Vacuum Effects of Ultra-low Mass Particle Account for Recent Acceleration of Universe

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In recent work, we showed that non-perturbative vacuum effects of a very low mass particle could induce, at a redshift of order 1, a transition from a matter-dominated to an accelerating universe. In that work, we used the simplification of a sudden transition out of the matter-dominated stage and were able to fit the Type Ia supernovae (SNe-Ia) data points with a spatially-open universe. In the present work, we find a more accurate, smooth spatially-flat analytic solution to the quantum-corrected Einstein equations. This solution gives a good fit to the SNe-Ia data with a particle mass parameter $m_h$ in the range $6.40 \times 10^{-33}$ eV to $7.25 \times 10^{-33}$ eV. It follows that the ratio of total matter density (including dark matter) to critical density, $\Omega_0$, is in the range 0.58 to 0.15, and the age $t_0$ of the universe is in the range $8.10 \, h^{-1}$ Gyr to $12.2 \, h^{-1}$ Gyr, where $h$ is the present value of the Hubble constant, measured as a fraction of the value 100 km/(s Mpc). This spatially-flat model agrees with estimates of the position of the first acoustic peak in the small angular scale fluctuations of the cosmic background radiation, and with light-element abundances of standard big-bang nucleosynthesis. Our model has only a single free parameter, $m_h$, and does not require that we live at a special time in the evolution of the universe.

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I. INTRODUCTION

Many attempts have been made to account for the unexpected behavior of the recent expansion of the universe. In a previous paper, we showed that non-perturbative quantum effects in the vacuum may account for the SNe-Ia data suggesting a recent acceleration of the expansion of the universe. We considered a free quantized scalar field (with possible coupling to the scalar curvature, $R$). The propagator of the field included an infinite sum of all terms having at least one factor of $R$. The effective action was given in, which can be referred to by the reader for a more complete exposition of the theory. We found an approximate solution to the effective Einstein gravitational field equations. In that solution, there was a sudden transition (at redshift $z$ of order 1) from an earlier matter dominated stage of the expansion to a mildly inflating de Sitter expansion. This sudden transition model required the universe to be spatially open in order to fit the cosmological data.

Here we find an improved analytic solution to the effective Einstein equations and determine its consequences. Surprisingly, we find that the analytic solution permits the SNe-Ia data to be fit with a spatially-flat model having an early inflationary stage.

In our present model, there is only one free parameter, namely, the ratio of the mass parameter $\overline{m}$ to the present value of the Hubble constant, $H_0$. It is convenient to express this parameter in the form

$$m_h \equiv \overline{m}/h,$$

(1)

where

$$h \equiv H_0/(100 \text{ km}/(\text{s Mpc})),$$

(2)

with $H_0$ expressed in km/(s Mpc). We find that a good fit to the SNe-Ia data is obtained when $m_h$ is in the range $6.40 \times 10^{-33} \text{ eV} < m_h < 7.25 \times 10^{-33} \text{ eV}$. For this parameter range, the ratio of the total matter density (including dark matter) to critical density, $\Omega_0$, is found to be in the range $0.58 > \Omega_0 > 0.15$, and the age $t_0$ of the universe is found to be in the range $8.10 h^{-1} \text{ Gyr} < t_0 < 12.2 h^{-1} \text{ Gyr}$. Also, we find that our model gives reasonable abundances for light elements formed during big-bang nucleosynthesis. In our model, no special coincidence at the present time is necessary to explain why the energy density of matter is of the same order as the vacuum energy density.

The organization of this paper is as follows. In Section II, we give the effective Einstein equation of the model, and summarize the solution of Ref. In Section III, we derive an analytic solution for the scale factor of the model. In Section IV, we compare the model to SNe-Ia data and obtain ranges for $m_h$, $\Omega_0$ and $t_0$. In Section V, we discuss the implications of the model for big-bang nucleosynthesis and for the spectrum of CMBR fluctuations. In Section VI, we show it is probable (independent of fitting the observations) that, at the present time, the vacuum energy density in our model is comparable to the matter density. Finally, our conclusions are given in Section VII.

II. OUR MODEL

In Ref., we consider a free, massive quantized scalar field of inverse Compton wavelength (or mass) $m$, and curvature coupling $\xi$. The effective action for gravity coupled to such a field is obtained by integrating out the vacuum fluctuations of the field. This effective action is the simplest one that gives the standard trace anomaly in the massless-conformally-coupled limit, and contains the explicit sum (in arbitrary dimensions) of all terms in the propagator having at least one factor of the scalar curvature, $R$.

