The functor of units of Burnside rings for $p$-groups

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Abstract: In this note I describe the structure of the biset functor $B^\times$ sending a $p$-group $P$ to the group of units of its Burnside ring $B(P)$. In particular, I show that $B^\times$ is a rational biset functor. It follows that if $P$ is a $p$-group, the structure of $B^\times(P)$ can be read from a genetic basis of $P$: the group $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number isomorphism classes of rational irreducible representations of $P$ whose type is trivial, cyclic of order 2, or dihedral.

1. Introduction

If $G$ is a finite group, denote by $B(G)$ the Burnside ring of $G$, i.e. the Grothendieck ring of the category of finite $G$-sets (see e.g. [2]). The question of structure of the multiplicative group $B^\times(G)$ has been studied by T. tom Dieck ([13]), T. Matsuda ([11]), T. Matsuda and T. Miyata ([12]), T. Yoshida ([16]), by geometric and algebraic methods.

Recently, E. Yalçın wrote a very nice paper ([14]), in which he proves an induction theorem for $B^\times$ for 2-groups, which says that if $P$ is a 2-group, then any element of $B^\times(P)$ is a sum of elements obtained by inflation and tensor induction from sections $(T,S)$ of $P$, such that $T/S$ is trivial or dihedral.

The main theorem of the present paper implies a more precise form of Yalçın’s Theorem, but the proof is independent, and uses entirely different methods. In particular, the biset functor techniques developed in [1], [4] and [6], lead to a precise description of $B^\times(P)$, when $P$ is a 2-group (actually also for arbitrary $p$-groups, but the case $p$ odd is known to be rather trivial). The main ingredient consists to show that $B^\times$ is a rational biset functor, and this is done by showing that the functor $B^\times$ (restricted to $p$-groups) is a subfunctor of the functor $\mathbb{F}_2 R_\mathbb{Q}^\times$. This leads to a description of $B^\times(P)$ in

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terms of a genetic basis of \( P \), or equivalently, in terms of rational irreducible representations of \( P \).

The paper is organized as follows: in Section 2, I recall the main definitions and notation on biset functors. Section 3 deals with genetic subgroups and rational biset functors. Section 4 gives a natural exposition of the biset functor structure of \( B^\times \). In Section 5, I state results about faithful elements in \( B^\times (P) \) for some specific \( p \)-groups \( P \). In Section 6, I introduce a natural transformation of biset functors from \( B^\times \) to \( \mathbb{F}_2 B^* \). This transformation is injective, and in Section 7, I show that the image of its restriction to \( p \)-groups is contained in the subfunctor \( \mathbb{F}_2 R^*_Q \) of \( \mathbb{F}_2 B^* \). This is the key result, leading in Section 8 to a description of the lattice of subfunctors of the restriction of \( B^\times \) to \( p \)-groups: it is always a uniserial \( p \)-biset functor (even simple if \( p \) is odd). This also provides an answer to the question, raised by Yalçın ([14]), of the surjectivity of the exponential map \( B(P) \to B^\times (P) \) for a 2-group \( P \).

2. Biset functors

2.1. Notation and Definition: Denote by \( \mathcal{C} \) the following category:

- The objects of \( \mathcal{C} \) are the finite groups.
- If \( G \) and \( H \) are finite \( p \)-groups, then \( \text{Hom}_\mathcal{C}(G, H) = B(H \times G^\text{op}) \) is the Burnside group of finite \((H, G)\)-bisets. An element of this group is called a virtual \((H, G)\)-biset.
- The composition of morphisms is \( \mathbb{Z} \)-bilinear, and if \( G, H, K \) are finite groups, if \( U \) is a finite \((H, G)\)-biset, and \( V \) is a finite \((K, H)\)-biset, then the composition of (the isomorphism classes of) \( V \) and \( U \) is the (isomorphism class) of \( V \times_H U \). The identity morphism \( \text{Id}_G \) of the group \( G \) is the class of the set \( G \), with left and right action by multiplication.

If \( p \) is a prime number, denote by \( \mathcal{C}_p \) the full subcategory of \( \mathcal{C} \) whose objects are finite \( p \)-groups.

Let \( \mathcal{F} \) denote the category of additive functors from \( \mathcal{C} \) to the category \( \mathbb{Z}\text{-Mod} \) of abelian groups. An object of \( \mathcal{F} \) is called a biset functor. Similarly, denote by \( \mathcal{F}_p \) the category of additive functors from \( \mathcal{C}_p \) to \( \mathbb{Z}\text{-Mod} \). An object of \( \mathcal{F}_p \) will be called a \( p \)-biset functor.

If \( F \) is an object of \( \mathcal{F} \), if \( G \) and \( H \) are finite groups, and if \( \varphi \in \text{Hom}_\mathcal{C}(G, H) \), then the image of \( w \in F(G) \) by the map \( F(\varphi) \) will generally be denoted by \( \varphi(w) \). The composition \( \psi \circ \varphi \) of morphisms \( \varphi \in \text{Hom}_\mathcal{C}(G, H) \) and \( \psi \in \text{Hom}_\mathcal{C}(H, K) \) will also be denoted by \( \psi \times_H \varphi \).
2.2. Notation: The Burnside biset functor (defined e.g. as the Yoneda functor $\text{Hom}_C(1, -)$), will be denoted by $B$. The functor of rational representations (see Section 1 of [4]) will be denoted by $R_{\mathbb{Q}}$. The restriction of $B$ and $R_{\mathbb{Q}}$ to $C_p$ will also be denoted by $B$ and $R_{\mathbb{Q}}$.

2.3. Examples: Recall that this formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences:

- If $H$ is a subgroup of $G$, then let $\text{Ind}_H^G \in \text{Hom}_C(H, G)$ denote the set $G$, with left action of $G$ and right action of $H$ by multiplication.
- If $H$ is a subgroup of $G$, then let $\text{Res}_H^G \in \text{Hom}_C(G, H)$ denote the set $G$, with left action of $H$ and right action of $G$ by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\text{Inf}_H^G \in \text{Hom}_C(H, G)$ denote the set $H$, with left action of $G$ by projection and multiplication, and right action of $H$ by multiplication.
- If $N \trianglelefteq G$, and $H = G/N$, then let $\text{Def}_H^G \in \text{Hom}_C(G, H)$ denote the set $H$, with left action of $H$ by multiplication, and right action of $G$ by projection and multiplication.
- If $\varphi : G \to H$ is a group isomorphism, then let $\text{Iso}_H^G = \text{Iso}_G^H(\varphi) \in \text{Hom}_C(G, H)$ denote the set $H$, with left action of $H$ by multiplication, and right action of $G$ by taking image by $\varphi$, and then multiplying in $H$.

