Global Existence and Large Time Asymptotic Behavior of Strong Solutions to the Cauchy Problem of 2D Density-Dependent Magnetohydrodynamic Equations with Vacuum*

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Abstract

This paper concerns the Cauchy problem of the two-dimensional (2D) nonhomogeneous incompressible Magnetohydrodynamic (MHD) equations with vacuum as far field density. We establish the global existence and uniqueness of strong solutions to the 2D Cauchy problem on the whole space $\mathbb{R}^2$, provided that the initial density and the initial magnetic decay not too slow at infinity. In particular, the initial data can be arbitrarily large and the initial density can contain vacuum states and even have compact support. Furthermore, we also obtain the large time decay rates of the gradients of velocity, magnetic and pressure.

Keywords: nonhomogeneous incompressible MHD equations; global strong solution; large time behavior; vacuum.

Math Subject Classification: 35Q35; 76D03; 76W05.

1 Introduction

Magnetohydrodynamics is concerned with the interaction between fluid flow and magnetic field. The governing equations of nonhomogeneous incompressible MHD can be stated as follows [8],

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= \mu \Delta u + H \cdot \nabla H - \frac{1}{2} \nabla |H|^2, \\
H_t - \nu \Delta H + u \cdot \nabla H - H \cdot \nabla u &= 0, \\
div u &= 0, \quad div H = 0.
\end{align*}
$$

Here, $t \geq 0$ is time, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is the spatial coordinate, and $\rho = \rho(x, t)$, $u = (u_1, u_2)(x, t)$, $H = (H^1, H^2)(x, t)$, and $p = p(x, t)$ denote the density, velocity, magnetic, and pressure of the fluid, respectively; $\mu > 0$ stands for the viscosity constant; the constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields.

Let $\Omega = \mathbb{R}^2$ and we consider the Cauchy problem for (1.1) with $(\rho, u, H)$ vanishing at infinity (in some weak sense) and the initial conditions:

$$
\rho(x, 0) = \rho_0(x), \quad pu(x, 0) = \rho_0 u_0(x), \quad H(x, 0) = H_0(x), \quad x \in \mathbb{R}^2,
$$

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for given initial data $\rho_0, u_0$ and $H_0$.

Magnetohydrodynamics studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. The dynamic motion of the fluid and the magnetic field interact strongly with each other, so the hydrodynamic and electrodynamic effects are coupled. If this motion occurs in the absence of magnetic field, that is, $H = 0$, the MHD system reduces to the Navier-Stokes equations, which have been discussed in numerous studies, please refer to [3][4][6][11][13][27][33][36]. In general, due to the similarity of the second equation and the third equation in (1.1), the study for MHD system has been along with that for Navier-Stokes one. However, the issues of well-posedness and dynamical behaviors of MHD system are rather complicated to investigate because of the strong coupling and interplay interaction between the fluid motion and the magnetic field.

First, let us give a short survey for the study of incompressible Navier-Stokes equations, that is, the system (1.1) with $H = 0$. In the case when $\rho_0$ is bounded away from zero, Kazhikov [22] established the global existence of weak solutions (see also [3]). Later, Antontsev-Kazhikov-Monakhov [11] gave the first result on local existence and uniqueness of strong solutions, and then proved the unique local strong solution is global in two dimensions. When the initial data may contain vacuum states, Simon [33] obtained the global existence of weak solutions, see also Lions [20] for the case of density-dependent viscosity. Choe-Kim [6] proposed a compatibility condition and established the local existence of strong solutions. Under some suitable smallness conditions, the global existence of strong solutions on bounded domains were established by Huang-Wang [18][19] and Zhang [36], respectively. Recently, the local and global (with general large data) existence of strong solutions to the 2D Cauchy problem with vacuum on the whole space $\mathbb{R}^2$ were established, by Liang [25] and Lü-Shi-Zhong [27], respectively.

Let’s go back to the MHD system (1.1). When $\rho$ is a constant, which means the fluid is homogeneous, the MHD system has been extensively studied. Duraut-Lions [11] constructed a class of weak solutions with finite energy and a class of local strong solutions (the local strong solution has been proved to be global in two dimensions, but only local in three dimensions except for small data), see also Sermange-Temam [32]. For the nonhomogeneous case, Gerbeau and Le Bris [13], Desjardins and Le Bris [10] studied the global existence of weak solutions with finite energy on 3D bounded domains and on the torus, respectively. In the presence of vacuum, Abidi-Hmidi [1] and Abidi-Paicu [2] established the local and global (with small initial data) existence of strong solutions in some Besov spaces, respectively. In the presence of vacuum, under the following compatibility conditions,

$$\text{div} u_0 = \text{div} H_0 = 0, \quad -\Delta u_0 + \nabla p_0 - (H_0 \cdot \nabla) H_0 = \rho_0^{1/2} g, \quad \text{in } \Omega,$$

where $(p_0, g) \in H^1 \times L^2$ and $\Omega = \mathbb{R}^3$, Chen-Tan-Wang [5] obtained the local existence of strong solutions to the 3D Cauchy problem. When $\Omega \subset \mathbb{R}^2$ is a bounded domain, Huang-Wang [17] investigated the global existence of strong solution with general large data when the initial density contains vacuum states and the initial data satisfy the compatibility conditions (1.3). Very recently, Lü-Xu-Zhong [28] established the local existence of strong solution to the 2D Cauchy problem (1.1) with vacuum as far field density. However, the global existence of strong solution with general large data to the 2D Cauchy problem (1.1) with vacuum as far field density is still open. In fact, this is the main aim of this paper.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For $R > 0$, set

$$B_R \triangleq \{ x \in \mathbb{R}^2 | |x| < R \}, \quad \int \cdot \, dx \triangleq \int_{\mathbb{R}^2} \cdot \, dx.$$

Moreover, for $1 \leq r \leq \infty$ and $k \geq 1$, the standard Lebesgue and Sobolev spaces are defined as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}. $$
Next, we give the definition of strong solution to (1.1) as follows:

**Definition 1.1** If all derivatives involved in (1.1) for \((\rho, u, p, H)\) are regular distributions, and equations (1.1) hold almost everywhere in \(\mathbb{R}^2 \times (0, T)\), then \((\rho, u, p, H)\) is called a strong solution to (1.1).

Without loss of generality, we assume that the initial density \(\rho_0\) satisfies

\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]

which implies that there exists a positive constant \(N_0\) such that

\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int \rho_0 dx = \frac{1}{2}.
\]

Our main result can be stated as follows:

**Theorem 1.1** In addition to (1.4) and (1.5), assume that the initial data \((\rho_0, u_0, H_0)\) satisfy for any given numbers \(a > 1\) and \(q > 2\),

\[
\begin{cases}
\rho_0 \geq 0, \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, H_0 \bar{x}^{a/2} \in L^2, \sqrt{\rho_0} u_0 \in L^2, \\
\nabla u_0 \in L^2, \nabla H_0 \in L^2, \operatorname{div} u_0 = \operatorname{div} H_0 = 0,
\end{cases}
\]

where

\[
\bar{x} \triangleq (e + |x|^2)^{1/2} \log^2(e + |x|^2).
\]

