Nodal solutions of second-order two-point boundary value problems

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Abstract

We shall study the existence and multiplicity of nodal solutions of the nonlinear second-order two-point boundary value problems,

\[ u'' + f(t, u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0. \]

The proof of our main results is based upon bifurcation techniques.

Mathematics Subject Classifications: 34B07; 34C10; 34C23.

Keywords: nodal solutions, bifurcation

1 Introduction

In [1], Ma and Thompson were considered with determining interval of \( \mu \), in which there exist nodal solutions for the boundary value problem (BVP)

\[ u''(t) + \mu w(t)f(u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0 \]  \hspace{1cm} (1.1)

under the assumptions:

(C1) \( w(\cdot) \in C([0, 1], [0, \infty)) \) and does not vanish identically on any subinterval of \([0, 1] \);

(C2) \( f \in C(\mathbb{R}, \mathbb{R}) \) with \( sf(s) > 0 \) for \( s \neq 0 \);

(C3) there exist \( f_0, f_\infty \in (0, \infty) \) such that

\[ f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s| \to \infty} \frac{f(s)}{s}. \]

It is well known that under (C1) assumption, the eigenvalue problem

\[ \varphi''(t) + \mu w(t)\varphi(t) = 0, \quad t \in (0, 1), \quad \varphi(0) = \varphi(1) = 0 \]  \hspace{1cm} (1.2)

has a countable number of simple eigenvalues \( \mu_k, k = 1, 2, \ldots \), which satisfy

\[ 0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots, \quad \text{and} \lim_{k \to \infty} \mu_k = \infty, \]

and let \( \mu_k \) be the \( k \)th eigenvalue of (1.2) and \( \phi_k \) be an eigenfunction corresponding to \( \mu_k \), then \( \phi_k \) has exactly \( k - 1 \) simple zeros in \((0,1)\) (see, e.g., [2]).

Using Rabinowitz bifurcation theorem, they established the following interesting results:

**Theorem A** (Ma and Thompson [[1], Theorem 1.1]). Let (C1)-(C3) hold. Assume that for some \( k \in \mathbb{N} \), either \( \frac{\mu_k}{f_\infty} < \mu < \frac{\mu_k}{f_0} \) or \( \frac{\mu_k}{f_0} < \mu < \frac{\mu_k}{f_\infty} \). Then BVP (1.1) has two
solutions \( u^+_k \) and \( u^-_k \) such that \( u^+_k \) has exactly \( k - 1 \) zeros in \((0, 1)\) and is positive near 0, and \( u^-_k \) has exactly \( k - 1 \) zeros in \((0, 1)\) and is negative near 0.

In [3], Ma and Thompson studied the existence and multiplicity of nodal solutions for BVP

\[
\begin{align*}
    u''(t) + w(t)f(u(t)) = 0, & \quad t \in (0, 1), \\
    u(0) = u(1) = 0.
\end{align*}
\]  

(1.3)

They gave conditions on the ratio \( \frac{f(s)}{s} \) at infinity and zero that guarantee the existence of solutions with prescribed nodal properties.

Using Rabinowitz bifurcation theorem also, they established the following two main results:

**Theorem B** (Ma and Thompson [1], Theorem 2). Let (C1)-(C3) hold. Assume that either (i) or (ii) holds for some \( k \in \mathbb{N} \) and \( j \in \{0\} \cup \mathbb{N} \);

(i) \( f_0 < \mu_k < \cdots < f_{k+j} < f_\infty \);

(ii) \( f_\infty < \mu_k < \cdots < f_{k+j} < f_0 \)

where \( \mu_k \) denotes the \( k \)th eigenvalue of (1.2). Then BVP (1.3) has \( 2(j+1) \) solutions \( u^+_{k+i}, u^-_{k+i}, i = 0, \ldots, j \), such that \( u^+_{k+i} \) has exactly \( k+i-1 \) zeros in \((0, 1)\) and are positive near 0, and \( u^-_{k+i} \) has exactly \( k+i-1 \) zeros in \((0, 1)\) and are negative near 0.

