Some open problems on permutation patterns

Einar Steingrímsson

Abstract

This is a brief survey of some open problems on permutation patterns, with an emphasis on subjects not covered in the recent book by Kitaev, Patterns in Permutations and words. I first survey recent developments on the enumeration and asymptotics of the pattern 1324, the last pattern of length 4 whose asymptotic growth is unknown, and related issues such as upper bounds for the number of avoiders of any pattern of length $k$ for any given $k$. Other subjects treated are the Möbius function, topological properties and other algebraic aspects of the poset of permutations, ordered by containment, and also the study of growth rates of permutation classes, which are containment closed subsets of this poset.

1 Introduction

The notion of permutation patterns is implicit in the literature a long way back, which is no surprise given that permutations are a natural object in many branches of mathematics, and because patterns of various sorts are ubiquitous in any study of discrete objects. In recent decades the study of permutation patterns has become a discipline in its own right, with hundreds of published papers. This rapid development has not only led to myriad new results, but also, and more interestingly, spawned several different research directions in the last few years. Also, many connections have been discovered between permutation patterns and other research areas, both inside and outside of combinatorics, showcasing the fundamental nature of patterns in permutations and other kinds of words.

Recently, Sergey Kitaev published a comprehensive reference work entitled Patterns in Permutations and words [31]. In the present paper I highlight some aspects of some of the most recent developments and some areas that have hardly been touched, but which I think deserve more attention. This is speculative, of course, and strongly coloured by my own preferences and by the limits of my own knowledge in the field. The topics dealt with here are mostly left out in [31] (due to their very recent appearance) so there is little overlap here with that book.

In Section 2 I briefly describe the different kinds of patterns that are prominent in the field. In Section 3 I treat one case of pattern avoidance, that of the pattern 1324, which is the shortest classical pattern whose avoidance is not known. Not even the asymptotics of the number of 1324-avoiders is known, even though significant effort has been put into this, resulting in ever better bounds. In fact, after a hiatus of a few years there have been several successively improved results in the last year, which also seem likely to be applicable in a wider context.

Section 4 describes conjectures about which patterns are easiest to avoid, that is, are avoided by more permutations than other patterns of the same length. All the

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evidence points in the same direction, namely, that for patterns of any given length \( k \) it is a layered pattern that is avoided by most permutations. It is still unclear, however, what form the most avoided layered pattern of length \( k \) has, although there are conjectures for particular families of values of \( k \) that seem reasonable, while others (in an abundance of conjectures, published and unpublished) have been shown false.

In Section 5 I discuss the poset (partially ordered set) \( P \) consisting of all permutations, ordered by pattern containment. This poset is the underlying object of all studies of pattern avoidance and containment. I mention the results so far on the Möbius function of \( P \), perhaps the most important combinatorial invariant of a poset, and some topological aspects of the order complexes of intervals in \( P \). Hardly anything is known so far on the topology of these intervals, but there are indications that large classes of them have a nice topological structure, whose understanding might shed light on various pattern problems.

In Section 6 I mention some algebraic properties of the set of mesh patterns, a generalisation encompassing all the kinds of patterns considered here, and also the ring of functions counting occurrences of vincular patterns in permutations. Neither of these aspects has been studied substantially, but there are reasons to believe that they might be interesting.

Finally, in Section 7 I treat permutation classes, which are classes of permutations avoiding a set, finite or infinite, of classical patterns. The emphasis here is on the growth rates of these classes, that is, the growth of the number of permutations of length \( n \) in a given class. Substantial progress has been made here in recent years, although this only begins to scratch the surface of what promises to be an interesting study of fundamental properties of pattern containment and avoidance. In particular, the study of permutation classes and their growth rates is intimately related to the types of generating functions enumerating these classes, and generating functions are among the most important tools in the study of permutation patterns, as in all of enumerative combinatorics.

2 Kinds of patterns

We write permutations in one-line notation, as \( a_1 a_2 \ldots a_n \), where the \( a_i \) are precisely the integers in \( [n] = \{1, 2, \ldots, n\} \). For terminology not defined here see [31].

An occurrence of a classical pattern \( p = p_1 p_2 \ldots p_k \) in a permutation \( \pi = a_1 a_2 \ldots a_n \) is a subsequence \( a_{i_1} a_{i_2} \ldots a_{i_k} \) of \( \pi \) whose letters appear in the same relative order of size as those in \( p \). For example, 1-3-2\(^2\) appears in 31542 as 142, 152, 154 and 354. Here, 354 is also an occurrence of the vincular pattern 1-32, because the 5 and 4 are adjacent in 31542, which is required by the absence of a dash between 3 and 2 in 1-32. Also, 142 is an occurrence of the bivincular pattern 1-32, because 4 and 2 are adjacent (in position) and 1 and 2 are adjacent in value, as required by the bar over the 1 in 1-32.

\(^2\)In this section I write classical patterns with dashes between all pairs of adjacent letters, to distinguish them from other vincular and bivincular patterns. In later sections, I will write classical patterns in the classical way, without any dashes, to keep the notation less cumbersome.
These three kinds of patterns are illustrated in Figure 1 where the shaded column between the second and third black dots of the diagram for 3-24-1 indicates that in an occurrence of 3-24-1 in a permutation $\pi$ no letter of $\pi$ is allowed to lie between the letters corresponding to the 2 and the 4 or, equivalently, that those letters have to be adjacent in $\pi$. Patterns thus represented by diagrams with entire columns shaded are called vincular patterns, but were called generalised patterns when they were introduced in [6]. Similarly, the shaded row in the diagram for $p = 3-24-\bar{1}$ indicates that in an occurrence of $p$ in a permutation $\pi$, there must be no letters in $\pi$ whose values lie between those corresponding to the 1 and the 2, that is, that the letters corresponding to the 1 and 2 must be adjacent in value, because of the bar over the 1 in $p$. Patterns with some rows and columns shaded are called bivincular, and were introduced in [14].

