Complex Burgers-type equations and functional equations for the Cauchy transforms, the $R$-transforms, and the $S$-transforms of log-gas measures

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Abstract

We study hydrodynamic limits of three kinds of one-dimensional stochastic log-gases known as Dyson’s Brownian motion model, its chiral version, and the Bru–Wishart process studied in dynamical random matrix theory. We define the measure-valued processes so that their Cauchy transforms solve the complex Burgers-type equations. We show that applications of the method of characteristic curves to these partial differential equations provide functional equations relating the Cauchy transforms of measures at an arbitrary time and those at the initial time. We rewrite the complex Burgers-like equations and functional equations for the Cauchy transforms of the log-gas measures to those for the $R$-transforms and the $S$-transforms of the measures which play central roles in free probability theory. Some of the results are discussed using the notion of free convolutions.

Keywords: stochastic log-gases, complex Burgers-type equations, Cauchy transforms, $R$-transforms, $S$-transforms, free probability and free convolutions

1 Introduction and Main Results

Among a variety of recent developments in random matrix theory [33, 21, 2, 1], we will study an intersection of two important topics in this paper; time-dependent random matrix models [19, 12, 30, 28, 27] and free probability theory [5, 39, 34].

The Gaussian unitary ensemble (GUE) and the chiral GUE are typical eigenvalue distributions on $\mathbb{R}$ of Hermitian random matrices and their dynamical extension are described by systems of stochastic differential equations (SDEs) called Dyson’s Brownian motion model [19] and its chiral version [13, 30, 28]. Let $\mathcal{P}^0(S)$ be a set of all Borel probability measures on $S = \mathbb{R}$ or $S = \mathbb{R}_+ := [0, \infty)$ with finite supports equipped with the weak topology. For $T > 0$, $C([0, T] \to \mathcal{P}^0(S))$ denotes the space of continuous processes defined in the time period $[0, T]$ realized in $\mathcal{P}^0(S)$. We consider

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the hydrodynamic limit of the Dyson model and that of the chiral Dyson model with an additional parameter \( \lambda \in \mathbb{R}_+ \) as elements of \( C([0, \infty) \to \mathcal{P}^0(\mathbb{R})) \), which are denoted by \((w_t)_{t \geq 0}\) and \((w_{\lambda,t})_{t \geq 0}\), respectively. We assume that the initial probability measure \( w_0 \) and \( w_{\lambda,0} \) are in \( \mathcal{P}^0(\mathbb{R}) \) and the Cauchy transforms of measures,

\[
G_\mu(z) := \int_S \frac{\mu(dx)}{z - x}, \quad z \in \mathbb{C}^+: = \{z \in \mathbb{C} : \Im z > 0\}, \tag{1.1}
\]

are well defined satisfying the condition \( \lim_{y \to \infty} \sqrt{-1} y G_\mu(\sqrt{-1} y) = 1 \) for \( \mu = w_t \) and \( w_{\lambda,t}, \ t \geq 0 \). It is known that [11, 8, 37, 38, 9, 11] given \( G_{w_0} \) and \( G_{w_{\lambda,0}} \) obtained by \( w_0 \) and \( w_{\lambda,0} \), respectively, \((G_{w_t}(z))_{t \geq 0}\) and \((G_{w_{\lambda,t}}(z))_{t \geq 0}\) are uniquely determined by the solutions of the following partial differential equations (PDEs),

\[
\frac{\partial G_{w_t}(z)}{\partial t} + G_{w_t}(z) \frac{\partial G_{w_t}(z)}{\partial z} = 0, \quad t \geq 0, \tag{1.2}
\]

\[
\frac{\partial G_{w_{\lambda,t}}(z)}{\partial t} + \left(G_{w_{\lambda,t}}(z) + \frac{\lambda - 1}{2z} \right) \frac{\partial G_{w_{\lambda,t}}(z)}{\partial z} + \frac{1 - \lambda}{2z^2} G_{w_{\lambda,t}}(z) = 0, \quad t \geq 0. \tag{1.3}
\]

Equation (1.2) is known as the complex Burgers equation in the inviscid limit. It is obvious that when \( \lambda = 1 \) (1.3) is reduced to (1.2) and hence, if \( w_0 = w_{1,0} \), then

\[
w_t = w_{1,t}, \quad t \geq 0. \tag{1.4}
\]

In other words, the process \((w_{\lambda,t})_{t \geq 0}\) is the one-parameter \((\lambda \in \mathbb{R}_+)\) extension of \((w_t)_{t \geq 0}\).

For \( p > 0 \) and \( \mu \in \mathcal{P}^0(\mathbb{R}) \), the \( p \)-th push-forward measure \( \mu^{(p)} \) is defined by

\[
\mu^{(p)}(B) := \int_{\mathbb{R}} 1_B(|x|^p) \mu(dx), \quad B \in \mathcal{B}((0, \infty)), \tag{1.5}
\]

where \( \mathcal{B}((0, \infty)) \) denotes the \( \sigma \)-algebra of Borel sets on \((0, \infty)\) [10].

We define

\[
m_{\lambda,t} := w_{\lambda,t}^{(2)} \in \mathcal{P}^0(\mathbb{R}_+), \quad t \geq 0, \tag{1.6}
\]

provided the matching of initial measures in the sense that \( m_{\lambda,0} = w_{\lambda,0}^{(2)} \). Note that combining this definition with (1.4) we have the equality

\[
w_t^{(2)} = m_{1,t}, \quad t \geq 0. \tag{1.7}
\]

We can show that by (1.6) the PDE (1.3) is transformed to the following equation for the Cauchy transform \( G_{m_{\lambda,t}}(z) \) of \( m_{\lambda,t} \),

\[
\frac{\partial G_{m_{\lambda,t}}(z)}{\partial t} + \left(2zG_{m_{\lambda,t}}(z) + \lambda - 1 \right) \frac{\partial G_{m_{\lambda,t}}(z)}{\partial z} + G_{m_{\lambda,t}}(z)^2 = 0, \quad t \geq 0. \tag{1.8}
\]

It should be noted that this PDE is obtained when we consider the hydrodynamic limit of the system of SDEs [14] known as the Bru–Wishart process in multivariate stochastic calculus [13, 28] and as the Laguerre process in dynamical random matrix theory [30]. The PDEs (1.3) and (1.8) describe the large-number limits of colors of the systems with the chiral symmetry in the quantum chromodynamics (QCD) in high energy physics [26, 8, 37, 38, 10, 31]. We call (1.3) and (1.8)
The simplest initial probability measure in $\mathcal{P}^0(S)$, $S = \mathbb{R}$ or $\mathbb{R}_+$ is the single delta measure on the origin $\delta_0$. We regard the solution of the Burgers-type equation starting from $G_{\delta_0}(z) = 1/z$ as the fundamental solution and denote the obtained measure-valued process as $(\mu_t^0)_{t \geq 0}$ with a superscript 0. We can show that (see, for instance, [34])

$$G_{w^0}(z) = \frac{1}{2t}\left[\frac{z - \sqrt{z^2 - 4t}}{z}\right], \quad t \geq 0,$$

$$G_{m^0_{\lambda,t}}(z) = \frac{1}{2tz}\left[z + t(1 - \lambda) - \sqrt{(z - x_{\lambda,t}^+)(z - x_{\lambda,t}^-)}\right] \quad \text{with} \quad x_{\lambda,t}^\pm = t(1 \pm \sqrt{\lambda})^2, \quad t \geq 0,$$

and they determine the time-dependent measures as

$$w_t^0(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2} 1_{[-2\sqrt{t}, 2\sqrt{t}]}(x)dx, \quad t \geq 0,$$

$$m_{\lambda,t}^0(dx) = \max(0, 1 - \lambda)\delta_0(dx) + \frac{1}{2\pi tx}\sqrt{(x - x_{\lambda,t}^-)(x_{\lambda,t}^+ - x)} 1_{[x_{\lambda,t}^-, x_{\lambda,t}^+]}(x)dx, \quad t \geq 0,$$

respectively. Here, for $B \in \mathcal{B}(\mathbb{R})$, $1_B(x) = 1$ if $x \in B$, $1_B(x) = 0$ otherwise. The measure (1.9) is known as the centered Wigner’s semicircle distribution with variance $t$ and (1.10) is the two-parametric Marcenko–Pastur distribution with parameters $\lambda$ and $t$ [9, 11, 20]. Here we emphasize the fact that the equality $(w_t^0)^2 = m_{1,t}^0$, $t \geq 0$ is well known and the equality (1.7) mentioned above generalizes it for any initial probability measure with finite support satisfying $w_0^2 = m_{1,0}$.

