Primordial Fluctuations from Inflation with a Triad of Background Gauge Fields

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Abstract

We study the linear perturbation of the recently proposed model of inflation where a uniform gauge-kinetic coupling of the inflaton to multiple vector fields breaks the cosmic no-hair conjecture while maintaining the isotropy. We derive the general quadratic action for the perturbation and calculate the power spectra of scalar and tensor modes at the end of inflation by in-in formalism. It is shown that the model predicts slightly red spectra and the tensor-to-scalar ratio tends to be suppressed. The comparison with the data from WMAP 7-year does not impose strong constraints on the parameters and both weak- and strong- gauge-field regimes are consistent with the current observations.

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I. INTRODUCTION

Over the past decade, the prospect of unveiling minute details of density fluctuations through observations of the Cosmic Microwave Background (CMB) and large scale structure has driven a growing interest in inflationary scenarios beyond the single-field slow-roll model. Despite its simplicity and the remarkable success in matching the observed density power spectrum, it is hard to believe that a single scalar field is entirely responsible for the dynamics of the early universe. Along with multi-scalar models such as hybrid inflation \[1\], attempts have been made to incorporate vector fields \[2–4\], which have proven to be largely unsuccessful due to various instabilities in realizing accelerated expansion \[5–9\]. Recently, a viable model of vector inflation has been proposed by Soda and his collaborators \[10\] where the gauge-kinetic coupling with inflaton, originating from supergravity, maintains the amplitude of the vector and results in non-trivial signatures in the primordial fluctuations, e.g. statistical anisotropy and scalar-tensor correlations \[11\]. This "inflation with vector-hair" has been subsequently scrutinized \[12–17\] and its stability has been widely established \[18–20\]. Moreover, it was found that the inclusion of additional vector fields tends to reduce anisotropy in the background through a generic dynamical mechanism \[21\]. In the case of uniform gauge-kinetic coupling for three or more vectors, the trajectories converge to a universal isotropic attractor with non-vanishing vector energy density, whose amplitude is determined by the strength of the coupling. Very similar dynamical behaviors have been observed in the models employing non-Abelian gauge fields \[22–25\], which indicates that the tendency towards isotropy is generic in the inflation with multiple vectorial degrees of freedom. This special case is indistinguishable from the single-field slow-roll model at the level of classical background dynamics as long as it provides a sufficient e-folding number. As such, it is necessary to investigate its linear perturbation in order to impose observational constraints and decide its viability, which is the aim of this article.

Incidentally, these gauge-kinetic models can be viewed as the classical counterpart for the particle production effects in preheating scenarios. In this context, couplings between the inflaton and gauge fields have often been discussed in the attempts of generating primordial magnetic fields \[26, 27\], gravitational waves \[28\], and non-Gaussianity \[29\]. The crucial conceptual leap of the inflationary models considered in this article is the breach of the cosmic-no-hair conjecture whereby typical preheating scenarios assume vanishing classical
background values of the fields excited by the inflaton, which forces the analysis to go beyond linear order and makes it complicated. The presence of vector-hair in inflation illustrates that the non-vanishing background energy density of (potentially anisotropic) auxiliary fields does not necessarily mean a break down of the inflationary regime and the accelerated expansion may continue without wiping out classical "hair," being the attractor solution in the phase space at the same time. Therefore, it is interesting to ask whether the existence of the background gauge fields can produce any of the features mentioned above within the well-established linear perturbation around the isotropic Friedmann-Lemaître-Robertson-Walker space-time and quantum field theory in the quasi-de-Sitter background, and how large is the amplitude if any.

In the present article, we calculate the spectra of the scalar curvature in uniform density slicing and the gravitational waves generated through the interaction of the perturbative variables with the background triad of the gauge fields. The result obtained is very much analogous to what was found for the anisotropic cases [11, 30, 31]. We find that the scalar power spectrum acquires a double logarithmic scale dependence with the magnitude of the correction being given by the fractional energy density of the background gauge fields with respect to the scalar kinetic energy. The effect is not slow-roll suppressed due to the steepness of the gauge-kinetic function that is needed to maintain the background gauge fields in the accelerated expansion. While the tensor mode receives similar corrections, the magnitude is smaller than that for the scalar mode by a factor of slow-roll parameter, which was also seen in the anisotropic cases. However, since the effect is completely isotropic, the restrictions coming from the observational data are much weaker than the anisotropic single-gauge-field models. Although the correction terms involve e-folding numbers, which resulted in strong constraints on the energy density of the gauge field in the anisotropic models, the large face-value of the e-folding numbers can only affect the overall amplitude of the power spectra in this isotropic setup. Since the amplitudes of the quantum fluctuations are always normalized by the energy scale of inflation, the potentially large corrections to these amplitudes can be absorbed into this normalization, whence do not immediately pose a problem.

The paper is organized as follows. In section II, we present the background equations and introduce the slow-roll parameters to characterize the inflationary dynamics. Section III explains the structure of the second order action for the gauge fields, which lead to an extended notion of scalar-vector-tensor decomposition. In section IV, the power spectrum
of the scalar mode is computed by employing in-in formalism on the de-Sitter space-time. In section V, a similar calculation is performed for the tensor mode and observables such as spectral tilt and tensor-to-scalar ratio are obtained. Section VI summarizes the results and discusses the outlook.

II. THE ISOTROPIC BACKGROUND DYNAMICS OF THE INFLATION WITH GAUGE-KINETIC COUPLING

We consider the universe described by the following action;

\[ S = \int dx^4 \sqrt{-g} \left( \frac{M^2_{pl}}{2} R - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) - \frac{f(\varphi)^2}{4} F^{(m)}_{\mu\nu} F^{(m)\mu\nu} \right). \]  

(1)

The scalar field \( \varphi \) acts as the inflaton and interacts with the gauge fields through the gauge-kinetic function \( f(\varphi) \). Although we assume for simplicity that \( F^{(m)}_{\mu\nu}, (m = 1, 2, 3) \) are three copies of Abelian gauge field

\[ F^{(m)}_{\mu\nu} = (dA^{(m)})_{\mu\nu}, \]  

(2)

there is strong evidence for the fact that the dynamics of an \( SU(2) \) gauge field with a gauge-kinetic coupling in an inflationary regime is well described by the action [15]. The generalization for \( m \geq 4 \) is also straightforward. We adopt the ADM formalism and follow the notation of the ref. [32] to write the metric as

\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k N_j & N_j \\ N_i g_{ij} & g_{ij} \end{pmatrix}, \]  

(3)

where

\[ g^{ik} g_{kj} = \delta^i_j, \quad N^i = g^{ij} N_j. \]  

(4)

It is well known that in terms of the normalized extrinsic curvature

\[ E_{ij} = -\frac{1}{2} \left( g_{ij} - 2N_{(ij)} \right) \]  

(5)

and the intrinsic scalar curvature of the constant time slice

\[ 3R = (g_{ij,kl} + g_{mn} \Gamma^m_{ij} \Gamma^n_{kl}) \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right), \]  

(6)

the gravitational part of the Lagrangian is written as

\[ \frac{1}{M^2_{pl}} \mathcal{L}_g = \frac{\sqrt{g}}{2N} \left( E_{ij} E^{ij} - E^2 \right) + \frac{1}{2} N \sqrt{g} 3R. \]  

(7)
We introduce the electric fields by
\[ E^{(m)}_i = F^{(m)}_{0i} \]  
for the 1 + 3 decomposition and write the gauge-field Lagrangian in the form
\[ \mathcal{L}_M = \frac{\sqrt{g}}{2N} f^2 g^{ik} \left( E^{(m)}_i + F^{(m)}_{ij} N^j \right) \left( E^{(m)}_k + F^{(m)}_{kl} N^l \right) - \frac{1}{4} N \sqrt{g} f^2 g^{ik} g^{jl} F^{(m)}_{ij} F^{(m)}_{kl}. \]  

As usual, the scalar Lagrangian is given by
\[ \mathcal{L}_\varphi = \frac{\sqrt{g}}{N} \left( \frac{1}{2} \dot{\varphi}^2 - \varphi \dot{\varphi}_i N^i + (\varphi, i N^i)^2 \right) - N \sqrt{g} \left( \frac{1}{2} g^{ij} \dot{\varphi}_i \dot{\varphi}_j + V(\varphi) \right). \]  

For the background, we use the ansatz
\[ N = N(t), \quad N_i = 0, \quad g_{ij} = a(t)^2 \delta_{ij}, \quad \varphi = \bar{\varphi}(t), \quad A^{(m)}_0 = 0, \quad A^{(m)}_i = A(t) \delta^m_i. \]

