Renormalization-scheme variation of a QCD perturbation expansion with tamed large-order behaviour

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The renormalization-scheme and scale dependence of the truncated QCD perturbative expansions is one of the main sources of theoretical error of the Standard Model predictions, especially at intermediate energies. Recently, a class of renormalization schemes, parametrized by a single real number $C$, has been defined and investigated in the frame of the standard perturbation expansions in powers of the coupling. In the present paper we investigate the $C$-scheme variation of a modified QCD perturbation series, which implements information about the large-order divergent character of perturbation theory by means of an optimal conformal mapping of the Borel plane. In these new expansions, the powers of the strong coupling are replaced by a set of expansion functions with properties which resemble those of the expanded correlators, having in particular a singular behaviour at the origin of the complex coupling plane. On the other hand, the new expansions have a tamed behaviour at high orders, as demonstrated by previous studies in the $\overline{\text{MS}}$ renormalization scheme. Using as examples the Adler function and the hadronic decay width of the $\tau$ lepton, we find good convergence properties of the new expansions also in the $C$-scheme, which is a further argument in favour of this version of QCD perturbation theory at intermediate energies.

PACS numbers: 12.38.Cy, 13.35.Dx, 11.10.Hi

I. INTRODUCTION

In the standard QCD perturbation theory, the finite-order approximations of physical quantities are renormalization-scale ($\mu$) and renormalization-scheme (RS) dependent. For a truncated expansion of order $N$, the scale and scheme variations, being in principle $O(\alpha_s^{N+1}(\mu))$ corrections, are expected to be small at large scales due to asymptotic freedom. However, since the perturbative expansion in QCD is a divergent series, with coefficients growing factorially at large orders, the scale and scheme variation might be in practice quite large, especially at intermediate energies where the strong coupling $\alpha_s$ is not very small.

The quest for in some sense “optimal” scale and scheme is important for meaningful applications. There are several recipes \cite{1–6} how to do that. The one proposed in \cite{1} and known as the “principle of minimal sensitivity” (PMS) selects the scale and scheme by the condition of local scale and scheme invariance. Therefore, the PMS selects the point where the truncated approximant has locally the property which the all-order summation must have globally. A different, process-dependent recipe, known as “effective charge approach”, was proposed in \cite{2}, while the method advocated in \cite{5, 6}, denoted as “principle of maximum conformality”, chooses the scale such as to absorb in the coupling all the non-conformal dependence of the perturbative coefficients. Since the problem is difficult and has so far no generally accepted solution, perturbative computations are performed mainly in convenient schemes like the modified minimal subtraction $\overline{\text{MS}}$ \cite{7}.

Recently, a new class of process-independent renormalization schemes depending on a single real parameter $C$ was defined in \cite{8}. In Refs. \cite{8–10}, the properties of these schemes have been investigated using the perturbation expansion of the QCD Adler function and the $\tau$ hadronic width, and in \cite{11} the class of $C$-scheme was discussed from the point of view of the maximum conformality principle \cite{6}.

In the present paper, we shall investigate the $C$-scheme in connection with the fact that the perturbation expansions in QCD are divergent series, the expansion coefficients of typical correlators growing factorially at large orders \cite{12–15}. This is related to the fact that the QCD correlators as functions of $\alpha_s$ are singular at the origin of the complex $\alpha_s$ plane \cite{12}. Therefore the radius of convergence of the expansions in powers of the strong coupling $\alpha_s$ should be zero.

Starting from the divergent character of the QCD perturbation series, a modified perturbation expansion was defined in \cite{16} and was further investigated in \cite{17–23} (for a recent review see \cite{24}). In this approach, instead of the powers of the strong coupling, a new set of expansion functions is used, defined by means of an optimal (i.e. ensuring the best asymptotic rate of convergence) conformal mapping of the Borel complex plane. The properties of the new expansion functions resemble those of the expanded correlators, by exhibiting in particular a singular behaviour at the origin of the $\alpha_s$ plane. On the other hand, the new expansions have a tamed behavior...
at high orders. The good convergence properties of the new expansions have been demonstrated in [19, 21, 23] in the MS renormalization scheme on mathematical models that simulate the physical Adler function.

In the present paper we investigate the properties of these modified perturbation expansions in the class of renormalization schemes defined in [8]. As in [8, 10], we use for illustration the Adler function and the hadronic width of the τ lepton. We start by recalling, in the next section, a few facts about the calculation of these quantities in perturbative QCD. In Sec. [11] we briefly review, following [8], the C-scheme definition of the QCD coupling. In Sec. [17] we introduce the modified, non-power perturbative expansions defined in [16] and investigated further in [17, 23], and in Sec. [18] we rewrite them in the C-scheme. In Sec. [19] we investigate the C-scheme variation of the modified perturbative expansions of the Adler function and the hadronic τ decay width. We also investigate the large-order behaviour of the expansions, using for generating the large-order expansion coefficients a model of the Adler function proposed in [25], which we review for completeness in the Appendix. The last section contains a summary and our conclusions.

II. ADLER FUNCTION AND τ HADRONIC WIDTH IN PERTURBATIVE QCD

We recall that the Adler function is the logarithmic derivative of the invariant amplitude of the two-current correlation tensor, \( D(s) = -s d\Pi(s)/ds \), where \( s \) is the momentum squared. As in Ref. [8] we shall consider the reduced function \( \hat{D}(s) \) defined as:

\[
\hat{D}(s) \equiv 4s^2 D(s) - 1.
\]

From general principles of field theory, it is known that \( \hat{D}(s) \) is an analytic function of real type (i.e. it satisfies the Schwarz reflection \( \hat{D}(s^*) = \hat{D}^*(s) \)) in the complex \( s \) plane cut along the timelike axis for \( s > 4m^2 \).

At large spacelike momenta \( s < 0 \), the function \( \hat{D} \) is given by the QCD perturbative expansion

\[
\hat{D}(\mu^2) = \sum_{n \geq 1} a_n \frac{\mu^{n}}{k} c_{n,k} \left( \ln(-s/\mu^2) \right)^{k-1},
\]

where \( a_n \equiv a_n(\mu^2)/\pi \) is the renormalized strong coupling in a certain RS at an arbitrary scale \( \mu \). As in [8], we emphasize from now on the fact that \( \hat{D} \) is a function of the QCD coupling, the dependence on the momentum squared \( s \) being assumed implicitly.

