Asymptotic geometry and growth of conjugacy classes of nonpositively curved manifolds

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Abstract

Let $X$ be a Hadamard manifold and $\Gamma \subset \text{Isom}(X)$ a discrete group of isometries which contains an axial isometry without invariant flat half plane. We study the behavior of conformal densities on the limit set of $\Gamma$ in order to derive a new asymptotic estimate for the growth rate of closed geodesics in not necessarily compact or finite volume manifolds.

1 Introduction

Let $M$ be a complete Riemannian manifold of nonpositive sectional curvature, and denote by $P(t)$ the number of primitive closed geodesics in $M$ of period $\leq t$ modulo free homotopy. Our main interest in this paper is the asymptotic behavior of this function.

In the case of a negative upper bound on the sectional curvature of $M$, there is only one closed geodesic in each free homotopy class. Using the ergodic theory of the geodesic flow, G. A. Margulis ([M], [MS]) proved that for compact manifolds of pinched negative curvature with volume entropy $h$

$$P(t)hte^{-ht} \to 1 \quad \text{as} \quad t \to \infty.$$ 

Recently, M. Coornaert and G. Knieper established an analogous generalization of Margulis’ result for compact quotients of Gromov hyperbolic metric spaces ([CK, Theorem 1.1]).

G. Knieper ([K1], [K2], [K3]) extended the theory to compact geometric rank one manifolds $M$. If $h$ denotes the volume entropy of $M$, and $P_{hyp}(t)$ the number of closed geodesics of period $\leq t$ which do not admit a perpendicular parallel Jacobi field, he proved the existence of constants $a > 1$ and $t_0 > 0$ such that

$$\frac{1}{a} t e^{ht} \leq P_{hyp}(t) \leq P(t) \leq a t e^{ht}$$

for $t > t_0$ ([K3, Theorem 5.6.2]).

The purpose of this paper is a partial generalization of this result to a larger class of manifolds $M$. Let $X$ be the Riemannian universal covering manifold of $M$, and $\Gamma$ the group of deck transformations. Then $X$ is a Hadamard manifold, $M = X/\Gamma$, and $\Gamma$ is a discrete torsion free subgroup of the isometry group $\text{Isom}(X)$ of $X$. Let $\partial X$ denote the geometric boundary of $X$ endowed with the cone topology. We will only require that $\Gamma$ contains an axial isometry which translates a geodesic without flat half plane in $X$ (see [B1] for precise definitions) and does not possess a global fixed point in $\partial X$. We emphasize that we do not assume the manifold $M$ to be compact or of finite volume. Instead of the volume entropy, we will therefore consider the critical exponent $\delta(\Gamma)$ of $\Gamma$ which is defined as the exponent of convergence of the Poincaré series $P_\times(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}$. Furthermore, if $X$ does
not admit a quotient of finite volume, then the rank rigidity result of Ballmann ([B2]) and Burns-Spatzier ([BS]) does not require $M$ to be a geometric rank one manifold, hence every geodesic in $X$ may bound a flat strip of finite width.

If $W$ is an open set in $X$, we denote $\mathcal{P}(W)$ the set of free homotopy classes of closed geodesics in $M$ which possess a representative with a lift to $X$ intersecting $W$ nontrivially, and $\mathcal{P}_h(W) \subseteq \mathcal{P}(W)$ the subset of free homotopy classes containing a representative with a lift which intersects $W$ and does not bound a flat half plane. Notice that due to the possible occurrence of flat strips along each geodesic in $X$ there can be infinitely many closed geodesics in every free homotopy class of closed geodesics in $M$. However, all closed geodesics in the same free homotopy class have the same period. We will be interested in the number of elements of period $\leq t$ in $\mathcal{P}(W)$ and $\mathcal{P}_h(W)$ which we denote by $P(t; W)$ and $P_h(t; W)$ respectively.

Our main result is the following:

**Theorem 1** Let $W$ be a sufficiently large bounded open set in $X$, and $\Gamma$ as above. Then

$$\delta(\Gamma) = \lim_{t \to \infty} \left(\frac{1}{t} \log P(t; W)\right) = \lim_{t \to \infty} \left(\frac{1}{t} \log P_h(t; W)\right).$$

If $\Gamma$ is “weakly cocompact” (see Definition 3.9), then there exist constants $b > 1$, $R > 0$ such that for $t > R$

$$\frac{1}{b} e^{\delta(\Gamma)t} \leq P_h(t; W) \leq P(t; W) \leq be^{\delta(\Gamma)t}.$$ 

Although we extend some of the methods from [K1] and [K2] to noncompact manifolds, our proof of the lower bound in Theorem 1 avoids the use of Lemma 2.7 in [K1]. Instead of that we make use of Theorem 2 and Corollary 4.3 below.

Fix $o \in X$ and put $N_\Gamma(R) := \# \{ \gamma \in \Gamma \mid d(o, \gamma o) < R \}$. A large part of the present paper is devoted to the study of the behavior of conformal densities on the geometric boundary $\partial X$ and their relation to $N_\Gamma(R)$. Generalizing Lemma 4.4 in [K2], we derive that every $\alpha$-dimensional conformal density satisfies $\alpha \geq \delta(\Gamma)$. Using this fact and similar arguments as T. Roblin in [R], we are able to prove

**Theorem 2** If $\Gamma$ is as above, then

$$\lim_{R \to \infty} \left(\frac{1}{R} \log N_\Gamma(R)\right)$$

exists and equals $\delta(\Gamma)$.

This theorem extends the main theorem in [R] to manifolds of nonpositive curvature as above which are not necessarily CAT$(-1)$. Furthermore, its Corollary 4.3 will be one of the key ingredients in our proof of the lower bound of Theorem 1.

The paper is organized as follows: In section 2 we recall some basic facts about Hadamard manifolds and discrete groups of isometries which contain an axial isometry without flat half plane. In section 3 we introduce the concept of conformal densities and prove a so-called shadow lemma, Theorem 3.6. This theorem gives an idea of the local behavior of conformal densities and allows to deduce asymptotic bounds on the exponential growth rate of the number of orbit points of $\Gamma$. Section 4 is devoted to the proof of Theorem 2 and its corollaries, which will be a key ingredient in the proof of the lower bound of Theorem 1. In section 5, we investigate the asymptotic growth rate of geometrically distinct closed geodesics modulo free homotopy in a complete Riemannian manifold of nonpositive curvature and prove Theorem 1.

## 2 Axial isometries of Hadamard manifolds

In this section we recall a few properties of Hadamard manifolds which possess a geodesic without flat half plane. Most of the material can be found in ([B1]).

Let $X$ be a complete simply connected Riemannian manifold of nonpositive sectional curvature. The geometric boundary $\partial X$ of $X$ is the set of equivalence classes of asymptotic
geodesic rays endowed with the cone topology (see e.g. [B3, chapter II]). This boundary is homeomorphic to the unit tangent space of an arbitrary point in $X$, and $\overline{X} := X \cup \partial X$ is homeomorphic to a closed $N$–ball in $\mathbb{R}^N$, where $N = \dim X$. Moreover, the isometry group of $X$ has a natural action by homeomorphisms on the geometric boundary.

All geodesics are assumed to have unit speed. For $x \in X$ and $z \in \overline{X}$ we denote by $\sigma_{x,z}$ the unique unit speed geodesic emanating from $x$ which contains $z$. Let $D := \{(x, x) \mid x \in X\}$ denote the diagonal in $X$. For later use we introduce the continuous projection

$$pr : \overline{X} \times X \setminus D \rightarrow \partial X$$

$$(z, x) \mapsto \sigma_{x,z}(-\infty).$$

We say that two points $\xi, \eta \in \partial X$ can be joined by a geodesic, if there exists a geodesic $\sigma$ with extremities $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta$. A geodesic $\sigma : \mathbb{R} \rightarrow X$ is said to bound a flat strip of width $c \geq 0$ if there exists a totally geodesic isometric embedding $i : [0, c] \times \mathbb{R} \rightarrow X$ such that $i(0, t) = \sigma(t)$ for any $t \in \mathbb{R}$.

**Definition 2.1** A geodesic $\sigma$ in $X$ is called **hyperbolic** if it does not bound a flat strip of infinite width. In this case we call

$$c(\sigma) := \sup\{c \geq 0 \mid \sigma \text{ bounds a flat strip of width } c\}$$

the **width** of the hyperbolic geodesic $\sigma$.

Notice that the definition of hyperbolic geodesics given in [K2] is more restrictive than Definition 2.1 above, since we allow that hyperbolic geodesics bound a flat strip of finite width. We refer to geodesics satisfying Knieper's hyperbolicity condition as rank one geodesics and remark that the width of a rank one geodesic is zero. The following lemma is a direct consequence of Lemma 2.1 in [B1] and its proof.