Then the problem (1.1) - (1.2) has a unique global strong solution \((\rho, u, p, H)\) satisfying that for any \(0 < T < \infty\),

\[
\begin{align*}
0 &\leq \rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \\
\rho \bar{x}^a &\in L^\infty(0, T; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{\rho} \nabla P, \sqrt{\rho} \nabla^2 u &\in L^\infty(0, T; L^2), \\
H, H \bar{x}^{a/2}, \nabla H, \sqrt{\rho} \nabla H, \sqrt{\rho} \nabla^2 H &\in L^\infty(0, T; L^2), \\
\nabla u &\in L^2(0, T; H^1) \cap L^{(q+1)/q}(0, T; W^{1,q}), \\
\nabla P &\in L^2(0, T; L^2) \cap L^{(q+1)/q}(0, T; L^q), \\
\nabla H &\in L^2(0, T; H^1), H_t, \nabla H \bar{x}^{a/2} &\in L^2(0, T; L^2), \\
\sqrt{\rho} \nabla u &\in L^2(0, T; W^{1,q}), \\
\sqrt{\rho} u_t, \sqrt{\rho} \nabla H \bar{x}^{a/2}, \sqrt{\rho} \nabla u_t, \sqrt{\rho} \nabla H_t, \sqrt{\rho} \bar{x}^{-1} u_t &\in L^2(\mathbb{R}^2 \times (0, T)),
\end{align*}
\]

and

\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho(x, t) dx \geq \frac{1}{4},
\]

for some positive constant \(N_1\) depending only on \(\|\rho_0\|_{L^1}, \|\rho_0^{1/2} u_0\|_{L^2}, N_0, \) and \(T\). Moreover, \((\rho, u, p, H)\) has the following decay rates, that is, for \(t \geq 1\),

\[
\begin{align*}
\|\nabla u(\cdot, t)\|_{L^2} + \|\nabla H(\cdot, t)\|_{L^2} &\leq C t^{-1/2}, \\
\|\nabla^2 u(\cdot, t)\|_{L^2} + \|\nabla p(\cdot, t)\|_{L^2} + \|H\|_{W^{1,2}} &\leq C t^{-1},
\end{align*}
\]

where \(C\) depends only on \(\mu, \nu, \|\rho_0\|_{L^1 \cap L^\infty}, \|\rho_0^{1/2} u_0\|_{L^2}, \|\nabla u_0\|_{L^2}, \) and \(\|H_0\|_{H^1}. \)
Remark 1.1 When there is no electromagnetic field effect, that is $H = 0$, (1.1) turns to be the incompressible Navier-Stokes equations, and Theorem 1.1 is similar to the results of [27]. Roughly speaking, we generalize the results of [27] to the incompressible MHD system. Furthermore, the large time decay rates (1.10) with $H = 0$ are the same as those in [27], hence the magnetic field has no influence on the large time behaviors of the velocity and the pressure.

Remark 1.2 Our Theorem 1.1 holds for arbitrarily large initial data which is in sharp contrast to Lü-Shi-Xu [30] where the smallness conditions on the initial energy is needed in order to obtain the global existence of strong solutions to the 2D compressible MHD equations.

Remark 1.3 Compared with [15,17], there is no need to impose the additional compatibility condition on the initial data for the global existence of the strong solution.

We now make some comments on the analysis of this paper. Note that for initial data in the class satisfying $\|\cdot\|_2$, the local existence and uniqueness of strong solutions to the Cauchy problem, (1.1)-(1.2), have been established recently in [28] (see Lemma 2.1). To extend the strong solution globally in time, one needs some global a priori estimates on strong solutions to (1.1)-(1.2) in suitable higher norms. It should be pointed out that, on the one hand, the crucial techniques of proofs in [17,18] cannot be adapted to the situation treated here, since their arguments only hold true for the case of bounded domains. On the other hand, it seems difficult to bound the $L^4(\mathbb{R}^2)$-norm of $u$ just in terms of $||\sqrt{\rho} u||_{L^2(\mathbb{R}^2)}$ and $||\nabla u||_{L^2(\mathbb{R}^2)}$. To this end, we try to adapt some basic ideas used in [27], where they investigated the global existence of strong solutions to 2D Cauchy problem of the density-dependent Navier-Stokes equations. However, compared with [27], for the incompressible MHD equations treated here, the strong coupling between the velocity field and the magnetic field, such as $u \cdot \nabla H$, will bring out some new difficulties.

To overcome these difficulties stated above, some new ideas are needed. First, we try to obtain the estimates on the $L^\infty(0,T;L^2(\mathbb{R}^2))^3$-norm of the gradients of velocity and magnetic. On the one hand, motivated by [14,20,24], multiplying (1.1) by the material derivatives $\dot{u} = u_t + u \cdot \nabla u$ instead of the usual $u_t$ (see [17,18]), the key point is to control the term $\int |p| |\nabla u|^2 dx$. Motivated by [27] (see also [9]), using some facts on Hardy and BMO spaces (see Lemma 2.5), we succeed in bounding the term $\int |p| |\nabla u|^2 dx$ by $||\nabla p||_{L^2}||\nabla u||_{L^2}^2$ (see (3.5)). On the other hand, the usual $L^2(\mathbb{R}^2 \times (0,T))$-norm of $H_t$ cannot be directly estimated due to the strong coupled term $u \cdot \nabla H$. Motivated by [30], multiplying (1.1) by $\Delta H$ instead of the usual $H_t$ (see [17]), the coupled term $u \cdot \nabla H$ can be controlled after integration by parts (see (5.12)). Next, using the structure of the 2D magnetic equation (see (3.31) and (3.34)-(3.33)), we multiply (1.1) by $H \Delta |H|^2$ and thus obtain some useful a priori estimates on $||H||_{L^2}$ and $||H|\Delta H||_{L^2}$, which are crucial in deriving the time-independent estimates on both the $L^\infty(0,T;L^2(\mathbb{R}^2))^3$-norm of $t^{1/2}p^{1/2}u$ and the $L^2(\mathbb{R}^2 \times (0,T))$-norm of $t^{1/2} \nabla u$ (see (3.22)). This together with some careful analysis on the spatial weighted estimates of the density (see (3.44)) indicates the desired $L^1(0,T;L^\infty)$-bound of the gradient of the velocity (see (3.50)) which in particular implies the bound on the $L^\infty(0,T;L^q)$-norm of the gradient of the density. With the a priori estimates stated above at hand, using the similar arguments as in [23,27,30], the next step is to bound the higher order derivatives of the solutions $(\rho, u, p, H)$. Finally, some useful spatial weighted estimates on both $H$ and $\nabla H$ (see (3.63) and (3.64)) are derived, such a derivation yields the estimate on the $L^2(\mathbb{R}^2 \times (0,T))$-norms of both $t^{1/2} \nabla u_t$ and $t^{1/2} \nabla H_t$, and simultaneously also the bound of the $L^\infty(0,T;L^2(\mathbb{R}^2))$-norm of $t^{1/2} \nabla^2 H$, see Lemma 3.3 and its proof.

The rest of this paper is organized as follows. In Section 2 we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a priori estimates. Finally, Theorem 1.1 is proved in Section 4.
2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later. We start with the local existence theorem of strong solutions whose proof can be found in \[28\].

Lemma 2.1 Assume that \((\rho_0, u_0, H_0)\) satisfies \((1.6)\). Then there exists a small time \(T > 0\) and a unique strong solution \((\rho, u, p, H)\) to the problem \((1.1) - (1.2)\) in \(\mathbb{R}^2 \times (0, T)\) satisfying \((1.8)\) and \((1.9)\).

Next, the following well-known Gagliardo-Nirenberg inequality (see \[31\]) will be used later.

Lemma 2.2 (Gagliardo-Nirenberg) For \(s \in [2, \infty), q \in (1, \infty),\) and \(r \in (2, \infty),\) there exists some generic constant \(C > 0\) which may depend on \(s, q,\) and \(r\) such that for \(f \in H^1(\mathbb{R}^2)\) and \(g \in L^q(\mathbb{R}^2) \cap \mathring{D}^{1,r}(\mathbb{R}^2),\) we have

\[
\|f\|_{L^s(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{2/q-2}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{2/q},
\]

\[
\|g\|_{C(\mathbb{R}^2)} \leq C\|g\|_{L^q(\mathbb{R}^2)}^{q/(2r+q(r-2))}\|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+q(r-2))}.
\]

The following weighted \(L^m\) bounds for elements in \(\mathring{D}^{1,2}(\mathbb{R}^2) \triangleq \{ v \in H^1_{\text{loc}}(\mathbb{R}^2) | \nabla v \in L^2(\mathbb{R}^2) \}\) can be found in \[26\] Theorem B.1.

