A Posteriori $L_\infty(L_2^) + L_2(H^1)$–Error Bounds in Discontinuous Galerkin Methods For Semidiscrete Semilinear Parabolic Interface Problems

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Abstract:
The aim of this paper is to derive a posteriori error estimates for semilinear parabolic interface problems. More specifically, optimal order a posteriori error analysis in the $L_\infty(L_2) + L_2(H^1)$- norm for semidiscrete semilinear parabolic interface problems is derived by using elliptic reconstruction technique introduced by Makridakis and Nochetto in (2003). A key idea for this technique is the use of error estimators derived for elliptic interface problems to obtain parabolic estimators that are of optimal order in space and time.

Key words: A posteriori error estimates, Discontinuous Galerkin methods, Interface semilinear parabolic problems.

Introduction:
Whilst the topic of a posteriori error estimation for linear and nonlinear parabolic problems is now relatively well understood for both conforming and nonconforming methods, see, e.g., (1-15), there are comparatively few results for semi linear parabolic interface problems (16-21). Particularly, Cangiani et al (17) used a nonstandard elliptic projection of Douglas and Dupont (16) to derive optimal order a priori error estimates for these problems in $L (L_2)$-norm and extended this work to the fully discrete setting in (18). Metcalfe (19) derived optimal order a posteriori error estimates in the $L_\infty(L_2) + L_2(H^1)$ norm for fully discrete parabolic interface problems. Gupta and Sinha (20) used elliptic reconstruction techniques to derive a posterior error estimates for semi linear parabolic interface problems with a locally-Lipschitz continuous nonlinearity in the forcing term. They used a Backward-Euler-Galerkin scheme to discretise in time with a conforming finite element method in space.

More recently, Sabawi (21) has derived an a posteriori error estimate for a class of nonlinear parabolic interface problems involving possibly curved interfaces, with flux balancing interface conditions, e.g., modelling mass transfer of solutes through semi-permeable membranes, in both the $L_\infty(L_2) + L_2(H^1)$ and $L (L_2)$ norms. Optimal order a posteriori error estimates $L_\infty(L_2) + L_2(H^1)$ were derived for semi and fully discrete nonlinear parabolic interface problems. The analysis revolved around a nonstandard elliptic reconstruction introduced by Douglas and Dupont (16).

The main contribution of this paper is to extend (21) to the case of semidiscrete semilinear parabolic interface problems in terms of $L_\infty(L_2) + L_2(H^1)$-norm. The main difficulty in constructing an optimal order a posteriori error estimator in $L_\infty(L_2) + L_2(H^1)$ is to deal with the nonlinear reaction term. These challenges are addressed by employing a continuation argument and the elliptic reconstruction technique, introduced by Makridakis and Nochetto (12) and extended to dG methods in (7).

It is worth noting the main reason of this technique is to lead us utilise ready elliptic interface a posteriori estimates that derived from elliptic interface problem (22-23) to bound the main part of the spatial error. There are some error estimators for semilinear parabolic problems available in the literature (24-30).
The rest of this paper is structured as follows. In Section 2, the model problem is introduced and discontinuous Galerkin method, with some necessary background results, are discussed. Section 3, $L_ω(L_2) + L_2(H^1)$ error bounds for semi discrete semilinear parabolic interface problems, are presented. Conclusions are given in Section 4.

