Cramér-type moderate deviation of normal approximation for unbounded exchangeable pairs

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In Stein’s method, the exchangeable pair approach is commonly used to estimate the approximation errors in normal approximation. In this paper, we establish a Cramér-type moderate deviation theorem of normal approximation for unbounded exchangeable pairs. As applications, Cramér-type moderate deviation theorems for the sums of local statistics and general Curie–Weiss model are obtained.

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1. Introduction

The exchangeable pair approach of Stein’s method is commonly used to estimate the convergence rates for distributional approximation. Using exchangeable pair approach, Chatterjee and Shao [7] and Shao and Zhang [26] provided a concrete tool to identify the limiting distribution of the target random variable as well as the $L_1$ bound of the approximation. Recently, Shao and Zhang [27] obtained a Berry–Esseen-type bound of normal and nonnormal approximation for unbounded exchangeable pairs. Specifically, let $W$ be the random variable of interest, and we say $(W, W')$ an exchangeable pair if $(W, W') \overset{d}= (W', W)$. Let $\Delta = W - W'$. It is often to assume that (see, e.g., Rinott and Rotar [24]) there exists a constant $\lambda > 0$ and a random variable $R$ such that

$$E \{\Delta \mid W\} = \lambda (W + R).$$

Under condition (1.1), Shao and Zhang [27] proved the following Berry–Esseen-type bound

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \frac{1}{2\lambda} E \{\Delta^2 \mid W\} + \frac{1}{\lambda} E |E \{\Delta^* \Delta \mid W\}| + E |R|,$$

where $\Phi(z)$ is the standard normal distribution function and where $\Delta^* := \Delta^*(W, W')$ is any random variable satisfying that $\Delta^*(W, W') = \Delta^*(W', W)$ and $\Delta^* \geq |\Delta|$. We refer to Stein [29], Rinott and
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Rotar [24], Chatterjee, Diaconis and Meckes [5], Chatterjee and Meckes [6] and Meckes [21] for other related results of $L_1$ bound and Berry–Esseen bound for the exchangeable pair approach.

While the $L_1$ bound and Berry–Esseen-type bound describe the absolute error for distributional approximations, the Cramér-type moderate deviation reflects the relative error in convergences in distribution. More precisely, let $\{Y_n, n \geq 1\}$ be a sequence of random variables that converge to $Y$ in distribution, the Cramér-type moderate deviation is

$$\frac{\mathbb{P}(Y_n > x)}{\mathbb{P}(Y > x)} = 1 + \text{error term} \to 1$$

for $0 \leq x \leq a_n$, where $a_n \to \infty$ as $n \to \infty$. Specially, let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables satisfying that $\mathbb{E} X_1 = 0$, $\mathbb{E} X_1^2 = 1$ and $\mathbb{E} e^{t_0 |X_1|} < \infty$ for some $t_0 > 0$, and put $W_n = n^{-1/2}(X_1 + \cdots + X_n)$. Then,

$$\frac{\mathbb{P}(W_n > x)}{1 - \Phi(x)} = 1 + O(1)n^{-1/2}(1 + x^3), \quad (1.3)$$

for $0 \leq x \leq n^{1/6}$. The range $0 \leq x \leq n^{1/6}$ and the order of the error term $n^{-1/2}(1 + x^3)$ are optimal for i.i.d. random variables. We refer to Chapter 8 of Petrov [22] for details.

The proof of Cramér-type moderate deviation theorems for independent random variables is based on the conjugate method and the Fourier transform. However, when dealing with dependent random variables, it is more common to use Stein’s method to estimate distributional approximation errors. Since introduced by Stein [28] in 1972, Stein’s method has been deeply developed in recent years, and shows its importance and power in estimating the approximation errors of normal and nonnormal approximation. We refer to Chen, Goldstein and Shao [11] and Chatterjee [4] for more details. Recent years have seen a rapid development of applying Stein’s method to prove moderate deviation results. For instance, using Stein’s method, Raič [23] proved the moderate deviation under certain local dependence structures. In the context of Poisson approximation, Barbour, Holst and Janson [2], Chen and Choi [8] and Barbour, Chen and Choi [1] applied Stein’s method to prove moderate deviation results for sums of independent indicators, whereas Chen, Fang and Shao [9] studied sums of dependent indicators. Moreover, Chen, Fang and Shao [10], Shao, Zhang and Zhang [25] and Fang, Luo and Shao [17] obtained the general Cramér-type moderate deviation results of normal and nonnormal approximation for dependent random variables whose dependence structure is defined in terms of a Stein identity under a boundedness assumption on $|\Delta|$.

However, in practice, it may not be easy to check the condition (1.1) in general, and the boundedness assumption on $|\Delta|$ is also too strict in applications. In this paper, our aim is to apply Stein’s method and the exchangeable pair approach to prove a Cramér-type moderate deviation result without assuming that $|\Delta|$ is bounded. The results are then applied to sums of local statistics and the general Curie–Weiss model to obtain the Cramér-type moderate deviation results with optimal ranges and convergence rates.
The rest of this paper is organized as follows. We present our main results in Section 2. In Section 3, we give some applications of our main result. The proof of Theorem 2.1 are put in Section 4. The proofs of other results are postponed to Section 5.

2. Main results

Let $X$ be a random variable valued on a measurable space $\mathcal{X}$, and let $W = \varphi(X) \in \mathbb{R}$ be an $\mathbb{R}$-valued random variable of interest. We consider the following condition:

(D1) Let $(X, X')$ be an exchangeable pair. Assume that there exists $D := \Psi(X, X')$, where $\Psi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is an antisymmetric function, satisfying that $E\{D \mid X\} = \lambda(W + R)$ for some constant $\lambda > 0$ and some random variable $R := r(X)$.

Remark 2.1. The operator of antisymmetric functions was introduced by Holmes and Reinert [18], and the condition (D1) has been considered by Chatterjee [3], who applied Stein’s method to prove concentration inequalities. The condition (D1) is a natural generalization of (1.1). Specially, if (1.1) is satisfied, we can simply choose $D = \Delta$.

Under the condition (D1), by antisymmetry, it follows that $E\{D(f(W) + f(W'))\} = 0$ for any absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying that $E|Df(W)| < \infty$. A direct rearranging yields

$$
0 = E\{D(f(W) + f(W'))\} \\
= 2E\{Df(W)\} - E\{D(f(W) - f(W'))\} \\
= 2\lambda E\{(W + R)f(W)\} - E\left\{D \int_{-\Delta}^{0} f'(W + u) \, du\right\}.
$$

Then,

$$
E\{W f(W)\} = \frac{1}{2\lambda} E\left\{D \int_{-\Delta}^{0} f'(W + u) \, du\right\} - E\{Rf(W)\}.
$$

(2.1)

Our main result Theorem 2.1 provides a Cramér-type moderate deviation theorem under the condition (D1) without the assumption that $|\Delta|$ is bounded:

Theorem 2.1. Let $(X, X')$ be an exchangeable pair satisfying the condition (D1), let $W = \varphi(X)$, $W' = \varphi(X')$ and $\Delta = W - W'$. Let $D^* := D^*(X, X')$ be any random variable such that $D^*(X, X') = D^*(X', X)$ and $D^* \geq |D|$. Assume that there exists a constant $\tau > 0$ such that

(A1) $E\{(1 + |D|)e^{tW}\} < \infty$,
(A2) $E\{|1 - \frac{1}{2\lambda} E\{D\Delta \mid X\}e^{tW}\} \leq \delta_1(t) Ee^{tW}$,
(A3) $E\left\{|\frac{1}{2\lambda} E\{D^*\Delta \mid X\}e^{tW}\right\} \leq \delta_2(t) Ee^{tW}$, and
(A4) $E\{|R|e^{tW}\} \leq \delta_3(t) Ee^{tW}$.
where for each \( j = 1, 2, 3 \), the function \( \delta_j(\cdot) \) is increasing and satisfies that \( \delta_j(\tau) < \infty \). For \( \theta > 0 \), let
\[ \tau_0(\theta) := \max\{0 \leq t \leq \tau : t^2 (\delta_1(t) + \delta_2(t)) / 2 + t \delta_3(t) \leq \theta \}. \]
Then, for any \( \theta > 0 \),
\[ \left| \frac{P(W > z)}{1 - \Phi(z)} - 1 \right| \leq 20 e^{\theta} (1 + z^2) (\delta_1(z) + \delta_2(z)) + (1 + z) \delta_3(z), \quad (2.2) \]
provided that \( 0 \leq z \leq \tau_0(\theta) \).

**Remark 2.2.** Recently, Zhang [30] proved the following Berry–Esseen bound under the condition (D1):
\[ \sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq E \left| 1 - \frac{1}{2\lambda} E \{ D \Delta \mid X \} \right| + \frac{1}{\lambda} E|E \{ D^* \Delta \mid X \}| + E|R|. \quad (2.3) \]
The expectation terms in conditions (A2)–(A4) can be understood as a change of measure of those on the right hand side of (2.3). To see this, for \( t \geq 0 \), let \( Y_t \) be a random variable having the distribution
\[ P(Y_t \in A) = \frac{E\{ e^{t\varphi(X)} 1_{\{X \in A\}} \}}{E e^{t\varphi(X)}}. \]
Let \( g_1 \) and \( g_2 \) be functions defined as
\[ g_1(x) = 1 - \frac{1}{2\lambda} E\{ D \Delta \mid X = x \}, \quad g_2(x) = \frac{1}{2\lambda} E\{ D^* \Delta \mid X = x \}, \]
and recall that \( R = r(X) \) as given in (D1). Then, conditions (A2)–(A4) can be replaced by
\[ E|g_1(Y_t)| \leq \delta_1(t), \quad E|g_2(Y_t)| \leq \delta_2(t), \quad E|r(Y_t)| \leq \delta_3(t). \]

### 3. Applications

#### 3.1. Sums of local statistics

Let \( J \) be an index set and let \( X = \{ X_\alpha : \alpha \in J \} \) be a field of independent random variables where \( X_\alpha \) is valued on a measurable space \( \mathcal{X} \). For any subset \( J \subset J \), denote \( X_J = \{ X_\alpha : \alpha \in J \} \). Let \( n \geq 1 \) and let \( [n] = \{ 1, \ldots, n \} \). For each \( i \in [n] \), let \( \xi_i = f_i(X_{J_i}) \), where \( J_i \subset J \) and \( f_i : \mathcal{X}^{|J_i|} \to \mathbb{R} \), satisfying that \( E \xi_i = 0 \) for each \( i \in [n] \) and \( \sum_{i=1}^n E \xi_i^2 = 1 \). Let \( N_\alpha = \{ i \in [n] : \alpha \in J_i \} \) for \( \alpha \in J \). For any set \( A \), let \( |A| \) be its cardinality. We consider the sum of local statistic \( W = \sum_{i=1}^n \xi_i \).

