Semiclassical limit of the scattering cross section as a distribution

E.L. Lakshtanov *

Abstract

We consider quantum scattering from a compactly supported potential \( q \). The semiclassical limit amounts to letting the wavenumber \( k \to \infty \) while rescaling the potential as \( k^2 q \) (alternatively, one can scale Planck’s constant \( \hbar \to 0 \)). It is well-known that, under appropriate conditions, for \( \omega \in S_{n-1} \) such that there is exactly one outgoing ray with direction \( \omega \) (in the sense of geometric optics), the differential scattering cross section \( |f(\omega, k)|^2 \) tends to the classical differential cross section \( |f_{cl}(\omega)|^2 \) as \( k \to \infty \). It is also clear that the same can not be true if there is more than one outgoing ray with direction \( \omega \) or for nonregular directions (including the forward direction \( \theta_0 \)). However, based on physical intuition, one could conjecture \( |f|^2 \to |f_{cl}|^2 + \sigma_{cl} \delta_{\theta_0} \) where \( |f_{cl}|^2 \) is the classical cross section and \( \delta_{\theta_0} \) is the Dirac measure supported at the forward direction \( \theta_0 \). The aim of this paper is to prove this conjecture.

Key words: wave scattering, high frequency limit, scattering amplitude, semiclassical approximation

1 Introduction

The semiclassical approximation in physics goes back to the work of Wentzel, Kramers and Brioullin (WKB) on the Schroedinger equation in 1926. A lot of mathematical work has been devoted to the subject and semiclassical scattering has grown into a discipline of its own. We sketch a simple setup which is central to our approach. We study the scattering of a quantum particle in \( \mathbb{R}^n \) at a localized potential \( q(x) \). The motion of the particle is governed by the Hamiltonian

\[
H = -\Delta + k^2 q(x)
\]

acting on wavefunctions \( \Psi(x) \) in \( L^2(\mathbb{R}^n) \).

One takes the initial momentum of the particle \( k \) to infinity which is compensated by scaling the potential as \( k^2 q \), so as to keep a balance between kinetic and potential energies. The basic intuition is that in this limit, the scattering problem is well approximated by a problem of Newtonian mechanics, namely, the scattering of a classical particle with momentum 1 at the potential \( q \). This was first made precise by Vainberg [1] who proved that for certain outgoing directions \( \omega \in S_{n-1} \),

\[
f(\omega, k) \approx \sum_j f_j(\omega)e^{i\theta_j k} + o(k^{-1}), \quad k \to \infty
\]
In this expression, \( f(\omega, k) \) is the quantum scattering amplitude at momentum \( k \). The index \( j \) labels different trajectories which yield the same outgoing direction \(-\omega\) for the classical scattering problem and the functions \( f_j(\omega) \) give the angular density of trajectories around each of those \( \omega \)-trajectories. This result has been refined by several authors, we mention [6, 7], [8, 9] and recently [1]. In [8], it was proven that under some quite general assumptions, the total scattering cross section

\[
\int_{S_{n-1}} d\mu(\omega) |f(\omega, k)|^2 \rightarrow 2\sigma_{cl} + o(k^0)
\]

where

\[
\sigma_{cl} = \int_{S_{n-1}} d\mu(\omega) |f_{cl}(\omega)|^2
\]

is the classical total cross section (eg [20, XI.2]).

The surprising factor of 2 is well known in the physical literature and we would like to comment on it. In fact, it appears because of a linguistic problem. Indeed, in [20] the classical total cross section is defined so as to measure the 'fraction' of the particles that interact with the scatterer. In quantum theory, the total cross section measures the defect between the field without scatterer (incident wave) and the field in the presence of the scatterer (function \( \Psi(x) \)). And these two definitions are not the same! In the shadow zone \( \Psi(x) \) vanishes for any fixed \( x \) as \( k \to \infty \) and mathematically this means that the defect between the wave without a scatterer and in the presence of the scatterer equals \(-1\) multiplied by the incident wave. So the contribution to the total cross section of the shadow zone equals the geometrical cross section of the support of the potential, which is \( \sigma_{cl} \) according to the classical definition. So, finally, if one defines the classical total cross section as the defect between densities of the free flow of particles and the flow of particles in the presence of the scatterer, then evidently, the total cross section also equals twice the geometrical cross section. In this article we will use the classical definition of the classical total cross section as it is in [20, XI.2], namely \( \sigma_{cl} \) equals the geometrical cross section of the support of the potential.

One of the aims of our article is to state and prove this property rigorously, namely: under the same assumptions (assumptions 2.1, 2.2) as those required for (1.3) the following property is valid (see lemma 3.2)

\[
\lim_{\delta \to 0} \lim_{k \to \infty} \int_{|\omega - \theta_0| < \delta} |f(\omega, k)|^2 dS = \sigma_{cl}.
\]

Our second aim is to join three facts (1.2), (1.3), (1.5) into one statement. Recall that the result (1.2) is only valid for certain directions \( \omega \in S_{n-1} \). The excluded directions are called nonregular, meaning that classical trajectories accumulate in those directions. The forward direction is always nonregular, (since all rays tangent to the boundary of the support of \( q \) have forward direction).

