Exponential Decay in a Timoshenko-type System of Thermoelasticity of Type III with Frictional versus Viscoelastic Damping and Delay

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Abstract

In this work, a Timoshenko system of type III of thermoelasticity with frictional versus viscoelastic under Dirichlet-Dirichlet-Neumann boundary conditions was considered. By exploiting energy method to produce a suitable Lyapunov functional, we establish the global existence, exponential decay of Type-III case.

Key words: linear dynamics, energy method, global existence, Lyapunov function.

1 Introduction

In the present paper, we are concerned with

\begin{align}
\begin{cases}
\rho_1 \varphi_{tt} - \kappa_1 (\varphi_x + \psi)_x + \mu \varphi_t &= 0, &\text{in} (0, 1) \times (0, +\infty) \\
\rho_2 \psi_{tt} - \kappa_2 \psi_{xx} + \int_0^t g(t-s) \psi_{xx}(s) ds + \kappa_1 (\varphi_x + \psi) + h(\psi_t) + \gamma \theta_{xt} &= 0, &\text{in} (0, 1) \times (0, +\infty) \\
\rho_2 \psi_{tt} + \gamma \psi_{tx} - \beta \psi_{txx} - \beta \theta_{txx} &= 0, &\text{in} (0, 1) \times (0, +\infty) \\
\varphi(1, t) = \varphi(0, t) = \psi(1, t) = \psi(0, t) = \theta_x(1, t) = \theta_x(0, t) = 0, \\
\varphi(., 0) = \varphi_0(0); \psi(., 0) = \psi_0(0); \theta(., 0) = \theta_0(0), \\
\varphi_t(., 0) = \varphi_1(., 0); \psi_t(., 0) = \psi_1(., 0); \theta_t(., 0) = \theta_1(., 0)
\end{cases}
\end{align}

(1.1)

in the case of equal speeds of propagation $\frac{\rho_1}{\rho_2} = \frac{\kappa_1}{\kappa_2}$. Therefore, without loss of generality, we take $\rho_1 = \kappa_i = \mu = \beta = \delta = \gamma = \beta = \alpha = 1$ and $L = 1$. We aim at investigating the effect of both frictional and viscoelastic dampings, where each one of them can vanish on the whole domain or in a part of it.

Before we state and prove our main result, let us recall some results regarding the Timoshenko system of wave equations.

In 1921, a simple system was proposed by Timoshenko [1]

\begin{align}
\begin{cases}
\rho_1 u_{tt} = \kappa (u_x - \varphi)_x, &\text{in} (0, L) \times (0, +\infty) \\
I_\rho \psi_{tt} = (EI \varphi_x)_x + \kappa (u_x - \psi), &\text{in} (0, L) \times (0, +\infty)
\end{cases}
\end{align}

(1.2)

which describes the transverse vibration of a beam of length $L$ in its equilibrium configuration. Here $t$ denotes the time variable, $x$ is the space variable along the beam, The coefficients $\rho, I_\rho, E, I$ and $K$ are respectively the density, the polar moment of inertia of a cross section and the shear modulus.

Together with boundary conditions of the form

$$EI \varphi(0, t)_x = EI \varphi(L, t)_x = 0, \kappa(u_x - \psi)(0, t) = \kappa(u_x - \psi)(L, t) = 0$$

is conservative, and so the total energy of the beam remains constant along the time.
Kim and Renardy considered together with two boundary controls of the form

\[ \kappa \varphi(L, t) - \kappa u(L, t)_x = \alpha u(L, t)_t, \forall t \geq 0 \]
\[ EI \varphi(L, t)_x = -\beta \varphi(L, t)_t, \quad \forall t \geq 0 \]

and used the multiplier techniques to establish an exponential decay result for the natural energy of system (1.2). They also provided numerical estimate to the eigenvalues of the operator which is associated with system (1.2).

Raposo et al. [2] studied following system

\[
\begin{align*}
\rho_1 u_{tt} - \kappa (u_x - \varphi)_x + u_x &= 0, \quad \text{in} \,(0, L) \times (0, +\infty) \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (u_x - \psi) + \psi_t &= 0, \quad \text{in} \,(0, L) \times (0, +\infty)
\end{align*}
\]

with homogeneous Dirichlet boundary conditions, and prove that the associated energy decays exponentially.

Soufyane and Wehbe [3] showed that it is possible to stabilize uniformly by using a unique locally distributed feedback. They studied

\[
\begin{align*}
\rho_1 u_{tt} &= \kappa (u_x - \varphi)_x, \quad \text{in} \,(0, L) \times (0, +\infty) \\
I_{\rho} \psi_{tt} &= (EI \varphi_x)_x + \kappa (u_x - \psi) - b \psi_t, \quad \text{in} \,(0, L) \times (0, +\infty) \\
u(0, t) &= u(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad \text{in} \,(0, L) \times (0, +\infty)
\end{align*}
\]

and prove that the uniform stability of (1.10) hold if and only if the wave speeds are equal \( \frac{\kappa}{\rho} = \frac{EI}{\rho}; \) otherwise only the asymptotic stability has been proved.

