On well-posedness of the Cauchy problem for the wave equation in static spherically symmetric spacetimes

Ricardo E Gamboa Saraví1,2, Marcela Sanmartino3 and Philippe Tchamitchian4

1 Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Casilla de Correo 67, 1900 La Plata, Argentina
2 IFLP, CONICET, La Plata, Argentina
3 Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Casilla de Correo 172, 1900 La Plata, Argentina
4 Aix-Marseille Université, CNRS, LATP (UMR 6632), 39, rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France

E-mail: quique@fisica.unlp.edu.ar, tatu@mate.unlp.edu.ar and philippe.tchamitchian@univ-amu.fr

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Abstract

We give simple conditions implying the well-posedness of the Cauchy problem for the propagation of classical scalar fields in general \((n+2)\)-dimensional static and spherically symmetric spacetimes. They are related to the properties of the underlying spatial part of the wave operator, one of which being the standard essentially self-adjointness. However, in many examples the spatial part of the wave operator turns out to be not essentially self-adjoint, but it does satisfy a weaker property that we call here \textit{quasi-essentially self-adjointness}, which is enough to ensure the desired well-posedness. This is why we also characterize this second property. We state abstract results, then general results for a class of operators encompassing many examples in the literature, and we finish with the explicit analysis of some of them.

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1. Introduction

Hawking and Penrose have shown that, according to general relativity, there must exist the singularities of infinite density and space-time curvature in many physically reasonable situations. This phenomenon occurs in the big bang scenery at the very beginning of time, and it would be an end of time for sufficiently massive collapsing bodies (see, for example, [1] and references therein). At these singularities all the known laws of physics and our ability to predict the future would break down.

However, in the case of black holes, any observer who remained outside the event horizon would not be affected by this failure of predictability, because neither light nor any other signal
could reach him from the singularity. This notable feature led Penrose to propose the weak
\textit{cosmic censorship hypothesis}: all singularities produced by the gravitational collapse occur
only in places, like black holes, where they are hidden from the outside view by an event
horizon [2].

The strong version of the cosmic censorship hypothesis states that any physically realistic
spacetime must be globally hyperbolic [3]. The concept of global hyperbolicity was introduced
for dealing with hyperbolic partial differential equations on a manifold [4]. A spacetime is
said to be globally hyperbolic if, given any two of its points, the set of of all causal curves
joining these points is compact (in a suitable topology). Only in this case there is a Cauchy
surface whose domain of dependence is the entire spacetime. This is a reasonable condition
to impose, for example, to ensure the existence and uniqueness of solutions of hyperbolic
differential equations [4, 5].

Nevertheless, the relevant physical condition to assure predictability is not global
hyperbolicity, but the well-posedness of the field equations. Indeed, there are many examples
of spacetimes that are not geodesically complete and violate cosmic censorship, but where
there is still a well-posed initial-value problem for test fields. Global hyperbolicity is sufficient,
but not necessary for this. This suggests that, in more general situations, we could find a weaker
condition to replace the notion of global hyperbolicity by making direct reference to test fields
[6–8].

The above considerations motivate a deeper study of the well-posedness of the initial-value
problem for fields in more general singular spacetimes.

This paper is a continuation of a previous one [9], tackling the well-posedness of Cauchy
problem for waves in static spacetimes. This subject has been launched by Wald in [6], and
further developed by, among others, the authors of [7, 10, 11].

The propagation of waves is, in such spaces, described by a classical equation of
the form

\[ \partial_t \phi + A \phi = 0, \]

where \( A \) is a self-adjoint extension of a given symmetric and positive operator \( A \) which reflects
the underlying geometry.

Our motivation relies on the following observation: although \( A \) may not be essentially
self-adjoint (e.s.a.), boundary conditions are not necessary to construct \( A \) in some geometries
of interest. Such a situation arises when, even if \( A \) has many self-adjoint extensions, only one
has its domain included in the energy space naturally associated with \( A \). Here we call this
property \textit{quasi-essentially self-adjoint} (q.e.s.a.).

We have shown in [9] that operators \( A \) given by the propagation of massless scalar fields
in static spacetimes with naked timelike singularities may be q.e.s.a. but not e.s.a.. Thus, in
such situations, demanding the finiteness of the energy is enough to select one self-adjoint
extension of \( A \), and only one; in addition, we proved that the solutions of the wave equation
may have a non-trivial trace at the boundary of the geometrical domain, even though this trace
is not imposed by any boundary condition at all. This phenomenon never happens with e.s.a.
operators.

Here, we deeply examine the case of general \((n + 2)\)-dimensional static and spherically
symmetric spacetimes. More precisely, the concrete setting is the following.

The domain is of the form \( I \times \mathcal{M} \), where \( I \subset (0, +\infty) \) is an open interval and \( \mathcal{M} \)
is a compact, oriented Riemannian manifold without boundary. The operator \( A \) is defined on
\( C_0^\infty(I \times \mathcal{M}) \) as

\[ A \phi(z, x) = \frac{1}{a(z)} \left\{ -\partial_z (b(z) \partial_z \phi(z, x)) - c(z) \Delta_\mathcal{M} \phi(z, x) + d(z) \phi(z, x) \right\}, \]
where $\Delta_r$ is the Laplace–Beltrami operator on $M$, and $a$, $b$, $c$ and $d$ are suitable positive coefficients only depending on the radial variable $z \in I$. No condition is prescribed on the coefficients at the boundary of the domain.

For this class of operators, we fully characterize e.s.a. and q.e.s.a. properties. More precisely, under rather general conditions on the coefficients, we give a necessary and sufficient condition for q.e.s.a. depending only on the integrability of the function $(\frac{1}{\beta(z)} + d(z) + a(z))$ at the boundary of $I$. We also give a necessary and sufficient condition for e.s.a.; in this case, the condition depends also on the integrability of the functions $a(z)$ and $\beta(z)^2 a(z)$ at the boundary of $I$, where $\beta(z)$ is a particular solution of the ordinary differential equation (ODE)

$$-(b(z) \beta'(z))' + d(z) \beta(z) = 0.$$ 

We then apply this analysis to scalar fields propagating in static spherically symmetric spacetimes of arbitrary dimension, solutions of the Einstein equations with cosmological constant and matter satisfying the dominant energy condition or vacuum. The criteria for e.s.a. and q.e.s.a. on the coefficients of the operator $A$ are then translated into criteria on the components of the metric tensor. This provides a systematic procedure to analyse the situations where boundary conditions are, or are not, necessary for the Cauchy problem to be well posed.

A significant physical result is stated in theorem 5.5: in the outer region of a static, spherically symmetric and asymptotically flat spacetime where the dominant energy condition holds, the operator $A$ is essentially self-adjoint, i.e. the Cauchy problem is well posed without any boundary conditions, if, and only if, an observer at infinity measures that it takes an infinite time for a photon to reach the boundary.

Finally, we directly apply the developed theory to the discussion of some exact vacuum solutions as explicit examples. We discuss the $(n + 2)$-dimensional Minkowski spacetime with a removed spatial point and the higher dimensional generalization of Schwarzschild and Reissner–Nordström geometries; we systematically describe the situations where boundary conditions are, or are not, necessary for the Cauchy problem to be well posed.

The outline of the paper is as follows. Section 2 is devoted to abstract results on e.s.a. and q.e.s.a. properties. In section 3, we completely characterize e.s.a. and q.e.s.a. properties of the operator given in (1). We show, in section 4, the well-posedness of the Cauchy problem when the operator $A$ is q.e.s.a. but not necessarily e.s.a.. In section 5, we apply our results to the study of propagation of scalar fields in general $(n + 2)$-dimensional static and spherically symmetric spacetime with $n \geq 1$. We close by discussing the examples in section 6.

2. Quasi essentially and essentially self-adjointness

Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain\(^5\) and $H$ a Hilbert space such that $C^\infty_c(\Omega)$ is dense in $H$, where $C^\infty_c(\Omega)$ is the space of the restrictions to $\Omega$ of $C^\infty_0(\mathbb{R}^{n+1})$. We consider an unbounded symmetric definite positive operator $A$, whose domain is $C^\infty_0(\Omega)$. We assume the existence of a Hilbert space $E$, continuously embedded in $H$, and a related bilinear symmetric form $b$ with domain $E$ having the following properties:

(i) if $\phi \in E$, $\|\phi\|^2_E = \|\phi\|^2_H + b(\phi, \phi)$;
(ii) $C^\infty_c(\Omega)$ is dense in $E$;
(iii) if $\phi, \psi \in C^\infty_c(\Omega)$, then $b(\phi, \psi) = (\phi, A\psi)$.

The reader should note that $A$ is defined only on $C^\infty_0(\Omega)$, and that consequently the relation between the form $b$ and the operator $A$ is only stated for functions in $C^\infty_0(\Omega)$ as well, although $C^\infty_c(\Omega)$ is dense in both spaces $H$ and $E$. This is motivated by the difficulties arising with

\(^5\) Being Lipschitz is not the weakest possible hypothesis on $\Omega$ for our results to hold, but it is enough for the examples we have in mind.
boundary conditions: whether they must be specified in advance or not is the question we consider in the subsequent theorem 2.2. We will show that there is a ‘natural’ self-adjoint extension of \( A \), defined without specifying any boundary condition, if and only if \( C_0^\infty(\Omega) \) is dense in \( \mathcal{E} \). We will also show that this density property is always true when \( A \) is essentially self-adjoint, but may occur even when \( A \) is not. Various examples are given at the end of the paper.

**Definition 2.1.** We shall say that \( A \), any given self-adjoint extension of \( A \), is of finite energy when \( D(A) \subset \mathcal{E} \), with continuous injection.

Calling \( \mathcal{E}^0 \) the closure of \( C_0^\infty(\Omega) \) in \( \mathcal{E} \), we have the following result.

**Theorem 2.2.** Under these hypotheses we have.

(i) The operator \( A \) has only one self-adjoint extension with finite energy, if and only if \( \mathcal{E}^0 = \mathcal{E} \). If this is the case, this extension is \( A_r \), the Friedrichs extension.

(ii) If \( \mathcal{E}^0 = \mathcal{E} \), then \( C_0^\infty(\Omega) \) is dense in \( D(A_r) \), if and only if \( A \) is essentially self-adjoint (e.s.a.), i.e., \( A \) has only one self-adjoint extension.

**Proof.**

(i) To prove this assertion, we begin with assuming that \( A \) has only one self-adjoint extension with finite energy. Let \( A \) be the self-adjoint operator associated with the energy form \( b \); let \( A_0 \) be the self-adjoint operator associated with the restriction of \( b \) to \( \mathcal{E}^0 \). Both are the extensions of \( A \) with domains included in \( \mathcal{E} \), and so, are equal. But then we must have \( D(A^0) = D(A_0^0), \) which is \( \mathcal{E} = \mathcal{E}^0 \).

Reciprocally, if \( \mathcal{E} = \mathcal{E}^0 \), the only self-adjoint extension of \( A \) with domain in \( \mathcal{E} \) is its Friedrichs extension, because the form \( b \) defined on \( \mathcal{E} \) is the closure of the form \( b \) defined on \( C_0^\infty(\Omega) \).

(ii) Recall that

\[
D(A^*) = \{ \varphi \in H : \exists C > 0 : \forall \psi \in C_0^\infty(\Omega), |\langle \varphi, A\psi \rangle| \leq C\|\psi\|_H \},
\]

and that

\[
D(A_r) = \{ \varphi \in \mathcal{E} : \exists C > 0 : \forall \eta \in \mathcal{E}, b(\varphi, \eta) \leq C\|\eta\|_H \}.
\] (2)

We assume first that \( C_0^\infty(\Omega) \) is dense in \( D(A_r) \). It is enough to see that \( D(A^*) \subset D(A_r) \).

Taking \( \phi_0 \in D(A^*) \) and \( \eta_0 = (A^* + I)\phi_0 \), we have for all \( \psi \in C_0^\infty(\Omega) \)

\[
\langle \phi_0, (A_r + I)\psi \rangle = \langle \phi_0, (A + I)\psi \rangle = |\langle \eta_0, \psi \rangle |.
\]

and then, since \( C_0^\infty(\Omega) \) is dense in \( D(A_r) \), for all \( \varphi \in D(A_r) \)

\[
\langle \phi_0, (A_r + I)\psi \rangle = |\eta_0, \varphi |.
\]

Taking into account that \( (A_r + I)^{-1} \) is defined on all \( H \), by calling \( \varphi_0 = (A_r + I)^{-1}\eta_0 \in D(A_r) \) we have

\[
\langle \eta_0, \varphi \rangle = \langle (A_r + I)(A_r + I)^{-1}\eta_0, \varphi \rangle = \langle \varphi_0, (A_r + I)\varphi \rangle \text{ for all } \varphi \in D(A_r),
\]

and then

\[
\langle \varphi_0 - \phi_0, (A_r + I)\varphi \rangle = 0 \text{ for all } \varphi \in D(A_r).
\]

Since \( \text{Im}(A_r + I) = H \), we have \( \varphi_0 = \phi_0 \). It implies \( D(A^*) \subset D(A_r) \) and so \( A^* = A_r \).