Mesh patterns, introduced in [15] are now defined by extending the above prohibitions determined by shaded columns and rows to a shading of an arbitrary subset of squares in the diagram. Thus, in an occurrence, in a permutation $\pi$, of the pattern $(3241, R)$ in Figure 1 there must, for example, be no letter in $\pi$ that precedes all letters in the occurrence and lies between the values of those corresponding to the 2 and the 3. This is required by the shaded square in the leftmost column. For example, in the permutation 415362, 5362 is not an occurrence of $(3241, R)$, since 4 precedes 5 and lies between 5 and 3 in value, whereas the subsequence 4362 is an occurrence of this mesh pattern.

![Pattern Diagrams](image)

Figure 1: The patterns 3-2-4-1 (classical), 3-24-1 (vincular), 3-24-\bar{1} (bivincular) and the mesh pattern $(3241, R)$, where $R = \{(0, 2), (1, 4), (3, 2)\}$.

The bivincular patterns were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in [14]. The original motivation behind their definition was to increase the symmetries of the vincular patterns, whose set is invariant under taking complement and reverse (corresponding to reflecting their diagrams horizontally and vertically), but not with respect to taking inverse, which corresponds to reflecting along the SW-NE diagonal. The study of the bivincular pattern 2-3\bar{1} was the catalyst of the paper [14], where permutations avoiding this pattern were shown to be in bijective correspondence with two other families of combinatorial objects, the $(2 + 2)$-free posets (see Figure 2) and the ascent sequences. An ascent sequence is a sequence $a_1a_2\ldots a_n$ of nonnegative integers where $a_1 = 0$ and, for all $i$ with $1 < i \leq n$, $a_i \leq \text{asc}(a_1a_2\ldots a_{i-1}) + 1$,

where $\text{asc}(a_1a_2\ldots a_k)$ is the number of ascents in the sequence $a_1a_2\ldots a_k$, that is, the number of places $j \geq 1$ such that $a_j < a_{j+1}$. An example of such a sequence is 0101312052, whereas 0012143 is not, because the 4 is greater than $\text{asc}(00121) + 1 = 3$.  

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This connection between \((2 + 2)\)-free posets (also known as interval orders) and ascent sequences led to the determination of the elegant generating function for these families of objects, given in \cite{14} Theorem 13 as

\[
\sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1 - t)^i).
\]

This generating function, equivalently an exact enumeration of the \((2+2)\)-free posets, had eluded researchers for a long time. This became tractable because the previously little studied ascent sequences are more easily amenable to an effective recursive decomposition than the posets. The paper \cite{14} has been followed by a great number of papers on these and bijectively related combinatorial objects, with no end in sight. The study of pattern avoidance by the ascent sequences in their own right was recently initiated \cite{24} and has been furthered in \cite{20, 35, 51}, each of which proves some of the conjectures made in \cite{24}.

![Figure 2: The poset \(2 + 2\), consisting of two disjoint chains, and a poset containing \(2 + 2\) in the subposet induced by \(b, c, e\) and \(x\). The poset on the right is \((2 + 2)\)-free if the vertex \(x\) is removed.](image)

We say that two patterns are \textit{Wilf equivalent}, and belong to the same \textit{Wilf class}, if, for each \(n\), the same number of permutations of length \(n\) avoids each. Of course, the symmetries mentioned above, corresponding to reflections of the diagram of a pattern, are (rather trivial) examples of Wilf equivalence, but there are many more, and they are often hard to prove. The smallest example of non-trivial Wilf equivalence is that for the classical patterns of length 3: The patterns \(1\underline{2}\underline{3}\) and \(3\underline{2}\underline{1}\) are trivially Wilf equivalent, and the same is true of the remaining four patterns of length 3, namely \(1\underline{3}\underline{2}\), \(2\underline{1}\underline{3}\), \(2\underline{3}\underline{1}\), and \(3\underline{1}\underline{2}\). All six of these patterns are Wilf equivalent, which is easy but non-trivial to prove; each is avoided by \(C_n\) permutations of length \(n\), where \(C_n\) is the Catalan number \(\frac{1}{n+1} \binom{2n}{n}\).

By extension, we say that two patterns \(p\) and \(q\) are \textit{strongly Wilf equivalent} if they have the same \textit{distribution} on the set of permutations of length \(n\) for each \(n\), that is, if for each nonnegative integer \(k\) the number of permutations of length \(n\) with exactly \(k\) occurrences of \(p\) is the same as that for \(q\). It is easy to see that the symmetry equivalences mentioned above imply strong Wilf equivalence. For example, \(p = 1\underline{3}\underline{2}\) is strongly Wilf equivalent to \(q = 2\underline{3}\underline{1}\), since the bijection defined by reversing a
permutation turns an occurrence of \( p \) into an occurrence of \( q \) and conversely. On the other hand, 1-3-2 and 1-2-3 are not strongly Wilf equivalent, although they are Wilf equivalent. For example, the permutation 1234 has four occurrences of 1-2-3, but there is no permutation of length 4 with four occurrences of 1-3-2.

Just as finding the distribution of occurrences of a pattern is in general harder than finding the number of avoiders, so there are still few results about strong Wilf equivalence. In fact, the only nontrivial such results I am aware of are recent results of Kasraoui [30], who gives an infinite family of strong Wilf equivalences for non-classical vincular patterns, including, for example, the equivalence of 3-421 and 421-3.

The mesh patterns, which may seem overly general at first sight, turn out to be the right level of generalisation for expressing a seemingly deep algebraic relationship (see Section 6) that encompasses all the kinds of patterns defined above. They also allow for simple expressions of some more cumbersome definitions, such as some of the so-called barred patterns, and they provide elegant expressions for some well-known statistics on permutations, such as the number of left-to-right maxima and the number of components in a permutation \( \pi \), that is, the maximum number of terms in a direct sum decomposition of \( \pi \), to be defined in Section 4.

