Di–jet asymmetry and wave turbulence

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Abstract
We describe a new physical picture for the fragmentation of an energetic jet propagating through a dense QCD medium, which emerges from perturbative QCD and has the potential to explain the di–jet asymmetry observed in Pb–Pb collisions at the LHC. The central ingredient in this picture is the phenomenon of wave turbulence, which provides a very efficient mechanism for the transport of energy towards the medium, via many soft particles which propagate at large angles with respect to the jet axis.

Keywords: Perturbative QCD, Heavy Ion Collisions, Jet Quenching, Wave Turbulence

1. Introduction
One of the most interesting discoveries of the heavy ion program at the LHC is the phenomenon known as di–jet asymmetry — a strong imbalance between the energies of two energetic back–to–back jets produced in an ultrarelativistic nucleus–nucleus collision. This is attributed to the effect of the interactions of one of the two jets with the dense QCD matter that it traverses, while the other leaves the system unaffected. Originally identified [1] as missing transverse energy within a conventionally defined ‘jet’ with small angular opening (the same as for the trigger jet), this phenomenon has been subsequently shown, via more detailed studies [2, 3], to consist in the transport of a part of the jet energy towards large angles and by soft particles. The total amount of energy thus transferred from small to large angles (about 10 to 20 GeV) is considerably larger than the typical transverse momentum, ∼ 1 GeV, of a parton in the medium, so in that sense the effect is large and potentially non–perturbative.

Yet, there exists a mechanism within perturbative QCD which naturally leads to energy loss at large angles: the BDMPSZ mechanism for medium–induced gluon radiation [4, 5]. Most previous studies within this approach have focused on the energy lost by the leading particle, as controlled by relatively hard emissions at small angles. More recently, in the wake of the LHC data, the attention has been shifted towards softer emissions (which occur at large angles) and, more generally, towards a global understanding of the in–medium jet evolution. This raised the difficulty of including the effects of multiple gluon branchings, which become important for the soft emissions. After first studies of interference phenomena, which exhibited the role of medium rescattering in destroying the colour coherence between partonic sources [6, 7], we have recently demonstrated [8] that the in–medium jet evolution can be reformulated (to the perturbative accuracy of interest) as a classical stochastic process. This allows for systematic numerical studies via Monte Carlo methods, like for jets fragmenting in the vacuum. It also allows for analytic studies, at least for particular problems, like the recent study of the energy flow throughout the cascade in Ref. [9]. This study revealed a remarkable phenomenon, which is new in the context of QCD and which, besides its conceptual interest, has also the potential to explain the LHC data for di-jet asymmetry: the wave turbulence. The developments in Refs. [8, 9] will be briefly reviewed in what follows, with emphasis on the physical picture of wave turbulence.

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2. Medium–induced radiation à la BDMPSZ

The BDMPSZ mechanism relates the radiative energy loss by an energetic parton propagating through a dense QCD medium (‘quark–gluon plasma’) to the transverse momentum broadening via scattering off the medium constituents. A central concept is the formation time $\tau_n(\omega)$ — the typical times it takes a gluon with energy $\omega \ll E$ to be emitted. ($E$ is the energy of the original parton, a.k.a. the ‘leading particle.’) The gluon starts as a virtual fluctuation which moves away from its parent parton via quantum diffusion: the transverse separation $b_\perp$ grows with time as $b^2_\perp \sim \Delta t/\omega$. The gluon can be considered as ‘formed’ when it loses coherence w.r.t to its source, meaning that $b_\perp$ is at least as large as the gluon transverse wavelength $\lambda_\perp = 1/k_\perp$. But the gluon transverse momentum $k_\perp$ is itself increasing with time, via collisions which add random kicks $\Delta k_\perp$ at a rate given by the jet quenching parameter $\hat{q}$: $\Delta k^2_\perp \sim \hat{q}\Delta t$. The ‘formation’ condition, $b_\perp \gtrsim 1/\Delta k_\perp$ for $\Delta t \gtrsim \tau_n$, implies

$$\tau_n(\omega) \approx \sqrt{\frac{2\omega}{\hat{q}}}, \quad k^2_\perp \approx \hat{q}\tau_n(\omega) \approx (2\omega \hat{q})^{1/2}, \quad \theta_n \approx \frac{k_n}{\hat{q}} \approx \left(\frac{2\hat{q}}{\omega}\right)^{1/4},$$

