Pareto-optimal energy sharing between battery-equipped renewable generators

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ABSTRACT

The inherent intermittency of renewable sources like wind and solar has resulted in a bundling of renewable generators with storage resources (batteries) for increased reliability. In this paper, we consider the problem of energy sharing between two such bundles, each associated with their own demand profiles. The demand profiles might, for example, correspond to commitments made by the bundle to the grid. With each bundle seeking to minimize its loss of load rate, we explore the possibility that one bundle can supply energy to the other at times of deficit, in return for a reciprocal supply from the other when it faces a deficit itself. We show that there always exist mutually beneficial energy sharing arrangements between the two bundles. Moreover, we show that Pareto-optimal arrangements involve at least one bundle transferring energy to the other at the maximum feasible rate at times of deficit. We illustrate the potential gains from such dynamic energy sharing via an extensive case study.

CCS CONCEPTS
• Applied computing → Economics; Decision analysis;

KEYWORDS
Renewable generation, energy storage, dynamic energy sharing, Pareto-optimality, bargaining solutions

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1 INTRODUCTION

Environmental concerns are driving a worldwide push towards the adoption of renewable sources for electricity generation. However, the primary challenge in increasing the penetration of renewable sources in the electricity grid is the intermittency and unpredictability of their generation. As a result, renewable sources are increasingly being bundled with storage devices (batteries) to smoothen out the temporal intermittency in generation, enabling the generator to supply power to the grid according to a contracted supply profile with high reliability. In this paper, we explore dynamic energy sharing between two such bundles (also referred to as agents), each consisting of a renewable generator and a battery, with the goal of enhancing the reliability of both bundles.

The idea behind dynamic energy sharing is that one can exploit statistical diversity between the net generation (supply minus demand) processes of the bundles, so that one agent can supply energy to the other (from its battery) in times of deficit. For example, a solar generator and a wind generator could opportunistically supply energy to one another, in order to meet their respective commitments to the grid. However, such a sharing arrangement would only work if it benefits both parties involved, i.e., the sharing arrangement must be mutually beneficial. Interestingly, naive complete pooling, wherein the two batteries are treated as one common resource, may not have this property. This motivates us to consider partial sharing mechanisms, wherein each agent supplies energy to the other to meet its deficit, but only up to a pre-specified drain rate (i.e., an upper bound on the rate of energy supply). One of our main contributions is to show that there always exist mutually beneficial partial sharing mechanisms of this kind.

Having established the existence of mutually beneficial sharing configurations, the next natural step is to capture the Pareto frontier of efficient sharing configurations. Remarkably, we are able to provide a precise characterization of this Pareto frontier; all Pareto-optimal configurations involve at least one agent allowing the other to draw energy at the maximum possible rate in times of deficit. Given this characterization of the Pareto frontier, one can capture the sharing arrangements that would emerge between the agents by appealing to the theory of bargaining [20].

Structurally, our work is related to the vast literature on resource pooling in service systems and networks. This body of work explores the sharing of a resource, such as service capacity, bandwidth, cache memory, etc., to exploit statistical economies of scale [2, 5, 8, 12, 13, 22, 23]. However, the primary style of pooling considered in this literature is complete pooling, wherein the entities ‘merge’ by pooling their resources completely, with the payoff of the coalition being split among its members using ideas from cooperative game theory. In contrast, our interest here is in the setting of non-transferable utility, wherein no side payments occur between the agents, implying that an agent would agree to a sharing arrangement only if doing so increases its own (private, non-transferable) utility. The only other work we are aware of...
that takes this approach is [21], which focuses on partial server sharing between two Erlang-B loss systems. However, the system modeling, sharing mechanism, and its analysis differ considerably between [21] and the present paper.

At its core, our model involves inventory pooling between two agents, each having its own supply and demand process. These agents might correspond to renewable generators connected to the grid with their own contracted supply commitments, prosumers within a smart microgrid (as in [24]), or even different remote microgrids that can help each another increase their reliability. Our main departure from the prior literature on these topics (described below) is the cooperative game theoretic framework of non-transferable utility. This is applicable in situations where agents cannot balance a certain loss of reliability with a monetary reward. Instead, we appeal to the theory of bargaining to balance the benefit of both agents from the sharing arrangement. For example, in the case of renewable generators participating in electricity markets, maintaining high reliability might be a pre-requisite for market participation. Similarly, remote microgrids might prefer not to set up a system of monetary payments, but to simply maintain equity in the benefits from energy sharing.

The remainder of this paper is organised as follows. After a quick review of the related literature below, we describe our system model and our energy sharing mechanism in Section 2. We establish key monotonicity properties of our sharing model in Section 3. Pareto-optimal sharing configurations are addressed in Section 4. We present a case study in Section 5, and conclude in Section 6.

Related literature
There are two broadly two strands of work on dynamic energy sharing between renewable generators (or prosumers). The first treats energy trading as an economic transaction, using prices to design market mechanisms for efficient energy transfer; papers that take this approach include [6, 7, 10, 11, 16–18].

The second strand of work treats energy sharing from the standpoint of a central optimizer, who is interested in maximizing social welfare. For example, [15] analyzes the tradeoff between sharing and the use of storage within smart microgrid. Another work in this space is [24], which focuses on scheduling algorithms for energy transfer within a microgrid for loss minimization.

In contrast to the first stream of work, the present paper considers the non-transferable utility setting, i.e., one where there are no side-payments between the agents. In contrast to the second strand, we still consider the agents as strategic in that they participate in energy sharing mechanisms with the sole objective to enhancing their own reliability.

2 MODEL AND PRELIMINARIES
We begin by describing the ‘standalone’ setting, i.e., in the absence of a sharing arrangement between the agents. We use the following notation throughout the paper: For $x \in \mathbb{R}$,

$$\begin{align*}
[x]_+ &= (x)_+ = \max(0, x), \\
[x]_- &= (x)_- = -\min(0, x).
\end{align*}$$

Also, when referring to any Agent $i$, we refer to the other agent as Agent $\neg i$.

