City shapes that maximize the number of walking-only trips based on Manhattan distance

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Abstract
Various modes of transportation are available when people travel within cities, and trips can be classified into two types depending on whether some type of vehicle is used. Compared to vehicular travel, trips conducted only by walking have the advantages of lower environmental impact and less space required for road networks. By assuming that the proportion of walking-only trips decreases exponentially with the distance traveled, we explore the problem of finding a city shape with a fixed land area that maximizes the number of walking-only trips based on Manhattan distance. For many-to-one travel with the city center as the destination, we show that the optimal city shape is a diamond. For many-to-many travel, a method is presented that expresses the number of walking-only trips as a double integral, originally formulated as a four-dimensional integral. Using this, an optimization problem is formulated whose variables are the vertex coordinates of a polygon, and approximate solutions for the optimal city shape under several different settings are obtained numerically. For many-to-many travel, it is shown that a large number of walking-only trips occur when the city shape is close to being circular, although the exact shape varies with the distance deterrence coefficient.

Keywords: Optimal city shape, Walking-only trips, Walking probability, Manhattan distance

1. Introduction

When people travel within a city, they often have the choice of various modes of transportation, such as walking, bicycles, vehicles, and trains. Broadly speaking, such trips can be divided into those that can be conducted only by walking and those that involve some type of vehicle. Of the two types of trips, walking trips have the advantages of being environmentally friendly and requiring relatively little space for road networks. Therefore, it is important to build mathematical models for estimating the number of walking trips in a city area. Furthermore, an important research topic to explore is the ideal shape of a city in which many trips are conducted by walking only. The purpose of this paper is to address this problem mathematically and obtain basic knowledge about optimal city shapes.

In general, a walking trip that uses no type of vehicle from origin to destination occurs when the travel distance between origin and destination is short enough. Hereinafter, we refer to this type of trip as a walking-only trip. However, as the travel distance increases, people tend to use other modes of transportation such as bicycles, cars, and trains. To describe this trend, we assume that the percentage of people who choose a walking-only trip decreases exponentially with the trip length. Under this assumption, we construct city models that have a rectangular grid network, and we explore the city shape that maximizes the number of walking-only trips when the travel distance is given by the Manhattan distance. The Manhattan distance has been extensively used in describing distances in urban road networks. It is frequently employed in developing mathematical models for transportation and locational analysis (Averbakh et al. 2015; Bender et al. 2008; Larson and Odoni 1981; Vaughan 1987). Especially, there are existing literature assuming the Manhattan distance that deals with optimal city shapes to minimize the total travel distance (e.g., Bender et al. 2008; Demaine et al. 2011). Thus,
it is valuable to construct mathematical models to explore the basic characteristic of the Manhattan distance by focusing on the number of walking-only trips within a city.

Various previous studies have evaluated the basic characteristics of road network patterns using simple city shapes such as circular or rectangular. Such studies often assume infinitely dense networks to allow analytical approaches and to derive concrete results. The expected distances and their distribution are well-studied research topics by assuming that trip origins and destinations are uniformly and independently distributed over the city area. This continuous description of urban areas is known as the continuous approximation approach. See for example Vaughan (1987) and Larson and Odoni (1981) for various continuous approximation models regarding urban analysis and transportation problems. Continuous approximation models often employ a geometrical probability approach (Mathai, 1999; Solomon, 1978). For example, the average Euclidean distances and their distributions have been calculated for various city shapes based on uniformly and independently distributed origins and destinations. The purpose of the continuous approximation approach is to use a boldly simplified model and obtain explicit results analytically. Using the continuous approximation approach, we explore how the city shape that maximizes the number of walking-only trips with a grid network changes depending on several input parameters.

Rather than specifying the city shape, some previous studies have sought the optimum shape. For example, Karp et al. (1975) considered the shapes of two-dimensional regions with a fixed land area in which many trips could be conducted efficiently. They obtained city shapes that minimized the average Manhattan distance and the average maximum distance between any two points uniformly and independently chosen within a city. They also showed that there is an approximate solution method for finding an optimal shape to a desired level of accuracy by solving a differential equation. The optimal shape obtained by their method was shown to be close to a circle for both the Manhattan distance and the maximum distance. Bender et al. (2004) proposed a similar problem focusing on the Manhattan distance, and they presented city shapes based on the variational method. Their method has been further extended so that it can be used to calculate optimal shapes based on a general $L_p$ metric (Bender et al. 2007). These are treated as problems in continuous space, but some studies have discussed similar problems in a discrete framework (Avin et al. 2015; Bender et al. 2008; Demaine et al. 2011; Fekete et al. 2014). Other studies have sought to minimize the average time required for the optimal form of three-dimensional cities and buildings assuming vertical movement (Johnson 1992; Suzuki 1993; Koshizuka 1995).

However, none of the aforementioned studies was focused explicitly on the mode of transportation used when traveling except that some studies set different speed for vertical and horizontal movement (e.g, Koshizuka 1995). The viewpoint of maximizing the number of walking-only trips in a city based on the walking probability is a new approach that can provide new insights into optimal city shapes. Herein, we use the approach of Bender et al. (2004) for the problem of maximizing the number of walking-only trips in a city of arbitrary shape. In addition to the problem of finding the optimal shape of an urban area, the proposed method has applications in various other situations. For example, building a new bridge in an area divided by a river will shorten some trips, and some people will switch from using a vehicle to walking. In doing so, important urban engineering issues must be addressed, such as how the number of walking trips changes depending on the bridge position and where is the optimal bridge position that maximizes the number of walking-only trips. The proposed framework may also be applied to estimating the required number of moving vehicles in large areas such as factories, theme parks, and event venues, and to designing areas with few such vehicles.

This paper is organized as follows. In Section 2, we describe city models with a grid-type network and introduce the walking probability function. In addition, by using person-trip survey data, we show that the walking probability in each distance range fits well with the proposed exponential deterrence model. In Section 3, we deal with the city model based on the many-to-one trip pattern. Given a city with a fixed area and assuming that all trips are to the center and originate from uniformly distributed points, we show that the city shape that maximizes the number of walking-only trips is a diamond regardless of the distance deterrence coefficient. In Sections 4–6, we deal with models that assume the many-to-many trip pattern. First, in Section 4, we derive the number of walking-only trips in the urban shapes often used in previous studies (i.e., a circle, a square, and a diamond), and we compare the results. In Section 5, we show how to derive the number of walking-only trips for an arbitrary city shape. The total number of walking trips is expressed originally as a four-dimensional integral with respect to the two-dimensional coordinates of the origin and destination, but such a formulation is difficult to treat even numerically. We therefore present a method for reducing this integral to a two-dimensional integral that is much easier to treat. In Section 6, we analyze the numerically obtained solutions in detail for the problem of maximizing the number of walking-only trips, assuming a general city shape. Finally, in Section 7 we summarize the present research and discuss future issues.
2. Basic assumptions and model description

2.1. City model

In this section, we explain the basic assumptions and describe the proposed model. Our approach and notation are based on those of Bender et al. (2004). As shown in Fig. 1, we consider a planar convex region $D$ of arbitrary shape with area $S$ and that has an infinitely dense rectangular grid network. The city boundary in the first quadrant ($x \geq 0, y \geq 0$) of this region is denoted by $w(x)$ ($0 \leq x \leq a$), and let $(b, b)$ be the intersection of $w(x)$ and $y = x$. We assume the following about the city:

(i) the city shape is symmetric with respect to the $x$ and $y$ axes and $y = x$ and $y = -x$, and denote by $h(x)$ and $g(x)$ the parts $0 \leq x \leq b$ and $b \leq x \leq a$, respectively, in the first quadrant;

(ii) the boundary function $w(x)$ is a strictly monotonic decreasing function with respect to $x$;

(iii) the region boundaries $h(x)$ and $g(x)$ are differentiable in the ranges $0 < x < b$ and $b < x < a$, respectively.