If all directions, other than the forward one, are regular, then obviously the forward peak has total intensity \( \sigma_{cl} \), and hence we might write, in the sense of distributions on \( S_{n-1} \),

\[
|f|^2 \rightarrow |f_{cl}|^2 + \sigma_{cl} \delta_{\theta_0}, \quad k \to \infty
\]

\[\text{In this book the classical total cross section is defined as a measure on } S^{n-1} \setminus \{\theta_0\}, \text{ that is, a sphere of directions without the forward direction. We understand by total cross section the full measure of } S^{n-1} \setminus \{\theta_0\}.\]
where $\delta_{\theta_0}$ is the Dirac distribution, centered at the forward direction. In case when there are two or more rays scattered into the same direction, the limit of $|f|^2$ does not exists. But if we consider $|f|^2$ as a measure on $\mathbb{S}^{n-1}$ and supposing that phases $\theta_j(\omega)$ in (1.2) are significantly not coincide (see assumption 2.3) then formula (1.6) is also valid due to the property of quickly oscillating measure to vanish in the limit. Particularly, it means that under our assumptions, impact of infinitesimally small neighborhoods of non regular directions into the total cross section, goes to zero, as $k$ goes to infinity.

We think that (1.6) is particularly interesting because it teaches us immediately that, in contrast to the total cross section, the transport cross section

$$
\int_{\mathbb{S}^{n-1}} d\mu(\omega) |f(\omega, k)|^2 (1 - \cos \omega)
$$

is equal to the classical expression (in the semiclassical limit. This is not at all obvious from the physics point of view, see e.g. [5, III.A] where one erroneously concludes that the transport cross section is also twice the classical value.

Our method of proof is based on the canonical Maslov operator, see e.g. [2].

In Section 2 we state the problem and result precisely. In Section 3 we present the proof.

## 2 Problem and result

Consider a potential $q \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$ (smooth functions with compact support). Choose a unit vector $\theta_0$ in $\mathbb{R}^n$ which is to be thought of as the direction of incoming particles. The projection of $x \in \mathbb{R}^n$ on this vector is denoted $x^n = <x, \theta_0>$ and we write $r := |x| = \sqrt{\sum_{i=1}^n (x_i)^2}$.

Define $\Psi(x, k)$ as the unique function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R})$ (see e.g. [12], [3, Add2., Col.2.1]) satisfying

1. The equation

$$
[\Delta_x + k^2 + k^2 q(x)]\Psi(x, k) = 0, \quad x \in \mathbb{R}^n,
$$

2. The radiation condition

$$
u(x, k) := \Psi(x, k) - e^{ikx^n} = f(\omega, k) r^{(1-n)/2} e^{ikr} (1 + O(r^{-1})), \quad r \to \infty, \quad \omega = \frac{x}{r} \in \mathbb{S}_{n-1}.
$$

for some function $f(\omega, k)$.

The function $f(\omega, k)$, commonly called the scattering amplitude, is uniquely determined by the potential $q$ and our results will concern its asymptotics as $k \uparrow \infty$. To formulate our assumptions, we introduce more notation.

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2For hard strictly convex obstacles, the formula (1.6) was remarked in [17], relying on [11].
2.1 Assumptions

Consider the (classical) Hamiltonian $H(x,p) = |p|^2 - q(x)$, $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and corresponding dynamical system

\[
\frac{dx}{ds} = 2ps, \quad \frac{dp}{ds} = \nabla q(x); \quad x_0 = (y,-a), \quad p_0 = \theta_0,
\]

with $y \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}^+$ such that $\sup_{x \in \text{Supp} q} (a + x^n) > 0$ where $\text{Supp} q \subset \mathbb{R}^n$ denotes the support of $q$. A solution $t \mapsto (x_t, p_t) = (x_t(x_0, p_0), p_t(x_0, p_0))$ of (2.3) is called a bicharacteristic and its projection to $\mathbb{R}^n_+$ (i.e. $(x,p) \mapsto x$) is a ray. Our first assumption expresses that the Hamiltonian system (2.3) satisfies a non-trapping condition, i.e.

**Assumption 2.1.** For any $c < \infty$, there is a $T$ such that for $s > T$ the rays of (2.3) with any $y \in \mathbb{R}^{n-1}$ are contained in the region $|x| > c$.

We denote by $\mathcal{I}$ the projection of $\text{Supp} q$ on the hyperplane $x^n = -a$. In accordance with Assumption 2.1, a ray of (2.3) with initial data $(y,-a)$ will reduce in finite time to a line, whose direction is characterized by the momentum $p_\infty((y,-a), p_0) = \lim_{t \to \infty} p_t((y,-a), p_0) \in \mathbb{S}_{n-1}$, since $|p_\infty(y,-a)| = |\theta_0| = 1$ by energy conservation. This defines a map

\[
J : \mathcal{I} \mapsto \mathbb{S}_{n-1} \subset \mathbb{R}^n \quad y \mapsto J(y) = p_\infty((y,-a), p_0).
\]

By $|\frac{DJ(y)}{Dy}|$, we denote the absolute value of Jacobian determinant of $J$. We call $\omega \in \mathbb{S}_{n-1}$ a regular direction iff. $J(y) = \omega$ implies $|\frac{DJ(y)}{Dy}| \neq 0$. Otherwise, we call $\omega$ nonregular.