Lemma 2.3 For \(m \in [2, \infty)\) and \(\theta \in (1 + m/2, \infty),\) there exists a positive constant \(C\) such that we have for all \(v \in \mathring{D}^{1,2}(\mathbb{R}^2),\)

\[
\left( \int_{\mathbb{R}^2} \frac{|v|^{m}}{e + |x|^2}(\log(e + |x|^2))^{-\theta} dx \right)^{1/m} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}.
\]

The combination of Lemma 2.2 and the Poincaré inequality yields the following useful results on weighted bounds, whose proof can be found in \[24\] Theorem 2.4.

Lemma 2.4 Let \(\bar{x}\) be as in \((1.7)\). Assume that \(\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\) is a non-negative function such that

\[
\int_{B_{N_1}} \rho dx \geq M_1, \quad \|\rho\|_{L^1(\mathbb{R}^2)} \geq M_2,
\]

for positive constants \(M_1, M_2,\) and \(N_1 \geq 1\) with \(B_{N_1} \subset \mathbb{R}^2.\) Then for \(\varepsilon > 0\) and \(\eta > 0,\) there is a positive constant \(C\) depending only on \(\varepsilon, \eta, M_1, M_2,\) and \(N_1\) such that every \(v \in \mathring{D}^{1,2}(\mathbb{R}^2)\) satisfies

\[
\|v^{1/2} - v\|_{L^{2+\eta}(\mathbb{R}^2)} \leq C\|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)},
\]

with \(\bar{\eta} = \min\{1, \eta\}.\)

Finally, let \(H^1(\mathbb{R}^2)\) and \(BMO(\mathbb{R}^2)\) stand for the usual Hardy and \(BMO\) spaces (see \[31\] Chapter IV\)). Then the following well-known facts play a key role in the proof of Lemma 2.5 in the next section.

Lemma 2.5 (a) There is a positive constant \(C\) such that

\[
\|E \cdot B\|_{H^1(\mathbb{R}^2)} \leq C\|E\|_{L^2(\mathbb{R}^2)}\|B\|_{L^2(\mathbb{R}^2)},
\]

for all \(E \in L^2(\mathbb{R}^2)\) and \(B \in L^2(\mathbb{R}^2)\) satisfying

\[
\text{div} E = 0, \quad \nabla \perp \cdot B = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2).
\]

(b) There is a positive constant \(C\) such that

\[
\|v\|_{BMO(\mathbb{R}^2)} \leq C\|\nabla v\|_{L^2(\mathbb{R}^2)},
\]

for all \(v \in D^1(\mathbb{R}^2)\).
Proof. (a) For the detailed proof, please see [7, Theorem II.1].
(b) It follows from the Poincaré inequality that for any ball $B \subset \mathbb{R}^2$
\[
\frac{1}{|B|} \int_B |v(x)|^2 \, dx \leq C \left( \int_B |
abla v|^2 \, dx \right)^{1/2},
\]
which directly gives (2.6). □

3 A Priori Estimates

In this section, we will establish some necessary a priori bounds for strong solutions $(\rho, u, p, H)$ to the Cauchy problem (1.1)-(1.2) to extend the local strong solutions guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and $(\rho, u, p, H)$ be the strong solution to (1.1)-(1.2) on $\mathbb{R}^2 \times (0, T]$ with initial data $(\rho_0, u_0, H_0)$ satisfying (1.3)-(1.6).

In what follows, we will use the convention that $C$ denotes a generic positive constant depending on $\mu, \nu, a$, and the initial data, and use $C(\alpha)$ to emphasize that $C$ depends on $\alpha$.

3.1 Lower Order Estimates

First, since $\text{div} u = 0$, we have the following estimate on the $L^\infty(0, T; L^1)$-norm of the density.

Lemma 3.1 ([26]) There exists a positive constant $C$ depending only on $\|\rho_0\|_{L^1 \cap L^\infty}$ such that
\[
\sup_{t \in [0, T]} \|\rho\|_{L^1 \cap L^\infty} \leq C. \tag{3.1}
\]

The following lemma concerns the time-independent estimates on the $L^\infty(0, T; L^2)$-norm of the gradients of the velocity and the magnetic.

Lemma 3.2 There exists a positive constant $C$ depending only on $\mu, \nu$, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{1/2} u_0\|_{L^2}$, $\|\nabla u_0\|_{L^2}$, and $\|H_0\|_{H^1}$ such that
\[
\sup_{t \in [0, T]} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|H\|_{L^4}^4)
+ \int_0^T \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 + \|\nabla H\|_{H^1}^2 \right) \, dt \leq C. \tag{3.2}
\]

Here $\dot{v} \triangleq \partial_t v + u \cdot \nabla v$. Furthermore, one has
\[
\sup_{t \in [0, T]} \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|H\|_{L^4}^4 \right)
+ \int_0^T \left( \|\rho^{1/2} \ddot{u}\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 + \|\nabla H\|_{H^1}^2 \right) \, dt \leq C. \tag{3.3}
\]

Proof. First, applying standard energy estimate to (1.1) gives
\[
\sup_{t \in [0, T]} \left( \|\rho^{1/2} u\|_{L^2}^2 + \|H\|_{L^2}^2 \right)
+ \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \, dt \leq C. \tag{3.4}
\]

Next, multiplying (1.2) by $\dot{u}$ and integrating the resulting equality over $\mathbb{R}^2$ lead to
\[
\int \rho |\dot{u}|^2 \, dx = \int \mu \Delta u \cdot \dot{u} \, dx - \int \nabla p \cdot \dot{u} \, dx - \frac{1}{2} \int \nabla |H|^2 \cdot \dot{u} \, dx + \int H \cdot \nabla H \cdot \dot{u} \, dx \triangleq I_1 + I_2 + I_3 + I_4. \tag{3.5}
\]
It follows from integration by parts and Gagliardo-Nirenberg inequality that

\[
I_1 = \int \mu \Delta u \cdot (u_t + u \cdot \nabla u) \, dx
\]
\[
= -\frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 - \mu \int \partial_t u^j \partial_i (u^k \partial_k u^j) \, dx
\]
\[
\leq -\frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^3}^3
\]
\[
\leq -\frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla^2 u \|_{L^2}. \tag{3.6}
\]

Integration by parts together with (1.1) gives

\[
I_2 = -\int \nabla p \cdot (u_t + u \cdot \nabla u) \, dx
\]
\[
= \int p \partial_j u^i \partial_i u^j \, dx
\]
\[
\leq C \| p \|_{BMO} \| \partial_j u^i \partial_i u^j \|_{H^1}, \tag{3.7}
\]

where one has used the duality of $H^1$ and $BMO$ (see [34, Chapter IV]) in the last inequality. Since $\text{div}(\partial_j u) = \partial_j \text{div} u = 0$ and $\nabla \cdot (\nabla u) = 0$, Lemma 2.3 yields

\[
|I_2| = \left| \int p \partial_j u^i \partial_i u^j \, dx \right| \leq C \| \nabla p \|_{L^2} \| \nabla u \|_{L^2}. \tag{3.8}
\]

For the term $I_3$, integration by parts together with (1.1) and (2.1) leads to

\[
I_3 = \frac{1}{2} \int |H|^2 \partial_i u^j \partial_j u^i \, dx
\]
\[
\leq C \| H \|_{L^6}^3 \| \nabla u \|_{L^3}^3
\]
\[
\leq C \| H \|_{L^2}^4 \| \nabla H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla^2 u \|_{L^2}. \tag{3.9}
\]

