STABILITY OF SOLUTIONS TO THE Riemann Problem FOR A THIN FILM MODEL OF A PERFECTLY SOLUBLE ANTI-SURFACTANT SOLUTION

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Abstract. In this article, we consider a quasilinear hyperbolic system of partial differential equations governing the dynamics of a thin film of a perfectly soluble anti-surfactant liquid. We construct elementary waves of the corresponding Riemann problem and study their interactions. Further, we provide exact solution of the Riemann problem along with numerical examples. Finally, we show that the solution of the Riemann problem is stable under small perturbation of the initial data.

1. Introduction. In this article, we are concerned with the solution to a Riemann problem and elementary wave interactions for a quasilinear hyperbolic system of partial differential equations (PDEs) governing the dynamics of a thin film of a perfectly soluble anti-surfactant. Anti-surfactants arise in many practical situations, including many salts, i.e., when common salt is added to water [15, 12], when water is added to short-chain alcohol [7]. Conn et al. [3] proposed a mathematical model for an anti-surfactant solution. Subsequently Conn et al. [4] obtained exact solutions to a family of Riemann problems for a reduced version of their model which describes a thin film of a perfectly wetting anti-surfactant solution (i.e., the case in which the surface concentration of anti-surfactant is identically zero). Specifically, Conn et al. [4] studied a governing system of PDEs given in Cartesian coordinates by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{3Ca} h^3 \frac{\partial^3 h}{\partial x^3} + \frac{1}{2} h^2 \frac{\partial c}{\partial x} \right) = 0, \quad (1)$$

$$h \frac{\partial c}{\partial t} + \left( \frac{1}{3Ca} h^3 \frac{\partial^3 h}{\partial x^3} + \frac{1}{2} h^2 \frac{\partial c}{\partial x} \right) \frac{\partial c}{\partial x} - \frac{1}{Pe} \frac{\partial}{\partial x} \left( h \frac{\partial c}{\partial x} \right) = 0, \quad (2)$$

where $x$ and $t$ are independent space and time coordinates and the dependent variables $h$ and $c$, respectively, denote the dimensionless film thickness and concentration of the solute. Here, $Pe$ is the Péclet number that indicates the ratio of advection to diffusion, while the Capillary number $Ca$ indicates the relative effect of viscous drag forces versus surface tension forces. If diffusion effects are neglected

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we have $Pe \to \infty$ and further in case of large Capillary number $Ca \to \infty$, the governing system of PDEs (1) and (2) reduces to

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} h^2 \frac{\partial c}{\partial x} \right) = 0,$$

(3)

$$\frac{\partial c}{\partial t} + \frac{1}{2} h \left( \frac{\partial c}{\partial x} \right)^2 = 0.$$  

(4)

Note that (3) represents the conservation of mass of the fluid whereas (4) represents the conservation of mass of the solute. Let us denote the gradient of concentration of the solute by $b(x, t)$, i.e., $\partial c/\partial x = b(x, t)$ and differentiate (4) with respect to $x$ to obtain a system of PDEs in a conservative form [4]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( h^2 b \right) = 0,$$

(5)

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x} \left( h b^2 \right) = 0.$$  

In case of non-uniform surface tension, the surface tension is a function of temperature or concentration or both. This phenomena is typically known as Marangoni effect [23]. It is known from literature that the surface tension, denoted by $\Gamma$, varies with concentration as

$$\Gamma = k_1 - k_2 c,$$  

(6)

In case of balancing the shear stress with the Marangoni effects, we typically have

$$[\varsigma] = -\nabla_s (\Gamma),$$  

(7)

where $\nabla_s$ denotes surface gradient operator and $[\varsigma]$ denotes jump in shear stress. Using (6) in (7), we have $[\varsigma] = k_2 b$. Therefore, the shear stress at the free surface of the film is positive if the value of $b$ is positive. Also due to the Marangoni effect, the fluid will flow from left to the right for positive values of $b$ and vice versa for negative values of $b$. One may see a similar type of anti-surfactant properties that arise in certain resins that are included in solvent-based paints [17, 29, 8].

Elementary wave interactions for Riemann problem is not only key mechanism to determine the qualitative properties of physical variables but it exhibits certain essential features of the solution. Further, understanding relevant properties based on wave interactions play a significant role in several practical applications. In particular, in the field of two-phase flows [16, 11], magnetogasdynamics [20, 13], nonlinear elasticity [22], traffic flow [1] etc. Hence, it is a subject of great interest both from the mathematical and physical point of view. Various methods to deal with such system of equations very much relate to the mathematical theory of quasilinear hyperbolic system of PDEs. Raja Sekhar et al. [19] discussed the interaction of elementary waves for shallow water equations. A similar analysis for blood flow equations in one dimension is done by Raja Sekhar et al. [18]. Using the method of characteristics, the interaction of elementary waves and stability of Riemann solution for Aw-Rascle model and chromatography equations are investigated in [27, 28, 25]. The stability of rarefaction waves for Navier-Stokes-Poisson equations may be found in [5]. Recently, the structural stability of Riemann solution for nonlinear elasticity has been discussed in [21]. More details on Riemann problem and interactions of elementary waves for various practical problems may be found in [24, 26, 10, 2, 9, 14] and the references cited therein.
In a fundamental work, Conn et al. [3] considered linear stability of an infinitely deep layer of initially quiescent fluid and discussed the occurrence of an instability driven by surface-tension gradients, which occurs for anti-surfactant solutions. Further, Conn et al. [4] derived a set of Riemann solutions to the system (5) for different Riemann initial data and discussed their properties.

