Submodule structure of $\mathbb{C}[s, t]$ over $Vir(0, b)$

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ABSTRACT: For any triple $(\lambda, \alpha, h(t))$ with $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$ and $h(t) \in \mathbb{C}[t]$, as free $U(L_0 \oplus W_0)$-module of rank one, $\Phi(\lambda, \alpha, h) := \mathbb{C}[s, t]$ (resp. $\Theta(\lambda, h) := \mathbb{C}[s, t]$) also carries the structure of a module over $Vir(0, b)$ (resp. $Vir(0, 1)$). By introducing two sequences of useful operators on $\mathbb{C}[s, t]$ we give all its submodules and also study the submodules of $\mathbb{C}[s, t]$ over the Virasoro algebra $Vir$, which are shown to be finitely generated if and only if $\deg h(t) \geq 1$. We prove that $\Phi(\lambda, \alpha, h)$ is an irreducible $Vir$-module if and only if $\deg h(t) = 1$ and $\alpha \neq 0$. And by taking the tensor products of a finite number of such irreducible $Vir$-modules $\Phi(\lambda_i, \alpha_i, h_i)$ with irreducible $Vir$-modules $V$ we obtain a family of new irreducible $Vir$-modules, where for $V$ there exists a nonnegative integer $R_V$ such that $L_m$ for all $m \geq R_V$ are locally finite on $V$.

Key words: submodule, non-weight module, Virasoro algebra, irreducible module

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1 Introduction

It is well-known that non-weight modules constitute an important ingredient in the representation theory of Lie algebras, the study of which is more challenging than that of weight modules. Recently these kinds of modules were extensively studied. For instance, non-weight modules over Virasoro algebra and related algebras (see, e.g., [BM, CG1, CG2, CG3, CHS, CHSY, LiuZ, LZ2, LLZ, MW, MZ, TZ1, TZ2, TZ3]), non-weight $g$-modules which are free $U(h)$-modules (see, e.g., [HCS, TZ3, N1, N2]), where $h$ is a subalgebra containing in some Cartan subalgebra of $g$.

Let us recall the non-weight modules studied in [HCS]. Let $\mathcal{B} = \{L_n, W_n, C_i \mid n \in \mathbb{Z}, i = 1, 2, 3\}$ be a linear independent set. For any pair $(a, b)$ of complex numbers, the infinite-dimensional space

$$\mathcal{Vir}(a, b) = \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus_{n \in \mathbb{Z}} \mathbb{C} W_n \oplus \mathbb{C} C_1 \oplus \sum_{n \in \mathbb{Z}} \mathbb{C} C_{2,n} \oplus \sum_{n \in \mathbb{Z}} \mathbb{C} C_{3,n}$$

carries the structure of a Lie algebra (see, e.g., [ON]) whose Lie brackets are given by

$$[L_n, L_m] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{n^3 - n}{12} C_1,$$

$$[L_n, W_m] = (a + m + bn)W_{m+n} + \delta_{m+n,0} C_{2,n},$$

$$[W_n, W_m] = \delta_{m+n, -2a} C_{3,n},$$

$$[C_1, \mathcal{Vir}(a, b)] = [C_{2,n}, \mathcal{Vir}(a, b)] = [C_{3,n}, \mathcal{Vir}(a, b)] = 0.$$
Theorem 1. 

The latter case. Then we have the following results (see [HCS]).

where

\[ C_{2,n} = \begin{cases} 
(n^2 + n)C_2 & \text{if } (a,b) = (0,0), \\
\frac{n^3 - n}{12}C_2 & \text{if } (a,b) = (0,1), \\
nc_2 & \text{if } (a,b) = (1,0), \\
0 & \text{otherwise}, 
\end{cases} \]

and \[ C_{3,n} = \begin{cases} 
nC_3 & \text{if } (a,b) = (0,0), \\
(2n + 1)C_3 & \text{if } (a,b) = (0,1), \\
0 & \text{otherwise}. 
\end{cases} \]

Note that the class of Lie algebras \( \mathcal{V}ir(a,b) \) contains several well-known infinite dimensional Lie algebras, such as the Virasoro algebra \( \mathfrak{V} := \text{span}\{L_m, C_1 \mid m \in \mathbb{Z}\} \), the Heisenberg-Virasoro algebra \( (=\mathcal{V}ir(0,0)) \), the \( W \)-algebra \( W(2,2) \ (=\mathcal{V}ir(0,-1)) \) and so on.

Fix any \( \lambda \in \mathbb{C}^*, \alpha, b \in \mathbb{C} \) and \( h(t) \in \mathbb{C}[t] \), set

\[ g(t) = \frac{h(t) - h(\alpha)}{t - \alpha} \quad \text{and} \quad h_m(t) = mh(t) - m\alpha (\delta_{b,-1}(m - 1) + \delta_{b,1}) g(t) \quad \text{for } m \in \mathbb{Z}. \]

and define the action of the set \( \mathcal{B} \) on \( \mathbb{C}[s, t] \) in the following two different ways:

\[ L_m f(s, t) = \lambda^m (s + h_m(t)) f(s - m, t) + bm\lambda^m (t - \delta_{b,-1}m\alpha - \delta_{b,1}\alpha) \frac{\partial}{\partial t} (f(s - m, t)), \]

\[ W_m f(s, t) = \lambda^m (t - \delta_{b,-1}m\alpha - \delta_{b,1}(1 - \delta_{m,0})\alpha) f(s - m, t), \]

\[ C_i f(s, t) = 0, \quad i = 1, 2, 3, \]

and

\[ L_m f(s, t) = \lambda^m (s + mh(t)) f(s - m, t) \]

\[ W_m f(s, t) = \delta_{m,0} tf(s - m, t) \]

\[ C_i f(s, t) = 0, \quad i = 1, 2, 3, \]

where \( f(s, t) \in \mathbb{C}[s, t] \). Denote \( \mathbb{C}[s, t] \) by \( \Phi(\lambda, \alpha, h) \) in the former case and by \( \Theta(\lambda, h) \) in the latter case. Then we have the following results (see [HCS]).

**Theorem 1.** Let notation be as above.

(1) All \( \Phi(\lambda, \alpha, h) \) are \( \mathcal{V}ir(0,b) \)-modules and all \( \Theta(\lambda, h) \) are \( \mathcal{V}ir(0,1) \)-modules;

(2) Suppose that there exists a \( \mathcal{V}ir(a,b) \)-module \( M \) such that it is a free \( U(\mathbb{C}L_0 \oplus \mathbb{C}W_0) \)-module of rank 1. Then \( a = 0, M \cong \Phi(\lambda, \alpha, h) \) or \( \Theta(\lambda, h) \) if \( b = 1 \), and \( M \cong \Phi(\lambda, \alpha, h) \) if \( b \neq 1 \) for some \( \alpha \in \mathbb{C}, \lambda \in \mathbb{C}^* \) and \( h(t) \in \mathbb{C}[t] \).

The aim of the present paper is to study the submodule structure of all these free \( U(L_0 \oplus W_0) \)-modules of rank 1. To be explicit, we are going to give all \( \mathcal{V}ir(0,b) \)-submodules of \( \Phi(\lambda, \alpha, h) \) and \( \Theta(\lambda, h) \) and also study under which conditions \( \Phi(\lambda, \alpha, h) \) or \( \Theta(\lambda, h) \) is irreducible as a \( \mathfrak{V} \)-module. And using these irreducible \( \mathfrak{V} \)-modules we get a family of new irreducible non-weight \( \mathfrak{V} \)-modules by considering the tensor product of which with \( \mathfrak{V} \)-modules.
studied in [MZ]. It is worthwhile to point out that two sequences of operators $S^j$ and $T^j$ for $j \geq 0$ (see Section 2 below) are introduced, which play a very important role in the present paper. These operators allows us, on the one hand, to determine easily whether a space is a submodule and on the other hand, to compute cyclic submodules.

The rest of the present paper is organized as follows. Section 2 is devoted to determining all $\mathcal{V}ir(0, b)$-submodules of $\Phi(\lambda, \alpha, h)$ and all $\mathcal{V}ir(0, 1)$-submodules of $\Theta(\lambda, h)$. We first introduce operators $S^j$ and $T^j$ for $j \in \mathbb{Z}_+$. Then it is divided into three subsections to give all submodules of $\Phi(\lambda, \alpha, h)$ and $\Theta(\lambda, h)$. In Section 3, all cyclic $\mathfrak{U}$-submodules are given. Moreover, all $\mathfrak{U}$-submodules are proved to be finitely generated. Among all $\mathfrak{U}$-modules $\Phi(\lambda, \alpha, h)$ and $\Theta(\lambda, h)$ we show that $\Theta(\lambda, h)$ is reducible and that $\Phi(\lambda, \alpha, h)$ is irreducible if and only if $\alpha \in \mathbb{C}^*$, $\deg h(t) = 1$ and $b = -1$. Using these irreducible $\mathfrak{U}$-modules $\Phi(\lambda, \alpha, h)$ we obtain a family of new irreducible $\mathfrak{U}$-modules in Section 4 by considering the tensor product of a finite number of which with the $\mathfrak{U}$-modules $V$ for which there exists $R_V \in \mathbb{Z}_+$ such that $L_m$ is locally finite on $V$ whenever $m \geq R_V$.

Throughout the paper, $\mathbb{Z}_+$, $\mathbb{C}$, $\mathbb{C}^*$ will be used to denote the sets of nonnegative integers, complex numbers and nonzero complex numbers, respectively.