Ammar-Khodja et al. [4] considered a linear Timoshenko-type system with memory of the form

\[
\begin{align*}
\rho_1 \varphi_{tt} - \kappa (\varphi_x + \psi)_x &= 0, \quad \text{in} \,(0, L) \times (0, +\infty) \\
\rho_2 \psi_{tt} - b \psi_{xx} + \int_0^t g(t-s) \psi_x(x)ds + \kappa (\varphi_x + \psi) &= 0
\end{align*}
\]

in \((0, L) \times (0, +\infty),\) together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal \( \frac{\kappa}{\rho_1} = \frac{I_{\rho_1}}{\rho_2}; \) \( g \) decays uniformly. Precisely, they proved an exponential decay if \( g \) decays in an exponential rate and polynomially if \( g \) decays in a polynomial rate. They also required some extra technical conditions on both \( g' \) and \( g'' \) to obtain their result.

For Timoshenko system in thermoelasticity, River and Racke [5] considered

\[
\begin{align*}
\rho_1 u_{tt} - \sigma (\varphi_x, \psi)_x &= 0, \quad \text{in} \,(0, L) \times (0, +\infty) \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (u_x + \psi) + \gamma \theta_t &= 0, \quad \text{in} \,(0, L) \times (0, +\infty) \\
\rho_3 \theta_t - \kappa \theta_{xx} + \gamma \psi_{tx} &= 0, \quad \text{in} \,(0, L) \times (0, +\infty)
\end{align*}
\]

where \( \varphi, \psi \) and \( \theta \) are functions of \((x, t)\) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of \( \sigma, \rho_1, b, \kappa, \gamma, \) they proved several exponential decay results for the linearized system and a non-exponential stability result for the case of different wave speeds.

Messaoudi et al. [6] studied the following problem

\[
\begin{align*}
\rho_1 u_{tt} - \sigma (\varphi_x, \psi)_x + \mu \varphi_t &= 0, \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (u_x + \psi) + \beta \theta_x &= 0, \\
\rho_3 \theta_t - \kappa \theta_{xx} + \gamma \psi_{tx} &= 0, \\
\tau_0 q_t + q + \kappa \theta_x &= 0
\end{align*}
\]
where \((x, t) \in (0, L) \times (0, +\infty)\) and \(\varphi = \psi(x, t)\) is the displacement vector, \(\psi = \psi(x, t)\) is the rotation angle of the filament, \(\theta = \theta(x, t)\) is the temperature difference, \(q = q(x, t)\) is the heat flux vector, \(\rho_1, \rho_2, \rho_3, b, \kappa, \gamma, \delta, \tau, \mu\) are positive constants. The nonlinear function \(\sigma\) is assumed to be sufficiently smooth and satisfies

\[
\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = \kappa \\
\sigma_{\varphi_x \psi_x}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\varphi \psi} = 0
\]

Several exponential decay results for both linear and nonlinear cases have been established.

Guesmia and Messaoudi [7] studied the following system

\[
\begin{cases}
\rho_1 \varphi_{tt} - \kappa_1 (\varphi_x + \psi) = 0, & \text{in} (0, L) \times \mathbb{R}_+ \\
\rho_2 \psi_{tt} - \kappa_2 \psi_{xx} + \int_0^t g(t - \tau)(a(x)\psi_x(\tau)) d\tau + \kappa_1 (\varphi_x + \psi) + b(x)h(\psi_t) = 0, & \text{in} (0, L) \times \mathbb{R}_+
\end{cases}
\]

with Dirichlet boundary conditions and initial data where \(a, b, g\) and \(h\) are specific functions and \(\rho, \kappa_1, \kappa_2\) and \(L\) are given positive constants. They establish a general stability estimate using multiplier method and some properties of convex functions. Without imposing any growth condition on \(h\) at the origin, they show that the energy of the system is bounded above by a quantity, depending on \(g\) and \(h\), which tends to zero as time goes to infinity.

Ouchenane and Rahamoune [8] considered a one-dimensional linear thermoelastic system of Timoshenko system

\[
\begin{cases}
\rho_1 u_{tt} - K (\varphi_x + \psi)_x + \mu \varphi_t = 0, \\
\rho_2 \psi_{tt} - \bar{b} \psi_{xx} + \int_0^t g(t - s)(a(x)\psi_x(s)) ds + K (u_x + \psi) + b(x)h(\psi_t) + \gamma \theta_x = 0, \\
\rho_3 \theta_t + \kappa q_x + \gamma \psi_{tt} = 0, \\
\tau_0 q_t + \delta q + \kappa \theta_x = 0
\end{cases}
\]

where the heat flux is given by Cattaneo’s law. They establish a general decay estimate where the exponential and polynomial decay rates are only particular cases.

