Tilting objects on some global quotient stacks

Saša Novaković

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Abstract. We prove the existence of tilting objects on some global quotient stacks that are obtained as quotients of smooth projective schemes by finite group actions. As a consequence we provide further evidence for the dimension conjecture of derived categories formulated by Orlov.

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1. Introduction

Geometric tilting theory started with the construction of tilting bundles on the projective space by Beilinson [9]. Later Kapranov [42], [43], [44] constructed tilting bundles for certain homogeneous spaces and further examples can be obtained from certain blowups and taking projective bundles [24], [25], [57]. Note that a smooth projective \(k\)-scheme admitting a tilting bundle satisfies very strict conditions, namely its Grothendieck group is a free abelian group of finite rank and the Hodge diamond is concentrated on the diagonal (in characteristic zero) [21]. However, it is still an open problem to give a complete classification of smooth projective \(k\)-schemes admitting a tilting bundle. In the case of curves one can prove that a smooth projective algebraic curve has a tilting bundle if and only if it is a one-dimensional Brauer–Severi variety. But already for smooth projective algebraic surfaces there is currently no classification of surfaces admitting such a tilting object. It is conjectured that smooth projective algebraic surfaces have a tilting bundle if and only if they are rational (see [20], [34], [35], [37], [38], [48] and [59] for results in this direction). In the present work we will focus on certain stacks and prove the existence of tilting objects for their derived category. Several examples of stacks admitting a tilting object are known (see [40], [41], [45], [53], [55] and [56]). But as in the case of schemes, we are still far from a classification of stacks admitting a tilting object. The first main result of the present paper is the following:

Theorem. (Theorem 4.1) Let \(X\) be a smooth projective and integral \(k\)-scheme, \(G\) a finite group acting on \(X\) and \(\text{char}(k) \nmid \text{ord}(G)\). Suppose that \(T\) is a tilting sheaf for \(D^b(X)\), admitting a \(G\)-equivariant structure. Let \(W_i\) be the irreducible representations of \(G\), then \(T_G = \bigoplus_i T \otimes W_i\) is a tilting sheaf for \(D^b([X/G])\) and one therefore has an equivalence

\[ \mathbb{R} \text{Hom}_G(T_G, -) : D^b([X/G]) \rightarrow D^b(\text{End}_G(T_G)). \]
This theorem enables us to find certain examples of quotient stacks admitting a tilting object. Notice that Elagin [31] proved the existence of tilting objects on some stacks $[X/G]$, induced from full strongly exceptional collections. Since there really exist $k$-schemes having tilting objects but not a full strongly exceptional collection (see Proposition 4.6), Theorem 4.1 indeed provides us with some new examples (see for instance Example 4.5). Moreover, exploiting derived McKay correspondence [17], Theorem 4.1 also provides us with some crepant resolutions that have tilting objects (see Theorem 4.10 and Corollary 4.11 and 4.12). Roughly, we proved that the quotient stack $[X/G]$ has a tilting object if $X$ admits one. However, the converse of this statement is not true for trivial reasons (see Proposition 4.9). Theorem 4.1 from above can be applied to prove the following result generalizing and harmonizing results of Bridgeland and Stern [19], Theorem 3.6 (see also [18], Proposition 4.1) and Brav [16], Theorem 4.2.1. For this, let $E$ be a $G$-equivariant locally free sheaf on $X$ and $\mathcal{A}(E)$ the affine bundle. Suppose $\text{char}(k) \nmid \text{ord}(G)$ and denote by $\pi : \mathcal{A}(E) \to X$ the projection and by $\mathcal{T}_G$ the tilting sheaf of the previous theorem. We then obtain:

**Theorem.** (Theorem 5.1) Let $X$ and $G$ be as above and $E$ a $G$-equivariant locally free sheaf of finite rank. Suppose $T$ is a tilting bundle for $\mathcal{D}^b(X)$ admitting a $G$-equivariant structure. If $H^i(X, T^\vee \otimes T \otimes S^l(E)) = 0$ for all $i \neq 0$ and all $l > 0$, then one has an equivalence

$$\mathcal{R}\text{Hom}(\pi^*\mathcal{T}_G, -) : \mathcal{D}^b(\mathcal{A}(E)/G) \xrightarrow{\sim} \mathcal{D}^b(\text{End}_G(\pi^*\mathcal{T}_G)).$$

Note that for $X$ being a Fano variety, $E = \omega_X$ and $G = 1$ one obtains the result of Bridgeland and Stern [19] (see also [18]) and if $X = \text{Spec}(\mathbb{C})$ the result of Brav [16]. In both cases the assumption $H^i(X, T^\vee \otimes T \otimes S^l(E)) = 0$ for all $i \neq 0$ and all $l > 0$ can be shown to be fulfilled. The proof of Theorem 5.1 also simplifies some proves given in [16]. In view of Theorem 5.1 it is also natural to consider projective bundles of $G$-equivariant locally free sheaves $E$ on $X$. In this case, exploiting Theorem 4.1, we prove the following generalization of a result of Costa, Di Rocco and Miró-Roig [24]:

**Theorem.** (Theorem 5.3) Let $X$ and $G$ be as above and suppose that $E$ is $G$-equivariant. Assume furthermore that $\mathcal{D}^b(X)$ admits a tilting bundle $T$ with $G$-equivariant structure. Then the quotient stack $[\mathbb{P}(E)/G]$ admits a tilting object too.

In Section 6 we shortly state some problems occurring when one considers quotients of schemes by arbitrary not necessary finite algebraic groups. In this situation one cannot expect to have a result as Theorem 4.1 but nevertheless, under some mild conditions, one always has semiorthogonal decompositions [31] and therefore a nice description of the bounded derived category of coherent sheaves. Finally, as an application of the above results we provide some further evidence for a conjecture firstly formulated by Orlov [58] for schemes and extended by Ballard and Favero [6] to certain Deligne–Mumford stacks. It is the following conjecture:

**Conjecture.** ([6]) Let $X$ be a smooth and tame Deligne–Mumford stack of finite type over $k$ with quasiprojective coarse moduli space, then $\dim \mathcal{D}^b(X) = \dim(X)$.

With the help of results obtained in [6] and the results proved in the present note we show the following theorem:

**Theorem.** (Theorem 7.8) The dimension conjecture holds in the following cases:

(i) quotient stacks of the form $[\mathbb{P}^2/G]$, where $G$ is a finite subgroup of $\text{PGL}_{m+1}(k)$, $k$ is perfect and $\text{char}(k) \nmid \text{ord}(G)$,

(ii) quotient stacks of the form $[X/G]$, where $G$ is a finite subgroup of $\text{Aut}(X)$, $\text{char}(k) \nmid \text{ord}(G)$ and $X$ is a Brauer–Severi variety over a perfect field $k$,

(iii) quotient stacks of the form $[\text{Grass}_d(d,n)/G]$, provided $2d \neq n$ and where the action of the finite group $G$ is induced by a homomorphism $G \to \text{PGL}_n(\mathbb{C})$,

(iv) $G$-Hilbert schemes $\text{Hilb}_G(\mathbb{P}^2)$ for $n \leq 3$ and $G$ a finite subgroup of $\text{PGL}_{m+1}(\mathbb{C})$ such that $\omega_{\mathbb{P}^2}$ is locally trivial as a $G$-equivariant sheaf.
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Conventions. Throughout this work \( k \) is an arbitrary field unless stated otherwise.

2. Generalities on quotient stacks

In this section we briefly recall the definition of a (quotient) stack and refer to [52] for all technical details. Moreover, we use the Appendix of [68], since it gives a nice and comprehensive summary of the main definitions needed in this section.

Let \( S \) be a scheme and \( Z \) a category over \( S \), fibered in groupoids. One then has a functor \( p : Z \to (\text{Sch}/S) \), called projection. Given any \( S \)-scheme \( T \), we denote by \( Z(T) \) the category whose objects are objects \( a \in Z \) such that \( p(a) = T \) and whose arrows are arrows \( f \) in \( Z \) with \( p(f) = \text{id} \). We call this category the fiber of \( Z \) over \( T \). Now \( Z \) is called a stack over \( S \) if the following hold (see [68], Definition 7.3):

(i) For any \( U \in (\text{Sch}/S) \) and any two objects \( a, b \in Z(U) \) the functor \( \mathcal{I}_{\forall j}(a, b) : (\text{Sch}/U) \to \text{Set} \), \( V \mapsto \{a|_V \sim b|_V \in Z(V)\} \) is a sheaf in the étale topology. Precisely this means: For all \( U \in (\text{Sch}/S) \) and all \( a, b \in Z(U) \) and for all open covers \( \{U_i \to U\} \) in the étale topology and all isomorphisms \( \alpha_i : a_{|U_i} \sim b_{|U_i} \), such that \( \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}} \), there is a unique isomorphism \( \alpha : a \sim b \), such that \( \alpha|_{U_i} = \alpha_i \).

(ii) For all open covers \( \{U_i \to U\} \) the descent datum is effective. This means the following: For all open covers \( \{U_i \to U\} \) in the étale topology and all \( a_i \in Z(U_i) \) and all isomorphisms \( \alpha_{ij} : a_{i|U_{ij}} \sim a_{j|U_{ij}} \) in \( Z(U_i \times_U U_j) \), such that \( \alpha_{ik} = \alpha_{jk} \circ \alpha_{ij} \), there is a \( a \in Z(U) \) and \( \alpha_i : a_{|U_i} \sim a_i \), such that \( \alpha_{ij} = \alpha_{ij|U_{ij}} \circ (\alpha_{i|U_{ij}})^{-1} \). A morphism of stacks is a functor \( F : Z_1 \to Z_2 \), such that for the projections \( p_{Z_1} : Z_1 \to (\text{Sch}/S) \) and \( p_{Z_2} : Z_2 \to (\text{Sch}/S) \) one has \( p_{Z_1} = p_{Z_2} \circ F \).

Example 2.1. A scheme \( X \) can be considered as a stack via its functor of points (see [68], Example 7.2). To be more precise, consider a \( S \)-scheme \( X \) with structure morphism \( \pi : X \to S \). Then consider the functor \( p : (\text{Sch}/X) \to (\text{Sch}/S) \) assigning to a \( X \)-scheme \( T \) the \( S \)-scheme \( T \to X \to S \). One can show that \( (\text{Sch}/X) \) is fibered in groupoids over \( S \). The fibers \( Z(U) \) over a \( S \)-scheme \( U \) are just \( \text{Hom}_S(U, X) \) as a set. Both conditions (i) and (ii) from above can be verified to hold true. Therefore, we can consider the \( S \)-scheme \( X \) as a stack via its functor of points. To every \( S \)-scheme \( T \) the corresponding fiber of the stack is given by \( \text{Hom}_S(T, X) \).

Note that one can also form the fiber product \( Z_1 \times_Z Z_2 \) of two morphisms of stacks \( Z_1 \to Z \) and \( Z_2 \to Z \) (see [68], Definition 7.9). The diagonal morphism \( \Delta_Z : Z \to Z \times_Z Z \) is given by the two identity morphisms. A morphism \( F : Z_1 \to Z_2 \) of stacks is called representable if for any \( S \)-scheme \( T \) and any morphism \( T \to Z_2 \) the fiber product \( Z_1 \times_{Z_2} T \) is a scheme (see [68], Definition 7.11). Let \( Z \) be an stack and \( T \) a \( S \)-scheme considered as a stack via its functor of points. An étale surjective morphism \( T \to Z \) is called atlas. As our further investigations important class of stacks is the class of Deligne–Mumford stacks (see [68], Definition 7.14).

Definition 2.2. A Deligne–Mumford stack is a stack \( Z \) satisfying the following conditions:

(i) The diagonal morphism \( \Delta_Z : Z \to Z \times_Z Z \) is representable, quasicompact and separated.

(ii) There is a scheme \( T \) and an étale surjective morphism \( T \to Z \).
Remark 2.3. A stack $Z$ is called Artin stack if it satisfies (i) from above and if one, instead of the existence of a scheme $T$ with an étale surjective morphism $T \to Z$, claims the existence of a scheme $T$ with a smooth surjective morphism $T \to Z$.

