A LOWER BOUND FOR THE NUMBER OF CENTRAL CONFIGURATIONS ON $\mathbb{H}^2$

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ABSTRACT. We study the indices of the geodesic central configurations on $\mathbb{H}^2$. We then show that central configurations are bounded away from the singularity set. With Morse’s inequality, we obtain a lower bound for the number of central configurations on $\mathbb{H}^2$.

Key Words: celestial mechanics; curved $N$-body problem; central configurations; Morse theory.

1. INTRODUCTION

The Newtonian $N$-body problem is the study of the dynamics of $N$ particles moving according to Newton’s laws of motion in $\mathbb{R}^n$, where $n$ is always 2 or 3. After the discovery of Non-Euclidean geometry in 19th century, geometers considered the possibility of a three-dimensional sphere, $\mathbb{S}^3$, and a three-dimensional hyperbolic sphere, $\mathbb{H}^3$ universe. Thus the dynamics of $N$ particles in $\mathbb{S}^3$ and $\mathbb{H}^3$, moving according to some attraction law, were considered. We call this problem the curved $N$-body problem. There have been many publications in this field before the rise of general relativity. This problem attracted attention later from the point of view of quantum mechanics [22] and the theory of integrable dynamical systems [16] [23]. Readers interested in its history may read [11] [4] [24]. On the topic of relative equilibria, researchers studied mainly the 2-dimensional ones [15] before Diacu’s work. Diacu wrote the equations of motion in extrinsic coordinates in $\mathbb{R}^4$ for $\mathbb{S}^3$, and the Minkowski space $\mathbb{R}^{3,1}$ for $\mathbb{H}^3$. In this set up, the matrix Lie group $SO(4)$ ($SO(3, 1)$) serves as the symmetry group, which makes the study the 3-dimensional relative equilibria easier. With this new approach, Diacu obtained many new results on relative equilibria [4] [5] [6] [9] and on other topics like singularity [2], homographic orbits [3], rotopulsators [7], stability of orbits [8], and the
relationship between the Newtonian and the curved $N$-body problem [11]. There are many following works like [10, 29] etc.

Based on Diacu’s works, especially [4, 6], the authors of [12] proposed to study central configurations. Roughly speaking, central configurations are special arrangements of the point particles such that the acceleration vector for each particle points toward a special geodesic, see [12, page 31]. Like what happens in the Newtonian $N$-body problem [17], central configurations are quite important in the study of the curved $N$-body problem. For instance, each central configuration gives rise to a one-parameter family of relative equilibria, and central configurations are the bifurcation points in the topological classification of the curved $N$-body problem [12]. The interested readers may read [12, 30, 31] for more detail.

In this paper, we concentrate on central configurations in $H^3$. Previous studies show that their properties are similar to the properties of the central configurations in $R^3$. For example, Moulton’s theorem [18] on geodesic central configurations in $R^3$ can be extended to $H^3$, while it can’t be extended to $S^3$ [12]; In $H^3$, only 2-dimensional central configurations give rise to relative equilibria [30], which is the same as in $R^3$ [28].

In this paper, we extend another interesting result of the Newtonian $N$-body problem to the curved $N$-body problem in $H^3$. Recall that Smale [27] and Palmore [20] have applied Morse theory to obtain a lower bound for the number of central configurations of the Newtonian $N$-body problem. Their idea is as follows. First characterise central configurations as critical points of a certain function on a certain manifold. Then find the indices of some known critical points and the Poincaré polynomial of the manifold. In the end, assuming the function is a Morse function, Morse inequality, which relates the critical points and the topology of the manifold, gives rise to an estimation for the number of the critical points. We apply the same idea to study central configurations in $H^3$.

The paper is organized as follows. In Section 2, we recall the basic setting of the curved $N$-body problem in $H^3$ and some basic facts about central configurations. In Section 3, we show that the central configurations in $H^3$ are critical points of a certain function on $S_c$ and discuss the computation of the Hessian. In Section 4, we study the indices some known critical points, namely, the $N!/2$ geodesic central configurations. We prove that the index of each of them is $N - 2$. In Section 5, we show that there is a neighbourhood of the singularity set which contains no central configurations, which is essential for the application of Morse theory. Then with the Poincaré polynomial of $S_c$, we apply Morse inequality to obtain a lower bound for the number of central configurations on $H^2$. 
2. THE CURVED $N$-BODY PROBLEM IN $\mathbb{H}^3$ AND CENTRAL CONFIGURATIONS

Vectors are all column vectors, but written as row vectors in the text. As done in [4, 6], the equations will be written in the Minkowski space $\mathbb{R}^{3,1}$. For two vectors, $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)$ and $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)$, the inner products are given by

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 - w_1 w_2.$$

We define the unit hyperbolic sphere $\mathbb{H}^3$ as

$$\mathbb{H}^3 := \{ (x, y, z, w) \in \mathbb{R}^{3,1} \mid x^2 + y^2 + z^2 - w^2 = -1, \ w > 0 \}.$$

Given the positive masses $m_1, \ldots, m_N$, whose positions are described by the configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in (\mathbb{H}^3)^N$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i), \ i = 1, \ldots, N$, we define the singularity set

$$\Delta = \cup_{1 \leq i < j \leq N} \{ \mathbf{q} \in (\mathbb{H}^3)^N ; \mathbf{q}_i = \mathbf{q}_j \}.$$

Let $d_{ij}$ be the geodesic distance between the point masses $m_i$ and $m_j$, which is computed by

$$\cosh d_{ij}(\mathbf{q}) = -\mathbf{q}_i \cdot \mathbf{q}_j.$$

The force function $U$ in $(\mathbb{H}^3)^N \setminus \Delta$ is

$$U(\mathbf{q}) := \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij}(\mathbf{q}).$$

Define the kinetic energy as $T(\dot{\mathbf{q}}) = \frac{1}{2} \sum_{1 \leq i \leq N} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i, \ \dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_N)$. Then the curved $N$-body problem in $\mathbb{H}^3$ is given by the Lagrange system on $T((\mathbb{H}^3)^N \setminus \Delta)$, with

$$L(\mathbf{q}, \dot{\mathbf{q}}) := T(\dot{\mathbf{q}}) + U(\mathbf{q}).$$

Using variational methods, it is easy to obtain the equations [12]:

$$\begin{cases} \ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_i m_j (\mathbf{q}_i - \cosh d_{ij} \mathbf{q}_j)}{\sinh^2 d_{ij}} + m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = -1, \quad i = 1, \ldots, N. \end{cases}$$

The first part of the acceleration is from the gradient of the force function, $U(\mathbf{q}) : (\mathbb{H}^3)^N \setminus \Delta \to \mathbb{R}$, and we will denote it by $F_i$. It is the sum of $F_{ij} := \frac{m_i m_j (\mathbf{q}_i - \cosh d_{ij} \mathbf{q}_j)}{\sinh^2 d_{ij}}$ for $j \neq i$, see [12] page 17 for the derivation.

**Definition 1.** A configuration $\mathbf{q} \in (\mathbb{H}^3)^N \setminus \Delta$ is called a central configuration if there is some constant $\lambda$ such that

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}), \quad i = 1, \ldots, N,$$

where $\nabla f$ is the gradient of a function $f : (\mathbb{H}^3)^N \setminus \Delta \to \mathbb{R}$, and $I(\mathbf{q})$ is the moment of inertia defined by $I(\mathbf{q}) = \sum_{i=1}^N m_i (x_i^2 + y_i^2)$. We will refer to these conditions as the central configuration equations.
A central configuration \( q \) with all \( q_i \) lying on one geodesic is called a geodesic central configuration.

