A Comment on the $\beta$-expansion of $s = \frac{1}{2}$ and $s = 1$ Ising Models

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Abstract

The purpose of the present work is to apply the method recently developed in reference [1] to the spin-1 Ising chain, showing how to obtain analytical $\beta$-expansions of thermodynamical functions through this formalism. In this method, we do not solve any transfer matrix-like equations. A comparison between the $\beta$-expansions of the specific heat and the magnetic susceptibility for the $s = \frac{1}{2}$ and $s = 1$ one-dimensional Ising models is presented. We show that those expansions have poorer convergence when the auxiliary function of the model has singularities.
1 Introduction

One dimensional (1D) lattice models have been widely studied for decades, motivated, for example, by the possibility of obtaining physical insights about more realistic theories and of their utilization as good laboratories for testing the applicability of new methods.

Several instances of chain models have been intensively investigated, mainly by virtue of their relevance in connection with Condensed Matter phenomena; among them, we can mention the one-dimensional (1D) Ising, the spin-1/2 XXZ Heisenberg\(^2\) and the Hubbard\(^3, 4\) models. Actually, the chain models mentioned above belong to the class of exactly solvable models, due to their integrability property\(^2\). Thus, the algebraic Bethe ansatz method has been widely applied in order to determine their thermodynamical functions by means of non-linear integral equations.

Recently\(^1\), Rojas et al. showed how to obtain a closed analytical expansion for the grand potential in the thermodynamic limit for any translationally invariant chain model from the high temperature expansion of the cumulant method\(^5\). Another point that we can stress is that the method developed in ref.\(^1\) can also be extended to models including nearest interacting neighbors (e.g. frustrated quantum Heisenberg models with spin-S\(^6, 7\)). For such quantum systems, the calculation of higher order terms in the \(\beta\)-expansion of thermodynamical functions allows one to study their properties at lower temperatures\(^7\). Namely, very good results have been obtained applying the method to the (integrable) spin-1/2 XXZ chain model\(^1\). Actually, we should stress that the applicability of such method extends further to either integrable or non-integrable models (the XXZ with spin-1-case being an example of the latter case), provided that the hamiltonian presents translational invariance, nearest-neighbor interaction, and periodic boundary conditions.

In this paper we intend to obtain an analytical \(\beta\)-expansion of the Helmholtz free energy for the spin-1 Ising model through the application of the results of reference \(^1\). Using the fact that this model is exactly solvable, the correctness of such expansion can be checked. The derivation of the Helmholtz free energy for any unidimensional chain model in the thermodynamical limit is obtained from the auxiliary function \(\varphi(\lambda)\), the latter representing the actual mathematical improvement in this approach, in relation to the literature (cf.\(^5\)). Moreover, the present paper is a first step towards the application of such approach to 1D Ising models with arbitrary spin\(^8\), as well as to more interesting models, such as the spin-1 XXZ Heisenberg chain\(^9\).

The importance and usefulness of the \(\beta\)-expansion of thermodynamic functions relies on the perspective of obtaining information for lower temperatures, as terms of higher order in the expansion are taken into account. Hence, knowing how fast or how slow such expansion converges is crucial to its applicability. We want to clarify how the characteristics (namely, the singularities) of the auxiliary function \(\varphi(\lambda)\) impact on the convergence of the \(\beta\)-expansion of any thermodynamic function. In order to do so, we have studied the \(\beta\)-expansion of the thermodynamic functions for the spin-1 and spin-1/2 Ising models. At this point, we should stress once more that our \(\beta\)-expansion results are analytic, non-perturbative and valid, in principle, for arbitrary values of the parameters.

