Invariant colorings of random planar maps

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Abstract. We show that every locally finite random graph embedded in the plane with an isometry-invariant distribution can be five-colored in an invariant and deterministic way, under some non-triviality assumption and a mild assumption on the tail of edge lengths. The assumptions hold for any Voronoi map on a point process that has no non-trivial symmetries almost surely, hence we improve and generalize previous results on six-coloring the Voronoi map on a Poisson point process (see Angel, Benjamini, Gurel-Gurevich, Mayerovitch and Peled [Stationary map coloring. Preprint, 2008]).

1. Introduction

We consider random graphs $G$ embedded in the plane such that the number of vertices in any bounded set is almost surely finite, and such that the distribution of the image of $G$ as a subset of the plane (which we also denote by $G$) is invariant with respect to some transitive group $\Gamma$ of isometries of the plane (e.g., all isometries, or all the translations). With a slight abuse of notation, we use $G$ both for the graph and for the embedded image of it in the plane (hence thinking about it as a 1-complex in the plane). We want to give a coloring of this graph with as few colors as possible, in such a way that the coloring is an equivariant and measurable function of $G$ with respect to $\Gamma$. In other words, if $B_R(x)$ is the disc of radius $R$ around $x$ in the plane, we would like to construct a mapping $c_G : V(G) \rightarrow \{1, \ldots, k\}$ such that $c_G$ assigns different values to adjacent vertices, and $c_G$ satisfies the following.

(i) $c_{g(G)}(g(x)) = c_G(x)$ for every $g \in \Gamma$.

(ii) With probability tending to 1 with $R$, $c_G(x)$ can be determined from $B_R(x) \cap G$, as a measurable function of $B_R(x) \cap G$.

We want to make $k$ as small as possible. The condition of ‘being an equivariant function of $G$’ is sometimes simply referred to as ‘equivariance’. Note that the assumption on measurability is important not only in order to avoid a trivial answer (since every infinite planar graph is four-colorable, as we will discuss later), but also in order to be able to talk about probabilistic properties of the coloring (i.e., the distribution of what one sees locally around some fixed point of the underlying space). This natural assumption is always needed when one studies equivariant functions of point processes (even though in some papers this assumption is not made explicitly). In this paper, for simplicity, when we say
that a function is equivariant, we always include that it is also measurable. Equivariant and measurable functions are often called factors in the literature.

We will impose the condition that the set of symmetries of \( G \) is almost always trivial, where by a symmetry we mean a graph automorphism which is achieved by an isometry in the plane. In particular, the condition holds for any random graph \( G \) where the graph has no non-trivial automorphism, or for \( G \) such that the only isometry of \( V(G) \) (as a point set in the plane) is the trivial one. We will also need a condition on \( G \), which is in brief a condition on the existence of relatively few long edges. Namely, say that a \( G \) invariant planar map has the regular decay property if for every \( r \) there is an \( a(r) \), with \( a(r) \to 0 \) as \( r \to \infty \), such that, for any \( R \geq r \), the probability that the \( R \)-neighborhood \( B_R \) of \( o \) in the plane contains the endpoints \( u, v \) of an edge in \( G \) such that \( \text{dist}_{\mathbb{R}^2}(u, v) \geq a(r)R/6 \) is smaller than \( a(r) \).

**Theorem 1.1.** Let \( G \) be a random graph in the plane, whose distribution is invariant with respect to some transitive group of isometries of the plane. Suppose that with probability 1, the only symmetry that \( G \) has is the trivial one and that \( G \) satisfies the regular decay property. Then there exists a five-coloring for \( G \) which is an equivariant function of \( G \).

In particular, every Voronoi map \( G \) on some point process with only the trivial symmetry has a five-coloring which is an equivariant function of the point process.

See Remark 4.4 for the case when the symmetry assumption on \( G \) does not hold, in which case \( G \) is a ‘quasi’ lattice with finitely many orbits. Such graphs either trivially have no equivariant coloring by any finite number of colors, or their ‘equivariant measurable’ chromatic number is seven or less (depending on the chromatic number of the factor graph \( G/\Gamma \)).

The famous four color theorem states that every finite planar map is four-colorable. This was first proved by Appel and Haken, then with a much smaller, but still significant amount of computer verification by Robertson, Sanders, Seymour and Thomas. (See [2] for a survey on the four color theorem, and further references.) This can be extended to infinite graphs, by standard compactness arguments. However, if one did this right away, one is likely to get a four-coloring that is neither equivariant, nor measurable.

The question that we address in Theorem 1.1 was asked by Benjamini, and an equivariant coloring by six colors is given for Voronoi tessellation on a Poisson point process by Angel et al [1]. The proof in [1] uses some explicit computations about the distribution of the number of neighbors of a region in this planar map, and the bounds attained are used to prove that by repeatedly removing every region of less than or equal to five neighbors from the graph, one gets only finite components, after a finite number of iterations. This need not be true for a general \( G \) that only satisfies the assumption in Theorem 1.1 (even not for every Voronoi map on a point process, see Example 4.3), hence the proof in [1] (whose second part is a ‘greedy’ coloring, as in the usual proof of the six color theorem), does not seem to fully generalize to our setup, even with six colors.

Theorem 1.1 will follow from Lemmas 2.1 and 3.8 right away. For some further relaxation of the condition on \( G \), see Remark 4.2.