Then \( A \) is essentially self-adjoint.
On the other hand, if $C_0^\infty(\Omega)$ is not dense in $D(A_r)$, there exists $\varphi \in D(A_r)$ such that $A_r\varphi \neq 0$ and

$$\langle A_r\varphi, A_r\psi \rangle = 0 \quad \forall \psi \in C_0^\infty(\Omega).$$

Let us call $\eta = A_r\varphi$. If $\eta \in \mathcal{E}$, then $b(\eta, \psi) = \langle \eta, A\psi \rangle = \langle \eta, A_r\psi \rangle = 0$ for all $\psi \in C_0^\infty(\Omega)$ and then by density of $C_0^\infty(\Omega)$ in $\mathcal{E}$, $b(\eta, \eta) = 0$. Since by hypothesis $\eta \neq 0$, we have $\eta \notin \mathcal{E}$.

Therefore, we have proved that there exists $\eta \in H$, such that $\eta \in \ker(A^*)$ but $\eta \notin \mathcal{E}$, so $A$ cannot be essentially self-adjoint.

\[ \square \]

**Definition 2.3.** Under the preceding hypotheses, the operator $A$ is quasi essentially self-adjoint (q.e.s.a.) if it has only one extension with finite energy.

**Lemma 2.4.** If $A$ is a q.e.s.a. operator, then $D(A_F) = D(A^*) \cap \mathcal{E}$.

**Proof.** Since $D(A_r) \subset D(A^*)$ by definition of $A^*$ and $D(A_F) \subset \mathcal{E}$ by definition of $A_F$, then $D(A_F) \subset D(A^*) \cap \mathcal{E}$.

Conversely, let $\varphi \in D(A^*) \cap \mathcal{E}$, then

$$b(\varphi, \psi) \leq C \|\psi\|_H \quad \forall \psi \in C_0^\infty(\Omega)$$

by definition of $D(A^*)$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{E}$ and $\varphi \in \mathcal{E}$, this inequality extends to any $\psi \in \mathcal{E}$, proving that $\varphi \in D(A_F)$.

\[ \square \]

**Lemma 2.5.** If $A$ is a q.e.s.a. operator, then the following three statements are equivalent:

(i) $A$ is not an e.s.a. operator.

(ii) There exists $\varphi \in D(A^*)$ but $\varphi \notin \mathcal{E}$.

(iii) There exists $\varphi \in D(A^*)$ non vanishing and such that $(A^* + I)\varphi = 0$.

**Proof.**

(i) $\Leftrightarrow$ (ii): observe that $A$ is an e.s.a. operator if and only if $A^* = A_F$; thus by lemma 2.4, $A$ is an e.s.a. operator if and only if $D(A^*) \subset \mathcal{E}$.

(ii) $\Leftrightarrow$ (iii) let $\varphi_0 \in D(A^*)$ and $\varphi_0 \notin \mathcal{E}$, and define $f = (A^* + I)\varphi_0 \in H$, $\varphi = (A_r + I)^{-1}f \in D(A_r)$. We have $(A_r + I)\varphi = (A^* + I)\varphi_0$ and since $\varphi \in D(A^*)$, this implies $A^*(\varphi_0 - \varphi) + (\varphi_0 - \varphi) = 0$. Finally $\varphi_0 - \varphi$ cannot identically vanish, since $\varphi_0 \notin \mathcal{E}$ while $\varphi \in \mathcal{E}$. Thus, (iii) holds.

Conversely, let $\varphi \neq 0$ a.e., $\varphi \in D(A^*)$ such that $(A^* + I)\varphi = 0$. If $\varphi \in \mathcal{E}$, by lemma 2.4, $\varphi \in D(A_F)$, then $\varphi = 0$ a.e. since $A_r + I$ is injective, which is a contradiction. Thus, $\varphi \notin \mathcal{E}$ and (ii) holds.

\[ \square \]

3. A characterization of some q.e.s.a. and e.s.a. divergence-type operators

Let $\mathcal{M}$ be a Riemannian manifold of dimension $n$ with a metric $(g_{ij})$. We also assume that $\mathcal{M}$ is compact, connected, without boundary and with a given orientation.

In local coordinates, for $u \in C^\infty(\mathcal{M})$ the Laplace–Beltrami operator is

$$\Delta_{\mathcal{M}} u = \text{div}(\nabla_{\mathcal{M}} u) = \sum_{i,j=1}^n \partial_i (\sqrt{g} g^{ij} \partial_j u),$$

where $g$ is the determinant of the metric. Let us consider in $\Omega = (0, +\infty) \times \mathcal{M}$, the operator $A$ given by

$$A\varphi(z, x) = \frac{1}{a(z)} \{-\partial_i (b(z) \partial_i \varphi(z, x)) - c(z) \Delta_{\mathcal{M}} \varphi(z, x) + d(z) \varphi(z, x)\}.$$  \[ (3) \]
for all $\varphi \in C^\infty_0(\Omega)$, where the functions $a$, $b$, $c$ and $d$ satisfy the following hypotheses:

- $a, c, d \in L^1_{\text{loc}}((0, +\infty))$ and $b \in C((0, +\infty))$,
- $a > 0$, $b > 0$, $c > 0$ and $d \geq 0$ in $(0, +\infty)$,
- $a^{-1}$, $b^{-1}$, $c^{-1} \in L^1_{\text{loc}}((0, +\infty))$.

Examples will be presented in the last two sections. Let us state in advance that the coefficient $d$ is nonvanishing only in the massive case. This is why we will call the case $d = 0$ massless.

We define the Hilbert spaces

$$H = \{\varphi \in L^2_{\text{loc}}(\Omega) : \int_\Omega |\varphi(z, x)|^2 a(z) \, d\omega_x \, dz < \infty\},$$

and the energy space

$$E = \{\varphi \in H \cap H^1_{\text{loc}}(\Omega) : b(\varphi, \varphi) < +\infty\},$$

where we denote $\omega_x$ the natural measure in $\mathcal{M}$, and

$$b(\varphi, \psi) = \int_\Omega b(z) \partial_z \varphi(z, x) \overline{\partial_z \psi(z, x)} \, d\omega_x \, dz + \int_\Omega c(z) \nabla_x \varphi(z, x) \cdot \nabla_x \overline{\psi(z, x)} \, d\omega_x \, dz + \int_\Omega d(z) \varphi(z, x) \overline{\psi(z, x)} \, d\omega_x \, dz,$$

for $\varphi, \psi \in C^\infty_0(\Omega)$.

Thus, $H$ and $E$ are Hilbert spaces equipped with their canonical norms:

$$\|\varphi\|_H^2 = \int_\Omega |\varphi(z, x)|^2 a(z) \, d\omega_x \, dz \quad \text{and} \quad \|\varphi\|_E^2 = \|\varphi\|_H^2 + b(\varphi, \varphi).$$

The operator $A$ is well defined on $C^\infty_0(\Omega)$ and it is symmetric in $H$ by definition.

We shall explore when $A$ is a q.e.s.a. operator by using theorem 2.2. Then, the question is to determine that under which conditions on the coefficients of $A$, $C^\infty_0(\Omega)$ is dense in $E$. A related one is whether $C^\infty_0(\Omega) \cap E$ is dense in $E$.

**Notation 3.1.** From now on, $\int_\mathcal{D}$ and $\int_1^z$, respectively, denote $\int_{-\infty}^z$ and $\int_{\mathcal{D}}^z$, for a positive and small enough $\varepsilon$. And $\int_{-\infty}^z < +\infty$ means that their exists $z > 0$ such that $\int_{\mathcal{D}}^z < +\infty$.

**Theorem 3.2.** Let $A$ be the operator defined in (3). Then,

(i) $C^\infty_0(\Omega) \cap E$ is dense in $E$ if and only if $\int_{-\infty}^z \left(\frac{1}{a(z)} + d(z) + a(z)\right) \, dz = +\infty$,

(ii) $A$ is a q.e.s.a. operator (i.e. $C^\infty_0(\Omega)$ is dense in $E$) if and only if $\int_{-\infty}^z \left(\frac{1}{a(z)} + d(z) + a(z)\right) \, dz = +\infty$ and $\int_0^\infty \left(\frac{1}{a(z)} + d(z) + a(z)\right) \, dz = +\infty$.

**Proof.** The proof goes through three steps: first reducing the problem to a one-dimensional case, second proving that compactly supported functions are dense under the given hypotheses, and finally getting the desired result.

**First step: reduction to the one-dimensional case.**

Let $(\lambda_k, k \geq 0)$ be the spectrum of $-\Delta_x$, with $\lambda_0 = 0$ and $\lambda_k$ an increasing sequence, and let $(\psi_k)_{k \geq 0}$ be an associated orthonormal basis of $L^2(\mathcal{M})$.

We define, for each $k \geq 0$,

$$A_k u(z) = \frac{1}{a(z)} \left( -(b(z)u'(z))' + (\lambda_k c(z) + d(z))u(z) \right),$$

(4)

for $u \in C^\infty_0((0, +\infty))$, with the underlying Hilbert space $H_0 = L^2((0, +\infty), a(z) \, dz)$ and energy spaces $E_k = \{u \in H_0 \cap H^1_{\text{loc}}((0, +\infty)) : b_k(u, u) < +\infty\}$, where

$$b_k(u, v) = \int_0^{+\infty} b(z)u'(z)v'(z) \, dz + \int_0^{+\infty} (\lambda_k c(z) + d(z))u(z)v(z) \, dz.$$
Then, we consider the Hilbert spaces $E_k$ with their natural norms
\[
\|u\|_E^2 = \int_0^{+\infty} b(z)|u'(z)|^2 \, dz + \int_0^{+\infty} (\lambda_k c(z) + d(z) + a(z))|u(z)|^2 \, dz.
\]

\[\square\]

**Lemma 3.3.** $C_c^\infty(\Omega) \cap E$ (respectively, $C_0^\infty(\Omega)$) is dense in $E$ if and only if $C^\infty([0, +\infty)) \cap E_k$ (respectively, $C_0^\infty(0, +\infty)$) is dense in $E_k$ for all $k \geq 0$.

**Proof.** Given $\varphi \in E$, it can be decomposed into a sum $\varphi = \sum_{k \geq 0} u_k \otimes \psi_k$, where $u_k \in E_k$ and
\[
\|\varphi\|_E^2 = \sum_{k \geq 0} \|u_k\|_{E_k}^2.
\]
So, density in $E$ implies density in each $E_k$.

For the reciprocal, given $\varphi \in E$, we first approximate it by the functions $\varphi_m = \sum_{k=0}^m u_k \otimes \psi_k$, and density in $E_k$ for all $k \geq 0$ implies that each $\varphi_m$ can be approximate by functions of $C^\infty_c(\Omega) \cap E$ (respectively, $C_0^\infty(\Omega)$).

\[\square\]

**Second step: density of compactly supported functions in $E_0$.**

Here, for convenience we shall restrict our attention at first to the case $k = 0$ and $d(z) \equiv 0$.

We define
\[
E_{0,e} = E_0 \cap \{ \text{functions with compact support in } [0, +\infty) \},
\]
\[
E_{0,0} = E_0 \cap \{ \text{functions with compact support in } (0, +\infty) \}.
\]

**Lemma 3.4.** $E_{0,e}$ is dense in $E_0$ if and only if $\int_0^{+\infty} \left( \frac{1}{b(z)} + a(z) \right) \, dz = +\infty$.

**Proof.** Assume first that $\int_0^{+\infty} \left( \frac{1}{b(z)} + a(z) \right) \, dz < +\infty$. If $u \in E_0$, then $u' \in L^1([z', +\infty))$ for any $z' > 0$, since $\int_0^{+\infty} \frac{1}{b(z)} \, dz < +\infty$ and using Hölder inequality. Moreover, $\lim_{z \to +\infty} u(z)$ exists and is not necessarily zero because $\int_0^{+\infty} a(z) < +\infty$. Thus, there exists a linear functional on $E_0$ which vanishes on $E_{0,e}$, but not everywhere, showing that $E_{0,e}$ is not dense in $E_0$. Such functional may be
\[
\lambda(u) = \int_0^{+\infty} (u(z) \eta(z))' \, dz,
\]
where $\eta(z)$ is a smooth function such that $\eta(z) = 0$ if $z \in [0, z']$ and $\eta(z) = 1$ if $z \geq 2z'$.

Assuming now that $\int_0^{+\infty} \left( \frac{1}{b(z)} + a(z) \right) \, dz = +\infty$, we shall see that $E_{0,e}$ is dense in $E_0$.

