A Constrained-Dictionary version of LZ78 asymptotically achieves the Finite-State Compressibility for any Individual Sequence.

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Abstract

The unrestricted LZ78 universal data-compression algorithm (as well as the LZ77 and LZW versions) achieves asymptotically, as the block-length tends to infinity, the FS compressibility, namely the best compression-ratio that may be achieved by any Information-lossless(IL) block-to-variable finite-state(FS) algorithm, for any infinitely-long individual sequence.

The encoder parses the sequence into distinct phrases where each newly generated phrase is a past phrase which is already stored in a dictionary, extended by one letter. The newly generated phrase is then added to the updated, ever-growing dictionary.

In practice, heuristics that adaptively constrain the the number of entries in the dictionary by a prefixed level, were proposed.

One such heuristic approach is the “Least Recently Utilized” (LRU) deletion approach, where only the most recent D entries are kept in the dictionary, thus yielding a constrained-dictionary version of LZ78, denoted by LZ78(LRU).

In this note, for the sake of completeness, it is demonstrated again via a simple proof that the unrestricted LZ78 algorithm asymptotically achieves the FS-Compressibility. Then, it is demonstrated that the LZ78(LRU) information-lossless data-compression algorithm also achieves the FS compressibility, as the dictionary size D tends to infinity. Although this is perhaps not surprising, it does nevertheless yield theoretical optimality argument for the popular LZ78(LRU) algorithm (and similarly, for the LZW(LRU) algorithm).
A. Introduction and Summary of Results:

Consider sequences \( x_k^1 = x_1, x_2, ..., x_k; x_i \in A; i = 1, 2, ..., k \) where \( |A| = A \).

Also, let \( x = x_1^\infty \).

The unconstrained LZ78 universal data compression algorithm has been introduced in [1], where it is shown that when applied to an individual sequence \( x^n_1 \), as \( n \) tends to infinity, it achieves the FS compressibility.

Let a finite-state encoder be denoted by the triple \((S, g, f)\) where \( S \) is a finite set of states, \( g: S \times A \rightarrow A \), and \( f: S \times A \rightarrow B^\ast \). For each starting state \( s_1 \), the triple defines a mapping from \( x \in A^\infty \) into \( y \in B^\infty \), where \( y_i = f(s_i, x_i) \) is a (possibly empty) binary word, \( s_{i+1} = g(s_i, x_i) \) is the next state and where \( i = 1, 2, \ldots \).

An information-lossless (IL) finite-state encoder is one for which for each \( n \), the sequence \( x^n_1 \) is determined by \( y^n_1 \), \( s_1 \) and \( s_{n+1} \).

The corresponding compression-ratio for \( x^n_1 \) is \( \frac{1}{n \log A} \sum l(y_i) \), where \( l(y_i) \) is the length in bits of the (possibly empty) binary word \( y_i \).

The minimum compression ratio for \( x^n_1 \) over all finite-state IL encoders with at most \( s \) states is denoted by \( FS_s(x^n_1) \).

Also, let \( FS_s(x) = \limsup_{n \rightarrow \infty} FS_s(x^n_1) \) and let the FS compressibility of \( x \) be defined by, \( FS(x) = \lim_{n \rightarrow \infty} FS_s(x) \).

Consider now the parsing of \( x^n_1 \) into some \( c \) (not necessarily distinct) phrases:

\[
x^n_1 = X_1, X_2, \ldots, X_j, \ldots, X_c; X_j = x^{i(j+1)-1}_{i(j)}; j = 1, 2, \ldots, c.
\]

Let \( Z_j; j = 1, 2, \ldots, k; k \leq c \) denote the \( k \) distinct substrings among the \( c \) phrases in \( x^n_1 \), where \( s_i \) denotes the start state and \( s_o \) denotes the end state of the phrase \( Z_j \).

Also, let \( L(Z_j | s_i, s_o) \) denote the length of the binary code-word that is generated by the IL FS encoder above, when fed with \( Z_j \), given the start state \( s_i \) and the end state \( s_o \).

Let \( p(Z_j | s_i, s_o) \) denote the empirical probability (i.e. fraction) of \( Z_j \) among all phrases that are characterized by a start state \( s_i \) and an end state \( s_o \).

