ON TOPOLOGICAL PROPERTIES OF POSITIVE COMPLEXITY ONE SPACES

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Abstract. Motivated by work of Fine and Panov, and of Lindsay and Panov, we prove that every closed symplectic complexity one space that is positive (e.g. positive monotone) enjoys topological properties that Fano varieties with a complexity one holomorphic torus action possess. In particular, such spaces are simply connected, have Todd genus equal to one and vanishing odd Betti numbers.

1. Introduction

A driving (meta-)question in symplectic topology is to understand how closed symplectic manifolds differ from smooth complex projective varieties. While there are examples of closed symplectic manifolds that cannot be Kähler (see, for instance, [30, 28]), it makes sense to consider refinements of the above problem. Largely inspired by work of Fine and Panov [7, 8] and of Lindsay and Panov [21], in this paper we prove that a class of symplectic manifolds with ‘sufficiently large’ torus symmetries share topological properties with their complex projective counterparts (see the Main Result below).

First, we introduce the class of symplectic manifolds that we consider. To this end, given a symplectic manifold \((M,\omega)\), we denote its first Chern class by \(c_1\).

Definition 1.1. A closed symplectic manifold \((M,\omega)\) is

- positive monotone if there exists \(\lambda > 0\) such that \(c_1 = \lambda [\omega]\), and
- symplectic Fano if there exists a compatible almost-complex structure \(J\) such that \(c_1[A] > 0\) for all non-zero \(A \in H_2(M)\) that can be represented by a \(J\)-holomorphic curve.

Remark 1.2 In some works in the literature, what we call ‘positive monotone’ is referred to as ‘monotone’ (sometimes imposing \(\lambda = 1\) in...
Definition 1.2), or ‘symplectic Fano’ (see, for instance, [21]). The above definition of symplectic Fano is taken from [27, Remark 11.1.1].

Remark 1.3 Observe that, given an almost complex structure $J$ compatible with $\omega$, $J$-holomorphic curves in $(M, \omega)$ are necessarily symplectic. Hence positive monotone implies symplectic Fano in Definition 1.1.

The class of manifolds introduced in Definition 1.1 can be thought of as the symplectic analog of smooth Fano complex varieties, namely those having an ample anticanonical bundle. In fact, a Fano variety $Y$, together with the symplectic form induced by pulling back the Fubini-Study form on projective space along the embedding given by ampleness of the anticanonical bundle, is necessarily positive monotone. Fano varieties have been extensively studied in differential, symplectic and algebraic geometry. For the purposes of this paper, we remark that they are are simply connected (see [14, Corollary 6.2.18] and [22, Remark 3.12]), and have Todd genus equal to one (see [13, Section 1.8] for a definition).

Next we introduce the symmetries that we allow. Throughout this paper, we denote a compact torus by $T$. Moreover, all actions are assumed to be effective, unless otherwise stated. On a positive monotone/symplectic Fano closed symplectic manifold $(M, \omega)$ we consider Hamiltonian $T$-actions, i.e., those for which there exists a $T$-invariant smooth map $\Phi : M \to (\text{Lie}(T))^*$, called moment map, such that, for all $\xi \in \text{Lie}(T)$, $\iota_{X_\xi} \omega = d\langle \Phi, \xi \rangle$, where $\text{Lie}(T)$ denotes the Lie algebra of $T$, $X_\xi \in \mathfrak{X}(M)$ is the vector field induced by $\xi$, and $\langle \cdot, \cdot \rangle$ is the natural pairing between $(\text{Lie}(T))^*$ and $\text{Lie}(T)$. A Hamiltonian $T$-space is a symplectic manifold $(M, \omega)$ endowed with an effective Hamiltonian $T$-action. Such a space, together with a choice of moment map $\Phi$, is denoted by $(M, \omega, T, \Phi)$. To make sense of when torus symmetries are ‘sufficiently large’, we introduce the following notion.

Definition 1.4. The complexity of a Hamiltonian $T$-space $(M, \omega, T, \Phi)$ is $\frac{1}{2} \dim M - \dim T$.

Intuitively, the smaller the complexity, the larger the symmetry. Moreover, a simple symplectic argument shows that the complexity of a Hamiltonian $T$-space is always non-negative. Henceforth, a Hamiltonian $T$-space of complexity $k$ is simply referred to as a complexity $k$ space. Complexity zero spaces are known as symplectic toric manifolds and it is known that positive monotone complexity zero spaces are $T$-equivariantly symplectomorphic to toric Fano varieties, i.e., Fano
varieties $Y$ endowed with a holomorphic $T_C$-action, where $T_C = T \otimes_R \mathbb{C}$ and $\dim_C Y = \dim_C T_C$.