**Assumption 2.2.** The set $\{ y \in \mathcal{I}, |\frac{DJ(y)}{Dy}| = 0 \}$ has measure zero and

\[
y is in the interior of $\mathcal{I}$ \quad y \in J^{-1}(\theta_0) \quad \Rightarrow \quad \left| \frac{DJ(y)}{Dy} \right| \neq 0 \quad (2.5)
\]

We denote by $\Lambda^n \subset \mathbb{R}^n_+ \times \mathbb{R}^n_p$ the Lagrangian manifold constructed as

\[
\Lambda^n = \bigcup_{\mathcal{I}} (x_\mathcal{I}((y,-a), \theta_0), p_\mathcal{I}((y,-a), \theta_0)) \quad (2.6)
\]

As global coordinates on $\Lambda^n$, one can choose $(y,t)$ with $y \in \mathbb{R}^{n-1}$. By solving (2.3), we obtain a function $S = S(x,p) \in C^\infty(\Lambda^n)$:

\[
S(x,p) = -a + \int_L p \, dx > , \quad (2.7)
\]

where $L$ - is the segment of a unique bicharacteristic in $\Lambda^n$ between the points $((y,-a), \theta_0)$ and $x, p$ for some $y \in \mathbb{R}^{n-1}$. Consider a regular direction $\omega_0$. We can find points (see lemma 3.4 below or Lemma 11 lemma 1) $y_1, \ldots, y_r \in \mathcal{I}$ with neighborhoods $\mathcal{M}_i$ such that $J(y_i) = \omega_0$ and $J$ is a diffeomorphism on $\mathcal{M}_i$. Hence on $J(\mathcal{M}_i)$, we can define the following map

\[
J(\mathcal{M}_i) \mapsto \mathbb{R} : \omega \mapsto F_i(\omega) := S(x,p = \omega) - \langle \omega, x \rangle, \quad (2.8)
\]

where $(x, p = \omega)$ is a point on the bicharacteristic starting from $\mathcal{M}_i$ and with $|x| > a$. Indeed, for $|x| > a$, the expression (2.8) is independent of $x$ since $\nabla_x S(x,p) = \omega$.

The next assumption should ensure there are not “too much” interference effects

**Assumption 2.3.** For any regular value $\omega_0$, the set of critical values of the functions $F_i - F_j$ on $J(\mathcal{M}_i) \cap J(\mathcal{M}_j)$ has measure zero.
2.2 Result

The classical differential cross section of the dynamical system (2.3), which we denote by \(|f_c|^2\), can be defined as a distribution on \(\mathbb{S}_{n-1}\) by the formula

\[
\int_{\mathbb{S}_{n-1}} \varphi(\omega)|f_c|^2(\omega)\,d\mu(\omega) = \int_{\mathcal{I}} \varphi(J(y))\,dy, \quad \varphi \in C^\infty(\mathbb{S}_{n-1}).
\]  

(2.9)

Note that we denote the Lebesgue measure on \(\mathbb{S}_{n-1}\) by \(d\mu(\cdot)\). By Assumption 2.2, \(|f_c|^2\) is actually a regular distribution (hence a function), which is known explicitly, see below in 3.3.

We will also need the classical total cross section

\[
\sigma_c := \int_{\mathbb{S}_{n-1}} \varphi(\omega)|f_c|^2(\omega)\,d\mu(\omega).
\]

(2.10)

From (2.9), it follows that \(\sigma_c = \text{meas}(\mathcal{I})\) (the Lebesgue measure of \(\mathcal{I}\) in \(\mathbb{R}^{n-1}\)). Our result reads

**Theorem 2.4.** Let the potential \(q(x)\) satisfy Assumptions 2.1, 2.2 and 2.3. Then we have, for all \(\varphi \in C(\mathbb{S}_{n-1})\),

\[
\int_{\mathbb{S}_{n-1}} \varphi(\omega)|f_c|^2(\omega)\,d\mu(\omega) \rightarrow \int_{\mathbb{S}_{n-1}} \varphi(\omega)|f_c|^2(\omega)\varphi(\omega) + \sigma_c\varphi(\theta_0), \quad k \rightarrow \infty,
\]

(2.11)

An announcement of this result was published in [19].

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3 Proof

3.1 Proof of Theorem 2.4

The proof of Theorem 2.4 goes through two lemmas, whose proofs are postponed to the next sections.

**Lemma 3.1.** Assume Assumptions 2.1 and 2.2 then

\[
\sigma = 2\sigma_c + o(k^0), \quad k \rightarrow \infty.
\]

(3.1)

**Lemma 3.2.** Assume Assumptions 2.1 and 2.2 then

\[
\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \int_{||\omega-\theta_0||<\delta} |f(\omega; k)|^2\,dS = \sigma_c.
\]

(3.2)

For a regular direction \(\omega\), we introduce an index \(j\) which labels the elements in \(J^{-1}(\omega)\). Remark that \(J^{-1}(\omega)\) is a finite set for regular directions \(\omega\) since \(\mathcal{I}\) is compact and \(J\) is continuous and therefore pre-images can not be concentrated near the caustic sets. However, the cardinality of \(J^{-1}(\omega)\) can change. We put

\[
|f_c|^2(\omega) := \sum_{j \in J^{-1}(\omega)} |f_j(\omega)|^2, \quad f_j(\omega) := \left|\frac{DJ(y_j)}{Dy_j}\right|.
\]

(3.3)
Pick a test function $\phi$ on $S_{n-1}$ and choose $\epsilon > 0$. Let $U_1(\epsilon), U_2(\epsilon), U_3(\epsilon)$ be neighborhoods of respectively 1) $\theta_0$, 2) the nonregular directions with $\theta_0$ excluded and 3) the regular directions $\omega$ which are critical points of $F_j - F_j'$ on $J(\mathcal{M}_j) \cap J(\mathcal{M}_{j'})$ (see Assumption 2.3). 3

Choose the neighborhoods such that $\text{meas}(U_{1,2,3}(\epsilon)) \leq \epsilon$. We have to prove that

$$
\int d\mu(\omega)\phi(\omega)|f(\omega, k)|^2 = \int \prod_{i=1}^{n} d\mu(\omega)\phi(\omega)|f(\omega, k)|^2 + \int_{S_{n-1} \setminus \prod_{i=1}^{n} U(\epsilon)} d\mu(\omega)\phi(\omega)|f(\omega, k)|^2
$$

By Theorem 2, the fact that $F_2(\omega) - F_1(\omega)$ has no critical points and (3.3), the pointwise limit of the integrand $|f(\omega, k)|^2$ in the second term in (3.4) gives $|f_{cl}|^2(\omega)$.