The trace of the Einstein equations obtained by variation of this effective action with respect to the metric tensor take the following form in a Friedmann-Robertson-Walker (FRW) spacetime (in units such that $c = 1$):

$$R + \frac{T_{cl}}{2\kappa_o} - 4\Lambda_o = \frac{\hbar m^2}{32\pi^2\kappa_o} \left\{ (m^2 + \xi R) \ln | 1 + \xi R m^{-2} | ight.$$  

$$- \frac{m^2 \xi R}{m^2 + \xi R} \left( 1 + \frac{3\pi}{2} - \frac{5}{2} \frac{R}{m^2} + \frac{1}{2} \frac{R^2}{m^4} (\xi^2 - (1080)^{-1}) + v \right) \right\},$$  

(3)

where $T_{cl}$ is the trace of the stress tensor of a classical, perfect fluid component containing mixed matter and radiation, $\Lambda_o$ is the cosmological constant, $\kappa_o = (16\pi G)^{-1}$ ($G$ is Newton’s constant), $\xi = \xi - 1/6$, and $v$ is a quantity that vanishes in de Sitter space.
\begin{equation}
  v = \frac{1}{180m^4} \left( \frac{1}{4}R^2 - R_{\mu\nu}R^{\mu\nu} \right).
\end{equation}

Here, the metric is

\begin{equation}
  ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right),
\end{equation}

where \( a(t) \) is the scale factor and \( k = 0, 1, -1 \) give spatially flat, closed and open universes respectively.

The right-hand-side (RHS) of Eq. (3) is proportional to the trace of the quantum contribution to the stress tensor. In Eq. (4), we have assumed that terms in the quantum stress tensor that depend on derivatives of the curvature are negligible. This assumption is consistent with the solution that we obtain.

In [4], we found that Eq. (3) admits a solution in which a matter-dominated FRW universe transits to a stage of the expansion in which the scalar curvature is nearly constant. This led us to construct an approximate cosmological model which makes a \textit{sudden} transition to a de Sitter universe out of a matter-dominated one. In that approximation, we needed a spatially-open cosmology in order to fit high-redshift SNe-Ia data, and to get a reasonable value of the present matter density.

In this paper, we eliminate the assumption of a sudden transition to a de Sitter expansion, by finding an analytic solution to the effective Einstein equations of our model. Consistency with cosmological observations requires that this improved analytic solution has zero (or nearly zero) spatial curvature. This solution consists of a matter-dominated universe smoothly, with continuous first and second derivatives of the scale factor (i.e., \( C^2 \)), joined to a constant scalar curvature expansion that asymptotes to a de Sitter universe at late times. The spatial flatness is consistent with estimates of the first acoustic peak of small angular scale fluctuations in the CMBR, which suggest that the universe is spatially flat (although this issue awaits an ultimate decision at the time of writing, see Ref. [10] for details).

We now turn to the derivation of a smooth solution for the cosmological scale factor, \( a(t) \).

**III. ANALYTIC SOLUTION**

In the following analysis, as in [4], we take \( \Lambda_o = 0 \). For \( \bar{\xi} < 0 \), and with a low value of \( \bar{m} \) \( (\simeq 10^{-33} \text{ eV}) \), we find in [4] that the evolution of the FRW universe is essentially unaffected by the quantum contributions at early times. A perturbative analysis in \( \hbar \) shows that quantum contributions to the stress tensor begin to have a significant effect at a time \( t_j \), when the classical scalar curvature has decreased to a value given roughly by

\begin{equation}
  R_{cl}(t_j) \equiv \frac{\rho_m(t_j)}{2\kappa_o} = \bar{m}^2,
\end{equation}

where

\begin{equation}
  \bar{m}^2 \equiv m^2/(-\bar{\xi}).
\end{equation}

This effect is shown to occur well into the matter-dominated stage of the evolution (therefore, for \( t > t_j \), \( T_{cl} \simeq -\rho_m \)). We further argue that, after the scalar curvature reaches the value \( R_{cl} \) (i.e. for \( t > t_j \)), it stays essentially constant during the later evolution of the universe. More precisely, for all times \( t > t_j \), we find that the scalar curvature has the form

\begin{equation}
  R(t) = \bar{m}^2(1 - \epsilon(t)),
\end{equation}

where

\begin{equation}
  \epsilon(t) = \frac{1}{2} \bar{\xi} \left( -\delta + (\delta^2 + 4\beta^2\bar{\xi}^{-1})^{1/2} \right),
\end{equation}

and

\begin{equation}
  \delta = -\frac{\rho_m(t)}{2m^2\kappa_o} - \bar{\xi}^{-1}, \quad \beta = \frac{r}{2\pi} \left( v(t) - \frac{1}{2160\bar{\xi}} \right).
\end{equation}

