Quantum signaling game

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Abstract
We present a quantum approach to a signaling game; a special kind of extensive game of incomplete information. Our model is based on quantum schemes for games in strategic form where players perform unitary operators on their own qubits of some fixed initial state and the payoff function is given by a measurement on the resulting final state. We show that the quantum game induced by our scheme coincides with a signaling game as a special case and outputs nonclassical results in general. As an example, we consider a quantum extension of the signaling game in which the chance move is a three-parameter unitary operator whereas the players’ actions are equivalent to classical ones. In this case, we study the game in terms of Nash equilibria and refine the pure Nash equilibria adapting to the quantum game the notion of a weak perfect Bayesian equilibrium.

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1. Introduction

The fifteen-year period of the development of quantum games has brought some of the ideas that tell us how special extensive form games might be played in the quantum domain, for example, the quantum model of Stackelberg duopoly [1] or games with multiple rounds [2]. However, previous results do not explain (even in a simple two-stage quantum game) how to identify behavioral strategies, information sets and other terms connected with extensive game theory. In our recent papers [3, 4] we have proposed a way of quantizing extensive games without chance moves through their normal representation which covers not only two-stage extensive games but also more complex games, including games with imperfect information. In this paper, with the use of a signaling game, we generalize our idea by allowing a chance mover to perform a quantum operation.
The key feature of our research is the study of the extensive structure of the quantum scheme so that we are able to introduce the notion of perfect Bayesian equilibrium—a Nash equilibrium refinement for games of incomplete information. For convenience, we consider the case where the chance mover and the players are equipped with unitary operations. Thus, we assume that the only interaction with the environment is by a quantum measurement. Certainly, a more general scheme could be constructed. According to [5, 6], the most natural generalization is to allow the players to use general quantum operations, i.e., trace-preserving, completely-positive maps. Our aim is not to construct the most general scheme but to show that quantum game theory can be developed by applying advanced terms from classical game theory.

To make the paper self-contained, we begin with recalling the notion of a signaling game and perfect Bayesian equilibrium.

2. Signaling game

The signaling game that we are going to study was introduced by In-Koo Cho and David M Kreps in [7]. The game begins with a chance move that determines the type of player 1. After player 1 is informed about her type, she chooses her action. Then player 2 observes this action and moves next. The extensive form of such a game is illustrated in figure 1.

In this game each player has got two sets of information—points of the game that describe the player’s knowledge about previous actions chosen in the game. Player 1’s information sets are represented by single nodes $t_1$ and $t_2$ since player 1 knows exactly her type. On the other hand, player 2’s information sets are determined by the actions of player 1. They are represented by the nodes connected by dashed lines that follow player 1’s actions. These information sets point out that player 2 learns about an action chosen by player 1. She does not know, however, the type of player 1. This lack of knowledge is a key feature of a signaling game. The only way for player 2 to find out player 1’s type is to analyze her chosen actions that might be a signal about this type.
**Solution concepts for a signaling game**

One of the most commonly used solution concepts for noncooperative games is a Nash equilibrium [8] (see also [9]). It is a strategy profile such that no player gains by unilateral deviation from the equilibrium strategy. The formal definition of a pure Nash equilibrium for a game in strategic form is as follows.

Let \((N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})\) be a game in strategic form, where \(N = \{1, 2, \ldots, n\}\), \(n \in \mathbb{N}\) is the set of players, \(S_i\) is the set of strategies of player \(i \in N\) and \(u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}\) is the payoff function of player \(i\) that assigns for every strategy profile \((s_1, s_2, \ldots, s_n)\) payoff \(u_i(s_1, s_2, \ldots, s_n)\).

