Frames of Reference and the Intrinsic Directional Information of a Particle With Spin

Daniel Collins
Group of Applied Physics, University of Geneva, 20, rue de l'Ecole-de-Médecine, CH-1211 Geneva 4, Switzerland

Sandu Popescu
H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL and Hewlett-Packard Laboratories, Stoke Gifford, Bristol BS12 6QZ, UK
(Dated: 18 January 2004)

"Information is physical", and here we consider the physical directional information of a particle with spin. We ask whether, in the presence of a classical frame of reference, such a particle contains any intrinsic directional information, i.e. information above that which can be transmitted by a classical bit. We show that when sending a large number of spins, the answer is asymptotically "no". For finite numbers of spins, N, we do not know the answer. We also show that any frame of reference which we can consider to be classical must use some resource which is exponentially large in N. This gives a quantitative meaning to the idea that classical objects are big.

I. INTRODUCTION

In the last 20 years a conceptual revolution has swept through physics and computer science, beginning with the idea that "information is physical". This is the idea that information is something which is encoded in the physical world, and has no existence without it. Since the physical world is in fact quantum mechanical, this must apply to information too, giving us quantum information.

It has been usual in quantum information theory to view the physical system in which the information is stored as unimportant, viewing it simply as a carrier of information living in an abstract n-dimensional Hilbert space. Spin-\(\frac{1}{2}\) particles, pairs of energy levels, and photon polarisations have all been treated in the same way, disregarding the fact that the first carries spatial information, the second carries time information, and the third a mixture of the two. Whilst it is often useful to ignore the differences, we feel it is time to treat the systems more carefully, taking into account the physical information they carry. We would like to understand exactly what is the content of the physical information, and what it can be used for. With this in mind, we looked at the spatial information carried in spin-\(\frac{1}{2}\) particles. We considered the following problem. How well can one specify a direction in space by sending spin-\(\frac{1}{2}\) particles? By this we mean, suppose Alice wishes to tell Bob a direction in space, and is allowed to send Bob N spin-\(\frac{1}{2}\) particles, in any state. How well can she do it? We shall compare:

- N directional qubits (spin \(\frac{1}{2}\) particles).
- N non-directional qubits.
- N non-directional classical bits.

Intuitively the difference should tell us about the intrinsic directional information of the spins.

One might want to consider also directional classical bits. The difficulty is that the very notion of a classical directional bit, although clear intuitively, is rather delicate and, as far as we know, is not well established. One can send classical directional information by sending an arrow pointing in the desired direction. This however transmits an infinite number of bits of information. A finite amount of information would be transmitted by a "noisy" arrow, i.e. by an arrow pointing in a direction according to some probability distribution around the desired direction. But there are many different distributions we could choose, such as a gaussian, or \(\cos^2\) distribution, and we do not see why to choose one distribution over another. For this reason we shall not consider classical directional bits here.

To make the comparison, we must also state whether Alice and Bob share any prior directional information. Consider first that Alice and Bob do not share any prior directional information. By this we mean that they do not begin with any shared frame of reference, i.e. that there are no distant stars from which to fix a direction, or any other clues. If they were only allowed to send non-directional classical bits encoded for example as holes or blanks in a punch tape, they will not be able to send any directional information whatsoever. Nor would sending qubits encoded as directionless energy levels help. However, since quantum spins point in some direction, they can be used to specify a direction in many different ways, and one can try to find the optimal method.

In the first part of this paper we consider the case of perfectly aligned frames of reference. Using this it is possible to use classical bits to specify a direction, e.g. by splitting the sphere into patches in advance, and using the classical bits to say within which patch the direction lies. The shared frame of reference also allows us to use the qubits as classical bits. Thus it is not necessary to use the directional information contained intrinsically within the spins in order to specify a direction. However, for
any finite number of bits/spins, the spins may be more efficient. That is, it may be possible, even with aligned frames, to use the intrinsic directional information contained within $N$ spins to more precisely point a direction than if we were to use the spins simply as $N$ classical bits.

