An alternative approach to heavy-traffic limits for finite-pool queues

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November 26, 2018

Abstract

We consider a model for transitory queues in which only a finite number of customers can join. The queue thus operates over a finite time horizon. In this system, also known as the $\Delta_{(i)}/G/1$ queue, the customers decide independently when to join the queue by sampling their arrival time from a common distribution. We prove that, when the queue satisfies a certain heavy-traffic condition and under the additional assumption that the second moment of the service time is finite, the rescaled queue length process converges to a reflected Brownian motion with parabolic drift. Our result holds for general arrival times, thus improving on an earlier result [2] which assumes exponential arrival times.

1 Introduction

The analysis of transient and time-dependent queueing models is of great relevance for numerous applications, such as call centres [4] and outpatient wards of hospitals where the server operates only over a finite amount of time [12, 13]. Besides their practical relevance, these systems provide a substantial mathematical challenge because the standard tools of renewal theory and ergodic theory are unsuited for their study. In other words, the steady-state distribution provides, if any, a poor approximation for the performance measures of transient queueing systems.

Here we focus on a particular class of transient queues, in which a finite (but large) number $n$ of customers can potentially join. As time passes, fewer customers can join the queue, so that eventually the queue length process will be identically zero and only its time-dependent behavior is of interest. We exploit ideas from the heavy-traffic approximation literature to prove that the queue length process can be approximated by a diffusion consisting of a Brownian motion with parabolic drift, reflected at zero.

The heavy-traffic approximation approach has been pioneered by Iglehart and Whitt [10] and has since been extended to a wide variety of settings where the time-dependent behavior is of interest, see [7] for an excellent overview. Indeed, our result should be contrasted with [10], where the queue length process is shown to converge to a reflected Brownian motion. The additional parabolic drift captures the effect of the diminishing pool of customers.

Transient queueing models have been studied lately by Honmappa et al. [8, 9]. However, interest in non-ergodic queues dates back to the pioneering work of Newell on the so-called $M_t/M_t/1$ queue [16, 17, 18]. Later, Keller [11] rederived Newell's heuristic results by methods of asymptotic expansion of the transition probabilities. Massey [15] expanded and formalized these earlier results by using operator techniques. More recently, Honmappa, Jain and Ward [8] introduced the $\Delta_{(i)}/G/1$ queue as a model for systems in which a finite number of customers can join and/or which operate only over a finite time window. In [8] the authors prove a Functional Law of Large Numbers (FLLN) and a Functional Central
Limit Theorem (FCLT) for the $\Delta_i/G/1$ queue under very mild assumptions. In [2], by exploiting a general martingale FCLT from [6], it is shown that, when the arrival times are exponentially distributed and under the additional assumption that the queue satisfies a certain heavy-traffic condition, the rescaled queue length process converges in distribution to a reflected Brownian motion with parabolic drift.

The martingale FCLT is a convenient and powerful tool, but comes at a high cost in terms of computations to verify technical conditions. On the other hand, both the pre-limit and the limit queue length processes are easily characterized through explicit formulas. This suggests that it should possible to prove the convergence result in [2] by using the “straightforward” approach to stochastic-process convergence, as detailed e.g. in [3]. As an example, assume a sequence of processes $(S_n(\cdot))_{n \geq 1}$ and a candidate limit $S(\cdot)$ are given. The “straightforward” approach consists in proving separately the tightness of the family $(S_n(\cdot))_{n \geq 1}$, seen as measures on a certain function space, and the convergence of the finite-dimensional distributions, that is, as $n \to \infty$,

$$
\mathbb{P}(S_n(t_1) \in A_1, \ldots, S_n(t_k) \in A_k) \to \mathbb{P}(S(t_1) \in A_1, \ldots, S(t_k) \in A_k),
$$

for each $k \geq 1$ and $t_1, \ldots, t_k$. Note that condition (1.1) characterizes the limit process uniquely.

By exploiting this method, we prove that the queue length process of the $\Delta_i/G/1$ queue converges in distribution to a Brownian motion with negative quadratic drift, reflected at zero. In particular, the proof we give is substantially simpler than the one in [2], requiring only the standard notions of stochastic-process convergence theory [3]. This approach has two advantages. First, we impose mild assumptions on the arrival time distribution, thus generalizing [2], where the arrival times were assumed to be exponentially distributed. Second, as a consequence of our main theorem, several results relating quantities of interest other than the queue length can be deduced. As an example of this, we prove a sample path Little’s Law.

The rest of the paper is organized as follows. In Section 2 we describe the $\Delta_i/G/1$ model, our assumptions, the processes of interest and state the main result. In Section 3 we prove the main theorem, by separately proving convergence of the terms appearing in the expression for the queue length process. In Section 4 we prove a transient version of Little’s Law by building on the techniques and results of Section 3. In Section 5 we summarize our result and sketch some interesting future research directions.

## 2 The model and the main result

We consider a population of $n$ customers. Each customer is assigned a clock $T_i$, with $i = 1, \ldots, n$. We assume $(T_i)_{i=1}^\infty$ to be a sequence of positive i.i.d. random variables with common density function $f_T(\cdot)$ and distribution function $F_T(\cdot)$. In particular, $F_T(\cdot)$ is continuous. Customers arrive at a single server with an infinite buffer and are served on a First-Come-First-Served basis. The number of arrivals in $[0, t]$ is then given by

$$
A^n(t) := \sum_{i=1}^n 1_{\{T_i \leq t\}}.
$$

Note that $A^n(t)/n$ is the empirical cumulative distribution function associated with $(T_i)_{i=1}^n$. The service times $(S_i)_{i=1}^\infty$ are i.i.d. random variables such that $\sigma^2 := \text{Var}(S) < \infty$. The corresponding (rescaled) renewal process is defined as

