THE 2+1 KEPLER PROBLEM AND ITS QUANTIZATION

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Abstract

We study a system of two pointlike particles coupled to three dimensional Einstein gravity. The reduced phase space can be considered as a deformed version of the phase space of two special-relativistic point particles in the centre of mass frame. When the system is quantized, we find some possibly general effects of quantum gravity, such as a minimal distances and a foaminess of the spacetime at the order of the Planck length. We also obtain a quantization of geometry, which restricts the possible asymptotic geometries of the universe.
Outline and summary

The Kepler system is the simplest realistic example of a coupled two body system, and belongs to the few systems that can be solved exactly within the framework of Newtonian gravity. It consists of two pointlike objects, characterized only by their masses, and interacting with the gravitational field. Unfortunately, within the framework of general relativity, the two body problem not only lacks of an exact solution. It is not even well defined, because Einstein gravity in four spacetime dimensions does not admit pointlike matter sources. Clearly, this makes general relativity so interesting. But the obvious drawback is the absence of a simple but still realistic toy model, which is sometimes very useful. The Kepler system is, in a sense, the hydrogen atom of gravity.

The situation is different in three spacetime dimensions, where Einstein gravity is not only a much simpler field theory [1, 2, 3]. It also admits pointlike matter sources [4, 5, 6, 7, 8, 9, 10], and even a more or less straightforward canonical quantization [1, 2, 11, 12, 13]. The vacuum Einstein equations in three dimensions require the spacetime to be flat outside the matter sources. There are neither gravitational waves, nor local gravitational forces. However, the spacetime becomes curved if matter is present. The simplest example of a non-trivial spacetime is the gravitational field of a massive pointlike particle. The spacetime is the direct product of a real line with a conical space. The particle is sitting at the tip, and the deficit angle of the cone is $8\pi G m$, where $m$ is the rest mass of the particle, and $G$ is Newton’s constant.

In units where the velocity of light is one, $G$ has the dimension of an inverse mass or energy. The inverse $M_{Pl} = 1/G$ is the Planck mass. It is a classical quantity in the sense that $\hbar$ is not involved in the definition. There is thus a certain amount of curvature on the world line of a point particle, which is proportional to its mass. But this is just a very simple conical singularity in an otherwise flat spacetime. There is a simple way to visualize a spacetime containing a single point particle. One starts from a flat three dimensional Minkowski space and cuts out a wedge. A wedge is a subset which is bounded by two timelike half planes, whose common boundary is a timelike geodesic. This geodesic becomes the world line of the particle.

The two half planes are mapped onto each other by a certain isometry of Minkowski space. It is a Lorentz rotation about the world line, and the angle of rotation is $8\pi G m$. The region inside the wedge is taken away, and the points on the two half planes are identified, according to the Lorentz rotation. The result is a locally flat spacetime with a conical singularity on the world line. For a massless particle, the same procedure can be applied to a pair of half planes, whose common boundary is a lightlike geodesic. They are then mapped onto each other by a null rotation. The Kepler spacetime contains two particles, and therefore we have to apply this procedure twice. The result is shown in figure [1].

Unless one of the particles has a deficit angle which is bigger than $\pi$, hence a mass above $M_{Pl}/8$, every possible two particle spacetime can be constructed in this way. The geometry of the spacetime only depends on the relative motion of the particles, which can be read off immediately from the relative orientation of the two world lines in Minkowski space. It is therefore possible to get an overview of all possible spacetime geometries very easily. However, there are also some problems with this simple construction. First of all, it does not work for masses bigger then $M_{Pl}/8$, although such spacetimes do exist. In this case, the half planes defining the boundary of the wedges must be replaced by curved surfaces, as otherwise the wedges overlap. But this is actually not a serious problem.
Figure 1: The Kepler spacetime can be constructed by cutting out two wedges from a flat Minkowski space. The faces are identified, such that two conical singularities arise in an otherwise flat spacetime. In the rest frame of the each particle, the deficit angle of the conical space is proportional to the mass of the particle.

The more serious problem has to do with the asymptotic structure of the spacetime at infinity. The region far away from the particles is split into two segments in figure 1. Each segment is a subset of Minkowski space. But on the wedges we have to apply non-trivial transition functions, relating the Minkowski coordinates on one side to those on the other side. To find out what the spacetime looks like at infinity, it would be nicer to have a single coordinate chart covering this region. There is in fact a particular reason why we are interested in the asymptotic structure of the Kepler spacetime. In order to quantize it in the end, we first have to set up a proper classical Hamiltonian formulation. This requires a proper definition of an action principle for the underlying field theory of Einstein gravity. And this again requires some kind of asymptotical flatness condition to be imposed on the metric at infinity [14].

The asymptotic structure of the Kepler spacetime depends crucially on the relative motion of the particles. If they are moving slowly, then far away from the particles the spacetime is also conical. It looks almost like the gravitational field of a single particle, whose mass is equal to the sum of the two masses of the real particles. The rest frame of this fictitious particle can be identified with the centre of mass frame of the universe. If the particles are moving faster, the apparent mass of the fictitious particle has to be replaced by the total energy of the system. It also receives a spin, which represents the total angular momentum. But still, the universe looks like a cone at infinity, and this cone defines the centre of mass frame.

Something strange happens when the relative motion of the particles exceeds a certain threshold [15]. The definition of a centre of mass frame then breaks down, and the asymptotic structure of the spacetime is no longer conical. Even more peculiar, the spacetime then contains closed timelike curves [16, 17, 18]. Clearly, these are very interesting features of such a simple two particle spacetime. But for our purpose we have to exclude them, again because we want to set up a proper Hamiltonian framework. This requires a well defined causal structure of the space-
time. Otherwise the Hamiltonian, or ADM formulation cannot be applied to general relativity. It requires the existence of a foliation of the spacetime by spacelike slices [19].

In order to get rid of these problems, we have to go over from the simple construction of the Kepler spacetime in figure 1, to a slightly more sophisticated one, which focuses more on the asymptotic structure of spacetime at infinity. The basic idea is, first to fix the asymptotic structure of the spacetime, and then insert the particles. In figure 1, the sequence is the other way around. The world lines are inserted first, then the actual spacetime is constructed, and finally the asymptotic structure can be read off by looking at the region at infinity. The transition from this picture to an alternative description of the Kepler spacetime, based on its asymptotic structure, is explicitly carried out in [20]. There we also give a comprehensive overview of all those spacetimes that admit the definition of a centre of mass frame.

Somewhat schematically, the alternative construction of a two particle spacetime is shown in figure 3. One starts from a big cone, cuts off the tip, and identifies the cut lines, which are two geodesics, such that a conical surface with two tips arises. The three dimensional version of this construction yields the Kepler spacetime, although the technical details are a little bit more involved. The advantage of this procedure is that the original cone immediately defines the centre of mass frame of the universe, as seen by an observer at infinity, and independent of the way the particles are inserted. Moreover, we will be able also introduce position and momentum coordinates of the particles, referring to the centre of mass frame defined by the big cone.

In this way, the Kepler system can effectively be treated like a simple two particle system in a fixed three dimensional background spacetime, although the technical details are again slightly more involved. The phase space becomes a finite dimensional manifold, and the kinematical and dynamical properties of the Kepler system are finally encoded in the usual way, in the symplectic structure, or the Poisson bracket, and the Hamiltonian. The phase space structure is very similar to a system of two special-relativistic point particles in a flat Minkowski space, restricted to the centre of mass frame. And in fact, it is possible to take the limit $G \to 0$, where the gravitational interaction is switched off, and then the Kepler system reduces to this free particle system.

The article is therefore organized as follows. In section 1, we shall only study the free particle system. Of course, there is nothing new about this to be learned, except perhaps the relativistic definition of a centre of mass frame, which is in this form not a very common concept. It is actually a centre of energy frame, but we shall stick to the more familiar notion of a centre of mass. After imposing the appropriate restriction on the phase space, we shall go through the usual canonical programme. Starting from the classical Hamiltonian framework, we perform a phase space reduction, derive and solve the classical equations of motion, and finally we quantize it, deriving the energy eigenstates and the spectra of certain interesting operators. The idea behind this preparation, using a well known and very simple toy model for our real toy model, is to keep the conceptual aspects apart from the technical aspects.

The conceptual aspects are, for example, the definitions of the various phase spaces, the way the mass shell constraints are imposed, the principle idea of the phase space reduction, and finally also the quantization methods. At the classical level, the free particle system also provides a nice toy model for the Hamiltonian formulation of general relativity within the ADM framework. This is indicated in figure 2. At the quantum level, we are going to consider two alternative quantization methods, the Schrödinger method applied to the reduced classical phase space, where all gauge symmetries are removed, and the Dirac method applied to an extended phase space, where
the dynamics of the system is defined by a generalized mass shell constraint.

All these concepts can then be applied to the Kepler system in the very same way. It is therefore useful first to explain them using a much simpler model. The more technical aspects are then the modifications that we have to make when the gravitational interaction is switched on. They are divided into a classical part in section 2, and a quantum part in section 3. In the classical section, we shall first look at the Kepler spacetime itself, in the way explained above and described in figure 3. We shall mainly focus on a geometric point of view, without going into any technical details and proofs. All these details can be found in the previously mentioned and more comprehensive article [20]. It includes, in particular, the precise transition from figure 1 to the alternative description in figure 3.

In a certain sense, the Kepler spacetime is a deformation of the free particle spacetime, with Newton’s constant being the deformation parameter. At any stage of the derivation we can always get back to the free particle system by taking the limit \( G \to 0 \). This can be used as a cross check at various points, and sometimes it is even possible to set up general rules, telling us how to deform the free particle system to obtain the corresponding structures of the coupled system. This also applies to the phase space of the Kepler system, which is the subject of the second part of section 2. It is a deformed version of the free particle phase space. The interesting point is thereby that gravity affects the symplectic structure of this phase space, and not only the Hamiltonian, which is otherwise the typical feature of interactions between point particles.

The actual derivation of the deformed symplectic structure will not be given in the article. We should however emphasize that this derivation is a very important point. We do not want to make any special or unmotivated assumptions. Instead, the only assumption that enters the definition of the Kepler system is the following. The kinematical and dynamical features of the gravitational field are completely defined by the Einstein Hilbert action. All the relevant phase space structures can then be derived from this action principle, by a straightforward phase space reduction. However, apart from the symplectic structure, all other features of the phase space can more or less be inferred from geometric considerations. We shall therefore restrict to these geometrical aspects here, and refer to [10] for the derivation of the symplectic structure for a more general multi particle system.

In the last part of section 2, we will go through the whole canonical programme once again, deriving and solving the classical equations of motion, and briefly describing the various kinds of trajectories. At this point, we can actually forget about the general relativistic nature of the system, and treat it as if it was a simple two particle system living in a three dimensional background spacetime. Or, if we do not want to give up the general relativistic point of view completely, we may at least stick to the ADM picture, and consider the Kepler system as a space which evolves in time. As shown in figure 4, the phase space variables define the geometry of space at a moment of time, and this geometry changes with time. Effectively, the particles are moving in a two dimensional space. We can also think of a typical scattering process, and define quantities like incoming and outgoing momenta, and scattering angles.

In the quantum section, we will first try to apply the same quantization methods that we also applied to the free particle system. We’ll find that the straightforward Schrödinger quantization fails, due to some peculiar features of the deformed classical phase space and its symplectic structure. This already indicates that there are some new effects to be expected, which are due to the gravitational interaction, and which are fundamentally different from other interactions. The
Dirac method however works. It is possible to set up a well defined operator representation, and to quantize and solve the constraint equation, which is a generalized Klein Gordon equation. A similar equation has also been found for a somewhat simpler single particle system [13].

We can solve this constraint equation, and finally we are able to express the energy eigenstates of the Kepler system explicitly as wave functions on a suitably defined configuration space. Suitably thereby means that the wave function has the usual physical interpretation as a probability amplitude for the particles in space. Or, once again, if we want to stick to the general relativistic point of view, it is a probability amplitude for certain geometries of space. We have in this sense a simple example for a truly quantized general relativistic system. The quantum state gives us probabilities for geometries. But nevertheless, it is somewhat more intuitive to consider the wave function as a probability amplitude in an ordinary space. The space is thereby the configuration space of the two particles in the centre of mass frame. And it is also useful to have in mind the usual picture of a scattering state in quantum mechanics.

The energy eigenstates are parameterized by two quantum numbers. The inverse radial wavelength at infinity represents the eigenvalue of the incoming and outgoing momenta of the particles. And the inverse angular wavelength represents the eigenvalue of the angular momentum. The latter is quantized in steps of $\hbar$, and the radial momentum turns out to have a positive continuous spectrum. The energy, or the frequency of the wave function, only depends on the radial quantum number. So far, these are the typical features of scattering states in quantum mechanics. However, the energy spectrum is bounded from below and from above. This is actually not surprising, because the total energy contained in a three dimensional universe is bounded from above by $M_{\text{Pl}}/4$, which corresponds to the maximal deficit angle $2\pi$ of a conical spacetime. The lower bound for the total energy is the sum of the rest masses of the particles, which is also not surprising.

What is remarkable, however, is that the radial momentum of the particles is nevertheless unbounded. For very large momenta of the particles, the total energy of the Kepler system approaches the upper bound $M_{\text{Pl}}/4$. This behaviour has also been found for a single particle, where the upper bound is half as big, thus $M_{\text{Pl}}/8$ [13]. The relation between the spatial momentum of the particles in the centre of mass frame, and the total energy of the system is shown in figure 5, where it is compared to the corresponding free particle energy. The energy of the Kepler system is always smaller than the free particle energy. The difference is a kind of gravitational binding energy. At the quantum level, this has the strange consequence that the wavelength of the wave function in space can be arbitrarily small, but the frequency is bounded from above.

Finally, we shall then look at the wave functions themselves. Unfortunately, it is hardly possible to read off any physically interesting information directly from the analytic expressions. They are somewhat complicated, involving hypergeometric functions. We shall therefore look at the graphical representations of some typical wave functions. They are shown in the figures through 8, both for the free particles and the Kepler system. We shall thereby find the following interesting features. At large distances, the gravitational interaction has almost no effect. We have the typical scattering wave function, which is a superposition of an ingoing and an outgoing radial wave. The only difference between the free and the coupled system is a phase shift, which indicates that some interaction takes place when the particles are closer to each other.

At small distances however, which are of the order of the Planck length $\ell = G\hbar$, the wave function changes drastically. We typically find that the particles avoid to be at the same point
in space. We even find that under certain very general circumstances, it is impossible for the particles to get closer to each other than a certain minimal distance. It is of the order of ten to hundred Planck lengths, and depends on the rest masses of the particles and their statistics. In three spacetime dimensions, there are not only bosons and fermions, but also anyons, and there is also a generalized statistics if the particles are not identical [21]. All this can be taken into account very easily when the quantization is performed. The finite lower bound for the distance of the particles in space arises whenever the particles are not two bosons, and it is maximal for two identical fermions.

Referring again to the scattering picture, it is reasonable to say that due to this minimal distance, it is impossible to probe the structure of spacetime at small length scales, even if we increase the momentum of the particles unboundedly. This is exactly the kind of limit that quantum gravity is expected to impose on the ability to look at small length scales in spacetime. Another feature of our toy model is closely related to this, but a little bit more general. Even if the particles are further apart than the minimal distance, it is still impossible to localize them within a box that is smaller than a certain size. More precisely, it is impossible to find a quantum state where the relative position of the particles in space is arbitrarily sharp at a given moment of time, even if the distance between the particles is many orders of magnitude above the Planck scale.

Both features indicate that the quantized spacetime in which the particles are living obtains a kind of foamy structure. Unfortunately, it is not possible to derive a more explicit and intuitive spacetime spectrum, like the one for the single particle system in [13]. But in principle, we have a very similar situation, and this is also expected to arise in a more realistic, or even in a fully consistent theory of quantum gravity in higher dimensions. Hence, although the Kepler system is only a very simple toy model, some principle effects of quantum gravity can be seen. These are really quantum gravity effects, because they disappear not only at the classical level, hence in the limit $\hbar \to 0$, whatever this precisely means, but also in the well defined limit $G \to 0$, where the gravitational interaction is switched off.

Finally, a nice feature of this toy model is that, once the reduction to a two particle system in three dimensions is accepted, everything else can be derived exactly and without any further assumptions. It is possible to keep the assumptions and simplifications clearly apart from the mathematics and the physical conclusions. There are no hidden points were additional, say, intuitive assumptions must made. The only assumption that enters the definition of the Kepler system as a toy model is the Einstein Hilbert action with the appropriate matter terms for the particles, which is assumed to define the dynamics of the gravitational field. This is the concept on which the derivation of a general multi particle phase space is based in [10]. The point where this assumption enters this article, is the definition of the symplectic structure in section 2.

1 The free particle system

Before we switch on the gravitational interaction, let us consider a system of two uncoupled relativistic point particles $\pi_k (k = 1, 2)$ in flat, three dimensional Minkowski space. As a vector space, we identify this with the spinor representation $\mathfrak{sl}(2)$ of the three dimensional Lorentz algebra. The twelve dimensional kinematical phase space is spanned by the positions $x_k = x_k^a \gamma_a$ and the momentum vectors $p_k = p_k^a \gamma_a$ of the particles, where $\gamma_a (a = 0, 1, 2)$ is an orthonormal
basis given by the usual gamma matrices \((\mathbf{A}, \mathbf{I})\). Further conventions and some useful formulas regarding the vector and matrix notation are given in the appendix.

We have the usual symplectic potential \(\tilde{\Theta}\), from which we read off the Poisson brackets,

\[
\tilde{\Theta} = \frac{1}{2} \sum_k \text{Tr}(\mathbf{p}_k \, d\mathbf{x}_k) \quad \Rightarrow \quad \{p^a_k, x^b_k\} = \eta^{ab}.
\]  

The Hamiltonian \(\tilde{H}\) is a linear combination of the two mass shell constraints \(C_k\), with Lagrange multipliers \(\zeta_k \in \mathbb{R}\) as coefficients,

\[
\tilde{H} = \sum_k \zeta_k C_k, \quad C_k = \frac{1}{4} \text{Tr}(\mathbf{p}_k^2) + \frac{1}{2} m_k^2.
\]

The physical phase space is a subset of the kinematical phase space, which is defined by the mass shell constraints and the positive energy conditions,

\[
\frac{1}{2} \text{Tr}(\mathbf{p}_k^2) = -m_k^2, \quad p^0_k = \frac{1}{2} \text{Tr}(\mathbf{p}_k \gamma^0) > 0.
\]

The world lines are parameterized by a common, unphysical time coordinate \(t\), and the Hamiltonian generates the time evolution with respect to this coordinate,

\[
\dot{p}_k = \{\tilde{H}, p_k\} = 0, \quad \dot{x}_k = \{\tilde{H}, x_k\} = \zeta_k \mathbf{p}_k.
\]

The freedom to choose the multipliers \(\zeta_k\) corresponds to the gauge freedom to reparameterize the world lines. Finally, there are some conserved charges which are of interest, namely the total momentum vector and the total angular momentum vector,

\[
P = \sum_k p_k, \quad J = \frac{1}{2} \sum_k [p_k, x_k].
\]

The associated rigid symmetries are the translations and Lorentz rotations of the world lines with respect to the reference frame, which is defined by the coordinates of the embedding Minkowski space.

**The centre of mass frame**

So far, this is the standard Hamiltonian formulation of a special-relativistic two particle system. Since we are only interested in the relative motion of the particles, we shall now impose some further restrictions on the phase space variables. Provided that the total momentum vector is positive timelike, there always exists a reference frame where

\[
P = M \gamma_0, \quad J = S \gamma_0, \quad M, S \in \mathbb{R}.
\]

We call this the centre of mass frame. There is only one special situation where a centre of mass frame does not exist. If both particles are massless and if they move with the velocity of light into the same direction, then the total momentum is lightlike. These special states are excluded in the following.

In the centre of mass frame, the rigid symmetries are reduced to a two dimensional group of time translations and spatial rotations about the \(\gamma_0\)-axis. The associated charges are the total
Figure 2: The free particle system in the centre of mass frame. The embedding Minkowski space is foliated by a family of equal time planes, labeled by an ADM time coordinate $t$. The relative position of the particles in space is defined by $X$ and $\phi$, and the spatial momentum by $K$ and $\beta$. The energies of the particles are $M_k$, and the clock $T$ represents the absolute time in the centre of mass frame, which can be regarded as a reference frame of some external observer.

Energy $M$ and the spatial angular momentum $S$. It is also allowed to speak about an absolute time, defined by the $\gamma_0$-axis, and an absolute space orthogonal to it, defined by the $\gamma_1,2$-axes in Minkowski space. Essentially, we have a non-relativistic system, although the dynamical properties of the particles are still relativistic. We can use this to impose a gauge condition, which restricts the way the world lines are parameterized. It is reasonable to choose the parameterization such that

$$x^0_1 = x^0_2. \quad (1.7)$$

At each moment of time $t$, the particles are then located on the same equal time plane, as indicated in figure 2. We can think of an ADM like foliation of the embedding Minkowski space by equal time planes. The planes are labeled by an ADM time coordinate $t$, and each plane represents an instant of time in the centre of mass frame. For the moment, we do not require the ADM time $t$ to be related in any way to the absolute time in the centre of mass frame. Hence, there is still one gauge degrees of freedom left, which is compatible with the gauge condition (1.7). This is a simultaneous reparameterization of both world lines.

