On a Generalization of Markowitz Preference Relation

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Abstract
Given two families of continuous functions $u = (u_p)_{p \in I}$ and $v = (v_q)_{q \in J}$ on a topological space $X$, we define a preorder $R = R(u, v)$ on $X$ by the condition that any member of $u$ is an $R$-increasing and any member of $v$ is an $R$-decreasing function. It turns out that if the topological space $X$ is quasi-compact and sequentially compact, then any element $x \in X$ is $R$-dominated by an $R$-maximal element $m \in X$: $x R m$. In particular, since the $(n-1)$-dimensional simplex is a compact subset of $\mathbb{R}^n$, then considering its members as portfolios consisting of $n$ financial assets, we obtain the classical 1952 result of Harry Markowitz that any portfolio is dominated by an efficient portfolio. Moreover, several other examples of possible application of this general setup are presented.

1 Markowitz Optimization

1.1 Return of a Portfolio
Let $\Delta_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$ be the $n-1$-dimensional simplex and let $[n] = \{1, \ldots, n\}$. The ordered pairs $([n], x), x \in \Delta_{n-1}$, are sample spaces with set of outcomes $[n]$ and probability assignment $x: [n] \to \mathbb{R}, x(i) = x_i, i = 1, \ldots, n$. The set of all sample spaces of this form can be identified with the $n-1$-dimensional simplex $\Delta_{n-1}$ and also are said to be $(n-1)$-dimensional lotteries or $(n-1)$-dimensional portfolios.

Given a sample space $S$ with probability $P$, let $s_1, \ldots, s_n$ be random variables on $S$ with expected values $\mu_1, \ldots, \mu_n$, respectively. For any portfolio $x \in \Delta_{n-1}$ the weighted sum $s(x) = x_1s_1 + \cdots + x_ns_n$ is a random variable with expected value $u(x) = E(s(x)) = x_1\mu_1 + \cdots + x_n\mu_n$ and the variance $v(x) = \text{Var}(s(x))$ is a non-negative quadratic form in $x_1, \ldots, x_n$.

Remark 1.1.1 Below we interpret $i \in [n]$ as financial assets, the sample space $S$ as a financial market, the random variables $s_i$ on $S$ as returns on asset $i$.

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on families of continuous real functions on \( X \). Then \( u(x) = E(s(x)) \) is the expected return and \( v(x) = \text{Var}(s(x)) \) is the risk (or, the volatility) of the portfolio \( x \) — see, for example, [2] 2.1.

### 1.2 Markowitz Preferences

Let \( x \in \Delta_{n-1} \) be a portfolio and \( u(x) = E(s(x)) \) and \( v(x) = \text{Var}(s(x)) \) be the expected return and the volatility of \( x \). The Markowitz’s approach to portfolio selection is based on the following definition of preference \( R \) on the set \( \Delta_{n-1} \) of portfolios: \( xRy \) if \( u(x) \leq u(y) \) and \( v(y) \leq v(x) \). Non-formally, \( xRy \) means that the portfolio \( y \) is at least as good as \( x \). The symmetric part \( E \) of the preorder \( R \) is

\[
E = \{ (x, y) \in \Delta_{n-1}^2 \mid u(x) = u(y) \text{ and } v(y) = v(x) \}
\]

and the asymmetric part \( F \) of \( R \) is \( F = R \setminus E \). Thus, \( xFy \) if and only if either \( u(x) < u(y) \) and \( v(y) \leq v(x) \) or \( u(x) \leq u(y) \) and \( v(y) < v(x) \). Non-formally, \( xFy \) means that the portfolio \( y \) is definitely better than the portfolio \( x \).

