Minimizing Symmetric Set Functions Faster

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Abstract

We describe a combinatorial algorithm which, given a monotone and consistent symmetric set function \(d\) on a finite set \(V\) in the sense of Rizzi [Riz00], constructs a nontrivial set \(S\) minimizing \(d(S, V \setminus S)\). This includes the possibility for the minimization of symmetric submodular functions. The presented algorithm requires at most as much time as the one in [Riz00], but depending on the function \(d\), it may allow several improvements.

Keywords: symmetric set function, symmetric submodular function, minimum cut, optimal bipartition

1 Introduction

Let \(V\) be a finite set. A symmetric set function \(d\) is a map assigning a real number to each pair \((S, T)\) of disjoint subsets of \(V\) satisfying \(d(S, T) = d(T, S)\). The function \(d\) is called monotone, if for every pair \(S, T\) of disjoint sets and \(T' \subseteq T\) the inequality \(d(S, T') \leq d(S, T)\) holds. It is called consistent, if for every triple \(R, S, T\) of pairwise disjoint sets \(d(S, R) \geq d(T, R)\) implies \(d(S, R \cup T) \geq d(S \cup R, T)\). As the examples below indicate, the construction of a minimum bipartition of \((V, d)\), ie. a nontrivial\(^1\) subset \(S\) of \(V\) minimizing \(d(S, V \setminus S)\), is important for many applications.

Example 1.1. If \(G = (V, E)\) is a (hyper-)graph with weighted edges, the weight \(w(S, T)\) of edges between two disjoint sets of vertices is a monotone and consistent symmetric set function. A minimum bipartition of \((V, d)\) is a minimum cut of the (hyper-)graph \(G\). [NI92, Fra94, SW97, KW95, Bri05]

Example 1.2. If \(f: 2^V \to \mathbb{R}\) is submodular, ie.

\[
    f(S) + f(T) \geq f(S \cup T) + f(S \cup T)
\]

for all subsets \(S, T \subseteq V\), then the generalized connectivity function

\[
    c_f(S, T) := f(S) + f(T) - f(S \cup T)
\]

is a monotone and consistent symmetric set function. If \(f\) is symmetric, ie. \(f(A) = f(V \setminus A)\), then a minimum bipartition of \(c_f\) is a nontrivial set minimizing \(f\). [Cam85, Que98, Riz06]

A local version of the minimum bipartition problem, consists in the detection of a set \(S\), such that \(d(S, V \setminus S)\) is minimal among all sets separating two given elements \(s\) and \(t\), ie. \(s \in S\) and \(t \notin S\). We define

\[
    \lambda_{(V,d)}(s, t) = \min \{d(S, V \setminus S) \mid s \in S, t \notin S\}
\]

\(^1\)Nontrivial means \(\emptyset \neq S \neq V\)
and

$$\lambda_{(V,d)}(s, t) = \min \{ \lambda_{(V,d)}(s, t) \mid s, t \in V \}.$$  

Hence, we want to find a set $S$, such that $d(S, V \setminus S) = \lambda_{(V,d)}$.

Based on the work of Nagamochi and Ibaraki [NI92], Stoer and Wagner in SW97, and independently Frank in Tra97, described an algorithm constructing a minimum cut of a weighted graph (exp. 1.1). This algorithm was generalized by Queyranne Que98 to the minimization of symmetric submodular functions (exp. 1.2), and subsequently by Nagamochi and Ibaraki to a wider class of functions, satisfying a less restrictive symmetry condition.

Another generalization was given by Rizzi in Riz00. He proved that the algorithm of Stoer/Wagner/Frank/Queyranne generalizes to the minimization of monotone and consistent symmetric set functions. Based on his work, we will describe a variation of the algorithm which provides possibilities to reduce the required time. Unfortunately, these improvements depend heavily on the function to be minimized. But we can guarantee, that the new algorithm requires at most the same time as that one described in Riz00.

One example for the possible improvement can be found in Br05. There, the author used the techniques presented in this paper to reduce the running time of the algorithm of Stoer and Wagner for the minimum cut of an integer weighted graph from $O(nm + n^2 \log n)$ to $O(\delta n^2)$, with $\delta$ being the minimum degree$^2$.

