On the Inequalities of Projected Volumes and the Constructible Region

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Abstract

We study the following geometry problem: given a $2^n-1$ dimensional vector $\pi = \{\pi_S\}_{S \subseteq [n], S \neq \emptyset}$, is there an object $T \subseteq \mathbb{R}^n$ such that $\log(\text{vol}(T_S)) = \pi_S$, for all $S \subseteq [n]$, where $T_S$ is the projection of $T$ to the subspace spanned by the axes in $S$? If $\pi$ does correspond to an object in $\mathbb{R}^n$, we say that $\pi$ is constructible. We use $\Psi_n$ to denote the constructible region, i.e., the set of all constructible vectors in $\mathbb{R}^{2^n-1}$. In 1995, Bollobás and Thomason showed that $\Psi_n$ is contained in a polyhedral cone, defined a class of so called uniform cover inequalities. We propose a new set of natural inequalities, called nonuniform-cover inequalities, which generalize the BT inequalities. We show that any linear inequality that all points in $\Psi_n$ satisfy must be a nonuniform-cover inequality. Based on this result and an example by Bollobás and Thomason, we show that constructible region $\Psi_n$ is not even convex, and thus cannot be fully characterized by linear inequalities. We further show that some subclasses of the nonuniform-cover inequalities are not correct by various combinatorial constructions, which refutes a previous conjecture about $\Psi_n$. Finally, we conclude with an interesting conjecture regarding the convex hull of $\Psi_n$.

1 Introduction

We use notations introduced in [3].

Let $T$ be an object in $\mathbb{R}_+^n$ and let $\{v_1, \cdots, v_n\}$ be the standard basis of $\mathbb{R}^n$. By an object, we mean a bounded compact subset of $\mathbb{R}_+^n$. We let $\text{Span}(S)$ denote the subspace spanned by $\{v_i \mid i \in S\}$. Given an index set $S \subseteq [n] = \{1, 2, \cdots, n\}$ with $|S| = d$, we denote by $T_S$ the orthogonal projection of $T$ onto $\text{Span}(S)$, and by $|T_S|$ its $d$-dimensional volume. We use $|T|$ to denote the $n$-dimensional volume of $T$. Given an $n$-dimensional object $T$, define $\pi(T)$ to be the log-projection vector of $T$, which is a $2^n-1$ dimensional vector with entries indexed by subsets of $[n]$ and $\pi(T)_S = \log |T_S|$ for all $S \subseteq [n]$ (we use the convention that $\log 0 = -\infty$). Whenever we refer to a $2^n-1$ dimensional vector $\pi$, we assume that the entries are indexed by the subsets of $[n]$ (i.e., $\pi_S$ is the entry index by $S \subseteq [n]$). We say that a $2^n-1$ dimensional vector $\pi$ is constructible if $\pi$ is the log-projection vector of some object $T$ in $\mathbb{R}^n$. Let us define the constructible region $\Psi_n$, the central subject studied in this paper, to be the set of all constructible vectors:

$$\Psi_n = \{\pi \in \mathbb{R}^{2^n-1} \mid \pi \text{ is constructible}\}.$$  

Having the above definitions, it is natural to ask the following questions:

1. Given a $2^n-1$ dimensional vector $\pi$, is there an algorithm to decide whether $\pi$ is in $\Psi_n$?

2. What does $\Psi_n$ look like? What property does $\Psi_n$ have?
In 1995, Bollobás and Thomason proposed a class of inequalities relating the projected volumes. Their result reads as follows. Let $\mathcal{A}$ be a family of subsets of $[n]$. We say $\mathcal{A}$ is a $k$-cover of $[n]$, if each element of $[n]$ appears exactly $k$ times in the multiset induced by $\mathcal{A}$. For example, $\{1,2\}, \{2,3\}, \{1,3\}$ is a 2-uniform cover of $\{1,2,3\}$.

**Theorem 1.** (Bollobás-Thomason (BT) uniform-cover inequalities) Suppose $T$ is an object in $\mathbb{R}^n$ and $\mathcal{A}$ is a $k$-cover of $[n]$. Then, we have that $|T|^k \leq \prod_{A \in A} |TA|$.

