Highly covariant quantum lattice gas model of the Dirac equation

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We revisit the quantum lattice gas model of a spinor quantum field theory—the smallest scale particle dynamics is partitioned into unitary collide and stream operations. The construction is covariant (on all scales down to a small length ℓ and small time τ = c ℓ) with respect to Lorentz transformations. The mass m and momentum p of the modeled Dirac particle depend on ℓ according to newfound relations \( m = m_0 \cos \frac{2\pi}{\lambda} \) and \( p = \frac{\hbar}{\tau} \sin \frac{2\pi}{\lambda} \), respectively, where \( \lambda \) is the Compton wavelength of the modeled particle. These relations represent departures from a relativistically invariant mass and the de Broglie relation—when taken as quantifying numerical errors the model is physically accurate when \( \ell \ll \lambda \). Calculating the vacuum energy in the special case of a massless spinor field, we find that it vanishes (or can have a small positive value) for a sufficiently large wave number cutoff. This is a marked departure from the usual behavior of such a massless field.

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I. INTRODUCTION

We consider the dynamics of a spinor quantum field where spacetime becomes discrete at scales smaller than some fundamental length. In particular, we revisit the quantum computational lattice representation known as the quantum lattice gas model, a dynamical Feynman chessboard model of the Dirac equation [1,2]. Variations, rediscoveries and improvements of the Feynman chessboard model have appeared over the years, including a model in 3+1 dimensions by Riazanov [3], an Ising spin chain representation by Jacobson and Schulman [4,5], a lattice Boltzmann model by Succi and Benzi quantum [8], a unitary model by Bialynicki-Birula [9], and quantum lattice gas models in 1+1 dimensions by Meyer [10] and in 3+1 dimensions by this author [11]. We presently consider a representation that retains 4-momentum conservation \( E^2 = (cp)^2 + (mc^2)^2 \) of special relativity down to a small length scale \( \ell \) and time scale \( \tau \). The low-energy limit of the lattice model is defined as the dynamical regime where the Compton wavelength \( \lambda \) of the quantum particle represented by an amplitude field \( \psi(x) \) is much larger than the small scale. \( \psi(x) \) is treated as continuous for \( \lambda \gg \ell \). Continuous derivatives emerge as effective quantum operators and the particle physics may be represented by the Dirac Lagrangian \( \mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi \), where the Dirac matrices are \( \gamma^\mu = (\gamma^0, \gamma^i) \), the spacetime derivatives are \( \partial_\mu = (\partial_0, \partial_i) \) for \( i = 1, 2, 3 \), and \( m_0 \) is the “invariant” particle mass (here expressed in natural units with \( \hbar = 1 \) and \( c = 1 \) for convenience).

This paper is organized as follows. We begin in Sec. II by formally introducing the quantum lattice gas model as a Lagrangian based theory. Then, in Sec. III we present a mapping procedure whereby the discrete dynamics of a quantum lattice gas model is made equivalent to the Dirac equation. This procedure leads to analytical form of the particle momentum that is a modification of the de Broglie relation. In Sec. IV we present a deviation of the quantum lattice gas stream and collide operators that form the basis of our quantum algorithm for the Dirac equation. In particular, we derive a unitary expression for the collision operator that is serves as a mechanism to give the spinor field its mass. Our primary intent is to show that the quantum lattice gas, taken as a numerical tool for this quantum computational physics application, provides a high degree of numerical accuracy. Then, in Sec. V we examine the newfound requirements to have the dynamical equation of motion of the quantum lattice gas model equal the Dirac equation at a selected small scale and explore the consequences of these requirements. We present a calculation of the vacuum energy density of a spinor field, treating the special case of a massless spinor field. Following a detailed analysis of the behavior of error terms, one finds an alternate theoretical purpose of the quantum lattice gas as a toy model. It provides an example where the vacuum energy of a massless spinor field can vanish or be very small. That is, one can take the quantum lattice gas as a toy model of Planckian scale physics and thus set the small scale sizes \( \ell \) and \( \tau \) to the Planck length \( \ell_p = \sqrt{\hbar G/c^3} \) and Planck time \( \tau_p = \ell/c \), providing a route for a small positive cosmological constant. In Sec. VI is a conclusion and summary.

