A new proof of the growth rate of the solvable Baumslag–Solitar groups

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Abstract
We exhibit a regular language of geodesics for a large set of elements of $BS(1, n)$ and show that the growth rate of this language is the growth rate of the group. This provides a straightforward calculation of the growth rate of $BS(1, n)$, which was initially computed by Collins et al. (AM (Basel) 62:1-11, 1994). Our methods are based on those we develop in Taback and Walker (JTA, to appear) to show that $BS(1, n)$ has a positive density of elements of positive, negative and zero conjugation curvature, as introduced by Bar-Natan et al. (JTA, 2020, https://doi.org/10.1142/S1793525321500096).

Keywords Baumslag-Solitar group · Growth rate · Geodesics

1 Introduction

In this paper we compute the growth rate of the solvable Baumslag–Solitar groups

$$BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle$$

for $n \geq 2$. The first computation of this growth rate is due to Collins, Edjvet and Gill in [5] who additionally exhibit the growth series for the group. Bucher and Talambutsa in [3] reprove the results in [5] for prime $n$; their methods involve understanding the action of the group on its Bass-Serre tree. Their goal is to show that the minimal exponential growth rates of the solvable Baumslag–Solitar group $BS(1, n)$ and the lamplighter group $L_n = \mathbb{Z}_n \wr \mathbb{Z}$ coincide for prime $n > 2$ but differ for $n = 2$. In both [3] and [5], as well as in the proofs below, the rate of growth is computed to be the reciprocal of a root of a particular polynomial. When $n > 2$ is even our polynomials are much simpler than those in [5], and they match those in both references in the remaining cases. We give a new proof of the following theorem, where the sets $O_n(N) \subset BS(1, n)$ are defined in Sect. 5.
Theorem 1.1 Let \( \alpha = \lfloor \frac{n}{2} \rfloor \). The growth rate of the sequence \( \{ |O_n(N)| \} \) for odd \( n \) is the reciprocal of the smallest magnitude root of

\[
1 - x - \sum_{i=1}^{\alpha} 2x^{i+1},
\]

and for even \( n \) is the reciprocal of the smallest magnitude root of

\[
1 - 2x - x^2 + 2x^{\alpha+1} - 2x^{\alpha+2} + 2x^{2\alpha+2}.
\]

In Table 1, we compute the growth rate of \( BS(1, n) \) for \( 2 \leq n \leq 8 \) and discuss the limiting case as \( n \) approaches infinity in the context provided by our methods.

Our proofs rely on a series of techniques developed by the authors in [8] to produce a geodesic representative for an element of \( BS(1, n) \) given in a standard normal form. In [8] we prove that \( BS(1, n) \) has a positive density of elements of positive, negative and zero conjugation curvature, as introduced by Bar-Natan, Duchin and Kropholler in [1]. A direct consequence of our methods for understanding geodesic words is the computation of the growth rate contained below. We present a concise version of the arguments in [8] and refer the reader to that paper for more details.

Briefly, our approach is as follows. Given any element of \( BS(1, n) \) in a standard normal form, we parametrize a set of paths representing the element which we show contains a geodesic. These paths come in four basic “shapes.” Focusing on geodesic paths of one particular shape, we prove that the set of all paths of this shape form a regular language, and exhibit a finite state automaton which accepts it. This allows us to analyze the growth rate of this set, which we show to be identical to the growth rate of the group \( BS(1, n) \). This is analogous to the work of Brazil in [2], who follows the same outline to show that \( BS(1, n) \) has rational growth for all \( n > 1 \). He remarks that one should be able to use his methods to calculate the exact growth rate but does not do so.

Also of interest for \( BS(1, n) \) is the conjugacy growth rate. In recent work, Ciobanu, Evetts and Ho in [4] show that the conjugacy growth rate for \( BS(1, n) \) is identical to the standard growth rate, using the presentation given above. If \( G \) is a group with finite generating set \( S \), the conjugacy growth function measures the number of conjugacy classes intersecting the ball of radius \( m \) in \( \Gamma(G, S) \). They show that the corresponding generating function, called the conjugacy growth series, is transcendental. These results provide positive evidence towards two conjectures: first, that the conjugacy and standard growth rates are identical in finitely generated groups, and second, that only virtually abelian groups have rational conjugacy growth series.

2 Representations of integers and geodesic paths

2.1 Background and approach

For \( n \in \mathbb{N} \) with \( n > 1 \), the solvable Baumslag–Solitar group \( BS(1, n) \) has presentation

\[
BS(1, n) = \langle a, t | tat^{-1} = a^n \rangle.
\]

We consider elements of \( BS(1, n) \) in the standard normal form, namely each \( g \in BS(1, n) \) can be written uniquely as \( t^{-u}a^v t^w \) where \( u, v, w \in \mathbb{Z} \) and \( u, w \geq 0 \), with the additional requirement that if \( n|v \) then \( uw = 0 \). If \( n|v \) but \( uw \neq 0 \) then the group relator can be applied...
to simplify the normal form expression. When we write $g = t^{-u}a^vt^w$ we will assume that these conditions are satisfied.

Our approach to finding geodesic words representing elements of $BS(1, n)$ builds on [6], where it is shown that any geodesic takes one of a small number of prescribed forms. We create a vector from the exponents of the generator $a$ in any one of these forms, which is related to the “horizontal” distance traveled by the path in the Cayley graph $\Gamma(BS(1, n), \{a, t\})$. We then develop criteria to determine when a vector of exponents corresponds to a geodesic path. This strategy is described more fully in the sections below, with proofs of the statements provided in [8]. We refer the reader to Sect. 2 of [8] for a more detailed description of the geometry of the Cayley graph of $BS(1, n)$ and the “tree” of coarsely hyperbolic planes on which $BS(1, n)$ acts.

### 2.2 The digit lattice

In order to produce a geodesic representative of a given group element $g = t^{-u}a^vt^w$, we begin with an investigation of finite length vectors with integer entries having bounded absolute value. We describe an algorithm to translate this vector, which we also refer to as a digit sequence, into a path in $\Gamma(BS(1, n), \{a, t\})$. Our goal is to impose simple conditions on the sequence of digits so that the resulting path will be geodesic.

Our approach is based on the fact that finding a geodesic representing $t^{-u}a^vt^w$ is closely related to finding the “most efficient” way of writing $v$ as a sum of powers of $n$, that is, finding the base $n$ expansion of $v$. However, there are many subtleties which arise in the geometry of the Cayley graph of $BS(1, n)$ and relate to the relative values of $u, v$ and $w$ which complicate the analogy of expressing $v$ simply as a base $n$ expression. These are described both in [8] and below.

A digit sequence has an associated integer number, which becomes the exponent of $a$ in the standard normal form for any path associated with this digit sequence. We formalize the concept of digit sequences using the direct sum $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, where we take the convention that $0 \in \mathbb{N}$. Given a vector $x = (x_0, x_1, \ldots) \in \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, define the function $\Sigma : \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$\Sigma(x) = \sum_{i \in \mathbb{N}} x_i n^i.$$ 

For any $v \in \mathbb{Z}$, let $L_v = \Sigma^{-1}(v)$ be the set of vectors $x \in \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ with $\Sigma(x) = v$. We establish some notation for the remainder of this paper. Given a vector $x$,

- the coordinates of $x$ will be written with a matching non-bold letter, for example, $x_i$,
- $k_x$ will denote the index of the final nonzero coordinate in $x$, and
- the length of the vector is $k_x + 1$.

Vector entries will be called either “coordinates” or “digits.” These vectors, while constructed in $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ or $L_v$, have a finite number of non-zero entries; we list only those non-zero entries and write, for example, $x = (x_0, \ldots, x_{k_x})$, assuming that $x_i = 0$ for $i > k_x$ and $x_{k_x} \neq 0$.

