ON DERIVATIONS AND LIE IDEALS OF SEMIRINGS

Madhu Dadhwal\(^1\) and Neelam\(^1\)

\(^1\) Department of Mathematics and Statistics, Himachal Pradesh University, Summer Hill, Shimla-171005, India

Communicated by Erhard Aichinger

Original Research Paper

Received: Aug 29, 2021 • Accepted: Mar 9, 2022

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ABSTRACT

In this paper, centralizing (semi-centralizing) and commuting (semi-commuting) derivations of semirings are characterized. The action of these derivations on Lie ideals is also discussed and as a consequence, some significant results are proved. In addition, Posner’s commutativity theorem is generalized for Lie ideals of semirings and this result is also extended to the case of centralizing (semi-centralizing) derivations of prime semirings. Further, we observe that if there exists a skew-commuting (skew-centralizing) derivation \(D\) of \(S\), then \(D = 0\). It is also proved that for any two derivations \(d_1\) and \(d_2\) of a prime semiring \(S\) with \(\text{char } S \neq 2\) and \(x^{d_1} x^{d_2} = 0\), for all \(x \in S\) implies either \(d_1 = 0\) or \(d_2 = 0\).

KEYWORDS
semirings, Lie ideals, additively regular semirings, derivations, inner derivations and pseudo inverses

MATHEMATICS SUBJECT CLASSIFICATION (2020)
Primary 16Y60; Secondary 16Y99

1. INTRODUCTION

The connection between derivations and Lie ideals of prime rings was first investigated by Bergen et al. [4]. Over the years, various algebraists have studied the commutativity of prime rings [1,2] along with the action of commutators on Lie ideals of prime rings involving the images of elements of Lie ideals under some appropriate mappings. In an attempt to extend Posner’s theorems, many mathematicians characterized the action of various additive maps on prime rings with some algebraic conditions. This motivates us to study and characterize the action of centralizing and semi-centralizing derivations on semiring and its Lie ideals.

However, commutators play a significant role in both ring theory and semiring theory, but in the absence of additive inverses in semirings, the structure of commutators differs from that of ring theory. Therefore, in order to explore this action on Lie ideals of semirings, we use a weaker version of an additive inverse, that is, the pseudo inverse which was introduced by Karvellas [11]. For more information of the enormous structure of semirings, one can refer to [16-19]. We recall from [10] that a semiring \(S\) is a nonempty set on which operations of addition and multiplication have been defined such that the following axioms are satisfied:

1. \((S, +)\) is a commutative monoid with identity element 0;
(2) \((S, \cdot)\) is a semigroup;
(3) Multiplication distributes over addition from either side;
(4) \(0s = 0 = s0\), for all \(s \in S\).

Moreover, an element \(a\) of a semiring \(S\) is said to be additively regular if and only if there exists an element \(a'\) of \(S\) which satisfies \(a + a' + a = a\) and \(a + a + a' = a'\). Further, \(S\) is called an additively regular semiring if and only if \(S = S' = \text{reg}(S)\), where \(\text{reg}(S)\) denotes the set of all additively regular elements of \(S\). For example, every additively idempotent semiring is an additively regular semiring with \(a' = a\). Furthermore, an additively regular semiring \(S\) is proper if for some \(a \in S\), we have \(a + a' \neq 0\), that is, if \(S\) is not a ring. Bandelt and Petrich [3] considered an additively regular semiring \(S\) which satisfies:

\[
\begin{align*}
(A_1) & \quad x(x + x') = x + x', \text{ for all } x \in S; \\
(A_2) & \quad y(x + x') = (x + x')y, \text{ for all } x, y \in S; \\
(A_3) & \quad x + (x + x')y = 1, \text{ for all } x, y \in S.
\end{align*}
\]

Throughout this paper, \(S\) is a proper additively regular semiring with \(A_2\)-condition. For any \(a, b \in S\), the Lie bracket \([a, b]\) represents the commutator \(ab + ba'\) or equivalently, \(ab + ba'\). Also, for any two subsets \(A\) and \(B\) of \(S\), the Lie bracket \([A, B]\) represents an additive submonoid \(S\) generated by all elements of the form \(ab + ba'\) or \(ab + ba'\), for any \(a \in A\) and \(b \in B\). An additive submonoid \(L\) of \(S\) is said to be a Lie ideal of \(S\) if \([L, S] \subseteq L\).

In addition, an additive map \(d : S \to S\) is called a derivation of \(S\), if \((xy)d = x^d y + xy^d\), for all \(x, y \in S\). If \(a \in S\), then the map \(d : S \to S\) defined by \(x^d = [a, x]\) is a derivation of \(S\), and it is called an inner derivation of \(S\) determined by \(a\). A mapping \(f : S \to S\) is said to be centralizing on \(S\), if \([x^d, x], y] = 0\), for all \(x, y \in S\) and \(f\) is called commuting, if \([x^d, x] = 0\), for all \(x \in S\). Furthermore, a mapping \(f : S \to S\) is called skew-centralizing on \(S\), if \([x^d, x + xx^d, y] = 0\), for all \(x, y \in S\) and it is called skew-commuting, if \(x^d x + xx^d = 0\), for all \(x \in S\). However, a mapping \(f : S \to S\) is said to be semi-centralizing, if either \(f\) is centralizing or skew-centralizing. Similarly, \(f\) is called semi-commuting, if either \(f\) is commuting or skew-commuting.

In this paper, Posner’s commutativity theorem for centralizing and semi-centralizing derivations of \(S\) is generalized. In the sequel, Posner’s commutativity theorem is also extended for centralizing derivations in the case of Lie ideals. As a consequence of this result, we also give a generalization to Mayne’s theorem [13] in the framework of derivations of \(S\). It is also observed that a non-zero derivation of \(S\) can not be skew-commuting and skew-centralizing and these results are proved in Section 3. In Section 4, we discuss some important actions of derivations on Lie ideals of semirings. This action was studied by Argac [2] on ideals of semiprime rings and we generalize some of his results for Lie ideals of a semiring. Moreover, Theorem 4.7 is inspired by a result proved by Bresar and Hvala (see [7, Theorem 2]) which asserts that if characteristic of a prime ring \(R\) is not 2 and \(f\) is an additive mapping with \(f(x^d) = x^d\), for all \(x \in R\), then either \(f = 1\) or \(f = -1\). Furthermore, the result proved by Bresar and Vukman (see [9, Corollary 2]) motivates us to prove: For any two derivations \(d_1\) and \(d_2\) of \(S\) with \(\text{char}(S) \neq 2\) such that \(x^{d_1} x^{d_2} = 0\) for all \(x \in S\) implies either \(d_1 = 0\) or \(d_2 = 0\). Further, we recall some basic results which are used frequently in proving many results of this paper.

**Lemma 1.1 ([10]).** If \(S\) is an additively regular semiring, then the following hold:

\[
\begin{align*}
(i) & \quad a^d = a; \\
(ii) & \quad a^d b^d = (a^d b^d) = (ab)^d = (ab)^d = ab; \\
(iii) & \quad (ab)^d = a^d b^d; \\
(iv) & \quad (a + b)^d = a^d + b^d, \text{ for all } a, b \in S.
\end{align*}
\]

For simplicity, we denote \(a_\cdot = a + a'\) so by \(A_2\)-condition, we have \(a \in Z(S)\), for all \(a \in S\).

**Lemma 1.2 ([10]).** If \(S\) is an additively regular semiring, then the following hold:

\[
\begin{align*}
(i) & \quad a_\cdot + a_\cdot = a_\cdot = a_\cdot; \\
(ii) & \quad a + a_\cdot = a_\cdot; \\
(iii) & \quad a_\cdot + a_\cdot = a_\cdot; \\
(iv) & \quad a_\cdot b_\cdot = ab_\cdot = (ab)_\cdot = a_\cdot b_\cdot = b_\cdot a_\cdot = (ba)_\cdot, \text{ for all } a, b \in S.
\end{align*}
\]
2. SOME AUXILIARY RESULTS AND EXAMPLES

In this section, we prove some results which play a vital role for the development of the theory of commutators in semirings. We start this section with the following examples:

Throughout our discussion, $S$ is prime and $L$ will denote a Lie ideal of $S$ with the property $LL \subseteq L$.

**EXAMPLE 2.1.** Let $S = \{0, 1, \alpha, \beta\}$, where $0, 1, \alpha, \beta$ are additively idempotent elements of $S$. We define binary operations as follows:

\[
\begin{array}{|c|c|c|}
\hline
\oplus & 0 & 1 & \alpha & \beta \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & \alpha & \beta \\
\alpha & \alpha & \alpha & \alpha & \beta \\
\beta & \beta & \beta & \beta & \beta \\
\hline
\end{array}
\]

From the Cayley tables, it is clear that $a' = a$, for all $a \in S$ and $S$ is a proper additively regular semiring with $A_2$-condition. Take $L = \{0, \alpha, \beta\}$. Clearly, $L$ is an additive submonoid of $S$ which satisfies $[L, S] \subseteq L$. Thus, $L$ is a Lie ideal of $S$ with $LL \subseteq L$. One can easily check that $S$ is prime.