In this work, our main purpose is to investigate all possible elementary wave interactions and discuss the stability of Riemann solution for the system of PDEs (5). To the best of our knowledge, this kind of interactions and stability of Riemann solutions for this system (5) have not been studied previously. In order to determine all possible cases of wave interactions for the system (5), we consider the following initial data with three piecewise constant states

\[
(h, b)(x, 0) = \begin{cases} 
(h_l, b_l), & \text{if } -\infty < x < -\epsilon, \\
(h_m, b_m), & \text{if } -\epsilon < x < \epsilon, \\
(h_r, b_r), & \text{if } \epsilon < x < \infty, 
\end{cases} \tag{8}
\]

where \( \epsilon > 0 \) is arbitrarily small. Note that, the initial data (8) is nothing but the local perturbation of the Riemann initial data

\[
(h, b)(x, 0) = \begin{cases} 
(h_l, b_l), & \text{if } x < 0, \\
(h_r, b_r), & \text{if } x > 0. 
\end{cases} \tag{9}
\]

Our objective is to study whether the solution of Riemann problem (5) together with the data (9) is the limit of the solution of (5) and (8) when the parameter \( \epsilon \to 0 \). We construct the elementary wave interactions case by case to resolve this problem.

This paper is structured as follows. In section 2, we briefly express the elementary waves and discuss their properties. We study the Riemann problem (5) and (9) along with numerical examples to see various wave patterns with different possible initial data in section 3. In section 4, we mainly focus on the interaction of elementary waves for all possible cases and investigate the global stability of Riemann solution under the small perturbation of initial data. Finally, we list brief conclusions in section 5.

2. Elementary waves. It may be noted that Conn et al. [4] considered the system of PDEs (5) subject to the Riemann data (9) and derived the corresponding Riemann solution. However, we emphasize here that they have followed the method of characteristics to obtain the solution with a restriction on the initial data, i.e., \( b_l < 0 \) and \( b_r < 0 \). In order to study the stability of the perturbed Riemann problem (5) and (8), we need to study the properties of the elementary waves. Hence, we investigate the behavior of the elementary waves in phase plane. Accordingly, we consider the quasilinear form of the system (5)

\[
\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = 0, \tag{10}
\]

where \( U \) and \( A(U) \) denote the primitive variable and the Jacobian matrix, respectively, which are given by

\[
U = \begin{pmatrix} h \\ b \end{pmatrix}, \quad A(U) = \begin{pmatrix} bh & \frac{k^2}{2} \\ \frac{k^2}{2} & bh \end{pmatrix}. \tag{11}
\]
The eigenvalues of the Jacobian matrix are $\lambda_1 = bh/2$ and $\lambda_2 = 3bh/2$ and the corresponding eigenvectors are given by $r_1 = (h, -b)^T$ and $r_2 = (h, b)^T$, where $T$ denotes the transposition. It can easily be seen that $\nabla \lambda_1 \cdot r_1 = 0$ and $\nabla \lambda_2 \cdot r_2 = 3bh$, i.e., 1-characteristic field is linearly degenerate and 2-characteristic field is genuinely nonlinear. Therefore, the solution corresponding to 1-family is always a contact discontinuity and the solution corresponding to the 2-family is either a shock wave or a rarefaction wave. Now, we briefly express these three elementary waves and discuss their properties (for details, we refer [24, 26, 6]).

2.1. Contact discontinuity. Let us denote the left hand and right hand states of an elementary wave by $U_l$ and $U$, respectively. Since the characteristics on either side of the contact discontinuity are parallel, we have

$$\lambda_1(U_l) = \tau = \lambda_1(U),$$

which implies that $b_l h_l / 2 = \tau = bh/2$, where $\tau$ denotes the speed of the contact discontinuity. Therefore, for the contact discontinuity we have

$$J(h_l, b_l) : = \begin{cases} 
  b = b_l h_l, \\
  \tau = \frac{b_l h_l}{2} = \frac{bh}{2},
\end{cases}$$

and the corresponding $J$ curve is shown in Figure 1.

2.2. Shock wave. The Rankine-Hugoniot jump conditions across the shock wave are given by

$$\sigma[h] = \frac{1}{2}[h^2 b],$$

$$\sigma[b] = \frac{1}{2}[hb^2],$$

where the speed of shock wave is denoted by $\sigma$ and we use the notation $[h] = h_l - h$ which denotes the jump in $h$ across the shock. Since, (14) consist of two equations in three unknowns $\sigma, h$ and $b$, we express $\sigma$ and $b$ in terms of $h$ as

$$b = \frac{b_l h_l}{h_l}, \quad \sigma = \frac{b_l (h_l^2 + h h_l + h_l^2)}{2 h_l}.$$  