2.1 Standalone setting
We consider two agents, Agent 1 and Agent 2, each associated with a stochastic supply and demand process, and equipped with a battery. The battery of Agent $i$ has capacity $B_i$. The energy content in the battery is modulated by each agent’s net generation (i.e., supply minus demand) process. Specifically, we denote the net (power) generation process of Agent $i$ by $r_i(t) = g_i(t) - d_i(t)$, where $g_i(t)$ and $d_i(t)$ denote, respectively, the supply and demand process of Agent $i$. For example, $g_i(t)$ might represent the power generation from a renewable generator, and $d_i(t)$ its contracted commitment to the grid. Thus, the energy content in the battery of Agent $i$, denoted by $b_i(t)$, evolves as follows:

$$\frac{db_i(t)}{dt} = \begin{cases} 0 & \text{if } b_i(t) = 0 \text{ and } r_i(t) < 0, \\
0 & \text{if } b_i(t) = B_i \text{ and } r_i(t) > 0, \\
r_i(t) & \text{otherwise.}
\end{cases}$$

The above dynamics capture the boundary conditions that a fully charged battery cannot be charged further, and an empty battery cannot be discharged further. Except for these boundary cases, the battery is charged/discharged at the net generation rate $r_i(t)$. Further, we assume that the net generation rates $r_1(t)$ and $r_2(t)$ are dependent on the state of a common background Markov process, which captures the supply and demand uncertainty of each agent.

Formally, let $\{X(t)\}$ denote the background Markov process. We assume that $\{X(t)\}$ is an irreducible continuous-time Markov chain (CTMC) over a finite state space $S$. The net generation rate of Agent $i$ is a function of the state of this background process, i.e., $r_i(t) = r_i(X(t))$.

To ensure regeneration of the buffer occupancy processes, we assume there exist in $S$, states $s_1$, $s_2$, $s_3$, and $s_4$ satisfying:

- $r_1(s_1), r_2(s_1) > 0$,
- $r_1(s_2), r_2(s_2) < 0$,
- $r_1(s_3) > 0$, $r_2(s_3) < 0$, and $r_2(s_3) \leq r_2(s) \forall s \in S$,
- $r_1(s_4) < 0$, $r_2(s_4) > 0$, and $r_1(s_4) \leq r_1(s) \forall s \in S$.

State $s_1$ (respectively, $s_2$) results in positive (respectively, negative) net generation for both agents. State $s_3$ results in a positive net generation for Agent 1 and a negative net generation for Agent 2, such that this is also the most negative net generation for Agent 2. State $s_4$ is similar, except the roles of the agents are reversed.

We note here that this model is quite general; the state of the background process can capture all factors that influence the supply and demand of each agent, including past and present weather conditions, as well as time of day. Moreover, the model allows for the net generation processes of both agents to be correlated in an arbitrary fashion. Finally, ramp constraints on battery charging/discharging can also be incorporated into the model; these would simply limit the values that the (battery modulating) net generation process can take.

The performance of each agent is measured via its loss of load rate (LLR), which is the long run average rate of lost load (unmet demand):
demand. Based on the battery dynamics (1), note that Agent $i$ is unable to cater to its demand when $b_i(t) = 0$ and $r_i(t) < 0$, i.e., the battery is empty and the instantaneous generation is insufficient to meet the instantaneous demand. Thus, the standalone loss of load rate of Agent $i$ is defined as
\[
\text{LLR}^{sa}_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t 1 \{ b_i(t) = 0 \} |r_i(t)| \, dt \quad \text{(almost surely)}.
\] (2)
Here, $1(A)$ equals 1 if $A$ is true and zero otherwise. Each agent seeks to minimize its loss of load rate. Indeed, it would be natural for the grid to penalize a renewable generator in terms of the LLR relative to its contracted supply curve [3, 4]. Note that the existence of the almost sure limit in (2) follows from the positive recurrence of the background Markov process.

For any agent, the evolution of its battery level can be modeled as a Markov modulated fluid queue, a well studied object in the queueing literature [1, 19]. In particular, it is easy to see that corresponding to Agent $i$, $(X(t), b_i(t))$ is a Markov process over state space $\mathcal{S} \times [0, B_i]$, whose invariant distribution can be computed by solving a system of ordinary differential equations (see [1, 19]). This invariant distribution can in turn be used to compute $\text{LLR}_i$ [9]. However, in the present paper, we are interested not in the ‘standalone’ behavior of each individual agent, but in dynamic energy sharing arrangements between the two. (This naturally couples the evolution of the two batteries, necessitating a joint analysis of both battery occupancies.) Our proposed mechanism for dynamic energy sharing is described next.

### 2.2 Sharing Mechanisms

For either agent, loss of load is undesirable (and might even result in penalty from the grid [3, 4]). The motivation for dynamic energy sharing is that when Agent $i$ faces loss of load, Agent $-i$ can supply energy to Agent $i$ from its battery to (either completely or partially) satisfy the unmet demand. However, if no constraints are placed on such energy transfer, i.e., if each agent can draw energy from the other’s battery without restriction, the resulting configuration might not be beneficial to both parties. Indeed, if Agent $i$ is considerably more likely to run a deficit than Agent $-i$, it is natural to expect that unconstrained energy sharing would actually be detrimental to Agent $-i$. This motivates us to explore constrained or partial energy sharing arrangements.

Our energy sharing arrangement is characterized by the tuple $(c_1, c_2)$. Informally, $c_1$ is the maximum energy drain rate allowed by Agent $i$ when Agent $-i$ faces loss of load. Additionally, we assume that there is a capacity constraint $c$ on the rate of energy transfer from either agent to the other. This captures any physical transmission constraints that limit the rate of energy transfer between the agents. Thus, $c_1, c_2 \leq c$.

Formally, if Agent $-i$ runs a deficit at time $t$, i.e., $b_{-i}(t) = 0$ and $r_{-i}(t) < 0$, then:

1. If $b_i(t) \in (0, B_i)$, then Agent $i$ transfers energy to Agent $-i$ at rate

   \[
   \text{etr}_i(t) = \min(c_i, -r_{-i}(t)).
   \]

   Here, \text{etr} stands for energy transfer rate. In other words, Agent $i$ helps cut the deficit rate of Agent $-i$ subject to a maximum transfer rate of $c_i$.