II. QUANTUM LATTICE GAS MODEL

The proposed high-energy (small scale) quantum lattice gas representation may be formally expressed by the Lagrangian density of the form

\[
\mathcal{L}^{\text{grid}} = \psi \left[ (-\gamma^0 \partial_0 - e^{-\tau (\gamma^0 \partial_0 + i\gamma^i \partial_i + im^0)}) - \frac{\tau}{\tau} \right] \psi = \mathcal{L}_{\text{Dirac}} + \mathcal{O}(\tau^2).
\]
By the least action principle, this Lagrangian density leads to the equation of motion of the form

\[ e^{i\hat{\partial}_t} \psi(x) = e^{-i\tau^\mu \gamma^\mu} e^{-i\sigma_m \gamma^0} \psi(x). \]  

(1b)

Equation (1b) is the equation of motion of a quantum lattice gas, a unitary model for a system of noninteracting Dirac particles. On the right-hand side of (1b), free chiral particle motion is given by a stream operator \( \hat{U}_s = e^{i\tau^\mu \gamma^\mu} \) with momentum operator \( p_i = -i\partial_i \). A mass-generating interaction that causes a lefthanded particle to flip into a right-handed particle (and vice versa) is given by a unitary collision operator \( \hat{U}_c = e^{-i\sigma_m \gamma^0} \). The product \( U_{QLG} = \hat{U}_s \hat{U}_c \) is the local evolution operator of a quantum lattice gas system acting on the spinor field \( \psi(x) = (\psi_L(x), \psi_R(x)) \). The lefthand side of (1b) is a newly computed state \( \psi'(x) = e^{i\hat{\partial}_t} \psi(x) \) at time \( t + \tau \), so (1b) may be written as a quantum algorithmic map

\[ \psi'(x) = U_s U_c \psi(x) \rightarrow \psi(x), \]  

(1c)

taken to be homogeneously applied at all points \( x \) of space and at all increments \( t \) of time. In natural units (\( \hbar = 1 \) and \( c = 1 \)), the quantum lattice gas model (1c) is specified in 1+1 dimensions by the following unitary operators:

\[ \hat{U}_s = e^{i p_i \gamma_i} \]  

(2a)

\[ \hat{U}_c = \sqrt{1 - m_\sigma^2 \tau^2} - i\sigma_x e^{i\sigma_y} e^{i\sigma_z} m_\sigma \tau, \]  

(2b)

where \( m_\sigma \) is the mass of the modeled Dirac particle in the low-energy limit \( \ell/\lambda \sim 0 \).

In the low-energy limit, the \( O(\tau^2) \) error terms on the right-hand side of (1b) become negligible, so \( \hat{L}^{\text{low}} \sim \hat{L}_{\text{Dirac}} \) is covariant with respect to Lorentz transformations in this limit. Yet, can \( \hat{L}^{\text{low}} \) be manifestly covariant at high-energies \( \ell/\lambda \sim 1? \) We consider how to achieve the covariance of (1a) at a small scale: it necessarily occurs when the high-energy equation (1b)—or equivalently the quantum lattice gas equation (1c)—has the form of the Dirac equation (\( \gamma^\mu p_\mu + m\psi(x) = 0 \)).

The model is an expression of the simple idea of a spacetime manifold that becomes discrete below a small scale \( \ell \). The prescriptions needed to make (1c) equivalent to the Dirac equation are derived in the following section.

### III. IMPOSING COVARIANCE AT THE SMALL SCALE

Here we show that (1c) in the high-energy limit can be made equivalent to the Dirac equation. We begin with a local evolution operator as a composition of “qubit rotations” \( U_{n_2} = e^{-i\beta_2 \gamma \cdot \sigma} \) and \( U_{n_1} = e^{-i\beta_1 \gamma \cdot \sigma} \):

\[ U_{n_2}(\beta_2) U_{n_1}(\beta_1) = e^{-i\beta_2 \gamma \cdot \sigma} e^{-i\beta_1 \gamma \cdot \sigma} \]  