In section 2 we present a summary of the results derived in reference \(^1\). In section 3 we obtain an analytical expression for the auxiliary function \(\varphi(\lambda)\) for the spin-1 Ising model for finite values of \(\beta\); in section 4 we discuss the rapidity of the convergence of the \(\beta\)-expansion of two thermodynamical functions, namely, the specific heat and the correlation between the \(z\)-component of spin between nearest neighbors, when the auxiliary function \(\varphi(\lambda)\) is/is not singular at \(\lambda = 1\). For the spin-1 Ising model, we restrict ourselves to the case \(h = 0\). In section 5 we present our conclusions.
2 A Survey of the Calculation of the Grand Potential of a Chain Model

In reference [1] Rojas et al. obtained, in the thermodynamic limit, a closed expression for the grand potential of any translationally invariant chain model with periodic boundary condition from an auxiliary function. Here we present a survey of the method and we refer to reference [1] for further details.

Let us consider a one-dimensional regular lattice (a periodic chain) with \( N \) sites, so that the Hilbert space of the chain model is simply \( \mathcal{H}^{(N)} = \otimes \mathcal{H}, \mathcal{H} \) being the irreducible representation at one site, including all its degrees to freedom. The dimension of this Hilbert space is \( \dim \mathcal{H}^{(N)} = \text{tr}_N(1) \).

The notation \( \text{tr}_N \) means the trace over all \( N \) sites and their internal degrees of freedom, e.g. spin.

The grand canonical partition function of a quantum system in the chain with \( N \) sites is given by

\[
Z_N(\beta, \mu) = \text{tr}_N(e^{-\beta K}),
\]

where \( K = H - \mu N \), with \( \mu \) being the chemical potential and \( N \) being an operator that commutes with the Hamiltonian of the system.

Let \( A \) be any operator that acts on \( \mathcal{H}^{(M)} \) where \( M \leq N \). We define \( \langle A \rangle \equiv \frac{\text{tr}_M(A)}{\text{tr}_M(1)} \), for any dimension of \( \mathcal{H}^{(M)} \). We call \( \langle A \rangle \) the normalized trace of operator \( A \).

Using the definition of normalized trace, eq.(1) becomes the expansion of \( Z_N(\beta, \mu) \) around \( \beta = 0 \),

\[
Z_N(\beta, \mu) = \text{tr}_N(1) \left\{ 1 + \sum_{n=1}^{\infty} (-\beta)^n \frac{\langle K^n \rangle}{n!} \right\}.
\]

In reference [1] we showed that the coefficients \( \langle K^n \rangle \) in eq.(1) can be written, for any translationally invariant Hamiltonian with interactions between first neighbors as

\[
\frac{\langle K^n \rangle}{n!} = \sum_{r=1}^{[n,N]} \sum_{m=r}^{[n,N]} N \binom{N-m-1}{r-1} K_{r,m}^{(n)}.
\]

The notation \([n,N]\) means the \( \min(n,N) \) and \( K_{r,m}^{(n)} \) is defined by

\[
K_{r,m}^{(n)} \equiv \sum_{\{n_i\}} \sum_{\{m_i\}} \prod_{j=1}^{r} K_{1,m_j}^{(n_j)},
\]

where \( \sum_{\{n_i\}} \) means the restriction: \( \sum_{i=1}^{m} n_i = n \) and \( n_i \neq 0 \) for \( i = 1, 2, \ldots, m \). We also use the notation \( \{n_i\} \equiv \{n_1, n_2, \ldots, n_m\} \) and \( \{m_i\} \equiv \{m_1, m_2, \ldots, m_m\} \). The function \( K_{1,m}^{(n)} \) is defined as

\[
K_{1,m}^{(n)} = \sum_{\{n_i\}} \prod_{i=1}^{m} \frac{K_{n_i+1}^{n_i}}{n_i!} / g,
\]
and each term on the r.h.s. of eq.\((\text{5})\) corresponds to the g-trace of an open connected sub-chain. 