The definition of central configurations of the curved \( N \)-body problem is based on the work of Smale [26, 27], see [12, page 25]. A central configuration gives rise to a one-parameter family of relative equilibria, see [12, page 29]. They also influence the topology of the integral manifolds [26].

Obviously, the two functions \( U \) and \( I \) are both invariant under \( SO(2) \times SO(1, 1) \). Let \( \chi = (\chi_1, \chi_2) \in SO(2) \times SO(1, 1) \). The action is defined by

\[
\chi q = (\chi q_1, ..., \chi q_N), \quad \chi q_i = (\chi_1(x_i, y_i)^T, \chi_2(z_i, w_i)^T).
\]

It is easy to see that if \( q \) is a central configuration, so is \( \chi q \). We call two such central configurations equivalent. In this paper, when we say a central configuration, we mean a class of central configurations as defined by the above equivalence relation.

For our purpose, we will need the following three results. It is not hard to verify Theorem 1 and Corollary 1 by using the central configuration equations (1).

**Theorem 1.** [12, page 48] Let \( q = (q_1, ..., q_N) \), \( q_i = (x_i, y_i, z_i, w_i) \), \( i = 1, ..., N \), be a central configuration in \( H^3 \). Then we have the relationships

\[
\sum_{i=1}^{N} m_i x_iz_i = \sum_{i=1}^{N} m_i x_i w_i = \sum_{i=1}^{N} m_i y_i z_i = \sum_{i=1}^{N} m_i y_i w_i = 0.
\]

Define \( H^2_{xyw} := \{(x, y, z, w) \in H^3 : z = 0\} \).

**Theorem 2.** \( \text{[30]} \). Each central configuration in \( H^3 \) is equivalent to some central configuration on \( H^2_{xyw} \).

Define \( H^1_{xw} := \{(x, y, z, w) \in H^3 : y = z = 0\} \).

**Corollary 1.** [12, page 37] Each geodesic central configuration in \( H^3 \) is equivalent to some central configuration on \( H^1_{xw} \).

Thus to study central configurations in \( H^3 \), it is enough to study them on \( H^2_{xyw} \) for non-geodesic ones, and on \( H^1_{xw} \) for geodesic ones. From now on, unless specified otherwise, we use \( H^2 \) to indicate \( H^2_{xyw} \in \mathbb{R}^{2,1} \), and use \( H^1 \) for \( H^1_{xw} \in \mathbb{R}^{1,1} \). We only study central configurations on \( H^2 \). Then two central configurations \( q, \tilde{q} \) on \( H^2 \) are equivalent if there is some element \( \chi \) in \( SO(2) \) such that \( \chi q = \tilde{q} \). The action is defined by \( \chi q = (\chi q_1, ..., \chi q_N), \chi q_i = (\chi(x_i, y_i)^T, w_i) \). It is easy to find that the expression of \( \nabla I \) is

\[
\nabla_q I(q) = 2m_i \left(x_i w_i^2, y_i w_i^2, w_i(x_i^2 + y_i^2)\right),
\]

see [12, page 26] for the derivation. Let \( r_i = (x_i^2 + y_i^2)^{1/2} \geq 0 \).
Proposition 1. A central configuration \( \bar{q} \) on \( \mathbb{H}^2 \) is a critical point of the function 
\[ U(q) - \lambda I(q) : (\mathbb{H}^2)^N \setminus \Delta \to \mathbb{R}, \] 
where \( \lambda \) is some constant depending on the configuration \( \bar{q} \).

The value of \( \lambda \) can be obtained in the following way. Let \( M \) be the matrix 
\[ \text{diag}(m_1, m_1, m_1, \ldots, m_N, m_N, m_N). \]
Introduce a metric in \((\mathbb{R}^{2,1})^N:\)
\[ \langle q, q \rangle = \sum_{i=1}^{N} m_i q_i \cdot q_i = q \cdot M q. \]

Proposition 2. Let \( q \) be a central configuration on \( \mathbb{H}^2 \), then the value of \( \lambda \) in the central configuration equation is 
\[ \frac{\langle M^{-1} \nabla U, M^{-1} \nabla I \rangle}{\langle M^{-1} \nabla I, M^{-1} \nabla I \rangle} \] 
and it is negative.

**Proof.** Using the explicit formula of \( \nabla q_i I \), we see that 
\[ \langle M^{-1} \nabla I, M^{-1} \nabla I \rangle = 0 \]
if and only if \( q_1 = \ldots = q_N = (0,0,1) \). Thus for a central configuration \( q \) that 
satisfies the equation \( \nabla q_i U = \lambda \nabla q_i I \), we have 
\[ \langle M^{-1} \nabla U, M^{-1} \nabla I \rangle \neq 0. \]
Since 
\[ \langle M^{-1} \nabla U, M^{-1} \nabla I \rangle = \lambda \langle M^{-1} \nabla I, M^{-1} \nabla I \rangle, \]
we obtain \( \lambda = \frac{\langle M^{-1} \nabla U, M^{-1} \nabla I \rangle}{\langle M^{-1} \nabla I, M^{-1} \nabla I \rangle}. \)

Direct computation leads to
\[ \langle M^{-1} \nabla I, M^{-1} \nabla I \rangle = \sum_{i=1}^{N} m_i (x_i^2 w_i^4 + y_i^2 w_i^4 - (x_i^2 + y_i^2)^2 w_i^2), \]
\[ \langle M^{-1} \nabla U, M^{-1} \nabla I \rangle = \sum_{i=1}^{N} \left( \sum_{j \neq i} m_j \frac{q_j - \cosh d_{ij} q_i}{\sinh^3 d_{ij}} \cdot (x_i w_i^2, y_i w_i^2, (x_i^2 + y_i^2) w_i) \right) \]
\[ = \sum_{1 \leq i < j \leq N} m_i m_j \frac{(w_i w_j - \cosh d_{ij})(w_i^2 + w_j^2) - w_i w_j (w_i^2 + w_j^2 - 2)}{\sinh^3 d_{ij}} \]
\[ = \sum_{1 \leq i < j \leq N} m_i m_j \frac{- \cosh d_{ij}(w_i^2 + w_j^2) + (w_i^2 + w_j^2) - (w_i^2 + w_j^2) + 2w_i w_j}{\sinh^3 d_{ij}} \]
\[ = \sum_{1 \leq i < j \leq N} m_i m_j \frac{(w_i^2 + w_j^2)(1 - \cosh d_{ij}) - (w_i - w_j)^2}{\sinh^3 d_{ij}} < 0. \]

Here we used the identities \( \cosh d_{ij} = w_i w_j - (x_i x_j + y_i y_j) \) and \( x_i^2 + y_i^2 - w_i^2 = -1 \). 
This remark completes the proof. \( \square \)

3. The gradient flow and the Hessian

In this section, we characterize central configurations as the restpoints of a 
certain gradient flow, i.e., the critical points of a certain function, then we discuss
the computation of the Hessian of these critical points. We denote by \( S_c \) the set \( \{ q \in (\mathbb{H}^2)^N \setminus \Delta | I(q) = c \} \).