**Theorem 3.1.** Assume that for each \( i \in [n] \),
\[ |\xi_i| \leq \sum_{\alpha \in J_i} g_{i\alpha}(X_\alpha), \quad (3.1) \]
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where \( \{g_{i\alpha} : i \in [n], \alpha \in J_i \} \) is an array of nonnegative functions, and there exist \( a > 0 \) and \( b \geq 1 \) such that

\[
\max_{i \in [n]} \max_{\alpha \in J_i} \mathbb{E} e^{ag_{i\alpha}(X_{i\alpha})} \leq b. \tag{3.2}
\]

Moreover, we further assume that there exist \( s \geq 1 \) and \( d \geq 1 \) such that

\[
\max_{i \in [n]} |J_i| \leq s, \quad \max_{\alpha \in J} |N_{i\alpha}| \leq d. \tag{3.3}
\]

Let \( \delta = n^{1/2}a^{-2}d^{3/2}s^{7/2}b^{1/2}(1 + n^{1/2}a^{-1}d^{1/2}s^{3/2}b^{1/2}) \). Then, we have

\[
\frac{|\mathbb{P}(W \geq z) - 1|}{1 - \Phi(z)} \leq C_0 \delta(1 + z^3) \tag{3.4}
\]

for \( 0 \leq z \leq \min\{a/(8dsb^2), \delta^{-1/3}\} \), where \( C_0 \) is an absolute positive constant.

**Remark 3.1.** We make some remarks on the conditions in Theorem 3.1. The condition (3.1) can be satisfied for a wide class of functions. For example, if \( f_i(x_i) = \prod_{j \in A_i} x_j \), then (3.1) is satisfied with \( g_{ij} = |x_j|^{J_{ij}/|J_i|} \) by Young’s inequality. Condition (3.2) is also known as the Cramér’s condition, which is commonly taken when proving Cramér-type moderate deviation theorems. The condition (3.3) is also assumed by Fang, Luo and Shao [17]. We remark that both \( d \) and \( s \) in (3.3) can also depend on \( n \).

**Remark 3.2.** In order to illustrate the range and convergence rate (3.4) are correct, consider sums of i.i.d. random variables. Let \( X_1, \ldots, X_n \) be i.i.d. random variables satisfying that \( \mathbb{E} X_i = 0, \mathbb{E} X_i^2 = 1 \) and \( \mathbb{E} e^{t_0 |X_i|} \leq b \) for some \( t_0 > 0 \) and \( b \geq 1 \). Let \( \xi_i = X_i/\sqrt{n} \) for \( i \in [n] \) and \( W = \sum_{i=1}^n \xi_i \). Then, we have (3.1)–(3.3) are satisfied with

\[
a = t_0 \sqrt{n}, \quad s = d = 1. \]

Then \( \delta = n^{-1/2}t_0^{-2}b^{1/2}(1 + t_0^{-1}b^{1/2}) \leq n^{-1/2}(1 + t_0^{-3}b) \). Therefore, Theorem 3.1 reduces to

\[
\frac{|\mathbb{P}(W \geq z) - 1|}{1 - \Phi(z)} \leq C_0 n^{-1/2}(1 + t_0^{-3}b)(1 + z^3).
\]

for \( 0 \leq z \leq \min\{t_0 b^{-2}n^{1/2}, t_0^{2/3}b^{-1/6}n^{1/6}\} \), which is as same as the optimal result (1.3).

**Remark 3.3.** Recently, Fang, Luo and Shao [17] proved a higher-order approximation relative error for the cases where \( \xi_i \)'s are bounded. Although we only consider low-order approximation in this subsection, the boundedness assumption is relaxed in Theorem 3.1. It would be interesting if one could prove a higher-order approximation error bound for unbounded cases.
3.2. The general Curie–Weiss model

The Curie–Weiss model of ferromagnetic interaction has been extensively studied in the past decades. Let $\rho$ be a probability measure on $\mathbb{R}$ satisfying that
\begin{equation}
\int_{-\infty}^{\infty} x \, d\rho(x) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 \, d\rho(x) = 1.
\end{equation}

The general Curie–Weiss model $\text{CW}(\rho)$ at inverse temperature $\beta$ is defined as the array of spin random variables $X = (X_1, \ldots, X_n)$ with joint distribution
\begin{equation}
dP_{n,\beta}(x) = Z_n^{-1} \exp \left( \frac{\beta}{2n} (x_1 + \cdots + x_n)^2 \right) \prod_{i=1}^{n} d\rho(x_i)
\end{equation}
for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ where $Z_n$ is the normalizing constant
\begin{equation}
Z_n = \int \exp \left( \frac{\beta}{2n} (x_1 + \cdots + x_n)^2 \right) \prod_{i=1}^{n} d\rho(x_i).
\end{equation}

The Curie–Weiss model $\text{CW}(\rho)$ is called “at the critical temperature” if $\beta = 1$. The total magnetization is defined by $S = \sum_{i=1}^{n} X_i$. The asymptotic behavior of the $S$ is well studied by Ellis and Newman [15, 16]. Stein’s method can be applied to estimate the convergence rate, for example, using the exchangeable pair approach, Eichelsbacher and Loewe [12], Chatterjee and Shao [7] obtained Berry–Esseen bounds for Curie–Weiss model with boundedly supported $\rho$, and Shao and Zhang [27] proved the Berry–Esseen bound for the general Curie–Weiss model with unboundedly supported $\rho$. We refer to Kirkpatrick and Meckes [19], Kirkpatrick and Nawaz [20], Eichelsbacher and Martschink [13, 14] for other normal approximation results of mean field models by Stein’s method. Moreover, Chen, Fang and Shao [10] and Shao, Zhang and Zhang [25] obtained the Cramér-type moderate deviation results for the cases where $\rho$ has a finite support.

In this subsection, we establish the Cramér-type moderate deviation result for the general Curie–Weiss model at noncritical temperature with infinitely supported probability measure $\rho$. Let $(X_1, \ldots, X_n)$ follow the joint distribution (3.6) with $0 < \beta < 1$ and $\rho$ satisfying (3.5) and
\begin{equation}
\int_{-\infty}^{\infty} e^{tx} \, d\rho(x) \leq e^{t^2/2} \quad \text{for all} \ t \in \mathbb{R}.
\end{equation}
Let $W = n^{-1/2} (1 - \beta)^{1/2} \sum_{i=1}^{n} X_i$ be the standardized version of the total magnetization. We have the following theorem.

**Theorem 3.2.** Under (3.5) and (3.7), we have
\begin{equation}
\left| \frac{\mathbb{P}(W > z)}{1 - \Phi(z)} - 1 \right| \leq C n^{-1/2} (1 + z^3) \quad \text{for} \ 0 \leq z \leq \sqrt{n}.
\end{equation}
The Berry–Esseen bound was obtained by Shao and Zhang [27] with the convergence rate $O(n^{-1/2})$. For the simplest Curie–Weiss model, where the magnetization is valued on $\{-1, 1\}$ with equal probability, Chen, Fang and Shao [10] proved the same convergence rate as (3.8) with convergence range $[0, n^{1/6}]$. However, Theorem 3.2 provides a wider convergence range.

4. Proof of main result

In this section, we give the proof of Theorem 2.1. In subsection 4.1, we prove a more general moderate deviation result, which might be of independent interest. In subsection 4.2, we prove a bound for $\mathbb{E} e^{tW}$ using Stein’s method. Our main result Theorem 2.1 follows from a combination of Propositions 4.1 and 4.4, and the details of the proof are put in subsection 4.3.

4.1. A general moderate deviation result

**Proposition 4.1.** Let $(X, X'), W, W', D, \Delta$ and $D^*$ be defined as in Theorem 2.1. Assume that there exists a constant $\tau_0 > 0$ such that for all $0 \leq t \leq \tau_0$,

(B1) $\mathbb{E}\left\{\left|1 - \frac{1}{2^X} \mathbb{E}\{D\Delta \mid X\}e^{tW}\right| \right\} \leq \kappa_1(t)e^{t^2/2}$,

(B2) $\mathbb{E}\left\{\left|\frac{1}{2^X} \mathbb{E}\{D^*\Delta \mid X\}e^{tW}\right| \right\} \leq \kappa_2(t)e^{t^2/2}$, and

(B3) $\mathbb{E}\{|R|e^{tW}\} \leq \kappa_3(t)e^{t^2/2}$.

where $\kappa_1(t), \kappa_2(t)$ and $\kappa_3(t)$ are nondecreasing functions satisfying that $\kappa_j(\tau_0) < \infty$ for $j = 1, 2, 3$. Then,

\[
\left|\frac{\mathbb{P}(W > z)}{1 - \Phi(z)} - 1\right| \leq 20\left(1 + z^2\right)(\kappa_1(z) + \kappa_2(z)) + (1 + z)\kappa_3(z),
\]

(4.1)

provided that $0 \leq z \leq \tau_0$.

We now make some remarks on Theorem 2.1 and Proposition 4.1. In practice, if we know little about the explicit distributional information of the random variable $X$, it might not be easy to obtain the $e^{t^2/2}$ term directly when calculating the expectation terms on the left hand of conditions (B1)–(B3). However, on the other hand, it is usually easier to obtain the self-bounding inequalities conditions (A2)–(A4) only based on the structure of $W$. After that, one can apply Stein’s method to prove the moment generating inequality $\mathbb{E} e^{tW} \leq Ce^{t^2/2}$, which further implies conditions (B1)–(B3). We refer to Proposition 4.4 (see below) for more details for the moment generating inequality.

Before proving Proposition 4.1, we first present some preliminary lemmas. In the proofs, we use the techniques in Chen, Fang and Shao [10, Lemmas 5.1–5.2] and Shao and Zhang [27, pp. 71–73].
Lemma 4.2. Let \( f \) be a nondecreasing function. Then,
\[
\left| \mathbb{E} \left\{ D \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\} \right| \leq \mathbb{E} \{ D^* \Delta f(W) \},
\] (4.2)
where \( D^* \) is as defined in Proposition 4.1.

Proof of Lemma 4.2. In this proof, we use the technique as in Shao and Zhang \cite{ShaoZhang2017}. Since \( f(\cdot) \) is nondecreasing, it follows that \( \Delta(f(W) - f(W')) \geq 0 \) and that
\[
0 \geq \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \geq -\Delta(f(W) - f(W')).
\] (4.3)
Let \( D^+ = D1_{\{D > 0\}} \) and \( D^- = -D1_{\{D < 0\}} \). Then, it follows that \( D = D^+ - D^- \) and \( |D| = D^+ + D^- \). By (4.3), we have
\[
D^+ \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \leq 0, \quad D^- \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \leq 0.
\]
Therefore,
\[
\mathbb{E} \left\{ D \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\} = \mathbb{E} \left\{ (D^+ - D^-) \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\}
\geq \mathbb{E} \left\{ D^+ \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\}
\geq -\mathbb{E} \{ D^+ \Delta (f(W) - f(W')) \}.
\]
and similarly
\[
\mathbb{E} \left\{ D \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\} \leq \mathbb{E} \{ D^- \Delta (f(W) - f(W')) \}.
\]
Recalling that \( W = \phi(X) \), \( D = F(X, X') \) is antisymmetric and \( D^* = F^*(X, X') \) is symmetric, as \((X, X')\) is exchangeable, we have
\[
\mathbb{E} \{ D^+ \Delta (f(W) - f(W')) \} = \mathbb{E} \{ D^- \Delta (f(W) - f(W')) \} \geq 0.
\]
Therefore, the LHS of (4.2) is bounded by
\[
\frac{1}{2\lambda} \left| \mathbb{E} \left\{ D \int_{-\Delta}^{0} (f(W + u) - f(W)) \, du \right\} \right| \leq \frac{1}{2\lambda} \mathbb{E} \{ D^- \Delta (f(W) - f(W')) \}.
\] (4.4)
In order to bound the RHS of (4.4), note that \( \mathbb{E} \{ D^* \Delta 1_{\{D = 0\}} (f(W) - f(W')) \} \geq 0 \) and that \( \mathbb{E} \{ D^* 1_{\{D = 0\}} \Delta f(W) \} = -\mathbb{E} \{ D^* 1_{\{D = 0\}} \Delta \phi(W') \} \), and then we have
\[
\mathbb{E} \{ D^* \Delta 1_{\{D = 0\}} f(W) \} \geq 0.
\] (4.5)
Moreover, note that $D^*$ is symmetric with respect to $X$ and $X'$, and by by exchangeability,

$$E\{D^*\Delta 1_{\{D<0\}}f(W')\} = E\{D^*\Delta 1_{\{D>0\}}f(W)\}. \quad (4.6)$$