Here,
\[ r = \frac{m^2}{m_{Pl}}, \]  

\( m_{Pl} \) being the Planck mass. As shown in [4], it is straightforward to verify that \( \epsilon(t) \) is of order \( r \) for \( t > t_j \) and of order \( \sqrt{\tau} \) at \( t = t_j \). For the low value of \( r \) under consideration, it follows that \( \epsilon(t) \ll 1 \) for all \( t \geq t_j \). Thus the scalar curvature stays nearly constant at the value \( \tilde{m}^2 \) for \( t \geq t_j \).

Despite the constancy of the scalar curvature, the quantum contribution to the stress tensor increases dramatically from the time \( t_j \) to the present time \( t_0 \). To see this, consider the trace of the quantum stress-tensor, \( T_q \), defined by \( T_q/(2\kappa_o) = -(\text{RHS of (3)}) \). It follows from (3) that (with \( \Lambda \equiv 0 \)),

\[
T_q = -2\kappa_o R - T_{cl} = -2\kappa_o \tilde{m}^2 + \rho_m + O(\epsilon),
\]

where the second approximate equality holds for \( t \geq t_j \). Assuming (as we find) that the transition time \( t_j \) occurs at a redshift \( z_j \) of order 1, and given that \( \rho_m(t_j) \simeq 2\kappa_o \tilde{m}^2 \) (from Eq. (3)), we obtain the matter density at the present time, \( \rho_m(t_0) \), as

\[
\rho_m(t_0) = -T_{cl}(t_0) = (a(t_j)/a(t_0))^3 \rho_m(t_j) \simeq (1 + z_j)^{-3}2\kappa_o \tilde{m}^2
\]

\[
\simeq \kappa_o \tilde{m}^2/4 + O(\epsilon).
\]

Thus, according to Eq. (12), \( T_q \) grows from \( T_q(t_j) \simeq -2\kappa_o \tilde{m}^2 + 2\kappa_o \tilde{m}^2 \simeq 0 \) at time \( t_j \) to \( T_q(t_0) \simeq -2\kappa_o \tilde{m}^2 + (1/4)\kappa_o \tilde{m}^2 \simeq -(7/4)\kappa_o \tilde{m}^2 \simeq 7T_{cl}(t_0) \) at the present time \( t_0 \). Thus, we find that the quantum contribution to the stress tensor grows from a negligible value at a redshift of order 1 to a value exceeding the classical contribution at the present time.

To obtain the scale factor \( a(t) \), we consider a spatially flat \((k = 0)\) cosmology, and find the constant scalar curvature solution for \( t \geq t_j \) that joins to the usual matter-dominated solution at time \( t_j \), with continuous first and second derivatives. The time \( t_j \) obtained from Eq. (3), using \( R_{cl}(t) = (4/3)t^{-2} \) for \( t \leq t_j \) in the spatially flat matter-dominated universe, is

\[
t_j = (2/\sqrt{3}) \tilde{m}^{-1}.
\]

For such a universe the Hubble constant \( H(t) = (2/3)t^{-1} \), has the value at \( t_j \) given by

\[
H(t_j) = \tilde{m}/\sqrt{3}.
\]

For \( t > t_j \), we now obtain the unique solution to the constant scalar curvature equation

\[
R = 6 \left( \frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \tilde{m}^2,
\]

that satisfies

\[
H(t_j) = \dot{a}(t_j)/a(t_j) = \tilde{m}/\sqrt{3}.
\]

Continuity of the scalar curvature at \( t_j \) ensures that the second derivative of the scale factor is also continuous. This solution is

\[
a(t) = a(t_j) \sqrt{\sinh \left( \frac{\tilde{m}}{\sqrt{3}} - \alpha \right) / \sinh \left( \frac{2}{3} - \alpha \right)}, \quad t > t_j,
\]

where

\[
\alpha = \frac{2}{3} - \tanh^{-1} \left( \frac{1}{2} \right) \approx 0.117.
\]

We have verified that this solution also satisfies the remaining Einstein equations up to terms of order \( \epsilon \). According to Eq. (15), the expansion approaches a de Sitter expansion at late times (i.e., \( t\tilde{m} \gg 1 \)). Furthermore, it has the property that the deceleration parameter \( q_0 \equiv -\dddot{a}/\dot{a}^2 \) is negative (i.e., the universe is accelerating) for all \( t > t_a \), where
Also, the solution joins in a smooth ($C^2$) manner to the usual spatially flat matter-dominated solution for $t < t_j$, i.e.

