Quantum Circuit-depth Lower Bounds For Homological Codes

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Abstract

We provide an $\Omega(\log(n))$ lower bound for the depth of any quantum circuit generating the unique groundstate of Kitaev’s spherical code. No circuit-depth lower bound was known before on this code in the general case where the gates can connect qubits even if they are far away; It is a known hurdle in computational complexity to handle general circuits, and indeed the proof requires introducing new techniques beyond those used to prove the $\Omega(\sqrt{n})$ lower bound which holds in the geometrical case. The lower bound is tight (up to constants) since a MERA circuit of logarithmic depth exists. To the best of our knowledge, this is the first time a quantum circuit-depth lower bound is given for a unique ground state of a gapped local Hamiltonian. Providing a lower bound in this case seems more challenging, since such systems exhibit exponential decay of correlations and standard lower bound techniques do not apply. We prove our lower bound by introducing the new notion of $\gamma$-separation, and analyzing its behavior using algebraic topology arguments.

We extend our methods also to a wide class of polygonal complexes beyond the sphere, and prove a circuit-depth lower bound whenever the complex does not have a small “bottle neck” (in a sense which we define). Here our lower bound on the circuit depth is only $\Omega(\log \log \log(n))$. We conjecture that the correct lower bound is at least $\Omega(\log \log(n))$, but this seems harder to achieve due to the possibility of hyperbolic geometry. For general simplicial complexes the lack of geometrical restriction on the gates becomes considerably more problematic than for the sphere, and we need to thoroughly modify the original argument in order to get a meaningful bound.

To the best of our knowledge, this is the first time the class of trivial quantum states is separated from the class of unique groundstates of gapped local Hamiltonians, improving our understanding of the hierarchy of global entanglement and topological order; we provide a survey of the current status of this hierarchy for completeness. We hope the tools developed here will be useful in various contexts in which quantum circuit depth lower bounds are of interest, including the study of topological order, quantum computational complexity and quantum algorithmic speed-ups.

1 Introduction

Since the early days of quantum mechanics, physicists have tried to understand and quantify entanglement in quantum systems. Whereas quantifying two-body entanglement is rather well understood, using Von Neuman entropy of subsystems, the question is far more involved when many-body entanglement is concerned. In [21] Hastings suggested a very interesting definition which captures perhaps the first step in understanding this question: He suggested to use, as a first order approximation of global entanglement or topological order, the requirement that a state be non-trivial, where a quantum state is trivial if it can be generated (or approximated) efficiently by a constant depth quantum circuit:

Definition 1. Trivial pure states (roughly) A family $\{\psi_n\}$ of pure states on $n$ qubits is called trivial if for every $n$, $\psi_n = U_n |0\rangle^n$ where $U_n$ is a quantum circuit of bounded depth $d_n = O(1)$ made of quantum gates acting on at most $O(1)$ qubits each.

One can include a notion of approximation in a natural way [21]. If one is interested in simulating local observable on such states, as is often the case in physics as well as in quantum complexity, then the term “trivial” is well justified. This is because sampling local observables of such a state $|\psi\rangle$ can be
done efficiently with a classical computer, given the classical description of the shallow quantum circuit that generated it. The way to do this is to notice that the size of the light cone of each output particle, namely the number of qubits in the input and throughout the evolution affecting its result, is $O(1)$. From a more physics-like point of view, trivial quantum states are those states that are adiabatically connected, via a path of gapped Hamiltonians, to a tensor product state \( |\psi\rangle \); this is the simplest type of a quantum states from the point of view of entanglement, and so this means that there is a constant time adiabatic evolution which generates this state starting from a tensor product state. In Physics language, this adiabatic connection without closing the gap is a way to partition the set of groundstates to equivalence classes, called “phases”; and so trivial states are viewed as belonging to the same “phase” as tensor product states, constituting the simplest possible states from the point of view of entanglement.

In some sense, all other states are viewed as having topological order (e.g., [42]).

We note that surprisingly, even such simple quantum states as trivial states can exhibit extremely interesting correlations, when considering measuring all of their particles \([3, 5]\). The correlations that can be generated even in such simple quantum evolutions can be computationally hard classically. Yet, in the most common situation in quantum complexity, in which local measurements are of interest, states in this class can indeed be regarded as “simple” from the point of view of entanglement.

We take this point of view, and ask whether a state (rather, a family of states) is trivial or not. From a physics point of view, the fact that a state is trivial provides strong intuition about how limited its global entanglement is, and is a basic question in the study of topological order \([21, 42]\). The question is of importance of course also in quantum computation theory (see \([6]\)): proving non-triviality of states means proving a lower bound on the minimal-depth of the circuit generating the state, and such lower bounds are a central topic of interest when understanding the complexity of quantum states. As a notable example, a study of Freedman and Hastings \([42]\) connects between the triviality of states and the major open problem of resolving the quantum PCP conjecture \([24, 25]\). One version of this conjecture, called the gapped qPCP version, states that deciding whether the ground energy of a local Hamiltonian with \( M \) local terms is 0 or at least a constant fraction of \( M \), is QMA hard. Freedman and Hastings noted that assuming qPCP holds (and assuming the class QMA not equal to \( NP \)), one needs to at least be able to point at a family of local Hamiltonians whose low energy states are all non-trivial; the existence of such family is called the NLTS conjecture \([42]\) (and it is still wide open, despite some recent progress \([29]\)). Thus, whether or not a family of quantum states is trivial, is of interest in both complexity and physical context.

### Simple arguments for quantum circuit depth lower bounds

There are very few known techniques for proving circuit lower bounds of quantum states, and very few classes of non-trivial states known. The easiest argument in this direction was given already in \([31]\) (see also \([2]\)) to show that the \( |CAT^+\rangle \) state \( \frac{1}{\sqrt{2}} (|0^n\rangle + |1^n\rangle) \) is non-trivial (albeit not in this terminology), and requires \( \Omega(\log(n)) \) circuit-depth. The argument goes by noting that any two qubits in \( |CAT^+\rangle \) are correlated (i.e., their reduced density matrix is far from a tensor product); while a a constant depth circuit correlates any given qubit with at most a constant number of other qubits. This simple correlation based argument, however, is rather limited, since states in which any two qubits are correlated are rather special. When correlations are more complicated (in particular, multipartite) this argument will not work.

We note that the above correlation-based argument can be adapted to provide an example of a non-trivial state which is also a unique groundstate of a local-Hamiltonian; while it is easy to see that \( |CAT^+\rangle \) itself is not the unique groundstate of any local Hamiltonian (since it agrees locally with \( |CAT^-\rangle \) ), using the by-now-standard method of the circuit-to-Hamiltonian construction \([33]\) (as suggested to us by Itai Arad \([26]\)) one can construct a state whose restriction to a non-negligible subsystem is very close to \( |CAT^+\rangle \) \( \varepsilon \), and yet the overall state is a unique groundstate of a local Hamiltonian (See Appendix B).

Another rather simple argument implying non-triviality, works for quantum error correcting code states. The argument uses the following simple fact:

**Fact 1.** **Trivial quantum states are unique ground states of gapped local Hamiltonians** Let \( |\psi_n\rangle \) be the output state of a constant depth circuit on the tensor state \( |0\rangle^n \), then \( |\psi_n\rangle \) is the unique ground state of a gapped local Hamiltonian \( H = \sum_{i=1}^{m} H_i \).

We will give a formal proof of this fact in Appendix A, however it would be beneficial to explain the argument here since its underlying idea constitutes a starting point for many results in our paper. The
argument starts by noting that by definition, a trivial state can be written as \( U |0^n\rangle \) for some constant depth \( U \). The idea is to notice that \( |0^n\rangle \) is the unique groundstate of the gapped local Hamiltonian \( H = \sum_i |1\rangle \langle 1| \) which projects each of the qubits on its \( |1\rangle \) state; to generate a local Hamiltonian whose unique groundstate is \( U |0^n\rangle \) we need to change the basis of \( H \) by \( U \), namely consider terms of the form \( U(|1\rangle \langle 1| \otimes I_n)U^\dagger \). The key point is that this term acts non-trivially only on a constant number of qubits, which are within the light cone on the \( i \)th qubit, since all gates in \( U \) outside of the light cone of the \( i \)th qubit, cancel with their corresponding inverse gate in \( U^{-1} \). For a detailed proof see Appendix A.

Let us now see how to deduce from this easy fact, that any state in any quantum error correcting code whose distance \( d \) is more than a constant, is non-trivial. Recall that any two states in such a QECC must have the same reduced density matrix on any set of \( k \) qubits, for \( k < d \) (See Fact 2). Hence, if one of the states in a QECC of distance \( d > k \) is a groundstate of a \( k \)-local Hamiltonian, so are all the others, and hence none of them can be the unique groundstate of that \( k \)-local Hamiltonian. By Fact 1 such a state cannot be a trivial state. This argument was essentially the same as the argument given in [6], except that they use the Lieb-Robinson bound (which is the physics continuous analog of the light cone notion), and this makes the argument somewhat less transparent for computer scientists.

To the best of our knowledge, these two simple arguments are the only currently known arguments for proving non-triviality (or more generally, circuit-depth lower bounds) of groundstates of local Hamiltonians.

**Geometrical non-triviality** While proving general non-triviality of groundstates in general seems rather difficult, the task becomes more accessible when we add a geometrical restriction on the gates of the quantum circuit \( U \). Namely, we will consider geometric trivial states, defined in the same fashion as triviality of states except for considering only quantum circuits \( U \) whose gates act on qubits lying within a constant distance from each other, where the metric is given by the interaction graph of the Hamiltonian (i.e., two qubits are connected if there is a term in the Hamiltonian acting on them).

**Definition 2. Geometrically Trivial pure states**: A family \( \{ |\psi_n\rangle \} \) of pure states defined on \( n \) qubits sitting on the edges of a graph is called geometrically trivial if for every \( n \), \( |\psi_n\rangle = U_n |0^n\rangle \) where \( U_n \) is a quantum circuit of bounded depth \( d_n = O(1) \) made of quantum gates acting on at most \( O(1) \) qubits each, where all of the qubits acted upon by the same gate are within \( O(1) \) distance from each other in the natural graph metric.

Under this geometrically restricted circuit setting, Bravyi [43] provided a beautiful argument that the toric code states cannot be generated by constant depth quantum circuit. This indeed follows already from the fact that the toric code is a QECC of distance \( \Omega(\sqrt{n}) \), by the argument sketched above; however, Bravyi’s argument is very different, and relies on a beautiful topological consideration.

Importantly for the focus of this paper, Bravyi’s argument (which was never published and so we provide it here for completeness; see Subsection 5) can be applied also in the case of Kitaev’s code on the sphere, which is defined similarly to the toric code but has dimension 1. For that reason, the above argument relying on large distance QECC does not hold whereas Bravyi’s argument does and directly implies geometrical non-triviality of the spherical code state. Freedman and Hastings [42] also used topological tools to argue geometrical non-triviality of certain other quantum groundstates.

The above proofs do work for unique ground states of gapped local Hamiltonians, but importantly, they are restricted to geometrical non-triviality (and in fact, clearly derive lower bounds which are too strong to apply when the geometrical restriction on the gates is relaxed).

**Moving towards general non-triviality of unique groundstates of gapped Hamiltonians** In light of all the above, one might ask: Could it be true that all unique groundstates of gapped local Hamiltonians are trivial, if one allows gates which are non-geometrically restricted? In other words, could it be that Fact 1 is “if and only if”? Hastings and Koma [41] proved that ground states of gapped Hamiltonians exhibit a phenomenon of decay of correlations; this suggests that the entanglement in such groundstates is severely limited, and further suggests that the possibility that such states are always trivial, at least cannot be ruled out immediately, since the argument sketched above of correlations implying non-triviality cannot be applied. Moreover, the argument of non-triviality for quantum error correcting code states cannot be applied either in this case.

In this paper we would indeed like to prove general circuit lower bounds in such cases. For a start, one would like to be able to prove a circuit-depth lower bound for the celebrated state of Kitaev’s spherical
code, in the non-geometrical setting.

The task of providing general lower bounds is notoriously hard in computational complexity; it seems that general methods for providing such lower bounds do not exist. Apart from the two simple arguments sketched above, which are suited for non-geometrical lower bounds, we are aware of only one technique for proving such quantum circuit depth lower bounds, given by Eldar and Harrow [29]. Eldar and Harrow provide non-geometrical circuit depth lower bounds for a certain class of quantum states, using a very innovative technique related to expansion of probability distributions; however their methods inherently do not apply for unique ground states of local Hamiltonians, since, just like in the CAT state and in the QECC codes arguments above, at the bottom of their argument lies the existence of two states which are globally orthogonal but locally similar; such an argument cannot be used when the groundstate is unique.

1.1 Results

In this paper, we prove quantum circuit depth lower bounds for a large class of quantum states, for general non-geometrically restricted circuits. In particular, we prove an $\Omega(\sqrt{N})$ lower bound for Kitaev’s spherical code state; we then extend the results to various groundstates of codes defined on polygonal complexes (as long as they don’t have small bottlenecks). To this end we develop various tools, mostly borrowed from algebraic topology.

To our knowledge, this work gives the first known example of non-trivial quantum states that are unique ground states of gapped Hamiltonians, in which the decay of correlations due to Hastings and Koma [41] holds.

Kitaev’s spherical code. We start with Kitaev’s surface codes. Instead of a surface code on a sphere, we work with the code set on a large 2-dimensional cube whose 6 faces are each tessellated to squares using an $n \times n$ grid, and the corresponding local Hamiltonian is the usual star and plaquettes Hamiltonian. It is well known that in this case the groundstate is unique. Bravyi’s [6] topology-based proof of non-triviality of the toric code state, holds also in this case.

**Theorem 1.** (Adapted from [43]) Let {$|\psi_i\rangle$}$_{i=1}^{\infty}$ be the family of cube states on $N_i$ qubits. Furthermore, let $U_i$ be a quantum circuit of depth $d_i$ using geometrically local gates on at most $c = O(1)$ qubits, such that $U_i|0^{N_i}\rangle = |\psi_i\rangle$. Then, $d_i = \Omega(\sqrt{N})$.

However, this method assumes that the circuits have the same geometry as the grid, and so this result only shows geometrical non-triviality (see definition 2).

Our first result is to show that the state is non-trivial regardless of how far from each other the qubits on which the c-local gates act are. In fact, we prove a logarithmic lower bound on any generating quantum circuit of the cube states:

**Theorem 2.** Let {$|\psi_i\rangle$}$_{i=1}^{\infty}$ be the family of cube states on $N_i$ qubits, and for every $i \geq 1$, let $U_i$ be a quantum circuit of depth $d_i$ using gates of locality $c = O(1)$, such that $U_i|0^{N_i}\rangle = |\psi_i\rangle$. Then, $d_i = \Omega(\log(N_i))$.

We believe this result was commonly assumed to be true, however, a proof did not exist. In fact, the proof is far from being straight forward; The non-geometrical gates pose considerable complications, and the proof turns out to be non-trivial (see overview of proofs).

We next proceed to prove a similar result for a much more general class of states. We note that on one hand, the cube states is a family of states which admits a simple geometric description - and therefore is a good candidate to illustrate some of the ideas in our more general proof. On the other hand, it also has the interesting property of yielding states which are unique ground states of gapped local Hamiltonians, making it resistant to the two simple methods sketched in the introduction, for proving lower bounds on general quantum circuit-depth; the cube states are subject to the exponential decay of correlations [41] and exhibit no obvious long range correlations; they are also not members of any QECC with non-constant distance.