Let $\mathcal{P}_s^0(\mathbb{R})$ be the set of all symmetric Borel probability measures on $\mathbb{R}$ (i.e., $\mu(B) = \mu(-B)$, $B \in \mathcal{B}((0, \infty))$) and define the symmetric Bernoulli delta measure with displacement $2a > 0$ as

$$d_a := \frac{1}{2}(\delta_{-a} + \delta_a) \in \mathcal{P}_s^0(\mathbb{R}).$$

The processes $(w_t^1)_{t \geq 0}$ and $(w_{\lambda,t})_{t \geq 0}$ starting from $d_a$, which are defined by the solutions of (1.2) and (1.3) with the initial condition $G_{d_a}(z) = z/(z^2 - a^2)$, were reported in [35, 36, 3] and in [20], respectively. These solutions show dynamical phase transitions, where the parameter $a$ controls a transition observed at a critical time. We have found that they are very interesting but much more complicated compared with the fundamental solutions (1.9) and (1.10). See [23] for the exact solutions for other initial probability measures. It has seemed to be rather difficult to discuss general properties of the present measure-valued processes by directly solving the Burgers-type equations under general initial probability measures.

In free probability theory [35, 44, 24, 39, 40, 4, 34], we can learn importance of other transformations of probability measures than the Cauchy transforms, which are known as the $R$-transform and the $S$-transform. They define new-types of convolutions of probability measures called free convolutions. Here we report the applications of these notions to the present time-dependent systems. Given a probability measure $\mu \in \mathcal{P}^0(S)$, let $\tau_n(\mu)$ represent the $n$-th moment of $\mu$, $n \in \mathbb{N}$ and define the moment generating function by

$$\Psi(\mu)(z) := \int_S \frac{xz}{1 - xz} \mu(dx) = \sum_{n=1}^{\infty} \tau_n(\mu)z^n.$$

(1.12)

The Cauchy transform $G_\mu$ of $\mu$ is related to $\Psi_\mu$ by

$$\Psi_\mu(z) = \frac{G_\mu(1/z)}{z} - 1 \iff G_\mu(1/z) = z(\Psi_\mu(z) + 1).$$

(1.13)
We write the inverse function of $G_\mu$ as $G_\mu^{(-1)}$. The R-transform of $\mu$ is then defined by

$$R_\mu(z) := zG_\mu^{(-1)}(z) - 1 \iff G_\mu^{(-1)}(z) = \frac{R_\mu(z) + 1}{z}. \quad (1.14)$$

This is the generating function of free cumulants $\kappa_n = \kappa_n(\mu), n \in \mathbb{N}$;

$$R_\mu(z) = \sum_{n=1}^{\infty} \kappa_n(\mu)z^n.$$ 

The relations between the moments $\{\tau_n(\mu)\}$ in (1.12) and the free cumulants $\{\tau_n(\mu)\}$ are given by

$$\begin{align*}
\kappa_1(\mu) &= \tau_1(\mu), \\
\kappa_2(\mu) &= \tau_2(\mu) - \tau_1(\mu)^2, \\
\kappa_3(\mu) &= \tau_3(\mu) - 3\tau_2(\mu)\tau_1(\mu) + 2\tau_1(\mu)^3, \\
\kappa_4(\mu) &= \tau_4(\mu) - 4\tau_3(\mu)\tau_1(\mu) - 2\tau_2(\mu)^2 + 10\tau_2(\mu)\tau_1(\mu)^2 - 5\tau_1(\mu)^4, \\
&\quad \ldots,
\end{align*}$$

which are generally different from the relations satisfied in the ‘classical probability theory’. For two probability measures $\mu, \nu$, the free additive convolution of them is denoted by $\mu \boxplus \nu$ and defined by [5]

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

Moreover, if $\mu$ and $\nu$ are free independent, then

$$\kappa_n(\mu \boxplus \nu) = \kappa_n(\mu) + \kappa_n(\nu).$$

For $\mu \in \mathcal{P}_s^0(\mathbb{R}_+)$ with $\mu(\{0\}) < 1$, the moment generating function $\Psi_\mu(z)$ defined by (1.13) has a unique inverse $\chi_\mu(z) := \Psi_\mu^{(-1)}(z)$ on the left-half plane $\sqrt{-1}C^+$ [5]. In this case the $S$-transform of $\mu$ is defined by

$$S_\mu(z) := \frac{1 + z}{z} \chi_\mu(z), \quad z \in \Psi_\mu(\sqrt{-1}C^+). \quad (1.15)$$

For two probability measures $\mu, \nu$ having $S$-transforms, the free multiplicative convolution $\boxtimes$ is defined as the probability measure $\mu \boxtimes \nu$ such that

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z)S_\nu(z)$$

for $z$ in a common region of $\Psi_\mu(\sqrt{-1}C^+) \cup \Psi_\mu(\sqrt{-1}C^+)$. \n
The definition of the $S$-transform can be extended for symmetric probability measures $\mu \in \mathcal{P}_s^0(\mathbb{R})$ [5] [10]. When $\mu \in \mathcal{P}_s^0(\mathbb{R})$ with $\mu(\{0\}) < 1$, $\Psi_\mu(z)$ has a unique inverse on $H := \{z \in C^- : |Rz| < |3z|\}$, $\chi_\mu : \Psi_\mu(H) \to H$ and a unique inverse on $\tilde{H} := \{z \in C^+ : |Rz| < |3z|\}$, $\tilde{\chi}_\mu : \Psi_\mu(\tilde{H}) \to H$, where $C^- := \{z \in C : 3z < 0\}$. Therefore, there are two $S$-transforms for $\mu$ given by

$$S_\mu(z) = \frac{1 + z}{z} \chi_\mu(z) \quad \text{and} \quad \tilde{S}_\mu(z) = \frac{1 + z}{z} \tilde{\chi}_\mu(z),$$

and they satisfy

$$S_\mu(z)^2 = \frac{1 + z}{z} S_{\mu(z)}(z) \quad \text{and} \quad \tilde{S}_\mu(z)^2 = \frac{1 + z}{z} S_{\mu(z)}(z).$$
It is known that for a probability measure $\mu \in \mathcal{P}^0(\mathbb{R}_+)$, there exists a unique symmetric probability measure $\mu_s \in \mathcal{P}_s^0$ such that
\[
\int_{\mathbb{R}} f(x^2) \mu_s(dx) = \int_{\mathbb{R}} f(x) \mu(1/2)(dx)
\]
for every compactly supported continuous function $f$. The probability measure $\mu_s$ is called symmetrization of a probability measure $\mu$. For details, see [25, page 134]. In the present paper, we will note that the $S$-transform of $d_a$ defined by (1.11) is given by
\[
S_{d_a}(z) = \frac{a}{a + z}, \quad a > 0.
\]
If the parameter $a = 1$, we use the notation $d$ for $d_1$ [40].

The following are the main results in this paper.

**Theorem 1.1**  
(i) Assume that $w_0 \in \mathcal{P}^0(\mathbb{R})$. Then
\[
R_{w_t}(z) = R_{w_0}(z) + R_{w_t^0}(z), \quad t \geq 0,
\]
where $(w_t^0)_{t \geq 0}$ is given by (1.9). It means that $w_t = w_0 \boxplus w_t^0$, $t \geq 0$.

(ii) Assume that $m_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}_+)$. Then
\[
R_{m_{\lambda,t}}(z) = \frac{1}{1 - tz} R_{m_{\lambda,0}} \left( \frac{z}{1 - tz} \right) + R_{m_{\lambda,t}^0}(z), \quad t \geq 0,
\]
where $(m_{\lambda,t}^0)_{t \geq 0}$ is given by (1.10).

(iii) Provided that $w_{\lambda,0} = m_{\lambda,0}^s$, the following equality holds,
\[
S_{w_{\lambda,t}}(z) = S_{d}(z) \sqrt{S_{m_{\lambda,t}}(z)}, \quad t \geq 0.
\]