The resulting equations of motion read
\[ M^2_{\text{pl}} \left( \frac{\dot{a}^2}{a^2} + 2 \frac{\dot{a}}{a} - 2 \frac{\dot{a} N}{a N} \right) + \frac{1}{2} \dot{\varphi}^2 - N^2 V + \frac{f^2}{2a^2} \dot{\varphi}^2 = 0, \]  
\[ 3M^2_{\text{pl}} \frac{\dot{a}^2}{a^2} - \frac{1}{2} \dot{\varphi}^2 - N^2 V - \frac{3f^2}{2a^2} \dot{\varphi}^2 = 0, \]  
\[ \ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} - \frac{N}{N} \dot{\varphi} + N^2 V - \frac{3f \dot{\varphi}}{a^2} = 0, \]  
\[ \frac{a f^2}{N} \dot{A} = \text{const.} \equiv c M_{\text{pl}}. \]  

The first integral \[^{[15]}\] will be used to eliminate \( \dot{A} \). In order to figure out the conditions for inflation, let \( t \) be the proper time coordinate by choosing \( N = 1 \) and define the Hubble expansion rate \( H = \dot{a}/a \). For the spatial slice to undergo accelerated expansion and for it to last more than one Hubble time, it is required that the parameters
\[ \epsilon_H = - \frac{\dot{H}}{H^2} = - \frac{\dot{\varphi}^2}{2M^2_{\text{pl}} H^2} + \frac{c^2}{a^4 f^2 H^2}, \]
\[ \eta_H = \frac{\epsilon_H}{H \epsilon_H} = 2 \left( \epsilon_H + \frac{\dot{\varphi}^2}{2 \epsilon_H M^2_{\text{pl}} H^2 H \dot{\varphi}} \right) - 2 \left( 1 - \frac{\dot{\varphi}^2}{2 \epsilon_H M^2_{\text{pl}} H^2 H \dot{\varphi}} \right) \left( 2 + \frac{f}{H f} \right) \]
be much less than unity. These two parameters characterize the evolution of the space-time: \( \epsilon_H \ll 1 \) guarantees accelerated expansion \( \ddot{a}/a > 0 \) and constancy of \( H \) over a few Hubble times and \( \eta_H \ll 1 \) ensures that this regime lasts for at least tens of e-foldings. Another
important element is the balance between the scalar kinetic energy and the amplitude of the
gauge fields. It proves to be convenient to use the following set of parameters;

\[ \epsilon_\varphi = \frac{\dot{\varphi}^2}{2M_{pl}^2H^2} < \epsilon_H, \]  

\[ \eta_\varphi = \frac{\dot{\epsilon}_\varphi}{H\epsilon_\varphi} = 2 \left( \epsilon_H + \frac{\ddot{\varphi}}{H\dot{\varphi}} \right). \]  

(18)

(19)

\[ \epsilon_\varphi \] represents the kinetic energy of inflaton, which is already much smaller than unity by the
conditions above. Its significance resides in the fact that it controls the balance between the
kinetic energy of the inflaton and the gauge fields through

\[ \frac{f^2 \dot{A}^2}{a^2M_{pl}^2H^2} = \frac{c^2M_{pl}^2}{a^4f^2H^2} = \epsilon_H - \epsilon_\varphi < \epsilon_H. \]  

(20)

In connection to this balance, it later proves to be useful to define another parameter

\[ I = \sqrt{\frac{\epsilon_H - \epsilon_\varphi}{\epsilon_\varphi}}, \]  

(21)

measuring the ratio of kinetic energy between the background gauge fields and the inflaton.

Note that there is no a priori constraint on \( I \) as long as \( \epsilon_\varphi < \epsilon_H \ll 1 \) is satisfied. Although it
would not be necessary to assume \( \eta_\varphi \ll 1 \), a large \( \eta_\varphi \) means rapidly varying \( \epsilon_\varphi \) and in turn a
rapid exchange of the energy between the gauge fields and the inflaton, which would require
a dedicated condition to be met by \( V \) and \( f \). In order to avoid unnecessary complications,
we do assume the smallness of \( \eta_\varphi \), which leads to the control over the gradients of \( V \) and \( f \) through

\[ \frac{V_\varphi}{M_{pl}H^2} = \frac{1}{\sqrt{2\epsilon_\varphi}} \left( -6\epsilon_H + 3\epsilon_H^2 - 2 \frac{\epsilon_H\eta_H}{\epsilon_\varphi} - \epsilon_H\epsilon_\varphi + \frac{1}{2}\epsilon_\varphi\eta_\varphi \right), \]  

(22)

and

\[ \frac{M_{pl}f_\varphi}{f} = \frac{1}{\sqrt{2\epsilon_\varphi}} \left( -2 + \epsilon_H - \frac{1}{2}\frac{\epsilon_H\eta_H - \epsilon_\varphi\eta_\varphi}{\epsilon_H - \epsilon_\varphi} \right). \]  

(23)

The slower \( \varphi \) rolls down the potential, the greater is the amplitude of the gauge fields and
the steeper is the slope of the gauge-kinetic function.

**III. THE SECOND ORDER ACTION AND SCALAR-VECTOR-TENSOR DE-
COMPOSITION**

Now let us perturb the background and write down the second order action. Our con-
vention for the perturbative variables are given as follows:
Metric:
\[ N = \mathcal{N}(1 + \phi), \quad N_i = \mathcal{N}_i \beta, \quad g_{ij} = a^2(\delta_{ij} + 2\gamma_{ij}). \]  

\( (24) \)

Inflaton:
\[ \varphi = \bar{\varphi} + \pi. \]  

\( (25) \)

Gauge fields:
\[ A_0^{(m)} = \sigma^m, \quad A_i^{(m)} = A\delta^m_i + \chi^m_i. \]  

\( (26) \)

Note that the definition of \( \phi \) is unconventional. The derivation for the gravity and scalar actions can be seen in any standard literature (e.g. ref. 33). To derive the gauge-field Lagrangian, it is convenient to define the perturbed electric fields
\[ X^m_i = \dot{\chi}^m_i - \sigma^m_i. \]  

\( (27) \)

Substituting
\[ E_i^{(m)} = \dot{A}\delta^m_i + X^m_i, \quad F_{ij}^{(m)} = -2\chi^m_{[i,j]} \]  

and
\[ E_i^{(m)} + F_{ij}^{(m)}N^j = \dot{A}\delta^m_i + X^m_i - 2\mathcal{N}_a\chi^m_{[i,j]}\beta_j + \text{higher order terms}, \]  

into the action \( (9) \), one obtains
\[ \mathcal{L}_M^{(2)} = \frac{af^2}{2\mathcal{N}}X^m_iX^m_i - \frac{af^2}{\mathcal{N}}\dot{A}(2\gamma_{ij}X^i_j - \gamma X^m_m) + \frac{a\dot{A}}{\mathcal{N}}(2ff_{\nu\pi} - f^2\phi)X^m_m \]
\[ -\frac{a}{\mathcal{N}}f^2\chi^m_{ij}\chi^m_{ij} - 2\dot{A}\chi^m_i\beta_j + \frac{3af^2}{2N}\dot{A}^2\phi^2 - \frac{a\dot{A}^2}{2N}\phi(f^2\gamma + 6ff_{\nu\pi}) \]  

\( (30) \)

where \( \gamma = \gamma_{ii} \). Combining this with the gravity and scalar Lagrangians, using the background equations and performing integration by parts, we derive the general quadratic Lagrangian as
\[ \mathcal{L}^{(2)} = \frac{a^3M^2_{pl}}{2\mathcal{N}}(\hat{\gamma}_{ij}\hat{\gamma}_{ij} - \gamma^2) + \frac{Naf^2}{2N}(2\gamma_{ij,j}\gamma_{ik,k} - \gamma_{ij,k}\gamma_{ij,k} + 2\gamma_{ij,ij} - \gamma_{ij,ii}) \]
\[ + \frac{af^2}{2N}\dot{A}^2(2\gamma_{ij}\gamma_{ij} - \gamma^2) + \frac{a^3}{2N}\hat{\pi}^2 - \frac{Na}{2}\pi_{,i} \pi_{,i} - \frac{1}{2}\left(Na^3V_{,\varphi \varphi} - \frac{3af_{\varphi \varphi}}{N}\dot{A}^2\right)\hat{\pi}^2 \]
\[ + \frac{af^2}{2N}X^i_j X^j_i - \frac{Naf^2}{a}\chi^k_{[i,j]}\chi^k_{[i,j]} - 2\frac{af}{N}\dot{A}\gamma_{ij}X^i_j + \frac{2af_{\varphi \pi}}{N}\dot{A}\pi X^m_m \]
\[ + \frac{NaM^2_{pl}}{2}(\beta_{(i,j)}\beta_{(i,j)} - \beta_{i}^2) + a^2M^2_{pl}\beta_i(\hat{\gamma}_{ij} - \hat{\gamma}_{i} + \frac{a}{a}\hat{\phi}_{,i}) \]  