**Proposition 3.** \( I^{-1}(c) \) is homeomorphic to a \((2N-1)\)-dimensional sphere for each positive value of \( c \).

**Proof.** Consider the homomorphism \( \pi : \mathbb{R}^2 \to \mathbb{H}^2, \pi(x, y) = (x, y, \sqrt{x^2 + y^2 + 1}) \). The map induces a homomorphism from \((\mathbb{R}^2)^N \) to \((\mathbb{H}^2)^N\), which we still denote by \( \pi \). Thus the function \( I : (\mathbb{H}^2)^N \to \mathbb{R} \) induces a function \( \bar{I} : (\mathbb{R}^2)^N \to \mathbb{R} \) by \( \bar{I}(\bar{q}) = I(\pi \bar{q}) \), where \( \bar{q} \) is a point in \((\mathbb{R}^2)^N\). It is easy to see that \( \bar{I}^{-1}(c) \) is homeomorphic to a \((2N-1)\)-dimensional sphere for each positive value of \( c \) and \( \bar{I}^{-1}(c) \) is homeomorphic to \( I^{-1}(c) \). This remark completes the proof. \( \square \)

Note that in the central configuration equation, the value \( \lambda \) can be also interpreted as Lagrange multiplier. More precisely, Consider the restricted function:

\[ U : S_c \to \mathbb{R} \]

**Proposition 4.** The vectorfield

\[ X = M^{-1}\nabla U - M^{-1}\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle} \nabla I \]

is the gradient of \( U|_{S_c} \), the restriction of \( U(q) \) on the set \( S_c \), with respect to the metric \( \langle \cdot, \cdot \rangle \). Moreover, the restpoints of this vectorfield are exactly the central configurations in \( S_c \).

**Proof.** Since \( \langle X, M^{-1}\nabla I \rangle = 0 \), the vector field is tangential to \( S_c \). For any \( v \in T_qS_c \), we have \( \langle v, M^{-1}\nabla I \rangle = 0 \), thus

\[ \langle X, v \rangle = \langle M^{-1}\nabla U, v \rangle = dUv, \]

where \( dU \) is the differential of \( U \). Thus \( X \) is the gradient flow of \( U \) on \( S_c \). The other statements are self-clear, a remark that completes the proof. \( \square \)

The critical points of \( U|_{S_c} \) are not isolated. Let \( q \) be a central configuration and \( \phi \) an element of \( SO(2) \). Then \( \phi q \) is also a central configuration. Thus it follows that the critical points of \( U|_{S_c} \) are not isolated, but rather occur as manifolds of critical points. This fact suggests that we can further look at the central configurations as critical points of \( U \) subject to a quotient manifold. Note that both \( U, I, \) and \( S_c \) are invariant under the \( SO(2) \) action. We thus have the following property.

**Proposition 5.** There is a one-to-one correspondence between the classes of central configurations on \( \mathbb{H}^2 \) and the critical points of \( U \) on the quotient set \( S_c / SO(2) \).
It is interesting to classify central configurations by their Morse index. Recall that if \( x \) is a critical point of a smooth function \( f \) on a manifold \( M \), there is a Hessian quadratic form on the tangent space \( T_x M \) that is given in local coordinates by the symmetric matrix of the second derivatives:

\[
H(x)(v) = v^T D^2 f(x) v.
\]

The Morse index \( \text{ind}(x) \) is the maximum dimension of a subspace of \( T_x M \) on which \( H(x) \) is negative-definite. The nullity is the dimension of

\[
\ker H(x) = \{ v : H(x)(v, u) = 0 \text{ for all } u \in T_x M \},
\]

where \( H(x)(v, u) = v^T D^2 f(x) u \) is the symmetric bilinear form associated to \( H(x) \).

We are interested in the function \( U |_{S_c} \) given by restricting the potential to the manifold \( S_c \), a \( 2N - 1 \)-dimensional sphere. Instead of using the local coordinates of \( S_c \), it is more convenient to use the coordinates of \( \mathbb{H}^2 \). Then the Hessian is given by a \( 2N \times 2N \) matrix, also called \( H(x) \), whose restriction to \( T_q S_c \) gives the correct values.

**Lemma 1.** Let \( M \) be a smooth manifold, \( N \) be a submanifold of \( M \), and \( f_1 \) be a smooth function on \( N \). Assume that \( f \) is a smooth function on \( M \) and \( f|_N = f_1 + c \), where \( c \) is some constant, and that \( x \in N \) is a critical point of \( f \) and \( f_1 \). Denote the Hessian of \( f \) and the Hessian of \( f_1 \) by \( H(x) \) and \( H_1(x) \) respectively. Then \( H(x)|_{T_x N} = H_1(x) \).

**Proof.** Let \( (x_1, \ldots, x_n) \) be a local coordinate system of \( N \) near \( x \). Extend this system to a local coordinate system of \( M \) near \( x \), \( (x_1, \ldots, x_n, y_1, \ldots, y_k) \). Since \( x \) is the critical point of \( f \) and \( f_1 \), we get

\[
H(x) = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j + \frac{\partial^2 f}{\partial x^i \partial y^j} dx^i \otimes dy^j + \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i \otimes dy^j,
\]

\[
H_1(x) = \frac{\partial^2 f_1}{\partial x^i \partial x^j} dx^i \otimes dx^j = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j,
\]

where we used Einstein’s summation. Now let \( v = a^i \frac{\partial}{\partial x^i} \in T_x N \), we obtain

\[
H(x)(v) = H_1(x)(v) \quad \text{since} \quad \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dy^j(v) = \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i \otimes dy^j(v) = 0.
\]

This remark completes the proof. \( \square \)

**Lemma 2.** Let \( q \) be a central configuration on \( \mathbb{H}^2 \), and \( I(q) = c \). Let \( \lambda \) be the value of \( \langle \lambda^{-1} \nabla U, \lambda^{-1} \nabla I \rangle \) at \( q \). Then the Hessian of \( U |_{S_c} \) at the critical point \( q \) is given by \( H(q)(v) = v^T H(q) v \), where \( H(q) \) is the \( 2N \times 2N \) matrix

\[
H(q) = D^2 U - \lambda D^2 I,
\]

and \( D^2 U \) and \( D^2 I \) are the second derivatives matrices of \( U \) and \( I \) in some coordinates of \( (\mathbb{H}^2)^N \).
Proof. Consider the manifold $(\mathbb{H}^2)^N \setminus \Delta$, the submanifold $S_c$, the smooth function $U - \lambda I : (\mathbb{H}^2)^N \setminus \Delta \to \mathbb{R}$, and the smooth function $U : S_c \to \mathbb{R}$. By Proposition 1 and Proposition 4, $q$ is a critical point of the two functions. Restricting the first function to $S_c$, we have $U - \lambda I = U - \lambda c$. Thus by the above lemma, we see that on $T_q S_c$, the two Hessians are the same, which is $H(q) = D^2 U - \lambda D^2 I$. This remark completes the proof. □

As noticed above, the critical points of $U|_{S_c}$ are not isolated, which implies that the central configuration are always degenerate as critical points. The following result describe the minimal degeneracy.

