On 2-closed abelian permutation groups

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\textbf{Abstract}

A permutation group $G \leq \text{Sym}(\Omega)$ is said to be 2-closed if no group $H$ such that $G < H \leq \text{Sym}(\Omega)$ has the same orbits on $\Omega \times \Omega$ as $G$. A simple and efficient inductive criterion for the 2-closedness is established for abelian permutation groups with cyclic transitive constituents.

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\section{Introduction}

The concept of $m$-closed permutation groups, $m \in \mathbb{N}$, was introduced by Wielandt \cite{10} in the framework of the method of invariant relations, developed by him to study group actions. He proved that the $m$-closed groups are exactly the automorphism groups of the relational structures formed by families of $m$-ary relations on the same set. The 1-closed groups are just the direct products of symmetric groups. In the present paper, we are interested in the 2-closed groups.

Let $\Omega$ be a finite set and $G \leq \text{Sym}(\Omega)$. Denote by $\text{Orb}_2(G)$ the set of all orbits of the induced action of $G$ on $\Omega \times \Omega$. The \textit{2-closure} of the permutation group $G$ is defined to be the largest subgroup $\overline{G} = G^{(2)}$ in $\text{Sym}(\Omega)$ that has the same 2-orbits as $G$, i.e.,

$$\text{Orb}_2(G) = \text{Orb}_2(\overline{G}).$$

The group $G$ is said to be \textit{2-closed} if $G = \overline{G}$. The set $\text{Orb}_2(G)$ forms a relational structure on $\Omega$, which is called a coherent configuration \cite{1}. Thus the 2-closed groups are the automorphism groups of coherent configurations. This fact makes the concept of 2-closed groups especially important for algebraic combinatorics \cite{5}.

Based on the definition alone, it is difficult to determine whether a given permutation group $G$ is 2-closed. In some special cases (mostly for transitive groups the degree of which is the product of a small number of primes), good criteria for the 2-closedness are obtained via the full classification of permutation groups of given degree, see, e.g., \cite{3}. To the best of our knowledge, efficient algorithms recognizing 2-closed groups are known only for a few cases, e.g., for odd order groups \cite{4} and for supersolvable groups \cite{8}.
The present paper is motivated by the lack of good criteria for the 2-closedness even for abelian groups. A criterion found in [7, Theorem 6.1] does not seem quite satisfactory, because it requires for a group $G$ to inspect all permutations in the direct product of the constituents $G^A$, $A \in \text{Orb}(G)$. Our first result reduces the 2-closedness question to the case of $p$-groups, and generalizes a Wielandt observation that the classes of $p$-groups and abelian groups are invariant with respect to taking the 2-closure.

**Theorem 1.1.** Let $G$ be a nilpotent permutation group. Then $\overline{G}$ is nilpotent. Moreover,

$$\overline{G} = \prod_{P \in \text{Syl}(G)} P.$$

**Corollary 1.2.** A nilpotent permutation group $G$ is 2-closed if and only if every Sylow subgroup of $G$ is 2-closed.

Probably, the first attempt to classify the 2-closed abelian groups was made by Zelikovskii in [11, Corollary 5]. However, the characterization appears to be wrong and infinitely many counterexamples were found in [6]. To explain the gap in Zelikovskii’s argument, let $G \leq \text{Sym}(\Omega)$. One can associate with $G$ a permutation group on $\Omega$, defined as follows:

$$\text{Zel}(G) = \prod_{\Delta} \bigcap_{\Delta' \neq \Delta} G^A_{\Delta'},$$

(2)

where $\Delta$ and $\Delta'$ run over the orbits of $G$, and $G^A_{\Delta} = (G^A_\Delta)^A$ is the restriction to $\Delta$ of the pointwise stabilizer of $\Delta'$ in $G$. In this notation, the necessary and sufficient condition claimed by Zelikovskii for an abelian group $G$ to be 2-closed is that

$$\text{Zel}(G)^\Delta = (G_{\Omega\setminus\Delta})^\Delta \quad \text{for all } \Delta \in \text{Orb}(G).$$

It is not hard to see that this condition is equivalent to the inclusion $\text{Zel}(G) \leq G$. Essentially, the only gap in Zelikovskii’s argument consists in the wrong statement that if $G$ is 2-closed, then so is the permutation group induced by the action of $G$ on the orbits of $\text{Zel}(G)$. In this way we arrive at the second result of the present paper.

**Theorem 1.3.** Let $G$ be an abelian permutation group and $Z = \text{Zel}(G)$. Then $G$ is 2-closed if and only if $Z \leq G$ and $G^{\text{Orb}(Z)}$ is 2-closed.