2. If $b_i(t) = 0$ and $r_i(t) > 0$, then Agent $i$ transfers energy to Agent $-i$ at rate

   \[
   \text{etr}_i(t) = \min(r_i(t), c_i, -r_{-i}(t)).
   \]

   In this case, since Agent $i$ cannot draw energy from its battery, its ability to transfer energy to Agent $-i$ is further limited by its own net generation rate $r_i(t)$. If $b_i(t) = 0$ and $r_i(t) \leq 0$, then Agent $i$ itself faces loss of load, and therefore cannot transfer energy to Agent $-i$, i.e., $\text{etr}_i(t) = 0$.

3. If $b_i(t) = B_i$, then Agent $i$ transfers energy to Agent $-i$ at rate

   \[
   \text{etr}_i(t) = \min(c_i, r_i(t), -r_{-i}(t)).
   \]

   In the former case, the nominal transfer rate of min$(c_i, -r_{-i}(t))$ exceeds the net generation rate of Agent $i$, and so Agent $i$ begins to discharge its battery to help Agent $-i$. In the latter case, the net generation rate of Agent $i$ exceeds the nominal transfer rate, and so Agent $i$ does not discharge its battery, but simply transfers its surplus generation to Agent $-i$ subject to the capacity constraint $c$.

Additionally, when Agent $i$ has a fully charged battery with a positive net generation rate, we assume that it transfers its overflow rate to Agent $-i$ (whether or not Agent $-i$ faces loss of load), again subject to the capacity constraint $c$. Of course, if both batteries are fully charged, then the overflow is lost. The above sharing mechanism is summarized in Table 1.

It is important to note that under the proposed mechanism, unless Agent $i$’s battery is full, it only transfers energy to Agent $-i$ when the latter faces loss of load. Moreover, the energy transfer in this case only covers (part of) the deficit rate, it does not actually charge the battery of Agent $-i$.

Next, we define the loss of load rate for each agent under the proposed sharing mechanism.

### 2.3 LLR characterization

Under the sharing configuration $(c_1, c_2)$, the loss of load rate of each agent is characterized as follows. When Agent $i$ faces loss of load, i.e., $b_i(t) = 0$ and $r_i(t) < 0$, then the instantaneous rate at which it loses load equals $[-r_i(t) - \text{etr}_i(t)]_+$. Thus, its loss of load rate is given by

\[
\text{LLR}_i(c_1, c_2) = \lim_{t \to \infty} \frac{1}{t} \int_0^t 1 \{ b_i(t) = 0, r_i(t) < 0 \} [-r_i(t) - \text{etr}_i(t)]_+ \, dt.
\]

As before, the above limit is in an almost sure sense, and its existence is guaranteed by the positive recurrence of the background process. It is important to note that the sharing configuration $(0, 0)$ is not equivalent to the standalone setting, since even under the $(0, 0)$ configuration, the agents supply (overflow) energy to one another when their batteries are full. In fact, it can be shown that $\text{LLR}_i(0, 0) < \text{LLR}^{sa}_i$ for $i = 1, 2$. 
Indeed, a closed form for the loss of load rate does not exist even for the ‘standalone’ setting. However, we are still remarkably able to analytically establish the following results.

- There exist mutually beneficial sharing configurations. Specifically, there exists a configuration \((c_1, c_2) \in [0, c]^2\) such that \(\text{LLR}_i(c_1, c_2) < \text{LLR}_i(0, 0) < \text{LLR}_i^{\text{max}}\) for \(i = 1, 2\).
- The Pareto frontier of efficient, mutually beneficial, sharing configurations is non-empty. All configurations on the Pareto frontier involve at least one agent sharing energy with the other at the maximum possible rate, i.e., \(c_i = c\) for at least one \(i \in \{1, 2\}\).

The above results are proved using monotonicity properties of the loss of load rates with respect to \(c_1\) and \(c_2\). These monotonicity properties, which are illuminating in their own right, are the focus of the following section.

### 3 MONOTONICITY PROPERTIES

Without loss of generality, we assume that

\[
 c_i \leq c_{i, \text{max}} := \min(c, \max\{r_i(s)\} - s)
\]

for \(i \in \{1, 2\}\). Given the sharing mechanism described in Section 2.2, it is easy to see that increasing \(c_i\) beyond \(c_{i, \text{max}}\) does not influence the realised energy sharing, since agents only help fulfill the other’s deficit rates (except under overflow, which is not constrained by \((c_1, c_2)\)).

The main result of this section is the following.

**Theorem 3.1. Under the proposed sharing mechanism,**

- \(\text{LLR}_1(c_1, c_2)\) is strictly increasing in \(c_1\) over \([0, c_{1, \text{max}}]\).
- \(\text{LLR}_2(c_1, c_2)\) is strictly decreasing in \(c_1\) over \([0, c_{1, \text{max}}]\).
- \(\text{LLR}_1(c_1, c_2) + \text{LLR}_2(c_1, c_2)\) is strictly decreasing in \(c_1\) over \([0, c_{1, \text{max}}]\).

The first two statements of Theorem 3.1 are intuitive: if the peak rate \(c_i\) of energy transfer from Agent \(i\) to Agent \(-i\) is increased, the loss of load rate of Agent \(i\) increases, while that of Agent \(-i\) decreases. The third statement shows that an increase in \(c_i\) decreases the overall loss of load rate across the two agents. In other words, an increase in \(c_i\) is detrimental to Agent \(i\), beneficial to Agent \(-i\), and beneficial to the overall system LLR. Immediate takeaways from Theorem 3.1 are the following. First, the ‘socially optimal’ configuration is \((c_{1, \text{max}}, c_{2, \text{max}})\). Second, interpreting \(c_i\) to be Agent \(i\)’s action, the only Nash equilibrium of the resulting two-player game is \((0, 0)\). This means that a non-cooperative setting does not yield efficient sharing configurations. Instead, mutually beneficial sharing configurations can only be sustained via binding agreements between the agents, in the spirit of bargaining theory.