(1)

where $k_n$ and $\theta_n$ are the typical values of the gluon transverse momentum and its emission angle at the time of formation. Eq. (1) applies as long as $\ell \ll \tau_n(\omega) < L$, where $L$ is the length of the medium and $\ell$ is the mean free path between successive collisions. The second inequality implies an upper limit on the energy of a gluon that can be emitted via this mechanism, and hence a lower limit on the emission angle: $\omega \lesssim \omega_c \equiv \hat{q}L^2/2$ and $\theta_n \gtrsim \theta_c \equiv 2/(\hat{q}L^3)^{1/3}$. The BDMPSZ regime corresponds to $\hat{q}L^3 \gg 1$ and hence $\theta_c \ll 1$. Choosing $\hat{q} = 1$ GeV$^2$/fm (the weak coupling estimate $[4]$ for a QGP with temperature $T = 250$ MeV) and $L = 4$ fm, one finds $\omega_c \approx 40$ GeV and $\theta_c \approx 0.05$.

It is furthermore easy to deduce a parametric estimate for the spectrum of the emitted gluons (at least for the relatively soft gluons with $\omega \ll \omega_c$): this is the product of the standard bremsstrahlung spectrum for the emission of a single gluon times the average number of emissions which can occur within the plasma, that is $L/\tau_n$:

$$\frac{dN}{d\omega} \omega \frac{dN}{d\omega} \approx \frac{\alpha_s N_c}{\pi} \frac{L}{\tau_n(\omega)} = \hat{\alpha} \sqrt{\frac{\omega_c}{\omega}},$$

(2)

with $\hat{\alpha} \equiv \alpha_s N_c/\pi$. Note that the number of emissions $L/\tau_n$ is much smaller than the number of collisions $L/\ell$, since several successive collisions can coherently contribute to a single emission; this is known as the LPM effect (Landau, Pomeranchuk, Migdal) and leads to the characteristic $\sim 1/\sqrt{\omega}$ dependence of the BDMPSZ spectrum $[2]$. By integrating this spectrum over all the energies $\omega \leq \omega_c$, one estimates to the total energy loss by the leading particle:

$$\Delta E_{\text{tot}} = \int_{\omega_c}^{\infty} d\omega \omega \frac{dN}{d\omega} \sim \hat{\alpha} \omega_c \sim \hat{\alpha} \hat{q}L^2.$$

(3)

The above integral is dominated by its upper limit: the total energy loss is controlled by the hardest possible emissions, those with energies $\omega \sim \omega_c$. Such hard emissions, however, propagate at small angles $\theta \sim \theta_c$ w.r.t. to the jet axis, so they remain a part of the conventionally defined ‘jet’ and thus cannot contribute to the di-jet asymmetry. On the other hand, the soft gluons with $\omega \ll \omega_c$ are emitted directly at large angles $\theta \gg \theta_c$ and, moreover, these angles are further enhanced after emission via rescattering in the medium: a gluon which crosses the medium over a distance $\sim L$ acquires a transverse momentum broadening $k^2_\perp \sim \hat{q}L \equiv Q^2$, which for $\omega \ll \omega_c$ is in fact larger than the respective momentum acquired during formation: $Q^2 \gg k^2_\perp(\omega)$. Accordingly, a soft gluon emerges at a typical angle $\theta(\omega) \sim Q/\omega$ which is even larger than $\theta_n(\omega)$ — and of course much larger than $\theta_c$. It is interesting to try and estimate the typical energy which would be transported in this way at angles larger than a given value $\theta_0$, with $\theta_0 \gg \theta_c$:

$$\Delta E(\theta > \theta_0) = \int_{\omega_0}^{\infty} d\omega \omega \frac{dN}{d\omega} \sim \hat{\alpha} \sqrt{\omega_0 \omega_0} \propto \frac{1}{\sqrt{\theta_0}} \quad \text{with} \quad \omega_0 \equiv \frac{Q}{\theta_0}.$$