**Definition 2.1.** A profile of strategies \((s_1^*, s_2^*, \ldots, s_n^*)\) is a pure Nash equilibrium in a strategic game \((N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})\) if for each player \(i \in N\) and for all \(s_i \in S_i\)

\[
u_i(s_i^*, s_{-i}^*) \geq \nu_i(s_i, s_{-i}^*), \quad \text{where} \quad s_{-i}^* = (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*). \tag{1}
\]

A Nash equilibrium is treated as a necessary condition for a strategy profile to be a reasonable solution of any noncooperative game and it may be very useful for determining a probabilistic outcome of a strategic game if the number of Nash equilibria is quite low. However, the significance of a Nash equilibrium may decline if one considers a game in extensive form. An extensive game may have a lot of Nash equilibria and/or some of the Nash equilibria may include actions which are not optimal off the equilibrium path (see for example [9, 10]). As an example, let us consider an extensive game in figure 1 with \(p = \frac{1}{2}\). One way to find pure Nash equilibria in this game is first to determine its normal form. It is a strategic game defined by the number of players, all the possible strategies of the extensive game and the payoffs corresponding to the strategy profiles. We recall that a player’s strategy in an extensive game is a function assigning an action to each information set of the player. As a result, the normal form with \(p = \frac{1}{2}\) is as follows:

\[
\begin{array}{cccc}
u & ud & du & dd \\
LL & (6, 6) & (6, 6) & (5, 1) & (5, 1) \\
LR & (5, 7) & (6, 6) & (4, 1) & (5, 0) \\
RL & (8, 4) & (6, 1) & (8, 5) & (6, 2) \\
RR & (7, 5) & (6, 1) & (7, 5) & (6, 1) \\
\end{array}
\tag{2}
\]

where, for example, strategy LR means that player 1 plays action L at her first information set and R at the second one, and \(uu\) means that player 2 chooses action \(u\) at both her information sets. Using the system of inequalities (1), pure Nash equilibria in (2) are then \((RL, du)\) and \((LL, ud)\). Although it might seem that these two equilibria are equally likely scenarios of the game, it is supposed that equilibrium strategy \(ud\) will not be chosen by a rational player. It follows from non-optimal action \(d\) off the equilibrium path. Indeed, note that action \(d\) is strictly dominated by action \(u\) at the right information set of player 2, i.e., it always gives a worse outcome than action \(u\) at this information set. Thus, in case player 2 specifies her action at the right information set, she ought to choose \(u\) instead.

Equilibrium refinements can exclude Nash equilibria containing non-optimal actions. For the signaling game it is sufficient to consider a (weak) perfect Bayesian equilibrium, where, for our needs, we restrict ourselves to pure strategies. Following Kreps and Wilson [11], let us first define an assessment to be a pair \((s, \mu)\) of a pure strategy profile \(s\) and a belief system \(\mu\), i.e., a map that assigns to each information set a probability distribution over the nodes of this information set. Thus, in our example in figure 1, player 2’s beliefs are probability distributions \((b_1, 1 - b_1)\) and \((b_2, 1 - b_2)\). In turn, player 1’s beliefs assign to her decision
nodes a probability equal to 1. Now, for any node \( x \) from an information set \( h \) let \( P(x) \) denote the probability that \( x \) is reached given \( s \) and chance moves, if any.

**Definition 2.2.** An assessment \((s, \mu)\) is Bayesian consistent if belief \( \mu(x) \) at node \( x \) is equal to
\[
P(x) / \sum_{x \in h} P(x)
\]
for all \( h \) for which \( \sum_{x \in h} P(x) > 0 \) and all \( x \in h \).

Denote by \( u_i(a|s, x) \) the expected payoff of player \( i \) from playing action \( a \), conditional on being at node \( x \) and strategy profile \( s \). Then, \( \sum_{x \in h} \mu(x) u_i(a|s, x) \) is the expected payoff of player \( i \) from playing the action \( a \), conditional on being at information set \( h \).

**Definition 2.3.** An assessment \((s, \mu)\) in an extensive game is sequentially rational if for each player \( i \in N \), each information set \( h \) of this player and action \( a \) from set the \( \Lambda(h) \) of available actions at \( h \), if \( a \) is consistent with \( s_i \), then
\[
\sum_{x \in h} \mu(x) u_i(a|s, x) = \max_{a' \in \Lambda(h)} \sum_{x \in h} \mu(x) u_i(a'|s, x).
\]

In other words, condition (3) requires for each player \( i \) that action \( a \) prescribed by strategy \( s_i \), is optimal given \( s = (s_i)_{i \in N} \) and beliefs \( \mu \). The two definitions above allow one to formulate the following equilibrium refinement.

**Definition 2.4.** An assessment \((s, \mu)\) in an extensive game is a (weak) perfect Bayesian equilibrium if it is sequentially rational and Bayesian consistent.