**Extension to polygonal complexes** We then generalize this result to a wide class of quantum states, defined as codes on what we call "closed surface complexes" (CSCs), where the code states are stabilized.
by the usual star and plaquette operators on these complexes. This is done under a topological condition on the complex, which we call \( \text{r-simply connectedness} \).

We start by showing that the geometrical non-triviality topology based proof of \([3]\) extends to geometrical non-triviality in the case of CSCs as well:

**Theorem 3.** Let \( \{C_i\}_{i=1}^{\infty} \) be a family of (possibly non degenerate) surface codes on \( N_i \) qubits defined on the closed surface complexes \( G_i(V_i, E_i, F_i)_{i=1}^{\infty} \) (see definition \([13]\)) such that \( \deg(G) = O(1) \), \( \overline{\deg}(G) = O(1) \) and for all \( i \geq 1 \), let \( |\psi_i\rangle \in C_i \). Furthermore, let \( U_i \) be a quantum circuit of depth \( d_i \) using geometrically local gates on at most \( c = O(1) \) qubits, such that \( U_i |0\rangle^{N_i} = |\psi_i\rangle \). Assume that \( G_i \) is \( f \)-simply connected for some \( f = O(\log(N_i)) \), then, \( d_i = \Omega(\log(N_i)) \).

Our main result is a generalization of this to general (non-geometrically restricted) quantum circuits:

**Theorem 4** (Surface states are non trivial). Let \( \{C_i\}_{i=1}^{\infty} \) be a family of (possibly non degenerate) surface codes on \( N_i \) qubits defined on the closed surface complexes \( G_i(V_i, E_i, F_i)_{i=1}^{\infty} \) such that \( \deg(G) \), \( \overline{\deg}(G) = O(1) \) and for all \( i \geq 1 \), let \( |\psi_i\rangle \in C_i \). Furthermore, let \( U_i \) be a quantum circuit of depth \( d_i \) using gates acting on \( c = O(1) \) qubits such that \( U_i |0\rangle^{N_i} = |\psi_i\rangle \). Assume that \( G_i \) is \( f \)-simply connected, for some \( f = O(\log(\log(N_i))) \) then, \( d_i = \Omega(\log(\log(N_i))) \).

The condition on the face and vertex degrees is just a way to restate the fact that our states should be ground states of \( \text{local} \) Hamiltonians only. Broadly speaking, \( \text{r-simply connectedness} \) means that the subcomplex induced by any ball of radius at most \( r \) should be simply connected (see Definition \([34]\)). In other words, \( G_i \) shouldn’t have any ”bottlenecks”. Observe that at first glance, this condition might seem weaker than simple connected of the whole complex; it also seems as if it is implied by it. However this is not the case, as can be seen in Figure \([3]\). This is quite subtle, and indeed, our proof does NOT work on some spaces which are simply connected, but have small bottle necks so they are NOT \( r \)-simply connected for some small \( r \).

The main difference between Theorem \([2]\) and Theorem \([3]\) lies in the different lower bounds they exhibit. In fact, this is a reflection of the fundamental difference between Euclidean and hyperbolic metric spaces, and more precisely between the ratios of the area of disks to their radii in both cases. We will further discuss this issue in section \([4]\).

### 1.2 Proofs overview

Our starting point for this paper is Bravyi’s topological argument for \( \Omega(\sqrt{n}) \)-circuit depth lower bound for toric code states \([15]\). This argument works essentially as is for Kitaev spherical code states. For completeness, we rederive this argument fully in this paper, since it was not published; in fact, we provide the proof for a slightly different set of states, which we call cube states, defined in Section \([5]\). Bravyi’s proof relies on a central lemma, Lemma \([1]\) which will be the basis point for all our proofs. This lemma is proved in Section \([4]\). The lowerbound for cube state itself is then derived in Section \([5]\).

We will then explain how starting from Bravyi’s approach we can provide a non-geometrical lower bound, and later we analyze both geometrical and non-geometrical lower bounds for the more general case of Closed simplicial complexes, CSCs (see definition \([13]\)).

#### 1.2.1 Bravyi’s argument

We now sketch the central lemma in Bravyi’s argument. Given a grid representing a closed surface, suppose \( U \) is a constant depth circuit generating the groundstate of some Hamiltonian set on the grid. We choose two distant edges \( e \) and \( f \) on the cube grid, and consider a path \( \gamma \) between them. The effective support \( B_{[\gamma],U} \) of the path \( \gamma \), relative to a generating circuit \( U \) is the following set of edges: Consider the set of all qubits that are non-trivially acted upon, if we conjugate the path \( \gamma \) by the circuit \( U \); if \( U \) has constant depth, this set takes the form of a ”thickening” (of constant thickness) of \( \gamma \), as in the argument behind the proof of fact \([1]\) (see Appendix A). We look at the intersection of these subsets when going over all paths \( \gamma \) connecting \( e \) to \( f \), and we call this the effective support of \( \gamma \) with respect to \( U \), denoted \( B_{[\gamma],U} \). This turns out to be the union of two small regions around \( e \) and \( f \). (See Figure \([1]\).)
Lemma 1 states that any operator which stabilizes the groundstate, and whose support lies completely outside of \( B_{\beta, U} \) for the path \( \beta \), must commute with any Pauli Z operator supported on \( \beta \), when acting on the groundspace. This relies on the fact that the circuit is constant depth, and thus under a suitable change of basis, all arguments about commutation relations between operators on different sets can essentially be made “classical”, by considering the operators supports except “thickened” by some constant. We next use this lemma to derive the geometrical lower bound.

1.2.2 Cube states: the geometric case

In section 5, we prove geometric non-triviality of cube states, which are easy-to-work-with variant of the spherical codes of Kitaev (see [13]). More specifically, We consider the quantum codes associated with regular tessellations of the cube, with the usual definition of star and plaquette operators (see Definition 25 in Subsection 2.3). Since the cube is simply connected, these codes have dimension 1 (see corollary 1). These unique ground states are the “cube states”.

In order to prove non triviality, we first show in Lemma 2 that for the cube states - assuming the generating circuits \( U_n \) are geometrically local and that they have constant depth - the effective supports (see definition 30) of any path \( \gamma \) is contained in the union of two balls of radius proportional to the depth of \( U \) and centred around the endpoints of \( \gamma \). This fact has a rather intuitive geometric explanation (see Figure 1).

Finally, using Lemma 1 one can derive a contradiction: We choose both endpoints of \( \gamma \) to be on opposite faces of the cube, and find a large closed copath \( \beta \) far from these end points (for now, think of this as a huge loop around the cube) cutting the cube into two connected components, each one containing one of the endpoints. This can be done such that any path between the endpoints intersects the copath an odd number of times since it starts from one connected component and ends up in the second one (see Figure 2). Now consider the following 2 operators: The first \( \gamma_Z \) applying \( Z \)'s on all the qubits in \( \gamma \), and the second, \( \beta_X \) applying \( X \)'s on all the qubits in \( \beta \). Since \( \gamma \) and \( \beta \) intersect in an odd number of edges, these global operators anticommute. However note that \( \beta_X \) stabilizes the groundstate, and its support is outside of \( B_{\gamma, U} \) since it is far from the endpoints of \( \gamma \). This finally yields a contradiction to Lemma 4 claiming those two operators should commute.

1.2.3 Cube states: the non geometric case

The key issue motivating this work is that the above arguments stop working when one drops the spatial locality assumption. Indeed, in that case, \( B_{\gamma, U} \) cannot be shown to be neither small, nor spatially close to the endpoints. In section 5, we solve that issue for the special case of the cube states, by showing that the very fact that \( B_{\beta, U} \) big can actually be used to prove the non triviality of the cube state. In fact the first step is to prove in lemma 5 that \( |B_{\gamma, U}| \) is actually lower bounded by the length of \( \gamma \). Indeed, suppose otherwise, then both endpoints \( e \) and \( f \) of \( \gamma \) would lie in a different connected component of \( B_{\gamma, U} \). One can now define an operator \( \beta_X \) on the coboundary \( \beta \) of the connected component \( T_e \) containing \( e \). Observe that \( \gamma \) and \( \beta \) intersect on an odd amount of edges: indeed, we can consider \( \gamma \) as an indexed finite sequence of vertices where each edge in \( \gamma \cap \beta \) is interpreted as getting “in” or “out” of \( T_e \). But since \( e \in T_e \) and \( f \notin T_e \), we immediately get \( |\gamma \cap \beta| = 0 \) mod 2. It follows that \( \gamma_Z \) and \( \beta_X \) must anticommute. On the other hand, from lemma 1 since \( \beta_X \) is supported on \( E \setminus B \), we expect \( \beta_X \) and \( \gamma_Z \) to commute, leading to a contradiction.

To this end, we introduce the notion of \( \gamma \)-separation: A set of edges is said to be \( \gamma \)-separating if any path \( \gamma' \in [\gamma] \) intersects it non trivially. We show in claim 3 that the upper light cone of any edge in \( \Lambda \) is \( \gamma \)-separating. Then, we proceed to prove in claim 4 and corollary 3 that W.l.o.g, those lightcones can be assumed to be connected. Indeed: given 2 different paths \( \gamma \) and \( \gamma' \) intersecting 2 distinct copath connected components, one can find a new path \( \beta \in [\gamma] \) such that \( \beta \) doesn’t intersects with either component, contradicting the assumption of \( \gamma \)-separation.

In lemma 4 we show that those copath connected \( \gamma \)-separating lightcones \( \{LC(a) | a \in A \} \) must lie within the union of 2 balls centered around \( e \) and \( f \) of radius \( |LC(A)| \). This is rather obvious for the cube complex since one can always find a ball \( C \) of radius \( |LC(a)| \) and containing \( LC(a) \) such that both endpoints of \( \gamma \) lie outside of \( C \). Then, removing \( C \) from the original cube complex yields a connected subcomplex where any path between \( e \) and \( f \) doesn’t intersect \( LC(a) \) leading to a contradiction.
Figure 1: The yellow areas represents the lower light cone of 3 distinct paths $\gamma_1$, $\gamma_2$ and $\gamma_3$ between $e$ and $f$. The blue area centered around $e$ and $f$ represent the intersection between those 3 areas when going through these 3 paths. The set $A$ of edges in the intersection of the yellow "fattenings" for all paths in $\Gamma_{e,f}$ is contained in the union of 2 balls centered around $e$ and $f$ and whose radii is upper bounded by some function of the circuit depth.

Figure 2: The co-path in blue separates the edge $e$ from $f$ on the other side of the cube: Both $\gamma_1$ and $\gamma_2$ have to intersect it non-trivially.
Figure 3: The yellow part $S$ represents a small disk on a bottleneck area of some simply connected surface. Even though $S$ is small, it is not simply connected as any loop around the surface lying inside this area is non trivial.

Finally, we prove Theorem 2 by using the fact that on one hand, $A$ must be a large subset from lemma 3 (namely, its size is bounded from above by the diameter of the cube), but on the other hand, $A$ is included inside the lower lightcone of the small balls around $e$ and $f$ defined earlier, whose radii depend of the depth of the generating circuits $U_n$. Therefore, we can extract a relation between the circuit depth and the diameter of the graph which is known to be $O(n)=O(\sqrt{N})$ for the cube where $N$ is the number of qubits in the system.

1.2.4 From cube states to complexes: the geometric case

In section 7.1, we develop the tools to prove Theorem 4, which generalizes the above result to more general closed surface complexes. The main issue with the previous approach comes from the fact that lemma 4 doesn’t hold in the general setting. Indeed, one could construct a CSC with small "bottlenecks", such that any path between $e$ and $f$ must pass through it. Therefore, the upper light cone of some edge in $A$ could lie in that small domain even though it could be far away from both $e$ and $f$ (see figure 3). This example is the main motivation behind the additional assumption of $r$-simple connectedness we require in order to avoid such complications. Namely, a CSC is said to be $r$-simply connected if the first homology and cohomology groups of any ball of radius $r$ around any edge of the complex vanishes. This can be shown to be enough to ensure that Theorem 6 - a more general version of lemma 4 - holds in the framework of $r$-simply connected CSCs. The proof of Theorem 6 relies on rather technical observations encompassed in lemma 5.

1.2.5 CSC states: the non geometric case

Finally, in section 7 we derive our main results, Theorems 3 and 4. Both proofs follows the same lines as in theorems 1 and 2 relying on Theorem 6 and the assumption of $r$-simple connectedness to ensure that the sets $L(a)$ for $a \in A$ lie inside a ball of small radius around the endpoints $e$ and $f$ as in the cube case.

1.2.6 Remarks

It should be noted that the main difference between the asymptotic lower bounds derived from the cube state and from more general CSC states lies in the geometric properties of the underlying topological space. More precisely, the maximal ratio between the radius of a ball and its area (the number of edges lying inside the ball) on the complex, is the main factor influencing the asymptotic behaviour
of the circuit depth lower bounds described in this paper. In the euclidean case (e.g the toric code or cube state), this ratio is quadratic in \( r \) leading to a stronger lower bound of \( \Omega((\log N)) \) (and \( \Omega(\sqrt{N}) \) in the geometrical case). On the other hand, if this ratio decays exponentially in \( r \) (as it is the case for hyperbolic surfaces, see figure 6) we can merely extract a \( \Omega(\log(\log(\log(N)))) \) bound (and \( \Omega(\log N) \) in the geometrical case, Theorem 3).

1.3 Further discussion

Our result is the first circuit depth lower bound for unique groundstates of gapped local Hamiltonians, in the challenging setting in which entanglement is limited by exponential decay [41].

We note that using Kitaev’s circuit-to-Hamiltonian construction one can construct a non-trivial state which is very close to the CAT state, and is a unique groundstate of a local Hamiltonian; however, this local Hamiltonian is not gapped, and thus the state does not exhibit exponentially decaying correlations. Indeed, it is exactly those non-decaying corrections which are used to prove its non-triviality.

The results can be viewed in the more general context of the hierarchy of topological order. More specifically, one can consider increasingly growing classes of states, which are more and more entangled; at the bottom lie the trivial states, then comes the class of unique groundstates of gapped local Hamiltonians (UGS), then unique groundstates of local (not necessarily gapped) Hamiltonians, and these are contained within the class of states determined by their local reduced density matrices among all mixed states (UDA [3]), and then among all pure states (UDP [3]). Our proof can be viewed as separating the second set from the third; despite previous attempts, it remains open to clarify whether the fourth is or is not equal to the fifth. This study is thus motivated also by the goal of clarifying the connection between local Hamiltonians, local reduced density matrices and the global entanglement determined by them. In the next subsection we provide a thorough description of this hierarchy, for completeness of the context.

Finally, we ask: can non-trivial circuit depth lower bound be proven for quantum groundstates which are non-homological? Of course, a major open question is whether a superlogarithmic lower bound can be proven for any quantum state.

1.4 The Heirarchy of Topological Order

Our work brings some new insights into the hierarchy of topologically ordered quantum states, namely, states with global entanglement. In particular, we survey our current understanding of states which are not topologically ordered.

At the bottom of the hierarchy lies the class of trivial states (see definition 1). In [6], the authors proved that trivial states are unique groundstates of gapped Hamiltonians, and we give an alternative proof in Appendix A. We can define the class of states with the aforementioned property:

**Definition 3.** \( k,\Delta - \text{Gapped-UGS} \) states Let \( k > 0 \). A pure quantum state \( |\psi\rangle \) is said to be \( k,\Delta - \text{Gapped-UGS} \) if there exists a \( k \)-local Hamiltonian \( H = \sum_{i=1}^{m} H_i^{\alpha} \) with energy gap at most \( \Delta \) such that \( |\psi\rangle \) is the unique ground state of \( H \).