The “only if” part of Theorem 1.3 can be illustrated by the two following examples in which we construct non-2-closed groups $G$ and $H$, respectively. In the first example, $\text{Zel}(G) \not\leq G$, whereas in the second one, $\text{Zel}(H) \leq H$ but $H^{\text{Orb}(\text{Zel}(H))}$ is not 2-closed.

**Example 1.** Let $p$ be a prime, and let $G \leq \text{Sym}(3p)$ be an elementary abelian group of order $p^2$. The action of $G$ is chosen so that (a) there are exactly three $G$-orbits, each of size $p$, and (b) for any two points $\alpha$ and $\beta$ belonging to different $G$-orbits, the stabilizers $G_\alpha$ and $G_\beta$ are different subgroups of $G$ of order $p$ (if $\alpha$ and $\beta$ belong to the same orbit, then, of course, $G_\alpha = G_\beta$). From Wielandt’s dissection theorem [10, Theorem 6.5], it follows that the group $\overline{G}$ equals the direct product of its transitive constituents; since $\overline{G}$ is abelian and $G \leq \overline{G}$, we conclude that $|\overline{G}| = p^3$. Thus, $G$ is not 2-closed and $\overline{G} = \text{Zel}(G)$.

**Example 2.** Let $\Omega_1$ and $\Omega_2$ be disjoint sets, and let $G_1 \leq \text{Sym}(\Omega_1)$ and $G_2 \leq \text{Sym}(\Omega_2)$ be two copies of the permutation group $G$ from Example 1. For any two permutations $g_1 \in G_1$ and $g_2 \in G_2$, corresponding to the same element of $G$, we define a permutation $g \in \text{Sym}(\Omega_1 \cup \Omega_2)$ such

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1The cited statement was formulated for arbitrary intransitive groups with 2-closed transitive constituents.
that \( g^\Omega = g_i, \ i = 1, 2 \). Let \( H \) be the group of all these permutations \( g \); this group is isomorphic to \( G \) as an abstract group and has the same point stabilizers as \( G \) (again as abstract groups). Then \( H^\Delta_A = 1 \) for any two orbits \( \Delta \) and \( \Delta' \) corresponding to the same orbit of \( G \). It follows that \( H \) is not 2-closed (see Lemma 5.1) and \( \text{Zel}(H) = 1 \).

Theorem 1.3 reduces the question on the 2-closedness of an abelian group \( G \) to the case when the group \( \text{Zel}(G) \) is trivial. This case can really happen even if \( G \) is nontrivial, see Example 2. In a large class of abelian permutation groups, one can continue the reduction by “removing” unessential orbits defined as follows. An orbit \( \Delta \) of a group \( G \leq \text{Sym}(\Omega) \) is said to be unessential if \( G \) is 2-closed if and only if \( G^\Delta_A \) is 2-closed.

**Theorem 1.4.** Let \( G \) be an intransitive group and \( p \) a prime. Assume that every transitive constituent of \( G \) is a cyclic \( p \)-group. Then \( \text{Zel}(G) \) is trivial only if every orbit of \( G \) is unessential.

Combining the obtained results, we arrive at the following simple and efficient inductive criterion of the 2-closedness for an (abelian) permutation group \( G \) with cyclic transitive constituents. First, there is nothing to do if \( G \) is transitive, because in this case \( G \) is 2-closed by [10, Example 5.13]. Second, Theorem 1.1 reduces the problem to the case when \( G \) is a \( p \)-group. At this point a further reduction is needed. Namely, depending on whether the group \( Z = \text{Zel}(G) \) is trivial, we continue with

\[
G^{\text{Orb}(Z)} \text{ or } G^\Omega_A,
\]

where \( \Delta \) is an arbitrarily chosen orbit of \( G \). The correctness is provided by Theorems 1.3 and 1.4, respectively. Each of these two reductions decreases the degree of a group and hence the test is finished after at most \( |\Omega| \) reductions.

The proofs of Theorems 1.1, 1.3, and 1.4 are presented in Sections 3–5, respectively. The relevant notation and definitions are collected in Section 2.

### 2. Permutation groups

Throughout the paper, \( \Omega \) is a finite set. For a permutation group \( G \leq \text{Sym}(\Omega) \), we use the notation

\[
\text{Orb}(G) = \text{Orb}(G, \Omega) = \{x^G : x \in \Omega\}
\]

for the set of \( G \)-orbits \( x^G = \{x^g : g \in G\} \). A set \( \Delta \subseteq \Omega \) is said to be \( G \)-invariant if \( \Delta^\delta = \Delta \) for all \( g \in G \). In this case, we denote by \( G^\Delta \) the permutation group on \( \Delta \), induced by the natural action of \( G \) on \( \Delta \); it consists of the permutations \( g^\Delta \subseteq \text{Sym}(\Delta), g \in G \), taking \( \delta \) to \( \delta^g \), \( \delta \in \Delta \).