The reason we need that there are no non-trivial symmetries is that then there is a so called index function from \( G \) (respectively from \( G \times G \)) to the reals, which is an
injective equivariant (respectively diagonally invariant) function of $G$. This enables one to take certain subsets of infinite point sets in an equivariant way, or make local choices, e.g., choose a vertex from each finite class of some equivariant partition, and still preserve equivariance. See [4] for more details. Hence, when we say ‘choose’, ‘fix’ etc some point of each element of some equivariant collection of finite subsets of $G$, it always means that we have some previously fixed rule, which makes the choice depend on the precise local configuration (using the index function), and makes it remain equivariant and a deterministic function of the configuration.

An induced subgraph of $G$ is a subgraph $H$ such that the set of edges of $G$ with both endpoint in $V(H)$ is equal to $H$. We call a subgraph of $G$ non-self-touching, if the graph that it induces in $G$ is itself. Two subgraphs of $G$ are called non-touching, if they are not adjacent: there is no edge with one endpoint in each. By a path we always mean a simple path, i.e., no multiple vertices are allowed.

The next two paragraphs summarize how the proof proceeds, in an informal way. First, for any infinite graph $G$, there is a way to five-color it using only ‘local information’ (and hence feasible for extension to equivariant deterministic colorings of random $G$), provided that there is a collection $C$ of cycles in $G$, with the following properties. First, $C$ is such that the pairwise distance between any two elements of it is at least four, and such that no two vertices in any of the cycles are adjacent by any other edge than the ones in the cycle. Suppose further that each cycle has even length, and that the complement of the union of these cycles consists of only finite components. Then, for each such finite component $\gamma$, consider the set $C_{\gamma}$ of cycles on the $G$-boundary of $\gamma$. Take the subgraph of $G$ induced by $\gamma \cup (\bigcup_{C \in C_{\gamma}} C)$, and contract each $C \in C_{\gamma}$ to one point. We can define a four-coloring of the resulting graph by the four color theorem. Thanks to the properties of $C$, we can then put these colorings together, by introducing an extra color and doing some local changes if necessary, to get a five coloring of $G$ (the method is illustrated on Figures 2 and 3). Now, if instead of a fixed $G$ and given $C$ we have a random invariant $G$, we need to find $C$, which we will call an even cycle exhaustion, as a deterministic and equivariant function of $G$. Once this is done, the above coloring procedure gives an equivariant five-coloring of $G$. By the fact that for any fixed point $x$ in the plane we can determine the cycle of $C$ surrounding or containing $x$ by looking at a large enough neighborhood, the four-coloring of the finite piece containing $x$ is also defined locally, and this is the same with the final five-coloring, because whether a vertex is recolored when we put together the pieces is determined from local information.

If, as by our assumptions, the random $G$ is ‘disordered enough’, and the sizes of regions have a tail that is not too thick, one can define a cycle exhaustion as follows. First we take an (equivariant, measurable) sequence of coarser and coarser partitions of the plane, which naturally define a partition of the vertex set. With some mild condition on the cells of a partition, the vertices on the boundary of a cell of the partition induce a cycle with no chordal edges (given that $G$ is triangulated, which we may assume). The cycles coming from a partition this way are disjoint, but they may intersect some cycle coming from a later partition. However, if the cells in the partitions grow fast enough, then the density of such cycles is small, so we can erase them, and still have a family of cycles such that almost surely every point of the plane is surrounded by some unerased cycle of some late
enough partition. By definition, all cycles in this family are pairwise disjoint, and a similar construction leads to such a family of cycles where all pairwise distances are at least four. Finally, by some local changes performed on the cycles, we can make them all have even lengths, without changing any of the other properties needed for a cycle exhaustion.

In §2 we present the combinatorial trick that reduces the question of coloring to finding a certain kind of exhaustion for $G$ (namely, the above mentioned cycle exhaustion). The existence of such an exhaustion is less sensitive to local changes than colorings. Section 3 contains the construction of such an exhaustion, with some complications because of the generality of our setup. Section 4 concerns some open questions and generalizations.

2. Five-coloring from induced cycles

Given a cycle $C$ in $G$, define $\text{int}(C)$ to be the subgraph induced in $G$ by the set of vertices in the bounded component of $G \setminus C$. For the next definition, note that, for any infinite tree with one end, one can define a parent to each vertex $w$ as the first vertex on the path from $w$ to infinity. If $v$ is the parent of $w$, we will write $w \rightarrow v$.

Say that $(T, \lambda)$ is an even cycle exhaustion of $G$, with corridors of width $c > 0$, if it is an equivariant function of $G$, and the following hold.

1. $T$ is an infinite tree with one-end.
2. $\lambda : V(T) \mapsto 2^G$ is such that, for every vertex $v$ of $T$, $\lambda(v)$ is a non-self-touching cycle of even length of $G$.
3. Any $\lambda(v)$ and $\lambda(w)$ have distance at least $c$ whenever $v \neq w$.
4. $\lambda(w) \subset \text{int}(\lambda(v))$ whenever $w \rightarrow v$.
5. $G \setminus \bigcup_{v \in V(T)} \lambda(v)$ has only finite components.

If we do not require the $\lambda(v)$ to have even lengths, then we simply call the above structure a cycle exhaustion.