2 $\mathcal{V}ir(0, b)$-submodules

Set
\[
\partial^j_s = \left(\frac{\partial}{\partial s}\right)^j, \quad \partial_t = \frac{\partial}{\partial t}, \quad T^j = \frac{t - \delta_{b,1} \alpha}{j!} \partial^j_s + \frac{\alpha}{(j-1)!} \partial^{j-1}_s \delta_{b,-1},
\]
\[
S^j = \frac{s}{j!} \partial^j_s - \frac{1}{(j-1)!} \partial^{j-1}_s h(t),
\]
and
\[
S^j = \frac{s}{j!} \partial^j_s - \frac{1}{(j-1)!} \partial^{j-1}_s \left(h(t) + \delta_{b,-1} \alpha g(t) - \delta_{b,1} \alpha g(t) + (bt - \delta_{b,1} \alpha) \partial_t\right)
\]
\[
- \frac{1}{(j-2)!} \partial^{j-2}_s \delta_{b,-1} \alpha (g(t) - \partial_t) \quad \text{for } j \in \mathbb{Z}_+.
\]

Here and in what follows we make the convention that $(\frac{\partial}{\partial s})^{-1} = 0$, $k! = 1$ for $k < 0$ and $(\frac{j}{j}) = 0$ for $j > i$ or $j < 0$.

**Lemma 2.1.** Let $W$ be a subspace of $\mathbb{C}[s, t]$. Then $W$ is a $\mathcal{V}ir(0, b)$-submodule of $\Phi(\lambda, \alpha, h)$ if and only if $W$ is stable under the operators $S^j$ and $T^j$ for all $j \in \mathbb{Z}_+$; $W$ is a $\mathcal{V}ir(0, 1)$-submodule of $\Theta(\lambda, h)$ if and only $W$ is stable under the operators $S^j_\Theta$ and the left multiplication of $t$ for $j \in \mathbb{Z}_+$. In particular, $W$ is a $\mathfrak{U}$-submodule of $\Phi(\lambda, \alpha, h)$ (resp. $\Theta(\lambda, h)$) if and only if $W$ is stable under the operators $S^j$ (resp. $S^j_\Theta$) for $j \in \mathbb{Z}_+$.

**Proof.** Here we only show the statement concerning $\Phi(\lambda, \alpha, h)$, similar argument can be applied to $\Theta(\lambda, h)$. For any $f(s, t) = \sum_{i=0}^{n} s^i f_i(t) \in \mathbb{C}[s, t]$, we have
\[ W_m f(s, t) = \lambda^m (t - \delta_{b,-1} m \alpha - \delta_{b,1} (1 - \delta_{m,0}) \alpha) \sum_{i=0}^{n} (s - m)^i f_i(t) \]

\[ = \lambda^m \sum_{j=0}^{n+1} (-m)^j \sum_{i=0}^{n} s^{i-j} \left\{ \binom{i}{j} (t - \delta_{b,1} (1 - \delta_{m,0}) \alpha) + \binom{i}{j-1} \delta_{b,-1} \alpha s \right\} f_i(t) \]

\[ = \lambda^m \sum_{j=0}^{n+1} (-m)^j T^j + \delta_{b,1} \delta_{m,0} \alpha f(s, t), \]

\[ L_m f(s, t) = \lambda^m (s + h_m(t)) f(s - m, t) + \lambda^m b m (t - \delta_{b,-1} m \alpha - \delta_{b,1} \alpha) \partial_t (f(s - m, t)) \]

\[ = \lambda^m \left\{ -m^2 \delta_{b,-1} \alpha (g(t) - \partial_t) + m (h(t) + \delta_{b,-1} \alpha g(t) - \delta_{b,1} \alpha g(t) + \right. \]

\[ (bt - \delta_{b,1} \alpha) \partial_t \left\} + s \right\} \sum_{i=0}^{n} (s - m)^i f_i(t) \]

\[ = \lambda^m \sum_{j=0}^{n+2} (-m)^j \sum_{i=0}^{n} s^{i+1-j} \left\{ \binom{i}{j} - \binom{i}{j-1} \right\} (h(t) + \delta_{b,-1} \alpha g(t) - \delta_{b,1} \alpha g(t) + \right. \]

\[ (bt - \delta_{b,1} \alpha) \partial_t \left\} - \binom{i}{j-2} \delta_{b,-1} \alpha s (g(t) - \partial_t) \right\} f_i(t) \]

\[ = \lambda^m \sum_{j=0}^{n+2} (-m)^j S^j f(s, t) \]

i.e.,

\[ L_m = \lambda^m \sum_{j=0}^{\infty} (-m)^j S^j \quad \text{and} \quad W_m = \lambda^m \sum_{j=0}^{\infty} (-m)^j (T^j + \delta_{b,1} \delta_{m,0} \alpha). \]

So \( W \) is a \( \Vir(0, b) \)-module if and only if \( S^j f(s, t), T^j f(s, t) \in W \) for any \( f(s, t) \in W \) and \( j \in \mathbb{Z}_+ \).

The aim of this section is to study the reducibilities of the \( \Vir(0, b) \)-modules \( \Phi(\lambda, \alpha, h) \) and the \( \Vir(0, 1) \)-modules \( \Theta(\lambda, h) \), and give all their submodules when they are reducible. Note that the expression of \( S^j \) depends on the value of \( b \). So the discussion is divided into the following three subsections.

For \( f(s, t) \in \mathbb{C}[s, t] \), we use the same notation \( S_{f(s,t)} \) to denote the \( \Vir(0, b) \)-submodule of \( \Phi(\lambda, \alpha, h) \) and the \( \Vir(0, 1) \)-submodule of \( \Theta(\lambda, h) \) generated by \( f(s, t) \).
2.1 \( \mathcal{V}ir(0, 0) \)-submodules

In this subsection we assume that \( b = 0 \). In this case \( T^j, S^j \) have the simplified form:

\[
T^j = \frac{t}{j!} \partial_s^j \quad \text{and} \quad S^j = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} h(t).
\]

It is easy to check that

\[
\mathcal{S}_{sf(t)} = \mathbb{C}[s, t]sf(t) + \mathbb{C}[s, t]tf(t) + \mathbb{C}[s, t]h(t) f(t), \quad \mathcal{S}_{f(t)} = \mathbb{C}[s, t]f(t) \quad \text{for} \ f(t) \in \mathbb{C}[t].
\]

Moreover, \( \mathcal{S}_{sf(t)} = \mathbb{C}[s, t]sf(t) + \mathbb{C}[s, t]tf(t) \) if \( h(0) = 0 \) and \( \mathcal{S}_{sf(t)} = \mathcal{S}_{f(t)} \) if \( h(0) \neq 0 \).

**Lemma 2.2.** Let \( f(s, t) = \sum_{i=0}^{n} s^i f_i(s, t) \in \mathbb{C}[s, t] \). Then

\[
\mathcal{S}_{f(s, t)} = \sum_{i=1}^{n} \mathcal{S}_{sf_i} + \mathcal{S}_{f_0}.
\]

**Proof.** Since \( \mathcal{S}_{sf_i} \) for \( i = 1, 2, \ldots, n \) and \( \mathcal{S}_{f_0} \) are all \( \mathcal{V}ir(0, 0) \)-submodules, so is \( \sum_{i=1}^{n} \mathcal{S}_{sf_i} + \mathcal{S}_{f_0} \), containing \( f(s, t) \). Thus, it suffices to show that \( \sum_{i=1}^{n} \mathcal{S}_{sf_i} + \mathcal{S}_{f_0} \subseteq \mathcal{S}_{f(s, t)} \), which is reduced to showing \( \mathcal{S}_{sf_n} \subseteq \mathcal{S}_{f(s, t)} \). But this is equivalent to \( sf_n(t) \in \mathcal{S}_{f(s, t)} \).

We first consider the case \( h(0) \neq 0 \). Note by Lemma 2.1 that \( sf(t) = S^0 S^n f(s, t) \in \mathcal{S}_{f(s, t)} \) and that

\[
s(h(t) - h(0)) f_n(t) = \frac{S^0(h(T^0) - h(0)) T^n f(s, t)}{t} \in \mathcal{S}_{f(s, t)}.
\]

Thus, \( sf_n(t) \in \mathcal{S}_{f(s, t)} \).

Now assume that \( h(0) = 0 \), i.e., \( t \mid h(t) \). It follows from Lemma 2.1 that \( sf_n(t) = S^0 T^n f(s, t) \in \mathcal{S}_{f(s, t)} \) and \( nstf_n(t) + tf_{n-1}(t) = T^{n-1} f(s, t) \in \mathcal{S}_{f(s, t)} \), which imply \( tf_n(t) \in \mathcal{S}_{f(s, t)} \). Then by our assumption \( t \mid h(t) \) and Lemma 2.1 again we have

\[
h(t) f_{n-1}(t) = h(T^0) f_{n-1}(t) \in \mathcal{S}_{f(s, t)}.
\]

These and \( sf_n(t) - nsf(t) f_n(t) - h(t) f_{n-1}(t) = S^n f(s, t) \in \mathcal{S}_{f(s, t)} \) immediately give \( sf_n(t) \in \mathcal{S}_{f(s, t)} \).

**Theorem 2.3.** The set

\[
\left\{ \mathcal{S}_{sf}, \mathcal{S}_f \mid f(t) \in \mathbb{C}[t] \right\}
\]

exhausts all \( \mathcal{V}ir(0, 0) \)-submodules of \( \Phi(\lambda, \alpha, h) \). In particular, all \( \mathcal{V}ir(0, 0) \)-submodules of \( \Phi(\lambda, \alpha, h) \) are cyclic.
Proof. Let $M$ be a nonzero proper $\mathcal{V}ir(0,0)$-submodule of $\Phi(\lambda, \alpha, h)$. Note by Lemma 2.2 that there exist nonzero polynomials $f(t), g(t) \in \mathbb{C}[t]$ such that $sf(t), g(t) \in M$. Without loss of generality, we assume that both $\deg f(t)$ and $\deg g(t)$ are minimal (here and in what follows we decree the degree of $0$ is $\infty$).