Example 2.4. Let $G$ be a smooth linear algebraic group over $k$ acting on a $k$-scheme $X$. Denote by $[X/G]$ the category fibered over the category of $k$-schemes (Sch$/k$), the fibers over a $k$-scheme $T$ being defined as the set of principal $G$-bundles $P \to T$ together with $G$-equivariant map $P \to X$. One can show that $[X/G]$ is an Artin stack and if the stabilizers of the geometric points of $X$ are finite and reduced, $[X/G]$ is a Deligne–Mumford stack (see [68], Example 7.17). In particular, if $G$ is a finite group acting on $X$ with $\text{char}(k) \nmid \text{ord}(G)$, the stack $[X/G]$ is a Deligne–Mumford stack. Stacks of the form $[X/G]$ are called global quotient stacks.

Now let $Z$ be a stack. A quasicoherent sheaf $\mathcal{F}$ on $Z$ consists of the following data (see [68], Definition 7.18).

(i) For each atlas $T \to Z$ one has a quasicoherent sheaf $\mathcal{F}_T$ on $T$.

(ii) For each morphism $\phi : T \to U$ of atlases there is an isomorphism $\alpha_\phi : \mathcal{F}_T \to \phi^* \mathcal{F}_U$.

The isomorphisms $\alpha_\phi$ are required to satisfy the cocycle condition that we do not want to reproduce here, referring to the literature for details. We recall the definition of a $G$-equivariant sheaf (see [54] and [64]). Let $X$ be an integral $k$-scheme and $G$ an algebraic group over $k$ acting on $X$. Now let $m : G \times G \to G$ be the multiplication morphism and $\sigma : G \times X \to X$ the action of $G$ on $X$. Denote the projections of $G \times G$, $G \times X$ or $G \times G \times X$ onto the $i$-th factor by $p_i$ and the projections of $G \times G \times X$ or $G \times G \times G$ on the product of the first two (last two) factors by $p_{12}$ (or $p_{23}$) respectively. A $G$-equivariant sheaf $\mathcal{F}$ on $X$ is a sheaf $\mathcal{F}$, together with an isomorphism $\theta : p_2^* \mathcal{F} \to \sigma^* \mathcal{F}$ of sheaves on $G \times X$, satisfying the following condition:

$$(m \times \text{id}_X)^* \theta = p_{23}^* \theta \circ (\text{id}_G \times \sigma)^* \theta.$$  

The isomorphism $\theta$ is called equivariant structure of $\mathcal{F}$. Note that in the literature such sheaves are also called $G$-linearized. If the $G$-equivariant sheaf $\mathcal{F}$ is quasicoherent (coherent or locally free), it is called $G$-equivariant quasicoherent (coherent or locally free) sheaf. A morphism of $G$-equivariant sheaves is defined to be compatible with the $G$-equivariant structure and the group of $G$-equivariant homomorphisms will be denoted by $\text{Hom}_G(\mathcal{F}, \mathcal{G})$. Note that one has a natural action of $G$ on $\text{Hom}(\mathcal{F}, \mathcal{G})$ and taking invariants yields $\text{Hom}_G(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{G})^G$. Considering $G$-equivariant (quasi-) coherent sheaves on $X$ one has abelian categories denoted by $\text{Qcoh}_G(X)$ and $\text{Coh}_G(X)$ respectively. We write $D_G(\text{Qcoh}(X))$ for the derived category of $G$-equivariant quasicoherent sheaves on $X$ and $\text{D}_G(X)$ for the bounded derived category of $G$-equivariant coherent sheaves.

Now let $X$ be a smooth projective and integral $k$-scheme and $G$ a smooth linearly reductive group over $k$ acting on $X$. Consider the quotient stack $[X/G]$. The atlas is given by the projection $X \to [X/G]$ (see [68], Example 7.17) and one has an isomorphism of groupoids between $X \times [X/G] X \rightrightarrows X$ and $X \times_S G \rightrightarrows X$ (see [68], Example 7.21). This isomorphism directly implies that a coherent sheaf $\mathcal{F}$ on the stack $[X/G]$ is by definition a $G$-equivariant coherent sheaf $\mathcal{F}$ on $X$. Note that the cocycle condition translates to the condition of $\mathcal{F}$ being $G$-equivariant and hence the categories $\text{Coh}_G(X)$ and $\text{Coh}([X/G])$ are equivalent. This implies $\mathcal{D}_G'(X) \simeq D^b([X/G])$. Summarizing, we hold on to the fact that $G$-equivariant (quasi-) coherent sheaves on $X$ are the same as (quasi-) coherent sheaves on the quotient stack $[X/G]$. For $X$ being a point the stack $[pt/G]$ is denoted by $BG$ and is called the classifying stack of $G$. Quasicoherent sheaves on this stack can be thought of as representations of the group $G$. We want to note that there is also a very useful characterization of linearly reductive groups including the category of quasicoherent sheaves on $[pt/G]$. A linear algebraic group $G$ over $k$ is called linearly reductive if the functor $(-)^G : \text{Qcoh}([pt/G]) \to \text{Qcoh}(pt)$, $V \mapsto V^G$ is exact (see [1], Definition 2.4).
Moreover, since we are considering algebraic groups over fields, $G$ is linearly reductive if and only if the functor $\text{Coh}([pt/G]) \to \text{Coh}(pt)$, $V \mapsto V^G$ is exact (see [1], Proposition 2.5).

If $X$ is a smooth projective and integral $k$-scheme and $G$ a finite group acting on $X$ with $\text{char}(k) \nmid \text{ord}(G)$, the quotient stack $[X/G]$ is a smooth, proper, tame and connected Deligne–Mumford stack with coarse projective moduli space. To see this, we first recall the notion of a tame stack. The inertia stack of a stack $Z$ over $k$ is defined to be $IZ := Z \times_{Z \times_k Z} Z$. If $IZ \to Z$ is finite, there exists a coarse moduli space $\rho : Z \to M$ and if furthermore $\rho_* : \text{Qcoh}(Z) \to \text{Qcoh}(M)$ is exact, then $Z$ is called tame (see [1] for details). Note that $X \to [X/G]$ is the atlas of $[X/G]$. Now [68], Proposition 2.11 implies that the quotient scheme $X//G$ is a coarse moduli space for $[X/G]$. Since $X$ is a smooth projective and integral $k$-scheme, the quotient $X//G$ is a projective scheme. Furthermore, since $X$ is smooth and projective, $G$ is finite and since $\text{char}(k) \nmid \text{ord}(G)$, the quotient stack $[X/G]$ is a smooth proper and tame Deligne–Mumford stack. To see why $[X/G]$ is a field, we refer to [1], Theorem 3.2. At this point one needs that the characteristic of $k$ does not divide the order of $G$ so that $G$ is linearly reductive.

We end up this section stating a very useful observation that will be needed later on quite frequently. It is well-known and can be found for instance in [7], Lemma 2.2.8.

**Lemma 2.5.** Let $X$ be smooth quasiprojective and integral $k$-scheme and $G$ a linearly reductive group acting on $X$. Then for arbitrary complexes of quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ the following holds:

\[
\text{Hom}_G(\mathcal{F}, \mathcal{G}[i]) \simeq \text{Hom}_X(\mathcal{F}, \mathcal{G}[i])^G.
\]

**Proof.** We sketch the proof since it can be found in [7]. Recall that there is an isomorphism of functors $\text{Hom}_G(-,-) \simeq \text{Hom}(-,-)^G$. Grothendieck spectral sequence for the composition of two functors applied to the two functors $\text{Hom}(-,-)$ and $(-)^G$ yields the desired isomorphism. Note that under the assumption on $G$ being linearly reductive, taking $G$-invariants is exact. This yields the above isomorphism. \(\square\)

### 3. Geometric tilting theory

In this section we recall some facts of geometric tilting theory. We start with the main definition (see [21]).

**Definition 3.1.** Let $k$ be a field, $X$ a noetherian quasiprojective $k$-scheme and $G$ an algebraic group over $k$ acting on $X$. An object $T \in \text{D}_G(\text{Qcoh}(X))$ is called tilting object for $\text{D}_G(\text{Qcoh}(X))$ if the following hold:

(i) $\text{Hom}_G(T, T[i]) = 0$ for $i \neq 0$.

(ii) If $\mathcal{N} \in \text{D}_G(\text{Qcoh}(X))$ satisfies $\mathbb{R}\text{Hom}_G(T, \mathcal{N}) = 0$, then $\mathcal{N} = 0$.

(iii) $\text{Hom}_G(T, -)$ commutes with direct sums.

**Remark 3.2.** If one has a tilting object $T$ for $\text{D}_G(\text{Qcoh}(X))$ one can form the smallest thick subcategory containing $T$, additionally being closed under direct sums. We denote this category by $\langle T \rangle$. One can show that condition (ii) from above is equivalent to $\langle T \rangle = \text{D}_G(\text{Qcoh}(X))$ (see [21], Remark 1.2). We say $T$ is generating the derived category $\text{D}_G(\text{Qcoh}(X))$. Furthermore, if $\text{D}_G(\text{Qcoh}(X))$ is compactly generated and the compact objects are all of $\text{D}_G^c(X)$, then to show that an object $T$ generates $\text{D}_G(\text{Qcoh}(X))$ is equivalent to show that it generates $\text{D}_G^c(X)$, i.e., that the smallest thick subcategory containing $T$ that additionally is closed under direct sums equals $\text{D}_G^c(X)$ (see [15], Theorem 2.1.2).

One has the following $G$-equivariant tilting correspondence proved in [16], Theorem 3.1.1. It is a direct application of a result of Keller [47], Theorem 8.5. Notice that $\text{Mod}(A)$ below is the category of right $A$-modules.
Theorem 3.3. Let $X$ be a noetherian quasiprojective $k$-scheme and $G$ a finite group acting on $X$ with char$(k) \nmid$ ord$(G)$. Suppose we are given a tilting object $T$ for $D_G(Qcoh(X))$ and let $A = \text{End}_G(T)$. Then the following hold:

(i) The functor $\psi = \mathbb{R}\text{Hom}_G(T, -) : D_G(Qcoh(X)) \to \text{D}(\text{Mod}(A))$ is an equivalence.

(ii) If $X$ is smooth, then the equivalence $\psi$ of (i) restricts to an equivalence $D_G^b(X) \xrightarrow{\sim} \text{perf}(A)$.

(iii) If the global dimension of $A$ is finite, then $\text{perf}(A) \simeq D^b(A)$.

Remark 3.4. Note that if $X$ is for instance a smooth projective and integral $k$-scheme and $G = 1$, the derived category $D(Qcoh(X))$ is compactly generated and the compact objects are exactly $D^b(X)$. In this case an object $T$ generates $D^b(X)$ if and only if it generates $D^b(X)$. Since the natural functor $D^b(X) \to D(Qcoh(X))$ is fully faithfull (see [39]), an object $T$ lying in the subcategory $D^b(X)$ is a tilting object if and only if $T$ generates $D^b(X)$ and $\text{Hom}_{D^b(X)}(T, T[i]) = 0$ for $i \neq 0$. If the tilting object $T$ is a sheaf, the above definition coincides with the definition of a tilting sheaf given in [5]. In this case the tilting object is called tilting sheaf on $X$ or in $D^b(X)$. If it is a locally free sheaf we simply say that $T$ is a tilting bundle. Theorem 3.3 then gives the classical tilting correspondence as firstly proved by Bondal [13] and later extended by Baer [5].

The next result shows that in the above theorem the smoothness of $X$ already implies the finiteness of the global dimension of $A$.

Proposition 3.5. Let $X$, $G$ and $T$ be as in Theorem 3.3. If $X$ is smooth projective and integral, then $A = \text{End}_G(T)$ has finite global dimension and therefore the equivalence (i) of Theorem 3.3 restricts to an equivalence $D_G^b(X) \xrightarrow{\sim} \text{D}^b(A)$.

Proof. Imitating the proof of Theorem 7.6 in [36], one can argue as follows: For arbitrary finitely generated right $A$-modules $M$ and $N$ we conclude with the equivalence $\psi : D_G^b(X) \to \text{perf}(A)$ (see Theorem 3.3 (ii)):

\[ \text{Ext}_A^i(M, N) \cong \text{Hom}_G(\psi^{-1}(M), \psi^{-1}(N)[i]) \cong \text{Hom}(\psi^{-1}(M), \psi^{-1}(N)[i])^G = 0 \]

for $i >> 0$, since $X$ is by assumption smooth, projective and integral. This is obtained from local-to-global spectral sequence, Grothendieck vanishing Theorem and Lemma 2.5. Furthermore, since $X$ is projective, $A = \text{End}_G(T)$ is a finite-dimensional $k$-algebra and hence a noetherian ring. But for noetherian rings the vanishing of $\text{Ext}_A^i(M, N)$ for $i >> 0$ for any two finitely generated $A$-modules $M$ and $N$ suffices to conclude that the global dimension of $A$ has to be finite.

