A note on the Grover walk and the generalized Ihara zeta function of the one-dimensional integer lattice

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December 17, 2021
2000 Mathematical Subject Classification: 60F05, 05C50, 15A15, 05C25.
Key words: zeta function, quantum walk, Grover walk, regular graph, integer lattice

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Abstract

Chinta, Jorgenson and Karlsson introduced a generalized version of the determinant formula for the Ihara zeta function associated to finite or infinite regular graphs. On the other hand, Konno and Sato obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a finite graph. In this paper, we focus on a relationship between the Grover walk and the generalized Ihara zeta function. That is to say, we treat the generalized Ihara zeta function of the one-dimensional integer lattice as a limit of the Ihara zeta function of the cycle graph.

1 Introduction

Ihara [9] introduced the Ihara zeta functions of graphs, and showed that the reciprocal of the Ihara zeta function of a regular graph is an explicit polynomial. Afterwards, the Ihara zeta function of a finite graph was studied in [12, 13, 14, 8, 1], and its determinant expressions were presented. Chinta, Jorgenson and Karlsson [2] gave a generalized version of the determinant formula for the Ihara zeta function associated to finite or infinite regular graphs.

A discrete-time quantum walk is a quantum analog of the classical random walk on a graph whose state vector is governed by a matrix called the time evolution matrix. The time evolution matrix of a discrete-time quantum walk in a graph is closely related to the Ihara zeta function of a graph. Ren et al. [11] gave a relationship between the discrete-time evolution matrix of a discrete-time quantum walk in a graph and the Ihara zeta function of a graph. Konno and Sato [10] obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a graph.

In this paper, we consider the relation between the Grover walk and the generalized Ihara zeta function, and present the generalized Ihara zeta function of the one-dimensional integer lattice as a limit of the Ihara zeta function of the cycle graph.

In Section 2, we state a review for the Ihara zeta function of a finite graph and the generalized Ihara zeta function of a finite or infinite vertex transitive graph. In Section 3, we deal with the Grover walk on a graph as a discrete-time quantum walk on a graph. In Section 4, we treat the generalized Ihara zeta function of $Z$ as a limit of the Ihara zeta function of the cycle graph $C_n$ with $n$ vertices.

2 The Ihara zeta function of a graph

All graphs in this paper are assumed to be simple. Let $G = (V(G), E(G))$ be a connected graph (without multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Let $D_G$ be the symmetric digraph corresponding to $G$. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. For $v \in V(G)$, the degree $\deg_G v = \deg v = d_v$ of $v$ is the number of vertices adjacent to $v$ in $G$.

A path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \ldots, e_n)$ of $n$ arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1}) (1 \leq i \leq n - 1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, n$, then we write $P = (v_0, v_1, \ldots, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is called an $(o(P), t(P))$-path. We say that a path $P = (e_1, \ldots, e_n)$ has a backtracking if $e_i^{-1} = e_i$ for some $i (1 \leq i \leq n - 1)$. A $(v, w)$-path is called a $v$-cycle (or $v$-closed path) if $v = w$. A cycle $C = (e_1, \ldots, e_r)$ has a tail if $e_r = e_1^{-1}$. A cycle $C$ is reduced if $C$ has neither a backtracking nor a tail. For a natural number $k \in \mathbb{N}$, let $N_k$ be the number of reduced cycles of length $k$ in $G$. 
The **Ihara zeta function** of a graph $G$ is a function of a complex variable $u$ with $|u|$ sufficiently small, defined by

$$Z(G, u) = Z_G(u) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} u^k\right).$$

Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. The **adjacency matrix** $A = A(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. The **degree** of a vertex $v_i$ of $G$ is defined by $\deg v_i = \deg_G v_i = \{v_j \mid v_i v_j \in E(G)\}$. If $\deg v = k(\text{constant})$ for each $v \in V(G)$, then $G$ is called $k$-regular.

**Theorem 1 (Ihara; Bass)** Let $G$ be a connected graph. Then the reciprocal of the Ihara zeta function of $G$ is given by

$$Z(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - uA(G) + u^2(D - \mathbf{I})), $$

where $r$ is the Betti number of $G$, and $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0, i \neq j, (V(G) = \{v_1, \ldots, v_n\})$.