### 3 Enumeration and asymptotics of avoiders: The case of 1324

For information about the state of the art in enumeration of permutations avoiding given patterns, refer to [31]. Here I will essentially only treat one unresolved case, that of the only classical pattern of length 4, up to Wilf equivalence, for which neither exact enumeration nor asymptotics have been determined. Of course, there is an endless list of problems left to solve when it comes to avoidance, until we find general theorems. Whether that will ever happen is likely to remain unknown for quite a while, given the slow progress so far. The situation is similar for vincular patterns, a short survey on which appears in [45], and even less has, understandably, been done when it comes to bivincular and mesh patterns. Although we are up against a major obstacle in furthering the knowledge about classical pattern avoidance, there are probably some reasonably easy, and interesting, results left to be found for the vincular, bivincular and mesh patterns.

Observe that in the remainder of the paper, unless otherwise noted, I am talking about classical patterns, which will be written in the classical way, that is, without any dashes to separate their letters (as should be done were they being considered as vincular patterns).

In 2004, Marcus and Tardos [36] proved the Stanley-Wilf conjecture, stating that, for any classical pattern \( p \), we have \( \text{Av}_p(n) < C^n \) for some constant \( C \) depending only on \( p \), where \( \text{Av}_p(n) \) is the number of permutations of length \( n \) avoiding \( p \). It had been shown earlier by Arratia [5] that this was equivalent to the existence of the limit

\[
\text{SW}(p) = \lim_{n \to \infty} \sqrt[n]{\text{Av}_p(n)},
\]

which is called the *Stanley-Wilf limit* for \( p \). It should be noted that this exponential
growth does not apply to vincular patterns in general. For example, the avoiders of length $n$ of the pattern 1-23 are enumerated by the Bell numbers $B_n$ [22], which count set partitions and grow faster than any exponential function. Namely, when $n$ goes to infinity, the quotient $B_n/B_{n-1}$ grows like $n/\log n$ [22, Prop. 2.6]. In fact, it seems likely (see [43, Section 8]) that there are no vincular patterns of length greater than 3 with exponential growth.

The Stanley-Wilf limit is 4 for all patterns of length 3, which follows from the fact that the number of avoiders of any one of these is the $n$-th Catalan number $C_n$, as mentioned above. This limit is known to be 8 for the pattern 1342 (see Bóna’s paper [12]). For the pattern 1234 the limit is 9. This is a special case of a result of Regev [39] (see also the more recent [38]), who provided a formula for the asymptotic growth of the number of standard Young tableaux with at most $k$ rows, pairs of which are in bijection, via the Robinson-Schensted correspondence, with permutations avoiding an increasing pattern of length $k + 1$. This limit can also be derived from Gessel’s general result [26] for the number of avoiders of an increasing pattern of any length.

The only Wilf class of patterns of length 4 for which the Stanley-Wilf limit is unknown is represented by 1324. A lower bound of 9.47 was established by Albert et al. [1], who used an interesting technique, the insertion encoding of a permutation, introduced in [3]. This encoding was used to analyse, in an efficient fashion, how a 1324-avoider can be built up by inserting the letters 1, 2, . . . , $n$ in increasing order. They then used this to show that a certain subset of 1324-avoiders has insertion encodings that are accepted by a particular finite automaton, from which they could deduce the lower bound.

Successively improved upper bounds have been established in several steps. The first reasonably small one, was given in [23], where 1324-avoiders were shown to inject into the set of pairs of permutations where one avoids 132 and the other avoids 213, which led to an upper bound of $4 \cdot 4 = 16$. Refining the method used in [23], Bóna [10] was able to reduce this bound to $7 + 4 \cdot \sqrt{3} \approx 13.9282$.

In [23] it is conjectured that $SW(1324) < e^{n\sqrt{\frac{2}{3}}} \approx 13.001954$. This would follow from another conjecture in [23], which says that the number of 1324-avoiders of length $n$ with a fixed number $k$ of inversions is increasing as a function of $n$. It is further conjectured that this holds for avoiders of any classical pattern other than the increasing ones, and there is some evidence that this in fact applies to all non-increasing vincular patterns.

Using Markov chain Monte Carlo methods to generate random 1324-avoiders, Madras and Liu [34] estimated that, with high likelihood, $SW(1324)$ lies in the interval [10.71, 11.83]. Recent computer simulations [4] I have done (unpublished), building on

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3 An inversion in a permutation $a_1a_2\ldots a_n$ is a pair $(i, j)$ such that $i < j$ and $a_i > a_j$.

4 The simulations were done in the following way: Using the method of [34, Sections 2 and 4], I first generated, in each of roughly 155 independent processes, $10^9$ 1324-avoiders of length 1,000 (the initialisation phase in the terminology of [34]). Using each of these 155 generated seed permutations, each in an independent process, I then generated a total of approx. $2.57 \cdot 10^{12}$ further avoiders, (the data collection phase). Using one in every ten thousand of those avoiders, I found the right end of the leftmost occurrence of the pattern 132. The number of that place equals the number of different places where $n + 1 = 1001$ can be inserted to obtain an avoider of length 1,001 from one of length
the random generation method of Madras and Liu, but estimating $SW(1324)$ in a different way, point to the actual limit being close to 11. Given how hard it seems to determine $SW(1324)$ makes it understandable that Doron Zeilberger is claimed to have said [25] that “Not even God knows the number of 1324-avoiders of length 1,000.” I’m not sure how good Zeilberger’s God is at math, but I believe that some humans will find this number in the not so distant future.

4 Are layered patterns the most easily avoided?

A layered permutation is a permutation that is a concatenation of decreasing sequences, each containing smaller letters than in any of the following sequences. An example of a layered permutation is 32145768, whose layers are displayed by 321 − 45 − 76 − 8. A layered pattern is thus a direct sum of decreasing permutations. The direct sum $\sigma \oplus \tau$ of two permutations $\sigma$ and $\tau$ is obtained by appending $\tau$ to $\sigma$ after adding the length of $\sigma$ to each letter of $\tau$. The skew sum $\sigma \ominus \tau$ of $\sigma$ and $\tau$ is obtained by prepending $\sigma$ to $\tau$ after adding the length of $\tau$ to each letter of $\sigma$. Thus, for example, we have $3142 \oplus 231 = 3142675$ and $3142 \ominus 231 = 6475231$.

