The Quantized $O(1, 2)/O(2) \times Z_2$ Sigma Model Has No Continuum Limit in Four Dimensions. I. Theoretical Framework.

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Abstract
The nonlinear sigma model for which the field takes its values in the coset space $O(1, 2)/O(2) \times Z_2$ is similar to quantum gravity in being perturbatively nonrenormalizable and having a noncompact curved configuration space. It is therefore a good model for testing nonperturbative methods that may be useful in quantum gravity, especially methods based on lattice field theory. In this paper we develop the theoretical framework necessary for recognizing and studying a consistent nonperturbative quantum field theory of the $O(1, 2)/O(2) \times Z_2$ model. We describe the action, the geometry of the configuration space, the conserved Noether currents, and the current algebra, and we construct a version of the Ward-Slavnov identity that makes it easy to switch from a given field to a nonlinearly related one. Renormalization of the model is defined via the effective action and via current algebra. The two definitions are shown to be equivalent. In a companion paper we develop a lattice formulation of the theory that is particularly well suited to the sigma model, and we report the results of Monte Carlo simulations of this lattice model. These simulations indicate that as the lattice cutoff is removed the theory becomes that of a pair of massless free fields. Because the geometry and symmetries of these fields differ from those of the original model we conclude that a continuum limit of the $O(1, 2)/O(2) \times Z_2$ model which preserves these properties does not exist.

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1 Introduction

Nonlinear sigma models in 4-dimensional spacetime are not perturbatively renormalizable. They are therefore often viewed as being at best phenomenological, incapable of representing physical reality at a fundamental level. However, perturbative nonrenormalizability has never been proved to imply nonrenormalizability, i.e., the impossibility of giving precise meaning to a model, even in the extreme ultraviolet limit, in some nonperturbative sense. The purpose of this and the paper following is to present evidence that it is indeed impossible to give such meaning to at least one model: the $O(1, 2)/O(2) \times Z_2$ sigma model.

Interest in this model stems from the following features that it shares with quantum gravity:

1. Each theory has a single scale constant $\mu$ having the dimensions of mass.
2. Each has a noncompact curved configuration space.
3. Corresponding Feynman graphs for each (in perturbation theory) have identical degrees of divergence.
4. Neither is perturbatively renormalizable.

Despite the negative result, experience gained in studying the $O(1, 2)/O(2) \times Z_2$ model is useful in approaching the more difficult problem of quantum gravity, not only in regard to methodology, but also in setting ranges for certain parameters and in discovering the limits of present supercomputers. Although a positive result, i.e., demonstration of the existence of a continuum limit for the $O(1, 2)/O(2) \times Z_2$ model, would have been of obvious interest, no conclusion can yet be drawn about the existence or nonexistence of a continuum limit for quantum gravity. In no part of our analysis have we found reasons to have expected a negative result a priori. Each theory requires separate investigation, and it must be stressed that quantum gravity differs in several important respects from a sigma model:

1. Its invariance group, the diffeomorphism group, is local, whereas the invariance group $O(1, 2)$ of the sigma model is global.
2. The ground state of the sigma model is degenerate, and spontaneous symmetry breaking occurs. There is no evidence that the vacuum state of quantum gravity is degenerate.
3. Quantization of the gravitational field smears the light cone. The light cone for the sigma model remains fixed and sharp.

No method is presently known for studying nonlinear sigma models nonperturbatively in the extreme ultraviolet other than that of direct assault via the functional integral of Feynman. In field theory no nonperturbative definition of the functional integral is known other than that of taking the continuum limit of the corresponding integral on a lattice. In this paper we describe the continuum theory, pretending that it exists, in order to motivate our later choice of lattice action and of things to compute. The theoretical framework that will guide the computations is given in some detail because there are important differences between noncompact sigma models and the more familiar compact ones. For example, noncompact models undergo no transition to a phase of unbroken symmetry. In the following paper we introduce a lattice action that is particularly well suited to models with curved configuration spaces, and we describe the lattice Monte Carlo simulation of the $O(1,2)/O(2) \times Z_2$ model, using this action.

2 Action

When we pass to the lattice simulation on the computer we shall need to work in Euclidean space, but for most of this paper we shall work in a Minkowski space with metric $(\eta_{\mu\nu}) = \text{diag}(-1,1,1,1)$. The action functional of a general sigma model has the form

$$S = -\frac{1}{2}\mu^2 \int g_{ij}(\phi(x)) \phi_i,\mu \phi_j,\mu \, d^4x,$$

(2.1)

where commas followed by Greek indices denote differentiation with respect to the Minkowski coordinates $x^\mu (\mu = 0, 1, 2, 3)$ and can be raised and lowered by means of the Minkowski metric. The $\phi_i$ are scalar fields that take their values in charts of configuration space, and the $g_{ij}$ are the components (in those charts) of the metric tensor of configuration space. The configuration space itself is always a coset space of a Lie group, and the metric tensor is the natural group invariant one. The configuration space is therefore a symmetric space with a covariantly constant curvature tensor $R_{ijkl}$ and a strictly constant curvature scalar $R (= R_{ij}^{ij})$. Only those coset spaces are chosen for which the natural metric is positive definite.

The fields $\phi_i$ will be taken dimensionless. In units with $\hbar = c = 1$ the constant $\mu$ has then the dimensions of mass. It is sometimes convenient to absorb the constant $\mu$ into the fields by redefining

$$\psi^i = \mu \phi^i,$$

(2.2)
and to follow the more standard convention of regarding scalar fields in four dimensions as having the dimensions of mass. This redefinition is useful, for example, in a chart in configuration space in which the metric takes the form

$$g_{ij} = \delta_{ij} - \frac{1}{3} R^0_{ikjl} \phi^k \phi^l + \cdots,$$  

(2.3)

$R^0_{ikjl}$ being the curvature tensor at $\phi = 0$ in this chart. The action (2.1) becomes

$$S = -\frac{1}{2} \int \left( \delta_{ij} - \frac{1}{3} \mu^{-2} R^0_{ikjl} \psi^i \psi^l + \cdots \right) \psi^i, \mu \psi^j, \mu d^4x,$$  

(2.4)

and the role of coupling constant in the theory is seen to be played by the inverse quantity $\mu^{-2}$. Of course, the Lagrangian of the actions (2.1) and (2.4) is generally nonpolynomial, and there is in fact an infinity of coupling constants: the powers of $\mu^{-2}$. We prefer to view $\mu$ as simply a scale parameter for the theory. In recognition of the model’s superficial resemblance to quantum gravity we shall call $\mu$ the bare Planck mass.

Many years ago Palais and Mostow \(^1\) showed that coset spaces of semisimple Lie groups can always be embedded in vector spaces in such a way that the group actions on each coset space can be represented as linear homogeneous transformations of the corresponding embedding space. This means that the field variables $\phi^i$ can be replaced by a larger set of variables $\phi^a$ together with a set of Lagrange multipliers $\lambda^A$ that enforce the embedding constraints. The $\lambda^A$ themselves remain invariant under the actions of the group. Since only a single chart is needed in the embedding space it is often convenient to present nonlinear sigma models in this form.

The $O(1,2)/O(2) \times Z_2$ model is defined, in this form, by the action

$$\bar{S} = -\frac{1}{2} \mu^2 \int \left[ \eta_{ab} \phi^a, \mu \phi^b, \mu + \lambda \left( \eta_{ab} \phi^a \phi^b + 1 \right) \right] d^4x, \quad a, b \in \{0, 1, 2\},$$  

(2.5)

$$\eta_{ab} = \text{diag}(-1, 1, 1),$$  

(2.6)

together with the constraint

$$\phi^0 \geq 1.$$  

(2.7)

The embedding space is seen to be a 3-dimensional Minkowski space on which the actions of $O(1,2)$ are Lorentz transformations. The Lagrange multiplier enforces the constraint

$$\eta_{ab} \phi^a \phi^b = -1$$  

(2.8)

\(^1\) R. S. Palais, J. Math. Mech. 6, 673 (1957). G. D. Mostow, Annals of Math. 65, 432 (1957).
which, together with (2.7), identifies the configuration space as the upper sheet of a spacelike hyperboloid in the embedding space.