**Model problem**

$$\frac{\partial u}{\partial t} - a\Delta u = f(u)$$

$$u = 0$$

$$a\nabla u_1, n^1 = C_{tr} (u_2 - u_1)$$

$$a\nabla u_2, n^2 = C_{tr} (u_1 - u_2)$$

$$u = u_i$$

$$\partial \Omega$$

$$\Omega_1$$

$$\Gamma_{tr}$$

$$\Omega_2$$

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded open polygonal/polyhedral domain. The interface $\Gamma_{tr}$ subdivides the domain $\Omega$ into two subdomains $\Omega_1, \Omega_2$, such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_{tr}$, with $\partial \Omega := (\partial \Omega_1 \cap \partial \Omega_2) \setminus \Gamma_{tr}$; sFig. 1 for an illustration. Considered the semilinear parabolic interface problem

![Figure 1. The interface $\Gamma_{tr}$ subdivides the domain $\Omega$ into two subdomains $\Omega_1, \Omega_2$.](image)

Here, $C_{tr} > 0$ is the interface transmission coefficient, and $u^i, i = 1, 2$ represent the concentration of two compounds present in $\Omega_1$ and $\Omega_2$ respectively. Let $f: \Omega_1 \cup \Omega_2 \to \mathbb{R}$ be a given data function, and let $u_i, i = 1, 2$ denote the respective outward unit normal vectors of $\Omega_i$. This model is presented in (17, 18) to describe the mass transfer of solutes through semi-permeable membranes. For simplicity in our analysis will make some abbreviations. The standard Lebesgue and Hilbertian Sobolev spaces are denoted by $L_p(\omega), 0 \leq p \leq \infty, L_p(\omega)$, $0 \leq p \leq \infty$ and $H^r(\omega), r \in \mathbb{R}$, respectively. In the special case when $r = 0$, will be donated by $L_2(\omega) \equiv H^0(\omega), \omega \subset \Omega$. The norm and standard $L_2(\omega)$-inner product will be denoted by $\| \cdot \| \equiv \| \cdot \|_{\Omega}$ and $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{\Omega}$ respectively when $\omega = \Omega$. Let the nonlinear forcing term $f(u)$ satisfy the following growth condition. There exist real numbers $C_{tr} > 0$ and $\sigma \geq 0$ such that

$$|f(u) - f(v)| \leq C_{tr} (1 + |u| + |v|)^\sigma |u - v|.$$  

Next, the definition of the norm is given by

$$\|v\|_{L_p(\Omega; X)} := \left( \int_0^T \|v\|_X^p \ dt \right)^{1/p}$$

$$\|v\|_{L_p(\Omega; X)} := \esssup_{0 \leq t \leq T} \|v\|_X$$

$$\|v\|_{L_p(\Omega; X)} := \sup_{0 \leq t \leq T} \|v\|_X$$

With this notation, picking $H^1(0, T; X) := \{ u \in L^2(0, T; X); \frac{\partial u}{\partial t} \in L^2(0, T; X) \}$.

Also, setting $\mathcal{H}^1 := H^1(\Omega_1 \cup \Omega_2)$, and

$$\mathcal{H}^1_0 := \{ v \in \mathcal{H}^1; v = 0 \text{ on } \partial \Omega \}.$$  

Multiply (1) by a test function $v \in \mathcal{H}^1_0$ and integration by parts on each sub-domain and applying the interface condition in (1), such that

$$\int_{\Omega_1} \frac{\partial u}{\partial t} \cdot v \ dx + \int_{\Omega_1} a\Delta u \cdot v \ dx = \int_{\Omega_1} f(u) \cdot v \ dx$$

$$\int_{\Omega_1} \frac{\partial u}{\partial t} \cdot v \ dx + \int_{\Omega_1, \Omega_2} a\nabla u \cdot \nabla v \ dx$$

$$- \sum_{i=1}^2 \int_{\Gamma_{tr}} n^i \cdot a\nabla u_i v_i \ ds = \int_{\Gamma_{tr}} f(u) \cdot v \ dx$$

$$\int_{\Omega_1} \frac{\partial u}{\partial t} \cdot v \ dx + \int_{\Omega_1, \Omega_2} a\nabla u \cdot \nabla v \ dx$$

$$- \int_{\Gamma_{tr}} n^1 \cdot a\nabla u_1 v_1 \ ds - \int_{\Gamma_{tr}} n^2 \cdot a\nabla u_2 v_2 \ ds$$

$$= \int_{\Gamma_{tr}} f(u) \cdot v \ dx$$

$$\int_{\Omega_1} \frac{\partial u}{\partial t} \cdot v \ dx + \int_{\Omega_1, \Omega_2} a\nabla u \cdot \nabla v \ dx$$

$$+ \int_{\Gamma_{tr}} C_{tr} \|u\| \|v\| \ ds = \int_{\Omega} f(u) \cdot v \ dx.$$  

