A GLOBAL VERSION OF THE KOON-MARSDEN JACOBIATOR FORMULA

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Abstract. In this paper we study the Jacobiator (the cyclic sum that vanishes when the Jacobi identity holds) of the almost Poisson brackets describing nonholonomic systems. We revisit the local formula for the Jacobiator established by Koon and Marsden in [16] using suitable local coordinates and explain how it is related to the global formula obtained in [1], based on the choice of a complement to the constraint distribution. We use an example to illustrate the benefits of the coordinate-free viewpoint.

Dedicated to the memory of J.E. Marsden

Contents

1. Introduction 1
2. Nonholonomic systems 2
2.1. The hamiltonian viewpoint 2
2.2. The nonholonomic bracket 3
2.3. The Jacobiator formula 4
3. The Koon-Marsden adapted coordinates 4
4. The coordinate version of the Jacobiator formula 5
4.1. Interpretation of the adapted coordinates 5
4.2. The Jacobiator in adapted coordinates 7
5. Example: the snakeboard 9
References 11

1. Introduction

The geometric approach to nonholonomic systems was among the many research interests of J. E. Marsden, and his contributions to this area were fundamental. A system with nonholonomic constraints can be geometrically described by an almost Poisson bracket [11, 14, 19], whose failure to satisfy the Jacobi identity, measured by the so-called Jacobiator, is precisely what encodes the nonholonomic nature of the system. There is a vast literature on the study of such nonholonomic brackets and their properties, starting with the early work of Chaplygin [6], see e.g. [2, 11, 13, 8, 5, 10, 12]. An explicit formula for the Jacobiator of nonholonomic brackets, expressed in suitable local coordinates, was obtained by Koon and Marsden in their 1998 paper [16]. In the present paper, we revisit the Koon-Marsden formula of [16] and explain how it can be derived from the coordinate-free Jacobiator formula for nonholonomic brackets obtained in [1].

We organize the paper as follows. In Section 2 we recall the hamiltonian viewpoint to systems with nonholonomic constraints. For a nonholonomic system on a configuration manifold $Q$, determined by a lagrangian $L : TQ \to \mathbb{R}$ and a nonintegrable distribution $D$ on $Q$ (the constraint distribution, defining
the permitted velocities of the system), we consider the induced nonholonomic bracket $\{\cdot,\cdot\}_\text{nh}$ defined on the submanifold $\mathcal{M} := \text{Leg}(\mathcal{D})$ of $T^*\mathcal{Q}$, where $\text{Leg} : T^*\mathcal{Q} \to \mathcal{Q}$ is the Legendre transform (see Section 2.2). In Section 2.3 (see Theorem 2.1) we recall the global formula for the Jacobiator of $\{\cdot,\cdot\}_\text{nh}$ from [1], which depends on the choice of a complement $\mathcal{W}$ of the constraint distribution $\mathcal{D}$ such that $T\mathcal{Q} = D \oplus \mathcal{W}$. As shown in [1], this formula is useful to provide information about properties of reduced nonholonomic brackets in the presence of symmetries.

In Section 3 we recall the choice of coordinates, suitably adapted to the constraints, used by Koon and Marsden in [10], and in terms of which their Jacobiator formula is expressed. We then compare the global and local viewpoints in Section 3, explaining how one can derive the local Jacobiator formula in [10] from the coordinate-free formula in [1].

Since the formula in [1] is coordinate free, it can be used in examples without specific choices of coordinates. We illustrate this fact studying the snakeboard, following [18] [16]; here the natural coordinates in the problem are not adapted to the constraints so, in principle, the local formula from [16] cannot be directly applied.

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2. Nonholonomic systems

2.1. The hamiltonian viewpoint. A nonholonomic system is a mechanical system on a configuration manifold $\mathcal{Q}$ with constraints on the velocities which are not derived from constraints in the positions. Mathematically, it is defined by a lagrangian $L : \mathcal{Q} \to \mathbb{R}$ of mechanical type, i.e., $L = \kappa - U$ where $\kappa$ is the kinetic energy metric and $U \in C^\infty(\mathcal{Q})$ is the potential energy, and a nonintegrable distribution $\mathcal{D}$ on $\mathcal{Q}$ determining the constraints, see [8] [7]. If $\mathcal{D}$ is an integrable distribution then the system is called holonomic.

In order to have an intrinsic formulation of the dynamics of nonholonomic systems, let us consider the Legendre transform $\text{Leg} : T^*\mathcal{Q} \to T^*\mathcal{Q}$ associated to the lagrangian $L$. The Legendre transform is a diffeomorphism since $\text{Leg} = \kappa^\flat$, where $\kappa^\flat : T^*\mathcal{Q} \to T^*\mathcal{Q}$ is defined by $\kappa^\flat(X)(Y) = \kappa(X, Y)$. We denote by $\mathcal{H} : T^*\mathcal{Q} \to \mathbb{R}$ the hamiltonian function associated to the lagrangian $L$.

We define the constraint submanifold $\mathcal{M}$ of $T^*\mathcal{Q}$ by $\mathcal{M} = \kappa^\flat(\mathcal{D})$. Note that $\mathcal{M}$ is a vector subbundle of $T^*\mathcal{Q}$. We denote by $\tau : \mathcal{M} \to \mathcal{Q}$ the restriction to $\mathcal{M}$ of the canonical projection $\tau : T^*\mathcal{Q} \to \mathcal{Q}$.

On $\mathcal{M}$ we have a natural 2-form $\Omega_\mathcal{M}$ given by $\Omega_\mathcal{M} := \iota^*\Omega_\mathcal{Q}$ where $\iota : \mathcal{M} \to T^*\mathcal{Q}$ is the inclusion and $\Omega_\mathcal{Q}$ is the canonical 2-form on $T^*\mathcal{Q}$. The constraints are encoded on a (regular) distribution $\mathcal{E}$ on $\mathcal{M}$ defined, at each $m \in \mathcal{M}$, by

\[ \mathcal{E}_m = \{ v \in T_m\mathcal{M} : T\tau(v) \in D_{\tau(m)} \}. \]

(2.1)

It was proven in [2] that the point-wise restriction of the 2-form $\Omega_\mathcal{M}$ to $\mathcal{E}$, denoted by $\Omega_\mathcal{M}|_\mathcal{E}$, is nondegenerate. That is, if $X \in \Gamma(\mathcal{E})$ is such that $\iota_X\Omega_\mathcal{M}|_\mathcal{E} \equiv 0$, then $X = 0$. Therefore, there is a unique vector field $X_{\text{nh}}$ on $\mathcal{M}$, called the nonholonomic vector field, such that $X_{\text{nh}}(m) \in \mathcal{E}_m$ and

\[ \iota_{X_{\text{nh}}}\Omega_\mathcal{M}|_\mathcal{E} = d\mathcal{H}_m|_\mathcal{E}, \]

(2.2)

where $\mathcal{H}_m := \iota^*\mathcal{H} : \mathcal{M} \to \mathbb{R}$. The integral curves of $X_{\text{nh}}$ are solutions of the nonholonomic dynamics [2].

