CROSSED FLUX HOMOMORPHISMS AND VANISHING THEOREMS FOR FLUX GROUPS

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ABSTRACT. We study the flux homomorphism for closed forms of arbitrary degree, with special emphasis on volume forms and on symplectic forms. The volume flux group is an invariant of the underlying manifold, whose non-vanishing implies that the manifold resembles one with a circle action with homologically essential orbits.

1. INTRODUCTION

1.1. Flux homomorphisms. Let $M$ be a closed smooth manifold and $\alpha$ a closed $p$-form on $M$. We shall denote by $\text{Diff}^\alpha$ the group of diffeomorphisms of $M$ which preserve $\alpha$, equipped with the $C^\infty$ topology. Let $\text{Diff}^\alpha_0$ be its identity component. The flux homomorphism associated to $\alpha$ is defined on the universal covering of $\text{Diff}^\alpha_0$ by the formula

$$\text{Flux}_\alpha : \widetilde{\text{Diff}}^\alpha_0 \longrightarrow H^{p-1}(M; \mathbb{R})$$

$$\varphi_t \longmapsto \int_0^1 [i_{\dot{\varphi}_t}, \alpha] \, dt$$

for any path $\varphi_t$ in $\text{Diff}^\alpha_0$ with $\varphi_0 = \text{Id}_M$. It is easy to see that the defining integral depends on the path only up to homotopy with fixed endpoints. Identifying an element of the fundamental group of $\text{Diff}^\alpha_0$ with the corresponding homotopy class of paths based at the identity in the universal covering one obtains a homomorphism

$$\text{Flux}_\alpha : \pi_1(\text{Diff}^\alpha_0) \longrightarrow H^{p-1}(M; \mathbb{R})$$

whose image is the flux group $\Gamma_\alpha$ associated with $\alpha$. The flux homomorphism descends to a homomorphism defined on $\text{Diff}^\alpha_0$, also called the flux:

$$\text{Flux}_\alpha : \text{Diff}^\alpha_0 \longrightarrow H^{p-1}(M; \mathbb{R})/\Gamma_\alpha$$

There are a number of general questions one can ask about this situation, such as whether the flux group $\Gamma_\alpha$ is trivial, or at least discrete, and whether the flux homomorphism can be extended from $\text{Diff}^\alpha_0$ to the whole group $\text{Diff}^\alpha$. As far as we know, these questions have only been considered in the literature in the case when $\alpha$ is a symplectic form, see for example [4, 5, 9, 22, 24, 26, 27, 29, 31, 33] and the papers cited there.

It is the aim of this paper to discuss these questions in some generality. In Section 2 we shall show how certain arguments used in [24] for the case of symplectic forms, mostly on surfaces, can be adapted to the general case, proving the following result:
Theorem 1. Suppose that the total space of every $M$-bundle with structure group $\text{Diff}^\alpha$ has a cohomology class restricting to $[\alpha]$ on the fiber. Then the flux homomorphism

$$\text{Flux}_\alpha : \widetilde{\text{Diff}}^\alpha_0 \rightarrow H^{p-1}(M; \mathbb{R})$$

vanishes on $\pi_1(\text{Diff}^\alpha_0)$ and extends to a crossed homomorphism

$$\widetilde{\text{Flux}}_\alpha : \text{Diff}^\alpha \rightarrow H^{p-1}(M; \mathbb{R}).$$

The crossed flux homomorphism $\widetilde{\text{Flux}}_\alpha$ is a cocycle representing a cohomology class with coefficients in $H^{p-1}(M; \mathbb{R})$ on the group $\text{Diff}^\alpha$ considered as a discrete group, which extends the cohomology class on $\text{Diff}^\alpha_0$ given by the flux homomorphism.

There are many other situations in which we only prove the vanishing of the flux group $\Gamma_\alpha$, without exhibiting a crossed homomorphism extending the flux homomorphism. An important instance of this occurs when $\alpha$ represents a bounded cohomology class, see Theorem 13.

1.2. Volume flux. In Sections 3 and 4 we shall study the flux homomorphism for volume forms $\mu$. We begin by showing that a smooth circle action gives rise to a non-trivial volume flux group if and only if its orbits are homologically essential in real homology. In fact, these are the only known examples of non-trivial volume flux, and it might be possible that there are no others. In dimensions 1 and 2 the only closed manifolds with non-trivial volume flux groups are $S^1$, respectively $T^2$. For these manifolds $\text{Diff}^\mu_0$ is homotopy equivalent to the manifold itself, and the loops in $\text{Diff}^\mu_0$ with non-trivial flux are generated by smooth circle actions. Modulo the Poincaré conjecture, this last statement is also true in dimension 3, as we will show in Section 3.3 by proving the following:

Theorem 2. Let $M$ be a closed oriented 3-manifold without any fake cells. If $M$ has non-zero volume flux group $\Gamma_\mu$, then $M$ is a Seifert fiber space and a multiple of every loop with non-zero volume flux is realized by a fixed-point-free circle action on $M$.

Our guiding paradigm then is to extend to manifolds with a non-trivial volume flux group the known restrictions on manifolds with homologically essential circle actions. The results we have described so far show that a non-trivial volume flux group implies the vanishing of all real characteristic numbers, see Corollary 16 and of the simplicial volume, see Corollary 17. In the case of fixed-point-free circle actions, these results are consequences of the vanishing of the minimal volume. Recall that the minimal volume, introduced by Gromov in [17], is defined by

$$\text{MinVol}(M) = \inf\{\text{Vol}(M, g) \mid g \in \text{Met}(M) \text{ with } |K_g| \leq 1\},$$

where $K_g$ denotes the sectional curvature of $g$. It is known that this is a very sensitive invariant of $M$, which depends on the smooth structure in an essential way. Even the vanishing or non-vanishing of the minimal volume depends subtly on the smooth structure [23]. In the presence of a fixed-point-free smooth circle action on $M$ the minimal volume vanishes because one can shrink a suitable invariant metric in the direction of the orbits, and thus collapse $M$ while keeping the sectional curvatures bounded. It is tempting to speculate that a non-trivial volume flux group is enough to imply the vanishing of the minimal volume, and this would follow if one could prove that circle actions also account for non-trivial volume flux groups in dimensions $> 3$. In order to describe further results in this direction, we need to recall certain notions of entropy.
1.3. **Flux groups and entropies.** Let $M$ be a connected closed oriented manifold of dimension $n$. Elaborating on ideas of Gromov [17], the following lower bounds for the minimal volume of $M$ have been proved, compare [6, 23, 41]:

\[
\frac{n^{n/2}}{n!} ||M|| \leq 2^n n^{n/2} T(M) \leq \lambda(M)^n \leq h(M)^n \leq (n-1)^n \text{MinVol}(M).
\]

Here $T(M)$ is the spherical volume introduced by Besson, Courtois and Gallot [6], and $h(M)$ is the minimal topological entropy of geodesic flows on $M$. We will have nothing to say about these two invariants, but shall be concerned with the other three quantities in (1), namely the simplicial volume $||M||$, the minimal volume entropy or asymptotic volume $\lambda(M)$, and the minimal volume $\text{MinVol}(M)$.

For a Riemannian metric $g$ on $M$ consider the lift $\tilde{g}$ to the universal covering $\tilde{M}$. For an arbitrary basepoint $p \in \tilde{M}$ consider the limit

\[
\lambda(M, g) = \lim_{R \to \infty} \frac{\log \text{Vol}(B(p, R))}{R},
\]

where $B(p, R)$ is the ball of radius $R$ around $p$ in $\tilde{M}$ with respect to $\tilde{g}$, and the volume is taken with respect to $\tilde{g}$ as well. After earlier work by Efremovich, Shvarts, Milnor [35] and others, Manning showed that the limit exists and is independent of $p$. It follows from [35] that $\lambda(M, g) > 0$ if and only if $\pi_1(M)$ has exponential growth. We call $\lambda(M, g)$ the volume entropy of the metric $g$, and define the minimal volume entropy or asymptotic volume of $M$ to be

\[
\lambda(M) = \inf \{ \lambda(M, g) \mid g \in \text{Met}(M) \text{ with } \text{Vol}(M, g) = 1 \}.
\]

This sometimes vanishes even when $\lambda(M, g) > 0$ for every $g$. The normalization of the total volume is necessary because of the scaling properties of $\lambda(M, g)$. Babenko proved that the minimal volume entropy $\lambda(M)$ is invariant under homotopy equivalences, and also under certain bordisms over $B\pi_1(M)$, see [2, 3].