**Lemma 2.2** Let $\sigma$ be a hyperbolic geodesic of width $c(\sigma) \geq 0$ with extremities $\sigma(-\infty)$ and $\sigma(\infty)$. Then for all $\varepsilon > 0$ there exist neighborhoods $U, V \subset \partial X$ of $\sigma(-\infty)$, $\sigma(\infty)$ with closures $\overline{U}$, $\overline{V}$ homeomorphic to closed balls and $\overline{U} \cap \overline{V} = \emptyset$ such that any pair of points $(\xi, \eta) \in U \times V$ can be joined by a geodesic. Furthermore, if $\tilde{\sigma}$ is a geodesic with extremities in $U$ and $V$, then $\tilde{\sigma}$ is hyperbolic and $d(\tilde{\sigma}, \sigma(0)) < c(\sigma) + \varepsilon$.

**Definition 2.3** An isometry $\gamma \neq \text{id}$ of $X$ is called **axial**, if there exists a constant $l = l(\gamma) > 0$ and a geodesic $\sigma$ such that $\gamma(\sigma(t)) = \sigma(t + l)$ for all $t \in \mathbb{R}$. We call $l(\gamma)$ the translation length of $\gamma$, and $\sigma$ an **axis** of $\gamma$. The boundary point $\gamma^+ := \sigma(\infty)$ is called the **attractive fixed point**, and $\gamma^- := \sigma(-\infty)$ the **repulsive fixed point** of $\gamma$. We further put $Ax(\gamma) := \{x \in X \mid d(x, \gamma x) = l(\gamma)\}$.

We remark that $Ax(\gamma)$ consists of the union of parallel geodesics translated by $\gamma$, and $\overline{Ax(\gamma)} \cap \partial X$ is exactly the set of fixed points of $\gamma$.

The following kind of axial isometries will play a crucial role in the sequel.

**Definition 2.4** An axial isometry is called **hyperbolic axial** if it possesses a hyperbolic axis. The width $c(\gamma)$ of a hyperbolic axial isometry $\gamma$ is defined by

$$c(\gamma) := \sup\{d(x, \sigma_y, \gamma^+) \mid x, y \in Ax(\gamma)\}.$$  

Notice that if $\gamma$ is hyperbolic axial, then $\gamma^+$ and $\gamma^-$ are the only fixed points of $\gamma$, and every axial isometry commuting with $\gamma$ possesses the same set of invariant geodesics as $\gamma$. Furthermore, if $\sigma$ is an axis of $\gamma$, then $c(\gamma) \leq 2c(\sigma)$.

Let us recall some further properties of hyperbolic axial isometries stated in Theorem 2.2 of [B1].
Lemma 2.5 Suppose γ is a hyperbolic axial isometry. Then

1. every point ξ ∈ ∂X \ {γ\+} can be joined to γ\+ by a geodesic, and all these geodesics are hyperbolic,
2. given neighborhoods U of γ\+ and V of γ\-, there exists N₀ ∈ N such that γⁿ(\(X \setminus V\)) ⊂ U and γ⁻ⁿ(\(X \setminus U\)) ⊂ V for all n ≥ N₀.

For a discrete subgroup Γ ⊂ Isom(\(X\)) the geometric limit set of Γ is defined by \(L_Γ := \Gamma \cdot x \cap ∂X\), where x ∈ X is arbitrary. We say that two points ξ, η ∈ ∂X are dual with respect to Γ if for all neighborhoods U of ξ and V of η there exists γ ∈ Γ such that γ(\(X \setminus U\)) ⊂ V and γ⁻¹(\(X \setminus V\)) ⊂ U. In this case, both ξ and η belong to \(L_Γ\).

From here on we will assume that Γ ⊂ Isom(\(X\)) is a discrete group which contains a hyperbolic axial isometry h and does not possess a global fixed point in ∂X. The following proposition recalls the part of Theorem 2.8 in [B1] which applies to our groups Γ.

**Proposition 2.6** For every neighborhood U of a limit point ξ ∈ \(L_Γ\) there exists γ ∈ Γ such that γh\+ ∈ U \ {ξ}. Moreover, the closure of Γ\·ξ equals \(L_Γ\), and any two points of \(L_Γ\) are dual with respect to Γ.

For x ∈ X and r > 0 we denote by \(B_x(r)\) the open ball of radius r centered at x. Our first result states that for ξ ∈ ∂X the projection \(pr_ξ := pr(ξ, \cdot)\) of a sufficiently large ball in X contains an open set in ∂X.

The main difficulty in generalizing the analogous statement Lemma 3.5 in [K2] to our situation consists in the fact that every hyperbolic geodesic may bound a flat strip. We therefore have to uniformly bound the width of such flat strips along the geodesics in question.

**Lemma 2.7** For each x ∈ X there exists a constant r > 0 such that for all ξ ∈ ∂X \(pr_ξ(B_x(r))\) contains an open set \(U ⊂ ∂X\) with \(U \cap L_Γ \neq \emptyset\).

**Proof.** Let x ∈ X arbitrary, fix a point y ∈ Ax(h) and let c(h) > 0 denote the width of h (see Definition 2.3). The idea is to construct a covering of ∂X by open sets.

Let ε > 0 and \(U^\pm\) be disjoint neighborhoods of \(h^\pm\) as in Lemma 2.2. That is, any two points in \(U^+, U^-\) can be joined by a geodesic, every such geodesic σ is hyperbolic and \(d(ξ, η) < c(ξ) + ε\). For each η ∈ ∂X \(\setminus (U^+ ∪ U^-)\) we denote by \(σ_η\) a hyperbolic geodesic connecting η and \(h^+\), and by \(c(σ_η)\) the width of \(σ_η\). Let \(V_η\) be a neighborhood of \(η\) in \(U^+\) that any two points in \(U_η, V_η\) can be joined by a geodesic, every such geodesic \(σ\) is hyperbolic and \(d(σ_0, σ) < c(σ_0) + ε\). Then
\[
∂X \subseteq U^+ \cup U^- \cup \bigcup_{η ∈ ∂X \setminus (U^+ ∪ U^-)} V_η,
\]
and, since ∂X is compact, there exist finitely many points η₁, η₂, ..., ηₙ ∈ ∂X such that
\[
∂X \subseteq U^+ \cup U^- \cup \bigcup_{i=1}^{n} V_{η_i}.
\]
If \(r := c(h) + ε + d(x, y) + \max_{1 \leq i \leq n} (c(σ_{η_i}) + d(x, σ_{η_i}(0)))\), then for any ξ ∈ ∂X, the projection \(pr_ξ(B_x(r))\) contains an open set in ∂X: If ξ ∈ \(V_{η_i}\) for some \(i \in \{1, 2, ..., n\}\), then \(U_{η_i}\) is the desired set, and \(h^+ \in U_{η_i}\) implies \(U_{η_i} \cap L_Γ \neq \emptyset\). If ξ ∈ ∅, then \(h^+ \in U^+ \subseteq pr_ξ(B_x(r))\), if ξ ∈ ∅, then \(h^- \in U^- \subseteq pr_ξ(B_x(r))\).

The following lemma states that the limit set can be covered by finitely many Γ–translates of an appropriate open set in ∂X.
Lemma 2.8 For any open subset \( A \subset \partial X \) with \( A \cap L_{\Gamma} \neq \emptyset \) there exists a finite set \( \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \subset \Gamma \) such that
\[
L_{\Gamma} \subseteq \bigcup_{i=1}^{m} \gamma_i A.
\]

Proof. Fix \( \xi \in A \cap L_{\Gamma} \). If \( \xi = h^- \), we pick a neighborhood \( U \) of \( \xi \) such that \( U \subseteq A \) and \( h^+ \not\in \overline{U} \). If \( L_{\Gamma} \subseteq \overline{U} \cup \{h^+\} \), we choose \( \gamma \in \Gamma \) such that \( \gamma h^+ \in U \setminus \{\xi\} \). Then \( h^+ \in \gamma^{-1} U \), hence \( L_{\Gamma} \subseteq U \cup \gamma^{-1} U \). If \( L_{\Gamma} \not\subseteq \overline{U} \cup \{h^+\} \), there exists \( \eta \in L_{\Gamma} \setminus \overline{U}, \eta \neq h^+ \). Since \( \eta \) and \( \xi \) are dual with respect to \( \Gamma \), for any neighborhood \( V \) of \( \eta \) there exists \( \gamma \in \Gamma \) such that \( \gamma^{-1}(X \setminus U) \subseteq \gamma U \). Choose neighborhoods \( V \) of \( \eta, W \) of \( h^+ \) sufficiently small so that the closures of the sets \( U, V, W \) are pairwise disjoint. By Lemma 2.5 (2) there exists \( n \in \mathbb{N} \) such that \( h^n(X \setminus U) \subseteq W \) and \( h^{-n}(X \setminus W) \subseteq U \), in particular \( X \setminus W \subseteq h^n U \). We conclude
\[
L_{\Gamma} \subseteq \overline{X} \setminus V \cup \overline{X} \setminus W \subseteq \gamma U \cup h^n U \subseteq \gamma A \cup h^n A.
\]
The case \( \xi = h^- \) is analogous.