\( (30) \)
\[-N a^3 V \phi^2 + \gamma \left[ \frac{a^3 \dot{\phi}}{N} \pi - \left( N a^3 V_{,\phi} - \frac{af f_{,\phi}}{N} \dot{A}^2 \right) \pi + \frac{af^2}{N} \dot{A} X^m_m \right] \]
\[-a^2 \beta_i \left( \dot{\phi} \pi, i + 2 \frac{f^2}{a} \dot{A} \chi^k_{[k,i]} \right) + M_{pl}^2 \phi \left[ \frac{2a^2 \dot{a}}{N} \dot{\gamma} + Na (\gamma_{ij,ij} - \gamma_{,ii}) \right] \]
\[+ \phi \left[ \frac{af^2}{N} \dot{A}^2 \gamma - \frac{a^3 \dot{\phi}}{N} \pi - \left( N a^3 V_{,\phi} + 3a \dot{A} f f_{,\phi} \right) \pi - \frac{af^2}{N} \dot{A} X^m_m \right]. \quad (31)\]

Although the upper indices originated from the label attached to each copy of gauge field, some of them are now contracted with spatial indices. This happened since the background respects the spatial \( O(3) \) symmetry and is also invariant under linear mixing of the three vector fields by \( O(3) \) "rotation," namely \( A^{(m)}_{i} \propto \delta^m_i \), which resulted in transferring spatial indices to the ones denoting species. Consequently, \( \chi^i_{,j} \) (and \( X^i_{,j} \)) looks as if it were a \( 3 \times 3 \) spatial tensor even though it represents a triplet of vectors. This suggests that one can formally extend the scalar-vector-tensor decomposition of the perturbed quantities as was first carried out in the ref. [22]. We use the following symbols:

\[ \beta_i = B_{,i} - S_i, \]
\[ \gamma_{ij} = -\phi \delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2} h_{ij}, \]
\[ \chi^i_{,j} = \alpha \delta_{ij} + \theta_{ij} + \epsilon_{ijk} \tau_k + \kappa_{(i,j)} + \epsilon_{ijk} \lambda_k + \omega_{ij}, \]
\[ \sigma^i = \mu_{,i} + \nu_i. \]

As usual, \( S_i, F_i, \kappa_i, \lambda_i \) and \( \nu_i \) are divergence-free vectors and \( h_{ij} \) and \( \omega_{ij} \) are traceless transverse tensors and \( \epsilon_{ijk} \) is the Levi-Civita symbol in three dimensions. Now all the indices are downstairs, which signals we are going to ignore the distinction between spatial and component indices. It can be easily seen that the Lagrangian indeed splits into three pieces each of which contains only scalars, vectors and tensors respectively, up to a surface term, and one can deal with each mode separately as far as linear perturbations are concerned.

Not all of the quantities are dynamical because of the gauge freedom of general relativity and the \( U(1) \) gauge symmetry. Let us first consider an infinitesimal space-time diffeomorphism

\[ t \to t + \eta, \quad x^i \to x^i + \xi^i + \xi_i. \]

It induces the following transformations:

\[ \mu \to \mu + A \xi, \quad \nu_i \to \nu_i + A \dot{\xi}_i, \]
\[ \alpha \to \alpha - \dot{A} \eta, \quad \theta \to \theta + A \xi \]

\[ \text{(37)}\]
\[ \kappa_i \rightarrow \kappa_i + A\xi_i, \quad \lambda_i \rightarrow \lambda_i + \frac{A}{2} \epsilon_{ijk} \xi_{j,k}. \]

Note that \( \tau \) as well as \( \omega_{ij} \) are gauge invariant. One is also allowed to perform the infinitesimal \( U(1) \) transformations
\[ A^{(m)}(\mu) \rightarrow A^{(m)}(\mu) + \partial_\mu \rho^{(m)} \] (38)
where \( \rho^{(m)} \) are arbitrary scalar functions. It is again convenient to decompose them as
\[ \rho^{(m)} = \rho^m + \rho \] (39)
and the transformation laws become
\[ \mu \rightarrow \mu + \dot{\rho}, \quad \nu_i \rightarrow \nu_i + \dot{\rho}_i, \]
\[ \theta \rightarrow \theta + \rho, \quad \kappa_i \rightarrow \kappa_i + \rho_i, \quad \lambda_i \rightarrow \lambda_i + \frac{1}{2} \epsilon_{ijk} \rho_{j,k}. \] (40)

It is instructive to count the numbers of degrees of freedom. There are four scalars from the metric, four from the gauge fields and another for the inflaton. Two from the metric \((\phi, B)\) and one from the gauge fields \((\mu)\) are non-dynamical. We have three arbitrary functions for gauge transformations, which results in three dynamical scalar degrees of freedom. For vector perturbations, there are two from the metric and three from the gauge-fields, among which we have one non-dynamical for each sector \((S_i \text{ and } \nu_i)\). Two more are redundant because of the gauge freedom and we are left with only one dynamical vector mode. Finally, there are two gauge-invariant dynamical tensor degrees of freedom.

IV. SCALAR MODES AND THE CURVATURE POWER SPECTRUM

In this section, we consider the scalar perturbation; \( \phi, \psi, B, E, \pi, \alpha, \theta, \tau \) and \( \mu \) and compute the curvature power spectrum. We fix the gauge by setting
\[ \psi = E = \dot{\theta} + \mu = 0, \] (41)
which means the constant-time hypersurfaces are flat and \( \pi, \alpha \) and \( \tau \) are the three dynamical variables. The convenient choice of time-coordinate is the conformal time \( \eta (N = a) \) and differentiation with respect to it is denoted by primes. We set \( M_{\text{pl}} = 1 \) in this section and the next to save the space.
A. The Lagrangian and the Curvature of the Uniform-density Hypersurfaces

The second order Lagrangian is written as

$$L^{(2)}_S = \frac{a^2}{2} \left( \pi'^2 - \pi_i \pi^i \right) - \frac{1}{2} \left( a^4 V_{\varphi \varphi} - 3 (f f_{,\varphi})_{,\varphi} A'^2 \right) \pi^2 + 6 f f_{,\varphi} A' \pi \alpha' + f^2 \left( \frac{3}{2} \alpha'^2 - \alpha_i \alpha^i + \tau_{,i} \tau_{,i} - \tau_{,ii} \tau_{,i} \right) - a^2 B \left( \frac{a'}{a} \phi_{,ii} - \phi' \pi_{,ii} - \frac{f^2}{a^2} A' \alpha_{,ii} \right)$$

$$- a^4 V \phi^2 - \phi \left[ a^2 \varphi' \pi' + (a^4 V_{,\varphi} + 3 f f_{,\varphi} A'^2) \pi + 3 f^2 A' \alpha' \right].$$

Varying $B$ gives

$$\phi = \frac{\varphi'}{2H} + \frac{f^2}{Ha^2} A' \alpha$$

where we defined

$$H = \frac{a'}{a}.$$

Its substitution into the action eliminates the non-dynamical variables $\phi$ and $B$ and leaves three scalar degrees of freedom. Using the canonically normalized variables

$$\hat{\pi} = a \pi,$$

$$\hat{\alpha} = \sqrt{3} f \alpha,$$

$$\hat{\tau} = \sqrt{2} f \tau,$$

and performing several integrations by parts, it yields

$$L^{(2)}_S = \frac{1}{2} \left( \hat{\pi}'^2 - (\nabla \hat{\pi})^2 - m_\pi^2 \hat{\pi}^2 \right) + \frac{1}{2} \left( \hat{\alpha}'^2 - \frac{2}{3} (\nabla \hat{\alpha})^2 - m_\alpha^2 \hat{\alpha}^2 \right)$$

$$- \sqrt{\frac{2\epsilon_{\varphi}(\epsilon_H - \epsilon_{\varphi})}{3}} \mathcal{H} \hat{\pi}' \hat{\alpha} + \sqrt{3(\epsilon_H - \epsilon_{\varphi})} \left( \frac{2 f_{,\varphi}}{f} - \sqrt{\frac{\epsilon_{\varphi}}{2}} \right) \mathcal{H} \hat{\pi} \hat{\alpha}' + g_{\pi \alpha} \hat{\pi} \hat{\alpha}$$

$$+ \frac{1}{2} \left( (\nabla \hat{\tau})^2 - (\nabla^2 \hat{\tau})^2 + \frac{f''}{f} (\nabla \hat{\tau})^2 \right),$$

where

$$m_\pi^2 = -(2 + 3 \epsilon_{\varphi} - \epsilon_H - \epsilon_{\varphi} \epsilon_H + \eta_{\varphi} \epsilon_{\varphi}) \mathcal{H}^2$$

