Quantum Memory Assisted Precision Rotation Sensing

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We propose to implement a solid-state rotation sensor by employing a many-body quantum spin system which takes the advantages of the easy controllability of the electron spin and the robustness provided by the collective nuclear spin state. The sensor consists of a central electron spin coupled to many surrounding nuclear spins. Previously, this central spin system has been suggested to realize a quantum memory. Here, we further utilize the collective nuclear spins, which store a certain quantum state, to detect the macroscopic rotation. Different from other nuclear spin based gyroscopes, our proposal does not directly manipulate nuclear spins via nuclear magnetic resonance technique. We also analytically and numerically investigate the effects of partial nuclear polarization and decoherence on the sensitivity in a full quantum mechanical fashion. Our proposal paves the way to the experimental realization of a compact solid-state, full-electrical and spin-based gyroscope.

Quantum spin systems are attractive candidates to implement a quantum gyroscope, such as the well-explored nuclear magnetic resonance gyroscopes (NMRG) 1–3, which have manifested excellent sensitivity in a laboratory. However, these systems are usually difficult to be minimized which limits their practical applications. Recently, solid-state quantum spin systems, such as nitrogen-vacancy (NV) centers in a diamond, have been proposed to realize a rotation sensor via the geometric phase 4–6. These solid-state spin gyroscopes are promising to be miniaturized and to reduce the power consumption. Moreover, the rotating solid-state quantum spin system has been investigated in recent experiments 7,8, and shown good prospect as a rotation sensor.

Unlike the spin-based magnetometry 9, the geometric phase change is independent of the gyromagnetic ratio in a spin-based gyroscope. Therefore, a nuclear spin is a better candidate than an electron spin as a rotation sensor, due to its stability to magnetic noise and long coherence time 5,6. However, because of the inefficient manipulation and measurement of nuclear spins, the gyroscope sensitivity is greatly limited in practice. An attractive way is to utilize the hyperfine interaction to manipulate and measure the nuclear spin state through the electron. Single electronic spin coupled with single nuclear spin has been suggested to implement such a gyroscope in diamond 10. However, the required pulse sequences of this protocol are rather complicated, consisting of both nuclear magnetic resonance (NMR) and electron spin resonance (ESR). In addition, the single nuclear spin proximal to the NV electronic spin is also very sensitive to external perturbations 11.

The long-lived quantum memory proposed by use of a semiconductor quantum dot, takes the advantages of the fast electron spin manipulation and the long coherence time provided by nuclei 12. Specifically, this protocol circumvents the difficulty of controlling single nuclear spins by utilizing collective nuclear spin state. In this Letter, we combine the quantum memory technique with the nuclear spin rotation sensing, to implement a quantum memory assisted rotation sensor. The most significant advantage of our protocol is that, it only needs the fast and high efficient electron spin manipulations and measurement instead of the slow and inefficient nuclear spin manipulations 13. Furthermore, we also show that, the sensitivity of the rotation sensor can be greatly enhanced when the nuclear spins are inhomogeneously polarized.

Rotation sensing protocol.— We illustrate our proposal by use of the many-spin system in a semiconductor quantum dot, as depicted in Fig. 1. The protocol includes three stages, the encoding, the sensing and the retrieval stage. At the beginning of the encoding stage, an electron in a certain spin state [here, the spin is polarized along x axis, $|\psi_e(0)\rangle = (1/\sqrt{2})(|\uparrow\rangle_e + |\downarrow\rangle_e)$] is injected into a quantum dot. With an external magnetic field properly tuned on resonance, the electron spin state is mapped into the nuclear spins’ collective state after a half period of Rabi oscillation 12. This state mapping helps to build up the coherence of the collective nuclear spin state. Then, the electron is ejected from the quantum dot. The nuclear spins experience a pseudo magnetic field due to the macroscopic rotation and undergo a coherent precession in the rotating reference frame during the sensing stage. After the rotation sensing, a new electron polarized along the $-z$ axis ($|\downarrow\rangle_e$) is injected, along with the external magnetic field tuned on resonance again. After another half period of Rabi oscillation, the collective nuclear spin state with rotational information encoded, is mapped back into the electron spin and the electron spin state is measured subsequently.