If there exists $z' > 0$ such that $\int_{z'}^{+\infty} \frac{1}{b(z)} \, dz < +\infty$, taking $u \in E_0$, we again have that $u' \in L^1([z', +\infty))$, but now $\lim_{z \to +\infty} u(z) = 0$ necessarily, since $\int_0^{+\infty} a(z) \, dz = +\infty$. Thus, we have
\[
u(z) = -\int_z^{+\infty} u'(s) \, ds.
\]
Hence, defining $\beta_0(z) = \int_z^{+\infty} \frac{1}{b(z)} \, dz$ and using Hölder inequality, we have
\[
\|u(z)\| \leq \sqrt{\beta_0(z)} \left( \int_z^{+\infty} b(z) |u'(z)|^2 \, dz \right)^{1/2}.
\]
(5)

Since $\|u\|_{E_0} < +\infty$, for $\varepsilon > 0$, there exists $z_0 > 0$ such that
\[
\int_{z_0}^{+\infty} (b(z)|u'(z)|^2 + a(z)|u(z)|^2) \, dz \leq \varepsilon.
\]
(6)
We define $\chi(z)$ on $[0, +\infty)$ by

$$
\chi(z) = \begin{cases} 
1 & \text{if } 0 \leq z \leq z_0, \\
\ln \left( \frac{\beta(z)}{\beta(0)} \right) & \text{if } z_0 \leq z \leq z_1, \\
0 & \text{if } z_1 \leq z \leq +\infty,
\end{cases}
$$

with $z_1$ given by the equation $\beta_0(z_1) = e^{-1} \beta_0(z_0)$. Then, we have

$$
\|u - u\chi\|_{E_0}^2 \leq \int_{z_0}^{+\infty} a(z)(1 - \chi(z))^2|u(z)|^2 \, dz + \int_{z_0}^{+\infty} b(z)(1 - \chi(z))^2|u'(z)|^2 \, dz
$$

$$
+ \int_{z_0}^{+\infty} b(z)\chi'(z)^2|u(z)|^2 \, dz.
$$

The first two terms are small by (6), and for the third one, we have from (5) and (6)

$$
\int_{z_0}^{+\infty} b(z)\chi'(z)^2|u(z)|^2 \, dz \leq \int_{z_0}^{+\infty} \frac{1}{b(z)\beta_0(z)^2} |u(z)|^2 \, dz
$$

$$
\leq \varepsilon \int_{z_0}^{+\infty} \frac{1}{b(z)\beta_0(z)} \, dz
$$

$$
\leq C\varepsilon.
$$

Since $u\chi \in E_{0,c}$, the density of $E_{0,c}$ in $E_0$ is proved.

For the case when $\int_{z_0}^{+\infty} \frac{1}{b(z)} \, dz = +\infty$, given $z' > 0$ we define $\beta_0(z) = \int_{z'}^{z} \frac{1}{b(s)} \, ds$, and we choose $z^*, z$ such that $z' \leq z^* \leq z$. We have

$$
|u(z) - u(z^*)| \leq \int_{z^*}^{z} |u'(s)| \, ds \leq \left( \int_{z^*}^{+\infty} b(s)|u'(s)|^2 \, ds \right)^{\frac{1}{2}} \sqrt{\beta_0(z)},
$$

hence

$$
|u(z)| \leq |u(z^*)| + \left( \int_{z^*}^{+\infty} b(s)|u'(s)|^2 \, ds \right)^{\frac{1}{2}} \sqrt{\beta_0(z)}.
$$

This implies

$$
\lim_{z \to +\infty} \frac{|u(z)|}{\sqrt{\beta_0(z)}} = 0. \tag{7}
$$

Now, by (7) for any $\varepsilon > 0$, there exists $z_0 > 0$ such that

$$
\frac{|u(z_0)|^2}{\beta_0(z_0)} + \int_{z_0}^{+\infty} (b(z)|u'(z)|^2 + a(z)|u(z)|^2) \, dz \leq \varepsilon.
$$

Then,

$$
|u(z)| \leq |u(z_0)| + \sqrt{\varepsilon \sqrt{\beta_0(z)}},
$$

when $z \geq z_0$. We define $\chi(z)$ by

$$
\chi(z) = \begin{cases} 
1 & \text{if } 0 \leq z \leq z_0, \\
\ln \left( \frac{\beta(z)}{\beta(0)} \right) & \text{if } z_0 \leq z \leq z_1, \\
0 & \text{if } z_1 \leq z \leq +\infty,
\end{cases}
$$

with $z_1$ given by the equation $\beta_0(z_1) = e^{-1} \beta_0(z_0)$, and we can prove, as above, that there exists a constant $C$ such that

$$
\|u - u\chi\|_{E_0}^2 \leq C\varepsilon.
$$

Thus, in this case also, $E_{0,c}$ is dense in $E_0$. \hfill \square
Lemma 3.5.

(i) The set of all \( u \in \mathcal{E}_0 \) which vanishes in some neighbourhood of 0 (depending on \( u \)) is dense in \( \mathcal{E}_0 \) if and only if \( \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \).

(ii) \( \mathcal{E}_{0,0} \) is dense in \( \mathcal{E} \) if and only if \( f^\infty \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \) and \( \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \).

Proof.

(i) We consider the transformation \( \phi(z) = \frac{1}{z} : (0, +\infty) \to (0, +\infty) \) and let

\[
\mathcal{E}_{\phi} = \left\{ u \in H_{loc}^1((0, +\infty)) : \|u\|^2_{\phi} = \int_0^{+\infty} \left( b_{\phi}(z)|u'(z)|^2 + a_{\phi}(z)|u(z)|^2 \right) \, dz < +\infty \right\},
\]

where \( b_{\phi}(z) = z^2 b(1/z) \) and \( a_{\phi}(z) = a(1/z)/z^2 \).

Then \( \mathcal{E}_\phi \) and \( \mathcal{E}_{0} \) are isomorphic, through the application \( \Phi : \mathcal{E}_0 \to \mathcal{E}_\phi \) given by \( \Phi(v) = u = v \circ \phi \).

By lemma 3.4, \( \mathcal{E}_{\phi,c} \) is dense in \( \mathcal{E}_\phi \) if and only if \( \int^{+\infty} \left( \frac{1}{y(z)} + a_{\phi}(z) \right) \, dz = \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \), and we observe that \( v \in \mathcal{E}_0 \) vanishes in a neighbourhood of 0 if and only if \( \Phi(v) \in \mathcal{E}_{\phi,c} \).

(ii) follows directly from both assertion (i) and lemma 3.4.

In this step, we assume that \( d = 0 \) and \( k = 0 \). When \( d \) or \( k \) are not vanishing, then it suffices to replace \( a(z) \) by \( a(z) + d(z) + k \) to obtain the appropriate versions of lemmas 3.4 and 3.5.

Third step: conclusion in the one-dimensional case.

Lemma 3.6.

(i) \( C^\infty_0((0, +\infty)) \cap \mathcal{E}_0 \) is dense in \( \mathcal{E}_0 \) if and only if \( f^{+\infty} \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \).

(ii) \( C^\infty_0((0, +\infty)) \) is dense in \( \mathcal{E}_0 \) if and only if \( f^{+\infty} \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \) and \( \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \).

Proof. (ii) Assume first \( C^\infty_0((0, +\infty)) \) is dense in \( \mathcal{E}_0 \), then \( \mathcal{E}_{0,0} \) must be dense too, and this implies, by lemma 3.5, \( f^{+\infty} \left( \frac{1}{y(z)} + a(z) \right) \, dz = \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \).

Reciprocally, if \( f^{+\infty} \left( \frac{1}{y(z)} + a(z) \right) \, dz = \int_0^1 \left( \frac{1}{y(z)} + a(z) \right) \, dz = +\infty \), by lemma 3.5, \( \mathcal{E}_{0,0} \) is dense in \( \mathcal{E}_0 \). Therefore, it suffices to prove that \( C^\infty_0((0, +\infty)) \) is dense in \( \mathcal{E}_{0,0} \). For this purpose, we will show that for any compact interval \( I = [z_0, z_1] \subset (0, +\infty) \), \( C^\infty_0(I) \) is dense in \( \mathcal{E}_I : = \{ u \in \mathcal{E}_0 : \text{supp} u \subset I \} \).

Let \( m = \int_I b(z) \, dz \) and define \( \phi : I \to J = [0, m] \) by \( \phi(z) = \int_0^z b(s) \, ds \). Then, \( L^2(I, b(z) \, dz) \) and \( L^2(J, ds) \) are isomorphic through the application \( \Phi : L^2(I, ds) \to L^2(J, b(z) \, dz) \) such that \( \Phi(v) = v \circ \phi \).

Let \( u \in \mathcal{E}_I \), and denote \( f = u' \) and \( g = f \circ \phi^{-1} \), \( g \in L^2(J, ds) \), then there exists a sequence \( (g_n)_{n \geq 0} \) such that \( g_n \in C^\infty_0(J) \) for all \( n \geq 0 \) and \( g_n \to g \) in \( L^2(J, ds) \). Let \( f_n = g_n \circ \phi \), then \( f_n \in C^\infty_0(I) \) and \( f_n \to f \) in \( L^2(I, b(z) \, dz) \), we also have that

\[
\int_I \left| f(z) - f_n(z) \right| \, dz \leq C \left( \int_I b(z) \left| f(z) - f_n(z) \right|^2 \, dz \right)^{1/2},
\]

\( C_0(J) \) is the space of continuous functions with compact support in \((0, m)\).
by Cauchy–Schwarz inequality and because \( \frac{1}{n} \in L_{\text{loc}}^1((0, \infty)) \). Since \( \int_I f(z) \, dz = 0 \), we deduce that
\[
\lim_{n \to \infty} \int_I f_n(z) \, dz = 0.
\]
We choose \( \chi \in C_0(I) \), such that \( \int_I \chi(z) \, dz = 1 \), and define
\[
\tilde{f}_n = f_n - \left( \int_I f_n(z) \, dz \right) \chi.
\]
Then, \( \int_I \tilde{f}_n(z) \, dz = 0 \), \( \tilde{f}_n \in C_0(I) \) and \( \tilde{f}_n \to f \) in \( L^2(I, b(z) \, dz) \):
\[
\int_I b(z)(f(z) - \tilde{f}_n(z))^2 \, dz \leq \int_I b(z)(f(z) - f_n(z))^2 \, dz + \left( \int_I f_n(z) \, dz \right)^2 \int_I b(z)\chi(z)^2 \, dz
\]
\[
\quad \to 0.
\]
We set
\[
\tilde{u}_n(z) = \int_{I_0} \tilde{f}_n(s) \, ds,
\]
since \( \int_I \tilde{f}_n(z) \, dz = 0 \), \( \tilde{u}_n(z) \in C_0(I) \) for all \( n \geq 0 \), and by (8),
\[
\lim_{n \to \infty} \int_I b(z)|u'(z) - \tilde{u}_n'(z)|^2 \, dz = 0,
\]
and
\[
\lim_{n \to \infty} \|u - \tilde{u}_n\|_{\infty} = 0
\]
because
\[
\lim_{n \to \infty} \int_I |f(z) - \tilde{f}_n(z)| \, dz = 0.
\]
Hence, we have
\[
\lim_{n \to \infty} \int_I a(z)|u(z) - \tilde{u}_n(z)|^2 \, dz = 0,
\]
so that, finally,
\[
\lim_{n \to \infty} \|u - \tilde{u}_n\|_{E_1} = 0.
\]
This proves the density of \( C^0_0(I) \) in \( \mathcal{E}_I \). To pass from \( C^0_0(I) \) to \( C^\infty_0(I) \), a classical regularization procedure is enough: it shows that \( C^\infty_0(I) \) is dense in \( C^0_0(I) \) for the topology given by the norm
\[
\sup_{z \in I} |u(z)| + \sup_{z \in I} |u'(z)|;
\]
since \( a \) and \( b \) are integrable on \( I \), this implies the same density for the topology induced by \( \mathcal{E}_I \), and part (ii) of the lemma is completely proved.

Regarding part (i), we will be sketchy. The necessity of the condition \( \int^{+\infty} (\frac{1}{|\mathcal{E}|} + a(z)) \, dz = +\infty \) follows from lemma 3.4. Its sufficiency needs only to be proved when \( C^\infty_0((0, \infty)) \) is not dense, that is to say when \( \int_0^1 (\frac{1}{|\mathcal{E}|} + a(z)) \, dz < +\infty \).

But then, the same proof as above works, even when \( I = [0, z_1] \).

**Proof of theorem 3.2.** Let us now prove theorem 3.2 (ii): if \( C^\infty_0(\Omega) \) is dense in \( \mathcal{E} \), by lemma 3.3 \( C^\infty_0((0, +\infty)) \) is dense in \( \mathcal{E}_I \) for all \( k \geq 0 \), in particular for \( k = 0 \), then by lemma 3.6, we have
\[
\int^{+\infty} (\frac{1}{|\mathcal{E}|} + d(z) + a(z)) \, dz = \int_0^1 (\frac{1}{|\mathcal{E}|} + d(z) + a(z)) \, dz = +\infty.
\]
Conversely, if \( \int_1^{+\infty} \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = \int_0^1 \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = +\infty \), we also have
\[
\int_1^{+\infty} \left( \frac{1}{b(z)} + d(z) + a(z) + \lambda_4 c(z) \right) \, dz = \int_0^1 \left( \frac{1}{b(z)} + d(z) + a(z) + \lambda_4 c(z) \right) \, dz = +\infty,
\]
then \( C_0^\infty((0, +\infty)) \) is dense in \( \mathcal{E}_k \) for all \( k \), we can see it changing \( a(z) \) by \( d(z) + a(z) + \lambda_4 c(z) \) in all the previous results, and again by lemma 3.3, \( C_0^\infty(\Omega) \) is dense in \( \mathcal{E} \).

The proof of (i) analogously follows. Theorem 3.2 is completely proved. \( \Box \)

**Remark 3.7.** Under different hypotheses, when the coefficients of the operator \( A \) depend on \((z, x)\), we have given a characterization of \( q.e.s.a. \) operators in [9]. Warning: in page 21 of that reference, the integrand of (43) was mistakenly written as \( \frac{1}{M_{n+1,n+1}(z, x)} \) instead of \( (M^{-1})_{n+1,n+1}(z, x) \).

**Essentially self-adjointness characterization**

The characterization of \( e.s.a. \) for the operator \( A \) defined in (3) will rely on the real-valued solutions of the ODE
\[
-b(z)u'(z)' + d(z)u(z) = 0
\]
on \((0, z')\) and on \((z', +\infty)\).