The group invariant metric on the hyperboloid is that induced by the metric $\eta_{ab}$ of the embedding space. If one parameterizes the configuration space by

$$
\phi^0 = \cosh s,
\phi^1 = \sinh s \cos \theta,
\phi^2 = \sinh s \sin \theta,
$$

(2.9)

then one may write the induced line element in the form

$$
-\left(\frac{d\phi^0}{\cosh s}\right)^2 + \left(\frac{d\phi^1}{\sinh s \cos \theta}\right)^2 + \left(\frac{d\phi^2}{\sinh s \sin \theta}\right)^2 = ds^2 + \sinh^2 s \, d\theta^2 ,
$$

(2.10)

and the action (2.1) becomes

$$
S = -\frac{1}{2} \mu^2 \int \left( s_{,\mu} s^{,\mu} + \sinh^2 s \, \theta_{,\mu} \theta^{,\mu} \right) d^4 x .
$$

(2.11)

The coordinates $s, \theta$ are here allowed to cover the configuration space an infinite number of times so that the fields $s, \theta$ may be taken differentiable wherever the $\phi^a$ are differentiable.

## 3 Geometry of the configuration space

The configuration space of the $O(1, 2)/O(2) \times Z_2$ model has constant negative curvature and is noncompact. In fact it is topologically $\mathbb{R}^2$ and can itself be covered by a single chart, for example by the components of the 2-vector

$$
\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} .
$$

(3.1)

The constraints (2.7) and (2.8) can be expressed by

$$
\phi^0 = \sqrt{1 + \phi^2} ,
$$

(3.2)

and it is not difficult to verify that the metric in this chart takes the form

$$
g_{ij} = \delta_{ij} - \frac{\phi^i \phi^j}{1 + \phi^2} , \quad i, j \in \{1, 2\} .
$$

(3.3)
The curvature tensor is also easily derived:

\begin{align}
R_{ijkl} &= -(g_{ik} g_{jl} - g_{il} g_{jk}), \\
R_{ij} &= R_{ikj}^k = -g_{ij}, \\
R &= R_i^i = -2.
\end{align}

Most nonlinear sigma models that have been studied have compact configuration spaces. We shall see later that noncompactness of its configuration space leads to profound differences in the behavior of the $O(1,2)/O(2) \times Z_2$ model from that of the compact models. This will reflect itself in the need to deal with the lattice simulation of the model by methods that will in part be unfamiliar. We note at this stage just one fact: The area of a large circle in the $O(1,2)/O(2) \times Z_2$ configuration space increases not quadratically but exponentially with radius. This means that the functional integral must probe an increasingly huge region of the space of field histories as the continuum limit is approached.

We shall need later an expression for the geodetic distance $\Delta$, in the configuration space, between two points having coordinates $(s, \theta)$ and $(s', \theta')$ respectively. The simplest way to compute this is to note first that $s$ is the geodetic distance from the point $\phi = 0$ to the point $(s, \theta)$. Therefore, since the geodetic distance between two points remains invariant under the actions of $O(1,2)$, one may apply a Lorentz boost that brings one of the two points to $\phi = 0$. This is most easily done in terms of the variables $\phi^a$, $a \in \{0,1,2\}$, and one finds

\begin{equation}
\Delta = \cosh^{-1} \left( -\eta_{ab} \phi^a \phi'^b \right).
\end{equation}

Substitution of expressions (2.9) into (3.7) yields

\begin{equation}
\Delta = \cosh^{-1} \left[ \cosh s \cosh s' - \sinh s \sinh s' \cos(\theta - \theta') \right].
\end{equation}

\section{Conserved currents}

The infinitesimal actions of the invariance group on the variables $\phi^i$ may be expressed in the form

\begin{equation}
\delta \phi^i = Q^i_\alpha(\phi) \delta \xi^\alpha,
\end{equation}

where the $\delta \xi^\alpha$ are infinitesimal group parameters and the $Q^i_\alpha$ are the components (in the configuration space chart) of a set of vector fields $Q_\alpha$ on the configuration space, which
satisfy the Lie bracket relations

\[ [Q_\alpha, Q_\beta] = -Q_\gamma c^{\gamma}_{\alpha\beta} \]

(4.2)

the \( c^{\gamma}_{\alpha\beta} \) being the structure constants of the group. The statement that the Lagrangian

\[ L = -\frac{1}{2} \mu^2 g_{ij} \phi^i , \mu \phi^j , \mu \]

(4.3)
is invariant under (4.1) is just the statement that \( g_{ij} \) is group invariant:

\[ \mathcal{L}_{Q_\alpha} g_{ij} = 0. \]

(4.4)

Using (4.1) together with the field equations

\[ \frac{\partial L}{\partial \phi^i} - \left( \frac{\partial L}{\partial \phi^i , \mu} \right) _\mu = 0, \]

(4.5)
one may rewrite the invariance condition in the form

\[ 0 = \delta L = \frac{\partial L}{\partial \phi^i} \delta \phi^i + \frac{\partial L}{\partial \phi^i , \mu} \delta \phi^i , \mu = j_\alpha^\mu , \mu \delta \xi^\alpha \]

(4.6)

where the \( j_\alpha^\mu \) are the Noether currents:

\[ j_\alpha^\mu = \frac{\partial L}{\partial \phi^i} Q^{i, \alpha} = -\mu^2 g_{ij} Q^{i, \alpha} \phi^j , \mu. \]

(4.7)

Expression (4.7) is valid for any sigma model. It is completely equivalent to the corresponding expression obtained when the alternative action is used, involving the linear embedding variables \( \phi^a \) and the Lagrange multipliers \( \lambda_A \). In this case the infinitesimal group transformation law takes the form

\[ \delta \phi^a = G^{a, \alpha}_{\alpha \beta} \phi^\beta \delta \xi^\alpha \]

(4.8)

or, with Latin indices suppressed,

\[ \delta \phi = G_\alpha \phi \delta \xi^\alpha, \]

(4.9)

where the matrices \( G_\alpha \) satisfy the commutation relation

\[ [G_\alpha, G_\beta] = G_\gamma c^{\gamma}_{\alpha\beta}. \]

(4.10)

Since the Lagrange multipliers are invariant under the group they play no role in the definition of the currents, and one finds

\[ j_\alpha^\mu = \frac{\partial L}{\partial \phi^i} G_\alpha \phi \]

(4.11)
where \( \bar{L} \) is the Lagrangian for the alternative action. Expression (4.11) is usually more convenient to use than (4.7).

Since the constants \( \delta \xi^\alpha \) in eq. (4.6) are arbitrary we have

\[
j_{\alpha}^{\mu, \mu} = 0,
\]

(4.12)

which implies conservation of the total charges

\[
q_{\alpha} = \int_{\Sigma} j_{\alpha}^{\mu} d\Sigma_{\mu}.
\]

(4.13)

Here \( \Sigma \) is any smooth deformation of a global spacelike Cauchy hypersurface in spacetime and \( d\Sigma_{\mu} \) is its surface element.

In the case of the \( O(1,2)/O(2) \times Z_2 \) model it is convenient to replace the single group index \( \alpha \) by a pair of indices \( ab \), so that equation (4.8) takes the simple form

\[
\delta \phi^a = \frac{1}{2} G_{\alpha \beta}^{ab} \phi^b \delta \xi^{\alpha \beta} = \delta \xi^a \phi^b.
\]

(4.14)

Here we have

\[
G_{\alpha \beta}^{ab} = \delta_{\alpha \beta}^{c} \eta_{cb} - \delta_{\alpha \beta}^{a} \eta_{cb}, \quad \delta \xi^{a} \phi^b = \eta_{bc} \delta \xi^{ac}.
\]

(4.15)

The group parameters \( \delta \xi^{ab} \) are those of an infinitesimal Lorentz transformation in the configuration space and satisfy

\[
\delta \xi^{ab} = -\delta \xi^{ba}.
\]

(4.16)

Reference to eqs. (2.5) and (4.11) shows that the currents are

\[
j_{ab}^{\mu} = -\mu^2 \eta_{cd} \phi^{c\mu} G_{\alpha \beta}^{ab} \phi^e = \mu^2 (\phi_a \phi_b^{\mu} - \phi_b \phi_a^{\mu})
\]

(4.17)

where \( \phi_a = \eta_{ab} \phi^b \). If the hypersurface \( \Sigma \) is chosen to be \( x^0 = \text{constant} \) then the corresponding charges can be expressed in the form

\[
q_{ab} = \int j_{ab}^{0} d^3x = -\mu^2 \int (\phi_a \phi_{b,0} - \phi_b \phi_{a,0}) d^3x.
\]