**Theorem C** (Ma and Thompson [1], Theorem 3). Let (C1)-(C3) hold. Assume that there exists an integer \( k \in \mathbb{N} \) such that

\[
\mu_{k-1} < \frac{f(s)}{s} < \mu_k
\]

where \( \mu_k \) denotes the \( k \)th eigenvalue of (1.2). Then BVP (1.3) has no nontrivial solution.

From above literature, we can see that the existence and multiplicity results are largely based on the assumption that \( t \) and \( u \) are separated in nonlinearity term. It is interesting to know what will happen if \( t \) and \( u \) are not separated in nonlinearity term? We shall give a confirm answer for this question.

In this article, we consider the existence and multiplicity of nodal solutions for the nonlinear BVP

\[
\begin{align*}
    u''(t) + f(t, u) = 0, & \quad t \in (0, 1), \\
    u(0) = u(1) = 0
\end{align*}
\]  

(1.4)

under the following assumptions:

\( (H_1) \) \( \lambda_k \leq a(t) \equiv \lim_{|s| \to +\infty} \frac{f(t, s)}{s} \) uniformly on \([0, 1]\), and the inequality is strict on some subset of positive measure in \((0, 1)\), where \( \lambda_k \) denotes the \( k \)th eigenvalue of

\[
\begin{align*}
    u''(t) + \lambda_n(t) = 0, & \quad t \in (0, 1), \\
    u(0) = u(1) = 0
\end{align*}
\]  

(1.5)

\( (H_2) \) \( 0 \leq \lim_{|s| \to 0} \frac{f(t, s)}{s} \equiv c(t) \leq \lambda_k \) uniformly on \([0, 1]\), and all the inequalities are strict on some subset of positive measure in \((0, 1)\), where \( \lambda_k \) denotes the \( k \)th eigenvalue of (1.5);

\( (H_3) \) \( f(t, s)s > 0 \) for \( t \in (0, 1) \) and \( s \neq 0 \).
Remark 1.1. From \((H_1)-(H_3)\), we can see that there exist a positive constant \(\varrho\) and a subinterval \([\alpha, \beta]\) of \([0, 1]\) such that \(\alpha < \beta\) and \(f(r,s) \geq \varrho\) for all \(r \in [\alpha, \beta]\) and \(s \neq 0\).

In the celebrated study [4], Rabinowitz established Rabinowitz’s global bifurcation theory [[4], Theorems 1.27 and 1.40]. However, as pointed out by Dancer [5,6] and López-Gómez [7], the proofs of these theorems contain gaps, the original statement of Theorem 1.40 of [4] is not correct, the original statement of Theorem 1.27 of [4] is stronger than what one can actually prove so far. Although there exist some gaps in the proofs of Rabinowitz’s Theorems 1.27, 1.40, and 1.27 has been used several times in the literature to analyze the global behavior of the component of nodal solutions emanating from \(u = 0\) in wide classes of boundary value problems for equations and systems [1,2,8,9]. Fortunately, López-Gómez gave a corrected version of unilateral bifurcation theorem in [7].

By applying the bifurcation theorem of López-Gómez [[7], Theorem 6.4.3], we shall establish the following:

**Theorem 1.1.** Suppose that \(f(t, u)\) satisfies \((H_1), (H_2),\) and \((H_3)\), then problem (1.4) possesses two solutions \(u_k^+\) and \(u_k^-\), such that \(u_k^+\) has exactly \(k - 1\) zeros in \((0, 1)\) and is positive near 0, and \(u_k^-\) has exactly \(k - 1\) zeros in \((0,1)\) and is negative near 0.