In [1] p. 82 H. Markowitz gives (up to notation) the following definition:

The portfolio \( x \) is said to be efficient if

\[
\begin{align*}
  u(x) &= \max_{y \in \Delta_{n-1}, v(y) \leq v(x)} u(y) \quad \text{and} \quad v(x) = \min_{y \in \Delta_{n-1}, u(y) \geq u(x)} v(y). \quad (1.2.1)
\end{align*}
\]

In other words, for any portfolio \( y \in \Delta_{n-1} \) the inequality \( v(y) \leq v(x) \) implies the inequality \( u(x) \geq u(y) \) and the inequality \( u(y) \geq u(x) \) implies the inequality \( v(x) \leq v(y) \). The negation of the last statement is: There exists \( y \in \Delta_{n-1} \) such that \( xFy \), that is, the portfolio \( x \) is not \( R \)-maximal.

Thus, we see that \( x \) is Markowitz’s efficient portfolio if and only if \( x \) is \( R \)-maximal — this is our setup.

### 2 Generalization

In this section we present a wide generalization of Markowitz’s preference relation, defined in [22]. Using Kuratowski-Zorn Theorem (equivalent to the Axiom of Choice), we show that any member of this preference structure is dominated by a maximal element (generalized efficient portfolio). In particular, the set of generalized efficient portfolios is not empty.

#### 2.1 A Preorder on a Topological Space

Let \( X \) be a topological space and let \( u = (u_p)_{p \in I} \) and \( v = (v_q)_{q \in J} \) be two families of continuous real functions on \( X \). We define a preorder \( R = R(u, v) \) on \( X \) in the following way:

\[
R = \{ (x, y) \in X^2 \mid u_p(x) \leq u_p(y) \text{ and } v_q(x) \geq v_q(y) \text{ for all } p \in I, q \in J \}. \quad (2.1.1)
\]
Then for the symmetric part $E$ of $R$ (an equivalence relation) one has

$$E = \{(x, y) \in X^2 \mid u_p(x) = u_p(y) \text{ and } v_q(x) = v_q(y) \text{ for all } p \in I, q \in J\}$$

and for the asymmetric part $F$ of $R$ (an asymmetric and transitive relation) one has $F = R \setminus E$. Thus, $xFy$ means $xRy$ and either there exists index $p_0 \in I$ with $u_{p_0}(x) < u_{p_0}(y)$ or there exists index $q_0 \in J$ with $v_{q_0}(x) > v_{q_0}(y)$.

On the account of repetitions of functions within one family and adding the negatives of functions from one family to the other, we can assume that both families have the same set of indices, $u = (u_p)_{p \in I}$, $v = (v_p)_{p \in I}$, without changing the corresponding preorder on $X$. Moreover, on the account of adding a third countable family of continuous functions on $X$ to both families, the corresponding preorder can be defined by two systems of inequalities and a system of equalities.

Below, if the opposite is in not stated, the families $u = (u_p)_{p \in I}$ and $v = (v_p)_{p \in I}$ have the same index set.

### 2.2 Maximal Elements

In order to fix the terminology, we remind several definitions. A topological space $X$ is called quasi-compact if every open covering of $X$ contains a finite open covering. The space $X$ is called compact if it is quasi-compact and Hausdorff, and sequentially compact if any infinite sequence of elements of $X$ has a converging subsequence.

It is well known (see, for example, [3, Sec. 1]) that any compact and first countable space is sequentially compact and that every Lindelöf, sequentially compact (and Hausdorff) space is quasi-compact (compact).

Given a preorder $R$ on the set $X$, a subset $C \subset X$ is said to be chain in $X$ if the induced preorder on $C$ is complete. A preorder set $X$ is called inductive if every chain in $X$ has an upper bound.

Below, if the opposite is not stated, we suppose that the topological space $X$ is furnished with the preorder $R$ produced by the families of continuous functions $u = (u_p)_{p \in I}$ and $v = (v_p)_{p \in I}$.

The sequence $(x_i)_{i=1}^{\infty}$, $x_i \in X$, is said to be $R$-increasing (respectively, strictly $R$-increasing) if $x_iRx_{i+1}$ (respectively, $x_iRx_{i+1}$) for all $i \geq 1$. By analogy, we define $R$-decreasing (respectively, strictly $R$-decreasing) sequences.

