ON MIXING AND ERGODICITY IN LOCALLY COMPACT MOTION GROUPS

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Abstract. Let $G$ be a semi-direct product $G = A \times K$ with $A$ Abelian and $K$ compact. We characterize spread-out probability measures on $G$ that are mixing by convolutions by means of their Fourier transforms. A key tool is a spectral radius formula for the Fourier transform of a regular Borel measure on $G$ that we develop, and which is analogous to the well-known Beurling–Gelfand spectral radius formula. For spread-out probability measures on $G$, we also characterize ergodicity by means of the Fourier transform of the measure. Finally, we show that spread-out probability measures on such groups are mixing if and only if they are weakly mixing.

1. Introduction

The purpose of this paper, which may be viewed as a sequel to [2], is to exploit methods of non-commutative harmonic analysis to study random walks on locally compact groups.

Our starting point is the following spectral radius formula for a regular (complex) Borel measure $\mu$ on a locally compact Hausdorff group $G$:

$$\lim_{n \to \infty} \|\mu^n\|^{1/n} = \sup_U \varrho(\hat{\mu}(U)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n},$$

(1.1)

where $\mu^n := \mu * \cdots * \mu$ denotes $n$-fold convolution of $\mu$ with itself, $(\mu^n)_s$ is the singular part of $\mu^n$ with respect to Haar measure $\lambda_G$ and $U$ runs through a complete set of continuous irreducible unitary representations of $G$ (also $a \vee b := \max\{a, b\}$); here, and throughout the paper, $\lambda_G$ is a fixed left Haar measure on $G$, $\hat{\mu}(U)$ denotes the Fourier transform of $\mu$ at the unitary representation $U$, and $\varrho(\hat{\mu}(U))$ its spectral radius. When $G$ is Abelian, this formula is a direct consequence of Gelfand theory for the commutative Banach algebra $M(G)$ of regular (complex) Borel measures on $G$, and for groups $G$ with a symmetric group algebra $L^1(G)$, (1.1) has been established by Palmer [23] for absolutely continuous measures $\mu$.

In [2] (1.1) was established for compact (Hausdorff) groups $G$ and then used to study random walks on such groups. In particular, one of its uses there was to characterize those regular Borel probability measures $\mu$ on the compact group $G$, for which $\mu^n \to \lambda_G$ in the total variation norm (here we assume that $\lambda_G$ has been chosen to satisfy $\lambda_G(G) = 1$). Of course when $G$ is non-compact, $\mu^n$ cannot converge to Haar measure for any probability measure $\mu$, and neither, of course,
can \((1/n) \sum_{k=0}^{n-1} \mu_k\). Two conditions that have been studied extensively instead are

\[
\lim_{n \to \infty} \| f \ast \mu^n \|_1 = 0 \quad \forall f \in L^1_0(G)
\]

and

\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \ast \mu_k \right\|_1 = 0 \quad \forall f \in L^1_0(G),
\]

where \(L^1_0(G)\) denotes the closed two-sided ideal of functions \(f \in L^1(G)\) with \(\int_G f \, d\lambda_G = 0\). We shall call probability measures \(\mu \in M(G)\) satisfying (1.2) mixing by convolutions, and those satisfying (1.3) ergodic by convolutions, adhering to terminology introduced by Rosenblatt in [25], in view of the fact that \(\mu\) satisfies (1.2) if and only if the associated random walk is mixing, and similarly for ergodicity.

The results of this paper are then as follows:

(A) We establish the spectral radius formula (1.1) for arbitrary regular Borel measures in motion groups (Theorem 4.2).

(B) Using (A), we show that in a motion group \(G = A \times \varphi K\) with \(G\) acting regularly on \(\hat{A}\), a spread-out probability measure \(\mu \in M(G)\) is mixing by convolutions if and only if

\[
\rho(\hat{\mu}(U)) < 1 \quad \forall [U] \in \hat{G} \setminus \{1_G\},
\]

where \(\hat{G}\) is the unitary dual of \(G\) and \(1_G\) designates the trivial representation of \(G\) (Theorem 5.1).

(C) Under the same conditions on \(G\), we show that a spread-out probability measure \(\mu \in M(G)\), is ergodic by convolutions if and only if

\[
1 \notin \sigma(\hat{\mu}(U)) \quad \forall [U] \in \hat{G} \setminus \{1_G\},
\]

where \(\sigma(\hat{\mu}(U))\) denotes the spectrum of the operator \(\hat{\mu}(U)\) (Theorem 6.3).

Finally, as a consequence of our approach, we are able to address a query in [25] (p. 33). We show that in a motion group \(G = A \times \varphi K\) with \(G\) acting regularly on \(\hat{A}\), a spread-out probability measure \(\mu \in M(G)\) is mixing by convolutions if and only if it satisfies the apparently weaker condition

\[
\frac{1}{n} \sum_{k=0}^{n-1} \left| \int_G (f \ast \mu^k) h \, d\lambda_G \right| \to 0 \quad \forall f \in L^1_0(G) \quad \forall h \in L^\infty(G)
\]

(Corollary 7.2). Adhering to standard terminology again, we shall call probability measures satisfying (1.6) weakly mixing by convolutions. Besides Abelian and compact groups, the only other groups we know of for which mixing has been shown to be equivalent to weak-mixing are groups possessing small invariant neighborhoods (SIN); this is done for arbitrary regular Borel probability measures in a recent paper by Jaworski ([14]).

Let us expand briefly and also motivate these results. By motion group we shall mean a group which is a semi-direct product \(G = A \times \varphi K\) with \(A\) Abelian and \(K\) compact; both \(A\) and \(K\) are assumed to be locally compact and Hausdorff here. In all our results concerning motion groups we shall also assume that \(G\) acts regularly on the dual group \(\hat{A}\) of \(A\) [7, p. 183]; this condition is automatically satisfied when \(G\) is second countable ([9]). Recall also that a probability measure \(\mu \in M(G)\) on
a locally compact Hausdorff group is called spread-out if not all of its convolution powers \( \mu^n \) are singular with respect to Haar measure \( \lambda_G \).

When \( G \) is an Abelian group, (1.4) reduces to \( |\hat{\mu}(\chi)| < 1 \) for all non-trivial characters \( \chi \) of \( G \), and (B) in this case is a result of Foguel [6]; also, for Abelian \( G \), (1.5) simply says that \( \hat{\mu}(\chi) \neq 1 \) for all non-trivial characters \( \chi \) of \( G \), and (C) in this case follows from the Choquet–Deny theorem [4] (see also [24] and [25]). On the other hand, when \( G \) is non-commutative, the Fourier transform \( \hat{\mu}(U) \) of a probability measure \( \mu \) is an operator on a Hilbert space, so it is not immediately clear what the appropriate generalizations of these conditions are, to begin with. A natural choice is to try to use the norms of the operators \( \hat{\mu}(U) \) to give conditions for mixing and ergodicity (see [15], [16]), but as it turns out, \( \| \hat{\mu}(U) \| \) does not characterize mixing nor ergodicity. Our proof of (B) uses the spectral radius formula (1.1), through which the spectral radius \( \rho(\hat{\mu}(U)) \) emerges naturally. Let us also remark that (B), (C), and the equivalence of mixing and weak mixing, hold without the spread-out assumption on \( \mu \) when \( G \) is either Abelian or compact, and that the results for \( G \) compact, although not explicitly appearing in the existing literature, once appropriately formulated, also follow from existing results, notably the work of Kawada and Ito [17] (see Section 8).

Our result (C) on ergodicity is closely related to a result of Jaworski [13], who shows that in a locally compact second countable group of polynomial growth, a spread-out probability measure is ergodic by convolutions if and only if it is adapted (see Section 8). In fact, in Section 8 we give a short direct argument showing how our Theorem 6.3 for second countable motion groups, may be obtained from Jarowski’s result, whose proof relies on structure theory for groups of polynomial growth. As adaptedness of a probability measure is known to not be equivalent to ergodicity in general groups however (see Rosenblatt [25]), condition (1.5) may well be worth considering, especially in view of the discussion of Rosenblatt’s example in Section 8. For the same reason, we have retained a proof of Theorem 6.3 relying solely on the methods of the present paper.

Finally, let us also mention that we in fact obtain stronger results in one direction in (B) and (C): (1.4) is necessary for weak mixing, and hence also for mixing, and (1.5) necessary for ergodicity, for spread-out probability measures in any CCR group (Proposition 7.2 and Corollary 6.2; see also Corollary 2.6).

In [24] Ramsey and Weit give different proofs of Foguel’s result on mixing and the Choquet–Deny theorem for Abelian groups, which are more illuminating from the point of view of harmonic analysis. We now briefly contrast the proof of the more involved direction of our result on mixing, namely the sufficiency of the condition (1.4) in (B) above, to the corresponding proof of Ramsey and Weit for Abelian groups \( G \). This will also indicate how the spectral radius formula (1.1) for measures, rather than functions in \( L^1 \), is relevant. The proof of Ramsey and Weit relies on the fact that if \( f \in L^1(G) \) is such that \( \hat{f} \) has compact support not containing 0, then \( f \) factorizes as \( f \ast h = f \), with \( h \in L^1(G) \) and such that \( h \) has compact support not containing 0. They then use the Beurling–Gelfand spectral radius formula for functions in \( L^1 \) and the fact that \( \{1_G\} \) is a set of synthesis to conclude the proof.

In our setting, we work with the representations \( \Lambda_\alpha \), \( \alpha \in \hat{A} \), where \( \Lambda_\alpha \) is obtained by inducing the character \( \alpha \) of the group \( A \) to \( G \). In the proof of Ramsey and Weit, it is crucial that the function \( h \) appearing in the factorization of \( f \) commutes with
\(\mu.\) In a motion group \(G = A \times_{\varphi} K\) however, the center of \(M(G)\) may not contain non-trivial elements of \(L^1(G)\). Yet, for certain \(f \in L^0_0(G)\), we are able to exhibit appropriate measures \(\nu\) in the center of \(M(G)\) which may be used in the place of \(h\) in the above argument. Then we use the spectral radius formula for measures \((1.1)\) to conclude that such \(f\) satisfy the mixing condition if \(\mu\) is spread-out and satisfies \((1.4)\). By a result of Ludwig on sets of spectral synthesis we then obtain that such \(f\) are dense in \(\ker(A_0)\). However, since \(\ker(A_0)\) may be strictly contained in \(L^0_0(G)\), an additional argument is required in order to treat the general \(f \in L^0_0(G)\).

Finally, we mention two more papers that are related. In \([10]\) Kaniuth considers more general groups \(G\), namely locally compact Hausdorff groups of polynomial growth and with a symmetric group-algebra \(L^1(G)\), but only central probability measures \(\mu \in M(G)\) on such groups; for such measures he gives the necessary and sufficient conditions \(\|\hat{\mu}(U)\| < 1\) and \(\hat{\mu}(U) \neq I\), for all non-trivial irreducible \(U\), for \(\mu\) to be mixing and ergodic by convolutions respectively. Also related, although more loosely, is the paper by Jones, Rosenblatt and Tempelman \([15]\), which, however, has a wider scope.

We close this Section by fixing some notation and recalling some terminology, to be used throughout the paper.

**Notation–Terminology.** We shall follow the terminology of \([7]\) regarding group-representations. In particular, by a unitary representation of a locally compact Hausdorff group \(G\) we shall always mean a group homomorphism from \(G\) into the group of unitary operators on some Hilbert space, which is continuous with respect to the strong operator topology. Irreducible will always mean topologically irreducible. Recall also that the unitary dual \(\widehat{G}\) of \(G\) consists of unitary equivalence classes of irreducible unitary representations of \(G\); for such a representation \(U\), we shall denote by \([U]\) the equivalence class in \(\widehat{G}\) to which \(U\) belongs, by \(\mathcal{H}_U\) the representation space of \(U\), and by \(d[U]\) the dimension of \(\mathcal{H}_U\).

Let \(G\) be a locally compact Hausdorff group. We shall denote by \(M(G)\) the Banach-* algebra of complex, regular Borel measures on \(G\). \(L^1(G)\) will stand for the sub-algebra of \(M(G)\) consisting of Haar-integrable Borel functions on \(G\) and \(L^0_0(G)\) for the closed two-sided ideal of \(f \in L^1(G)\) with \(\int_G f \, d\lambda_G = 0\). For a measure \(\mu \in M(G)\), the Fourier transform of \(\mu\) is the bounded linear operator \(\hat{\mu}(U) := \int_G U(x^{-1}) \, d\mu(x)\), defined, for any continuous unitary representation \(U\) of \(G\) on some Hilbert space \(\mathcal{H}_U\), weakly by \(\langle \hat{\mu}(U)u, v \rangle := \int_G \langle U(x^{-1})u, v \rangle \, d\mu(x)\) \((u, v \in \mathcal{H}_U)\). For such a representation \(U\), we shall also write \(U(\mu) := \int_G U(x) \, d\mu(x)\) for the *-representation that \(U\) induces on \(M(G)\); notice that \(\hat{\mu}(U) = \overline{U(\hat{\mu})^*}\), where * denotes adjoint and \(\hat{\mu}\) complex conjugation: \(\hat{\mu}(B) = \overline{\mu(B)}\) for Borel subsets \(B\) of \(G\).

