HEAT FLOW OUT OF A COMPACT MANIFOLD

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Abstract. We discuss the heat content asymptotics associated with the heat flow out of a smooth compact manifold in a larger compact Riemannian manifold. Although there are no boundary conditions, the corresponding heat content asymptotics involve terms localized on the boundary. The classical pseudo-differential calculus is used to establish the existence of the complete asymptotic series and methods of invariance theory are used to determine the first few terms in the asymptotic series in terms of geometric data. The operator driving the heat process is assumed to be an operator of Laplace type.

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1. Introduction

1.1. Basic assumptions. We adopt the following notational conventions. We let (M, g) be a smooth Riemannian manifold of dimension m. Let Ω ⊂ M be a smooth compact submanifold of M which has dimension m as well. We suppose that the boundary of Ω is non-empty and smooth. In this paper, we shall investigate the heat flow out of Ω into M \ Ω where (M, g) satisfies exactly one of the following three conditions:

(1) M is a compact and without boundary.
(2) (M, g) = (R^m, g_e) where g_e is the usual Euclidean inner metric on R^m.
(3) M is a compact submanifold of R^m with smooth boundary and g = g_e|_M.

The case of flat space is fundamental in this subject and hence we have concentrated on that setting in (2) and (3) to avoid technical difficulties. However, the results of this paper hold more generally.

1.2. Heat content. Let ∆_g be the associated Laplace-Beltrami operator acting on smooth functions on M. The heat equation on M takes the form

\[ \Delta_g u + \frac{\partial u}{\partial t} = 0, \ x \in M, \ t > 0, \]

with initial condition

\[ u(x; 0) = \phi(x), \ x \in M, \]

where \( \phi : M \to \mathbb{R} \) is continuous. Here \( u(x; t) \) represents the temperature at a point \( x \in M \) at time \( t \) if \( M \) has initial temperature profile \( \phi \). We say that \( K(x, \tilde{x}; t) \) is a fundamental solution if we have:

\[ u(x; t) = \int_M K(x, \tilde{x}; t)d\tilde{x}. \]

We say that \( (M, g) \) is stochastically complete if

\[ \int_M K(x, \tilde{x}; t)d\tilde{x} = 1 \text{ for all } x \in M, \ t > 0. \]

In order to guarantee that Equation (1.a) and Equation (1.b) have a unique solution, we have to impose some conditions on the geometry of M. These are summarized in the following result (see Chapter VII of [5] and the references therein):

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Theorem 1.1.

(1) If \((M, g)\) is compact without boundary or if \((M, g)\) is complete and with Ricci curvature bounded from below, then there exists a unique minimal positive fundamental solution \(K\). Moreover \((M, g)\) is stochastically complete.

(2) If \((M, g)\) is compact with non-empty smooth boundary \(\partial M\), then there exists a unique minimal positive fundamental solution \(K\). In this case \(K\) is the Dirichlet heat kernel for \(M\); \(u(x; t) = 0\) for all \(x \in \partial M\) and for all \(t > 0\).

Let \(\rho : M \to \mathbb{R}\) be the specific heat of \(M\); we suppose that \(\rho\) is continuous. Following [11] we define the heat content of \(\Omega\) in \(M\) by setting:

\[
\beta_{\Omega}(\phi, \rho, \Delta_g)(t) := \int_{\Omega} \int_{\Omega} K(x, \tilde{x}; t) \rho(\tilde{x}) d\tilde{x} dx.
\]  

(1.c)

Let \(\phi\) and \(\rho\) be smooth on \(\Omega\) and let \((M, g)\) satisfy one of the conditions described in Section [14]. Theorems 5.1 and Theorem 5.3 below show there exists a complete asymptotic series

\[
\beta_{\Omega}(\phi, \rho, \Delta_g)(t) \sim \sum_{n=0}^{\infty} t^n 2^n \tilde{\beta}_{n}^{\Omega}(\phi, \rho, \Delta_g) + \sum_{j=0}^{\infty} t^{(1+j)/2} \tilde{\beta}_{j}^{\Omega}(\phi, \rho, \Delta_g) \quad \text{as} \quad t \to 0^+,
\]

where \(\tilde{\beta}_{n}^{\Omega}\) and \(\tilde{\beta}_{j}^{\Omega}\) are the integrals of certain locally computable invariants over \(\Omega\) and \(\partial \Omega\), respectively. We extend \(\phi\) by 0 on \(M \setminus \Omega\), and note that \(\phi\) may be discontinuous on all or part of \(\partial \Omega\). In that case the initial condition is satisfied for \(x \in M \setminus \partial \Omega\)

The study of the heat content of \(\Omega\) in \(\mathbb{R}^m\) was initiated in [11]. It was shown [9] [13] [11] that if \(\Omega\) is an open set in \(\mathbb{R}^m\) with finite Lebesgue measure \(|\Omega|\) and with finite perimeter \(P(\Omega)\) then

\[
P(\Omega) = \lim_{t \to 0} \left( \frac{\pi}{4} \right)^{1/2} \int_{\Omega \times (\mathbb{R}^m \setminus \Omega)} K(x, \tilde{x}; t) d\tilde{x} dx,
\]

where \(K(x, \tilde{x}; t) = (4\pi t)^{-m/2} e^{-|x-\tilde{x}|^2/(4t)}\) is the heat kernel for \(\mathbb{R}^m\). It immediately follows that

\[
\beta_{\Omega}(1, 1, \Delta_g)(t) = |\Omega| - |\Omega|^{1/2} P(\Omega) t^{1/2} + o(t^{1/2}), \quad t \downarrow 0,
\]

(1.d)

where \(|\Omega| = \int_{\Omega} 1 dx\). We note that the main contribution beyond the constant term \(|\Omega|\) in Equation (1.4) comes from localization near \(\partial \Omega\). We shall see presently in Theorem 5.1 that only the geometry near \(\Omega\) plays a role in the heat content modulo an exponentially small error in \(t^{-1}\) as \(t \downarrow 0\) and which therefore plays no role in the asymptotic series. Even though the \(t \downarrow 0\) behaviour of the heat kernel is known for general \((M, g)\) [8], this explicit asymptotic behaviour does not give much insight, and is not helpful in the determination of the locally computable invariants of \(\Omega\) and of \(\partial \Omega\) respectively.

We must employ a more general formalism; even if we were only interested in the scalar Laplacian, it is a facet of the “method of universal examples” that one must examine this more general framework. Let \((M, g)\) be as described in Section [14]. Let \(D_M\) be an operator of Laplace type on a smooth vector bundle \(V\) over a complete Riemannian manifold \((M, g)\). Let \(\Omega\) be a compact manifold in \(M\) with smooth boundary. Let \(\phi \in \mathcal{L}^1(V|\Omega)\) represent the initial temperature, and let \(\rho \in \mathcal{L}^1(V^*|\Omega)\) represent the specific heat. As above we extend \(\phi\) and \(\rho\) to \(M\) to be zero on \(\Omega^c\); and we denote the resulting extensions by \(\phi_\Omega\) and \(\rho_\Omega\) to emphasize that they are supported on \(\Omega\). Similarly to Equation (1.4) we define the heat content of \(\Omega\) in \(M\) by

\[
\beta_{\Omega}(\phi, \rho, D_M)(t) = \beta_M(\phi_\Omega, \rho_\Omega, D_M)(t) = \int_{\Omega} \int_{\Omega} (K(x, \tilde{x}; t) \phi(x), \rho(\tilde{x})) d\tilde{x} dx.
\]
where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing between \( V \) and \( V^* \). As above we shall suppose that both \( \rho \) and \( \phi \) are smooth on the interior of \( \Omega \). However, we obtain additional information by permitting \( \phi \) to have a controlled singularity near \( \partial \Omega \). If \( r \) is the geodesic distance to the boundary, we shall suppose that \( r^{\alpha} \phi \) is smooth near the boundary for some fixed complex number \( \alpha \). We shall always assume \( \Re(\alpha) < 1 \) to ensure that \( \phi \in L^1(\Omega) \). The parameter \( \alpha \) controls the blow up (if \( \Re(\alpha) > 0 \) or decay (if \( \Re(\alpha) < 0 \)) of \( \phi \) near the boundary. We permit \( \alpha \) to be complex. Although this has no physical significance, it is useful in analytic continuation arguments as we shall see in the proof of Lemma 3.1. Let \( K_\alpha = K_\alpha(V) \) denote the resulting space of functions. We will establish the following result in Section 2:

**Theorem 1.3.** Let \( \partial \Omega = \{ \partial \Omega \} \) denote the resulting space.

In Section 5 we will use Theorem 1.2 to establish the following result:

**Theorem 1.4.** With the notation established above we may take

\[
\beta_n^\Omega(\phi, \rho, D_M)(t) \sim \sum_{n=0}^\infty t^n \beta_n^\Omega(\phi, \rho, D_M) + \sum_{j=0}^\infty t^{(1+j-\alpha)/2} \beta_{j,\alpha}^\partial(\phi, \rho, D_M),
\]

where \( \beta_n^\Omega \) and \( \beta_{j,\alpha}^\partial \) are integrals of certain locally computable invariants over \( \Omega \) and \( \partial \Omega \), respectively.

