Efficient Projection onto the $\ell_{\infty,1}$ Mixed-Norm Ball using a Newton root search method

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Abstract

Mixed norms that promote structured sparsity have numerous applications in signal processing and machine learning problems. In this work we present a new algorithm, based on a Newton root search technique, for computing the projection onto the $\ell_{\infty,1}$ ball, which has found application in cognitive neuroscience and classification tasks. Numerical simulations show that our proposed method is between 8 and 10 times faster on average, and of up to 20 times faster for very sparse solutions, than the previous state of the art. Tests on real functional magnetic resonance image data show that, for some data distributions, our algorithm can obtain speed improvements by a factor of more than 100.

Index Terms

Mixed norms, Structured sparsity, Projection, Regularization, Root-finding, Fibonacci search

I. INTRODUCTION

Mixed norms are important in modeling group correlations in applications such as genetics [1], electroencephalography [2] and signal processing [3]. In this work we consider mixed norms with non-overlapping groups applied to matrix-form data $A \in \mathbb{R}^{M \times N}$, where the rows $a_m \in \mathbb{R}^N$ represent the different groups. Following the notation of [3], we consider the definition of the $\ell_{p,q}$-norm of $A$ as

$$
\|A\|_{p,q} = \left( \sum_{m=1}^{M} \|a_m\|_p^q \right)^{1/q}.
$$

(1)

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We will focus on a special case, the $\ell_{\infty,1}$-norm:

$$\|A\|_{\infty,1} = \sum_{m=1}^{M} \|a_m\|_{\infty},$$  \hspace{1cm} (2)

where $\|u\|_{\infty} = \max_{n}\{|u_n|\}$ for $u \in \mathbb{R}^N$.

The main contribution of this work is a new, computationally efficient algorithm for computing the projection onto the $\ell_{\infty,1}$ ball

$$\text{proj}_{\|\cdot\|_{\infty,1}}(B, \tau) := \arg\min_{X} \frac{1}{2} \|X - B\|_F^2, \hspace{1cm} \text{s.t. } \|X\|_{\infty,1} \leq \tau.$$ \hspace{1cm} (3)

This $\ell_{\infty,1}$ constraint problem has been applied to image annotation [4], cognitive neuroscience [5] and least absolute shrinkage and selection operator (LASSO) regression [6]. We propose a novel approach for solving (3) that utilizes a root search approach based on a Newton method, in which the total number of major iterations for the root search is reduced by applying a simple scheme for choosing a feasible initial solution.

Our previous version of this algorithm used a Steffensen root finding method [7]. This manuscript presents a number of additional contributions:

(i) We present a new method for solving (3). Instead of a Steffensen root-finding procedure, we formally develop an approximated Newton method for the root search function.

(ii) We significantly expand the theoretical analysis of the initial point estimation and pruning.

(iii) We consider additional experiments conducted on real functional magnetic resonance imaging (fMRI) data in order to validate the usefulness of our proposed method.

The manuscripts is organized as follows: Section II summarizes existing approaches for solving (3), Section III presents some mathematical preliminaries needed for our derivations, Section IV describes our proposed method, Sections V and VI present our results on simulated and real data, and in Sections VI-D and VII, we discuss these results and present the conclusions of our work.

II. EXISTING APPROACHES

In this section we review some previous approaches for solving (3). These methods tend to focus on reinterpreting the problem via some form quadratic or linear programming (Sections II-A and II-B) or via root-finding (Sections II-C and II-D).
A. Solution via interior point methods

In [8], the authors presented an approach for solving problems of the type

\[
\min_X \frac{1}{2} \|QX - B\|_F^2 \quad (4)
\]

\[
\text{s.t. } \|X\|_{\infty,1} \leq \tau .
\]

where \(Q\) is a fixed matrix. For the particular case where \(Q = I\), their approach consisted of introducing the variables \(\rho_m, m \in \{1, \ldots, M\}\) and recasting the problem as a convex quadratic optimization:

\[
\min_{\{\rho_m\}} \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^M (|b_{nm}| - \rho_m)_+^2 \quad (5)
\]

\[
\text{s.t. } \sum_{m=1}^M \rho_m = \tau ,
\]

\[
\rho_m \geq 0 , \quad m \in \{1, \ldots, M\} ,
\]

where \((x)_+ = \max(x, 0)\). They showed that the \(\rho_l\) are piecewise linear functions of \(\tau\) and that they fulfill the Karush-Kuhn-Tucker conditions in each linear section. Their algorithm for solving (5) involves starting at 0 and finding the “knots” where the \(\rho_l\) change from one linear piece to another, until they encounter a interval containing \(\tau\).

B. Solution via linear programming

In [4], the authors derived an equivalent linear program given by:

\[
\text{find } \mu, \theta
\]

\[
\text{s.t. } \sum_n \mu_n = \tau ,
\]

\[
\sum_m (A_{nm} - \mu_n)_+ = \theta, \forall n \text{ such that } \mu_n > 0 ,
\]

\[
\sum_m A_{i,j} \leq \theta , \forall n \text{ such that } \mu_n = 0 ,
\]

\[
\mu_n \geq 0, \forall n ,
\]

\[
\theta \geq 0 . \quad (6)
\]
The variables $\mu_i$ correspond to the $\ell_\infty$-norm values of each row of the optimum, and $\theta$ is related to a shrinkage parameter associated with the projection onto the $\ell_1$-ball. $\theta$ and $\mu$ are found via a search procedure over a piece-wise linear function, similar to that of Section II-A.