Of course, these states are a subclass of unique groundstates of Hamiltonians which are not necessarily gapped:

**Definition 4.** \( k \)-UGS states Let \( k > 0 \). A pure quantum state \( |\psi\rangle \) is said to be \( k \)-UGS if there exists a \( k \)-local Hamiltonian \( H = \sum_{i=1}^{m} H_i^{\alpha} \) such that \( |\psi\rangle \) is the unique ground state of \( H \).

While \( k,\Delta \)-Gapped-UGS trivially implies \( k \)-UGS, the reverse implication doesn’t hold. In fact, in Appendix A, we provide a family of pure states \( \{|\phi_n\rangle\} \) which asymptotically approximate the family \( \{|\text{CAT}_{+}\rangle_n\} \) of CAT states. We show that \( \{|\phi_n\rangle\} \) is \( k \)-UGS but not \( k,\Delta \)-Gapped-UGS for any constant \( \Delta \).

The fact that states are unique groundstates of \( k \)-local Hamiltonians, stems from the question of whether their reduced local density matrices can be uniquely “lifted” to a single pure state. The following two definitions attempting to capture this notion in two slightly different ways were introduced in [40]:

**Definition 5.** \( k \)-UDA states Let \( k > 0 \). A pure quantum state \( |\psi\rangle \) is said to be \( k \)-UDA if given any mixed state \( \rho \) which satisfies that for all subsets \( A \) of qubits, \( |A| = k \)

\[ \text{Tr}_A(|\psi\rangle \langle \psi|) = \text{Tr}_A(|\phi\rangle \langle \phi|) \]
then $|\psi\rangle \langle \psi| = \rho$.

**Definition 6. k-UDP states** Let $k > 0$. A pure quantum state $|\psi\rangle$ is said to be $k$-UDP if given a pure state $|\phi\rangle$ which satisfies that if for all subsets $A$ of qubits, $|A| = k$

$$Tr_A(|\psi\rangle \langle \psi|) = Tr_A(|\phi\rangle \langle \phi|)$$

then $|\phi\rangle = |\psi\rangle$.

While $k$-UDA always implies $k$-UDP (namely, $k$-UDA is contained in $k$-UDP), the converse is not known to hold: in [36], the authors showed a 4 qubit state which is 2-UDP but not 2-UDA; extending this to a family of $n$ qubit states for growing $n$ is left open.

One can easily show that $k$-UGS states are also $k$-UDA (hence also $k$-UDP) since the energy of a state relative to a $k$-local hamiltonian depends only on its reduced density matrices on sets of $k$ qubits (see Appendix C). The reverse implication is still an open problem (though it was claimed at some point to hold, following from [45] which was then found wrong in [35]).

The above sequence of sets thus provides a hierarchy of states which are in some sense increasingly "more and more globally entangled"; As explained above, most of these containments were already known to be strict, as can be seen in Figure 4. Indeed, Theorem 2 can in fact be put in this context: it shows that the containment of trivial states inside Gapped-UGS is also strict, as the spherical code is a unique groundstate of the gapped Hamiltonian of the spherical code, while it is non-trivial. This containment might be somewhat surprising given that Gapped-UGS are known to have exponentially decaying correlations, and thus of limited global entanglement. This leaves one open question, regarding the strict containments in this heirarchy: Is there a $k$-UDA state which is not $k$-UGS? In addition, also the question of relation between UDP and UDA requires further clarification.

We mention that outside of this Heirarchy of lack of topological order, there is also a separate heirarchy of the more distinguished topologically ordered states. Such a heirarchy was suggested in [34], which defined the TQO-1 condition:

**Definition 7. TQO-1 states (see [34])** A family $\{ |\psi_n\rangle \}$ of quantum states on $n$ qubits is said to be TQO-1 if there exists a family of positive rate quantum codes $\{ C_n \}$ with macroscopic distance such that for all $n$, $|\psi_n\rangle \in C_n$.

Furthermore, the authors argued that TQO-1 alone could not ensure stability of the related local Hamiltonian under small perturbation (see [34] for a formal definition), and defined a more refined class of TO states which satisfy another condition called TQO-2 (see [34] for a formal definition); the two conditions together imply such a stable topological order (in Figure 4 this is denoted STQO).

We observe that by definition, TQO-1 states are not $k$-UDP. However, the sets do not complement each other: the CAT state is neither TQO-1 nor $k$-UDP. Indeed, the reduced density matrices of the 2 orthogonal CAT states on any proper subset of qubits are similar, but the code they span only has distance 1.

As in Figure 4, we can thus draw the picture as two hierarchies. Of course, these two hierarchies can be merged into one hierarchy by considering the complementing sets of either of them; but this leads to a less intuitive picture we think). In between those hierarchies, lie "intermediate" families of states such as the CAT states which are complicated enough to be outside of $k$-UDP and can exhibit long range correlations, but are not inside any non-vanishing distance quantum codes.

The overall hierarchy of topological orders can be visualized in Figure 4.

### 2 Background

#### 2.1 Polygonal complexes and graphs

**Definition 8. Paths in undirected graphs:** Let $G = (V, E)$ be an undirected simple graph and let $e, e' \in E$, we define a path between $e$ and $e'$ as any sequence of edges $\gamma = (e = e_0, e_1, \ldots, e_k = e')$ such that for all $0 \leq i \leq k - 1$, $e_i$ and $e_{i+1}$ share a vertex. A path is said to be closed, if $e = e'$. A path $\gamma$ is said to be simple if every vertex belongs to at most 2 edges in $\gamma$. 

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Figure 4: The hierarchy of topological orders. On the left side, the hierarchy of lack of quantum topological order. On the right side, the 2 TQO conditions. $|\text{CAT}\rangle$ states fall in the middle: They exhibit long range correlations but are not inside any macroscopic distance quantum code.
**Definition 9. polygonal complex** Let $G = (V, E)$ be a finite connected undirected simple graph. Let $F$ be a subset of closed simple paths in $G$ such that for every pair $f_1, f_2 \in F$, such that $f_1 \neq f_2$, one of the following conditions holds:

1. $f_1 \cap f_2 = \emptyset$
2. $f_1 \cap f_2 \in V$
3. $f_1 \cap f_2 \in E$

then the triplet $(V, E, F)$ is called a polygonal complex. $V$ is called the vertex set, $E$ is called the edge set and $F$ is called face set of $G$.

The following definition is dual to the notion of path in polygonal complexes:

**Definition 10. co-paths in a polygonal complex:** Let $G = (V, E, F)$ be a and let $e, e' \in E$, we define a copath between $e$ and $e'$ as any sequence of edges $\gamma = (e = e_0, e_1, ..., e_k = f)$ such that for all $0 \leq i \leq k - 1$, $e_i$ and $e_{i+1}$ belong to a common face $f \in F$. A copath is said to be closed, if $e = e'$. A copath is said to be simple if every face in $F$ contains at most 2 edges from $\gamma$.

We now define the vertex and face support of a set of edges:

**Definition 11. Vertex and face support of a set of edges:** Let $G = (V, E, F)$ a polygonal complex and let $X \subseteq E$. Define the vertex and face support of $X$ by:

$$V_X = \{v \in V \mid \exists e' \in X, v \in e'\}$$

and

$$F_X = \{f \in F \mid \exists e' \in X, e' \in f\}$$

In the same fashion, we have:

**Definition 12. Edge support of vertex and face subsets:** given subsets $Y \subseteq V$ or $Y' \subseteq F$ we can define the edge support of $Y$ and the edge support of $Y'$ to be:

$$E_Y = \{e \in E \mid \exists e' \in Y, e' \in e\}$$

$$E_{Y'} = \{e \in E \mid \exists f' \in Y', e \in f'\}$$

Similarly we define the face and vertex supports of $Y$ and $Y'$ to be:

$$F_Y = \{f \in F \mid \exists e \in E \exists v \in Y, v \in e, e \in f\}$$

$$V_{Y'} = \{v \in V \mid \exists e \in E \exists f' \in Y', v \in e, e \in f'\}$$

We will be interested in polygonal complexes that mimic compact surfaces with no boundary. In the continuous model, this condition has a local characterization: namely, that any points of the 2-dimensional manifold has some neighbourhood homeomorphic to the plane $\mathbb{R}^2$. The next definition gives a discrete analogue to this local condition:

**Definition 13. closed surface complex** A closed surface complex (or CSC) is a connected and simply connected polygonal complex $G = (V, E, F)$ such that for all $v \in V$, the following conditions hold:

1. $|E(v)| = |F(v)| \geq 2$
2. There are orderings $E(v) = (e_1, ..., e_k)$ and $F(v) = (f_1, ..., f_k)$ such that for all $i, f_i \cap f_{i+1} = e_i$ and $f_i \cap f_j = \emptyset$ for all $i, j$ such that $|i - j| \geq 2$ (Index summation is done modulo $k$).

See Figure 5 for a graphic illustration of the CSC condition.

**Claim 1.** Let $G = (V, E, F)$ be a CSC. Then for every edge $e \in E$, there are exactly 2 faces $f_1, f_2 \in F$ such that $e \in f_1$ and $e \in f_2$
Figure 5: The CSC condition implies that one can order all edges adjacent to \( v \) in a cyclic order such that every 2 consecutive edges share exactly one face.

**Proof.** Let \( v \in e \). Since \( G \) is a CSC, we have \( E_v = (e_1, \ldots, e_k) \) and there is a face between every 2 cyclically consecutive edges in the list. Since \( e \in E \), there exist some index \( i \leq k \) such that \( e = e_i \) and therefore there exist 2 unique faces \( f_1 \) and \( f_2 \) such that \( f_1 \) connects between \( e \) and \( e_{i-1} \) and \( f_2 \) connects between \( e \) and \( e_{i+1} \). But observe that if there was another face \( f_3 \) such that \( e \in f_3 \) then \( f_3 \) must contain an edge \( e' \neq e \) such that \( v \in e' \) and \( e' \in E_v \) contradicting the uniqueness part of the CSC condition. \( \square \)

**Definition 14. Vertex and face degree** Let \( G = (V, E, F) \) be a polygonal complex. We define the vertex and face degrees of \( G \) to be:

\[
\deg(G) = \max_{v \in V} |E_v| \\
\hat{\deg}(G) = \max_{f \in F} |E_f|
\]

Now, consider a polygonal complex \( G = (V, E, F) \). We define two natural metrics on \( E \):

**Definition 15. The set of all paths between two edges, \( \Gamma \):** We denote the set of all paths between \( e \) and \( f \) by \( \Gamma_{e,f} \), and the set of copaths between \( e \) and \( f \) as \( \hat{\Gamma}_{e,f} \). Note that \( |\gamma| = k \) is the number of edges in \( \gamma \).

The above definitions yield the following 2 discrete metrics on \( E \):

**Definition 16. Metrics:**

\[
\forall e, f \in E: d(e, f) = \min_{\gamma \in \Gamma_{e,f}} (|\gamma| - 1) \\
\forall e, f \in E: \hat{d}(e, f) = \min_{\gamma \in \hat{\Gamma}_{e,f}} (|\gamma| - 1)
\]

Note that while both metrics \( d \) and \( \hat{d} \) can be defined on any 2-dimensional polygonal complex, they need not be equivalent (related up to a constant) in general. On the other hand, one can easily show that on CSC’s with bounded vertex and face degrees, they are indeed equivalent. Indeed, assuming \( \deg(G) = D \), it can be shown that for all \( e, f \in E \):

\[
\hat{d}(e, f) \leq d(e, f) \leq D \cdot \hat{d}(e, f)
\]

We will not be making use of this claim, but give it just for intuition, hence we skip the proof here.

**Definition 17. Topological balls and diameter:** Let \( G = (V, E, F) \) be a CSC and let \( n \geq 0 \). Given some edge \( e \in E \) define the ball of radius \( n \) around \( e \) relative to the metrics \( d \) and \( \hat{d} \) by:

\[
Ball(e, n) = \{e' \in E \mid d(e, e') \leq n\} \\
\hat{Ball}(e, n) = \{e' \in E \mid \hat{d}(e, e') \leq n\}
\]

The resulting 2 diameters of \( G \) are defined as:

\[
diam(G) = \max_{e, e' \in E} d(e, e') \\
\hat{diam}(G) = \max_{e, e' \in E} \hat{d}(e, e')
\]

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We will also need to define the notions of boundary and coboundary:

**Definition 18. Boundary and coboundary:** Let \( G = (V, E, F) \) a CSC and \( E' \subseteq E \) be a subset of edges. The edge coboundary of \( E' \) in \( G \) is defined as:

\[
\partial(E') = \{ (u, v) \in E \mid \exists (u, v') \in E', \exists (u', v) \in E' \}
\]

Observe that from claim \([1]\) every edge \( e \in E \) is the intersection of 2 unique faces \( f_e \neq g_e \in F \). Therefore, we can define the edge boundary of \( E' \) as:

\[
\partial(E') = \{ e \in E \mid \exists e' \in E' : e' \in f_e, \exists e'' \in E' : e'' \in g_e \}
\]

### 2.2 Algebraic Topology

We recall some basic definitions from algebraic topology (52).

**Definition 19. Chain complex:** A chain complex \( C = (\{C_i\}, \{\partial_i\}) \) is a sequence of abelian groups \( C_i \) and group homomorphisms \( \partial_i : C_i \rightarrow C_{i-1} \) called *boundary maps* such that for \( i \geq 1 \), we have

\[
\partial_{i-1} \circ \partial_i = 0
\]

**Definition 20. Cochain complex:** A cochain complex \( C^* = (\{C^*_i\}, \{\partial^*_i\}) \) is a sequence of abelian groups \( C^*_i \) and group homomorphisms \( \partial^*_i : C^*_i \rightarrow C^*_{i+1} \) called *coboundary maps* such that for \( i \geq 0 \), we have

\[
\partial^{i+1} \circ \partial^i = 0
\]

**Definition 21. Chain and cochain complexes from CSCs:** Let \( G = (V, E, F) \) be a CSC. There exists a natural way to define both a chain and a cochain complex from \( G \) the following way: Define \( C_0 = < V >, C_1 = < E > \) and \( C_2 = < F > \) to be the free abelian groups generated by the sets \( V, E, F \) respectively. There are 2 well defined group homomorphisms \( \partial_2 : C_2 \rightarrow C_1 \) and \( \partial_1 : C_1 \rightarrow C_0 \) corresponding to "taking the boundary": more precisely, we can define \( \partial_2 \) and \( \partial_1 \) on basis elements the following way:

\[
\partial_1(e) = v_1 + v_2
\]

\[
\partial_2(f) = \sum_{e \in f} e
\]

Define \( C = ((C_i), (\partial_i))_{i \in \{1, 2\}} \). It can be easily shown that the chain complex condition \( \partial_1 \circ \partial_2 = 0 \) holds for \( C \). Indeed, if \( f = (e_1, \ldots, e_k) \) then \( \partial_2(f) = e_1 + \ldots + e_k \) and assuming \( e_i \) shares the vertex \( v_i \) with \( e_i + 1 \) cyclically, then

\[
\partial_1 \circ \partial_2(f) = \partial_1(e_1) + \ldots + \partial_1(e_k) = 2(v_1 + \ldots + v_k) = 0
\]

In a similar fashion, one can naturally define a cochain complex structure on \( G \):

Let \( i \in 0, 1, 2 \), and let \( C^*_i = Hom(C_i, Z_2) \) where \( Hom(C_i, Z_2) \) is the group of homomorphisms from \( C_i \) to the 2 elements group \( Z_2 \). Note that for every element \( x \in C_i \) in a generating set \( X_i \) of \( C_i \), one can define the following element \( f_x, \in C^*_i \):

\[
f_x(x) = \begin{cases} 0 & \text{if } x_i = x \\ 1 & \text{if } x \neq x \end{cases}
\]

In fact, it can be easily checked that the set \( \{f_x | x \in X_i\} \) generates the whole group \( C^*_i \). The boundary maps \( \partial_1 \) and \( \partial_2 \) on \( C_1 \) and \( C_2 \) induce coboundary maps \( \partial^1 : C_0^* \rightarrow C_1^* \) and \( \partial^2 : C_1^* \rightarrow C_2^* \).