**Remark 1**  Consider a matrix-valued Brownian motion $(M_t)_{t \geq 0}$ which is given by a time-evolution of a Hermitian $N \times N$ matrix starting from a Hermitian matrix $M_0$. When $M_0$ is a null matrix, we write this process as $(M_t^0)_{t \geq 0}$. Then we have
\[
M_t = M_0 + M_t^0, \quad t \geq 0.
\]
We assume that the empirical eigenvalue distribution of $M_0$ converges to $w_0$ as $N \to \infty$. As an eigenvalue process of (1.19), we can obtain Dyson’s Brownian motion model with $\beta = 2$ starting from the eigenvalues of $M_0$. (See Section 2.1 below.) Moreover, for any $\beta > 0$, we can obtain the same Cauchy transform. The process $(w_t)_{t \geq 0}$ is obtained as the time-evolution of the limit empirical measure of Dyson’s Brownian motion model. We can show that $M_0$ and $M_t^0$ are asymptotically free for any $t \geq 0$. Thus at each time $t \geq 0$, the limiting eigenvalue distribution $w_t$ of $M_t$ converges to $w_0 \boxplus w_t^0$. That is, the assertion (i) of Theorem 1.1 is consistent with the asymptotic freeness of a random matrix in the GUE and an arbitrary deterministic Hermitian matrix. Such an interpretation of the assertion (ii) of Theorem 1.1 is required.
Remark 2 The process \((w^0_t)_{t \geq 0}\) given by (1.9) is identified with the free Brownian motion studied in free probability theory [7]. Here we would like to consider the process \((w_t)_{t \geq 0}\) determined by the Burgers equation (1.2) as a generalization of the free Brownian motion, since initial probability measure \(w_0\) is now arbitrary in \(\mathcal{P}^0(\mathbb{R})\). Capitaine and Donati-Martin [15] introduce the free Wishart processes based on the Marcenko–Pastur distribution (1.10). Our process \((m_{\lambda,t})_{t \geq 0}\) is defined by the Burgers-type equation (1.8) specified by its initial probability measure \(m_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}_+)^N\) and is different from the free Wishart process. The equality (1.8) in Theorem 1.1 (ii) characterize our process \((m_{\lambda,t})_{t \geq 0}\). To the best of our knowledge, the chiral GUE and its time evolution \((w_{\lambda,t})_{t \geq 0}\) determined by (1.3) have not been systematically studied in free probability theory. Theorem 1.1 (iii) first reveals the simple but useful relationship among \((w_{\lambda,t})_{t \geq 0}\), \((m_{\lambda,t})_{t \geq 0}\) and \(d\).

The present paper is organized as follows. In Section 2 we explain how the complex Burgers-type equations are derived in hydrodynamic limits of stochastic log-gases. Then we prove fundamental relations between a measure \(\mu \in \mathcal{P}^s(\mathbb{R})\) and its second push-forward measure \(\mu^{(2)} \in \mathcal{P}^0(\mathbb{R}_+)^N\). Section 3 is denoted to solving the present three kinds of complex Burgers equations (1.2), (1.3), and (1.8) by the method of characteristic curves [16, 17]. Proof of Theorem 1.1 is given in Section 4 and concluding remarks are in Section 5. In Appendix A functional equations of the \(S\)-transforms are derived. The \(R\)-transforms and the \(S\)-transforms of \((w_t)_{t \geq 0}\) starting from \(d_u, a > 0\) and \((m_{\lambda,t})_{t \geq 0}\) starting from \(\delta_b, b > 0\) are shown in Appendices B and C respectively, as applications of Theorem 1.1.

2 Preliminaries

2.1 From matrix-valued processes to complex Burgers-type equations through hydrodynamic limit

For \(N \in \mathbb{N} := \{1, 2, \ldots\}\), let \(H_N\) and \(U_N\) be the space of \(N \times N\) Hermitian matrices and the group of \(N \times N\) unitary matrices, respectively. We consider complex-valued continuous semi-martingale processes \((M^{ij}_t)_{t \geq 0}, 1 \leq i, j \leq N\) with the condition \(\overline{M^{ij}_t} = M^{ij}_t\), where \(\overline{z}\) denotes the complex conjugate of \(z \in \mathbb{C}\), and define an \(H_N\)-valued process by \(M_t = (M^{ij}_t)_{1 \leq i, j \leq N}\). For \(S = \mathbb{R}\) or \(\mathbb{R}_+\), define the Weyl chambers as \(\mathcal{W}_N(S) := \{x = (x^1, \ldots, x^N) \in S^N : x^1 < \cdots < x^N\}\), and write their closures as \(\overline{\mathcal{W}_N(S)} = \{x \in \overline{S^N} : x^1 \leq \cdots \leq x^N\}\). For each \(t \geq 0\), there exists \(U_t = (U^{ij}_t)_{1 \leq i, j \leq N} \in U_N\) such that it diagonalizes \(M_t\) as \(U^\dagger_t M_t U_t = \text{diag}(\Lambda^1_t, \ldots, \Lambda^N_t)\) with the eigenvalues \(\{\Lambda^i\}_{i=1}^N\) of \(M_t\), where \(U^\dagger_t\) is the Hermitian conjugate of \(U_t\), \((U^{ij}_t)^\dagger = U^{ji}_t\), \(1 \leq i, j \leq N\), and we assume \(\Lambda_t := (\Lambda^1_t, \ldots, \Lambda^N_t) \in \overline{\mathcal{W}_N(\mathbb{R})}, t \geq 0\). For \(dM_t := (dM^{ij}_t)_{1 \leq i, j \leq N}\), define a set of quadratic variations,

\[
\Gamma^{ij,kl}_t := \mathbb{E}\left(\left|(U^\dagger_t dMU)^{ij}_t\right|, (U^\dagger_t dMU)^{kl}_t\right), \quad 1 \leq i, j, k, \ell \leq N, \quad t \geq 0.
\]

We denote by \(\mathbf{1}_{(E)}\) the indicator function of an event \(E\); \(\mathbf{1}_{(E)} = 1\) if \(E\) occurs, and \(\mathbf{1}_{(E)} = 0\) otherwise. As a special case, Kronecker’s delta is defined by \(\delta^{ij} := \mathbf{1}_{(i=j)}, i, j \in \mathbb{N}\). Note that \(1_B(x) = \mathbf{1}_{(x \in B)}\) for \(B \in \mathcal{B}(\mathbb{R})\). The following is proved [12, 28, 27]. See Section 4.3 of [2] for details of proof.

Proposition 2.1 The eigenvalue process \((\Lambda_t)_{t \geq 0}\) satisfies the following system of SDEs,

\[
d\Lambda^i_t = dM^i_t + dL^i_t, \quad t \geq 0, \quad 1 \leq i \leq N,
\]
where \((\mathcal{M}^i_t)_{t \geq 0}, 1 \leq i \leq N\) are martingales with quadratic variations \(\langle d\mathcal{M}^i, d\mathcal{M}^j \rangle_t = \Gamma^{ij,ji}_{t} dt, t \geq 0,\)
and \((J^i_t)_{t \geq 0}, 1 \leq i \leq N\) are the processes with finite variations given by
\[
dJ^i_t = \sum_{j=1}^N 1_{\{\Lambda^i_t \neq \Lambda^j_t\}} \Gamma^{ij,ji}_t dt + d\Upsilon^i_t.
\]
Here \(d\Upsilon^i_t\) denotes the finite-variation part of \((U^i_t d\mathcal{M}^i_t)^i\), \(t \geq 0, 1 \leq i \leq N\).

We will show two basic examples of \(\mathcal{M}_t \in \mathbb{H}_N, t \geq 0\) and applications of Proposition 2.1. Let \(\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\) and \((B^i_{t^j})_{t \geq 0}, 1 \leq i \leq N + \nu, 1 \leq j \leq N\) be independent one-dimensional standard Brownian motions. For \(1 \leq i \leq j \leq N\), put
\[
S^i_{t^j} = \begin{cases} B^i_{t^j}/\sqrt{2}, & (i < j), \\ B^i_{t^j}, & (i = j), \\ \end{cases} \quad A^i_{t^j} = \begin{cases} \tilde{B}^i_{t^j}/\sqrt{2}, & (i < j), \\ 0, & (i = j), \end{cases}
\]
and let \(S^i_{t^i} = S^i_{t^i}\) (symmetric) and \(A^i_{t^j} = -A^j_{t^i}\) (anti-symmetric), \(t \geq 0\) for \(1 \leq j < i \leq N\).

**Example 1** Put
\[
\mathcal{M}_t = (\mathcal{M}^i_{t^j}) := (S^i_{t^j} + \sqrt{-1} A^i_{t^j})_{1 \leq i, j \leq N}, \quad t \geq 0.
\]
By definition \(\langle d\mathcal{M}^i, d\mathcal{M}^j \rangle_t = \delta_{ij} \delta_{kl} dt, t \geq 0, 1 \leq i, j, k, \ell \leq N\). Hence, by unitarity of \(U_t \in \mathbb{U}_N, t \geq 0\), we see that \(\Gamma^{ij,k\ell} = \delta_{ik} \delta_{j\ell}\), which gives \(\langle d\mathcal{M}^i, d\mathcal{M}^j \rangle_t = \Gamma^{ij,ji}_t dt = \delta_{ij} dt\) and \(\Gamma^{ij,ji}_t \equiv 1, t \geq 0, 1 \leq i, j \leq N\). Then Proposition 2.1 proves that the eigenvalue process \((\Lambda^i_t)_{t \geq 0}\), satisfies the following system of SDEs with \(\beta = 2\),
\[
d\Lambda^i_t = dB^i_t + \frac{\beta}{2} \sum_{1 \leq j \leq N, j \neq i} \frac{dt}{\Lambda^i_t - \Lambda^j_t}, \quad t \geq 0, \quad 1 \leq i \leq N. \quad (2.1)
\]
Here \((B^i_{t^j})_{t \geq 0}, 1 \leq i \leq N\) are independent one-dimensional standard Brownian motions, which are different from \((B^i_{t^j})_{t \geq 0}\) and \((\tilde{B}^i_{t^j})_{t \geq 0}\) used to define \((S^i_{t^j})_{t \geq 0}\) and \((A^i_{t^j})_{t \geq 0}\). If \(\beta = 2\) and the initial configuration is \(N \delta_0\), that is, all \(N\) particles are at the origin, then at each time \(t > 0\), \(\Lambda_t = (\Lambda^1_t, \ldots, \Lambda^N_t)\) gives a point process on \(\mathbb{R}\) which is equal in distribution with the GUE eigenvalue point process with variance \(t\) \([19, 28]\). For \(\beta > 0\), we call the solution of (2.1) the \(N\)-particle system of *Dyson’s Brownian motion model* with parameter \(\beta\) \([19]\), and write it as \((\Lambda^i_t)_{t \geq 0}\).