**1.3. 1-dimensional geometry.** What happens on the line is in many ways crucial to our analysis as we shall “bootstrap” our way from that setting to the higher dimensional setting. Let \( S^1 = [0, 2\pi] \) where we identify \( 0 \sim 2\pi \). Let \( \Omega = [0, \pi] \subset S^1 \). Let \( V = V^* \) be the trivial bundles and let \( D = -\partial_x^2 \). Near the boundary point \( x = 0 \), we may expand \( \phi \) and \( \rho \) in modified Taylor series for \( x > 0 \):

\[
\phi(x) \sim x^{-\alpha} \{ \phi_0 + \phi_1 x + \phi_2 x^2 + \ldots \} \quad \text{and} \quad \rho(x) \sim \rho_0 + \rho_1 x + \rho_2 x^2 + \ldots
\]

There is a similar expansion near \( x = \pi \) as well: \( \partial \Omega = \{ 0, \pi \} \) and the boundary \( dy \) is simply counting measure in this instance. We will establish the following result in Section 2:

**Lemma 1.4.** With the notation established above we may take

\[
\beta_n^\Omega = \frac{(-1)^n}{n!} \int_\Omega \phi \cdot D^n \rho dx, \quad \text{and} \quad \beta_n^{\partial \Omega} = \sum_{k+\ell=\alpha} \int_{\partial \Omega} c_{k,\ell,\alpha} \phi_k \rho_\ell dy,
\]

where \( dy \) is the Riemannian volume element on the boundary of \( \partial \Omega \), and where

\[
c_{k,\ell,\alpha} = (-1)^{\ell+1} \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty e^{-\frac{(u+v)^2}{4}} u^{k-\alpha-\ell} v^\ell du dv.
\]
1.4. **Local invariants.** We return to the general setting. By considering the case in which $\Omega = M$, we see that $\beta^\partial_n$ can be taken to be

$$\beta^\partial_n(\phi, \rho, D_M) = \left(\frac{(-1)^n}{n!}\right)^{\frac{1}{2}} \int_\partial \phi, D^\partial_M \rho dx,$$

where $D^\partial_M$ is the dual operator on $V^\ast$. We shall not differentiate $\phi$ on the interior owing to the lack of smoothness in $\phi$ as we approach the boundary. This is a crucial point: were we to examine the doubly singular setting, we would need to regularize the interior integrals. Thus attention is focused on the boundary in variants of the terms involving the second fundamental form.

1.5. **A Bochner formalism.** Although in Sections 2 and 3 we will use a coordinate formalism for describing the invariants $\beta^\partial_{j,\alpha}$, it is useful to introduce an invariant tensorial formalism at this point. There is a unique connection $\nabla$ on $V$ and a unique endomorphism $E$ of $V$ so that we may express $D_M$ using a Bochner formalism:

$$D_M \phi = -\{g^{ij} \phi_{,ij} + E \phi\}.$$

Here $\phi_{,ij}$ denotes the components of the second covariant derivative $\nabla^2 \phi$ of $\phi$. We adopt the *Einstein* convention and sum over repeated indices (see Gilkey [6] for details). Let $\Gamma$ denote the Christoffel symbols of the metric $g$. If we express $D_M$ in a coordinate system in the form:

$$D_M = -\{g^{ij} \partial_x \partial_{x_j} + A^i \partial_{x_i} + B\},$$

then the connection 1-form $\omega$ of $\nabla$ and the endomorphism $E$ are given by

$$\omega_i = \frac{1}{2}(g_{ij} A^k + g^{kl} \Gamma_{kli} \text{Id})$$

and $E = B - g^{ij}(\partial_x \omega_j + \omega_i \omega_j - \omega_k \Gamma_{ij}^k)$.

The question always arises as to why it is necessary (or desirable) to consider these more general operators of Laplace type when in practice one is usually only interested in the scalar Laplacian. In Section 3.4 we will evaluate the coefficients of the terms involving the second fundamental form $L_{ab}$ and the Ricci curvature $Ric_{nm}$ in $\beta^\partial_{1,\alpha}$ and $\beta^\partial_{2,\alpha}$. We will use warped products and operators which are not the scalar Laplacian. The fact that we are working with quite general operators will be crucial. It is typical in this subject that to obtain formulas for the scalar Laplacian, one must invoke the more general formalism and derive formulas in a very general context.

Let $\nabla^*\!$ be the dual connection on $V^\ast$, and let the associated connection 1-form be given by $-\omega^*\!$. We let indices $\{i,j,k,l\}$ range from 1 to $m$. Let $R_{ijkl}$ be the components of the metric tensor, let $Ric_{ij}$ be the components of the Ricci tensor, and let $\tau$ be the scalar curvature. Let indices $\{a,b,c\}$ range from 1 to $m - 1$. Near the boundary, we choose a local orthonormal frame $\{e_i\}$ for the tangent bundle so that $e_m$ is the inward unit geodesic normal. Let $L_{ab}$ denote the components of the second fundamental form. Let

$$\phi_j := \nabla^e_{e_m} \phi|_{\partial M} \quad \text{and} \quad \rho_j := (\nabla^e_{e_m})^i \rho|_{\partial M}.$$

One may use dimensional analysis to see that $\beta^\partial_{j,\alpha}$ is homogeneous of order $j$ in the derivatives of the structures involved. This leads to the following observation.

**Lemma 1.5.** There exist universal constants $\varepsilon_{\nu,\alpha}$ which depend holomorphically on $\alpha$ so that:

$$\beta^\partial_{0,\alpha}(\phi, \rho, D_M) = \int_{\partial M} \varepsilon_{0,\alpha}(\phi_0, \rho_0) dy;$$

$$\beta^\partial_{1,\alpha}(\phi, \rho, D_M) = \int_{\partial M} \left\{\varepsilon_{1,\alpha}(\phi_1, \rho_0) + \varepsilon_{2,\alpha}(L_{aa} \phi_0, \rho_0) + \varepsilon_{3,\alpha}(\phi_0, \rho_1)\right\} dy.$$
Corollary 1.7. Arise from the Dirichlet realization.

Because 1.6. Normalizing constants. Following the discussion in [4], we define:

\[ c_\alpha := 2^{1-\alpha} \Gamma \left( \frac{2 - \alpha}{2} \right) \frac{1}{\sqrt{\pi}(\alpha - 1)}. \]

Because \( s \Gamma(s) = \Gamma(s+1) \), we have the recursion relations

\[ c_\alpha = -\frac{\alpha - 3}{2(\alpha - 1)(\alpha - 2)} c_{\alpha-2} \quad \text{and} \quad c_{\alpha+1} = -\frac{\alpha - 2}{2\alpha(\alpha - 1)} c_{\alpha-1}. \]

We set \( \alpha = 0 \) to see:

\[ c_0 = -\frac{2}{\sqrt{\pi}}, \quad c_{-1} = -1, \quad c_{-2} = -\frac{8}{3\sqrt{\pi}}. \]

1.7. Some terms in the asymptotic series. In Section 3 we will establish the following result by determining the universal constants in Lemma 1.5. This together with Theorem 1.6 are the two main results of this paper.

Theorem 1.6.

\[ \beta^{[\alpha]}_{2,0}(\phi, \rho, D_M) = c_\alpha \int_{\partial \Omega} \frac{1}{2} \langle \phi_0, \rho_0 \rangle dy. \]

\[ \beta^{[\alpha]}_{1,1}(\phi, \rho, D_M) = c_{\alpha-1} \int_{\partial \Omega} \left\{ \frac{1}{2} \langle \phi_0, \rho_0 \rangle + \frac{\alpha}{4(1-\alpha)} \langle L_{aa} \phi_0, \rho_0 \rangle + \frac{1}{2(\alpha - 1)} \langle \phi_0, \rho_1 \rangle \right\} dy. \]

\[ \beta^{[\alpha]}_{2,0}(\phi, \rho, D_M) = c_{\alpha-2} \int_{\partial \Omega} \left\{ \frac{1}{2} \langle \phi_0, \rho_0 \rangle + \frac{\alpha - 1}{4(2-\alpha)} \langle L_{aa} \phi_1, \rho_0 \rangle + \frac{3 - \alpha}{4(1-\alpha)(2-\alpha)} \langle E \phi_0, \rho_0 \rangle \right. \]

\[ + \frac{1}{(1-\alpha)(2-\alpha)} \langle \phi_0, \rho_2 \rangle - \frac{\alpha + 1}{4(2-\alpha)} \langle L_{aa} \phi_0, \rho_1 \rangle - \frac{1 - \alpha}{8(2-\alpha)} \langle \text{Ric}_{mm} \phi_0, \rho_0 \rangle \]

\[ - \frac{\alpha - 3}{8(2-\alpha)} \langle \phi_0, \rho_0, \rho_0 \rangle + 0 \langle \tau \phi_0, \rho_0 \rangle + \frac{1}{4(2-\alpha)} \langle \phi_1, \rho_1 \rangle \} dy. \]

We specialize Theorem 1.6 to the smooth setting by setting \( \alpha = 0 \) to obtain:

Corollary 1.7.