2 Preliminaries

In order to prove our main result we formulate the following hypotheses

\begin{enumerate}
\item[(H1)] \(h : \mathbb{R} \to \mathbb{R}\) is a differentiable nondecreasing function such that there exists constants \(\epsilon', \epsilon', \epsilon'' > 0\) and a convex and increasing function \(H : \mathbb{R} \to \mathbb{R}\) of class \(C^1(\mathbb{R}) \cap C^2(0, \infty)\) satisfying \(H(0) = 0\) and \(H\) is linear on \([0, \epsilon']\) or \(H'(0) = 0\) and \(H'' > 0\) on \((0, \epsilon]\) such that

\[
\begin{cases}
\epsilon' \mid s \leq \mid h(s) \leq \epsilon'' \mid s \mid, & \text{if} \ s \mid \geq \epsilon' \\
\mid s^2 + h^2(s) \leq H^{-1}(sh(s)), & \text{if} \ s \mid \geq \epsilon'
\end{cases}
\]

\item[(H2)] \(g: \mathbb{R}_+ \to \mathbb{R}_+\) is a differentiable function such that

\[
g(0) > 0, \quad 1 - \int_0^{+\infty} g(s) ds = l > 0
\]

\item[(H3)] There exists a non-increasing differentiable function \(\xi : \mathbb{R}_+ \to \mathbb{R}_+\) satisfying

\[
g'(s) \leq -\xi(s)g(s), \ \forall s \geq 0
\]
\end{enumerate}

Except all of the above, we also need the following lemmas to prove our results. See, e.g., Zheng[9].
Lemma 2.1. Let $A$ be a linear operator defined in a Hilbert space $H$, $D(A) \subset H \to H$. Then the necessary and sufficient conditions for $A$ being maximal accretive operator are

1. $\Re(\langle Ax, x \rangle) \leq 0$, $\forall x \in D(A)$;
2. $R(I - A) = H$.

Lemma 2.2. Suppose that $A$ is m-accretive in a Banach space $B$, and $U_0 \in D(A)$. Then problem (1.1) has a unique classical solution $U$ such that

$U \in C([0, +\infty), D(A)) \cap C^1([0, +\infty), B)$

In proving the stability results of global solution, the next lemma plays a key role. See e.g., Moñoz Rivera[10].

Lemma 2.3. Suppose that $y(t) \in C^1(R^+)$, $y(t) \geq 0$, $\forall t > 0$, and satisfies

$y'(t) \leq -C_0 y(t) + \lambda(t)$, $\forall t > 0$,

where $0 \leq \lambda(t) \leq L^1(R^+)$ and $C_0$ is a positive constant. Then we have

$\lim_{t \to \infty} y(t) = 0$.

Furthermore,

1. If $\lambda(t) \leq C_1 e^{-\delta t}$, $\forall t > 0$, with $C_1 > 0, \delta > 0$ being constants, then

$y(t) \leq C_2 e^{-\delta t}$, $\forall t > 0$

with $C_2 > 0, \delta > 0$ being constants.
2. If $\lambda(t) \leq C_3 (1 + t)^{-p}$, $\forall t > 0$, with $p > 1, C_3 > 0$ being constants, then

$y(t) \leq C_4 (1 + t)^{-p+1}$, $\forall t > 0$

with a constant $C_4 > 0$.

Lemma 2.4. If $1 \leq p \leq \infty$ and $a, b \geq 0$, then

$(a + b)^p \leq 2^{p-1}(a^p + b^p)$.

See e.g., Adams[11].

3 Global Existence and Exponential stability

In this section, we establish the global existence and exponential estimate for the generalized solutions in $H^1$ to problem (1.1) and then complete the proof of lemma 2.1 in terms of a series of lemmas. We start the vector function $U = (\varphi, u, v, \psi, \theta, w)^T$, where $u = \varphi_t$, $v = \psi_t$, $w = \theta_t$. We introduce as in [12]

$L^2_*(0, 1) \equiv \{ w \in L^2(0, 1) | \int_0^1 w(s)ds = 0 \}$

$H^1_*(0, 1) \equiv \{ w \in H^1(0, 1) | w_x(0) = w_x(1) = 0 \}$

$H^2_*(0, 1) \equiv \{ w \in H^2(0, 1) | w_x(0) = w_x(1) = 0 \}$

In order to use the Poincaré inequality for $\theta$, we set

$\bar{\theta} \equiv \theta(x, t) - t \int_0^1 \theta_1 dx - \int_0^1 \theta_0 dx$
then by (1.1) we have
\[ \int_0^1 \theta = 0 \]