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In weak form, the above equation becomes the following: find such that for almost every $t \in [0, T]$, reads
\[
\frac{\partial u}{\partial t} + D(u, v) = (f(u), v) \\
\forall v \in \mathcal{H}_0^1, \quad u(., 0) = u_i, \tag{3}
\]
with
\[
D(u, v) := \int_{\Omega_{0,2}} a \nabla u \cdot \nabla v \, dx + \int_{\Gamma} c_{tr} [v] [v] \, ds,
\]
\[
(f(u), v) = \int_{\Omega} f(u) v \, dx,
\] where $D(u, v)$ is a bilinear form and $[u] = u_1|_{K} \cdot n_1^2 + u_2|_{K} \cdot n_2^2$ is the jump across the interface. The Cauchy-Schwarz inequality provides the coercivity and continuity of the bilinear form $D_viz.$
\[
D(v, v) := \int_{\Omega_{1,2}} a |\nabla v|^2 \, dx + \int_{\Gamma} c_{tr} ||v||^2 \, ds \leq \leq \leq \leq \leq ||v||^2 \quad \forall v \in \mathcal{H}_0^1 \tag{5}
\]
\[
D(v, w) \leq ||v|| \cdot ||w||, \quad \forall v \in \mathcal{H}_0^1.
\]

Discontinuous Galerkin method. Given a mesh $\mathcal{T} = \{K\}$ (with $K$ representing a generic element), the discontinuous finite element space $V_h^p$ is constructed by Fig.2
\[
V_h^p = \{L_2(\Omega): v|_{K} \in P_p(K)\}, \tag{6}
\]

where $P_p(K)$ denotes the space of polynomials of total degree $p$ on an element $K$. Suppose that $K_1$ and $K_2$ are two elements sharing the same face $E \in \Gamma^\text{int} \cup \Gamma^\text{tr}$, with $E \subset \partial K_1 \cap \partial K_2$ with $n_{K_1}$ and $n_{K_2}$ denoting the outward unit normal vectors on $E$ of $\partial K_1$ and $\partial K_2$, respectively. Then, subdividing $\Gamma$ the mesh skeleton into three disjoint subsets $\Gamma = \partial \Omega \cup \Gamma^\text{int} \cup \Gamma^\text{tr}$, where $\Gamma^\text{int} = \Gamma \setminus (\partial \Omega \cup \Gamma^\text{tr})$ is the interior points.

Let $v$ be a discontinuous function across $\Gamma$. Setting $v_i = v|_{K_i}$ and defining its jump $[v_h]$ and average $\{v_h\}$ across $E$ by
\[
[v_h] = v_h|_{K_1} + v_h|_{K_2}, \quad \{v_h\} = \frac{v_h|_{K_1} + v_h|_{K_2}}{2}.
\]

Similarly, for a vector valued function $w_h$, piecewise smooth on $\mathcal{T}$ with $w_i = w|_{K_i}$, such that $[w_h] = w|_{K_1} + w|_{K_2}, \quad \{w_h\} = \frac{w|_{K_1} + w|_{K_2}}{2}$.

