Extending the Observational Frequency Range for Gravitational Waves in a Pulsar Timing Array

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Abstract
We provide an observation method for gravitational waves using a pulsar timing array to extend the observational frequency range up to the rotational frequency of pulsars. For this purpose, we perform an analysis of a perturbed electromagnetic wave in perturbed spacetime from the field perspective. We apply the analysis to the received electromagnetic waves in a radio telescope, which partially composes the periodic electromagnetic pulse emitted by a pulsar. For simple observation, two frequency windows are considered. For each window, we propose gauge-invariant quantities and discuss their observations.

Unified Astronomy Thesaurus concepts: Gravitational wave detectors (676); Pulsar timing method (1305); Radio telescopes (1360)

1. Introduction
A pulsar is a stably rotating compact star that periodically emits an electromagnetic pulse (EMP) due to its magnetic axis being tilted to the rotation axis. Because the rotation of a pulsar is extremely stable, the period of the EMP generally remains roughly constant. However, when an EMP passes through a perturbed spacetime, its period will be changed, and the perturbation will contain information about gravitational waves (GWs).

In practice, GW observation by a single pulsar measurement is not possible because various noises are larger than the GW signal. Instead, the correlation between measurements from different pulsars can amplify a stochastic gravitational wave (SGW) signal by increasing the measurement time if the noises of the measurements are independent of each other. In this way, observations are being performed to detect the SGW for the first time by pulsar timing arrays (PTAs) (Shannon et al.2015; Babak et al.2016; Arzoumanian et al.2020). The method of extracting an SGW signal from the correlation was proposed by Hellings & Downs (1983).

Let us look at the analysis for the effect of GW on the EMP period. When we consider an electromagnetic wave (EMW) in a perturbed spacetime with GWs such that the frequency of the EMW is much larger than those of the GWs, we can introduce geometrical optics. By an approximation of geometrical optics, the wavevector of the EMW is null and follows null geodesic (Misner et al.2017). From the solution of the perturbed null geodesic equation with appropriate boundary conditions, we obtain information of the perturbed EMW, e.g., redshift and angular deflection as in Book & Flanagan (2011).

On the other hand, the time interval between emission and arrival time of the EMP is obtained by the perturbation of the null condition as in Maggiore (2018). Additionally, assuming that GW frequency is much smaller than the EMP frequency, we obtain the well-known redshift formula given by Estabrook & Wahlquist (1975) and Detweiler (1979). Note that we distinguish between the frequency of EMP and EMW. The EMP frequency is defined from the periodic intensity of the EMWs, which is quadratic to the electromagnetic field. Therefore, a periodic EMP can be composed of EMWs with relatively high frequencies. For example, a radio telescope with a GHz frequency band of EMW measures a periodic EMP in KHz.

In this paper, we determine an analysis of perturbed EMWs that applies to the observation of GWs with frequencies comparable to the EMP frequency of a pulsar. To do this, we first obtain a general solution of the vacuum Maxwell equation in the presence of GWs without using the geometric optical approximation. The general solution consists of a particular solution and a homogeneous solution. The particular solution is determined by GWs, and the homogeneous solution is determined to satisfy the appropriate boundary condition for the electromagnetic field. In Montanari (1998), they performed an analysis of EMWs with GWs by solving the Maxwell equation introducing the Fermi normal coordinate. Instead, we will give a covariant analysis without introducing a coordinate system.

For measurement of EMWs, we introduce spatial scalars of the electromagnetic field independent of the spatial frame in a given observer, e.g., $E^2$, $B^2$, and $E \cdot B$. When the measurement is given in components that depend on a spatial frame, the analysis of its perturbation is challenging because we need the frame perturbation that reflects the structure of the device. In contrast, spatial scalars are given only by the electric field, the magnetic field, and the spatial metric.

Finally, we restrict frequencies of GWs to much less than that of EMWs to introduce the geometrical optics approximation. On the other hand, the frequency of a GW can be comparable to that of an EMP. With these GWs, we obtain the perturbed spatial scalars of a periodic EMP composed by high frequency EMWs. As an application, two frequency windows for spatial scalars are considered. For each window, we propose gauge-invariant quantities and discuss their observations.

In this paper, we introduce the geometrized unit ($c = 1$, $G = 1$) for the spacetime, and the Gaussian unit ($\epsilon_0 = 1/4\pi$, $\mu_0 = 4\pi$) for the electromagnetism. Indices
represented by italic lowercase Latin letters starting from $a$ are abstract indices, as in Wald (1984).