Similarly, we also have the following:

**Theorem 1.2.** Suppose that \(f(t, u)\) satisfies \((H_3)\) and

\[
(H_1') \lambda_k \geq a(x) \equiv \lim_{|s| \to +\infty} \frac{f(t,s)}{s} \geq 0 \text{ uniformly on } [0,1], \text{ and all the inequalities are strict on some subset of positive measure in } (0, 1), \text{ where } \lambda_k \text{ denotes the } k\text{th eigenvalue of (1.5)}; \\
(H_2') \lim_{|s| \to 0} \frac{f(t,s)}{s} \equiv c(x) \geq \lambda_k \text{ uniformly on } [0,1], \text{ and the inequality is strict on some subset of positive measure in } (0, 1), \text{ where } \lambda_k \text{ denotes the } k\text{th eigenvalue of (1.5)}, \text{ then problem (1.4) possesses two solutions } u_k^+ \text{ and } u_k^-, \text{ such that } u_k^+ \text{ has exactly } k - 1 \text{ zeros in } (0,1) \text{ and is positive near 0, and } u_k^- \text{ has exactly } k - 1 \text{ zeros in } (0,1) \text{ and is negative near 0.}
\]

**Remark 1.2.** We would like to point out that the assumptions \((H_1)\) and \((H_2)\) are weaker than the corresponding conditions of Theorem A. In fact, if we let \(f(t, s) \equiv \mu(t)f(s),\) then we can get \(\lim_{|s| \to +\infty} \frac{f(t,s)}{s} \equiv \mu w(t)f_\infty := a(t)\) and \(\lim_{|s| \to 0} \frac{f(t,s)}{s} \equiv \mu w(t)f_0 := c(t).\) By the strict decreasing of \(\mu_k(f)\) with respect to weight function \(f\) (see [10]), where \(\mu_k(f)\) denotes the \(k\)th eigenvalue of (1.2) corresponding to weight function \(f,\) we can show that our condition \(c(t) \leq \lambda_k \leq a(t)\) is equivalent to the condition \(\frac{\mu_k}{f_\infty} < \mu < \frac{\lambda_k}{f_0}.\) Similarly, our condition \(c(t) \geq \lambda_k \geq a(t)\) is equivalent to the condition \(\frac{\mu_k}{f_0} < \mu < \frac{\lambda_k}{f_\infty}.\) Therefore, Theorem A is the corollary of Theorems 1.1 and 1.2.

Using the similar proof with the proof Theorems 1.1 and 1.2, we can obtain the more general results as follows.

**Theorem 1.3.** Suppose that \((H_4)\) holds, and either (i) or (ii) holds for some \(k \in \mathbb{N}\) and \(j \in \{0\} \cup \mathbb{N}:\)
(i) \( 0 \leq c(t) \equiv \lim_{|s| \to 0} \frac{f(t, s)}{s} \leq \lambda_k < \cdots < \lambda_{k+1} \leq a(t) \equiv \lim_{|s| \to +\infty} \frac{f(t, s)}{s} \) uniformly on \([0, 1]\), and the inequalities are strict on some subset of positive measure in \((0, 1)\), where \(\lambda_k\) denotes the \(k\)th eigenvalue of \((1.5)\);

(ii) \( 0 \leq a(t) \equiv \lim_{|s| \to +\infty} \frac{f(t, s)}{s} \leq \lambda_k < \cdots < \lambda_{k+1} \leq c(t) \equiv \lim_{|s| \to 0} \frac{f(t, s)}{s} \) uniformly on \([0, 1]\), and the inequality is strict on some subset of positive measure in \((0, 1)\), where \(\lambda_k\) denotes the \(k\)th eigenvalue of \((1.5)\).

Then BVP \((1.4)\) has \(2j + 1\) solutions \(u_{k+i}^+, u_{k+i}^-\), \(i = 0, \ldots, j\), such that \(u_{k+i}^+\) has exactly \(k + i - 1\) zeros in \((0,1)\) and are positive near 0, and \(u_{k+i}^-\) has exactly \(k + i - 1\) zeros in \((0,1)\) and are negative near 0.

Using Sturm Comparison Theorem, we also can get a non-existence result when \(f\) satisfies a non-resonance condition.

**Theorem 1.4.** Let \((H_3)\) hold. Assume that there exists an integer \(k \in \mathbb{N}\) such that

\[
\lambda_{k-1} < \frac{f(t, u)}{u} < \lambda_k
\]

for any \(t \in [0, 1]\), where \(\lambda_k\) denotes the \(k\)th eigenvalue of \((1.5)\). Then BVP \((1.4)\) has no nontrivial solution.