\[ \beta^{[0]}_{0}(\phi, \rho, D_M) = -\frac{1}{\sqrt{\pi}} \int_{\partial \Omega} \frac{1}{2} \langle \phi_0, \rho_0 \rangle dy. \]

\[ \beta^{[0]}_{1,1}(\phi, \rho, D_M) = -\int_{\partial \Omega} \left\{ \frac{1}{2} \langle \phi_1, \rho_0 \rangle + 0 \langle L_{aa} \phi_0, \rho_0 \rangle - \frac{1}{2} \langle \phi_0, \rho_1 \rangle \right\} dy. \]

\[ \beta^{[0]}_{2,0}(\phi, \rho, D_M) = -\frac{2}{\sqrt{\pi}} \int_{\partial \Omega} \left\{ \frac{1}{2} \langle \phi_2, \rho_0 \rangle - \frac{1}{2} \langle L_{aa} \phi_1, \rho_0 \rangle + \frac{3}{2} \langle E \phi_0, \rho_0 \rangle \right. \]

\[ + \frac{1}{2} \langle \phi_0, \rho_2 \rangle - \frac{1}{2} \langle L_{aa} \phi_0, \rho_1 \rangle - \frac{1}{16} \langle \text{Ric}_{mm} \phi_0, \rho_0 \rangle - \frac{1}{16} \langle L_{aa} L_{bb} \phi_0, \rho_0 \rangle \]

\[ - \frac{1}{10} \langle L_{ab} L_{ab} \phi_0, \rho_0 \rangle - \frac{3}{4} \langle \phi_{0,\alpha}, \rho_{0,\alpha} \rangle + 0 \langle \tau \phi_0, \rho_0 \rangle - \frac{1}{2} \langle \phi_1, \rho_1 \rangle \} dy. \]
1.8. Dirichlet and Robin boundary conditions. Let $B_D := f|_{\partial \Omega}$ be the Dirichlet boundary operator and let $B_S := (\nabla_m f + S f)|_{\partial \Omega}$ be the Robin boundary operator; here $S$ is an auxiliary endomorphism of $V|_{\partial \Omega}$. We use $S^*$ and the dual connection to define the dual boundary operator $\tilde{B}$ on $C^\infty(V^*)$. Let $D_B$ denote the Dirichlet or Robin realization of $D_M$. In this instance, the ambient manifold $M$ plays no role and defines:

$$\beta(\phi, \rho, D_M, B) = \int_{\Omega} (e^{-tD_B} \phi, \rho) dx.$$  

The the existence of an analogous asymptotic series in this setting has been established in [4]. After adjusting the notation suitably using Equation (1.6) from that in [4], the results of [4] yield:

**Theorem 1.8.**

1. $\beta_{0,0}^M(\phi, \rho, D_M, B_D) = c_0 \int_{\partial \Omega} \langle \phi_0, \rho_0 \rangle dy$.

2. $\beta_{1,0}^M(\phi, \rho, D_M, B_D) = c_{-1} \int_{\partial \Omega} \langle \phi_1, \rho_0 \rangle dy$.

3. $\beta_{1,0}^M(\phi, \rho, D_M, B_D) = c_{-2} \int_{\partial \Omega} \langle \phi_2, \rho_0 \rangle$,

$$- \frac{2}{2(\alpha - 1)(\alpha - 2)} E \phi_0, \rho_0 \rangle + \frac{2}{2(\alpha - 1)(\alpha - 2)} \langle \phi_0, \rho_2 \rangle - \frac{1}{2} \langle L_a \phi_0, \rho_1 \rangle$$
$$+ \frac{2}{2(\alpha - 1)(\alpha - 2)} \langle \phi_0, \rho_0, \rho_0 \rangle - \frac{1}{2} \langle \text{Ric}_{mm}, \phi_0 \rangle$$
$$+ \frac{1}{2} \langle L_a L_a \phi_0, \rho_0 \rangle - \frac{1}{2} \langle L_{ab} L_{ab} \phi_0, \rho_0 \rangle dy.$$

4. $\beta_{0,0}^M(\phi, \rho, D_M, B_R) = 0$.

5. $\beta_{1,0}^M(\phi, \rho, D_M, B_R) = - \frac{a - 2}{2(\alpha - 1)} c_{-1} \int_{\partial \Omega} \langle \phi_0, \tilde{B} \rho \rangle dy$

$$= - \frac{2}{2(\alpha - 1)} c_{-1} \int_{\partial \Omega} \langle \phi_0, \tilde{B} \rho \rangle dy$$

$$= - \frac{1}{2(\alpha - 1)} c_{-1} \int_{\partial \Omega} \langle \phi_0, \tilde{B} \rho \rangle dy.$$

6. $\beta_{1,0}^M(\phi, \rho, D_M, B_R) = - \frac{2}{2(\alpha - 1)} c_{-1} \int_{\partial \Omega} \langle (1 - \alpha) \phi_1 + S \phi_0 - \frac{1}{2} L_a \phi_0, \tilde{B} \rho \rangle dy$

$$= - \int_{\partial \Omega} \langle (1 - \alpha) \phi_1 + S \phi_0 - \frac{1}{2} L_a \phi_0, \tilde{B} \rho \rangle dy.$$

2. Proof of Theorem 1.2

We will follow the discussion of the pseudo-differential calculus based on the work by Seeley [12, 13], and refer to Gilkey [6]. Throughout this section, we assume the ambient Riemannian manifold $(M, g)$ is compact and without boundary.

By using a partition of unity, we may assume that $\rho$ and $\phi$ are supported within coordinate systems. Since the kernel of the heat equation decays exponentially in $t^{-1}$ for $\text{dist}_g(x, \tilde{x}) \geq \epsilon > 0$, we may assume that $\rho$ and $\phi$ have support within the same coordinate system. There will, of course, be three different types of coordinate systems to be considered - those which touch the boundary of $\Omega$, those which are contained entirely within the interior of $\Omega$, and those which are contained in the exterior of $\Omega$; those contained in the exterior of $\Omega$ play no role as they contribute an exponentially small error in $t^{-1}$. In Section 2.1 we establish notational conventions and prove a technical result. In Section 2.2 we review the notion of a pseudo-differential operator; in Section 2.3 we construct the resolvent, and in Section 2.4 we construct an approximation to the kernel of the heat equation. The new material begins in Section 2.5 where we begin the examination of the heat content. In Section 2.6 we establish the existence of two different kinds of asymptotic series. In Section 2.7 we use the results of Section 2.6 to discuss coordinate systems contained in the interior of $\Omega$ and Section 2.8 we use the results of Section 2.6 to study coordinate systems near the boundary. The fact that $D_M$ is of Laplace type plays a central role in the discussion.
2.1. Notational conventions. Let \( x = (x^1, \ldots, x^m) \in \mathbb{R}^m \) be coordinates on an open set \( \mathcal{O} \subset M \). Let \( (x, \xi) \) be the induced coordinate system on the cotangent space \( T^*\mathcal{O} \) where we expand a 1-form \( \omega \in T^*\mathcal{O} \) in the form:

\[
\omega = \xi dx^i \quad \text{to define} \quad \xi = (\xi_1, \ldots, \xi_m).
\]

We let \( x \cdot \xi \) be the natural pairing

\[
x \cdot \xi := x^i \xi_i.
\]

If \( \alpha = (a_1, \ldots, a_m) \) is a multi-index, set

\[
|\alpha| = a_1 + \ldots + a_m, \quad \alpha! = a_1! \ldots a_m!, \quad \partial^\alpha = \partial_{x^1}^{a_1} \ldots \partial_{x^m}^{a_m},
\]

\[
\phi^\alpha := \partial^\alpha \phi, \quad \rho^\alpha := \partial^\alpha \rho, \quad D^\alpha := \sqrt{-1} |\alpha| d^\alpha_x,
\]

\[
d^\alpha = \partial_{\xi_1}^{a_1} \ldots \partial_{\xi_m}^{a_m}, \quad x^\alpha = x_1^{a_1} \ldots x_m^{a_m}, \quad \xi^\alpha = \xi_1^{a_1} \ldots \xi_m^{a_m}.
\]

For example, with these notational conventions, Taylor’s theorem becomes

\[
O = \sum_{|\alpha| \leq n} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^\alpha + O(|x - x_0|^{n+1}). \tag{2.a}
\]

Let \( d\nu_x, d\nu_\xi \) denote Lebesgue measure on \( \mathbb{R}^m \). Let \( L^2_\mathcal{O} \) denote \( L^2(\mathbb{R}^m) \) with respect to Lebesgue measure. Let \( g(x) = g_{ij}(x)dx^i \circ dx^j \) be a Riemannian metric on \( \mathcal{O} \) and define:

\[
||\xi||_{g^*(x)}^2 = g^{ij}(x) \xi_i \xi_j \quad \text{and} \quad ||x - \tilde{x}||_{g(x)}^2 := g_{ij}(x)(x^i - \tilde{x}^i)(x^j - \tilde{x}^j).
\]

We shall always be restricting to compact \( x \) and \( \tilde{x} \) subsets. Let \( (\cdot, \cdot)_e \) and \( ||\cdot||_e \) denote the usual Euclidean inner product and norm, respectively:

\[
(x, y)_e := x_1 y_1 + \ldots + x_m y_m \quad \text{and} \quad ||(x_1, \ldots, x_m)||_e^2 := (x, x)_e = x_1^2 + \ldots + x_m^2.
\]

We have estimates:

\[
C_1 ||\xi||^2_e \leq ||\xi||_{g^*(x)}^2 \leq C_2 ||\xi||^2_e \quad \text{and} \quad C_1 ||x - \tilde{x}||^2_{g(x)} \leq ||x - \tilde{x}||_{g(x)}^2 \leq C_2 ||x - \tilde{x}||^2_e
\]

for some constants \( C_1 > 0 \) and \( C_2 > 0 \). Let \( \theta = \sqrt{g} \); \( \theta \) is a symmetric metric so that \( \theta_{ij} \theta_{jk} = g_{ik} \). We then have

\[
||\xi||_{g^*(x)}^2 = ||\theta^{-1}(x)\xi||^2_e \quad \text{and} \quad ||(x - \tilde{x})||_{g(x)}^2 = ||\theta(x)(x - \tilde{x})||^2_e.
\]