From the phase space point of view, the various restrictions can be regarded as additional constraints. All together, we have seven constraints. The two mass shell constraints (1.3), the gauge condition (1.7), and four independent spatial components of the definition (1.6) of the center of mass frame. Note that the time components of these equations are definitions of the phase space functions $M$ and $S$. Since we have an odd number of constraints, it is clear that at least one of them is a first class constraint, which generates the simultaneous time evolution of
the two particles as a gauge symmetry. The other six are second class constraints, as we are now going to show.

To easiest way to do this is to eliminate six of the seven constraints, and to show that the reduced symplectic structure is still non-degenerate. We do this by introducing new phase space variables, which are similar to the usual non-relativistic centre of mass coordinates and momenta of a two particle system. The general solution to the equation \( \mathbf{P} = M \gamma_0 \) can obviously be written as

\[
p_1 = M_1 \gamma_0 - K \gamma(\beta), \quad p_2 = M_2 \gamma_0 + K \gamma(\beta).
\]

The new variables \( M_k \) are the energies of the particles, and \( K \) and \( \beta \) are polar coordinates defining the spatial momentum and its direction. The rotating unit vector \( \gamma(\beta) \) is introduced in (A.6). It defines the angular direction \( \beta \) in Minkowski space, thus in this case the direction of motion of the particles in the centre of mass frame.

To express the mass shell constraints in terms of these variables, it is convenient to replace the mass parameters \( m_1 \) and \( m_2 \) by a total mass \( \mu \) and a relative mass \( \nu \),

\[
\mu = m_2 + m_1, \quad \nu = m_2 - m_1, \quad 0 \leq \nu \leq \mu.
\]

Without loss of generality, we assume that \( m_2 \geq m_1 \). The special cases are \( \nu = 0 \), where both particles have the same mass, and \( \nu = \mu \), where at least one particle is massless. The same redefinition can be applied to the energy variables,

\[
M = M_2 + M_1, \quad V = M_2 - M_1, \quad |V| < M,
\]

where the inequality represents the positive energy condition for both particles. The mass shell constraints (L.2) are then given by

\[
C_1 = \frac{K^2}{2} - \frac{(M - V)^2 - (\mu - \nu)^2}{8}, \quad C_2 = \frac{K^2}{2} - \frac{(M + V)^2 - (\mu + \nu)^2}{8}.
\]

They can be simplified by taking the following linear combinations,

\[
\mathcal{D} = C_2 - C_1 = \frac{\mu \nu - MV}{2}, \quad \mathcal{E} = C_2 + C_1 = K^2 + \frac{M^2 + V^2 - \mu^2 - \nu^2}{4}.
\]

Let us solve the constraint \( \mathcal{D} = 0 \) for \( V \), so that \( V \) becomes a simple function of \( M \),

\[
V = \frac{\mu \nu}{M}.
\]

What remains is a single mass shell constraint, which can be written as

\[
\mathcal{E} = K^2 - F(M) \approx 0, \quad \text{where} \quad F(M) = \frac{(M^2 - \mu^2)(M^2 - \nu^2)}{4M^2}.
\]

We’ll see later one that this is just a somewhat unusual way to write the familiar relation between the momentum and the energy of two relativistic point particles. The positive energy condition becomes a non-trivial condition to be imposed on \( M \), namely

\[
|V| < M \iff \frac{\mu \nu}{M} < M \iff M > \sqrt{\mu \nu}.
\]
Note that this implies $M \neq 0$, so that the various places where $M$ appears in the denominator are not problematic. All together, the momentum vectors are parameterized by three independent variables $M, K, \beta$. They have an immediate physical interpretation as the total energy, the spatial momentum, and the direction of motion of the particles in the centre of mass frame.

The positions $x_k$ are subject to the constraints $J = S\gamma_0$ and $x_1^0 = x_2^0$. The general solution can be parameterized by three other variables $T, X, \phi$, so that

$$T = x_1^0 = x_2^0, \quad z = x_2 - x_1 = X \gamma(\phi).$$

The radial coordinate $X \geq 0$ represents the relative position of the particles in space, and $\phi$ is the spatial orientation. It is also useful to introduce the relative position vector $z$, which is going to have a generalization for the interacting particles later on. The clock $T$ represents the absolute time in the centre of mass frame. It is the $\gamma_0$-coordinate of the equal time plane in figure 2. So far, it is an arbitrary function of the ADM time $t$, because we are still free to choose the parameterization of the world lines, and thus the labeling of the equal time planes by the unphysical coordinate $t$.

Using all this, we can solve the equation $J = S\gamma_0$, and express the result in terms of the new phase space variables,

$$x_1 = T \gamma_0 - \frac{M + V}{2M} X \gamma(\phi), \quad x_2 = T \gamma_0 + \frac{M - V}{2M} X \gamma(\phi).$$

The relativistic version of the centre of mass frame is thus actually a centre of energy frame. At each moment of time, the centre of energy is located on the $\gamma_0$-axis in figure 2. But nevertheless, let us stick to the notion centre of mass frame. The crucial point is that the positions of the particles are, like the momenta, specified by three independent variables. We have the clock $T$, the relative position $X$, and the orientation $\phi$.

To see that the six eliminated constraints were second class constraints, we have to compute the reduced symplectic potential. Inserting (1.8) and (1.17) into (1.1) gives

$$\tilde{\Theta} = K \cos(\beta - \phi) dX + X K \sin(\beta - \phi) d\phi - M dT.$$ (1.18)

To simplify this, we replace the polar momentum coordinates $K$ and $\beta$ by Cartesian coordinates

$$Q = X K \cos(\beta - \phi), \quad S = X K \sin(\beta - \phi).$$ (1.19)

They define the radial and angular momentum. One can easily verify that $S$ indeed satisfies $J = S\gamma_0$, so it coincides with the previous definition. Moreover, inserting this into (1.8) gives

$$p_1 = \frac{M - V}{2} \gamma_0 - \frac{Q \gamma(\phi)}{X} + S \gamma'(\phi), \quad p_2 = \frac{M + V}{2} \gamma_0 + \frac{Q \gamma(\phi)}{X} + S \gamma'(\phi).$$ (1.20)

This tells us that $Q/X$ is the component of the momentum parallel to the relative position, and $S/X$ is the component orthogonal to it. This is the usual definition of a radial and angular momentum. If we use $Q$ and $S$ as the basic phase space variables, then the expression to be inserted into the mass shell constraint (1.14) is

$$K^2 = \frac{Q^2 + S^2}{X^2}.$$ (1.21)
And finally, the symplectic potential simplifies to
\[
\tilde{\Theta} = X^{-1}Q \, dX + S \, d\phi - M \, dT.
\] (1.22)

This defines a non-degenerate symplectic structure \( \tilde{\Omega} = d\tilde{\Theta} \), and we read off the following non-vanishing Poisson brackets,
\[
\{ M, T \} = -1, \quad \{ Q, X \} = X, \quad \{ S, \phi \} = 1.
\] (1.23)

From this it is immediately obvious that \( M \) is the charge associated with time translations \( T \rightarrow T - \Delta T \), and \( S \) is the charge associated with spatial rotations \( \phi \rightarrow \phi + \Delta \phi \).

What remains from the Hamiltonian is a single mass shell constraint \( E \), and a multiplier \( \zeta \), which is some not further interesting linear combination of the original multipliers \( \zeta_k \),
\[
\tilde{H} = \zeta \, E.
\] (1.24)

It is still an unphysical Hamiltonian. Its value is zero for physical states, and it generates the time evolution with respect to the unphysical ADM time \( t \), which is formally a gauge transformation. Except for the reduced symmetry group, this looks very much like the Hamiltonian description of a single relativistic point particle. We have a six dimensional extended, or kinematical phase space
\[
\tilde{\mathcal{P}} = \{ (M, Q, S; T, X, \phi) \mid X \geq 0, \quad \phi \equiv \phi + 2\pi \},
\] (1.25)
and a single mass shell constraint \( \mathcal{E} \), which defines the physical subspace and an associated gauge symmetry. It is this feature that we are going to exploit in the following, treating the two particle system in the centre of mass frame as if it was a single particle system. Of course, this is a well known way to solve a two particle system in non-relativistic mechanics, and we’ll see that it also works for this almost trivial relativistic system.

**Complete reduction**

We can also go over to an effectively non-relativistic formulation, where no constraint and no gauge symmetry is left. We just have to impose another gauge condition. The most natural one is to require the ADM time \( t \) to coincide with the absolute time \( T \) in the centre of mass frame. This provides another constraint, and together with the mass shell constraint we get a new pair of second class constraints,
\[
K^2 = F(M), \quad T = t.
\] (1.26)

Let us assume that the function \( F \) can be inverted. We can then define a four dimensional reduced phase space
\[
\mathcal{P} = \{ (Q, S; X, \phi) \mid X \geq 0, \quad \phi \equiv \phi + 2\pi \}.
\] (1.27)

It is just the usual non-relativistic phase space of the relative motion of two particles in a plane. To derive the symplectic structure and the Hamiltonian on \( \mathcal{P} \), we have to take into account that the constraints (1.26) are explicitly time dependent. This implies a mixing of the Hamiltonian and the symplectic structure when we perform the reduction. We have to consider the extended symplectic potential on \( \tilde{\mathcal{P}} \),
\[
\tilde{\Theta} - \tilde{H} \, dt = X^{-1}Q \, dX + S \, d\phi - M \, dT - \zeta \mathcal{E} \, dt.
\] (1.28)
To obtain the reduced structures on $\mathcal{P}$, we insert the solutions to the equations $K^2 = F(M)$ and $T = t$, and write the result as a combination of the reduced symplectic potential $\Theta$ and the reduced Hamiltonian $H$,

$$\Theta - H \, dt = X^{-1} Q \, dX + S \, d\phi - F^{-1}(K^2) \, dt.$$  \hspace{1cm} (1.29)

Note that the last term in (1.28) vanishes, because the constraint $E = 0$ is now identically satisfied. What comes out is

$$\Theta = X^{-1} Q \, dX + S \, d\phi \quad \Rightarrow \quad \{Q, X\} = X, \quad \{S, \phi\} = 1. \hspace{1cm} (1.30)$$

The Poisson brackets of the remaining variables are unchanged, and they are the usual non-relativistic ones.

So, we now have an unconstrained Hamiltonian formulation of the free particle system, where the Hamiltonian represents the physical energy of the system, and generates the time evolution with respect to the absolute time in the centre of mass frame. Explicitly, one finds that

$$H = F^{-1}(K^2) = \sqrt{K^2 + m_1^2} + \sqrt{K^2 + m_2^2}. \hspace{1cm} (1.31)$$

Not surprisingly, this is just the relativistic energy of two point particles with the same spatial momentum $K$. The signs of the two square roots are fixed by the positive energy condition. A minus sign for one of the square roots would violate (1.15), as one of the particles would then be in an antiparticle state. To express $H$ as a function of the reduced phase space variables, we have to insert (1.21).

It is also quite instructive to look at the range of $H$. The right hand side of (1.31) is obviously minimal for $K = 0$, and it increases unboundedly with $K$. Thus, we have $H \geq M_{\min}$, where

$$M_{\min} = m_1 + m_2. \hspace{1cm} (1.32)$$

For $K = 0$ the particles are at rest with respect to each other, and thus also with respect to the centre of mass. There is one exception, however, where such a state cannot be realized. If one of the particles is massless, then we have $\nu = \mu$, and the positive energy condition (1.15) requires that $M > M_{\min}$. In this case, the states with $K = 0$ are excluded, and we have the stronger condition $H > M_{\min}$. Clearly, this is because a massless particles cannot be at rest, and consequently a state with vanishing momentum $K$ does not exist.

Let us also consider the non-relativistic limit. For small momenta $K$, the energy $H$ either starts off linearly or quadratically with $K$, depending on whether a massless particle is present or not. Let us consider the case where both particles are massive. If we then expand the right hand side of (1.31) up to the second order in $K$, we get

$$H \approx M_{\min} + \frac{1}{2m} K^2, \quad \text{where} \quad m = \frac{m_1 m_2}{m_1 + m_2}. \hspace{1cm} (1.33)$$

This is the usual non-relativistic relation between the spatial momentum $K$ and the energy $H$ in the centre of mass frame. The parameter $m$ is the reduced mass of the two particles system. The difference between the total energy $H$ and the total rest mass $M_{\min}$ is the non-relativistic kinetic energy. For large momenta $K$, on the other hand, we find the usual relativistic behaviour $H \approx 2K$. The energy becomes a linear function of the spatial momentum. The factor of two arises because we have two particles with the same momentum. For some typical mass parameters, the relations between $K$ and $H$ are shown as the broken lines in figure 3.
Trajectories

There are now two alternative ways to describe the two-particle system. We can either use the constrained Hamiltonian formulation based on the extended phase space $\hat{\mathcal{P}}$. Or we may use the unconstrained formulation, based on the reduced phase space $\mathcal{P}$. Consider first the constrained formulation. In this case we have six independent phase space variables, and the Hamiltonian is given by $\hat{H} = \zeta \mathcal{E}$, where $\mathcal{E}$ is the mass shell constraint and $\zeta$ is an arbitrarily chosen function of the unphysical ADM time $t$. It is not difficult to derive the resulting time evolution equations and to solve them.

The energy $\hat{M}$ and the angular momentum $\hat{S}$ are of course preserved charges. The same holds for the spatial momentum $\hat{K}$, as it is only this combination of the phase space variables $\hat{Q}, \hat{S}$, and $\hat{X}$ that enters the Hamiltonian. Thus,

$$\dot{\hat{M}} = \{\hat{H}, M\} = 0, \quad \dot{\hat{S}} = \{\hat{H}, S\} = 0, \quad \dot{\hat{K}} = \{\hat{H}, K\} = 0. \quad \text{(1.34)}$$

For the relative position $\hat{X}$ and the conjugate radial momentum $\hat{Q}$, we find

$$\dot{\hat{X}} = \{\hat{H}, \hat{X}\} = 2\zeta \frac{\hat{Q}}{\hat{X}}, \quad \dot{\hat{Q}} = \{\hat{H}, \hat{Q}\} = 2\zeta \frac{\hat{Q}^2 + \hat{S}^2}{\hat{X}^2} = 2\zeta K^2. \quad \text{(1.35)}$$

And finally, the brackets of $\hat{H}$ with $\phi$ and $T$ are given by

$$\dot{\phi} = \{\hat{H}, \phi\} = 2\zeta \frac{\hat{S}}{\hat{X}^2}, \quad \dot{T} = \{\hat{H}, T\} = \zeta F'(\hat{M}). \quad \text{(1.36)}$$

It is clear that the same equations of motion arise in the unconstrained formulation, except that $\hat{M}$ and $\hat{T}$ are not independent variables, $\hat{H}$ is replaced by $\hat{H}$, and instead of the multiplier $\zeta$ the function $1/F'(\hat{H})$ appears, which formally implies that $\hat{T} = 1$, which is consistent with the gauge condition $T = t$.

We can easily solve these differential equations step by step. Those for $\hat{M}$ and $\hat{S}$ are trivial, stating that $M(t) = M_0$ and $S(t) = S_0$ for some constants $M_0$ and $S_0$. Moreover, $K(t) = K_0$ is also a constant of motion. It is related to $M_0$ by the constraint $K_0^2 = F(M_0)$. Thus $K_0$ and $M_0$ are not independent, and they are subject to the previously derived restrictions. For massive particles, we have $M_0 \geq M_{\text{min}}$ and $K_0 \geq 0$. If at least one massless particle is present, then we have the stronger condition $M_0 > M_{\text{min}}$ and $K_0 > 0$.

Using this, it is easy to solve the equation of motion for $\hat{Q}$. The general solution is $Q(t) = Q_0 + \epsilon(t)K_0^2$, where $Q_0$ is some integration constant. The function $\epsilon(t)$ is determined up to a constant by $\dot{\epsilon}(t) = 2\zeta(t)$. For $K_0 > 0$ we can obviously choose this free constant so that $Q_0 = 0$. On the other hand, $K_0 = 0$ implies, by definition (1.24), that both $Q$ and $S$ must be zero. In this case we also have $Q_0 = 0$. Therefore, the general solutions found so far are

$$M(t) = M_0, \quad S(t) = S_0, \quad K(t) = K_0, \quad Q(t) = \epsilon(t)K_0^2. \quad \text{(1.37)}$$

If we insert this into the time evolution equations for $X$ and $\phi$, then we can integrate them. The result is

$$X(t) = \sqrt{R_0^2 + \epsilon^2(t)K_0^2}, \quad \phi(t) = \phi_0 + \arctan\left(\epsilon(t)\frac{K_0^2}{S_0}\right). \quad \text{(1.38)}$$
where $R_0 \geq 0$ and $\phi_0$ are two more integration constants. And finally, we can also solve the equation of motion for $T$, which gives

$$T(t) = T_0 + \epsilon(t) \frac{F'(M_0)}{2}.$$  \hspace{1cm} (1.39)

These are the most general solutions to the time evolution equations on $\tilde{P}$, provided by the Hamiltonian $\tilde{H} = \zeta \mathcal{E}$. They are parameterized by an arbitrary function $\epsilon(t)$, representing the gauge freedom, and six integration constants $M_0, K_0, S_0, T_0, R_0, \phi_0$. Only four of them are independent. The constraint relates the momentum $K_0$ to the energy $M_0$, and additionally there is a relation between $S_0, X_0$ and $K_0$, which follows from the definition (1.21),

$$K_0^2 = F(M_0), \quad K_0 R_0 = |S_0|. \hspace{1cm} (1.40)$$

There are two ways to look at these solutions. If we keep $t$ fixed and vary $\epsilon(t)$, then we pass along a gauge orbit generated by the constraint $\mathcal{E}$. If we instead fix the function $\epsilon(t)$ and vary $t$, then we see a trajectory in the phase space $\tilde{P}$, which describes the time evolution generated by the Hamiltonian $\tilde{H}$ in a particular gauge.

The derivation of the trajectories on the reduced phase space $P$ yields the same result, with the same integration constants, however with the gauge function replaced by

$$\epsilon(t) = 2 \frac{t - T_0}{F'(M_0)}, \hspace{1cm} (1.41)$$

which obviously implies $T(t) = t$. The inverse of $F'(M_0)$ is always well defined because, within the range of $M_0$ allowed by the positive energy condition, the function $F(M)$ is monotonically increasing with $M$. So, the trajectories on $P$ are parameterized by the same four independent integration constants, but we no longer have any gauge freedom.

Let us now look a little bit closer at the various trajectories. Consider first the case where $S_0 \neq 0$. The second relation in (1.40) then implies that both $K_0 > 0$ and $R_0 > 0$, and (1.38) describes a straight line in polar coordinates. The particles approach each other from a direction $\phi_{\text{in}}$. At the time $T_0$ they reach a minimal distance $R_0$ with spatial orientation $\phi_0$. And finally they separate again, moving into a spatial direction $\phi_{\text{out}}$. The directions $\phi_{\text{in}}$ and $\phi_{\text{out}}$ are always antipodal, and both are orthogonal to $\phi_0$,

$$\phi_{\text{in}} = \phi_0 - \frac{\pi}{2} \text{sgn} S_0, \quad \phi_{\text{out}} = \phi_0 + \frac{\pi}{2} \text{sgn} S_0 \quad \Rightarrow \quad \phi_{\text{out}} = \phi_{\text{in}} + \pi \text{sgn} S_0. \hspace{1cm} (1.42)$$

The sign of the angular momentum $S$ tells us on which side the particles pass each other. This is a trivial scattering process, because the directions $\phi_{\text{in}}$ and $\phi_{\text{out}}$ always differ by $\pi$. Nevertheless, the scattering picture is quite useful. There will by a real scattering when the gravitational interaction is switched on.

The constant of motion $K = K_0$ has a useful interpretation in this picture. It represents the momentum which is canonically conjugate to the distance between the particles, when this distance is very large. More precisely, consider the pair of canonically conjugate variables

$$P = Q/X, \quad R = X \quad \Rightarrow \quad \{P, R\} = 1, \hspace{1cm} (1.43)$$
replacing $Q$ and $X$. Thus, $R$ also represents the distance between the particles in space, but $P$ is now the momentum which is canonically conjugate to it, in contrast to $Q$ which is not exactly conjugate. For the trajectories derived above, we have

$$R(t) = \sqrt{R_0^2 + \epsilon^2(t) K_0^2}, \quad P(t) = \frac{\epsilon(t) K_0^2}{\sqrt{R_0^2 + \epsilon^2(t) K_0^2}}. \quad (1.44)$$

In the limit $t \to \pm \infty$, or equivalently $\epsilon(t) \to \pm \infty$, we have $R(t) \to \infty$ and $P(t) \to \pm K_0$. The constant of motion $K = K_0$ is equal to the momentum $P$ when the particles are far apart. This will be useful to know when the system is quantized later on. The eigenvalue of $K$ is then the inverse radial wavelength of the wave function at spatial infinity. At the classical level, $K_0$ is the momentum of the incoming and outgoing particles in the scattering process.

A special situation arises when $S_0 = 0$. Then, according to (1.40), we have either $K_0 = 0$ or $R_0 = 0$. The first case, $S_0 = 0, K_0 = 0$, and $R_0 > 0$, is the trivial one. In (1.38), we may replace $K_0^2/S_0$ by $S_0/R_0^2$. Then we get the simple trajectory $X(t) = R_0$ and $\phi(t) = \phi_0$. The particles are at rest, at a spatial distance $R_0$ and with angular orientation $\phi_0$. The integration constant $T_0$ is in this case redundant, and the energy $M_0 = M_{\text{min}}$ is equal to the sum of the rest masses. Of course, such a trajectory only exists when both particles are massive, as otherwise we must have $K_0 > 0$, because massless particles cannot be at rest.