2. Ansatz

We consider a one-parameter family of perturbed spacetime $(M_\epsilon, g(\epsilon))$, where $\epsilon$ is a perturbation parameter and $(M_0, g)$ is the Minkowski background spacetime. We need a diffeomorphism $\phi_\epsilon$ to identify points between $M_0$ and $M_\epsilon$. A perturbed quantity is defined by the pull-back of a quantity from $M_\epsilon$ to $M_0$ denoted with the left superscript of $\epsilon$, e.g., $\tilde{Q} \equiv \phi_\epsilon^* Q(\epsilon)$ for a geometrical quantity $Q(\epsilon)$. This style of perturbation in a coordinate invariant manner was introduced in Stewart et al. (1974) and Stewart (1990).

Perturbed metric $\tilde{g}$ is expanded as Taylor series by

$$
\epsilon \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab} + O(\epsilon^2),
$$

where $g$ is the Minkowski background metric, and $h$ is the first-order metric perturbation. We impose gauge conditions on $h$ as

$$
\nabla^a h_{ab} = 0,
$$

$$
h^a_{\; a} = 0,
$$

where $\nabla$ is the Levi-Civita connection associated with $g$. Then, the perturbation of the vacuum Einstein equation becomes the wave equation

$$
\nabla^a \nabla_a h_{ab} = 0.
$$

Let us consider geodesic observers with four-velocity vector field $\nu$ over perturbed spacetime given by

$$
\nu^n = n^a + \epsilon \delta n^a + O(\epsilon^2),
$$

where $n^a \equiv -g^{ab} \nabla_b t$ is the normal vector of globally inertial time coordinate $t$ in Minkowski spacetime, and $\delta n$ is the first-order perturbation. Then, the perturbed geodesic equation becomes

$$
\delta n^b \nabla_b n^a + n^b \nabla_b \delta n^a + C^a_{bc} n^b \delta n^c = 0,
$$

where $C^a_{bc} \equiv \frac{1}{2}(\nabla_a h_{bc} + \nabla_b h_{ac} - \nabla_c h_{ab})$. Additionally imposing a gauge condition on $h$ as

$$
h_{ab} n^b = 0,
$$

we obtain $n^b \nabla_b \delta n^a = 0$ from Equation (6). Assuming observers with $\delta n^a = 0$ before arrival of GW, their four-velocity perturbations remain as $\delta n^a = 0$. We only consider these observers. Note that the observers of our choice are identical to the observers with fixed TT coordinate (Misner et al. 2017).

Metric perturbation $h$ is superposed by monochromatic plane GW over all directions and all frequencies. It is given by

$$
h_{ab}(t,x) = \int d^2 \kappa \int_{-\infty}^{\infty} \frac{d \omega_g}{2\pi} \hat{h}_{ab}(\omega_g, \kappa) e^{iP(t,x;\omega_g,\kappa)},
$$

where $\kappa$ is the spatial unit vector, $d^2 \kappa$ is the solid angle element of three-dimensional unit sphere, $\omega_g$ is the frequency, $\hat{h}$ is the amplitude of GW with $(\omega_g, \kappa)$, and $P(t,x;\omega_g, \kappa) \equiv \omega_g(-t + \kappa \cdot x)$ is the phase. Note that Equation (8) does not cover all solutions of $h$. For example, gravitational memory effect (Zel’’dovich & Polnarev 1974; Smarr 1977) cannot be expressed in the form of Equation (8). Let $\hat{h}$ be zero when $\omega_g = 0$. This corresponds to a gauge choice of $h$. Wavevector is defined by

$$
k^a(\omega_g, \kappa) \equiv g^{ab} \nabla_b P,
$$

having $3 + 1$ decomposition (Gourgoulhon 2012) as

$$
k^a = \omega_g (n^a + \kappa^a),
$$

We assume that the electromagnetic potential $A$ has first-order strength in $\epsilon$, i.e.,

$$
\epsilon A_a = \epsilon A_a + \epsilon^2 X_a + O(\epsilon^3),
$$

where $A$ is leading order value, and $X$ is its next order perturbation. In the field strength,

$$
\epsilon F_{ab} = 2\nabla_{[a} \epsilon A_{b]} = \epsilon F_{ab} + \epsilon^2 Y_{ab} + O(\epsilon^3),
$$

where $F_{ab} = 2\nabla_{[a} A_{b]}$ and $Y_{ab} = 2\nabla_{[a} X_{b]}$. Additionally, we assume no electromagnetic source. Then, its stress–energy

$$
\epsilon T_{ab} = \frac{1}{4\pi} \left( F_{cd} F_{ab} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)
$$

$$
= \epsilon^2 T_{ab} + O(\epsilon^3),
$$

having $O(\epsilon^2)$ does not influence metric perturbation $h$ by the perturbed Einstein equation. We summarize perturbation orders of major quantities in Table 1.

