however, that the entropy of an extremal black hole could agree with the Bekenstein-Hawking entropy, provided it is not interpreted as a statistical (Boltzmann) entropy.

### IV. SUMMARY

We have shown that the Bekenstein-Hawking thermodynamic relation, if applied to extremal black holes, leads to violation of the generalized second law of thermodynamics. This motivates the conjecture that the standard area-entropy relation is not valid for extremal black holes, as first suggested (from a completely different point of view) by Hawking et al. [2], and by Teitelboim [5].

Moreover, we have shown that in a quantum framework, the assimilation of a particle by an extremal black hole always results with a final non extremal black hole (under the plausible assumption of cosmic censorship). This final result, together with a null entropy
for extremal black holes restore the validity of the GSL to our gedanken experiment because
the huge increase in black-hole entropy (from zero to $A/4\hbar$) compensate for the loss of the
object’s entropy.

We have further shown that for extremal black holes the area-entropy *thermodynamic*
relation is inconsistent with *quantum* and *statistical* physics arguments. The later imply
that the entropy of extremal black holes should equal a constant. This should be con-
trasted with the corresponding situation in the physics of non extremal black holes, where
area quantization, statistical physics arguments, and the Bekenstein-Hawking area-entropy
thermodynamic relation are all compatible [27,28].

We finally note that a zero entropy for extremal black holes is in agreement with the
interpretation of black-hole entropy as the logarithm of the number of quantum mechanically
distinct ways in which the particular black hole could have been made through successive
excitations [29,30], because it is has been shown that a non extremal black hole cannot be
transformed into an extremal one [31,32] (and the present paper also reveals, that within
a quantum framework an extremal black hole cannot be transformed into another extremal
black hole). These results therefore imply $S_{\text{ext}} = \ln 1 = 0$ for the entropy of extremal black
holes.

**ACKNOWLEDGMENTS**

I thank Jacob D. Bekenstein and Avraham E. Mayo for discussions. This research was
supported by a grant from the Israel Science Foundation.
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This state of affairs should be *contrasted* with the corresponding one involving nonextremal black holes. For non-extremal black holes, the increase in black-hole surface area (entropy) ensures the validity of the GSL [22–25].
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