2.4. Definition: A section of the group $G$ is a pair $(T, S)$ of subgroups of $G$ such that $S \trianglelefteq T$.

2.5. Notation: If $(T, S)$ is a section of $G$, set

$$\text{Ind}_{T/S}^G = \text{Ind}_T^G \text{Inf}_{T/S}^T \quad \text{and} \quad \text{Def}_{T/S}^G = \text{Def}_T^T \text{Res}_T^S$$

Then $\text{Ind}_{T/S}^G \cong G/S$ as $(G, T/S)$-biset, and $\text{Def}_{T/S}^G \cong S\setminus G$ as $(T/S, G)$-biset.

2.6. Notation: Let $G$ and $H$ be groups, let $U$ be an $(H, G)$-biset, and let $u \in U$. If $T$ is a subgroup of $H$, set

$$T^u = \{ g \in G \mid \exists t \in T, \ tu = ug \}$$

This is a subgroup of $G$. Similarly, if $S$ is a subgroup of $G$, set

$$u^S = \{ h \in H \mid \exists s \in S, \ us = hu \}$$

This is a subgroup of $H$. 

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2.7. Lemma: Let $G$ and $H$ be groups, let $U$ be an $(H,G)$-biset, and let $S$ be a subgroup of $G$. Then there is an isomorphism of $H$-sets

$$U/G = \bigsqcup_{u \in [H \setminus U]/S} H^u S,$$

where $[H \setminus U]/S$ is a set of representatives of $(H,S)$-orbits on $U$.

Proof: Indeed $H \setminus U/S$ is the set of orbits of $H$ on $U/S$, and $^u S$ is the stabilizer of $uS$ in $H$.

2.8. Opposite bisets: If $G$ and $H$ are finite groups, and if $U$ is a finite $(H,G)$-biset, then let $U^{\text{op}}$ denote the opposite biset: as a set, it is equal to $U$, and it is a $(G,H)$-biset for the following action

$$\forall h \in H, \forall u \in U, \forall g \in G, g.u.h \text{ (in $U^{\text{op}}$)} = h^{-1}ug^{-1} \text{ (in $U$)}.$$

This definition can be extended by linearity, to give an isomorphism

$$\varphi \mapsto \varphi^{\text{op}} : \text{Hom}_C(G,H) \to \text{Hom}_C(H,G).$$

It is easy to check that $(\varphi \circ \psi)^{\text{op}} = \psi^{\text{op}} \circ \varphi^{\text{op}}$, with obvious notation, and the functor

$$\left\{ \begin{array}{c} G \mapsto G \\
\varphi \mapsto \varphi^{\text{op}} \end{array} \right.$$ is an equivalence of categories from $C$ to the dual category, which restricts to an equivalence of $C_p$ to its dual category.

2.9. Example: if $G$ is a finite group, and $(T,S)$ is a section of $G$, then

$$(\text{Ind}_{T/S}^G)^{\text{op}} \cong \text{Defres}_{T/S}^G$$
as $(T/S,G)$-bisets.

2.10. Definition and Notation: If $F$ is a biset functor, the dual biset functor $F^*$ is defined by

$$F^*(G) = \text{Hom}_\mathbb{Z}(F(G), \mathbb{Z}),$$

for a finite group $G$, and by

$$F^*(\varphi)(\alpha) = \alpha \circ F(\varphi^{\text{op}}),$$

for any $\alpha \in F^*(G)$, any finite group $H$, and any $\varphi \in \text{Hom}_C(G,H)$.
2.11. Some idempotents in $\text{End}_C(G)$ : Let $G$ be a finite group, and let $N \trianglelefteq G$. Then it is clear from the definitions that

$$\text{Def}^G_{G/N} \circ \text{Inf}^G_{G/N} = (G/N) \times_G (G/N) = \text{Id}_{G/N} .$$

It follows that the composition $e^G_N = \text{Inf}^G_{G/N} \circ \text{Def}^G_{G/N}$ is an idempotent in $\text{End}_C(G)$. Moreover, if $M$ and $N$ are normal subgroups of $G$, then $e^G_N \circ e^G_M = e^G_{NM}$. Moreover $e^G_1 = \text{Id}_G$.

2.12. Lemma : (Lemma 2.5) If $N \trianglelefteq G$, define $f^G_N \in \text{End}_C(G)$ by

$$f^G_N = \sum_{\substack{M \trianglelefteq G \ N \subseteq M}} \mu(M \trianglelefteq G, N, M) e^G_M ,$$

where $\mu(M \trianglelefteq G)$ denotes the Möbius function of the poset of normal subgroups of $G$. Then the elements $f^G_N$, for $N \trianglelefteq G$, are orthogonal idempotents of $\text{End}_C(G)$, and their sum is equal to $\text{Id}_G$.

Moreover, it is easy to check from the definition that for $N \trianglelefteq G$,

$$(2.13) \quad f^G_N = \text{Inf}^G_{G/N} \circ f^G_1 \circ \text{Def}^G_{G/N} ,$$

and

$$e^G_N = \text{Inf}^G_{G/N} \circ \text{Def}^G_{G/N} = \sum_{\substack{M \trianglelefteq G \ M \supseteq N}} f^G_M .$$

2.14. Lemma : If $N$ is a non trivial normal subgroup of $G$, then

$$f^G_1 \circ \text{Inf}^G_{G/N} = 0 \quad \text{and} \quad \text{Def}^G_{G/N} \circ f^G_1 = 0 .$$

Proof: Indeed by (2.13)

$$f^G_1 \circ \text{Inf}^G_{G/N} = \sum_{\substack{M \trianglelefteq N \ M \supseteq N}} f^G_M \text{Inf}^G_{G/N} = 0 ,$$

since $M \neq 1$ when $M \supseteq N$. The other equality of the lemma follows by taking opposite bisets.

2.15. Remark : It was also shown in Section 2.7 of [6] that if $P$ is a $p$-group, then

$$f^P_1 = \sum_{N \trianglelefteq \Omega_1 Z(P)} \mu(1, N) P/N ,$$

where $\mu$ is the Möbius function of the poset of subgroups of $N$, and $\Omega_1 Z(P)$ is the subgroup of the centre of $P$ consisting of elements of order at most $p$. 
2.16. Notation and Definition: If $F$ is a a biset functor, and if $G$ is a finite group, then the idempotent $f_1^G$ of $\text{End}_C(G)$ acts on $F(G)$. Its image

$$\partial F(G) = f_1^G F(G)$$

is a direct summand of $F(G)$ as $\mathbb{Z}$-module: it will be called the set of faithful elements of $F(G)$.