We impose gauge conditions on $A$ as

$$
\nabla^a A_a = 0,
$$

$$
A_a n^a = 0.
$$

Then, the Maxwell equation in leading order becomes the wave equation

$$
\nabla^b \nabla_b A_a = 0.
$$

We consider a plane EMW propagating to spatial unit vector $\lambda$

$$
A_a(t, x) = \int_{-\infty}^{\infty} \frac{d \omega_e}{2\pi} \hat{A}_a(\omega_e) e^{iQ(t,x;\omega_e, \lambda)},
$$

where $\omega_e$ is the frequency, $\hat{A}$ is the amplitude of EMW with $(\omega_e, \lambda)$, and $Q(t,x;\omega_e, \lambda) \equiv \omega_e (-t + \lambda \cdot x)$ is the phase. Let $\hat{A}$ be zero when $\omega_e = 0$. This corresponds to a gauge choice of $A$. Wavevector is defined by

$$
l^a(\omega_e, \lambda) \equiv g^{ab} \nabla_b Q,\n$$
having $3 + 1$ decomposition as

$$l^a = \omega_c (n^a + \chi^a).$$  \hspace{1cm} (19)

For measurement of an EMW, we introduce basic spatial scalars defined by

$$\epsilon I = \epsilon_{\gamma ab} E_a^i E_b^i$$ \hspace{1cm} (20)

$$\epsilon B = \epsilon_{\gamma ab} B_a^i B_b^i$$ \hspace{1cm} (21)

$$\epsilon I^h = \epsilon_{\gamma ab} E_a^i E_b^i$$ \hspace{1cm} (22)

where $E_a^i = \epsilon F_{ab}^i n^b$ is the electric field, $B_a^i = \frac{1}{2} \epsilon^{abc} a^i E_bc$ is the magnetic field, $\epsilon_{\gamma}$ is the spatial metric, and $\epsilon$ is the spatial Levi-Civita tensor. All spatial scalars that are composed only by electromagnetic quantities ($E$, $B$) and spacetime quantities ($\epsilon_{\gamma}$, $\epsilon$) can be expressed by a combination of $I_1, I_2,$ and $I_3$. For example, the magnitude of the Poynting vector is given by

$$\epsilon I = \epsilon I^I + \epsilon I^B + O(\epsilon^4),$$ \hspace{1cm} (23)

for $i \in \{1, 2, 3\}$.

### 3. Perturbation of EMW

The perturbation of electromagnetic potential $X$ is determined by the perturbed Maxwell equation, i.e.,

$$\nabla^b \nabla_a X_a = 2 \epsilon^b \nabla_b \lambda + \omega \nabla_b \lambda \nabla_a$$

$$= \int_V dV (2 \epsilon^b \nabla_b \lambda \nabla_a - \nabla_b \lambda \nabla_a) e^{i(\omega t - k \cdot x)},$$ \hspace{1cm} (24)

where $\tilde{C}^a_{bc} = \frac{1}{2} (k_b \tilde{h}_c + k_c \tilde{h}_b - k_b \tilde{h}_c)$, and the integration over domain $V$ is defined by

$$\int_V dV = \int_{k=\lambda} dV \int_{\omega \neq 0} d\omega \pi \int_{\omega \neq 0} d\omega \pi,$$ \hspace{1cm} (25)

excluding the case of $\kappa = \lambda$ from $V$, in which the right-hand side of Equation (24) vanishes. The solution $X$ consists of homogeneous solution $X^h$ and particular solution $X^p$. As shown in Park & Kim (2021), the particular solution is obtained by

$$X^p = \int_V dV \tilde{X}^p_a e^{i(\omega t - k \cdot x)},$$ \hspace{1cm} (26)

where $X^p$ is

$$\tilde{X}^p_a = \frac{1}{k^a} (2i \tilde{C}^c_{ab} l^b \tilde{A}_c + \tilde{h}_{bc} l^b \tilde{F}_{ab}).$$ \hspace{1cm} (27)

We define its field strength as

$$\tilde{Y}^p_{ab} = 2 \nabla_a X^h_b = \int_V dV \tilde{Y}^p_{ab} e^{i(\omega t - k \cdot x)},$$ \hspace{1cm} (28)

where $Y^p$ is

$$\tilde{Y}^p_{ab} = 2i (k_a l_b + l_a k_b) \tilde{X}^p_{ab}. $$ \hspace{1cm} (29)