This paper begins the study of the relation between positive monotone (respectively symplectic Fano) complexity one spaces and Fano varieties $Y$ equipped with an effective holomorphic $T_C$-action satisfying $\dim_C Y = \dim_C T_C + 1$. To the best of our knowledge, this is an unexplored problem except for low real dimensions, namely 2 and 4. While the two-dimensional case is not particularly interesting, a positive monotone 4-dimensional complexity one space is $S^1$-equivariant symplectomorphic to a del Pezzo surface (i.e., a Fano surface) endowed with a holomorphic $\mathbb{C}^*$-action. This can be proved using techniques that underpin the classification of closed Hamiltonian $S^1$-spaces in dimension 4 (see [15]).

The main result of this paper is the following:

**Main Result.** If $(M, \omega, T, \Phi)$ is a closed complexity one space that is either positive monotone, or symplectic Fano with respect to a compatible $T$-invariant almost-complex structure, then $M$ is simply connected, its odd Betti numbers vanish, and $(M, \omega)$ has Todd genus equal to 1.

**Remark 1.5** As mentioned above, Fano varieties are necessarily simply connected and have Todd genus equal to 1. It can be checked that a Fano variety $Y$ that is endowed with an effective holomorphic $T_C$-action satisfying $\dim_C Y = \dim_C T_C + 1$ has vanishing odd Betti numbers.

The above result is very much inspired by work of Fine and Panov [7, 8] and of Lindsay and Panov [21], and should be placed in the context of the broader question of studying the relation between closed positive monotone symplectic manifolds and Fano varieties in the presence of a torus action. Without assuming the existence of a Hamiltonian torus action, in real dimension 4 every closed positive monotone symplectic manifold is diffeomorphic to a del Pezzo surface, i.e., a Fano two-fold (see [25, 10, 29]). However, this need not hold in higher dimensions (see [7, 28] for a counterexample). However, in [8] Section 1.2 and 7] it is conjectured that a closed positive monotone manifold of real dimension 6 with a non-trivial Hamiltonian $S^1$-action must be diffeomorphic to a Fano threefold. In [21] Lindsay and Panov make some important steps towards proving the above conjecture as they show that, like Fano varieties, such a symplectic manifold is simply connected and its Todd genus is one. Moreover, under various additional hypotheses either on the topology of the manifold or on the type of the action, there is evidence that real six-dimensional positive monotone symplectic manifolds with a Hamiltonian $S^1$-action are either diffeomorphic or
$S^1$-equivariantly symplectomorphic to Fano three-folds endowed with a holomorphic $\mathbb{C}^*$-action (see [3, 4, 5, 8, 9, 21, 24, 31]).

Our Main Result should be compared with (some of) the main results in [21, 22]. If, on the one hand, the hypotheses in [21] are weaker than those of our Main Result, in that they deal with complexity two spaces, the results therein are specific to the real 6-dimensional case, whereas our result applies to all dimensions. Moreover, while we are able to conclude that in our case the odd Betti numbers vanish, the corresponding statement in the real 6-dimensional case with a Hamiltonian $S^1$-action only holds by imposing further mild conditions (see [22, Theorem 14.4]). (In fact, there exist complexity one Fano 3-folds with $b_3 > 0$, see [22, Example 14.8].) Finally, it is important to remark that the techniques in [21] are significantly more sophisticated than those used in this paper, seeing as, for instance, [21] uses Seiberg-Witten theory (cf. Section 4 below).

The proof of the above result comes from combining several well-known properties of closed complexity one spaces under the assumption that the space be ‘positive’ (see Definition 4.1 for details). Closed complexity one spaces as in the hypothesis of our Main Result satisfy this positivity condition (see Lemma 4.5). Our strategy is simple: we prove that the assumption of positivity on a closed complexity one space limits the topology of the connected components of the fixed point set of the action (see Theorem 4.5) and this is the key ingredient in the proof of our Main Result. To prove Theorem 4.5 we use the Duistermaat-Heckman function, the fact that its minimum need be attained at a vertex for closed complexity one spaces (see Theorem 3.7 and Corollary 3.8), and a topological restriction in the case in which the vertex that attains the minimum of this function is the image of a 2-dimensional component of the fixed point set (see Lemma 3.9).

The paper is structured as follows. In Section 2 we recall the basics of (closed) Hamiltonian $T$-spaces. Section 3 deals with (closed) complexity one spaces and its aim is to prove Lemma 3.9. While most results contained therein are standard, there are a few observations that we could not find elsewhere in the literature, including Lemma 3.9. Most (if not all) of the material in Sections 2 and 3 are probably well-known to experts, and it is included for completeness. The notion of positivity as well as the proof of our Main Result can be found in Section 4.
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2. Basic properties of (closed) Hamiltonian $T$-spaces

Throughout this section, given a compact torus $T$, its Lie algebra and the lattice therein are denoted by $t$ and $\ell = \ker(\exp : t \to T)$ respectively. The aim of this section is to recall a few fundamental facts about (closed) Hamiltonian $T$-spaces, i.e., symplectic manifolds endowed with an effective Hamiltonian $T$-action.