Of course, reversing a layered pattern $p$, or taking its complement, gives a pattern that is Wilf equivalent to $p$, and such a pattern/permutation might be called up-layered, since each layer is increasing. Clearly, an up-layered pattern is the skew sum of increasing permutations. To simplify the discussion in this section, without changing the traditional definition of layered permutations, I will abuse notation by letting “layered” refer both to layered and up-layered patterns.

Evidence going back at least to Julian West’s thesis [50, Section 3.3] supports the conjecture that among all patterns of length $k$ the pattern avoided by most permutations of a sufficiently large length $n$ is a layered pattern. This conjecture has been around for a long time, and several variations on it, ranging from asymptotic dominance to the conjecture that $Av_\sigma(n) \leq Av_\tau(n)$ for all $n$ if $\sigma$ and $\tau$ have the same length and $\tau$ is layered but $\sigma$ isn’t. The latter conjecture here is false, as pointed out by Vít Jelínek [28], who noted that, by [13, Theorem 4.2], the non-layered pattern obtained as the direct sum $p \oplus q$ of the layered pattern $p$ with layer sizes $(1, 2, 1, 2, \ldots, 1, 2, 1)$ and $q = 231$ has a larger Stanley-Wilf limit than the layered pattern 12...$k$, if $k$ is sufficiently large. One of the weakest of these conjectures, widely believed to be true, is made explicit in [23, Conjecture 1] (although it may have been made before) and says that among all patterns of a given length $k$, the largest Stanley-Wilf limit is attained by some layered pattern.

The first published instances of such conjectures I am aware of appear in Burstein’s thesis [18, Conjectures 9.5], where several conjectures are listed, both by Burstein and others, some of which have been refuted, while others have yet to be proved right or wrong. I think these questions are quite interesting, apart from their intrinsic interest.

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1,000: In the limit where $n \to \infty$ the average of the number of this place over all avoiders of a given length $n$ would give the S-W limit. The average I obtained was approximately 11.01146. However, the convergence may be very slow, so, as pointed out to me by Josef Cibulka [21], this may be close to the correct value for $n = 1,000$ although the actual limit might be significantly greater.
because any results about them are likely to be accompanied by quite general results about the enumeration and asymptotics of pattern avoidance.

Although the evidence is strong in support of the conjecture that the most easily avoided pattern of any given length is a layered pattern, there is currently no general conjecture that fits all the known data about the particular layered patterns with the most avoiders. However, there are some ideas about what form the most avoided layered patterns ought to have, and specific conjectures that have not been shown to be false.

As is mentioned in Burstein’s thesis [18, see Conjectures 9.10, 9.11 and 9.12] Kézdy and Snevily had made some conjectures about this, and Burstein did too. In all these cases, which fall into three classes, depending on the congruence class modulo 4 of $k$, the patterns conjectured to have the maximal number of avoiders have small layers, and are highly symmetric. For example, a conjecture of Kézdy and Snevily [13, Conj. 9.10] that has not been refuted is that for patterns of even length $k$ the most avoided pattern is the one with layers $(1, 2, 2, \ldots, 2, 1)$. Also, Burstein’s Conjecture 9.11 in [18] that for $k \equiv 1 \pmod{4}$ the most avoided pattern of length $k$ has symmetric layers $(1, 2, \ldots, 2, 3, 2, \ldots, 2, 1)$ still stands. Data computed by Vít Jelínek [28] support these conjectures. Jelínek provided all such data mentioned in this section. These computations range over all patterns of lengths up to 10 and permutations of lengths up to 14, although they are not exhaustive within these ranges.

For patterns of length $k \equiv 3 \pmod{4}$, however, the situation is not quite that simple. Burstein’s Conjecture 9.12 in [18], has the symmetric pattern $(1,2,\ldots,2,1,2,\ldots,2,1)$ as most avoided, but among permutations of lengths 11 and 12 the most avoided pattern of length 7 is 1432657, with layers $(1,3,2,1)$. The next pattern in this respect is 2143657, with layers $(2,2,2,1)$, and only in third place comes 1324657, with the symmetric layers $(1,2,1,2,1)$. The respective numbers of avoiders of these patterns, for permutations of length 12, are 457657176, 457656206 and 457655768, differing in their last four digits.

But, strange things do occasionally happen in this field, so Burstein’s pretty conjecture for $k \equiv 3 \pmod{4}$ cannot be written off entirely yet as far as asymptotic growth is concerned. Namely, Stankova and West [42, Figure 9] unearthed a somewhat irregular behaviour in pattern avoidance that gives pause here. They computed data that show, among other examples, that although $\text{Av}_p(n)(53241) < \text{Av}_p(n)(43251)$ when $7 \leq n \leq 12$, the inequality is switched for $n = 13$. It is still unknown whether such a switch occurs twice or more for any pair of patterns, but Stankova and West conjecture [42, Conjecture 2] that this does not happen. They also make the weaker conjecture that for any pair of patterns $\sigma$ and $\tau$ there is an $N$ such that either $\text{Av}_\sigma(n) < \text{Av}_\tau(n)$ for all $n > N$ or else the opposite inequality holds for all $n > N$. In other words, that two patterns do not switch places infinitely often.

It was shown in [23, Corollary 7] that the Stanley-Wilf limit of a layered pattern of length $k$ is at most $4k^2$. In a recent preprint [11] Bóna, building on results from [23] and [11], proves that this bound can be reduced to $2.25k^2$. If the conjecture is true that the most avoided pattern of length $k$, for any given $k$, is a layered pattern then
this would be an upper bound for the Stanley-Wilf limit of any pattern of length $k$, but it is not known whether this bound is the best possible.