**Lemma 2.1.** We use \( \theta^2 = g \) to lower indices and regard \( \theta^2(x - \tilde{x}) \) as a vector \( (X - \tilde{X})_i := g_{ij}(x - \tilde{x})^j. \) With this identification, we have:

\[
||\xi||_{g^*(x)}^2 + \sqrt{-1}(x - \tilde{x}) \cdot \xi/\sqrt{t}
\]

\[
= ||\xi + \frac{1}{2}\sqrt{-1}(X - \tilde{X})/\sqrt{t}||_{g^*(x)}^2 + ||(x - \tilde{x})||_{g(x)}^2/(4t). \tag{4a}
\]

**Proof.** We expand:

\[
||\xi||_{g^*(x)}^2 + \sqrt{-1}(x - \tilde{x}) \cdot \xi/\sqrt{t}
\]

\[
= (\theta^{-1}(x)\xi, \theta^{-1}(x)\xi)_e + \sqrt{-1}(\theta^{-1}(x)\theta^2(x)(\tilde{x} - x)/\sqrt{t}, \theta^{-1}(x)\xi)_e
\]

\[
= (\theta^{-1}(x)[\xi + \frac{1}{2}\sqrt{-1}(X - \tilde{X})/\sqrt{t}], \theta^{-1}(x)_e
\]

\[
= ||\theta^{-1}(x)||^2_e[\xi + \frac{1}{2}\sqrt{-1}\theta^2(x)(\tilde{x} - x)/\sqrt{t}]||^2_e + ||\theta(x)(\tilde{x} - x)||^2_{g(x)}/(4t)
\]

\[
= ||\xi + \frac{1}{2}\sqrt{-1}(X - \tilde{X})/\sqrt{t}||_{g^*(x)}^2 + ||(x - \tilde{x})||_{g(x)}^2/(4t). \tag{4b} \]
2.2. **Pseudo-differential operators.** If $P$ is a pseudo-differential operator with symbol $p(x, \xi)$, then $P$ is characterized by following identity for all $\phi \in C_0^\infty(V)$ and $\rho \in C_0^\infty(V^*)$:

$$\langle P \phi, \rho \rangle_{L^2} = (2\pi)^{-m} \iiint e^{-\sqrt{-1} \langle (x-\tilde{x}), \xi \rangle} \langle p(x, \xi) \phi(x), \rho(\tilde{x}) \rangle dv_x dv_{\tilde{x}} dv_{\xi}.$$  \hfill (2.2)

The integrals in question here are iterated integrals - the convergence is not absolute and the $dv_{\tilde{x}}$ integral has to be performed before the $dv_{\xi}$ integral. However, if $p(x, \xi)$ decays rapidly enough in $\xi$, then the integrals are in fact absolutely convergent and we can interchange the order of integration to see following [6] Lemma 1.2.5] that $P$ is given by a kernel:

$$\langle P \phi, \rho \rangle_{L^2} = \int (K_P(x, \tilde{x}) \phi(x), \rho(\tilde{x})) dv_x dv_{\tilde{x}}$$

where

$$K_P(x, \tilde{x}) := (2\pi)^{-m} \int e^{-\sqrt{-1} \langle (x-\tilde{x}), \xi \rangle} p(x, \xi) dv_{\xi}.$$ \hfill (2.3)

2.3. **The resolvent.** Let $D_M$ be an operator of Laplace type on $C^\infty(V)$ over $M$. In a system of local coordinates $(x^1, ..., x^n)$ on an open subset $\mathcal{O}$ of $M$, we may change notation slightly from that employed in Equation (1.e) and expand:

$$D_M = a_{ij}^\alpha(x) D_{x_i} D_{x_j} + a_1^\alpha(x) D_{x_\alpha} + a_0(x).$$

We ensure that Equation (2.1) defines the operator $D_M$ by defining:

$$p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) + p_0(x, \xi)$$

where

$$p_2(x, \xi) := |\xi|^2 g_\alpha(x), \quad p_1(x, \xi) := a_1^\alpha(x) \xi_\alpha, \quad p_0(x, \xi) := a_0(x).$$

Let $\mathcal{R} \subset \mathbb{C}$ be the complement of a cone of angle $\varepsilon_{1,\alpha}$ about the positive real axis and a ball of radius $\varepsilon_{2,\alpha}$ about the origin where $\varepsilon_{2,\alpha} = \varepsilon_{2,\alpha}(\varepsilon_{1,\alpha})$ is chosen so that $D_M$ has no eigenvalues in $\mathcal{R}$. We let $\lambda \in \mathcal{R}$ henceforth. Following the discussion of [6] Lemma 1.7.2], we define $r_n(x, \xi; \lambda)$ for $(x, \xi) \in T^*(\mathcal{O})$ and $\lambda \in \mathcal{R}$ inductively by setting:

$$r_0(x, \xi; \lambda) := (|\xi|^2 g_\alpha(x) - \lambda)^{-1},$$

$$r_n := -r_0 \sum_{|\alpha| + 2 + k = n, j < n} \partial_{\xi}^\alpha p_k \cdot D_x^\alpha r_j \text{ for } n > 0.$$ \hfill (2.4)

Define

$$\text{ord}(\partial_{\xi}^\alpha p_2) = |\alpha|, \quad \text{ord}(\partial_{\xi}^\alpha p_1) = |\alpha| + 1, \quad \text{ord}(\partial_{\xi}^\alpha p_0) = |\alpha| + 2, \quad \text{weight}(\lambda) = 2, \quad \text{weight}(\xi) = 1.$$ 

The following lemma follows immediately by induction from the recursive definition in Equation (2.4):

**Lemma 2.2.**

1. $r_n$ is homogeneous of order $n$ in the derivatives of the symbol of $D_M$.
2. $r_n$ has weight $-n - 2$ in $(\xi, \lambda)$.
3. There exist polynomials $r_{n,j,\alpha}(x, D_M)$ for $n \leq j \leq 3n$ which are homogeneous of order $n$ in the derivatives of the symbol of $D_M$ so:

$$r_n(x, \xi; \lambda) = \sum_{2j - |\alpha| = n} r_{n, j, \alpha}(x, D_M) (|\xi|^2 g_\alpha(x) - \lambda)^{-j-1} \xi_\alpha.$$ 

We use Equation (2.1) to define the pseudo-differential operator $R_n(\lambda)$ with symbol $r_n$ so that

$$\langle R_n(\lambda) \phi, \rho \rangle_{L^2} = (2\pi)^{-m} \iiint e^{-\sqrt{-1} \langle (x-\tilde{x}), \xi \rangle} \langle r_n(x, \xi; \lambda) \phi(x), \rho(\tilde{x}) \rangle dv_x dv_{\tilde{x}} dv_{\xi}.$$
Let $||-||_{k,k}$ be the norm of a map from the Sobolev space $H_{-k}$ to the Sobolev space $H_k$. By [6, Lemma 1.7.3] we have that if $\lambda \geq \lambda(k)$ and if $n \geq n(k)$, then:

$$||(D_M - \lambda)^{-1} - R_0(\lambda) - ... - R_n(\lambda)||_{k,k} \leq C_k(1 + |\lambda|)^{-k}.$$ 

2.4. **An approximation to the kernel of the heat equation.** Let $\gamma$ be the boundary of $\mathcal{R}$ oriented suitably. We use the operator valued Riemann integral to define

$$e^{-tD_M} := \frac{1}{2\pi} \int_{\gamma} e^{-t\lambda}(D_M - \lambda)^{-1} d\lambda.$$ 

We use [6, Lemma 1.7.5] to see that this is the fundamental solution of the heat equation and belongs to $\text{Hom}(H_{-k}, H_k)$ for any $k$. We now let

$$e_n(x, \xi; t) = \frac{1}{2\pi} \int_{\gamma} e^{-t\lambda} r_n(x, \xi; \lambda) d\lambda$$

(2.e)

define the pseudo-differential operator

$$E_n(t, D_M) := \frac{1}{2\pi} \int_{\gamma} e^{-t\lambda} R_n(\lambda) d\lambda.$$ 

(2.f)

We use Lemma 2.2 (3) and Cauchy's integral formula to rewrite Equation (2.e) as:

$$e_n(x, \xi; t) = \sum_{2j-|\alpha|=n} t^j \sum_{j} \xi^\alpha e^{-t|\xi|^{2}_{\gamma(x)} r_{n,j,\alpha}(x, D_M)}.$$ 

(2.g)

We now use Equation (2.c) and Equation (2.g) to see the operator $E_n$ of Equation (2.h) is given by the smooth kernel

$$K_n(x, \tilde{x}; t) = \sum_{2j-|\alpha|=n} (2\pi)^{-m} t^j \int_{\mathbb{R}^n} e^{-t|\xi|^{2}_{\gamma(x)}-\sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha r_{n,j,\alpha}(x, D_M) d\xi.$$ 

(2.h)

Let $||k||_{C^k}$ denote the $C^k$ norm. Given any $k \in \mathbb{N}$, there exists $n(k)$ so that if $n \geq n(k)$ and if $0 < t < 1$, then [6, Lemma 1.8.1] implies:

$$||e^{-tD_M} - \sum_{n=0}^{n(k)} E_n(t, D_M)||_{-k,k} \leq C_k t^k$$ 

This gives rise to a corresponding estimate (after increasing $n(k)$ appropriately):

$$||K(t, x, \tilde{x}, D_M) - \sum_{n=0}^{n(k)} K_n(t, x, \tilde{x}, D_M)||_{C^k} \leq C_k t^k.$$ 

(2.i)

2.5. **Examining the heat content.** We use Equation (2.d), Equation (2.h), and Equation (2.i) to expand

$$\beta(\phi, \rho, D_M)(t) = \sum_{2j-|\alpha|=0} (2\pi)^{-m} t^j \int_{\mathbb{R}^n} \int e^{-t|\xi|^{2}_{\gamma(x)}-\sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha \
\times \langle r_{n,j,\alpha}(x, D_M)\phi(x), \rho(\tilde{x}) \rangle dv_x dv\xi dv\tilde{x} + O(t^k).$$ 