The problem (1.1) can be written as the following
\[
\begin{cases}
\frac{dU}{dt} = AU, & t > 0 \\
U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1)^T
\end{cases}
\]

where the operator \( A \) is defined by
\[
AU = \begin{pmatrix}
u \\
(\varphi_x + \psi)_x - \varphi_t \\
\psi_{xx} - \int_0^t g(t-s)\varphi_{xx}(s)ds - (\varphi_x + \psi) - h(\psi_t) - \theta_xt \\
-w \\
-\psi_{tx} + \theta_{xx} + \theta_{txx}
\end{pmatrix}
\]

Let
\[
\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2_*(0,1)
\]

be Hilbert space, for \( U = (u_1, u_2, u_3, u_4, u_5, u_6) \), \( V = (v_1, v_2, v_3, \) ), there is inner product
\[
(U, V)_H = \frac{1}{2} \left( \int_0^1 \left[ u_2 v_2 + u_4 v_4 + u_6 v_6 + (u_1 x + u_3)(v_1 x + v_3) + (1 - \int_0^t g(s)ds)u_{3x}v_{3x}ight.ight.
\]
\[
+ u_{5x}v_{5x} + \int_0^t g(t-s)(u_{3x}(t) - u_{3x}(s))(v_{3x}(t) - v_{3x}(s))ds \right) dx
\]

The domain of \( A \) is
\[
D(A) = \{ U \in \mathcal{H}, \ \varphi, \psi \in H^1_0(0,1) \cap H^2(0,1), \ \theta \in H^1_0(0,1) \cap L^2_*(0,1), \ u \in L^2(0,1), \ v \in H^1(0,1), \ w \in H^2_* (0,1) \}
\]

we have the following global existence result.

**Theorem 3.1.** Let \( U_0 \in \mathcal{H}, \) then problem (1.1) has a unique classical solution, that verifies
\[
(\varphi, \psi, \theta) \in C([0, +\infty), H^2(0,1) \times H^2(0,1) \cap H^1_0(0,1)) \times H^2(0,1) \cap H^1_0(0,1) \times H^2_* (0,1)
\]
\[
\cap C^1([0, +\infty), H^1_0(0,1) \times H^1_0(0,1) \times H^2(0,1)) \times H^2(x,0,1)
\]
\[
\cap C^2([0, +\infty), L^2(0,1) \times L^2(0,1) \times L^2_*(0,1))
\]

**Proof.** The result follows from Theorem 3.1 provided we prove that \( A \) is a maximal accretive operator. In what follows, we prove that \( A \) is monotone. For any \( U \in D(A), \) and using the inner product, we obtain
\[
(AU, U)_H = - \int_0^1 \varphi_t^2 dx - \int_0^1 h(\psi_t)\psi_t dx - \int_0^1 \theta_t^2 dx - \frac{1}{2} \int_0^1 g(t)\psi_x^2 dx
\]
\[
+ \frac{1}{2} \int_0^1 \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2 ds dx
\]

Using (H1), (H2) and (H3), we have
\[
(AU, U)_H \leq 0
\]
it follows that $\text{Re}(AU, U) \leq 0$, which implies that $A$ is monotone.

Next, we prove that the operator $I - A$ is subjective. Given $B = (b_1, b_2, b_3, b_4, b_5, b_6)^T \in \mathcal{H}$, we prove that there exists $U = (u_1, u_2, u_3, u_4, u_5, u_6) \in D(A)$ satisfying

$$U - AU = B$$

that is,

$$\begin{aligned}
\varphi - u &= b_1, \\
2u - (\varphi_x + \psi)_x &= b_2, \\
\psi - v &= b_3, \\
v - \varphi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + (\varphi_x + \psi) + h(v) + w_x &= b_4, \\
\theta - w &= b_5, \\
w + v_x - \theta_{xx} - w_{xx} &= b_6, \\
\end{aligned}$$

In order to solve (3.1), we consider the following variational formulation

$$F((\varphi, \psi, \theta), (\varphi_1, \psi_1, \theta_1)) = G(\varphi_1, \psi_1, \theta_1)$$

where $B : [H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1)]^2 \to \mathcal{R}$ is the bilinear form defined by

$$F((\varphi, \psi, \theta), (\varphi_1, \psi_1, \theta_1)) = 2 \int_0^1 \varphi \varphi_1 dx + \int_0^1 (\varphi_x + \psi)(\varphi_{1x} + \psi_1) dx + \int_0^1 \psi \psi_1 dx$$

$$+ \int_0^1 \int_0^t g(t-s)\psi_{xx}(s)ds\psi_1 dx + \int_0^1 h(\psi_1)\psi_1 dx + \int_0^1 \theta_x \psi_1 dx$$

$$+ \int_0^1 \psi_x \theta_1 dx + 2 \int_0^1 \theta_1 dx + 2 \int_0^1 \theta_x \theta_1 dx + \int_0^1 \psi_x \psi_{1x} dx$$

and $G : [H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1)] \to \mathcal{R}$ is the linear functional given by

$$G(\varphi_1, \psi_1, \theta_1) = \int_0^1 (2b_1 + b_2)\varphi_1 dx + \int_0^1 (b_3 + b_4 + b_5)\psi_1 dx + \int_0^1 (b_3 - b_5)\theta_1 dx$$