Claim 2. $M = \mathcal{S}_{sf} + \mathcal{S}_g$.

It suffices to show that $M \subseteq \mathcal{S}_{sf} + \mathcal{S}_g$, since the converse inclusion is obvious. For any $p(s, t) = \sum_{i \geq 0} s^i p_i(t) \in M$, we know by Lemma 2.2 that $sp_i(t), p_0(t) \in M$ for $i \geq 1$. Now by the minimality of $\deg f(t)$ and $\deg g(t)$, $f(t) | p_i(t)$ and $g(t) | p_0(t)$ for $i \geq 1$. Thus, $p(s, t) \in \mathcal{S}_{sf} + \mathcal{S}_g$, proving the claim.

Since $sg(t), tf(t) \in M$, by the minimality of $\deg f(t)$ and $\deg g(t)$ we have $f(t) | g(t)$ and $g(t) | tf(t)$. Thus, $g(t) = f(t)$ or $g(t) = tf(t)$ (up to a scalar), and accordingly, $M = \mathcal{S}_g$ or $M = \mathcal{S}_{sf}$.

Now given a $\mathcal{V}ir(0,0)$-submodule of $\Phi(\lambda, \alpha, h)$ we can describe all its maximal submodules and determine when $\mathcal{S}_{f(s,t)} = \mathcal{S}_{sg}$ or $\mathcal{S}_g$.

Proposition 2.4. (1) Let $0 \neq f(t) \in \mathbb{C}[t]$. Then every maximal $\mathcal{V}ir(0,0)$-submodule of $\mathcal{S}_{sf}$ (resp. $\mathcal{S}_f$) has the form $\mathcal{S}_{spf}$ or $\mathcal{S}_{sf}$ (resp. $\mathcal{S}_{pf}$ or $\mathcal{S}_s$ if $f = 1$ and $h(0) = 0$), where $p(t) \in \mathbb{C}[t]$ is irreducible.

(2) Let $f(s, t) = \sum_i f_i(t)s^i \in \mathbb{C}[s, t]$ and $g(t) \in \mathbb{C}[t]$ be monic. Then $\mathcal{S}_{f(s,t)} = \mathcal{S}_{sg}$ is if and only if

$$
\begin{align*}
&\begin{cases}
 tg(t) | f_0(t) & \text{and} \quad (f_0(t), f_1(t), \cdots, f_n(t)) = g(t) \quad \text{when } h(0) = 0, \\
 (f_0(t), f_1(t), \cdots, f_n(t)) = g(t) & \quad \text{when } h(0) \neq 0,
\end{cases}
\end{align*}
$$

and $\mathcal{S}_{f(s,t)} = \mathcal{S}_g$ if and only if

$$
\begin{align*}
&\begin{cases}
 (f_0(t), tf_1(t), \cdots, f_n(t)) = g(t) & \text{and} \quad g(t) | f_i(t) \quad \text{for } i \geq 1 \quad \text{when } h(0) = 0, \\
 (f_0(t), f_1(t), \cdots, f_n(t)) = g(t) & \quad \text{when } h(0) \neq 0.
\end{cases}
\end{align*}
$$

Proof. (1) We only focus on the case $\mathcal{S}_{sf}$, the other case can be treated similarly. Assume that $M$ is a maximal submodule of $\mathcal{S}_{sf}$. Then either $M = \mathcal{S}_{sg}$ or $M = \mathcal{S}_g$ for some $g(t) \in \mathbb{C}[t]$ by Theorem 2.3. In fact, either $f(t) | g(t)$ or $tf(t) | g(t)$. So in either case we can write $g(t) = p(t)f(t)$ for some $p(t) \in \mathbb{C}[t]$. Then $p(t)$ must be irreducible, since any nontrivial factor $q(t)$ of $p(t)$ would produce a larger submodule $\mathcal{S}_{sqf}$ (resp. $\mathcal{S}_{qf}$) than $\mathcal{S}_{sg}$ (resp. $\mathcal{S}_g$).

Note that if $M = \mathcal{S}_g = \mathcal{S}_{pf} \subseteq \mathcal{S}_{sf}$, we must have $pf | tf$. That is, $p | t$ and thus $M = \mathcal{S}_{tf}$.

(2) Note by Lemma 2.2 that
Lemma 2.6. Let us first determine the $S_{f(s,t)} = S_{sg}$

\[
S_{f(s,t)} = S_{sg} \iff \begin{cases}
\sum_{i=1}^{n} (\mathbb{C}[s,t]sf_i(t) + \mathbb{C}[s,t]tf_i(t)) + \mathbb{C}[s,t]f_0(t) \\
= \mathbb{C}[s,t]sg(t) + \mathbb{C}[s,t]tg(t) \\
\sum_{i=1}^{n} \mathbb{C}[s,t]f_i(t) + \mathbb{C}[s,t]f_0(t) = \mathbb{C}[s,t]g(t)
\end{cases}
\]

if $h(0) = 0$,

\[
\sum_{i=0}^{n} \mathbb{C}[s,t]f_i(t) = \mathbb{C}[s,t]g(t)
\]

if $h(0) \neq 0$.

\[
\exists q_i(t) \in \mathbb{C}[t] \text{ such that } sg(t) = \sum_{i=0}^{n} q_i(t)f_i(t)
\]

and $tg(t) | t f_i(t)$ for $i \geq 1$, $tg(t) | f_0(t)$ if $h(0) = 0$,

\[
\exists q_i(t) \in \mathbb{C}[t] \text{ such that } g(t) = \sum_{i=0}^{n} q_i(t)f_i(t)
\]

and $g(t) | f_i(t)$ for $i \geq 0$ if $h(0) \neq 0$.

\[
(f_0(t), f_1(t), \ldots, f_n(t)) = g(t) \text{ and } tg(t) | f_0(t) \quad \text{if } h(0) = 0,
\]

\[
(f_0(t), f_1(t), \ldots, f_n(t)) = g(t) \quad \text{if } h(0) \neq 0.
\]

and also that

\[
S_{f(s,t)} = S_{g}
\]

\[
\iff \begin{cases}
\sum_{i=1}^{n} (\mathbb{C}[s,t]sf_i(t) + \mathbb{C}[s,t]tf_i(t)) + \mathbb{C}[s,t]f_0(t) = \mathbb{C}[s,t]g(t) \\
\sum_{i=0}^{n} \mathbb{C}[s,t]f_i(t) = \mathbb{C}[s,t]g(t)
\end{cases}
\]

if $h(0) = 0$,

\[
\sum_{i=0}^{n} \mathbb{C}[s,t]f_i(t) = \mathbb{C}[s,t]g(t)
\]

if $h(0) \neq 0$.

\[
\exists q_i(t) \in \mathbb{C}[t] \text{ such that } g(t) = \sum_{i=0}^{n} q_i(t)tf_i(t) + q_0(t)f_0(t)
\]

and $g(t) | f_i(t)$ for $i \geq 0$ if $h(0) = 0$,

\[
\exists q_i(t) \in \mathbb{C}[t] \text{ such that } g(t) = \sum_{i=0}^{n} q_i(t)f_i(t)
\]

and $g(t) | f_i(t)$ for $i \geq 0$ if $h(0) \neq 0$.

\[
(f_0(t), f_1(t), \ldots, tf_n(t)) = g(t) \text{ and } g(t) | f_i(t) \quad \text{for } i \geq 1 \quad \text{if } h(0) = 0,
\]

\[
(f_0(t), f_1(t), \ldots, f_n(t)) = g(t) \quad \text{if } h(0) \neq 0.
\]

\[\square\]

As a consequence of Proposition 2.4, we have the following result.

Corollary 2.5. Every $\mathcal{V}ir(0,0)$-submodule of $\Phi(\lambda, \alpha, h)$ has a composition series.

2.2 $\mathcal{V}ir(0,1)$-submodules

Let us first determine the $\mathcal{V}ir(0,1)$-submodules $S_{f(s,t)}$ of $\Theta(\lambda, h)$.

Lemma 2.6. (1) $S_{\epsilon f(t)} = \mathbb{C}[s,t]s^{1-\delta_{n,0}} f(t) + \mathbb{C}[s,t]h(t)f(t)$ for all $n \in \mathbb{Z}_+$.

(2) $S_{sf_i(t)+f_0(t)} = \mathbb{C}[s,t](sf_i(t) + f_0(t)) + S_{h(t)f_i(t)} + S_{(1+h(t))f_0(t)} = S_{sf_i} + S_{f_0}$.

(3) For $f(s,t) = \sum_{i=0}^{n} s^i f_i(t) \in \mathbb{C}[s,t]$,

\[
S_{f(s,t)} = \sum_{i=1}^{n+1} S_{s(1+h(t))f_i(t)-h(t)f_{i-1}(t)} + S_{(1+h(t))f_0(t)} (f_{n+1}(t) = 0).
\]
Proof. (1) It suffices to show that $S_{s^n f(t)}$ includes $C[s, t]$ and $s^n f(t) + C[s, t] h(t)f(t)$, since it is easy to check that the latter is a $S$-module containing $s^n f(t)$. Now consider $n \geq 1$, since the case $n = 0$ is trivial. By Lemma 2.1,

$$S_{s^n f(t)} \ni S_{s^n f(t)} = \left( \frac{s^n}{n!} \partial_s^n - \frac{1}{(n-1)!} \partial_s^{n-1} h(t) \right) s^n f(t) = s f(t) - nsh(t)f(t)$$

and $S_{s^n f(t)} \ni -S_{s^n f(t)} = \frac{1}{n!} \partial_s^n h(t)s^n f(t) = h(t)f(t)$. Then, $s f(t) \in S_{s^n f(t)}$, as desired.