**Proposition 6.** Let $q \in S_c$ be a central configuration. Then the nullity of $q$ as a critical point of $U|_{S_c}$ satisfies

$$\text{null}(q) \geq 1.$$  

**Proof.** Consider the curve in $S_c$, $q(t) = B(t)q$, where $B(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$. They are also central configurations in $S_c$ with the same value of $\lambda$. Thus we have the equation $\nabla_{q(t)} U(q(t)) = \lambda \nabla_{q(t)} (q(t))$, $i = 1, ..., N$. Taking the derivative with respect to $t$ at $t = 0$, we get

$$(D^2 U - \lambda D^2 I)\dot{B}(0)q = 0.$$  

Since $B(t)q \in S_c$, we obtain that $\dot{B}(0)q \in T_q S_c$. Thus the nullity of the Hessian is at least one. □

For central configurations, it is natural to call a critical point nondegenerate if its nullity is as small as possible given the rotational symmetry.

**Definition 2.** A central configuration on $\mathbb{H}^2$ is nondegenerate if the nullity is one; a central configuration on $\mathbb{H}^1$ is nondegenerate if the nullity is zero.

4. THE GEODESIC CENTRAL CONFIGURATIONS AND THEIR INDICES

In [12], we have proved that existence of $N!/2$ geodesic central configurations. We study their indices now.

**Theorem 3.** [12, page 60] Given masses $m_1, \ldots, m_N > 0$ on $\mathbb{H}^1$, for any $c > 0$, there are exactly $N!/2$ geodesic central configurations on $S_c$, one for each ordering of the masses along $\mathbb{H}^1$.

The number of ordering of $N$ masses is $N!$. Since a $180^\circ$ rotation changes the ordering, which means that we counted each case twice, so there are exactly $N!/2$ classes of geodesic central configurations.
Now we study the Hessian of the $N!/2$ geodesic central configurations on $S_c = \{ q \in (H^2)^N \setminus \Delta | I(q) = c \}$. For our purpose, we use the coordinate system of $H^2$, 

$$(x, y, w) = (\sinh \theta, \cosh \theta \sin \varphi, \cosh \theta \cos \varphi), \quad \theta, \varphi \in \mathbb{R}.$$ 

Then $H^1$ corresponds to $\varphi = 0$. This coordinates system gives a homomorphism between $\mathbb{R}^2 = (\theta, \varphi)$ and $H^2$. Then 

$$U = \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij}, \quad I = \sum_{i=1}^N m_i (x_i^2 + y_i^2) = \sum_{i=1}^N m_i (\sinh^2 \theta_i + \cosh^2 \theta_i \sinh^2 \varphi_i).$$ 

Order the coordinates as $(\theta_1, ..., \theta_N, \varphi_1, ..., \varphi_N)$. For a geodesic central configuration $(\theta_1, ..., \theta_N, 0, ..., 0)$, direct computations lead to 

$$H(q) = D^2U - \lambda D^2I = \begin{bmatrix} \frac{\partial^2 U}{\partial \theta_i \partial \theta_j} & 0 \\ 0 & \frac{\partial^2 U}{\partial \varphi_i \partial \varphi_j} \end{bmatrix} - \lambda \begin{bmatrix} \frac{\partial^2 I}{\partial \theta_i \partial \theta_j} & 0 \\ 0 & \frac{\partial^2 I}{\partial \varphi_i \partial \varphi_j} \end{bmatrix}.$$ 

Thus it is enough to study the upper left block $H_\theta := [\frac{\partial^2 U}{\partial \theta_i \partial \theta_j} - \lambda \frac{\partial^2 I}{\partial \theta_i \partial \theta_j}]_{N \times N}$ and the lower right block $H_\varphi := [\frac{\partial^2 U}{\partial \varphi_i \partial \varphi_j} - \lambda \frac{\partial^2 I}{\partial \varphi_i \partial \varphi_j}]_{N \times N}$. 

The upper left block $H_\theta$ is positive-definite, see P.61 of [12]. More precisely, this block acts on $T_q(H^1)^N$, which is spanned by the $N$ vectors $\frac{\partial}{\partial \theta_1}, ..., \frac{\partial}{\partial \theta_N}$. In this $N$-dimensional space, there is a 1-dimensional subspace that is not in $T_q S_c$. It is generated by 

$$\nabla I = \sum_{i=1}^N m_i \sinh 2\theta_i \frac{\partial}{\partial \theta_i},$$

which is actually orthogonal to $S_c$. Thus we see that there is an $(N - 1)$-dimensional subspace in the geodesic directions of $T_q S_c$ on which $H(q)$ is positive-definite.

The lower right block acts on the complementary subspace spanned by the $N$ vectors $\frac{\partial}{\partial \varphi_1}, ..., \frac{\partial}{\partial \varphi_N}$. At the geodesic central configuration $q$, it is easy to see that 

$$dI(\frac{\partial}{\partial \varphi_i}) = 0 \text{ for each } i \text{ since } \varphi_i = 0.$$ 

Thus the $N$-dimensional subspace belongs to $T_q S_c$. Explicitly, this block is

$$H_\varphi = \begin{bmatrix} \sum_{j=1, j \neq 1}^N \frac{-m_j m_{ij} \coth \theta_i \cosh \theta_j}{\sinh^3 d_{ij}} & \frac{m_1 m_2 \coth \theta_1 \cosh \theta_2}{\sinh^3 d_{12}} & \cdots & \frac{m_1 m_N \coth \theta_1 \cosh \theta_N}{\sinh^3 d_{1N}} \\ \frac{m_2 m_1 \coth \theta_2 \cosh \theta_1}{\sinh^3 d_{12}} & \sum_{j=1, j \neq 2}^N \frac{-m_j m_{ij} \cosh \theta_i \cosh \theta_j}{\sinh^3 d_{ij}} & \cdots & \frac{m_2 m_N \coth \theta_2 \cosh \theta_N}{\sinh^3 d_{2N}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{m_N m_1 \coth \theta_N \cosh \theta_1}{\sinh^3 d_{1N}} & \cdots & \cdots & \sum_{j=1, j \neq N}^N \frac{-m_j m_{ij} \coth \theta_i \cosh \theta_j}{\sinh^3 d_{Nj}} \end{bmatrix}$$
First, notice that there is a null vector of $H(q)$ in this subspace. Proposition 6 shows, by the $SO(2)$ symmetry, that there is at least one null vector for the Hessian of any central configuration on $\mathbb{H}^2$. In the $xyw$-coordinates, obviously, the null vector is

$$v = \sum_{i=1}^{N} -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} = \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i}.$$ 

Expressed in the $\theta, \varphi$ coordinates, it is in the subspace spanned by the $N$ vectors $\frac{\partial}{\partial \varphi_1}, \ldots, \frac{\partial}{\partial \varphi_N}$, and

$$v = \sum_{i=1}^{N} -\sinh \theta_i \cosh \varphi_i \sinh \theta_i - \frac{\partial}{\partial \varphi_i} = - \left( 0, \ldots, \frac{\sinh \theta_1}{\cosh \theta_1}, \ldots, \frac{\sinh \theta_N}{\cosh \theta_N} \right).$$

Thus $v$ is a null vector of $H_\varphi$.

We will need the following inequalities on distance.