We examine a typical term in the sum setting:

$$\beta_{n,j,\alpha}(\phi, \rho)(t) := (2\pi)^{-m} t^j \int_{\mathbb{R}^n} \int e^{-t|\xi|^{2}_{\gamma(x)}-\sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha \
\times \langle r_{n,j,\alpha}(x)\phi(x), \rho(\tilde{x}) \rangle dv_x dv\xi dv\tilde{x}.$$ 


Here all integrals are over $\mathbb{R}^m$ and converge absolutely for $t > 0$; $\phi$ and $\rho$ have compact support. We change variables setting $\xi := t^{1/2} \xi$ to express:

$$
\beta_{n,j,o}(\phi, \rho)(t) = \frac{t^{j/m - 1/2} |\alpha|}{j!} (2\pi)^{-m} \iint e^{-\|\xi\|^2_{g(x)}} \sqrt{r(x, \bar{x})} / \sqrt{r(x, \bar{x})} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}}.
$$

Note that $\frac{1}{2} n = j - \frac{1}{2} |\alpha|$. We adopt the notation of Lemma 2.1 and make a complex change of coordinates setting:

$$
\eta = \xi + \frac{1}{2} \sqrt{-1} (X - \bar{X}) / \sqrt{t}.
$$

We then apply Lemma 2.3. and the binomial theorem to express:

$$
\beta_{n,j,o}(\phi, \rho)(t) = \frac{(2\pi)^{-m}}{j!} \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \frac{1}{\alpha_1! \alpha_2!} \int \cdots \int e^{-\|\eta\|^2_{g(x)}} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}}.
$$

The $d\nu_x$ integral is over the complex domain $\eta \in \mathbb{R} + \frac{1}{2} \sqrt{-1} \frac{X - \bar{X}}{\sqrt{t}}$. But we can deform that domain back to the real domain $\eta \in \mathbb{R}$. Set

$$
c_{\alpha_1, \alpha_2, j} := \frac{(2\pi)^{-m}}{j!} \int \cdots \int e^{-\|\eta\|^2_{g(x)}} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}}.
$$

The sum ranges over $|\alpha_1|$ even as otherwise $\alpha_{1,2}$ vanishes. Thus $|\alpha_2| \equiv |\alpha| \equiv n$ mod 2. This reduces the proof to considering expressions of the form:

$$
f_{n,j,o,\alpha_2}(t) := \int \cdots \int e^{-\|\eta\|^2_{g(x)}} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}},
$$

where $|\alpha_2| \equiv n$ mod 2 and $\text{ord}(r_{n,j,o}(x, D_M)) = n$.

### 2.6. Asymptotic series

Before proceeding further with our analysis of Equation (2.3), we must establish the existence of asymptotic series in certain quite general contexts:

**Lemma 2.3.** Let $\Phi \in L^1(\mathbb{R}^m)$, let $\rho \in C^\infty(\mathbb{R}^m)$ have compact support in an open subset $\mathcal{O} \subset \mathbb{R}^m$, and let $(X - \bar{X}) := g_{ij}(x - \bar{x})$. Let

$$
F(t) := \frac{t^{(n-m)/2}}{\int \cdots \int e^{-\|\eta\|^2_{g(x)}} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}},
$$

and

$$
G(t) := \frac{t^{(n-1)/2}}{\int \cdots \int e^{-\|\eta\|^2_{g(x)}} \int \cdots \int (r_{n,j,o}(x, D_M) \phi(x), \rho(\bar{x})) d\nu_x d\nu_{\bar{x}},
$$

Then

1. There exist smooth coefficients $c_{\alpha_1, \alpha_2} = c_{\alpha_1, \alpha_2}(g(x))$ so that there is a complete asymptotic series as $t \downarrow 0^+$ of the form

$$
F(t) \sim \sum_{|\sigma| = 0}^{\infty} t^{(n+|\sigma|)/2} \int_{\mathcal{O}} c_{\sigma, \alpha_2} \phi(\sigma)(x) d\nu_x.
$$
We make the change of variables $\tilde{x} = x + u$ and dually $\tilde{X} = X + U$ where $U_i = g_{ij}u^j$ to express

$$F(t) = t^{(n-m)/2} \int e^{-||u||^2_{g(x)}/(4t)} \left( \frac{t}{\sqrt{t}} \right)^{\sigma} \langle \Phi(x), \rho(x + u) \rangle dv_u dv_x. $$

The $dv_u$ integral decays exponentially for $|u| > t^{1/4}$ so we may assume the $dv_u$ integral is localized to $|u| < t^{1/4}$. For $u$ small, we use Equation (2.3) to express:

$$\rho(x + u) \sim \sum_{|\sigma| \leq N} \frac{1}{\sigma!} u^\sigma d_x^\sigma \rho(x) + O(u^N),$$

$$F(t) = t^{(n-m)/2} \sum_{|\sigma| \leq N} \frac{1}{\sigma!} \int e^{-||u||^2_{g(x)}/(4t)}$$

$$\times \left\{ \left( \frac{4t}{\sigma^2} \right)^{\sigma^2} u^\sigma \langle \Phi(x), \rho^{(\sigma)}(x) \rangle + O(|u|^N) \right\} dv_u dv_x.$$

We set $\tilde{u} = u/\sqrt{t}$ and $\tilde{U} = U/\sqrt{t}$ to express

$$F(t) \sim \sum_{|\sigma| \leq N} \frac{1}{\sigma!} t^{(n+|\sigma|)/2} \int e^{-||\tilde{u}||^2_{\tilde{g}(x)}/4} \tilde{U}^{m2} \tilde{u}^\sigma \langle \Phi(x), \rho^{(\sigma)}(x) \rangle dv_\tilde{u} dv_x. $$

The $dv_\tilde{u}$ integral remains an integral over $\mathcal{O}$. But as $t \downarrow 0$, the $dv_\tilde{u}$ integral expands to $\mathbb{R}^m$ and defines the coefficients $c_{\sigma, \alpha_2} = c_{\sigma, \alpha_2}(\tilde{g})$. This establishes Assertion (1).

Let $m = 1$. We note that $G$ decays exponentially for $x \geq \varepsilon > 0$ or $\tilde{x} \geq \varepsilon > 0$. On the small square, we expand

$$\Phi(x) \sim \sum_{i \geq 0} C_i x^i \text{ and } \rho(\tilde{x}) \sim \tilde{x}^\varepsilon \rho^{(\varepsilon)}(0).$$

We then make the change of variables with $u = x/\sqrt{t}$ and $\tilde{u} = \tilde{x}/\sqrt{t}$ to express

$$G(t) \sim t^{(n+1)/2} \sum_{i+j \leq N} t^{(i+j-\alpha)/2} c_{i,j,\alpha} \langle C_i, \rho^{(j)}(0) \rangle $$

where

$$c_{i,j,\alpha} := \int_0^\infty \int_0^\infty e^{-\frac{1}{2} |u + \tilde{u}|^2} u^{i-\alpha} \tilde{u}^j dv_u dv_\tilde{u}. \quad \square$$

2.7. **The interior terms in Theorem 1.2** We apply Lemma 2.3 (1) to the case $\Phi = r_{n,j,\alpha} \phi$ in Equation (2.3). By assumption $r_{n,j,\alpha}$ is of order $n$ in the derivatives of the total symbol of $D_M$. We have $\rho^{(\sigma)}$ is of order $|\sigma|$ in the derivatives of $\rho$. Thus we have expressions which are of order $n + |\sigma|$ in the derivatives of the symbol of $D_M$ and in the derivatives of $\rho$. Furthermore, the $dv_\tilde{u}$ integral vanishes unless $|\sigma| + |\alpha_2|$ is even. Since $|\alpha_2| \equiv n \bmod 2$, this implies $|\sigma| + n$ is even so terms involving fractional powers of $t$ vanish as claimed. This leads to exactly the sort of interior expansion described in Theorem 1.2.
2.8. The heat content on a chart near the boundary of $\Omega$. We now assume the coordinate chart meets the boundary. Again, we examine Equation (2.7). We set $x = (r, y)$; the $dr_r$ integral ranges over $0 \leq r < \infty$ and the $dy_y$ integral ranges over $y \in \mathbb{R}^{m-1}$. The $dy_y$ and $dj_j$ integrals are handled using the analysis of Lemma 1.3 (1). We therefore suppress these variables and concentrate on the $dr_r$ integrals and in essence assume that we are dealing with a 1-dimensional problem; we can always choose the coordinates so $ds^2 = dr^2 + g_{xy}(r,y)dy_ydy_y$. We resume the computation with Equation (2.3) where we do not perform the integrals in the two variables normal to the boundary. We suppress other elements of the notation to examine an integral of the form:

$$f(t) := t^{(n-1)/2} \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-\frac{||x-\bar{x}||^2}{4t}} \left( \frac{x}{\sqrt{t}} \right)^{\alpha_2} \langle r_{n,j,\alpha}(x, D_M)\phi(x), \rho(\tilde{x}) \rangle dv_x dv_y.$$

Here $g$ has compact support in $(x, \tilde{x})$ and is homogeneous of degree $n$ in the derivatives of the symbol of $D_M$, in the derivatives of $\phi$, and in the derivatives of $\rho$; there is no trouble with convergence. We suppress the role of $|g|$ in the tangential integrals which can also depend on the normal parameter. A crucial point is that the extra power of $-\frac{n-1}{2}$ occurs in applying Lemma 1.3 to $\mathbb{R}^{n-1}$. We set

$$f_1(t) := t^{(n-1)/2} \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-\frac{||x-\bar{x}||^2}{4t}}$$

$$\times \left( \frac{x}{\sqrt{t}} \right)^{\alpha_2} \langle r_{n,j,\alpha}(x, D_M)\phi(x), \rho(\tilde{x}) \rangle dv_x dv_y.$$

Again, there is no trouble with convergence. The sum $f(t) + f_1(t)$ can then be handled as in Section 2.7 and gives rise to the interior term we have been studying. Thus everything new comes from $f_1(t)$ and this is handled by Lemma 1.3 (2) with $\alpha = 0$ after we replace $\tilde{x}$ by $\bar{x}$. The terms multiplying $t^{(n+1+|\alpha_1|+|\alpha_2|)/2}$ have degree $n + |\alpha_1| + |\alpha_2|$ in the derivatives of the symbol of $D_M$, of the derivatives of $\phi$, and of the derivatives of $\rho$. After setting $j = n + |\alpha_1| + |\alpha_2|$ and summing, we obtain the boundary terms of Theorem 1.2. We start out at $t^{(n-1)/2}$ but then we have two factors of $t^{1/2}$ arising from the $x$ and $\tilde{x}$ change of variable.