![Fig. 1 Arbitrarily shaped city model with rectangular grid network.](image)

Assumptions (i) and (ii) mean that $h(x)$ and $g(x)$ are mutually inverse:

$$h(x) = g^{-1}(x).$$  \(1\)

Furthermore, because of the symmetry, the total area of the city is eight times that of the $y \geq x$ part of the first quadrant:

$$S = 8 \int_0^b [h(x) - x] \, dx.$$  \(2\)

2.2. Walking probability

Usually, various transportation modes are available for traveling between two locations in a city. For short journeys, people tend to walk, whereas they tend to use vehicles such as bicycles and cars for longer journeys. Therefore, in general, the probability of a person selecting a walking-only trip among several alternative modes of transportation can be expressed as a decreasing function of the travel distance. Herein, we use an exponential function to describe this decreasing trend. More concretely, we denote the probability $p$ as a function of the travel distance $r$ as

$$p(r) = \exp(-\lambda r), \quad (r \geq 0).$$  \(3\)

In the following, we refer to this function as the walking probability. As shown in Fig. 2, when the travel distance is small enough, almost all trips are walking trips, and the probability of choosing a walking-only trip decreases exponentially with the travel distance. The degree of decrease is determined by $\lambda$: when $\lambda$ is large, people dislike walking far and tend to select a vehicle as the mode of transportation, whereas small $\lambda$ allows many people to walk a long distance.

In a previous study, Tirachini (2015) focused on walking as a mode of transportation when traveling in a city, and analyzed the trip-length distributions of walking based on the results of origin–destination surveys conducted in several cities around the world. Tirachini showed that the exponential distribution

$$f(r) = \lambda \exp(-\lambda r), \quad (r \geq 0)$$  \(4\)
well-approximates the actual data of trips which are conducted by walking only. Using the exponential distribution, he compared the estimated values of $\lambda$ for various cities. In Eq. (4), $1/\lambda$ represents the expected value of the walking distance.

The above result by Tirachini (2015) is pertinent to the present study in that the former shows that more people walk when the travel distance is small. However, note that Eqs. (3) and (4) have different meanings. The walking probability $p(r)$, which is the present focus, represents the ratio of people with travel distance $r$ who choose to walk all the way from origin to destination. In this paper, we assume that the walking probability depends only on the travel distance.

Before formulating the number of walking-only trips, it is worthwhile examining how the proposed model function $p(r)$ describes the trend of the rate of walking-only trips in each distance range using actual travel data. Here, we use the results of the 2015 nationwide person-trip survey (Ministry of Land, Infrastructure, Transport and Tourism 2015). These include actual travel data on selection of mode of transportation in each distance range based on the results of a large-scale questionnaire survey conducted on both weekdays and holidays. The target cities are the major metropolitan areas of Tokyo, Osaka, and Nagoya, as well as large cities in other areas. The number of households surveyed was around 47,300, the questionnaire collection rate was 29.2%, and all distances were aggregated to the nearest 0.1 km. Using the data, we can classify trips into two types, namely, (i) those in which only walking was used and (ii) those in which a different mode of transportation or a combination of several modes of transportation was used. By taking the ratio of the number of trips of type (i) to the sum of both types in each distance range, we can evaluate the actual decreasing trend of walking-only trips as the distance range increases.

Figure 3 shows the proportion of walking-only trips for each distance range for weekday and holiday trips in the three metropolitan areas, Tokyo, Osaka and Nagoya (Three MAs), and large cities in other areas (Large CAs). The large cities include cities with high population such as Sapporo, Sendai and Hiroshima and large local cities such as Kanazawa, Hirosaki and Takasaki. For more concrete information, see the data description of the 2015 nationwide person-trip survey (URL: http://www.mlit.go.jp/common/001229430.pdf). In the figure, we also show the graphs of $p(r)$ for which $\lambda$ was estimated by applying the least-squares method to the actual data. From Fig. 3, the decreasing trend of the proportion of actual walking-only trips fits well with the model function $p(r)$. Therefore, we use the exponential model as a macroscopic description of walking-only trips.

Note that many trip-choice behavioral models have been proposed to date, such as logit models. However, an important characteristic of the model function $p(r)$ is that it is much simpler than other such transportation-mode choice models. Because of the goodness of fit and the simplicity of the model, using the walking probability $p(r)$ is well suited to the purpose of the present study.

3. Optimal city shape for many-to-one travel pattern

In this section, we consider the many-to-one travel pattern in which journeys originate from uniformly distributed points in the region and end at a fixed destination at the center. We denote a region by $D$ that has a fixed area $S$, and we seek the optimal city shape that maximizes the number of walking-only trips. This problem arises, for example, when designing the shape of the catchment area for a central facility that minimizes the required parking space (because it minimizes the number of car drivers who access the facility). This many-to-one travel pattern also applies when designing a grid-network city that has a fixed population and a central business district. The problem can be regarded as that of finding the city shape that minimizes the number of vehicle trips. We denote the coordinates of the trip origin by $(x, y)$ and assume the following:
Fig. 3 Proportion of walking-only trips in each distance range for 2015 person-trip survey and estimated \( p(r) \): (a) weekdays; (b) holidays.

| Weekdays | Weekdays | Weekdays | Weekdays | Weekdays | Holidays | Holidays | Holidays | Holidays |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| dist \( r \) | Three MAs | Large CA | dist \( r \) | Three MAs | Large CA | dist \( r \) | Three MAs | Large CA |
| 0.1km~   | 0.8338   | 0.8015   | 0.1km~   | 0.8371   | 0.8214   | 0.2km~   | 0.8024   | 0.6233   |
| 0.2km~   | 0.7743   | 0.6598   | 0.2km~   | 0.8024   | 0.6233   | 0.5km~   | 0.6573   | 0.5229   |
| 0.5km~   | 0.5247   | 0.3292   | 0.5km~   | 0.2117   | 0.1377   | 1.0km~   | 0.4555   | 0.3638   |
| 1.0km~   | 0.3547   | 0.3292   | 1.0km~   | 0.3768   | 0.2305   | 1.5km~   | 0.2062   | 0.1717   |
| 1.5km~   | 0.0922   | 0.0842   | 1.5km~   | 0.0797   | 0.0523   | 2.0km~   | 0.0449   | 0.0350   |
| 2.0km~   | 0.0166   | 0.0133   | 2.0km~   | 0.0224   | 0.0108   | 3.0km~   | 0.0475   | 0.0277   |
| 3.0km~   | 0.0166   | 0.0133   | 3.0km~   | 0.0224   | 0.0108   | 4.0km~   | 0.0475   | 0.0277   |
| 4.0km~   | 0.0166   | 0.0133   | 4.0km~   | 0.0224   | 0.0108   | 5.0km~   | 0.0166   | 0.0133   |

(i) The trip origins are distributed uniformly and continuously over region \( D \);

(ii) A trip maker chooses a shortest path to the destination, that is, the distance from the origin \((x, y)\) to the center is given by the Manhattan distance \(|x| + |y|\).