(3a)

\[ = \cos \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) - \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \sigma_x - i \left( \sin \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) \right) \sin \left( \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \right) \sigma_y \]

\[ - i \left( \sin \left( \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \sin \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) \right) \right) \sigma_z, \]  

(3b)

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is a vector of Pauli matrices, \( \hat{n}_1 \) and \( \hat{n}_2 \) are unit vectors specifying the respective principal axes of rotation, and \( \beta_1 \) and \( \beta_2 \) are real-valued rotation angles.\(^1\) Let us take \( U^*_\sigma = e^{-i\beta_2 \gamma \cdot \sigma} \) as our stream operator and \( U_c = e^{-i\beta_1 \gamma \cdot \sigma} \) as our collision operator. Without loss of generality, we may choose the principle axis of rotation along the \( \hat{z} \) to generate streaming,

\[ U^*_\sigma = e^{i p_z \gamma \cdot \sigma / \hbar} = e^{-i\beta_2 \gamma \cdot \sigma}, \]  

(4a)

and treat the quantum algorithmic map as if it were applied in 1+1 dimensions.\(^2\) In this frame a general collision operator is

\[ U_c = e^{-i\beta_1 \gamma \cdot \sigma}, \]  

(4b)

where \( \alpha, \beta \), and \( \gamma \) are real valued components subject to the constraint \( \alpha^2 + \beta^2 + \gamma^2 = 1 \). The unitary operators (4) are applied locally and homogeneously at all the points in the system. That is, we consider a construction whereby the two principal unit vectors specifying the axes of rotation are

\[ \hat{n}_1 = (\alpha, \beta, \gamma) \]  

\[ \hat{n}_2 = (0, 0, 1). \]  

(5)

With this choice, \( \hat{n}_1 \times \hat{n}_2 = (\beta, -\alpha, 0) \) and \( \hat{n}_1 \cdot \hat{n}_2 = \gamma \), so (4b) is a quite general representation of a quantum lattice gas evolution operator

\[ U^*_\sigma U_c \]  

\[ \cos \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) - \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \sigma_x - i \left( \sin \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) \right) \sin \left( \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \right) \sigma_y \]

\[ - i \left( \sin \left( \frac{\beta_1}{2} \sin \left( \frac{\beta_2}{2} \right) \right) \sin \left( \frac{\beta_1}{2} \cos \left( \frac{\beta_2}{2} \right) \right) \right) \sigma_z. \]  

(6)

\(^1\) In (3b) we used the identity \( (a \cdot \sigma) \cdot (b \cdot \sigma) = a \cdot b + i (a \times b) \cdot \sigma \).

\(^2\) The reduction from 3+1 to 1+1 dimensions is allowed because the algorithm has the product form \( \psi'(x) = U_s U_c \psi(x) \rightarrow \psi(x) \), where \( U_s = e^{-i\sigma \gamma \cdot \sigma} U^*_\sigma \) and \( U_c = e^{i \gamma \cdot \sigma} \), with Dirac matrices \( \gamma^0 = \sigma_0 \otimes 1 \) and \( \gamma^i = i \sigma_i \otimes \sigma_i \) in the chiral representation (3b). Streaming in each of the spatial directions occurs independently, so for simplicity we can choose to consider a Dirac wave moving along \( \hat{z} \).


The Dirac equation for a relativistic quantum particle of mass \(m_o\) may be written as
\[
\imath \hbar \partial \psi = -c p_x \sigma_x \psi + m_o c^2 \sigma_x \psi.
\] (7)

Its time-difference form may be written as
\[
\psi' = \left( 1 + \frac{ic p_x \tau}{\hbar} \sigma_z - \frac{im_o c^2 \tau}{\hbar} \sigma_x \right) \psi,
\] (8)
for small \(\tau\) and for momentum operator \(p_x = -i \hbar \partial_x\). We may view the unitary operator acting on the right-hand side of (8) as the effective low-energy operator obtained from the quantum lattice gas operator (4).