Now, for $V = H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1)$ equipped with the norm

$$\|\varphi, \psi, \theta\|^2_V = \|\varphi + \psi\|^2_2 + \|\varphi\|^2_2 + \|\varphi_x\|^2_2 + \|\theta_x\|^2_2$$

Using integration by parts, we have,

$$F((\varphi, \psi, \theta), (\varphi, \psi, \theta)) = 2 \int_0^1 \varphi^2 dx + \int_0^1 (\varphi_x + \psi)^2 dx + \int_0^1 \psi^2 dx + \int_0^1 \psi^2 dx + \int_0^1 h(\psi_1)\psi_1 dx$$

$$+ \int_0^1 \int_0^t g(t-s)\psi_{xx}(s)ds\psi_1 dx + 2 \int_0^1 \theta^2 dx + 2 \int_0^1 \theta_x^2 dx dx \geq \alpha_0 \|\varphi, \psi, \theta\|^2_V$$

for some $\alpha_0 > 0$. Thus, $B$ is coercive. By Cauchy-Schwarz and Poincaré's inequalities, we can easily get

$$F((\varphi, \psi, \theta), (\varphi, \psi, \theta_1)) \leq c'' \|\varphi, \psi, \theta\|_V \|\varphi_1, \psi_1, \theta_1\|_V$$

Similarly

$$G(\varphi_1, \psi_1, \theta_1) \leq c'' \|\varphi_1, \psi_1, \theta_1\|_V$$
According to Lax-Milgram Theorem, we can easily obtain unique

\[(\varphi, \psi, \theta) \in H_0^1(0,1) \times H_0^1(0,1) \times H_1^1(0,1)\]

satisfying

\[F((\varphi, \psi, \theta), (\varphi_1, \psi_1, \theta_1)) = G(\varphi_1, \psi_1, \theta_1), \quad \forall (\varphi_1, \psi_1, \theta_1) \in V.\]

Applying the classical elliptic regularity, it follows from (3.1) that

\[(\varphi, \psi, \theta) \in H_0^1(0,1) \cap H^2(0,1) \times H_0^1(0,1) \cap H^2(0,1) \times H_1^1(0,1) \cap L^2(0,1)\]

satisfying

\[F((\varphi, \psi, \theta), (\varphi_1, \psi_1, \theta_1)) = G(\varphi_1, \psi_1, \theta_1) \in V.\]

Applying the classical elliptic regularity, it follows from (3.1) that

\[(\varphi, \psi, \theta) \in H_0^1(0,1) \cap H^2(0,1) \times H_0^1(0,1) \cap H^2(0,1) \times H_1^1(0,1) \cap L^2(0,1)\]

The existence result has been proved.

**Theorem 3.2.** Now, we introduce the energy functional defined by

\[E(t) = \frac{1}{2} \int_0^t \varphi_t^2 + \psi_t^2 + \theta_t^2 + (\varphi_x + \psi)^2 + \theta_x^2 + [1 - \int_0^t g(s)ds] \psi_x^2 + \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 ds dx\]

which satisfies

\[E(t) \leq C_0 e^{-\delta_0 t} \quad \text{and} \quad \lim_{t \to \infty} E(t) = 0 \quad (3.2)\]

where \(C_0\) and \(\delta_0\) are positive constants.

To prove Theorem 3.2, we will use the energy method to produce a suitable Lyapunov functional. This will be established through several lemmas. We have the following results.

**Lemma 3.1.** Let \((\varphi, \psi, \theta)\) be the solution of problem (1.1) and assume (H1)-(H3) hold. Then the energy \(E\) is non-increasing function and satisfies, \(\forall t \geq 0\),

\[E'(t) = - \int_0^t \varphi_t^2 dx - \frac{1}{2} \int_0^t g(t) \psi_x^2 dx - \int_0^t \theta_t^2 dx - \int_0^t h(\psi_t) \psi_t dx + \frac{1}{2} \int_0^t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2 ds dx \leq 0\]