To prove (2) and (3) we first proceed induction on $n$ to show for each $k \in \mathbb{Z}_+$ that

$$S_{s^n f(t)} = \sum_{i=0}^{n+1} s^{i-k} \binom{i}{k} (s(1 + h(t))f_i(t) - h(t)f_{i-1}(t)) \quad (2.1)$$

It is easily to check that it is true for the case $n = 0$. By the definition of $S_{s^n f(t)}$ for $k \in \mathbb{Z}_+$, we have

$$S_{s^n f(t)} = \sum_{i=0}^{n+1} s^{i-k} \binom{i}{k} (s(1 + h(t))f_i(t) - h(t)f_{i-1}(t))$$

Then by this and inductive assumption,

$$S_{s^n f(t)} = S_{s^n f(t)} + S_{s^n f(t)} \sum_{i=0}^{n-1} s^i f_i(t)$$

$$= S_{s^n f(t)} - s^{n-k} \binom{n}{k} f_{n-1}(t) + \sum_{i=0}^{n-1} \binom{i}{k} (s(1 + h(t))f_i(t) - h(t)f_{i-1}(t))$$

$$= \sum_{i=0}^{n+1} s^{i-k} \binom{i}{k} (s(1 + h(t))f_i(t) - h(t)f_{i-1}(t)),$$

that is, (2.1) holds for $n$. It follows from (2.1) that for each $k \in \mathbb{Z}_+$,

$$S_{s^n f(t)} = s(1 + h(t))f_k(t) - h(t)f_{k-1}(t) \quad (mod \sum_{i=k+1}^{n+1} S_{s(1+h(t))f_i(t) - h(t)f_{i-1}(t)}).$$

Now (3) follows by noting

$$f(s, t) \equiv (1 + h(t))f_0(t) \quad (mod \sum_{k=1}^{n+1} S_{s(1+h(t))f_k(t) - h(t)f_{k-1}(t)}).$$
And the first equality in (2) follows from (3) by noting
\[ S_{h(t)f_1(t)} + S_{s(1+h(t))f_1(t)-h(t)f_0(t)} + S_{(1+h(t))f_0(t)} = C[s, t](s f_1(t) + f_0(t)) + S_{h(t)f_1(t)} + S_{(1+h(t))f_0(t)}. \]

It remains to establish the equality: \( S_{sf_1(t)+f_0(t)} = S_{sf_1} + S_{f_0} \). Note that
\[
S_{sf_1(t)+f_0(t)} \ni S_{f_1}^{1}(sf_1(t) + f_0(t)) = sf_1(t) - h(t)(sf_1(t) + f_0(t))
\]
and \( h(t)(sf_1(t) + f_0(t)) \in S_{sf_1(t)+f_0(t)} \) by Lemma 2.1. Thus, \( sf_1(t), f_0(t) \in S_{sf_1(t)+f_0(t)} \). Now the desired equality clearly holds.

**Theorem 2.7.** The set \( \{S_{f(t)} + S_{sg(t)} \mid f(t), g(t) \in C[t]\} \) exhausts all \( \text{Vir}(0,1) \)-submodules of \( \Theta(\lambda, h) \).

**Proof.** Let \( W \) be a nonzero \( \text{Vir}(0,1) \)-submodule of \( \Theta(\lambda, h) \). Take \( f(t) \in W \cap C[t] \) and \( sg(t) \in W \cap sC[t] \) such that deg \( f(t) \) and deg \( g(t) \) are minimal (here deg \( f \) may be \( \infty \)). Then \( W \cap C[t] \subseteq S_{f(t)} \) by the choice of \( f(t) \) and Lemma 2.6(1). While for any \( sp_1(t) + p_0(t) \in W \), we similarly have \( sp_1(t) + p_0(t) \in S_{sg(t)} + S_{f(t)} \). These together with Lemma 2.6(2) and (3) force any element of \( W \) lie in \( S_{sg(t)} + S_{f(t)} \). Thus, \( W = S_{sg(t)} + S_{f(t)} \).

Note in the case \( b = 1 \) that
\[
T^j = t^{-\alpha} \frac{\partial^j}{\partial s^j} = \\
S^j = \frac{\partial^j}{\partial s^j} - \frac{1}{(j-1)!} \partial_s^{j-1}(G(t) + (t - \alpha) \partial_t) \text{ with } G(t) = h(t) - \alpha g(t).
\]

Then all the \( \text{Vir}(0,1) \)-submodules \( S_{f(s,t)} \) of \( \Phi(\lambda, \alpha, h) \) can be determined as follows.

**Lemma 2.8.** (1) \( S_{(t-\alpha)^n} = C[s, t - \alpha](t - \alpha)^n \) for any \( n \in \mathbb{Z}_+ \).

(2) For \( f(t) = \sum_{i \geq 0} a_i (t - \alpha)^i \in C[t] \) with \( a_n \neq 0 \), \( S_{f(t)} = S_{s^m f(t)} = S_{t^{(t-\alpha)^n}} \) for any \( 1 \leq m \in \mathbb{Z}_+ \).

(3) For \( f(s, t) = \sum_{i=p}^q \sum_{j=n}^{m} a_{i,j} j^i (t - \alpha)^j \) with \( a_{p,n} \neq 0 \), \( S_{f(s,t)} = S_{(t-\alpha)^n} \).

**Proof.** It is easy to check that \( C[s, t - \alpha](t - \alpha)^n \) is \( S^j \) and \( T^j \)-invariant for any \( j \in \mathbb{Z}_+ \). Thus, by Lemma 2.1 \( C[s, t - \alpha](t - \alpha)^n \) is a \( \text{Vir}(0,1) \)-module and (1) follows.

(2) For the equality \( S_{f(t)} = S_{(t-\alpha)^n} \), it suffices to show that \( (t - \alpha)^n \in S_{f(t)} \). It follows from \( S_{f(t)} \ni -S^1 f(t) = (G(t) + (t - \alpha) \partial_t) f(t) \) that \( (t - \alpha) \partial_t f(t) \in S_{f(t)} \). Inductively, one has \( (t - \alpha) \partial_t^k f(t) \in S_{f(t)} \) for \( k \geq 1 \), from which we deduce that \( (t - \alpha)^n \in S_{f(t)} \), as desired. For the equality \( S_{s^m f(t)} = S_{(t-\alpha)^n} \), it suffices to show \( (t - \alpha)^n \in S_{sf(t)} \). Note that \( C[t - \alpha](t - \alpha)f(t) \subseteq S_{s^m f(t)} \) and \( S_{s^m f(t)} \ni -S^{m+1} s^m f(t) = (G(t) + (t - \alpha) \partial_t) f(t) \). Then by \( (t - \alpha) | (G(t) - G(\alpha)) \), \( (G(\alpha) + (t - \alpha) \partial_t) f(t) \in S_{s^m f(t)} \), which in turn implies \( (G(\alpha) + (t - \alpha) \partial_t^k f(t) \in S_{s^m f(t)} \) for any \( k \geq 1 \). Note that \( (G(\alpha) + (t - \alpha) \partial_t^k f(t) = \sum_{i \geq n} (G(\alpha) + i)^k a_i (t - \alpha)^i \). Thus, \( (t - \alpha)^n \in S_{s^m f(t)} \).
Proof. Note that \( f(t) = \sum_{i=0}^n a_i t^n \). Then it follows from the proof of \( S \subseteq S_{(t-\alpha)} \) in (2) that \( s^q f(t) \in \mathcal{S}_{(s,t)} \). Thus, inductively one can show that \( s^q f_p(t) \in \mathcal{S}_{(s,t)} \). Then by (2), \( \mathcal{S}_{(t-\alpha)} \subseteq \mathcal{S}_{(s,t)} \) and (3) follows. □

**Theorem 2.9.** The set \( \{ \mathcal{S}_{(t-\alpha)} \mid n \in \mathbb{Z}_+ \} \) exhausts all nonzero \( \mathcal{V}ir(0,1) \)-submodules of \( \Phi(\alpha, \lambda, h) \).

**Proof.** Assume that \( \mathcal{M} \) is a nonzero \( \mathcal{V}ir(0,1) \)-submodule of \( \Phi(\alpha, \lambda, h) \). Choose \( n \in \mathbb{Z}_+ \) minimal such that \( (t-\alpha)^n \in \mathcal{M} \). Let \( f(s,t) = \sum_{i=0}^a f_i \in M \) with \( a_{p,q} \neq 0 \). Then by Lemma 2.8(3), \( (t-\alpha)^l \in \mathcal{S}_{(t-\alpha)} \subseteq \mathcal{M} \). Now by the minimality of \( n \), \( f(s,t) \in \mathcal{S}_{(t-\alpha)} \subseteq \mathcal{S}_{(t-\alpha)} \subseteq \mathcal{M} \). Thus, \( \mathcal{M} = \mathcal{S}_{(t-\alpha)} \). □

### 2.3 \( \mathcal{V}ir(0, b) \)-submodules when \( b = -1 \) or \( b \neq \pm 1, 0 \)

Notice in the case \( b = -1 \) that

\[
T^j = \frac{j}{j!} \partial_s^j + \frac{\alpha}{(j-1)!} \partial_s^{j-1},
\]

and

\[
S^j = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} (t g(t) - \partial_t) + h(\alpha) \}
\]

Thus, repeating this process (if necessary) leads us to obtain

Thus, repeating this process (if necessary) leads us to obtain

This shows that \( \Phi(\lambda, \alpha, h) \) is irreducible when \( \alpha \neq 0 \), proving (1).

