Three-manifolds of positive Ricci curvature and convex weakly umbilic boundary

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Abstract

In this paper we study a boundary value problem in manifolds with weakly umbilic boundary (the Second Fundamental form of the boundary is a constant multiple of the metric). We show that if we start with a metric of positive Ricci curvature and convex boundary (positive Second Fundamental form), the flow uniformizes the curvature.

1 Introduction

In this paper we consider the problem of deforming metrics in manifolds with boundary via the Ricci flow. The Ricci flow in manifolds with boundary was first considered by Shen in [She]. The problem he considered was the following

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -2 \text{Ric}(g) \quad \text{in} \quad M \times [0,T) \\
h_g &= \kappa g \quad \text{on} \quad \partial M \times [0,T) \\
g|_{t=0} &= \bar{g}
\end{align*}
\]  

(1)

where \(h_g\) is the Second Fundamental Form of the boundary with respect to the outward unit normal, and \(\kappa\) is a constant. We also ask for the following compatibility condition (which is required in order to show that the solution has enough regularity),

\[h(\cdot,0) = \kappa \bar{g}(\cdot).\]

The proof of the basic short time existence result is given in [She]. Here we state it with a little more specifications which are necessary to justify certain applications of the Maximum Principle.

Theorem 1.1 The Boundary Value Problem (1) has a unique continuous solution for a short time. This solution is smooth in \(\bar{M} \times (0,T)\), and \(\text{Ric}(g)\) is continuous in \(\bar{M} \times [0,T)\).

Also, in the same paper, the following result is proved
Theorem 1.2 ([She]) Let \((M, g)\) be a compact three-dimensional Riemannian manifold with totally geodesic boundary and with positive Ricci curvature. Then \((M, g)\) can be deformed to \((M, g_{\infty})\) via the Ricci flow such that \((M, g_{\infty})\) has constant positive curvature and totally geodesic boundary.

Here we consider the normalized version of (1). The normalization goes as follows. Let \(\psi(t)\) be such that for \(\tilde{g} = \psi g\) we have \(Vol(M) = 1\). Then we change the time scale by letting

\[\tilde{t}(t) = \int_0^t \psi(t) \, d\tau,\]

then \(\tilde{g}(\tilde{t})\) satisfies the equation

\[
\begin{cases}
\frac{\partial}{\partial \tilde{t}} \tilde{g} = \frac{\tilde{r}}{n} \tilde{g} - 2Ric(\tilde{g}) & \text{in } M \times [0,T) \\
h_{\tilde{g}} = \kappa(\tilde{t}) \tilde{g} & \text{on } \partial M \times [0,T) 
\end{cases}
\]

where

\[\tilde{r} = \int_M \tilde{R} \, dV.\]

and \(\tilde{R}\) is the scalar curvature of the metric \(\tilde{r}\).

Our main result is

**Theorem 1** If \(\kappa \geq 0\) the solution of (2) exists for all time and converges exponentially to a metric of constant sectional curvature and totally geodesic boundary.

We conjecture that

**Conjecture 1** If \(h = \kappa g\), for \(\kappa\) any nonnegative function, then the metric \(g_0\) can be deformed via the Ricci flow to a manifold of constant curvature and totally geodesic boundary.

This paper is organized as follows. In Section 2 we present the basic Maximum Principle for tensors used in this work. In section 4.1, we prove the basic pinching estimates, and we give an argument to prove Theorem 1. Finally, in Section 5 we sketch a method to produce bounds on derivative of the curvature up to the boundary from bounds in the curvature.

### 2 Maximum Principle

The Hopf Maximum Principle for the Ricci Flow is due to Shen ([She]). We restate it here in a slightly different way, very convenient for our purposes. First a definition.
Definition 2.1 Let $M_{ij}$ be a tensor and let $N_{ij} = p(M_{ij}, g_{ij})$ be a polynomial in $M_{ij}$ formed by contracting products of $M_{ij}$ with itself using the metric $g_{ij}$. We say that $N$ satisfies the **null eigenvector condition** if whenever $v^i$ is a null eigenvector of $M_{ij}$, the we have $N_{ij}v^iv^j \geq 0$

We say that $M$ satisfies the **normal derivative condition** at a point $p \in \partial M$, if for any null-eigenvector $v$ of $M_{ij}$,

$$(M_{ij}; v) v^i v^j \geq 0$$

Theorem 2.2 ([She], [H1]) Let $(M, g)$ be a Riemannian manifold such that $R_{ij} \geq - (n-1) \omega g_{ij}$. Suppose we have

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij}$$

for some constant $\omega$, where $N = P(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition, and $M$ satisfies the normal derivative condition. Then the condition $M_{ij} > 0$ is preserved under the flow.