A typical case is when \( \int_0^1 a(z) \, dz < +\infty \), but \( \int_1^{+\infty} a(z) \, dz = +\infty \). Then, since we may assume \( A \) to be \( q.e.s.a. \) (otherwise it cannot be \( e.s.a. \)), we have \( \int_0^1 \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty \). In such a case, we will show that there is a unique solution of (9), denoted by \( \alpha \), such that
\[
\alpha \text{ is a solution of (9) in } (0, z'), \\
\alpha(z') = 1, \\
\int_0^{z'} b(z)\alpha'(z)^2 + d(z)\alpha(z)^2 \, dz < +\infty.
\]

Then, we define \( \beta(z), z \in (0, z') \), by
\[
\beta(z) = \alpha(z) \int_0^z \frac{1}{b(s)\alpha(s)^2} \, ds.
\]

Note that, by construction, \( \beta \) is another solution of (9) in \((0, z')\). We shall prove that: \( A \) is \( e.s.a. \), if and only if \( \int_0^1 \beta(z)^2 a(z) \, dz = +\infty \).

In the case where the role of 0 and \(+\infty\) are exchanged, the result is similar. We will show that there exists a unique function \( \alpha \) such that
\[
\alpha(z) \text{ is a solution of (9) in } (z', +\infty), \\
\alpha(z') = 1, \\
\int_{z'}^{+\infty} b(z)\alpha'(z)^2 + d(z)\alpha(z)^2 \, dz < +\infty.
\]

Then, we define \( \beta(z), z \in (z', +\infty) \), by
\[
\beta(z) = \alpha(z) \int_{z'}^{+\infty} \frac{1}{b(s)\alpha(s)^2} \, ds,
\]
and we shall prove that \( A \) is \( e.s.a. \) if and only if \( \int_{z'}^{+\infty} \beta(z)^2 a(z) \, dz = +\infty \).

Note that, when \( d(z) \equiv 0 \), the problem considerably simplifies since, in this case, \( \alpha \equiv 1 \) and \( \beta(z) \) turns out to be either \( \beta_0(z) = \int_0^{z'} \frac{1}{b(s)} \, ds \) or \( \beta_0(z) = \int_{z'}^{+\infty} \frac{1}{b(s)} \, ds \), respectively.

**Notation 3.8.** We denote \((\alpha(z), \beta(z))\) the above couples of solutions of (9); the context will indicate whether \( z \in (0, z') \), in which case \((\alpha(z), \beta(z))\) are given by (10) and (11), or \( z \in (z', +\infty) \), where \((\alpha(z), \beta(z))\) are given by (12) and (13).

With this notation, the result is the following.
Theorem 3.9. Assume the operator $A$ given in (3) to be q.e.s.a. that is to say
\[
\int_0^1 \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = \int^{+\infty} \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = +\infty.
\]
Then,
(i) If $\int_0^1 a(z) \, dz = \int^{+\infty} a(z) \, dz = +\infty$, then $A$ is e.s.a.;
(ii) if $\int_0^1 a(z) \, dz < +\infty$ and $\int^{+\infty} a(z) \, dz = +\infty$, then $A$ is e.s.a. if and only if
\[
\int^{+\infty} b(z)^2 a(z) \, dz = +\infty;
\]
(iii) if $\int_0^1 a(z) \, dz = +\infty$ and $\int^{+\infty} a(z) \, dz < +\infty$, then $A$ is e.s.a. if and only if
\[
\int^{+\infty} b(z)^2 a(z) \, dz = +\infty;
\]
(iv) if $\int_0^1 a(z) \, dz < +\infty$ and $\int^{+\infty} a(z) \, dz < +\infty$, then $A$ is e.s.a. if and only if
\[
\int_0^{+\infty} b(z)^2 a(z) \, dz = +\infty.
\]

Remark 3.10. Take care of the uniqueness of $\alpha$ (and thus the meaningfulness of the definitions above): it holds when $\int_0^1 \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty$ or $\int^{+\infty} \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty$, according to where the variable $z$ lives.

Preliminary step: study of solutions of (9)

Lemma 3.11. Let $u(z)$ be a solution of (9) in some interval $I \subset (0, +\infty)$. Then, the function $b(z)u(z)u'(z)$ is increasing in $I$.

Proof. From (9) we obtain
\[-(b(z)u(z)u'(z))' + b(z)u'(z)^2 + d(z)u(z)^2 = 0,
\]
showing that $(b(z)u(z)u'(z))'$ is nonnegative. 

Lemma 3.12. Let $u(z)$ be a solution of (9) in $(0, z')$. Then,
\[
\int_0^z (b(z)u'(z)^2 + d(z)u(z)^2) \, dz = +\infty
\]
if and only if
\[
\lim_{z \to 0^+} b(z)u'(z)u(z) = -\infty.
\]

Proof. Since $u(z')$ and $u'(z')$ exist, the proof follows immediately from the fact that, for $0 < z_0 < z'$, we have
\[
\int_{z_0}^{\zeta} (b(z)u'(z)^2 + d(z)u(z)^2) \, dz = \int_{z_0}^{z'} (b(z)u(z)u'(z))' \, dz = b(z')u'(z')u(z') - b(z_0)u'(z_0)u(z_0).
\]
Lemma 3.13. Let \( z' > 0 \) be chosen.

(i) There exists at least one solution \( \alpha(z) \) of (9), in the interval \((0, z')\), such that
\[
\alpha(z') = 1
\]
and
\[
\int_0^{z'} (b(z)\alpha'(z)^2 + d(z)\alpha(z)^2) \, dz < +\infty.
\]
This solution is positive and increasing in \((0, z')\), satisfying
\[
\lim_{z \to 0^+} b(z)\alpha'(z)\alpha(z) = 0.
\]
(ii) If in addition \( \int_0^{z'} \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty \), this solution is unique.

Proof. Let \( L^2((0, z')) \) be the space of measurable functions \( f(z) \) such that
\[
\int_0^{z'} b(z)f(z)^2 \, dz < +\infty.
\]
We define, for any \( f \) in this space, the function \( T \) by
\[
Tf(z) = 1 - \int_z^{z'} f(s) \, ds,
\]
so that \( T \in C((0, z')) \cap H^1_{\text{loc}}((0, z')) \), with \( (Tf)'(z) = f(z) \). Let
\[
q(f) = \int_0^{z'} (b(z)f(z)^2 + d(z)(Tf(z))^2) \, dz,
\]
taking values in \((0, +\infty]\), and
\[
q_0 = \inf_{f \in L^2((0, z'))} q(f).
\]
Note that \( q_0 \) is finite since, for example, for \( f(z) = \frac{1}{z - z_0} \mathbf{1}_{[z_0, z']}((z) \) for some \( 0 < z_0 < z' \), \( q(f) < +\infty \). We shall show that \( q_0 \) is in fact a minimum. To this end, let \( (f_n)_{n \in \mathbb{N}} \) be a minimizing sequence
\[
\lim_{n \to +\infty} q(f_n) = q_0.
\]
Then, by construction,
\[
\sup_{n \in \mathbb{N}} \|f_n\|_{L^2} < +\infty,
\]
so that (up to extracting a subsequence) we may suppose that the sequence \( (f_n) \) has a weak limit \( f_0 \) in \( L^2((0, z')) \). Let us prove that \( q(f_0) = q_0 \).

For any \( z_0 \in (0, z') \) and for all \( z \geq z_0 \)
\[
|Tf_n(z)| \leq 1 + \left( \int_{z_0}^{z'} \frac{1}{b(z)} \, dz \right)^{1/2} \|f_n\|_{L^2} \leq C(z_0),
\]
and
\[
Tf_0(z) \mathbf{1}_{[z_0, z']} = \lim_{n \to +\infty} Tf_n(z) \mathbf{1}_{[z_0, z']}.
\]
So, by dominated convergence, we have
\[
\lim_{n \to +\infty} \int_{z_0}^{z'} d(z)(Tf_n(z))^2 \, dz = \int_{z_0}^{z'} d(z)(Tf_0(z))^2 \, dz.
\]
Also we know that
\[
\int_{z_0}^{z} b(z) f_0(z)^2 \, dz \leq \liminf_{n \to +\infty} \int_{z_0}^{z} b(z) f_n(z)^2 \, dz,
\]
since \( f_0 = \text{w-lim} \, f_n \) in \( L^2((0, z')) \) as well. From these two facts, we deduce
\[
\int_{z_0}^{z} (b(z) f_0(z)^2 + d(z) (T f_0(z))^2) \, dz
\leq \liminf_{n \to +\infty} \int_{z_0}^{z} (b(z) f_n(z)^2 + d(z) (T f_n(z))^2) \, dz \leq q_0.
\]
Letting \( z_0 \to 0^+ \), we obtain \( q(f_0) \leq q_0 \), and thus \( q(f_0) = q_0 \) as desired.

Let now \( \alpha(z) = T f_0(z) \). For any \( u \in C((0, z')) \bigcap H^1_{\text{loc}}((0, z')) \), with \( u(z') = 1 \), we define
\[
Q(u) = q(u') = \int_{0}^{z} (b(z) u'(z)^2 + d(z) u(z)^2) \, dz.
\]
We have proved that
\[
Q(\alpha) = \min_{u} Q(u).
\]
We define
\[
\alpha^+ = \begin{cases} \alpha & \text{if } \alpha \geq 0 \\ 0 & \text{if not} \end{cases}
\quad \text{and} \quad
\alpha^- = \begin{cases} -\alpha & \text{if } \alpha \leq 0 \\ 0 & \text{if not,} \end{cases}
\]
so that \( \alpha = \alpha^+ - \alpha^- \) and \( \alpha^+ \alpha^- = 0 \). Then, we have that \( Q(\alpha^+) \leq Q(\alpha) \) with strict inequality if and only if \( \alpha^- \neq 0 \), and since \( \alpha^+ \in C((0, z')) \bigcap H^1_{\text{loc}}((0, z')) \) with \( \alpha^+(z') = 1 \), we must have
\[
Q(\alpha^+) = Q(\alpha),
\]
and \( \alpha^+ = \alpha \), i.e., \( \alpha \) is positive in \((0, z']\).

If \( \psi \in C((0, z')) \bigcap H^1_{\text{loc}}((0, z')) \) is such that \( Q(\alpha + t\psi) < +\infty \) for all \( t \in \mathbb{R} \) and \( \psi(z') = 0 \), we must have
\[
Q(\alpha) \leq Q(\alpha + t\psi),
\]
and this implies
\[
\int_{0}^{z} (b(z) \alpha(z) \psi(z)' + d(z) \alpha(z) \psi(z)) \, dz = 0.
\]
This, in particular, is true for all \( \psi \in C^0_{\text{loc}}((0, z')) \), implying that
\[
-(b(z) \alpha(z) \psi(z)' + d(z) \alpha(z) = 0
\]
in \((0, z']\).

But then, this means that
\[
\int_{0}^{z} (b(z) \alpha(z) \psi(z)' + (b(z) \alpha(z))' \psi(z)) \, dz = 0
\]
for all \( \psi \in C((0, z')) \bigcap H^1_{\text{loc}}((0, z')) \) with \( Q(\alpha + t\psi) < +\infty \) and \( \psi(z') = 0 \). Therefore,
\[
\lim_{z \to 0^+} b(z) \alpha'(z) \psi(z) = 0.
\]
Choosing \( \psi = a \eta \), where \( \eta \in \mathcal{C}^\infty(0, +\infty) \), \( \eta = 1 \) near 0 and \( \eta = 0 \) near \( z' \), we obtain
\[
\lim_{z \to 0^+} b(z) = 0.
\]

With lemma 3.11, this shows that (recall that \( \alpha \) is positive) \( \alpha' \eta \) and hence \( \alpha \) are both increasing in \((0, 1)\). Thus, part (i) is entirely proved.

(ii) Let
\[
\beta(z) = \alpha(z) \int_z^{z'} \frac{1}{b(s)\alpha(s)^2} \, ds.
\]
Then, \( \beta(z) \) is another solution of (9) in \((0, z')\), so that any solution writes \( \lambda \alpha(z) + \mu \beta(z), \lambda, \mu \in \mathbb{R} \). The uniqueness of \( \alpha(z) \) will follow from the proof of
\[
\int_0^z (b(z)\beta'(z)^2 + d(z)\beta(z)^2) \, dz = +\infty.
\]  

(14)

A direct calculation shows that \( \beta'(z') = 0 \) and \( \beta'(z) = -\frac{1}{b(z)} \). Thus, from the ODE (9), we obtain
\[
-\beta'(z) = \frac{1}{b(z)} + \frac{1}{b(z)} \int_z^{z'} d(s)\beta(s) \, ds.
\]
Since \( \beta \) is positive by construction, it turns out to be decreasing in \((0, z')\), with
\[
|\beta'(z)| \geq \frac{1}{b(z)}, \quad 0 < z \leq z',
\]
and
\[
\beta(z) \geq \int_{z'}^z \frac{1}{b(s)} \, ds =: \beta_0(z).
\]  

(15)

Hence, there exists a constant \( C \) such that \( \beta(z) \geq C \) if \( z \leq z'/2 \), and we obtain
\[
\int_0^z (b(z)\beta'(z)^2 + d(z)\beta(z)^2) \, dz \geq \int_0^z \frac{1}{b(z)} \, dz + C^2 \int_0^{z'/2} d(z) \, dz = +\infty.
\]

The lemma is proved. \( \square \)

Lemma 3.13 has an analogous counterpart near \(+\infty\), which is the following.