We illustrate the protocol described above using the central spin model, which can describe a central elec-
tron spin coupled to many surrounding nuclear spins via hyperfine interaction in a semiconductor quantum dot. During the short encoding and retrieval stages, the Hamiltonian is,

$$H = g_e \mu_B B_0 S_z + g_n \mu_n B_0 \sum_{j=1}^{N} I_{jz} + \sum_{k=1}^{N} A_k \mathbf{S} \cdot \mathbf{I}_k. \quad (1)$$

The first term and the second term are the Zeeman energies of the electron spin and nuclear spins, respectively, where $g_e$ ($g_n$) is the Landé g-factor of the electron (nucleus), $\mu_B$ ($\mu_n$) is the Bohr magneton (nuclear magneton), and $B_0$ is the magnitude of the external magnetic field applied along $z$ axis. The third term is the hyperfine contact interaction of the electron with $N$ surrounding nuclei, where $\mathbf{S}$ ($\mathbf{I}_k$) is the spin operator of the electron ($k$-th nucleus). Typically, the hyperfine coupling coefficient $A_k$ is nonuniform, for example, in a quantum dot, $A_k \propto |\psi(r_k)|^2$, where $|\psi(r_k)|^2$ is the electron profile density at site $r_k$ of the $k$-th nucleus. This nonuniformity ($A_k \neq A$) intrinsically limits the fidelity of quantum memory and will also limit the sensitivity of the rotation sensing proposed in this Letter. Specifically, for the encoding and retrieval processes to work, an external magnetic field needs to be applied to bring the system into resonance (spin-exchange interaction in a semiconductor quantum dot). This term becomes $\Omega \sum_{k=1}^{N} I_{kz}$, where $\Omega$ is the rotational vector, in the frame rotating at an angular frequency $\Omega$. For simplicity, we assume the rotational vector to direct along the $z$ axis, so the first term becomes $\Omega \sum_{k=1}^{N} I_{kz}$. The second term describes the dipole-dipole interaction between nuclear spins, where the coefficient $\Gamma_{jk}$ is the dipolar coupling strength. This term is neglected in $H$ (Eq. 1), since the rate of state transfer ($\sim$GHz) is many orders of magnitude faster than the typical decoherence rate ($\sim$kHz) induced by the nuclear dipolar interaction [12]. However, during the rotation sensing stage, we cannot neglect the nuclear dipolar interaction any more, since for high sensitivity purpose, the sensing time usually needs to be as long as possible, which is ultimately restricted by the decoherence.

To demonstrate the entire rotation sensing protocol, as shown in Fig. 1 we first consider the simplest case that nuclear spins are perfectly polarized, namely, $P = 1$. In this case, the dipolar interaction between nuclear spins becomes greatly suppressed [10], so we can temporarily neglect the nuclear dipole-dipole term, $H_S \approx \Omega \sum_{k=1}^{N} I_{kz}$. Because of the conservation of the total spin, $[J_z, H] = [J_z, H_S] = 0$, where $J_z = S_z + \sum_{k=1}^{N} I_{kz}$, the Hamiltonians can be expanded in a set of orthonormal basis states [17]: $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$, and $\langle\psi_0\rangle$, $\langle\psi_1\rangle$, $\langle\psi_2\rangle$. The Hamiltonian $H$ can be represented as,

$$H = \frac{\sqrt{M_2}}{2} \begin{bmatrix} \theta & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} \delta \\ 0 & 0 & \sqrt{2} \delta & 0 \end{bmatrix},$$

where $\delta = ||u'||/\sqrt{2}$ and $\theta = \sqrt{M_4/M_2}$. In realistic physical systems, for example, in a semiconductor quantum dot, $A_k$ is often supposed in a Gaussian distribution, then $\delta, \theta \sim 1/\sqrt{N}$, which are small quantities when $N$ is large. The initial state of the compound system is $\rho(0) = \langle\Psi(0)| \langle\Psi(0)$, where $|\Psi(0)\rangle = |\psi_e(0)\rangle \otimes |\phi_0\rangle$. The
where \( E = e^{-iHt} \), \( S = e^{-iH_{21}t} \), and \( UR = e^{-iH_{21}t} \) are the evolution operators during the encoding, the sensing and the retrieval stage, respectively. The encoding (retieval) time \( t_{E(R)} = \pi/\sqrt{M_2} \), and \( t_S \) is the rotation sensing time.