Let $L_0 = \Sigma^{-1}(0)$; in [8] we show that $L_0$ is a lattice spanned by the following set of vectors. Define vectors $\{w(i)\}_{i \in \mathbb{N}}$ whose coordinates $w_j^{(i)}$ are given by

$$w_j^{(i)} = \begin{cases} 
1 & \text{if } j = i + 1 \\
-n & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}$$
Fig. 1 Schematics of the different path shapes. Note that the vertical segments may lie in different planes in $X_n$, the "tree" of coarsely hyperbolic planes on which $BS(1,n)$ acts. The distance $v$ represents the number of horizontal edges in $X_n$ measured at the height of the lowest point on the path.

That is,

$$w^{(i)} = (0, \ldots, 0, w_i^{(i)}, w_{i+1}^{(i)}, 0, \ldots) = (0, \ldots, 0, -n_i, 1, 0, \ldots)$$

where we indicate the index of each entry in the second expression. In [8] we show that $L_v$ is an affine lattice, for $v \in \mathbb{Z}$. We use these vectors $\{w^{(i)}\}_{i \in \mathbb{N}}$ to describe a deterministic algorithm which produces a geodesic representative for $g \in BS(1,n)$.

Given a group element $g = t^{-u}a^v t^w$, we define a map $\eta_{u,v,w} : L_v \rightarrow \{a^\pm, t^\pm\}^*$ which takes a vector $x = (x_0, \ldots, x_k) \in L_v$ to a word $\eta_{u,v,w}(x)$ representing $g$ in the following way:

$$\eta_{u,v,w}(x) = \begin{cases} t^{-u}a^{x_0}t^{x_1} \cdots t^{x_k}t^{-k} & \text{if } k_x \leq \max(u,w) \text{ (shapes 1 and 2)} \\ t^{k - u}a^{x_0}t^{x_1} \cdots t^{x_k}t^{-k} & \text{if } u < k_x \leq u \text{ (shape 2)} \\ t^{-u}a^{x_0}t^{x_1} \cdots t^{x_k}t^{-k} & \text{if } u \leq w < k_x \text{ (shape 3)} \\ t^{k - u}a^{x_0}t^{x_1} \cdots t^{x_k}t^{-k} & \text{if } w < u < k_x \text{ (shape 4)} \end{cases}$$

Paths of shapes 1 and 3 and shapes 2 and 4 have identical expressions up to the signs of certain exponents. Following [8] we additionally denote those geodesics of shape 1 for which $k_x < w$ as geodesics of strict shape 1. Fig. 1 gives a schematic for each path shape.

We now show that the length of each path above is given by one of two expressions. Here, $|\cdot|$ denotes the length of the string rather than the word length with respect to the generating set $\{a^\pm, t^\pm\}$. We repeat the proof from [8] as this lemma is crucial to subsequent results.

Lemma 2.1 ([8], Lemma 3.7) For $x = (x_0, \ldots, x_k) \in L_v$, we have

$$|\eta_{u,v,w}(x)| = \begin{cases} \|x\| + u + w & \text{if } k_x \leq \max(u,w) \text{ (shapes 1 and 2)} \\ \|x\| + 2k_x - |u - w| & \text{otherwise (shapes 3 and 4)} \end{cases}$$
Proof To prove the lemma, we sum the absolute values of the exponents in the above expressions for \( \eta_{u,v,w}(x) \). Accounting for the signs of the expressions, the first formula follows immediately.

For the second case, we compute the length of a path of shape 3:

\[
|\eta_{u,v,w}(x)| = \|x\|_1 + u + k_x + k_x - w
\]

and for shape 4:

\[
|\eta_{u,v,w}(x)| = \|x\|_1 + (k_x - u) + k_x + w.
\]

Considering the relative magnitudes of \( u, \), \( w, \) and \( k_x, \), we see that the two expressions combine into the second formula of the lemma. \( \square \)

When \( k_x = \max(u, w) \), it follows easily that the two expressions for word length given in Lemma 2.1 agree. To verify this, when \( k_x = \max(u, w) \) we have

\[
2k_x - |u - w| = 2\max(u, w) - |u - w|
\]

\[
= 2\max(u, w) - \max(u, w) + \min(u, w)
\]

\[
= u + w
\]

The following lemma is straightforward, and proven in [8]. We include the proof for clarity.

Lemma 2.2 ([8], Lemma 3.9) If \( x \in \mathcal{L}_v \) is such that \( |\eta_{u,v,w}(x)| \) is minimal, then \( \eta_{u,v,w}(x) \) is a geodesic representing the group element \( g = t^{-u}a^vt^w \).

Proof This is a corollary of [6], Proposition 2.3, where it is shown that there must be a geodesic representing \( g \) which has one of shapes 1–4. For the geodesic \( \xi \) guaranteed by [6], there is a vector \( y \in \mathcal{L}_v \) so that \( \eta(y) = \xi \).

For a given \( x \in \mathcal{L}_v \), one can form two possible induced paths which are potentially geodesic: one in shape 1 or 3, depending on whether \( w < k_x \), and one in shape 2 or 4. The word length formulas in Lemma 2.1 allow us to define the shorter of these paths as the output of \( \eta_{u,v,w} \), and hence \( \eta_{u,v,w} \) describes a geodesic path to \( g \). \( \square \)

If we are given \( g = t^{-u}a^vt^w \) and want to find a geodesic for \( g \), then by Lemma 2.2, it suffices to find a vector \( x \in \mathcal{L}_v \) such that \( |\eta_{u,v,w}(x)| \) is minimal; we will refer to such an \( x \) as a minimal vector. As \( \mathcal{L}_v \) is an affine lattice, this is equivalent to minimizing \( |\eta_{u,v,w}(x + z)| \), where \( x \in \mathcal{L}_v \) is any vector and \( z \in \mathcal{L}_0 \). Lemma 2.3 shows that some vectors \( x \) are easily altered in this way to reduce \( |\eta_{u,v,w}(x)| \). We will refer to the change from \( x \) to \( x + z \) in this way as reducing \( x \).

For \( n \geq 3 \), let \( \mathcal{B}^{u,w}_v \subseteq \mathcal{L}_v \) be defined as the set of \( x = (x_0, \ldots, x_k) \in \mathcal{L}_v \) satisfying the following conditions.

1. If \( i < k_x \), then \( |x_i| \leq \left\lfloor \frac{n}{2} \right\rfloor \).
2. If \( i = k_x < \max(u, w) \), then \( |x_i| \leq \left\lfloor \frac{n}{2} \right\rfloor \).
3. If \( i = k_x = \max(u, w) \), then \( |x_i| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

When \( n = 2 \), we define \( \mathcal{B}^{u,w}_v \) as above, replacing the third inequality with \( |x_i| \leq \left\lfloor \frac{n}{2} \right\rfloor + 2 \).

Note that the entries of vectors in \( \mathcal{B}^{u,w}_v \) are uniformly bounded in absolute value by \( \left\lfloor \frac{n}{2} \right\rfloor \) with the exception of the final coordinate, which in some cases can be slightly larger in absolute value. With the goal of finding minimal vectors in \( \mathcal{B}^{u,w}_v \), this modified bound on the final digit results from certain examples where a larger final digit produces a shorter path. Figure 2 shows an example of this in \( BS(1, 6) \).
In Lemma 2.3 below we show that given \( x \in \mathcal{L}_v \), we can find a vector \( y \in B^u_w \subseteq \mathcal{L}_v \) so that \( |\eta_{u,v,w}(y)| \leq |\eta_{u,v,w}(x)| \). Since \( \eta_{u,v,w}(x) \) and \( \eta_{u,v,w}(y) \) represent the same group element, this implies that finding a geodesic for a group element is equivalent to searching for a minimal vector within \( B^u_w \). For such \( x \) and \( y \), we will write \( x \leq_{u,w} y \) to mean that \( |\eta_{u,v,w}(y)| \leq |\eta_{u,v,w}(x)| \). Note that although \( \leq_{u,w} \) is transitive, it is not a partial order because it is not antisymmetric. However, it still makes sense to refer to vectors as being minimal with respect to the relation.

