Geometrically nonlinear calculation of thin shells taking into account shear deformations when using the form of interpolation of the sought quantities

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Annotation. A finite element model for the analysis of geometrically nonlinear deformation of a thin-walled shell-type structure based on the principles of the Timoshenko type shear theory is proposed. As the basis of this model, we consider a fragment of the surface of the object under study in the form of a curved quadrilateral with nodes that coincide with its vertices. The desired unknowns at the nodes of the curved quadrilateral were the increments of the components of the displacement vector and the partial derivatives of these increments with respect to the natural coordinates of the surface of the shell object under study, as well as the increments of the components of the vector of the angles of rotation of the normal. To obtain interpolation expressions for the desired values, we implemented a fundamentally different vector form of the interpolation procedure from the standard one. The principal distinguishing feature of the above-mentioned form of interpolation is the compilation of interpolation dependencies not for each desired variable parameter as an isolated scalar value, but for the increment of the displacement vector and the increment of the vector of the angles of rotation of the normal. To obtain interpolation expressions for the desired values, we implemented a fundamentally different vector form of the interpolation procedure from the standard one. The principal distinguishing feature of the above-mentioned form of interpolation is the compilation of interpolation dependencies not for each desired variable parameter as an isolated scalar value, but for the increment of the displacement vector and the increment of the vector of the angles of rotation of the normal, which act as interpolation objects. As a result, in a curved coordinate system, original interpolation dependencies were obtained for the increments of the components of the displacement vectors and the angles of rotation of the normal at an arbitrary point of the quadrilateral, which are functions of the nodal values of all the increments of the components of the above-mentioned vectors, and not just the increments of the components of one particular direction.

1. Introduction

Large-span structures made of shells or their parts at the present stage of the development of the technosphere are becoming increasingly widespread in many engineering industries. The curved shape of the surface of such objects allows you to fully realize the strength properties of the material used with minimal weight and sufficient stability. Digitalization of the economy in general and the construction industry in particular, actualizes the development of numerical methods for analyzing nonlinear deformation processes of thin-walled structures made of shells or their parts [1-3]. Among other numerical methods, the finite element method (FEM) is currently considered the most promising [4-9]. The fundamental aspect in the finite element method is the interpolation procedure of the
desired unknowns through their values at the nodal points. At the present stage, in the FEM, the interpolation procedure has become almost ubiquitous, based on the compilation of an interpolation expression for each component of the displacement vector or for the increment component of this vector, if we are talking about nonlinear analysis, by means of nodal values of the same component. The specified interpolation technique is quite justified in the Cartesian rectangular coordinate system. However, due to the curvilinearity of the surface of the technosphere objects made of shells or their parts, the most acceptable in this case should be considered curved coordinate systems. But when using curved coordinate systems in the finite element analysis of objects with curved boundaries, it is necessary to take into account the displacements of the finite element as a rigid whole. Accounting for such displacements is possible only when implementing the interpolation of displacement vectors or their increments, which involves the compilation of interpolation expressions for vector quantities, and not their individual components [10].

The aim of the study is to develop an algorithm for the finite element analysis of the processes of deformation of objects in the shape of a shell in a geometrically nonlinear formulation, taking into account the shear deformations when implementing the interpolation of the increments of the displacement vectors and the increments of the vectors of the angles of rotation of the normal.

2. Materials and methods

2.1. Geometric relations

The surface of the shell structure or its fragment, which is equidistant from the inner and outer boundaries, can be given by the radius vector formula

$$\vec{R}^0 = x(\theta^1, \theta^2)\hat{i} + y(\theta^1, \theta^2)\hat{j} + z(\theta^1, \theta^2)\hat{k},$$

where $\theta^1$, $\theta^2$ is the curvilinear coordinates of the surface equidistant from the outer and inner boundaries.