The operators that describe electron ejection and injection are defined as, \( P_\uparrow = |\uparrow⟩⟨\downarrow| \) and \( S_{\pm} = S_x ± i S_y \). Then the expectation values of the electron spin can be calculated, \( ⟨S_i⟩ = Tr[τ_e S_i] \), with \( i = \alpha, \beta, \gamma \), where \( \tau_e = Tr_n \rho_F \) is the reduced density matrix of the electron spin. Usually, the electron spin level population is measured in experiment and the corresponding signal (here the probability that \( |\uparrow⟩ \) is populated after applying an electron spin \( \pi/2 \) pulse) is,

\[
S = \frac{1}{2} + \text{Tr}[e^{-i\frac{\pi}{2}S_y \tau_e} e^{i\frac{\pi}{2}S_y S_z}]
\]

\[
= \frac{1}{2} - \sin^2 (\frac{\lambda}{2}) \left[ 1 + (2\lambda^2 - 1) \cos (\frac{\pi}{2}) \right] \cos (\pi \theta - \Omega t)
\]

where \( \lambda = \sqrt{1+\delta^2} \). This analytical result demonstrates that the retrieved electron spin signal indeed oscillates at the rotational angular frequency \( \Omega \). The finite value of \( \delta \) prevents complete transfer of spin state, just as in the quantum memory protocol. Specifically, we note that the finite value of \( \theta \) introduces an extra phase in the oscillation. This analytical result is plotted in Fig. 2(d), which shows great agreement with numerical simulations (see the following part).

**Effects of partial polarization and decoherence.** — However, in realistic situations, the nuclear polarization is far from perfect \( (P < 1) \), which limits the state transfer fidelity and reduces the signal contrast. At the same time, the nuclear dipolar interaction induced decoherence also intrinsically limits the sensitivity. In order to study these imperfections, we use exact numerical method, based on the Chebyshev polynomial expansion of the evolution operator [18], to simulate the dynamics of this many-spin system. Here, we follow the configurations used in previous work [19], where \( N = 20 \) nuclei are placed on a \( 4 \times 5 \) 2D lattice, with the value of \( A_k \)'s in the range of 0.31 to 0.96, corresponding to a Gaussian \( \{\psi(r)\}^2 \), and obtain \( \Gamma_{jk} \)'s from uniformly distributed random numbers in the range of -0.01 to 0.01.

Specifically, we will consider two nuclear polarization circumstances in detail. Thermal nuclear polarization, which usually needs applying a large static magnetic field [15], results in homogeneous nuclear polarization and the initial nuclear state is \( \rho_n(0) = (1/Z) \exp(-7 \sum_k I_{2k}) \), where \( Z \) is the partition function and \( \gamma = 2 \tanh^{-1}(P) \). On the other hand, dynamic nuclear polarization (DNP), for example, by passing a series of spin-polarized electrons through a quantum dot, leads to inhomogeneous nuclear polarization [20, 21] and the initial nuclear state is \( \rho_n(0) = \otimes_{k=1}^N \rho_{nk}(0) \), with \( \rho_{nk}(0) = (1/2)I + p_k I_{zz} \), where \( p_k = 1 - \exp(-2\beta A_k^2) \) is the polarization of the \( k \)-th nuclear spin \( \beta \) is a parameter reflecting an effective spin temperature) and the average nuclear polarization is defined as, \( P = (1/N) \sum_{k=1}^N p_k \). The evolution of the entire rotation sensing process can be calculated following Eq. 4 and here the initial state is \( \rho(0) = |\psi_e(0)⟩⟨\psi_e(0)| \otimes \rho_n(0) \).

The numerically calculated electron spin population \( S \) versus sensing time \( t_S \) is plotted in Fig. 2(a) and (b). Indeed, the nuclear dipole-dipole interaction results in decay of the signal, restricting the available sensing time and eventually limiting the sensitivity. In Fig. 2(c), we plot the Fourier analysis of the signals in (a). (d) The analytical result (solid line) and numerical simulation result (circles) for the case of perfect nuclear polarization with \( N = 20 \). The phase shift is due to the finite value of \( \theta \).
with the same average nuclear polarization, the inhomogeneous case presents larger oscillation amplitude than the homogeneous case, indicating higher signal-to-noise ratio in experiment. These features are similar to that of the quantum memory protocol \textsuperscript{10}, since the increase of the state transfer fidelity naturally leads to an improvement of the signal contrast and the sensitivity.