To establish a correspondence between (8) and (4), we simply choose the real-valued components of \(\vec{n}_1\) to satisfy the following three conditions:
\[
\begin{align*}
\alpha \sin \frac{\beta_1}{2} \cos \frac{\beta_2}{2} - \beta \sin \frac{\beta_1}{2} \sin \frac{\beta_2}{2} &= \frac{m_o c^2 \tau}{\hbar} \quad (10\text{a}) \\
\beta \sin \frac{\beta_1}{2} \cos \frac{\beta_2}{2} + \alpha \sin \frac{\beta_1}{2} \sin \frac{\beta_2}{2} &= 0 \quad (10\text{b}) \\
\gamma \sin \frac{\beta_1}{2} \cos \frac{\beta_2}{2} + \cos \frac{\beta_1}{2} \sin \frac{\beta_2}{2} &= -\frac{c p_x \tau}{\hbar} \quad (10\text{c})
\end{align*}
\]

Additionally, we should respect the reality condition that \(\vec{n}_1\) have unit norm\(^3\)
\[
\alpha^2 + \beta^2 + \gamma^2 = 1 \quad (10\text{d})
\]
that we established above with the collision operator (4). For the sake of simplicity, let us start with a specialized construction whereby \(\vec{n}_1\) is perpendicular to \(\vec{n}_2\).

The solution of (10) in this special case is
\[
\begin{align*}
\alpha &= \cos \frac{\beta_2}{2} \\
\beta &= -\sin \frac{\beta_2}{2} \\
\gamma &= 0
\end{align*}
\] (11)

Inserting (11) into (10a) gives \(\sin \frac{\beta_2}{2} = \frac{m_o c^2 \tau}{\hbar}\), and in turn (10a) is \(1 - \left( \frac{m_o c^2 \tau}{\hbar} \right)^2 \sin \frac{\beta_2}{2} = -\frac{c p_x \tau}{\hbar}\). In (11b) we chose \(-\ell p_x / \hbar = \beta_2 / 2\), so in turn we have
\[
\sqrt{1 - \left( \frac{m_o c^2 \tau}{\hbar} \right)^2 \sin \frac{\ell p_x}{\hbar}} = \frac{c p_x \tau}{\hbar}.
\] (12)

This is a grid equation that relates the cell sizes \(\ell\) and \(\tau\) to the mass and momentum of the quantum particle in an intrinsic way. Equation (12) can be interpreted as a rather fundamental relativistic relationship between particles and points. In place of the theory of special relativity for classical particle dynamics in a continuum, here we have constructed a lattice-based version of special relativity for particle dynamics emerging at a small scale where the spacetime foam has a regular structure.

Let us consider some implications of (12). Squaring gives
\[
\left( \frac{\hbar}{\tau} \sin \frac{\ell p_x}{\hbar} \right)^2 - \left( \frac{m_o c^2 \sin \frac{\ell p_x}{\hbar}}{\hbar} \right)^2 = \left( c p_x \right)^2.
\] (13a)

Then adding \(m_o c^4\) to both sides, we have
\[
\left( \frac{\hbar}{\tau} \sin \frac{\ell p_x}{\hbar} \right)^2 + \left( m_o c^2 \sin \frac{\ell p_x}{\hbar} \right)^2 = (c p_x)^2 + (m_o c^2)^2.
\] (13b)

This is a candidate grid-level relativistic energy equation that leads us to define a grid momentum \(p_{\text{grid}}^x\) and a grid mass \(m\) dependent on \(\ell\) as follows:
\[
p_{\text{grid}}^x = \frac{\hbar}{c \tau} \sin \frac{\ell p_x}{\hbar} \quad m = m_o \cos \frac{\ell p_x}{\hbar}.
\] (14)

Hence, the lefthand side of (13b) can be interpreted as a redefinition of the Dirac particle’s kinetic and rest energies. Inserting the de Broglie relation \((p_x = \hbar / \lambda\) momentum eigenvalue\)) the grid mass and momentum become
\[
p_{\text{grid}}^x = \frac{\hbar}{c \tau} \sin \frac{2 \pi \ell}{\lambda} \quad m = m_o \cos \frac{2 \pi \ell}{\lambda}.
\] (15)

In the low-energy limit defined by \(\lambda \gg 2 \pi \ell\), expanding (13b) to first order implies that the space and time cell sizes are linearly related by the speed light \(\ell = c \tau\), an intuitive relationship that we expect to hold. In the low-energy limit, (15) reduces to
\[
p_{\text{grid}}^x = \frac{h}{c \tau} \sin \frac{2 \pi \ell}{\lambda} \quad m = m_o \cos \frac{2 \pi \ell}{\lambda}.
\] (16)

That is, the low-energy limit of (15) corresponds to a usual quantum particle with an invariant mass that is entirely independent of the particle’s momentum, and the quantum particle acts like a wave according to standard quantum mechanics. However, there is a marked departure from standard quantum mechanics in the high-energy limit in the region \(\lambda \lesssim 20 \ell\) as shown in Fig. 1.