To formulate the number of walking-only trips, we introduce the origin density function \( \mu(x, y) \). This gives the number of trips arising from a unit area (in a unit time) around \((x, y)\), and \( \mu(x, y) \) satisfies

\[
\int_{(x,y) \in D} \mu(x, y) \, dx \, dy = N,
\]

where \( N \) is the total number of trips within the entire region \( D \).

Herein, we focus on the most basic and important uniform case as assumed above, which gives the trip density as

\[
\mu(x, y) = \mu_{uni} = \frac{N}{S}.
\]

This uniform distribution has been used most extensively in the literature (Larson and Odoni 1981; Vaughan 1987), which often gives us the tractable results and important implications.

Using this, we formulate the number of walking-only trips, \( F[w(x)] \). The walking probability from \((x, y)\) is given by \( e^{-\lambda(|x| + |y|)} \) because the Manhattan distance from the center is \(|x| + |y|\). \( F[w(x)] \) is formulated as

\[
F[w(x)] = \mu_{uni} \left[ \int_{0}^{a} \int_{-a}^{b} e^{-\lambda(|x| + |y|)} \, dy \, dx + \int_{-a}^{0} \int_{-a}^{b} e^{-\lambda(|x| + |y|)} \, dy \, dx \right] = 4\mu_{uni} \int_{0}^{a} \int_{0}^{b} e^{-\lambda x + \gamma y} \, dy \, dx.
\]

The second equation holds because by symmetry \( F[w(x)] \) is four times the number of walking-only trips whose origins are in the first quadrant. The number of walking-only trips can be rewritten as follows using \( g(x) \) and \( h(x) \):

\[
F[h(x), g(x)] = 4\mu_{uni} \left[ \int_{0}^{b} \int_{0}^{a} e^{-\lambda x + \gamma y} \, dy \, dx + \int_{0}^{a} \int_{0}^{b} e^{-\lambda x + \gamma y} \, dy \, dx \right].
\]
Next, we exploit the symmetry about \( y = x \) as mentioned in Section 1, that is, \( g \) is the inverse of \( h \). By denoting \( g(x) = \gamma \), we obtain \( x = g^{-1}(\gamma) = h(\gamma) \) and \( dx/dy = h'(\gamma) \). Using this relationship, the number of walking-only trips is written as

\[
F[h] = 4\mu_{uni} \left[ \int_0^b \left( e^{-\lambda(\gamma+x)} + 1 \right) e^{-\lambda x} d\gamma \right] dx + \int_0^b e^{-\lambda(\gamma+y)}h'(\gamma) d\gamma dy
\]

\[
= 4\mu_{uni} \left[ \int_0^b \left( e^{-\lambda(\gamma+x)} + 1 \right) e^{-\lambda x} d\gamma \right] dx + \int_0^b \left( e^{-\lambda(\gamma+y)} + \frac{1}{\lambda} e^{-\lambda y}h'(\gamma) \right) d\gamma dy
\]

\[
= 4\mu_{uni} \left[ -2 \int_0^b e^{-\lambda(\gamma+x)} d\gamma - \frac{1}{\lambda} e^{-2\lambda b} + \frac{1}{\lambda} \right].
\]  

(9)

Using the above notation, the problem of seeking the shape of region \( D \) that maximizes the number of walking-only trips from uniformly distributed points to the center for a fixed area \( S \) can be formulated as follows.

**Problem 1: Optimal shape problem for many-to-one travel pattern**

\[
\text{maximize } F[h] = 4\mu_{uni} \left[ -2 \int_0^b e^{-\lambda(\gamma+x)} d\gamma - \frac{1}{\lambda} e^{-2\lambda b} + \frac{1}{\lambda} \right]
\]  

subject to \( 8 \int_0^b \{h(x) - x\} dx = S \)

(10) (11)

This problem is a variational problem concerning \( h(x) \) with a fixed area constraint. Concerning the optimal solution of this problem, the following theorem holds.

**Theorem 1:** The optimal solution to Problem 1 is a diamond (45° rotated square) regardless of the value of \( \lambda \).

**Proof:** Using a Lagrangian multiplier \( \beta \), the objective function \( F[h(x)] \) can be written as

\[
\Psi[h(x)] = F[h(x)] + \beta \left( 8 \int_0^b \{h(x) - x\} dx - S \right).
\]

(12)

The optimal solution of this problem satisfies the Euler equation for \( h(x) \) in Eq. (12). The Euler equation is expressed as follows (e.g., Troutman, 1996):

\[
\frac{\partial \Psi[h(x)]}{\partial h(x)} = 8\mu_{uni} e^{-\lambda h(x)+x} + 8\beta = 0.
\]

(13)

Differentiating Eq. (13) with respect to \( x \) gives

\[
-8\mu_{uni} h'(x) + 1 - e^{-\lambda h(x)+x} = 0.
\]

(14)

Because \( \mu_{uni} > 0 \), \( \lambda > 0 \), and \( e^{-\lambda h(x)+x} > 0 \), we obtain

\[ h'(x) = -1. \]

(15)

Using the initial condition \( h(0) = a \), we obtain

\[ h(x) = -x + a. \]

(16)

This shows that the solution satisfying the Euler equation is a diamond.

Furthermore, we can show that this solution also satisfies a sufficient condition of optimality, that is, Eq. (16) is an optimal solution of Problem 1. We have that

\[
\Psi[h(x)] = F[h(x)] + 8\beta \int_0^b \{h(x) - x\} dx
\]

\[
= \int_0^b \left[ -8\mu_{uni} e^{-\lambda h(x)+x} + 8\beta \{h(x) - x\} \right] dx - \frac{4\mu_{uni}}{\lambda^2} e^{-2\lambda b} + \frac{4\mu_{uni}}{\lambda^2} = -8\mu_{uni} e^{-\lambda h(x)+x} + 8\beta \{h(x) - x\}.
\]

(17)

Here, we focus on the integrand in Eq. (17), namely, \( \psi[h(x)] = -8\mu_{uni} e^{-\lambda h(x)+x} + 8\beta \{h(x) - x\} \). Because \( \mu_{uni} > 0 \) and \( \lambda > 0 \), we have

\[
\frac{d^2 \psi[h(x)]}{dh(x)^2} = -8\mu_{uni} e^{-\lambda h(x)+x} < 0.
\]

(18)

This means that \( \Psi[h(x)] \) is concave as is \( h(x) \). Because Problem 1 is equivalent to minimizing \( -\Psi[h(x)] \), we arrive at the result that Eq. (16) satisfies a sufficient condition of optimality (e.g., Troutman, 1996).

The above discussion shows analytically that the region shape that maximizes the number of walking-only trips is a diamond when the trip origins are uniformly distributed over the area and the destination is the origin. It is interesting to note that the optimal shape does not depend on the distance deterrence coefficient \( \lambda \) of the walking probability.
4. Results for cities with specific shapes

In Sections 4 and 5, we assume the many-to-many trip pattern and derive the total number of walking-only trips. In the present section, we focus on cities with the specific shapes shown in Fig. 4, namely, a circle, a square, and a diamond, and we calculate the number of walking-only trips. The results presented in the present section are compared in Section 5 when more-general city shapes are assumed.