The leading coefficients \( c_{n,1} \) in (2) are evaluated by Feynman diagrams, while the remaining ones, \( c_{n,k} \) with \( k > 1 \) are obtained in terms of \( c_{n,1} \) with \( m < n \) and the coefficients \( \beta_n \) of the \( \beta \) function, which governs the variation of the QCD coupling with the scale in each RS:

\[
-\mu \frac{da_n}{d\mu} = \beta(a_n) = \sum_{n \geq 1} \beta_n a_n^{n+1}.
\]

We recall that in mass-independent renormalization schemes the first two coefficients \( \beta_1 \) and \( \beta_2 \) are scheme invariant, depending only on the number \( n_f \) of active flavours, while \( \beta_n \) for \( n \geq 3 \) depend on the renormalization scheme. In MS, the known coefficients for \( n_f = 3 \) are (cf. [26] and references therein):

\[
\beta_1 = \frac{9}{2}, \quad \beta_2 = 8, \quad \beta_3 = 20.12, \quad \beta_4 = 54.46, \quad \beta_5 = 268.16.
\]

The Adler function was calculated in MS to order \( \alpha_s^4 \), which makes it one of the most precisely known Green functions in QCD. For \( n_f = 3 \) the leading coefficients \( c_{n,1} \) have the values (cf. [27] and references therein):

\[
c_{1,1} = 1, \quad c_{2,1} = 1.640, \quad c_{3,1} = 6.371, \quad c_{4,1} = 49.076.
\]

In the applications done in [8, 10], an additional term was included, \( c_{5,1} = 283 \), based on the estimate made in [25].

We shall consider also the perturbative expansion of the total \( \tau \) hadronic width. The central observable is the ratio \( R_\tau \) of the total hadronic branching fraction to the electron branching fraction, which can be expressed as

\[
R_\tau = 3 S_{EW} (|V_{ud}|^2 + |V_{us}|^2) (1 + \delta^{(0)} + \ldots),
\]

where \( S_{EW} \) is an electroweak correction, \( V_{ud} \) and \( V_{us} \) are CKM matrix elements, and \( \delta^{(0)} \) is the perturbative QCD contribution. As shown in [28–31], \( \delta^{(0)} \) can be expressed, using analyticity, by an integral involving the values of the Adler function in the complex \( s \) plane. In our normalization, this relation is [25]:

\[
\delta^{(0)} = \frac{1}{2\pi i} \oint \frac{ds}{s} \left( 1 - \frac{s}{m^2} \right)^3 \left( 1 + \frac{s}{m^2} \right) \hat{D}(\mu^2),
\]

For the evaluation of \( \delta^{(0)} \) one can either insert in the integral (7) the expansion (2) at a fixed scale and perform the integration of the coefficients with respect to \( s \) along the circle, which gives in particular for \( \mu = m_\tau \):

\[
\delta^{(0)}(\mu^2) = a_\mu + 5.2 a_\mu^2 + 26.37 a_\mu^3 + 127.1 a_\mu^4 + 873.8 a_\mu^5 + \ldots
\]

Alternatively, as proposed in [31], one can take the variable scale \( \mu^2 = -s \) in (2) and insert in the integral (7) the expansion

\[
\hat{D}(\mu^2) = \sum_{n \geq 1} c_{n,1} a_\mu^n, \quad \mu^2 = -s,
\]

with the running coupling calculated by the numerical integration of the renormalization-group equation (3) along the circle \( |s| = m_\tau^2 \), starting from the spacelike point \( s = -m_\tau^2 \). These alternatives are known as “fixed order perturbation theory” (FOPT) and “contour improved”

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1 Another approach, proposed in [22], includes all the terms available from renormalization-group invariance and can be expressed as an effective expansion in powers of the one loop solution of Eq. (4). This summation was investigated in the case of the Adler function in [27, 28].
perturbation theory” (CIPT). As remarked first in [25], contrary to the naive expectations, at the scale set by \( m_\tau \) the difference between the predictions of these two summation procedures increased when an additional, five loop term, calculated in [27], was included in the expansion of the \( \tau \) hadronic width.

Actually, the fixed-order series (2) is expected to have a poor convergence for \( s \) near the timelike axis, since the \( s \)-dependent expansion coefficients become quite large there. However, fortuitous cancellations of terms in the integral and the suppressing effect of the weight function in (3) might favour the fixed-order series, leading to better results for \( \delta(0) \) calculated in FOPT than in CIPT. Many studies have been devoted to the difference between FO and CI summands, including the analysis of specific models for the Adler function and the practical implications on the extraction of \( \alpha_s(m_\tau^2) \) from data on hadronic \( \tau \) decays [19–25].

### III. THE C-SCHHEME QCD COUPLING

As discussed in [8], one can define a new coupling \( \hat{\alpha}_\mu \) by using the relation

\[
\frac{1}{\hat{\alpha}_\mu} + \frac{\beta_2}{\beta_1} \ln \hat{\alpha}_\mu - \frac{\beta_1 C}{2} \equiv \beta_1 \ln \frac{\mu}{\Lambda} \equiv \frac{1}{\beta(a)} \ln \frac{a}{\beta(a)} + \frac{\beta_2}{\beta_1} \ln \hat{\alpha}_\mu - \frac{\beta_1}{\beta_1} \int_0^a \frac{da}{\beta(a)} = \frac{1}{\beta(a)} = \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a^2}.
\]

(10)

where \( \Lambda \) is the scale-invariant QCD parameter and

\[
\frac{1}{\beta(a)} = \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a^2}.
\]

(11)

From (10) it is seen that \( C \) incorporates the effects of all scheme-dependent terms \( \beta_n \) with \( n \geq 3 \), contained in the function \( \beta(a) \). Eq. (10) implies also that the running of \( \hat{\alpha}_\mu \) is scheme independent:

\[
- \mu \frac{d\hat{\alpha}_\mu}{d\mu} \equiv \hat{\beta}(\hat{\alpha}_\mu) = \frac{\beta_1 \hat{\alpha}_\mu^2}{(1 - \frac{25}{\beta_1 \hat{\alpha}_\mu})}.
\]

(12)

Starting from the coupling \( a_\mu \) and the coefficients \( \beta_n \) in a certain RS, one can find the \( \hat{\alpha}_\mu \) coupling in the C-scheme, at the same scale \( \mu \) and a definite value of \( C \), either by solving numerically Eq. (10) or from the perturbative relation between \( a_\mu \) and the C-scheme coupling \( \hat{\alpha}_\mu \), written formally as

\[
\hat{\alpha}_\mu(a_\mu) = \sum_{n \geq 1} \xi_n(C) a_\mu^n,
\]

(13)

and the inverse relation

\[
a_\mu(\hat{\alpha}_\mu) = \sum_{n \geq 1} \xi_n(C) \hat{\alpha}_\mu^n.
\]

(14)

For the \( \overline{\text{MS}} \) scheme, using the coefficients [4], the explicit forms of the expansions (13) and (14) are given in Eqs. (7) and (8) of [8]. From the comparison with the full solution it follows, as remarked in [8], that the perturbative expansions break below roughly \( C = -2 \).