The vectors of the local basis point $N^0$ of this surface can be determined by the formulas

$$\vec{a}_1^0 = \vec{R}^0_1, \quad \vec{a}_2^0 = \vec{R}^0_2, \quad \vec{a}^0 = \left(\vec{a}_1^0 \times \vec{a}_2^0\right)/\left|\vec{a}_1^0 \times \vec{a}_2^0\right|,$$

where the comma indicates the differentiation operation, and the subscripts 1 and 2 after the comma indicate that the differentiation is performed by the curvilinear coordinates $\theta^1$ and $\theta^2$, respectively.

When developing computational algorithms in a geometrically nonlinear formulation, as a rule, the procedure of sequential loading in steps is used. When implementing this loading procedure, the points $N^0$ and $\zeta N^0$ (spaced from the point $N^0$ at a distance $\zeta$ along the normal) will move in $j$ steps to the points $N^j$ and $\zeta N^j$, respectively, and after the $(j+1)$-th loading step, the above points will occupy the positions $N^{\ast}$ and $\zeta N^{\ast}$. The total and step vectors of displacements of a point of an equidistant surface $N^0$ can be represented by components of the local basis (2) of this point

$$\vec{v} = v^\rho \vec{a}_\rho^0 + v^0 \vec{a}_0^0, \quad \vec{w} = w^\rho \vec{a}_\rho^0 + w^0 \vec{a}_0^0,$$

where the Greek indices $\rho$ consistently take the values 1, 2.

The movements of the point $N^{0\zeta}$ for $j$ and for $(j+1)$-th loading steps can be determined by the corresponding vectors

$$\vec{V} = \vec{v} + \zeta \vec{G}; \quad \vec{W} = \vec{w} + \zeta \vec{\gamma},$$

where $\vec{G}$ and $\vec{\gamma}$ are the total and step vectors of the normal rotation.

The positions of the point of an arbitrary shell layer $N^{0\zeta}$ after $j$ steps and after the $(j+1)$-th loading step can be fixed by the following radius vectors

$$\vec{R}^\zeta = \vec{R}^{0\zeta} + \vec{V}; \quad \vec{R}^{\zeta\ast} = \vec{R}^\zeta + \vec{W},$$
where $\tilde{R}^\zeta = \tilde{R}^0 + \zeta \tilde{a}^0$.

By applying the differentiation operation along the surface coordinates and along the normal to the surface (5), we can obtain the vectors of the basis point of an arbitrary layer $N^\zeta$ in the initial and deformed states after $j$ steps and after the $(j+1)$-th loading step

$$\tilde{g}_0^0 = \tilde{R}^\zeta; \quad \tilde{g}_\rho = \left(\tilde{R}^\zeta + \tilde{V}\right)_\rho;$$

$$\tilde{g}_3^0 = \tilde{R}^\zeta; \quad \tilde{g}_3 = \left(\tilde{R}^\zeta + \tilde{V}\right)_3; \quad \tilde{g}_3^* = \left(\tilde{R}^\zeta + \tilde{W}\right)_3,$$

where the subscript $\zeta$ after the decimal point denotes the operation of differentiation along the normal to the surface of the shell structure.

The total deformations after $j$ loading steps and their increments at the $(j+1)$-th loading step are equal to the half-differences of the components of the metric tensors [11]

$$\varepsilon^\zeta_{mn} = \left(g_{mn}^0 - g_{mn}^0\right)/2; \quad \Delta \varepsilon^\zeta_{mn} = \left(g_{mn}^* - g_{mn}^0\right)/2,$$

where $g_{mn}^0 = \tilde{g}_m \cdot \tilde{g}_n^0$, $g_{mn} = \tilde{g}_m \cdot \tilde{g}_n^*$, $g_{mn}^* = \tilde{g}_m \cdot \tilde{g}_n^*$, the lower indexes $m$, $n$ take the values 1, 2, 3 sequentially.

2.2. A sampling element in the form of a curved quadrilateral and an interpolation procedure for increments of displacement vectors increments of vectors of rotation angles of the normal.