The following lemma follows immediately.

**Lemma 2.3** ([8], Lemma 3.10) If \( x \notin B^u_w \), then there exists \( z \in \mathcal{L}_0 \) so that \( x + z \in B^u_w \) and

\[
x + z \leq_{u,w} x.
\]

Consequently, if \( x \in B^u_w \) is minimal in \( B^u_w \), then \( \eta_{u,v,w}(x) \) is geodesic.

In the sections below, we will give some simple conditions to certify that \( x \) is minimal.

### 2.3 Minimal vectors for \( n \) odd

Let \( g = t^{-u}a^v t^w \in BS(1, n) \) for \( n \) odd. Lemma 2.3 shows that \( B^u_w \) is nonempty and that if \( x \in \mathcal{L}_v \) is a minimal vector in \( B^u_w \), then \( \eta_{u,v,w}(x) \) is geodesic for \( g \). The next lemma shows that when \( n \) is odd, the set \( B^u_w \) contains at most two vectors.

**Lemma 2.4** ([8], Lemma 3.13) Let \( n \geq 3 \) be odd and \( x \in B^u_w \).

1. If \( k_x < \max(u, w) \), then \( |B^u_w| = 1 \).
2. If \( k_x \geq \max(u, w) \), then \( |B^u_w| \leq 2 \). If \( |B^u_w| = 2 \), then \( B^u_w \) has the form \( B^u_w = \{x, x + \epsilon w^{(k_x)}\} \),

where \( \epsilon \in \{-1, 1\} \). Moreover, \( y \in B^u_w \) is not minimal if and only if \( k_y > \max(u, w) \) and the final digits of \( y \) are \((\delta \left\lfloor \frac{n}{2} \right\rfloor, -\delta)\), where \( \delta \in \{\pm 1\} \).

Lemma 2.4 presents a direct algorithm for producing a geodesic representative of \( g = t^{-u}a^v t^w \) in \( BS(1, n) \) when \( n \) is odd. Begin with any vector in \( \mathcal{L}_v \); if it does not lie in \( B^u_w \), reduce its digits as described above so that it does. Then inspect the final two digits to assess minimality, adding a basis vector as specified by the theorem if necessary.

### 2.4 Minimal vectors for \( n \) even

When \( n \) is even, \( B^u_w \), as defined above, may contain more than two minimal vectors. In order to make a consistent choice among them, we add a constraint on the absolute values...
of the digits. For $n$ even, we will say that $\mathbf{x}$ is minimal if $|\eta_{u,v,w}(\mathbf{x})|$ is minimal and the vector of absolute values of $\mathbf{x}$ is lexicographically minimal among all such vectors. For this lexicographic order, smaller-index digits are considered more significant.

As there may be many more vectors in $B_{u,v}^{u,w}$ to consider when $n$ is even, the question of deciding whether a vector is minimal is more complicated. For us it will suffice to characterize minimality for a subset of all vectors in $B_{u,v}^{u,w}$, namely, those vectors $\mathbf{x} \in B_{u,v}^{u,w}$ with $k_\mathbf{x} < w$. These vectors correspond to paths $\eta_{u,v,w}(\mathbf{x})$ of strict shape 1, and Lemma 2.5 below describes when such vectors are minimal.

We subdivide this case by $n = 2$ and $n > 2$ as the definition of $B_{u,v}^{u,w}$ is slightly different. Our goal is to exhibit simple local conditions to determine whether a vector $\mathbf{x} \in B_{u,v}^{u,w}$ with $k_\mathbf{x} < w$ is minimal. Such a minimal vector will correspond to a geodesic $\eta_{u,v,w}(\mathbf{x})$ of strict shape 1. We will compute the growth rate of the language of geodesics of strict shape 1, and show that this is the same as the growth rate of $BS(1,n)$.

We condense Lemmas 3.22 and 3.29 of [8] into the following lemma. While the result is identical for $n = 2$ and $n > 2$, the methods of proof are slightly different. We refer to reader to [8] for these two proofs.

**Lemma 2.5 ([8], Lemmas 3.22 and 3.29)** Let $n$ be even and $\mathbf{x} \in B_{u,v}^{u,w}$ with $k_\mathbf{x} < \max(u,w)$. Then $\mathbf{x}$ is not minimal if and only if one of the following holds, for $\delta \in \{ \pm 1 \}$.

- There are two adjacent digits of the form $(\frac{u}{2}, \frac{n}{2})$.
- There are two adjacent digits of the form $(\frac{n}{2}, x_i)$ with $\text{sign}(x_i) = -\text{sign}(\delta)$.

### 3 Regular languages

Given $u$, $w$, and $\mathbf{x}$ with $k_\mathbf{x} < \max(u,w)$, Lemmas 2.4 and 2.5 provide a straightforward way to determine whether $\mathbf{x} \in B_{u,v}^{u,w}$ is minimal, that is, whether $\eta_{u,v,w}(\mathbf{x})$ is a geodesic, by examining the digits of $\mathbf{x}$. Recall that if $\eta_{u,v,w}$ has shape 1 and $k_\mathbf{x} < \max(u,w)$, we say that $\eta_{u,v,w}(\mathbf{x})$ has strict shape 1.

In this section, we show that the set of vectors $\mathbf{x}$ for which there are $u$, $w$ so that $\eta_{u,v,w}(\mathbf{x})$ is geodesic and has strict shape 1 forms a regular language. We then show that the set of geodesics of strict shape 1 is also a regular language and construct automata accepting both of these languages. This will allow us, in Sect. 5, to count the geodesics of strict shape 1 with a given length and determine the growth rate of $BS(1,n)$.

#### 3.1 The language of strict shape 1 vectors

Let $D_n$ be the language of minimal vectors $\mathbf{x} \in B_{u,v}^{u,w}$ for some $u$, $v$, $w$ where $\eta_{u,v,w}(\mathbf{x}) \in BS(1,n)$ has strict shape 1.

**Lemma 3.1** For all $n \geq 2$, the language $D_n$ is regular. These languages are accepted by the finite state automata shown in Figs. 3 and 4.

**Proof** Note that $D_n$ is the subset of words in $\{-\left\lfloor \frac{n}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}^*$ which satisfy no condition of Lemmas 2.4, or 2.5 and do not end with a 0. The conditions of Lemmas 2.4 and 2.5 are local conditions which are therefore regular, and the finite state automata shown in Figs. 3 and 4 accept only digit strings which do not satisfy any of them. \qed

We remark that $D_n$ is the language of digit sequences which are minimal for some $u$, $w$. Thus the pattern $(\{\left\lfloor \frac{n}{2} \right\rfloor, -\delta\}$, for $\delta \in \{1,-1\}$ is permitted at the end of a sequence, because if $w$ is sufficiently large, this pattern can exist in a minimal vector.
Fig. 3 A finite state automaton accepting the regular language $D_n$ when $n$ is odd. An edge label for a range of $x$ values represents that many single edges with labels in the appropriate interval. The state $s_0$ is the start and accept state.

$x > 0$
$x = -\frac{n}{2}$
$x < 0$

For technical reasons, it will be helpful in Sect. 3.2 to consider a language very closely related to $D_n$: the language of strict shape 1 vectors which are allowed to end with a string of 0 digits. That is, the language of vectors which satisfy no condition of Lemma 2.4 or 2.5. This simply relaxes the last condition from the definition of $D_n$. We denote this new language by $D'_n$. Note that $D_n \subseteq D'_n$. Finite state automata which accept $D'_n$ are shown in Fig. 5. These simpler automata are obtained by merging the state keeping track of the 0 digit (that is, $s_1$ for $n$ odd and $s_3$ for $n$ even) into the start state $s_0$. 