Note in the case \( (b, \alpha) = (-1, 0) \) that

\[
T^j = \frac{t}{j!} \partial_s^j, \quad S^j = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} (t g(t) + h(0) - t \partial_t),
\]

and in the case \( b \neq \pm 1, 0 \) that

\[
T^j = \frac{t}{j!} \partial_s^j, \quad S^j = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} (h(t) + bt \partial_t)
\]

Then (2) follows from Lemma 2.8 and Theorem 2.9. □
As a consequence of Theorems 2.3, 2.7, 2.9 and 2.10 we have (see also HCS Proposition 2.3(iii)):

**Corollary 2.11.** For \( \lambda \in \mathbb{C}^* \), \( \alpha \in \mathbb{C} \) and \( h(t) \in \mathbb{C}[t] \), the \( \text{Vir}(0,1) \)-module \( \Theta(\lambda, h) \) is reducible and the \( \text{Vir}(0,b) \)-module \( \Phi(\lambda, \alpha, h) \) is irreducible if and only if \( \alpha \neq 0 \) and \( b = -1 \).

## 3 \( \mathfrak{U} \)-submodules

In this section we use the same notation \( \Psi_{f(s,t)} \) to denote the \( \mathfrak{U} \)-submodule of \( \Phi(\lambda, \alpha, h) \) and \( \Theta(\lambda, h) \) generated by \( f(s,t) \in \mathbb{C}[s,t] \).

A pair \((n,i) \in \mathbb{Z}_+^2\) is called minimal if \( n \leq m \) for which there exists \( j \) such that \((m,j) \in \mathbb{Z}_+^2\) and \( m(k-1)+j = n(k-1)+i \), where \( k = \deg h(t) \).

**Lemma 3.1.** Assume that \( b = -1 \) and \( \alpha \neq 0 \).

1. \( \Psi_{f(t)} = \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{C}[s]t^i(g(t) - \partial_t)^mf(t) \).
2. \( \Psi_{s^n f(t)} = \mathbb{C}[s]sf + \Psi_{(g(t)-\partial_t)f} + \Psi_{(t(g(t)-\partial_t)+h(\alpha))f} \) for all \( n \geq 1 \).
3. If \( 2 \leq \deg h(t) < \infty \), then \( \Psi_{f(t)} \) has a basis \( \mathcal{B} = \{s^lt^i(g(t) - \partial_t)^n(f) \mid l \in \mathbb{Z}_+, i \leq n, (n,i) \in \mathbb{Z}_+^2 \) is minimal\} and a maximal submodule linearly spanned by \( \mathcal{B} - \{f\} \). In particular, \( \Psi_{f(t)} \) is a proper submodule in this case.

**Proof.** Set \( F = g(t) - \partial_t \). Note that \((g(t) - \partial_t)t = t(g(t) - \partial_t) - 1 \) on \( \mathbb{C}[s,t] \). Then it is easy to check that the subspace on the right hand side of the desired formula in (1) is \( S^i \)-invariant for any \( j \in \mathbb{Z}_+ \) (see (2.2)), and thereby is a \( \mathfrak{U} \)-submodule and the equality in (1) holds.

Applying \( S^{n+2} \) and \( S^{n+1} \) to \( s^nf \) one has \( Ff \in \Psi_{s^n f} \) and \( (tF+h(\alpha))f \in \Psi_{s^n f} \). It follows from these and \( S^n(s^nf) \in \Psi_{s^n f} \) that \( sf \in \Psi_{s^n f} \). Thus, \( \Psi_{s^n f} = \Psi_{s^n f} \) for any \( n \geq 1 \). So it suffices to determine \( \Psi_{sf} \). Also observe that \( \mathbb{C}[s]sf + \Psi_{Ff} + \Psi_{(tF+h(\alpha))f} \) is a submodule of \( \Psi_{sf} \) containing \( sf \). So (2) follows.

To show that \( \mathcal{B} \) is a basis of \( \Psi_{f(t)} \), it is sufficient to show that \( \mathfrak{M} = \{t^i(g(t) - \partial_t)^n(f) \mid i \leq n, (n,i) \in \mathbb{Z}_+^2 \) is minimal\} is a basis of \( V(f) := \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{C}[t]t^i(g(t) - \partial_t)^mf(t) \). Note first that the set \( \mathfrak{M} \) is linear independent, since elements in \( \mathfrak{M} \) have distinct degrees. Let \( V_n(f) = \text{span}\{t^i(g(t) - \partial_t)^m f(t) \mid 0 \leq i \leq m, m(k-1)+i \leq n\} \).

Assume that each element in \( V_n(f) \) is a linear combination of \( \mathfrak{M} \). We show that this is also true for \( n+1 \). Note that \( V_{n+1}(f) = V_n(f) \oplus \text{span}_\mathbb{C}\{t^i(g(t) - \partial_t)^s f(t) \mid 0 \leq i \leq s, s(k-1) + i = n+1\} \).

Let \((s_0, i_0)\) be minimal among the set \( \{(s,i) \mid s(k-1) + i = n+1\} \). Note that for any \( t^i(g(t) - \partial_t)^s(f) \) such that \( s(k-1)+i = n+1 \), there exists \( \zeta \in \mathbb{C} \) such that either \( \deg(t^i g(t) - \partial_t)^s(f) \leq \deg(t^{i_0} g(t) - \partial_t)^s(f) \).

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\( \partial_t^s(f) - \zeta t^{i_0}(g(t) - \partial_t)^{s_0}(f) \leq n + \deg f(t) \) or \( t^i(g(t) - \partial_t)^s(f) - \zeta t^{i_0}(g(t) - \partial_t)^{s_0}(f) = 0 \). So in either case, \( t^i(g(t) - \partial_t)^s(f) - \zeta t^{i_0}(g(t) - \partial_t)^{s_0}(f) \in V_n \). Then \( t^i(g(t) - \partial_t)^s(f) \) is a linear combination of \( \mathcal{M} \) by inductive hypothesis, as desired.

Finally, it is easy to see that the linear span of \( \mathcal{B} - \{f\} \) is \( S^j \)-invariant for \( j \in \mathbb{Z}_+ \), and thus is a \( \mathcal{V} \)-submodule by Lemma 2.1. The maximality is trivial.

**Theorem 3.2.** The \( \mathcal{V} \)-module \( \Phi(\lambda, \alpha, h) \) is irreducible if and only if \( b = -1, \alpha \neq 0 \) and \( \deg h(t) = 1 \).

**Proof.** Assume that \( b = -1 \) and \( \alpha \neq 0 \). In view of Corollary 2.11, the situation that \( \Phi(\lambda, \alpha, h) \) is irreducible as a \( \mathcal{V} \)-module can only occur under this case. Assume first that \( h(t) \in \mathbb{C} \), in which case \( g(t) = 0 \) and one can check that the subspace \( \mathbb{C}[s] \) is invariant under \( S^j \) for any \( j \in \mathbb{Z}_+ \). Thus, by Lemma 3.1(1), \( \mathbb{C}[s] = \Psi_1 \) is a proper \( \mathcal{V} \)-submodule of \( \Phi(\lambda, \alpha, h) \). The reducibility of \( \Phi(\lambda, \alpha, h) \) follows from Lemma 3.1(3) if \( 2 \leq \deg h(t) < \infty \).

It remains to show that \( \Phi(\lambda, \alpha, h) \) is irreducible if \( \deg h(t) = 1 \). Let \( W \) be a nonzero \( \mathcal{V} \)-submodule of \( \Phi(\lambda, \alpha, h) \). Note that \( \eta := g(t) \in \mathbb{C} \setminus \{0\} \) and also \( 0 \neq W \cap \mathbb{C}[t] \). Take \( 0 \neq f(t) \in W \cap \mathbb{C}[t] \) such that \( \deg f(t) \) is minimal. Then \( f(t) \in \mathbb{C} \), since by Lemma 3.1(1), \( \partial_t f(t) = -(\eta - \partial_t) f(t) + \eta f(t) \in W \cap \mathbb{C}[t] \). And again by Lemma 3.1(1),

\[
W \supseteq \Psi_{f(t)} = \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{C}[s] t^i (g(t) - \partial_t)^m f(t) = \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{C}[s] t^i = \mathbb{C}[s, t].
\]

So \( \Phi(\lambda, \alpha, h) \) is irreducible when \( \deg h(t) = 1 \).

**Remark 3.3.** \( \mathcal{V} \)-submodules, especially maximal submodules of \( \Phi(\lambda, \alpha, h) \) were studied in detail in [CG1]; Theorem 3.2 was also obtained there and Lemma 3.1(3) solves one of the problems listed there.

Note when \( b \neq -1 \) that \( S^j \) and \( S^j_0 \) have the same form:

\[
S^j_0 = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} H(t) \quad \text{with} \quad H(t) = h(t),
\]

\[
S^j = \frac{s}{j!} \partial_s^j - \frac{1}{(j-1)!} \partial_s^{j-1} H(t) \quad \text{with} \quad H(t) = \left( h(t) + \delta_{b,-1} \alpha g(t) - \delta_{b,1} \alpha g(t) + (bt - \delta_{b,1} \alpha) \partial_t \right).
\]

The following results can be obtained by following the proof of Lemma 2.6

**Proposition 3.4.** Assume that \( b \neq -1 \). Then

1. \( \Psi_{s^n f(t)} = \sum_{i=1}^{\infty} \mathbb{C}[s](H(t))^i f(t) + \mathbb{C}[s] s^{1-\delta_{b,0}} f(t) \) for all \( n \in \mathbb{Z}_+ \);
2. \( \Psi_{s f_1(t) + f_0(t)} = \mathbb{C}[s](s f_1(t) + f_0(t)) + \Psi_{H(t) f_1(t)} + \Psi_{(1+H(t)) f_0(t)} \);
3. for \( f(s, t) = \sum_{i=0}^{n} s^i f_i(t) \in \mathbb{C}[s, t] \),

\[
\Psi_{f(s, t)} = \sum_{i=1}^{n+1} \Psi_{s(1+H(t)) f_i(t) - H(t) f_i-1(t)} + \Psi_{(1+H(t)) f_0(t)} \quad (f_{n+1}(t) = 0).
\]
Theorem 3.5. Each \( \mathfrak{V} \)-submodule of \( \Phi(\lambda, \alpha, h) \) and \( \Theta(\lambda, h) \) is finitely generated if and only if \( \deg h(t) \geq 1 \).

