Optimal Variational Principle for Backward Stochastic Control Systems Associated with Lévy Processes *

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Abstract

The paper is concerned with optimal control of backward stochastic differential equation (BSDE) driven by Teugel’s martingales and an independent multidimensional Brownian motion, where Teugel’s martingales are a family of pairwise strongly orthonormal martingales associated with Lévy processes (see Nualart and Schoutens [14]). We derive the necessary and sufficient conditions for the existence of the optimal control by means of convex variation methods and duality techniques. As an application, the optimal control problem of linear backward stochastic differential equation with a quadratic cost criteria (called backward linear-quadratic problem, or BLQ problem for short) is discussed and characterized by stochastic Hamilton system.

Keywords: stochastic control, stochastic maximum principle, Lévy processes, Teugel’s martingales, backward stochastic differential equations

1 Introduction

It is well known that the maximum principle for a stochastic optimal control problem involves the so-called adjoint processes which solve the corresponding adjoint equation. In fact, the adjoint equation is in general a linear backward stochastic differential equation (BSDE) with a specified a random terminal condition on the state. Unlike a forward stochastic differential equation, the solution of a BSDE is a pair of adapted solutions. Thus, in order to obtain the maximum principle, we need first obtain the existence and uniqueness theorem for the pair of adapted solutions of adjoint equation.

*This work is partially supported by the National Basic Research Program of China (973 Program) (Grant No.2007CB814904), the National Natural Science Foundation of China (Grants No.10325101, 11071069), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No.20090071120002) and the Innovation Team Foundation of the Department of Education of Zhejiang Province (Grant No.T200924).
The linear BSDE was first proposed by Bismut [4] in 1973. This research field developed fast after the pioneer work of Pardoux and Peng [16] in 1990 got the existence and uniqueness theorem for the solution of nonlinear BSDE driven by Brownian motion under Lipschitz condition. Now BSDE theory has been playing a key role not only in dealing with stochastic optimal control problems, but in mathematical finance, particularly in hedging and nonlinear pricing theory for imperfect market (see e.g. [7]).

As for BSDE driven by the non-continuous martingale, Tang and Li [20] first discussed the existence and uniqueness theorem of the solution of BSDE driven by Poisson point process and consequently proved the maximum principle for optimal control of stochastic systems with random jumps. In 2000, Nualart and Schoutens [14] got a martingale representation theorem for a type of Lévy processes through Teugel’s martingales, where Teugel’s martingales are a family of pairwise strongly orthonormal martingales associated with Lévy processes. Later, they proved in [15] the existence and uniqueness theory of BSDE driven by Teugel’s martingales. The above results are further extended to the one-dimensional BSDE driven by Teugel’s martingales and an independent multi-dimensional Brownian motion by Bahlali et al [1]. One can refer to [8, 9, 17, 18] for more results on such kind of BSDEs.

In the mean time, the stochastic optimal control problems related to Teugel’s martingales were studied. In 2008, a stochastic linear-quadratic problem with Lévy processes was considered by Mitsui and Tabata [13], in which they established the closeness property of multi-dimensional backward stochastic Riccati differential equation (BSRDE) with Teugel’s martingales and proved the existence and uniqueness of solution to such kind of one-dimensional BSRDE, moreover, in their paper an application of BSDE to a financial problem with full and partial observations was demonstrated. Motivated by [13], Meng and Tang [12] studied the general stochastic optimal control problem for the forward stochastic systems driven by Teugel’s martingales and an independent multi-dimensional Brownian motion, of which the necessary and sufficient optimality conditions in the form of stochastic maximum principle with the convex control domain are obtained.

However, [12] and [13] are only concerned with the optimal control problem of the forward controlled stochastic system. Since a BSDE is a well-defined dynamic system itself and has important applications in mathematical finance, it is necessary and natural to consider the optimal control problem of BSDE. Actually, there has been much literature on BSDE control system driven by Brownian motion (see e.g. [2, 3, 5, 11, 10]). But to our best knowledge, there is no discussion on the optimal control problem of BSDE driven by Teugel martingales and an independent Brownian motion, which motizations us to write this paper.

In this paper, by means of convex variation methods and duality techniques, we will give the necessary and sufficient conditions for the existence of the optimal control for BSDE system driven by Teugel martingales and an independent multi-dimensional Brownian motion. As an application, the optimal control for linear backward stochastic differential equation with a quadratic cost criteria or called backward linear-quadratic (BLQ) problem is discussed in details. The optimal control of BLQ problem will be characterized by stochastic Hamilton systems. In this case, the stochastic Hamilton system is a linear forward-backward stochastic differential equation driven by Teugel’s martingales and an independent multi-dimensional Brownian motion, consisting of the state equation, the adjoint equation and the dual presentation of the optimal control.
The rest of this paper is organized as follows. In section 2, we introduce useful notation and some existing results on stochastic differential equations (SDEs) and BSDEs driven by Teugel’s martingales. In section 3, we state the optimal control problem we study, give needed assumptions and prove some preliminary results on variational equation and variational inequality. In section 4, we prove the necessary and sufficient optimality conditions for the optimal control problem put forward in section 3. As an application, the optimal control for BLQ problem is discussed in section 5.

2 Notation and preliminaries

Let $\langle \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P \rangle$ be a complete probability space. The filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is right-continuous and generated by a $d$-dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ and a one-dimensional Lévy process $\{L(t), 0 \leq t \leq T\}$. It is known that $L(t)$ has a characteristic function of the form

$$E e^{i\theta L(t)} = \exp \left[ ia\theta t - \frac{1}{2} \sigma^2 \theta^2 t + t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x I_{\{|x|<1\}}) v(dx) \right],$$

where $a \in \mathbb{R}$, $\sigma > 0$ and $v$ is a measure on $\mathbb{R}$ satisfying (i) $\int_0^T (1 \wedge x^2) v(dx) < \infty$ and (ii) there exists $\varepsilon > 0$ and $\lambda > 0$, s.t. $\int_{\{-\varepsilon,\varepsilon\}} e^{\lambda|x|} v(dx) < \infty$. These settings imply that the random variables $L(t)$ have moments of all orders. Denote by $P$ the predictable sub-$\sigma$ field of $\mathcal{B}([0, T]) \times \mathcal{F}$, then we introduce the following notation used throughout this paper.

- $\mathcal{H}$: a Hilbert space with norm $\| \cdot \|_{\mathcal{H}}$.
- $\langle \alpha, \beta \rangle$: the inner product in $\mathbb{R}^n$, $\forall \alpha, \beta \in \mathbb{R}^n$.
- $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm of $\mathbb{R}^n$.
- $\langle A, B \rangle = \text{tr}(AB^T)$: the inner product in $\mathbb{R}^{n \times m}$, $\forall A, B \in \mathbb{R}^{n \times m}$.
- $|A| = \sqrt{\text{tr}(AA^T)}$: the norm of $\mathbb{R}^{n \times m}$.
- $l^2$: the space of all real-valued sequences $x = (x_n)_{n \geq 0}$ satisfying

$$\|x\|_{l^2} \triangleq \sqrt{\sum_{i=1}^{\infty} x_i^2} < +\infty.$$

- $l^2(H)$: the space of all $H$-valued sequence $f = \{f^i\}_{i \geq 1}$ satisfying

$$\|f\|_{l^2(H)} \triangleq \sqrt{\sum_{i=1}^{\infty} \|f^i\|_{H}^2} < +\infty.$$