Theoretical estimation of the sensitivity. — We quantify the performance of this rotation sensing protocol by calculating the quantum Fisher information (QFI), which is defined \textsuperscript{22} as, \( F_Q = \text{Tr}[\tau_cL^2] \), where \( L \) is the symmetric logarithmic derivative (determined by \( \partial \tau_c/\partial \Omega = (1/2)[\tau_cL + L\tau_c] \)). The QFI as a function of average nuclear polarization with fixed sensing time is depicted in Fig. 3(c). Apparently, the QFI increases as the average nuclear polarization increases, indicating an improvement in measurement precision. In addition, under the same \( P \), the inhomogeneous case exhibits much larger QFI than the homogeneous case, almost 2 times larger when \( P = 0.7 \). We also find that the behavior of the homogeneous case is in accordance with the analytical model (by assuming \( A_k = A \text{\textsuperscript{17}} \)), whereas the inhomogeneous case shows nontrivial characteristics, which can be explained by the utilization of the nonuniformity of the hyperfine coupling \textsuperscript{19}.

Next, we focus on the effects of decoherence on the sensitivity. Using the quantum Cramér-Rao bound \textsuperscript{22}, we can estimate the sensitivity from the QFI, \( \delta \Omega = \sqrt{T_S + t_M/(C \sqrt{F_Q})} \), where \( C \) is the coefficient measuring the readout efficiency of the electron spin state; \( t_M \) is the dead time which is required for initialization, transfer, and readout of the electron spin state. In Fig. 3(d), we present the estimated sensitivity. As we can see, there indeed exists an optimal sensing time, \( t_S \approx T_2^* / 2 \), where \( T_2^* \) can be obtained by fitting to an exponential decay of the corresponding signal in Fig. 2(b). Again, the inhomogeneous case always exhibits an enhanced sensitivity compared with the homogeneous case.

Our proposal can be implemented using a semiconductor quantum dot, where the measurement time \( t_M \) of the electron spin can be as short as 100 ns \textsuperscript{24}, the readout visibility can be as high as 80\% \textsuperscript{22}, and the maximum average nuclear polarization achieved so far is about 65\% \textsuperscript{26}. We estimate the nuclear decoherence time \( T_2^* \approx 3 \text{ ms} \textsuperscript{12, 27} \) and the sensitivity is estimated \( \delta \Omega \approx 50 \text{ rad s}^{-1} \text{ Hz}^{-1/2} \) per sensor unit (or per qubit).

Discussion. — Compared to proposals using NV centers in diamond \textsuperscript{2, 10}, our proposal shows substantial advantages. First, in a semiconductor quantum dot, single-shot readout of electron spin state can be realized via spin-to-charge conversion \textsuperscript{25, 28}, with a much higher readout efficiency (larger \( C \)) and much faster measurement speed (\( t_M \ll t_S \)) than the optical readout in diamond. Second, the single proximal nuclear spin in diamond has relatively short coherence time because of its strong interaction with the NV electronic spin \textsuperscript{24}. On the contrary, the electron can be removed from the quantum dot, so the hyperfine interaction can be turned off completely, resulting in a remarkably long nuclear coherence time \textsuperscript{31}. Third, our proposal circumvents direct nuclear manipulations via NMR technique by exploiting the coherent spin state transfer, while it is widely known that the nuclear spin manipulation (especially for single nuclear spins) is usually inefficient and always brings extra noise into the system \textsuperscript{13}. Fourth, by use of a gate-defined quantum dot \textsuperscript{31}, a full-electrical spin-based rotation sensor seems feasible, since nuclear spins can be electrically polarized by DNP \textsuperscript{21, 32} and the electron spin state can be electrically readout via spin-to-charge conversion \textsuperscript{28}. The full-electrical solutions can be much more compact and easier to integrate with other quantum devices than the optical solutions.

