Symplectic SUSY Gauge Theories with Antisymmetric Matter

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Abstract

We investigate the confining phase vacua of supersymmetric $Sp(2N_C)$ gauge theories that contain matter in both fundamental and antisymmetric representations. The moduli spaces of such models with $N_F = 3$ quark flavors and $N_A = 1$ antisymmetric field are analogous to that of SUSY QCD with $N_F = N_C + 1$ flavors. In particular, the forms of their quantum superpotentials are fixed by classical constraints. When mass terms are coupled to $W_{(N_F=3,N_A=1)}$ and heavy fields are integrated out, complete towers of dynamically generated superpotentials for low energy theories with fewer numbers of matter fields can be derived. Following this approach, we deduce exact superpotentials in $Sp(4)$ and $Sp(6)$ theories which cannot be determined by symmetry considerations or integrating in techniques. Building upon these simple symplectic group results, we also examine the ground state structures of several $Sp(4) \times Sp(4)$ and $Sp(6) \times Sp(2)$ models. We emphasize that the top-down approach may be used to methodically find dynamical superpotentials in many other confining supersymmetric gauge theories.
I. INTRODUCTION

During the past few years, dramatic progress has been made in understanding nonperturbative aspects of $N = 1$ supersymmetric gauge theories. Recent work initiated by Seiberg and collaborators has addressed questions that previously seemed intractably difficult [1]. For example, the phase structures of several supersymmetric theories are now known, and dynamical mechanisms for various phase transitions have been explored [2]. Highly nontrivial exact superpotentials describing low energy limits have also been derived in many cases [3]. Results from SUSY model investigations have shed light upon such interesting nonperturbative phenomena as confinement and chiral symmetry breaking. General insights gleaned from their study will hopefully be applicable to nonsupersymmetric gauge theories as well.

Many of the new ideas about $N = 1$ supersymmetric models have been developed within the context of SUSY QCD [4,5]. The low energy structure of this theory crucially depends upon its number of colors $N_C$ and quark flavors $N_F$. For $N_F \leq N_C + 1$, SUSY QCD confines in the far infrared. In the particular case when $N_F = N_C + 1$, the quantum description of its moduli space of vacua is concisely summarized by the dynamically generated superpotential

$$W_{N_F = N_C + 1} = \bar{B}MB - \det M \Lambda^{2N_C - 1}$$

(1.1)

The meson and baryon superfields appearing in this expression represent gauge invariant coordinates on the moduli space. The equations of motion obtained by varying $W$ with respect to $M$, $B$ and $\bar{B}$ reproduce the constraints

$$M^i_j B^j = \bar{B} M^i_j = 0$$

$$\frac{1}{N_C!} \epsilon^{i_1 \cdots j_{N_F}} \epsilon_{i_1 \cdots i_{N_F}} M^{i_1} \cdots M^{i_{N_C}} = \bar{B} M^{i_{N_F}} B^{i_{N_F}}$$

(1.2)

which characterize the moduli space manifold. It is important to note that the form of the superpotential in (1.1) can be deduced by requiring that these constraints be recovered. As we shall see, this observation provides the key to unlocking the vacuum structure of many other confining supersymmetric gauge theories that are more complicated than SUSY QCD.

Once $W_{N_F = N_C + 1}$ is known, it is straightforward to add tree level mass terms and flow down to theories with fewer numbers of quark flavors. After integrating out the first quark, one finds that no dynamically generated superpotential exists in $N_F = N_C$ SUSY QCD. Nonetheless,
nonperturbative effects do transmute the classical moduli space relation \( \det M - B\bar{B} = 0 \) into the quantum constraint

\[
\det M - B\bar{B} = \Lambda^{2N_C}. \tag{1.3}
\]

When additional quark fields are integrated out, the superpotential for the resulting low energy effective field theory with \( N_F < N_C \) flavors takes the form

\[
W_{N_F < N_C} = (N_C - N_F) \left[ \frac{\Lambda^{3N_C - N_F}}{\det M} \right]^{1/(N_C - N_F)}. \tag{1.4}
\]

The vacua in this case may be stabilized by adding mass terms to the dynamically generated superpotential.

The ground state structures of other confining supersymmetric gauge theories with matter in only defining representations are qualitatively similar to that for SUSY QCD. Symmetry and holomorphy considerations fix the functional forms of the low energy superpotentials in such models. On the other hand, the quantum moduli spaces of theories with more complicated matter content are generally much harder to uncover. Dynamical superpotentials for \( SU(N_C) \) gauge theories with \( N_F \leq 3 \) fundamentals, \( N_C + N_F - 4 \) antifundamentals and one antisymmetric tensor were derived in ref. \[8\]. The vacuum structure for some confining \( SO(N_C) \) models with matter in both vector and spinor representations have also been studied \[9,10\]. But the list of solved multimatter theories is not very long.

In this article, we will analyze the moduli spaces for a class of symplectic models that contain matter in both fundamental and antisymmetric representations. We focus upon the low energy descriptions of their confining phase sectors. The methods we develop to construct the superpotentials for these symplectic theories can be applied to other simple group models. They also expand the number of known product group moduli spaces. These double-matter models consequently provide useful laboratories for exploring new aspects of supersymmetric gauge theories.

Our paper is organized as follows. We first review some basic elements of symplectic group theory in section 2. We then study the vacuum structure of \( Sp(2N_C) \) models which contain

\[1\] Other phases of such theories, in the presence of tree level superpotentials, have been studied in ref. \[11\].
antisymmetric matter in section 3. We discuss in detail the superpotentials for \( Sp(4) \) and \( Sp(6) \) theories with \( N_F \leq 3 \) quark flavors and one antisymmetric field. In section 4, we use these simple group superpotentials to map the moduli spaces of several \( Sp(4) \times Sp(4) \) and \( Sp(6) \times Sp(2) \) product group models. Finally, we close in section 5 with a summary of our findings and some thoughts on possible future extensions of this work.