The surface of the shell structure, which is equidistant from the inner and outer boundaries, is modeled by a set of curved quadrilaterals with nodes located at their vertices. The unknown components of the step vector of displacement and their first-order partial derivatives in surface coordinates, as well as the components of the step vector of the angle of rotation of the normal, are determined at the $(j+1)$-th loading step

$$\{W^L\}^T_{1=44} = \left\{\{W^L\}^T_{1=12}, \{W^L\}^T_{1=2}, \{W^L\}^T_{1=1}, \{W^L\}^T_{1=2}\right\};$$

$$\{W^G\}^T_{1=44} = \left\{\{W^G\}^T_{1=12}, \{W^G\}^T_{1=2}, \{W^G\}^T_{1=1}, \{W^G\}^T_{1=2}\right\},$$

where the Superscript $L$ and $G$ indicates the local $-1 \leq \zeta, \eta \leq 1$, and global curvilinear $\Theta^1, \Theta^2$ coordinate systems.

The row submatrices in (8) and (9) have the following structure

$$\{q^L\}^T_{1=12} = \{q^L_1 q^L_2 q^L_3 q^L_4 q^L_5 q^L_6 q^L_7 q^L_8 q^L_9 q^L_{10} q^L_{11} q^L_{12}\}; \quad \{q^G\}^T_{1=12} = \{q^L_1 q^L_2 q^L_3 q^L_4 q^L_5 q^L_6 q^L_7 q^L_8 q^L_9 q^L_{10} q^L_{11} q^L_{12}\};$$

$$\{\gamma^\rho\}^T_{1=4} = \{\gamma^\rho_1 \gamma^\rho_2 \gamma^\rho_3 \gamma^\rho_4\};$$

where $i, j, k, l$ are nodes of the quadrilateral.

In the introduction, attention was drawn to the fact that the currently generally accepted approach to the implementation of the interpolation procedure in the FEM [12-17] is to compile interpolation expressions for individual components of the step displacement vector (the step vector of the normal slope angle) independently of each other

$$q = \{q^L\}^T_{1=12} \{q^L\}^T_{1=12}, \quad \gamma^\rho = \{\gamma^\rho\}^T_{1=4} \{\gamma^\rho\}^T_{1=4},$$

where $\{q^L\}^T_{1=12} = \{\phi_1 \phi_2 ... \phi_{12}\}$ is a matrix-a string of form functions from third-order Hermite products;

$$\{\gamma^\rho\}^T_{1=4} = \{\nu \nu_2 \nu_3 \nu_4\}$$

is a matrix-a string of bilinear functions of local coordinates $\xi, \eta$.
The implementation of the vector version of the interpolation procedure involves applying the interpolation expression not to the individual components of the step displacement vector and the step vector of the normal slope angle, but to the step vectors themselves
\[
\{ \vec{w} \} = \{ \phi \}^T \{ w \}, \quad \{ \vec{\gamma} \} = \{ \psi \}^T \{ \gamma \},
\]
where

\[
\begin{align*}
\{ \vec{w} \} & = \{ \vec{w}_1 \} = \{ \vec{w}_{12} \}, \\
\{ \vec{\gamma} \} & = \{ \vec{\gamma}_{14} \}.
\end{align*}
\]

The phage vectors of the nodal points of the quadrilateral and their derivatives included in (12) can be represented by the components of the nodal basis vectors of the surface equidistant from the outer and inner boundaries
\[
\{ \vec{w}_m \} = w^{\rho s} a^{0}_s, \quad \{ \vec{w}_m \} = Z^{\rho s} a^{0}_s + Z^s a^{0}_s; \quad \{ \vec{\gamma} \} = \gamma^{\rho s} a^{0}_s,
\]
where the superscript \( s \) indicates a particular node of the quadrilateral.