The more interesting case is $S_0 = 0, R_0 = 0$, and $K_0 > 0$. The argument of the $\arctan$ is then ill defined. But we can still consider the trajectory as a limit. In the limit $S_0 \to 0$ and $R_0 \to 0$, with $K_0$ and all other integration constants fixed, we get

$$X(t) = |\epsilon(t)| K_0, \quad \phi(t) = \phi_0 + \frac{\pi}{2} \text{sgn } S_0 \text{ sgn } \epsilon(t). \quad (1.45)$$

The crucial point is now that the limit still depends on $\text{sgn } S_0$, thus on the direction of the limit. But if we change the definition of the integration constants slightly, we may also write

$$\phi(t) = \begin{cases} \phi_{\text{in}} & \text{for } T < T_0, \\ \phi_{\text{out}} & \text{for } T > T_0, \end{cases} \quad \phi_{\text{out}} = \phi_{\text{in}} \pm \pi, \quad (1.46)$$

where the sign no longer matters. Clearly, this is just the definition of a straight line through the origin of the polar coordinate system. The particles approach each other from a direction $\phi_{\text{in}}$, touch each other at $t = T_0$ without interaction, and separate again into the opposite direction $\phi_{\text{out}}$. The case $S_0 = 0$ is only special because of a coordinate singularity of the phase space variables. We could avoid this by introducing a globally well defined chart on $\mathcal{P}$ and $\tilde{\mathcal{P}}$, replacing the polar coordinates $X$ and $\phi$ by Cartesian coordinates, and the radial and angular momenta $Q$ and $S$ by the appropriate conjugate variables.

**Schrödinger quantization**

Let us now, finally, perform the quantization of the free particle system. The most straightforward way is to start from the reduced classical phase space $\mathcal{P} = \{(Q, S; X, \phi)\}$ with symplectic potential (1.30),

$$\Theta = X^{-1} Q \text{ d}X + S \text{ d}\phi. \quad (1.47)$$
This is an unconstrained formulation. We can apply the standard Schrödinger quantization procedure. We use a position representation, where the wave function is given by \( \psi(x, \varphi) \), with \( x \) and \( \varphi \) being the eigenvalues of the operators \( \hat{X} \) and \( \hat{\phi} \),

\[
\hat{X} \psi(x, \varphi) = x \psi(x, \varphi), \quad \hat{\phi} \psi(x, \varphi) = \varphi \psi(x, \varphi),
\]

(1.48)

with \( x \geq 0 \) and \( \varphi \equiv \varphi + 2\pi \). Actually, the second equation is of course only well defined with \( \phi \) replaced by a periodic function of \( \phi \), as only those functions are well defined on the classical phase space. Concerning the coordinate singularity at \( x = 0 \), we shall follow the concept in [22]. It can later also be applied to the interacting system without any essential modification. We impose a periodicity condition on the wave function,

\[
\psi(x, \varphi + 2\pi) = e^{2\pi i \lambda} \psi(x, \varphi),
\]

(1.49)

and we require the wave function to be finite at \( x = 0 \). The parameter \( \lambda \) is some fixed real number. It represents a quantization ambiguity that typically arises when the classical phase space is not simply connected [23]. A special case occurs when the two particles have the same mass. They can then be regarded as identical, and replacing \( \varphi \) with \( \varphi + \pi \) already takes us back to the same state. In this case, we have the stronger relation

\[
\psi(x, \varphi + \pi) = e^{\pi i \lambda} \psi(x, \varphi),
\]

(1.50)

and the parameter \( \lambda \) defines the statistics of the particles. If \( \lambda \) is an even integer, then the particles are called bosons, and for odd integers they are called fermions. Note that this only refers to the statistics, not to the internal structure of the particles. There is no spin statistics theorem for this simple toy model.

For non-integer values of \( \lambda \), the particles are usually called anyons [21]. Anyons can only arise in three spacetime dimensions. In higher dimensions, the fundamental group of the phase space of two identical particles is \( \mathbb{Z}_2 \) rather than \( \mathbb{Z} \), which implies that \( \lambda \) has to be an integer, which is even for bosons and odd for fermions. Another special feature in three spacetime dimensions is that the non-trivial phase factor in (1.49) also shows up when the particles are not identical. Slightly abusing the language, we shall then also refer to \( \lambda \) as the statistics parameter. We should also note that for identical particles the real number \( \lambda \) is defined modulo two, otherwise modulo one.

For \( \psi(x, \varphi) \) to represent the usual probability amplitude in polar coordinates, we define the scalar product to be

\[
\langle \psi_1 | \psi_2 \rangle = \int x \, dx \, d\varphi \, \bar{\psi}_1(x, \varphi) \psi_2(x, \varphi).
\]

(1.51)

The momentum operators are defined so that the commutators are \(-i\hbar\) times the Poisson brackets,

\[
[\hat{Q}, \hat{X}] = -i\hbar \hat{X}, \quad [\hat{S}, \hat{\phi}] = -i\hbar.
\]

(1.52)

They become self-adjoint operators if we set

\[
\hat{Q} \psi(x, \varphi) = -i\hbar \frac{\partial}{\partial x} x \psi(x, \varphi), \quad \hat{S} \psi(x, \varphi) = -i\hbar \frac{\partial}{\partial \varphi} \psi(x, \varphi).
\]

(1.53)
Note the extra factor of $x$ in the operator representation of $Q$. It has to appear in the given ordering, since otherwise the resulting operator is not self-adjoint with respect to the inner product (1.51). We should also note that the operators for $Q$ and $S$ are uniquely defined by the commutation relation, up to redefinitions that can be compensated either by a phase transformation of the wave function, or by redefining the periodicity condition (1.49). In other words, the statistics parameter $\lambda$ is the only relevant quantization ambiguity.

To derive the energy eigenstates let us first consider the operator $K^2$, of which the Hamiltonian $H = F^{-1}(K^2)$ is a function. We choose the Hermitian ordering

$$\hat{K}^2 = \hat{X}^{-1}(\hat{Q}^2 + \hat{S}^2)\hat{X}^{-1}. \tag{1.54}$$

In the representation (1.53), $K^2$ is then proportional to the usual Laplacian in the polar coordinates,

$$\hat{K}^2 \psi(x, \varphi) = -\hbar^2 \Delta \psi(x, \varphi), \quad \text{where} \quad \Delta = x^{-1} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} + x^{-2} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi}. \tag{1.55}$$

To make $K^2$ self-adjoint, a boundary condition at $x = 0$ is needed. Following [22], we choose this condition so that the wave function remains finite at $x = 0$. The normalized eigenfunctions are then parametrised by two quantum numbers, $k$ and $s$, and given by the Bessel functions of the first kind,

$$\chi(k, s; x, \varphi) = \frac{1}{\sqrt{2\pi}} e^{i s \varphi} J_{|s|}(kx). \tag{1.56}$$

They are normalized so that

$$\int x \, dx \, d\varphi \, \bar{\chi}(k_1, s_1; x, \varphi) \chi(k_2, s_2; x, \varphi) = k^{-1} \delta(k_2 - k_1) \delta_{s_2 - s_1}. \tag{1.57}$$

As expected, the quantum number $k$ is the inverse radial wavelength of the wave function at infinity. We already expected the eigenvalues of $K$ to have this property, because for large distances of the particles the momentum $K$ is equal to the canonical radial momentum $P = Q/X$. The quantum number $s$ is the inverse angular wavelength, and of course it represents the eigenvalue of the angular momentum. The possible values of $s$ can be inferred from the periodicity condition (1.49). It implies

$$e^{i s(\varphi + 2\pi)} = e^{2\pi i \lambda} e^{i s \varphi} \iff s \in \lambda + \mathbb{Z}. \tag{1.58}$$

For identical particles, the stronger condition (1.50) implies

$$e^{i s(\varphi + \pi)} = e^{\pi i \lambda} e^{i s \varphi} \iff s \in \lambda + 2\mathbb{Z}. \tag{1.59}$$

It follows that the eigenvalues of $K$ and $S$ are

$$K = \hbar k, \quad S = \hbar s. \tag{1.60}$$

The spectrum of $K$ is continuous and positive, and $S$ is quantized in steps of $\hbar$, or $2\hbar$ for identical particles. The actual values of $S$ are determined by $\lambda$. In the special case of two identical bosons, for example, the total angular momentum is an even multiple of $\hbar$, and for two identical
fermions it is an odd multiple of $\hbar$. This is what we usually find for bosons and fermions in higher dimensions as well. The peculiar feature of anyons is that the spectrum of the angular momentum is shifted by a non-integer multiple $\lambda \hbar$.

The completeness relation of the eigenfunctions reads

$$\sum_s \int k \, dk \, \bar{\chi}(k, s; x_1, \varphi_1) \chi(k, s; x_2, \varphi_2) = x^{-1} \delta(x_2 - x_1) \delta(\varphi_2 - \varphi_1),$$

where $s$ takes the values given above, and $\delta_\lambda$ is a periodic delta function. It fulfills the same periodicity condition as the wave functions, hence (1.49) or (1.50), depending on whether the particles are identical or not.

The momentum eigenstates $\chi(k, s; x, \varphi)$ are also the energy eigenstates. The Hamiltonian is a function $H = F^{-1}(K^2)$ of $K$, so that the energy becomes a function of the quantum number $k$. It is useful to introduce a dispersion relation, which represents the classical relation between the momentum $K$ and the energy $H$ as a quantum relation between the inverse radial wavelength $x$ at infinity and the frequency $\omega$ of the wave function,

$$\omega(k) = \hbar^{-1} F^{-1}(\hbar^2 k^2). \quad \Rightarrow \quad \hat{H} \chi(k, s; x, \varphi) = \hbar \omega(k) \chi(k, s; x, \varphi).$$

It follows that the spectrum of $H$ coincides with its classical range $M_{\text{min}} \leq H < \infty$. And finally, we can write down the general solution to the time dependent Schrödinger equation, which is a superposition of energy eigenstates with appropriate frequencies,

$$\psi(t; x, \varphi) = \sum_s \int k \, dk \, e^{-i \omega(k)t} \hat{\psi}(k, s) \chi(k, s; x, \varphi).$$

The time-independent function $\psi(k, s)$ represents the probability amplitude in momentum space. Clearly, this is a general superposition of plane waves, written down in polar coordinates using the Bessel functions. The Schrödinger quantization of the free particles system is completely straightforward and gives the expected result.

**Dirac quantization**

There is not much more to be said about this very simple dynamical system. However, in case of the Kepler system it turns out that a simple Schrödinger quantization like this is not possible. The reasons are not immediately obvious at this point, so let us not discuss them here. Instead, let us look for an alternative method that leads to the same result. A possible alternative quantization is the Dirac procedure, based on the six dimensional extended phase space $\tilde{\mathcal{P}} = \{(M, Q, S; T, X, \phi)\}$, with symplectic potential (1.22), thus

$$\tilde{\Theta} = X^{-1} Q \, dX + S \, d\phi - M \, dT.$$ 

To quantize this phase space, we choose a wave function $\psi(\tau, x, \varphi)$, which additionally depends on the eigenvalue $\tau$ of the clock $T$. In addition to the operators (1.48) and (1.53) we have

$$\hat{T} \psi(\tau, x, \varphi) = \tau \psi(\tau, x, \varphi), \quad \hat{M} \psi(\tau, x, \varphi) = i\hbar \frac{\partial}{\partial \tau} \psi(\tau, x, \varphi).$$
All the basic operators are then self adjoint with respect to the scalar product
\[ \langle \psi_1 | \psi_2 \rangle = \int d\tau \, dx \, d\varphi \, \bar{\psi}_1(\tau, x, \varphi) \psi_2(\tau, x, \varphi). \] (1.66)

On this extended Hilbert space, we have to impose the constraint \( \mathcal{E} = K^2 - F(M) \), and the positive energy condition \( M > \sqrt{\mu \nu} \). The constraint is given as a function of \( K \) and \( M \), so it is useful to diagonalize these operators first. Clearly, the eigenstates are again given by the same Bessel function,
\[ \chi(\omega, k, s; \tau, x, \varphi) = \frac{1}{\sqrt{2\pi}} e^{-i\omega \tau} e^{is\varphi} J_{|s|}(kx). \] (1.67)

The normalization now reads
\[ \int d\tau \, dx \, d\varphi \, \bar{\chi}(\omega_1, k_1, s_1; \tau, x, \varphi) \chi(\omega_2, k_2, s_2; \tau, x, \varphi) = 2\pi \delta(\omega_2 - \omega_1) k_1^{-1} \delta(k_2 - k_1) \delta_{s_2 - s_1}. \] (1.68)

The spectrum of \( K \) and \( S \) is the same as before, and that of \( M \) is real and continuous. The eigenvalues are
\[ M = \hbar \omega, \quad K = \hbar k, \quad S = \hbar s. \] (1.69)

It is then very easy to solve the constraint. The states which are annihilated by the operator \( \mathcal{E} = K^2 - F(M) \) and satisfy the positive energy condition \( M > \sqrt{\mu \nu} \) are those where the quantum numbers are related by the dispersion relation \( \omega = \omega(k) \). A general physical state is given by
\[ \psi(\tau, x, \varphi) = \sum_s \int k \, dk \, \bar{\psi}(k, s) \chi(\omega(k), k, s; \tau, x, \varphi), \] (1.70)

where \( \psi(k, s) \) is again the wave function in momentum space. It is useful to split off a radial wave function and write the result as
\[ \psi(\tau, x, \varphi) = \sum_s \int k \, dk \, e^{-i\omega(k)\tau} e^{is\varphi} \bar{\psi}(k, s) \zeta(k, s; x), \] (1.71)

where
\[ \zeta(k, s; x) = \frac{1}{\sqrt{2\pi}} J_{|s|}(kx). \] (1.72)

We shall later compare this radial wave function to that of the Kepler system, and from this we will be able to read off the basic effects of quantum gravity in this toy model.

Formally, the physical wave functions (1.70) in the Dirac approach are exactly the same as (1.63) in the Schrödinger approach. There is only the following conceptual difference. The wave function in the Dirac approach is not a solution to the time dependent Schrödinger equation, but a solution to a generalized Klein Gordon equation, thus a constraint equation. In the Schrödinger formulation, the wave function depends by definition on the physical time \( t \), whereas in the Dirac formulation, the time dependence is encoded implicitly in the dependence of the wave function on the eigenvalue \( \tau \) of the clock, which is one of the classical phase space variables.

In the Dirac approach, there is no time evolution of the state with respect to the ADM time \( t \), because this is an unphysical coordinate. The Hamiltonian annihilates all physical states, and the Schrödinger equation is void. To say something about the physical time evolution in the Dirac quantization, we have to refer to an operator that represents a clock. The free particle system is in this context a nice toy model to explain how the problem of time arises in quantum gravity, and how it can be solved by considering operators representing physical clocks [24].
2 The classical Kepler system

Now we are going to switch on the gravitational interaction. As already mentioned in the introduction, we shall first consider an alternative way to construct the Kepler spacetime, which includes the definition of a centre of mass frame, based on the asymptotic structure of the spacetime far away from the particles. We shall thereby focus on the geometrical aspects, and skip all technical details and proofs. They can be found in [24]. We shall then introduce a set of phase space variables. They provide position and momentum coordinates, similar to those of the free particles. But at the same time they also specify the geometry of space at a moment of time, thus providing the of ADM variables of general relativity.

On the phase space spanned by these variables, we can then introduce a Hamiltonian and a symplectic potential. The former can be inferred from some straightforward geometric considerations. The latter can only be derived from the full theory of Einstein gravity as a field theory. As this is not part of this article, we can here only refer to the general derivation of the phase space structures for a multi particle model in [14]. On the given phase space, we can then derive and solve the equations of motion, and finally we shall briefly discuss the various trajectories and compare them to the free particle trajectories.

The static cone

To explain the definition of the centre of mass frame, and to show how the phase space variables of the Kepler system specify the geometry of space, it is useful first to consider a simplified example. Let us assume that the particles are at rest with respect to each other, so that the spacetime is static. It is then possible to introduce a globally defined absolute time coordinate $\tilde{T}$, and the spacetime can be foliated by the surfaces of constant $\tilde{T} = T$. In the following, a letter with a tilde always denotes a coordinate on the spacetime, whereas all other symbols are configuration or phase space variables, or functions thereof.

In the static spacetime, the surface $\tilde{T} = T$ represents an instant of time, like the equal time planes for the free particle system in figure 2. All slices have the same geometry, and the spacetime is just the direct product of a fixed space with a real line. The Einstein equations require this space to be locally flat, and there must be two conical singularities, representing the particles. In other words, the space is a conical surface with two tips. We denote them by $\pi_k$, with $k = 1, 2$. The deficit angles at the tips are equal to $8\pi G m_k$. What we would like to find is a suitable set of configuration variables, describing the geometry of such a surface. And we are also looking for an appropriate definition of a centre of mass frame.

For this purpose, we embed the conical surface in a special way into a Euclidean plane, as shown in figure 3. The first step is to introduce a geodesic $\lambda$, which connects the two tips $\pi_1$ and $\pi_2$ on the conical surface. There is always a unique such geodesic, because apart from the two tips there is no curvature. When the conical space is cut along the line $\lambda$, it becomes a surface with a hole in the middle. The points $\pi_k$ are located on a circular boundary. As shown in figure 3(a), the resulting surface is a subset of an ordinary cone, with a certain region around the tip taken away. The total deficit angle of the big cone is the sum of the deficit angles at the tips $\pi_k$, thus $8\pi G (m_1 + m_2)$.

Let us introduce on the big cone a polar coordinate system $(\tilde{R}, \tilde{\phi})$, so that $\tilde{R}$ defines the radial distance from the tip, and $\tilde{\phi}$ is an angular coordinate with a period of $2\pi$. Including the absolute
Figure 3: A double cone with two tips can be constructed by cutting off the tip from a single cone. The cut lines $\lambda$, which are afterwards identified, are two geodesics connecting the points $\pi_1$ and $\pi_2$. If the spacetime is static, then it is the direct product of this space with a real line. If the conical surface is cut not only along the geodesic $\lambda$, but also along two radial lines $\eta_1$ and $\eta_2$, then the two half spaces $\Delta_\pm$ can be embedded symmetrically into a Euclidean plane.

time $\tilde{T}$ as a third coordinate, the spacetime metric can be written as

$$ds^2 = -d\tilde{T}^2 + d\tilde{R}^2 + (1 - 4G(m_1 + m_2))^2 \tilde{R}^2 d\tilde{\phi}^2. \quad (2.1)$$

This defines a static cone with a total deficit angle of $8\pi G(m_1 + m_2)$. The conical surface in figure 3(a) becomes a surface of constant $\tilde{T} = T$ in this spacetime. Roughly speaking, we can say that the static Kepler spacetime is a cone with a tip cut off, and with the cut lines identified.

From the physical point of view, the situation is as follows. Far away from the particles, say at spatial infinity, the spacetime looks like the gravitational field of a single particle with mass $m_1 + m_2$. It is reasonable to identify the rest frame of this fictitious particle with the centre of mass frame of the universe. The centre of mass frame is thus defined by a certain property of the spacetime metric at spatial infinity, which does not directly refer to the particles. It is a kind of asymptotical flatness, which is explicitly defined in terms of the conical coordinates $(\tilde{T}, \tilde{R}, \tilde{\phi})$.

The fictitious tip of the big cone in figure 3(a) is the apparent location of the centre of mass, as seen by an observer at infinity. In analogy to the free particle system, let us identify the centre of mass frame with the reference frame of such an external observer. With respect to this reference frame, we define the absolute positions of the particles in spacetime as the conical coordinates $(T_k, R_k, \phi_k)$ of the points $\pi_k$ on the big cone. Again in analogy to the free particles, these absolute coordinates are not independent. They are specified by three independent relative coordinates.

At each moment of time, both points $\pi_k$ are located on the same surface of constant $\tilde{T} = T$. Moreover, for the cut line $\lambda$ to be of the same length on both sides of the big cone, the points $\pi_k$ must be located on two opposite, or antipodal radial lines. Hence, we must have

$$T_1 = T, \quad T_2 = T, \quad \phi_1 = \phi \pm \pi, \quad \phi_2 = \phi. \quad (2.2)$$
The variables $T$ and $\phi$ are straightforward generalizations of the corresponding free particle variables. The former defines a clock, the latter represents the angular orientation of the particles with respect to the reference frame. As a third independent coordinate, we introduce the distance $R$ between the particles. It is the length of the geodesic $\lambda$. It is a simple exercise in conical geometry to compute this. The result can be expressed as a function of the conical radial coordinates $R_k$ and the total deficit angle,

$$R^2 = R_1^2 + R_2^2 + 2 R_1 R_2 \cos(4\pi G (m_1 + m_2)). \quad (2.3)$$

Finally, there is one more consistency condition. The deficit angles at the points $\pi_k$ must be equal to $8\pi G m_k$. So far we have only fixed the total deficit angle of the big cone, which is the sum of the two individual deficit angles. We still have to fix the way this deficit angle is distributed over the two tips. This yields another relation between the radial coordinates $R_k$. It is again a simple exercise in conical geometry to show that

$$R_1 = \frac{\sin(4\pi G m_2)}{\sin(4\pi G (m_1 + m_2))} R \quad R_2 = \frac{\sin(4\pi G m_1)}{\sin(4\pi G (m_1 + m_2))} R. \quad (2.4)$$

So, what is the conclusion? There are obvious three independent variables $T$, $R$, and $\phi$. They define the geometry of the space manifold at a moment of time, and the way this space manifold is embedded into the spacetime manifold. In this sense, they provide some kind of discretized ADM variables of general relativity. On the other hand, from the particle point of view, the same variables define the relative position of the particles with respect to each other, as well as their absolute positions with respect to the centre of mass frame. The definition of the relative coordinates is very similar to (1.16) for the free particles.