### IV. THE ALGORITHM IN NATURAL UNITS

For expediency, let us now switch our dimensional convention to the natural units, \(\hbar = 1\) and \(c = 1\).\(^4\) We can

\[^3\] Alternatively, instead of (10d), we could impose the condition that \(\cos \frac{\beta_1}{2} \cos \frac{\beta_2}{2} - \gamma \sin \frac{\beta_1}{2} \sin \frac{\beta_2}{2} = 1\), forcing (8) to be identical to (9). However, in this case, the resulting solution for components of \(\vec{n}_1\) has \(\alpha\) imaginary, and this breaks the unitarity of \(U_C\). So, we impose (10d) to strictly enforce unitarity.

\[^4\] In the natural units \(\hbar = 1\) and \(c = 1\), length and time have like dimension of length (i.e. \([\ell] = [\tau] = L\)) while mass, momen-
write $\{12\}$ as
\[
\sqrt{1 - m_{c}^{2} \tau^{2}} \sin(lp_{z}) = p_{z} \tau \quad \text{(17a)}
\]
\[
\sqrt{1 - m_{c}^{2} \tau^{2}} \cos(lp_{z}) = \sqrt{1 - E^{2} \tau^{2}} \quad \text{(17b)}
\]
or equivalently as $e^{itp_{z}} = \exp \left[ i \cos^{-1} \left( \frac{1 - E^{2} \tau^{2}}{1 - m_{c}^{2} \tau^{2}} \right) \right]$. Furthermore, our solution $\{11\}$ implies that the rotation axis is $\hat{n}_{1} \cdot \sigma = \sigma_{x} \cos lp_{z} + \sigma_{y} \sin lp_{z} = \sigma_{x} e^{i\sigma_{z} lp_{z}}$. Hence, since $\sin \frac{\pi}{2} = m_{c} \tau$, we can explicitly represent the collision operator $U_{C} = e^{-i\frac{\pi}{2} \hat{n}_{1} \cdot \sigma}$ in terms of the mass and momentum of the quantum particle as
\[
U_{C} = \sqrt{1 - m_{c}^{2} \tau^{2}} - im_{c} \tau 
\]
Multiplying by the stream operator $U_{\mathcal{S}} = e^{itp_{z} \sigma_{z}}$, the evolution operator $\{10\}$ can now be explicitly calculated
\[
U_{\mathcal{S}}U_{C} = e^{itp_{z} \sigma_{z}} \sqrt{1 - m_{c}^{2} \tau^{2}} - etp_{z} \sigma_{x} e^{i\sigma_{z} \tau} \sin \tau \quad \text{(19a)}
\]
\[
= \sqrt{1 - m_{c}^{2} \tau^{2}} \cos lp_{z} \tau + i\sigma_{x} \sqrt{1 - m_{c}^{2} \tau^{2}} \sin lp_{z} \tau 
\]
\[
- i\sigma_{x} \sin m_{c} \tau 
\]
\[
\exp \left[ i \cos^{-1} \left( \frac{1 - E^{2} \tau^{2}}{1 - m_{c}^{2} \tau^{2}} \right) \hat{n}_{12} \cdot \sigma \right] \quad \text{(20a)}
\]
\[
\exp \left[ i \cos^{-1} \left( \frac{1 - E^{2} \tau^{2}}{E} \right) \right] \quad \text{(20b)}
\]
\[
\approx e^{-im_{c} \tau} \quad \text{(20c)}
\]
where in the last line we made the identification
\[
\cos(E\tau) = \sqrt{1 - E^{2} \tau^{2}} \quad \text{(22)}
\]

which is exact to one part in $10^{139}$ (i.e. accurate to 4th order in $\ell$). This is equivalent to identifying the gate angle with the ratio of the small scale length to the particle energy, $-\beta_{1} / E = \ell / \tau$. Then, since $U_{S} U_{C} \equiv e^{-i\hbar \text{evd} \tau}$, the high-energy Hamiltonian may be written as
\[
\hbar \text{evd} \ell \tau \left[ -p_{z} \sigma_{z} + m_{o} \sigma_{x} \right] \quad \text{(23)}
\]