We proceed as follows. First, we introduce lax-back orders, generalizing the max-back orders required by Rizzi in Riz00. Following that, we prove that the lax-back orders provide the possibility to identify several elements, hence reducing the size of $V$, without losing optimal sets. Then these results are used to describe and prove a new algorithm, bearing several possible improvements of the ones given in Riz00 and Que98.

## 2 Lax-Back Orders

In the following, for each finite set $V$ we write $n = |V|$ and a singleton set $\{v\} \subseteq V$ will simply be denoted by $v$.

Before we proceed, we make a simple and useful observation.

\textbf{Lemma 2.1.} Let $d$ be a symmetric set function on $V$ and $u, v, w$ elements in $V$ and $\tau \in \mathbb{R}$, then $\lambda_{(V,d)}(u, v), \lambda_{(V,d)}(v, w) \geq \tau$ implies $\lambda_{(V,d)}(u, w) \geq \tau$.

\textbf{Proof.} Let $S$ be a subset of $V$ with $u \in S$ and $w \notin S$, such that $\lambda_{(V,d)}(u, w) = d(S, V \setminus S)$. If $v \in S$, $S$ separates $v$ and $w$ and hence $d(S, V \setminus S) \geq \lambda_{(V,d)}(v, w) \geq \tau$. If $v \notin S$ we have $d(S, V \setminus S) \geq \lambda_{(V,d)}(u, v) \geq \tau$. \hfill $\square$

Let $V$ be a finite set and $d$ a symmetric set function on $V$. An ordered pair $(s, t)$ of elements of $V$ is called good, if $d(t, V \setminus t) = \lambda_{(V,d)}(s, t)$. As Rizzi proved in Riz00, such a pair $(v_{n-1}, v_n)$ can be found by constructing a total order $v_1, \ldots, v_n$ on $V$, such that

$$d(v_i, \{v_1, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, \ldots, v_{j-1}\}) \quad \text{for } 1 \leq i < j \leq n.$$  

An order of this type is called max-back order for $(V,d)$.

\textbf{Lemma 2.2 (Rizzi Riz00).} Let $v_1, \ldots, v_n$ be a max-back order for $(V,d)$. Then $(v_{n-1}, v_n)$ is good for $(V,d)$.

Instead of using the original function $d$, we adapt it and introduce a threshold $\tau$.

\textbf{Lemma 2.3.} Let $d$ be a monotone and consistent symmetric set function and $\tau$ a real number. Then $d(S, T) = \min \{ \tau, d(S, T) \}$ is a monotone and consistent symmetric set function.

$^2$‘Reduction’ is a bit dangerous, since the running time is $O(\delta n^2)$ and hence pseudo polynomial. Nonetheless, if the edge weights are reasonable, one can expect the new algorithm to be faster than the old one.
Proof. Obviously $\hat{d}$ is symmetric. Now let $S, T$ be two disjoint sets and $T' \subseteq T$. Then we have

$$\hat{d}(S, T') = \min\{\tau, d(S, T')\} \leq \min\{\tau, d(S, T)\} = \hat{d}(S, T).$$

Hence, $\hat{d}$ is monotone.

Now let $R, S, T$ be three pairwise disjoint sets such that $\hat{d}(S, R) \geq \hat{d}(T, R)$. If $d(S, R) < \tau$, then $d(T, R) \leq d(S, R)$. Due to the consistency of $d$, this leads to

$$\hat{d}(S, R \cup T) = \min\{\tau, d(S, R \cup T)\} \geq \min\{\tau, d(S \cup R, T)\} = \hat{d}(S \cup R, T).$$

If on the other hand $d(S, R) \geq \tau$, we have

$$\hat{d}(S, R \cup T) = \min\{\tau, d(S, R \cup T)\} \geq \min\{\tau, d(S, R)\} = \tau.$$

Furthermore,

$$\hat{d}(S \cup R, T) = \min\{\tau, d(S \cup R, T)\}\begin{cases} \tau & \text{if } d(S \cup R, T) \geq \tau \\ d(S \cup R, T) & \text{if } d(S \cup R, T) < \tau \end{cases} \leq \tau \leq \hat{d}(S, R \cup T),$$

concluding the proof of the consistency of $\hat{d}$. \qed

A max-back order $v_1, \ldots, v_n$ for $(V, \hat{d})$ satisfies

$$\min\{\tau, d(v_i, \{v_1, \ldots, v_{i-1}\}\}) \geq \min\{\tau, d(v_j, \{v_1, \ldots, v_{j-1}\}\}) \quad \text{for } 1 \leq i < j \leq n.$$

Such an order is called lax-back order with threshold $\tau$ for $(V, d)$.