Similarly, let \( p(Z_j) \) denote the empirical probability of \( Z_j \) among the \( c \) phrases in \( x^n_1 \) and let \( p(s_i, s_o) \) denote the empirical probability of the pair of states \((s_i, s_o)\) among the \( c \) phrases.
The corresponding compression-ratio for \( x^n \) is

\[
\frac{1}{n \log A} \sum_{1}^{c} L(Z_j | s_i, s_o) = \frac{c}{n \log A} \sum_{s_{i}=s(1)}^{s_{o}=s(1)} \sum_{1}^{k} p(s_i, s_o) \sum_{1}^{k} p(Z_j | s_{i}.s_{o}) L(Z_j | s_i, s_o)
\]

where \( s(t); t = 1, 2, \ldots, s \) are the distinct states that appear at the start or at the end of any of the \( c \) phrases (at most \( s \) such states)

**Lemma 1** Consider an arbitrary parsing of \( x^n \) into \( c \) substrings (phrases). Then,

\[
FS_s(x^n) \geq \frac{c}{n \log A} \sum_{1}^{k} p(Z_j) \log \left( \frac{1}{p(Z_j)} \right) - 2 \log s.
\]

**Proof:**

For a given states pair \( s_i, s_o \), an IL FS encoder outputs a distinct binary code-word for each of the \( c \) phrases that start with the state \( s_i \) and end with \( s_o \).

Thus, by the Huffman IL coding argument [6],

\[
\sum_{1}^{k} p(Z_j | s_i, s_o) L(Z_j | s_i, s_o) \geq \sum_{1}^{k} p(Z_j | s_i, s_o) \log \left( \frac{1}{p(Z_j | s_i, s_o)} \right)
\]

Lemma 1 follows immediately by observing that \(- \log p(Z_j | s_i, s_o) \geq - \log p(Z_j) - \log p(s_i, s_o)\).

Now, in the case of LZ78 [1], all the \( c \) phrases that are generated for \( x^n \), are all distinct (except perhaps of the last phrase). For example, in the case of the LZ78 algorithm, each new phrase is either an extension of a previous phrase by one letter, or a single letter that is not identical to any of the past single-letter phrases. The code length for each phrase is bounded by \( \log C_n(LZ78) + 1 + \log A \), where \( C_n(LZ78) \) is the number of distinct phrases that are generated by LZ78.

Therefore,

**Lemma 2** For any individual sequence \( x \)

\[
FS(x) \geq \limsup_{n \to \infty} \frac{1}{n \log A} \left[ (C_n(LZ)) \log(C_n(LZ78)) \right].
\]

The main result in [1] follows from Lemma 1 and Lemma 2 as follows:

The compression-ratio that is achieved for an individual sequence \( x^n \) that is parsed into \( C_n(LZ78) \) distinct phrases by LZ78 is upper-bounded by

\[
\frac{1}{n \log A} \left[ (C_n(LZ78)) (\log C_n(LZ78) + 1 + \log A) \right].
\]
Thus,

**Lemma 3** The LZ78 universal IL data-compression algorithm asymptotically achieves $F_S(x)$. Similarly, it follows that Lemma 3 holds for LZW [2] and LZ77 [3] as well.

In practice, in order to avoid the ever growing size of the dictionary that contains all the past phrases that are generated by LZ78 (or similarly, by LZW), heuristic constrained-dictionary versions has been proposed.

Apparently, the preferred heuristics is the Last-Recently-Used (LRU) method [4]. In this case, only the most recent phrases (no larger than some preset number $D$) are kept in the dictionary.

This approach is analyzed below, and is shown to asymptotically achieve $F_S(x)$ as well.

Consider a constrain-dictionary LZ78 algorithm, where the dictionary has $D$ entries, each no longer than $L_{\text{max}} = (\log D)^2$ letters. Each newly generated phrase is a copy of one of previous $D$ phrases, extended by the next incoming letter. If no match is found with any of the phrases in the dictionary, then the first incoming letter is the next phrase.

The new phrase is then included in the dictionary and the last recently used phrase is removed from the dictionary, except for the case where the newly generated phrase is of length $L_{\text{max}} + 1$, in which case the dictionary is not updated.

The code length for each successive phrase is $\log D + 1 + \log A$. Denote this algorithm by LZ78(LRU).

**Theorem 1** The compression-ratio that is achieved by LZ78(MLRU) when applied to an individual $x$ converges asymptotically to $F_S(x)$ as $D$ tends to infinity.

**Proof:**

Let $c(n)$ denote the number of phrases that are generated by LZ78(LRU) when applied to $x_1^n$, and let $c(n|L_{\text{max}} + 1)$ denote the number of phrases of length $L_{\text{max}} + 1$.