(4.18)

5 Functional integrals. Ward-Slavnov identities.

Since we shall ultimately attempt to define the quantum theory of the \( O(1,2)/O(2) \times Z_2 \) model by means of the functional integral of Feynman, it is appropriate that we discuss
quantum problems in functional-integral language from the outset. In principle the functional integral gives a direct representation of transition amplitudes:

\[
\langle \text{out} | \text{in} \rangle = N \int e^{iS[\phi]} \mu[\phi] \, d\phi \\
= \bar{N} \int e^{i\bar{S}[\phi,\lambda]} \, d\phi \, d\lambda. \tag{5.1a}
\]

Here \( |\text{in}\rangle \) and \( |\text{out}\rangle \) are vectors corresponding to arbitrary “in” and “out” states, the integrations are over field histories satisfying boundary conditions appropriate to those states, and it is understood that expressions (2.1) and (2.5) for \( S \) and \( \bar{S} \) may need to be made more precise by imposition of integration limits as well as addition of boundary terms. \( N \) and \( \bar{N} \) are normalization constants and \( \mu[\phi] \) is the configuration space measure:

\[
\mu[\phi] = \prod_x g^{1/2}(\phi(x)), \quad g = \det (g_{ij}). \tag{5.2}
\]

In principle a measure functional \( \bar{\mu}[\phi,\lambda] \) should be inserted in the integrand of (5.1b), but it would be simply a constant because the embedding space is flat. The volume elements of (5.1a) and (5.1b) are respectively

\[
d\phi = \prod_{i,x} d\phi_i(x), \tag{5.3a}
\]

\[
d\phi \, d\lambda = \prod_x \prod_a d\phi^a(x) \prod_A d\lambda_A(x). \tag{5.3b}
\]

Although expressions (5.2), (5.3a) and (5.3b) are purely formal they have a ready interpretation on a lattice.

The functional integral may also be used to represent “in-out” matrix elements of chronologically ordered operators:

\[
\langle \text{out} | T(A[\phi]) | \text{in} \rangle = N \int A[\phi] e^{iS[\phi]} \mu[\phi] \, d\phi \\
= \bar{N} \int A[\phi] e^{i\bar{S}[\phi,\lambda]} d\phi \, d\lambda. \tag{5.4a}
\]

The symbol \( \phi \) on the left of these equations is to be understood as a quantum operator, on the right as an integration variable.

Suppose now the dummy variables of integration \( \phi^i \) in eq. (5.4a) are replaced by

\[
\phi'^i = \phi^i + Q^i_\alpha(\phi) \delta \xi^\alpha, \tag{5.5}
\]
where the $\delta \xi^\alpha$, instead of being constants as in eq. (4.1), are now scalar functions of compact support in spacetime. The replacement (5.5) leaves (5.4a) unchanged, but it produces the following changes in the individual factors under the integral sign:

$$\delta A[i\phi] = \int \frac{\delta A[i\phi]}{\delta \phi^\alpha(x)} Q^i_\alpha(\phi(x)) \delta \xi^\alpha(x) \, d^4x,$$  

(5.6)

$$\delta e^{iS[\phi]} = i e^{iS[\phi]} \int \left[ \frac{\partial L}{\partial \phi^i} Q^i_\alpha \delta \xi^\alpha + \frac{\partial L}{\partial \phi^i_\mu} \left( Q^i_\alpha \delta \xi^\alpha \right)_\mu \right] \, d^4x$$

$$= i e^{iS[\phi]} \int j^i_\alpha \mu \delta \xi^\alpha_\mu \, d^4x$$

$$= -i e^{iS[\phi]} \int j^i_\alpha \mu \delta \xi^\alpha_\mu \, d^4x,$$  

(5.7)

$$\delta (\mu[\phi] \, d\phi) = \mu[\phi'] \, d\phi' - \mu[\phi] \, d\phi$$

$$= \{ \mu[\phi'] - \mu[\phi] + \mu[\phi'] \left[ \partial(\phi')/\partial(\phi) - 1 \right] \} \, d\phi$$

$$= \mu[\phi] \, d\phi \sum_x g^{-1/2} \left[ \left( g^{1/2}/\partial \phi^i \right) Q^i_\alpha + g^{1/2} \partial Q^i_\alpha/\partial \phi^i \right] \delta \xi^\alpha.$$  

(5.8)

The second line of (5.7) follows from the observation that the Lagrangian would be invariant under (5.5) if the $\delta \xi^\alpha$ were constants. Expression (5.8) actually vanishes by virtue of the following immediate corollary of eq. (4.4):

$$0 = \mathcal{L}_{Q^i_\alpha} g^{1/2} = \partial \left( g^{1/2} Q^i_\alpha \right) / \partial \phi^i.$$  

(5.9)

Taking note of the arbitrariness of the $\delta \xi^\alpha$ one may therefore infer

$$0 = N \int \left\{ \frac{\delta A[i\phi]}{\delta \phi^\alpha(x)} Q^i_\alpha(\phi(x)) - i A[i\phi] j^i_\mu(x) \right\} e^{iS[\phi]} \mu[\phi] \, d\phi$$

$$= \langle \text{out} | T \left( \frac{\delta A[i\phi]}{\delta \phi^\alpha(x)} Q^i_\alpha(\phi(x)) - i A[i\phi] j^i_\mu(x) \right) | \text{in} \rangle.$$  

(5.10)

Because the “in” and “out” states are themselves arbitrary one can, in fact, infer the operator identity

$$\frac{\partial}{\partial x^\mu} T (A[i\phi] j^\mu(x)) = -i T \left( \frac{\delta A[i\phi]}{\delta \phi^\alpha(x)} Q^i_\alpha(\phi(x)) \right).$$  

(5.11)

Note here that, because the functional integral of the difference of two functionals is the difference of their individual integrals, the chronological ordering that eq. (5.4a) defines
commutes with the operation of differentiation with respect to the spacetime coordinates. Equation (5.11) is a slightly generalized version of the standard Ward-Slavnov identity.

The Ward-Slavnov identity can be extended to the embedding variables $\phi^a$ by working with expression (5.4b) instead of (5.4a) and replacing (5.5) by

$$\phi' = (1 + G_{\alpha} \delta \xi^\alpha) \phi.$$  \hspace{1cm} (5.12)

The invariance of the volume element $d\phi$ under this transformation follows from

$$\text{tr} G_{\alpha} = 0,$$ \hspace{1cm} (5.13)

which in turn follows from the semisimplicity of the invariance group. The extended Ward-Slavnov identity takes the form

$$\frac{\partial}{\partial x^\mu} T (A[\phi] j_\alpha^\mu(x)) = -iT \left( \frac{\delta A[\phi]}{\delta \phi(x)} G_{\alpha} \phi(x) \right).$$ \hspace{1cm} (5.14)

6 Current algebra

Setting $A[\phi] = 1$ in eq. (6.14) one gets the operator conservation law

$$j_\alpha^\mu,\mu = 0.$$ \hspace{1cm} (6.1)

On the other hand, choosing $A[\phi] = \phi(x')$, writing

$$T (\phi(x') j_\alpha^\mu(x)) = \theta \left(x'^0 - x^0\right) \phi(x') j_\alpha^\mu(x) + \theta \left(x^0 - x'^0\right) j_\alpha^\mu(x) \phi(x')$$ \hspace{1cm} (6.2)

where $\theta$ is the step function, and making use of (5.11), one gets

$$\left[ \phi(x'), j_\alpha^0(x) \right] \delta \left(x'^0 - x^0\right) = iG_{\alpha} \phi(x) \delta (x' - x),$$ \hspace{1cm} (6.3)

which is equivalent to the equal-time commutator

$$\left[ \phi(x), j_\alpha^0(x') \right] = iG_{\alpha} \phi(x) \delta (x - x'), \hspace{1cm} x^0 = x'^0.$$ \hspace{1cm} (6.4)

