A NOTE ON FRAGMENTABILITY AND WEAK-$G_\delta$ SETS.

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Abstract. In terms of fragmentability, we describe a new class of Banach spaces which do not contain weak-$G_\delta$ open bounded subsets. In particular, none of these spaces is isomorphic to a separable polyhedral space.

1. Introduction and Preliminaries

All Banach spaces under consideration in this note are assumed to be real and infinite-dimensional.

According to a well known theorem of Lindenstrauss and Phelps [LP], if $X$ is a reflexive space then every closed convex and bounded body in $X$ has uncountably many extreme points. The first named author has obtained different generalisations of this result. In particular, in [F1], it is proved that every infinite-dimensional Banach space $X$ which is not $c_0$-saturated does not admit a countable boundary. Moreover, if $X$ is not $c_0$-saturated then [F2]

(a) $X$ does not contain an open, bounded weak-$G_\delta$ set.

In [F3] it is shown that if a separable space $X$ does not contain $c_0$ then

(b) the polar $A^\circ$ of any closed convex and bounded body $A \subset X$ with $0 \in \text{int}A$ contains uncountably many $w^*$-exposed points.

Recall that a set $B \subseteq B_X^*$ is said to be a boundary for $X$ if, for every $x \in X$, there is $f \in B$ such that $f(x) = ||x||$. Assume that $\{f_n\}_{n=1}^\infty$ is a countable boundary for $X$ (the case: $X$ is polyhedral [F4]), then

$$\text{int} B_X = \bigcap_{n=1}^\infty \{x \in X : f_n(x) < 1\}.$$  

Thus $\text{int} B_X$ is an open, bounded weak-$G_\delta$ set. Next let $G \subset X$ be a bounded, convex, closed body and $0 \in \text{int}G$. A point $x \in \partial G$ is said to be smooth if the Minkowski functional $p$ of $G$ is Gâteaux differentiable at $x$. A point $f$ of a subset

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A ⊂ X∗ of the dual space X∗ is said to be a w*-exposed point of A if there is x ∈ X such that f(x) > g(x) for every g ∈ A, g ≠ f. Moreover, we say that this x w*-exposes f. Let us recall also the following well known fact.

**Fact 1.** A point x ∈ G is smooth for G if and only if x w*-exposes some point f in the polar G° of G, with f(x) = 1.

In this note we describe (by means of special fragmentable sets) a new class K of Banach spaces X which have both properties (a) and (b) and which may be c₀-saturated.

**Definition 2** (Namioka [N]). A set M in a Banach space X is said to be fragmentable if, for any subset A of M and any ε > 0, there is a weak open set V which meets A and diam(A ∩ V) < ε.

Additionally, a set M is dentable if, for any ε > 0, there is an open half space H which meets M and diam(A ∩ H) < ε. Clearly, if every subset of a set M is dentable then M is fragmentable. It is known that if M is a weakly compact subset of a Banach space, or a bounded subset of a dual space of an Asplund space, then every subset of M is dentable (see e.g. [B, p. 31, 60 and 91]).

We define the class K as follows.

**Definition 3.** A Banach space X belongs to K if X contains a non-empty fragmentable set M ⊂ int B_X satisfying the following condition

(*) for any ε > 0, any weak open set V and any x₀ ∈ V ∩ M, there is a finite sequence \{x_i\}_{i=1}^n ⊂ V ∩ M such that \|x_i - x_{i-1}\| < ε, i = 1, . . . , n, and \|x_n\| ≥ 1 - ε.

Our main result is the following

**Theorem 4.** If X ∈ K then X does not contain open bounded w-Gδ sets. Moreover, if X ∈ K is separable and G is a convex bounded open set then the set of all smooth points of clG cannot be covered by a countable union of weak closed sets which does not meet G. In particular, if 0 ∈ G then the set w*-exp G° is uncountable.

The following corollary complements the main result from [F3].

**Corollary 5.** Assume that a separable Banach space X contains a subspace with the Radon-Nikodým property (e.g. reflexive or l₁). If F ⊂ X is a bounded closed convex body with 0 ∈ int F then the set w*-exp F° is uncountable.

Hitherto, there were no c₀-saturated Banach spaces known to satisfy the conclusions of Theorem 4. Despite the fact that the proof of Theorem 4 uses some ideas from [F3], we give examples of c₀-saturated Banach spaces which admit sets satisfying the hypotheses of Theorem 4.

2. **Proof of Theorem 4**

The proof of the following fact is standard.
Fact 6. Let $K$ be a weak compact subset of a weak open subset $V$ of a Banach space $X$. Then there is a non-empty weak open neighbourhood $W$ of the origin such that $K + W \subset V$.