**Example 2** Consider an \((N + \nu) \times N\) rectangular-matrix-valued process given by
\[
K_t := (B^i_{t^j} + \sqrt{-1} \tilde{B}^i_{t^j})_{1 \leq i \leq N + \nu, 1 \leq j \leq N}, \quad t \geq 0,
\]
and define an \(\mathbb{H}_N\)-valued process by
\[
\mathcal{M}_t = K^\dagger(t) K(t), \quad t \geq 0.
\]
The matrix \(\mathcal{M}_t\) is positive semi-definite and hence the eigenvalues are non-negative; \(\Lambda^i_t \in \mathbb{R}_+, t \geq 0, 1 \leq i \leq N\). We see that the finite-variation part of \(d\mathcal{M}^i_t\) is equal to \(2(N + \nu) \delta_{ij} dt, t \geq 0, \quad \langle d\mathcal{M}^i, d\mathcal{M}^j \rangle_t = 2(\mathcal{M}^i_t \delta^{jk} + \mathcal{M}^j_t \delta^{ik}) dt, t \geq 0, 1 \leq i, j, k, \ell \leq N\), which implies that
\[ dY^i_t = 2(N + \nu) dt, \quad \Gamma^{ij}_t = 2(\Lambda^i_t + \Lambda^j_t), \quad \text{and} \quad (dM^i, dM^j)_t = \Gamma^{ii,jj}_t dt = 4\Lambda^i_t \delta^{ij} dt, \quad t \geq 0, \quad 1 \leq i, j \leq N. \]

Then we have the following SDEs with \( \beta = 2 \) for the eigenvalue process of \((M_t)_{t \geq 0}\),

\[
d\Lambda^i_t = 2\sqrt{\Lambda^i_t} d\tilde{B}^i_t + \beta \left[(\nu + 1) + 2\Lambda^i_t \sum_{1 \leq j \leq N, j \neq i} \frac{1}{\Lambda^j_t - \Lambda^i_t}\right] dt, \quad t \geq 0, \quad 1 \leq i \leq N, \tag{2.2}
\]

where \((\tilde{B}^i_t)_{t \geq 0}, 1 \leq i \leq N\) are independent one-dimensional standard Brownian motions, which are different from \((B^i_t)_{t \geq 0}\) and \((\tilde{B}^i_t)_{t \geq 0}, 1 \leq i, j \leq N\), used above to define the rectangular-matrix-valued process \((K_t)_{t \geq 0}\). The parameter \(\nu\) can be extended to \(\nu > -1\), in which if \(\nu \in (-1, 0)\), a reflecting wall is put at the origin \([28]\). We call the solution of (2.2) the \(N\)-particle system of the Bru–Wishart process with parameters \((\beta, \nu)\) \([13]\), and write it as \(\Lambda^{\text{BW}}(\beta, \nu)_{t \geq 0}\).

The positive roots of eigenvalues of \(M_t\) give the singular values of the rectangular matrix \(K_t\), which are denoted by

\[
S^i_t := \sqrt{\Lambda^i_t}, \quad t \geq 0, \quad 1 \leq i \leq N. \tag{2.3}
\]

The system of SDEs for them is readily obtained from (2.2) as

\[
dS^i_t = d\tilde{B}^i_t + \frac{\beta(\nu + 1) - 1}{2S^i_t} dt + \frac{\beta}{2} \sum_{1 \leq j \leq N, j \neq i} \left(\frac{1}{S^j_t - S^i_t} + \frac{1}{S^i_t + S^j_t}\right) dt, \tag{2.4}
\]

\(t \geq 0, 1 \leq i \leq N\) with \(\beta = 2\) and \(\nu > -1\). If \(\beta = 2\) and the initial configuration is \(N\delta_0\), then at each time \(t > 0\), \(S_t = (S^1_t, \ldots, S^N_t)\) on \(\mathbb{R}_+^N\) gives the chiral GUE point process with parameter \(\nu\) and variance \(t\) studied in random matrix theory for high energy physics \([42, 45, 44, 21, 1]\). For \(\beta > 0\), we call the solution of (2.4) the chiral version of Dyson’s Brownian motion model with parameters \((\beta, \nu)\), and write it as \((S^{\text{chD}}(\beta, \nu))_{t \geq 0}\).

At each time \(t > 0\), the point processes \(\Lambda^{\text{D}}(N, \beta), \Lambda^{\text{BW}}(N, \beta, \nu), \text{and } S^{\text{chD}}(\beta, \nu)\) are known as typical examples of one-dimensional log-gases \([21]\). Therefore, we will call the solutions of the SDEs (2.1), (2.2), and (2.4) stochastic log-gases. Note that, when \(\beta = 2\), (2.2) and (2.4) can be regarded as the \(N\)-variable extensions of the \(2(\nu + 1)\)-dimensional squared Bessel process and the Bessel process with parameter \(\nu > -1\), respectively \([27]\).

For \(\Lambda^i_t(N, \beta) = (\Lambda^i_t(N, \beta)_1, \ldots, \Lambda^i_t(N, \beta)_N), \quad t \geq 0\), we regard the time evolution of empirical measures

\[
\Xi^\text{D}(N, \beta)_t(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{\Lambda^i_t(N, \beta)_1(\cdot)}, \quad t \in [0, T],
\]

as an element of \(C([0, T] \to \mathcal{P}^0(\mathbb{R}))\). For \((\Lambda^{\text{BW}}(N, \beta, \nu))_{t \geq 0}\), and \((S^{\text{D}}(N, \beta, \nu))_{t \geq 0}\), let

\[
\lambda := \frac{N + \nu}{N} \iff \nu = (\lambda - 1)N,
\]

and consider

\[
\Xi^{\text{BW}}(N, \beta, (\lambda - 1)N)_t(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{\Lambda^{\text{BW}}(N, \beta, (\lambda - 1)N)_1(\cdot)}, \quad t \in [0, T]
\]

as an element of \(C([0, T] \to \mathcal{P}^0(\mathbb{R}_+))\), and with (2.3) consider

\[
\Sigma^{\text{chD}}(N, \beta, (\lambda - 1)N)_t(\cdot) := \frac{1}{2N} \sum_{i=1}^N \left\{ \delta_{S^{\text{chD}}(N, \beta, (\lambda - 1)N)_1(\cdot)} + \delta_{-S^{\text{chD}}(N, \beta, (\lambda - 1)N)_1(\cdot)} \right\}, \quad t \in [0, T],
\]

8
Theorem 2.2 Assume that for any $N \in \mathbb{N}$, the initial measures $\Xi_t^{D(N,\beta)}$, $\Xi_t^{BW(N,\beta,(\lambda-1)N)}$, and $\Sigma_t^{chD(N,\beta,(\lambda-1)N)}$ have finite supports, where $\Sigma_t^{chD(N,\beta,(\lambda-1)N)}(2) = \Xi_t^{BW(N,\beta,(\lambda-1)N)}$ is satisfied, and in $N \to \infty$ they converge weakly to the measures $w_0 \in \mathcal{P}_0(\mathbb{R})$, $m_{\lambda,0} \in \mathcal{P}_0^0(\mathbb{R})$, and $w_{\lambda,0} \in \mathcal{P}_s^0(\mathbb{R})$, respectively. Then for any fixed $T < \infty$,

\[
(\Xi_t^{D(N,\beta)}(\cdot))_{t \in [0,T]} \Rightarrow (w_t(\cdot))_{t \in [0,T]} \quad \text{a.s. in } C([0,T] \to \mathcal{P}_0^0(\mathbb{R})),
\]

\[
(\Xi_t^{BW(N,\beta,(\lambda-1)N)}(\cdot))_{t \in [0,T]} \Rightarrow (m_{\lambda,t}(\cdot))_{t \in [0,T]} \quad \text{a.s. in } C([0,T] \to \mathcal{P}_0^0(\mathbb{R})),
\]

\[
(\Sigma_t^{chD(N,\beta,(\lambda-1)N)}(\cdot))_{t \in [0,T]} \Rightarrow (w_{\lambda,t}(\cdot))_{t \in [0,T]} \quad \text{a.s. in } C([0,T] \to \mathcal{P}_s^0(\mathbb{R})),
\]

and the Cauchy transforms of the limit measures satisfy the PDEs \(\{1.2\}, \{1.3\}\), and \(\{1.3\}\), respectively.