Assuming the existence of a tilting object $T \in D_G(Qcoh(X))$, where $X$ is smooth projective and integral, Theorem 3.3 and Proposition 3.5 gives an equivalence

\[ \mathbb{R}\text{Hom}(T, -) : D_G^b(X) \to \text{D}^b(A), \]

with $A = \text{End}_G(T)$ being a finite dimensional $k$-algebra. If the field $k$ is supposed to be algebraically closed, any finite-dimensional $k$-algebra $A$ admits a complete set of primitive orthogonal idempotents $e_1, \ldots, e_n$ (see [3], I.4). Idempotents $e_1, \ldots, e_n$ are called orthogonal if $e_ie_j = e_je_i = 0$ for $i \neq j$ and complete if $e_1 + \ldots + e_n = 1$. Furthermore, an idempotent $e$ is called primitive if it cannot be written as a sum of two non-zero orthogonal idempotents. Now let $e_1, \ldots, e_n$ be the complete set of primitive orthogonal idempotents of the above endomorphism algebra $A$. Associated to $A$, there is a finite-dimensional $k$-algebra $A'$ with a complete set of primitive orthogonal idempotents $e'_1, \ldots, e'_r$ such that $e'_iA' = e'_jA'$ as right $A$-modules only if $i = j$ (see [3], I.6, Definition 6.3). There is an equivalence of categories between $\text{mod}(A)$ and $\text{mod}(A')$ (see [3], I.6, Corollary 6.10). Now to every such algebra $A'$, with $e'_iA' = e'_jA'$ as right $A$-modules only if $i = j$, one can associate a quiver with relations $(Q, R)$ as follows: The set $Q_0$ is given by the set $e'_1, \ldots, e'_r$ and the number of arrows from $e'_i$ to $e'_j$ is given by $\text{Ext}_A^1(S_i, S_j)$, where $S_l = e'_lA'/e'_l\text{rad}(A')$ (see...
to a direct summand $E$.

Moreover, the quiver $(Q, R)$ is uniquely determined up to isomorphism by $A'$ and the path algebra of $(Q, R)$ is isomorphic to $A'$ (see [4], III Theorem 1.9, Corollary 1.10). This yields an equivalence between $\text{mod}(A)$ and $\text{mod}(kQ/(R))$ and hence between $D^b_G(X)$ and $D^b(kQ/(R))$. Under this equivalence, each projective module $e_i A'$ is mapped to a direct summand $E_i$ of the tilting object $T$ and the direct sum $T' = \bigoplus_{i=1}^r E_i$ is again a tilting object for $D^b_G(X)$. The difference between $T$ and $T'$ is that $T$ may contain several copies of $E_i$. This equivalence between $D^b_G(X)$ and $D^b(kQ/(R))$ now enables one to apply representation-theoretical techniques to investigate the derived category of equivariant coherent sheaves on $X$. As a classical example we consider the tilting bundle $\mathcal{T} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on the projective line $\mathbb{P}^1$. The corresponding quiver consists of two vertices and two arrows from the first vertex to the second $1 \rightarrow 2$ and the representations of this quiver were studied by Kronecker and are well-known. For details and further examples of quivers related to tilting objects we refer to [5], [13], [18], [19], [26], [27], [53] and [60].

In the literature (in view of the Krull–Schmidt decomposition), instead of the tilting object one often studied the set $E_1, ..., E_n$ of its indecomposable, pairwise non-isomorphic direct summands. There is a special case where all the summands form a so-called full strongly exceptional collection. Closely related to the notion of a full strongly exceptional collection is that of a semiorthogonal decomposition. We recall the definition of an exceptional collection and a semiorthogonal decomposition respectively. We follow the definition given in [39] and refer to the work of Bondal and Orlov [14] for further details.

**Definition 3.6.** Let $X$ and $G$ be as in Definition 3.1 with the difference of $G$ being flat over $k$. An object $E \in D^b_G(X)$ is called exceptional if $\text{End}_G(E) = k$ and $\text{Hom}_G(E, E[l]) = 0$ for all $l \neq 0$. An exceptional collection is a collection of exceptional objects $E_1, ..., E_n$, satisfying

(i) $\text{Hom}_G(E_i, E_j[l]) = k$ for $l = 0$ and $i = j$,

(ii) $\text{Hom}_G(E_i, E_j[l]) = 0$ for all $l \neq 0$ and $i = j$,

(iii) $\text{Hom}_G(E_i, E_j[l]) = 0$ for all $l \in \mathbb{Z}$ if $i > j$.

An exceptional collection is called full if the collection generates $D^b_G(X)$, i.e., if the smallest thick subcategory containing $E_1, ..., E_n$ that additionally is closed under direct sums equals $D^b_G(X)$. If in addition $\text{Hom}_G(E_i, E_j[l]) = 0$ for all $i, j$ and $l \neq 0$ the collection is called strongly exceptional.

As a generalization one has the notion of a semiorthogonal decomposition of $D^b_G(X)$ (see [14] or [39]).

**Definition 3.7.** Let $X$ and $G$ be as above. A collection $D_1, ..., D_r$ of full triangulated subcategories is called a semiorthogonal decomposition for $D^b_G(X)$ if the following properties hold:

(i) The inclusion $D_i \subset D^b_G(X)$ has a right adjoint $p : D^b_G(X) \rightarrow D_i$.

(ii) $D_j \subset D_i^\perp = \{B \in D^b_G(X) | \text{Hom}_G(A, B) = 0, \forall A \in D_i\}$ for $i > j$.

(iii) The collection $D_i$ generates $D^b_G(X)$, i.e., the smallest thick subcategory containing all $D_i$ that additionally is closed under direct sums equals $D^b_G(X)$.

For a semiorthogonal decomposition of $D^b_G(X)$ we simply write $D^b_G(X) = \langle D_1, ..., D_r \rangle$.

**Example 3.8.** If we have a full exceptional collection $E_1, ..., E_n$ in $D^b_G(X)$, then by Lemma 1.58 in [39] the inclusion $\langle E_i \rangle \rightarrow D^b_G(X)$ has a right adjoint. Furthermore, condition (ii) is fulfilled for $D_i = \langle E_i \rangle$ and since the collection $E_1, ..., E_n$ is full, the collection $\langle E_1 \rangle, ..., \langle E_n \rangle$ generates $D^b_G(X)$. Hence a full exceptional collection $E_1, ..., E_n$ gives rise to a semiorthogonal decomposition $D^b_G(X) = \langle \langle E_1 \rangle, ..., \langle E_n \rangle \rangle$. 

[3], II.3, Definition 3.1, see also [4], p.52 and Proposition 1.14. Note that this quiver does not depend on the choice of the set complete idempotents (see [3], II.3, Lemma 3.2).
Clearly, the direct sum of the exceptional objects in a full strongly exceptional collection gives rise to a tilting object but the converse is not true in general. However, it is easy to verify that, after possibly reordering, the pairwise non-isomorphic indecomposable direct summands of a tilting object form a full strongly exceptional collection, provided the summands are invertible. Exceptional collections and semiorthogonal decompositions were intensively studied and we know quite a lot of examples of schemes admitting full exceptional collections or semiorthogonal decompositions. For a comprehensive overview we refer the reader to [14] and [51].

The above described connection between representation theory and the derived category of coherent sheaves is not the only motivation to study derived categories with exceptional collections or tilting objects. Another motivation comes from Kontsevich’s Homological Mirror Symmetry conjecture [49], see also [39], 13.2. Moreover, a conjecture of Dubrovin [28] states that the quantum cohomology of a smooth projective variety \( X \) is generically semisimple if and only if there exists a full exceptional collection in \( D^b(X) \) and the validity of this conjecture would also provide evidence for the Homological Mirror Symmetry conjecture. Although this conjecture turned out to be wrong in general, it seems that there is still a relationship between the existence of full exceptional collections and its quantum cohomology (see [8]). Motivated by the Mirror Symmetry, in the recent past full strongly exceptional collections have also been considered in physics in the context of string theory, concretely in studying so-called \( D \)-branes (see for instance [2]). Particular interest in exceptional collections also comes from local Calabi–Yau varieties. Consider the total space \( \pi: A(\omega_X) \to X \) for the canonical bundle \( \omega_X \). This is a local Calabi–Yau variety and it follows from results of Bridgeland [18] that a full strongly exceptional collection \( E_i \) on \( X \) can be extended to a cyclic strongly exceptional collection if and only if the pullbacks \( \pi^*E_i \) give rise to a tilting object on the total space \( A(\omega_X) \). In this situation the endomorphism algebra of the tilting object for \( A(\omega_X) \) gives an example of a noncommutative resolution in the sense of [67].

Smooth projective and integral \( k \)-schemes admitting tilting objects include projective spaces [9], flag varieties of type \( A_n \) (over the complex numbers) [44], Grassmannians over arbitrary fields [22], certain toric varieties [23], [61], rational complex surfaces [38] and certain iterated projective bundles and fibrations [24], [25]. Furthermore, tilting bundles are presumed to exist on rational homogeneous varieties (see [12] and several results in this direction have been proved (see [50] for details). In the case of stacks, tilting objects are known to exist for weighted projective lines [53], certain toric Deligne–Mumford stacks [45], some quotient stacks obtained from Fermat varieties [40] or for any two-dimensional smooth toric weak Fano stack [41]. Moreover, Orlov conjectured that a tilting object should also exist for any smooth projective toric Deligne–Mumford stack [29].

4. Tilting objects on \([X/G]\)

We now prove the first main result of the present note. It simplifies and generalizes arguments given in [16]. It is the following theorem.

**Theorem 4.1.** Let \( X \) be a smooth projective and integral \( k \)-scheme, \( G \) a finite group acting on \( X \) and \( \text{char}(k) \nmid \text{ord}(G) \). Suppose that \( T \) is a tilting sheaf for \( D^b(X) \), admitting a \( G \)-equivariant structure. Let \( W_j \) be the irreducible representations of \( G \). Then \( T_G = \bigoplus_j T \otimes W_j \) is a tilting object for \( D^b([X/G]) \) and one therefore has an equivalence

\[
\mathbb{R}
\text{Hom}_G(T_G, -) : D^b([X/G]) \to D^b(\text{End}_G(T_G)).
\]

**Proof.** Recall that \( D^b_G(X) \simeq D^b([X/G]) \). Note that for every \( i \) one has canonical isomorphisms on \( X \)

\[
\text{Ext}^i(T \otimes W_i, T \otimes W_m) \simeq \text{Ext}^i(T, T) \otimes \text{Hom}(W_i, W_m).
\]
Since $T$ is supposed to be $G$-equivariant, the coherent sheaf $T \otimes W_i$ is $G$-equivariant too. By assumption, the characteristic of $k$ does not divide the order of $G$ and hence $G$ is linearly reductive. In this situation Lemma 2.5 applies and we have with (1):

$$\text{Ext}^i_G(T \otimes W_i, T \otimes W_m) \simeq (\text{Ext}^i(T, T) \otimes \text{Hom}(W_i, W_m))^G.$$ 

Since $T$ is by assumption a tilting sheaf for $D^b(X)$ we have $\text{Ext}^i(T, T) = 0$ for $i \neq 0$ and hence

$$\text{Ext}^i_G(T \otimes W_i, T \otimes W_m) = 0 \text{ for } i \neq 0.$$ 

This implies $\text{Ext}^i_G(T_G, T_G) = 0$ for $i \neq 0$ and hence the vanishing of Ext holds true. To see that $T_G$ generates $D_G(\text{Qcoh}(X))$, we note that the quotient stack $[X/G]$ is a quasicompact and separated Deligne–Mumford stack with coarse moduli space being the quotient scheme $X//G$. Hence by [65], Corollary 4.2 the derived category $D_G(\text{Qcoh}(X))$ is compactly generated. So by Remark 3.2 it suffices to prove that $T_G$ generates the subcategory of compact objects of $D_G(\text{Qcoh}(X))$. Since $X$ is smooth projective and integral, the compact objects of $D_G(\text{Qcoh}(X))$ are all of $D_G^c(X)$. So we assume $F \in D_G^c(X)$ and $\text{RHom}_G(T_G, F) = 0$. This implies

$$\text{Hom}_G(T_G, F[i]) = 0 \text{ for all } i \in \mathbb{Z}.$$ 

With Lemma 2.5 we get

$$\text{Hom}(T_G, F[i])^G = 0 \text{ for all } i \in \mathbb{Z}.$$ 

By the construction of $T_G$ we conclude

$$\text{Hom}(T \otimes W_m, F[i])^G = 0$$

for all $i \in \mathbb{Z}$ and all irreducible representations $W_m$. But then $\text{Hom}(T, F[i])$ contains no copy of any irreducible representation $W_m$ and so must be zero. Since $T$ is a tilting sheaf for $D^b(X)$ and therefore generates $D^b(X)$, $\text{Hom}(T, F[i]) = 0$ for all $i \in \mathbb{Z}$ implies $F = 0$. This shows that $T_G$ generates $D^b(X)$ and hence $D_G(\text{Qcoh}(X))$. To finish the proof, we apply Theorem 3.3, together with Proposition 3.5 to get that $T_G$ is a tilting object and that $\text{RHom}_G(T_G, -) : D_G^b(X) \to D^b(\text{End}_G(T_G))$ is an equivalence. 