Lax entropy conditions across the shock wave are given by

$$\lambda_1(U_l) < \sigma < \lambda_2(U), \quad \lambda_2(U) < \sigma,$$

which follow that $b_l h_l / 2 < \sigma < 3b_l h_l / 2$ and $3bh/2 < \sigma$. One can observe that, these inequalities hold only when $b_l > 0$ as $h$ is always positive. Therefore, without loss of generality, we assume $b_l > 0$ throughout the article. Since, $3bh/2 < \sigma < 3b_l h_l / 2$ holds across the shock wave, using (15) we obtain $h^2 < h_l^2$, i.e., $h < h_l$ and consequently we obtain $b < b_l$ as both $h$ and $h_l$ are always positive. Therefore, the shock curve $S$ is given by (see, Figure 1)

$$S(h_l, b_l) : = \begin{cases} 
  b = \frac{b_l h_l}{h_l}, \\
  \sigma = \frac{b_l (h_l^2 + h h_l + h_l^2)}{2 h_l}, \\
  h < h_l.
\end{cases}$$
Figure 1. Elementary wave curves passing through a fixed state $(h_l, b_l)$ in the $(b, h)$-plane. Three elementary wave curves are identified, namely a shock wave curve (labelled as $S$), a rarefaction wave curve (labelled as $R$) and a contact discontinuity curve (labelled as $J$).

2.3. Rarefaction wave. The Riemann invariants corresponding to the 1- and 2-characteristic fields are $\Pi_1 = bh$ and $\Pi_2 = b/h$, respectively. Since the Riemann invariants are constant across the rarefaction wave, we have $b/h = b_l/h_l$. Further, the characteristic speed increases from left to right across the rarefaction wave, i.e., $\lambda_2(U_l) \leq \lambda_2(U)$. Therefore, the rarefaction wave curve ($R$) can be expressed as (see, Figure 1).

$$R(h_l, b_l) : = \begin{cases} b = \frac{b_l}{h_l} h, \\ \frac{dx}{dt} = \lambda_2 = \frac{3bh}{2}, \\ h \geq h_l. \end{cases} \quad (18)$$

We summarize the properties of elementary waves as follows:

**Theorem 2.1.** Across 2-shock corresponding to the system $(5)$ $b < b_l$ and $h < h_l$, while across 2-rarefaction wave $b \geq b_l$ and $h \geq h_l$.

**Theorem 2.2.** Contact discontinuity curve corresponding to the system $(5)$ is monotonically decreasing and convex while the curve of shock and rarefaction waves are monotonically increasing straight lines, respectively.

The proofs of Theorem 2.1 and Theorem 2.2 follow immediately from (13), (17), (18) and the properties discussed above.

3. Riemann problem. In this section, we consider the Riemann problem $(5)$ and $(9)$ and present the corresponding solution. We have already seen from the characteristic analysis that the solution of Riemann problem associated with 1-wave is always a contact discontinuity and the solution corresponding to 2-wave is either a shock or a rarefaction. Therefore, these elementary waves separate the $(x, t)$-plane into at most three constant states $(h_l, b_l)$, $(h_r, b_r)$ and $(h_\ast, b_\ast)$.
initial data generates two types of wave patterns as shown in Figure 2. Since the

\[
(\dot{h}_l, b_l) (\dot{h}_r, b_r)
\]

\[
(\dot{h}_e, b_e)
\]

Figure 2. Solution structure of Riemann problem in the \((x,t)\)-plane. Three constant states, namely \((h_l, b_l)\), \((h_e, b_e)\) and \((h_r, b_r)\) are separated by the elementary waves

state \((h_l, b_l)\) is connected to the unknown state \((h_e, b_e)\) by a contact discontinuity, from (13) we have

\[
h_l b_l - h_e b_e = 0 \tag{19}
\]

From (17) and (18), one can observe that the shock curve and the rarefaction curve coincide with each other and in either of the cases the unknown state \((h_e, b_e)\) is connected to \((h_r, b_r)\) by

\[
\frac{h_e}{b_e} = \frac{h_r}{b_r} \tag{20}
\]

Combining (19) and (20), we obtain the unknown state \((h_e, b_e)\) given by

\[
h_e = \sqrt{\frac{b_l h_l b_r}{b_r}}, \quad b_e = \sqrt{\frac{b_l h_l b_r}{h_r}}
\]

We can determine the solution inside the rarefaction wave fan by using the slope of the characteristic joining \((0,0)\) to \((x,t)\). The corresponding characteristic speed is given by

\[
\frac{dx}{dt} = \frac{x}{t} = \frac{3bh}{2} \tag{21}
\]

Therefore, from (20) and (21), we obtain the solution inside the rarefaction fan as

\[
h = \sqrt{\frac{2x b_r}{3th_r}}, \quad b = \sqrt{\frac{2x b_r}{3th_r}}
\]

One can observe that, if the solution of the Riemann problem consists of contact discontinuity followed by a shock wave then \(h_l b_l = h_e b_e\), where \(h_e > h_r\) and \(b_e > b_r\) which implies that \(b_l h_l > b_r h_r\). Conversely, if \(b_l h_l > b_r h_r\) then we obtain \(h_e > h_r\) and \(b_e > b_r\), i.e., solution consists of a shock wave. Similarly, the solution of the Riemann problem consists of contact discontinuity followed by a rarefaction wave if and only if \(b_l h_l \leq b_r h_r\). Therefore, depending upon the choice of the initial data (9), there are two possible wave patterns of the solution to the Riemann problem (5) and (9) which are described as follows:

**Case A:** If \(b_l h_l > b_r h_r\), then the solution of the Riemann problem consists of
contact discontinuity \( J \) followed by a shock wave \( S \) and the corresponding solution at \( (x, t) \) is given by

\[
(h, b)(x, t) = \begin{cases} 
(h_l, b_l), & x < \tau t, \\
(h_*, b_*), & \tau t < x < \sigma t, \\
(h_r, b_r), & x > \sigma t,
\end{cases}
\]  \hspace{1cm} (22)

where \( \tau = b_l h_l / 2 \) and \( \sigma = (b_*(h_*^2 + h_* h_r + h_r^2)) / 2h_* \) are, respectively, the speeds of propagation of contact discontinuity and shock wave.