C. Projection onto the $\ell_{p,1}$ ball by root search

The general $\ell_{p,1}$-ball projection problem was solved by means of a root search technique in [6]. This approach relies on the fact that the proximal operator of the $\ell_{p,1}$ norm, defined as

$$\text{prox}_{\|\cdot\|_{p,1}}(B, \lambda) := \arg\min_X \frac{1}{2}\|X - B\|_F^2 + \lambda\|X\|_{p,1},$$

has a simpler solution than the projection onto the $\ell_{p,1}$-ball (3), since (7) can be computed by solving independent $\ell_p$-norm proximity subproblems [6] of the form

$$\arg\min_{x_m} \frac{1}{2}\|x_m - b_m\|_F^2 + \lambda\|x_m\|_p.$$  

One of the contributions of [6] was to propose a method to take advantage of the separability of (7) in order to solve (3). Let $\mathcal{L}(X, \theta)$ be the Lagrangian of (3), i.e,

$$\mathcal{L}(X, \theta) = \frac{1}{2}\|X - B\|_F^2 + \theta(\|X\|_{\infty,1} - \tau)$$

and let $\theta^*$ be the optimal dual variable. As long as $\tau > 0$, (3) satisfies Slater’s conditions for strong duality [9]. Therefore, the primal optimal variable $X^*(\theta^*) = \arg\min_X \mathcal{L}(X, \theta^*)$ can be obtained by computing

$$X^*(\theta^*) = \arg\min_X \frac{1}{2}\|X - B\|_F^2 + \theta^*(\|X\|_{p,1} - \tau).$$

It can be shown [6, Lemma 1] that the scalar function

$$g(\theta) = \|X(\theta)\|_{p,1} - \tau,$$

where $X(\theta) = \text{prox}_{\|\cdot\|_{p,1}}(B, \theta)$, satisfies the Fourier conditions [10], i.e. in the interval $[0, \theta_{\text{max}}]$

(i) $g(0) > 0 > g(\theta_{\text{max}})$
(ii) $g'(\theta) < 0$
(iii) $g''(\theta) \geq 0$.

Additionally, there exists a unique solution for $g(\theta) = 0$, and $\theta^*$ coincides with this unique root. Thus, $\theta^*$ can be found by using a root finding method.

The authors of [6] recommended a root finding method combining bisection, inverse quadratic interpolation, and the secant method. This root finding based solution for projections onto the
ℓp,1 ball was extended to the more general ℓp,q case in a follow-up article [11]. Computational performance comparisons indicated [11, Section 3.1] that this extended algorithm was both much more accurate, and twice as fast as that of [4] for projections onto the ℓ∞,1 ball.

D. Solution using Steffensen’s root search method

We have previously presented a technique for solving the ℓ1,∞-constraint problem via a Steffensen root-finding method [7]. Steffensen’s method is a quasi-Newton root finding algorithm [12] that is useful when an analytical expression of the derivative is not available. The drawback is that it requires two function evaluations, and is usually more expensive than Newton’s method. Given the function \( f(x) \), Steffensen’s original iterations consist of the update

\[
x_{n+1} := x_n + \frac{x_n}{\delta F(x_n, y_n)},
\]

where

\[
\delta F(x_n, y_n) = \frac{f(y_n) - f(x_n)}{y_n - x_n},
\]

\[
y_n = x_n + f(x_n).
\]

Steffensen’s method tends to exhibit convergence problems if the initial \( x_0 \) is too far from the actual root. Therefore, [7] used the modified version proposed in [12]:

\[
y_n = x_n + \alpha_n |f(x_n)|.
\]

Here \( \alpha_n \) is an adaptive parameter that is recommended to take values that satisfy

\[
\text{tol}_c \ll \frac{\text{tol}_u}{2|f(x_n)|} < |\alpha_n| < \frac{\text{tol}_u}{|f(x_n)|},
\]

where \( \text{tol}_c \) is chosen in accordance with the computer precision used in the implementation, and \( \text{tol}_u \) is a user-defined parameter.

To the best of our knowledge, the two fastest methods for solving the ℓ∞,1-ball projection problem are [6] and [7]. Accordingly, these two algorithms will be used as benchmark for all our comparisons.

III. Preliminaries

A. Notation

We will denote matrices with non-bold upper case font and vectors with bold lower case. Additionally, if \( A \in \mathbb{R}^{M \times N} \) is a matrix, we will denote the \( m^{\text{th}} \) row of \( A \) by \( a_m \), and the \( i^{\text{th}} \) element of the \( m^{\text{th}} \) row of \( A \) by \( a_{im} \).
B. Projection onto the $\ell_1$-ball

For our proposed solution to (3), we will need to solve the closely related problem of projection onto the $\ell_1$-ball, which is defined as

$$\text{proj}_{\|\cdot\|_1}(u, \tau) = \min_x \frac{1}{2}\|x - u\|_2^2 \quad \text{s.t. } \|x\|_1 \leq \tau,$$

where $x, u \in \mathbb{R}^N$. The solution to this equation is given by [13], [14], [15], [11], [16]

$$x^* = \begin{cases} 
  u & \text{if } \|u\|_1 < \tau \\
  \text{shrink}(x, \lambda(\tau)) & \text{if } \|u\|_1 \geq \tau,
\end{cases}$$

where

$$\text{shrink}(x, \lambda(\tau)) = \text{sign}(x) \odot \max(|x| - \lambda(\tau), 0),$$

$\lambda(\tau)$ is a shrinkage parameter that depend on $\tau$, and $\odot$ is the element-wise (Hadamard) vector product.

To solve problem (16) we must find $\lambda^*$ such that $f(\lambda^*) = 0$, where

$$f(\lambda) = \sum_n \max(|u_n| - \lambda, 0) - \tau.$$

Clearly, if $|u_n| > \lambda^*$ then this element contributes to the sum defined in (19); thus by sorting $|u|$ in decreasing order, a simple search will lead to the solution: i.e. let $v = \text{sort}(|u|)$ and define

$$L = \max \left\{ l \left| l^{-1} \left( \sum_{n=0}^{l} v_n - \tau \right) < v_l \right. \right\},$$

then

$$\lambda^* = L^{-1} \left( \sum_{n=0}^{L} v_k - \tau \right).$$

This solution was originally described in [17], with several improvements also reported in [18], [19], [13].