In order to write (2.2) in local coordinates, suppose that the constraint distribution $\mathcal{D}$ is described (locally) by the annihilators of 1-forms $\epsilon^a$ for $a = 1, \ldots, k$, that is $\mathcal{D} = \{ (q, \dot{q}) : \epsilon^a(q, \dot{q}) = 0 \text{ for all } a = \ldots \}$. 


1, ..., k}. If we consider canonical coordinates \((q^i, p_i)\) on \(T^*Q\) then the constraints are given by

\[ \epsilon^a_i(q) \frac{\partial \xi}{\partial p_i} = 0, \quad \text{for } a = 1, ..., k, \]

and (2.2) becomes

\[ \dot{q}^i = \frac{\partial \xi}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \xi}{\partial q^i} + \lambda_a \epsilon^a, \]

where \(\lambda_a\) are functions (called the Lagrange multipliers) which are uniquely determined by the fact that the constraints are satisfied.

### 2.2. The nonholonomic bracket.

Recall that an almost Poisson bracket on \(M\) is an \(\mathbb{R}\)-bilinear bracket \(\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)\) that is skew-symmetric and satisfies the Leibniz condition:

\[ \{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \text{for } f, g, h \in C^\infty(M). \]

If \(\{\cdot, \cdot\}\) satisfies the Jacobi identity, then the bracket is called Poisson. The Hamiltonian vector field \(X_f\) on \(M\) associated to a \(f \in C^\infty(M)\) is defined by

\[ X_f = \{\cdot, f\} \tag{2.3} \]

and the characteristic distribution of \(\{\cdot, \cdot\}\) is a distribution on the manifold \(M\) whose fibers are spanned by the Hamiltonian vector fields. If the bracket is Poisson, then its characteristic distribution is integrable. However, the converse is not always true.

From the Leibniz identity it follows that there is a one-to-one correspondence between almost Poisson brackets \(\{\cdot, \cdot\}\) and bivector fields \(\pi \in \Lambda^2(TM)\) given by

\[ \{f, g\} = \pi(df, dg), \quad f, g \in C^\infty(M). \tag{2.4} \]

Let us denote by \(\pi^\sharp : T^*M \rightarrow TM\) the map defined by \(\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)\). Then, using (2.3), the Hamiltonian vector field \(X_f\) is also given by \(X_f = -\pi^\sharp(df)\) and the characteristic distribution of \(\pi\) is the image of \(\pi^\sharp\). The Schouten bracket \([\pi, \pi]\) (see [17]) measures the failure of the Jacobi identity of \(\{\cdot, \cdot\}\) through the relation

\[ \frac{1}{2}[\pi, \pi](df, dg, dh) = \{f, \{g, h\}\} + \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \tag{2.5} \]

for \(f, g, h \in C^\infty(M)\). So we refer to the trivector \(\frac{1}{2}[\pi, \pi]\) as the Jacobiator of \(\pi\), which is zero when \(\pi\) is a Poisson bivector.

Coming back to our context, consider a nonholonomic system on a manifold \(Q\) defined by a Lagrangian \(L\) and a constraint distribution \(D\). Due to the nondegeneracy of \(\Omega_M|_\xi\), there is an induced bivector field \(\pi_{nh} \in \Lambda^2(TM)\) defined at each \(\alpha \in T^*M\) by

\[ \pi_{nh}^\sharp(\alpha) = X \quad \text{if and only if} \quad i_X \Omega_M|_\xi = -\alpha|_\xi. \tag{2.6} \]

The characteristic distribution of \(\pi_{nh}\) is the distribution \(\xi\) defined in (2.1). Since \(\xi\) is not integrable, \(\pi_{nh}\) is not Poisson.

The bivector field \(\pi_{nh}\) is called the nonholonomic bivector field [19, 14, 11] and it describes the dynamics in the sense that

\[ \pi_{nh}^\sharp(d\xi_M) = -X_{nh}. \tag{2.7} \]

By (2.4), the nonholonomic bivector \(\pi_{nh}\) defines uniquely an almost Poisson bracket \(\{\cdot, \cdot\}_{nh}\) on \(M\), called the nonholonomic bracket. From (2.6) we observe that

\[ \{f, g\}_{nh} = \Omega_M(X_f, X_g) \quad \text{for } f, g \in C^\infty(M), \]

where \(X_f = -\pi_{nh}^\sharp(df)\) and \(X_g = -\pi_{nh}^\sharp(dg)\). The nonholonomic vector field (2.7) is equivalently defined through the equation \(X_{nh} = \{\cdot, \xi_M\}_{nh}\).
2.3. The Jacobiator formula. Recall that $\mathcal{C}$ is a smooth distribution on $\mathbb{M}$. Choose a complement $W$ of $\mathcal{C}$ on $\mathbb{T}M$ such that, for each $m \in \mathbb{M}$,

$$T_m\mathbb{M} = \mathcal{C}_m \oplus W_m. \tag{2.8}$$

Let $P_\mathcal{C} : \mathbb{T}M \to \mathcal{C}$ and $P_W : \mathbb{T}M \to W$ be the projections associated to the decomposition (2.8). Since $P_W : \mathbb{T}M \to W$ can be seen as a $W$-valued 1-form, following [1], we define the $W$-valued 2-form $K_W$ given by

$$K_W(X,Y) = -P_W([P_\mathcal{C}(X),P_\mathcal{C}(Y)]) \quad \text{for } X, Y \in \mathfrak{X}(\mathbb{M}). \tag{2.9}$$

Once a complement $W$ of $\mathcal{C}$ is chosen, we obtain a coordinate-free formula for the Jacobiator of the nonholonomic bracket.

**Theorem 2.1.** [1] The following holds:

$$\frac{1}{2}[\pi_{nh},\pi_{nh}]_{\{\alpha,\beta,\gamma\}} = \Omega_M(K_W(\pi_{nh}^\alpha,\pi_{nh}^\beta),\pi_{nh}^\gamma) - \gamma(K_W(\pi_{nh}^\beta,\pi_{nh}^\gamma)) + \text{cyclic}. \tag{2.10}$$

for $\alpha, \beta, \gamma \in T^*\mathbb{M}$.

In fact, a more general formula appeared in [1], valid for any bivector field $\pi_B$ gauge related to $\pi_{nh}$. In that context, this formula was used to understand under which circumstances the reduction of $\pi_B$ by symmetries had an integrable characteristic distribution (even if it was not Poisson).

We will now show how this formula recovers the coordinate Jacobiator formula obtained in [16].