Let us consider, for example, Nash equilibrium \((LL, ud)\) with a view to a perfect Bayesian equilibrium. If probability distribution \((p, 1 - p) = (1/2, 1/2)\) determines the chance moves, the Bayesian consistency on player 2’s beliefs requires \( b_1 = 1/2 \) whereas beliefs \((b_2, 1 - b_2)\) are not forced by the consistency requirement. In this case action \( u \) at the left information set is optimal. However, given arbitrary beliefs \((b_2, 1 - b_2)\), action \( d \) at the right information set gives player 2 a lower payoff than action \( u \). As a consequence, the profile \((LL, ud)\) is not a perfect Bayesian Nash equilibrium. A similar analysis would show that the profile \((RL, du)\) together with player 2’s beliefs \( b_1 = 0 \) and \( b_2 = 1 \) satisfies Bayesian consistency and sequential rationality.

### 3. Quantum model for a signaling game

In paper [4] we introduced a model for describing extensive games, where we focused on the normal form and studied Nash equilibria in the resulting quantum game. Here, we are going to justify our scheme with respect to the dynamic nature of an extensive game.

**Motivation for the model construction**

Let us consider the generalized Eisert–Wilkens–Lewenstein (EWL) quantum approach to an \( n \)-player strategic game with two-element strategy sets [5] (we encourage readers who are not familiar with the EWL scheme to first see [12]). According to an alternative notation for the EWL scheme introduced in [13] and generalized in [5], the quantum protocol is defined by 4-tuple
\[
\{\mathcal{H}, |\Psi_0\rangle, SU(2), \{M_i\}_{i \in \{1, 2, \ldots, n\}}\},
\]
where the components specify the game in the following way:
\begin{itemize}
\item $\mathcal{H}$ is a Hilbert space $(\mathbb{C}^2)^{\otimes n}$ with basis $\{|\Psi_{x_1x_2...x_n}\rangle\}_{x_1x_2...x_n \in \{0,1\}^n}$ defined for all $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ by the formula

$$\langle \Psi_{x_1x_2...x_n} \rangle = \frac{|x_1x_2...x_n\rangle + i|x_1\bar{x}_2...\bar{x}_n\rangle}{\sqrt{2}},$$

where $\bar{x}_i$ is the negation of $x_i$.
\item $|\Psi_{00}\rangle$ is called the initial state, and $|\Psi_{ii}\rangle = |\Psi_{00...0}\rangle$.
\item $\text{SU}(2)$ defines the unitary operators available for each player. The matrix representation of the operators from $\text{SU}(2)$ (with respect to the computational basis) can be written as follows:

$$U_i(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\beta} \cos \frac{\theta}{2} & i e^{i\beta} \sin \frac{\theta}{2} \\ i e^{-i\beta} \sin \frac{\theta}{2} & e^{-i\beta} \cos \frac{\theta}{2} \end{pmatrix}.$$ \hspace{0.5cm} (6)

\item $M_i$ for each $i \in \{1, \ldots, n\}$ is an observable given by the formula

$$M_i = \sum_{x_i, x_{i+1}, \ldots, x_n \in \{0,1\}} m_{x_1x_2...x_n} |\Psi_{x_1x_2...x_n}\rangle \langle \Psi_{x_1x_2...x_n}|.$$ \hspace{0.5cm} (7)

The measurement is performed on the final state $|\Psi_f\rangle = \bigotimes_{i=1}^n U_i(\theta, \alpha, \beta_i)|\Psi_{in}\rangle$. The possible outcomes are $m_{x_1x_2...x_n} \in \mathbb{R}$ of the measurement corresponding to player $i$’s payoffs.