One might also compare the spins and classical bits to directionless qubits, such as energy levels, or time-bins. The idea would be that the difference between spins and energy levels tells us about the intrinsic directional information, whereas the difference between the energy levels and the classical bits tells us about the difference between quantum and classical.

Our main results are that, with perfectly aligned frames:

- $N$ directional qubits are completely equivalent to $N$ non-directional qubits (assuming that the appropriate non-directional frames of reference are perfectly aligned).
- $N$ qubits are, asymptotically in block coding, no better than $N$ non-directional classical bits.

This is surprising in one sense, since the spin can be placed in many possible states, and so seems to contain many bits of information. However it is not so surprising in the light of Holevo’s theorem, since this tells us that a qubit cannot be used to transmit more than one classical bit of information (if there is no shared entanglement).

In the second part of the paper we will consider the case of non-perfectly aligned frames. For example, the frames could be specified by a fixed number of spins held by Alice, and similarly for Bob. We shall show that for the frames to behave classically when we send $N$ further spins to specify a direction, the number of spins in the frames must be exponential in $N$. In other words, for a frame of reference to be considered classical, it must be exponentially big.

II. DIRECTIONAL AND NON-DIRECTIONAL QUBITS ARE EQUIVALENT

We shall now prove that, with perfectly aligned frames, $N$ directional qubits are equivalent to $N$ non-directional qubits. For clarity, consider a non-directional qubit encoded in the energy levels of an atom. Furthermore, suppose that the energy frames are also aligned. By this we mean that Alice and Bob can agree on what the state $(\alpha |E_1\rangle + e^{i\phi} \beta |E_2\rangle)$ is. It is simple to agree on whether the qubit is in state $|E_1\rangle$ or $|E_2\rangle$. However in order to determine the relative phase, Alice and Bob will need to synchronize their clocks.

With all frames aligned, Alice and Bob can agree upon a one-to-one mapping between the direction and energy frames. They then simulate the results of sending any spin qubit by sending an energy qubit.

Once the frames are all aligned, a qubit is a qubit and there is no difference whether it is encoded as energy, or time, or direction. However, if we do not have perfectly aligned frames, this is no longer the case. Also, we are still left with the difference between the qubit and the classical bit, which we shall focus upon for the next part of this paper.

III. QUBITS ARE ASYMPTOTICALLY EQUIVALENT TO BITS

In order to state our second result properly, we must define what we mean by sending a direction precisely. For simplicity, we shall assume that Alice tries to send a direction $\hat{n}$ chosen from a finite set of directions, $\hat{n}$, according to probability $p(\hat{n})$. We will insist that Bob must, at the end of the protocol, guess which direction was sent: we shall call his guess $\hat{m}$, chosen from a set $\hat{m}$. We shall give him some (bounded) score to say how well they do in any run, $f(\hat{n}, \hat{m})$, and try to maximize the average score, averaged over many runs. We want to know whether they can get a better average score by sending $N$ spin-$\frac{1}{2}$ particles than they can by sending $N$ classical bits.

To allow for more generality, we allow the set of guessed directions to be different to the set of sent directions. One might wonder why one would guess a direction which was not sent. It could be useful to Bob if he is not sure which of two directions was sent, and so he guesses a direction inbetween the two, so that it is not too far from either.

As is common in information theory, we shall find it simplest to perform the analysis in a block coding scenario. By block coding we mean that, rather than taking one input direction, sending $N$ spins/bits, and making one guess, we shall have Alice take a large number, $K$, of directions, $\hat{n}^k; k = 1..K$, each chosen from $\hat{n}$, according to $p(\hat{n})$, send $KN$ spins/bits together, and have Bob make $K$ guesses, $\hat{m}^k; k = 1..K$. We shall denote Alice’s block of $K$ directions by $\hat{n}^\otimes K$, and similarly Bob’s block of guessed directions by $\hat{m}^\otimes K$. We will then have $K$ pairs, $(\hat{n}^k, \hat{m}^k)$, and some probability distribution, $p(\hat{n}^\otimes K, \hat{m}^\otimes K)$, of input and output blocks of length $K$. The score will still be given by the single copy fidelity, depending only on pairs of directions, $(\hat{n}^k, \hat{m}^k)$, not on pairs of blocks, $(\hat{n}^\otimes K, \hat{m}^\otimes K)$. We shall be interested in the asymptotic limit of arbitrarily large block size, ie. $K \to \infty$.