Integration of this equation over $x'$ yields

$$[\phi, q_\alpha] = iG_{\alpha} \phi$$ \hspace{1cm} (6.5)
which, by virtue of the group invariance of the Lagrangian $\bar{L}$, in turn yields

$$\left[ \frac{\partial \bar{L}}{\partial \phi_{,\mu}}, q_\alpha \right] = -i \frac{\partial \bar{L}}{\partial \phi_{,\mu}} G_\alpha, \quad (6.6)$$

and hence

$$[j_\alpha^\mu, q_\beta] = i \frac{\partial \bar{L}}{\partial \phi_{,\mu}} [G_\alpha, G_\beta] \phi = i j_\gamma^\mu c^\gamma_{\alpha\beta}, \quad (6.7)$$

$$[q_\alpha, q_\beta] = i q_\gamma c^\gamma_{\alpha\beta}. \quad (6.8)$$

Using (4.15) one easily finds, for the $O(1, 2)/O(2) \times Z_2$ model,

$$[\phi^0(x), j_{ij}^0(x')] \delta \left(x^0 - x'^0\right) = 0, \quad (6.9)$$

$$[\phi^k(x), j_{ij}^0(x')] \delta \left(x^0 - x'^0\right) = i \left[ \delta^k_i \phi_j(x) - \delta^k_j \phi_i(x) \right] \delta(x - x'), \quad (6.10)$$

$$[\phi^0(x), j_{0i}^0(x')] \delta \left(x^0 - x'^0\right) = i \phi_i(x) \delta(x - x'), \quad (6.11)$$

$$[\phi^k(x), j_{0i}^0(x')] \delta \left(x^0 - x'^0\right) = -i \delta^k_i \phi_0(x) \delta(x - x')$$

$$\quad = i \delta^k_i \phi^0(x) \delta(x - x'), \quad (6.12)$$

where $i, j, k \in \{1, 2\}$.

It is sometimes convenient to use chart coordinates in configuration space other than the $\phi^i$. For the $O(1, 2)/O(2) \times Z_2$ model these will be chosen so as to transform like the $\phi^i$ under the $O(2)$ subgroup. The most general such coordinates are

$$\varphi^i = f (|\phi|) \phi^i, \quad (6.13)$$

where $f(|\phi|)$ is a positive-valued smooth function such that $|\phi| f(|\phi|)$ has everywhere positive slope. For example, if

$$f(|\phi|) = \frac{\sinh^{-1} |\phi|}{|\phi|} \quad (6.14)$$

then the $\varphi^i$ are the Riemann normal coordinates $\sigma^i$:

$$\begin{cases} 
\sigma^1 = s \cos \theta \\
\sigma^2 = s \sin \theta.
\end{cases} \quad (6.15)$$

It is readily verified that in general we have

$$[\varphi^k(x), j_{ij}^0(x')] \delta \left(x^0 - x'^0\right) = i \left[ \delta^k_i \varphi_j(x) - \delta^k_j \varphi_i(x) \right] \delta(x - x'), \quad (6.16)$$

$$[\varphi^k(x), j_{0i}^0(x')] \delta \left(x^0 - x'^0\right) = \frac{\partial \varphi^k(x)}{\partial \phi^i(x)} \phi_0(x) \delta(x - x'). \quad (6.17)$$
7 Vacuum and 1-particle states

The classical energy corresponding to the action (2.1) is nonnegative and vanishes for constant fields. Each different constant field corresponds to a different classical vacuum state. Since $O(1,2)$ acts transitively on the configuration space the classical vacua are obtainable from one another by group operations. One expects the degeneracy of the classical ground state to be reflected in a corresponding degeneracy of the quantum ground state. If $|\text{vac}\rangle$ is the vector corresponding to one of the quantum vacua then the vectors corresponding to the others are obtained through multiplication by unitary operators $\exp(iq_\alpha \xi^\alpha)$.

Since all points of configuration space are equivalent under $O(1,2)$ one can always choose a Lorentz frame in the embedding space so that the classical vacuum state is $\phi^1 = \phi^2 = 0$. The corresponding quantum vacuum is fixed by the conditions

$$\bar{\phi}^1 = \bar{\phi}^2 = 0, \quad \bar{\phi}^a = \langle \text{vac} | \phi^a | \text{vac} \rangle, \quad a \in \{0, 1, 2\}.$$ (7.1)

Here the normalization

$$\langle \text{vac} \mid \text{vac} \rangle = 1$$ (7.2)

will be assumed. Each quantum vacuum will also be assumed to be stable.

The degeneracy of the quantum ground state implies that the basic particles of the theory are two Goldstone bosons. The masslessness of these particles is already suggested by the form of the first term in the expansion (2.4) of the classical action. Since the classical action generates no bare vertices having fewer than four prongs, perturbation theory, if it were valid, would imply that these particles, despite being massless, are stable. Since perturbation theory fails one must either assume that these particles are stable or else assume that 2-point functions exist and that the quantum vacuum, like the classical vacuum, has an $O(2)$ invariance that is never broken. In the latter case Goldstone’s theorem will guarantee that these particles are stable, for, as will be shown later, the degeneracy of the quantum ground state is never removed.

Since the $S$-matrix of the theory is expected to be independent of the choice of interpolating field one should in principle be able to construct particle creation and annihilation operators out of any of the fields (6.13) including $\phi^i$ itself. Asymptotic arguments imply

\[\text{Phase space restrictions forbid the decay of a single massless particle into three or more others. The derivative coupling also suppresses the decay.}\]
that the creation operators are given by
\[
    a^i(p)^* = i\mu_\varphi \int_{x^\infty} u(x, p) \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x'_\mu} \right) \varphi^i(x) \, d\Sigma_\mu,
\]
(7.3)
where “+∞” and “−∞” denote Cauchy hypersurfaces in the remote future and past respectively, either asymptotic region being usable because of the stability of the particles. Here \( u \) is a mode function that should strictly have the form of a wave packet but which, if the packet is big enough, may effectively be represented by a plane wave:
\[
\begin{align*}
    u(x, p) &= (2\pi)^{-3/2}(2\omega)^{-1/2}e^{ip \cdot x}, \\
    \omega &= p^0 = |p|, \\
    p^2 = 0.
\end{align*}
\]
(7.4)
In the Appendix it is shown that if the 1-particle state vectors are defined by
\[
    |i, p\rangle = a^i(p)^* |\text{vac}\rangle
\]
(7.5)
then the normalization
\[
    \langle i, p | j, p' \rangle = \delta_{ij} \delta(p - p')
\]
(7.6)
is secured by choosing the coefficient \( \mu_\varphi \) in (7.3) to be the square root of the reciprocal of the residue at the “pole” at \( p^2 = 0 \) in the Fourier transform of the 2-point function:
\[
    \langle \text{vac} | T \left( \varphi^i(x)\varphi^j(x') \right) |\text{vac}\rangle = -\frac{i}{(2\pi)^4} \int \frac{\delta_{ij} e^{ip \cdot (x-x')}}{\mu_\varphi^2 (p^2 - i0) + \cdots} d^4p.
\]
(7.7)
If \( \varphi^i \) were replaced by \( \phi^i \) and if all the terms in the integrand of (2.4) other than the first were missing, so that the theory were “free,” then \( \mu_\varphi \) would be simply the bare Planck mass \( \mu \).

For the nonlinear theory it must be computed. This, in fact, is a major goal of the numerical simulation.

Strictly speaking, the singularity of the integrand of (7.7) at \( p^2 = 0 \) cannot be a pole when the massless particles interact but must be a branch point. The coefficient \( \mu_\varphi^2 \) can nevertheless be identified because the unwritten terms “…” in the denominator of the integrand generally vanish more rapidly than \( p^2 \) as \( p^2 \to 0 \). For example, in perturbation theory the first unwritten term behaves like \( p^6 \ln p^2 \).

The proof in the Appendix, of the normalization (7.6), makes use of the mode-function orthonormality relations
\[
\begin{align*}
    -i \int_{\Sigma} u(x, p)^* \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x'_\mu} \right) u(x, p') \, d\Sigma_\mu &= \delta(p - p') \\
    -i \int_{\Sigma} u(x, p) \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x'_\mu} \right) u(x, p') \, d\Sigma_\mu &= 0,
\end{align*}
\]
(7.8)
and of the defining condition
\[ a^i(p) |\text{vac}\rangle = 0, \quad \langle \text{vac} | a^i(p)^* = 0, \quad (7.9) \]
for the vacuum state vector. Equations (7.3) and (7.8), together with the completeness of the mode functions, imply that \( \varphi^i(x) \) may, in the remote past or future, be effectively represented by
\[ \varphi^i(x) = \int [u(x, p)a^i(p) + u(x, p)^*a^i(p)^*] \, d^3p, \quad x^0 \to \pm \infty. \quad (7.10) \]
This in turn implies
\[ \langle \text{vac} | \varphi^i(x) |\text{vac}\rangle = 0 \quad (7.11) \]
for \( x^0 \to \pm \infty \). Because the action is an even functional of the fields eq. (7.1) in fact holds for all \( x \) (cf. eq. (7.1)).