**Proof.** Multiplying (1.1)_1, (1.1)_2 and (1.1)_3 by \(\varphi_t, \psi_t\) and \(\theta_t\), respectively, and integrating over \((0,1)\), summing them up, then using integration by parts and the boundary conditions, we obtain

\[\frac{1}{2} \int_0^1 \int_0^t \left[ \varphi_t^2 + \psi_t^2 + \theta_t^2 + (\varphi_x + \psi)^2 + \theta_x^2 + \psi_x^2 \right] ds dx + \int_0^1 \psi_t \int_0^t g(t-s) \psi_{xx}(s) ds dx = - \int_0^1 \varphi_t^2 dx - \int_0^1 \theta_t^2 dx - \int_0^1 h(\psi_t) \psi_t ds dx\]
calculating the term
\[
\int_0^1 \psi_t \int_0^t g(t-s)\psi_{xx}(s)dsdx
\]
\[
= \int_0^1 \int_0^t g(t-s)(\psi_{xx}(x,s) - \psi_{xx}(t))(\psi(t) - \psi(s))sdtdx + \int_0^1 \int_0^t g(t-s)\psi_{xx}(t)\psi_t dsdx
\]
\[
= \int_0^1 \int_0^t g(t-s)(\psi_{xx}(t) - \psi_x(t))(\psi_x(t) - \psi_x(s))dsdx - \int_0^1 \int_0^t g(t-s)\psi_x(t)\psi_xt(t)dsdx
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2dsdx - \frac{1}{2} \int_0^1 \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2dsdx
\]
\[
- \frac{1}{2} \frac{d}{dt} \int_0^1 (t-s)d\psi_x^2ds + \frac{1}{2} \int_0^1 \int_0^t g'(t-s)d\psi_x^2ds + g(0) \int_0^1 \psi_x^2ds
\]

then we have
\[
E'(t) = - \int_0^1 \psi_t^2ds - \int_0^1 \theta_{tX}^2ds - \int_0^1 h(\psi_t)\psi_t ds - \frac{1}{2} g(t) \int_0^1 \psi_x^2ds
\]
\[
+ \frac{1}{2} \int_0^1 \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t))^2dsdx
\]
\[
\leq - \int_0^1 \psi_t^2ds - \int_0^1 \theta_{tX}^2ds - c' \int_0^1 \psi_x^2ds - \frac{1}{2} g(t) \int_0^1 \psi_x^2ds
\]
\[
+ \frac{1}{2} \int_0^1 \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t))^2dsdx
\]

using (H1), (H2) and (H3), we get
\[
E'(t) \leq 0 \tag{3.3}
\]

the Lemma 3.1 has been proved.

**Lemma 3.2.** Let \((\varphi, \psi, \theta)\) be the solution of problem (1.1), the functional

\[
I_1(t) \equiv - \int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx
\]

satisfies the estimate

\[
I_1'(t) \leq -[\int_0^t g(s)ds - \epsilon] \int_0^1 \psi_t^2ds + \epsilon \int_0^1 (\varphi_x + \psi)^2dx + \epsilon \int_0^1 \psi_x^2dx + \epsilon \int_0^1 \theta_{tX}^2dx
\]
\[
+ \frac{c}{\epsilon} \int_0^1 \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2dsdx + c(\epsilon + \frac{1}{\epsilon}) \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2dsdx
\]
Proof. By using (1.1)_2, we get

\[
I'_1(t) = - \int_0^1 [\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - (\varphi_x + \psi) - h(\psi_t) - \theta_{xt}] \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx
- \int_0^1 \int_0^t g(s)ds\psi^2_t dx - \int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))dsdx
= \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx - \int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx
+ \int_0^1 (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx + \int_0^1 h(\psi_t) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx
+ \int_0^1 \theta_{xt} \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx - \int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))dsdx
\]

By using Young’s inequality and Poincaré inequality, we obtain, \( \forall \varepsilon_1 > 0, \)

\[
\int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx \\
\leq \varepsilon_1 \int_0^1 \psi^2_x dx + \frac{c_1}{\varepsilon_1} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx
\]
similarly, we have

\[
\int_0^1 (\varphi_x + \psi) \int_0^t g'(t-s)(\psi(t) - \psi(s))dsdx \\
\leq \varepsilon_2 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c_2}{\varepsilon_2} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx
\]

\[
\int_0^1 h(\psi_t) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx \\
\leq (c''')^2 \varepsilon_3 \int_0^1 \psi^2_t dx + \frac{c_3}{\varepsilon_3} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx
\]

\[
- \int_0^1 \varphi_t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))dsdx \\
\leq \varepsilon_4 \int_0^1 \varphi^2_t dx + \frac{c_4}{\varepsilon_4} \int_0^1 \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx
\]

\[
- \int_0^1 \theta_{xt} \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx \\
\leq \varepsilon_5 \int_0^1 \theta^2_{xt} dx + \frac{c_5}{\varepsilon_5} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 dsdx
\]
calculate the term

\[- \int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx \]

\[\leq \epsilon_6 \int_0^1 (\int_0^t g(t-s)(\psi_x(s) - \psi_x(t))ds) dx + \frac{c_6}{\epsilon_6} \int_0^t \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds dx\]