**Remark 1.3.** Similarly to Remark 1.2, we note that the assumptions (i) and (ii) are weaker than the corresponding conditions of Theorem 1.4. In fact, if we let \(f(t, s) = w(t)\) \(f(s)\), then we can get \(\lim_{|s| \to +\infty} \frac{f(t, s)}{s} \equiv w(t) f_{\infty} := a(t)\) and \(\lim_{|s| \to 0} \frac{f(t, s)}{s} \equiv w(t) f_0 := c(t)\). By the strict decreasing of \(\mu_k(f)\) with respect to weight function \(f\) (see [11]), where \(\mu_k(f)\) denotes the \(k\)th eigenvalue of \((1.2)\) corresponding to weight function \(f\), we can show that our condition \(c(t) \leq \lambda_k < \cdots < \lambda_{k+1} \leq a(t)\) is equivalent to the condition \(f_0 \mu_k < \cdots < \mu_{k+1} < f_{\infty}\). Similarly, our condition \(a(t) \leq \lambda_k < \cdots < \lambda_{k+1} \leq c(t)\) is equivalent to the condition \(f_\infty < \mu_k < \cdots < \mu_{k+1} < f_0\). Therefore, Theorem 1.4 is also the corollary of Theorem 1.3.

**2 Preliminary results**

To show the nodal solutions of the BVP \((1.4)\), we need only consider an operator equation of the following form

\[
u = \lambda Au.
\]

Equations of the form \((2.1)\) are usually called nonlinear eigenvalue problems. López-Gómez [7] studied a nonlinear eigenvalue problem of the form

\[
u = G(r, u),
\]

where \(r \in \mathbb{R}\) is a parameter, \(u \in X, X\) is a Banach space, \(\theta\) is the zero element of \(X\), and \(G: \mathbb{R} \times X \to X\) is completely continuous. In addition, \(G(r, u) = r Tu + H(r, u)\), where \(H(r, u) = o(||u||)\) as \(||u|| \to 0\) uniformly on bounded \(r\) interval, and \(T\) is a linear completely continuous operator on \(X\). A solution of \((2.2)\) is a pair \((r, u) \in \mathbb{R} \times X\), which satisfies the equation \((2.2)\). The closure of the set nontrivial solutions of \((2.2)\) is denoted by \(C\), let \(\Sigma(T)\) denote the set of eigenvalues of linear operator \(T\). López-Gómez [7] established the following results:
Lemma 2.1 [[7], Theorem 6.4.3]. Assume $\Sigma(T)$ is discrete. Let $\lambda_0 \in \Sigma(T)$ such that 
$\text{ind}(0, \lambda_0 T)$ changes sign as $\lambda$ crosses $\lambda_0$, then each of the components $C^+_{\lambda_0}, v \in \{+, -\}$ satisfies $(\lambda_0, \theta) \in C^+_{\lambda_0}$, and either

(i) meets infinity in $X$,
(ii) meets $(\tau, \theta)$, where $\tau \neq \lambda_0 \in \Sigma(T)$ or
(iii) $C^+_{\lambda_0}, v \in \{+, -\}$ contains a point

$(t, y) \in \mathbb{R} \times (V \setminus \{0\})$,

where $V$ is the complement of $\text{span}\{\varphi_{\lambda_0}\}, \varphi_{\lambda_0}$ denotes the eigenfunction corresponding to eigenvalue $\lambda_0$.

Lemma 2.2 [[7], Theorem 6.5.1]. Under the assumptions:
(A) $X$ is an order Banach space, whose positive cone, denoted by $P$, is normal and has a nonempty interior;
(B) The family $\Upsilon(r)$ has the special form

$$
\Upsilon(r) = I_X - rT,
$$

where $T$ is a compact strongly positive operator, i.e., $T(P \setminus \{0\}) \subseteq \text{int} P$;
(C) The solutions of $u = rTu + H(r, u)$ satisfy the strong maximum principle.