**Lemma 3.15.** Let \( z' > 0 \) be chosen.

(i) There exists at least one solution \( \alpha(z) \) of (9), in the interval \((z', +\infty)\), such that
\[
\alpha(z') = 1
\]
and
\[
\int_{z'}^{+\infty} (b(z)\alpha'(z)^2 + d(z)\alpha(z)^2) \, dz < +\infty.
\]

This solution is positive and decreasing in \((z', +\infty)\), satisfying
\[
\lim_{z \to +\infty} b(z)\alpha'(z)\alpha(z) = 0.
\]

(ii) If in addition \( \int_{z'}^{+\infty} \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty \), this solution is unique.

**Proof.** By making the change of variable \( z \mapsto \frac{z}{z'} \), the proof immediately follows from the previous lemma. \( \square \)

**Remark 3.15.** The function \( \alpha(z) \) given in \((0, z')\) (respectively, in \((z', +\infty)\)) by lemma 3.13 (respectively, lemma 3.14) is not a solution of (9) on \((0, +\infty)\), but of
\[
-(b(z)\alpha'(z)')' + d(z)\alpha(z) = \lambda \delta_{z'}(z),
\]
where \( \delta_z(z) \) is the Dirac measure at \( z = z' \), and \( \lambda = \int_0^{+\infty} (b(z)\alpha'(z)^2 + d(z)\alpha(z)^2) \, dz \).
Main step: e.s.a. characterization in dimension 1

Let us consider now the operator
\[ A_0 u(z) = \frac{1}{a(z)} \left( -(b(z)u'(z))' + d(z)u(z) \right) \]
defined as in (4) with
\[ \int_0^1 \left( \frac{1}{b(z)} + d(z) \right) \, dz = \int_0^{+\infty} \left( \frac{1}{b(z)} + d(z) \right) \, dz = +\infty. \]  \( (16) \)

Lemma 3.16. If \( \int_0^1 a(z) \, dz = \int_0^{+\infty} a(z) \, dz = +\infty \), \( A_0 \) is an e.s.a. operator.

Proof. Assume \( A_0 \) is not e.s.a.. By lemma 2.5 there exists \( u \in H_0 \) such that
\[ -(b(z)u'(z))' + d(z)u(z) + a(z)u(z) = 0, \]
and \( u \notin E_0 \), i.e., either \( \int_0^1 b(z)u'(z)^2 + (d(z) + a(z)) \, d(z) = +\infty \) or \( \int_0^{+\infty} (b(z)u'(z)^2 + (d(z) + a(z)) \, d(z) = +\infty \) (or both).

If \( \int_0^1 b(z)u'(z)^2 + (d(z) + a(z)) \, d(z) = +\infty \), by lemma 3.12 (changing \( d \) in \( d + a \)), we have
\[ \lim_{z \to 0} b(z)u'(z)u(z) = -\infty. \]
In particular, \( u'(z)u(z) < 0 \) for \( z \leq z_0 \), for some \( z_0 > 0 \), so that \( u^2 \) is decreasing in \( (0, z_0] \). But since \( \int_0^{+\infty} a(z) \, dz < +\infty \), this implies \( \int_0^1 a(z) \, dz < +\infty \), which is a contradiction.

If \( \int_0^{+\infty} (b(z)u'(z)^2 + (d(z) + a(z)) \, d(z) = +\infty \), a change of variable reduces the proof to the preceding case. \( \square \)

Lemma 3.17. Assume \( \int_0^1 a(z) \, dz < +\infty \) and \( \int_0^{+\infty} a(z) \, dz = +\infty \). Then, \( A_0 \) is an e.s.a. operator if and only if \( \int_0^1 b(z)^2a(z) \, dz = +\infty \).

Proof. We first assume that \( \int_0^1 b(z)^2a(z) \, dz < +\infty \). We set \( u(z) = \beta(z) \eta(z) \) with \( \eta \in C^\infty(0, +\infty) \), \( \eta = 1 \) near 0 and \( \eta = 0 \) for \( z \geq \varepsilon \). Then, \( u \in H_0 \) and \( A_0^*u \in H_0 \). But by the hypotheses \( (16) \), \( u \notin E_0 \) (see \( (14) \) in the proof of lemma 3.13). Thus, \( A_0 \) is not e.s.a.. Reciprocally, assume that \( A_0 \) is not e.s.a.. Then, there exists \( u \in H_0 \) such that
\[ -(b(z)u'(z))' + d(z)u(z) + a(z)u(z) = 0, \]
and \( u \notin E_0 \).
Since \( \int_0^{+\infty} a(z) \, dz = +\infty \) and \( \int_0^{+\infty} u(z)^2a(z) \, dz < +\infty \), the same argument as in lemma 3.16 shows that necessarily
\[ \int_0^{+\infty} (b(z)u'(z)^2 + d(z)u(z)^2) \, dz < +\infty. \]
Thus, we must have
\[ \int_0^1 (b(z)u'(z)^2 + d(z)u(z)^2) \, dz = +\infty. \]
By lemma 3.12, \( \lim_{z \to 0^+} b(z)u'(z)u(z) = -\infty \), and in particular, \( u^2 \) is decreasing in \( (0, z_0] \) for some \( z_0 > 0 \). We may assume that \( u(z_0) > 0 \) and \( u'(z_0) < 0 \) (up to changing \( u \) in \( -u \)). Let \( C_1 \)
and $C_2$ be two constants such that
\[
\begin{align*}
C_1 \alpha(z_0) + C_2 \beta(z_0) &= u(z_0), \\
C_1 \alpha'(z_0) + C_2 \beta'(z_0) &= u'(z_0).
\end{align*}
\]
They exist because we know that the Wronskian $b(z)(\alpha(z)\beta'(z) - \alpha'(z)\beta(z))$ is never vanishing. Moreover, we must have $C_2 \neq 0$, otherwise $u(z_0)$ and $u'(z_0)$ would have the same sign (recall that $\alpha$ is positive and increasing, by lemma 3.13). We even have $C_2 > 0$.\(^8\)

Let $v(z) = u(z) - C_1 \alpha(z) - C_2 \beta(z)$. We have
\[ -(b(z)v'(z))' + d(z)v(z) + a(z)u(z) = 0, \]
with $v(z_0) = v'(z_0) = 0$, $u > 0$ in $(0, z_0]$. By classical arguments, $v$ must be positive and decreasing in $(0, z_0]$.

- It is so in some neighbourhood of $z_0$, because $(b(z)v'(z))' > 0$ near $z_0$ and $v'(z_0) = 0$, so that $v'(z) < 0$ in $(z_0 - \epsilon, z_0)$;
- it cannot change its sense of variation in $(0, z_0) (v(z_1) > 0, v'(z_1) = 0, v''(z_1) \leq 0$ at some $z_1 < z_0$ is impossible).

Hence, since $C_2 > 0$, we have
\[ \beta(z) \leq \frac{1}{C_2} (u(z) - C_1 \alpha(z)) \]
in $(0, z_0]$. Since $\alpha$ is bounded, \( \int_0 a(z) \, dz < +\infty \) and $u \in H_0$, this implies
\[ \int_0 \beta(z)^2 a(z) \, dz < +\infty, \]
and the proof is finished. \( \square \)

**Lemma 3.18.** Assume \( \int_0 a(z) \, dz = +\infty \) and \( \int^{+\infty} a(z) \, dz < +\infty \). Then, $A_0$ is an e.s.a. operator if and only if \( \int_0 \beta(z)^2 a(z) \, dz = \int^{+\infty} \beta(z)^2 a(z) \, dz = +\infty \).

**Proof.** The result follows by a change of variable and the preceding lemma. \( \square \)

**Lemma 3.19.** Assume \( \int_0 a(z) \, dz < +\infty \) and \( \int^{+\infty} a(z) \, dz < +\infty \). Then, $A_0$ is an e.s.a. operator if and only if \( \int^{+\infty} \beta(z)^2 a(z) \, dz = +\infty \).

**Proof.** If $A_0$ is not e.s.a., there exists $u \in H_0$ solution of
\[ -(b(z)u'(z))' + d(z)u(z) + a(z)u(z) = 0, \]
and either \( \int_0 (b(z)u'(z) + d(z)u(z)^2) \, dz = +\infty \) or \( \int^{+\infty} (b(z)u'(z)^2 + d(z)u(z)^2) \, dz = +\infty \). Use the arguments of lemmas 3.17 or 3.18, depending on the case.

Reciprocally, as we have done in lemma 3.17, we consider $u(z) = \beta(z) \eta(z)$ for a suitable $\eta$ and the result follows. \( \square \)

---

\(^7\) In fact, it is a constant, equal to $-1$.

\(^8\) $C_2 = b(z_0)(u(z_0)\alpha'(z_0) - u'(z_0)\alpha(z_0)) > 0$. 

Final step: reduction to the one-dimensional case

Defining the operators $A_k$ as in (4), i.e.,

$$A_k u(z) = \frac{1}{a(z)} \left( -(b(z)u'(z))' + (\lambda_k c(z) + d(z))u(z) \right),$$

we have the following result.

**Lemma 3.20.** $A$ is an e.s.a. operator if and only if for all $k \geq 0$ $A_k$ is an e.s.a. operator.

**Proof.** We use the notation introduced in the first step of the proof of theorem 3.2. By lemma 2.5, if $A_k$ is not e.s.a., there exists $u \in H_{0,k} \subset D(A^*_k)$ but $u \notin E_k$. This implies that $\varphi = u \otimes \psi_k \in D(A^*_k)$ and $\varphi \notin E$, so that $A$ is not e.s.a.

Reciprocally, if $A$ is not e.s.a., there exists $\varphi \in H$ nonvanishing, such that

$$A^* \varphi + \varphi = 0.$$

Decomposing

$$\varphi = \sum_{k \geq 0} u_k \otimes \psi_k$$

there exists $k$ such that $u_k \neq 0$. If $\psi \in C^\infty_0((0, \infty))$, we have

$$0 = \langle \varphi, A(\psi_k \otimes \phi) + \phi \otimes \psi_k \rangle_H = \langle u_k, A_k \phi + \phi \rangle_{H_k},$$

which means that $A^*_k u_k + u_k = 0$. Thus, $A$ is not e.s.a. by lemma 2.5 again. □

**Proof of theorem 3.9.** (i) If $\int_0^a(z) \, dz = +\infty$ and $\int^+\infty a(z) \, dz = +\infty$, then $\int_0^a(z) + \lambda_k c(z) \, dz = +\infty$ and $\int^+\infty (a(z) + \lambda_k c(z)) \, dz = +\infty$, for all $k \geq 0$. Therefore, $A_k$ is e.s.a. by lemma 3.16 with $a$ changed in $a + \lambda_k c(z)$, and by lemma 3.20 $A$ is e.s.a.

In the cases (ii), (iii) and (iv), if $A$ is e.s.a. it follows by lemma 3.20 that in particular $A_0$ is e.s.a.. Then, lemmas 3.17, 3.18 and 3.19 give the result.

For the converse, let us take the case (ii). If $A$ is not e.s.a., by lemma 3.20, there exists $k \geq 0$ such that $A_k$ is not e.s.a.. Then, by lemma 3.17

$$\int_0^\beta_k (z)^2 a(z) \, dz < +\infty,$$

(17)

where $\beta_k$ is the solution of

$$-(b(z)u'(z))' + (c(z)\lambda_k + d(z))u(z) = 0$$

on $(0, z')$ with Cauchy data $u(z') = 0$ and $u'(z') = -\frac{1}{\partial z}$. A classical comparison principle, applied to the functions $\beta_k$ and $\beta$, defined in (11), gives us $0 \leq \beta \leq \beta_k$ on $(0, z')$. Then, (17) implies

$$\int_0^\beta (z)^2 a(z) \, dz < +\infty,$$

as desired.

The other cases are analogous.

Theorem 3.9 is completely proved. □
Remark 3.21. The precise definition of the function $\beta(z)$ is needed only for the sufficiency of the condition
\[ \int_0^z \beta(z)^2 a(z) \, dz < +\infty \]
for $A$ to be e.s.a. This is not used in the reciprocal, where the ‘massless-$\beta$’
\[ \beta_0(z) = \int_z^\infty \frac{1}{b(s)} \, ds \]
would have worked as well (see (15)). But, for the sufficiency, if we choose $u(z) = \beta_0(z) \eta(z)$ in lemma 3.17, with $\eta \in C^\infty([0, +\infty))$, $\eta = 1$ near 0 and $\eta = 0$ for $z \geq \frac{1}{2}$, then
\[ A_0 u(z) = \frac{1}{a(z)} (-b(z)\beta_0(z)\eta'(z) + d(z)\beta_0(z)\eta(z)), \]
and this belongs to $H_0$ only when
\[ \int_0^1 d(z)^2 \beta_0(z)^2 \frac{1}{a(z)} \, dz < +\infty. \]

This gives a necessary and sufficient condition for e.s.a. in terms of $\beta_0(z)$ only, not $\beta(z)$, when $\frac{d(z)}{a(z)}$ is bounded.

Corollary 3.22. When $\frac{d(z)}{a(z)}$ is bounded near 0, $\int_0^1 a(z) \, dz < +\infty$ and $\int_0^1 a(z) \, dz = +\infty$, $A$ is e.s.a. if and only if $\int_0^1 \beta_0(z)^2 a(z) \, dz = +\infty$.

There are similar statements in the other cases.