**Corollary 2.4.** Let $d$ be a symmetric set function on $V$ and $v_1, \ldots, v_n$ a lax-back order with threshold $\tau$ for $(V, d)$. Then the pair $(v_{n-1}, v_n)$ is $\tau$-good for $(V, d)$, i.e. for each set $T$ with $v_n \in T$ and $v_{n-1} \notin T$, we have

$$\min\{\tau, d(\{v_n\}, V \setminus \{v_n\}\}) = \min\{\tau, \lambda_{(V, d)}(v_n, v_{n-1})\}.$$

The proof of the following lemma is straightforward.

**Lemma 2.5.** Let $d$ be a monotone and consistent symmetric set function on $V$.

1. If $v_1, \ldots, v_n$ is a max-back order for $(V, d)$, then it is a lax-back order with threshold $\tau$ for every $\tau$.

2. If $v_1, \ldots, v_n$ is a lax-back order for $(V, d)$ with threshold $\tau$, then it is a lax-back order with threshold $\tau'$ for any $\tau' \leq \tau$.

As the preceding lemma shows, we can interpret a max-back order as a lax back-order with threshold $\infty$.

## 3 Contraction and Lax-Back Orders

In the following let $d$ be a fixed monotone and consistent symmetric set function on $V$. In [Riz00], Rizzi used a max-back order for $(V, d)$ to identify one good pair $(v_{n-1}, v_n)$. He then identified these two elements, obtaining a monotone and consistent symmetric set function $d'$ on a smaller set $V'$. This process of identification or contraction may easily be extended to arbitrary partitions of $V$.
Let $\mathcal{V} = \{U_1, \ldots, U_k\}$ be an arbitrary partition of $V$. For each element $v \in V$, the class of $v$ is denoted by $[v]$, i.e., $[v] = U_i$ for $v \in U_i$. If, on the other hand, $\mathcal{S}$ is a set of classes of $\mathcal{V}$, we write

$$\cup \mathcal{S} = \bigcup_{U_i \in \mathcal{V}} U_i \subseteq V$$

for the set of members of classes in $\mathcal{S}$.

If $d$ is a symmetric set function, we define the induced function $d_{\mathcal{V}}$ on $\mathcal{V}$ as

$$d_{\mathcal{V}}(\mathcal{S}, \mathcal{T}) := d(\cup \mathcal{S}, \cup \mathcal{T}).$$

It is easy to check, that $d_{\mathcal{V}}$ is monotone and consistent if $d$ is.

**Lemma 3.1.** Let $d$ be a monotone and consistent symmetric set function on $d$. If $v_1, \ldots, v_n$ is a lax-back-order with threshold $\tau$ for $(V, d)$, then

$$\min \{ \tau, \lambda(V, d)(v_{i-1}, v_i) \} \geq \min \{ \tau, d(v_i, \{v_1, \ldots, v_{i-1}\}) \}$$

for $2 \leq i \leq n$.

**Proof.** Obviously, the restriction $d_i$ of $d$ to $V_i := \{v_1, \ldots, v_i\}$ is a monotone and consistent symmetric set function on $V_i$ and hence $v_1, \ldots, v_i$ is a lax-back-order with threshold $\tau$ for $(V_i, d_i)$. By corollary 224 we have

$$\min \{ \tau, d_i(v_i, V_{i-1}) \} = \min \{ \tau, \lambda(V_i, d_i)(v_{i-1}, v_i) \}.$$

Now let $\mathcal{S}$ be an arbitrary subset of $V$ with $v_i \in \mathcal{S}$ and $v_{i-1} \notin \mathcal{S}$. Then $S_i := \mathcal{S} \cap V_i$ separates $v_i$ and $v_{i-1}$ in $V_i$ and hence, due to the monotony of $d$,

$$\lambda(V_i, d_i)(v_i, v_{i-1}) \leq d_i(S_i, V_i \setminus S_i) = d(S_i, V_i \setminus S_i) \leq d(S, V \setminus S),$$

and thus

$$\lambda(V_i, d_i)(v_i, v_{i-1}) \leq \lambda(V, d)(v_i, v_{i-1}).$$

In combination with the observation made above, this leads to

$$\min \{ \tau, d_i(v_i, V_{i-1}) \} = \min \{ \tau, \lambda(V_i, d_i)(v_{i-1}, v_i) \} \leq \{ \tau, \lambda(V, d)(v_i, v_{i-1}) \}.$$