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The language \( D_n \) is a subset of the full set of minimal digit sequences. Note that \( D_n \) is not a language of geodesic words in \( BS(1, n) \); if \( x \) is an accepted string in \( D_n \) then \( \eta_{u,v,w}(x) \) is a word in \( BS(1, n) \), where for appropriate choices of \( u, w \), it follows that \( \eta_{u,v,w}(x) \) is a geodesic. Let \( O_n \) be the language of strict shape 1 geodesics in \( BS(1, n) \). Using \( D'_n \) and the finite state automaton accepting it, we will show that \( O_n \) is regular and exhibit a finite state automaton accepting it.

For simplicity, we will abuse notation and write \( D'_n \) for both the language and the finite state automaton which accepts it. We now derive a finite state automaton which accepts any number of paths in \( D_n \) for all \( n \).

One nice feature of \( D_n \) and \( D'_n \) is that the number of states is independent of \( n \). This is not the case for \( O_n \), so we cannot exhibit a general structure analogous to Figs. 3, 4, and 5. Instead, we describe a simple expansion rule to derive a finite state automaton accepting \( O_n \) from \( D'_n \). We will use \( O_n \) to refer to both the language of strict shape 1 geodesics and the finite state automaton which accepts this language.

In order to motivate the expansion of \( D'_n \) to \( O_n \), consider the structure of geodesic paths in \( BS(1, n) \) of strict shape 1 representing \( n \) as a sequence of digits. Specifically, for each \( n \) of strict shape 1 requiring that \( 0 \leq k_x < w \) and hence \( w - k_x > 0 \), that is, the string ends with a positive power of \( t \). For example, \( tat \) has strict shape 1 with \( u = 0, w = 2 \) and \( k_x = 1 \). As a strict shape 1 geodesic is allowed to end with an arbitrarily large power of \( t \), we are no longer concerned with making certain that the digit sequence in question does not end with a 0 digit. Any such extraneous 0 digits correspond in the geodesic to a higher power of \( t \), that is, to \( t^{w-k_x} \).

To construct a finite state automaton \( O_n \) which accepts these geodesics, we must allow initial strings of \( t^{-1}s \), require nonempty final strings of \( t's \), and expand each digit of \( x \) into a sequence of copies of \( a^{\pm 1} \), separated by \( t \).

For both even and odd \( n \), define the \( \alpha \)-digit expansion of a state \( s_i \) in \( D'_n \), where \( \alpha = \lfloor \frac{n}{2} \rfloor \), to be the collection of states and transitions shown in Fig. 6. Specifically, for each \( i \), the state \( s_i \) in \( D'_n \) is replaced with the collection of states

\[ \{s_i,-\alpha,s_i,-\alpha+1,\ldots,s_i,0,\ldots,s_i,\alpha-1,s_i,\alpha\} \]

connected as follows.

1. For all \( 0 \leq j < \alpha \), there is an edge labeled \( a \) from \( s_{i,j} \) to \( s_{i,j+1} \).
2. For all \( -\alpha < j \leq 0 \), there is an edge labeled \( a^{-1} \) from \( s_{i,j} \) to \( s_{i,j-1} \).
3. All edges outgoing from \( s_i \) to another state \( s_j \) are replaced by edges outgoing from \( s_{i,\ell} \) to the state \( s_{j,0} \) in the \( \alpha \)-digit expansion of \( s_j \), with label \( t \).

We define \( O_n \) to be the finite state automaton which is obtained by performing the \( \alpha \)-digit expansion on every state in \( D'_n \) and prepending states start and \( s_{i,-1} \), in order to allow for any number of initial \( t^{-1} \) letters. It is unnecessary to append a special state accepting the final sequence of powers of \( t \); such sequences will be “interpreted” by \( O_n \) as a sequence of digits.
consisting only of zeros and accepted. Since an accepted string must end with the letter \( t \), we designate the states \( s_{j,0} \) for \( j \in \{0, 1, 2\} \) as accept states. Except for the start state, these are exactly the states with an incoming edge labeled \( t \).

**Theorem 3.2** The finite state automaton \( O_n \) accepts exactly the language of geodesic paths of strict shape 1 in \( BS(1, n) \).

**Proof** This is an immediate consequence of Lemma 3.1 together with the observations above deriving the language of strict shape 1 geodesics from the language of reduced paths accepted by \( D_n \). \( \square \)

### 3.3 Example automata for \( n = 2 \)

To illustrate the derivation of \( O_n \) from \( D'_n \), we construct these automata when \( n = 2 \). Figure 7 illustrates \( D_2 \) and \( D'_2 \), while Fig. 8 illustrates \( O_2 \), the result of performing the digit expansion on each state in \( D'_2 \).

### 4 Exponential growth

In this section we present two lemmas about growth rates which are frequently referenced in Sect. 5. Recall that the growth rate of a sequence \( \{f(N)\}_{N=1}^{\infty} \) is \( \lambda \) if and only if

\[
\lim_{N \to \infty} \frac{\log f(N)}{N \log \lambda} = 1.
\]
Fig. 8 The finite state automaton $O_2$ derived from $D'_2$ by expanding each state and prepending states to allow an initial sequence of $r^{-1}$ letters. Accept states are indicated with a double circle. Each accepted string corresponds to an infinite family of geodesics of strict shape 1 in $BS(1,2)$. We have omitted the expanded states $s_{1,\pm1}$ and $s_{2,\pm1}$ which are unreachable and have rearranged the states for clarity.

Equivalently, we write $f(N) = \Theta(\lambda^N)$; that is, there are constants $A, B > 0$ such that

$$A\lambda^N \leq f(N) \leq B\lambda^N$$

for sufficiently large $N$.

**Lemma 4.1** Suppose that $f(N) = \Theta(\lambda^N)$ with $\lambda > 1$.

1. Both $f(N + k)$ and $\sum_{i=1}^{N} f(i)$ are $\Theta(\lambda^N)$.
2. If $f(N)$ and $g(N)$ are $\Theta(\lambda^N)$, there are $N_0, d > 0$ so that $f(N)/g(N) > d$ for $N > N_0$.

**Proof** It is clear that $f(N + k) = \Theta(\lambda^{N+k}) = \Theta(\lambda^N)$. We now show that $\sum_{i=1}^{N} f(i) = \Theta(\lambda^N)$. As discussed above, there are $C_1, C_2, M > 0$ so that for all $N > M$ we have

$$C_1\lambda^N \leq \sum_{i=1}^{N} f(i) \leq C_2\lambda^N.$$

Let $D = \sum_{i=1}^{M} f(i)$; note that $D$ is constant. The inequalities

$$C_1 \sum_{i=M+1}^{N} \lambda^i + D \leq \sum_{i=1}^{N} f(i) \leq C_2 \sum_{i=M+1}^{N} \lambda^i + D,$$

combined with the expansion $\sum_{i=1}^{N} \lambda^i = \lambda(\lambda^N - 1)/(\lambda - 1) = \Theta(\lambda^N)$, yield additional constants $C_3, C_4 > 0$ so that

$$C_3\lambda^N \leq C_1 \sum_{i=M+1}^{N} \lambda^i + D \leq \sum_{i=1}^{N} f(i) \leq C_2 \sum_{i=M+1}^{N} \lambda^i + D \leq C_4\lambda^N.$$
for sufficiently large $N$. Thus $\sum_{i=1}^{N} f(i) = \Theta(\lambda^N)$, as desired.