If \(T\) is a bounded linear operator on a Hilbert or Banach space, we shall denote by \(\sigma(T)\) its spectrum and by \(\varrho(T)\) its spectral radius.

If \(\mathcal{H}\) is a Hilbert space, \(B(\mathcal{H})\) shall denote the space of bounded linear operators from \(\mathcal{H}\) to \(\mathcal{H}\).

If \(E\) is a set in a space \(X\), we shall denote by \(1_E\) the function which is 1 on \(E\) and 0 elsewhere; thus, in particular, if \(G\) is a locally compact Hausdorff group, \(1_G\) identifies with the trivial representation of \(G\).

When \(G\) is compact, we will always assume that \(\lambda_G\) has been chosen to satisfy \(\lambda_G(G) = 1\).
Finally, all groups considered in the paper will be assumed to have Hausdorff topologies, without further notice.

2. A Necessary Condition for Mixing in General Groups

In this Section we give some necessary conditions for mixing by convolutions for general groups $G$. We shall use the notion of a quasi-compact operator, and recall the definition here. This class of operators was introduced by Kryloff and Bogoliúboff ([19] [20]).

Definition 2.1. A linear operator $T$ on a Banach space $X$ is quasi-compact if there exists $n \in \mathbb{N}$ and a compact operator $Q$ on $X$ such that $\|T^n - Q\| < 1$.

We single out the following property of quasi-compact operators (see the Remarks following Theorem 2.2.8 and the discussion following Theorem 2.2.7 of [18]).

Lemma 2.2 (Yosida–Kakutani [20]). Let $T$ be a quasi-compact operator on a Banach space $X$ such that $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$. Then either $\sigma(T) \subseteq \{0\}$ or $\{z \in \sigma(T): |z| = 1\}$ contains only eigenvalues of $T$.

The following Lemma sheds then some light on the role played by the spread-out condition:

Lemma 2.3. Let $G$ be a locally compact CCR group. If $\mu$ is a spread-out probability measure in $M(G)$, then $\widehat{\mu}(U)$ is quasi-compact for any $[U] \in \widehat{G}$.

Proof. Write $\mu^n = (\mu^n)_{a.c.} + (\mu^n)_s$ with $(\mu^n)_s \perp \lambda_G$ and $(\mu^n)_{a.c.} \ll \lambda_G$, $n \in \mathbb{N}$. If $\mu^n \neq (\mu^n)_s$ for some $n$, then

$$\|\widehat{\mu}(U)^n - (\mu^n)_{a.c.}(U)\| = \|\widehat{(\mu^n)_s(U)}\| \leq \|(\mu^n)_s\| < 1,$$

and $(\mu^n)_{a.c.}(U)$ is compact.

We shall also use the following fact:

Lemma 2.4. Let $G$ be a locally compact group. Then, for any $[U] \in \widehat{G} \setminus \{1_G\}$, there exists $h \in \mathcal{H}_U$ such that the set $\{U(f)h: f \in L^1_0(G)\}$ is dense in $\mathcal{H}_U$.

Proof. For any $h \neq 0$, the subspace $\{U(f)h: f \in L^1_0(G)\}$ of $\mathcal{H}_U$ is invariant under $U$, and hence it is either $\{0\}$ or dense; thus, we only have to exclude the possibility that it is $\{0\}$ for every $h$. Since $L^1_0(G)$ has co-dimension one in $L^1(G)$, if these subspaces are all trivial, then $U$ is one-dimensional, and the representation $f \mapsto U(f)$ of $L^1(G)$ has kernel $L^1_0(G)$. But the only one-dimensional representation of $G$ for which the corresponding representation of $L^1(G)$ has kernel $L^1_0(G)$ is $1_G$.

Proposition 2.5. Let $G$ be a locally compact group, and let $\mu$ be a probability measure in $M(G)$ which is mixing by convolutions. Then:

(i) $\widehat{\mu}(U)^n \to 0$ in the strong operator topology, for any $[U] \in \widehat{G} \setminus \{1_G\}$.

(ii) $\sigma(\widehat{\mu}(U)) \subseteq \{0\}$ for any $[U] \in \widehat{G} \setminus \{1_G\}$ for which $\widehat{\mu}(U)$ is quasi-compact.

Proof. (i) Let $[U] \in \widehat{G} \setminus \{1_G\}$ and fix $h \in \mathcal{H}_U$. Let also $\epsilon > 0$ be given. By Lemma 2.4 there exists $f \in L^1_0(U)$ and $h' \in \mathcal{H}_U$ such that $\|U(f)h' - h\| < \epsilon/2$. Set $g(x) := \Delta_G(x^{-1})f(x^{-1})$, where $\Delta_G$ is the modular function of $G$. Then $\widehat{g}(U) = U(f)$, whence

$$\|\widehat{\mu}(U)^n h\| \leq \|\widehat{\mu}(U)^n \widehat{g}(U)h'\| + \|\widehat{\mu}(U)^n(\widehat{g}(U)h' - h)\| < \|g * \mu^n\|_1 + \epsilon/2,$$

Finally, all groups considered in the paper will be assumed to have Hausdorff topologies, without further notice.
and this is \( < \epsilon \) for all sufficiently large \( n \), since \( g \in L^1(G) \).

(ii) Fix \([U] \in \hat{G} \setminus \{1_G\}\) and suppose that \( \hat{\mu}(U) \) is quasi-compact. By (i), \( \hat{\mu}(U) \) cannot have eigenvalues of modulus one. It then follows from Lemma 2.2 that \( g(\hat{\mu}(U)) < 1 \). \( \square \)

Note that as a result of Proposition 3.2, and Lemma 2.3 one immediately obtains the necessity of condition (1.4) for mixing of spread-out measures on CCR groups:

**Corollary 2.6.** Let \( G \) be a locally compact CCR group. If \( \mu \) is a spread-out probability measure in \( M(G) \) which is mixing by convolutions, then \( g(\hat{\mu}(U)) < 1 \) for all \([U] \in \hat{G} \setminus \{1_G\}\).

Corollary 2.6 will in fact be subsumed by the stronger result of Proposition 7.1.

### 3. Unitary Representations of Motion Groups

#### 3.1. Locally Compact Motion Groups

Let \( K \) be a compact group, \( A \) an Abelian group, and consider the semi-direct product \( G = A \times \varphi K \). So \( \varphi: K \to \text{Aut}(A) \) is assumed to be a group homomorphism, and we shall write \( \varphi_\kappa \in \text{Aut}(A) \) for the image of the element \( \kappa \in K \) under \( \varphi \). The group operation on \( G \) is given by

\[
(a_1, \kappa_1) \cdot (a_2, \kappa_2) := (a_1 + \varphi_{\kappa_1}(a_2), \kappa_1\kappa_2),
\]

and the mapping \( (a, \kappa) \mapsto \varphi_\kappa(a) \) is assumed to be continuous.

Left Haar measure \( \lambda_G \) on \( G \) is the product \( \lambda_A \otimes \lambda_K \) of the Haar measures on \( A \) and \( K \) respectively, owing to the fact that we assume \( K \) compact and \( A \) abelian [15, 15.29]. Furthermore, using standard arguments, it is not hard to see that each \( \varphi_\kappa \) must be measure-preserving; i.e., if \( \varphi_\kappa(\lambda_A) = \lambda_A \circ \varphi_\kappa^{-1} \) denotes the measure defined by \( \varphi_\kappa(\lambda_A)(B) := \lambda_A(\varphi_\kappa^{-1}(B)) \) for Borel subsets \( B \) of \( A \), then

\[
\varphi_\kappa(\lambda_A) = \lambda_A \quad \forall \kappa \in K
\]

(see [15, 15.29] again).

Throughout, we shall be using additive notation for Abelian groups and multiplicative notation for other groups. 1 shall denote the neutral element of the group \( K \) and 0 the neutral element of \( A \). Finally, if \( a \in A \) and \( \alpha \in \hat{A} \) is a character of the Abelian group \( A \), we shall also use the notation

\[
\langle a, \alpha \rangle := \alpha(a).
\]

#### 3.2. Unitary Representations of Motion Groups

The action \( \varphi \) of the group \( K \) on \( A \) determines an action of \( K \) on the dual group \( \hat{A} \) of \( A \) through \( \alpha \mapsto \varphi_\alpha(\alpha) := \alpha \circ \varphi_\alpha^{-1} \).

For \( \alpha \in \hat{A} \), we shall denote by \( A_\alpha \) the induced representation \( \text{ind}^{\hat{G}}_{\hat{A} \times \{1\}}(\alpha) \) on \( G \). This may be realized on \( L^2(K) \) as follows. For \((a, \kappa) \in G \) and \( \phi \in L^2(K) \),

\[
[A_\alpha(a, \kappa)\phi](\kappa') = \langle a, \varphi_{\kappa'}(\alpha) \rangle \cdot [L_K(\kappa)\phi](\kappa') = \langle a, \varphi_{\kappa'}(\alpha) \rangle \cdot \phi(\kappa^{-1}\kappa'),
\]

\( \kappa' \in K \), where \( L_K \) denotes the left regular representation of \( K \) on \( L^2(K) \). Observe that \( A_\alpha \) and \( A_{\alpha'} \) are unitarily equivalent when \( \alpha \) and \( \alpha' \) belong to the same orbit, i.e., when \( \alpha' = \varphi_{\kappa'}(\alpha) \) for some \( \kappa' \in \hat{K} \); in fact

\[
A_{\varphi_{\kappa'}(\alpha)}(a, \kappa) = R_K(\kappa') A_\alpha(a, \kappa) R_K(\kappa')^{-1} \quad \forall (a, \kappa) \in G
\]

for all \( \alpha \in \hat{A} \), \( \kappa' \in K \), where \( R_K \) denotes the right regular representation of \( K \) on \( L^2(K) \).
In all our results concerning motion groups, we shall be assuming that $G$ acts regularly on the dual group $\hat{A}$ of $A$. The irreducible unitary representations of $G = A \times K$ are then as follows.

**Proposition 3.1.** Let $G = A \times K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then any irreducible unitary representation of $G$ is unitarily equivalent to a sub-representation of $\Lambda_\alpha$ for some $\alpha \in \hat{A}$. Furthermore, each $\Lambda_\alpha$ is the direct sum of irreducible unitary representations of $G$. Finally, two irreducible unitary representations $U_1, U_2$ of $G$ are unitarily equivalent only if they are equivalent to sub-representations of $\Lambda_{\alpha_1}$ and $\Lambda_{\alpha_2}$ respectively, with $\alpha_1$ and $\alpha_2$ belonging to the same orbit, i.e., with $\alpha_2 = \varphi_\kappa(\alpha_1)$ for some $\kappa \in K$.

**Proof.** This follows from [7, Theorem 6.42] and induction by stages [7, Theorem 6.14].

We shall also need an analogue of the Riemann–Lebesgue lemma; as we were unable to locate the following version in the literature, we give a proof in the Appendix.

**Theorem 3.2** (Riemann–Lebesgue Lemma). Let $G = A \times K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then, for any $f \in L^1(G)$, one has the following:

(i) Given $\varepsilon > 0$, there exists a compact set $\hat{C} \subseteq \hat{A}$, such that $\|\hat{f}(\Lambda_\alpha)\| < \varepsilon$ for all $\alpha \in \hat{A} \setminus \hat{C}$.

(ii) Given $\varepsilon > 0$ and $\alpha \in \hat{A}$, if $\Lambda_\alpha = \bigoplus_{i \in I} U_i$ is a direct-sum decomposition of $\Lambda_\alpha$ into irreducible unitary representations of $G$, then $\|\hat{f}(U_i)\| < \varepsilon$ for all but finitely many $i \in I$.

Finally, we shall also use the following result, a proof of which is also given in the Appendix.

**Theorem 3.3.** Let $G = A \times K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then, for any $\mu \in M(G)$, the operator-valued function $\alpha \mapsto \hat{\mu}(\Lambda_\alpha)$ is uniformly continuous on $\hat{A}$ with respect to the norm topology on $B(L^2(K))$.