3 Shen’s Parabolic Simon Identity

In this section we prove an important parabolic identity for the Second Fundamental form of the boundary of $(M, g)$. This identity was derived by Shen (with what it seems to us some mistakes), in [She] using Cartan’s formalism for his computations. We recast and rederive this formula using classical tensor notation. First we fix some notation.

**Notation.** The metric $g$ restricted to $\partial M$ will be denoted by $g$. $\nabla$ will denote the connection of $g$, whereas $\not\nabla$ will denote the (Levi-Civita) connection of $g$. Covariant differentiation with respect to $\nabla$ will be denoted by a vertical bar ($|$). Covariant differentiation with respect to $\not\nabla$ will be denoted by a semicolon ($;$). By $h_{\alpha\beta}$ we will denote the Second Fundamental form of $\partial M$. Finally, greek numerals (except $\nu$, which we have chosen to represent the outward unit normal) denote quantities in the boundary.

We are ready to establish,

**Proposition 3.1** (Shen’s Simon parabolic identity)

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \Delta h_{\alpha\beta} - H_{\alpha\beta} - R_{\alpha\beta\gamma\delta} n_{\gamma} - HR_{\alpha\beta n \nu} - H g^{\alpha\omega} h_{\beta\rho} n_{\omega} + |A|^2 h_{\alpha\beta}$$

$$-g^{\gamma\delta} g^{\rho\omega} (R_{\beta\gamma\delta\rho} h_{\omega \alpha} - R_{\delta\alpha\rho} h_{\beta\gamma})$$

$$-g^{\gamma\delta} g^{\rho\omega} (h_{\beta\rho} h_{\delta\omega \alpha} h_{\gamma} - h_{\beta\gamma} h_{\delta\omega \rho})$$

(3)
Proof. We want to find an expression for \( h_{\alpha\beta;\gamma \delta} \). We start our calculations,

\[
\begin{align*}
    h_{\alpha\beta;\gamma \delta} &= h_{\alpha\gamma;\delta \beta} + R_{\beta\gamma n \alpha \delta} \\
    &= h_{\alpha\gamma;\delta \beta} + g^{\rho \omega} \left( R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma} \right) + R_{\beta\gamma n \alpha \delta} \\
    &= (h_{\delta \gamma; \alpha} + R_{\alpha \delta \gamma n})_{\beta} \\
    &+ g^{\rho \omega} \left( R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma} \right) + R_{\beta\gamma n \alpha \delta} \\
    &= h_{\delta \gamma; \alpha \beta} + R_{\alpha \delta \gamma n \beta} \\
    &+ g^{\rho \omega} \left( R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma} \right) + R_{\beta\gamma n \alpha \delta},
\end{align*}
\]

then by Gauss equation,

\[
\begin{align*}
    R_{\beta\delta \gamma \rho} &= R_{\beta\delta \gamma \rho} + h_{\beta \rho} h_{\delta \gamma} - h_{\beta \gamma} h_{\delta \rho} \\
    R_{\beta\delta \alpha \rho} &= R_{\beta\delta \alpha \rho} + h_{\beta \alpha} h_{\delta \rho} - h_{\beta \rho} h_{\delta \alpha}
\end{align*}
\]

from which we obtain

\[
\begin{align*}
    h_{\alpha\beta;\gamma \delta} &= h_{\delta \gamma; \alpha \beta} + R_{\alpha \delta \gamma n \beta} + R_{\beta\gamma n \alpha \delta} \\
    &+ g^{\rho \omega} \left( R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma} \right) \\
    &+ g^{\rho \omega} \left( h_{\beta \rho} h_{\delta \gamma} h_{\omega \alpha} + h_{\beta \gamma} h_{\delta \alpha} h_{\omega \rho} \\
    &- h_{\beta \gamma} h_{\delta \rho} h_{\omega \alpha} - h_{\beta \alpha} h_{\delta \rho} h_{\omega \gamma} \right).
\end{align*}
\]

Using the following fact,

\[
\begin{align*}
    R_{\alpha \delta \gamma n \beta} &= R_{\alpha \delta \gamma n \beta} - h_{\alpha \beta} R_{n \delta \gamma n} - h_{\delta \beta} R_{\alpha n \gamma n} + g^{\rho \omega} h_{\beta \rho} R_{\alpha \delta \gamma \omega} \\
    &+ R_{\beta \gamma n \alpha \delta} - h_{\beta \gamma} h_{\delta \rho} h_{\omega \alpha} - h_{\beta \alpha} h_{\delta \rho} h_{\omega \gamma}.
\end{align*}
\]

and (4) we get

\[
\begin{align*}
    h_{\alpha\beta;\gamma \delta} &= h_{\delta \gamma; \alpha \beta} + R_{\alpha \delta \gamma n \beta} + R_{\beta\gamma n \alpha \delta} \\
    &+ g^{\rho \omega} \left( R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma} \right) \\
    &+ g^{\rho \omega} \left( h_{\beta \rho} h_{\delta \gamma} h_{\omega \alpha} + h_{\beta \gamma} h_{\delta \alpha} h_{\omega \rho} \\
    &- h_{\beta \gamma} h_{\delta \rho} h_{\omega \alpha} - h_{\beta \alpha} h_{\delta \rho} h_{\omega \gamma} \right).
\end{align*}
\]