To prove the second statement in the lemma, observe that for sufficiently large $N$ and new constants $C_i > 0$ we have

$$C_1 \lambda^N \leq f(N) \leq C_2 \lambda^N \text{ and } C_3 \lambda^N \leq g(N) \leq C_4 \lambda^N,$$

hence $\frac{f(N)}{g(N)} \geq \frac{C_1}{C_4} > 0$.

We will be interested in determining the growth rate of the function which counts the number of accepted paths of a given length in a finite state automaton.

**Lemma 4.2** Let $F$ be a finite state automaton with state set $S$. Let $f(N)$ denote the number of accepted paths in $F$ of length $N$, and for each $s \in S$, let $f_s(N)$ denote the number of paths of length $N$ from $s$ to an accept state. Let $F_1, \ldots, F_c$ be the strongly connected components of $F$.

1. For each $i$, the growth rate of $\{f_s(N)\}_{N \in \mathbb{N}}$ is constant over all $s \in F_i$.
2. The growth rate of $\{f(N)\}_{N \in \mathbb{N}}$ is the maximum of the growth rates of the $F_i$.

**Proof** Let $\lambda$ be the growth rate of the sequence $\{f(N)\}_{N \in \mathbb{N}}$, and $\lambda_s$ the growth rate of the sequence $\{f_s(N)\}_{N \in \mathbb{N}}$ for any $s \in S$. Let $s, s' \in S$ be states. If there is an edge from $s$ to $s'$, then $f_s(N + 1) \geq f_{s'}(N)$. It follows from Lemma 4.1 that $\lambda_s \geq \lambda_{s'}$. Iterating this argument shows that if there is a path of any length from $s$ to $s'$, then $\lambda_s \geq \lambda_{s'}$ and hence $\lambda_s$ is constant over a strongly connected component in $F$.

Next observe that if $s$ is any state in $F$, then $f_s(N) = \sum_{s \rightarrow s'} f_{s'}(N - 1)$, where $s \rightarrow s'$ denotes the existence of an edge from $s$ to $s'$. Therefore, $\lambda_s = \max_{s \rightarrow s'} \lambda_{s'}$. Iterating this argument shows that $\lambda_s$ is the maximum $\lambda_{s'}$ over all $s$ which are reachable from $s$. Applying this argument to the start state proves the second statement of the lemma. \hfill $\square$

5 The growth rate of $BS(1, n)$

Let $S_n(N)$ denote the sphere of radius $N$ in $BS(1, n)$. The growth rate of a finitely generated group $G$ is defined to be the growth rate of the sequence $\{|S_n(N)|\}_{n \in \mathbb{N}}$. In this section we compute the growth rate of $BS(1, n)$ for all $n > 1$ using the finite state automaton constructed in Sect. 3 which accepts $O_n$, the language of geodesic paths of strict shape 1.

In order to obtain bounds on the number of minimal vectors producing geodesic paths of any shape, we construct a map from the set of all minimal vectors to the set of minimal vectors corresponding to geodesic paths of strict shape 1 and bound the degree of this map. The difficulty is that there are certain digits allowed in a minimal vector $x$ which are not permitted as exponents in a geodesic path of strict shape 1. Specifically, when $k_x \geq \max(u, w)$, the final digit of $x$ is allowed to have absolute value greater than $\left\lfloor \frac{n}{2} \right\rfloor$, but a minimal vector corresponding to a geodesic of strict shape 1 must satisfy $k_x < w$ and have $|x_i| \leq \frac{n}{2}$ for all $i \leq k_x < w$.

The idea of our map is not complicated: start with $x \in B_v^{u,w}$. If $\eta_{u,w}(x)$ is strict shape 1, the map returns $x \in B_v^{u,w}$. If not, so that $k_x \geq w$, we modify $x$ to produce a vector $x'$ where each digit has absolute value at most $\left\lfloor \frac{n}{2} \right\rfloor$, and alter $u$ and/or $w$ to $u'$ and/or $w'$, ensuring that $k_{x'} < w'$. The map then returns $x' \in B_v^{u',w'}$.

We now define a map $c : \mathcal{L}_v \rightarrow \mathcal{L}_v$ which implements the algorithm described above. The subsequent modification of $u$ and/or $w$ is a secondary step. Take $x \in B_v^{u,w}$. If it exists, let $j \leq k_x$ be the minimal index such that $|x_i| \geq \frac{n}{2}$ for $j \leq i \leq k_x$.

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It is easy to verify that for \( j \leq i < k_x \), the digits \( x_i \) have constant sign. Suppose \((x_i, x_{i+1}) = (\delta \frac{n}{2}, -\delta \frac{n}{2})\). Let \( y = x + \delta w^{(i)} \). Then \( |k_x - k_y| \leq 1 \) which implies that we can use the same length formula from Lemma 2.1 to determine whether \( |\eta_{u', v', w'}(y)| < |\eta_{u, v, w}(x)| \).

It is easy to verify that \( \|y\|_1 < \|x\|_1 \), and thus, regardless of which length formula is required, \( |\eta_{u', v', w'}(y)| < |\eta_{u, v, w}(x)| \). Thus \( x \) can be reduced, contradicting the fact that it is minimal.

Note that the digits of \( c(x) \) and \( x \) are identical for indices less than \( j \).

When \( n \) is odd it follows from by Lemma 2.4 that the only possibility we must consider is \( j = k_x \) with \( x_{k_x} = \delta(\frac{n}{2} + 1) \), where \( \delta \in \{\pm 1\} \). When \( n \geq 2 \) is even, \((x_j, \cdots, x_{k_x})\) is a maximal sequence where all but the final digit is \( \delta \frac{n}{2} \), and the final digit might be \( \delta(\frac{n}{2} + 1) \).

When \( n = 2 \), the final digit is chosen from the set \([\delta, \delta 2, \delta 3]\). Note that this sequence of digits is likely quite short as the number of repetitions of the digit \( \frac{n}{2} \) is always limited, as shown in [8].

The map \( c \) requires an alternate definition when \( n = 2 \) to account for the differing bounds on the final digit of a vector in \( B^n_u w \). Let “condition (A)” denote the case \( n = 2 \) and either

- \( x_{k_x} = \delta 3 \) or
- \( x_{k_x} = \delta 2 \) and \( j < k_x \).

If condition (A) holds, define

\[
    c(x) = x + \delta \sum_{i=j}^{k_x-1} w^{(i)} + \delta 2w^{(k_x)} + \delta w^{(k_x+1)}.
\]

Otherwise, if \( x_{k_x} = \delta(\frac{n}{2} + 1) \) or \( j < k_x \), that is, the last digit has absolute value greater than \( \frac{n}{2} \) or the sequence has length at least 2, define

\[
    c(x) = x + \delta \sum_{i=j}^{k_x} w^{(i)}.
\]

When \( j \) is undefined or there is a single \( \delta \frac{n}{2} \) at the end of \( x \), define \( c(x) = x \).

When \( n \) is odd, note that \( c(x) \) replaces a final digit in \( x \) of \( \delta(\frac{n}{2} + 1) \) with the final digits \((-\delta(\frac{n}{2} + 1), \delta)\). When \( n \) is even, the definition of \( c(x) \) depends on the configuration of the final digits of \( x \), as prescribed by Lemma 2.5. As we will need to refer to this computation later, we explain it in detail here. The simplest case is that there is a sequence of at least two \( \delta \frac{n}{2} \) digits at the end of \( x \). We compare the digits of \( x \) and \( c(x) \).

\[
    x = (x_0, \cdots, x_{j-1}, \delta \frac{n}{2}, \delta \frac{n}{2}, \cdots, \delta \frac{n}{2})
\]

\[
    c(x) = (x_0, \cdots, x_{j-1}, -\delta \frac{n}{2}, -\delta (\frac{n}{2} - 1), \cdots, -\delta (\frac{n}{2} - 1), 1) \tag{1}
\]

We now describe the slight variations in the other cases.