Let \( i \in 1, 2 \) and let \( f \in C^*_i \) and \( x \in C_{i+1} \). Define:

\[
[\partial^i(f)](x) = f(\partial^{i+1}(x))
\]

or in other words, \( \partial^i = f \circ \partial^{i+1} \). Now, define \( C^* = ((C^*_i), (\partial^i))_{i \in \{1, 2\}} \). It is easily seen that the boundary condition \( \partial_1 \circ \partial_2 = 0 \) induces the coboundary condition on \( (\partial^1, \partial^2) \):

\[
\partial^{i+1} \circ \partial^i = 0
\]

giving \( C^* \) a structure of cochain complex.
In order to extract meaningful invariants from a surface chain complex, one needs to look at the actual homology and cohomology groups derived from $C$ and $C^*$:

**Definition 22. Homology groups:** Let $C = (C_i, \partial_i)_{i=0}^{n}$ be a chain complex. The $i$-th homology group $H_i(C)$ of $C$ is defined as:

$$H_i(C) = \text{Ker}(\partial_i)/\text{Im}(\partial_{i+1})$$

Elements of $\text{Im}(\partial_{i+1})$ are called $i$-boundaries and elements of $\text{Ker}(\partial_i)$ are called $i$-cycles. When $C$ is derived from a CSC $G$, we use the notation $H_i(G) = H_i(C)$.

**Definition 23. Cohomology groups:** Let $C^* = (C^i, \partial^i)_{i=0}^{n}$ be a cochain complex. The $i$-th cohomology group $H^i(C^*)$ of $C^*$ is defined as:

$$H^i(C) = \text{Ker}(\partial^i)/\text{Im}(\partial^{i-1})$$

Elements of $\text{Im}(\partial^{i-1})$ are called $i$-coboundaries and elements of $\text{Ker}(\partial_i)$ are called $i$-cocycles. When $C^*$ is derived from a CSC $G$, we use the notation $H^i(G) = H^i(C)$.

One can interpret the rank of $H_i$ as the number of $i$-dimensional ”holes” in $S$. A classic example is the torus $T$. The first homology group $H_1(T)$ of $T$ has rank 2. A natural basis for $H_1(T)$ is the one spanned by a lateral and a longitudinal non-trivial loops around $T$.

**Definition 24. Homology class:** Let $G$ be a CSC and let $\gamma$ be a path. Define the homology class $[\gamma]_G$ of $\gamma$ to be the set of chains $\gamma' \in C_1(G)$ such that $\gamma + \gamma'$ is a 1-boundary (or equivalently, $\gamma + \gamma'$ vanishes in $H_1(G)$). When the underlying complex $G$ is known from context we just write $[\gamma]$ instead of $[\gamma]_G$.

In the rest of this paper, we will look at paths (and copaths) as both subsets of the edge set $E$ and chains in $C_1$. With that in mind, the expression $\alpha + \beta$ should be understood either as the sum of 2 chains in $C_1$ or equivalently, as the symmetric difference between two sets.

**Remark 1.** The fact we used the same term $\partial$ for definitions 18 and 22 is not coincidental: indeed, it can easily be shown that for any set $E' \subseteq E$ of edges:

$$\partial(E') = \partial_2(F_{E'})$$

and

$$\tilde{\partial}(E') = \partial^0(V_{E'})$$

**2.3 Quantum surface codes**

First recall the definition of stabilizer codes:

**Definition 25. Stabilizer formalism:** Denote by $P_n$ the Pauli group acting on the Hilbert space $(H_n = \mathbb{C}^2)^{\otimes n}$ of $n$ qubits and let $\mathcal{G} \subset P_n$ be an abelian subgroup of $P_n$ such that $-1 \notin \mathcal{G}$. Then the set

$$C_\mathcal{G} = \{ |\psi\rangle \in H_n | \forall g \in \mathcal{G} : g \cdot |\psi\rangle = |\psi\rangle \}$$

is the $\mathcal{G}$-stabilized subspace of $H_n$ and has dimension $2^{n-dim(\mathcal{G})}$ where $dim(\mathcal{G})$ is the dimension of $\mathcal{G}$ as a vector space over $\mathbb{F}_2$.

We will also need the following fact about Quantum Error Correcting codes:

**Fact 2. adapted from [14] p.436:** For any two orthogonal states in a quantum error correcting code whose distance is $d_n$, we have for any operator $O_n$ of support on a set $K \subseteq [n]$ of size $|K| < d_n$

$$\langle \psi_n | O_n | \phi_n \rangle = 0$$

and from the properties of the partial trace, we get:

$$\text{Tr}_K (|\psi_n\rangle \langle \psi_n|) = \text{Tr}_K (|\phi_n\rangle \langle \phi_n|)$$
One of the most famous examples of stabilizer codes is the toric code, introduced by Kitaev \[13\].

Consider the lattice $\mathbb{Z}_n \times \mathbb{Z}_n$ where $\mathbb{Z}_n$ is the cyclic group of order $n$. For every edge in the lattice define a 2-dimensional site (namely, a qubit). Moreover, for each vertex $v$ in the grid, we associate a star Pauli operator $A_v = \sigma^X_1 \sigma^X_2 \sigma^X_3 \sigma^X_4$ where $\{\sigma^X_i\}^{4}_{i=1}$ are applied on the 4 sites (edges) connected to $v$. Similarly, for each face $f$ (any basic 1 × 1 square), we associate a plaquette Pauli operator $B_f = \sigma^Z_1 \sigma^Z_2 \sigma^Z_3 \sigma^Z_4$ where $\{\sigma^Z_i\}^{4}_{i=1}$ are applied on the 4 sites connected to $f$. Kitaev proved that the subgroup generated by $A_v, B_f$ satisfies the above 2 conditions, and has dimension $n$ (see \[13\]). Hence, the corresponding stabilized subspace has dimension $2^{n-(n-2)} = 4$ and can encode 2 qubits. One can naturally extend the definition of surface codes on any CSC in the following way:

**Definition 26. Stabilizer formalism on CSCs:** Let $G = (V,E,F)$ be a CSC. Associate a qubit with each edge $e \in E$. The underlying Hilbert space $H$ is the tensor product of all $n = |E|$ sites. For each vertex $v \in V$, define the following Pauli operator:

$$A_v = \prod_{e \in E, e \in v} \sigma^X_e$$

And for each face $f \in F$, define the following Pauli operator:

$$B_f = \prod_{f \in F, e \in f} \sigma^Z_e$$

Since each vertex has either 0 or 2 common edges with a given face, the above operators commute. Therefore, $\mathfrak{S} = \text{Span}\{A_v, B_f | v \in V, f \in F\}$ is an abelian subgroup of $P_n$, and since $-1 \notin \mathfrak{S}$, the stabilized subspace of $\mathfrak{S}$ is a quantum code $C_G$ of dimension $2^{n - \text{dim}(\mathfrak{S})}$.

The dimension of $C_G$ is directly related to the first homology class and cohomology classes $H_1(G)$ and $H^1(G)$ of $G$ through the following theorem (see \[44\] for further details):

**Theorem 5.** Let $G = (V,E,F)$ be a CSC, then $\text{dim}(C_G) = |V| - |E| + |F| + 2 = 2 - \chi(G)$.

**Proof.** From the general theory of the Pauli formalism, $\text{dim}(C_G) = n - \text{rank}(G)$ where $n$ is the number of physical qubits, and we get $n = |V|$. By definition, we have $G = \langle A_v, B_f | v \in V, f \in F \rangle$. Observe that any non trivial relation between those generators must involve only one type stabilizers: either star X operators or Z plaquette operators. Let $V' \subseteq V$, and assume that $\prod_{v' \in V'} A_{v'} = I$. We claim that $V' = V$ or $V' = \emptyset$. Indeed, suppose $V' \neq \emptyset$. Then there exists some $v_0 \in V'$. Now let $v_1 \in V$. From the connectedness of CSCs, there is a path $\gamma = e_0, ..., e_k$ joining $v_0$ to $v_1$. We can also look at $\gamma$ as a sequence of vertices $v_0 = v_0, ..., v_{k+1} = v_1$. We prove by induction on the path that $v_i \in V'$ for every $i \leq k + 1$.

By assumption, $v_0 \in V'$. Now assume $v_i \in V'$ for some $i \leq k$. Since $P = \prod_{v \in V} A_v = I$, the restriction of $P$ to the qubit on the edge $e_i = (v_i, v_{i+1})$ is identity and in particular, there must be an even amount of plaquettes $A_{v'}, v' \in V'$ acting non trivially on $e_i$. Since $A_{v_i}$ already acts non trivially on $e_i$, the plaquette $A_{v_{i+1}}$ must participate in $P$, or equivalently, $v_{i+1} \in V'$.

The same analysis also holds for star generators, therefore, the only non-trivial relations between the original generators are:

$$\prod_{v \in V} A_v$$

$$\prod_{f \in F} B_f$$

It follows that $\text{rank}(G) = |V| + |F| - 2$ and we get

$$\text{dim}(C_G) = n - \text{rank}(G) = |E| - |V| - |F| + 2 = 2 - \chi(G)$$

\[\square\]

**Corollary 1.** Let $G = (V,E,F)$ be a CSC, then $\text{dim}(C_G) = \text{dim}(H_1(G)) = \text{dim}(H^1(G))$
Proof. It is well known that for any hypergraph $G$, $\chi(G) = dim(H_0(G)) - dim(H_1(G)) + dim(H_2(G))$ (see [32]). Since $G$ is connected, we have both $dim(H_0(G)) = 1$ and $dim(H_2(G)) = 1$. Therefore from the previous theorem, we get:

$$dim(C_G) = 2 - \chi(G) = 2 - (2 - dim(H_1(G))) = dim(H_1(G))$$

Furthermore, from Poincare duality, since CSCs are 2 dimensional complexes, we have $dim(H^2(G)) = dim(H_{2-1}(G)) = dim(H_1(G))$ finishing the proof. □

Finally, we will make use of the following notation in the rest of this paper:

**Definition 27.** Pauli operators on subsets of edges: Let $G = (V,E,F)$ be a CSC and let $E' \subseteq E$. We define $E'_X = \prod_{e \in E'} \sigma^X_e \otimes I_{E \setminus X}$ and $E'_Z = \prod_{e \in E'} \sigma^Z_e \otimes I_{E \setminus X}$. In particular, if $\gamma$ is a path in $G$, then $\gamma_Z$ is defined as the Pauli operator which applies a $\sigma_Z$ operator on each site lying on $\gamma$ and identity everywhere else, and likewise for $\gamma_X$.

### 3 Notations and definitions

Before we proceed further, we need to state some definitions that will be relevant for the rest of the proof:

**Definition 28.** Circuit induced directed graph: Let $U$ be a unitary circuit acting on $H=(\mathbb{C}^2)^{\otimes n}$ we define the following directed graph $G_U$ associated with $U$: the edges are the wires of $U$ (directed according to the arrow of time) and the vertices are the quantum gates together with a set of $n$ input vertices and $n$ output vertices of degree one. An edge $e$ connecting the gate $v$ and $u$ if and only if there exists a wire in $U$ connecting the gate $u$ to the gate $v$. For the $i$-th qubit $q$ we can associate two edges in $G_U$: $in(q)$ and $out(q)$ corresponding to the $i$-th input and $i$-th output wire.

With that definition in hand, we can now formally define the upper and lower light cones of any subset of edges from $E$:

**Definition 29.** Upper and lower light cone: Let $U$ be a unitary circuit acting on and Hilbert space $H=(\mathbb{C}^2)^{\otimes n}$ of the set $S$ of $n$ qubits. Let $S' \subseteq S$ be a subset of the qubits. We define the upper and lower light cones of $S'$ to be:

$$L^+(S') = \{ q \in S | \exists q' \in S', \Gamma_{in(q'),out(q)} \neq \emptyset \}$$

$$L^-(S') = \{ q \in S | \exists q' \in S', \Gamma_{in(q),out(q')} \neq \emptyset \}$$

The following easy fact follows:

**Fact 3.** For two sets of qubits, $W$ and $V$, we have

$$L^+(W) \subseteq V \iff W \subseteq L^+(V)$$

$$V \subseteq L^-(W) \iff L^-(V) \subseteq W$$

**Proof.** Observe that $L^+(W) \subseteq V$ iff for all input qubit $w \in W$, there is a path in the graph from $w$ to an output $v \in V$ and this is iff $W \subseteq L^+(V)$. The other direction is similar. □

We are now ready to define the effective support of a given path $\gamma$ in the CSC; We treat $\gamma$ as a subset of edges, but also as a subset of the qubits.

**Definition 30.** The effective supports: A and B: Let $G = (V,E,F)$ be a CSC, and $U$ be a unitary circuit acting on $E$ such that $U|0\rangle \otimes |\gamma\rangle = |\psi\rangle$ where $|\psi\rangle \in C_G$ (Here, $C_G$ is the quantum code space defined on $G$ as in definition [26]). Let $e, f \in E$. For each homology class $[\gamma]$ in $\Gamma_{e,f}$, define the lower effective support of $[\gamma]$ under the action of $U$ to be:

$$A_{[\gamma],U} = \bigcap_{\gamma' \in [\gamma]} L^+(\gamma')$$

We also define the upper effective support $B_{[\gamma],U}$ to be the upper light cone of $A$ under the action of $U$:

$$B_{[\gamma],U} = L^+(A_{[\gamma],U})$$
Observe that by definition, $A_{[\gamma],U}$ is a function of the whole homology class $[\gamma]$ and doesn’t depend on any particular choice of representative chain $\gamma' \in [\gamma]$. For clarity, when both $\gamma \in \Gamma_{e,f}$ and $U$ are fixed, we will write $A = A_{[\gamma],U}$ and $B = B_{[\gamma],U}$. We can now proceed to prove a weak version of our main result.

4 The starting point: Bravyi’s commuting operators lemma

We now prove a lemma that we will be using multiple times in the rest of this paper. The lemma is the main idea underlying Bravyi’s unpublished proof of a $\Omega(\sqrt{n})$ lowerbound for the circuit depth of toric code states, in the geometric case [6].

We stress that the lemma does not rely on any assumption on the geometry of the quantum circuit.