The perturbative QCD expansions in the C-scheme have been investigated in Refs. [8–10] for the Adler function and \( \tau \) hadronic width. In particular, the series (9) is rewritten as an expansion in powers of the C-scheme coupling \( \hat{\alpha}_\mu \):

\[
\hat{D}(\hat{\alpha}_\mu) = \sum_{n \geq 1} \hat{c}_{n,1}(C) \hat{a}_\mu^n, \quad \mu^2 = -s,
\]

(15)

with C-dependent coefficients \( \hat{c}_{n,1}(C) \). Using in (9) the coefficients [3] and the estimate \( c_{5,1} = 283 \), this expansion reads [8]

\[
\hat{D}(\hat{\alpha}_\mu) = \hat{\alpha}_\mu + (1.64 + 2.25C) \hat{\alpha}_\mu^2 + (7.68 + 11.38C + 5.06C^2) \hat{\alpha}_\mu^3 + (61.06 + 72.08C + 47.40C^2 + 11.4C^3) \hat{\alpha}_\mu^4 + (348.5 + 677.7C + 408.6C^2 + 162.5C^3 + 25.6C^4) \hat{\alpha}_\mu^5 + \ldots
\]

(16)

In the same way, from [8] it follows that the expansion of \( \delta_{FO}(0) \) in powers of the C-scheme coupling \( \hat{\alpha}_\mu \) at the scale \( \mu = m_\tau \) is [8]

\[
\delta_{FO}(0, \hat{\alpha}_\mu) = \hat{\alpha}_\mu + (5.20 + 2.25C) \hat{\alpha}_\mu^2 + (27.7 + 27.4C + 5.1C^2) \hat{\alpha}_\mu^3 + (148.4 + 235.5C + 101.5C^2 + 11.4C^3) \hat{\alpha}_\mu^4 + (789.6 + 1754.4C + 1240.4C^2 + 324.8C^3 + 25.6C^4) \hat{\alpha}_\mu^5 + \ldots
\]

(17)

In Refs. [9–10] the high-order behaviour of the perturbation expansions in the C-scheme has been also investigated, using for generating the perturbative coefficients the model defined in Section 5 of Ref. [25], which expresses the Adler function in terms of several renormalons.

In these studies, a region of stability in the variation of the various quantities with the parameter \( C \) was identified, as required by the “principle of minimum sensitivity” [1], and an optimal value of \( C \) was further defined as the value which ensures the vanishing of the last coefficient in the expansion, in the spirit of asymptotic series. The studies led to results comparable to those in the \( \overline{\text{MS}} \) scheme. In particular, the difference between the FOPT and CIPT predictions for \( \delta(0) \) is not resolved by the C-scheme, and the higher-order divergence of the expansions manifests itself to the same or even to a larger extent.

In what follows we shall investigate the C-scheme for the new perturbative expansions with tamed large-order behaviour, defined by the technique of conformal mapping of the Borel plane in [16, 19, 21]. In the next section we review the construction of this new type of expansion.
IV. PERTURBATION EXPANSIONS WITH TAMED HIGH-ORDER BEHAVIOUR

The starting point is the remark that the perturbation expansion (9) of the Adler function has a zero radius of convergence, the coefficients $c_n$, increasing like $n!$ at large $n$. It is convenient to define the Borel transform by

$$B(u) = \sum_{n=0}^{\infty} b_n u^n, \quad b_n = \frac{c_{n+1,1}}{\beta_0 n!},$$  \hspace{1cm} (18)

where we used the standard notation $\beta_0 = \beta_1/2$. From (18) one can derive the formal Lagrange-Borel integral representation

$$D(a_n) = \frac{1}{\beta_0} \int_{0}^{\infty} \exp\left(\frac{-w}{\beta_0 a_n}\right) B(u) \, du.$$  \hspace{1cm} (19)

As it is known [13, 15], the large-order increase of the coefficients of the perturbation series is encoded in the singularities of the Borel transform $B(u)$ in the complex $u$ plane. In the particular case of the Adler function, $B(u)$ has singularities on the semiaxis $u \geq 2$, denoted as infrared (IR) renormalons, and for $u \leq -1$, denoted as ultraviolet (UV) renormalons. The names indicate the regions in the Feynman integrals responsible for the appearance of the corresponding singularities. Other singularities, at larger values on the positive real axis, are due to specific field configurations known as instantons. Apart from the two cuts along the lines $u \geq 2$ and $u \leq -1$, it is assumed that no other singularities are present in the complex $u$ plane [13].

Due to the singularities of $B(u)$ for $u \geq 2$, the Laplace-Borel integral (19) is not defined and requires a regularization. The principal value (PV) prescription, the most natural choice in mathematics, has been adopted also for the summation of perturbative QCD [14, 15]. As discussed in [22], this prescription is most suitable from the point of view of the momentum-plane analyticity properties that must be satisfied by the QCD correlation functions.

The singularities of $B(u)$ set a limitation on the convergence region of the power expansion (13): this series converges only inside the circle $|u| = 1$ which passes through the nearest singularity, namely the first UV renormalon. As shown for the first time in [15], the domain of convergence of a power series can be increased by expanding in powers of another variable, which performs the conformal mapping of the complex plane of the original variable (or a part of it) onto a disk (chosen for convenience to be of radius equal to unity) in the transformed plane. The new series converges in a larger region, well beyond the disk of convergence of the original expansion, and also has an increased asymptotic convergence rate at points lying inside this disk. An important result proved in [15] (see also [22] for a detailed proof) is that the asymptotic convergence rate is maximal if the new variable maps the entire holomorphy domain of the expanded function onto the unit disk. This special conformal mapping is called “optimal”.