4. **Spectral Radius Formulae in Motion Groups**

4.1. **The Analogue of the Group $C^*$-Algebra for Measures.** We shall need to consider the analogue of the group-$C^*$-algebra $C^*(G)$ for measures on $G$. Let $G$ be an arbitrary locally compact group. For $\mu \in M(G)$, define

$$\|\mu\|_* := \sup_{|U| \in G} \|U(\mu)\|,$$

where $U(\mu) := \int_G U(x) \, d\mu(x)$. One verifies easily that $\mu \mapsto \|\mu\|_*$ is a norm on $M(G)$ (the implication $\|\mu\|_* = 0 \implies \mu = 0$ follows from the injectivity of the Fourier transform [5, 18.2.3]). We shall then denote by $D^*(G)$ the completion of the unital Banach algebra $M(G)$ with respect to this norm. Then, $D^*(G)$ is a unital $C^*$-algebra, and the group-$C^*$-algebra $C^*(G)$ is a closed sub-algebra of $D^*(G)$. For compact groups, $D^*(G)$ has also been considered in [2].

Any $*$-representation of $M(G)$ extends uniquely to a $*$-representation of $D^*(G)$, so in particular, if $U$ is any irreducible unitary representation of $G$, $\mu \mapsto U(\mu)$ extends to a $*$-representation of $D^*(G)$. The Fourier transform $\mu \mapsto \hat{\mu}(U)$ then also extends to $D^*(G)$. 
Finally, we shall also use the following fact (see [11] Theorem 22.11).

**Fact 4.1.** Let $G$ be a locally compact group and denote by $L_G$ its left regular representation on $L^2(G)$. Then $\hat{\mu}(L_G) = 0 \implies \mu = 0$.

4.2. **Spectral Radius Formulae.** Recall that the spectral radius of an element $a$ in a Banach algebra $A$ may be defined by

$$\varrho(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}. \tag{4.1}$$

One then has the following spectral radius formula for measures in motion groups:

**Theorem 4.2.** Let $G = A \times_\varphi K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then, for any $\mu \in M(G)$, one has that

$$\varrho(\mu) = \lim_{n \to \infty} \|\mu^n\|^{1/n} = \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n}$$

$$= \sup_{\alpha \in \hat{A}} \varrho(\hat{\mu}(A_\alpha)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n},$$

where $\varrho(\mu)$ denotes the spectral radius of $\mu$ in the unitary Banach algebra $M(G)$, and $(\mu^n_s)$, the singular part of $\mu^n$ with respect to Haar measure $\lambda_G$.

**Note.** For numbers $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$.

**Lemma 4.3.** Let $G = A \times_\varphi K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then, for any $\mu \in M(G)$,

$$\lim_{n \to \infty} \|\mu^n\|^{1/n} \leq \|\mu_s\| \vee \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)) = \|\mu_s\| \vee \sup_{\alpha \in \hat{A}} \varrho(\hat{\mu}(A_\alpha)). \tag{4.2}$$

**Proof of Theorem 4.2.** Let $\mu \in M(G)$. Since

$$\|\mu^n\| \geq \|\hat{\mu}(U)^n\| \geq \left[\varrho(\hat{\mu}(U))\right]^n$$

for all $[U] \in \hat{G}$ and $n \in \mathbb{N}$, it is clear that $\varrho(\mu) \geq \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U))$. Since also

$$\|\mu^n\| = \|(\mu^n)_{a.c.}\| + \|(\mu^n)_s\| \geq \|(\mu^n)_s\|$$

(cf. [11] Theorem 14.22 for the equality), it follows that

$$\varrho(\mu) \geq \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n}. \tag{4.2}$$

For the reverse inequality apply Lemma 4.3 to the powers $\mu^n$ of $\mu$. For each $n \in \mathbb{N}$,

$$\varrho(\mu^n) \leq \|(\mu^n)_s\| \vee \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)^n),$$

and since in any Banach algebra $\varrho(a^m) = [\varrho(a)]^m$ for all $a$ and $m$, it follows that

$$[\varrho(\mu)]^n = \varrho(\mu^n) \leq \|(\mu^n)_s\| \vee \inf_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)^n).$$

This, together with (4.2) show that

$$\varrho(\mu) = \sup_{[U] \in \hat{G}} \varrho(\hat{\mu}(U)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n}. \tag{4.2}$$

The equality

$$\varrho(\mu) = \sup_{\alpha \in \hat{A}} \varrho(\hat{\mu}(A_\alpha)) \vee \inf_{n \in \mathbb{N}} \|\mu^n_s\|^{1/n}$$

is established in the same way. \qed
Proof of Lemma 4.3. As $L^1(G)$ is a symmetric Banach $*$-algebra [8], Remark 3 of [23] yields the formula

$$\varrho(f) = \lim_{n \to \infty} \|f^n\|_1^{1/n} = \lim_{n \to \infty} \|f^n\|_*^{1/n}$$

for $f \in L^1(G)$, where $\| \|_*$ is the norm

$$\|\mu\|_* := \sup_{|U| \in G} \|\hat{\mu}(U)\|$$

on $M(G)$, defined in Subsection 4.1. Write also

$$\varrho_*(\mu) := \lim_{n \to \infty} \|\mu^n\|_*^{1/n} \quad (\mu \in M(G)),$$

and recall that $D^*(G)$ is the completion of $M(G)$ with respect to the norm $\| \|_*$. Since $D^*(G)$ and $M(G)$ are unital Banach algebras, we have that

$$\varrho_*(\mu) = \sup\{|z|: z \in \mathbb{C} \text{ and } z\delta_e - \mu \text{ is not invertible in } D^*(G)\}$$

and

$$\varrho(\mu) = \sup\{|z|: z \in \mathbb{C} \text{ and } z\delta_e - \mu \text{ is not invertible in } M(G)\}$$

for $\mu \in M(G)$, where $e$ is the neutral element in $G$ and $\delta_e$ the Dirac point-mass at $e$. We shall first show that

$$\varrho(\mu) \leq \varrho_*(\mu) \vee \varrho(\mu_\alpha)$$

for any $\mu \in M(G)$. Indeed, fix $\mu \in M(G)$. If $z \in \mathbb{C}$ with $|z| > \varrho_*(\mu) \vee \varrho(\mu_\alpha)$, then $\delta_e - z^{-1}\mu_\alpha$ is invertible in $D^*(G)$ and $\delta_e - z^{-1}\mu_\alpha$ is invertible in $M(G)$. Set

$$\nu := (\delta_e - z^{-1}\mu_\alpha)^{-1} \ast \mu_{a.e.},$$

and notice that, because $L^1(G)$ is an ideal in $M(G)$, $\nu \in L^1(G)$. Since

$$\varrho_*(\nu) = \varrho_*(\delta_e - z^{-1}\mu_\alpha)^{-1} \ast \mu_{a.e.}$$

the right-hand side is invertible in $D^*(G)$, because the left side is. This shows that $\varrho_*(\nu) \leq \varrho_*(\mu) \vee \varrho(\mu_\alpha)$. By [4.3], then, $\varrho(\nu) = \varrho_*(\nu) \leq \varrho_*(\mu) \vee \varrho(\mu_\alpha)$, since $\nu \in L^1(G)$. Thus if $z \in \mathbb{C}$ with $|z| > \varrho_*(\mu) \vee \varrho(\mu_\alpha)$, then $|z| > \varrho(\nu)$ and the inverse $(\delta_e - z^{-1}\nu)^{-1}$, whose existence in $D^*(G)$ is guaranteed by [4.5], must actually belong to $M(G)$. Now [4.3] also yields that

$$\varrho_*(\nu) = \varrho_*(\delta_e - z^{-1}\nu)^{-1} \ast (\delta_e - z^{-1}\mu_\alpha)^{-1},$$

whence $(\delta_e - z^{-1}\mu)^{-1} \in M(G) \ast M(G) = M(G)$. This shows [4.4].

In order to establish the Lemma, we now only need to show that

$$\varrho_*(\mu) \leq \sup_{|U| \in \hat{G}} \varrho(\hat{\mu}(U)) \vee \|\mu_\alpha\| = \sup_{|U| \in \hat{G}} \varrho(\hat{\mu}(A_\alpha)) \vee \|\mu_\alpha\|$$

Let $\mu_{a.c.}$ denote the absolutely continuous part of $\mu$, and let $\epsilon > 0$ be given. By the Riemann–Lebesgue lemma (Theorem [5.2]), there exists a compact set $\hat{C} \subseteq \hat{A}$ such that $\|\hat{\mu}_{a.c.}(A_\alpha)\| < \epsilon$ for all $\alpha$ in $\hat{A} \setminus \hat{C}$.

Next set

$$r_\alpha(\alpha) := \|\hat{\mu}(A_\alpha)^n\|^{1/n} \quad \text{and} \quad r(\alpha) := \varrho(\hat{\mu}(A_\alpha)).$$
The inequality
\[
\|\hat{\mu}(A_\alpha^n - \hat{\mu}(A_\beta^n)\| \leq \|\hat{\mu}(A_\alpha) - \hat{\mu}(A_\beta)\| \sum_{k=0}^{n-1}\|\hat{\mu}(A_\alpha)^k\|\|\hat{\mu}(A_\beta)\|^{n-k-1} \\
\leq n \|\mu\|^{n-1}\|\hat{\mu}(A_\alpha) - \hat{\mu}(A_\beta)\|
\]
together with the norm-continuity of the operator-valued function \(\alpha \mapsto \hat{\mu}(A_\alpha)\) (Theorem 3.3), show that each \(r_n\) is a continuous function on \(\hat{A}\). The norm-continuity of \(\alpha \mapsto \hat{\mu}(A_\alpha)\) also implies that \(r\) is upper semi-continuous; hence it attains its maximum on \(C\), and let \(r^* = \max_{\alpha \in C} r(\alpha)\). Since the mapping \(\alpha \mapsto r_n(\alpha) - r^*\) is continuous, the sets \(\hat{C}_n := \{\alpha \in \hat{C}: r_n(\alpha) - r^* \geq \epsilon\}\) are compact, and since \(r_n(\alpha) \downarrow r(\alpha) \leq r^*\) for each \(\alpha\), by (4.1), \(\bigcap_{n=1}^\infty \hat{C}_n = \emptyset\); it follows that for some \(n(\epsilon) \in \mathbb{N}\), \(\hat{C}_{n(\epsilon)} = \emptyset\). Then, for \(n \geq n(\epsilon)\), one has that
\[
(4.7) \sup_{\alpha \in \hat{A}}\|\hat{\mu}(A_\alpha)\| \leq \sup_{\alpha \in \hat{C}}\|\hat{\mu}(A_\alpha)\| + \|\hat{\mu}(A_\alpha)\|^{n(\epsilon) + 1} \leq \sup_{\alpha \in \hat{A}}\|\hat{\mu}(A_\alpha)\| + \|\mu_s\|^{n(\epsilon) + 1}.
\]
Since, by Proposition 3.4, any irreducible unitary representation \(U\) of \(G\) is unitarily equivalent to a sub-representation of \(A_\alpha\) for some \(\alpha \in \hat{A}\), (4.7) then implies that
\[
\sup_{[U] \in \hat{G}}\|\hat{\mu}(U)\|^n \leq \sup_{\alpha \in \hat{A}}\|\hat{\mu}(A_\alpha)\| + \|\mu_s\|^{n(\epsilon) + 1},
\]
whence
\[
\|\mu^n\| \leq \sup_{\alpha \in \hat{A}}[\mu(\mu_s) + \|\mu_s\|^{n(\epsilon) + 1}]
\]
for all \(n \geq n(\epsilon)\). Since \(\epsilon\) was arbitrary, this shows that
\[
\mu_s(\mu) \leq \sup_{\alpha \in \hat{A}}[\mu(\mu_s) + \|\mu_s\|^{n(\epsilon) + 1}].
\]
We shall next show that
\[
(4.8) \sup_{\alpha \in \hat{A}}[\mu(\mu_s) + \|\mu_s\|^{n(\epsilon) + 1}],
\]
which will show (4.9) and thus complete the proof.