Thus, setting $\{v_h\} = v$, $[v_h] = vn$ and $[w_h] = w_h \cdot n$ with $n$ denoting the outward unit normal on the boundary $\partial \Omega$. For the mesh size, using the function $h: \Omega \rightarrow \mathbb{R}$, where $h|_{K} = h_K, K \in \mathcal{T}$ and $h = \{h\}$ on each $(d - 1)$ dimensional open face $E \subset \Gamma$. Further, assuming that $h_{\text{max}} = \max_{x \in \Omega} h$ and $h_{\text{min}} = \min_{x \in \Omega} h$. Without loss of generality, assuming that $h_{\text{max}}$ remains uniformly bounded throughout this work, to avoid having estimation constants depend on $\max\{1, h_{\text{max}}\}$. To introduce the interior penalty discontinuous Galerkin method, Multiplying (1) by a test function $v \in \mathcal{H}_0^1 + V_h^p(\mathcal{T})$ and, integrate over each subdomain, so that
\[
\int_{\Omega} \frac{\partial u}{\partial t} v - \int_{\Omega} a \Delta u \cdot v \, dx = \int_{\Omega} f(u) \cdot v \, dx.
\]

Then, splitting the integral into element contributions and integrate by parts:
\[
\int_{\Omega} \frac{\partial u}{\partial t} v \, dx + \sum_{K \in \mathcal{T}} \int_{\partial K} a \nabla u \cdot v \, ds = \sum_{K \in \mathcal{T}} \int_{\partial K} f(u)v \, ds.
\]

The next step is to decompose the face integrals:
\[
\sum_{K \in \mathcal{T}} \langle a n \cdot \nabla u, v \rangle = \sum_{E \in \Gamma^\text{int} \cap \Gamma^\text{tr}} \langle a n \cdot \nabla u, v \rangle_{\partial K_1 \cap \partial E} + \sum_{E \in \Gamma^\text{tr} \cap \Gamma^\text{tr}} \langle a n \cdot \nabla u, v \rangle_{\partial K_2 \cap \partial E} \tag{7}
\]

Figure 2. The mesh skeleton $\Gamma$ discretises the domain into $\partial \Omega$, $\Gamma^\text{int}$, $\Gamma^\text{tr}$.
Finally, it is ready to introduce the interior penalty discontinuous Galerkin method for (3), which reads: find $u_h \in V^p_h$ such that
\begin{equation}
\left( \frac{\partial u_h}{\partial t}, v_h \right) + D_h(t; u_h, v_h) = (f(u_h), v_h), \tag{7}
\end{equation}
for all $v_h \in V^p_h$.

where $K_1$ and $K_2 \in \mathcal{T}$, for $E \in \partial K_1 \cap \partial K_2$ $n$ is a corresponding unit normal on $E$ (exterior to $K_2$).

The estimator is derived using the elliptic reconstruction techniques in (2003). For each $t \in [0, T]$ the elliptic reconstruction $\mathcal{R}_h(u_h) \in \mathcal{H}_0^1$ to be the unique solution of the problem
\begin{equation}
D(t; \mathcal{R}_h(u_h), \varphi) = \left( f(u_h) - \frac{\partial u_h}{\partial t}, \varphi \right) \forall \varphi \in \mathcal{H}_0^1 \tag{8}
\end{equation}

\textbf{Lemma 3.1.} Let $\mathcal{R}_h(u_h) \in \mathcal{H}_0^1$ be the exact solution of elliptic problem (8). Then, the following a posteriori bound holds:
\begin{equation}
||| \mathcal{R}_h(u_h) - u_h ||| \leq C \left( \eta_{S\ell}(u_h) + \eta_{S\sigma}(u_h) \right),
\end{equation}
where
\begin{align}
\eta_{S\ell}(u_h) := \left( \sum_{K \in \mathcal{T}} \left( \frac{1}{v} a \sqrt{h} \left( f(u_h) - \frac{\partial u_h}{\partial t} \right) \right) + \frac{1}{2} \left( \frac{1}{h} a \|u_h\|^2 \right)_{\mathcal{H}_0^1(\Gamma_{int})} \right) \quad \text{and} \\
\eta_{S\sigma}(u_h) := \left( C_1 \left( \frac{1}{\sqrt{h^{-1}}} \left( \mathcal{R}_h(u_h) \right)_{\mathcal{H}_0^1(\Gamma_{int})} \right) \right)^{1/2}.
\end{align}

\textbf{Proof.} See (21, Lemma 6.2).