The homogeneous solution $X^h$ is governed by the wave equation $\nabla^b \nabla_a X^h_a = 0$. Accordingly, we consider a solution form of

$$X^h_a (x, t) = \int d^2 \omega \int_{\omega \neq 0} \frac{d\omega}{2 \pi} X^h_a (\omega_b, \mu) e^{iR(t, x; \omega_b, \mu)},$$ \hspace{1cm} (30)

where $\mu$ is the spatial unit vector, $d^2 \mu$ is the solid angle element of three-dimensional unit sphere, $\omega_b$ is the frequency, $X^h$ is the

amplitude of $(\omega_b, \mu)$, and $R(t, x; \omega_b, \mu) \equiv \omega_b (-t + \mu \cdot x)$ is the phase. The field strength of the homogeneous solution $Y^h_{ab}$ is defined by

$$Y^h_{ab} = 2 \nabla_a X^h_b = \int d^2 \mu \int_{\omega \neq 0} \frac{d\omega}{2 \pi} \tilde{Y}^h_{ab} (\omega_b, \mu) e^{iR},$$ \hspace{1cm} (31)

where the amplitude $\tilde{Y}^h_{ab}$ is determined by a boundary condition.

We impose a boundary condition on a plane that is perpendicular to $\lambda$ and located at the source of the EMW such that the field strength perturbation $Y_{ab} = Y^p_{ab} + Y^h_{ab}$ vanishes. It means that all geodesic observers located on the plane emit electromagnetic waves identical to Equation (17) in their own frame. Explicitly, the boundary condition is described as $Y_{ab}(t, y) = 0$ for all $(t, y)$ such that $\lambda \cdot y = d$, where $d$ is distance between the source plane and the origin. Assuming that a receiver is in the positive $\lambda$ side from the plane, directions of waves $\mu$ incident to the receiver from the source plane is restricted to $\mu \cdot \lambda > 0$. We only consider these waves. As shown by Appendix A, the solution that meets all conditions we imposed is given by

$$Y^h_{ab} = -\int dV \tilde{Y}^P_{ab} e^{i(\omega + \Omega + \Delta)},$$ \hspace{1cm} (32)

where

$$\Delta (x) \equiv \left\{ \sqrt{\omega^2 - 2\omega \omega_k \omega_e + (\omega_g h^a \lambda_a)^2} - \omega_e - \omega_g h^a \lambda_a \right\} (\lambda \cdot x + d).$$ \hspace{1cm} (33)

Collecting $Y^p$ in Equation (28) and $Y^h$ in Equation (32), we obtain the field strength perturbation $Y$ as

$$Y_{ab} = \int_V dV \tilde{Y}^P_{ab} e^{i(\mu + \Omega) (1 - e^{\Omega})}.$$ \hspace{1cm} (34)

Then, the perturbations of electric field and magnetic field are given by

$$\delta E_a = Y_{ab} n^b,$$ \hspace{1cm} (35)

$$\delta B_a = \frac{1}{2} e^{bc} Y_{bc} + e^{bc} h^b F_{bc}.$$ \hspace{1cm} (36)

Finally, the perturbations of spatial scalars are obtained by

$$\delta I_1 = 2 g^{ab} \delta E_a E_b - h^{ab} E_a E_b,$$ \hspace{1cm} (37)

$$\delta I_2 = 2 g^{ab} \delta B_a B_b - h^{ab} B_a B_b,$$ \hspace{1cm} (38)

$$\delta I_3 = g^{ab} \delta E_a B_b + g^{ab} \delta B_a E_b - h^{ab} E_a B_b.$$ \hspace{1cm} (39)

### 4. Application to PTA

For GW observation, we consider the square of the electric field $I_1$. For example, a static conducting wire can only be powered by an electric field. Using three conducting wires oriented in different directions, we can measure three components of the electric field. Eventually, $I_1$ is constructed by these components.

Let us consider an idealized pulsar rotating with exact frequency $\omega_0$. A plane EMW propagating to $\lambda$ has an electric field satisfying the periodic condition,

$$E_a (u + T) = E_a (u),$$ \hspace{1cm} (40)
where \( u(t, x) \equiv t - \lambda \cdot x \) and \( T \equiv 2\pi/\omega_0 \). Then, the electric field is decomposed into the Fourier series by

\[
E_a(u) = \sum_{n=-\infty}^{\infty} \tilde{E}_a^{(n)} e^{-i\omega_0 n u},
\]

where \( \tilde{E}_a^{(n)} \) are Fourier coefficients satisfying \( \tilde{E}_a^{(-n)} = \tilde{E}_a^{(n)*} \). Meanwhile, in the case of an actual pulsar, the electric field will deviate from the above. We deal with this effect as an intrinsic noise. The derived noise in an observation can be reduced by a correlation method with a PTA, which will be introduced later.