**Example 4.2.** Let $X = \mathbb{P}^n$ and $T = \bigoplus_{i=0}^n \mathcal{O}_X(i)$ the tilting bundle obtained by Beilinson [9]. Obviously, the invertible sheaves $\mathcal{O}_X(i)$ are preserved by automorphisms of $X$. For a finite subgroup $G$ of $\text{Aut}_k(X) \simeq \text{PGL}_{n+1}(k)$, the sheaf $T$ is a tilting object for $D^b(X)$ admitting a $G$-equivariant structure. Theorem 4.1 applies and yields a tilting object for the stack $[\mathbb{P}^n/G]$ (see also [16], Theorem 3.2.1). Next, we consider the Grassmannian $\text{Grass}_k(d,n)$ over a field $k$ of characteristic zero. For $2d \neq n$ the automorphisms of $\text{Grass}_k(d,n)$ are exactly $\text{PGL}_n(k)$ and the full strongly exceptional collection obtained by Kapranov [42] is preserved under the action of a group $G$ induced by a homomorphism $G \to \text{PGL}_n(k)$ (see [31], 3.4). If $G$ is finite, Theorem 4.1 applies and we get a tilting object for the stack $[\text{Grass}_k(d,n)/G]$. 

For $X$ and $G$ as above, let $k \subseteq L$ be a finite Galois extension and $X_L = X \otimes_k L$ the base change of $X$. Since $G$ acts on $X$, $G_L$ acts on $X_L$. Suppose there is a tilting sheaf $T$ for $D^b(X_L)$ admitting a $G_L$-equivariant structure and suppose furthermore that $T$ descends to a sheaf $T'$ on $X$. Then $T'$ admits a $G$-equivariant structure (see [10], Remark 5.17). Below we prove that in fact $T'$ is a tilting sheaf on $[X/G]$. First we need the following lemma, essentially proved in [11]. For convenience to the reader we give a proof.

**Lemma 4.3.** Let $X$ be a smooth projective and integral $k$-scheme and $k \subseteq L$ a field extension. Now for a given object $R \in D^b(X)$, suppose that $R \otimes_k L$ is a tilting object for $D^b(X \otimes_k L)$. Then $R$ is a tilting object for $D^b(X)$. 


Proof. Consider the projection \( v : X \otimes_k L \to X \). By assumption \( T = v^* R \) is a tilting object for \( D^b(X \otimes_k L) \). We claim that \( R \) is a tilting object for \( D^b(X) \). To prove the claim, we first calculate \( \Hom(R, R[l]) \). For this, consider the following base change diagram

\[
\begin{array}{c}
X \otimes_k L \xrightarrow{w} X \\
\downarrow q \quad \quad \downarrow p \\
\Spec(L) \xrightarrow{u} \Spec(k)
\end{array}
\]

Under the above assumption on \( X \), an object \( F \in D^b(X) \) is quasi-isomorphic to a complex of locally free sheaves. For two arbitrary complexes of locally free sheaves \( F, G \in D^b(X) \), flat base change (see [39], p.85 (3.18)) yields isomorphisms of functors

\[
u^* (\mathbb{R} \Hom(F, G)) \simeq u^* \mathbb{R} \Hom(F, G) \simeq \mathbb{R}q_* v^* \mathbb{R} \Hom(F, G) \simeq \mathbb{R}q_* v^* (F^\vee \otimes^L G) \simeq \mathbb{R}q_* \mathbb{R} \Hom(v^* F, v^* G) \simeq \mathbb{R} \Hom(v^* F, v^* G)
\]

This now implies

\[
\Hom(v^* R, v^* R[l]) \simeq \Hom(T, T[l]) \simeq \Hom(R, R[l]) \otimes_k L = 0
\]

for \( l \neq 0 \), since \( T \) is a tilting object for \( D^b(X \otimes_k L) \). Hence \( \Hom(R, R[l]) = 0 \) for \( l \neq 0 \) and therefore Ext-vanishing holds. For the generation property we take an object \( F \in D^b(X) \) and assume \( \mathbb{R} \Hom(R, F) = 0 \). The above equivalences obtained from flat base change yield

\[
0 = u^* (\mathbb{R} \Hom(R, F)) \simeq \mathbb{R} \Hom(v^* R, v^* F).
\]

Since \( v^* R \simeq T \) is a tilting object for \( D^b(X \otimes_k L) \) we conclude \( v^* F = 0 \). Since \( v \) is a flat, this implies \( F = 0 \) and hence \( R \) generates \( D^b(X) \). Finally, since \( X \) is smooth, the global dimension of \( \text{End}(R) \) is finite. This completes the proof.

**Proposition 4.4.** For \( X \) and \( G \) as above, let \( k \subseteq L \) be a finite Galois extension and \( X_L = X \otimes_k L \) the base change of \( X \). Suppose \( \text{char}(k) \nmid \text{ord}(G) \) and that there is a tilting sheaf \( T \) for \( D^b(X_L) \), admitting a \( G_L \)-equivariant structure. Suppose furthermore that \( T \) descends to a sheaf \( T' \) on \( X \). Then \( T' \) is a tilting sheaf on \([X/G]\).

**Proof.** Since \( T = T' \otimes_k L \) is a tilting sheaf on \( X_L \) by assumption, \( T' \) is a tilting sheaf on \( X \) according to Lemma 4.3. As mentioned above, \( T' \) admits a \( G \)-equivariant structure and hence Theorem 4.1 yields the assertion. \( \square \)

**Example 4.5.** Blank [11] constructed tilting bundles for Brauer–Severi varieties. A Brauer–Severi variety \( X \) is a smooth projective \( k \)-scheme that becomes isomorphic to some \( \mathbb{P}^n_L \) after base change to a finite Galois field extension \( L \). In loc.cit it is proved that \( T' = \bigoplus_{i=0}^n V^\otimes i \), where \( V \) is the tautological sheaf on \( X \), is a tilting bundle for \( D^b(X) \).

For the sheaves \( V^\otimes i \) one has \( V^\otimes i \otimes_k L \simeq \mathbb{O}_{\mathbb{P}^n_L}(-i)^{(n+1)i} \). Therefore the sheaf \( T = T' \otimes_k L \) is a tilting bundle on \( \mathbb{P}^n_L \). Now let \( G \) be finite subgroup of \( \text{Aut}(X) \) with \( \text{char}(k) \nmid \text{ord}(G) \).

From Example 4.2 we conclude that \( T \) is a tilting bundle for \( D^b(\mathbb{P}^n_L) \), admitting a \( G_L \)-equivariant structure. Thus Proposition 4.4 applies and yields that \( T' \) is a tilting bundle on \([X/G]\). Note that the same arguments show that for \( X \) being a generalized Brauer–Severi variety and \( G \) a certain subgroup of the automorphisms, the stack \([X/G]\) admits a tilting object too.
The next proposition shows that Example 4.5 cannot be obtained from the results given in [31], where tilting bundles on $[X/G]$ are constructed from full strongly exceptional collections. In particular it shows that there really exist $k$-schemes admitting a tilting object, but not a full strongly exceptional collection.

**Proposition 4.6.** Let $X \neq \mathbb{P}^1$ be a 1-dimensional Brauer–Severi variety. Then $D^b(X)$ does not admit a full strongly exceptional collection.

**Proof.** We first prove that $D^b(X)$ does not admit a full strongly exceptional collection consisting of coherent sheaves. For this, we suppose there is a full strongly exceptional collection $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of coherent sheaves on $X$. From Example 4.5 we know that $\mathcal{T} = \mathcal{O}_X \oplus \mathcal{V}$ is a tilting bundle on $X$ and hence $K_0(X)$ is a free abelian group of rank two. Since the full strongly exceptional collection $\mathcal{E}_1, \ldots, \mathcal{E}_n$ has to be a basis for $K_0(X)$, we conclude $m = 2$. By assumption we have $\text{End}(\mathcal{E}_1) = k$ which means that $\mathcal{E}_1$ and $\mathcal{E}_2$ have to be simple. Clearly, $\mathcal{E}_1$ remains simple after base change to some splitting field $L$ of $X$, since $\text{End}(\mathcal{E}_1 \otimes_k L) \simeq L$. Simple coherent sheaves on the projective line $X \otimes_k L \simeq \mathbb{P}^1$ are known to be invertible sheaves or skyscraper sheaves supported on a closed point. Therefore the simple sheaves $\mathcal{E}_1 \otimes_k L$ have to be isomorphic to either $\mathcal{O}_{\mathbb{P}^1}(n)$ or $L(x)$. Since the two exceptional sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ form a full strongly exceptional collection, their direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ gives a tilting sheaf. This tilting sheaf remains tilting after base change to $L$ and hence $(\mathcal{E}_1 \otimes_k L) \oplus (\mathcal{E}_2 \otimes_k L)$ is a tilting sheaf for $D^b(\mathbb{P}^1)$. Note that every invertible sheaf on $X \otimes_k L \simeq \mathbb{P}^1$ coming from an invertible sheaf on the Brauer–Severi variety $X$ is of the form $\mathcal{O}_{\mathbb{P}^1}(2n)$. We now show that $(\mathcal{E}_1 \otimes_k L) \oplus (\mathcal{E}_2 \otimes_k L)$ cannot be a tilting sheaf on $\mathbb{P}^1$.

We have three cases that have to be considered:

(i) Assume both $\mathcal{E}_1 \otimes_k L$ and $\mathcal{E}_2 \otimes_k L$ are invertible sheaves. In this case $\mathcal{E}_1 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2n)$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2m)$. Without loss of generality, we can assume $\mathcal{E}_1 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2)$. But then we have

$$\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \neq 0$$

for $l > 0$, since $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) = H^1(X, \mathcal{O}_{\mathbb{P}^1}(-2)) = L$. Hence the vanishing of $\text{Ext}$ fails to hold.

(ii) We now assume $\mathcal{E}_1 \otimes_k L = L(x)$ and $\mathcal{E}_2 \otimes_k L = L(y)$. Again, considering $\text{Ext}^1(L(x) \oplus L(y), L(x) \oplus L(y))$ we find

$$\text{Ext}^1(L(x), L(x)) \simeq T_x,$$

where $T_x$ is the tangent space in $x$ (see [39], Example 11.9) that clearly is non-zero. Again the $\text{Ext}$ vanishing fails to hold.

(iii) It remains the case $\mathcal{E}_1 \otimes_k L = L(x)$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2n)$. For $\mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x)$ we find

$$\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x), \mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x)) \neq 0$$

for $l > 0$, since $\text{Ext}^1(L(x), L(x)) \simeq T_x$ and hence is non-zero.

Hence $D^b(X)$ does not admit a full strongly exceptional collection consisting of coherent sheaves. To conclude that $D^b(X)$ does not admit a full strongly exceptional collection consisting of arbitrary objects, we note that the abelian category $\text{Coh}(X)$ is hereditary. According to [47], 2.5, every object $\mathcal{G} \in D^b(X)$ is of the form $\bigoplus_{i \in \mathbb{Z}} H^i(\mathcal{G})[i]$. Since exceptional objects are simple and therefore indecomposable, the exceptional objects in $D^b(X)$ are just shifts of exceptional coherent sheaves. With the arguments from above this finally implies that $D^b(X)$ does not admit a full strongly exceptional collection.