We cannot say anything about the momentum variables at this point, because we are only considering the static states. We still have to generalize these concepts for moving particle. But let us stick to the static case for a moment, and let us introduce a slightly different representation of the same conical geometry of space. Let us not only cut the space manifold along the geodesic $\lambda$, but also along two antipodal radial lines $\eta_k$, extending from the points $\pi_k$ to infinity. Hence, $\eta_k$ is a line of constant $\tilde{T} = T_k$ and $\tilde{\phi} = \phi_k$ in the big cone, which lies inside the surface of constant conical time $\tilde{T} = T$. The conical surface is then divided symmetrically into two half spaces, which we denote by $\Delta_+$ and $\Delta_-$. Each half space $\Delta_\pm$ is bounded by three edges, denoted by $\eta_{1\pm}$, $\lambda_\pm$, and $\eta_{2\pm}$, and it has two corners, called $\pi_{1\pm}$ and $\pi_{2\pm}$. The two half spaces are flat and simply connected. They can be embedded into a Euclidean plane in the following unique way, which is shown in figure 3(b). The apparent position of the fictitious centre of mass, hence the tip of the big cone, is mapped onto the origin of the plane. Moreover, the edges $\lambda_\pm$ are mapped onto two parallel straight lines, with angular direction $\phi$. This fixes the embedding of the half spaces into the plane completely.

To describe the embedding more explicitly, it is convenient to think of the Euclidean plane as an equal time plane in Minkowski space. We choose it to be the plane with $\gamma_0$-coordinate $T$. The geometry of the two half spaces can then be specified as follows. The edges $\lambda_\pm$ are represented by two spacelike Minkowski vectors $z_\pm$. They are given by

$$z_\pm = R \gamma(\phi). \quad (2.5)$$

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The end points of the edges $\lambda_{\pm}$ are the corners $\pi_{k\pm}$. They are located at the following points in Minkowski space,

$$x_{1\pm} = T \gamma_0 - R_1 \gamma (\phi \mp 4\pi G m_1), \quad x_{2\pm} = T \gamma_0 + R_2 \gamma (\phi \pm 4\pi G m_2), \quad (2.6)$$

where the coordinates $R_k$ given by (2.4). One can easily verify that this implies $z_{\pm} = x_{2\pm} - x_{1\pm}$.

And finally, the edges $\eta_{k\pm}$ are radial lines, extending from the corners $\pi_{k\pm}$ to infinity. A radial line in Minkowski space is a geodesic which is orthogonal to the $\gamma_0$-axis, and intersects with it if it is extended beyond its end point.

The advantage of figure 3(b) is that it uses less dimensions to visualize the conical geometry of space. In contrast to figure 3(a), we do not need an auxiliary third Euclidean dimension to embed the big cone. This will be useful when we now consider the moving particles, because then the space is no longer flat. We can also read off the deficit angles of the particles directly from figure 3(b). The deficit angle of the particle $\pi_1$ is the angle between the edges $\eta_{1+}$ and $\eta_{1-}$ in the plane. According to (2.6), this angle is equal to $8\pi G m_1$. The deficit angle of the particle $\pi_2$ is the angle between the edges $\eta_{2-}$ and $\eta_{2+}$ on the other side, which is equal to $8\pi G m_2$.

The disadvantage of figure 3(b) is that the definition of the centre of mass frame is less obvious. However, it is actually only the relation between the geometry of space and the centre of mass frame, which we need in the following. And this can still be read off from figure 3(b). First, we need to know which time slice is represented by the given space. This is specified by the clock $T$, and according to (2.6), this is also the $\gamma_0$-coordinate of the plane shown figure 3(b). And secondly, we need to know the spatial orientation of the particles with respect to the reference frame. This is specified by the symmetry axis of figure 3(b). It is the solid line in the middle, which is a radial line with angular direction $\gamma(\phi)$.

The spinning cone

When the particles are moving, we have to generalize two aspects of this construction. The geometry of space is no longer constant in time. The previously introduced relative position co-ordinates become time dependent, and we also have to introduce conjugate momentum variables. The time evolution will finally be provided by some Hamiltonian, which we still have to derive. Moreover, we also have to replace the conical metric (2.1) of the big cone by a more general conical geometry. Let us first consider this second aspect.

If the particles are in motion with respect to each other, then the fictitious centre of mass particle receives a variable mass $M$ and a variable spin $S$, where $M$ is the total energy and $S$ is the total angular momentum of the system. The gravitational field of such a particle is a spinning cone [5]. Using the same conical coordinates as before, the metric of a spinning cone can be written as

$$ds^2 = -(d \tilde{T} + 4GS d \tilde{\phi})^2 + d \tilde{R}^2 + (1 - 4GM)^2 \tilde{R}^2 d \tilde{\phi}^2. \quad (2.7)$$

This is obviously a generalized version of (2.1). The previously considered static case is recovered if we set $M = m_1 + m_2$ and $S = 0$. But nevertheless, we can still use the conical coordinates $(\tilde{T}, \tilde{R}, \tilde{\phi})$ to define the centre of mass frame of the universe, and we may also identify this with the reference frame of some external observer sitting at infinity.

To visualize the geometry of the spinning cone, let us consider a surface of constant conical time $\tilde{T} = T$. We can still say that each such surface represents an instant of time in the centre of
mass frame. Moreover, all such surfaces have the same geometry, as long as we do not insert the moving particles. The spinning cone is no longer static, but still stationary. A surface of constant conical time is not flat, but it can locally be embedded into Minkowski space. The following local isometry maps a segment of the spinning cone into Minkowski space. We fix some angular direction \( \phi_0 \) and define

\[
(\tilde{T}, \tilde{R}, \tilde{\phi}) \mapsto (\tilde{T} + 4GS (\tilde{\phi} - \phi_0)) \gamma_0 + \tilde{R} \gamma (\tilde{\phi} - 4GM (\tilde{\phi} - \phi_0)).
\]

(2.8)

One can easily check that this is indeed an isometry. It is only a local isometry, because it does not respect the periodicity of \( \tilde{\phi} \). A surface of constant \( \tilde{T} = T \) of the spinning cone is mapped onto a screw surface in Minkowski space.

A screw surface in Minkowski space is defined by a family radial lines, where the \( \gamma_0 \)-coordinate increases linearly with the angular direction. The slope of this screw surface is determined by the parameters \( M \) and \( S \). If we pass once around the spinning cone, thus if we increase \( \tilde{\phi} \) by \( 2\pi \), then the total angle covered by the screw surface in Minkowski space is \( 2\pi - 8\pi GM \), and the amount that we move forward in time is \( 8\pi GS \). The spinning cone has a deficit angle of \( 8\pi GM \) and a time offset of \( 8\pi GS \). In units where the velocity of light is one, the latter is a length or time if \( S \) has the dimension of an angular momentum.

The central axis of the spinning cone at \( \tilde{R} = 0 \) is mapped onto the \( \gamma_0 \)-axis in Minkowski space. This is the apparent world line of the centre of mass, as seen by an observer at infinity. There is a critical radius \( R_0 = 4GS/(1 - 4GM) \). For \( \tilde{R} < R_0 \), the screw surface is timelike, at \( \tilde{R} = R_0 \) it is lightlike, and for \( \tilde{R} > R_0 \) it is spacelike. There are thus closed timelike curves in the spinning cone, passing through the region \( \tilde{R} < R_0 \). When we insert the particles, this critical region will be cut away, so that the actual Kepler spacetime has no closed timelike curves. Outside the critical region the causal structure of the spinning cone is well defined.

To insert the particles, we have to generalize the construction in figure 5(b). Remember that the plane in which this construction took place was an equal time plane in Minkowski space, with \( \gamma_0 \)-coordinate \( T \). And remember also that the symmetry axis was a radial line in this plane, with angular direction \( \gamma(\phi) \). We shall now lift this picture into the third dimension. The half spaces \( \Delta_{\pm} \) are tilted in a certain way, while the symmetry of the figure is preserved. The deformation is so that finally the half spaces become two segments of the previously considered screw surface. Hence, they fit into the spinning cone as a surface of constant conical time \( \tilde{T} = T \).

As indicated in figure 4, the deformation is as follows. The corners \( \pi_{k \pm} \) are raised and lowered symmetrically by \( 2\pi GS \). Moreover, the mass parameters \( m_k \) appearing in (2.7), which define the deficit angles of the particles, are replaced by two independent energy variables \( M_k \). This is because for a moving particle the deficit angle is actually not proportional to its mass, but proportional to its energy \([13]\). We shall look at this more closely in a moment. As we want the symmetry axis of the figure to be preserved, the locations of the corners \( \pi_{k \pm} \) are then given by

\[
\begin{align*}
x_{1\pm} &= (T \pm 2\pi GS) \gamma_0 - R_1 \gamma (\phi \mp 4\pi GM_1), \\
x_{2\pm} &= (T \mp 2\pi GS) \gamma_0 + R_1 \gamma (\phi \pm 4\pi GM_2).
\end{align*}
\]

(2.9)

The radial coordinates \( R_k \) are still to be determined. We recover the static case (2.7) if we set \( M_k = m_k \) and \( S = 0 \). The edges \( \eta_{k \pm} \) are still defined as the unique radial lines in Minkowski space which pass through the corners \( \pi_{k \pm} \). As they are now located at different time levels in the
embedding Minkowski space, each half space becomes a segment of a screw surface. The time offset of each half space is $4\pi GS$, and the angle in Minkowski space covered by each half space is $\pi - 4\pi GM$, where $M = M_1 + M_2$.

Let us glue the two half spaces together along the edges $\eta_{k\pm}$. The result is a screw surface with a deficit angle of $8\pi GM$ and a time offset of $8\pi GS$. We can still use figure 3(a) as a schematic representation of this surface. It is a conical surface with a hole in the middle or, roughly speaking, a cone with the tip cut off. It can be embedded into a spinning cone, as a surface of constant conical time $\tilde{T} = T$. Thus it represents an instant of time in the centre of mass frame, and the clock $T$ represents the absolute time. The angular orientation of this embedding is fixed in the same way as before. The geodesics $\eta_k$ are two antipodal radial lines with conical coordinates $\tilde{t} = T_k$ and $\tilde{\phi} = \phi_k$, where

$$T_1 = T, \quad T_2 = T, \quad \phi_1 = \phi \pm \pi, \quad \phi_2 = \phi.$$  \hspace{1cm} (2.10)

The edges $\lambda_{\pm}$ are the geodesics in Minkowski space, connecting the corners $\pi_{k\pm}$. As these are in general not contained in the screw surfaces, we actually have to deform the half spaces slightly in the neighbourhood of these edges. But this does not affect the embedding into the spinning cone. We can still assign two Minkowski vectors $z_{\pm}$ to the edges $\lambda_{\pm}$. They must be spacelike for the half spaces to be spacelike surfaces. And they can still be chosen so that both have the same direction $\phi$, parallel to the symmetry axis. Moreover, we already know that the $\gamma_0$-components
of these vectors must be equal to the time offsets $4\pi GS$ of the two half spaces. Hence, we have

$$z_\pm = \sqrt{R^2 + (4\pi GS)^2} \gamma(\phi) \mp 4\pi GS \gamma_0,$$

where $R$ is again the distance between the particles. On the other hand, also have $z_\pm = x_{2\pm} - x_{1\pm}$, which implies that the radial coordinates $R_k$ are given by the following generalization of (2.4),

$$R_1 = \frac{\sin(4\pi GM_2)}{\sin(4\pi G(M_1 + M_2))} \sqrt{R^2 + (4\pi GS)^2},$$
$$R_2 = \frac{\sin(4\pi GM_1)}{\sin(4\pi G(M_1 + M_2))} \sqrt{R^2 + (4\pi GS)^2}.$$

(2.12)

So, after this somewhat technical construction, what did we get? There are now six independent configuration variables $M_1, M_2, S, T, R,$ and $\phi$. From the point of view of general relativity, they specify the geometry of space at a moment of time, providing a kind of discretized ADM variables. From the particle point of view, they define both the relative position of the particles with respect to each other, as well as their absolute positions with respect to the centre of mass frame.

The relative position of the particles is specified by the geodesic distance $R$ and the angular orientation $\phi$. The absolute positions are given by the conical coordinates $(T_k, R_k, \phi_k)$ of the points $\pi_k$ in the spinning cone. All this is analogous to the previously considered static case, and also to the free particles, where the relative coordinates are $X$ (or $R$) and $\phi$, and the absolute positions are given by (1.17). What makes the definitions a little bit more complicated here is that the reference frame is a conical frame, and not a Minkowski frame. What we do not see so far is in which sense the energy and momentum variables are conjugate to the position coordinates, how they evolve in time, and what is also still missing is a variable that replaces the radial momentum $Q$.

**Transition functions**

Let us have a closer look at the way the edges in figure 4 are glued together. The embedding Minkowski space can locally be identified with the spacetime manifold, if the world lines are excluded. More precisely, it provides an atlas of the spacetime manifold in the neighbourhood of the embedded space manifold at the given moment of time. The atlas consists of two charts, and each chart contains one half space as a spacelike surface. There are three overlap regions between the two charts, corresponding to the edges $\eta_1\pm, \lambda_\pm,$ and $\eta_2\pm$. The transition functions in these overlap regions are isometries of the embedding Minkowski space, hence Poincaré transformations.

Since in the following only the rotational components are of interest, let us ignore the translations that are involved. We can then say that the edge $\eta_1-$ is mapped onto the edge $\eta_1+$ by a certain Lorentz rotation. It can be represented by an element $h_1 \in SL(2)$ of the spinor representation of the three dimensional Lorentz group. Similarly, the edge $\eta_2-$ is also mapped onto the edge $\eta_2+$ by some Lorentz rotation, which is given by $h_2 \in SL(2)$. And finally, $\lambda_-$ is mapped onto $\lambda_+$ by yet another Lorentz rotation $g \in SL(2)$. The transition functions tell us how the half
spaces are to be glued together along the edges. The question is thus, are they already determined by the variables introduced so far, or is there still some ambiguity?

Consider first the edges \( \eta_{1}^{-} \) and \( \eta_{1}^{+} \), on the right hand side of figure 4. They are obviously mapped onto each other by a rotation about the \( \gamma_{0} \)-axis by \( 8 \pi GM_{1} \) in clockwise direction. Similarly, \( \eta_{2}^{-} \) is mapped onto \( \eta_{2}^{+} \) by a rotation about the \( \gamma_{0} \)-axis by \( 8 \pi GM_{2} \) in counter clockwise direction. It is shown in the appendix that the group elements representing these rotations are

\[
h_{1} = e^{4 \pi GM_{1} \gamma_{0}}, \quad h_{2} = e^{-4 \pi GM_{2} \gamma_{0}}.
\] (2.13)

To be precise, there are more general Lorentz rotations mapping the given edges onto each other. We may, for example, multiply \( h_{1} \) from the left with a boost whose axis is parallel to the edge \( \eta_{1}^{-} \), or from the right with boost whose axis is parallel to the edge \( \eta_{1}^{+} \). However, there is yet another consistency condition. The two half spaces have to fit together as two halves of a single screw surface, without a kink, and this is only the case if the transition functions are pure rotations.

Regarding the transition function \( g \), which specifies how the edges \( \lambda^{+} \) and \( \lambda^{-} \) are glued together, there is no such additional restriction. The only consistency condition is that the Lorentz rotation \( g \) maps the Minkowski vector \( z^{-} \) onto the vector \( z^{+} \), hence

\[
z^{+} = g^{-1} z^{-} g \quad \iff \quad z^{-} = g z^{+} g^{-1}.
\] (2.14)

There is a one parameter family of such group elements, and therefore there is one additional degree of freedom that has to be fixed. As an ansatz, we expand the matrix \( g \) in terms of the unit and gamma matrices, using \( \gamma_{0} \), \( \gamma(\phi) \) and \( \gamma'(\phi) \) as an orthonormal basis,

\[
g = u 1 + w \gamma_{0} + q \gamma(\phi) + s \gamma'(\phi).
\] (2.15)

This basis is adapted to the definition (2.11) of the vectors \( z_{\pm} \), so that the equations (2.14) for the coefficients \( u, w, q, s \) become as simple as possible. Additionally, we have the equation \( \det(g) = 1 \). The resulting system of quadratic equations can be easily solved. There is a one-parameter family of solutions, which is given by \( w = 0 \) and

\[
u = \frac{\sqrt{R^2 + (4 \pi GS)^2}}{\sqrt{R^2 - (4 \pi GQ)^2}}, \quad q = \frac{4 \pi GQ}{\sqrt{R^2 - (4 \pi GQ)^2}}, \quad s = \frac{4 \pi GS}{\sqrt{R^2 - (4 \pi GQ)^2}}.
\] (2.16)

The reason for choosing the parameter \( Q \) in this particular way will become clear in a moment. The motivation is to have a certain symmetry between the spatial components \( q \) and \( s \) of the matrix \( g \) on one side, and the variables \( Q \) and \( S \) on the other side.

Due to the square roots in the denominator, there is obviously a restricted range \(-R < 4 \pi GQ < R\) for the new parameter. To avoid such a non-trivial restriction on the range of the phase space variables later on, it is useful to replace the variable \( R \) by a new variable \( X \), so that

\[
R^2 = X^2 + (4 \pi GQ)^2.
\] (2.17)

The expression for the transition function \( g \) then simplifies slightly and becomes

\[
g = U \ 1 + 4 \pi G \frac{Q \gamma(\phi) + S \gamma'(\phi)}{X},
\] (2.18)
where $U \geq 0$ is defined so that the $g$ has a unit determinant,

$$U^2 = 1 + (4\pi G)^2 \frac{Q^2 + S^2}{X^2}. \quad (2.19)$$

Here we should note the similarity to the formula (1.20), defining the momentum vectors of the free particles. If the group element $g$ is considered as the spatial momentum of the particles, the $Q$ and $S$ are generalized versions of the radial and angular momentum.

To see what kind of group element $g$ it is, let us rewrite it in the following alternative way. We replace the Cartesian coordinates $Q$ and $S$ by polar coordinates $K$ and $\beta$, so that

$$4\pi G Q = X \sinh(4\pi G K) \cos(\beta - \phi), \quad 4\pi G S = X \sinh(4\pi G K) \sin(\beta - \phi). \quad (2.20)$$

This is obviously a generalization of (1.19), to which it reduces in the limit $G \to 0$. It follows that $U = \cosh(4\pi G K)$, and the transition function becomes

$$g = e^{4\pi G K \gamma(\beta)} = \cosh(4\pi G K) \mathbf{1} + \sinh(4\pi G K) \gamma(\beta). \quad (2.21)$$

It represents a boost with rapidity $4\pi G K$ and angular direction $\beta$. There is thus a certain relation between the transition functions $h_k$ and the energies $M_k$ of the particles on one side, and the transition function $g$ and the spatial momentum $K$ of the particles on the other side. Of course, we thereby assume that $K$ still defines the spatial momentum, in the same way as previously for the free particles.

**Mass shell constraints**

To see that $M_k$ are in fact the energies, and $K$ is the spatial momentum of the particles, let us now have a closer look at the conical singularities, which arise at the corners of the half spaces in figure 4 when they are glued together. We somehow have to ensure that the spacetime metric in the neighbourhood of a world line represents the gravitational field of a point particle with a fixed mass $m_k$. It has to be a cone with a deficit angle of $8\pi G m_k$, when this angle is measured in the rest frame of the particle. In the static case, we can read off this deficit angle directly from the geometry of space in figure 4, or figure 3(b). In general, however, we first have to transform to the rest frame of the particle, as otherwise the deficit angle is proportional to the energy and not to the rest mass.

An alternative way to derive the deficit angle is to consider the holonomy of the particle. This is, by definition, the Lorentz rotation that acts on a spacetime vector which is transported once around the particle, say, in clockwise direction. If the deficit angle of the particle $\pi_k$ is $8\pi G m_k$, then the holonomy is a rotation by $8\pi G m_k$ about some timelike axis. The axis is parallel to the world line, and thus the holonomy also defines the direction of motion of the particle. Massless particles can also be included. The holonomy is then a null rotation, and the world line is lightlike. The holonomy is in this sense a generalized, group valued momentum of the particle [13].

If we think of the embedding Minkowski space in figure 4 as a spacetime atlas consisting of two charts, then a spacetime vector is simply represented by a Minkowski vector, which is attached to some point on one of the two half spaces. As long as we stick to this half space, the
parallel transport is trivial. But whenever we pass across one of the edges $\eta_k$ or $\lambda$ from $\Delta_-$ to $\Delta_+$, we have to act on the vector with the appropriate transition function, hence $h_k$ or $g$. And when we pass from $\Delta_+$ to $\Delta_-$, we have to act on the vector with the inverse transition function. There are then all together four holonomies that can be defined. We can choose the particle $\pi_k$ to be surrounded, and we can choose the half space $\Delta_{\pm}$ in which the path begins and ends.