**EXAMPLE 2.2 ([10]).** Let $R = (\mathbb{Z}_+, \oplus, 0, \otimes)$, where $\mathbb{Z}_+$ denotes the set of all positive integers and the binary operations are defined as:

\[
a \oplus b = \begin{cases} 
\text{lcm}(a, b), & \text{if } a, b \in \mathbb{Z}_+ \\
\infty, & \text{if } a = \infty \text{ or } b = \infty 
\end{cases}
\]

and

\[
a \otimes b = \begin{cases} 
\text{gcd}(a, b), & \text{if } a, b \in \mathbb{Z}_+ \\
a, & \text{if } b = \infty \\
b, & \text{if } a = \infty.
\end{cases}
\]

Then $S = \mathbb{Z} \times R$ is an additively regular commutative semiring in which $(p, q)' = (-p, q)$. Moreover, $L = \{(0, r) : r \in R\}$ is a Lie ideal of $S$ with $LL \subseteq L$.

Next example concludes that every additively regular semiring need not satisfy $A_2$-condition.

**EXAMPLE 2.3.** Let $R$ be a non-commutative ring and $S$ be a non-commutative additively regular semiring. Take $K = R \times S = \{(a, \alpha) : a \in R, \alpha \in S\}$. Then $K$ is a non-commutative proper additively regular semiring with operations pointwise addition and pointwise multiplication. We define the pseudo inverse of an element of $K$ as $(a, \alpha)' = (-a, \alpha')$. One can easily check that $K$ does not satisfy $A_2$-condition.

**EXAMPLE 2.4.** Let $B$ be a boolean semiring and $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in B \right\}$. Clearly, $S$ is an additively regular semiring under the usual addition and multiplication of matrices. Define $d : S \to S$ by

\[
d \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},
\]

then $d$ is a derivation of $S$.

**LEMMA 2.5.** If $S$ is an additively regular semiring, then the following identities hold:

(i) $[a, bc] = [a, b]c + b[a, c]$;
(ii) $[ab, c] = a[b, c] + [a, c]b$;
(iii) $[a + b, c] = [a, c] + [b, c]$;
(iv) $[a, [b, c]] + [b, [c, a]] = [[a, b], c]$. (Jacobi Identity)

**LEMMA 2.6.** If $a + b = 0$, for any $a, b \in S$, then $b = a'$.

**LEMMA 2.7.** If $I$ is a non-zero left ideal of $S$, then $I + [S, S] = S$.

Proof. Firstly, we will prove that $IS \subseteq I + [I, S]$. Now, if $i \in I, s \in S$, then

\[is = is + is = is + s = si + (is + s'i) \in I + [I, S].\]

Hence, $S = IS \subseteq I + [I, S] \subseteq I + [S, S]$. This proves the lemma.

**LEMMA 2.8.** If $L \neq (0)$ is a Lie ideal of $S$ with $aL = (0)$, for any $a \in S$, then $a = 0$. 
Proof. Let $I = \{ x \in S : xL = (0) \}$. Then, clearly $I$ is a left ideal of $S$. It suffices to prove that $I = (0)$. If possible, let $I \neq (0)$. Then Lemma 2.7 yields $I + [S, S] = S$. Further, let $i \in I, a \in [S, S], l \in L$, then

\[
(ia + d')l = ial + d'l = ial
= ial + i'al
= i(al + l'a) + ila
= i(a + l'a)
\in I[S, S, L] \subseteq I[S, L] \subseteq IL' = (0), \quad \text{as } IL = (0).
\]

Thus, $[I, [S, S]] \subseteq I$. This leads to

\[
[i, [r, s]]l = 0, \quad \text{for all } i \in I, l \in L, r, s \in S.
\]

This infers that

\[
i[r, s]l + [r, s]'il = 0, \quad \text{for all } i \in I, l \in L, r, s \in S
\]

which implies

\[
i[r, s]l = 0, \quad \text{for all } i \in I, l \in L, r, s \in S.
\]

Hence, $I[S, S] \subseteq I$. Therefore, $S = IS = I(I + [S, S]) \subseteq I$, which gives $SL = (0)$, a contradiction. This completes the proof. \hfill \Box

**COROLLARY 2.9.** If $I \neq (0)$ is a Lie ideal of $S$ which satisfies $a[b = (0)$, for any $a, b \in S$, then either $a = 0$ or $b = 0$.

**Proof.** By the given hypothesis, we have $a[l, s]b = 0$, for all $l \in L, s \in S$. This infers that

\[
alb + a's'lb = 0, \quad \text{for all } l \in L, s \in S.
\]

Applying Lemma 2.6 on equation (2.1), we get

\[
alb = a's'lb, \quad \text{for all } l \in L, s \in S.
\]

Using equation (2.2) in (2.1), we obtain

\[
al(s + s')b = 0, \quad \text{for all } l \in L, s \in S.
\]

This is equivalent to

\[
al s, b = (0), \quad \text{for all } s \in S
\]

or

\[
al (sb), = (0), \quad \text{for all } s \in S.
\]

This infers that

\[
al S(sb), = (0), \quad \text{for all } s \in S.
\]

By primeness of $S$, we have either $al = (0)$ or $(sb) = 0$, for all $s \in S$. If $al = (0)$, then Lemma 2.8 concludes that $a = 0$. On the other hand, if $(sb) = 0$, for all $s \in S$, then $s.b = 0$. This leads to $(s + s')sb = (0)$, for all $s \in S$. Again, by primeness of $S$, we obtain $b = 0$, as $S$ is proper. This proves the result. \hfill \Box

**PROPOSITION 2.10.** Let $d$ be a derivation of $S$ and $a$ be any arbitrary element of $S$. If $ax^d = 0$, for all $x \in S$, then either $a = 0$ or $d = 0$.

**Proof.** By the given hypothesis, we have $a(xy)^d = 0$, for all $x, y \in S$ which is equivalent to $axy^d + ax^d y = 0$, for all $x, y \in S$. Again, given hypothesis infers that $axy^d = 0$, for all $x, y \in S$, that is, $aSy^d = (0)$, for all $y \in S$. Therefore, by primeness of $S$, either $a = 0$ or $d = 0$. \hfill \Box

The next example shows that the restriction of primeness imposed on hypothesis of the Proposition 2.10 is not redundant.
**Example 2.11.** Let \( R \) be any additively regular semiring. Then \( S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in R \right\} \) is an additively regular semiring. Further, we claim that \( S \) is not prime. For this, we take two non-zero elements of \( S \) namely, \( \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( \beta = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \). However, it is easy to check that \( \alpha \mathcal{S} \beta = (0) \). Thus, \( S \) is not prime. Now, define \( d : S \to S \) by \( \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \), which is clearly a non-zero derivation of \( S \). But if we take \( a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), then \( ax^d = 0 \), for any \( x \in S \). Thus, the hypothesis of primeness is crucial in the above proposition.

**Lemma 2.12.** If \( d \) is a derivation of \( S \) and \( L \) is a non-zero Lie ideal of \( S \) with \( L^d = (0) \), then \( d = 0 \).

**Proof.** By the given hypothesis, we have \( [I, s]^d = 0 \), for all \( l \in L, s \in S \). This is equivalent to \( [I^d, s] + [I, s^d] = 0 \), for all \( l \in L, s \in S \). Again, given hypothesis implies that \( [I, s^d] = 0 \), for all \( l \in L, s \in S \), that is,

\[
ls^d + s^d l' = 0, \text{ for all } l \in L, s \in S. \tag{2.3}
\]

Using Lemma 2.6 in equation (2.3), we get that \( ls^d = s^d l \), for all \( l \in L, s \in S \). This reduces equation (2.3) to

\[
(l + l')s^d = 0, \text{ for all } l \in L, s \in S. \tag{2.4}
\]

By the Proposition 2.10, we obtain either \( l + l' = 0 \), for all \( l \in L \) or \( s^d = 0 \), for all \( s \in S \). But if \( l + l' = 0 \), for all \( l \in L \). Then, \( l(s + s') = 0 \), for all \( l \in L, s \in S \) which is equivalent to \( s + s' = 0 \), for all \( l \in L, s \in S \). This concludes that \( s + s' = 0 \), for all \( s \in S \). Then Lemma 2.8 infers that \( s + s' = 0 \), for all \( s \in S \), which is not true. Hence, \( d = 0 \).

**Remark 2.13.** In view of Corollary 2.9 and Lemma 2.12, it is observed that if \( d \) is a derivation of \( S \) with \( \text{char } S \neq 2 \) and \( L \) is a non-zero Lie ideal of \( S \) which satisfies \( L^d = (0) \), then \( d = 0 \).

**Proposition 2.14.** Let \( L \) be a non-zero Lie ideal of \( S \) and \( a, b \in S \). Then the following statements are equivalent:

(i) \( alb = 0 \), for all \( l \in L \).

(ii) \( bla = 0 \), for all \( l \in L \).

(iii) \( alb + bla = 0 \), for all \( l \in L \).

**Proof.** Let \( alb = 0 \), for all \( l \in L \). Then \( (alb)ma = 0 \), for all \( l, m \in L \). Equivalently, \( a(lbm) = 0 \), for all \( m \in L \). Therefore, Corollary 2.9 gives that either \( a = 0 \) or \( bma = 0 \), for all \( m \in L \) and in both cases (ii) holds.