- $l^2_{\mathcal{F}}(0, T, H)$: the space of all $l^2(H)$-valued and $\mathcal{F}_t$-predictable processes $f = \{f^i(t, \omega), (t, \omega) \in [0, T] \times \Omega\}_{i \geq 1}$ satisfying

$$\|f\|_{l^2_{\mathcal{F}}(0, T, H)} \triangleq \sqrt{E \int_0^T \sum_{i=1}^{\infty} \|f^i(t)\|_{H}^2 dt} < \infty.$$
\[ M^2_{\mathcal{F}}(0, T; H) : \text{the space of all } H\text{-valued and } \mathcal{F}_t\text{-adapted processes } f = \{ f(t, \omega), (t, \omega) \in [0, T] \times \Omega \} \text{ satisfying} \]
\[
\|f\|_{M^2_{\mathcal{F}}(0, T; H)} \triangleq \sqrt{E \int_0^T \|f(t)\|_H^2 dt} < \infty.
\]

\[ S^2_{\mathcal{F}}(0, T; H) : \text{the space of all } H\text{-valued and } \mathcal{F}_t\text{-adapted càdlàg processes } f = \{ f(t, \omega), (t, \omega) \in [0, T] \times \Omega \} \text{ satisfying} \]
\[
\|f\|_{S^2_{\mathcal{F}}(0, T; H)} \triangleq \sqrt{E \sup_{0 \leq t \leq T} \|f(t)\|_H^2 dt} < +\infty.
\]

\[ L^2(\Omega, \mathcal{F}, P; H) : \text{the space of all } H\text{-valued random variables } \xi \text{ on } (\Omega, \mathcal{F}, P) \text{ satisfying} \]
\[
\|\xi\|_{L^2(\Omega, \mathcal{F}, P; H)} \triangleq E\|\xi\|_H^2 < \infty.
\]

We denote by \( \{H^i(t), 0 \leq t \leq T\}_{i=1}^{\infty} \) the Teugel's martingales associated with the Lévy process \( \{L(t), 0 \leq t \leq T\} \). \( H^i(t) \) is given by
\[
H^i(t) = c_{i,1}Y^{(1)}(t) + c_{i,2}Y^{(2)}(t) + \cdots + c_{i,1}Y^{(1)}(t),
\]
where \( Y^{(i)}(t) = L^{(i)}(t) - E[L^{(i)}(t)] \) for all \( i \geq 1 \), \( L^{(i)}(t) \) are so called power-jump processes with \( L^{(1)}(t) = L(t) \), \( L^{(i)}(t) = \sum_{0<s \leq t} (\Delta L(s))^i \) for \( i \geq 2 \) and the coefficients \( c_{ij} \) correspond to the orthonormalization of polynomials \( 1, x, x^2, \cdots \) w.r.t. the measure \( \mu(dx) = x^2v(dx) + \sigma^2\delta_0(dx) \). The Teugel’s martingales \( \{H^i(t)\}_{i=1}^{\infty} \) are pathwise strongly orthogonal and their predictable quadratic variation processes are given by
\[
\langle H^i(t), H^j(t) \rangle = \delta_{ij}t
\]

For more details of Teugel’s martingales, we invite the reader to consult Nualart and Schoutens [14, 15].

In what follows, we will state some basic results on SDE and BSDE driven by Teugel’s martingales \( \{H^i(t), 0 \leq t \leq T\}_{i=1}^{\infty} \) and the \( d \)-dimensional Brownian motion \( \{W(t), 0 \leq t \leq T\} \).

Consider SDE:
\[
X(t) = a + \int_0^t b(s, X(s))ds + \sum_{i=1}^d \int_0^t g^i(s, X(s))dW^i(s) + \sum_{i=1}^\infty \int_0^t \sigma^i(s, X(s-))dH^i(s), \quad t \in [0, T], \tag{2.1}
\]
where \( (a, b, g, \sigma) \) are given mappings satisfying the assumptions below.

**Assumption 2.1.** Random variable \( a \) is \( \mathcal{F}_0\)-measurable and \((b, g, \sigma)\) are three random mappings
\[
b : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n,
\]

For more details of Teugel’s martingales, we invite the reader to consult Nualart and Schoutens [14, 15].
\[ g \equiv (g^1, g^2, \cdots, g^d) : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}, \]
\[ \sigma \equiv (\sigma^i)_{i=1}^{\infty} : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \ell^2(\mathbb{R}^n) \]
satisfying
(i) \( b, g \) and \( \sigma \) are \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \) measurable with \( b(\cdot, 0) \in M^2(0, T; \mathbb{R}^n) \), \( g(\cdot, 0) \in M^2(0, T; \mathbb{R}^{n \times d}) \) and \( \sigma(\cdot, 0) \in \ell^2(0, T; \mathbb{R}^n) \).
(ii) \( b, g \) and \( \sigma \) are uniformly Lipschitz continuous w.r.t. \( x \), i.e. there exists a constant \( C > 0 \) s.t. for all \( (t, x, \bar{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \) and a.s. \( \omega \in \Omega \),
\[ |b(t, x) - b(t, \bar{x})| + |g(t, x) - g(t, \bar{x})| + |||\sigma(t, x) - \sigma(t, \bar{x})|||_{\ell^2(\mathbb{R}^n)} \leq C|x - \bar{x}|. \]

Lemma 2.1 ([19], Existence and Uniqueness Theorem of SDE). If coefficients \( (a, b, g, \sigma) \) satisfy Assumption 2.1, then SDE (2.1) has a unique solution \( x(\cdot) \in S^2(0, T; \mathbb{R}^n) \).

Lemma 2.2 ([12], Continuous Dependence Theorem of SDE). Assume coefficients \( (a, b, g, \sigma) \) and \( (\bar{a}, \bar{b}, \bar{g}, \bar{\sigma}) \) satisfy Assumption 2.1. If \( x(\cdot) \) and \( \bar{x}(\cdot) \) are the solutions to SDE (2.1) corresponding to \( (a, b, g, \sigma) \) and \( (\bar{a}, \bar{b}, \bar{g}, \bar{\sigma}) \), respectively, then we have
\[
E \sup_{0 \leq t \leq T} |x(t) - \bar{x}(t)|^2 \leq K \left[ |a - \bar{a}|^2 + E \int_0^T |b(t, \bar{x}(t)) - \bar{b}(t, \bar{x}(t))|^2 dt + E \int_0^T |g(t, \bar{x}(t)) - \bar{g}(t, \bar{x}(t))|^2 dt + E \int_0^T |||\sigma(t, \bar{x}(t)) - \bar{\sigma}(t, \bar{x}(t))|||_{\ell^2(\mathbb{R}^n)} dt \right],
\]
where \( K \) is a positive constant depending only on \( T \) and the Lipschitz constant \( C \).

In particular, for \( (\bar{a}, \bar{b}, \bar{g}, \bar{\sigma}) = (0, 0, 0, 0) \), we have
\[
E \sup_{0 \leq t \leq T} |x(t)|^2 \leq K \left[ |a|^2 + E \int_0^T |b(t, 0)|^2 dt + E \int_0^T |g(t, 0)|^2 dt + E \int_0^T |||\sigma(t, 0)|||_{\ell^2(\mathbb{R}^n)} dt \right] < +\infty.
\]

Now we consider BSDE:
\[
y(t) = \xi + \int_t^T f(s, y(s), q(s), z(s)) ds - \sum_{i=1}^d \int_t^T q^i(s) dw^i(s) - \sum_{i=1}^\infty \int_t^\infty z^i(s) dH^i(s), \quad t \in [0, T], \tag{2.2}
\]
where coefficients \( (\xi, f) \) are given mappings satisfying the assumptions below.