**Conjecture.** Let $X \neq \mathbb{P}^n$ be a $n$-dimensional Brauer–Severi variety. Then $D^b(X)$ does not admit a full strongly exceptional collection.

**Remark 4.7.** Results of Hille and Perling [37], [38] show that there are also smooth projective complex surfaces admitting tilting bundles, but not a full strongly exceptional
collection consisting of invertible sheaves. To be precise, every smooth projective complex surface admits a tilting bundle, but smooth complete toric surfaces $X \neq \mathbb{P}^2$ obtained by blowing up toric fixed points in at most two steps do not admit full strongly exceptional collections consisting of invertible sheaves. It would be interesting to figure out if there are smooth projective complex surfaces that do not have full strongly exceptional collections consisting of arbitrary objects.

Theorem 4.1 above states more or less that the stack $[X/G]$ admits a tilting object if $X$ admits one. Below we show that in general the converse of this statement does not hold. For this, we roughly recall the derived McKay correspondence and refer to the celebrated work of Bridgeland, King and Reid [17] for details. Let $k$ be an algebraically closed field of characteristic zero and $X$ a quasiprojective $k$-scheme. Furthermore, let $G$ be a finite subgroup of $\text{Aut}(X)$ acting on $X$. Note that the quotient scheme $X//G$ is usually singular. The main idea of McKay correspondence is to find a certain "nice" resolution of $X//G$ and to relate the geometry of the resolution to that of $X//G$. Recall, a resolution of singularities $\tilde{X} \to X$ of a given non-singular $X$ is called crepant if $\omega_{\tilde{X}}$ is the pullback of $\omega_X$. Whether such resolutions exist is a difficult question and closely related to the minimal model program. In the situation described above, the $G$-Hilbert scheme of $X$ exists and we denote it by $\text{Hilb}_G(X)$ (see [10] for details on $G$-Hilbert schemes). Now take $Y \subset \text{Hilb}_G(X)$ to be the irreducible component containing the free orbits. Suppose that $G$ acts on $X$ such that $\omega_X$ is locally trivial as a $G$-equivariant sheaf and write $Z \subset X \times Y$ for the universal closed subscheme. Then there is a commutative diagram of schemes

$$
\begin{array}{ccc}
Z & \overset{q}{\longrightarrow} & X \\
\downarrow{\rho} & & \downarrow{\pi} \\
Y & \overset{r}{\longrightarrow} & X//G
\end{array}
$$

such that $q$ an $\tau$ are birational and $p$ and $\pi$ finite. Moreover $p$ is flat. Under the assumptions on $X$ and $G$ made above, one has the derived McKay correspondence proved by Bridgeland, King and Reid (see [17], Theorem 1.1):

**Theorem 4.8.** Let $X$, $G$ and $Y$ be as above and suppose that $\omega_X$ is locally trivial as a $G$-equivariant sheaf. Suppose furthermore, that $\dim(Y \times_{X//G} Y) < \dim(X) + 1$, then the functor $\mathbb{R}q_* \circ p^*$ is an equivalence $\mathbb{R}q_* \circ p^* : D^b(Y) \to D^b_G(X)$

and $\tau : Y \to X//G$ is a crepant resolution.

Moreover, if $\dim(X) \leq 3$, Bridgeland, King and Reid proved that $\text{Hilb}_G(X)$ is irreducible and hence one has an equivalence $\mathbb{R}q_* \circ p^* : D^b(\text{Hilb}_G(X)) \to D^b_G(X)$ and in this case $\text{Hilb}_G(X) \to X//G$ is a crepant resolution (see [17], Theorem 1.2). To my best knowledge, there is no generalization of that result for arbitrary fields. Blume [10] generalized the classical McKay correspondence to arbitrary fields of characteristic zero. We now prove the following observation.

**Proposition 4.9.** Let $C$ be an elliptic curve over an algebraically closed field of characteristic zero with $j \neq 0$ and $j \neq 1728$ and let $G = \{id, -id\} = \text{Aut}(C)$ act on $C$. Furthermore, consider the induced action of $G' = G \times G$ on $C \times C$. Then both $D^b_G(C)$ and $D^b_G(C \times C)$ admit a tilting object.

**Proof.** By assumption on $C$ and $G$, the quotient scheme $C//G$ exists and is isomorphic to $\mathbb{P}^1$. Hence $(C \times C) // G' \simeq (C//G) \times (C//G) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\omega_G = O_C$ and $\omega_{C \times C} = O_{C \times C}$ and hence they are locally trivial as $G$-equivariant and $G'$-equivariant sheaf respectively. Since $\dim(C) = 1$ we get a crepant resolution $\text{Hilb}_G(C) \to C//G$ and hence $\text{Hilb}_G(C) \simeq \mathbb{P}^1$. Derived McKay correspondence gives
and since \( \mathbb{P}^1 \) admits a tilting object, \( D^b_{\mathbb{P}^1}(\mathcal{C}) \) admits a tilting object too. For \( D^b_{C}(\mathcal{C}) \) the argument is the same. Since \( \dim(C \times C) = 2 \) we get a crepant resolution \( \text{Hilb}_{C}(C \times C) \rightarrow (C \times C)/G \) and hence \( \text{Hilb}_{C}(C \times C) \) is a rational surface. Again we have derived McKay correspondence

\[
D^b(\text{Hilb}_{C}(C \times C)) \sim D^b_{\mathbb{P}^1}(C \times C)
\]

and since smooth projective rational surfaces admit tilting objects (see [38], Theorem 1.1) we conclude that \( D^b_{\mathbb{P}^1}(C \times C) \) admits a tilting object too. This completes the proof. \( \square \)

Following the idea of exploiting derived McKay correspondence to obtain further examples of schemes having tilting objects, we also state the following useful application of Theorem 4.1.

**Theorem 4.10.** Let \( X, G \) and \( Y \) be as above and suppose that \( \omega_X \) is locally trivial as a \( G \)-equivariant sheaf. Suppose furthermore that \( \dim(Y \times_{X/G} Y) < \dim(X) + 1 \) and that \( D^b(X) \) admits a tilting bundle \( \mathcal{T} \) that has a \( G \)-equivariant structure. Then \( D^b(Y) \) admits a tilting object.

**Proof.** Since \( \mathcal{T} \) is a tilting bundle for \( D^b(X) \) admitting a \( G \)-equivariant structure, we conclude with Theorem 4.1 that \( D^b_G(X) \) admits a tilting object. Under the above assumptions on \( X \) and \( G \) and since \( \dim(Y \times_{X/G} Y) < \dim(X) + 1 \), derived McKay correspondence yields an equivalence

\[
D^b(Y) \sim D^b_G(X).
\]

Since \( D^b_G(X) \) admits a tilting object, \( D^b(Y) \) admits one too. This completes the proof. \( \square \)

**Corollary 4.11.** For \( n \leq 3 \), let \( G \subset \text{Aut}(\mathbb{P}^n) \) be a finite subgroup such that \( \omega_{\mathbb{P}^n} \) is locally trivial as a \( G \)-equivariant sheaf. Then \( D^b(\text{Hilb}_{C}(\mathbb{P}^n)) \) admits a tilting object.

**Proof.** The sheaves of the full strongly exceptional collection \( \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), ..., \mathcal{O}_{\mathbb{P}^n}(n) \) are preserved under automorphisms of \( \mathbb{P}^n \) (see Example 4.2). Theorem 4.10 applies and yields, together with the discussion right after Theorem 4.8, that \( D^b(\text{Hilb}_{C}(\mathbb{P}^n)) \) admits a tilting object. \( \square \)

**Corollary 4.12.** Let \( X = \text{Grass}(d,n) \) be the Grassmannian over \( C \) and \( 2d \neq n \). Let \( G \) be a finite group with action on \( X \) induced by a homomorphism \( G \rightarrow \text{PGL}_n(k) \) such that \( \omega_X \) is locally trivial as a \( G \)-equivariant sheaf. Let furthermore \( Y \subset \text{Hilb}_{C}(X) \) be the irreducible component containing the free orbits and suppose that \( \dim(Y \times_{X/G} Y) < \dim(X) + 1 \). Then \( D^b(Y) \) admits a tilting object.

5. **Tilting objects on \([A(E)/G]\) and \([\mathbb{P}(E)/G]\)**

In this section we apply the result of the last section in order to prove the other two main results stated in the introduction. We start with a generalization of a result of Brav [16] and Bridgeland and Stern [19]. In loc.cit. the authors investigated certain total spaces and proved the existence of tilting objects for their bounded derived categories. The existence of tilting objects on certain total spaces also led Weyman and Zhao [69] to a construction of non-commutative desingularizations. In the spirit of the present note we study total spaces with finite group actions.

Let \( G \) be a finite group acting on some smooth projective and integral \( k \)-scheme \( X \). Suppose that the characteristic of \( k \) does not divide the order of \( G \). Furthermore, let \( \mathcal{E} \) be a \( G \)-equivariant locally free sheaf of finite rank. Consider the total space \( A(\mathcal{E}) = \text{Spec}(S^*(\mathcal{E})) \), where \( S^*(\mathcal{E}) = \text{Sym}(\mathcal{E}) \). Since \( \mathcal{E} \) is \( G \)-equivariant, the group \( G \) acts on \( A(\mathcal{E}) \)
in the natural way. Note that the total space comes equipped with an affine structure.

\[ \text{morphism } \pi : \mathcal{A}(\mathcal{E}) \to X \text{ that is compatible with the action of } G. \] Assuming the existence of a tilting sheaf \( \mathcal{T} \) on \( X \) with \( G \)-equivariant structure, the question arises if the stack \([\mathcal{A}(\mathcal{E})/G]\) admits a tilting object too. There is a natural candidate for a tilting object on \([\mathcal{A}(\mathcal{E})/G]\). Take the tilting sheaf \( \mathcal{T} \) for \( D^b(X) \). Then from Theorem 4.1 we know that \( \mathcal{T}_G = \bigoplus \mathcal{T} \otimes W_j \) is a tilting object for the stack \([X/G]\). It is a \( G \)-equivariant coherent sheaf on \( X \) and pulling it back to \( \mathcal{A}(\mathcal{E}) \) yields the object \( \pi^* \mathcal{T}_G \). This is a coherent sheaf on \( \mathcal{A}(\mathcal{E}) \) that has a natural \( G \)-equivariant structure. In order to prove that \( \pi^* \mathcal{T}_G \) is a tilting object, there is one problem that occurs. Since \( \mathcal{A}(\mathcal{E}) \) is not projective over \( k \), Proposition 3.5 cannot be applied. Therefore we first have to investigate what happens to the global dimension of \( \text{End}_G(\pi^* \mathcal{T}_G) \). Note that by adjointness of \( \pi^* \) and \( \pi_* \) (see [10] for \( G \)-equivariant adjointness) we have

\[
\text{End}_G(\pi^* \mathcal{T}_G) = \text{Hom}_G(\pi^* \mathcal{T}_G, \pi^* \mathcal{T}_G) \cong \text{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes S^*(\mathcal{E})),
\]

since \( \pi_* \pi^* \mathcal{T}_G \cong S^*(\mathcal{E}) \otimes \mathcal{T}_G \). Hence \( A = \text{End}_G(\pi^* \mathcal{T}_G) \) is a graded \( k \)-algebra which of course is infinite-dimensional. Note that the proof of Proposition 3.5 also works if the endomorphism algebra is required to be a noetherian ring. In fact, the algebra \( A \) is noetherian so that the arguments of the proof of Proposition 3.5 can be applied. To see why \( A \) is noetherian, note that \( \mathcal{A}(\mathcal{E}) \) is a noetherian scheme, since \( X \) is noetherian. In this situation the pullback \( \pi^* \mathcal{T}_G \) is a coherent sheaf on \( \mathcal{A}(\mathcal{E}) \). Now it is a fact that the endomorphism algebra \( \text{End}_G(\mathcal{F}) \) of a coherent sheaf \( \mathcal{F} \) on a noetherian scheme \( Y \) is again noetherian. This can be seen by considering the algebra \( \text{End}_G(\mathcal{F}) \) as finitely generated module over the global sections \( B = \Gamma(Y, \mathcal{O}_Y) \). Note that since \( Y \) is noetherian, \( B \) is a noetherian ring. If a finite group \( G \) is acting on \( Y \), it is well-known that \( B \) is a finitely generated module over \( B^G \). Therefore \( \text{End}_G(\mathcal{F}) \) is noetherian for a \( G \)-equivariant coherent sheaf \( \mathcal{F} \) on \( X \). This implies that the endomorphism algebra \( \text{End}_G(\pi^* \mathcal{T}_G) \) is a noetherian ring.