In general, when \( G \) acts on a set \( \Delta \) (which is not necessarily a subset of \( \Omega \)), we denote by \( G^\Delta \) the subgroup of \( \text{Sym}(\Delta) \), induced by the corresponding action.

For an arbitrary \( \Delta \subseteq \Omega \), we denote by \( G_\Delta \) and \( G_{(\Delta)} \) the pointwise and setwise stabilizer of \( \Delta \) in \( G \), respectively. Thus, if \( \Delta \) is \( G \)-invariant, then \( G_\Delta \) is the kernel of the restriction homomorphism \( G \to G^\Delta \), whereas \( G_{(\Delta)} = G \).

The groups \( G, H \leq \text{Sym}(\Delta) \) are said to be 2-equivalent if \( \text{Orb}_2(G) = \text{Orb}_2(H) \). Formula (1) implies that \( G \) and \( \overline{G} \) are 2-equivalent. In fact, the 2-closure \( \overline{G} \) is the largest group in the class of all groups 2-equivalent to \( G \).

All other undefined terms and notation are standard and can be found in [2].

### 3. Proof of Theorem 1.1

Let \( G \leq \text{Sym}(\Omega) \) be a nilpotent group of degree \( n = |\Omega| \). For a prime \( p | n \), the largest \( p \)-power divisor of \( n \) is denoted by \( n_p \); if \( \pi \) is a set of prime divisors of \( n \), then we put \( n_\pi := \prod_{p \in \pi} n_p \). The set of all prime divisors of the order of \( G \) is denoted by \( \pi(G) \).
Lemma 3.1. Assume that $G$ is transitive and $H$ a Hall subgroup of $G$. Then

1. the size of every $H$-orbit is equal to $n_\pi$, where $\pi = \pi(H)$,
2. $G$ acts on $\text{Orb}(H)$; moreover, the kernel of this action is equal to $H$.

Proof. The nilpotency of $G$ implies that $H \trianglelefteq G$. Therefore, $G$ permutes the orbits of $H$ and hence acts on $\Delta = \text{Orb}(H)$. Denote by $L$ the kernel of this action.

The transitivity of $G$ and normality of $H$ imply that all $H$-orbits have the same size $m$. In particular, $m$ divides both $n$ and $|H|$. Taking into account that $\gcd(n, |H|)$ divides $n_\pi$, we conclude that

$$m \text{ divides } n_\pi.$$  
(3)

Next, the group $G^\Delta \leq \text{Sym}(\Delta)$ is transitive. Consequently, $|\Delta|$ divides $|G^\Delta|$. However the numbers $|G^\Delta|$ and $|H|$ are coprime, because $H \leq L$ and $H$ is a Hall subgroup of $G$. Since also $|\Delta|$ divides $n$, this implies that $|\Delta|$ divides $n_\pi'$, where $\pi'$ is the set of prime divisors of $n$, not belonging to $\pi$. Together with (3) this shows that

$$n = m \cdot |\Delta| = n_\pi \cdot n_\pi' = n,$$
whence $m = n_\pi$. This proves statement (1).

To prove statement (2), assume on the contrary that $H < L$. Then there exists an element $g \in L$ of prime order $q \not\in \pi$. Denote by $\Gamma$ an orbit of $L$ such that $g^\Gamma \neq \text{id}_\Gamma$. Then the Sylow $q$-subgroup $Q$ of $L^\Gamma$ is nontrivial. Since $L^\Gamma$ is transitive and nilpotent, we can apply statement (1) to $G = L^\Gamma$ and $H = Q$. Then

$$|\Gamma| = |\Gamma|_q \cdot |\text{Orb}(Q)|.$$

On the other hand, we can apply statement (1) to $G = L^\Gamma$ and $H = H^\Gamma$. Since $\Gamma$ is also an orbit of $H$, this statement implies that

$$|\Gamma| = |\Gamma|_\pi \cdot |\text{Orb}(H^\Gamma)| = |\Gamma|_\pi.$$ 

Thus, $|\Gamma|_q \cdot |\text{Orb}(Q)| = |\Gamma|_\pi$. However, the numbers $|\Gamma|_q$ and $|\Gamma|_\pi$ are relatively prime. Therefore $|\Gamma|_q = 1$ and hence $|\Gamma| = |\text{Orb}(Q)|$. It follows that $Q = 1$, a contradiction.  