One can easily prove (ii) implies (i) by the same argument.

Now, assume that \( alb = 0 \), for all \( l \in L \), then \( bla = 0 \), for all \( l \in L \). Hence, \( alb + bla = 0 \), for all \( l \in L \). Conversely, suppose that \( alb + bla = 0 \), for all \( l \in L \). Then Lemma 2.6 yields \( alb = bla' \), for all \( l \in L \). This concludes that \( alb + abl' = 0 \), for all \( l \in L \). This infers that \( abla = 0 \), which leads to \( albs = 0 \), for all \( l \in L, s \in S \). Henceforth, \( albs = 0 \), for all \( l \in L, s \in S \). Then, Corollary 2.9 concludes that \( ab = 0 \), as \( S \) is proper. This proves the result.

The next result can easily be proved by using the similar argument as in Proposition 2.14.

**Proposition 2.15.** For any additively regular semiring \( S \) and \( a, b \in S \), the following statements are equivalent:

(i) \( aSb = (0) \).

(ii) \( bSa = (0) \).

(iii) \( asb + bsa = (0) \), for all \( s \in S \).

**Theorem 2.16.** If \( L \) is a non-zero Lie ideal of \( S \) and \( d \neq 0 \) is a derivation of \( S \) which satisfies \( [I, S]L^d = (0) \), for all \( l \in L \), then \( S \) is commutative.
**Proof.** By given hypothesis, we have \([l, S]L^d = 0\), for all \(l \in L\), therefore by Corollary 2.9, either \([l, S] = 0\) or \(L^d = 0\), for all \(l \in L\). Thus, for any element \(l \in L\), we have either \([l, S] = 0\) or \(L^d = 0\). But by Lemma 2.12, \(L^d = 0\), for all \(l \in L\) infers that \(d = 0\), which is absurd. Hence, there exists some \(l_k \in L\) such that \(L_{L_k}^d \neq 0\) and \([l_k, S] = 0\). Now, we will prove that \([l, S] = 0\), for all \(l \in L\). To prove this, let \(m(\neq l_k) \in L\) with \([m, S] \neq 0\). Then \(L^d = 0\). Further, \([l_k + m, S] = [l_k, S] + [m, S] = [m, S] \neq 0\). Therefore, \((l_k + m)^d\) must be equal to zero but \((l_k + m)^d = L_{L_k}^d \neq 0\), which is a contradiction. Hence, \([l, S] = 0\), for all \(l \in L\). (2.5)

By using definition of \(L\) and equation (2.5), we obtain
\[
[l, s] + s[l, t] = 0, \text{ for all } l \in L, s, t \in S
\]
which infers that
\[
[l, s] + l[s, t] + s[l, t] + [s', t]l = 0. \tag{2.6}
\]

Then by applying equation (2.5) on equation (2.6), we are left with
\[
l[s, t] + [s, t]l' = 0, \text{ for all } l \in L, s, t \in S. \tag{2.7}
\]

Then Lemma 2.6 and equation (2.7), yields \([l, s] = [s, t]l\) and this reduces equation (2.7) to
\[
(l + l')[s, t] = 0, \text{ for all } l \in L, s, t \in S. \tag{2.8}
\]

Further, replacing \(s\) by \(rs\), with \(r \in S\) and then using the above equation, we have \((l + l')S[s, t] = 0\), for all \(l \in L, s, t \in S\). By primeness of \(S\), either \(l + l' = 0\) or \(l \in L\) or \([s, t] = 0\), for all \(s, t \in S\). But if \(l + l' = 0\), for all \(l \in L\), then as shown in Lemma 2.12, we obtain \(s + s' = 0\), for all \(s \in S\), which is absurd. Hence, \([s, t] = 0\), for all \(s, t \in S\), which gives the desired conclusion. \(\sqcup\)

**THEOREM 2.17.** If \(L \neq (0)\) is a Lie ideal of \(S\) with \(\text{char } S \neq 2\) and \([L, L] = (0)\), then \(S\) is commutative.

**Proof.** By definition of \(L\), we have \([l, st] \in L\), for all \(l \in L, s, t \in S\). Thus, given hypothesis implies that
\[
[l, [l, st]] = 0, \text{ for all } l \in L, s, t \in S.
\]

Equivalently,
\[
l[l, st] + l[s, l]t + s[l, t]l' + [l, s]lt' = 0, \text{ for all } l \in L, s, t \in S. \tag{2.9}
\]

Again by given hypothesis, we get \([l, [l, s]] = 0\), for all \(l \in L, s \in S\) and hence
\[
l[l, s] + [l, s]l' = 0, \text{ for all } l \in L, s \in S. \tag{2.10}
\]

Using Lemma 2.6 on the above equation, we obtain \([l, s]l = [l, s]l' = [l, s]l' = l'[l, s]\). Thus, equation (2.9) becomes
\[
l[l, l'] + [l, s]lt + s[l, t]l' + [l, s]lt' = 0, \text{ for all } l \in L, s, t \in S.
\]

This gives that
\[
2[l, s][l, t] = 0, \text{ for all } l \in L, s, t \in S
\]
which infers that
\[
[l, s][l, t] = 0, \text{ for all } l \in L, s, t \in S. \tag{2.11}
\]

Replacing \(s\) by \(sr\), with \(r \in S\), we get
\[
[l, s]r[l, t] + s[l, r][l, t] = 0, \text{ for all } l \in L, r, s, t \in S.
\]

Using equation (2.11), we obtain
\[
[l, s][l, t] = 0, \text{ for all } l \in L, s, t \in S.
\]

By primeness of \(S\), we have \([l, S] = (0)\), for all \(l \in L\). Further, by the same argument as in Theorem 2.16, we can easily prove that \(S\) is commutative. \(\sqcup\)

As an immediate consequence of the above result, we have

**COROLLARY 2.18.** If \(L \neq (0)\) is a Lie ideal of \(S\) with \(\text{char } S \neq 2\) and \([L, L] = (0)\), then \(L \subseteq Z(S)\).
3. CENTRALIZING AND SEMI-CENTRALIZING DERIVATIONS

We proceed this section with the generalization of some well known results of [12-14] which enable us to determine the commutativity of prime semirings. For all of the following, $S$ shall denote a proper, prime additively regular semiring. In Theorem 3.12, we prove that zero is the only skew-centralizing (skew-commuting) derivation on prime semirings. Firstly, we start this section with an alternate proof of ([15, Theorem 6]) which is given as follows:

**THEOREM 3.1.** If $d$ is a commuting derivation of $S$, then either $d = 0$ or $S$ is commutative.

**Proof.** By the given hypothesis, we have

$$[d, x] = 0, \text{ for all } x \in S. \quad (3.1)$$

By linearizing (3.1), we have $[(x + y)^d, x + y] = 0$, for all $x, y \in S$ and using definition of $d$, Lemma 2.5 and equation (3.1), we get that

$$[d, y] + [y^d, x] = 0, \text{ for all } x, y \in S. \quad (3.2)$$

Now, replacing $x$ by $xy$ in the above equation, we have

$$[d, y + xy^d, y] + [y^d, xy] = 0, \text{ for all } x, y \in S.$$ 

This implies that

$$[d, y]y + x^d[y, y] + [x, y]^d + x[y^d, y] + [y^d, x]y + x[y^d, y] = 0.$$ 

After applying equations (3.1) and (3.2) in the above equation, we are left with

$$x^d[y, y] + [x, y]^d = 0, \text{ for all } x, y \in S. \quad (3.3)$$

Equivalently,

$$0 = x^dy^d + y^d + (xy + y^d)x^d$$

$$= x^dy^d + x^dy^d + y^d + x^dy^d + [x, y]^d$$

$$= x^dy^d + y^d + x^dy^d + y^d + x^dy^d + [x, y]^d.$$ 

In other words,

$$(x^dy + x^dy) + x^dy + y^d + y^d + y^d + y^d + x^dy^d + x^dy^d + y^d + [x, y]^d = 0$$

or

$$(x^dy + [y^d, x]) + (x^dy + [y^d, x])y^d + [x, y]^d = 0.$$ 

Thus equation (3.2) implies that

$$[x, y]^d = 0, \text{ for all } x, y \in S. \quad (3.4)$$

Further, by replacing $x$ by $xz$, with $z \in S$, in equation (3.4), we get that

$$[x, y]^d + z^d + x[z, y]^d = 0, \text{ for all } x, y, z \in S.$$ 

Then equation (3.4) infers that $[x, y]^d = 0$, for all $x, y \in S$ and by primeness of $S$, we have either $y^d = 0$ or $[x, y] = 0$, for all $x, y \in S$. Thus, for any $y \in S$, either $y^d = 0$ or $[S, y] = 0$. If $d = 0$, then we get the desired result. If $d \neq 0$, then there exists some $x \in S$ with $x^d \neq 0$ and $[S, x] = 0$. We further claim that $[S, y] = 0$, for all $y \in S$. If possible, let $y(x^d) = 0$, then clearly $y^d = 0$. Henceforth, $[S, x] = 0$, and $(x + y)^d = 0$ holds simultaneously, which is not true. Therefore, $[S, y] = 0$, for all $y \in S$.