The remainder of this section is devoted to the proof of Theorem 3.1. We are able to prove monotonicity properties of the loss of load rates without any explicit expressions of the same using sample path techniques. Specifically, we consider two identical instances of our model, one operating under sharing configuration \((c_1, c_2)\), where \(c_1 < c_{1, \text{max}}\) and the other operating under sharing configuration \((c_1 + \epsilon, c_2)\), where \(\epsilon > 0\) such that \(c_1 + \epsilon < c_{1, \text{max}}\). We refer to the former system as the ‘original system’ and the latter one as the ‘˜ system’; we denote parameters pertaining to this latter system with a ˜ accent. To prove Theorem 3.1, it suffices to show that

\[
\text{LLR}_1 > \text{LLR}_1, \quad \text{LLR}_2 < \text{LLR}_2, \quad \text{LLR}_1 + \text{LLR}_2 < \text{LLR}_1 + \text{LLR}_2.
\]

We prove these inequalities by coupling the sample paths of the background process across these systems. In other words, both systems see exactly the same net generation at all times, and all that distinguishes these systems is the upper bound on the energy transfer rate from Agent 1 to Agent 2 \((c_1\) in the original system versus \(c_1 + \epsilon\) in the ˜ system). Our first result compares the battery levels of the two agents across the two systems.

**Lemma 3.2. On any sample path, under the coupling between the original and ˜ system described above, if \(b_1(0) = \hat{b}_1(0), b_2(0) = \hat{b}_2(0)\), then for all \(t \geq 0,\)

\[
\hat{b}_1(t) \leq b_1(t), \quad \hat{b}_2(t) \leq b_2(t).
\]

Lemma 3.2 states that in the ˜ system, which permits a higher rate of energy transfer from Agent 1 to Agent 2 relative to the original system, the battery occupancies of both agents get reduced. The intuition behind this result is the following. Since Agent 1 transfers more energy to Agent 2 in the ˜ system, its battery occupancy gets reduced relative to the original system. Importantly however, this increased energy transfer from Agent 1 to Agent 2 is utilized purely in the following sense:

\[
\hat{b}_1(t), \hat{b}_2(t) \text{ get modulated by the same net generation process } r_1(t), \text{ while battery levels } b_1(t), b_2(t) \text{ get modulated by same net generation process } r_2(t).
\]
to cut the loss of load suffered by Agent 2, and not to charge its battery. Over time, this causes a reduction in the battery occupancy of Agent 2 as well, due to (i) reduced energy transfer from Agent 1 in the form of battery overflow (which can be used to charge Agent 2’s battery), and (ii) increased energy transfer to Agent 1 when it faces loss of load. This intuition is formalized in our proof of Lemma 3.2, which can be found in the appendix.

An immediate consequence of Lemma 3.2 is the following lemma. Let \( \tilde{O}_i(t) \) (respectively, \( \tilde{\ell}_i(t) \)) denote the cumulative energy lost due to battery overflow in the interval \([0, t]\) from the battery of Agent \( i \) in the original system (respectively, in the \( \sim \) system). Also, let \( \ell_i(t) \) (respectively, \( \tilde{\ell}_i(t) \)) denote the cumulative lost load by Agent \( i \) in the original system (respectively, in the \( \sim \) system). Specifically, note that for \( i \in \{1, 2\} \),

\[
\text{LLR}_i = \lim_{t \to \infty} \frac{\ell_i(t)}{t}, \quad \text{LLR}_i = \lim_{t \to \infty} \frac{\tilde{\ell}_i(t)}{t}.
\]

**Lemma 3.3.** On any sample path, under the coupling between the original and \( \sim \) system described above, if \( b_1(0) = b_1(0), b_2(0) = b_2(0) \), then for all \( t \geq 0 \),

\[
\tilde{\ell}_i(t) \geq \ell_i(t), \quad \tilde{O}_i(t) \leq O_i(t) \quad \forall \ i \in \{1, 2\}.
\]

Lemma 3.3 states that the \( \sim \) system ‘wastes’ less energy due to overflow from either battery, as compared to the original system. Further, it shows that in the \( \sim \) system, Agent 1 incurs greater loss of load compared to the original system.

**Proof of Lemma 3.3.** Since \( \tilde{b}_1(t) \leq b_1(t) \) (Lemma 3.2), if Agent 1 faces loss of load in the original system at any time \( t \), then it also faces loss of load in the \( \sim \) system at that time. Moreover, since (i) \( b_2(t) \leq \tilde{b}_2(t) \) (also by Lemma 3.2), and (ii) the bound \( c_2 \) on the rate of energy transfer from Agent 2 to Agent 1 is the same in both systems, it follows that the instantaneous rate of lost load in the \( \sim \) system exceeds that in the original system at all times. This implies (4). It is important to note that a similar argument does not hold for Agent 2, since it enjoys a higher rate of energy transfer from Agent 1 in the \( \sim \) system.

A similar line of reasoning can also be used to prove the inequalities for cumulative energy lost due to overflow (5). Since \( \tilde{b}_1(t) \leq b_1(t) \) (Lemma 3.2) and given that the batteries are driven by the same net generation process across both systems, an overflow out of Battery \( i \) in the \( \sim \) system at any time implies an overflow at at least the same rate out of Battery \( i \) in the original system. \( \square \)

We are now ready to give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** From Lemma 3.3, it follows that

\[
\lim_{t \to \infty} \frac{\tilde{\ell}_1(t)}{t} \geq \lim_{t \to \infty} \frac{\ell_1(t)}{t} \quad \Rightarrow \quad \text{LLR}_1 \geq \text{LLR}_1.
\]

That the latter inequality is strict follows from a straightforward renewal reward argument, which we sketch here. Given our coupling between the original and \( \sim \) systems, consider a renewal process, where the renewal instants correspond to hitting times of the configuration \( X(t) = g_2 \) and \( b_2(t) = 0 \) for \( i = 1, 2 \). Let \( \ell_i(n) \) (respectively, \( \tilde{\ell}_i(n) \)) denote the lost load in the \( n \)th renewal cycle by Agent \( i \) in the original (respectively, the \( \sim \) system. Thus, by the renewal reward theorem,

\[
\text{LLR}_i = \mathbb{E} \left[ \frac{\ell_i(1)}{T} \right], \quad \text{LLR}_i = \mathbb{E} \left[ \frac{\tilde{\ell}_i(1)}{T} \right].
\]

where \( T \) denotes the length of a typical renewal cycle. Using the above characterization, it suffices to show that \( \mathbb{E} \left[ \tilde{\ell}_i(1) \right] > \mathbb{E} \left[ \ell_i(1) \right] \).