It turns out that we can adapt scheme (4) for any extensive game with two actions at each information set. Since operators $U_i(\theta_i, 0, 0)$ represent classical moves in the EWL scheme for a $2 \times 2$ bimatrix game, it is natural to assume that they correspond to classical moves in any quantum game defined by the generalized scheme. The argumentation is as follows. The final state $|\Psi_f\rangle$ after each player performs her unitary operator $U_i(\theta_i, 0, 0)$ is as follows:

$$|\Psi_f\rangle = \bigotimes_{i=1}^n U_i(\theta_i, 0, 0)|\Psi_{in}\rangle$$

$$= \bigotimes_{i=1}^n \left( \cos \frac{\theta_i}{2} |0\rangle + i \sin \frac{\theta_i}{2} |1\rangle \right) + i \bigotimes_{i=1}^n \left( i \sin \frac{\theta_i}{2} |0\rangle + \cos \frac{\theta_i}{2} |1\rangle \right)$$

$$= \sum_{x_1, x_2, \ldots, x_n \in \{0,1\}} \sum_{j=1}^n x_j \cos \left( \frac{x_1 \pi - \theta_1}{2} \right) \cdots \cos \left( \frac{x_n \pi - \theta_n}{2} \right) |\Psi_{x_1...x_n}\rangle.$$ \hspace{0.5cm} (8)

Let us denote by $x_i \in \{0, 1\}$ an action at the $i$th information set and by

$$P_\sigma := \sum_{x_i \in \{0,1\}, k \neq i} |\Psi_{x_1x_2...x_n}\rangle \langle \Psi_{x_1x_2...x_n}|$$

$$P_{x_1x_2...x_n} := \sum_{x_k \in \{0,1\}, k \neq i} |\Psi_{x_1x_2...x_n}\rangle \langle \Psi_{x_1x_2...x_n}|$$

$$\vdots$$

$$P_{x_1x_2...x_n} := |\Psi_{x_1x_2...x_n}\rangle \langle \Psi_{x_1x_2...x_n}|$$ \hspace{0.5cm} (9)

the projectors onto the respective subspaces of $(\mathbb{C}^2)^{\otimes n}$. Let us assign to each action $x_i$ projection $P_{x_i}$ of the state vector $|\Psi_f\rangle$. Then, $\langle \Psi_f|P_{x_i}|\Psi_f\rangle = \cos^2 \frac{\theta_i}{2}$ and $\langle \Psi_f|P_{\bar{x}_i}|\Psi_f\rangle = \sin^2 \frac{\theta_i}{2}$. Taking $p := \cos^2 \frac{\theta_i}{2}$, we obtain a probability distribution over the actions equivalent to one given by a classical behavioral strategy $b = (p, 1 - p)$. Thus, in particular, $U_i(0, 0, 0)$ and $U_i(\pi, 0, 0)$ represent pure actions. In general, let us assign to a sequence of actions $x_1, x_2, \ldots, x_k$ the product of projections $\prod_{j=1}^k P_{x_j} = P_{x_1x_2...x_k}$. Then $\langle \Psi_f|P_{x_1x_2...x_k}|\Psi_f\rangle = \prod_{j=1}^k \cos^2 \frac{\theta_j}{2}$ and this corresponds to the product of probabilities given by applying a sequence $(b_j)$ of classical behavioral strategies $b_j = (\cos^2 \frac{\theta_j}{2}, \sin^2 \frac{\theta_j}{2})$ at the $j$th information set, where $j = 1, \ldots, k$.\end{itemize}
As a result, we have obtained the procedure to describe an extensive game in terms of the mathematical methods of quantum information. At the same time, we have obtained a scheme that places an extensive game in quantum domain whenever the set of unitary operators of at least one player is \( \text{SU}2 \).

**Quantum model**

A detailed description of the quantum scheme for a game in figure 1 is as follows. This is a 6-tuple

\[
(\mathcal{H}, N \cup \{C\}, |\Psi_\text{in}\rangle, \xi, \text{SU}(2), \{M_i\}_{i \in N}),
\]