**Case B:** If \( b_l h_l \leq b_r h_r \), then the solution of Riemann problem consists of contact discontinuity \( J \) followed by a rarefaction wave \( R \) and the solution at \( (x, t) \) is given by

\[
(h, b)(x, t) = \begin{cases} 
(h_l, b_l), & x < \tau t, \\
(h_*, b_*), & \tau t < x < \lambda_2(U_l)t, \\
\left( \sqrt{\frac{2x h_r}{3 h_r}}, \sqrt{\frac{2x b_r}{3 h_r}} \right), & \lambda_2(U_l)t \leq x \leq \lambda_2(U_r)t, \\
(h_r, b_r), & x > \lambda_2(U_r)t.
\end{cases}
\]  \hspace{1cm} (23)

In order to analyze different types of wave patterns for the Riemann problem we consider two test cases. The initial data and the analytical solution for the Riemann problem in the unknown region for the two test cases are listed in Table 1. The analytical solution of the Riemann problem consists of a left contact discontinuity and a right rarefaction wave for test 1. The solution profiles at time \( t = 0.95 \) are shown in Figure 3. For test 2, the solution consists of a left contact discontinuity and right shock wave, the solution profiles at time \( t = 0.95 \) are depicted in Figure 4.

**Table 1.** Initial data and solution for the Riemann problem

| Test | \( h_l \) | \( b_l \) | \( h_r \) | \( b_r \) | \( b_* \) | \( h_* \) | Result  |
|------|--------|--------|--------|--------|--------|--------|--------|
| 1    | 1.0    | 0.8    | 1.0    | 1.8    | 1.20   | 0.667  | \( J + R \) |
| 2    | 1.0    | 0.8    | 0.5    | 0.3    | 0.693  | 1.155  | \( J + S \) |

**Figure 3.** Exact solution of thickness parameter \( h \) and concentration gradient \( b \) at \( t = 0.95 \) with \( b_l = 0.8 \), \( h_l = 1.0 \), \( b_r = 1.8 \) and \( h_r = 1.0 \).
We have observed that depending on the choice of the initial data, there are two kind of solutions to the Riemann problem. Further, we have presented the analytical expression of the solution for the Riemann problem in (22) and (23). In the next section, we use these solutions to study the wave interactions and stability of solution to the Riemann problem.

4. Interaction of elementary waves. Here, we consider the system of PDEs (5) with the initial data (8) to study the elementary wave interactions. Let us assume that \((h_\epsilon, b_\epsilon)\) be the solution of perturbed Riemann problem (5) and (8). Now, we wish to investigate the stability of the solution of the Riemann problem, i.e., whether the solution of the Riemann problem (5) and (9) is the limit of \((h_\epsilon, b_\epsilon)\) as the perturbation parameter \(\epsilon \to 0\).

Since, we have two local Riemann problems for the initial data (8) at \((-\epsilon, 0)\) and \((\epsilon, 0)\), we realize that four different combinations of the elementary waves are possible. These are given by \(J^+S\) and \(J^+S\), \(J^+R\) and \(J^+S\), \(J^+S\) and \(J^+R\). We now discuss these interactions.

Case 1: \(J + S\) and \(J + S\)

In this case, we consider the interactions between a contact discontinuity followed by a shock wave originating from \((-\epsilon, 0)\) and a contact discontinuity followed by a shock wave originating from \((\epsilon, 0)\). This case occurs only when the initial data (8) satisfies the inequality \(b_1 h_1 > b_m h_m > b_r h_r\). For a small enough time \(t\), the solution of the initial value problem (5) and (8) can be expressed as follows: \((h_1, b_1) + J_1 + (h_1, b_1) + S_1 + (h_m, b_m) + J_2 + (h_2, b_2) + S_2 + (h_r, b_r)\), where “+” denotes “followed by”, and \((h_1, b_1), (h_2, b_2)\) are solutions of the unknown states of first and second Riemann problems, respectively. One can easily compute that

\[
\sigma_1 - \tau_2 = \frac{b_1 (h_1^2 + h_1 h_m + h_m^2)}{2h_1} - \frac{b_m h_m}{2} = \frac{b_1 (h_1^2 + h_1 h_m)}{2h_1} > 0,
\]

where \(\sigma_1 = (b_1 (h_1^2 + h_1 h_m + h_m^2)) / 2h_1\) and \(\tau_2 = b_m h_m / 2\) are propagation speeds of \(S_1\) and \(J_2\), respectively. Hence, \(S_1\) and \(J_2\) interact after a finite time, say, \(t_1\) (see,
(a) Interaction of elementary waves in the \((x, t)\)-plane when solution of both the local Riemann problems consist of a contact discontinuity followed by a shock wave

