Subminimal Paths on a Stochastic Graph

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Abstract
A simple model of a frustrated disordered system is presented. Apart from the (very different) physical interpretation, the model shares many features with that of Sherrington-Kirkpatrick for spin glasses, but, as a consequence of its relative simplicity, its ground state can be exactly determined by numerical methods. This fact allows us to test experimentally some theoretical predictions, based on a specialization of the “cavity method” developed for the SK model, which is presently limited to a “non-frustrated” approximation, corresponding to some extent to the replica-symmetric one for the SK model.
1. Introduction

The Sherrington-Kirkpatrick model for spin glasses ([1], SK model in the following) was proposed in 1975 as a simple and exactly solvable model of a disordered frustrated system. This is a model with very long range interaction, for which the mean field theory is supposed to hold exactly, and therefore it should be easy to handle. But the hopes of the authors were soon disappointed: the model is exactly solvable, but its solution turned to be much more complex than what initially thought. Only in 1980 the work of Parisi [2] and the complicated and mysterious “breaking of the replica symmetry” solved the problem, and only a few years later the so called cavity method [3,4,5,6] revealed the physics hidden behind the intricate mechanism of the replica formalism.

Two features appear to be essential in the model, both linked to a highly non-trivial decomposition of the Boltzmann measure in pure states:
1) the pure states are organized in an ultrametric scheme [3,4];
2) the free energies at each level of this organization are Poissonian random variables with exponential density, and the levels of organization are connected through a probability cascade à la Ruelle [5,7].

It is not yet clear whether this picture can be applied to other frustrated disordered models, such as 3-dimensional spin glasses with nearest neighbours interaction [8,9].(*) For this reason, we have considered a model which preserves many features of the SK model, but is much simpler to be analyzed numerically. Our hope is twofold: to clarify if and how one can apply the same “ultrametric” picture as the SK model, and to study the nearest neighbours interaction case, taking advantage of a better numerical tractability. This letter represents a first step in this direction.

The outline of the paper is as follows. In section 2 we define the model, in section 3 we clarify the concept of frustration in the context of our model, in section 4 we propose two approximate analytical solutions of the model, in section 5 we show some numerical results, as a test for the analytical ones.

2. Definition of the model

We consider a set of points \( V \), connected through links of a given length, defining the distance between each pair of points of \( V \). We start by stating the problem with the greatest generality, with the points of \( V \) not necessarily distributed in the real space, and their relative distances not necessarily being the euclidean ones. To be precise, let us recall a few definitions. A \emph{weighted graph} is a triple \( G = (V, A, d) \) where \( V \) is a finite set of \emph{nodes} or \emph{vertices}, \( A \subset V \times V \) is the set of the \emph{arcs} (i.e. the links) which connect the nodes of \( V \) and the map \( d : A \rightarrow \mathbb{R}^+ \) gives the length (or weight) of an arc, i.e. the distance between

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(*) Models like the REM and the GREM of Derrida [10,11], though very instructive, are rather artificial.
the nodes of an (ordered) pair. A path of \( n \) steps in the graph starting from \( u \) and leading to \( v \) is a sequence of \( n + 1 \) nodes \( \{u_i\}_{i=0}^{n} \) such that

\[
    u_0 = u, \quad u_n = v, \quad u_i \neq u_j \quad \forall i, j = 0, \ldots, n \quad i \neq j
\]

\[
    (u_i, u_{i+1}) \in A \quad \forall i = 0, \ldots, n - 1
\]

(1)

Let now \( C_{u,v} \) be the set of the paths joining \( u \) to \( v \), irrespective of the number of their steps. We suppose that the graph is connected, i.e. that there is at least one path joining \( u \) to \( v \). The length of paths in \( C_{u,v} \) is represented by a map \( l \) defined on \( C_{u,v} \) in a natural way, i.e. \( l : C_{u,v} \to \mathbb{R}^+ \) such that

\[
    c \in C_{u,v}, \quad c = \{u_i\}_{i=0}^{n}, \quad l[c] \equiv \sum_{i=0}^{n-1} d(u_i, u_{i+1})
\]

(2)

Our system has phase space

\[
    \Omega = C_{s,t}
\]

with \( s \) and \( t \) fixed in advance, and Hamiltonian

\[
    \mathcal{H} = l
\]

The ground state of the system is the shortest path in \( G \) between \( s \) and \( t \). Using a standard algorithm, the Dijkstra’s algorithm (see for instance [12] and references therein), it can be determined in a time which grows not too fast with the size of the problem, for any realization of the distance matrix \( d(u,v) \). On the contrary, to find the ground state of the SK model is a so called \( NP \)-complete problem (see [12] and also [13]) and therefore it is very unlike that a “good” algorithm can be found for it.

The distances \( d(u,v) \) play the same role of the couplings \( J_{ij} \) among the spins in a generic model of interacting Ising spins. In the case of our interest they will be random variables and will represent the “disorder” of the model. How frustration arises in this context will be discussed in the next section. As anticipated, our main interest in studying the problem is to gain informations about the behaviour of disordered and frustrated systems, in a context which has a rather rich structure, but nevertheless is simpler to handle than spin systems. Apart from this rather abstract motivation, the study of our problem could be useful in the framework of high-temperature expansion for disordered systems of interacting Ising spins. Moreover, also the problem of directed polymers in a random medium (see, for some recent results, the work in [14] and references therein) could be stated as a particular case of our problem. In fact, a directed polymer is a path in a particular graph which has a privileged direction of motion: that is to say, with the previously introduced notation, that if the link \( (u,v) \in A \) points in the privileged direction then \( (v,u) \notin A \) (the reversed link is absent). With our notation the random medium is represented by a distance matrix written in terms of a single random function \( \phi \), the local disorder,

\[
    d(u,v) = \frac{\phi(u) + \phi(v)}{2}
\]
so that equation (2) for the length of a path $c$ of $n$ steps becomes

$$l[c] = \sum_{i=0}^{n} \phi(u_i) - \frac{\phi(u_0) + \phi(u_n)}{2}$$

which is the usual potential energy term in the Hamiltonian of a polymer in a local disorder field, apart from a contribution vanishing in the limit $n \gg 1$.