Of all these invariants, only the simplicial volume is known to be multiplicative in coverings. As it will be convenient to allow ourselves passage to finite coverings, we make the following definition in the spirit of [36], compare also [6].

**Definition 3.** Let $I$ be an invariant of $n$-dimensional closed manifolds. Then define

\[
I^*(M) = \inf \left\{ \frac{I(N)}{d} \mid N \text{ a } d \text{-sheeted covering of } M \right\},
\]

where the infimum is taken over all finite coverings of $M$.

Clearly $I^*(M) \leq I(M)$, and if $I(M) \leq J(M)$ for all $M$, then $I^*(M) \leq J^*(M)$. Thus, (1) implies

\[
\frac{n^{n/2}}{n!} ||M|| \leq 2^n n^{n/2} T^*(M) \leq \lambda^*(M)^n \leq h^*(M)^n \leq (n-1)^n \text{MinVol}^*(M).
\]

As we mentioned already, the existence of a smooth circle action without fixed points on $M$ implies that its minimal volume vanishes. Therefore, all the quantities in (1) and (2) vanish. In the case of a non-trivial volume flux group, we are not able to prove the vanishing of the minimal volume, but, in Section 4, we shall prove the following weaker result:

**Theorem 4.** Let $M$ be a closed oriented manifold with non-vanishing volume flux group $\Gamma_{\mu}$. Then $M$ has a finite covering $\tilde{M}$ whose volume entropy $\lambda(M)$ vanishes. In particular, $\lambda^*(M) = 0$. 
The proof uses in an essential way the bordism invariance of the minimal volume entropy $\lambda(M)$, proved by Babenko [3].

1.4. Symplectic flux. In Section 5 we shall consider symplectic forms and their powers, for which we obtain generalizations of some results previously proved in [31] [22] [24]. Unlike in the case of volume forms, where the flux group is always discrete for purely topological reasons, the discreteness of the symplectic flux group was an open problem until very recently. This issue, first raised by Banyaga, has just been resolved by Ono’s proof [38] using methods of hard symplectic topology. Our results in Section 5, like those of some of the references mentioned above, show that very often the symplectic flux group actually vanishes.

2. General properties of the flux and crossed flux homomorphism

For cyclic isotopies representing elements of the fundamental group of $\text{Diff}^\alpha_0$ we can reformulate the definition of the flux as follows.

**Lemma 5.** If $\varphi_t$ is a closed loop representing an element of $\pi_1(\text{Diff}^\alpha_0)$ and $c$ is a cycle representing a homology class in $H_{p-1}(M, \mathbb{Z})$, then up to sign we have

$$\langle \text{Flux}_\alpha(\varphi_t), [c] \rangle = \langle [\alpha], [\varphi_t(c)] \rangle,$$

where $\varphi_t(c)$ denotes the cycle swept out by the loop of diffeomorphisms $\varphi_t$ applied to $c$.

**Proof.** This is immediate from the definition of the flux; compare [31].

As every real cohomology class can be detected by mapping closed oriented manifolds $\Sigma$ into $M$, this lemma shows that the flux group of $\alpha$ is detected by the evaluation of the pullback of $[\alpha]$ on products $S^1 \times \Sigma$. More precisely, define $\phi: S^1 \times \Sigma \to M$ by $\phi(t, x) = \varphi_t(f(x))$, where $f: \Sigma \to M$ is a representative for (a multiple of) $[c]$. Then

$$\langle \text{Flux}_\alpha(\varphi_t), [\Sigma] \rangle = \langle \phi^*[\alpha], [S^1 \times \Sigma] \rangle.$$

One should not be misled by this discussion into thinking that the flux group depends only on the cohomology class of $[\alpha]$, because which loops in $\text{Diff}_0$ can be deformed to essential loops in $\text{Diff}^\alpha_0$ depends on $\alpha$ itself, and not just on its cohomology class.

**Remark 6.** Let $[\varphi_t] \in \pi_1(\text{Diff}^\alpha_0)$, and denote by $E_{\varphi_t} \to S^2$ the $M$-bundle corresponding to $\varphi_t$ via the clutching construction. It is known [29] [40] that the flux homomorphism $\text{Flux}_\alpha: \pi_1(\text{Diff}^\alpha_0) \to H^{p-1}(M, \mathbb{R})$ is equal to the evaluation on $[\alpha]$ of the differential

$$\partial^*_\varphi: H^*(M; \mathbb{R}) \to H^{*+1}(M; \mathbb{R})$$

in the cohomology spectral sequence associated to $E_{\varphi_t}$. More precisely,

$$\langle \text{Flux}_\alpha(\varphi_t), c \rangle = \langle \partial^*_\varphi[\alpha], c \rangle$$

for all $c \in H_{p-1}(M)$.

The diffeomorphism group $\text{Diff}^\alpha$ acts by conjugation on itself and on the universal covering of its identity component. It also acts on cohomology, and this latter action factors through the mapping class group $\mathcal{M}_\alpha$ with respect to $\alpha$, defined to be the quotient group $\text{Diff}^\alpha / \text{Diff}^\alpha_0$. Our first observation is that the flux is equivariant with respect to these actions:
Lemma 7. The flux homomorphism $\text{Flux}_\alpha: \tilde{\text{Diff}}_0^\alpha \rightarrow H^{p-1}(M; \mathbb{R})$ is equivariant with respect to the natural actions of $\text{Diff}^\alpha$. In other words, for any two elements $\psi \in \text{Diff}^\alpha$ and $\varphi_t \in \tilde{\text{Diff}}_0^\alpha$, we have the identity

$$\text{Flux}(\psi \varphi_t \psi^{-1}) = \tilde{\psi}(\text{Flux}(\varphi_t))$$

where $\tilde{\psi} \in M_\alpha$ denotes the mapping class of $\psi$ and $M_\alpha$ acts on $H^{p-1}(M; \mathbb{R})$ from the left by the rule $\tilde{\psi}(w) = (\psi^{-1})^*(w)$ for $w \in H^{p-1}(M; \mathbb{R})$.

This follows immediately from the definition of the flux and the chain rule.

The lemma suggests that one should not expect an extension of the flux to $\text{Diff}^\alpha$ to exist as a homomorphism, but rather as a crossed homomorphism with respect to this action of $\text{Diff}^\alpha$ on cohomology. Indeed, in certain situations we shall prove the existence of an extension as a crossed homomorphism.

Consider the extension

$$1 \rightarrow \text{Diff}_0^\alpha \rightarrow \text{Diff}^\alpha \rightarrow M_\alpha \rightarrow 1$$

and its associated exact sequence of cohomology groups of discrete groups:

$$0 \rightarrow H^1(M_\alpha; H^{p-1}(M; \mathbb{R})/\Gamma_\alpha) \rightarrow H^1(\text{Diff}^\alpha; H^{p-1}(M; \mathbb{R})/\Gamma_\alpha) \rightarrow H^1(\text{Diff}_0^\alpha; H^{p-1}(M; \mathbb{R})/\Gamma_\alpha) \rightarrow 0$$

Lemma 7 shows that we can think of the flux homomorphism as an element

$$\text{Flux}_\alpha \in H^1(\text{Diff}_0^\alpha; H^{p-1}(M; \mathbb{R})/\Gamma_\alpha).$$

Extending the flux to $\text{Diff}^\alpha$ as a crossed homomorphism is equivalent to the vanishing of $\delta(\text{Flux}_\alpha)$ in the above exact sequence. We now examine this issue in detail.

For a foliated $M$-bundle $E \rightarrow B$ whose total holonomy is contained in $\text{Diff}^\alpha$ we have a transverse invariant class $a \in H^p(E; \mathbb{R})$ defined as follows. Pulling back $\alpha$ from $M$ to $\tilde{B} \times M$, we obtain an invariant form $\tilde{\alpha}$ which descends to $E = (\tilde{B} \times M)/\pi_1(B)$ as a closed form. We denote its cohomology class by $a$.