Lemma 3.2. Theorem 1.1 holds if $G$ is transitive.

Proof. If $G$ is a $p$-group, then $G$ is a $p$-group [10, Exercise 5.28] and the required statement is true. Thus we may assume that $G = P \times H$, where $P$ and $H$ are the Sylow $p$-subgroup and Hall $p'$-subgroup of $G$, respectively. Let $\Delta \in \text{Orb}(P)$ and $\Gamma \in \text{Orb}(H)$. By Lemma 3.1(1), we have

$$|\Delta| = n_\rho \text{ and } |\Gamma| = n_{\rho'},$$
where $\rho' = \pi(H)$. Further, $\Delta$ and $\Gamma$ are blocks of the transitive group $G$. Therefore, the intersection $\Delta \cap \Gamma$ is either empty or is a block the size of which divides both $|\Delta|$ and $|\Gamma|$. Thus, $|\Delta \cap \Gamma| \leq 1$.

Each point $\alpha \in \Omega$ lies in exactly one $P$-orbit, say $\Delta_\alpha$, and in exactly one $H$-orbit, say $\Gamma_\alpha$. By the above, $|\Delta_\alpha \cap \Gamma_\alpha| = 1$. Consequently, the mapping

$$\rho : \Omega \to \text{Orb}(P) \times \text{Orb}(H), \quad \alpha \mapsto (\Delta_\alpha, \Gamma_\alpha)$$

is a bijection. Denote by $P'$ and $H'$ the permutation groups induced by the actions of $G$ on $\text{Orb}(H)$ and on $\text{Orb}(P)$, respectively. By Lemma 3.1(2), we have

$$P' = P' \times 1 \text{ and } H' = 1 \times H'.$$
Thus the group $G$ can be identified with the direct product $P' \times H'$ acting on $\text{Orb}(H) \times \text{Orb}(P)$.
From [4, Proposition 3.1(2)], it follows that 
\[(P' \times H')^{(2)} = (P')^{(2)} \times (H')^{(2)},\]
which completes the proof by induction on the number \(|\pi(G)|\) with taking into account that \(\pi(G) = \pi(P) \cup \pi(H)\).

\[\textit{Proof.}\] The first part of the statement follows from the monotonicity of the 2-closure operator; in particular, \(P \leq \overline{P} \leq Q\). Thus, to prove the second part, it suffices to verify that each \(P\)-orbit \(\Delta\) is a \(Q\)-orbit. Denote by \(\Gamma\) the \(G\)-orbit containing \(\Delta\). Then \(\Gamma\) is also a \(\overline{G}\)-orbit. It follows that there exists a \(Q\)-orbit \(\Delta'\) such that 
\[\Delta \subseteq \Delta' \subseteq \Gamma.\]
The groups \(G^\Gamma\) and \(\overline{G}^\Gamma\) are transitive and nilpotent. By Lemma 3.1(1), this implies that \(|\Delta| = |\Gamma|_p = |\Delta'|\). This shows that \(\Delta = \Delta'\), as required.

\[\textit{Lemma 3.3.} \text{ Let } P \text{ and } Q \text{ as above. Then } \overline{P} \leq Q. \text{ Moreover, } \text{Orb}(P) = \text{Orb}(Q). \]

\[\textit{Proof.}\] The first part of the statement follows from the monotonicity of the 2-closure operator; in particular, \(P \leq \overline{P} \leq Q\). Thus, to prove the second part, it suffices to verify that each \(P\)-orbit \(\Delta\) is a \(Q\)-orbit. Denote by \(\Gamma\) the \(G\)-orbit containing \(\Delta\). Then \(\Gamma\) is also a \(\overline{G}\)-orbit. It follows that there exists a \(Q\)-orbit \(\Delta'\) such that 
\[\Delta \subseteq \Delta' \subseteq \Gamma.\]
The groups \(G^\Gamma\) and \(\overline{G}^\Gamma\) are transitive and nilpotent. By Lemma 3.1(1), this implies that \(|\Delta| = |\Gamma|_p = |\Delta'|\). This shows that \(\Delta = \Delta'\), as required.