Let $v_1, \ldots, v_n$ be a lax-back order with threshold $\tau$ for $(V, d)$. If $\lambda(V, d) < \tau$, then no pair $(v_{i-1}, v_i)$ of elements with $d(v_i, V_{i-1}) \geq \tau$ can be separated by a set $S$ with $d(S, V \setminus S) = \lambda(V, d)$. Hence we may identify $v_i$ and $v_{i-1}$, without increasing $\lambda(V, d)$. More precisely, we have the following result.

**Lemma 3.2.** Let $d$ be a monotone and consistent symmetric set function on $V$ and $\tau$. If $\mathcal{V}$ is a partition of $V$, satisfying $\lambda(V, d)(u, v) \geq \tau$ for each pair $u, v$ of elements with $[u] \neq [v]$, then for each pair $s, t$ of elements of $V$ with $[s] \neq [t]$, we have

$$\min \{ \tau, \lambda(V, d)(s, t) \} = \min \{ \tau, \lambda(V, d)(s, t) \},$$

and

$$\min \{ \tau, \lambda(V, d)(s, t) \} = \min \{ \tau, \lambda(V, d)(s, t) \}.$$
Algorithm 1: Lax-Back-Order

**Input:** A consistent symmetric set function \( d \) on a finite set \( V \), given by a lax oracle \( F \) and a threshold \( \tau \)

**Output:** A lax-back order \( L = (v_1, \ldots, v_n) \) with threshold \( \tau \) for \((V, d)\)

**Data:** An ordered list \( L \) and a subset \( U \) of \( V \)

1. \( L = (v_1) \);
2. \( U = V \setminus v_1 \);
3. while \(|U| \geq 1\) do
   4. \( x := 0 \);
   5. forall \( u' \in U \) do
      6. if \( F(u, V \setminus U; \tau) \geq \tau \) then
         7. \( L := (L, u), U := U \setminus u; \) // Append \( u \) to \( L \).
         8. \( x := \tau \);
      9. else if \( F(u, V \setminus U; \tau) > x \) then
         10. \( x := F(u, V \setminus U; \tau); \)
         11. \( u := u'; \)
   12. end
   13. if \( x < \tau \) then
      14. \( L := (L, u), U := U \setminus u; \) // Append \( u \) to \( L \).
      15. end
   16. end
   17. end

Now assume that \( \lambda_{(V,d)}(s,t) < \tau \). In this case, no set \( S \) separating \( s \) and \( t \) with \( d(S, V \setminus S) = \lambda_{(V,d)}(s,t) \), can separate two elements \( u \) and \( v \) with \([u] = [v] \), because \( \lambda_{(V,d)}(u,v) \geq \tau \). Hence each partition \([u]\) is contained in either \( S \) or \( V \setminus S \). Therefore, \( S \) induces a subset \( S \) of \( V \) with \( \bigcup S = S \) and hence \( \lambda_{(V,d)}([s],[t]) \leq \lambda_{(V,d)}(s,t) \), implying

\[
\min \{ \tau, \lambda_{(V,d)}([s],[t]) \} = \lambda_{(V,d)}([s],[t]) \leq \lambda_{(V,d)}(s,t) \leq \min \{ \tau, \lambda_{(V,d)}(s,t) \} \leq \min \{ \tau, \lambda_{(V,d)}([s],[t]) \}
\]

The second part of the lemma is a direct consequence of the first part and the definition of \( \lambda_{(V,d)} \).

Now let \( v_1, \ldots, v_n \) be a lax-back order with threshold \( \tau \) of \((V,d)\). Then define \( V \) as the partition of \( V \) consisting of the classes of the transitive and symmetric closure of the relation \( v_i \sim v_{i-1} \) iff \( d(v_i, \{v_1, \ldots, v_{i-1}\}) \geq \tau \). By lemma 2.1 \( V \) satisfies the condition of 3.2 and hence

\[
\min \{ \tau, \lambda_{(V,d)} \} = \min \{ \tau, \lambda_{(V,d)} \}.
\]

This simple observation is the main tool for the algorithm presented in the next section.