**Proposition 7.** On $\mathbb{H}^1 = (\sinh \theta, \cosh \theta)$, for $N$ distinct points with $\theta_1 < \theta_2 < \cdots < \theta_N$, we have the following inequalities:

1. if $k < i < j$, then $\frac{1}{\sinh^4(\theta_j - \theta_k) \cosh \theta_j} < \frac{1}{\sinh^4(\theta_i - \theta_k) \cosh \theta_i};$
2. if $i < j < k$, then $\frac{1}{\sinh^4(\theta_k - \theta_j) \cosh \theta_j} > \frac{1}{\sinh^4(\theta_k - \theta_i) \cosh \theta_i}.$

**Proof.** If $k < i < j$, what we need to show is

$$\sinh^3(\theta_j - \theta_k) \cosh \theta_j - \sinh^3(\theta_i - \theta_k) \cosh \theta_i > 0.$$ 

View this as a function of $\theta_j$, i.e., $f(x) = \sinh^3(x - \theta_k) \cosh x - \sinh^3(\theta_i - \theta_k) \cosh \theta_i$, $x > \theta_i$. Then $f(\theta_i) = 0$, and

$$f'(x) = \sinh^3(x - \theta_k) \sinh x + 3 \sinh^2(x - \theta_k) \cosh(x - \theta_k) \cosh x - \sinh^2(x - \theta_k) \cosh x + 3 \cosh(x - \theta_k) \cosh x = \sinh^2(x - \theta_k) \cosh(2x - \theta_k) + 2 \sinh^2(x - \theta_k) \cosh(x - \theta_k) \cosh x > 0.$$ 

So for $k < i < j$, we always have $\frac{1}{\sinh^4(\theta_j - \theta_k) \cosh \theta_j} < \frac{1}{\sinh^4(\theta_i - \theta_k) \cosh \theta_i}.$ The proof of the other inequality is similar. □

The following theorem extends the result on the indices of geodesic central configurations in $\mathbb{R}^3$ [17]. As done in [19], the essential idea of the proof is due to Conley.
Theorem 4. Every geodesic central configuration $q$ on $\mathbb{H}^2$ is nondegenerate with $	ext{null}(q) = 1$ and $\text{ind}(q) = N - 2$. In the geodesic tangent directions, which are $(N - 1)$-dimensional, $H(q)$ is positive definite, while in the normal directions it is positive on a 1-dimensional subspace, zero on another 1-dimensional subspace, negative-definite on the rest $(N - 2)$-dimensional subspace.

Proof. We first simplify the form of the matrix $H_\varphi$. Introduce the following three $N \times N$ matrices:

$C := \text{diag}\{\cosh \theta_1, \cdots, \cosh \theta_N\}$,

$\bar{M} := \text{diag}\{m_1, \cdots, m_N\}$,

$A := \begin{bmatrix}
    \sum_{j=1, j \neq 1}^{N} \frac{-m_j \cosh \theta_j}{\sinh^2 d_{1j}} & \frac{m_2}{\sinh^2 d_{12}} & \cdots & \frac{m_N}{\sinh^2 d_{1N}} \\
    \frac{m_2}{\sinh^2 d_{12}} & \sum_{j=1, j \neq 2}^{N} \frac{-m_j \cosh \theta_j}{\sinh^2 d_{2j}} & \cdots & \frac{m_N}{\sinh^2 d_{2N}} \\
    \cdots & \cdots & \cdots & \cdots \\
    \frac{m_1}{\sinh^2 d_{1N}} & \cdots & \cdots & \sum_{j=1, j \neq N}^{N} \frac{-m_j \cosh \theta_j}{\cosh \theta_N \sinh^2 d_{Nj}}
\end{bmatrix}$.

Then it is easy to check that $H_\varphi = \bar{M}(A - 2\lambda)C$. Thus to study the eigenvalues of $H_\varphi$ is equivalent to studying the eigenvalues of $A - 2\lambda$. Precisely, note that $\bar{M}^{-\frac{1}{2}}$ and $C^{-\frac{1}{2}}$ are well defined. Then $H_\varphi$ is congruent to $G_1 := (C^{-\frac{1}{2}})^TH_\varphi C^{-\frac{1}{2}}$, which is similar to $C^\frac{1}{2}G_1 C^{-\frac{1}{2}} = H_\varphi C^{-1} = \bar{M}(A - 2\lambda)$. Similarly, we can obtain rid of $\bar{M}$. By Sylvester’s law of inertia [13], we have

$n_0(H_\varphi) = n_0(A - 2\lambda)$, $n_-(H_\varphi) = n_-(A - 2\lambda)$, $n_+(H_\varphi) = n_+(A - 2\lambda)$,

where $n_0(\ast)$ is the number of zero eigenvalues of matrix $\ast$, $n_-(\ast)$ the number of negative eigenvalues, and $n_+(\ast)$ the number of positive eigenvalues.

To study the eigenvalues of $A - 2\lambda$, it is enough to study the eigenvalues of $A$ and compare them with the negative number $2\lambda$. First, notice that there are two obvious eigenvectors of $A$:

$v_1 = (\cosh \theta_1, \cdots, \cosh \theta_N)$, \hspace{1cm} $Av_1 = 0v_1$,

$v_2 = (\sinh \theta_1, \cdots, \sinh \theta_N)$, \hspace{1cm} $Av_2 = 2\lambda v_2$.

The first vector can be obtained by inspecting the matrix $A$. The second vector $v_2$ equals $-Cv$, where $v = (-\frac{\sinh \theta_1}{\cosh \theta_1}, \cdots, \frac{\sinh \theta_N}{\cosh \theta_N})$ is the null vector of $\mathbb{H}_\varphi$, see [4]. Since $H_\varphi v = \bar{M}(A - 2\lambda)Cv = 0$, we have

$ACv = 2\lambda Cv$, \hspace{0.5cm} $Av_2 = 2\lambda v_2$. 
Now we employ the idea of Conley to show that all other eigenvalues of $A$ are smaller than $2\lambda$. The idea is to consider the linear system in $\mathbb{R}^N$:

$$\dot{u} = Au, \quad u = (u_1, \ldots, u_N) \in \mathbb{R}^N.$$ 

Conley observed that to show that all other eigenvalues of $A$ are smaller than $2\lambda$ is equivalent to showing that, in the flow on $\mathbb{R}^N$, the line determined by $v_2$ is an attractor. It is enough to find a "cone", $K$, around $v_2$ that is carried strictly inside itself by the flow (except for the origin).

**Figure 1.** The linear flow in $\mathbb{R}^N$

Suppose that the ordering of the geodesic central configuration $q$ is $\theta_1 < \theta_2 < \cdots < \theta_N$. Then let

$$K = \left\{ u \in \mathbb{R}^N | \sum_{i=1}^N m_i \cosh \theta_i u_i = 0, \quad \frac{u_1}{\cosh \theta_1} \leq \frac{u_2}{\cosh \theta_2} \leq \cdots \leq \frac{u_N}{\cosh \theta_N} \right\}.$$ 

Endow $\mathbb{R}^N$ with the metric by the matrix $\tilde{M}$. Then the cone $K$ is in the $(N-1)$-dimensional subspace perpendicular to $v_1$. Note that $v_2 \in K$. First, by equation (2), we have $0 = \sum_{i=1}^N m_i x_i w_i = \sum_{i=1}^N m_i \sinh \theta_i \cosh \theta_i$. Second, since $\tanh \theta$ is an increasing function, we have $\frac{\sinh \theta_1}{\cosh \theta_1} \leq \frac{\sinh \theta_2}{\cosh \theta_2} \leq \cdots \leq \frac{\sinh \theta_N}{\cosh \theta_N}$. The boundary $\partial K$ consists of points for which one or more equalities hold. However, except for the origin, at least one inequality must hold (otherwise $u = k(\cosh \theta_1, \cdots, \cosh \theta_N) = k v_1$).