Note that dependence on the parameter $\beta$ vanishes in the limit $N \to \infty$. By the construction mentioned above, the relation

\[
(\Sigma_t^{chD(N,\beta,(\lambda-1)N)}(2)) = (\Xi_t^{BW(N,\beta,(\lambda-1)N)}), \quad t \geq 0,
\]

holds, and then Theorem 2.2 proves the equality $w_{\lambda,t}^{(2)} = m_{\lambda,t}$, $t \geq 0$. This is consistent with the definition \(\{1.6\}\).

2.2 Expressions of second push-forward measures

We can prove the following.

Lemma 2.3 For $\mu \in \mathcal{P}_s^0(\mathbb{R})$ and $\nu \in \mathcal{P}_0^0(\mathbb{R})$, the following four statements are equivalent with each other,

\[
\begin{align*}
\text{(i)} & \quad \mu^{(2)} = \nu, \\
\text{(ii)} & \quad G_\mu(z) = z G_\nu(z^2), \\
\text{(iii)} & \quad R_\mu(z) = R_\nu \left( \frac{z^2}{R_\mu(z) + 1} \right), \\
\text{(iv)} & \quad S_\mu(z) = S_d(z) \sqrt{S_\nu(z)}. 
\end{align*}
\]

Proof Assume that $\mu(dx)$ (resp. $\nu(dx)$) has a probability density function $\rho_\mu(x)$ (resp. $\rho_\nu(x)$). Let $B \in \mathcal{B}((0,\infty))$. Then

\[
\nu(B) = \int_{\mathbb{R}} 1_B(x) \rho_\nu(x) dx.
\]

On the other hand, by definition \(\{1.5\}\), $\mu^{(2)}(B) = \int_{\mathbb{R}} 1_B(x^2) \rho_\mu(x) dx$. Hence

\[
\begin{align*}
\text{(i)} & \quad \iff \rho_\mu(x) = \rho_\nu(x^2) |x|, \quad x \in \mathbb{R}. 
\end{align*}
\]

(2.5)
Since (2.5) implies the symmetry $\rho_\mu(-x) = \rho_\mu(x)$, $x \in \mathbb{R}$, we see that

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x} \, dx$$

$$= \frac{1}{2} \left\{ \int_{\mathbb{R}} \frac{\rho_\mu(-x)}{z-x} \, dx + \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x} \, dx \right\}$$

$$= \frac{1}{2} \left\{ \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z+x} \, dx + \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z-x} \, dx \right\} = z \int_{\mathbb{R}} \frac{\rho_\mu(x)}{z^2-x^2} \, dx.$$

Then when (2.5) is satisfied,

$$G_\mu(z) = z \int_{\mathbb{R}} \frac{\rho_\mu(x^2)|x|}{z^2-x^2} \, dx = z \int_{\mathbb{R}_+} \frac{\rho_\nu(x^2)}{z^2-x^2} \, dx^2 = zG_\nu(z^2) \iff \text{ (ii)}.$$

When (ii) is satisfied,

$$G_\mu \left( \frac{R_\mu(z)+1}{z} \right) = \frac{R_\mu(z)+1}{z} G_\nu \left( \left( \frac{R_\mu(z)+1}{z} \right)^2 \right)$$

holds. By (1.14), this implies

$$z = \frac{R_\mu(z)+1}{z} G_\nu \left( \left( \frac{R_\mu(z)+1}{z} \right)^2 \right) \iff G_\nu \left( \left( \frac{R_\mu(z)+1}{z} \right)^2 \right) = \frac{z^2}{R_\mu(z)+1}$$

$$\iff \left( \frac{R_\mu(z)+1}{z} \right)^2 = G_\nu^{(-1)} \left( \frac{z^2}{R_\mu(z)+1} \right) = \frac{R_\mu(z)+1}{z^2} \left\{ R_\nu \left( \frac{z^2}{R_\mu(z)+1} \right) + 1 \right\},$$

where we used (1.13) again. This is equivalent with (iii).

When (ii) is satisfied,

$$G_\mu \left( \frac{R_\mu(z)+1}{z} \right) = \frac{R_\mu(z)+1}{z} G_\nu \left( \left( \frac{R_\mu(z)+1}{z} \right)^2 \right)$$

holds. By (1.13) and (1.15) with $\chi_\mu(z) := \Psi_\mu^{(-1)}(z)$, we can prove the equality

$$G_\mu \left( \frac{z+1}{zS_\mu(z)} \right) = \frac{z+1}{zS_\mu(z)} G_\nu \left( \left( \frac{z+1}{zS_\mu(z)} \right)^2 \right) \tag{2.6}$$

holds. Then (2.6) gives

$$G_\nu \left( \left( \frac{z+1}{zS_\mu(z)} \right)^2 \right) = (z+1) \left( \frac{zS_\mu(z)}{z+1} \right)^2. \tag{2.7}$$

By (1.13), the LHS of (2.7) is equal to $[\Psi_\nu(\{zS_\mu(z)/(z+1)\}^2) + 1]\{zS_\mu(z)/(z+1)\}^2$. Hence we obtain the equalities $z = \Psi_\nu(\{zS_\mu(z)/(z+1)\}^2) \iff \chi_\nu(z) = \{zS_\mu(z)/(z+1)\}^2$. By (1.15), this gives

$$S_\mu(z)^2 = \frac{1+z}{z} S_\nu(z).$$

We use (1.16) with $a = 1$ and then (iv) is obtained. Hence the proof is complete. ■
3 General Solutions of Complex Burgers-type Equations

Let $t \in [0, \infty)$ and $z \in \mathbb{C}^+$ be independent variables and consider a PDE for a complex function $g = g(t, z) \in \mathbb{C}$ in the form,

\[
A(t, z, g) \frac{\partial g}{\partial t} + B(t, z, g) \frac{\partial g}{\partial z} = C(t, z, g).
\]  

(3.1)

We regard the solution of (3.1) as a surface $g = g(t, z)$ in the space $[0, \infty) \times \mathbb{C}^+ \times \mathbb{C}$. Then (3.1) is interpreted as a geometrical statement that the vector field $(A(t, z, g), B(t, z, g), C(t, z, g))$ is tangent to the surface at every point. This statement means that the graph of solution is given by a union of integral curves of this vector field. They are called the characteristic curves \[16\] of (3.1) and satisfy the Lagrange–Charpit equation (see \[17\] and references therein),

\[
\frac{dt}{A(t, z, g)} = \frac{dz}{B(t, z, g)} = \frac{dg}{C(t, z, g)}.
\]  

(3.2)

Here we consider the special case such that $A(t, z, g) \equiv 1$. Then (3.2) is written as

\[
\begin{align*}
\frac{dz}{dt} &= B(t, z, g), \\
\frac{dg}{dt} &= C(t, z, g).
\end{align*}
\]  

(3.3)

We show that the solutions of (3.3) for (1.2), (1.3) and (1.8) are obtained in the forms of functional equations. The formula (3.4) is well known and found in literature, but the formulas (3.5) and (3.6) seem to be new.

**Proposition 3.1**

(i) Given the Cauchy transform $G_{w_0}(z)$ of the initial measure $w_0 \in \mathcal{P}^0(\mathbb{R})$, the solution of (1.2) satisfies the functional equation,

\[
G_{w_1}(z) = G_{w_0}(z - tG_{w_1}(z)), \quad t \geq 0.
\]  

(3.4)

(ii) Given the Cauchy transform $G_{m_{\lambda,0}}(z)$ of the initial measure $m_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}^+)$, the solution of (1.8) satisfies the functional equation,

\[
\frac{1}{G_{m_{\lambda,t}}(z)} = t + \frac{1}{G_{m_{\lambda,0}}(1 - tG_{m_{\lambda,t}}(z))\{(1 - \lambda)t + (1 - tG_{m_{\lambda,t}}(z))z\}}, \quad t \geq 0.
\]  

(3.5)

(iii) Given the Cauchy transform $G_{w_{\lambda,0}}(z)$ of the initial measure $w_{\lambda,0} \in \mathcal{P}^0(\mathbb{R}^+)$, the solution of (1.3) satisfies the functional equation,

\[
\frac{1}{G_{w_{\lambda,t}}(z)} = t + \frac{\sqrt{(1 - \frac{t}{z}G_{w_{\lambda,t}}(z))\{(1 - \lambda)t + (1 - \frac{t}{z}G_{w_{\lambda,t}}(z))z^2\}}}{zG_{w_{\lambda,0}}(\sqrt{(1 - \frac{t}{z}G_{w_{\lambda,t}}(z))\{(1 - \lambda)t + (1 - \frac{t}{z}G_{w_{\lambda,t}}(z))z^2\}})}, \quad t \geq 0.
\]  

(3.6)

**Proof**

(i) Consider the PDE (1.2) for $g(t, z) = G_{w_1}(z)$. In this case (3.3) becomes

\[
\begin{align*}
\frac{dz(t)}{dt} &= g(t, z(t)), \\
\frac{dg(t, z(t))}{dt} &= 0.
\end{align*}
\]  

(3.7)  
(3.8)
By (3.8), we can conclude that
\[ g(t, z(t)) = g(0, z(0)) \quad \forall t \geq 0. \tag{3.9} \]

Therefore, (3.7) is integrated as
\[ z(t) = z(0) + g(0, z(0))t = z(0) + g(t, z(t))t \quad \iff \quad z(0) = z(t) - tg(t, z(t)), \quad t \geq 0. \]

Inserting this into (3.9), we obtain (3.4) for \( g(t, z) = G_{w_1}(z) \).