The use of a conformal mapping of the Borel plane was suggested in [14] in order to reduce or eliminate the ambiguities (power corrections) due to the large momenta in the Feynman integrals. As shown in [16], the particular conformal mapping proposed in [14] was not optimal in the sense defined above. The optimal mapping, which ensures the convergence of the corresponding power series in the entire doubly-cut Borel plane, was found in [16] and was further investigated in [17, 21]. It is given by the function:

$$\tilde{w}(u) = \frac{\sqrt{1 + u} - \sqrt{1 - u/2}}{\sqrt{1 + u} + \sqrt{1 - u/2}}.$$  \hspace{1cm} (20)

One can check that (20) maps the complex $u$ plane cut along the real axis for $u \geq 2$ and $u \leq -1$ onto the interior of the circle $|w| = 1$ in the complex plane $w = \tilde{w}(u)$, such that the origin $u = 0$ of the $u$ plane corresponds to the origin $w = 0$ of the $w$ plane, and the upper (lower) edges of the cuts are mapped onto the upper (lower) semicircles in the $w$ plane. By the mapping (20), all the singularities of the Borel transform, the UV and IR renormalons, are pushed on the boundary of the unit disk in the $w$ plane, all at equal distance from the origin. Therefore, the expansion of $B(u)$ in powers of the variable $w = \tilde{w}(u)$:

$$B(u) = \sum_{n \geq 0} c_n w^n, \quad w = \tilde{w}(u),$$  \hspace{1cm} (21)

converges in a larger domain that the original series (18).

By inserting the expansion (21) in the Borel-Laplace integral (19), one can define a new expansion of the Adler function, in terms of a set of expansion functions given by the Laplace-Borel transform of the powers $(\tilde{w}(u))^n$. As shown in [16], for the Adler function one can further exploit the fact that in this case the nature of the leading singularities in the Borel plane is known: near the first branch points, $u = -1$ and $u = 2$, $B(u)$ behaves like

$$B(u) \sim \frac{r_1}{(1 + u)^{\gamma_1}} \quad \text{and} \quad B(u) \sim \frac{r_2}{(1 - u/2)^{\gamma_2}},$$  \hspace{1cm} (22)

respectively, where the residues $r_1$ and $r_2$ are not known, but the exponents $\gamma_1$ and $\gamma_2$ have been calculated using renormalization-group invariance [14, 23, 44]. For $n_f = 3$, their values are

$$\gamma_1 = 1.21, \quad \gamma_2 = 2.58.$$  \hspace{1cm} (23)

This additional information can be used to improve the expansion (21). As discussed in [19, 21], while the optimal conformal mapping (20) is unique, there is no unique prescription to implement the knowledge provided by (22). Several possible “softening factors” have been investigated in [21]. Here we shall adopt one of these suitable factors and expand $B(u)$ as [19]

$$B(u) = \frac{1}{(1 + w)^{2\gamma_1}(1 - w)^{2\gamma_2}} \sum_{n \geq 0} d_n w^n.$$  \hspace{1cm} (24)
By inserting this expansion in the Borel-Laplace integral [19], we can define a new perturbative series for the Adler function:

\[ \tilde{D}(a_\mu) = \sum_{n \geq 0} d_n W_n(a_\mu), \]  

(25)
in terms of the expansion functions

\[ W_n(a_\mu) = \frac{1}{\beta_0} \text{PV} \int_0^\infty \frac{e^{-\frac{u}{\beta_0}} (\tilde{w}(u))^n}{(1 + \tilde{w}(u))^{2\gamma_1}(1 - \tilde{w}(u))^{2\gamma_2}} du, \]  

(26)

We emphasize that the definition [25] implies the permutation of the summation and the integration, which is not a trivial step, therefore [25] represents a genuinely new perturbation expansion in QCD.

By construction, when reexpanded in powers of \( a_\mu \), the series [25] reproduces the expansion [9] with the coefficients \( c_{n,1} \) known from Feynman diagrams. The expansion functions \( W_n(a_\mu) \) are singular at \( a_\mu = 0 \), and therefore have divergent expansions when expanded in powers of \( a_\mu \), resembling the expanded function \( \tilde{D}(a_\mu) \) itself [18]. On the other hand, the expansions have a tamed behaviour at high orders. As discussed in [17], under certain conditions, the expansion [25] may even converge in a domain of the s-plane.

We presented above the steps leading to the new expansion starting from the renormalization-group improved expansion [9], but similar steps can be followed starting from the fixed-order expansion [2]. The explicit formulae are given in [21].

Since the expansion functions \( W_n(a_\mu) \) defined in (26) are no longer powers of the coupling \( a_\mu \), the new expansion [25] can be viewed as a “non-power perturbation theory” (NPPT) [21]. We shall also refer to it as “Borel-improved” expansion, to emphasize the fact that [25] is defined by the analytic continuation of the Borel series [18] outside the original convergence disk in the whole cut Borel plane up to its cuts.

V. NON-POWER EXPANSIONS IN THE C-SCHHEME

The non-power expansions defined above have been investigated up to now in the \( \overline{\text{MS}} \)-renormalization scheme. However, the construction presented in the previous section is general and can be performed in any renormalization scheme. Starting from the expansion [15] of the Adler function in powers of the C-scheme coupling \( a_\mu \), we define the corresponding Borel transform as

\[ \hat{B}(u, C) = \sum_{n=0}^\infty b_n(C) u^n, \quad b_n = \frac{\hat{c}_{n+1,1}(C)}{\beta_0^n n!}, \]  

(27)

and obtain the formal Laplace-Borel integral representation

\[ \tilde{D}(a_\mu) = \frac{1}{\beta_0} \text{PV} \int_0^\infty \exp \left( \frac{-u}{\beta_0 a_\mu} \right) \hat{B}(u, C) du. \]  

(28)

A useful remark is that the position and the nature of the first singularities of the Borel transform in the \( u \) plane depend only on the first two coefficients, \( \beta_1 \) and \( \beta_2 \), of the \( \beta \) function, which are scheme-independent [14, 25, 44]. It follows that, for every \( C \), the first singularities of the function \( \hat{B}(u, C) \) are situated at \( u = -1 \) and \( u = 2 \), and the nature of the singularities is given by [22]. Therefore, we can use for \( \hat{B}(u, C) \) the expansion

\[ \hat{B}(u, C) = \frac{1}{(1 + w)^{2\gamma_1}(1 - w)^{2\gamma_2}} \sum_{n \geq 0} d_n(C) w^n, \]  

(29)
similar to [24], the only difference being that now the coefficients \( d_n \) depend on \( C \). By inserting this expansion into [25], we define the Borel-improved expansion of the Adler function in the C-scheme by

\[ \tilde{D}(a_\mu) = \sum_{n \geq 0} \hat{d}_n(C) W_n(a_\mu), \]  

(30)

where the expansions functions

\[ \hat{W}_n(a_\mu) = \frac{1}{\beta_0} \text{PV} \int_0^\infty \frac{e^{-\frac{u}{\beta_0}} (\tilde{w}(u))^n}{(1 + \tilde{w}(u))^{2\gamma_1}(1 - \tilde{w}(u))^{2\gamma_2}} du, \]  

(31)

formally similar to [26], depend on \( C \) through the C-dependent coupling \( a_\mu \).

