The asymptotic collapsed fraction in an eternal universe

Hugo Martel\textsuperscript{1} and Paul R. Shapiro\textsuperscript{1}

\textsuperscript{1} Department of Astronomy, University of Texas, Austin, TX 78712, USA

E-mail: hugo@simplicio.as.utexas.edu (HM); shapiro@astro.as.utexas.edu (PRS)

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ABSTRACT

We calculate the maximum fraction of matter which is able to condense out of the expanding background universe by gravitational instability – the asymptotic collapsed fraction – for any universe which is unbound and, hence, will expand forever. We solve this problem by application of a simple, pressure-free, spherically symmetric, nonlinear model for the growth of density fluctuations in the universe. This model includes general kinds of Friedmann universes, such as the open, matter-dominated universe and those in which there is an extra, uniform background component of energy-density (e.g. the cosmological constant or so-called “quintessence”), perturbed by Gaussian random noise matter-density fluctuations. These background universes all have the property that matter-domination eventually gives way either to curvature-domination or domination by the positive energy density of the additional background component. When this happens, gravitational instability is suppressed and, with it, so is the growth of the collapsed fraction.

Our results serve to identify a limitation of the well-known Press-Schechter approximation for the time-dependent mass function of cosmological structure formation. In the latter approximation, the mass function determined from the predicted collapse of positive density fluctuations is multiplied by an ad hoc correction factor of 2 based upon an assumption that every positive density fluctuation which is fated to collapse will simultaneously accrete an equal share of additional matter from nearby regions of compensating negative density fluctuation. The model presented here explicitly determines the actual value of the factor by which any positive density fluctuation which ever collapses will asymptotically increase its mass by accreting from a compensat-
ing underdensity which surrounds it. We show that, while the famous factor of 2 adopted by the Press-Schechter approximation is correct for an Einstein-de Sitter universe, it is not correct when the “freeze-out” of fluctuation growth inherent in the more general class of background universes described above occurs. When “freeze-out” occurs, the correction factor reduces to unity and the standard Press-Schechter approximation must overestimate the collapsed fraction.

To illustrate this effect, we apply our model to currently viable versions of the Cold Dark Matter (CDM) model for structure formation, with primordial density fluctuations in accordance with data on cosmic microwave background anisotropy from the COBE satellite DMR experiment. For $H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$ and matter-density parameter $\Omega_0 = 0.3$, the open, matter-dominated CDM model and the flat CDM model with nonzero cosmological constant yield asymptotic collapsed fractions on the galaxy cluster mass-scale of $10^{15}M_\odot$ and above of 0.0361 and 0.0562, respectively, only 55% of the values determined by the Press-Schechter approximation. These results have implications for the use of the latter approximation to compare the observed space density of X-ray clusters today with that predicted by cosmological models.

**Key words**: cosmology: theory — galaxies: clusters: general — galaxies: formation — gravitation — large-scale structure of the universe

1 INTRODUCTION

Explaining the origin and evolution of galaxies and large-scale structure and determining the fundamental properties of the background universe are the primary goals of modern cosmology. The most common assumption is that the structure we observe today (density structures such as galaxies, clusters, and voids, as well as velocity structures such as the Virgocentric Infall or that associated with the Great Attractor), results from the growth, by gravitational instability, of small-amplitude, primordial density fluctuations present in the universe at early times. These fluctuations are normally assumed to originate from a Gaussian random process. In this case, they can be described as a superposition of plane-wave density fluctuations with random phases. One important property of these initial conditions is that overdense and underdense regions occupy equal volumes (in other words, their filling factors are 1/2). Since the density is nearly uniform at early times, overdense and underdense regions also contain the same mass.

The gravitational instability scenario makes the following predictions: overdense regions,
because of their larger gravitational field, will decelerate faster than the background universe, resulting in an increase of their density contrast relative to the background. If this deceleration is large enough, these regions will turn back and recollapse on themselves, resulting in the formation of positive density structures such as galaxies and clusters. The opposite phenomenon occurs in underdense regions. These regions decelerate more slowly than the background universe, thus getting more underdense, and eventually become the cosmic voids we observe today.

In this paper, we investigate the asymptotic collapsed fraction, defined as the fraction of the matter in the universe that will eventually end up inside collapsed objects. Obviously, this makes sense only in an unbound universe. Naively, we might think that the asymptotic collapsed fraction will be equal to 1/2, since half the matter is located in overdense regions at early times. This ignores two important effects. First, some overdense regions might be unbound, and second, matter located inside underdense regions could be accreted by collapsed objects. The importance of these effects depends upon the particular background universe in which these structures form. Consider, for instance, an Einstein-de Sitter universe. In this case, the background density is exactly equal to the critical density, and therefore all overdense regions are bound, and will eventually collapse. Furthermore, it can easily be shown that any mass element located inside an underdense region is gravitationally bound to at least one overdense region. Consequently, all the matter inside underdense regions will eventually be accreted by collapsed objects, and the asymptotic collapsed fraction is unity. This is not true, however, for a background universe with mean density below that of an Einstein-de Sitter universe.

Interest in models of the background universe in which the matter density is less than the critical value for a flat, matter-dominated universe is now particularly strong, on the basis of several lines of evidence which can be reconciled most economically if $\Omega_0 < 1$, where $\Omega_0$ is the present mean matter density in units of the critical value. (For reviews and references, see, e.g., Ostriker & Steinhardt 1995; Turner 1998; Krauss 1998; Bahcall 1999). Arguments in favor of a flat universe with $\Omega_0 < 1$ in which a nonzero cosmological constant makes up the difference between the matter density and the critical density have been significantly strengthened recently by measurements of the redshifts and distances of Type Ia SNe, which are best explained if the universe is expanding at an accelerating rate, consistent with $\Omega_0 = 0.3$ and $\lambda_0 = 0.7$, where $\lambda_0$ is the vacuum energy density in units of the critical density at present (Garnavich et al. 1998a; Perlmutter et al. 1998). When combined
with measurements of the angular power spectrum of the cosmic microwave background (CMB) anisotropy, these Type Ia SN results can be used to restrict further the range of models for the mass-energy content of the universe. In particular, while the SN data alone are better fit by a flat model with $\Omega_0 < 1$ and a positive cosmological constant than by an open, matter-dominated model with no cosmological constant (e.g. Perlmutter et al. 1998), the combined information from Type Ia SNe and the CMB significantly strengthens the case for a flat model with cosmological constant over that for an open, matter-dominated model (e.g. Garnavich et al. 1998b). Exotic alternatives to the well-known cosmological constant which might also contribute positively to the total cosmic energy density and thereby similarly affect the mean expansion rate have also been discussed, sometimes referred to as “quintessence” models (e.g. Turner & White 1997; Caldwell, Dave, & Steinhardt 1998). Such models can also explain the presently accelerating expansion rate indicated by the Type Ia SNe, while satisfying several other constraints which suggest that $\Omega_0 < 1$. The results from Type I SNe and CMB anisotropy combined can be used to constrain the range of equations of state allowed for this other component of energy density $\rho_x$, with pressure $p_x = w_x \rho_x c^2$. The current results favor a flat universe with $\Omega_0 < 1$ and an equation of state for the second component with a value of $w_x \approx -1$ (where $w_x = -1$ for a cosmological constant) favored over larger values of $w_x$ (such as would describe topological defects like domain walls, strings, or textures), although the restriction of the range allowed for $w_x$ is not yet very precise (Garnavich et al. 1998b).

Consider now an unbound universe with a matter density parameter $\Omega$ with present value $\Omega_0 < 1$. In such a universe, the critical density exceeds the mean density, and therefore some overdense regions are unbound. The asymptotic collapsed fraction could still be unity if all the matter in overdense, unbound regions plus all the matter in underdense regions is accreted. This will never be the case, however. In such a universe, the density parameter $\Omega$ is near unity at early times, and structures can grow. Eventually $\Omega$ drops significantly below unity, and a phenomenon known as “freeze-out” occurs. In this regime, density fluctuations do not grow unless their density is already significantly larger than the background density. After freeze-out, accretion by collapsed objects will be very slow, and most of the unaccreted matter will remain unaccreted. The asymptotic collapsed fraction will therefore be less than unity.