\textbf{Lemma 3.2.} (Error relation) Let $u$ and $u_h$ are the exact and approximate solutions defined by (3) and (7) respectively, and let $\mathcal{R}_h(u_h)$ be given by (8), and set
\begin{align}
\varepsilon &:= \rho \varepsilon, \quad \rho := u - \mathcal{R}_h(u_h), \quad \varepsilon := \mathcal{R}_h(u_h) - u_h, \quad \varepsilon := \mathcal{R}_h(u_h) - u_h, \quad u_h^0 := u_h - u_h.
\end{align}

So that, the identity
\begin{equation}
\left( \frac{\partial \varepsilon}{\partial t}, v \right) + \mathcal{D}(\varepsilon, v) = \left( f(u_h) - f(u_h), \varepsilon \right) + D(t, \varepsilon, v) + \left( \frac{\partial u_h}{\partial t}, v \right), \tag{9}
\end{equation}

\textbf{Proof:} Going back in the definition (8), lead to
\begin{equation}
\left( \frac{\partial u_h}{\partial t}, v \right) + D(t; \mathcal{R}_h(u_h), v) = \left( f(u_h), v \right) \quad \forall \ v \in \mathcal{H}_0^1 \tag{10}
\end{equation}

Subtracting (3) from above equation, gives
\begin{equation}
\left( \frac{\partial u_h}{\partial t} - \frac{\partial u}{\partial t}, v \right) + D(t; \mathcal{R}_h(u_h) - u, v) = \left( f(u_h) - f(u), v \right)
\end{equation}
\[
\begin{align*}
\left\{ \frac{\partial u_h}{\partial t} + \frac{\partial u_h^c}{\partial t} - \frac{\partial u_h^c}{\partial t}, v \right\} \\
+ D(t; \mathcal{R}_h(u_h) - u_h^c + u_h^c - u, v) \\
= (f(u_h) - f(u), v) \\
+ \left\{ \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}, v \right\} \\
+ D(t; u_h^c - u, v) \\
= (f(u_h) - f(u), v) + D(t; \mathcal{R}_h(u_h) - u_h^c, v) \\
+ \left\{ \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}, v \right\}.
\end{align*}
\]

Using \( \frac{\partial e_c}{\partial t} = \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}, e_c = \mathcal{R}_h(u_h) - u_h^c \), \( \frac{\partial u_h}{\partial t} = \frac{\partial u_h^c}{\partial t} \), the result follows.

**Theorem 3.3.** \((L_2(H^1) + L_\infty (L_2))\)-norm. For each \( t \in [0, T] \), let \( r \) be as in, (17, lemma 5.1), with \( h_{\text{max}} \) small enough. Then, the error bound is
\[
\begin{align*}
\theta^2(\varepsilon) &= \left( \|u_0 - u_h(0)\| + C_0 \sqrt{\int_\Omega \|u_h(0)\|^2} \right)^2 \\
&+ C \int_0^T \left( \eta_2^2(u_h) + \eta_2^2(u_h) \right) dt \\
&+ \left( \eta_2^2(u_h) + \eta_2^2(u_h) \right) dt \\
&+ \sqrt{\int_\Omega \|u_h^c\|^2} + \sqrt{\mathcal{R}_h(u_h)} \right)^2 dt \\
&+ \sqrt{\int_\Omega \|u_h^c\|^2} + \sqrt{\mathcal{R}_h(u_h)} \right)^2 dt \\
&+ \int_\Omega \|u_h^c\|^2 + \|u_h\|^2 \alpha^2(t) \left( \|\varepsilon\|^2 + \|\varepsilon\|^2 \right) dt \\
&+ \int_\Omega \|u_h^c\|^2 + \|u_h\|^2 \alpha^2(t) \left( \|\varepsilon\|^2 + \|\varepsilon\|^2 \right) dt \\
&+ \int_\Omega \|u_h^c\|^2 + \|u_h\|^2 \alpha^2(t) \left( \|\varepsilon\|^2 + \|\varepsilon\|^2 \right) dt.
\end{align*}
\]

