Numerical Method for One-Dimensional Convection-diffusion Equation Using Radical Basis Functions

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Abstract

In this paper, the meshless method is employed for the numerical solution of the one-dimensional (1D) convection-diffusion equation based on radical basis functions (RBFs). Coupled with the time discretization and the collocation method, the proposed method is a truly meshless method which requires neither domain nor boundary discretization. The algorithm is very simple so it is very easy to implement. The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme.

Keywords: Meshless method; Radical basis function (RBF); Numerical solution; Convection-diffusion equation

Introduction

Whenever we consider mass transport of a dissolved species (solute species) or a component in a gas mixture, concentration gradients will cause diffusion. If there is bulk fluid motion, convection will also contribute to the flux of chemical species. Therefore, we are often interested in solving for the combined effect of both convection and diffusion.

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection [1].

The general convection-diffusion equation has the following form [2,3]:

$$\frac{\partial u(x,t)}{\partial t} = \nabla \cdot ( \nu \nabla u(x,t) ) - \nabla (\gamma u(x,t)) + R(x,t).$$

In the above equation, four terms represents transient, convection, diffusion and source term respectively. Where $u(x, t)$ is the variable of interest (species concentration for mass transfer, temperature for heat transfer), $D$ is the diffusivity (also called diffusion coefficient), $\nu$ is the average velocity that the quantity is moving, $R(x,t)$ is source term represents capacity of internal sources, $\gamma$ represents gradient and $\nabla \cdot \nabla$ represents divergence.

This paper is devoted to the numerical computation of the one-dimension (1D) convection-diffusion equation:

$$u_t(x,t) + \alpha u_x(x,t) + \beta u(x,t) = \gamma u_{xx} + f(x,t), \quad a \leq x \leq b, \quad t \geq 0,$$  \hspace{1cm} (1.1)

With the initial conditions:

$$u(x,0) = h(x), \quad a \leq x \leq b,$$  \hspace{1cm} (1.2)

And Dirichlet boundary conditions:

$$u(a,t) = g_1(t), u(b,t) = g_2(t), \quad 0 \leq t,$$  \hspace{1cm} (1.3)

Where $\alpha$, $\beta$ and $\gamma$ are known constant coefficients, $h(x)$ and $g_i(t)$ ($i = 0, 1$) are known continuous functions.

Recently, much attention has been given to the development, analysis, and implementation of stable methods for the numerical solution of the convection-diffusion equations (see [4] and the reference therein). Jim Douglas, et al. [5] combine definite element and finite difference methods based on the method of characteristic for solving the convection-diffusion problems. Chen and Hon [6] consider the 2D and 3D Helmholtz and convection-diffusion equation using boundary knot method. The meshless local Petro-Galerkin method for convection-diffusion equation was considered in [7]. A new finite difference method described by Ram P. Manohar and John W. Stephenson [8].

In this article, we present a numerical scheme to solve the convection-diffusion equation using the collocation method with Radial Basis Function (RBF). The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme.

In last 25 years, the radial basis functions (RBFs) method is known as a powerful tool for scattered data interpolation problem. The use of RBFs as a meshless procedure for numerical solution of partial differential equations is based on the collocation scheme. Because of the collection technique, this method does not need to evaluate any integral. The main advantage of numerical procedures which use RBFs over traditional techniques is mesh-less property of these methods. RBFs are used actively for solving partial differential equations. The examples see [9-11]. In the last decade, the development of the RBFs as a truly meshless method for approximating the solutions of PDEs has drawn the attention of many researchers in science and engineering [12-14]. Meshless method has become an important numerical computation method, and there are many academic monographs are published [15-17].

The layout of the article is as follows: In section 2, we introduce the collocation method and apply this method on the convection-diffusion...
The results of numerical experiments are presented in section 3. Section 4 is dedicated to a brief conclusion. Finally, some references are introduced at the end.

The Collocation Method with Radical Basis Function

Radial basis function approximation

The approximation of a distribution \( u(x) \), using RBF may be written as a linear combination of \( N \) radial functions, usually it takes the following form:

\[
u(x) = \sum_{j=1}^{N} \lambda_j \phi(x, x_j) + \psi(x), \quad \text{for } x \in \Omega \subseteq R^d \tag{2.1}
\]

Where \( N \) is the number of data points, \( x=(x_1, x_2, ..., x_d) \), \( d \) is the dimension of the problem, the \( \lambda_j \)'s are coefficients to be determined and \( \phi \) is the radial basis function. Eq. (2.1.1) can be written without the polynomial \( \psi \).