Contract in the indices \( \gamma \) and \( \delta \). Denoting by \( H = g^{\alpha \beta} h_{\alpha \beta} \), by \( |A|^2 \) the norm of the tensor \( h_{\alpha \beta} \), and using the Second Bianchi identity, produce

\[
\begin{align*}
    \Delta h_{\alpha \beta} &= H_{\alpha \beta} + R_{\alpha n | \beta} + R_{\beta n | \alpha} - R_{\beta \alpha | n} + R_{n \alpha \beta | n} \\
    &- h_{\alpha \beta} R_{n n} + H R_{n n} \\
    &+ g^{\gamma \delta} g^{\rho \omega} h_{\beta \rho} R_{\alpha \delta \gamma \omega} + g^{\gamma \delta} g^{\rho \omega} h_{\delta \rho} R_{\beta \gamma \alpha \omega} \\
    &+ H g^{\rho \omega} h_{\beta \rho} h_{\omega \alpha} - |A|^2 h_{\alpha \beta} \\
    &+ g^{\gamma \delta} g^{\rho \omega} (R_{\beta\delta \gamma \rho \omega} h_{\omega \alpha} + R_{\beta\delta \alpha \rho \omega} h_{\omega \gamma}) \\
    &+ g^{\gamma \delta} g^{\rho \omega} (h_{\beta \rho} h_{\delta \alpha} h_{\omega \gamma} - h_{\beta \gamma} h_{\delta \rho} h_{\omega \alpha}).
\end{align*}
\]
Finally, we use the fact

\[ \frac{\partial}{\partial t} h_{\alpha\beta} = -R_{nn} h_{\alpha\beta} + R_{\alpha n|\beta} + R_{\beta n|\alpha} - R_{\beta\alpha|n}, \tag{7} \]

to finally show the formula.

Contracting the previous formula in the indices \( \alpha \) and \( \beta \) yields,

**Corollary 3.2**

\[ R_{\nu\nu|\nu} = -g^{\alpha\beta} \frac{\partial}{\partial t} h_{\alpha\beta} - HR_{\nu\nu} = 2\kappa \left[ g^{\alpha\beta} R_{\alpha\beta} - R_{\nu\nu} \right]. \tag{8} \]

**Proposition 3.3**

\[ \nabla_{\nu} R_{\alpha\beta} = R_{\nu\nu} h_{\alpha\beta}. \tag{9} \]

**Proof.** Let \( \frac{\partial}{\partial \eta} \) be the outward normal vector to \( \partial M \) at \( t = 0 \). Then, as a consequence of the Codazzi equations, \( \frac{\partial}{\partial \eta} \) remains normal to \( \partial M \) throughout the flow. In this case we have that

\[ h_{\alpha\beta} = \frac{1}{2} \left( g_{\eta\eta} \right)^{1/2} \frac{\partial g_{\alpha\beta}}{\partial \eta}. \]

A straightforward computation gives,

\[ h'_{\alpha\beta} = R_{\nu\nu} h_{\alpha\beta} - \frac{\partial R_{\alpha\beta}}{\partial \nu}. \]

Recalling that \( h_{\alpha\beta} = \kappa g_{\alpha\beta} \), we obtain

\[ -2\kappa R_{\alpha\beta} = R_{\nu\nu} h_{\alpha\beta} - \frac{\partial R_{\alpha\beta}}{\partial \nu}, \]

which can be written as,

\[ \frac{\partial R_{\alpha\beta}}{\partial \nu} = 2\kappa R_{\alpha\beta} + R_{\nu\nu} h_{\alpha\beta}. \tag{10} \]

On the other hand we have,

\[ \nabla_{\nu} R_{\alpha\beta} = \frac{\partial R_{\alpha\beta}}{\partial \nu} - g^{\omega\nu} (R_{\alpha\rho} h_{\omega\beta} + R_{\beta\rho} h_{\omega\alpha}), \]

which in the special case we are studying becomes,

\[ \nabla_{\nu} R_{\alpha\beta} = \frac{\partial R_{\alpha\beta}}{\partial \nu} - \kappa g^{\rho\nu} (R_{\alpha\rho} g_{\omega\beta} + R_{\beta\rho} g_{\omega\alpha}) \]
\[ = R_{\nu\nu} h_{\alpha\beta} + 2\kappa R_{\alpha\beta} - 2\kappa R_{\alpha\beta} \]
\[ = R_{\nu\nu} h_{\alpha\beta}. \]

\[ \square \]
Remark 3.4 The previous computation is valid even when the metric is not rotationally symmetric.