Let us define $u_{k\pm} \in \mathbb{SL}(2)$ to be the Lorentz rotation acting on a vector which is first defined in the half space $\Delta_{\pm}$, and then transported in clockwise direction around the particle $\pi_k$. Sorting out the factor ordering and the signs, we find that

$$u_{1+} = g^{-1} h_1, \quad u_{1-} = h_1 g^{-1}, \quad u_{2+} = h_2^{-1} g, \quad u_{2-} = g h_2^{-1}.$$  \quad (2.22)

For $u_{k\pm}$ to be a rotation by $8\pi G m_k$ in clockwise direction, it has to be an element of the conjugacy class of $e^{4\pi G m_k \gamma_0}$ in $\mathbb{SL}(2)$. This special element represents a clockwise rotation by $8\pi G m_k$ about the $\gamma_0$-axis. All others are obtained by acting on this with a proper Lorentz rotation, hence a conjugation with some other element of the Lorentz group. The conjugacy class is defined by the following mass shell and positive energy condition \cite{13,10}

$$u_k = \frac{1}{2} \text{Tr}(u_{k\pm}) = \cos(4\pi G m_k), \quad \frac{1}{2} \text{Tr}(u_{k\pm} \gamma^0) > 0.$$  \quad (2.23)

Since the trace of the holonomy is independent of the factor ordering, this is one equation to be satisfied for each particle. Actually, each particle has only one holonomy $u_k$. The two different values $u_{k+}$ and $u_{k-}$ arise because the same physical object is represented in two coordinate charts.

If the holonomy is interpreted as a generalized momentum, then the mass shell constraints are obviously generalizations of the free particle constraints \cite{13}. There is now, however, a restriction on the mass parameters $m_k$, which has no counterpart for the free particles. The deficit angle of a cone must be smaller then $2\pi$, and consequently the rest mass of a particle is bounded from above by $1/4G = M_{pl}/4$, where $M_{pl} = 1/G$ is the Planck mass. At this upper bound, the cosine in (2.23) takes its minimum $-1$, and the holonomy becomes a full rotation by $2\pi$. For a massless particle, on the other hand, the cosine is equal to one, which means that the holonomy is a null rotation.

The same restriction applies to the total energy $M$, which defines the total deficit angle of universe. We’ll see later on that the energy $M$ is in fact also bounded from below by $M_{\text{min}} = m_1 + m_2$, which is the total energy for a static state. Hence, the allowed range for the mass parameters is

$$m_k \geq 0, \quad m_1 + m_2 < M_{pl}/4.$$  \quad (2.24)

To see that the free particle mass shell constraints are recovered in the limit $G \to 0$, let us write the holonomies as functions of the configuration variables. It is thereby useful to introduce the following notation. We define a modified set of trigonometric functions with rescaled arguments,

$$\cs \varrho = \cos(2\pi G \varrho), \quad \sn \varrho = \frac{\sin(2\pi G \varrho)}{2\pi G}, \quad \tn \varrho = \frac{\tan(2\pi G \varrho)}{2\pi G}.$$  \quad (2.25)

Some useful properties of these functions are that the relation $\tn \varrho = \sn \varrho / \cs \varrho$ remains valid, and in the limit $G \to 0$ we have $\cs \varrho \to 1, \sn \varrho \to \varrho, \tn \varrho \to \varrho$. We also introduce an analogous
set of hyperbolic functions, but this time it is useful to rescale the argument by a different factor,

\[
\csh \varrho = \cosh(4\pi G \varrho), \quad \snh \varrho = \frac{\sinh(4\pi G \varrho)}{4\pi G}, \quad \tanh \varrho = \frac{\tanh(4\pi G \varrho)}{4\pi G}.
\] (2.26)

Again, we have \( \tanh \varrho = \snh \varrho / \csh \varrho \), and in the limit \( G \to 0 \) we get \( \csh \varrho \to 1, \snh \varrho \to \varrho \), \( \tanh \varrho \to \varrho \). As an example for the application of these rescaled function, consider the definition (2.20), relating \( Q \) and \( S \) to \( K \) and \( \beta \). They simplify to

\[
Q = X \snh K \cos(\beta - \phi), \quad S = X \snh K \sin(\beta - \phi),
\] (2.27)

and \( K \) can be written as a function of \( Q, S, \) and \( X \),

\[
\snh^2 K = \frac{Q^2 + S^2}{X^2}.
\] (2.28)

According to the general rule \( \snh \varrho \to \varrho \), this reduces to (2.21) in the limit \( G \to 0 \). Similar rules apply to all other formulas below. They always reduce to the free particle counterparts in the limit where the gravitational interaction is switched off.

To express the holonomies and finally the mass shell constraints in terms of the energy and momentum variables, it is useful to make the same redefinition that we previously also made for the free particles. We replace the mass parameters \( m_k \) by a total mass \( \mu \) and a relative mass \( \nu \), assuming without loss of generality that \( m_2 \geq m_1 \),

\[
\mu = m_2 + m_1, \quad \nu = m_2 - m_1, \quad 0 \leq \nu \leq \mu < M_{Pl}/4.
\] (2.29)

Like for the free particles, the special cases are \( \nu = 0 \), where both particles have the same mass, and \( \nu = \mu \), where at least one of the particles is massless. The only new feature is the upper bound for \( \mu \), which goes to infinity in the limit \( G \to 0 \). Similarly, we also replace the energy variables \( M_k \) by the total energy \( M \) and a relative energy \( V \), so that

\[
M = M_2 + M_1, \quad V = M_2 - M_1, \quad |V| < M < M_{Pl}/4.
\] (2.30)

The upper bound on \( M \) follows from the fact that the total deficit angle of the spinning cone must be smaller than \( 2\pi \). The restriction on \( V \) is the positive energy condition, which implies that \( M_k > 0 \) for both particles.

Using all this, we can finally express the transition functions (2.13) and (2.21) in terms of the new energy and momentum variables. What we get is

\[
h_1 = \cs(M - V) 1 + 2\pi G \sn(M - V) \gamma_0, \\
h_2 = \cs(M + V) 1 - 2\pi G \sn(M + V) \gamma_0, \\
g = \csh K 1 + 4\pi G \snh K \gamma(\beta).
\] (2.31)

It is then not difficult to evaluate the traces of the holonomies, and to express the mass shell constraints in terms of the same variables. They are given by

\[
u_1 = \csh K \cs(M - V) = \cs(\mu - \nu), \quad \nu_2 = \csh K \cs(M + V) = \cs(\mu + \nu).
\] (2.32)
This is the same as (2.36) in [13], if $K$ is interpreted as the common spatial momentum of both particles in the centre of mass frame, and $(M \pm V)/2$ are the energies of the particles, which are in general different. Finally, we have to rescale the actual mass shell constraints by an appropriate power of $G$, in order to get the correct limit $G \to 0$,
\[
C_1 = \frac{\cosh K \cos(M - V) - \cos(\mu - \nu)}{(4\pi G)^2}, \quad C_2 = \frac{\cosh K \cos(M + V) - \cos(\mu + \nu)}{(4\pi G)^2}.
\] (2.33)

Expanding this up to the second order in $G$, we recover the free particle constraints (1.11) in the limit $G \to 0$. And the same applies to the positive energy condition (2.30), where the upper bound on $M$ goes to infinity.

We can then proceed in the same way as before. We use one of the mass shell constraints to eliminate the relative energy $V$. There is then a single mass shell constraint left, which is going to be the generator of the time evolution. And we also get an even dimensional phase space which is spanned by six independent variables. The first step is again to define the linear combinations
\[
D' = C_2 - C_1 = \frac{\cosh K \sin M \sin V - \sin \mu \sin \nu}{2}, \quad E' = C_2 + C_1 = \frac{\cosh K \cos M \cos V - \cos \mu \cos \nu}{2(2\pi G)^2}.
\] (2.34)

Here we used some trigonometric identities to simplify the results. In the limit $G \to 0$, these constraints are equal to (1.12). However, it is now a little bit more complicated to decouple the variables $K$ and $V$. We have to take another linear combination, namely
\[
D = D' + (2\pi G)^2 \tan M \tan V E' = \frac{\tan \mu \tan \nu - \tan M \tan V}{2}.
\] (2.35)

Note that the functions $1/\cos \mu$ and $1/\cos \nu$, and similarly $\tan M$ and $\tan V$ are well defined within the range (2.29) and (2.30). The first poles of these functions are just outside the allowed range of the arguments. Now, the positive energy condition implies $\tan M > 0$, and the function $\tan$ is invertible within the allowed range of $V$. We can solve the equation $D = 0$ for $V$, setting
\[
V = \tan^{-1} \left( \frac{\tan \mu \tan \nu}{\tan M} \right).
\] (2.36)

The function $\tan^{-1}$ is the inverse of $\tan$,
\[
\tan^{-1} \varrho = \frac{\arctan(2\pi G \varrho)}{2\pi G} \Rightarrow -M_{Pl}/4 < \tan^{-1} \varrho < M_{Pl}/4,
\] (2.37)

and in the limit $G \to 0$ we have $\tan^{-1} \varrho \to \varrho$. Using this and the fact that $\tan$ and $\tan^{-1}$ are both monotonically increasing, the positive energy condition becomes a non-trivial condition to be imposed on $M$, namely
\[
|\tan V| = \frac{\tan \mu \tan \nu}{\tan M} < \tan M \quad \Leftrightarrow \quad \tan^{-1} \left( \sqrt{\tan \mu \tan \nu} \right) < M < M_{Pl}/4.
\] (2.38)

This is obviously a generalized version on (1.15), with all mass parameters and energy variables replaced by their rescaled tangent. This is in fact a rather general rule, which tells us how to
obtain the various structures of the Kepler system from those of the free particle system. Consider for example the mass shell constraint that remains after eliminating $V$ from (2.34). To derive this, we have to take yet another linear combination, namely

$$
E = \cosh K \frac{cs}{2cs^2 M} \mathcal{E}' + \frac{cosh K \sn}{2sn^2 M} \mathcal{D}'.
$$  (2.39)

Again, the coefficients are well defined, because the range (2.30) of $M$ is exactly the interval where both $sn M$ and $cs M$ are different from zero. Evaluating this linear combination, $V$ drops out, and after some trigonometric simplifications we get

$$
\mathcal{E} = \text{snh}^2 K - \Lambda^2 F(M) \approx 0.
$$  (2.40)

Here, $0 < \Lambda \leq 1$ is a constant, which only depends on the mass parameters $\mu$ and $\nu$, and $F$ is the same function that also appears in (1.14), but once again with the masses $\mu$, $\nu$, and the energy $M$ are replaced by $tn \mu$, $tn \nu$, and $tn M$, respectively,

$$
\Lambda = cs \mu cs \nu, \quad F(M) = \frac{(tn^2 M - tn^2 \mu)(tn^2 M - tn^2 \nu)}{4 tn^2 M}.
$$  (2.41)

Roughly speaking, the rule is to replace all mass and energy quantities $\rho$ by $tn \rho$, and to replace the momentum $K$ by $\text{snh} K$. Then we obtain the mass shell constraints of the Kepler system from those of the free particle system, and also, for example, the relation (2.28) between the radial momentum $Q$, the angular momentum $S$, and the total spatial momentum $K$ of the particles. As all these modifications become identities in the limit $G \to 0$, we can say that the Kepler system is a deformation of the free particle system, and Newton’s constant is the deformation parameter. We’ll now see that this also applies to the various phase space structures, such as the symplectic potential, the Poisson brackets, and the Hamiltonian.

The phase space

So far, we have seen that all physically relevant quantities, including the geometry of space, the positions of the particles with respect to the centre of mass frame, as well as the holonomies and the mass shell constraints, can be expressed as a function of six independent variables, the energy and momentum variables $M$, $Q$, $S$, and the time and position variables $T$, $X$, $\phi$. In analogy to the free particles, we define the extended phase space

$$
\tilde{\mathcal{P}} = \{ (M, Q, S; T, X, \phi) \mid X \geq 0, \quad \phi \equiv \phi + 2\pi \}.
$$  (2.42)

Note that we do not impose any restriction on the energy $M$ at this point. The positive energy condition (2.38) is imposed together with the constraint (2.40), and this defines the physical subspace. The Hamiltonian on $\tilde{\mathcal{P}}$ is again proportional to the mass shell constraint. Thus if we introduce a multiplier $\zeta$, we have in analogy to (1.24)

$$
\tilde{H} = \zeta \mathcal{E}.
$$  (2.43)

To derive the equations of motion, we also have to know the symplectic structure on $\tilde{\mathcal{P}}$. But so far, there is no natural way to define it, and there is also no reason why it should be the same
as that of the free particle system. Since we do not want to make any additional assumptions, we can only derive the symplectic structure from the underlying field theory of Einstein gravity. Thus, we have to apply a straightforward phase space reduction to the Einstein Hilbert action.

The actual derivation is rather involved and technical, but on the other hand it can be carried out without further complications for a general multi particle model. It is therefore given in a separate article [10]. We are here not going to say anything in detail about this derivation. We just take the general result, and adapt it to the special case of a two particle system. The general expression (3.9) in [10], adapted to the special situation given in figure 4, is

\[ \tilde{\Theta} = - \sum_k M_k \left( dT_k + 4GS d\phi_k \right) - \frac{1}{8\pi G} \text{Tr}(g^{-1}dg z_+). \]

(2.44)

Here, \( M_k \) are the energies of the particles \( \pi_k, T_k \) and \( \phi_k \) are their absolute positions with respect to the conical reference frame, \( z_+ \) is the relative position vector of the particles in the half space \( \Delta_+ \), and \( g \) is the transition function mapping the edge \( \lambda_- \) onto \( \lambda_+ \).

This expression does not look symmetric, but in fact it is, which can be seen by replacing \( z_+ \) and \( g \) with \( z_- \) and \( g^{-1} \). To express the symplectic potential in terms of our six independent phase space variables, we have to insert the expressions (2.10) for \( T_k \) and \( \phi_k \), (2.11) for \( z_\pm \), and (2.18) for \( g \). The result is very simple and reads, up to a total derivative that can be neglected,

\[ \tilde{\Theta} = X^{-1}Q \, dX + (1 - 4GM) \, S \, d\phi - M \, dT. \]

(2.45)

This is almost the same as the free particle expression (1.22). There is only one modification. The angular momentum \( S \), which is conjugate to the orientation \( \phi \) of the particles, is rescaled by a factor \( 1 - 4GM \). This factor takes values between zero and one, and obviously it has to do with the conical geometry of the spacetime at spatial infinity. As a consequence, we have the following almost canonical Poisson brackets

\[ \{ M, T \} = -1, \quad \{ Q, X \} = X, \]

(2.46)

but the following non-canonical brackets involving the angular momentum \( S \),

\[ \{ S, \phi \} = \frac{1}{1 - 4GM}, \quad \{ S, T \} = -\frac{4GS}{1 - 4GM}. \]

(2.47)

The last one results from an off-diagonal term in the symplectic two-form \( \tilde{\Omega} = d\tilde{\Theta} \), involving a product of \( dM \) and \( d\phi \). The brackets can most easily be derived if we notice that the canonically conjugate angular momentum is actually

\[ J = (1 - 4GM) \, S \quad \Rightarrow \quad \{ J, \phi \} = 1. \]

(2.48)

All other brackets with \( J \) are zero. The brackets (2.47) are then easily found by expressing \( S \) as a function of \( J \) and \( M \). But let us nevertheless stick to \( S \) as one of the basic phase space variables, because it has an immediate geometric interpretation, defining the time offset of the spinning cone. It also shows up in the transition function \( g \) defined in (2.18), and in the mass shell constraint (2.40), implicitly through the definition (2.28) of \( K \).

Now we have all the structures at hand which we need to describe the kinematical and dynamical features of the Kepler system. In particular, we can now derive the equations of motion,
study the classical trajectories, and finally we can quantize the Kepler system. At each step, the result will be a deformed version of the corresponding free particle result. The most interesting feature is thereby the deformation of the symplectic structure by the factor in front of the angular momentum. Apparently, this is only a marginal modification. However, it turns out that it is responsible for some unexpected effects at the quantum level. For example, the particles can no longer be localized in space, and they cannot come closer to each other than a specific distance.

But for the moment we shall stick to the classical phase space. Some basic features of the Kepler system can be inferred immediately from the given phase space structures. For example, we have a two dimensional rigid symmetry group of time translations \( T \mapsto T - \Delta T \) and spatial rotations \( \phi \mapsto \phi + \Delta \phi \). The Hamiltonian and the symplectic potential are both invariant. At the spacetime level, these are the Killing symmetries of the spinning cone. The rigid symmetries are the possible translations and rotations of the universe with respect to the reference frame \([14]\). The conserved charges associated with these symmetries are obviously \( M \) and \( J \).

Actually, we expected \( S \) to be the angular momentum. It is a function of \( M \) and \( J \), thus also a conserved charge, but the associated symmetry is a combination of a time translation and a spatial rotation. It is a screw rotation of the spinning cone. The Killing vector of this symmetry is a linear combination of the Killing vectors of time translations and spatial rotations. It is orthogonal to the Killing vector of time translations, whereas the rotational Killing vector associated with \( J \) has closed orbits of affine length \( 2\pi \). Unless the spinning cone is static, this is not the same. The definition of an angular momentum is therefore somewhat ambiguous.

**Complete reduction**

We can also define a reduced phase space \( \mathcal{P} \), in the very same way as before. Instead of a mass shell constraint, the time evolution is then provided by a Hamiltonian that represents the physical energy. To go over from the constrained to the unconstrained formulation, we have to solve the mass shell constraint \( E = 0 \), and additionally we have to impose a suitable gauge condition. As all this is completely analogous to the free particles. We can choose the same natural gauge condition, requiring the ADM time \( t \) to be equal to the absolute time \( T \) in the centre of mass frame. What we get is a pair of second class constraints,

\[
\text{snh}^2 K = \Lambda^2 F(M), \quad T = t. \tag{2.49}
\]

Since \( F \) is more or less the same function as before, just with some modified mass parameters, it can still be inverted, defining \( M \) as a function of \( K \). The explicit solution is given in (2.53) below. It is thus possible to remove the canonical pair \((M, T)\), and to go over to a four dimensional, completely reduced phase space

\[
\mathcal{P} = \{ (Q, S; X, \phi) \mid X \geq 0, \quad \phi \equiv \phi + 2\pi \}. \tag{2.50}
\]

To derive the symplectic potential \( \Theta \) and the Hamiltonian \( H \) on \( \mathcal{P} \), we have to start from the extended symplectic potential on \( \tilde{\mathcal{P}} \), again because the gauge condition above is explicitly time dependent. According to (2.43) and (2.45), it is given by

\[
\tilde{\Theta} - \tilde{H} \, dt = X^{-1} Q \, dX + (1 - 4GM) S \, d\phi - M \, dT - \zeta \mathcal{E} \, dt. \tag{2.51}
\]
On the reduced phase space, this becomes

\[ \Theta - H \, dt = X^{-1}Q \, dX + (1 - 4 GH) \, S \, d\phi - H \, dt. \quad (2.52) \]

To derive the reduced Hamiltonian \( H \), we have to solve the equation (2.49) for \( M \), so that the positive energy condition (2.38) is satisfied. The unique solution is

\[ H = F^{-1} \left( \Lambda^{-2} \, \text{snh}^2 K \right) = \text{atn} \left( \sqrt{\Lambda^{-2} \, \text{snh}^2 K + \tilde{m}_1^2 + \sqrt{\Lambda^{-2} \, \text{snh}^2 K + \tilde{m}_2^2}} \right), \quad (2.53) \]

where \( K \) is given by (2.28) as a function of the \( Q \), \( S \), and \( X \). The deformed mass parameters \( \tilde{m}_k \) are just some useful abbreviations. They are given by

\[ \tilde{m}_1 = \frac{\text{tn} \, \mu - \text{tn} \, \nu}{2}, \quad \tilde{m}_2 = \frac{\text{tn} \, \mu + \text{tn} \, \nu}{2}. \quad (2.54) \]

So, we find a somewhat deformed Hamiltonian, as compared to the free particle definition (1.31). It still depends only on the momentum \( K \), and not, for example, on the relative position \( X \). This we might have expected as a kind of gravitational potential. However, in three dimensional Einstein gravity, there are no local gravitational forces, and therefore we do not have a gravitational potential. But nevertheless, there is something like a gravitational binding energy. To see this, let us briefly discuss the range of \( H \), and the way it depends on the momentum \( K \). First of all, we easily see that \( H \) is minimal for \( K = 0 \), where it takes the value

\[ M_{\text{min}} = m_1 + m_2. \quad (2.55) \]

As for the free particles, the static states with \( K = 0 \) are excluded if at least one of the particles is massless. In this case, the positive energy condition (2.38) requires \( M > M_{\text{min}} \). Moreover, for small momenta \( K \), the energy also increases linearly if a massless particle is present, and quadratically if both particles are massive. For massive particles, we can also consider a non-relativistic limit, where \( K \) is small and the particles are moving slowly compared to the speed of light. Expanding (2.53) up to the second order in \( K \) gives

\[ M \approx M_{\text{min}} + \frac{1}{2\tilde{m}} \, K^2, \quad \text{with} \quad \tilde{m} = \frac{\text{cs}^2 \, \nu \, \text{tn}^2 \, \mu - \text{sn}^2 \, \nu}{4 \, \text{tn} \, \mu}. \quad (2.56) \]

Apart from the deformed expression for the reduced mass \( \tilde{m} \), we have the usual non-relativistic behaviour of the kinetic energy. It can be shown that \( \tilde{m} > m \), thus the reduced mass of the coupled system is always larger than the reduced mass of the free particles in (1.33). This is the only effect of the gravitational coupling for low momenta. For a given value of \( K \), the energy of the coupled system is slightly smaller than the free energy. There is still no gravitational potential. In particular, we do not recover Newtonian gravity. In three spacetime dimensions, this is not the non-relativistic limit of Einstein gravity. The relation between the two theories only exists in higher dimensions.