\[\textit{Lemma 3.4.} \text{ Let } \Delta, \Gamma \in \text{Orb}(P). \text{ Then } (G_{\{\Delta\}} \cap G_{\{\Gamma\}})^{\Delta \cup \Gamma} \leq P^{\Delta \cup \Gamma}. \]

\[\textit{Proof.}\] The statement is trivial if \(G = P\). Now let \(G = P \times H\), where \(H\) is a Hall subgroup of \(G\). It follows that each \(g \in G\) can be written as \(g = xy\) with \(x \in P\) and \(y \in H\). Assume that 
\[g \in G_{\{\Delta\}} \cap G_{\{\Gamma\}},\]
i.e., \(g\) leaves \(\Delta\) and \(\Gamma\) fixed (as sets). Then the permutation \(x^{-1} \in P\) leaves the sets \(\Delta\) and \(\Gamma\) fixed, because they are \(P\)-orbits. Thus, \(\Delta' = \Delta x^{-1}g = \Delta \Delta\) and similarly, \(\Gamma' = \Gamma\).

We claim that 
\[y^\Delta = \text{id}_\Delta \text{ and } y^\Gamma = \text{id}_\Gamma. \]

Let us prove the first equality; the second one is proved analogously. The permutation \(y^\Delta\) belongs to the centralizer \(Z\) of the transitive group \(P^\Delta\) in \(\text{Sym}(\Delta)\). According to [9, Exercise 4.5'], the group \(Z\) is semiregular. In particular, \(|Z|\) divides \(|\Delta|\) which is a \(p\)-power. Therefore, \(Z\) is a \(p\)-group. Consequently, the order of \(y^\Delta\) is a \(p\)-power and hence \(y^\Delta \in P^\Delta\). Taking into account that \(P^\Delta \cap H^\Delta = 1\), we conclude that the first equality in (5) holds.

Using equalities (5), we have 
\[g^{\Delta \cup \Gamma} = (xy)^{\Delta \cup \Gamma} = x^{\Delta \cup \Gamma} y^{\Delta \cup \Gamma} = x^{\Delta \cup \Gamma} \in P^{\Delta \cup \Gamma},\]
as required.
To complete the proof of the theorem, we note that $\overline{Q}$ is a $p$-subgroup of $\overline{G}$, containing the Sylow $p$-subgroup $Q$. Therefore, $Q$ is 2-closed. Thus it suffices to verify that given $\alpha, \beta \in \Omega$ and $g \in Q$, there exists $h \in P$ such that

$$(\alpha, \beta)^g = (\alpha, \beta)^h.$$ 

The 2-equivalence of $G$ and $\overline{G}$ implies that $(\alpha, \beta)^g = (\alpha, \beta)^\delta$ for some $g \in G$. Denote by $\Delta$ and $\Gamma$ the $Q$-orbits containing the points $\alpha$ and $\beta$, respectively. Then obviously, $\alpha^\delta = \alpha^\delta$ belongs to $\Delta$ and $\beta^\delta = \beta^\delta$ belongs to $\Gamma$. Therefore,

$$\alpha, \beta^\delta \in \Delta \text{ and } \beta, \beta^\delta \in \Gamma.$$ 

In view of equality (4), $\Delta$ and $\Gamma$ are also $P$-orbits. Since the group $G$ permutes the $P$-orbits, it follows that $g \in G_{(\Delta)} \cap G_{(\Gamma)}$. By Lemma 3.4(1), there exists $h \in P$ such that $h^{\Delta \cup \Gamma} = g^{\Delta \cup \Gamma}$. Thus,

$$(\alpha, \beta)^g = (\alpha, \beta)^\delta = (\alpha, \beta)^h,$$

as required. Theorem 1.1 is completely proved.

**Proof of Corollary 1.2.** Let $G$ be a nilpotent permutation group. Assume that $G$ is 2-closed. Then $G = \overline{G}$ and hence $\text{Syl}(G) = \text{Syl}(\overline{G})$. It remains to note that by Theorem 1.1, any Sylow subgroup of $\overline{G}$ is 2-closed. Conversely, assume that $P = \overline{P}$ for each $P \in \text{Syl}(G)$. Then again by Theorem 1.1, we have

$$\overline{G} = \prod_{P \in \text{Syl}(G)} \overline{P} = \prod_{P \in \text{Syl}(G)} P = G,$$

i.e., $G$ is 2-closed. $\square$

### 4. Proof of Theorem 1.3

It is well known that a transitive abelian group is regular [9, Proposition 4.4]. Therefore every abelian permutation group $G$ is quasiregular, i.e., every transitive constituent of $G$ is regular. Thus Theorem 1.3 is an immediate consequence of Theorem 4.2. To formulate the latter, we need an auxiliary lemma.