If \( \xi \not\in \{h^+, h^-\} \), we choose a neighborhood \( U \subseteq A \) of \( \xi \) such that \( h^+, h^- \not\in \overline{U} \). Pick \( \gamma \in \Gamma \) such that \( \gamma h^+ \in U \setminus \{\xi\} \). Then \( \gamma h^{-1} \) is hyperbolic axial, and by the discreteness of \( \Gamma \) (see [B1, Lemma 2.9]), \( \gamma h^- \neq h^+ \). Hence there exist neighborhoods \( V \) of \( \gamma h^- \) and \( W \) of \( h^+ \) such that the closures of \( U, V, W \) are pairwise disjoint. Since \( h^+ \) and \( \gamma h^- \) are dual, there exists \( g \in \Gamma \) such that \( g(X \setminus W) \subseteq V \). Furthermore, there exists \( n \in \mathbb{N} \) such that \( (\gamma h^{-1} g)^n(X \setminus V) \subseteq U \). We conclude
\[
L_{\Gamma} \subseteq \overline{X} \setminus W \cup \overline{X} \setminus V \subseteq g^{-1} U \cup (\gamma h^{-1} g)^{-1} U \subseteq g^{-1} A \cup h^n A.
\]
\( \square \)

3 Conformal densities

Let \( X \) be a Hadamard manifold with Riemannian distance \( d, o \in X \) a fixed base point, and \( \Gamma \subset \text{Isom}(X) \) a discrete infinite subgroup. For \( x, y \in X, s \in \mathbb{R} \) we denote by
\[
P^s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}
\]
the Poincaré series. Its exponent of convergence \( \delta(\Gamma) \) is independent of \( x, y \) by the triangle inequality, and is called the critical exponent of \( \Gamma \). If
\[
N_{\Gamma}(R) := \#\{\gamma \in \Gamma \mid d(o, \gamma o) < R\}, \quad \Delta N_{\Gamma}(R) := N_{\Gamma}(R) - N_{\Gamma}(R - 1),
\]
then an easy calculation shows that
\[
\delta(\Gamma) = \limsup_{R \to \infty} \left( \frac{1}{R} \log N_{\Gamma}(R) \right) = \limsup_{R \to \infty} \left( \frac{1}{R} \log \Delta N_{\Gamma}(R) \right).
\]
For \( z \in X \) we consider the continuous map
\[
B_z : \quad X \times X \to \mathbb{R} \\
x \mapsto d(x, z) - d(y, z).
\]
This map extends continuously to the boundary via
\[
B_\eta(x, y) := \lim_{s \to \infty} \left( d(x, \sigma(s)) - d(y, \sigma(s)) \right),
\]
where \( \sigma \) is an arbitrary ray in the class of \( \eta \in \partial X \). For \( \xi \in \partial X, y \in X \), the function
\[
B_\xi(\cdot, y) : \quad X \to \mathbb{R} \\
x \mapsto B_\xi(x, y)
\]
is called the Busemann function centered at $\xi$ based at $y$. It is independent of the chosen ray $\sigma$ in the class of $\xi$ (see also [B3, chapter II]).

**Definition 3.1** Let $M^+(\partial X)$ denote the cone of positive finite Borel measures on $\partial X$, and $\alpha > 0$. An $\alpha$-dimensional conformal density is a continuous map

$$\mu : X \rightarrow M^+(\partial X)$$

with the properties

(i) $\text{supp}(\mu_0) \subseteq L_\Gamma$,

(ii) $\gamma \ast \mu_x = \mu_{\gamma^{-1}x}$ for any $\gamma \in \Gamma$, $x \in X$,

(iii) $\frac{d\mu_x}{d\mu_0}(\eta) = e^{\alpha B_\eta(x, \xi)}$ for any $x \in X$, $\eta \in \text{supp}(\mu_0)$.

The existence of a $\delta(\Gamma)$-dimensional conformal density $\mu$ was proved by G. Knieper ([K2, Lemma 2.2]) for Hadamard manifolds and arbitrary discrete infinite isometry groups. From his construction it follows that $\mu_x(L_\Gamma) > 0$ for all $x \in X$. Our goal in this section is a generalization of Lemma 4.4 in [K2] which is valid for discrete groups with compact quotient of geometric rank one. We will only require that $\Gamma$ is a discrete isometry group which contains a hyperbolic axial isometry $h$ and does not globally fix a point in $\partial X$. Given Lemma 2.8 the following lemma and its corollary are straightforward from Lemma 4.1 in [K2].

**Lemma 3.2** Let $\mu$ be a conformal density and $x \in X$. Then $\mu_x(L_\Gamma) > 0$ implies $\text{supp}(\mu_x) = L_\Gamma$.

**Proof.** Suppose $\xi \in L_\Gamma$, $\xi \notin \text{supp}(\mu_x)$. Let $U$ be a neighborhood of $\xi$ such that $\mu_x(U) = 0$. By Lemma 2.8 there exists $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ such that $L_\Gamma \subseteq \bigcup_{i=1}^m \gamma_i U$. Hence

$$\mu_x(L_\Gamma) \leq \sum_{i=1}^m \mu_x(\gamma_i U) = \sum_{i=1}^m \mu_{\gamma_i^{-1}x}(U) = 0,$$

because $\mu_{\gamma_i^{-1}x}$, $1 \leq i \leq m$, is absolutely continuous with respect to $\mu_x$. \hfill $\square$

**Corollary 3.3** If $\mu$ is a $\delta(\Gamma)$-dimensional conformal density, then for any $x \in X$ $\text{supp}(\mu_x) = L_\Gamma$.

For $x \in X$, $\xi \in \partial X$ and $\varepsilon > 0$, we put $C^\varepsilon_{x, \xi} := \{z \in \overline{X} | \angle_x(z, \xi) < \varepsilon\}$.

The following two lemmata are easy generalizations of Lemma 4.2 and Proposition 3.6 of [K2] to the noncompact case.

**Lemma 3.4** Fix $x \in X$ and let $\mu$ be a conformal density with $\mu_x(L_\Gamma) > 0$. Then for any $\varepsilon > 0$ there exists a constant $q = q(x, \varepsilon) > 0$ such that $\mu_x(C^\varepsilon_{x, \xi}) > q$ for all $\xi \in L_\Gamma$.

**Proof.** Suppose the contrary is true. Then there exists $\varepsilon > 0$ and a sequence $(\xi_n) \subseteq L_\Gamma$ such that $\mu_x(C^\varepsilon_{x, \xi_n}) \rightarrow 0$ as $n \rightarrow \infty$. Passing to a subsequence if necessary, we may assume that $\xi_n$ converges to a point $\xi \in \partial X$. Then there exists $N_0 \in \mathbb{N}$ such that $C^\varepsilon_{x, \xi} \subseteq C^\varepsilon_{x, \xi_n}$ for $n \geq N_0$, hence $\mu_x(C^\varepsilon_{x, \xi}) = 0$. Since the limit set is closed, we further have $\xi \in L_\Gamma$. Arguing as in the proof of Lemma 3.2 we obtain a contradiction to $\mu_x(L_\Gamma) > 0$. \hfill $\square$
Lemma 3.5 For $x \in X$ there exist constants $c_0 > 0$ and $\varepsilon > 0$ such that for any $c > c_0$ and $y \in X \setminus B_x(c)
abla$, $$\text{pr}_y(B_x(c)) \supseteq C^{\varepsilon}_{x,\xi} \cap \partial X \quad \text{for some } \xi \in L_\Gamma.$$ Proof. Fix $x \in X$ and suppose the assertion is not true. Then for all $n \in \mathbb{N}$ there exists a point $y_n \in X \setminus B_x(n)$ such that $\text{pr}_{y_n}(B_x(n)) \nsubseteq C^{1/n}_{x,\xi} \cap \partial X$ for all $\xi \in L_\Gamma$. Passing to a subsequence if necessary, we assume that $(y_n)$ converges to a point $y \in \partial X$.