II. SP(2N) BASICS

The definition of the classical group of symplectic transformations stems from geometrical considerations similar to those for the more familiar orthogonal and unitary groups. The fundamental representations of \( SO(N) \), \( SU(N) \) and \( Sp(2N) \) leave invariant the inner products

\[
\langle v_1, v_2 \rangle = v_1^T \cdot v_2 \quad (2.1a)
\]

\[
\langle z_1, z_2 \rangle = z_1^\dagger \cdot z_2 \quad (2.1b)
\]

\[
\langle q_1, q_2 \rangle = \bar{q}_1^T \cdot q_2 \quad (2.1c)
\]

defined on the real, complex and quaternionic vector spaces \( \mathbb{R}^N \), \( \mathbb{C}^N \) and \( \mathbb{H}^N \). \( Sp(2N) \) is thus isomorphic to the group of \( N \times N \) matrices with quaternionic elements which preserve the dotproduct in (2.1d).

Just as any complex number may be regarded as an ordered pair of two real numbers, so may any quaternion be viewed as an ordered pair of two complex numbers. An element \( q \in \mathbb{H} \) decomposes over \( \mathbb{C}^2 \) and \( \mathbb{R}^4 \) as

\[
q = z_1 + jz_2 \quad z_1, z_2 \in \mathbb{C}
\]

\[
= (\alpha + i\beta) + j(\gamma - i\delta) \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}
\]

\[
= \alpha + i\beta + j\gamma + k\delta
\]

where the symbols \( i, j \) and \( k \) satisfy the relations \( i^2 = j^2 = k^2 = -1 \) and \( ij = -ji = k \) plus cyclic permutations. Quaternionic vectors in \( \mathbb{H}^N \) can similarly be rewritten as elements of \( \mathbb{C}^{2N} \):

\[
q_1 \equiv s + jt
\]

\[
q_2 \equiv u + jv. \quad (2.3)
\]

Recalling that \( i, j \) and \( k \) are mapped into their negatives under conjugation, we see that the
inner product (2.1c) is expressible as

\[ \langle q_1, q_2 \rangle = (s_1^* t_1^* \cdots s_N^* t_N^*) I_{2N \times 2N} + j (s_1^T t_1^T \cdots s_N^T t_N^T) J \]  

where \( J = 1_{N \times N} \times i\sigma_2 \). Any rotation \( U \) satisfying \( U^\dagger U = 1 \) and \( U^T J U = J \) preserves the RHS of eqn. (2.4). The set of all such complex \( 2N \times 2N \) matrices forms the fundamental irreducible representation of \( Sp(2N) \).

\( Sp(2N) \) is a rank \( N \) subgroup of \( U(2N) \) with \( 2N^2 + N \) generators. One convenient basis for the symplectic group’s generators in the \( 2N \) dimensional representation is schematically given by the tensor products \( S_N \times \sigma \) and \( A_N \times 1_{2 \times 2} \) where \( S \) and \( A \) respectively denote symmetric and antisymmetric hermitian generators of \( SU(N) \) \([12]\]. In this basis, one can readily check that the similarity transformation \(-T_a^* = JT_aJ^{-1} \) holds for all \( 2N^2 + N \) generators of the \( 2N \) and \( 2N \) representations. The fundamental irrep of \( Sp(2N) \) is consequently pseudoreal. All other representations formed by taking tensor products of fundamentals are also either real or pseudoreal. No essential distinction therefore exists between matter and antimatter in symplectic gauge theories.

Although \( Sp(2N) \) may seem more foreign than \( SO(N) \) and \( SU(N) \), symplectic group theory is actually easier than its orthogonal and unitary analogs. This fact simplifies the analysis of the symplectic models that we shall study in this article.

### III. SYMPELCTIC MODELS WITH ANTISYMMETRIC MATTER

The number of different classes of supersymmetric gauge theories which possess nontrivial confining phases is surprisingly small. Such models must first be asymptotically free. As is well known, only a limited number of theories with matter in relatively low dimension representations exhibit asymptotic freedom. Additional conditions that confining models must satisfy significantly restrict their number. As a result, only a handful of confining simple group classes exist.
In order to determine whether a SUSY model confines, it is useful to consider the R-charge associated with its strong interaction scale. Recall that any $N = 1$ supersymmetric gauge theory with zero tree level superpotential can be rendered invariant under a $U(1)_R$ symmetry which rotates the Grassmann $\theta$ parameter by a phase. By definition, $\theta$ has one unit of R-charge. The overall R-charges of matter superfields are a priori unspecified. If they are all set to zero, the strong interaction scale must be assigned

$$R(\Lambda^0) = K(\text{Adj}) - \sum_{\text{matter reps } \rho} K(\rho)$$

(3.1)

to cancel a global $U(1)_R$ anomaly in the quantized theory. In this expression, $K(\rho)$ denotes the group theory index of representation $\rho$ with $K(\text{fundamental}) \equiv 1$. This last normalization choice fixes the one-loop beta function coefficient

$$b_0 = \frac{1}{2}[3K(\text{Adj}) - \sum_{\text{matter reps } \rho} K(\rho)]$$

(3.2)

that appears on the LHS of (3.1).

Any dynamically generated superpotential $W_{\text{dyn}}$ must have $R = 2$ in order for the supersymmetric action to be $U(1)_R$ invariant. The strong interaction scale dependence of $W_{\text{dyn}}$ is consequently determined in simple group models since only $\Lambda^0$ carries nonvanishing R-charge. In SUSY QCD, $\Lambda^0$ enters into the denominator of superpotential (1.1) when $R(\Lambda^0) = 2(N_C - N_F) = -2$. The numerator is then a simple polynomial in the meson and baryon fields. If one attempts to construct $W_{\text{dyn}}$ for $N_F = N_C + 2$ flavors, a square root branch point is encountered at the moduli space origin. Such a singularity indicates that new phenomena emerge in $N_F = N_C + 2$ SUSY QCD which are absent in the $N_F = N_C + 1$ theory. Indeed, it is now known that supersymmetric QCD ceases to confine at this juncture and enters into the free magnetic phase [5]. SUSY QCD therefore binds together colored partons into colorless hadrons only so long as $R(\Lambda^0) \geq -2$.