(ii) Consider the PDE (1.8) for \( g(t, z) = G_{m_{\lambda,1}}(z) \). In this case (3.3) becomes
\[
\begin{align*}
\frac{dz(t)}{dt} & = 2zg(t, z(t)) + \lambda - 1, \quad (3.10) \\
\frac{dg(t, z(t))}{dt} & = -g(t, z(t))^2. \quad (3.11)
\end{align*}
\]

The solution of (3.11) is given by
\[ g(t, z(t)) = \frac{1}{t + 1/g(0, z(0))}. \tag{3.12} \]

Then (3.10) is written as
\[ \frac{dz(t)}{dt} = \frac{2z(t)}{t + 1/g(0, z(0))} + \lambda - 1. \]

This is integrated as
\[ z(t) = \left( t + \frac{1}{g(0, z(0))} \right) \left\{ -\lambda + 1 + C \left( t + \frac{1}{g(0, z(0))} \right) \right\}, \tag{3.13} \]

where \( C \) is an integral constant. By setting \( t = 0 \) in this equation, we see that
\[ C = g(0, z(0))\{z(0)g(0, z(0)) + \lambda - 1\}. \]

Using this and (3.12), (3.13) is rewritten as
\[ z(t) = \frac{1}{g(t, z(t))} + \frac{\lambda tg(t, z(t)) - 1}{(1 - tg(t, z(t)))g(t, z(t))} + \frac{z(0)}{(1 - tg(t, z(t)))^2}, \]

which gives
\[ z(0) = (1 - tg(t, z(t))\{(1 - \lambda)t + (1 - tg(t, z(t)))z(t)\}, \quad t \geq 0. \]

If we insert this expression of \( z(0) \) into (3.12) and replace \( z(t) \) by \( z, g(t, z(t)) \) by \( G_{m_{\lambda,1}}(z) \), and \( g(0, \cdot) \) by \( G_{m_{\lambda,0}}(\cdot) \), then we obtain (3.5).

(iii) By Lemma 2.3 (ii), (3.5) is transformed into (3.6).
4 Proof of Theorem 1.1

4.1 Proof of assertion (i)

We put $z = G_{w_1}^{(-1)}(\zeta)$ in (3.4). Then we have $\zeta = G_{w_0}(G_{w_1}^{(-1)}(\zeta) - t\zeta)$. Next we apply $G_{w_0}^{(-1)}$ on the both sides and obtain

$$G_{w_0}^{(-1)}(\zeta) = G_{w_1}^{(-1)}(\zeta) - t\zeta \iff \zeta G_{w_0}^{(-1)}(\zeta) - 1 = (\zeta G_{w_1}^{(-1)}(\zeta) - 1) - t\zeta^2.$$

By the definition (1.14), this implies the following equation between $R$-transforms

$$R_{w_0}(\zeta) = R_{w_1}(\zeta) - t\zeta^2, \quad t \geq 0.$$

The assertion (i) of Theorem 1.1 is concluded by the following lemma [24].

**Lemma 4.1** The $R$-transform of $w_0^t$ is given by

$$R_{w_0}^t(z) = t z^2.$$

**Proof** Since $G_{w_1}^t(z) = 1/z$, the functional equation (3.4) becomes

$$G_{w_0}^t(z) = \frac{1}{z - t G_{w_0}^t(z)}.$$

Put $z = G_{w_0}^{(-1)}(\zeta)$. Then we have $G_{w_0}^{(-1)}(\zeta) = 1/\zeta + t\zeta$. By the definition (1.14), the lemma is proved. 

4.2 Proof of assertion (ii)

We put $z = G_{m_{\lambda,t}}^{(-1)}(\zeta)$ in (3.5). Then we have

$$\frac{1}{\zeta} = t + \frac{1}{G_{m_{\lambda,0}}((1-t\zeta)\{(1-\lambda)t + (1-t\zeta)G_{m_{\lambda,t}}^{(-1)}(\zeta)\})} \iff G_{m_{\lambda,0}}((1-t\zeta)\{(1-\lambda)t + (1-t\zeta)G_{m_{\lambda,t}}^{(-1)}(\zeta)\}) = \frac{\zeta}{1-t\zeta}.$$

We apply $G_{m_{\lambda,0}}^{(-1)}$ on the both sides and obtain

$$(1-t\zeta)\{(1-\lambda)t + (1-t\zeta)G_{m_{\lambda,t}}^{(-1)}(\zeta)\} = G_{m_{\lambda,0}}^{(-1)}\left(\frac{\zeta}{1-t\zeta}\right) \iff -\lambda t\zeta + (1-t\zeta)\left[\zeta G_{m_{\lambda,t}}^{(-1)}(\zeta) - 1\right] = \frac{\zeta}{1-t\zeta} G_{m_{\lambda,0}}^{(-1)}\left(\frac{\zeta}{1-t\zeta}\right) - 1.$$

By the definition (1.14), this implies the following equations between $R$-transforms,

$$-\lambda tz + (1-tz)R_{m_{\lambda,t}}(z) = R_{m_{\lambda,0}}\left(\frac{z}{1-tz}\right) \iff R_{m_{\lambda,t}}(z) = \frac{1}{1-tz} R_{m_{\lambda,0}}\left(\frac{z}{1-tz}\right) + \frac{\lambda tz}{1-tz}.$$
Since \( R_{m_{\lambda,t}}(z) = \lambda t z / (1 - t z) \) \([24, 9]\), the assertion (ii) is proved.

**Remark 3** Applying Lemma 2.3 (iii) to (1.6), (1.18) in Theorem 1.1 gives the following for the \( R \)-transform of \( w_{\lambda,t} \),

\[
R_{w_{\lambda,t}}(z) + \frac{(1 - \lambda) t z^2}{R_{w_{\lambda,t}}(z) + 1} = R_{w_{\lambda,0}}(z) + \frac{(1 - \lambda) t z^2}{R_{w_{\lambda,0}}(z) + 1} + R_{w_{\lambda,0}} \left( z \sqrt{1 - (1 - \lambda) \left\{ \frac{1}{R_{w_{\lambda,t}}(z) + 1} - \frac{1}{R_{w_{\lambda,t}}(z) + 1 - t z^2} \right\}} \right), \quad t \geq 0.
\]

This seems to be so complicated, but this clearly shows that if \( \lambda = 1 \), the equality is reduced to (1.17) in Theorem 1.1 as expected by (1.4).

### 4.3 Proof of assertion (iii)

Since \( m_{\lambda,t} \in \mathcal{P}_0(\mathbb{R}^+) \) is defined by (1.6), Lemma 2.3 (iv) proves the assertion (iii) of Theorem 1.1.

Hence the proof of Theorem 1.1 is complete.

### 5 Concluding Remarks

We list out some concluding remarks.