**Assumption 2.2.** The terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) \) and \( f \) is a random mapping
\[
f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \ell^2(\mathbb{R}^n) \rightarrow \mathbb{R}^n
\]
satisfying
(i) \( f \) is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d}) \otimes \ell^2(\mathbb{R}^n) \) measurable with \( f(\cdot, 0, 0, 0) \in M^2(0, T; \mathbb{R}^n) \).
(ii) \( f \) is uniformly Lipschitz continuous w.r.t. \( (y, q, z) \), i.e. there exists a constant \( C > 0 \)
s.t. for all $(t, y, q, z, \bar{y}, \bar{q}, \bar{z}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n)$ and a.s. $\omega \in \Omega$,

$$|f(t, y, q, z) - f(t, \bar{y}, \bar{q}, \bar{z})| \leq C\left[|y - \bar{y}| + |q - \bar{q}| + \|z - \bar{z}\|_{l^2(\mathbb{R}^n)}\right].$$

**Lemma 2.3** ([1], Existence and Uniqueness of BSDE). If coefficients $(\xi, f)$ satisfy Assumption [2.3], then BSDE (2.2) has a unique solution

$$(y(\cdot), q(\cdot), z(\cdot)) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^2_{\mathbb{F}}(0, T; \mathbb{R}^{n \times d}) \times l^2_{\mathbb{F}}(0, T; \mathbb{R}^n).$$

**Lemma 2.4** ([1], Continuous Dependence Theorem of BSDE). Assume that coefficients $(\xi, f)$ and $(\bar{\xi}, \bar{f})$ satisfy Assumption [2.2]. If $(y(\cdot), q(\cdot), z(\cdot))$ and $(\bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))$ are the solutions to BSDE (2.2) corresponding to $(\xi, f)$ and $(\bar{\xi}, \bar{f})$, respectively, then we have

$$E \sup_{0 \leq t \leq T} |y(t) - \bar{y}(t)|^2 + E \int_0^T |q(t) - \bar{q}(t)|^2 dt + E \int_0^T \|z(t) - \bar{z}(t)\|_{l^2(\mathbb{R}^n)}^2 dt \leq K \left[E|\xi - \bar{\xi}|^2 + E \int_0^T |f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t)) - \bar{f}(t, \bar{y}(t), \bar{q}(t), \bar{z}(t))|^2 dt\right],$$

where $K$ is a positive constant depending only on $T$ and the Lipschitz constant $C$.

In particular, if $(\xi, \bar{\xi}) = (0, 0)$, we have

$$E \sup_{0 \leq t \leq T} |y(t)|^2 + E \int_0^T |q(t)|^2 dt + E \int_0^T \|z(t)\|_{l^2(\mathbb{R}^n)}^2 dt \leq K \left[E|\xi|^2 + E \int_0^T |f(t, 0, 0, 0)|^2 dt\right]. \quad (2.3)$$

In view of Assumptions [2.1][2.2], Lemmas [2.1][2.4] follow from an application of Itô’s formula, Gronwall’s inequality and Burkholder-Davis-Gundy inequality. One can refer to [1], [12] and [19] for details.

### 3 Formulation of the problem and preliminary lemmas

Let the admissible control set $U$ be a nonempty convex subset of $\mathbb{R}^m$. An admissible control process $u(\cdot)$ is defined as a $\mathcal{F}_t$-predictable process with values in $U$ s.t. $E \int_0^T |u(t)|^2 dt < +\infty$. We denote by $\mathcal{A}$ the set including all admissible control processes.

For any given admissible control $u(\cdot) \in \mathcal{A}$, we consider the following controlled nonlinear BSDE driven by multi-dimensional Brownian motion $W$ and Teugel’s martingales $\{H^i\}_{i=1}^\infty$:

$$y(t) = \xi + \int_t^T f(s, y(s), q(s), z(s), u(s))ds - \sum_{i=1}^d \int_t^T q^i(s)dW^i(s) - \sum_{i=1}^\infty \int_t^T z^i(s)dH^i(s), \quad t \in [0, T]$$ \quad (3.1)
with the cost functional

$$J(u(\cdot)) = E\left[\int_0^T l(t, y(t), q(t), z(t), u(t))dt + \Phi(y(0))\right], \quad (3.2)$$

where

$$\xi : \Omega \longrightarrow \mathbb{R}^n, \quad f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times U \longrightarrow \mathbb{R}^n,$$

$$l : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times U \longrightarrow \mathbb{R}^1$$

and

$$\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^1$$

are given coefficients.

Throughout this paper, we introduce the following basic assumptions on coefficients $(\xi, f, l, \phi)$.

**Assumption 3.1.** The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ and the random mapping $f$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d}) \otimes \mathcal{B}(l^2(\mathbb{R}^n)) \otimes \mathcal{B}(U)$ measurable with $f(\cdot, 0, 0, 0, 0) \in M^2(0, T; \mathbb{R}^n)$. For almost all $(t, \omega) \in [0, T] \times \Omega$, $f(t, \omega, y, p, z, u)$ is Fréchet differentiable w.r.t. $(y, p, z, u)$ and the corresponding Fréchet derivatives $f_y, f_p, f_z, f_u$ are continuous and uniformly bounded.

**Assumption 3.2.** The random mapping $l$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n \times d}) \otimes \mathcal{B}(l^2(\mathbb{R}^n)) \otimes \mathcal{B}(U)$ measurable and for almost all $(t, \omega) \in [0, T] \times \Omega$, $l$ is Fréchet differentiable w.r.t. $(y, p, z, u)$ with continuous Fréchet derivatives $l_y, l_p, l_z, l_u$. The random mapping $\phi$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n)$ measurable and for almost all $(t, \omega) \in [0, T] \times \Omega$, $\phi$ is Fréchet differentiable w.r.t. $y$ with continuous Fréchet derivatives $\phi_y$. Moreover, for almost all $(t, \omega) \in [0, T] \times \Omega$, there exists a constant $C$ s.t. for all $(p, q, z, u) \in \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times U$, \n
$$|l| \leq C(1 + |y|^2 + |q|^2 + |z|^2 + |u|^2), \quad |\phi| \leq C(1 + |y|^2),$$

$$|l_y| + |l_p| + |l_z| + |l_u| \leq C(1 + |y| + |q| + |z| + |u|) \text{ and } |\phi_y| \leq C(1 + |y|).$$

Under Assumption 3.1, we can get from Lemma 2.3 that for each $u(\cdot) \in \mathcal{A}$, the system (3.1) admits a unique strong solution. We denote the strong solution of (3.1) by $(y^n(\cdot), q^n(\cdot), z^n(\cdot))$, or $(y(\cdot), q(\cdot), z(\cdot))$ if its dependence on admissible control $u(\cdot)$ is clear from context. Then we call $(y(\cdot), q(\cdot), z(\cdot))$ the state processes corresponding to the control process $u(\cdot)$ and call $(u(\cdot); y(\cdot), q(\cdot), z(\cdot))$ the admissible pair. Furthermore, by Assumption 3.2 and a priori estimate (2.3), it is easy to check that

$$|J(u(\cdot))| < \infty.$$

Then we put forward the optimal control problem we study.

**Problem 3.1.** Find an admissible control $\bar{u}(\cdot)$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot)).$$
Any \( \bar{u}(\cdot) \in \mathcal{A} \) satisfying above is called an optimal control process of Problem 3.1 and the corresponding state processes \((\bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) are called the optimal state processes. Correspondingly \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) is called an optimal pair of Problem 3.1.