With the above discussion we now obtain the following result:

**Theorem 5.1.** Let \( X \) and \( G \) be as above and \( \mathcal{E} \) a \( G \)-equivariant locally free sheaf of finite rank. Suppose \( \mathcal{T} \) is a tilting bundle for \( D^b(X) \) and suppose furthermore that \( \mathcal{T} \) is \( G \)-equivariant. If \( H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0 \) for all \( i \neq 0 \) and all \( l > 0 \), then one has an equivalence

\[
\mathbb{R} \text{Hom}_G(\pi^* \mathcal{T}_G, -) : D^b([\mathcal{A}(\mathcal{E})/G]) \xrightarrow{\sim} D^b(\text{End}_G(\pi^* \mathcal{T}_G)).
\]

**Proof.** We first show that \( \text{Hom}_G(\pi^* \mathcal{T}_G, \pi^* \mathcal{T}_G[i]) = 0 \) for \( i \neq 0 \). By adjointness of \( \pi^* \) and \( \pi_* \) we have

\[
\text{Hom}_G(\pi^* \mathcal{T}_G, \pi^* \mathcal{T}_G[i]) \cong \text{Hom}_G(\mathcal{T}_G, \pi_* \pi^* \mathcal{T}_G[i]).
\]

Projection formula now implies \( \pi_* \pi^* \mathcal{T}_G \cong \mathcal{T}_G \otimes S^*(\mathcal{E}) \) and hence

\[
\text{Hom}_G(\pi^* \mathcal{T}_G, \pi^* \mathcal{T}_G[i]) \cong \text{Hom}_G(\mathcal{T}_G, S^*(\mathcal{E}) \otimes \mathcal{T}_G[i]).
\]

Since \( X \) is a smooth projective and integral \( k \)-scheme and \( S^*(\mathcal{E}) \) quasicoherent on \( X \), Lemma 2.5 can be applied and we obtain

\[
\text{Hom}_G(\mathcal{T}_G, S^*(\mathcal{E}) \otimes \mathcal{T}_G[i]) \cong \text{Hom}(\mathcal{T}_G, S^*(\mathcal{E}) \otimes \mathcal{T}_G[i]).
\]

Notice that \( \mathcal{T}_G = \bigoplus \mathcal{T} \otimes W_j \), where \( W_j \) are the irreducible representations of \( G \). Now for a fixed \( l > 0 \) one has for irreducible representations \( W_r \) and \( W_s \) a canonical isomorphism on \( X \)

\[
\text{Ext}^i(\mathcal{T} \otimes W_r, S^l(\mathcal{E}) \otimes \mathcal{T} \otimes W_s) \cong \text{Ext}^i(\mathcal{T} \otimes S^l(\mathcal{E}) \otimes \mathcal{T}) \otimes \text{Hom}(W_r, W_s).
\]

By assumption \( \text{Ext}^i(\mathcal{T}, S^l(\mathcal{E}) \otimes \mathcal{T}) \cong H^i(X, (\mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E}))) = 0 \) for all \( i \neq 0 \) and all \( l > 0 \), what therefore implies that...
\text{Hom}(\mathcal{T}_G, S^*(\mathcal{E}) \otimes \mathcal{T}_G[i])^G = 0

\text{for } i \neq 0. \text{ Thus } \text{Hom}_G(\pi^*\mathcal{T}_G, \pi^*\mathcal{T}_G[i]) = 0 \text{ for } i \neq 0. \text{ It remains to prove that } \mathcal{R} = \pi^*\mathcal{T}_G \text{ generates } D^b_G(\text{Qcoh}(\mathcal{A}(\mathcal{E}))). \text{ The argument is exactly the same as in Theorem 4.1. First note that the stack } [\mathcal{A}(\mathcal{E})/G] \text{ is a quasicompact and separated Deligne–Mumford stack with coarse moduli being the quotient scheme } \mathcal{A}(\mathcal{E})/G. \text{ Thus [65], Corollary 4.2 implies that } D^b_G(\text{Qcoh}(\mathcal{A}(\mathcal{E}))) \text{ is compactly generated. The compact objects are all of } D^b_G(\mathcal{A}(\mathcal{E})) \text{ and hence it suffices to prove that } \mathcal{R} \text{ generates } D^b_G(\mathcal{A}(\mathcal{E})). \text{ So we take an object } F \in D^b_G(\mathcal{A}(\mathcal{E})) \text{ and assume } \mathcal{R}\text{Hom}_G(\pi^*\mathcal{T}_G, F) = 0. \text{ Adjoint property of } \pi^* \text{ and } \pi_* \text{ implies } \mathcal{R}\text{Hom}_G(\mathcal{T}_G, \pi_* F) = 0. \text{ The same argument that shows the generating property in the proof of Theorem 4.1 now implies that } \pi_* F = 0. \text{ Since } \pi \text{ is affine, we conclude } F = 0 \text{ and hence } \pi^*\mathcal{T}_G \text{ generates } D^b_G(\mathcal{A}(\mathcal{E})). \text{ Now since } \text{End}_G(\pi^*\mathcal{T}_G) \text{ is noetherian, the argument in the proof of Proposition 3.5 shows that the global dimension of } \text{End}_G(\pi^*\mathcal{T}_G) \text{ is finite (notice that the noetherian property in the arguments of the proof of Proposition 3.5 is enough). This establishes the desired equivalence}

\mathcal{R}\text{Hom}_G(\pi^*\mathcal{T}_G,-) : D^b(\mathcal{A}(\mathcal{E})/G) \to D^b(\text{End}_G(\pi^*\mathcal{T}_G))

\text{and completes the proof. \hfill \Box}

In general, one can show that for } \mathcal{E} = \mathcal{L}^\otimes n \text{ with } \mathcal{L} \text{ being } G\text{-equivariant and ample on } X \text{ and } n \gg 0 \text{ the assumption } H^i(X, \mathcal{L}^\otimes l \otimes S^j(\mathcal{E})) = 0 \text{ for all } i \neq 0 \text{ and all } l > 0 \text{ is always fulfilled. In this case the stack } [\mathcal{A}(\mathcal{L}^\otimes n)/G] \text{ admits a tilting object. Note that for } X \text{ being a Fano variety with } \mathcal{E} = \omega_X \text{ and } G = 1 \text{ one obtains the result of Bridgeland and Stern [19], Theorem 3.6 (see also [18], Proposition 4.1) and if } X = \text{Spec}(\mathbb{C}) \text{ the result of Brav [16], Theorem 4.2.1. In both cases the assumption } H^i(X, \mathcal{L}^\otimes l \otimes S^j(\mathcal{E})) = 0 \text{ for all } i \neq 0 \text{ and all } l > 0 \text{ can be shown to be fulfilled. The arguments in the proof of Theorem 5.1 also unify and simplify the arguments given in the proofs of Theorem 4.2.1 and 5.3.1 in [16]. From representation-theoretic point of view it would also be of interest to figure out for which } G\text{-equivariant locally free sheaves } \mathcal{E} \text{ the endomorphism algebra } \text{End}_G(\pi^*\mathcal{T}_G) \text{ is Koszul.}

In view of Theorem 5.1 above, it is also very natural to consider projective bundles with group actions. A semiorthogonal decomposition for the equivariant derived category of the projective bundles was proved to exist by Elagin [30]. Below we will prove that if the base scheme } X \text{ admits a } G\text{-equivariant tilting bundle, the stack } [\mathbb{P}(\mathcal{E})/G] \text{ admits a tilting object too. We start with some preliminary notations and observations. Let } X \text{ be a smooth projective and integral } k\text{-scheme and } G \text{ a finite group acting on } X. \text{ Suppose again } \text{char}(k) \nmid \text{ord}(G). \text{ Let } \mathcal{E} \text{ be a } G\text{-equivariant locally free sheaf of rank } r \text{ on } X. \text{ This provides us with a projective bundle } \mathbb{P}(\mathcal{E}) \text{ on which } G \text{ acts naturally. The structural morphism } \pi : \mathbb{P}(\mathcal{E}) \to X \text{ is a } G\text{-equivariant morphism and Elagin [30], Theorem 4.3 proved that one has a semiorthogonal decomposition}

D^b_G(\mathbb{P}(\mathcal{E})) = \langle \pi^*D^b_G(X), \pi^*D^b_G(X) \otimes \mathcal{O}_X(1), \ldots, \pi^*D^b_G(X) \otimes \mathcal{O}_X(r-1) \rangle.

\text{Suppose that } \mathcal{T} \text{ is a tilting bundle for } D^b(X) \text{ admitting a } G\text{-equivariant structure. Theorem 4.1 yields that } \mathcal{T}_G = \bigoplus_j \mathcal{T} \otimes W_j \text{ is a tilting bundle for } D^b_G(X). \text{ In view of the above semiorthogonal decomposition one easily verifies that } \mathcal{R} = \bigoplus_{i=0}^{r-1} \pi^*\mathcal{T}_G \otimes \mathcal{O}_X(i) \text{ generates } D^b_G(\mathbb{P}(\mathcal{E})).

\text{Lemma 5.2. Let } X, G \text{ and } \mathcal{E} \text{ be as above and suppose that a coherent sheaf } \mathcal{A} \text{ generates } D^b_G(X). \text{ Then } \bigoplus_{i=0}^{r-1} \pi^*\mathcal{A} \otimes \mathcal{O}_X(i) \text{ generates } D^b_G(\mathbb{P}(\mathcal{E})).

\text{Proof. First we note that } D^b_G(X) \text{ is derived equivalent to } \pi^*D^b_G(X) \otimes \mathcal{O}_X(j) \text{ via the functor } \mathcal{F} \mapsto \pi^*\mathcal{F} \otimes \mathcal{O}_X(j). \text{ Indeed, by adjunction of } \pi^* \text{ and } \pi_* \text{ we have}

\text{Hom}_G(\pi^*\mathcal{F} \otimes \mathcal{O}_X(j), \pi^*\mathcal{G} \otimes \mathcal{O}_X(j)) \simeq \text{Hom}_G(\mathcal{F}, \mathcal{R}\pi_*\pi^*(\mathcal{G})) \simeq \text{Hom}_G(\mathcal{F}, \mathcal{G}),
since $\mathbb{R}_m \pi^*(G) \simeq \mathbb{R}_m \mathcal{O}_X \otimes G$ according to the projection formula and due to the fact that $\mathbb{R}_m \mathcal{O}_X \simeq \mathcal{O}_X$. Since $A$ generates $D^b(G)(X)$ and $D^b(G)(X)$ is derived equivalent to $\pi^*(D^b_G(X)) \otimes \mathcal{O}_E(j)$, we conclude that $\pi^*A \otimes \mathcal{O}_E(j)$ generates $\pi^*(D^b_G(X)) \otimes \mathcal{O}_E(j)$. Hence $\bigoplus_{i=0}^1 \pi^*A \otimes \mathcal{O}_E(i)$ generates $D^b_G(\mathbb{P}(E))$ in view of the fact that $(\pi^*(D^b(G)(X)), \pi^*D^b_G(X) \otimes \mathcal{O}_E(1), \ldots, \pi^*D^b_G(X) \otimes \mathcal{O}_E(r-1))$ is a semiorthogonal decomposition for $D^b_G(\mathbb{P}(E))$. □

With this lemma we now prove the following result.

**Theorem 5.3.** Let $X$, $G$ and $E$ be as above and assume that $D^b(X)$ has a tilting bundle $T$, admitting a $G$-equivariant structure. Then the quotient stack $[\mathbb{P}(E)/G]$ admits a tilting object too.