This concludes that $[x, y] = 0$, for all $x, y \in S$ which leads to $xy + yx = xy$, for all $x, y \in S$. Thus $A_2$-condition and Lemma 1.2 infers that $xy = yx$, for all $x, y \in S$. This completes the proof. □

**COROLLARY 3.2.** If $d$ is a commuting derivation of $S$, then either $d = 0$ or $[x, y] = 0$, for all $x, y \in S$ and this gives the desired conclusion.

**Proof.** As proved in above theorem, if $d$ is a commuting derivation of $S$, then either $d = 0$ or $[x, y] = 0$, for all $x, y \in S$ and this gives the desired conclusion. □
An immediate consequence of Theorem 3.1 is

**COROLLARY 3.3.** If $d$ is a non zero commuting derivation of $S$, then $d$ maps $S$ into $Z(S)$. 

**PROPOSITION 3.4.** If $f$ is a centralizing (skew-centralizing) mapping of $S$, then $f$ is commuting (skew-commuting).

**Proof.** We will prove the result for a centralizing mapping. One can easily prove the result for a skew-centralizing mapping by using the similar argument.

Let $f$ be a centralizing mapping of $S$. Then by definition of $f$, we have

$$[[x^f, x], y] = 0, \text{ for all } x, y \in S.$$ 

This is equivalent to

$$[x^f, x]y + y'[x^f, x] = 0, \text{ for all } x, y \in S.$$ 

This infers that

$$[x^f, x]y + (y' + y)[x^f, x] = y[x^f, x], \text{ for all } x, y \in S.$$ 

Furthermore, $A_2$-condition yields

$$[x^f, x]y = y[x^f, x], \text{ for all } x, y \in S.$$ 

Hence, $[x^f, x] \in Z(S)$, for all $x \in S$. Again, by using definition of $f$, we have

$$0 = [[x^f, x], y] = [x^f, x]y + y'[x^f, x], \text{ for all } x, y \in S.$$ 

As $[x^f, x] \in Z(S)$, therefore

$$[x^f, x](y + y') = 0, \text{ for all } x, y \in S.$$ 

This concludes that

$$[x^f, x]f(y + y') = (0), \text{ for all } x, y \in S.$$ 

Now, by using primeness of $S$, we obtain either $[x^f, x] = 0$, for all $x \in S$ or $y + y' = 0$, for all $y \in S$. But by definition of $S$, there exists some $a \in S$ with $a + a' \neq 0$. Hence, $[x^f, x] = 0$, for all $x \in S$. Hence, every centralizing mapping on a prime additively regular semiring $S$ is commuting. □

The next result follows by combining Theorem 3.1 and Proposition 3.4. Also, it is an extension to Posner’s commutativity theorem as well as an extension to ([15, Theorem 7]).

**THEOREM 3.5.** If $d$ is a non zero centralizing derivation of $S$, then $S$ is commutative.

**COROLLARY 3.6.** If $d$ is a non zero centralizing derivation of $S$, then $d$ maps $S$ into $Z(S)$.

The following proposition is a direct consequence of Corollary 3.2.

**PROPOSITION 3.7.** Let $d$ be a commuting derivation of $S$ and $\delta$ be an inner derivation of $S$ determined by $a \in S$. Then $d\delta$ and $\delta d$ are derivations of $S$.

**Proof.** Clearly, $d\delta$ is an additive mapping of $S$. Now,

$$((xy)^{\delta})^d = [a, xy]^d = [(a, x)y + x[a, y]]^d = [a, x]^dy + [a, x]^dy^d + x^d[a, y] + x[a, y]^d.$$ 

Then Corollary 3.2 yields

$$((xy)^{\delta})^d = [a, x]^dy + x[a, y]^d = (x^d)^y + x(y^d)^d.$$ 

Hence, $d\delta$ is a derivation of $S$. Similarly, we can prove that $d\delta$ is also a derivation of $S$. □

The next theorem is a generalization of Theorem 3.1, as well as a generalization of ([14, Lemma 3]).

**THEOREM 3.8.** Let $L \neq (0)$ be a Lie ideal of $S$ and char $S \neq 2$. If $d$ is a non-zero derivation of $S$ which is commuting on $L$, that is, $[[d, I], I] = 0$, for all $l \in L$, then $S$ is commutative.
The substitution of $km$ for $m$ with $k \in L$, in the above equation gives

$$\langle k^d m + km^d \rangle [l, l] + [km, l]^d = 0, \text{ for all } l, m, k \in L.$$  
(3.6)

which is equivalent to

$$k^d m[l, l] + km^d[l, l] + [k, l] ml^d + k[l, l]^d = 0, \text{ for all } l, m, k \in L.$$  

By using equation (3.5), we get

$$k^d m[l, l] + [k, l] ml^d = 0, \text{ for all } l, m, k \in L,$$

that is,

$$0 = [k, l] ml^d + k ml,$$

$$= (kl + l')k ml^d + l km^d + k^d l ml$$

$$= [k, l] ml^d + (k^d l + k^d l') ml + l km^d + k km^d$$

$$= [k, l] ml^d + [k^d, l] ml + [k, l^d] ml + l km^d + l km^d$$

$$= [k, l] ml^d + [k^d, l] ml + [l, k^d] ml + [l, l^d] ml + (l^d k + l^d k') ml$$

$$= [k, l] ml^d + (k^d, l) ml + (l^d k + l^d k') ml$$

As proved in equation (3.2) of Theorem 3.1, we can similarly prove

$$[l^d, m] + [m^d, l] = 0, \text{ for all } l, m \in L.$$  
(3.7)

Thus, equation (3.7) reduces equation (3.6) to

$$[k, l] ml^d = 0, \text{ for all } l, m, k \in L.$$  

In other words, $[k, l] L l^d = (0)$, for all $l, k \in L$.

By using Corollary 2.9, we have either $l^d = 0$ or $[k, l] = 0$ for all $l, k \in L$. Thus, for any $l \in L$, $l^d = 0$ or $[k, l] = 0$, for all $k \in L$. But $L^d \neq (0)$, as $d$ is a non-zero derivation. Hence, there exists some $l_k \in L$ such that $l_k^d \neq 0$ and $[l, l_k] = (0)$. Let $m$ be another element of $L$ with $[l, m] \neq (0)$. Then, $m^d = 0$. Henceforth, $[L, l_k + m] = [L, l_k] + [L, m] = [L, m] \neq (0)$. This implies that $(l_k + m)^d = 0$, but $(l_k + m)^d = l_k^d \neq 0$, a contradiction. This leads to $[k, l] = 0$, for all $l, k \in L$. Hence, Theorem 2.17 concludes that $S$ is commutative. \hfill \Box

The next result is a generalization of Posner’s commutativity theorem as well as a generalization of ([12, Theorem 1]) in case of Lie ideals of prime additively regular semirings and it follows by combining Proposition 3.4 and Theorem 3.8.

**THEOREM 3.9.** Let $L \neq (0)$ be a Lie ideal of $S$ and char $S \neq 2$. If $d$ is a non-zero derivation of $S$ which is centralizing on $L$, then $S$ is commutative.

**COROLLARY 3.10.** Let $L \neq (0)$ be a Lie ideal of $S$ with char $S \neq 2$. If there exists a non trivial centralizing derivation of $S$, then $d$ maps $S$ into $Z(S)$.

The next theorem is somehow an extension of ([13, Theorem]) for a derivation of additively regular semirings and it is a direct consequence of Theorem 3.9.

**THEOREM 3.11.** Let $d$ be a derivation of $S$ which is centralizing and non-trivial on a Lie ideal $L$ of $S$. If char $S \neq 2$, then $L \subseteq Z(S)$.

In 1993, Bresar ([6, Theorem 2]) proved that there does not exist any non-trivial skew-commuting map on a 2-torsion free semiprime ring. This result is an inspiration for the upcoming theorem.
THEOREM 3.12. If $D$ is a skew-commuting (skew-centralizing) derivation of $S$, then $D = 0$.

Proof. We will prove the result for a skew-commuting derivation of $S$ and it holds for a skew-centralizing derivation of $S$ by virtue of Proposition 3.4. Let $D$ be a skew-commuting derivation of $S$. Then by definition of $D$, we have
\[ x^D x + xx^D = 0, \quad \text{for all } x \in S. \quad (3.8) \]
In order to reach the desired conclusion, it suffices to prove that $D = 0$. Henceforth, linearization of equation (3.8) gives
\[ (x + y)^D (x + y) + (x + y)(x + y)^D = 0 \]
or
\[ x^D x + x^D y + y^D x + yy^D x + xy^D + yx^D + yx^D + yy^D = 0. \]
Then by using (3.8), we get that
\[ x^D y + y^D x + xy^D + yx^D = 0, \quad \text{for all } x, y \in S. \quad (3.9) \]
Further, by replacing $y$ by $yx$ in (3.9), we have
\[ x^D yx + y^D x^2 + yyx^D x + yxx^D + yxx^D = 0, \]
for all $x, y \in S$.