Given an $R$-chain $C \subset X$, for any $p \in I$ and any real number $r \in \mathbb{R}$ we set:

$$M_p = \sup_{x \in C} u_p(x), \quad m_p = \inf_{x \in C} v_p(x),$$

$$C_p = \{x \in C \mid u_p(x) = M_p\}, \quad C_p^{(-)} = \{x \in C \mid u_p(x) < M_p\},$$

$$C_p = \{x \in C \mid v_p(x) = m_p\}, \quad C_p^{(+)p} = \{x \in C \mid v_p(x) > m_p\}.$$  

Finally, we denote $C_p^* = \{x \in X \mid u_p(x) = M_p\}$, $c_p = \{x \in X \mid v_p(x) = m_p\}$, so $C_p \subset C_p^{(*)}$ and $c_p \subset c_p^{(*)}$. Note that $C = C_p \cup C_p^{(-)} = c_p \cup c_p^{(+)}$. 

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Lemma 2.2.1 Let \( p, q \in I \).

(i) One has \( c_p \subset C_p \) or \( C_p \subset c_p \).

(ii) One has \( c_p \cap C_p \subset c_q \cap C_q \) or \( c_q \cap C_q \subset c_p \cap C_p \).

Proof: (i) If \( v_p(x) = m_p \) for all \( x \in C_p \), then \( C_p \subset c_p \). Otherwise, there exists \( x \in C_p \) with \( v_p(x) > m_p \) and, hence, \( v_p(y) < v_p(x) \) for all \( y \in c_p \). Since any \( y \in c_p \) is \( R \)-comparable with \( x \), we have \( u_p(y) \geq u_p(x) = M_p \), that is, \( y \in C_p \). In other words, \( c_p \subset C_p \).

(ii) If \( v_q(x) = m_q \) and \( u_q(x) = M_q \) for all \( x \in c_q \cap C_p \), then \( c_q \cap C_p \subset c_p \cap C_q \). Otherwise, there exists \( x \in c_q \cap C_p \) with \( v_q(x) > m_q \) or \( u_q(x) < M_q \). If \( v_q(x) > m_q \) (respectively, \( u_q(x) < M_q \)), then \( v_q(y) < v_q(x) \) (respectively, \( u_q(y) < u_q(x) \)) for all \( y \in c_q \cap C_p \). Since \( x \) and \( y \) are \( R \)-comparable, in both cases we have \( u_p(y) \geq u_p(x) = M_p \) and \( m_p = v_p(x) \geq v_p(y) \). In other words, \( y \in c_p \cap C_p \) for all \( y \in c_q \cap C_q \).

Let us fix a positive integer \( s \) and a finite subset \( \{p_1, \ldots, p_s\} \subset I \).

Using Lemma 2.2.1, (i), (ii), and induction, we obtain immediately the following:

Corollary 2.2.2 The intersection \( C_{p_1} \cap C_{p_1} \cap \ldots \cap C_{p_k} \) is equal to one of the sets \( C_{p_1}, C_{p_1}, \ldots, C_{p_k}, C_{p_k} \) for all \( k \leq s \).

Given an \( s \geq 1 \), in accord with Lemma 2.2.1, (i), (ii), and eventual renumbering of the pairs of functions \( u_{p_k}, v_{p_k} \), we order the intersections \( C_{p_k} \cap C_{p_k} \), \( k \leq s \), with respect to inclusion from smallest to largest:

\[
c_{p_1} \cap C_{p_1} \subset \ldots \subset c_{p_k} \cap C_{p_k} \subset c_{p_{k+1}} \cap C_{p_{k+1}} \subset \ldots \subset c_{p_s} \cap C_{p_s},
\]

(2.2.1)

where \( c_{p_i} = \emptyset \) or \( C_{p_i} = \emptyset \), \( 1 \leq i \leq \ell \), and \( c_{p_{\ell+1}} \cap C_{p_{\ell+1}} \neq \emptyset \). Below, if the opposite is not stated, after fixing \( \{p_1, \ldots, p_s\} \subset I \), we assume that (2.2.1) holds.