$$
S^n(t) := \sup \left\{ m \geq 1 \mid \sum_{i=1}^m S_i \leq nt \right\}.
$$

2
We further assume that at time zero the system obeys the heavy-traffic condition
\[ f_T(0) = \sup_{t \geq 0} f_T(t), \]  
and that
\[ \mathbb{E}[S]f_T(0) = 1, \]
which can be interpreted as follows. The number of arrivals in the interval \([0, dt]\) is approximately \(n(F_T(0 + dt) - F_T(0)) \approx n f_T(0) dt\). Consequently, \(\lambda_n = n f_T(0)\) represents the instantaneous arrival rate in zero. On the other hand, because of the time scaling in (2.2), the service rate is \(\mu_n = n/\mathbb{E}[S]\). The heavy-traffic condition is then equivalent to assuming that
\[ \frac{\lambda_n}{\mu_n} = 1. \]  
More generally, condition (2.5) could be replaced by \(\lambda_n + \varepsilon_n = 1\), for some \(\varepsilon_n \to 0\), but we refrain from doing it here. For a detailed explanation of the condition (2.4), see [2]. Our main object of interest is the queue length process, defined as
\[ Q^n(t) = A^n(t) - S^n(B^n(t)). \]
Here \(B^n(t)\) is a continuous process that increases at rate 1 if the server is working, and is constant otherwise. Note that \(A^n(t)\) and \(S^n(t)\) are independent as they only depend respectively on \((T_i)_{i \geq 1}\) and \((S_i)_{i \geq 1}\). They interact through the time-change \(t \mapsto B^n(t)\), which depends on both \((T_i)_{i \geq 1}\) and \((S_i)_{i \geq 1}\). The diffusion-scaled heavy-traffic queue length process is defined as
\[ \hat{Q}^n(t) := \frac{Q^n(t)}{n^{1/3}}. \]
Recall that the Skorokhod reflection map is the functional defined by
\[ \psi(f)(t) = \inf_{s \leq t} (f(s))^-, \]
\[ \phi(f)(t) = f(t) + \psi(f)(t). \]
We are now able to state our main theorem.

**Theorem 1** (Scaling limit of the queue length process). As \(n \to \infty\),
\[ \hat{Q}^n(t) \overset{d}{\to} \phi(\hat{X})(t), \quad \text{in } (\mathcal{D}, J_1), \]  
where
\[ \hat{X}(t) = B_1(f_T(0)t) - \frac{\sigma}{\mathbb{E}[S]^{3/2}} B_2(t) - \frac{f'_T(0)}{2} t^2, \]
and \(B_1(\cdot), B_2(\cdot)\) are two independent standard Brownian motions.

**Notation.** Here \(\mathcal{D}(\mathbb{R}) = \mathcal{D}\) denotes the space of càdlàg functions with values in \(\mathbb{R}\), that is of functions \(f(\cdot) : \mathbb{R}^+ \to \mathbb{R}\) which are continuous from the right at every point and such that \(\lim_{s \to t^-} f(s)\) exists for all \(t > 0\). \(\mathcal{D}\) is endowed with the usual Skorokhod \(J_1\) topology. For a sequence of stochastic processes \((X_n)_{n \geq 1}\), \(X_n \overset{d}{\to} X\) in \((\mathcal{D}, J_1)\) means that \((X_n)_{n \geq 1}\), seen as a sequence of random variables on \(\mathcal{D}\), converges to \(X\) in distribution, when \(\mathcal{D}\) is endowed with the \(J_1\) topology. Analogously, \(X_n \overset{d}{\to} X\) in \((\mathcal{D}, U)\) means that \((X_n)_{n \geq 1}\) converges to \(X\) in distribution, uniformly over compact subsets. Recall that, for a sequence \((x_n)_{n \geq 1} \subset \mathcal{D}\), if \(x_n \to x\) in \((\mathcal{D}, J_1)\) as \(n \to \infty\), and \(x\) is continuous, then \(x_n \to x\) in \((\mathcal{D}, U)\), see [3, p. 124]. When dealing with vectors of functions we make use of the weak \(J_1\) topology \(\mathcal{W}_1\). This coincides with the product topology on \(\mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D} = \mathcal{D}^k\). Given two (possibly
random) functions, either on the real numbers or on the integers, $f, g$ the notation $f \sim g$ means $\lim_{x \to \infty} f(x)/g(x) = 1$, where $x \in \mathbb{R}$ or $x \in \mathbb{N}$. The notation $f(x) = o_{f}(g(x))$ means that $f(x)/g(x) \to 0$ as $x \to \infty$. The notation $f(x) = \Theta(g(x))$ means $f(x) = O(g(x))$ and $g(x) = O(f(x))$. Finally, $f(x)^+ = \max\{0, f(x)\}$ and $f(x)^- = \max\{0, -f(x)\}$ denote the positive and negative part of a function $f(\cdot)$ respectively.

The cumulative busy time process. We now give an explicit analytical characterization of $B^n(\cdot)$. To this end, we need to introduce several auxiliary processes.

The cumulative input process is defined as

$$C^n(t) := \sum_{i=1}^{A^n(t)} \frac{S_i}{n}. \quad (2.12)$$

$C^n(t)$ can be seen as the (rescaled) total amount of work that has entered the queue by time $t$. Assuming that the server works at speed one, the net-put process $N^n(t)$ is defined as

$$N^n(t) := C^n(t) - t. \quad (2.13)$$

The workload process is then defined as

$$L^n(t) := \phi(N^n)(t) = N^n(t) - \inf_{s \leq t} (N^n(s))^-. \quad (2.14)$$

Note that $L^n(t)$ is positive if and only if

$$C^n(t) \geq t + \inf_{s \leq t} (N^n(s))^-= t - \psi(N^n)(t). \quad (2.15)$$

By construction, $\psi(N^n)(t)$ increases (linearly) if and only if the server is idling, and is constant otherwise. In other words, $I^n(t) := \psi(N^n)(t)$ can be interpreted as the cumulative idle time process. Consequently the term on the right-hand side of (2.15) can be interpreted as the cumulative busy time process, and we define it as

$$B^n(t) := t - \psi(N^n)(t). \quad (2.16)$$

Note that $B^n(t)$ increases only if the server is working, and is constant otherwise. With this definition, (2.15) reads

$$C^n(t) \geq B^n(t), \quad (2.17)$$

so that the workload is positive if and only if the cumulative input up to time $t$ is larger than the total time the server has spent processing jobs, and in that case it decreases linearly in time.