where

- \( \mathcal{H} \) and \( |\Psi_\text{in}\rangle \) are the special case of those from (4) for a Hilbert space \( (\mathbb{C}^2)^{\otimes 5} \);
- \( N = \{1, 2\} \) is a set of players and \( C \) is a chance mover;
- \( \text{SU}(2) \) specifies the players’ and the chance mover’s actions. It is assumed that a unitary operation performed by the chance mover is known to the players;
- \( \xi \) is a map that relates qubits to players and the chance mover. It is a map \( \xi : \{1, 2, \ldots, 5\} \to N \cup \{C\} \) given by formula
  \[
  \xi(j) = \begin{cases} 
  C & \text{if } j = 1 \\
  1 & \text{if } j \in \{2, 3\} \\
  2 & \text{if } j \in \{4, 5\},
  \end{cases}
  \]
  (11)
  that assigns to each index \( j \in \{1, 2, \ldots, 5\} \) of \( x_j \) in \( |\Psi_\text{1,2,3,4,5}\rangle \) a player or the chance mover;
- \( M_i \) is an observable that describes a measurement on the final state \( |\Psi_f\rangle \),
  \[
  M_i = m_1^i P_001 + m_2^i P_010 + m_3^i P_100 + m_4^i P_110 + m_5^i P_011 + m_6^i P_101,
  \]
  (12)
  Then the average value \( E_i \) of measurement \( M_i \),
  \[
  E_i = \langle \Psi_f | M_i | \Psi_f \rangle
  \]
  (13)
determines a payoff for player \( i \in N \).

Thus, the quantum model for the signaling game in figure 1 requires a five-qubit state. The chance mover’s action is represented by a unitary operation on the first qubit. In turn, a unitary operation \( U_2 \otimes U_3 \) on the second and third qubit, and a unitary operation \( U_4 \otimes U_5 \) on the fourth and fifth one are player 1’s and player 2’s strategies, respectively. The form of observable (12) is based on our motivation for the scheme construction. Following this line of thought, and then the link between projections and actions as it is given in figure 2, each term in \( M_i \) corresponds to measurement that the state of the game is in the respective end node.

**4. A signaling game with a quantum chance mover**

In the literature of quantum games one can find many examples that show the advantages of quantum strategies over classical ones. The same could be done for the quantum signaling game if, for example, one player’s strategy set were extended to the full range of unitary operators. We are going to consider another case where the players’ actions are still classical ones, i.e. they are in the form of \( U(\theta, 0, 0) \), and the full set \( \text{SU}(2) \) is available only for the chance mover. In this case, we obtain an interesting example, where the normal form of the resulting quantum game has the same dimension as the classical game. As it is shown below, this feature makes the classical and quantum game easy to compare. Moreover, the fact that
the players are equipped with operators $U(\theta, 0, 0)$ enables us, easily, to refine Nash equilibria in the quantum game by using the notion of perfect Bayesian equilibrium.

More precisely, let us consider 6-tuple (10) in which the set $SU(2)$ is available only for the chance mover, and the players are equipped with the set $[U(\theta) := U(\theta, 0, 0) : \theta \in [0, \pi]]$. The possible measurement $M'_i$'s outcomes $m'_j$, for $j = 1, \ldots, 8$ of measurement $M_i$ correspond to payoffs from the game in figure (1), i.e.,

- $(m^1_1, m^2_1) = (6, 12)$
- $(m^1_2, m^2_2) = (4, 0)$
- $(m^1_3, m^2_3) = (6, 0)$
- $(m^1_4, m^2_4) = (6, 2)$
- $(m^1_5, m^2_5) = (10, 8)$
- $(m^1_6, m^2_6) = (6, 2)$
- $(m^1_7, m^2_7) = (4, 2)$
- $(m^1_8, m^2_8) = (6, 0)$

Let us assume that the chance mover specifies $U_1(\pi/2, \pi/6, \pi/3)$ as her move. We recall that a chance mover's action $U_1(\pi/2, 0, 0)$ corresponds to probability distribution $(1/2, 1/2)$ over her classical actions and non zero coordinates $\alpha_1$ and $\beta_1$ in $U_1(\theta_1, \alpha_1, \beta_1)$ place the game into the quantum domain. The final state after the first and second player specify $U_2(\theta_2) \otimes U_3(\theta_3)$ and $U(\theta_4) \otimes U(\theta_5)$, respectively, is in the form

$$|\Psi_i\rangle = \left(U_1 \left(\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}\right) \otimes U_2(\theta_2) \otimes U_3(\theta_3) \otimes U_4(\theta_4) \otimes U_5(\theta_5)\right)|\Psi_{00000}\rangle.$$  