The average travel distances and their distribution for circles and squares have been dealt with extensively in the literature. Mathematical models that assume travel on a continuous plane using simple city shapes are summarized in detail by Vaughan (1987). In particular, with regard to the Manhattan distance, there is accumulated research on the circle, square, and diamond shapes as shown in Fig. 4.

In the following, we introduce the existing research that derives the distribution of the Manhattan distance assuming a uniform trip between any two points in these three city models. We then describe the method for deriving the total number of walking-only trips using the distance distribution, and we present some results.

![City models with specific shapes](image)

Fig. 4 City models with specific shapes: (a) circle; (b) square; (c) diamond.

In the following, we denote the origin by $P(x_1, y_1)$ and the destination by $Q(x_2, y_2)$, and we assume the following:

(i) The origins and destinations of trips are uniformly and independently distributed within the city;

(ii) A traveler chooses the shortest path on a rectangular grid network, and thus the Manhattan distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $|x_1 - x_2| + |y_1 - y_2|$.

We introduce the trip density $\rho(x_1, y_1, x_2, y_2)$ to describe the total number of walking-only trips, which gives the number of trips originating from a small zone with area $dx_1 dy_1$ at $P(x_1, y_1)$ and terminating in a small zone with area $dx_2 dy_2$ at $Q(x_2, y_2)$ as

$$\rho(x_1, y_1, x_2, y_2)dx_1 dy_1 dx_2 dy_2.$$  \hspace{1cm} (19)

The total number of trips in the city, $N$, is related to the trip density by

$$\int_{(x_1,y_1)\in D} \int_{(x_2,y_2)\in D} \rho(x_1, y_1, x_2, y_2)dy_2 dx_2 dy_1 dx_1 = N. \hspace{1cm} (20)$$

Using the trip density, the total number of walking-only trips is given as

$$F[w] = \int_{(x_1,y_1)\in D} \int_{(x_2,y_2)\in D} \rho(x_1, y_1, x_2, y_2)e^{-|x_1 - x_2| + |y_1 - y_2|}dy_2 dx_2 dy_1 dx_1.$$  \hspace{1cm} (21)

As noted in assumption (i), we consider the most basic and important uniform case for the trip density:

$$\rho(x_1, y_1, x_2, y_2) = \rho_{uni} = \frac{N}{S^2}. \hspace{1cm} (22)$$

The case in which trip origins and destinations are uniformly and independently distributed is the one that is used most often in existing city models. In particular, a circular city is known as a Smeed city after Ruben Smeed, who contributed greatly to the development of continuous transportation modeling (Vaughan, 1987).

Meanwhile, if the distance distribution is already known, as it is in the present section, we need only perform the one-dimensional integration described below instead of the general expression of Eq. (21). The distance distribution refers to the probability density function of the distance traveled within a city based on the continuous trip density. In other words, for the distance distribution $\varphi(r)$, the proportion of travelers whose travel distance lies in the small range $[r, r + dr]$ is
given by \( \varphi(r)dr \). If the distance distribution \( \varphi(r) \) is known for a particular city model, then \( N \times p(r) \times \varphi(r)dr \) represents the number of trips whose travel distance is in the range \([r, r + dr]\) and the travel is conducted by walk only. Therefore, the overall number of walking-only trips can be represented as

\[
F = N \int_0^\infty p(r) \varphi(r)dr.
\]  

(23)

Kurita (2001) derived the distribution of the Manhattan distance for a circular city of radius \( R \) as

\[
\varphi_{\text{cir}}(r) = \begin{cases} 
\frac{4}{\pi R^2} \frac{g(u)}{\sqrt{4u^2 - r^2}} & (0 \leq r \leq 2R), \\
\frac{4}{\pi R^2} \frac{g(u)}{\sqrt{4u^2 - r^2}} & (2R \leq r \leq 2\sqrt{2}R),
\end{cases}
\]

(24)

where \( g(u) \) is the distribution of the Euclidean distance for two points uniformly and independently distributed within a circular city (Vaughan, 1987; Mathai, 1999), namely,

\[
g(u) = \frac{4r}{\pi R^2} \arccos\left( \frac{r}{2R} \right) - \frac{r^2}{\pi R^4} \sqrt{4R^2 - r^2} \quad (0 \leq r \leq 2R).
\]

(25)

The distance distribution for a square city with side length \( L \) is given as

\[
\varphi_{\text{sq}}(r) = \begin{cases} 
\frac{1}{3L^2} (2r^3 - 12Lr^2 + 12L^2r) & (0 \leq r \leq L), \\
\frac{2}{3L^2} (2L - r)^3 & (L \leq r \leq 2L),
\end{cases}
\]

(26)

and Tanaka et al. (2007) derived the distance distribution in a diamond city as

\[
\varphi_{\text{dia}}(r) = \frac{1}{L^2} \left( r^3 - 3\sqrt{2}Lr^3 + 4L^3r \right) \quad (0 \leq r \leq \sqrt{2}L).
\]

(27)

Here, by applying Eq. (23) for the number of walking-only trips within a city to the distance distributions of Eqs. (26) and (27), we obtain the total numbers of walking-only trips for the square (\( F_{\text{sq}} \)) and diamond (\( F_{\text{dia}} \)) cities as

\[
F_{\text{sq}} = N \left\{ \frac{4}{3L^2} \left( 1 + e^{-2\lambda L} - 2e^{-\lambda L} \right) - \frac{8}{3L^2} \left( 1 - e^{-\lambda L} \right) + \frac{4}{L^2} \right\},
\]

(28)

\[
F_{\text{dia}} = N \left\{ \frac{6}{L^2} \left( 1 - e^{-\sqrt{2}\lambda L} \right) - \frac{6 \sqrt{2}}{L^2} \left( 2 - e^{-\sqrt{2}\lambda L} \right) + \frac{2}{L^2} \right\}.
\]

(29)

For the circular city, obtaining the total number of walking-only trips, \( F_{\text{cir}} \), involves numerical integration because the distance distribution is expressed in integral form as shown in Eq. (24).

Let us compare the total numbers of walking-only trips \( F_{\text{cir}}, F_{\text{sq}}, \) and \( F_{\text{dia}} \) for the three different city shapes with the same area. We set the area of each city as \( S = \pi \), that is, the square and the diamond have side length \( \sqrt{\pi} \approx 1.77 \), and the circle has unit radius. We set \( N = 100 \) and focus on four different distance deterrence values: \( \lambda = 0.5, 1.0, 1.5, \) and \( 2.0 \). The results for the total numbers of walking-only trips for the three cities are summarized in Table 2, from which it can be seen that the total number of walking-only trips decreases with \( \lambda \). Of the three shapes, the circular city realizes the largest number of walking-only trips. In Section 3, we showed that the optimal shape for many-to-one trips to the center is a diamond, whereas Table 2 shows that the circular city performs best for all four values of \( \lambda \). It is interesting that for the many-to-many trip pattern the circular city performs better than the diamond city. Looking at the details, the diamond city is more advantageous than the square when \( \lambda = 0.5 \), but the square is better than the diamond for other values of \( \lambda \). As such, the desirable city shape changes depending on the value of the distance deterrence coefficient, which is an interesting finding of the proposed model.