By using equation (3.9), this is equivalent to
\[ (xy + yx)^D = 0, \quad \text{for all } x, y \in S. \quad (3.10) \]
Furthermore, by Proposition 2.10, we have either $xy + yx = 0$, for all $x, y \in S$ or $x^D = 0$, for all $x \in S$. Now, if $xy + yx = 0$, for all $x, y \in S$, then Lemma 2.6 infers that $xy = y'x$. Hence $y'x + yx = 0$, for all $x, y \in S$. Equivalently, $(y' + y)S = (0)$, for all $y \in S$. Then primeness of $S$ concludes that $y' + y = 0$, for all $y \in S$, which is absurd. Hence, $D = 0$. □

THEOREM 3.13. If $D$ is a nontrivial semi-commuting derivation of $S$, then $S$ is commutative.

Proof. If $D$ is a nontrivial semi-commuting derivation of $S$, then either $D$ is commuting or $D$ is skew-commuting. But by Theorem 3.12, $D$ can not be a skew-commuting derivation of $S$. Thus, $D$ is commuting derivation of $S$ and hence Theorem 3.1 concludes that $S$ is commutative. □

The upcoming result is an extension of Posner’s commutativity theorem in case of semi-centralizing derivations of $S$ and it is a direct consequences of Theorems 3.5 and 3.12.

THEOREM 3.14. If $D$ is a nontrivial semi-centralizing derivation of $S$, then $S$ is commutative.

THEOREM 3.15. Let $d$ be a derivation of $S$. If the mapping $x \mapsto [x^d, x]$ is skew-commuting on $S$, then either $d = 0$ or $S$ is commutative.

Proof. By given hypothesis, the mapping $x \mapsto [x^d, x]$ is skew-commuting on $S$, therefore
\[ [x^d, x]x + x[x^d, x] = 0, \quad \text{for all } x \in S. \quad (3.11) \]
Now, by applying Lemma 2.6 on (3.11), we get that $[x^d, x]x = x'[x^d, x]$ and using this in equation (3.11), we obtain
\[ (x' + x)[x^d, x] = 0, \quad \text{for all } x \in S. \]
This infers that
\[ (x' + x + x' + x)x^d [x^d, x] = 0, \quad \text{for all } x \in S. \]
Equivalently,
\[ (x, x^d + x^d x)[x^d, x] = 0, \quad \text{for all } x \in S. \]
This concludes that
\[ (x, x^d + x^d x)[x^d, x] = 0, \quad \text{for all } x \in S \]
or
\[ ([x^d, x] + [x^d, x'])[x^d, x] = 0, \quad \text{for all } x \in S. \]
In other words,

\[ [x^d, x], [x^d, x] = 0, \text{ for all } x \in S. \]

This implies that

\[ [x^d, x], z[x^d, x] = 0, \text{ for all } x, z \in S. \]

This is equivalent to

\[ [x^d, x], S[x^d, x] = (0), \text{ for all } x \in S. \]

Primeness of \( S \) gives \([x^d, x] = 0 \) or \([x^d, x] = 0, \) for all \( x \in S. \) Now, we will show that if \([x^d, x] = 0, \) for all \( x \in S, \) then \([x^d, x] = 0. \) Let \([x^d, x] = 0, \) for all \( x \in S. \) This gives \([x^d, x]s = 0, \) for all \( x, s \in S. \)

Thus, \([x^d, x]s = (0), \) for all \( x, s \in S. \) Again, by primeness of \( S, \) we obtain \([x^d, x] = 0, \) for all \( x \in S, \) as \( S \) is proper. Hence, in both cases, \([x^d, x] = 0, \) for all \( x \in S. \) Therefore by Theorem 3.1, either \( d = 0 \) or \( S \) is commutative.

The upcoming results are direct consequences of Theorem 3.15.

**COROLLARY 3.16.** Let \( d \) be a derivation of \( S. \) Suppose that the mapping \( x \mapsto [x^d, x] \) is skew-commuting on \( S, \) then \( d \) maps \( S \) into \( Z(S). \)

**COROLLARY 3.17.** Let \( d \) be a derivation of \( S. \) Suppose that the mapping \( x \mapsto [x^d, x] \) is skew-centralizing on \( S, \) then either \( d = 0 \) or \( S \) is commutative.

**COROLLARY 3.18.** Let \( d \) be a derivation of \( S. \) Suppose that the mapping \( x \mapsto [x^d, x] \) is skew-centralizing on \( S, \) then \( d \) maps \( S \) into \( Z(S). \)

### 4. RESULTS ON LIE IDEALS AND DERIVATIONS

This section deals with the behaviour of derivations on Lie ideals of semirings and generalization to some results of [2] in the framework of Lie ideals of semirings. Finally, we generalize some well known results of [7,9]. We now proceed with the following result which is an extension to ([14, Lemma 3]), as well as a partial extension to ([5, Theorem 4.1]).

**THEOREM 4.1.** Let \( L \neq (0) \) be a Lie ideal of \( S \) with char \( S \neq 2 \) and \( d_1, d_2 \) be any two derivations of \( S \) such that atleast one of them is non-zero. If \((l^d_1 l + l' l^d_2) = 0, \) for all \( l \in L, \) then \( S \) is commutative.

**Proof.** Without loss of generality, we may assume that \( d_2 \neq 0 \) and by linearizing \((l^d_1 l + l' l^d_2) = 0, \) we get

\[ l^d_1 m + m^d_1 l + l'm^d_2 + m' l^d_2 = 0, \text{ for all } l, m \in L. \]  

(4.1)

Replacing \( l \) by \( lm \) in the above equation, we obtain

\[ (lm)^d_1 m + m^d_1 lm + l'm^d_2 m + m' l^d_2 m = 0, \text{ for all } l, m \in L \]

or

\[ l^d_1 m + m^d_1 m' l + l'm^d_2 m + m' l^d_2 m = 0, \text{ for all } l, m \in L. \]

Thus, for all \( l, m \in L, \)

\[ (l^d_1 m + m^d_1 l + l'm^d_2) m + m' l^d_2 m + l'm m^d_2 = 0. \]  

(4.2)

Now, by applying Lemma 2.6 on equation (4.1), we have

\[ lm^d_2 = l^d_1 m + m^d_1 l + m' l^d_2. \]  

(4.3)

Equation (4.3) reduces equation (4.2) to

\[ lm^d_2 m + m' l m^d_2 + lm^d_1 m + lm^d_2 = 0, \text{ for all } l, m \in L \]

which is equivalent to

\[ lm^d_2 m + m' l m^d_2 + l(m^d_1 m + m' m^d_2) = 0, \text{ for all } l, m \in L. \]

Further, given hypothesis infers that

\[ lm^d_2 m + m' l m^d_2 = 0, \text{ for all } l, m \in L, \]
that is,

\[ [lm^d, m] = 0, \text{ for all } l, m \in L. \]

This implies that

\[ [l, m]m^d + l[m^d, m] = 0, \text{ for all } l, m \in L. \quad (4.4) \]

Again replacing \( l \) by \( nl \) with \( n \in L \) in equation (4.4), we have

\[ [n, m][lm^d] + n[l, m]m^d + nl[m^d, m] = 0, \text{ for all } l, m \in L. \]

By using equation (4.4), we are left with

\[ [n, m]m^d = 0, \text{ for all } l, m, n \in L. \]

Equivalently,

\[ [n, m][lm^d] = 0, \text{ for all } l, m, n \in L. \]

Then Corollary 2.9 leads to \([n, m] = 0 \text{ or } m^d = 0, \text{ for all } m, n \in L\). Thus, for any \( m \in L \), either \( m^d = 0 \) or \([L, m] = (0)\). But, \( L^d \neq (0) \), as \( d^2 \neq 0 \). Henceforth, there exists some \( l_k \in L \) with \( l_k^d \neq 0 \) and \([L, l_k] = (0)\). Now, we claim that \([L, l] = (0), \text{ for all } l \in L\). For this, let \( k \neq l_k \in L \) with \([L, k] = (0)\). This implies that \( k^d = 0 \). Moreover, \([L, l_k + k] = [L, k] = (0)\) and \((l_k + k)^d = l_k^d \neq 0\), which is absurd, as \( l_k + k \in L\). This concludes that \([L, l] = (0), \text{ for all } l \in L\). Hence, Theorem 2.17 gives the desired conclusion. \( \square \)

**THEOREM 4.2.** Let \( L = (0) \) be a Lie ideal of \( S \) with char \( S \neq 2 \) and \( d \) be a non-zero derivation of \( S \). If one of the following statements hold:

(i) \([L, m]^d + [m, l] = 0, \text{ for all } l, m \in L\);
(ii) \([L, m]^d + [l, m] = 0, \text{ for all } l, m \in L\);

then, either \( L \subset Z(S) \) or \( d \) is commuting on \( L \). Moreover, \( S \) is commutative.