With the above notations, we define the polyhedron cone

$$BT_n = \{ \pi \in \mathbb{R}^{2^n-1} \mid k\pi_S \leq \sum_{A \in A} \pi_A, \text{ for all } k \text{ and } \mathcal{A} \text{ that } k\text{-covers } S, S \subseteq [n] \}.$$ 

BT inequalities essentially assert that every constructible vector is in $BT_n$, or equivalently $\Psi_n \subseteq BT_n$. In the very same paper, they also presented a non-constructible point in $BT_4$, which immediately implies that $\Psi_4 \not\subseteq BT_4$. However the above result does not rule out the possibility that $\Psi_n$ is convex, or even can be characterized by a finite set of linear inequalities.

### 1.1 Our Results

Except the results mentioned above, very little is known about $\Psi_n$ and the main goal of this paper is to deepen our understanding about its structure. First, we propose a new class of natural inequalities, called nonuniform-cover inequalities, which generalize the BT uniform-cover inequalities. We need a few notations first.

Let $\mathcal{A} = \{A_i\}_{i=1}^k, \mathcal{B} = \{B_j\}_{j=1}^m$ be two families of subsets of $[n]$, where $A_i$s and $B_j$s are subsets of $[n]$. We say $\mathcal{A}$ covers $\mathcal{B}$ if the following properties hold:

1. The disjoint union of $\{A_i\}_{i=1}^k$ is the same as the disjoint union of $\{B_j\}_{j=1}^m$. In other words, for every element $e \in [n], |\{i \mid e \in A_i\}| = |\{j \mid e \in B_j\}|$.

2. Let $\Sigma = \{(A_i,t) \mid t \in A_i\}$ and $\Gamma = \{(B_j,s) \mid s \in B_j\}$, and there is an one-to-one mapping $f$ between $\Sigma$ and $\Gamma$ such that: for any $(A_i,t) \in \Sigma$ with $(B_j,s) = f(A_i,t), t = s$ and $A_i \subset B_j$.

**Definition 1.** (Nonuniform-Cover (NC) inequalities) $x$ is a $2^n-1$ dimensional vector. Suppose $\mathcal{A}$ covers $\mathcal{B}$. A nonuniform-cover inequality is of the following form:

$$\prod_{A_i \in \mathcal{A}} x_{A_i} \geq \prod_{B_j \in \mathcal{B}} x_{B_j}.$$ 

**Example 1.** Let $\mathcal{A} = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and $\mathcal{B} = \{\{1,2,3\}, \{2,3,4\}\}$. We can see $\mathcal{A}$ covers $\mathcal{B}$. The corresponding NC inequality is $x_{\{1,2\}} \cdot x_{\{2,3\}} \cdot x_{\{3,4\}} \geq x_{\{1,2,3\}} \cdot x_{\{2,3,4\}}$. Here is another example: $x_{\{1\}} \cdot x_{\{1,2\}} \cdot x_{\{2,3\}} \cdot x_{\{3,4\}} \cdot x_{\{2,4\}} \geq x_{\{1,2,3\}} \cdot x_{\{2,3,4\}} \cdot x_{\{1,2,4\}}$.

When the context is clear, we refer to a linear inequality of the form $\sum_{B} \pi_{B_j} \leq \sum_{A} \pi_{A_i}$ as an NC inequality as well. It is easy to see that that every BT inequality is an NC inequality. But the converse may not be true. For example, $x_{\{1,2\}} \cdot x_{\{2,3\}} \cdot x_{\{3,4\}} \geq x_{\{1,2,3\}} \cdot x_{\{2,3,4\}}$. (We alert the reader that we do not claim such inequalities are always true. We will discuss it in detail in Section [4])

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1 A subset of $[n]$ may appear more than one times in $\mathcal{A}$ or $\mathcal{B}$
Similar to $\mathcal{BT}_n$, we define $\mathcal{NC}_n$ to be the set of all points that satisfies all NC inequalities: Formally, it is the following polyhedron cone:

$$\mathcal{NC}_n = \{ \pi \in \mathbb{R}^{2^n-1} \mid \sum_{B_i \in \mathcal{B}} \pi_{B_i} \leq \sum_{A_i \in \mathcal{A}} \pi_{A_i}, \text{ for all } \mathcal{A}, \mathcal{B} \text{ such that } \mathcal{A} \text{ covers } \mathcal{B} \}.$$ 

Our first result states that all correct linear inequalities should be in this class.

**Theorem 2.** If all points in $\Psi_n$ satisfy a certain linear inequality $\sum_{S \subseteq [n]} \alpha_S \pi_S \leq 0$, the linear inequality must be an NC inequality, or a positive combination of NC inequalities.