(b) Elementary wave interactions in \((b, h)\)-plane with the notations \((l) = (h_l, b_l), (r) = (h_r, b_r), (m) = (h_m, b_m), (1) = (h_1, b_1), (2) = (h_2, b_2)\) and \((3) = (h_3, b_3)\)

**Figure 5.** Wave interactions when \(b_l h_l > b_m h_m > b_r h_r\). Figure 5(a)) and the point of interaction \((x_1, t_1)\) is determined by

\[
\begin{align*}
   x_1 + \epsilon &= \sigma_1 t_1, \\
   x_1 - \epsilon &= \tau_2 t_1,
\end{align*}
\]

which yields

\[
(x_1, t_1) = \left( \frac{\sigma_1 + \tau_2}{\sigma_1 - \tau_2} \epsilon, \frac{2 \epsilon}{\sigma_1 - \tau_2} \right).
\]

Therefore, at time \(t = t_1\) we have a new Riemann problem with initial data \((h_1, b_1) = (h_1, b_1)\) and \((h_r, b_r) = (h_2, b_2)\) and the solution of this new Riemann problem consists of a new contact discontinuity \(J_3\) and a new shock \(S_3\). We see that the propagation speeds of the contact discontinuities \(J_1\) and \(J_3\) are equal (i.e., \(\tau_1 = \tau_3\)). Hence, \(J_1\) and \(J_3\) are parallel and never interact. Now, for \(t > t_1\) the
propagation speeds of $S_2$ and $S_3$ are respectively given by
\[
\sigma_2 = \frac{(b_2(h_2^2 + h_r h_2 + h_r^2))/2h_2, \sigma_3 = (b_3(h_3^2 + h_3 h_r + h_r^2))/2h_3,}
\]
with $h_r < h_2 < h_3$ and $b_r < b_2 < b_3$. Straight forward calculations yield
\[
\sigma_3 - \sigma_2 = \frac{b_2(h_2 + h_3 + h_r)(h_3 - h_r)}{2h_2} > 0.
\]
Therefore, $S_3$ overtakes $S_2$ after a finite time, say, $t_2$ and the interaction point $(x_2, t_2)$ satisfies the following equations
\[
\begin{aligned}
&x_2 - \epsilon = \sigma_2 t_2, \\
x_2 - x_1 = \sigma_3 (t_2 - t_1).
\end{aligned}
\]
Solving (26), we obtain
\[
(x_2, t_2) = \left(\frac{\sigma_2 x_1 + \sigma_2 \sigma_3 t_1 - \sigma_3 \epsilon}{\sigma_2 - \sigma_3}, \frac{x_1 - \epsilon - \sigma_3 t_1}{\sigma_2 - \sigma_3}\right).
\]
Therefore, at $t = t_2$ again a new Riemann problem forms with the initial data $(h_1, b_1) = (h_3, b_3)$ and $(h_r, b_r)$. Since $b_2/h_3 = b_2/h_2 = b_r/h_r$, and $b_r h_r < b_2 h_3$, then $(h_3, b_3)$ and $(h_r, b_r)$ can be connected only by a shock wave (see, Figure 5(b)) and hence, for $t > t_2$, a new shock $S_4$ occurs (see, Figure 5(a)) with a speed of propagation $\sigma_4 = (b_3(h_3^2 + h_3 h_r + h_r^2))/2h_3$. Further, $\tau_3 - \sigma_4 = -(b_3(h_3 h_r + h_r^2))/2h_3 < 0$ implies that contact discontinuity $J_3$ can not overtake $S_4$. Therefore, for $t > t_2$, the solution can be expressed as (see, Figure 5(a))
\[
(h_1, b_1) + J_1 + (h_1, b_1) + J_3 + (h_3, b_3) + S_4 + (h_r, b_r).
\]
One can observe that, when the perturbation parameter $\epsilon \to 0$ then both $(x_1, t_1)$ and $(x_2, t_2)$ tend to $(0, 0)$ and the curves $J_1$ and $J_3$ coincide. Hence, we can conclude that for a sufficiently large time $t$, the results of interaction can be expressed as $(h_1, b_1) + J + (h_3, b_3) + S_4 + (h_r, b_r)$. This result indicates that the limit of the solution of initial value problem (5) and (8) is exactly the corresponding Riemann solution of (5) and (9). Therefore, the solution of Riemann problem (5) and (9) is globally stable under small perturbation.