Before we deduce the necessary and sufficient conditions for the optimal control of Problem 3.1, we need do some preparations. Since the control domain \( U \) is convex, the classical method to get necessary conditions for optimal control processes is the so-called convex perturbation method. More precisely, assuming that \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) is an optimal pair of Problem 3.1 for any given admissible control \( u(\cdot) \), we define an admissible control in the form of convex variation

\[
u^{\varepsilon}(\cdot) = \bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot)),\]

where \( \varepsilon > 0 \) can be chosen sufficiently small. Denoting by \((y^{\varepsilon}(\cdot), q^{\varepsilon}(\cdot), z^{\varepsilon}(\cdot))\) the state processes of the control system (3.1) corresponding to the control process \( u^{\varepsilon}(\cdot) \), we obtain the variational inequality

\[
J(u^{\varepsilon}(\cdot)) - J(\bar{u}(\cdot)) \geq 0.\]

In what follows, we do some estimates on the optimal pair and the convex variable pair.

Lemma 3.2. Under Assumptions 3.1-3.2, we have

\[
E \sup_{0 \leq t \leq T} |y^{\varepsilon}(t) - \bar{y}(t)|^2 + E \int_0^T |q^{\varepsilon}(t) - \bar{q}(t)|^2 dt + E \int_0^T ||z^{\varepsilon}(t) - \bar{z}(t)||^2_{L^2(\mathbb{R}^n)} dt = O(\varepsilon^2).
\]

Proof. By continuous dependence theorem of BSDE (Lemma 2.4) and the uniformly bounded property of Fréchet derivative \( f_u \), we have

\[
E \sup_{0 \leq t \leq T} |y^{\varepsilon}(t) - \bar{y}(t)|^2 + E \int_0^T |q^{\varepsilon}(t) - \bar{q}(t)|^2 dt + E \int_0^T ||z^{\varepsilon}(t) - \bar{z}(t)||^2_{L^2(\mathbb{R}^n)} dt \\
\leq KE \int_0^T |f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), u^{\varepsilon}(t)) - f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t))|^2 dt \\
\leq KE \int_0^T |u^{\varepsilon}(t) - \bar{u}(t)|^2 dt \\
= KE \int_0^T |(\bar{u}(t) + \varepsilon(u(t) - \bar{u}(t))) - \bar{u}(t)|^2 dt \\
= KE \int_0^T |u(t) - \bar{u}(t)|^2 dt = O(\varepsilon^2).
\]

Here and in the rest of this paper, \( K \) is a generic positive constant and might change from line to line. \qed

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Lemma 3.3. Under Assumption 3.1, by Lemma 2.3 we know that BSDE (3.3) has a unique solution

\[
\begin{align*}
    dY_t &= - \left[ f_y(t, \hat{y}(t), \hat{q}(t), \hat{z}(t), \bar{u}_t)Y(t) + f_q(t, \hat{y}(t), \hat{q}(t), \hat{z}(t), \bar{u}_t)Q_t \\
    &\quad + f_z(t, \hat{y}(t), \hat{q}(t), \hat{z}(t), \bar{u}_t)Z(t) + f_u(t, \hat{y}(t), \hat{q}(t), \hat{z}(t), \bar{u}_t)(u(t) - \bar{u}(t)) \right] dt \\
    &\quad + \sum_{i=1}^d \int_t^T Q^i(s) dW^i(s) + \sum_{i=1}^\infty Z^i(t) dH^i(t) \\
Y(T) &= 0.
\end{align*}
\]  

(3.3)

Under Assumption 3.1, by Lemma 2.3, we know that BSDE (3.3) has a unique solution \((Y, Q, Z) \in S^{\infty}_F(0, T; \mathbb{R}^n) \times M^{\infty}_F(0, T; \mathbb{R}^{n \times d}) \times l^2_F(0, T; \mathbb{R}^n)\).

**Lemma 3.3.** Under Assumptions 3.1, 3.2, it follows that

\[
E \sup_{0 \leq t \leq T} |y^\varepsilon(t) - \hat{y}(t) - \varepsilon Y(t)|^2 + E \int_0^T |q^\varepsilon(t) - \hat{q}(t) - \varepsilon Q(t)|^2 dt \\
+ E \int_0^T ||z^\varepsilon(t) - \hat{z}(t) - \varepsilon Z(t)||_{L^2(\mathbb{R}^n)}^2 dt = o(\varepsilon^2).
\]

**Proof.** Firstly, one can check that

\[
\begin{align*}
y^\varepsilon(t) - \hat{y}(t) &= \int_t^T \left[ f_y^\varepsilon(s)(y^\varepsilon(s) - \hat{y}(s)) + f_q^\varepsilon(s)(q^\varepsilon(s) - \hat{q}(s)) \\
    &\quad + f_z^\varepsilon(s)(z^\varepsilon(s) - \hat{z}(s)) + f_u^\varepsilon(s)(u^\varepsilon(s) - \bar{u}(s)) \right] ds \\
    &\quad - \sum_{i=1}^d \int_t^T \left( q^{\varepsilon i}(s) - \hat{q}^{\varepsilon i}(s) \right) dW^i(s) - \sum_{i=1}^\infty \int_t^T \left( z^{\varepsilon i}(s) - \hat{z}^{\varepsilon i}(s) \right) dH^i(s)
\end{align*}
\]

and

\[
\begin{align*}
\varepsilon Y(t) &= \int_t^T \left[ f_y(s)\varepsilon Y(s) + f_q(s)\varepsilon Q(s) + f_z(s)\varepsilon Z(s) + f_u(s)\varepsilon (u(s) - \bar{u}(s)) \right] ds \\
    &\quad - \sum_{i=1}^d \int_t^T \varepsilon Q^i(s) dW^i(s) - \sum_{i=1}^\infty \int_t^T \varepsilon Z^i(s) dH^i(s),
\end{align*}
\]

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where we have used the abbreviations for $\varphi = f, l$ as follows:

$$
\begin{align*}
\varphi_y(t) &= \varphi_y(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}_t), \\
\varphi_z(t) &= \varphi_z(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}_t), \\
\varphi_q(t) &= \varphi_q(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}_t), \\
\varphi_u(t) &= \varphi_u(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}_t), \\
\tilde{\varphi}_y^\varepsilon(t) &= \int_0^1 \varphi_y(t, \bar{y}(t) + \lambda(y^\varepsilon(t) - \bar{y}(t)), \bar{z}(t) + \lambda(z^\varepsilon(t) - \bar{z}(t)), \bar{u}(t) + \lambda(u^\varepsilon(t) - u(t))) d\lambda, \\
\tilde{\varphi}_z^\varepsilon(t) &= \int_0^1 \varphi_z(t, \bar{y}(t) + \lambda(y^\varepsilon(t) - \bar{y}(t)), \bar{z}(t) + \lambda(z^\varepsilon(t) - \bar{z}(t)), \bar{u}(t) + \lambda(u^\varepsilon(t) - u(t))) d\lambda, \\
\tilde{\varphi}_q^\varepsilon(t) &= \int_0^1 \varphi_q(t, \bar{y}(t) + \lambda(q^\varepsilon(t) - \bar{y}(t)), \bar{z}(t) + \lambda(q^\varepsilon(t) - \bar{z}(t)), \bar{u}(t) + \lambda(u^\varepsilon(t) - u(t))) d\lambda, \\
\tilde{\varphi}_u^\varepsilon(t) &= \int_0^1 \varphi_u(t, \bar{y}(t) + \lambda(y^\varepsilon(t) - \bar{y}(t)), \bar{z}(t) + \lambda(z^\varepsilon(t) - \bar{z}(t)), \bar{u}(t) + \lambda(u^\varepsilon(t) - u(t))) d\lambda.
\end{align*}
$$