Proof. Assume first that \( h(t) \in \mathbb{C} \). Note by Proposition 3.4 that each \( \mathfrak{V} \)-submodule \( \Psi_{f(s,t)} \) contains polynomials in \( \mathbb{C}[t] \) of only finitely many different degrees. Thus, in particular, both \( \Phi(\lambda, \alpha, h) \) and \( \Theta(\lambda, h) \) are not finitely generated. Consider now \( 1 \leq k = \deg h(t) < \infty \). Let \( M \) be a nonzero \( \mathfrak{V} \)-submodule of \( \mathbb{C}[s,t] \). We need to show that \( M \) is finitely generated. Here we only prove this for the case \( b \neq -1 \), and similar idea can be applied to the case \( b = -1 \) (or see [CG1]). Set

\[
I = \{ \deg f \mid 0 \neq f \in \mathbb{C}[t] \cap M \}, \quad J = \{ \deg f \mid sf(t) + g(t) \in (s\mathbb{C}[t] + \mathbb{C}[t]) \cap M, f(t) \neq 0 \}.
\]

Note that there exist \( d_1, d_2, \ldots, d_n \in I \) such that \( I = \bigcup_{i=1}^{\infty} \{ d_i + nk \mid n \in \mathbb{Z}_+ \} \). Choose \( p_i \in \mathbb{C}[t] \cap M \) such that \( \deg p_i = d_i \) for \( i = 1, 2, \ldots, n \). Similarly, we can choose \( sq_i(t) + r_i(t) \in (s\mathbb{C}[t] + \mathbb{C}[t]) \cap M \) for \( i = 1, 2, \ldots, m \) such that \( J = \bigcup_{i=1}^{\infty} \{ \deg q_i + nk \mid n \in \mathbb{Z}_+ \} \). Let \( W \) be the \( \mathfrak{V} \)-submodule of \( M \) generated by \( \{ p_i(t), sq_j(t) + r_j(t) \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \} \).

We are going to show that \( \mathbb{C}[t] \cap M \subseteq W \) and any element of form \( sf_1(t) + f_0(t) \) in \( M \) lies in \( W \). Then by Proposition 3.4, \( M = W \).

Take any \( 0 \neq f(t) \in \mathbb{C}[t] \cap M \). Then by induction on \( \deg f \) one can show that \( f(t) \in \sum_{i=1}^{n} \sum_{j=0}^{\infty} CH(t)^{2}p_i(t)) \). Thus, \( f(t) \in \sum_{i=1}^{n} \Psi_{p_i(t)} \subseteq W \) by Proposition 3.4 as desired. Let \( sf_1(t) + f_0(t) \in M \). Similarly, there exists \( F(t) \in \mathbb{C}[t] \) such that \( sf_1(t) + f_0(t) \in F(t) + \sum_{i=1}^{m} \Psi_{sq_i(t)+r_i(t)} \). Note from this and the previous argument that \( F(t) \in \mathbb{C}[t] \cap M \subseteq W \).

Thus, \( sf_1(t) + f_0(t) \in W \) and we are done.

\[ \square \]

4 New irreducible \( \mathfrak{V} \)-modules

Motivated by [TZ2] we consider the tensor products \( \bigotimes_{i=1}^{n} \Phi(\lambda_i, \alpha_i, h_i) \otimes V \) of \( \mathfrak{V} \)-modules in this section, and the case \( n = 1 \) was studied in [GWL]. We first recall all various known irreducible non-weight \( \mathfrak{V} \)-modules, and then show that modules under our considerations are irreducible and also give the necessary and sufficient conditions under which these modules are isomorphic. Finally we compare these modules with the known non-weight modules.

Recall from [LZ2] that the \( \mathfrak{V} \)-module structure on \( \mathbb{C}[t] \) which is given as:

\[
C_1 t^i = 0, \quad L_n t^i = \lambda^n (t-n)^i (t+n(b-1)) \quad \text{for} \quad i \in \mathbb{Z}_+, \ n \in \mathbb{Z},
\]

where \( \lambda \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). After equipping this action we write \( \Omega(\lambda, b) \) rather than \( \mathbb{C}[t] \), which is irreducible if and only if \( b \neq 1 \).

Let \( \mathfrak{V}_+ \) denote the subalgebra of \( \mathfrak{V} \) spanned by \( L_i \) for \( i \in \mathbb{Z}_+ \). For any \( c \in \mathbb{C} \) and \( \mathfrak{V}_+ \)-module \( N \), form the induced module \( \text{Ind}(N) := U(\mathfrak{V}) \otimes_{U(\mathfrak{V}_+)} N \) and denote \( \text{Ind}_c(N) := \text{Ind}(N)/(C_1-c)\text{Ind}(N) \). For any \( b' \in \mathbb{C} \), recall from [LZ2] the irreducible \( \mathfrak{V} \)-module structure
on $A$, the associative algebra $\mathbb{C}[t^{\pm 1}, t\partial_t]$, on which the action of $\mathfrak{V}$ is given by

$$C_1 w = 0, \quad L_n w = (t^n (t\partial_t) + nb't^n)w \quad \text{for } n \in \mathbb{Z}, \ w \in \mathbb{C}[t^{\pm 1}, t\partial_t].$$

Let $r \in \mathbb{Z}_+$, $\mathfrak{V}_r$ be the ideal of $\mathfrak{V}$ spanned by $\{L_i \mid i > r\}$ and $M$ an irreducible $\mathfrak{a}_r := \mathfrak{V}_r/\mathfrak{V}_r$-module such that $L_r$ is injective on $M$, where $L_i = L_i + \mathfrak{V}_r$. For any $\gamma(t) \in \mathbb{C}[t] \setminus \mathbb{C}$, recall from [LLZ] that the linear tensor product, denoted by $\mathcal{N}(M, \gamma(t))$, of an $\mathfrak{a}_r$-module $M$ and the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$, carries the structure of an irreducible $\mathfrak{V}$-module if the action of $\mathfrak{V}$ on $\mathcal{N}(M, \gamma(t))$ is defined by

$$C_1(v \otimes t^k) = 0, \quad L_n(v \otimes t^k) = \left( kv + \sum_{i=0}^{r} \frac{n^{i+1}}{(i+1)!} \tilde{L}_i v \right) \otimes t^{n+k}v \otimes t^n \gamma(t) \quad \text{for } k, n \in \mathbb{Z}, v \in M.$$

We call a $\mathfrak{V}_r$-module $V$ is locally finite if for any $v \in V$ the dimension of $U(\mathfrak{V}_r)v$ is finite and call $V$ is locally nilpotent if for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $L_{i_1}L_{i_2}\cdots L_{i_n}v = 0$ for any $L_{i_1}, L_{i_2}, \ldots, L_{i_n} \in \mathfrak{V}_r$.

Finally, we also need to recall irreducible highest weight modules over $\mathfrak{V}$ (see [KR]). For any $c, h \in \mathbb{C}$, let $I(c, h)$ be the left idea of $U(\mathfrak{V})$, the universal enveloping algebra of $\mathfrak{V}$, generated by the set $\{L_0 - h, C_1 - c, d_i \mid i \geq 0\}$. Form the quotient $M(c, h) := U(\mathfrak{V})/I(c, h)$, which is called the Verma module with highest weight $(c, h)$. Let $W(c, h)$ be the unique maximal proper submodule of $M(c, h)$ and $L(c, h)$ denote the irreducible highest weight module $M(c, h)/W(c, h)$ with highest weight $(c, h)$.

**Theorem 4.1.** [MZ] Let $N$ and $V$ be irreducible $\mathfrak{V}$-modules.

1. If there exists $k \in \mathbb{Z}_+$ satisfying the following two conditions:
   - (a) $L_k$ acts injectively on $N$;
   - (b) $L_i N = 0$ for all $i > k$.
   Then for any $c \in \mathbb{C}$ the $\mathfrak{V}$-module $\text{Ind}_c(N)$ is irreducible.

2. The following conditions are equivalent:
   - (i) There exists $k \in \mathbb{Z}_+$ such that $V$ is a locally finite $\mathfrak{V}_k$-module;
   - (ii) There exists $n \in \mathbb{Z}_+$ such that $V$ is a locally nilpotent $\mathfrak{V}_n$-module;
   - (iii) $V \cong L(c, h)$ or $\text{Ind}_c(W)$ for some $c, h \in \mathbb{C}$, irreducible $\mathfrak{V}_+$-module $W$ satisfying the conditions in (1).

**Remark 4.2.** If $V$ is an irreducible $\mathfrak{V}$-module for which there exists $R_V \in \mathbb{Z}_+$ such that $L_i$ for all $i > R_V$ are locally finite on $V$. Then by Theorem 4.1 there exists $n \in \mathbb{Z}_+$ such that $L_n$ locally nilpotent on $V$. In particular, for any $v \in V$, there exists $K_v \in \mathbb{Z}_+$ such that $L_i v = 0$ for all $i \geq K_v$.