Thus, we have successfully demonstrated that the quantum lattice gas equation $\{11\}$ is equivalent to the Dirac equation as the Hamiltonian generating its unitary dynamics is the Dirac hamiltonian, even at the small scale.

V. CONSEQUENCES OF THE MODIFIED DE BROGLIE RELATION

Imposing covariance on the high-energy representation $\{11\}$ leads to two departures $\{15\}$ that we derived in the previous section, from the correct behavior given by relativistic quantum field theory. First, the momentum of the quantum particle must obey a small length $\ell$ dependent momentum relation
\[
p = \frac{\hbar}{\ell} \sin \frac{2\pi \ell}{\lambda} \quad \text{(24a)}
\]
in place of the de Broglie relation $p = \hbar / \lambda$. Second, the mass of the Dirac particle is no longer taken as an invariant quantity—it must depend on the small length as well as the particle’s Compton wavelength
\[
m = m_{o} \cos \frac{2\pi \ell}{\lambda} \quad \text{(24b)}
\]

where $m_{o}$ is a fixed constant, otherwise interpreted in the low-energy limit as the invariant particle mass. Plots of $\{24\}$ are given in Fig. $\{1\}$ for $m_{o}$ set to the electron mass. Notice that $\{24a\}$ vanishes for $\lambda = \ell$ and $\lambda = 2\ell$ and oscillates about zero as $\lambda \to 0$. Also notice that $\{24b\}$ vanishes at $\lambda = 4\ell$ and is in fact negative for $\lambda = 2\ell$ and $\lambda = 3\ell$, again oscillating as $\lambda \to 0$. These departures from standard quantum mechanics and special relativity theory are rather consequential at very small scales, departing on scales $\lesssim 20\ell$ that can be viewed as a region where the errors in the lattice gas model are dominate.

Yet, at the relevant larger scales (far above the small scale $\ell$), the physics of the quantum lattice gas model is indistinguishable from that predicted by the relativistic quantum field theory representation of Dirac fields.

Yet, in the context of the toy model, we should use $\{24\}$ to calculate the vacuum energy associated with a spin-$\frac{1}{2}$ fermion. When we integrate over all space to determine the total density contained in a Dirac field we have
\[
\rho_{\text{tot}} = \int \frac{d^{3}k}{c^{2}(2\pi)^{3}} \sqrt{(pc)^{2} + (mc^{2})^{2}} \quad \text{(25a)}
\]
\[
\frac{\hbar}{2\pi^{2}c\ell^{2}} \int_{0}^{k_{\ell}} \rho(k\ell)(k\ell)^{2} \sqrt{\sin^{2}k\ell + \ell^{2}} \cos^{2}k\ell \quad \text{(25b)}
\]
where the quantity \( \epsilon \equiv m_c \ell / h \) is small when \( m_c \) is much less than the mass, \( m_0 \ll h (\ell^2 / \tau)^{-1} \). If we take \( \epsilon = 0 \) to model a massless relativistic particle, then performing the integration of (25a) yields

\[
\rho_{\text{vac}}^{\text{theory}} = \frac{h}{2 \pi^2 c^4 \ell^3} \left( 2k_\ell \sin k \ell - (k^2 \ell^2 - 2) \cos k \ell \right) \bigg|_{k_\ell}^{k_c \ell},
\]

which can be either positive, zero, or negative depending sensitively on the value of the wave number cutoff \( k_c \) as well as on the value of the small scale \( \ell \).