First, we recall the local normal form near fixed points of the $T$-action, which is a special case of a more general result due to Marle, and Guillemin and Sternberg (see [23, 12]). Given a Hamiltonian $T$-space $(M, \omega, T, \Phi)$, the set of fixed points of the action is denoted by $M^T$ and endowed with the subspace topology. Fix a Hamiltonian $T$-space $(M, \omega, T, \Phi)$, set $\dim M = 2n$, fix $p \in M^T$, and a $T$-invariant compatible almost complex structure $J$. Since $p$ is a fixed point, there is a $\mathbb{C}$-linear $T$-action on $T_p M$ that is isomorphic to a $T$-action on $\mathbb{C}^n$ determined by an injective homomorphism $\rho : T \to (S^1)^n$, where $(S^1)^n \subset \text{GL}(n; \mathbb{C})$ is the subgroup of diagonal matrices whose entries have norm one. The homomorphism $\rho$ is known as the (symplectic) slice representation.

**Definition 2.1.** Let $(M, \omega, T, \Phi)$ be a Hamiltonian $T$-space and let $p$ be a fixed point. The differential at the identity of the components of the slice representation determines elements $\alpha_1, \ldots, \alpha_n \in \ell^*$, called the isotropy weights of the $T$-action at $p$.

**Remark 2.2**

- Since the slice representation is injective, it follows that the $\mathbb{Z}$-span of the isotropy weights at a fixed point is $\ell^*$.
- Since both $T$ and $(S^1)^n$ are connected, the isotropy weights $\alpha_1, \ldots, \alpha_n$ (up to permutation) determine the slice representation $\rho$.
- Let $F \subset M^T$ be a connected component. Then for any $p, p' \in F$ the isotropy weights of the $T$-action at $p$ and $p'$ are equal. Thus it makes sense to talk about the isotropy weights of the $T$-action at $F$. 
Suppose that $\alpha_1, \ldots, \alpha_n \in \mathfrak{t}^*$ are the isotropy weights at $p$. Endowing $\mathbb{C}^n$ with the standard symplectic structure $\omega_{\text{st}}$, the linear $T$-action on $\mathbb{C}^n$, determined by $\alpha_1, \ldots, \alpha_n$ as above, is Hamiltonian and one of the moment maps is given by

\begin{equation}
\Phi_{\text{lin}}(z_1, \ldots, z_n) = \frac{1}{2} \sum_{j=1}^{n} \alpha_j |z_j|^2 + \Phi(p).
\end{equation}

We henceforth refer to the above Hamiltonian $T$-space as the \textit{linear model} at the fixed point $p$. The following result is a (very!) particular case of the local normal form for Hamiltonian group actions by compact Lie groups due to Marle, Guillemin and Sternberg (see [12, 23]).

\textbf{Theorem 2.3} (Local normal form at fixed points). Let $(M, \omega, T, \Phi)$ be a Hamiltonian $T$-space and let $p \in M^T$ be a fixed point. Then there exist $T$-invariant open neighborhoods $U \subset M$ and $V \subset \mathbb{C}^n$ of $p$ and $0$ respectively, and a $T$-equivariant symplectomorphism $\Psi : (U, \omega) \to (V, \omega_{\text{st}})$ such that $\Phi = \Phi_{\text{lin}} \circ \Psi$, where $\Phi_{\text{lin}}$ is as in (2.1).

Next we state without proof the following basic, yet important, result (see [26, Lemma 5.53]).

\textbf{Lemma 2.4}. Let $(M, \omega, T, \Phi)$ be a Hamiltonian $T$-space. For any closed $H \subset T$, each connected component of the set of points that are fixed by $H$ is a symplectic submanifold of $(M, \omega)$.

We conclude this section by recalling two results (without proof) concerning \textit{closed} Hamiltonian $T$-spaces. The first one is the well-known milestone due to Atiyah, Guillemin and Sternberg (see [1, 11]).

\textbf{Theorem 2.5}. Let $(M, \omega, T, \Phi)$ be a closed Hamiltonian $T$-space. Then the fibers of $\Phi$ are connected and $\Phi(M) \subset \mathfrak{t}^*$ is the convex hull of the image of the connected components of $M^T$.