The asymptotic collapsed fraction is a quantity which is relevant to modern attempts to interpret observations of cosmic structure in at least two ways. For one, anthropic rea-
soning can be used to calculate a probability distribution for the observed values of some fundamental property of the universe, such as the cosmological constant, in models in which that property takes a variety of values with varying probabilities (Efstathiou 1995; Vilenkin 1995; Weinberg 1996; Martel, Shapiro, & Weinberg 1998, hereafter MSW). Examples of such models include those in which a state vector is derived for the universe which is a superposition of terms with different values of the fundamental property (e.g. Hawking 1983, 1984; Coleman 1988) and chaotic inflation in which the observed big bang is just one of an infinite number of expanding regions in each of which the fundamental property takes a different value (Linde 1986, 1987, 1988). In models like these, the probability of observing any particular value of the property is conditioned by the existence of observers in those “subuniverses” in which the property takes that value. This probability is proportional to the fraction of matter which is destined to condense out of the background into mass concentrations large enough to form observers – i.e. the asymptotic collapsed fraction for collapse into objects of this mass or greater. MSW used this approach to offer a possible resolution of the infamous “cosmological constant problem,” one of the most serious crises of quantum cosmology. Estimates of the size of a relic vacuum energy density $\rho_V$ from quantum fluctuations in the early universe suggest a value which is many orders of magnitude larger than the cosmic mass density today, and no cancellation mechanism has yet been identified which would reduce this to zero, let alone one so finely tuned as to leave the small but nonzero value suggested by recent astronomical observations (i.e. where the net $\rho_V$ is the sum of a contribution from quantum fluctuations and a term $\Lambda/8\pi G$, where $\Lambda$ is the cosmological constant which appears in Einstein’s field equations) (Weinberg 1989; Carroll, Press, & Turner 1992). MSW calculated the relative likelihood of observing any given value of $\rho_V$ within the context of the flat CDM model with nonzero cosmological constant, with the amplitude and shape of the primordial power spectrum in accordance with current data on the CMB anisotropy. Underlying this calculation was the notion that values of $\rho_V$ which are large are unlikely to be observed since such values of $\rho_V$ tend to suppress gravitational instability and prevent galaxy formation. MSW found that a small, positive cosmological constant in the range suggested by astronomical evidence is actually a reasonably likely value to observe, even if the a priori probability distribution that a given subuniverse has some value of the cosmological constant does not favor such small values. Similar reasoning can, in principle, be used to assess the probability of our observing some range of values for other properties of the universe, too, in the absence of a theory which uniquely determines their values (e.g.
the value of $\Omega_0$; Garriga, Tanaka, & Vilenkin 1998). In such calculations, the asymptotic collapsed fraction is a fundamental ingredient.

Aside from its importance in anthropic probability calculations like these, in which one needs to know the state of the universe in the infinite future, the asymptotic collapsed fraction is also relevant as an approximation to the present universe, for the following reason. In an Einstein-de Sitter universe, in which there is no freeze-out, the asymptotic collapsed fraction is unity. In any other unbound universe, there will be a freeze-out at some epoch. If we live in such a universe, the freeze-out epoch could be either in the future or in the past. However, if recent attempts to reconcile a number of the observed properties of our universe with theoretical models of the background universe and of structure formation by invoking an unbound universe with $\Omega_0 < 1$ are correct, then the freeze-out epoch is much more likely to be in the past. If it were in the future, then the matter density parameter today would still be close to unity, e.g. $\Omega_0 > 0.9$ or $\Omega_0 > 0.99$. If so, then the observable consequences of the eventual departure of the background model from Einstein-de Sitter would be largely in the future, as well. As such, the strong motivation for considering models with $\Omega_0 < 1$ in order to explain a number of the observed properties of our universe as described above would vanish. In short, the current interest in a universe with $\Omega_0 < 1$ is consistent with a value of $\Omega_0$ small enough that the epoch of freeze-out is largely in the past. In that case, the asymptotic collapsed fraction should be a good approximation to the present collapsed fraction. This quantity is of interest, for example, since, by combining it with the observed luminosity density of the universe, we can get a handle on the average mass-to-light ratio of the universe, and the amount of dark matter. The complementary quantity, the uncollapsed fraction, is of interest, too, since it determines the amount of matter left behind as the intergalactic medium, observable in absorption and by its possible contributions to background radiation. A knowledge of the amount of matter left uncollapsed is also necessary in order to interpret observations of gravitational lensing of distant sources by large-scale structure. In addition, as we shall see, the dependence of the asymptotic collapsed fraction on the equation of state

* This is a subjective notion, since there is no precise definition of the freeze-out epoch. For a flat, universe with positive cosmological constant, for example, spherical density fluctuations must have fractional overdensity $\delta = (\rho - \bar{\rho})/\bar{\rho} \geq (729 \rho_V/500 \bar{\rho})^{1/3}$ in order to undergo gravitational collapse, where $\rho_V$ is the vacuum energy density and $\bar{\rho}$ is the mean matter density (Weinberg 1987). In this case, once $\bar{\rho}$ drops to a value of the order of $\rho_V$ or less, only density enhancements which are already nonlinear will remain gravitationally bound. As such, the “freeze-out” epoch corresponds roughly to the time when $\bar{\rho} \approx \rho_V$. Recent estimates from measurements of distant Type Ia SNe, however, suggest values which, if interpreted in terms of this model, are closer to $\bar{\rho} \lesssim \rho_V/2$ (Garnavich et al. 1998a; Perlmutter et al. 1998), so “freeze-out” began in the past for this model.
of the background universe will imply that theoretical tools, such as the Press-Schechter
approximation, require adjustment in order to take proper account of the effect of “freeze-
out” on the rate of cosmic structure formation.

In this paper, we compute the asymptotic collapsed fraction for unbound universes,
using an analytical model involving spherical top-hat density perturbations surrounded by
shells of compensating underdensity, applied statistically to the case of Gaussian random
noise density fluctuations, a model introduced by MSW for the particular case of a flat
universe with a cosmological constant. We consider a generic cosmological model with 2
components, a nonrelativistic component whose mean energy density varies as $\bar{\rho} \propto a^{-3}$,
where $a$ is the FRW scale factor, and a uniform, nonclumping component whose energy
density varies as $\rho_X \propto a^{-n}$, where $n$ is non-negative. In terms of the equations of state for
these two components, we can write this as $p_i = w_i \rho_i c^2$, where $\rho_i$ and $p_i$ are the mean energy
density and pressure contributed by component $i$. For the nonrelativistic matter component,
$w = 0$, while for component $X$, $-1 \leq w \leq 0$ is the physically allowed range in models in
which the universe had a big bang in its past and the energy of component $X$ was not more
important in the past than that of matter, which corresponds to $n = 3(1 + w)$ and the range
$0 \leq n \leq 3$. The latter condition is necessary in order to be consistent with observations of
cosmic structure and the CMB anisotropy today. Special cases of this model include models
with a cosmological constant ($n = 0$), domain walls ($n = 1$), infinite strings ($n = 2$), massive
neutrinos ($n = 3$), and radiation background ($n = 4$) (although, as explained above, we shall
exclude values of $n > 3$ in our treatment here). This generic model, or similar ones, have
been discussed previously by many authors (e.g. Fry 1985; Charlton & Turner 1987; Silveira
& Waga 1994; Martel 1995; Dodelson, Gates, & Turner 1996; Turner & White 1997; Martel
& Shapiro 1998). Recently, such models have been referred to as “quintessence” models (e.g.
Caldwell, Dave, & Steinhardt 1998) or as models involving “dark energy.”

The Friedmann equation for this model is

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[ (1 - \Omega_0 - \Omega_{X0}) \left(\frac{a}{a_0}\right)^{-2} + \Omega_0 \left(\frac{a}{a_0}\right)^{-3} + \Omega_{X0} \left(\frac{a}{a_0}\right)^{-n}\right],$$

(1)

† We note that in some models, the X component is not entirely nonclumping: For massive neutrinos, for example, the
assumption that the X-component is nonclumping is a very good approximation only for fluctuations of wavelength smaller
than the “free-streaming,” or “damping,” length of the neutrinos and for epochs such that longer wavelength fluctuations are
still in the linear amplitude phase.
where $H$ is the Hubble constant, $\Omega = \bar{\rho}/\rho_c$, $\Omega_X = \rho_X/\rho_c$, $\rho_c = 3H^2/(8\pi G)$, and subscripts zero indicate present values of time-varying quantities.