By construction, $p(Z_j) \geq \frac{1}{D}$ for any phrase $Z_j$ among the $c(n)$ phrases that are no longer than $L_{\text{max}}$ since the number of phrases in between any such phrase and it’s most recent previous appearance is at least $D$ (since it is not included in the dictionary).

Let $\rho_{LZ78(MLRU)}(x_1^n) = \frac{c(n)}{n \log A}(\log D + 1 + \log A)$ denote the compression-ratio that is achieved by LZ78(MLRU) when applied to $x_1^n$. 

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By Lemma 1,
\[
\rho_{LZ78(RLU)}(x^n_1) = \frac{C(n)}{n \log A} (\log D + 1 + \log A) \leq 
\]
\[
FS_s(x^n_1) + \frac{c(n)}{n \log A} (2 \log s + \log D + 1 + \log A + \frac{1}{n} c(n|L_{max} + 1) \log D.
\]

Therefore,
\[
\rho_{LZ78(RLU)}(x^n_1)(1 - \frac{2 \log s}{\log D}) \leq FS_s(x^n_1) + \frac{c(n)}{n \log A} (\log D + 1 + \log A + \frac{1}{n} c(n|L_{max} + 1) \log D.
\]

Observe that \( n \geq c(n|L_{max} + 1)(L_{max} + 1) \) and that \( L_{max} = (\log D)^2 \).

Thus,
\[
\limsup_{D \to \infty} \limsup_{n \to \infty} \rho_{LZ78(RLU)}(x^n_1)x + \frac{2 \log s}{\log D} = \limsup_{D \to \infty} \limsup_{n \to \infty} \rho_{LZ78(RLU)}(x^n_1) = FS(x)
\]
which proves Theorem 1.

The same result holds for LZW(LRU) as well as for a sliding version of LZ77 where the window is set at \( L_{max} \) and where the phrase length is constrained to be no larger than \( L_{max} \).

The fact that a sliding window version of LZ77, where the phrase is not constrained to be no longer than \( L_{max} \), yields a compression ratio that is equal to \( FS(x) \) was already established by P. Shields [5].

It should also pointed out that while LZ77, LZ78 and LZW are not finite-state algorithms, LZ78(RLU), LZW(RLU) and the sliding-window version of the LZ77 algorithm are all elements of the class for which \( FS(x) \) is defined.

Acknowledgment

Helpful comments by Marcelo Weinberger are acknowledged with thanks.

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Abstract—The unrestricted LZ78 universal data-compression algorithm (as well as the LZ77 and LZW versions) achieves asymptotically, as the block-length tends to infinity, the FS compressibility, namely the best compression-ratio that may be achieved by any Information-lossless(IL) block-to-variable finite-state(FS) algorithm, for any infinitely-long individual sequence.

The encoder parses the sequence into distinct phrases where each newly generated phrase is a past phrase which is already stored in a dictionary, extended by one letter. The newly generated phrase is then added to the updated, ever-growing dictionary.

In practice, heuristics that adaptively constrain the the number of entries in the dictionary by a prefixed level, were proposed.

One such heuristic approach is the “Least Recently Utilized” (LRU) deletion approach, where only the most recent D entries are kept in the dictionary, thus yielding a constrained-dictionary version of LZ78, denoted by LZ78(LRU).

In this note, for the sake of completeness, it is demonstrated again via a simple proof that the unrestricted LZ78 algorithm asymptotically achieves the FS-Compressibility. Then, it is demonstrated that the LZ78(LRU) information-lossless data-compression algorithm also achieves the FS compressibility, as the dictionary size D tends to infinity. Although this is perhaps not surprising, it does nevertheless yield theoretical optimality argument for the popular LZ78(LRU) algorithm (and similarly, for the LZW(LRU) algorithm).

I. INTRODUCTION AND SUMMARY OF RESULTS:

Consider sequences \( x^k = x_1, x_2, ..., x_k; x_i \in A \); \( i = 1, 2, ..., k \) where \( |A| = |A| \).

Also, let \( x = x_1^\infty \).

The unconstrained LZ78 universal data compression algorithm has been introduced in [1], where it is shown that when applied to an individual sequence \( x^k \), as \( n \) tends to infinity, it achieves the FS compressibility.