Similar heuristic arguments suggest that adding sufficient matter into any SUSY model causes its would-be superpotential to develop a branch point at the origin of moduli space which signals the end of the confining phase. If this hypothesis is accepted, it is straightforward to check that the number of confining model classes is quite limited.

One interesting set of theories which does possess a nontrivial confining phase is based upon the symmetry group.
\[ G = \text{Sp}(2N_C)_{\text{local}} \times [SU(2N_F) \times U(1)_Q \times U(1)_A \times U(1)_R]_{\text{global}} \]  

(3.3)

with microscopic matter

\[ Q \sim (2N_C; 2N_F; 1, 0, 0) \]
\[ A \sim \left[ \left( \frac{2N_C}{2} \right) - 1; 1; 0, 1, 0 \right] \]
\[ \Lambda^{bo} \sim (1; 1; 2N_F, 2(N_C - 1), 4 - 2N_F). \]  

(3.4)

Several points about the field content of these symplectic theories should be noted. Firstly, in order to avoid a global Witten anomaly, these models must involve an even number \( 2N_F \) of quarks in the fundamental irrep of \( \text{Sp}(2N_C) \) [13]. In the limit of zero tree level superpotential, the supersymmetric action remains invariant under a global \( U(2N_F) \cong SU(2N_F) \times U(1)_Q \) symmetry that rotates the quarks among themselves. Secondly, the \( A \) field transforms according to the “traceless” antisymmetric representation of \( \text{Sp}(2N_C) \). Its inner product with the skew metric \( J \) vanishes. Finally, we regard the strong interaction scale \( \Lambda^{bo} \) with \( b_0 = 2N_C - N_F + 4 \) as a background “spurion” field [14]. Its abelian charges are chosen so that all global \( U(1) \) factors in \( G \) are nonanomalous. Looking at the R-charge assignment for \( \Lambda^{bo} \), we see that these models confine so long as their number of quark flavors does not exceed \( N_F = 3 \).

The classical moduli space of vacua for the symplectic theories is simplest to analyze when \( N_F = 0 \). The flat directions along which the scalar potential vanishes are then determined by the antisymmetric field expectation values satisfying \( \text{Tr}(T_a AA^\dagger) = 0 \). Working with the basis of fundamental irrep \( \text{Sp}(2N_C) \) generators introduced in section 2, we find that the general solution to this D-flatness condition is given by a linear combination of \( U(2N_C)/\text{Sp}(2N_C) \) coset space generators:

\[ AA^\dagger = S_{N_C \times N_C} \times 1_{2 \times 2} + \tilde{A}_{N_C \times N_C} \times \tilde{\sigma}. \]  

(3.5)

Since the quaternions \( (1, i, j, k) \) are isomorphic to the \( 2 \times 2 \) Pauli matrices \( (1_{2 \times 2}, -i\tilde{\sigma}) \), the matrix \( AA^\dagger \) may be regarded as a general hermitian element of \( H^{N_C \times N_C} \). It can be diagonalized by some unitary matrix built out of quaternions that is equivalent to an element of \( \text{Sp}(2N_C) \):

\[ AA^\dagger \rightarrow D_{N_C \times N_C} \times 1_{2 \times 2}. \]  

(3.6)

\[ ^2 \text{We thank Howard Georgi for this group theory insight.} \]
The vev of $A$ itself looks like

$$\langle A \rangle = \begin{pmatrix} z_1 \\ \vdots \\ z_{N_C} \end{pmatrix} \times i\sigma_2 \quad \text{with} \quad \sum_{n=1}^{N_C} z_n = 0 \quad (3.7)$$

up to a local gauge transformation. The classical moduli space is thus $N_C - 1$ complex dimensional.

The flat directions in the $N_F = 0$ theory are labeled by the gauge invariant operators

$$O_n = \text{Tr}[(AJ)^n], \quad n = 2, 3, \cdots, N_C. \quad (3.8)$$

In the models with $N_F = 1, 2$ and 3 quark flavors, additional meson fields

$$M_{ij} = Q_i^T J Q_j$$
$$N_{ij} = Q_i^T J A J Q_j$$
$$P_{ij} = Q_i^T J A J A J Q_j$$
$$\vdots$$
$$R_{ij} = Q_i^T J (AJ)^{N_C-1} Q_j \quad (3.9)$$

are needed to act as moduli space coordinates. The characteristic polynomial for the matrix $AJ$ truncates their number. Only $N_C$ such meson operators are therefore independent for an $Sp(2N_C)$ color group.

The simplest symplectic models that support antisymmetric matter have gauge group $Sp(4)$. The vacuum structure of these theories is most readily analyzed if we start with $N_F = 3$ quark flavors. The hadron superfield charge assignments for this case are listed in Table 1. Looking at the $R = -2$ entry for the spurion field $\Lambda^b_0$, we see that the $N_F = 3, N_A = 1$ $Sp(4)$ model is analogous to $N_F = N_C + 1$ SUSY QCD. Its classical and quantum moduli spaces are the same, and the constraint equations that define its moduli space manifold are polynomials in the gauge invariant hadron fields. Symmetry considerations significantly limit the possible terms in the dynamical superpotential

$$W_{\text{dyn}} = \frac{\text{Some polynomial in } M_{ij}, N_{ij} \text{ and } O_2}{\Lambda^5} \quad (3.10)$$

which generates these constraints. In order for $W_{\text{dyn}}$ to be $U(1)_Q \times U(1)_A$ invariant, the numerator in eqn. $(3.10)$ must have abelian charges $Q = 6$ and $A = 2$. Moreover, the numerator’s
mass dimension must equal 8 so that \( W_{\text{dyn}} \) has dimension 3. The form of the superpotential consistent with all these restrictions is unique:

\[
W_{\text{dyn}} = -\frac{\epsilon^{ijklmn}}{48\Lambda^5} [M_{ij} M_{kl} M_{mn} O_2 + \alpha N_{ij} N_{kl} M_{mn}]. \tag{3.11}
\]