The reason for this name is that any element $u \in F(G)$ which is inflated from a proper quotient of $G$ is such that $F(f_1^G)u = 0$. From Lemma 2.14, it is also clear that

$$\partial F(G) = \bigcap_{1 \neq N \triangleleft G} \ker \text{Def}^G_{G/N}.$$

3. Genetic subgroups and rational $p$-biset functors

The following definitions are essentially taken from Section 2 of [7]:

3.1. Definition and Notation: Let $P$ be a finite $p$-group. If $Q$ is a subgroup of $P$, denote by $Z_P(Q)$ the subgroup of $P$ defined by

$$Z_P(Q)/Q = Z(N_P(Q)/Q).$$

A subgroup $Q$ of $P$ is called genetic if it satisfies the following two conditions:

1. The group $N_P(Q)/Q$ has normal $p$-rank 1.
2. If $x \in P$, then $Q^x \cap Z_P(Q) \subseteq Q$ if and only if $Q^x = Q$.

Two genetic subgroups $Q$ and $R$ are said to be linked modulo $P$ (notation $Q \leftrightarrow_p R$), if there exist elements $x$ and $y$ in $P$ such that $Q^x \cap Z_P(R) \subseteq R$ and $R^y \cap Z_P(Q) \subseteq Q$.

This relation is an equivalence relation on the set of genetic subgroups of $P$. The set of equivalence classes is in one to one correspondence with the set of isomorphism classes of rational irreducible representations of $P$. A genetic basis of $P$ is a set of representatives of these equivalence classes.

If $V$ is an irreducible representation of $P$, then the type of $V$ is the isomorphism class of the group $N_P(Q)/Q$, where $Q$ is a genetic subgroup of $P$ in the equivalence class corresponding to $V$ by the above bijection.

3.2. Remark: The definition of the relation $\leftrightarrow_p$ given here is different from Definition 2.9 of [7], but it is equivalent to it, by Lemma 4.5 of [6].

The following is Theorem 3.2 of [6], in a slightly different form:
3.3. **Theorem:** Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Let $F$ be a $p$-biset functor. Then the map

$$\mathcal{I}_G = \oplus_{Q \in G} \text{Indinf}_{N_P(Q)/Q}(Q) : \oplus_{Q \in G} \partial F(N_P(Q)/Q) \rightarrow F(P)$$

is split injective.

3.4. **Remark:** There are two differences with the initial statement of Theorem 3.2 of [6]: here I use genetic subgroups instead of genetic sections, because these two notions are equivalent by Proposition 4.4 of [6]. Also the definition of the map $\mathcal{I}_G$ is apparently different: with the notation of [6], the map $\mathcal{I}_G$ is the sum of the maps $F(a_Q)$, where $a_Q$ is the trivial $(P, P/P)$-biset if $Q = P$, and $a_Q$ is the virtual $(P, N_P(Q)/Q)$-biset $P/Q - P/\hat{Q}$ if $Q \neq P$, where $\hat{Q}$ is the unique subgroup of $Z_P(Q)$ containing $Q$, and such that $|\hat{Q} : Q| = p$. But it is easy to see that the restriction of the map $F(P/\hat{Q})$ to $\partial F(N_P(Q)/Q)$ is actually 0. Moreover, the map $F(a_Q)$ is equal to $\text{Indinf}_{N_P(Q)/Q}$. So in fact, the above map $\mathcal{I}_G$ is the same as the one defined in Theorem 3.2 of [6].

3.5. **Definition:** A $p$-biset functor $F$ is called rational if for any finite $p$-group $P$ and any genetic basis $G$ of $P$, the map $\mathcal{I}_G$ is an isomorphism.

It was shown in Proposition 7.4 of [6] that subfunctors, quotient functors, and dual functors of rational $p$-biset functors are rational.

4. The functor of units of the Burnside ring

4.1. **Notation:** If $G$ is a finite group, let $B^x(G)$ denote the group of units of the Burnside ring $B(G)$.

If $G$ and $H$ are finite groups, if $U$ is a finite $(H, G)$-biset, recall that $U^{op}$ denotes the $(G, H)$-biset obtained from $U$ by reversing the actions. If $X$ is a finite $G$-set, then $T_U(X) = \text{Hom}_G(U^{op}, X)$ is a finite $H$-set. The correspondence $X \mapsto T_U(X)$ can be extended to a correspondence $T_U : B(G) \rightarrow B(H)$, which is multiplicative (i.e. $T_U(ab) = T_U(a)T_U(b)$ for any $a, b \in B(G)$), and preserves identity elements (i.e. $T_U(G/G) = H/H$). This extension to $B(G)$ can be built by different means, and the following is described in Section 4.1 of [3]: if $a$ is an element of $B(G)$, then there exists a finite $G$-poset $X$ such that $a$ is equal to the Lefschetz invariant $\Lambda_X$. Now $\text{Hom}_G(U^{op}, X)$ has a natural structure of $H$-poset, and one can set $T_U(a) = \Lambda_{\text{Hom}_G(U^{op}, X)}$. It is an element of $B(H)$, which does not depend of the choice of the poset $X$.
such that $a = \Lambda_X$, because with Notation 2.6 and Lemma 2.7 for any subgroup $T$ of $H$ the Euler-Poincaré characteristics $\chi \left( \text{Hom}_G(U^{\text{op}}, X)^T \right)$ can be computed by

$$\chi \left( \text{Hom}_G(U^{\text{op}}, X)^T \right) = \prod_{u \in T \setminus U / G} \chi(X^{T^u})$$

and the latter only depends on the element $\Lambda_X$ of $B(G)$. As a consequence, one has that

$$|T_U(a)^T| = \prod_{u \in T \setminus U / G} |a^{T^u}|.$$  

It follows in particular that $T_U(B^X(G)) \subseteq B^X(H)$. Moreover, it is easy to check that $T_U = T_{U'}$ if $U$ and $U'$ are isomorphic $(H,G)$-bisets, that $T_{U_1 U_2}(a) = T_{U_1}(a) T_{U_2}(a)$ for any $(H,G)$-bisets $U_1$ and $U_2$, and any $a \in B(G)$.

It follows that there is a well defined bilinear pairing

$$B(H \times G^{\text{op}}) \times B^X(G) \to B^X(H)$$

extending the correspondence $(U, a) \mapsto T_U(a)$. If $f \in B(H \times G^{\text{op}})$ (i.e. if $f$ is a virtual $(H,G)$-biset), the corresponding group homomorphism $B^X(G) \to B^X(H)$ will be denoted by $B^X(f)$.

Now let $K$ be a third group, and $V$ be a finite $(K,H)$-set. If $X$ is a finite $G$-set, there is a canonical isomorphism of $K$-sets

$$\text{Hom}_H(V^{\text{op}}, \text{Hom}_G(U^{\text{op}}, X)) \cong \text{Hom}_G((V \times_H U)^{\text{op}}, X)$$

showing that $T_V \circ T_U = T_{V \times_H U}$.

It follows more generally that $B^X(g) \circ B^X(f) = B^X(g \times_H f)$ for any $g \in B(K \times H^{\text{op}})$ and any $f \in B(H \times G^{\text{op}})$. Finally this shows:

**4.2. Proposition:** The correspondence sending a finite group $G$ to $B^X(G)$, and an homomorphism $f$ in $\mathcal{C}$ to $B^X(f)$, is a biset functor.