Since \( \tilde{\ell}_1(t) \geq \ell_1(t) \) on all sample paths, one has to simply argue that with positive probability, \( \tilde{\ell}_1(t) > \ell_1(t) \). This is not hard to show.

Thus, we have

\[
\text{LLR}_1 \geq \text{LLR}_1.
\]

It therefore follows more generally that \( \text{LLR}_1(c_1, c_2) \) is strictly increasing in \( c_i \).

Next, recall that under our coupling of the background process for the original system and the \( \sim \) system, the total energy received in original system within any time interval \([0, t]\) is equal to the total energy received in the \( \sim \) system within the same time interval. Thus,

\[
b_1(t) + b_2(t) + \ell_1(t) + \ell_2(t) + O_1(t) + O_2(t) = \tilde{b}_1(t) + \tilde{b}_2(t) + \tilde{\ell}_1(t) + \tilde{\ell}_2(t) + \tilde{O}_1(t) + \tilde{O}_2(t).
\]

where \( \ell_1(t) \) (respectively, \( \ell_2(t) \)) is the cumulative load catered (i.e., demand supplied) by Agent 1 over the interval \([0, t]\) in the original system (respectively, the \( \sim \) system). Since \( O_1(t) \geq \tilde{O}_1(t) \) (by Lemma 3.3),

\[
O_1(t) + O_2(t) \geq \tilde{O}_1(t) + \tilde{O}_2(t).
\]

Similarly, since \( b_1(t) \geq \tilde{b}_1(t) \) (by Lemma 3.2), we have

\[
b_1(t) + b_2(t) \geq \tilde{b}_1(t) + \tilde{b}_2(t).
\]

Therefore, using (8) and (9) in (7), we get

\[
\ell_1(t) + \ell_2(t) \leq \tilde{\ell}_1(t) + \tilde{\ell}_2(t)
\]

i.e., the total cumulative load served or demand supplied over \([0, t]\) in the original system is less than or equal to that in the \( \sim \) system. It follows now that within the interval \([0, t]\), the cumulative lost load in original system is greater than or equal to that in \( \sim \) system, i.e., \( \ell_1(t) + \ell_2(t) \geq \tilde{\ell}_1(t) + \tilde{\ell}_2(t) \), which implies

\[
\left( \lim_{t \to \infty} \frac{\ell_1(t)}{t} + \frac{\ell_2(t)}{t} \right) \leq \left( \lim_{t \to \infty} \frac{\tilde{\ell}_1(t)}{t} + \frac{\tilde{\ell}_2(t)}{t} \right) \Rightarrow \text{LLR}_1 + \text{LLR}_2 \leq \text{LLR}_1 + \text{LLR}_2.
\]

Again, it can be shown that the above inequality is strict via a renewal reward argument, so that

\[
\text{LLR}_1 + \text{LLR}_2 < \text{LLR}_1 + \text{LLR}_2.
\]

This shows that more generally, \( \text{LLR}_1(c_1, c_2) + \text{LLR}_2(c_1, c_2) \) is strictly decreasing with respect to any \( c_i \).

Finally, it follows from (6) and (10) that \( \text{LLR}_2 < \text{LLR}_2 \), which establishes that \( \text{LLR}_i(c_1, c_2) \) is strictly decreasing in \( c_i \). \( \square \)

Armed with Theorem 3.1, we address the existence of mutually beneficial sharing configurations, and the structure of the Pareto frontier of efficient sharing configurations in the following section. We conclude this section by noting that \( \text{LLR}_i(\cdot, \cdot) \) is a continuous function, specifically Lipschitz continuous.

**Lemma 3.4.** For \( i \in \{1, 2\} \), \( \text{LLR}_i(c_1 + \epsilon, c_2) = \text{LLR}_i(c_1, c_2) \leq \epsilon. \)
We also define which implies that 4 alone operation (i.e., without overflow sharing).

An immediate consequence of Lemma 3.4 is that \( LLR_i(\cdot, \cdot) \) is differentiable almost everywhere.

4 PARETO-OPTIMAL SHARING

In this section, our goal is to shed light on meaningful sharing configurations that the agents might agree upon from a game theoretic standpoint. Our first step is to show that there exist sharing configurations that are beneficial to both agents.

4.1 Existence of mutually beneficial configurations

Recall that \( c_i \) denotes the maximum rate at which Agent \( i \) shares energy with Agent \( i \) when the latter faces loss of load. Moreover, \( c_i \leq c_{i, \max} \); this constraint incorporates any physical transmission constraints that limit the flow of energy from one agent to the other. Thus, the space of possible sharing configurations is given by

\[
X = [0, c_{1, \max}] \times [0, c_{2, \max}].
\]

We also define

\[
X^o = [0, c_{1, \max}] \times [0, c_{2, \max}].
\]

Note that \( X^o \) excludes those configurations where either or both of the parameters \( c_i \) take their maximum value.

Our first result shows that given any configuration in \( X^o \), it is possible to perturb it in such a manner that both agents are comparatively better off. Specializing this result to the configuration \((0, 0)\), we conclude that mutually beneficial configurations are guaranteed to exist.\(^6\) Remarkably, this is true no matter how asymmetric the net generation processes might be across the agents; it is always possible to find sharing arrangements that enable the agents to help one another.

**Lemma 4.1.** For any sharing configuration \((c_1, c_2) \in X^o\), there exists a direction \((1, \theta)\), where \( \theta > 0 \), such that the gradients \( \nabla LLR_i(c_1, c_2) \) satisfy \( \nabla LLR_i(c_1, c_2) \cdot (1, \theta) < 0 \) \( \forall i \in \{1, 2\} \).