8 Renormalization via the effective action

Renormalization is most easily discussed in terms of the effective action, denoted here by \( \Gamma [\bar{\phi}] \) or \( \bar{\Gamma} [\bar{\phi}, \bar{\lambda}] \) according as the classical action is taken to be \( S[\phi] \) or \( \bar{S}[\phi, \lambda] \). Consider first \( \bar{\Gamma} \). The traditional way to define it is to introduce sources \( J_a, J_A \) coupled to the \( \phi^a \) and \( \lambda^A \) respectively and to set
\[ e^{i\bar{W}[J]} = \langle \text{out}, \text{vac} | in, \text{vac} \rangle = N \int e^{i[S[\phi, \lambda] + \int (J_a \phi^a + J_A \lambda^A) \, dx]} \, d\phi \, d\lambda, \quad (8.1) \]
where, because of the presence of the sources (which are assumed to have compact support in spacetime), one must now distinguish between “in” and “out” vacua. The functional \( \bar{W}[J] \) is used to define the following quantities:
\[ \bar{\phi}^a = \frac{\delta \bar{W}}{\delta J_a}, \quad \bar{\lambda}^A = \frac{\delta \bar{W}}{\delta J_A}, \quad (8.2) \]
\[ G^{a_1...a'_r A''_1...A'''_s} = \frac{\delta}{\delta J_{a_1}(x)} \cdots \frac{\delta}{\delta J_{a_r}(x')} \frac{\delta}{\delta \lambda_{A_1}(x'')} \cdots \frac{\delta}{\delta \lambda_{A_s}(x'''')} \bar{W}. \quad (8.3) \]
It is straightforward to show that these quantities are related to the so called \( n \)-point functions:
\[ \langle \phi^a \rangle = \bar{\phi}^a, \quad \langle \lambda^A \rangle = \bar{\lambda}^A, \quad (8.4) \]
\[ \langle \phi^a \phi^b \rangle = \bar{\phi}^a \bar{\phi}^b - iG^{ab'}, \text{ etc.}, \quad (8.5) \]
the $n$-point functions themselves being special cases of the general average

$$\langle A[\phi, \lambda] \rangle = \frac{\langle \text{out, vac} | T(A[\phi, \lambda]) | \text{in, vac} \rangle}{\langle \text{out, vac} | \text{in, vac} \rangle}$$

$$= \frac{\int A[\phi, \lambda] e^{i[S[\phi, \lambda] + \int (J_a \phi^a + J_A \lambda^A) d^4x]} d\phi d\lambda}{\int e^{i[S[\phi, \lambda] + \int (J_a \phi^a + J_A \lambda^A) d^4x]} d\phi d\lambda}. \quad (8.6)$$

$\tilde{\phi}^a$ and $\tilde{\lambda}^A$ are called *mean fields* and the $G$’s are known as *correlation functions*.

The effective action is derived from the functional $\tilde{W}[J]$ by the Legendre transformation

$$\tilde{\Gamma} [\tilde{\phi}, \tilde{\lambda}] = \tilde{W}[J] - \int \left( J_a \tilde{\phi}^a + J_A \tilde{\lambda}^A \right) d^4x. \quad (8.7)$$

It is not difficult to verify that $\tilde{\Gamma}$ satisfies the equations

$$\frac{\delta \tilde{\Gamma}}{\delta \tilde{\phi}^a} = -J_a, \quad \frac{\delta \tilde{\Gamma}}{\delta \tilde{\lambda}^A} = -J_A. \quad (8.8)$$

Functional differentiation of these equations yields

$$\int \left( \frac{\delta^2 \tilde{\Gamma}}{\delta \tilde{\phi}^a \delta \tilde{\phi}^b} G^{ab'} + \frac{\delta^2 \tilde{\Gamma}}{\delta \tilde{\phi}^a \delta \tilde{\lambda}^b} G^{Aa'} \right) d^4x'' = -\delta_a^b \delta(x - x'), \text{ etc.}, \quad (8.9)$$

which reveals $G^{ab'}$, $G^{Aa'}$, $G^{Aa'}$, $G^{AAB'}$ as Green’s functions of the second functional derivative of $\Gamma$. These are the *full propagators* of the theory. Further functional differentiation of eqs. (8.9) enables one to express the higher order correlation functions in terms of these propagators together with the functional derivatives of $\tilde{\Gamma}$ of order three and higher, which are known as the *full vertex functions*. The resulting relations have a tree graph structure that allows one to recognize $\tilde{\Gamma}$ as the generator of the 1-particle irreducible amplitudes of the theory.

The effective action $\Gamma [\tilde{\phi}]$ associated with the classical action $S[\phi]$ is constructed in a different way. The reason for this is that although the variables $\phi^a$ transform linearly under the invariance group the variables $\phi^i$ transform nonlinearly among themselves. The direct coupling of the latter variables to sources would lead to a $\Gamma$ which has no simple transformation law under the group. What one does instead is first note that the above construction of $\tilde{\Gamma}$ is completely equivalent to defining it implicitly (or recursively) by

$$e^{i\tilde{\Gamma} [\tilde{\phi}, \tilde{\lambda}]} = N \int \exp \left\{ S[\phi, \lambda] + \int \left[ \frac{\delta \tilde{\Gamma} [\tilde{\phi}, \tilde{\lambda}]}{\delta \tilde{\phi}^a} (\tilde{\phi}^a - \phi^a) + \frac{\delta \tilde{\Gamma} [\tilde{\phi}, \tilde{\lambda}]}{\delta \tilde{\lambda}^A} (\tilde{\lambda}^A - \lambda^A) \right] d^4x \right\} d\phi d\lambda. \quad (8.10)$$

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From this it is easy to see that $\bar{\Gamma}$ must be linear in the $\bar{\lambda}^A$:

$$\bar{\Gamma} \left[ \bar{\phi}, \bar{\lambda} \right] = \tilde{\Gamma} \left[ \bar{\phi} \right] + \int \bar{\lambda}^A C_A \left[ \bar{\phi} \right] \, d^4x.$$  \hspace{1cm} (8.11)

One also notes that because the $\bar{\phi}^a$, $\bar{\lambda}^A$ transform linearly under the group so do the $\bar{\phi}^a$, $\bar{\lambda}^A$:

$$\delta \bar{\phi}^a = G_{ab}^a \bar{\phi}^b \delta \xi^\alpha, \quad \delta \bar{\lambda}^A = 0.$$ \hspace{1cm} (8.12)

The sources $J_a$, $J_A$ may be assumed to transform contragradiently, and $\tilde{\Gamma} \left[ \bar{\phi} \right]$ and the $C_A \left[ \bar{\phi} \right]$ are therefore group invariant.

The effective action $\Gamma \left[ \bar{\phi} \right]$ is now constructed from $\bar{\Gamma} \left[ \bar{\phi}, \bar{\lambda} \right]$ in the same way as the classical action $S[\phi]$ is constructed from $\tilde{S}[\phi, \lambda]$. One sets the sources $J_A$ equal to zero and solves the constraint equations

$$\frac{\delta \bar{\Gamma} \left[ \bar{\phi}, \bar{\lambda} \right]}{\delta \bar{\lambda}^A} \equiv C_A \left[ \bar{\phi} \right] = 0$$ \hspace{1cm} (8.13)

for the superfluous embedding variables (or rather their barred forms) in terms of the chart variables in the coset space, and then substitutes into $\tilde{\Gamma} \left[ \bar{\phi} \right]$ to get $\Gamma \left[ \bar{\phi} \right]$. In the case of the $O(1, 2)/O(2) \times Z_2$ model, if the $\bar{\phi}^a$ are constant fields (e.g., when the sources $J_a$ vanish) equation (8.13) is necessarily equivalent to

$$\eta_{ab} \bar{\phi}^a \bar{\phi}^b = -Z^2$$ \hspace{1cm} (8.14)

for some constant $Z$.