Taking into account (13), the relations (12) can be written as a product of row-matrix, column-matrix, and column-matrix
\[
\begin{align*}
\{ \vec{w} \} & = \{ \phi \}^T \{ n \} \{ A_w \} \{ t \} = \{ \phi \}^T \{ n \} \{ A_w \} \{ \gamma \} \{ G \} \{ t \} \\
\{ \vec{\gamma} \} & = \{ \psi \}^T \{ A \} \{ \gamma \} = \{ \psi \}^T \{ A \} \{ \gamma \} \{ T \} \{ \gamma \} \{ G \} \{ t \}
\end{align*}
\]
where
\[
\begin{align*}
\{ \vec{t} \} & = \{ \vec{t}_{14} \}, \\
\{ \vec{\gamma} \} & = \{ \vec{\gamma}_{18} \}.
\end{align*}
\]

The matrices \( \{ A_w \} \) and \( \{ A \} \) included in (14) and (15) contain the submatrices
\[
\begin{align*}
\{ a_{w0} \} & = \{ a_{10} \} \{ a_{20} \} \{ a_{30} \} \{ a_{40} \}, \\
\{ a_{0} \} & = \{ a_{01} \} \{ a_{02} \} \{ a_{03} \} \{ a_{04} \},
\end{align*}
\]
respectively, as structural elements.

The data of the submatrix \( \{ a_{w0} \} \) and \( \{ a_{0} \} \) without fundamental difficulties can be expressed in terms of the ors of the Cartesian coordinate system
\[
\begin{align*}
\{ a_{w0} \} & = \{ b \} \{ a \}, \\
\{ a_{0} \} & = \{ d \} \{ a \}.
\end{align*}
\]

Performing the inversion operation for the basis vectors of the inner point \( N^0 \) of the quadrilatera
\[
\{ \vec{b} \} = \{ b \}^{-1} \{ a \},
\]
the matrix relations (16) can be arranged in the form
\[
\begin{align*}
\{ a_{w0} \} & = \{ b \} \{ a \} \{ a \}^{-1} \{ a \}, \\
\{ a_{0} \} & = \{ d \} \{ a \} \{ a \}^{-1} \{ a \}.
\end{align*}
\]

Containing the node vectors of the bases of the matrix \( \{ A_w \} \) and \( \{ A \} \) taking into account (16) can be replaced by the sums of scalar matrices multiplied by the basis vectors of the point \( N^0 \)
\[
\begin{align*}
\{ A_w \} & = a_{w0} \{ A_{w0} \} + a_{w1} \{ A_{w1} \} + a_{w2} \{ A_{w2} \}, \\
\{ A \} & = a_{00} \{ A_{00} \} + a_{01} \{ A_{01} \} + a_{02} \{ A_{02} \} + a_{03} \{ A_{03} \} + a_{04} \{ A_{04} \}.
\end{align*}
\]

As a result of substituting (19) into (14) and (15), the latter, taking into account (4), can be converted to the form
\[ w^1 \vec{a}_1 + w^2 \vec{a}_2 + w \vec{a}^0 = \left\{ \phi^T \right\}_{1\times 12} \begin{bmatrix} \vec{a}_1^0 & \vec{a}_2^0 & \vec{a}_w^0 \end{bmatrix} \begin{bmatrix} A_1^0 & A_2^0 & A_w^0 \end{bmatrix} \begin{bmatrix} Y_w \end{bmatrix} ; \] (20)

\[ \gamma^1 \vec{a}_1^0 + \gamma^2 \vec{a}_2^0 = \left\{ \psi^T \right\}_{1\times 4} \begin{bmatrix} a_1^0 & a_2^0 & a_w^0 \end{bmatrix} \begin{bmatrix} A_1^0 & A_2^0 & A_w^0 \end{bmatrix} \begin{bmatrix} Y_w \end{bmatrix} ; \] (21)

where \( \{ Y_w \} = \begin{bmatrix} G & W^G \end{bmatrix} \); \( \{ Y_j \} = \begin{bmatrix} G & W^G \end{bmatrix} \begin{bmatrix} \{ Y^1 \} & \{ Y^2 \} \end{bmatrix} \).