Symmetry principles leave undetermined the overall numerical prefactor that multiplies \( W_{\text{dyn}} \). Its value is rather arbitrary since it may be altered by a redefinition of the strong interaction scale. For later convenience, we have set this normalization factor equal to \(-1/48\). On the other hand, the value of the relative coefficient \( \alpha \) between the two terms in (3.11) is fixed by consistency requirements. The equations of motion obtained by varying \( W_{\text{dyn}} \) with respect to \( M_{ij} \), \( N_{ij} \) and \( O_2 \) yield the classical constraints

\[
\epsilon^{ijklmn} [3O_2 M_{kl} M_{mn} + \alpha N_{kl} N_{mn}] = 0 \tag{3.12a}
\]
\[
\epsilon^{ijklmn} N_{kl} M_{mn} = 0 \tag{3.12b}
\]
\[
\epsilon^{ijklmn} M_{ij} M_{kl} M_{mn} = 0 \tag{3.12c}
\]

which must be satisfied when the hadron fields are decomposed in terms of their parton constituents. Inserting \( M_{ij} = Q_i^T J Q_j \), \( N_{ij} = Q_i^T J A J Q_j \) and \( O_2 = \text{Tr}[(A J)^2] \) into these relations, we find (3.12(b) and (3.12(c) are classical identities whereas (3.12(a) is satisfied only if \( \alpha = 12 \).

In order to probe the structure of the \( N_F = 3, N_A = 1 \) Sp(4) theory’s moduli space, it is instructive to couple tree level sources to the dynamical superpotential in (3.11). We choose to add the \( Q \) and \( A \) field mass terms

\[
W_{\text{tree}} = \frac{1}{2} \mu^{ij} M_{ji} + mO_2 \tag{3.13}
\]
which lift all flat directions while preserving the gauge group. The hadron operators then develop the expectation values

\[
\langle M_{ij} \rangle = m^\frac{3}{2}(\text{Pf} \mu)^\frac{1}{2}\Lambda^\frac{3}{2}(\mu^{-1})_{ij}
\]

\[
\langle N_{ij} \rangle = 0
\]

\[
\langle O_2 \rangle = \frac{1}{6}m^{-\frac{3}{2}}(\text{Pf} \mu)^\frac{3}{2}\Lambda^\frac{3}{2}
\]

in the presence of the sources. If we substitute these vevs back into the superpotential, we can methodically integrate out quark flavors and flow down to \(Sp(4)\) theories with smaller values of \(N_F\). The tower of resulting dynamical superpotentials is displayed below:

\[
W_{(N_F=3,N_A=1)} = -\frac{\epsilon^{ijklmn}}{48\Lambda^5_{(3,1)}}[M_{ij}M_{kl}M_{mn}O_2 + 12N_{ij}N_{kl}M_{mn}]
\]

\[
W_{(N_F=2,N_A=1)} = X\epsilon^{ijkl}[M_{ij}M_{kl}O_2 + 4N_{ij}N_{kl} - 8\Lambda^6_{(2,1)}] + Y\epsilon^{ijkl}M_{ij}N_{kl}
\]

\[
W_{(N_F=1,N_A=1)} = \frac{\text{Pf} M}{O_2(\text{Pf} M)^2 - 4(\text{Pf} N)^2} \Lambda^7_{(1,1)}
\]

\[
W_{(N_F=0,N_A=1)} = 2\left[\frac{\Lambda^8_{(0,1)}}{O_2}\right]^\frac{3}{2}
\]

\[
W_{(N_F=0,N_A=0)} = 3[\Lambda^9_{(0,0)}]^\frac{3}{2}.
\]

All the strong interaction scales appearing in this hierarchy are related by matching conditions that ensure continuity of the running \(Sp(4)\) gauge coupling across heavy particle thresholds. These matching relations are most conveniently expressed in the basis where the quark mass matrix

\[
\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \times i\sigma_2
\]

is block diagonalized. The matching conditions then form the simple chain

\[
\Lambda^9_{(0,0)} = m\Lambda^8_{(0,1)} = m\mu_1\Lambda^7_{(1,1)} = m\mu_1\mu_2\Lambda^6_{(2,1)} = m\mu_1\mu_2\mu_3\Lambda^6_{(3,1)}.
\]
A number of checks on the renormalization group flow between the different $Sp(4)$ theories in eqn. (3.13) can be performed. Firstly, we can at any stage integrate out the antisymmetric $A$ field and recover the superpotential

$$W_{(N_F,N_A=0)} = (3 - N_F) \left[ \frac{\Lambda^{9-N_F}_{(N_F,0)}}{\text{PFM}} \right]^{1/(3-N_F)} (3.18)$$

for $Sp(4)$ theory with $2N_F$ flavors of quarks $[7]$. We used the normalization of this known result along with the strong interaction scale matching relations to fix the overall numerical prefactor in eqn. (3.11). Secondly, we can verify that the quantum constraints

$$\epsilon^{ijkl} [M_{ij} M_{kl} O_2 + 4N_{ij} N_{kl}] = 8\Lambda^{6}_{(2,1)}$$

$$\epsilon^{ijkl} M_{ij} N_{kl} = 0 (3.19)$$

obtained by varying $W_{(N_F=2,N_A=1)}$ with respect to its $X$ and $Y$ Lagrange multiplier superfields yield valid classical relations in the $\Lambda_{(2,1)} \rightarrow 0$ limit. $N_F = 2$, $N_A = 1$ $Sp(4)$ theory is analogous to $N_F = N_C$ SUSY QCD inasmuch as $R(\Lambda^b_0) = 0$ in both models. But whereas the moduli space of the latter is defined by just the single constraint in eqn. (1.3), the moduli space of the former involves the two relations in (3.19). In general, $N_C$ different equations are needed to characterize the $N_F = 2$, $N_A = 1$ $Sp(2N_C)$ moduli manifold. Finally, the two $W_{(N_F=0,N_A=1)}$ superpotentials for $Sp(4)$ theory with matter in only the 5-dimensional antisymmetric irrep must agree with those for $SO(5)$ theory with matter in the vector representation since the Lie algebras of $Sp(4)$ and $SO(5)$ are identical. A quick check confirms that they do $[3]$.