Thus, the existence of \( k \leq s \) with \( c_{p_k} = \emptyset \) or \( C_{p_k} = \emptyset \) after renumbering implies \( \ell \geq 1 \), that is, \( c_{p_1} = \emptyset \) or \( C_{p_1} = \emptyset \).

Lemma 2.2.3 Let \( X \) be a sequentially compact space and let \( C_{p_1} = \emptyset \) (respectively, \( c_{p_1} = \emptyset \)).

(i) There exists a strictly \( R \)-increasing and divergent sequence

\[
(x_i)_{i=1}^{\infty},
\]

(2.2.2)

with \( x_i \in C \) and limit \( x^* \in X \), such that the sequence of real numbers \( (u_{p_1}(x_i))_{i=1}^{\infty} \) is strictly increasing and diverges to \( u_{p_1}(x^*) = M_{p_1} \) and every sequence of real numbers \( (v_q(x_i))_{i=1}^{\infty} \), \( q \in I \), is decreasing and diverges to \( v_{p_1}(x^*) = m_p \) (respectively, the sequence of real numbers \( (v_{p_1}(x_i))_{i=1}^{\infty} \) is strictly decreasing and diverges to \( v_{p_1}(x^*) = m_p \) and every sequence of real numbers \( (u_q(x_i))_{i=1}^{\infty}, q \in I \), is increasing and diverges to \( u_q(x^*) = M_q \)).

(ii) Let for the sequence (2.2.2) from part (i) one has \( u_{p_1}(x^*) = M_{p_1}, u_{p_2}(x^*) = M_{p_2}, \ldots, u_{p_k}(x^*) = M_{p_k} \) (respectively, \( v_{p_1}(x^*) = m_{p_1}, v_{p_2}(x^*) = m_{p_2}, \ldots, v_{p_k}(x^*) = m_{p_k} \)), for some \( k < s \). Then either there exists \( y \in \)
\(\cap_{\lambda \in I} \cup_{\lambda \in I} C_{\lambda} \cap \cup_{\lambda \in I} C_{\lambda} \cap \cup_{\lambda \in I} C_{\lambda} \cap \cap_{\lambda \in I} C_{\lambda} \) (respectively, \(y \in c_{p_{1}} \cap \ldots \cap c_{p_{k}} \cap c_{p_{k+1}} \cap \cap_{\lambda \in I} C_{\lambda} \), or there exists a strictly R-increasing and divergent sequence \((y_{k})_{k=1}^{\infty}\), with \(y_{k} \in C\) and limit \(y^{*} \in X\), such that \(u_{p_{1}}(y^{*}) = M_{p_{1}}, u_{p_{2}}(y^{*}) = M_{p_{2}}, \ldots, u_{p_{k}}(y^{*}) = M_{p_{k}}\), and \(v_{q}(y^{*}) = m_{q}, q \in I\) (respectively, \(v_{p_{1}}(y^{*}) = m_{p_{1}}, v_{p_{2}}(y^{*}) = m_{p_{2}}, \ldots, v_{p_{k}}(y^{*}) = m_{p_{k}}\), and \(u_{q}(x^{*}) = M_{q}, q \in I\)), the sequence of real numbers \((u_{p_{k+1}}(y_{k}))_{k=1}^{\infty}\) is strictly increasing and diverges to \(u_{p_{k+1}}(y^{*}) = M_{p_{k+1}}\) and every sequence of real numbers \((v_{q}(y_{k}))_{k=1}^{\infty}\), \(q \in I\), is decreasing and diverges to \(v_{q}(y^{*}) = m_{q}\) (respectively, the sequence of real numbers \((v_{p_{k+1}}(y_{k}))_{k=1}^{\infty}\) is strictly decreasing and diverges to \(v_{p_{k+1}}(y^{*}) = m_{p_{k+1}}\) and every sequence of real numbers \((u_{q}(y_{k}))_{k=1}^{\infty}\), \(q \in I\), is increasing and diverges to \(u_{q}(y^{*}) = M_{q}\).)