By using the Lemma 2.4, we get

\[\epsilon_6 \int_0^1 (\int_0^t g(t-s)(\psi_x(s) - \psi_x(t))ds) dx \leq 2\epsilon_6 \int_0^1 (\int_0^t g(t-s)(\psi_x(s) - \psi_x(t))ds) dx + 2\epsilon_6 \int_0^1 (\int_0^t g(t-s)(\psi_x(t))ds) dx\]

then we have

\[- \int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx \leq 2\epsilon_6 \int_0^1 \int_0^t g(s)ds(\psi_x(t))^2 dx + (2\epsilon_6 + \frac{c_6}{\epsilon_6}) \int_0^1 \int_0^t g(t-s)(\psi_x(s) - \psi_x(t))^2 ds dx\]

There exists \(c \geq \max\{c_1, c_2, c_3, c_4, c_5, (c')^2\}, \epsilon \geq \max\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, 2\epsilon_6\}\) that satisfies

\[\frac{c}{\epsilon} \geq \max\{\frac{c_1}{\epsilon_1} + \frac{c_2}{\epsilon_2} + \frac{c_3}{\epsilon_3} + \frac{c_5}{\epsilon_5} + (2\epsilon_6 + \frac{c_6}{\epsilon_6}), \frac{c_4}{\epsilon_4}\}\]

by combining all the above estimates, the Lemma 3.2 is proved.

**Lemma 3.3.** Let \((\varphi, \psi, \theta)\) be the solution of problem (1.1) and assume (H1)-(H3) holds. Then we have the functional

\[I_2(t) = \int_0^1 \psi_t(\varphi_x + \psi)dx + \int_0^1 \psi_x \varphi_t dx - \int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)ds dx\]

satisfies the estimate

\[I_2'(t) \leq [\psi_x - \int_0^t g(t-s)\psi_x(s)ds] \varphi_x \big|_{x=0} - (1 - \epsilon_7) \int_0^1 (\varphi_x + \psi)^2 dx\]

\[+ \frac{c_7}{\epsilon_7} \int_0^1 \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))^2 ds dx + c_7(\epsilon_7 + \frac{1}{\epsilon_7}) \int_0^1 \psi_x^2 dx\]

\[+ \frac{c_7}{\epsilon_7} \int_0^1 \psi_t^2 + \frac{c_7}{\epsilon_7} \int_0^1 \theta_{xt}^2 dx + c_7(\epsilon_7 + \frac{1}{\epsilon_7}) \int_0^1 \varphi_t^2 dx\]
**Proof.** By exploiting (1.1)_2, (1.1)_2 and repeating the same procedure as in the above, we have

\[
I'_2(t) = \int_0^1 (\varphi_x + \psi)[\dot{\psi}_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - (\varphi_x + \psi) - h(\psi_t) - \theta_{xt}]dx
\]

\[
+ \int_0^1 (\varphi_x + \psi)\psi_t dx + \int_0^1 \psi_{xt}\varphi_t dx + \int_0^1 \psi_x\varphi_{tt} dx - g(0) \int_0^1 \varphi_t \psi_x dx
\]

\[
- \int_0^1 [(\varphi_x + \psi) - \varphi_t] \int_0^t g(t-s)\psi_x(s)ds dx - \int_0^t \varphi_t \int_0^t g'(t-s)\psi_x(s)ds dx
\]

\[
= [(\psi_x - \int_0^t g(t-s)\psi_x(s))\varphi_x]_{x=0}^t - \int_0^1 (\varphi_x + \psi)^2 dx - \int_0^1 h(\psi_t)(\varphi_x + \psi) dx
\]

\[
- \int_0^1 \theta_{tx}(\varphi_x + \psi) dx + \int_0^1 \psi_t^2 dx - [g(t) + 1]\int_0^1 \psi_x\varphi_t dx + \int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)ds dx
\]

\[
- \int_0^1 \varphi_t \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t))ds dx
\]

By using the Young’s inequality, we have the Lemma 3.3.

**Lemma 3.4.** Let \((\varphi, \psi, \theta)\) be the solution of problem (1.1) and assume (H1)-(H3) hold. Then we have the functional

\[
I_3(t) = - \int_0^1 \theta_t dx
\]

satisfies the estimate

\[
I'_3(t) \leq - \int_0^1 \theta_t^2 dx + \epsilon_8 \int_0^1 \psi_t dx + (\epsilon_8 + \frac{c_8}{\epsilon_8}) \int_0^1 \theta_{tx}^2 dx
\]

**Proof.** By exploiting (1.1)_3, we have

\[
I'_3(t) = - \int_0^1 \theta_t^2 dx - \int_0^1 \theta_{tt} dx
\]

\[
= - \int_0^1 \theta_t^2 dx - \int_0^1 \theta_{tx} \psi_t dx + \int_0^1 \theta_{tt}^2 dx + \int_0^1 \theta_{tx} \theta_{tx} dx
\]

By using Young’s inequality, we prove the Lemma 3.4.