Lemma 1. Operators out of effective support of $\gamma$ commute with operators on $\gamma$ Let $G$ be a CSC, and let $U$ be a quantum circuit such that $U |0\rangle^n = |\psi\rangle \in C_G$. Let $e, f \in E$, let $\gamma \in \Gamma_{e,f}$ and let $B = B_{[\gamma],U}$. Then, for every operator $P$ supported on $E \setminus B$ that stabilizes $|\psi\rangle$, we have:

$$P_{\gamma}\gamma|\psi\rangle = \gamma Z P|\psi\rangle.$$

Proof. We first prove that in the $U$ basis, applying $\gamma$ on the groundstate $|\psi\rangle$ can be replaced by applying an operator whose support is confined to $A = A_{[\gamma],U}$. In other words, $U|\gamma Z U|0\rangle^n = |0\rangle_{E \setminus A} \otimes |\phi\rangle_A$ for some state $|\phi\rangle$. To show this, let $e$ be an edge such that $e \notin A$. Then by definition, there exists a path $\alpha \in [\gamma]$ such that $e \notin L^\uparrow(\alpha)$. Since $\gamma$ and $\alpha$ are both in $[\gamma]$, we have $\gamma Z \alpha Z \in \Phi$ and therefore:

$$\gamma Z \alpha Z |\psi\rangle = |\psi\rangle$$

so

$$\alpha Z |\psi\rangle = (\gamma Z)^{-1} |\psi\rangle = \gamma Z |\psi\rangle$$

$$U \alpha Z U |0\rangle^n = U \gamma Z U |0\rangle^n$$

Since $U \alpha Z U$ has support on $L^\uparrow(\alpha)$, and $e \notin L^\uparrow(\alpha)$, it leaves $|0\rangle_e$ intact. Therefore:

$$U \gamma Z U |0\rangle^n = |0\rangle_{E \setminus A} \otimes |\phi\rangle_A$$

Now since $P$ is supported on $E \setminus B$, $U^\uparrow P U$ is supported on $L^\downarrow(E \setminus B) = L^\downarrow(E \setminus L^\uparrow(A)) \subseteq E \setminus A$ (where the last inequality follows by Fact 3 from $E \setminus L^\uparrow(A) \subseteq E \setminus A \subseteq L^\downarrow(E \setminus A)$). So $U^\uparrow P U$’s support is contained in $E \setminus A$.

Since $P|\psi\rangle = |\psi\rangle$ we have $U^\uparrow P U |0\rangle^n = |0\rangle^n$. Since $U^\uparrow P U$ has support on $E \setminus A$, it also holds that for every pure state $|\phi\rangle_A$ of the qubits in $A$,

$$U^\uparrow P U (|0\rangle_{E \setminus A} \otimes |\phi\rangle_A) = |0\rangle_{E \setminus A} \otimes |\phi\rangle_A$$

Therefore,

$$U^\uparrow P \gamma Z |\psi\rangle = (U^\uparrow P U)(U \gamma Z U) |0\rangle^n$$

$$= (U^\uparrow P U) (|0\rangle_{E \setminus A} \otimes |\phi\rangle_A)$$

$$= |0\rangle_{E \setminus A} \otimes |\phi\rangle_A$$

$$= U \gamma Z |\psi\rangle$$

and it follows that

$$P \gamma Z |\psi\rangle = \gamma Z |\psi\rangle = \gamma Z P|\psi\rangle.$$
5 Geometric non-triviality of the cube states

In order to understand the ideas behind the proof of the general case, we will first prove a weaker version of the main theorem on a simple CSC: the cube. There are two motivations for this choice: On the one hand, there exists a fairly simple and natural way to define a CSC family on the cube as we shall soon see. On the other hand, the cube has a trivial first homology group. Hence, the stabilized subspace arising from the surface code construct mentioned earlier has dimension 1. Therefore, that unique state doesn’t satisfy the usual Topological Order condition (see definition 13). Furthermore, there are no long range correlations in the cube state as every logical operator is in fact trivial and measurement of the logical qubits doesn’t yield any information on the state nor modifies it. Hence, we have to use additional tools in order to show non-triviality of the cube states.

Definition 31. The cube states: Let $S_n = (V_n, E_n, F_n)$ be the CSC such that

$$V_n = [n] \times [n]$$

$$E_n = \{(i, j), (i', j')\} \mid (i = i' \text{ and } |j - j'| = 1) \text{ or } (j = j' \text{ and } |i - i'| = 1)$$

and, $F_n$ is defined to be the set of all 1 by 1 squares with edges from $E_n$.

Now, take 6 copies of $S_n$ and "glue" them together to get a cube, when we identify edges and vertices from different copies of $S_n$ if and only if they coincide after the gluing process.

We call the resulting CSC: $T_n$. For each edge in $T_n$ we associate a qubit, and we define $C_n$ the surface code associated to $T_n$ according to the previous quantum code construction (see definition 26).

Let $N = O(n^2)$ to be the total number of qubits in $C_n$. Since the first homology group of the cube vanishes, it follows from [13] that $C_n$ contains a single state $|\psi_n\rangle$. We define $|\psi_n\rangle$ as the $n$-th cube state defined on $N$ qubits.

In order to prove Theorem 1 we consider the cube state $|\psi_n\rangle$ defined on the CSC $T_n$ as defined above. Given 2 edges $e$ and $f$, we first show that the effective support of any path $\gamma \in \Gamma_e, f$ is bounded inside a small region close to either $e$ or $f$.

Lemma 2. Shallow circuit implies small effective support for operators on $\gamma$ Let $n \geq 0$ and let $|\psi_n\rangle$ the $n$-th cube state as defined above. Let $e, f \in E_n$ and $\gamma \in \Gamma_e, f$. Furthermore, assume there exists a quantum circuit $U_n$ of depth $d_n$ using geometrically local gates of locality $c$ (namely, two qubits acted upon by the same gate are within distance $c$), such that $U_n |0^N\rangle = |\psi_n\rangle$. Let $A = A(\gamma), U_n$ and $B = B(\gamma), U_n$ be the lower and upper effective supports on $T_n$ associated with $U_n$ and $\gamma$. Then for large enough $n$, and assuming $d = o(n)$, we have $A \subseteq \hat{\text{Ball}}(e, cd_n) \cup \hat{\text{Ball}}(f, cd_n)$ and $B \subseteq \hat{\text{Ball}}(e, 2cd_n) \cup \hat{\text{Ball}}(f, 2cd_n)$.

Proof. Let $\gamma' \in [\gamma]$. Since $U$ only makes use of geometrically local gates, the lower light cone $L^\downarrow(\gamma')$ of $\gamma'$ satisfies:

$$L^\downarrow(\gamma') \subseteq \{e' \in E | d(e', \gamma') \leq c \cdot d\}$$ (9)

Now let $e' \in E$, and assume that $d(e, e') > cd$ and $d(f, e') > cd$. The ball $\hat{\text{Ball}}(e', cd)$ doesn’t contain neither $e$ or $f$. But from the geometry of the cube removing a ball of radius $d = o(n)$ from the original cube leaves it connected. Therefore, there exists some path $\gamma'' \in [\gamma]$ such that $d(\gamma'', e') > cd$ and then from Equation (6), $e' \notin L^\downarrow(\gamma'')$. But obviously, since the cube is simply connected, $\gamma'' \in [\gamma]$.

From the definition of $A$, we conclude that $e' \notin A$. This proves the first part of the claim: $A \subseteq \hat{\text{Ball}}(e, cd_n) \cup \hat{\text{Ball}}(f, cd_n)$. Since $B = L^\uparrow(A)$, by definition any qubit in $B$ is within distance at most $cd_n$ from any qubit in $A$ and thus $B$ is contained in the set of all qubits of deistance at most $cd_n$ from $\hat{\text{Ball}}(e, cd_n) \cup \hat{\text{Ball}}(f, cd_n)$: this set is contained in $\hat{\text{Ball}}(e, 2cd_n) \cup \hat{\text{Ball}}(f, 2cd_n)$.

Note that from the above lemma, and assuming $d_n = o(n) = o(\sqrt{N})$, $B$ has to be a $o(\sqrt{N})$-size set.

We can now provide the proof of Theorem 1.

proof of Theorem 1: Choose two edges $e, f$ lying on two opposite faces of the cube $T_n$, and choose some path $\gamma \in \Gamma_e, f$. Obviously, since $e$ and $f$ lie on two opposites faces we have $d(e, f) \geq n$. Now assume by contradiction that there exists a depth $d_n = o(\sqrt{N})$ geometrically local quantum circuit $U_n$ such that $U_n |0^N\rangle = |\psi_n\rangle$. W.l.o.g, we assume $U_n$ makes use of quantum gates on at most $c$ qubits. Since $e$ and $f$
lie on opposite faces \( F_e \) and \( F_f \) of the cube \( T_n \), consider the remaining four faces cyclically glued to each other. We can now consider the closed co-path cutting all of those four faces in half and in the middle, and call it \( \zeta \). Since \( \zeta \) is a closed co-path it is a coboundary. Note that \( \zeta \cap B = \emptyset \); indeed, by construction, for every edge \( e' \) lying in \( F_e \cup F_f \), \( d(\zeta, e') \geq \sqrt{2} \). But from Lemma 2, \( B \subseteq Ball(e, c_{dn}) \cup Ball(f, c_{dn}) \), and since \( d_{eu} = o(n) \), we get \( d(B, \zeta) > 0 \). The operator \( \zeta_X \) is thus supported on \( E_n \setminus B \). Since \( \zeta_X \) is an \( X \) operator on a coboundary, it follows that \( \zeta_X \) stabilizes \( |\psi_n\rangle \). Therefore, we can apply Lemma 1 using \( P = \zeta_X \) to get:

\[
\zeta_X \gamma Z |\psi_n\rangle = \gamma Z |\psi_n\rangle
\]

(10)

But observe that \( \zeta \) actually separates \( T_n \) into two disconnected components, one containing \( e \) and the other one containing \( f \)! Therefore, any path \( \gamma' \) connecting \( e \) to \( f \) has to cut through \( \zeta \) on an odd number of edges. Hence, \( \zeta_X \gamma Z = -\gamma Z \zeta_X \). Inserting this equality in equation (10) we get:

\[
-\gamma Z |\psi_n\rangle = -\gamma Z \zeta_X |\psi_n\rangle = \zeta_X \gamma Z |\psi_n\rangle = \gamma Z |\psi_n\rangle
\]

and we get \( |\psi_n\rangle = 0 \) which obviously doesn’t hold. This concludes the proof of Theorem 1.

6 Non triviality of the cube states

In this subsection we shall generalize our result from the previous chapter and prove Theorem 2. Again, we consider the family of cube states \( \{ |\psi_n\rangle \} \) defined on the cubes \( \{ T_n \} \), but this time we will consider quantum circuits based on gates whose only restriction is to have bounded support size. In particular, we don’t require from the quantum gates in a generating circuits \( \{ U_n \} \) to have support on qubits that are close to each other in the CSC metric.

We start by explaining in more detail why the previous proof doesn’t work in this more general non-geometrically-local setting. The main issue with the previous proof lies in Lemma 2: without the geometrical case, a similar argument to that of Lemma 2 only implies that \( |\gamma| \) is in some sense “small”, as it can be confined in the union of two small radius balls around \( e \) and \( f \), as in Lemma 2. This was used for the construction of the path \( \zeta \) outside of \( B \), required to apply Lemma 1. In the non-geometrical case, a similar argument to that of Lemma 2 only implies that \( |B| = O(\text{diam}(T_n)) = O(n) = O(\sqrt{N}) \); this bound is not strong enough to guarantee a path \( \gamma \) separating \( e \) from \( f \) which does not intersect \( B \), needed for the application of Lemma 1. In fact, we later show that we can assume that \( A \) is contained in the union of two small balls around \( e \) and \( f \) - this is done later in Claim 2 and requires much more work than the analogous Lemma 2 in the geometrical case. However, the method of generating a path around one of these balls would not work in the non-geometrical case, since the balls no longer have a nice geometrical location. We will need to derive a contradiction via a different argument.

6.1 A different approach towards a contradiction: A lower bound on \( |A| \)

To derive a contradiction, we prove the following lemma, Lemma 3, stating that regardless of the depth of the circuit, and without relying on any geometrical restrictions on the gates, the effective support \( B \), as well as \( A \), must in fact be large. This is what will lead to a contradiction with the above mentioned Claim 2 stating that \( A \) is small.

Lemma 3. Let \( G = (V, E, F) \) be a CSC such that \( |E| = N \) and assume \( U |0\rangle^N \in C_G \) where \( U \) is a quantum circuit. Let \( e, f \) be 2 edges in \( E \), \( \gamma \in \Gamma_{e,f} \) and \( B = B_{[\gamma],U} \). Then there exists \( \gamma' \in [\gamma] \) which is contained in \( B \); in particular, \( |B| \geq d(e,f) \).

Note that since this lemma holds also in the special case of geometrically restricted gates, we derive an alternative proof to Theorem 1 using the fact that Lemma 3 put together with Lemma 2 leads to a contradiction.

Proof. Suppose by contradiction that \( B \) does not contain any \( \gamma \in [\gamma] \). Note that \( e, f \in B \) since \( e, f \in \gamma' \) for all \( \gamma' \in [\gamma] \). By the assumption, \( e \) and \( f \) belong to a different path-connected component of \( B \). Let
Let $G = (V, E, F)$ be a CSC and let $X \subseteq E$. Define $\mu = \hat{\partial}(B_e)$. We first prove that $\mu_X = \prod_{v \in V_A} A_v \in \emptyset$. Indeed, by definition, if $e'$ is in $\hat{\partial}(B_e)$, then $e'$ is connected to exactly one vertex $v$ in $V_B$ and therefore, $(\prod_{v \in V_B} A_v)_{e'} = \sigma_X$. On the other hand, if $e' \in E \backslash \hat{\partial}(B_e)$ then $e'$ is connected to either 0 or 2 vertices from $V_B$ and therefore, $(\prod_{v \in V_B} A_v)_{e'} = I$. Therefore, $\mu_X = \prod_{v \in V_B} A_v$, and it is obviously in $\emptyset$.

Now we prove that $\mu \cap B = \emptyset$: Indeed, by construction, $\mu \cap B_e = \emptyset$. Assume by contradiction that there exists $e' \in B \backslash B_e \cap \mu$. Since $e' \in \mu$, $e'$ has a vertex in $V_B$. But then, it follows that $e' \in B_e$ and we get a contradiction.

Since $\mu_X \in \emptyset$ we have $\mu_X \| \psi = \| \psi$, and moreover, $\mu_X$ is contained in $E \backslash B$ since $\mu_X \cap B = \emptyset$. Therefore we can apply lemma \ref{lemma:gamma-separation-bound} with $P = \mu_X$, and for all $\gamma \in \Gamma_{e,f}$ we get:

$$\mu_X \gamma \| \psi = \gamma \| \psi$$

But note that since $e, f$ are in different connected components of $B$, then $|\mu \cap \gamma|$ is odd. Indeed, let $\gamma = (e_1, \ldots, e_k)$. Then we can extract from $\gamma$ a sequence of vertices: $(v_1, \ldots, v_{k+1})$ where $e = (v_1, v_2)$, $f = (v_k, v_{k+1})$ and $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq k$. But note that $e_i \in \mu$ if and only if exactly one of the vertices $v_i, v_{i+1}$ is in $V_B$. Therefore, $|\mu \cap \gamma|$ counts the number of times we get in/out of $V_B$ in the sequence $(v_1, \ldots, v_{k+1})$. Since $v_1 \in V_{B_e}$ and $v_{k+1} \notin V_{B_e, r}$ $|\mu \cap \gamma|$ has to be odd.

It follows that

$$\mu_X \gamma \| \psi = -\gamma \| \mu_X \| \psi = -\gamma \| \psi$$

and we get a contradiction. \hfill $\Box$

A simple corollary is that provided the circuit depth is small, then $A$ is also large:

**Corollary 2.** Assume the depth of the generating circuit $U_n$ of $\| \psi_n \|$ is $d_n$, then $|A| > \frac{d(e, f)}{e^{d_n}}$.

**Proof.** Since $L^1(A) = B$, we have $e^{d_n}|A| \geq |B|$, and from lemma \ref{lemma:gamma-separation-bound}$|B| \geq d(e, f)$. Therefore:

$$|A| \geq \frac{d(e, f)}{e^{d_n}}.$$

To derive a contradiction in the non-geometrical case, we need the non-geometrical analogue of Lemma \ref{lemma:gamma-separation-bound} providing an upper bound on $|A|$. This requires developing some tools, which we do in the next subsection.