For illustration, we list the first coefficients \( \hat{d}_n(C) \) appearing in (30):

\[ \hat{d}_0 = 1, \quad \hat{d}_1 = -0.80 + 2.67 C, \]
\[ \hat{d}_2 = 1.33 + 2.46 C + 3.56 C^2, \]
\[ \hat{d}_3 = 10.69 + 23.1 C + 8.15 C^2 + 3.16 C^3, \]
\[ \hat{d}_4 = 1.15 + 23.4 C + 8.37 C^2 + 11.02 C^3 + 2.11 C^4. \]

In a similar way, starting from the expansion [17] of \( \delta^{(0)}_{\text{FO}} \) in the C-scheme, we can define the improved series in terms of the same set of functions \( W_n(a_\mu) \), since from the definition [5] it follows that the position and the nature of the first singularities of the Borel transform of \( \delta^{(0)}_{\text{FO}} \) are the same as those of the Adler function, So, we can write

\[ \delta^{(0)}_{\text{CI}}(a_\mu) = \sum_{n \geq 0} \delta_n(C) W_n(a_\mu), \]  

(33)

where \( \mu = m_{\tau} \) and the first coefficients \( \delta_n(C) \) are

\[ \delta_0 = 1, \quad \delta_1 = 3.42 + 2.67 C, \]
\[ \delta_2 = 6.62 + 13.72 C + 3.56 C^2, \]
\[ \delta_3 = 4.96 + 31.82 C + 23.16 C^2 + 3.16 C^3, \]
\[ \delta_4 = -14.22 + 29.32 C + 65.63 C^2 + 24.36 C^3 + 2.11 C^4. \]

In CIPT, the calculation of \( \delta^{(0)}_{\text{CI}}(a_\mu) \) involves the numer-
where the expansion functions $\hat{W}_n$ and the coefficients $\hat{d}_n$ are given in Eqs. (31) and (32), respectively. Inside the integrals, the running coupling $\hat{a}_\mu$ at the scale $\mu^2 = m^2_\tau \exp(\phi)$, $\phi \in (0, 2\pi)$, is calculated along the circle by integrating the renormalization-group equation (12) in the $C$-scheme, starting from a given value at the scale $\mu = m_\tau$. Therefore, the whole expansion depends only on the $C$-scheme coupling $\hat{a}_\mu$ at the scale $\mu = m_\tau$.

VI. RESULTS

We calculate first the Adler function using the Borel-improved expansion in the $C$-scheme, given in Eqs. (30)-(32). As in [8], we take $s = -m_\tau^2$ and the scale $\mu = m_\tau$, using the value $\alpha_s(m_\tau^2) = 0.316 \pm 0.010$ in the MS scheme, which follows from the PDG value of the strong coupling at the scale $\mu = m_Z$ [45]. The corresponding $C$-dependent value of the coupling $\hat{a}_m$, was obtained by numerically solving equation (10).

In Fig. 1 we show the variation with $C$ of the Borel-improved expansions, for $C$ in the range $(-2, 2)$ considered in [8]: the central black line represents the expansion to order $\alpha_s^2$, and the lines delimiting the yellow region are obtained by either removing or doubling the coefficient $\hat{d}_4$ given in Eq. (32).

![FIG. 1: Borel-improved expansions in the C-scheme for the Adler function. The central line is the expansion to order $\alpha_s^2$. The yellow band is obtained by either removing or doubling the last term. In red are marked the points where the last term of the expansion vanishes and the five loop term is taken as uncertainty.](image1)

The three curves exhibit a plateau where the expansions are stable with respect to the variation of $C$. There are two points, represented in red, where the three expansions coincide: the rightmost point, $C_0 = -0.05$, is the only real solution of the equation $\hat{d}_4 = 0$ in the range $(-2, 2)$. At the other point, $C_0 = -0.56$, the expansion function $\hat{W}_4$ itself vanishes, a phenomenon which can take place since the expansion functions are no longer simple powers of the coupling $\hat{a}_\mu$.

![FIG. 2: $\delta_{\text{Cl}}^{(0)}(\hat{a}_m_\tau)$ as a function of $C$. The yellow band arises from either removing or doubling the fifth-order term. In the red dots, the last term vanishes, and the previous one is taken as uncertainty.](image2)

As in [8], we define the truncation error as the magnitude of the last nonvanishing term, $\hat{d}_4W_3$, and choose $C_0 = -0.05$ as the optimal point, where the error shown in Fig. 1 is smallest. This leads to the prediction

$$\hat{D}(\hat{a}_\mu) = 0.1360 \pm 0.0080 \pm 0.0061,$$

where the first error is the truncation error and the second is due to the uncertainty in $\alpha_s(m_\tau^2)$. For comparison we note that to five loop the standard expansions give for the same quantity the values $0.1287 \pm 0.0050 \pm 0.0055$ in the MS scheme and $0.1343 \pm 0.0070 \pm 0.0067$ in the $C$-scheme [8], while the Borel-improved expansion defined in Sec. IV in the MS scheme gives $0.1357 \pm 0.0067 \pm 0.0061$. We note that the Borel-improved expansions to five loop give very close results in the MS and $C$-scheme, which is a first nice feature of these expansions.