Then the following assertions are true:
(1) $\text{Spr}(T)$ is a simple eigenvalue of $T$, having a positive eigenfunction denoted by $\psi_0 > 0$, i.e., $\psi_0 \in \text{int} P$, and there is no other eigenvalue of $T$ with a positive eigenfunction;
(2) For every $y \in \text{int} P$, the equation

$$
u - rTu = y$$

has exactly one positive solution if $r < \frac{1}{\text{Spr}(T)}$, whereas it does not admit a positive solution if $r \geq \frac{1}{\text{Spr}(T)}$.

Lemma 2.3 [[10], Theorem 2.5]. Assume $T : X \to X$ is a completely continuous linear operator, and 1 is not an eigenvalue of $T$, then

$$
\text{ind}(I - T, \theta) = (-1)^{\beta},
$$

where $\beta$ is the sum of the algebraic multiplicities of the eigenvalues of $T$ large than 1, and $\beta = 0$ if $T$ has no eigenvalue of this kind.

Let $Y = C[0, 1]$ with the norm $\|u\|_{\infty} = \max_{t \in [0, 1]} |u(t)|$. Let

$$
E = \{u \in C^1[0, 1] | u(0) = u(1) = 0\}
$$

with the norm

$$
\|u\|_E = \max_{t \in [0, 1]} |u| + \max_{t \in [0, 1]} |u'|.
$$

Define $L : D(L) \to Y$ by setting

$$
Lu := -u''(t), \quad t \in [0, 1], \quad u \in D(L),
$$
where
\[ D(L) = \{ u \in C^2[0, 1] | u(0) = u(1) = 0 \}. \]

Then \( L^{-1} : Y \to E \) is compact. Let \( E = \mathbb{R} \times E \) under the product topology. For any \( C^1 \) function \( u \), if \( u(x_0) = 0 \), then \( x_0 \) is a simple zero of \( u \), if \( u'(x_0) \neq 0 \). For any integer \( k \in \mathbb{N} \) and \( \nu \in \{+, -\} \), define \( S^\nu_k \subset C^1[0, 1] \) consisting of functions \( u \in C^1[0, 1] \) satisfying the following conditions:

(i) \( u(0) = 0 \), \( \nu \) \( u'(0) > 0 \);
(ii) \( u \) has only simple zeros in \([0, 1]\) and exactly \( n - 1 \) zeros in \((0,1)\).

Then sets \( S^\nu_k \) are disjoint and open in \( E \).

Finally, let \( \varphi^\nu_k = \mathbb{R} \times S^\nu_k \).

Furthermore, let \( \zeta \in C[0, 1] \times \mathbb{R} \) be such that
\[ f(t, u) = c(t)u + \varphi(t, u) \]
with
\[ \lim_{|u| \to 0} \frac{\varphi(t, u)}{u} = 0 \quad \text{and} \quad \lim_{|u| \to \infty} \frac{\varphi(t, u)}{u} = a(t) - c(t) \quad \text{uniformly on } [0, 1]. \] (2.3)

Let
\[ \bar{\varphi}(t, u) = \max_{0 \leq |s| \leq u} |g(t, u)| \quad \text{for } t \in [0, 1], \]
then \( \bar{\varphi} \) is nondecreasing with respect to \( u \) and
\[ \lim_{u \to 0^+} \frac{\bar{\varphi}(t, u)}{|u|} = 0. \]

If \( u \in E \), it follows from (2.3) that
\[ \frac{\varphi(t, u)}{|u|} \leq \bar{\varphi}(t, |u|) \leq \frac{\bar{\varphi}(t, \|u\|_E)}{|u|} \leq \frac{\bar{\varphi}(t, \|u\|_E)}{|u|} \to 0, \quad \text{as } \|u\|_E \to 0 \]
uniformly for \( t \in [0, 1] \).

Let us study
\[ Lu - \mu c(t)u = \mu \varphi(t, u) \] (2.4)
as a bifurcation problem from the trivial solution \( u \equiv 0 \).