Remark 2.4. It is clear from the construction that the coefficients in the boundary asymptotic expansion depend holomorphically on the complex parameter $\alpha$ for $\Re(\alpha) < 1$; the fact that the constants $\varepsilon_{\nu,\alpha}$ of Lemma 1.5 are holomorphic as well now follows.

3. The proof of Theorem 1.6

We will establish Theorem 1.6 by evaluating the universal constants which appear in Lemma 1.5. In Section 3.1 we establish Lemma 1.3 which relates to the heat content asymptotics on the line. This result is then used in Section 3.2 to determine the constants $\{\varepsilon_0,\varepsilon_1,\varepsilon_3,\varepsilon_4,\varepsilon_7,\varepsilon_{14}\}$. Then in Section 3.3 we use product formulas to determine $\{\varepsilon_6,\varepsilon_{12,\alpha},\varepsilon_{14,\alpha}\}$. We complete the computation in Section 3.4 using warped products.

3.1. The proof of Lemma 1.3. We apply the analysis of Section 2 to the 1-dimensional setting. We work in the scalar setting and set $D = -\partial_x^2$. Consequently

$$p_2(x, \xi) = \xi^2, \quad p_1(x, \xi) = 0, \quad p_0(x) = 0,$$

$$r_0(x, \xi; \lambda) = (\xi^2 - \lambda)^{-1}, \quad r_n(x, \xi; \lambda) = 0 \text{ for } n \geq 1,$$

$$c_0(x, \xi; t) = e^{-\xi^2t}, \quad c_n(x, \xi; t) = 0 \text{ for } n \geq 1.$$
Consequently we have $K_n = 0$ for $n \geq 1$ while

$$
K_0(x, \tilde{x}; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-t} (x - \tilde{x})} \xi e^{-t \xi^2} d\xi
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x - \tilde{x})^2/(4t)} \int_{-\infty}^{\infty} e^{t|\xi|^2} d\xi
$$

$$
= \frac{1}{\sqrt{4\pi t}} e^{-(x - \tilde{x})^2/(4t)}.
$$

This is, of course, not surprising as this is the heat kernel in flat space. Let

$$
f_1(t) := \frac{1}{\sqrt{4\pi t}} \int_{x=0}^{\infty} \int_{\tilde{x}=-\infty}^{\infty} e^{-(x - \tilde{x})^2/(4t)} \phi(x) \rho(\tilde{x}) d\tilde{x} dx,
$$

$$
f_2(t) := \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x + \tilde{x})^2/(4t)} \phi(x) \rho(-\tilde{x}) d\tilde{x} dx.
$$

We may then express $\beta(\phi, \rho, D_M)(t) = f_1(t) - f_2(t)$. We change variables setting $\tilde{x} = x + u$ to express

$$
f_1(t) \sim \frac{1}{\sqrt{4\pi t}} \int_{x=0}^{\infty} \int_{u=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \phi(x) \rho(k)(x) u^k e^{-u^2/(4t)} du dx.
$$

In the sum, we must have $k = 2k$ is even. We integrate by parts $k$ times to evaluate the constants which appear. Alternatively we change variables $u^2 = 4tv$ and use standard formulae for the $\Gamma$-function to obtain that

$$
= \frac{1}{\sqrt{4\pi t}} \int_{x=0}^{\infty} \frac{1}{(2k)!} u^{2k} e^{-u^2/(4t)} du = \frac{2}{\sqrt{4\pi t}(2k)!} \int_{0}^{\infty} u^{2k} e^{-u^2/(4t)} du = \frac{2^{2k} \Gamma(k + \frac{1}{2})}{\sqrt{\pi} (2k)!} = \frac{\Gamma(k)}{k!}.
$$

The interior terms arise from expanding

$$
f_1(t) \sim \sum_{k} \frac{t^k}{k!} \int_{0}^{\infty} \phi(x) \rho(2k)(x) dx \sim \sum_{k} (-1)^k \frac{t^k}{k!} \int_{0}^{\infty} \phi(x) D^k \rho(x) dx.
$$

Next we evaluate $f_2$ (and we have to subtract this term). We expand

$$
\phi(x) \sim x^{-\alpha} \sum_{i} \phi_i x^i \quad \text{and} \quad \rho(\tilde{x}) \sim \sum_{j} \rho_j \tilde{x}^j.
$$

We do not put in the factorials so $\rho_j = \frac{1}{j!} \rho^{(j)}(0)$.

$$
f_2(t) \sim \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x + \tilde{x})^2/(4t)} \sum_{i,j} \phi_i \rho_j x^{-\alpha} \tilde{x}^j dv_x dv_{\tilde{x}}.
$$

We change variables to set $x = \sqrt{t} u$ and $\tilde{x} = \sqrt{t} \tilde{u}$ to complete the proof of Lemma [24] by expressing

$$
f_2(t) \sim \frac{1}{\sqrt{4\pi}} \int_{u}^{\infty} \int_{\tilde{u}}^{\infty} e^{-(u + \tilde{u})^2/4t} \sum_{i,j} (-1)^{i+j+\alpha+1/2} \phi_i \rho_j \times \int_{0}^{\infty} \int_{0}^{\infty} e^{-(u + \tilde{u})^2/4t} u^{i-\alpha} \tilde{u}^j du d\tilde{u}.
$$

3.2. Evaluating the constants for the 1-dimensional case. We use e.g. Mathematica [13] to compute the coefficients of Lemma [24]

$$
\varepsilon_{0, \alpha} = -\frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} \int_{0}^{\infty} x^{-\alpha} e^{-(x+y)^2} dx dy = \frac{2^{1-\alpha} \Gamma(\frac{2-\alpha}{2})}{(1+\alpha)\sqrt{\pi}} = c_{\alpha}.
$$
for $\phi$ permits us to conclude that interior terms are given by Lemma 1.4. Equating the two asymptotic series then decouple,

There are, of course, no boundary terms in the asymptotic series for $\phi$. We consider the product manifold. Let $(M, g_M) := (N \times S^1, g_N + dx^2)$, $\Sigma := [0, \pi]$,

\[
\phi(x, z) = \phi_S(x)\phi_N(z), \quad \rho(x, z) = \rho_S(x)\rho_N(z)
\]

for $\phi_N \in C^\infty(N)$, $\rho_N \in C^\infty(N)$, $\phi_S \in K_\alpha(\Sigma)$, and $\phi_S \in C^\infty(\Sigma)$. As the structures decouple,

\[
e^{-tD_M} = e^{-tD_S}e^{-tD_N}
\]

so

\[
\beta\Omega(\phi, \rho, D_M)(t) = \beta\Sigma(\phi, \rho, D_S)(t) \cdot \beta N(\phi_N, \rho_N, D_N)(t).
\]

There are, of course, no boundary terms in the asymptotic series for $\beta N$, and the interior terms are given by Lemma [14]. Equating the two asymptotic series then permits us to conclude that $\beta\Sigma^M_{\alpha}(\phi, \rho, D_M) = \mathcal{E}_2 + \mathcal{E}_0$ where

\[
\mathcal{E}_2 := \int_{\partial\Sigma} \beta\Sigma^M_{\alpha}(\phi_S, \rho_S, D_S)(x)dx \cdot \int_N \phi_N(z)\rho_N(z)dz \quad \text{and}
\]

\[
\mathcal{E}_0 := \frac{1}{2}c_{\alpha-2} \int_{\Sigma} \phi_S(0)(y)\rho_S(0)(y)dy \cdot \int_N -\phi_N(z) \cdot D_N\rho_N(z)dz.
\]

We suppress terms not of interest to equate

\[
\int_{\partial\Sigma} \int_N \phi_S(0)(x)\rho_S(0)(x)
\]

\[\cdot \{\epsilon_{6, \alpha}E_N(z)\phi_N(z)\rho_N(z) + \epsilon_{12, \alpha}\phi_N\rho_{N, d} + \epsilon_{13, \alpha\tau}\phi_N\rho_N\}
\]

\[= \frac{1}{2}c_{\alpha} \int_{\partial\Sigma} \phi_S(0)(x)\rho_S(0)(x)dx \cdot \int_N \phi_N(z)(\rho_{N, aa} + E)\rho_N(z)dz.
\]

Integrating by parts on $N$ permits us to see

\[\int_N \phi_N(z)\rho_{N, aa}(z)dz = - \int_N \phi_N(z)\rho_{N, a}(z)dz.
\]