Combining now Lemma’s 3.1 and 3.2, one ends the proof.

### 3.2 Preliminaries

For any $\omega \in S_{n-1}$, we have the representation (see [1]):

$$f(\omega, k) = \gamma_n \int_{RS_{n-1}} \left[ \frac{\partial u}{\partial r} + ik \left( \frac{\omega}{r} \right) u \right] e^{-ik < x, x >} d\mu(x),$$

where $u = u(x, k)$ was defined in (3.2), $RS_{n-1} \subset \mathbb{R}^n$ is the sphere of radius R and

$$\gamma_n = \gamma_n(\kappa) = -\frac{1}{4\pi} \left( \frac{k}{2\pi i} \right)^{(n-3)/2}.$$ 3

Recall the optical theorem (which could be easy derived using Green formula from (3.16)):

$$\text{Im} f(\theta_0, k) = -\gamma_n k \sigma, \quad \forall \ k \geq 0.$$ 3

We will need the canonical Maslow operator, acting from $C^\infty(\Lambda^n)$ to $C^\infty(\mathbb{R}^n)$. We follow the conventions introduced in [1].

#### 3.2.1 The canonical Maslow operator

If the manifold $\Lambda^n$ can be equipped with the chart $\mathbb{R}^n_x$, i.e. if $x \mapsto (x, p = p(x))$ is a diffeomorphism from $\mathbb{R}^n$ to $\Lambda^n$, then we can define the canonical Maslow operator $K_{\Lambda^n} : C^\infty(\Lambda^n) \to C^\infty(\mathbb{R}^n)$ as

$$K_{\Lambda^n}[\varphi] = I^{-1/2} \varphi \exp(ikS)|_{p = p(x)}, \quad I = \frac{1}{2} \left| \frac{D(x)}{D(y, s)} \right|, \quad \varphi \in C^\infty(\Lambda^n),$$

where $(y, s)$ are global coordinates on $\Lambda^n$, introduced in Section 2.1.

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3 Note that the index range $j$ and the sets $\mathcal{M}_j$ in general depend on $\omega$. However, locally the functions $F_j$ are well-defined. By Assumption 2.3, the set of $\omega$ which are critical points of some $F_j - F_j'$, has measure zero as a countable union of sets of measure zero.
It is not always possible to choose \( \mathbb{R}^n_x \) as global coordinates since rays can cross. However, we can fix a locally finite covering \((\Omega_j)\) of \(\Lambda^n\) such that for each \(j\), the manifold \(\Omega_j\) projects homeomorphically on the subset of cartesian product of a \(l\)-dimensional subspace of \(\mathbb{R}_x^n\) and a \((n-l)\)-dimensional subspace of \(\mathbb{R}_p^n\). The coordinates in the respective spaces are denoted as \(x_\alpha, \ldots, x_\alpha\) and \(p_\beta, \ldots, p_\beta\) (\(l = l_j\) here). Hence the coordinates in the chart corresponding to \(\Omega_j\) are \((x_\alpha, p_\beta)\) and the function

\[
I_j = \frac{1}{2} \left| \frac{D(x_\alpha, p_\beta)}{D(y, s)} \right|
\]

is bounded away from 0. Of course, the functions \(x_\beta = x_\beta(x_\alpha, p_\beta), p_\alpha = p_\alpha(x_\alpha, p_\beta)\) can still be defined.

Let \(\{e_j\}\) be a resolution of unity on \(\Lambda^n\) such that \(e_j \in C_0^\infty(\Omega_j)\), and let \(\{g_j \in C^\infty(\mathbb{R}_x^n)\}\) be such that \(g_j(x) = 1\) in a neighborhood of \(\Omega_j, x\), the projection of \(\Omega_j\) on \(\mathbb{R}_x^n\), and such that each \(x \in \mathbb{R}_x^n\) belongs to Supp \(g_j\) at most a finite number of \(j\). The points in \(\Lambda^n\) for which no neighborhood projects diffeomorphically on \(\mathbb{R}_x^n\), are called \textit{singular}. The projection on \(\mathbb{R}_x^n\) of the singular points is called the \textit{caustic set}.

We now define for each \(j\) operator \(K_{\Omega_j}: C^\infty(\Omega_j) \to C^\infty(\Omega_j, x)\) as

\[
K_{\Omega_j}[\varphi] = \left( \frac{k}{-2\pi i} \right)^{\frac{|\beta|}{2}} g_j \int_{\Omega_j, p_\beta} e_j e^{ik[G_j(p_\beta) - i\frac{\pi}{2} \nu_j] I_j^{-1/2}} \varphi dp_\beta,
\]

(3.10)

where the function \(G_j(p_\beta)\) is defined by

\[
G_j(x, p_\beta) = S(x(x_\alpha, p_\beta), p(x_\alpha, p_\beta)) - <x_\beta(x_\alpha, p_\beta), p_\beta> + <x_\beta, p_\beta>,
\]

(3.11)