If the model is to have a consistent continuum limit the effective action must, because of symmetry requirements, reduce at low energies to a local action of the classical form (2.1), but with possibly a new value of $\mu$. Thus

$$\Gamma \left[ \bar{\phi} \right] = -\frac{1}{2} \bar{\mu}^2 \int \bar{g}_{ij} \left( \bar{\phi}(x) \right) \bar{\phi}_i \bar{\phi}_j \, d^4x + \Delta \Gamma \left[ \bar{\phi} \right],$$ \hspace{1cm} (8.15)

where $\Delta \Gamma$, which includes nonlocal contributions, becomes important at higher energies.\[ 3\]

The metric $\bar{g}_{ij}$ is group invariant (i.e., it satisfies eq. (4.4)) but, because the $\bar{\phi}^a$ are constrained by (8.14) to a generally different hyperboloid than the original classical variables $\phi^a$, $\bar{g}_{ij}$ is not equal to the metric (3.3) but is instead given by

$$\bar{g}_{ij} = \delta_{ij} - \frac{\bar{\phi}^i \bar{\phi}^j}{Z^2 + \bar{\phi}^2}.$$ \hspace{1cm} (8.16)

\[ 3\] The clean separation of $\Gamma$ into “leading” and “subleading” parts is implied by the same power-counting arguments that show the theory to be perturbatively nonrenormalizable.
The original metric is restored if the quantum operator fields are replaced by renormalized fields:

\[ \phi^a_R = Z^{-1}\phi^a. \]  

(8.17)

The effective action then takes the form

\[ \Gamma = -\frac{1}{2}\mu_R^2 \int g_{ij} \left( \bar{\phi}_R(x) \right) \bar{\phi}_{R,\mu}^i \phi_{R,\mu}^j d^4x + \Delta \Gamma \]  

(8.18)

where

\[ \mu_R = Z \bar{\mu}. \]  

(8.19)

\(\mu_R\) is the renormalized or “experimentally observed” Planck mass. Note that because the Lagrangian is nonpolynomial the replacement (8.17) renormalizes an infinite number of vertex functions at once. Note also that although (8.17) looks like a standard “wave function renormalization” the renormalization constant \(Z\) is not obtained from the residue of the pole of the 2-point function. Rather, it is the constant \(\bar{\mu}_\phi\) of eq. (7.7) that is defined by the residue.

When the sources \(J_a\) vanish and the “in” and “out” vacua become identical, the vacuum-fixing conditions (7.1) imply

\[ Z = \bar{\phi}^0 = \langle \text{vac} | \phi^0 | \text{vac} \rangle , \]  

(8.20)

\[ \langle \text{vac} | T \left( \phi^i_R(x) \phi^j_R(x') \right) | \text{vac} \rangle = -iG_R^{ij} , \]  

(8.21)

(cf. eq. (8.15)). \(G_R^{ij}\) is the Green’s function of the second functional derivative of the effective action with respect to the renormalized fields. The structure of the local part of expression (8.18) implies that the low-energy behavior of this Green’s function is given by

\[ G_R^{ij} = \frac{1}{(2\pi)^4} \int \frac{\delta_{ij} e^{ip(x-x')}}{\mu_R^2 (p^2 - i0) + \cdots} d^4p. \]  

(8.22)

Comparison of eqs. (7.7), (8.21) and (8.22) allows one to infer

\[ \mu_R = Z \bar{\mu}_\phi. \]  

(8.23)

Comparison of eq. (8.19) and (8.23) in turn informs one that the \(\bar{\mu}_\phi\) of eq. (8.15) is just the \(\bar{\mu}_\phi\) of eq. (7.7) when \(\varphi = \phi\).

9 Renormalization via current algebra

The renormalization analysis above makes special use of the variables \(\phi^a\), which transform linearly under the group. Determination of the renormalized Planck mass \(\mu_R\), however,
should not depend on which field variables are used. The generality of eq. (7.7) suggests that the choice of variables should, in fact, be irrelevant and that for every $\bar{\mu}_\varphi$ there should be an easily computable $Z_\varphi$ such that

$$\mu_R = Z_\varphi \bar{\mu}_\varphi.$$  \hspace{1cm} (9.1)

In this section we show that this is indeed the case.

Consider the vacuum average $\langle \text{vac}| T (\varphi^i(x)j_{ab}^{\mu}(x)) |\text{vac}\rangle$ where $\varphi^i$ is one of the fields (6.13). Because of the displacement invariance of the theory this average must be a function of the differences $x'^\nu - x^\nu$. It must also transform as a vector under Lorentz transformations. Its Fourier transform must therefore have the form

$$\int e^{ip\cdot(x'-x)} \langle \text{vac}| T (\varphi^i(x)j_{ab}^{\mu}(x)) |\text{vac}\rangle d^4x' = p^\mu X^i_{ab}(p^2)$$  \hspace{1cm} (9.2)

where the $X^i_{ab}$ are certain scalar functions of $p^2$. These functions may be determined by multiplying eq. (9.2) by $p_\mu$ and invoking the Ward-Slavnov identity (5.14):

$$p^2 X^i_{ab} = -i \int e^{ip\cdot(x'-x)} \frac{\partial}{\partial x'^\mu} \langle \text{vac}| T (\varphi^i(x)j_{ab}^{\mu}(x)) |\text{vac}\rangle d^4x'$$

$$= -i \int e^{ip\cdot(x'-x)} \langle \text{vac}| T (\varphi^i(x)j_{ab}^{\mu}(x)) |\text{vac}\rangle d^4x'$$

$$= - \int e^{ip\cdot(x'-x)} \delta(x' - x) \langle \text{vac}| \frac{\partial\varphi^i}{\partial\phi^c} G_{abd}^c \phi^d |\text{vac}\rangle d^4x'$$

$$= - \langle \text{vac}| \left( \frac{\partial\varphi^i}{\partial\phi^b} \phi_a - \frac{\partial\varphi^i}{\partial\phi^a} \phi_b \right) |\text{vac}\rangle.$$  \hspace{1cm} (9.3)

The integration by parts and subsequent replacement of $\partial/\partial x'^\mu$ by $-\partial/\partial x^\mu$, leading to the second line, assumes that the vacuum average falls off sufficiently rapidly at infinity. Omission of the chronological ordering symbol in the last two lines is allowed by the ultralocality of the operators involved.

$X^i_{ab}$ is seen to have a simple $1/p^2$ behavior. The pole at $p^2 = 0$ can be removed by taking the Laplacian of the original vacuum average and integrating by parts. We do this for the special case $a = 0, b = j$ and note that, because the vacuum average depends only on the differences $x'^\nu - x^\nu$, application of $\Box'$ is equivalent to application of $\Box$:

$$\int e^{ip\cdot(x'-x)} \Box' \langle \text{vac}| T (\varphi^i(x)j_{0j}^{\mu}(x)) |\text{vac}\rangle d^4x'$$

$$= -p^\mu p^2 X^i_{0j} = p^\mu \langle \text{vac}| \left( \frac{\partial\varphi^i}{\partial\phi^0} \phi_j - \frac{\partial\varphi^i}{\partial\phi^j} \phi_0 \right) |\text{vac}\rangle$$

$$= \delta_{ij} p^\mu \langle \text{vac}| \left[ f (|\phi|) + \frac{1}{2} |\phi| f' (|\phi|) \right] \phi^0 |\text{vac}\rangle.$$  \hspace{1cm} (9.4)
The last line is obtained by invoking the explicit form (6.13) and the fact that the vacuum average must be proportional to $\delta_{ij}$ since, as far as the indices $i$ and $j$ are concerned it is an O(2) invariant tensor. This follows from the fact that the vacuum is O(2) invariant, being a zero-eigenvalue eigenstate of the charge $q_{ij}$.