A radio telescope measures an EMW with a frequency band from 100 MHz to 100 GHz (Boll et al. 2019). However, the frequency of pulsar \( \omega_0 \) is at most \( \sim 1 \)KHz (Hessels 2006). Accordingly, we only consider the partial frequency range of EMW inside the radio telescope frequency band as

\[
E_a(u) = \sum_{|n|=n_1}^{n_2} \tilde{E}_a^{(n)} e^{-i\omega_0 n u},
\]

where \( n_1 \) and \( n_2 \) are positive integers much larger than 1, and the summation with vertical bars is defined by

\[
\sum_{|n|=n_1}^{n_2} \equiv \sum_{n=n_1}^{n_2} + \sum_{n=-n_2}^{n_1}.
\]

To introduce the geometrical optics, we need the condition that \( \omega_0 = n_1 \omega_\theta \gg \omega_\phi \). In our situation, it is already the case that \( n \gg 1 \). Therefore, GWs of frequencies comparable to \( \omega_\phi \) are in the geometrical optics regime. Keeping the leading order \( O(\omega_\phi/\omega_\theta) \) and the next order \( O(1) \) in Equation (33), we obtain

\[
\Delta(x) \approx \omega_\phi (1 - \kappa^2 \lambda_\phi) (\lambda \cdot x + d).
\]

Note that \( \Delta \) does not depend on \( \omega_0 \) because the leading order vanishes. By substitution of Equation (42) into Equation (37), keeping the leading order and the next order as well, we obtain

\[
\begin{align*}
\delta I_1 &= -\int_{\kappa=\lambda} d^2\kappa \int_{\omega_0}^{\omega_2} \frac{d\omega_\phi}{2\pi} \sum_{|n|=n_1}^{n_2} \sum_{|m|=m_1}^{m_2} \tilde{E}_a^{(n)} \tilde{E}_b^{(m)} \times \\
&\times \left\{ \frac{1}{\Theta} \left[ 1 + n \frac{\omega_0}{\omega_\phi} \right] \tilde{h}_{1+} \chi^2 \lambda^2 \tilde{p}_{1+}^{ab} (1 - e^{i\Delta}) + \tilde{h}_{1-} e^{i\Delta} \right\} \times e^{i\Delta} e^{-i(n+m+\omega)u} ,
\end{align*}
\]

where \( (\omega_1, \omega_2) \) is the GW frequency range, \( \Theta \equiv 1 - \kappa^2 \lambda_\phi \), and \( \tilde{p}_{1+}^{ab} \equiv g^{ab} + n^a n^b - \lambda \chi^2 \lambda \) is the projection orthogonal to \( n \) and \( \lambda \). The integral with vertical bars is defined by

\[
\int_{\omega_0}^{\omega_2} \frac{d\omega_\phi}{\omega_\phi} \equiv \int_{\omega_1}^{\omega_2} \frac{d\omega_\phi}{\omega_\phi} + \int_{\omega_2}^{\omega_1} \frac{d\omega_\phi}{\omega_\phi}.
\]

5. Frequency Window

We set a frequency window of range \( (0, \omega_0/2) \) for \( \delta I_1 = \varphi(t) I_1 + \varphi(t) \delta I_1 + O(\epsilon^4) \). We also assume for simplicity that the GW frequency range is below \( \omega_0/2 \). Then, we obtain \( I_1 \) and \( \delta I_1 \) as

\[
\begin{align*}
\delta I_1 &= -2 \int_{\kappa=\lambda} d^2\kappa \int_{\omega_0}^{\omega_2} \frac{d\omega_\phi}{2\pi} \sum_{n=-n_1}^{n_2} \tilde{E}_a^{(n)} \tilde{E}_b^{(n)*} \times \\
&\times \left\{ \frac{1}{\Theta} \tilde{h}_{1+} \chi^2 \lambda \tilde{p}_{1+}^{ab} (1 - e^{i\Delta}) + \tilde{h}_{1-} e^{i\Delta} \right\} e^{i\Delta}.
\end{align*}
\]

Note that \( \delta I_1 \) is gauge-invariant because \( I_1 \) is constant scalar. This statement is proved by the lemma in Stewart et al. (1974).