Informally, an even cycle exhaustion is a collection of non-self-touching cycles of even lengths, such that their pairwise distances are at least $c$, every point of the plane is surrounded by at least one (and hence infinitely many) of these cycles, and the relation ‘surrounding’ defines a natural tree structure on these cycles. Note that by defining the collection of cycles, the tree structure is uniquely defined as well. This is how we prefer to think about $(T, \lambda)$.

**Lemma 2.1.** Let $(T, \lambda)$ be an even cycle exhaustion of $G$ of corridor width 4. Then there is an equivariant five-coloring of $G$.

**Proof.** First, for every cycle $\lambda(v)$, which is a bipartite cycle, fix one of the two classes of the bipartition, and say that its elements are the odd elements, while the elements in the other class are the even elements. Do it so that the choices are invariant with respect to $\Gamma$.

For every $v \in V(T)$ consider the finite graph $H_v$ induced by $\text{int}(\lambda(v)) \cup \lambda(v)$ in $G$. Then in $H_v$, for every $w \rightarrow v$, contract $H_w$ (which is naturally sitting in $H_v$). Call the resulting vertex $p_v(w)$. Also contract $\lambda(v)$ to one vertex $p_v(v)$ in $H_v$. The graph obtained from $H_v$ after these contractions is called $G_v$. The vertices of $G_v$ that did not come to existence by contraction, but were present in $H_v$, are called ordinary vertices. See Figure 1 for an example (note, however, that the example there does not satisfy the condition on the corridor width). We refer to ordinary vertices and their identical copy in $G$ under the same
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Figure 1. $H_v$ (left) and $G_v$ (right). We simplified the picture by making $v$ have only one child, and the distance between $\lambda(v)$ and $\lambda(w)$ be only two.

Figure 2. The coloring of $\lambda(v) \cup \text{int}(\lambda(v))$ (left) coming from $\gamma_v$ of $G_v$ (right, upper), and $\gamma_w$ for $G_w$ (right, lower), when $\gamma_v(p_v(v)) = \gamma_w(p_w(w))$. Black stands for color 0.

name. Since $G_v$ is finite planar, we can fix some four-coloring $\gamma_v : V(G_v) \mapsto \{1, 2, 3, 4\}$ of $G_v$, making the choices invariant under $\Gamma$.

To get a coloring of $G$, do the following. For every $v \in V(t)$, and every ordinary $x \in G_v$ assign $x$ the color given to it by $\gamma_v$. This coloring of $G \setminus \bigcup_{v \in V(t)} \lambda(v)$ will be called $\gamma$.

Now, every cycle $\lambda(w)$ was contracted into vertex $p_w(w)$ in $G_w$, and into vertex $p_v(v)$ in $G_v$, where $w \rightarrow v$. If $\gamma_v(p_v(v)) = \gamma_w(p_w(w))$, color every even vertex of $\lambda(w)$ with color $\gamma_v(p_v(v))$, and every odd vertex with color 0. See Figure 2 for an illustration of this case.

Otherwise, if $\gamma_v(p_v(v)) \neq \gamma_w(p_w(w))$, color every even vertex of $\lambda(w)$ with $\gamma_v(p_v(v))$, and every odd one with $\gamma_w(p_w(w))$. 
Call the resulting assignment of colors to $V(G)$ (which we obtain by extending $\gamma$ from $G\setminus \cup \lambda(w)$ to $G$ as just described) $\gamma'$. This $\gamma'$ is typically not a good coloring yet. There may be $G$-neighbors of identical color in two possible ways: either an element of $\lambda(w)$ and an ordinary vertex in $G_w$ both got color $\gamma_v(p_v(w))$, or an element of $\lambda(w)$ and an ordinary vertex in $G_u$ both got color $\gamma_w(p_w(w))$. For all such pairs, recolor the point not in $\gamma(w)$ by assigning it color 0. Doing this for all pairs of neighbors that had the same color by $\gamma'$, we obtain a coloring $\Gamma$ of $V(G)$, which we claim to be a good coloring. For this, one only has to check that a vertex $x$ that was recolored to 0 in this last step, has no recolored neighbor $y$, and no neighbor $z$ that had color 0 by $\gamma'$. The existence of a $z$ as above is not possible by the condition that the $\gamma(w)$ are at distance at least 5 from each other, and every recolored vertex is at distance one from some $\gamma(w)$. The existence of an $y$ as above is not possible because, if there existed such a $y$, then one would have $\gamma(x) = \gamma(y)$, and thus $\gamma_u(x) = \gamma_u(y)$ with the appropriate $u(x, y \in G_u)$, which would contradict that $\gamma_u$ is a good coloring. See Figure 3 for this case. This finishes the proof that $\Gamma$ is a five-coloring as desired. \hfill $\Box$

3. Existence of a cycle exhaustion
We shall assume that $G$ is triangulated. This is not a restriction: we can triangulate every face of $G$ in some equivariant deterministic way, also respecting the conditions on $G$. An equivariant coloring of the new, triangulated graph is also a coloring for the original one.

Given a subgraph $H$ of $G$, let $\partial H$ be the outer boundary of $H$, that is, the set of vertices in $G\setminus H$ that are adjacent to $H$. Let $\partial_r H$ be the set of vertices in $G\setminus H$ at distance at most $r$ from $H$. 

![Figure 3. The coloring of int(\lambda(v)) (left) coming from \gamma_v on G_v (right, upper), and \gamma_w for G_w (right, lower), when \gamma_v(p_v(v)) \neq \gamma_w(p_w(v)).](image-url)
We will need the following graph theoretic observation later.