\(\Omega_j, p_\beta\) is the projection of \(\Omega_j\) on \(\mathbb{R}_p^n\) and \(\nu_j\) are the Morse-Maslow-Keller indices (see details in [1]). In [1], it is shown that there exists a sequence \(\eta_{j, m} \in C^\infty(\Lambda^n), m = 0, \ldots, \infty, \eta_{j, 0} \equiv 1\) such that

\[
\Psi_N(k, x) = \sum_j \Psi_{N,j} \quad \Psi_{N,j} = K_{\Omega_j} \left[ \sum_{m=0}^N (ik)^{-m} \eta_{j, m} \right].
\]

(3.12)

is an approximative solution of (2.1): for a compact \(V \subset \mathbb{R}_x^n\), one has

\[
|D_x^\nu[\Psi(k, x) - \Psi_N(k, x)]| < C(V, N, \nu)k^{-N-1+|\nu|+n/2}, \quad k > 1,
\]

(3.13)

If \(V\) does not contain points of the caustic set, than \(n/2\) can be omitted in the RHS of (3.13).

Using (3.13), one can prove the following celebrated Theorem

**Theorem 3.3.** [Vainberg] Let \(q(x)\) satisfy Assumption 2.1 and let \(\omega_0\) be a regular direction, then, for \(\omega\) a certain neighborhood of \(\omega_0\)

\[
f(\omega, k) = \sum_{j \in J^{-1}(\omega)} \left| \frac{D J(y_j)}{D y_j} \right|^{-1/2} e^{ik(F_j(\omega) - \frac{\pi}{2} \nu_j)} + O(k^{-1}),
\]

(3.14)

where \(\nu_j\) are the Morse-Maslow-Keller indices and the points \(y_j\) make up \(J^{-1}(\omega)\). The functions \(J, F\) were defined earlier.
The following lemma is an evident generalization of Theorem 3.3 which will be used in our proofs.

**Lemma 3.4.** Let \( y_1, \ldots, y_v \) be points in the interior of \( I \), such that \( \frac{\partial J(y_j)}{\partial y_j} \neq 0 \) and \( J(y_j) = \omega_0 \) for some \( \omega_0 \in S_{n-1} \) (possibly nonregular). Then

1) There are neighborhoods \( \mathcal{M}_j \) of \( y_j \) and \( R > 0 \) such that \( J \) is a diffeomorphism from \( \mathcal{M}_j \) to \( J(\mathcal{M}_j) \) and such that on the bicharacteristics starting from \( \mathcal{M}_j \) and with \( |x| > R \), the functions

\[
\mathcal{M}_j \times \mathbb{R}^+ \to \mathbb{R}^+ : (y_j, s) \mapsto \left| \frac{D(x)}{D(y_j, s)} \right|
\]  

exist and are bounded away from zero.

2) Define

\[
f_j(\omega, k) := \gamma_n \int_{\mathbb{R}^n} \left[ \frac{\partial \Psi_j}{\partial r} + ik \left( \omega, \frac{x}{r} \right) \Psi_j \right] e^{-ik<\omega,x>} d\mu(x), \quad \Psi_j(x) = \sum_{n \geq 0} \Psi_{n,j}(x).
\]

For \( \omega \) in a certain neighborhood of \( \omega_0 \)

\[
f_j(\omega, k) = \left| \frac{DJ(y_j)}{Dy_j} \right|^{-1/2} e^{ikF_j(\omega)(\omega)} + O(k^{-1}),
\]

Statement (1) is an easy analogue of Lemma [1, Lemma 1]. The only difference is that, where Vainberg assumes \( \omega_0 \) to be regular, we simply cut out some bicharacteristics to make the direction \( \omega_0 \) regular. Statement (2) follows from (1) in the same way that Theorem 3.3 follows from Lemma [1, Lemma 1]. When \( \omega_0 \) is a regular direction, Lemma 3.4 reduces to Theorem 3.3. In Theorem 3.3 there is however no need of introducing the functions \( f_j \).

### 3.3 Proof of Lemma 3.1

Let \( h_R \) be a \( C^\infty \) function on \( \mathbb{R} \) with support contained in the interval \([R, R+1]\) and \( \int_{\mathbb{R}} h_R = 1 \).

We will estimate

\[
f(\theta_0) = \gamma_n \int_{\mathbb{R}^n} h_R \left[ \frac{\partial u}{\partial r} + ik \left( \theta_0, \frac{x}{r} \right) u \right] e^{-ik<\theta_0,x>} dx,
\]

for a certain \( R > a \).

Let \( J^{-1}(\theta_0) = \{y_1, \ldots, y_v\} \) and recall (Assumption 2.2) that \( J^{-1}(\theta_0) \) lies in the interior of \( I \). We choose the neighborhoods \( \mathcal{M}_j \subset I, y_j \in \mathcal{M}_j, j = 1, \ldots, v \) and \( R \) such as in Lemma 3.4. We now fix the covering \( \Omega_{j \in \mathbb{N}} \), as required in Section 3.2.1.

Let for \( j = 1, \ldots, v \), \( \{\Omega_j\} \) be the parts of \( \Lambda^n \) defined by \( y \in \mathcal{M}_j, |x| > R \). Let \( \Omega^n_0 \) be the part of \( \Lambda^n \) which contains all bicharacteristics with initial data outside \( I \) and let \( \Omega_0 \) be a neighbourhood of \( \Omega_0^n \), such that \( \Omega_0 \) does not contain singular points. This is possible, since the Jacobian \( I(x, p) \) is a smooth function and it equals one for points from \( \Omega_0^n \), so there exists a neighborhood where \( I(x, p) \) is not equal to zero. Hence \( \beta = 0 \) for \( j = 0, 1, \ldots, v \). The rest
of the covering is chosen arbitrarily, but the functions $g_j$ are chosen such that for $|x| > R$, the rays starting from $y_j, j \leq v$ are not in $\text{Supp}\, g_j, j > v$. This is possible by Lemma 3.4.