\textbf{Remark 2.6} Observe that, under the hypotheses of Theorem 2.5, $M^T$ has finitely many connected components. Thus the moment map image of a closed Hamiltonian $T$-space is a convex compact polytope in $\mathfrak{t}^*$.

The last result is a special case of a theorem of Li (see [20]).

\textbf{Theorem 2.7}. Let $(M, \omega, T, \Phi)$ be a closed Hamiltonian $T$-space. For any $\alpha \in \Phi(M)$, $\pi_1(M) \cong \pi_1(M_\alpha)$, where $M_\alpha = \Phi^{-1}(\alpha)/T$ denotes the reduced space at $\alpha$. 
3. SOME PROPERTIES OF (CLOSED) COMPLEXITY ONE SPACES

In this section, we specialize to (closed) complexity one spaces and we prove results that are needed in the proofs of Section 4. Throughout this section, \((M,\omega,T,\Phi)\) denotes a complexity one space unless otherwise stated.

3.1. Fixed surfaces. We begin by showing that the complexity of \((M,\omega,T,\Phi)\) being one restricts the possible dimensions of the connected components of \(M^T\).

**Corollary 3.1.** Let \((M,\omega,T,\Phi)\) be a complexity one space. The connected components of \(M^T\) are either points or symplectic surfaces.

**Proof.** Let \(F \subset M^T\) be a connected component and fix \(p \in F\). By Lemma 2.4, we know that \(F\) is a symplectic submanifold so we only have to bound its dimension. By Theorem 2.3, it suffices to consider the linear model, i.e., \(p = 0 \in \mathbb{C}^n\) and the \(T\)-action is given by an injective representation \(\rho : T \to (S^1)^n\). Observe that \(T_pF = \bigcap_{t \in T} \ker(id - \rho(t))\) (see [26, proof of Lemma 5.53]). Since \(\rho\) is injective and \(n = \dim T + 1\), it follows that \(\dim T_pF\) equals either 0 or 2.

Next we deduce properties of the linear model at (and, hence, of the local behavior of the \(T\)-action near) a fixed point lying on a 2-dimensional connected component of \(M^T\). Such a connected component is henceforth referred to as a fixed surface. Fix such a surface \(\Sigma \subset M^T\) and a point \(p \in \Sigma\). As in the proof of Corollary 3.1, assume, without loss of generality, that \(M = \mathbb{C}^n\), \(p = 0\) and the \(T\)-action is given by an injective representation \(\rho : T \to (S^1)^n\). Since \(p\) lies on a fixed surface, it follows that one of the isotropy weights of the action is 0, i.e., the \(T\)-action fixes a complex line in \(\mathbb{C}^n\). Therefore, there exists an isomorphism \(\mathbb{C}^n \cong \mathbb{C} \times \mathbb{C}^{n-1}\) so that for all \(t \in T\) and all \((z,w) \in \mathbb{C} \times \mathbb{C}^{n-1}\),

\[
(3.1) \quad t \cdot (z,w) = (z,\hat{\rho}(t)w)
\]

for some isomorphism \(\hat{\rho} : T \to (S^1)^{n-1}\). In what follows, we ignore the zero isotropy weight at \(p\) and refer to \(\alpha_1,\ldots,\alpha_{n-1} \in \ell^*\) as the isotropy weights at \(p\).

**Remark 3.2** In the above description, for any \(z_0 \in \mathbb{C}\), the subspace \(\{z_0\} \times \mathbb{C}^{n-1}\) is a symplectic subspace of \(\mathbb{C} \times \mathbb{C}^{n-1}\) whose induced symplectic form is denoted by \(\omega_0\) (and is symplectomorphic to the standard symplectic form on \(\mathbb{C}^{n-1}\)). Let \(\Phi_0\) denote the moment map of the
effective Hamiltonian $T$-action on $(\{z_0\} \times \mathbb{C}^{n-1}, \omega_0)$ given by restricting the $T$-action on $\mathbb{C}^n$. The complexity of the Hamiltonian $T$-space $(\{z_0\} \times \mathbb{C}^{n-1}, \omega_0, \Phi_0)$ is zero, i.e., $T$ acts on $(\{z_0\} \times \mathbb{C}^{n-1}, \omega_0)$ in a toric fashion.

In the following simple result, we use the above discussion in the case in which $(M, \omega, T, \Phi)$ is a closed complexity one space.

**Lemma 3.3.** Let $(M, \omega, T, \Phi)$ be a closed complexity one space and let $\Sigma \subset M^T$ be a fixed surface. Then $\Sigma$ is the preimage of a vertex of $\Phi(M)$.