The model we shall consider in the following is defined by:

1) $(u, v) \in A$ for every $u, v \in V$, with $u \neq v$;
2) the $d(u, v)$’s are independent identically distributed random variables (with $d(u, v) = d(v, u)$).

Statements 1) and 2) are the counterparts of the defining characterization of the SK model, the ones by which mean field theory turns out to hold exactly. This analogy makes us confident that one could find an exact solution for our model, although the meaning of “mean field theory” in our context is not clear. Moreover, the fact that from a numerical point of view the problem is much simpler to deal with than the SK model, lead us to hope that the same happens also in an analytical approach. We did not succeed in finding such an exact solution, nevertheless thanks to the 1) and 2) above we can obtain an approximate solution (see section 4) which fits well enough the numerical data (see section 5).

We choose as probability distribution for the $d$

$$p_\alpha(x) = (\alpha + 1) \cdot x^\alpha, \quad x \in [0, 1]$$

In the analytical computation of section 4, $\alpha$ remains a free parameter, whereas in the numerical simulation of section 5 we fix, just for concreteness, $\alpha = 0$ and $\alpha = 1$, i.e. we consider, respectively, uniform and linear probability distribution of the distances.

3. Frustration

It is not immediate to understand how the usual concept of frustration, developed in the framework of spin systems, applies to this context. What naturally characterizes frustration, in a context-free way, is the existence of a lot of local minima, hampering the selection of the global one. Although in our model we can not precisely identify the local minima, we propose a generic definition of frustration, which easily implies that our model is frustrated.

It is now useful to introduce another model, structurally similar to ours. Let us recall that a tree is a connected graph without cycles and a spanning tree of a graph is a tree which has the same set of nodes as the graph. The phase space of this second model is the set $ST$ of all the spanning trees of $G$, and the Hamiltonian is the map $weight \ w : ST \rightarrow \mathbb{R}^+$ naturally defined by

$$T \in ST, \ T = (V, A_T), \ w[T] \equiv \sum_{(u,v) \in A_T} d(u,v)$$
In the following we shall call \textit{minimal spanning tree} (MST) the ground state of this system. As a matter of fact, the Dijkstra’s algorithm, which determines the shortest path from \( s \) to \( t \), gives also the shortest path from \( s \) to any other \( u \in V \). These paths form a tree, the so called \textit{shortest path tree} (SPT). Thus, to a certain extent, the SPT problem and the MST problem are similar, as their solutions are both represented by trees on the graph (they do not coincide, however, as it is easily shown by a counterexample). On the other hand the algorithms which solve them are rather different. The Dijkstra’s algorithm is somewhat tricky. On the contrary, the algorithm which solves the MST problem is much simpler:

\[
\text{MST} := \emptyset \; ; \; \; A' := A \; ; \\
\text{repeat} \\
\quad \text{let a the arc in } A' \text{ of minimal length} \; ; \\
\quad A' := A' \setminus \{a\} \; ; \\
\quad \text{if } \text{MST} + a \text{ is a tree then } \text{MST} := \text{MST} + a \; ; \\
\text{until } A' = \emptyset ;
\]

(5)

This algorithm is called \textit{greedy} for obvious reasons.

Systems for which the greedy algorithm to find the ground state works are called \textit{matroids}. Though this seems to be an informal characterization, it turns to be equivalent to a purely algebraic one (see [12]). For matroids there exists a natural way to give a notion of neighbourhood between the points in the phase space, that for the spanning trees of a graph results in:

\[
\text{If } T \in \text{ST} \text{ then the neighbours of } T \text{ are all the } T' \in \text{ST} \text{ obtained from } T \text{ as follows: add an arc to the tree } T, \text{ producing a cycle; then delete one other arc on the cycle.}
\]

(6)

If we consider the local minima of the Hamiltonian \( w \) with respect to this “topology”, it is an amusing exercise [12, exercise 3 chap. 1] to show that, apart from accidental degenerations, there exists a unique local minimum, which therefore is a global one. Thus the following \textit{steepest descent} algorithm is equivalent to the greedy one, in order to determine the MST:

\[
\text{let } T \in \text{ST} \; ; \\
\text{repeat} \\
\quad \text{let } T' \text{ the neighbour of } T \text{ of minimal weight} \; ; \\
\quad \text{if } T' \neq T \text{ then } T := T' \; ; \\
\text{until } T' = T \; ; \\
\text{MST} := T ;
\]

(7)

Now, it seems natural to consider the matroids as the archetypical example of a non-frustrated system, in contrast with the frustrated ones, where, as it is well known, the huge number of local minima causes the steepest descent method to fail. We remark that, to adapt the greedy algorithm to a spin system, it is necessary to orient subsequently the spins of the system, so that the couplings with the already oriented ones are satisfied. It is not difficult to make the above procedure formal, and to verify that it works if and only if for every \( n \)-tuple of lattice sites \( \{i_1, \ldots, i_n\} \) the following condition holds:

\[
J_{i_1 i_2} \cdot J_{i_2 i_3} \cdot \ldots \cdot J_{i_n i_1} \geq 0 ,
\]
where $J_{ij}$ is the coupling constant between the spin $\sigma_i$ and the spin $\sigma_j$. This is nothing but the usual definition of absence of frustration. In conclusion, two facts lead us to conjecture that matroids are the most general non-frustrated systems:

- there exists a unique local minimum, which therefore is a global one;
- the spin systems which are matroids are the non-frustrated ones, in the usual meaning. From this point of view our model is frustrated, and the MST problem becomes its non-frustrated counterpart. To prove that our system is frustrated it is sufficient to show that it is not a matroid, and this can be done in two ways: either by its algebraic characterization, or by showing with a counterexample that to be greedy does not work. Roughly speaking, in order to find the shortest path between $s$ and $t$, it is not sufficient in general to proceed naively, i.e. by choosing always the shortest arc when going out from a node.