The basic strategy we followed to construct the tower of low energy $Sp(4)$ models can be applied to other symplectic theories in the same class with greater numbers of colors. This procedure generally yields highly nontrivial superpotentials for complicated quantum moduli spaces. Yet its starting point depends only upon classical physics. The structure of the $N_F = 3$, $N_A = 1$ superpotential in $Sp(2N_C)$ theory is

$$W_{(N_F=3,N_A=1)} = \frac{P(M_{ij},N_{ij},\cdots;O_2,O_3,\cdots)}{\Lambda^{b_0}} (3.20)$$

---

$^3$ Our normalization for $\Lambda^{b_0}$ in the pure quark theory differs by a factor of $2^{N_C-1}$ from the one adopted in ref. [7]
where $P$ denotes some holomorphic function of the color-singlet superfields. Symmetry considerations do not prevent $P$ from being riddled with poles and branch cuts. But such singularities would possess no clear physical interpretation. Moreover, duality arguments suggest that a magnetic dual to the $N_F = 3, N_A = 1$ $Sp(2N_C)$ theory is very weakly coupled in the far infrared and does not generate any superpotential singularities [3]. As a result, it is reasonable to assume that $P$ is a polynomial. The equations of motion obtained by varying $P$ with respect to its arguments then yield classical constraints which must be satisfied when the hadron operators are decomposed in terms of their underlying $Q$ and $A$ constituents. This requirement fixes polynomial $P$. The superpotentials for other models with smaller $N_F$ or $N_A$ values can subsequently be obtained from (3.20) by systematically integrating out matter degrees of freedom.

To illustrate the utility of this approach, we consider $Sp(6)$ theory with $N_F = 0$ and $N_A = 1$. We recall that the scalar potential in this model has two independent flat directions which are labeled by the gauge invariant operators $O_2 = \text{Tr}[(AJ)^2]$ and $O_3 = \text{Tr}[(AJ)^3]$. The dimensionless and chargeless ratio $R \equiv -12 O_3^2 / O_2^3$ can be constructed from these two operators. Symmetry places no restrictions on the superpotential’s dependence upon $R$. Furthermore, “integrating in” techniques fail to determine the functional form of $W_{(N_F=0,N_A=1)}$ [3,15]. But if we start with the $N_F = 3, N_A = 1$ $Sp(6)$ model and progressively integrate out quark flavors, we can in fact deduce the dynamical superpotential in the pure antisymmetric theory. The tower of confining phase $Sp(6)$ superpotentials is displayed below:
The low energy $Sp(6)$ effective theories satisfy several consistency checks. Known $Sp(6)$ models with just quark matter are recovered when the antisymmetric field is integrated out. ’t Hooft anomaly matching conditions are also satisfied at various points on the moduli manifolds. For instance, the global $SU(4) \times U(1)_Q \times U(1)_A \times U(1)_R$ symmetry in the $N_F = 2, N_A = 1$ model is broken down to $Sp(4) \times U(1)_{Q-A} \times U(1)_R$ at the point $M_{ij} = N_{ij} = O_2 = O_3 = 0$, $P_{ij} = (\Lambda_{(2,1)}^4/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ on the quantum moduli space. One can straightforwardly check that parton and hadron level calculations of the $Sp(4)^2U(1)_{Q-A}$, $Sp(4)^2U(1)_R$, $U(1)^2_{Q-A}U(1)_R$ and $U(1)_R^3$ anomalies agree. The singularity structure of the $Sp(6)$ superpotentials provides another test of their validity. Branch points and poles generally signal points of enhanced gauge symmetry in the moduli space. For example, $Sp(6)$ breaks to $Sp(4) \times Sp(2)$ in the $N_F = 0, N_A = 1$ effective theory when the antisymmetric field develops the expectation value

$$\langle A \rangle = v \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times i\sigma_2.$$  

(3.22)
Expanding about this point in moduli space, we find that the nonvanishing $Sp(6)$ superpotential behaves as

$$W_{(N_F=0,N_A=1)} \sim \left[ \frac{\Lambda_{01}^{10}}{O_2^{(4)}} \right]^{\frac{1}{2}}$$

(3.23)

where $O_2^{(4)} = \text{Tr}[(A_4J_4)^2]$ denotes the canonically normalized $O_2$ operator in the pure antisymmetric $Sp(4)$ theory. The inverse squareroot singularity in the $N_F = 0, N_A = 1$ superpotential in (3.15) is therefore recovered from (3.21).

We could continue to apply our superpotential algorithm to theories with larger symplectic gauge groups. While it becomes algebraically more difficult to implement as $N_C$ increases, the method does provide a general means for mapping the ground state structure of any model of type (3.3). But we will instead turn to consider the implications of our $Sp(4)$ and $Sp(6)$ findings for theories built out of products of these groups. We take up this topic in the next section.

**IV. PRODUCT GROUP MODELS**

The moduli spaces of supersymmetric theories based upon product gauge groups $G = \prod G_i$ are generally more complicated than those for simple group models. The nonperturbative superpotentials that summarize their structure involve the strong interaction scales for each factor in $G$. The dependence of $W_{\text{dyn}}$ upon $\Lambda_i^{h_{1i}}$ is not fixed by $U(1)_R$ invariance since scale ratios with zero R-charge can arise. Symmetry principles also do not completely constrain the matter fields in $W_{\text{dyn}}$ even when they all transform according to only fundamental or singlet irreps of $G_i$. But by considering different limits in which the product group theory reduces to some known simple group model, one can often reconstruct its full superpotential. This approach was used in refs. [3] and [16] to study the vacua of several instructive $Sp(2) \times Sp(2)$ and $Sp(4) \times Sp(2)$ theories. Following these works, we will broaden the scope of known symplectic product group models to include a number of interesting $Sp(4) \times Sp(4)$ and $Sp(6) \times Sp(2)$ theories.