In the definition of the function \(K_{1,m}^{(n)}\) we have the g-trace which means

\[
\langle K_{i_1,i_1+1}^{n_1} K_{i_2,i_2+1}^{n_2} \ldots K_{i_m,i_m+1}^{n_m} \rangle_g \equiv \frac{1}{n!} \sum_{\mathcal{P}} \langle \mathcal{P}(K_{i_1,i_1+1}^{n_1}, K_{i_2,i_2+1}^{n_2}, \ldots, K_{i_m,i_m+1}^{n_m}) \rangle,
\]

where \(\sum_{i=1}^{m} n_i = n\) with \(n_i \neq 0\) and the indices \(i_k, k = 1..m\) are distinct among themselves. The notation \(\langle \mathcal{P}(K_{i_1,i_1+1}^{n_1}, K_{i_2,i_2+1}^{n_2}, \ldots, K_{i_m,i_m+1}^{n_m}) \rangle\) represents all the distinct permutations of the \(m\) operators \(\{K_{i_1,i_1+1}^{n_1}, K_{i_2,i_2+1}^{n_2}, \ldots, K_{i_m,i_m+1}^{n_m}\}\).

We show in reference\([1]\), that the grand potential per site \(\mathcal{W}(\beta, \mu)\), in the thermodynamic limit is written as

\[
\mathcal{W}(\beta, \mu) = -\frac{1}{\beta} \{\ln(\text{tr}_1(1)) + \ln(1 + \xi)\}.
\]

where

\[
\xi = \sum_{n=0}^{\infty} \frac{d^n}{d\lambda^n} \left(\frac{\varphi(\lambda)^{n+1}}{(n+1)!}\right) \bigg|_{\lambda=1}
\]

with \(\lambda\) being a parameter, the auxiliary function \(\varphi(\lambda)\) is equal to

\[
\varphi(\lambda) = \sum_{m=1}^{\infty} \frac{\Gamma_m}{\lambda^m},
\]

and

\[
\Gamma_m \equiv \sum_{n=m}^{\infty} (-\beta)^n K_{1,m}^{(n)}.
\]

From eqs.\((\text{6})\)-\((\text{9b})\) we see that the grand potential per site, in the thermodynamic limit, can be derived only from the open connected sub-chains. The weight of each sub-chain in the \(\beta\)-expansion of \(\mathcal{W}(\beta, \mu)\) is already presented in eq.\((\text{7})\).

### 3 The Helmholtz Free Energy of the Spin-1 Ising model

The hamiltonian of the spin-1 Ising model (with single-ion anisotropy) is:

\[
\mathcal{H} = \sum_{j=1}^{N} \Delta S_j^z S_{j+1}^z - h S_j^z + D(S_j^z)^2,
\]

where \(h\) is the external magnetic field in the z-direction, while the parameters \(\Delta\) and \(D\) are the exchange and single-ion anisotropy, respectively. The chain has \(N\) sites, and periodic boundary conditions are assumed.
It is worth noticing that the partition function for the model above may be obtained from the transfer matrix approach by calculating the eigenvalues of the Hamiltonian (10). These eigenvalues are solutions of a polynomial of third degree. Nevertheless, for arbitrary values of the parameters $\Delta$, $h$ and $D$, such an approach only gives numerical solutions for the thermodynamic functions, which allow us to verify the applicability of the method presented in reference [1] to one exactly solvable limiting case of a non-integrable model[2]. The validity of the present approach has already been verified for the integrable XXZ Heisenberg spin-1/2 model[1] and its limiting cases[10].

In order to obtain the Helmholtz free energy, we must calculate at first its functions $H_{i,m}^{(n)}$. Due to the fact that all terms that contribute to the Hamiltonian of the Ising model commute among themselves, we may write

$$H_{i,m}^{(n)} = \sum_{\{n_i\}} \langle \prod_{i=1}^{m} H_{i,i+1}^{n_i} \rangle, \quad \text{with } m \leq n. \quad (11)$$