Remark 3.23. The previous results can be readily extended to a more general domain $(z_0, z_1) \times \mathcal{M}$, $0 \leq z_0 \leq z_1 \leq \infty$. Let us consider $\Omega = (z_0, z_1) \times \mathcal{M}$ and a differential operator $A$ defined as in (3) by
\[ A\varphi(z, x) = \frac{1}{a(z)} \left\{ -\partial_z(b(z)\partial_z\varphi(z, x)) - c(z)\Delta_x\varphi(z, x) + d(z)\varphi(z, x) \right\}, \]
for all $\varphi \in C_0^\infty(\Omega)$, where the functions $a$, $b$, and $c$ satisfy the following hypotheses:
\begin{itemize}
  \item $a, c, d \in L_1^{\infty}((z_0, z_1))$, $b \in C((z_0, z_1))$;
  \item $a > 0$, $b > 0$, $c > 0$ and $d \geq 0$ in $(z_0, z_1)$;
  \item $a^{-1}, b^{-1}, c^{-1} \in L_1^{\infty}((z_0, z_1)).$
\end{itemize}

The previous results straightforwardly generalize to such a case. For the convenience of the reader, we state the two main theorems.

Theorem 3.24. $A$ is a q.e.s.a. operator in $H$ if and only if $\int_0^1 \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = +\infty$ and $\int_{z_0}^z \left( \frac{1}{b(z)} + d(z) + a(z) \right) \, dz = +\infty$.

Theorem 3.25. We assume $A$ is a q.e.s.a. operator. Then,
\begin{enumerate}
  \item[(i)] If $\int_{z_0}^1 a(z) \, dz = \int_0^1 a(z) \, dz = +\infty$, then $A$ is e.s.a.;
  \item[(ii)] If $\int_{z_0}^1 a(z) \, dz < +\infty$ and $\int_0^1 a(z) \, dz = +\infty$, then $A$ is e.s.a. if and only if $\int_0^1 \beta(z)^2 a(z) \, dz = +\infty$;
  \item[(iii)] If $\int_{z_0}^1 a(z) \, dz = +\infty$ and $\int_0^1 a(z) \, dz < +\infty$, then $A$ is e.s.a. if and only if $\int_0^1 \beta(z)^2 a(z) \, dz = +\infty$;
  \item[(iv)] If $\int_{z_0}^1 a(z) \, dz < +\infty$ and $\int_0^1 a(z) \, dz < +\infty$, then $A$ is e.s.a. if and only if $\int_0^1 \beta(z)^2 a(z) \, dz = \int_0^1 \beta(z)^2 a(z) \, dz = +\infty$.
\end{enumerate}
A typical situation where these results apply is when\( f^i(\frac{1}{b(z)} + d(z) + a(z)) dz = +\infty \) but \( f^i(\frac{1}{b(z)} + d(z) + a(z)) dz < +\infty \). Then, \( C^\infty_0(\Omega) \) is not dense in \( E \), but the only non-trivial linear forms continuous on \( E \), vanishing on \( C^\infty_0(\Omega) \), are supported on \( \{z_0\} \times \mathcal{M} \). This means that a boundary condition must be chosen at \( z = z_0 \), but not at \( z = z_1 \).

Moreover, if we have, for example, \( f^i a(z) dz = +\infty \), the self-adjoint extension \( \tilde{A} \), defined from \( A \) with an appropriate boundary condition at \( z = z_0 \), will be unique. In particular, considering null Dirichlet boundary condition, \( A \) will be the self-adjoint extension of \( A \) constructed from the restriction of the bilinear form to \( E^0 \).

### 4. Well-posedness of the Cauchy problem

Let \( A \) and \( \Omega \) be as in the previous section. We assume \( A \) to be at least q.e.s.a. but not necessarily e.s.a.; we denote in the same way its unique self-adjoint extension with finite energy. We take functions \( f \in E \) and \( g \in H \) and consider the Cauchy problem

\[
(P) \begin{cases}
\partial_t \phi + A \phi &= 0, \\
\phi(0, \cdot) &= f, \\
\partial_t \phi(0, \cdot) &= g.
\end{cases}
\]

**Theorem 4.1.** Under the hypotheses above, the problem \( (P) \) has a unique solution

\[
\phi \in C([0, \infty); E) \cap C^1([0, \infty); H),
\]

and there exists a constant \( C > 0 \) such that

\[
\forall t > 0 \quad \|\phi(t, \cdot)\|_E + \|\partial_t \phi(t, \cdot)\|_H \leq C(\|f\|_E + \|g\|_H).
\]

In this case, the energy

\[
E(\phi, t) = \frac{1}{2} \int_{\Omega} (a(z)(\partial_t \phi)^2 + b(z)(\partial_\nu \phi)^2 + c(z)|\nabla \phi|^2 + d(z)|\phi|^2) d\mu
\]

is well defined and conserved:

\[
\forall t > 0 \quad E(\phi, t) = \frac{1}{2} (\|g\|_H^2 + b(f, f)).
\]

**Proof.** Let \( D \) be the domain defined in (2), given \( f \in D \) and \( g \in E \), the solution of \( (P) \) is given by (see, for example, [6] and references therein)

\[
\phi(t, \cdot) = \cos(tA^2)f + A^{-\frac{1}{2}} \sin(tA^2)g.
\]

(18)

Taking into account that \( D(A^2) = E \), we have \( \phi(t, \cdot) \in D \) and \( \partial_t \phi(t, \cdot) \in E \). That \( \phi(t, \cdot) \) and \( \partial_t \phi(t, \cdot) \) are continuous vector-valued functions (in \( D \) and in \( E \), respectively) rely on a classical density argument we only sketch. For \( \varepsilon > 0 \), we set \( f_\varepsilon = (I + \varepsilon A)^{-1}f \) and \( \phi_\varepsilon = (I + \varepsilon A)^{-1} \phi \). Then, \( \partial_t \phi_\varepsilon(t, \cdot) \in D \) and \( \partial_t \phi_\varepsilon(t, \cdot) \in E \), with their norms uniformly bounded in \( t \), while \( \phi_\varepsilon(t, \cdot) \to \phi(t, \cdot) \) in \( D \) and \( \partial_t \phi_\varepsilon(t, \cdot) \to \partial_t \phi(t, \cdot) \in E \) when \( \varepsilon \to 0 \). The conclusion readily follows.

When \( f \in E \) and \( g \in H \), we define \( \phi(t, \cdot) \) by (18). Then, \( \phi(t, \cdot) \in E \) and \( \partial_t \phi(t, \cdot) \in H \).

The continuity results are obtained by density arguments in the same way as above.

The reader should notice that in this case we have \( \partial_t \phi(t, \cdot) + A \phi(t, \cdot) = 0 \) in \( E' \), where \( E' \) is the dual space of \( E \); hence, \( \phi \) is a weak solution of \( (P) \). Regarding the conservation of the energy, although the argument here is standard, we recall it for its convenience. We assume first that \( f \in D \) and \( g \in E \). Then, \( \phi(t, \cdot) \) is a strong solution of \( (P) \) and we have

\[
\int_{t_1}^{t_2} \int_{\Omega} (a(z)(\partial_t \phi + A \phi) dt d\mu = 0.
\]

(19)
We consider each term separately, obtaining for the first one
\[
\int_\Omega \int_{\mathcal{H}_1} a(z) \partial_\delta \phi \partial_\beta \phi \, dr \, d\mu = \frac{1}{2} \int_\Omega a(z) (\partial_\delta \phi)^2 \, d\mu\bigg|_{\mathcal{H}_1},
\] (20)
and for the second one (see for instance [12])
\[
\int_{\mathcal{H}_1} \int_\Omega (\partial_\phi, A\phi) \, d\mu \, dt = \int_{\mathcal{H}_1} b(\phi, \partial_\phi) \, dt = \frac{1}{2} \int_\Omega \left(a(z) (\partial_\delta \phi)^2 + b(z) (\partial_\epsilon \phi)^2 + c(z) |\nabla \phi|^2 + d(z)|\phi|^2\right) \, d\mu\bigg|_{\mathcal{H}_1}.
\] (21)

Now, by (19), adding (20) and (21), we have for all \(t > 0\)
\[
E(\phi, t) = \frac{1}{2} \int_\Omega \left(a(z) (\partial_\delta \phi)^2 + b(z)(\partial_\epsilon \phi)^2 + c(z) |\nabla \phi|^2 + d(z) |\phi|^2\right) \, d\mu = \frac{1}{2} \left(\|\rho\|^2_{L^2} + b(f, f)\right).
\]
Again, by a density argument as before, this result remains true when \(f \in \mathcal{E}\) and \(g \in H\). □

5. Propagation of classical scalar fields in static spherically symmetric spacetimes

We consider an \((n + 2)\)-dimensional static and spherically symmetric spacetime with \(n \geq 1\) and metric signature \((-+\cdots+)\). Due to the required isometries the more general line element can be written as
\[
\mathrm{d}s^2 = -F(r) \, \mathrm{d}t^2 + G(r) \, \mathrm{d}r^2 + r^2 \, \mathrm{d}^2 S^2,
\] (22)
where \(\mathrm{d}^2 S^2\) is the metric on the unit \(n\)-sphere \(S^n\) and \(r \in (0, +\infty)\). For a non-degenerate Lorentzian metric \(g_{\alpha\beta}\) (22) makes sense only for those values of \(r\) such that \(0 < F(r)G(r) < +\infty\). On the other hand, since \(g_{\alpha\beta}(\partial_\epsilon)^\alpha(\partial_\delta)^\beta = -F\), the Killing vector field \(\partial_\epsilon\) is timelike only in the region \(F(r) > 0\), and so spacetime is static only in this region. Therefore, without loss of generality, from now on we shall restrict ourselves to the region where \(F(r)\) and \(G(r)\) are both finite and positive. In addition we shall assume that \(F\) and \(G\) are such that the condition \(0 < F(r)\), \(G(r) < +\infty\) holds in a finite union of disjoint non empty open subintervals \((r^-_i, r^+_i)\) of \((0, +\infty)\) and \(F, F', G \in C^4(\bigcup_{i=1}^m (r^-_i, r^+_i))\). If the spacetime is asymptotically flat, in the outer region \((r^-_m, +\infty)\), we can find coordinates such that \(\lim_{r \to +\infty} F(r) = \lim_{r \to +\infty} G(r) = 1\).

Due to the required symmetries the more general energy–momentum tensor can be written as
\[
T^b_\alpha = \text{diag}\{-\rho(r), p_r(r), p_\theta(r), \ldots, p_\theta(r)\},
\] (23)
where \(\rho(r)\) is the energy density, and \(p_r(r), p_\theta(r)\) are the principal pressures. We shall assume that \(\rho(r)\) is bounded and the dominant energy condition\(^9\) is satisfied, which, in this case, is equivalent to
\[
|p_r(r)|, |p_\theta(r)| \leq \rho(r) < +\infty.
\] (24)

\(^9\) See for example [1].
From (22) and (23), we obtain that Einstein’s equations, i.e., $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, become
\begin{equation}
G_r = -\frac{n}{2r^2} \left( (n-1) \left( 1 - \frac{1}{G(r)} \right) + r\frac{G'(r)}{G(r)^2} \right) = -8\pi \rho(r) - \Lambda, \tag{25}
\end{equation}
\begin{equation}
G^r_r = -\frac{n}{2r^2} \left( \frac{rF'(r)}{F(r)G(r)} + (n-1) \left( \frac{1}{G(r)} - 1 \right) \right) = 8\pi \rho_r(r) - \Lambda, \tag{26}
\end{equation}
\begin{equation}
G^\theta_\theta = \frac{F''(r)}{2F(r)G(r)} - \frac{F'(r)G'(r)}{2F(r)G(r)^2} + \frac{(n-1)F'(r)}{2rF(r)G(r)} - \frac{F'(r)^2}{4F(r)^2G(r)} - \frac{1}{r^2} \left( \frac{n}{2} - 1 \right) \left( \frac{\rho}{G(r)} + \frac{\rho_r}{r} \right), \tag{27}
\end{equation}
where $\Lambda$ is the cosmological constant. Furthermore, the local energy–momentum conservation $(\nabla_a T^{ab} = 0)$ gives
\begin{equation}
\rho(r) + \frac{r}{2} \frac{\rho'(r)}{F(r)} - \frac{n}{r} \left( \frac{\rho(r)}{r} - \frac{\rho_r}{r^2} \right). \tag{28}
\end{equation}
Of course, due to Bianchi’s identities, (25)–(28) are not independent. These are a system of three parameters, includes the higher dimensional generalization of Schwarzschild, de Sitter geometries.

From (25) and (26), we can write down a more handleable set of two equivalent independent equations
\begin{equation}
\rho^{\mu-1} \left( 1 - \frac{1}{G(r)} \right) + \frac{2\rho^{\mu}}{n} (8\pi \rho(r) + \Lambda), \tag{29}
\end{equation}
\begin{equation}
\ln'(F(r)G(r)) = \frac{16\pi}{n} (\rho(r) + \rho_r(r)) r G(r), \tag{30}
\end{equation}
which in the vacuum cases, leads readily to the solution.