In §2, we derive the conditions that the cosmological parameters must satisfy in order for the background universe to qualify as an eternal, unbound universe. In §3, we compute the critical density contrast $\delta_c$, defined as the minimum density contrast a spherical perturbation must have in order to be bound. In §4, we derive the asymptotic collapse fraction $f_{c,\infty}$ in an unbound universe, using the model introduced by MSW involving compensated spherical top-hat density fluctuations. In §5, we compute $f_{c,\infty}$ using the Press-Schechter approximation, instead. In §6, we compare the predictions of the two models. As we shall see, this comparison points up a fundamental limitation to the validity of the ad hoc, overall correction factor of 2 by which the Press-Schechter integral over positive initial density fluctuations is traditionally multiplied so as to recover a total collapsed factor which takes account of the accretion of mass initially in underdense regions. In particular, we shall derive this factor of 2 for the Einstein-de Sitter case, but show that the same factor of 2 in the Press-Schechter formula overestimates the asymptotic collapsed fraction for an unbound universe. To illustrate the importance of these results for currently viable models of cosmic structure formation, we apply our model in §6 to two examples of the Cold Dark Matter (CDM) model, with $\Omega_0 = 0.3$ and $H_0 = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$, the open, matter-dominated model and the flat model with cosmological constant.

2 CRITERIA FOR AN UNBOUND UNIVERSE

The Friedmann equation (1) describes the time-evolution of the scale factor $a(t)$. The solutions of this equation can be grouped into four categories, according to their asymptotic behavior at late times. If the derivative $\dot{a}$, which is initially positive, remains positive at all times, never dropping to zero, then the universe is unbound. This is the case, for instance, in a matter-dominated universe with $\Omega_0 < 1$. If, instead, $\dot{a}$ drops to zero as $a \to \infty$, then the universe is marginally bound. This is the case for the Einstein-de Sitter universe ($\Omega_0 = 1$, $\Omega_{X0} = 0$). If $\dot{a}$ drops to zero at a finite value $a = a_t$, then two situations can occur: If the second derivative $\ddot{a}$ is negative at $a = a_t$, the universe will turn back and recollapse. This is the case for a matter-dominated universe with $\Omega_0 > 1$. However if both $\dot{a}$ and $\ddot{a}$ are zero at $a = a_t$, then the universe asymptotically approaches an equilibrium state with $a = a_t$ at

\[ \text{The asymptotic value of } \dot{a} \text{ in the limit } a \to \infty \text{ can be either finite or infinite} \]
late times. This is the case of the de Sitter universe with a positive cosmological constant, which initially expands, and asymptotically becomes an Einstein static universe.

To determine in which category a particular model falls, we need to study the properties of the Friedmann equation (1). For convenience, we rewrite this equation as

\[ g(y) \equiv H_0^2 y y^2 = (1 - \Omega_0 - \Omega_{X0}) y + \Omega_0 + \Omega_{X0} y^{3-n}, \]

where \( y \equiv a/a_0 = 1/(1+z) \). Only non-negative values of \( g \) are physically allowed. Since \( y > 0 \) after the big bang, the condition \( g = 0 \) is equivalent to \( \dot{y} = 0 \) (or \( \dot{a} = 0 \)). The first term in the right-hand-side of equation (2) can be either positive or negative, while the last two terms cannot be negative. If \( n = 2 \) or \( \Omega_{X0} = 0 \), then the quantity \( \Omega_{X0} \) cancels out in equation (2). This merely illustrates the fact that a universe with a uniform component whose density varies as \( a(t)^{-2} \) (cf. a universe with infinite strings) behaves exactly like a matter-dominated universe. Such a universe is bound, marginally bound, or unbound if \( \Omega_0 > 1 \), \( \Omega_0 = 1 \), or \( \Omega_0 < 1 \), respectively. The case in which \( \Omega_{X0} \neq 0 \) and \( n = 3 \) is exactly the same as that with \( \Omega_{X0} = 0 \), except that \( \Omega_0 \) is everywhere replaced by \( \Omega_0 + \Omega_{X0} \). In that case, the universe is bound, marginally bound, or unbound according to whether \( \Omega_0 + \Omega_{X0} > 1 \), \( \Omega_0 + \Omega_{X0} = 0 \), or \( \Omega_0 + \Omega_{X0} < 1 \), respectively. Let us now consider cases with \( \Omega_{X0} \neq 0 \) for which \( n \neq 2 \) and \( n \neq 3 \).

For \( 1 - \Omega_0 - \Omega_{X0} \geq 0 \), the universe cannot be bound. Clearly, if \( 1 - \Omega_0 - \Omega_{X0} > 0 \), then \( g(y) > 0 \) for all \( y \), and the universe is unbound for any value of \( n \). If \( 1 - \Omega_0 - \Omega_{X0} = 0 \), then the last term in equation (2) will eventually dominate (since we assume \( n < 3 \)). Two situations can then occur. If \( n > 2 \), then \( g(y) \) grows more slowly than \( y \), implying that \( \dot{y}^2 = g(y)/H_0^2 y \) decreases as \( y \) increases, reaching zero as \( y \to \infty \). This is the case of a marginally bound universe. If \( n < 2 \), \( \dot{y}^2 \) will eventually increase with \( y \). The universe is then unbound.

Let us now focus on the case \( 1 - \Omega_0 - \Omega_{X0} < 0 \). If \( n > 2 \), then at small \( y \), \( g(y) > 0 \), but as \( y \) increases, the first term in equation (2) will eventually dominate the other terms, giving \( g(y) < 0 \). There will therefore be a change of sign of \( g(y) \) at some finite value \( y = y_t \), where \( g(y_t) = 0 \). That corresponds to a bound universe. This leaves the interesting case of a universe with \( 1 - \Omega_0 - \Omega_{X0} < 0 \) and \( n < 2 \). Since the slope of \( g(y) \) at early times for any \( n < 2 \) is \( (1 - \Omega_0 - \Omega_{X0}) < 0 \), while at late times it is \( (3-n)\Omega_{X0} y^{2-n} > 0 \), \( g(y) \) has a minimum at some intermediate value of \( y \). When \( g(y) \) is zero at that intermediate value,
Figure 1. Schematic plot of function $g(y)$ versus $y$ for three different universes if $1 - \Omega_0 - \Omega_{X0} < 0$ and $n < 2$: an unbound universe, a marginally bound universe, and a bound universe. The marginally bound case is characterized by the existence of a point $y_t$ where $g = dg/dy = 0$. This corresponds to the case of a marginally bound universe. The various possibilities for the cases with $1 - \Omega_0 - \Omega_{X0} < 0$ and $n < 2$ are shown in Figure 1. The top curve shows a case for which $g(y) > 0$ for all $y$, that is, an unbound universe. The bottom curve shows a case for which $g(y)$ drops to zero at a finite value of $y$. In this case, the universe turns back and recollapses. It is therefore bound\(^4\). The transition between these two cases, a marginally bound universe, is illustrated by the middle curve in Figure 1, which is tangent to the $y$-axis. At $y = y_t$, both the function $g(y)$ and its first derivative $dg/dy$ vanish. The condition for having a marginally bound universe is, therefore, given by the following simultaneous equations,

\begin{align}
(1 - \Omega_0 - \Omega_{X0})y_t + \Omega_0 + \Omega_{X0}y_t^{3-n} &= 0, \quad (3) \\
(1 - \Omega_0 - \Omega_{X0}) + (3-n)\Omega_{X0}y_t^{2-n} &= 0. \quad (4)
\end{align}

We can solve equation (4) for $y_t$, and substitute this $y_t$ into equation (3). We get, after some algebra,

\(^4\) At large $y$, the function $g(y)$ becomes positive again, indicating that there are possible solutions for $y$ large. These are “catenary universes,” sometimes referred as “no big bang solutions.” In such models, the universe contracts from an infinite radius, turns back, and reexpands forever. These solutions are not considered to be physically interesting.
\[
\left( \frac{\Omega_0 + \Omega_{X0} - 1}{3 - n} \right)^{3-n} = \left( \frac{\Omega_0}{2 - n} \right)^{2-n} \Omega_{X0}.
\]  
(5)

We can easily check some limiting cases. For a matter-dominated universe \( (\Omega_{X0} = 0) \), equation (5) gives \( \Omega_0 = 1 \) as the condition for a marginally bound universe, as expected. For a universe with a nonzero cosmological constant \( (n = 0) \), equation (5) reduces to

\[
(\Omega_0 + \lambda_0 + 1)^3 = \frac{27}{4}\lambda_0\Omega_0^2,
\]

(6)

where we have replaced \( \Omega_{X0} \) by \( \lambda_0 \). This is actually a well-known result (see, for instance, Glanfield 1966; Felten & Isaacman 1986; Martel 1990).