Let a finite-state encoder be denoted by the triple \((S, g, f)\) where \( S \) is a finite set of states, \( g: S \times A \rightarrow S \), and \( f: S \times A \rightarrow B^* \). For each starting state \( s_1 \), the triple defines a mapping from \( x \in A^\infty \) into \( y \in B^\infty \), where \( y_i = f(s_i, x_i) \) is a (possibly empty) binary word, \( s_{i+1} = g(s_i, x_i) \) is the next state and where \( i = 1, 2, ..., n \).

An information-lossless (IL) finite-state encoder is one for which for each \( n \), the sequence \( x_n^k \) is determined by \( y_1^n \), \( s_1 \) and \( s_{n+1} \).

The corresponding compression-ratio for \( x_n^k \) is \( \frac{1}{n \log |A|} \sum_{i=1}^{n} l(y_i) \), where \( l(y_i) \) is the length in bits of the (possibly empty) binary word \( y_i \).

The minimum compression ratio for \( x_n^k \) over all finite-state IL encoders with at most \( s \) states is denoted by \( FS_s(x_n^k) \).

Also, let \( FS_s(x) = \limsup_{n \rightarrow \infty} FS_s(x_n^k) \) and let...
the FS compressibility of x be defined by,
\[ FS(x) = \lim_{n \to \infty} FS_n(x). \]

Consider now the parsing of \( x_i^n \) into some c (not necessarily distinct) phrases:
\[ x_i^n = X_1, X_2, \ldots, X_j, \ldots, X_c; \quad X_j = x_i^{|l(j)|-1}; \]
\[ j = 1, 2, \ldots, c. \]
Let \( Z_{ij} \) denote the \( k \) distinct substrings among the \( c \) phrases in \( x_i^n \), where \( s_i \) denotes the start state and \( s_o \) denotes the end state of the phrase \( Z_{ij} \).

Also, let \( L(Z_{ij}|s_i, s_o) \) denote the length of the binary code-word that is generated by the IL FS encoder above, when fed with \( Z_{ij} \), given the start state \( s_i \) and the end state \( s_o \).

Let \( p(Z_{ij}|s_i, s_o) \) denote the empirical probability (i.e. fraction) of \( Z_{ij} \) among all phrases that are characterized by a start state \( s_i \) and an end state \( s_o \).

Similarly, let \( p(Z_{ij}) \) denote the empirical probability of \( Z_{ij} \) among the \( c \) phrases in \( x_i^n \) and let \( p(s_i, s_o) \) denote the empirical probability of the pair of states \( (s_i, s_o) \) among the (initial, end) pairs of states of the \( c \) phrases.

The corresponding compression-ratio for \( x_i^n \) is
\[ \frac{1}{n \log A} \sum_1^c L(Z_{ij}|s_i, s_o) = \frac{c}{n \log A} \sum_1^c \sum_{s_i-s(1)}^s \sum_{s_o-s(1)} s p(s_i, s_o) \]
\[ \sum_1^k p(Z_{ij}|s_i, s_o) L(Z_{ij}|s_i, s_o) \]
where \( s(t); t = 1, 2, \ldots, s \) are the distinct states that appear at the start or at the end of any of the \( c \) phrases (at most \( s \) such states).

**Lemma 1:** Consider an arbitrary parsing of \( x_i^n \) into \( c \) substrings (phrases). Then,
\[ FS_n(x^n_i) \geq \frac{c}{n \log A} \left[ \sum_1^k p(Z_{ij}) \log \left( \frac{1}{p(Z_{ij})} \right) - 2 \log s \right] \]

**Proof:**

For a given states pair \( s_i, s_o \), an IL FS encoder outputs a distinct binary code-word for each of the \( c \) phrases that start with the state \( s_i \) and end with \( s_o \).

Thus, by the Huffman IL coding argument [6],
\[ \sum_{1}^{k} p(Z_{ij}|s_i, s_o) L(Z_{ij}|s_i, s_o) \]
\[ \geq \sum_{1}^{k} p(Z_{ij}|s_i, s_o) \log \left( \frac{1}{p(Z_{ij}|s_i, s_o)} \right) \]

**Lemma 1** follows immediately by observing that
\[ - \log p(Z_{ij}|s_i, s_o) \geq - \log p(Z_{ij}) - \log p(s_i, s_o). \]

Now, in the case of LZ78 [1], all the \( c \) phrases that are generated for \( x_i^n \), are all distinct (except perhaps of the last phrase). For example, in the case of the LZ78 algorithm, each new phrase is either an extension of a previous phrase by one letter, or a single letter that is not identical to any of the past single-letter phrases. The code length for each phrase is bounded by \( \log C_n(LZ78) + 1 + \log A \), where \( C_n(LZ78) \) is the number of distinct phrases that are generated by LZ78.