Using (1.1) and (2.1), one deduces from integration by parts and (2.1) that

\[
I_4 = \int H \cdot \nabla H \cdot u_t \, dx + \int H \cdot \nabla H \cdot (u \cdot \nabla u) \, dx
\]
\[
= -\frac{d}{dt} \int H \cdot \nabla u \cdot H \, dx + \int H_t \cdot \nabla u \cdot H \, dx + \int H \cdot \nabla u \cdot H \, dx
\]
\[
- \int H^i \partial_i u^j \partial_j u^k H^k \, dx - \int H \partial_i u^j \partial_j u^k H^k \, dx
\]
\[
= -\frac{d}{dt} \int H \cdot \nabla u \cdot H \, dx + \int (\nu \Delta H - u \cdot \nabla H + H \cdot \nabla u) \cdot \nabla u \cdot H \, dx
\]
\[
+ \int H \cdot \nabla u \cdot (\nu \Delta H - u \cdot \nabla H + H \cdot \nabla u) \, dx - \int H^i \partial_i u^j \partial_j u^k H^k \, dx
\]
\[
+ \int w^j \partial_j H^i \partial_i u^k H^k \, dx + \int H^i \partial_i u^k \partial_j H^k \, dx
\]
\[
\leq -\frac{d}{dt} \int H \cdot \nabla u \cdot H \, dx + \frac{\nu}{2} \| \Delta H \|_{L^2}^2 + C \| H \|_{L^2}^2 \| \nabla H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla^2 u \|_{L^2}. \tag{3.10}
\]

Hence, inserting (3.6) and (3.8), (3.10) into (3.5) indicates that

\[
\frac{d}{dt} \left( \frac{\mu}{2} \| \nabla u \|_{L^2}^2 + \int H \cdot \nabla u \cdot H \, dx \right) + \| \sqrt{\mu} \dot{u} \|_{L^2}^2
\]
\[
\leq \frac{\nu}{2} \| \Delta H \|_{L^2}^2 + C \| H \|_{L^2}^2 \| \nabla H \|_{L^2}^2 + C \left( \| \nabla^2 u \|_{L^2} + \| \nabla p \|_{L^2} \right) \| \nabla u \|_{L^2}^2. \tag{3.11}
\]
Now, multiplying (1.13) by $\Delta H$ and integrating the resulting equality by parts over $\mathbb{R}^2$, it follows from Hölder’s and Gagliardo-Nirenberg inequalities that

$$
\frac{d}{dt} \int |\nabla H|^2 dx + 2\nu \int |\Delta H|^2 dx \\
\leq C \int |\nabla u| |\nabla H|^2 dx + C \int |\nabla u||H||\Delta H| dx \\
\leq C ||\nabla u||L^3 ||\nabla H||^\frac{2}{3} L^\frac{4}{3} ||\Delta H||^\frac{2}{3} L^\frac{4}{3} + C ||\nabla u||L^3 ||H||L^2 ||\nabla^2 H||L^2 \\
\leq C ||\nabla u||L^2 ||\nabla^2 u||L^2 + C(1 + ||H||^\frac{2}{3} L^\frac{4}{3}) ||\nabla H||L^2 + \frac{\nu}{2} ||\Delta H||^2 L^2, \tag{3.12}
$$

which together with (3.11) and (3.4) gives

$$
\frac{d}{dt} \left( \frac{\mu}{2} ||\nabla u||^2 L^2 + ||\nabla H||^2 L^2 + \int H \cdot \nabla u \cdot H dx \right) + ||\sqrt{\rho u}||^2 L^2 + \nu ||\Delta H||^2 L^2 \\
\leq C ||\nabla H||^\frac{4}{2} L^2 + C (||\nabla^2 u||L^2 + ||\nabla p||L^2) ||\nabla u||^2 L^2. \tag{3.13}
$$

On the other hand, since $(\rho, u, p, H)$ satisfies the following Stokes system

$$
\begin{cases}
-\mu \Delta u + \nabla p = -\rho \dot{u} + H \cdot \nabla H - \frac{1}{2} \nabla |\nabla H|^2, & x \in \mathbb{R}^2, \\
\text{div} u = 0, & x \in \mathbb{R}^2, \\
u \nabla u \rightarrow 0, & |x| \rightarrow \infty, \tag{3.14}
\end{cases}
$$

applying the standard $L^r$-estimate to (3.14) (see 3.5) yields that for any $r > 1$,

$$
||\nabla^2 u||_{L^r} + ||\nabla p||_{L^r} \leq C ||\rho \dot{u}||_{L^r} + C ||H|| ||\nabla H||_{L^r} \leq C ||\sqrt{\rho u}||_{L^r} + C ||H|| ||\nabla H||_{L^r}, \tag{3.15}
$$

where in the last inequality one has used 3.11.

Thus, it follows from (3.13) and (3.15) that

$$
\frac{d}{dt} B(t) + ||\sqrt{\rho u}||^2 L^2 + \nu ||\Delta H||^2 L^2 \\
\leq C ||\nabla H||^\frac{4}{3} L^2 + C ||\nabla u||^\frac{4}{3} L^2 + \frac{\nu}{2} ||\Delta H||^2 L^2, \tag{3.16}
$$

where

$$
B(t) \triangleq \frac{\mu}{2} ||\nabla u||^2 L^2 + ||\nabla H||^2 L^2 + \int H \cdot \nabla u \cdot H dx \tag{3.17}
$$

satisfies

$$
\frac{\mu}{4} ||\nabla u||^2 L^2 + ||\nabla H||^2 L^2 - C_1 ||H||^\frac{4}{3} L^2 \leq B(t) \leq C ||\nabla u||^2 L^2 + C ||\nabla H||^2 L^2 \tag{3.18}
$$

owing to (2.1), (3.3) and the following estimate

$$
\int |H \cdot \nabla u \cdot H| dx \leq \frac{\mu}{4} ||\nabla u||^2 L^2 + C_1 ||H||^\frac{4}{3} L^2. \tag{3.19}
$$

Next, multiplying (1.13) by $|H||H|^2$ and integrating the resulting equality by parts over $\mathbb{R}^2$ lead to

$$
\frac{1}{4} (||H||^2 L^2),_t + \nu ||\nabla H|| ||H||^2 L^2 + \frac{\nu}{2} ||\nabla |H||^2 L^2 \\
\leq C ||\nabla u||L^2 |||H||^2 L^2 + \frac{\nu}{2} ||\nabla |H||^2 L^2 \\
\leq C ||\nabla u||L^2 |||H||^2 L^2 + \nabla |H|| ||H||L^2 \\
\leq \frac{\nu}{4} ||\nabla |H||^2 L^2 + C ||\nabla u||^2 L^2 + C ||\nabla H||^2 L^2. \tag{3.20}
$$
due to Gagliardo-Nirenberg inequality and (3.4).

Now, adding (3.20) multiplied by 4\((C_1 + 1)\) to \(3.16\) and choosing \(\varepsilon\) suitably small, we obtain after using (3.18) that

\[
\frac{d}{dt} \left( B(t) + (C_1 + 1)\|H\|_{L^4}^4 \right) + \frac{1}{2}\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \nu \|\Delta H\|_{L^2}^2 + 2\nu \|\nabla H\|_{L^2}^2 \\
\leq C \left( \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) (B(t) + (C_1 + 1)\|H\|_{L^4}^4). \tag{3.21}
\]

This combined with (3.4), (3.18), and Gronwall’s inequality gives (3.2).