Equation (2.4) can be converted to the equivalent equation
\[ u(t) = \mu L^{-1}[c(t)u(t)] + \mu L^{-1}[\varphi(t, u(t))]. \]

Further we note that \( ||L^{-1}[\varphi(t, u(t))]||_E = o(||u||_E) \) for \( u \) near 0 in \( E \).

**Lemma 2.4.** For each \( k \in \mathbb{N} \) and \( \nu \in \{+, -\} \), there exists a continuum \( C^\nu_k \subset \varphi^\nu_k \) of solutions of (2.4) with the properties:

(i) \( \lambda_k, \theta \in C^\nu_k \);
(ii) \( C^\nu_k \setminus \{\lambda_k, \theta\} \subset \varphi^\nu_k \);
(iii) \( C^\nu_k \) is unbounded in \( E \), where \( \lambda_k \) denotes the \( k \)-th eigenvalue of (1.5).

**Proof.** It is easy to see that the problem (2.4) is of the form considered in [7], and satisfies the general hypotheses imposed in that article.
Combining Lemma 2.1 with Lemma 2.3, we know that there exists a continuum $C_k^\nu \subset E$ of solutions of (2.4) such that:

(a) $C_k^\nu$ is unbounded and $(\lambda_k, \theta) \in C_k^\nu \setminus \{(\lambda_k, \theta)\} \subset \phi_k^\nu$;
(b) or $(\lambda_j, \theta) \in C_k^\nu$, where $j \in \mathbb{N}$, $\lambda_j$ is another eigenvalue of (1.5) and different from $\lambda_k$;
(c) or $C_k^\nu$ contains a point $(\iota, y) \in \mathbb{R} \times (V \setminus \{0\})$, where $V$ is the complement of $\text{span}\{\phi_k\}$, $\phi_k$ denotes the eigenfunction corresponding to eigenvalue $\lambda_k$.

We finally prove that the first choice of the (a) is the only possibility.

In fact, all functions belong to the continuum sets $C_k^\nu$ have exactly $k - 1$ simple zeros, this implies that it is impossible to exist $(\lambda_j, \theta) \in C_k^\nu, j \in \mathbb{N}$.

Next, we shall prove (c) is impossible, suppose (c) occurs, then $C_k^\nu$ is bounded and without loss of generality, suppose there exists a point $(\iota, y) \in \mathbb{R} \times (V \setminus \{\theta\}) \cap C_k^\nu$.

Moreover, it follows from Lemma 2.1 that

$$C_k^\nu \cap \{(\lambda, \theta) : \lambda \in \mathbb{R}\} = \{(\lambda_k, \theta)\}.$$ 

Note that as the complement $V$ of $\text{span}\{\phi_k\}$ in $E$, we can take

$$V := \mathbb{R}[I_E - \lambda_k L].$$

Thus, for this choice of $V$, the component $C_k^\nu$ cannot contain a point $(\iota, y) \in \mathbb{R} \times (V \setminus \{\theta\}) \cap C_k^\nu$.

Indeed, if

$$(\iota, y) \in \mathbb{R} \times (V \setminus \{\theta\}) \cap C_k^\nu.$$ 

then $y > 0$ in $(0, a_0)$, where $a_0$ denotes the first zero point of $y$, and there exists $u \in E$ for which

$$u - \lambda_k L u = y > 0, \quad \text{in} \quad (0, a_0).$$

Thus, for each sufficiently large $\alpha > 0$, we have that $u + \alpha \phi_k >> 0$ in $(0, a_0)$ and

$$u + \alpha \phi_k - \lambda_k L (u + \alpha \phi_k) = y > 0, \quad \text{in} \quad (0, a_0).$$

Define

$$P = \{u \in E | u(t) \geq 0, \quad t \in [0, a_0]\}.$$ 

Hence, according to Lemma 2.2

$$\text{Spr}(\lambda_k L) < 1,$$

which is impossible since $\text{Spr}(L) = \frac{1}{\lambda_{kf0}}$.

**Lemma 2.5.** If $(\mu, u) \in \mathbb{E}$ is a non-trivial solution of (2.4), then $u \in S_k^\nu$ for $\nu$ and some $k \in \mathbb{N}$.
Proof. Taking into account Lemma 2.4, we only need to prove that $C_k^u \subset \Phi_k^u \cup \{(\lambda_k, \theta)\}$.