Let us calculate the expected values $E_1$ and $E_2$ for each $\theta_2, \theta_3, \theta_4, \theta_5 \in \{0, \pi\}$; values corresponding to pure strategy profiles. For example, quadruple $(0, 0, \pi, \pi)$ implies the following final state:

$$|\Psi_i\rangle = -\frac{\sqrt{6}}{4}|\Psi_{00011}\rangle - \frac{\sqrt{2}}{4}|\Psi_{11000}\rangle - \frac{\sqrt{2}}{4}|\Psi_{10011}\rangle + \frac{\sqrt{6}}{4}|\Psi_{01010}\rangle.$$  

Then, the pair $\langle |\Psi_i\rangle |M_1|\Psi_i\rangle, \langle |\Psi_i\rangle |M_2|\Psi_i\rangle$ equals $(6.5, 3.5)$. By calculating the average values of measurements $M_1$ and $M_2$ for the other quadruples we obtain the following bimatrix:

$$
\begin{pmatrix}
U_4(0) \otimes U_5(0) & U_4(0) \otimes U_5(\pi) & U_4(\pi) \otimes U_5(0) & U_4(\pi) \otimes U_5(\pi) \\
U_3(0) \otimes U_4(0) & U_3(0) \otimes U_4(\pi) & U_3(\pi) \otimes U_4(0) & U_3(\pi) \otimes U_4(\pi) \\
(6, 5.25) & (7.25, 7.75) & (5.25, 1) & (6.5, 3.5) \\
(7.5, 7.75) & (5, 1) & (6.75, 3) & (6.75, 3) \\
(6.75, 3) & (5, 1) & (7.5, 7.75) & (5.75, 5.75) \\
(6.75, 3) & (5, 1) & (7.25, 7.75) & (6, 5.25) \\
\end{pmatrix}.
$$

![Diagram](image-url)
As a result, quantum scheme (10) provides the players with a quite different bimatrix compared with (2). In particular, the classical game and the quantum counterpart have two pure Nash equilibria but differ in payoff outcomes. Indeed, in contrast to the classical case, profiles \((U_2(0) \otimes U_3(\pi)), (U_4(0) \otimes U_3(\pi))\) and \((U_2(\pi) \otimes U_3(0)), (U_4(\pi) \otimes U_3(0))\) are Nash equilibria with the same payoff outcome \((7.5, 7.75)\).

**Perfect Bayesian-type equilibria**

Let us study profile \(((U_2(\pi) \otimes U_3(0)), (U_4(\pi) \otimes U_3(0)))\). According to definition 2.1, no player gains by unilaterally deviating from unitary operator \(U(\pi) \otimes U(0)\). It turns out that it can be said more about the profile in terms of Bayesian consistency and sequential rationality. First, let us carry out perfect Bayesian equilibrium analysis for player 1. Following figure 2, the probability that the game reaches the upper node given chance move \(U_1(\pi/2, \pi/6, \pi/3)\) and player 2’s strategy \((U_4(\pi) \otimes U_3(0))\) equals \(|\Psi_i[P_1]\rangle = 3/4\) where

\[
|\Psi_i\rangle = \left(U_1\left(\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}\right) \otimes \mathbb{I} \otimes \mathbb{I} \otimes U_4(\pi) \otimes U_3(0)\right)|\Psi_{00000}\rangle
\]

\[
= \frac{\sqrt{6}}{4}|\Psi_{00010}\rangle + \frac{\sqrt{2}}{2}|\Psi_{11011}\rangle - \frac{\sqrt{3}}{4}|\Psi_{10010}\rangle + \frac{\sqrt{6}}{4}|\Psi_{01101}\rangle,
\]

and \(\mathbb{I}\) means the identity operator on \(\mathbb{C}^2\). Since player 1’s information sets are singletons, reaching the upper node is equivalent to reaching the corresponding information set, so her belief of being at the upper node attaches probability 1 to this node. Therefore, after player 1 learns that the state of the chance mover’s qubit corresponds to \(P_0\), she believes that with probability 1 faces the following state:

\[
|\Psi_i\rangle = \frac{P_0|\Psi_i\rangle}{\sqrt{|\Psi_i[P_1]\rangle}} = \frac{\sqrt{2}}{2}|\Psi_{00010}\rangle + \frac{\sqrt{2}}{2}|\Psi_{01101}\rangle.
\]