Thus by Lemma 2.4 again, we get

$$
E \sup_{0 \leq t \leq T} |y^\varepsilon(t) - \bar{y}(t) - \varepsilon Y(t)|^2 + E \int_0^T |q^\varepsilon(t) - \bar{q}(t) - \varepsilon Q(t)|^2 dt \\
+ E \int_0^T \|z^\varepsilon(t) - \bar{z}(t) - \varepsilon Z(t)\|^2_{\mathbb{R}^n} dt \\
\leq K\varepsilon^2 \left[ E \int_0^T \left| (\tilde{f}_y^\varepsilon(t) - f_y(t))Y(t) + (\tilde{f}_q^\varepsilon(t) - f_q(t))Q(t) + (\tilde{f}_z^\varepsilon(t) - f_z(t))Z(t) \\
+ (\tilde{f}_u^\varepsilon(t) - f_u(t))(u(t) - \bar{u}(t)) \right|^2 dt \right] \\
= K\varepsilon^2 \cdot \alpha(\varepsilon),
$$

where

$$
\alpha(\varepsilon) = E \int_0^T \left| (\tilde{f}_y^\varepsilon(t) - f_y(t))Y(t) + (\tilde{f}_q^\varepsilon(t) - f_q(t))Q(t) \\
+ (\tilde{f}_z^\varepsilon(t) - f_z(t))Z(t) + (\tilde{f}_u^\varepsilon(t) - f_u(t))(u(t) - \bar{u}(t)) \right|^2 dt.
$$

Consequently, using Lemma 3.2 and Assumption 3.1 by the dominated convergence theorem we can deduce

$$
\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0.
$$

Then the lemma follows from above and (3.5). $\square$

**Lemma 3.4.** Under Assumptions 3.1-3.2 using the abbreviations (3.4) we have

$$
J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) = \varepsilon E\varphi_y(\bar{y}(0))Y(0) + \varepsilon E \int_0^T l_y(t)Y(t) dt + \varepsilon E \int_0^T l_q(t)Q(t) dt \\
+ \varepsilon E \int_0^T l_z(t)Z(t) dt + \varepsilon E \int_0^T l_u(t)(u(t) - \bar{u}(t)) dt + o(\varepsilon).
$$
Proof. After a first order development, we have

\[ J(\bar{u}(\cdot)) - J(\bar{u}(\cdot)) \]
\[ = E \int_0^1 \phi_y(\bar{y}(0) + \lambda(\bar{u}(0) - \bar{y}(0))(\bar{y}(0) - \bar{y}(0)))d\lambda \]
\[ + E \int_0^T \bar{r}_y(t)(\bar{u}(t) - \bar{y}(t))dt + E \int_0^T \bar{r}_q(t)(\bar{q}(t) - \bar{q}(t))dt \]
\[ + E \int_0^T \bar{r}_z(t)(\bar{z}(t) - \bar{z}(t))dt + E \int_0^T \bar{r}_u(t)(\bar{u}(t) - \bar{u}(t))dt \]
\[ = \varepsilon E \phi_y(\bar{y}(0))Y(0) + E \phi_y(\bar{y}(0))(\bar{y}(0) - \bar{y}(0) - \varepsilon Y(0)) \]
\[ + E \int_0^1 \left[ \phi_y(\bar{y}(0) + \lambda(\bar{u}(0) - \bar{y}(0))) - \phi_y(\bar{y}(0)) \right] (\bar{y}(0) - \bar{y}(0))d\lambda \]
\[ + \varepsilon E \int_0^T l_y(t)Y(t)dt + E \int_0^T l_y(t)(\bar{u}(t) - \bar{q}(t) - \varepsilon Y(t))dt \]
\[ + E \int_0^T (\bar{l}_y(t) - l_y(t))(\bar{y}(t) - \bar{q}(t))dt \]
\[ + \varepsilon E \int_0^T l_q(t)q(t)dt + E \int_0^T l_q(t)(\bar{q}(t) - \bar{q}(t) - \varepsilon Q(t))dt \]
\[ + E \int_0^T (\bar{l}_q(t) - l_q(t))(\bar{q}(t) - \bar{q}(t))dt \]
\[ + \varepsilon E \int_0^T l_z(t)Z(t)dt + E \int_0^T l_z(t)(\bar{z}(t) - \bar{z}(t) - \varepsilon Z(t))dt \]
\[ + E \int_0^T (\bar{l}_z(t) - l_z(t))(\bar{z}(t) - \bar{z}(t))dt \]
\[ + E \int_0^T l_u(t)\varepsilon(\bar{u}(t) - \bar{u}(t))dt + E \int_0^T (\bar{l}_u(t) - l_u(t))\varepsilon(\bar{u}(t) - \bar{u}(t))dt \]
\[ = \varepsilon E \phi_y(\bar{y}(0))Y(0) + \varepsilon E \int_0^1 l_y(t)Y(t)dt + \varepsilon E \int_0^1 l_q(t)Q(t)dt \]
\[ + \varepsilon E \int_0^1 l_z(t)Z(t)dt + \varepsilon E \int_0^1 l_u(t)(\bar{u}(t) - \bar{u}(t))dt + \beta(\varepsilon), \]
where $\beta(\varepsilon)$ is given by
\[
\beta(\varepsilon) = E\phi_y(\bar{y}(0))(y^\varepsilon(0) - \bar{y}(0) - \varepsilon Y(0))
+ E\int_0^T \left[ \phi_y(\bar{y}(0) + \lambda (y^\varepsilon(0) - \bar{y}(0))) - \phi_y(\bar{y}(0)) \right] (y^\varepsilon(0) - \bar{y}(0)) d\lambda
+ E\int_0^T \lambda (y^\varepsilon(t) - \bar{y}(t)) dt + E\int_0^T (\bar{\ell}^\varepsilon(t) - \ell_y(t))(y^\varepsilon(t) - \bar{y}(t)) dt
+ E\int_0^T (\bar{l}^\varepsilon(t) - l_y(t))(q^\varepsilon(t) - \bar{q}(t)) dt + E\int_0^T (\bar{l}^\varepsilon(t) - l_q(t))(q^\varepsilon(t) - \bar{q}(t)) dt
+ E\int_0^T \bar{l}_z(t)(z^\varepsilon(t) - \bar{z}(t) - \varepsilon Z(t)) dt + E\int_0^T (\bar{l}^\varepsilon(t) - l_z(t))(z^\varepsilon(t) - \bar{z}(t)) dt
+ E\int_0^T (\bar{l}^\varepsilon_u(t) - l_u(t)) \varepsilon (u(t) - \bar{u}(t)) dt.
\]

Thus combining Lemma 3.2, Lemma 3.4 and Assumption 3.2 by the dominated convergence theorem we conclude that $\beta(\varepsilon) = o(\varepsilon)$.  

By Lemma 3.4 and the fact that $\lim_{\varepsilon \to 0^+} \frac{J(u^\varepsilon) - J(\bar{u})}{\varepsilon} \geq 0$, we can further deduce

**Corollary 3.5.** Under Assumptions 3.1-3.2 we have the variation inequality below
\[
E\phi_y(\bar{y}(0))Y(0) + E\int_0^T l_y(t)Y(t) dt + E\int_0^T l_y(t)Y(t) dt
+ E\int_0^T l_z(t)Z(t) dt + E\int_0^T l_u(t)(u(t) - \bar{u}(t)) dt \geq 0.
\]

### 4 Necessary and sufficient optimality conditions

We first introduce the adjoint equation corresponding to the variational equation (3.3):

\[
\begin{cases}
dk(t) = - \left[ - f^*_y(t)k(t) + l_y(t) \right] dt - \sum_{i=1}^d \left[ - f^*_q(t)k(t) + l_q(t) \right] dW^i(t) \\
\quad - \sum_{i=1}^\infty \left[ - f^*_z(t)k(t) + l_z(t) \right] dH^i(t) \\
k(0) = -\phi_y(\bar{y}(0)), \quad 0 \leq t \leq T,
\end{cases}
\]

(4.1)

where $f^*_y, f^*_q$ and $f^*_z$ are the dual operators of $f_y, f_q$ and $f_z$, respectively.