$$+ a^2 \left( V_{\varphi \varphi} + \sqrt{2 \epsilon_{\varphi} V_{,\varphi} + \epsilon_{\varphi} V} \right) - 3(\epsilon_H - \epsilon_{\varphi}) \left( \frac{f_{,\varphi}}{f} + \frac{f^2_{,\varphi}}{f^2} - \sqrt{2 \epsilon_{\varphi}} \frac{f_{,\varphi}}{f} \right) \mathcal{H}^2,$$

$$m_\alpha^2 = - \frac{f''}{f} + (\epsilon_H - \epsilon_{\varphi}) (3 - \epsilon_H) \mathcal{H}^2 \frac{2}{3} (\epsilon_H - \epsilon_{\varphi}) a^2 V,$$

$$g_{\pi \alpha} = \sqrt{\frac{2\epsilon_{\varphi}(\epsilon_H - \epsilon_{\varphi})}{3}} (\mathcal{H}^2 - a^2 V) - \sqrt{3(\epsilon_H - \epsilon_{\varphi})} \left( \frac{2 f_{,\varphi}}{f} - \sqrt{\frac{\epsilon_{\varphi}}{2}} \right) \frac{f'}{f} \mathcal{H}$$

$$- \sqrt{\frac{\epsilon_H - \epsilon_{\varphi}}{3}} \left( a^2 V_{,\varphi} + 3 \frac{f_{,\varphi}}{f} (\epsilon_H - \epsilon_{\varphi}) \mathcal{H}^2 \right).$$
We notice that $\tilde{\tau}$ is decoupled from the other degrees of freedom since it is a pseudo-scalar representing the magnetic fields, which are zero in the background and hence gauge invariant. As its evolution is trivial and only affects the curvature perturbation at a non-linear order, we ignore this contribution. It should be noted, however, that it may well play an important role in the three-point or higher order correlation functions. On the other hand, $\tilde{\alpha}$ makes a physical contribution to the scalar curvature at linear order directly (see the equation (49)) and through the interaction with $\tilde{\pi}$, even though its origin is the vectorial gauge fields. It is not surprising since this $\tilde{\alpha}$-mode represents the modulation of the background value of the gauge triad which already affects the background expansion. $\tilde{\alpha}$ is a superposition of three mutually orthogonal vector modes, which also explains why its propagation velocity is reduced from the speed of light to $c^2 = 2/3$.

Our aim is to compute the curvature perturbation in the uniform density gauge, which is given by

$$-\zeta = \psi + \mathcal{H} \frac{\delta \rho}{\delta' \rho'},$$

in the linear order according to Malik and Wands [34]. This quantity is not conserved during the inflationary period in the present model, even on the super-horizon scales because of the multi-field interaction and we will calculate its spectrum at the end of inflation quantum mechanically by tree-level perturbative expansion of the in-in formalism. Although it might be modified even after inflation by the entropy perturbations potentially generated by the gauge fields, we will not discuss it as the details depend on reheating. The definition of the total energy perturbation $\delta \rho$ here is that of Kodama and Sasaki [35]. At linear order, it is just the $0-0$ component of the energy-momentum tensor and we have

$$\delta \rho = -\delta T^0_0$$

$$= a^{-2} \bar{\phi'} \pi' + \left[ -\frac{\bar{\phi}^3}{2a^2} + \frac{3c^2}{f^2} \left( \frac{f_{\phi}}{f} - \frac{\bar{\phi}'}{2H} \right) + V_{\phi} \right] \pi + \frac{3c}{a^2} \alpha' - \frac{c}{\mathcal{H}a^4} \left( \bar{\phi}^2 + \frac{3c^2}{a^2 f^2} \right) \alpha$$

$$= a^{-3} \sqrt{2c_{\phi}} \mathcal{H} \pi' + a^{-3} \left[ -\sqrt{2c_{\phi}(1 - c_{\phi})H^2} + 3(c_{\phi} - c_{\phi}) \left( \frac{f_{\phi}}{f} - \sqrt{\frac{c_{\phi}}{2}} \right) H^2 + a^2 V_{\phi} \right] \pi$$

$$+ a^{-3} \sqrt{3(c_{\phi} - c_{\phi})H \alpha'} + a^{-3} \sqrt{3(c_{\phi} - c_{\phi})H^2} \left( -\frac{3f_{\phi}}{f} \mathcal{H} + (3c_{\phi} - c_{\phi})H^2 \right) \tilde{\alpha}.$$
which is reasonable, the curvature perturbation is expressed as
\[
\zeta = \frac{1}{3H\alpha e_H} \sqrt{\frac{\epsilon}{2}} \left[ \hat{\pi}' + \frac{4 + 6\mathcal{I}^2}{\eta} \hat{\pi} + \sqrt{\frac{3}{2}\mathcal{I}} \left( \hat{\alpha}' - \frac{2}{\eta} \hat{\alpha} \right) \right],
\]
whose evolution is governed by the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \hat{\pi}'^2 - \frac{1}{2} (\nabla \hat{\pi})^2 + \frac{1 + 6\mathcal{I}^2}{\eta^2} \hat{\pi}^2 + \frac{1}{2} \hat{\alpha}'^2 - \frac{1}{3} (\nabla \hat{\alpha})^2 + \frac{1}{\eta^2} \hat{\alpha}^2 + \frac{2\sqrt{6\mathcal{I}}}{\eta^2} \hat{\pi} \hat{\alpha}' - \frac{4\sqrt{6\mathcal{I}}}{\eta^2} \hat{\pi} \hat{\alpha}. \tag{52}
\]
The steepness of the gauge-kinetic function compensates the suppressions coming from the small background density of the inflaton and the gauge fields as well as the flatness of the inflaton potential, hence results in the appearance of the parameter \(\mathcal{I}\) defined in \([21]\), which is potentially of order unity.

**B. Calculation of the Power Spectrum**

In order to calculate the two-point function, we treat the terms involving \(\mathcal{I}\) as small perturbations. Then, \(\hat{\pi}\) and \(\hat{\alpha}\) are free particles in the de-Sitter background at the leading order. We impose the Bunch-Davies vacuum condition and write them as
\[
\hat{\pi}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left( u_k(\eta) a_k e^{i\mathbf{k} \cdot \mathbf{x}} + u_k^*(\eta) a_k^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}} \right),
\]
\[
\hat{\alpha}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left( \tilde{u}_k(\eta) b_k e^{i\mathbf{k} \cdot \mathbf{x}} + \tilde{u}_k^*(\eta) b_k^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}} \right). \tag{54}
\]
In the above expressions and all the following, the field symbols denote the corresponding quantities in the interaction picture. The vacuum mode functions are given by
\[
u_k(\eta) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} \right) e^{-ik\eta},
\]
\[
\tilde{u}_k(\eta) = \frac{1}{\sqrt{2c^3_k}} \left( 1 - \frac{i}{c^3_k\eta} \right) e^{-ic^3_k\eta}
\]
with \(c^3_s = 2/3\) being the propagation speed of \(\hat{\alpha}\), and the creation and annihilation operators satisfy
\[
[a_p, a_q^\dagger] = [b_p, b_q^\dagger] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad [a_p, b_q] = 0, \quad \text{etc.} \tag{57}
\]
The interaction Hamiltonian is given by
\[
H_I(\eta) = \int d^3x \left( -\frac{6\mathcal{I}^2}{\eta^2} \hat{\pi}^2 + \frac{2\sqrt{6\mathcal{I}}}{\eta^2} \hat{\pi}' \hat{\alpha}' + \frac{2\sqrt{6\mathcal{I}}}{\eta^2} \hat{\pi} \hat{\alpha} \right). \tag{58}
\]
Note that we performed integration by parts for simplifying the calculations below. In interpreting the expression, normal ordering is understood in order to eliminate vacuum bubbles. In terms of the de-Sitter vacuum state defined by
\[ a_k |0\rangle = b_k |0\rangle = 0, \]  
the in-state is written as
\[ |\text{in}\rangle = T \exp \left( -i \int_{-\infty}^{\eta} d\tilde{\eta} H_I(\tilde{\eta}) \right) |0\rangle \]
where the symbol $T \exp$ denotes the time-ordered exponential. While the lower limit of the time integral should physically be the beginning of inflation, it does not affect the following calculations as long as it is far enough in the past. The quadratic variation of the curvature perturbation during inflation is defined by
\[ (2\pi)^3 \delta(k + p) \langle \text{in} | \zeta_k^2(\eta) |\text{in}\rangle = \int d^3 x \ e^{i k \cdot x} \langle \text{in} | \zeta(\eta, x) \zeta(\eta, 0) |\text{in}\rangle = \int \frac{d^3 p}{(2\pi)^3} \langle \text{in} | (\zeta_k + \zeta_k^\dagger)(\zeta_p + \zeta_p^\dagger) |\text{in}\rangle \]
where we used
\[ \zeta(\eta, x) = \int \frac{d^3 k}{(2\pi)^3} (\zeta_k(\eta) e^{-ik \cdot x} + \zeta_k^\dagger(\eta) e^{-k \cdot x}). \]
Comparing this with (51), we need to compute
\[ \langle \text{in} | \left( g_k(\eta) a_k + g_k^*(\eta) a_k^\dagger \right) \left( g_p(\eta) a_p + g_p^*(\eta) a_p^\dagger \right) |\text{in}\rangle, \]
and
\[ \langle \text{in} | \left( j_k(\eta) b_k + j_k^*(\eta) b_k^\dagger \right) \left( j_p(\eta) b_p + j_p^*(\eta) b_p^\dagger \right) |\text{in}\rangle, \]
where $g_k(\eta)$ and $j_k(\eta)$ are linear combinations of $u_k(\eta), \tilde{u}_k(\eta)'$ and their derivatives given by
\[ g_k(\eta) = u_k'(\eta) + \frac{4 + 6 I^2}{\eta} u_k(\eta) \]
\[ j_k(\eta) = \tilde{u}_k'(\eta) - \frac{2}{\eta} \tilde{u}_k(\eta). \]
Let us start from (63). Expanding the time ordered exponential up to second order in $I$, we need to keep the following contributions:
\[ |g_k(\eta)|^2 (2\pi)^3 \delta(k + p) \]
We obtain