Therefore, by (4.5) and (4.6),

$$\text{RHS of (4.4)} \leq \frac{1}{2\lambda} E\{D^* 1_{\{D<0\}} \Delta (f(W) - f(W'))\}$$

$$= \frac{1}{2\lambda} E\{D^* \Delta (1_{\{D>0\}} + 1_{\{D<0\}}) f(W)\}$$

$$\leq \frac{1}{2\lambda} E\{D^* \Delta f(W)\}.$$

This completes the proof. \qed

**Lemma 4.3.** Under the conditions of Proposition 4.1, we have for $0 \leq z \leq \tau_0$,

$$E\left\{\left|1 - \frac{1}{2\lambda} E\{D \Delta |W|\} We^{W^2/2} 1_{\{0 \leq W \leq z\}}\right|\right\} \leq 4(1 + z^2)\kappa_1(z), \quad (4.7)$$

$$\frac{1}{2\lambda} E\left\{\left|E\{D^* \Delta |W|\} We^{W^2/2} 1_{\{0 \leq W \leq z\}}\right|\right\} \leq 4(1 + z^2)\kappa_2(z), \quad (4.8)$$

$$E\left\{|R| e^{W^2/2} 1_{\{0 \leq W \leq z\}}\right\} \leq 2(1 + z)\kappa_3(z). \quad (4.9)$$

**Proof of Lemma 4.3.** We apply the idea of Chen, Fang and Shao [10, Lemma 5.2] in this proof. For $a \in \mathbb{R}_+$, denote $[a] = \max\{n \in \mathbb{N} : n \leq a\}$. By condition (B1), and recalling that the function $\kappa_1(\cdot)$ is increasing, for any $0 \leq x \leq z \leq \tau_0$,

$$e^{-z^2/2} E\left\{\left|1 - \frac{1}{2\lambda} E\{D \Delta |W|\} e^{xW}\right|\right\} \leq \kappa_1(x) \leq \kappa_1(z). \quad (4.10)$$

We have

$$\text{RHS of (4.7)} = \sum_{j=1}^{[z]} E\left\{\left|1 - \frac{1}{2\lambda} E\{D \Delta |W|\} We^{W^2/2} 1_{\{j-1 < W < j\}}\right|\right\}$$

$$+ E\left\{1 - \frac{1}{2\lambda} E\{D \Delta |W|\} We^{W^2/2} 1_{\{z \leq W \leq z\}}\right\}$$

$$\leq \sum_{j=1}^{[z]} je^{(j-1)^2/2-j(j-1)} E\left\{\left|1 - \frac{1}{2\lambda} E\{D \Delta |W|\} e^{jW} 1_{\{j-1 < W < j\}}\right|\right\}$$

$$+ ze^{[z]^2/2-[z]z} E\left\{1 - \frac{1}{2\lambda} E\{D \Delta |W|\} e^{zW} 1_{\{z \leq W \leq z\}}\right\}$$

$$\leq 2 \sum_{j=1}^{[z]} je^{-j^2/2} E\left\{1 - \frac{1}{2\lambda} E\{D \Delta |W|\} e^{jW} 1_{\{j-1 < W < j\}}\right\}$$
\[ + 2ze^{-z^2/2} \mathbb{P} \left\{ 1 - \frac{1}{2\lambda} \mathbb{E} \{ D\Delta \mid W \} \left| e^{zW} 1_{\{|z| \leq W \}} \right. \right\} \]
\[ \leq 2\kappa_1(z) \left( \sum_{j=1}^{\lfloor z \rfloor} j + z \right) \leq 4(1 + z^2)\kappa_1(z), \]

where we used (4.10) in the last line. This proves (4.7). The inequalities (4.8) and (4.9) can be shown similarly.

Now we are ready to give the proof of Proposition 4.1. In the proof, we combine the ideas in Shao and Zhang [27, Section 4] and Chen, Fang and Shao [10, Section 6].

**Proof of Proposition 4.1.** Let \( z \geq 0 \) be a fixed real number, and let \( f_z \) be the solution to the Stein equation

\[ f'(w) - wf(w) = 1_{\{w \leq z\}} - \Phi(z), \quad (4.11) \]

where \( \Phi(\cdot) \) is the standard normal distribution function. It is well known that (see, e.g., Chen, Goldstein and Shao [11]) \( f_z \) is given by

\[ f_z(w) = \begin{cases} \frac{\Phi(w) \{1 - \Phi(z)\}}{p(w)}, & w \leq z, \\ \frac{\Phi(z) \{1 - \Phi(w)\}}{p(w)}, & w > z, \end{cases} \quad (4.12) \]

where \( p(w) = (2\pi)^{-1/2} e^{-w^2/2} \) is the standard normal density function.

By (4.11) and applying (2.1) by taking \( f(w) = f_z(w) \) yields

\[ \mathbb{P}(W > z) - \{1 - \Phi(z)\} = \mathbb{E}\{f'_z(W) - Wf_z(W)\} = J_1 - J_2 + J_3, \quad (4.13) \]

where

\[ J_1 = \mathbb{E}\left\{ f'_z(W) \left( 1 - \frac{1}{2\lambda} \mathbb{E} \{ D\Delta \mid W \} \right) \right\}, \]
\[ J_2 = \frac{1}{2\lambda} \mathbb{E}\left\{ D \int_{-\Delta}^{0} (f'_z(W + u) - f'_z(W)) \, du \right\}, \]
\[ J_3 = \mathbb{E}\{Rf_z(W)\}. \]

Without loss of generality, we only consider \( J_2 \), because \( J_1 \) and \( J_3 \) can be bounded similarly.
For $J_2$, observe that $f'_z(w) = wf(w) - 1_{\{w > z\}} + \{1 - \Phi(z)\}$, and both $wf_z(w)$ and $1_{\{w > z\}}$ are increasing functions (see, e.g., Chen, Goldstein and Shao [11, Lemma 2.3]), by Lemma 4.2.

\[
|J_2| \leq \frac{1}{2\lambda} \mathbb{E} \left[ D \int_{-\Delta}^{0} \left\{ (W + u)f_z(W + u) - W f'_z(W) \right\} du \right] \\
+ \frac{1}{2\lambda} \mathbb{E} \left[ D \int_{-\Delta}^{0} \left\{ 1_{\{W + u > z\}} - 1_{\{W > z\}} \right\} du \right] \\
\leq \frac{1}{2\lambda} \mathbb{E} \left| \mathbb{E} \{ D^* \Delta | W \} \right| \mathbb{E} \{ |W f_z(W)| + 1_{\{W > z\}} \} = J_{21} + J_{22},
\]

where

\[
J_{21} = \frac{1}{2\lambda} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} \cdot |W f_z(W)| \right\}, \quad J_{22} = \frac{1}{2\lambda} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} 1_{\{W > z\}} \right\}.
\]

For any $w > 0$, it is well known that $(1 - \Phi(w))/p(w) \leq \min \{1/w, \sqrt{2\pi}/2\}$. Then, for $w > z$,

\[
|f_z(w)| \leq \frac{\sqrt{2\pi}}{2} \Phi(z), \quad |wf_z(w)| \leq \Phi(z), \quad (4.15)
\]

and by symmetry, for $w < 0$,

\[
|f_z(w)| \leq \frac{\sqrt{2\pi}}{2} \{1 - \Phi(z)\}, \quad |wf_z(w)| \leq 1 - \Phi(z). \quad (4.16)
\]

For $J_{21}$, by (4.12), (4.15) and (4.16), we have

\[
J_{21} \leq \frac{1}{2\lambda} \{1 - \Phi(z)\} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} 1_{\{W < 0\}} \right\} \\
+ \frac{\sqrt{2\pi}}{2\lambda} \{1 - \Phi(z)\} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} \mathbb{E} W^2/2 1_{\{0 \leq W \leq z\}} \right\} \\
+ \frac{1}{2\lambda} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} 1_{\{W > z\}} \right\}.
\]

Thus, by (4.14) and (4.17),

\[
|J_2| \leq \frac{1}{2\lambda} \{1 - \Phi(z)\} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} 1_{\{W < 0\}} \right\} \\
+ \frac{\sqrt{2\pi}}{2\lambda} \{1 - \Phi(z)\} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} \mathbb{E} W^2/2 1_{\{0 \leq W \leq z\}} \right\} \\
+ \frac{1}{2\lambda} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} 1_{\{W > z\}} \right\}.
\]

For the second term of (4.18), by Lemma 4.3, we have

\[
\frac{\sqrt{2\pi}}{2\lambda} \mathbb{E} \left\{ \mathbb{E} \{ D^* \Delta | W \} \mathbb{E} W^2/2 1_{\{0 \leq W \leq z\}} \right\} \leq 4\sqrt{2\pi}(1 + z^2)\kappa_2(z). \quad (4.19)
\]
It is well known that for $z > 0$,
\[ e^{-z^2/2} \leq \sqrt{2\pi}(1+z)\{1-\Phi(z)\} \leq \frac{3\sqrt{2\pi}}{2}(1+z^2)\{1-\Phi(z)\}. \]

For the third term of (4.18), by condition (B2), for $0 \leq z \leq \tau_0$,
\[ \frac{1}{\lambda} \mathbb{E} \left\{ |\mathbb{E} \{ D^* \Delta | W\} |1_{\{W > z\}} \right\} \leq 2\kappa_2(z)e^{-z^2/2} \leq 3\sqrt{2\pi}(1+z^2)\kappa_2(z)\{1-\Phi(z)\}. \quad (4.20) \]

Therefore, combining (4.18)–(4.20), for $0 \leq z \leq \tau_0$, we have
\[ |J_2| \leq (7\sqrt{2\pi}+1)(1+z^2)\kappa_2(z)(1-\Phi(z)) \leq 20(1+z^2)\kappa_2(z)(1-\Phi(z)). \]

Similarly,
\[ |J_1| \leq 20(1+z^2)\kappa_1(z)(1-\Phi(z)), \quad |J_3| \leq 20(1+z)\kappa_3(z)(1-\Phi(z)). \]

This completes the proof together with (4.13).

\[ \square \]

### 4.2. A moment generating function bound

The following proposition provides a moment generating function bound.