The solution of problem (16) can be cast as a root-finding problem, as the function defined in (19) satisfies the Fourier conditions in the interval $[0, u_{\text{max}}]$ and thus has a single root in this interval [14], [20].

The application of the Newton root-finding method [21, Section 11.1]: i.e.

$$\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}$$

leads to the so-called Michelot algorithm [22], [23], [14], [20]. More recently, [16], [24], [25] have proposed further improvements to [22].
Since it will be helpful in the derivation of our proposed algorithm (see Section IV), we show how the Michelot’s algorithm can be derived from a Newton’s root-search method applied to (19). This equation can be rewritten as [20, eq. (32)],[24, eq. (7)]:

\[ f(\lambda) = z^T u - \lambda z^T z - \tau, \quad (21) \]

where

\[ z = \text{sign}(u) \odot I_{|u| < \lambda}, \quad (22) \]

where \( I(\cdot) \) is the indicator function defined for a set \( A \) as [26, Chapter 2]:

\[ I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases} \quad (23) \]

Applying the Newton method to (21) by temporarily disregarding the dependence of \( z \) on \( \lambda \), leads to the Michelot algorithm iterations:

\[ \lambda_{k+1} = \lambda_k - \frac{z_k^T u - \lambda_k z_k^T z_k - \tau}{-z_k^T z_k} = \frac{z_k^T u - \tau}{z_k^T z_k}, \quad (24) \]

where \( z_k^T u \) is the \( \ell_1 \) norm of the subset of \( |u| \) whose elements are greater than \( \lambda_k \), and \( z_k^T z_k \) is the number of elements of this subset (equal to the number of non-zero elements in \( z_k \)). Furthermore it can also be shown that

\[ 0 < \lambda_k \leq \lambda_{k+1} \leq \lambda^* \forall k, \quad (25) \]

where \( \lambda^* \) is the parameter that solves (16). Thus at each iteration we can discard or prune all the elements in \( u \) such that \( |u_n| \leq \lambda_k \). Following the guidelines given in [24, Section 3.2.2], a careful implementation of such a pruning strategy can lead to a very efficient computational performance of the Michelot algorithm, as empirically shown in [25, Tables I and II].

C. Dual norm

In this section, we summarize additional theoretical results from [6] that will be useful in the analysis and derivation of the proposed algorithm.

**Definition 1.** [9] If \( \| \cdot \| \) is a norm on \( \mathbb{R}^m \), then the associated dual norm, \( \| \cdot \|_* \), is defined as

\[ \|z\|_* \triangleq \sup \left\{ z^T x \mid \|x\| \leq 1 \right\}. \quad (26) \]

**Lemma 1** (see [6, Lemma 1]). Let \( q \geq 1 \) and let \( p^* \) be its conjugate exponent satisfying

\[ \frac{1}{p} + \frac{1}{p^*} = 1. \]

Then, the norm \( \| \cdot \|_{p^*, \infty} \) is dual to \( \| \cdot \|_{p, 1} \).
It can be shown via Moreau’s decomposition [27], [6] that the dual problem of
\[
\text{prox}_{\|\cdot\|_{p^*,\infty}}(B,\lambda) := \arg\min_{X} \frac{1}{2} \|X - B\|_F^2 + \lambda \cdot \|X\|_{p^*,\infty},
\]  
where
\[
\|X\|_{p^*,\infty} = \max \{\|x_k\|_{p^*}\},
\]
is the projection onto the \(\ell_{p,1}\) ball with radius \(\lambda\), i.e.,
\[
\text{proj}_{\|\cdot\|_{p,1}}(B,\lambda) := \arg\min_{X} \frac{1}{2} \|X - B\|_F^2
\]
\[
\text{s.t. } \|X\|_{p,1} \leq \lambda.
\]

IV. PROPOSED METHOD

A. Leveraging \(\text{prox}_{\|\cdot\|_{1,\infty}}(\cdot)\), the dual of \(\text{proj}_{\|\cdot\|_{\infty,1}}(\cdot)\)

As described in Section II-C, the authors of [6] solved (28) using a root finding approach over (3). Here we consider an alternative reinterpretation that allows us to derive several improvements in the proposed algorithm, as explained in the following section. By the results of Section III-C, the proximal operator of \(\ell_{1,\infty}\) is the dual of the projection on the \(\ell_{\infty,1}\) ball and vice-versa, then \(X^* = \text{proj}_{\|\cdot\|_{\infty,1}}(B,\tau)\), can be written as \(X^* = B - A^*\), where
\[
A^* = \text{prox}_{\|\cdot\|_{1,\infty}}(B,\tau)
\]
\[
= \arg\min_{A} \frac{1}{2} \|A - B\|_F^2 + \tau \cdot \|A\|_{1,\infty}.
\]
Now, if \(A^*\) is known, we can define
\[
\gamma^* = \|A^*\|_{1,\infty} = \max \{\|a^*_m\|_1\},
\]
and thus, after simple algebraic manipulation, (29) can be written as
\[
\arg\min_{\{a_m\}} \frac{1}{2} \sum_m \|a_m - b_m\|_2^2 \text{ s.t. } \|a_m\|_1 \leq \gamma^*, \forall m.
\]
Clearly, (30) is separable in \(a_m\), with the individual problems corresponding to a projection on the \(\ell_1\)-ball (see Section III-B). Accordingly, if we devise a method for obtaining the optimal \(\gamma^*\) value, then the solution to (29), and therefore to (3), can be easily calculated. The \(\gamma^*\) value can be found by a root finding method, as described in the following section.
B. Search function and solution by Newton’s method