Proof. Testing with \( v = e_c \) in (9), gives
\[
\begin{align*}
\vec{\text{d}} e_c - \vec{\text{d}}(e_c, e_c) + \vec{\text{d}}(t, e_c, e_c) \\
= (f(u_h) - f(u_h), e_c) + D(t, e_c, e_c) \\
+ \left\{ \frac{\partial u_h}{\partial t} - \frac{\partial u_h^c}{\partial t}, e_c \right\}.
\end{align*}
\]
To simplify the left-hand side and using the identity
\[
\begin{align*}
\frac{\partial e_c}{\partial t} - (e_c, e_c) = \int_\Omega \frac{d e_c}{d t} - (e_c, e_c) dx = \frac{1}{2} \frac{d}{d t} \left( \int_\Omega (e_c)^2 dx \right) \\
= \frac{1}{2} \frac{d}{d t} \|e_c\|^2.
\end{align*}
\]
Therefore,
\[
\int_\Omega (\|\varepsilon\|^2\|\rho\|^{2r} + \|\varepsilon\|^{2r}\|\rho\|) \\
\leq \left( \frac{1}{r+1} + \frac{r}{r+1} \right) \|\rho\|^{\frac{2r}{r+1}} + \|\varepsilon\|^{\frac{2r}{r+1}} \\
\leq C\left(\|\varepsilon\|^{\frac{2r}{r+1}} + \|\rho\|^{\frac{2r}{r+1}} + \|\varepsilon\|^{\frac{2r}{r+1}} + \|\rho\|^{\frac{2r}{r+1}}\right) \\
\leq C\|\varepsilon\|^{\frac{2r}{r+1}} + \|\rho\|^{\frac{2r}{r+1}}. 
\]

Applying the Hölder’s inequality with \( m = \frac{2}{2-r}, n = \frac{r}{2}, \frac{1}{m} + \frac{1}{n} = 1 \), along with the Sobolev imbedding inequality \( \|\rho\|_{L^{2m}} \leq C_{\rho} \|\rho\|_1 \), on the first term on the above equation, which gives

\[
\|e^{\varepsilon}\|^{\frac{2r}{r+1}} = \int_\Omega |e^{\varepsilon}|^{2r+2} dx = \int_\Omega |e^{\varepsilon}|^r |e^{\varepsilon}|^2 dx \\
\leq C_{\rho} \|e^{\varepsilon}\|^{2r} \|e^{\varepsilon}\|_1^2 \]

Hence,

\[
\int_\Omega (\|\varepsilon\|^2\|\rho\|^{2r} + \|\varepsilon\|^{2r}\|\rho\|) dx \\
= C_{\rho} \|e^{\varepsilon}\|^{2r} \|e^{\varepsilon}\|^2 \\
+ C \|u^d_h\|^{2+2r} + \|\varepsilon\|^{2+2r}, 
\]

valid for all \( \rho \in L^\infty \) and \( \rho \leq 2 \) for \( d = 2 \) and \( r = \frac{4}{3} \) for \( d = 3 \). Substituting this into our earlier inequality (12), this leads to

\[
\int_\Omega \|f(u_h) - f(u_h)\| \leq \alpha^2(t) \left( \|\beta^2(t)\| \|e^{\varepsilon}\| + \beta^2(t) \|d^e_h\|^2 \right) \\
+ \|\beta^2(t)\| \|e^{\varepsilon}\| + 2 \|\|e^{\varepsilon}\| + \|d^e_h\|^2 \| \\
\leq \alpha^2(t) \left( \beta^2(t) \|e^{\varepsilon}\| + \beta^2(t) \|d^e_h\|^2 \right) \\
+ \|\beta^2(t)\| \|e^{\varepsilon}\| + 2 \|\|e^{\varepsilon}\| + \|d^e_h\|^2 \|. 
\]