Consider a boundary point with

$$\frac{u_1}{\cosh \theta_1} \leq \cdots \leq \frac{u_i}{\cosh \theta_i} = \cdots = \frac{u_j}{\cosh \theta_j} \leq \cdots \leq \frac{u_N}{\cosh \theta_N}.$$
To prove that at this point the flow is pointing inwards, see Figure 1 we need to show \( \frac{\dot{u_j}}{\cosh \theta_j} - \frac{\dot{u_i}}{\cosh \theta_i} > 0 \). Direct computation shows that \( \frac{\dot{u_j}}{\cosh \theta_j} - \frac{\dot{u_i}}{\cosh \theta_i} \) is

\[
\sum_{k=1, k \neq j}^{N} \frac{m_k}{\cosh \theta_j \sinh^2 d_{kj}} \left( u_k - \frac{u_j \cosh \theta_k}{\cosh \theta_j} \right) - \sum_{k=1, k \neq i}^{N} \frac{m_k}{\cosh \theta_i \sinh^2 d_{ki}} \left( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \right)
\]

\[
= \sum_{k=1, k \neq i, j}^{N} m_k \left( \frac{u_k}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{u_j \cosh \theta_k}{\cosh^2 \theta_j} - \frac{u_k}{\sinh^3 d_{ki} \cosh \theta_i} + \frac{u_i \cosh \theta_k}{\cosh^2 \theta_i} \right)
\]

\[
+ \frac{m_i}{\sinh^3 d_{ij} \cosh \theta_j} \left( u_i - \frac{u_j \cosh \theta_i}{\cosh \theta_j} \right) - \frac{m_j}{\sinh^3 d_{ij} \cosh \theta_i} \left( u_j - \frac{u_i \cosh \theta_j}{\cosh \theta_i} \right).
\]

Since \( \frac{u_i}{\cosh \theta_i} = \frac{u_j}{\cosh \theta_j} \), the last two terms are zero, and the first part can be nicely written as

\[
\sum_{k=1, k \neq i, j}^{N} m_k \left( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \right) \left( \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} \right).
\]

Every term in this sum is non-negative:

1. If \( k < i \), then \( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \leq 0 \) and \( \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} < 0 \) by Proposition 7

2. If \( i \leq k \leq j \), then \( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} = 0 \).

3. If \( i < k \), then \( u_k - \frac{u_i \cosh \theta_k}{\cosh \theta_i} \geq 0 \) and \( \frac{1}{\sinh^3 d_{kj} \cosh \theta_j} - \frac{1}{\sinh^3 d_{ki} \cosh \theta_i} > 0 \) by Proposition 7

Moreover, at least one term is strictly positive since at least one inequality in the definition of the cone must hold. Thus we have proved \( \frac{\dot{u_j}}{\cosh \theta_j} - \frac{\dot{u_i}}{\cosh \theta_i} > 0 \) on \( \partial K \) and the boundary point moves into the interior of the cone as required.

This proves that all the other eigenvalues of \( A \) are smaller than \( 2 \lambda \), thus the \( N \) eigenvalues of \( A - 2 \lambda I \) are \( -2 \lambda > 0 \). \( \lambda_3 < 0 \), \( \lambda_4 < 0 \), \( \cdots \), \( \lambda_N < 0 \). Therefore we get

\[
\begin{align*}
n_0(H_\varphi) &= n_0(A-2 \lambda) = 1, \\
n_-(H_\varphi) &= n_-(A-2 \lambda) = N-2, \\
n_+(H_\varphi) &= n_+(A-2 \lambda) = 1.
\end{align*}
\]

This remark completes the proof. \( \square \)

5. THE COMPACTNESS OF CENTRAL CONFIGURATIONS AND A LOWER BOUND FOR THE NUMBER OF CENTRAL CONFIGURATIONS

In this section, we employ Morse inequality to obtain a lower bound for the number of central configurations. The Morse inequality holds for compact manifolds. But the manifold we got is \( S_c \), which is not compact. In \( \mathbb{R}^3 \), this difficulty is overcome by a result known as Shub’s Lemma, which shows that there are no central configurations near the singularity set \( \Delta \) in \( S_c \) for given masses. We first need to extend this result.
Recall that \( S_c = \{ \mathbf{q} \in \mathbb{R}^2 \} \). Let \( X \) be a point in \( \Delta \), \( X = (q_1, \ldots, q_k, q_{k+1}, \ldots, q_{k+2}, \ldots, q_N) \), where \( q_1 \) = \( \ldots \) = \( q_k \), \( q_{k+1} = \ldots = q_{k+2} \), \( q_i = \ldots = q_N \). Assume that \( \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) = c \), i.e., \( X \) belongs to \( I^{-1}(c) \). Our purpose is to show that there is some neighbourhood \( U \) of \( X \) in \( I^{-1}(c) \), such that there are no central configurations in \( S_c \cap U \). We represent a point in such a neighbourhood of \( X \) by \( \mathbf{q} = (q_1, q_2, \ldots, q_N) \),
\[
q_i = (x_i' + \delta_{i1}, y_i' + \delta_{i2}, w_i' + \delta_{i3}),
\]
with \( q_i \in \mathbb{R}^2 \), \( q_i \notin \Delta \), and \( I(q) = c \). The configuration \( X \) defines a partition of the bodies into clusters, where \( m_i, m_j \) are in the same cluster if there is an \( l \) such that \( k_l < i < j \leq k_{l+1} \) or \( * \leq i < j \leq N \).

We can assume that there are at least two clusters away from \( (0, 0, 1) \). If there is no such cluster, then \( q_1 = \ldots = q_N = (0, 0, 1) \). This contradicts with the fact that \( \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) = c \). If there is only one such cluster, say, \( (x_1', y_1', w_1') \neq (0, 0, 1) \), and \( q_{k+1} = \ldots = q_N = (0, 0, 1) \), then equation (2), \( \sum_{i=1}^{N} m_i x_i w_i = 0 \), can’t be satisfied for \( q \) sufficiently close to \( X \), since
\[
\sum_{i=1}^{N} m_i x_i w_i \approx \sum_{i=1}^{k_1} m_i x_i' w_i' + \sum_{i=k_1+1}^{N} m_i x_i' w_i' = x_1' w_1' \left( \sum_{i=1}^{k_1} m_i \right).
\]
Thus \( q \) can’t be a central configuration.

**Theorem 5.** For fixed masses \( m_1, \ldots, m_N \) on \( \mathbb{R}^2 \), there is a neighbourhood of \( \Delta \) in \( S_c \) that contains no central configurations.

**Proof.** We need to show that the function \( U = \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij} \) restricted to the \((2N-1)\)-sphere \( I^{-1}(c) \) has no critical points in a neighbourhood of \( X \in \Delta \cap I^{-1}(c) \). Let \( X \) be the point as defined above. We have showed that we can assume that there are at least two clusters away from \( (0, 0, 1) \). Thus, we may require \( k_1 \geq 2 \), \( q_1' \neq q_1 \), and \( q_N' \neq (0, 0, 1) \).