First, let us discretize Eq. (1.1) according to the following conditions due to Eq. (2.4) are written as:

\[
\sum_{j=1}^{N} \lambda_j^N \phi(x_j) = 0.
\]

In a similar representation as Eq. (2.1), for any linear partial differential operator \( \ell \), \( \ell u \) can be approximated by:

\[
\ell u(x) \approx \sum_{j=1}^{N} \lambda_j \ell \phi(x, x_j) + \ell \psi(x).
\]

The convection-diffusion equation

\[
\ell u(x, t) = g(x, t),
\]

where \( g = \ell u + \psi \).

(1 + \theta \cdot \beta \cdot r) \cdot u^{n+1} - \theta \cdot \beta \cdot r \cdot \nabla u^{n+1} = (1 - (1 - \theta) \cdot \beta \cdot r) \cdot u^{n+1} - \theta \cdot \beta \cdot r \cdot \nabla u^{n+1} + r \cdot f^n.
\]

Assuming that there are \( N - 2 \) interpolation points, \( u^n(x) \) can be approximated by:

\[
u^n(x) = \sum_{j=1}^{N-2} \lambda_j^N \phi(x_j), \quad j = 1, 2, \cdots, N - 2.
\]

To guarantee the positive definition, here we use the following approximation:

\[
u^n(x) = \sum_{j=1}^{N-2} \lambda_j^N \phi(x_j) + \lambda_j^N x_j = 0.
\]

Writing Eq. (2.9) together with Eq. (2.10) in a matrix form we have:

\[
[u^n] = A[\lambda^n],
\]

\[
A = a_{ij}, \quad 1 \leq i, j \leq N
\]

Assuming that there are \( p < N - 2 \) internal points and \( N - 2 - p \) boundary points, then the \( N \times N \) matrix \( A \) can be split into:

\[
A = A_1 + A_2 + \hat{A},
\]

where

\[
A_1 = [a_{ij}, \quad 1 \leq i, j \leq N - 2]
\]

and

\[
A_2 = [a_{ij}, \quad 1 \leq i, j \leq N - 2, \quad p + 1 \leq i, j \leq N - 1]
\]

Using the notation \( \ell A \) to designate the matrix of the same dimension as \( A \) and containing the elements \( \hat{a}_{ij} \) where \( \hat{a}_{ij} = a_{ij}, \quad 1 \leq i, j \leq N \), then Eq. (2.2.1) together with the boundary conditions Eq. (1.3) can be written in matrix form as:

\[
B\lambda^{n+1} + \ell \lambda^{n+1} + [G^T]^{n+1} = [f^n],
\]

where

\[
C = (1 - \beta \cdot (1 - \theta) \cdot r) A_1 + (1 - \theta) \cdot \beta \cdot r \cdot \alpha \cdot \nabla A_1,
\]

\[
B = (1 + \beta \cdot r) A_2 - \beta \cdot \alpha \cdot \nabla A_2 + A_1 + A_2,
\]

\[
[G^T]^{n+1} = [0 \cdots 0 \quad g^{n+1}_{1} \cdots g^{n+1}_{p} \quad 1]^{T},
\]

\[
[f^n] = [f^n_{1} \cdots f^n_{p} \quad 0 \cdots 0]^{T}
\]

Eq. (2.14) is obtained by combining Eq. (2.6), which applies to the
domain points, while Eq. (1.3) applies to the boundary points. Together with the initial condition Eq. (1.2) and Eq. (2.14), we can get all λ's, thus we can get the numerical solutions.

Since the coefficient matrix is unchanged in time steps, we use the LU factorization to the coefficient matrix only once and use this factorization in our algorithm.

Remark: Although Eq. (2.14) is valid for any value of \( \theta \in [0,1] \), we will use \( \theta = \frac{1}{2} \) (The famous Crank-Nicolson scheme).

Numerical Examples

In this section, we present several numerical results to confirm the efficiency of our algorithm for solving the 1D convection-diffusion equation.

Example 1

In this example, we consider the convection-diffusion Eq. (1.1) in \([0,1]\) with \( \alpha = 0.1, \beta = 0, \varepsilon = 0.01 \), with the boundary conditions:

\[
    u(0,t) = \exp(0.1t), \quad u(1,t) = \exp(-1 + 0.1t), \quad t > 0,
\]

And the initial condition

\[
    u(x,0) = \exp(-x) \quad 0 \leq x \leq 1,
\]

Then the analytical solution of the equation is

\[
    u(x,t) = \exp(-x + 0.1t).
\]

The right side functions

\[
    f(x,t) = 0.
\]

We use MQ radial basis function for the computation, the \( L_\alpha, L_\beta \) and RMS errors and Root-Mean-Square (RMS) of errors are obtained in Table 1 for \( T = 0.1, 0.3, 0.5, 0.7, 1.0 \) with time steps \( \tau = 0.001 \) and \( dx = 0.001 \).