- When subsequence has length at least 2 and the final digit of \( x \) is \( \delta (\frac{n}{2} + 1) \), then the penultimate digit of \( c(x) \) is \(-\delta(\frac{n}{2} - 2) \) and all other digits are as in Eq. (1).

- When there are no \( \delta \frac{n}{2} \) digits, a final digit of \( \delta (\frac{n}{2} + 1) \) is replaced by the sequence of digits \((-\delta (\frac{n}{2} - 1), 1)\).

- When \( n = 2 \), we use the symbol \( | \) to mark a location in the vector so that the change in digits is clearly depicted. When \( x \) ends with the maximal subsequence:
  - \( (\delta, \delta, \cdots, \delta, | \delta 3) \), it is replaced with \((-\delta, 0, \cdots, 0, 0, 0, \delta) \), where the \( \delta 3 \) is replaced by \((0, 0, \delta) \).
  - \( (\delta 3) \), it is replaced with \((-\delta, 0, \delta) \).
  - \( (\delta, \delta, \cdots, \delta, | \delta 2) \), it is replaced with \((-\delta, 0, \cdots, 0, | - \delta, 0, \delta) \).
  - \( (\delta 2) \), it is replaced with \((0, \delta) \).
A computation shows that when condition (A) holds we have $k_{c(x)} = k_x + 2$. In all other cases, we have $k_{c(x)} = k_x + 1$. The change in $\ell'$ norm depends on the length $k_x - j + 1$ of the digit sequence $(x_j, \cdots, x_k)$ and the value of the final digit. Specifically, we observe that

$$
\|c(x)\|_1 = \begin{cases} 
\|x\|_1 & \text{if } n \text{ is odd or } c(x) = x \\
\|x\|_1 - (k_x - j - 1) & \text{if } |x_{k_x}| = \frac{n}{2} \\
\|x\|_1 - (k_x - j - 1) & \text{if } n = 2 \text{ with } |x_{k_x}| = 2 \text{ and } j < k_x \\
\|x\|_1 - (k_x - j + 1) & \text{otherwise.}
\end{cases}
$$

(2)

**Lemma 5.1** Let $x \in B_{v,w}^{u}$ and define the map $c$ as above. Then $\|c(x)\|_1 \leq \|x\|_1$.

**Proof** When $c(x) = x$ or $k_x - j \geq 1$, the lemma is immediate. If $k_x = j$, it follows from the definition of $c(x)$ that $|x_{k_x}| > \frac{n}{2}$, so we are in the final case of Eq. 2 above. Here $k_x - j + 1 > 0$, so the lemma follows. \qed

In order to compare geodesic length before and after our alteration of the vector $x$, define

$$
\beta(x) = 3 + \|x\|_1 - \|c(x)\|_1.
$$

It follows immediately from Lemma 5.1 that $\beta(x) \geq 3$.

Define

$$
\Phi(u, w, x) = \begin{cases} 
(u, w + \beta(x), c(x)) & \text{if } \eta_{u,v,w}(x) \text{ has shape 1} \\
(w, u + \beta(x), c(x)) & \text{if } \eta_{u,v,w}(x) \text{ has shape 2} \\
(u, 2k_x - w + \beta(x), c(x)) & \text{if } \eta_{u,v,w}(x) \text{ has shape 3} \\
(w, 2k_x - u + \beta(x), c(x)) & \text{if } \eta_{u,v,w}(x) \text{ has shape 4}
\end{cases}
$$

**Lemma 5.2** Let $x \in B_{v,w}^{u}$. If $(u', w', c(x)) = \Phi(u, w, x)$, then

$$
|\eta_{u',v',w'}(c(x))| = |\eta_{u,v,w}(x)| + 3,
$$

where $v'$ is determined by $c(x)$ and $\eta_{u',v',w'}(c(x))$ has strict shape 1. Moreover, if $x$ is minimal, then $c(x) \in B_{v',w'}^{u'}$ is also minimal.

**Proof** We begin with the assumption that $\eta_{u',v',w'}(c(x))$ has strict shape 1 and first prove that $|\eta_{u',v',w'}(c(x))| - |\eta_{u,v,w}(x)| = 3$.

Note that if $\eta$ is a geodesic of strict shape 1, we use the first length formula in Lemma 2.1 to compute its length. It follows that $|\eta_{u',v',w'}(c(x))| = \|x\|_1 + u' + w'$. When $\eta_{u,v,w}(x)$ has shape 1 or 2, we again use the first length formula in Lemma 2.1 to compute its length. Recalling the definition of $\beta$, we compute

$$
|\eta_{u',v',w'}(c(x))| = \|c(x)\|_1 + u' + w' \\
= \|c(x)\|_1 + u + w + \beta(x) \\
= \|x\|_1 + u + w + 3 \\
= |\eta_{u,v,w}(x)| + 3.
$$

When $\eta_{u,v,w}(x)$ has shape 3 we must have $u < k_x$, and we use the second formula in Lemma 2.1 to compute $|\eta_{u,v,w}(x)| = \|x\|_1 + 2k_x - |u - w|$. It follows that

$$
|\eta_{u',v',w'}(c(x))| = \|c(x)\|_1 + u' + w' \\
= \|c(x)\|_1 + u + 2k_x - w + \beta(x) \\
= \|x\|_1 + 3 + 2k_x - |u - w| \\
= |\eta_{u,v,w}(x)| + 3.
$$
An analogous computation yields the same conclusion when $\eta_{u,v,w}(x)$ has shape 4. Thus it remains to show that $\eta_{u',v',w'}(c(x))$ has strict shape 1 and $c(x) \in B_{u',v'}$ is minimal, that is, $\eta_{u',v',w'}(c(x))$ is a geodesic.

We begin by proving that $\eta_{u',v',w'}(c(x))$ has strict shape 1. Suppose that $\eta_{u,v,w}(x)$ has shape 1 or 2. From the definition of $c(x)$ we know that $k_{c(x)} \leq k_x + 2$. If $\eta_{u,v,w}(x)$ has shape 1, then $k_x \leq w$ and hence

$$k_{c(x)} \leq k_x + 2 < k_x + 3 \leq w + 3 \leq w + \beta(x) = w'.$$

That is, $\eta_{u',v',w'}(c(x))$ is a geodesic of strict shape 1. An analogous argument holds when $\eta_{u,v,w}(x)$ has shape 2.

When $\eta_{u,v,w}(x)$ has shape 3 we have $w < k_x$, so $2k_x - w > k_x$. To verify that $\eta_{u',v',w'}(c(x))$ has strict shape 1 we must check that $k_{c(x)} < w'$. Since $\beta(x) \geq 3$ and $k_{c(x)} \leq k_x + 2$, it follows that $k_{c(x)} < k_x + \beta(x)$ and

$$k_{c(x)} < k_x + \beta(x) < 2k_x - w + \beta(x) = w'.$$

A similar relation holds when $\eta_{u,v,w}(x)$ has shape 4.

By construction, $c(x)$ satisfies the digit bounds on $B_{u',v'}$. It remains to show that $c(x) \in B_{u',v'}$ is minimal in all cases. When $n$ is odd, the fact that $k_{c(x)} < \max(u', w')$ together with Lemma 2.4 imply that $c(x)$ is minimal.

Next let $n$ be even. Note that for $i < j$ we have $|c(x)_{j-1}| = |x_i|$; additionally we have $|c(x)_{j-1}| = |x_{j-1}| < \frac{n}{2}$. Suppose towards a contradiction that $c(x)$ is not minimal. It follows from Lemma 2.5 that $c(x)$ contains a digit subsequence $(c(x)_{i+1}, c(x)i+1) = (\delta_{\frac{n}{2}}, \delta_{\frac{n}{2}})$ or $(c(x)_i, c(x)_{i+1}) = (\delta_{\frac{n}{2}}, \delta_{\frac{n}{2}}), c(x)_{i+1})$ where $\text{sign}(c(x)_{i+1}) = -\text{sign}(\delta)$.