Set $\Omega'_0 := \Omega_0 \setminus \Omega''_0$ and let $\tilde{\Omega}', \tilde{\Omega}''$ be the projections on $\mathbb{R}_x^n$ of resp. $\Omega'_0, \Omega''_0$.

Using $\Psi = \sum_j \Psi_j$, we split the integral \textbf{(3.18)} (changing $\langle \theta, \frac{x}{r} \rangle$ into $\frac{x}{r}$)

\[
\gamma_n^{-1} f(\theta_0) = \left( \sum_{j \geq 0} \int_{\mathbb{R}^n} h_R \left[ \frac{\partial \Psi_j}{\partial r} + ik \frac{x_n}{r} \Psi_j \right] e^{-ikx_n} \, dx \right) \tag{3.19}
\]

\[
- \int_{\mathbb{R}^n} h_R \left[ \frac{\partial e^{ikx_n}}{\partial r} + ik \frac{x_n}{r} e^{ikx_n} \right] e^{-ikx_n} \, dx \tag{3.20}
\]

\[
\int_{\tilde{\Omega}''} h_R \left[ \frac{\partial \Psi_0}{\partial r} + ik \frac{x_n}{r} \Psi_0 \right] e^{-ikx_n} \, dx - \int_{\tilde{\Omega}''} h_R \left[ \frac{\partial e^{ikx_n}}{\partial r} + ik \frac{x_n}{r} e^{ikx_n} \right] e^{-ikx_n} \, dx \tag{3.21}
\]

\[
\int_{\tilde{\Omega}''} h_R \left[ \frac{\partial \Psi_0}{\partial r} + ik \frac{x_n}{r} \Psi_0 \right] e^{-ikx_n} \, dx + \left( \sum_{j>0} \int_{\mathbb{R}^n} h_R \left[ \frac{\partial \Psi_j}{\partial r} + ik \frac{x_n}{r} \Psi_j \right] e^{-ikx_n} \, dx \right) \tag{3.22}
\]

\[
- \int_{\mathbb{R}^n \setminus \tilde{\Omega}''} h_R \left[ \frac{\partial e^{ikx_n}}{\partial r} + ik \frac{x_n}{r} e^{ikx_n} \right] e^{-ikx_n} \, dx \tag{3.23}
\]

A first observation is that by application of \textbf{(3.13)} with $N = 0$, the sum of both expressions in \textbf{(3.22)} is of order $-1, k \to \infty$, since $\beta = 0$ for $\Omega_0$ and $\Psi_{0,0}|_{\hat{\Omega}''} = e^{ikx_n}$ (see beginning of \textbf{3.21}).

The term \textbf{(3.21)} is easily seen to give $-2\gamma_n \text{meas}(\mathcal{I})k$. Hence, we are left with the two terms of \textbf{3.22}. By using \textbf{3.13}, these terms are recast in the form

\[
k \int_{\Omega_0} h_R \eta_0,0 e^{ik(S(x) - x_n)} \, dx + O(1) \tag{3.25}
\]

\[
+ k \sum_{j>0} \sum_{m=0}^{\ell_j} k(|\beta| + n - 1)/2 - m \int_{\Omega_0} h_R \eta_{j,m} e^{ik(G_j(x,p_j) - x_n)} \, dp_\beta \, dx, \quad k \uparrow \infty \tag{3.26}
\]

where $\eta_{j,m} \in C^\infty(\bar{\Omega}_{j,x} \times \bar{\Omega}_{j,p})$ and the index $\ell_j$ is high enough so as to make the exponent in the error term or order $O(k^0)$ (since the error term comes from the estimate in \textbf{3.13}, which can be made arbitrarily small by increasing $N$ and hence $\ell_j$.) To show that the term in \textbf{3.25} has order $O(k)$ and the term \textbf{3.26} has order $O(k^0)$, it suffices to note that the critical points of the exponent $S(x,p) - x_n$ have measure zero, and critical points of the exponents $G_j(x,p_\beta) - x_n$ have isolated critical points only. This is shown now.

Using \textbf{(2.7)}, one calculates

\[
d(S(x,p = p(x)) - x_n) = \langle p, dx \rangle = -\langle \theta, dx \rangle \tag{3.27}
\]

which shows that $S(x) - x_n$ has critical points only at the boundary of $\Omega'_0$, i.e. for $p = \theta_0$. For $j \leq v$, the function $G_j(x,p_\beta) - x_n$ equals $S(x,p) - x_n$ (since $\beta = 0$) and the critical points $p = \theta_0$ are isolated points in $\Omega_j$. The terms in \textbf{3.26} could be calculated explicitly through the stationary phase method, moreover their leading asymptotics are given by theorems \textbf{3.3}.

Note that Vainberg showed in \textbf{[1]} that the determinant of the Hessian of $G_j(x,p_\beta) - x_n$ in
the isolated critical points equals \( r^{n-1} \left| \frac{D^2 y_j}{D x^j} \right| + O(r^{n-2}) \) which is not equal to zero according to Assumption 2.2.

For \( j > v \) we calculate (for details we refer to \[1\])

\[
d(G_j(x, p_\beta) - x^n) = < p, dx > - < \theta, dx > + < x_\beta - x_\beta(x_\alpha, p_\beta) >
\]

and we find that critical points must again satisfy\( \hat{O} \ST \). Note that one can continue the expansion up to \( \ell \).