**Proof of Lemma 4.1.** Since the total loss of load \( LLR_1(c_1, c_2) + LLR_2(c_1, c_2) \) strictly decreases with \( c_1 \) (see Theorem 3.1), differentiating with respect to \( c_1 \) and \( c_2 \), we get

\[
\frac{\partial LLR_1}{\partial c_1} < -\frac{\partial LLR_2}{\partial c_1}, \quad \frac{\partial LLR_2}{\partial c_2} < -\frac{\partial LLR_1}{\partial c_2}.
\]

\[
\Rightarrow \frac{\partial LLR_1}{\partial c_1} \frac{\partial LLR_2}{\partial c_2} < (-\frac{\partial LLR_2}{\partial c_1})(-\frac{\partial LLR_1}{\partial c_2}) \Rightarrow \frac{\partial LLR_1}{\partial c_1} < \frac{\partial LLR_2}{\partial c_2}.
\]

Therefore, there exists a \( \theta \) such that 5

\[
\frac{\partial LLR_1}{\partial c_1} < \theta < \frac{\partial LLR_2}{\partial c_2}
\]

which implies that \( \nabla LLR_1 \cdot (1, \theta) < 0 \) and \( \nabla LLR_2 \cdot (1, \theta) < 0 \). \( \square \)

Next, we turn to the set of Pareto-optimal sharing configurations.

4.2 Pareto frontier

We begin by defining Pareto-optimal configurations.

**Definition 4.2.** An energy sharing configuration \((c_1, c_2) \in X\) is Pareto-optimal if there does not exist \((\tilde{c}_1, \tilde{c}_2) \in X\) such that \( LLR_i(\tilde{c}_1, \tilde{c}_2) \leq LLR_i(c_1, c_2) \) for all \( i \in \{1, 2\} \), the inequality being strict for at least one \( i \).

The set of Pareto-optimal sharing configurations is denoted by \( \mathcal{P} \), and is referred to as the Pareto frontier. Pareto-optimal sharing configurations are efficient, in the sense that there do not exist configurations that dominate them. In other words, over the Pareto frontier, the LLR of any agent can be only improved by increasing the LLR of the other.

**Lemma 4.1.** There are no Pareto-optimal configurations in \( X^o \). The following theorem shows that the Pareto frontier is in fact \( X \setminus X^o \). Moreover, the Pareto frontier contains mutually beneficial configurations.

**Theorem 4.3.** The Pareto frontier \( \mathcal{P} = X \setminus X^o \). Moreover, there exists points \((c_1, c_2) \in \mathcal{P}\) satisfying \( LLR_i(c_1, c_2) < LLR_i^{sa}\) for all \( i \in \{1, 2\} \).

**Theorem 4.3** shows that all efficient sharing configurations involve at least one agent transferring energy to the other at the maximum possible rate when the latter faces loss of load. Moreover, there exist efficient configurations that are mutually beneficial relative to standalone operation. Intuitively, if Agent \( i \) is considerably more deficit prone than Agent \(-i\), then efficient, mutually beneficial would involve \( c_i = c_{i, \max} \). This way, Agent \( i \) transfers energy to Agent \(-i\) at the maximum possible rate at those relatively rare instances when the latter faces loss of load. In return, Agent \(-i\) agrees to a modest rate of energy transfer to Agent \( i \) at those (relatively more often) times when it faces deficit.

**Proof of Theorem 4.3.** That the Pareto frontier consists of all points in \( X \setminus X^o \) follows by noting that these configurations cannot be dominated by any other configuration in \( X \). To see this, suppose that \((c_1, c_2) \in X \setminus X^o \). From Theorem 3.1, it follows that no other configuration in \( X \setminus X^o \) dominates \((c_1, c_2)\). For the purpose of obtaining a contradiction, suppose that there exists \((\tilde{c}_1', \tilde{c}_2') \in X^o \) that dominates \((c_1, c_2)\). But Lemma 4.1 shows that there exists \((c_1', c_2') \in X \setminus X^o \) that further dominates \((\tilde{c}_1', \tilde{c}_2')\), and thus also dominates \((c_1, c_2)\), yielding a contradiction.

To argue that there exist mutually beneficial configurations in the Pareto frontier, consider the optimization

\[
\max_{(c_1, c_2) \in X} [LLR_1(0, 0) - LLR_1(c_1, c_2)] + [LLR_2(0, 0) - LLR_2(c_1, c_2)].
\]

Since this is the maximization of a continuous function over a compact set, an optimal solution, say \((c_1', c_2')\), exists. Moreover, in light of Lemma 4.1, the objective value at \((c_1', c_2')\) is strictly positive. This means \((c_1', c_2')\) is Pareto-optimal, and also satisfies \( LLR_i(c_1, c_2) < LLR_i(0, 0) < LLR_i^{sa}\) for both \( i \in \{1, 2\} \). \( \square \)

4.3 Bargaining solutions

Having proved that there exist configurations on the Pareto frontier that benefit both agents relative to standalone operation, the next
We consider the net generation processes of the two agents as
To validate our theoretical findings, in this section, we present a
The egalitarian solution is
5.1 Toy example
5 CASE STUDY
To validate our theoretical findings, in this section, we present a
simulation study using a toy model comprising two independent
operation) of the agents. If feasible, the egalitarian solution would
in this paper, which tries to balance the benefits of the agents as
as far as possible.
Formally, the egalitarian solution \((c_1^{eg}, c_2^{eg})\) is defined by
\[
(c_1^{eg}, c_2^{eg}) = \arg\max_{(c_1, c_2) \in P} \min\{LLR_i(c_1, c_2)_{+}\}.
\]
The egalitarian solution is fair, in that it maximizes the minimum
benefit (measured by the reduction in LLR relative to standalone
operation) of the agents. If feasible, the egalitarian solution would
in fact equalize these benefits. In the following section, we illustrate
the benefits that can be achieved in practice with our proposed
sharing mechanism under the egalitarian solution.
5 CASE STUDY
To validate our theoretical findings, in this section, we present a
simulation study using a toy model comprising two independent
Markov net generation processes, and a case study involving a solar
generator and a wind generator using real-world data traces.
5.1 Toy example
We consider the net generation processes of the two agents as
independent, two-state CTMCs. Specifically, the rate matrix cor-
responding to each agent is taken as \(Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\). This means
that for each agent, state transitions occur after exponentially dis-
tributed intervals of unit mean length. We consider three cases for
the net generation values:
- Symmetric case: \(r_1(t), r_2(t) \in [-1.5, 2]\)
- Asymmetric case 1: \(r_1(t) \in [-1.5, 2]\) and \(r_2(t) \in [-1.5, 2.15]\)
- Asymmetric case 2: \(r_1(t) \in [-1.5, 2]\) and \(r_2(t) \in [-1.5, 2.5]\)