**Proposition 4.4.** Under the assumptions in Theorem 2.1. For $0 \leq t \leq \tau$, we have
\[ \mathbb{E} e^{tW} \leq \exp \left\{ \frac{t^2}{2} (1 + \delta_1(t) + \delta_2(t)) + \delta_3(t)t \right\} \text{ for } 0 \leq t \leq \tau. \quad (4.21) \]

**Proof of Proposition 4.4.** The proof is based on a standard technique of proving exponential bounds in Stein’s method. Specifically, let $h(t) = \mathbb{E} e^{tW}$. In order to bound $h(t)$, we need to find an upper bound for $h'(t)$ using Stein’s method, and then obtain a bound for $(\log h(t))'$. This technique was firstly considered by Chatterjee [3], who proved a concentration inequality for exchangeable pairs. Chen, Fang and Shao [10], Shao, Zhang and Zhang [25], Fang, Luo and Shao [17] also used this technique to prove Cramér-type moderate deviation theorems.

By conditions (A1) and (A4), $\mathbb{E} \{|D| e^{tW}\} < \infty$ and $\mathbb{E} \{|R| e^{tW}\} < \infty$. Since $\mathbb{E} \{D|X\} = \lambda(W + R)$, we have $\mathbb{E} \{|W| e^{tW}\} < \infty$. Applying (2.1) with $f(w) = e^{tw}$, and by (2.1), we have
\[
h'(t) = \mathbb{E} \{W e^{tW}\} = \frac{t}{2\lambda} \mathbb{E} \left\{ D \int_{-\Delta}^{0} e^{t(W+u)} \, du \right\} - \mathbb{E} \{Re^{tW}\} \]
\[
\leq t \mathbb{E} e^{tW} + t \mathbb{E} \left\{ \frac{1}{2\lambda} \mathbb{E} \{D\Delta |X\} - 1 \right\} e^{tW} \right\} + t \mathbb{E} \left\{ \mathbb{E} \left\{ D \int_{-\Delta}^{0} (e^{t(W+u)} - e^{tW}) \, du \right\} \right\} + \mathbb{E} \{|R| e^{tW}\} \]
\[
\leq t \mathbb{E} e^{tW} + t \mathbb{E} \left\{ \frac{1}{2\lambda} \mathbb{E} \{D\Delta |X\} - 1 \right\} e^{tW} \right\} + \mathbb{E} \left\{ \mathbb{E} \left\{ D \Delta |X\right\} e^{tW}\right\} + \mathbb{E} \{|R| e^{tW}\},
\]
where we used Lemma 4.2 in the last line. By conditions (A2)–(A4), we have for \(0 \leq t \leq \tau\),
\[
h'(t) \leq \{t[1 + \delta_1(t) + \delta_2(t)] + \delta_3(t)\} h(t),
\]
which is
\[
(\log h(t))' \leq t[1 + \delta_1(t) + \delta_2(t)] + \delta_3(t).
\]
(4.22)

Note that \(h(0) = 1\) and that \(\delta_1(t), \delta_2(t), \delta_3(t)\) are increasing, we have
\[
\log h(t) \leq \int_0^t \{u[1 + \delta_1(u) + \delta_2(u)] + \delta_3(u)\} du \leq \frac{t^2}{2}[1 + \delta_1(t) + \delta_2(t)] + t\delta_3(t),
\]
which further implies (4.21). \(\square\)

### 4.3. Proof of Theorem 2.1

With the help of Proposition 4.1 and Proposition 4.4, we are ready to give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By Proposition 4.4, we have \(\mathbb{E} e^{tW} \leq e^{\theta t^2/2}\) for \(0 \leq t \leq \tau_0(\theta)\). By conditions (A2)–(A4), we have conditions (B1)–(B3) are satisfied with \(\tau_0 = \tau_0(\theta)\), and
\[
\kappa_1(t) = e^{\theta} \delta_1(t), \quad \kappa_2(t) = e^{\theta} \delta_2(t), \quad \kappa_3(t) = e^{\theta} \delta_3(t).
\]
This proves Theorem 2.1 by applying Proposition 4.1. \(\square\)

### 5. Proof of other results

#### 5.1. Proof of Theorem 3.1

We apply Proposition 4.1 to prove the result. In subsection 5.1.1, we construct an exchangeable pair \((X, X')\), and give several exponential moment inequalities in Propositions 5.1 and 5.2 (see below). The main proof of Theorem 3.1 is put in subsection 5.1.2, and we provide the proof of Proposition 5.2 in subsection 5.1.3. In what follows, we use \(i, j, i', j', k, l\) to represent elements in \([n]\) and use \(\alpha, \alpha'\) to represent elements in \(\mathcal{J}\). Moreover, throughout this section, we denote by \(C\) an absolute constant, which may take different values in different places.
5.1.1. Exchangeable pairs and exponential moment inequalities

To begin with, we construct an exchangeable pair for $X$. Let $X^* = \{X^*_\alpha : \alpha \in \mathcal{J}\}$ be an independent copy of $X$. For each $J \subset \mathcal{J}$, let $X^J = \{X^J_\alpha : \alpha \in J\}$, where

$$X^J_\alpha = \begin{cases} X^*_\alpha & \text{if } \alpha \in J, \\ X_\alpha & \text{if } \alpha \not\in J. \end{cases}$$

For each $1 \leq i \leq n$, define the random vector $(\xi^{(i)}_1, \ldots, \xi^{(i)}_n)$ by $\xi^{(i)}_j = f_j(X^J_j)$. Let $I$ be a random variable uniformly chosen from $\{1, \ldots, n\}$, which is independent of all others. Let $X' = X^J_I$ and $W' = \sum_{j=1}^n \xi^{(I)}_j$. Then, it follows that $(X, X')$ is exchangeable, and therefore so is $(W, W')$. Let $A_i = \{j : J_j \cap J_i \neq \emptyset\}$. Then, for any $i$, we have $\xi^i_j = \xi^{(i)}_j$ for all $j \not\in A_i$. Define

$$D := \xi_I - \xi^{(I)}_I, \quad D^* := |\xi_I - \xi^{(I)}_I|, \quad \Delta := W - W' = \sum_{j \in A_I} (\xi_j - \xi^{(I)}_j). \quad (5.1)$$

Note that $\xi^{(i)}_i$ is independent of $X$ and that $\mathbb{E} \xi_i = 0$, then it follows that

$$\mathbb{E} \{D \mid X\} = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \{\xi_i - \xi^{(i)}_i \mid X\} = \frac{1}{n} \sum_{i=1}^n (\xi_i - \mathbb{E} \xi_i) = \frac{1}{n} W.$$

Therefore, the condition (D1) holds with $\lambda = 1/n$ and $R = 0$. Applying (2.1) with $f(w) = w$, we have $\mathbb{E}\{D\Delta\}/(2\lambda) = \mathbb{E} W^2 = 1$. Moreover,

$$\frac{1}{2\lambda} \mathbb{E} \{D\Delta \mid X, X^*\} - 1 = \frac{1}{2\lambda} [\mathbb{E} \{D\Delta \mid X, X^*\} - \mathbb{E} \{D\Delta\}]$$

$$= \frac{1}{2} \sum_{i=1}^n [(\xi_i - \xi^{(i)}_i)(\xi_j - \xi^{(i)}_j) - \mathbb{E}((\xi_i - \xi^{(i)}_i)(\xi_j - \xi^{(i)}_j))] \quad (5.2)$$

$$\frac{1}{\lambda} \mathbb{E} \{D^*\Delta \mid X, X^*\} = \sum_{i=1}^n |\xi_i - \xi^{(i)}_i|(\xi_j - \xi^{(i)}_j).$$

To verify conditions (A1)–(A4), we need to prove the following proposition. The first proposition is important in verifying condition (A1).

**Proposition 5.1.** Assume that (3.1)–(3.3) hold, we have $\mathbb{E} e^{tW} < \infty$ for $0 \leq t \leq a/d$.

The proof of Proposition 5.1 is based on (3.1) and (3.2) and independence, and the details is given in the Supplementary Material [31]. The following proposition plays an important role in verifying conditions (A2)–(A4).
Proposition 5.2. For each $i, j \in [n]$, let $\eta_{ij} = (\xi_i - \xi_i^{(i)})(\xi_j - \xi_j^{(i)}) - \mathbb{E}(\xi_i - \xi_i^{(i)})(\xi_j - \xi_j^{(i)})$ or $\eta_{ij} = |\xi_i - \xi_i^{(i)}||\xi_j - \xi_j^{(i)}|$. Under the assumptions of Theorem 3.1, we have for $0 \leq t \leq a/(8dsb^2)$,

$$
\mathbb{E}\left\{ \left( \sum_{i \in [n]} \sum_{j \in A_i} \eta_{ij} \right)^2 e^{tW} \right\} \leq C n a^{-4} d^3 s^7 b (1 + na^{-2} ds^3 bt^2) \mathbb{E} e^{tW}.
$$

(5.3)

The proof of Proposition 5.2 is put in subsection 5.1.3.

5.1.2. Proof of Theorem 3.1

Now, we are ready to prove Theorem 3.1 with the help of Propositions 5.1 and 5.2.

Proof of Theorem 3.1. Recall that condition (D1) is satisfied with $\lambda = 1/n$ and $R = 0$. As $D = \xi_I - \xi_I^{(I)}$, by Cauchy’s inequality and recalling the assumption that $\sum_{i=1}^{n} \mathbb{E} \xi_i^2 = 1$, it follows that

$$
\mathbb{E}\{|D|e^{tW}\} \leq (\mathbb{E}|\xi_I - \xi_I^{(I)}|^2)(\mathbb{E} e^{2tW})^{1/2} \leq 2(\mathbb{E} e^{2tW})^{1/2}.
$$

By Proposition 5.1, and recalling that $s \geq 1, d \geq 1, b \geq 1$, we have condition (A1) is satisfied for $0 \leq t \leq a/(8dsb^2)$.

Moreover, by Jensen’s inequality and H"older’s inequality, we have

$$
\mathbb{E}\{|1 - \frac{1}{2\lambda} \mathbb{E}\{D\Delta | X\}|e^{tW}\} \leq \mathbb{E}\{|1 - \frac{1}{2\lambda} \mathbb{E}\{D\Delta | X, X^*\}|e^{tW}\},
$$

$$
\leq (\mathbb{E} e^{tW})^{1/2} \left( \mathbb{E}\{|1 - \frac{1}{2\lambda} \mathbb{E}\{D\Delta | X, X^*\}|^2 e^{tW}\} \right)^{1/2}.
$$

Let $\tau_0 = \min\{a/(8dsb^2), \delta^{-1/3}\}$. By (5.2) and Proposition 5.2, we have condition (A2) in Theorem 2.1 are satisfied with $\delta_1(t) = C\delta(1 + t)$. Similarly, condition (A3) is also satisfied with $\delta_2(t) = C\delta(1 + t)$. Applying Theorem 2.1 yields the desired result.

5.1.3. Proof of Proposition 5.2

Before giving the proof of Proposition 5.2, we need to prove some preliminary lemmas, which provide moment inequalities for $g_{ia}(X_{\alpha})$ and $\xi_i$. The proofs are given in the Supplementary Material [31].