**Lemma 4.1.** If $G$ is a quasiregular permutation group and $Z = \text{Zel}(G)$, then $Z \leq \overline{G}$, and also $Z^\Delta \leq G^\Delta$ for all $\Delta \in \text{Orb}(\overline{G})$.

**Proof.** The group $Z$ is generated by the permutations $z$ satisfying the following condition: there exists $\Delta \in \text{Orb}(G)$ such that

$$z^\Delta \in Z^\Delta \text{ and } z^{\Omega \setminus \Delta} = \text{id}_{\Omega \setminus \Delta}.$$ 

(6)

Thus to prove the inclusion $Z \leq \overline{G}$, it suffices to verify that each such $z$ belongs to $\overline{G}$, or equivalently that $z^\delta = z$ for every $s \in \text{Orb}_2(G)$.

Denote by $\Delta'$ and $\Delta''$ the $G$-orbits such that $s \subseteq \Delta' \times \Delta''$. If $\Delta' \neq \Delta \neq \Delta''$, then $s^\delta = s$ by the second equality in (6), whereas if $\Delta = \Delta' = \Delta''$, then $s \in \text{Orb}_2(G\Delta)$ and again $s^\delta = s$, because $Z^\Delta \leq G^\Delta$. Thus without loss of generality, we may assume that $\Delta = \Delta'' \neq \Delta'$. Then by the definition of $Z$ (see formula (2)), there exists $g \in G_{\Delta'}$ such that

$$z^{\Delta \cup \Delta'} = g^{\Delta \cup \Delta'}.$$ 

Thus, $s^\delta = s^\delta = s$, as required.
Let us prove that \( Z^\Delta \leq G^\Delta \). We have \( G^\Delta \leq G \) for every \( \Delta' \in \text{Orb}(G) \). Therefore, \( G^\Delta \leq G^\Delta \). Consequently, the intersection of all \( G^\Delta \) taken over all \( \Delta' \neq \Delta \) is also normal in \( G^\Delta \). Since this intersection coincides with \( Z^\Delta \) and \( G^\Delta = G^\Delta \), we are done. \( \square \)

From Lemma 4.1, it follows that for every \( \Delta \in \text{Orb}(G) \), the group \( G^\Delta \) acts on the set \( \text{Orb}(Z^\Delta) \). Therefore, \( G \) acts on the union \( \text{Orb}(Z) \) of all \( \text{Orb}(Z^\Delta) \).

**Theorem 4.2.** Let \( G \) be a quasiregular permutation group and \( Z = \text{Zel}(G) \). Then \( G \) is 2-closed if and only if \( Z \leq G \) and \( G^{\text{Orb}(Z)} \) is 2-closed.

**Proof.** Let \( \Delta \) be a \( G \)-orbit. Then the group \( G^\Delta \) is regular and hence 2-closed. Since \( G \) is contained in the direct product of the 2-closures of the groups \( G^\Delta \), this shows that \( G \) is quasiregular. In what follows, we assume that \( G \leq \text{Sym}(\Omega) \).

Let \( \bar{\rho}: \bar{G} \to G^{\text{Orb}(Z)} \) be the epimorphism corresponding the action of \( G \) on \( \text{Orb}(Z) \). Then obviously \( Z \) is a subgroup of \( L := \ker(\bar{\rho}) \). Moreover, if \( \Delta \) is a \( \bar{G} \)-orbit, then

\[
L^\Delta = Z^\Delta,
\]

because \( L^\Delta \) and \( Z^\Delta \) are subgroups of the regular group \( \bar{G}^\Delta \) (recall that the group \( \bar{G} \) is quasiregular) that have the same orbits. Since \( L \) is contained in the direct product of the \( Z^\Delta \), the definition of \( Z \) implies that \( L = Z \). This proves the following statement. \( \square \)

**Lemma 4.3.** \( \ker(\bar{\rho}) = Z \).

The epimorphism \( \bar{\rho} \) induces the action of \( G \leq \bar{G} \) on the set \( \text{Orb}(Z) \); denote by \( \rho \) the corresponding epimorphism from \( G \) to \( G^{\text{Orb}(Z)} \). It should be noted that while the group \( Z \) is not, in general, a subgroup of \( G \), the permutation group \( G^{\text{Orb}(Z)} \) is well defined.