Lemma 8. Let $I = [0, 1]$. For any $\varphi \in \text{Diff}^\alpha$ let $\pi: M_\varphi \rightarrow S^1$ be the foliated $M$-bundle over $S^1$ with monodromy $\varphi$. It is the quotient of $M \times I$ by the equivalence relation $(p, 0) \sim (\varphi(p), 1)$. By assumption, there is an isotopy $\varphi_t \in \text{Diff}_0^\alpha$ such that $\varphi_0 = \text{Id}$ and $\varphi_1 = \varphi$. Let $f: M_\varphi \rightarrow M \times S^1$ be the induced diffeomorphism given by the correspondence

$$M_\varphi \ni (p, t) \mapsto (\varphi_t^{-1}(p), t) \in M \times S^1.$$  

Then the transverse invariant class $a \in H^p(M_\varphi; \mathbb{R})$ is equal to

$$[\alpha] + \text{Flux}_\alpha(\varphi_t) \otimes \nu \in H^p(M \times S^1; \mathbb{R})$$

$$\cong H^p(M; \mathbb{R}) \oplus (H^{p-1}(M; \mathbb{R}) \otimes H^1(S^1; \mathbb{R}))$$

under the above isomorphism, where $\nu \in H^1(S^1; \mathbb{R})$ denotes the fundamental cohomology class of $S^1$.

Proof. The horizontal foliation on $M_\varphi$ is induced from the trivial foliation on $M \times I$. Hence the transverse invariant class $a$ is represented by the form $p_1^* \alpha$ on $M \times I$, where $p_1: M \times I \rightarrow M$ denotes the projection to the first factor. It is clear that the $H^p(M; \mathbb{R})$-component of $a$ is equal to $[\alpha]$, so that we only need to prove that for any $(p-1)$-cycle $c \subset M$, the value of $a$ on the cycle
\( f^{-1}(c \times S^1) \subset M_\varphi \) is equal to \( \text{Flux}(\varphi_\ell)([c]) \) where \([c] \in H_{p-1}(M; \mathbb{Z})\) denotes the homology class of \( c \). Now on \( M \times I \), the above cycle is expressed as the image of the map
\[
c \times I \ni (q, t) \mapsto (\varphi_\ell(q), t) \in M \times I
\]
because \( f^{-1}(q, t) = (\varphi_\ell(q), t) \ ( (q, t) \in M \times S^1 ) \). Hence the required value is equal to the integral of \( \alpha \) over the image of the map
\[
c \times I \ni (q, t) \mapsto \varphi_\ell(q) \in M.
\]
But this is exactly equal to the value of \( \text{Flux}(\varphi_\ell) \) on the homology class represented by the cycle \( c \subset M \). This completes the proof. \( \square \)

For the formulation of our result about extensions of the flux homomorphism as a crossed homomorphism we use the following definition:

**Definition 9.** Let \( M \) be a closed manifold and \( G \subset \text{Diff}(M) \) a subgroup. We say that a cohomology class \( c \in H^*(M) \) extends \( G \)-universally if there is a class \( b \) on the total space of any \( M \)-bundle with structure group \( G \) restricting to \( c \) on the fibers.

With this terminology we have the following precise version of Theorem 1 from the introduction:

**Theorem 10.** Let \( G \subset \text{Diff}^\omega \) be a subgroup. If \( [\alpha] \) extends \( G \)-universally, then the flux homomorphism
\[
\text{Flux}_\alpha : \widetilde{G}_0 \to H^{p-1}(M; \mathbb{R})
\]
vanishes on \( \pi_1(G_0) \) and extends to a crossed homomorphism
\[
\widetilde{\text{Flux}}_\alpha : G \to H^{p-1}(M; \mathbb{R}).
\]

**Proof.** By assumption \([\alpha] \) extends \( G \)-universally, so in particular it extends to the \( M \)-bundle over \( S^2 \) given by the clutching construction for a loop in \( G_0 \). Therefore, \( \text{Flux}_\alpha \) vanishes on \( \pi_1(G_0) \), cf. Remark 6.

Let \( BG^\delta \) be the classifying space of \( G \) considered as a discrete group, and denote by
\[
\pi : EG^\delta \longrightarrow BG^\delta
\]
the universal foliated \( M \)-bundle over \( BG^\delta \) with total holonomy group in \( G \). Let \( b \in H^p(EG^\delta; \mathbb{R}) \) be a universal extension of \([\alpha] \), and consider the difference
\[
u = a - b \in H^p(EG^\delta; \mathbb{R}),
\]
where \( a \) denotes the transverse invariant class represented by the global \( p \)-form \( \tilde{\alpha} \) on \( EG^\delta \) which restricts to \( \alpha \) on each fiber. The restriction of \( u \) to the fiber vanishes, so that, in the spectral sequence \( \{ E^{p,q}_r \} \) for the real cohomology, we have the natural projection
\[
P : \ker \left( H^p(EG^\delta; \mathbb{R}) \to H^p(M; \mathbb{R}) \right) \ni u
\]
\[
\longrightarrow P(u) \in E^{1,p-1}_\infty \subset E^{1,p-1}_2 = H^1(BG^\delta; H^{p-1}(M; \mathbb{R})).
\]
Now Lemma 6 implies that the restriction of \( P(u) \) to the identity component of \( G \) coincides with the flux homomorphism:
\[
P(u) = \text{Flux}_\alpha : G_0 \longrightarrow H^{p-1}(M; \mathbb{R}).
\]
Thus we see that \( P(u) \) defines an extension of the flux homomorphism as a cohomology class whose representing cocycles are crossed homomorphisms. \( \square \)
A particular case where the class $[\alpha]$ does extend universally is when it represents some characteristic class for $M$:

**Corollary 11.** Suppose $[\alpha] \in H^p(M; \mathbb{R})$ is a non-zero multiple of a polynomial in the Euler and Pontryagin classes of $M$. Then the flux group $\Gamma_{\alpha}$ vanishes and the flux homomorphism extends as a crossed homomorphism

$$\widetilde{\text{Flux}}_{\alpha} : \text{Diff}^\alpha \to H^{p-1}(M; \mathbb{R}).$$

**Proof.** For any $M$-bundle $E \to B$ consider the tangent bundle along the fibers. Its characteristic classes extend the characteristic classes of $TM$ from the fiber to the total space. □

If $\alpha$ defines a geometric structure on $M$, then $\text{Diff}^\alpha$ acts by automorphisms of this structure and preserves its characteristic classes. We shall consider the case of a symplectic structure in Section 5 below. Another instance of this is the case of foliations:

**Example 12.** Suppose $\alpha$ is of constant rank, and $TF \subset TM$ is its kernel. Then $\text{Diff}^\alpha$ preserves $TF$, and its characteristic classes extend $\text{Diff}^\alpha$-universally. Therefore, the flux group vanishes and the flux homomorphism extends as a crossed homomorphism if $[\alpha]$ is a non-zero multiple of a polynomial in the Euler and Pontryagin classes of $TF$ and of $TM/TF$.

In general one cannot expect that the extension of the flux homomorphism to a crossed homomorphism is unique (if it exists at all). However, it was proved in [24] that for the case of a symplectic form, equivalently an area form, on a surface of genus $\geq 2$, the extension is unique.

There is another mechanism which can force the vanishing of the flux group $\Gamma_{\alpha}$, stemming from Gromov’s notion of bounded cohomology [17]. As usual, we say that a real cohomology class $[\alpha] \in H^p(M; \mathbb{R})$ is a bounded class, if it has a representative which is bounded as a functional on the set of singular simplices. This means that the class in the image of the comparison map from bounded to usual cohomology:

$$H^p_b(M; \mathbb{R}) \to H^p(M; \mathbb{R}).$$

**Theorem 13.** Suppose the closed $p$-form $\alpha$ represents a bounded cohomology class. Then the flux group $\Gamma_{\alpha}$ vanishes.

**Proof.** Suppose the flux group $\Gamma_{\alpha}$ does not vanish. Then according to Lemma 5 there is a smooth map $\phi : S^1 \times \Sigma^{p-1} \to M$ for which $\phi^* \alpha$ has non-zero integral over $S^1 \times \Sigma$. As $[\alpha]$ is assumed to be a bounded class, so is $\phi^*[\alpha]$. This means that the cohomology generator in top degree on $S^1 \times \Sigma$ is bounded, equivalently the simplicial volume of $S^1 \times \Sigma$ is non-zero. This is clearly false, as $S^1 \times \Sigma$ maps to itself with arbitrarily large degree. □

In this case we obtain the vanishing of the flux group although we do not have a crossed homomorphism extending the flux homomorphism.

