Infinite series of magnetization plateaus in the frustrated dimer-plaquette chain

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The dimer-plaquette chain undergoes a first order transition driven by frustration to an orthogonal dimer ground state. We analyze the magnetization curve in this orthogonal dimer regime. Besides of the recently found magnetization plateaus at $m = 1/4$ and the $m = 1/2$ we find an infinite sequence of plateaus between $m = 1/4$ and $m = 1/2$. These additional plateaus belong to magnetizations $m(n) = n/(2n + 2)$, where $n$ is a positive integer number.

The frustrated dimer-plaquette chain (or orthogonal dimer chain) introduced in [1] has attracted attention because of its first order quantum phase transition to a product dimer ground state and its close relation to the two-dimensional spin model for SrCu$_2$(BO$_3$)$_2$ [2]. Motivated by the recent discovery of magnetization plateaus in SrCu$_2$(BO$_3$)$_2$ Koga et. al. [3] investigated the magnetization curve for the one-dimensional model. They found magnetization plateaus at $m = 1/4$ and $m = 1/2$ of the full moment near the phase transition point using exact diagonalization and DMRG methods. In order to reproduce their results we studied the magnetization curve using large-size exact diagonalization (up to $N=40$) and analytical considerations based on the product nature of a certain class of eigenstates of the orthogonal dimer chain [4]. We found clear evidence of further magnetization plateaus between $m = 1/4$ and $m = 1/2$. In what follows we will explain our results for parameters $J=1$ and $j=J'/J=0.7$ (for notations see Fig. 1 and Refs. [2][3]), i.e., in the zero-field regime the ground state is a dimer product state, but the argumentation is more general and can be applied in principle to the (b)-phase of the phase diagram presented in Fig. 9 of [3].

To show that we have not only a single jump between the $m = 1/4$ and $m = 1/2$ plateaus we need two facts.

First we have to know the lowest energy of eigenstates with $m = 1/4$ and $m = 1/2$. The energy for $m = 1/4$ is exactly known as $4E/N = (E_1 - 3/4)/2 = -1.25595$ (see table II), because the corresponding eigenstate is a simple product state. For $m = 1/2$ the ground-state energy can be estimated very accurately to $4E/N = -0.68059(1)$ by finite-size extrapolation. The table II shows exact diagonalization (ED) results for $E(m=1/2)$ up to $N = 40$ sites. Obviously, the energy per site increases with increasing chain length for an even number of plaquettes, whereas the energy per site decreases with increasing chain length for an odd number of plaquettes. Both energies meet each other at $N \to \infty$. We note, that for the considered maximal odd and even chain lengths of $N = 36$ and $N = 40$ already the first five digits of $E(m=1/2)$ coincide.

Second we have to find at least one state with magnetization $m'$ between $m = 1/4$ and $m = 1/2$ having an energy below the straight line connecting the $m = 1/4$ and $m = 1/2$ points in the $E(m)$ diagram. Then with increasing field the system would occupy next after the state with $m = 1/4$ the state with $m'$ and not that one with $m = 1/2 > m'$. According to [2] low-lying states of certain $m$ belong to the class of product eigenstates, for which in every $(k+1)$-th plaquette the spins $S_1$ and $S_3$ form a vertical dimer singlet $(S_1 + S_3)^2 = 0$ (i.e., the quantum number $S_{13}$ in Koga’s notation is zero). The finite strips between two vertical dimer singlets contain $k$ vertical dimer triplets ($S_{13} = 1$). Since these strips are decoupled from each other we call them fragments of length $k$ here where such a fragment consist of $N_k = 4k + 2$ spins. The energies $E$ of those eigenstates can be expressed by the energies of the finite fragments $E_k$ which can be calculated exactly up to a maximum fragment length of $k = 9$ (see table II). For instance, the energy $E$ of states consisting of $N/(4k+4)$ fragments of identical length $k$ separated by one vertical dimer singlet each is $4E/N = (E_k - 3/4)/(k+1)$.

From the exact numerical data of table II it can be found that all of these states have energies below the straight line connecting the $m = 1/4$ and $m = 1/2$ points in the $E(m)$ diagram. That clearly shows, that we have not only a single jump between the $m = 1/4$ and the $m = 1/2$ plateau.