**4.3. Remark and Notation:** The restriction and inflation maps for the functor $B^X$ are the usual ones for the functor $B$. The deflation map $\text{Def}_{G/N}^G$ corresponds to taking fixed points under $N$ (so it does not coincide with the usual deflation map for $B$, which consist in taking orbits under $N$).

Similarly, if $H$ is a subgroup of $G$, the induction map from $H$ to $G$ for the functor $B^X$ is sometimes called multiplicative induction. I will call it tensor induction, and denote it by $\text{Ten}_H^G$. If $(T, S)$ is a section of $G$, I will also set $\text{Teninf}_T^P = \text{Ten}_T^P \text{Inf}_T^P$. 

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5. Faithful elements in $B^\times(G)$

5.1. Notation and definition: Let $G$ be a finite group. Denote by $[s_G]$ a set of representatives of conjugacy classes of subgroups of $G$. Then the elements $G/L$, for $L \in [s_G]$, form a basis of $B(G)$ over $\mathbb{Z}$, called the canonical basis of $B(G)$.

The primitive idempotents of $\mathbb{Q}B(G)$ are also indexed by $[s_G]$: if $H \in [s_G]$, the correspondent idempotent $e^G_H$ is equal to
\[
e^G_H = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H)G/K ,
\]
where $\mu(K, H)$ denotes the Möbius function of the poset of subgroups of $G$, ordered by inclusion (see [10], [15], or [2]).

Recall that if $a \in B(G)$, then $a \cdot e^G_H = |a^H| e^G_H$ so that $a$ can be written as
\[a = \sum_{H \in [s_G]} |a^H| e^G_H .\]

Now $a \in B^\times(G)$ if and only if $a \in B(G)$ and $|a^H| \in \{\pm 1\}$ for any $H \in [s_G]$, or equivalently if $a^2 = G/G$. If now $P$ is a $p$-group, and if $p \neq 2$, since $|a^H| \equiv |a|$ $(p)$ for any subgroup $|H|$ of $P$, it follows that $|a^H| = |a|$ for any $H$, thus $a = \pm P/P$. This shows the following well known

5.2. Lemma: If $P$ is an odd order $p$-group, then $B^\times(P) = \{\pm P/P\}$.

5.3. Remark: So in the sequel, when considering $p$-groups, the only really non-trivial case will occur for $p = 2$. However, some statements will be given for arbitrary $p$-groups.

5.4. Notation: If $G$ is a finite group, denote by $F_G$ the set of subgroups $H$ of $G$ such that $H \cap Z(G) = 1$, and set $[F_G] = F_G \cap [s_G]$.

5.5. Lemma: Let $G$ be a finite group. If $|Z(G)| > 2$, then $\partial B^\times(G)$ is trivial.

Proof: Indeed let $a \in \partial B^\times(G)$. Then $\text{Def}_{G/N}^G a$ is the identity element of $B^\times(G/N)$, for any non-trivial normal subgroup $N$ of $G$. Now suppose that $H$ is a subgroup of $G$ containing $N$. Then
\[|a^H| = |\text{Def}_{G/N}^G a| = |\text{Iso}_{G/N}^{N(G)(H)/H} \text{Def}_{G/N}^G| \text{Def}_{G/N}^G a| = 1 .\]

In particular $|a^H| = 1$ if $H \cap Z(G) \neq 1$. It follows that there exists a subset $A$ of $[F_G]$ such that
\[a = G/G - 2 \sum_{H \in A} e^G_H .\]
If \( A \neq \emptyset \), i.e. if \( a \neq G/G \), let \( L \) be a maximal element of \( A \). Then \( L \neq G \), because \( Z(G) \neq 1 \). The coefficient of \( G/L \) in the expression of \( a \) in the canonical basis of \( B(G) \) is equal to

\[
-2 \frac{|L| \mu(L, L)}{|N_G(L)|} = -2 \frac{|L|}{|N_G(L)|}.
\]

This is moreover an integer, since \( a \in B \times (G) \). It follows that \(|N_G(L) : L| \) is equal to 1 or 2. But since \( L \cap Z(G) = 1 \), the group \( Z(G) \) embeds into the group \( N_G(L)/L \). Hence \(|N_G(L) : L| \geq 3 \), and this contradiction shows that \( A = \emptyset \), thus \( a = G/G \).

5.6. Lemma: Let \( P \) be a finite 2-group, of order at least 4, and suppose that the maximal elements of \( F_P \) have order 2. If \(|P| \geq 2|F_P|\), then \( \partial B^\times(P) \) is trivial.

Proof: Let \( a \in \partial B^\times(P) \). By the argument of the previous proof, there exists a subset \( A \) of \([F_P]\) such that

\[
a = P/P - 2 \sum_{H \in A} e_H^P.
\]

The hypothesis implies that \( \mu(1, H) = -1 \) for any non-trivial element \( H \) of \([F_P]\). Now if \( 1 \in A \), the coefficient of \( P/1 \) in the expression of \( a \) in the canonical basis of \( B(P) \) is equal to

\[
-2 \frac{1}{|P|} + 2 \sum_{H \in A - \{1\}} \frac{1}{|N_P(H)|} = -2 \frac{1}{|P|} + 2 \sum_{H \in A - \{1\}} \frac{1}{|P|} = \frac{-4 + 2|\overline{A}|}{|P|},
\]

where \( \overline{A} \) is the set of subgroups of \( P \) which are conjugate to some element of \( A \). This coefficient is an integer if \( a \in B(P) \), so \(|P| \) divides \( 2|\overline{A}| - 4 \). But \(|\overline{A}| \) is always odd, since the trivial subgroup is the only normal subgroup of \( P \) which is in \( \overline{A} \) in this case. Thus \( 2|\overline{A}| - 4 \) is congruent to 2 modulo 4, and cannot be divisible by \(|P|\), since \(|P| \geq 4\).

So \( 1 \notin A \), and the coefficient of \( P/1 \) in the expression of \( a \) is equal to

\[
2 \sum_{H \in A} \frac{1}{|N_P(H)|} = \frac{2|\overline{A}|}{|P|}.
\]

Now this is an integer, so \( 2|\overline{A}| \) is congruent to 0 or 1 modulo the order of \( P \), which is even since \(|P| \geq 2|F_P| \geq 2 \). Thus \( 1 \notin A \), and \( 2|\overline{A}| \) is a multiple of \(|P|\). But \( 2|\overline{A}| < 2|F_P| \) since \( 1 \notin A \). So if \( 2|F_P| \leq |P| \), it follows that \( \overline{A} \) is empty, and \( A \) is empty. Hence \( a = P/P \), as was to be shown.
5.7. **Corollary**: Let $P$ be a finite 2-group. Then the group $\partial B^\times(P)\times(P)$ is trivial in each of the following cases:

1. $P$ is abelian of order at least 3.
2. $P$ is generalized quaternion or semi-dihedral.