Then either for all $r > 0$, $\varepsilon > 0$ and $\xi \in L_\Gamma$ $C^{\varepsilon}_{x,\xi} \cap \partial X \nsubseteq \text{pr}_y(B_x(r))$ in contradiction to Lemma 2.7, or there exist constants $q > q_0$ such that $C^{\varepsilon}_{x,\xi} \cap \partial X \subseteq \text{pr}_y(B_x(r))$. However, the continuity of the map pr : $\overline{X} \times X \setminus D \to \partial X$ would then imply the existence of $N_0 \in \mathbb{N}$ such that $C^{\varepsilon}_{x,\xi} \subseteq \text{pr}_{y_n}(B_x(r))$ for any $n > N_0$, in contradiction to the choice of $y_n$ for $n > r$ and $1/n < \varepsilon$. \hfill \Box

We are finally able to prove the main theorem of this section.

Theorem 3.6 Let $\alpha > 0$ and $\mu$ an $\alpha$–dimensional conformal density of positive and finite total mass. Then there exists a constant $c_0 > 0$ with the following property: For $c > c_0$ there exists a constant $D(c) > 1$ such that for all $\gamma \in \Gamma$ with $d(o, \gamma o) > c$ we have $$\frac{1}{D(c)} e^{-\alpha d(o, \gamma o)} \leq \mu_o(\text{pr}_o(B_{\gamma o}(c))) \leq D(c) e^{-\alpha d(o, \gamma o)}.$$ Proof. Fix $c_0 > 0$ as in the previous lemma. By Lemma 3.5 there exists a constant $q > 0$ such that for all $y \in X \setminus B_o(c)$ we have $\mu_o(\text{pr}_y(B_o(c))) > q$. Hence for any $\gamma \in \Gamma$ with $d(o, \gamma o) > c$ $$q < \mu_o(\text{pr}_{\gamma^{-1}}(B_o(c))) < \mu_o(\partial X).$$ Furthermore, if $\eta \in \text{pr}_o(B_{\gamma o}(c))$ then $0 \leq d(o, \gamma o) - B_o(\gamma o) \leq 2c$ by elementary geometric estimates. For $\gamma \in \Gamma$ we abbreviate $S_\gamma := \text{pr}_o(B_{\gamma o}(c))$ and conclude $$q < \mu_o(\gamma^{-1}S_\gamma) = \mu_o(S_\gamma) = \int_{S_\gamma} d\mu_o = \int_{S_\gamma} e^{\alpha B_o(\gamma o)} d\mu_o(\eta) \leq e^{\alpha d(o, \gamma o)} \mu_o(S_\gamma).$$ Similarly we have $$\mu_o(\partial X) \geq \mu_o(\gamma^{-1}S_\gamma) = \int_{S_\gamma} e^{\alpha B_o(\gamma o)} d\mu_o(\eta) \geq e^{-2\alpha c} e^{\alpha d(o, \gamma o)} \mu_o(S_\gamma)$$ and summarize $$e^{-\alpha d(o, \gamma o)} q < \mu_o(S_\gamma) \leq e^{-\alpha d(o, \gamma o)} e^{2\alpha c} \mu_o(\partial X). \hfill \Box

The following proposition will be crucial in order to apply the methods developed by T. Roblin in [R].

Proposition 3.7 If an $\alpha$–dimensional conformal density $\mu$ of positive and finite total mass exists, then $\alpha \geq \delta(\Gamma)$.

Proof. Suppose $\mu$ is an $\alpha$–dimensional conformal density of positive and finite total mass. Let $c > c_0$ with $c_0$ as in Theorem 5.4 and $R > c$ arbitrary. Since $\Gamma$ is discrete, every ball of radius $c + 1$ in $X$ contains at most $M = M(c)$ orbit points $\gamma o$. Hence every point in $\partial X$ is covered by at most $M$ sets $\text{pr}_o(B_{\gamma o}(c))$, $R - 1 \leq d(o, \gamma o) < R$, and therefore $$\sum_{\gamma \in \Gamma \atop R - 1 \leq d(o, \gamma o) < R} \mu_o(\text{pr}_o(B_{\gamma o}(c))) \leq M \mu_o(\partial X).$$
Recall from (2) the definition of $\Delta N_\Gamma(R)$. We conclude

$$
\Delta N_\Gamma(R) \frac{1}{D(c)} e^{-\alpha R} \leq \sum_{\gamma \in \Gamma, R-1 \leq d(o,\gamma o) < R} \frac{1}{D(c)} e^{-\alpha d(o,\gamma o)}
$$

$$
\leq \sum_{\gamma \in \Gamma, R-1 \leq d(o,\gamma o) < R} \mu_o(\text{pr}_o(B_{\gamma o}(c))) \leq M \mu_o(\partial X),
$$

hence $\delta(\Gamma) = \limsup_{R \to \infty} \frac{1}{R} \log(\Delta N_\Gamma(R)) \leq \alpha$. □

**Corollary 3.8** There exists a constant $b > 0$ such that $N_\Gamma(R) \leq b e^{\delta(\Gamma) R}$ for sufficiently large $R > 0$.

**Proof.** We compute as in the proof of the previous proposition

$$
\Delta N_\Gamma(R) \frac{1}{D(c)} e^{-\delta(\Gamma) R} \leq \sum_{\gamma \in \Gamma, R-1 \leq d(o,\gamma o) < R} \frac{1}{D(c)} e^{-\delta(\Gamma) d(o,\gamma o)}
$$

$$
\leq \sum_{\gamma \in \Gamma, R-1 \leq d(o,\gamma o) < R} \mu_o(\text{pr}_o(B_{\gamma o}(c))) \leq M \mu_o(\partial X),
$$

hence $\Delta N_\Gamma(R) \leq M D(c) \mu_o(\partial X) e^{\delta(\Gamma) R}$. Furthermore, if $n$ denotes the smallest integer greater than $R$, we have

$$
N_\Gamma(R) \leq \sum_{j=1}^{n} \Delta N_\Gamma(j) \leq M D(c) \mu_o(\partial X) \sum_{j=1}^{n} (e^{\delta(\Gamma)})^{j}
$$

$$
= M D(c) \mu_o(\partial X) \frac{e^{\delta(\Gamma)n} - 1}{e^{\delta(\Gamma)} - 1}
$$

and the assertion follows with $b = M D(c) \mu_o(\partial X) e^{2\delta(\Gamma)}/(e^{\delta(\Gamma)} - 1)$. □

We next introduce a class of groups for which we will be able to derive stronger results.

**Definition 3.9** A discrete subgroup $\Gamma \subset \text{Isom}(X)$ is called weakly cocompact if there exists a $\delta(\Gamma)$-dimensional conformal density $\mu$ and constants $b > 0$, $c_\Gamma > 0$ and $r > 0$ such that for all $c \geq c_\Gamma$

$$
\liminf_{R \to \infty} \mu_o\left( \bigcup_{R-r \leq d(o,\gamma o) < R} \text{pr}_o(B_{\gamma o}(c)) \right) \geq b.
$$

Notice that for discrete isometry groups of Hadamard manifolds, cocompact implies weakly cocompact, because in this case

$$
0 < \mu_o(\partial X) = \mu_o\left( \bigcup_{R-1 \leq d(o,\gamma o) < R} \text{pr}_o(B_{\gamma o}(c)) \right)
$$

if $c \geq \text{diam}(X/\Gamma)$ and $R > c$. Further examples of weakly cocompact groups are convex cocompact isometry groups of real hyperbolic spaces, and radially cocompact isometry groups (see [L] for a definition) of symmetric spaces.

For weakly cocompact groups, we have the following lower bound for $N_\Gamma(R)$. 

---

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**Lemma 3.10** Let $\Gamma \subset {\text{Isom}}(X)$ be a weakly cocompact discrete group which contains a hyperbolic axial isometry. Then there exists $a > 0$ and $R_0 > 0$ such that for all $R \geq R_0$

$$N_\Gamma(R) \geq ae^{\delta(\Gamma)R}.$$ 

**Proof.** Let $c > \max\{c_0, c_1\}$ with $c_0 > 0$ as in Theorem 3.6, $b > 0$, $c_1 > 0$ and $r > 0$ the constants from the definition. Then there exists $R_0 > r$ such that for all $R \geq R_0$

$$b \leq \mu_o\left( \bigcup_{R-r \leq d(o, \gamma o) < R} \text{pr}_o(B_{\gamma o}(c)) \right) \leq \sum_{\gamma \in \Gamma} \mu_o(\text{pr}_o(B_{\gamma o}(c)))$$

$$\leq D(c) \sum_{R-r \leq d(o, \gamma o) < R} e^{-\delta(\Gamma)d(o, \gamma o)} \leq D(c)N_\Gamma(R)e^{-\delta(\Gamma)(R-r)},$$

hence $N_\Gamma(R) \geq ae^{\delta(\Gamma)R}$ for $a = be^{-r\delta(\Gamma)/D(c)}$. 

\[\square\]

4 The critical exponent

In this section we are going to prove Theorem 4.1 and some corollaries, which will be necessary to derive the lower bound in Theorem 5.8. As before, $X$ will denote a Hadamard manifold, $o \in X$ a fixed base point, and $\Gamma \subset \text{Isom}(X)$ a discrete group which contains a hyperbolic axial isometry and possesses infinitely many limit points. Given Proposition 3.7, the arguments of T. Roblin in the context of $\text{CAT}(-1)$-spaces ([R]) remain valid in our setting. We include the proofs for the convenience of the reader.