**Proposition 3.1.** Let $H$ be some connected subgraph of the infinite, triangulated planar graph $G$. Then the set $O(H)$ of vertices $x$ in $\partial H$ that are visible from infinity (i.e., there is an infinite path from $x$ in $G\setminus(\partial H\setminus\{x\})$) induces a non-self-touching cycle in $G$.

**Proof.** There is a natural cyclic ordering on $O(H)$ (defined as we ‘walk along’ $O(H)$ in $G\setminus(H \cup \partial H)$), and any two vertices following each other in this ordering are adjacent, because $G$ is triangulated. Thus there is a cycle with vertex set $O(H)$, and we only have to prove that $O(H)$ induces no edges other than these. Now, if $O(H)$ induced some other edge $\{x, y\}$, then the graph induced in $G$ by $O(H)\setminus\{x, y\}$ would have at least two components, because $x$ and $y$ are both visible from $H$ and from infinity. This contradicts the fact that any two vertices of $O(H)$ can be joined by a path with every inner vertex in $H$, which should be true since $H$ is connected and $O(H)$ is in its boundary. \qed

Call the set of vertices in $\partial H$ visible from infinity the **exterior boundary** of $H$.

A much stronger version of the next lemma was proved in [5], for Poisson point processes. There one wanted the $P_i$ to be a sequence of coarser and coarser partitions, and also one needed some extra properties for the distribution of configuration points in the cells of $P_i$, which makes the proof lengthier (and restriction to Poisson point processes somehow necessary).

Fix a point $o$ of the plane.

**Lemma 3.2.** Let $\omega$ be a point process such that the only isometry for the configuration is the identity almost surely. Let $\epsilon_i \to 0$ be arbitrary. Then there is a sequence of partitions $P_i$ of the plane, defined as equivariant functions of $\omega$, and such that

$$P[o \in C, C \in P_i, C \text{ is a } 2^i \times 2^i \text{ square}] \geq 1 - \epsilon_i.$$ 

**Proof.** Choose an equivariant subset $\omega_n \subset \omega$ such that any two elements of $\omega_n$ are at distance at least $n$ from each other. See [4, Corollary 3.2] for such a choice. Let $\mathcal{V}_n$ be the Voronoi tessellation on $\omega_n$. Then, as shown in [4], the probability that a point $x$ is in the $r$-neighborhood of the boundary of some cell in $\mathcal{V}_n$ is at most $cr/n$ with some universal constant $c$. Now, let $n(i)$ be a sequence of integers that tends to infinity fast enough, and for each $C \in \mathcal{V}_{n(i)}$, subdivide $C$ by a copy of the $2^i \times 2^i$ square grid, whose position is determined by some deterministic rule (which tells, for example, in which corner of $C$ one should put the origin of the grid, and which incident edge should be ‘covered’ by the horizontal axis of the grid). Let the set of cells resulting from this subdivision be $P_i$. We have that

$$P[o \in C, C \in P_i, C \text{ is not a } 2^i \times 2^i \text{ square}] \leq P[o \text{ is in the } 2^{i+1}\text{-neighborhood of the boundary of some cell in } \mathcal{V}_{n(i)}] \leq c2^i/n(i).$$

This is arbitrarily small, if $n(i)$ grows fast enough, proving the claim. \qed

The next example shows a translation invariant random planar map that does not have the regular decay property.
**Example 3.3.** For simplicity, we construct a partition of \( \mathbb{Z}^d \) that is invariant with respect to translations of \( \mathbb{Z}^d \). One can easily modify this by random rotations to get an isometry-invariant partition of the plane.

For each \( i \in \mathbb{Z} \), let \( \xi_i \) be a geometric random variable with parameter \( 1/2 \). Partition the vertical line \( \{(i, j) \mid j \in \mathbb{Z}\} \) to intervals of length \( 2^{2^i} \) each, by choosing one of the \( 2^{2^i} \) such partitions uniformly, independently for the different \( i \).

Similar but more complicated constructions lead to examples that are invariant under planar isometries, and look ‘more two-dimensional’.

**Proposition 3.4.** Let \( \omega \) be a point process. Then the graph \( G \) defined on \( \omega \) by the Voronoi tessellation satisfies the regular decay property.

In particular, the Poisson–Voronoi map has the regular decay property.

**Proof of Proposition 3.4.** Suppose that the statement is false. Then there is an \( a > 0 \) such that for every \( r \) there is an \( R \geq r \) such that with probability at least \( a \), \( B_R(\omega) \) contains \( x, y \in \omega \) with adjacent Voronoi cells and such that \( \text{dist}_{\mathbb{R}^2}(x, y) \geq aR/6 \). Now, consider the square \( S \) over diagonal \( xy \) and the two triangles that the diagonal \( xy \) divides \( S \) into. It is easy to check that if both these triangles contain a configuration point in their interiors, then \( x \) and \( y \) cannot have adjacent Voronoi cells. Therefore one of them has to be empty, and consequently \( S \) contains an empty square of diagonal half of that of \( S \). We conclude that the probability that \( B_R(\omega) \) contains \( x, y \in \omega \) with adjacent Voronoi cells and such that \( \text{dist}_{\mathbb{R}^2}(x, y) \geq aR/6 \), is smaller than the probability that it contains an empty square \( D \) of area \((Ra/6)^2/4\). Covering \( B_R \) of \( \omega \) by \( ca^2 \) many squares of area \((Ra/6)^2/16\), one of them thus has to be empty (one that is inside \( D \)). Summing up the probabilities for this, we get

\[
\mathbb{P}[B_R \text{ contains a pair of adjacent vertices at distance } aR/6] \leq ca^{-2}\mathbb{P}[\text{a fixed square of area } (aR)^2/576 \text{ is empty}].
\]