\[ 24 \mathcal{I}^2 \int d\eta_1 \frac{1}{\eta_1^2} \Im \left( u_k(\eta_1)^2 g^*_k(\eta_1)^2 \right) = 4 \mathcal{I}^2 k \mathcal{M} \]  

(81)

where the amplitude \( \mathcal{M} \) is given by

\[ \mathcal{M} = - (P \cos 2x + 2Q \sin 2x) \text{Ci}(-2x) + (P \sin 2x - 2Q \cos 2x) \text{Si}(-2x) - \frac{1}{x^2} P + \frac{1}{x^3} + \frac{1}{x} Q. \]  

(82)

The first term in (58), which can be written in momentum space as

\[- \frac{6 \mathcal{I}^2}{\eta^2} \int d^3 \mathbf{x} \pi^2 = - \frac{6 \mathcal{I}^2}{\eta^2} \int d^3 k (2\pi)^3 \left( u_k^2 a_k a_{-k} + 2 |u_k|^2 a_k^\dagger a_{-k} + u_k^* a_k^\dagger a_{-k}^\dagger \right), \]  

(74)

contributes to the second and third lines only at the order \( \mathcal{I}^2 \). This modified mass effect reduces to

\[ (69) + (70) = 24 \mathcal{I}^2 (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \int d\eta_1 \frac{1}{\eta_1^2} \Im \left( u_k(\eta_1)^2 g^*_k(\eta_1)^2 \right), \]  

(75)

where \( \Im \) means taking the imaginary part. The integration can be readily carried out by using the cosine- and sine-integrals that are defined by

\[ \text{Ci}(x) = \int_{-\infty}^{x} dt \frac{\cos t}{t} = \gamma + \ln x + O(x^2), \]  

(76)

\[ \text{Si}(x) = - \int_{-\infty}^{x} dt \frac{\sin t}{t} = \frac{\pi}{2} + O(x). \]  

(77)

It is convenient to introduce \( x = k\eta \) and the rational functions

\[ P(x) = \left( 1 + \frac{3(1 + 2\mathcal{I}^2)}{x^2} \right)^2 - \frac{9(1 + 2\mathcal{I}^2)^2}{x^2}, \]  

(78)

\[ Q(x) = \frac{3(1 + 2\mathcal{I}^2)}{x} \left( 1 + \frac{3(1 + 2\mathcal{I}^2)}{x^2} \right), \]  

(79)

by which \( g^*_k(\eta)^2 \) is expressed as

\[ g^*_k(\eta)^2 = \frac{k}{2} (-P(x) + 2iQ(x)) e^{2ix}. \]  

(80)

We obtain

\[ 24 \mathcal{I}^2 \int d\eta_1 \frac{1}{\eta_1^2} \Im \left( u_k(\eta_1)^2 g^*_k(\eta_1)^2 \right) = 4 \mathcal{I}^2 k \mathcal{M} \]  

(81)
The cross terms in the Hamiltonian (58) in Fourier space become

\[
\frac{2\sqrt{6} I}{\eta} \int d^3x \left( \hat{\pi} \hat{\alpha} + \frac{1}{\eta} \hat{\pi} \hat{\alpha} \right) = \frac{2\sqrt{6} I}{\eta} \int \frac{d^3k}{(2\pi)^3} \left( v_k \bar{u}_k a_k b_{-k} + v_k \bar{u}_k b_k^\dagger a_k + v_k^* \bar{u}_k a_k^\dagger b_k + v_k^* \bar{u}_k^\dagger a_k^\dagger b_{-k} \right),
\]

where

\[
v_k(\eta) = u_k'(\eta) + \frac{1}{\eta} u_k(\eta).
\]

They do not affect (69) and (70) at tree-level but are responsible for the remaining terms. After eliminating the contributions corresponding to disconnected diagrams, they yield

\[
(71) + (72) + (73) = -48 I^2 (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \times \int \frac{d\eta_1}{\eta_1} \int \frac{d\eta_2}{\eta_2} \frac{1}{\eta_1 \eta_2} (v_k(\eta_2) u_k(\eta_2) v_k(\eta_1) u_k^*(\eta_1) g_k^*(\eta)^2 + (c.c.))
\]

\[
+ 48 I^2 (2\pi)^3 \delta(\mathbf{k} + \mathbf{p}) \int \frac{d\eta_1}{\eta_1} \int \frac{d\eta_2}{\eta_2} v_k(\eta_1) u_k^*(\eta_1) v_k(\eta_2) u_k^*(\eta_2) |g_k(\eta)|^2.
\]

The integrals are not as bad as they look once the relations

\[
v_k(\eta_1) \bar{u}_k^*(\eta_1) = \frac{-i}{2 \sqrt{c_s}} \left( 1 + \frac{i}{c_s k \eta_1} \right) e^{-i(1-c_s)k \eta_1},
\]

and

\[
v_k(\eta_2) \bar{u}_k(\eta_2) = \frac{-i}{2 \sqrt{c_s}} \left( 1 - \frac{i}{c_s k \eta_2} \right) e^{-i(1+c_s)k \eta_2}
\]

are taken into account. The result is

\[
(71) + (72) + (73) = 6 I^2 k A (2\pi)^3 \delta(\mathbf{k} + \mathbf{p})
\]

with the amplitude

\[
A = \frac{4(3 + I)^2}{c_s^3 x^3} \left[ \frac{1}{2x} - \text{Ci}(-1 + c_s x) \sin(1 + c_s x) + \text{Si}(-1 + c_s x) \cos(1 + c_s x) \right]
\]

\[
+ \frac{8}{c_s^2} Q(x) \left[ \frac{1}{2x} - \text{Ci}(-2x) \sin 2x + \text{Si}(-2x) \cos 2x \right] - \frac{4}{c_s^2} P(x) [\text{Ci}(-2x) \cos 2x + \text{Si}(-2x) \sin 2x]
\]

\[
- \frac{4}{c_s^2} P(x) [\text{Ci}(-1 + c_s x) \cos(1 + c_s x) + \text{Si}(-1 + c_s x) \sin(1 + c_s x) x]
\]

\[
- \frac{2}{c_s^2} \int dy \int \frac{dy}{z} \left[ P(x) \sin(2x - y - z) - 2Q(x) \cos(2x - y - z) \right] \sin c_s(y - z)
\]

\[
+ \frac{1}{c_s^2} (P(x) \sin 2x - 2Q(x) \cos 2x) [\text{Ci}(-1 + c_s x) \text{Si}(-1 + c_s x) - \text{Si}(-1 + c_s x) \text{Ci}(-1 + c_s x)]
\]

\[
+ \frac{1}{c_s^2} (P(x) \cos 2x + 2Q(x) \sin 2x) [\text{Ci}(-1 + c_s x) \text{Ci}(-1 + c_s x) - \text{Si}(-1 + c_s x) \text{Si}(-1 + c_s x)]
\]

\[
+ \frac{1}{c_s^2} \left( P(x) + \frac{2(3 + I^2)^2}{x^2} \right) (\text{Ci}(-1 + c_s x)^2 + \text{Si}(-1 + c_s x)^2).
\]
The $\alpha$-$\alpha$ correlation \text{(64)} is needed only at the leading order and given by