Applying the recursion relation of Equation [14] then yields:

\[\epsilon_{12, \alpha} = \epsilon_{6, \alpha} = \frac{1}{2}c_{\alpha} = - \frac{\alpha - 3}{2(\alpha - 1)(\alpha - 2)} \cdot \frac{1}{2}c_{\alpha-2}, \quad \text{and} \quad \epsilon_{13, \alpha} = 0.
\]
3.4. Warped products. We now determine the coefficients involving $L$ and $\text{Ric}$. The fact that we are working with quite general operators is now crucial. We extend the formalism of Section 3.3 and consider warped products. Let 
$$\Sigma = [0, \pi], \quad S = [0, 2\pi] \setminus 0 \sim 2\pi, \quad D_S = -\partial_\theta^2.$$ 
Let $T^{m-1}$ be the torus with periodic parameters $(\theta_1, ..., \theta_{m-1})$, let $M = T^{m-1} \times S^1$, and let $\Omega = T^{m-1} \times S$. Let $f_\alpha \in C^\infty(S)$ be a collection of smooth functions satisfying $f_\alpha(0) = 0$ and $f_\alpha \equiv 0$ near $x = \pi$. Let $\delta_\alpha \in \mathbb{R}$. Set

$$M := T^{m-1} \times S, \quad ds^2_M = \sum_{a=1}^{m-1} e^{2f_\alpha(x)} d\theta_a \circ d\theta_a + dx \circ dx,$$

$$\Omega := T^{m-1} \times [0, \pi], \quad D_M := -\sum_{a=1}^{m-1} e^{-2f_\alpha(x)} (\partial^2_{\theta_a} + \delta_\alpha \partial_{\theta_a}) - \partial_x^2.$$

Let $\phi_\Sigma \in \mathcal{R}_\Sigma(\Sigma)$ with $\phi_\Sigma$ vanishing identically near $\pi$ and let $\rho_\Sigma \in C^\infty([0, \pi])$ with $\rho_\Sigma$ vanishing identically near $\pi$ as well. Set:

$$\phi_\Omega(x, y) = \phi_\Sigma(x) \quad \text{and} \quad \rho_\Omega(x, y) = \rho_\Sigma(x) e^{-\sum_\alpha f_\alpha(x)}.$$

**Lemma 3.1.** $\beta^{\partial M}_{\alpha}(\phi_\Omega, \rho_\Omega, D_M) = \beta^{\partial \Sigma}_{\theta,\rho}(\phi_\Sigma, \rho_\Sigma, D_S) \text{vol}(T^{m-1})$ for $j \geq 0$.

**Proof.** Note that $x$ is the geodesic distance to $\{0\}$ in $\Sigma$ and that $x$ is the geodesic distance to $(0 \times T^{m-1})$ in $\Omega$; the component where $x = \pi$ plays no role as $\phi$ and $\rho$ vanish identically near this component. Since $\phi_\Theta$ is independent of $y \in T$, the problem decouples and

$$\{ e^{-tD_M \phi_\Omega} \} (x, y; t) = \{ e^{-tD_S \phi_\Sigma} \} (x; t).$$

The Riemannian measure on $M$ takes the form:

$$dv_M = \sqrt{\det g_{ij}} dxdy = e^{\sum_\alpha f_\alpha(x)} dx dy dx.$$ 

Since $\rho_\Omega dv_\Omega = \rho_\Sigma dx dy$, we have:

$$\beta^{\partial \Sigma}_{\theta,\rho}(\phi_\Sigma, \rho_\Sigma, D_S)(t) = \beta^{\partial \Sigma}_{\theta,\rho}(\phi_\Sigma, \rho_\Sigma, D_S)(t) \cdot \text{vol}(T^{m-1}).$$

**Lemma 3.1** now follows for $\alpha \not\in \mathbb{Z}$ since the interior invariants and the boundary invariants do not interact. Since the invariants $\beta^{\partial \Sigma}_{\theta,\rho}$ are analytic in $\alpha$, the desired conclusion also follows for $\alpha \in \mathbb{Z}$. Thus even if one were only interested in the case $\alpha = 0$, it is convenient to have more general values of $\alpha$ available. □

We apply Lemma 3.1. Although the structures are flat on $\Sigma$, they are not flat on $\Omega$ and this makes all the difference. We determine the relevant tensors as follows:

$$\Gamma_{abm} = -f'_a \delta_{ab} e^{2f_\alpha}, \quad \Gamma_{ab}^m = -f'_a e^{2f_\alpha} \delta_{ab},$$

$$\Gamma_{amb} = f'_a \delta_{ab} e^{2f_\alpha}, \quad \Gamma_{am}^b = f'_a \delta_{ab},$$

$$L_{ab} = \Gamma_{ab}^m |_{\partial M} = -f'_a \delta_{ab},$$

$$\omega_a = \frac{1}{2} e^{2f_\alpha} \delta_a, \quad \omega_m = -\omega_a = \frac{1}{2} e^{2f_\alpha} \delta_a,$$

$$\omega_\epsilon = \frac{1}{2} \sum_a f'_a, \quad \omega_m = -\omega_\epsilon = \frac{1}{2} \sum_a f'_a.$$ 

Consequently:

$$R_{ambm} = g((\nabla_a \nabla_m - \nabla_m \nabla_a) e_b, e_m) = \Gamma_{ac}^m \Gamma_{mb} e - \partial_m \Gamma_{amb}^m$$

$$= - (f''_a + f'_a)^2 e^{2f_\alpha} \delta_{ab},$$

$$\text{Ric}_{mm} = -\sum_a \{ f''_a + f'_a \},$$

$$E|_{\partial M} = -\partial_m \omega_m - \omega_a^2 - \omega_m^2 = -\partial_m \Gamma_{abm}^m$$

$$= \frac{1}{2} \sum_a f''_a - \frac{1}{2} \sum_a \delta_a^2 - \frac{1}{2} \sum_a \partial_m f'_a f'_b + \frac{1}{2} \sum_a \partial_m f'_a f'_b$$

$$= \frac{1}{2} \sum a f''_a - \frac{1}{2} \sum a \delta_a^2 + \frac{1}{2} \sum a \partial_m f'_a f'_b.$$
We introduce the notation $\phi_\Omega$ and $\rho_\Omega$ to emphasize we are computing with the structures on $M$ and not on $S$. We evaluate on the component $x = 0$:

\[
\phi_{\Omega,0}|_{x=0} = \phi_{\Sigma,0}(0),
\]

\[
\phi_{\Omega,1}|_{x=0} = \{(\partial_x - \frac{1}{2} \sum_a \left. \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0) \right)\}_{x=0} = -\frac{1}{2} \sum_a \left. \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0) \right),
\]

\[
\phi_{\Omega,2}|_{x=0} = \left. (\partial_x - \frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0)) \right|_{x=0} = \frac{1}{2} \left( (\partial_x - \frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0)) \right).
\]

\[
\rho_{\Omega,0}|_{x=0} = \rho_{\Sigma,0}(0),
\]

\[
\rho_{\Omega,1}|_{x=0} = \{(\partial_x + \frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0)) \}_{x=0} = -\frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0),
\]

\[
\rho_{\Omega,2}|_{x=0} = \left. (\partial_x + \frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0)) \right|_{x=0} = \frac{1}{2} \left( (\partial_x + \frac{1}{2} \sum_a \phi_{\Sigma,0}(0) + \phi_{\Sigma,1}(0)) \right).
\]

\[
\phi_{\Omega,0,\alpha}\rho_{\Omega,0,\alpha} = -\frac{1}{2} \sum_a \delta_a^2.
\]

The structures defined by $f'_a$, $f''_a$, and $\delta_a$ do not appear in $\beta_{j,a}^{\beta_{\Sigma}}$ and thus these terms must give zero in $\beta_{j,a}^{\beta_{\Omega}}$. By considering the monomial $\sum_{a,b} f'_a f'_b \phi_{\Sigma,0}(0) \rho_{\Sigma,0}(0)$ in $\beta_{j,a}^{\beta_{\Omega}}$, we obtain the relation

\[
\frac{1}{2} \delta_{4,a} + \frac{1}{2} \delta_{5,a} + \frac{1}{2} \delta_{6,a} + \frac{1}{2} \delta_{7,a} + \frac{1}{2} \delta_{8,a} + \frac{1}{2} \delta_{10,a} + \frac{1}{2} \delta_{14,a} = 0.
\]

We obtain other relations by considering suitable monomials:

| Relation | Monomial |
|----------|----------|
| $-\frac{1}{2} \delta_{1,a} - \delta_{2,a} - \frac{1}{2} \delta_{3,a} = 0$, | $f'_a \phi_{\Sigma,0}(0) \rho_{\Sigma,0}(0)$, |
| $-\frac{1}{2} \delta_{6,a} + \delta_{12,a} = 0$, | $\phi_{\Sigma,0}(0) \rho_{\Sigma,0}(0)$, |
| $-\delta_{9,a} + \delta_{11,a} = 0$, | $\phi_{\Sigma,0}(0) \rho_{\Sigma,0}(0)$, |
| $-\frac{1}{2} \delta_{4,a} - \delta_{5,a} = 0$, | $\phi_{\Sigma,1}(0) \rho_{\Sigma,0}(0)$, |
| $-\frac{1}{2} \delta_{14,a} - \delta_{8,a} = 0$, | $\phi_{\Sigma,0}(0) \rho_{\Sigma,1}(0)$, |
| $-\frac{1}{2} \delta_{4,a} + \delta_{6,a} = 0$, | $\phi_{\Sigma,0}(0) \rho_{\Sigma,0}(0)$, |

Theorem 1.6 follows from these equations and the relations established previously.