(4)

This is only a small fraction $(\theta_0/\theta_c)^{1/2}$ of the total energy loss $[3]$, but it is lost at large angles, so it counts for the energy loss by the jet. Yet, Eq. (4) does not show the right trend to explain the LHC data: this estimate decreases quite

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1. The ‘transverse directions’ refer to the 2–dimensional plane orthogonal to the 3–momentum of the leading particle (the ‘longitudinal axis’).
fast with increasing $\theta_0$, thus predicting that most of the energy loss should lie just outside the jet cone (and thus be easily recovered when gradually increasing the jet angular opening). This contradicts the results of a detailed analysis by CMS [2], which show that most of the ‘missing’ energy is deposited at very large angles $\theta > 0.8$.

However, the previous argument misses an important ingredient: the gluon spectrum [2] is a measure of the probability for emitting a gluon via the BDMPSZ mechanism. For $\omega \sim \omega_c$, this probability if of $O(\bar{\alpha})$, showing that hard emissions are relatively rare events. But when $\omega \sim \bar{\alpha}^2 \omega_c$, this probability becomes of $O(1)$, meaning that the soft emissions with $\omega \lesssim \bar{\alpha}^2 \omega_c$ can occur abundantly, event-by-event. For such small energies, the result [2] must be corrected to account for multiple emissions and, especially, multiple branchings of the soft emitted gluons [3,9].

3. Democratic branchings and wave turbulence

Multiple soft emissions by the leading particle have already been discussed by BDMPSZ [10]: they change the energy distribution of the leading hadron, but not the inclusive spectrum [2] for the soft radiation, nor its (unrealistic) prediction for the angular distribution of the radiation, Eq. [4]. What is more important for the present purposes, is the fate of the soft gluons after being emitted. The probability for a gluon with energy $\omega$ to split into two daughter gluons with energy fractions $x$ and $1 - x$ is obtained by replacing $\omega \rightarrow x(1 - x)\omega$ in Eq. [2]. Hence, when $\omega \sim \bar{\alpha}^2 \omega_c$, this probability is of $O(1)$ for generic values of $x$ : the soft parent gluon is certain to split and its branching is quasi–democratic (i.e. unbiased towards the endpoints at $x = 0$ and $x = 1$ of the distribution in $x$) [9]. For even softer energies, $\omega \ll \bar{\alpha}^2 \omega_c$, the lifetime $\Delta \tau$ of a gluon generation, i.e. the time interval between two successive branchings, is considerably smaller than the medium size $\Delta \tau \sim (1/\bar{\alpha})\tau_{\omega_c}(\omega) \ll L$. Hence, such soft gluons undergo successive branchings leading to gluon cascades. Being quasi–democratic, these branchings efficiently degrade the energy to smaller and smaller values of $x$. And since the gluons produced by these branchings are softer and softer, they get easily deviated by the collisions in the medium to larger and larger angles (see Fig. 1). Thus, the quasi–democratic and quasi–deterministic cascade provides a very efficient mechanism for transporting energy at large angles. This mechanism is a manifestation of a phenomenon well known in other fields of physics: the wave turbulence [11][12].

Before we characterize this new phenomenon in mode detail, let us describe the formalism which allows us to treat multiple branching [8]. In principle, one can construct a parton cascade by iterating the $1 \rightarrow 2$ ‘vertex’ for parton splitting, which here is the BDMPSZ spectrum [2]. This would certainly be the correct procedure for a classical branching process. It turns out to that this is also the right procedure for the quantum problem at hand, but in this case such a procedure is highly non-trivial, as it could be invalidated by interference phenomena. Recall e.g. the evolution of a jet via successive parton branching in the vacuum: the daughter partons produced by one splitting remain ‘color-coherent’ with each other (their total color charge is fixed to be equal to the respective charge of the parent parton) until the next splitting of any of them. This coherence implies interference effects between the emissions by the two daughter partons, which in that context are well known to be important: they lead to the angular ordering of successive emissions, which ultimately favors jet collimation [13].