The queue length process. It is more convenient to express $Q^n(t)$ as a reflection of a simpler process $X^n(t)$. We will refer to $X^n(t)$ as the free process. To do so, we rewrite (2.6) as

$$Q^n(t) = \left( A^n(t) - S^n(B^n(t)) + \frac{B^n(t)}{E[S]} - f_\tau(0)t \right) - \left( \frac{B^n(t)}{E[S]} - f_\tau(0)t \right)$$

$$= \left( A^n(t) - S^n(B^n(t)) + \frac{B^n(t)}{E[S]} - f_\tau(0)t \right) + f_\tau(0)I^n(t), \quad (2.18)$$

where we used (2.4) in the second equality. We define

$$X^n(t) = A^n(t) - S^n(B^n(t)) + \frac{B^n(t)}{E[S]} - f_\tau(0)t. \quad (2.19)$$
We recall that, for a given process $X^n(t)$, the Skorokhod problem associated with $X^n(t)$ consists in finding two processes $P(t)$ and $R(t)$ such that $P(t) = X^n(t) + R(t) \geq 0$, $R(t)$ is increasing, and $\int_0^\infty X^n(t) dR(t) = 0$. Note that $I^n(\cdot)$ is increasing and, by definition of $Q^n(t)$ and $I^n(t)$,
\begin{equation}
\int_0^\infty Q^n(t) dI^n(t) = 0.
\end{equation}
Then $Q^n(t)$ and $I^n(t)$ are a solution to the Skorokhod problem associated with $X^n(t)$ and, by applying \cite[Proposition 2.2, p.251]{1}, we have the representation
\begin{equation}
Q^n(t) = X^n + \psi(X^n)(t) = \phi(X^n)(t),
\end{equation}
where
\begin{equation}
\psi(X^n)(t) = -\left( \frac{B^n(t)}{E[S]} - f_\tau(0)t \right).\end{equation}

The fluid and diffusive scaling regimes. The fluid-scaled heavy-traffic queue length process is defined as
\begin{equation}
\bar{Q}^n(t) := \frac{Q^n(tn^{-1/3})}{n^{2/3}} = n^{1/3} \left( \frac{A^n(tn^{-1/3})}{n} - S^n(B^n(tn^{-1/3})) \right).
\end{equation}
Correspondingly, $\bar{X}^n(t)$ is defined as
\begin{equation}
\bar{X}^n(t) := n^{1/3} \left( \frac{A^n(tn^{-1/3})}{n} - S^n(B^n(tn^{-1/3})) \right) + n^{1/3} \frac{B^n(tn^{-1/3})}{E[S]} - f_\tau(0)t
\end{equation}
\begin{equation}
+ (n^{1/3} F_\tau(tn^{-1/3}) - f_\tau(0)tn^{-1/3}).
\end{equation}
where in the second equality we have added and subtracted $F_\tau(t)$ in order to rewrite $\bar{X}^n(t)$. It can be shown through an application of the functional Law of Large Numbers that, as $n \to \infty$, the fluid-scaled process $\bar{Q}^n(\cdot)$ converges to a deterministic process $\bar{Q}(\cdot)$. However, under our heavy-traffic assumption the process $\bar{Q}(\cdot)$ is identically zero. Because of this, the diffusion-scaled queue length process can be rewritten as
\begin{equation}
\bar{Q}^n(t) = n^{1/3} \bar{Q}^n(t) = n^{1/3}(Q^n(t) - \bar{Q}(t)).
\end{equation}
Accordingly, $\bar{X}^n(t)$ is defined as
\begin{equation}
\bar{X}^n(t) := n^{1/3} \bar{X}^n(t)
\end{equation}
\begin{equation}
= n^{2/3} \left( \frac{A^n(tn^{-1/3})}{n} - F_\tau(tn^{-1/3}) \right) - n^{2/3} \left( \frac{S^n(B^n(tn^{-1/3}))}{n} - \frac{B^n(tn^{-1/3})}{E[S]} \right)
\end{equation}
\begin{equation}
+ n^{2/3} (F_\tau(tn^{-1/3}) - f_\tau(0)tn^{-1/3}).
\end{equation}
In order to prove Theorem 1 we will rely on an analogous result for $\bar{X}^n(\cdot)$. In fact, Theorem 1 is a straightforward consequence of the following:

**Theorem 2** (Scaling limit of the free process). As $n \to \infty$,
\begin{equation}
\bar{X}^n(t) \xrightarrow{d} \bar{X}(t), \quad \text{in } (\mathcal{D}, J_1),
\end{equation}
where $\bar{X}(\cdot)$ is given by
\begin{equation}
\bar{X}(t) = B_1(f_\tau(0)t) - \frac{\sigma}{E[S]^{1/2}} B_2(t) - \frac{f_\tau'(0)}{2} t^2,
\end{equation}
and $B_1(\cdot), B_2(\cdot)$ are two independent standard Brownian motions.
The scaling exponents. Let us now give an heuristic motivation for the scaling exponents in (2.26). Define the general time scaling exponent as $-\alpha$ and the spatial scaling exponent as $\beta$, for some $\alpha, \beta > 0$ to be determined, so that $\hat{X}^n(t)$ is given by

$$
\hat{X}^n(t) = n^{\beta} \left( \frac{A^n(tn^{-\alpha})}{n} - F_T(tn^{-\alpha}) \right) + n^{\beta} \left( \frac{S^n(B^n(tn^{-\alpha}))}{n} \right) - B^n(tn^{-\alpha}) + n^{\beta} \left( F(tn^{-\alpha}) - f_T(0)tn^{-\alpha} \right).
$$

(2.29)

For the deterministic drift to converge to a non-trivial limit it is necessary that $\alpha, \beta$ be such that $2\alpha = \beta$. Indeed, replacing $F_T(tn^{-\alpha})$ with its Taylor expansion up to the second term, we get

$$
n^{\beta}(F(tn^{-\alpha}) - f_T(0)tn^{-\alpha}) = n^{\beta} \left( \frac{f_T'(0)}{2} t^2 n^{-2\alpha} + o(n^{-\alpha}) \right).
$$

(2.30)

Moreover, a necessary condition for $\hat{A}^n(\cdot)$ in (2.29) to converge to a non-trivial random process is that, for fixed time $t > 0$, the variance of $\hat{A}^n(t)$ be $\Theta(1)$. This is given by

$$
\text{Var}(\hat{A}^n(t)) = n^{2\beta} \text{Var}(1_{T \leq tn^{-\alpha}}) = n^{2\beta} \mathbb{P}(T \leq tn^{-\alpha})(1 - \mathbb{P}(T \leq tn^{-\alpha})) = n^{2\beta} (f_T(0)tn^{-\alpha} + o(n^{-\alpha})).
$$

(2.31)

Then, $\alpha$ and $\beta$ should be such that

$$
n^{2\beta - \alpha} = O(1),
$$

(2.32)

which, together with $\beta = 2\alpha$, imply that $\alpha = 1/3$ and $\beta = 2/3$.