Suppose $c(x)$ contains the digit subsequence $(c(x)_i, c(x)_{i+1}) = (\delta_{\frac{n}{2}}, \delta_{\frac{n}{2}})$. The definition of $c(x)$ precludes any digit $|c(x)_m| \geq \frac{n}{2}$ for any $m$ with $j+1 \leq m < k_{c(x)}$ and $|c(x)_{j-1}| < \frac{n}{2}$. Therefore, the indices of both digits in the subsequence must be strictly less than $j - 1$. Thus $(c(x)_i, c(x)_{i+1}) = (x_i, x_{i+1})$ is contained in $x$. Including the subsequent digit, we have $(x_i, x_{i+1}, x_{i+2}) = (\delta_{\frac{n}{2}}, \delta_{\frac{n}{2}}, x_{i+2})$, and $|x_{i+2}| < \frac{n}{2}$.

Let $y = x + \delta(w^i + w^j - 1)$. It is easily verified that $\|y\|_1 \leq \|x\|_1$, as $(x_i, x_{i+1}, x_{i+2}) = (\delta_{\frac{n}{2}}, \delta_{\frac{n}{2}}, x_{i+2})$ and $(y_i, y_{i+1}, y_{i+2}) = (-\delta_{\frac{n}{2}}, -\delta_{\frac{n}{2}} - 1, x_{i+2} + \delta)$. If there is equality between the two $L^1$ norms, note that the change from $x_{i+1}$ to $y_{i+1}$ is a lexicographic decrease.

Suppose $c(x)$ contains the digit subsequence $(c(x)_i, c(x)_{i+1}) = (\delta_{\frac{n}{2}}, c(x)_i)$, where $\text{sign}(c(x)_i) = -\text{sign}(\delta)$. It follows that $i + 1 \leq j - 1$ and thus $(c(x)_i, c(x)_{i+1}) = (x_i, x_{i+1})$. Let $y = x + w^i$. It is easily verified that $\|y\|_1 < \|x\|_1$.

In both cases, as $k_x - k_y \leq 1$ we can use the same length formula from Lemma 2.1 to determine whether $\eta_{u',v',w'}(y) < \eta_{u,v,w}(x)$'. It follows from the comparison of $\|y\|_1$ and $\|x\|_1$ that $\eta_{u',v',w'}(y) < \eta_{u,v,w}(x)$, contradicting the fact that $x$ is minimal. Thus it must be the case that $c(x) \in B_{u',v'}$ is minimal.

Let $O_n(N)$ denote the set of elements of the language $O_n$ which are of length $N$. The next lemma allows us to compute the degree of the map $\Phi$, which will be crucial to the proof of Corollary 5.4, where we show that the growth rates of the sequences $|O_n(N)|_{N \in \mathbb{N}}$ and $|\{S_n(N)\}|_{n \in \mathbb{N}}$ are identical.

**Lemma 5.3** For any minimal vector $y \in B_{u',v'}$ the maximal number of minimal vectors $x \in B_{u',v'}$ such that $c(x) = y$ is

- 5 if $n = 2$,
- 3 if $n > 2$ is even, and
forms can overlap, so we conclude that y of Lemmas 5.2 and 5.3. However, our result is sufficient to prove Theorem 1.1.

In the notation above, we have
\[ |\mathcal{O}_n(N)| \leq |S_n(N)| \leq 20|\mathcal{O}_n(N + 3)|. \]

Consequently, the growth rates of the sequences \(|\mathcal{O}_n(N)|\}_{n \in \mathbb{N}} and \(|S_n(N)|\}_{n \in \mathbb{N}} are identical.

Proof The proof reduces to considering the digit comparison in Eq. (1) and its variants, and observing the relationship between the final digits of \(c(x)\) and the final digits of \(x\).

First let \(n > 2\) be even. In the vector subsequences below, any terms in square brackets may be omitted from the expression. If the vector \(y\) ends with a subsequence \((y_j, \ldots, y_{k_y})\) of the form

- \((-\delta n_2, -\delta(n_2 - 1), \ldots, -\delta(n_2 - 1), -\delta(n_2 - 2), 1)\), then a preimage under \(c\) is \(x = y - \sum_{i=j}^{k_y-1} w(i)\).
- \((-\delta n_2, -\delta(n_2 - 1), \ldots, -\delta(n_2 - 1), -\delta(n_2 - 2), 1)\), then a preimage under \(c\) is \(x = y - \sum_{i=j}^{k_y-1} w(i)\).
- \((-\delta(n_2 - 1), 1)\), then a preimage under \(c\) is \(x = y - w(k_y-1)\).

For all \(y\), it might be the case that \(x = y\) is a preimage. By construction, the preimages listed above are the only other possibilities. Only two sequences of the above forms above can overlap. That is, \(y\) can only contain at most two subsequences of the forms above. Therefore, \(y\) can have at most 3 preimages if \(n > 2\) is even. When \(n\) is odd, only the final bullet above applies, and we conclude that \(y\) has at most two preimages in this case.

We perform the same analysis when \(n = 2\). If the vector \(y\) ends with a subsequence \((y_j, \ldots, y_{k_y})\) of the form

- \((-\delta, 0, \delta)\), then a preimage under \(c\) is \(x = y - 2\delta w(k_y-2) - \delta w(k_y-1)\).
- \((-\delta, [0, \ldots, 0], 0, 0, \delta)\), then a preimage under \(c\) is
  \[ x = y - \delta \left( \sum_{i=j}^{k_y-3} w(i) + 2w(k_y-2) + w(k_y-1) \right). \]
- \((-\delta, [0, \ldots, 0], -\delta, 0, \delta)\), then a preimage under \(c\) is
  \[ x = y - \delta \left( \sum_{i=j}^{k_y-3} w(i) + 2w(k_y-2) + w(k_y-1) \right). \]
- \((0, \delta)\), then a preimage under \(c\) is \(x = y - \delta w(k_y-1)\).
- \((-\delta, [0, \ldots, 0], 0, \delta)\), then a preimage under \(c\) is \(x = y - \delta \sum_{i=j}^{k_y-1} w(i)\).

Again it might be the case that \(x = y\) is a preimage. At most four subsequences of these forms can overlap, so we conclude that \(y\) can have at most five preimages under \(c\). □

We now show that the cardinality of the set of all geodesics is within a uniform constant multiple of the cardinality of the set of geodesics of strict shape 1. By carefully considering geodesics of different shapes, we could obtain a stronger inequality in the following corollary of Lemmas 5.2 and 5.3. However, our result is sufficient to prove Theorem 1.1.

Corollary 5.4 In the notation above, we have
\[ |\mathcal{O}_n(N)| \leq |S_n(N)| \leq 20|\mathcal{O}_n(N + 3)|. \]

Consequently, the growth rates of the sequences \(|\mathcal{O}_n(N)|\}_{n \in \mathbb{N}} and \(|S_n(N)|\}_{n \in \mathbb{N}} are identical.
Proof Clearly $|O_n(N)| \leq |S_n(N)|$. It follows from Lemma 5.2 that $\Phi$ maps the set $S_n(N)$ to $O_n(N + 3)$, so to prove the rightmost inequality we must show that the degree of $\Phi$ is bounded above by 20. First we apply Lemma 5.3 to conclude that the degree of $c$ is at most 5. As $\Phi$ is defined in four cases, we see that any output triple $(u', w', x')$ could arise from at most five preimages of $x'$ in each of the four cases of $\Phi$. Thus the degree of $\Phi$ is at most 20, as desired.