3 THE CRITICAL DENSITY CONTRAST

Consider, at some initial redshift \( z_i \gg 1 \), a spherical perturbation of density \( \rho_i = \bar{\rho}_i (1 + \delta_i) \) in an otherwise uniform background of density \( \rho_i \). Let us focus on positive density perturbations \( (\delta_i > 0) \). Clearly, if the background universe is bound or marginally bound, then the perturbation is bound. However, if the background universe is unbound, then the perturbation can be either bound or unbound depending upon the value of the initial density contrast \( \delta_i \). Our goal in this section is to derive the critical density contrast \( \delta_{i,c} \), which is defined as the minimum value of \( \delta_i \) for which the perturbation is bound. To compute \( \delta_{i,c} \), we make use of the Birkhoff theorem, which implies that a uniform, spherically symmetric perturbation in an otherwise smooth Friedmann universe evolves like a separate Friedmann universe with the same mean energy density and equation of state as the perturbation.\[P\]

Pursuing this analogy, a perturbation with \( \delta_i > \delta_{i,c} \) behaves like a bound universe, a perturbation with \( \delta_i < \delta_{i,c} \) behaves like an unbound universe, and a perturbation with \( \delta_i = \delta_{i,c} \) behaves like a marginally bound universe. We can then use the results of the previous section to compute \( \delta_{i,c} \).

First, we need to derive expressions for the “effective cosmological parameters” of the perturbation. Notice first that an overdense perturbation has been decelerating relative to the background between the big bang and the initial redshift \( z_i \). Hence, at \( z = z_i \), the perturbation is expanding with an “effective Hubble constant” \( H'_i \) which is smaller than the Hubble constant \( H_i \) of the background universe. Assuming that the redshift \( z_i \) is small enough for linear theory to be accurate and for the universe to resemble an Einstein-de Sitter\[P\]

\[P\]

Note: For a nonuniform spherically symmetric perturbation, every spherical mass shell evolves as it would in a universe with the same mean energy density and equation of state as that of the average of the sphere bounded by that shell.
universe (\(\Omega_i \approx 1, \Omega_{Xi} \ll 1\)), but late enough to allow us to neglect the linear decaying mode, we can easily compute the relationship between \(H'_i\) and \(\delta_i\),

\[
H'_i = H_i \left(1 - \frac{\delta_i}{3}\right),
\]

(see, for instance, Lahav et al. 1991). The effective density parameters of the perturbation are then given by

\[
\Omega'_i = \frac{8\pi G \rho_i}{3H_i^2} = \frac{8\pi G \bar{\rho}_i (1 + \delta_i)}{3H_i^2 (1 - \frac{\delta_i}{3})^2} = \Omega_i (1 + \delta_i) \left(1 - \frac{\delta_i}{3}\right)^2,
\]

\[
\Omega'_{Xi} = \frac{8\pi G \rho_{Xi}}{3H_i^2} = \frac{8\pi G \rho_{Xi}}{3H_i^2 (1 - \frac{\delta_i}{3})^2} = \frac{\Omega_{Xi}}{(1 - \frac{\delta_i}{3})^2}.
\]

Next, we need to find combinations of \(\Omega'_i\) and \(\Omega'_{Xi}\) that correspond to "effective" marginally bound universes. For the cases for which \(n < 2\), this condition is given by equation (5). We now replace \(\Omega_0\) and \(\Omega_{Xi}\) by \(\Omega'_i\) and \(\Omega'_{Xi}\) in equation (5)\(^{**}\) and replace \(\delta_i\) by \(\delta_{i,c}\). This equation becomes

\[
\left[\frac{\Omega_i (1 + \delta_{i,c}) + \Omega_{Xi} - (1 - \frac{\delta_{i,c}}{3})^2}{3 - n}\right]^{3-n} = \left[\frac{\Omega_i (1 + \delta_{i,c})}{2 - n}\right]^{2-n} \Omega_{Xi}.
\]

Since \(\delta_{i,c} \ll 1\), we can expand this expression in powers of \(\delta_{i,c}\) and keep only leading terms. We can then simplify this expression further by using the approximation \(\Omega_i \approx 1\). Equation (10) reduces to

\[
\left[\frac{\Omega_i + \Omega_{Xi} - 1 + 5\delta_{i,c}/3}{3 - n}\right]^{3-n} = \frac{\Omega_{Xi}}{(2 - n)^{2-n}}.
\]

Notice that we had to keep the term \(\Omega_i\) in the left hand side because of the presence of the term \(-1\), and that we cannot expand the left hand side in powers of \(\delta_{i,c}\) because the quantity \(\Omega_i + \Omega_{Xi} - 1\) might be as small as \(\delta_{i,c}\). We now solve this equation for \(\delta_{i,c}\), and get

\[
\delta_{i,c} = \frac{3}{5} \left[\frac{3 - n}{(2 - n)^{(2-n)/(3-n)}} + 1 - \Omega_i - \Omega_{Xi}\right].
\]

This gives the critical density contrast as a function of the initial density parameters \(\Omega_i\) and \(\Omega_{Xi}\). We can reexpress it as a function of the present density parameters \(\Omega_0\) and \(\Omega_{X0}\) and the initial redshift, as follows: The initial density parameters are given by \(\Omega_i = 8\pi G \bar{\rho}_i / 3H_i^2 = 8\pi G \bar{\rho}_0 (1 + z_i)^3(H_0/H_i)^2/3H_0^2 = \Omega_0 (1 + z_i)^3(H_0/H_i)^2\) and \(\Omega_{Xi} = 8\pi G \rho_{Xi} / 3H_i^2 = 8\pi G \rho_{X0} (1 + z_i)^n(H_0/H_i)^2/3H_0^2 = \Omega_{X0} (1 + z_i)^n(H_0/H_i)^2\). The ratio \((H_0/H_i)^2\) is given directly by equation (1) (with \(a_0/a_i = 1 + z_i\)). We substitute these expressions into equation (12), and, using the fact that \(z_i \gg 1\), we keep only the leading terms in \((1 + z_i)^{-1}\). Equation (12) reduces to

\(^{**}\) That equation was derived using the present values of the density parameters, but it is of course valid at any epoch.
\[ \delta_{i,c} = \frac{3}{5(1 + z_i)} \left[ \frac{(3 - n)}{(2 - n) \gamma(3 - n)} \left( \frac{\Omega_{X0}}{\Omega_0} \right)^{1/(3 - n)} + \frac{1 - \Omega_0 - \Omega_{X0}}{\Omega_0} \right]. \]  

(13)

For the particular cases of a matter-dominated universe \((\Omega_{X0} = 0)\) or a flat universe with a nonzero cosmological constant \((n = 0, 1 - \Omega_0 - \Omega_{X0} = 0)\), we recover the results derived by Weinberg (1987) and Martel (1994, eqs. [7] and [8]).

For cases in the range \(2 \leq n < 3\), the condition for a marginally bound universe is \(1 - \Omega_0 - \Omega_{X0} = 0\). We substitute equations (8) and (9) into this expression, make the same approximations as above, and get

\[ \delta_{i,c} = \frac{3}{5}(1 - \Omega_i - \Omega_{Xi}). \]  

(14)

In terms of the present density parameters and the initial redshift, this expression reduces to

\[ \delta_{i,c} = \frac{3(1 - \Omega_0 - \Omega_{X0})}{5\Omega_0(1 + z_i)}. \]  

(15)

The case \(n = 3\) differs from all others in that the energy density of the X component does not diminish relative to that of the ordinary matter component as we go back in time. As such, we are never free to assume that the early behavior of the top-hat is the same as it would be in the absence of the X component. We shall, therefore, for simplicity, exclude this case \(n = 3\) from further consideration here.