Case 2: J+R and J+R

Here, we consider the initial data (8) such that the solution of both the local Riemann problems consist of contact discontinuity and rarefaction wave only. Therefore, for sufficiently small time $t$, the solution of (5) and (8) can be expressed as $(h_1, b_1) + J_1 + (h_1, b_1) + R_1 + (h_m, b_m) + J_2 + (h_2, b_2) + R_2 + (h_r, b_r)$. This situation occurs only when the initial data (8) satisfies the condition $b_2 h_1 \leq b_m h_m \leq b_r h_r$. The propagation speeds of head of rarefaction wave $R_1$ and contact discontinuity $J_2$ are respectively given by $\omega_1 = 3b_m h_m/2$ and $\tau_2 = b_m h_m/2$. It is clear that $\omega_1 > \tau_2$ and they interact at some point, say, at $(x_1, t_1)$ (see, Figure 6(a)) which is determined by
\[
\begin{aligned}
&x_1 + \epsilon = \omega_1 t_1, \\
x_1 - \epsilon = \tau_2 t_1.
\end{aligned}
\]
A direct computation yields
\[
(x_1, t_1) = \left(\frac{\omega_1 + \tau_2}{\omega_1 - \tau_2}, \frac{2\epsilon}{\omega_1 - \tau_2}\right).
\]
For $t > t_1$, the contact discontinuity $J_2$ is at the onset of interaction with the
(a) Interaction of elementary waves in \((x, t)\)-plane when solution of both the local Riemann problems consists of a contact discontinuity followed by a rarefaction wave

\[
J_1 \quad (h_1, b_1) \quad R_1 \quad J_2 \quad (x_1, t_1) \quad (h_2, b_2) \quad R_2 \quad J_3 \quad (x_2, t_2) \quad (h_3, b_3) \quad R_3
\]

\((-\epsilon, 0) \quad (\epsilon, 0)\)

(b) Elementary wave interactions in the \((b, h)\)-plane

**Figure 6.** Wave interactions when \(b_l h_1 \leq b_m h_m \leq b_r h_r\)

rarefaction wave \(R_1\). Accordingly, during the process of penetration, the curve of contact discontinuity \(J_2\) is determined by

\[
\frac{dx}{dt} = \frac{bh}{2}, \quad (30a)
\]

\[
x + \epsilon = \frac{3bh}{2} t, \quad (30b)
\]

\[
b = \frac{b_m h_m}{h}, \quad (30c)
\]

\[
x(t_1) = x_1, \quad h_1 \leq h \leq h_m. \quad (30d)
\]

From (30a), we obtain

\[
\frac{d^2x}{dt^2} = \frac{1}{2} \left( \frac{bh}{dt} + \frac{db}{dt} \right). \quad (31)
\]
Differentiating (30b) with respect to \( t \), we get
\[
\frac{dx}{dt} = \frac{3}{2} bh + \frac{3t}{2} \left( \frac{dh}{dt} + h \frac{dB}{dt} \right).
\]
Using (30a) and (31) in (32), we obtain
\[
\frac{d^2x}{dt^2} = -\frac{bh}{3t} = -\frac{b m}{3h_m} h^2 < 0,
\]
which implies that the contact discontinuity \( J_2 \) will decelerate during the process of penetration. Further, combining (30a) and (30b), we obtain
\[
\frac{dx}{dt} = \frac{x + \epsilon}{3t}, \quad x(t_1) = x_1.
\]
The solution of the initial value problem (34) is given by
\[
x(t) = (x_1 + \epsilon) \left( \frac{t}{t_1} \right)^{\frac{1}{2}}.
\]
It may be noted that the curve in equation (35) denotes contact discontinuity during the process of penetration. Also, the equation of the tail of the rarefaction wave \( R_1 \) curve is given by
\[
x + \epsilon = \frac{3b_1 h_1 t}{2}.
\]
The intersection of these two curves will give the end point of penetration. Accordingly, we solve (35) and (36) to get the end point of penetration, say, \( (x_2, t_2) \) given by
\[
\begin{align*}
x_2 &= \sqrt{\frac{2(x_1 + \epsilon)^3}{3b_1 h_1 t_1}}, \\
t_2 &= \left( \frac{2(x_1 + \epsilon)}{3b_1 h_1 t_1^2} \right)^{\frac{1}{2}}
\end{align*}
\]
Therefore, the contact discontinuity \( J_2 \) sweeps the entire rarefaction fan completely after a finite time \( t_2 \). As \( b_1 h_1 \geq b_2 h_2 \), for \( t > t_2 \), a new contact discontinuity \( J_3 \) and a new rarefaction wave \( R_3 \) can occur (see, Figure 6) as a solution of new Riemann problem. Since \( 3b_3 h_3/2 = 3b_1 h_1/2 \) and \( 3b_m h_m/2 = 3b_2 h_2/2 \), which implies that the head and tail of the rarefaction wave \( R_3 \) coincides with the head and tail of the rarefaction wave \( R_1 \). Further, with \( \tau_1 = \tau_3 \), the contact discontinuities \( J_1 \) and \( J_3 \) are parallel.

In the limit \( \epsilon \to 0 \), the points \( (x_1, t_1) \) and \( (x_2, t_2) \) approach \( (0, 0) \) and \( J_1 \) and \( J_3 \) coincide each other. Furthermore, in this limit the speed of propagation of the head of rarefaction wave \( R_3 \) and the propagation speed of the tail of rarefaction wave of \( R_2 \) are equal and hence \( R_3 \) and \( R_2 \) coincide each other in this limit. Therefore, for sufficiently large \( t \), the result of interaction can be expressed as \( (h_i, b_i) + J + (h_3, b_3) + R + (h_r, b_r) \). Hence, the solution is stable under small perturbation.