It follows from Lemma 4.1 that the growth rates of $\{|O_n(N)|\}_{N \in \mathbb{N}}$ and $\{|S_n(N)|\}_{n \in \mathbb{N}}$ are identical.

Recall that the growth series of a function $f(N)$ is an infinite series $R(x) = \sum_{i=0}^{\infty} f(i)x^i$. We will be interested in growth series which records the number of paths in a finite automata starting at a given state.

It follows from Corollary 5.4 that to determine the growth rate of $BS(1, n)$ it suffices to determine the growth rate of the sequence $\{|O_n(N)|\}_{N \in \mathbb{N}}$. As discussed in Sect. 3, it is nontrivial to do this in general because the number of states in $O_n$ is not uniformly bounded. Our solution is to formally define a growth series for every state in $\mathcal{D}'_n$ which counts the number of paths in $O_n$ starting in that state. We can then write down a matrix equation of fixed size in these series and determine their growth rates. That is, we trade a computation with arbitrarily large matrices over the integers (computing an eigenvalue) for a computation with fixed size matrices whose entries are infinite series. Also note that for the purpose of computing the growth rate of the sequence $\{|O_n(N)|\}_{N \in \mathbb{N}}$, we will ignore the state $s_{r-1}$; it follows from Lemma 4.2 that the growth rate of the strongly connected component containing the $\alpha$-digit expansions of all the states $s_i$ will determine the growth rate of the sequence $\{|O_n(N)|\}_{N \in \mathbb{N}}$.

Theorem 1.1 Let $\alpha = \lfloor \frac{9}{2} \rfloor$. The growth rate of the sequence $\{|O_n(N)|\}_{N \in \mathbb{N}}$, and hence $BS(1, n)$, for $n$ odd is the reciprocal of the smallest magnitude root of

$$1 - x - \sum_{i=1}^{\alpha} 2x^{i+1},$$

and for $n$ even is the reciprocal of the smallest magnitude root of

$$1 - 2x - x^2 + 2x^{\alpha+1} - 2x^{\alpha+2} + 2x^{2\alpha+2}.$$

Proof First assume that $n$ is odd. Denote by $R(x)$ the growth series of paths starting at state $s_{0,0}$. By consulting Fig. 5, we see that for each $i$ with $-\alpha \leq i \leq \alpha$, there is a path of length $|i| + 1$ in $O_n$ which returns to state $s_{0,0}$ consisting of $|i|$ edges with label $a^{\pm 1}$ followed by $t$. Thus we can write the recurrence

$$R(x) = \left[x + 2 \sum_{i=1}^{\alpha} x^{i+1}\right] R(x) + P(x),$$

where $P(x)$ is a polynomial of degree at most $\alpha + 1$. This “error” polynomial arises because not all ways of returning to state $s_{0,0}$ are possible if the remaining allowed path is shorter than length $\alpha + 1$. Thus we have

$$R(x) = P(x) \left[1 - x - \sum_{i=1}^{\alpha} 2x^{i+1}\right]^{-1}.$$
example, [7] §IV. Solving the recurrence above for $R(x)$ and applying this fact proves the theorem in this case.

Now assume that $n$ is even. Here $\alpha = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$. For each state $s_i$ in $D_n$, denote the growth series of paths starting at the state $s_i$ in $O_n$ by $R_i(x)$. We will express each of the $R_i(x)$ as a polynomial combination of all the $R_j(x)$. We can then solve a matrix equation to find these growth rates. First consider $R_0(x)$. By consulting Fig. 5, we see that for each $i$ with $-\alpha < i < \alpha$, there is a path of length $|i| + 1$ in $O_n$ which returns to state $s_{0,0}$ consisting of $|i|$ edges with label $a^\pm$ followed by $t$. There is also a path of length $\alpha + 1$ to state $s_{1,0}$ and a path of length $\alpha + 1$ to state $s_{2,0}$. Thus we can write the recurrence

$$R_0(x) = x + \sum_{i=1}^{\alpha-1} 2x^{i+1} R_0(x) + x^{\alpha+1} R_1(x) + x^{\alpha+1} R_2(x) + P_0(x),$$

where $P_0(x)$ is a polynomial of degree at most $\alpha + 1$. This “error” polynomial arises because not all ways of returning to $s_{0,0}$ or transiting to the other states are possible if the remaining allowed path is shorter than length $\alpha + 1$.

If we start in state $s_{1,0}$, then for each $i$ with $0 \leq i < \alpha$, there is a path of length $|i| + 1$ to $s_{0,0}$. Therefore,

$$R_1(x) = \left[ \sum_{i=0}^{\alpha-1} x^{i+1} \right] R_0(x) + P_1(x).$$

The computation for $R_2(x)$ is similar.

These computations yield the matrix equation

$$\begin{bmatrix} R_0(x) \\ R_1(x) \\ R_2(x) \end{bmatrix} = \begin{bmatrix} x + 2 \sum_{i=1}^{\alpha-1} x^{i+1} x^{\alpha+1} x^{\alpha+1} \\ \sum_{i=0}^{\alpha-1} x^{i+1} 0 0 \\ \sum_{i=0}^{\alpha-1} x^{i+1} 0 0 \end{bmatrix} \begin{bmatrix} R_0(x) \\ R_1(x) \\ R_2(x) \end{bmatrix} + \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \end{bmatrix}.$$ 

Letting $R(x)$ be the column vector of growth series, $P(x)$ the column vector of polynomials of degree at most $\alpha + 1$, and $M(x)$ the matrix, we wish to solve the equation $(I - M(x))R(x) = P(x)$. Cramer’s rule implies that $\det(I - M(x))$ is the denominator for each generating function and thus the reciprocal of its smallest root is the growth rate. A computation to simplify $\det(I - M(x))$ yields the polynomial stated in the theorem.

It follows immediately from Corollary 5.4 that the growth rate of $BS(1, n)$ is also given by the reciprocal of the root of smallest magnitude of the above polynomials.

The statement and proof of Theorem 1.1 involve relatively simple polynomials. Due to the necessary reliance on Corollary 5.4, these polynomials only count elements of the subset $O_n(N)$ of $BS(1, n)$, not all elements of $BS(1, n)$ of length $N$, so we have not recovered the growth series of $BS(1, n)$. The “error” polynomials in the proof quantify the small-scale behavior of the finite state automaton $O_n$, as opposed to just the growth arising from its recurrent component.

### 5.1 Growth rate examples and the limiting case

Using Theorem 1.1, it is straightforward to compute the growth rate of $BS(1, n)$ for small values of $n$. These are shown in Table 1.
Table 1 lists a growth rate for \( n = \infty \), indicating the limit of the growth rates for \( BS(1, n) \) as \( n \to \infty \). We can compute this quantity both independently and as a double-check on the polynomials in Theorem 1.1. In the odd case, note that on any open disk of radius less than 1, as \( n \to \infty \) the sequence of polynomials in Theorem 1.1 converges uniformly to the power series

\[
1 - x - 2 \sum_{i=1}^{\infty} x^{i+1} = 1 - x - \frac{2x^2}{1 - x},
\]

whose smallest magnitude root is \( \sqrt{2} - 1 \). In the even case, the sequence of polynomials converges uniformly to \( 1 - 2x - x^2 \), which has the same roots as the power series above, so the even and odd cases agree in the limit.

As an independent check on the limiting case, consider what a geodesic of strict shape 1 would look like for “infinite” \( n \): there would be no bound on the powers of \( a \) and \( a^{-1} \), so the set of geodesics of strict shape 1 would be a regular language on the three symbols \( \{a, a^{-1}, t\} \), subject to the condition that \( a \) and \( a^{-1} \) are never adjacent. We ignore here the possible initial power of \( t^{-1} \), which does not affect the growth rate. A finite automata accepting this language has adjacency matrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

whose largest eigenvalue is \( \sqrt{2} + 1 \).

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