Comparison with known results. We conclude by drawing a connection between Theorem 1 and the analogous result in [2]. There, the queue length process is shown to converge to $\phi(X)(t)$, where $X(t) = \sigma B(t) - t^2/2$, where $\sigma^2 = \mathbb{E}[S^2]/\mathbb{E}[S]^3$ and $B(t)$ is a standard Brownian motion. The random process consisting of the sum of two Brownian motions in (2.11) is equivalent in distribution to a single Brownian motion with variance equal to

$$
f_T(0) + \frac{\mathbb{E}[S^2] - \mathbb{E}[S]^2}{\mathbb{E}[S]^3}.
$$

(2.33)

By the heavy-traffic condition (2.4) this can be simplified to

$$
\frac{\mathbb{E}[S^2] + \mathbb{E}[S^2] - \mathbb{E}[S]^2}{\mathbb{E}[S]^3} = \frac{\mathbb{E}[S^2]}{\mathbb{E}[S]^3}.
$$

(2.34)

Therefore, the two limits are equal in distribution, as expected.

3 Proof of Theorem 1

3.1 Overview of the proof

The proof of Theorem 1 proceeds in several steps. These consist in proving convergence of the three terms in (2.7) to the respective terms in (2.11) separately. The first term in (2.7) is the centred and rescaled empirical distribution function of the sequence $(T_i)_{i \geq 1}$. Therefore, its convergence to $B_1(f_T(0)t)$ can be seen as a ‘local Donsker’s Theorem’, in which the limiting Brownian Bridge is replaced by a Brownian motion. The second term in (2.7)
is a time-changed, centred and rescaled renewal process and thus converges by a random time-change theorem and the FCLT for renewal processes. The third term also converges trivially to the limiting quadratic drift. Then, the convergence (2.10) follows immediately from (2.27) by the continuity of the Skorokhod reflection \( \phi(x) \) in all \( x \in \mathcal{C} \), the space of real-valued continuous functions, see [20, Theorem 13.5.1].

### 3.2 A local Donsker’s Theorem

For sake of simplicity, let us define

\[
\hat{A}^n(t) := n^{2/3} \left( \frac{A^n(tn^{-1/3})}{n} - F_\nu(tn^{-1/3}) \right)
\]

(3.1)

and

\[
\hat{A}(t) := B_1(f_\nu(0)t).
\]

(3.2)

The goal of this section is to prove the following:

**Lemma 1** (Convergence of the arrival process). As \( n \to \infty \),

\[
\hat{A}^n(t) \xrightarrow{d} \hat{A}(t), \quad \text{in } (\mathcal{D}, J_1).
\]

(3.3)

**Proof.** The proof proceeds in two steps. First, we prove convergence of the finite-dimensional distributions. This characterizes the limit uniquely. Second, we prove tightness of the family \((\hat{A}^n(t))_{n \geq 1}\) in the space of measures on the Polish space \( \hat{D} \) of càdlàg functions. By definition, we say that the finite-dimensional distributions of \( \hat{A}^n(\cdot) \) converge to the finite-dimensional distributions of \( \hat{A}(\cdot) \) if, for every \( n \in \mathbb{N} \) and for each choice of \((t_i)_{i=1}^n\) such that \( 0 < t_1 < t_2 < \ldots < t_n < \infty \) it holds that, as \( n \to \infty \),

\[
(\hat{A}^n(t_1), \ldots, \hat{A}^n(t_n)) \xrightarrow{d} (\hat{A}(t_1), \ldots, \hat{A}(t_n)).
\]

(3.4)

For simplicity we shall prove (3.4) for \( t_1 < t_2 \), the generalization to an arbitrary choice of \((t_i)_{i=1}^n\) being straightforward. We then aim to show that, as \( n \to \infty \),

\[
(\hat{A}^n(t_1), \hat{A}^n(t_2)) \xrightarrow{d} (\hat{A}(t_1), \hat{A}(t_2)).
\]

(3.5)

Let \( \mathcal{N}(m, v) \) denote a normally distributed random variable with mean \( m \) and covariance matrix \( v \). Then \((\hat{A}(t_1), \hat{A}(t_2)) \sim \mathcal{N}(m, V_{t_1, t_2})\), with mean \( m = (0, 0) \) and covariance matrix \( V_{t_1, t_2} \) given by

\[
V_{t_1, t_2} = f_\nu(0) \begin{pmatrix} t_1 & t_1 \wedge t_2 \\ t_1 \wedge t_2 & t_2 \end{pmatrix},
\]

(3.6)

where \( a \wedge b = \min\{a, b\} \). To show joint convergence, we apply the Cramér-Wold device. Given an arbitrary vector \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2 \), we aim to show that, as \( n \to \infty \),

\[
\gamma_1 \hat{A}^n(t_1) + \gamma_2 \hat{A}^n(t_2) \xrightarrow{d} \gamma_1 \hat{A}(t_1) + \gamma_2 \hat{A}(t_2).
\]

(3.7)

This is done through the following straightforward generalization of the Lindeberg-Feller CLT.

**Theorem 3** (Lindeberg-Feller CLT [14]). Let \( (X_{n,l})_{l=1}^n \) be an array of random variables such that \( \mathbb{E}[X_{n,l}] = 0 \) for all \( n \geq 1 \) and \( l \leq n \) and \( \sum_{l=1}^n \mathbb{Var}(X_{n,l}) \to 1 \). Define

\[
S_n := X_{n,1} + \ldots + X_{n,n}.
\]

(3.8)

Assume that the Lindeberg condition holds, i.e. for \( \varepsilon > 0 \),

\[
\frac{1}{\mathbb{Var}(S_n)} \sum_{l=1}^n \mathbb{E}[X_{n,l}^2 I_{\{X_{n,l}^2 > \varepsilon^2 \mathbb{Var}(S_n)\}}] \to 0, \quad n \to \infty.
\]

(3.9)

Then \( (S_n)_{n \geq 1} \) converges in distribution to a standard normal random variable.
We remark that in the usual formulation of the Lindeberg-Feller CLT it is assumed that \( \sum_{i=1}^{n} \text{Var}(X_{n,i}) = 1 \). The proof of the theorem, as presented e.g. in [14] can be directly generalized to accommodate for the assumption that \( \sum_{i=1}^{n} \text{Var}(X_{n,i}) \to 1 \). We now take \( X_{n,l} \) to be

\[
X_{n,l} = \gamma_1 \frac{I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3})}{n^{1/3} v_{t_1,t_2}} + \gamma_2 \frac{I(T_{\leq t_2} n^{-1/3}) - F_T(t_2 n^{-1/3})}{n^{1/3} v_{t_1,t_2}},
\]

(3.10)

where \( v_{t_1,t_2} \) is a normalizing constant and is given by

\[
v_{t_1,t_2} = \sqrt{f_T(0)(\gamma_1^2 t_1 + \gamma_2^2 t_2 + 2 \gamma_1 \gamma_2 t_1)}.
\]