The space-time graph of analytical and numerical solutions for \( t = 1 \) are given in Figure 1. Note that we cannot distinguish the exact solution from the estimated solution in Figure 1.

### Table 1: For \( T = 0.1, 0.3, 0.5, 0.7, 1.0 \) with time steps \( \tau = 0.001 \) and \( dx = 0.001 \).

| \( T \) | 0.1 | 0.3 | 0.5 | 0.7 | 1.0 |
|---|---|---|---|---|---|
| \( L_\alpha \)-errors | 1.547×10^{-5} | 2.371×10^{-4} | 3.599×10^{-4} | 4.639×10^{-4} | 5.909×10^{-4} |
| \( L_\beta \)-errors | 6.788×10^{-5} | 1.787×10^{-4} | 3.453×10^{-4} | 3.453×10^{-4} | 3.472×10^{-4} |
| RMS-errors | 6.778×10^{-5} | 1.784×10^{-4} | 3.448×10^{-4} | 3.448×10^{-4} | 4.365×10^{-4} |

Example 2

In this example, we consider Eq. (1.1) with \( \alpha = 1, \beta = 0, \varepsilon = 1 \) and the boundary conditions:

\[
    u(0,t) = \exp\left(\frac{3t}{4}\right), \quad u(1,t) = \exp\left(-\frac{2 + 3t}{4}\right).
\]

And the analytical solution of the equation is given as:

\[
    u(x,t) = \exp\left(-\frac{2x + 3t}{4}\right).
\]

We get the initial conditions from the exact solution. The right side functions

\[
    f(x,t) = 0.
\]

The \( L_\alpha, L_\beta \) and RMS errors and Root-Mean-Square (RMS) of errors are obtained in Table 2 for \( T = 0.1, 0.25, 0.5, 0.75, 1.0 \) with time steps \( \tau = 0.001 \) and \( dx = 0.001 \).

Similar to the previous example, the space-time graph of analytical and estimated solutions for \( t = 1 \) are presented in Figure 2.

### Example 3

We consider the convection-diffusion equation Eq. (1.1) with \( \alpha = -1, \beta = 10, \varepsilon = 1 \) in the interval \([0,1]\), the exact solution is given as

\[
    u(x,t) = t \sin(\pi x) \exp(-x^2).
\]

The boundary conditions are:

\[
    u(0,t) = 0, \quad u(1,t) = 0, \quad t \geq 0.
\]

The right side functions

\[
    f(x,t) = 0.
\]

We extract the initial conditions from the exact solution.

These results are obtained with \( dx = 0.001, \tau = 0.001 \). Similar to the previous examples, the \( L_\alpha \) and \( L_\beta \) error and RMS errors for \( t = 0.5, 0.75, 1.0, 1.25 \) and \( 1.5 \) are presented in Table 3.
Example 4

In this example, we consider the convection-diffusion equation in $[0,1]$ has the following form:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 2u = \frac{\partial u}{\partial x} + f(x,t)$$

The right side functions $f(x,t)=x^2 + 2x + 2\exp(-t)$, with the boundary condition:

$$u(0,t) = 0, \quad u(1,t) = \exp(-t), \quad t > 0$$

Then the analytical solution of the equation is $u(x,t) = x^2 \exp(-t)$, we get the initial conditions from the exact solution.

In this case, we use the radial basis functions MQ for the discussed scheme. These results are obtained for $dx = 0.001$, $\tau = 0.001$. The graph of analytical and numerical solution for $t=0.1$, $0.2$, $0.3$, $0.4$ and $0.5$ is given in Figure 4. The results obtained show the very good accuracy and efficiency of the new approximate scheme. Note that we cannot distinguish the exact solution from the estimated solution in Figure 4.

We also give the difference between exact solutions and numerical solutions in Figure 5.

Conclusion

In this paper, the collocation method is employed for the numerical solution of convection-diffusion equation based on radial basis functions (RBFs). Coupled with the time discretization, the proposed method is a truly meshless method which requires neither domain nor boundary discretization. The results of numerical experiments are presented, and are compared with analytical solutions confirmed the good accuracy of the presented scheme.

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