Lemma 5.3. We have for all $i \in [n]$ and $\alpha \in J_i$,

$$
\mathbb{E}|g_{ia}(X_{\alpha})|^6 \leq 120a^{-6}b,
$$

(5.4)

$$
\mathbb{E}|\xi_i|^6 \leq 120a^{-6} sb^6.
$$

(5.5)

Lemma 5.4. For any $i \in [n]$ and $\alpha \in J_i$, we have for $0 \leq t \leq a/(4dsb^2)$,

$$
\mathbb{E}\{g_{ia}(X_{\alpha})^6 e^{tW}\} \leq Ca^{-6}b^2 \mathbb{E} e^{tW}.
$$
Proof of Proposition 5.2. In this proof, $C$ denotes an absolute positive constant, and $O(a)$ denotes a quantity satisfying that $|O(a)| \leq Ca$. Expanding the square term in the left hand side of (5.3) yields

$$\text{LHS of (5.3)} = \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \mathbb{E}\left\{ \eta_{i,j} \eta_{i',j'} e^{tW} \right\}.$$  
(5.6)

In what follows, we focus on estimating each summand on the right hand side of (5.6), which is based on independence, Taylor’s expansion and exponential moment bounds.

In order to calculate the summand on the right hand side of (5.6), we need to introduce some notation. Let $\tilde{X} = (\tilde{X}_m : m \in J)$ be an independent copy of $X$, which is also independent of $X^*$. Again, for any $J \subset J$, let $\tilde{X}^J = (\tilde{X}_m : m \in J)$ be defined as

$$\tilde{X}_m^J = \begin{cases} \tilde{X}_m & \text{if } m \in J, \\ X_m & \text{if } m \notin J. \end{cases}$$

Let $J_{ij,i'j'} = J_i \cup J_j \cup J_{i'} \cup J_{j'}$. For each $1 \leq k \leq n$, let $\xi_k^{(i,j,i',j')}$ be defined as $\tilde{X}_m^{J_{ij,i'j'}}$ and let $\tilde{W}^{(i,j,i',j')} = \sum_{k \in [n]} \xi_k^{(i,j,i',j')}$. Then, we have $\tilde{W}^{(i,j,i',j')}$ is independent of $(\eta_{ij}, \eta_{i'j'})$ and has the same distribution as $W$. For $1 \leq i \leq n$, let $A_i = \{ j : J_i \cap J_j \not= \emptyset \}$. From (3.3),

$$|A_i| \leq \min\{sd, n\}.$$  
(5.7)

For any $i, j, i', j'$, let $A_{ij,i'j'} = A_i \cup A_j \cup A_{i'} \cup A_{j'}$. Moreover, it follows that $W - \tilde{W}^{(i,j,i',j')} = \sum_{k \in A_{ij,i'j'}} (\xi_k - \tilde{\xi}_k^{(i,j,i',j')})$. Applying Taylor’s expansion to right hand side of (5.6), we obtain

$$\text{RHS of (5.6)} = \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \mathbb{E}\left\{ \eta_{i,j} \eta_{i',j'} e^{t\tilde{W}^{(i,j,i',j')}} \right\}$$

$$+ t \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,i'j'}} \mathbb{E}\left\{ \eta_{i,j} \eta_{i',j'} \xi_k e^{t\tilde{W}^{(i,j,i',j')}} \right\}$$

$$- t \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,i'j'}} \mathbb{E}\left\{ \eta_{i,j} \eta_{i',j'} \tilde{\xi}_k^{(i,j,i',j')} e^{t\tilde{W}^{(i,j,i',j')}} \right\}$$

$$+ O(t^2) \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \mathbb{E}\left\{ |\eta_{i,j} \eta_{i',j'}| \left( \sum_{k \in A_{ij,i'j'}} (\xi_k - \tilde{\xi}_k^{(i,j,i',j')}) \right)^2 e^{t\tilde{W}^{(i,j,i',j')}} \right\}$$

$$+ O(t^2) \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \mathbb{E}\left\{ |\eta_{i,j} \eta_{i',j'}| \left( \sum_{k \in A_{ij,i'j'}} (\xi_k - \tilde{\xi}_k^{(i,j,i',j')}) \right)^2 e^{tW} \right\}$$

$$= H_1 + H_2 - H_3 + H_4 + H_5.$$
For $H_1$, recalling that $\tilde{\mathcal{W}}^{(i,j,j',j')}$ is independent of $(\eta_{ij}, \eta_{i'j'})$ and has the same distribution as $W$, we have

$$
\mathbb{E}\{\eta_{ij}\eta_{i'j'}e^{i\tilde{\mathcal{W}}^{(i,j,j',j')}}\} = \mathbb{E}\{\eta_{ij}\eta_{i'j'}\} \mathbb{E} e^{tW}.
$$

(5.8)

Now, note that $\eta_{ij}$ is independent of $\eta_{i'j'}$ when $i' \in A'_{i,j}$ and $j' \in A'_{i',j}$, and that $\mathbb{E} \eta_{ij} = \mathbb{E} \eta_{i'j'} = 0$, thus,

$$
\mathbb{E}\{\eta_{ij}\eta_{i'j'}\} = 0 \text{ if } i' \in A'_{i,j} \text{ and } j' \in A'_{i',j}.
$$

(5.9)

Moreover, by Lemma 5.3 and Hölder’s inequality,

$$
\mathbb{E}|\eta_{ij}|^2 \leq C(\mathbb{E} \xi_t^4 + \mathbb{E} \xi_s^4) \leq Ca^{-4}s^4b^{2/3}.
$$

(5.10)

By (5.7) and (3.3),

$$
|\{i': j' \in A_{i'}\}| = |\{i': J_{i'} \cap J_{j'} \neq \emptyset\}| = |A_{i'}| \leq \min\{sd, n\},
$$

(5.11)

and then we obtain

$$
|H_1| \leq \mathbb{E}e^{tW} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \mathbb{E}|\eta_{ij}\eta_{i'j'}|1_{\{i' \in A_{ij} \text{ or } j' \in A_{ij}\}}
$$

$$
\leq Ca^{-4}s^4b^{2/3} \mathbb{E}e^{tW} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} 1_{\{i' \in A_{ij} \text{ or } j' \in A_{ij}\}}
$$

$$
\leq Ca^{-4}s^4b^{2/3} \mathbb{E}e^{tW} \left( \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in A_{ij}} \sum_{j' \in A_{i'}} 1 + \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in A_{ij}} \sum_{j' \in A_{i'}} 1 \right)
$$

$$
\leq Ca^{-4}s^4b^{2/3}(n^d s^3 \mathbb{E}e^{tW} \leq Cna^{-4}d^3s^7b^{2/3} \mathbb{E}e^{tW}.
$$

(5.12)

To calculate $H_2$, we need to introduce some more notation. For fixed $i, j, i', j', k$, let $J_{ij,j',k} = J_{ij} \cup J_{i'j'} \cup J_k$, and for each $1 \leq l \leq n$, let $\tilde{\zeta}_l^{(i,j,i',j',k)} = f_l(\tilde{X}_{ij}^{(i,j,j',k)})$ and $\tilde{\mathcal{W}}^{(i,j,i',j',k)} = \sum_{l \in [n]} \tilde{\zeta}_l^{(i,j,i',j',k)}$. Then, we have $\tilde{\mathcal{W}}^{(i,j,i',j',k)}$ is independent of $(\eta_{ij}, \eta_{i'j'}, \xi_k)$ and has the same distribution as $W$, and thus,

$$
\mathbb{E}\{\eta_{ij}\eta_{i'j'}\xi_k e^{i\tilde{\mathcal{W}}^{(i,j,i',j',k)}}\} = \mathbb{E}\{\eta_{ij}\eta_{i'j'}\xi_k e^{i\tilde{\mathcal{W}}^{(i,j,j',k)}}\} = \mathbb{E}\{\eta_{ij}\eta_{i'j'}\xi_k\} \mathbb{E} e^{tW}.
$$

(5.13)

Moreover, note that

$$
\tilde{\mathcal{W}}^{(i,j,i',j',k)} - \tilde{\mathcal{W}}^{(i,j,i',j',k)} = \sum_{l \in A_k} (\tilde{\zeta}_l^{(i,j,i',j')} - \tilde{\zeta}_l^{(i,j,j',k)}).
$$

(5.14)
By Taylor’s expansion again and by (5.14), we have

$$H_2 = t \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,j'}} \mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k e^{i \tilde{W}_{ij,j'}} \right\}$$

$$+ O(t^2) \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,j'}} \mathbb{E} \left\{ \left| \eta_{ij} \eta_{ij'} \| \xi_k \right| e^{i \tilde{W}_{ij,j'}} \right\}$$

$$+ O(t^2) \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,j'}} \mathbb{E} \left\{ \left| \eta_{ij} \eta_{ij'} \| \xi_k \right| e^{i \tilde{W}_{ij,j'}} \right\}$$

$$= H_{21} + H_{22} + H_{23}.$$

For $H_{21}$, by (5.13) and by the fact that $A_{ij,j'} = A_{ij} \cup A_{i,j'}$, we have

$$|H_{21}| \leq t \mathbb{E} e^{W} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,j'}} \left\| \mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k \right\} \right\|$$

$$\leq t \mathbb{E} e^{W} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \left( \sum_{k \in A_{ij}} \left\| \mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k \right\} \right\| + \sum_{k \in A_{i,j'}} \left\| \mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k \right\} \right\| \right)$$

$$= 2t \mathbb{E} e^{W} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} \left\| \mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k \right\} \right\|,$$

where the last line is based on symmetry. If $i', j' \in A_{ij,k}$, then $\eta_{ij,j'}$ is independent of $\eta_{ij} \| \xi_k$, and thus $\mathbb{E} \left\{ \eta_{ij} \eta_{ij'} \| \xi_k \right\} = \mathbb{E} \left\{ \eta_{ij} \| \xi_k \right\} \mathbb{E} \| \xi_{ij} \| = 0$. Moreover, by Hölder’s inequality and Lemma 5.3,

$$\mathbb{E} |\eta_{ij} \eta_{ij'} \| \xi_k| \leq C(\mathbb{E} |\xi_i|^5 + \mathbb{E} |\xi_j|^5 + \mathbb{E} |\xi_{ij'}|^5 + \mathbb{E} |\xi_{ij}|^5 + \mathbb{E} |\xi_{k}|^5) \leq C \alpha^{-5} s^5 b^{5/6}.$$

Therefore, by (5.7) and (5.11),

$$|H_{21}| \leq C t \alpha^{-5} s^5 b^{5/6} \mathbb{E} e^{W} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} \mathbf{1}_{\{i' \in A_{ij,k} \text{ or } j' \in A_{ij,k}\}}$$

$$\leq C t \alpha^{-5} s^5 b^{5/6} \mathbb{E} e^{W} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{k \in A_{ij}} \left( \sum_{i' \in A_{ij,k}} \sum_{j' \in A_{ij,k}} \mathbf{1}_{\{j' \in A_{ij,k}\}} + \sum_{j' \in A_{ij,k}} \sum_{i' \in A_{ij,k}} \mathbf{1}_{\{i' \in A_{ij,k}\}} \right)$$

$$\leq C t \alpha^{-5} s^5 b^{5/6} \left( n \min \{\alpha, s\} \right)^4 \mathbb{E} e^{W}$$

$$\leq C n a^{-4} d^3 s^7 b \left( 1 + t^2 n a^{-2} d^2 s b \right) e^{W},$$

where we used Cauchy’s inequality and the fact that $b \geq 1$ in the last line.
For $H_{22}$, by Young’s inequality, we have
\[ \mathbb{E}\left\{ |\eta_{i,j}^{t}\eta_{i,j}'^{t}\xi_k^{t}\xi_{k}^{t(i,j,i',j')} - \xi_{k}^{t(i,j,i',j')}|e^{t\overline{W}(i,j,i',j')} \right\} \]
\[ \leq \mathbb{E}\left\{ |\eta_{i,j}^{t}\eta_{i,j}'^{t}\xi_k^{t}\xi_{k}^{t(i,j,i',j')}|e^{t\overline{W}(i,j,i',j')} \right\} + \mathbb{E}\left\{ |\eta_{i,j}^{t}\eta_{i,j}'^{t}\xi_k^{t}\xi_{k}^{t(i,j,i',j')}|e^{t\overline{W}(i,j,i',j')} \right\} \]
\[ \leq \frac{2}{3} \mathbb{E}\{ |\eta_{i,j}^{t}|^3 e^{t\overline{W}(i,j,i',j')} \} + \frac{2}{3} \mathbb{E}\{ |\eta_{i,j}'^{t}|^3 e^{t\overline{W}(i,j,i',j')} \} + \frac{1}{3} \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \} + \frac{1}{3} \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \}. \] (5.17)