**Proof.** We prove that $\mathcal{R} = \bigoplus_{i=0}^1 \pi^*T \otimes \mathcal{O}_E(i)$ from above is a tilting object. $G$-equivariant adjunction of $\pi^*$ and $\pi_*$ and projection formula yields for $0 \leq r_1, r_2 \leq r-1$:

$$\text{Hom}_G(\pi^*T \otimes \mathcal{O}_E(r_1), \pi^*T \otimes \mathcal{O}_E(r_2)[m]) \simeq \text{Hom}_G(T, T \otimes \mathbb{R}_m \mathcal{O}_E(r_2-r_1)[m]).$$

If $r_1 = r_2$ we have $\mathbb{R}_m \mathcal{O}_E(r_2-r_1) \simeq \mathcal{O}_X$ and hence

$$\text{Hom}_G(\pi^*T \otimes \mathcal{O}_E(r_1), \pi^*T \otimes \mathcal{O}_E(r_2)[m]) \simeq \text{Ext}^0_G(T, T) = 0$$

for $m > 0$ since $T$ is a tilting sheaf for $D^b_G(X)$ according to Theorem 4.1. If $0 \leq r_2 < r_1 \leq r-1$ we have $r_2 - r_1 > -r$ and hence $\mathbb{R}_m \mathcal{O}_E(r_2-r_1) = 0$ (see [33]) what implies

$$\text{Hom}_G(\pi^*T \otimes \mathcal{O}_E(r_1), \pi^*T \otimes \mathcal{O}_E(r_2)[m]) \simeq \text{Ext}^0_G(T, T) = 0$$

for all $m \geq 0$. It remains the case $0 \leq r_1 < r_2 \leq r-1$. In this case we have for $l = r_2 - r_1$, $\mathbb{R}_m \mathcal{O}_E(r_2-r_1) \simeq S^l(E)$ (see [33]) and hence

$$\text{Hom}_G(\pi^*T \otimes \mathcal{O}_E(r_1), \pi^*T \otimes \mathcal{O}_E(r_2)[m]) \simeq \text{Ext}^m_G(T, T \otimes S^l(E)) \simeq H^m(X, T \otimes T \otimes S^l(E))^G.$$

To achieve the vanishing of the above cohomology, we take a $G$-equivariant ample invertible sheaf $L$ on $X$. Such a $L$ always exist by the following argument: Take a projective embedding $X \to \mathbb{P}^N$ and construct a $G$-equivariant embedding $X \to \mathbb{P}^N \times \ldots \times \mathbb{P}^N$ by permuting the factors in the product, where we take $\text{ord}(G)$ copies of $\mathbb{P}^N$. Following this by a Segre embedding leads to a projective embedding $\iota : X \to \mathbb{P}^N \times \ldots \times \mathbb{P}^N \to \mathbb{P}^M$ that is compatible with a linear action of $G$ on $\mathbb{P}^M$. Then take $\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$ to be the $G$-equivariant ample invertible sheaf $L$ on $X$. Since $X$ is projective there is for a fixed $l > 0$ a natural number $n_l > > 0$ such that

$$H^m(X, T \otimes T \otimes S^l(E \otimes \mathcal{O}^{\otimes n_l})) \simeq H^m(X, T \otimes T \otimes S^l(E \otimes \mathcal{O}^{\otimes n_l})) = 0$$

for $m > 0$. Since $0 < l \leq r-1$, we have only finitely many $l > 0$ and we can choose $n > \max\{n_l\} | 0 < l \leq r-1\}$ so that for $\mathcal{O}^{\otimes n_l}$ one has

$$H^m(X, T \otimes T \otimes S^l(E \otimes \mathcal{O}^{\otimes n_l})) \simeq H^m(X, T \otimes T \otimes S^l(E \otimes \mathcal{O}^{\otimes n_l})) = 0$$

for $m > 0$ and all $0 < l \leq r-1$. This implies that $\mathcal{R}' = \bigoplus_{i=0}^1 \pi^*T \otimes \mathcal{O}_E(i)$ is a tilting object for $\mathbb{P}(E')$ with $E' = E \otimes \mathcal{O}^{\otimes n_l}$. Note that the generating property of $\mathcal{R}'$ follows from Lemma 5.2 and the finiteness of the global dimension of $\text{End}_G(\mathcal{R}')$ from the proof of Proposition 3.5. Finally, since $\mathbb{P}(E') \simeq \mathbb{P}(E)$ as $G$-schemes, we conclude that $D^b_G(\mathbb{P}(E))$ admits a tilting object. This completes the proof. □
Example 5.4. According to Example 4.2, the tilting bundle $T = \bigoplus_{i=0}^n\mathcal{O}_P(i)$ on $P^n$ is $G$-

equivariant if for instance $G$ is a finite subgroup of $\text{PGL}_{n+1}(k)$. Suppose $\text{char}(k) \n\n\not\mid \text{ord}(G)$. Then for any $G$-equivariant locally free sheaf $E$ on $P^n$ the stack $[P(E)/G]$ admits a tilting object. The same argument shows that for $X = \text{Grass}(d, n)$ with $2d \neq n$ and $G$ a finite group such that the action of $G$ on $X$ is induced by an homomorphism $G \to \text{PGL}_n(C)$, the quotient stack $[P_X(E)/G]$ admits a tilting object for any $G$-equivariant locally free sheaf $E$ on $X$.

Example 5.5. Let $X$ be a Brauer–Severi variety over $k$ and $G$ a finite subgroup of $\text{Aut}(X)$ with $\text{char}(k) \n\n\not\mid \text{ord}(G)$. The locally free sheaf $T'$ of Example 4.5 is a tilting bundle on $X$ admitting a $G$-equivariant structure. Now Theorem 5.3 applies and yields the existence of a tilting object on $[P(E)/G]$ for any $G$-equivariant locally free sheaf $E$ on $X$. The same statement holds also for $X$ being a generalized Brauer–Severi variety over a field of characteristic zero and $G$ a certain finite subgroup of the automorphisms.

6. The case of arbitrary algebraic groups

In this section we state some problems we are faced with while considering arbitrary, not necessary finite, algebraic groups acting on $X$.

For arbitrary algebraic groups we have the following tilting correspondence:

**Theorem 6.1.** Let $X$ be a projective and integral $k$-scheme and $G$ an algebraic group over $k$ acting on $X$. Suppose we are given a compact generator $T$ for the derived category $D_G(\text{Qcoh}(X))$ and let $A = \text{RHom}_G(T, T)$. Denote by $D_G^b(\text{Qcoh}(X))$ the subcategory of compact objects. Then the following hold:

(i) The functor $\mathbb{R}\text{Hom}_G(T, -) : D_G(\text{Qcoh}(X)) \to D(\text{Mod}(A))$ is an equivalence.

(ii) The equivalence of (i) restricts to an equivalence $D_G^b(\text{Qcoh}(X)) \sim \text{perf}(A)$.

(iii) If the global dimension of $A$ is finite then $\text{perf}(A) \cong D^b(A)$.

**Proof.** The equivalence of (i) is just an application of [47], Theorem 8.5. To prove (ii), we note that the equivalence $\psi = \mathbb{R}\text{Hom}_G(T, -)$ from (i) restricts to an equivalence between the subcategories of compact objects. This can be seen as follows: Since $\psi$ is an equivalence it admits as a right adjoint, being the inverse $\psi^{-1}$ (see [39], Proposition 1.26). Hence for compact object $C \in D_G(\text{Qcoh}(X))$ we have

$$\text{Hom}_A(\psi(C), \bigoplus_i \mathcal{F}_i) \simeq \text{Hom}_G(C, \bigoplus_i \psi^{-1}(\mathcal{F}_i))$$

$$\simeq \bigoplus_i \text{Hom}_G(C, \mathcal{F}_i) \simeq \bigoplus_i \text{Hom}_A(\psi(C), \mathcal{F}_i).$$

The compact objects of $D(\text{Mod}(A))$ are all of $\text{perf}(A)$ (see [47], Theorem 8.2). This establishes the equivalence $D_G^b(\text{Qcoh}(X)) \sim \text{perf}(A)$.

Finally, if the global dimension of $A$ is finite, one has $\text{perf}(A) \cong D^b(A)$ (see for instance [47], Theorem 8.6). This completes the proof. □

**Remark 6.2.** Note that in the above situation $\mathbb{R}\text{Hom}_G(T, T)$ is a dg-algebra (see [46] for details on dg-categories). If one furthermore assumes $\text{Hom}_G(T, T[i]) = 0$ for $i \neq 0$, the dg-algebra becomes an algebra.

To get the desired equivalence $D_G^b(X) \sim D^b(A)$ one has to prove that $D_G^b(\text{Qcoh}(X))$ is all of $D_G^b(X)$ and that the global dimension of $A$ is finite. Usually, the equivalence between $D_G^b(\text{Qcoh}(X))$ and $D_G^b(X)$ is proved as follows: Firstly, one has to guarantee
that the compact objects are the same as the bounded complexes of $G$-equivariant locally free sheaves of finite rank. But exactly this is in general not possible (see [32] for stacks where this holds). Secondly, provided $X$ is smooth, one has to show that any coherent $G$-equivariant sheaf $\mathcal{F}$ admits a finite $G$-equivariant resolution of locally free sheaves of finite rank. This is usually proved by applying results of Thomason [64] that also do not hold for arbitrary algebraic groups. By the work of Totaro [66], the existence of resolutions of coherent sheaves by locally free ones somehow imply that the group has to be affine. So in general it is not clear how to get the equivalence between $D^b_c(Qcoh(X))$ and $D^b_c(X)$ and furthermore, how to prove the finiteness of the global dimension of $A$. For quotient stacks $[X/G]$ obtained as quotients of smooth quasiprojective and integral $k$-schemes $X$ by actions of affine algebraic groups $G$, the derived category $D^b_c(Qcoh(X))$ should be all of $D^b_c(X)$ (this should follow from [32] and [64]). It is sensible to believe that in this geometric situation tilting object should exist for certain quotient stacks. For instance the Deligne–Mumford stack $[(\text{Spec} x_0, x_1, x_2 \setminus 0)/\mathbb{G}_m]$, obtained by the $\mathbb{G}_m$ action of weight $(1, 1, n)$, admits $\mathcal{T} = \bigoplus_{i=0}^{n+2} O(i)$ as tilting bundle. So there are quotient stacks obtained by non-finite group actions that awfully well admit tilting objects. It is an open problem what kind of quotient stacks (or more generally ”nice” Deligne–Mumford stacks) may possess tilting objects. One firstly should investigate for what kind of quotient stacks (Deligne–Mumford stacks) the derived categories of (quasi-) coherent sheaves admits a compact generator. But this is a rather delicate issue (see for instance [32] and [65] and references therein for discussions and results in this direction). In general, tilting objects for quotient stacks cannot exist. For instance the category $D^b(\langle \text{pt}/G \rangle)$ has a generating object if the group has finitely many irreducible representations. For a list of groups such that $D(Qcoh(\langle \text{pt}/G \rangle))$ admits a compact generator we refer to [32] and references therein. Notice that in loc.cit. Hall and Rydh proved the existence of compact generators for certain stacks including the class of quasicompact tame Deligne–Mumford stacks with affine diagonal. This includes a huge class of global quotient stacks, all of them being candidates for studying the existence of tilting objects. Although many quotient stacks $[X/G]$ cannot have tilting objects, their derived categories have semiorthogonal decompositions. We note that Elagin [30], [31] considered these cases and established semiorthogonal decompositions for schemes with actions of arbitrary algebraic groups. Finally we note, that considering full strongly exceptional collections instead of tilting objects, the theorem of Keller [47] gives a quite satisfactory answer. To be precise, for a full strongly exceptional collection $\mathcal{E}_1, ..., \mathcal{E}_n$ in a $k$-linear triangulated category $T$ that admits an enhancement, the category $T$ is equivalent to $D^b(A)$, where $A = \text{End}_T(\bigoplus_{i=1}^n \mathcal{E}_i)$.

### 7. Application: Orlov’s dimension conjecture

As an application of some of the results proved in the previous sections we provide some further evidence for a conjecture formulated by Orlov [58].

Let $T$ be a triangulated category. For a subcategory $M$ we denote by $\langle M \rangle$ the full triangulated subcategory of $T$ whose objects are isomorphic to summands of finite coproducts of shifts of objects in $M$. Concretely this means that $\langle M \rangle$ is the smallest full triangulated subcategory containing $M$ that is closed under isomorphisms, shifting and taking finite coproducts and summands (see [6], [62]). For two triangulated subcategories $M$ and $N$ of $T$ we want to denote by $M \star N$ the smallest full subcategory consisting of objects $R$ such that there exists a distinguished triangles of the form

$$X_1 \rightarrow R \rightarrow X_2 \rightarrow X_1[1],$$

where $X_1 \in M$ and $X_2 \in N$. Then set $M \circ N = \langle M \star N \rangle$. We inductively define $\langle M \rangle_n = \langle M \rangle_{n-1} \circ (M)$ and set $(M)_0$ to be $\langle M \rangle$.