Now, putting \( \beta(t) = \sqrt{1 + 4r\|u^e_h\|^{2r}} \), this becomes

\[
\int_\Omega \|f(u_h) - f(u_h)\| \leq \alpha^2(t) \left( \beta^2(t) \|e^{\varepsilon}\| + \beta^2(t) \|d^e_h\|^2 \right) \\
+ \|\beta^2(t)\| \|e^{\varepsilon}\| + 2 \|\|e^{\varepsilon}\| + \|d^e_h\|^2 \|. 
\]

Refusing (11) and Young’s inequality, so that

\[
\|e^{\varepsilon}\| + \|e^{\varepsilon}\| \leq C_{\rho} \|e^{\varepsilon}\|_1^2 \\
+ \frac{1}{2} \|e^{\varepsilon}\|^2 . 
\]

Setting

\[
F(t) = 2C_{\rho} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{4}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 \\
+ \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 . 
\]

Integrating the time variable over \( s \in (0, t) \), and set

\[
C_{\rho} = 2C_{\rho}^2 C_{\rho}^2 \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 . 
\]

Taking the time variable over \( s \in (0, t) \), and set

\[
C_{\rho} = 2C_{\rho}^2 C_{\rho}^2 \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 + \frac{2}{t} \|e^{\varepsilon}\|^2 . 
\]

Now, putting \( \beta(t) = \sqrt{1 + 4r\|u^e_h\|^{2r}} \), this becomes

\[
\int_\Omega \|f(u_h) - f(u_h)\| \leq \alpha^2(t) \left( \beta^2(t) \|e^{\varepsilon}\| + \beta^2(t) \|d^e_h\|^2 \right) \\
+ \|\beta^2(t)\| \|e^{\varepsilon}\| + 2 \|\|e^{\varepsilon}\| + \|d^e_h\|^2 \|. 
\]

Letting \( \theta(e^{\varepsilon}) = \sqrt{1 + 4r\|u^e_h\|^{2r}} \), and implies that

\[
\|e^{\varepsilon}\|^2 + 2 \int_0^t \|e^{\varepsilon}\|^2 ds \leq \theta(e^{\varepsilon})^2 . 
\]
\[
+ \mathcal{M}(t) \int_0^t || e^c ||^2 ds \\
+ C_2(\| e^c(s) \|_{L^2})^{1+r}.
\]  

(17)

To deal with final term on the right-hand side of (17), applying the inequality \( z_1^2 z_2 \leq (z_1^2 + z_2)^{1+\frac{r}{2}} \) and assume that the mesh-size \( h_{\text{max}} \) is small enough that

\[
\theta(e^c) \leq \frac{1}{(C_2)^r} (4e^{M(t)T})^{-1+\frac{r}{T}} \Rightarrow \theta(e^c)^r \\
\leq \frac{1}{C_2} (4e^{M(t)T})^{-1+\frac{r}{T}} \\
\Rightarrow C_2 \theta(e^c)^r (4e^{M(t)T})^{1+r} \leq 1
\]

\[
\Rightarrow C_2 (4\theta(e^c)^2 e^{M(t)T})^{1+r} \leq \theta(e^c)^2.
\]

Since the left-hand side of (17) depends continuously on \( t \), which implies that the set