**Proof.** Compactness of $M$ implies compactness of $\Sigma$ as it is closed. Fix $p \in \Sigma$. Theorem 2.3 and the above discussion imply that there exists an open neighborhood of $U_p$ of $p$ such that $\Phi(U)$ is the image of an open neighborhood $V_p$ of $0 \in \mathbb{C}^n \simeq \mathbb{C} \times \mathbb{C}^{n-1}$ under the map

\[
\Phi_{\text{lin}}(z, w_1, \ldots, w_{n-1}) = \frac{1}{2} \sum_{j=1}^{n-1} \alpha_j |w_j|^2 + \Phi(p),
\]

(cf. formula (2.1)). Since the isotropy weights at a fixed surface are well-defined (see Remark 2.2), the above statement holds for all $p \in \Sigma$. Since $\Sigma$ is compact, there exists finitely many $p_1, \ldots, p_N$ such that $\Sigma$ is covered by $U_{p_1}, \ldots, U_{p_N}$. Set $U := \bigcup_{l=1}^N U_{p_l} \subset M$ and $V := \bigcap_{l=1}^N V_{p_l} \subset \mathbb{C}^n$. Observe that $V$ is a $T$-invariant open neighborhood of $\mathbb{C}^n \simeq \mathbb{C} \times \mathbb{C}^{n-1}$ and that $\Phi_{\text{lin}}(V)$ is an open subset of $\Phi(\Sigma) + \mathbb{R}_{\geq 0} \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$. Moreover, by construction, $\Phi_{\text{lin}}(V) \subset \Phi(U)$. Since $\Phi$ is proper, it follows that $U$ contains an open neighborhood of $\Sigma$ saturated by the fibers of $\Phi$ which are connected. Hence, possibly changing $U$ with this open neighborhood $U'$ saturated by $\Phi$, the vertex of $\Phi(U')$, whose preimage is $\Sigma$, must be a vertex of $\Phi(M)$. \qed

**Remark 3.4** Lemma 3.3 can be generalized to closed complexity $k$ spaces by substituting fixed surfaces with connected components of the fixed point set of dimension $2k$. However, in complexity $k \geq 2$, it is not true that 2-dimensional connected components of the fixed point set are either level sets of the moment map or that they are contained only in the preimages of vertices of the moment map image.

Using Lemma 3.3, given a closed complexity one space $(M, \omega, T, \Phi)$, we characterize the preimage of an edge of $\Phi(M)$ that is incident to a vertex whose preimage is a fixed surface. More precisely, the following result holds.
**Lemma 3.5.** Let \((M, \omega, T, \Phi)\) be a closed complexity one space and suppose that a vertex \(v \in \Phi(M)\) is the image of a fixed surface. Then the preimage of any closed edge incident to \(v\) is a closed 4-dimensional symplectic submanifold of \((M, \omega)\) endowed with an effective Hamiltonian \(S^1\)-action.

**Proof.** Set \(\Sigma := \Phi^{-1}(v)\). The proof of Lemma 3.3 shows that the image of \(\Phi\) near \(v\) is contained in \(\Phi(\Sigma) + \mathbb{R}_{\geq 0}(\alpha_1, \ldots, \alpha_{n-1})\), where \(\alpha_1, \ldots, \alpha_{n-1}\) are the isotropy weights at the fixed surface \(\Sigma\). Let \(e_1, \ldots, e_{n-1}\) denote the closed edges incident to \(v\) so that, for all \(i = 1, \ldots, n - 1\), \(e_i \subset \Phi(\Sigma) + \mathbb{R}_{\geq 0}\alpha_i\). Fix \(i = 1, \ldots, n - 1\) and a point \(p \in \Sigma\). Using the local normal form of Theorem 2.3, together with the discussion preceding Lemma 3.3, we may identify an open neighborhood \(U\) of \(p\) with an open neighborhood of \(0 \in \mathbb{C}^n \cong \mathbb{C} \times \mathbb{C}^{n-1}\) with \(T\)-action and moment map given as in (3.1) and (3.2) respectively. Under this identification, \(\Phi^{-1}(e_i) \cap U\) is given by the 4-dimensional subspace
\[
\{(z, w_1, \ldots, w_{n-1}) \mid w_1 = 0, \ldots, w_{i-1} = 0, w_{i+1} = 0, \ldots, w_{n-1} = 0\}.
\]
Moreover, setting \(h_i := \ker \alpha_i\) and \(H_i := \exp(h_i)\), we have that \(q \in \Phi^{-1}(e_i) \cap U\) if and only if it is fixed by \(H_i\). (Note that the weights \(\alpha_1, \ldots, \alpha_{n-1}\) are linearly independent since their \(\mathbb{Z}\)-span must be \(\ell^*\), otherwise the effectiveness of the \(T\)-action would be contradicted).
Since \(\Phi^{-1}(e_i)\) is connected (as the fibers of \(\Phi\), and \(e_i\), are), Lemma 2.4 implies that \(\Phi^{-1}(e_i) =: M_i\) is a 4-dimensional symplectic submanifold of \((M, \omega)\) whose points are fixed by \(H_i\). Moreover, \(M_i\) is closed as it is the preimage of a compact subset under the proper map \(\Phi\). Set \(\omega_i := \omega|_{M_i}\). It remains to show that \((M_i, \omega_i)\) is endowed with an effective Hamiltonian \(S^1\)-action. Since \(\alpha_1, \ldots, \alpha_{n-1} \in \ell^*\) are a basis (see Remarks 2.2 and 3.2), consider the dual basis \(a_1, \ldots, a_{n-1} \in \ell\). By construction, \(\exp((a_i)) \subset T\) is isomorphic to \(S^1\) and it acts in a Hamiltonian fashion on \((M_i, \omega_i)\). To check that this action is effective, observe that, by construction, it is effective locally near \(p\); this can be checked directly in the linear model at \(p\). \(\square\)