The sums in $\Gamma_m$ (see eq.(9b)), including the restricted sums over the indices $\{m_i\}$, are more tractable if we recognize that they can be substituted, in the thermodynamic limit, by $m$ independent sums with each index $\{m_i\}$ varying from 1 to $\infty$. For this model, $\Gamma_m$ becomes

$$\Gamma_m = \sum_{n_1,n_2,\ldots,n_m=1}^{\infty} \langle \prod_{i=1}^{m} \frac{(-\beta)^{n_i} H_{i,i+1}^{n_i}}{n_i!} \rangle \quad (12)$$

where $H_{i,i+1} = \Delta S_i^z S_{i+1}^z + B_i^z$, with $B_i^z \equiv -h S_i^z + D(S_i^z)^2$. The two operators in $H_{i,i+1}$ commute; consequently, we can apply Newton’s multinomial formula to obtain the expansion of $H_{i,i+1}^{n_i}$,

$$H_{i,i+1}^{n_i} = \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} (\Delta)^{n_i-j_i} (S_i^z)^{n_i-j_i} (B_i^z)^{j_i} (S_{i+1}^z)^{n_i-j_i}, \quad (13)$$

where $\binom{n_i}{j_i}$ are the multinomial coefficients. After averaging over the space sites from $i = 1$ up to $i = m + 1$, we obtain

$$\Gamma_m = \frac{1}{3} \prod_{i=1}^{m} \left( \frac{1}{3} \sum_{n_i=1}^{\infty} \frac{(-\beta)^{n_i}}{n_i!} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} \Delta^{n_i-j_i} ((D-h)^{j_i} + (-1)^{n_i-j_i}(D+h)^{j_i}) \right) \times \times [1 + (-1)^{n_m-j_m} + \delta_{j_m,n_m}]. \quad (14)$$

It is tedious but simple to show that

$$\frac{1}{3} \sum_{n_m=1}^{\infty} \frac{(-\beta)^{n_m}}{n_m!} \sum_{j_m=0}^{n_m} \binom{n_m}{j_m} \Delta^{n_m-j_m} ((D-h)^{j_m} + (-1)^{n_m-j_m}(D+h)^{j_m}) \times \times [1 + (-1)^{n_m-j_m} + \delta_{j_m,n_m}] = (-1)^{n_m-1-j_m-1} a_1 + b_1, \quad (15)$$

where the constants $a_1$ and $b_1$ have been defined as
\begin{align*}
a_1 & \equiv p^+ + q^- + r^- \quad \text{and} \quad b_1 \equiv p^- + q^+ + r^+ \quad (16) \\
p^\pm & \equiv \frac{1}{3} \left( e^{\pm \beta (\Delta - h \pm D)} - 1 \right), \quad q^\pm \equiv \frac{1}{3} \left( e^{\pm \beta (\Delta + h \mp D)} - 1 \right) \quad \text{and} \quad r^\pm \equiv \frac{1}{3} \left( e^{\pm \beta (h \mp D)} - 1 \right). \quad (17)
\end{align*}
Performing the sums in the product on the r.h.s. of eq.(14), we get a recursive solution
\[ \Gamma_m = \frac{a_m + b_m}{3}, \quad (18) \]
where
\begin{align*}
a_i &= b_{i-1} p^+ + a_{i-1} q^- \quad \text{and} \quad b_i = b_{i-1} p^- + a_{i-1} q^+, \quad i = 2, 3, \cdots, m. \quad (19)
\end{align*}
Eq.(19) gives us the relation between the auxiliary function \( \varphi(\lambda) \) and \( \Gamma_m \) which, for this particular model, is
\[ \varphi(\lambda) = \frac{1}{3} \sum_{m=1}^{\infty} \frac{a_m}{\lambda^m} + \frac{1}{3} \sum_{m=1}^{\infty} \frac{b_m}{\lambda^m}. \quad (20) \]
Defining \( \varphi(\lambda) \equiv \phi_a(\lambda) + \phi_b(\lambda) \) with
\begin{align*}
\phi_a(\lambda) & \equiv \frac{1}{3} \sum_{m=1}^{\infty} \frac{a_m}{\lambda^m} \quad \text{and} \quad \phi_b(\lambda) \equiv \frac{1}{3} \sum_{m=1}^{\infty} \frac{b_m}{\lambda^m}. \quad (21)
\end{align*}
we may use the relations (15) to rewrite \( \phi_a(\lambda) \) and \( \phi_b(\lambda) \) as
\begin{align*}
\phi_a(\lambda) &= \frac{a_1}{3\lambda} + \frac{p^+}{\lambda} \phi_b(\lambda) + \frac{q^-}{\lambda} \phi_a(\lambda) \quad (22a) \\
\phi_b(\lambda) &= \frac{b_1}{3\lambda} + \frac{p^-}{\lambda} \phi_b(\lambda) + \frac{q^+}{\lambda} \phi_a(\lambda) \quad (22b)
\end{align*}
which yields
\[ \varphi(\lambda) = \frac{1}{3} \left( p^+ + p^- + q^+ + q^- + r^+ + r^- \right) \lambda + 2 \left( p^+ q^+ - p^- q^- \right) + r^+ \left( p^+ - q^- \right) + r^- \left( q^+ - p^- \right) \]
\[ \lambda^2 - \left( p^- + q^- \right) \lambda - \left( p^+ q^+ - p^- q^- \right). \quad (23) \]
The Helmholtz free energy of the Ising model is then obtained by substituting eq.(23) in eq.(7), that is,
\[ W(\beta) = -\frac{1}{\beta} \left\{ \ln(3) + \ln \left( 1 + \sum_{n=0}^{\infty} \frac{d^n}{d\lambda^n} \left( \varphi(\lambda)^{n+1} \right) \bigg|_{\lambda=1} \right) \right\}, \] (24)