**Definition 7.1.** ([6], [62]) Let $A$ be an object of a triangulated category $T$. If there is an integer $n \geq 0$ with $\langle A \rangle_n = T$ we define the generation time $\text{gt}(A)$ of $A$ to be
min\{n|(A)_n = T\}. Otherwise, we set gt(A) = ∞. If gt(A) is finite, we say A is a strong generator, otherwise A is called generator. The dimension of the triangulated category T is now defined to be the minimal generation time among strong generators and is denoted by dimT. It is set to be ∞ if there are no strong generators.

Orlov [58] investigated the dimension of triangulated categories coming from geometry and conjectured the following:

**Conjecture.** If X is a smooth integral and separated scheme of finite type over k, then \(\dim D^b(X) = \dim(X)\).

In loc.cit. it is proved that the conjecture holds for smooth projective curves C of genus \(g \geq 1\). For curves of genus \(g = 0\) this is clear and well known. Therefore one has \(\dim D^b(C) = 1\) for all smooth projective curves C. Additionally, the conjecture is known to be true in the following cases:

- affine schemes of finite type over k, certain flags and quadrics [62].
- del Pezzo surfaces, certain Fano three-folds, Hirzebruch surfaces, toric surfaces with nef anti-canonical divisor and certain toric Deligne–Mumford stacks over \(\mathbb{C}\).

There are also lower and upper bounds for the dimension of the bounded derived category \(D\)-schemes \(X\), \(\dim D^b(X) \geq 2\dim(X)\) (see [62], Proposition 7.17) and that for smooth quasiprojective \(k\)-schemes \(X\), \(\dim D^b(X) \leq 2\dim(X)\) (see [62], Proposition 7.9).

**Remark 7.2.** Note that the above conjecture also makes sense for certain Deligne–Mumford stacks over a field \(k\) (see [6]). This is due to the fact that for smooth tame Deligne–Mumford stacks \(X\) with coarse moduli space being quasiprojective one also has \(\dim D^b(X) \leq 2\dim(X)\) (see [6], Lemma 2.20). Furthermore, as for reduced separated schemes, for tame Deligne–Mumford stack with coarse moduli being reduced and separated one has \(\dim D^b(X) \geq \dim(X)\) (see [6], Lemma 2.17).

In view of Remark 7.2, Ballard and Favero [6] extended the above conjecture of Orlov and formulated:

**Conjecture.** ([6]) Let \(X\) be a smooth and tame Deligne–Mumford stack of finite type over \(k\) with quasiprojective coarse moduli space, then \(\dim D^b(X) = \dim(X)\).

We now want to apply results of Ballard and Favero [6] to produce some more examples where the above conjecture holds true. We start with the quotient stack \([X/G]\) obtained by an action of a finite group with \(\text{char}(k) \not\mid \text{ord}(G)\).

**Proposition 7.3.** Let \(k\) be a perfect field and \(X\) and \(G\) as in Theorem 4.1. Suppose \(X\) is connected and \(T\) is a tilting sheaf for \(D^b(X)\) admitting a \(G\)-equivariant structure. Suppose furthermore that \(\text{Ext}^i(T, T \otimes \omega_X) = 0\) for \(i > 0\). Then the quotient stack \([X/G]\) satisfies \(\dim([X/G]) = \dim D^b_C(X)\).

**Proof.** According to Theorem 4.1, \(T_G = \bigoplus_i T \otimes W_i\), with \(W_i\) being the irreducible representations of \(G\), is a tilting sheaf for \(D^b_C(X)\). As mentioned in Section 2, \([X/G]\) is a smooth proper tame and connected Deligne–Mumford stack with projective coarse moduli space. In this case, to determine the dimension of \(D^b_C(X)\) by [6], Theorem 3.2 we only have to calculate the largest \(i\) for which \(\text{Hom}_C(T_G, T_G \otimes \omega_{[X/G]}[i]) \neq 0\).

Since \(\omega_X\) has a \(G\)-equivariant structure and as a sheaf on \(X\) gives rise to the Serre functor \(- \otimes \omega_X[\dim(X)]\), we conclude with [6], Lemma 2.26 f. that \(\omega_{[X/G]} \simeq \omega_X\) on \([X/G]\), since the Serre functor is unique up to isomorphism (see [39], p.10). Therefore we have
Now this implies

\[ \text{Ext}^i(T_G, T_G \otimes \omega_X^i) \simeq (\bigoplus_j \text{Ext}^i(T, T \otimes \omega_X^j) \otimes \text{Hom}(W_j, W_j)). \]

But by assumption we have

\[ \text{Ext}^i(T, T \otimes \omega_X^i) = 0 \text{ for } i > 0. \]

Now this implies

\[ \text{Hom}(T_G, T_G \otimes \omega_X^i) = 0 \]

for \( i > 0 \) and hence [6], Theorem 3.2 yields \( \dim([X/G]) = \dim D^b_C(X). \)

**Corollary 7.4.** Let \( k \) be a perfect field and \( G \) a finite subgroup of \( \text{PGL}_{n+1}(k) \) acting on \( \mathbb{P}^n \). Suppose \( \text{char}(k) \nmid \text{ord}(G) \). Then \( \dim([\mathbb{P}^n/G]) = \dim D^b([\mathbb{P}^n/G]). \)

**Proof.** Recall that \( T = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i) \) is a tilting bundle for \( \mathbb{P}^n \) (see [9]). Since \( \mathcal{O}_{\mathbb{P}^n}(i) \) is preserved by automorphisms, the tilting bundle \( T \) is \( G \)-equivariant. Furthermore, we have

\[ \text{Ext}^i(T, T \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)) = 0 \text{ for } i > 0. \]

The rest follows from Proposition 7.3.

The statement of Corollary 7.4 also holds for Brauer–Severi varieties.

**Corollary 7.5.** Let \( X \) be a Brauer–Severi variety over a perfect field \( k \) and \( G \) a finite subgroup of \( \text{Aut}(X) \). Suppose \( \text{char}(k) \nmid \text{ord}(G) \). Then \( \dim D^b([X/G]) = \dim([X/G]). \)

**Proof.** Let \( k \subset L \) be a Galois splitting field for \( X \), i.e., \( X \otimes_k L \simeq \mathbb{P}^n_L \). Since \( X \) admits a tilting bundle \( T' = \bigoplus_{i=0}^n V^{n+i} \) with \( G \)-equivariant structure (see Example 4.5), Proposition 7.3 tells us that we only have to verify \( \text{Ext}^l(T', T' \otimes \omega_X^{i+1}) = 0 \) for \( l > 0 \). Notice that \( V^{n+i} \otimes_k L \simeq \mathcal{O}_{\mathbb{P}^n_L}(-i)^{(n+1)i} \) and \( \omega_X = \mathcal{O}_X(-n-1) \). One easily verifies

\[ \text{Ext}^l(T', T' \otimes \omega_X^{i+1}) \otimes_k L = 0 \text{ for } l > 0 \]

on \( \mathbb{P}^n_L \) and thus

\[ \text{Ext}^l(T', T' \otimes \omega_X^{i+1}) = 0 \text{ for } l > 0 \]

on \( X \). Now Proposition 7.3 yields the assertion.

**Corollary 7.6.** Let \( k \) be an algebraically closed field of characteristic zero and \( G \) a finite group. Suppose \( 2d \neq n \) and that we are given an action of \( G \) on \( X = \text{Grass}(d, n) \) induced by a homomorphism \( G \to \text{PGL}_n(k) \). Then \( \dim([X/G]) = \dim D^b([X/G]). \)

**Proof.** We mentioned in Example 4.2 that under the above assumptions Kapranov’s tilting bundle \( T = \bigoplus_i \Sigma^i(S) \), where \( S \) is the tautological sheaf of \( X \) and \( \Sigma^i \) the Schur functor (see [42]), is \( G \)-equivariant. Note that \( \Lambda^i(S) \simeq \mathcal{O}_X(-1) \) and \( \omega_X = \mathcal{O}_X(-n) \). To apply Proposition 7.3 we only have to verify \( \text{Ext}^i(T, T \otimes \mathcal{O}_X(n)) = 0 \) for \( i > 0 \). By the construction of \( T \), it is enough to verify

\[ \text{Ext}^i(\Sigma^i(S), \Sigma^i(S) \otimes \mathcal{O}_X(n)) \simeq H^i(X, \Sigma^i(S)^* \otimes \Sigma^i(S) \otimes \mathcal{O}_X(n)) = 0 \]

for \( i > 0 \). By the Littlewood–Richardson rule the partitions \( \gamma \) of irreducible summands \( \Sigma^i(S) \) of \( \text{Hom}(\Sigma^i(S), \Sigma^i(S)) \simeq \Sigma^i(S)^* \otimes \Sigma^i(S) \) satisfy \( \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d \geq -(n-d) \) (see [44], 3.3). So we can restrict ourselves to show

\[ H^i(X, \Sigma^i(S) \otimes \mathcal{O}_X(n)) = 0 \text{ for } i > 0. \]
Since $\Sigma^n(S) \otimes \mathcal{O}_X(n) \simeq \Sigma^{n+1}(S) \simeq \Sigma^{-\gamma-n}(S^\vee)$, where $\gamma + n = (\gamma_1 + n, ..., \gamma_d + n)$, we have $\gamma_1 + n \geq \gamma_2 + n \geq ... \geq \gamma_d + n \geq d$ and by Kapranov’s calculation of the cohomology of $\Sigma^n(S^\vee)$ (see [42], Lemma 2.2 or [44], Lemma 3.2) we conclude:

$$H^i(X, \Sigma^{n+1}(S)) = 0 \text{ for } i > 0.$$ 

Now Proposition 7.3 applies and we get the desired equality $\dim([X/G]) = \dim D^b([X/G])$. □

As an further application of Proposition 7.3 we obtain the next result.

**Proposition 7.7.** Let $k$, $X$, $G$ and $Y \subset \text{Hilb}_G(X)$ be as in Theorem 4.10 (note that $\omega_X$ is locally trivial as a $G$-equivariant sheaf). Suppose that $D^b(X)$ has a tilting sheaf $T$ admitting a $G$-equivariant structure. Furthermore, suppose that $\dim(Y \times_{X//G} Y) < \dim(X) + 1$ and that $\text{Ext}^1(T, T \otimes \omega_X^Y) = 0$ for $i > 0$. Then $\dim(Y) = \dim D^b(Y)$.

**Proof.** Note that by assumption $k$ is an algebraically closed field of characteristic zero and hence perfect. Under the above assumptions Proposition 7.3 applies and we have $\dim([X/G]) = \dim D^b_G(X)$. Furthermore, derived McKay correspondence applies and provides us with a birational morphism $Y \to X//G$ and the McKay equivalence $D^b(Y) \cong D^b_G(X)$. Since $\dim(Y) = \dim(X//G) = \dim(X) = \dim([X/G]) = \dim D^b_G(X)$, we conclude with the McKay equivalence that $\dim(Y) = \dim D^b(Y)$. This completes the proof. □

Summarizing the above observations we obtain the following theorem.

**Theorem 7.8.** The dimension conjecture holds in the following cases:

- (i) quotient stacks of the form $[\mathbb{P}^n_G]$, where $G$ is a finite subgroup of $\text{PGL}_{n+1}(k)$, $k$ is perfect and $\text{char}(k) \nmid \text{ord}(G)$.
- (ii) quotient stacks of the form $[X/G]$, where $G$ is a finite subgroup of $\text{Aut}(X)$, $\text{char}(k) \nmid \text{ord}(G)$ and $X$ is a Brauer–Severi variety over a perfect field $k$.
- (iii) quotient stacks of the form $[\text{Grass}(d, n)_G]$, provided $2d \neq n$ and where the action of the finite group $G$ is induced by a homomorphism $G \to \text{PGL}_n(\mathbb{C})$.
- (iv) $G$-Hilbert schemes $\text{Hilb}_G(\mathbb{P}^n_G)$ for $n \leq 3$ and $G$ a finite subgroup of $\text{PGL}_{n+1}(\mathbb{C})$ such that $\omega_{\mathbb{P}^n_G}$ is locally trivial as a $G$-equivariant sheaf.

Finally, we note that exploiting the above techniques we can also prove some further examples where the dimension conjecture is hold to be true. This will be the content of a forthcoming paper of the author.

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MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT 40225 DÜSSELDORF, GERMANY
E-mail adress: novakovic@math.uni-duesseldorf.de