From Corollary 3.2 and Proposition 3.3 we get the following formula for the normal derivative of the scalar curvature

\[ \frac{\partial}{\partial \nu} R = 2 \kappa g^{\alpha \beta} R_{\alpha \beta}. \] (11)

4 Proof of the Main Theorem

We show that the estimates used in [H1] are still valid in the case we are studying in this paper.

4.1 Pinching Estimates

The first consequence of the previous computations is the following result that we state without proof.

Lemma 4.1 If \( \text{Ric} > 0 \) at time \( t = 0 \), it remains so as long as the solution to the flow exists.

Using the results of the previous section we can show the following pinching estimates. We adopt the following notation: \( \mu \leq \lambda \) are the eigenvalues of \( R_{\alpha \beta} \) and \( \nu = R_{\nu \nu} \) on \( \partial M \).

Lemma 4.2 Let \( \eta > 0 \) be such that \( \lambda < (1 + \eta) \mu \) throughout the flow, and assume that at \( t = 0 \) we have \( R_{ij} > \epsilon R g_{ij} \) for \( \epsilon < \frac{1}{2(2+\eta)} \). This condition is preserved under the flow.

Proof. The equation satisfied in the interior by the tensor

\[ T_{ij} = \frac{R_{ij}}{R} - \epsilon g_{ij} \]

are computed in [H1]. All we have to see is what happens with the normal covariant derivative of \( T \) at a point in the boundary when it gets a null eigenvalue. By the decomposition of the Ricci tensor at the boundary (consequence of Codazzi equations), we have to consider two cases separately: when the eigenvector is normal to the boundary, and when it is tangent.

For the first case we must have

\[ R_{\alpha \beta} \geq \epsilon R g_{ij} \quad \text{and} \quad R_{\nu \nu} = \epsilon R. \]
If we compute the relevant part (it is clear that if $v \perp \partial M$, then $(\nabla_v T_{\alpha\beta}) v^\alpha v^\beta = 0$),

$$
\nabla_v \left( \frac{R_{\alpha\beta}}{R} - \epsilon \right) = 2\kappa g^{\alpha\beta} \frac{R_{\alpha\beta}}{R} - 2\kappa \frac{R_{\nu\alpha\beta}}{R} - \frac{1}{R} \left( 2\kappa g^{\alpha\beta} R_{\alpha\beta} \right) R_{\nu\nu} - 2\kappa g^{\alpha\beta} \frac{R_{\nu\alpha\beta}}{R} - 2\kappa \frac{R_{\nu\nu}}{R} \\
\geq 2\kappa g^{\alpha\beta} \frac{R_{\alpha\beta}}{R} - 2\kappa g^{\alpha\beta} \frac{R_{\alpha\beta}}{R} - 2\kappa \epsilon \\
= 2\kappa (1 - \epsilon) g^{\alpha\beta} R_{\alpha\beta} - 2\kappa \epsilon \\
\geq 2\kappa (1 - \epsilon) g^{\alpha\beta} (\epsilon g_{\alpha\beta}) - 2\kappa \epsilon \\
= 4\kappa (1 - \epsilon) \epsilon - 2\kappa \epsilon \\
= 2\kappa \epsilon (1 - 2\epsilon) > 0 \text{ if } \epsilon \leq \frac{1}{2}.
$$

The second case is taken care of by the following computation (in this case notice that if $v \in T_p \partial M$, then $(\nabla_v T_{\nu\alpha\beta}) v^n = 0$ - because $v^n = 0$). In this case we assume $R_{\alpha\beta} = \epsilon R g_{\alpha\beta}$ (here we use the hypothesis on the rotational symmetry of the metric) and $R_{\nu\nu} \geq \epsilon R$.

$$
\nabla_v \left( \frac{R_{\alpha\beta}}{R} - \epsilon g_{\alpha\beta} \right) = \frac{\nabla_v R_{\alpha\beta}}{R} - \frac{1}{R^2} \frac{\partial R}{\partial v} R_{\alpha\beta} \\
= \frac{R_{\nu\nu}}{R} h_{\alpha\beta} - \frac{1}{R} 2\kappa g^{\sigma\rho} R_{\rho\sigma} R_{\alpha\beta} \\
= \frac{R_{\nu\nu}}{R} h_{\alpha\beta} - 2\kappa g^{\sigma\rho} R_{\rho\sigma} \frac{R_{\alpha\beta}}{R} \\
\geq \epsilon h_{\alpha\beta} - 2\kappa (2 + \eta) \epsilon \cdot \epsilon g_{\alpha\beta} \\
= \kappa \epsilon g_{\alpha\beta} - 4\kappa \epsilon^2 g_{\alpha\beta} = \kappa \left[ \epsilon - 2(2 + \eta) \epsilon^2 \right] \\
\geq 0 \text{ if } \epsilon \leq \frac{1}{2(2+\eta)}.
$$