4 THE ASYMPTOTIC COLLAPSED FRACTION

Our goal is to compute the fraction of the matter in the universe that will eventually end up inside collapsed objects (the asymptotic collapsed fraction). Clearly, this question only makes sense in unbound or marginally bound universes. In general, the answer depends upon the mass scale of the collapsed objects being considered. For cosmological models with Gaussian random noise initial conditions (the usual assumption), the density contrast \(\delta(\lambda)\) for fluctuations of comoving length scale \(\lambda\) is of order \([k^3P(k)]^{1/2}\), where \(k = 2\pi/\lambda\) is the wavenumber, and \(P(k)\) is the power spectrum. For a model such as Cold Dark Matter, for instance, the power spectrum decreases more slowly than \(k^{-3}\) at large \(k\). Thus the density contrast diverges at small scale. Normally, we eliminate small-scale perturbations from the calculation by filtering the power spectrum at the mass scale of interest, typically the mass required to form a galaxy. The density fluctuations at that scale have a variance \(\sigma^2\) given by

\[ \sigma^2 = \frac{1}{2\pi^2} \int_0^\infty P(k) \hat{W}^2(kR) k^2 dk, \]  

(16)
where $\hat{W}$ is a window function, and $R$ is the comoving radius of a sphere enclosing a mass in the unperturbed density field which is equal to the mass scale of interest. Assuming that the initial conditions are Gaussian, the fluctuation distribution for positive values of $\delta$ is given by

$$N(\delta) = \frac{2^{1/2}}{\pi^{1/2}\sigma} e^{-\delta^2/2\sigma^2}.$$  \hfill(17)

Our problem consists of computing the asymptotic collapsed fraction involving initially positive density fluctuations of mass equal to that contained on average by a sphere of comoving radius $R$, together with the additional mass which eventually accretes onto these positive density fluctuations from initially underdense regions, starting from initial conditions described by equations (16) and (17). In this section, we consider the analytical model introduced by MSW. In the next section, we will consider the well-known Press-Schechter approximation, instead.

Consider, at some early time $t_i$, a spherical, top-hat matter-density fluctuation of volume $V$ and density contrast $\delta_i$, surrounded by a compensating shell of volume $U$ and negative density contrast, such that the average density contrast of the system top-hat + shell vanishes. This model is parametrized by the shape parameter $s \equiv V/U$. If $\delta_i \geq \delta_{i,c}$, the top-hat core will collapse. Furthermore, a fraction of the matter located outside the top-hat, inside the shell, initially occupying a volume $U' \leq U$, will be accreted by the top-hat. Since the density is nearly uniform at early times, the asymptotic collapsed mass fraction of this system is simply $(V + U')/(V + U)$. We now approximate the initial conditions for the whole universe as an ensemble of these compensated top-hat perturbations, with a distribution of top-hat core positive density fluctuations given by equation (17), and we neglect the interaction between perturbations. As discussed in MSW, the value of $s = 0$ corresponds to the limit in which each positive fluctuation is isolated, surrounded by an infinite volume of compensating underdensity (at a total density infinitessimally below the mean value $\bar{\rho}$). For a flat universe with nonzero cosmological constant, this case was treated by Weinberg (1996). The case $s = \infty$ corresponds to the limit of “no infall” in which the additional mass associated with the compensating underdense volume $V$ is negligible compared with that of the initial top-hat. This case was considered for the flat universe with $\lambda_0 \neq 0$ by Weinberg (1987). If $s = 1$, however, the volume occupied by every positive fluctuation is surrounded by an equal volume of compensating negative density fluctuation. This is the case most relevant to the problem at hand, involving a Gaussian-random distribution of linear density

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fluctuations, since the latter ensures that the volumes initially occupied by positive and negative density fluctuations of equal amplitude are exactly equal. The full range of values of \( s, 0 \leq s \leq \infty \), was treated by MSW for the flat universe with \( \lambda_0 \neq 0 \), with a special focus on \( s = 1 \) as the case corresponding to Gaussian-random noise initial conditions. The insensitivity of the results for the anthropic probability calculations presented there to the value assumed for \( s \) suggests that the relative amount of total collapsed fraction in universes with different values of \( \rho_V \) may not be sensitive to the crudeness of the treatment of the effect of one fluctuation on another. However, we will also present results here for the full range of values of \( s \), while noting that the value \( s = 1 \) is the most relevant to the case at hand of Gaussian random density fluctuations.

Under these assumptions, the asymptotic collapsed fraction for the whole universe is given by

\[
f_{c,\infty} = \frac{2^{1/2}s}{\pi^{1/2}\sigma_i} \int_{\delta_{i,c}}^{\infty} \frac{\delta e^{-\delta^2/2\sigma_i^2} d\delta}{\delta_{i,c} + s\delta},
\]

where \( \sigma_i \) is the value of \( \sigma \) at time \( t_i \) (MSW). For bound and marginally bound universes (including, in particular, the Einstein-de Sitter universe), \( \delta_{i,c} = 0 \), and equation (18) reduces trivially to \( f_{c,\infty} = 1 \) for all values of \( s \). Hence, the MSW model predicts that, in an Einstein-de Sitter universe, all the matter will eventually end up in collapsed objects. For unbound universes, we change variables from \( \delta \) to \( x \equiv \delta^2/2\sigma_i^2 \). Equation (18) reduces to

\[
f_{c,\infty} = \frac{s}{\pi^{1/2}} \int_{\beta}^{\infty} \frac{e^{-x} dx}{sx^{1/2} + \beta^{1/2}},
\]

where

\[
\beta \equiv \frac{\delta_{i,c}^2}{2\sigma_i^2}.
\]

This equation shows that the collapsed fraction \( f_{c,\infty} \) is unity only when \( \beta = 0 \), which requires \( \delta_{i,c} = 0 \). However, in an unbound universe, \( \delta_{i,c} \) is always positive. Hence, according to the MSW model, the collapsed fraction in an unbound universe is always less than unity. Notice that the dependence upon the cosmological parameters is entirely contained in the parameter \( \beta \). For any cosmological model, we can compute \( \sigma_i \) using equation (16) and \( \delta_{i,c} \) using either equation (13) or (15). Since \( \sigma_i \propto (1 + z_i)^{-1} \) at large \( z_i \) for any universe with \( n < 3 \), the dependence on \( z_i \) cancels out in the calculation of \( \beta \), as it should: The asymptotic collapsed fraction should not depend upon the initial epoch chosen for the calculation.

The size of the asymptotic collapse parameter \( \beta \) determines not only how large or small the collapsed fraction is but also how important the increase of collapsed fraction is due to
accretion from the surrounding underdense regions. For small values of \( \beta \), the asymptotic collapsed fraction is close to unity because both the typical positive initial density fluctuation and its fair share of the matter in surrounding regions of compensating underdensity collapse out before the effects of “freeze-out” suppress fluctuation growth. Hence, in this limit of small \( \beta \), “freeze-out” is unimportant and the results resemble that for an Einstein-de Sitter universe. For values of \( \beta \gtrsim 1 \), however, the typical collapse occurs after “freeze-out” has begun to limit the growth of density fluctuations. The large \( \beta \) limit, in fact, is that in which only a rare, much-higher-than-average, positive density fluctuation is able to collapse out of the background before “freeze-out” prevents it, and very little of the compensating underdense matter condenses out along with it. For this large \( \beta \) limit, equation (19) can be shown to reduce to the following simple formula (see Appendix A),

\[
 f_{c,\infty}(\beta \gg 1) = \left( \frac{s}{s+1} \right) \frac{e^{-\beta}}{(\pi \beta)^{1/2}}.
\]

(21)

5 THE ASYMPTOTIC LIMIT OF THE PRESS-SCHECHTER APPROXIMATION

In the Press-Schechter approximation (Press & Schechter 1974; henceforth, “PS”), the collapsed fraction at time \( t \) is estimated as follows: Consider a spherical top-hat perturbation with an initial linear density contrast \( \delta_i \) chosen such that this perturbation collapses precisely at time \( t \). The density contrast of that perturbation is infinite at time \( t \). However, if we estimate the density contrast at that epoch using linear perturbation theory, we obtain instead a finite value \( \delta = \Delta_c \), because linear theory underestimates the growth of positive fluctuations. The value of \( \Delta_c \) is usually taken to be \( (3/5)(3\pi/2)^{2/3} = 1.6865 \), though this result is strictly correct only for the Einstein-de Sitter universe (cf. Shapiro, Martel, & Iliev 1999, and references therein). A larger perturbation would collapse earlier, and linear theory would predict that its density contrast exceeds \( \Delta_c \) at time \( t \). To compute the collapsed fraction at time \( t \), we simply need to integrate over all perturbations whose density contrast predicted by linear theory would exceed \( \Delta_c \) at time \( t \), using the distribution given by equation (17). The resulting expression, after multiplication by a factor of “2” to correct for the fact that half the mass was initially in underdense regions outside the positive density fluctuations, is

\[
 f_{c,PS} = \frac{2^{1/2}}{\pi^{1/2}\sigma(t)} \int_{\Delta_c(t)}^{\infty} e^{-\delta^2/2\sigma(t)^2} d\delta .
\]