In principle, the rotation sensing technique proposed in this Letter is widely suited to various quantum systems, that can be implemented as a quantum memory, such as atomic ensembles \textsuperscript{33} or other solid-state spin systems \textsuperscript{34, 35}. Via the coherent state transfer, en-
tanglement may be generated in this many-spin system, which can also be used to improve the sensitivity [36, 37]. Besides, this protocol combines a quantum sensor with a quantum memory, so it may allow us to implement the quantum non-demolition (QND) measurement to increase the sensitivity even further [38, 39].

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I. ANALYTICAL RESULT FOR THE CASE WITH PERFECT NUCLEAR POLARIZATION (\(P = 1\)).

When the nuclear spins are perfectly polarized, namely, \(P = 1\), the analytic result can be obtained. At the start of the encoding stage, the initial state, namely, \(|\psi_0\rangle = (1/\sqrt{2})(|\uparrow\rangle_e + |\downarrow\rangle_e) \otimes |\ldots\rangle_{n'}\) Here, we first define two basis states, \(|\psi_0\rangle = |\rangle_e \otimes |\ldots\rangle_{n'}\) and \(|\psi_1\rangle = |\uparrow\rangle_e \otimes |\ldots\rangle_{n'}\). Obviously, \(|\psi_0\rangle\) is the eigenstate of the Hamiltonian, namely,

\[ H |\psi_0\rangle = -\frac{g_e B}{2}(B_0 + B_{\text{ovh}}) |\psi_0\rangle, \]

where \(B_{\text{ovh}} = -\sum_k A_k/2\) is the Overhauser field. Since \(g_e \mu_n \ll g_e B\), we neglect the nuclear Zeeman term in \(H\). Similarly,

\[ H |\psi_1\rangle = \frac{g_e B}{2}(B_0 + B_{\text{ovh}} + M_2/M_0) |\psi_1\rangle + b/2 |\psi_2\rangle, \tag{2} \]

where \(|\psi_2\rangle = (1/\sqrt{M_2}) \sum_k A_k |\rangle_e \otimes (I_+^k |\ldots\rangle_{n'})\) is the third basis state and we define the moment of the distribution of the coupling strength, \(M_n = \sum_k A_k^n\) and \(b = \sqrt{M_2}\) is the Rabi frequency. When applying the Hamiltonian on \(|\psi_2\rangle\), we get,

\[ H |\psi_2\rangle = -\frac{g_e B}{2}(B_0 + B_{\text{ovh}} + M_2/M_0) |\psi_2\rangle + b/2 |\psi_1\rangle + b/2 |u'\rangle, \tag{3} \]

where

\[ |u'\rangle = -\frac{1}{\sqrt{2}M_2} \sum_k A_k |\downarrow\rangle_e \otimes (I_+^k |\ldots\rangle_{n'}) - M_3/M_2 |\psi_2\rangle, \]

and \(||u'||^2 = (M_4/M_2^2 - M_3^2/M_2^2)\) is the norm of \(|u'\rangle\). When the external field is applied on resonance, \(B_0 = -B_{\text{ovh}} - M_3/(2M_2),\) the Hamiltonian \(H\) after subtracting a constant term, \(\tilde{H} = H + M_3/(2M_2),\) can be expanded in the basis \(|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle,\) and \(|\psi_3\rangle = |u'\rangle / ||u'||\), as,

\[ \tilde{H} = \frac{b}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2}\delta \\ \theta & 0 & 0 & 0 \end{bmatrix}, \tag{4} \]

where \(\delta = ||u'|| / \sqrt{2}\) and \(\theta = M_3/(M_2^{3/2})\).

II. ANALYTICAL SOLUTION FOR THE UNIFORM COUPLING CASE (\(A_k = A\)).

In order to analytically investigate the effects of average nuclear polarization on sensitivity, we consider the case that the electron spin has a uniform coupling with nuclear spins, namely, \(A_k = A\), so the Hamiltonian during the encoding and retrieval stage can be simplified as,

\[ H = g_e \mu_B B_0 S_z + A S \cdot I, \tag{5} \]

where \(I = \sum_{k=1}^N I_k\) is the collective nuclear spin operator. For simplicity, we also neglect the nuclear dipole-dipole interaction during the sensing stage,

\[ H_S \approx \Omega I_z. \tag{6} \]