\begin{equation}
(64) = |j_k(\eta)|^2(2\pi)^3\delta(k + p).
\end{equation}

Finally, the cross correlation \text{(65)} is calculated as

\begin{equation}
(65) = i\langle 0| \int_0^{\eta} d\eta_1 H_I(\eta_1) \left(g_k a_{\mathbf{k}} + g_k^* a_{-\mathbf{k}}^*\right) j_{p}^* b_{-\mathbf{p}}|0\rangle + (\text{h.c.})
\end{equation}

\begin{equation}
\quad + i\langle 0| \int_0^{\eta} d\eta_1 H_I(\eta_1) \left(j_k b_{\mathbf{k}} + j_k^* b_{-\mathbf{k}}^*\right) g_{p}^* a_{-\mathbf{p}}|0\rangle + (\text{h.c.})
\end{equation}

\begin{equation}
= 4\sqrt{6}i \int_0^{\eta} \frac{d\eta_1}{\eta_1} \left[v_k^*(\eta_1) \tilde{u}_k^*(\eta_1) g_k(\eta) j_k(\eta) - (\text{c.c.})\right] \equiv 2\sqrt{6}i kB
\end{equation}

with $B$ given by

\begin{equation}
B = \frac{3 + T^2}{c_s x^2} \left(1 - \frac{3}{c_s^2 x^2}\right) - \frac{3}{c_s^2 x} \left(1 + \frac{3 + T^2}{x^2}\right)
\end{equation}

\begin{equation}
- \frac{1}{c_s} \left\{3\left(1 + \frac{T^2}{c_s^2 x^2}\right) + \left(1 + \frac{3 + T^2}{x^2}\right) \left(1 - \frac{3}{c_s^2 x^2}\right)\right\}
\end{equation}

\begin{equation}
\times \left(\text{Ci}(-1 + c_s x) \cos(1 + c_s x) - \text{Si}(-1 + c_s x) \sin(1 + c_s x)\right)
\end{equation}

\begin{equation}
+ \frac{1}{c_s} \left\{3 + \frac{T^2}{x} \left(1 - \frac{3}{c_s^2 x^2}\right) - \frac{3}{c_s x} \left(1 + \frac{3 + T^2}{x^2}\right)\right\}
\end{equation}

\begin{equation}
\times \left(\text{Si}(-1 + c_s x) \cos(1 + c_s x) + \text{Ci}(-1 + c_s x) \sin(1 + c_s x)\right).
\end{equation}

Combining all the pieces, the curvature power spectrum is given by

\begin{equation}
\langle \ln|\zeta_k(\eta)|^2|\ln\rangle = \frac{H^2 \epsilon_{\varphi}}{18 \epsilon_{H}^2} \eta^4 \left(|g_k(\eta)|^2 + 4 T^2 k M + \frac{3 T^2}{2} |j_k(\eta)|^2 + 6 T^2 kA + 12 T^2 kB\right),
\end{equation}

where we used $a = -1/H \eta$. In the standard calculation for single-field models, one evaluates this quantity at the horizon crossing since $\zeta$ is conserved, even at non-linear order. Here in contrast, $\zeta$ still evolves beyond the horizon scale and therefore we are going to evaluate its value at the end of inflation $\eta = \eta_f$. The scales of cosmological interest should be well outside the horizon at the end of inflation, which means $-k \eta_f \ll 1$. In the corresponding limit $x \to 0$, the power spectrum reduces to

\begin{equation}
\langle \ln|\zeta_k(\eta_f)|^2|\ln\rangle \rightarrow \frac{\epsilon_{\varphi} H^2}{\epsilon_{H}^2} \frac{H^2}{4 k^3} \left(1 + 18 \sqrt{6} \frac{T^2}{k} (\ln|k \eta_f|)^2\right).
\end{equation}

The disadvantage in this strategy is to introduce extra errors by neglecting time-variation of $H$ and the slow-roll parameters over a number of Hubble times. We assume that these corrections are subdominant since they are suppressed by $\epsilon_H$ and the other small parameters. In fact, the leading correction coming from varying $H$ is proportional to $\epsilon_H \ln|k \eta|$, which can
be safely discarded compared to $I^2(\ln |k\eta|)^2$. It is difficult to estimate the effect of varying $I$ as it is already a correction from the non-adiabatic evolution of $\zeta$. It is expected to be proportional to

$$\frac{\mathcal{I}}{H \mathcal{I}} = \frac{\epsilon_H \eta_H - \eta_G}{2 \epsilon_H - \epsilon_G}$$

multiplied by several powers of $\ln |k\eta|$. Thus the approximation might break down when the e-folding number is too large.

V. PRIMORDIAL GRAVITATIONAL WAVES AND PHENOMENOLOGICAL IMPLICATIONS

A. Tensor Power Spectrum

Let us now turn to the tensor perturbations

$$\gamma_{ij} = \frac{1}{2} h_{ij}, \quad X^i_j = \omega_{ij}$$

where both $h_{ij}$ and $\omega_{ij}$ are trace- and divergence-free. Our quadratic action is

$$\mathcal{L}_T^{(2)} = \frac{a^2}{8} (h'_{ij} h'_{ij} - h_{ij,k} h_{ij,k}) + \frac{c^2}{4f^2} h_{ij} h_{ij} + \frac{f^2}{2} (\omega'_{ij} \omega'_{ij} - \omega_{ij,k} \omega_{ij,k}) - c h_{ij} \omega'_{ij}. \quad (97)$$

Normalizing the variables by

$$\hat{h}_{ij} = \frac{1}{2} a h_{ij}, \quad \hat{\omega}_{ij} = f \omega_{ij}, \quad (98)$$

it becomes

$$\mathcal{L}_T^{(2)} = \frac{1}{2} \left[ \hat{h}_{ij} \hat{h}_{ij} - \hat{h}_{ij,k} \hat{h}_{ij,k} + \left( \frac{a''}{a} + \frac{2c^2}{f^2 a^2} \right) \hat{h}_{ij} \hat{h}_{ij} \right]$$

$$+ \frac{1}{2} \left( \hat{\omega}_{ij} \hat{\omega}_{ij} - \hat{\omega}_{ij,k} \hat{\omega}_{ij,k} + \frac{f''}{f} \hat{\omega}_{ij} \hat{\omega}_{ij} \right) - \frac{2c}{af} \left( \hat{\omega}_{ij} - \frac{f'}{f} \hat{\omega}_{ij} \right) \hat{h}_{ij}. \quad (100)$$

They are decomposed according to their polarization as

$$\hat{h}_{ij}(\eta, \mathbf{x}) = \sum_{s=1,2} \epsilon_{ij}^s \hat{h}^s(\eta, \mathbf{x}), \quad \epsilon_{ij}^s \epsilon_{ij}^{s'} = \delta_{ss'}, \quad (101)$$

and a similar decomposition applies to $\hat{\omega}_{ij}$ as well. Applying Fourier transform as before,

$$\hat{h}^s(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{h}^s_k(\eta) e^{i \mathbf{k} \cdot \mathbf{x}} + \hat{h}^s_k(\eta) e^{-i \mathbf{k} \cdot \mathbf{x}} \right), \quad (102)$$
what one wants to compute here is the amplitude of the gravitational waves
\[
(2\pi)^3 \delta(k+p) \langle \text{in}|h_k(\eta)^2|\text{in}\rangle = \frac{4}{a^2} \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3} \langle \text{in}|(\hat{h}^s_k(\eta) + \hat{h}^{s\dagger}_k(\eta))(\hat{h}^s_p(\eta) + \hat{h}^{s\dagger}_p(\eta))|\text{in}\rangle.
\]
(103)

For this purpose, we only have to keep the leading order terms in \(\hat{\omega}_{ij}\) and therefore
\[
\frac{f'}{f} \sim -2\mathcal{H}, \quad \frac{f''}{f} \sim 2\mathcal{H}^2.
\]
(104)

After discarding higher order corrections, the problem becomes a quantum field theory for the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \sum_{s=1,2} \left[ (\dot{h}^s)^2 - (\nabla \dot{h}^s)^2 + \frac{2 - \epsilon_{\varphi} \hat{h}^s}{\eta^2} + ((\dot{\omega}^s)^2 - (\nabla \dot{\omega}^s)^2 + \frac{2}{\eta^2} (\dot{\omega}^s)^2 \right] + \frac{2}{\eta} \sqrt{\epsilon_H - \epsilon_{\varphi}} \sum_{s=1,2} \left( (\hat{\omega}^s)' - \frac{2}{\eta} \hat{\omega}^s \right) \hat{h}^s.
\]
(105)

One notices that the two polarization decouple from each other and the Lagrangian is of the same form as the scalar modes with different coupling constants and twice many fields.