Indeed, if we for instance set $\rho(r) = -\rho_r(r) = \rho_0(r)$, from (28) we immediately obtain that
\begin{equation}
\rho_r(r) = -\rho(r) = -\frac{C_1}{r^{2n}},
\end{equation}
where the constant $C_1$ must be positive by (24). Then, we find from (29) that
\begin{equation}
\frac{1}{G(r)} = 1 - \frac{C_2}{r^{2n-1}} + \frac{16\pi C_1}{n(n-1)r^{2n-2}} - \frac{2\Lambda r^2}{n(n+1)},
\end{equation}
where $C_2$ is a new arbitrary constant. And (30) immediately gives $F(r)G(r) = C_3$, and we can always set the constant $C_3 = 1$ by scaling the time. This family of solutions, depending on three parameters, includes the higher dimensional generalization of Schwarzschild, de Sitter and Reissner–Nordström geometries.

For future use, we shall prove the following result.

**Lemma 5.1.** If $0 < F(r), G(r) < +\infty$ in some interval $(r_i^-, r_i^+)$, then
(i) $F(r)G(r)$ is a nondecreasing function of $r$ in $(r_i^-, r_i^+)$, and then bounded in a neighbourhood of $r_i^-$. 
(ii) In the outer region of an asymptotically flat spacetime, $F(r)G(r)$ is bounded.
Proof.

(i) As a consequence of the dominant energy condition (24) the right-hand side of (30) cannot be negative, then

\[ F(r)G(r) \leq 1 \text{ since } \lim_{r \to +\infty} F(r) = \lim_{r \to +\infty} G(r) = 1. \]

□

In these spacetimes, we shall consider the propagation of a scalar field \( \psi \) with Lagrangian density

\[ L = -\frac{1}{2} \nabla^a \psi \nabla_a \psi - \frac{m^2}{2} \psi^2, \]  

where the constant \( m \) is the mass of the field and \( \nabla \) denotes the covariant derivative (Levi-Civita connection).

As usual, we obtain the field equations by requiring that the action

\[ S = \int L(\nabla_a \psi, \psi, g_{ab}) \sqrt{|g|} \, dt \, d\mu \]

be stationary under arbitrary variations of the fields \( \delta \psi \) in the interior of any compact region, but vanishing at its boundary. Thus, we have the Euler–Lagrange equation

\[ \nabla_a \left( \frac{\partial L}{\partial \nabla_a \psi} \right) = \frac{\partial L}{\partial \psi}, \]

which, in our case, becomes the Klein–Gordon equation

\[ \nabla_a \nabla^a \psi = \Box \psi = \frac{\partial_a \left( \sqrt{|g|} g^{ab} \partial_b \psi \right)}{\sqrt{|g|}} = m^2 \psi. \]  

(31)

Therefore, we obtain from (22) and (31) that the field equation may be written as

\[ \partial_{tt} \psi = -A \psi, \]

where

\[ A = -\frac{1}{r^2} \sqrt{\frac{F(r)}{G(r)}} \left( \partial_r \left( r^2 \frac{F(r)}{G(r)} \partial_r \psi \right) + r^{n-2} \sqrt{F(r)G(r)} \Delta_{S^n} \psi \right) + m^2 F(r) \psi, \]  

(32)

where \( \Delta_{S^n} \) is the Laplacian on the unit \( n \)-sphere. Then, by comparing with the operator defined in (3), we obtain the identification of the coefficients

\[ a(r) = r^n \sqrt{\frac{G(r)}{F(r)}}; \quad b(r) = r^n \frac{F(r)}{G(r)}; \quad c(r) = r^{n-2} \sqrt{F(r)G(r)}; \quad d(r) = m^2 r^n \sqrt{F(r)G(r)}. \]  

(33)

Remark 5.2. From (22), we obtain that radial null geodesics satisfy \( \frac{\partial \psi}{\partial r} = \pm \sqrt{\frac{G(r)}{F(r)}} \). Then, if \( r_0 \) and \( r \) belong to the closure of a connected region where \( 0 < F(s), G(s) < +\infty \), we find from (33) that the coordinate time \( t \) a radial photon takes to travel from \( r \) to \( r_0 \) is

\[ T(r \to r_0) = \left| \int_r^{r_0} \frac{G(s)}{F(s)} \, ds \right| = \left| \int_r^{r_0} \frac{a(s)}{b(s)} \, ds \right|. \]  

(34)

We shall see that it is actually this time which plays a crucial role in the analysis of e.s.a. when there is a horizon at \( r_0 \) (\( r_0 = r^+_0 \) or \( r_0 = r^-_0 \)) in the spacetime, i.e., \( T(r \to r_0) = +\infty \).
Lemma 5.3. In the outer region of an asymptotically flat spacetime one has $\int^{+\infty} a(r) \, dr = +\infty$.

Proof. Since $\lim_{r \to +\infty} F(r) = \lim_{r \to +\infty} G(r) = 1$ we get from (33) that $\lim_{r \to +\infty} \frac{a(r)}{r^{n}} = 1$, and then $\int^{+\infty} a(r) \, dr = +\infty$. \qed

Lemma 5.4. If $0 < F(r), G(r) < +\infty$ in $(r^{-}, r^{+})$, with $r^{-} > 0$, the three following statements are equivalent

$\int_{r^{-}}^{r^{+}} \frac{1}{b(r)} \, dr = +\infty$, $\int_{r^{-}}^{r^{+}} a(r) \, dr = +\infty$ and $\int_{r^{-}}^{r^{+}} \sqrt{\frac{a(r)}{b(r)}} \, dr = +\infty$.

On the other hand, if $r^{+}$ is finite, the three following statements are equivalent

$\int_{r^{-}}^{r^{+}} \frac{1}{b(r)} \, dr = +\infty$, $\int_{r^{-}}^{r^{+}} a(r) \, dr = +\infty$ and $\int_{r^{-}}^{r^{+}} \sqrt{\frac{a(r)}{b(r)}} \, dr = +\infty$.

Proof. By (33), we have that $a(r)b(r) = r^{2n}$. For $r_{*} < r < r^{*} < +\infty$, we readily obtain the inequalities

$$\frac{r^{2n}}{b(r)} < a(r) < \frac{r^{2n}}{b(r)}$$

and

$$r^{2n} \sqrt{\frac{a(r)}{b(r)}} < a(r) < r^{2n} \sqrt{\frac{a(r)}{b(r)}}$$

Now, by integrating these expressions between $r_{*}$ and $r^{*}$, we obtain the result. \qed

Observe that by the properties of the functions $F$ and $G$, under the hypotheses of lemma 5.4 we have

- $a, b, c, d \in C^{1}((r^{-}, r^{+}))$,
- $a, b, c > 0$ and $d \geq 0$ in $(r^{-}, r^{+})$,
- $a^{-1}, b^{-1}, c^{-1} \in L_{loc}^{1}((r^{-}, r^{+}))$.

Then, if we consider the operator defined by (32) in $\Omega = (r^{-}, r^{+}) \times S^{n}$, we have

Theorem 5.5. For $0 < r_{m} < +\infty$, let $A$ be the operator corresponding to the propagation of a scalar field in $\Omega = (r_{m}, +\infty) \times S^{n}$ in a static, spherically symmetric and asymptotically flat spacetime where the dominant energy condition holds. The three following statements are equivalent.

(i) The time $T(r \to r_{m})$ is infinite.
(ii) $A$ is a q.e.s.a. operator.
(iii) $A$ is an e.s.a. operator.

Or, in other words, $A$ is e.s.a. if and only if a radial photon needs an infinite amount of time to get $r_{m}$.

Proof. (i) $\Rightarrow$ (ii) and (iii): by lemma 5.3, we have that $\int^{+\infty} a(r) \, dr = +\infty$. On the other hand, if $T(r \to r_{m}) = +\infty$, it follows by (34) that $\int_{r_{m}}^{r} \sqrt{\frac{a(r)}{b(r)}} \, dr = +\infty$, and then from lemma 5.4 we have $\int_{r_{m}}^{r} a(z) \, dz = +\infty$. Therefore, it follows from theorem 3.24 that the operator $A$ is q.e.s.a. and from theorem 3.25 (i) that the operator $A$ is e.s.a.

(ii) $\Rightarrow$ (i): conversely, assume that $T(r \to r_{m}) < +\infty$, then $\int_{r_{m}}^{r} \sqrt{\frac{a(r)}{b(r)}} \, dr < +\infty$. And it immediately follows from lemma 5.4 that $\int_{r_{m}}^{r} a(r) \, dr < +\infty$ and $\int_{r_{m}}^{r} \frac{1}{b(r)} \, dr < +\infty$. On the
other hand, since $F(r)G(r)$ is bounded by lemma 5.1, \( \int_{r_m}^{a} d(r) \, dr = m^2 \int_{r_m}^{r_0} r^2 \sqrt{F(r)G(r)} \, dr < +\infty \). Therefore, it follows from theorem 3.24 that the operator $A$ is not q.e.s.a.

(iii) $\Rightarrow$ (ii): this is obvious by definition.

\[ \square \]

**Remark 5.6.** Note that the boundedness of $F(r)G(r)$ is only used in the proof of the sufficiency of the condition $T(r \to r_m) = +\infty$, to guarantee that $d(r)$ is integrable at $r_m$. Therefore, for massless fields, since in this case $d(r) \equiv 0$, the theorem follows without invoking any energy condition.

Similar results also follow from remark 3.23 and lemma 5.4 at internal horizons.

6. **Examples**

6.1. \((n+2)\)-dimensional punctured Minkowski spacetime

Here, we consider the flat $(n + 2)$-dimensional Minkowski spacetime with a removed spatial point. We chose the origin of coordinates at this point and then the line element can be written as

\[ ds^2 = -dt^2 + dr^2 + r^2 \, dl_n^2, \]

where $-\infty < t < +\infty$ and $0 < r < +\infty$. This spacetime has a time-like singular boundary along the $t$ axis. In this case, $\Omega = (0, \infty) \times S^n$ and $F(r) = G(r) = 1$, so the coefficients in (33) are $a(r) = b(r) = r^n$, $c(r) = r^{n-2}$ and $d(r) = m^2 \, r^n$. The operator $A$ in (32) turns out to be

\[ A\psi = -\frac{1}{r^n} \frac{\partial}{\partial r} (r^n \partial_r \psi) - \frac{1}{r} \Delta_{\Omega} \psi + m^2 \psi, \]

which formally is nothing but $-\Delta + m^2$.

Now, for $n \geq 1$, we have that \( \int^{+\infty} a(r) \, dr = +\infty \) and \( \int_{0}^{r_m} \frac{dr}{b(r)} = +\infty \). Then, it immediately follows from theorem 3.2 that $A$ is a q.e.s.a. operator for every $m^2 \geq 0$ and every $n \geq 1$.

We turn now to explore whether $A$ is an e.s.a. operator too. Taking into account that $d(z)/a(z) = m^2$, \( \int_{0}^{r_m} a(z) \, dz = \int_{0}^{1} r^n \, dr < +\infty \) and \( \int^{+\infty} a(z) \, dz = +\infty \), we can apply corollary 3.22.

Now, for $0 < r_1 < +\infty$, we have

\[ \beta_0(r) = \int_{r}^{r_1} \frac{du}{b(u)} = \begin{cases} -\ln \left( \frac{r}{r_1} \right) & \text{if } n = 1, \\ \frac{r^{1-n} - r_1^{1-n}}{n-1} & \text{if } n \geq 2. \end{cases} \]

Thus,

\[ \int_{0}^{r_1} \beta_0^2(r) a(r) \, dr < +\infty \]

if and only if $n = 1, 2$. Therefore, it immediately follows from corollary 3.22 that $A$ is an e.s.a. operator only if $n \geq 3$. This is a well-known result, see for instance [13, 14].

6.2. \((n+2)\)-dimensional anti-Schwarzschild \((M < 0)\) spacetime

Here, we consider the $(n + 2)$-dimensional spacetime with line element

\[ ds^2 = -\left( 1 + \frac{r^{n-1}}{\Omega_n} \right) \, dr^2 + \left( 1 + \frac{r_1^{n-1}}{\Omega_n} \right)^{\frac{1}{n-1}} \, dr^2 + r^2 \, d\Omega_n^2, \]

25
where $-\infty < t < +\infty$, $0 < r < +\infty$, $r_s$ is a positive constant and $n \geq 2$. This spacetime has a naked timelike singularity at $r = 0$, where some components of the Weyl tensor diverge.

In this case, $\Omega = (0, \infty) \times S^n$ and we obtain from (33) that the coefficients of the operator $A$ are

$$a(r) = \frac{r^{2n-1}}{r^{n-1} + r_s^{n-1}}, \quad b(r) = r(r^{n-1} + r_s^{n-1}), \quad c(r) = r^{n-2} \quad \text{and} \quad d(r) = m^2 r^n.$$

We obtain therefore

$$\int_0^r \frac{dr}{b(r)} = +\infty \quad \text{and} \quad \int_0^{+\infty} a(r) \, dr = +\infty.$$

Then, it immediately follows from theorem 3.2 that $A$ is a q.e.s.a. operator for every $m^2 \geq 0$ and every $n \geq 2$.

For $m = 0$ and $n = 2$, we have already proved in [9] that $A$ is not an e.s.a. operator. Here, we shall analyze the general case.

We first consider the case $m = 0$. Taking into account that

$$\int_0 a(r) \, dr < +\infty, \quad \int_0^{+\infty} a(r) \, dr = +\infty \quad \text{and} \quad d(z) = 0,$$

we can apply corollary 3.22.