The behaviour of \( H \) for large momenta \( K \) is changed more drastically, as compared to the free particle system. Asymptotically, the energy is no longer linear in \( K \). The right hand side in (2.53) is bounded from above by \( M_{\text{pl}}/4 \). The energy asymptotically approaches the upper bound for large momenta \( K \). This is also a non-trivial result, because we might have expected...
that the upper bound $M_{Pl}/4$ for the total energy $M$ imposes some upper bound on the spatial momentum $K$ as well. But it seems that the gravitational interaction knows about this upper bound, and takes it into account in the relation between momentum and energy. To compare the energy momentum relation for the free and the coupled particles, we have plotted them for some typical mass parameters in figure 5.

For low momenta, the behaviour is very similar, we just have a slightly modified reduced mass $\tilde{m} > m$ for massive particles. This implies that the energy of the coupled system is a little bit smaller than the free particle energy. For large momenta, however, we see that the energy of the coupled system approaches $M_{\text{max}} = M_{Pl}/4$, whereas the energy of the free particles increases unboundedly. We can think of the difference between the free particle energy and the total energy of the coupled system as the negative energy of the gravitational field. This obviously compensates for the increasing kinetic energy of the particles, so that the total energy remains below the upper bound, which a quarter of the Planck energy.

If the momentum $K$ is of the order of one tenth of the Planck scale, then the negative binding energy is already of the same order of magnitude as the free particle kinetic energy. For momenta which are larger than the Planck scale, the energy has more or less saturated at the upper bound. The same behaviour of the energy can be found for a single particle system [13], where the

---

Figure 5: The energy momentum relation for the Kepler system (solid lines), and the free particles (broken lines). For low momenta, the behaviour is almost the same. For large momenta, the energy of the free particle system increases linearly with the momentum, whereas that of the coupled system approaches the Planck energy from below. All quantities are given in Planck units $M_{Pl} = 1/G$, and the upper bound for the total energy is $M_{\text{max}} = M_{Pl}/4$. 

---
upper bound is \( M_{\text{Pl}}/8 \), thus half of the maximal energy here. This is a generic feature of three dimensional general relativity. It has to do with its topological nature. The gravitational field does not fall off at spatial infinity, which means that the spacetime is not asymptotically flat. Instead, it is asymptotically conical, and therefore it is not possible for the universe to contain more energy than \( M_{\text{max}} = M_{\text{Pl}}/4 \).

But now, let us turn to the more interesting object on the reduced phase space \( P \), the symplectic structure. It follows from (2.52) that the reduced symplectic potential is given by

\[
\Theta = X^{-1} Q \, dX + (1 - 4GH) \, S \, d\phi. \tag{2.57}
\]

The somewhat curious feature of this expression is the Hamiltonian \( H \), which appears at the place where originally the energy \( M \) has shown up. Since \( H \) depends on \( Q \), \( S \) and \( X \), the resulting Poisson brackets become very complicated, because the two form \( \tilde{\Omega} = d\tilde{\Theta} \) is far from being diagonal in the given variables.

It is therefore more appropriate first to look for a set of canonically conjugate variables and then to define the Poisson brackets. Clearly, what we have to do is to replace the variable \( S \) by the canonical angular momentum \( J = (1 - 4GH)S \). Doing so, the symplectic potential and the resulting Poisson brackets become

\[
\Theta = X^{-1} Q \, dX + J \, d\phi \quad \Rightarrow \quad \{ Q, X \} = X, \quad \{ J, \phi \} = 1. \tag{2.58}
\]

All other brackets of \( Q, J, X \), and \( \phi \) are zero. We can further replace \( X \) and \( Q \) by the geodesic distance \( R \) of the particles, given by (2.17), and its canonically conjugate momentum \( P \). The transformation reads

\[
R^2 = X^2 + (4\pi GQ)^2, \quad \text{snh} \, P = \frac{Q}{X}, \tag{2.59}
\]

and the inverse transformation is

\[
X = \frac{R}{\text{csh} \, P}, \quad Q = R \, \text{tanh} \, P. \tag{2.60}
\]

Inserting this into the symplectic potential gives

\[
\Theta = P \, dR + J \, d\phi \quad \Rightarrow \quad \{ P, R \} = 1, \quad \{ J, \phi \} = 1. \tag{2.61}
\]

The chart \( (P, J; R, \phi) \) is thus canonical on \( P \), and the coordinates have a straightforward physical interpretation. This is also the canonical chart introduced in [20].

In the chart \( (Q, J; X, \phi) \) the Hamiltonian \( H \) is determined implicitly as the solution to

\[
\Lambda^2 F(H) = \text{snh}^2 K = \frac{Q^2 + S^2}{X^2} = \frac{Q^2 + (1 - 4GH)^{-2}J^2}{X^2}, \tag{2.62}
\]

and the corresponding equation in the chart \( (R, \phi, P, J) \) follows by the substitution (2.60). As \( F(H) \) involves the rescaled tangent of \( H \), the equation is transcendental in either chart. The price for simplifying the symplectic potential (2.57) into (2.58) or into the fully canonical form (2.61) therefore is that \( H \) cannot be expressed as an elementary function of the variables.
Trajectories

To find the classical trajectories, we return to the constrained formulation based on the six-di-
dimensional phase space $\tilde{P}$, with the additional independent variables $M$ and $T$, the symplectic
structure (2.45), and the Hamiltonian constraint (2.43). As $\tilde{P}$ is the gravitating version of the
special-relativistic phase space of section I, we can follow the analysis of section I with the
appropriate deformations.

The starting point is the Hamiltonian, which generates the time evolution with respect to the
unphysical ADM time $t$. It is given by the mass shell constraint (2.40), and a multiplier $\zeta$, which
is some arbitrary function of $t$,

$$\tilde{H} = \zeta \mathcal{E}, \quad \mathcal{E} = \frac{Q^2 + S^2}{X^2} - \Lambda^2 F(M).$$

(2.63)

This is almost identical to (1.14), and we can now use the brackets (2.46) to derive the time
evolution equations. The total energy $M$ and the angular momentum $S$ are of course conserved
charges, and the same holds for the total spatial momentum $K$,

$$\dot{M} = \{\tilde{H}, M\} = 0, \quad \dot{S} = \{\tilde{H}, S\} = 0 \quad \dot{K} = \{\tilde{H}, K\} = 0.$$  

(2.64)

The time evolution equations for the radial coordinates are almost the same as (1.35),

$$\dot{X} = \{\tilde{H}, X\} = 2 \zeta \frac{Q}{X}, \quad \dot{Q} = \{\tilde{H}, Q\} = 2 \zeta \frac{Q^2 + S^2}{X^2} = 2 \zeta \text{snh}^2 K.$$  

(2.65)

The only essential modifications arise when we derive the evolution equations for $\phi$ and $T$. They
differ from (1.36), because of the non-canonical brackets (2.47) of $S$ with $\phi$ and $T$. For the
angular orientation, we find

$$\dot{\phi} = \{\tilde{H}, \phi\} = 2 \zeta \frac{S}{(1 - 4GM)X^2},$$

(2.66)

and for the clock $T$ we get

$$\dot{T} = \{\tilde{H}, T\} = \zeta \left( \Lambda^2 F'(M) - \frac{8G}{1 - 4GM} \frac{S^2}{X^2} \right).$$

(2.67)

At first sight, these equations of motion are somewhat complicated, but it is still possible to solve
them explicitly, in the same way as in section I. First, we observe that $M, S$ and $K$ are constants
of motion. Then, we introduce a function $\epsilon(t)$ so that $\dot{\epsilon}(t) = 2\zeta(t)$. Using the same arguments
that led to (1.37), we conclude that

$$M(t) = M_0, \quad S(t) = S_0, \quad K(t) = K_0, \quad Q(t) = \epsilon(t) \text{snh}^2 K_0,$$

(2.68)

where $M_0, S_0$ and $K_0$ are integration constants. It is then straightforward to solve the equation
of motion for $X$. The solution is

$$X(t) = \sqrt{R_0^2 + \epsilon^2(t) \text{snh}^2 K_0},$$

(2.69)
where \( R_0 \) is some integration constant. And yet another constant \( \phi_0 \) arises when we then solve the equation of motion for \( \phi \),

\[
\phi(t) = \phi_0 + \frac{1}{1 - 4G M_0} \arctan \left( \epsilon(t) \frac{\sinh^2 K_0}{S_0} \right). \tag{2.70}
\]

And finally, we can also solve the equation of motion for \( T \). The solution is

\[
T(t) = T_0 + \epsilon(t) \frac{\Lambda^2 F'(M_0)}{2} - \frac{4G S_0}{1 - 4G M_0} \arctan \left( \epsilon(t) \frac{\sinh^2 K_0}{S_0} \right), \tag{2.71}
\]

where \( T_0 \) is yet another integration constant. All together, the resulting trajectory is parameterized by the gauge function \( \epsilon(t) \), and six integration constants \( M_0, K_0, S_0, T_0, R_0, \phi_0 \). Again, only four of them are independent. It follows from the constraint \( E = 0 \), and the definition of \( K \) as a function of \( Q, S, \) and \( X \) that

\[
\sinh^2 K_0 = \Lambda^2 F(M_0), \quad R_0 \sinh K_0 = |S_0|. \tag{2.72}
\]

These relations and all the formulas above reduce to their free particle counterparts in the limit \( G \to 0 \). What applies to the phase space structure, also applies to the classical trajectories of the Kepler system. They reduce to those of the free particles when the gravitational interaction is switched off. This is also a non-trivial statement. It implies, for example, that there are no bound states, again as a consequence of the absence of local gravitational forces.

As seen in section \[1\], the trajectories on the reduced phase space \( \mathcal{P} \) are obtained from those on \( \tilde{\mathcal{P}} \) by imposing the gauge condition \( T(t) = t \). We shall now verify that this gauge is always accessible and unique. By construction, the trajectories on \( \mathcal{P} \) then solve Hamilton’s equations with the Hamiltonian defined implicitly by (2.62). We have to show is that the right hand side of (2.71) is a monotonically increasing function of \( \epsilon(t) \), and that its range is the whole real line. If this is the case, then the equation \( T(t) = t \) can always be solved for \( \epsilon(t) \).

That the range is the whole real line can be seen quite easily. In the range of \( M_0 \) which is allowed by the positive energy condition, \( F'(M_0) \) is positive. Hence, if \( \epsilon(t) \) is large, then the \( \arctan \) term can be neglected as compared to the linear term, which is unbounded. So, what remains to be shown is that the derivative of the right hand side of (2.71) with respect to \( \epsilon(t) \) is positive. We know already that, up to a factor of \( 2\zeta = \dot{\epsilon} \), this derivative is equal to the right hand side of (2.67). This is how it was derived. Hence, we have to show that for all physical states

\[
\Lambda^2 F'(M) > \frac{8G}{1 - 4GM} \frac{S^2}{X^2}. \tag{2.73}
\]

The constraint (2.63) implies that

\[
\frac{S^2}{X^2} \leq \frac{S^2 + Q^2}{X^2} = \Lambda^2 F(M). \tag{2.74}
\]

It is therefore sufficient to show that

\[
F'(M) > \frac{8G}{1 - 4GM} F(M). \tag{2.75}
\]
This can be simplified further, using the following estimation. For \( 0 < \alpha < \pi \) we have \( \pi - \alpha > \sin \alpha \). With \( \alpha = 4\pi GM \), this implies

\[
1 - 4GM > \frac{\sin(4\pi GM)}{\pi} = 4G \text{sn} M \text{ cs} M \quad \Rightarrow \quad \frac{8G}{1 - 4GM} < \frac{2}{\text{sn} M \text{ cs} M}.
\]  

(2.76)

Hence, it is also sufficient to show

\[
F'(M) > \frac{2F(M)}{\text{sn} M \text{ cs} M (2.77)}
\]

Now, all we have to do is to insert the definition (2.41) for \( F \) and compute its derivative. It is then very easy to verify that this inequality is indeed satisfied, provided that the positive energy condition holds, that is \( \text{tn} M > \sqrt{\text{tn} \mu \text{tn} \bar{\mu}} \). Thus, we conclude that a gauge function \( \epsilon(t) \) exists, so that \( T(t) = t \), although we do not know it explicitly.

**Scattering**

Let us now describe the physical properties of the trajectories, in particular the way they deviate from the free particle trajectories. Since the relative position \( X \) is still an auxiliary variable, we should first replace it by the actual geodesic distance \( R \) between the particles. And it is also useful to consider the previously defined canonically conjugate momentum \( P \). Using (2.59), we find that on the given trajectories we have

\[
R(t) = \sqrt{R_0^2 + \frac{1}{4}\epsilon^2(t) \text{snh}^2(2K_0)}, \quad \text{snh} P(t) = \frac{\epsilon(t) \text{snh}^2 K_0}{\sqrt{R_0^2 + \epsilon^2(t) \text{snh}^2 K_0}}.
\]  

(2.78)

Here, we can first of all see that the assumption was correct that \( K \) represents the ingoing and outgoing momentum of the particles. If \( R(t) \) is large, then the canonically conjugate momentum \( P(t) \) approaches \( \pm K_0 \). As for the free particles, we expect the eigenvalues of \( K \) to specify the inverse radial wavelength at infinity, when the system is quantized.

But now, let us consider the actual trajectories. First we assume that \( S_0 \neq 0 \). According to (2.72), this implies that \( R_0 > 0 \) and \( K_0 > 0 \). In this case, the particles approach each other from infinity from some direction \( \phi_{\text{in}} \), they reach a minimal distance \( R_0 \) at the absolute time \( T_0 \) and with angular orientation \( \phi_0 \), and afterwards the separate again, moving to infinity into a direction \( \phi_{\text{out}} \). The interpretation of the integration constants \( M_0 \) and \( S_0 \) is also obvious. These are the total energy and the angular momentum, which are constant along the trajectory. So far, everything is the same as before.

However, there is now a crucial difference, because now we have a real scattering. The angular directions \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) are given by

\[
\phi_{\text{in}} = \phi_0 - \frac{\pi}{2} \frac{\text{sgn} S_0}{1 - 4GM_0}, \quad \phi_{\text{out}} = \phi_0 + \frac{\pi}{2} \frac{\text{sgn} S_0}{1 - 4GM_0}.
\]  

(2.79)

It is no longer so that \( \phi_{\text{out}} = \phi_{\text{in}} \pm \pi \). Instead, the scattering angle is

\[
\phi_{\text{out}} = \phi_{\text{in}} + \pi \frac{\text{sgn} S_0}{1 - 4GM_0}.
\]  

(2.80)
The scattering angle depends on the deficit angle of the spinning cone, thus on the energy of the particles, and the direction of the scattering depends on the sign of the angular momentum, thus on which side the two particles pass each other. This is not a surprise, because it is the typical behaviour of a pair geodesics on a cone, and of course also on a spinning cone.

In fact, the world lines of the particles can effectively be considered as a pair of geodesics on antipodal sides of the spinning cone \([20]\). However, the crucial point is that the geometry of this spinning cone depends on the energy and the angular momentum of the particles. The particles are not just moving on a fixed background cone. It is also interesting to note that for large momenta \(K_0\), thus for \(M_0 \approx M_{\text{max}}\), the factor in the denominator in (2.80) becomes arbitrarily small. In this case, the particles wind around each other many times before they separate again. We also see that the scattering angle is always larger than \(\pi\). This means, in a sense, that gravity is attractive, even though there are no local forces.

The case \(S_0 = 0\) is again somewhat special. According to (2.72), we either have \(K_0 = 0\) or \(R_0 = 0\). For \(K_0 = 0\), we have a static state. The particles are at rest, with a fixed distance \(R_0 > 0\), and a fixed angular orientation \(\phi_0\). The geometry of space is that of figure [3], and it is constant in time. The total energy is \(M_0 = M_{\text{min}}\), and the integration constant \(T_0\) is redundant. The more interesting case is \(S_0 = 0\) and \(R_0 = 0\). Let us once again consider the limit \(S_0 \to 0\) and \(R_0 \to 0\), with \(K_0 > 0\) and all other integration constant fixed. What we find is similar to (1.45),

\[
R(t) = |\epsilon(t)| \frac{\sinh(2K_0)}{2}, \quad \phi(t) = \phi_0 + \frac{\pi}{2} \frac{\text{sgn} S_0}{1 - 4GM_0} \text{sgn} \epsilon(t).
\]

(2.81)

Again, the particles approach each other on a radial line with direction \(\phi_{\text{in}}\), then they hit in a head on collision at the absolute time \(T_0\), and after that they continue on a radial line with direction \(\phi_{\text{out}}\). However, there is now a crucial difference to the free particles. The relation between \(\phi_{\text{in}}\) and \(\phi_{\text{out}}\) is no longer independent of the sign of \(S_0\), and thus the limit is actually not well defined. We have

\[
\phi(t) = \begin{cases} 
\phi_{\text{in}} & \text{for } T < T_0, \\
\phi_{\text{out}} & \text{for } T > T_0,
\end{cases} \quad \phi_{\text{out}} = \phi_{\text{in}} \pm \frac{\pi}{1 + 4GM_0},
\]

(2.82)

which depends on the sign, unless \(1/(1 - 4GM_0) \in \mathbb{Z}\). These are very special cases where the deficit angle of the universe takes a value in a certain discrete set.

Clearly, this is the typical behaviour of a geodesic on a cone. If the geodesic hits the tip, then it does not have a unique continuation. If it is considered as a limit of a family of geodesics missing the tip, then its continuation depends on the way we take the limit. To get a unique time evolution, we have to say explicitly what the particles have to do when they hit each other. Quite remarkably, this ambiguity disappears at the quantum level. The following semi-classical argument explains why. The problem only arises when the angular momentum \(S\) vanishes. But then, the angular orientation \(\phi\) is smeared out uniformly over all directions. Hence, for \(S = 0\) there is anyway no well defined scattering angle, because both \(\phi_{\text{in}}\) and \(\phi_{\text{out}}\) are maximally uncertain.

Apart from this ambiguity, the classical dynamics of the Kepler system is well defined on the extended phase space \(\tilde{P}\), and the trajectories are easy to derive. Implicitly, this also defines the trajectories on the reduced phase space \(P\). We know that they exist, and of course they have the same physical properties, but we cannot write them down explicitly. To go back to the spacetime picture, we have to perform the construction in figure [4] at each moment of ADM time \(t\). What we get is a space that evolves in time, and this provides a foliation of the spacetime manifold.
by slices of constant absolute time in the centre of mass frame. We can either take this general relativistic point of view, but we may also stick to a simplified point of view where we just think of two particles moving in a two dimensional space.

3 The quantum Kepler system

With all the preparations made in the previous sections, we can now, finally, quantize the Kepler system. As an ansatz, we shall first try the Schrödinger method, based on the reduced phase space \( \mathcal{P} \). This turns out to fail, due to a somewhat peculiar feature of the classical phase space. Its global structure is not as simple as it appears, which implies that a straightforward choice of a set of commuting configuration variables does not provide a proper basis of the quantum Hilbert space. We already noticed that the definition of the Poisson brackets and the Hamiltonian was technically somewhat involved, and we were not able, for example, to define the Hamiltonian explicitly as a function of a set of canonically conjugate variables.

This technical problem turns into a principle one at the quantum level. The way out is to use the Dirac quantization method, based on the extended phase space \( \tilde{\mathcal{P}} \). On this phase space, the mass shell constraint can easily be expressed as a function of a set of canonically conjugate variables. Instead of solving the Schrödinger equation, we have to impose a constraint. It becomes a generalized Klein Gordon equation and defines the physical Hilbert space. Just like the classical phase space, the quantum Hilbert space turns out to be a deformed version of the free particle Hilbert space. Finally, we are going to write down the energy eigenstates explicitly as wave functions in space and time, and from them we read off some interesting effects of quantum gravity.