5 The pattern poset, its Möbius function and topology

The set of all permutations forms a poset $\mathcal{P}$ with respect to classical pattern containment. That is, a permutation $\sigma$ is smaller than $\pi$, denoted $\sigma \leq \pi$, if $\sigma$ occurs as a pattern in $\pi$. This poset is the underlying object of all studies of pattern avoidance and containment. In the following two subsections I treat the Möbius function of $\mathcal{P}$, perhaps the most studied invariant of a poset, and then the topological aspects of the simplicial complexes naturally associated to intervals in $\mathcal{P}$.

5.1 The Möbius function of the permutation poset

An interval $[x, y]$ in a poset $\mathcal{P}$ is the set of all elements $z \in \mathcal{P}$ such that $x \leq z \leq y$. An interval is thus a subposet with unique minimum and maximum elements, namely $x$ and $y$, respectively, except when $x \not\leq y$, in which case $[x, y]$ is empty. The Möbius function of an interval is defined recursively as follows: For all $x$, we set $\mu(x, x) = 1$ and

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z).$$

Thus, the Möbius function is uniquely defined by setting its sum over any interval $[x, y]$ to 1 if $x = y$ and to 0 otherwise. Figure 3 gives an example of an interval from $\mathcal{P}$, and the values of the Möbius function on this interval, computed from bottom to top using the above recursive definition, showing that $\mu(321, 316254) = -1$.

![Diagram of the permutation poset and Möbius function values]

Figure 3: The interval $\mathcal{I} = [321, 316254]$ in $\mathcal{P}$ and the values of $\mu(321, \tau)$ on its elements $\tau$.

The Möbius function of intervals in the permutation poset exhibits a great variety in values, even for permutations of small length, and is seemingly very hard to determine.
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in the general case. The first such results were found by Sagan and Vatter [40], who
found a formula for intervals of layered permutations.

In [46] the problem was solved for some special cases, and in [19] an effective (poly-
nomial time) formula was given for all separable permutations. These are the permuta-
tions that can be built from singletons by combinations of direct sums and skew
sums. The permutation 31254 is separable, since $31254 = (1 \ominus (1 \oplus 1)) \oplus (1 \ominus 1)$,
whereas 3142 is not. In fact, the separable permutations are precisely those that avoid
both 3142 and 2413 (the two simple permutations of length 4; see definition later in
this section). This fact is usually attributed to folklore, and it is straightforward to
prove.

The formula in [19] for the Möbius function of an interval of separable permutations
$[\sigma, \tau]$ is based on the representation of separable permutations by rooted trees, and
then certain embeddings of (the tree for) $\sigma$ in (the tree for) $\tau$ are counted, with a sign,
to obtain $\mu(\sigma, \tau)$. More precisely, the reduced separating tree of a separable permuta-
tion $\pi$ describes exactly how $\pi$ is composed by sums and skew sums. The separating
tree is defined recursively by letting the children of the root be the summands in the (skew) sum of $\pi$ decomposed into the maximal number of summands (see [19]
Figure 1). The embeddings of the tree for $\tau$ into the tree for $\sigma$ that are counted
to compute the Möbius function are the so called normal embeddings, defined by a
rather technical condition that we won’t go into here. Each embedding corresponds
to a unique occurrence of $\sigma$ in $\tau$. The formula for the Möbius function $\mu(\sigma, \tau)$ is then
given by

$$\mu(\sigma, \tau) = \sum_{f \in N(\sigma, \tau)} \text{sgn}(f),$$  \hspace{1cm} (5.1)

where $N(\sigma, \tau)$ is the set of normal embeddings of $\sigma$ in $\tau$ and sgn$(f)$ is a sign associated
to the embedding $f$, depending on which leaves of the tree for $\tau$ belong to the
embedding of $\sigma$ in $\tau$. Again, this is determined by a rather technical condition that
we omit, but we refer the reader to [19 Section 4].

Formula (5.1) implies, among many other things, that the absolute value of $\mu(\sigma, \tau)$
cannot exceed the number of occurrences of $\sigma$ in $\tau$ (which is far from true in the
general case), because each of the embeddings in question corresponds to a unique
occurrence of $\sigma$ in $\tau$.

This formula also allows for many easy computations of values of the Möbius function,
such as this example: If $\pi_i = 1, 3, 5, \ldots, 2i - 1, 2i, \ldots, 4, 2$, then

$$\mu(\pi_k, \pi_n) = \binom{n + k - 1}{n - k}.$$

For example, $\mu(\pi_2, \pi_4) = \mu(1342, 13578642) = \binom{4+2-1}{4-2} = 10$.

It is conjectured in [19] Conjecture 30 that the maximum value of $\mu(\sigma, \pi)$ for any
separable $\pi$ of length $n \geq 3$ is obtained by a permutation of this form, for $k$ that is
roughly $n/2$ (but whose exact formula depends on the parity of the length of $\pi$).

In [19] a recursive formula for the Möbius function was also given in the case of decomposable permutations, those that can be written non-trivially as sums or skew
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sums \((241365 = 2413 \oplus (1 \ominus 1)\) is decomposable, whereas 2413 is not). This reduces the problem to the indecomposable permutations, for which it seems unlikely there will be a general formula anytime soon. However, solutions for large classes of these are reasonable to hope for. Moreover, indications are that the results for separable permutations, which are characterised by avoiding 2413 and 3142, can be extended to other more complicated permutation classes. That would allow us to find the maximum (absolute) value of the Möbius function on infinite permutation classes by computing it on a small finite set of permutations in the class.

An occurrence of a consecutive (vincular) pattern \(p\) in a permutation \(\pi\) is an occurrence of \(p\) in \(\pi\) whose letters are consecutive in \(\pi\), such as the occurrence 364 of the pattern 132 in 5136472. An effective formula for computing the Möbius function of the poset of permutations ordered by containment as consecutive patterns was given in [5]. Sagan and Willenbring [41] later provided another proof, and used discrete Morse theory to determine the homotopy type of intervals of this poset. This poset is rather simple; its Möbius function is restricted to 0, 1 and \(-1\), and it was shown in [41] that its intervals are either contractible or else homotopy equivalent to a single sphere (see the next subsection, on topology of the permutation poset).