$$a(t) = a(t_j) (t/t_j)^{2/3} = \left(\frac{\sqrt{3} m}{2}\right)^{2/3}, \quad t < t_j.$$  \hfill (21)

This solution, given by Eqs. (15), (16) and (21), depends on only one parameter, $m$. It is more accurate than the approximation of a sudden transition to a de Sitter expansion, that we used in [4]. Our analytic solution takes a long time (of order $\overline{m}^{-1}$) to approach a de Sitter expansion. This is the reason that the spatially flat model gives agreement with observations (as we show below). In the approximation of a sudden transition to de Sitter, consistency with observations required a negative spatial curvature. We further note that our analytic solution is not the same as that of a mixed matter-cosmological constant model, for which $a(t) \propto (\sinh(\sqrt{3} \Lambda t/2))^{2/3}$. Further improvements in the accuracy of our solution can be made by an iterative procedure in which the present scale factor $a_0$ is replaced by $a_0^0$.

IV. COMPARISON WITH OBSERVATIONS

For comparison with observations of high-redshift Type Ia supernovae (SNe-Ia), we calculate the luminosity distance-redshift relation for the model defined by Eqs. (18) and (21), and, from it, the difference in apparent and absolute magnitudes as a function of the redshift $z$ of a source. This difference is given by

$$m - M = 5 \log_{10} d_L + 25,$$  \hfill (22)

where $d_L$ is the luminosity distance to the source in Mpc, defined as

$$d_L = (1+z)a_0 r_1,$$  \hfill (23)

where $a_0$ is the present scale factor, and $r_1$ is the comoving coordinate distance from a source at redshift $z$ to a detector at redshift 0. For a spatially flat FRW universe, $r_1$ is given by

$$r_1 = c a_0^{-1} \int_0^z \frac{dz'}{H(z')},$$  \hfill (24)

where $H(z)$ is the Hubble constant as a function of $z$, and the speed of light $c$ is now shown explicitly. It is convenient to introduce the parameter $\bar{m} \equiv H_0/(100 \text{ km/s Mpc})$, where $H_0$ is the present value of the Hubble constant in km/s Mpc. It is also useful to work with the rescaled mass parameter $m_h$ of Eq. (10). We note that $m_h$, being an inverse length, can be measured in units of Mpc$^{-1}$.

For our model, we define $z_j$ as the redshift at time $t_j$. Then we obtain, using Eqs. (18) and (24),

$$h r_{1<}(z) = a_0^{-1} \sqrt{12} m_h^{-1} \chi \int_1^{1+z} \frac{dx}{\sqrt{x^3 + \chi^2}},$$  \hfill (25)

where $r_{1<}(z)$ denotes $r_1(z)$ for $z < z_j$, and

$$\chi = \sinh \left( \frac{cht_0 m_h}{\sqrt{3}} - \alpha \right).$$  \hfill (26)

For $z > z_j$, we use Eqs. (21) and (24) to obtain

$$h r_{1>}(z) = h r_{1<}(z_j) + 2 \sqrt{3} a_0^{-1} m_h^{-1} (1+z_j)^{3/2} \left( (1+z_j)^{-1/2} - (1+z)^{-1/2} \right).$$  \hfill (27)

Eq. (24) gives the luminosity distance for this model, $d_{L1}(z)$, defined by

$$d_{L1}(z) = (1+z)a_0 r_{1<}(z), \quad z < z_j$$

$$= (1+z)a_0 r_{1>}(z), \quad z > z_j.$$  \hfill (28)
The above range on \( m \) and \( m_h \) in the above equations, \( h d_{L1}(z) \) does not depend on \( a_0 \) (for a spatially flat universe). The three parameters that occur in \( h d_{L1} \) are \( h t_0 \), \( z_j \) and \( m_h \). However, we may differentiate Eq. (18) to obtain

\[
H(t) = \frac{\rho_m}{\sqrt{12}} \coth \left( \frac{ct \rho_m}{\sqrt{3}} \right),
\]

The above equation, evaluated at the present cosmic time \( t_0 \), can be rewritten in the form

\[
h t_0 = (3.26 \times 10^6 \text{ yr/Mpc}) m_h^{-1} \left( \tanh^{-1} \left( (865.4 \text{ Mpc}) m_h + \alpha \right) \right),
\]

which gives \( h t_0 \) in years. Also, Eq. (18) yields

\[
1 + z_j \equiv a(t_0)/a(t_j) = \sqrt{\sinh \left( \frac{c h t_0 m_h}{\sqrt{3}} \right) / \sinh \left( \frac{2}{3} - \alpha \right)}.
\]