In this asymptotic block coding scenario, we shall show that the best average score we can obtain with $N$ spins can also be (arbitrarily closely) obtained using $N$ classical bits. We shall prove this in three stages. We shall first show that we can use block coding of classical bits and the shared classical randomness to arbitrarily closely simulate the probability of each pair of directions $(\hat{n}_i, \hat{m}_j)$ that occurs in any single copy protocol using $N$ spins. We show from this that the block coded classical protocol can obtain a fidelity arbitrarily close to the opti-
We know this because Holevo’s theorem [6] tells us that to send \(2^N\) classical bits, which is more useful than classical bits.

IV. SIMULATING THE QUANTUM PROBABILITIES

Before giving the classical block coding protocol which arbitrarily closely simulates the single copy quantum probabilities, we define the typical set. The key element of Shannon and Schumacher compression is that if we look at many samples from a distribution, we are almost certain to find a sequence which lies in the weakly typical (entropy typical) set. This is the set of all sequences for which the logarithm of the probability is close to the entropy of the distribution. This set is much smaller than the set of all possible sequences, and so elements within it can be described by much fewer bits than would be required to describe an arbitrary sequence, giving us compression.

In order to prove that our protocol works, we need to use a different sort of typical set, the strongly (or frequency) typical set. This is the set of all sequences which have frequencies of each outcome very close to the probability of that outcome. Intuitively, one expects that if we take many samples from a distribution, the samples will almost certainly form a sequence within this frequency typical set. This is indeed the case. This set is quite similar to the weakly typical set, but is slightly smaller. The compression properties of the two sets are asymptotically the same. We only use the strongly typical set because having the frequencies of all the outcomes close to the probabilities is very useful. Though well known in classical information theory, this kind of set is only beginning to appear in quantum information theory (see eg. [4]).

More precisely, the strongly typical set, \(A_K^\epsilon\), is the set of blocks \((\hat{n}^K, \hat{m}^K)\) such that

\[
\left\{ \frac{\#(\hat{n}_i, \hat{m}_j)}{K} - p_Q(\hat{n}_i, \hat{m}_j) \right\} < \epsilon \quad \forall (i, j),
\]

where \(\#(\hat{n}_i, \hat{m}_j)\) is the number of pairs of directions \((\hat{n}_i, \hat{m}_j)\) in the pair of blocks \((\hat{n}^K, \hat{m}^K)\) which point in the directions \((\hat{n}_i, \hat{m}_j)\), and \(|i|, |j|\) are the numbers of directions \(\hat{n}_i\) and \(\hat{m}_j\) respectively.

Now, we shall give the protocol. Later we shall explain why it works.

Step 1: Alice and Bob agree in advance a large table, which is created by random sampling. Each entry in the table is actually an ordered list of \(K\) guessed directions, which are independently identically distributed (i.i.d.) according to \(p_Q(\hat{n}_i, \hat{m}_j) = \sum_A p_A(\hat{n}_i, \hat{m}_j)\). The table has \(2^K(I(\hat{n}, \hat{m}) + c)\) entries, where \(I(\hat{n}, \hat{m})\) is the mutual information between \(\hat{n}\) and \(\hat{m}\) in the quantum protocol. Thus it will take \(K I(\hat{n}, \hat{m})\) bits for Alice to specify to Bob a particular entry in the table. Since \(I(\hat{n}, \hat{m}) \leq N\) this will take at most \(K N\) bits, which is precisely the number of classical bits she is allowed to send.

Step 2: Alice is given some list of \(K\) directions, \(\hat{n}^K\), which are independently identically distributed according to \(p(\hat{n})\).

Step 3: She looks at the table to see if there exists some entry, \(\hat{n}^K\), such that \((\hat{n}^K, \hat{m}^K) \in A_K^\epsilon(\hat{n}, \hat{m})\).