Under Assumptions 3.1-3.2 by Lemma 2.1, it is easy to see that the above adjoint equation has a unique solution $k(\cdot) \in S^2_y(0, T; \mathbb{R}^n)$. Then we define the Hamiltonian function $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times U \times \mathbb{R} \to \mathbb{R}$ by

\[
H(t, y, q, z, u, k) = \langle k, -f(t, y, q, z, u) \rangle + l(t, y, q, z, u)
\]

(4.2)
and rewrite the adjoint equation in the Hamiltonian system form:

\[
\begin{aligned}
    dk(t) &= -H_y(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t))dt \\
    & \quad - \sum_{i=1}^{d} H_{\gamma}^i(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t))dW^i(t) \\
    & \quad - \sum_{i=1}^{\infty} H_{\zeta}^i(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t))dH^i(t) \\
    k(0) &= -\phi_y(\bar{y}(0)).
\end{aligned}
\]  

(4.3)

Now we are ready to give the necessary conditions for an optimal control of Problem 3.1.

**Theorem 4.1.** Under Assumptions 3.1-3.2, if \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) is an optimal pair of Problem 3.1 then we have

\[
H_u(t, \bar{y}(t-), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t-))(u - \bar{u}(t)) \geq 0, \quad \forall u \in U, \quad a.e. \ a.s.,
\]

(4.4)

where \(k(\cdot)\) is the solution to the adjoint equation (4.1).

**Proof.** By (3.3) and (4.1), applying Itô formula to \(\langle Y(t), k(t) \rangle\) we have

\[
E\phi_y(\bar{y}(0))Y(0) + E \int_0^T l_y(t)Y(t)dt + E \int_0^T l_z(t)Z(t)dt + E \int_0^T l_u(t)(u(t) - \bar{u}(t))dt
\]

\[
= -E \int_0^T \langle k(t), f_u(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t))(u(t) - \bar{u}(t)) \rangle dt + E \int_0^T l_u(t)(u(t) - \bar{u}(t))dt.
\]

Then noticing the definition of Hamilton function (4.2) and the variational inequality (3.6), for any \(u(\cdot) \in A\), we have

\[
E \int_0^T H_u(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t))(u(t) - \bar{u}(t))dt \geq 0,
\]

which implies (4.4). \(\square\)

We then consider the sufficient conditions for an optimal control of Problem 3.1.

**Theorem 4.2.** Under Assumptions 3.1-3.2, let \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) be an admissible pair and \(k(\cdot)\) be the unique solution of the corresponding adjoint equation (4.3). Assume that for almost all \((t, \omega) \in [0, T] \times \Omega\), \(H(t, y, q, z, u, k(t))\) and \(\phi(y)\) are convex w.r.t. \((y, q, z, u)\) and \(y\), respectively, and the optimality condition

\[
H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)) = \min_{u \in U} H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), u, k(t))
\]

holds, then \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) is an optimal pair of Problem 3.1.

**Proof.** Let \((u(\cdot); y(\cdot), q(\cdot), z(\cdot))\) be an arbitrary admissible pair. It follows from the form of the cost functional (3.2) that

\[
J(u(\cdot)) - J(\bar{u}(\cdot)) = E \int_0^T \left[ l(t, y(t), q(t), z(t), u(t)) - l(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)) \right] dt + E \left[ \phi(y(0)) - \phi(\bar{y}(0)) \right]
\]

\[
= I_1 + I_2,
\]

(4.5)
where
\[
I_1 = E \int_0^T \left[ l(t, y(t), q(t), z(t), u(t)) - l(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)) \right] dt
\]
and
\[
I_2 = E \left[ \phi(y(0)) - \phi(\bar{y}(0)) \right].
\]
Due to the convexity of \( \phi \), applying Itô formula to \( \langle k(t), y(t) - \bar{y}(t) \rangle \), we have
\[
I_2 = E[\phi(y(0)) - \phi(\bar{y}(0))] \geq E[\phi(y(0)) - y(0) - \bar{y}(0)] = -E[\langle k(0), y(0) - \bar{y}(0) \rangle]
\]
\[
= -E \int_0^T \langle H_y(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), y(t) - \bar{y}(t) \rangle dt
\]
\[
- \sum_{i=1}^{d} E \int_0^T \langle H^i_q(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), q^i(t) - \bar{q}^i(t) \rangle dt
\]
\[
- E \int_0^T \langle f(t, y(t), q(t), z(t), u(t)) - f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)), k(t) \rangle dt
\]
\[
= -J_1 + J_2,
\]
where
\[
J_1 = E \int_0^T \langle H_y(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), y(t) - \bar{y}(t) \rangle dt
\]
\[
+ \sum_{i=1}^{d} E \int_0^T \langle H^i_q(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), q^i(t) - \bar{q}^i(t) \rangle dt
\]
\[
+ \sum_{i=1}^{\infty} E \int_0^T \langle H^i_z(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), z^i(t) - \bar{z}^i(t) \rangle dt
\]
and
\[
J_2 = -E \int_0^T \langle f(t, y(t), q(t), z(t), u(t)) - f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)), k(t) \rangle dt.
\]
Using the definition of the Hamiltonian function \([4.2]\) again, we have
\[
I_1 = E \int_0^T \left[ l(t, y(t), q(t), z(t), u(t)) - l(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)) \right] dt
\]
\[
= E \int_0^T \left[ H(t, y(t), q(t), z(t), u(t), k(t)) - H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)) \right] dt
\]
\[
+ E \int_0^T \langle f(t, y(t), q(t), z(t), u(t)) - f(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t)), k(t) \rangle dt
\]
\[
= J_3 - J_2,
\]
where
\[
J_3 = E \int_0^T \left[ H(t, y(t), q(t), z(t), u(t), k(t)) - H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)) \right] dt.
\]
Since \( H(t, y, q, z, u, k(t)) \) is convex w.r.t. \((y, q, z, u)\) for almost all \((t, \omega) \in [0, T] \times \Omega\), it turns out that
\[
H(t, y(t), q(t), z(t), u(t), k(t)) - H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)) \\
\geq \langle H_y(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), y(t) - \bar{y}(t) \rangle
\]
\[
+ \sum_{i=1}^{d} \langle H_q^i(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), q^i(t) - \bar{q}^i(t) \rangle
\]
\[
+ \sum_{i=1}^{\infty} \langle H_z^i(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), z^i(t) - \bar{z}^i(t) \rangle
\]
\[
+ \langle H_u(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), u(t) - \bar{u}(t) \rangle, \ a.s. \ a.e.
\] (4.9)

On the other hand, for almost all \((t, \omega) \in [0, T] \times \Omega\), \(u \rightarrow H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), u, k(t))\) takes its minimal value at \(\bar{u}(t)\) in the domain \(U\), thus
\[
\langle H_u(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), \bar{u}(t), k(t)), u(t) - \bar{u}(t) \rangle \geq 0, \ a.s. \ a.e.
\] (4.10)

Therefore, by (4.8)–(4.10) we first have
\[
J_3 \geq J_1.
\] (4.11)

By (4.11), together with (4.5)–(4.7), it follows that
\[
J(u(\cdot)) - J(\bar{u}(\cdot)) = I_1 + I_2 = (J_3 - J_2) + (-J_1 + J_2) \geq (J_1 - J_2) + (-J_1 + J_2) = 0.
\]

Due to the arbitrariness of \(u(\cdot)\), we conclude that \(\bar{u}(\cdot)\) is an optimal control process and thus \((\bar{u}(\cdot); \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) is an optimal pair.