**Lemma 4.4.** \( \text{im}(\bar{\rho}) = \bar{G}^{\text{Orb}(Z)} \).

**Proof.** The groups \( G^{\text{Orb}(Z)} \) and \( \bar{G}^{\text{Orb}(Z)} \) are 2-equivalent [8, Lemma 2.1(1)]. Therefore,

\[
\text{im}(\bar{\rho}) = \bar{G}^{\text{Orb}(Z)} \leq G^{\text{Orb}(Z)}.
\]

Conversely, we need to verify that for every \( \bar{g} \in \bar{G}^{\text{Orb}(Z)} \), there exists \( g \in G \) such that

\[
\bar{\rho}(g) = \bar{g}.
\]

(7)

Let \( \Delta \in \text{Orb}(G) \). The quasiregularity of \( \bar{G} \) implies that the group \( \bar{G}^\Delta = G^\Delta \) is regular. Since also \( Z^\Delta \leq G^\Delta \) (Lemma 4.1), the group \( (\bar{G}^\Delta)^{\text{Orb}(Z^\Delta)} \) is also regular and hence 2-closed. It follows that

\[
\bar{G}^{\text{Orb}(Z)} = \bar{G}^{\text{Orb}(Z^\Delta)} = (G^\Delta)^{\text{Orb}(Z^\Delta)} = (\bar{G}^\Delta)^{\text{Orb}(Z^\Delta)} = (\bar{G}^\Delta)^{\text{Orb}(Z^\Delta)},
\]

where \( \bar{\Delta} \) is the \( \bar{G}^{\text{Orb}(Z)} \)-orbit the points of which are the \( Z \)-orbits contained in \( \Delta \). Thus for every \( \Delta \in \text{Orb}(\bar{G}) \), there exists a permutation \( g_\Delta \in \bar{G}^\Delta \) such that

\[
\bar{\rho}_\Delta(g_\Delta) = \bar{g}^\Delta,
\]

where \( \bar{\rho}_\Delta \) is the epimorphism from \( \bar{G}^\Delta \) to \( (\bar{G}^\Delta)^{\text{Orb}(Z^\Delta)} \), induced by \( \bar{\rho} \). Now if the product \( g \) of all the \( g_\Delta \) lies in \( \bar{G} \), then

\[
\bar{\rho}(g) = \bar{\rho} \left( \prod_\Delta g_\Delta \right) = \prod_\Delta \bar{\rho}_\Delta(g_\Delta) = \prod_\Delta \bar{g}^\Delta = \bar{g},
\]

which proves equality (7).
It remains to verify that \( g \in G \). To this end, let \( s \in \text{Orb}_2(G) \). Then \( s \in \text{Orb}_2(\overline{G}) \) and hence \( s \subseteq \Delta \times \Gamma \) for some \( \Delta, \Gamma \in \text{Orb}(\overline{G}) \). Now if \( \Delta = \Gamma \), then \( s^g = s^{g_\Delta} = s \), because \( g_\Delta \in G^\Delta \). Assume that \( \Delta \neq \Gamma \). Then by Lemma 4.3 and the definition of \( Z \), we have \( Z^\Delta \times Z^\Gamma \subseteq \overline{G}^{\Delta \times \Gamma} \). It follows that

\[
(\alpha, \beta) \in s \iff \overline{\alpha} \times \overline{\beta} \subseteq s,
\]

where \( \overline{\alpha} = \alpha^g \) and \( \overline{\beta} = \beta^g \). Furthermore, the set \( \overline{3} = \{ (\overline{\alpha}, \overline{\beta}) : (\alpha, \beta) \in s \} \) is a 2-orbit of the group \( G^{\text{ Orb}(\overline{Z})} \) and hence \( s^g = \overline{3} \). Thus,

\[
\overline{3} = \left( \bigcup_{(\overline{\alpha}, \overline{\beta}) \in \overline{3}} \overline{\alpha} \times \overline{\beta} \right)^g = \bigcup_{(\overline{\alpha}, \overline{\beta}) \in \overline{3}} \overline{\alpha}^g \times \overline{\beta}^g = \bigcup_{(\overline{\alpha}, \overline{\beta}) \in \overline{3}} \overline{\alpha} \times \overline{\beta} = \bigcup_{(\overline{\alpha}, \overline{\beta}) \in \overline{3}} \overline{\alpha} \times \overline{\beta} = s
\]
as required.

To prove the “only if” part, assume that the group \( G \) is 2-closed. Then by Lemma 4.1, we have \( Z \leq \overline{G} = G \), whereas by Lemma 4.4, we have

\[
G^{\text{ Orb}(\overline{Z})} = \overline{G}^{\text{ Orb}(\overline{Z})} = \text{im}(\overline{\rho}) = \overline{G^{\text{ Orb}(\overline{Z})}};
\]
i.e., the group \( G^{\text{ Orb}(\overline{Z})} \) is 2-closed, as required.