Let us now present a more detailed analysis of the magnetization process between $m = 1/4$ and $m = 1/2$. We start from the $m = 1/4$ eigenstate, which is a product state of identical fragments of length $k = 1$ each having total spin $S_{k=1} = 1$ leading to a total spin of the chain $S_{chain} = S_k \cdot N/(4k+4) = N/8$. To increase the magnetization we have to insert one fragment of length $k = 2$ with total spin $S_{k=2} = 2$. The next eigenstates with higher magnetizations can be generated by replacing further fragments of length $k = 1$ by fragments of length $k = 2$. Due to the product nature of these eigenstates the energy necessary to increase the magnetization is constant and defines the critical field for leaving the $m = 1/4$ plateau. After all fragments of length $k = 1$ are replaced by fragments of length $k = 2$ the system has reached a state with $S_{chain} = S_k \cdot N/(4k+4) = N/6$, i.e., a $m = 1/3$ plateau. The energy of this $m = 1/3$ eigenstate is $4E/N = (E_2 - 3/4)/3 = -1.076056$ (see table II). This fragmented structure of the relevant eigenstates is also responsible for the finite-size effects, since for finite
chains the splitting of the chain in fragments must be compatible with the total length of the chain. To get higher magnetizations all fragments of length \( k = 2 \) and \( S_{k=2} = 2 \) must now be replaced successively by fragments of length \( k = 3 \) and \( S_{k=3} = 3 \) and so on. This scenario generates a series of plateaus at \( m = k/(2k + 2) \) where the width of the plateaus decreases rapidly, because the energy difference between fragments of length \( k \) and \( k+1 \) is decreasing with increasing \( k \). The critical field between two plateaus can be calculated from the energies of the fragments \( E_k \)

\[
h_c(k) = kE_k - (k + 1)E_{k-1} + 3/4.
\]  

The energies of the fragments \( E_k \) can be found in table I. Figure 2 gives an overview about the resulting magnetization curve. In contrast to the corresponding Fig. 4 in [1] we get a staircase like infinite series of plateaus. We emphasize, according to the linear programming theorem (see, [4] and references therein), fragmented states with nonuniform fragment lengths have higher energies than corresponding states built by fragments of identical length. Furthermore our numerical data clearly show that the energy of the above considered fragmented states is significantly below the energies of competing low-lying non-fragmented states (i.e. states where all quantum numbers \( S_{13} = 1 \)).

We notice, that our result is in accordance with the general rule of Oshikawa et al. [5] that \( n(s - \bar{m}) \) has to be an integer. In our case the period of the ground state \( n \) is \( 4(k+1) \), the spin \( s \) is one half and the magnetization per spin \( \bar{m} \) is \( k/(4k + 4) \) (the magnetization per site \( \bar{m} \) corresponds to on half of the quantity \( m \) used in this comment).

Finally we mention, that the magnetic phase diagram on the \((j-h)\) plane presented in Fig. 9 in [3] has to be extended taking into account the additional plateaus at \( m = k/(2k + 2) \).

In conclusion, we have found an infinite series of magnetization plateaus in a frustrated one-dimensional quantum spin system which is closely related to the possibility of fragmentation of the considered chain. Though the possibility of fragmentation has been observed also for frustrated two-leg spin ladder [6,7], in the spin ladder system only very simple ground states seem to be relevant for its magnetization curve. Hence the sequence of plateaus with rational \( m(k) = k/(2k + 2) \) found in this paper seems to be a novel property of quantum spin systems, not found so far.

The curve \( m > 1/2 \) is only a guide for the eyes. The dashed line is the exact diagonalization result of the finite system \( N = 32 \).

TABLE I. Finite-size dependence of the energy of \( m = 1/2 \) states \( E(m = 1/2) \) for \( j = 0.7 \) up to \( N = 4N_p = 40 \) sites. \( N_p \) is the number of plaquettes (vertical dimers) in the chain with periodic boundary conditions.

| \( N_p \) | \( E/N_p \) | \( N_p \) | \( E/N_p \) |
|----------|----------|----------|----------|
| 1        | -0.50000000 | 2        | -0.70313092 |
| 3        | -0.67602844 | 4        | -0.68187420 |
| 5        | -0.68023225 | 6        | -0.68071087 |
| 7        | -0.68055572 | 8        | -0.68060683 |
| 9        | -0.68058916 | 10       | -0.68059535 |

[1] N.B. Ivanov, and J. Richter, Phys.Lett.A 232, 308 (1997); J. Richter, N.B. Ivanov, and J. Schulenburg, J. Phys.: Condens. Matter 10, 3639 (1998)

[2] K. Ueda and S. Miyahira, J. Phys.: Condens. Matter 11, L175 (1999).

[3] A. Koga, K. Okunishi, and N. Kawakami, Phys. Rev. B 62, 5558 (2000)

[4] H. Niggemann, G. Uimin, and J. Zittartz J. Phys.: Condens. Matter 9, 9031 (1997)

[5] M. Oshikawa, M. Yamanaka, I. Affleck, Phys. Rev. Letter 78, 1984 (1997)

[6] F. Mila, Eur. Phys. J. B 6, 201 (1998)

[7] A. Honecker, F. Mila and M. Troyer, Eur. Phys. J. B 15, 227 (2000)
TABLE II. Energies of fragments of the orthogonal-dimer chain of length $k$ and total spin $S_k = k$ for $j = 0.7$ and up to $N_k = 4k + 2 = 38$ sites.

| $k$ | $E_k$     | $k$ | $E_k$     |
|-----|-----------|-----|-----------|
| 0   | -0.7500000| 5   | -4.5289991|
| 1   | -1.7619002| 6   | -5.2096756|
| 2   | -2.4781677| 7   | -5.8902882|
| 3   | -3.1658665| 8   | -6.5708863|
| 4   | -3.8480439| 9   | -7.2514810|