Suppose $C_k^u \not\subset \Phi_k^u \cup \{(\lambda_k, \theta)\}$. Then there exists $(\mu^*, u) \in C_k^u \cap (\mathbb{R} \times \delta S_k^u)$ such that $(\mu^*, u) \neq (\lambda_k, \theta), u \not\in S_k^u$, and $(\mu_j, u_j) \to (\mu^*, u)$ with $(\mu_j, u_j) \in C_k^u \cap (\mathbb{R} \times S_k^u)$. Since $u \in \delta S_k^u$, so $u \equiv 0$. Let $c_j := \frac{u_j}{\|u_j\|_E}$, then $c_j$ should be a solution of problem,

$$c_j = \mu_j L^{-1} \left[ c(t) c_j(t) + \frac{\xi(t, u_j(t))}{\|u_j\|_E} \right]. \tag{2.5}$$

By (2.3), (2.5) and the compactness of $L^1$, we obtain that for some convenient subsequence $c_j \to c_0 \neq 0$ as $j \to +\infty$. Now $c_0$ verifies the equation

$$-c_0''(t) = \mu^* c(t)c_0(t), \quad t \in (0, 1)$$

and $|c_0|_E = 1$. Hence $\mu^* = \lambda_i$, for some $i \neq k, i \in \mathbb{N}$. Therefore, $(\mu_j, u_j) \to (\lambda_i, \theta)$ with $(\mu_j, u_j) \in C_k^u \cap (\mathbb{R} \times S_k^u)$. This contradicts to Lemma 2.3.

3 Proof of main results

Proof of Theorems 1.1 and 1.2. We only prove Theorem 1.1 since the proof of Theorem 1.2 is similar. It is clear that any solution of (2.4) of the form $(1, u)$ yields a solution $u$ of (1.4). We shall show $C_k^u$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

By the strict decreasing of $\mu_k(c(t))$ with respect to $c(t)$ (see [11]), where $\mu_k(c(t))$ is the $k$th eigenvalue of (1.2) corresponding to the weight function $c(t)$, we have $\mu_k(c(t)) > \mu_k(\lambda_k) = 1$.

Let $(\mu_j, u_j) \in C_k^u$ with $u_j \not\equiv 0$ satisfies

$$\mu_j + \|u_j\|_E \to +\infty.$$ 

We note that $\mu_j > 0$ for all $j \in \mathbb{N}$, since $(0,0)$ is the only solution of (2.4) for $\mu = 0$ and $C_k^u \cap \{0\} \times E = \emptyset$.

Step 1: We show that if there exists a constant $M > 0$, such that

$$\mu_j \subset (0, M)$$

for $j \in \mathbb{N}$ large enough, then $C_k^u$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

In this case it follows that

$$\|u_j\|_E \to \infty.$$ 

Let $\xi \in C([0, 1] \times \mathbb{R})$ be such that

$$f(t, u) = a(t)u + \xi(t, u)$$

with

$$\lim_{|u| \to +\infty} \frac{\xi(t, u)}{u} = 0 \quad \text{and} \quad \lim_{|u| \to 0} \frac{\xi(t, u)}{u} = c(t) - a(t), \text{ uniformly on } [0, 1]. \tag{3.1}$$
We divide the equation

\[ Lu_j - \mu_j a(t) u_j = \mu_j \xi(t, u_j) , \]  

(3.2)

set \( \bar{u}_j = \frac{u_j}{\| u_j \|_E} \). Since \( \bar{u}_j \) is bounded in \( C^2 [0, 1] \), after taking a subsequence if necessary, we have that \( \bar{u}_j \to \bar{u} \) for some \( \bar{u} \in E \) with \( \| u \|_E = 1 \). By (3.1), using the similar proof of (2.3), we have that

\[ \lim_{j \to +\infty} \frac{\xi(t, u_j(t))}{\| u_j \|_E} = 0 \quad \text{in } Y. \]

By the compactness of \( L \) we obtain

\[ -\bar{u}'' + \bar{\mu} a(t) \bar{u} = 0, \]

where \( \bar{\mu} = \lim_{j \to +\infty} \mu_j \), again choosing a subsequence and relabeling if necessary.

