Carr-Nadtochiy’s weak reflection principle for Markov chains on $\mathbb{Z}^d$

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Abstract

The present paper establishes a discrete version of the result obtained by P. Carr and S. Nadtochiy in [4] for 1-dimensional diffusion processes. Our result is for Markov chains on $\mathbb{Z}^d$.

1 Introduction

Let $X^x$ be a diffusion process on $\mathbb{R}$ starting from $x$, and $\tau_y$ be the first time when $X$ visits $y$, which is smaller than $x$, that is,

$$\tau_y := \inf\{s > 0 : X^x_s \leq y\}.$$

In [4], it is proven that, $X$ being in quite a general class, for a measurable function $f$ which is zero on $(-\infty, y]$ with some regularity conditions, there exists a function $g$ which is zero on $[y, \infty)$ such that for any $t > 0$,

$$\mathbb{E}[f(X^x_t)1_{\{\tau_y > t\}}] = \mathbb{E}[g(X^x_t)].$$ (1)

We call the correspondence

$$f \mapsto g$$

Carr-Nadtochiy transform.

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The financial meaning of Carr-Nadtochiy transform is as follows. We consider a down and out option whose payoff at the maturity $T$ is given by

$$f(X_T^y)I_{\{\tau_y > T\}},$$

where $X_T$ is the stock price at the maturity, $y \in (0, X_0)$ is a knock-out (lower) boundary, and $f$ is a payoff function of this option. We choose a portfolio of European type options with payoff $f$ and $-g$, where $f \mapsto g$ is given by Carr-Nadtochiy transform. Then it holds that, for any $0 < t < T$,

$$E[e^{-r(T-t)}f(X_T^y)I_{\{\tau_y > T\}} | F_t] = I_{\{t \leq T \land \tau_y \}} E[e^{-r(T-t)}f(X_T^y)I_{\{\tau_y > T\}} | F_t] - I_{\{t \leq T \land \tau_y \}} E[e^{-r(T-t)}g(X_T^y) | F_t],$$

where $r$ is the risk-free interest rate. The equation (2) shows that the down and out option can be hedged by the static portfolio of the European plain option with pay-off $f$ and that with $-g$ since

- if $X$ never hit $y$ until the maturity, the portfolio at the maturity pays $f$, which hedges the down and out option that is active.

- Once $X$ hits $y$, the hedger should liquidate the portfolio at $\tau_y$. Thanks to (1) with strong Markov property of $X$, it costs zero. since the pay-off at the maturity is also zero, it is hedged.

In [4] an analytic form of $g$ in (1) for a class of 1-dimensional diffusion processes whose volatility coefficients $\sigma$ and drift coefficients $\mu$, with some regularity conditions. Without loss of generality the knock out (or knock in) boundary can be $\{0\}$.

**Theorem 1** (Proposition 1 in [2] and Theorem 2.7 in [4]). Let $X$ be a diffusion process with a regularity condition, and $f$ be a function with $\text{supp } f = [0, \infty)$ whose derivative is locally integrable. Then there exists a continuous and exponentially bounded function $g$ with $\text{supp } g = (-\infty, 0]$ satisfying (1), which is given by

$$g(x) = \frac{2}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{w \psi_1(x, w)}{\partial_x \psi_1(o, w) - \partial_x \psi_2(o, w)} \left( \int_{-\infty}^{0} \frac{\psi_1(z, w)}{\sigma^2(z)} \exp \left( -2 \int_{0}^{z} \frac{\mu(y)}{\sigma^2(y)} dy \right) f(z) \, dz \, dw \right)$$
for any large enough $\gamma > 0$, where $\psi_1$ and $\psi_2$ are the fundamental solution of the following Strum-Liouville equation
\[
\frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} \psi(x, w) + \mu(x) \frac{\partial}{\partial x} \psi(x, w) - w^2 \psi(x, w) = 0
\]
such that $\psi_1$ is square integrable function on $(-\infty, 0)$, $\psi_2$ is square integrable function on $(0, \infty)$ and $\psi_1(0, w) = \psi_2(0, w) = 1$.