**Proof:** Below, when \(c_{p_{1}} = \emptyset\), we replace \(u_{q}\) with \(-v_{q}\), \(v_{q}\) with \(-u_{q}\), and use the corresponding proofs in case \(c_{p_{1}} = \emptyset\).

(i) Let \(C_{p_{1}} = \emptyset\). Then \(M_{p_{1}} = \sup_{x \in C_{p_{1}}} u_{p_{1}}(x)\) and we choose \((x_{i})_{i=1}^{\infty}\) to be a sequence of members of \(C = C_{p_{1}}\) such that the sequence of real numbers \((u_{p_{1}}(x_{i}))_{i=1}^{\infty}\) is strictly increasing with \(\lim_{i \to \infty} u_{p_{1}}(x_{i}) = M_{p_{1}}\). Since the elements \(x_{i}, i \geq 1\), are pairwise R-comparable, it turns out that the sequences of real numbers \((u_{q}(x_{i}))_{i=1}^{\infty}\), \(q \in I\), \(q \neq p_{1}\), are increasing and \((v_{p}(x_{i}))_{i=1}^{\infty}\), \(q \in I\), are decreasing. Thus, the sequence \((x_{i})_{i=1}^{\infty}\) is strictly R-increasing. In accord with the sequential compactness of the topological space \(X\), we can suppose that \((x_{i})_{i=1}^{\infty}\) diverges to a point \(x^{*} \in X\). Thus, \(u_{p_{1}}(x^{*}) = M_{p_{1}}\). For any \(q \in I\) we set \(m_{q}' = \lim_{i \to \infty} v_{q}(x_{i})\). Let us suppose \(m_{q}' < m_{q}^{'}\) for some \(q_{0} \in I\) and let \(y \in C\) be such that \(v_{q_{0}}(y) < m_{q_{0}}^{'}\). In particular, \(v_{q_{0}}(y) < v_{q_{0}}(x_{i})\), hence \(u_{p_{1}}(y) \geq u_{p_{1}}(x_{i})\) for all \(i \geq 1\). Taking the limit we obtain \(u_{p_{1}}(y) \geq M_{p_{1}}\), that is, \(y \in C_{p_{1}}\), which is a contradiction. Therefore \(m_{q} = m_{q}^{'}\) and \(v_{q}(x^{*}) = m_{q}\) for all \(q \in I\).

(ii) Let \(M_{p_{k+1}^{*}} = \lim_{i \to \infty} u_{p_{k+1}}(x_{i})\). We have \(M_{p_{k+1}^{*}} \leq M_{p_{k+1}}\) and if \(M_{p_{k+1}^{*}} = M_{p_{k+1}}\), then \(u_{p_{k+1}}(x^{*}) = M_{p_{k+1}}\). In other words, \(x^{*} \in \cap_{\lambda=1}^{\infty} C_{\lambda}^{*} \cap \cup_{\lambda=1}^{\infty} C_{\lambda} \cap \cup_{\lambda=1}^{\infty} C_{\lambda} \). Now, let \(M_{p_{k+1}^{*}} < M_{p_{k+1}}\).

In case \(C_{p_{k+1}} \neq \emptyset\), we choose \(y \in C_{p_{k+1}}\) and since \(x_{i}'\)s and \(y\) are R-comparable, the inequalities \(u_{p_{k+1}}(x_{i}) \leq M_{p_{k+1}^{*}} < u_{p_{k+1}}(y)\) yield

\[u_{q}(x_{i}) \leq u_{q}(y)\]

for all \(q \in I\), \(q \neq p_{k+1}\), and

\[v_{q}(x_{i}) \geq v_{q}(y)\]

for all \(q \in I\). Taking the limit \(i \to \infty\) in (2.2.3) for all \(q = p_{1}, \ldots, p_{k}\) and in (2.2.4) for all \(q \in I\), we obtain \(y \in \cap_{\lambda=1}^{\infty} C_{\lambda} \cap \cup_{\lambda=1}^{\infty} C_{\lambda} \cap \cup_{\lambda=1}^{\infty} C_{\lambda} \).