As a corollary we get

**Corollary 4.3** In the rotationally symmetric case if $R_{ij} \geq \epsilon R g_{ij}$ at time $t = 0$, with $\epsilon < \frac{1}{4}$, then it remains so for all time.

**Proof.** Apply Lemma 4.2 with $\eta = 0$.

**Corollary 4.4** The scalar curvature blows up in finite time.

**Proof.** Notice that the scalar curvature satisfies the following differential inequality

$$
\begin{cases}
\frac{\partial R}{\partial t} \geq \Delta R + \epsilon^2 R^2 \\
\frac{\partial R}{\partial v} \geq 0
\end{cases}
$$

The Maximum Principle can be applied now to show the statement of the corollary.
Remark 4.5 From the proof of Lemma 4.2 we see that a sufficient condition to have the pinching estimate is that the following holds. There is $\delta > 0$ such that

$$R_{\nu\nu} \geq \delta g^{\alpha\beta} R_{\alpha\beta} \quad \text{on} \quad \partial M \times (0, T) \quad (12)$$

More exactly we have

Proposition 4.6 Assume inequality (12) holds as long as the solution to the Ricci flow exists. Then if at $t = 0$ we have $R_{ij} \geq \epsilon R_{ij}$ with $\epsilon \leq \frac{\delta}{2}$, it remains so.

Proof. The computations in Lemma 4.2 the first (in time) null eigenvector of the tensor

$$T_{ij} = \frac{R_{ij}}{R} - \epsilon g_{ij}$$

cannot happen in the normal direction. So if $v$ is the first null eigenvector of $T_{ij}$ to occur, then $v^\nu = 0$. We compute as follows,

$$\nabla_\nu \left( \frac{R_{\alpha\beta}}{R} - \epsilon g_{\alpha\beta} \right) v^\alpha v^\beta \geq \left[ \delta g^{\rho\sigma} R_{\rho\sigma} R_{\alpha\beta} - 2 \kappa g^{\rho\sigma} R_{\rho\sigma} R_{\alpha\beta} \right] v^\alpha v^\beta \geq \kappa g^{\rho\sigma} R_{\rho\sigma} \left[ \delta g_{\alpha\beta} - 2 \epsilon g_{\alpha\beta} \right] v^\alpha v^\beta \geq 0 \quad \text{if} \quad \epsilon \leq \frac{\delta}{2}.$$

\[\square\]

4.2 Preserving pinching.

We continue to use the following notation: $\lambda \geq \mu$ are the eigenvalues of $R_{\alpha\beta}$, and $\nu = R_{\nu\nu}$.

Lemma 4.7 Let $\eta > 0$ be such that for any $P \in \partial M$, we have

either $\quad \lambda \leq (1 + \eta) \mu \quad \text{or} \quad \lambda \leq (1 + \eta) \nu$.

Then a pinching condition holds for $\epsilon \leq \frac{1}{2(2 + \eta)}$.

Proof. Choose $0 < \epsilon < \frac{1}{2(2 + \eta)}$ such that the pinching condition $R_{ij} > \epsilon R_{ij}$ holds at $t = 0$. Then if there is a time $t_0$ where this pinching ceases to hold, there should be a point $p \in \partial M$ where the tensor $T_{ij} = \frac{R_{ij}}{R} - \epsilon g_{ij}$ achieves its first zero eigenvalue. By the computations in Lemma 4.2 we know that the eigenvector corresponding to this eigenvalue is tangent to the boundary. If at this point $\lambda < (1 + \eta) \mu$ holds, the calculations in Lemma 4.2 show that

$$\nabla_\nu \left( \frac{R_{\alpha\beta}}{R} - \epsilon g_{\alpha\beta} \right) \geq \epsilon \kappa - 2 \kappa (2 + \eta) \epsilon^2 \geq 0.$$

In the case that $\lambda < (1 + \eta) \nu$ holds then $R_{\nu\nu} > 2 \epsilon g^{\alpha\beta} R_{\alpha\beta}$ holds, and at this point there cannot be a 0 eigenvalue of $T_{ij}$ by Proposition 4.6.
Consider the function
\[ f = \frac{S}{R^2} = \frac{\lambda^2 + \mu^2 + \nu^2}{(\lambda + \mu + \nu)^2} \]
then we have,

**Lemma 4.8** If there is \( \delta > 0 \) such that \( f \leq 1 - \delta \), then there is \( \eta > 0 \) such that the hypothesis of Lemma 4.7 holds.