Note that \( c \) was a constant independent of \( r \) and \( R \), so this upper bound tends to 0 as \( R \) tends to infinity. This contradicts the assumption on \( a \). \( \Box \)

**Proposition 3.5.** If \( G \) has the regular decay property, then there is a cycle exhaustion of width 6 for \( G \).

**Proof.** As before, \( \omega \) is a point of the plane.

Let \( \mathcal{P}_i \) be a sequence of partitions of the plane such that

\[
\mathbb{P}[\omega \in C, \ C \in \mathcal{P}_i \text{ is an } r_i \times r_i \text{ square}] \geq 1 - 2^{-i},
\]

as given by Proposition 3.4, setting \( \epsilon_i = 2^{-i} \) for simplicity. The \( r_i \) will be chosen later, to increase fast enough. Let \( E_i \) be the set of edges in \( G \) that intersect the boundary of some cell in \( \mathcal{P}_i \), and let \( E_i^1 \) be the set of edges in \( G \) at distance no more than \( j \) from \( E_i \) (hence \( E_i^0 \) is \( E_i \), \( E_i^1 \) is the set of edges of \( G \) with an endpoint in \( E_i \), etc). Let \( G_i := G \setminus E_i^4 \).

For a subset \( A \) of the plane, let \( \partial A \) be the set of points at Euclidean distance at most \( r \) from \( A \). Say that \( C \in \mathcal{P}_i \) is **good**, if it is an \( r_i \times r_i \) square and there is no path of length no more than 4 in \( G \) that connects the complement of \( C \) with \( C^o := C \setminus \partial_{4r_i} r_i C \). Here \( a(r) \) is the function from the definition of the regular decay property. By the assumption on \( \mathcal{P}_i \)
and using the definition of the positive decay property, we obtain
\[
P[x \in C, C \in \mathcal{P}_j \text{ is good}] \geq 1 - 2^{-i} - a(r_i).
\] (3.6)

Now, if \( C \) is good, then all vertices in \( C^o \) are contained in the same connected component of \( G_i \): otherwise the graph induced by \( E_i^4 \cup (G \setminus C) \) would separate them, which implies that some edge of \( E_i^5 \) would cross \( C^o \). Then there would be a path of length at most 11 containing this edge and crossing the boundary of \( C \) by both its first and last edge; in particular one of the edges in this path would have length at least \( 2a(r_i)r_i/11 > a(r_i)r_i/6 \), contradicting the assumption that \( C \) is good. We have obtained that for a fixed point \( x \) of the plane
\[
P[x \in C^o, C \in \mathcal{P}_j, C^o \cap V(G) \text{ is in one connected component of } G_i] \geq 1 - 2^i - 3a(r_i)
\] (3.7)
using (3.6) and the generous upper bound \( 2a(r_i) \) on the probability that \( x \in C \setminus C^o \). From this it is easy to see that the probability that \( x \) is surrounded by a cycle of \( G_i \) also tends to 1 as \( i \) tends to infinity.

Note that by definition every component \( \gamma \) of \( G_i \) is inside some set \((\text{cell})\) of the partition \( \mathcal{P}_j \). Call this \( C(\gamma) \). Take \( G_i^{\text{good}} \) to be the union of connected components \( \gamma \) of \( G_i \) such that every vertex inside \( \gamma \cap C(\gamma)^o \) is in the same component of \( G_i \). By (3.7) and the remark after it we know that \( P[x \text{ is surrounded by a cycle in } G_i^{\text{good}}] \) tends to 1 with \( i \).

By definition of \( G_i \), every two connected components of \( G_i^{\text{good}} \) have distance at least 8 (the 4-neighborhood of \( E_i \) is in between two such components). Hence, for \( i \) fixed, the set \( B_i \) of external boundaries of the components of \( G_i^{\text{good}} \) as in Proposition 3.1 forms a family of non-self-touching cycles at distances at least 6 from each other. Observe that every cycle in \( B_i \) is contained in \( C \setminus C^o \) for some good \( C \in \mathcal{P}_j \), since it is in the boundary of a graph that contains \( C^o \cap G \), but does not contain any element of \( E_i^4 \). We have seen that \( x \) is surrounded by one cycle of \( B_i \) with probability arbitrary close to 1 if \( r_i \) is large enough. The next assertion is another consequence of the fact that every cycle \( O \in B_i \) is contained in some \( C(O) \setminus C(O)^o \), \( C(O) \in \mathcal{P}_j \). For \( j > i \), \( O_j \in B_j \), \( O_i \in B_i \), \( C(O_i) \) can intersect the 5-neighborhood of \( C(O_j) \cap G \) in \( G \) only if the Euclidean distance of \( C(O_i) \) from the boundary of \( C(O_j) \) is less then \( 5a(r_j)r_j \) (using that \( C(O_j) \) is good). If \( r_j \) was chosen to grow fast enough, the probability that the \( C(O_i) \) containing \( x \) is such for some \( j > i \) tends to zero. That is, if we delete every \( O \) with this property, then the probability that \( o \) is contained in some cycle \( O \in B_i \) that was not deleted tends to 1 with \( i \) arbitrarily fast by a suitable choice of \( r_i \). Hence we can finish the construction as described in the next paragraph.