Finally, applying Gronwall’s inequality to (3.21) multiplied by \(t\), together with (3.3) and (3.18) yields (3.3) and finishes the proof of Lemma 3.2. \(\square\)

**Lemma 3.3** There exists a positive constant \(C\) depending only on \(\mu, \nu, \|\rho_0\|_{L^1 \cap L^\infty}, \|\rho_1^{1/2} u_0\|_{L^2}, \|\nabla u_0\|_{L^2}, \text{ and } \|H_0\|_{H^1}\) such that for \(i = 1, 2,\)

\[
\sup_{t \in [0, T]} t^i \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|H\|_{L^\infty}^2 \right) + \int_0^T t^i \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\Delta H\|_{H^1}^2 \right) dt \leq C, \tag{3.22}
\]

and

\[
\sup_{t \in [0, T]} t^i \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \right) \leq C. \tag{3.23}
\]

**Proof.** Motivated by [14, 20, 24], operating \(\partial_t + u \cdot \nabla\) to (1.1)\(^i\), one gets by some simple calculations that

\[
\partial_t (\rho \dot{u}^i) + \text{div}(\rho u \dot{u}^i) - \mu \Delta \dot{u}^i \\
= -\mu \partial_1 (\partial_1 u \cdot \nabla u^i) - \mu \text{div}(\partial_1 u \partial_1 u^i) - \partial_1 \partial_2 p - (u \cdot \nabla) \partial_2 p \\
- \frac{1}{2} \partial_2 (\partial_2 |H|^2) - \frac{1}{2} u \cdot \nabla (\partial_2 |H|^2) + \partial_2 (H \cdot \nabla H^j) + u \cdot \nabla (H \cdot \nabla H^j), \tag{3.24}
\]

which multiplied by \(\dot{u}^i\), together with integration by parts and (1.1)\(^i\), leads to

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx = -\int \mu \partial_1 (\partial_1 u \cdot \nabla u^i) \dot{u}^i dx - \int \mu \text{div}(\partial_1 u \partial_1 u^i) \dot{u}^i dx \\
- \int (\dot{u}^i \partial_1 \partial_2 p + \dot{u}^i (u \cdot \nabla) \partial_2 p) dx \\
- \frac{1}{2} \int \dot{u}^i (\partial_2 (\partial_2 |H|^2) + u \cdot \nabla (\partial_2 |H|^2)) dx \\
+ \int \dot{u}^i (\partial_2 (H \cdot \nabla H^j) + u \cdot \nabla (H \cdot \nabla H^j)) dx
\]

\[
\triangleq \sum_{i=1}^{5} J_i. \tag{3.25}
\]

We estimate each term on the right-hand side of (3.25) as follows.

First, by the same arguments as in [24, Lemma 3.3], one has

\[
\sum_{i=1}^{3} J_i \leq \frac{d}{dt} \int p \partial_1 u^i \partial_1 u^i dx + C \left( \|p\|_{L^1}^2 + \|\nabla u\|_{L^4}^4 \right) + \frac{\mu}{6} \|\nabla \dot{u}\|_{L^2}^6. \tag{3.26}
\]

Next, it follows from integration by parts, (1.1)\(^i\), (1.1)\(^5\), and (2.1) that

\[
J_4 = \int \partial_2 \dot{u}^i H \cdot H dx - \frac{1}{2} \int \dot{u}^i u^i \partial_2 |H|^2 dx
\]
Integrating (3.31) multiplied by 4
\[\nu\]
it is easy to deduce from (1.1)

Substituting (3.26)-(3.28) into (3.25) gives

Similar to (3.27), we also have

Next, as in [29, 30], for \(a_1, a_2 \in \{-1, 0, 1\}\), denote

it is easy to deduce from (1.13) that

Integrating (3.31) multiplied by \(4\nu^{-1}\bar{H}\triangle\bar{H}^2\) by parts over \(\mathbb{R}^2\) leads to

Noticing that

and

with

it thus follows from (3.32) multiplied by \((C_2 + 1)\) that

\[\frac{d}{dt}(\nu^{-1}(C_2 + 1)G(t)) + (C_2 + 1)\|\Delta H\|_{L^2}^2 \leq C\|\nabla u\|_{L^4}^4 + C\|\nabla H\|_{L^4}^4 + C\|H\|_{L^4}^4.\]
This combined with (3.29) yields that
\[
\frac{d}{dt} F(t) + \frac{\mu}{2} \| \sqrt{p}u \|^2_{L^2} + \| \Delta H \| H \|_{L^2}^2 \leq C \| p \|_{L^4}^4 + C \| \nabla u \|_{L^4}^4 + C \| \nabla H \|_{L^4}^4 + C \| H \|_{L^4}^2, \tag{3.36}
\]
where
\[
F(t) \equiv \frac{1}{2} \| \sqrt{p}u \|^2_{L^2} + \nu^{-1}(C_2 + 1)G(t) - \int p \partial_j u^j \partial_i u^i \, dx \tag{3.37}
\]
satisfies
\[
\frac{1}{4} \| \sqrt{p}u \|^2_{L^2} + \frac{\nu^{-1}(C_2 + 1)}{2} G(t) \leq F(t) \leq C \| \sqrt{p}u \|^2_{L^2} + CG(t) + C \| \nabla u \|_{L^2}^4 \tag{3.38}
\]
owing to the following estimate
\[
\left| \int p \partial_j u^j \partial_i u^i \, dx \right| \leq C(\| \sqrt{p}u \|^2_{L^2} + \| H \|_{L^2}^2) \| \nabla u \|_{L^2}^2 \leq \frac{1}{2} \| \sqrt{p}u \|^2_{L^2} + \frac{\nu^{-1}(C_2 + 1)}{2} G(t) + C \| \nabla u \|_{L^2}^4, \tag{3.39}
\]
which is deduced from (3.8), (3.15), (3.34) and Young’s inequality.

Now, we shall estimate the terms on the right-hand side of (3.36). On the one hand, it follows from Sobolev’s inequality, (3.15), (2.1), (3.1), and (3.4) that
\[
\| p \|^4_{L^4} + \| \nabla u \|^4_{L^4} \leq C \| \nabla p \|^4_{L^4/3} + C \| \nabla u \|^4_{L^4/3} \leq C \| \sqrt{p}u \|^4_{L^4/3} + C \| H \|_{L^4}^4 \| \nabla H \|^4_{L^4/3} \leq C \| \sqrt{p}u \|^4_{L^2} + C \| H \|_{L^2}^4 \| \nabla H \|^4_{L^2}. \tag{3.40}
\]
On the other hand, it holds from (2.1) and (3.4) that
\[
\| \nabla H \|^4_{L^4} + \| H \|^2_{L^4} \leq C \| \nabla H \|^2_{L^2} \| \nabla^2 H \|^2_{L^2} + C \| \nabla H \|^2_{L^2} \| H \|_{L^2} \| \nabla H \|^2_{L^2}. \tag{3.41}
\]
Thus, putting (3.40)–(3.41) into (3.39), together with (3.31) and (3.38), one has
\[
\frac{d}{dt} F(t) + \frac{\mu}{2} \| \sqrt{p}u \|^2_{L^2} + \| \Delta H \| H \|_{L^2}^2 \leq C \left( \| \sqrt{p}u \|^2_{L^2} + \| \nabla H \|^2_{L^2} \right) (F(t) + \| \nabla u \|^4_{L^2}) + C \| \nabla H \|^2_{L^2} \| \nabla^2 H \|^2_{L^2}. \tag{3.42}
\]
Then, applying Gronwall’s inequality to (3.42) multiplied by \( t^i \) (\( i = 1, 2 \)), it follows from (3.31), (3.38), (3.42), (3.3), and (3.3) that
\[
\sup_{t \in [0,T]} \left( t^i F(t) \right) + \int_0^T t^i \| \sqrt{p}u \|^2_{L^2} dt + \int_0^T t^i \| \nabla H \|^2_{L^2} \| H \|^2_{L^2} dt \leq C \int_0^T t^{i-1} F(t) dt + C \int_0^T t^i \| \nabla H \|^2_{L^2} \| \nabla^2 H \|^2_{L^2} dt + C \int_0^T \left( \| \sqrt{p}u \|^2_{L^2} + \| \nabla H \|^2_{L^2} \right) t^i \| \nabla u \|^4_{L^2} dt \leq C \int_0^T t^{i-1} \left( \| \sqrt{p}u \|^2_{L^2} + \| H \| \| \nabla H \|^2_{L^2} \right) dt + C \sup_{t \in [0,T]} (t^{i-1} \| \nabla u \|^2_{L^2}) \int_0^T \| \nabla u \|^2_{L^2} dt \leq C. \tag{3.43}
\]
This together with (3.31), (3.38), and (3.3) yields the desired result (3.22), which combined with (3.15) implies (3.23). The proof of Lemma 3.3 is completed. \( \square \)
3.2 Higher order estimates

The following spatial weighted estimate on the density plays an important role in deriving the bounds on the higher order derivatives of the solutions \((\rho, u, p, H)\).