For the first two terms of the right hand side of (5.17), recalling that $\overline{W}(i,j,i',j')$ is independent of $(\eta_{i,j}, \eta_{i,j}')$ and has the same distribution as $W$, and by the definition of $\eta_{i,j}$ and by Lemma 5.3, we have
\[ \mathbb{E}\{ |\eta_{i,j}^{t}|^3 e^{t\overline{W}(i,j,i',j')} \} = \mathbb{E}\{ |\eta_{i,j}^{t}|^3 e^{tW} \} \leq C \mathbb{E} e^{tW} (\mathbb{E}|\xi|^{6} + \mathbb{E}|\xi|^{6}) \leq Ca^{-6} s^{6} b \mathbb{E} e^{tW}, \]
\[ \mathbb{E}\{ |\eta_{i,j}'^{t}|^3 e^{t\overline{W}(i,j,i',j')} \} = \mathbb{E}\{ |\eta_{i,j}'^{t}|^3 e^{tW} \} \leq C \mathbb{E} e^{tW} (\mathbb{E}|\xi|^{6} + \mathbb{E}|\xi|^{6}) \leq Ca^{-6} s^{6} b \mathbb{E} e^{tW}. \] (5.18)

For the third term of the right hand side of (5.17), by (3.1),
\[ \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \} \leq s^{5} \sum_{m \in J_{k}} \mathbb{E}\{ g_{km}(X_{m})^{6} e^{t\overline{W}(i,j,i',j')} \}. \] (5.19)

Now, if $m \in J_{k} \setminus J_{ij,j'}$, then $X_{m}$ is independent of $\overline{W}(i,j,i',j')$, and thus
\[ \mathbb{E}\{ g_{km}(X_{m})^{6} e^{t\overline{W}(i,j,i',j')} \} = \mathbb{E}\{ g_{km}(X_{m})^{6} e^{tW} \} \leq Ca^{-6} b \mathbb{E} e^{tW}, \] (5.20)
where we used Lemma 5.3 in the last inequality. If $m \in J_{k} \cap J_{ij,j'}$, then $(X_{m}, \overline{W}(i,j,i',j'))$ has the same distribution as $(X_{m}, W)$ by the construction of $\overline{W}(i,j,i',j')$, and thus
\[ \mathbb{E}\{ g_{km}(X_{m})^{6} e^{t\overline{W}(i,j,i',j')} \} = \mathbb{E}\{ g_{km}(X_{m})^{6} e^{tW} \} \leq Ca^{-6} b^{2} \mathbb{E} e^{tW}, \] (5.21)
where we used Lemma 5.4 in the last inequality. By (5.20) and (5.21) and recalling that $b \geq 1$, we have (5.19) can be further bounded by
\[ \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \} \leq Ca^{-6} s^{6} b^{2} \mathbb{E} e^{tW}. \] (5.22)

Using a similar argument to (5.22), we have
\[ \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \} \leq Ca^{-6} s^{6} b^{2} \mathbb{E} e^{tW}, \] (5.23)
\[ \mathbb{E}\{ e^{6} e^{t\overline{W}(i,j,i',j')} \} \leq Ca^{-6} s^{6} b^{2} \mathbb{E} e^{tW}. \] (5.24)

Combining (5.17), (5.18), (5.22) and (5.23), and by (5.7), we have
\[ |H_{22}| \leq Ct^{2} a^{-6} s^{6} b^{2} (n^{2} d^{4} s^{4}) \mathbb{E} e^{tW} \leq Ct^{2} n^{2} a^{-6} d^{4} s^{10} b^{2} \mathbb{E} e^{tW}. \] (5.24)
Similarly,

\[ |H_{23}| \leq C t^2 n^2 a^{-6} d^4 s^{10} b^2 E e^{tW}. \]  \hfill (5.25)

Combining (5.15), (5.24) and (5.25), we have

\[ |H_2| \leq C t n a^{-5} d^4 s^{9/2} E e^{tW} + C t^2 n^2 a^{-6} d^4 s^{10} b^2 E e^{tW} \]  \hfill (5.26)

For \( H_3 \), note that \( \left( \xi_k^{(i,j,i',j')}, \tilde{W}^{(i,j,i',j')} \right) \) is independent of \( (\eta_{i,j}, \eta_{i',j'}) \) and has the same distribution as \( (\xi_k, W) \). By (5.9) and (5.10), we have

\[
|H_3| \leq t \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,i'j'}} \mathbb{E}[|\eta_{ij} \eta_{ij'}| \mathbb{E}\{\xi_k e^{tW}\} 1_{\{i' \in A_{ij} \text{ or } j' \in A_{ij'}\}}]
\leq C t n a^{-4} s^{4} b^{2/3} \sum_{i \in [n]} \sum_{j \in A_i} \sum_{i' \in [n]} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij,i'j'}} \mathbb{E}\{|\xi_k| e^{tW}\} 1_{\{i' \in A_{ij} \text{ or } j' \in A_{ij'}\}}
\leq C t n a^{-4} s^{4} b^{2/3} \min\{n, sd\}^4 \max_{k \in [n]} \mathbb{E}\{|\xi_k| e^{tW}\}. \tag{5.27}
\]

By Hölder’s inequality, (3.1), (3.3) and Lemma 5.4, for all \( k \in [n] \),

\[
\mathbb{E}\{|\xi_k| e^{tW}\} \leq \sum_{m \in J_k} \mathbb{E}\{g_{km}(X_m)e^{tW}\}
\leq \sum_{m \in J_k} \left( \mathbb{E}\{g_{km}(X_m)^6 e^{tW}\} \right)^{1/6} \left(\mathbb{E} e^{tW}\right)^{5/6}
\leq C a^{-1} s^{1/3} E e^{tW}. \tag{5.28}
\]

Substituting (5.28) to (5.27) yields

\[
|H_3| \leq C t n a^{-5} s^5 b \min\{n, sd\}^4 E e^{tW} \leq C n a^{-4} d^4 s^7 b (1 + t^2 n a^{-2} d s^3 b) E e^{tW}.
\]

For \( H_4 \) and \( H_5 \), similar to (5.17)–(5.24), we have

\[
H_4 + H_5 = O(t^2 n^2 a^{-6} d^4 s^{10} b^2 E e^{tW}). \tag{5.29}
\]

Combining (5.6), (5.8), (5.26), (5.27) and (5.29) yields (5.3). This completes the proof. \( \square \)

5.2. Proof of Theorem 3.2

In this subsection, we denote by \( C \) a general constant that depends only on \( \beta \), where \( 0 < \beta < 1 \). Let \( \mathcal{X} \) be the sigma field generated by \((X_1, \ldots, X_n)\). For each \( 1 \leq i \leq n \), let \( \mathcal{X}'_i \) be conditionally independent of \( X_i \) with the conditional distribution of \( X_i \) given \( \{X_j, j \neq i\} \). Let \( I \) be a random index that is
uniformly distributed over \( \{1, \ldots, n\} \) and independent of all others. Define \( S'_n = S_n - X + X'_n \); then \((S_n, S'_n)\) is an exchangeable pair.

For \( n \leq 16 \max\{1, \beta/(1 - \beta)\} \), and for \( 0 \leq z \leq \sqrt{n} \), we have \( z \leq z_\beta := 4\sqrt{\max\{1, \beta/(1 - \beta)\}} \).

By Shao and Zhang [27, Theorem 3.2], for \( 0 \leq z \leq z_\beta \),

\[
|\mathbb{P}(W > z) - (1 - \Phi(z))| \leq Cn^{-1/2} \leq Cn^{-1/2}(1 - \Phi(z_\beta)) \leq Cn^{-1/2}(1 - \Phi(z)).
\]

Hence (3.8) holds. For \( n > 16 \max\{1, \beta(1 - \beta)\} \), we apply Proposition 4.1 to prove the moderate deviation result. To this end, we need to prove the following propositions.

**Proposition 5.5.** Let \( \bar{X} = S_n/n \) and let

\[
R_1 = \mathbb{E}\{S_n - S'_n \mid \mathcal{F}\} - (1 - \beta)\bar{X}.
\]

For \( n > 16 \max\{1, \beta/(1 - \beta)\} \) and \( 0 \leq t \leq \sqrt{n} \), we have

\[
\mathbb{E}e^{tW} \leq Ce^{t^2/2},
\]

\[
\mathbb{E}\{|X|e^{tW}\} \leq Cn^{-1/2}(1 + t)e^{t^2/2}.
\]

\[
\mathbb{E}\{R_1e^{tW}\} \leq Cn^{-1}(1 + t^2)e^{t^2/2},
\]

\[
\mathbb{E}\left\{\mathbb{E}\{S_n - S'_n\}^2 \mid \mathcal{F}\} - 2\mathbb{E}\{W\}\right\} \leq Cn^{-1/2}(1 + t)e^{t^2/2},
\]

\[
\mathbb{E}\left\{\mathbb{E}\{S_n - S'_n\}^3 \mid \mathcal{F}\} \mathbb{E}\{e^{tW}\}\right\} \leq Cn^{-1/2}(1 + t)e^{t^2/2}.
\]

With the help of Proposition 5.5, we can check the conditions (B1)–(B3) immediately. To see this, let \( W' = n^{-1/2}(1 - \beta)^{1/2}S'_n \), \( D = \Delta = W - W' \) and \( D^* = n^{-1/2}(1 - \beta)^{1/2} + n^{1/2}(1 - \beta)^{-1/2}D^2 \). Then, it follows that \( |D| \leq D^* \) and \( D^* \) is symmetric with respect to \( W \) and \( W' \). Observe that

\[
\mathbb{E}\{D \mid \mathcal{F}\} = \mathbb{E}\{W - W' \mid \mathcal{F}\} = n^{-1/2}(1 - \beta)^{1/2} \mathbb{E}\{S_n - S'_n \mid \mathcal{F}\} = \lambda(W + R),
\]

where \( \lambda = (1 - \beta)/n \) and \( R = n^{1/2}(1 - \beta)^{1/2}R_1 \). Moreover,

\[
\frac{1}{2\lambda} \mathbb{E}\{D\Delta \mid \mathcal{F}\} - 1 = \frac{1}{2\lambda} \mathbb{E}\{(W - W')^2 \mid \mathcal{F}\} - 1 = \frac{1}{2} \left( \mathbb{E}\{(S_n - S'_n)^2 \mid \mathcal{F}\} - 2 \right),
\]

and

\[
\frac{1}{\lambda} \mathbb{E}\{D^*\Delta \mid \mathcal{F}\} = \frac{n^{-1/2}(1 - \beta)^{1/2}}{\lambda} \mathbb{E}\{W - W' \mid \mathcal{F}\} + \frac{n^{1/2}(1 - \beta)^{-1/2}}{\lambda} \mathbb{E}\{(W - W')^3 \mid \mathcal{F}\}
\]

\[
= \mathbb{E}\{S_n - S'_n \mid \mathcal{F}\} + \mathbb{E}\{(S_n - S'_n)^3 \mid \mathcal{F}\}.
\]

Hence, by Proposition 5.5, we have that conditions (B1)–(B3) are satisfied with \( \tau_0 = n^{1/2}, \delta_1(t) = \delta_2(t) = Cn^{-1/2}(1 + t) \) and \( \delta_3(t) = Cn^{-1}(1 + t^2) \). This completes the proof of Theorem 3.2 by applying Proposition 4.1.
It suffices to prove Proposition 5.5: to this end, we need to show some preliminary lemmas.