The above analysis showed that the circular shape is the most advantageous of the three city shapes. A natural question that arises here is whether there is a better shape than the circle. In Section 5, to answer this question, we analyze the number of walking-only trips for more-general city shapes, assuming uniform trips between any two points in the city.
5. Optimal city shape for many-to-many travel patterns

We formulate the total number of walking-only trips in a region for the case in which trips occur between any two points in the city with equal frequency. This uniform trip distribution has been used most extensively in the literature as the most basic mathematical assumption in the continuous travel demand models (Bender et al. 2008; Larson and Odoni 1981; Vaughan 1987). The number of walking-only trips is given by a four-dimensional integral with respect to the \((x, y)\) coordinates of the origin and destination. The procedure for expressing this as a two-dimensional integral is described in detail. When formulating the problem of finding the shape that maximizes the number of walking-only trips, this operation makes it easier to compute that number than when four-dimensional integration is involved. We denote the origin by \(P(x_1, y_1)\) and the destination by \(Q(x_2, y_2)\), and we assume the following:

(i) the trip origins and destinations are uniformly and independently distributed within the city area;

(ii) travelers choose the route with the shortest distance from origin to destination over the rectangular grid network, that is, the Manhattan distance is given by \(|x_1 - x_2| + |y_1 - y_2|\).

According to assumption (i), the trip density \(\rho_{uni}\) is \(N/S^2\). The probability that a given traveler chooses a walking-only trip is given by \(e^{-\lambda(|x_1-x_2|+|y_1-y_2|)}\). The total number of walking-only trips can be obtained by integrating \(\rho_{uni} \times e^{-\lambda(|x_1-x_2|+|y_1-y_2|)}\) with respect to all origins and destinations. The problem we consider in the paper is shown below:

Problem 2: Optimal shape problem for many-to-many travel pattern

\[
\text{maximize } F[w] = \rho_{uni} \int_{(x_1,y_1)\in D} \int_{(x_2,y_2)\in D} e^{-\lambda(|x_1-x_2|+|y_1-y_2|)} dy_2 dx_2 dy_1 dx_1 \\
\text{subject to } 8 \int_0^b [h(x) - x] \, dx = S
\]

By exploiting the symmetry of the four quadrants, Eq. (30) can be expressed four times when the origin is in the first quadrant of the region. Using the city boundary \(w(x)\) in the first quadrant, \(F[w]\) is represented by

\[
F[w(x)] = 4\rho_{uni} \int_0^{w(0)} \int_{-w(x)}^{w(x)} e^{-\lambda(|x_1-x_2|+|y_1-y_2|)} dy_2 dx_2 + \int_{-\infty}^{w(0)} \int_{-\infty}^{-w(x)} e^{-\lambda(|x_1-x_2|+|y_1-y_2|)} dy_2 dx_2 dx_1.
\]

As Eq. (32) shows, the number of walking-only trips can be described by the four-dimensional integral \(F[w(x)]\). In the case of the average distance treated by Bender et al. (2004), the distance part of the integrand can be decomposed into a horizontal movement part and a vertical movement part. By exploiting this property, they reduced the dimension of integration and greatly simplified the calculation.

Meanwhile, for the number \(F[w(x)]\) of walking-only trips considered herein, the Manhattan distance between two points appears in the exponent of the exponential function. This prevents us from taking a similar simplification approach to that of Bender et al. (2004), thereby making the problem more difficult. Herein, we show that the four-dimensional integration of Eq. (32) can be reduced to a two-dimensional integration \(F[h]\) through multiple steps. This approach allows us to compute the original objective function much more easily and devise a method for formulating the problem of optimizing the city shape, which is an important contribution of the paper. The expression involving only a double integral of the walking-only trips is shown in the following theorem.

| City model | \(\lambda = 0.5\) | \(\lambda = 1.0\) | \(\lambda = 1.5\) | \(\lambda = 2.0\) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| Circular  | 58.33584        | 36.50879        | 24.29893        | 17.04157        |
| Square    | 57.75371        | 35.99172        | 23.92463        | 16.77964        |
| Diamond   | 57.86143        | 35.96591        | 23.82505        | 16.66300        |

Table 2: Total numbers of walking-only trips for three city shapes
Theorem 2: The total number of walking-only trips (32) assuming uniform and independent point pairs under the Manhattan distance can be calculated by performing the following two-dimensional integration:

\[
F[h] = 4 \rho_{\text{uni}} \left[ \frac{4}{\Delta^3} \int_0^{b} \int_0^{c} \int_0^{d} \left( e^{-h(x+y)+u+v} - e^{-h(x+y)+u-v} - e^{-h(x+y)+u+v} - e^{-h(x+y)+u-v} \right) \, dx \, dy \, dz \\
+ \frac{2}{\Delta^2} \int_0^{b} \int_0^{c} \int_0^{d} \left( e^{-h(x+y)+u-v} + e^{-h(x+y)+u+v} + 2e^{-h(x+y)+u+v} + e^{-h(x+y)+u-v} \right) \, dx \, dy \, dz \\
+ \frac{4}{\Delta^2} \int_0^{b} \int_0^{c} \int_0^{d} \left( e^{-h(x+y)+u+2b} + e^{-h(x+y)+u+2b} - e^{-h(x+y)+u} - e^{-h(x+y)+u} \right) \, dx \, dy \, dz \\
+ \frac{8}{\Delta^2} \int_0^{b} \int_0^{c} \int_0^{d} \left[ 1 - e^{-2u} \right] h(u) \, dx \, dy \, dz \right] \\
+ \frac{1}{\Delta^2} \left( e^{-4b} - 2e^{-4b} + 1 \right) + \frac{4b}{\Delta^3} \left[ 1 - e^{-2b} \right] - \frac{4b^2}{\Delta^2}.
\] (33)

In the following, we show how to obtain this result according to the following five steps.

Step 1 Express all different expressions of the integrand in Eq. (32) without using absolute-value functions based on the magnitudes of the \(x\) and \(y\) coordinates.

Step 2 Delete the inverse function \(w^{-1}\) appearing in the integration range by variable transformation.

Step 3 Express \(F\) using \(g(x)\) and \(h(x)\) in Fig. 1 instead of \(w(x)\).

Step 4 Express \(F\) using only \(h(x)\) based on the relationship that \(g(x)\) is the inverse of \(h(x)\).

Step 5 Delete \(h'(x)\) by carrying out the integration of the composite function, and express \(F\) as a function of \(h\) only.

In the following, the total number of walking-only trips is calculated from Eq. (32) to arrive at Eq. (33) by executing the above steps.

5.1. Step 1

We denote the \(i\)-th quadrant by \(R_i\) (\(i = 1, 2, 3, 4\)) and begin with the case in which both the \(x\) and \(y\) coordinates of origin are non-negative (i.e., the origin is in the first quadrant \(R_1\)). We denote the number of walking-only trips as \(F_i[w]\) (\(i = 1, 2, 3, 4\)) when the destination is in \(R_i\) (\(i = 1, 2, 3, 4\)) as shown in Fig. 5. Note that \(0 < x < b\) and \(b < x < a\) parts of \(w(x)\) are denoted by \(g(x)\) and \(h(x)\) respectively, as introduced in Fig. 1. Using the same argument when the origin is in a quadrant other than \(R_1\), the following relation holds:

\[
F[w] = 4 \left[ F_1[w] + F_2[w] + F_3[w] + F_4[w] \right].
\] (34)

Concretely, \(F_i[w]\) (\(i = 1, 2, 3, 4\)) can be expressed as Eqs. (35)–(38).