We use \((x, y)\) as the local coordinates of \( \mathbb{R}^2 \) and let \( \mathbf{q}_i = (x_i, y_i) \). We proceed as follows: It is easy to find that the differential of \( U \) is
\[
dU = \sum_{i=1}^{N} \frac{\partial U}{\partial x_i} dx_i + \frac{\partial U}{\partial y_i} dy_i = \sum_{i=1}^{N} \left( \sum_{j=1, j \neq i}^{N} m_i m_j \frac{x_j - \frac{w_j}{w_i} x_i}{\sinh^2 d_{ij}} dx_i + m_i m_j \frac{y_j - \frac{w_j}{w_i} y_i}{\sinh^2 d_{ij}} dy_i \right).
\]
For a point \( \mathbf{q} \in S_c \) that approaches \( X \), we will pick a bounded vector \( \mathbf{v}(\mathbf{q}) = (v_1, v_2, \ldots, v_N) \in T_q S_c \) such that \( dU \mathbf{v} \rightarrow -\infty \).

If this is done, then we can conclude that any point \( \mathbf{q} \in S_c \) sufficiently close to \( X \) can’t be a critical point of \( U|_{S_c} \). Let \( \mathbf{v}_i = (v_{i1}, v_{i2}) = v_{i1} \frac{\partial}{\partial x_i} + v_{i2} \frac{\partial}{\partial y_i} \). We do this by letting
(1) \( v_i = w_i^2 q_i = w_i^2(x_i, y_i), 1 \leq i \leq k_1, \)
(2) \( v_{k_1+1} = v_{k_1+2} = \ldots = v_{s-1} = 0, \)
(3) \( v_i = w_i v_0, 0 \leq i \leq N, \)

where \( v_0(\sum_{i=1}^N m_i w_i x_i) + v_02(\sum_{i=1}^N m_i w_i y_i) = -\sum_{i=1}^{k_1} m_i v_i r_i^2. \) Note that this is a linear equation of \((v_01, v_02)\) for any given \( q \) and that the coefficients have the property

\[
(\sum_{i=1}^N m_i w_i x_i, \sum_{i=1}^N m_i w_i y_i) \approx (\sum_{i=1}^N m_i)(w_i x_i, w_i y_i) \neq (0, 0),
\]

for \( q \) sufficiently close to \( X. \) So we can always find such a \( v_0 = (v_01, v_02). \) The vector \( v \) constructed in this way is bounded and it is in \( T_q S_c, \) since

\[
dIv = \sum_{i=1}^N m_i (x_i v_i + y_i v_i^2) = \sum_{i=1}^{k_1} m_i v_i r_i^2 + v_01(\sum_{i=1}^N m_i w_i x_i) + v_02(\sum_{i=1}^N m_i w_i y_i) = 0.
\]

Let us show that \( dUv \to -\infty \) for \( q \) sufficiently close to \( X. \) Note that we can write \( (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) = \sum_{j=1}^N \frac{m_j m_{ij}}{\sinh^3 d_{ij}} (q_j - w_i q_i). \) Let \( \cdot \) be the inner product in \( \mathbb{R}^2. \) Then

\[
dUv = \sum_{i=1}^{k_1} (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot v_i + \sum_{i=k_1+1}^{s-1} (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot v_i + \sum_{i=s}^N (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot v_i.
\]

The first sum goes to \(-\infty\) when \( q \to X. \) Explicitly, \( \sum_{i=1}^{k_1} (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot v_i \) is

\[
= \sum_{1 \leq i < j \leq k_1} \frac{m_i m_j}{\sinh^3 d_{ij}} (q_j - w_i q_i) \cdot v_i + \sum_{i=1}^{k_1} \sum_{j=k_1+1}^N \frac{m_i m_j}{\sinh^3 d_{ij}} (q_j - w_i q_i) \cdot v_i
\]

\[
= \sum_{1 \leq i < j \leq k_1} \frac{m_i m_j}{\sinh^3 d_{ij}} ((w_i^2 + w_j^2) q_i \cdot q_j - w_i w_j (q_i \cdot q_i + q_j \cdot q_j)) + O(1),
\]

where \( O(1) \) means a bounded term. Note that \((w_i^2 + w_j^2) q_i \cdot q_j - w_i w_j (q_i \cdot q_i + q_j \cdot q_j)\) is

\[
(x_i x_j + y_i y_j)(w_i^2 + w_j^2) - w_i w_j (r_i^2 + r_j^2)
\]

\[
= (w_i w_j - \cosh d_{ij})(w_i^2 + w_j^2) - w_i w_j (w_i^2 + w_j^2 - 2)
\]

\[
= - \cosh d_{ij}(w_i^2 + w_j^2) + (w_i^2 + w_j^2) - (w_i^2 + w_j^2) + 2 w_i w_j
\]

\[
= (w_i^2 + w_j^2)(1 - \cosh d_{ij}) - (w_i - w_j)^2 \leq (w_i^2 + w_j^2)(1 - \cosh d_{ij}).
\]
When \( \mathbf{q} \) approaches \( X \), \( d_{ij} \) approaches 0 for \( 1 \leq i < j \leq k_1 \). Thus
\[
\sum_{i=1}^{k_1} \left( \frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i} \right) \cdot \mathbf{v}_i \leq \sum_{1 \leq i < j \leq k_1} \frac{m_im_j}{\sinh^3 d_{ij}} (w_i^2 + w_j^2)(1 - \cosh d_{ij}) + O(1)
\]
\[
\leq \sum_{1 \leq i < j \leq k_1} -\frac{m_im_j}{\sinh^3 d_{ij}} (w_i^2 + w_j^2)(\frac{d_{ij}^2}{2} + \frac{d_{ij}^4}{4!} + \ldots) \to -\infty.
\]

The second sum is obviously zero. The third sum is bounded. Explicitly, \( \sum_{i=1}^{N} (\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i \) is
\[
\sum_{i=1}^{N} \sum_{j=1}^{N-1} \frac{m_im_j}{\sinh^3 d_{ij}} (\mathbf{q}_j - \frac{w_j}{w_i} \mathbf{q}_i) \cdot \mathbf{v}_i + \sum_{1 \leq i < j \leq N} \frac{m_im_j}{\sinh^3 d_{ij}} \left[ (\mathbf{q}_j - \frac{w_j}{w_i} \mathbf{q}_i) \cdot \mathbf{v}_i + (\mathbf{q}_i - \frac{w_i}{w_j} \mathbf{q}_j) \cdot \mathbf{v}_j \right]
\]
\[
= O(1) + \sum_{1 \leq i < j \leq N} \frac{m_im_j}{\sinh^3 d_{ij}} (w_i \mathbf{q}_j \cdot \mathbf{v}_0 - w_j \mathbf{q}_i \cdot \mathbf{v}_0 + w_j \mathbf{q}_i \cdot \mathbf{v}_0 - w_i \mathbf{q}_j \cdot \mathbf{v}_0)
\]
\[
= O(1).
\]

We have shown that \( dU\mathbf{v} \to -\infty \) for \( \mathbf{q} \) sufficiently close to \( X \), where \( \mathbf{v} \in T_{\mathbf{q}} S_c \) is bounded. Thus \( \mathbf{q} \) can’t be a critical point of \( U|_{S_c} \) when \( \mathbf{q} \) is sufficiently close to \( X \), a remark that completes the proof. \( \square \)

We can now extend the result of Smale \[27\] and Palmore \[20\] on the number of central configurations in the Newtonian \( N \)-body problem. We will make use of Morse theory and assume that, for generic masses, the central configurations has nullity 1, the minimum value compatible with the symmetry, see Proposition 6.