Since \(\{I^2, H\} = \{I^2, H_S\} = 0\), the magnitude of the collective nuclear spin angular momentum \(I_0\) is the constant of motion during the entire rotation sensing protocol. We first calculate the spin dynamics with the collective nuclear spin state, \(|I_0, M_0\rangle\), where \(I_0(I_0 + 1)\) is the eigenvalue of \(I^2\) and \(M_0\) is the eigenvalue of \(I_z\), and then average over the distribution of \(I_0\) and \(M_0\) corresponding to the thermal nuclear spin state due to partial nuclear polarization. The Hamiltonian can be expanded in the basis \(|\uparrow\rangle \otimes |I_0, M_0\rangle, |\downarrow\rangle \otimes |I_0, M_0 + 1\rangle, |\downarrow\rangle \otimes |I_0, M_0\rangle\) and \(|\downarrow\rangle \otimes |I_0, M_0 - 1\rangle\), as,

\[ H = \begin{bmatrix} p_1 & 0 & q_2 & 0 \\ 0 & p_1 - \frac{A}{2} & 0 & q_1 \\ q_2 & 0 & -p_1 - \frac{A}{2} & 0 \\ 0 & q_1 & 0 & -p_1 \end{bmatrix}, \tag{7} \]

where,

\[ q_1 = \frac{A}{2} \sqrt{(I_0 + M_0)(I_0 - M_0 + 1)}, \]

\[ q_2 = \frac{A}{2} \sqrt{(I_0 - M_0)(I_0 + M_0 + 1)}, \]

\[ p_1 = \frac{A}{2} M_0 + \frac{1}{2} g_e \mu_B B_0. \]

The spin dynamics during the entire rotation sensing procedure can be calculated as follows:
1. The initial state of the system is,

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |I_0, M_0\rangle. \quad (8) \]

2. After the encoding process, the state evolves into,

\[ |\psi(t_E)\rangle = e^{-iH_{t_E}}|\psi(0)\rangle, \]

and we denote this state as,

\[ |\psi(t_E)\rangle = |\uparrow\rangle \otimes |v_1\rangle + |\downarrow\rangle \otimes |v_2\rangle, \quad (9) \]

where \( |v_1\rangle = \alpha_1|I_0, M_0\rangle + \alpha_2|I_0, M_0 - 1\rangle \) and \( |v_2\rangle = \alpha_3|I_0, M_0 + 1\rangle + \alpha_4|I_0, M_0\rangle. \]

3. After the ejection of the electron spin, the compound system evolves into a mixed state,

\[ \rho(t_E) = |\uparrow\rangle \langle \uparrow| \otimes |v_1\rangle \langle v_1| + |\downarrow\rangle \langle \downarrow| \otimes |v_2\rangle \langle v_2|. \quad (10) \]

4. During the retrieval stage, the nuclear spins precess in the common pseudo magnetic field, and after the rotation sensing the state evolves into,

\[ |v'_1\rangle = e^{-iH_{t_R}}|v_1\rangle = \alpha_1 e^{-iM_0 \Omega t} |I_0, M_0\rangle + \alpha_2 e^{-i(M_0 - 1) \Omega t} |I_0, M_0 - 1\rangle, \]

\[ |v'_2\rangle = e^{-iH_{t_R}}|v_2\rangle = \alpha_3 e^{-iM_0 \Omega t} |I_0, M_0 + 1\rangle + \alpha_4 e^{-i(M_0 + 1) \Omega t} |I_0, M_0 + 1\rangle. \quad (11) \]

5. After the injection of a new polarized electron spin, the state of the compound system becomes,

\[ \rho'(t_R) = |\downarrow\rangle \langle \downarrow| \otimes (|v'_1\rangle \langle v'_1| + |v'_2\rangle \langle v'_2|). \quad (12) \]