In case \(C_{p_{k+1}} = \emptyset\), there exists a sequence \((y_{k})_{k=1}^{\infty}\), \(y_{k} \in C\), such that \(M_{p_{k+1}} < u_{p_{k+1}}(y_{k}) < M_{p_{k+1}}, k \geq 1\), the sequence of real numbers \((u_{p_{k+1}}(y_{k}))_{k=1}^{\infty}\) is strictly increasing and diverges to \(M_{p_{k+1}}\). In particular, \(u_{p_{k+1}}(x_{i}) < u_{p_{k+1}}(y_{k})\) for all \(i, k \geq 1\). Since \(x_{i}\)s and \(y_{k}\)s are R-comparable, we obtain for all \(i, k \geq 1\) the inequalities

\[u_{q}(x_{i}) \leq u_{q}(y_{k}) \leq M_{q}\]

(2.2.5)
for all $q \neq p_{k+1}$, and

$$v_q(x_i) \geq v_q(y_\kappa) \geq m_q \quad (2.2.6)$$

for all $q \in I$. Since the topological space $X$ is sequentially compact, we can assume that $(y_\kappa)_{\kappa=1}^\infty$ diverges with limit $y^* \in X$, so $u_{p_{k+1}}(y^*) = M_{p_{k+1}}$. Taking consecutively the limits $\iota \to \infty$, $\kappa \to \infty$, in (2.2.5) for all $q = p_1, \ldots, p_k$ and in (2.2.6) for all $q \in I$, we obtain $y^* \in \cap_{\lambda=1}^\infty C^*_\lambda \cap C^*_{p_1} \cap \cdots \cap C^*_{p_k} \cap C^*_{p_{k+1}}$.

**Proposition 2.2.4** Let $X$ be a sequentially compact space endowed with the preorder $R$ from (2.1.1) and let $C \subset X$ be a chain.

(i) For any finite subset $\{p_1, \ldots, p_s\} \subset I$ one has

$$\cap_{i=1}^s C^*_p \cap c^*_p \neq \emptyset. \quad (2.2.7)$$

(ii) If $X$ is, in addition, quasi-compact, then

$$\cap_{p \in I} C^*_p \cap c^*_p \neq \emptyset. \quad (2.2.8)$$

**Proof:** (i) If $C$ is a finite $R$-chain, then its largest element is a member of the intersection $\cap_{i=1}^t C^*_i \cap c^*_i$.

Now, let us suppose that the $R$-chain $C$ is infinite. In case all sets $c_1$, $C_1$, $\ldots$, $c_s$, $C_s$, are nonempty Corollary 2.2.2 implies that their intersection is not empty, hence (2.2.8) holds. Otherwise, using Lemma 2.2.3 and induction with respect to $k$, we are done.

(ii) Since $X$ is quasi-compact, part (i) implies part (ii).

**Corollary 2.2.5** If $X$ is a quasi-compact and sequentially compact space, then the preordered set $X$ is inductive.

**Proof:** Every element $x^* \in \cap_{p \in I} C^*_p \cap c^*_p$ is an upper bound of the $R$-chain $C$, hence the preordered set $X$ is inductive.

Now, Corollary 2.2.5 and Kuratowski-Zorn Theorem yield the following:

**Theorem 2.2.6** Let $X$ be a quasi-compact and sequentially compact space. For any element $x \in X$ there exists an $R$-maximal element $y \in X$ with $xRy$.

### 2.3 Examples

Since the $(n - 1)$-dimensional simplex $\Delta_{n-1}$ is a compact set in $\mathbb{R}^n$, it is a quasi-compact and sequentially compact topological space. In case the family $u$ consists of one function $u(x)$ — the expected return of the portfolio $x$ and the family $v$ consists of one function $v(x)$ — its volatility, using Theorem 2.2.6 we obtain the existence of Markowitz efficient portfolios and something more: Any portfolio is $R$-dominated by a Markowitz efficient portfolio.