We consider now the calculation of the QCD correction $\delta^{(0)}$ to the $\tau$ hadronic width [6]. The Borel-improved FO expansion of $\delta^{(0)}$ in the $C$-scheme is given in Eqs. (33)-(34). In Fig. 2 we show the result of the calculation to order $\alpha_s^2$, as central line, and the curves obtained by either doubling or removing the last term, which delimit the yellow band. As seen from the figure, there are three points where the curves coincide: the leftmost and the rightmost ones are the solutions of the equation $\hat{d}_4 = 0$, while the middle point is the solution of the equation $\hat{W}_4 = 0$, encountered already in Fig. 1. It turns out that the uncertainty, defined as the magnitude of the last nonvanishing term, $\hat{d}_3W_3$, is minimal for the leftmost point, $C_0 = -1$, which we adopt as optimal. This leads...
to the prediction
\[ \delta_{\text{FO}}^{(0)}(\hat{a}_{m_r}) = 0.2163 \pm 0.0039 \pm 0.0195, \]  
(37)

where, as above, the first error is the truncation error and the second accounts for the uncertainty in the coupling. We mention that the standard expansions predict 0.1991 \pm 0.0061 \pm 0.0119 in the \( \overline{\text{MS}} \) scheme and 0.2047 \pm 0.0054 \pm 0.0133 in the \( C \)-scheme, while the Borel-improved expansion in the \( \overline{\text{MS}} \) scheme to five loop gives 0.2136 \pm 0.0003 \pm 0.0133. Again, the results of the Borel-improved expansions in the \( \overline{\text{MS}} \) and \( C \)-scheme are closer than the corresponding predictions of the standard expansion. We note that the small truncation error of the Borel-improved expansion in the \( \overline{\text{MS}} \) scheme is due to an accidentally small value of the fourth-order term.

\[ \delta_{\text{CI}}^{(0)}(\hat{a}_{m_r}) = \frac{\delta_{\text{CI}}^{(0)}(\hat{a}_{m_r})}{\delta_{\text{CI}}^{(0)}(\hat{a}_{m_r})}, \]  
(38)

where, as above, the first error is the truncation error and the second accounts for the uncertainty in the coupling. We quote for completeness the values predicted by the standard expansions, 0.0\pm0.1793 \pm 0.0071 \pm 0.0080 in the \( \overline{\text{MS}} \) scheme and 0.1840 \pm 0.0062 \pm 0.0084 in the \( C \)-scheme, while the Borel-improved expansion in the \( \overline{\text{MS}} \) scheme gives 0.1978 \pm 0.00174 \pm 0.0115. The large truncation error of the last result is due to an accidentally large value of the fourth-order coefficient in this case.

It is of interest to investigate also the high-order behaviour of the Borel-improved expansions in the \( C \)-scheme. The high-order behaviour of the standard QCD expansions in the \( C \)-scheme was considered in Refs. [9, 10], where a renormalon-based model of the Adler function was adopted for generating the higher-order coefficients. We use the same model (presented for completeness in the Appendix) for assessing the quality of the Borel-improved expansions in the \( C \)-scheme.

In Fig. 4 we present the variation with \( C \) of the Borel-improved perturbative expansions of the Adler function \( \hat{D}(\hat{a}_{\mu}) \) at the spacelike point \( s = -m_{\tau}^2 \) and \( \mu = m_r \), for increasing orders of perturbation theory (\( N \) denotes the number of terms kept in the expansion). The figure shows a clear region of stability with respect with \( \overline{\text{MS}} \) and the remarkable convergence of the truncated expansions, which, as we shall see below, approach actually the exact result given by the model. We note also that the big variations which appear when \( C \) decreases towards the lower limit of the chosen range are due to the fact that in this range the perturbative connection between the QCD coupling in \( \overline{\text{MS}} \) and \( C \)-scheme breaks down, so the use of perturbation theory is not legitimate.

For a detailed comparison of the various expansions, we present in Table I the Adler function calculated for \( s = -m_{\tau}^2 \) and \( a_{\mu}(m_{\tau}^2) = 0.316 \) with the standard series in powers of the coupling in \( \overline{\text{MS}} \) and in \( C \)-scheme, and with the Borel-improved expansions in \( \overline{\text{MS}} \) and in \( C \)-scheme, respectively. The calculations in the \( C \)-scheme have been done, as in Refs. [9, 10], using for all \( N \) the same value of \( C \); taken to be the optimal value determined for the expansion truncated at \( N = 5 \) (i.e. \( C = -0.78 \) for the standard expansion \( \overline{\text{MS}} \), and \( C = -0.05 \) for the Borel-improved expansion). Of course, an optimal \( C \) can be defined for each truncation order \( N \) in practical applications. But the purpose of the present study is the comparison of two definite schemes, namely \( \overline{\text{MS}} \) and a particular \( C \)-scheme. Actually, as seen in Fig. 4, the Borel-improved expansions exhibit a common region of stability for all orders \( N \), so the choice of a single \( C \) is reasonable in this case.

From Table I one can see the divergent behaviour of the standard expansions (the first two columns), which is even more pronounced for the \( C \)-scheme. Due to the
FIG. 5: Real part (left) and imaginary part (right) of the Adler function in the model [25] and its truncated Borel-improved expansions in the C-scheme along the circle \(|s| = m^2 \exp(i\phi)|.

TABLE I: Adler function \(\hat{D}\) of the model [25] calculated with perturbative expansions truncated at order \(N\). First two columns: standard expansions. Last two columns: Borel-improved expansions. Exact value \(\hat{D} = 0.1354\). For details see the text.

| \(N\) | MS | C-scheme | MS | C-scheme |
|-------|----|-----------|----|-----------|
| 3     | 0.1237 | 0.1273 | 0.1289 | 0.1280 |
| 4     | 0.1287 | 0.1343 | 0.1357 | 0.1360 |
| 5     | 0.1316 | 0.1343 | 0.1360 | 0.1360 |
| 6     | 0.1350 | 0.1414 | 0.1359 | 0.1360 |
| 7     | 0.1369 | 0.1377 | 0.1354 | 0.1357 |
| 8     | 0.1410 | 0.1567 | 0.1353 | 0.1351 |
| 9     | 0.1420 | 0.1283 | 0.1354 | 0.1350 |
| 10    | 0.1509 | 0.2279 | 0.1354 | 0.1352 |
| 11    | 0.1453 | -0.0195 | 0.1354 | 0.1351 |
| 12    | 0.1816 | 0.8446 | 0.1354 | 0.1351 |
| 13    | 0.1164 | -2.004 | 0.1354 | 0.1352 |
| 14    | 0.3707 | 8.982 | 0.1354 | 0.1353 |
| 15    | -0.3412 | -34.61 | 0.1354 | 0.1353 |
| 16    | 2.3152 | 154.94 | 0.1354 | 0.1353 |
| 17    | -7.027 | -711.57 | 0.1354 | 0.1353 |
| 18    | 30.26 | 3522.7 | 0.1354 | 0.1353 |

factorial increase of the coefficients, the independence of the results on the renormalization scheme is not manifest for these expansions at large orders. On the other hand, for the Borel-improved expansions with tamed increase at large orders, the results in the two schemes are very close and converge to the exact result. This proves the renormalization-group independence of the perturbative expansions in QCD, once the divergent character is properly treated.