As originally proposed in [6], we use \text{prox}_{\|\cdot\|_{\infty,1}}(\cdot) to solve \text{proj}_{\|\cdot\|_{\infty,1}}(\cdot) in (11). Thus, we replace \(X\) by \(B - A\) in (11) and after simple algebraic manipulations, we obtain

\[
f(\gamma) = \sum_{m=1}^{M} \|b_m - a_m(\gamma)\|_{\infty} - \tau,
\]

defined for \(\gamma \geq 0\). Furthermore, since (31) is equivalent to (11), it also satisfies the Fourier conditions, and thus it has a unique root at \(\gamma^*\). For a given \(\gamma\), \(a_m\) is computed using the approach described in (30). As each \(a_m(\gamma)\) corresponds to a projection onto the \(\ell_1\)-ball, we apply (38) and obtain

\[
a_m(\gamma) = \begin{cases} 
  b_m & \text{if } \|b_m\|_1 < \gamma \\
  \text{sign}(b_m) \odot \max(|b_m| - \lambda_m(\gamma), 0) & \text{if } \|b_m\|_1 \geq \gamma.
\end{cases}
\]

By substituting (32) into (31), we note that only the terms corresponding to the \(\|b_m\|_1 \geq \tau\) contribute to the sum. Accordingly, at each evaluation of the search function, we can prune the rows of \(B\) that do not fulfill this condition, and only perform the projections specified in (32) on the remaining rows. Our numerical experiments show that this pruning strategy can reduce the computational time by half or more. Based on this remark, we can rewrite the search function as:

\[
f(\gamma) = \sum_{m \in M} \|b_m - \text{sign}(b_m) \odot \max(|b_m| - \lambda_m(\gamma), 0)\|_{\infty} - \tau,
\]

where \(M\) denotes the set of indexes \(m\) where \(\|b_m\|_1 \geq \gamma\). We can reduce this expression further by noting that \(b_m\) can be rewritten in the form \(\text{sign}(b_m) \odot |b_m|\) and factorizing:

\[
\begin{align*}
f(\gamma) &= \sum_{m \in M} \||b_m| - \text{sign}(b_m) \odot \max(|b_m| - \lambda_m(\gamma), 0)\|_{\infty} - \lambda \\
f(\gamma) &= \sum_{m \in M} \||b_m| - \max(|b_m| - \lambda_m(\gamma), 0)\|_{\infty} - \tau.
\end{align*}
\]

Now, we turn to the analysis of

\[
\beta(b_m) = \| |b_m| - \max(|b_m| - \lambda_m(\gamma), 0)\|_{\infty}.
\]
We will denote by $b_{im}$ the $i^{th}$ component of the vector $b_m$. As all the components of this vector are positive, we can write

$$\beta(b_m) = \max_i(|b_{im}| - \max(|b_{im}| - \lambda_m(\gamma), 0)) . \quad (37)$$

For each component, we have:

$$|b_{im}| - \max(|b_{im}| - \lambda_m(\gamma), 0) = \begin{cases} 
\lambda_m(\gamma) & \text{if } |b_{im}| > \lambda_m(\gamma) \\
|b_{im}| & \text{if } |b_{im}| \leq \lambda_m(\gamma) 
\end{cases} . \quad (38)$$

Now, we assert that there exists at least one element $b_{jm}$ such that $|b_{jm}| > \lambda_m(\gamma)$. To prove this, suppose that $|b_{im}| \leq \lambda_m(\gamma)$ for all $i$. Substituting this assumption into (38), we would have that $a_m^*$ is zero. As we are only considering terms corresponding to $\|b_m\|_1 \geq \gamma$ we arrive to a contradiction.

Then, as there exist $b_{jm}$ such that $|b_{jm}| > \lambda_m(\gamma)$

$$|b_{jm}| - \max(|b_{jm}| - \lambda_m(\gamma), 0) = \lambda_m(\gamma) .$$

All other elements are in turn less or equal to $\lambda_m(\gamma)$. From this, we conclude that $\beta(b_m) = \lambda_m(\gamma)$. Thus, we rewrite (35) as

$$f(\gamma) = \sum_{m \in M} \lambda_m(\gamma) - \tau . \quad (39)$$

As outlined in equation (24) (note that here the sub-indexes have a different interpretation), $\lambda_m(\gamma)$ can be expressed as

$$\lambda_m(\gamma) = \frac{z_k^Tb_m - \gamma}{z_k^Tz_k} , \quad (40)$$

where

$$z_m = \text{sign}(b_m) \odot I_{|b_m|<\lambda_m(\gamma)} ,$$

so that (39) becomes

$$f(\gamma) = \sum_{m \in M} \frac{z_m^Tb_m - \lambda}{z_m^Tz_m} - \tau . \quad (41)$$

Both $z_m$ and $M$ depend on $\gamma$, however, similar to the derivation for the Michelot algorithm for $\ell_1$-ball projection [28] presented in Section III-B, we temporarily disregard these dependences and approximate the derivative of $f$ as

$$\frac{\partial f(\gamma)}{\partial \gamma} \approx -\sum_{m \in M} \frac{1}{z_m^Tz_m} . \quad (42)$$
Thus, the updates of the root-finding procedure can be performed in a Newton-like fashion by setting:

$$\gamma_{n+1} := \gamma_n + \frac{f(\gamma)}{\sum_{m \in \mathcal{M}} z_m^m z_m}.$$  

(43)

Our numerical tests suggest that, if we update $z_k$ at each iteration, this approximation is good enough for use in a Newton root search method. Similarly to the re-derivation of the Michelot algorithm presented in [28], our method can be understood as a quasi-Newton method in the broad-sense of the term. However, we note that it cannot be readily derived from the application of classical quasi-Newton schemes for root-finding, such as Broyden’s or Brent’s method [29].