1. In the present paper, we have defined the three-kinds of measure-valued processes \((\mu_t)_{t \geq 0} = (w_t)_{t \geq 0}, (w_{\lambda,t})_{t \geq 0}, \text{ and } (m_{\lambda,t})_{t \geq 0}\) with \( \lambda > 0 \) so that the solutions of the complex Burgers-type equations \((1.2), (1.3)\), and \((1.8)\) determine their Cauchy transformations. We solved the PDEs for general initial probability measures with finite supports and expressed dependence of solutions \(G_{\mu_t}, t \geq 0\) on initial Cauchy transforms \(G_{\mu_0}\) by functional equations (Proposition 3.1). Then we transformed the formulas of \((G_{\mu_t})_{t \geq 0}\) to those of \((R_{\mu_t})_{t \geq 0}\) and of \((S_{\mu_t})_{t \geq 0}\). We have obtained addition formulas for \((R_{w_t})_{t \geq 0}\) and \((R_{m_{\lambda,t}})_{t \geq 0}\) expressed by sums of the \( R \)-transforms of initial measures and those of fundamental solutions starting from \( \delta_0 \) (Theorem 1.1 (i) and (ii)), and a multiplication relation among \((S_{w_{\lambda,t}})_{t \geq 0}\), \((S_{m_{\lambda,t}})_{t \geq 0}\) and \(S_d \) (Theorem 1.1 (iii)). As a matter of course, we can transform directly the PDEs of \((G_{\mu_t})_{t \geq 0}\) to those of \((R_{\mu_t})_{t \geq 0}\) and of \((S_{\mu_t})_{t \geq 0}\). The results are given as follows; for \((R_{\mu_t})_{t \geq 0}\),

\[
\begin{align*}
\frac{\partial R_{w_t}(z)}{\partial t} - z^2 &= 0, \\
\frac{\partial R_{w_{\lambda,t}}(z)}{\partial t} + \frac{(\lambda - 1) z^3}{2(R_{w_{\lambda,t}}(z) + 1)^2} \frac{\partial R_{w_{\lambda,t}}(z)}{\partial z} - z^2 - \frac{(\lambda - 1) z^2}{R_{w_{\lambda,t}}(z) + 1} &= 0, \\
\frac{\partial R_{m_{\lambda,t}}(z)}{\partial t} - z^2 \frac{\partial R_{m_{\lambda,t}}(z)}{\partial z} - z(R_{m_{\lambda,t}}(z) + \lambda) &= 0,
\end{align*}
\]

(5.1)
and for \((S_{\mu,t})_{t \geq 0}\),
\[
\begin{align*}
\frac{\partial S_{w_1}(z)}{\partial t} + z^2 S_{w_1}(z)^2 \frac{\partial S_{w_1}(z)}{\partial z} + z S_{w_1}(z)^3 &= 0, \\
\frac{\partial S_{\lambda,t}(z)}{\partial t} + z^2 S_{\lambda,t}(z)^2 \frac{\partial S_{\lambda,t}(z)}{\partial z} + \left\{ z + \frac{(\lambda - 1)z}{2(1 + z)} \right\} S_{\lambda,t}(z)^3 &= 0, \\
\frac{\partial S_{m_{\lambda,t}}(z)}{\partial t} + z(z + \lambda) S_{m_{\lambda,t}}(z) \frac{\partial S_{m_{\lambda,t}}(z)}{\partial z} + (2z + \lambda) S_{m_{\lambda,t}}(z)^2 &= 0. \quad (5.2)
\end{align*}
\]

We find that the equation of \((R_{w_1})_{t \geq 0}\) is extremely simple and it corresponds to the asymptotic freeness of a random matrix in the GUE and a deterministic Hermitian matrix (see the assertion (i) of Theorem 1.1 and Remark 1). On the other hand, the ‘external-force terms’ include both of the coordinate \(z\) and the ‘fields’ in other equations and seem to be more complicated than the equation (1.8) for \((G_{m_{\lambda,t}})_{t \geq 0}\) whose external-force term is simply given by \(G_{m_{\lambda,t}}(z)^2\). The initial-value problems of all these PDEs given by (5.1) and (5.2) have been transformed into functional equations as reported in Theorem 1.1 (i), (ii), Remark 3, and Propositions A.1, A.2, A.3 in Appendix A given below.

(2) In addition to the free Brownian motion [7] and the free Wishart process [15], Demni introduced the free Jacobi process in [18]. This process has two parameters \(\lambda\) and \(\theta\). He derived the following PDE for the Cauchy transform of the measure-valued process \((k_{\lambda,\theta,t})_{t \geq 0}\),
\[
\begin{align*}
\frac{\partial G_{k_{\lambda,\theta,t}}(z)}{\partial t} + \left[ 2\lambda \theta (1-z) G_{k_{\lambda,\theta,t}}(z) + \left\{ (2\lambda \theta - 1)z + \theta (1 - \lambda) \right\} \right] \frac{\partial G_{k_{\lambda,\theta,t}}(z)}{\partial z} &+ \left\{ \lambda \theta (1 - 2z) G_{k_{\lambda,\theta,t}}(z) + (2\lambda \theta - 1) \right\} G_{k_{\lambda,\theta,t}}(z) = 0. \quad (5.3)
\end{align*}
\]

Demni showed that this equation has the stationary measure given by
\[
k_{\lambda,\theta}(dx) = \max \left( 0, 1 - \frac{1}{\lambda} \right) \delta_0(dx) + \max \left( 0, 1 - \frac{1 - \theta}{\lambda \theta} \right) \delta_1(dx) + g(x)1_{[-\infty,\infty]}(x)dx,
\]
where
\[
g(x) = \sqrt{\frac{(x - x_-)(x_+ - x)}{2\lambda \theta \pi x(1 - x)}} \quad \text{with} \quad x_{\pm} = \left( \sqrt{\theta(1 - \lambda \theta)} \pm \sqrt{\lambda \theta(1 - \theta)} \right)^2.
\]

The distribution with the density \(g(x)\) is known as the Kesten–McKay law [29, 32]. Is it possible to solve the initial-value problem for (5.3) as we did in this paper? Another PDE for measure-valued process was reported in [33]
\[
\begin{align*}
\frac{\partial G_{t_{0,c,\ell}}(z)}{\partial t} - \left\{ 2cz G_{t_{0,c,\ell}}(z) + (z + 2 - \alpha) \right\} \frac{\partial G_{t_{0,c,\ell}}(z)}{\partial z} - z^2 \frac{\partial^2 G_{t_{0,c,\ell}}(z)}{\partial z^2} &- \left\{ c G_{t_{0,c,\ell}}(z) + 1 \right\} G_{t_{0,c,\ell}}(z) = 0,
\end{align*}
\]

where \(\alpha\) and \(c\) are positive parameters. This describes the hydrodynamic limit of the Bru–Wishart (Laguerre) process in a high temperature regime (see also [56]). How can we solve this viscous Burgers-type equation?
Forrester and Grela [22] studied the hydrodynamic limits of the circular ensemble as well as the Jacobi ensemble. In the former case, they considered the following type of Cauchy transform,
\[ G^\circ_\mu(z) = \frac{1}{2} \oint \cot \left( \frac{z - x}{2} \right) \mu(dx). \]

This seems to be a trigonometric extension of the usual Cauchy transform (1.1), since if we introduce a parameter \( r > 0 \), then we see \( \frac{1}{2r} \cot \left( \frac{z - x}{2r} \right) \to \frac{1}{z-x} \) as \( r \to \infty \). Some elliptic extensions of Cauchy-type transform have been also considered in a recent study of elliptic integrable systems [6]. Is it meaningful to consider trigonometric and elliptic extensions of free probability?

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A Functional Equations for S-Transforms

Here we consider the S-transforms of the present measure-valued processes. For \((w^0_t)_{t \geq 0}\) starting from \( \delta_0 \), it is easy to verify that (see, for instance, [40])
\[ S_{w^0_t}(z) = \sqrt{\frac{1}{tz}}, \quad t \geq 0. \]  

Proposition A.1 Assume that the S-transform of the initial probability measure \( S_{w^0_t}(z) \) is well defined. Then the S-transform of \( w_t \), \( t \geq 0 \) starting from \( w^0 \) satisfies the functional equation,
\[ \frac{S_{w_t}(z)}{1 - (S_{w_t}(z)/S_{w^0_t}(z))^2} = S_{w^0_t} \left( z \left\{ 1 - (S_{w_t}(z)/S_{w^0_t}(z))^2 \right\} \right), \quad t \geq 0. \]  

Proof We start from (3.4) in Proposition 3.1 (i); that is, \( G_{w^0_t}(z) = G_{w^0_t}(z - tG_{w^0_t}(z)) \), \( t \geq 0 \). Replace \( z \) by \( 1/z \) and then apply (1.13). We have
\[ z(\Psi_{w^0_t}(z) + 1) = \frac{1}{1/z - t\Psi_{w^0_t}(z) + 1} \left\{ \Psi_{w^0_t} \left( \frac{1}{1/z - t\Psi_{w^0_t}(z) + 1} \right) + 1 \right\}. \]

Put \( z = \Psi_{w^0_t}^{-1}(\zeta) =: \chi_{w^0_t}(\zeta) \). Then
\[ \chi_{w^0_t}(\zeta)(\zeta + 1) = \frac{1}{1/\chi_{w^0_t}(\zeta) - t\chi_{w^0_t}(\zeta)(\zeta + 1)} \left\{ \Psi_{w^0_t} \left( \frac{1}{1/\chi_{w^0_t}(\zeta) - t\chi_{w^0_t}(\zeta)(\zeta + 1)} \right) + 1 \right\}. \]

By (1.13), the above is written as follows,
\[ \zeta S_{w^0_t}(\zeta) = \frac{1}{\zeta S_{w^0_t}(\zeta)} - t\zeta S_{w^0_t}(\zeta) \left\{ \Psi_{w^0_t} \left( \frac{1}{\zeta S_{w^0_t}(\zeta)} - t\zeta S_{w^0_t}(\zeta) \right) + 1 \right\} \]
\[ \iff \zeta \left\{ 1 - t\zeta S_{w^0_t}(\zeta)^2 \right\} = \Psi_{w^0_t} \left( \frac{\zeta S_{w^0_t}(\zeta)}{\zeta + 1 - t\zeta^2 S_{w^0_t}(\zeta)^2} \right). \]
Now we apply \( \chi_w \) on the both sides and obtain

\[
\chi_w \left( \zeta \left\{ 1 - t \zeta S_{w_t}(\zeta)^2 \right\} \right) = \frac{\zeta S_{w_t}(\zeta)}{\zeta + 1 - t^2 \zeta S_{w_t}(\zeta)^2}.
\]

Again we use (1.15) and replace the variable \( \zeta \) by \( z \). Then we obtain

\[
\frac{S_{w_t}(z)}{1 - tzS_{w_t}(z)^2} = S_w(\zeta \left\{ 1 - tzS_{w_t}(z)^2 \right\}), \quad t \geq 0.
\]

which is written as (A.2). The proof is complete.