The functions \( g_j \) and the functions \( h_j \) of Supp \( h_j \) are defined in Supp \( h_j \). For future use in the proof of Lemma 3.2, we note that one can continue the expansion up to \( \ell_j + 1 \) to conclude that for \( j > v \), the last term in \[3.23\] is of order \( O(k^{-\infty}) \).

3.4 Proof of Lemma 3.2

Choose the covering \( \{ \Omega_j \} \) and \( R > 0 \) as defined in the previous section with the additional constraint about the \( \hat{O} \) that measure of the set \( \{ (y, -a) \in \mathcal{I} : \exists s : (y, s) \in \hat{O} \} \) is smaller than \( \delta > 0 \). This is possible due to the assumption 2.2.

\[
\gamma_n f(\omega) = \int_{\hat{O}_0} h_R \left[ \frac{\partial \Psi_0}{\partial r} + ik \left< \omega, \frac{x}{r} \right> \Psi_0 \right] e^{-ik(\omega,x)} dx
\]

\[
\int_{\hat{O}_0} h_R \left[ \frac{\partial e^{ikx^n}}{\partial r} + ik \left< \omega, \frac{x}{r} \right> e^{ikx^n} \right] e^{-ik(\omega,x)} dx
\]

\[
= \left( \sum_{j>0} \int_{\mathbb{R}^n} h_R \left[ \frac{\partial \Psi_j}{\partial r} + ik \left< \omega, \frac{x}{r} \right> \Psi_j \right] e^{-ik(\omega,x)} dx \right)
\]

\[
= \left( \sum_{j>0} \int_{\mathbb{R}^n} h_R \left[ \frac{\partial e^{ikx^n}}{\partial r} + ik \left< \omega, \frac{x}{r} \right> e^{ikx^n} \right] e^{-ik(\omega,x)} dx \right)
\]

As in the proof of Lemma 3.1, the sum of terms \(3.29\) and \(3.30\) is dominated by a constant, independent of \( \omega \), and hence these terms vanish upon integration over a small neighbourhood of \( \omega = \theta_0 \). Omitting \(3.29\) and \(3.30\), the above representation defines the functions \( \{ f_j(\omega) \}, f^a(\omega), f^b(\omega) \) corresponding to respectively \(3.31\), \(3.32\) and \(3.33\) such that

\[
f(\omega) = \sum_{j>0} f_j(\omega) + f^a(\omega) + f^b(\omega).
\]

Call \( U(\delta) = \{ \omega \in \mathbb{S}_{n-1}, \| \omega - \theta_0 \| \leq \delta \} \). We need to prove that

\[
\lim_{k \to \infty} \int_{U(\delta)} |f(\omega)|^2 dS(\omega) = \sigma_0 + o(\delta^0)
\]

Since each \( x \in RS_{n-1} \subset R_x \) belongs to Supp \( g_j \) for a finite number of \( j \) and \( RS_{n-1} \) is compact, only a finite number of terms are nonzero in \(3.34\). We will show that

\[
\lim_{k \to \infty} \int_{U(\delta)} |f^b(\omega)|^2 d\mu(\omega) = \sigma_0 + o(1), \quad \delta \to 0
\]

\[
\lim_{k \to \infty} \int_{U(\delta)} |f^j(\omega)|^2 d\mu(\omega) = o(1), \quad \delta \to 0, \quad j > 0,
\]

\[
\lim_{k \to \infty} \int_{U(\delta)} |f^a(\omega)|^2 d\mu(\omega) = o(1), \quad \delta \to 0
\]
From which \((3.35)\) will follow by the Cauchy-Schwarz inequality.

3.4.1 Proof of \((3.37)\)

If \(\delta\) is small enough, the set \(\Omega_j, j > p\) does not contain bicharacteristics originating from \(U(\delta)\). The claim then follows by the remark at the end of \((3.3)\). For \(0 < j \leq p\), Lemma \(3.4\) yields \(|f_j(\omega)| < O(k^0)\).

3.4.2 Proof of \((3.38)\)

By applying \((3.13)\) (with \(\beta = 0\)) one obtains

\[
f^\alpha(\omega) = \gamma_n(k) \int_{\tilde{\Omega}_0} h_R(x) \tilde{I}(x) e^{-ik\omega, x} dx + o(k^0) \tag{3.39}
\]

for \(\eta_{0,m} \in C^\infty(\Omega_0), \eta_{0,0} \equiv 1\). Here \(\tilde{I}(x)\) is

\[
\tilde{I}(x) = \left( \frac{d}{dr} + ik \omega, \frac{x}{r} \right) \left[ I^{-1/2}(x) e^{ikS(x)} \left( \sum_{m=0}^{1} k^{-m} \eta_{0,m} \right)(x) \right]
\]

\[
e^{ikS(x)} \left[ i k I^{-1/2}(x) \left( \frac{ds(x)}{dr} + \langle \omega, \frac{x}{r} \rangle \right) + \varphi(x) \right] = e^{ikS(x)} \left[ i k I^{-1/2}(x) \left( \langle p(x), \frac{x}{r} \rangle + \langle \omega, \frac{x}{r} \rangle \right) + \varphi(x) \right].
\]

Here

\[
\varphi(x) = \frac{d}{dr} \left( \sum_{m=0}^{1} k^{-m} \eta_{0,m} \right)(x) + \frac{1}{k} I^{-1/2}(x) \frac{d\eta_{0,1}}{dr}(x)
\]

finally

\[
f^\alpha(\omega) = \gamma_n(k) \int_{\tilde{\Omega}_0} h_R(x) \left[ ik I^{-1/2}(x) \left( p(x) + \omega, \frac{x}{r} \right) + \varphi(x) \right] e^{ik(S(x) - \omega, x)} dx + o(k^0) \tag{3.40}
\]