### 3. Volume-preserving Diffeomorphisms

In this section we consider the flux of $\alpha$ when $\alpha = \mu$ is a volume form on $M$. In this case Moser’s celebrated result [37] implies that $\text{Diff}^\mu$ is weakly homotopy equivalent to the full diffeomorphism group $\text{Diff}(M)$ of $M$. Moreover, applying Lemma 5 to a homotopy of volume forms of equal total volume, we deduce that, up to normalization, $\Gamma_{\mu}$ is independent of $\mu$. Furthermore, in this situation the mapping class group $\mathcal{M}_{\mu}$ does not depend on $\mu$, as it equals $\text{Diff}(M)/\text{Diff}_0(M)$.

To see some examples and get a feel for what results to expect, we first consider loops of diffeomorphisms generated by circle actions.
3.1. Circle actions. Suppose we are given a smooth effective circle action on an oriented closed manifold $M$. By averaging we can always construct an invariant volume form $\mu$, so that we have a non-trivial homomorphism $S^1 \rightarrow \text{Diff}_0^\mu$. We can easily characterize the non-triviality of the flux on the image of this homomorphism:

**Proposition 14.** A circle action gives rise to a nonzero element in $\Gamma_\mu$ if and only if its orbits are nonzero in real homology.

*Proof.* Let $X$ be the vector field generating the $S^1$-action. Then $L_X \mu = 0$, and $i_X \mu$ is a closed $S^1$-invariant $(n-1)$-form representing the volume flux evaluated on the image of $\pi_1(S^1)$. This form is also a defining form for the (singular) foliation defined by the orbits.

Choose closed $S^1$-invariant 1-forms $\alpha_1, \ldots, \alpha_k$ representing a basis for the first de Rham cohomology. Then the wedge products $\alpha_i \wedge i_X \mu$ are $S^1$-invariant as well, and their cohomology classes vanish for all $i \in \{1, \ldots, k\}$ if and only if the flux vanishes in cohomology. But $\alpha_i \wedge i_X \mu$ is constant along each orbit, and vanishes if and only if $\alpha_i(X)$ vanishes along the orbit. As all the orbits are homologous to each other and fill out the manifold, the flux can only vanish if $\alpha_i(X)$ vanishes identically for all $i$, which is equivalent to the orbits being null-homologous. (Note that it is not possible that $\alpha_i \wedge i_X \mu$ is exact but not identically zero.)

If the orbits are non-trivial in homology, then the action has no fixed points. A closed one-form $\alpha_i$ representing an integral class and evaluating non-trivially on an orbit has $\alpha_i(X) \neq 0$ everywhere, and therefore defines a smooth fibration over $S^1$ with fibers transverse to the circle action. The finiteness of the isotropy groups of the circle action implies finiteness of the monodromy of the fibration over $S^1$. Thus our manifold has a finite cover which splits off $S^1$ smoothly and equivariantly, with the standard circle action. This gives a differential-geometric proof of the Conner–Raymond theorem [12], originally proved by topological means. (Compare [13] for similar arguments.)

Given any fixed-point-free circle action on $M$, Gromov [17] showed that the minimal volume of $M$ vanishes. *A fortiori*, the simplicial volume and the real characteristic numbers of $M$ vanish. Specializing the general theorems of the previous section to volume forms allows us to extend these vanishing results from circle actions to non-trivial volume flux groups. Further, it is well-known that the orbits of circle actions represent central elements of the fundamental group acting trivially on homotopy groups, see for example Browder–Hsiang [8] or Appendix 2 in [17]. We shall generalize these statements in Theorem 15.

In the case of 3-manifolds there are more precise results. Namely, it was proved by Epstein that any fixed-point-free circle action on a closed 3-manifold occurs by rotating the fibers of a Seifert fibration. Moreover, if the orbits are non-trivial in real homology, it is easy to see that the Euler class of the fibration is trivial, so that $M$ is finitely covered by a product of a surface with the circle. We shall show in Theorem 18 below that these circle fibrations account for all non-trivial volume flux elements on 3-manifolds.

3.2. Topological consequences of non-vanishing volume flux groups. We begin with a characterization of the non-triviality of the volume flux group, together with some homotopical constraints.

**Theorem 15.** Let $M$ be any closed $n$-manifold with volume form $\mu$.

(1) The volume flux group $\Gamma_\mu$ is trivial if and only if the evaluation map $\text{ev}: \text{Diff}_0^\mu \rightarrow M$ induces the trivial map on the first real homology.
If $\Gamma_\mu \neq 0$, then $ev_* : \pi_1(\text{Diff}_0^\mu) \to \pi_1(M)$ has an infinite image, which acts trivially on the homotopy groups of $M$. In particular, the center of $\pi_1(M)$ is infinite.

Proof. Suppose $\varphi_t \in \pi_1(\text{Diff}_0^\mu)$ and denote by $M \to E_{\varphi_t} \to S^2$ the bundle associated to $\varphi_t$ by the clutching construction. According to Remark 6, the non-vanishing of $\text{Flux}_\mu(\varphi_t)$ is equivalent to the non-vanishing of $\partial_{\varphi_t}(\mu)$. Applying Poincaré duality in $E_{\varphi_t}$, this in turn is equivalent to the non-triviality of the differential $\partial_{\varphi_t}: H_0(M) \to H_1(M)$. On the other hand, $\partial_{\varphi_t}[pt] = ev_*(\varphi_t)$, where $ev_*: H_1(\text{Diff}_0^\mu) \to H_1(M)$ is the map induced by the evaluation at the point $pt \in M$. This proves the first claim.

Considering the evaluation on the fundamental group, we conclude from what we proved above that it has infinite image. It is a general property of the image of the evaluation that it acts trivially on all homotopy groups. This was first noticed by Gottlieb [15], compare also Theorem 2.2 in [40]. □

In the case of volume forms, Corollary 11 gives the following:

**Corollary 16.** Let $M$ be a closed oriented manifold of dimension $2n$, and $\mu$ a volume form on $M$. If $M$ has a nonzero real characteristic number, then the flux group $\Gamma_\mu$ is trivial, and the flux homomorphism $\text{Flux}_\mu$ extends to a crossed homomorphism $\tilde{\text{Flux}}_\mu : \text{Diff}_0^\mu \to H^{2n-1}(M; \mathbb{R})$.

It was proved in Theorem 2 of [24] that the cohomology class of the extension $\tilde{\text{Flux}}_\mu$ is uniquely determined in $H^1(\text{Diff}_0^\mu; H^1(M; \mathbb{R}))$ if $\mu$ is a volume form on a surface of genus $g \geq 2$. When $M$ has dimension at least 4, and has two different nonzero characteristic numbers, for example the Euler characteristic and the signature, then it may happen that these two choices give rise to different extensions of the flux associated with a volume form. In cohomology, the difference between any two such extensions is in $H^1(M; H^{2n-1}(M; \mathbb{R}))$, where $M = \text{Diff}(M)/\text{Diff}_0(M)$ is the mapping class group of $M$.

Corollary 16 only applies to even-dimensional manifolds. In all dimensions, we have the following special case of Theorem 13:

**Corollary 17.** Let $M$ be a closed oriented manifold with nonzero simplicial volume. Then the volume flux group $\Gamma_\mu$ is trivial for every volume form $\mu$.

### 3.3. The case of 3-manifolds.

We can now prove a precise version of Theorem 2 mentioned in the introduction:

**Theorem 18.** Let $M$ be a closed oriented 3-manifold without any fake cells. If $M$ has nonzero volume flux group $\Gamma_\mu$, then $M$ is a Seifert fiber space. In particular, its minimal volume vanishes. Moreover, a multiple of every loop $\varphi_t \in \pi_1(\text{Diff}_0^\mu)$ with nonzero volume flux is realized by a fixed-point-free circle action on $M$.