\footnote{This estimate for $\Delta \tau$ follows from the condition that the emission probability $P(\omega) = \bar{\alpha}(\Delta \tau/\tau_{\omega_c}(\omega))$ become of $O(1)$.}
Remarkably, the situation in that respect appears to be simpler for parton branching in the medium [6–8]: the daughter partons efficiently randomize their color charges via rescattering in the medium and thus lose their mutual color coherence already during the formation process [8]. Accordingly, the interference effects are suppressed (as compared to the independent branchings) by a phase–space factor $\tau_s(\omega)/L$, which is small whenever $\omega \ll \omega_c$. This implies that the successive medium–induced emissions can be effectively treated as independent of each other and taken into account via a probabilistic branching process, in which the BDMPSZ spectrum plays the role of a branching rate. Such a process has already been used in applications to phenomenology, albeit on a heuristic basis [10, 14, 15].

The general branching process is a Markovian process in 3+1 dimensions which describes the gluon distribution in energy ($\omega$) and transverse momentum ($k$, ), and its evolution when increasing the medium size $L$ (see Ref. [8] for details). This process is well suited for numerical studies via Monte-Carlo simulations. But analytic results have been obtained too [9], for a simplified process in 1+1 dimensions which describes the energy distribution alone. These results lead to an interesting physical picture, that of wave turbulence, that we shall now describe.

To that aim, it is convenient to focus on the gluon spectrum $D(x, \tau) \equiv x(dN/dx)$, where $x \equiv \omega/E$ is the energy fraction carried by a gluon from the jet and the ‘evolution time’ $\tau$ is the medium size in dimensionless units, as defined in the equation below. The quantity $D(x, \tau)$ obeys a ‘rate’ equation [9, 12, 15], which reads, schematically,

$$\frac{\partial D(x, \tau)}{\partial \tau} = I[D(x, \tau)] = \text{Gain}[D] - \text{Loss}[D], \quad \text{with} \quad \tau = \bar{a} \sqrt{\frac{\alpha}{E}} L. \quad (5)$$

where the ‘collision term’ $I[D]$ (a linear functional of $D(x, \tau)$) is the difference between a ‘gain’ term and a ‘loss’ term, as illustrated in Fig. 2. The ‘gain’ term describes the increase in the number of gluons with a given $x$ via radiation from gluons with a larger $x' = x/z$, with any $x < z < 1$. The ‘loss’ term expresses the decrease in the number of gluons at $x$ via their decay $x \rightarrow z, (1 - z)x$, with any $0 < z < 1$. By construction, the first iteration of this equation coincides with the BDMPSZ spectrum [2], which in our new notations reads (for relatively soft gluons with $x \ll 1$)

$$D^{(1)}(x \ll 1, \tau) \approx \frac{\tau}{\sqrt{x}}. \quad (6)$$

This approximation breaks down when $D^{(1)}(x, \tau) \sim O(1)$, meaning for $x \lesssim \tau^2$ (the familiar condition $\omega \lesssim \bar{a}^2 \omega_c$ in these new notations). In this non–perturbative regime at small $x$, one needs an exact result which resums multiple branchings to all orders. Such a solution has been presented in [9] and reads (for $x \ll 1$ once again)

$$D(x \ll 1, \tau) \approx \frac{\tau}{\sqrt{x}} e^{-\pi \tau^2}. \quad (7)$$

Formally, one can read Eq. (7) as ‘BDMPSZ spectrum by the leading particle $x$ survival probability for the latter’. However, unlike Eq. (6), the spectrum (7) also includes the effects of multiple branchings. That is, the energy in a given bin with $x \ll 1$ is produced both via direct radiation by the leading particle, and via energy transfer from the higher bins at $x' > x$, through successive splittings. The persistence of the scaling spectrum $D_s \equiv 1/\sqrt{x}$ under this evolution demonstrates that this spectrum is a fixed point of the collision kernel: $I[D_s](x) = 0$ for $x \ll 1$. In turn, this means that the rate for energy transfer from one parton generation to the next one is independent of the generation (i.e. of $x$). This property is the distinguished signature of wave turbulence [11, 12]: via successive splittings, the energy flows from large $x$ to small $x$ without accumulating at any intermediate value of $x$. It rather accumulates into a condensate at $x = 0$.