3. The Koon-Marsden adapted coordinates

In this section we will recall the Koon-Marsden approach to writing the Jacobiator of a nonholonomic bracket, based on a suitable choice of coordinates of the manifold $Q$. After this, we will write the objects presented in Section 2 (such as the 2-forms $\Omega_M$ and $K_W$, and the bivector $\pi_{nh}$) in such local coordinates in order to see the equivalence between the local and global viewpoints.

We start by recalling the coordinates chosen in [16]. Consider a nonholonomic system given by a lagrangian $L$ and a nonintegrable distribution $D$. Let $\epsilon^a$ for $a = 1, \ldots, k$ be 1-forms that span the annihilator of $D$, i.e., $D^\circ = \text{span}\{\epsilon^a\}$. The authors in [16] introduce local coordinates $(q^i) = (r^a, s^a)$ on $Q$ for which each 1-form $\epsilon^a$ has the form

$$\epsilon^a = ds^a + A^a_b(r,s)dr^b, \tag{3.11}$$

where $A^a_b$ are functions on $Q$ for $\alpha = 1, \ldots, n-k$ and $a = 1, \ldots, k$. During the present paper, we refer to the coordinates $(r^a, s^a)$ such that (3.11) is satisfied as coordinates adapted to the constraints.

These coordinates induce a (local) basis of $D$ given by $\{X_\alpha := \frac{\partial}{\partial r^\alpha} - A^a_\alpha \frac{\partial}{\partial s^a}\}$. We complete the basis $\{X_\alpha\}$ and $\{\epsilon^a\}$ in order to obtain dual basis on $TQ$ and $T^*Q$, that is

$$TQ = \text{span} \left\{ X_\alpha, \frac{\partial}{\partial s^a} \right\} \quad \text{and} \quad T^*Q = \text{span}\{dr^a, \epsilon^a\}.$$

Let $(\tilde{p}_\alpha, \tilde{p}_a)$ be the coordinates on $T^*Q$ associated to the basis $\{dr^a, \epsilon^a\}$. Since $M = \text{span}\{\kappa_\alpha(X_\alpha)\} \subset T^*Q$ then

$$M = \{ (q^i, \tilde{p}_a, \tilde{p}_\alpha) : \tilde{p}_a = [\kappa_\alpha][\kappa_{\alpha\beta}]^{-1}\tilde{p}_\beta = J^{\beta}_a\tilde{p}_\beta \}, \tag{3.12}$$

where $[\kappa_{\alpha\beta}]$ denotes the $(k \times (n-k))$-matrix with entries given by $\kappa_{\alpha\beta} = \kappa(\frac{\partial}{\partial s^\alpha}, X_\beta)$, $[\kappa_{\alpha\beta}]^{-1}$ is the inverse matrix associated to the invertible $(n-k) \times (n-k)$-matrix with entries given by $\kappa_{\alpha\beta} = \kappa(X_\alpha, X_\beta)$ and $J^{\alpha}_a$ are the functions on $Q$ representing the entries of the matrix $[\kappa_{\alpha\beta}]^{-1}$. Therefore, each element $(r^a, s^a; \tilde{p}_a)$ represents a point on the manifold $M$. 

In [16] the Jacobiator formula is written in terms of the curvature of an Ehresmann connection. The local coordinates \((r^a, s^a)\) induce a fiber bundle with projection given by \(v(r^a, s^a) = r^a\). Let us call \(W\) the vertical distribution defined by this projection.

The Ehresmann connection \(A\) on \(v : Q = \{r^a, s^a\} \rightarrow R = \{r^a\}\) is chosen in such a way that its horizontal space agrees with the distribution \(D\). The connection \(A\) is represented by a vector-valued differential form given, at each \(X \in TQ\), by

\[
A(X) = e^a(X) \frac{\partial}{\partial s^a}. \tag{3.13}
\]

The *curvature* associated to this connection is a vector-valued 2-form \(K_W\) defined on \(X, Y \in \mathfrak{X}(Q)\) by

\[
K_W(X, Y) = -A([P_D(X), P_D(Y))]), \tag{3.14}
\]

where \(P_D : TQ \rightarrow TQ\) is the projection to \(D\) given by \(P_D(X) = dr^a(X)X_a\).

In coordinates, the curvature \(K_W\) is given by the following formula [16] Sec. 2.1:

\[
K_W(X, Y) = de^a(P_D(X), P_D(Y)) \frac{\partial}{\partial s^a},
\]

hence, locally,

\[
de^a|_D = C^{a\beta}_{\alpha\delta} dr^\alpha \wedge dr^\beta|_D , \tag{3.15}
\]

where \(C^{a\beta}_{\alpha\delta}(r, s) = \frac{\partial A^a_\beta}{\partial r^\alpha} - A^b_\alpha \frac{\partial A^a_\beta}{\partial s^b}\). Let us define

\[
K^{a\beta}_{\alpha\beta} = C^{a\beta}_{\alpha\beta} - C^{a\beta}_{\alpha\gamma}. \tag{3.16}
\]

For each \(a = 1, ..., k\) the coefficients \(K^{a\beta}_{\alpha\beta}\) are skew-symmetric and \(de^a|_D = K^{a\beta}_{\alpha\beta} dr^\alpha \wedge dr^\beta|_D\), for \(\alpha < \beta\). Therefore, if \(X, \dot{X} \in D\) then \(de^a(X, \dot{X}) = K^{a\beta}_{\alpha\beta} \dot{v}^\alpha \dot{v}^\beta\) where \(\dot{X} = v^a X_a\) and \(\dot{X} = \dot{\bar{v}}^\beta X_\beta\).

**Remark 3.1.** Observe that in [16], the 1-forms \(e^a\) where denoted by \(\omega^a\) while \(K_W\) was denoted by \(B\) and the coefficients \(K^{a\beta}_{\alpha\beta}\) were \(-B^{a\beta}_{\alpha\gamma}\). In this case, for \(\dot{q} \in D\) then \(\omega^a|_D = -B^{a\beta}_{\alpha\gamma} \dot{v}^\alpha \dot{v}^\beta|_D\) (observe the correction in the sign with respect to the equation in [16] Sec. 2.1]).

Finally, in [16] Theorem 2.1] the almost Poisson bracket \(\{\cdot, \cdot\}_M\) describing the dynamics of a nonholonomic system was written following [19] but in local coordinates on \(Q\) adapted to the constraints \((3.11)\). That is, \(\{\cdot, \cdot\}_M\) was computed from the canonical Poisson bracket on \(T^*Q\) but written in terms of the adapted coordinates \((r^a, s^a, \bar{p}_a, \bar{p}_a)\). As a result, the almost Poisson bracket \(\{\cdot, \cdot\}_M\) on \(M\), written in local coordinates \((r^a, s^a, \bar{p}_a)\), has the following form [16]

\[
\{q^i, q^j\}_M = 0, \quad \{r^a, \bar{p}_\beta\}_M = \delta^a_\beta, \quad \{s^a, \bar{p}_\alpha\}_M = -A^a_\alpha, \quad \{\bar{p}_\alpha, \bar{p}_\beta\}_M = K^{a\gamma}_{\alpha\beta} \bar{p}_\gamma \tag{3.17}
\]

4. The coordinate version of the Jacobiator formula

4.1. Interpretation of the adapted coordinates. In this section, we will relate the choice of the coordinates proposed in [19] with the choice of a complement \(W\) done in [11] (see \((3.11)\) and \((2.8)\), respectively). We will also connect the curvature \((3.14)\) with the 2-form \((2.9)\), and the nonholonomic bivector \(\pi_{nh}\) with the bracket \(\{\cdot, \cdot\}_M\) given in \((2.6)\) and \((3.17)\), respectively.