The following rather technical proposition will be our main tool.

Proposition 7. Let $G$ be an open bounded subset of a Banach space $X$ and assume $0 \in K \subset G$, where $K$ is compact. Put $G_K = \{ x \in X : x + K \subset G \}$. Assume that $M$ is a non-empty fragmentable subset of $\text{cl}G_K$, such that for any weak open set $U$ with $U \cap M \neq \emptyset$, for any weak closed subset $E$ with $E \cap G = \emptyset$, and for any $\varepsilon > 0$, there is $y \in M \cap U$ such that $(y + K) \cap E = \emptyset$ and $d(y, \partial G_K) < \varepsilon$. Then, for any $w$-$F_\sigma$ set $F$ with $F \cap G = \emptyset$, there is $x \in X$ such that $x + K \subset \text{cl}G$, $(x + K) \cap F = \emptyset$, and $(x + K) \cap \partial G \neq \emptyset$.

Proof. Let $F = \bigcup_{n=1}^\infty F_n$, where $\{F_n\}$ is an increasing sequence of weak closed sets. Set $F_0 = \emptyset$ and let $\{\varepsilon_n\}_{n=0}^\infty$ be a sequence of positive numbers tending to 0, where $\varepsilon_0 > \text{diam}(G_K)$. We construct a sequence $\{x_n\}_{n=0}^\infty \subset M$ and decreasing sequences of $w$-open sets $\{U_n\}_{n=0}^\infty$ and $\{V_n\}_{n=0}^\infty$ with the following properties

1. $x_n \in U_n$ and $x_n + K \subset V_n$;
2. $w$-$\text{cl}V_n \cap F_n = \emptyset$;
3. $\text{diam}(U_n \cap M) < \varepsilon_n$;
4. $d(x_n, \partial G_K) < \varepsilon_n$

for all $n$. To begin, let $x_0 \in M$ be arbitrary and $U_0 = V_0 = X$. Assume we have constructed $x_n$, $U_n$ and $V_n$. By Fact 6 we can take a weak open neighbourhood $W$ of $x_n$ such that $W + K \subset V_n$. Since $x_n \in U_n \cap W \cap M$ and $M$ is fragmentable, there exists weak open $U_{n+1} \subset W \cap U_n$ such that $U_{n+1} \cap M$ is non-empty and $\text{diam}(U_{n+1} \cap M) < \varepsilon_{n+1}$. From our hypothesis, there exists $x_{n+1} \in U_{n+1} \cap M$ with the property that $(x_{n+1} + K) \cap F_{n+1} = \emptyset$. Since $x_{n+1} + K \subset U_{n+1} + K \subset W + K \subset V_n$ and $x_{n+1} + K \subset X \setminus F_{n+1}$, again by Fact 6 we can pick a weak open neighbourhood $W'$ of $x_{n+1}$, satisfying $w$-$\text{cl}W' + K \subset V_n \setminus F_{n+1}$. Define $V_{n+1} = W' + K$ to complete the construction.

From the conditions above, it follows that $\{x_n\}$ is a Cauchy sequence. Let $x = \lim_{n \to \infty} x_n$. We have $x + K \subset \bigcap_{n=0}^\infty w$-$\text{cl}V_n$ and $x \in \partial G_K$. Hence $x + K \subset \text{cl}G$ and $(x + K) \cap F = \emptyset$. Since $K$ is a compact set and $x \in \partial G_K$, it follows that $(x + K) \cap \partial G \neq \emptyset$. The proof is complete.

Recall that a Banach space $X$ is called polyhedral [K] if the unit ball of any of its finite-dimensional subspace is a polytope. It was proved in [F4] that a separable polyhedral space admits a countable boundary. The next assertion shows that fragmentable subsets of the unit sphere of a separable polyhedral space are quite small.

Corollary 8. Let $G$ be an open bounded subset of a Banach space $X$, and let $M \subset \partial G$ be a fragmentable set such that for any weak open set $U$ with $U \cap M \neq \emptyset$, and for any weak closed set $F$ with $F \cap G = \emptyset$, we have $(U \cap M) \setminus F \neq \emptyset$. Then $G$ is not a weak $G_\delta$ set. In particular, if $X$ is polyhedral then, for every fragmentable
set \( M \subset S_X \), there is a weak open set \( U \) which meets \( M \) and a finite number of hyperplanes \( \{H_i\}_{i=1}^m \) in \( X \), such that \( U \cap M \subset \bigcup_{i=1}^m H_i \).