Let $G = (V(G), E(G))$ be a vertex transitive $(q + 1)$-regular graph and $x_0 \in V(G)$ a fixed vertex. Then the **generalized Ihara zeta function** $\zeta_G(u)$ of $G$ is defined by

$$\zeta_G(u) = \exp(\sum_{m=1}^{\infty} \frac{N_0^m}{m} u^m),$$

where $N_0^m$ is the number of reduced $x_0$-cycles of length $m$ in $G$. Note that, for a finite graph, the classical Ihara zeta function is just the above Ihara zeta function raised to the power equaling the number of vertices. Furthermore, the **Laplacian** of $G$ is given by

$$\Delta = \Delta(G) = D - A(G).$$

A formula for the generalized Ihara zeta function of a vertex transitive graph is given as follows:

**Theorem 2 (Chinta, Jorgenson and Karlsson)** Let $G$ be a vertex transitive $(q + 1)$-regular graph with spectral measure $\mu$ for the Laplacian. Then

$$\zeta_G(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp(\int \log(1 - (q + 1 - \lambda)u + qu^2) d\mu(\lambda)).$$

### 3 The Grover walk on a graph

Let $G$ be a connected graph with $n$ vertices and $m$ edges. Set $V(G) = \{v_1, \ldots, v_n\}$ and $d_j = d_{v_j} = \deg v_j, j = 1, \ldots, n$. For $u \in V(G)$, let $D(u) = \{e \in D(G) \mid t(e) = u\}$. Furthermore, let $\alpha_u, u \in V(G)$ be a unit vector with respect to $D(u)$, that is,

$$\alpha_u(e) = \begin{cases} \text{non zero complex number} & \text{if } e \in D(u), \\ 0 & \text{otherwise}, \end{cases}$$

where $\alpha_u(e)$ is the entry of $\alpha_u$ corresponding to the arc $e \in D(G)$.

Now, a $2m \times 2m$ matrix $C$ is given as follows:

$$C = 2 \sum_{u \in V(G)} |\alpha_u\rangle \langle \alpha_u| - I_{2m}.$$
The matrix $C$ is the *coin operator* of the considered quantum walk. Note that $C$ is unitary. Then the *time evolution matrix* $U$ is defined by

$$U = SC,$$

where $S = (S_{ef})_{e,f \in D(G)}$ is given by

$$S_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $S$ is called the *shift operator*.

The time evolution of a quantum walk on $G$ through $U$ is given by

$$\psi_{t+1} = U\psi_t.$$

Here, $\psi_{t+1}, \psi_t$ are the states. Note that the state $\psi_t$ is written with respect to the initial state $\psi_0$ as follows:

$$\psi_t = U^t \psi_0.$$

A quantum walk on $G$ with $U$ as a time evolution matrix is called a *coined quantum walk* on $G$.

If $\alpha_u(e) = \frac{1}{\sqrt{d_u}}$ for $e \in D(u)$, then the time evolution matrix $U$ is called the *Grover matrix* of $G$, and a quantum walk on $G$ with the Grover matrix as a time evolution matrix is called a *Grover walk* on $G$. Thus, the *Grover matrix* $U = U(G) = (U_{ef})_{e,f \in D(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} 2/d_0(e) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/d_0(e) - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise} \end{cases}$$

Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then the $n \times n$ matrix $T_n(G) = (T_{uv})_{u,v \in V(G)}$ is given as follows:

$$T_{uv} = \begin{cases} 1/(\deg_G u) & \text{if } (u,v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the matrix $T(G)$ is the transition probability matrix of the simple random walk on $G$.

**Theorem 3 (Konno and Sato)** Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges. Then the characteristic polynomial for the Grover matrix $U$ of $G$ is given by

$$\det(\lambda I_{2m} - U) = (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)I_n - 2\lambda T(G))$$

$$= \frac{(\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)D - 2\lambda A(G))}{d_{v_1} \cdots d_{v_n}}.$$

From Theorem 3, the following equation for the Grover matrix on a graph is obtained.

**Corollary 1** Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges. Then the characteristic polynomial for the Grover matrix $U$ of $G$ is given by

$$\det(I_{2m} - uU) = (1 - u^2)^{m-n} \det((1 + u^2)I_n - 2u T(G))$$

$$= \frac{(1-u^2)^{m-n} \det((1+u^2)D - 2u A(G))}{d_{v_1} \cdots d_{v_n}}.$$
4 The generalized Ihara zeta function of \( \mathbb{Z} \)

Let the cycle graph \( C_n \) be the connected 2-regular graph with \( n \) vertices. If \( n \to \infty \), then the limit of \( C_n \) is the one-dimensional integer lattice \( \mathbb{Z} \). Then we consider the Grover walk on \( \mathbb{Z} \). This quantum walk is a free quantum walk.

Let \( U_n^{(s)} \) be the Grover matrix on \( C_n \) and \( P_n^{(s)} \) the transition probability matrix of the simple random walk on \( C_n \). Then we have

\[
P_n^{(s)} = \frac{1}{2} A(C_n).
\]

By Corollary 1, we have

\[
det(I_{2n} - u U_n^{(s)}) = (1-u^2)^n - n \det((1+u^2)I_n - 2u P_n^{(s)}) = (1-u^2)^n - n \det(I_n - uA(C_n) + u^2 I_n).
\]

That is,

\[
Z(C_n, u)^{-1} = det(I_{2n} - u U_n^{(s)}).
\]