The interaction Hamiltonian for the each polarization mode is given by
\[
H^s_I = \frac{\epsilon_{\varphi}}{2\eta^2} \int d^3x (\dot{h}^s)^2 + \frac{2 \sqrt{\epsilon_H - \epsilon_{\varphi}}}{\eta} \int d^3x \left( (\dot{h}^s)'^2 + \frac{1}{\eta} \dot{h}^s \dot{\omega}^s \right).
\]
(106)

Hence, if we focus on the corrections up to first order in \(\epsilon_H\) and \(\epsilon_{\varphi}\), we only have to repeat the calculations in the previous section with replacements
\[
\frac{\epsilon_{\varphi}}{2\eta^2} \int \frac{d^3k}{(2\pi)^3} \left( u_k^2 a_k^s a_k^{s\dagger} + 2|u_k|^2 a_k^s a_k^{s\dagger} + v_k^2 a_k^{s\dagger} a_k^s + v_k^2 a_k^{s\dagger} a_k^{s\dagger} \right) \quad \text{for} \quad (104)
\]
and
\[
\frac{2 \sqrt{\epsilon_H - \epsilon_{\varphi}}}{\eta} \int \frac{d^3k}{(2\pi)^3} \left( v_k u_k a_k^s b_k^{s\dagger} + v_k u_k^* b_k^{s\dagger} a_k^s + v_k^* u_k a_k^{s\dagger} b_k^s + v_k^* u_k^* b_k^{s\dagger} a_k^{s\dagger} \right) \quad \text{for} \quad (83)
\]
and multiply the result by a factor of two to add up the polarizations. We also substitute \(u_k(\eta)\) for \(g_k(\eta)\), which simplifies the calculations. In the end, one obtains
\[
\langle \text{in}|h_k(\eta)^2|\text{in}\rangle = \frac{4H^2 \eta^2}{k} \left( 1 + \frac{1}{x^2} - \frac{2}{3} \epsilon_{\varphi} \hat{\mathcal{M}} + 2(\epsilon_H - \epsilon_{\varphi}) \hat{\mathcal{A}} \right)
\]
(109)

with the amplitudes given by
\[
\hat{\mathcal{M}} = \frac{2}{x^2} - \frac{1 - x^2}{x^2} \left( \text{Ci}(-2x) \cos 2x - \text{Si}(-2x) \sin 2x \right).
\]
(110)
\[ -\frac{2}{x} (\text{Ci}(-2x) \sin 2x + \text{Si}(-2x) \cos 2x), \]

\[ \tilde{A} = \frac{2}{x^2} - \frac{2(1 - x^2)}{x^3} \sin 2x + \frac{4}{x} \cos 2x + \frac{1 + x^2}{x^2} (\text{Ci}(-2x)^2 + \text{Si}(-2x)^2) \]

\[ + \left( \frac{1 - x^2}{x^2} \ln |x| - \frac{8}{x^2} + 4 \right) (\text{Ci}(-2x) \cos 2x + \text{Si}(-2x) \sin 2x) \]

\[ + \frac{2}{x} (\ln |x| - 6) (\text{Ci}(-2x) \sin 2x - \text{Si}(-2x) \cos 2x) \]

\[ + 2 \int \frac{dy}{y} \int \frac{dz}{z} \left[ \frac{2}{x} \cos(2x - y - z) - \frac{1 - x^2}{x^2} \sin(2x - y - z) \right] \sin(y - z). \]

In the limit \( x \to 0 \), it becomes

\[ \langle |h_k(\eta_f)^2| \rangle \to \frac{4H^2}{k^3} \left[ 1 + 4(\epsilon_H - \epsilon_f) (\ln |k\eta_f|)^2 \right]. \]

**B. Vector Mode**

Before discussing the observational implications, we shall briefly look at the vector perturbation

\[ \beta_i = -S_i, \quad \gamma_{ij} = F_{(ij)}, \]

\[ \sigma^i = \nu_i, \quad \chi_{ij} = \kappa_{(ij)} + \epsilon_{ijk}\lambda_k, \]

for completeness. As before, we take the flat slicing \( F_i = 0 \). The Lagrangian is given by

\[ \mathcal{L}^{(2)} = +\frac{f^2}{4} \kappa_{i,j} \kappa'_{i,j} - \frac{f^2}{8} \kappa_{i,jk} \kappa_{i,jk} + f^2 \lambda'_{k} \lambda'_k - \frac{f^2}{2} \lambda_{i,j} \lambda_{i,j} \]

\[ - \frac{f^2}{2} \epsilon_{ijk} \lambda_{k,l} \kappa_{i,jl} + \frac{a^2}{4} S_{i,j} S_{i,j} - a^2 \left( \frac{f^2}{a} A' \kappa_{i,jj} + \frac{2f^2}{a} A' \epsilon_{ijk} \lambda_{j,k} \right) \]

\[ + \frac{f^2}{2} \nu_{i,j} \nu_{i,j} + \nu_{i,j} \left( \frac{f^2}{a} \kappa'_{(i,j)} + f^2 \epsilon_{ijk} \lambda'_k \right). \]

Varying \( S_i \) yields

\[ \nabla^2 S_i = -\frac{2f^2}{a} A' (\nabla^2 \kappa_i + 2 \text{curl} \lambda_i). \]

Similarly, \( \nu_i \) is non-dynamical and solved as

\[ \nabla^2 \nu_i = -\frac{1}{2} \nabla^2 \kappa'_i + \text{curl} \lambda'_i. \]

Perhaps there are two promising choices of the gauge for the vector fields; \( \lambda_i = 0 \) and \( \kappa_i = 0 \). The action becomes

\[ \mathcal{L} = \frac{f^2}{4} (\nabla \kappa'_i) \cdot (\nabla \kappa'_i) - \frac{f^2}{8} (\nabla^2 \kappa_i) (\nabla^2 \kappa_i) \]
for the former and
\[ \mathcal{L} = f^2 \chi_k \chi_k' - \frac{f^2}{2} (\nabla \chi_i) \cdot (\nabla \chi_i) \]  \hspace{1cm} (119)
for the latter. In either way, we have a free massless vector field with the propagation speed 
\[ c_s = 1/\sqrt{2}. \]

C. Phenomenological Consequences

If we assume instantaneous reheating whereby all the energy of the inflaton and gauge fields is dumped into a single relativistic fluid, the scalar curvature \( \zeta \) and the gravitational wave \( h_{ij} \) are conserved until re-entering inside the Hubble horizon and are observable through the CMB. The vector perturbation generated by the quantum fluctuation will quickly decay away and not be observed. For the scalar mode, the relevant parameter is the spectral tilt \( n_S - 1 \). From (94), this model predicts
\[ n_S - 1 = \frac{d}{d \ln k} \ln \left( k^3 \langle \epsilon_k^2 \rangle \right) = \frac{36 \sqrt{6} I^2 \ln |k \eta_f|}{1 + 18 \sqrt{6} I^2 (\ln |k \eta_f|)^2}, \]  \hspace{1cm} (120)
which is negative for \( |k \eta_f| \ll 1 \), whence the spectrum is red. There is no contribution from the time-variation of \( H, \epsilon_{H, \phi} \) and \( I \) since the spectrum was evaluated at the end of inflation, not the time each mode crossed the horizon. It can be seen that
\[ |n_S - 1| \leq \min \left( -36 \sqrt{6} I^2 \ln |k \eta_f|, -\frac{2}{\ln |k \eta_f|} \right), \]  \hspace{1cm} (121)
which means the spectral tilt doesn’t necessarily impose a stringent constraint on the value of \( I \). We note that \( \ln |k \eta_f| \) is the e-folding number counted from horizon exit of the mode with wavenumber \( k \) until the end of inflation, whose value is model-dependent. If, for instance, we take \( \ln |k \eta_f| \sim -50 \), it yields
\[ |n_S - 1| \sim \frac{1}{25} \]  \hspace{1cm} (122)
which is nicely consistent with WMAP 7-year [36], even with \( I \sim 1 \). The running of \( n_S \) can also be computed as
\[ \frac{d}{d \ln k} (n_S - 1) = \frac{n_S - 1}{\ln |k \eta_f|} - (n_S - 1)^2, \]  \hspace{1cm} (123)
which is safely small as long as \( n_S - 1 \) is small. Similarly, the tilt of the tensor spectrum is given as
\[ n_T = \frac{d}{d \ln k} \ln \left( k^3 \langle h^2_k \rangle \right) = \frac{8 (\epsilon_H - \epsilon_\phi) \ln |k \eta_f|}{1 + 4 (\epsilon_H - \epsilon_\phi) (\ln |k \eta_f|)^2}. \]  \hspace{1cm} (124)
Thus, the spectrum of gravitational wave is also red. However, it is much closer to scale invariance than that of scalar if $\mathcal{I} \ll 1$.