Consider a nonholonomic system on a manifold \(Q\) given by a lagrangian \(L\) and a nonintegrable distribution \(D\). Let us consider local coordinates \((r^a, s^a)\) adapted to the constraints as in \((3.11)\).
The projection \( P_W : TQ \to W \) associated to the decomposition \((4.18)\) is interpreted in \([16]\) as the Ehresmann connection \( A \) \((3.13)\). In this context we compare the curvature \( K_W \) defined in \((3.11)\) (see \([16]\)) with the \( W \)-valued 2-form \( K_W \) defined in \((3.14)\).

Recall that the submanifold \( M = \kappa^\tau(D) \subset T^*Q \) is described by local coordinates \((r^\alpha, s^a, \tilde{p}_a)\) (see \((3.12)\)). Locally \( T^*M \) is generated by the basis \( \mathcal{B}_{T^*M} = \{dr^\alpha, e^a, d\tilde{p}_a\} \). During the rest of the paper, when there is no risk of confusion, we will use the same notation for 1-forms on \( Q \) and their pull back to \( M \) and \( T^*Q \), (i.e., \( \tau^*dr^\alpha = dr^\alpha \) and \( \tau^*e^a = e^a \) where \( \tau : M \to Q \) is the canonical projection).

Since \( \tau \)-projectable vector fields generate \( TM \) at each point, we can consider a complement \( W \) of \( \mathcal{C} \) generated by \( \tau \)-projectable vector fields \( Z_a \) such that \( T\tau(Z_a) \in W \). That is,

\[
\mathcal{C} = \text{span} \left\{ X_\alpha, \frac{\partial}{\partial p_\alpha} \right\} \quad \text{and} \quad W = \text{span} \left\{ Z_a : T\tau(Z_a) = \frac{\partial}{\partial s^a} \right\}. \tag{4.19}
\]

**Lemma 4.2.** Let \( \mathcal{W} \) be a complement of \( \mathcal{C} \) as in \((4.19)\) where \( W \) is the complement of \( D \) induced by the coordinates \((r^\alpha, s^a)\) as in Lemma 3.1.

1. The \( \mathcal{W} \)-valued 2-form \( K_W \) and the curvature \( K_W \), defined in \((3.14)\) and \((3.11)\) respectively, are related, at each \( X, Y \in TM \), by \( K_W(T\tau(X), T\tau(Y)) = T\tau(K_W(X,Y)) \). In local coordinates \((r^\alpha, s^a)\) adapted to the constraints \((3.11)\), the following holds:

\[
K_W|_\mathcal{C} = (C^a_{\alpha\beta} dr^\alpha \wedge dr^\beta|_\mathcal{C}) \otimes Z_a.
\]

2. Let \( \tilde{\mathcal{W}} \) be a different complement of \( \mathcal{C} \) such that \( T\tau(\tilde{\mathcal{W}}) = T\tau(\mathcal{W}) = W \). For \( X, Y \in \Gamma(\mathcal{C}) \) we have

\[
K_W(X, Y) - K_W(X, Y) \in \Gamma(\mathcal{C}).
\]

**Proof.** (i) During this proof and to avoid confusion, we will work with the basis \( \{\tau^*dr^\alpha, \tau^*e^a, d\tilde{p}_a\} \) of \( T^*M \), keeping \( dr^\alpha \) and \( ds^a \) to denote 1-forms on \( Q \). Let us consider the basis \( \mathcal{B} = \{X_\alpha, \frac{\partial}{\partial p_\alpha}, Z_a\} \) of \( TM \) adapted to \( \mathcal{C} \oplus W \) and its dual \( \mathcal{B}^* = \{\tau^*dr^\alpha, \Psi_\alpha, \tau^*e^a\} \) where \( \Psi_\beta(X_\alpha) = \Psi_\beta(Z_a) = 0 \) and \( \Psi_\beta(\frac{\partial}{\partial p_\alpha}) = \delta_{\alpha\beta} \). Then, for \( X, Y \in \Gamma(\mathcal{C}) \),

\[
K_W(X, Y) = -P_W([X, Y]) = -\tau^*e^a([X, Y])Z_a = dr^a e^a(X, Y) Z_a = de^a(T\tau(X), T\tau(Y)) Z_a.
\]

Therefore, \( T\tau(K_W(X, Y)) = de^a(T\tau(X), T\tau(Y)) \otimes \frac{\partial}{\partial x^a} = K_W(T\tau(X), T\tau(Y)) \). Finally, since \( T\tau(X), T\tau(Y) \in \Gamma(D) \) (see \((2.1)\)) and using \((3.15)\) we obtain

\[
K_W|_\mathcal{C} = (C^a_{\alpha\beta} \tau^*dr^\alpha \wedge \tau^*dr^\beta|_\mathcal{C}) \otimes Z_a.
\]

Using our simplified notation \((\tau^*dr^\alpha = dr^\alpha)\) we obtain the desired formula.

(ii) Let \( \mathcal{B} \) and \( \mathcal{B}^* \) be the basis as in item (i). Consider also \( \mathcal{B} = \{X_\alpha, \frac{\partial}{\partial p_\alpha}, Z_a\} \) a basis of \( TM \) adapted to \( \mathcal{M} = \mathcal{C} \oplus \tilde{\mathcal{W}} \) such that \( T\tau(Z_a) = \frac{\partial}{\partial s^a} \) and its dual \( \mathcal{B}^* = \{dr^\alpha, \Psi_\alpha, e^a\} \), such that \( \Psi_\beta(X_\alpha) = \Psi_\beta(Z_a) = 0 \) and \( \Psi_\beta(\frac{\partial}{\partial p_\alpha}) = \delta_{\alpha\beta} \). Then we have that, for \( X, Y \in \mathcal{C} \),

\[
K_W(X, Y) = -P_W([X, Y]) = e^a([X, Y])\tilde{Z}_a = K_W(X, Y) + e^a([X, Y]) \otimes (\tilde{Z}_a - Z_a).
\]

Since \( \tilde{Z}_a - Z_a \in \text{Ker} T\tau \subset \mathcal{C} \) then \( K_W(X, Y) - K_{\tilde{W}}(X, Y) \in \mathcal{C} \). \( \square \)
Remark 4.3. Note that the coordinates description of $K_W$ shows that it is semi-basic with respect to the bundle projection $\tau: M \to Q$, i.e., $\iota_\tau K_W = 0$ if $T\tau(X) = 0$. This is in agreement with [16, Prop. 3.1].