Therefore,

**Lemma 2:** For any individual sequence \( x \)
\[ FS(x) \geq \lim_{n \to \infty} \frac{1}{n \log A} [(C_n(LZ78)) \log(C_n(LZ78))]. \]

The main result in [1] follows from Lemma 1 and Lemma 2 as follows:

The compression-ratio that is achieved for an individual sequence \( x^n_i \) that is parsed into \( C_n(LZ78) \) distinct phrases by LZ78 is upper-bounded by
\[ \frac{1}{n \log A} [(C_n(LZ78)) (\log C_n(LZ78) + 1 + \log A)] \]

**Thus,**

**Lemma 3:** The LZ78 universal IL data-compression algorithm asymptotically achieves \( FS(x) \).

Similarly, it follows that Lemma 3 holds for LZW [2] and LZ77 [3] as well. In practice, in order to
avoid the ever growing size of the dictionary that contains all the past phrases that are generated by LZ78 (or similarly, by LZW), heuristic constrained-dictionary versions has been proposed.

Apparently, the preferred heuristics is the Last-Recently -Used (LRU) method [3]. In this case, only the most recent phrases ( no larger than some preset number D) are kept in the dictionary.

This approach is analyzed below, and is shown to asymptotically achieve $FS(x)$ as well.

Consider a constrain-dictionary LZ78 algorithm, where the dictionary has D entries, each no longer than $L_{max}=(log D)^2$ letters. Each newly generated phrase is a copy of one of previous D phrases, extended by the next incoming letter. If no match is found with any of the phrases in the dictionary, then the first incoming letter is the next phrase.

The new phrase is then included in the dictionary and the last recently used phrase is removed from the dictionary, except for the case where the newly generated phrase is of length $L_{max}+1$, in which case the dictionary is not updated.

The code length for each successive phrase is $log D + 1 + log A$. Denote this algorithm by LZ78(LRU).

**Theorem 1:** The compression-ratio that is achieved by LZ78(LRU) when applied to an individual $x$ converges asymptotically to $FS(x)$ as D tends to infinity.

**Proof:** Let $c(n)$ denote the number of phrases that are generated by LZ78(LRU) when applied to $x_1^n$, and let $c(n)L_{max}+1$ denote the number of phrases of length $L_{max} + 1$.

By construction, $p(Z_j) \geq \frac{1}{D}$ for any phrase $Z_j$ among the $c(n)$ phrases that are no longer than $L_{max}$ since the number of phrases in between any such phrase and it’s most recent previous appearance is at least D (since it is not included in the dictionary).

Let $\rho_{LZ78(RLU)}(x_1^n) = \frac{C(n)}{n log A}(log D + 1 + log A)$ denote the compression-ratio that is achieved by LZ78(RLU) when applied to $x_1^n$.

By Lemma 1,

$$\rho_{LZ78(RLU)}(x_1^n) = \frac{C(n)}{n log A}(log D + 1 + log A)$$

$$\leq FS_s(x_1^n) + \frac{c(n)}{n log A}(2 log s + log D + 1 + log A) + \frac{1}{n}c(n)L_{max}+1)log D.$$

Therefore,

$$\rho_{LZ78(RLU)}(x_1^n)(1 - \frac{2 log s}{log D})$$

$$\leq FS_s(x_1^n) + \frac{c(n)}{n log A}(log D + 1 + log A) + \frac{1}{n}c(n)L_{max}+1)log D$$

where $c(n)L_{max}+1)$ denotes the number of phrases among the $c(n)$ phrases, of length $L_{max} + 1$.

Observe that $n \geq c(n)L_{max}+1)(L_{max}+1)$ and that $L_{max}=(log D)^2$. Thus,

$$\lim sup_{D \to \infty} \lim sup_{n \to \infty} \rho_{LZ78(RLU)}(x_1^n)x = \frac{2 log s}{log D} = \lim sup_{D \to \infty} \lim sup_{n \to \infty} \rho_{LZ78(RLU)}(x_1^n) = FS(x)$$

which proves Theorem 1.

The same result holds for LZW(LRU) as well as for a sliding version of LZ77 where the window is set at $L_{max}$ and where the phrase length is constrained to be no larger than $L_{max}$.

The fact that a sliding window version of LZ77, where the phrase is not constrained to be no longer than $L_{max}$, yields a compression ratio that is equal to $FS(x)$ was already established by P. Shields [5].

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