The first model we investigate has symmetry group

$$G = [Sp(4)_L \times Sp(4)_R]_{\text{local}} \times [SU(4)_L \times SU(2)_R \times U(1)_Q \times U(1)_L \times U(1)_R \times U(1)_R']_{\text{global}}$$

(4.1)
and matter content

\[ Q_{\alpha\dot{\alpha}} \sim (4, 4; 1; 1, 0, 0, 0) \]
\[ L_{\alpha i} \sim (4; 1, 4; 1, 0, 0, 0) \]
\[ R_{\dot{\alpha}I} \sim (1, 4; 1, 2, 0, 0, 1, 0) \]
\[ \Lambda_L^5 \sim (1, 1, 1; 4, 4, 0, -2) \]
\[ \Lambda_R^6 \sim (1, 1, 1; 4, 0, 2, 0). \]  

(4.2)

We add a prime onto the \( U(1) \) in (4.1) that counts R-charge to distinguish it from the abelian factor which tallies \( Sp(4)_R \) flavor number. We also adopt the nomenclature of ref. [16] and let \( \alpha, \beta \) and \( \dot{\alpha}, \dot{\beta} \) respectively denote \( Sp(4)_L \) and \( Sp(4)_R \) color indices. We use \( i, j \) and \( I, J \) as \( SU(4)_L \) and \( SU(2)_R \) flavor indices.

The moduli space of the chiral \( Sp(4)_L \times Sp(4)_R \) gauge theory is most conveniently analyzed in the \( \Lambda_L \gg \Lambda_R \) limit. Its dynamics in the intermediate energy range \( \Lambda_R < \mu < \Lambda_L \) is then described by an effective \( Sp(4)_L \) field theory in which \( Sp(4)_R \) plays the role of a weakly coupled external gauge group. The strong left-handed force confines the \( Q_{\alpha\dot{\alpha}} \) and \( L_{\alpha i} \) quarks into the mesons appearing inside the matrix

\[ M_{8\times8} = \begin{pmatrix} Q^T JQ & Q^T JL \\ L^T JQ & L^T JL \end{pmatrix}, \]  

(4.3)

while the right handed fields remain unbound. The \( Sp(4)_L \) color-singlet hadrons

\[ O_1 = \text{Tr}(QJQ^T J) \quad (V_L)_{\dot{\alpha}i} = (Q^T JL)_{\dot{\alpha}i} \]
\[ A_{\dot{\alpha}\dot{\beta}} = [Q^T JQ + \frac{1}{4} O_1 J]_{\dot{\alpha}\dot{\beta}} \quad (M_L)_{ij} = (L^T JL)_{ij} \]  

(4.4)

along with the \( R_{\dot{\alpha}I} \) quarks thus represent the active matter degrees of freedom in the energy interval between the two \( Sp(4) \) scales.

The quantum numbers of the partons in (4.2) were intentionally chosen so that the \( Sp(4)_L \) gauge group would act upon \( N_{FL} = N_{CL} + 2 = 4 \) flavors of fundamental quartets. This theory generates the superpotential \( W_L = -\text{Pf}M_{8\times8}/\Lambda_L^5 \) which is analogous to \( W_{NF=NC+1} \) in eqn. (1.1) for SUSY QCD [7]. After decomposing the Pfaffian of the \( 8 \times 8 \) matrix in terms of its \( 4 \times 4 \) blocks, we obtain the superpotential

\[ W_L = -\frac{1}{16\Lambda_L^5} \left[ (O_1^2 + 16\text{Pf}A)\text{Pf}M_L + 16\text{Pf}(V_L^T J V_L) + \epsilon^{ijkl}(M_L)_{ij}(O_1 V_L^T J V_L - 4V_L^T J A J V_L)_{kl} \right] \]  

(4.5)
for the intermediate effective theory.

At energies below the $\Lambda_R$ scale, the $Sp(4)_R$ force among the three flavors of right handed quartets in $(V_L)_{\alpha i}$ and $R_{\alpha I}$ and the antisymmetric field $A_{\dot{\alpha}\dot{\beta}}$ grows strong. As we know from our results for $N_F = 3, N_A = 1$ $Sp(4)$ theory, the nonperturbative $Sp(4)_R$ dynamics confines the right handed partons inside the colorless mesons

$M_{6 \times 6} = \begin{pmatrix}
V^T_L JV_L & V^T_L JR \\
R^T JV_L & R^T JR
\end{pmatrix}$ \quad \text{and} \quad
N_{6 \times 6} = \begin{pmatrix}
V^T_L JAJV_L & V^T_L JAJR \\
R^T JAJV_L & R^T JAJR
\end{pmatrix}

(4.6)

and produces the superpotential

$W_R = -\frac{\epsilon^{ijklmn}}{48\Lambda^2_L \Lambda^2_R} [M_{ij} M_{kl} M_{mn} O_2 + 12 N_{ij} N_{kl} M_{mn}]$. \quad (4.7)

When the $6 \times 6$ matrices are decomposed in terms of their $4 \times 4$, $4 \times 2$ and $2 \times 2$ block components, the following set of $Sp(4)_L \times Sp(4)_R$ invariant operators naturally emerges:

$O_1 = \text{Tr}(QJQ^T J)$ \quad \quad $O_2 = \text{Tr}(AJ)^2$

$(M_L)_{ij} = (L^T JL)_{ij}$ \quad \quad $(M_R)_{IJ} = (R^T JR)_{IJ}$

$(N'_L)_{ij} = (L^T JQJQ^T JL)_{ij} = -(V_L^T JV_L)_{ij}$ \quad \quad $(N_R)_{IJ} = (R^T JAJR)_{IJ}$

$(P_L)_{ij} = (L^T JQJAJQ^T JL)_{ij} = -(V_L^T JAJV_L)_{ij}$

$S_{iI} = (L^T JQJR)_{iI} = -(V_L^T JAR)_{iI}$

$T_{iI} = (L^T JQJAJR)_{iI} = -(V_L^T JAJR)_{iI}$. \quad (4.8)

These operators serve as moduli space coordinates for the theory with $N_{FL} = 2$ and $N_{FR} = 1$ flavors of $L_\alpha$ and $R_\alpha$ quarks and $N_Q = 1$ $Q_{\alpha\dot{\alpha}}$ field.