For the process \((m_{\lambda,t}^0)_{t \geq 0}\) starting from \(\delta_0\), we have

\[
S_{m_{\lambda,t}^0}(z) = \frac{1}{t(z + \lambda)}, \quad t \geq 0.
\]

**Proposition A.2** Assume that the \( S \)-transform of the initial measure \( S_{m_{\lambda,0}}(z) \) is well defined. Then the \( S \)-transform of \( m_{\lambda,t}, t \geq 0 \) starting from \( m_{\lambda,0} \) satisfies the functional equation,

\[
\frac{S_{m_{\lambda,t}}(z)}{1 - S_{m_{\lambda,t}}(z)/S_{w^2}(z)^2} = S_{m_{\lambda,0}} \left( z \left\{ 1 - S_{m_{\lambda,t}}(z)/S_{m_{\lambda,t}^0}(z) \right\} \right) \quad t \geq 0.
\]

(A.3)

Since the proof is similar to that given above for Proposition [A.1] we omit it. We just mentioned that by (iv) of Lemma 2.3 [A.3] is transformed into the following.

**Proposition A.3** Assume that the \( S \)-transform of the initial measure \( S_{w_{\lambda,0}}(z) \) is well defined. Then the \( S \)-transform of \( w_{\lambda,t}, t \geq 0 \) starting from \( w_{\lambda,0} \) satisfies the functional equation,

\[
\frac{1 - t^2 \{S_{w_{\lambda,t}}(z)/S_{w_{\lambda,t}^0}(z)^2\}}{1 - t^2 \{S_{w_{\lambda,t}}(z)/S_{w_{\lambda,t}^2}(z)^2\}} \frac{S_{w_{\lambda,t}}(z)}{1 - S_{w_{\lambda,t}}(z)/S_{w_{\lambda,t}^2}(z)^2} = S_{w_{\lambda,0}} \left( z \left\{ 1 - S_{w_{\lambda,t}}(z)/S_{w_{\lambda,t}^0}(z)^2\right\} \right), \quad t \geq 0.
\]

(A.4)

Since \( w_{1,t} = w_t \), it is easy to verify that if \( \lambda = 1 \) (A.4) is reduced to (A.2).

**B** \( R \)-Transform and \( S \)-Transform of \( w_t^a \)

We consider the process \((w_t)_{t \geq 0}\) starting from the symmetric Bernoulli delta measure \( d_a \) with displacement \( 2a \geq 0 \) given by (1.11). Here we write this process as \((w_t^a)_{t \geq 0}\).

**Lemma B.1** The \( R \)-transform and the \( S \)-transform of \( w_t^a, t \geq 0 \) are given by

\[
R_{w_t^a}(z) = \frac{1}{2} \left[ 2tz^2 - 1 + \sqrt{1 + 4a^2 z^2} \right], \quad t \geq 0,
\]

(B.1)

\[
S_{w_t^a}(z) = \frac{1}{(t z)^{1/2}} \left[ 1 + \frac{1}{2z} + \frac{a^2}{2 tz} - \frac{1}{2z} \left( 1 + \frac{a^2}{t} \right) \sqrt{1 + \frac{4a^2/t}{(1 + a^2/t)^2} z^2} \right]^{1/2}, \quad t \geq 0.
\]

(B.2)
\textit{Proof} It is easy to confirm that
\begin{align*}
R_{\omega_t^0}(z) &= tz^2, \\
R_{\delta_a}(z) &= \frac{1}{2}\left[\sqrt{1+4a^2z^2} - 1\right].
\end{align*}

Then (B.1) is immediately concluded from Theorem 1.1 (i). We have already obtained $S_{\delta_a}(z)$ as (1.16) and $S_{\omega_t^0}(z)$ as (A.1). Then (A.2) of Proposition A.1 becomes
\begin{align*}
S_{\omega_t}(z) - tzS_{\omega_t}(z)^2 &= \frac{1}{a}\left[\frac{z+1-tz^2S_{\omega_t}(z)^2}{z-tz^2S_{\omega_t}(z)^2}\right].
\end{align*}

This is written as
\begin{align*}
t^2z^3S_{\omega_t}(z)^4 - z\{2tz + t + a^2\}S_{\omega_t}(z)^2 + (z + 1) = 0,
\end{align*}
which is solved by (B.2).

Note that (B.1) determines the free cumulants of $\omega_t^a$ as
\begin{align*}
\kappa_n(\omega_t^a) = \begin{cases} 
  t + a^2, & n = 2, \\
  -(-1)^{n/2}(n-3)!!d^{n/2-1}a^n, & n \in \{4, 6, 8, \ldots \}, \ t \geq 0, \\
  0, & \text{otherwise},
\end{cases}
\end{align*}

\section{C $R$-Transform and $S$-Transform of $m_{\lambda,t}^b$}

We consider the process $(m_{\lambda,t}^b)_{t \geq 0}$ starting from $\delta_b$ with $b > 0$. Here we write this process as $(m_{\lambda,t}^b)_{t \geq 0}$. If the variable $x$ is replaced by $x/\lambda$, the parameters $r$ by $1/\lambda$, and $a$ by $b/\lambda$ in the three parametric Marcenko–Pastur measure studied in [20], we obtain the present probability measure $m_{\lambda,t}^b$, $t \geq 0$.

\textbf{Lemma C.1} \textit{The $R$-transform and the $S$-transform of $m_{\lambda,t}^b$ are given by}
\begin{align}
R_{m_{\lambda,t}^b}(z) &= \frac{z\{(\lambda t + b) - \lambda t^2z\}}{(1-tz)^2}, \quad t \geq 0, \\
S_{m_{\lambda,t}^b}(z) &= \frac{1}{t(\lambda + z)} \left[1 + \frac{\lambda}{2z} + \frac{b}{2tz} - \frac{1}{2z}\left(\lambda + \frac{b}{t}\right)\sqrt{1 + \frac{4b/t}{(\lambda + a/t)z}}\right], \quad t \geq 0.
\end{align}

\textit{Proof} It is easy to confirm that
\begin{align*}
R_{m_{\lambda,t}^0}(z) &= \frac{\lambda t z}{1-tz} = \lambda \left(\frac{1}{1-tz} - 1\right), \\
R_{\delta_b}(z) &= bz.
\end{align*}

Then (C.1) is immediately concluded from Theorem 1.1 (ii). We can see that $S_{\delta_b}(z) = 1/b$. Then (A.3) of Proposition A.2 becomes
\begin{align*}
\frac{S_{m_{\lambda,t}^b}(z)}{(1-tzS_{m_{\lambda,t}^b}(z))(1-t(\lambda + z)S_{m_{\lambda,t}^b}(z))} &= \frac{1}{b}.
\end{align*}
This is written as
\[ t^2(z + \lambda)S_{\mu}(z)^2 - (2tz + t\lambda + b)S_{\mu}(z) + 1. \]
which is solved by (C.2).

Note that (C.1) determines the free cumulants of \( m_{\lambda,t}^b \) as
\[ \kappa_m(m_{\lambda,t}^b) = (\lambda t + bn)^{n-1}, \quad n \in \mathbb{N}, \quad t \geq 0. \]

Comparing (B.2) and (C.2), we obtain the equality,
\[ S_{\omega t}^q(z) = \sqrt{1+\frac{z}{z}} \sqrt{S_{m_{\lambda,t}^2}^q(z)}. \quad (C.3) \]

It is readily confirmed by the definition (1.5) that \( d^{(2)}_a = \delta_{a,2}, a > 0. \) Hence as a special case of (1.7),
\( (w_t^2)^{(2)} = m_{\lambda,t}^2, \quad t \geq 0, \quad a > 0. \) Therefore, (C.3) can be regarded as a special case of the assertion (iii) of Theorem 1.1.

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