The Carr-Nadtochiy transform is looked upon as a generalization of the reflection principle. Let us explain the reason why. We say that a 1-dimensional strong Markov process satisfies Put-Call symmetry at $y$ if the law of $X_t$ and that of $2y - X_t$ coincide for any $t > 0$ when $X$ starts from $y$. If $X$ is with the put-call symmetry, Carr-Nadtochiy transform is given by $f(2y - x)$. The most important example with the Put-Call symmetry is 1-dimensional Brownian motion. Actually, Brownian motion satisfies Put-Call symmetry for any point thanks to the reflection principle.

As we have seen, the put-call symmetry or the Carr-Nadtochiy transform allow us to construct a static hedge of barrier option. A multi-dimensional extension becomes very difficult, mainly because the boundary is not anymore a single point. Construction of Carr-Nadtochiy transform for a multi-dimensional process is, to say nothing of, among the hardest. The main result (Theorem 7) of the present paper therefore could be a step forward since the case where the boundary is an infinite set is managed. The proof only needs basic toolkits from linear algebra. The result is actually a discovery rather than an invention.

From a practical point of view, the explicit expression (Proposition 9) will be useful to construct a specific static hedge in a multi-dimensional setting including stochastic volatility environment, which, though, we do not discuss in detail in the present paper.

The organization of the present paper is quite simple. In the following section, after a preliminary subsection where a key lemma is proven, a proof

\footnote{Put-Call symmetry of a diffusion process was discussed in \cite{3}. A sufficient condition of Put-Call symmetry is that the diffusion and the drift coefficients are symmetric with respect to the boundary. Most of strong Markov processes including the ones used in financial modeling, however, do not satisfy the Put-Call symmetry for any points. In \cite{5}, introduced is a scheme for symmetricization, with which a diffusion is transformed to one with Put-Call symmetry. The scheme gives a numerical framework to calculate the price of a barrier option.}
to the main theorem, based on an explicit construction of the transformation, is given.

2 Multi-dimensional Discrete Carr-Nadtochiy Transform

2.1 Preliminaries

Let $d \in \mathbb{N}$. We set $B = \{x = (x_1, \cdots, x_d) \in \mathbb{Z}^d; \ x_d > 0\}$ and $\partial B = \{x \in \mathbb{Z}^d; \ x_d = 0\}$. We will obtain a discrete-time analogue of Carr-Nadtochiy transform when the process is a random walk, meaning that it is Markovian and at each time the process can move in the direction of the unit vector $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, $i = 1, \cdots, d$, or to be precise, it is a discrete-time time-homogeneous Markov chain on $\mathbb{Z}^d$ with starting at $x \in \mathbb{Z}^d$, which we will denote by $(Z^x_t)_{t \in \mathbb{Z}^+}$, with

$$\sum_{i=1}^d P(Z^x_1 = x + e_i) + \sum_{i=1}^d P(Z^x_1 = x - e_i) = 1,$$

for any $x \in \mathbb{Z}^d$. Here we further assume that

$$P(Z^x_1 = x + e_i), \ P(Z^x_1 = x - e_i) > 0 \ (i = 1, \cdots, d) \quad (3)$$

for any $x \in \mathbb{Z}^d$.

Let $\tau_{\partial B}$ be the first time when $X$ visit $\partial B$, that is,

$$\tau_{\partial B} := \inf\{s > 0 : Z^x_s = \partial B\}. \quad (4)$$

$Z^x_t \in S(t, x)$, where

$$S(t, x) = \{(s, y) \in \mathbb{N} \times \partial B : s \in \{1, \cdots, t\}, \ ||y|| = \sum_{i=1}^{d-1} |k_i| \leq t - s, \ \ \ \ \ \ \ t - s - ||y - x|| \ \text{is an even number}\},$$

for each $t \in \mathbb{N}$ and $x \in \partial B$.

Lemma 2. For any $(t, x)$, $(s, y) \in \mathbb{N} \times \partial B$,

$$P(Z^x_t = y \pm se_d) > 0$$

if and only if $(s, y) \in S(t, x)$.
Proof. It is immediate from (3).