(22)
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The introduction of this ad hoc correction factor of “2” in equation (22) is based on some assumption about the amount of matter located in unbound regions, either underdense or overdense, which is destined to be accreted onto collapsed perturbations. Consider, for instance, the case of an Einstein-de Sitter universe at late time. The critical density contrast distinguishing a bound from an unbound density fluctuation is zero, and therefore all overdense perturbations are bound and will eventually collapse. Since, for Gaussian perturbations, the overdense regions initially contain only half the mass of the universe, the asymptotic collapsed fraction, without taking accretion into account, would be $f_{c,\infty} = 1/2$. However, it can easily be shown that in an Einstein-de Sitter universe, all matter in the universe will eventually end up inside bound objects. Hence, for this particular case, the proper way to handle accretion is to multiply the collapsed fraction by a factor of 2. Equation (22) is derived by assuming that this factor of 2 is valid, not only for the asymptotic limit of the Einstein-de Sitter universe, but for all universes and at all epochs. Hence, the PS approximation assumes that the total mass accreted by collapsed positive density fluctuations is instantaneously equal to the total mass of these collapsed objects themselves.

What is the asymptotic collapsed fraction according to this PS approximation? We now change variables from $\delta$ to $x = \delta^2/2\sigma^2$. Equation (22) reduces to

$$f_{c,\infty}^{\text{PS}} = \frac{1}{\pi^{1/2}} \beta_{\text{PS}} \int_{-\infty}^{\infty} e^{-x^2} dx,$$

(23)

where

$$\beta_{\text{PS}}(t) \equiv \frac{\Delta_c(t)^2}{2\sigma(t)^2}.$$

(24)

To compute the asymptotic collapsed fraction, $f_{c,\infty}^{\text{PS}}$, we need to take the limit of equations (23) and (24) as $t \to \infty$. Consider a bound spherical perturbation, with the values of its initial density contrast $\delta_i$ at initial time $t_i$ chosen so that it collapses at $t = \infty$. Call this value of $\delta_i$, $\delta_{i,\infty}$. By definition, the quantity $\Delta_c$ at $t = \infty$ is given by

$$\Delta_c(\infty) = \delta_{i,\infty} \frac{\delta_+(\infty)}{\delta_+(t_i)},$$

(25)

where $\delta_+(t)$ is the linear growing mode. Since this spherical perturbation collapses at $t = \infty$, the initial density contrast $\delta_{i,\infty}$ must be equal to the critical density contrast $\delta_{i,c}$. If $\delta_{i,\infty}$ was less than $\delta_{i,c}$ the perturbation would not collapse at all, while if it was greater, the perturbation would collapse at a finite time. We can therefore replace $\delta_{i,\infty}$ by $\delta_{i,c}$ in equation (25). Finally, we notice that the quantity $\sigma$ also evolves according to linear theory,
\[ \sigma(\infty) = \frac{\delta_+(\infty)}{\delta_+(t_i)}. \]  

(26)

Combining these results, we get

\[ \beta_{\text{PS}}(\infty) = \frac{[\delta_{i,c}\delta_+(\infty)/\delta_+(t_i)]^2}{2[\sigma_i\delta_+(\infty)/\delta_+(t_i)]^2} = \frac{\delta_{i,c}^2}{2\sigma_i^2} = \beta, \]  

(27)

(see eq. [20]). Hence, the PS \( \beta_{\text{PS}} \) parameter reduces to the MSW \( \beta \) parameter in the limit \( t \to \infty \). Now, comparing equations (19) and (23), we see immediately that these equations are identical in the limit \( s \to \infty \). Notice that the product \( \delta_{i,c}\delta_+(\infty) \) takes the undetermined form \( 0 \cdot \infty \) in the case of an Einstein-de Sitter universe. In this case, the quantity \( \delta_{i,c}\delta_+(\infty)/\delta_+(t_i) \) is equal to \( (3/5)(3\pi/2)^{2/3} \), or 1.6865, at all times. In the Einstein-de Sitter case, \( \beta = \beta_{\text{PS}}(\infty) = 0 \), and equations (19) and (23) are the same for all values of \( s \); the asymptotic collapsed fractions in that case are all equal to unity.

In the limit of large \( \beta \), \( f_{c,\infty}^{\text{PS}} \) in equation (23), with \( \beta_{\text{PS}} \) replaced by \( \beta \), according to equation (27), can be shown to reduce to the following simple formula (see Appendix A):

\[ f_{c,\infty}^{\text{PS}}(\beta \gg 1) = \frac{e^{-\beta}}{(\pi \beta)^{1/2}}. \]  

(28)

A comparison of equations (21) and (28) reveals that the asymptotic collapsed fraction \( f_{c,\infty}^{\text{PS}} \) according to the PS approximation is just a factor of \( (s + 1)/s \) times \( f_{c,\infty} \) according to the MSW model, in the limit of large \( \beta \).

6 DISCUSSION AND CONCLUSION

Our analytical result in equations (19) and (20) for the asymptotic collapsed fraction in an eternal universe can be evaluated for any background universe which satisfies the conditions given in §2 which identify it as an unbound universe. We need only specify the background universe and the power spectrum of primordial density fluctuations, in order to evaluate \( \beta \). Before we do this for a few illustrative cases, however, it is instructive to evaluate the asymptotic collapsed fraction \( f_{c,\infty} \) in general as a function of \( \beta \) and \( s \), and compare \( f_{c,\infty} \) to the prediction of the PS approximation, \( f_{c,\infty}^{\text{PS}} \), according to equations (23) and (27).

We have shown above that the asymptotic collapsed fraction predicted for an eternal universe by the PS approximation differs from that predicted here by the spherical model of MSW (as generalized to other background universe cases) for \( s = 1 \), the value of the shape parameter appropriate for Gaussian random initial density fluctuations, with the exception of the Einstein-de Sitter universe, for which \( f_{c,\infty}^{\text{PS}} = f_{c,\infty} = 1 \). For any eternal universe other
than Einstein-de Sitter, in fact, the two approaches predict the same asymptotic collapsed fraction only if $s = \infty$, instead. The fact that the two approaches generally predict different asymptotic collapsed fractions for $s = 1$ is not surprising, since the PS approximation never concerns itself with the fraction of matter which is inside some gravitationally bound region and is, hence, fated to collapse out, as the MSW model explicitly does. Instead, the PS approximation assumes that, as long as the matter is located within a region of average density which is high enough to make it collapse according to the spherical top-hat model, it will not only collapse but will also take with it an equal share of the matter outside this region which was not initially overdense. This latter assumption is not correct if the underdense matter is not all gravitationally bound to some overdense matter. What is perhaps more surprising than this disagreement between the two approaches for $s = 1$ is the fact that they do agree for all models if $s = \infty$.

The fact that in the limit $s \to \infty$ the MSW model reduces to the asymptotic limit of the PS approximation is significant, because the two models are based on different assumptions. In the case of the PS approximation, a factor of 2 is introduced to take accretion into account. In the MSW model, the limit $s \to \infty$ corresponds to perturbations surrounded by underdense shells of negligible volume and mass. In this limit, there is essentially no accretion. However, the volume filling factor of overdense regions, which is $1/2$ in the PS approximation, approaches unity in the limit $s \to \infty$ for the MSW model, resulting once again in a factor of 2 in the expression for the collapsed fraction, but for a different reason.