We conclude our discussion of fixed surfaces of closed complexity one spaces with a simple observation regarding the case in which there is one with positive genus.

**Lemma 3.6.** Let \((M, \omega, T, \Phi)\) be a closed complexity one space. If there exists a fixed surface \(\Sigma_0 \subset M^T\) whose genus \(g(\Sigma_0)\) is positive, then, for any vertex \(v \in \Phi(M)\), \(\Phi^{-1}(v)\) is a fixed surface of genus \(g(\Sigma_0)\).

**Proof.** By Lemma 3.3, \(\Phi(\Sigma_0)\) is a vertex of \(\Phi(M)\), say \(v_0\). Then, by Theorem 2.7, \(\pi_1(M) \cong \pi_1(M_{v_0}) \cong \pi_1(\Sigma_0)\), where \(M_{v_0}\) is the reduced
space at \( v_0 \). By assumption, \( \pi_1(\Sigma_0) \) is not trivial. Since the first isomorphism above holds for any vertex \( v \) of \( \Phi(M) \), it follows that the preimage of any other vertex is a fixed surface whose genus equals that of \( \Sigma_0 \) as desired.

\[ \square \]

### 3.2. The Duistermaat-Heckman function and its minimum.

Let \( DH : \Phi(M) \to \mathbb{R} \) denote the Duistermaat-Heckman function associated to a closed complexity one space \( (M, \omega, T, \Phi) \), namely \( DH(\alpha) \) is the symplectic volume of the reduced space at \( \alpha \) (see [6]). First, we state the following result due to Cho and Kim without proof (see [2]).

**Theorem 3.7.** The Duistermaat-Heckman function of a closed complexity one space is log-concave, i.e., \( \log DH \) is a concave function.

Combining Theorems 2.5 and 3.7, we obtain the following:

**Corollary 3.8.** The minimum of the Duistermaat-Heckman function of a closed complexity one space is attained at a vertex of the moment map image.

**Proof.** Let \( (M, \omega, T, \Phi) \) be a closed complexity one space and let \( DH : \Phi(M) \to \mathbb{R} \) denote its Duistermaat-Heckman function. Theorem 3.7 asserts that \( \log DH \) is concave. Thus, to prove the result, it suffices to show that \( \log DH \) attains its minimum at a vertex of \( \Phi(M) \). This follows at once by convexity of \( \Phi(M) \) (see Theorem 2.5), since a concave function on a compact convex polytope must attain its minimum at a vertex.

The next result describes a topological restriction on a fixed surface whose image corresponds to the minimum of the Duistermaat-Heckman function (see Lemma 3.3 and Corollary 3.8).

**Lemma 3.9.** Let \( (M, \omega, T, \Phi) \) be a closed complexity one space and let \( v \in \Phi(M) \) be a vertex that attains the minimum of the Duistermaat-Heckman function \( DH \). If \( \Phi^{-1}(v) \) is a fixed surface, then

\[
\begin{align*}
  c_1(N)[\Phi^{-1}(v)] & \leq 0, \\
\end{align*}
\]

where \( N \) and \( c_1(N) \) denote the normal bundle to \( \Phi^{-1}(v) \) and its first Chern class respectively.