As can be seen from (112), the amplitude of the tensor mode itself no longer provides the unambiguous information about the energy scale of inflation as it receives a potentially significant correction. The tensor-to-scalar ratio $r$ can be used to determine $\epsilon_H$ and $\epsilon_\varphi$. Recalling $\mathcal{I}^2 = (\epsilon_H - \epsilon_\varphi)/\epsilon_\varphi$, it is given by

$$r = \frac{\langle h_k^2 \rangle}{\langle \zeta_k^2 \rangle} = 16\epsilon_H^2 \frac{1 + 4(\epsilon_H - \epsilon_\varphi) (\ln |k\eta_f|)^2}{\epsilon_\varphi + 18\sqrt{6}(\epsilon_H - \epsilon_\varphi) (\ln |k\eta_f|)^2}.$$  \hspace{1cm} (125)

Hence, the tensor to scalar ratio is suppressed compared to the single-field slow roll inflation. The suppression is stronger when the e-folding number is greater and the scalar kinetic energy is subdominant.

In summary, we have seen that there are several different regimes that are consistent with the observations made so far, as far as the above three quantities are concerned. They are classified in the following.

$\mathcal{I}^2 \ll 1$: Recalling the formula (23), this occurs when the energy density of the gauge fields is much smaller than the scalar kinetic energy density. If the e-folding number experienced by the modes at CMB scales is of order hundred or so, the model predicts a slightly red scalar power spectrum and almost scale invariant gravitational waves. Since it means $\epsilon_H \sim \epsilon_\varphi$, the tensor-to-scalar ratio is unchanged from the ordinary single-field slow-roll inflation.

$\mathcal{I} \sim 1$: Although it means the dominant contribution to the scalar power spectrum comes from the terms proportional to $\mathcal{I}^2$, whose origin is the interaction between the inflaton and the gauge fields, the model still predicts an observationally consistent spectral tilt. On the other hand, even though the background energy density of the gauge fields is comparable to the scalar kinetic energy, the spectrum of gravitational waves is not very much affected for $-\ln |k\eta_f| \sim 50$. The tensor-to-scalar ratio is suppressed. Of course, it should be noted that the perturbative approach cannot be trusted and non-linear effects may significantly modify the results.

$-\ln |k\eta_f| \gg 50$: Surprisingly, this regime is viable regardless of the value of $\mathcal{I}$. The spectral tilt of scalar curvature is $\sim -1/\ln |k\eta_f|$ and the tensor mode is suppressed by a factor
VI. CONCLUSION

In the present article, we investigated the linear perturbation of the inflationary model with a triad of background gauge fields coupled to the inflaton. We characterized the accelerated expansion by introducing four parameters $\epsilon_H, \epsilon_\phi, \eta_H$ and $\eta_\phi$ which are generalizations of the usual slow-roll parameters. The general second order Lagrangian was derived and irreducible mode decomposition was carried out according to the transformation property under the spatial rotations, and the internal "rotation" of the triad of gauge fields. The scalar- and tensor-mode power spectra were computed by employing the in-in formalism on the de-Sitter background. We found the scalar fluctuation is characterized by the parameter $I$, which is potentially of order unity, while its tensor counterparts are smaller by an order of $\epsilon_\phi$. The enhancement in the scalar correction is due to the steep gauge-kinetic function. The observational implications were studied by looking at the spectral tilt and tensor-to-scalar ratio. The generic prediction is that the spectra are red with the stronger effect for the scalar mode and the tensor-to-scalar ratio tends to be suppressed. The magnitudes of tilt and the suppression depend on $I$ and the e-folding number. The structure of the corrections is such that even accurate measurements of those quantities are unable to impose strong constraints on $I$ or the other small parameters because of the involvement of the e-folding number. As it stands, both weak and strong background gauge fields in the unit of inflaton kinetic energy are consistent with the results from WMAP.

There are several implications for the preheating of auxiliary particles by the inflaton. The coupling needed to produce those particles back-reacts onto the scalar perturbation and can modify the power spectrum significantly. While gravitons can also be produced even in this isotropic background, which could affect the determination of the energy scale of inflation by measuring the amplitude of tensor mode, the effect is much smaller than that on the scalar mode, at least in the present model.

Since it appears that the model passes the observational tests at linear order, it will be worth looking at the higher order corrections, namely non-Gaussianity. A recent work [29]
suggests that the gauge-kinetic coupling can lead to a strong signal, although the analysis was done for a vanishing background gauge field. In the above calculations, we have ignored the entropy modes and vector mode, which should affect the result at non-linear order. It is certainly interesting to carry out a thorough analysis taking into account all the modes involved, which should be possible for this isotropic model, and will be presented in the near future.

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[1] A. Linde, Phys. Rev. D 49, 215748 (1994)
[2] L. H. Ford, Phys. Rev. D 40, 415967 (1989)
[3] T. S. Koivisto and D. F. Mota, J. Cosmol. Astropart. Phys. 0808, 021 (2008)
[4] A. Golovnev, V. Mukhanov and V. Vanchurin, J. Cosmol. Astropart. Phys. 06, 009 (2008)
[5] B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. Lett. 102, 111301 (2009)
[6] T. S. Koivisto, D. F. Mota and C. Pitrou, J. High Energy Phys. 0909, 092 (2009)
[7] B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. D 80, 123530 (2009)
[8] A. Golovnev, Phys. Rev. D 81, 023514 (2010)
[9] G. Esposito-Farese, C. Pitrou and J. -P. Uzan, Phys. Rev. D 81, 063519 (2010)
[10] M. -a. Watanabe, S. Kanno and J. Soda, Phys. Rev. Lett. 102, 191302 (2009)
[11] M. a. Watanabe, S. Kanno and J. Soda, Prog. Theor. Phys. 123, 1041 (2010)
[12] S. Kanno, J. Soda and M. a. Watanabe, J. Cosmol. Astropart. Phys. 0912, 009 (2009)
[13] S. Kanno, J. Soda and M. a. Watanabe, J. Cosmol. Astropart. Phys. 1012, 024 (2010)
[14] P. V. Moniz and J. Ward, Class. Quant. Grav. 27, 235009 (2010)
[15] K. Murata and J. Soda, J. Cosmol. Astropart. Phys. 1106, 037 (2011)
[16] T. Q. Do, W. F. Kao and I. C. Lin, Phys. Rev. D 83, 123002 (2011)
[17] R. Emami, H. Firouzjahi, S. M. Sadegh Movahed and M. Zarei, J. Cosmol. Astropart. Phys. 1102 005 (2011)
[18] J. M. Wagstaff and K. Dimopoulos, Phys. Rev. D 83, 023523 (2011)
[19] S. Hervik, D. F. Mota and M. Thorsurd, J. High Energy Phys. 1111, 146 (2011)
[20] T. Q. Do and W. F. Kao, Phys. Rev. D 84, 123009 (2011)
[21] K. Yamamoto, M. a. Watanabe and J. Soda, arXiv:1201.5309 [hep-th]
[22] A. Maleknejad and M. M. Sheikh-Jabbari, Phys. Rev. D 84, 043515 (2011)
[23] P. Adshead and M. Wyman, arXiv:1202.2366 [hep-th]
[24] P. Adshead and M. Wyman, arXiv:1203.2264 [hep-th]
[25] M. M. Sheikh-Jabbari, arXiv:1203.2265 [hep-th]
[26] B. Ratra, Astrophys. J. 391, L1 (1992)
[27] V. Demozzi, V. Mukhanov and H. Rubinstein, J. Cosmol. Astropart. Phys. 0908, 025 (2009)
[28] J. L. Cook and L. Sorbo, Phys. Rev. D 85, 023534 (2012)
[29] N. Barnaby, R. Namba and M. Peloso, arXiv:1202.1469 [astro-ph.CO]
[30] T. R. Dulaney and M. I. Gresham, Phys. Rev. D 81, 103532 (2010)
[31] A. E. Gumrukcuoglu, B. Himmetoglu and M. Peloso, Phys. Rev. D 81, 063528 (2010)
[32] R. Arnowitt, S. Deser and C. W. Misner, Gravitation: an introduction to current research (Wiley 1962), chapter 7, pp 227 [arXiv:0405109 [gr-qc]]
[33] H. Feldman, V. Mukhanov and R. Brandenberger, Phys. Rept. 215, 56203 (1992)
[34] K. Malik and D. Wands, Phys. Rept. 475, 1-51 (2009)
[35] H. Kodama and M. Sasaki, Prog. Theor. Phys. 78 1 (1984)
[36] E. Komatsu et. al, Astrophys. J. Suppl. 192, 17 (2011)