A recent paper by McNamara and Sagan [37] established similar and further results in the more wide ranging case of the poset of generalised subword order. In particular, they determined the Möbius function for all intervals of this poset, and exhibited a family of intervals that are homotopic to wedges of spheres. Given the similarity in definition of this poset to the poset \(\mathcal{P}\) of permutations with the classical pattern containment order, it seems reasonable to hope that the methods developed in [37] can be adapted to obtain results for large classes of intervals in \(\mathcal{P}\).

Let 1 be the permutation of length 1. It is shown in [19, Corollary 24] that for any separable permutation \(\pi\) the only possible values of \(\mu(1, \pi)\) are 0, 1 and \(-1\). Although nobody has bothered proving this yet, it appears certain that for arbitrary permutations \(\pi\) the absolute value \(\mu(1, \pi)\) is unbounded. For example, it seems a safe guess (based on computed data) that

\[
\mu(1, 2468 \ldots (2n)135 \ldots (2n - 1)) = -\left(\frac{n + 1}{2}\right).
\]

As an example, \(\mu(1, 24681357) = -\left(\frac{4+1}{2}\right) = -10\).

The maximum absolute value of \(\mu(1, \pi)\) is known for all \(\pi\) of length at most 11, as mentioned in [19, Section 5]. Recent computations I have done, although not exhaustive, suggest that for \(n = 12\) the maximum is attained only by the permutation \(\pi = 4\ 7\ 2\ 10\ 5\ 1\ 12\ 8\ 3\ 11\ 6\ 9\) and its symmetric equivalents, with \(\mu(1, \pi) = -261\). That would match the results for \(n < 12\), mentioned in [19], namely that the maximum is in each case attained by only one permutation (up to trivial symmetries), and that that permutation is simple\(^5\). A permutation \(\pi = a_1a_2 \ldots a_n\) is simple if it has no segment \(a_ia_{i+1} \ldots a_{i+k}\), where \(0 < k < n - 1\), that consists of a segment of values, that is, such that \(\{a_i, a_{i+1}, \ldots, a_{i+k}\} = \{\ell, \ell + 1, \ldots, \ell + k\}\) for some \(\ell\). For example, 315264 is simple, but 461325 is not, since 132 is a segment of consecutive

\(^5\)Except when \(n = 3\), for which there are no simple permutations
values that is nontrivial, that is, neither a singleton nor the entire permutation. A nice survey on simple permutations, which are important in the study of permutation classes, is found in [16].

It is worth noting here that separable permutations and simple permutations are in some imprecise sense each others’ opposites; the separable ones decompose very nicely, while being simple is an obstruction to such decomposition. Thus, it would not be surprising if the maximum of $|\mu(1, \pi)|$ over all $\pi$ of length $n$ turns out to be attained by a simple permutation. It seems less certain that there is, for all $n$, a unique permutation, up to trivial symmetries, that attains the maximum value for each $n$. The sequence of values of $\mu(1, \pi)$ for which $|\mu(1, \pi)|$ is maximised (for each length $n$, starting at $n = 1$) begins with

$$1, -1, 1, -3, 6, -11, 15, -27, -50, -58, 143, -261, \ldots$$

(the last entry still conjectural, as mentioned above) but no nontrivial upper bound on its $n$-th term is known.

Although many families of intervals $[\sigma, \tau]$ with $\mu(\sigma, \tau) = 0$ are described in [19] [46], it is an open problem to characterise such intervals completely. It might also be interesting to characterise those intervals for which $|\mu(\sigma, \pi)|$ equals the number of occurrences of $\sigma$ in $\pi$.

Another question raised in [19] is whether it is possible to find a bound on $|\mu(\sigma, \tau)|$ that depends only on the number of occurrences of $\sigma$ in $\tau$. As mentioned above, it has been shown that for separable $\sigma$ and $\tau$, the value $|\mu(\sigma, \tau)|$ cannot exceed the number of occurrences of $\sigma$ in $\tau$, but nothing similar is known for the general case.

In conclusion, although a general formula for the Möbius function of an interval will likely remain untractable for a while it seems reasonable to expect much further progress. In particular, the fact that there are families of intervals whose Möbius function has a “nice” formula, such as binomial coefficients, gives hope that these intervals have a structure that can be understood and exploited, and used to elicit the topology of these intervals, which is treated in the next section.

5.2 Topology of the permutation poset

Another important aspect of the poset of permutations, as for any combinatorially defined poset, is the topology of (the order complexes of) its intervals. (For terminology not defined in this section, see [43].) The order complex of a poset $P$, denoted $\Delta(P)$, is the abstract simplicial complex consisting of the chains of $P$. A chain in a poset $P$ is a set of elements in $P$ that are pairwise comparable, and thus totally ordered. Any subset of a chain forms a chain, so the set of chains is closed in this respect, which is a defining property of simplicial complexes. To study the topology of an interval $I$ we remove the maximum and minimum element of $I$, and take the order complex of the remaining “interior” of $I$, which we denote by $\overline{I}$.

There are some indications that large classes of intervals in the permutation poset $P$ may have a “nice” topology, meaning that the topology can be simply described, and
thus that the homology, and the homotopy type, of these intervals can be well understood. Gaining such understanding may well lead to significant progress in answering other questions, for example about the Möbius function. This is because, in some sense, the topological properties of a complicated interval give a much clearer picture of the important overall structure, sweeping aside irrelevant details that obscure the view. One well known connection to the Möbius function is that the Möbius function of a poset is equal to the reduced Euler characteristic of its order complex, which is a topological invariant (see [44, Proposition 3.8.6]).