and from it we derive a $\beta$-expansion of $W(\beta)$ with arbitrary value of $n$ through an algebraic computational language such as MAPLE.

Depending on the values of $\Delta$, $h$ and $D$, the function $\varphi(\lambda)$ may be singular at $\lambda = 1$ for some values of $\beta$. This is also true for powers of this function at $\lambda = 1$ which contribute to eq.(24). However, we know that there is no phase transition at finite $\beta$ in any one-dimensional Ising model. Consequently, those singularities in $\varphi(1)$ (i.e., $\varphi(\lambda = 1)$) must be non-physical. We point out that a similar behavior is shown by the auxiliary function for the spin-1/2 Ising model (see reference [10]). A question yet to be answered is whether the presence of non-physical singularities in $\varphi(1)$ could indicate a poorer convergence of high temperature expansions of the thermodynamical quantities.

In the next section we compare the high temperature expansions for the Ising models with spin-1/2 and spin-1, when their respective auxiliary functions have a singularity at $\lambda = 1$, and when they do not.

### 4 Comparison of the High Temperature Expansions of $s = \frac{1}{2}$ and $s = 1$ Ising models

The auxiliary function $\varphi(\lambda)$ appears for the first time in the literature in our reference [1]. Therefore calculating it for exactly solvable models is useful to the understanding of its properties, including the impact of its singularities on the convergence of the $\beta$-expansion of thermodynamic quantities.

The function $\varphi_1(\lambda)$ of the spin-1 Ising model (see eq.(23)) can be written as

\[ \varphi_1(\lambda) = \frac{A(\beta)}{\lambda - \lambda_+} + \frac{B(\beta)}{\lambda - \lambda_-}, \] (25a)

where $\lambda_{\pm}$ are the roots of the polynomial of second degree in $\lambda$ in the denominator of eq.(23), namely,

\[ \lambda_{\pm} = \frac{1}{2} \left( q^- + p^- \pm \sqrt{(q^- - p^-)^2 + 4q^+p^+} \right). \] (25b)

The constants $A(\beta)$ and $B(\beta)$ are easily obtained substituting eq.(25b) in eq.(23).