**Lemma 3.4** There exists a positive constant \(C\) depending on \(T\) such that

\[
\sup_{t \in [0, T]} \|\rho \bar{x}^a\|_{L^1} \leq C(T). \tag{3.44}
\]

**Proof.** First, for \(N > 1\), let \(\varphi_N \in C_0^\infty(B_N)\) satisfy

\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = \begin{cases} 
1, & |x| \leq N/2, \\
0, & |x| \geq N,
\end{cases} \quad |\nabla \varphi_N| \leq C N^{-1}. \tag{3.45}
\]

It follows from (1.1) that

\[
dt \int \rho \varphi_N dx = \int \rho u \cdot \nabla \varphi_N dx
\]

\[
\geq -CN^{-1} \left( \int \rho dx \right)^{1/2} \left( \int \rho |u|^2 dx \right)^{1/2} \geq -\tilde{C} N^{-1} \tag{3.46}
\]

owing to (3.1) and (3.4). Integrating (3.46) and choosing \(N = N_1 = 2N_0 + 4 \tilde{C} T\), we obtain after using (1.5) that

\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho dx \geq \inf_{0 \leq t \leq T} \int \rho \varphi_{N_1} dx
\]

\[
\geq \int_{B_{N_0}} \rho_0 dx - \tilde{C} T \frac{2N_0 + 4 \tilde{C} T}{2N_0 + 4 \tilde{C} T}
\]

\[
\geq 1/4. \tag{3.47}
\]

Hence, it follows from (3.47), (3.1) and (2.5) that for any \(v \in \tilde{D}^{1,2}(\mathbb{R}^2)\),

\[
\|v \bar{x}^{-\eta}\|_{L^\infty} \leq C(\eta, s)(\|\rho^{1/2} v\|_{L^2} + \|\nabla v\|_{L^2}), \tag{3.48}
\]

where \(\eta \in (0, 1]\) and \(s > 2\). In particular, we deduce from (3.48), (3.4), and (3.2) that

\[
\|u \bar{x}^{-\eta}\|_{L^\infty} \leq C(\|\rho^{1/2} u\|_{L^2} + \|\nabla u\|_{L^2}) \leq C. \tag{3.49}
\]

Multiplying (1.1) by \(\bar{x}^a\) and integrating the resulting equation by parts over \(\mathbb{R}^2\) yield that

\[
\frac{d}{dt} \int \rho \bar{x}^a dx \leq C \int \rho |u| \bar{x}^{a-1} \log(e + |x|^2) dx
\]

\[
\leq C \|\rho \bar{x}^{a-1} \frac{\bar{x}}{s+a} \|_{L^{s+a}} \|u \bar{x}^{-\frac{1}{s+a}}\|_{L^{s+a}}
\]

\[
\leq C \int \rho \bar{x}^a dx + C, \tag{3.50}
\]

which along with Gronwall’s inequality gives (3.44) and finishes the proof of Lemma 3.4.

**Lemma 3.5** There exists a positive constant \(C\) depending on \(T\) such that

\[
\sup_{t \in [0, T]} \|\rho\|_{H^1 \cap W^{1,a}} + \int_0^T \left( \|
abla^2 u\|_{L^2}^2 + \|
abla^2 u\|_{L^q}^{q+1} + t\|
abla^2 u\|_{L^{2q} \cap L^q}^2 \right) dt
\]

\[
+ \int_0^T \left( \|
abla p\|_{L^2}^2 + \|
abla p\|_{L^q}^{q+1} + t\|
abla p\|_{L^{2q} \cap L^q}^2 \right) dt \leq C(T). \tag{3.51}
\]
Proof. First, it follows from the mass equation (1.1) that $\nabla \rho$ satisfies for any $r \geq 2$,

$$\frac{d}{dt} \|\nabla \rho\|_{L^r} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^r}. \quad (3.52)$$

Next, one gets from Gagliardo-Nirenberg inequality (2.2), (3.2), and (3.14) that for $q > 2$,

$$\|\nabla u\|_{L^q} \leq C(q) \|\nabla u\|_{L^{2(q-1)}} \|\nabla^2 u\|_{L^{2(q-1)}}^{\frac{q-2}{q}} \leq C \left( \|\rho \dot{u}\|_{L^q}^{\frac{q-2}{q}} + ||H|| \|\nabla H\|_{L^q}^{\frac{q-2}{q}} \right). \quad (3.53)$$

Notice that, it is easy to deduce from (3.1), (3.48), and (3.44) that for any $s > 2$,

$$\|\rho^s v\|_{L^q} \leq C \|\rho\|_{L^\infty}^{\frac{3n}{q}} \|v\|_{L^q}^{\frac{3n}{q}} \leq C \left( \|\rho\|_{L^\infty}^{\frac{3n}{q}} + \|\nabla u\|_{L^2} \right) \quad (3.54)$$

which together with Gagliardo-Nirenberg inequality yields that

$$\|\rho \dot{u}\|_{L^q} \leq C \|\rho \|_{L^\infty}^{\frac{2(q-1)}{q^2-2}} \|\rho \dot{u}\|_{L^2}^{\frac{q(q-2)}{q^2-2}} \leq C \|\sqrt{\rho} \|_{L^2} \left( \sqrt{\rho} \|\dot{u}\|_{L^2} + \|\nabla u\|_{L^2} \right) \quad (3.55)$$

This combined with (3.2) and (3.22) implies that

$$\int_0^T \left( \|\rho \dot{u}\|_{L^q}^{\frac{q+1}{q}} + t \|\rho \dot{u}\|_{L^q}^{2} \right) dt \leq C \int_0^T \left( \|\rho \dot{u}\|_{L^2}^2 + t \|\nabla u\|_{L^2}^2 + t \frac{3^2 q^2 - 2q + 1}{q^2 - 2q + 1} + 1 \right) dt \leq C. \quad (3.56)$$

Furthermore, it follows from (3.2) and (3.3) that

$$\int_0^T \left( \|H||\nabla H||_{L^q}^{\frac{q+1}{q}} + t \|H||\nabla H||_{L^2}^2 \right) dt \leq C \int_0^T \left( \|\nabla^2 H||_{L^2}^{2-1/q} + t \|\nabla^2 H||_{L^2}^{2-2/q} \right) dt \leq C \int_0^T \left( 1 + t^q + \|\nabla^2 H||_{L^2}^2 \right) dt \leq C, \quad (3.57)$$

where one has used the following estimate

$$\|H||\nabla H||_{L^q} \leq C(q) \|H||_{L^2}^{\frac{1}{q}} \|\nabla H||_{L^2} \|\nabla^2 H||_{L^2}^{\frac{q-1}{q}} \leq C \|\nabla^2 H||_{L^2}^{\frac{q-1}{q}} \quad (3.58)$$

owing to (2.1), (3.2), and (3.3).
The combination of \((3.53)\), \((3.56)\), and \((3.57)\) gives
\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C. \tag{3.59}
\]
Thus, applying Gronwall’s inequality to \((3.52)\) yields
\[
\sup_{t \in [0,T]} \|\nabla \rho\|_{L^2 \cap L^q} \leq C(T). \tag{3.60}
\]
Finally, it is easy to deduce from \((3.15)\), \((3.56)\), \((3.57)\), \((3.2)\), and \((3.3)\) that
\[
\int_0^T \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{q+1} + t\|\nabla^2 u\|_{L^2 \cap L^q} \right) dt \leq C,
\]
which along with \((3.1)\) and \((3.60)\) gives \((3.51)\), and finishes the proof of Lemma 3.5. \(\Box\)

Now, we give the following spatial weighted estimate on the gradient of the density, which has been proved in \([27, \text{Lemma 3.6}]\). We omit the detailed proof here for simplicity.