\section{Applications in BLQ problems}

In this section, we will apply our stochastic maximum principle to the so-called BLQ problem, i.e. minimize the following quadratic cost functional over \(u(\cdot) \in A\):
\[
J(u(\cdot)) := E\langle My(0), y(0) \rangle + E \int_0^T \langle E(s)y(s), y(s) \rangle ds + \sum_{i=1}^{d} E \int_0^T \langle F^i(s)q^i(s), q^i(s) \rangle ds
\]
\[
+ \sum_{i=1}^{\infty} E \int_0^T \langle G^i(s)z^i(s), z^i(s) \rangle ds + E \int_0^T \langle N(s)u(s), u(s) \rangle ds,
\] (5.1)

where the state processes \((y(\cdot), q(\cdot), z(\cdot))\) are the solution to the controlled linear backward stochastic system as follows:
\[
\begin{align*}
\begin{cases}
dy(t) = - \left[ A(t)y(t) + \sum_{i=1}^{d} B^i(t)q^i(t) + \sum_{i=1}^{\infty} C^i(t)z^i(t) + D(t)u(t) \right] dt \\
+ \sum_{i=1}^{d} q^i dW^i(t) + \sum_{i=1}^{\infty} z^i dH^i(t)
\end{cases}
y(T) = \xi.
\end{align*}
\] (5.2)

To study this problem, we need the assumptions on the coefficients below.
Lemma 5.1. The \( \{\mathcal{F}_t, 0 \leq t \leq T\}\)-predictable matrix processes \( A : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, B^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, i = 1, 2, \ldots, d, C^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, i = 1, 2, \ldots, D : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, E : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, F^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, i = 1, 2, \ldots, G^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}, i = 1, 2, \ldots, N : [0, T] \times \Omega \rightarrow \mathbb{R}^{m \times m} \) and the \( \mathcal{F}_T \)-measurable random matrix \( M : \Omega \rightarrow \mathbb{R}^{n \times n} \) are uniformly bounded.

Assumption 5.2. The state weighting matrix processes \( E, F^i, G^i \), the control weighting matrix process \( N \) and the random matrix \( M \) are a.e. a.s. symmetric and nonnegative. Moreover, \( N \) is a.e. a.s. uniformly positive, i.e. \( N \geq \delta I \) for some positive constant \( \delta \) a.e. a.s.

Assumption 5.3. There is no further constraint imposed on the control processes, i.e.

\[
\mathcal{A} = \left\{ u(\cdot) \mid u(\cdot) \text{ is } \mathcal{F}_t \text{-predictable with values in } \mathbb{R}^m \text{ and } E \int_0^T |u(t)|^2 dt < \infty \right\}.
\]

From Assumption 5.3 we know that \( \mathcal{A} \) is a Hilbert space. If we denote the norm of \( \mathcal{A} \) by \( \| \cdot \|_\mathcal{A} \), then for any control process \( u(\cdot) \in \mathcal{A} \), \( \| u(\cdot) \|_\mathcal{A} = E \int_0^T |u(t)|^2 dt \).

Under Assumptions 5.1 by Lemma 2.3 we first know that the linear BSDE 5.2 in BLQ problem has a unique solution and thus the BLQ problem is well-defined. Then, under Assumptions 5.1-5.3, we will demonstrate that BLQ problem has a unique optimal control.

Lemma 5.1. Under Assumptions 5.1-5.3 the cost functional \( J \) is strictly convex over \( \mathcal{A} \) and \( \lim_{\| u(\cdot) \|_\mathcal{A} \rightarrow \infty} J(u(\cdot)) = \infty \).

Proof. The convexity of the cost functional \( J \) over \( \mathcal{A} \) is obvious. Actually, since the weighting matrix process \( N \) is uniformly positive, \( J \) is strictly convex. In view of the nonnegative property of \( M, E, F^i, G^i \) and the strictly positive property of \( N \), we have

\[
J(u(\cdot)) \geq \delta E \int_0^T |u(t)|^2 dt = \delta \| u(\cdot) \|^2_\mathcal{A}.
\]

Therefore, \( \lim_{\| u(\cdot) \|_\mathcal{A} \rightarrow \infty} J(u(\cdot)) = \infty. \)

Lemma 5.2. Under Assumptions 5.1-5.3 the cost functional \( J \) is Fréchet differentiable over \( \mathcal{A} \) and its Fréchet derivative \( J' \) at any admissible control process \( u(\cdot) \in \mathcal{A} \) is given by

\[
\langle J'(u(\cdot)), v(\cdot) \rangle = 2E \int_0^T \langle E(t)y^u(t), Y^v(t) \rangle dt + 2 \sum_{i=1}^d E \int_0^T \langle F^i(t)q^{iu}(t), Q^{iv}(t) \rangle dt + 2 \sum_{i=1}^\infty E \int_0^T \langle G^i(t)z^{iu}(t), Z^{iv}(t) \rangle dt + 2E \langle My^u(0), Y^v(0) \rangle,
\]

where \( v(\cdot) \in \mathcal{A} \) is arbitrary, \( (Y^v, Q^v, Z^v) \) is the solution of BSDE 5.2 corresponding to the control process \( v(\cdot) \in \mathcal{A} \) and the terminal value 0, and \( (y^u(\cdot), q^{iu}(\cdot), z^{iu}(\cdot)) \) are the state processes corresponding to the control process \( u(\cdot) \).

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Proof. For any $v(\cdot) \in \mathcal{A}$, we set
\[
\Delta J = J(u(\cdot) + v(\cdot)) - J(u(\cdot)) - 2E \int_0^T \langle E(t)u(t), Y^v(t) \rangle dt \\
- 2 \sum_{i=1}^d E \int_0^T \langle F^i(t)q^{iu}(t), Q^{iv}(t) \rangle dt - 2 \sum_{i=1}^\infty E \int_0^T \langle G^i(t)z^{iu}(t), Z^{iv}(t) \rangle dt \\
- 2E \int_0^T \langle N(t)u(t), v(t) \rangle dt - 2E \langle My^u(0), Y^v(0) \rangle.
\]

By the definition of cost functional (5.1), we have
\[
\Delta J = E \langle MY^v(0), Y^v(0) \rangle + E \int_0^T \langle E(s)Y^v(s), Y^v(s) \rangle ds + \sum_{i=1}^d E \int_0^T \langle F^i(s)Q^{iv}(s), Q^{iv}(s) \rangle ds \\
+ \sum_{i=1}^\infty E \int_0^T \langle G^i(s)Z^{iv}(s), Z^{iv}(s) \rangle ds + E \int_0^T \langle N(s)v(s), v(s) \rangle ds.
\]

Then it follows from Assumption 5.1 and a priori estimate (2.3) that
\[
|\Delta J| \leq K \left[ E \sup_{0 \leq s \leq T} |Y^v(t)|^2 + E \int_0^T |Q^i(t)|^2 dt + E \int_0^T \|Z^{iv}(t)\|^2_{L^2(\mathbb{R}^N)} dt + E \int_0^T |v(t)|^2 dt \right] \\
\leq KE \int_0^T |v(t)|^2 dt = K\|v(\cdot)\|_A^2.
\]

Consequently, we have
\[
\lim_{\|v(\cdot)\|_A \to 0} \frac{|\Delta J|}{\|v(\cdot)\|_A} = 0,
\]
which implies that $J$ is Fréchet differentiable and its Fréchet derivative $J'$ is given by (5.3).

The strict convexity and the Fréchet differentiability of $J$ deduced from Lemmas 5.1-5.2 lead to the lower semi-continuity of $J$, thus the following lemma is applicable to $J$ and $\mathcal{A}$ in our BLQ problem.