**Proof.** We can assume that \( 0 \in G \). If we put \( K = \{0\} \) and apply Proposition \( 7 \) we see that \( G \) is not a weak \( G_\delta \) set. If \( X \) is polyhedral then \([F4]\) it has a countable boundary and hence there is a sequence of hyperplanes \( \{H_i\}_{i=1}^\infty \) in \( X \) with \( S_X \subset \bigcup_{i=1}^\infty H_i \). Setting \( F_n = \bigcup_{i=1}^n H_i \) for \( n \geq 1 \), using the proof of Proposition \( 7 \) we find a weak open set \( U \) and \( m \in \mathbb{N} \) such that \( (M \cap U) \setminus F_m = \emptyset \) and \( M \cap U \neq \emptyset \).

\[ \blacksquare \]

**Proof of Theorem \( 4 \).** Let \( M \subset X \) be as in Definition \( 3 \). It will help to assume that \( 0 \in M \). If necessary, this can be done by replacing \( M \) with the set

\[ \left( \frac{M - z}{1 - ||z||} \right) \cap \text{int}B_X \]

where \( z \in M \) is arbitrary. Assume that \( G \subset X \) is an open bounded set and \( 0 \in K \subset G \), with \( K \) a compact set which we specify later. We will check the conditions of Proposition \( 7 \). First of all \( 0 \in M \cap G_K \). Now let \( G_K, U, E, \) and \( \varepsilon > 0 \), be as in Proposition \( 7 \). Pick \( x_0 \in U \cap M \cap G_K \) and by using the condition (*), find \( \{x_i\}_{i=1}^n \subset U \cap M \) with \( ||x_i - x_{i-1}|| < \varepsilon, \ i = 1, \ldots, n, \ ||x_n|| \geq 1 - \varepsilon \). Assume that \( x_n \in M \cap G_K \). Then since \( ||x_n|| \geq 1 - \varepsilon \) and \( x_n \in M \cap G_K \subset G_K \subset G \subset B_X \), it follows that \( d(x_n, \partial G) < \varepsilon \). If \( x_n \not\in M \cap G_K \) then there is \( m < n \) with \( x_m \in M \cap G_K \) and \( x_{m+1} \not\in M \cap G_K \). Since \( ||x_m - x_{m+1}|| < \varepsilon \), it follows that \( d(x_m, \partial G_K) < \varepsilon \). Set \( y = x_m \) if \( x_n \not\in M \cap G_K \), and \( y = x_n \) otherwise. Hence \( d(y, \partial G_K) < \varepsilon \). Since \( y \in M \cap G_K \subset G_K \) we get that \( y \in G \). Having in mind that \( E \cap G = \emptyset \), we get \( (y + K) \cap E = \emptyset \).

Now assume to contrary that \( G \) is a weak \( G_\delta \) set. Put \( F = X \setminus G \). Then \( F \) is a weak \( F_\sigma \) set and by Proposition \( 7 \) there is \( x \in X \) such that \( x + K \subset \text{cl}G \), \( (x + K) \cap \partial G = \emptyset \), and \( (x + K) \cap \partial G \neq \emptyset \), contradicting \( \partial G \subset F \).

The proof of the second part of the theorem uses an idea from the proof of \([F3, \text{Theorem 2}]\). Given a separable Banach space \( X \) and a convex, bounded open set \( G \) with \( 0 \in G \), we let \( K = T(B(\ell_2)) \), where \( T: \ell_2 \rightarrow X \) is a linear, compact operator with dense range, and chosen so that \( K \) is contained in the interior of \( G \). If \( F \) is a weak \( F_\sigma \) set with \( F \cap G = \emptyset \) then by Proposition \( 7 \) we obtain \( x \in \text{cl}G \) satisfying

\[ x + K \subset \text{cl}G, \ (x + K) \cap \partial G = \emptyset, \ (x + K) \cap \partial G \cap F = \emptyset. \]

Now assume to the contrary that \( w^*\text{-exp} \ G^\circ \) is countable. Then by Fact \( 1 \) the set \( \text{sm}(\text{cl}G) \) of all smooth points of \( \text{cl}G \) is \( w - F_\sigma \). Put \( F = \text{sm}(\text{cl}G) \) and apply (2.1).

We get a point \( z \in (x + K) \cap (\partial G \setminus F) \). However by using that \( K = T(B(\ell_2)) \) and \( \text{cl}\text{span}K = X \) it is easy to see that \( z \in F \), a contradiction. The proof is complete.