**Theorem 4.1** If $\Gamma$ is a discrete isometry group of a Hadamard manifold $X$ which contains a hyperbolic axial isometry and possesses infinitely many limit points, and $o \in X$ a fixed base point, then $\lim_{R \to \infty} r\log N_\Gamma(R)$ exists and equals $\delta(\Gamma)$.

**Proof.** Assume that $\liminf_{R \to \infty} r\log N_\Gamma(R) < \delta(\Gamma)$. Then there exists a sequence $(R_k) \subset \mathbb{R}$, $R_k \to \infty$, and $0 < \alpha < \delta(\Gamma)$ such that $N_\Gamma(R_k) \leq e^{\alpha R_k}$ for all $k \in \mathbb{N}$. We are going to construct an $\alpha$-dimensional conformal density in order to obtain a contradiction to Proposition 3.7.

Let $\delta$ denote the unit Dirac point measure. For $R > 0$ and $x \in X$ we put

$$\nu_x^R := \sum_{\gamma \in \Gamma, d(x, \gamma o) \leq R} e^{-\alpha d(x, \gamma o)} \delta(\gamma o) / \left( \sum_{\gamma \in \Gamma, d(o, \gamma o) \leq R} e^{-\alpha d(o, \gamma o)} \right).$$

From $\|\nu_o^R\| = 1$ and the Theorem of Banach-Alaoglu it follows that for any $R > 0$ there exists a sequence $k_n(r) \subset \mathbb{N}$, $k_n(r) \to \infty$, such that $\nu_{o, R_k_n(r)}^R$ converges weakly to a probability measure $\nu_o^R$. Furthermore, the support of $\nu_o^R$ equals $L_\Gamma$, because, since $\alpha < \delta(\Gamma)$, the series in the denominator diverges as $R \to \infty$. We denote by $\mu^\alpha$ the $\alpha$-dimensional conformal density induced by $\nu_o^R$. Our aim is to prove that for any $x \in B_o(r)$ the measures $\nu_x^{R_k_n(r)}$ converge weakly to $\mu^\alpha_x$. That is, we have to show that for every bounded and continuous function $f$ on $X$

$$\lim_{n \to \infty} \int_X f(z) d\nu_x^{R_k_n(r)}(z) = \int_X f(z) d\mu^\alpha_x(z).$$

From the definition of $\mu^\alpha_x$ and $\nu_o^R$ it follows that

$$\int_X f(z) d\mu^\alpha_x(z) = \int_X f(z) e^{\alpha B_z(o, x)} d\nu_o^R(z) = \lim_{n \to \infty} \int_X f(z) e^{\alpha B_z(o, x)} d\nu_o^{R_k_n(r)}(z).$$

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with \( B_z(o, x) = d(o, z) - d(x, z), z \in \overline{X} \), as in [3]. We put \( \|f\| := \sup_{x \in \overline{X}} |f(x)| \),

\[
\Psi_n := \sum_{\gamma \in \Gamma} e^{-\alpha d(o, \gamma o)}
\]

and compute for all \( n \in \mathbb{N} \)

\[
\left| \int_{\overline{X}} f(z) d\nu_x^{R_{k_n(o(r)} - r}(z) - \int_{\overline{X}} f(z) e^{\alpha B_z(o, x)} d\nu_o^{R_{k_n(o(r)} - r}(z) \right| \leq \frac{\|f\|}{\Psi_n} \sum_{\gamma \in \Gamma} e^{-\alpha d(x, \gamma o)} - \sum_{\gamma \in \Gamma} e^{(d(o, \gamma o) - d(x, \gamma o))} e^{-\alpha d(o, \gamma o)}.
\]

Consider the number

\[
g_x(R) := \left| \sum_{\gamma \in \Gamma} e^{-\alpha d(x, \gamma o)} - \sum_{\gamma \in \Gamma} e^{-\alpha d(o, \gamma o)} \right|.
\]

Then for \( x \in B_o(r) \) we have

\[
g_x(R) \leq \sum_{\gamma \in \Gamma} e^{-\alpha d(x, \gamma o)} \leq \sum_{\gamma \in \Gamma} e^{-\alpha d(o, \gamma o) + \alpha d(o, x)}
\]

\[
\leq \sum_{\gamma \in \Gamma} e^{-\alpha R + 2\alpha d(o, x)} \leq N_{\Gamma}(R + r) e^{-\alpha R + 2\alpha r}.
\]

We conclude

\[
\left| \int_{\overline{X}} f(z) d\nu_x^{R_{k_n(o(r)} - r}(z) - \int_{\overline{X}} f(z) e^{\alpha B_z(o, x)} d\nu_o^{R_{k_n(o(r)} - r}(z) \right| \leq \frac{\|f\|}{\Psi_n} g_x(R_{k_n(o(r)} - r) - r) \leq \frac{\|f\|}{\Psi_n} N_{\Gamma}(R_{k_n(o(r)} - r) e^{-\alpha R_{k_n(o(r)} - r + 3\alpha r} \leq \frac{\|f\|}{\Psi_n} e^{3\alpha r} \to 0,
\]

because \( \Psi_n \) is unbounded as \( n \to \infty \). This implies that \( \nu_x^{R_{k_n(o(r)} - r} \) converges weakly to \( \mu_x^r \) for all \( x \in B_o(r) \).

Obviously, we have \( \gamma \ast \nu_x^R = \nu_x^{R_{\gamma^{-1}x}} \) for all \( x \in X, \gamma \in \Gamma \). If \( d(o, x) < r \) and \( d(o, \gamma o) < r \), this implies \( \gamma \ast \mu_x^r = \mu_x^{R_{\gamma^{-1}x}} \).

We finally consider a sequence \( (r_j) \subset \mathbb{R}, r_j \to \infty \), such that \( \mu_y^{r_j} \) converges weakly to a probability measure \( \mu_y \). Let \( \mu_{x} \) be the conformal density induced by \( \mu_y \). Then \( \mu_y^{r_j} \) converges weakly to \( \mu_y \) for all \( x \in X \). Furthermore \( \gamma \ast \mu_y^r = \mu_y^{R_{\gamma^{-1}x}} \) for all \( x \in X, \gamma \in \Gamma \). This yields the desired contradiction.

\[
\square
\]

If \( A \subset \partial X, z \in \overline{X} \), we let \( \angle_o(z, A) := \inf_{\eta \in A} \angle_o(z, \eta) \) and

\[
N_\Gamma(R; A) := \# \{ \gamma \in \Gamma \mid d(o, \gamma o) < R, \angle_o(\gamma o, A) = 0 \}.
\]

**Corollary 4.2** If \( A \subset \partial X \) is an open set with \( A \cap L_\Gamma \neq \emptyset \), then

\[
\lim_{R \to \infty} \left( \frac{1}{R} \log N_\Gamma(R; A) \right) = \delta(\Gamma).
\]
For weakly cocompact $\Gamma$ there exist $b > 1$ and $R_0 > 0$ such that for all $R > R_0$

$$\frac{1}{b}e^{\delta(\Gamma)R} \leq N_\Gamma(R; A) \leq be^{\delta(\Gamma)R}.$$  

**Proof.** Let $U \subset \partial X$ be an open set with $U \cap L_\Gamma \neq \emptyset$ such that $\overline{U} \subset A$. By Lemma 3.8 there exist $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ such that $L_\Gamma \subseteq \bigcup_{i=1}^m \gamma_iU$. Let $M \subseteq \bigcup_{i=1}^m \gamma_iU$ be an open set which contains $L_\Gamma$. Then $\angle_o(\gamma_i, M) = 0$ for all but finitely many $\gamma_i \in \Gamma$ by the definition of the limit set, hence $N_\Gamma(R; M) \geq N_\Gamma(R) - j$ for some constant $j \in \mathbb{N}$.