If one takes the quantum lattice gas as a simplistic representation of Planckian scale physics (choosing \( \ell = \ell_p \)) where (24) is interpreted as physical behavior instead of numerical grid error, then its prediction of an allowable small value of the vacuum energy of a massless spinor field may be viewed as a new physical mechanism. That is, if we use (24) to calculate the vacuum energy associated with a spin-\( \frac{1}{2} \) fermion, then the toy model can avoid the cosmological constant problem—it is not necessarily 10^{121} times too large. Taking \( k_c \) as a parameter in (25a), at \( k_c = (4.08557...)/\ell \), we find that \( \rho_{\text{vac}}^{\text{theory}} = 0 \), as shown in Fig. 2. The experimentally observed value, \( \rho_{\text{vac}}^{\text{obs}} = 9.9 \times 10^{-27} \text{kg/m}^3 \) (such as obtained by the Wilkinson Microwave Anisotropy Probe) is obtained at a slightly smaller wave number cutoff, corresponding to a grid scale about four times smaller than \( \ell \), a sub-Planckian length scale.\(^5\)

Alternatively, one can choose the Planck scale to be smaller than the grid scale, \( \ell_p \ll \ell \ll \lambda \). In this case, we still have \( \rho_{\text{vac}}^{\text{theory}}(4.08557k_c \ell) = 0 \). There exists a real-valued number \( C \lesssim 4.08557 \) for which \( \rho_{\text{vac}}^{\text{theory}}(Ck_c \ell) = \rho_{\text{vac}}^{\text{obs}} \), although we have not predicted this number and thus do not address the fine-tuning problem. Considerations regarding an additional fundamental length scale, in addition to the Planck scale, have recently appeared in Ref. 12, including references therein.

VI. CONCLUSION

We revisited the quantum lattice gas model with a unitary evolution operator \( U_\ell U_c \) applied at a small scale \( \ell \) that advances a Dirac spinor field, represented on a grid, forward by a small time scale increment \( \tau \). We derived the conditions for which the generator of evolution is the Dirac Hamiltonian, \( U_\ell U_c \cong e^{-i(L - \sigma_3 p_3 + \sigma_5 m_0 \cdot \mathbf{p})} \). We quantified the error of the quantum lattice gas model as a departure from standard quantum mechanical behavior for the particle momentum going as \( p = (\hbar / \ell) \sin(2\pi / \lambda) \) and as its departure from a relativistically invariant particle mass going as \( m = m_0 \cos(2\pi / \lambda) \). In this regard, the quantum lattice gas model (2) is numerically accurate only to scales \( \gtrsim 20\ell \), even though it retains covariant...

\(^5\) Our simplistic estimate is further simplified by not considering the inflationary epoch of space under an extremely high vacuum energy density, just the dynamics of a fermionic field when the spacetime is flat with a small positive cosmological constant when its discreteness below the Planck scale becomes relevant.
behavior down for scales $\gtrsim \ell$. Yet, the numerical error of the model can be taken as a good feature—providing a mechanism for a small positive cosmological constant.

There have been a number of theoretical attempts employing, for example, supersymmetry [12], string theory [13], and the anthropic principle [14] to bridge the known chasm between the large quantum field theory prediction of $\rho_{\text{qft vac}} \sim 10^{110}\text{eV}^4$ for the late-time, zero-temperature vacuum energy density of “empty” space and the observed value of $\rho_{\text{vac}} \sim 10^{-11}\text{eV}^4$ associated with a small positive cosmological constant. The toy model presented herein gives a value of $k_c$ for which the vacuum energy vanishes. This is the root of the equation

$$2k_c\ell \sin k_c\ell - (k_c^2\ell^2 - 2) \cos k_c\ell - 2 = 0.$$ 

However, the theoretical considerations presented above do not tell us why the wave number cutoff should be fined tuned so that $\rho_{\text{vac}}^\text{theory} = \rho_{\text{vac}}^\text{obs}$. Nevertheless, the quantum lattice gas model appears to be one potential route to reconcile a discretized quantum field theory, at least a version modified at a small scale by [24], with the well accepted experimental observation of a positive cosmological constant by employing a plausible wave number cutoff parameter that corresponds to a fundamental grid scale. In a subsequent paper, we will numerically evaluate the novel unitary collision operator [13D] employed in a quantum algorithmic simulation of the dynamical behavior of a system of Dirac particles.

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