Lemma 5.3. (Proposition 1.2 of Chapter II in [6]) Let $\mathcal{A}$ be a reflexive Banach space and $J : \mathcal{A} \mapsto \mathbb{R}^1$ be a convex function. Assume that $J$ is lower semi-continuous and proper, and consider the minimization problem
\[
\inf_{u \in \mathcal{A}} J(u).
\]

If the function $J$ is coercive over $\mathcal{A}$, i.e.
\[
\lim_{\|u\|_A \to \infty} J(u) = \infty,
\]
then the minimization problem has at least one solution. Moreover, if $J$ is strictly convex over $\mathcal{A}$, then the minimization problem has a unique solution.
By Lemma 5.3 we can immediately conclude

**Theorem 5.4.** Under Assumptions \textit{5.1-5.3}, BLQ problem has a unique optimal control.

In what follows, we will utilize the stochastic maximum principle to study the dual representation of the optimal control to BLQ problem and construct its stochastic Hamilton system. As in section 4, we first introduce the adjoint forward equation corresponding to an admissible pair \((u(\cdot); y(\cdot), q(\cdot), z(\cdot))\):

\[
\left\{
\begin{array}{l}
    \displaystyle dk(t) = \left( A^*(t)k(t) - 2E(t)y(t) \right) dt + \sum_{i=1}^{d} \left( B^{i*}(t)k^i(t) - 2F^i(t)q^i(t) \right) dW^i(t) \\
    \quad + \sum_{i=1}^{\infty} \left( C^{i*}(t)k(t) - 2G^i(t)z^i(t) \right) dH^i(t) \\
    k(0) = -2M y(0).
\end{array}
\right.
\tag{5.4}
\]

Also we define the Hamiltonian function \(H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2(\mathbb{R}^n) \times U \times \mathbb{R}^n \to \mathbb{R}^1\) by

\[
H(t, y, q, z, u, k) = \langle k, A(t)y + \sum_{i=1}^{d} B^i(t)q^i + \sum_{i=1}^{\infty} C^i(t)z^i + D(t)u \rangle + \langle E(t)y, y \rangle + \sum_{i=1}^{d} \langle F^i(t)q^i, q^i \rangle + \sum_{i=1}^{\infty} \langle G^i(t)z^i, z^i \rangle + \langle N(t)u, u \rangle.
\tag{5.5}
\]

Then the adjoint equation can be rewritten as a Hamiltonian form:

\[
\left\{\begin{array}{l}
    \displaystyle dk(t) = -H_y(t, y(t), q(t), z(t), u(t), k(t)) dt - \sum_{i=1}^{d} H_q^i(t, y(t), q(t), z(t), u(t), k(t)) dB^i(t) \\
    \quad - \sum_{i=1}^{\infty} H_z^i(t, y(t), q(t), z(t), u(t), k(t)) dH^i(t) \\
    k(0) = -2M y(0).
\end{array}\right.
\tag{5.6}
\]

Under Assumption \textit{5.1} for each admissible pair \((u(\cdot); y(\cdot), q(\cdot), z(\cdot))\), by Lemma 2.1 the adjoint equation \((5.6)\) has a unique solution \(k(\cdot)\).

It is time to give the the dual characterization of the optimal control.

**Theorem 5.5.** Under Assumptions \textit{5.1-5.3}, BLQ problem has a unique optimal control and the optimal control is given by

\[
u(t) = -\frac{1}{2} N^{-1}(t) D^*(t) k(t), \quad \text{a.e. a.s.,}
\tag{5.7}
\]

where \(k(\cdot)\) is the unique solution of the adjoint equation \((5.4)\) (or equivalently, \((5.6)\)) corresponding to the optimal pair \((u(\cdot); y(\cdot), q(\cdot), z(\cdot))\).
Proof. By Theorem 5.1, we know the existence and uniqueness of optimal control to BLQ problem and denote the optimal control by \( u(\cdot) \). We only need to prove \( u \) has an expression as in (5.7). For this, let \((y(\cdot), q(\cdot), z(\cdot))\) be the optimal state processes corresponding to \( u(\cdot) \) and \( k(\cdot) \) be the unique solution of the adjoint equation (5.6) corresponding to the optimal pair \((u(\cdot); y(\cdot), q(\cdot), z(\cdot))\). By the necessary optimality condition (1.4) and Assumption 5.3, we have

\[
H_u(t, y(t-), q(t), z(t), u(t), k(t-)) = 0, \quad \text{a.e. a.s.}
\]

Noticing the definition of \( H \) in (5.3), we get

\[
2N(t)u(t) + D^*(t)k(t-) = 0, \quad \text{a.e. a.s.}
\]

Then the claim that the unique optimal control \( u(\cdot) \) satisfies (5.7) follows.

Finally we introduce the so-called stochastic Hamilton system which consists of the state equation (5.2), the adjoint equation (5.4) (or equivalently, (5.6)) and the dual representation (5.7):

\[
\begin{aligned}
dy(t) &= -\left(A(t)y(t) + \sum_{i=1}^{d} B^i(t)q^i(t) + \sum_{i=1}^{\infty} C^i(t)z^i(t) + D(t)u(t)\right) dt \\
&\quad + \sum_{i=1}^{\infty} q^i dW^i(t) + \sum_{i=1}^{\infty} z^i dH^i(t) \\
y(T) &= \xi, \\
dk(t) &= \left(A^*(t)k(t) - 2E(t)y(t)\right) dt + \sum_{i=1}^{d} \left(B^{*i}(t)k^i(t) - 2F^i(t)q^i(t)\right) dW^i(t) \\
&\quad + \sum_{i=1}^{\infty} \left(C^{*i}(t)k(t) - 2G^i(t)z^i(t)\right) dH^i(t)
\end{aligned}
\]

Clearly this is a fully coupled forward-backward stochastic differential equation (FBSDE) driven by \( d \)-dimensional Brownian motion \( W \) and Teugel’s martingales \( \{H^i\}_{i=1}^{\infty} \), and its solution is a stochastic processes quaternary \((k(\cdot), y(\cdot), q(\cdot), z(\cdot))\).

**Theorem 5.6.** Under Assumptions 5.1, 5.3, the stochastic Hamilton system (5.8) has a unique solution \((k(\cdot), y(\cdot), q(\cdot), z(\cdot)) \in S_2^2(0, T; \mathbb{R}^n) \times S_2^2(0, T; \mathbb{R}^n) \times M_2^2(0, T; \mathbb{R}^{n \times d}) \times l_2^2(0, T; \mathbb{R}^n), \) where \( u(\cdot) \) is the optimal control of BLQ problem and \((y(\cdot), q(\cdot), z(\cdot))\) are its corresponding optimal state. Moreover,

\[
E \sup_{0 \leq t \leq T} |k(t)|^2 + E \sup_{0 \leq t \leq T} |y(t)|^2 + E \int_0^T |q(t)|^2 dt + E \int_0^T ||z(t)||^2_{l_2^2(\mathbb{R}^n)} dt \leq KE|\xi|^2.
\]

**Proof.** The existence result follows from Theorem 5.5 and the uniqueness result is obvious once a priori estimate (5.9) holds. But noticing Assumptions 5.1, 5.3 and using Lemmas 2.2 and 2.4, we can deduce (5.9) immediately.
In summary, the stochastic Hamilton system (5.8) completely characterize the optimal control of BLQ problem in this section. Therefore, solving BLQ problem is equivalent to solving the stochastic Hamilton system, moreover, the unique optimal control of the stochastic Hamilton system can be given explicitly by (5.7).

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