In order to write the nonholonomic bivector $\pi_{nh}$ using (2.6) but in local coordinates $(r^a, s^a; \tilde{p}_\alpha)$ on $M$ we study the local description of the 2-section $\Omega_M|c$.

The canonical 1-form $\Theta_Q$ on $T^*Q$ is given, in local coordinates $(r^a, s^a; \tilde{p}_\alpha, \tilde{p}_a)$, by $\Theta_Q = \tilde{p}_a dr^a + \tilde{p}_\alpha d\epsilon^\alpha$. Then, it is straightforward to see that the canonical 2-form $\Omega_Q$ is written locally as

$$\Omega_Q = dr^a \wedge d\tilde{p}_a + \epsilon^a \wedge d\tilde{p}_\alpha - \tilde{p}_a d\epsilon^a.$$  

Recall that $\iota: M \to T^*Q$ is the natural inclusion, so the pull back of $\Omega_Q$ to $M$ is given by

$$\Omega_M = \iota^* \Omega_Q = dr^a \wedge d\tilde{p}_a + \epsilon^a \wedge d\tilde{p}_\alpha - \tilde{p}_a d\epsilon^a,$$

where $dr^a$ and $d\tilde{p}_a$ are considered as 1-forms on $M$.

Therefore,

$$\Omega_M|c = dr^a \wedge d\tilde{p}_a - \iota^*(\tilde{p}_a)(d\epsilon^a)|c$$

$$= dr^a \wedge d\tilde{p}_a - J_a^b \tilde{p}_b C^a_{\alpha\beta} dr^\alpha \wedge dr^\beta|c,$$

where in the last equation we use (3.12) and the coordinate version of $de|D$ given in (3.15). Applying (2.6) to the 2-form $\Omega_M$ and $\epsilon$, given in (4.20) and (4.19) respectively, we compute the nonholonomic bivector field $\pi_{nh}$ on $M$:

$$\pi^{\dot{r}}_{nh}(dr^a) = \frac{\partial}{\partial \tilde{p}_a}, \quad \pi^{s^a}_{nh}(ds^a) = -A^a_\alpha \frac{\partial}{\partial \tilde{p}_\alpha}, \quad \pi^{\tilde{p}_\alpha}_{nh}(d\tilde{p}_a) = -X_\alpha + J_a^b \tilde{p}_b K^a_{\alpha\beta} \frac{\partial}{\partial \tilde{p}_\beta}.$$  

Lemma 4.4. The almost Poisson bracket $\{\cdot, \cdot\}_M$ given in (3.17) (see [16, Theorem 2.1]) is the coordinate version of the nonholonomic bracket $\{\cdot, \cdot\}_{nh}$ associated to the bivector field $\pi_{nh}$ obtained from (2.6).

4.2. The Jacobiator in adapted coordinates. Consider a nonholonomic system on a manifold $Q$ given by a lagrangian $L$ and a constraint distribution $D$ such that $\epsilon^a$, for $a = 1, \ldots, k$, are 1-forms generating $D^\circ$. Consider local coordinates $(r^a, s^a)$ on $Q$ adapted to the constraints as in (3.11). Let $(r^\alpha, s^\alpha; \tilde{p}_\alpha)$ be the coordinates on the manifold $M = \epsilon^a(D)$. By Lemma 4.4 the almost Poisson bracket $\{\cdot, \cdot\}_M$ (3.17) is the coordinate version of the bivector field $\pi_{nh}$ given in (2.6), and thus Koon-Marsden formula for the Jacobiator can be written directly with respect to $\{\cdot, \cdot\}_{nh}$.

Theorem 4.5. [16, Sec. 2.5] The Jacobiator of the nonholonomic bracket $\{\cdot, \cdot\}_{nh}$ in coordinates $(r^a, s^a; \tilde{p}_\alpha)$ on $M$, is given by the following formula

$$\{\tilde{p}_\gamma, \{r^\alpha, \tilde{p}_\beta\}_{nh}\}_{nh} + \text{cyclic} = J_b^a K^b_{\alpha\gamma},$$

$$\{\tilde{p}_\beta, \{s^a, \tilde{p}_\alpha\}_{nh}\}_{nh} + \text{cyclic} = -K^a_{\alpha\beta} - A^a_\gamma J_b^\gamma K^b_{\alpha\beta},$$

$$\{\tilde{p}_\gamma, \{\tilde{p}_\alpha, \tilde{p}_\beta\}_{nh}\}_{nh} + \text{cyclic} = \tilde{p}_\gamma J^b_a \frac{\partial A^a_\beta}{\partial s^b} K^b_{\alpha\gamma} + \tilde{p}_\beta J^b_a K^b_{\gamma\alpha} - \tilde{p}_\gamma K^b_{\alpha\beta} \left( \frac{\partial J_b^a}{\partial r^\gamma} - A^a_\gamma \frac{\partial J_b^\gamma}{\partial s^a} \right) + \text{cyclic},$$

with all other combinations equal to zero and where $J^a_b$, $K^a_{\alpha\beta}$ and $A^a_\gamma$ are the functions on $Q$ defined in (3.12), (3.16) and (3.11), respectively.

The next result relates the coordinate formula (4.23) of the Jacobiator with the coordinate-free formula given in Theorem 2.1.
Theorem 4.6. Let \((r^a, s^a)\) be coordinates on \(Q\) adapted to the constraints as in (3.11) and let \(W\) be the complement of \(D\) induced by the coordinates (Lemma 4.4). The Koon-Marsden Jacobiator formula (4.23) for the nonholonomic bracket \(\{\cdot, \cdot\}_n\) is the coordinate version of the Jacobiator formula given in Theorem 2.2 for \(W\) any complement of \(C\) as in (3.11).

Proof. In order to prove the equivalence we write the Schouten bracket \([\pi_{nh}, \pi_{nh}]\) using Theorem 2.2 evaluated on the elements \(\{\varphi^a, ds^a, dp_{\alpha}\}\).

First, observe that by Remark 4.3, the 2-form \(K_W\) defined in (2.9) is annihilated by any of the elements \(\pi_{nh}^a(dr^a)\) or \(\pi_{nh}^a(ds^a)\) (see (4.22)). On the other hand, by Lemma 4.4(ii), we have that \(K_W(\pi_{nh}^a(dp_{\alpha}), \pi_{nh}^a(dp_{\beta})) = K_{\alpha}^a Z_a\), where \(Z_a \in TM\) such that \(T\tau(Z_a) = \frac{\partial}{\partial s^a}\). Moreover, observe that \(e^a(Z_b) = \delta_{ab}\) and \(dr^a(Z_a) = 0\).