3. Discussion

From (20) and (21), we can obtain the interpolation dependencies necessary for the arrangement of the Cauchy relations at the loading step for the components of the step vector of displacement and the components of the step vector of the angle of rotation of the normal.

\[ \{ Y_w \} = \begin{bmatrix} G & W^G \end{bmatrix} ; \] (22)

The matrix-column of the covariant components of the strain increment tensor, arranged on the basis of (22), can be represented in a matrix way through the column of the desired unknowns at the \((j+1)\)-th loading step.

\[ \{ \Delta \varepsilon_{mn}^e \} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} U^G \end{bmatrix} \] (23)

where \( \{ \Delta \varepsilon_{mn}^e \} = \begin{bmatrix} \Delta \varepsilon_{11}^e & 2 \Delta \varepsilon_{12}^e & \Delta \varepsilon_{13}^e & \Delta \varepsilon_{22}^e & 2 \Delta \varepsilon_{23}^e \end{bmatrix} \).

By applying the operation of minimizing the functional that expresses the equality of possible work of external and internal forces on possible step displacements, it is possible to perform the procedure for composing the stiffness matrix and the column of nodal forces of the quadrangular sampling element at the \((j+1)\) stage of step loading.

\[ F_L = \int_{1\times 5} \left\{ \Delta \varepsilon_{mn}^e \right\}_1^5 \left\{ \delta_{mn} \right\}_1^5 + \left\{ \Delta \sigma_{mn} \right\}_1^5 \left\{ \sigma_{mn} \right\}_1^5 \left\{ \sigma_{mn} \right\}_1^5 dV - \int_{1\times 3} \left\{ W^T \right\}_1^3 \left\{ P \right\}_1^3 + \left\{ \Delta P \right\}_1^3 dF, \] (24)

where \( \{ \Delta \varepsilon_{mn}^e \} = \begin{bmatrix} \Delta \varepsilon_{11}^e & 2 \Delta \varepsilon_{12}^e & \Delta \varepsilon_{13}^e & \Delta \varepsilon_{22}^e & 2 \Delta \varepsilon_{23}^e \end{bmatrix} \); \( \{ \sigma_{mn} \} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{22} & \sigma_{23} \end{bmatrix} \); \( \{ \Delta \sigma_{mn} \} = \begin{bmatrix} \Delta \sigma_{11} & \Delta \sigma_{12} & \Delta \sigma_{13} & \Delta \sigma_{22} & \Delta \sigma_{23} \end{bmatrix} \); \( \{ P \}_1^3 = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix} \); \( \{ \Delta P \}_1^3 = \begin{bmatrix} \Delta P^1 & \Delta P^2 & \Delta P^3 \end{bmatrix} \); the stresses accumulated during the \( j \) of the previous loading steps and their increments at the \((j+1)\)-m loading step; \( \{ P \}_1^3 = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix} \); \( \{ \Delta P \}_1^3 = \begin{bmatrix} \Delta P^1 & \Delta P^2 & \Delta P^3 \end{bmatrix} \); the total external load accumulated during the \( j \) of the previous loading steps and its increment at the \((j+1)\)-m loading step.

Further transformations of the functional (24) are carried out according to the standard FEM method [15-17].

4. Conclusions

Performing a comparative analysis of the obtained interpolation expressions (22) for the components of the step vector of displacement and the components of the step vector of the angle of rotation of the normal in comparison with the approaches generally accepted in the FEM [6-10, 12-17] to interpolate the desired variable parameters, we can conclude that the above-mentioned components of the step vectors of a point located in the inner zone of a quadrangular fragment are functions of the nodal values of all components of these vectors. By this circumstance, the obtained interpolating expressions (22) differ in principle from the classical isolated interpolation in the FEM of each component of the displacement vector or the increment component of a given vector, if we are talking about nonlinear approaches, through the values at the nodes of the same component, as follows from (11).
5. References

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