When \(P(x_1, y_1) \in R_1\) and \(Q(x_2, y_2) \in R_1\), we have

\[
F_1[w] = \rho_{\text{uni}} \left\{ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+u+y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 + \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+u-y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \\
+ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+u+y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 + \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+u-y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \\
+ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+u+y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \right\}. 
\] (35)

When \(P(x_1, y_1) \in R_1\) and \(Q(x_2, y_2) \in R_2\), we have

\[
F_2[w] = \rho_{\text{uni}} \left\{ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+y+y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \\
+ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+y-y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \\
+ \int_0^b \int_0^{x_1} \int_0^{y_1} e^{-h(x_1+y_1+y-y)} \, dy_2 \, dx_2 \, dy_1 \, dx_1 \right\}. 
\] (36)
When $P(x_1, y_1) \in R_1$ and $Q(x_2, y_2) \in R_3$, we have

$$F_3[w] = \rho_{\text{uni}} \int_0^a \int_0^{w(x_1)} \int_{-a}^0 \int_{-a}^0 e^{-\lambda(x_1-x_2+y_1-y_2)} dy_2 dx_2 dy_1 dx_1.$$  \hspace{1cm} (37)

When $P(x_1, y_1) \in R_1$ and $Q(x_2, y_2) \in R_4$, we have

$$F_4[w] = \rho_{\text{uni}} \left\{ \int_0^a \int_0^{w(x_1)} \int_0^0 \int_{-a}^0 e^{-\lambda(x_1-x_2+y_1-y_2)} dy_2 dx_2 dy_1 dx_1$$
$$+ \int_0^a \int_0^{w(x_1)} \int_0^a \int_{-a}^0 e^{-\lambda(x_2-x_1+y_1-y_2)} dy_2 dx_2 dy_1 dx_1 \right\}.$$  \hspace{1cm} (38)

Fig. 5 Four cases based on destination location when origin is in $R_1$. 

(a) $Q(x_2, y_2) \in R_1$

(b) $Q(x_2, y_2) \in R_2$

(c) $Q(x_2, y_2) \in R_3$

(d) $Q(x_2, y_2) \in R_4$
5.2. Step 2
In Eqs. (35)–(38), \( t = w^{-1}(y_1) \) leads to \( y_1 = w(t) \) and \( dy_1/dt = w'(t) \). Using these relations to organize these expressions, the objective function \( F[w] \) is expressed as

\[
F[w, w'] = 4P_{\text{uni}} \left[ \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ -e^{-\lambda(x_1-t)} - e^{-\lambda(\alpha+t)} + e^{-\lambda(x_1-t)} + e^{-\lambda(\alpha+t)} \right\} \left( \frac{\lambda}{\beta} \right) \right] d\lambda \] 

\[
+ \frac{1}{A} \int_{0}^{\alpha} \int_{x_1}^{b} \left\{ 2e^{-\lambda(x_1-t)} + e^{-\lambda(\alpha+t)} - e^{-\lambda(x_1-t)} - e^{-\lambda(\alpha+t)} \right\} f(x) dx \] 

\[
+ \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ 2\lambda u(x_1) + 2\lambda u(\alpha) + e^{-\lambda(x_1-t)} + e^{-\lambda(\alpha+t)} \right\} g(x) dx \] 

\[
+ \frac{1}{A} \int_{0}^{\alpha} \int_{x_1}^{b} \left\{ e^{-\lambda(x_1-t)} - e^{-\lambda(\alpha+t)} \right\} h(x) dx \] 

(39)

For convenience of explanation, we denote the six integral terms inside the parentheses in Eq. (39) by \( F_A[w], F_B[w], F_C[w], F_D[w], F_E[w], \) and \( F_F[w] \), then Eq. (39) can be expressed as

\[
F[w] = 4P_{\text{uni}} \left[ F_A[w] + F_B[w] + F_C[w] + F_D[w] + F_E[w] + F_F[w] \right]. 
\] 

(40)

In Step 3, \( F[w] \) represented by \( w(x) \) defined in the entire first quadrant is expressed as a functional of \( g(x) \) and \( h(x) \). Space limitations mean that this method is described for \( F_A[w] \) only, but \( F_B[w] \) through \( F_F[w] \) can be obtained by the same procedure.

5.3. Step 3
Based on the above, \( F[w] \) can be expressed using \( g \) and \( h \). The boundary function \( w \) can be described as \( h \) for the integral variable \( t \) in \([0, b]\), and \( g \) in \([b, a] \). The same is true for the integral variable \( x_1 \). By carrying out the above process, we obtain

\[
F_A[g, g', h, h'] = \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ -e^{-\lambda(h(x)-t)} - e^{-\lambda(h(x)+t)} + e^{-\lambda(h(x)-t)} + e^{-\lambda(h(x)+t)} \right\} h'(t) dx \] 

\[
+ \frac{1}{A} \int_{0}^{\alpha} \int_{x_1}^{b} \left\{ -e^{-\lambda(g(x)-t)} - e^{-\lambda(g(x)+t)} + e^{-\lambda(g(x)-t)} + e^{-\lambda(g(x)+t)} \right\} g'(t) dx \] 

\[
+ \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ e^{-\lambda(h(x)-t)} - e^{-\lambda(h(x)+t)} \right\} h'(t) dx. 
\] 

(41)

5.4. Step 4
\( F[g, h, g', h'] \) is expressed as a functional with only \( h \). We show how to conduct this process by focusing on \( F_A[g, h, g', h'] \) in Eq. (41). Due to the symmetry about \( y = x \) in the first quadrant, if we express \( g(x_1) \) by \( y \), then the relations \( x_1 = g^{-1}(y) = h(y) \) and \( dx_1 /dy = h'(y) \) hold. Similarly, expressing \( g(t) \) by \( \alpha \) leads to \( t = g^{-1}(\alpha) = h(\alpha) \) and \( dt /d\alpha = h'(\alpha) \). Using these relations, \( F_A[g, h] \) can be expressed with only \( h \):

\[
F_A[h, h'] = \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ -e^{-\lambda(h(x)-t)} - e^{-\lambda(h(x)+t)} + e^{-\lambda(h(x)-t)} + e^{-\lambda(h(x)+t)} \right\} h'(t) dx \] 

\[
+ \frac{1}{A} \int_{0}^{\alpha} \int_{x_1}^{b} \left\{ e^{-\lambda(h(x)-t)} - e^{-\lambda(h(x)+t)} \right\} h'(t) dy \] 

\[
= \frac{1}{A} \int_{x_1}^{b} \int_{0}^{\alpha} \left\{ -e^{-\lambda(h(x)-t)} - e^{-\lambda(h(x)+t)} + e^{-\lambda(h(x)-t)} + e^{-\lambda(h(x)+t)} \right\} h'(t) dx. 
\] 

(42)

We can apply the same procedure for \( F_B[g, h, g', h'] \) through \( F_F[g, h, g', h'] \).
5.5. Step 5

We show how to arrive at the final results by deleting \( h'(x) \) by carrying out the integration of the composite function. We take Eq. (42) as an example to show the method:

\[
F_A[h] = \frac{1}{A^2} \int_0^b \left\{ e^{-h(x_1)+2} + e^{-h(x_1)+4} + e^{-h(x_1)+6} + e^{-h(x_1)+8} \right\} dx_1
+ \frac{1}{A^2} \int_0^b \left\{ e^{-h(x_1)+2} + e^{-h(x_1)+4} + e^{-h(x_1)+6} + e^{-h(x_1)+8} \right\} dx_1
+ \frac{1}{A^2} \int_0^b \left\{ e^{-h(x_1)+2} + e^{-h(x_1)+4} + e^{-h(x_1)+6} + e^{-h(x_1)+8} \right\} dx_1
+ \frac{1}{A^2} \int_0^b \left\{ e^{-h(x_1)+2} + e^{-h(x_1)+4} + e^{-h(x_1)+6} + e^{-h(x_1)+8} \right\} dx_1
\]

By applying a similar procedure, we can obtain the expression for the number of walking-only trips as shown in Eq. (33), thereby concluding the proof.