Top to bottom, note that Agent 2 becomes increasingly ‘generative’ relative to Agent 1 in the above scenarios. The battery sizes are set as \(B_1 = B_2 = 10\). For each sharing configuration \((c_1, c_2)\), the CTMCs \(\{r_1(t)\}\) and \(\{r_2(t)\}\) are simulated for a suitably long duration and the temporal evolution of battery and the incurred loss of load for each agent is captured. The loss of load rate (LLR\(_i\)) for Agent \(i\) is computed as the cumulative loss of load over the simulation horizon, divided by the horizon length. Figure 1 depicts the variation of the LLR of each agent over the Pareto frontier for all three settings. For convenience of presentation, we have ‘flattened’ the Pareto frontier to be the horizontal axis in these figures; i.e., the independent variable ranges from \((0, c_{2_{max}})\) to \((c_{1_{max}}, 0)\) via \((c_{1_{max}}, c_{2_{max}})\) covering entire range of the Pareto frontier \(X \setminus X^0\). The constraint parameter \(c\) is set as 1.5 (and hence it only restricts the overflow), so that \(c_{1_{max}} = c_{2_{max}} = 1.5\).

The monotonicity properties of LLR (Theorem 3.1) are evident from Figure 1. For the symmetric case (panel (a)), each agent experiences the same standalone LLR\(_{1_{sa}}\) and the same value of LLR\(_1(0, 0)\). For the asymmetric cases, the standalone loss of load rates of the two agents LLR\(_{1_{sa}}\) are different (see panels (b) and (c)). As expected, the egalitarian bargaining solution for the symmetric case is found to be full sharing, i.e., \((1.5, 1.5)\), resulting in an 85% reduction in LLR for each agent relative to the standalone setting. In asymmetric case 1, where Agent 2 becomes more generative, the bargaining solution shifts ‘right’ to \((1.5, 0.75)\), i.e., Agent 2 reduces its peak energy transfer rate to Agent 1. In asymmetric case 2, where Agent 2 becomes even more generative, the egalitarian solution shifts further ‘right’ to \((1.5, 0)\), i.e., Agent 1 shares energy with Agent 2 at the maximum rate when the latter faces loss of load, but Agent 2 only shares its overflow energy with Agent 1. Intuitively, this is because Agent 2 is substantially less likely to face loss of load in this example, and substantially more likely to have an energy overflow. Thus, Agent 1 obtains a considerable benefit from just receiving this overflow energy, which it reciprocates by sharing energy with Agent 1 at the maximum rate when the latter faces a deficit.

Figure 1: Toy example all case: LLR variation with sharing configuration \((c_1, c_2) \in X \setminus X^0\) when the net generation processes of two agents are symmetric(left) and asymmetric (middle, right). \(B_1 = B_2 = 10\)

(a) \(r_1(t), r_2(t) \in [-1.5, 2]\)
(b) \(r_1(t) \in [-1.5, 2]\) and \(r_2(t) \in [-1.5, 2.15]\)
(c) \(r_1(t) \in [-1.5, 2]\) and \(r_2(t) \in [-1.5, 2.5]\)
5.2 Energy sharing between wind and solar generator

We collected six years of wind generation data corresponding to a location in New Mexico, USA. The data is obtained from the Wind Integration National Dataset (WIND) Toolkit, which has been made public by National Renewable Energy Laboratory (NREL) [14]. The data obtained is of five-minute temporal resolution and ranges from 0-16 megawatt of wind power. For the same location, we obtained the hourly solar generation data trace from the software tool SAM, also available on the NREL website. Assuming the solar power to be constant over each hour, this hourly data trace of solar generation, which ranges from 0-16 megawatt of power, is converted into a time series with a finer temporal resolution of five minutes.

We consider Agent 1 to be a wind generator, and Agent 2 to be a solar generator, with the above generation traces. To capture demand, for Agent 1, we fix a constant demand that is 90% of the time average wind generation. Since solar generation is only available from around 7 am to 5 pm (total 10 hours), we chose a demand curve which is non-zero only during this interval i.e., 7 am to 5 pm. The demand value during this interval is chosen to be 90 percent of the average solar generation, where the averaging is only performed over the same interval. Each agent is equipped with a battery capacity 500 kWh.

Over the six years of time series data, the standalone LLR$^*$ of Agent 1 (the wind generator) is computed to be 1.8873 MW, whereas that of Agent 2 (the solar generator) equals 0.7218 MW. The considerably higher standalone LLR for the wind generator suggests that the wind generation is much more ‘variable’ than the solar generation. The loss of load rate of both agents over the (flattened, as before) Pareto frontier is plotted in Figure 2. Note that there is a substantial reduction in LLR (relative to standalone setting) for the wind generator, but only a modest reduction for the solar generator (again, consistent with the considerably higher variability of wind generation). Thus, the egalitarian solution corresponds to $(c_1^*, c_2^*) = (c_1\text{max}, 0)$, i.e., the wind generator shares energy with the solar generator at the peak rate when the latter faces a deficit, whereas the solar generator only shares its overflow with the wind generator. The associated reduction in load rate (compared to the standalone setting) is 70% for the wind generator and 22% for the solar generator.

6 CONCLUDING REMARKS

This work motivates extensions along various directions. A natural first step would be to generalize the sharing mechanism to multiple (more than two) agents, and characterize the Pareto frontier of the achievable reliability vectors. Another useful direction is to consider other reliability metrics, potentially capturing (i) shorter timescales, (ii) a non-linear cost/penalty associated with increasing loss of load.

Another interesting line of questions pertains to network formation: Given a collection of renewable generators, which of them should come together to enter into an energy sharing agreement? Since statistical diversity lies at the core of effective energy sharing, it is also important to consider policy interventions that would encourage energy sharing agreements between geographically separated renewable generators. Indeed, such generators would have to be allowed to fulfill a deficit in generation at one bus on the grid with a surplus injection at another (of course subject to grid stability considerations).