By the fact that, for a finite graph, the classical Ihara zeta function is just the above Ihara zeta function raised to the power equaling the number of vertices, we have

\[
\zeta_{C_n}(u)^{-1} = Z(C_n, u)^{-1/n} = det(I_{2n} - u U_n^{(s)})^{1/n}
\]

\[
= \{ det((1+u^2)I_n - 2u P_n^{(s)}) \}^{1/n}
\]

\[
= \{ \prod_{\lambda \in \text{Spec}(P_n^{(s)})} ((1+u^2) - 2u \lambda) \}^{1/n}
\]

\[
= \exp[\log(\{ \prod_{\lambda \in \text{Spec}(P_n^{(s)})} ((1+u^2) - 2u \lambda) \}^{1/n})]
\]

\[
= \exp[\frac{1}{n} \sum_{\lambda \in \text{Spec}(P_n^{(s)})} \log((1+u^2) - 2u \lambda)].
\]

Now, since

\[
det(I_{2n} - u U_n^{(s)}) = Z(C_n, u)^{-1} = (1-u^2)^n,
\]

we have

\[
det(\lambda I_{2n} - U_n^{(s)}) = (\lambda^2 - 1)^n.
\]

Thus,

\[
\text{Spec}(U_n^{(s)}) = \{ e^{i\theta_0}, e^{i\theta_1}, \ldots, e^{i\theta_{n-1}} \}.
\]

where \( \text{Spec}(F) \) is the spectra of a square matrix \( F \), and

\[
\theta_k = \frac{2\pi k}{n} \quad (k = 0, 1, \ldots, n-1).
\]

Furthermore, by Theorem 3 (Konno-Sato Theorem), we obtain the following spectral mapping theorem:

\[
det(\lambda I_{2n} - U_n^{(s)}) = (2\lambda)^n (\lambda^2 - 1)^0 \det(\frac{\lambda + 1}{2\lambda} I_n - P_n^{(s)}) = (2\lambda)^n \det(\frac{\lambda + \lambda}{2} I_n - P_n^{(s)}).
\]

If \( \lambda = e^{i\theta_k} (k = 0, 1, \ldots, n-1) \), then we obtain

\[
\frac{\lambda + \overline{\lambda}}{2} = \cos \theta_k.
\]
That is,
\[ \text{Spec}(\mathbf{P}_n^{(s)}) = \{ [\cos \theta_k]^1 \mid k = 0, 1, \ldots, n - 1 \}. \]

Thus, we have
\[
\zeta_{C_n}(u)^{-1} = \exp \left[ \sum_{k=0}^{n-1} \log \left( (1 + u^2) - 2u \cos \left( \frac{2\pi k}{n} \right) \right) \frac{1}{2\pi} \times \frac{2\pi}{n} \right].
\]

When \( n \to \infty \), then we have
\[
\lim_{n \to \infty} \zeta_{C_n}(u)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( (1 + u^2) - 2u \cos x \right) \frac{dx}{2\pi} \right] = \frac{u^2 + 1 + |u^2 - 1|}{2} = \begin{cases} 
1 & \text{if } |u| < 1, \\
u^2 & \text{if } |u| \geq 1.
\end{cases}
\]

On the other hand, we see that
\[ \zeta_Z(u) = 1, \]
since \( Z \) has no reduced cycle. Therefore we obtain the following result.

**Theorem 4**

\[ \lim_{n \to \infty} \zeta_{C_n}(u) = \zeta_Z(u) \text{ for } |u| < 1. \]

From now on we consider a relation between Theorem 1.3 given by Chinta et al. [2] and Theorem 3.

Their result for \( Z \) case gives
\[ \zeta_Z^{-1} = (1 - u^2)^{(1-1)/2} \exp \left( \int \log(1 - 2u + u^2 + u\lambda) \, d\mu(\lambda) \right) \]
\[ = \exp \left( \int \log(1 - 2u + u^2 + \lambda u) \, d\mu(\lambda) \right) \quad (1). \]

Noting that \( \Delta(C_n) = D - A(C_n) = 2(I_n - \mathbf{P}_n^{(s)}) \), we have
\[
\zeta_{C_n}(u)^{-1} = Z(C_n, u)^{-1/n} = \det(I_{2m} - u\mathbf{U}_n^{(s)})^{1/n}
\]
\[ = \left\{ \det((1 - 2u + u^2)I_n + u\Delta(C_n)) \right\}^{1/n}. \]

Thus a similar argument in the proof of Theorem implies
\[
\lim_{n \to \infty} \zeta_{C_n}(u)^{-1} = \exp \left[ \int_0^{2\pi} \log((1 - 2u + u^2) + 2u(1 - \cos x)) \frac{dx}{2\pi} \right].
\]

Remark that the right-hand side of this equality is noting but that of Eq. (1). Then we have

**Corollary 2**

\[ \lim_{n \to \infty} \zeta_{C_n}(u)^{-1} = \zeta_Z(u)^{-1} = \exp \left( \int \log(1 - 2u + u^2 + \lambda u) \, d\mu(\lambda) \right), \]

where
\[ \lambda d\mu(\lambda) \sim 2(1 - \cos x) \frac{dx}{2\pi} \text{ on } [0, 2\pi). \]
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