### 6.2 Upper bound on $|A|$ using $\gamma$-separation

Here we prove the following claim, which essentially replaces Lemma \ref{lemma:gamma-separation-bound} in the geometrical case, stating that the effective support $A$ is contained in two small balls surrounding $e$ and $f$, except here the size of the balls is exponentially bigger, due to the lack of geometrical restriction, but it is still bounded by a constant. The proof is significantly more complex.

**Claim 2.** Assume that $e$ and $f$ are two edges on opposite sides in $T_n$ so that $d(e, f) \geq n$. Then $A \subseteq L^1(\overline{B}(e, e^{d_n}) \cup \overline{B}(f, e^{d_n}))$.

To prove this claim, we need to define a new notion:

**Definition 32.** $\gamma$-separation: Let $G = (V, E, F)$ be a CSC and let $X \subseteq E$. Let $e, f \in E$ and $\gamma \in \Gamma_{e,f}$. $X$ is called $\gamma$-separating if for all $\gamma' \in [\gamma], \gamma' \cap X \neq \emptyset$.

The main motivation behind this definition is encompassed in the following two claims:

**Claim 3.** The upper light cone of any edge in the lower effective support $A$ is $\gamma$-separating.

Let $G = (V, E, F)$ be a CSC and let $e, f \in E$ and $\gamma \in \Gamma_{e,f}$. Assume that $U_0 \cap A' \subseteq C_G$. If $g \in A = A_{[\gamma], U}$, then $L^1(g)$ is $\gamma$-separating.
Proof. Assume there exists a path \( \gamma' \in [\gamma] \) such that \( \gamma' \cap L^j(g) = \emptyset \). Therefore, we have \( g \notin L^j(\gamma') \) but since \( g \in A = \bigcap_{j \in [\gamma]} L^j(\gamma') \), it follows that \( g \in L^j(\gamma') \) and we get a contradiction. \( \square \)

**Claim 4.** \( \gamma - \) separating sets can always be reduced to be copath connected: Let \( G = (V,E,F) \) be a CSC and let \( e,f \in E \) and \( \gamma \in \Gamma_{e,f} \). Let \( X,X' \subseteq E \) such that \( X \) and \( X' \) are not copath connected to each other (within \( X \cup X' \)). Furthermore, assume that \( X \cup X' \) is \( \gamma - \) separating and let \( \gamma' \in [\gamma] \).

Then either \( X \) or \( X' \) intersects both \( \gamma \) and \( \gamma' \) non trivially.

**Proof.** Since \( X \cup X' \) is \( \gamma - \) separating, we can assume w.l.o.g.
that \( X \cap \gamma \neq \emptyset \) and \( X' \cap \gamma' \neq \emptyset \). Since \( \gamma' \in [\gamma] \), \( \gamma + \gamma' \) is a boundary, and therefore there is a subset \( E' \subseteq F \) such that \( \gamma + \gamma' = \sum_{f \in E'} \partial_F(f) \).

Define \( J = F' \cap F_X \) and \( \alpha = \sum_{f \in J} \partial_F(f) \).

By definition, \( \alpha \) is a boundary (as a sum of boundaries) and therefore, \( \beta = \gamma + \alpha \). Therefore, we conclude that \( X \cap \gamma = \emptyset \) and \( X' \cap \gamma' = \emptyset \). We will show that \( \beta \cap (X \cup X') = \emptyset \) contradicting the assumption that \( X \cup X' \) is \( \gamma - \) separating:

Observe that \( i \in \beta \) if \( i \in \gamma \cap \alpha \) or \( i \in \alpha \cap \gamma \). Also observe that given an edge \( i \in E \), \( i \) belongs to 1 face exactly in \( F' \) if \( i \in \gamma + \gamma' \). We have 2 cases:

1. First assume that \( i \in \gamma \cap \alpha \). Since \( X' \cap \gamma = \emptyset \), we have \( i \notin X' \). Now if \( i \notin \gamma' \) then obviously, \( i \notin X \), so we can assume that \( i \notin \gamma \cap \gamma' \). From the above observation, since \( i \notin \gamma + \gamma' \), \( i \) belongs to a unique face \( f_i \in F' \). But, if \( i \in X \) then we also have \( f_i \in F_X \), and we get \( f_i \in F' \cap F_X = J \). From the uniqueness of \( f_i \), we get \( i \notin \sum_{f \in J} \partial_F(f) = \alpha \) contradicting the assumption. Therefore we conclude that \( i \notin X \), and overall, \( i \notin X \cup X' \).

2. Now assume that \( i \notin \alpha \cap \gamma \). Since \( \gamma \) and \( \gamma' \) are not co-path connected to each other, it follows immediately that \( i \notin X' \). (otherwise, \( X \) and \( X' \) would be connected through \( f_i \).) On the other hand, assume \( i \in X \). Since \( i \notin \alpha \), \( i \) belongs to a unique face \( f_i \in J \) and since \( i \in X \), \( i \) belongs to exactly 2 faces \( f_i,f_i' \in F_X \). But \( f_i' \notin F_X \setminus J \Rightarrow f_i' \notin F' \). Therefore, \( i \) belongs to a unique face \( f_i \) in \( F' \) from the above observation, we get \( i \notin \gamma + \gamma' \). But by assumption, \( i \notin \gamma \) and since \( X \cap \gamma' = \emptyset \), and \( i \notin X \), also \( i \notin \gamma' \). Therefore, \( i \notin \gamma + \gamma' \) and we get a contradiction to the assumption that \( i \notin X \). Overall, we proved that \( i \notin X \cup X' \).

To conclude, we showed that \( \beta \cap (X \cup X') = \emptyset \) and therefore \( X \cup X' \) is not \( \gamma - \) separating. It follows that either \( X \cap \gamma' \neq \emptyset \) or \( X' \cap \gamma \neq \emptyset \) which proves the claim. \( \square \)

**Corollary 3.** Let \( G = (V,E,F) \) be a CSC and let \( e,f \in E \) and \( \gamma \in \Gamma_{e,f} \). If \( X \subseteq E \) is \( \gamma - \) separating then there exists a copath-connected component \( X' \subseteq X \) that is also \( \gamma - \) separating.

**Proof.** Let \( X = \bigcup_{m=1}^\infty X_m \) where \( X_m \) are the co-path connected components of \( X \). We proceed by induction on the number of co-path connected components of \( X \):

If \( m = 1 \), the statement is trivial.

Now, assume \( m \geq 2 \): if \( X_1 \) is \( \gamma - \) separating then we are done. Otherwise there exists some \( \gamma' \in [\gamma] \) such that \( X_1 \cap \gamma' = \emptyset \). Let \( \gamma'' \in [\gamma] \). Since \( Y = \bigcup_{m=2}^\infty X_m \) and \( X_1 \) are not co-path connected and \( X = Y \cup X_1 \) is \( \gamma' - \) separating, we can infer from claim 4 that either \( X_1 \) or \( Y \) intersects both \( \gamma' \) and \( \gamma'' \) non trivially. From the assumption, \( X_1 \cap \gamma' = \emptyset \) and therefore, \( Y \cap \gamma'' \neq \emptyset \). Therefore, since this holds for all \( \gamma'' \in [\gamma] \), we deduce that \( Y \) is \( \gamma - \) separating. \( \square \)

From the above corollary, one can always extract a connected, \( \gamma - \) separating set from a non connected one.

Our main tool for the cube state case as well as for the more general case, is the following lemma which shows that any small connected \( \gamma - \) separating set of edges lies within a small distance from either \( e \) or \( f \).

**Lemma 4.** Small connected \( \gamma - \) separators are close to end points Let \( T_n = (V_n,E_n,F_n) \) be \( n \)-th cube CSC. Let \( X_n \subseteq E_n \) be copath connected and \( \gamma - \) separating and suppose \( |X_n| < n \), then \( X_n \cap (\overline{\text{Ball}(e,|X_n|)} \cup \overline{\text{Ball}(f,|X_n|)}) \neq \emptyset \).

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Proof of lemma 2. Let \( x_0 \in X_n \). Since \( X_n \) is copath connected, for all \( x \in X_n \), there exists a copath \( \delta \in \Gamma_{x_0,x} \) such that \( \delta \subseteq X_n \). Thus, \( \delta(x_0, x) \leq |\delta| - 1 < |X_n| \) and therefore, \( X_n \subseteq \overline{Ball}(x_0, |X_n|) \).

Assume by contradiction that \( X_n \cap \left( \overline{Ball}(e, |X_n|) \cup \overline{Ball}(f, |X_n|) \right) = \emptyset \). Hence, \( \delta(e, x_0), \delta(f, x_0) > |X_n| \) which implies \( e, f \notin \overline{Ball}(x_0, |X_n|) \).

We now need a claim, which will be very simple to prove in the cube case:

**Claim 5.** Let \( e \in E_n \) and \( r < n \). Then \( T_n \setminus \overline{Ball}(e, r) \) is path connected.

**Proof.** This claim is trivial for the case of the cube we are now handling; we do not provide the details, since the more general case will be proven later in full. \( \square \)

Hence, we have that \( T_n \setminus \overline{Ball}(x_0, |X_n|) \) is path connected. Since \( e, f \notin \overline{Ball}(x_0, |X_n|) \), there exists a path \( \gamma' \in \Gamma_{e,f} \) such that \( \gamma' \cap \overline{Ball}(x_0, |X_n|) = \emptyset \). Hence, \( \gamma' \cap X_n = \emptyset \). But since the cube has a trivial first homology group, all paths in \( \Gamma_{e,f} \) are in the same homology class, and \( \gamma' \in [\gamma] \) contradicting the assumption that \( X_n \) is \( \gamma \)-separating.

We can now prove Claim 2. To do this, recall that we showed in Claim 3 that the upper light cone of any element in \( A \) satisfies the conditions of the lemma 2; namely it (or a subset of it) is both \( \gamma \)-separating and co-path connected. Hence we can apply Lemma 4 for a subset of the upper light cone of any element in \( A \), to deduce that this subset intersects a small ball around one of the end points; this will allow us to deduce that \( A \) is contained in two small balls around the end points (Claim 2).

**Proof of Claim 2.** Assume there exists some edge \( g \in A \) such that \( g \notin L^\gamma(\overline{Ball}(e, c^{dn})) \cup \overline{Ball}(f, c^{dn})) \). From claim 3 \( L^\gamma(g) \) is \( \gamma \)-separating and from Corollary 2 we can conclude that \( L^\gamma(g) \) contains a set \( X_n \) which is \( \gamma \)-separating and copath connected. Since each layer in \( U \) increases the size of the upper light cone of \( g \) by a multiplicative factor of at most \( c \), we have \( |X_n| \leq |L^\gamma(g)| < c^{dn} \). Therefore, from Lemma 4 we get:

\[
X_n \cap \left( \overline{Ball}(e, c^{dn}) \cup \overline{Ball}(f, c^{dn}) \right) \neq \emptyset
\]

which implies

\[
L^\gamma(g) \cap \left( \overline{Ball}(e, c^{dn}) \cup \overline{Ball}(f, c^{dn}) \right) \neq \emptyset
\]

or equivalently, \( g \in L^\gamma(\overline{Ball}(e, c^{dn})) \cup \overline{Ball}(f, c^{dn})) \) and we get a contradiction. \( \square \)

### 6.3 Deducing the Theorem using \( \gamma \)-separations

We now use the above Claim 2 upper bounding \( |A| \), together with the fact that \( A \) must be large (Corollary 2) to derive a contradiction.

We first prove a simple fact bounding the number of edges in a ball on the cube \( T_1 \):

**Fact 4.** Let \( e \) be an edge of the cube CSC \( T_i \), and let \( d \geq 1 \). Then,

\[
|\overline{Ball}(e, d)| \leq 10d^2.
\]

**Proof.** Observe that the size of a radius \( d \) ball on the cube \( T_1 \) can always be bounded from above by the size of a ball of the same radius on the infinite grid \( \mathbb{Z} \times \mathbb{Z} \). But it is also clear that such a ball is contained inside a \( 2d \times 2d \) square on the grid, which contains less than \( 10d^2 \) edges. \( \square \)

We are now ready to prove Theorem 2

**Proof of Theorem 2.** Assume that \( e \) and \( f \) are two edges on opposite sides in \( T_n \) so that \( d(e, f) \geq n \). We now separate the set of natural numbers \( n \) to two sets. In the first set, we have \( n < 2d^{dn} \) (and so \( d_n > \log(n)/\log(2c) \)). For the other values of \( n \), we have \( d(e, f) \geq n \geq 2d^{dn} \). We will derive a
contradiction if there are infinitely many \( n \)'s of the latter type. For any such \( n \) we have, applying Claim 2 and Fact 4:

\[
|A| \leq |L^1(Ball(e, c^{d_n}) \cup Ball(f, c^{d_n}))| \\
\leq c^{d_n} |Ball(e, c^{d_n}) \cup Ball(f, c^{d_n})| \\
= c^{d_n} \left(|Ball(e, c^{d_n})| + |Ball(f, c^{d_n})|\right) \\
\leq 2c^{d_n} \cdot 10c^{2d_n} \\
= 20c^{3d_n}.
\] (11)

Now, from Corollary 2 we have \( |A| > \frac{d(e,f)}{c^n} \geq \frac{1}{c^n} \). Combining those equations together we get:

\[
\frac{n}{c^n} < 20c^{3d_n}
\]

from which we get \( d_n \geq \log(n/20)/4\log(c) \) for those \( d_n \)'s in the second set. Altogether, \( d_n \) for all \( n \) is greater than the minimum of \( \log(n/20)/4\log(c) \) and \( \log(n)/\log(2c) \), and hence \( d_n = \Omega(\log(n)) \).

**Remark:** Observe that this proof heavily relies on fact 4 which provides a polynomial (and even quadratic) upper bound on the amount of edges inside a ball on the cube grid. Unfortunately, this result doesn’t hold in the general framework of CSCs, and so this proof cannot be carried over to the more general set of complexes we would like to consider. Indeed, in the general case, the best bound that can be proved is exponential in the radius of the ball. This kind of behaviour is characteristic of hyperbolic surfaces where the area of a disk of radius \( r \) is proportional to \( \sinh(r) \) instead of the usual \( r^2 \) in flat euclidean manifolds like the sphere or the torus. This fact alone leads to another \( \log \) in the lower bound; but we will in fact only derive a \( \log\log\log(n) \) lower bound due to another phenomenon, as we shall see in the next sections.

### 7 General CSCs

In this Section we will prove Theorems 3 and 4, generalizing Theorems 1 and 2 to general CSCs. While most of the proofs - both in the geometrical and non geometrical case - translate to the framework of arbitrary CSCs, observe that we made use of the geometry of the cube both in Lemma 2 and lemma 4; in both lemmas, we strongly used the fact that removing any ball of small radius from \( T_n \) leaves the cube connected (see Claim 5). The difficulty in the case of CSCs is how to guarantee the existence of a path between the two edges \( e \) and \( f \) which does not intersect a set of small radius. To this end, we introduce the notion of \( r \)-simple connectedness, and use it to prove Theorem 6 below; this replaces the analogous simple fact about the cube, stated in Claim 5.