4 Minimizing Symmetric Set Functions

4.1 Construction of Lax-Back Orders

Our algorithm for the calculation of a minimum bipartition of \((V,d)\), requires the construction of a lax-back order as a subroutine. To achieve this, we assume, that we have access to a \(^3\)Equivalently, we may say, that the classes of \( V \) are the maximal subsequences \( v_i, \ldots, v_k \) such that 
\( d(v_j, V_i) \geq \tau \) for \( i < j \leq k \).
a program which returns \( \min \{ \tau, d(S,T) \} \) for any input of two disjoint subsets of \( V \) and a threshold \( \tau \).

The correctness of the algorithm is intuitively clear. Nevertheless we will prove it in detail. We show that, if an element \( u \) is added to \( L \) in lines 7 or 15, it satisfies the inequality
\[
\min \{ \tau, d(u,V \setminus U) \} \geq \min \{ \tau, d(u',V \setminus U) \}
\]
for every \( u' \in U \). Since \( V \setminus U \) contains all elements already in \( L \), this implies that \( L \) is an lax-back order with threshold \( \tau \) for \( (V,d) \) at the end of the algorithm.

If \( u \) is appended to \( L \) in line 7, we have \( d(u,V \setminus U) \geq \tau \) and hence
\[
\min \{ \tau, d(u,V \setminus U) \} = \tau \geq \min \{ \tau, d(u',V \setminus U) \}
\]
for each \( u' \in U \).

If \( u \) is appended to \( L \) in line 15, no element was appended to \( L \) in line 7 in this round. Hence \( d(u,V \setminus U) \) is obvious maximum among all \( u' \in U \). This concludes the proof of the correctness of algorithm 1.

In each round of the while loop, exactly \( |U| \) calls of the oracle are required. Since in each round at least one element is removed from \( U \), and since \( U \) begins with \( |V| - 1 \) elements (line 2), at most \( \frac{|V|}{\delta} \) calls to the oracle are made with \( n = |V| \). Since it is possible, that more than one element is removed from \( U \) in a round, this bound may be very conservative, depending on the function \( d \). As a result, a lax-back order can be calculated in time \( O(n^2T_F) \), where \( T_F \) is an upper bound for the execution time of the lax-oracle \( F \).

But, depending on the lax oracle, the runtime required for the construction of a lax-back order may be smaller. For example, as proven in [SW97], the construction of a maximum adjacency order on a weighted, undirected graph \( G = (V,E) \), the analogue of a max-back order, requires time \( O(m+n \log n) \), where \( n = |V| \) and \( m = |E| \). In [Bri05] this was ‘reduced’ to time \( O(m+\delta n) \) for the first, and \( O(\delta n) \) for each subsequent lax-back order with a threshold \( t \leq \delta \). Here \( \delta \) is the minimum degree of \( G \).

As the above example indicates, it is possible in many situations to calculate \( d(u,V_i) \) from \( d(u,V_{i-1}) \) much faster, than computing \( d(u,V_i) \) directly. In these cases a variation of algorithm 1 in the fashion of the algorithm described in [Que98] is more appropriate.

In algorithm 2 we use a priority queue \( Q \) with threshold \( \tau \), i.e. \( Q \) contains value-key-pairs \( (v,k) \in V \times \mathbb{R} \) and provides three operations.

- The insert operation adds a pair \((v,k)\) to \( Q \).
- The del\_max operation removes a pair \((v,k)\) from \( Q \) such that
  \[
  k \geq \min \{ \tau, \max\{k' \mid (v',k') \in Q\} \}.
  \]
- The update\_key operation updates the key \( k = d(u,V_i) \) to \( k = d(u,V_{i+1}) \) if \( v_i \) was extracted from \( Q \).

The time required by algorithm 2 obviously depends on the type of thresholded priority queue used and the time required by update\_key. But, since we are looking at the most general case, we simply assume that each lax-back order can be calculated in time \( T_{LOB} \in O(n^2T_F) \).