Schrödinger approach

The most straightforward way to quantize the Kepler system seems to be a Schrödinger quantization, based on the completely reduced phase space \( \mathcal{P} \) defined in (2.50), and spanned by the variables \( Q, S, X, \) and \( \phi \). Apparently, the only technical problem is to define the Hamiltonian \( H \) explicitly as a quantum operator, which is given by (2.53). However, a first obstacle already arises when we look at the symplectic potential (2.57), which explicitly depends on this function,

\[
\Theta = X^{-1} Q \, dX + (1 - 4GH) \, S \, d\phi. \tag{3.1}
\]

This we have to take into account when we set up the operator representation. We saw, however, that we may equally well replace the variable \( S \) by

\[
J = (1 - 4GH) \, S \quad \Rightarrow \quad \Theta = X^{-1} Q \, dX + J \, d\phi, \tag{3.2}
\]

and consequently we have the canonical brackets

\[
\{Q, X\} = X, \quad \{J, \phi\} = 1. \tag{3.3}
\]

Now the quantization is straightforward. We choose a basis where \( X \) and \( \phi \) are diagonal. The wave functions is \( \psi(x, \varphi) \), with \( x \geq 0 \), and \( \varphi \) has a period of \( 2\pi \) if the particles are distinguishable, or the period is \( \pi \) if the particles are identical. We introduce a statistics parameter \( \lambda \), in the
same way as previously for the free particles. We may also define the same scalar product,

\[ \langle \psi_1 | \psi_2 \rangle = \int x \, dx \, d\varphi \, \bar{\psi}_1(x, \varphi) \psi_2(x, \varphi), \]  

(3.4)

which implies that \( \psi(x, \varphi) \) is the usual probability amplitude in polar coordinates. And it is also straightforward to define the operator representation, which is given by

\[ \hat{\mathbf{X}} \psi(x, \varphi) = x \psi(x, \varphi), \quad \hat{\varphi} \psi(x, \varphi) = \varphi \psi(x, \varphi), \]  

(3.5)

for the position operators, and the momentum operators are

\[ \hat{Q} \psi(x, \varphi) = -i\hbar \frac{\partial}{\partial x} \psi(x, \varphi), \quad \hat{J} \psi(x, \varphi) = -i\hbar \frac{\partial}{\partial \varphi} \psi(x, \varphi). \]  

(3.6)

So far, everything is the same as before. It seems that the only technical problem is to express the Hamiltonian \( H \) in terms of \( X, Q, \) and \( J \). It is now implicitly defined by the equation (2.62), thus

\[ \frac{Q^2 + (1 - 4GH)^{-2}J^2}{X^2} = \Lambda^2 F(H). \]  

(3.7)

Since we could not even solve this equation explicitly at the classical level, it is of course also impossible to derive an explicit operator representation for \( H \) at the quantum level. Nevertheless, since we know that the Hamiltonian is well defined on the classical phase space, this only seems to be a technical problem. But if we look at all this more closely, we find that the problem is actually not a technical, but a principle one. In fact, we made a wrong assumption in the above derivation.

To set up a quantum representation like the one given above, it is not sufficient to pick out a complete set of commuting phase space variables like \( X \) and \( \phi \), and to define the wave function \( \psi(x, \varphi) \) to be a function on this configuration space. There is a second consistency condition, which is sometimes overlooked because it is usually immediately obvious. The classical phase space has to be the cotangent bundle of the configuration space, equipped with its canonical symplectic structure. At each point \( (X, \phi) \) of the configuration space, the canonically conjugate momenta, which are \( (Q/X, J) \) in our case, have to span a vector space. Splitting the phase space into a configuration and a momentum space like this is called a polarization [23].

Without going into any details, let us roughly show what goes wrong when the second criterion is not satisfied. Given a Hilbert space such that to each point \( x \) in the configuration space there corresponds a basis state \( \psi_x \), then for every bounded function \( F \) on the configuration space there exists a bounded self adjoint operator, whose eigenstates are \( \psi_x \) with eigenvalues \( F(x) \). This is just usual quantum mechanics, for example in the position representation. Consequently, there is also a family of unitary operators \( e^{iF/\hbar} \), representing the flow of \( F \). On the classical phase space, this flow is a translation in the conjugate momentum space. Therefore, the whole construction is consistent only if the momentum space at each point \( x \) of the configuration space is at least an affine space. It can be made a vector space by choosing some origin.

Now, coming back to the Kepler system, let us check whether the second criterion is satisfied or not. We have to find out whether for every fixed point in the configuration space \( (X, \phi) \), the
canonically conjugate momenta \((Q/X, J)\) span a vector space. Unfortunately, this is not the case. To see this, consider the following estimation,

\[
\frac{J^2}{X^2} = (1 - 4GH)^2 \frac{S^2}{X^2} \leq (1 - 4GH)^2 \frac{S^2 + Q^2}{X^2} = (1 - 4GH)^2 \Lambda^2 F(H). \tag{3.8}
\]

The first factor can be estimated further by

\[
1 - 4GH < \frac{1}{\pi^2 G \operatorname{tn} H}, \tag{3.9}
\]

which follows from \(\cot \alpha > (\pi/2 - \alpha)\) for \(0 < \alpha < \pi/2\), and with \(\alpha = 2\pi GH\). Using this, we find that

\[
\frac{J^2}{X^2} \leq \frac{\Lambda^2 F(H)}{\pi^4 G^2 \operatorname{tn}^2 H} < \frac{\Lambda^2}{4\pi^4 G^2}. \tag{3.10}
\]

The last inequality follows from the following property of the function \(F\) defined in (2.41). For \(\operatorname{tn} H \geq \operatorname{tn} \mu \geq \operatorname{tn} \nu\), which follows from the positive energy condition, we have

\[
\frac{F(H)}{\operatorname{tn}^2 H} = \frac{(\operatorname{tn}^2 H - \operatorname{tn}^2 \mu)(\operatorname{tn}^2 H - \operatorname{tn}^2 \nu)}{4 \operatorname{tn}^4 H} < \frac{1}{4}. \tag{3.11}
\]

Hence, all together it follows that

\[
|J| < J_{\text{max}}, \quad \text{where} \quad J_{\text{max}} = \frac{\Lambda}{2\pi^2 G} X. \tag{3.12}
\]

The canonical angular momentum \(J\) is restricted to some interval around zero, whose size is determined by the configuration variable \(X\). Thus, at a fixed point \((X, \phi)\) of the configuration space, the canonical momenta \((Q/X, J)\) do not span a vector space. We have to conclude that the above definition of a position representation is not the correct quantization of the given classical phase space.

Of course, we could now argue that we simply picked out the wrong configuration space. We just have to find an appropriate configuration space to perform the correct quantization. But then we would lose the simple physical interpretation of the configuration variables, and consequently the direct interpretation of the wave function as a probability amplitude in the usual sense. Therefore, to avoid this unnecessary complication, we shall not try to quantize the Kepler system bases on the reduced phase space \(\mathcal{P}\)

**Dirac approach**

The whole problem can be circumvented when we use the Dirac method instead. The extended phase space \(\tilde{\mathcal{P}}\) defined in (2.42) has the additional independent variables \(M\) and \(T\). It is a proper cotangent bundle with its canonical symplectic potential (2.45),

\[
\tilde{\Theta} = X^{-1}Q \, dX + (1 - 4GM) \, S \, d\phi - M \, dT. \tag{3.13}
\]

The configuration space is spanned by \((T, X, \phi)\), and the canonically conjugate momenta \((-M, Q/X, (1 - 4GM)S)\) have an unrestricted range for fixed \(T, X,\) and \(\phi\). The quantization is
straightforward. We have a wave function $\psi(\tau, x, \phi)$, with $x \geq 0$, and regarding the periodicity in $\phi$ we have the same relations as before. If the particles are distinguishable, we have

$$\psi(\tau, x, \phi + 2\pi) = e^{2\pi i\lambda} \psi(\tau, x, \phi), \quad (3.14)$$

and for identical particles the stronger relation is

$$\psi(\tau, x, \phi + \pi) = e^{\pi i\lambda} \psi(\tau, x, \phi), \quad (3.15)$$

where the statistics parameter $\lambda$ is a fixed real number. The position operators are given by

$$\hat{X} \psi(\tau, x, \phi) = x \psi(\tau, x, \phi), \quad \hat{\phi} \psi(\tau, x, \phi) = \phi \psi(\tau, x, \phi), \quad (3.16)$$

where the last equation is again to be understood with $\phi$ replaced by a periodic function thereof.

For the clock and the canonically conjugate energy we have

$$\hat{T} \psi(\tau, x, \phi) = \tau \psi(\tau, x, \phi), \quad \hat{M} \psi(\tau, x, \phi) = i\hbar \frac{\partial}{\partial \tau} \psi(\tau, x, \phi). \quad (3.17)$$

And finally, we need the momentum operators $Q$ and $S$. As $S$ can be expressed as a simple function of $M$ and $J$, let us first define

$$\hat{Q} \psi(\tau, x, \phi) = -i\hbar \frac{\partial}{\partial x} x \psi(\tau, x, \phi), \quad \hat{J} \psi(\tau, x, \phi) = -i\hbar \frac{\partial}{\partial \phi} \psi(\tau, x, \phi). \quad (3.18)$$

The operator for $S$ can then be expressed as

$$\hat{S} \psi(\tau, x, \phi) = -i\hbar \left( 1 - 4i\ell \frac{\partial}{\partial \tau} \right) \frac{1}{2} \frac{\partial}{\partial \phi} \psi(\tau, x, \phi). \quad (3.19)$$

The constant $\ell = G\hbar$ is the Planck length, which is here actually interpreted as a Planck time. So, we have a somewhat unusual operator representation for the angular momentum $S$, which is non-local in $\tau$. But this is not a serious problem. Finally, we have the following scalar product, with respect to which all these operators are self adjoint, and which is formally the same as (1.66),

$$\langle \psi_1 | \psi_2 \rangle = \int d\tau dxd\phi \bar{\psi}_1(\tau, x, \phi) \psi_2(\tau, x, \phi). \quad (3.20)$$

This completes the definition of the extended Hilbert space. Now we have to impose the mass shell constraint to define the physical Hilbert space. The constraint is given by (2.63), so we should first try to diagonalize the operator

$$\text{snh}^2 \hat{K} = \hat{X}^{-1} \left( \hat{Q}^2 + \hat{S}^2 \right) \hat{X}^{-1}. \quad (3.21)$$

It acts on a wave function as

$$\text{snh}^2 \hat{K} \psi(\tau, x, \phi) = -\hbar^2 \triangle \psi(\tau, x, \phi), \quad (3.22)$$

where $\triangle$ is a deformed Laplacian,

$$\triangle = x^{-1} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} + x^{-2} \left( 1 - 4i\ell \frac{\partial}{\partial \tau} \right) \frac{1}{2} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi}. \quad (3.23)$$
Now, suppose that the operator $1 - 4GM$ in parenthesis is also diagonal. Then $\triangle$ is the Laplacian on a cone with a deficit angle of $8\pi GM$. This is not surprising. We found that the classical particles are effectively moving on such a cone. The eigenfunctions are still Bessel function, however with slightly modified indices and arguments. The normalized eigenstates are parameterized by three quantum numbers $\omega$, $k$, and $s$, and they are explicitly given by

$$\chi(\omega, k, s; \tau, x, \varphi) = \sqrt{\frac{\sn h(2hk)}{4\pi h}} e^{-i\omega\tau} e^{is\varphi} J_{\frac{|s|}{1-4\ell\omega}} \left(\frac{\sn h(hk)}{h} x\right).$$

(3.24)

This reduces to (1.67) in the limit $G \to 0$, which implies $\ell \to 0$ and $\sn h q \to q$. We are here imposing the same regularity condition at $x = 0$ that we also imposed in section 1. Following the arguments in [22], we require the wave function to stay finite. This makes (3.22) a well defined self adjoint operator. The eigenstates are normalized in the same way as (1.68), thus

$$\int d\tau x dx d\varphi \chi(\omega_1, k_1, s_1; \tau, x, \varphi) \chi(\omega_2, k_2, s_2; \tau, x, \varphi) = 2\pi \delta(\omega_2 - \omega_1) k_1^{-1} \delta(k_2 - k_1) \delta_{s_2 - s_1}.$$  

(3.25)

The quantum numbers $k > 0$ and $\omega$ are continuous, and $s$ is discrete. It takes the values $s \in \mathbb{Z} + \lambda$ or $s \in 2\mathbb{Z} + \lambda$, depending on whether the particles are distinguishable or not. This follows again from the periodicity condition imposed on the wave function. Taking this into account when summing over $s$, we have the following completeness relation for the eigenstates,

$$\sum_s \int d\omega dk \chi(\omega, k, s; \tau_1, x_1, \varphi_1) \chi(\omega, k, s; \tau_2, x_2, \varphi_2) = 2\pi \delta(\tau_2 - \tau_1) x_1^{-1} \delta(x_2 - x_1) \delta(\varphi_2 - \varphi_1),$$

(3.26)

where $\delta_\lambda$ is again the periodic delta function satisfying the (3.14) or (3.15). So, we now have a basis of the extended Hilbert space, where the energy and momentum operators are diagonal, with eigenvalues

$$K = \hbar k, \quad M = \hbar \omega, \quad J = \hbar s, \quad S = \frac{\hbar s}{1-4\ell\omega}.$$  

(3.27)

We see that the spectrum of $J$ is the one which is quantized in steps of $\hbar$, like the angular momentum of the free particles in (1.69). This is reasonable. On the classical phase space it is the charge $J$ which generates a rotation with closed orbits, and a period of $2\pi$. The spectrum of $S$ is also discrete, but the eigenvalues additionally depend on the eigenvalues of $M$. The closer the energy gets to the maximal physical energy $M_{\text{max}} = M_{\text{Pl}}/4 = 1/4G$, the larger the steps are between the eigenvalues of $S$. We shall discuss this spectrum and its physical implications at the very end of this section.

It is now straightforward to impose the mass shell constraint. We can simply repeat all the steps from section 1. The constraint acts on the energy momentum eigenstates as

$$\hat{E} \chi(\omega, k, s; \tau, x, \varphi) = \left(\sn h^2(hk) - \Lambda^2 F(\hbar\omega)\right) \chi(\omega, k, s; \tau, x, \varphi).$$

(3.28)

It is again useful to introduce a dispersion relation

$$\omega(k) = \hbar^{-1} F^{-1} \left(\frac{\sn h^2(hk)}{\Lambda^2}\right).$$

(3.29)
It is the quantum version of the classical relation between the momentum \( K \) and the energy \( M \), which is shown in figure 5. We conclude that the physical states are those energy momentum eigenstates, where the quantum numbers satisfy \( \omega = \omega(k) \). These states are annihilated by the constraint (3.28). The most general physical state is a superposition

\[
\psi(\tau, x, \varphi) = \sum_s \int k \, dk \, \psi(k, s) \chi(\omega(k), k; \tau, x, \varphi),
\]

(3.30)

where \( \psi(k, s) \) is again the wave function in momentum space. Let us once again split off the dependence on the radial coordinate, writing

\[
\psi(\tau, x, \varphi) = \sum_s \int k \, dk \, e^{-i\omega(k)\tau} e^{is\varphi} \psi(k, s) \zeta(k, s; x).
\]

(3.31)

The radial wave function is then given by

\[
\zeta(k, s; x) = \frac{1}{4\pi} \, \sqrt{\frac{\sinh(8\pi\ell k)}{\ell k}} \, J_{|s|} \left( \frac{\sinh(4\pi\ell k)}{4\pi\ell} x \right).
\]

(3.32)

Here we replaced the rescaled hyperbolic functions by the usual ones, to see that only the Planck length \( \ell \) appears as a dimensionful constant. In the limit \( \ell \to 0 \), we still recover the free particle expression (1.72).

**Position representation**

Before we read off any physical information from the wave functions, we have to transform to a representation where the geodesic distance \( R \) between the particles is diagonal, and not the auxiliary variable \( X \). So, the task is to find the eigenstates of \( R \) and then to transform the wave functions into this representation. According to (2.17), we have \( R^2 = X^2 + (4\pi GQ)^2 \), thus

\[
\hat{R}^2 \psi(\tau, x, \varphi) = \left( x^2 - (4\pi\ell)^2 \frac{\partial}{\partial x} x \frac{\partial}{\partial x} x \right) \psi(\tau, x, \varphi).
\]

(3.33)

The eigenfunctions of this operator are modified Bessel functions of the second kind,

\[
\zeta(r; x) = \frac{\sqrt{\sinh(r/4\ell)}}{2\sqrt{2\pi^2\ell x}} K_{i|s|} \left( \frac{x}{4\pi\ell} \right) \Rightarrow \hat{R} \zeta(r; x) = r \zeta(r; x).
\]

(3.34)

The eigenvalues \( r \) are positive, as it should be, and continuous. The eigenstates are orthonormal,

\[
\int x \, dx \, \eta(r_1; x) \, \eta(r_2; x) = r^{-1}\delta(r_2 - r_1),
\]

(3.35)

and complete

\[
\int r \, dr \, \eta(r_1; x) \, \eta(r_2; x) = x^{-1}\delta(x_2 - x_1).
\]

(3.36)

The transformation of a wave function from the \( X \)-representation to the \( R \)-representation is given by

\[
\psi(\tau, r, \varphi) = \int x \, dx \, \eta(r; x) \psi(\tau, x, \varphi).
\]

(3.37)
For simplicity, we use the same symbol for the wave function in both representations. This transformation is known as the Kontorovich Lebedev transform \[26\]. The scalar product in the \(R\)-representation becomes

\[
\langle \psi_1 | \psi_2 \rangle = \int d\tau \, r \, dr \, d\varphi \, \bar{\psi}_1(\tau, r, \varphi) \, \psi_2(\tau, r, \varphi),
\]

so that the transformed wave function \(\psi(\tau, r, \varphi)\) represents the correct probability amplitude in the relative position space of the particles. We can finally write down the most general physical state as a wave function in the position representation. All we have to do is to transform the radial wave functions (3.32) from the \(X\)-representation to the \(R\)-representation, thus

\[
\zeta(k, s; r) = \int x \, dx \, \eta(r; x) \, \zeta(k, s; x),
\]

This convolution can be carried out explicitly, and the result can be expressed in terms of a hypergeometric function \[25\],

\[
\zeta(k, s; r) = \frac{\Gamma(A) \, \Gamma(\bar{A})}{\Gamma(A + \bar{A})} \, \frac{\sqrt{\sinh(r/4\ell) \, \sinh(8\pi\ell k)}}{(2\pi)^2 \, \sqrt{2\ell \, k}} \times \\
\times \sinh^{A + \bar{A} - 1}(4\pi\ell k) \, _2F_1 \left( A, \bar{A}; A + \bar{A}; -\sinh^2(4\pi\ell k) \right),
\]

where

\[
A = \frac{1}{2} \left( 1 + \frac{|s|}{1 - 4\ell \omega(k)} + \frac{ir}{4\pi\ell} \right), \quad \bar{A} = \frac{1}{2} \left( 1 + \frac{|s|}{1 - 4\ell \omega(k)} - \frac{ir}{4\pi\ell} \right).
\]

So, we finally have a rather complicated but explicit representation of the physical energy eigenstates as wave functions in the relative position space of the particles. The operators for \(R\) and \(\phi\), representing the distance and the orientation of the particles, are diagonal in this representation. It is therefore possible to read off the physical properties of the states directly from the wave functions.

**Wave functions**

We shall in the following study the radial wave functions \(\zeta(k, s; r)\), representing the physical states with eigenvalues

\[
M = \hbar \omega(k), \quad K = \hbar k, \quad J = \hbar s, \quad S = \frac{\hbar s}{1 - 4\ell \omega(k)}.
\]

We shall compare them to the corresponding radial wave functions \(\zeta(k, s; r)\) of the free particle system, as given by \[1.72\]. There we may simply identify the radial coordinate \(x\) with \(r\), as in the limit \(G \to 0\) the transformation from the \(X\)-representation to the \(R\)-representation is trivial. In both cases, the physical Hilbert space is spanned by a two parameter family of energy momentum eigenstates, and the quantum numbers \(k\) and \(s\) have the same interpretation. The former is the inverse radial wavelength at infinity, representing the momentum of the incoming and outgoing particles, and the latter is the inverse angular wavelength, representing the angular momentum.
Actually, we haven’t check this yet for the Kepler system. For the free particles, it is a well
know feature of the Bessel function (1.72). For large \( r \) it falls off with \( \sqrt{r} \), oscillating with a
wavelength of \( 2\pi/k \). It is not at all obvious that the function (3.40) has the same property. But
we can use the following semi-classical argument. We have seen in (2.78) that for the classical
trajectories, the canonically conjugate momentum \( P \) of the distance \( R \) is equal to \( \pm K \) for large
\( R \). Therefore, we expect the eigenfunction of \( K \) with eigenvalue \( \hbar k \) to be a superposition of an
ingoing and an outgoing radial wave, with wavelength \( 2\pi/k \) for large \( r \). This argument holds for
both the free particles, where the corresponding classical relation is (1.44), and for the coupled
particles.

An alternative way to see this is as follows. This argument can be made rigorous by taking
into account the correct operator ordering. But for simplicity let us stick to a semi-classical level.
Consider the eigenvalue equation for \( K \) in the \( R \)-representation. As a phase space function, we
have
\[
\sinh^2 K = \frac{Q^2 + S^2}{X^2} = \frac{R^2 \tanh^2 P + S^2}{R^2/cosh^2 P} = \sinh^2 P + \frac{S^2}{R^2} \cosh^2 P.
\]
Hence, for large \( R \) we have \( P \approx \pm K \). Since \( P \) is canonically conjugate to \( R \), and thus repre-
scated by the operator \( -i\hbar \partial/\partial r \), it follows that an eigenfunction of \( K \) with eigenvalue \( \hbar k \) has
a radial wavelength of \( 2\pi/k \) for large \( r \). At this point, we see what the advantage of the aux-
iliary variable \( X \) was. Of course, we could have used \( R \) and \( P \) as phase space variables from
the very beginning. Then, however, we had to solve the eigenvalue equation for \( (3.43) \), with
the appropriate operator ordering. Obviously, this is much more difficult than solving the eigen-
value equation for \( (3.21) \), because the hyperbolic functions become a non-local operators in the
complex \( r \)-plane.