Previous investigations [19, 21, 25] considered also the large-order behaviour of the expansions of the quantity \(\delta^{(0)}\) defined in [8], which involves the values of the Adler function in the complex \(s\) plane. For the C-scheme the problem was studied in [9, 10]. As discussed in Sec. 11, this quantity allows one to extract only indirect information about the perturbation expansion of the Adler function itself along the circle.

In order to assess in a straightforward way the quality of the expansions in the complex plane, we compare in Fig. 5 the values of the Adler function calculated with the model presented in the Appendix for complex values \(s = m^2 \exp(i\phi)\), and its Borel-improved approximants in the C-scheme. We restricted \(\phi\) to the range \(\phi \in (0, \pi)\), the values on the semicircle in the lower half-plane being obtained by using the reality property \(\hat{D}(s^*) = \hat{D}^*(s)\). As above, we took the optimal \(C = -0.05\). One can remark the impressive convergence of the Borel-improved expansions in the C-scheme along the whole circle, up to higher orders. As shown in [19, 21], a similar behaviour is obtained for the Borel-improved expansions in \(\overline{\text{MS}}\) scheme, while the standard expansions show big oscillations far from the true values.

VII. DISCUSSION AND CONCLUSIONS

In the present paper we investigated the renormalization scheme variation of an improved perturbation expansion in QCD, with tamed behaviour at large orders, defined by means of the optimal conformal mapping of the Borel plane. Detailed studies performed in previous works [19, 21] demonstrated the good properties of these Borel-improved expansions in the \(\overline{\text{MS}}\) scheme. The analysis was extended now to a class of renormalization schemes, denoted as C-scheme, proposed in [8] and investigated in the frame of standard QCD expansions in [8, 10]. Our purpose was not to advocate the advantage of a particular scheme, but to study the variation with the renormalization scheme of a quantity known to be renormalization-scheme invariant. We performed our study using the perturbation expansion of the Adler function and the \(\tau\) hadronic width.

For the expansions to order \(\alpha_s^5\), we found, as in [8], a
range of stability with respect to the variation of the $C$ parameter, shown in Figs. [4]. We selected an optimal value of $C$ using the same prescription as in [5], namely the vanishing of the highest term in the expansion, the last nonzero term being taken as a conservative estimate of the truncation error. While there is of course some arbitrariness in defining the optimal $C$ and the truncation error, this prescription is in the spirit of asymptotic expansions and the recipes for scheme fixing proposed in the literature [1] [2]. However, as noticed in the previous section, it can lead in some cases to unnatural estimates of the truncation uncertainty, due to accidentally large or small values of the last nonzero coefficient.

As concerns the central values, we obtained quite close predictions of the Borel-improved expansions to five loop in the $\overline{\text{MS}}$ and $C$-scheme. This property, found for both the Adler function and the phenomenological parameter $\delta^{(4)}$, announces the scheme-independence exhibited by the higher-order Borel-improved expansions evaluated in these schemes.

Indeed, the good properties of the Borel-improved expansions manifest themselves in an impressive way at large orders. In our study, for generating the higher perturbative coefficients we used as in [9] [10] a theoretical model for the Adler function, proposed in [25]. The results shown in Figs. 3 and 4 prove the remarkable independence of the QCD perturbation theory, once the model are determined such as to generate the low-order divergence is properly treated.

The results shown in Figs. 3 and 4 prove the remarkable independence of the theory of the Adler function, proposed in [25]. We give below the values of the coefficients used in the calculations presented in Sec. VI:

$$c_6 = 3275.45, \quad c_7 = 18758.4, \quad c_8 = 388446, \quad c_9 = 919119, \quad c_{10} = 8.36 \times 10^7,$$

$$c_{11} = -5.19 \times 10^8, \quad c_{12} = 3.38 \times 10^{10}, \quad c_{13} = -6.04 \times 10^{11}, \quad c_{14} = 2.34 \times 10^{13},$$

$$c_{15} = -6.52 \times 10^{14}, \quad c_{16} = 2.42 \times 10^{16}, \quad c_{17} = -8.46 \times 10^{17}, \quad c_{18} = 3.36 \times 10^{19}. \quad (A6)$$

One can note the dramatic increase of the coefficients, which implies that the perturbation series of the Adler function in this model is divergent.

For $\alpha_s(m_t^2) = 0.316 \pm 0.010$, the Adler function at $s = -m_t^2$ and $\mu = m_t$ given by this model has the value $\hat{D}(a_{n_r}) = 0.1354 \pm 0.0127 \pm 0.0058$, where the first error comes from renormalon ambiguity and the second from the uncertainty of the coupling.

Appendix A: Model of the Adler function

For testing the convergence of the various expansions, we considered the model proposed in [25], which expresses the Adler function by means of the PV-regulated Laplace-Borel integral:

$$\hat{D}(a_n) = \frac{1}{\beta_0} \text{PV} \int_0^\infty e^{-\frac{u}{\beta_0}} B(u) \, du, \quad (A1)$$

with a Borel transform $B(u)$ parametrized in terms of a few UV and IR renormalons. Specifically, in the model proposed in [25], the function $B(u)$ is expressed as

$$B(u) = B_1^{UV}(u) + B_2^{IR}(u) + B_3^{IR}(u) + d_0^{PO} + d_1^{PO} u, \quad (A2)$$

where

$$B_p^{IR}(u) = \frac{d_p^{IR}}{(p-u)^\gamma} \left[ 1 + \bar{b}_1(p-u) + \ldots \right], \quad (A3)$$

$$B_p^{UV}(u) = \frac{d_p^{UV}}{(p+u)^\gamma} \left[ 1 + b_1(p+u) + \ldots \right]. \quad (A4)$$

The free parameters of the model are the residues $d_1^{UV}, d_2^{IR}, d_3^{IR}$ of the first renormalons and the coefficients $d_0^{PO}, d_1^{PO}$ of the polynomial in $A2$, determined in [25] as

$$d_1^{UV} = -1.56 \times 10^{-2}, \quad d_2^{IR} = 3.16, \quad d_3^{IR} = -13.5, \quad d_0^{PO} = 0.781, \quad d_1^{PO} = 7.66 \times 10^{-3}, \quad (A5)$$

by the requirement to reproduce the perturbative coefficients $c_{n_1}$ in the $\overline{\text{MS}}$ scheme for $n \leq 4$, given in [5], and the estimate $c_{5,1} = 283$.