C. Initial Point

From here on, we suppose that

$$\|B\|_{\infty,1} = \sum_{m=1}^{M} \|b_m\|_\infty > \tau .$$  

(44)

If $\|B\|_{\infty,1} \leq \tau$ in (3) then the optimal solution is trivial, $X^* = B$. We try to find a point $\gamma_0$ such that $f(\gamma_0) > 0$ in (31). Then, as $f(\cdot)$ satisfies the Fourier conditions and is therefore non-increasing in the $[0, \gamma^*]$ interval, we can conclude that $0 \leq \gamma_0 \leq \gamma^*$.

We start by assuming that the $\ell_1$-norm of the $j$th row of the solution $A^*$ coincides with $\|A^*\|_{1,\infty}$, i.e., $\max_m \{\|a_m\|_1\} = \|a_j\|_1$. Then, via (29), we can find $a_j$ as:

$$a_j = \arg \min_a \frac{1}{2} \|a - b_j\|^2_2 + \tau \|a\|_1 = \text{shrink}(b_j, \tau) .$$  

(45)

We define $\gamma_0 = \|\text{shrink}(b_j, \tau)\|_1$ and proceed to show that $f(\gamma_0) > 0$. Separating the sum and using the definition of the shrinkage operator in (31), we can write:

$$f(\gamma_0) = \sum_{m=1}^{M} \|b_m - a_m(\gamma)\|_\infty - \tau$$  

(46)

$$f(\gamma_0) = \|b_j - \text{sign}(b_j) \odot (|b_j| - \max(|b_j| - \tau, 0))\|_\infty$$  

$$+ \sum_{m \neq j} \|b_m - \text{proj}_{\|\cdot\|_1}(b_m, \gamma_0)\|_\infty - \tau .$$  

(47)

We can reduce this expression further by noting that $b_j$ can be rewritten in the form $\text{sign}(b_k) \odot |b_k|$ and factorize:

$$f(\gamma_0) = \tau_j(b_j) + \sum_{m \neq j} \|b_m - \text{proj}_{\|\cdot\|_1}(b_m, \gamma_0)\|_\infty - \tau ,$$  

(48)
where
\[ \tau_j(b_j) = \| |b_j| - \max(|b_j| - \tau, 0) \|_\infty. \tag{49} \]

We now turn to the analysis of \( \tau_j(b_j) \). As all the components involved in \( \tau_j(b_j) \) are positive, we can write the norm as the maximum of all the components of the vector. For each component, we have:
\[ \| b_j^{(i)} | - \max(|b_j^{(i)}| - \tau, 0) = \begin{cases} \tau & \text{if } |b_{ij}| > \tau \\ |b_{ij}| & \text{if } |b_{ij}| \leq \tau. \end{cases} \tag{50} \]

If we know assume that \( \| b_j \|_\infty > \tau \), as this is a simple algorithmic check included in our method (See Section IV-D), then at least one of the components of \( b_j \) must fulfill the first condition in (50). For at least this component, we have
\[ |b_{ij}| - \max(|b_{ij}| - \tau, 0) = \tau, \tag{51} \]
and all the other components are less than or equal to \( \tau \). Accordingly,
\[ \| |b_j| - \max(|b_j| - \tau, 0) \|_\infty = \tau. \]
Replacing this in (48), we obtain
\[ f(\gamma_0) = \sum_{m \neq j} \| b_m - \text{proj}_{\| \cdot \|_1}(b_m, \gamma_0) \|_\infty > 0. \tag{52} \]

Thus, \( f(\gamma_0) > 0 \) and, instead of starting the root search from 0, we can start from \( \gamma_0 \), which is a better initial guess of \( \gamma^* \). A priori, we do not know which \( j \) is closer to the real maximum. In order to obtain the initial point \( \gamma_0 \), we solve (45) for every row and then take among these solutions the one with the maximum \( \ell_1 \)-norm.

\section*{D. Proposed method}

The full proposed method is presented in Algorithm 1 and depicted for a specific example in Figure 1. In line 2, \( B \) is checked to see if it is already in the \( \ell_{\infty, 1} \)-ball. Lines 4 and 5 compute the initial guess of the solution as outlined in Section IV-C and the initialization Block of Figure 1. Note that if \( \| B \|_{\infty, \infty} = \max_{i,j} b_{i,j} < \tau \), lines 4 and 5 do not need to be evaluated and \( \gamma \) is assigned an initial value of 0. Likewise, the shrinkage operation is performed solely for the rows whose \( \ell_{\infty} \)-norm is greater than \( \tau \).

Line 10 corresponds to the pruning step described in Section IV-B, where all rows of \( B \) with \( \ell_1 \) norm less than the current \( \gamma \) are discarded. This is also illustrated in the bottom part of the
In initialization, $\eta = \{||b_k||_1\}$ and $\gamma_0 = \max\{||\text{shrink}(b_k, \tau)||_1\}$ for $k < 0$.

- Prune $\eta_k < \gamma_0$
- Prune $\eta_k < \gamma_1$
- Prune $\eta_k < \gamma_2$

Newton updates are done via Newton's method, converging to the root of the function.