(3.11)

Recall that \( t_1 < t_2 \) by assumption. In order to deduce the desired convergence (3.7) we are left to check the conditions of Theorem 3. Trivially, \( \mathbb{E}[X_{n,l}] = 0 \). Moreover, it is possible to explicitly compute \( \text{Var}(X_{n,l}) \) as follows:

\[
\text{Var}(X_{n,l}) = \frac{\gamma_1^2}{n^{2/3} v_{t_1,t_2}^2} \left( F_T(t_1 n^{-1/3}) - F_T(t_1 n^{-1/3})^2 \right) + \frac{\gamma_2^2}{n^{2/3} v_{t_1,t_2}^2} \left( F_T(t_2 n^{-1/3}) - F_T(t_2 n^{-1/3})^2 \right) + \frac{2 \gamma_1 \gamma_2}{n^{2/3} v_{t_1,t_2}^2} \left( F_T(t_1 n^{-1/3}) - F_T(t_1 n^{-1/3}) F_T(t_2 n^{-1/3}) \right) = \frac{f_T(0)}{v_{t_1,t_2}^2} \left( \gamma_1^2 t_1 + \gamma_2^2 t_2 + 2 \gamma_1 \gamma_2 t_1 \right) + O(n^{-4/3}),
\]

(3.12)

where in the second equality the distribution function \( F_T(\cdot) \) was Taylor expanded. In particular,

\[
\sum_{l=1}^{n} \text{Var}(X_{n,l}) = 1 + O(n^{-1/3}).
\]

(3.13)

The Lindeberg condition is also satisfied, since

\[
\sum_{l=1}^{n} \frac{1}{n^{2/3} v_{t_1,t_2}^2} \mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^2 \right] \mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^2 \left( I(T_{\leq t_2} n^{-1/3}) - F_T(t_2 n^{-1/3}) \right) \right] \\
= \frac{n^{1/3}}{v_{t_1,t_2}^2} \mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^2 \right] \mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^2 \left( I(T_{\leq t_2} n^{-1/3}) - F_T(t_2 n^{-1/3}) \right) \right] \\
\leq \frac{n^{1/3}}{v_{t_1,t_2}^2} \sqrt{\mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^4 \right] \mathbb{E} \left[ (I(T_{\leq t_1} n^{-1/3}) - F_T(t_1 n^{-1/3}))^4 \right] \mathbb{E} \left[ (I(T_{\leq t_2} n^{-1/3}) - F_T(t_2 n^{-1/3}) \right) \geq \varepsilon n^{1/3})},
\]

(3.14)

by the Cauchy-Schwartz inequality. The first term is of the order \( O(n^{-1/3}) \), while the second is identically zero for \( n \) large enough.

By Theorem 3,

\[
\frac{1}{v_{t_1,t_2}} (\gamma_1, \gamma_2) \cdot (\hat{A}^n(t_1), \hat{A}^n(t_2))^T \xrightarrow{d} \mathcal{N}(0,1),
\]

(3.15)

where \( \cdot \) denotes the usual scalar product and \( q^T \) denotes the transpose of a vector \( q \). However, since

\[
(\gamma_1, \gamma_2) \cdot V_{t_1,t_2} (\gamma_1, \gamma_2)^T = v_{t_1,t_2}^2,
\]

(3.16)

then

\[
\mathcal{N}(0,1) \overset{d}{=} \frac{1}{v_{t_1,t_2}} (\gamma_1, \gamma_2) \cdot \mathcal{N}((0,0), V_{t_1,t_2}),
\]

(3.17)
and this together with (3.15) implies (3.7). By an application of the Cramér-Wold device, joint convergence follows.

The last step of the proof is to show that $(\hat{A}^n(\cdot))^\infty_{n=1}$ is a tight family of random variables on $\mathcal{D}$. By [3, Theorem 13.5], in particular equation (13.14), it is enough for $(\hat{A}^n(\cdot))^\infty_{n=1}$ to satisfy the following condition. For every $T > 0$,

$$\mathbb{E}[|\hat{A}^n(t) - \hat{A}^n(t_1)|^2 | \hat{A}^n(t_2) - \hat{A}^n(t)|^2] \leq (f_{\text{inc}}(t_2) - f_{\text{inc}}(t_1))^2,$$

(3.18)

for $0 \leq t_1 \leq t \leq t_2 \leq T$ and $f_{\text{inc}}(\cdot)$ is a non-decreasing function. Checking (3.18) amounts to computing the mean appearing on the left side of the equation. Define

$$p_1 := F_T(tn^{-1/3}) - F_T(t_1n^{-1/3}),$$

$$p_2 := F_T(t_2n^{-1/3}) - F_T(tn^{-1/3}).$$

(3.19)

Define also

$$\alpha_i := \begin{cases} 1 - p_1, & \text{if } \frac{i}{n} \not\in (t_1, t], \\ -p_1, & \text{if } \frac{i}{n} \not\in (t_1, t], \end{cases}$$

(3.20)

and

$$\beta_i := \begin{cases} 1 - p_2, & \text{if } \frac{i}{n} \not\in (t_2, t], \\ -p_2, & \text{if } \frac{i}{n} \not\in (t_2, t], \end{cases}$$

(3.21)

where we have omitted dependencies on $n$ to avoid cumbersome notation. Note that $\mathbb{E}[\alpha_1] = \mathbb{E}[\beta_1] = 0$. With the help of these definitions, (3.18) can be rewritten in the following form:

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} \alpha_i \right)^2 \left(\sum_{i=1}^{n} \beta_i \right)^2\right] \leq n^{4/3}(f_{\text{inc}}(t_2) - f_{\text{inc}}(t_1))^2.$$

(3.22)

We will take $f_{\text{inc}}(t) = \sqrt{Kt}$ for a certain constant $K > 0$. By definition $\alpha_i$ (resp. $\beta_i$) is independent from $\alpha_j$ and $\beta_j$ for $j \neq i$, so that the left side of (3.22) can be simplified as

$$n\mathbb{E}[\alpha_i^2 \beta_i^2] + n(n-1)\mathbb{E}[\alpha_i^2]\mathbb{E}[\beta_i^2] + 2n(n-1)\mathbb{E}[\alpha_i \beta_i]\mathbb{E}[\alpha_j \beta_j].$$

(3.23)