For the case $h = 0$, eqs.(25) reduce to

\[ \varphi_1^{(0)}(\lambda) = \frac{A(\beta)}{\lambda - (p^+ + p^-)}, \] (26a)

with

\[ A(\beta) = \frac{2}{3}(p^+ + p^- + r^-). \] (26b)

For $h = 0$, the type of $\lambda$-dependence in eq.(26a) gives rise to an alternating series to the Helmholtz free energy, in the variable $A(\beta)/(1 - \lambda_+)^2$. 
For $-D < \Delta < D$ and $D > 0$, the function $\varphi_1^{(0)}(1)$ has no singularity; otherwise, there is one real and positive value of $\beta$ for which it is singular.

The auxiliary function (26a) has the same $\lambda$-dependence as the auxiliary function of the spin-1/2 Ising model, that was summed up in reference [10]. Substituting eq.(26a) in eq.(23), and following the steps described in [10], we obtain the Helmholtz free energy of the spin-1 Ising model at $h = 0$,

$$W_1(\beta) = -\frac{1}{\beta} \ln \left[ \frac{1}{2} \left( 1 + p^+ + p^- + \sqrt{(1 - p^+ - p^-)^2 + \frac{3}{8}(p^+ + p^- + r^-)} \right) \right],$$

valid for finite values of $\beta$. This result coincides with the one derived from the transfer matrix approach, as well as with those obtained by numerical analysis.

We derived in reference [10] the auxiliary function $\varphi_1^{(1)}(\lambda)$ for the spin-1/2 Ising model. For arbitrary value of the external magnetic field $h$ we got

$$\varphi_1^{(1)}(\lambda) = e^{2\beta h} \left( \frac{1}{4} \left( \frac{1}{\lambda - e^{\beta h} - \frac{1}{2}} \right) \right).$$

Only for $h \neq 0$ his function has one real and positive value of $\beta$ where $\varphi_1^{(1)}(1)$ is singular. The Helmholtz free energy of the spin-1/2 Ising model, for arbitrary value of $h$ is

$$W_1^{(1)}(\beta) = -\frac{1}{\beta} \ln \left[ e^{-\frac{\beta \Delta}{4}} \cosh(\beta h) + \sqrt{e^{-\frac{\beta \Delta}{2}} \sinh^2(\beta h) + e^{\frac{\beta \Delta}{2}}} \right],$$

that is also valid for finite values of $\beta$.

In both equations (29) and (27), the respective Helmholtz free energies are $\beta$-expansions of infinite range. We are interested in the impact of the singularities of $\varphi(\lambda)$ on the rapidity of convergence of the $\beta$-expansion of thermodynamical quantities. For both spin=1 and spin-1/2 models, we have taken the specific heat $C_v$ and the $z$-component of spin correlation $S_i^z S_{i+1}^z$ between nearest neighbors as examples, and have chosen two suitable sets of values for the parameters $(\Delta, D, h)$ so that the singular and non-singular $\varphi$ cases can be compared, as far as the rapidity of convergence of $\beta$-expansions is concerned. In what follows, approximate curves will refer to those obtained by truncation of the $\beta$-expansion at 80th order in $\beta$, in contrast to the exact curves.