Recall that a central configuration on \( \mathbb{H}^2 \) is a critical point of \( U : S_c \to \mathbb{R} \) and that \( S_c = \{ \mathbf{q} \in (\mathbb{H}^2)^N \setminus \Delta | I(\mathbf{q}) = c \} \). The group \( SO(2) \) acts freely on \( S_c \), which reduces \( U \) as a smooth function on the quotient manifold \( \mathcal{M} = (S_c)/SO(2) \).

A Morse function is such a function that all its critical points are nondegenerate. Thus assuming that the all central configurations for generic masses are nondegenerate is the same as assuming that all critical points of \( U : \mathcal{M} \to \mathbb{R} \) are nondegenerate, that is, \( U : \mathcal{M} \to \mathbb{R} \) is a Morse function. Recall that the critical points of \( U : \mathcal{M} \to \mathbb{R} \) correspond to classes of central configurations in a 1-1 manner. Thus the counting of central configurations is the same as the counting of critical points of \( U \) restricted to the quotient manifold.

The Morse inequality is most easily expressed in terms of polynomial generating functions. Define a Morse polynomial
\[
M(t) = \sum_k \gamma_k t^k, \quad \gamma_k = \text{number of critical points of index } k,
\]
and the Poincaré polynomial $P(t) = \sum_k \beta_k t^k$, where $\beta_k$ is the $k$-th Betti number of the manifold. By the Betti numbers, we mean the ranks of the homology groups $H_k(\mathcal{M}, \mathbb{R})$ with real coefficients. Then the Morse inequalities can be written as

$$M(t) = P(t) + (1 + t)R(t),$$

where $R(t)$ is some polynomial with non-negative integer coefficients \cite{14}. Thus the Poincaré polynomial can be used to obtain an estimate of the number of critical points.

The above Morse inequality holds for a compact manifold. The manifold we are interested is $S_c$, a non-compact manifold. Recall that Proposition 5 shows that the critical point set of $U|_{S_c}$ is compact and that, near the boundary of $S_c$, $U$ approaches $+\infty$. Thus we can restrict to a compact set of the form $K = \{q \in S_c : U(q) \leq U_0\}$ for some sufficiently large $U_0$. Therefore, Morse inequality applies.

**Proposition 8.** For the curved $N$-body problem in $\mathbb{H}^2$, the Poincaré polynomial of $S_c/\text{SO}(2)$ is

$$P(t) = (1 + 2t)(1 + (N - 1)t).$$

**Proof.** Consider the homomorphism $\pi : \mathbb{R}^2 \to \mathbb{H}^2$, $\pi(x, y) = (x, y, \sqrt{x^2 + y^2 + 1})$. The map induces a homomorphism from $(\mathbb{R}^2)^N$ to $(\mathbb{H}^2)^N$, which we still denote by $\pi$. Thus the function $I : (\mathbb{H}^2)^N \to \mathbb{R}$ induces a function $\bar{I} : (\mathbb{R}^2)^N \to \mathbb{R}$ by $\bar{I}(q) = I(\pi q)$, where $q$ is a point in $(\mathbb{R}^2)^N$. Note that $\pi$ is also a homomorphism between the singularity set $\Delta$ in $(\mathbb{H}^2)^N$ and $\bar{\Delta} = \cup_{1 \leq i < j \leq N} \{q \in (\mathbb{R}^2)^N : q_i = q_j\}$. Then $S_c \simeq \bar{I}^{-1}(c) \setminus \bar{\Delta}$. Also $\pi$ commutes with the $\text{SO}(2)$ action on $\mathbb{R}^2$. Thus we obtain

$$S_c/\text{SO}(2) = \mathcal{M} \simeq (\bar{I}^{-1}(c) \setminus \bar{\Delta})/\text{SO}(2).$$

It has been proved that the Poincaré polynomial of $(\bar{I}^{-1}(c) \setminus \bar{\Delta})/\text{SO}(2)$ is $(1 + 2t)(1 + (N - 1)t)$ \cite{17, 20}, thus we see that the Poincaré polynomial of $S_c/\text{SO}(2)$ is $P(t) = (1 + 2t)(1 + (N - 1)t)$. \hfill \Box

The following proof is the same as in the Newtonian $N$-body problem case \cite{17}, we reproduce it here just for completeness.

**Theorem 6.** Suppose that all of the central configurations are nondegenerate for a certain choice of masses in the curved $N$-body problem in $\mathbb{H}^2$. Then on $S_c$ ($c > 0$), there are at least

$$\frac{(3N - 4)(N - 1)!}{2}$$

central configurations, of which at least

$$\frac{(2N - 4)(N - 1)!}{2}$$

are non-geodesic.
Proof. Obviously, the compact subset $\mathcal{K} = \{ q \in S_c : U(q) \leq U_0 \}$ is homotopic to $S_c$, where $U_0$ is sufficiently large. Thus the Poincaré polynomial of $\mathcal{K}/SO(2)$ is $P(t) = (1 + 2t)\cdots(1 + (N - 1)t)$. We have assumed that $U$ is a Morse function on $\mathcal{K}/SO(2)$. We further assume that its Morse polynomial is

$$M(t) = \sum_{k=0}^{2N-2} \gamma_k t^k, \quad \gamma_k = \text{number of critical points of index } k.$$ 

Thus the Morse inequality implies that there is some $R(t)$ with non-negative integer coefficients such that $M(t) = P(t) + (1 + t)R(t)$. The simplest estimate is obtained by setting $t = 1$,

$$\sum_{k=0}^{2N-2} \gamma_k \geq P(1) \geq N!/2,$$

which predicts the existence of the $N!/2$ geodesic central configurations. Theorem 4 shows that the index of each geodesic central configuration in $\mathcal{K}/SO(2)$ is $N - 2$. Then $\gamma_{N-2}$ is at least $N!/2$. On the other hand, the coefficient of $t^{N-2}$ in $P(t)$ is $2 \cdot 3 \cdots (N - 1) = (N - 1)!$.

Let $R(t) = \sum_k r_k t^k$. Then the coefficient of $t^{N-2}$ in $(1 + t)R(t)$ is $r_{N-2} + r_{N-3}$. So

$$r_{N-2} + r_{N-3} + (N - 1)! \geq N!/2.$$

Setting $t = 1$ in the Morse inequality now gives

$$\sum_{k=0}^{2N-2} \gamma_k \geq N!/2 + 2(r_{N-2} + r_{N-3}) \geq 3N!/2 - 2(N - 1)! = \frac{(3N - 4)(N - 1)!}{2}.$$ 

Subtracting $N!/2$ gives the non-geodesic estimate. This remark completes the proof. □

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A Lower Bound for the Number of Central Configurations on $\mathbb{H}^2$

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