\( \mathcal{A} = \{ t \in [0, T] : \| e^c(s) \|^2 \leq 4 \theta(e^c)^2 e^{M(t)T} \} \) is non-empty and closed. Therefore, setting \( t^* = \max \mathcal{A} \) and assuming that \( t^* < T \), so that

\[
\| e^c(t) \|^2 + 2 \int_0^t || e^c ||^2 ds \leq 2 \theta(e^c)^2 + \mathcal{M}(t) \int_0^t || e^c ||^2 ds,
\]

and using Gronwall’s inequality, such that

\[
\| e^c(t) \|^2 + 2 \int_0^t || e^c ||^2 ds \leq 2 \theta(e^c)^2 e^{M(t)T} \\
< 4 \theta(e^c)^2 e^{M(t)T}.
\]

(19)

This leads to contradiction with hypothesis \( t^* < T \) because of the continuity of the left-hand side of (19). Hence, \( t^* = T \). Setting \( t^* = t \), and \( \| e^c(t^*) \| = \| e^c \|_{L^\infty(0, t^*, L^2(\Omega))} \) due to the continuity with respect to \( t \), (19) implies

\[
\| e^c(t) \|^2_{L^\infty(0, t^*, L^2(\Omega))} \leq \| e^c(t^*) \|^2 + \int_0^{t^*} || e^c ||^2 ds \\
\leq 2 \theta(e^c)^2 e^{M(t^*)T} \\
\leq 2 \theta(e^c)^2 e^{M(t)T}.
\]

With \( T \) being the final time. Combining this with (19) for \( t = T \), to arrive

\[
\| e^c \|^2 = \| e^c \|^2_{L^\infty(0, T, L^2(\Omega))} + \int_0^T || e^c ||^2 ds \\
\leq 2 \theta(e^c)^2 e^{M(T)T}
\]

The triangle inequality along with Lemma 3.1, now implies

\[
\| e^c \|^2_\ast \leq 2 || e^c ||^2 + 2 || u_h^d ||^2 \leq 4 \theta(e^c)^2 e^{M(T)T} + 2 || u_h^d ||^2,
\]

where

\[
\norm{u_h^d}^2 = \norm{u_h^d}_{L^\infty(0, T, L^2(\Omega))} + \int_0^T \norm{u_h^d}^2 ds
\]

The result therefore, follows from the last three inequalities.

**Conclusion**

This paper aims to derive an optimal order a posteriori error estimates in the \( L^\infty(L^2) + L^2(H^1) \)-norm for semidiscrete semilinear parabolic interface problems. An important factor in our analysis to derive this estimator is to use the elliptic reconstruction framework of Makridakis and Nochetto (12) although, crucially, a number of challenges have to be overcome due to non-linearity on the forcing term depending on Gronwall’s Lemma and Sobolev embedding through continuation argument. The main use for these bounds is in designing an efficient adaptive scheme, and consequently leading to a reduction in the computational cost of the scheme. In the future, this work can be extended to the fully discrete case for semilinear parabolic interface problems in \( L(L^2) \) and \( L(H^1) \) norms.

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**Author’s declaration:**

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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تقدير خطا الابعد في كالكرین غير مستمرة لمسائل ذات الوسط البناء لشبه متكافئة خطية لشبه مقطعة

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قسم تربية الرياضيات، كلية التربية، جامعة تشك الدولية، أربيل، العراق.

الخلاصة:
ان الهدف من هذا البحث هو استقاف الخطأ الابعد لمسائل ذات الوسط البناء لشبه متكافئة شبه متقطعة. وبشكل أكثر تحديدًا، تم تحليل الخطأ الابدب الأمثل لمسائل ذات الوسط البناء لشبه خطية متكافئة متقطعة باستخدام تقنية إعادة الإهليلجية المقدمة من مارك داكس وأنيكيتو (2003). الفكرة الأساسية لهذه التقنية هي استخدام مقدرات الخطأ المكتشفة من مشاكل الواجهة الإهليلجية للحصول على مقدرات مكافئة ذات ترتيب أمثل في المجال والزمان.

الكلمات المفتاحية: تقدير الخطأ الابعد، طرق كالكرين غير المستمرة، مسائل ذات الوسط البناء لشبه خطية متكافئة شبه مقطعة.