**Case 3: J+R and J+S**

This case occurs when the initial data (8) satisfies the inequalities \( b_l h_l \leq b_m h_m \) and \( b_l h_r < b_m h_m \). For small enough time \( t \), the solution of the initial value problem (5) and (8) can be represented as follows:

\[
(b_1, h_1) + J_1 + (h_1, b_1) + R_1 + (h_m, b_m) + J_2 + (h_2, b_2) + S_1 + (h_r, b_r).
\]

As discussed in **Case 2**, the interaction of \( R_1 \) and \( J_2 \) produce a contact discontinuity \( J_3 \) and a rarefaction wave \( R_2 \). Moreover, the propagation speeds of \( J_1 \) and \( J_3 \) are equal as \( \tau_1 = \tau_3 \) and hence \( J_3 \) is parallel to \( J_1 \).
(a) Interaction of elementary waves in \((x, t)\)-plane when solution of first local Riemann problem consists of a contact discontinuity followed by a rarefaction wave and the second local Riemann problem consists of a contact discontinuity followed by a shock wave.

Also the interaction point \((x_1, t_1)\) of \(R_1\) and \(J_2\) is given in (29) and the end point of penetration \((x_2, t_2)\) of \(R_1\) and \(J_2\) satisfies (37).

The propagation speed of the head of rarefaction wave fan \(R_2\) and \(S_1\) are respectively given by \(\omega_2 = 3b_2h_2/2\) and \(\sigma_2 = (b_2(h_2^2 + h_r h_2 + h_r^2))/2h_2\), where \(h_r < h_2\). Straight forward calculations yield

\[
\omega_2 - \sigma_2 = \frac{b_2(h_2^2 - h_r^2) + h_2(h_2 - h_r)}{2h_2} > 0.
\] (38)
Therefore, $R_2$ and $S_1$ interact at a point $(x_3, t_3)$ which can be determined by

$$
\begin{align*}
    x_3 - x_1 &= \omega_2(t_3 - t_1), \\
    x_3 - \epsilon &= \sigma_1 t_3.
\end{align*}
$$

From (39), we obtain

$$(x_3, t_3) = \left( \frac{x_1 - \epsilon - \omega_2 t_1}{\sigma_1 - \omega_2}, \frac{\sigma_1(x_1 - \epsilon - \omega_2 t_1)}{\sigma_1 - \omega_2} + \epsilon \right).$$

Therefore, for $t > t_3$, $S_1$ begins to penetrate $R_2$ and during the process of penetration, $S_1$ propagates with a varying speed which can be determined by
\[
\frac{dx}{dt} = \frac{b(h^2 + hh_x + h^2)}{2h}, \quad (40a)
\]
\[
x + \epsilon = \frac{3bh}{2}t, \quad (40b)
\]
\[
b = \frac{b_2}{h_2}h, \quad (40c)
\]
\[
x(t_3) = x_3, \quad h_3 \leq h \leq h_2, \quad b_3 \leq b \leq b_2. \quad (40d)
\]

From (40a) and (40b), we obtain
\[
\frac{3}{2} \left( \frac{bh}{dt} + \frac{db}{dt} \right) t = b \left( hh_x + h^2 - 2h^2 \right). \quad (41)
\]

Usage of (40c) in (41) yields
\[
\frac{dh}{dt} = \frac{bh^2(hh_x + h^2)}{3h(hb_2 + bh_2)t}. \quad (42)
\]

Using the properties of the rarefaction wave (please refer Theorem 2.1), it may be noted that \( \frac{dh}{dt} < 0 \). Now differentiating (40a) and combining with (40c), we get
\[
\frac{d^2x}{dt^2} = \frac{b_r(h_r + 2h)}{2h_r} \frac{dh}{dt} < 0. \quad (43)
\]

The estimate (43) indicates that \( S_1 \) decelerates during the process of penetration. Integrating (42), we obtain
\[
\int_{h_3}^{h} \frac{h}{hh_x + h^2 - 2h^2} dh = \ln \frac{t}{t_3}. \quad (44)
\]

Hence, for \( t > t_3 \), \( S_1 \) sweeps the entire rarefaction fan when \( h_r < h_3 \) (see, Figure 7) after a finite time \( t_4 \) given by
\[
t_4 = t_3 \exp \left( \int_{h_3}^{h_r} \frac{h}{hh_x + h^2 - 2h^2} dh \right).
\]

This indicates that \( S_1 \) can never overtake the entire \( R_2 \) when \( h_3 \leq h_r \) (see, Figure 8) since \( t \to \infty \) as \( h \to h_r \) and the propagation speed of the shock is \( 3b_r h_r/2 \) as \( t \to \infty \).

Therefore, in the limit \( \epsilon \to 0 \), for a sufficiently large time \( t \), when \( h_r b_r < h_l b_l \), the solution can be expressed as follows \( (h_l, b_l) + J + (h_3, b_3) + S + (h_r, b_r) \) (see, Figure 7), whereas when \( h_r b_r \geq h_l b_l \), the solution can be expressed as \( (h_l, b_l) + J + (h_3, b_3) + R + (h_r, b_r) \) (see, Figure 8). Further, for a sufficiently large time, the shock wave has \( x + \epsilon = 3b_r h_r t/2 \) as its asymptote in the latter case (i.e., when \( h_r b_r \geq h_l b_l \)). Hence, the solution of the Riemann problem (5) and (9) is stable under small perturbation.