Therefore, using the coordinate version of \(\Omega_M\) (4.20) in Theorem 2.2 we obtain

\[
\frac{1}{2}[\pi_{nh}, \pi_{nh}](dr^a, dp_{\beta}, dp_{\gamma}) = \Omega_M(K_W(\pi_{nh}^a(dp_{\beta}), \pi_{nh}^a(dp_{\gamma})), \pi_{nh}^a(dr^a)) - dr^a(K_W(\pi_{nh}^a(dp_{\beta}), \pi_{nh}^a(dp_{\gamma}))) = J_a^b K_{b}^{\gamma}. 
\]

\[
\frac{1}{2}[\pi_{nh}, \pi_{nh}](ds^a, dp_{\alpha}, dp_{\beta}) = \Omega_M(K_W(\pi_{nh}^a(dp_{\alpha}), \pi_{nh}^a(dp_{\beta})), \pi_{nh}^a(ds^a)) - ds^a(K_W(\pi_{nh}^a(dp_{\alpha}), \pi_{nh}^a(dp_{\beta}))) = -K_{\alpha}^b A^a_{\gamma} J^\gamma_b - K_{\alpha}^a.
\]

Finally, let \(Y_a := Z_a - \frac{\partial}{\partial s^a} \in \text{Ker} \tau C\). Then, we have that

\[
\frac{1}{2}[\pi_{nh}, \pi_{nh}](dp_{\alpha}, dp_{\beta}, dp_{\gamma}) = \Omega_M(K_{\alpha}^a Z_a, \pi_{nh}^a(dp_{\gamma})), -dp_{\gamma}(K_{\alpha}^a Z_a) + \text{cyclic} = \Omega_M(K_{\alpha}^a \frac{\partial}{\partial s^a}, \pi_{nh}^a(dp_{\gamma})), + \Omega_M(K_{\alpha}^a Y_a, \pi_{nh}^a(dp_{\gamma})), -dp_{\gamma}(K_{\alpha}^a Y_a) + \text{cyclic} = \Omega_M(K_{\alpha}^a \frac{\partial}{\partial s^a}, \pi_{nh}^a(dp_{\gamma})), + \text{cyclic} = \tilde{p}_r J_a^b \frac{\partial A^a}{\partial s^b} K_{\alpha}^b + \tilde{p}_r J_a^b K_{\alpha}^b - \tilde{p}_r K_{\alpha}^b \left( \frac{\partial J_a^b}{\partial s^a} - A^a \frac{\partial J_a^b}{\partial s^b} \right) + \text{cyclic}.
\]

The Jacobiator on the other combinations of elements of the basis \(\{dr^a, ds^a, dp_{\alpha}\}\) is zero. Thus, the relation (2.5) implies that the Jacobiator formula in Theorem 2.2 evaluated in coordinates (3.11) gives the Koon-Marsden formula (4.23).

Observe that in this proof we are implicitly using Lemma 4.4 for \(W\) and \(W_0 = \text{span} \{\frac{\partial}{\partial s^a}\}\).

Remark 4.7. From (2.10) it is straightforward to see that if the 2-form \(K_W\) is zero then the bivector \(\pi_{nh}\) is Poisson. On the other hand, it was observed in [15] that if the curvature \(K_W\) is zero then the Jacobiator identity of \(\{\cdot, \cdot\}_n\) is satisfied. Using the equivalence between \(K_W\) and \(K_{W}\) (Lemma 4.4(i)) we see that both 2-forms are zero when \(D\) is involutive, i.e., the system is holonomic.

Remark 4.8. (Symmetries) If the nonholonomic system admits a group of symmetries \(G\) then \(\pi_{nh}\) is \(G\)-invariant with respect to the induced (lifted) action on \(M\). As a consequence, the orbit projection \(M \to M/G\) induces a reduced bivector field \(\pi_{red}\) on \(M/G\) describing the reduced dynamics. Consider \((r^a, s^a)\) adapted coordinates to the constraints as in (3.11) and \(W\) the induced complement of \(D\) in \(TQ\) given by Lemma 4.4. Let \(V\) (respectively \(V\)) be the tangent space to the orbit of the \(G\)-action on \(Q\) (respect. on \(M\)). If \(W \subset V\) then there is a unique choice of the complement \(W\) contained in \(V\):

\[
W := (T\tau|_V)^{-1}(W).
\]
With this choice of $W$, Theorem 2.1 induces a formula for the Jacobiator of the reduced bivector $\pi_{\mathrm{red}}^{\text{nh}}$ (see [1, Sec.4]).

There are a number of examples of systems verifying that the complement $W$ induced by the coordinates adapted to the constraints (3.11) (as in Lemma 4.1) is vertical with respect to a $G$-symmetry, including the vertical rolling disk, the nonholonomic particle and the Chaplygin sphere, see [1, Sec. 7].

On the other hand, it may happen that a given example is described in coordinates that are not adapted to the constraints. Then, it is better to use the coordinate free formula of Theorem 2.1.

5. Example: the snakeboard

The snakeboard describes the dynamics of a board with two sets of actuated wheels, one on each end of the board. A human rider generates forward motion by twisting his body back and forth, and thus producing a movement on the wheels. This effect is modeled as a momentum wheel which sits in the middle of the board and is allowed to spin about the vertical axis. The configuration of the snakeboard is given by the position and orientation of the board in the plane, the angle of the momentum wheel and the angles of the back and front wheels. Therefore, the configuration manifold $Q$ is given by $Q = SE(2) \times (-\pi/2, \pi/2) \times S^1$ with local coordinates $q = (x, y, \theta, \psi, \phi)$, where $(x, y, \theta)$ represents the position and orientation of the center of the board, $\psi$ is the angle of the momentum wheel relative to the board and $\phi$ is the angle of the front and back wheel as in [18] (for details see [4] and [16]).

The Lagrangian is given by

$$L(q, \dot{q}) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{m r^2}{2} \dot{\theta}^2 + J_0 \dot{\psi}^2 + J_0 \dot{\phi} + J_1 \dot{\phi}^2,$$

where $m$ the total mass of the board, $r$ is the distance between the center of the board and the wheels, $J_0$ is the inertia of the rotor and $J_1$ is the inertia of each wheel.

The (nonintegrable) constraint distribution $D$ is given by the annihilator of the following 1-forms:

$$\epsilon^1 = -\sin(\theta + \phi) dx + \cos(\theta + \phi) dy - r \cos \phi d\theta$$

$$\epsilon^2 = -\sin(\theta - \phi) dx + \cos(\theta - \phi) dy + r \cos \phi d\theta. \quad (5.24)$$

Remark 5.1. The coordinates $(x, y, \theta, \psi, \phi)$ on $Q$ are not adapted to the 1-forms of constraints $\epsilon^1, \epsilon^2$. In [15] a simplified version is considered where, taking $\phi \neq 0$, it is possible to write the 1-forms of constraints in such a way that $(x, y, \theta, \psi, \phi)$ are adapted coordinates as in (3.11). In this paper, we will work with the 1-forms given in (5.24), so, our coordinates in $Q$ are not adapted to the constraints, even though these are the coordinates chosen in [16] to study the reduction by the group of symmetries $SE(2)$.