The first term $n\mathbb{E}[\alpha_i^2 \beta_i^2]$ is of lower order, so we focus on the remaining two. A simple computation gives

$$\mathbb{E}[\alpha_i^2] = p_1(1-p_1) \leq p_1,$$

$$\mathbb{E}[\beta_i^2] = p_2(1-p_2) \leq p_2,$$

$$\mathbb{E}[\alpha_i \beta_i] = -p_1p_2,$$

(3.24)

so that, since $p_1 \leq (p_1 + p_2)$ and $p_2 \leq (p_1 + p_2),$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} \alpha_i \right)^2 \left(\sum_{i=1}^{n} \beta_i \right)^2\right] \leq C_0n^2p_1p_2 \leq C_0n^2(p_2 + p_1)^2$$

$$= C_0n^2(F_T(t_2n^{-1/3}) - F_T(t_1n^{-1/3}))^2$$

$$\leq C_1 n^{4/3} f_{\text{inc}}(t_2 - t_1)^2,$$

(3.25)

for a sufficiently large $C_1 > 0$. Therefore, we have verified (3.22) with $f_{\text{inc}}(t) = \sqrt{C_1 f_T(0)t}$. \qed

9
3.3 A functional CLT for renewal processes

We define

\[ \hat{S}_n(t) := \frac{n^{2/3}}{3} \left( S_n(t) - \frac{1}{E[S]} tn^{-1/3} \right) \]  

and

\[ \hat{S}(t) := \frac{\sigma}{E[S]^{3/2}} B_2(t), \]  

where \( \sigma^2 \) is the variance of \( S \). The goal of this section is then to prove the following:

**Lemma 2** (Convergence of the service process). As \( n \to \infty \),

\[ \hat{S}_n(t) \xrightarrow{a.s.} \hat{S}(t), \quad \text{in } (\mathcal{D}, J_1). \]  

**Proof.** Note that \( S_n(tn^{-1/3}) = S_n^{2/3}(t) \). Moreover,

\[ n^{2/3} \left( \frac{S_n(tn^{-1/3})}{n} - \frac{1}{E[S]} tn^{-1/3} \right) = \frac{S_n^{2/3}(t) - E[S]^{-1} tn^{2/3}}{n^{1/3}}. \]  

Therefore, the claim (3.28) can be proven by directly applying a FCLT for renewal processes, see e.g. [3, Theorem 14.6]. \( \square \)

3.4 Convergence of the cumulative busy time

In this section we exploit Lemma 2 and the random time change theorem to prove that the rescaled service process in (2.7) converges. First, we prove some scaling limits for the arrival process. Define the fluid-scaled arrival process as

\[ \bar{A}_n(t) := A_n(tn^{-1/3}). \]  

The following straightforward generalization of the Chebyshev inequality is useful when proving the strong Law of Large Numbers:

**Lemma 3** (Generalized Chebyshev inequality). For any \( p = 1, 2, \ldots \), and any random variable \( X \) such that \( E[|X|^p] < \infty \),

\[ P(|X| \geq \varepsilon) \leq \frac{E[|X|^p]}{\varepsilon^p}. \]  

By using Lemma 3 together with the Borel-Cantelli lemma, we can prove the following:

**Lemma 4** (LLN for the arrival process). As \( n \to \infty \),

\[ |\bar{A}_n(t) - f_r(0)| \xrightarrow{a.s.} 0, \]  

for fixed \( t \geq 0 \).

**Proof.** First, we rewrite

\[ \bar{A}_n(t) - f_r(0) = \frac{1}{n} \sum_{i=1}^n (n^{1/3} 1_{T_i \leq tn^{-1/3}} - n^{1/3} F_r(tn^{-1/3})). \]  

In order to apply the Borel-Cantelli lemma, we compute

\[ P\left( \left| \sum_{i=1}^n Y_i \right| \geq \varepsilon n \right) \leq \frac{E[|\sum_{i=1}^n Y_i|^4]}{n^4 \varepsilon^4} \]

\[ = \frac{n E[|Y_1|^4] + 3n(n-1) E[|Y_1|^2]^2}{n^4 \varepsilon^4}. \]  

10
It is immediate to see that the leading orders of the expectation values are
\[
E[|Y_1|^4] = O(n^{4/3}P(T_i \leq tn^{-1/3})) = O(tn),
\]
\[
E[|Y_1|^2] = O(n^{2/3}P(T_i \leq tn^{-1/3})) = O(tn^{1/3}).
\] (3.35)

We conclude that, for a large constant \( C_1 > 0 \),
\[
P\left(\left\| \sum_{i=1}^{n} Y_i \right\| \geq \varepsilon n \right) \leq C_1 \frac{tn^2 + 3tn^{8/3}}{n^4}.
\] (3.36)

Define the event \( A := \{|\sum_{i=1}^{n} Y_i| \geq \varepsilon n \text{ for infinitely many } n \}\). Since
\[
\sum_{n=1}^{\infty} P\left(\left\| \sum_{i=1}^{n} Y_i \right\| \geq \varepsilon n \right) \leq C_2 \sum_{n=1}^{\infty} \frac{1}{n^{4/3}n^4} < \infty,
\] (3.37)

for some large constant \( C_2 > 0 \), by the Borel-Cantelli lemma,
\[
P(A) = 0.
\] (3.38)

Since \( \varepsilon > 0 \) is arbitrary, this concludes the proof of (3.32).

We are now interested in obtaining a Glivenko-Cantelli-type theorem which extends the convergence (3.32) to uniform convergence over compact subsets of the positive half-line. This is summarized in the following lemma.

**Lemma 5** (Glivenko-Cantelli Theorem for the arrival process). As \( n \to \infty \),
\[
\bar{A}^n(t) \overset{a.s.}{\to} f_T(0) t, \quad \text{in } (D, U).
\] (3.39)

Consequently, as \( n \to \infty \),
\[
n^{1/3}C_n (tn^{-1/3}) \overset{a.s.}{\to} t \quad \text{in } (D, U).
\] (3.40)

**Proof.** Let \( T > 0 \) be arbitrary. The claim (3.39) is then equivalent to
\[
\lim_{n \to \infty} \sup_{t \leq T} \left| \bar{A}^n(t) - f_T(0) t \right| = 0, \quad \text{a.s.}
\] (3.41)

Let \( N \) be a large but arbitrary natural number and define
\[
t_j := \frac{1}{f_T(0)} j N, \quad j = 1, \ldots, N,
\] (3.42)

so that \( f_T(0) t_j = \frac{j}{N} T \). The idea is that both \( A^n(t) \) and \( f_T(0) t \) are increasing, so for \( t \in (t_{j-1}, t_j) \) the difference of the two can be bounded by their values in \( t_{j-1} \) and \( t_j \). Then, we have convergence because of Lemma 4 and because \( N \) is fixed. Formally, define the error as
\[
E_{n,N} := \max_{j=1,\ldots,N} \{|A^n(t_j n^{-1/3})/n^{2/3} - f_T(0) t_j| + |A^n(t_j n^{-1/3})/n^{2/3} - f_T(0) t_j|\}.
\] (3.43)