The total superpotential that characterizes the far infrared structure of this $Sp(4)_L \times Sp(4)_R$ model simply equals the sum of the left and right handed sector components \[16\]:

$W_{(N_Q=1, N_{FL}=2, N_{FR}=1)} = W_L + \beta W_R$. \quad (4.9)

When expressed as functions of the operators in (4.8), the two terms on the RHS become

\[15\]

$\text{We add a prime on } N'_L \text{ as a reminder that it differs from the canonical } N \text{ meson in eqn. (3.9).}$
\[ W_L = -\frac{1}{16\Lambda_L^6} \left[ (O_1^2 - 4O_2)\text{Pf} M_L + 16\text{Pf} N'_L + \epsilon^{ijkl} (M_L)_{ij} (4P_L - O_1 N'_L)_{kl} \right] \]
\[ W_R = -\frac{1}{16\Lambda_L^6 \Lambda_R^6} \left\{ 4O_2 \text{Pf} N'_L \text{Pf} M_R + 16\text{Pf} P_L \text{Pf} M_R + 4\text{Pf} N_R \epsilon^{ijkl} (N'_L)_{ij} (P_L)_{kl} + \epsilon^{ijkl} \epsilon^{IJ} S_{ij} S_{Jl} (N'_L)_{kl} + 8S_{ii} T_{jj} (P_L)_{kl} \right\}. \] (4.10)

The relative coefficient \( \beta \) standing between these terms is convention dependent and \textit{a priori} unknown. But as we shall shortly see, its value \( \beta = 1/4 \) is fixed by threshold factor independent matching relations and parity considerations in the \( N_Q = N_{FL} = N_{FR} = 1 \) theory. We should also note that although the superpotential in (4.9) was derived in the \( \Lambda_L \gg \Lambda_R \) limit, its validity transcends this special case. \( W_{(N_Q=1,N_{FL}=2,N_{FR}=1)} \) describes the entire low energy moduli space for arbitrary values of the \( Sp(4)_L \) and \( Sp(4)_R \) scales [16].

If we couple sources to the dynamically generated superpotential in (4.9) and remove heavy degrees of freedom, the \( N_Q = 1, N_{FL} = 2, N_{FR} = 1 \) model flows down to theories with fewer matter fields. We first add a tree level mass term which decouples one flavor of left handed quarks and yields the \( N_Q = N_{FL} = N_{FR} = 1 \) model whose moduli space is defined by three quantum constraints:

\[ (O_1^2 - 4O_2)\text{Pf} M_L - 4O_1 \text{Pf} N'_L + 16\text{Pf} P_L = 16\Lambda_L^6 \] (4.11a)
\[ \text{Pf} N'_L \text{Pf} N_R + \text{Pf} M_R \text{Pf} P_L + \epsilon^{ijkl} \epsilon^{IJ} S_{ij} T_{jj} = -\text{Pf} M_L \Lambda_R^6 \] (4.11b)
\[ O_2 (\text{Pf} N'_L \text{Pf} M_R + \det S) + 4(\text{Pf} P_L \text{Pf} N_R + \det T) = (O_1 \text{Pf} M_L - 4\text{Pf} N'_L) \Lambda_R^6. \] (4.11c)

The \( \Lambda_L \) scale in (4.11a) is related to its counterpart in (4.10) by a simple matching condition, while the \( \Lambda_R \) scales in the upstairs and downstairs theories are exactly the same. All mesons appearing in these formulae represent \( 2 \times 2 \) matrices in flavor space.

The three quantum relations in (4.11) do not show any sign of a discrete parity reflection in the \( N_Q = N_{FL} = N_{FR} = 1 \) model whose left and right handed sectors are precisely parallel. But appearances can be deceiving. After eliminating the \( P_L \) field via the first constraint and performing the superfield redefinitions

\[ N'_R = R^T J Q^T J Q J R = N_R + \frac{1}{4} O_1 M_R \]
\[ \Delta = \det Q = \text{Pf} Q^T J Q = \frac{1}{16} (O_1^2 - 4O_2), \] (4.12)

we find two new quantum relations which we incorporate within the superpotential.
\[ W_{(N_Q=N_{FL}=N_{FR}=1)} = X \left[ \Delta \text{Pf} M_L \text{Pf} M_R - \text{Pf} N'_L \text{Pf} N'_R - \epsilon^{ij} \epsilon^{IJ} S_{iI} T_{jJ} - \text{Pf} M_R \Lambda^6_L - \text{Pf} M_L \Lambda^6_R \right] \\
+ Y \left[ \Delta (\text{Pf} M_L \text{Pf} N'_R + \text{Pf} M_R \text{Pf} N'_L) - \frac{1}{2} O_1 \text{Pf} N'_L \text{Pf} N'_R + (\Delta - \frac{1}{16} O^2_1) \det S - \det T \right] \\
- \frac{1}{4} O_1 \epsilon^{ij} \epsilon^{IJ} S_{iI} T_{jJ} - \text{Pf} N'_R \Lambda^6_L - \text{Pf} N'_L \Lambda^6_R \right]. \] (4.13)

\[ \text{Mirabile dictu, this expression is manifestly left-right symmetric!} \]