Delete every cycle of \( B_i \) that intersects the 5-neighborhood (in \( G \) of any cycle in \( \mathcal{P}_j \), \( j > i \) arbitrary. Call the set of remaining cycles \( \tilde{B}_i \). If the \( r_i \) grew fast enough, the probability that the cycle of \( B_i \) surrounding \( x \) (conditioned on that there is such a cycle) intersects the cycle of some \( \mathcal{P}_j \), \( j > i \), is at most \( 2^{-i} \). The probability that a cycle of \( \tilde{B}_i \) surrounds \( x \) is at least \( 1 - 2^{-i} \).

We conclude that \( \cup \tilde{B}_i \) is a cycle exhaustion. The corresponding tree \( T \) and labeling of the vertices of \( T \), is uniquely determined by the construction (see the comment after the definition of a cycle exhaustion).
Figure 4. The construction of $\nu(O_1)$. Here $P_1 := Q_1$ and $P_2 := Q_2$.

**Lemma 3.8.** Let $G$ be a random triangulated planar map that satisfies the regular decay property, and suppose that $G$ has only the trivial symmetry almost surely. Then there exists an equivariant function of $G$ that is an even cycle exhaustion of corridor width 4.

**Proof.** To prove the existence of a cycle exhaustion with even cycles can be obtained as a modification of the cycle exhaustion constructed in Proposition 3.5. Note that if the set $\{v \in V(T) \mid \lambda(v) \text{ is even}\}$ has a complement in $T$ with only finite components, then keeping only the even $\lambda(v)$, we would obtain an even cycle exhaustion. Hence, if this is not the case, one may keep only the odd cycles and get a cycle exhaustion. Therefore, consider a cycle exhaustion with only odd cycles. Call the set of cycles corresponding to the leaves of the tree in the cycle exhaustion $\mathcal{L}_1$, those corresponding to neighbors of the leaves that are not leaves $\mathcal{L}_2$, and so on. Call the set of cycles in the exhaustion $\mathcal{C}$. That is, $\mathcal{C} = \{\lambda(x) \mid x \in V(T)\}$. We will keep notation $\text{int}(O)$ when $O \in \mathcal{C}$, to denote the bounded component of $G \setminus O$. Now, we will show that one is able to modify any cycle $O_1 \in \mathcal{C}$ and some $O_0 \in \mathcal{C}$ inside $\text{int}(O_1)$, to get an even cycle $\nu(O_1)$, preserving the property that the $\nu(O_1)$ are at distance at least 4 from each other.

Let $O_1$ be an arbitrary odd cycle in $\mathcal{C}$, such that there is an $O_0 \in \mathcal{C}$ contained in $\text{int}(O_1)$, chosen in a later defined way. We will find a way to remove a small arch of $O_1$, and connect the remaining arch of $O_1$ to an arch of $O_0$ by two paths in such a way that the resulting graph is a non-self-touching even cycle, it still has distance at least 4 from the other cycles of $\mathcal{C}$ (or their modified version, if we have already modified them in the way we are modifying $O_1$), and, further, it surrounds ‘almost’ as many points as $O_1$ did, so condition (5) of a cycle exhaustion is preserved by the modified cycles. (See Figure 4. for an illustration of what follows.) More precisely, we will find the following.

(I) Paths $P_1$ and $P_2$ in $\text{int}(O_1) \setminus O_0$, such that there is an endpoint $x_i$ for $P_i$ that is adjacent to $O_0$, the other endpoint $y_i$ of $P_i$ is adjacent to $O_1$, and the number of vertices in $O_0$ that are adjacent to $x_0$ and to $x_1$, respectively, have the same parity.

(II) $P_1$ and $P_2$ are not self-touching, they do not touch each other, and none of their inner vertices is adjacent to $O_0 \cup O_1$.

(III) $P_1 \cup P_2$ has distance at least 4 from all $\tilde{O} \in \mathcal{C} \setminus \{O_0, O_1\}$.

(IV) Every child $O' \neq O_0$ of $O_1$ is in the same connected component of $G \setminus (P_1 \cup P_2 \cup O_0 \cup O_2)$.

Suppose we can find the above described objects. Let $\ell_1$ and $r_1$ ($\ell_2$ and $r_2$) be the ‘extremal’ neighbors of $P_1$ ($P_2$) on $O_0$. By extremal we mean that there is no neighbor...
of $x_1$ in one of the arches of $O_0$ from $\ell_1$ to $r_1$ (and similarly for $x_2$). Index them so that the cyclic order of these four points on the cycle $O_0$ is $\ell_1, r_1, \ell_2, r_2$. Let the arch between $r_1$ and $\ell_2$ (respectively $r_2$ and $\ell_1$) that does not contain the other two points be $A_1$ (respectively $A_2$). Finally, let $A$ be the longer of the two arches on $O_1$ between a neighbor of $P_1$ and a neighbor of $P_2$ such that $A$ does not contain any other neighbor of $P_1 \cup P_2$. Note that $(-1)^{|A_1|+|A_2|} = (-1)^{|O_0|} = -1$, where the first equation is by (I) and the second is by the assumption that every cycle in $C$ is odd. Hence one of $A \cup P_1 \cup P_2 \cup A_1$ and $A \cup P_1 \cup P_2 \cup A_2$ is even (since they have opposite parity); call this $v(O_1)$. Note that $v(O_1)$ is a non-self-touching cycle, by (II), (III) and the assumption that $O_0$ and $O_1$ had distance at least 6. Now, if we consider $v(O_1)$ for every cycle $O_1 \in L_{2k}, k \in \mathbb{Z}^+$, then no cycle of $C$ is used as $O_1$ or $O_0$ for more than one $v(O_1)$. On the other hand, condition (IV) guaranteed that the interior of $v(O_1)$ contains every element of $C \setminus \{O_0\}$ that int($O_1$) contained, so (5) remains valid for $\{v(O_1)\}$. Hence the resulting set $\{v(O_1) \mid O_1 \in L_{2k}, k \in \mathbb{Z}^+\}$ is a cycle exhaustion.