To give an understandable picture, we simplify the EMW as a monochromatic plane wave of \( (\omega_\phi, \lambda) \) having the form of

\[
E_a = 2\Re \left( (E_{1+}) p_a + i E_{1-} \delta_{a1} e^{i(\Omega + \Delta t)} \right),
\]

where \( \delta_{a1} \) is determined to make \( E_{1+} \) and \( E_{1-} \) Real, and to make \( \{ n, p, s, \lambda \} \) an orthonormal basis. The ellipticity of the EMW polarization ellipse is defined by

\[
\tan \chi \equiv \frac{E_{1+}}{E_{1-}}.
\]

Likewise, an amplitude of GW \( \tilde{h}_{ab}(\omega_\phi, 0) \) has the form of

\[
\tilde{h}_{ab}(\psi) = (H_+ e_+^a + i H_\times e_+^b) e^{i\psi},
\]

where \( \psi \) is determined to make \( H_+ \) and \( H_\times \), real, and to make \( \{ e^+, e^\times \} \) an orthonormal basis satisfying

\[
\begin{align*}
0 &= e^+_a e^+_b, \\
0 &= e^+_a e^\times_b, \\
\delta_{AB} &= e^+_a e^+_b + e^\times_a e^\times_b,
\end{align*}
\]

for \( A, B \in \{ +, \times \} \). The set of all possible orthonormal bases orthogonal to \( \kappa \) is parameterized by \( \psi \) as

\[
\begin{align*}
e^+_a(\psi) &= \cos (\psi) e^+_a(0) + \sin (\psi) e^\times_a(0), \\
e^\times_a(\psi) &= -\sin (\psi) e^+_a(0) + \cos (\psi) e^\times_a(0),
\end{align*}
\]

where \( \{ e^+(0), e^\times(0) \} \) is a reference basis. We define \( \theta \) and \( \phi \) as the polar and azimuthal angle, respectively, of \( \kappa \) with respect to the spatial frame \( \{ p, s, \lambda \} \) such that \( \kappa^2 \equiv \sin \theta \cos \phi p^a + \sin \theta \sin \phi e^a + \cos \theta e^\lambda \).

Substituting Equations (49) and (51) into Equations (47) and (48), the amplitude of observation \( \tilde{H}(\omega_\phi, \kappa) \) and the complex detector tensor \( \tilde{F}_{ab}^{\omega_\phi}(\omega_\phi, \kappa) \) are defined as

\[
\begin{align*}
\delta I_1 &= \int_{\kappa=\lambda} d^2\kappa \int_{\omega_0}^{\omega_2} \frac{d\omega_\phi}{2\pi} \tilde{h}_{ab} e^{i\Delta}, \\
\tilde{H} &= \tilde{h}_{ab} \tilde{F}_{ab}^{\omega_\phi}, \\
\tilde{F}_{ab}^{\omega_\phi} &= -(\cos^2 \chi^2 \lambda^2 p^a p^b + \sin^2 \chi^2 \delta^a \delta^b) e^{i\Delta} - \frac{1}{\Theta} \chi^2 \delta^a (1 - e^{i\Delta}).
\end{align*}
\]

We consider the complex pattern functions \( F^A \) for \( A \in \{ +, \times \} \) from

\[
\tilde{H} = H_A F^A,
\]

where we used the Einstein summation convention for index \( A \). The strength of observation \( |\tilde{H}|^2 \) is given by

\[
|\tilde{H}|^2 = H_A H_B F^{AB},
\]

where \( F^{AB} \equiv \Re (F^A F^{*B}) \). To see the angular dependency of the strength, we consider the angle average of \( |\tilde{H}|^2 \) over \( \Delta \) and \( \psi \) fixing \( H_A \) because they can be any values from different pulsar
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In order to consider a practical observation, we introduce a noise term \( N \) explicitly in the observation \( \delta l_i / I_1 \) at the origin as

\[
\frac{\delta l_i}{I_1} = H(t, x = 0) + N(t),
\]

where \( H(t, x) \) is right-hand side of Equation (57). The noise \( N \) comes from various sources, e.g., the intrinsic noise of the pulsar, the fluctuation of pulse by the interstellar medium, and the detector noise. We assume that the noise \( N \) is stationary Gaussian and has zero expectation that is achieved by a value shift of the observation. Also, we suppose that noises from two observations are uncorrelated. Then, the correlation between two observations is simplified as

\[
\left\langle \frac{\delta l_i}{I_1} \frac{\delta l_i'}{I_1'} \right\rangle = \int_{\omega_0}^{\omega_2} \frac{d\omega_g}{2\pi} S_0 \int_{\nu = \lambda, \nu'} \frac{d^2 \delta}{4\pi} D^{\kappa \kappa'}_{\nu \nu'} p_{\nu \nu'}^{(c)},
\]

where quantities with prime are defined from a different pulsar. Notice that two pulsars have different frequencies \( \omega_0 \) and \( \omega_0' \). Therefore, we need restriction of the frequency window by \( 0, \min (\omega_0, \omega_0') / 2 \). Accordingly, the observational frequency of GWs is also limited to the same range. In the case of PTA, the observational frequency range for GWs is determined by the lowest rotational frequency among pulsars.