We have computed the collapsed fraction predicted by the MSW model as a function of the parameter $\beta$, for various values of $s$, by numerically evaluating equation (19). The results are plotted in Figure 2. In addition, the analytical expression in equation (21) which is valid in the large $\beta$ limit is plotted in Figure 2 for the case $s = 1$. The analytical expression provides an excellent fit to the exact results for the important case of $s = 1$, not only for large $\beta$, but for all $\beta \gtrsim 1$. The error even at $\beta = 1$, for example, is only 15%, while at $\beta = 5$, the error is reduced to 4.5%. For comparison, we also show the prediction of the PS approximation, according to equations (23) and (27). This curve is identical to the curve for $f_{c,\infty}$ for the case $s = \infty$. The point $\beta = 0$, $f_{c,\infty} = 1$ corresponds to the Einstein-de Sitter universe. As we see, all curves go through this point, indicating that the MSW model predicts the correct asymptotic limit in this case, for any value of $s$. The $s = 1$ case is particularly important, since it is the only one which gives equal filling factors to overdense
Figure 2. Asymptotic collapsed fraction $f_{c,\infty}$ versus $\beta$, calculated using the MSW model for various values of the shape parameter $s$ (solid curves). The curve for $s = \infty$ is identical to $f_{c,\infty}^{PS}$, the asymptotic limit ($t \to \infty$) of the Press-Schechter approximation. Also plotted is the simple algebraic formula of the MSW model in equation (21), derived for the large $\beta$ limit, for the case $s = 1$ (dashed curve).

and underdense perturbations, a requirement for describing realistic Gaussian random initial conditions.

Figure 2 shows that for finite values of $s$ and for $\beta > 0$, the asymptotic collapsed fraction $f_{c,\infty}$ predicted by the MSW model is always less than the asymptotic collapsed fraction $f_{c,\infty}^{PS}$ predicted by the PS approximation. For $s < 1$, this is not surprising, since in this limit the filling factor of the overdense regions is below the value of 1/2 assumed by the PS approximation. However, if $s > 1$, then the filling factor of overdense regions exceeds 1/2, indicating that the bound perturbations contain more mass in the MSW model than in the PS approximation. In spite of this, we still have $f_{c,\infty} < f_{c,\infty}^{PS}$. This is caused by their different treatments of accretion. The MSW model includes a detailed calculation of the amount of matter accreted by a spherical top-hat, while the PS approximation simply assumes that the accreted mass equals the initially overdense mass, for all cosmological models. Figure 2 suggests that this approximation can be quite crude in some situations and greatly overestimate the amount of matter actually accreted.

To estimate this effect, we have computed, for the MSW model, the “accretion factor,” $F_{acc}$, defined as the ratio of the total asymptotic collapsed fraction $f_{c,\infty}$ divided by the asymptotic collapsed fraction $f_{c,\infty}^{*}$ that we would obtain if accretion were neglected. (This
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Figure 3. Accretion factor $F_{\text{acc}} \equiv f_{c,\infty}(\text{accretion included})/f_{c,\infty}(\text{no accretion})$ versus $\beta$ for the MSW model, for various values of the parameter $s$. The dashed line indicates the value of 2 which is used in the Press-Schechter approximation.

The factor $F_{\text{acc}}$ is 2 for the PS approximation. We can easily compute $f_{c,\infty}^*$ by going back to the derivation of MSW and dropping the term in equation (19) which represents the accreted matter. The resulting expression is

$$f_{c,\infty}^*(s, \beta) = \frac{1}{\pi^{1/2}} \left( \frac{s}{s+1} \right) \int_0^\infty \frac{e^{-x} dx}{x^{1/2}} = \left( \frac{s}{s+1} \right) f_{c,\infty}(s = \infty, \beta). \quad (29)$$

Hence, the accretion factor is

$$F_{\text{acc}}(s, \beta) = \left( \frac{s+1}{s} \right) \frac{f_{c,\infty}(s, \beta)}{f_{c,\infty}(\infty, \beta)}. \quad (30)$$

In the large $\beta$ limit, equations (21) and (30) indicate that $F_{\text{acc}}(s, \beta \gg 1) = 1$; in this limit, none of the matter in the compensating underdense regions is able to condense out.

In Figure 3, we plot this accretion factor $F_{\text{acc}}$ as a function of $\beta$, for various values of $s$. The factor $F_{\text{acc}}^{\text{PS}} = 2$ for the PS approximation is indicated by the dashed line. For the MSW model, the accretion factor depends mostly on the amount of matter available in the shell surrounding the top-hat, which goes to zero in the limit $s \to \infty$ and to infinity in the limit $s \to 0$. For the interesting case $s = 1$ (underdense and overdense regions with equal filling factors), we recover the PS limit $F_{\text{acc}} = 2$ at small $\beta$, but the value departs rapidly from 2 at larger $\beta$. At $\beta = 1$, for example, the accretion factor drops to 1.125, indicating that the PS approximation overestimates the amount of matter being accreted by a factor of 8!

To demonstrate the importance of this effect for actual cosmological models, we consider...
two variations of the Cold Dark Matter (CDM) model: (a) open, matter-dominated CDM ($\Omega_{X_0} = 0$), and (b) flat CDM with nonzero cosmological constant ($\Omega_{X_0} = \lambda_0 = 1 - \Omega_0, n = 0$), both with an untilted primordial Harrison-Zel’dovich power spectrum\(^{\dagger\dagger}\). The primordial density fluctuation power spectrum for this model, consistent with the standard inflationary cosmology and the measured anisotropy of the cosmic microwave background according to the COBE DMR experiment, is described in great detail in Bunn & White (1997, and references therein). In the absence of tilt, this power spectrum (extrapolated to the present according to linear theory) is given by

$$P(k) = 2\pi^2 \left(\frac{c}{H_0}\right)^4 \delta^2_H k^n T^2_{\text{CDM}}(k).$$

where $c$ is the speed of light and $T_{\text{CDM}}$ is the transfer function, given by

$$T_{\text{CDM}}(q) = \ln(1 + 2.34q)^2 \ln(1 + 2.34q)^2$$

(Bardeen et al. 1986), with $q$ defined by

$$q = \left(\frac{k}{\text{Mpc}^{-1}}\right) \alpha^{-1/2} (\Omega_0 h^2)^{-1} \Theta_{2.7}^2,$$

$$\alpha = a_1^{-\Omega_b/\Omega_0} a_2^{-(\Omega_b/\Omega_0)^3},$$

$$a_1 = (46.9 \Omega_0 h^2)^{0.670} [1 + (32.1 \Omega_0 h^2)^{-0.532}],$$

$$a_2 = (12.0 \Omega_0 h^2)^{0.424} [1 + (45.0 \Omega_0 h^2)^{-0.582}]$$

(Hu & Sugiyama 1996, eqs. [D-28] and [E-12]), where $\Omega_b$ is the density parameter of the baryons, and $\Theta_{2.7}$ is the temperature of the cosmic microwave background in units of 2.7K.

The quantity $\delta_H$ is given by

$$\delta_H = \begin{cases} 
1.95 \times 10^{-5} \Omega_0^{-0.35 - 0.19 \ln \Omega_0}, & \lambda_0 = 0, \text{ no tilt;} \\
1.94 \times 10^{-5} \Omega_0^{-0.785 - 0.05 \ln \Omega_0}, & \lambda_0 = 1 - \Omega_0, \text{ no tilt;}
\end{cases}$$

Once the power spectrum is specified, we can compute the variance $\sigma^2$ of the present density contrast (i.e. as extrapolated to the present using linear theory) as a function of the comoving length scale or, equivalently, mass scale over which the density field is smoothed, using equation (16). We then compute the parameter $\beta$ using equation (20), where $\delta_{i,c}$ is given by either equation (13) or (15), and $\sigma_i = \sigma \delta_+(z_i)/\delta_+(0)$. After some algebra, we get

$$\beta = \frac{9}{50 \sigma^2 \eta^2(\Omega_0, \lambda_0, z_i)} \left[3 \left(\frac{\lambda_0}{4 \Omega_0}\right)^{1/3} + \frac{1 - \Omega_0 - \lambda_0}{\Omega_0}\right]^2,$$

where the function $\eta(\Omega_0, \lambda_0, z)$ is defined by

\(^{\dagger\dagger}\) The exponent of the primordial power spectrum, which is unity in the absence of tilt, is usually designated by the letter $n$. It should not be confused by the exponent $n$ used in this paper, which is introduced in equation (1).
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\[ \eta(\Omega_0, \lambda_0, z) = (1 + z) \frac{\delta_+(z)}{\delta_+(0)} \]  