The distribution $D$ on $Q$ is given by

$$D = \text{span} \left\{ X_\psi := \frac{\partial}{\partial \psi}, \ X_\phi := \frac{\partial}{\partial \phi}, \ X_\theta := -2r \cos^2 \phi \cos \theta \frac{\partial}{\partial x} - 2r \cos^2 \phi \sin \theta \frac{\partial}{\partial y} + \sin(2\phi) \frac{\partial}{\partial \theta} \right\}.$$

We choose the complement $W$ of $D$ generated by \{X_1, X_2\} so that $\epsilon^a(X_b) = \delta^a_b$ for $a, b = 1, 2$, that is

$$W = \text{span} \left\{ X_1 := -\frac{1}{2} \sin \theta \sec \phi \frac{\partial}{\partial x} + \frac{1}{2} \cos \theta \sec \phi \frac{\partial}{\partial y} - \frac{1}{2r} \sec \phi \frac{\partial}{\partial \theta}, \ X_2 := -\frac{1}{2} \sin \theta \sec \phi \frac{\partial}{\partial x} + \frac{1}{2} \cos \theta \sec \phi \frac{\partial}{\partial y} + \frac{1}{2r} \sec \phi \frac{\partial}{\partial \theta} \right\}.$$
Consider the dual basis $\mathfrak{B}_{TQ} = \{X_\psi, X_\phi, X_S, X_1, X_2\}$ and $\mathfrak{B}_{T^*Q} = \{d\psi, d\phi, \alpha_S, \epsilon^1, \epsilon^2\}$ where
\[ \alpha_S = -\frac{1}{2r} \cos \theta \sec^2 \phi dx - \frac{1}{2r} \sin \theta \sec^2 \phi dy. \]

Let us denote by $(q; v_\psi, v_\phi, v_S, v_1, v_2)$ the coordinates on $TQ$ associated with the basis $\mathfrak{B}_{TQ}$ while $(q; \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_S, \tilde{p}_1, \tilde{p}_2)$ denote the coordinates on $T^*Q$ associated with $\mathfrak{B}_{T^*Q}$.

The submanifold $M = \kappa^5(D) = \text{span}\{\kappa^5(X_\psi), \kappa^5(X_\phi), \kappa^5(X_S)\}$ is defined in coordinates by
\[ M = \{(q; \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_S, \tilde{p}_1, \tilde{p}_2) : \tilde{p}_1 = -\tilde{p}_2 = J_1(\phi)\tilde{p}_S + J_2(\phi)\tilde{p}_\psi\}, \tag{5.25} \]
where
\[ J_1(\phi) = \frac{mr}{4(r^2m - J_0 \sin^2 \phi)} \sin \phi \sec^2 \phi \quad \text{and} \quad J_2(\phi) = -J_1(\phi) \sin(2\phi). \]

In order to compute the nonholonomic bivector $\pi_{\text{nh}}$ describing the dynamics, we write the 2-form $\Omega_M$ and the 2-section $\Omega_M|_\mathcal{E}$ in our local coordinates. The canonical 1-form $\Theta_Q$ on $T^*Q$ is given by
\[ \Theta_Q = \tilde{p}_\psi d\psi + \tilde{p}_\phi d\phi + \tilde{p}_S \alpha_S + \tilde{p}_a e^a. \]

Then,
\[ \Omega_Q = d\psi \wedge \tilde{p}_\psi + d\phi \wedge \tilde{p}_\phi + \alpha_S \wedge d\tilde{p}_S - \tilde{p}_S d\alpha_S + \epsilon^1 \wedge d\tilde{p}_1 + \epsilon^2 \wedge d\tilde{p}_2 - \tilde{p}_1 d\epsilon^1 - \tilde{p}_2 d\epsilon^2, \]

Let us consider the basis $\mathfrak{B}_{T^*M} = \{d\phi, d\psi, \alpha_S, \epsilon^1, \epsilon^2, d\tilde{p}_\phi, d\tilde{p}_S, d\tilde{p}_S\}$ of $T^*M$ (here we are using the same notation for the pullbacks of the forms to $M$). Recall that, on $M$, $\tilde{p}_1$ and $\tilde{p}_2$ are given by $\tilde{p}_1 = \tilde{p}_\psi$ and $\tilde{p}_2 = \tilde{p}_\phi$ and denoting $J_i = J_i(\phi)$ for $i = 1, 2$ we obtain
\[ \Omega_M = d\psi \wedge \tilde{p}_\psi + d\phi \wedge \tilde{p}_\phi + \alpha_S \wedge d\tilde{p}_S - J_1 \epsilon^1 \wedge d\tilde{p}_1 + J_2 \epsilon^1 \wedge d\tilde{p}_2 + \tilde{p}_S \alpha_S \wedge (J_1 \epsilon^2 \wedge d\tilde{p}_2 - J_2 \epsilon^2 \wedge d\tilde{p}_1 - \tilde{p}_1 d\epsilon^1 - \tilde{p}_2 d\epsilon^2). \tag{5.26} \]

On $TM$ consider the dual basis $\mathfrak{B}_{TM} = \{X_\psi, X_\phi, X_S, X_1, X_2, \frac{\partial}{\partial \tilde{p}_\psi}, \frac{\partial}{\partial \tilde{p}_\phi}, \frac{\partial}{\partial \tilde{p}_S}\}$ associated to $\mathfrak{B}_{T^*M}$. Therefore, we can decompose $TM = \mathcal{E} \oplus \mathcal{W}$ such that
\[ \mathcal{E} = \text{span}\left\{X_\psi, X_\phi, X_S, \frac{\partial}{\partial \tilde{p}_\psi}, \frac{\partial}{\partial \tilde{p}_\phi}, \frac{\partial}{\partial \tilde{p}_S}\right\}, \quad \mathcal{W} = \text{span}\left\{X_1, X_2\right\}. \tag{5.27} \]

Therefore, using that $d e^a|_\mathcal{E} = (-1)^a 2r \cos \phi \alpha_S \wedge d\phi|_\mathcal{E}$ for $a = 1, 2$ and that $d\alpha_S|_\mathcal{E} = 2 \tan \phi d\phi \wedge \alpha_S|_\mathcal{E}$, the 2-section $\Omega_M|_\mathcal{E}$ is given by
\[ \Omega_M|_\mathcal{E} = d\psi \wedge \tilde{p}_\psi + d\phi \wedge \tilde{p}_\phi + \alpha_S \wedge d\tilde{p}_S - \tilde{p}_S 2 \tan \phi d\phi \wedge \alpha_S + (J_1 \tilde{p}_S + J_2 \tilde{p}_\psi) 4r \cos \phi \alpha_S \wedge d\phi|_\mathcal{E}. \]