5.8. **Remark**: Case 1 follows easily from Matsuda’s Theorem (11). Case 2 follows from Lemma 4.6 of Yalcin (14).

**Proof**: Case 1 follows from Lemma 5.5. In Case 2, if $P$ is generalized quaternion, then $F_P = \{1\}$, thus $|P| \geq 2|F_P|$. And if $P$ is semidihedral, then there is a unique conjugacy class of non-trivial subgroups $H$ of $P$ such that $H \cap Z(P) = 1$. Such a group has order 2, and $N_P(H) = HZ(P)$ has order 4. Thus $|F_P| = 1 + \frac{|P|}{4}$, and $|P| \geq 2|F_P|$ also in this case.

5.9. **Corollary**: [Yalcin (14) Lemma 4.6 and Lemma 5.2] Let $P$ be a $p$-group of normal $p$-rank 1. Then $\partial B^\times(P)\times(P)$ is trivial, except if $P$ is

- the trivial group, and $\partial B^\times(P)$ is the group of order 2 generated by $v_P = -P/P$.
- cyclic of order 2, and $\partial B^\times(P)$ is the group of order 2 generated by $v_P = P/P - P/1$.
- dihedral of order at least 16, and then $\partial B^\times(P)$ is the group of order 2 generated by the element $v_P = P/P + P/1 - P/I - P/J$,

where $I$ and $J$ are non-central subgroups of order 2 of $P$, not conjugate in $P$.

**Proof**: Lemma 5.2 and Lemma 5.5 show that $\partial B^\times(P)$ is trivial, when $P$ has normal $p$-rank 1, and $P$ is not trivial, cyclic of order 2, or dihedral: indeed then, the group $P$ is cyclic of order at least 3, or generalized quaternion, or semi-dihedral.

Now if $P$ is trivial, then obviously $B(P) = Z$, so $B^\times(P) = \partial B^\times(P) = \{\pm P/P\}$. If $P$ has order 2, then clearly $B^\times(P)$ consists of $\pm P/P$ and $\pm(P/P - P/1)$, and $\partial B^\times(P) = \{P/P, P/P - P/1\}$. Finally, if $P$ is dihedral, the set $F_P$ consists of the trivial group, and of two conjugacy classes of subgroups $H$ of order 2 of $P$, and $N_P(H) = HZ$ for each of these, where $Z$ is the centre of $P$. Thus

$$|F_P| = 1 + 2 \frac{|P|}{4} = 1 + \frac{|P|}{2}.$$
Now with the notation of the proof of Lemma 5.6, one has that $2|\overline{A}| \equiv 0 (|P|)$, and $2|\overline{A}| < |F_p| = 2 + |P|$. So either $A = \emptyset$, and in this case $a = P/P$, or $2|\overline{A}| = |P|$, which means that $\overline{A}$ is the whole set of non-trivial elements of $F_p$.

In this case

$$a = P/P - 2(e_I^P + e_J^P),$$

where $I$ and $J$ are non-central subgroups of order 2 of $P$, not conjugate in $P$. It is then easy to check that

$$a = P/P + P/1 - (P/I + P/J),$$

so $a$ is indeed in $B(P)$, hence in $B^\times(P)$. Moreover $\text{Def}_{p/2}^P a$ is the identity element of $B^\times(P/Z)$, so $a = f_1^P a$, and $a \in \partial B^\times(P)$. This completes the proof.

\[\boxed{}\]

6. A morphism of biset functors

If $k$ is any commutative ring, there is an obvious isomorphism of biset functor from $kB^* = k \otimes_{\mathbb{Z}} B^*$ to $\text{Hom}(B,k)$, which is defined for a group $G$ by sending the element $\alpha = \sum_i \alpha_i \otimes \psi_i$, where $\alpha_i \in k$ and $\psi_i \in B^*(G)$, to the linear form $\tilde{\alpha} : B(G) \to k$ defined by $\tilde{\alpha}(G/H) = \sum_i \psi_i(G/H)\alpha_i$.

\[\boxed{}\]

6.1. Notation : Let $\{\pm 1\} = \mathbb{Z}^\times$ be the group of units of the ring $\mathbb{Z}$. The unique group isomorphism from $\{\pm 1\}$ to $\mathbb{Z}/2\mathbb{Z}$ will be denoted by $u \mapsto u_+.$

If $G$ is a finite group, and if $a \in B^\times(G)$, then recall that for each subgroup $S$ of $G$, the integer $|a^S|$ is equal to $\pm 1$. Define a map $\epsilon_G : B^\times(G) \to \mathbb{F}_2 B^*(G)$ by setting $\epsilon_G(a)(G/S) = |a^S|_+$, for any $a \in B^\times(G)$ and any subgroup $S$ of $G$.

\[\boxed{}\]

6.2. Proposition : The maps $\epsilon_G$ define a injective morphism of biset functors

$$\epsilon : B^\times \to \mathbb{F}_2 B^*.$$

\[\boxed{}\]

Proof: The injectivity of the map $\epsilon_G$ is obvious. Now let $G$ and $H$ be finite groups, and let $U$ be a finite $(H,G)$-biset. Also denote by $U$ the corresponding element of $B(H \times G^{op})$. If $a \in B^\times(G)$, and if $T$ is a subgroup of $H$, then

$$|B^\times(U)(a)^T| = \prod_{u \in T \setminus U/G} |a^{T_u}|.$$

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Thus

\[ \epsilon_H \left( B^\times(U)(a) \right)(H/T) = \left( \prod_{u \in T \setminus U/G} |a^{T^u}| \right)_+ \]

\[ = \sum_{u \in T \setminus U/G} |a^{T^u}|_+ \]

\[ = \sum_{u \in T \setminus U/G} \epsilon_G(a)(G/T^u) \]

\[ = \epsilon_G(a)(U^{op}/T) \]

\[ = \epsilon_G(a)(U^{op} \times_H H/T) \]

\[ = \mathbb{F}_2 B^*(U)(\epsilon_G(a))(H/T) \]

Thus \( \epsilon_H \circ B^\times(U) = \mathbb{F}_2 B^*(U) \circ \epsilon_G \). Since both sides are additive with respect to \( U \), the same equality holds when \( U \) is an arbitrary element of \( B(H \times G^{op}) \), completing the proof.