In the following, we shall fix our attention on the shortest path between two fixed nodes in a graph, which we denote with $s$ and $t$. The size of the graph, i.e. the number of its nodes, will be denoted by $N$, and we shall be interested in the asymptotic properties in the limit $N \to \infty$. Moreover, as it will be clear in the following, we shall study, together with the shortest path, also the paths in the graph between $s$ and $t$ which are next to the shortest one, i.e. we order the paths by increasing length, and we shall be interested not only in the first one, but also in the second one, the third one, and so on. These will be called “subminimal” or “suboptimal” paths, whereas the shortest path is the minimal, i.e. the optimal, one. We do not attempt to give a rigorous definition of this notion, i.e. we do not say how close a path should be to the shortest one, to be called a subminimal one. However, if we order by increasing length all the paths between $s$ and $t$ in the graph, it is clear that the $h$-th one, with $h$ fixed, is always subminimal in the limit $N \to \infty$, since it becomes infinitely close to the shortest one. In fact, as far as we could say about our problem, by a partial analytical solution and a numerical one, all the shortest $k$ paths in the graph, with $k$ fixed, in the limit $N \gg 1$ share the same leading behaviour versus $N$ (see (12) below, and section 5). Thus we could take as an informal characterization of the subminimal paths the fact that they all have the same asymptotic behaviour versus $N$, when $N$ goes to infinity. It is clear that the number of subminimal paths must grow with $N$, but we do not know at all how.

We believe that this characterization, though informal, is not trivial. Roughly speaking, we do not know exactly which paths are the subminimal ones, but we can argue which of them are surely not. In fact, let us suppose we are looking for the shortest path between $s$ and $t$ which visits also any other node in the graph. Such a path is the so called Hamilton path; to find it is a problem very close to the well known travelling salesman problem (TSP), which is a so called NP-complete problem (see [12]), i.e. so hard to solve as to find the ground state of the SK model. On the other hand, our problem, i.e. the problem of finding the shortest path between two nodes in a graph, is one of the so called class-P problems, and it is much easier to solve numerically. It is commonly believed (see [12]) that NP-complete problems are drastically different from class-P ones, at least as far as the cardinality of the continuum (i.e. the number of points on the real line), is drastically different from the cardinality of natural numbers. Thus we argue that the class of subminimal paths, i.e. the paths close to the shortest one, is drastically different from the class of the paths close to the Hamilton path.
4. Two partial solutions

The following heuristic argument gives some insight on the ground state of our model; but, in fact, it is easy to see that the argument should work only in the non-frustrated case, i.e. in the MST problem. Nevertheless we will see later that the resulting relations fit the numerical data well enough (see (8) and next section). We try now to estimate the length of the shortest path between $s$ and $t$. Starting from $s$, we select the outgoing arcs shorter than $\epsilon$ (to be determined), and so on, until we reach $t$. At each step the average number of selected outgoing arcs is

$$N_\epsilon = N \int_0^\epsilon p_\alpha(x) \, dx$$

where $N = |V|$ is the number of nodes in $G$. After $n$ steps, the number of reached nodes is about $N_\epsilon^n$ (if $N_\epsilon \gg 1$). The probability of reaching $t$ in $n$ steps is one if $N_\epsilon^n = N$, i.e.

$$n = \frac{\ln N}{\ln N_\epsilon}$$

The length of the path thus constructed is approximately

$$l = n \cdot \frac{\int_0^\epsilon x p_\alpha(x) \, dx}{\int_0^\epsilon p_\alpha(x) \, dx}$$

Taking the minimum as $\epsilon$ varies, we get

$$n = \frac{\ln N}{\alpha + 1}$$

$$\frac{l}{n} = \frac{K}{N^{1/((\alpha + 1))}} \quad (K = \frac{\alpha + 1}{\alpha + 2} e)$$

(8)

We have performed a numerical analysis, measuring the average length of the shortest path, and the average number of its arcs, for various values of $N$ and both uniform and linear probability distribution of the distances, i.e. for $\alpha = 0$ and $\alpha = 1$. The result of the analysis is shown in next section: for the time being we stress that the numerical data can be fitted with the following functional form, for $N \gg 1$ (see figures 1 and 2)

$$n \approx c \cdot \ln N + C$$

$$\frac{l}{n} \approx \frac{K}{N^s}$$

(8')

This is in close agreement with (8), and, fairly surprisingly, also the numerical values of $c$ and $s$ agree with the ones in (8), i.e. $c = s \simeq 1/(\alpha + 1)$. This fact will be discussed in greater detail in the following.