Now, we compute the nonholonomic bracket $\pi_{\text{nh}}$ using (2.6)
\[ \pi_{\text{nh}} = \left(\frac{\partial}{\partial \tilde{p}_\psi} \wedge \frac{\partial}{\partial \tilde{p}_\phi} + \frac{\partial}{\partial \tilde{p}_\phi} \wedge \frac{\partial}{\partial \tilde{p}_S} + X_S \wedge \frac{\partial}{\partial \tilde{p}_S} - (\tilde{p}_S 2 \tan \phi + 4r(J_1 \tilde{p}_S + J_2 \tilde{p}_\psi) \cos \phi) \frac{\partial}{\partial \tilde{p}_S} \wedge \frac{\partial}{\partial \tilde{p}_\phi}\right). \tag{5.28} \]

Therefore, the hamiltonian vector fields are
\[ \pi_{\text{nh}}^\sharp(d\psi) = \frac{\partial}{\partial \tilde{p}_\psi}, \quad \pi_{\text{nh}}^\sharp(d\phi) = \frac{\partial}{\partial \tilde{p}_\phi}, \]
\[ \pi_{\text{nh}}^\sharp(\alpha_S) = \frac{\partial}{\partial \tilde{p}_S}, \quad \pi_{\text{nh}}^\sharp(\epsilon^1) = 0, \quad \pi_{\text{nh}}^\sharp(d\tilde{p}_\phi) = -\frac{\partial}{\partial \tilde{p}_\phi}, \]
\[ \pi_{\text{nh}}^\sharp(d\tilde{p}_S) = \frac{\partial}{\partial \tilde{p}_S} - (\tilde{p}_S 2 \tan \phi + 4r \cos \phi (J_1 \tilde{p}_S + J_2 \tilde{p}_\psi)) \frac{\partial}{\partial \tilde{p}_S} + (2 \tan \phi \tilde{p}_S + 4r \cos \phi (J_1 \tilde{p}_S + J_2 \tilde{p}_\psi)) \frac{\partial}{\partial \tilde{p}_\phi}. \tag{5.29} \]
In order to apply Theorem 2.11 to compute the Jacobiator of \( \pi_{\text{nh}} \) we study the \( \mathbb{W} \)-valued 2-form \( \mathbf{K}_\mathbb{W} \) defined in (2.9) for \( \mathbb{W} \) in (5.27). For \( X, Y \in \mathfrak{c} \) and using the dual basis \( \mathfrak{B}_{TM} \) and \( \mathfrak{B}_{T^*M} \) we have that

\[
\mathbf{K}_\mathbb{W}(X, Y) = -P_{\mathbb{W}}([X, Y]) = -\epsilon^i([X, Y])X_1 - \epsilon^2([X, Y])X_2 = \epsilon^1(X, Y)X_1 + \epsilon^2(X, Y)X_2.
\]

Therefore,

\[
\mathbf{K}_\mathbb{W}|_c = -2\epsilon^r \cos(\phi)(\alpha_S \wedge d\phi|_c) \otimes (X_1 - X_2). \tag{5.30}
\]

Finally, we consider the 2-forms \( \Omega_M \) and \( \mathbf{K}_\mathbb{W} \), described in (5.26) and (5.30) and the vector fields (5.29), to obtain, by (2.10), that

\[
[\pi_{\text{nh}}, \pi_{\text{nh}}](\tilde{d}\tilde{p}_\phi, \tilde{d}\tilde{p}_S, d\psi) = 4r \cos(\phi)J_2(\phi),
\]

\[
[\pi_{\text{nh}}, \pi_{\text{nh}}](\tilde{d}\tilde{p}_\phi, \tilde{d}\tilde{p}_S, \alpha) = 4r \cos(\phi)J_1(\phi),
\]

\[
[\pi_{\text{nh}}, \pi_{\text{nh}}](\tilde{d}\tilde{p}_\phi, \tilde{d}\tilde{p}_S, \epsilon^i) = (-1)^i 2r \cos(\phi), \quad i = 1, 2,
\]

while on other combination of elements the Jacobiator is zero.

This example admits a symmetry given by the Lie group \( SE(2) \), see (10). The reduced manifold \( \mathcal{M}/G \) is \( S^1 \times S(-\pi/2, -\pi/2) \times \mathbb{R}^3 \) and the nonholonomic bivector field \( \pi_{\text{nh}} \) is invariant by the orbit projection \( \rho : \mathcal{M} \to \mathcal{M}/G \). Thus, on \( \mathcal{M}/G \) we have the reduced nonholonomic bivector defined at each \( \alpha \in T^*(\mathcal{M}/G) \) by

\[
(\pi_{\text{red}}^2)^i(\alpha) = T\rho \pi_{\text{nh}}^2(\rho^* \alpha).
\]

The Jacobiator of the reduced nonholonomic bivector field \( \pi_{\text{red}}^2 \) satisfies

\[
[\pi_{\text{red}}^2, \pi_{\text{red}}^2](\alpha, \beta, \gamma) = T\rho ([\pi_{\text{nh}}, \pi_{\text{nh}}](\rho^* \alpha, \rho^* \beta, \rho^* \gamma))
\]

for \( \alpha, \beta, \gamma \in T^*(\mathcal{M}/G) \). So, in our example it is simple to compute the Jacobiator of \( \pi_{\text{red}}^2 \). Taking into account that, in local coordinates, the orbit projection \( \rho : \mathcal{M} \to \mathcal{M}/G \) is given by \( \rho(\psi, \phi, \theta, x, y, \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_S) = (\psi, \phi, \tilde{p}_\phi, \tilde{p}_S) \), the Jacobiator of the reduced bivector \( \pi_{\text{red}}^2 \) describing the dynamics is given by

\[
[\pi_{\text{red}}^2, \pi_{\text{red}}^2](\tilde{d}\tilde{p}_\phi, \tilde{d}\tilde{p}_S, d\psi) = 4r \cos(\phi)J_2(\phi)
\]

while on other elements of \( T^*(\mathcal{M}/G) \) is zero.

Just to complete the example we can write, in our coordinates, the reduced bivector field \( \pi_{\text{red}}^2 \) on \( \mathcal{M}/G \):

\[
\pi_{\text{red}}^2 = \frac{\partial}{\partial \psi} \wedge \frac{\partial}{\partial \tilde{p}_\phi} + \frac{\partial}{\partial \tilde{p}_\phi} \wedge \frac{\partial}{\partial \tilde{p}_\psi} - (\tilde{p}_S 2 \tan \phi + 4r(J_1 \tilde{p}_S + J_2 \tilde{p}_\phi) \cos(\phi)) \frac{\partial}{\partial \tilde{p}_S} \wedge \frac{\partial}{\partial \tilde{p}_\phi}.
\]

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