Proof. As $\pi_1(M)$ has non-trivial center by Theorem 15, it is indecomposable as a free product. Therefore, in the Kneser–Milnor prime decomposition [34] of $M$, all summands but one are simply connected. As $M$ contains no fake cells by assumption, we conclude that it is prime. Thus, either $M$ is $S^1 \times S^2$, which is a Seifert fibration in the obvious way, or it is irreducible.
If $M$ is irreducible, then because its first Betti number is positive, it is also sufficiently large\(^1\), meaning that $M$ contains an incompressible surface, cf. \cite{21} p. 35. Now it is a theorem of Waldhausen \cite{45} that a closed irreducible sufficiently large 3-manifold $M$ such that $\pi_1(M)$ has non-trivial center is Seifert fibered.

By shrinking a suitable invariant metric in the direction of the circle action, one sees that the minimal volume vanishes, cf. \cite{17}.

On a Seifert manifold every element of the center of the fundamental group is, up to a multiple, represented by the fiber of a Seifert fibering, cf. \cite{21} p. 92/93 or \cite{43}, and therefore by a circle action. Thus a multiple of the evaluation of a loop $\varphi_\ell \in \pi_1(Diff^*_0)$ with non-trivial volume flux is homotopic in $M$ to the orbits of a circle action. It remains to show that $\varphi_\ell$ and the loop given by the circle action are homotopic in $Diff^*_0$, before we apply the evaluation. For this we distinguish the two cases we encountered above: either $M$ is $S^1 \times S^2$, or it is irreducible.

If $M$ is $S^1 \times S^2$, we can use the work of Hatcher \cite{18} \cite{19}, who determined the homotopy type of $Diff(S^1 \times S^2)$ completely. This shows in particular that the free part of the fundamental group of $Diff_0$, and therefore of $Diff^*_0$, is generated by the circle action $S^1 \hookrightarrow Diff_0(S^1 \times S^2)$ given by rotation of the first factor.

If $M$ is irreducible, consider the space $HEquiv(M)$ of self-homotopy equivalences of $M$, endowed with the compact-open topology, and the following composition of maps:

$$Diff^*_0(M) \xrightarrow{i} Diff_0(M) \xrightarrow{j} HEquiv_0(M) \xrightarrow{ev} M,$$

where $HEquiv_0(M)$ denotes the connected component of the identity. The first map is a weak homotopy equivalence by Moser’s theorem \cite{37}. For an irreducible sufficiently large 3-manifold $M$, Laudenbach \cite{28} proved that $j_*$ is an isomorphism on fundamental groups\(^2\). An irreducible 3-manifold with infinite fundamental group is aspherical by the sphere theorem, see \cite{34}, and it is true for any aspherical manifold that $ev: HEquiv_0(M) \rightarrow M$ induces an isomorphism between $\pi_1(HEquiv_0(M))$ and the center of $\pi_1(M)$, compare \cite{15}. Thus, two loops in $Diff^*_0$ having homotopic evaluations in $M$ are homotopic in $Diff^*_0$. This completes the proof of the theorem.

\begin{remark}
The Seifert fibered 3-manifolds occuring in the theorem carry Thurston geometries of type $S^2 \times \mathbb{R}$ in the case of $S^1 \times S^2$, and of type $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ in the irreducible case, compare \cite{43}. The center of $\pi_1(M)$ is $\mathbb{Z}$ if $M$ is $T^3$, and is $\mathbb{Z}$ otherwise. For Seifert manifolds which are not circle bundles over a surface, the generator of the center can not be realised by a circle action, so that passing to multiples is unavoidable. We shall see a similar phenomenon in higher dimensions in Example \cite{22} below.
\end{remark}

### 3.4. Back to higher dimensions

The proof of Theorem \cite{18} has a partial generalization to higher dimensions:

\begin{theorem}
Let $M$ be a closed $n$-manifold with $\Gamma_\mu \neq 0$. If $M$ is homotopy equivalent to a connected sum $M_1 \# M_2$ then one of the $M_i$ is a homotopy sphere.
\end{theorem}

\begin{proof}
As $\pi_1(M)$ has non-trivial center by Theorem \cite{15} it is indecomposable as a free product. Therefore, one of the $M_i$, say $M_1$, is simply connected. In particular, $H_1(M_1; \mathbb{Z}) = 0$. If $M_1$ is not a homotopy sphere, then there is a smallest $k \leq n - 2$ for which $H_k(M_1; \mathbb{Z})$ does not vanish. By the Hurewicz theorem $\pi_k(M_1) \cong H_k(M_1; \mathbb{Z}) \neq 0$.
\end{proof}

\(^1\)Thus $M$ is “Haken”.

\(^2\)Laudenbach \cite{28} assumed the Smale conjecture, subsequently proved by Hatcher \cite{19}.
Now the universal cover of $M$ is obtained from the universal cover of $M_2$ by connected summing with infinitely many copies of $M_1$. Every non-trivial element of $\pi_1(M) \cong \pi_1(M_2)$ acts non-trivially on $H_k(M)$ by permuting the different summands coming from the different copies of $M_1$. This shows that every element of the fundamental group acts non-trivially on $\pi_k(M)$, contradicting the second part of Theorem 15.

Remark 21. Theorem 20 is new even in the case when the non-triviality of the volume flux group arises from a circle action. See [13] for partial results in this direction.

The following example shows that Theorem 20 is sharp.

Example 22. Let $M = (S^1 \times S^6) \# \Sigma^7$, with $\Sigma$ a homotopy 7-sphere. Now every $\Sigma$ is a twisted sphere, i.e. it is of the form $D^7 \cup_{\psi} D^7$ for some $\psi$ in the mapping class group of $S^6 = \partial D^7$. But then $M$ is just the mapping torus of $\psi$, and as the mapping class group of $S^n = \partial D^{n+1}$ is finite (of order 28), we conclude that $M$ fibers over $S^1$ with finite monodromy ($= \psi$). There is a fixed-point-free circle action transverse to this fibration generating a non-trivial volume flux group by Proposition 14.

Note that the generator of $\pi_1(M) \cong \mathbb{Z}$ cannot always be realized by the orbits of a circle action on $M$, so that passing to multiples is unavoidable. Indeed, if an $S^1$-action on $M$ surjects $\pi_1(S^1)$ onto $\pi_1(M)$, then all the orbits have trivial stabilizer, because their homotopy classes are primitive. Then the quotient map $M \to M/S^1$ is a smooth circle bundle over a homotopy 6-sphere. This bundle is trivial, and as there are no exotic 6-spheres we conclude that $M$ is diffeomorphic to $S^1 \times S^6$. But according to Browder [7], Corollary 2.8, we can choose $\Sigma$ in such a way that $M$ is not diffeomorphic to $S^1 \times S^6$.

This example shows that the topological manifold $S^1 \times S^6$ has several distinct smooth structures, all of which admit fixed-point-free circle actions. There are also pairs of homeomorphic manifolds for which one has a free smooth circle action, and the other one has no smooth circle action at all:

Example 23. Let $M = T^7 \# \Sigma^7$, with $\Sigma$ a homotopy 7-sphere. Whenever $\Sigma$ is not the standard sphere, Assadi and Burghelea [11] showed that $M$ admits no effective smooth circle action.

We do not know whether the volume flux group is non-trivial in this case, or not. See Section 6.1 below for further discussion.

4. Entropy and Volume Flux

We have seen that a non-trivial volume flux group forces the vanishing of the simplicial volume, and the vanishing of all real characteristic numbers. In the case when the volume flux comes from a smooth circle action, these vanishing results are consequences of the vanishing of the minimal volume, compare [11]. One might therefore speculate that the non-vanishing of the volume flux may imply the vanishing of the minimal volume. As we are not able to prove this, we consider the intermediate invariants from [11], which interpolate between the simplicial volume and the minimal volume. The strongest vanishing result we can prove about them is Theorem 4. The remainder of this section is occupied by the proof of this theorem.

If the volume flux group $\Gamma_\mu$ for $(M, \mu)$ is non-trivial, then by Theorem 15 the evaluation at a point of the corresponding loop $\varphi_t$ in $\text{Diff}_\mu$ gives us a loop which is of infinite order in the center of $\pi_1(M)$ and in $H_1(M; \mathbb{Z})$. After replacing $M$ by a finite cover, we may assume that this element is primitive in $H_1(M; \mathbb{Z})/\text{tor}$, so that the fundamental group of $M$ splits as a direct product $\pi_1(M) \cong \mathbb{Z} \times \pi$ with the generator of the first factor corresponding to the evaluation
of our loop of diffeomorphisms, compare [39]. In this situation, Gottlieb [16] and independently Oprea [39] proved a homotopical analogue of the Conner–Raymond splitting theorem, showing that \( M \) is homotopy equivalent to \( S^1 \times Y \), where \( Y \) is the homotopy fiber of a map \( f: M \to S^1 \) inducing the projection \( p_1: \pi_1(M) \to \mathbb{Z} \) onto the first factor of the fundamental group. If the homotopy type \( Y \) can be represented by a closed oriented \((n - 1)\)-manifold, then we conclude the proof of Theorem 3 by noting that \( \lambda(S^1 \times Y) \) vanishes because of the obvious circle action, and the minimal entropy is known to be homotopy-invariant by a result of Babenko [2].