Once the parameters are fixed, the model predicts all the higher order perturbative coefficients $c_{n,1}$ for $n > 5$. We give below the values of the coefficients used in the calculations presented in Sec. [VI].

$$c_{6,1} = 3275.45, \quad c_{7,1} = 18758.4, \quad c_{8,1} = 388446, \quad c_{9,1} = 919119, \quad c_{10,1} = 8.36 \times 10^7,$$

$$c_{11,1} = -5.19 \times 10^8, \quad c_{12,1} = 3.38 \times 10^{10}, \quad c_{13,1} = -6.04 \times 10^{11}, \quad c_{14,1} = 2.34 \times 10^{13},$$

$$c_{15,1} = -6.52 \times 10^{14}, \quad c_{16,1} = 2.42 \times 10^{16}, \quad c_{17,1} = -8.46 \times 10^{17}, \quad c_{18,1} = 3.36 \times 10^{19}. \quad (A6)$$

Acknowledgments

I thank D. Boito for useful discussions and suggestions on the manuscript. This work was supported by the Romanian Ministry of Research and Innovation, Contract PN 18090101/2018.
[1] P.M. Stevenson, Phys. Rev. D 23, 2916 (1981).
[2] G. Grunberg, Phys. Rev. D 29, 2315 (1984).
[3] S. Brodsky, P. Lepage and P. Mackenzie, Phys. Rev. D 28, 228 (1983).
[4] J. Chyla, A. Kataev and S. Larin, Phys. Lett. 267, 269 (1991).
[5] S.J. Brodsky and L. Di Giustino, Phys. Rev. D 86, 085026 (2012).
[6] J.M. Shen, X.G. Wu, B.L. Du and S.J. Brodsky, Phys. Rev. D 95, 094006 (2017).
[7] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D 18, 3998 (1978).
[8] D. Boito, M. Jamin and R. Miravitllas, Phys. Rev. Lett. 117, 152001 (2016).
[9] D. Boito, M. Jamin and R. Miravitllas, EPJ Web of Conferences 137, 05007 (2017).
[10] D. Boito, M. Jamin and R. Miravitllas, Nucl. Part. Phys. Proc. 287, 77 (2017).
[11] X.-G. Wu, J.-M. Shen, B.-L. Du and S.J. Brodsky, Phys. Rev. D 97, 094003 (2018).
[12] G. ’t Hooft, Can we make sense out of Quantum Chromodynamics? in: The WHys of Subnuclear Physics, Proceedings of the 15th International School on Subnuclear Physics, Erice, Sicily, 1977, edited by A. Zichichi (Plenum Press, New York, 1979), p. 943.
[13] A.H. Mueller, Nucl. Phys. B250, 327 (1985).
[14] A.H. Mueller, in QCD - twenty years later, Aachen 1992, edited by P. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1992).
[15] M. Beneke, Phys. Rep. 317, 1 (1999).
[16] I. Caprini and J. Fischer, Phys. Rev. D 60, 054014 (1999).
[17] I. Caprini and J. Fischer, Phys. Rev. D 62, 054007 (2000).
[18] I. Caprini and J. Fischer, Eur. Phys. J. C 24, 127 (2002).
[19] I. Caprini and J. Fischer, Eur. Phys. J. C 64, 35 (2009).
[20] I. Caprini and J. Fischer, Nucl. Phys. B Proc. Suppl., 218, 128 (2011).
[21] I. Caprini and J. Fischer, Phys. Rev. D 84, 054019 (2011).
[22] G. Abbas, B. Ananthanarayan, I. Caprini and J. Fischer, Phys. Rev. D 87, 014008 (2013).
[23] G. Abbas, B. Ananthanarayan, I. Caprini and J. Fischer, Phys. Rev. D 88, 034026 (2013).
[24] I. Caprini, J. Fischer, G. Abbas, B. Ananthanarayan, Perturbative Expansions in QCD Improved by Conformal Mappings of the Borel Plane, in Perturbation Theory: Advances in Research and Applications, Nova Science Publishers (2018), pp. 211-254.
[25] M. Beneke and M. Jamin, JHEP 09, 044 (2008).
[26] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 118, 082002 (2017).
[27] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 101, 012002 (2008).
[28] S. Narison and A. Pich, Phys. Lett. B211, 183 (1988).
[29] E. Braaten, Phys. Rev. Lett. 60, 1606 (1988).
[30] E. Braaten, S. Narison and A. Pich, Nucl. Phys. B373, 581 (1992).
[31] F.L. Diberder and A. Pich, Phys. Lett. B286, 147 (1992).
[32] M.R. Ahmady et al., Phys. Rev. D 67, 034017 (2003).
[33] G. Abbas, B. Ananthanarayan and I. Caprini, Phys. Rev. D 85, 094018 (2012).
[34] S. Menke, [arXiv:0904.1796 [hep-ph]].
[35] A. Pich, Acta Phys. Polon. Supp. 3, 165 (2010).
[36] S. Descotes-Genon and B. Malaescu, [arXiv:1002.2968 [hep-ph]].
[37] A. Pich, Nucl. Phys. B Proc. Suppl., 218, 89 (2011).
[38] A. Pich, Workshop on Precision Measurements of αs, ed. S. Bethke et al, page 18, [arXiv:1110.0016 [hep-ph]].
[39] M. Beneke, D. Boito, M. Jamin, JHEP 1301, 125 (2013).
[40] A. Pich, A. Rodríguez-Snchez, Phys.Rev. D 94, 034027 (2016).
[41] D. Boito, M. Golterman, K. Maltman, S. Peris, Phys. Rev. D 95, 034024 (2017).
[42] I. Caprini and M. Neubert, JHEP 03, 007 (1999).
[43] S. Ciulli and J. Fischer, Nucl. Phys. B24, 465 (1961).
[44] M. Beneke, V.M. Braun and N. Kivel, Phys. Lett. B404, 315 (1997).
[45] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).