It is clear that \( \bar{u} \in C^\infty_k \subseteq C^k_k \) since \( C^k_k \) is closed in \( \mathbb{R} \times E \). Therefore, \( \bar{\mu}(a(t)) \) is the \( k \)th eigenvalue of

\[ u''(t) + \mu a(t) u(t) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0. \]

By the strict decreasing of \( \bar{\mu}(a(t)) \) with respect to \( a(t) \) (see [11]), where \( \bar{\mu}(a(t)) \) is the \( k \)th eigenvalue of (1.2) corresponding to the weight function \( a(t) \), we have \( \bar{\mu}(a(t)) < \bar{\mu}(\lambda_k) = 1 \). Therefore, \( C^\infty_k \) crosses the hyperplane \( \{1\} \times E \) in \( \mathbb{R} \times E \).

Step 2: We show that there exists a constant \( M \) such that \( \mu_j \in (0, M] \) for \( j \in \mathbb{N} \) large enough.

On the contrary, we suppose that

\[ \lim_{j \to +\infty} \mu_j = +\infty. \]

On the other hand, we note that

\[ -u_j'' = \mu_j \frac{f(t, u_j)}{u_j} \]

In view of Remark 1.1, we have \( \mu_j \frac{f(t, u_j)}{u_j} > \lambda_k \) on \( [\alpha, \beta] \) and for \( j \) large enough and all \( t \in [0, 1] \). By Lemma 3.2 of [12], we get \( u_j \) must change its sign more than \( k \) times on \( [\alpha, \beta] \) for \( j \) large enough, which contradicts the act that \( u_j \in S^\alpha_k \).

Therefore,

\[ \mu_j \leq M \]

for some constant number \( M > 0 \) and \( j \in \mathbb{N} \) sufficiently large.

Proof of Theorem 1.3. Repeating the arguments used in the proof of Theorems 1.1 and 1.2, we see that for \( \nu \in \{+, -\} \) and each \( i \in \{k, k + 1, ..., k + j\} \)

\[ C^\nu_i \cap \{(1) \times E\} \neq \emptyset. \]

The results follows.
Proof of Theorem 1.4. Assume to the contrary that BVP (1.4) has a solution \( u \in E \), we see that \( u \) satisfies

\[ u''(t) + b(t)u(t) = 0, \quad t \in (0, 1), \]

where \( b(t) = \frac{f(t, u)}{u} \).

Note that \( c(t) = \lim_{s \to 0} \frac{f(t, s)}{s} \leq \lambda_{k+1} \) and hence \( f(t, u) \) can be regarded as a continuous function on \( \mathbb{R} \). Thus we get \( b(\cdot) \in C[0, 1] \). Also, notice that a nontrivial solution of (1.4) has a finite number of zeros. From (2.8) and the above fact \( \lambda_k < b(t) < \lambda_{k+1} \) for all \( t \in [0, 1] \).

We know that the eigenfunction \( \phi_k \) corresponding to \( \lambda_k \) has exactly \( k-1 \) zeros in \([0, 1]\). Applying Lemma 2.4 of [13] to \( \phi_k \) and \( u \), we see that \( u \) has at least \( k \) zeros in \( I \). By Lemma 2.4 of [13] again to \( u \) and \( \phi_{k+1} \), we get that \( \phi_{k+1} \) has at least \( k + 1 \) zeros in \([0, 1]\). This is a contradiction.

Acknowledgements
The authors were very grateful to the anonymous referees for their valuable suggestions. This study was supported by the NSFC (No. 11061030, No. 10971087) and NWNU-LKQN-10-21.

Authors’ contributions
GD conceived of the study, and participated in its design and coordination and helped to draft the manuscript. BY drafted the manuscript. RM participated in the design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 16 August 2011 Accepted: 10 February 2012 Published: 10 February 2012

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