Therefore it only remains to show the existence of $x_1, x_2$ and $P_1, P_2$ that satisfy (I)–(IV); so let $O_1 \in C$ be given, and let $Q$ be a non-self-touching path with endpoints adjacent to $O_1$ and $O_0$ respectively, where $O_0 \subset \text{int}(O_1)$ is chosen so that $Q \subset G \setminus \bigcup_{O \in C, O \neq O_0} (O \cup \partial_{c-2} O)$. By switching to a subpath of $Q$ if necessary, we may assume that none of the inner vertices of $Q$ is adjacent to $O_1 \cup O_0$. If we take the outer boundary of $\partial Q$ in $G$, then it contains two non-self-touching paths $Q_1$ and $Q_2$, between $O_0$ and $O_1$. (Make them non-self-touching by choosing them to have minimal length.) By switching to subpaths of $Q_1$ and $Q_0$ if necessary, we may assume that none of the inner vertices of $Q_1$ or $Q_0$ is adjacent to $O_1 \cup O_0$. Since the parity condition in (I) has to be satisfied by at least two of $Q_1, Q_2, Q$, we can chose those two to be $P_1$ and $P_2$. One can easily check that the other requirements are also satisfied. \hfill $\square$

4. Concluding remarks, further directions

In this section we discuss how necessary the conditions in Theorem 1.1 are. We characterize the case when there is some non-trivial symmetry. Whether the conclusion of Theorem 1.1 holds when we do not assume the regular decay property is not clear. By Lemma 3.8, it would follow from a positive answer to the next question.

**Question 4.1.** Let $G$ be a random graph in the plane such that the distribution of $G$ is invariant with respect to some transitive group of isometries of the plane. Suppose that almost surely the only symmetry that $G$ has is the trivial one. Is there an even cycle exhaustion of corridor width 4 for $G$?

**Remark 4.2.** We assumed that $G$ has only trivial symmetries. This assumption can be slightly weakened, since we only need the lack of symmetries in order to construct the sequence of partitions $P_i$ in the plane, and to make ‘local choices’ etc. Therefore, suppose that $G$ is an arbitrary graph embedded in the plane, with an invariant distribution, and that there is some invariant point process $P$, which may not be independent of $G$. Then one may look at colorings of $G$ that are equivariant measurable functions of the pair $(P, G)$. We usually assume that one of $P$ and $G$ is an equivariant function of the other. One type of
example is when $G$ is an equivariant function of $P$, such as the graph given by the Voronoi map on $P$; another class of examples is when we have $G$, a priori, and then define $P$ as an equivariant function of this, e.g., $P$ is the set of vertices in $G$. If $P$ and $V$ are independent, that corresponds to the case when one can use local extra randomness while coloring $G$. See the next example as an illustration of this more general setup (and a case where there is no equivariant four-coloring).

**Example 4.3.** Let $\omega$ be the point process obtained as follows. Let $H$ be the triangular grid of unit edge lengths, with an extra vertex added in the center of each triangle and connected to the three nodes of the triangle. Translate $H$ by a uniformly chosen vector from the union of the six triangles of the original triangular lattice incident to some vertex. We get a point set $\omega'$ that is invariant with respect to translations. Now, relocate every point of $\omega$ uniformly in its neighborhood of radius $1/100$. The resulting $\omega$ has only trivial isometry almost surely, and the Voronoi tessellation on $\omega$ as a map is isomorphic to $H$. Hence its chromatic number is four, and our method gives a five-coloring that is an equivariant measurable function of $\omega$. On the other hand, four colors do not suffice for this, since up to permutation of colors there is a unique four-coloring for $H$.

**Remark 4.4.** Consider now the general case when $G$ does have some non-trivial symmetry with positive probability.

It follows that each ergodic component where $G$ has some non-trivial symmetry is the $\Gamma$-translate of some quasi-transitive graph $H$. Consider $H/\Gamma$. The subgroup of $\Gamma$ of elements whose natural action on the torus defines an automorphism for the $H/\Gamma$ embedded in the torus is trivial. Hence, if $H/\Gamma$ is colorable by $k$ colors, then that extends to an equivariant measurable coloring of $H$. We get the color of each $x \in G$ by simply identifying which vertex of $H/\Gamma$ the factor map maps $x$ into (which can be determined from a large enough neighborhood of $x$), and taking the color of that vertex by $c$. Otherwise there is no coloring by any number of colors (this is the case when $H$ has a loop edge).