Fix \(\alpha \in \hat{A}\). By Proposition 3.4, \(A_\alpha\) is a direct sum of irreducible unitary representations of \(G\), say \(A_\alpha = \bigoplus_{i \in I} U_i\). Then, for any \(n \in \mathbb{N}\),
\[
(4.9) \quad [\mu(\mu_s)]^n \leq \sup_{i \in I} [\mu(\mu_s)]^n = \sup_{i \in I} [\mu(U_i)]^n.
\]
Let \(\epsilon > 0\). By Theorem 3.3, there exists a finite set \(I_1 \subseteq I\) such that \(\|\mu_s(\mu_s)\| < \epsilon\) for all \(i \in I_2 := I \setminus I_1\). Choose \(n(\epsilon) \in \mathbb{N}\) so that \(\|\mu(U_i)\|^n \leq [\mu(U_i) + \epsilon]^n\) for
all \( n \geq n(\varepsilon) \) and all \( i \in I_1 \). Then, for \( n \geq n(\varepsilon) \), one has that

\[
\sup_{i \in I} \|\hat{\mu}(U_i)^n\| \leq \sup_{i \in I_1} \|\hat{\mu}(U_i)^n\| \vee \sup_{i \in I_2} \|\hat{\mu}(U_i)^n\|
\leq \sup_{i \in I_1} \left[ g(\hat{\mu}(U_i)) + \varepsilon \right]^n \vee \sup_{i \in I_2} \left( \|\hat{\mu}_n(U_i)\| + \|\hat{\mu}_n(U_i)\| \right)^n
\leq \sup_{[U] \in \hat{G}} \left[ g(\hat{\mu}(U)) + \varepsilon \right]^n \vee (\varepsilon + \|\mu_n\|)^n,
\]

whence by (4.9)

\[ g(\hat{\mu}(A_\alpha)) \leq \sup_{[U] \in \hat{G}} \left[ g(\hat{\mu}(U)) + \varepsilon \right] \vee (\varepsilon + \|\mu_n\|). \]

As \( \alpha \) and \( \varepsilon \) were arbitrary, this establishes (4.8).

We close this Section with a result from \cite{ref22} that we shall use in the sequel (Section 6):

\begin{proposition}
Suppose that \( G = A \times K \) is a motion group with \( G \) acting regularly on \( \hat{A} \). For \( f \in L^1(G) \), let \( \sigma_{M(G)}(f) \) denote the spectrum of \( f \) as an element of the unital Banach algebra \( M(G) \) and \( \sigma_{D^*(G)}(f) \) the spectrum of \( f \) as an element of the unital Banach algebra \( D^*(G) \). Then:

\[ \sigma_{D^*(G)}(f) \cup \{0\} = \sigma_{M(G)}(f) \cup \{0\} \quad \forall f \in L^1(G). \]

\end{proposition}

\begin{proof}
This follows from \cite{ref22} Proposition 10.4.6 (b).
\end{proof}

5. Mixing in Motion Groups

In this Section we prove the following result:

\begin{theorem}
Let \( G = A \times K \) be a motion group with \( G \) acting regularly on \( \hat{A} \). Then, if \( \mu \in M(G) \) is a spread-out probability measure, \( \mu \) is mixing by convolutions if and only if \( g(\hat{\mu}(U)) < 1 \forall [U] \in \hat{G} \setminus \{1_G\} \).

Observe that one direction of the Theorem follows directly from Corollary 2.6 as any motion group with \( G \) acting regularly on \( \hat{A} \) is CCR \cite{ref22} 4.5.2.1.

For the converse, first recall the standard facts that any motion group has a) polynomial growth \cite{ref10} Theorem 1.4, and b) a symmetric group algebra \( L^1(G) \) \cite{ref8}.

Given a probability measure \( \mu \in M(G) \), set

\[ I_\mu := \{ f \in L^1(G) : \| f * \mu^n \|_1 \to 0 \text{ as } n \to \infty \}; \]

\( I_\mu \) is clearly a closed left ideal in \( L^1(G) \), contained in \( L^1_0(G) \). Notice however, that it is not \textit{a priori} clear that \( I_\mu \) is a two-sided ideal in \( L^1(G) \); for this reason, one cannot directly refer to spectral synthesis, as is the case when \( G \) is an Abelian group (cf. \cite{ref23}) or when \( \mu \) is a central measure (cf. \cite{ref16}), and we shall have to use Lemma 5.3 below instead.

To prove Theorem 5.1 we will first show that, if \( \mu \) is spread-out and satisfies \( g(\hat{\mu}(U)) < 1 \forall [U] \in \hat{G} \setminus \{1_G\} \), then \( \ker(A_0) \subseteq I_\mu \), and then deduce from this that all of \( L^1_0(G) \) is contained in \( I_\mu \).

Let \( \hat{G}_0 \) denote the collection of all compact subsets of \( \hat{A} \) not containing \( 0 \in \hat{A} \), and set

\[ I := \bigcup_{\hat{C} \in \hat{G}_0} \{ f \in L^1(G) : \hat{f}(A_\alpha) = 0 \forall \alpha \in \hat{A} \setminus \hat{C} \}. \]
Lemma 5.3. Given a compact subset \( \hat{C} \) of \( \hat{A} \) not containing \( 0 \in \hat{A} \), there exists \( \nu \in Z(M(G)) \) such that \( \hat{\nu}(A_\alpha) = \hat{h}(\alpha) I_{L^2(K)} \), where \( I_{L^2(K)} \) is the identity operator on \( L^2(K) \), and where \( h: A \to \mathbb{C} \) is a (necessarily) \( K \)-invariant \( L^1 \)-function whose Fourier transform satisfies:

\[
\begin{align*}
(i) \quad & \hat{h}(\alpha) = 1 \ \forall \alpha \in \hat{C}; \\
(ii) \quad & \hat{h}(\alpha) = 0 \text{ for all } \alpha \text{ outside a compact set not containing } 0 \in \hat{A}; \\
(iii) \quad & 0 \leq \hat{h} \leq 1.
\end{align*}
\]

Note. \( Z(M(G)) \) denotes the center of \( M(G) \), i.e., those elements of \( M(G) \) which commute with every other element of \( M(G) \).

Proof of Lemma 5.2. Have a \( K \)-invariant function \( h \in L^1(A) \) whose Fourier transform \( h \) satisfies properties \((i)\)–\((iii)\), and let \( \nu \in M(G) \) be the measure on \( G \) defined by \( \nu := (h \, d\lambda_A) \otimes \delta_{\{1\}} \), where \( \otimes \) denotes product-measure and \( \delta_{\{1\}} \) denotes a point-mass at the identity \( 1 \in K \). Then, by (3.2),

\[
[\hat{\nu}(A_\alpha) \phi](\kappa) = \int_A (-a, \phi_\kappa(a)) \phi(\kappa) \, d\lambda_A(a) = \phi(\kappa) \hat{h}(\phi_\kappa(\alpha))
\]

for any \( \alpha \in \hat{A}, \kappa \in K \), and \( \phi \in L^2(K) \) (recall that the representation space of each \( \Lambda_\alpha \) is \( L^2(K) \)). Since \( h \) and \( \lambda_A \) are \( K \)-invariant, we have that \( \hat{h}(\phi_\kappa(\alpha)) = \hat{h}(\alpha) \), and so we finally deduce that

\[
\hat{\nu}(A_\alpha) = \hat{h}(\alpha) I_{L^2(K)} \quad (\alpha \in \hat{A})
\]

where \( I_{L^2(K)} \) is the identity operator on \( L^2(K) \).

\[\square\]

Lemma 5.3. Let \( G \) be as in Theorem 5.1 and let \( \mu \in M(G) \) be a spread-out probability measure with \( \varrho(\hat{\mu}(U)) < 1 \forall [U] \in \hat{G} \setminus \{1_G\} \). Then \( I \subseteq I_\mu \).

Proof. Fix \( f \in I \). Then \( \hat{f}(A_\alpha) = 0 \) for all \( \alpha \in \hat{A} \setminus \hat{C} \) for some compact set \( \hat{C} \subset \hat{C}_0 \), which we fix. Fix a central measure \( \nu \) as in Lemma 5.2 for this \( \hat{C} \), and denote the support of the corresponding function \( \hat{h} \) by \( \hat{S} \). Then \( \hat{f}(A_\alpha) = \hat{f}(A_\alpha) \hat{\nu}(A_\alpha) = \hat{\nu}(A_\alpha) \hat{f}(A_\alpha) \) for all \( \alpha \in \hat{A} \), and since each irreducible unitary representation of \( G \) is contained as a direct summand in some \( \Lambda_\alpha \), it follows that \( \hat{f}(U) \hat{\nu}(U) = \hat{\nu}(U) \hat{f}(U) = \hat{f}(U) \) for any irreducible unitary representation \( U \) of \( G \). Hence \( f \ast \nu = \nu \ast f = f \), by the injectivity of the Fourier transform on \( M(G) \) (\( \| \cdot \| \), defined in Subsection 4.1), is a norm on \( M(G) \). Since \( \nu \) is also central, we then have that

\[
\|f \ast \mu^n\|_1 = \|f \ast (\nu \ast \mu^n)\|_1 = \|f \ast (\nu \ast \mu)\|_1 \leq \|f\|_1 \|\nu \ast \mu^n\|.
\]

On the other hand, by the spectral radius formula of Theorem 4.2

\[
\lim_{n \to \infty} \|(\nu \ast \mu^n)\|^{1/n} = \sup_{\alpha \in \hat{A}} \varrho(\hat{\mu}(A_\alpha)\hat{\nu}(A_\alpha)) \vee \inf_{n \in \mathbb{N}} \|(\nu \ast \mu^n)\|^{1/n}
\]

\[
= \sup_{\alpha \in \hat{A}} \left[ \hat{h}(\alpha) \varrho(\hat{\mu}(A_\alpha)) \right] \vee \inf_{n \in \mathbb{N}} \|(\nu \ast \mu^n)\|^{1/n}
\]

\[
= \sup_{\alpha \in \hat{S}} \left[ \hat{h}(\alpha) \varrho(\hat{\mu}(A_\alpha)) \right] \vee \inf_{n \in \mathbb{N}} \|(\nu \ast \mu^n)\|^{1/n},
\]

and this is < 1 because we are assuming that \( \varrho(\hat{\mu}(U)) < 1 \) for all \([U] \in \hat{G} \setminus \{1_G\}\), and because \( \mu \) is spread-out and \( 0 \leq \hat{h} \leq 1 \). Indeed, since

\[
\|(\nu^n \ast \mu^n)\| \leq \|(\nu^n)* (\mu^n)\| \leq \|\mu^n\|.
\]
and \( \mu \) is spread-out,
\[
\inf_{n \in \mathbb{N}} \| (\nu^n * \mu^n)_n \| < 1.  
\]  

On the other hand, recall from the proof of Lemma 4.3 that the function \( \alpha \mapsto r(\alpha) := \varrho(\hat{\mu}(A_\alpha)) \) is upper semi-continuous and therefore attains its maximum on the compact set \( \hat{\mathcal{S}} \). Fix \( \xi \in \hat{\mathcal{S}} \) such that \( r(\xi) = \max_{\alpha \in \hat{\mathcal{S}}} r(\alpha) \), and observe that \( \xi \neq 0 \) as \( 0 \notin \hat{\mathcal{S}} \). Now recall the argument proving (4.9). By Proposition 3.1, \( A_\xi = \bigoplus_{i \in \mathcal{I}} U_i \) with each \( U_i \) an irreducible unitary representation of \( G \). Assume first that \( \mu \) is not singular with respect to Haar measure. Have \( \epsilon > 0 \) with \( \epsilon < 1 - \| \mu_n \| \), and then a finite set \( \mathcal{I}_1 \subseteq \mathcal{I} \) such that \( \| \mu_{\mathcal{I}_1}(U_i) \| < \epsilon \) for all \( i \in \mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1 \) (use the Riemann–Lebesgue lemma (Theorem 3.2)). Then, by (4.10), and as in (5.4),
\[
\left[ \varrho(\hat{\mu}(A_\xi)) \right]^n \leq \max_{i \in \mathcal{I}_1} \| \hat{\mu}(U_i) \| \lor (\epsilon + \| \mu_n \|)^n
\]
for all \( n \in \mathbb{N} \), and since \( \varrho(\hat{\mu}(U_i)) < 1 \) for each \( i \), because \( \xi \neq 0 \) and therefore \( U_i \neq 1_G \) for all \( i \in \mathcal{I} \), it follows that \( \left[ \varrho(\hat{\mu}(A_\xi)) \right]^n < 1 \) for some sufficiently large \( n \). Hence \( \varrho(\hat{\mu}(A_\xi)) < 1 \), and therefore
\[
\sup_{\alpha \in \hat{\mathcal{S}}} \left[ \hat{h}(\alpha) \varrho(\hat{\mu}(A_\alpha)) \right] \leq r(\xi) = \varrho(\hat{\mu}(A_\xi)) < 1. 
\]

Finally, if \( \mu \perp \lambda_G \), then \( \mu^n \perp \lambda_G \) for some \( n \in \mathbb{N} \), because \( \mu \) is spread-out, and the above argument with \( \mu^n \) in place of \( \mu \) gives again (5.7), since also \( \varrho(x^n) = [\rho(x)]^n \) for any element \( x \) in a Banach algebra. 

It follows from (5.4), (5.5), (5.6) and (5.7), that \( f \in I_{\mu} \).