7. **Restriction to \( p \)-groups**

The additional result that holds for finite \( p \)-groups (and not for arbitrary finite groups) is the Ritter-Segal theorem, which says that the natural transformation \( B \to R_Q \) of biset functors for \( p \)-groups, is surjective. By duality, it follows that the natural transformation \( i : kR^*_Q \to kB^* \) is injective, for any commutative ring \( k \). The following gives a characterization of the image \( i(kR^*_Q) \) inside \( kB^* \):

**7.1. Proposition:** Let \( p \) be a prime number, let \( P \) be a \( p \)-group, let \( k \) be a commutative ring. Then the element \( \varphi \in kB^*(P) \) lies in \( i(kR^*_Q(P)) \) if and only if the element \( \text{Defres}_{T/S}^P \varphi \) lies in \( i(kR^*_Q(T/S)) \), for any section \( T/S \) of \( P \) which is

- elementary abelian of rank 2, or non-abelian of order \( p^3 \) and exponent \( p \), if \( p \neq 2 \).
- elementary abelian of rank 2, or dihedral of order at least 8, if \( p = 2 \).

**Proof:** Since the image of \( kR^*_Q \) is a subfunctor of \( kB^* \), if \( \varphi \in i(kR^*_Q(P)) \), then \( \text{Defres}_{T/S}^P \varphi \in i(kR^*_Q(T/S)) \), for any section \( (T, S) \) of \( P \).

Conversely, consider the exact sequence of biset functors over \( p \)-groups

\[ 0 \to K \to B \to R_Q \to 0 \]
Every evaluation of this sequence at a particular \( p \)-group is a split exact sequence of (free) abelian groups. Hence by duality, for any ring \( k \), there is an exact sequence
\[
0 \to kR^*_Q \to kB^* \to kK^* \to 0 .
\]
With the identification \( kB^* \cong \text{Hom}_{\mathbb{Z}}(B, k) \), this means that if \( P \) is a \( p \)-group, the element \( \varphi \in RB^*(P) \) lies in \( i(kR^*_Q(P)) \) if and only if \( \varphi(K(P)) = 0 \). Now by Corollary 6.16 of [7], the group \( K(P) \) is the set of linear combinations of elements of the form \( \text{Ind}_{T/S}^P(\theta(\kappa)) \), where \( T/S \) is a section of \( P \), and \( \theta \) is a group isomorphism from one of the group listed in the proposition to \( T/S \), and \( \kappa \) is a specific element of \( K(T/S) \) in each case. The proposition follows, because
\[
\varphi(\text{Ind}_{T/S}^P(\theta(\kappa))) = (\text{Defres}_{T/S}^P)(\theta(\kappa)) ,
\]
and this is zero if \( \text{Defres}_{T/S}^P \) lies in \( i(kR^*_Q(T/S)) \).

7.2. Theorem: Let \( p \) be a prime number, and \( P \) be a finite \( p \)-group. The image of the map \( \epsilon_P \) is contained in \( i(\mathbb{Z}_2 R^*_Q(P)) \).

Proof: Let \( a \in B^*(P) \), and let \( T/S \) be any section of \( P \). Since
\[
\text{Defres}_{T/S}^P(i_P(a)) = i_T/S\text{Defres}_{T/S}^P(a) ,
\]
by Proposition 7.1 it is enough to check that the image of \( \epsilon_P \) is contained in \( i(\mathbb{Z}_2 R^*_Q(P)) \), when \( P \) is elementary abelian of rank 2 or non-abelian of order \( p^3 \) and exponent \( p \) if \( p \) is odd, or when \( P \) is elementary abelian of rank 2 or dihedral if \( p = 2 \).

Now if \( N \) is a normal subgroup of \( P \), one has that
\[
f_{P/N}^P(i_P(a)) = \text{Inf}_{P/N}^P(i_{P/N}(f_{1/N}^P(\text{Def}_{P/N}^P(a)))) .
\]
Thus by induction on the order of \( P \), one can suppose \( a \in \partial B^*(P) \). But if \( P \) is elementary abelian of rank 2, or if \( P \) has odd order, then \( \partial B^*(P) \) is trivial, by Lemma 5.2 and Corollary 5.7. Hence there is nothing more to prove if \( p \) is odd. And for \( p = 2 \), the only case left is when \( P \) is dihedral. In that case by Corollary 5.9 the group \( \partial B^*(P) \) has order 2, generated by the element
\[
\nu_P = \sum_{H \in [s_P]-\{I,J\}} e_{H}^P - (e_{I}^P + e_{J}^P) ,
\]
where \( [s_P] \) is a set of representatives of conjugacy classes of subgroups of \( P \), and where \( I \) and \( J \) are the elements of \( [s_P] \) which have order 2, and are non central in \( P \). Moreover the element \( \theta(\kappa) \) mentioned above is equal to
\[
(P/I - P/I'Z) - (P/J - P/J'Z) ,
\]
where $Z$ is the centre of $P$, and $I'$ and $J'$ are non-central subgroups of order 2 of $P$, not conjugate in $P$. Hence up to sign $\theta(\kappa)$ is equal to

$$\delta_P = (P/I - P/IZ) - (P/J - P/JZ).$$

Since $\epsilon_P(v_P)(P/H)$ is equal to zero, except if $H$ is conjugate to $I$ or $J$, and then $\epsilon_P(v_P)(P/H) = 1$, it follows that $\epsilon_P(v_P)(\delta_P) = 1 - 1 = 0$, as was to be shown. This completes the proof. $\square$

7.3. **Corollary:** The $p$-biset functor $B^\times$ is rational.

**Proof:** Indeed, it is isomorphic to a subfunctor of $\mathbb{F}_2 R_Q^* \cong \text{Hom}_\mathbb{Z}(R_Q, \mathbb{F}_2)$, which is rational by Proposition 7.4 of [6]. $\square$

7.4. **Theorem:** Let $P$ be a $p$-group. Then $B^\times(P)$ is an elementary abelian 2-group of rank equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is trivial, cyclic of order 2, or dihedral. More precisely:

1. If $p \neq 2$, then $B^\times(P) = \{\pm 1\}$.
2. If $p = 2$, then let $\mathcal{G}$ be a genetic basis of $P$, and let $\mathcal{H}$ be the subset of $\mathcal{G}$ consisting of elements $Q$ such that $N_P(Q)/Q$ is trivial, cyclic of order 2, or dihedral. If $Q \in \mathcal{H}$, then $\partial B^\times(N_P(Q)/Q)$ has order 2, generated by $v_{N_P(Q)/Q}$. Then the set

$$\{\text{T}e\text{n}i\text{n}f^{P}_{N_P(Q)/Q} v_{N_P(Q)/Q} | Q \in \mathcal{H}\}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

**Proof:** This follows from the definition of a rational biset functor, and from Corollary 5.9. $\square$

7.5. **Remark:** If $P$ is abelian, then there is a unique genetic basis of $P$, consisting of subgroups $Q$ such that $P/Q$ is cyclic. So in that case, the rank of $B^\times(P)$ is equal 1 plus the number of subgroups of index 2 in $P$: this gives a new proof of Matsuda’s Theorem ([11]).