**Lemma 3.6** There exists a positive constant \(C\) depending on \(T\) such that
\[
\sup_{t \in [0,T]} \|\rho \bar{x}^a\|_{L^1 \cap H^1 \cap W^{1,q}} \leq C(T). \tag{3.62}
\]

Next, by the similar arguments as in \([29, 30]\), we shall show the following spatial weighted estimates of \(H\) and \(\nabla H\), which are crucial to derive the estimates on the gradients of both \(u_t\) and \(H_t\).

**Lemma 3.7** There exists a positive constant \(C\) depending on \(T\) such that
\[
\sup_{t \in [0,T]} \|H \bar{x}^{a/2}\|_{L^2}^2 + \int_0^T \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 dt \leq C(T), \tag{3.63}
\]
and
\[
\sup_{t \in [0,T]} \left( t\|\nabla H \bar{x}^{a/2}\|_{L^2}^2 \right) + \int_0^T t\|\Delta H \bar{x}^{a/2}\|_{L^2}^2 dt \leq C(T). \tag{3.64}
\]

**Proof.** First, multiplying \((1.1)_3\) by \(H \bar{x}^a\) and integrating the resulting equality by parts over \(\mathbb{R}^2\) indicate that
\[
\frac{1}{2} \left( \|H \bar{x}^{a/2}\|_{L^2}^2 \right)_t + \nu \|\nabla H \bar{x}^{a/2}\|_{L^2}^2
= \frac{\nu}{2} \int |H|^2 \Delta \bar{x}^a dx + \int H \cdot \nabla u \cdot H \bar{x}^a dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \bar{x}^a dx
\]
\[
\triangleq \hat{J}_1 + \hat{J}_2 + \hat{J}_3, \tag{3.65}
\]
where
\[
\hat{J}_1 \leq C \int |H|^2 \bar{x}^a \bar{x}^{-2} \log(e + |x|^2) dx \leq C \|H \bar{x}^{a/2}\|_{L^2}^2,
\]
\[
\hat{J}_2 \leq C \int |\nabla u| |H|^2 \bar{x}^a dx \leq C \|\nabla u\|_{L^2} \|H \bar{x}^{a/2}\|_{L^4}^2
\]
\[
\leq C \|H \bar{x}^{a/2}\|_{L^2} \left( \|\nabla H \bar{x}^{a/2}\|_{L^2} + \|H \nabla \bar{x}^{a/2}\|_{L^2} \right)
\]
Using Gagliardo-Nirenberg inequality, (3.2), (3.63), and (3.49), it holds
\begin{align*}
\mathcal{J}_3 & \leq C\|\mathcal{H}^{\alpha/2}\|_{L^4}^2 + \frac{\nu}{4}\|\nabla\mathcal{H}^{\alpha/2}\|_{L^2}^2,
\mathcal{J}_2 & \leq C\|\mathcal{H}^{\alpha/2}\|_{L^4}^2 \cdot \|\mathcal{H}^{\alpha/2}\|_{L^2}^2 + \nu\|\nabla\mathcal{H}^{\alpha/2}\|_{L^2}^2 \cdot \nu^\alpha/2 + 1) \end{align*}
due to (3.41), (3.42), and (3.49). Then, substituting the above estimates into (3.65), together with Gronwall’s inequality, gives (3.63).

Now, multiplying (3.61) by $\Delta \mathcal{H}^{\alpha}$ and integrating the resultant equality by parts over $\mathbb{R}^2$ lead to
\begin{equation}
\frac{1}{2} \left(\|\nabla \mathcal{H}^{\alpha/2}\|_{L^2}^2 \right)_t + \nu\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2 \\
\leq C \int \nabla H |\nabla u| |\nabla \mathcal{H}^{\alpha}| dx + C \int \nabla H |\nabla u|^2 |\nabla \mathcal{H}^{\alpha/2}| dx + C \int \nabla H |\Delta \mathcal{H}^{\alpha/2}| dx
\end{equation}
where $\Delta \mathcal{H}^{\alpha} = \sum_{i=1}^5 \tilde{J}_i$.

Using Gagliardo-Nirenberg inequality, (3.2), (3.63), and (3.49), it holds
\begin{align*}
\tilde{J}_1 & \leq C\|\mathcal{H}^{\alpha/2}\|_{L^4}^2 + C\|\nabla u\|_{L^4}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2
\leq C\|\mathcal{H}^{\alpha/2}\|_{L^4}^2 \left(\|\nabla \mathcal{H}^{\alpha/2}\|_{L^2}^2 + \|\mathcal{H}^{\alpha/2}\|_{L^2}^2 \right) + C\|\nabla u\|_{L^4}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2
\leq C + C\|\nabla u\|_{L^4}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2,
\tilde{J}_2 & \leq C\|\nabla H |\nabla \mathcal{H}^{\alpha/2}|\|_{L^4}^2 \leq C\|\nabla H |\nabla \mathcal{H}^{\alpha/2}|\|_{L^4}^2 + C\|\nabla H |\nabla \mathcal{H}^{\alpha/2}|\|_{L^4}^2
\leq C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + \frac{\nu}{4}\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2,
\end{align*}
\begin{align*}
\tilde{J}_3 + \tilde{J}_4 & \leq \nu\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^2
\leq \nu\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^2
\leq C\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^2
\leq C\left(1 + \|\nabla u\|_{L^4}^{(q+1)/q}\right)\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2.
\end{align*}

Inserting the above estimates into (3.66) implies that
\begin{equation}
\left(\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 \right)_t + \nu\|\Delta \mathcal{H}^{\alpha/2}\|_{L^2}^2 \\
\leq C \left(1 + \|\nabla u\|_{L^4}^{(q+1)/q}\right)\|\nabla H \mathcal{H}^{\alpha/2}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^2 + 1),
\end{equation}
which multiplied by $t$, together with Gronwall’s inequality, (3.63), and (3.41) yields (3.64). The proof of Lemma 3.7 is finished. 

\begin{lemma}
There exists a positive constant $C$ depending on $T$ such that
\begin{equation}
\sup_{t \in [0,T]} \left(\|\rho^{1/2} u\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2\right) + \int_0^T \left(t\|\nabla u\|_{L^2}^2 + t\|\nabla H_t\|_{L^2}^2\right) dt \leq C(T).
\end{equation}
\end{lemma}
Indeed, on the one hand, owing to (2.1) and (3.69). On the other hand, (1.1) combined with (2.1) and (3.2) leads to

\[
\|H_t\|_{L^2}^2 \leq C\|\Delta H\|_{L^2}^2 + \|H\|\|\Delta u\|_{L^2}^2 + \|u\|\|\Delta H\|_{L^2}^2 \\
\leq C\|\Delta H\|_{L^2}^2 + \|H\|_{L^4}\|\Delta u\|_{L^4}^2 + \|u\|\|\Delta H\|_{L^2}^2 \\
\leq C\|\Delta H\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2 + C\|\Delta^2 u\|_{L^2}^2 + C\|\Delta H \bar{x}^{\alpha/2}\|_{L^2}^2,
\]

where in the last inequality one has used

\[
\|u\|\|\Delta u\|_{L^2}^2 \leq C\|u\bar{x}^{-\alpha/4}\|_{L^4}^4\|\Delta H\|_{L^2}^2 + C\|\Delta H \bar{x}^{\alpha/2}\|_{L^2}^2 \\
\leq \frac{1}{2}\|\Delta H\|_{L^2}^2 + C\|\Delta H \bar{x}^{\alpha/2}\|_{L^2}^2
\]

due to (3.69) and (2.1). Hence, (3.71) is a direct consequence of (3.72), (3.73), (3.2), (3.4), (3.51), and (3.63).