Since there is only a finite amount of energy available (the energy $E$ of the leading particle),

![Figure 2. The change in the gluon spectrum $D(x, \tau) \equiv x(dN/dx)$ due to one additional branching $g \rightarrow gg$.](image-url)
it follows that the total energy which is contained in the spectrum (i.e. in the bins at $0 < x \leq 1$) must decrease with time. Indeed, a direct calculation yields [9]

$$
\int_0^1 dx D(x, \tau) = e^{-\pi \tau^2} \implies E_{\text{flow}}(\tau) \equiv 1 - \int_0^1 dx D(x, \tau) = 1 - e^{-\pi \tau^2}.
$$

The quantity $E_{\text{flow}}(\tau)$ is the energy fraction carried away by the flow and which formally ends up in the condensate. As we shall shortly discuss, this energy is in fact transferred to the medium, at very large angles.

These considerations are illustrated in Fig. 3, which shows the spectrum for various values of $\tau$. At small $\tau \ll 1/\sqrt{\pi}$, the small-$x$ part of the spectrum rises linearly with $\tau$, as shown by Eq. (6) (see the full lines in Fig. 3). At the same time, the leading–particle peak, which originally was a $\delta$–function at $x = 1$, moves at $1 - x = \pi \tau^2$ and becomes broader. For larger times $\tau \gtrsim 1/\sqrt{\pi}$, the source disappears and the spectrum is globally suppressed by the Gaussian factor in (7), yet, the scaling behavior $D \approx 1/\sqrt{\pi}$ is still visible at small $x$ (see the dotted lines in Fig. 3).

4. Energy loss at large angles

The emergence of a flow component $E_{\text{flow}}(\tau)$ in the energy transport down the cascade explains one of the main characteristics of (wave) turbulence: this is a very efficient mechanism for transferring energy between two widely separated scales --- here, from $x = 1$ down to $x = 0$. To see this, let us compute the energy transferred after time $\tau$ below a given value $x_0 \ll 1$. This includes two components: the energy which is contained in the spectrum, in the bins at $0 < x < x_0$, and the flow energy, which is independent of $x_0$ (since accumulated at $x = 0$). Thus,

$$
E(x \leq x_0, \tau) = 2\tau \sqrt{x_0} e^{-\pi \tau^2} + (1 - e^{-\pi \tau^2}) \approx 2\tau \sqrt{x_0} + \pi \tau^2,
$$

where the second, approximate, equality holds for $\pi \tau^2 \ll 1$. Note that, even for small times, the flow component dominates over the non–flow one provided $x_0 < \tau^2$, that is, in the non–perturbative regime at small $x$ where the multiple branching becomes important. For larger times $\tau \gtrsim 1/\sqrt{\pi}$, the flow piece dominates for any $x_0$ and approaches unity, meaning that the whole energy can be lost towards arbitrarily soft quanta, which propagate at arbitrarily large angles.

To understand how remarkable this situation is, let us compare it with the more familiar example of the DGLAP evolution (say, for a jet in the vacuum), where there is no flow. (The DGLAP equation too can be viewed as a ‘rate equation’), cf. Eq. (5), with the logarithm of the virtuality playing the role of the ‘evolution time’.) In that case, the splittings are typically asymmetric ($x \to 0$ or $x \to 1$), leading to a rapid increase in the number of gluons at small $x$. Yet most of the energy remains in the few partons with relatively large values of $x$. Indeed, for the DGLAP cascade, the energy is fully contained within the spectrum (no flow) and the energy sum-rule $\int_0^1 dx D(x, \tau) = 1$ is dominated by the higher values of $x$ in the support of the function $D(x, \tau)$ at time $\tau$. Conversely, one can show that a necessary condition for the emergence of (turbulent) flow is quasi–democratic branching [11].
So far, we have assumed that the evolution remains unchanged down to \( x = 0 \), but physically this is not the case: when the gluon energies become as low as the typical energy scale in the medium — the ‘temperature’ \( T \sim 1 \) GeV —, the gluons ‘thermalize’ and disappear from the jet. The energy which is thus transferred to the medium (and hence lost by the jet) can be evaluated by replacing \( x_0 \to x_0 \equiv T/E \) in Eq. (9). This energy loss is independent of the details of the thermalization mechanism and even of the medium temperature (since dominated by the flow component, as we shall shortly see). This \textit{universality} is the hallmark of turbulence: the rate for energy transfer at the lower end of the cascade is fixed by the turbulent flow alone, and thus is independent of the specific mechanism for dissipation.