**Lemma 5.6.** For $0 \leq \theta < 1$, we have

$$\mathbb{E} e^{\theta \xi^2/2} \leq C_\theta,$$  \hfill (5.36)

where $C_\theta > 0$ is a constant depending on $\theta$.

**Lemma 5.7.** For $1 \leq m \leq n$, let $T_m = \xi_1 + \cdots + \xi_m$ and assume that $n \geq 16 \max\{1, \beta/(1-\beta)\}$. We have for all $m \in [n]$ and $0 \leq t \leq \sqrt{n}$,

$$\mathbb{E} \exp \left( \left( \frac{\beta}{2n} + \frac{2\beta}{n^2} \right) T_m^2 + \left( \frac{1-\beta}{n} \right)^{1/2} t T_m \right) \leq Ce^{t^2/2},$$  \hfill (5.37)

and

$$\mathbb{E} \left\{ T_m^2 \exp \left( \left( \frac{\beta}{2n} + \frac{2\beta}{n^2} \right) T_m^2 + \left( \frac{1-\beta}{n} \right)^{1/2} t T_m \right) \right\} \leq Cn(1+t^2)e^{t^2/2},$$  \hfill (5.38)

where $C > 0$ is a constant depending only on $\beta$.

Recall that for each $1 \leq i \leq n$, given $\{X_j, j \neq i\}$, $X'_i$ is conditionally independent of $X_i$ with the conditional distribution of $X_i$. Also, recall the normalizing constant $Z_n = \mathbb{E} \exp \{ \beta (\xi_1 + \cdots + \xi_n)^2/(2n) \}$.

**Lemma 5.8.** For $0 < \beta < 1$, we have

$$1 \leq Z_n \leq C,$$  \hfill (5.39)

and for $n > 4\{1, \beta/(1-\beta)\}$ and $0 \leq t \leq \sqrt{n}$,

$$\mathbb{E} |X_i|^6 e^{tW} \leq Ce^{t^2/2},$$  \hfill (5.40)

and

$$\mathbb{E} |X'_i|^6 e^{tW} \leq Ce^{t^2/2}.$$  \hfill (5.41)

The following lemma provides an upper bound of $|\mathbb{E} \{ (X^k_i - \mu_k)(X^k_j - \mu_k)e^{tW} \}|$, whose proof is similar to Lemma 5.7 of Shao and Zhang [27].

**Lemma 5.9.** For $i \neq j \in [n]$ and $k = 1, 2, 3$, we have for $n > 16 \max\{1, \beta/(1-\beta)\}$ and $0 \leq t \leq \sqrt{n}$,

$$|\mathbb{E} \{ (X^k_i - \mu_k)(X^k_j - \mu_k)e^{tW} \}| \leq Cn^{-1}(1+t^2)e^{t^2/2},$$

where $\mu_k = \mathbb{E} \xi^k$.

The details of proofs of Lemmas 5.6–5.9 are put in the Supplementary Material [31].

Now we are ready to prove Proposition 5.5.
Proof of Proposition 5.5. Let $\xi, \xi_1, \ldots, \xi_n$ be i.i.d. random variables with probability measure $\rho$. Let $E_\xi$ denote the expectation with respect to $\xi$ conditional on other random variables. Recall that $\bar{X} = (X_1 + \cdots + X_n)/n$. For each $i \in [n]$, let $\bar{X}_i = \bar{X} - X_i/n$. In what follows, we fix $n > 16 \max\{1, \beta/(1 - \beta)\}$ and $0 \leq t \leq \sqrt{n}$. Again, let $\alpha_n(t) = n^{-1/2}(1 - \beta)^{1/2}t$. We now prove (5.31)–(5.35) one by one.

(i). Proof of (5.31). By (3.6), (5.37) and (5.39), we have
\[
E e^{tW} = \frac{1}{Z_n} E \exp\left(\frac{\beta}{2n}T_n^2 + \alpha_n(t)T_n\right) \leq C e^{t^2/2}.
\]

(ii). Proof of (5.32). Let $T_n = \xi_1 + \cdots + \xi_n$. By (3.6), (5.38) and (5.39), we have
\[
E \{\bar{X}^2 e^{tW}\} = \frac{1}{n^2 Z_n} E \left\{T_n^2 \exp\left(\frac{\beta}{2n}T_n^2 + \alpha_n(t)T_n\right)\right\} \leq C n^{-1} (1 + t^2) e^{t^2/2}.
\]

By Hölder’s inequality, (5.31) and (5.42), we have
\[
E \{|\bar{X}| e^{tW}\} \leq (E e^{tW})^{1/2} (E \{\bar{X}^2 e^{tW}\})^{1/2} \leq C n^{-1/2} (1 + t) e^{t^2/2}.
\]

(iii). Proof of (5.33). By the definition of $(S_n, S'_n)$, we have
\[
E \{S_n - S'_n \mid \mathcal{F}\} = \frac{1}{n} \sum_{i=1}^n E \{X_i - X'_i \mid \mathcal{F}\} = \bar{X} - \frac{1}{n} \sum_{i=1}^n \frac{E_\xi \{\xi e^{\beta\bar{X}_i} + \beta X_i\xi\}}{E_\xi \{e^{\beta\bar{X}_i} + \beta X_i\xi\}}.
\]

Observe that
\[
\frac{E_\xi \{\xi e^{\beta\bar{X}_i} + \beta X_i\xi\}}{E_\xi \{e^{\beta\bar{X}_i} + \beta X_i\xi\}} = h(\bar{X}_i) + r_{1i},
\]

where
\[
h(s) = \frac{E_\xi \{\xi e^{\beta s} + \beta X_i\xi\}}{E_\xi \{e^{\beta s} + \beta X_i\xi\}}, \quad r_{1i} = \frac{E_\xi \{\xi e^{\beta\bar{X}_i} + \beta X_i\xi\}}{E_\xi \{e^{\beta\bar{X}_i} + \beta X_i\xi\}} - \frac{E_\xi \{\xi e^{\beta\bar{X}_i} + \beta X_i\xi\}}{E_\xi \{e^{\beta\bar{X}_i} + \beta X_i\xi\}}.
\]

Note that $n > 8\{1, \beta/(1 - \beta)\}$, and thus $\beta/(2n) \leq 1/16$. Moreover, it is easy to see that $x^3 \leq 10e\beta x^2/8$ for all $x > 0$. Therefore,
\[
|E_\xi \{\xi e^{\beta\bar{X}_i} + \beta X_i\xi\} - E_\xi \{\xi e^{\beta\bar{X}_i}\}| \leq n^{-1} E_\xi \left\{ |\xi| \beta e^{\beta\bar{X}_i} + \beta X_i \xi \right\} \leq n^{-1} e^{\beta\bar{X}_i^2} E_\xi \left\{ |\xi|^3 e^{\beta\bar{X}_i^2 + \beta X_i^2/4} \right\} \leq C n^{-1} e^{\beta\bar{X}_i^2} e^{3\bar{X}_i^2/8} \leq C n^{-1} e^{\beta\bar{X}_i^2},
\]

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where we used Hölder’s inequality in the third line and where we used Lemma 5.6 in the last line. Similarly,

\[ |E_\xi \{ \xi e^{\frac{\beta \xi^2}{2\alpha_n(t)}} \} - E_\xi \{ \xi e^{\beta X_i \xi} \} | \leq C n^{-1} e^{\beta X_i^2}. \]

As \( E_\xi = 0 \), it follows from the Jensen inequality that \( E_\xi e^{\beta s \xi} \geq 1 \) for all \( s \in \mathbb{R} \). Hence,

\[ |r_{11}| \leq C n^{-1} e^{\beta X_i^2}. \tag{5.45} \]

By Taylor’s expansion,

\[ h(X_i) = \beta X_i - \frac{\beta}{n} X_i + \int_0^{X_i} h''(t)(X_i - t) \, dt. \tag{5.46} \]

By Shao and Zhang \[27\], Eq. (5.41),

\[ \left| \int_0^{X_i} h''(t)(X_i - t) \, dt \right| \leq C |X_i|^2 e^{\beta X_i^2}. \tag{5.47} \]

It follows from (5.44)–(5.47) that

\[ r_{2i} := \left| \frac{E_\xi \{ \xi e^{\frac{\beta \xi^2}{2\alpha_n(t)}} \} - \beta \bar{X}}{E_\xi \{ \xi e^{\beta X_i \xi} \}} \right| \leq \frac{\beta}{n} |X_i| + C |\bar{X}|^2 e^{\beta X_i^2}. \tag{5.48} \]

By (5.43) and (5.48),

\[ |R_1| \leq C n \sum_{i=1}^n \left\{ \beta n^{-1} |X_i| \right\} + |\bar{X}|^2 e^{\beta X_i^2}. \tag{5.49} \]

Next we prove the bound of \( E |R_1| e^{tW} \). Note that \( \alpha_n(t) \leq 1 \) for \( 0 \leq t \leq \sqrt{n} \), and by Hölder’s inequality and by Lemma 5.6,

\[ E e^{\beta \xi_1^2/2 + \alpha_n(t) \xi_1} \leq E e^{\alpha_n(t)(t^2/(1-\beta))} E e^{\beta \xi_1^2/2 + (1-\beta) \xi_1^2/4} \leq C. \tag{5.50} \]

Let \( M_1 = S_n - \xi_1 \). By Lemma 5.7 with \( m = n - 1 \) and by (5.50),

\[ \mathbb{E} \{ |\bar{X}_i|^2 e^{\beta X_i^2 + tW} \} \leq \frac{1}{n^2 Z_n} \mathbb{E} \left\{ M_1^2 \exp \left( \left( \frac{\beta}{2n} + \frac{\beta}{n} \right) M_1 + \frac{\beta}{2} \xi_1^2 + \alpha_n(t)(\xi_1 + M_1) \right) \right\} \]

\[ = \frac{1}{n^2 Z_n} \mathbb{E} e^{\beta \xi_1^2/2 + \alpha_n(t) \xi_1} \mathbb{E} \left\{ M_1^2 \exp \left( \left( \frac{\beta}{2n} + \frac{\beta}{n} \right) M_1^2 + \alpha_n(t) M_1 \right) \right\} \]

\[ \leq C n^{-1} (1 + t^2) e^{t^2/2}. \tag{5.51} \]

By (5.40),

\[ \mathbb{E} \{ |X_i|^2 e^{tW} \} \leq C e^{t^2/2}. \tag{5.52} \]
Combining (5.51) and (5.52), we complete the proof of (5.33).