To prove the “if” part, assume that \( Z \leq G \) and the group \( G^{\text{ Orb}(\overline{Z})} \) is 2-closed. By Lemma 4.3, the first condition implies that

\[
Z \leq \ker(\rho) \leq \ker(\overline{\rho}) = Z,
\]
in particular, \( \ker(\rho) = \ker(\overline{\rho}) \). Furthermore, by Lemma 4.4 the second condition implies that \( G^{\text{ Orb}(\overline{Z})} = G^{\text{ Orb}(\overline{Z})} \) and hence

\[
\text{im}(\rho) = G^{\text{ Orb}(\overline{Z})} = \overline{G^{\text{ Orb}(\overline{Z})}} = \text{im}(\overline{\rho}).
\]

Thus,

\[
|G| = |\ker(\rho)| \cdot |\text{im}(\rho)| = |\ker(\overline{\rho})| \cdot |\text{im}(\overline{\rho})| = |\overline{G}|.
\]

Since \( G \leq \overline{G} \), this means that \( G = \overline{G} \), i.e., \( G \) is 2-closed.

\[
\square
\]

5. Proof of Theorem 1.4

We begin with a sufficient condition for an orbit of quasiregular permutation group to be unessential. The proof is based on a special result from the theory of coherent configurations [1].

Lemma 5.1. Let \( G \) be a quasiregular permutation group and \( \Delta \in \text{Orb}(G) \). Assume that \( G^\Delta = 1 \) for some \( G \)-orbit \( \Delta' \neq \Delta \). Then the orbit \( \Delta \) is unessential.

Proof. The quasiregularity of \( G \) implies that \( G_{\delta'} = G_{\Delta'} \) for each \( \delta' \in \Delta' \). It follows that if \( \delta, \lambda \in \Delta \), then

\[
(\delta', \delta) \in (\delta', \lambda)^G \Rightarrow \delta = \lambda.
\]

Indeed, if \( (\delta', \delta) = (\delta', \lambda)^g \) for some \( g \in G \), then \( g \in G_{\delta'} = G_{\Delta'} \). Since \( G^\Delta = 1 \), this implies that \( \delta = \delta^g = \lambda \).

Denote by \( \Omega \) the point set of \( G \) and put \( S = \text{Orb}_2(G) \). Then the pair \( \mathcal{X} = (\Omega, S) \) is a coherent configuration and \( \overline{G} = \text{Aut}(\mathcal{X}) \) is the automorphism group of \( \mathcal{X} \), see [1]. Formula (8) implies that the condition (3.3.14) from [1] is satisfied for the coherent configuration \( \mathcal{X} \) and the set \( \Delta \) equal to

\[
\Omega' := \Omega \setminus \Delta.
\]
By [1, Lemma 3.3.20(1)], the restriction homomorphism \( \text{Aut}(\mathcal{X}) \to \text{Aut}(\mathcal{X}_\mathcal{Y}) \) is an isomorphism; in particular,
\[
G^\mathcal{Y} = \text{Aut}(\mathcal{X})^\mathcal{Y} = \text{Aut}(\mathcal{X}_\mathcal{Y}) = G^\mathcal{Y}.
\] (9)
and \(|G| = |G^\mathcal{Y}|.

To prove that the orbit \( \Delta \) is unessential, first assume that the group \( G \) is 2-closed. Then formula (9) shows that the group \( G^\mathcal{Y} = G^\mathcal{Y} \) is also 2-closed. Conversely, assume that \( G^\mathcal{Y} \) is 2-closed. Then \( G^\mathcal{Y} = G^\mathcal{Y} \). Consequently,
\[
|G| \leq |G^\mathcal{Y}| = |G^\mathcal{Y}| = |G^\mathcal{Y}| \leq |G|,
\]
whence \(|G| = |G^\mathcal{Y}|\). Since \( G \leq G^\mathcal{Y} \), this implies that \( G = G^\mathcal{Y} \), i.e., \( G \) is 2-closed.

Turn to the proof of Theorem 1.4. Assume that the group \( \text{Zel}(G) \) is trivial. Then for each \( \Delta \in \text{Orb}(G) \), we have
\[
\bigcap_{\Delta' \neq \Delta} G^\Delta_{\Delta'} = 1.
\]
On the other hand, by the theorem hypothesis, \( G^\Delta_{\Delta'} \leq G^\Delta \) is a cyclic \( p \)-group for all \( G \)-orbits \( \Delta' \). Thus, \( G^\Delta_{\Delta'} = 1 \) for at least one \( \Delta' \). By Lemma 5.1, this implies that the orbit \( \Delta \) is unessential.

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