This approximate computation can be replaced with a more precise analysis as follows. Consider the set $C_{s,t}$ of all the paths from $s$ to $t$, ordered by increasing length, and denote with $P(h)(n, \{x_i\}_{i=1}^n)$ the probability distribution that the $h$-th path is composed of $n$ arcs
of lengths $x_1, \ldots, x_n$. We shall need the following notations: let $a_k$ be the length of the arc $k \in A$; given a path $c \in C_{s,t}$, let $l_c$ be its length, $n_c$ the number of its arcs, and \( \{k^i_c\}_{i=1}^{n_c} \subset A \) the sequence of arcs which it is made of (so that $l_c = \sum_{i=1}^{n_c} a^i_c$); finally we need the characteristic function
\[
I^h_c = \sum_{c_1, \ldots, c_{h-1}} \delta(c_{h-1}) \prod_{c' \neq c_1, \ldots, c_{h-1}, c} \Theta(l_{c'} - l_c) \cdot \Theta(l_c - l_{c_{h-1}}) \cdots \Theta(l_{c_2} - l_{c_1})
\]
(which is 1 if $c$ is the $h$-th path, 0 elsewhere). Then it is immediate to see that:
\[
P^{(h)}(n, \{x_i\}) = \int \left[ \sum_c \delta_{n,n_c} \prod_{i=1}^{n_c} \delta(x_i - a^i_c) I^h_c \right] \prod_{k \in A} p_\alpha(a_k) \, da_k
\]
(10)

In order to make the computation of the integral in (10) feasible, we shall make the additional assumption
\[
\prod_{k \in A} p_\alpha(a_k) = \prod_{c \in C_{s,t}} \prod_{i=1}^{n_c} p_\alpha(a^i_c),
\]
(11)
i.e. two different paths do not have common arcs. Assumption (11) should be given a suitable sense “in the average” for $N \gg 1$ (since, as it stands, it is evidently false). However this is a minor problem, firstly because we could not find better simplifying assumptions; secondly, because we believe that to neglect the correlations between the paths (as in (11)) corresponds to a kind of non-frustrated approximation, which do not suitably fits our problem. We hope that this point will become clearer in the following. We omit here the cumbersome algebra, and give only the final results: for $N \gg 1$ with $h$ fixed\(^\star\)
\[
\langle n \rangle^{(h)} = \frac{\ln N}{\alpha + 1} + \frac{1}{\alpha + 1} \left[ \ln(\alpha + 1) + 1 - \gamma_E + \sum_{n=1}^{h-1} \frac{1}{n} \right] + O\left( \frac{\ln N}{N} \right)
\]
\[
\langle l \rangle^{(h)} = \frac{\alpha + 1}{\chi N^{1/(\alpha+1)}} \left[ 1 - \frac{1}{(\alpha + 1)\langle n \rangle^{(h)}} \right], \quad \chi = (\alpha + 1)!^{1/(\alpha+1)}
\]
and
\[
\frac{\sigma_n^2}{\langle n \rangle^2} = O\left( \frac{1}{\ln N} \right), \quad \frac{\sigma_l^2}{\langle l \rangle^2} = O\left( \frac{1}{\ln^2 N} \right)
\]
where $\langle n \rangle$ and $\langle l \rangle$ denote the mean value, respectively, of $n_c$ and $l_c$ in the distribution (10).

The relations (12) give to leading order the non-frustrated behaviour of (8). In other words, this more refined computation with the simplifying assumption (11) gives again, as far as the leading behaviour is concerned, the result of the previous heuristic argument, i.e. a functional form of the type $(8')$ with $c = s = 1/(\alpha + 1)$, which, as we have already pointed out, fits well enough the numerical data (see next section and figures 1 and 2). An interesting feature of the form (12) is that the leading behaviour versus $N$, $N \gg 1$, of the number of arcs and of the length of the $h$-th path, is not dependent on $h$: this is
\[^{\star}\] $\gamma_E = 0.5772157 \ldots$ is the Euler’s gamma constant.
evident from the fact that the dependence on $h$ in the right hand side of the (12) is only in the next-to-leading terms. This feature is very well confirmed by the numerical data (see again next section and figures 1 and 2), so that, as we anticipated at the end of previous section, we feel authorized to study all the subminimal paths as a whole. As far as the complete probability distribution of $l_c$ is concerned, it results:

$$P^{(h)}(l) = \frac{H'_\alpha(l)}{(h-1)!} [H_\alpha(l)]^{h-1} \cdot \exp -H_\alpha(l)$$

$$H_\alpha(l) = \frac{e^{Nl}}{N(\alpha + 1)} \quad \mathcal{N} = (\alpha + 1)!^{1/(\alpha + 1)} N^{1/(\alpha + 1)}$$

The plots of the distributions $P^{(h)}(l)$, for $N = 100$ and $h = 1, 2$ and 3, are shown in figure 3, whereas in figure 5 a comparison with the experimental results is exhibited. One can see that the qualitative shapes of the distributions (13) and the experimental ones are rather different, though their mean values are very close. For this reason we believe that the assumption (11), that we called “non-frustrated” and that was at the basis of this computation, is not completely satisfactory to fit our problem.

Another approach to the problem is suggested by the cavity method of M´ezard, Parisi and Virasoro [6]. We can not pursue this approach beyond a kind of non-frustrated approximation, to some extent equivalent to the (11) above, and very reminiscent of the replica-symmetric approximation for the SK model. We shall describe briefly the procedure. Let $G_N = (V_N, A_N)$ be a graph with $N$ nodes, and fixed $s \in G_N$ let $\{l_u\}_{u \in V_N \setminus \{s\}}$ be the lengths of the shortest paths in $G_N$ from $s$ to $u$. Let us denote with $P_N(\{l_u\})$ their joint probability distribution. Let now $G_{N+1} = (V_{N+1}, A_{N+1})$ be the graph with $N + 1$ nodes obtained by adding a node $u_0$ to $V_N$, and all the arcs between $u_0$ and the nodes of $V_N$ to $A_N$; finally, let $\{L_u\}_{u \in V_N}$ be the lengths of these new arcs, with distribution $P(\{L_u\}) = \prod p_\alpha(L_u)$. The shortest path from $s$ to $u_0$ in $G_{N+1}$ has length

$$l_0 = \min \{L_s, \min_{u \in V_N \setminus \{s\}} (l_u + L_u)\}$$

As above, it is straightforward to write the exact form of the probability distribution of $l_0$, but there is no point in using it without suitable simplifying assumptions. Namely, we shall assume the $\{l_u\}$ to be uncorrelated, that is to say