4. Further functorial properties

In Section 4.1 we examine dimension shifting and in Section 4.2 we relate the Dirichlet and Neumann heat content asymptotics to the asymptotics we have been studying. In addition to providing useful crosschecks on our work, these properties are worth noting as they promise to be useful in other contexts.

4.1. Dimension shifting. Let $R(\alpha) << 0$. If $\phi \in K_{\alpha-1}$, then we may regard $\phi$ as defining an element $\bar{\phi} \in K_{\alpha}$. If we expand $\phi = r^{-\alpha+1}(\phi_0 + r\phi_1 + ...)$, then $\bar{\phi} = r^{-\alpha}(0 + r\phi_0 + r^2\phi_1 + ...)$. Consequently $\phi_i = \phi_{i-1}$ for $i \geq 1$ and

\[
\beta_{j,a}^{\beta_{\Omega}}(\phi, \bar{\rho}, D_M) = \beta_{j-1,a}^{\beta_{\Omega}}(\phi, \rho, D_M).
\]

Examining the formulas of Lemma 4.5 then yields the relations:

\[
\epsilon_{1,a} = \epsilon_{0,a-1}, \quad \epsilon_{4,a} = \epsilon_{0,a-2}, \quad \epsilon_{5,a} = \epsilon_{2,a-1}, \quad \epsilon_{14,a} = \epsilon_{4,a-1}.
\]

These hold, of course, only if $R(\alpha) < 0$; we use analytic continuation to derive the general result. Once again, it is convenient to have values of $\alpha$ other than $\alpha = 0$. 

4.2. The Dirichlet and Neumann heat content asymptotics. There is a useful relationship between the heat content asymptotics being studied at present and the ones studied previously.

**Lemma 4.1.** Let \((M, g)\) be a closed Riemannian manifold. Let \(T\) be an isometric involution of \(M\) with \(\text{Fix}(T) = N\) a totally geodesic submanifold of co-dimension 1. Assume \(M - N\) decomposes as the union of two open sub manifolds \(M_+ \cup M_-\) which are interchanged by \(T\). Let \(\Omega := M_+ \cup N\). Then
\[
\beta_\Omega(\phi, \rho, \Delta)(t) = \frac{1}{2}\{\beta_D(\phi, \rho, \Delta)(t) + \beta_N(\phi, \rho, \Delta)(t)\}.
\]

**Proof.** We can use the \(Z_2\) involution \(T\) to choose a spectral resolution
\[
\{\lambda_N, \phi_{N,n}\}_{n=1}^\infty \cup \{\lambda_D, \phi_{D,n}\}_{n=1}^\infty
\]
for \(L^2(M)\) so \(T^*\phi_{N,n} = -\Phi_{N,n}\) and \(T^*\phi_{D,n} = \Phi_{D,n}\). Then \(\{\lambda_N, \sqrt{\Phi_{N,n}}\}_{n=1}^\infty\) is a spectral resolution for the Neumann Laplacian on \(\Omega_+\) and \(\{\lambda_D, \sqrt{\Phi_{D,n}}\}_{n=1}^\infty\) is a spectral resolution for the Dirichlet Laplacian on \(\Omega_+\). We compute that
\[
e^{-t\Delta} \phi = 2 \sum_n e^{-t\lambda_n} \phi_{N,n} L^2 \phi_{N,n},
\]
\[
e^{-t\Delta} \phi = 2 \sum_n e^{-t\lambda_n} \phi_{D,n} L^2 \phi_{D,n},
\]
\[
e^{-t\Delta} \phi = \sum_n e^{-t\lambda_n} \phi_{N,n} L^2 \phi_{N,n} + \sum_n e^{-t\lambda_n} \phi_{D,n} L^2 \phi_{D,n},
\]
\[
= \frac{1}{2} (e^{-t\Delta} \phi + e^{-t\Delta} \phi).
\]
The desired result now follows by taking the inner product with \(\rho\) and integrating over the support of \(\rho\) which is \(\Omega_+\). \(\square\)

This is not useful for studying the terms which involve the second fundamental form \(L_{ab}\). However, if we average the remaining terms for Dirichlet and Neumann boundary conditions in Theorem 1.8 we get the analogous terms in Theorem 1.6

5. Reduction to the closed setting

In Section 2 we used the classic calculus of pseudo-differential operators depending on a complex parameter which was developed by Seeley [12, 13]. That formalism is valid only for compact and closed manifolds. In this section, we will derive Theorem 1.3 where \((M, g) = (\mathbb{R}^m, g_c)\) or where \((M, g)\) is a compact subset of \(\mathbb{R}^m\) of dimension \(m\) from Theorem 1.2 which dealt with compact manifolds without boundary.

We adopt the following notational conventions. Let \(\Omega \subset \text{Int}(M) \subset \tilde{M} \subset M\) where \(\Omega\) and \(\tilde{M}\) are compact manifolds of dimension \(m\) with smooth boundaries. Let \(\epsilon := \text{dist}_g(\partial \Omega, \partial \tilde{M}) > 0\). Let \(\beta^\Omega_M\) be the heat content of \(\Omega\) in \(\tilde{M}\) and let \(\beta^\tilde{M}_M\) be the heat content of \(\tilde{M}\) in \(M\).

**Theorem 5.1.** Assume that \((M, g)\) is complete with non-negative Ricci curvature. Let \(\rho\) be continuous on \(M\) and let \(\phi \in L^1(\Omega)\). Then:
\[
|\beta^\Omega_M(\phi, \rho, \Delta_g)(t) - \beta^\tilde{M}_M(\phi, \rho, \Delta_g)(t)| \leq 2(2^m+1)^2 \|\phi\|_{L^1(\Omega)} \|\rho\|_{L^\infty(\Omega)} e^{-\epsilon^2/8t}.
\]

**Proof.** Let \(K\) be the Dirichlet heat kernel for \(\tilde{M}\). By minimality,
\[
0 \leq K(x, \tilde{x}; t) \leq K(x, \tilde{x}; t)
\]
for all \(x \in \tilde{M}, \tilde{x} \in \tilde{M}, t > 0\). Since \(M\) is stochastically complete we have that
\[
1 = \int_M K(x, \tilde{x}; t) d\tilde{x}.
\]
Moreover since $M$ and hence $\tilde{M}$ have non-negative Ricci curvature we have by Theorem 3.5.3 in [7] (see also Lemma 5 in [2]) that

\[
\int_{\tilde{M}} \tilde{K}(x, \tilde{x}; t) d\tilde{x} \geq 1 - 2^{(2+m)/2} e^{-\text{dist}_g(x, \partial \tilde{M})^2/(8t)} \\
= \int_{M} K(x, \tilde{x}; t) d\tilde{x} - 2^{(2+m)/2} e^{-\text{dist}_g(x, \partial \tilde{M})^2/(8t)} \\
\geq \int_{M} K(x, \tilde{x}; t) d\tilde{x} - 2^{(2+m)/2} e^{-x^2/(8t)}, \quad x \in \Omega, \ t > 0. \quad (5.a)
\]

So

\[
|\beta_\Omega(\phi, \rho, \Delta_g)(t) - \tilde{\beta}_\Omega(\phi, \rho, \Delta_g)(t)| \\
= \left| \int_{\Omega} \int_{\Omega} (K(x, \tilde{x}; t) - \tilde{K}(x, \tilde{x}; t)) \phi(x) \rho(\tilde{x}) dxd\tilde{x} \right| \\
\leq \|\phi\|_{L^\infty(\Omega)} \int_{\Omega} \int_{\Omega} (K(x, \tilde{x}; t) - \tilde{K}(x, \tilde{x}; t)) \rho(\tilde{x}) dxd\tilde{x} \\
\leq \|\phi\|_{L^\infty(\Omega)} \int_{\Omega} \|\rho\|_{L^\infty(\Omega)} \int_{\Omega} \int_{M} (K(x, \tilde{x}; t) - \tilde{K}(x, \tilde{x}; t)) d\tilde{x} \\
\leq \|\phi\|_{L^\infty(\Omega)} \int_{\Omega} \|\rho\|_{L^\infty(\Omega)} \int_{\Omega} \int_{M} K(x, \tilde{x}; t) d\tilde{x} - \int_{M} \tilde{K}(x, \tilde{x}; t) d\tilde{x}.
\]

Theorem 5.1 now follows by Equation (5.a). \hfill \square

Suppose that $\mathcal{N} = (\mathbb{R}^m, g_e)$ or that $\mathcal{N}$ is a compact subset of dimension $m$ with smooth boundary in $(\mathbb{R}^m, g_e)$. The following lemma will permit us to deduce Theorem 1.2 for $\mathcal{N}$ from the corresponding assertion for closed ambient manifolds by using Theorem 5.1 to localize matters to a small neighbourhood $\tilde{M}$ of $\Omega$.

**Lemma 5.2.** Let $(\tilde{M}, g_e)$ be a compact smooth manifold of dimension $m$ which is contained in $\mathbb{R}^m$. Then $(\tilde{M}, g_e)$ is isometric to a compact smooth manifold of a flat $m$-dimensional torus.

**Proof.** Let $B_n(0)$ be the ball of radius 0 about the origin in $\mathbb{R}^m$. Since $\tilde{M}$ is compact, $\tilde{M}$ is contained in $B_n(0)$ for some positive integer $n$. Let $\Gamma := \{2n\mathbb{Z}\}^m$ be the rescaled integer lattice and let $T = \mathbb{R}^m/\Gamma$ be a flat torus. Then $\tilde{M} \subset B_n(0)$ embeds isometrically in $T^m$. \hfill \square

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