Regardless what the homotopy fiber \( Y \) is, we can proceed as follows. Choose a smooth map \( f: M \to S^1 \) with \( f_* = p_1 \) and let \( F' \) be a regular fiber. Then \( \pi_1(F') \) surjects onto \( \pi \cong \text{Ker } f_* \), and we can modify \( F' \) by ambient surgery\(^3\) inside \( M \) to obtain an embedded submanifold \( F \subset M \) in the same homology class, such that \( \pi_1(F) \cong \pi \), and the inclusion induces an isomorphism between \( \pi_1(F) \) and \( 0 \times \pi \subset \mathbb{Z} \times \pi \cong \pi_1(M) \). Consider then the map \( \Phi \) given by the composition

\[
S^1 \times F \to S^1 \times M \xrightarrow{I_d \times \phi} S^1 \times M \xrightarrow{\pi_2} M
\]

\[
(t, x) \mapsto (t, x) \mapsto (t, \varphi_t(x)) \mapsto \varphi_t(x)
\]

where the first map is the inclusion, and the composition of the second and third maps is the evaluation.

**Lemma 24.** The map \( \Phi \) has the following properties:

1. It induces an isomorphism on fundamental groups.
2. It has degree one.
3. It pulls back the tangent bundle of \( M \) to the tangent bundle of \( S^1 \times F \).

**Proof.** The first claim is clear from the construction of \( F \) and \( \Phi \). The second claim follows from the fact that \( F \subset M \) has algebraic intersection number \( 1 \) with the evaluation loop.

For the third claim, consider the factorization \( \Phi = \pi_2 \circ (I_d \times \varphi) \circ i \). The diffeomorphism \( I_d \times \varphi \) pulls back \( \pi_1^*TM \) to itself. But this bundle restricts to the image of \( i \) as \( \mathbb{R} \oplus TF \), which proves the claim. \( \square \)

Let \( c: M \to B\pi_1(M) \) be the classifying map for the universal cover of \( M \), and consider the classes \([M, c]\) and \([S^1 \times F, c \circ \Phi]\) in the bordism group \( \Omega_n(B\pi_1(M)) \). If these bordism classes agree, then there is a bordism \([W', \alpha']\) between them. It follows that \( \alpha': \pi_1(W') \to \pi_1(M) \) is surjective, and we can modify the bordism by surgery in the interior of \( W' \) so as to obtain a new bordism \([W, \alpha]\) for which \( \alpha: \pi_1(W) \to \pi_1(M) \) is an isomorphism. This new bordism has the property that the inclusion of each boundary component into \( W \) induces an isomorphism on fundamental groups, which is the defining property of an \( R \)-cobordism in the terminology of Babenko [3]. The result of [3] is that \( R \)-cobordant manifolds have the same minimal volume entropy. As before, the volume entropy of \( S^1 \times F \) vanishes because its minimal volume vanishes courtesy of the circle action, cf. [1].

It remains to prove that the bordism classes \([M, c]\) and \([S^1 \times F, c \circ \Phi]\) agree—or to deal with their failure to do so. Consider first the case when the integral homology of \( B\pi_1(M) \) has no odd-order torsion. Then the bordism spectral sequence for \( B\pi_1(M) \) is trivial, and we have an isomorphism

\[
\Omega_n(B\pi_1(M)) \cong \bigoplus_{i=0}^{n} H_i(B\pi_1(M); \Omega_{n-i}(\ast))
\]

\(^3\)At this point it is useful to assume that \( \text{dim}(F') > 2 \), equivalently \( \text{dim}(M) \geq 4 \). This is no loss of generality, as we have a much stronger conclusion for small dimensions by a different argument, see Theorem 18.
compare Theorem 15.2 in [11], or [44]. The elements of the summands on the right-hand side are detected by the collection of all Pontryagin and Stiefel-Whitney numbers twisted by cohomology classes on $B\pi_1(M)$. Lemma 24 shows that these twisted characteristic numbers agree for $[M, c]$ and $[S^1 \times F, c \circ \Phi]$, so these two bordism classes in $\Omega_n(B\pi_1(M))$ agree.

In the general case, when the homology of $B\pi_1(M)$ is allowed to have odd-order torsion, we can still find the required bordism between $[M, c]$ and $[S^1 \times F, c \circ \Phi]$ after passing to a suitable finite cover induced from a finite cover of $S^1$. For this we only have to prove that on such a cover the map $\Phi: S^1 \times F \to M$ is bordant to the identity of $M$.

**Proposition 25.** In the above situation $M$ has a finite covering induced from a finite covering of $S^1$ via the map $M \to S^1$, such that the lift of $\Phi$ to the corresponding covering of $S^1 \times F$ by itself is bordant to the identity of the target.

**Proof.** We shall use the language of surgery theory. First we recall a few basic facts from this theory, see e.g. [30].

Let $O = \lim_{n \to \infty} O_n$ denote the infinite orthogonal group and let $BO$ denote its classifying space. Also let $G_n$ denote the group consisting of homotopy equivalences of $S^{n-1}$ equipped with the compact-open topology. Let $BG$ be the classifying space of $G = \lim_{n \to \infty} G_n$, which classifies stable isomorphism classes of spherical fibrations. The homotopy groups $\pi_i(G)$ can be canonically identified with the stable homotopy groups of spheres $\pi_i(G) \cong \lim_{k \to \infty} \pi_{i+k}(S^k)$.

Hence $\pi_i(G)$ is finite for all $i$. There exists a canonical map $BO \to BG$, which corresponds to associating the unit sphere bundle to a vector bundle. We have a fibration

$$G/O \to BO \to BG$$

which can be extended to the left as

$$G \to G/O \to BO \to BG.$$

Now let $X$ be a closed oriented smooth manifold and let $\nu(X)$ be its stable normal bundle. A commutative diagram

$$\begin{array}{ccc}
\nu(N) & \xrightarrow{\tilde{f}} & \nu(X) \\
\downarrow & & \downarrow \\
N & \xrightarrow{f} & X,
\end{array}$$

where $f: N \to X$ is a degree 1 map from a closed oriented manifold $N$ to $X$ and $\tilde{f}: \nu(N) \to \nu(X)$ is a bundle map, is called a normal map over $X$. There exists a notion of normal cobordism, which is an equivalence relation on the set of all normal maps over $X$. The set of all the normal cobordism classes of normal maps over $X$ is denoted by $NM_O(X)$. The map

$$\sigma: NM_O(X) \to [X, G/O]$$

is called the surgery map. It is known to be a bijection by the Pontryagin–Thom construction, see [30], Theorem 2.23. A normal map $(\tilde{f}, f)$ as in (4) is normally cobordant to the identity map of $X$ if and only if $\sigma$ sends the corresponding class $[(\tilde{f}, f)] \in NM_O(X)$ to the homotopy class of the constant map in $[X, G/O]$. 

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Now we apply this general framework to our map \( \Phi : S^1 \times F \to M \). First of all, we already know from Lemma 24 that \( \Phi \) is a tangential equivalence, so that it is covered by a bundle map \( \tilde{\Phi} : T(S^1 \times F) \to TM \). Taking the stable normal bundles instead of the tangent bundles, we obtain a normal map

\[
\begin{array}{ccc}
\nu(S^1 \times F) & \xrightarrow{\tilde{\Phi}^\nu} & \nu(M) \\
\downarrow & & \downarrow \\
S^1 \times F & \xrightarrow{\Phi} & M
\end{array}
\]

over \( M \). Consider the surgery map

\[\sigma : NM_0(M) \to [M, G/O]\]

and let \( \alpha \in NM_0(M) \) be the element represented by the normal map (5). Our task is to show that, after passing to a suitable finite cover, the image \( \sigma(\alpha) \) is the homotopy class in \([M, G/O]\) represented by the constant map.