Eq. (30) shows that \( h t_0 \) is a function of \( m_h \) alone. Therefore, the redshift \( z_j \) as given by Eq. (31) is also a function of \( m_h \) alone. This implies that the function \( h d_{L1}(z) \) depends on a single parameter, \( m_h \). The present ratio of the matter density to critical density, \( \Omega_0 \), is also a function of the same parameter. To see this, we use \( \rho_m \propto a^{-3} \). Then one has

\[
\Omega_0 \equiv \frac{8\pi G}{3H_0^2} \rho_m(t_0) = \frac{8\pi G}{3H(t_j)^2} \rho_m(t_j) \left( \frac{H(t_j)}{H_0} \right)^2 \left( \frac{a(t_j)}{a_0} \right)^3.
\]

Continuity with the spatially flat matter-dominated universe at \( t = t_j \) requires that \( (8\pi G \rho_m(t_j))/(3H(t_j)^2) = 1 \). Therefore

\[
\Omega_0 = (H(t_j)/H_0)^2 (a(t_j)/a_0)^3
\]

\[
= (2.996 \times 10^6 \text{ Mpc}^2) m_h^2 \left( \sinh \left( \frac{c h t_0 m_h}{\sqrt{3}} \right) / \sinh \left( \frac{2}{3} - \alpha \right) \right)^{-3/2},
\]

where we have used Eqs. (15) and (18) in arriving at the last equality. Note that the quantity \( h t_0 \) appearing in the above equation is itself a function of \( m_h \) (see Eq. (29)). Therefore \( \Omega_0 \) is also a function of the parameter \( m_h \), which is the only adjustable parameter in our model.

If the time \( t_j \) is less than the present age \( t_0 \), the monotonic behavior of Eq. (29) implies that \( H(\infty) < H_0 < H(t_j) \). With \( H(\infty) \) and \( H(t_j) \) obtained from Eq. (29), the previous inequality gives

\[
\overline{\rho} > \sqrt{3} H_0/c = 5.78 \times 10^{-4} \text{ Mpc}^{-1},
\]

\[
\overline{\rho} < \sqrt{12} H_0/c = 1.16 \times 10^{-3} \text{ Mpc}^{-1}.
\]

We therefore find that the rescaled mass parameter \( m_h \) is constrained by the model to lie in the range

\[
5.78 \times 10^{-4} \text{ Mpc}^{-1} < m_h < 1.16 \times 10^{-3} \text{ Mpc}^{-1}.
\]

The above range on \( m_h \) can be also expressed in electron-volts (eV), as

\[
3.69 \times 10^{-33} \text{ eV} < m_h < 7.39 \times 10^{-33} \text{ eV}.
\]

We emphasize that the above range of values of \( m_h \) is a consequence of our model, if \( t_0 > t_j \), independent of any fit to cosmological observations.

Plots of \( \Omega_0 \) and \( h t_0 \) vs. \( m_h \), using Eqs. (33) and (30) respectively, are given in Fig. 1 for the range of \( m_h \) values of Eq. (37) above. For the same range of \( m_h \) values, Fig. 2 is a plot of \( \Omega_0 \) vs. \( h t_0 \).

We now use the luminosity distance of Eq. (28) to fit the SNe-Ia data. To do so, it is convenient to normalize the difference \( m - M \) of Eq. (24) to its value in an open universe with \( \Omega_0 = 0.2 \). We define

\[
\Delta(m - M)(z) = 5 \log_{10} \left( \frac{d_{L1}(z)}{d_{L2}(0.2, z)} \right) = 5 \log_{10} \left( \frac{h d_{L1}(z)}{h d_{L2}(0.2, z)} \right),
\]

where
\[
d_{L2}(\Omega_0, z) = 2H_0^{-1}c\Omega_0^{-2}\left(\Omega_0z + (\Omega_0 - 2)\left(\sqrt{1 + \Omega_0z} - 1\right)\right)
= h^{-1}(5995.8\text{ Mpc})\Omega_0^{-2}\left(\Omega_0z + (\Omega_0 - 2)\left(\sqrt{1 + \Omega_0z} - 1\right)\right).
\]

Since \(hd_{L2}(z)\) is a function of \(m_h\) and \(z\), and \(hd_{L2}(0.2, z)\) is a function of \(z\) alone, it follows that \(\Delta(m - M)\) is a function of \(m_h\) and \(z\).