Step 4: If she finds such an entry, she sends its index to Bob. If there is more than one, she picks the first such one. If there is no such entry, she sends the index 1 to Bob.

Step 5: Bob uses the list of \(K\) guessed directions which Alice has pointed out to him.

Because the fidelity is single copy, we are interested in the single copy probabilities of pairs of directions which this classical procedure produces. By single copy probability we mean the probability of the pair of directions at the \(k\)th position in the block being \((\hat{n}_i, \hat{m}_j)\), ignoring (ie. summing over) the possible outcomes in all the other positions in the block. Since the protocol is symmetric between positions in the block, it does not matter which \(k\) we look at. We shall show that, by taking \(K\) sufficiently large, this procedure gives classical single copy probabilities arbitrarily close to the quantum ones. ie.

\[
|p_Q(\hat{n}_i, \hat{m}_j) - p_Q(\hat{n}_i, \hat{m}_j)| < \epsilon \quad \forall (i, j),
\]
where $p_C(\hat{n}_i, \hat{m}_j)$ is the classical single copy probability.

The details of the proof that the protocol does this follow closely Section 13.6 of Cover and Thomas [10]. We just give a sketch here.

The proof is based upon the idea that pairs of blocks $(\hat{n}^\otimes K, \hat{m}^\otimes K) \in A'_K(\hat{n}, \hat{m})$ have frequencies of pairs $(\hat{n}, \hat{m})$ within $\epsilon$ of the quantum probabilities, $p_Q(\hat{n}, \hat{m})$. Thus, if we are almost certain that the protocol will give us input and output blocks of directions which are in the typical set, then the probability of a pair of directions $(\hat{n}, \hat{m})$ appearing will be within $\epsilon$ of the quantum probabilities.

Now, the list of directions, $\hat{n}^\otimes K$, which Alice is given is almost certain to be a strongly typical sequence, i.e. a sequence in the set

$$A'_K(\hat{n}) = \left\{ \hat{n}^\otimes K : \left| \frac{\#\hat{n}_i}{K} - p(\hat{n}_i) \right| < \frac{\epsilon}{|n|}, \forall i \right\},$$

(5)

where $\#\hat{n}_i$ is the number of directions $\hat{n}^i$ in the sequence $\hat{n}^\otimes K$ which point in the directions $\hat{n}_i$, and $|i|$ is the numbers of directions $\hat{n}_i$.

Next, for any $\hat{n}^\otimes K \in A'_K(\hat{n})$, if we add a sequence of guessed directions $\hat{m}^\otimes K$ which are independently distributed according to $p_Q(\hat{m}_j)$, then the probability that the pair forms a jointly typical sequence, $(\hat{n}^\otimes K, \hat{m}^\otimes K) \in A'_K(\hat{n}, \hat{m})$, is approximately $2^{-Kf(\hat{n}, \hat{m})}$.

Finally, if we take $2^{K(I(\hat{n}, \hat{m})+\epsilon)}$ sequences of guessed directions in our table, we are almost certain to find one which is jointly typical with $\hat{n}^\otimes K$.

Thus our protocol indeed gives $|p(\hat{n}_i, \hat{m}_j) - p_Q(\hat{n}_i, \hat{m}_j)| < \frac{\epsilon}{|n||m|}, \forall (i,j)$. The difference between the quantum fidelity and our classical fidelity is given by:

$$|\tilde{f}_Q - \tilde{f}_C| \leq \sum_{i,j} |f(\hat{n}_i, \hat{m}_j)| |p_Q(\hat{n}_i, \hat{m}_j) - p_C(\hat{n}_i, \hat{m}_j)|$$

$$\leq \epsilon f_{\text{max}}.$$  

(6)

Thus our classical block coding protocol gives an average fidelity arbitrarily close to the optimal single copy quantum one.