### 4.2 Constructing a Minimum Bipartition

Algorithm 3 constructs a minimum bipartition, using algorithm 1 as a subroutine. Before we begin with the analysis of the algorithm, three remarks are appropriate:

- In the given form, algorithm 3 first builds a max-back order, since \( \tau = \infty \). Alternatively, the initial threshold may be set to
  \[
  \min \{ d(v,S \setminus v) \mid v \in V \},
  \]
resulting in a lax-back order, even in the first round. But in the worst case this requires \( n \) additional calls of the oracle.
Algorithm 2: Lax-Back-Order with Priority Queue

Input: A consistent symmetric set function \( d \) on a finite set \( V \), given by a lax oracle \( F \) and a threshold \( \tau \)

Output: A lax-back order \( L = (v_1, \ldots, v_n) \) with threshold \( \tau \) for \((V, d)\)

Data: An ordered list \( L \) and a priority queue \( Q \) with threshold \( \tau \).

1. forall \( v \in V \) do \( Q.\)insert\((v, -\infty)\);
2. while \(|Q| \geq 0\) do  
3. \hspace{1em} \( v := Q.\)del\_max\();
4. \hspace{1em} \( L := (L, v); \)  // Append \( u \) to \( L \).
5. forall \( u \in Q \) do update\_key\((u)\);
6. end

Algorithm 3: OptimalSet

Input: A consistent symmetric set function \( d \) on a finite set \( V \)

Output: A subset \( S \) of \( V \) with \( d(S, V \setminus S) = \lambda_{(V, d)} \)

Data: A partition \( \mathcal{V} \) of \( V \) and a real number \( \tau \)

1. \( S := \emptyset; \)
2. \( \mathcal{V} := V; \)
3. \( \tau := \infty; \)
4. while \(|\mathcal{V}| \geq 2\) do  
5. \hspace{1em} \( (\{v_1\}, \ldots, \{v_k\}) := \text{Lax-Back-Order}(\mathcal{V}, d_{\mathcal{V}, \tau}); \)
6. \hspace{1em} if \( d([v_k], \mathcal{V} \setminus [v_k]) < \tau \) then  
7. \hspace{2em} \( S := [v_k]; \)
8. \hspace{2em} \( \tau := d([v_k], \mathcal{V} \setminus [v_k]); \)
9. end
10. for \( i = 2, \ldots, k \) do  
11. \hspace{1em} if \( d([v_i], \{[v_1], \ldots, [v_{i-1}]\}) \geq \tau \) then  
12. \hspace{2em} Join \([v_{i-1}]\) and \([v_i]\);
13. end
14. end
15. end

- Of course, we assume in lines 6, 8 and 11, that the \( d([v_i], \{[v_1], \ldots, [v_{i-1}]\}) \) were stored during the construction of the lax-back order.
- We assume that lines 10-14 require \( O(n) \) time.

Now we turn our attention to the correctness of algorithm 3. First, we prove that at the end of each execution of the body of the exterior loop (lines 5-14), we have

\[
\lambda_{(V, d)} \leq \tau = d(S, V \setminus S) \quad \text{and} \quad \forall u, v \in V: [u]_{\mathcal{V}} = [v]_{\mathcal{V}} \Rightarrow \lambda_{(V, d)} \geq \tau, \quad (*)
\]

where \([u]_{\mathcal{V}}\) denotes the equivalence class of \( u \in V \) in \( \mathcal{V} \).

Observe that after the first execution of the body of the while loop, we have \( S = \{v_n\} \) and \( \tau = d(v_n, V \setminus v_n) \). Hence \((*)\) is satisfied.

Now assume that \( S, \tau \) and \( \mathcal{V} \) are the ‘values’ before the execution of the loop and that they satisfy \((*)\). If \( d([v_k], \mathcal{V} \setminus [v_k]) \geq \tau \), the values of \( S \) and \( \tau \) aren’t changed and hence the first part of \((*)\) is still valid at the end of the loop-body. If \( d([v_k], \mathcal{V} \setminus [v_k]) < \tau \) we obtain the new values \( \tau' = d([v_k], \mathcal{V} \setminus [v_k]) < \tau \) and \( S' = [v_k] \). Since \( d_{\mathcal{V}}([v_k], \mathcal{V} \setminus [v_k]) = d(S', V \setminus S') = \tau' \), we obviously have \( \lambda_{(V, d)} \leq \tau' = d(S', V \setminus S') \).