Assuming, \( \omega_0 d \gg 1, \omega_0 d' \gg 1 \), and that distance between two pulsars is much larger than wavelength of the SGW, the correlation at the origin approximately reduces to

\[
\left\langle \frac{\delta l_i}{I_1} \frac{\delta l_i'}{I_1'} \right\rangle \simeq C(\lambda, \lambda') \int_{\omega_0}^{\omega_2} \frac{d\omega_g}{2\pi} S_0,
\]

where \( C(\lambda, \lambda') \) is given by

\[
C(\lambda, \lambda') \equiv \frac{1}{3} \cos \alpha + 4(1 - \cos \alpha) \ln \left( \frac{\sin \alpha}{2} \right).
\]

which is identical to the Hellings & Downs (1983) curve up to factor of 4.

### 6. Frequency Window II

We set a frequency window of range \( (\omega_0 / 2, 3\omega_0 / 2) \) for \( I_1 \). We also assume that the GW frequency range is below to \( \omega_0 / 2 \). Then, we obtain \( I_1 \) and \( \delta l_1 \) as

\[
I_1 = 4\pi \mathcal{R} \left\{ \sum_{n = m_1}^{m_2 - 1} \sum_{\pi = 0}^{\pi'} \mathcal{E}_a^{(n)} \mathcal{E}_b^{(n + 1)} e^{-i\omega_n u} \right\},
\]

\[
\delta l_1 = -2\pi \mathcal{R} \left\{ \sum_{\nu = \lambda} \sum_{n = m_1}^{m_2 - 1} \frac{d^2 \delta}{4\pi} \left[ \frac{d\omega_g}{2\pi} \sum_{n = m_1}^{m_2 - 1} \mathcal{E}_a^{(n)} \mathcal{E}_b^{(n + 1)} \right] \right\}
\]

Note that \( \delta l_1 \) is not gauge-invariant because \( I_1 \) is oscillating with frequency \( \omega_0 \).
To find a gauge-invariant quantity, we consider peak time \( t_k \) of \( I_1 \) for integer \( k \) satisfying
\[
I_1(t_k) = 0,
\]
(73)
where \( \delta \) is the derivative \( n^a \nabla_a \) and \( t_k = t_0 + kT \) because of the periodicity. Referring to Appendix C, the perturbation of the peak time is given by
\[
\delta t_k = -\frac{\delta(\dot{I}_1)(t_k)}{\dot{I}_1(t_k)},
\]
(74)
where \( \delta(\dot{I}_1) \) is easily obtained from the identity \( \delta(\dot{I}_1) = n^a \nabla_a \delta I_1 \) because \( \delta n^a = 0 \). Then, quantity \( \delta T_k = \delta t_{k+1} - \delta t_k \) is gauge-invariant because its background value is constant \( T \). Because of its complexities, we think that \( \delta T_k \) does not have many advantages compared to Equation (48) in frequency window I. However, in limit \( \omega_\perp \ll \omega_0 \), it is simplified drastically and becomes
\[
\frac{\delta T_k}{T} = -\int_{\omega = \lambda} d^2 \kappa \int_{|\omega| = \omega_0} \frac{d\omega}{2\pi} \frac{\tilde{h}_{ab} \tilde{n}^a \tilde{n}^b}{2\Theta} \times (1 - e^{i\Delta}) e^{-i\omega_\perp (\nu - \kappa \xi)},
\]
(75)
which is used in the current GW observation by PTA.

7. Conclusions

We have obtained the perturbation of a plane EMW with arbitrary GWs. In the analysis, we directly solved the vacuum Maxwell equation imposing the boundary condition on the plane perpendicular to the propagating direction of the EMW located at a pulsar. This result was applied to the received EMW in a radio telescope, which partially composes the GWs. Because we did not impose the smallness of the GW frequency compared to the EMP frequency, the observational frequency range of PTA is extended by our results.

For the observation of GW, we introduced two frequency windows. In frequency window I, we proposed a gauge-invariant quantity that was the square of the electric windows. In frequency window I, we proposed a gauge-invariant quantity in frequency window I. However, when the GW frequency is much smaller than the frequency of the pulsar, it is simplified considerably and becomes the formula used in the current GW observation by PTA.