The following result was proved in [LZ1] Proposition 7], which is cited as a lemma here.
Lemma 4.3. Let $P$ be a vector space over $\mathbb{C}$ and $P_1$ a subspace of $P$. Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_s \in \mathbb{C}^*$ are pairwise distinct, $v_{i,j} \in P$ and $f_{i,j}(t) \in \mathbb{C}[t]$ with $\deg f_{i,j}(t) = j$ for $i = 1, 2, \cdots, s, j = 0, 1, 2 \cdots, k$. If

$$
\sum_{i=1}^{s} \sum_{j=0}^{k} \lambda_i^m f_{i,j}(m)v_{i,j} \in P_1 \quad \text{for } m \in \mathbb{Z}_+,
$$

then $v_{i,j} \in P_1$ for all $i, j$.

Throughout this section, all $\mathfrak{U}$-modules of form $\Phi(\lambda, \alpha, h)$ are assumed to be simple.

Theorem 4.4. Let $\lambda_i, \alpha_i \in \mathbb{C}^*$, $h_i(t) \in \mathbb{C}[t]$ for $i = 1, 2, \cdots, n$ such that these $\lambda_i$ are pairwise distinct and that all $\deg h_i(t)$ are equal to 1, and $V$ be an irreducible $\mathfrak{U}$-module for which there exists $R_V \in \mathbb{Z}_+$ such that all $L_k$ for $k \geq R_V$ are locally finite on $V$. Then the tensor product $\bigotimes_{i=1}^{n} \Phi(\lambda_i, \alpha_i, h_i) \otimes V$ of $\mathfrak{U}$-modules $\Phi(\lambda_i, \alpha_i, h_i)$ for $1 \leq i \leq n$ and $V$ is an irreducible $\mathfrak{U}$-module. In particular, $\bigotimes_{i=1}^{n} \Phi(\lambda_i, \alpha_i, h_i)$ is an irreducible $\mathfrak{U}$-module. Furthermore, $\bigotimes_{i=1}^{n} \Phi(\lambda_i, \alpha_i, h_i)$ is also irreducible as a $\mathfrak{U}_r$-module for any $r \in \mathbb{Z}_+$.

Proof. Set $T = \bigotimes_{i=1}^{n} \Phi(\lambda_i, \alpha_i, h_i)$ and $\eta_i = \frac{h_i(t) - h_i(\alpha_i)}{t - \alpha_i}$ for $i = 1, 2, \cdots, n$. Let $W$ be a nonzero $\mathfrak{U}_r$-submodule of $T \otimes V$ and $\{u_i \mid i \in \mathbb{Z}_+\}$ a basis of $V$. Take

$$
0 \neq u = \sum_{I=(i_1,\cdots, i_n) \in \Gamma_1, J=(j_1,\cdots, j_n) \in \Gamma_2, l \in \Gamma_3} a_{I,J,l}s^{i_1}t^{j_1} \otimes s^{i_2}t^{j_2} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l \in W
$$

such that $a_{I,J,l} \neq 0$ for all $I \in \Gamma_1, J \in \Gamma_2, l \in \Gamma_3$ and all terms in the sum are linearly independent. Let $K = \max\{r, K_{u_l} \mid l \in \Gamma_3\}$ (see Remark 4.2). Then for any $m \geq K$,

$$
W \ni L_m u = \sum_{I,J,l,k} a_{I,J,l}s^{i_1}t^{j_1} \otimes \cdots \otimes \sum_{j=0}^{\infty} \lambda_i^m(-m)^j S^j s^{i_k}t^{j_k} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l
$$

$$
= \sum_{i=1}^{n} \sum_{j=0}^{\infty} \lambda_i^m(-m)^j u_{i,j},
$$

where

$$
u_{k,j} = \sum_{I,J,l} a_{I,J,l}s^{i_1}t^{j_1} \otimes \cdots \otimes S^j s^{i_k}t^{j_k} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l.
$$

Now applying Lemma 4.3 we see that $u_{k,j} \in W$ for any $k, j$. For a fixed $k$, let $(i_k^0, j_k^0)$ be maximal in alphabetical order on $\mathbb{Z}_+^2$ among all $(i_k, j_k)$, where $i_k$ (resp. $j_k$) is the $k$-th component of $I$ (resp. $J$) and $I, J$ range respectively over $\Gamma_1, \Gamma_2$. Then

$$
u_{k,i_k^0+2} = -\alpha_k \sum_{J,l,l=(i_1,\cdots, i_k=i_k^0,\cdots,i_n) \in \Gamma_1} a_{I,J,l}s^{i_1}t^{j_1} \otimes \cdots \otimes (\eta_i - \partial_i)t^{j_k} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l
$$
and
\[ \sum_{J,J,L=(i_1,\ldots,i_k=0,\ldots,i_n)\in\Gamma_1} a_{J,J,L}s^{i_1}t^{j_1} \otimes \cdots \otimes (\eta_i - \partial_t)^2 t^{j_k} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l \in W. \]

It follows from these that
\[ \sum_{J,J,L=(i_1,\ldots,i_k=0,\ldots,i_n)\in\Gamma_1} a_{J,J,L}s^{i_1}t^{j_1} \otimes \cdots \otimes (\eta_i - \partial_t)^{j_{k-1}} \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l \in W. \]

Repeating the above procedure finitely many times gives
\[ 0 \neq \sum_{I=(i_1,\ldots,i_k=0,\ldots,i_n)\in\Gamma_1,J=(j_1,\ldots,j_k=0,\ldots,j_n)\in\Gamma_2,l\in\Gamma_3} a_{I,J,L}s^{i_1}t^{j_1} \otimes \cdots \otimes 1 \otimes \cdots \otimes s^{i_n}t^{j_n} \otimes u_l \in W. \]

That is, for a fixed \( k \), we can start with a nonzero element of \( W \) to get another nonzero element of \( W \) whose \( k \)-th factors all equal to 1. Similarly, doing this for the other factors yields \( 0 \neq 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w \in W \).

It is worthwhile to point out that in the previous argument we, in fact, have shown \( 1 \otimes \cdots \otimes 1 \otimes S^j 1 \otimes 1 \otimes 1 \otimes w \in W \) for any \( 1 \leq i \leq n \) and \( j \in \mathbb{Z}_+ \). In view of this, the simplicity of \( \Phi(\lambda_i,\alpha_i,h_i) \) and Lemma 24 it is not hard to see \( T \otimes w \subseteq W \). Now the irreducibility of \( T \) as a \( \mathfrak{M}_r \)-module follows by taking \( V \) to be the trivial \( \mathfrak{M} \)-module \( L(0,0) = \mathbb{C} \), and the irreducibility of \( T \otimes V \) follows from that of \( V \).

**Theorem 4.5.** Let \( \lambda_i^{(j)},\alpha_i^{(j)} \in \mathbb{C}^* \), \( h_i^{(j)}(t) \in \mathbb{C}[t] \) such that these \( \lambda_i^{(j)} \) are pairwise distinct and that \( \deg h_i^{(j)}(t) = 1 \), and \( V_j \) be an irreducible \( \mathfrak{M} \)-module for which there exists \( R_j \in \mathbb{Z}_+ \) such that all \( L_k \) for \( k \geq R_j \) are locally finite on \( V_j \) for \( i \in \mathbb{Z}_+ \) and \( j = 1,2 \). Then for any \( n_1,n_2 \in \mathbb{Z}_+ \)
\[ \bigotimes_{i=1}^{n_1} \Phi(\lambda_i^{(1)},\alpha_i^{(1)},h_i^{(1)}) \otimes V_1 \cong \bigotimes_{i=1}^{n_2} \Phi(\lambda_i^{(2)},\alpha_i^{(2)},h_i^{(2)}) \otimes V_2 \]
as \( \mathfrak{M} \)-modules if and only if
\[ n_1 = n_2, V_1 \cong V_2 \text{ as } \mathfrak{M} \text{-modules and } (\lambda_i^{(1)},\alpha_i^{(1)},h_i^{(1)}) = (\lambda_i^{(2)},\alpha_i^{(2)},h_i^{(2)}) \]
for some \( \sigma \in S_{n_1} \) (the \( n_1 \)-th symmetric group). In particular, \( \mathfrak{M} \)-modules of form \( \Phi(\lambda,\alpha,h) \) for \( \lambda \in \mathbb{C}^* \), \( \alpha \in \mathbb{C}^* \) and \( h(t) \in \mathbb{C}[t] \) with \( \deg h(t) = 1 \) are inequivalent to each other.

**Proof.** The second statement follows from a special case of the first one:
\[ n_1 = n_2 = 1 \text{ and } V_1 = V_2 = L(0,0). \]
It remains to show the “only if” part of the first statement. Without loss of generality we may assume that $n_2 \geq n_1$. Set $\eta^{(j)} = \frac{h^{(j)}(t) - h^{(j)}(\alpha)}{t - \alpha} \in \mathbb{C}^*$ and $T^{(j)} = \bigotimes_{i=1}^{n_j} \Phi(\lambda^{(j)}_i, \alpha^{(j)}_i, h^{(j)}_i)$ for $i \in \mathbb{Z}_+$ and $j = 1, 2$. Let
\[
\phi : T^{(1)} \otimes V_1 \to T^{(2)} \otimes V_2
\]
be an isomorphism of $\mathfrak{M}$-modules.

**Claim 3.** $n_1 = n_2$ and $\phi(1 \otimes V_1) = 1 \otimes V_2$, where the $i$-th 1 = $\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_n$.