Final solution $X^*$

V. MULTI-TASK LASSO

The Multi-task LASSO (MTL) problem will be used for testing our method on an application involving real data. Given $K$ tasks, each of length $N$ ordered in a $NK \times 1$ vector $b$, and a coefficient matrix $W \in \mathbb{R}^{N \times M}$, we want to find the matrix $X \in \mathbb{R}^{M \times K}$ of features that solves

$$\min_X \frac{1}{2} ||b - P\text{vec}(X)||_2^2$$

subject to $||X||_{1,\infty} \leq \tau$, (53)

where $P = I_N \otimes W$. The solution of this problem will tend to have few non-zero rows, i.e. selected features. It can be solved by means of projected gradient descent (PGD) [30, Chapter
Algorithm 1: Proposed method via root-finding

Input: matrix $B$, $\tau$, maxIter, tolerance

1. if $\|B\|_{\infty,1} \leq \tau$ then
2. return $B$
3. if $\|B\|_{\infty,\infty} > \tau$ then
4. Compute $\alpha_k = \|\text{shrink}(b_k, \tau)\|_1$ for each row of $B$.
5. Define $\gamma = \max_k(\alpha_k)$
6. else
7. $\gamma = 0$.
8. end

9. for $k = 1 : \text{maxIter}$ do
10. Prune the rows of $B$ that have $\ell_1$-norm less than $\gamma$.
11. Obtain $f(\gamma)$ as defined in (31) and $\frac{\partial f(\gamma)}{\partial \gamma}$ as defined in (42).
12. if $|f(\gamma)| < \text{tolerance}$ then
13. break.
14. end
15. Update $\gamma$ using Newton method.

16. Solve for $A$ in (30) with the obtained $\gamma$.
17. Return $B - A$

3], which is shown in Algorithm 2. The method consists of alternating unconstrained gradient descent steps and projections into the $\|X\|_{1,\infty} \leq \tau$ ball. (53) is a convex problem and thus it can be solved using other, more general, optimization methods such as interior point methods [9] or methods based on the augmented Lagrangian function [31].

In this case, projection operator will be a routine that solves (3) by means of our proposed method or via one of the methods in the literature.

VI. Results

All tests presented below were computed using single-threaded Matlab code running on an Intel i7-4770K CPU (8 cores, 3.50 GHz, 32GB RAM). In our simulations with synthetic data (Sections VI-A and VI-B), matrix $B$ was generated using a uniform distribution $[-0.5, 0.5]$, and $\tau$, the constraint used in (2), was taken such that $\tau = \alpha \|B\|_{\infty,1}$, where $\alpha$ is a small constant.
Algorithm 2: Projected gradient descent

**Input:** vector \( b \), `maxIter`, tolerance \( \varepsilon \), initial solution \( X^{(0)} \)

1. for \( k = 1 : \maxIter \) do
   2. Compute the current gradient \( g = P^T(P_{\text{vec}}(x^{(k-1)}) - b) \)
   3. Select step size \( \alpha \)
   4. Compute \( z^{(k)} := z^{(k-1)} - \alpha g \)
   5. \( X^{(k)} := \) Solve for \( X \) in (3) considering \( B = \text{vec}^{-1}(z^{(k-1)}) \) and using the procedure of Algorithm 1.
   6. if \( \|X^{(k)} - X^{(k-1)}\|_F < \varepsilon \) then
      7. break

Specific sizes of \( B \) and values of \( \alpha \) are mentioned below. Our Matlab code [32] can be used to reproduce our experimental results.

A. Impact of initial point

In order to study the impact of the initial point \( \gamma_0 \) on the performance of the algorithm, we constructed 100 different realizations of a \( 2000 \times 100 \) \( B \) matrix, considering \( \alpha \in [10^{-4}, 10^{-3}] \). For each value of \( \tau \), the values for the initial point \( \gamma_0 \) and the optimal value \( \gamma^* \) were averaged across the 100 realizations. These average values for each \( \tau \) are shown in Figure 2(a). It is observed that, at low \( \alpha \) values, \( \gamma_0 \) is very close to the optimal value, but it goes rapidly to zero as \( \tau \) increases. On the other hand, Figure 2(b) shows a comparison of the average number of iterations that the proposed method needs for arriving to the optimal value starting from either zero or \( \gamma_0 \). The number of iterations and computational time for the different \( \tau \) values, for the proposed method without pruning, along with the improvements provided by using \( \gamma_0 \), are listed in Table I.

B. Simulations

We compare both of our proposed method (denoted Proposed) against [6], denoted as general root-finding (GRF), and [7], [32], denoted as Steffensen. Unfortunately, we could not find a

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1These sizes and sparsity values are typical for known applications of (2) [4], [5], [6]. Results for larger values of \( \alpha \) can be obtained with our source code [32].
### Table I
Computational results comparing the effect of the initial point $\gamma_0$. The percent change from the zero-start case is shown in parenthesis for the $\gamma_0$ case. See Section VI-A.

| $\alpha \times 10^{-3}$ | Starting at zero | Starting at $\gamma_0$ |
|-------------------------|------------------|-----------------------|
| sparsity(%)             | num iter | time(s)  | num iter | time(s)  |
| 0.1 / 1.02              | 12.6     | 0.5      | 9.4      | (-25.0%) | 0.18     | (-65.0%) |
| 0.2 / 1.92              | 12.2     | 0.5      | 9.7      | (-19.9%) | 0.24     | (-52.4%) |
| 0.3 / 2.68              | 11.9     | 0.5      | 10.0     | (-16.7%) | 0.28     | (-44.5%) |
| 0.4 / 3.40              | 11.7     | 0.5      | 10.2     | (-12.8%) | 0.34     | (-32.2%) |
| 0.5 / 4.17              | 11.5     | 0.5      | 11.3     | (-2.0%)  | 0.48     | (-5.0%)  |
| 0.6 / 4.94              | 11.3     | 0.5      | 11.3     | (0.0%)   | 0.52     | (0.3%)   |
| 0.7 / 5.53              | 11.1     | 0.5      | 11.1     | (0.0%)   | 0.52     | (0.7%)   |
| 0.8 / 6.28              | 11.1     | 0.5      | 11.1     | (0.0%)   | 0.52     | (0.7%)   |
| 0.9 / 6.89              | 11.0     | 0.5      | 11.0     | (0.0%)   | 0.52     | (0.6%)   |
| 1.0 / 7.53              | 11.0     | 0.5      | 11.0     | (0.0%)   | 0.52     | (0.6%)   |