Now

\[
\int_{U(\delta)} |f^\alpha(\omega)|^2 d\mu(\omega) = \int_{\tilde{\Omega}_0} h_R(w) e^{-ikS(w)} t(w) dw + o(k^0) \tag{3.41}
\]

where for \(w \in \tilde{\Omega}_0\)

\[
t(w) := (\gamma_n k)^2 \int_{\tilde{\Omega}_0} h_R(x) \left[ I^{-1/2}(w) \left( p(w) + \omega, \frac{w}{r} \right) - (i/k) \varphi(w) \right]
\]

\[
\left[ I^{-1/2}(x) < p(x) + \omega, \frac{x}{r} > -(i/k) \varphi(x) \right] \int_{U(\delta)} e^{ik[S(x) - \omega, w]} dx d\mu(\omega). \tag{3.42}
\]

We can change the order in the integral due to integrability of the density \(I^{-1}\) on \(\tilde{\Omega}_0\) (this value is bounded by \(\delta\) due to the choice of \(\tilde{\Omega}_0\) in the beginning of \(3.4\)).

We write \(x = rz\) where \(r = |x|\) and \(z \in S_{n-1}\) and we perform the integration over the \(z\) and \(\omega\) coordinates. Call \(\tilde{S} = \tilde{S}_{w,r}(z, \omega)\) the restriction of \(S(x) - \omega, x >\) to \((z, \omega) \in S_{n-1} \times S_{n-1}\). Since

\[
d(S(x) - \omega, x >) = (dx, p - \omega) - (x - w, d\mu(\omega)) \tag{3.43}
\]
one sees that $d\tilde{S}$ vanishes whenever both $p - \omega$ and $x - w$ are orthogonal to the tangent plane of $S_{n-1}$ in resp. $z$ and $\omega$, leading to the critical point $z^* = z^*(w, r), \omega^* = \omega^*(w, r)$ where $z^* = \frac{\omega}{r} \text{ and } \omega^* = p(w)$. By choosing $\delta$ small enough, this is the unique critical point.

The Hessian matrix of $\tilde{S}$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is a block matrix of the form

$$\text{Hess } \tilde{S} = \begin{pmatrix} A & 1 \\ 1 & 0 \end{pmatrix}$$

(3.45)

where $A$ is the restriction of $\frac{\partial^2 \tilde{S}}{\partial x^2}$ to $|x| = r$ and 1 is the identity matrix. Since all blocks in that matrix commute, it is easy to see that this matrix has determinant $-1$ and hence the unique critical point is nondegenerate. ($\tilde{S}$ is a Morse function)

Since $\tilde{S}$ is a Morse function with Hessian uniformly bounded from zero for all values of the parameters $\Sigma, r \in [R, R + 1], w \in \hat{\Omega}_0$, there exists a smooth change of variables $(\tilde{z}, \tilde{\omega}) = (\tilde{z}_{r,w}(z, \omega), \tilde{\omega}_{r,w}(z, \omega))$ ([15] v. 1 8.2, 8.3, 8.5) which transforms $\tilde{S}$ to a pure quadratic form in the neighborhood $\Sigma = \Sigma_{r,w}$ of the critical point $(z^*, \omega^*)$. Since that map is smooth, and $r, w$ vary over a bounded set, one can bound

$$\left| \frac{D(z, \omega)}{D(\tilde{z}_{r,w}(z, \omega), \tilde{\omega}_{r,w}(z, \omega))} \right| \leq C, \quad (z, \omega) \in \Sigma, r \in [R, R + 1], w \in \hat{\Omega}_0. \quad (3.46)$$

Applying stationary phase method to the expression (3.42) and using that $w = rz^*(w, r)$, we get

$$t(w) = (2\gamma_n k)^2 I^{-1} \left( w) h_R(w) \right)$$

$$\int_R^{R+1} \int_R^R e^{i(c(S(w) + \frac{2k}{k} Sgn H(w, r)) + O(\frac{1}{k}))} \quad (3.47)$$

where $Sgn$ stands for ($\frac{n}{2}$ positive eigenvalues - $\frac{n}{2}$ negative eigenvalues). The term $O(1/k)$ in the last (3.47) is bounded uniformly on $w \in \hat{\Omega}_0$ due to (3.46) and absence of the dependence of the phase on the parameter $r$ and $w$.

Since $(2\gamma_n k)^2 (2\pi/k)^n - 1 = 1$ and $|\text{Hess } \tilde{S}_{w,r}(z^*, \omega^*)| = 1$, we have that (3.47) equals

$$\int_R^{R+1} \int_R^R e^{i(c(S(w) + \frac{2k}{k} Sgn H(w, r)) + O(\frac{1}{k}))} \quad (3.48)$$

Plugging this into (3.44) and using $\eta_{t,0} \equiv 1$ (see (3.2.1)), we get

$$\int_U^\delta |f^t(\omega)|^2 d\mu(\omega) \leq \int \int_R^{R+1} \int_R^R (h_R(w))^2 I^{-1}(w) dy + o(k^0) \leq \delta \left( \max_{[R, R + 1]} h^2(r) \right) + o(k^0) \quad (3.49)$$

Now, (3.38) is proved since $\delta$ could be chosen arbitrary small.

The proof of the statement (3.36) is a straightforward application of the stationary phase method.

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$^4$see also [13] 4.8 where are families of functions and germs deformations are connected
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