$$P_N(\{l_u\}) = \prod_{u \in V_N \setminus \{s\}} P_N(l_u)$$

At first glance it is not clear which is the relation between this approximation and the (11) above: the latter consists in neglecting the correlations among the paths between the same pair of nodes, the first one ignores the correlations between the shortest paths connecting a fixed starting node with all the other ones of the graph. However, as we shall see, the resulting behaviour for the mean length of the shortest path is exactly the same, though the two complete probability distributions differ qualitatively. Now, using the (15) one can
translate the recursive relation (14) in a recursive relation for the probability distribution of the shortest path: writing

\[ F_N(l) = 1 - \int_0^l P_N(t) \, dt \]

we obtain, in a more or less straightforward way,

\[ F_{N+1}(l) = [1 - \int_0^l p_\alpha] \cdot \exp(-N \left( \int_0^l p_\alpha - \int_0^l p_\alpha(l - x)F_N(x) \, dx \right)) \]  

(16)

If we now assume, apart from negligible terms,

\[ F_N^\infty(l) \simeq F_{N+1}(l) \simeq F_N(l) \quad \text{for } N \gg 1, \]

(17) becomes an integral (non-linear) equation in \( F_N^\infty \). It is possible to justify \( \text{(17)} \) \text{a posteriori.}

If \( \alpha = 0 \) the equation reduces to

\[ F_N^\infty(l) = (1 - l) \cdot \exp(-N(l - \int_0^l F_N^\infty(x) \, dx)) \]

which is equivalent to

\[ \begin{cases} 
\frac{d}{dl} F_N^\infty(l) = -\frac{F_N^\infty}{1 - l} + F_N^\infty \cdot N[F_N^\infty - 1] \\
F_N^\infty(0) = 1 
\end{cases} \]

(18)

and can be linearized by putting \( T = 1/F \). At the end we get

\[ F_N^\infty(l) = \frac{1}{1 + \exp(Nl) - \frac{1}{N(1 - l)}} \quad , \quad P_N^\infty(l) = -\frac{d}{dl} F_N^\infty(l) \]

(19)

If \( \alpha \neq 0 \) the equation (16), again under the hypothesis (17), must be solved numerically, thus giving \( F_N^\infty(l) \) for each fixed value of \( N \) and \( \alpha \). The resulting distributions \( P_N^\infty(l) \) differs qualitatively from the \( P_N^{(h)}(l) \)'s of equation (13) above, even for \( h = 1 \). Nevertheless, as we shall see soon, their mean values coincide. A comparison between these two distributions and the experimental one is shown in figure 5, for \( N = 100 \) and both uniform and linear probability distribution of the distances, \( \text{i.e. for } \alpha = 0 \) and \( \alpha = 1 \).

As far as the mean value of \( l \) is concerned, if \( \alpha = 0 \) the (19) can be easily integrated and gives

\[ \langle l \rangle = \int lP_N^\infty(l) \, dl \simeq \frac{\ln N}{N} \]

(20)
in perfect agreement with what is predicted by equation (12). If $\alpha \neq 0$ we must solve numerically the integral equation (16) for some selected values of $N$. Then we compute $\langle l \rangle$ and we fit the resulting data with a functional form of the type

$$\langle l \rangle = K' \frac{\ln N}{N} + \text{subleading terms}$$

We have performed such an analysis for $\alpha = 0$ and $\alpha = 1$, computing $\langle l \rangle$ for the same values of $N$ of the experimental data, thus giving to the finite-size effects the same relevance. The details are given in next section. For $\alpha = 0$ we obtain

$$K' = 1 \quad s = 1$$

with negligible corrections, in perfect agreement with (20), thus showing that the selected values of $N$ are not too small. For $\alpha = 1$, fairly surprisingly, we have found again a very good agreement with what is predicted by equation (12), i.e.

$$K' = \sqrt{2}/2 \quad s = 1/2$$

Thus this approach (suggested by the cavity method of [6] with the assumption (15)), as to the leading behaviour of the length of the shortest path, is equivalent at all to the one which leads to equation (12), i.e. the direct one with the assumption (11).

As anticipated, in next section we shall see that the non-frustrated behaviour of equation (12), shared by both our analytical approaches, fits well enough the experimental data for $\langle l \rangle$ and $\langle n \rangle$. As we shall see, however, the agreement is not beyond any doubts, so we believe that it would be essential to improve the rough approximation (15), thus obtaining a more refined prediction than equation (12). An idea for a future work could be the following.

Let us assume that the Dijkstra’s algorithm has given the SPT for the graph $G_N$ with $N$ nodes; then we dispose of the exact numerical values of the lengths $l_u$ of the shortest paths from $s$ to any other node $u \in V_N$. If now we add the node $u_0$, obtaining $G_{N+1}$, the length $l_0$ of the shortest path from $s$ to $u_0$ is given by (14). What can be said now about the other lengths $\bar{l}_u$ of the shortest paths from $s$ to the nodes of $V_N$? Of course, they will be affected by the addition of $u_0$, in analogy to what happens in the SK model when applying the cavity method. In fact the addition of one spin to an $N$-spin system produces a rearrangement of the relative weights of the configurations in a pure state, thence a minor rearrangement of the relative weights of the pure states in a cluster and so on. The cavity method allows to keep track of this cascade of rearrangements (see [6] for details). The crucial observation was the discovery of the cluster structure of the pure states in the SK model, the so called ultrametric organization (see [4]). We do not yet know how to implement an analogous procedure in our model, though it seems to be natural that an improvement of the rough approximations (15) or (11) should be required. In some sense the approximations we used here correspond to a “replica-symmetric” solution, which does not take into account the rearrangement of the pure states. This would be
exact if the system was not frustrated (i.e. with only one pure state, apart from accidental degenerations).