### Table II
Results for simulations with matrices of different size and the three tested methods. Error (Err.), number of iterations (N.I.) and running times are shown for each of them. Speedup with respect to GRF is shown for Steffensen and Proposed.

| Matrix Size | $\alpha$ / sp | GRF [6] | Steffensen [7] | Proposed |
|-------------|---------------|---------|----------------|----------|
|             | Err. | N.I. | Time(s) | Err. | N.I. | Time(s) | Speedup | Err. | N.I. | Time(s) | Speedup |
| 2000 x 100  | 0.0001/ 1.02 | 3.4e-11 | 9.6 | 3.99 | 2.6e-12 | 9.4 | 0.28 | 14.25 | 2.0e-16 | 9.4 | 0.18 | 22.17 |
|             | 0.0005/ 4.13 | 1.6e-10 | 13.2 | 4.11 | 1.4e-12 | 11.3 | 0.83 | 4.98 | 7.4e-16 | 11.3 | 0.48 | 8.65 |
|             | 0.001/ 7.51 | 5.1e-10 | 14.5 | 4.23 | 1.3e-12 | 11.0 | 0.80 | 5.31 | 14.25 | 11.0 | 0.51 | 8.63 |
| 5000 x 200  | 0.0001/ 1.37 | 1.8e-10 | 16.9 | 13.60 | 1.6e-12 | 10.6 | 1.28 | 10.63 | 1.0e-15 | 10.6 | 0.78 | 17.44 |
|             | 0.0005/ 5.59 | 3.5e-10 | 17.9 | 13.59 | 7.9e-13 | 12.0 | 2.41 | 5.64 | 2.3e-15 | 12.0 | 1.54 | 8.82 |
|             | 0.001/ 10.03 | 7.8e-10 | 17.9 | 13.46 | 6.3e-13 | 11.1 | 2.45 | 5.49 | 4.6e-15 | 11.1 | 1.56 | 8.63 |
| 10000 x 300 | 0.0001/ 1.62 | 5.0e-10 | 11.4 | 29.41 | 9.8e-14 | 13.0 | 5.76 | 5.11 | 1.9e-15 | 13.0 | 3.32 | 8.86 |
|             | 0.0005/ 6.6  | 2.2e-09 | 14.5 | 30.28 | 7.0e-13 | 12.0 | 5.66 | 5.35 | 9.0e-15 | 12.0 | 3.63 | 8.34 |
|             | 0.001/ 11.88 | 3.3e-09 | 15.7 | 30.67 | 2.8e-12 | 11.4 | 5.78 | 5.31 | 1.9e-12 | 11.4 | 3.71 | 8.27 |
| 10000 x 8000| 0.0001/ 4.24 | 6.0e-10 | 20.0 | 149.21 | 3.4e-14 | 13.5 | 25.5 | 5.84 | 4.3e-15 | 12.99 | 15.57 | 9.59 |
|             | 0.0005/ 16.58 | 5.7e-10 | 19.0 | 141.05 | 9.0e-14 | 12.3 | 23.7 | 5.95 | 2.0e-12 | 11.95 | 16.20 | 8.71 |
|             | 0.001/ 28.51 | 7.0e-10 | 18.9 | 140.19 | 2.9e-14 | 12.0 | 24.9 | 5.63 | 5.5e-14 | 11.01 | 16.69 | 8.40 |
| 10000 x 8000| 0.0001/ 6.18 | 7.0e-09 | 20.1 | 319.13 | 5.6e-14 | 13.4 | 53.80 | 5.93 | 1.5e-12 | 13.0 | 33.72 | 9.46 |
|             | 0.0005/ 23.72 | 2.3e-08 | 19.2 | 316.35 | 1.0e-13 | 12.4 | 53.88 | 5.87 | 2.4e-12 | 12.0 | 37.54 | 8.43 |
|             | 0.001/ 39.90 | 2.4e-08 | 18.0 | 317.97 | 4.0e-14 | 12.0 | 58.01 | 5.48 | 5.1e-14 | 11.0 | 39.46 | 8.06 |
public implementation of [6], and thus, we coded our own Matlab version using the \texttt{fzero}
function as root search method as suggested in [6].

We chose $\text{tol}_c = 10^{-12}$ and $\text{tol}_u = 10^{-8}$ in (15) for the Steffensen root search method. The projections onto the $\ell_1$-ball, needed for the evaluation of the search function in [6] and in our algorithm, are implemented using the Michelot algorithm [33]. This algorithm was chosen since it can be implemented efficiently [24, Section 3.2.2], [25, Section 3.1] and, for small size projections, we have empirically observed that it has better computational performance than the alternatives mentioned in Section III-B.

We simulated five different sizes for the matrix $B$, namely $\times 100$, $5000 \times 200$, $10000 \times 300$, $10000 \times 3000$ and $10000 \times 8000$. 100 realizations of $B$ were taken. We also considered $\alpha \in \{10^{-4}, 5 \times 10^{-4}, 10^{-3}\}$, to experimentally obtain approximate sparsity percentages (percentage of non-zero rows) of 1, 5 and 10%, respectively.

For all simulations we recorded the error, measured as the constraint violation $\|X\|_{\infty,1} - \tau$, the number of iterations, and the total time until convergence. We additionally computed a sparsity value as the percentage of non-zero rows. As all methods arrive to the same value of sparsity, only a single value is shown for each $\alpha$. The results averaged over the 100 realizations for the different matrix dimensions are shown in Table II and the average speedups are shown in Figure 3.