Fix $g \in \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ and suppose $\angle_o(\gamma_i, gU) = 0$ and $\angle_o(g^{-1}\gamma_i, A) > 0$ for infinitely many $\gamma_i \in \Gamma$. Let $(\gamma_k) \subset \Gamma$ be a sequence with this property. Passing to a subsequence if necessary, we may assume that $\gamma_k \to o$ converges to a point $\eta \in gU$. Since $gU \subset gA$, this implies $\angle_{go}(\gamma_k, gA) = 0$ for almost all $k \in \mathbb{N}$ in contradiction to $\angle_{go}(\gamma_k, gA) = \angle_o(g^{-1}\gamma_k, A) > 0$.

Hence

$$c(g) := \#\{\gamma \in \Gamma | \angle(o, \gamma, gU) = 0 \text{ and } \angle_o(g^{-1}\gamma, A) > 0\} < \infty,$$

and $N_\Gamma(R; gU) \leq N_\Gamma(R + d(o, go); A) + c(g)$. Put $r := \max_{1 \leq i \leq m} d(o, \gamma_i o)$ and $c := \sum_{i=1}^m c(\gamma_i)$.

Then

$$N_\Gamma(R - r; A) - j \leq N_\Gamma(R - r) - j \leq N_\Gamma(R - r; M) \leq \sum_{i=1}^m N_\Gamma(R - r; \gamma_i U) \leq \sum_{i=1}^m (N_\Gamma(R - r + d(o, \gamma_i o); A) + c(\gamma_i)) \leq mN_\Gamma(R; A) + c,$$

which proves the assertion. The claim for weakly cocompact $\Gamma$ follows with Corollary 4.3 and Lemma 5.10.\qed

The second corollary of Theorem 4.1 estimates the numbers

$$N_\Gamma(R; A, B) := \#\{\gamma \in \Gamma | d(o, \gamma o) < R, \angle(o, \gamma, A) = 0, \angle_o(\gamma^{-1}o, B) = 0\},$$

where $A, B \subseteq \partial X$ are open sets.

**Corollary 4.3** If $A, B \subset \partial X$ are open sets with $A \cap L_\Gamma \neq \emptyset, B \cap L_\Gamma \neq \emptyset$, then

$$\lim_{R \to \infty} \left(\frac{1}{R} \log N_\Gamma(R; A, B)\right) = \delta(\Gamma).$$

For weakly cocompact $\Gamma$ there exist $b > 1$ and $R_0 > 0$ such that for all $R > R_0$

$$\frac{1}{b}e^{\delta(\Gamma)R} \leq N_\Gamma(R; A, B) \leq be^{\delta(\Gamma)R}.$$
and $N_{\Gamma}(R; A, gV) \leq N_{\Gamma}(R + d(o, go); A, B) + c(g)$. We put
\[ r := \max_{1 \leq i \leq m} d(o, \gamma_i o) \text{ and } c := \sum_{i=1}^{m} c(\gamma_i) \text{ and conclude} \]
\[
N_{\Gamma}(R - r; A, B) - j \leq N_{\Gamma}(R - r; A) - j \leq N_{\Gamma}(R - r; A, M) \leq \sum_{i=1}^{m} N_{\Gamma}(R - r; A, \gamma_i V) \leq mN_{\Gamma}(R; A, B) + c,
\]
which yields the assertion. The claim for weakly cocompact $\Gamma$ follows with Corollary 3.8 and Lemma 3.10.

\[ \Box \]

5 Growth of conjugacy classes

Let $M$ be a complete Riemannian manifold of nonpositive sectional curvature with universal Riemannian covering manifold $X$, and $\Gamma \subset \text{Isom}(X)$ the group of deck transformations of the covering projection $X \to M$. It is well known that $X$ is a Hadamard manifold and $\Gamma$ a discrete and torsion free group isomorphic to the fundamental group of $M$. In this section, we will derive a new asymptotic estimate for the growth rate of geometrically distinct closed geodesics modulo free homotopy in $M$. Notice that due to the occurrence of flat strips there can be infinitely many closed geodesics in one free homotopy class.

We will only require that the group of deck transformations $\Gamma$ of $M$ contains a hyperbolic axial element and possesses infinitely many limit points. Since we do not assume the manifolds to be compact or of finite volume, we face certain difficulties which do not occur in the case of compact manifolds treated in [K2]. First, closed geodesics in $M$ may have arbitrarily small length. Moreover, our weaker notion of hyperbolic geodesics includes the treatment of manifolds $M$ which are not necessarily of geometric rank one, i.e. every geodesic in $X$ may bound a flat strip. In particular, we do not have a uniform upper bound on the width of hyperbolic axial isometries.

For these two reasons we encounter difficulties when trying to estimate the number of elements in $\Gamma$ which correspond to the same free homotopy class of closed geodesics. In particular, for lack of a uniform upper bound on the width of hyperbolic axial isometries, the argument in Lemma 5.4 of [K2] cannot be directly adapted to our case.

**Definition 5.1** $\gamma, \gamma' \in \Gamma$ are said to be equivalent if and only if there exist $n, m \in \mathbb{Z}$ and $\varphi \in \Gamma$ such that $(\gamma')^m = \varphi \gamma^n \varphi^{-1}$. An element $\gamma_0 \in \Gamma$ is called primitive if it cannot be written as a proper power $\gamma_0 = \varphi^n$, where $\varphi \in \Gamma$ and $n \geq 2$.

Each equivalence class can be represented as
\[ [\gamma] = \{ \varphi \gamma_0 \varphi^{-1} \mid \gamma_0 \in \Gamma, \gamma_0 \text{ primitive}, k \in \mathbb{Z}, \varphi \in \Gamma \} . \]

It is easy to see that the set of equivalence classes of axial elements in $\Gamma$ is in one to one correspondence with the set of geometrically distinct closed geodesics modulo free homotopy. If $\gamma_0$ is a primitive axial isometry representing $[\gamma]$, we have
\[ l([\gamma]) := \min\{ l(\varphi) \mid \varphi \in [\gamma] \} = l(\gamma_0) . \]

Then $P(t) := \#\{ [\gamma] \mid \gamma \in \Gamma \text{ axial, } l([\gamma]) \leq t \}$ counts the number of closed geometrically distinct geodesics of period $\leq t$ modulo free homotopy, and
\[ P_h(t) := \#\{ [\gamma] \mid \gamma \in \Gamma \text{ hyperbolic axial, } l([\gamma]) \leq t \} \leq P(t) \]

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the number of closed geometrically distinct hyperbolic geodesics of period \( \leq t \) modulo free homotopy. If \( W \subset X \) is an open set, we put

\[
\begin{align*}
P(t; W) &:= \#\{[\gamma] \mid \gamma \in \Gamma \text{ axial, } \text{Ax}(\gamma) \cap W \neq \emptyset, \ l([\gamma]) \leq t\}, \quad \text{and} \\
P_h(t; W) &:= \#\{[\gamma] \mid \gamma \in \Gamma \text{ hyperbolic axial, } \text{Ax}(\gamma) \cap W \neq \emptyset, \ l([\gamma]) \leq t\}.
\end{align*}
\]

For the remainder of this section we further fix a base point \( o \in X \). Our first lemma gives an easy upper bound for \( P(t; W) \).

**Lemma 5.2** Let \( W \subset X \) be a bounded open set. Then there exists a constant \( b > 1 \) such that

\[
P(t; W) \leq be^{d(\Gamma) t}.
\]

**Proof.** Let \( \gamma \in \Gamma \) be a primitive axial isometry, and \( x \in \text{Ax}(\gamma) \cap W \). If \( r := \sup_{y \in W} d(o, y) \), then

\[
d(o, \gamma o) \leq d(o, x) + d(x, \gamma x) + d(\gamma x, \gamma o) < l(\gamma) + 2r.
\]

We conclude \( P(t, W) \leq N_{\Gamma}(t + 2r) \) and, by Corollary 3.3,\[ P(t; W) \leq be^{r(\Gamma)}e^{d(\Gamma) t}.
\]

From here on we fix a hyperbolic axial isometry \( h \in \Gamma \) and let \( c_0 := c(h) \geq 0 \) denote the width of \( h \) (see Definition 2.4). We further assume that \( W \subset X \) is a bounded open set which contains the closure of a ball of radius \( c_0 \) centered at a point \( y \) on an axis of \( h \).

In order to bound \( P_h(t; W) \) from below, we will need a few preliminary lemmata. The first one gives a straightforward lower bound for \( P_h(t; W) \).