Now consider the matrix element $\langle \text{vac} | j_{0j}^\mu (x) | i, \mathbf{p} \rangle$. Since $|\text{vac}\rangle$ and $|i, \mathbf{p}\rangle$ are normalized physical state vectors, and since $j_{ab}^\mu$ is the physical current that yields the physically observable charges $q_{ab}$ (which generate group transformations), this matrix element must depend only on observable renormalized quantities. Moreover, its value must be independent of what field variables one uses to calculate it. Making use of eqs. (7.3), (7.5), (7.9) and (9.4), one gets

\[
\langle \text{vac} | j_{0j}^\mu (x) | i, \mathbf{p} \rangle = i \bar{\mu}_\phi \int_{-\infty}^\infty u(x', \mathbf{p}) \left( \frac{\partial}{\partial x'_\nu} - \frac{\hat{\partial}}{\partial x'_\nu} \right) \langle \text{vac} | j_{0j}^\mu (x') \varphi^i (x') | \text{vac} \rangle \ d\Sigma'_\nu
\]

\[
= -i \bar{\mu}_\phi \left( \int_{+\infty}^\infty - \int_{-\infty}^0 \right) u(x', \mathbf{p}) \left( \frac{\partial}{\partial x'_\nu} - \frac{\hat{\partial}}{\partial x'_\nu} \right) \langle \text{vac} | T \left( \varphi^i (x') j_{0j}^\mu (x) \right) | \text{vac} \rangle \ d\Sigma'_\nu
\]

\[
= -i \bar{\mu}_\phi (2\pi)^{-3/2} (2\omega)^{-1/2} e^{ip\cdot x} \int e^{ip\cdot (x'-x)} \left( \hat{\mathbf{D}} - \mathbf{D} \right) \langle \text{vac} | T \left( \varphi^i (x') j_{0j}^\mu (x) \right) | \text{vac} \rangle \ d^4x'
\]

\[
= -i \bar{\mu}_\phi \langle \text{vac} \left[ f (|\phi|) + \frac{1}{2} |\phi| f' (|\phi|) \right] \phi^0 | \text{vac} \rangle \delta_{ij} p^\mu u(x, \mathbf{p}) .
\]  

(9.5)

In the special case $f (|\phi|) = 1$, $\varphi^i = \phi^i$, $\bar{\mu}_\phi = \bar{\mu}_\phi = \bar{\mu}$, the final vacuum average is just $Z$ (see eq. (8.21)), and it follows from eq. (8.23) that

\[
\langle \text{vac} | j_{0j}^\mu (x) | i, \mathbf{p} \rangle = -i \mu_R \delta_{ij} p^\mu u(x, \mathbf{p})
\]

(9.6)

showing that this matrix element does indeed depend only on observable quantities. Moreover, eq. (9.1) now follows, with the identification

\[
Z_\phi = \langle \text{vac} \left[ f (|\phi|) + \frac{1}{2} |\phi| f' (|\phi|) \right] \phi^0 | \text{vac} \rangle .
\]

(9.7)

This identification allows eq. (8.21) (with $G_{ij}^\mu_R$ given by (8.22)) to be generalized:

\[
\langle \text{vac} | T \left( \varphi^i_R (x) \varphi^j_R (x') \right) | \text{vac} \rangle = -\frac{i}{(2\pi)^4} \int \frac{\delta_{ij} e^{ip\cdot (x-x')}}{\mu_R^2 (p^2 - i0) + \cdots} d^4p ,
\]

(9.8)

where

\[
\varphi^i_R = Z_\phi^{-1} \varphi^i .
\]

(9.9)
Note that since the fields $\phi^i_R$ and $\varphi^i_R$ are not identical the unwritten terms in the denominators of the integrands of expressions (8.22) and (9.3) will not generally be the same. Only the coefficients of $p^2$ will be identical.

We now have two completely independent ways of computing $\mu_R$. One is to compute $Z_\varphi$ by (9.7), carry out the field renormalization (9.9), and determine the residue of the pole at $p^2 = 0$ in the Fourier transform of the 2-point function (9.8). The other is to compute the matrix element (9.6).

The first method can immediately be translated to Euclidean space and adapted for the computer. The second requires a little modification. Instead of working directly with the matrix element (9.6) we arrange for it to appear in a sum over intermediate states:

$$\langle \text{vac} | T (j_0^\mu (x) j_0^\nu (x')) | \text{vac} \rangle = \langle \text{vac} | j_0^\mu (x) | k, p \rangle \langle k, p | j_0^\nu (x') | \text{vac} \rangle \sum_{k=1}^{\infty} \int \left[ \theta (x^0 - x'^0) \langle \text{vac} | j_0^\mu (x) | k, p \rangle \langle k, p | j_0^\nu (x') | \text{vac} \rangle + \theta (x'^0 - x^0) \langle \text{vac} | j_0^\nu (x') | k, p \rangle \langle k, p | j_0^\mu (x) | \text{vac} \rangle \right] d^3p$$

Here we assume $\mu \neq \nu$ so that we may use the naïve definition of chronological ordering without worrying about Schwinger terms. In the final sum the omission of the vacuum as an intermediate state follows from the Lorentz invariance and $O(2)$ invariance of the vacuum, which implies

$$\langle \text{vac} | j_0^\mu (x) | \text{vac} \rangle = 0 \text{ .} \quad (9.11)$$

The other unwritten terms in (9.10) are sums over intermediate states involving two or more particles. These are expected to be of higher order in $p^2$ than the written term when the Fourier transform is taken.

Fourier transforms will here be defined by

$$\tilde{f}(p) = (2\pi)^{-2} \int f(x) e^{ipx} d^4x \text{ .} \quad (9.12)$$

To obtain the Fourier transform of (9.10) first insert (9.6) into the sum, obtaining

$$\langle \text{vac} | T (j_0^\mu (x) j_0^\nu (x')) | \text{vac} \rangle$$
\begin{align*}
= (2\pi)^{-3} \mu_R^2 \delta_{ij} \int \frac{p^\mu p^\nu}{2\omega} \left[ \theta \left( x^0 - x'^0 \right) e^{ip(x-x')} + \theta \left( x'^0 - x^0 \right) e^{ip'(x-x')} \right] d^3 p + \cdots \\
= -\frac{i}{(2\pi)^4} \mu_R^2 \delta_{ij} \int \frac{p^\mu p^\nu}{p^2 - i0} e^{ip(x-x')} d^4 p + \cdots, \quad \mu \neq \nu,
\end{align*}

which immediately yields

\begin{equation}
\langle \text{vac} | T (\tilde{\phi}^{\mu}(p) \tilde{\phi}^{\nu}(p')) | \text{vac} \rangle = -\mu_R^2 \delta_{ij} \frac{p^\mu p^\nu}{p^2 - i0} \delta(p + p') + \cdots, \quad \mu \neq \nu.
\end{equation}

Since the unwritten terms are of higher order in \( p^2 \) it is evident that \( \mu_R^2 \) may in principle be determined by examining (9.14) in the limit \( p^2 \to 0 \).

## 10 Euclidean space

To pass to Euclidean space one makes the variable replacements

\begin{equation}
x^0 = -ix^4, \quad p_0 = ip_4,
\end{equation}

and carries out 90° rotations in the complex planes of these variables. In the functional integrals (5.1a) and (5.4a), as well as in expressions (5.2) and (5.3a) for the functional volume element, the \( x^\mu \) are simply labels, or generalized indices, on the dummy integration variables. Hence the replacements (10.1) would have no effect on the integrals themselves were it not for the fact that the integrands have an explicit dependence on \( x^0 \) through the dependence of the action functional (2.1) on the volume element \( d^4 x \), which gets multiplied by \(-i\). This dependence has the consequence that (5.1a) and (5.4a) get replaced by

\begin{align*}
\langle \text{out} | \text{in} \rangle &= N \int e^{-S_E[\phi]} \mu[\phi] d\phi, \\
\langle \text{out} | A[\phi] | \text{in} \rangle &= N \int A[\phi] e^{-S_E[\phi]} \mu[\phi] d\phi,
\end{align*}

where \( S_E \) is the Euclidean action,

\begin{equation}
S_E[\phi] = \frac{1}{2} \mu^2 \int g_{ij} (\phi(x)) \phi^i,\mu \phi^j,\mu d^4 x,
\end{equation}

and the Greek indices now run from 1 to 4. In eq. (10.3) there may also be additional dependence on \( x^0 \) and/or \( p_0 \) if \( A[\phi] \), \( |\text{in} \rangle \) or \( |\text{out} \rangle \) depends explicitly on one or both of these variables. In this case the corresponding dependence on \( x^4 \) and/or \( p_4 \) is obtained by analytic continuation.

It will be noted in eq. (10.4) that the Minkowski metric \( \eta_{\mu\nu} \) has been replaced by the Euclidean \( \delta_{\mu\nu} \). Moreover, in eq. (10.3) the chronological ordering symbol \( T \) has been dropped.
as irrelevant, the quantity on the left being now simply regarded as defined by the integral on the right.