**Case 4: J+S and J+R**

For a sufficiently small time \( t \), the solution of (5) and (8) can be expressed as \( (h_1, b_1) + J_1 + (h_3, b_3) + S_1 + (h_m, b_m) + J_2 + (h_2, b_2) + R_3 + (h_r, b_r) \) under the assumptions \( b_l b_t > b_m b_m \) and \( b_m b_m \leq b_r h_r \). As discussed in **Case 1**, \( S_1 \) and \( J_2 \) interact after a finite time, say, at \( (x_1, t_1) \) (see, Figure 10(a)) which satisfies (25). This interaction leads to a new contact discontinuity \( J_3 \) and a new shock wave \( S_2 \). The propagation speeds of shock wave \( S_2 \) and the tail of rarefaction wave \( R_1 \)
are respectively given by \( \sigma_2 = \left( h_3^2 + h_3 h_2 + h_2^3 \right) / 2h_3 \) and \( \omega_1 = 3b_2 h_2 / 2 \) with \( h_2 < h_3 \). Accordingly, we have the following comparison

\[
\sigma_2 - \omega_1 = \frac{b_2}{2h_2} \left( h_3^2 - h_2^2 + h_2(h_3 - h_2) \right) > 0.
\]

Therefore, \( S_2 \) interacts with \( R_2 \) after a finite time, say, \((x_2, t_2)\) (see, Figure 10(a)) which can be determined from the following equations

\[
\begin{align*}
x_2 - x_1 &= \sigma_2(t_2 - t_1), \\
x_2 - \epsilon &= \omega_1 t_2.
\end{align*}
\]
For $t > t_2$, the shock wave $S_2$ begins to penetrate $R_1$ with a varying speed of propagation given by

$$\frac{dx}{dt} = \frac{b_3(h_3^2 + hh_3 + h^2)}{2h_3}, \quad (46a)$$

$$x - \epsilon = \frac{3bh}{2}t, \quad (46b)$$

$$b = \frac{b_3}{h_3}h, \quad (46c)$$

$$x(t_2) = x_2, \ h_2 \leq h < h_3, \ b_2 \leq b < b_3. \quad (46d)$$

Further, from (46a) and (46c) we get

$$\frac{d^2 x}{dt^2} = \frac{b_3}{2h_3} (2h + h_3) \frac{dh}{dt} \quad \text{and} \quad \frac{db}{dt} = \frac{b_2}{h_2} \frac{dh}{dt}. \quad (47)$$
Using (46a) and (46b), we get
\[
\frac{dh}{dt} = \frac{(h_3^2 - h^2 + h(h_3 - h))}{3ht} > 0.
\]
Hence, we see that \(d^2x/dt^2 > 0\) indicating that the shock wave accelerates during the penetration process. Therefore, as done in Case 3, for a sufficiently large time \(t\), when \(h_r b_r \geq h_l b_l\) (i.e., when \(h_r \geq h_3\)), the solution can be expressed as follows \((h_1, b_1) + J + (h_3, b_3) + R + (h_r, b_r)\) (see, Figure 9), whereas when \(h_r b_r \geq h_l b_l\) (i.e., when \(h_r < h_3\)), the solution can be expressed as \((h_1, b_1) + J + (h_3, b_3) + S + (h_r, b_r)\) (see, Figure 10) as the limit \(\epsilon \to 0\). Hence, the solution of Riemann problem (5) and (9) is stable under such a small perturbation.

We summarize our results as in Theorem 4.1:

**Theorem 4.1.** If the perturbation parameter \(\epsilon \to 0\), then the limiting solution of the perturbed Riemann problem (5) and (8) exactly matches with the solution of the corresponding Riemann problem (5) and (9). Moreover, the initial states \((h_1, b_1)\) and \((h_r, b_r)\) completely govern the asymptotic behavior of the perturbed Riemann solution which implies that the solution of Riemann problem (5) and (9) is stable under the small local perturbation in the initial data (9) of the Riemann problem.

5. **Conclusions.** We discussed all possible elementary wave interactions using characteristic analysis method and the solution of the perturbed Riemann problem is constructed globally. By letting the limit \(\epsilon \to 0\), it is observed that the solution of perturbed Riemann problem (5) and (8) exactly matches with the corresponding Riemann solution of (5) and (9). Therefore, the limiting solution of perturbed Riemann problem consists of contact discontinuity followed by a rarefaction wave or a shock wave. In the first case, the solutions for \(h\) and \(b\) are continuous everywhere except for contact discontinuity between regions \(U_l\) and \(U_r\), which propagate towards right into the region \(x > 0\) at a constant speed \(\tau\). However, in the latter case the solutions for both \(h\) and \(b\) are uniform everywhere except across contact discontinuity and shocks, which propagate towards right into the region \(x > 0\) with constant speeds \(\tau\) and \(\sigma\), respectively. Moreover, in this case the concentration gradient \((b_*)\) always lies between \(b_l\) and \(b_r\) and the value of \(h_*\) is always greater than both \(h_l\) and \(h_r\), i.e. the film is always thickest in the middle region between the contact discontinuity and the shock. Also, it is noticed that due to positive initial values of \(b\), concentration gradient is always positive which drives the fluid to right. Hence, the asymptotic behavior of solution is completely governed by the initial states \((h_1, b_1)\) and \((h_r, b_r)\).

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