#### 7.1 \( r \)-simple connectedness

**Definition 33.** Let \( G = (V, E, F) \) be a CSC and let \( e \in E \) and \( r > 0 \). Define:

\[
K'(e, r) = (V', E', F')
\]

where

\[
F' = \{ f \in F | \forall x \in f, d(x, e) \leq r \}
\]

\[
V' = V_{F'}
\]

\[
E' = E_{F'}
\]

Now for each face \( f \in F \setminus F' \), add \( f \) to \( F' \) if and only if \( \partial_2(f) \subseteq \partial_2(F') \). (Note that this doesn’t change the definition of \( E' \) and \( V' \)) Call the resulting set of faces \( F' \), and define:

\[
K(e, r) = (V', E', F')
\]

Note that although \( K = K(e, r) = (V', E', F') \) is usually not a CSC as it could have a non empty boundary, it is always a polygonal complex.
Definition 34. r-simple connectedness Let $G = (V, E, F)$ be a CSC, and let $r > 0$. $G$ is called \textit{r-simply connected} if for every $e \in E$: $H^1(K(e, r)) = H_1(K(e, r)) = 0$.

In other words, one can think of an $r-\text{simply connected}$ complex as one that doesn’t contain any "bottleneck" as illustrated in figure 3.

The main goal of this section is to prove the following theorem, replacing Claim 5 by a proof which holds for general CSCs:

Theorem 6. Let $G$ be a CSC. Let $r \geq 0$, $e, f, x_0 \in E$ and $\gamma \in \Gamma_{e,f}$ such that $e, f \notin B = \text{Ball}(x_0, r)$, and $K = K(x_0, r + 1)$ is simply connected. Then, there exists $\gamma' \in [\gamma]$ such that $\gamma' \cap B = \emptyset$.

First we show that the boundary of any such subcomplex $K$ is path connected, provided its first homology group vanishes:

Lemma 5. Let $G = (V, E, F)$ be a CSC, let $e \in E$, $r > 0$ and let $K = K(e, r) = (V', E', F')$. Assume that $K$ is copath connected and $H_1(K) = 0$. Then $\partial_2(F')$ is path connected.

\begin{proof}
Assume by contradiction that $\partial_2(F')$ is not path connected. Let $S_1$ and $S_2$ be 2 distinct path connected components of $\partial_2(F')$. Observe that $S_1 \subseteq \text{Ker}(\partial_1)$. Indeed, for every $v \in V$, $|E_v \cap S_1| = 0 \mod 2$ since $E_v \cap S_1 = \sum_{f \in F': e \in E_f} E_v \cap \partial_2(f)$ and for any $f \in F'$, $|E_v \cap \partial_2(f)|$ is either 0 or 2, since by definition\[^{13}\] there are no self-edges. Therefore, $S_1 \subseteq \text{Ker}(\partial_1)$ and since by assumption $H_1(K) = 0$, it follows that $S_1 \subseteq \text{Im}(\partial_2)$ and we can write $S_1 = \sum_{f \in H \subseteq F'} \partial_2(f)$ for some subset $H$ of faces in $F'$. Now let $e_1 \in S_1$ and $e_2 \in S_2$. By assumption, $K$ is copath connected, hence we can find a copath $(e_1 = x_1, x_2, ..., x_k = e_2)$ where there is a unique edge $f_i$ connecting $x_i$ to $x_{i+1}$ for every $1 \leq i \leq k - 1$. The uniqueness of $f_i$ stems from the fact that the intersection of 2 faces in a polygonal complex is always a single edge. Since $x_1 \in \partial_2(F')$, we must have $f_1 \in K$. But observe that $x_2 \in f_1$ and $x_2 \notin S_1$ (since $x_2$ belongs to 2 faces). It follows that $f_2 \in H$: indeed, if $f_1 \in H$ and $f_2 \notin H$ then $x_2 \in S_1$ and we get a contradiction. By following the same inductive argument, we get that $f_i \in H$ for every $1 \leq i \leq k - 1$. But then it immediately follows that $x_k = e_2$ is in $S_1$ since $f_k \in H$ and $e_2 \in S_2$ by assumption. Therefore, $S_1$ and $S_2$ are copath connected in $K$ and hence also path connected in $K$, contradicting the assumption that $S_1$ and $S_2$ are distinct connected component $\partial_2(F')$.
\end{proof}

We are ready to prove Theorem 6.

\begin{proof}[proof of Theorem 6]
Let $K = K(x_0, r + 1) = (V', E', F')$. Define $B = \text{Ball}(x_0, r)$ and $C = E' \setminus (\partial_2(F'))$. Note $B \subseteq C$. Indeed, let $x \in B$: since $\hat{d}(x_0, x) \leq r$, there exist a copath $(x_0 = e_1, ..., e_k = x)$ of size $k \leq r + 1$. Let $f \in F$ such that $e_{k-1} \in f$ and $e_k \in f$. Obviously, for all other edges $e' \in f$, it also holds that $\hat{d}(x_0, e') \leq r \leq r + 1$ so that $f \in F'$, and $x \in E_{V_f'} = E'$. Let $f' \in F'$ be the face satisfying $f' \neq f$ and $x \in f'$. For every $y \in f'$, $(x_0 = e_1, ..., e_k = x, y)$ is a copath of size $k + 1 \leq r + 2$ and therefore, $\hat{d}(x_0, y) \leq r$. We conclude that $f' \in F'$, and since we already proved that $f \in F'$, it follows that $x \notin \partial_2(F')$. Similarly, $x \notin \partial_2'(V_{F'})$ since for any vertex $v \in x, v \in V_f \subset V_{F'}$.

Now let $\gamma = (e = e_1, ..., e_k = f)$. We can partition $\gamma$ into $\gamma = \bigcup_{i=1}^{s} \gamma_i$ where $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$, $\gamma_i = (e_{t_i}, ..., e_{t_{i+1}})$, $t_{i+1} = t_i + k_i + 1$ in such a way that for every even index, $\gamma_i \subseteq C$ and for every odd index, $\gamma_i \cap C = \emptyset$. Indeed, since $e, f \notin B$, we have $\hat{d}(x_0, e), \hat{d}(x_0, f) \geq r + 2$ and $e, f \notin C$ and therefore $s$ is odd and the partition defined above is well defined. Let $2 \leq i \leq s - 1$ be an even index.

Since $e_{t_{i-1}} \notin \gamma_{i-1}, e_{t_{i+1}} \notin C$ and on the other hand, $e_{t_i} \in C$.

Since $G$ is a CSC, we can order the set of edges adjacent to the common vertex of $e_{t_{i-1}}$ and $e_{t_i}$, and walk from $e_{t_{i-1}}$ and $e_{t_i}$ through connecting faces. Since $e_{t_{i-1}} \notin C$, it is not connected to a face in $F'$ but $e_{t_i} \in C$ is connected to at least one face from $F'$, i.e., it is a boundary of $F'$. Therefore, there must be some edge in the ordering between those 2 edges connected to exactly one face of $F'$, i.e., it is a boundary of $F'$.

Therefore $e_{t_i} \cap x_i \neq \emptyset$ for some $x_i \in \partial_2(F')$. Similarly, we get $e_{t_{i+1}} \cap x_j \neq \emptyset$ for some $x_j \in \partial_2(F')$.

From lemma 5, $\partial_2(F')$ is path connected. Hence, there exist a path $\theta_i \in \Gamma_{x_i, x_j}$ such that $\theta_i \subseteq \partial_2(F')$.
But $\gamma_i + \theta_i \subseteq E'$ is a closed path in $K$ and therefore, is a 1-cycle. By assumption, $K$ is simply connected so $H_1(K) = 0$ and $\gamma_i + \theta_i = \partial_2(J_i)$ for some subset $J_i \subseteq F'$. Now define:

$$\gamma' = \sum_{i=1}^{\theta+1} \gamma_{2i-1} + \sum_{i=1}^{\theta+1} \theta_{2i}$$

We get:

$$\gamma + \gamma' = \sum_{i=1}^{\theta+1} \gamma_i + \sum_{i=1}^{\theta+1} \gamma_{2i-1} + \sum_{i=1}^{\theta+1} \theta_{2i}$$

$$= \sum_{i=1}^{\theta+1} \sum_{i=1}^{\theta+1} \gamma_{2i} + \sum_{i=1}^{\theta+1} \theta_{2i}$$

$$= \sum_{i=1}^{\theta+1} \gamma_{2i} + \theta_{2i}$$

$$= \sum_{i=1}^{\theta+1} \partial_2(J_i)$$

$$= \partial_2(\sum_{i=1}^{\theta+1} J_i)$$

and we conclude that $\gamma' \in [\gamma]$.

Finally, for every odd $i$, $\gamma_i \cap C = \emptyset$ and since $B \subseteq C$, we get $\gamma_i \cap B = \emptyset$. On the other hand, for every even $i$, $\theta_i \subset \partial_2(F')$ and since $C \cap \partial_2(F') = \emptyset$, we also get $\theta_i \cap B = \emptyset$. Therefore,

$$\gamma' \cap B = \sum_{i=1}^{\theta+1} (\gamma_{2i-1} \cap B) + \sum_{i=1}^{\theta+1} (\theta_{2i} \cap B) = \emptyset$$

This concludes the proof of Theorem 6.

### 7.2 The geometrically local case

We will now prove Theorem 7 providing a lower bound on the circuit depth when the circuit is geometrically restricted, when the underlying CSC is $o(\log(n))$-simply connected. For that purpose we first prove a more general version of Lemma 2, whose proof is essentially almost identical, except for making use of Theorem 6.

**Lemma 6.** Let $G$ be a CSC and let $U$ be a geometrically-local circuit, whose gates each act on qubits of distance at most $c$ apart, and whose depth is at most $d$, satisfying $U |0^n\rangle = |\psi\rangle \in C_G$. Let $A$ and $B$ be the lower and upper effective supports with respect to $U$ for some $e, f \in E$ and $\gamma \in \Gamma_{e, f}$. Assume $G$ is $c \cdot d$-simply connected. Then for all $n$, $A \subseteq \text{Ball}(e, cd) \cup \text{Ball}(f, cd)$ and $B \subseteq \text{Ball}(e, 2cd) \cup \text{Ball}(f, 2cd)$

**Proof.** We start in the same fashion as in the proof of lemma 2. Let $\gamma' \in [\gamma]$, and we have

$$L^i(\gamma') \subseteq \{e' \in E | d(e', \gamma') \leq c \cdot d\}$$

Let $e' \in E$ and assume that $e' \notin A \subseteq \overline{\text{Ball}(e, cd)} \cup \overline{\text{Ball}(f, cd)}$, namely that $d(e, e') > cd$ and $d(f, e') > cd$. Observe that $\text{Ball}(e', cd)$ doesn’t contain neither $e$ or $f$. Therefore, from the assumption of $c \cdot d$-simple connectedness and from Theorem 6 there exists some path $\gamma'' \in [\gamma]$ such that $d(\gamma'', e') > cd$ and then from Equation 21, $e' \notin L^i(\gamma'')$, and hence it is not in $A$. This proves the first part of the claim. Since $B = L^1(A)$, by definition any qubit in $B$ is within distance at most $cd_n$ from any qubit in $A$ and thus $B$ is contained in the set of all qubits at distance at most $cd_n$ from $\text{Ball}(e, cd_n) \cup \text{Ball}(f, cd_n)$; this set is contained in $\text{Ball}(e, 2cd_n) \cup \text{Ball}(f, 2cd_n)$ and the second part of the claim follows.
Figure 6: The yellow path represents the circle of radius $r=3$ around the edge $e$. In hyperbolic geometry, both the circumference of a circle and the area it defines grow exponentially with the radius.

As we stated in the last section, Fact 4 doesn’t hold in the most general non-euclidean case, as can be seen in figure 6. What we have instead is:

Lemma 7. Let $G$ be a simple graph with $n$ edges and bounded degree $\text{deg}(G) \leq D$. Then for every edge $e$ in $G$, $|\text{Ball}(e, n)| \leq D^{n+1}$

Proof. Define $C_n = \{ g \in E | d(e, g) = n \}$:

Observe that $|C_n| \leq 2D^n$ from a simple union bound. Therefore we get:

\begin{align*}
|\text{Ball}(e, n)| &= |\bigcup_{i=0}^{n} C_i| \\
&= 1 + \sum_{i=1}^{n} |C_i| \\
&\leq 1 + \sum_{i=1}^{n} 2D^i \\
&\leq 1 + 2D \cdot \frac{D^n - 1}{D - 1} \\
&\leq 1 + D \cdot (D^n - 1) \\
&\leq D^{n+1}
\end{align*}

Corollary 4. Let $G$ be a simple graph with $n$ edges and bounded degree $\text{deg}(G) \leq D$. Then $\text{diam}(G) \geq \log_D(n) - 1$

Proof. Let $d = \text{diam}(G)$ and let $e \in E$. We have $E = \text{Ball}(e, d)$ and from the previous lemma, we get $|E| = n \leq D^{d+1}$. The corollary follows.
We now give a formal proof of theorem 4.

**Proof of Theorem 4.** For each \( n \), choose 2 edges \( e, f \) verifying \( d(e, f) = \text{diam}(G_i) \), and choose some path \( \gamma \in \Gamma_{e,f} \). From corollary 4 we have \( d(e, f) > \log(N_i) \). Now assume by contradiction that there exists a depth \( d_i \neq \Omega(\log(N_i)) \) geometrically local quantum circuit \( U_i \) such that \( U_i |0 \rangle |N_i \rangle = |\psi_i \rangle \). We can extract a subsequence \( d_{i,j} = o(\log(N_i)) \). From lemma 4, since \( G_{i,j} \) is \( O(\log(N_i)) \)-simply connected, we get \( B = B_{[\gamma],U_{i,j}} \subseteq \text{Ball}(e, 2cd_{i,j}) \cup \text{Ball}(f, 2cd_{i,j}) \). But for large enough \( j \), \( \text{Ball}(e, 2cd_{i,j}) \cap \text{Ball}(f, 2cd_{i,j}) = \emptyset \), and therefore, \( B \) doesn’t contain any path in \( [\gamma] \) contradicting lemma 3.

### 7.3 The non geometrically local case

Here we prove our main result, Theorem 4, providing a lower bound on circuit depth for the non-geometrical case, for general CSCs with sufficient \( r \)-simple connectedness. We first prove a modified version of Lemma 4 which generalizes it to any \( r \)-simply connected CSCs for suitable \( r \).

**Lemma 8.** Let \( G = (V, E, F) \) be a CSC. Let \( X \subseteq E \) be copath connected and \( \gamma \)-separating. Define \( r = |X| \) and assume \( G \) is \( r+1 \)-simply connected, then \( X \cap \left( \text{Ball}(e, r) \cup \text{Ball}(f, r) \right) \neq \emptyset \).

**Proof.** Let \( x_0 \in X \). Since \( X \) is copath connected, for all \( x \in X \), there exists a copath \( \tilde{\gamma} \in \tilde{F}_{x_0,x} \) such that \( \tilde{\gamma} \subseteq X \). Thus, \( \tilde{d}(x_0, x) \leq |	ilde{\gamma}| - 1 \leq r \) and therefore, \( X \subseteq \text{Ball}(x_0, r) \). Assume by contradiction that \( X \cap \left( \text{Ball}(e, r) \cup \text{Ball}(f, r) \right) = \emptyset \). Hence, \( \tilde{d}(x_0, e), \tilde{d}(x_0, f) \geq r \) and so \( e, f \notin \text{Ball}(x_0, r) \). We can now use Theorem 3 to find a path \( \gamma' \) that satisfies \( \gamma' \in [\gamma] \) and \( \gamma' \cap \text{Ball}(x_0, r) = \emptyset \). Since \( X \subseteq \text{Ball}(x_0, r) \) we also get \( \gamma \cap X = \emptyset \) and that’s a contradiction to the assumption that \( X \) is \( \gamma \)-separating.