For $0 < r < r_s$, we have

$$\beta_0(r) = \int_r^{r_s} \frac{ds}{b(s)} = -\frac{1}{r_s^{n-1}(n-1)} \ln \left(\frac{2r^{n-1}}{r^{n-1} + r_s^{n-1}}\right),$$

and

$$\int_0^{r_s} \beta_0^2(r) a(r) \, dr < +\infty.$$

Thus, in the massless case, $A$ is not an e.s.a. operator for every $n \geq 2$, thanks to the corollary 3.22.

For $m^2 > 0$, we cannot apply corollary 3.22 since $d(z)/a(z)$ is not bounded near 0. Nevertheless, the ODE (9), satisfied by the function $\alpha(z)$ of lemma 3.13, becomes in this case

$$-(r(r^{n-1} + r_s^{n-1})\alpha'(r)) + m^2 r^n \alpha(r) = 0,$$

and a straightforward computation shows that

$$\alpha(z) = \alpha(0) \left(1 + \frac{m^2 r_s^2}{(n+1)^2} \left(\frac{r}{r_s}\right)^{n+1} - \frac{m^2 r_s^2}{2n(n+1)} \left(\frac{r}{r_s}\right)^{2n} + \cdots\right)$$

near 0. Furthermore, since by lemma 3.13 $\alpha(r)$ is positive and increasing in $(0, r_s)$, and by definition $\alpha(r_s) = 1$, we obtain that $0 < \alpha(0) < 1$.

Therefore,

$$\beta(r) = \alpha(r) \int_r^{r_s} \frac{ds}{b(s)\alpha(s)^2} < \frac{1}{\alpha(r)} \int_r^{r_s} \frac{ds}{b(s)} < \frac{1}{\alpha(0)} \beta_0(r)$$

and

$$\int_0^{r_s} \beta^2(r) a(r) \, dr < \frac{1}{\alpha(0)^2} \int_0^{r_s} \beta_0^2(r) a(r) \, dr < +\infty.$$ 

It follows from theorem 3.9 (ii) that $A$ is not an e.s.a. operator for every $n \geq 2$ and $m^2 \geq 0$. 

\textsuperscript{10} The case $n = 1$ is three-dimensional Minkowski spacetime already discussed in 6.1.
Remark 6.1. Note that the estimate
\[
\beta(z) = \alpha(z) \int_{z}^{1} \frac{ds}{b(s)\alpha(s)^2} < \frac{1}{\alpha(z)} \beta_0(z),
\]
when \( \alpha(0) \neq 0 \), also gives a necessary and sufficient condition for e.s.a. in terms of \( \beta_0(z) \) only.

For analytic \( b(z) \) and \( d(z) \), as in our example, \( \alpha(0) \neq 0 \) if one of the roots of the indicial polynomial of (9) is zero and the other non positive, which requires that
\[
\lim_{z \to 0^+} \frac{z^2 d(z)}{b(z)} = 0 \quad \text{and} \quad \lim_{z \to 0^+} \frac{zb'(z)}{b(z)} \geq 1.
\]

6.3. \((n+2)\)-dimensional Schwarzschild–Tangherlini spacetime

Here, we consider the \((n+2)\)-dimensional spacetime with line element
\[
d s^2 = - \left(1 - \frac{r_n^{n-1}}{r_s^{n-1}} \right) d\tau^2 + \left(1 - \frac{r_n^{n-1}}{r_s^{n-1}} \right)^{-1} dr^2 + r^2 d\Omega_n^2,
\]
where \( r_s \) is a positive constant, \(-\infty < t < +\infty, 0 < r < r_s \) or \( r_s < r < +\infty \) and \( n \geq 2 \). This spacetime has a spacelike irremovable singularity at \( r = 0 \) where some components of the Riemann tensor diverge and an event horizon at \( r = r_s \), the latter may be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [15]. As already mentioned, our wave formulation only makes sense in the static region \( r_s < r < +\infty \), and we will use it to explore the properties of the wave equation (31) in this region.

Thus, we consider the operator \( A \) given by (32) in \( \Omega = (r_s, \infty) \times S^n \), and we see from (33) that
\[
a(r) = \frac{r_s^{2n-1}}{r_s^{n-1} - r_n^{n-1}}, \quad b(r) = r(r^{n-1} - r_s^{n-1}) \quad \text{and} \quad d(r) = m^2 r^2.
\]
Now, we obtain from (34) that
\[
T(r \to r_s) = \int_{r_s}^{r} \left( \frac{a(s)}{b(s)} \right)^{\frac{1}{2}} ds = \int_{r_s}^{r} \frac{s^{n-1}}{s^{n-1} - r_n^{n-1}} ds = +\infty.
\]
Therefore, it immediately follows from theorem 5.5 that \( A \) is an e.s.a. operator in \( \Omega = (r_s, \infty) \times S^n \) for every \( n \geq 2 \) and any \( m^2 \geq 0 \), and the Cauchy problem is well posed without requiring any boundary condition at the event horizon.

6.4. \((n+2)\)-dimensional Reissner–Nordström spacetime

Here, we consider the \((n+2)\)-dimensional spacetime with line element
\[
d s^2 = - \left(1 - \frac{r_s^{n-1}}{r_s^{n-1} + q^2 2n-2} \right) d\tau^2 + \left(1 - \frac{r_s^{n-1}}{r_s^{n-1} + q^2 2n-2} \right)^{-1} dr^2 + r^2 d\Omega_n^2,
\]
where \( r_s \) and \( q^2 \) are positive constants and \( n \geq 2 \).

If \( q^2 > r_s^2 \), the metric is nonsingular everywhere except for the timelike irremovable repulsive singularity at \( r = 0 \). If \( q^2 \leq r_s^2 \), the metric also has singularities at \( r_s \) and \( r_- \), where \( r_+ = (r_s^{n-1} + \sqrt{r_s^{2n-2} - q^2 2n-2})/2 \); it is regular in the regions defined by \( \infty > r > r_+ \), \( r_+ > r > r_- \) and \( r_- > r > 0 \) (if \( q^2 = r_s^2 \) then only the first and the third regions exist). As in the Schwarzschild case, these singularities may be removed by introducing suitable coordinates and extending the manifold to obtain a

\[\text{[11] The case } n = 1 \text{ is again three-dimensional Minkowski spacetime already discussed in 6.1.}\]
maximal analytic extension \cite{16, 17}. The first and the third regions are both static, whereas the second region \( (r_0, \infty) \times S^n \) is spatially homogeneous but not static.

We shall study the properties of the wave equation in the static regions. For convenience, we shall analyse separately the three cases. Note that, in the three cases this spacetime is asymptotically flat.

**6.4.1. Case \( q^2 > r_2^2 \).** This spacetime has only a naked timelike irremovable repulsive singularity at \( r = 0 \). In this case, we consider the operator \( A \) given by \eqref{eq:3.2} in \( \Omega = (0, \infty) \times S^n \), and from \eqref{eq:33} we have

\[
a(r) = \frac{r^n}{1 - \frac{r^{n-1}}{r_0^n} + \frac{q^{n-2}}{4n-2}}, \quad b(r) = r^n - r_2^n + \frac{q^{2n-2}}{4n-2} \quad \text{and} \quad d(r) = m^2 r^p.
\]

Then,

\[
\int_0^1 \left( \frac{1}{b(r)} + d(r) - a(r) \right) \, dr < +\infty.
\]

Hence, it follows from theorem 3.2 (ii) that \( A \) is not even a q.e.s.a. operator in this case, for every \( n \geq 2 \) and any \( m^2 \geq 0 \). Therefore, in contrast to the anti-Schwarzschild case, in order to have a well-posed Cauchy problem a boundary condition at the singularity must be given.

**6.4.2. Case \( r_2^2 = q^2 \) (extreme case).** This spacetime also has a removable singularity at \( r_* = 2^{\frac{1}{n}} r_0 \). In this case, we consider the operator \( A \) given by \eqref{eq:32} in two regions \( \Omega = (0, r_0) \times S^n \) or \( \Omega = (r_*, \infty) \times S^n \).

We obtain from \eqref{eq:33} that

\[
a(r) = \frac{r_2^{2n-2}}{(r_0^{n-1} - r_*^{n-1})^2}, \quad b(r) = \frac{(r_0^{n-1} - r_*^{n-1})^2}{r_2^{n-2}} \quad \text{and} \quad d(r) = m^2 r^p.
\]

We first consider the outer region \( (r_0 < r < +\infty) \). In this case, we obtain from \eqref{eq:34} that

\[
T(r \rightarrow r_*) = \int_{r_*}^r \left( \frac{a(s)}{b(s)} \right)^{\frac{1}{2}} \, ds = \int_{r_*}^r \frac{s^{2n-2}}{(r_0^{n-1} - r_*^{n-1})^2} \, ds = +\infty.
\]

Therefore, it follows from theorem 5.5 that \( A \) is an e.s.a. operator in \( \Omega = (r_*, \infty) \times S^n \) for every \( n \geq 2 \) and any \( m^2 \geq 0 \), and the Cauchy problem is well posed without requiring any boundary condition at the event horizon.

Regarding the inner region \( 0 < r < r_0 \), we obtain that

\[
\int_0^r \left( \frac{1}{b(r)} + d(r) - a(r) \right) \, dr < +\infty.
\]

Hence, it follows from theorem 3.24 that \( A \) is not even a q.e.s.a. operator, for every \( n \geq 2 \) and any \( m^2 \geq 0 \).

However, we have

\[
\int_{r_0}^{r_*} a(r) \, dr = \int_{r_0}^{r_*} \frac{r_2^{2n-2}}{(r_0^{n-1} - r_*^{n-1})^2} \, dr = +\infty,
\]

so it follows from remark 3.23 that in order to have a well-posed Cauchy problem in \( \Omega = (0, r_0) \times S^n \) a boundary condition at the singularity \( (r = 0) \) must be given but not at the horizon \( (r = r_0) \).
6.4.3. Case $r_2 > q^2$. This spacetime has, besides the timelike irremovable repulsive singularity at $r = 0$, two removable singularities at $r_+$ and $r_-$. In this case, we consider the operator $A$ given by (32) in two regions $\Omega = (0, r_-) \times S^n$ or $\Omega = (r_+, \infty) \times S^n$, by abuse of notation we call $A$ these two different operators.

From (33) we can write

$$a(r) = \frac{r^{2n-2}}{(n-1) (n-1) (n-1)}, \quad b(r) = \frac{(r^{2n-1} - r_+^{2n-1}) (r^{2n-1} - r_-^{2n-1})}{r^{n-2} - r_+^{n-2}},$$

and

$$d(r) = m^2 r^n.$$

We first consider the outer region ($r_+ < r < +\infty$). In this case, we obtain from (34) that

$$T(r \to r_+) = \int_{r_+}^r \left( \frac{a(s)}{b(s)} \right)^{1/3} ds = \int_{r_+}^r \frac{s^{2n-2}}{(s^{n-1} - r_-^{n-1}) (s^{n-1} - r_+^{n-1})} ds = +\infty.$$

Therefore, it follows from theorem 5.5 that $A$ is an e.s.a. operator in $\Omega = (r_+, \infty) \times S^n$ for every $n \geq 2$ and any $m^2 > 0$, and the Cauchy problem is well posed without requiring any boundary condition at the event horizon.

Regarding the inner region $0 < r < r_-$, we obtain

$$\int_0^r \left( \frac{1}{b(r)} + d(r) + a(r) \right) dr < +\infty.$$

Hence, it follows from theorem 3.24 that $A$ is not even a q.e.s.a. operator, for every $n \geq 2$ and any $m^2 > 0$.

However, we have

$$\int_{r_-}^{r_+} a(r) dr = \int_{r_-}^{r_+} \frac{r^{2n-2}}{(n-1) (n-1) (n-1) (n-1) (n-1)} dr = +\infty,$$

so it follows from remark 23.23 that in order to have a well-posed Cauchy problem in $\Omega = (0, r_-) \times S^n$ a boundary condition at the singularity ($r = 0$) must be given but not at the horizon ($r = r_-$).

References

[1] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[2] Penrose R 1969 Rivista del Nuovo Cimento, Numero Speciale I 252
[3] Penrose R 1979 Singularities and time asymmetry General Relativity. An Einstein Centenary Survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[4] Leray J 1953 Hyperbolic Differential Equations (Princeton, NJ: Institute for Advanced Studies)
[5] Choquey-Bruhat Y 1968 Hyperbolic partial differential equations on a manifold Battelle Recontres ed C M DeWitt and J A Weeler (New York: Benjamin)
[6] Wald R M 1980 J. Math. Phys. 21 2802
[7] Horowitz G T and Marolf D 1995 Phys. Rev. D 52 5670
[8] Clarke C J S 1998 Class. Quantum Grav. 15 975–84
[9] Gamboa Saraví R E, Sanmartino M and Tchamitchian P 2010 Class. Quantum Grav. 27 215016
[10] Seggev I 2004 Class. Quantum Grav. 21 2851
[11] Stalker J G and Shadi Tahvildar-Zadeh A 2004 Class. Quantum Grav. 21 2831
[12] Kato T 1966 Perturbation Theory for Linear Operators (Berlin: Springer)
[13] Berezin F A and Shubin M A 1991 The Schrödinger Equation (Dordrecht: Kluwer)
[14] Reed M and Simon B 1975 Methods of Modern Mathematical Physics: II. Fourier Analysis, Self-Adjointness (New York: Academic)
[15] Kreuzk M D 1969 Phys. Rev. 119 1743
[16] Graves J C and Brill D R 1969 Phys. Rev. 120 1507
[17] Carter B 1966 Phys. Lett. 21 423