Now, differentiating (1.1) with respect to \( t \) gives

\[
\rho u_t + \rho u \cdot \nabla u_t - \mu \Delta u_t + \nabla p_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t.
\]

Multiplying (3.75) by \( u_t \) and integrating the resulting equality by parts over \( \mathbb{R}^2 \), we obtain after using (1.1) and (1.1) that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx \\
\leq C \int \rho |u_t|(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|) dx + C \int \rho |u_t|^2 |\nabla u||\nabla u_t| dx \\
+ C \int \rho |u_t|^2 |\nabla u| dx - \int H_t \cdot \nabla u_t \cdot H dx - \int H \cdot \nabla u_t \cdot H_t dx
\]
\[\triangleq \sum_{i=1}^5 \tilde{J}_i.\]

We estimate each term on the right-hand side of (3.76) as follows.
It follows from\,(3.54),\,(3.69),\,(3.2),\,(2.1), and Hölder’s inequality that
\[
\mathcal{J}_1 \leq C\sqrt{\mu} \|u\|_{L^6} \|\sqrt{\mu} u_t\|_{L^6}^{1/2} \|\sqrt{\mu} u_t\|_{L^6}^{1/2} \left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}\right) \\
+ C\|\rho^{1/4} u\|_{L^4}^2 \|
\sqrt{\mu} u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} \\
\leq C\|\sqrt{\mu} u_t\|_{L^2} \left(\|\sqrt{\mu} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right) \left(\|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2}\right) \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\sqrt{\mu} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) . \tag{3.77}
\]

Next, Hölder’s inequality,\,(3.69), and\,(3.54) imply
\[
\mathcal{J}_2 + \mathcal{J}_3 \leq C\|\sqrt{\rho} u_t\|_{L^6} \|\nabla u\|_{L^4} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|\sqrt{\mu} u_t\|_{L^6} \|\sqrt{\mu} u_t\|_{L^2}^{1/2} \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\sqrt{\mu} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\right) . \tag{3.78}
\]

For the terms\,\(\mathcal{J}_4\) and\,\(\mathcal{J}_5\), using\,(3.2) and\,(2.1), we obtain that
\[
\mathcal{J}_4 + \mathcal{J}_5 \leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|H\|_{L^4}^2 \|H_t\|_{L^2}^2 \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|H\|_{L^4}^2 \|H_t\|_{L^2}^2 \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|H_t\|_{L^2}^2 \|\nabla H\|_{L^2} \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|H_t\|_{L^2}^2 + \frac{\mu \nu}{4(C_3 + 1)} \|\nabla H_t\|_{L^2}^2 . \tag{3.79}
\]

where\,\(C_3\) is defined in the following\,(3.82). Submitting\,(3.77)–(3.79) into\,(3.76) gives
\[
\frac{d}{dt} \left\| \rho^{1/2} u_t \right\|_{L^2}^2 + \mu \|\nabla u_t\|_{L^2}^2 \leq C \left( \left\| \rho^{1/2} u_t \right\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) \\
+ \frac{\mu \nu}{2(C_3 + 1)} \|\nabla H_t\|_{L^2}^2 + C \left( \|\nabla^2 u\|_{L^2}^2 + 1 \right) . \tag{3.80}
\]

Next, differentiating\,(1.1.3) with respect to\,\(t\) shows
\[
H_t - H_t \cdot \nabla u - H \cdot \nabla u_t + u_t \cdot \nabla H + u \cdot \nabla H_t = \nu \Delta H_t . \tag{3.81}
\]

Multiplying\,(3.81) by\,\(H_t\) and integrating the resulting equality by parts over\,\(\mathbb{R}^2\), it follows from\,(1.1.4),\,(2.1),\,(3.2),\,(3.48), and\,(3.63) that
\[
\frac{1}{2} \frac{d}{dt} \int |H_t|^2 \, dx + \nu \int |\nabla H_t|^2 \, dx \\
= \int H \cdot \nabla u_t \cdot H_t \, dx + \int H_t \cdot \nabla u \cdot H_t \, dx + \int u_t \cdot \nabla H_t \cdot H_t \, dx \\
\leq C \|H_t\|_{L^4} \|H\|_{L^4} \|\nabla u_t\|_{L^2} + \|H_t\|_{L^4} \|\nabla u\|_{L^2} \\
+ C \|H_t\|_{L^4} \|H\|_{L^4} \|\nabla u\|_{L^2} \\
\leq C \|H_t\|_{L^4}^2 + C \|\nabla u_t\|_{L^2}^2 + C \left( \left\| \rho^{1/2} u_t \right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \|\nabla H_t\|_{L^2}^2 \\
\leq \frac{\nu}{2} \|\nabla H_t\|_{L^2}^2 + C \left( \|H_t\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 \right) + \frac{C_3}{2} \|\nabla u_t\|_{L^2}^2 . \tag{3.82}
\]

Now, multiplying\,(3.80) by\,\(\mu^{-1}(C_3 + 1)\) and adding the resulting inequality into\,(3.82), we have
\[
\frac{d}{dt} \left( \mu^{-1}(C_3 + 1) \|\rho^{1/2} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + \|\nabla u_t\|_{L^2}^2 + \frac{\nu}{2} \|\nabla H_t\|_{L^2}^2 ,
\]

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\[ \leq C \left( \| \rho^{1/2} u_t \|_{L^2}^2 + \| H_t \|_{L^2}^2 \right) + C \left( \| \nabla^2 u \|_{L^2}^2 + 1 \right), \tag{3.83} \]

which multiplied by \( t \), together with Gronwall’s inequality, \( (3.70) \), and \( (3.51) \) leads to

\[ \sup_{t \in [0, T]} \left( \| \rho^{1/2} u_t \|_{L^2}^2 + \| H_t \|_{L^2}^2 \right) + \int_0^T \left( t \| \nabla u_t \|_{L^2}^2 + t \| \nabla H_t \|_{L^2}^2 \right) dt \leq C(T). \tag{3.84} \]

Finally, it follows from \( (1.1)_3 \), \( (2.1) \), \( (3.2) \), and \( (3.74) \) that

\[ \| \nabla^2 H \|_{L^2} \leq C \| H_t \|_{L^2}^2 + C \| H \|_{L^2}^2 \| \nabla u \|_{L^2}^2 + C \| \nabla^2 u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2, \tag{3.85} \]

which combined with \( (3.84) \), \( (3.23) \), and \( (3.64) \) indicates \( (3.68) \) and finishes the proof of Lemma \( 3.8 \). \( \blacksquare \)

4 Proof of Theorem 1.1

With all the a priori estimates in Section 3 at hand, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.1, there exists a \( T_* > 0 \) such that the problem \( (1.1)-(1.2) \) has a unique strong solution \( (\rho, u, p, H) \) on \( \mathbb{R}^2 \times (0, T_0) \]. Now, we will extend the local solution to all time.

Set \( T^* = \sup \{ T \mid (\rho, u, p, H) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T] \}. \tag{4.1} \]

First, for any \( 0 < \tau < T_* \leq T^* \) with \( T \) finite, one deduces from \( (3.2) \), \( (3.4) \), \( (3.23) \), and \( (3.68) \) that for any \( q \geq 2 \),

\[ \nabla u, \nabla H, H \in C([\tau, T]; L^2 \cap L^q), \tag{4.2} \]

where one has used the standard embedding

\[ L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C(\tau, T; L^q) \text{ for any } q \in [2, \infty). \]

Moreover, it follows from \( (3.51) \), \( (3.62) \), and [26, Lemma 2.3] that

\[ \rho \in C([0, T]; L^1 \cap H^1 \cap W^{1/q}). \tag{4.3} \]

Finally, we claim that

\[ T^* = \infty. \tag{4.4} \]

Otherwise, if \( T^* < \infty \), it follows from \( (1.2) \), \( (1.3) \), \( (3.2) \), \( (3.4) \), \( (3.62) \), and \( (3.63) \) that

\[ (\rho, u, H)(x, T^*) = \lim_{t \to T^*} (\rho, u, H)(x, t) \]

satisfies the initial conditions \( (1.6) \) at \( t = T^* \). Thus, taking \( (\rho, u, H)(x, T^*) \) as the initial data, Lemma 2.1 implies that one could extend the local strong solutions beyond \( T^* \). This contradicts the assumption of \( T^* \) in \( (4.1) \). The proof of Theorem 1.1 is completed. \( \square \)
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