**Lemma 5.3** There exist open neighborhoods \( U, V \subset \partial X \) of \( h^+ \), \( h^- \) with closures \( \overline{U}, \overline{V} \) homeomorphic to closed balls and \( \overline{U} \cap \overline{V} = \emptyset \) such that for all \( t > 0 \)

\[
P_h(t; W) \geq \#\{[\gamma] \mid \gamma \in \Gamma \text{ hyperbolic axial, } \gamma^- \in U, \ \gamma^+ \in V, \ l([\gamma]) \leq t\}.
\]

**Proof.** Recall that \( y \) is a point on an axis of \( h \) and \( W \supset \overline{B_y(c_0)} \). Let \( \varepsilon \in (0, 1) \) be arbitrarily small with the property \( \overline{B_y(c_0 + \varepsilon)} \subseteq W \), and \( U, V \subset \partial X \) the corresponding neighborhoods of \( h^- \), \( h^+ \) as in Lemma 2.2. Then if \( \gamma \) is hyperbolic axial with \( l(\gamma) = l([\gamma]) \leq t \), \( \gamma^- \in U \) and \( \gamma^+ \in V \), every axis \( \sigma \) of \( \gamma \) satisfies \( d(y, \sigma) < c_0 + \varepsilon \), hence \( \sigma \cap W \neq \emptyset \). We conclude that \( [\gamma] \) is contained in the set \( \{[\gamma] \mid \gamma \in \Gamma \text{ hyperbolic axial, } \text{Ax}(\gamma) \cap W \neq \emptyset, \ l([\gamma]) \leq t\} \).

The following lemma generalizes Lemma 5.4 in [K2]. It gives the necessary upper bound for the number of \( \gamma \in \Gamma \) which belong to the same equivalence class.

**Lemma 5.4** Let \( W \subset X \), \( y \in \text{Ax}(h) \), \( c_0 = c(h) \) and \( U, V \subset \partial X \) as in the previous lemma. Put

\[
\rho := \frac{1}{4} \min_{\gamma \in \Gamma \setminus \{id\}} \left( \inf_{x \in B_y(c_0 + 1)} d(x, \gamma x) \right).
\]

Let \( x \in X \), and \( \gamma_0 \in \Gamma \) a primitive hyperbolic axial element. Then there exists a constant \( b > 0 \) depending only on \( \rho \), \( c_0 \), \( U \), \( V \) and \( \text{N} = \dim X \) such that for all \( t > 0 \)

\[
\#\{ \gamma = \varphi \gamma_0^{k} \varphi^{-1} \mid \varphi \in \Gamma, \ k \in \mathbb{Z}, \ \varphi \gamma_0 \in U, \ \varphi \gamma_0^{-1} \in V, \ \text{Ax}(\gamma_0) \cap B_\rho(\rho) \cap B_y(c_0 + 1) \neq \emptyset, \ l(\gamma) \leq t\} \leq b \cdot t.
\]

**Proof.** Let \( t_0 := l(\gamma_0) \) and \( t \geq t_0 \). Then for \( \gamma = \varphi \gamma_0^{k} \varphi^{-1} \) with \( l(\gamma) \leq t \) we have \( t \geq l(\gamma_0^k) = |k|t_0 \), hence \( |k| \leq t/t_0 \).
We remark that if \( \varphi_{\gamma_0}^n \in U \) and \( \varphi_{\gamma_0}^+ \in V \), then every axis \( \sigma \) of \( \gamma_0 \) satisfies \( d(y, \varphi_\sigma) < c_0 + 1 \) by choice of \( U \) and \( V \).

If \( k \in \mathbb{Z} \setminus \{0\} \) is fixed, then \( \varphi_{\gamma_0}^k \varphi^{-1} \neq \beta_{\gamma_0}^k \beta^{-1} \) implies that \( \beta^{-1} \varphi \) does not belong to the centralizer of \( \gamma_0 \), in particular \( \varphi \text{Ax}(\gamma_0) \neq \beta \text{Ax}(\gamma_0) \).

Let \( F_0 \subset \text{Ax}(\gamma_0) \) be a fundamental domain for the action of \( \langle \gamma_0 \rangle \) on \( \text{Ax}(\gamma_0) \). If
\[
\varphi_{\gamma_0}^- \in U, \ \varphi_{\gamma_0}^+ \in V, \ \varphi \text{Ax}(\gamma_0) \cap B_x(\rho) \cap B_y(c_0 + 1) \neq \emptyset, \quad \text{and} \quad \beta_{\gamma_0}^- \in U, \ \beta_{\gamma_0}^+ \in V, \ \beta \text{Ax}(\gamma_0) \cap B_x(\rho) \cap B_y(c_0 + 1) \neq \emptyset,
\]
there exist \( p, q \in F_0 \) and \( n, m \in \mathbb{Z} \) such that \( \varphi_{\gamma_0}^n p, \beta_{\gamma_0}^m q \in B_x(\rho) \cap B_y(c_0 + 1) \). Furthermore \( g := \varphi_{\gamma_0}^n \neq \beta_{\gamma_0}^m =: f \) implies \( d(p, q) \geq 2\rho(W) \) since
\[
2\rho(W) \geq d(gp, fp) \geq d(gp, fp) - d(fp, fq) = d(gp, fp) - d(p, q) = d(gp, fp) - d(p, q) = 2\rho(W) - d(p, q).
\]
If \( d = \dim \text{Ax}(\gamma_0) \), then \( \text{vol}(F_0) = (2c_0 + 2)^{d-1} \cdot t_0 \leq 2^N(c_0 + 1)^{N-1} \cdot t_0 \) and
\[
\text{vol}(B_p(\rho) \cap \text{Ax}(\gamma_0)) = \omega_d \cdot \rho^d \quad \forall \rho \in \text{Ax}(\gamma_0).
\]
Put \( \omega := \min\{\omega_d \mid 1 \leq d \leq N\} \) and notice that \( \rho^d \geq \min\{1, \rho^N\} \). Since the balls of radius \( \rho \) centered at points in \( F_0 \) corresponding to different elements \( \varphi \in \Gamma \) are disjoint, there are at most
\[
\frac{\text{vol}(F_0)}{\omega_d \cdot \rho^d} \leq \frac{2^N(c_0 + 1)^{N-1} \cdot t_0}{\omega \cdot \min\{1, \rho\}^N}
\]
different elements of the form \( \varphi_{\gamma_0}^k \varphi^{-1} \) such that \( \varphi \text{Ax}(\gamma_0) \cap B_x(\rho) \cap B_y(c_0 + 1) \neq \emptyset \).

The assertion now follows from \( \#\{k \in \mathbb{Z} \mid |k| \leq t/t_0\} \leq 2t/t_0 \).

**COROLLARY 5.5** There exists a constant \( c > 0 \) depending only on \( c_0, U, V \) and \( N = \dim X \) such that for all \( t > 0 \)
\[
\#\{\gamma = \varphi_{\gamma_0}^k \varphi^{-1} \mid \varphi \in \Gamma, \ k \in \mathbb{Z}, \ \varphi_{\gamma_0}^- \in U, \ \varphi_{\gamma_0}^+ \in V, \ l(\gamma) \leq t\} \leq c \cdot t.
\]

**Proof.** We use the notations from the previous lemma and notice that by choice of \( U, V \) the conditions \( \varphi_{\gamma_0}^- \in U \) and \( \varphi_{\gamma_0}^+ \in V \) imply that every axis \( \sigma \) of \( \gamma_0 \) satisfies \( \varphi_\sigma \cap B_y(c_0 + 1) \neq \emptyset \).

Since \( B_y(c_0 + 1) \subset X \) is compact, there exist finitely many balls \( B_x(\rho), 1 \leq i \leq m \), such that
\[
\overline{B_y(c_0 + 1)} \subset \bigcup_{i=1}^m B_x(\rho).
\]
Hence if \( \varphi_{\gamma_0}^- \in U \) and \( \varphi_{\gamma_0}^+ \in V \), there exists \( j \in \{1, 2, \ldots, m\} \) such that \( \varphi \text{Ax}(\gamma_0) \cap (B_{x_j}(\rho) \cap B_y(c_0 + 1)) \neq \emptyset \). We conclude
\[
\#\{\gamma = \varphi_{\gamma_0}^k \varphi^{-1} \mid \varphi \in \Gamma, \ k \in \mathbb{Z}, \ \varphi_{\gamma_0}^- \in U, \ \varphi_{\gamma_0}^+ \in V, \ l(\gamma) \leq t\} \leq m \cdot a \cdot t.
\]

We will now state two more lemmata in order to relate \( P_h(t; W) \) to \( N_T(R; A, B) \) for appropriate sets \( A, B \subset \partial X \).