Conversely, any finite graph $F$ embedded in the torus can be lifted to define a quasi-transitive graph $H$ embedded in the plane, and a random translate can be used to define $G$. The chromatic number of $F$ is either between one and seven or $F$ is not colorable by any number of colors, see [3]. Hence, if $F$ is embedded in the torus so that there is a coloring by $k$ colors, then $H$ is equivariantly $k$-colorable, otherwise it is not.

We have obtained the following.

**Theorem 4.5.** Let $G$ be an ergodic random graph in the plane such that the distribution of $G$ is invariant with respect to some transitive group of isometries of the plane, and suppose that $\Gamma$ acts quasitransitively on $G$. Define $\Delta := G/\Gamma$ (which belong to one graph-isomorphism class almost surely). Then the minimal number of colors needed for an equivariant coloring of $G$ is the chromatic number of $\Delta$. This can be any number between one and seven; or, if $\Delta$ has a loop-edge, then $G$ is not colorable in an equivariant measurable way by any number of colors.

**Question 4.6.** Is it true that for any random planar graph $G$ that is invariant with respect to some transitive group $\Gamma$ of isometries of the plane, and that has no non-trivial symmetries, there is a four-coloring as an equivariant measurable function of $G$?
A necessary condition for a positive answer is that $G$ has infinitely many four-colorings, which is, to our knowledge, also open in graph theory.

Let us call a partition of the vertex set of a graph $G$ into classes $\{K_1, \ldots, K_k\}$ such that each of the $K_i$ is an independent set a blind $k$-coloring. Note that the color classes of every $k$-coloring give rise to a blind $k$-coloring. However, there is an invariant graph in one dimension (actually, every invariant connected graph, a bi-infinite path, is such) that has an invariant blind two-coloring, but no invariant two-coloring. Take, for example, a Poisson point process on the line, and let $G$ be the corresponding Voronoi map. There is no way to two-color $G$ in an equivariant way, because that would contradict ergodicity of the point process. On the other hand, there is a trivial equivariant blind two-coloring: let the interval of the origin and all intervals at an even distance from it form $K_1$, and the other intervals $K_2$. The way we chose $K_1$ is of course not invariant, but the set $\{K_1, K_2\}$ is. Perhaps surprisingly, it would be a lot easier to show the existence of a blind five-coloring for the case of Theorem 1.1, than it was to show the existence of a five-coloring, for the reason the we sketch in the next few paragraphs.

The last part of the proof of Lemma 3.8 consisted of showing that one can find a cyclic exhaustion consisting of even cycles. If we were satisfied with a blind five-coloring, this last step could be omitted by some modification of Lemma 2.1.

**Lemma 4.7.** Let $(T, \lambda)$ be a cycle exhaustion of $G$. Then there is an equivariant blind five-coloring of $G$.

The proof proceeds similarly to that of Lemma 2.1, with the following differences. When we define the $G_v$ we contract all but one vertex of each odd $\lambda(v)$. Then, for $w \to v$, we ‘match’ the colorings of $\lambda(w)$ determined by $\gamma_v$ and by $\gamma_w$ (in a way defined shortly) by permuting the colors assigned by $\gamma_w$. This means infinitely many permutations of colors as $v \in V(T)$ goes to infinity, hence we lose the color and can only detect whether two points are in the same or different color classes. This is exactly a blind five-coloring. The only thing missing from this sketch is how $\gamma_v$ would tell the color of an odd $\lambda(w)$ (before any potential permutations): color every second vertex, and the vertex that was not contracted, similarly to what $\gamma_v$ colored them (or their image after the identification), and color the remaining vertices with 0.

There are some questions of a similar flavor that we would like to mention to finish with. The first one is purely deterministic.

**Question 4.8.** Does every quasi-transitive planar graph $G$ admit a periodic four-coloring?

This was first asked by Bowen and Lyons. They observed that when the graph is Euclidean, there exists a five-coloring, by the following argument. It was shown by Thomassen [3] that every graph embedded on a surface of genus $g > 0$ with all non-contractible cycles long enough, can be colored by five colors. If $\Gamma$, as before, is the fixed transitive group of isometries of the plane that also act as automorphisms of $G$, the quotient of the plane by $\Gamma$ is a torus, with an embedded graph $H = G/\Gamma$. Any coloring of $H$ lifts to a periodic coloring of $G$. We may assume that the non-contractible cycles of $H$ are long enough for the assumption of Thomassen’s theorem (and hence the conclusion that $G$ has a periodic five-coloring), otherwise replace $H$ by the $H'$ obtained when lifting $H$ to a torus that covers that of $H$ $k$ times ($k$ large enough).
Question 4.9. Does every infinite quasi-transitive graph have an invariant random coloring with as many colors as its chromatic number?

A graph is called quasi-transitive if its set of automorphisms have finitely many orbits on the vertices. Invariance of a coloring is understood with respect to this group. This question was asked by Lyons and Schramm. The latter has shown that a coloring by $d + 1$ colors, where $d$ is the maximal degree in the graphs, is always possible.

Question 4.10. How many colors are needed to have a mixing invariant random coloring? What if the coloring has to be an equivariant function of independent and identically distributed (i.i.d.) Unif[0, 1] random labels on the vertices?

This question originated from Lyons. It is easy to see (by arguments similar to what we used when talking about blind-colorability and colorability in one dimension) that transitive trees can be invariantly colored by two colors, but one needs three to get a mixing two-coloring.

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