**Lemma 3.5.** Let \((\varphi, \psi, \theta)\) be the solution of problem (1.1) and assume (H1)-(H3) hold. Then we have the functional

\[
I_4(t) = - \int_0^1 (\psi \psi_t + \varphi \varphi_t) dx
\]

satisfies the estimate

\[
I'_4(t) \leq - \int_0^1 \psi_t^2 dx - (1 - \epsilon_9) \int_0^1 \varphi_t^2 dx + \int_0^1 (\varphi_x + \psi)^2 dx + \epsilon_9 \int_0^1 \psi_x^2 dx + \epsilon_9 \int_0^1 \theta_x^2 dx
\]

\[
+ \epsilon_9 \int_0^1 \int_0^t g(t-s)(\psi_x(s) - \psi_x(t))^2 ds dx + \frac{c_9}{\epsilon_9} \int_0^1 \varphi_x^2 dx
\]
At this point, we chose our constants carefully. First, let us take $N$ then we select $c$ Finally, we choose $\xi$ Lemma 3.6. For $N$ sufficiently large, the functional defined by

$$\mathcal{L}(t) \equiv NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t)$$

where $N$ and $N_i$ are positive real numbers to be choose appropriately later, satisfies

$$\mathcal{L}(t) \leq C_0 e^{-\delta_0 t}, \quad \forall t \geq 0$$

Proof. It is easily to get, $\forall t \geq 0$,

$$\mathcal{L}(t) \sim E(t) \quad (3.4)$$

Combining Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5, (H3), we obtain

$$\begin{align*}
\mathcal{L}'(t) & \leq -[N - N_1 c'(\epsilon + \frac{1}{\epsilon}) + N_4 (1 - \epsilon_8)] \int_0^1 \psi^2 dx \\
& \quad -N_1 \epsilon - N_2 \epsilon_7 - N_3 (\epsilon_8 + \frac{c_8}{\epsilon_8}) \int_0^1 \theta_1 dx 2 dx \\
& \quad -\{Nc' + N_1 [\int_0^t g(s)ds - \epsilon] - N_2 c_7 - N_3 (\epsilon_8 + \epsilon_7) \int_0^1 \psi^2 dx \\
& \quad -N_1 c' + N_2 (1 - \epsilon_7) - N_3 \int_0^1 (\psi_x + \varphi)^2 dx - N_3 \int_0^1 \theta_1^2 dx \\
& \quad -[\xi (\frac{N}{2} + N_1 c_7 + \frac{N_2 c_7}{\epsilon_7}) - (N_1 c_7 + \frac{1}{\epsilon}) + N_4 \epsilon_9] \int_0^1 \int_0^t g(t - s) (\varphi_x(t) - \varphi_x(s))^2 dx ds dx
\end{align*}$$

At this point, we chose our constants carefully. First, let us take $N_3 > 0$, then pick $N, N_2, \epsilon_7, c_7, \epsilon_8$ so that

$$N - N_2 c_7 (\epsilon + \frac{1}{\epsilon}) + N_4 (1 - \epsilon_8) > 0$$

then we select $N_1, \epsilon, c_8$ such that

$$N - N_1 \epsilon - N_2 \frac{c_7}{\epsilon_7} - N_3 (\epsilon_8 + \frac{c_8}{\epsilon_8}) > 0$$

Finally, we choose $c_9, \epsilon_9, N_4, \epsilon'$ such that

$$Nc' + N_1 [\int_0^t g(s)ds - \epsilon] - \frac{N_2 c_7}{\epsilon_7} - N_3 \epsilon_8 + N_4 > 0$$
and
\[
\frac{Ng(t)}{2} + N_1\epsilon - N_2c_7(\epsilon + 1) + N_3c_9 > 0
\]
and
\[
\xi\left(\frac{N}{2} + \frac{N_1\epsilon}{\epsilon} + \frac{N_2c_7}{\epsilon}\right) - (N_1c(\epsilon + 1) + N_4\epsilon) > 0
\]
and
\[-N_1\epsilon + N_2(1 - \epsilon_7) - N_4 > 0\]
combining all above inequalities, there exist positive \(\delta_1 > 0\) such that
\[
\mathcal{L}'(t) \leq -\delta_1 E(t)
\]
then we have
\[
\mathcal{L}'(t) \leq -\delta_0 \mathcal{L}(t)
\]
easily we can get
\[
\mathcal{L}(t) \leq C_0e^{-\delta_0 t}
\]
up to now, Lemma 3.6 has been proved.
Exploiting (3.4) there is
\[
E(t) \leq C_0e^{-\delta_0 t}
\]
combining (3.3), (3.5), Lemma 2.3, we prove the Theorem 3.2.

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