Passing to finite coverings along \( S^1 \) means that we consider \( \Phi_k : S^1 \times F \to M_k \), where the domain is the product of \( F \) with the standard \( k \)-fold cover of the circle, and \( M_k \) is the corresponding covering of \( M \) pulled back from the circle. Since \( \Phi_k \) has degree 1, we see that

\[\Phi_k^* : H^i(M_k) \to H^i(S^1 \times F)\]

is injective with any coefficients. Moreover, by restriction we have an injection

\[\Phi_k^* : H^i(M_k, F) \to H^i(S^1 \times F, pt. \times F) \cong \nu \otimes H^{i-1}(F),\]

where \( \nu \) denotes the generator of \( H^1(S^1) \) and \( F \) sits inside \( M_k \) in the obvious way for any \( k \).

Now \( \Phi \) is a tangential equivalence, and therefore a map \( M \to G/O \) representing \( \sigma(\alpha) \) can be lifted to \( g : M \to G \). Moreover, this map is constant on \( F \subset M \) because \( \Phi \) is the identity on \( F \). It suffices to show that \( g \) is homotopic to the constant map, at least after we pass to a suitable finite covering. The obstructions to \( g \) being null-homotopic are contained in

\[H^i(M, F; \pi_i(G)) \subset \nu \otimes H^{i-1}(F; \pi_i(G)).\]

But the group \( \pi_i(G) \) is finite for every \( i \), and if we pass to the finite covering \( M_k \), then the classifying map of the new surgery problem is just the composition

\[M_k \to M \xrightarrow{g} G,\]

as can be seen by inspecting the construction of the surgery map, cf. pp. 42-43 of [30]. Therefore we can kill the obstructions by taking suitable finite covers along \( S^1 \). In detail, the coverings send \( \nu \) to \( \nu_k \), and since the obstructions lie in \( \nu \otimes H^{i-1}(F) \) with finite coefficients, we can kill all the obstructions. This means that the normal map \( S^1 \times F \to M_k \) is normally bordant to the identity of \( M_k \).

The proof of Proposition 25 shows that \( S^1 \times F \) and \( M_k \) are bordant over \( B\pi_1(M_k) \). Together with the preceding discussion, this completes the proof of Theorem 4.

Remark 26. An alternative approach to the general case proceeds by observing that (3) always holds after tensoring with \( \mathbb{Q} \), cf. [11, 44]. As \([M, c]\) and \([S^1 \times F, c \circ \Phi]\) have the same twisted Pontryagin numbers, their difference is rationally zero-bordant. This means that for some \( k > 0 \) there is a bordism between the \( k \)-fold connected sums \( kM \) and \( k(S^1 \times F) \) endowed with the corresponding maps to \( B\pi_1(M) \). These maps induce the diagonal map \( D : \pi_1(M) \times \ldots \pi_1(M) \to \ldots\)
$\pi_1(M)$ on fundamental groups. Unfortunately it is unclear whether this can be arranged to be an $R$-cobordism in the sense of Babenko [3]. If this is possible, then a slightly different proof of Theorem 4 can be given as follows. Babenko’s theorem [3] implies that the minimal asymptotic exponential volume growth rates of the covers of $kM$ and of $k(S^1 \times F)$ with fundamental groups $\text{Ker}(D)$ are equal. (These are not the minimal volume entropies, because these covers are not the universal covers.) Now by a result of Paternain and Petean [41], Theorem 5.9, the circle action on $S^1 \times F$ gives rise to a $T$-structure on the connected sum $k(S^1 \times F)$. Another result of the same authors, Theorem A in [41], shows that the minimal topological entropy $h$ vanishes for any manifold with a $T$-structure. A fortiori, the minimal volume entropy of $k(S^1 \times F)$ vanishes, compare (1). This implies that the intermediate cover of $k(S^1 \times F)$ with fundamental group $\text{Ker}(D)$ also has slow volume growth. By the above discussion we have this conclusion also for the cover of $kM$ with fundamental group $\text{Ker}(D)$. This cover essentially contains a copy of the universal cover of $M$, which therefore has small minimal asymptotic exponential volume growth rate. Thus, $\lambda(M) = 0$.

5. Powers of a symplectic form

In this section we consider the case when $\alpha$ is a power $\omega^k$ of a symplectic form $\omega$ on $M$, with $M$ of dimension $2n$. It is clear that $\text{Diff}^\alpha$ contains the symplectomorphism group $\text{Symp} = \text{Diff}^\omega$, but is usually strictly larger when $k > 1$. In order to obtain a result parallel to Corollary 16 we want to use the Chern classes of the tangent bundle along the fibers in the universal foliated $M$-bundle. This means that instead of $\text{Diff}^\alpha$ we should only consider a smaller group which preserves the homotopy class of an almost complex structure compatible with $\omega$. We will simply take the symplectomorphism group and consider the $k$-flux

$$\text{Flux}_k : \text{Symp}_0 \rightarrow H^{2k-1}(M; \mathbb{R}) ,$$

which is the restriction of the flux with respect to $\omega^k$ to the symplectomorphism group. We denote by $\Gamma_k$ the image of $\pi_1(\text{Symp}_0)$ under the $k$-flux.

The groups $\Gamma_k$ for different values of $k$ are related to each other by the equation

(6)  $$\text{Flux}_k(\varphi) = k \cdot \text{Flux}_1(\varphi) \wedge \omega^{k-1} ,$$

which is immediate from the definition of the flux and the identity $i_X(\omega^k) = k \cdot i_X \omega \wedge \omega^{k-1}$. Thus $\Gamma_k$ is the image of the usual symplectic flux group $\Gamma_1 = \Gamma_\omega$, under multiplication by $k\omega^{k-1}$. Taking $k = n$, we can use this to draw consequences about the symplectic flux group from our results about the volume flux. Note that $\Gamma_n$ is not really the volume flux group, because we are only considering $\text{Symp}$, and not the usually larger $\text{Diff}^\omega$. Nevertheless, the same arguments apply to prove the following:

**Theorem 27.** Let $(M, \omega)$ be a closed symplectic manifold of dimension $2n$ that satisfies one of the following conditions:

1. the evaluation map $ev : \text{Symp}_0 \rightarrow M$ induces the trivial map on the first real homology, or
2. the fundamental group $\pi_1(M)$ has finite center, or
3. $M$ has a nonzero real characteristic number, or has nonzero renormalized minimal volume entropy $\lambda^*(M)$, or
4. $M$ is homotopy equivalent to a connected sum in which neither summand is a homotopy sphere.


Then the symplectic flux group $\Gamma_\omega \subset H^1(M; \mathbb{R})$ is in the kernel of the multiplication map

$$[\omega]^{n-1} : H^1(M; \mathbb{R}) \to H^{2n-1}(M; \mathbb{R}).$$

In particular, if $(M, \omega)$ also satisfies the hard Lefschetz condition, then the symplectic flux group vanishes.

Instead of taking the maximal power of the symplectic form, we can consider smaller powers, and we can also use the Chern classes of an almost complex structure compatible with the symplectic form. Theorem 10 has the following immediate consequence:

**Corollary 28.** Suppose that $[\omega^k] \in H^{2k}(M; \mathbb{R})$ is proportional to a polynomial in the Chern classes of $(M, \omega)$. Then $\Gamma_k = 0$, and $\text{Flux}_k$ extends to a crossed homomorphism

$$\widetilde{\text{Flux}}_k : \text{Symp} \to H^{2k-1}(M; \mathbb{R}).$$

This result has an antecedent in McDuff’s paper [31]. The case $k = 1$ was proved in [24]. As before, using (6) we obtain the following consequence. The case when $k = n$ was previously proved in [22].

**Corollary 29.** Suppose that $[\omega^k] \in H^{2k}(M; \mathbb{R})$ is proportional to a polynomial in the Chern classes of $(M, \omega)$. Then the symplectic flux group $\Gamma_1 = \Gamma_\omega \subset H^1(M; \mathbb{R})$ is in the kernel of the multiplication map

$$[\omega]^{k-1} : H^1(M; \mathbb{R}) \to H^{2k-1}(M; \mathbb{R}).$$

If $(M, \omega)$ satisfies a weak form of the Lefschetz property, namely if multiplication by $[\omega]^{k-1}$ is injective, then the usual symplectic flux group is trivial.