The first condition, in order to start the program sketched above, is to find out what plays the role of the SK pure states in our model. Let us therefore consider the following argument. After adding the node $u_0$ to the graph $G_N$ obtaining $G_{N+1}$, one can apply the Dijkstra’s algorithm to find the $SPT$ in $G_{N+1}$. Take now a node $u$ in $V_N$. If the shortest path in $G_{N+1}$ from $s$ to $u$ does not pass trough $u_0$, then it will coincide with the old shortest path in $G_N$. If, on the contrary, it passes through $u_0$, it is interesting to see how much “close” (in a suitable sense) it is to the old shortest path in $G_N$. In general, there is no reason to expect these two paths to be close, because of the frustration of the model, but one can reasonably suppose that the new path runs close to an old subminimal path in $G_N$ from $s$ to $u$. Therefore in our numerical analysis we have pointed out our interest on the suboptimal paths a little longer than the minimal one: can we interpret the suboptimal paths as the pure states (in a suitable sense) of our model? Let us stress that to leading order for $N \gg 1$ the lengths $\langle l \rangle^{(h)}$ of the suboptimal paths coincide, as it results from (12) and is confirmed by numerical analysis (see next section). Otherwise the identification between subminimal paths and pure states will fail: in the SK model the free energies are equal in the thermodynamical limit.

5. Numerical results

As pointed out above, we have performed some numerical computations to test the validity of the scaling laws (8), and to measure the probability distribution for the lengths of the subminimal paths. The algorithm used is a clever generalization of the Dijkstra’s algorithm [15]. It determines, for a given realization of the distance matrix $d(u, v)$, the first $k$ paths of $C_{s,t}$ ordered by increasing length. We have considered only $k = 3$, because the CPU-time spent in the computation grows rather fast with $k$. We made several runs for $\alpha = 0$ and $\alpha = 1$, i.e. with the probability distribution in (3) respectively uniform and linear, and for different values of $N$, the size of the graph, ranging from $N = 10$ to $N = 500$. In each run we measure the length and the number of arcs of the paths for a large number of independent realizations of the distance matrix. More precisely we made 200,000 iterations for each value of $\alpha$ for $N = 10, 20, 50, 100$; 100,000 iterations for $N = 200$; 40,000 iterations for $N = 500$. The runs required about 150 hours of CPU on a DEC VAX 6000-520, running VAX Pascal. The runs to measure the probability distribution, being purely qualitative and not requiring a high precision, were made only for $N = 100$ and low statistics: 30,000 iterations for each value of $\alpha$.

In figure 1 and 2 we plot respectively $\langle n \rangle^{(h)}$ and $\langle l \rangle^{(h)}/\langle n \rangle^{(h)}$ versus the size of the graph $N$, for $h = 1, 2, 3$ and both probability distributions for the distances ($\alpha = 0$ and $\alpha = 1$). The scaling laws that qualitatively appear in these plots are the ones anticipated in equation (8'), i.e. for $N \gg 1$

$$\langle n \rangle^{(h)} \approx c \cdot \ln N + C^{(h)}$$

$$\frac{\langle l \rangle^{(h)}}{\langle n \rangle^{(h)}} \approx \frac{K}{N^s} \quad \text{independent on } h$$
We want to stress once more that to leading order in $N$ these scaling laws do not depend on $h$.

To be sure that the leading behaviour for the “physical” quantities $\langle n \rangle$ and $\langle l \rangle$ is the non-frustrated one of equation (8) and (12), we should be more quantitative, and show that

$$c = s = \frac{1}{\alpha + 1}$$  \hspace{1cm} (21)

We cannot simply perform a naïve least-squares fit, as our data could not have reached their asymptotic behaviour, because of the low involved values of $N$. Therefore our data have been fitted with curves of the form

$$\langle n \rangle = c \cdot \ln N + C + B \frac{\ln N}{N}$$

$$\frac{\langle l \rangle}{\langle n \rangle} = \frac{K}{N^s} \left(1 + \frac{B}{\ln N}\right)$$  \hspace{1cm} (22)

for various fixed values of $B$, applying the flatness criterion advocated in [16, sections 4.2 and 5.3]. The $B$-dependent term should take into account the corrections to the leading behaviour, and its form was suggested by equation (12). Although we believe that this technique has been now generally accepted, in the appendix we shall spend some words of explanation. The result of the analysis for $\langle n \rangle$ and $\langle l \rangle/\langle n \rangle$ is given in table 1 and 2. From these data we see that relation (21) is verified well enough. Sufficiently well verified is also the fact that $K$ is independent on $h$, at least for $\alpha = 1$. As to the value of $C(h)$ the agreement with equation (12) is only qualitative.

Nevertheless the fitted values for $c$ and $s$ in table 1 and 2 are different from what is predicted by equation (21) by more than their error, though this should be a 95% confidence limit. Thus we conclude that, as we anticipated at the end of the previous section, the use of the non-frustrated leading behaviour (12) to fit our experimental data is not completely satisfactory. We strongly believe that the finite size errors are not underestimated, because the same procedure of fitting performed on the theoretical data leads to confirm the validity of (21) exactly, i.e. at 95% confidence level. More precisely we have proceeded as follows. We have generated a set of values for $\langle l \rangle$ integrating the theoretical distributions of equation (13) and (16), for the same values of $N$ of the real data, and with fictitious statistical errors equal to the real ones. Then we have performed in parallel the same fit, with the same procedure of the previous analysis, on the experimental data and the theoretical ones, which so work as a placebo. The result is given in table 3: the data have been fitted with the functional form

$$\langle l \rangle = K' \frac{\ln N}{N^s} \left(1 + \frac{B}{\ln N}\right)$$  \hspace{1cm} (22')

where again the corrections are suggested by equation (12). We see that the theoretical data give the expected values well within their error, with some troubles for $h = 1$, not only for the exponent $s$ (i.e. $1/(\alpha + 1)$), but also for the constant $K'$ (i.e. 1 for $\alpha = 0$ and $\sqrt{2}/2$ for $\alpha = 1$, see (12)). As to the experimental data, though the flatness region
in $B$ is larger, thus producing a systematic error larger, the fitted value for $s$ differs from $1/(\alpha + 1)$ by more than its error, and the discrepancy becomes stronger when $h$ increases.