Figure 4 shows the interval $I = [321, 316254]$, and the order complex of $\bar{I}$. This complex is homotopy equivalent to a sphere, since contracting the edge between 1432 and 21543 leaves a 1-dimensional sphere (topologically speaking) consisting of the edges of the rectangle in the figure. Note that $\mu(321, 21543) = 0$ and so removing 21543 from $I$ does not affect the Möbius function of $I$. The reduced Euler characteristic of a 1-dimensional sphere is $-1$, which, of course, equals $\mu(321, 316254)$. An obvious question (see below) is whether it is common for order complexes arising in this way from intervals of $P$ to have similarly nice properties, such as being homotopy equivalent to wedges of spheres of the same dimension.

The elements (which are sets) in a simplicial complex are called faces. A facet of a simplicial complex is a face that is maximal with respect to containment. A simplicial complex is pure if all its facets have the same dimension. A property that implies a pure simplicial complex is homotopy equivalent to a wedge of spheres is being shellable. Informally, this means that the complex can be built up, one facet at a time, such that each facet that is added, apart from the first one, intersects the union of the previous ones in a pure subcomplex of the maximum possible dimension (which is one less than the dimension of the facets). The complex in Figure 4 is shellable. One shelling order of its facets (edges) is $a, b, c, d, e$, whereas beginning with $a, c, \ldots$ can not give a shelling since the edge $c$ does not intersect $a$. As mentioned before, a shellable complex is necessarily homotopy equivalent to a wedge of spheres, and showing shellability is probably the most common tool used to determine the topology of combinatorially defined complexes.

As mentioned in the previous section, Sagan and Willenbring [41] used discrete Morse theory to determine the homotopy type of intervals of the poset of consecutive patterns, whose intervals are either contractible or else homotopy equivalent to a single sphere, and it is reasonable to expect that results using similar techniques will show large classes of intervals in $P$ to be homotopy equivalent to wedges of spheres. The Möbius function of such intervals is, up to a sign, just the number of spheres in the wedge.

In [19, Question 31] the following questions were raised, where, given an interval $I = [\sigma, \pi] \in P$, we let $\Delta(\sigma, \pi)$ be the order complex of $\bar{I}$:

1. For which $\sigma$ and $\pi$ does $\Delta(\sigma, \pi)$ have the homotopy type of a wedge of spheres?

2. Let $\Gamma$ be the subcomplex of $\Delta(\sigma, \pi)$ induced by those elements $\tau$ of $[\sigma, \pi]$ for which $\mu(\sigma, \tau) \neq 0$. Is $\Gamma$ a pure complex, that is, do all its maximal simplices (with respect to inclusion) have the same dimension?
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316254
15243 25143 21543
4132 1432 321

$\Delta(\bar{I})$

Figure 4: The interval $I = [321, 316254]$ in $P$ and the order complex of $\bar{I}$. Note that this is a deceptively simple example, since intervals of a higher rank have order complexes of higher dimension, which are not so easy to depict.

3. If $\sigma$ occurs precisely once in $\pi$, and $\mu(\sigma, \pi) = \pm 1$, is $\Delta(\sigma, \pi)$ homotopy equivalent to a sphere?

4. For which $\sigma$ and $\pi$ is $\Delta(\sigma, \pi)$ shellable?

An example where $\Delta(\sigma, \pi)$ is not shellable is given by $\sigma = 231$ and $\pi = 231564$, since $\Delta(\sigma, \pi)$ in this case consists of two disjoint components, each of which is contractible. If, however, we remove from $[231, 231564]$ all those elements $\tau$ for which $\mu(231, \tau) = 0$, we get a shellable complex, namely a boolean algebra of rank 2. For questions 2 and 3 above we don’t know any counterexamples. In fact, we know no counterexamples to question 3 even without the condition of just one occurrence. However, since we have so far only examined intervals of small rank, our evidence is weak.

6 Other algebraic aspects

In addition to the Möbius function (and the underlying incidence algebra of $P$) there are some algebraic aspects of permutation patterns that have been little studied, but which might harbour some interesting things. I mention two here: The ring of functions of vincular patterns, and Brändén and Claesson’s reciprocity theorem for mesh patterns.

Vincular patterns can be regarded as functions from the set of all permutations to the ring of integers, counting occurrences of themselves in a permutation. For example, $2\cdot 31(416253) = 2$, corresponding to the 462 and 453, which are all the occurrences of $2\cdot 31$ in 416253. Linear combinations of vincular patterns were used in [6] to classify the Mahonian permutation statistics, which are those that are equidistributed with the number of inversions. Other such combinations played a crucial role in [17],
where they were used to record the distribution of various statistics on the filled
Young tableaux treated there.

No further work seems to have been done along these lines, although it is almost
certain that there are many equivalences to be found of the kind discussed in [6]. A
promising indication in this context is that computer experiments suggest that the
distribution of two linear combinations of patterns is the same for all \( n \) provided
that it is the same for all \( n \) smaller than some (small) constant depending only
on the length of the patterns involved. As an example, exhaustive computer search
shows that if two linear combinations of three vincular patterns of length 3 diverge for
\( n < 10 \), then they diverge already for \( n = 6 \). For vincular patterns of length 4 all such
combinations with different distributions for \( n = 9 \) differ already for \( n = 8 \). Of course,
it is possible that divergence will occur again for greater \( n \), but this seems unlikely.
A general theorem to this effect, guaranteeing that checking such equidistributions
for small \( n \) is sufficient to establish equidistribution for all \( n \), would be a major
breakthrough, provided the values that need to be checked are small enough, since
that would give automatic proofs of various theorems. More importantly, proving
such a theorem would undoubtedly require a general understanding we lack today,
and thus likely lead to significant other progress.

Seen as functions, as described above, the set of vincular patterns constitutes a ring
of functions. (It is a tedious but straightforward exercise to verify that the product
of two vincular patterns can be expressed as a sum of vincular patterns.) One relation
is known in this ring, namely the \textit{upgrading} mentioned in [6, Equation (2)], an example
of which is

\[
(21-3) = (21-43) + (21-34) + (31-24) + (32-14) + (213).
\]

Since this ring contains all linear combinations of vincular patterns, it might be
worthwhile to study its algebraic structure further. In particular, it would be interest-
ing to know if there are other relations in this ring.