Fig. 3 is a plot of \(\Delta(m - M)\) vs. \(z\), along with a plot of SNe-Ia data acquired from Ref. [5]. A good fit is obtained by any curve that lies between the two dashed curves shown, where curve (a) has \(m_h = 6.40 \times 10^{-33}\text{ eV}\), and curve (b) has \(m_h = 7.25 \times 10^{-33}\text{ eV}\). This gives a value of \(\Omega_0\) in the range
\[
0.15 < \Omega_0 < 0.58,
\]
and a value of \(ht_0\) in the range
\[
8.10\text{ Gyr} < ht_0 < 12.2\text{ Gyr}.
\]

Estimates of \(h\) give \(0.55 < h < 0.75\). For \(h = 0.65\), Eq. (41) gives a range for the age of the universe, namely
\[
12.5\text{ Gyr} < t_0 < 18.8\text{ Gyr}.
\]

A representative solid curve (c) has the value \(m_h = 6.93 \times 10^{-33}\text{ eV}\), which gives \(\Omega_0 = 0.346\) and \(t_0 = 14.8\text{ Gyr}\) (with \(h = 0.65\)). A plot of the scale factor \(a(t)\) for this curve is shown in Fig. 4.

V. RECOMBINATION, NUCLEOSYNTHESIS AND CMBR FLUCTUATIONS

In this section, we first calculate the time of recombination in our model, compare it to the recombination time in a standard open universe, and discuss the implications for big-bang nucleosynthesis. We then show that the apparent angular size of CMBR fluctuations in our model is somewhat smaller than in a spatially flat \(\Omega_0 = 1\) model, although consistent with available data.

In our model, we find that recombination occurs in the matter-dominated era. The redshift at recombination is given by the following expression, discussed in Ref. [12],
\[
z_r = 1048 \left(1 + 0.0124(\Omega_b h^2)^{-0.738}\right) \left(1 + g_1(\Omega_b h^2)^{g_2}\right),
\]
where
\[
g_1 = 0.0783(\Omega_b h^2)^{-0.238}(1 + 39.5(\Omega_b h^2)^{0.763})^{-1},
\]
\[
g_2 = 0.560(1 + 21.1(\Omega_b h^2)^{1.81})^{-1},
\]
and \(\Omega_b\) is the ratio of baryon density and critical density at the present time. For the ranges \(0.0025 < \Omega_b h^2 < 0.25\), \(0.15 < \Omega_0 < 0.60\) and \(0.55 < h < 0.75\), we find,
\[
1055.73 < z_r < 1301.80.
\]

The redshift at matter-radiation equality, \(z_{eq}\), is given by
\[
1 + z_{eq} = \Omega_0 h^2 / (\Omega_r h^2),
\]
where \(\Omega_r\) is the ratio of radiation energy density to critical density at the present time. With a CMBR temperature of 2.726 K at the present time [13], we obtain \(\Omega_r h^2 = 4.16 \times 10^{-5}\). Using the same range of values as above for \(\Omega_0\) and \(h\), we find
\[
1090.14 < z_{eq} < 8115.90.
\]

Therefore, in our model, recombination occurs in the matter-dominated era.

We now invert Eqs. (18) and (21) to obtain a range for the time of recombination, \(t_r\), corresponding to the range of \(z_r\) in Eq. (46). A lower bound on \(t_r\) is obtained with the higher value of \(z_r\) and the lowest value of \(m_h\) consistent with a fit to the SNe-Ia data, i.e., \(m_h = 6.40 \times 10^{-33}\text{ eV}\), and an upper bound is obtained with the lower value of \(z_r\) and the highest value of \(m_h\) consistent with the same data, i.e., \(m_h = 7.25 \times 10^{-33}\text{ eV}\). We find
We compare the above range with the corresponding range of $ht_r$ in a standard, spatially open matter-dominated model, with the same ranges of $\Omega_0$, $h$ and $\Omega_b h^2$ (and therefore the same range of $z_r$). The scale factor for an open matter-dominated model [14] leads to the parametrized equations

$$1 + z_r = \frac{2(1 - \Omega_0)}{\Omega_0} (\cosh \psi_r - 1)^{-1},$$

$$ht_r = (4.89 \times 10^8 \text{ yr}) \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} (\sinh \psi_r - \psi_r),$$

(50)

$\psi_r$ being a dimensionless parameter. The above equations give the following range for $ht_r$ in an open matter-dominated universe,

$$1.79 \times 10^5 \text{ years} < ht_r < 4.89 \times 10^5 \text{ years}.$$

(51)

Comparison of Eqs. (49) and (51) reveals that the range of recombination times for the two models differ by only about 1.7 percent. Furthermore, the scale factors prior to recombination are almost identical in both models (the effect of spatial curvature in the open model is negligible for times prior to the recombination time). Therefore, it is expected that nuclear reaction calculations in big-bang nucleosynthesis would give nearly identical results for the abundances of light elements in both models. Since these calculations are known to give results in agreement with observation for the standard open model, nucleosynthesis calculations in our model would also give results that agree with observations, for the same values of $\Omega_0$, $h$ and $\Omega_b h^2$.