**Proof.** If the hypothesis of this Lemma holds, then we must have
\[ c \leq \frac{\mu}{\lambda} + \frac{\nu}{\lambda} < C, \]
and the conclusion of the Lemma follows.

The previous two Lemmas show that if the pinching ceases to hold, it is because \( f \) approaches 1 as \( t \to T \). Let us compute \( \nabla_\nu f \) at \( \partial M \).

**Lemma 4.9** We have
\[ \nabla_\nu f = \frac{2 \kappa}{R^3} \left\{ \nu (\lambda + \mu + \nu) [3 (\lambda + \mu) - 2 \nu] - 2 (\lambda + \mu) (\lambda^2 + \mu^2 + \nu^2) \right\} \]

A direct analysis of the previous expression shows the following,

**Lemma 4.10** There exists \( \rho > 0 \) small enough such that if \( \frac{\mu}{\lambda} + \frac{\nu}{\lambda} < \rho \) then \( \nabla_\nu f \leq 0 \). Also if \( \frac{\lambda}{\nu} \to 0 \), the same conclusion holds.

From [HI] we borrow the following Lemma,

**Lemma 4.11** \( f \) satisfies the following differential inequality
\[ \frac{\partial}{\partial t} f \leq \Delta f + u_k \partial_k f \]
where \( u_k = \frac{2}{R^2} g^{kl} \partial_l R \).

We show now that the maximum of \( f \) cannot approach 1 as \( t \to T \). Indeed, if it does, we must have that either for every \( \rho > 0 \) there is a time \( t \) where
\[ \frac{\mu}{\lambda} + \frac{\nu}{\lambda} < \rho \quad \text{or} \quad \frac{\nu}{\lambda} \to \infty \]

But at such point \( f \) cannot achieve a maximum, since this point must be in \( \partial M \) by Lemma 4.11 and Lemma 4.10 would lead to a contradiction. Hence we have shown

**Theorem 4.12** There is an \( \epsilon > 0 \) small enough, such that if \( R_{ij} > \epsilon R_{ij} \) holds at time \( t = 0 \), then it continues to hold for all time.
4.3 Pinching the eigenvalues

In this section we show that as it is the case in closed manifolds of positive Ricci curvature, the eigenvalues of the Ricci tensor approach each other as the scalar curvature blows up.

**Theorem 4.13** We can find a $\delta > 0$ and a constant $C$ depending only on the initial metric such that on $0 \leq t < T$ we have

$$S - \frac{1}{3} R^2 \leq CR^{2-\delta}.$$

**Proof.** As we now have proved the pinching estimates, Hamilton’s computations for manifolds without boundary, carries over to the interior of our manifold. If we define

$$f = \frac{S}{R^\gamma} - \frac{1}{3} R^{\gamma-2}, \quad \gamma = 2 - \delta, \quad \delta \leq 2\epsilon^2$$

we know that $f$ satisfies an inequality

$$\frac{\partial}{\partial t} f \leq \Delta f + u_k \partial_k f.$$

All we have to show then, to finally prove the Theorem is that for $\delta > 0$ small enough, $\nabla_\nu f \leq 0$ outside a compact set of values of $(\lambda_\nu, \mu_\nu, 1)$. We do this in a series of Lemmas.

**Lemma 4.14**

$$\nabla_\nu f = 2\kappa \left\{ \frac{\nu [3(\lambda + \mu + \nu) - 2\nu]}{(\lambda + \mu + \nu)^\gamma} - \gamma \frac{(\lambda^2 + \mu^2 + \nu^2)(\lambda + \mu)}{(\lambda + \mu + \nu)^\gamma + 1} + \delta \frac{\nu}{2} (\lambda + \mu + \nu)^{\gamma-3}(\lambda + \mu) \right\}$$

**Lemma 4.15** For $\delta > 0$ small enough and $M > 0$ big enough, if $\frac{\lambda}{\nu} + \frac{\mu}{\nu} \geq M$ then $\nabla_\nu f < 0$.