Let \( \pi_K : G \to K \) be the natural projection \( \pi_K(a, \kappa) = \kappa \). Being continuous (hence Borel measurable), \( \pi_K \) induces a mapping \( \pi_K : M(G) \to M(K) \), given explicitly by \( [\pi_K(\mu)](B) = \mu(\pi_K^{-1}(B)) (B \in \mathcal{B}(K)) \). Given a function \( f : G \to \mathcal{C} \) and \( \kappa \in K \), write \( f_\kappa \) for the function \( f_\kappa : A \to \mathcal{C} \) given by \( f_\kappa(a) = f(a, \kappa) \); then the restriction of \( \pi_K : M(G) \to M(K) \) to \( L^1(G) \) is given by
\[
\pi_K(f)(\kappa) = \int_A f_\kappa d\lambda_A = \hat{f}_\kappa(0) \quad \text{for } \lambda_K\text{-a.e. } \kappa \in K. 
\]

Observe that, since \( A_0(a, \kappa) = L_K(\kappa) \) for all \( a \in A \) and \( \kappa \in K \), where \( L_K \) is the left regular representation of \( K \) on \( L^2(K) \) (see (3.2)), for any measure \( \mu \in M(G) \) we have that
\[
\hat{\mu}(A_0) = \pi_K(\mu)(L_K),
\]
where \( \pi_K(\mu)(L_K) = \int_K L_K(\kappa)^* d\pi_K(\mu)(\kappa) \) is the Fourier transform of the measure \( \pi_K(\mu) \in M(K) \) at the representation \( L_K \) of \( K \). Observe further that one also has that
\[
\pi_K(\mu * \nu) = \pi_K(\mu) * \pi_K(\nu)
\]
for \( \mu, \nu \in M(G) \). Indeed,
\[
\pi_K(\mu * \nu)(L_K) = (\mu * \nu)(A_0) = \hat{\nu}(A_0) \hat{\mu}(A_0) = \pi_K(\nu)(L_K) \pi_K(\mu)(L_K),
\]
and this implies (5.9), by Fact 4.1. Finally observe that the kernel of the mapping \( \pi_K : L^1(G) \to L^1(K) \) coincides with the \( L^1 \)-kernel of \( A_0 \):
\[
\ker(\pi_K) = \ker(A_0).
\]
This also follows directly from (5.8) and Fact 4.1.

**Notation.** If $U$ is any unitary representation of $G$, the $L^1$-kernel of $U$ is $\ker([U]) = \{f \in L^1(G): U(f) = 0\}$. Note that, in particular for $A_0$, we also have that $\ker(A_0) = \{f \in L^1(G): \hat{f}(A_0) = 0\}$.

**Lemma 5.4.** Let $G$ be as in Theorem 5.1 and let $I$ be the ideal defined by (5.2). Then $I$ is dense in the ideal $\ker(A_0)$.

**Proof.** It is shown in [21, Theorem 2] that the hull

$$
\ker(\pi_K) = \{\ker([U]) : [U] \notin \hat{G}, \ker(\pi_K) \subseteq \ker([U])\}
$$

of $\ker(\pi_K)$ is a set of synthesis (see also the Remark following Lemma 3 of [21]). Therefore, by (5.10), it suffices to show that the hull of $I$ is contained in the hull of $\ker(\pi_K)$. Stated in more direct terms, it suffices to show that, if $[U] \in \hat{G}$ and $I \subseteq \ker([U])$, then also $\ker(\pi_K) \subseteq \ker([U])$ (see [21], last line of the proof of Theorem 2 and Theorem 1).

Let $[U] \in \hat{G}$ and suppose that $\ker([U])$ does not contain $\ker(\pi_K)$. For an irreducible unitary representation $V$ of $K$ let $U_V$ denote the (irreducible unitary) representation of $G$ defined by $U_V(a, \kappa) := V(\kappa)$ for all $(a, \kappa) \in G$, and observe that

$$
(5.11) \ker(\pi_K) = \bigcap_{[V] \in \hat{K}} \ker([U_V]),
$$

by (5.8) and the uniqueness of the Fourier transform on $K$. Now $U$ is equivalent to a sub-representation of some $A_\alpha$, by Proposition 3.1 and by (5.11) and our assumption that $\ker(\pi_K) \nsubseteq \ker([U])$ we have that $\alpha \neq 0$, since $A_0$ is the direct sum of copies of the $U_V$, $[V] \in \hat{K}$. There exists $f \in L^1(G)$ with $U(f) \neq 0$. Have a neighborhood $\hat{W}$ of $\alpha$ in $\hat{A}$ such that the closure $\hat{C}$ of $\hat{W}$ is compact and does not contain $0 \in \hat{A}$, and let $\nu \in M(G)$ be a measure as in Lemma 5.2 for this $\hat{C}$. Then $f * \check{\nu} \in I \setminus \ker([U])$, since $A_\alpha(\check{\nu}) = \check{\nu}(A_\alpha)^* \neq 0$ and similarly for $U$. 

**Corollary 5.5.** Let $G$ be as in Theorem 5.1 and $\ker(A_0)$ as in Lemma 5.4. If $\mu \in M(G)$ is a spread-out probability measure with $\rho(\hat{\mu}(U)) < 1 \forall [U] \in \hat{G} \setminus \{1_G\}$, then $\ker(A_0) \subseteq I_\mu$. 

We are now ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** By the Peter–Weyl theorem, $L^2(K)$ decomposes into the direct sum

$$
L^2(K) = \bigoplus_{[V] \in \hat{K}} \mathcal{E}_{[V]},
$$

where $\mathcal{E}_{[V]}$ is the finite-dimensional subspace of $L^2(K)$ spanned by the representative functions $V_{ij} : (V(\kappa)e_j, e_i)$ of any representation $V$ in the equivalence class $[V]$ with respect to any basis $\{e_1, \ldots, e_{d_{[V]}}\}$ of its representation space $\mathcal{H}_V$, and each subspace $\mathcal{E}_{[V]}$ is invariant under the right regular representation $R_K$ of $K$ on $L^2(K)$. Furthermore, if $R_K^{[V]}$ denotes the sub-representation

$$
R_K^{[V]}(\kappa) := R_K(\kappa)|_{\mathcal{E}_{[V]}}
$$
of $R_K$, then $R_K^{[V]}$ may be decomposed into a direct sum of $d_{[V]}$ copies from $[V]$. 
Next fix a Jordan normal form decomposition of the finite-dimensional operator $\pi_K(\mu)(R_K^{[V]}) : E[V] \to E[V]$:

$$E[V] = \text{span}(\phi_{i1}^{[V]}, \ldots, \phi_{ir_1}^{[V]}) + \cdots + \text{span}(\phi_{p1}^{[V]}, \ldots, \phi_{pr_p}^{[V]})$$

$(p$ and $r_1, \ldots, r_p$ depending on $[V])$, where $\phi_{ij}^{[V]}$, $i = 1, \ldots, p$, $j = 1, \ldots, r_i$, are a basis of $E[V]$ and satisfy

$$\pi_K(\mu)(R_K^{[V]})\phi_{ij}^{[V]} = \lambda_i^{[V]}\phi_{ij}^{[V]} + \phi_{ij-1}^{[V]} \quad (1 \leq i \leq r_i, \phi_{00}^{[V]} = 0),$$

where the $\lambda_i^{[V]}$ are eigenvalues of the operator $\pi_K(\mu)(R_K^{[V]})$. Observe that, since $R_K^{[V]}$ is a direct sum of $d_{[V]}$ copies of $[V]$, the eigenvalues $\lambda_i^{[V]}$ are the eigenvalues of $\pi_K(\mu)(V)$. Finally, observe also that, since $\phi_{ij}^{[V]} \in E[V]$, we have that

$$\int_K \phi_{ij}^{[V]} d\lambda_K = 0$$

for all $i, j$ whenever $V \neq 1_K$, by the Shur orthogonality relations.

To prove the Theorem we need to show that $L^1_0(G) \subseteq I_\mu$. By the preceding paragraph, and the Peter–Weyl theorem, the finite linear combinations of the functions $\phi_{ij}^{[V]}$ are dense in $C(K)$ and every $L^p(K)$; hence, by standard arguments, the functions of the form

$$(a, \kappa) \mapsto g_0(a) + \sum_{[V] \in F} \sum_{i,j} g_{ij}^{[V]}(a) \phi_{ij}^{[V]}(\kappa),$$

where $F$ is a finite subset of $K$ not containing the trivial representation $1_K$ and $g_0, g_{ij}^{[V]} \in L^1(A)$ for all $[V] \in F$ and all $i, j$, are dense in $L^1(G)$. From this, it follows that the functions of the form \eqref{eq:6.14} and with $\int_A g_0 \, d\lambda_A = 0$ are dense in $L^1_0(G)$ and $\lambda_0^{[V]} = 0$ belongs to $I_\mu$. Since the function $f(a, \kappa) := g_0(a)$ is in $\ker(A_0)$ if $\int_A g_0 \, d\lambda_A = 0$, and hence in $I_\mu$ by Corollary 5.5, it is then enough to show that any function of the form $f(a, \kappa) = g(a) \phi_{ij}^{[V]}(\kappa)$ with $g \in L^1(A)$ and $V \neq 1_K$ belongs to $I_\mu$. Finally, since $f \in \ker(A_0)$ if $\int_A g \, d\lambda_A = 0$, and therefore $f \in I_\mu$ by Corollary 5.5 again, it suffices to only consider the case $\int_A g \, d\lambda_A \neq 0$, and then we may as well assume that $\int_A g \, d\lambda_A = 1$.

Fix $[V] \in F$ with $V \neq 1_K$, and let us suppress the dependence of $\phi_{ij}^{[V]}$ and $\lambda_i^{[V]}$ on $[V]$ and write $\phi_{ij}$ and $\lambda_i$ instead. We will show that, if $f_{ij} \in L^1_0(G)$ are any functions with $\pi_K(f_{ij}) = \phi_{ij}$, then $f_{ij} \in I_\mu$, which, by the above discussion, proves the Theorem. This we will do for fixed $i$ by induction on $j$. Fix $i$ and recall from \eqref{eq:5.12} that the $\phi_{ij}$ satisfy $\phi_{ij} \in E[V]$ and

$$\pi_K(\mu)(R_K)\phi_{ij} = \lambda_i \phi_{ij} + \phi_{ij-1} \quad (1 \leq j \leq r_m, \phi_{00} = 0),$$

where $R_K$ is the right regular representation of $K$ on $L^2(K)$. Obviously $f_{i0} \in \ker(A_0)$, since $\phi_{00} = 0$. Assume next that $f_{ij-1} \in \ker(A_0)$ ($j \geq 1$). Since for any measure $\nu \in M(K)$, $\bar{\nu}(R_K) \phi = \phi \ast \nu$ for any $\phi \in L^2(K)$, we conclude from \eqref{eq:5.14} and \eqref{eq:5.9} that

$$\pi_K(f_{ij} \ast \mu - \lambda_i f_{ij} - f_{ij-1}) = 0,$$
Proposition 6.1. ergodicity. First, the analogue of Proposition 2.5 is the following: observing that results analogous to those for mixing of Section 2 also hold for \( \lambda \). Recall, however, that

\[
\| f_{ij} * \mu - \lambda_i f_{ij} \|_1 \to 0 \quad (n \to \infty).
\]

(5.16)

Now observe that the numerical sequence \( \| f_{ij} * \mu^n \|_1 \) is non-increasing and bounded by \( \| f_{ij} \|_1 \). It has therefore a limit, \( a \) say, and by (5.16), \( a \) must satisfy \( |\lambda_i| a = a \). Recall, however, that \( \lambda_i \) belongs to the spectrum of the operator \( \pi_K(\mu)(R_K^{(V)}) \), which is the same as the spectrum of \( \pi_K(\mu)(V) \), where \( U^V \) is the irreducible unitary representation of \( G \) defined by \( U(a, \kappa) = V(\kappa) \) for all \( (a, \kappa) \in G \), and so by our condition \( g(\mu(U)) < 1 \ \forall [U] \in \hat{G} \setminus \{1_G\} \) we have that \( |\lambda_i| < 1 \). It follows that \( a = 0 \), and hence \( f_{ij} \in I_\mu \). This concludes the proof of the Theorem. \( \square \)

6. Ergodicity by Convolutions

The main result of this Section is Theorem 6.3. Let us begin, however, by observing that results analogous to those for mixing of Section 2 also hold for ergodicity. First, the analogue of Proposition 2.5 is the following:

**Proposition 6.1.** Let \( G \) be a locally compact group, and let \( \mu \) be a probability measure in \( M(G) \) which is ergodic by convolutions. Then, for any \( [U] \in \hat{G} \setminus \{1_G\} \), \( n^{-1} \sum_{k=0}^{n-1} \hat{g}(U)^k \to 0 \) in the strong operator topology. In particular, the number 1 cannot be an eigenvalue of \( \hat{g}(U) \), for any \( [U] \in \hat{G} \setminus \{1_G\} \).