8. **The functorial structure of $B^\times$ for $p$-groups**

In this section, I will describe the lattice of subfunctors of the $p$-biset functor $B^\times$. 
8.1. The case $p \neq 2$. If $p \neq 2$, there is not much to say, since $B^\times(P) \cong F_2$ for any $p$-group $P$. In this case, the functor $B^\times$ is the constant functor $\Gamma_{F_2}$ introduced in Corollary 8.4 of [8]. It is also isomorphic to the simple functor $S_1 \times (P) \cong F_2$. In this case, the results of [6] and [7] lead to the following remarkable version of Theorem 11.2 of [8]:

8.2. Proposition: If $p \neq 2$, the inclusion $B^\times \to F_2^R \star Q$ leads to a short exact sequence of $p$-biset functors

$$0 \to B^\times \to F_2^R_{\star Q} \to D_{\text{tors}} \to 0,$$

where $D_{\text{tors}}$ is the torsion part of the Dade $p$-biset functor.

8.3. The case $p = 2$. There is a bilinear pairing

$$\langle \ , \rangle : F_2^R_{\star Q} \times F_2^R_{\star Q} \to F_2.$$ 

This means that for each 2-group $P$, there is a bilinear form

$$\langle \ , \rangle_P : F_2^R_{\star Q}(P) \times F_2^R_{\star Q}(P) \to F_2,$$

with the property that for any 2-group $Q$, for any $f \in \text{Hom}_{C_p}(P,Q)$, for any $a \in F_2^R_{\star Q}(P)$ and any $b \in F_2^R_{\star Q}(Q)$, one has that

$$\langle F_2^R_{\star Q}(f)(a), b \rangle_Q = \langle a, F_2^R_{\star Q}(f^{op})(b) \rangle_P.$$ 

Moreover this pairing is non-degenerate: this means that for any 2-group $P$, the pairing $\langle \ , \rangle_P$ is non-degenerate. In particular, each subfunctor $F$ of $F_2^R_{\star Q}$ is isomorphic to $F_2^R_{\star Q}/F_\perp$, where $F_\perp$ is the orthogonal of $F$ for the pairing $\langle \ , \rangle$.

In particular, the lattice of subfunctors of $F_2^R_{\star Q}$ is isomorphic to the opposite lattice of subfunctors of $F_2^R_{\star Q}$. Now since $B^\times$ is isomorphic to a subfunctor of $F_2^R_{\star Q}$, its lattice of subfunctors is isomorphic to the opposite lattice of subfunctors of $F_2^R_{\star Q}$ containing $B^\sharp = (B^\times)^\perp$. By Theorem 4.4 of [4], any subfunctor $L$ of $F_2^R_{\star Q}$ is equal to the sum of subfunctors $H_Q$ it contains, where $Q$ is a 2-group of normal 2-rank 1, and $H_Q$ is the subfunctor of $F_2^R_{\star Q}$ generated by the image $\Phi_Q$ of the unique (up to isomorphism) irreducible rational faithful $\mathbb{F}_Q$-module $\Phi_Q$ in $F_2^R_{\star Q}$.

In particular $B^\sharp$ is the sum of the subfunctors $H_Q$, where $Q$ is a 2-group of normal 2-rank 1 such that $\Phi_Q \in B^\sharp(Q)$. This means that $\langle a, \Phi_Q \rangle_Q = 0$, for any $a \in B^\times(Q)$. Now $\Phi_Q = f_1 \Phi_Q$ since $\Phi_Q$ is faithful, so

$$\langle a, \Phi_Q \rangle_Q = \langle a, f_1^Q \Phi_Q \rangle_Q = \langle f_1^Q a, \Phi_Q \rangle_Q.$$ 

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because $f_1^Q = (f_1^Q)^{op}$. Thus $\Phi_{\mathcal{Q}} \in B^\times(Q)$ if and only if $\Phi_{\mathcal{Q}}$ is orthogonal to $\partial B^\times(Q)$. Since $Q$ has normal 2-rank 1, this is always the case by Corollary 5.9 except maybe if $Q$ is trivial, cyclic of order 2, or dihedral (of order at least 16). Now $H_1 = H_{\mathbb{C}_2} = \mathbb{F}_2R_{\mathcal{Q}}$ by Theorem 5.6 of [4]. Since $B^\times$ is not the zero subfunctor of $\mathbb{F}_2R_{\mathcal{Q}}$, it follows that $H_{\mathcal{Q}} \not\subseteq B^\times$, if $Q$ is trivial or cyclic of order 2. Now if $Q$ is dihedral, then $\Phi_{\mathcal{Q}}$ is equal to $(Q/Q/I - Q/Q/IZ)$. Now

$$
\epsilon_Q(v_Q)(i(\Phi_{\mathcal{Q}})) = \epsilon_Q(v_Q)(Q/I - Q/IZ) = 1 - 0 = 1
$$

It follows that $H_{\mathcal{Q}} \not\subseteq B^\times$ if $Q$ is dihedral. Finally $B^\times$ is the sum of all subfunctors $H_{\mathcal{Q}}$, when $Q$ is cyclic of order at least 4, or generalized quaternion, or semi-dihedral.

Recall from Theorem 6.2 of [4] that the poset of proper subfunctors of $\mathbb{F}_2R_{\mathcal{Q}}$ is isomorphic to the poset of closed subsets of the following graph:

![Graph Image]

The vertices of this graph are the isomorphism classes of groups of normal 2-rank 1 and order at least 4, and there is an arrow from vertex $Q$ to vertex $R$ if and only if $H_R \subseteq H_Q$. The vertices with a filled $\bullet$ are exactly labelled by the groups $Q$ for which $H_{\mathcal{Q}} \subseteq B^\times$, and the vertices with a $\circ$ are labelled by dihedral groups.

By the above remarks, the lattice of subobjects of $B^\times$ is isomorphic to the opposite lattice of subfunctors of $\mathbb{F}_2R_{\mathcal{Q}}$ containing $B^\times$. Thus:

**8.4. Theorem:** The $p$-biset functor $B^\times$ is uniserial. It has an infinite strictly increasing series of proper subfunctors

$$
0 \subset L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots
$$

where $L_0$ is generated by the element $v_1$, and $L_i$, for $i > 0$, is generated by the element $v_{D_2^{2i+3}}$ of $B^\times(D_{2i+3})$. The functor $L_0$ is isomorphic to the simple
functor $S_{1,F_2}$, and the quotient $L_i/L_{i-1}$, for $i \geq 1$, is isomorphic to the simple functor $S_{D_{2i+3},F_2}$.