**Proof.** Assume for a moment that $\delta = 0$. Then the expression that determines the sign of $\nabla_\nu f$ is

$$(\lambda + \mu + \nu) [3(\lambda + \mu) - 2\nu] - 2 \left( \lambda^2 + \mu^2 + \nu^2 \right)(\lambda + \mu).$$

Dividing by $\nu$ and writing $x = \frac{\lambda}{\nu}, y = \frac{\mu}{\nu}$, we must analyze the behavior of the function

$$h(x, y) = (x + y + 1) [3(x + y) - 2] - 2 \left( x^2 + y^2 + 1 \right)(x + y).$$

Notice that if $x = 0$ or $y = 0$, there is $\rho > 0$ such that $h(x, y) \leq -\rho$. Now we show that for a rectangle $Q$ big enough with sides parallel to the coordinate axes,
we must have that if \((x, y) \in \mathbb{R}^2 \setminus Q\), then \(h(x, y) < 0\). Indeed, if we maximize the function \(h\) subject to the constraint \(x + y = \epsilon\), then we obtain that the maximum is reached when \(x = \frac{\epsilon}{2} = y\). Evaluating at this point we obtain

\[
h\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) = -\epsilon^3 + 3\epsilon^2 - \epsilon - 2
\]

it is easy to see that if \(\epsilon > 3\) and \(\epsilon < 1\), then \(h\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) < 0\). This together with the previous observation shows the assertion.

Observe that the term

\[
\frac{(\lambda + \mu + \nu)^7 (\lambda + \mu)}{(\lambda + \mu + \nu)^3}
\]

is bounded if \(R \geq \rho > 0\). This shows that if \(\delta > 0\) is small enough, then \(\nabla_{\nu} f < 0\). This also finishes the proof of Theorem 4.13.

\[\square\]

### 4.4 The gradient of the Scalar Curvature.

Now we can use a compactness argument as in [H2] to show the following

**Theorem 4.16** For every \(\theta > 0\) we can find a constant \(C(\theta)\) depending only on \(\theta\) and the initial value of the metric, such that on \(0 \leq t < T\) we have

\[
\max_{P} \max_{t \leq \tau} |D R m (P, t)| \leq \theta \max_{t \leq \tau} \max_{P} |R m (P, \tau)|^3 + C(\theta)
\]

Of course, to produce this compactness argument we need to bound derivatives of the curvature in terms of the curvature. The rest of the proof of long time existence and exponential convergence then follows the same arguments as in [H1], again as long as we learn how to bound derivatives of the curvature in terms of bounds in the curvature, and this is the purpose of the last section of this work.

### 5 Bounding derivatives of the curvature

In this section we sketch a procedure to produce bounds on the derivatives of the curvature tensor from bounds on the curvature. We skip the first order derivatives since the computations of Theorem 7.1 in [H2] (see also Theorem 6.1 in [C1]), can be easily adapted to our case and show how to produce bounds on certain second derivatives, hoping that the reader will find easy to convince himself that this method extends to bound any number of derivatives.

First of all, fix a collar of the boundary and fix Fermi coordinates \((x^1, \ldots, x^n)\) with respect to the initial metric, where \(x^n\) is the distance to the boundary. The
vector fields $\partial_\alpha$, $\alpha = 1, \ldots, n - 1$ remain tangent to the boundary (of course when restricted to the boundary) whereas the vector field $\partial_n$ remains normal (recall that we are working with a weakly umbilic boundary it remains).

By the interior derivative estimates of Shi, on the inner boundary of the collar for $t > 0$ we can assume bounds on all the derivatives of the curvature in terms of a bound of the curvature. All these said, we set ourselves to the task of estimating derivatives of the form $\nabla_\nu \nabla \text{Ric}$, $\nu = 1, \ldots, n$ where we denote by $\nu := \frac{1}{g_{nn}} \partial_n$, which coincides with the outward unit normal when restricted to the boundary.

From now on we assume bounds $|\text{Ric}| \leq M_0$ and $|\nabla \text{Ric}| \leq M_1$ on $M \times [0, T]$, $T < \infty$.

The evolution equations for the Ricci tensor are given by

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} R_{\alpha\beta} = \Delta R_{\alpha\beta} - Q_{\alpha\beta} \quad \text{in} \quad M \times (0, T) \\
\nabla_\nu R_{\alpha\beta} = \kappa R_{\nu\nu} g_{\alpha\beta} \quad \text{on} \quad \partial M \times (0, T) 
\end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} R_{\nu\nu} = \Delta R_{\nu\nu} - Q_{\nu\nu} + 2 (R_{\nu\nu})^2 \quad \text{in} \quad M \times (0, T) \\
\nabla_\nu R_{\nu\nu} = 2 \kappa \left[ g^{\alpha\beta} R_{\alpha\beta} - R_{\nu\nu} \right] \quad \text{on} \quad \partial M \times (0, T) \tag{14} \end{array} \right.$$  

Borrowing from Lemma 13.1 in [H1] we find that the first covariant derivative of the Ricci tensor satisfies the following system of equations

**Proposition 5.1**

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} (\nabla_\nu \text{Ric}) = \Delta (\nabla_\nu \text{Ric}) + \text{Ric} \ast \nabla_\nu \text{Ric} \quad \text{in} \quad M \times (0, T) \\
\nabla_\nu \text{Ric} = g \ast \text{Ric} \quad \text{on} \quad \partial M \times (0, T) \end{array} \right. \tag{15}$$

as in [H1], if $A$ and $B$ are two tensors we write $A \ast B$ for any linear combination of tensors formed by contraction on $A_{i \ldots j} B_{k \ldots l}$ using the metric $g^{ik}$. Notice that in the expression $g \ast \text{Ric}$ we have absorbed the constant $\kappa$.