**Proof.** (i) Since \([L, m]^d + [m, l] = 0 \text{ and } [L, m]^d = [l^d, m] + [l, m^d]\), therefore

\[ [l^d, m] + [l, m^d] + [m, l] = 0, \text{ for all } l, m \in L. \quad (4.5) \]

Replacing \( m \) by \( mn \), with \( n \in L \), we get that

\[ [l^d, m]n + m[l^d, n] + [l, m^d]n + m[n, l] = 0, \]

which is equivalent to

\[ ([l^d, m] + [m, l])n + m([l^d, n] + [l, n^d] + [n, l]) + m^d[l, n] + [l, m]n^d = 0, \text{ for all } l, m, n \in L. \]

Thus equation (4.5) yields

\[ m^d[l, n] + [l, m]n^d = 0, \text{ for all } l, m, n \in L. \quad (4.6) \]

Further, replacing \( l \) by \( l k \), with \( k \in L \), then

\[ m^d[l, n]k + m^d[l, k] + l[m, k]n^d + l[k, m]n^d = 0. \quad (4.7) \]

However, equation (4.6) and Lemma 2.6 concludes

\[ [k, m]n^d = (m^d)'[k, n] \text{ and } m^d[l, n] = [l, m](n^d)'. \]

This reduces equation (4.7) as

\[ [l, m][k, n^d] + m^d[l, [k, n]] = 0. \]

Equivalently,

\[ [l, m][k, n^d] + [m^d, l][k, n] = 0, \text{ for all } l, m, n, k \in L. \quad (4.8) \]

In (4.8), replace \( k \) with \( kn \), we get that

\[ [l, m][k, n^d] + [l, m]k[n, n^d] + [m^d, l][k, n]n + [m^d, l]k[n, n] = 0. \]
Hence, equation (4.8) gives that

\[ [l, m]k[n, n^d] + [m^d, l]k[n, n] = 0, \text{ for all } l, m, n, k \in L. \]

Thus, for any \( l, m, n, k \in L \),

\[
0 = (lm + m'l)k(nn^d + n^d'n) + mlkn^d + m^dlkn_n + l'm^dknn_n
\]

\[
= [l, m]k[n, n^d] + m^dnnl + nmn^dlk + nm'n^dlk + l'm^dknn_n
\]

\[
= [l, m]k[n, n^d] + m^dnnl + nmn^dlk + nm'n^dlk + l'm^dknn_n
\]

\[
= [l, m]k[n, n^d] + (m^dnnl + nm'n^dlk + mn^nlk + nm'n^dkl + l'm^dknn_n)
\]

\[
= [l, m]k[n, n^d] + (m^dnnl + nm'n^dlk + mn^nlk + nm'n^dkl + l'm^dknn_n)
\]

\[
= [l, m]k[n, n^d] + (m^dnnl + nm'n^dlk + mn^nlk + nm'n^dkl + l'm^dknn_n)
\]

In other words,

\[
[l, m]k[n, n^d] + (m^dnnl + nm'n^dlk + mn^nlk + nm'n^dkl + l'm^dknn_n)lkk = 0, \quad \text{for all } l, m, n, k \in L.
\]

But by equation (4.6),

\[
m^dnnl + nm'n^dlk + mn^nlk + nm'n^dkl + l'm^dknn_n = 0
\]

Therefore, equation (4.9) becomes

\[
[l, m]k[n, n^d] = 0, \quad \text{for all } l, m, n, k \in L,
\]

that is,

\[
[l, m]l[n, n^d] = (0), \quad \text{for all } l, m, n \in L.
\]

In accordance with Corollary 2.9, either \([l, m] = 0\), for all \( l, m \in L \) or \([n, n^d] = 0\), for all \( n \in L \). In case, \([l, m] = 0\), for all \( l, m \in L \), then Corollary 2.18 yields \( L \subset Z(S) \) and in other case, \([n, n^d] = 0\), for all \( n \in L \), then \( d \) is commuting on \( L \). Moreover, if \([L, l] = (0)\), then by Theorem 2.17, \( S \) is commutative. Also, if \( d \) is commuting on \( L \), then in accordance with Theorem 3.8, \( S \) is again commutative.

Similarly, (ii) can be proved by the same technique as in (i).

**THEOREM 4.3.** Let \( L \) be a non-zero Lie ideal of \( S \) with \( \text{char } S \neq 2 \) and \( d \) be a non-zero derivation of \( S \). If one of the following statements hold:

(i) \( (lm)^d + l'm = 0 \), for all \( l, m \in L \);

(ii) \( (lm)^d + m'l = 0 \), for all \( l, m \in L \);

then, either \( L \subset Z(S) \) or \( d \) is commuting on \( L \). Moreover, \( S \) is commutative.

**Proof.** (i) By given hypothesis, \((lm)^d + l'm = 0\) and \((ml)^d + m'l = 0\) for all \( l, m \in L \). Thus, \((m'l)^d + m'l = 0\) for all \( l, m \in L \) and hence \((lm)^d + (m'l)^d + m'l = 0\), for all \( l, m \in L \). Henceforth, \([l, m]^d + [m, l] = 0\), for all \( l, m \in L \). Then Theorem 4.2(i) gives the desired conclusion.

(ii) Similarly one can easily prove (ii) by using Theorem 4.2(ii).

Recall that for any \( a, b \in S \), the Jordan product \( a \circ b \) is defined by \( ab + ba \).

**THEOREM 4.4.** Let \( L \neq (0) \) be a Lie ideal of \( S \) with \( \text{char } S \neq 2 \) and \( d \) be a non-zero derivation of \( S \). If one of the following statements hold:

(i) \( (l-m)^d + (l-m)' = 0 \), for all \( l, m \in L \);

(ii) \( (l-m)^d + l-m = 0 \), for all \( l, m \in L \);

then \( L \subset Z(S) \). Moreover, \( S \) is commutative.
Proof. (i) By given hypothesis \((l-m)^d + (l-m)' = 0\), which infers that
\[
l^d m + l m^d + m^d l + ml^d + l' m + m' l = 0, \text{ for all } l, m \in L. \tag{4.10}
\]
Now, replacing \(m\) by \(ml\) in (4.10), we get that
\[
l^d ml + l m^d + ml^d + ml l^d + l' ml + m' l^2 = 0.
\]
Further, by equation (4.10), we are left with
\[
l ml^d + ml l^d = 0, \text{ for all } l, m \in L.
\]
Equivalently,
\[
(Im + ml)l^d = 0, \text{ for all } l, m \in L. \tag{4.11}
\]
Again replacing \(m\) by \(nm\), with \(n \in L\), we have
\[
(lnm + nmml)l^d = 0, \text{ for all } l, m, n \in L. \tag{4.12}
\]
Moreover, left multiplying equation (4.11) by \(n'\) and afterwards adding it to equation (4.12), we get that
\[
[l, n]ml^d + n.mll^d = 0, \text{ for all } l, m, n \in L.
\]
This is equivalent to
\[
0 = (ln + n')ml^d + n.ml l^d + n.ml l^d \\
= [l, n]ml^d + n.lml^d + n' lml^d + nml l^d + n ml l^d \\
= [l, n]ml^d + n(lm + ml)l^d + n'(lm + ml)l^d, \text{ for all } l, m, n \in L.
\]
Thus, by using equation (4.11), we have
\[
[l, n]ml^d = 0, \text{ for all } l, m, n \in L.
\]
In other words,
\[
[l, n]l^d = (0), \text{ for all } l, n \in L.
\]
Then by adopting the same techniques as in Theorem 3.8 after equation (4.12), we can similarly prove \([L, L] = (0)\). Hence, Theorem 2.17 and Corollary 2.18 conclude that \(S\) is commutative and \(L \subset Z(S)\). □

**COROLLARY 4.5.** Let \(L \neq (0)\) be a Lie ideal of \(S\) with char \(S \neq 2\) and \(d\) be a non-zero derivation of \(S\). If \((I^2)^d + (I^2)' = 0\), for all \(I \in L\), then \(L \subset Z(S)\). Moreover, \(S\) is commutative.

**Proof.** The given hypothesis \((l^2)^d + (l^2)' = 0\), for all \(l \in L\) leads to
\[
((l + m)^2)^d + ((l + m)^2)' = 0, \text{ for all } l, m \in L.
\]
This concludes that
\[
(l^2 + m^2 + lm + ml)^d + (l^2 + m^2 + lm + ml)' = 0.
\]
The given hypothesis reduces the above equation to
\[
(lm + ml)^d + (lm + ml)' = 0, \text{ for all } l, m \in L.
\]
Equivalently,
\[
(l\cdot m)^d + (l\cdot m)' = 0, \text{ for all } l, m \in L.
\]
Hence by Theorem 4.4(i), we reached the desired conclusion. □