Before using this result to show that classical block coding is as good as quantum block coding, we note that this classical procedure for producing joint probabilities is very general, and may be useful outside this context. It is a protocol where Alice takes one of various inputs, and sends Bob a sample of one of various probability distributions, which one depending upon Alice’s input. In this sense it is somewhat like remote state preparation, and for this reason we call it, ”remote distribution preparation”. Our protocol only sends $I(\hat{n}, \hat{m})$ classical bits in order to do this. This amount is optimal since Bob learns $I(\hat{n}, \hat{m})$ bits about Alice’s source from his output, and so any procedure using less than $I(\hat{n}, \hat{m})$ bits would allow us to send information faster than light.

We note also that the shared table is quite large, and was created randomly. Thus one could ask if we might be using shared randomness, an additional resource. The difference between shared randomness and shared instructions is that we can use the shared instructions many times without problems, whereas shared randomness is used up. A simple example is an unknown bit: the first time we look at it it is random, the second time it is the same as the first time, and so is no longer random. What happens if we use our table many times, i.e. to mimic many sets of $KN$ qubits? If we just look at the statistics of the spins individually, i.e. of the individual pairs $(\hat{n}, \hat{m})$, we will not see any problems. If however we look at the statistics of the blocks, $(\hat{n}^\otimes K, \hat{m}^\otimes K)$, we shall see that the blocks are correlated between one run and the next. We may need many runs to see this, but it will happen eventually. This correlation would not exist if we took a fresh table each time, i.e. had shared randomness. Fortunately we are only interested in the statistics of the individual pairs, and so can use one shared instruction table again and again without problem.

As the table of shared instructions is very big, we also wonder if the same task can be performed with a smaller table. We do not know whether this is possible.

To deal with quantum block coding first note that since the fidelity is bounded, there must be a quantum block code of finite length, $M$ say, which gives an average fidelity within $\epsilon$ of the optimal (infinite length) one. For this $M$, we can take a classical block code of length $KM$ which gives an average fidelity within $\epsilon$ of the quantum code of length $M$. Thus we have a finite classical code which gives an average fidelity arbitrarily close to the infinite length quantum block coded one, making the quantum and classical schemes asymptotically equivalent.

V. RELATED QUESTIONS

We have shown that when 2 parties have pre-aligned frames, the intrinsic directional information contained within spin-$\frac{1}{2}$ particles is of no use for pointing a direction in space, at least if we allow block coding. We may as well use the spins as classical bits. An open question is whether or not there is a difference for finite numbers of bits.

Another question is whether the presence of unlimited shared entanglement would help us to use the directional part of the quantum spins. It will certainly help us specify the direction more precisely: using superdense coding[12] we can send 2 classical bits of information with just 1 spin-$\frac{1}{2}$ particle. If we use classical bits to specify the direction entanglement will not help us at all. However, this is just the normal superdense coding, and is not intrinsically directional. To see whether there is anything more we should compare $N$ spins with $2N$ classical bits (both in the presence of shared prior entanglement). Doing this, we can use the fact that $N$ spins and shared entanglement cannot create more than $2N$ bits of mutual information between Alice and Bob’s directions, combined with our earlier results, to show that $2N$ classical bits can get an equally good average score.
Thus the presence of entanglement does not help unlock any directional information. Once we have aligned our frames of reference, we may as well treat spins as directionless qubits.

VI. CLASSICAL FRAMES ARE BIG

Finally, we shall argue that classical objects, in particular those which specify frames of reference, need to be very big. Recall that with a shared frame of reference, one can specify a direction in space using \( N \) spin-\( \frac{1}{2} \) particles with a fidelity \( F_N \sim 1 - \frac{\text{const.}}{N} \). If one has no shared frame of reference, \( N \) spins only specify a direction up to \( F_N \sim 1 - \frac{\text{const.}}{N^2} \). Put another way, without a shared frame of reference one needs around \( 2^\frac{N}{2} \) spins to do the same job as \( N \) spins would do with a shared frame. Now, the classical objects which define the frame can be considered to be \( M \) spin-\( \frac{1}{2} \) quantum mechanical particles with Alice, and \( M \) with Bob. We assume that Alice’s object is not entangled with Bob’s. Since we could try sending a direction without a pre-aligned frame by sending \( N + M \) spins, it must be that \( M \geq 2^\frac{N}{2} - N \). Hence a “classical” frame used to extract the full directional information from \( N \) spins must itself consist of around \( 2^\frac{N}{2} \) spins: an exponential number.