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3. Examples

Let $X$ be a Banach space with a normalized shrinking basis $\{e_i\}$, such that there is a sequence of numbers $\{t_i\}$, $\lim_i t_i = 0$, with two further properties:

(a) $\sup_n ||\sum_{i=1}^n t_i e_i|| = \infty$;
(b) for any subsequence $\{t_{i_k}\}$ such that $\sup_n ||\sum_{k=1}^n t_{i_k} e_{i_k}|| < \infty$, the series $\sum_{k=1}^\infty t_{i_k} e_{i_k}$ converges.

We show there exists a relatively weakly compact subset $M \subset B_X$, satisfying condition (*) of Theorem [4].

Let $\{e_i^*\}$ be the biorthogonal sequence for $\{e_i\}$ and

$$P_n x = \sum_{i=1}^n e_i^*(x) e_i, \quad x \in X, \quad n = 1, 2, \ldots$$

Denote

$$M = \{x = \sum_{i \in \sigma} t_i e_i : \sigma \subset \mathbb{N}, ||\sigma|| < \infty, ||P_n x|| \leq 1, \quad n = 1, 2, \ldots\}.$$ 

Now pick $x_0 = \sum_{i \in \sigma_0} t_i e_i \in M$, $||x_0|| < 1 - \varepsilon$, and a weak open set $V$ containing $x_0$. Find $\delta > 0$ and $m \in \mathbb{N}$ such that

$$x_0 \in U = \{u \in X : |e_i^*(x_0 - u)| < \delta : i = 1, \ldots, m\} \subset V.$$ 

Given $\varepsilon > 0$, find $l \in \mathbb{N}$ such that $|t_i| < \varepsilon$ for $i > l$. Denote $i_0 = \max \sigma$ and pick $j > \max \{i_0, l, m\}$. Set

$$x_{k+1} = x_0 + \sum_{i=j}^{j+k} t_i e_i, \quad k = 0, 1, \ldots$$

Clearly, $\{x_k\} \subset U$, $||x_k - x_{k+1}|| < \varepsilon$, $k = 0, 1, \ldots$, and $\lim_k ||x_k|| = \infty$. Let $n$ be the minimal index for which $||x_n|| < 1$ and $||x_{n+1}|| \geq 1$. Then $||x_k|| < 1$, $x_k \in M$, $k = 1, \ldots, n$, and $||x_n|| \geq ||x_{n+1}|| - ||x_n - x_{n+1}|| > 1 - \varepsilon$.

Next we show that $M$ is relatively weakly compact. Given a sequence $\{y_l\} \subset M$, we have finite $\sigma_l \subset \mathbb{N}$ such that $y_l = \sum_{i \in \sigma_l} t_i e_i$ and $\sup_n ||P_n y_l|| \leq 1$ for each $n$ and $l$. By taking a subsequence, we can find $\sigma \subset \mathbb{N}$ such that $\lim_l \sigma_l = \sigma$ in the pointwise topology of the power set of $\mathbb{N}$. We enumerate $\sigma$ as a strictly increasing sequence $\{i_k\}$. Clearly $\lim_l P_{i_k} y_l = \sum_{k=1}^\infty t_{i_k} e_{i_k}$ for each $n$, so by (b), $y = \sum_{k=1}^\infty t_{i_k} e_{i_k}$ converges in $X$. Since $\{e_i\}$ is shrinking, it is evident that $w\text{-}\lim y_l = y$.

Example 9. There is a separable Banach space $X$ with shrinking basis which is $c_0$-saturated but does not contain a bounded, open weak-$G_\delta$ set. Moreover, for any equivalent norm $|||\cdot|||$ on $X$, the set $\exp B_{(X, |||\cdot|||)}$ is uncountable.

Indeed, in [4] a non-degenerate Orlicz function $M$ is constructed such that there is a sequence $\{t_i\}$, $\lim_i t_i = 0$, with

$$\sup_i \frac{M(K t_i)}{M(t_i)} < \infty,$$
for any $K > 0$, and
\[
\alpha_M = \sup \{ q : \sup_{0 < \lambda, t \leq 1} \frac{M(\lambda t)}{M(\lambda) t^q} < \infty \} = \infty.
\]

From \cite[p. 143]{LT}, it follows that the space $h_M$ is $c_0$-saturated. By repeating some of the $t_i$ if necessary, we may assume that $\sum_i M(t_i) = \infty$. Then the unit vector basis $\{ e_i \}$ of $h_M$ and the sequence $\{ t_i \}$ satisfy the conditions (a) and (b). Let us mention that in \cite{L}, it is shown that such $h_M$ has no countable boundary for any equivalent norm.

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