For \( A \subset \mathbb{Z}^d \), denote by \( \mathcal{M}^d_A \) the set of all functions \( \mathbb{Z}^d \to \mathbb{R} \) with \( \text{supp} f \subset A \). Let \( W^+_y : \mathcal{M}^d_B \to \mathcal{M}^d_{\mathbb{N}} \) and \( W^-_y : \mathcal{M}^d_{B^c \setminus \partial B} \to \mathcal{M}^d_{\mathbb{N}} \) be defined by

\[
W^+_x h(t) = E[h(Z^x_t)],
\]

and

\[
W^-_x h(t) = E[h(Z^x_t)].
\]

By Lemma 2, \( W^+_x f(t) \) is expressed by

\[
W^+_x h(t) = \sum_{(s,y) \in S(t,x)} h(y + se_d) P(Z^x_t = y + se_d),
\]

and

\[
W^-_x h(t) = \sum_{(s,y) \in S(t,x)} h(y - se_d) P(Z^x_t = y - se_d).
\]

For each \( (t,x) \in \mathbb{N} \times \partial B \), we associate “square matrices” \( W^\pm_{t,x} \) as follows. Let

\[
\mathcal{L}^+_{(t,x)} := \{ \{ h(z \pm ue_d) \}_{(u,z) \in S(t,x)} \in \mathbb{R}^{\sharp S(t,x)} : h \in \mathcal{M}^d_B \}
\]

and

\[
\mathcal{L}^-_{(t,x)} := \{ \{ h(z + u e_d) \}_{(u,z) \in S(t,x)} \in \mathbb{R}^{\sharp S(t,x)} : h \in \mathcal{M}^d_{B^c \setminus \partial B} \}.
\]

Define \( W^\pm_{t,x} \) by

\[
W^\pm_{t,x} h(s,y) = W^\pm_y h(s) = \sum_{(u,z) \in S(s,y)} h(z \pm u e_d) P(Z^y_z = z \pm u e_d)
\]

for \( (s,y) \in S(t,x) \). Since \( P(Z^y_z = z + u e_d) = 0 \) for \( (u,z) \not\in S(s,y) \), we can regard \( W^\pm_{t,x} \) as a linear map from \( \mathcal{L}^\pm_{(t,x)} \simeq \mathbb{R}^{\sharp S(t,x)} \) to \( \mathcal{M}^{d+1}_{\mathbb{S}_{(t,x)}} \simeq \mathbb{R}^{\sharp S(t,x)} \), and therefore, we can identify it with a square matrix, which is in fact invertible.

Lemma 3. For any \( (t,x) \in \mathbb{N} \times \partial B \), the determinant of \( W^\pm_{t,x} \) is given by

\[
\det W^\pm_{t,x} = \prod_{(s,y) \in S(t,x)} P(Z^y_z = y \pm se_d) > 0. \tag{5}
\]

To prove Lemma 3 we need the following
Lemma 4. For a finite subset $\mathcal{A} \subset \mathbb{N} \times \partial B$, we denote by $\text{Sym}(\mathcal{A})$ the symmetric group over $\mathcal{A}$, and for $\sigma \in \text{Sym}(\mathcal{A})$, we write

$$\sigma(s, y) = (\sigma_1(s, y), \sigma_2(s, y)), \quad (s, y) \in \mathcal{A}.$$ 

For any $\sigma \in \text{Sym}(\mathcal{A})$, the following conditions are equivalent:

(i) \[\prod_{(s, y) \in \mathcal{A}} P(Z^y_s = \sigma_2(s, y) + \sigma_1(s, y)e_d) > 0.\]

(ii) \[\prod_{(s, y) \in \mathcal{A}} P(Z^y_s = \sigma_2(s, y) - \sigma_1(s, y)e_d) > 0.\]

(iii) $\sigma(s, y) \in S(s, y)$ for all $(s, y) \in \mathcal{A}$.

(iv) $\sigma$ is the identity, i.e., $\sigma(s, y) = (s, y)$ for $(s, y) \in \mathcal{A}$.

Proof. By Lemma 2, (i), (ii) and (iii) are equivalent. Since $\sigma(m, w) \in S(m, w)$ for any $(s, y) \in \mathbb{N} \times \partial B$, (iv) leads to (iii). It then remains to prove that (iii) implies (iv). Suppose that (iii) is satisfied but $\sigma$ is not the identity. Then the set $A := \{(s, y) \in \mathcal{A} : \sigma(s, y) \neq (s, y)\}$ is not empty. Let $m := \max_{(s, y) \in A} s$. We will show that for any $(m, w) \in A$, there is no $(s, y) \in A$ such that $\sigma(s, y) = (m, w)$, which is a contradiction. Since $\sigma(m, w) \in S(m, w) \setminus (m, w)$ for any $w$ with $(m, w) \in A$, we have that $\sigma_1(m, w) \neq m$. Moreover, for $(\tilde{m}, w) \in A$ with $\tilde{m} \neq m$, since $\sigma(\tilde{m}, w) \in S(m, w)$ is assumed, we see that $\sigma_1(\tilde{m}, w) \leq \tilde{m}$. Therefore $\sigma_1(s, y) \neq m$ for any $(s, y) \in A$.