Of course, this is not the only way to specify a frame: one could also use two atoms a large distance apart to specify the direction. However we still have an exponential use of resources, either in the increasing distance between the two atoms, or in the increasing momentum of the atoms (which must be very uncertain to precisely define the position of the atoms). This makes sense: every time we double the precision we require twice as many resources.

For current measurements the exponential growth is not a problem. In astronomy, angles in the x-y plane can be measured with a precision of \( 10^{-11} \) radians, and can be specified using a shared frame of reference and 34 classical bits. To specify such an angle using spins, we need a total angular momentum \( L \geq \Delta L \geq \frac{\hbar}{2} \). Since \( N \) spins have \( L = N \frac{\hbar}{2} \) we only need around \( 10^{11} \) spins. If we made the frame from two atoms, they could each be localised to 10 atom widths (=10\(^{-10}\) m), and located 100 meters apart.

Whilst these frames are relatively small, they are leaving the realm of the microscopic. As we probe the physical world more and more closely, we can expect to see our probes getting bigger and bigger.

Acknowledgements We thank Asher Peres and Andreas Winter for helpful discussions.

Note Added: there are several other papers \([3, 13, 15]\) which contain protocols closely related to the classical protocol in this paper. For example \([3]\) contains the related result that one can, using block coding and shared randomness, arbitrarily closely simulate a classical noisy channel of capacity \( C \) using a classical noiseless one of the same capacity. A noisy channel is defined by \( p(y|x) \), which is the probability that the channel gives output \( y \) when the input is \( x \). Its capacity is given by

\[
C = \max_x I(x, y),
\]

where the maximum is over our choice of how to use the channel, i.e. the probabilities of the various inputs \( p(x) \). This is essentially the same as the problem of Alice taking some input directions according to some distribution \( p(x) \), and trying to give Bob the output directions according to \( p(y|x) \), sending only \( I(x, y) \) bits of classical information (down a perfect channel). Despite the similarities, the aim our paper (to investigate the intrinsic directional information of spin-\( \frac{1}{2} \) particles), and theirs (to investigate the classical capacity of quantum channels) were quite different.

[1] S. Massar and S. Popescu, Phys. Rev. Lett. 74, 1259, (1995).
[2] N. Gisin and S. Popescu, Phys. Rev. Lett 83, 432 (1999).
[3] S. Massar, Phys. Rev. A 62, 040101 (2000).
[4] E. Bagan, M. Baig, A. Brey, R. Munoz-Tapia and R. Tarrach, Phys. Rev. Lett. 85, 5230 (2000); Phys. Rev. A 63, 052309 (2001).
[5] A. Peres and P.F. Scudo, Phys. Rev. Lett. 86, 4160 (2001); Phys. Rev. Lett 87 167901 (2001); J. Modern Optics 49, 1235 (2002); quant-ph/0201017.
[6] A.S. Kholevo, Problemy Peredachi Informatsii 9, 3 (1973); for an English translation see Problems of Information Transmission 9, 177 (1973).
[7] C.E. Shannon, Bell System Technical Journal 27, 379 (1948).
[8] B. Schumacher, Phys. Rev. A 51, 2738 (1995).
[9] C.H. Bennett, P. Shor, J.A. Smolin and A.V. Thapliyal, quant-ph/0106052.
[10] T. M. Cover, J. A. Thomas, Elements of Information Theory, Wiley-Interscience, Section 13.6.
[11] C.H. Bennett, D.P. DiVincenzo, P.W. Shor, J.A. Smolin, B.M.Terhal, W.K.Wooters, Phys. Rev. Lett. 87 077902 (2001).
[12] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wooters, Phys. Rev. Lett. 70, 1895-1899 (1993)
[13] C.H. Bennett and S. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[14] W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A 64, 022308 (2001).
[15] A. Winter, quant-ph/0208134.