Moreover, replacing the simplex $\Delta_{n-1}$ with a closed ball $B_{n-1}$ in the affine hyperplane $\sum_{i=1}^n x_i = 1$ in $\mathbb{R}^n$, such that $\Delta_{n-1} \subset B_{n-1}$, we admit bounded
negative \(x_i\)’s (that is, constrained short sales) and again Theorem 2.2.6 assures existence of Markowitz efficient portfolios which dominate any given portfolio.

Below, we remind some notions from statistics and give examples of application of Theorem 2.2.6.

Given the integer \(\ell \geq 2\), the \(\ell\)-th central moment of the random variable \(s(x)\) is \(E((s(x) - E(s(x)))^\ell)\). The standard variance is the second central moment \(v(x) = E((s(x) - E(s(x)))^2)\) of \(s(x)\) and it is a quadratic form in \(x_1, \ldots, x_n\). The third central moment \(E((s(x) - E(s(x)))^3)\) is a cubic form and the fourth central moment \(E((s(x) - E(s(x)))^4)\) is a form of degree 4 in \(x_1, \ldots, x_n\).

Given \(x \in \Delta_{n-1}\) and \(t \in \mathbb{R}\), we set \(F_x(t) = P(\{m \in S \mid s(x)(m) < t\})\), so \(F_x: \mathbb{R} \to [0, 1]\) is the cumulative distribution function of the random variable \(s(x)\). We assume that \(s(x)\) is a continuous random variable with density function \(f_x(t)\), so \(F_x(t) = \int_{-\infty}^{t} f_x(\tau)d\tau\) and \(F_x'(t) = f_x(t)\). In particular, the functions \(F_x(t)\) are continuous.

We define recursively \(D_x^{(1)}(t) = F_x(t), D_x^{(2)}(t) = \int_{-\infty}^{t} F_x(\tau)d\tau, \ldots, D_x^{(\ell)}(t) = \int_{-\infty}^{t} D_x^{(\ell-1)}(\tau)d\tau, \ldots\)

The portfolio \(x \in \Delta_{n-1}\) is said to be \(\ell\)-th order stochastically dominated by portfolio \(y \in \Delta_{n-1}\) if \(D_y^{(\ell)}(t) \leq D_x^{(\ell)}(t)\) for all \(t \in \mathbb{R}\). In case the previous inequalities hold and \(D_y^{(\ell)}(t) < D_x^{(\ell)}(t)\) for some \(t \in \mathbb{R}\), \(x\) is said to be \(\ell\)-th order strictly stochastically dominated by \(y\).

We set

\[
\text{Skew}(s(x)) = \frac{E((s(x) - E(s(x)))^3)}{\text{Var}(s(x))^{3/2}}
\]

to be the skewness and

\[
\text{Kurt}(s(x)) = \frac{E((s(x) - E(s(x)))^4)}{\text{Var}(s(x))^2} - 3
\]

to be the kurtosis, or, excess kurtosis of the random variable \(s(x)\).

If the random variable \(s(x)\) is normal, then \(\text{Skew}(s(x)) = \text{Kurt}(s(x)) = 0\).

Example 2.3.1 In case \(I = \{1\}, J = \emptyset\), the function \(u = u_1\) can be considered as an utility function on \(\Delta_{n-1}\) and \(R\) is the corresponding preference relation with negatively transitive asymmetric part \(F\).

Example 2.3.2 In case \(I = \{1\}, J = \{1\}, u_1(x) = E(s(x)), v_1(x) = \text{Var}(s(x))\), we obtain the classical Markowitz setup.