**Proof.** Fix \( [U] \in \hat{G} \setminus \{1_G\} \), \( h \in H_U \), and \( \epsilon > 0 \). As in the proof of (i) of Proposition 2.5, there exist \( g \in L_0^1(G) \) and \( h' \in H_U \) for which \( \| \hat{g}(U)h' - h \| < \epsilon/2 \). It follows that

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} \hat{g}(U)^k h \right\| \leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} \hat{g}(U)^k h' \right\| + \left\| \frac{1}{n} \sum_{k=0}^{n-1} \left( \hat{g}(U)^k h' - h \right) \right\| + \epsilon/2,
\]

and this is \( < \epsilon \) for all sufficiently large \( n \), since \( g \in L_0^1(G) \) and \( \mu \) is ergodic by convolutions. \( \square \)

Again, one can say more about spread-out measures. Combining Proposition 6.1 with Lemma 2.2 and Lemma 2.3 one obtains the following:

**Corollary 6.2.** Let \( G \) be a locally compact CCR group, and let \( \mu \) be a spread-out probability measure in \( M(G) \) which is ergodic by convolutions. Then \( 1 \notin \sigma(\hat{\mu}(U)) \) for any \( [U] \in \hat{G} \setminus \{1_G\} \).

Our main result concerning ergodicity by convolutions is the analogue of Theorem 7.1 for ergodicity, namely that the necessary condition of the above Corollary is also sufficient for ergodicity in motion groups.

**Theorem 6.3.** Let \( G = A \times \varphi K \) be a motion group with \( G \) acting regularly on \( \hat{A} \). Then, if \( \mu \in M(G) \) is a spread-out probability measure, \( \mu \) is ergodic by convolutions if and only if \( 1 \notin \sigma(\hat{\mu}(U)) \) for any \( [U] \in \hat{G} \setminus \{1_G\} \).
To prove the Theorem we only have to show that, for a spread-out probability measure $\mu \in M(G)$, $1 \notin \bigcup_{U \in \tilde{C} \setminus \{1_G\}} \sigma(\tilde{\mu}(U))$ implies ergodicity, the other direction being a consequence of Corollary 6.2. Let

$$J_\mu := \left\{ f \in L^1(G) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f * \mu^k \right\}.$$ 

Then $J_\mu$ is a closed left ideal in $L^1(G)$ contained in $L^1_0(G)$, and we have to show that $J_\mu = L^1_0(G)$. As in the case of mixing, we will first show that $\ker(A_0) \subseteq J_\mu$ and then deduce from this that all of $L^1_0(G)$ is contained in $J_\mu$. To prove the first assertion, we will use the following Lemma, whose proof we postpone to the end of this Section.

**Lemma 6.4.** Let $\mu$ be a probability measure in $M(G)$, with $\mu \not\perp \lambda_G$ and for which $1 \notin \sigma(\tilde{\mu}(U)) \forall U \in \tilde{G} \setminus 1_G$. Let also $\nu \in Z(M(G))$ be a measure as in Lemma 6.2 for some $\tilde{C} \in \tilde{C}_0$. Then $\delta_e - \mu * \nu$ is invertible in $D^*(G)$, where $\delta_e$ is the Dirac point-mass at the neutral element $e$ of $G$.

**Lemma 6.5.** Let $G$ be as in Theorem 6.3 and $\ker(A_0)$ as in Lemma 5.4. If $\mu \in M(G)$ is a spread-out probability measure for which $1 \notin \sigma(\tilde{\mu}(U)) \forall U \in \tilde{G} \setminus 1_G$, then $\ker(A_0) \subseteq J_\mu$.

**Proof.** Since $J_\mu$ is closed, it suffices, by Lemma 5.4, to show that $I \subseteq J_\mu$, where $I$ is defined in (5.2). Assume first that $\mu$ is not singular with respect to Haar measure $\lambda_G$. Next observe that

$$\frac{1}{n} \sum_{k=0}^{n-1} (f - f * \mu) \mu^k \right\} = \frac{1}{n} \|f - f * \mu\|_1 \leq \frac{2}{n} \|f\|_1 \forall f \in L^1(G),$$

and therefore it suffices to show that, for each $g \in I$, there exists $f \in L^1(G)$ such that $g = f - f * \mu$.

Fix $g \in I$. Then there exists a compact set $\tilde{C} \in \tilde{C}_0$ such that $\tilde{g}(A_\alpha) = 0$ for $\alpha$ not in $\tilde{C}$. Fix a measure $\nu \in Z(M(G))$ as in Lemma 5.2 for this $\tilde{C}$, and set $\nu':= \mu * \nu$. Since $\mu \not\perp \lambda_G$ we also have that $\mu' \not\perp \lambda_G$. Now recall the argument in the proof of Lemma 4.3. Write $\mu' = \mu_{\alpha,c} + \mu'_s$. Then there exists $\delta_e - \mu'_s$ is an invertible element of $M(G)$, because $\|\mu'_s\| < 1$. By the preceding lemma we also have that $\delta_e - \mu'$ is invertible in $D^*(G)$. Set $\nu':= (\delta_e - \mu'_s)^{-1} * \mu_{\alpha,c}$; since

$$\delta_e - \mu'_s)^{-1} * (\delta_e - \mu') = \delta_e - \nu'$$

and the left side is invertible in $D^*(G)$, $\delta_e - \nu'$ is invertible in $D^*(G)$. But $\nu' \in L^1(G)$, and therefore $\delta_e - \nu'$ is invertible in $D^*(G)$ if and only if it is invertible in $M(G)$, by Proposition 6.4. Therefore $\delta_e - \nu' \in M(G)$. It follows from this and (6.1) that

$$(\delta_e - \nu')^{-1} = (\delta_e - (\mu' - \nu'))^{-1} \in M(G) \ast M(G) = M(G).$$

Now set $f := g * (\delta_e - \mu')^{-1} \in L^1(G)$. Then $f - f * \mu' = g$. Furthermore, since $\tilde{f}(A_\alpha) = \delta_e(A_\alpha) - (\mu'(A_\alpha))^{-1} \tilde{g}(A_\alpha) = 0$ for $\alpha \in \tilde{A} \setminus \tilde{C}$ and $\tilde{g}(A_\alpha) = I_{L^2(K)}$ on $\tilde{C}$, it follows that $f * \nu = f$, and therefore $f - f * \mu' = f - f * \mu$. If $\mu$ is singular with respect to Haar measure, then some power $\mu^m$ of $\mu$ is not singular, because $\mu$ is assumed to be spread-out. Replacing $\mu$ by $\mu^m$ in the above
argument yields, for a given \( g \in I \), a function \( f \in L^1(G) \) for which \( g = f - f \ast \mu^m \).

But then
\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} g \ast \mu^k \right\|_1 = \left\| \frac{1}{n} \sum_{k=0}^{n-1} (f \ast f \ast \mu^m) \ast \mu^k \right\|_1
= \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \ast \mu^k - \sum_{k=n}^{n+m-1} f \ast \mu^k \right\|_1 \leq \frac{2m}{n} \| f \|_1,
\]
and therefore \( g \in J_\mu \) again.

**Proof of Theorem 6.3.** By the argument in the proof of Theorem 6.1 it suffices to show that, if \( f_{ij} \in L^1_0(G) \) are any functions with \( \pi_K(f_{ij}) = \phi_{ij} \) for some \( V \in \hat{K} \setminus \{1_K\} \), then \( f_{ij} \in J_\mu \), where the notation is as in that proof. Let us suppress the dependence on \([V]\) and write \( \lambda_i \) and \( \phi_{ij} \) instead of \( \lambda_{ij}^{[V]} \) and \( \phi_{ij}^{[V]} \) again. As shown in the proof of Theorem 5.1, \( f_{ij} \ast \mu - \lambda_i f_{ij} \) is in \( \ker(A_0) \), for all \( j \in \{1, \ldots, r\} \), so by Lemma 6.5
\[
\frac{1}{n} \left\| \sum_{k=0}^{n-1} \left( \lambda_i f_{ij} - f_{ij} \ast \mu \right) \ast \mu^k \right\|_1 \to 0 \quad (n \to \infty),
\]
and hence
\[
\frac{1}{n} \left\| (\lambda_i - 1) \sum_{k=1}^{n-1} f_{ij} \ast \mu^k + \lambda_i f_{ij} - f_{ij} \ast \mu^k \right\|_1 \to 0 \quad (n \to \infty),
\]
for all \( j \). From this, it follows that
\[
\frac{1}{n} |\lambda_i - 1| \left\| \sum_{k=1}^{n-1} f_{ij} \ast \mu^k \right\|_1 \to 0 \quad (n \to \infty),
\]
and as \( \lambda_i \neq 1 \), because we are assuming that \( 1 \notin \sigma(\hat{\mu}(U)) \forall [U] \in \hat{G} \setminus \{1_G\} \) and that \( V \neq 1_K \), we must have that \( f_{ij} \in J_\mu \).

**Proof of Lemma 6.4.** Fix a complete set of mutually inequivalent, irreducible, unitary representations of \( G \), denote it by \( U_G \), and consider the unital \( C^* \)-algebra
\[
C(\hat{G}) := \left\{ (T_U)_{U \in U_G} : T_U \in \mathcal{B} (\mathcal{H}_U), \sup_{U \in U_G} \|T_U\| < \infty \right\},
\]
with norm \( \sup_{U \in U_G} \|T_U\| \), and pointwise operations. Since \( \iota : D^*(G) \to C(\hat{G}) \) given by \( \iota(x) := (U(x))_{U \in U_G} \) is an isometry, \( \delta_e - \mu \ast \nu \) is invertible in \( D^*(G) \) if and only if \( \iota(\delta_e - \mu \ast \nu) \) is invertible in \( C(\hat{G}) \) [7, Proposition 1.23].

Recall Proposition 6.1 and write \([U] \in (A_\alpha) \) if \( U \) is unitarily equivalent to a sub-representation of \( A_\alpha \). Let \( \hat{S} \) denote the support of \( \hat{h} \), which, recall, is compact and does not contain \( 0 \in \hat{A} \). Then \( A_\alpha(\delta_e - \mu \ast \nu) = I_{L^2(K)} - \hat{h}(\alpha)A_\alpha(\mu) = I_{L^2(K)} \) for \( \alpha \) outside \( \hat{S} \), and therefore
\[
[U](\delta_e - \mu \ast \nu)^{-1} = I_{\mathcal{H}_U} \quad \text{for } [U] \in (A_\alpha) \text{ with } \alpha \in \hat{A} \setminus \hat{S}.
\]

Next let \( B'(L^2(K)) \) denote the invertible elements of \( B(L^2(K)) \), and recall that the mapping \( x \mapsto x^{-1} \) is continuous on \( B'(L^2(K)) \) [7, Theorem 1.4]. Since the mapping \( j : \hat{A} \to B(L^2(K)) \) with \( j(\alpha) := A_\alpha(\delta_e - \mu \ast \nu) \) is also continuous (Theorem
By Aaronson et al. [1], weak mixing is equivalent to the following condition:

\[
\text{Proof. Assume that } \alpha \in \Lambda_u \text{ also } \lambda \in \Lambda_u \text{ and hence bounded on the compact } S; \text{ thus we will have that }
\]

\[
\left\| \left[ A_\alpha A_\mu \right] \right\|_{L^2(K)} \leq \sup_{\kappa \in A} \left\| \left[ A_\alpha A_\mu \right] \right\| < \infty \quad \text{for } [U] \in \Lambda_u \text{ and } \alpha \in \hat{S},
\]

and this together with (6.2) will show that \([U(\delta_e - \mu * \nu)]_{U \in \mathcal{M}G} \in \mathcal{C}(G).