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In line 10-14, sequences \([v_1]_V, \ldots, [v_l]_V\) of classes in \(V\) are joined. This results in a partition \(V'\), such that each class \([u]_V'\) of \(V'\) is contained in a class of \(V\).

Now assume that \([u]_V' = [v]_V'\) for two elements \(s, t\) of \(V\). If \([u]_V = [v]_V\), we have \(\lambda_{(V,d)}(u, v) \geq \tau\), since (*) was satisfied before the sets were joined.

If \([u]_V \neq [v]_V\), we assume that \(S\) separates \(u\) and \(v\) with \(u \in S\) and \(\lambda_{(V,d)}(u, v) = d(S, V \setminus S)\). If \(S\) also separates two elements \(s\) and \(t\) with \([s]_V = [t]_V\), we have \(d(S, V \setminus S) \geq \tau\), by induction.

If on the other hand, each class of \(V\) lies either in \(S\) or in \(V \setminus S\), w.l.o.g. we have a sequence \([u]_V = [v_1]_V, \ldots, [v_l]_V = [v_l]_V\) of classes in \(V\), such that

\[
d([v_1]_V, \{[v_1]_V, \ldots, [v_{l-1}]_V\}) \geq \tau\quad\text{for } i < l \leq j.
\]

Therefore, lemma 3.1 implies

\[
\lambda_{(V,d)}([v_1]_V, [v_{l-1}]_V) \geq \tau\quad\text{for } i < l \leq j.
\]

By lemma 3.2 this implies

\[
\lambda_{(V,d)}(v_1, v_{l-1}) \geq \tau\quad\text{for } i < l \leq j
\]

and hence, by lemma 2.4 \(\lambda_{(V,d)}(u, v) \geq \tau\), proving the validity of (*) at the end of the loop.

Since \(\tau \leq d_{V,d}([v_k]_V, V \setminus [v_k]_V)\) after line 9, at least the sets \([v_{k-1}]_V\) and \([v_k]_V\) are joined in lines 10-14. Hence, the number of partitions in \(V\) decreases each round and at most \(|V| - 1\) rounds, we have \(|V| = 1\), and the algorithm terminates.

In this situation (*) implies \(\lambda_{(V,d)} \leq \tau = d(S, V \setminus S)\) and since \([u] = [v]\) for all \(u, v \in V\), we have \(\lambda_{(V,d)}(u, v) \geq \tau\), leading to

\[
\lambda_{(V,d)} = \tau = d(S, V \setminus S).
\]

**Theorem 4.1.** Given a lax oracle \(F\) for a monotone and consistent symmetric set function \(d\) on \(V\)

1. The algorithm \(LBO\) returns a set \(S\) of \(V\) with \(d(S, V \setminus S) = \lambda_{(V,d)}\) in time \(O(n^3T_F)\), where \(T_F\) is an upper bound for the time required by the oracle \(F\).

2. If \(T_{LBO}\) is an upper bound for the construction of a lax-back order with threshold \(\tau \leq \min\{d(v, V \setminus v) \mid v \in V\}\) for \((V, d)\), then a set \(S\) with \(d(S, V \setminus S) = \lambda_{(V,d)}\) can be constructed in time \(O(n(T_F + T_{LBO}))\).

**5 Conclusion**

We observed that for each monotone and consistent symmetric set function \(d\) on a finite set \(V\), the function \(d(S, T) = \min\{\tau, d(S, T)\}\) is monotone and consistent, too. This fact was used to weaken the conditions on the oracle required for the construction of a set \(\emptyset \subset S \subset V\), minimizing \(d(S, V \setminus S)\). Instead of using a (strict) oracle \(F\), we only required a lax oracle providing \(\min\{\tau, d(S, T)\}\) for a given threshold \(\tau\). This allowed several improvements:

- Depending on \(d\), a lax oracle may require less time than a strict oracle (an example can be found in [Bri05]).

- The usage of lax-back orders instead of max-back orders and the fact that \(d\) is monotone, allows a possibly faster construction of the order, since the number of oracle calls per element is reduced.

- Since more than one identification is possible per round, the total number of rounds may be reduced.

Unfortunately, all three improvements heavily depend on the function \(d\). Hence, only a detailed analysis of special cases, may lead to a guaranteed improvement. In full generality, we can only guarantee, that the runtime of the algorithm presented in this paper is at least as fast as the one in [Riz0]
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