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Appendix A

Homogeneous Solution of the Perturbed Maxwell Equation

From the form of homogeneous solution \( \tilde{X}_h \) in Equation (30), the field strength \( Y^h \) is given by
\[
Y^h_{\ab} = 2\nabla_{[a} \tilde{X}_h^{b]} = \int_{\rho \lambda \mu} d^2 \mu \int_{-\infty}^{\infty} d\omega \frac{d^2 \tilde{r}_\ab_{\ab} (\omega, \mu) e^{i(\rho \lambda \mu + \omega_x x + \omega_y y + \omega_z z)}}{2\pi},
\]
(A1)
where \( \tilde{r}_\ab_{\ab} = 2i\mu [a \tilde{X}_h^b] \). Then, the boundary condition becomes
\[
0 = Y^P_{\ab}(t, y) + \tilde{Y}^h_{\ab}(t, y) = \int_y dV \tilde{Y}^P_{\ab} e^{i(p + \phi)} + \int_{\rho \lambda \mu} d\rho d^2 \mu \int_{-\infty}^{\infty} d\omega \frac{d^2 \tilde{Y}^h_{\ab} e^{i(\rho \lambda \mu)}}{2\pi},
\]
(A2)
for all \( (t, y) \) such that \( \lambda \cdot y = -d \). We introduce a spatial Cartesian coordinate \([x, y, z]\) such that unit vector \( \hat{z} \) is identical to \( \lambda \). Therefore, the boundary is described by
\[
y = x\hat{x} + y\hat{y} - d\hat{z}.
\]
(A3)

By the Fourier transformation for variables \((t, x, y)\), we obtain \( \tilde{r}_\ab \) as
\[
\tilde{Y}^h_{\ab}(\omega, \mu) = -\left(\frac{\omega}{\Theta}\right)^2 \lambda \cdot \mu e^{i\omega_\perp \lambda \mu d} \int_{-\infty}^{\infty} dt e^{i\omega_\perp t} \times \int_{-\infty}^{\infty} dx e^{-i\omega_x x} \int_{-\infty}^{\infty} dy e^{-i\omega_y y} \tilde{Y}^P_{\ab} e^{i\omega_\perp(-1+\kappa)\hat{z} \cdot \omega_\perp x} e^{i\omega_\perp(-1-\kappa)\hat{z} \cdot \omega_\perp x}.
\]
(A4)

Using the Fourier transformation of the Dirac delta function, \( \tilde{Y}^h \) becomes
\[
\tilde{Y}^h_{\ab} = -2\pi \omega_\perp^2 \lambda \cdot \mu \int_y dV \tilde{Y}^P_{\ab} e^{-i\omega_\perp \lambda d} e^{-i\omega_\perp \mu d} \times \delta(\omega_\perp - \omega_\perp) \times \delta(\hat{z} \cdot (\omega_\perp x - \omega_\perp \mu)) = \delta(\hat{z} \cdot (\omega_\perp \mu - \omega_\perp \mu)),
\]
(A5)
where \( \delta \) is Dirac delta function. By substitution of the above into Equation (A1), we obtain Equation (32).

Appendix B

Antenna Pattern of the Observation with Frequency Window I

\[
\left< \mathcal{F}_{\psi, \Delta} \right> = \frac{1}{2} \left( \cos^4 \chi - 2(2 \cos^4 \chi - \cos^2 \chi) \sin^2 \phi 
+ (4 \cos^4 \chi - 4 \cos^2 \chi + 1) \sin^4 \phi \right) \sin^4 \theta 
- [(2 \cos^2 \chi - 1) \sin^2 \phi - \cos^2 \chi] \cos \theta 
+ 2 \cos^4 \chi - 2 \cos^2 \chi + 1) \sin^2 \theta 
+ (2(2 \cos^2 \chi - 1) \sin^2 \phi - 2 \cos^2 \chi + 3) \cos \theta 
+ 2(2 \cos^2 \chi - 1) \sin^2 \phi 
+ 2 \cos^4 \chi - 4 \cos^2 \chi + \frac{7}{2}
\]
(B1)
Appendix C

Perturbation of Peak Time in the Observation with Frequency Window II

In the perturbed spacetime, \( \epsilon \tilde{t}_k \) is determined from

\[
\epsilon \tilde{t}_k = 0. \tag{C1}
\]

Then,

\[
0 = \epsilon \tilde{\dot{t}}_k - \tilde{\dot{t}}_k + \epsilon \tilde{\ddot{t}}_k - \tilde{\ddot{t}}_k
\]

\[
= \epsilon \tilde{\dot{t}}_k (\tilde{t}_k - t_k) + \epsilon \delta \tilde{t}_k + O(\epsilon^2)
\]

\[
= \epsilon (\tilde{t}_k - t_k) \delta + O(\epsilon^2). \tag{C2}
\]

From above, we obtain Equation (74).

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