For the spin-1 Ising model, we have taken the first set of conditions to be $(\Delta = 1, D = 1, h = 0)$, which yields a non-singular $\varphi_1^{(0)}(1)$; by promoting a slight variation in $D$, a second set $(\Delta = 1, D = 0.99, h = 0)$ can be defined, yielding a singular $\varphi_1^{(0)}(1)$. For both sets, exact and approximate curves of the specific heat $C_v$ are shown in figure 1 and 2. We expect that both exact curves be slightly apart from each other, in some finite interval of sufficiently small $\beta > 0$. They differ, indeed, by a relative error inferior to 1%, for $0 \leq \beta < 2.5$. If we compare the behavior of the corresponding approximate curves, however, we observe that convergence in the singular $\varphi$ case is much worse than that of the non-singular case. In the non-singular case, the approximate and exact curves differ by less than 1% for $0 \leq \beta < 1.16$, whereas in the singular case they differ by less than 1% for a much smaller interval $0 \leq \beta < 0.54$. Similar behavior can be observed in the correlation function $S_i^z S_{i+1}^z$ (see figures 3 and 4). Exact curves in both singular and non-singular cases differ by less than 2% in the interval $0 \leq \beta < 1.5$; however, in the non-singular case the approximate and exact curves differ by less than 1% in $0 \leq \beta < 1.23$, whereas in the singular case this interval is much shorter, namely, $0 \leq \beta < 0.58$. 
Similar discussion can be carried out for the spin-1/2 Ising model. Here, the conditions ($\Delta = -1, D = 1, h = 0$) and ($\Delta = -1, D = 1, h = 0.01$) yield non-singular and singular cases, respectively. (Observe the slight variation in $h$, only.) For both sets, exact and approximate curves of the specific heat $C_v$ are shown in figure 5 and 6. The exact curves differ by less than 1% in the interval $0 \leq \beta < 2.6$. In the non-singular case, the approximate and exact curves differ by less than 1% for $0 \leq \beta < 5.2$, whereas in the singular case the same difference holds for a much smaller interval $0 \leq \beta < 1.8$. The same behavior is also exhibited by the correlation function $\langle S_i^z S_{i+1}^z \rangle$ (see figures 7 and 8). Exact curves in both singular and non-singular cases differ by less than 0.05% in the interval $0 \leq \beta < 1.8$; however, in the non-singular case the approximate and exact curves differ by less than 1% in $0 \leq \beta < 1.23$, whereas in the singular case this interval is much shorter, $0 \leq \beta < 0.58$.

Those results indicate that the presence of singularities in the auxiliary function $\varphi$ yields poor convergence of $\beta$-expansions of thermodynamical functions.

5 Conclusions

In reference [1] we obtained, in the thermodynamical limit, a closed analytical series for the grand potential for any unidimensional chain model with periodic boundary conditions, from the high-temperature expansion of the cumulant method [5]. In order to derive the grand potential, we must calculate the auxiliary function $\varphi(\lambda)$ for the particular model of interest. Recently, the method presented by Rojas et al. was applied successfully to the spin-1/2 XXZ Heinsenberg model [1] and its limiting cases [10]. In the present work we consider the exactly solvable spin-1 Ising model, whose numerical solutions can be easily attained and used in the verification of the correctness of the results derived from reference [1], as well as in the study of the properties of the auxiliary function $\varphi(\lambda)$ of this model. From eqs. (23) and (24) we obtain the $\beta$-expansion for the Helmholtz free energy $W(\beta)$ for arbitrary values of the parameters $\Delta$, $D$ and $h$. The analyticity of this $\beta$-expansion allows its use as input to the perturbative study of thermodynamical properties of uniaxial Ising-like models. For one set of values of the parameters, and $n = 40$ (where $n$ is the leading order of the $\beta$-expansion) we plotted those expansions and confirmed that our results match the numerical results for this thermodynamic function, in an interval of $\beta$, including situations in which $h \neq 0$. For $h = 0$ we are able to sum the terms in the expansion (24) of $W_1(\beta)$ for arbitrary values of $\Delta$ and $D$ at any finite $\beta$. In this case, the auxiliary function $\varphi_1^{(0)}(\lambda)$ is given by eq. (24), and it has no singularity at $\lambda = 1$ for $-D < \Delta < D$ and $D > 0$; otherwise, there is a value of $\beta$ for which $\varphi_1^{(0)}(\lambda)$ is singular. Such singularity for the spin-1 model is of the same type as that of spin-1/2 Ising model [10]. In both models the singularities are non-physical, since no unidimensional Ising model has phase transitions at finite $\beta$. In order to see if the existence of a singularity in the auxiliary function of both models could influence the rapidity of convergence of the $\beta$-expansion of the thermodynamic functions, for each model we considered two distinct sets of parameter values: in one of them the respective auxiliary function is singular, and in the other one it has no singularities, at $\lambda = 1$. For each model, the two sets differ by the value of only one parameter, and the difference in value is very small, in such a way that the results for those sets could be related to one another by perturbation theory, at least in a finite range of $\beta$.