In Minkowski spacetime the Fourier components of $\phi^i(x)\langle \text{vac}\rangle$ behave like $e^{i\omega x^0}$ in the remote past and those of $\langle \text{vac}\rvert \phi^i(x)\rangle$ behave like $e^{-i\omega x^0}$ in the remote future (see eq. (7.19)). The corresponding behaviors in Euclidean space are $e^{\omega x^4}$ and $e^{-\omega x^4}$ respectively, which imply that to get Euclidean vacuum averages one must integrate, in eqs. (10.2) and (10.3), over fields that vanish at infinity. On the computer, Euclidean space will be replaced by a 4-torus $T^4$, the fields will satisfy periodic boundary conditions, and only their integrals over the 4-torus will be required to vanish. The corresponding “Euclidean vacuum averages” will be indicated by the abbreviated notation.

\[
\langle A[\phi] \rangle = \frac{\int A[\phi] e^{-S_E[\phi]} \mu[\phi] d\phi}{\int e^{-S_E[\phi]} \mu[\phi] d\phi} .
\]

(10.5)

It is not possible to use a Euclideanized version of the action (2.5) on a computer because a straightforward Euclideanization of (2.5) leads to an action that is not bounded from below. One can use a Euclideanized version for theoretical purposes if the Lagrange multiplier field $\lambda$ is itself subjected to a rotation in the complex plane:

\[
\lambda = -i\lambda_E .
\]

(10.6)

A Euclideanized effective action $\bar{\Gamma}_E [\bar{\phi}, \bar{\lambda}_E]$ can then be introduced, as well as a corresponding $\Gamma_E [\bar{\phi}]$ having the form

\[
\Gamma_E [\bar{\phi}] = \frac{1}{2} \bar{\mu}^2 \int \bar{g}_{ij} \left( \bar{\phi}(x) \right) \bar{\phi}^i,\mu \bar{\phi}^j,\mu d^4x + \Delta \Gamma_E [\bar{\phi}]
\]

(10.7)

(cf. eq. (8.15)). It will be left as an exercise for the reader to rewrite section 8 in Euclidean language. We remark only that the renormalization constants all remain unchanged, and equations (8.17) to (8.20) take the forms

\[
\phi^a_R = Z^{-1} \phi^a ,
\]

(10.8)

\[
\Gamma_E = \frac{1}{2} \mu_R^2 \int g_{ij} \left( \bar{\phi}_R(x) \right) \bar{\phi}^{i,\mu}_R \bar{\phi}^i,\mu_R d^4x + \Delta \Gamma_E ,
\]

(10.9)

\[
\mu_R = Z \bar{\mu} ,
\]

(10.10)

\[
Z = \bar{\phi}^0 = \langle \phi^0 \rangle .
\]

(10.11)

More generally, one may write

\[
\mu_R = Z_{\phi} \bar{\mu}_{\phi}
\]

(10.12)
where
\[ Z_\varphi = \langle f(\phi) + \frac{1}{2} |\phi| f'(\phi) \rangle \phi^0 \] (10.13)
and where \( \bar{\mu}_\varphi \) may be identified from the relation
\[ \langle \varphi^i(x) \varphi^i(x') \rangle = \frac{1}{(2\pi)^4} \int \frac{\delta_{ij} e^{ip \cdot (x-x')}}{\bar{\mu}_\varphi^2 p^2 + \cdots} d^4 p, \] (10.14)
which follows from (7.7) and the fact that, in the transition from Minkowski spacetime to Euclidean space, the momentum volume element \( d^4 p \) gets multiplied by \( i \). In the special case in which the \( \varphi^i \) are chosen to be the Riemann normal coordinates \( \sigma^i \) (see eqs. (6.14) and (6.15)) it is straightforward to verify that
\[ Z_\sigma = \frac{1}{2} \left( 1 + \frac{s \cosh s}{\sinh s} \right). \] (10.15)
The variables \( \sigma^i \) are particularly convenient to work with on the computer, and in the following paper an account will be given of the determination of \( \mu_R \) based on the equation
\[ \langle \sigma^i_R(x) \sigma^j_R(x') \rangle = \frac{1}{(2\pi)^4} \int \frac{\delta_{ij} e^{ip \cdot (x-x')}}{\mu_R^2 p^2 + \cdots} d^4 p, \] (10.16)
\[ \sigma^{i}_R = Z_\sigma^{-1} \sigma^i. \] (10.17)
We note finally the Euclideanized versions of eqs. (9.13) and (9.14):
\[ \langle j_{0\mu}(x) j_{0\nu}(x') \rangle = \frac{\mu^2 R \delta_{ij}}{(2\pi)^4} \int \frac{p_\mu p_\nu e^{ip \cdot (x-x')}}{p^2} d^4 p + \cdots, \quad \mu \neq \nu, \] (10.18)
\[ \langle \tilde{j}_{0\mu}(p) \tilde{j}_{0\nu}(p') \rangle = \mu^2 R \delta_{ij} \frac{p_\mu p_\nu}{p^2} \delta(p + p') + \cdots, \quad \mu \neq \nu. \] (10.19)
The Fourier transforms are here defined as in eq. (9.12), and the currents themselves are defined by
\[ j_{ab\mu}(x) = \mu^2 (\phi_a \phi_{b,\mu} - \phi_b \phi_{a,\mu}) \] (10.20)
(cf. eq. (1.17)).

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Appendix

Equations (7.3) and (7.5) imply

\[
\langle \text{vac} | \varphi^i(x) | j, p' \rangle = i\bar{\mu}_\varphi \int_{-\infty}^{\infty} u(x', p') \left( \frac{\partial}{\partial x'_\mu} - \frac{\partial}{\partial x_\mu} \right) \langle \text{vac} | \varphi^i(x) \varphi^j(x') | \text{vac} \rangle \, d\Sigma'_\mu. \tag{A.1}
\]

This may be rewritten in the form

\[
-\bar{\mu}_\varphi \left( \int_{+\infty}^{-\infty} - \int_{-\infty}^{+\infty} \right) u(x', p') \left( \frac{\partial}{\partial x'_\mu} - \frac{\partial}{\partial x_\mu} \right) \langle \text{vac} | T \left( \varphi^i(x) \varphi^j(x') \right) | \text{vac} \rangle \, d\Sigma'_\mu
\]

\[
= -i\bar{\mu}_\varphi \int u(x', p') \left( \hat{D} - \hat{D}' \right) \langle \text{vac} | T \left( \varphi^i(x) \varphi^j(x') \right) | \text{vac} \rangle \, d^4x' \tag{A.2}
\]

since the insertion of the integral over the hypersurface +\infty contributes nothing in view of (7.3) and the kinematics of the chronological product. Inserting expressions (7.4) and (7.7) into (A.2), and using the fact that \( u(x', p') \hat{D} = 0 \), one obtains

\[
\langle \text{vac} | \varphi^i(x) | j, p' \rangle = (2\pi)^{-3/2}\bar{\mu}_\varphi (2\omega')^{-1/2} \int d^4x' \int d^4p e^{ip\cdot x'} \frac{1}{(2\pi)^4} \frac{\delta_{ij} e^{ip_\varphi (x-x')}}{\bar{\mu}_\varphi^2 + \cdots} \]

\[
= (2\pi)^{-3/2}\bar{\mu}_\varphi (2\omega')^{-1/2} \int \delta_{ij} \delta(p - p') e^{ip_\varphi x} \frac{1}{\bar{\mu}_\varphi^2 + \cdots} d^4p \]

\[
= \delta_{ij} \bar{\mu}_\varphi^{-1} u(x, p') \tag{A.3}
\]

To reach the last line one makes use of the fact that the \( \delta \)-function in the line above enforces the constraint \( p = p', \, p^2 = p'^2 = 0 \), and hence the unwritten terms in the denominator of the integrand vanish.

The adjoints of eqs. (7.3) and (7.5) now yield

\[
\langle i, p | j, p' \rangle = -i\bar{\mu}_\varphi \int_{\pm \infty} u(x, p) \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x'_\mu} \right) \langle \text{vac} | \varphi^i(x) | j, p' \rangle \, d\Sigma_\mu
\]

\[
= -i\delta_{ij} \int_{\pm \infty} u(x, p) \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x'_\mu} \right) u(x, p') \, d\Sigma_\mu \]

\[
= \delta_{ij} \delta(p - p') \tag{A.4}
\]