For any $x^{(i)} = \sum_l f^{(i)}_{1l} \otimes f^{(i)}_{2l} \otimes \cdots \otimes f^{(i)}_{nl} \otimes u^{(i)}_l \in T^{(i)} \otimes V_i$, denote
\[
V^{KI^{(i)}} = \text{span} \left\{ L_m x^{(i)} \mid m \geq K^{(i)} := \max\{K^{(i)}_u \mid l\} \right\}.
\]
Take any fixed $0 \neq u \in V_1$ and write
\[
\phi(1 \otimes u) = \sum_l f_{1l} \otimes f_{2l} \otimes \cdots \otimes f_{nl} \otimes u_l
\]
for some $f_{il} \in \Phi(\lambda^{(2)}_i, \alpha^{(2)}_i, h^{(2)}_i), u_l \in V_2$. Set
\[
K = \max\{K_u, K_{u_l} \mid l\}.
\]
Since
\[
L_m \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_n \otimes u = \sum_{k=1}^{n_1} \underbrace{1 \otimes \cdots \otimes 1}_{k-1} L_m 1 \otimes 1 \otimes \cdots \otimes 1 \otimes u
\]
\[
= \sum_{k=1}^{n_1} \sum_{j=0}^{\infty} (\lambda^{(1)}_k)^m (-m)^j \underbrace{1 \otimes \cdots \otimes S^j 1 \otimes 1 \otimes \cdots \otimes 1 \otimes u}_{k-1}
\]
$1 \otimes \cdots \otimes S^j 1 \otimes 1 \otimes \cdots \otimes 1 \otimes u \in V_{KI_{1 \otimes u}}$ by Lemma 4.3. It is easy to check that for any $0 \neq f(s, t) \in \mathbb{C}[s, t], \dim(\text{span}\{S^j f(s, t) \mid j \in \mathbb{Z}_+\}) \geq 3$ and the equality holds if and only if $f(s, t) \in \mathbb{C}^*$. So
\[
\dim V_{KI_{1 \otimes u}} = n_1 \dim(\text{span}\{S^j 1 \mid j \in \mathbb{Z}_+\}) = 3n_1.
\]
Similarly, a direct computation shows that $\dim V_{\phi(1 \otimes u)} \geq 3n_2$, and the equality holds if and only if $\phi(1 \otimes u) \in 1 \otimes V_2$. Note that $\phi$ sends $V_{KI_{1 \otimes u}}$ isomorphically onto $V_{\phi(1 \otimes u)}$ and also that $n_2 \geq n_1$, which force $n_1 = n_2$ and $\phi(1 \otimes u) \in 1 \otimes V_2$.

Note that Claim 3 allows us to define a linear map $\tau : V_1 \to V_2$ via $\phi(1 \otimes v) = 1 \otimes \tau(v)$ for $v \in V_1$. 

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Claim 4. There exists $\sigma \in S_{n_1}$ such that $(\lambda^{(1)}_{i_1}, \alpha^{(1)}_{i_1}, h^{(1)}_{i_1}) = (\lambda^{(2)}_{\sigma_{i_1}}, \alpha^{(2)}_{\sigma_{i_1}}, h^{(2)}_{\sigma_{i_1}})$ for all $1 \leq i \leq n_1$.

Suppose that there exists $n_1 > k \in \mathbb{Z}_+$ such that $\lambda^{(1)}_i = \lambda^{(2)}_i$ for $i \leq k$ and $\lambda^{(1)}_i \neq \lambda^{(2)}_j$ for all $k + 1 \leq i, j \leq n_1$. Then it follows from $\phi(L_m \otimes u) = L_m \otimes \tau(u)$ ($m \geq K$) that

$$\sum_{i=1}^{k} \sum_{j=0}^{\infty} (\lambda^{(1)}_i)^m (-m)^j \phi(1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes u) - 1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes \tau(u)$$

$$+ \sum_{i=k+1}^{n_1} \sum_{j=0}^{\infty} (\lambda^{(1)}_i)^m (-m)^j \phi(1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes u)$$

$$- \sum_{i=k+1}^{n_1} \sum_{j=0}^{\infty} (\lambda^{(2)}_i)^m (-m)^j 1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes \tau(u) = 0.$$

Now applying Lemma 4.3 gives $1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes \tau(u) = 0$ for $k + 1 \leq i \leq n_1$ and $j \in \mathbb{Z}_+$, which is impossible. Thus, $k = n_1$, namely, $\lambda^{(1)}_i = \lambda^{(2)}_i$ for all $1 \leq i \leq n_1$. Note from the preceding argument that

$$\phi(1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes u) = 1 \otimes \cdots \otimes S^j 1 \otimes 1 \cdots 1 \otimes \tau(u)$$

for $j \in \mathbb{Z}_+, 1 \leq i \leq n_1$.

Continuing in this way and by Lemma 2.1 we have

$$\phi(1 \otimes \cdots \otimes a_1 \otimes 1 \cdots 1 \otimes u) = 1 \otimes \cdots \otimes a_1 \otimes 1 \cdots 1 \otimes \tau(u)$$

for $a \in U(\mathfrak{V}), 1 \leq i \leq n_1$.

Then it follows from the irreducibility of the $\mathfrak{V}$-module $\Phi(\lambda^{(1)}_i, \alpha^{(1)}_i, h^{(1)}_i)$ (see Theorem 3.2) that

$$\phi(1 \otimes \cdots \otimes f \otimes 1 \cdots 1 \otimes u) = 1 \otimes \cdots \otimes f \otimes 1 \cdots 1 \otimes \tau(u)$$

for any $f \in \Phi(\lambda^{(1)}_i, \alpha^{(1)}_i, h^{(1)}_i)$ and $1 \leq i \leq n_1$. It follows from this,

$$(\lambda^{(1)}_i)^{-m} L_m - (\lambda^{(1)}_{i'})^{-m'} L_{m'}$$

$$= (m - m') (h^{(1)}_i (\alpha^{(1)}_i) - (m + m') \alpha^{(1)}_i \eta^{(1)}_i) + (m - m') \eta^{(1)}_i t$$

and

$$\phi(1 \otimes \cdots \otimes ((\lambda^{(1)}_i)^{-m} L_m - (\lambda^{(1)}_{i'})^{-m'} L_{m'}) 1 \otimes 1 \cdots 1 \otimes u)$$

$$= 1 \otimes \cdots \otimes ((\lambda^{(1)}_i)^{-m} L_m - (\lambda^{(1)}_{i'})^{-m'} L_{m'}) 1 \otimes 1 \cdots 1 \otimes \tau(u)$$

for any $1 \leq i \leq n_1$. It follows from this,
we see that \( \eta_i^{(1)} = \eta_i^{(2)} \), \( \alpha_i \eta_i^{(1)} = \alpha_i \eta_i^{(2)} \) and \( h^{(1)}(\alpha_i) = h^{(2)}(\alpha_i) \) for all \( i \). Thus, 
\( (\lambda_i^{(1)}, \alpha_i^{(1)}, h_i^{(1)}) = (\lambda_i^{(2)}, \alpha_i^{(2)}, h_i^{(2)}) \) for \( 1 \leq i \leq n_1 \).

**Claim 5.** \( \tau : V_1 \to V_2 \) is an isomorphism of \( \mathfrak{U} \)-modules.

Clearly, \( \tau \) is a linear isomorphism. Since \( \phi(a1 \otimes u) = a1 \otimes \tau(u) \) for any \( a \in U(\mathfrak{U}_K) \) and \( T^{(1)} \) is an irreducible \( \mathfrak{U}_K \)-module (see Theorem 4.3), \( \phi(f \otimes u) = f \otimes \tau(u) \) for any \( f \in T^{(1)} \). It follows from this and \( \phi(L_m(1 \otimes u)) = L_m(1 \otimes \tau(u)) \) we can easily deduce that 
\( \tau(L_m u) = L_m \tau(u) \) for any \( m \in \mathbb{Z} \), proving this claim. The proof is complete.

**Theorem 4.6.** Let \( \lambda_i, \alpha_i \in \mathbb{C}^*, h_i(t) \in \mathbb{C}[t] \) such that the \( \lambda_i \) are pairwise distinct and \( \deg h_i(t) = 1 \) for \( 1 \leq i \leq n \) and \( V \) an irreducible \( \mathfrak{U} \)-module for which there exists \( R_V \in \mathbb{Z}_+ \) such that all \( L_k \) for \( k \geq R_V \) are locally finite on \( V \). Then \( X := \bigotimes_{i=1}^n \Phi(\lambda_i, \alpha_i, h_i) \otimes V \) is not isomorphic to any of the following irreducible \( \mathfrak{U} \)-modules:

\[ V', N(M, \gamma(t)), A_{\gamma'} \text{ or } \bigotimes_{i=1}^m \Omega(\mu_i, b_i) \otimes V', \]

where \( 1 \leq m \in \mathbb{Z}_+, b', 1 \neq b \in \mathbb{C}, \mu_i \in \mathbb{C}^*, \gamma(t) \in \mathbb{C}[t] \setminus \mathbb{C}, M \text{ is an irreducible } \mathfrak{a}_r\text{-module, } \]

\( V' \) there exists \( R_{V'} \in \mathbb{Z}_+ \) such that \( L_i V' = 0 \) for \( i \geq R_{V'} \).

**Proof.** If \( X \cong \bigotimes_{i=1}^m \Omega(\mu_i, b_i) \otimes V' \), then following from the proof of Theorem 4.5 we see that \( \Phi(\lambda_1, \alpha_1, h_1) \cong \Omega(\mu_i, b_i) \) as \( \mathfrak{U} \)-module for some \( i \). This is impossible, since they have different ranks as free \( U(L_0) \)-modules. And the rest non-isomorphisms follows from the similar argument as in the proof of [TZ1, Corollary 4].

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