Proof of Lemma 3. By the definition of the determinant of a matrix, we have that

$$\det W_{t,x} = \sum_{\sigma \in \text{Sym}(S(t,x))} \text{sgn}(\sigma) \prod_{(s, y) \in S(t,x)} P(Z^y_s = \sigma_2(s, y) \pm \sigma_1(s, y)e_d).$$

Now (5) is clear by Lemma 4.
2.2 Construction of the transform

Since the matrix $W_{t,x}^\pm$ is invertible by Lemma 3, we can define a square matrix

$$N_{t,x} := (W_{t,x}^-)^{-1}W_{t,x}^+,\]$$

which is seen as a linear map from $L^+_{(t,x)}$ to $L^-_{(t,x)}$. The following lemma is essential for our result.

**Lemma 5.** For $(t, x)$ and $(\tilde{t}, \tilde{x}) \in N \times \partial B$ with $(\tilde{t}, \tilde{x}) \in S(t, x)$, it holds that

$$N_{t,x} = N_{\tilde{t},\tilde{x}}, \quad \text{on } L^+_{(\tilde{t},\tilde{x})}.$$

**Proof.** Let $h^\pm \in L^\pm_{(\tilde{t},\tilde{x})}$ and $(s, y) \in S(\tilde{t}, \tilde{x})$. By the definition of $W_y^\pm$ and $W_{t,x}^\pm$, it holds that

$$W_{t,x}^\pm h^\pm(s, y) = W_y^\pm h^\pm(s) = W_{\tilde{t},\tilde{x}}^\pm h_{\tilde{t},\tilde{x}}^\pm(s, y).$$

Therefore we see that $(W_{t,x}^\pm)^{-1} = (W_{\tilde{t},\tilde{x}}^\pm)^{-1}$ on $M^{d+1}_S(\tilde{t}, \tilde{x})$, and hence

$$N_{t,x} = (W_{t,x}^-)^{-1}W_{t,x}^+ = (W_{\tilde{t},\tilde{x}}^-)^{-1}W_{\tilde{t},\tilde{x}}^+. \]$$

$$N_{t,x} = N_{\tilde{t},\tilde{x}}.$$  

Since $L^+_{(t,x)}$ (resp. $L^-_{(t,x)}$) is projected from $M^d_B$ (resp. $M^d_{B \setminus \partial B}$), we can extend $N_{t,x}$ to a map from $M^d_B$ to $M^d_{B \setminus \partial B}$. Define $N : M^d_B \to M^d_{B \setminus \partial B}$ by

$$Nh(x - te_d) := N_{t,x}h(x - te_d) \quad \text{for } (t, x) \in N \times \partial B.$$  

**Lemma 6.** For $(t, x)$ and $(\tilde{t}, \tilde{x}) \in N \times \partial B$ such that $(\tilde{t}, \tilde{x}) \in S(t, x)$, we have that

$$N: h(\tilde{x} - \tilde{e}_d) = N_{t,x}h(\tilde{x} - \tilde{e}_d).$$  

**Proof.** It is immediate by Lemma 5. 

The following is our main result.