Denote by \(|\Psi''_i\rangle\) the state obtained when player 1 performs unitary strategy \(U_2(\theta_2) \otimes U_3(\theta_3)\) on \(|\Psi_i\rangle\). Then

\[
|\Psi''_i\rangle = (\mathbb{I} \otimes U_2(\theta_2) \otimes U_3(\theta_3) \otimes \mathbb{I} \otimes \mathbb{I})|\Psi_i\rangle
\]

\[
= \frac{\sqrt{2}}{2} \sum_{x_2,x_3 \in \{0,1\}} i^{x_2+x_3} c(x_2, x_3)(i|\Psi_{01010}\rangle + |\Psi_{00010}\rangle),
\]

where \(c(x_2, x_3) = \cos\left(\frac{\pi x_2 - \theta_2}{4}\right)\cos\left(\frac{3\pi x_2 - \theta_2}{4}\right)\). Then, \(U_2(\pi) \otimes U_3(0)\) is optimal given the belief (19) about the quantum state if

\[
U_2(\pi) \otimes U_3(0) \in \arg\max_{U_2(\theta_2) \otimes U_3(\theta_3)} \langle \Psi''_i|M_1|\Psi''_i\rangle.
\]

Indeed,

\[
\max_{U_2(\theta_2) \otimes U_3(\theta_3)} \langle \Psi''_i|M_1|\Psi''_i\rangle = \max_{U_2(\theta_2)} \left(8 - 3\cos^2\frac{\theta_2}{2}\right) = 8.
\]

Thus, condition (21) is satisfied.

Similar computation for the case when player 1 learns that the state of the chance mover’s qubit corresponds to \(P_1\), proves that \(U_2(\pi) \otimes U_3(0)\) is also optimal on state

\[
\frac{P_1|\Psi_i\rangle}{\sqrt{|\Psi_i[P_1]\rangle}} = \frac{\sqrt{2}}{2}|\Psi_{11011}\rangle - \frac{\sqrt{2}}{2}|\Psi_{01010}\rangle.
\]

As a result, player 1’s strategy \(U_2(\pi) \otimes U_3(0)\) is sequentially-type rational given her beliefs.
Let us consider now player 2’s strategy $U_4(\pi) \otimes U_5(0)$ in the terms of perfect Bayesian equilibrium. The state after the chance mover and player 1 use operators $U_1\left(\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}\right)$ and $U_2(\pi) \otimes U_3(0)$, respectively, is as follows

$$|\Psi_2\rangle = \left(U_1\left(\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}\right) \otimes U_2(\pi) \otimes U_3(0) \otimes 1 \otimes 1\right)|\Psi_{00000}\rangle$$

$$= \frac{\sqrt{6}i}{4}|\Psi_{00000}\rangle + \frac{\sqrt{2}i}{4}|\Psi_{01111}\rangle - \frac{\sqrt{2}}{4}|\Psi_{12001}\rangle + \frac{\sqrt{6}}{4}|\Psi_{01011}\rangle. \quad (24)$$

Then the probability that the left information set is reached is equal to $\langle \Psi_2|P_{00,0}|\Psi_2\rangle + \langle \Psi_2|P_{11,0}|\Psi_2\rangle = 1/2$. By Bayesian consistency, player 2’s beliefs of being at the upper and lower node at the left information set are

$$\frac{\langle \Psi_2|P_{00,0}|\Psi_2\rangle}{\langle \Psi_2|P_{00,0}|\Psi_2\rangle + \langle \Psi_2|P_{11,0}|\Psi_2\rangle} = \frac{3}{4} \quad \text{and} \quad \frac{\langle \Psi_2|P_{11,0}|\Psi_2\rangle}{\langle \Psi_2|P_{00,0}|\Psi_2\rangle + \langle \Psi_2|P_{11,0}|\Psi_2\rangle} = \frac{1}{4}. \quad (25)$$

As a consequence, specifying her beliefs, player 2 faces post-measurement state $P_{00,0}|\Psi_2\rangle/\sqrt{\langle \Psi_2|P_{00,0}|\Psi_2\rangle} = |\Psi_{00111}\rangle$ with probability 3/4, and state $P_{11,0}|\Psi_2\rangle/\sqrt{\langle \Psi_2|P_{11,0}|\Psi_2\rangle} = -|\Psi_{11000}\rangle$ with probability 1/4. In other words, player 2 is faced with the following mixed state

$$\rho_2 = \frac{3}{4}|\Psi_{00111}\rangle\langle\Psi_{00111}| + \frac{1}{4}|\Psi_{11000}\rangle\langle\Psi_{11000}|. \quad (26)$$