In figure 3, 4 and 5 we show the various probability distributions for the length of the subminimal paths for $N = 100$ and both $\alpha = 0$ and $\alpha = 1$. In figure 3 we plotted the distributions $P^{(h)}(l)$ of equation (13) for $h = 1, 2$ and 3. Figure 4 presents the experimental distributions for $h = 1, 2$ and 3, obtained by smoothing the histogram of the experimental data. More precisely we filled a histogram with the experimental data for $l$, for different realizations of the distance matrix (200 channels for 30,000 data, no one of the data falls out of the considered range), then we averaged the counting of each channel with the $H$ proceeding and the $H$ following it ($H = 5$ for $\alpha = 0$ and $H = 2$ for $\alpha = 1$), thus obtaining a smoother plot. Lastly, as far as the length of the shortest path is concerned, in figure 5 we compare the distribution $P^{(h)}(l)$ with $h = 1$ of figure 3, the experimental distribution of figure 4, and the distribution of the “cavity method”, i.e. the one obtained by solving equation (16), exactly for $\alpha = 0$ (see (19)), and numerically for $\alpha = 1$.

As a comment to figure 5 we remark again (this fact has been yet noted in the previous section) that the two theoretical distributions and the experimental one differ qualitatively from each others, at $N$ fixed, though the first two distributions share the same leading behaviour versus $N$ as to their mean value, whereas the third one gives a significantly different behaviour.

Appendix

In this appendix we sketch briefly the contents of the analysis of the corrections to the scaling à la Berretti-Sokal [16]. In our case we measure $\langle l \rangle$ and $\langle n \rangle$ for some values of $N$, the size of the graph, and we would extrapolate from the data the scaling behaviour versus $N$. Because of the involved relatively small values of $N$, we can not be sure that the data have reached their asymptotic behaviour, so we must take into account possible corrections to the scaling. Firstly we make an Ansatz for the corrections to the leading behaviour, as in (22). The coefficient $B$ of the corrections in (22) is not subject to a fit, although this would be easily made by well known non-linear methods of fitting, because this would not solve our problem, i.e. to understand to what extent the fitted parameters are reliable as true asymptotic values, and not merely as effective values, changing as the range in $N$ increases. In fact we do not know exactly which the subleading corrections are, and also, if we knew them, we could not include in a fit other but the first relevant ones, as our data are only for a few values of $N$ (6, in our case).

Thus we fix the value of $B$, we fix also a value $N_{\text{cut}}$ for $N$, and we fit the data with the curves in (22) only for $N > N_{\text{cut}}$. Varying $N_{\text{cut}}$ at $B$ fixed, if we observe a systematic dependence on $N_{\text{cut}}$ of the fitted parameters (e.g. a systematic decrease of the fitted exponent $s$), we say that the chosen value of $B$ can not take into account the corrections to the scaling. Now, by varying $B$, we select the range in $B$ for which the fitted parameters are flat (i.e. do not vary in a systematic way), with respect to $N_{\text{cut}}$. We say, following [16], that for $B$ in this range equation (22) takes into account the corrections to the scaling as an effective correction, even if the exact form for the corrections is different from that we imposed, and even if we rule out some other (more irrelevant) terms. In this range of $B$,
which we call the “flatness region”, for each fitted parameter we select the maximum and
the minimum value. The best fit for the parameter will be simply the arithmetic mean
of these values, whereas their difference will be considered as the systematic error due
to unconsidered corrections to the scaling, or to imperfect knowledge of the form of the
corrections (95% subjective confidence limit as defined in [16, footnote 17]). In addition
we quote the usual statistical error for the fit, at 95% confidence level (2\(\sigma\)).

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Captions for the figures and tables

**Figure 1.** $\langle n \rangle^{(h)}$ versus $N$, with uniform ($a$) and linear ($b$) probability distribution for the distances.

**Figure 2.** The same as in figure 1 for $\langle l \rangle^{(h)}/\langle n \rangle^{(h)}$.

**Figure 3.** Distributions $P^{(h)}(l)$ for the length of the $h$-th path as in equation (13), for $N = 100$ and uniform ($a$) and linear ($b$) probability distribution for the distances.

**Figure 4.** The same as in figure 3 for the measured distributions $P^{(h)}_{\text{exp}}(l)$, obtained by smoothing the histogram of the experimental data.

**Figure 5.** Three probability distributions for the length of the shortest path, with $N = 100$ and uniform ($a$) and linear ($b$) probability distribution for the distances. $P^{(1)}(l)$ is the distribution quoted in (13), as in figure 3, for $h = 1$; $\text{exp}$ denotes the experimental distribution, as in figure 4, for $h = 1$; $\text{cav}$ denotes the distribution of the “cavity method” (see equation (16)).

**Table 1.** Best estimates for $c$ and $C$ in the fit $\langle n \rangle^{(h)} = c \cdot \ln N + C$, with corrections to the leading behaviour inserted (see (22)). The first error is the systematic error due to unconsidered corrections to the scaling (95% subjective confidence limit as defined in [16, footnote 17]) and the second error is the usual statistical error (95% confidence interval). It is also quoted the range in $B$ which obeys to the flatness criterion.

**Table 2.** Best estimates as in table 1 for $s$ and $K$ in the fit $\langle l \rangle^{(h)}/\langle n \rangle^{(h)} = K/N^s$, with corrections to the leading behaviour inserted (see (22)).