But now, let us come to the actual physical questions. We wanted to find out what the basic
differences are between the radial wave functions \( \zeta(k,s;r) \) for the free particles and those for
the coupled particles. So far, we know that both represent a scattering state, where the incoming
and outgoing particles have a radial wavelength of \( 2\pi/k \), and a fixed angular momentum. In
other words, the wave functions are almost equal for large \( r \), where large means some orders of
magnitude above the Planck length \( \ell \). The interesting question is what happens at small distances.
To see this, we have plotted some typical wave functions, for different values of \( k \) and \( s \).

In each of the following figures, the value of \( s \) is fixed, and that of \( k \) varies. The upper part
of the figure always shows the free particle wave functions, the lower one those for the Kepler
system, both as a function of the distance \( r \), which is measured in units of the Planck length \( \ell \).
To distinguish the wave functions for different momenta \( k \), we use the following rule. The larger
\( k \) is, the smaller are the gaps in the curves. The broken curve with the largest gaps represents the
smallest momentum, and a solid curve corresponds to the largest momentum in the figure. The
quantum numbers \( k \) are given in units of \( 1/\ell \), so that \( k = 1 \) corresponds to a radial wavelength
of \( 2\pi\ell \) at infinity, and to a momentum in physical units of \( 1/G = M_{Pl} \).

The mass parameters \( m_k \) are given in units of the Plank mass \( M_{Pl} \). They only have a marginal
influence on the qualitative behaviour of the wave functions. We therefore keep them fixed in
the following, and we choose two particles with the same mass \( m_1 = m_2 = 0.02 M_{Pl} \), thus still
far below the maximally allowed rest mass. For simplicity, we shall also restrict to states with
integer \( s \), thus particles with conventional statistics. There are no principally different effects
occurring for anyons, only the numerical values of some quantities considered below are slightly

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different. We shall briefly comment on this at the appropriate place.

As a first example, consider the states with \( s = 0 \) in figure 6. In the upper part we see the Bessel functions \( J_0(kr) \). They take a maximum at \( r = 0 \), and fall off with the square root of \( r \) for large \( r \), oscillating with a wavelength of \( 2\pi/k \). Except for a small region near the origin, which is of the order of ten Planck lengths, the wave functions of the Kepler system are almost the same. For a vanishing angular momentum, the coupled particles behave almost like the free particles. There is not even a phase shift between the ingoing and outgoing particles, so that far away from the origin the interaction of the particles cannot even be detected. However, the wave function of the coupled particles goes to zero at \( r = 0 \), so that the probability to find the two particles at the same point in space vanishes.

Obviously, the coupled particles try to avoid being both in a region of space which is of the order of the Planck scale. This becomes even more obvious when we look at the states with \( s = 1 \) in figure 7. For the free particles, we now have the Bessel functions \( J_1(kr) \). They are zero at \( r = 0 \), and the first maximum is approximately at \( r \approx 1/k \). For large \( r \), they still fall off with the square root of \( r \), oscillating with a wavelength of \( 2\pi/k \). Apart from a different phase of the oscillation, this is also the large \( r \) behaviour of the wave functions of the Kepler system. To see this more clearly, we have plotted the same wave functions for large distances in figure 8. The phase shift indicates that an interaction between the ingoing and outgoing particles has taken place.

The more interesting feature is again the behaviour of the wave function for small \( r \). For the
free particles, the maximum of the wave function is getting closer to the origin with increasing momentum \( k \). For the coupled particles however, the maximum never gets beyond a certain minimal distance \( R_{\text{min}} \). In fact, we already found such a minimal distance at the classical level. Remember that the quantum number \( s \) specifies the eigenvalue \( \bar{h}s \) of the canonical angular momentum \( J \). On the other hand, we have seen that the classical phase space function \( J \) is restricted by (3.12). We used this to show that a Schrödinger quantization, in a representation where \( X \) and \( \phi \) are diagonal, is not possible.

Turning the argument around, it follows that for a fixed \( J \), there is a minimal \( X \), and thus also a minimal \( R \), since \( R^2 = X^2 + (4\pi GQ)^2 \geq X^2 \). Consequently, we have the semi-classical relation

\[
R > R_{\text{min}}(s) = \frac{2\pi^2 \ell}{\Lambda |s|}.
\]  

(3.44)

It follows directly from (3.12) with \( J = \bar{h}s \). For a fixed angular momentum, there is thus a minimal distance between the particles. It is of the order of the Planck length \( \ell \) if the angular momentum is of the order of \( \hbar \), and it increases linearly with the angular momentum. The numerical factor in (3.44) depends on the masses of the particles, defining the number \( 0 < \Lambda \leq 1 \) in (2.41). We see in figure 7 that beyond the point \( R_{\text{min}} \) the wave function is no longer oscillating, but falls off exponentially. In the classically forbidden region, the wave function shows the typical tunneling behaviour.

Another peculiar observation is the following. For large momenta \( k \), the wave function of the coupled particles freezes at small distances \( r \). More precisely, for small \( r \) and large \( k \) the radial
wave function $\zeta(k; s; r)$ no longer depends on $k$. The limiting wave function is shown as the dotted curve in figure 7. This is quite strange, because it means that the short distance behaviour of the wave function will not change anymore with increasing momentum, whereas for large distances we still have an oscillation with a wavelength of $2\pi/k$. One can show that the limiting wave function is $\eta(r; R_{\text{min}})$, as defined in (3.34). It is the $R$-representation of the eigenstate of $X$ with eigenvalue $x = R_{\text{min}}$. Hence, for small $r$ and large $k$ we have

$$\zeta_{\omega(k)}(k; s; r, \varphi) \approx e^{is\varphi} \chi(r; R_{\text{min}}), \quad (3.45)$$

with $R_{\text{min}}$ given by (3.44). Let us only briefly sketch how this can be proven. If we return from the $R$-representation to the $X$-representation, then the radial wave functions are given by the Bessel functions (3.32). In the limit $k \to \infty$, one can show that these functions provide an approximation of the delta function $\delta(x - R_{\text{min}})$. Transforming again to the $R$-Representation, this implies (3.45). But let us not go into any details of the proof here. Numerically it can be verified easily that (3.45) is in fact a good approximation.

What does it mean physically that the wave function freezes at small $r$, and that the first maximum never goes beyond the point $r = R_{\text{min}}$? Consider again the scattering process described by these wave functions. In case of the free particles, we can make the following statement. The larger we make the momentum of the incoming particles, the closer the particles get to each other. In other words, to probe small distances we need large momenta. This is a well known rule in elementary particle physics or actually quantum physics in general. It is one way to express Heisenberg’s uncertainty relation. However, this statement is no longer true when gravity is
switched on. We can make the momentum as large as we like, we never get beyond the minimal distance $R_{\text{min}}$. Apparently, there is some repulsion that keeps the particles apart.

To see that this is a real quantum gravity effect, observe that $R_{\text{min}}$ is of the order of the Planck length, which involves both $G$ and $\hbar$. Hence, the effect disappears both if we switch off gravity, as then we have the wave functions of the free particles, and it also disappears in the classical limit. There is no restriction on the integration constant $R_0$ for the classical trajectories in section 2, thus the particles can reach any arbitrarily small distance. But what does it mean that the minimal distance $R_{\text{min}}$ depends on the quantum number $s$? First of all, one could argue that in order to bring the particles closer together we just have to consider states with smaller $s$. And in fact, we have seen in figure 6, that no such minimal distance exists for $s = 0$, in agreement with (3.44).

But now assume that the particles are two identical fermions with masses $m_1 = m_2 = m$. Then we have $s \in 1 + 2\mathbb{Z}$, and consequently the minimal value of $|s|$ is 1. In this case, we have

$$R > R_{\text{min}} = \frac{2\pi^2 \ell}{\Lambda} = \frac{2\pi^2 \ell}{\cos(4\pi G m)} ,$$

(3.46)

and this is an absolute minimum for the distance, which can never be reached by any wave function. This absolute minimum only depends on the mass $m$ of the particles. For $m = 0$ we have $R_{\text{min}} = 2\pi^2 \ell \approx 20\ell$. With increasing mass it increases unboundedly. For $m \to M_{\text{Pl}}/8$, which is the maximal mass of two identical particles, the cosine goes to zero. For anyons the absolute minimum is smaller, because the minimal $|s|$ lies between zero and one, and for bosons this curious effect disappears.

So, we find the following remarkable physical effect of quantum gravity. If the statistics parameter $\lambda$ is different from zero, then the particles are no longer able to approach each other closer than some minimal distance, which is of the order of the Planck length. But this is not the only feature of the wave functions, from which we can learn something about the short scale structure of spacetime. A second effect also results from the relation (3.44), and this is even independent of the statistics. Consider the wave functions for larger angular momenta, for example $s = 5$ in figure 8. For large $r$, outside the figure, the free particle wave functions and those of the coupled particles are still of the same form, just with a phase shift, as in figure 8.

The minimal distance $R_{\text{min}}$ is now however five times as large as in figure 7. Now, suppose that we want to localize the particles relative to each other, not at the origin, but at some finite distance $r = r_0$, and with some angular orientation $\phi = \varphi_0$. Clearly, for this purpose we need a superposition of states with different quantum numbers $k$ and $s$. We need large quantum numbers $s$ to get a sharp peak in the angular orientation $\phi$. And we need large quantum numbers $k$ to get a sharp peak in the distance $R$. However, we have just seen that for large angular momenta $s$, there is also a large minimal distance $R_{\text{min}}$. And moreover, for large momenta $k$ the wave function freezes when the distance is of the order of this minimal distance.

Roughly speaking, we can say that once we have a sharp angular direction $\phi$ of the particles, we do no longer have enough radial states to superpose, in order to get a sharp distance $R$. Of course, this is a somewhat heuristic argument, so let us make it more precise. Consider first the following analogy. Suppose that we have a box potential in flat space, which is zero outside and positive inside the box. There are then two types of energy eigenstates, those tunneling through the box, and those with sufficiently high energy passing the box without tunneling. Here we have the situation that actually only the tunneling states exist, because the region $r < R_{\text{min}}$
Figure 9: The wave functions for \( s = 5 \). The minimal distance \( R_{\text{min}} \) increases linearly with \( s \). As a consequence, it is not possible to localize the particles relative to each other, by superposing states with different \( k \) and \( s \).

is classically forbidden for all physical states. The physical Hilbert space only consists of the tunneling states.

But this subspace does not provide a complete basis of states in the position space of the system. More precisely, it is not possible to superpose the states, so that an arbitrarily sharp peak in position space comes out. To make this a little bit more explicit, let us try to define a physical state, which describes a localized pair of particles. In case of the free particles, there is no problem. In the Schrödinger quantization, we simply have to consider the special wave function

\[
\psi(r, \varphi) = \frac{1}{r} \delta(r - r_0) \delta(\varphi - \varphi_0).
\]

(3.47)

It can of course be written a superposition of the energy momentum eigenstates. We can even say explicitly what the corresponding wave function \( \psi(k, s) \) in momentum space is, which has to be inserted into (1.63). It is

\[
\psi(k, s) = e^{i s \varphi_0} \zeta(k, s; r_0),
\]

(3.48)

because we have the completeness relation (1.61). The corresponding wave function in the Dirac quantization is obtained by inserting the same wave function \( \psi(k, s) \) in momentum space into the formula (1.71). The resulting wave function \( \psi(\tau, r, \varphi) \) is equal to (3.47) for \( \tau = 0 \), thus describing a localized pair of particle at a given moment of time. As this is not an energy eigenstate, the particles will of course be localized only at one moment of time.
So far, this is just ordinary quantum mechanics of free particles. But now, let us ask whether such a localized state also exists for the coupled particles. Are there physical states describing a localized pair of particles, with a wave function of the form (3.47)? We could try to insert the same wave function in momentum space (3.48) into the general superposition (3.31) of physical states. Doing so, we get

$$\psi(\tau, r, \varphi) = \sum_s \int k \, dk \, e^{-i\omega(k)\tau} e^{is(\varphi-\varphi_0)} \zeta(k, s; r_0) \zeta(k, s; r).$$  \hspace{1cm} (3.49)$$

Now, what about the functions $\zeta(k, s; r)$ for fixed $s$? Do they define an orthonormal basis of the space of all radial wave function? If this was the case, then we would immediately recover the wave function (3.47) for $\tau = 0$, thus we have a localized pair of particles at that moment of time. Unfortunately, however, the functions $\zeta(k, s; r)$ do not provide an orthonormal basis. This is not obvious from the representation (3.40), but it can be seen by going back to the $X$-representation (3.32). The Bessel functions provide an orthonormal basis if and only if the index is fixed. For the free particles this is the case, since in (1.72) the index is $|s|$. In (3.40), however, the index is $|s|/(1 - 4\ell\omega(k))$, and thus not only the argument but also the index of the Bessel function depends on $k$. The functions $\zeta(k, s; r)$ for fixed $s$ in (3.49) are not orthonormal, and therefore the resulting wave function for $\tau = 0$ is not of the form (3.47), describing a localized state. Now, one could argue that there might be another superposition of physical states describing a localized pair of particles. However, if there were such localized states, then we could use them as a basis of the physical Hilbert space, and this would be exactly the basis that we defined in the beginning of this section, when we tried to set up a Schrödinger quantization. We saw, however, that due to the global structure of the classical phase space such a basis does not exist.

So, what is the conclusion? Obviously, the spacetime in which the quantized particles are moving has some kind of foamy structure. Maybe it is possible to analyze this structure more explicitly, by having a closer look at the wave functions $\zeta(k, s; r)$. For a single particle system, such an analysis has been carried out in some detail in [13], finding a semi-discrete structure of spacetime and an explicit uncertainty relation, which forbids the particle to be localized. Here things are more involved and therefore it is more difficult to make the idea of a foamy spacetime more precise. But at least we can see that it is impossible to localize the particles at a point in space, and this is already a nice result. It gives us a hint to what kind of influence a real theory of quantum gravity might have on the local structure of spacetime in general.

The quantized cone

Finally, let us consider yet another strange result, which has actually not very much to do with the point particles. There is quantized conical geometry of the spacetime at spatial infinity. Let us once again look at the spinning cone (2.7). It defines the asymptotic structure of the spacetime at infinity. We saw that it has the following global geometric features. There is a deficit angle of $8\pi GM$, and a time offset of $8\pi GS$. Now, consider the energy momentum eigenstates spanning the physical Hilbert space. They are the eigenstates of $M$ and $S$, with eigenvalues

$$M = \hbar \omega, \quad S = \frac{\hbar s}{1 - 4\ell\omega}, \quad 0 < \omega < \frac{1}{4\ell}, \quad s \in \mathbb{Z}. \hspace{1cm} (3.50)$$
For simplicity, we assumed that the particles are massless, distinguishable, and that the statistics parameter is $\lambda = 0$. There is nothing essentially different in a more general case, except that the formulas below are slightly different. In the context of quantized general relativity, the energy momentum eigenstates can also be regarded as states with a *sharp geometry* of the spacetime at infinity, but with the locations of the particles in space smeared out. However, the geometry of the spinning cone is thereby not arbitrary. There is an obvious relation between the eigenvalues of $M$ and $S$, namely

$$ (1 - 4GM) S \in \mathbb{Z} \hbar. \quad (3.51) $$

As this results from the quantization of the canonical angular momentum $J$, we expect that this is a general result, which does not depend, for example, on the number of particles present. In fact, one can show that for a general multi particle model there is the same relation between the canonical angular momentum $J$ and the parameters of the spinning cone $M$ and $S$, and the same is expected for any kind of matter included \[10\].

Thus, we conclude that the geometry of the conical infinity itself is quantized. Just like in a quantized atom, where not every classically allowed combination of energy and angular momentum can be realized, the conical geometry of the spacetime at infinity is restricted by this quantum condition. It becomes a very simple relation when it is written in geometric units, that is lengths and angles. We define the total angle of the spinning cone to be $\alpha = 2\pi - 8\pi GM$. It is the circumference of a unit circle around the central axis. And we define the time offset to be $\tau = 8\pi GS$. Then the quantization condition becomes

$$ \alpha \tau \in (4\pi)^2 \mathbb{Z} \ell. \quad (3.52) $$

Only those conical spacetimes can be realized where the product of the total angle and the time offset is an integer multiple of $(4\pi)^2 \ell$. Note that $\alpha$ is dimensionless and $\tau$ is a length, or time, so that necessarily the Planck length shows up. If $\alpha$ is close to $2\pi$, then $\tau$ is quantized is units of $8\pi \ell$, which is a kind of time quantization. If we assume that this result is independent of the kind of matter being present inside the universe, then what we have found is a real quantization of geometry.

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**Appendix**

Here we summarize some facts about the spinor representation of the three dimensional Lorentz algebra $\mathfrak{sl}(2)$ of traceless $2 \times 2$ matrices, and the associated Lie group $\text{SL}(2)$. A more comprehensive collection of formulas, using the same notation, can be found in \[13\]. As a vector space, $\mathfrak{sl}(2)$ is isometric to three dimensional Minkowski space. An orthonormal basis is given by the gamma matrices

$$ \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.1) $$
They satisfy the algebra
\[ \gamma_a \gamma_b = \eta_{ab} \mathbf{1} - \varepsilon_{abc} \gamma^c, \quad (A.2) \]
where \( a, b = 0, 1, 2 \), the metric \( \eta_{ab} \) has signature \((-++,++)\), and for the Levi Civita symbol \( \varepsilon^{abc} \) we have \( \varepsilon^{012} = 1 \). Expanding a generic matrix in terms of these gamma matrices, we obtain an isomorphism of \( \mathfrak{sl}(2) \) and Minkowski space,
\[ v = v^a \gamma_a \quad \Leftrightarrow \quad v^a = \frac{1}{2} \text{Tr}(v \gamma^a). \quad (A.3) \]

Some useful relations are that the scalar product of two vectors is equal to the trace norm of the corresponding matrices, and the vector product is essentially given by the matrix commutator,
\[ \frac{1}{2} \text{Tr}(vw) = v_a w^a, \quad \frac{1}{2}[v, w] = -\varepsilon^{abc} v_a w_b \gamma_c. \quad (A.4) \]

Sometimes it is useful to introduce cylindrical coordinates in Minkowski space, writing
\[ v = \tau \gamma_0 + \rho \gamma(\varphi), \quad (A.5) \]
where \( \tau \) and \( \rho \geq 0 \) are real, and \( \varphi \) is an angular direction. The vector \( \gamma(\varphi) \) and its derivative \( \gamma'(\varphi) \) form a rotated set of spacelike unit vectors, pointing into the direction of \( \varphi \) and the orthogonal direction,
\[ \gamma(\varphi) = \cos \varphi \gamma_1 + \sin \varphi \gamma_2, \quad \gamma'(\varphi) = \cos \varphi \gamma_2 - \sin \varphi \gamma_1. \quad (A.6) \]

Useful relations are
\[ \gamma_0 \gamma(\varphi) = \gamma'(\varphi), \quad \gamma_0 \gamma'(\varphi) = -\gamma(\varphi), \quad (A.7) \]
and
\[ \gamma(\varphi_1) \gamma(\varphi_2) = \cos(\varphi_1 - \varphi_2) \mathbf{1} + \sin(\varphi_1 - \varphi_2) \gamma_0. \quad (A.8) \]

The Lie group \( \text{SL}(2) \) consists of matrices \( u \) with unit determinant. The group acts on the algebra in the adjoint representation, so that
\[ v \mapsto uu^{-1}. \quad (A.9) \]

This provides a proper Lorentz rotation of the vector \( v \). The conjugacy classes of the algebra are characterized by the invariant length \( \frac{1}{2} \text{Tr}(v^2) \). For timelike vectors \( v \) with \( \frac{1}{2} \text{Tr}(v^2) < 0 \), we distinguish between positive timelike vectors with \( v^0 = \frac{1}{2}v^0 \gamma^0 > 0 \), and negative timelike vectors with \( v^0 = \frac{1}{2}v^0 \gamma^0 < 0 \). And similar for lightlike vectors.

A special group element, which represents a clockwise rotations about the \( \gamma_0 \)-axis by an angle \( \alpha \), is
\[ u = e^{\alpha \gamma_0/2} = \cos(\alpha/2) \mathbf{1} + \sin(\alpha/2) \gamma_0. \quad (A.10) \]

This implies
\[ \gamma_0 \mapsto u^{-1} \gamma_0 u = \gamma_0, \quad \gamma(\varphi) \mapsto u^{-1} \gamma(\varphi) u = \gamma(\varphi - \alpha). \quad (A.11) \]

A boost is specified by a rapidity \( \chi > 0 \) and a direction \( \beta \),
\[ u = e^{i \gamma(\beta)} = \cosh \chi \mathbf{1} + \sinh \chi \gamma(\beta). \quad (A.12) \]

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A generic element $u \in \text{SL}(2)$ can be expanded in terms of the unit and the gamma matrices, defining a scalar $u$ and a vector $p \in \mathfrak{sl}(2)$,

$$u = u^1 + p^a \gamma_a \quad \Rightarrow \quad p = p^a \gamma_a.$$  \hfill (A.13)

The scalar $u$ is basically the trace of $u$, and the vector $p$ is called the projection of $u$. The determinant condition implies that

$$u^2 = p^a p_a + 1.$$  \hfill (A.14)

According to the property of the vector $p$, we distinguish between timelike, lightlike and spacelike group elements $u$. For a timelike element we have $-1 < u < 1$, and it represents a rotation about some timelike axis, which is then specified by the vector $p$. The angle of rotation $\alpha$ is given by $u = \cos(\alpha/2)$, and the direction is given by the sign of $p^0$.

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