We now turn to the calculation of the apparent angular size of a fluctuation of a fixed proper size $D$ at the surface of last scattering. This apparent angular size in our model will be compared to the apparent angular size of the same fluctuation in a spatially flat FRW model having $\Omega_0 = 1$, since the latter model is known to yield a CMBR fluctuation spectrum consistent with data.

For simplicity, we take the representative values $m_h = 6.93 \times 10^{-33}$ eV, $h = 0.65$, and $\Omega_b h^2 = 0.025$ in the discussion that follows. The chosen value of $m_h$ corresponds to the intermediate curve (c) of Fig. 3, and gives $\Omega_0 = 0.346$. Eq. (43) then gives the redshift at last scattering, as $z_r = 1089$.

The apparent angular size $\theta$ of a fluctuation of fixed proper size $D$ at last scattering is given by

$$\theta = D / (a(t_r)r_1) = D (1 + z_r)^2 / d_{L1}(z_r),$$

(52)

where $d_{L1}(z_r)$ is found from Eq. (28), which also gives

$$hd_{L1}(1089) = 1.017 \times 10^7 \text{ Mpc}.$$  

(53)

Therefore, with $h = 0.65$,

$$\theta = (7.594 \times 10^{-2} \text{ Mpc}^{-1})D$$

(54)

in our model.

In a spatially flat $\Omega_0 = 1$ model, taking $h = 0.65$ and $\Omega_b h^2 = 0.025$, Eq. (43) gives $z_r = 1105$.

The luminosity distance $d_L$ for a spatially flat $\Omega_0 = 1$ cosmology is given by

$$hd_L(z) = (5995.8 \text{ Mpc}) (1 + z) \left(1 - (1 + z)^{-1/2}\right),$$

(55)

which gives

$$hd_L(1105) = 6.433 \times 10^6 \text{ Mpc}.$$  

(56)

Therefore, with $h = 0.65$, and using Eq. (72), we obtain

$$\theta = (1.236 \times 10^{-1} \text{ Mpc}^{-1})D.$$  

(57)

Comparison of Eqs. (54) and (57) reveals that the apparent angular size of a fluctuation of a given proper size at last scattering is about 1.63 times less in our model than in a spatially flat $\Omega_0 = 1$ model. If the first acoustic peak in the small angular scale CMBR spectrum arises from a fluctuation of fixed proper size in all models, then the above result implies that, in our model, this peak would be shifted to a higher mode number relative to that in a $\Omega_0 = 1$ spatially flat model. In the latter model, the first peak is known to occur at mode number $l \simeq 200$, and therefore it would occur at $l \simeq 326$ in our model. These two possibilities are both consistent with the existing data on small angular scale CMBR fluctuations (see, for example, [2] and references therein).
VI. THE QUESTION OF FINE-TUNING

Recent observations of Type-Ia supernovae evidently indicate that the universe has been in an accelerating phase from a redshift of order 1 up to the present time, which implies that the contribution of matter to the total energy density is of the same order of magnitude as the contribution of vacuum energy density at the present time (in spatially flat models, this means that $\Omega_0/(1-\Omega_0)$ is of order 1). Why should this be the case? In mixed matter and cosmological constant models, this question requires an explanation of why the cosmological constant must be fine-tuned to a precise, non-zero value. As pointed out in [4], our model is relatively insensitive to the value of the cosmological constant term. However, in our model, it would appear that, within its allowed range of Eq. (37), the parameter $m_h$ must be finely tuned to give values for $z_j$ and $\Omega_0/(1-\Omega_0)$ that are within an order of magnitude of 1. It should be noted that the allowed range of values of $m_h$ given by Eq. (37) does not by itself constrain $z_j$ and $\Omega_0$. The lowest allowed value of $m_h$ gives $z_j = 0$ and $\Omega_0 = 1$, and the highest allowed value gives $z_j = \infty$ and $\Omega_0 = 0$.

Here, we argue that, in our model, values within an order of magnitude of 1 for $z_j$ and $\Omega_0/(1-\Omega_0)$ are more likely than other values. The argument that follows rests on two assumptions: (i) $t_0 > t_j$, and (ii) all values of $m_h$ within the allowed range given by Eq. (37) have equal a priori probability. Assumption (i) implies the range of $m_h$ values of Eq. (37), and assumption (ii) is reasonable in lieu of a detailed fundamental theory that predicts the value of $m_h$.