**Table 3.** Best estimates as in table 1 for $s$ and $K'$ in the fit $\langle l \rangle^{(h)} = K' \cdot \ln N/N^s$, with corrections to the leading behaviour inserted (see (22')). The fit has been performed for both the experimental data (denoted by $\text{exp}$) and the theoretical ones. These last were produced by integrating the two theoretical probability distributions for $l$, the one of equation (13) and the one of the cavity equation (16).
Table 1.

|       | $c$                           | $C$                           | $B$                           |
|-------|-------------------------------|-------------------------------|-------------------------------|
| $\alpha = 0$ |                                |                               |                               |
| $h = 1$  | $1.0070 \pm 0.0079 \pm 0.0023$ | $-0.475 \pm 0.041 \pm 0.008$ | $0.87 \div 0.99$             |
| $h = 2$  | $1.0363 \pm 0.0098 \pm 0.0024$ | $0.316 \pm 0.051 \pm 0.009$  | $-0.20 \div -0.05$           |
| $h = 3$  | $1.0627 \pm 0.0179 \pm 0.0125$ | $0.650 \pm 0.095 \pm 0.060$  | $-0.60 \div -0.40$           |
| $\alpha = 1$ |                                |                               |                               |
| $h = 1$  | $0.5051 \pm 0.0108 \pm 0.0025$ | $0.283 \pm 0.061 \pm 0.010$  | $-0.55 \div -0.30$           |
| $h = 2$  | $0.5051 \pm 0.0087 \pm 0.0074$ | $0.781 \pm 0.051 \pm 0.036$  | $-0.50 \div -0.30$           |
| $h = 3$  | $0.5138 \pm 0.0129 \pm 0.0026$ | $0.971 \pm 0.073 \pm 0.011$  | $-0.30 \div 0.00$            |

Table 2.

|       | $s$                           | $K$                           | $B$                           |
|-------|-------------------------------|-------------------------------|-------------------------------|
| $\alpha = 0$ |                                |                               |                               |
| $h = 1$  | $1.0191 \pm 0.0118 \pm 0.0021$ | $1.246 \pm 0.131 \pm 0.012$  | $0.25 \div 0.50$             |
| $h = 2$  | $0.9914 \pm 0.0106 \pm 0.0016$ | $0.844 \pm 0.094 \pm 0.006$  | $1.70 \div 2.10$             |
| $h = 3$  | $0.9780 \pm 0.0070 \pm 0.0032$ | $0.675 \pm 0.049 \pm 0.010$  | $2.90 \div 3.20$             |
| $\alpha = 1$ |                                |                               |                               |
| $h = 1$  | $0.4981 \pm 0.0102 \pm 0.0018$ | $1.361 \pm 0.119 \pm 0.011$  | $-0.08 \div 0.10$            |
| $h = 2$  | $0.4920 \pm 0.0076 \pm 0.0013$ | $1.265 \pm 0.084 \pm 0.008$  | $0.20 \div 0.35$             |
| $h = 3$  | $0.4948 \pm 0.0076 \pm 0.0012$ | $1.290 \pm 0.086 \pm 0.007$  | $0.20 \div 0.35$             |
|               | $s$                     | $K'$                    | $B$                  |
|---------------|-------------------------|-------------------------|----------------------|
| $\alpha = 0$  |                         |                         |                      |
| $\exp h = 1$  | $1.0077 \pm 0.0077 \pm 0.0013$ | $1.098 \pm 0.074 \pm 0.007$ | $0.20 \div 0.35$    |
| $\exp h = 2$  | $0.9874 \pm 0.0081 \pm 0.0008$ | $0.814 \pm 0.075 \pm 0.003$ | $2.50 \div 2.90$    |
| $\exp h = 3$  | $0.9667 \pm 0.0044 \pm 0.0005$ | $0.571 \pm 0.034 \pm 0.001$ | $5.50 \div 5.90$    |
| eq. (16)      |                         |                         |                      |
| eq. (13) $h = 1$ | $0.9935 \pm 0.0038 \pm 0.0007$ | $0.932 \pm 0.030 \pm 0.003$ | $-0.36 \div -0.43$ |
| eq. (13) $h = 2$ | $0.9998 \pm 0.0011 \pm 0.0003$ | $0.997 \pm 0.009 \pm 0.001$ | $0.415 \div 0.445$ |
| eq. (13) $h = 3$ | $0.9999 \pm 0.0013 \pm 0.0002$ | $0.999 \pm 0.012 \pm 0.001$ | $0.910 \div 0.935$ |
| $\alpha = 1$  |                         |                         |                      |
| $\exp h = 1$  | $0.5063 \pm 0.0041 \pm 0.0007$ | $0.763 \pm 0.027 \pm 0.002$ | $0.16 \div 0.24$    |
| $\exp h = 2$  | $0.4990 \pm 0.0035 \pm 0.0004$ | $0.692 \pm 0.025 \pm 0.001$ | $1.51 \div 1.63$    |
| $\exp h = 3$  | $0.4931 \pm 0.0033 \pm 0.0004$ | $0.638 \pm 0.023 \pm 0.001$ | $2.40 \div 2.55$    |
| eq. (16)      |                         |                         |                      |
| eq. (13) $h = 1$ | $0.4951 \pm 0.0025 \pm 0.0006$ | $0.670 \pm 0.013 \pm 0.002$ | $0.25 \div 0.29$    |
| eq. (13) $h = 2$ | $0.4999 \pm 0.0016 \pm 0.0002$ | $0.706 \pm 0.010 \pm 0.001$ | $1.10 \div 1.14$    |
| eq. (13) $h = 3$ | $0.5000 \pm 0.0010 \pm 0.0001$ | $0.707 \pm 0.007 \pm 0.001$ | $1.60 \div 1.63$    |