It remains to show that \(\hat{A} \subseteq B'(L^2(K))\), i.e., that \(j(\alpha) = A_\alpha(\delta_e - \mu * \nu)\) is invertible for each \(\alpha\), and by the line preceding (6.2) it suffices to only consider \(\alpha \neq 0 \in \hat{S}\). Fix such an \(\alpha\). Then for each \([U] \in \Lambda_u\) we have that \(1 \notin \sigma(U(\mu * \nu))\); for if \(0 \leq h(\alpha) < 1\) then \(\|U(\mu * \nu)\| = h(\alpha)\|U(\mu)\| < 1\), and if \(h(\alpha) = 1\) then \(U(\mu * \nu) = U(\mu)\) and \(1 \notin \sigma(U(\mu))\) by hypothesis, since \(\alpha \neq 0\) implies that \(U \neq 1_G\). Thus \(U(\delta_e - \mu * \nu)\) is invertible for each \([U] \in \Lambda_u\). By Proposition 7.1 \(A_\alpha = \oplus_{i \in I} U_i\) with each \(U_i\) an irreducible unitary representation of \(G\). Recall also that we are assuming that \(\mu\) is not singular with respect to Haar measure. Have \(\epsilon > 0\) with \(\epsilon \leq 1 - \|\mu\|\), and then a finite set \(I_1 \subseteq I\) such that \(\|U_i(\mu_{a,e})\| < \epsilon\) for all \(i \in I_2 := I \setminus I_1\) (use the Riemann–Lebesgue lemma (Theorem 5.1)). Then

\[
\|U_i(\mu * \nu)\| = h(\alpha)\|U_i(\mu)\| \leq \|U_i(\mu)\| \leq \|U_i(\mu_{a,c})\| + \|U_i(\mu_{a,c})\| < \epsilon + \|\mu_s\|
\]

for each \(i \in I_2\), and therefore, if we set \(U := \bigoplus_{i \in I_2} U_i\), then

\[
\|U(\mu * \nu)\| \leq \epsilon + \|\mu_s\| < 1.
\]

It follows that \(U(\delta_e - \mu * \nu)\) is invertible on \(\mathcal{H}_U\), and since

\[
A_\alpha(\delta_e - \mu * \nu) = \left[ \bigoplus_{i \in I_1} U_i(\delta_e - \mu * \nu) \right] \oplus U(\delta_e - \mu * \nu)
\]

is a finite sum with each summand invertible, it follows that \(A_\alpha(\delta_e - \mu * \nu)\) is invertible. \(\square\)

7. Weak Mixing

In [24] Rosenblatt observes that, by the work of Foguel [6], weak mixing is actually equivalent to mixing in Abelian groups, and asks whether this remains true for more general groups. The answer turns out to be affirmative for spread-out measures on motion groups with \(G\) acting regularly on \(\hat{A}\). To prove this, it suffices to show the following result and then refer to Theorem 5.1.

**Proposition 7.1.** Let \(G\) be a locally compact CCR group, and let \(\mu\) be a spread-out probability measure in \(\mathcal{M}(G)\) which is weakly mixing by convolutions. Then \(g(\hat{\mu}(U)) < 1\) for any \([U] \in \tilde{G} \setminus \{1_G\}\).

**Proof.** By Aaronson et al. [1], weak mixing is equivalent to the following condition: \(\mu * h = \lambda h\) with \(h \in L^\infty(G)\) and \(|\lambda| = 1\) implies that \(\lambda = 1\) and \(h\) is constant \(\lambda_G\)-a.e. Assume that \(\lambda \in \sigma(\hat{\mu}(U))\) for some \([U] \in \tilde{G} \setminus \{1_G\}\) and \(\lambda \in \mathbb{C}\) with \(|\lambda| = 1\). Then also \(\tilde{\lambda} \in \sigma(U(\mu))\). Since \(\mu\) is spread-out, and hence \(U(\mu)\) is quasi-compact, \(\tilde{\lambda}\) must be an eigenvalue of \(U(\mu)\) (see Lemma 2.2). Let \(u \in \mathcal{H}_U\) be an eigenvector for \(\tilde{\lambda}\) with \(\|u\| = 1\), i.e., assume that \(U(\mu)u = \tilde{\lambda}u\), and set \(h(x) := \langle U(x)u, u \rangle\). Then \(\mu * h = \lambda h\), so if \(\mu\) is weakly mixing we must have that \(\lambda = 1\) and that \(h\) is constant (since it is also continuous). But if \(U \neq 1_G\), \(h(x) = \langle U(x)u, u \rangle\) can not be constant. \(\square\)
Corollary 7.2. Let $G = A \times \varphi K$ be a motion group with $G$ acting regularly on $\hat{A}$. Then, if $\mu \in M(G)$ is a spread-out probability measure, $\mu$ is mixing by convolutions if and only if it is weakly mixing by convolutions.

8. Final Remarks

A probability measure $\mu \in M(G)$ on a locally compact group $G$ is adapted if it is not concentrated on a closed proper subgroup of $G$, and strictly aperiodic if it is not concentrated on a coset of a normal, closed, proper subgroup of $G$.

Consider the following conditions for a probability measure $\mu \in M(G)$ on a locally compact group $G$:

(E) $\mu$ is ergodic.

(M) $\mu$ is mixing.

(ASA) $\mu$ is aperiodic, i.e., adapted and strictly aperiodic.

(S) $1 \notin \sigma(\widehat{\mu}(U)) \forall [U] \in \hat{G} \setminus \{1_G\}$.

(SR) $\varphi(\widehat{\mu}(U)) < 1 \forall [U] \in \hat{G} \setminus \{1_G\}$.

In Abelian and compact groups, it is known that (E) $\Leftrightarrow$ (A) $\Leftrightarrow$ (S), and that (M) $\Leftrightarrow$ (ASA) $\Leftrightarrow$ (SR) (see [4], [6], [24, Theorems 2 and 1] and [25, Proposition 1.2 and Theorem 1.4] for Abelian groups, and [17], [28], [26, Theorem V.5.2] and [12, 2.5.14] for compact groups). The equivalence (E) $\Leftrightarrow$ (A) is also known for spread-out measures in locally compact, compactly generated, second countable groups of polynomial growth ([13]). In fact, adaptedness is necessary for ergodicity, and aperiodicity necessary for mixing, in any locally compact group (see [24, p. 33 and p. 38]). On the other hand, the example on p. 40 of [25] shows that these conditions are no longer sufficient for ergodicity and mixing, respectively, in arbitrary groups. Here, we make the observation that, for this example, there are in fact non-trivial irreducible unitary representations of the underlying group for which condition (S) fails.

Example (Rosenblatt [25]). The underlying group in this example is the semi-direct product $G = \mathbb{Z}^2 \times \varphi \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ through the automorphisms $\varphi_k(n_1, n_2) = (n_1, n_2)\Gamma$, where $\Gamma = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Set $a = (0, 0, 1)$, $b = (1, 2, 1)$, and $c = (2, 3, 1)$, and consider the probability measure $\mu := \frac{1}{4}(\delta_a + \delta_b + \delta_{ac} + \delta_{ca})$. It is shown in [25] that $\mu$ is aperiodic, yet neither mixing nor ergodic. Note that $\mu$ is certainly spread-out, as $G$ is discrete. For $(t_1, t_2) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{Z}^2$, let $A_{(t_1, t_2)}$ be the representation of $G$ on $\ell^2(\mathbb{Z})$ given by

$$[A_{(t_1, t_2)}(n_1, n_2, k)](m) = \exp(2\pi i(t_1, t_2)\Gamma^{-m}(n_1, n_2)) \phi(m - k)$$

with the dash denoting transpose, i.e., $(n_1, n_2)' = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. The representation $A_{(t_1, t_2)}$ is easily seen to be unitary, and it is also irreducible, as can be seen using Shur’s lemma. Furthermore, direct computation shows that

$$[A_{(t_1, t_2)}(\mu)](m) = \frac{1}{4}[1 + \exp(2\pi i(t_1, t_2)\Gamma^{-m}(1, 2)')] \phi(m - 1) + \frac{1}{4}[\exp(2\pi i(t_1, t_2)\Gamma^{-m}(9, 15)') + \exp(2\pi i(t_1, t_2)\Gamma^{-m}(10, 16)')] \phi(m - 2)$$
(\phi \in \ell^2(\mathbb{Z}), \ m \in \mathbb{Z}). \text{ So, if } \phi_n \in \ell^2(\mathbb{Z}) \text{ is the function with }

\phi_n(m) = \begin{cases} 
1/\sqrt{n}, & 0 \leq m < n \\
0, & \text{otherwise},
\end{cases}

then \|\phi_n\| = 1, \text{ and, by direct computation again, }

\|A_{(t_1, t_2)}(\mu)\phi_n - \phi_n\|^2 = \frac{1}{n} + \frac{1}{16n} |1 + \exp(2\pi i(t_1, t_2)\Gamma^{-1}(1, 2)' - 4|^2 \\
+ \frac{1}{16n} |\exp(2\pi i(t_1, t_2)\Gamma^{-n-1}(9, 15)') + \exp(2\pi i(t_1, t_2)\Gamma^{-n-1}(10, 16)')|^2 \\
+ \frac{1}{16n} |1 + \exp(2\pi i(t_1, t_2)\Gamma^{-n}(1, 2)' + \exp(2\pi i(t_1, t_2)\Gamma^{-n}(9, 15)') + \exp(2\pi i(t_1, t_2)\Gamma^{-n}(10, 16)')|^2 \\
+ \frac{1}{16n} \sum_{j=2}^{n-1} |1 + \exp((2\pi i(t_1, t_2)\Gamma^{-n}(1, 2)' + \exp(2\pi i(t_1, t_2)\Gamma^{-n}(9, 15)') + \exp(2\pi i(t_1, t_2)\Gamma^{-n}(10, 16)') - 4|^2.

Now observe that \( \lambda = \sqrt{5} - 2 \) is an eigenvalue of \( \Gamma^{-1} \), and choose \((t_1, t_2)\) to be any eigenvector corresponding to \( \lambda \). Then, since \(|\lambda| < 1\),

\( \exp(2\pi i(t_1, t_2)\Gamma^{-n}(n_1, n_2)') = \exp(2\pi i\lambda^n(t_1, t_2)(n_1, n_2)') \to 1 \)

as \( n \to \infty \) for any \((n_1, n_2) \in \mathbb{Z}^2\), and hence \( \|A_{(t_1, t_2)}(\mu)\phi_n - \phi_n\|^2 \to 0 \) as \( n \to \infty \). Since \( \|\phi_n\| = 1 \), this shows that \( 1 \in \sigma(A_{(t_1, t_2)}(\mu)) \).

Concluding, let us also mention the following facts in relation to the above: in a locally compact CCR group, strict aperiodicity is equivalent to the condition (SR) for adapted spread-out measures. This may be proved along the lines of Lemma 4.3 of [2], using also Lemma 2.2 and Lemma 2.3 of the present paper, and the fact that, in any locally compact group \( G \), if \( H \) is a closed, normal, proper subgroup of \( G \), then there exists a non-trivial irreducible unitary representation \( U \) of \( G \) which is identically equal to the identity operator when restricted to the subgroup \( H \). One can also prove along the lines of [2, Lemma 4.3] that: in a CCR group, (A) \( \Rightarrow \) (S) for spread-out measures. Hence also, the equivalences (E) \( \Leftrightarrow \) (A) \( \Leftrightarrow \) (S) and (M) \( \Leftrightarrow \) (ASA) \( \Leftrightarrow \) (SR) also hold for spread-out measures in motion groups like the ones studied in this paper. In fact there is a direct argument showing that (S) \( \Rightarrow \) (A) for spread-out measures on motion groups, which we now present; this, when combined with Jaworski’s theorem [13] Corollary 3.8 and the aforementioned implication (A) \( \Rightarrow \) (S) for spread-out measures in CCR groups, yields another proof of Theorem 0.3 when the motion group in question is second countable (see also Remark 3.9 of [13]).

**Lemma 8.1.** Let \( G = A \times_\phi K \) be a motion group with \( G \) acting regularly on \( \hat{A} \), and let \( H \) be an open proper subgroup of \( G \). Then there exists \( [U] \in \hat{G} \setminus \{1_G\} \) such that the restriction of \( U \) to \( H \) has a fixed vector: \( \exists u \in \mathcal{H}_U \) s.t. \( U(x)u = u \ \forall x \in H \).

**Proof.** For notational simplicity we shall identify \( A \) and \( A \times \{1\} \) in what follows. Assume first that \( HA \neq G \). Since \( H \) is open, \( HA \) is open, hence its projection \( \pi_K(HA) = \{a \in K: (a, \kappa) \in HA \text{ for some } a \in A\} \) onto \( K \) is an open subgroup of \( K \); thus \( \pi_K(HA) \) is also a closed subgroup of \( K \). By the Frobenious reciprocity
theorem, there exists a non-trivial, irreducible unitary representation \([V] \in \hat{K}\) which has a fixed vector when restricted to \(\pi_K(HA)\). Then \(U(a, \kappa) := V(\kappa)\) for all \(a \in A\) and \(\kappa \in K\) is the sought for representation of \(G\).