Thus, mixed state (26) after player 2 uses her unitary operator $U_4(\theta_4) \otimes U_5(\theta_5)$ takes the form

$$\rho'_2 = \left(1^{\otimes 3} \otimes U_4(\theta_4) \otimes U_5(\theta_5)\right)\rho_2\left(1^{\otimes 3} \otimes U_4(\theta_4) \otimes U_5(\theta_5)\right)^\dagger \quad (27)$$

and player 2’s expected payoff is given by $\text{tr}(\rho'_2 M_2)$. In order to prove that, given her beliefs, $U_4(\pi) \otimes U_5(0)$ is optimal for player 2 let us determine $\arg\max_{U_4(\theta_4) \otimes U_5(\theta_5)}\text{tr}(\rho'_2 M_2)$,

$$\arg\max_{U_4(\theta_4) \otimes U_5(\theta_5)}\text{tr}(\rho'_2 M_2) = \arg\max_{U_4(\theta_4) \otimes U_5(\theta_5)}\text{tr} \left( \sum_{x_4, x_5 \in \{0, 1\}} c^2(x_4, x_5) \left(\frac{3}{4}|\Psi_{001\pi_{i,5}}\rangle\langle\Psi_{001\pi_{i,5}}| \right. \right.$$

$$\left. + \frac{1}{4}|\Psi_{110_{i,5}}\rangle\langle\Psi_{110_{i,5}}| \right) \right)$$

$$= \arg\max_{U_4(\theta_4) \otimes U_5(\theta_5)} \frac{19}{2} \sin \frac{\theta_5}{2} = [\pi] \times [0, \pi]. \quad (28)$$

Result (28) shows that $U_4(\pi) \otimes U_5(0)$ is also sequentially-type rational given player 2’s beliefs at the left information set. In a similar way, we can prove sequential-type rationality of $U_4(\pi) \otimes U_5(0)$ at the right information set.

As a result, strategy profile $\{(U_2(\pi) \otimes U_3(0)), (U_4(\pi) \otimes U_5(0))\}$ consists of strategies that are optimal with respect to unilateral deviation in both cases: when only the payoff measurement is performed (a Nash equilibrium) and when a player performs the additional measurement before her move (sequential rationality). It can be shown that the other pure equilibrium given by bimatrix (17) is also a perfect Bayesian-type equilibrium.

5. Conclusion and further research

The purpose of the research was to translate signaling games into the formalism of quantum information and to examine how playing the game would then change. We showed that there exists a quantum approach to a signaling game that constitutes a generalization of the classical game. In particular, we proved (with the use of Eisert et al quantum scheme for strategic games) that the special one-parameter unitary strategies are equivalent to classical moves in...
the game and a broader range of unitary operators affects the game. The key result of our work was to show that optimal strategy analysis in quantum games can go beyond the concept of Nash equilibrium. A player measuring the state after the other players operate but before her move gives rise to a new solution concept in quantum games that can be treated as a counterpart of a perfect Bayesian equilibrium in classical game theory. It is worth noting that there is more than one way to define a quantum counterpart of a perfect Bayesian equilibrium consistent with the classical term. For example, when solving optimization problems (21) and (28) it does not matter whether we maximize over $U_2(\theta_2) \otimes U_3(\theta_3)$ and $U_4(\theta_4)$ or over $U_2(\theta_2)$ and $U_4(\theta)$. However, it may have great importance when the full set of unitary operators is also available for players and may constitute an independent subject of research. Another interesting problem would be to specify how the players’ strategic positions change when one or both players are provided with the set SU(2). In particular, studying player 2’s position in the game seems significant. In the classical game, she is deprived of knowing the type of player 1 and we suppose that the access to quantum strategies may improve player 2’s strategic position. Finally, one may investigate the relation between Nash equilibrium and perfect Bayesian equilibrium. We suppose that, in contrast to the classical case, the perfect Bayesian conditions in the way we have presented in the paper may not imply Nash equilibrium. This would point out another distinction between classical and quantum game theory.

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