Originally, the number of walking-only trips within a city required a four-dimensional integral. As shown in Eq. (33), this can be reduced to a two-dimensional integral in the variables \( u \) and \( v \), thereby allowing us to compute the objective function much more easily.

Next, we confirm that the number of walking-only trips in a city of a specific shape can be computed using Eq. (33). In Section 4, we presented values for the number of walking-only trips for three城市 shapes using the existing distance distributions. Here, we focus on two city shapes: circular and diamond. The boundary functions of these city models, \( h_{\text{cir}}(x) \) and \( h_{\text{dia}}(x) \), are given as

\[
h_{\text{cir}}(x) = \frac{\pi - x^2}{\pi}, \quad (44)
\]

\[
h_{\text{dia}}(x) = \frac{\pi - x}{2}, \quad (45)
\]

where \( S \) is the area of the city. We can compute \( F_{\text{cir}} \) and \( F_{\text{dia}} \) by applying \( h_{\text{cir}}(x) \) and \( h_{\text{dia}}(x) \) to \( h \) in Eq. (33). We take the case of \( S = \pi \) and \( \lambda = 1.0 \) as an example, and carry out numerical integration using Eq. (33) using Wolfram Mathematica 11.2. We obtain the numbers of walking-only trips as 36.50879 for the circle, and 35.96591 for the diamond. These results match exactly those obtained using the distance distributions of Eqs. (24) and (27). Note that for the square city, the boundary function parallel to the \( y \) axis has no inverse, and this approach cannot be used. However, in this special case we have already obtained the analytical result of Eq. (28), which was computed from the distance distribution.

We have already shown that unlike the case of many-to-one trips, the circular city is superior to the square and diamond cities in terms of the number of walking-only trips. In Section 6, we use Eq. (33) to explore city shapes that are even better that the circular one.

6. Numerical results

In this section, we address numerically the problem of maximizing the number of walking-only trips using Eq. (33) derived in Section 5. As for the boundary functions, we deal with two types: the one-parameter function treated by Bender et al. (2004) and a convex polygon.

6.1. One-parameter boundary function: \( h(x) = (a^2 - x^2)^{1/2} \)

Bender et al. (2004) used the following one-parameter boundary function in search of the city shape that minimizes the average Manhattan distance between every uniform point pair:

\[
h(x) = (a^2 - x^2)^{1/2}, \quad (0 \leq x \leq b), \quad (46)
\]

By varying the value of \( d \), Eq. (46) describes various city shapes. Important special cases are \( d = 1 \) for the diamond, \( d = 2 \) for the circle, and \( d = \infty \) for the square, which are the cases treated in Section 5. We denote the number of walking-only trips by \( F_d[h] \) when the border function (46) is employed. For a fixed value of \( d \), we can compute \( F_d[h] \) by carrying out the numerical integral as described in Eq. (33) by applying \( h(x) = (a^2 - x^2)^{1/2} \). We show some numerical results for \( S = \pi \) and \( N = 100 \). Figure 6 shows the values of \( F_d[h] \) obtained by numerical integration for various values of \( d \) in the four cases of \( \lambda = 0.5, 1.0, 1.5, \) and 2.0. As each graph shows, there is a value \( d^* \) that maximizes \( F_d[h] \).

Figure 6 shows that the larger the value of \( \lambda \), the larger the value of \( d^* \). It is interesting to note that the circle is in between \( \lambda = 0.5 \) and \( \lambda = 1.0 \). More concretely, when \( \lambda = 0.5, d^* \approx 1.95 \), which is almost a circle but distorted slightly.
toward a diamond. For the other three cases, we have $d^* > 2$, namely, $d^* \approx 2.08$ for $\lambda = 1.0$, $d^* \approx 2.18$ for $\lambda = 1.5$, and $d^* \approx 2.26$ for $\lambda = 2.0$. Also in these cases, the optimal shapes are very close to a circle but distorted slightly toward a square. In summary, while the value of $\lambda$ has some effect on the city shape, the optimal shape remains very close to a circle. This result is similar to those given by the average-distance models of Karp et al. (1975) and Bender et al. (2004). It is interesting to note that while the many-to-one optimal shape is a diamond, the many-to-many results are close to a circle. The optimum shape for many-to-many trips depends on $\lambda$, which is an interesting finding of the present paper.

In Fig. 7, the optimal shapes for $\lambda = 0.2$ and $\lambda = 5.0$ are compared to a circle. The latter relatively large value of $\lambda$ was chosen to show the difference in shape more clearly. While the shapes that are shown are not perfect circles, they are close to being so. In summary, the results show that a circular area performs well in terms of maximizing the number of walking-only trips within a city.

![Fig. 6](image1.png)  
(a) $\lambda = 0.5$  
(b) $\lambda = 1.0$  
(c) $\lambda = 1.5$  
(d) $\lambda = 2.0$  

\[ F_d[h] \] as a function of $d$ for $\lambda = 0.5, 1.0, 1.5, \text{ and } 2.0$.  

![Fig. 7](image2.png)  
Optimal shapes for $\lambda = 0.2 \text{ and } 5.0$.  

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6.2. Convex polygon

To explore city shapes that are better than those that arise with the one-parameter model, we consider a boundary given by a convex polygon. In a different model, Demaines et al. (2011) showed that the optimal shape is convex when the average distance is used as the objective function. There are two advantages of approximating a city with a polygon:

(i) an analytical expression for the number of walking-only trips can be obtained by integrating Eq. (33) directly;

(ii) with a sufficient number of vertices, a flexible shape can be expressed.

Advantage (i) above is due to using a simple (piecewise) linear function for (convex) polygons. The problem of seeking the optimal city shape can be regarded as an optimization problem with the vertex coordinates of the polygon as variables. Therefore, advantage (i) above has practical merit in that it enables us to solve the problem using a general-purpose nonlinear optimization solver. Equation (33) allows us to use the approach of reducing the four-dimensional integration to a two-dimensional one. Figure 8 illustrates the situation. On the boundary function \( h(x) \), the coordinates of the vertices are given by

\[
P_0(0, a), P_1(s_1, t_1), \ldots, P_{n-1}(s_{n-1}, t_{n-1}), P_n(b, b),
\]

and \( n \) segments \( h_1(x), h_2(x), \ldots, h_n(x) \) are set between adjacent points. These segments are expressed as

\[
h_1(x) = \frac{t_1 - a}{s_1} x + a \quad (0 \leq x \leq s_1), \tag{48}
\]

\[
h_j(x) = \frac{t_j - t_{j-1}}{s_j - s_{j-1}} (x - s_{j-1}) + t_{j-1} \quad (s_{j-1} \leq x \leq s_j), \quad j = 2, \ldots, n - 1, \tag{49}
\]

\[
h_n(x) = \frac{b - t_{n-1}}{b - s_{n-1}} (x - s_{n-1}) + t_{n-1} \quad (s_{n-1} \leq x \leq b). \tag{50}
\]

By describing the boundary line \( h(x) \) by a convex polygon, the exponential function multiplier included in the objective function becomes a linear expression, thereby giving the expression for the number of walking-only trips (33). However, because there exist multiple cases at the vertex coordinates, the expression is quite complicated.