\section{Comparisons on fMRI LASSO application}

We tested the computational improvements of our algorithm in the cognitive task described in [34]. The applications consists of predicting the neural functional magnetic resonance image (fMRI) response associated to a particular word based on co-occurrence features of this word with a dictionary of words whose response is already know. In [34], the co-occurrence with a
Fig. 3. Bar plot showing the average speedup with respect to GRF for Steffensen (blue) and the proposed method (orange) for the different matrix size and constraint values.

Fig. 4. Time (in seconds) per PGD iteration for each of the methods for solving the $\ell_{\infty,1}$ projection problem.

### TABLE III

| $\alpha$ / sp | GRF [6]  | Steffensen [7] | Proposed  |
|---------------|-----------|----------------|-----------|
|               | Time(s)   | Time(s)       | speedup   | Time(s)   | speedup   |
| 0.03/12.5%    | 869.52    | 6.39          | 136.08    | 6.60      | 131.75    |
| 0.05/31.4%    | 895.87    | 6.65          | 134.72    | 6.54      | 136.98    |
| 0.1/ 58.1%    | 808.15    | 6.58          | 122.82    | 6.54      | 123.57    |
hand-crafted set of 25 verbs was used as features in the prediction problem, whereas [5] showed improvements by using a larger dictionary and MTL to select the best features.

For our tests, we selected the 18 noun words with their corresponding fMRIs and used co-occurrence values of these with a dictionary of 10000 words gathered from Wikipedia and BBC, which was obtained from [35]. These set of 18 words was selected because the other words of the dataset were not present in the corpus used for the co-occurrence matrix calculation.

We subsampled the fMRIs by half resulting in approximate problem dimension of $K = 10000$, $N = 18$ and $M = 10000$. $\alpha$ values of 0.03, 0.05 and 0.1 were considered in order to obtain sparsity values of 12.5%, 31.4% and 58.1%, respectively.

The improvements that can be obtained with the use of mixed norms in MTL has already been demonstrated [5], so we focus only on computational metrics. We compare the time needed to solve (53) by using GRF, Steffensen and the proposed method as the projection operators in algorithm 2. For PGD, we use the minConf Matlab library [36], [37] which uses an Armijo inexact line search [21] for choosing the step size. We consider a fixed maximum of 40 iterations (due to the long computational times). The results per each of the iterations are shown in Figure 4, and the median time and speedup are shown in Table III.

D. Discussion

As can be observed from the results of Section VI-A, the initial point $\gamma_0$ has impact only at low $\alpha$ (and therefore $\tau$) values. This is easily explainable, as at high $\tau$ values the solution of (45) is zero. Accordingly, as suggested in Section IV-D, it is better to first evaluate the conditions on $\max_{n,k} \{b_{n,k}\}$ and $\|b_i\|_\infty$ to avoid unnecessary shrinkage operations, as these comparisons do not incur a great computational cost. When the initial point is different from zero, we see that the number of iterations and the computational time are slightly reduced. For our tests, when $\gamma_0$ is not zero, an average reductions of 2.2 iterations ($-15\%$) and of 65% in time are achieved.

As indicated by the results of Section VI-B, the proposed Newton method tends to require fewer iterations than GRF [6], although this is not directly comparable as the stopping criteria of the fzero function is different from the one we are using in Algorithm 1, and because our proposed algorithms are obtaining smaller errors. Additionally, the Newton-based method performs the same number of iterations as the Steffensen-based one [7], but the computational time is substantially lower. This is explained by the fact that in the Newton-based method, only one function evaluation is performed compared to the two evaluations performed for Steffensen.
Overall, the proposed Newton-based algorithm obtains speedups of 8 or more with respect to Sra (compared to the average of 5 obtained with Steffensen). Higher speedups are obtained at low $\alpha$ (sparser) values, as this is where the initial point guess is more effective [7]. For those cases, the speedup of our proposed method goes up to $14 \sim 25$.

Regarding the fMRI experiments, we observe that both our previous Steffensen-based method and our novel Newton-based methods obtain considerable speedups of $120 \sim 130$. This speedup reduced the total computational time for the whole PGD method from around 10 hours (GRF) to approximately 3 minutes (Steffensen and Proposed). The difference in performance with respect to the simulations is explained by the data distribution. As observed in Figure 5, the 1-norms of the rows of the data follow a Laplacian-like distribution, while the simulations considered uniformly distributed data. Accordingly, in the fMRI data, the pruning strategy mentioned in Section IV-B is able to greatly reduce the number of rows to be processed. Furthermore, the initial guess $\gamma_0$ obtained with the Steffensen and proposed methods is very close to the optimal $\gamma^*$, and so we noticed that only around two root-search iterations were needed for each projection. Although for this particular dataset the proposed method performs comparably to Steffensen’s, the simulations showed that for setting where the data is more uniformly distributed the difference in computation can be considerable.

VII. CONCLUSION

We have presented a new algorithm for efficient projection onto the $\ell_{\infty,1}$-norm ball which exploits the particular structure of this problem to improve over previous methods. The algorithm, based on a Newton root-finding approach, capitalizes on an approximation of the derivative of
the search function obtained from Moreau’s decomposition and duality theory. Our proposed method obtains speedups of eight or more with respect to previous state-of-the-art methods, while achieving smaller constraint violation errors in our simulations. These simulations can reproduced with our publicly available code [32]. When we applied our proposed algorithm to a multi-task Lasso (MTL), which used real fMRI data, we obtained considerable speedups of $120 \sim 130$. In [32], we also provide the data and code necessary for reproducing the experiment. Furthermore, the MTL test also highlights the impact of two key aspects of our proposed algorithm, namely our initial guess and pruning steps. When the distribution of the data is favorable, such steps have a very significant positive impact on the overall computational performance of our proposed algorithm.

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