If we continue to add quark mass terms, we can flow down to the \(Sp(4)_L \times Sp(4)_R\) model that has only \(Q_{a\dot{a}}\) matter. Since no source terms in the \(N_Q = N_{FL} = N_{FR} = 1\) theory transform like \((2, 2)\) under the global \(SU(2)_L \times SU(2)_R\) chiral symmetry group, the \(S\) and \(T\) meson fields cannot develop nonzero expectation values. When we evaluate the equations of motion for the remaining \(M_{L,R}\) and \(N'_{L,R}\) mesons along with the \(X\) and \(Y\) Lagrange multipliers, we find two distinct branches for the dynamical superpotential in the \(N_Q = 1, N_{FL} = N_{FR} = 0\) theory:

\[ W_{(N_Q=1,N_{FL}=N_{FR}=0)} = \left\{ \begin{array}{c} \left[ \Lambda^7_L \pm \Lambda^7_R \right]^2 \\
\Delta \\
\frac{1}{\Delta} \left[ \Lambda^7_L + \Lambda^7_R + \frac{O_1}{\sqrt{O^2_1 - 16\Delta}} \Lambda^7_L \Lambda^7_R \right] \end{array} \right\} \] (4.14)

This result exhibits several interesting features. Firstly, it correctly reduces to the instanton generated \(N_F = 2\) \(Sp(4)\) superpotential in (3.18) when either \(\Lambda_L\) or \(\Lambda_R\) vanishes. Secondly, its dependence upon the dimensionless and chargeless ratio \(R = 16\Delta/O^2_1\) could not have been determined by symmetry considerations or integrating in techniques. We also note that the full superpotential cannot be obtained by considering the \(\Lambda_L \gg \Lambda_R\) limit of the pure \(Q_{a\dot{a}}\) matter theory. Instead, it is necessary to start from a theory with a genuine moduli space of vacua and integrate out matter to obtain eqn. (4.14). Finally, the dichotomy in \(W_{(N_Q=1,N_{FL}=N_{FR}=0)}\) directly reflects the two \(N_F = 0, N_A = 1\) \(Sp(4)\) superpotential branches in eqn. (3.15). When \(Sp(4)_L \times Sp(4)_R\) is broken to its diagonal subgroup by the expectation value \(\langle Q \rangle = v 1_{4 \times 4}\), the fluctuations about this vev which survive in the low energy \(Sp(4)_{L+R}\) theory transform according to the 5 dimensional antisymmetric representation. Expanding about this point, we find the operator relations

\[ O_1 = -4v^2 - O^{(4)}_2 \]
\[ \Delta = \det Q = v^4 + O(v^2 O^{(4)}_2) \] (4.15)
and matching condition $\Lambda_L^2 \Lambda_R^2 = v^3 \Lambda_{L+R}^4$. When these expressions are inserted into the terms inside the curly brackets in (1.14), both $Sp(4) W_{(N_F=0, N_A=1)}$ superpotentials are recovered as limits of $W_{(N_Q=1, N_{FL}=N_{FR}=0)}$ in the $Sp(4)_L \times Sp(4)_R$ theory.

Techniques similar to those which we have used to analyze these $Sp(4) \times Sp(4)$ models can be applied to other product group theories as well. For example, we briefly sketch the outline of a model based upon $Sp(6) \times Sp(2)$ with $(6,2)$ matter that incorporates our simple $Sp(6)$ group findings. It is again easiest to first consider the limit $\Lambda_2 \gg \Lambda_6$. At energies well above the $\Lambda_6$ scale, the $Sp(2)$ gauge dynamics confine the elementary fields into $Sp(2)$ singlet mesons $A_{\alpha\beta}$ and generate the superpotential $W_2 = -\text{Pf} A / \Lambda_2^3$. For $\mu < \Lambda_6$, the strong $Sp(6)$ force binds together the composite antisymmetric fields into completely colorless $O_2$ and $O_3$ combinations and produces a dynamic superpotential $W_6$ which can be read off from eqn. (3.21). The total superpotential $W = W_2 + \gamma W_6$ for the low energy $Sp(6) \times Sp(2)$ effective theory equals the sum of the two separate contributions with a relative coefficient $\gamma$ that depends upon the normalization conventions for $\Lambda_2$ and $\Lambda_6$. Variations on this model with additional $Sp(6)$ quark fields can be worked out along similar lines.

V. CONCLUSION

In this article, we have examined the confining phase vacua of several symplectic SUSY gauge theories with matter in fundamental and antisymmetric representations. These models exhibit interesting nonperturbative features such as multiple quantum constraints, intricate superpotentials depending upon chargeless operator ratios and singularity structures that reproduce underlying theories at points of enhanced gauge symmetry. Our approach to studying these particular $Sp(2N_C)$ models can be profitably applied to other confining supersymmetric theories. In simple gauge group models, it is often possible to adjust microscopic matter contents so that the dynamically generated superpotentials are proportional to polynomials in gauge invariant fields. These polynomials are determined by requiring that they yield hadronic equations of motion which reduce to classical constraints among parton constituents. When tree level masses are added and heavy fields are integrated out, nonperturbative superpotentials for models with fewer matter degrees of freedom can be derived. This top-down algorithm yields complete towers of low energy effective theories. In contrast, the bottom-up approach which
starts with pure glue theory and successively integrates in matter frequently fails at various rungs on the effective theory ladder. While the utility of integrating in techniques has been demonstrated in certain cases [3,15], our experience with the symplectic models in this paper leads us to believe the top-down approach is more generally useful.

Once the ground state structure of a confining simple group model is known, its impact upon product group theories is largely determined. By considering various limits in which a product group model reduces to a known simple group theory, one can reconstruct its full superpotential. Simple group models thus serve as building blocks for arbitrarily complicated theories.

In closing, we mention some extensions of this work which would be interesting to pursue. The top-down method we have followed yielded exact superpotentials for $Sp(4)$ and $Sp(6)$ theories. In principle, it can be used to construct the low energy effective theory for any specified value of $N_C$. But deriving closed form expressions for confining phase superpotentials in symplectic theories with arbitrary numbers of colors would be preferable. Unfortunately, we have not yet found a clear pattern among the special cases we have solved so far. It would similarly be interesting to examine entire sequences of product group theories. Large $N_C$ limits of such models might reveal unexpected surprises. Finally, constructing dual descriptions of multimatter symplectic theories in nonconfining phases with zero tree level superpotentials remains an important outstanding problem. We look forward to investigating these issues in the future.

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