In the $\beta$-expansion of thermodynamic functions we kept $n = 80$ for both models and both cases (singular and non-singular $\varphi$’s, spin-1/2 and spin-1). These results strongly indicate that the presence of singularities in the auxiliary function allows one to infer that the $\beta$-expansions will have poor convergence, even though the series (23) has infinite range in $\beta$.

Our analytical results at $h = 0$ can be used as the starting point for analytical perturbative expansions for $h \neq 0$, and also for other models.
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[11] There is a misprint in reference [10]: in the expression of the Helmholtz free energy, the constant $\Delta$ should be replaced by $\frac{\Delta}{2}$. 


Specific heat ($s = 1$)

$h = 0$

Figure 1: The full line is the exact result of the specific heat for $\Delta = 1$ and $D = 1$. The dotted line is its expansion up to order $n=80$ in $\beta \in [0, 1.22]$.

Correlation between nearest neighbors ($s = 1$)

$h = 0$

Figure 3: The full line is the exact result of the correlation between nearest neighbors for $\Delta = 1$ and $D = 1$. The dotted line is its expansion up to order $n=80$ in $\beta \in [0, 1.3]$.

Specific heat ($s = 1$)

$h = 0$

Figure 2: The full line is the exact result of the specific heat for $\Delta = 1$ and $D = 0.99$. The dotted line is its expansion up to order $n=80$ in $\beta \in [0, 0.57]$.

Correlation between nearest neighbors ($s = 1$)

$h = 0$

Figure 4: The full line is the exact result of the correlation between nearest neighbors for $\Delta = 1$ and $D = 0.99$. The dotted line is its expansion up to order $n=80$ in $\beta \in [0, 0.62]$. 

$\beta$

$0.2$ $0.4$ $0.6$ $0.8$ $1$ $1.2$

$C_v$

$0$ $0.1$ $0.2$ $0.3$

$S_i^z S_{i+1}^z$

$0$ $-0.1$ $-0.5$

$\beta$

$0.2$ $0.4$ $0.6$ $0.8$ $1$

$\beta$

$0.2$ $0.4$ $0.6$ $0.8$ $1$ $1.2$

$C_v$

$0$ $0.1$ $0.2$ $0.3$

$S_i^z S_{i+1}^z$

$0$ $-0.1$ $-0.5$
Specific heat \((s = 1/2)\)

\[ C_v \]

\[ h = 0 \]

Specific heat \((s = 1/2)\)

\[ C_v \]

\[ h = 0.01 \]

Correlation between nearest neighbors \((s = 1/2)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0 \]

Correlation between nearest neighbors \((s = 1/2)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0.01 \]

Correlation between nearest neighbors \((s = 1/2)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0.01 \]

Specific heat \((s = 1)\)

\[ C_v \]

\[ h = 0 \]

Correlation between nearest neighbors \((s = 1)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0 \]

Correlation between nearest neighbors \((s = 1)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0.01 \]

Correlation between nearest neighbors \((s = 1)\)

\[ \beta \]

\[ 0 \]

\[ S_i^z S_{i+1}^z \]

\[ h = 0.01 \]

Figure 5: The full line is the exact result of the specific heat for \(\Delta = -1\) and \(D = 1\). The dotted line is its expansion up to order \(n=80\) in \(\beta \in [0, 5.85]\).

Figure 7: The full line is the exact result of the correlation between nearest neighbors for \(\Delta = -1\) and \(D = 1\). The dotted line is its expansion up to order \(n=80\) in \(\beta \in [0, 6.3]\).

Figure 6: The full line is the exact result of the specific heat for \(\Delta = -1\) and \(D = 1\), but for \(h = 0.01\). The dotted line is its expansion up to order \(n=80\) in \(\beta \in [0, 1.85]\).

Figure 8: The full line is the exact result of the correlation between nearest neighbors for \(\Delta = -1\) and \(D = 1\) but for \(h = 0.01\). The dotted line is its expansion up to order \(n=80\) in \(\beta \in [0, 1.9]\).