Example 2.3.3 In case

\[
u_1(x) = E(s(x)),
\]
\[ v_1(x) = \text{Var}(s(x)), \quad v_2(x) = \text{Skew}^2(s(x)), \]
we simultaneously maximize the expected return \( E(s(x)) \) and minimize the volatility \( \text{Var}(s(x)) \) and the absolute value of the skewness \( \text{Skew}(s(x)) \) of the return \( s(x) \) of the portfolio \( x \).

**Example 2.3.4** In case
\[
 u_1(x) = E(s(x)), \\
 v_1(x) = \text{Var}(s(x)), \quad v_2(x) = \text{Kurt}^2(s(x)),
\]
we simultaneously maximize the expected return \( E(s(x)) \) and minimize the volatility \( \text{Var}(s(x)) \) and the absolute value of the kurtosis \( \text{Kurt}(s(x)) \) of the return \( s(x) \), thus balancing the tails of its distribution.

**Example 2.3.5** In case
\[
 u_1(x) = E(s(x)), \\
 v_1(x) = \text{Var}(s(x)), \quad v_2(x) = \text{Skew}^2(s(x)), \quad v_3(x) = \text{Kurt}^2(s(x)),
\]
we simultaneously maximize the expected return \( E(s(x)) \) and minimize the volatility \( \text{Var}(s(x)) \), the the absolute value of the skewness \( \text{Skew}(s(x)) \), and the absolute value of the kurtosis \( \text{Kurt}(s(x)) \) of the return \( s(x) \). In this way we balance both the tails of the distribution of \( s(x) \) and ”round” the maximum of its density function \( f_x(t) \).

**Example 2.3.6** In case
\[
 v_1(x) = D_x^{(t)}(t), \quad t \in \mathbb{R},
\]
we maximize the \( \ell \)-th order stochastic dominance, \( \ell \geq 1 \).

**Example 2.3.7** In case
\[
 u(x) = E(s(x)), \\
 v(x) = \text{Var}(s(x)), \quad v_1(x) = D_x^{(t)}(t), \quad t \in \mathbb{R},
\]
we simultaneously maximize the expected return \( E(u(x)) \) and the \( \ell \)-th order stochastic dominance, \( \ell \geq 1 \), and minimize the volatility \( \text{Var}(s(x)) \).

**Example 2.3.8** Let \( X \) be a quasi-compact and sequentially compact space and let \( f: X \times X \to \mathbb{R} \) be a continuous real function. For any \( p \in X \) we set
\[
 u_p(x) = f(x, p), \quad x \in X, \\
 v_p(y) = f(p, y), \quad y \in X.
\]
Further, for any \( x \in X \) we set
\[
 U_x^{(2)} = \{ y \in X \mid f(y, p) \geq f(x, p) \text{ for all } p \in X \},
\]
\[ V_x^{(\leq)} = \{ y \in X \mid f(p, y) \leq f(p, x) \text{ for all } p \in X \}, \]

and for any \( x, p \in X \) we set

\[ U_x^{(\hat{p}; \geq)} = \{ y \in X \mid f(y, q) \geq f(x, q) \text{ for all } q \in X, q \neq p \}, \]

\[ V_x^{(\hat{p}; \leq)} = \{ y \in X \mid f(q, y) \leq f(q, x) \text{ for all } q \in X, q \neq p \}. \]

Note that \( U_x^{(\geq)}, V_x^{(\leq)}, U_x^{(\hat{p}; \geq)}, V_x^{(\hat{p}; \leq)} \) are closed subsets of \( X \) and that

\[ x \in U_x^{(\geq)} \subset U_x^{(\hat{p}; \geq)}, \quad x \in V_x^{(\leq)} \subset V_x^{(\hat{p}; \leq)} \]

for all \( x, p \in X \). According to Theorem 2.2.6, there exists an element \( m \in X \), such that for any \( p \in X \) one has

\[ f(m, p) = \max_{y \in U_m^{(\hat{p}; \geq)} \cap V_m^{(\leq)}} f(y, p) \]

and

\[ f(p, m) = \min_{y \in U_m^{(\hat{p}; \geq)} \cap V_m^{(\hat{p}; \leq)}} f(p, y). \]

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