The set of mesh patterns also forms a ring of functions that should be further inves-
tigated for its properties. It is in this ring that the striking Reciprocity Theorem of
Brändén and Claesson lives [15]. The Reciprocity Theorem expresses any mesh pat-
ttern (including the classical patterns) as a (possibly infinite) linear combination of
classical patterns whose coefficients are obtained from values of the \textit{dual} pattern on
permutations. To express that theorem, let \( p = (\pi, R) \) be a mesh pattern, and let \( R^c \)
be the complement of \( R \), that is, \( R^c = [0, n]^2 \setminus R \), where \( n \) is the length of \( \pi \). We then
define the dual pattern of \( p \) as \( p^* = (\pi, R^c) \), and define \( \lambda(\sigma) \) by
\( \lambda(\sigma) = (-1)^{n-k} p^*(\sigma) \), where \( k \) is the length of \( \sigma \). The Reciprocity Theorem is then the following identity,
where the sum is over all classical patterns \( \sigma \):

\[
p = \sum_{\sigma \in S} \lambda(\sigma) \sigma.
\]

It seems likely that much can be gained from this theorem, due to its universal
nature.
7 Growth rates of permutation classes

A permutation class is a set of permutations that is closed with respect to containment. That is, if $\pi \in C$ for a class $C$, and $\sigma$ occurs as a pattern in $\pi$, then $\sigma \in C$. The set of permutations avoiding any classical pattern, or set of such patterns, is easily seen to be a class (which is not true for vincular, bivincular or mesh patterns) and every permutation class is characterised by a unique antichain of permutations that are avoided by all elements of $C$. That antichain is called the basis of $C$. Note that there are infinite antichains of permutations (see Brignall [17] for the most general construction to-date), so bases can be infinite.

Studies of the poset of permutations have in recent years yielded many results about the diverse collection of permutation classes, parallelli ng work being done on other types of object (surveyed in Bollobás [9]). One of the most active and successful avenues of investigation has been into the growth rates of permutation classes. Given a class $C$, where $C_n$ is a set of permutations in $C$ of length $n$, the growth rate of $C$ is defined as

$$\text{gr}(C) = \limsup_{n \to \infty} \sqrt[n]{|C_n|}.$$ 

To connect this terminology with that of Section 3, note that the Stanley-Wilf limit of the (classical) pattern $p$ is the growth rate of the class of $p$-avoiding permutations. Thus the Stanley-Wilf Conjecture in this context states that all proper permutation classes have finite growth rates. One of the most natural open questions is whether the limit superior above in the definition can be replaced by a limit; this is known to be possible in the case of singleton-based classes by Arratia [5].

The line of research on growth rates attempts to determine both which growth rates are possible and where notable phase transitions take place in this spectrum. The first answers were provided by Kaiser and Klazar [29], who characterised the growth rates up to 2. At the smallest end of the scale, it is clear that 0 and 1 are growth rates of permutation classes, and that no classes have growth rates between these two numbers. Kaiser and Klazar showed that the next growth rate is the golden ratio and established the stronger result that if $|C_n| < F_n$ (the $n$th Fibonacci number) for any $n$, then $|C_n|$ is eventually polynomial (the structural properties of such classes were later explored by Huczynska and Vatter [27]). Between the golden ratio and 2, Kaiser and Klazar showed that all growth rates are roots of $x^k - x^{k-1} - \cdots - x - 1$ for some $k$; note that this makes 2 the least accumulation point of growth rates.

Vatter [49] extended the characterisation of growth rates up to $\kappa \approx 2.21$, the unique positive root of $x^3 - 2x^2 - 1$. Moreover, $\kappa$ represents a sharp phase transition: There are only countably many permutation classes of growth rate less than $\kappa$, but because infinite antichains of permutations begin to appear at this growth rate, there are uncountably many permutation classes of growth rate $\kappa$. Viewed on the number-line of growth rates, $\kappa$ also lies in an interesting place, as it is the least accumulation point of accumulation points of growth rates. Recent work by Albert, Ruškuc, and Vatter [4] has established another threshold at $\kappa$: Every permutation class of growth rate less than $\kappa$ has a rational generating function, while there are (by an elementary counting argument using the existence of infinite antichains) permutation classes of
growth rate $\kappa$ whose generating functions are not even holonomic. (A function on the natural numbers is holonomic if it satisfies a linear homogeneous recurrence relation with polynomial coefficients.)

Working in the closely related context of ordered graphs, Balogh, Bollobás, and Morris [7] characterised the growth rates up to 2 and made two conjectures which would have implied that growth rates of permutation classes are always algebraic integers and that the set of growth rates contains no accumulation points from above. These conjectures were both disproved by Albert and Linton [2], who constructed an uncountable set of growth rates. It remains open if, as suggested by Klazar [33], the conjectures of Balogh, Bollobás, and Morris hold when restricted to finitely based permutation classes.

Building on the work of Albert and Linton, Vatter [48] showed that every real number greater than or equal to $\lambda \approx 2.48$, the unique positive root of $x^5 - 2x^4 - 2x^2 - 2x - 1$, is the growth rate of a permutation class, and conjectured that $\lambda$ is best possible.

Thus a striking problem remains: To characterise the growth rates between $\kappa$ and $\lambda$. While it may well be impossible to describe the set of growth rates once it becomes uncountable (and before it consists of all real numbers), one could perhaps hope to describe it up to this point. The work of Vatter [48] implies that this happens at or before $\xi \approx 2.32$, the unique positive root of $x^5 - 2x^4 - x^2 - x - 1$. Thus, just as $\kappa$ represents the transition from countably many to uncountably many permutation classes, $\xi$ may represent the transition from countably many to uncountably many growth rates.

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