**LEMMA 5.6** Let \( \varepsilon > 0 \) and \( U, V \subseteq \partial X \) be the corresponding disjoint neighborhoods of \( h^+, h^- \) as in Lemma 5.5. Then there exist \( \alpha > 0 \) and \( R > 0 \) such that every \( \gamma \in \Gamma \) with \( d(\alpha, \gamma_0) \geq R \), \( \angle_\alpha(\gamma_0, h^+) < \alpha/2 \) and \( \angle_\alpha(\gamma^{-1}_0, h^-) < \alpha/2 \) satisfies \( \gamma U \cap V = \emptyset \).
Proof. Let $y \in \text{Ax}(h)$, put $c := c(h) + \varepsilon$ and choose $\delta > 0$ such that $C_{x,h,+}^\delta \subset U$ and $C_{y,h,-}^\delta \subset V$.

From Lemma 2.5 and Lemma 2.8 in [EO] it follows that for any $x \in X$ there exist $T(x) > 0$ and $\alpha(x) > 0$ such that

$$C_{x,h,+}^\alpha \setminus B_x(T(x)) \subseteq C_{y,h,+}^{\delta/2} \quad \text{and} \quad C_{x,h,-}^\alpha \setminus B_x(T(x)) \subseteq C_{y,h,-}^{\delta/2}.$$  

Since $T$ and $\alpha$ depend continuously on $x$, for $T := \max\{T(x)|x \in \{o\} \cup \overline{B_y(c)}\}$ and $\alpha := \min\{\alpha(x)|x \in \{o\} \cup \overline{B_y(c)}\}$ we have

$$C_{x,h,+}^\alpha \setminus B_o(T) \subseteq C_{y,h,+}^{\delta/2} \quad \text{and} \quad C_{x,h,-}^\alpha \setminus B_x(T) \subseteq C_{y,h,-}^{\delta/2} \quad \forall x \in \overline{B_y(c)} \cup \{o\}.$$  

Moreover, since for $\gamma \in \Gamma$ and $x \in \overline{B_y(c)}$ we have $d(\gamma x,\gamma o) = d(x,o) \leq c + d(y,o)$ and $d(\gamma^-1 y,\gamma^-1 o) = d(y,o)$, there exists $R > T + d(y,o) + c$ such that every $\gamma \in \Gamma$ with $d(o,\gamma o) > R$, $\angle_o(\gamma^-1 o,\gamma^-1 h^-) < \alpha/2$ and $\angle_o(\gamma^-1 o,\gamma^-1 h^-) < \alpha/2$ satisfies

$$\angle_y(\gamma^-1 o,\gamma^-1 h^-) < \delta/2 \quad \text{and} \quad \angle_x(\gamma^-1 y,\gamma^-1 h^-) < \delta/2 \quad \forall x \in \overline{B_y(c)}.$$  

Now for $\xi \in U$ arbitrary there exists a hyperbolic geodesic $\sigma$ joining $h^-$ to $\xi$ with $d(\gamma^\sigma,\xi) \leq c$. If $x \in \overline{B_y(c)}$ is the orthogonal projection of $y$ to $\sigma$, then $\angle_x(\gamma^-1 y,\xi) = \pi - \angle_x(\gamma^-1 y,\gamma^-1 h^-)$. Considering the triangle with vertices $\gamma^-1 y, x$ and $\xi$, we further have $\angle_{\gamma^-1 y}(\xi,x) + \angle_x(\gamma^-1 y,\xi) \leq \pi$.

We conclude

$$\angle_y(\gamma \xi,\gamma x) = \angle_{\gamma^-1 y}(\xi,x) \leq \angle_x(\gamma^-1 y,\gamma^-1 h^-) < \delta/2,$$

and therefore $\angle_y(\gamma \xi,\gamma h^-) \leq \angle_y(\gamma \xi,\gamma x) + \angle_y(\gamma x,\gamma h^-) < \delta$. In particular $\gamma \xi \in U$, which proves $\gamma U \cap V \subseteq U \cap V = \emptyset$. \hfill \Box

The following lemma is due to G. Knieper ([K1, Lemma 2.6]).

**Lemma 5.7** Let $U,V \subset \partial X$ be neighborhoods of $h^+, h^-$ with closures $\overline{U}, \overline{V}$ homeomorphic to closed balls and $\overline{U} \cap \overline{V} = \emptyset$. Then there exists a constant $\tau > 0$, $n \in \mathbb{N}$ such that for all $\gamma \in \Gamma$ with $d(o,\gamma o) < t$ and $\gamma U \cap V = \emptyset$, the isometry $h^n \gamma h^n$ possesses an axis with extremities in $U$ and $V$ and $l(h^n \gamma h^n) \leq t + \tau$.

**Proof.** Let $n \in \mathbb{N}$ such that $h^n(\overline{X} \setminus V) \subset U$ and $h^{-n}(\overline{X} \setminus U) \subset V$. Since $\gamma U \cap V = \emptyset$, we have $h^n \gamma h^n(\overline{U}) \subseteq h^n \gamma U \subseteq h^n(\overline{X} \setminus V) \subset U$ and $h^{-n} \gamma h^{-n}(\overline{V}) \subseteq h^{-n} \gamma h^{-n}(\overline{V}) \subseteq h^{-n} \gamma^{-1} \subseteq h^{-n} \gamma^{-1} \subseteq h^{-n} \gamma^{-1} \subseteq h^{-n} \gamma^{-1} \subseteq \overline{V}$. Since $\overline{U}$ and $\overline{V}$ are each homeomorphic to a closed ball, Brouwer’s fixed point theorem implies that $h^n \gamma h^n$ assumes its fixed points in $U$ and $V$. Furthermore,

$$d(o, h^n \gamma h^n o) \leq d(h^{-n} o, o) + d(o, \gamma o) + d(\gamma o, h^n o) \leq 2d(o, h^n o) + t.$$ \hfill \Box

**Theorem 5.8** Let $M$ be a complete Riemannian manifold of nonpositive sectional curvature with universal Riemannian covering manifold $X$, and $\Gamma \subset \text{Isom}(X)$ the group of deck transformations of the covering projection. Suppose $\Gamma$ contains a hyperbolic axial isometry $h$ and does not globally fix a point in $\partial X$. Let $W \subset X$ be a bounded open set which contains a closed ball of radius $c(h) \geq 0$ (as in Definition 2.7) centered at a point on an axis of $h$. Then

$$\delta(\Gamma) = \lim_{t \to \infty} \left( \frac{1}{t} \log P(t;W) \right) = \lim_{t \to \infty} \left( \frac{1}{t} \log P_h(t; W) \right).$$

For weakly cocompact $\Gamma$ there exist $b > 0$ and $R > 0$ such that for $t > R$

$$\frac{1}{bt} e^{b(\delta(\Gamma) t)} \leq P_h(t; W) \leq P(t; W) \leq b e^{b(\delta(\Gamma) t)}.$$
**Proof.** We choose \( y \in Ax(h) \) such that \( W \supset B_y(c(h)) \) and open subsets \( U, V \subseteq \partial X \) of \( h^+, h^- \) with closures \( \overline{U}, \overline{V} \) homeomorphic to closed balls and \( U \cap \overline{V} = \emptyset \) as in Lemma 5.3. By Lemma 5.7 there exists a constant \( \tau > 0 \) such that for any \( t > 0 \)

\[
\#\{ \gamma \in \Gamma \mid d(o, \gamma o) \leq t, \gamma U \cap V = \emptyset \} \leq \#\{ \gamma \in \Gamma \mid \gamma \text{ hyperbolic axial with } \gamma^- \in U, \gamma^+ \in V, l(\gamma) \leq t + \tau \}.
\]

By Lemma 5.6 there exist sets \( A \subset U, B \subset V \) and a constant \( R > 0 \) such that for all \( t > R \)

\[
\#\{ \gamma \in \Gamma \mid d(o, \gamma o) \leq t, \gamma U \cap V = \emptyset \} \geq N_\Gamma(t; A, B) - N_\Gamma(R; A, B).
\]

Using Lemma 5.3 and Corollary 5.5 we conclude that for \( t > \tau + R \)

\[
P_h(t; W) \geq \frac{1}{e \cdot t} \#\{ \gamma \in \Gamma \mid \gamma \text{ hyperbolic axial with } \gamma^- \in U, \gamma^+ \in V, l(\gamma) \leq t \}
\]

\[
\geq \frac{1}{e \cdot t} (N_\Gamma(t - \tau; A, B) - b e^{\delta(\Gamma) R}),
\]

which, together with Corollary 4.3 and Lemma 5.2 proves the assertion. \( \square \)

We remark that for compact manifolds \( M \) we may choose a bounded open set \( W \subset X \) which contains a fundamental domain for the action of \( \Gamma \). Hence Theorem 5.8 implies Theorem B of G. Knieper ([K2]).

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