**THEOREM 4.6.** Let \(L \neq (0)\) be a Lie ideal of \(S\) with char \(S \neq 2\) and \(d \neq 0\) be a derivation of \(S\). If one of the following conditions hold:
(i) \([l^d, m^d] + [m, l]^d = 0\), for all \(l, m \in L\);
(ii) \([l^d, m^d] + [l, m]^d = 0\), for all \(l, m \in L\);
then \(d\) is commuting on \(L\). Moreover, \(S\) is commutative.
Proof. (i) The given hypothesis gives that
\[ [l^d, m^d] + [m, l]^d = 0, \text{ for all } l, m \in L. \] (4.13)
This is equivalent to
\[ [l^d, m^d] + [m, l]^d = 0, \text{ for all } l, m \in L. \] (4.14)
Replacing \( m \) by \( ml \) in equation (4.13), we get that
\[
0 = [l^d, m^d l + ml^d] + (m[l, l] + [m, l]^d)\]
\[= ([l^d, m^d] + [m, l]^d) l + m^d[l^d, l] + [m, l]l^d + [l^d, m]l^d + m[l^d, l] + m[l, l]^d,
\]
for all \( l, m \in L \).
Therefore, equation (4.13) implies that
\[ m^d[l^d, l] + [m, l]l^d + [l^d, m]l^d + m^d l^d l + m[l, l]^d = 0. \]
This deduces that
\[ m^d[l^d, l] + (m[l, l]l^d + m^d l^d l + m^d l^d l + m[l, l]^d + m[l, l]^d)] = 0, \text{ for all } l, m \in L. \]
Now, \( A_2 \)-condition infers that
\[ m^d[l^d, l] + (m[l, l]l^d + m^d l^d l + m[l, l]^d + m[l, l]^d)] = 0, \text{ for all } l, m \in L.
\]
This is equivalent to
\[ (m^d[l^d, l] + m^d l^d l) + ([m, l]l^d + m[l, l]^d) + [l^d, m]l^d + m^d l^d l = 0. \]
Then Lemma 1.2(i) implies
\[ m^d[l^d, l] + [m, l]l^d + [l^d, m]l^d + m^d l^d l = 0, \text{ for all } l, m \in L. \]
Again by using \( A_2 \)-condition, we have
\[ 0 = m^d[l^d, l] + [m, l]l^d + [l^d, m]l^d + m^d l^d l + m[l, l]^d + m[l, l]^d]
\[= m^d[l^d, l] + [m, l]l^d + [l^d, m]l^d + ([l^d, m] + [m, l]^d) l + ([l^d, m] + [m, l]^d)] l'.
\]
Further, equation (4.14) concludes that
\[ m^d[l^d, l] + [l^d, m]l^d + [m, l]l^d = 0, \text{ for all } l, m \in L. \] (4.15)
Furthermore, replacing \( m \) by \( ml \) in the above equation, we get that
\[ l^d m[l^d, l] + lm[l^d, l] + [l^d, l]m^d l + [l^d, m]l^d + [m, l]l^d + [l, l]ml^d = 0. \]
Then equation (4.15) yields
\[ l^d m[l^d, l] + [l^d, l]ml^d + [l, l]ml^d = 0, \text{ for all } l, m \in L. \]
Equivalently,
\[ l^d m[l^d, l] + [l^d, l]ml^d + l, ml^d + l, ml^d = 0, \text{ for all } l, m \in L. \]
By \( A_2 \)-condition, we have
\[ 0 = l^d m[l^d, l] + [l^d, l]ml^d + l^d(l^d), ml + l^d ml + l^d ml]
\[= (l^d m[l^d, l] + [l^d, l]ml^d + ([l^d, l] + [l, l] + [l^d, l])ml. \]
Therefore, equation (4.14) leads to
\[ l^d m[l^d, l] + [l^d, l]ml^d = 0, \text{ for all } l, m \in L, \]
Again, given hypothesis concludes that $L^d L[l^d, l] + [L^d, l] L^d = (0)$, for all $l \in L$. Henceforth, Proposition 2.14 yields

$$L^d L[l^d, l] = (0), \text{ for all } l \in L. \tag{4.16}$$

Now,

$$L^d L[l^d, l] = L^d L[l^d, l] L^d \subseteq L^d L[l^d, l] \subseteq L^d L[l^d, l] = (0).$$

Thus,

$$L^d L[l^d, l] = (0), \text{ for all } l \in L. \tag{4.17}$$

Adding equations (4.16) and (4.18), we get

$$L^d L[l^d, l] = (0), \text{ for all } l \in L. \tag{4.18}$$

Hence, Corollary 2.9 infers that $[L^d, l] = 0$, for all $l \in L$. In other words, $d$ is commuting on $L$. Moreover, by Theorem 3.8, $S$ is commutative. \hfill \Box

The upcoming theorem is motivated by ([7, Theorem 2]).

**Theorem 4.7.** If $d$ is a derivation of $S$ which satisfies $(x^d)^2 + x^2 = 0$, for all $x \in S$, then $d = 0$.

**Proof.** Linearization of $(x^d)^2 + x^2 = 0$ gives

$$((x + y)^d)^2 + (x + y)^2 = 0, \text{ for all } x, y \in S.$$ 

This leads to

$$(x^d)^2 + (y^d)^2 + x^d y^d + y^d x^d + x^2 + y^2 + xy + yx = 0.$$ 

Thus, gives hypothesis infers that

$$x^d y^d + y^d x^d + xy + yx = 0, \text{ for all } x, y \in S. \tag{4.19}$$

Replacing $y$ by $xy$, we get

$$x^d xy^d + (x^d)^2 y + x^d yx^d + xy^d x^d + x^2 y + xyx = 0.$$ 

Again, given hypothesis concludes that

$$x^d xy^d + x^d yx^d + xy^d x^d + xyx = 0, \text{ for all } x, y \in S.$$ 

Equivalently,

$$x^d xy^d + x^d yx^d + x(y^d x^d + yx) = 0, \text{ for all } x, y \in S. \tag{4.20}$$

Now, by using Lemma 2.6 on equation (4.19), we obtain

$$y^d x^d + yx = (x^d)^2 y + xy, \text{ for all } x, y \in S. \tag{4.21}$$

Using equation (4.21) in equation (4.20), we have

$$x^d xy^d + x^d yx^d + x'(x^d y^d + xy) = 0, \text{ for all } x, y \in S,$n

that is,

$$[x^d, x] y^d + x^d yx^d + x'xy = 0, \text{ for all } x, y \in S.$$ 

This is equivalent to

$$x^d yx^d + (x^d)^2 y^d + x' x^d y^d + x' xy = 0. \tag{4.22}$$

Now, applying Lemma 2.6 on equation (4.20), we have

$$x^d xy^d = x^d yx^d + x'(y^d x^d + xy), \text{ for all } x, y \in S. \tag{4.23}$$

Using (4.23) in (4.22), we get

$$x^d yx^d + (x^d y^d x^d + x' y^d x^d + x' yx + x' x^d y^d + x' xy) = 0.$$
This can be written as
\[ x^d(y + y')x^d + x'(y^dx^d + yx x^d y^d + xy) = 0. \]

Then, equation (4.24) leads to
\[ x^dyx^d = 0, \text{ for all } x, y \in S. \]

Further, Lemma 1.2(iv) yields
\[ (x^d), yx^d = 0, \text{ for all } x, y \in S. \]

In other words,
\[ (x^d), Sx^d = (0), \text{ for all } x \in S. \]

Primeness of \( S \) concludes that either \( (x^d) = 0 \) or \( x^d = 0 \), for all \( x \in S \). Furthermore, we will show that if \( (x^d) = 0 \), for all \( x \in S \), then \( x^d = 0 \). For this, let \( (x^d) = 0 \), for all \( x \in S \). Then \( s(x^d) = 0 \), for all \( s, x \in S \) which is equivalent to \( s x^d = 0 \), for all \( s, x \in S \). Hence, \( (s + s')Sx^d = (0) \), for all \( s, x \in S \). Again by primeness of \( S \), we obtain \( x^d = 0 \), for all \( x \in S \), as \( s + s' \neq 0 \), for some \( s \in S \). Therefore, in both cases, we obtain \( x^d = 0 \), for all \( x \in S \). This proves the result.

As proved in the above theorem, the condition \( (x^d)^2 + x^2 = 0 \), for all \( x \in S \) leads to \( (x^d)^2 + y^d x^d + (xy + yx) = 0 \), for all \( x, y \in S \), that is, \( x^d + y^d + x + y = 0 \), for all \( x, y \in S \). Henceforth, Theorem 4.7 can be regarded as a Jordan analogue of a Lie type result of ([8, Theorem 1]).

The next theorem can easily be proved by using the same technique of the proof of Theorem 4.7, so we omit the proof.

**Theorem 4.8.** If \( L \) is a non-zero Lie ideal of \( S \) and \( d \) is a derivation of \( S \) with \( (L^d)^2 + l^2 = 0 \), for all \( l \in L \), then \( d = 0 \).

Now, we have all the required information to prove the last theorem of this section.

**Theorem 4.9.** If \( d_1 \) and \( d_2 \) are any two derivations of \( S \) with char \( S \neq 2 \) and \( x^{d_1} x^{d_2} = 0 \), for all \( x \in S \), then either \( d_1 = 0 \) or \( d_2 = 0 \).