Finally we turn to the proof of our main result:

**Proof of Theorem 4.** This proof will be almost identical to the proof of Theorem 4. Assume that \( e \) and \( f \) are 2 edges in \( E_i \) such that \( d(e, f) = \text{diam}(G_i) \). We first prove that \( A \subseteq L^1(\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}})) \). Indeed, assume by contradiction that there exists some edge \( g \in A \) such that \( g \notin L^1(\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}})) \). From claim 3, \( L^1(g) \) is \( \gamma \)-separating and from corollary 3, we may assume \( L^1(g) \) is copath connected.

Since each layer in \( U \) increases the size of the upper light cone of \( g \) by a multiplicative factor of at most \( c \), we have \( |L^1(g)| < c^{d_{i,j}} \). By assumption, \( G_i \) is \( f(N_i) \)-simply connected for all \( f = o(\log(\log(N_i))) \)

Now if \( D_i = \Omega(\log(\log(\log(N_i)))) \), we are done. Assume otherwise. Then we can extract a subsequence \( d_{i,j} = o(\log(\log(\log(N_i)))) \) and \( |L^1(g)| < c^{d_{i,j}} = o(\log(\log(N_i))) \). In this case, we can apply lemma 5 to get:

\[
L^1(g) \cap \left( \text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}}) \right) \neq \emptyset
\]

Hence, \( g \in L^1(L^1(g) \cap \left( \text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}}) \right)) \leq L^1(\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}})) \) and we get a contradiction. We deduce that \( A \subseteq L^1(\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}})) \).

Let’s bound the size of \( A \) from above:

\[
|A| \leq |L^1(\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}}))| \leq c^{d_{i,j}} |\text{Ball}(e, c^{d_{i,j}}) \cup \text{Ball}(f, c^{d_{i,j}})| \leq 2c^{d_{i,j}} \cdot D^{d_{i,j}+1}
\]

where we used lemma 5 in the last inequality, and \( D = \max(\text{deg}(G), \text{deg}(f)) \) (see Definition 14). Now, from Corollary 4, we have \( |A| > \frac{d(e,f)}{c^{d_{i,j}}} \geq \frac{\text{diam}(G_i)}{c^{d_{i,j}}} \). From corollary 4 we get \( |A| > \frac{\log_D(N_i)}{c^{d_{i,j}}} \). Combining those equations together we get:

\[
\frac{\log_D(N_i)}{c^{d_{i,j}}} - 1 < 2c^{d_{i,j}} \cdot D^{d_{i,j}+1}
\]
We argue that by showing by induction that the operators

\[ P \]

Indeed, observe that:

\[ \text{Lemma 9.} \]

We will first show that \( H_n \) act on a constant number of qubits, namely:

\[ \text{Proof.} \] Assume \( \supp(P) = k \). Then \( |supp(UPU^d)| \leq c^d k \) where \( c \) is the maximal size of the support of any gate in \( U \).

Now assume \( Supp(P_i) = K_i \) where \( |K_i| = c^d k \). Observe that \( P_{i+1} = U_{i+1} P_i U_{i+1}^\dagger \). Define

\[ K_{i+1} = \bigcup_{S_{i+1,j} \cap K_i \neq \emptyset} S_{i+1,j} \]

We argue that \( Supp(P_{i+1}) = K_{i+1} \). Indeed, We can write:

\[ U_{i+1} = \bigotimes_{S_{i+1,j} \cap K_i \neq \emptyset} V_{i+1}^j \bigotimes_{S_{i+1,j} \cap K_i = \emptyset} V_{i+1}^j \]

Define \( U_{i+1}^{(0)} = \bigotimes_{S_{i+1,j} \cap K_i \neq \emptyset} V_{i+1}^j \) and \( U_{i+1}^{(1)} = \bigotimes_{S_{i+1,j} \cap K_i = \emptyset} V_{i+1}^j \). We can write \( P_i = \tilde{P}_i \otimes I_{[n] \backslash K_i} \) where \( \tilde{P}_i \) acts on qubits in \( K_i \) only. Therefore:

\[
P_{i+1} = U_{i+1} P_i U_{i+1}^\dagger \\
= (U_{i+1}^{(0)} \otimes U_{i+1}^{(1)}) (\tilde{P}_i \otimes I_{[n] \backslash K_i}) (U_{i+1}^{(0)} \otimes U_{i+1}^{(1)}) \\
= (U_{i+1}^{(0)} \tilde{P}_i U_{i+1}^{(0)}) (U_{i+1}^{(1)} I_{[n] \backslash K_i} U_{i+1}^{(1)}) \\
= (U_{i+1}^{(0)} \tilde{P}_i U_{i+1}^{(0)}) I_{[n] \backslash K_{i+1}}
\]

This shows that indeed, \( Supp(P_{i+1}) = K_{i+1} \). Finally we are left to show that \( |K_{i+1}| \leq c \cdot |K_i| \).

Indeed, observe that:

\[
|K_{i+1}| = \bigcup_{S_{i+1,j} \cap K_i \neq \emptyset} |S_{i+1,j}| \\
= \sum_{S_{i+1,j} \cap K_i \neq \emptyset} |S_{i+1,j}| \\
\leq \sum_{l_{i} \in K_i, l_{i+1} \in S_{i+1,j}} |S_{i+1,j}| \\
\leq |K_i| \cdot \max_j |S_{i+1,j}| \\
\leq c \cdot |K_i|
\]

8 Appendix A: trivial states are UGS of gapped Hamiltonians

Assume \( \{|\psi_n\rangle\} \) is a family of trivial states on \( n \) qubits. Then we can find a family of constant depth circuits \( \{U_n\} \) such that \( U_n |0^n\rangle = |\psi_n\rangle \). Let \( H_n = U_n Z^n U_n^\dagger \) where \( Z^i \triangleq z^i \otimes I |\psi_n\rangle \langle \psi_n| \) and \( z^i \) is the \( z \) Pauli operator on the \( i \)-th qubit. Now, define the following Hamiltonian:

\[ K_n = -\sum_{i=1}^{n} H_n^i \]

This shows that indeed, \( \tilde{\psi}_n \) represents a quantum gate acting on a subset \( S_i \) and it follows that \( \tilde{\psi}_n \) is a family of trivial states on \( n \) qubits. Then we can find a family of constant depth circuits \( \{V_n\} \) such that \( V_n |0^n\rangle = |\tilde{\psi}_n\rangle \). Define \( V_{i+1} = V_{i+1} \otimes I_{[n] \backslash K_i} \) where \( V_{i+1} \) acts on qubits in \( K_i \) only. Therefore:

\[
P_{i+1} = U_{i+1} P_i U_{i+1}^\dagger \\
= (U_{i+1}^{(0)} \otimes U_{i+1}^{(1)}) (\tilde{P}_i \otimes I_{[n] \backslash K_i}) (U_{i+1}^{(0)} \otimes U_{i+1}^{(1)}) \\
= (U_{i+1}^{(0)} \tilde{P}_i U_{i+1}^{(0)}) (U_{i+1}^{(1)} I_{[n] \backslash K_i} U_{i+1}^{(1)}) \\
= (U_{i+1}^{(0)} \tilde{P}_i U_{i+1}^{(0)}) I_{[n] \backslash K_{i+1}}
\]

or:

\[
\log D(N_i) < 2e^{2d_{ij}} \cdot D^{d_{ij}+1} + 1
\]

and it follows that \( d_{ij} \neq o(\log_d (\log D(N_i))) \) leading to a contradiction.
We now turn to the proof of the main theorem:

**Theorem 7.** \{K_n\} is a family of non degenerate gapped commuting local Hamiltonians. Furthermore, for all \( n \geq 1 \), \(|\psi_n\rangle\) is the unique ground state of \( K_n \).

**Proof.** Let \( E_n = \langle \psi_n | K_n | \psi_n \rangle \). For all \( 1 \leq i \leq n \), \(|Z_i| = 1\) (the operator norm of \( Z_i \)), so \(|H_i^n| = 1\) since conjugation by a unitary operator preserves the norm. Therefore,

\[
|K_n| \leq \sum_{i=1}^{n} |H_i^n| = n
\]

Observe that

\[
E_n = -\left\langle \psi_n \bigg| \sum_{i=1}^{n} H_i^n \bigg| \psi_n \right\rangle
\]

\[
= -\sum_{i=1}^{n} \left\langle \psi_n \right| H_i^n \left| \psi_n \right\rangle
\]

\[
= -\sum_{i=1}^{n} \left\langle \psi_n \right| U_i^n Z_i U_i^{n\dagger} \left| \psi_n \right\rangle
\]

\[
= -\sum_{i=1}^{n} \langle 0^n | Z_i | 0^n \rangle
\]

\[
= -n
\]

Therefore, \(|\psi_n\rangle\) is in the ground space of \( K_n \).

On the other hand assume that another state \(|\psi'_n\rangle\) also had energy \( E'_n = \langle \psi'_n | K_n | \psi'_n \rangle = -n \). This can only happen if each of the \( n \) terms \( H_i^n \) has energy \( E_i^n = \langle \psi'_n | H_i^n | \psi'_n \rangle = 1 \) or equivalently, \( \langle \phi_n | Z_i | \phi_n \rangle = 1 \) where we define \( |\phi_n\rangle = U_n^\dagger |\psi'_n\rangle \). We conclude that for every \( i \), \(|\phi_n\rangle\) is a +1 eigenstate of \( Z_i \). Therefore, \(|\phi_n\rangle = |0^n\rangle\) and by applying the unitary \( U_n \) on both sides we get \(|\psi'_n\rangle = U_n |\phi_n\rangle = U_n |0^n\rangle = |\psi_n\rangle\).

Observe that the terms \( \{H_i^n\} \) are pairwise commuting since:

\[
[H_i^n, H_j^n] = [Z_i, Z_j] = (-1)^{i\cdot j}
\]

We now prove that \( \{K_n\} \) is gapped: Indeed, the second energy level of \( \{K_n\} \) is met when exactly one local term \( H_i^n \) has energy -1. In such a case, \( E_n^{(2)} = -[(n - 1) \cdot 1 + 1 \cdot (-1)] = n - 2 \) and we get a constant spectral gap \( \Delta E = n - (n - 2) = 2 \).

\( \square \)

### 9 Appendix B: Non-trivial unique groundstates

Here we show a generic construction (proposed to us by I. Arad [26]) of a non-gapped Hamiltonian with a unique non-trivial groundstate.

We use the quantum verifier to Hamiltonian construction of Kitaev in [33] applied on a family of circuits \( \{U_i\} \) generating the i-th CAT state:

\[
|CAT^+\rangle = \frac{1}{\sqrt{2}} (|0^n\rangle + |1^n\rangle)
\]

The n-th circuit \( U_n \) can be described as a product of \( n^2 \) layers of unitaries \( U_n = V_n^{n^2} \cdot \ldots \cdot V_n^1 \) where we define:

\[\forall i, 1 \leq i \leq n : V_n^i = CNOT_{i,i+1} \otimes I_{[n] \setminus \{i,i+1\}}\]
∀ i, n + 1 \leq i \leq n^2 : V_n^i = I[n]

Observe that this is a depth \( n^2 \) circuit that generates the \(|CAT^+\rangle\) state on the input \(|\text{init}\rangle_n = |+\rangle \otimes |0\rangle^{n-1}\) The actual state we will be looking at is the computation history state defined on \( n + \log(n^2) \) qubits as:

\[ |\varphi_n\rangle = \frac{1}{n} \sum_{i=1}^{n^2} U_n^i |\text{init}_n\rangle \otimes |i\rangle \]

where \( U_n^i = V_n^i \cdot \ldots \cdot V_n^1 \). This state is the unique ground state of the \( O(1) \)-local Hamiltonian:

\[ H = H_{out} + J_{in} H_{in} + J_{prop} H_{prop} \]

defined in chapter 4 of [33]. It is straightforward from Kitaev’s construction that \( H \) has unique ground state \(|\varphi_n\rangle\) (since it is the only state satisfying the local constraints of \( H \) all together, describing the unique computation of \( U_n \) on \(|\text{init}_n\rangle\). Kitaev also proves that the spectral gap of \( U \) satisfies \( \Delta(H) = O(\frac{1}{n}) \) so that \( H \) is not gapped. We show that \(|\varphi_n\rangle\) is also non trivial: To see that, we look at the extended CAT state: \(|CAT_n^+\rangle = |CAT_n^+\rangle \otimes |\theta_n\rangle\) where \(|\theta_n\rangle\) is the state \(|\theta_n\rangle = \frac{1}{\sqrt{n^2-n}} \sum_{i=n+1}^{n^2} |i\rangle\) on \( \log(n^2) \) qubits.

\[
|\langle \varphi_n|CAT^+\rangle| = \left| \frac{1}{n} \sum_{i=1}^{n^2} \langle \text{init}_n| U_n^i \otimes |i\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \right| \geq \left| \frac{1}{n} \sum_{i=n+1}^{n^2} \langle \text{init}_n| U_n^i \otimes |i\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \right| - \left| \frac{1}{n} \sum_{i=1}^{n} \langle \text{init}_n| U_n^i \otimes |i\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \right| \]

\[
= \frac{1}{n} \sum_{i=n+1}^{n^2} \langle \text{init}_n| U_n^i \otimes |i\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \]

\[
= \frac{1}{n} \langle \text{init}_n| U_n^i \otimes \sum_{i=n+1}^{n^2} |i\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \]

\[
= \frac{1}{n} \langle \text{init}_n| U_n^i \otimes \sqrt{n^2 - n} \langle \theta_n\rangle \langle CAT_n^+ \otimes |\theta_n\rangle \rangle \]

\[
= \frac{1}{n} \langle \text{init}_n| U_n^i \otimes CAT_n^+ \rangle \]

\[
= \frac{\sqrt{n^2-n}}{n} \langle \text{init}_n| CAT_n^+ \rangle \]

And therefore, \( \lim_{n \to \infty} \langle \varphi_n|CAT^+\rangle = 1 \). Since \( CAT^+ \) exhibit long range correlations between its first \( n \) qubits (as it is basically the usual \(|CAT^+\rangle\) state on those qubits) so does \(|\psi_n\rangle\). Since we know that the output of a constant depth circuit exhibits no such correlations, we conclude that \(|\psi_n\rangle\) is non trivial.

## 10 Appendix C: k-UDA implies k-UDP

Assume the \( n \) qubit state \(|\psi\rangle\) is the unique ground state of some \( k \)-local hamiltonian \( H = \sum_i H_i \), and suppose by contradiction that \( \rho \neq |\psi\rangle \langle \psi| \) satisfies that for all \( K \subseteq [n], |K| \leq k \). \( Tr_K |\psi\rangle \langle \psi| = Tr_K \rho \) then:
\[
\text{Tr}(H_n \rho) = \text{Tr}(\sum_i H_i \rho) = \sum_i \text{Tr}(H_i \rho) = \sum_i \text{Tr}(H_i \text{Tr}_K \rho) = \sum_i \text{Tr}(H_i |\psi \rangle \langle \psi|) = \text{Tr}(H_n |\psi \rangle \langle \psi|) \tag{55-60}
\]

Therefore, \(\rho \neq |\psi \rangle \langle \psi|\) have the same energy. Now write \(\rho = \sum_i p_i |\psi_i \rangle \langle \psi_i|\) where \(p_i \geq 0\) and \(\sum_i p_i = 1\). We have:

\[
\text{Tr}(\rho H) = \sum_i p_i \langle \psi_i |H| \psi_i \rangle
\]

Since \(\rho\) has minimal energy, every element \(\langle \psi_i |H| \psi_i \rangle\) in the above weighted sum must be minimal and it follows that \(|\psi_i \rangle = |\psi \rangle\) for all \(i\). Hence, \(\rho = |\psi \rangle \langle \psi|\) and we get a contradiction.

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