6. The evolution during the retrieval stage is again unitary,

\[ \rho(t_R) = e^{-iH_{t_R}} \rho'(t_E) e^{iH_{t_R}} = |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|, \]

where \( |\psi_1\rangle = e^{-iH_{t_R}} |\downarrow\rangle \otimes |v'_1\rangle \) and \( |\psi_2\rangle = e^{-iH_{t_R}} |\downarrow\rangle \otimes |v'_2\rangle \). For the calculation of \( |\psi_1\rangle \), theHamiltonian can be expanded in the basis, \(|\uparrow\rangle \otimes |I_0, M_0 - 1\rangle, |\uparrow\rangle \otimes |I_0, M_0 - 2\rangle, |\downarrow\rangle \otimes |I_0, M_0\rangle \) and \(|\downarrow\rangle \otimes |I_0, M_0 - 1\rangle \), and the matrix has the same form as Eq. 10 except changing the entries by \( M_0 \rightarrow M_0 + 1 \); For the calculation of \( |\psi_2\rangle \), the Hamiltonian can be expanded in the basis, \(|\uparrow\rangle \otimes |I_0, M_0 + 1\rangle, |\uparrow\rangle \otimes |I_0, M_0\rangle, |\downarrow\rangle \otimes |I_0, M_0 + 2\rangle \) and \(|\downarrow\rangle \otimes |I_0, M_0\rangle \), and the matrix has the same form as Eq. 10 except changing the entries by \( M_0 \rightarrow M_0 - 1 \).

After the final state is obtained, the expectation value of the electron spin can be calculated,

\[ \langle S_i \rangle' = \langle S_i \rangle = \langle \psi_1|S_i|\psi_1\rangle + \langle \psi_2|S_i|\psi_2\rangle, \quad (13) \]

with \( i = x, y, z \) and \( S_x = |\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow| \), \( S_y = -i |\uparrow\rangle \langle \downarrow| + i |\downarrow\rangle \langle \uparrow| \) and \( S_z = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|. \) The last step is the average over the distribution of \( I_0 \) and \( M_0 \),

\[ \langle S_i \rangle = \sum_{I_0 = 0}^{N/2} \sum_{M_0 = -I_0}^{I_0} w(I_0, M_0) \langle S_i \rangle', \quad (14) \]

where \( w(I_0, M_0) \) is the probability distribution of \( I_0 \) and \( M_0 \). For the thermal state with average nuclear polarization \( P \),

\[ w(I_0, M_0) = C_N^{N/2-I_0} \left( \frac{1 + P}{2} \right)^{N/2-M_0} \left( \frac{1 - P}{2} \right)^{N/2+M_0} \frac{2I_0 + 1}{N/2 + I_0 + 1}, \quad (15) \]

where \( C_N^{N/2-I_0} \) is the binomial coefficient. For the situation of large nuclear polarization \( P \sim 1 \) and many nuclear spins \( (N \rightarrow \infty) \), the contribution of \( I_0 = -M_0 \) dominates, and the distribution \( w(I_0, M_0) \) can be approximated by,

\[ w(M_0) \approx \frac{1}{\sqrt{2 \pi \sigma}} e^{-\frac{(M_0 - \bar{M})^2}{2 \sigma^2}} \quad (16) \]

with \( \bar{M} = \frac{N}{2} P \) and \( \sigma^2 = \frac{N}{2} (1 - P^2) \). Under this approximation, Eq. 14 becomes,

\[ \langle S_z \rangle = \lim_{N \to \infty} \int_{-N/2}^{N/2} \frac{1}{\sqrt{2 \pi \sigma}} e^{-\frac{(M_0 - \bar{M})^2}{2 \sigma^2}} 2M_0 \cos \left( \frac{(M_0 - \bar{M})\pi}{2\sqrt{N}} - \Omega \right) \sin^2 \left( \sqrt{\frac{M_0 + M_0(M_0 + 1) + \bar{M}^2}{2\sqrt{N}}} \right) dM_0, \quad (17) \]
\(\langle S_y \rangle\) and \(\langle S_z \rangle\) can be obtained similarly. After obtaining these electron spin expectation values, we can calculate the quantum Fisher information \([1]\),

\[
F_Q = \begin{cases} 
|\partial_\Omega v|^2 + \frac{(v \cdot \partial_\Omega v)^2}{1 - |v|^2} & \text{if } |v| < 1, \\
|\partial_\Omega v|^2 & \text{if } |v| = 1.
\end{cases}
\]  

(18)

where \(v = (\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle)\) is the Bloch vector.

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[1] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori, Phys. Rev. A 87, 022337 (2013)