For \( t \in (t_{j-1}, t_j) \) we upper bound \( \bar{A}^n(t) \) as follows
\[
\bar{A}^n(t) \leq \bar{A}^n(t_j^+) \leq f_T(0) t_j^- + E_{n,N} \leq f_T(0) t_j^- + E_{n,N} + \frac{T}{N},
\] (3.44)
where in the last inequality we used the bound $|f_r(0)t_j - f_r(0)t_{j-1}| \leq T$. Analogously, for the lower bound

$$\bar{A}^n(t) \geq \bar{A}^n(t_{j-1}) \geq f_r(0)t_{j-1} - E_n,N \geq f_r(0)t - E_n,N - \frac{T}{N}. \quad (3.45)$$

Summarizing the two bounds, since $E_n,N$ and $T/N$ do not depend on the choice of the sequence $(t_j)_{j=1}^N$,

$$\sup_{t \leq T} |\bar{A}^n(t) - f_r(0)t| \leq E_n,N + \frac{T}{N}. \quad (3.46)$$

Since $N$ is fixed, almost surely

$$\lim_{n \to \infty} E_n,N = 0, \quad (3.47)$$

by Lemma 4. Letting $N \to \infty$, we obtain (3.39).

The convergence (3.40) follows from (3.39). Indeed, by the functional strong Law of Large Numbers [5, Theorem 5.10] we have that

$$\sum_{i=1}^{tn^{2/3}} \frac{S_i}{n^{2/3}} \xrightarrow{a.s.} E[S]t \quad \text{in } (\mathcal{D}, U). \quad (3.48)$$

Since $\bar{A}^n(t)$ converges to a deterministic limit, we also have the joint convergence

$$\left( \sum_{i=1}^{tn^{2/3}} \frac{S_i}{n^{2/3}}, \bar{A}^n(t) \right) \xrightarrow{a.s.} (E[S]t, f_r(0)t), \quad \text{in } (\mathcal{D}^2, WJ_1). \quad (3.49)$$

Note that $A^n(t)$ is non-decreasing. Then, by a time-change theorem [3, Lemma p.151],

$$\sum_{i=1}^{tn^{2/3}} \frac{S_i}{n^{2/3}} \xrightarrow{a.s.} E[S]f_r(0)t \quad \text{in } (\mathcal{D}, U). \quad (3.50)$$

Recall that convergence in $(\mathcal{D}, J_1)$ to a continuous function implies convergence in $(\mathcal{D}, U)$. Moreover, $E[S]f_r(0) = 1$ by the heavy-traffic condition (2.4), and this concludes the proof of (3.40).

Since $t \mapsto f_r(0)t$ is not a proper distribution function, Theorem 5 should also be interpreted as a local version of the usual Glivenko-Cantelli Theorem. Let us now define the fluid-scaled cumulative busy time process as

$$\bar{B}^n(t) := n^{1/3}B^n(tn^{-1/3}). \quad (3.51)$$

We are able to prove the following lemma:

**Lemma 6** (Convergence of the time-changed service process). With assumptions as above, as $n \to \infty$,

$$\bar{B}^n(t) \xrightarrow{a.s.} t, \quad \text{in } (\mathcal{D}, U), \quad (3.52)$$

**Proof.** Recall that $B^n$ can be rewritten as

$$B^n(t) = t + \Psi(N^n)(t) = t + \inf_{s \leq t} (C^n(s) - s)^-. \quad (3.53)$$

By Lemma 5, $n^{1/3}(C^n(tn^{-1/3}) - tn^{-1/3}) \xrightarrow{a.s.} 0$ in $(\mathcal{D}, U)$. Moreover, the null function is a continuity point of $\Psi(\cdot)$ with probability one [20, Lemma 13.4.1]. The claim then follows from the Continuous Mapping Theorem [20, Theorem 3.4.3].
3.5 Proof of Theorem 1

Since $\hat{B}^n(\cdot)$ converges to a deterministic limit, we have

$$
(\hat{A}^n(t), \hat{S}^n(t), \hat{B}^n(t)) \xrightarrow{d} (\hat{A}(t), \hat{S}(t), t), \quad \text{in } (\mathcal{D}^3, WJ_1).
$$

(3.54)

Note also that $\hat{A}^n(\cdot)$ and $\hat{S}^n(\cdot)$ are independent processes, so that $\hat{A}(\cdot)$ and $\hat{S}(\cdot)$ are also independent. Applying the random time-change theorem [3, Lemma p.151], we get

$$
(\hat{A}^n(t), \hat{S}^n(\hat{B}^n(t))) \xrightarrow{d} (\hat{A}(t), \hat{S}(t)), \quad \text{in } (\mathcal{D}^2, WJ_1).
$$

(3.55)

Since the limit points are continuous, by [19, Theorem 4.1] addition is also continuous, so that

$$
\hat{A}^n(t) - \hat{S}^n(t) + n^{2/3}(F_\tau(tn^{-1/3}) - f_\tau(0)tn^{-1/3}) \xrightarrow{d} \hat{A}(t) - \hat{S}(t) - \frac{f_\tau(0)}{2}t^2, \quad \text{in } (\mathcal{D}, J_1),
$$

(3.56)

which is the first claim (2.27). By [20, Theorem 13.5.1], the reflection map $\phi(\cdot)$ is continuous when $\mathcal{D}$ is endowed with the $J_1$ topology, from which the second claim (2.10) follows. $\square$

4 Sample path Little’s Law

In this section we apply the ideas and results from the previous sections to derive a sample path version of Little’s Law. The standard formulation of Little’s Law relates the expected waiting time $E[W]$, to the expected queue length $E[L_q]$ as $E[L_q] = \lambda E[W]$, where $\lambda$ is the rate at which customers arrive. We will work instead with the virtual waiting time $W^n(t)$, defined as

$$
W^n(t) := C^n(t) - B^n(t).
$$

(4.1)

Accordingly, we define the diffusion-scaled virtual waiting time as

$$
\hat{W}^n(t) := n^{2/3}(C^n(tn^{-1/3}) - B^n(tn^{-1/3})) = n^{1/3} \left( \sum_{i=1}^{A^n(tn^{-1/3})} \frac{S_i}{n^{2/3}} - \hat{B}^n(t) \right).
$$

(4.2)