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when the background process Agent 1 to Agent 2 in this case, in the original system as well as in
Throughout the interval and hence
Therefore, within any time slot \( t_k, t_{k+1} \), the net generation is constant for both agents.
The proof proceeds via induction. We assume Lemma 3.2 holds at time \( t_k \). We will verify that in each of the possible ways in which the dynamics of the original system and the \( \tilde{b}_i \) can evolve, Lemma 3.2 holds \( \forall t \in (t_k, t_{k+1}) \). We define instances \( i_1, i_2, f_1, f_2 \in \{t_k, t_{k+1} \} \) as the instants when \( b_i(e_i) = 0 \), \( \dot{b}_i(e_i) = 0 \), \( f_1(i) = B_1 \) and \( \dot{b}_1(i) = B_1 \), respectively. Occasionally, for simplicity, we refer to the battery (not battery level) of Agent \( i \) in the original system (respectively, in the \( \tilde{b}_i \) system) by \( b_i \) (respectively, by \( \dot{b}_i \)). Let \( r_i \) denote the net generation rate of Agent \( i \) within the slot \( t_k, t_{k+1} \). There are the following four cases to consider.

Case 1: \( r_1 > 0, \ r_2 > 0 \)
There is no energy sharing in this case, so it is easy to see that the ordering of battery occupancies between the two systems continues to hold for \( t \in [t_k, t_{k+1}] \).

Case 2: \( r_1 < 0, \ r_2 > 0 \)
Throughout the interval \( t_k, t_{k+1} \), there is no energy transfer from Agent 1 to Agent 2 in this case, in the original system as well as in the \( \tilde{b}_i \) system. Therefore the extra \( \epsilon \) in the sharing configuration of Agent 1 in the \( \tilde{b}_i \) system is inconsequential and the induction step follows as before.

Case 3: \( r_1 < 0, \ r_2 < 0 \)
Due to space constraints, we only consider the case where \( \dot{b}_i \) gets fully discharged before \( \tilde{b}_i \), i.e., \( t_k < \tilde{e}_i < \tilde{e}_2 \). The complementary case can be handled on similar lines. The two possible cases are:
(1) \( \dot{b}_i \) discharges before \( \tilde{b}_2 \) which leads to the only possible sequence of events as \( \tilde{e}_1 < e_1 < \tilde{e}_2 < e_2 \) (see Figure 3).
(2) \( \tilde{b}_2 \) discharges before \( \dot{b}_1 \) which leads to the possible sequence of events as shown in Figure 4.

\[ \begin{align*}
\tilde{b}_1 &= \dot{b}_1 = r_1 \\
\tilde{b}_2 &= \dot{b}_2 = r_2 \\
\dot{b}_1 &= \dot{b}_2 - r_2 - \min(c_2, r_2) \quad \text{or} \quad \dot{b}_2 = \dot{b}_1 - \min(c_2, r_2) \\
\tilde{b}_1 &= \tilde{b}_2 = 0
\end{align*} \]

Figure 4: \( r_1 < 0, \ r_2 < 0 \), \( \tilde{b}_2 \) discharges before \( \dot{b}_1 \)

Again, it is clear from Figure 4 that Lemma 3.2 holds \( \forall t \in [t_k, t_{k+1}] \).

Case 4: \( r_1 > 0, \ r_2 < 0 \)
In this, there are two major cases to be considered, each one needs a separate analysis for \( r_1 \geq |r_2| \) and for \( r_1 < |r_2| \).

Subcase 1: \( b_1 \) becomes full before \( \tilde{b}_2 \) drains to zero \( (t_k < f_1 < \tilde{e}_2) \).

1a: \( r_1 \geq |r_2| \)
\[ \begin{align*}
\dot{b}_1 &= \dot{b}_2 = r_2 \\
\dot{b}_1 &= \dot{b}_2 = r_2 \\
\dot{b}_1 &= \dot{b}_2 = r_2 \quad \text{with } \dot{b}_1 = B_1
\end{align*} \]

Figure 5: \( r_2 < 0 < r_1, \ r_1 \geq |r_2| \)
From time \( f_1 \) onwards, \( b_1 \) gives energy to \( b_2 \) with rate \( r_1 \) due to its overflow. Since the overflow is greater than or equal to the absolute value of the discharging rate of \( b_2 \), \( b_2 \) will either charge or remain at the same level whereas \( b_1 \) will continue to charge and won’t discharge even after \( \tilde{b}_2 \) discharges to zero. Once \( b_1 \) is full, rate of change of battery levels in \( b_2 \) and \( \tilde{b}_2 \) is the same (if \( b_2 \) is not fully charged). Therefore from \( f_1 \) onwards, either \( b_2 = B_2 \) or \( \tilde{b}_2 = B_2 \) which results in \( b_2 \geq \tilde{b}_2 \) \( \forall t \in [f_1, t_{k+1}] \). Also note that from \( f_1 \) onwards, \( b_1 = B_1 \). Therefore together we conclude that Lemma 3.2 holds for \( \forall t \in [t_k, t_{k+1}] \).

1b: \( |r_1| < |r_2| \)
In this there are three further cases to consider:

1b-A: When full, both \( b_1 \) and \( \tilde{b}_2 \) allows discharge i.e., \( r_1 < \min(c_1, -r_2) \) and hence \( r_1 < \min(c_1 + \epsilon, -r_2) \). Possible event sequences are:
(1) \( f_1 < \tilde{e}_1 < e_1 < \tilde{e}_2 \)
(2) \( f_1 < \tilde{e}_2 < e_1 < \tilde{e}_1 \)
(3) \( f_1 < \tilde{e}_1 < \tilde{e}_2 < e_2 < \tilde{e}_1 < e_1 \)
It is not hard to verify that, in each case, the battery induction argument (3) holds along the sequence of events. Sequence (4) is depicted in the figure below. Other sequences can easily be verified analogously.

$$\begin{align*}
&b_1 = \tilde{b}_1 = r_1 \\
b_2 = \tilde{b}_2 = r_2 \\
b_1 = \tilde{b}_1 = B_1 \\
b_2 = \tilde{b}_2 = B_1 + r_2 \\
b_1 = \tilde{b}_1 = B_1 \\
b_2 = \tilde{b}_2 = B_2
\end{align*}$$

Figure 6: $r_2 < 0 < r_1$, $r_1 < |r_2|$; when full, both $b_1$ and $\tilde{b}_1$ allows discharge

Subcase 2: $\tilde{b}_2$ drains to zero before $b_1$ becomes full ($t_k < \tilde{e}_2 < f_1$)