**Theorem 7.** For $(t, x) \in N \times \partial B$ and $f \in M_B$, we have that

$$E[f(Z_t^x)] = E[Nf(Z_t^x)].$$  

(6)
Proof. By the definition of $W_x^\pm$ and $W_{t,x}^\pm$, we know that for $h^+ \in \mathcal{M}_B^d$ and $h^- \in \mathcal{M}_{B^c \setminus \partial B}^d$, 

$$
\mathbf{E}[h^+(Z^w_t)] = W_x^\pm h^+(t) = W_{t,x}^\pm h^+(t, x). 
$$

(7)

Since 

$$
W_{t,x}^+ = W_{t,x}^- (W_{t,x}^-)^{-1} W_{t,x}^+, \quad \text{on } L^+_{t,x}
$$

we have that 

$$
\mathbf{E}[f(Z^w_t)] = W_{t,x}^+ f(t, x) 
= W_{t,x}^- (W_{t,x}^-)^{-1} W_{t,x}^+ f(t, x) 
= W_{t,x}^{-} \mathcal{N}_{t,x} f(t, x).
$$

(8)

On the other hand, by (7) for $W_{t,x}^-$, we then obtain 

$$
W_{t,x}^{-} \mathcal{N}_{t,x} f(t, x) = W_{t,x}^{-} \mathcal{N}_{t,x} f(t) = \mathbf{E}[\mathcal{N}_{t,x} f(Z^w_t)].
$$

(9)

Thanks to Lemma 6 we notice that 

$$
\mathbf{E}[\mathcal{N}_{t,x} f(Z^w_t)] = \sum_{(s, y) \in S(t, x)} \mathcal{N}_{t,x} f(y - se_d) P(Z^w_t = y - se_d) 
= \sum_{(s, y) \in S(t, x)} \mathcal{N} f(y - se_d) P(Z^w_t = y - se_d).
$$

(10)

By combining (8), (9) and (10), we have the assertion.

\[\square\]

2.3 Uniqueness

The map $\mathcal{N}$ could be called Carr-Nadtochiy transform for the Markov chain $Z$. We can prove the uniqueness of the transform:

Theorem 8. If a map $\mathcal{N}' : \mathcal{M}_B^d \to \mathcal{M}_{B^c \setminus \partial B}^d$ satisfies (6) for any $(t, x) \in \mathbb{N} \times \partial B$, then $\mathcal{N}' = \mathcal{N}$.

Proof. Fix arbitrary $(t, x) \in \mathbb{N} \times \partial B$ and $f \in \mathcal{M}_B^d$. Since $\mathcal{N}$ and $\mathcal{N}'$ satisfy (6), we have that 

$$
W_{t,x}^\prime f(t, x) = \mathbf{E}[\mathcal{N}' f(Z^w_t)] = \mathbf{E}[f(Z^w_t)],
$$

8
and
\[ W_{t,x}^- N f(t, x) = E[N f(Z_t^x)] = E[f(Z_t^x)]. \]

Therefore we obtain that
\[ W_{t,x}^- N' f(t, x) = W_{t,x}^- N f(t, x). \]

Since the matrix \( W_{t,x}^- \) is invertible by Lemma 3, we conclude that
\[ N' f(x - t e_d) = (W_{t,x}^-)^{-1} W_{t,x}^- N f(t, x) = N f(x - t e_d). \]

\[ \square \]

2.4 An explicit form

An explicit form of \( N \) is given as follows:

**Proposition 9.** We have that
\[ N f(x - t e_d) = \sum_{(s, y) \in S(t, x)} c_{t,x}(s, y) f(y + s e_d), \quad (t, x) \in N \times \partial B \text{ and } f \in M_{B^c \setminus \partial B}, \]

where
\[
c_{t,x}(s, y) = \frac{1}{\prod_{(t,w) \in S(t, x)} P(Z_t^w = w - le_d)} \times \sum_{\sigma \in \text{Sym}(S(t, x))} \text{sgn}(\sigma) P(Z_{\sigma_1(t,x)}^w = y + s e_d) \times \prod_{(h,z) \in S(t,x) \setminus \{(t,x)\}} P(Z_{\sigma_2(h,z)} = z - he_d),
\]

for \((s, y) \in N \times \partial B\).

**Remark 10.** Thanks to Lemma 2, (11) is well-defined since \( \prod_{(t,w) \in S(t, x)} P(Z_t^w = w - le_d) \) is not zero.

**Proof of Proposition 9.** Recall that
\[ N f(x - t e_d) = (W_{t,x}^-)^{-1} W_{t,x}^+ f(t, x). \]

Here we note that the matrices are given by
\[ W_{t,x}^\pm = \{ P(Z_t^y = y' \pm s' e_d) : (s, y), (s', y') \in S(t, x) \}. \]

By Lemma 3 and Cramer’s rule, we have the assertion. \[ \square \]
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