(MSW). In the limit \( z \gg 1 \), which we assume here, the function \( \eta \) becomes independent of \( z \). For flat models \( (\Omega_0 + \lambda_0 = 1) \), MSW derived the following expression:

\[ \eta(\Omega_0, 1 - \Omega_0, z \gg 1) = \frac{6\lambda_0^{5/6}}{5\Omega_0^{1/3}} \left[ \int_0^{\lambda_0/\Omega_0} \frac{dw}{w^{1/6}(1 + w)^{3/2}} \right]^{-1}. \]

(40)

For matter-dominated models, we can easily compute the function \( \eta \) using the expressions given in Peebles (1980). For open models, we get

\[ \eta(\Omega_0, 0, z \gg 1) = \frac{2(1 - \Omega_0)}{5\Omega_0} \left[ 1 + \frac{3\Omega_0}{1 - \Omega_0} + \frac{3\Omega_0}{(1 - \Omega_0)^{3/2}} \ln \left[ \frac{1 - (1 - \Omega_0)^{1/2}}{\Omega_0^{1/2}} \right] \right]^{-1}. \]

(41)

The fraction of matter eventually collapsed into objects created by positive density fluctuations of mass greater than or equal to some mass \( M \) is entirely specified by the parameter \( \beta \) evaluated for this mass scale as the density field filter mass. Once \( \beta \) is known, we can compute the asymptotic collapsed mass fractions \( f_{c,\infty} \) and \( f_{PS,\infty} \) using equations (19) and (23), respectively. We consider models with \( H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1} (h = 0.7), \Omega_b = 0.015h^{-2} \) (Copi, Schramm, & Turner 1995), and \( \Theta_{2.7} = 1 \). We have computed \( \sigma^2 \) using equation (16) with a top-hat window function,

\[ W(kR) = \frac{3}{(kR)^3} (\sin kR - kR \cos kR). \]

(42)

In Figure 4, we plot the variation of the asymptotic collapse parameter \( \beta \) with the filter mass \( M \) (which corresponds to the length scale \( R \) in equation (42) according to \( M = 4\pi R^3 \rho_c \Omega_0 / 3 \), or \( M/M_\odot = 1.163 \times 10^{12} R_{\text{Mpc}}^3 h^2 \Omega_0 \) for two cases of interest: (a) open, matter-dominated, \( \Omega_0 = 0.3 \), and (b) flat with cosmological constant, \( \Omega_0 = 0.3 = 1 - \lambda_0 \). The value \( \beta = 1 \) for these two cases corresponds to the mass scales \( M/M_\odot = 3.651 \times 10^{14} \) (open) and \( 5.778 \times 10^{14} \) (flat), respectively. For \( H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1} \) and density parameter \( \Omega_0 = 0.3 \) (assuming the shape parameter \( s = 1 \), as required for Gaussian random noise density fluctuations), the open, matter-dominated CDM model and the flat CDM model with nonzero cosmological constant yield mass fractions asymptotically collapsed into objects created by positive density fluctuations of mass greater than or equal to the galaxy cluster mass-scale \( 10^{15} M_\odot \) of 0.0361 and 0.0562, respectively. These values of the asymptotic collapsed fraction are only 55% of the values determined by the Press-Schechter approximation. These results have implications for the use of the latter approximation to compare the observed space density of X-ray clusters today with that predicted by cosmological models.

We have also calculated the asymptotic collapsed fractions \( f_{c,\infty} \) and \( f_{PS,\infty} \) as a function
Figure 4. Asymptotic collapse parameter $\beta$ versus the filter mass scale $M$ for two COBE-normalized CDM models of interest (with $\Omega_0 h^2 = 0.015$, $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and a Harrison-Zel'dovich power spectrum): (a) open, matter-dominated ($\Omega_0 = 0.3$, $\lambda_0 = 0$) and (b) flat, with cosmological constant ($\Omega_0 = 0.3$, $\lambda_0 = 0.7$).

of $\Omega_0$ (assuming $s = 1$), for four different filter mass scales $M/M_{\odot} = 10^6$, $10^9$, $10^{12}$, and $10^{15}$ (notice that the length scale $R$ corresponding to a given mass scale varies with $\Omega_0$). The results are shown in Figure 5. In Figure 5a, $f_{c,\infty}$ and $f_{c,\infty}^{\text{PS}}$ are each plotted separately, while in Figure 5b, we plot the ratio $f_{c,\infty}^{\text{PS}}/f_{c,\infty}$ to demonstrate the extent to which the Press-Schechter approximation overestimates the collapsed fraction, especially for cluster mass objects and above.

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Figure 5. (a) (top panels) Asymptotic collapsed fraction $f_{c,\infty}$ versus $\Omega_0$ for COBE-normalized CDM models with a Harrison-Zel’dovich power spectrum, $H_0 = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$, and $\Omega_b h^2 = 0.015$: (i) open, matter-dominated models (top panel), and (ii) flat models with a nonzero cosmological constant (bottom panel). The solid curves show the results obtained using the MSW model. The dashed curves show the results obtained using the Press-Schechter approximation. Each panel has four curves of each type, corresponding to filter mass scales $M/M_\odot$ of $10^6$ (top curves), $10^9$, $10^{12}$, and $10^{15}$ (bottom curves); (b) (bottom panels) The ratio of the asymptotic collapsed fractions calculated by the Press-Schechter approximation to those calculated using the MSW model, for the cases shown in Fig. 5a. Curves are labelled with the filter mass scales in solar mass units.

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APPENDIX A: THE LARGE $\beta$ LIMIT

A1 The MSW Model

The asymptotic collapsed fraction according to the MSW model is given by

$$f_{c,\infty} = \frac{s}{\pi^{1/2}} \int_{\beta}^{\infty} \frac{e^{-x}dx}{s x^{1/2} + \beta^{1/2}}. \tag{A1}$$

If we change variables using $x = \beta(1 + w)$, then equation (A1) reduces to

$$f_{c,\infty} = \frac{s \beta^{1/2} e^{-\beta}}{\pi^{1/2}} \int_{0}^{\infty} \frac{e^{-\beta w}dw}{s(w + 1)^{1/2} + 1}. \tag{A2}$$

In the limit $1/\beta \ll 1$, we can always find a number $\alpha$ such that $1/\beta \ll \alpha \ll 1$. Since $\alpha \beta \gg 1$, we can truncate the integral in equation (A2) at $w = \alpha$, because the exponential $e^{-\beta w}$ is negligible for larger values of $w$. Hence

$$f_{c,\infty} \approx \frac{s \beta^{1/2} e^{-\beta}}{\pi^{1/2}} \int_{0}^{\alpha} \frac{e^{-\beta w}dw}{s(w + 1)^{1/2} + 1}. \tag{A3}$$

Since $\alpha \ll 1$, the integration variable $w$ is always much smaller than unity, and we can replace $w + 1$ by 1 in the denominator. The resulting integral yields
The asymptotic collapsed fraction in an eternal universe

\[ f_{c,\infty} \approx \left( \frac{s}{s+1} \right) \frac{e^{-\beta}(1 - e^{-\beta\alpha})}{(\pi\beta)^{1/2}}. \]  \hspace{1cm} (A4)

Since \( \beta\alpha \gg 1 \), the term \( e^{-\beta\alpha} \) is negligible. The final expression is

\[ f_{c,\infty}(\beta \gg 1) \approx \left( \frac{s}{s+1} \right) \frac{e^{-\beta}}{(\pi\beta)^{1/2}}. \]  \hspace{1cm} (A5)

A2 The PS Approximation

The asymptotic collapsed fraction according to the PS approximation is given by

\[ f_{c,\infty}^{PS} = \frac{1}{\pi^{1/2}} \int_{\beta}^{\infty} \frac{e^{-x}}{x^{1/2}} \, dx. \]  \hspace{1cm} (A6)

A change of variables to \( w = x^{1/2} \) allows us to rewrite equation (A6) as follows:

\[ f_{c,\infty}^{PS} = \frac{2}{\pi^{1/2}} \int_{\beta^{1/2}}^{\infty} e^{-w^2} \, dw = 1 - \text{erf}(\beta^{1/2}). \]  \hspace{1cm} (A7)

For large \( \beta \),

\[ \text{erf}(\beta^{1/2}) \approx 1 - \frac{e^{-\beta}}{(\pi\beta)^{1/2}}. \]  \hspace{1cm} (A8)

Combining equations (A7) and (A8), we find

\[ f_{c,\infty}^{PS}(\beta \gg 1) \approx \frac{e^{-\beta}}{(\pi\beta)^{1/2}}. \]  \hspace{1cm} (A9)