**Proof.** Set \( \Sigma := \Phi^{-1}(v) \) and fix a \( T \)-invariant almost complex structure \( J \). Let \( \alpha_1, \ldots, \alpha_{n-1} \in \ell^* \) be the isotropy weights of \( \Sigma \) (see Definition 2.1 and Remark 2.2). Since \( \alpha_1, \ldots, \alpha_{n-1} \) form a basis of \( \ell^* \) (see Remarks 2.2 and 3.2), the normal bundle \( N \) to \( \Sigma \) splits as the sum of \( n - 1 \) complex line bundles \( L_1, \ldots, L_{n-1} \), each \( L_i \) corresponding to exactly one \( \alpha_i \in \ell^* \), for \( i = 1, \ldots, n - 1 \). By additivity of the first Chern class.
class, the result is proved if we show that, for all $i = 1, \ldots, n-1$, $c_1(L_i)[\Sigma] \leq 0$.

Let $e_1, \ldots, e_{n-1}$ denote the (closed) edges of $\Phi(M)$ incident to $v$ so that, for all $i = 1, \ldots, n-1$, $e_i \subset v + \mathbb{R}_{\geq 0} \alpha_i$. For any $i = 1, \ldots, n-1$, $L_i$ is the normal bundle of $\Sigma$ inside the closed 4-dimensional symplectic submanifold $(M_i, \omega_i)$ of $M$ given by $\Phi^{-1}(e_i)$ and $\omega_i = \omega|_{M_i}$. By Lemma 3.5, for each $i = 1, \ldots, n-1$, there exists an injective homomorphism $\chi_i: S^1 \to T$ such that $\Phi_i := \chi_i^* \circ \Phi$ is the moment map of a Hamiltonian $S^1$-action on $(M_i, \omega_i, \Phi_i)$, where $\chi_i^*: \text{Lie}(T)^* \to \mathbb{R} \cong \text{Lie}(S^1)^*$ is the homomorphism induced by $\chi_i$. Fix $i = 1, \ldots, n-1$ and, without loss of generality, suppose that $\Phi_i(\Sigma) = 0$ and $\Phi_i(M_i) \subset \mathbb{R}_{\geq 0}$. For all $t > 0$ sufficiently small, the reduced space $\Phi_i^{-1}(t)/S^1$ is symplectomorphic to the reduced space $\Phi^{-1}(v + t\alpha_i)/T$, since, by construction, $\Phi_i^{-1}(v + t\alpha_i) = \Phi_i^{-1}(t)$. Since $v$ is assumed to be a minimum of the Duistermaat-Heckman function for $(M, \omega, T, \Phi)$, it follows that the Duistermaat-Heckman function $DH_i$ for $(M_i, \omega_i, \Phi_i)$ is a non-decreasing function in the interval $(0, t_0)$, for $t_0 > 0$ sufficiently small. Using [15, Lemma 2.12], it follows that $c_1(L_i)[\Sigma] \leq 0$. Since $i = 1, \ldots, n-1$ is arbitrary, this proves the desired result.

4. Positivity and the proof of our main result

The aim of this section is to provide a proof of our Main Result by showing that its conclusions hold if the closed complexity one space is assumed to be ‘positive’ in the following sense.

Definition 4.1. A closed complexity one space $(M, \omega, T, \Phi)$ is positive if, for any fixed surface $\Sigma \subset M^T$, $c_1[\Sigma] > 0$, where $c_1$ is the first Chern class of $(M, \omega)$.

Example 4.2 If all the connected components of $M^T$ of a closed complexity one space $(M, \omega, T, \Phi)$ are isolated fixed points then $(M, \omega, T, \Phi)$ is positive. This can be used to show that positive does not imply positive monotone or symplectic Fano in the sense of Definition 1.1. For instance, it is not hard to construct a closed complexity one space of dimension four all of whose fixed points are isolated that has a symplectic sphere of self-intersection equal to -2, and which is $J$-holomorphic with respect to an $S^1$-invariant compatible almost complex structure. The existence of such a sphere prevents the symplectic manifold from being positive monotone or symplectic Fano.

The next result illustrates why we introduce positivity.
Lemma 4.3. A closed complexity one space that is either positive monotone, or symplectic Fano with respect to a compatible $T$-invariant almost-complex structure, is positive.

Proof. Since fixed surfaces are symplectic submanifolds (see Lemma 2.4), it follows that a positive monotone closed complexity one space is positive. On the other hand, suppose that a closed complexity one space is symplectic Fano with respect to a $T$-invariant compatible almost complex structure. By [26, Proof of Lemma 5.53] it follows that any fixed surface of $(M, \omega, T, \Phi)$ is $J$-invariant and, therefore, $J$-holomorphic as it is two-dimensional. □

Using Lemma 4.3 our Main Result is a simple consequence of the following:

Theorem 4.4. Let $(M, \omega, T, \Phi)$ be a positive closed complexity one space. Then $M$ is simply connected, its odd Betti numbers vanish and the Todd genus of $(M, \omega)$ equals one.