Using the above notation, the problem of maximizing the number of walking-only trips can be described as

\[
\text{maximize } F_{\text{poly}}(a, b, s_1, s_2, \ldots, s_{n-1}, t_1, t_2, \ldots, t_{n-1}) \tag{51}
\]

subject to

\[
8 \left[ \int_0^{s_1} (h_1(x) - x) \, dx + \sum_{j=2}^{n-1} \int_{s_{j-1}}^{s_j} (h_j(x) - x) \, dx + \int_{s_{n-1}}^{b} (h_n(x) - x) \, dx \right] = S. \tag{52}
\]

We can obtain an explicit expression for Eq. (51) by carrying out the integral in Eq. (33). The first two terms of Eq. (33) involve double integrals, which can be conducted over the region shown in Fig. 9. We can also execute the integral in Eq. (52) to express the area for the polygon model.

In the following, the process for obtaining the solution of this polygon model is explained and the results are analyzed. Although both the objective function and the constraint equation can be expressed as functions of the polygonal vertex coordinates, because they are complex nonlinear functions, it is difficult to find a global optimal solution for this problem.
Our present approach is to use a general-purpose nonlinear solver and obtain several local optimal solutions. We used the FindMaximum function in Wolfram Mathematica 11.2. As an initial solution, we used a convex polygon with a random error added to each coordinate of each vertex of the regular polygon. More concretely, we obtained solutions by the following procedure.

Step 1 Set the coordinates of a regular polygon in the range $0 \leq x \leq b$ as $(r \cos \theta_i, r \sin \theta_i)$, where $\theta_i = \pi/2 - \bar{\theta}, \bar{\theta} = \pi/(4n)$ for $i = 1, \ldots, n$.

Step 2 For $i = 1, \ldots, n - 1$, generate $\epsilon_i$ randomly in the range $[-\bar{\theta}/2, \bar{\theta}/2]$.

Step 3 For $i = 1, \ldots, n - 1$, a regular polygon is distorted using $\epsilon_i$ to obtain a new polygon whose vertices are $(r \cos(\theta_i + \epsilon_i), r \sin(\theta_i + \epsilon_i))$.

Step 4 Adjust the size of the polygon to satisfy the area constraint.

Step 5 Input the resulting polygon to the solver as an initial solution of the problem.

Next, we show the results of applying the above process when the city is given as a 56-sided polygon ($n = 7$) with $S = \pi$ and $N = 100$. Figure 10 shows the solutions for $\lambda = 0.2$ and 5.0. For each example, we start with 30 different initial solutions and use the best one as the solution. A circle is also drawn for the purpose of comparison. The vertex positions of the polygon model are indicated by small dots. From this figure, the following features can be seen:

1. $\lambda = 0.2$ gives a circular form that is distorted slightly toward a diamond;
2. $\lambda = 5.0$ gives a circular form that is distorted slightly toward a square;
3. both forms are quite close to a circle.

This result is similar to the conclusion obtained with the one-parameter $d$ model. To compare the obtained results with the previous ones, Table 3 summarizes the results of calculating the number of walking-only trips for the diamond and circle and the $d$ and polygon models using five different values of $\lambda$. To compare the shapes obtained with the $d$ and polygon models, Fig. 11 shows the optimal solutions for $\lambda = 0.2$ and 2.0, which appear to be almost identical.

As described above, the vertices of the initial polygon are distributed on the circumference of a circle, but the obtained results are almost the same as the $d$-model solution whose vertices are away from the circumference. Judging from these results, the solutions obtained with both models are fairly good solutions. Nevertheless, comparing the numbers of walking-only trips from both models in Table 3, the result with the polygon model outperforms that with the $d$-model for all values of $\lambda$, although the differences are admittedly extremely small. For a polygon with a medium number of vertices, we reason that the result is more than that with the $d$-model because of the higher degree of freedom of the polygon model.
Fig. 10 Solutions for polygon model ($\lambda = 0.2$ and 5.0) and a circle.

Table 3 Comparison of solutions for various city models.

| $\lambda$ | Diamond | Circle | $d$-model | Polygon ($n = 7$) |
|-----------|---------|--------|-----------|------------------|
| 0.2       | 79.6250112 | 79.8943626 | 79.8962420 | 79.8962448 |
| 0.8       | 43.1476236 | 43.6906507 | 43.6907989 | 43.6908015 |
| 1.4       | 25.7505648 | 26.2423639 | 26.2455464 | 26.2455876 |
| 2.0       | 16.6630038 | 17.0415706 | 17.0469903 | 17.0470538 |
| 5.0       | 3.9711575  | 4.0553191  | 4.0576322  | 4.0576542  |

Fig. 11 Comparison of solutions with $d$-model and polygon model ($\lambda = 0.2$ and $\lambda = 2.0$).
7. Conclusion and future research perspectives

People traveling in a city have a choice to modes of transportation, which broadly speaking can be divided into walking and vehicles. Trips that involve only walking have the advantages of lower environmental impact and taking up less city space compared to travel using vehicles. In general, walking-only trips are made only when trip length is short, and the proportion of vehicular use increase with the travel distance. To describe this point, we assumed that the proportion of walking-only trips decreases exponentially with the trip length.

We assumed city models with a typical rectangular grid network, and we considered the problem of determining the city shape that maximizes the number of walking-only trips for many-to-one and many-to-many trip patterns. In the case of the many-to-one type where the destination is the city center, we used the variational method to show that the optimal city shape is a diamond. For the many-to-many type, we presented a method for expressing the number of walking-only trips by a double integral that was formulated originally as a four-dimensional integral.

Using this method, we formulated an optimization problem with the vertex coordinates of a polygon as the variables, and we obtained numerically an approximate solution for the optimal shape. As a result, we found that while the many-to-one optimal shape is a diamond, the many-to-many results are close to a circle. In addition, although the former case does not depend on the value of $\lambda$, the latter case depends on the value. We have also found that values of the number of walking-only trips are fairly stable regarding the deviation of shape from a circle and the value of $\lambda$. This result suggests that as long as the city shape does not deviate greatly from the circle and is roughly convex, the number of walking trips is not much different from the maximum value. Also even when the value of $\lambda$ changes due to, for example, the progress of aging society, desirable city shapes do not change dramatically. As we have shown in the numerical results, our proposed formulation enables us how the values of $\lambda$ affects the desirable city shapes, which have not been treated in the existing average distance (or time) minimization approach.

Maximizing the number of walking trips is equivalent to minimizing the number of trips using a vehicle. Therefore, the proposed approach is useful in reducing the space required for parking or relieving traffic jam within a city, which are important issues in urban planning. Based on the present framework, different problem formulations can be considered. For example, the problem of minimizing the distance traveled by a vehicle is also an interesting and important problem to explore. It is directly related to environment issues such as reducing the emission (e.g., Nox) from vehicles in cities. This problem can be addressed by performing an integral calculation for vehicle users considering the weighted distance traveled by vehicles. The number of walking-only trips can also be formulated in the network space where origin and destination of trips are continuously distributed along links of the network. It would also be interesting to pursue calculation methods and network shapes for maximization of the total number of walking-only trips.

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