**Proof:** Indeed $L_0 = B^2 + H_{D_{2i+4}}$ is the unique maximal proper subfunctor of $\mathbb{F}_2 R_Q$. Thus $L_0$ is isomorphic to the unique simple quotient of $\mathbb{F}_2 R_Q$, which is $S_{1,F_2}$ by Proposition 5.1 of [4]. Similarly for $i \geq 1$, the simple quotient $L_i/L_{i-1}$ is isomorphic to the quotient

$$(B^2 + H_{D_{2i+3}})/B^2 + H_{D_{2i+4}})$$

which is a quotient of

$$(B^2 + H_{D_{2i+3}})/B^2 \cong H_{D_{2i+3}}/(B^2 \cap H_{D_{2i+3}})$$

But the only simple quotient of $H_{D_{2i+3}}$ is $S_{D_{2i+3},F_2}$, by Proposition 5.1 of [4] again.

**8.5. Remark:** Let $P$ be a 2-group. By Theorem 5.12 of [4], the $\mathbb{F}_2$-dimension of $S_{1,F_2}(P)$ is equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is $1$ or $C_2$, whereas the $\mathbb{F}_2$-dimension of $S_{D_{2i+3},F_2}(P)$ is the number of isomorphism classes of rational irreducible representations of $P$ whose type is isomorphic to $D_{2i+3}$. This gives a way to recover Theorem 7.4: the $\mathbb{F}_2$-dimension of $B^\times(G)$ is equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is trivial, cyclic of order 2, or dihedral.

**8.6. The surjectivity of the exponential map.** Let $G$ be a finite group. The exponential map $\exp_G : B(G) \to B^\times(G)$ is defined in Section 7 of Yalçın’s paper ([14]) by

$$\exp_G(x) = (-1)^x$$

where $-1 = -1/1 \in B^\times(1)$, and where the exponentiation

$$(y,x) \in B^\times(G) \times B(G) \to B^\times(G)$$

is defined by extending the usual exponential map $(Y,X) \mapsto Y^X$, where $X$ and $Y$ are $G$-sets, and $Y^X$ is the set of maps from $X$ to $Y$, with $G$-action given by $(g \cdot f)(x) = gf(g^{-1}x)$.

It’s possible to give another interpretation of this map: indeed $B(G)$ is naturally isomorphic to Hom$_C(1,G)$, by considering any $G$-set as a $(G,1)$-biset. It is clear that if $X$ is a finite $G$-set, and $Y$ is a finite set, then

$$T_X(Y) = Y^X$$

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This can be extended by linearity, to show that for any \( x \in B(G) \)
\[
(-1)^x = B^\times(x)(-1)
\]
In particular the image \( \text{Im}(\exp_G) \) of the exponential map \( \exp_G \) is equal to \( \text{Hom}_C(1,G)(-1) \). Denoting by \( I \) the sub-biset functor of \( B^\times \) generated by \(-1 \in B^\times(1) \), it it now clear that \( \text{Im}(\exp_G) = I(G) \) for any finite group \( G \).

Now the restriction of the functor \( I \) to the category \( C_2 \) is equal to \( L_0 \), which is isomorphic to the simple functor \( S_{1,F_2} \). Using Remark 5.13 of [4], this shows finally the following :

**8.7. Proposition**: Let \( P \) be a finite 2-group. Then :

1. The \( F_2 \)-dimension of the image of the exponential map

\[
\exp_P : B(P) \to B^\times(P)
\]

is equal to the number of isomorphism classes of absolutely irreducible rational representations of \( P \).

2. The map \( \exp_P \) is surjective if and only if the group \( P \) has no irreducible rational representation of dihedral type, or equivalently, no genetic subgroup \( Q \) such that \( N_P(Q)/Q \) is dihedral.

**8.8. Proposition**: Let \( p \) be a prime number. There is an exact sequence of \( p \)-biset functors :

\[
0 \to B^\times \to \mathbb{F}_2 R^*_Q \to \mathbb{F}_2 D^\Omega_{\text{tors}} \to 0
\]
where \( D^\Omega_{\text{tors}} \) is the torsion part of the functor \( D^\Omega \) of relative syzygies in the Dade group.

**Proof**: In the case \( p \neq 2 \), this proposition is equivalent to Proposition 8.2, because \( \mathbb{F}_2 D^\Omega_{\text{tors}} = \mathbb{F}_2 D_{\text{tors}} \cong D_{\text{tors}} \) in this case. And for \( p = 2 \), the 2-functor \( D^\Omega_{\text{tors}} \) is a quotient of the functor \( R^*_Q \), by Corollary 7.5 of [6].

To prove the proposition in this case, is is enough to show that the image of \( B^\times \) in \( \mathbb{F}_2 R^*_Q \) is contained in the kernel of \( \mathbb{F}_2 \pi \), and that for any 2-group \( P \), the \( \mathbb{F}_2 \)-dimension of \( \mathbb{F}_2 R^*_Q(P) \) is equal to the sum of the \( \mathbb{F}_2 \)-dimensions of
$B^\times(P)$ and $\mathbb{F}_2D^\Omega_{\text{tors}}(P)$: but by Corollary 7.6 of [6], there is a group isomorphism

$$D^\Omega_{\text{tors}}(P) \cong (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P},$$

where $a_P$ is equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is generalized quaternion, and $b_P$ equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is cyclic of order at least 3, or semi-dihedral. Thus

$$\dim_{\mathbb{F}_2} \mathbb{F}_2D^\Omega_{\text{tors}}(P) = a_P + b_P.$$

Now since $\dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of $P$ whose type is cyclic of order at most 2, or dihedral, it follows that $\dim_{\mathbb{F}_2} \mathbb{F}_2D^\Omega_{\text{tors}}(P) + \dim_{\mathbb{F}_2} B^\times(P)$ is equal to the number of isomorphism classes of rational irreducible representations of $P$, i.e. to $\dim_{\mathbb{F}_2} \mathbb{F}_2 R^*_Q(P)$.

So the only thing to check to complete the proof, is that the image of $B^\times$ in $\mathbb{F}_2 R^*_Q$ is contained in the kernel of $\mathbb{F}_2 \pi$. Since $B^\times$, $\mathbb{F}_2 R^*_Q$ and $\mathbb{F}_2D^\Omega_{\text{tors}}$ are rational 2-biset functors, it suffices to check that if $P$ is a 2-group of normal 2-rank 1, and $a \in \partial B^\times(P)$, then the image of $a$ in $\partial \mathbb{F}_2 R^*_Q(P)$ lies in the kernel of $\mathbb{F}_2 \pi$. There is nothing to do if $P$ is generalized quaternion, or semi-dihedral, or cyclic of order at least 3, for in this case $\partial B^\times(P) = 0$ by Corollary 5.7. Now if $P$ is cyclic of order at most 2, then $D^\Omega(P) = \{0\}$, and the result follows. And if $P$ is dihedral, then $D^\Omega(P)$ is torsion free by Theorem 10.3 of [9], so $D^\Omega_{\text{tors}}(P) = \{0\}$ again.

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