To make contact with the phenomenology, we notice that for a jet with \( E = 100 \) GeV \( \approx 2\omega_c \), Eq. (5) implies \( \tau \equiv \bar{\alpha} \sqrt{2\omega_c/\bar{E}} \equiv \bar{\alpha} \approx 0.3 \), which is quite small. The flow piece in Eq. (9), which is independent of \( x_0 \), dominates over the non–flow piece for any \( x_0 < \tau^2 \approx 0.1 \), a value much larger than the thermalization scale \( x_0 \approx 0.01 \). Thus, in evaluating the energy loss via thermalization, one can keep only the flow component in the small–\( \tau \) version of Eq. (9), as anticipated. Returning to physical units, one finds

\[
\Delta E_{th} \approx E \Delta \tau \approx \nu \bar{\alpha}^2 \omega_c, \tag{10}
\]

where \( \nu \) would be equal to 2\( \pi \) according to Eq. (9), but a more precise calculation yields \( \nu \approx 4.96 \) [9]. This is formally suppressed by an additional power of \( \bar{\alpha} \) as compared to the total energy loss by the leading particle, Eq. (3), which we recall is controlled by hard gluon emissions (\( \omega \sim \omega_c \)) at small angles (\( \theta \sim \tau \)). However, the flow contribution in Eq. (10) is numerically quite large (because \( \nu \) is a reasonably large number) and moreover this is associated with soft emissions at large angles. It thus has the potential to explain the LHC data for di–jet asymmetry.

With \( \omega_c \approx 40 \) GeV, Eq. (10) predicts \( \Delta E_{th} \approx 20 \) GeV, a value that compares well with the experimental observations. This energy is carried by the relatively soft quanta at the lower end of the cascade (\( x \sim x_0 \)), that is, by particles whose energies are comparable to the ‘temperature’ \( T \) of the medium. Precisely because they are so soft, these particles propagate at very large angles with respect to the jet axis. To obtain a parametric estimate for these angles, we recall that a gluon with energy \( \omega \leq \bar{\alpha}^2 \omega_c \) has a lifetime \( \Delta \tau \sim (1/\bar{\alpha})\tau_c(\omega) \leq L \), during which it accumulates a transverse momentum broadening \( k_T^2 \sim \bar{\alpha}^2 \Delta \tau = (1/\bar{\alpha})k_T^2 \), via collisions in the medium. Accordingly, this gluon should emerge at an angle (compare to Eq. (1))

\[
\theta(\omega) \sim \frac{1}{\bar{\alpha}} \theta_c(\omega) \sim \frac{1}{\sqrt{\bar{\alpha}}} \left(\frac{2\bar{\alpha}}{\bar{\omega}}\right)^{1/4}. \tag{11}
\]

A \textit{lower limit} on this angle is obtained by choosing \( \omega \sim \bar{\alpha}^2 \omega_c \approx 4 \) GeV (the non–perturbative energy scale below which develops the turbulent cascade); this yields \( \theta \approx 0.5 \). But for a typical gluon with \( \omega \sim T \sim 1 \pm 2 \) GeV, this angle is even larger: \( \theta \sim O(1) \). This is in qualitative and even quantitative agreement with the detailed analyses of the data by CMS [2] and ATLAS [3], which show that most of the ‘missing’ energy lies at very large angles \( \theta \geq 0.8 \).

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