Given these two assumptions, one may compute the probability distributions for $z_j$ and $\Omega_0$ in a straightforward manner, since both quantities are functions of $m_h$ alone. By assumption (ii), the probability distribution function for $m_h$, $P(m_h)$, has the form

$$P(m_h) = P_0, \quad 3.69 \times 10^{-33}\text{eV} < m_h < 7.39 \times 10^{-33}\text{eV}$$

$$= 0, \quad \text{otherwise.} \quad (58)$$

Normalization of $P(m_h)$ yields $P_0 = 2.70 \times 10^{32}/\text{eV}$. We then obtain the probability distribution functions $P(z_j)$ and $P(\Omega_0/(1-\Omega_0))$ for $z_j$ and $\Omega_0/(1-\Omega_0)$ respectively, as

$$P(z_j) = P_0/\left\mid \frac{dz_j}{dm_h} \right\mid,$$ 

$$P(\Omega_0/(1-\Omega_0)) = P_0/\left\mid \frac{d(\Omega_0/(1-\Omega_0))}{dm_h} \right\mid, \quad (59) \quad (60)$$

where $z_j(m_h)$ and $\Omega_0(m_h)$ are given by Eqs. (31) and (33) respectively. We now compute the probability that $z_j$ and $\Omega_0/(1-\Omega_0)$ lie between 0.1 and 10 (i.e., they are within an order of magnitude of 1). We find

$$P[0.1 < z_j < 10] = \int_{0.1}^{10} dz_j P(z_j) = 0.851,$$

$$P[0.1 < \Omega_0/(1-\Omega_0) < 10] = \int_{0.1}^{10} d\left(\frac{\Omega_0}{1-\Omega_0}\right) P\left(\frac{\Omega_0}{1-\Omega_0}\right) = 0.619. \quad (61)$$

It is therefore more likely for $z_j$ and $\Omega_0/(1-\Omega_0)$ to be within an order of magnitude of 1 rather than outside that range, assuming that all allowed values of $m_h$ have equal a priori probability.

VII. CONCLUSIONS

We have shown that vacuum effects of a free scalar field of very low mass can account for the observed acceleration (i.e., the SNe-Ia data), while at the same time predicting reasonable values for the age of the universe and the total matter density. Evidently, our model also predicts reasonable light element abundances, and as a consequence of spatial flatness is in agreement with current data on small angular scale CMBR fluctuations.

Better SNe-Ia data would be able to distinguish between our model and mixed matter-cosmological constant models, as well as quintessence models [3]. The curves of $\Delta(m-M)$ vs $z$ are different in our model from those of the other models. Future observations of small angular size CMBR fluctuations may also distinguish between these models.

We emphasize that our model is based on a free renormalizable quantum field and does not require that we live at a very special time in the evolution of the universe. We also note that a graviton field of very low mass may give rise to vacuum effects similar to those of the scalar field we considered here. In contrast, a scalar particle of very high mass with similar characteristics to the present one, may contribute to a stage of early inflation, with reheating and exit from inflation caused by particle production from the vacuum and possibly from other inflationary exit mechanisms.
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FIG. 1. Plots of $\Omega_0$ and $ht_0$ versus $m_h$, with the range of $m_h$ values $3.69 \times 10^{-33}$ eV < $m_h$ < $7.40 \times 10^{-33}$ eV.
FIG. 2. A plot of $\Omega_0$ versus $ht_0$, with the range of $m_h$ values $3.69 \times 10^{-33}$ eV < $m_h$ < $7.40 \times 10^{-33}$ eV.
FIG. 3. A plot of the difference between apparent and absolute magnitudes, as functions of redshift $z$, normalized to an open universe with $\Omega_0 = 0.2$ and zero cosmological constant. The points with vertical error bars represent SNe-Ia data obtained from Ref.[5]. The two dashed curves represent the values (a) $m_h = 6.40 \times 10^{-33}$ eV (lower dashed curve), and (b) $m_h = 7.25 \times 10^{-33}$ eV (upper dashed curve). The solid curve represents the intermediate value (c) $m_h = 6.93 \times 10^{-33}$ eV.
FIG. 4. A plot of $a(t)/a(t_j)$ versus $ht$ in our model universe, for the value $m_h = 6.93 \times 10^{-33}$ eV. The graph terminates at the $x$-coordinate value $ht_0$, $t_0$ being the present age of the universe.