First, we rewrite the expression for $\hat{W}^n(t)$ as

$$
\hat{W}^n(t) = n^{1/3} \left( \sum_{i=1}^{A^n(t)n^{2/3}} \frac{S_i}{n^{2/3}} - E[S]A^n(t) \right) + n^{1/3}E[S](A^n(t) - n^{1/3}F_\tau(tn^{-1/3}))
$$

$$
+ n^{1/3}E[S](F_\tau(tn^{-1/3}) - f_\tau(0)t) + n^{1/3}E[S](f_\tau(0)t - B^n(t)/E[S]).
$$

(4.3)

By (2.22), $n^{1/3}(f_\tau(0)t - B^n(t)/E[S]) = \psi(\hat{X}^n)(t)$, so that (4.3) can be further simplified as

$$
\hat{W}^n(t) = E[S]\hat{Q}^n(t) + n^{1/3} \left( \sum_{i=1}^{A^n(t)n^{2/3}} \frac{S_i}{n^{2/3}} - E[S]\hat{A}^n(t) \right) + E[S]\hat{S}^n(B^n(t)).
$$

(4.4)

We now focus on the second and third terms in (4.4). Let us ignore the time change $t \mapsto \hat{A}^n(t)$ and $t \mapsto \hat{B}^n(t)$ for the moment. Then, the second and third terms in (4.4) represent the difference between the diffusion-scaled partial sums and the (negative) diffusion-scaled counting process associated with the sequence of random variables $(S_i)_{i \geq 1}$. These converge to the same limiting Brownian motion, so that their contribution to $\hat{W}^n(t)$ vanishes in the limit. We now aim to make this reasoning rigorous.
Theorem 4 (Diffusion sample path Little’s Law). As \( n \to \infty \),

\[
\hat{W}^n(t) \xrightarrow{d} \hat{W}(t), \quad \text{in } (D, J_1),
\]

where

\[
\hat{W}(t) := \mathbb{E}[S]\hat{Q}(t).
\]

Proof. Define the diffusion-scaled partial sum process as

\[
\hat{P}^n(t) = n^{1/3} \left( \sum_{i=1}^{tn^{2/3}} \frac{S_i}{n^{1/3}} - \mathbb{E}[S]\bar{A}^n(t) \right).
\]

By [20, Theorem 7.3.2], \( \hat{P}^n(\cdot) \) and \( \hat{S}^n(\cdot) \) jointly converge as follows:

\[
(\hat{P}^n(t), \hat{S}^n(t)) \xrightarrow{d} (-\mathbb{E}[S]\bar{S}(\mathbb{E}[S]t), \bar{S}(t)), \quad \text{in } (D^2, WJ_1),
\]

where \( \bar{S}^n(t) \) is the same as in (3.28). Since \( \bar{A}^n(t) \) is independent from \( \hat{P}^n(t) \) and \( \hat{S}^n(t) \),

\[
(\bar{A}^n(t), \hat{P}^n(t), \hat{S}^n(t)) \xrightarrow{d} (\bar{A}(t), -\mathbb{E}[S]\bar{S}(\mathbb{E}[S]t), \bar{S}(t)), \quad \text{in } (D^3, WJ_1).
\]

Moreover, since \( \bar{A}^n(\cdot) \) and \( \bar{B}^n(\cdot) \) converge to deterministic limits, by [20, Theorem 11.4.5] the above convergence can be strengthened to

\[
(\bar{A}^n(t), \hat{P}^n(t), \hat{S}^n(t), \bar{A}^n(t), \bar{B}^n(t)) \xrightarrow{d} (\bar{A}(t), -\mathbb{E}[S]\bar{S}(\mathbb{E}[S]t), \bar{S}(t), f_{\bar{A}}(0)t, t), \quad \text{in } (D^4, WJ_1).
\]

It follows that

\[
(\bar{A}^n(t), \hat{P}^n(\bar{A}^n(t)), \mathbb{E}[S]\bar{S}^n(\bar{B}^n(t))) \xrightarrow{d} (\bar{A}(t), -\mathbb{E}[S]\bar{S}(\mathbb{E}[S]t), \mathbb{E}[S]\bar{S}(t)), \quad \text{in } (D^3, WJ_1),
\]

by the heavy-traffic assumption (2.4). The limit functions are continuous with probability one, and thus their sums converge to the sums of the limits. This observation, together with the Continuous Mapping Theorem and (4.11) imply that

\[
\mathbb{E}[S]\hat{Q}^n(t) + \hat{P}^n(\bar{A}^n(t)) + \mathbb{E}[S]\bar{S}^n(\bar{B}^n(t)) \xrightarrow{d} \hat{Q}(t), \quad \text{in } (D, J_1),
\]

as \( n \to \infty \), as desired. \( \square \)

By our assumptions, \( \mathbb{E}[S] = 1/f_{\bar{A}}(0) \) so that we retrieve the usual form of Little’s Law as

\[
\hat{Q}(t) = f_{\bar{A}}(0)\hat{W}(t).
\]

Note that \( f_{\bar{A}}(0) = \lambda \) when \( T \) is exponentially distributed with mean \( 1/\lambda \).

Theorem 4 should be contrasted with the analogous result in [8, Proposition 4]. There, an extra diffusion term appears. This term is a function of the fluid limit of the queue length process. However, in our setting, this limit is the zero process, as can be seen in (2.7), where no centering is needed and thus the term disappears.

5 Conclusions

While the \( \Delta(\cdot)/G/1 \) queue originated as a simple model for the study of general time-inhomogeneous queueing systems, it has very recently gained much attention since it represents the standard model for queues in which only a finite number of customers request for service [9]. In this paper we have shown how techniques from the theory of stochastic-process
limits, and more specifically heavy-traffic diffusion approximations, can be successfully employed to prove convergence results for the $\Delta(i)/G/1$ queue. In particular, we have proven that when suitably rescaled (according to a non-standard scaling) the queue length process converges in distribution to a reflected Brownian motion with downwards parabolic drift. Our result is a generalization of [2], where the arrival times are assumed to follow an exponential distribution. Therefore, more demanding embedding and martingale techniques were used. Therefore, the techniques we have introduced offer a significant computational and conceptual advantage, allowing one to easily study other quantities of interest of the $\Delta(i)/G/1$ model other than the queue length process. As an example of the strength of our approach, we have proven a sample path diffusion Little’s Law, which relates the virtual waiting time and the queue length processes.

Acknowledgments

The author is very grateful to Debankur Mukherjee and Jori Selen for suggesting many improvements to the manuscript and numerous helpful discussions.

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