Next assume that \(HA = G\). Then \(H \cap A\) is normal in \(G\). Set \(\hat{G} := G/H \cap A\), \(\hat{A} := A/H \cap A\), and \(\hat{H} := H/H \cap A\). Note that \(\hat{A}\) and \(\hat{H}\) are closed subgroups of \(\hat{G}\), and that \(\hat{A}\) is normal in \(\hat{G}\) and \(\hat{H} \cap \hat{A}\) is trivial. It follows that \(\hat{G}\) is the semi-direct product \(\hat{G} = \hat{A} \ltimes \hat{H}\), in fact with the factor \(\hat{H}\) compact, since \(\hat{H}\) is open and \(\hat{H} = H/H \cap A \simeq HA/A = G/A\). Now let \(\hat{\alpha}\) be a non-trivial character of \(\hat{A}\), and let \(\hat{A}_\alpha = \text{ind}_{\hat{A}}^{\hat{G}}(\hat{\alpha})\) be the representation of \(\hat{G}\) on \(L^2(\hat{H})\) obtained as in (3.2). It is readily seen that the constant function \(1_{\hat{H}}\) is a fixed vector for the restriction of \(\hat{A}_\alpha\) on \(\hat{H}\): \(\hat{A}_\alpha(x)1_{\hat{H}} = 1_{\hat{H}}\) for all \(x \in \hat{H}\). It follows from Proposition 3.1 that \(\hat{U}(x)^* \phi = \phi \forall x \in \hat{H}\) for some non-zero \(\phi \in L^2(\hat{H})\), for some irreducible sub-representation of \(\hat{A}_\alpha\), and since \(\hat{\alpha}\) is non-trivial \(\hat{U}\) is non-trivial, again by Proposition 3.1 (note that \(\hat{G}\) acts regularly on the characters of \(\hat{A}\) because \(G\) acts regularly on \(A\)). Now lift \(\hat{U}\) to \(G\): the representation \(U(a, \kappa) := \hat{U}((a, \kappa)(H \cap A))\) is the desired representation of \(G\).

To obtain the asserted implication \((S) \Rightarrow (A)\) for a spread-out measure \(\mu\) on a motion group \(G = A \times \phi K\) with \(G\) acting regularly on \(\hat{A}\), we then argue as follows. It is easy to see that, when \(\mu\) is spread-out, some power \(\mu^n\) of \(\mu\) must dominate a positive multiple of Haar measure \(\lambda_G\) on some open set; from this it follows that the smallest closed subgroup \(H\) of \(G\) with \(\mu(H) = 1\) is also open. Thus if \(\mu\) is not adapted, there exists a non-trivial irreducible unitary representation \(U\) of \(G\) which has a non-trivial fixed vector \(u\) say when restricted to \(H\). It follows that \(\tilde{\mu}(U)u = u\), since \(\mu(H) = 1\), and thus \(1 \in \sigma(\tilde{\mu}(U))\).

APPENDIX

In this Appendix, we give the proofs of Theorems 3.2 and 3.3.

Proof of Theorem 3.2 (i) First, since \(\hat{f}(A_\alpha) = A_\alpha(\hat{f})^*\) for any \(f \in L^1(G)\), we may as well consider \(A_\alpha(f)\) instead of \(\hat{f}(A_\alpha)\). Second, since \(A_\alpha\) is a unitary representation, and hence
\[
\|A_\alpha(f) - A_\alpha(g)\| \leq \|f - g\|_1 \quad (f, g \in L^1(G), \alpha \in \hat{A}),
\]
it suffices to only consider functions of the form

(A.1) \[ g(a, \kappa) = \sum_{V \in \mathfrak{F}} \sum_{i,j=1}^{d_{[V]}} g_{ij}^V(a) V_{ij}(\kappa), \]

where \(\mathfrak{F}\) is a finite set of mutually inequivalent, irreducible, unitary representations of \(K\), the \(V_{ij}\) are the representative functions of a \(V \in \mathfrak{F}\) with respect to some basis of \(\mathcal{H}_V\), and \(g_{ij}^V \in L^1(A)\) for all \(V \in \mathfrak{F}\) and \(1 \leq i, j \leq d_{[V]}\), because the functions of the form (A.1) are dense in \(L^1(G)\). Fix a \(g\) as in (A.1).
For an arbitrary function \( h \in L^1(G) \) write \( h_\kappa \) for the function \( h_\kappa(a) := h(a, \kappa) \) on \( A \); then \( h_\kappa \in L^1(A) \) for \( \lambda_K \)-a.e. \( \kappa \) in \( K \), and for \( \phi \in L^2(K) \),

\[
[A_\alpha(h) \phi](\tilde{\kappa}) = \int_K \int_A h(a, \kappa) \cdot (a, \varphi_\kappa(a)) \cdot \phi(\kappa^{-1} \tilde{\kappa}) \, d\lambda_A(a) \, d\lambda_K(\kappa)
\]

\[
= \int_K \hat{h}_\kappa(-\varphi_\kappa(\alpha)) \cdot \phi(\kappa^{-1} \tilde{\kappa}) \, d\lambda_K(\kappa),
\]

where \( \hat{h}_\kappa(\varphi_\kappa(\alpha)) \) is the Fourier transform of the function \( h_\kappa : A \to \mathbb{C} \) on \( A \) at the character \( \varphi_\kappa(\alpha) \) of \( A \). Hence

\[
\|A_\alpha(h)\phi\|_{L^2(K)}^2 = \int_K \int_K |\hat{h}_\kappa(-\varphi_\kappa(\alpha)) \phi(\kappa^{-1} \tilde{\kappa})|^2 \, d\lambda_K(\kappa) 
\]

\[
\leq \|\phi\|_{L^2(K)}^2 \int_K \int_K |\hat{h}_\kappa(-\varphi_\kappa(\alpha))|^2 \, d\lambda_K(\kappa) \, d\lambda_K(\kappa),
\]

and therefore

\[
\|A_\alpha(h)\| \leq \left( \int_K \int_K |\hat{h}_\kappa(-\varphi_\kappa(\alpha))|^2 \, d\lambda_K(\kappa) \, d\lambda_K(\kappa) \right)^{1/2}. \tag{A.3}
\]

For the function \( g \) defined by \((A.1)\),

\[
\hat{g}_\kappa(\alpha) = \sum_{V \in \mathfrak{V}} \sum_{i,j=1}^{d_{[V]}} \langle g^V_{ij}(\kappa) \rangle V_{ij}(\kappa),
\]

whence by \((A.3)\)

\[
\|A_\alpha(g)\| \leq \sum_{V \in \mathfrak{V}} \sum_{i,j=1}^{d_{[V]}} d_{[V]}^{-1} \int_K |\hat{g}^V_{ij}(\kappa)|^2 \, d\lambda_K(\kappa). \tag{A.5}
\]

Fix \( \epsilon > 0 \), and set \( ||V|| := \sum_{i,j=1}^{d_{[V]}} d_{[V]} \). By the Riemann–Lebesgue lemma for Abelian groups \([7, Proposition 4.13]\), there exists a symmetric compact subset \( C \) of \( \hat{A} \) such that

\[
\left| \langle g^V_{ij}(\kappa) \rangle \right| < \frac{\epsilon}{||V||^{1/2}} \quad \text{if} \quad \alpha \in \hat{A} \setminus C,
\]

for all \( V \in \mathfrak{V} \) and \( 1 \leq i, j \leq d_{[V]} \). Then

\[
\left| \langle g^V_{ij}(\kappa) \rangle \right| < \frac{\epsilon}{||V||^{1/2}} \quad \text{if} \quad \alpha \in \hat{A} \setminus \hat{C},
\]

for all \( V \in \mathfrak{V} \), \( 1 \leq i, j \leq d_{[V]} \), and all \( \kappa \in K \), where \( \hat{C} := \bigcup_{\kappa \in K} \varphi_\kappa(C) \); furthermore, since \( C \) is compact so is \( \hat{C} \), by the continuity of the mapping \( (\alpha, \kappa) \mapsto \varphi_\kappa(\alpha) \). Assertion \((i)\) now follows from \((A.5)\).

\((ii)\) Fix \( \epsilon > 0 \) and \( \alpha \in \hat{A} \). First, it suffices to only consider functions \( g \) of the form \((A.1)\) again. Fix such a function \( g \) and write \( \mathcal{E}_\mathfrak{V} \) for the finite-dimensional subspace of \( L^2(K) \) spanned by the functions \( V_{ij}, \, i,j = 1, \ldots, d_{[V]}, \, V \in \mathfrak{V} \). We then claim that \( A_\alpha(g)(\mathcal{E}_\mathfrak{V}) = \{0\} \). Indeed, inserting \((A.4)\) into \((A.2)\) yields that

\[
[A_\alpha(g)\phi](\kappa) = \sum_{V \in \mathfrak{V}} \sum_{i,j=1}^{d_{[V]}} \langle g^V_{ij}(\kappa) \rangle \cdot (V_{ij} \ast \phi)(\kappa) \quad (\kappa \in K)
\]

for any \( \phi \in L^2(K) \), whence \( [A_\alpha(g)\phi] = 0 \) when \( \phi \) is a representative function of any irreducible unitary representation of \( K \) not equivalent to a representation in \( \mathfrak{V} \), by
the Shur orthogonality relations. Suppose then that \( A_{\alpha} = \bigoplus_{i \in \mathcal{I}} U_i \) with each \( U_i \) an irreducible unitary representation of \( G \). Then \( A_{\alpha}(g) = \bigoplus_{i \in \mathcal{I}} U_i(g) \), and since \( \mathcal{E}_\Xi \) is finite-dimensional, one must have that \( U_i(g) = 0 \) for all but finitely many \( i \).

**Proof of Theorem 3.3.** If \( \alpha, \beta \in \hat{A} \) are characters of \( A \), and \( a \in A \), then

\[
|\langle a, \alpha \rangle - \langle a, \beta \rangle|^2 = |1 - (a, \beta - \alpha)|^2.
\]

Thus, if \( \mu \in M(G) \) and \( \phi \in L^2(K) \), then

\[
\left| \int_{A \times K} |\langle a, \varphi_{\kappa\tilde{k}}(\beta - \alpha) \rangle|^2 |\phi(\kappa\tilde{k})|^2 d\mu(a, \kappa) \right| \leq \|\mu\| \left( \int_{A \times K} |1 - \langle a, \varphi_{\kappa\tilde{k}}(\beta - \alpha) \rangle|^2 |\phi(\kappa\tilde{k})|^2 d\mu(a, \kappa) \right)
\]

for all \( \tilde{k} \in K \), whence

\[
\|\widehat{\phi} \|_{L^2(K)}^2 = \left| \int_{A \times K} |\langle a, \varphi_{\kappa\tilde{k}}(\beta - \alpha) \rangle|^2 |\phi(\kappa\tilde{k})|^2 d\mu(a, \kappa) \right| \leq \|\mu\| \left( \int_{A \times K} |1 - \langle a, \varphi_{\kappa\tilde{k}}(\beta - \alpha) \rangle|^2 d\mu(a, \kappa) \right)
\]

for \( \alpha, \beta \in \hat{A} \) and \( \phi \in L^2(K) \), the first equality by the left-invariance of Haar measure \( \lambda_K \).

Next choose a compact set \( A_1 \subseteq A \) such that \( |\mu|((A_1 \times K)^c) < \epsilon \). Define \( \Phi: A \times K \to A \times K \) by \( \Phi(\alpha, \kappa) := (\varphi_{\kappa}^{-1}(\alpha), \kappa) \), which by the continuity of the mapping \( (a, \kappa) \mapsto \varphi_{\kappa}^{-1}(a) \) is continuous, and set \( A_2 := \pi_A(\Phi(A_1 \times K)) \) and \( A_3 := \bigcap_{\kappa \in K} \varphi_{\kappa}(A_2) \), where \( \pi_A \) is the projection \( \pi_A: A \times K \to A \). Observe that \( A_3 \supseteq A_1 \), whence

\[
|\mu|((A_3 \times K)^c) < \epsilon,
\]

and that \( A_2 \) is compact. Finally, let

\[
V_\epsilon := \{ a \in \hat{A} : |\langle a, \alpha \rangle| < \epsilon \ \forall \ a \in A_2 \};
\]

by the compactness of \( A_2, V_\epsilon \) is an open neighborhood of \( 0 \in \hat{A} \) [27 §1.2.6]. For \( a \in A_3 \) and \( \alpha - \beta \in V_\epsilon \), one then has that

\[
|1 - \langle a, \varphi_{\kappa}(\alpha - \beta) \rangle| = |1 - \langle \varphi_{\kappa}^{-1}(a), \alpha - \beta \rangle| < \epsilon \quad \text{for all} \ \kappa \in K,
\]

whence

\[
\int_{A \times K} |1 - \langle a, \varphi_{\kappa}(\beta - \alpha) \rangle|^2 d\mu(a, \kappa) \leq \epsilon^2 |\mu|(A_3 \times K) + |\mu|((A_3 \times K)^c) \leq \epsilon^2 |\mu| + \epsilon
\]

for all \( \kappa \in K \) when \( \alpha - \beta \in V_\epsilon \), which by (A.6) completes the proof. \( \square \)
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