**Proof.** If either \( d_1 \) or \( d_2 \) maps \( S \) into \( Z(S) \), then the result is obvious. Suppose that neither \( d_1 \) nor \( d_2 \) maps \( S \) into \( Z(S) \). Now, by the given hypothesis, we have
\[ x^{d_1} x^{d_2} = 0, \text{ for all } x \in S. \quad (4.24) \]

By linearizing \((4.24)\), we get that
\[ x^{d_1} x^{d_2} + x^{d_1} y^{d_2} + y^{d_1} x^{d_2} + y^{d_1} y^{d_2} = 0, \text{ for all } x, y \in S. \]

Applying equation \((4.24)\) we have
\[ x^{d_1} y^{d_2} + y^{d_1} x^{d_2} = 0, \text{ for all } x, y \in S. \quad (4.25) \]

Now, replacing \( y \) by \( yz \), with \( z \in S \), in the above equation, we get that
\[ x^{d_1} y^{d_2} z + x^{d_1} y z^{d_2} + y^{d_1} x z^{d_2} + y z^{d_1} x^{d_2} = 0, \text{ for all } x, y \in S. \quad (4.26) \]

Further by using Lemma 2.6 on equation \((4.25)\), we have \( x^{d_1} y^{d_2} = y^{d_1} (x^{d_2} y') \) and using this equality in equation \((4.26)\), we obtain
\[ y^{d_1} (x^{d_2})' z + x^{d_1} y z^{d_2} + y^{d_1} z x^{d_2} + y z^{d_1} x^{d_2} = 0, \text{ for all } x, y, z \in S. \]

This is equivalent to
\[ y^{d_1} [z, x^{d_2}] + x^{d_1} y z^{d_2} + y z^{d_1} x^{d_2} + x^{d_1} y z^{d_1} = 0, \text{ for all } x, y, z \in S. \]

Then \( A_2 \)-condition infers that
\[ y^{d_1} [z, x^{d_2}] + x^{d_1} y z^{d_2} + y z^{d_1} x^{d_2} + y x^{d_1} z^{d_1} = 0, \text{ for all } x, y, z \in S. \]

Equivalently,
\[ y^{d_1} [z, x^{d_2}] + x^{d_1} y z^{d_2} + y x^{d_1} z^{d_2} + y z^{d_1} x^{d_2} + y z^{d_1} x^{d_2} = 0. \]
Then, equation (4.25) yields
\[ y^{d_1}[z, x^{d_2}] + [x^{d_1}, y]z^{d_2} = 0, \text{ for all } x, y, z \in S. \]  \hfill (4.27)
Replacing \( y \) by \( x^{d_1} \), we get that
\[ x^{d_1}[z, x^{d_2}] + [x^{d_1}, x^{d_1}]z^{d_2} = 0, \text{ for all } x, z \in S. \]  \hfill (4.28)
Now, \([x^{d_1}, x^{d_1}]z^{d_2} = x^{d_1}(x^{d_1})(z^{d_2}) = x^{d_1}(x^{d_1})_1 (z^{d_2}) + x^{d_1}(x^{d_1})_2 (z^{d_2}).\) In other words,
\[ [x^{d_1}, x^{d_1}]z^{d_2} = x^{d_1}_1 x^{d_1}_2 z^{d_2} + (x^{d_1}_1)' x^{d_1}_2 z^{d_2} + x^{d_1}_1 x^{d_1}_2 (x^{d_1}_2)' + (x^{d_1}_1)' (x^{d_1}_2)' z^{d_2}. \]  \hfill (4.29)
Again by using Lemma 2.6 on equation (4.25), we have \( x^{d_1}_1 z^{d_2} = x^{d_1}(x^{d_1})_1 \) and using this equality in equation (4.29), we obtain
\[ [x^{d_1}, x^{d_1}]z^{d_2} = x^{d_1}_1 x^{d_1}_2 z^{d_2} + (x^{d_1}_1)' x^{d_1}_2 z^{d_2} + x^{d_1}_1 (x^{d_1})_2 (x^{d_1}_2)' + (x^{d_1}_1)' (x^{d_1}_2)' z^{d_2}. \]
by using equation (4.28).
Thus equation (4.28) reduces to
\[ x^{d_1}[z, x^{d_2}] = 0, \text{ for all } x, z \in S. \]  \hfill (4.30)
Replacing \( z \) by \( zy \), we get that
\[ x^{d_1}[(z, x^{d_1})y + z[y, x^{d_1}]] = 0, \text{ for all } x, y, z \in S. \]
Then, equation (4.30) concludes that
\[ x^{d_1}S[y, x^{d_2}] = (0), \text{ for all } x, y \in S. \]
By using primeness of \( S \), we have, for any \( x \in S \), either \( x^{d_1} = 0 \) or \( [y, x^{d_2}] = 0 \), for all \( y \in S \). Now, if \([y, x^{d_2}] = 0 \), for all \( x, y \in S \), then \( x^{d_2} \in Z(S) \), for all \( x \in S \), that is, \( d_2 \) maps \( S \) into \( Z(S) \), which is a contradiction. Thus, there exists some \( x_k \in S \) with \([S, x^{d_2}_k] \neq (0) \). Then, \( x^{d_2}_k = 0 \). We will prove that \( x^{d_2} = 0 \), for all \( x \in S \). If possible, let there exists some \( y \in S \) with \( y^{d_2} \neq 0 \). Thus, \([S, y^{d_2}] = (0) \). Moreover, \([S, (x_k + y)^{d_2}] \neq (0) \) and \((x_k + y)^{d_2} \neq 0 \) hold simultaneously, which is again a contradiction. Hence,
\[ x^{d_2} = 0, \text{ for all } x \in S. \]
This infers that
\[ (xy)^{d_2} = 0, \text{ for all } x, y \in S, \]
that is,
\[ 2x^{d_1}y^{d_1} = 0, \text{ for all } x, y \in S. \]
Therefore, \( x^{d_1}y^{d_1} = 0 \), for all \( x, y \in S \), as \( char \ S \neq 2 \). Again replacing \( y \) by \( yz \), we get that \( x^{d_1}y^{d_1}z + x^{d_1}yz^{d_1} = 0 \), for all \( x, y, z \in S \). This implies that \( x^{d_1}Sz^{d_1} = (0), \) for all \( x, z \in S \) and by primeness of \( S \), \( d_1 = 0 \).

5. CONCLUSIONS

This paper characterized the action of different kinds of derivations on Lie ideals of semirings along with the theory of commutators. Further, it is observed that there does not exist any non-trivial skew-commuting (skew-centralizing) derivation of \( S \) and generalizing thereby Posner’s commutativity theorem in the case of semi-centralizing (centralizing) derivations of \( S \). An extension to Posner’s commutativity theorem for Lie ideals of \( S \) is also established which enables us to extend Mayne’s theorem. Furthermore, it is shown that for any two derivations \( d_1 \) and \( d_2 \) of \( S \) with \( char S \neq 2 \) and \( x^{d_1}x^{d_2} = 0 \), for all \( x \in S \), either \( d_1 = 0 \) or \( d_2 = 0 \).

ACKNOWLEDGEMENTS

The second author gratefully acknowledges the financial assistance by CSIR-UGC.
REFERENCES

[1] A/ll.sc/b.sc/a.sc/s.sc, E. and A/r.sc/g.sc/a.sc/c.sc, N. Generalized derivations of prime rings. *Algebra Colloquium* 11, 3 (2004), 399–410.

[2] A/r.sc/g.sc/a.sc/c.sc, N. On prime and semiprime rings with derivations. *Algebra Colloquium* 13, 3 (2006), 371–380.

[3] B/andet, H. J. and Petrich, M. Subdirect products of rings and distributive lattices. *Proc. Edin Math. Soc.* 25, 2 (1982), 135–171.

[4] B/ergen, J. and Herstein, I. N. and Kerr, J. W. Lie ideals and derivations of prime rings. *J. Algebra* 7, 1 (1981), 259–267.

[5] B/resar, M. Centralizing mappings and derivations in prime rings. *J. Algebra* 156 (1993), 385–394.

[6] B/resar, M. On skew-commuting mapping of rings. *Bull. Austral. Math. Soc.* 47 (1993), 291–296.

[7] B/resar, M. and H/vala, B. On additive maps of prime rings. *Bull. Austral. Math. Soc.* 51 (1995), 377–381.

[8] B/resar, M. and Miers, C. R. Strong commutativity preserving maps of semiprime rings. *Canad. Math. Bull.* 37, 4 (1994), 457–460.

[9] B/resar, M. and V/ukman, J. Orthogonal derivations and an extension of a theorem of Posner. *Radovi Mat.* 5 (1989), 237–246.

[10] G/olan, J. S. *Semirings and their applications*. Kluwer Academic Publishers, Dordrecht 1999.

[11] K/arvellas, P. H. Inversive semirings. *J. Austral. Math. Soc.* 18 (1974), 277–288.

[12] M/ayne, J. H. Centralizing mappings of prime rings. *Canad. Math. Bull.* 27, 1 (1984), 122–126.

[13] M/ayne, J. H. Centralizing automorphisms of Lie ideals in prime rings. *Canad. Math. Bull.* 35, 4 (1992), 510–514.

[14] P/osner, E. C. Derivations in prime rings. *Proc. Amer. Math. Soc.* 8 (1957), 1093–1100.

[15] S/hafiq, S. and A/slam, M. On Posner’s second theorem in additively inverse semirings. *Hacet. J. Math. Stat.* 48, 4 (2019), 996–1000.

[16] S/harma, R. P. and M/adhu, On connes subgroups and graded semirings. *Vietnam Journal of Mathematics* 38, 3 (2010), 287–298.

[17] S/harma, R. P. and S/harma, R. and M/adhu, Radicals of semirings. *Asian-European Journal of Mathematics* 13, 7 2020, 2050138 (16 pages).

[18] S/harma, R. P. and S/harma, R. and M/adhu and K/ar, S. On the prime decomposition of $k$-ideals and fuzzy $k$-ideals in semirings. *Fuzzy information and Engineering* 13, 2 (2021), 223–235.

[19] S/harma, R. P. and S/harma, T. R. and J/oseph, R. Primary ideals in non-commutative semirings. *Southeast Asian Bulletin of Mathematics (Springer-Verlag)* 35, 2 (2011), 345–360.

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