We have to compute the Laplacian when acting on 3-tensors. We have the following formula

**Proposition 5.2**

$$\nabla_j \nabla_i \Psi_{lmk} = \frac{\partial^2}{\partial x^i \partial x^j} \Psi_{lmk} - \Psi_{pmk} \frac{\partial}{\partial x^j} (\Gamma^p_{il}) - \Psi_{lpk} \frac{\partial}{\partial x^j} (\Gamma^p_{im}) - \Psi_{lmp} \frac{\partial}{\partial x^j} (\Gamma^p_{ik}) - \Gamma^p_{il} \Psi_{pmk,j} - \Gamma^p_{im} \Psi_{lpk,j} - \Gamma^p_{ik} \Psi_{lmp,j} \tag{16}$$

Therefore, taking $\Psi = \text{Ric}$ in Proposition 5.2, system (15), becomes

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \nabla_\nu \text{Ric} - \Delta_e \nabla_\nu \text{Ric} = \mathcal{W} \quad \text{in} \quad M \times (0, T) \\
\nabla_\nu \text{Ric} = g \ast \text{Ric} \quad \text{on} \quad \partial M \times (0, T) \end{array} \right.$$  

where

$$\Delta_e = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \quad \text{and}$$
\[ W = \Gamma \ast \nabla^2 \text{Ric} + \text{Ric} \ast \nabla \text{Ric} + \partial \Gamma \ast \nabla \text{Ric}. \]

Let \( F_{ij} \) be the extension of the boundary quantity that appears in (13) and (14). Then we have the following formulae,

\[
\left\{ \begin{array}{l}
\frac{\partial F}{\partial t} = \text{Ric} \ast \text{Ric} + g \ast \nabla^2 \text{Ric} \\
\Delta F = g \ast \nabla^2 \text{Ric}
\end{array} \right.
\]

Hence, if we define \( U = \nabla \text{Ric} - F \), we find that it satisfies the following system

\[
\left\{ \begin{array}{l}
\frac{\partial U}{\partial t} - \Delta_e U = \overline{W} \quad \text{in} \quad M \times (0, T) \\
U = 0 \quad \text{on} \quad \partial M \times (0, T)
\end{array} \right.
\]

where \( \overline{W} = W - \frac{\partial F}{\partial t} - g \ast \nabla^2 \text{Ric} \).

Via the Ricci flow, and taking into account that we are considering a bounded time interval, it can be shown that \( |\partial \Gamma| \leq C |\nabla^{i+1} \text{Ric}| \), and hence we can bound \( \overline{W} \) in terms of bounds on \( |\nabla^i \text{Ric}| \) (\( i = 1, 2 \)).

By the local interior estimates of Shi, we can assume that \( U = 0 \) on the part of the boundary of the collar which is contained in the interior of the manifold. We must point out that by assuming \( U = 0 \) we are introducing in the estimates for the bounds terms depending on the size of the collar of the boundary (at time \( t = 0 \)) where we are estimating.

By the theory of the first boundary value problem, we can represent \( U \) as

\[
U = \int_0^\theta \left( \int \Gamma \overline{W} \right) \, d\tau + \int \Gamma (\nabla \text{Ric} + \text{Ric} \ast g) \big|_{t=0}
\]

where \( \Gamma \) is the fundamental solution of \( \Delta_e \). The integral sign without specified region of integration refers to spatial integration.

Notice that \( \nabla U = \nabla^2 \text{Ric} \). Therefore, if \( M_2 = \max |\nabla^2 \text{Ric}| \) we get from (17)

\[
M_2 \leq M_2 \int_0^\theta \left( \int \nabla \Gamma \right) + \frac{M_1 \cdot M_0}{\sqrt{\theta}}
\]

here \( M_i = \max |\nabla^i \text{Ric}|, \ i = 0, 1 \)

By making \( \theta > 0 \) small we obtain

\[
M_2 \leq \frac{M_1 M_0}{\sqrt{\theta}}
\]

The smallness of \( \theta \) depends on Lipschitz bounds on the coefficients of \( \Delta_e \) which in turn only depend on \( M_0 \) and \( M_1 \).
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