Thus it remains to prove Theorem 4.4. To this end, we first prove the following:

Theorem 4.5. Let $(M, \omega, T, \Phi)$ be a positive closed complexity one space. The connected components of $M^T$ are either points or spheres.

Remark 4.6 By Corollary 3.1 the statement of Theorem 4.5 is equivalent to the following:

The connected components of the fixed point set of a positive closed complexity one space are simply connected.

Proof of Theorem 4.5. Suppose that the statement does not hold. By Corollary 3.1 $M^T$ contains a fixed surface of positive genus $g$. By Lemma 3.6 the preimage of any vertex of $\Phi(M)$ is a fixed surface of genus $g$. Fix a vertex $v \in \Phi(M)$ and let $\Sigma \subset M^T$ be its preimage under $\Phi$. If $N$ denotes the normal bundle to $\Sigma$, then the first Chern class of $N$ must satisfy $c_1(N)[\Sigma] > 0$, for

\[ c_1(N)[\Sigma] = c_1[\Sigma] - c_1(\Sigma)[\Sigma] = c_1[\Sigma] + 2g - 2 > 0, \]

where $c_1(\Sigma)$ is the first Chern class of the tangent bundle to $\Sigma$, and the inequality follows by positivity of $(M, \omega, T, \Phi)$ and $g > 0$.

To derive a contradiction we use the Duistermaat-Heckman function $DH$. By Corollary 3.8 the minimum of $DH$ is attained at a vertex $m$ of $\Phi(M)$. However, this is impossible by Lemma 3.9. □
Remark 4.7 Following [17, 18], we say that a closed complexity one space is tall if all its reduced spaces are two-dimensional topological manifolds. If a closed complexity one space \((M, \omega, T, \Phi)\) is not tall, [18, Corollary 2.4] states that the set of \(\alpha \in \Phi(M)\) such that \(M_\alpha = \Phi^{-1}(\alpha)/T\) is a point is the union of closed faces. Take one such closed face \(\Delta \subset \Phi(M)\); since it is closed, it contains a vertex, say \(\alpha\). Arguing as above and using Theorem 2.7, we have that the fundamental group of any reduced space is trivial and, therefore, all connected components of \(M^T\) are simply connected (see Lemma 3.3). Therefore, Theorem 4.5 holds without the positivity assumption if \((M, \omega, T, \Phi)\) is not tall. As such, Theorem 4.5 should be seen as a statement about tall closed complexity one spaces (classified in [16, 17, 18]) under the assumption of positivity in the sense of Definition 4.1.

We conclude this section by proving Theorem 4.4 and, consequently, our Main Result.

Proof of Theorem 4.4. Let \((M, \omega, T, \Phi)\) be a positive closed complexity one space. By Theorem 4.5, the connected components of \(M^T\) are simply connected. Therefore, arguing as above, we have that the reduced space at any vertex of \(\Phi(M)\) is simply connected. Using Theorem 2.7, simple connectedness of \(M\) follows.

To prove the remaining statements, choose a generic \(S^1 \subset T\) with the property that \(M^{S^1} = M^T\) (this can be done because \(M\) is compact), and let \(F_1, \ldots, F_N\) be the connected components of \(M^{S^1} = M^T\). To prove that the Todd genus equals one, we use the same techniques as in [21, Corollary 1.4]; the argument is included below for completeness. Recall the following formula (see [13, Section 5.7]):

\[
\chi_y(M) = \sum_{j=1}^N (-y)^{d_j} \chi_y(F_j),
\]

where \(\chi_y\) is the Hirzebruch genus and, for all \(j = 1, \ldots, N\), \(d_j\) is the number of negative isotropy weights for the \(S^1\)-action (counted with multiplicity) at \(F_j\) (see Definition 2.1 and Remark 2.2). Observe that, if \(F_j\) is the component corresponding to the minimum of the moment map of the \(S^1\)-action, then \(\Phi(F_j)\) is a vertex of \(\Phi(M)\). By Theorem 4.5, \(F_j\) is necessarily a sphere or a point and, in both cases, the Todd genus is one. Evaluating (4.2) at \(y = 0\) and observing that \(\chi_0\) is precisely the Todd genus (by definition of the generating functions of the Hirzebruch and Todd genera, see [13, Sections 1.8 and 5.4]), we obtain that the Todd genus of \(M\) is 1 as desired.
To see that the odd Betti numbers vanish, we combine Theorem 4.5 with the well-known formula

\[(4.3) \quad H^\ast (M; \mathbb{R}) = \bigoplus_{j=1}^{N} H^{* - 2d_j} (F_j; \mathbb{R}),\]

where \(d_j\) is as above. Formula (4.3) is a consequence of the fact that the moment map for the \(S^1\)-action is perfect Morse-Bott (see [19]). □

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