(iv) Proof of (5.34). Observe that

$$E \left\{ (S_n - S'_n)^2 \mid X \right\} = 2 + R_2 + R_3 + R_4,$$  

(5.53)

where

$$R_2 = \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - 1), \quad R_3 = -\frac{1}{n} \sum_{i=1}^{n} 2X_i \frac{E_\xi \{ \xi e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}{E_\xi \{ e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}, \quad R_4 = \frac{1}{n} \sum_{i=1}^{n} \frac{E_\xi \{ \xi^2 e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}{E_\xi \{ e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}} - 1.$$

For $R_2$, applying Lemma 5.9 with $k = 2$, and by (5.31) and the Cauchy inequality that

$$E \{ |R_2| e^{iW} \} \leq (E e^{iW})^{1/2} (E \{ |R_2|^2 e^{iW} \})^{1/2} \leq Cn^{-1/2} (1 + t)e^{t^2/2}.$$  

(5.54)

For $R_3$, note that by (5.48),

$$|R_3| \leq 2\beta \bar{X}^2 + \frac{2}{n} \sum_{i=1}^{n} |X_i| r_2 \leq 2\beta \bar{X}^2 + \frac{2}{n} \sum_{i=1}^{n} X_i^2 + Cn^{-1} \sum_{i=1}^{n} |X_i| X_i^2 e^{\beta \bar{X}^2}.$$

Similar to (5.51), we have

$$E \{ |X_i \bar{X}_i^2| e^{\beta \bar{X}^2 + iW} \} \leq \frac{1}{n^2 Z_n} E \{ |\xi| e^{\beta \xi^2/2 + \alpha(t) \xi_1} \} E \{ M_1^2 \exp \left( \frac{\beta}{2n} + \frac{\beta}{n^2} M_1^2 + \alpha(t) M_1 \right) \} \leq Cn^{-1} (1 + t^2) e^{t^2/2}.$$  

(5.55)

By (5.40), (5.42) and (5.55), we obtain

$$E |R_3| \leq Cn^{-1} (1 + t^2) e^{t^2/2}.$$  

(5.56)

For $R_4$, note that

$$\frac{E_\xi \{ \xi e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}{E_\xi \{ e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}} - 1 = \frac{E_\xi \{ (\xi^2 - 1) e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}{E_\xi \{ e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}} = \frac{E_\xi \{ (\xi^2 - 1) e^{\beta X_i \xi} \}}{E_\xi \{ e^{\beta X_i \xi} \}} + r_{3i},$$

where

$$r_{3i} = \frac{E_\xi \{ (\xi^2 - 1) e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}}{E_\xi \{ e^{\frac{\beta \xi^2}{2n} + \beta X_i \xi} \}} - \frac{E_\xi \{ (\xi^2 - 1) e^{\beta X_i \xi} \}}{E_\xi \{ e^{\beta X_i \xi} \}}.$$

Similar to (5.45), we have $|r_{3i}| \leq Cn^{-1} e^{\beta X_i^2}$. Applying (3.7) with $t = \pm 1$ implies $E e^{\xi} \leq E e^{\xi} + E e^{-\xi} \leq 2e^{1/2}$, then $E |\xi|^3 \leq 1.4 E e^{\xi}$, $|\xi|^3 \leq 1.4 E e^{\xi} \leq 15$. Since $E \xi = 0$, it follows that $E_\xi \{ e^{\beta X_i \xi} \} \geq 1$ and

$$|E_\xi \{ (\xi^2 - 1) e^{\beta X_i \xi} \}| \leq |E \{ \xi^2 - 1 \}| + |\beta X_i e \{ \xi (\xi^2 - 1) \}| + C X_i^2 E_\xi \{ (\xi^2 - 1) \xi^2 |e^{\beta |X_i \xi|} \}$$

$$\leq C |X_i| + C X_i^2 e^{\beta X_i^2}.$$
Therefore,

\[ |R_4| \leq Cn^{-1} \sum_{i=1}^{n} \left( |\bar{X}_i| + (n^{-1} + \bar{X}_i^2)e^\beta \bar{X}_i^2 \right). \]

By (5.32) and (5.40), we have

\[ E\{ |\bar{X}_i|e^{tW} \} \leq E\{ |\bar{X}|e^{tW} \} + \frac{1}{n} E\{ |X_i|e^{tW} \} \leq Cn^{-1/2}(1 + t)e^{t^2/2}. \]

Similar to (5.51),

\[ E\{ (n^{-1} + \bar{X}_i^2)e^\beta \bar{X}_i^2 + tW \} \leq Cn^{-1}(1 + t^2)e^{t^2/2}. \]

Therefore,

\[ E\{ |R_4|e^{tW} \} \leq Cn^{-1/2}(1 + t)e^{t^2/2}. \]

This completes the proof of (5.34) by combining (5.53)–(5.57).

(v). Proof of (5.35). Observe that

\[ E\{ (S_n - S'_n)^3 \mid \mathcal{X} \} = R_5 + R_6 + R_7 + R_8, \]

where

\[ R_5 = \frac{1}{n} \sum_{i=1}^{n} (X_i^3 - E\xi^3), \quad R_6 = -\frac{1}{n} \sum_{i=1}^{n} \frac{3X_i^2 E\xi \left( \frac{\beta X_i \xi}{2n} + \beta X_i \xi \right)}{E\xi \left( e^{\frac{\beta X_i \xi}{2n} + \beta X_i \xi} \right)}, \]

\[ R_7 = \frac{1}{n} \sum_{i=1}^{n} \frac{3X_i E\xi \left( \frac{\beta X_i \xi}{2n} + \beta X_i \xi \right)}{E\xi \left( e^{\frac{\beta X_i \xi}{2n} + \beta X_i \xi} \right)}, \quad R_8 = -\frac{1}{n} \sum_{i=1}^{n} \left( \frac{E\xi \left( \frac{\beta X_i \xi}{2n} + \beta X_i \xi \right)}{E\xi \left( e^{\frac{\beta X_i \xi}{2n} + \beta X_i \xi} \right)} - E\xi^3 \right). \]

Similar to (iv), the inequality (5.35) holds.

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References

[1] BARBOUR, A., CHEN, L. H. and CHOI, K. (1995). Poisson approximation for unbounded functions, I: Independent summands. *Stat. Sin.* 5 749–766.

[2] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson approximation*. 2. The Clarendon Press, Oxford University Press, New York. Oxford Science Publications.

[3] CHATTERJEE, S. (2007) Stein’s Method for Concentration Inequalities. *Probab. Theory Relat. Fields* 138 305–321.

[4] CHATTERJEE, S. (2014). A short survey of Stein’s method. In *Proceedings of the International Congress of Mathematicians—Seoul 2014* (S. Y. Jang, Y. R. Kim, D.-W. Lee and I. Yie, eds.) 4 1–24.

[5] CHATTERJEE, S., DIACONIS, P. and MECKES, E. (2005). Exchangeable pairs and Poisson approximation. *Probab. Surveys* 2 64–106.

[6] CHATTERJEE, S. and MECKES, E. (2008). Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.* 4 257–283.

[7] CHATTERJEE, S. and SHAO, Q.-M. (2011). Nonnormal approximation by Stein’s method of exchangeable pairs with application to the Curie–Weiss model. *Ann. Appl. Probab.* 21 464–483.

[8] CHEN, L. H. Y. and CHOI, K. P. (1992). Some asymptotic and large deviation results in poisson approximation. *Ann. Probab.* 20 1867–1876.

[9] CHEN, L. H. Y., FANG, X. and SHAO, Q.-M. (2013). Moderate deviations in Poisson approximation: a first attempt. *Stat. Sin.* 23 1523–1540.

[10] CHEN, L. H. Y., FANG, X. and SHAO, Q.-M. (2013). From Stein identities to moderate deviations. *Ann. Probab.* 41 262–293.

[11] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein’s Method*. Probability and its Applications. Springer Berlin Heidelberg, New York.

[12] EICHELSBACHER, P. and LOEWE, M. (2010). Stein’s Method for Dependent Random Variables Occurring in Statistical Mechanics. *Electron. J. Probab.* 15 962–988.

[13] EICHELSBACHER, P. and MARTSCHINK, B. (2015). On rates of convergence in the Curie–Weiss–Potts model with an external field. *Ann. Henri Poincaré B.* 51 252–282.

[14] EICHELSBACHER, P. and MARTSCHINK, B. (2016). Rates of convergence in the Blume–Emery–Griffiths model. *J. Stat. Phys.* 154 1483–1507.

[15] ELLIS, R. S. and NEWMAN, C. M. (1978). Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete.* 44 117–139.

[16] ELLIS, R. S. and NEWMAN, C. M. (1978). The statistics of curie-weiss models. *Journal of Statistical Physics.* 19 149–161.
17. Fang, X., Luo, L. and Shao, Q.-M. (2020). A refined Cramér-type moderate deviation for sums of local statistics. *Bernoulli*. 26 2319–2352.

18. Holmes, S. and Reinert, G. (2004). Stein’s method for the bootstrap. In *Stein’s Method: Expository Lectures and Applications* 46 93–132. Institute of Mathematical Statistics, Hayward, CA.

19. Kirkpatrick, K. and Meckes, E. (2013). Asymptotics of the mean-field Heisenberg model. *J. Stat. Phys.* 152 54–92.

20. Kirkpatrick, K. and Nawaz, T. (2016) Asymptotics of mean-field $O(N)$ model. *J. Stat. Phys.* 165 1114–1140.

21. Meckes, E. (2009). On Stein’s Method for Multivariate Normal Approximation. In *High Dimensional Probability V: The Luminy Volume* 5 153–178. IMS, Beachwood, OH.

22. Petrov, V. V. (1975). *Sums of Independent Random Variables*. Springer Berlin Heidelberg.

23. Račič, M. (2007). CLT-related large deviation bounds based on stein’s method. *Adv. Appl. Probab.* 39 731–752.

24. Rinott, Y. and Rotar, V. (1997). On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted $U$-statistics. *Ann. Appl. Probab.* 7 1080–1105.

25. Shao, Q.-M., Zhang, M. and Zhang, Z.-S. (2020). Cramér-type moderate deviation theorems for nonnormal approximation. To appear in *Ann. Appl. Probab.*

26. Shao, Q.-M. and Zhang, Z.-S. (2016). Identifying the limiting distribution by a general approach of Stein’s method. *Sci. China Math.* 59 2379–2392.

27. Shao, Q.-M. and Zhang, Z.-S. (2019). Berry–Esseen bounds of normal and nonnormal approximation for unbounded exchangeable pairs. *Ann. Probab.* 47 61–108.

28. Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* 2 583–602. University of California Press, Berkeley.

29. Stein, C. (1986). *Approximate Computation of Expectations*. 7. IMS, Hayward, CA.

30. Zhang, Z.-S. (2021a). Berry–Esseen bounds for generalized $U$-statistics. Available at arXiv 2104.03419.

31. Zhang, Z.-S. (2021b). Supplement to “Cramér-type moderate deviation of normal approximation for unbounded exchangeable pairs”.