**Example 30.** Consider $M = F \times S^2$, where $F$ is a surface of genus $g \neq 1$. Then every cohomology class with nonzero square in $H^2(M; \mathbb{R})$ is realised by a split symplectic form, with the symplectic area of the factors scaled suitably. For all these symplectic forms the Chern classes are the same, namely $c_1 = (2 - 2g)P.D.[S^2] + 2P.D.[F]$ and $c_2 = (4 - 4g)P.D.[M]$. For those symplectic forms $\omega$ whose cohomology class is a multiple of $c_1$, the case $k = 1$ of Corollary 28 implies the triviality of the flux group $\Gamma_\omega$. When $[\omega]$ is not a multiple of $c_1$, we can use the case $k = 2$ and the fact that $c_2^2$ and $c_2$ are nonzero to conclude that $\Gamma_2$ is trivial⁴. As $M$ satisfies the hard Lefschetz property for every $\omega$, we again conclude the vanishing of $\Gamma_\omega$.

**Example 31.** Let $M = F \times S^2$ as before, with $g \neq 1$. Then the non-vanishing of the Chern numbers $c_2^2$ and $c_2$ gives rise to two potentially different extensions

$$\widetilde{\text{Flux}}_2 : \text{Symp} \to H^3(M; \mathbb{R}).$$

The difference between them corresponds to the difference $c_2^2(\xi) - 2c_2(\xi) \in H^4(E\text{Symp}; \mathbb{R})$, which restricts trivially to the fiber $M$. There are symplectic bundles with fiber $M$ which show that this difference class is non-trivial if $g = g(F) \geq 3$. Namely, let $X \to B$ be an $F$-bundle with nonzero signature. Then $X \times S^2$ is an $M$-bundle over $B$ for which $c_2^2(\xi) - 2c_2(\xi) \neq 0 \in H^4(X \times S^2; \mathbb{R})$.

However, the two extensions of $\text{Flux}_2$ are essentially the same. As $M$ satisfies the hard Lefschetz property, these extensions of $\text{Flux}_2$ are given by extensions of the usual flux homomorphism multiplied by the symplectic form. But the extensions of the usual flux homomorphism here come from $F$, where we know that the extension is unique as a cohomology class, see [24].

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⁴Alternatively we can use the fact that $\pi_1(M)$ has trivial center.
Example 32. Consider now $M = T^2 \times S^2$ with a split symplectic form. Then clearly the flux group is non-trivial. However, if we pass from $M$ to its blowup $\hat{M} = M \# \mathbb{CP}^2$, then $\hat{M}$ is reducible (and has nonzero Chern numbers), so that $\Gamma_2$ must vanish. As $\hat{M}$ satisfies the hard Lefschetz property, we conclude that the usual flux group $\Gamma_1 = \Gamma_\omega$ also vanishes.

Theorem 13 has the following consequence for the symplectomorphism groups.

Corollary 33. Suppose that $[\omega]^k$ is a bounded cohomology class. Then $\Gamma_k = 0$. In particular, if $\omega$ represents a bounded class, then the usual symplectic flux group is trivial.

It is interesting to compare the above vanishing results for the symplectic flux group with the following:

Proposition 34. (cf. [26, 42]) Assume that $(M, \omega)$ is symplectically aspherical, i.e. $\omega|_{\pi_2(M)} = 0$, and that $\pi_1(M)$ has finite center. Then $\Gamma_\omega = 0$.

Proof. As the center of $\pi_1(M)$ is finite, and the flux is multiplicative when we replace a loop by a multiple, we may assume that the evaluation of a loop $\varphi_t$ in $\text{Symp}_0$ whose flux we want to test bounds a 2-disk $D$ in $M$. If $\gamma \subset M$ is any closed loop, the degree 2 homology class of $\varphi_t(\gamma)$ can be represented by a 2-sphere $S$ obtained by surgering the torus along a meridian, using two copies of $D$. Now $\omega|_{\pi_2(M)} = 0$ implies that

$$\langle \text{Flux}_\omega(\varphi_t), [\gamma] \rangle = \int_{\varphi_t(\gamma)} \omega = \int_S \omega = 0.$$  

Example 35. Consider a surface bundle $X$ over a surface $B$, such that both the base $B$ and the fiber $F$ have genus $\geq 2$. Then the second Chern number is nonzero, and the center of $\pi_1(X)$ is trivial, so that $\Gamma_2 = 0$ by Theorem 27. On the one hand, there are many such $X$ which cannot satisfy the hard Lefschetz property (for any $\omega$), so that we cannot conclude the vanishing of $\Gamma_\omega$ from Theorem 27. On the other hand, it is always possible to choose $\omega$ in such a way that it represents a bounded cohomology class, in which case Corollary 33 implies the vanishing of the flux group.

Again this last argument does not cover all cases, because for suitable surface bundles $X$ one can also choose $\omega$ so that its cohomology class is not bounded. Nevertheless, Proposition 34 always applies, because $X$ is aspherical and its fundamental group has trivial center.

6. Final comments and remarks

6.1. Does the volume flux group depend on the smooth structure? So far we do not know whether the non-triviality of the volume flux group depends on the smooth structure, or not. Examples 22 and 23 are the closest we have come to seeing a dependence on the smooth structure, but these example are not conclusive. Most of the information we have derived from the non-triviality of the volume flux group, about the fundamental group, homotopical irreducibility, simplicial volume, and about the minimal volume entropy, is homotopy invariant. However, this is not known for the minimal topological entropy, and is definitely false for the minimal volume. Theorem 2 of [23] shows that in dimension 4 there exist homeomorphic manifolds such that one has vanishing minimal volume and the other one does not. It is also known that the minimal volume depends on the smooth structure in higher dimensions.

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5This is implicit in [20], where it was used to show that $X$ always has positive simplicial volume.
6.2. **Remarks on Gottlieb groups.** Recall that the Gottlieb group $G(M)$ of a manifold $M$ is the image of the evaluation homomorphism $ev_* : \pi_1(M^M) \to \pi_1(M)$, see [15] and [29, 40]. It will be clear to the experts that some of our arguments concerning volume flux groups depend only on the fact that a loop of diffeomorphisms having non-trivial volume flux gives an element of infinite order in the Gottlieb group. Indeed, the non-triviality of the Gottlieb group is enough to conclude that the Euler characteristic of $M$ vanishes, see [15], and for the irreducibility conclusion of Theorem 20. It is not clear whether the other consequences of a non-trivial volume flux follow from Gottlieb theory alone. If $M$ has a Gottlieb element whose image under the Hurewicz map has infinite order in homology, then the simplicial volume of $M$ vanishes. However our proof of the vanishing of the minimal volume entropy certainly does not apply in this generality.

Once again the situation is better for 3-manifolds. If a closed 3-manifold without any fake cells has non-trivial Gottlieb group, then it is Seifert fibered, and up to multiples the elements of the Gottlieb group are represented by circle actions. This follows from the Seifert fiber space conjecture, the final cases of which were settled by Casson–Jungreis [10] and Gabai [14] independently. Our proof of Theorem 18 did not need these deep results, because the existence of non-trivial volume flux implies that the manifold is Haken.

6.3. **Further developments.** Extended flux homomorphisms arose first in [24] for the case of monotone symplectic forms. There, a vanishing theorem for flux groups was proved as a byproduct of the search for extended flux homomorphisms, whereas in the present paper we obtain many more vanishing theorems in the general situation, where an extended flux homomorphism may not necessarily exist.

The results of [24, 25] illustrate how extended flux homomorphisms can help in understanding the homology of the groups $\text{Diff}^\alpha$ as discrete groups. For many of the situations where we have proved the existence of extensions of the flux as a crossed homomorphism in this paper, one can try to imitate the constructions of [25] in particular in order to find new non-trivial cohomology classes on diffeomorphism groups made discrete.

If an extended flux homomorphism $\widetilde{\text{Flux}}_\alpha$ exists, then its kernel is an interesting subgroup of $\text{Diff}^\alpha$ intersecting all connected components. In the special case when $\alpha$ is a symplectic form, this subgroup was recently studied by McDuff in [32]. She gives both an alternative proof of Theorem 1 for symplectic forms, and some elaborations on it.

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