In order to prove the above theorem, we introduce a class of objects called rectangular flowers. We let $\mathcal{RF}_n$ to denote all possible log-projection vectors that can be generated by rectangular flowers (see the definition in Section 2). We show that for any linear inequality that is not an NC inequality, we can construct a rectangular flower which violates the inequality. It is simple to show that a log-projection vector of a rectangular flower in $\mathbb{R}^n$ satisfies all nonuniform cover inequalities (i.e., it is in $\mathcal{NC}_n$). Moreover, we show that for every point $\pi \in \mathcal{NC}_n$, there is a rectangular flower in $\mathbb{R}^n$ whose log-projection vector is $\pi$. Therefore, we can prove the following theorem.

**Theorem 3.** For all $n \geq 1$, $\mathcal{NC}_n = \mathcal{RF}_n \subseteq \Psi_n$.

Given Theorem 3 it is natural to ask whether $\mathcal{NC}_n = \Psi_n$. If the answer was yes, $\Psi_n$ would have a compact description and deciding whether a point is in $\Psi_n$ can be done using linear programming (see Section 2 for the details). However, the answer is not that simple. In fact, using Theorem 2 we can show our next result which states that $\Psi_n$ is not even convex for $n \geq 4$. We note that for $n = 1, 2, 3$, $\Psi_n = \mathcal{BT}_n$, thus convex. For completeness, we provide a proof in Appendix A.

**Theorem 4.** (Non-convexity of $\Psi_n$) For $n \geq 4$, $\Psi_n$ is not convex.

Theorem 4 implies that there exist certain constructible vector in $\mathbb{R}^{2^n-1}$ which violates some NC constraint. In other words, $\mathcal{NC}_n \subsetneq \Psi_n$. Thus it would be interesting to know which NC inequalities are true and which are false (we already know BT inequalities are true). In Section 4 we provide several methods for constructing counterexamples for different subclasses of NC inequalities. However, we have not been able to disprove all NC inequalities that are not BT inequalities, nor prove any of them. This leads us to conjecture the following.

**Conjecture 1.** If all points in $\Psi_n$ satisfy a certain linear inequality $\sum_j \beta_j \pi_{B_j} \leq \sum_i \alpha_i \pi_{A_i}$, the linear inequality must be a BT inequality or a positive combination of several BT inequalities. Moreover, $\mathcal{BT}_n = \text{Conv}(\Psi_n)$, the convex hull of $\Psi_n$.

At the end of the introduction, we summarize our results in the following chain:

$$\mathcal{RF}_n = \mathcal{NC}_n \subsetneq \Psi_n \subsetneq \text{Conv}(\Psi_n) \subsetneq \mathcal{BT}_n,$$

and we conjecture that $\text{Conv}(\Psi_n) = \mathcal{BT}_n$. 

3
1.2 A Motivating Problem from Databases

Our problem is closely related to the data generation problem [1] studied in the area of databases, which is in fact our initial motivation for studying the problem. Generating synthetic relation under various constraints is a key problem for testing data management systems. A relation $R(A_1, \ldots, A_n)$ is essentially a table, where each row is one record about some entity, and each column $A_i$ is an attribute. One of the most important operations in relational databases is the projection operation to a subset of attributes. One can think of the projection to subset $S$ of attributes, denoted as $\pi_S(R)$, as the table $R$ first restricted to columns in $S$, and then with duplication removed. To see the connection between the database problem and geometry, we can think a relation $R(A_1, \ldots, A_n)$ with $n$ attributes as an $n$-dimensional object $T$ in $\mathbb{R}^n$: A tuple (i.e., a row) $(t_1, t_2, \ldots, t_m)$ can be thought as a unit cube $[t_1 - 1, t_1] \times \ldots \times [t_m - 1, t_m]$. Then, $T_S$, the projection of $T$ to $\text{Span}(S)$, corresponds to exactly the projected relation $\pi_S(R)$.

**Example 2.** The following table shows the information of course registration. 5 items in the table correspond to unit square in the coordinate system. In this way, a table is represented by an object in Euclidean space.

| Rank | Name | Course |
|------|------|--------|
| 1    | Alice| Math   |
| 2    | Alice| Physics|
| 3    | Alice| Biology|
| 4    | Bob  | Math   |
| 5    | Bob  | Physics|

In the data generating problem with projection constraints, we are given the cardinalities $|\pi_S(R)|$ for a set of subsets $S \subseteq [n]$. The goal is to construct a relation $R$ that is consistent with the given cardinalities. We can see it is a discrete version of our geometry problem. Moreover, if the given cardinalities (after taking logarithm) is not in $\Psi_n$, or violate any projection inequality, there is obviously no solution to the data generation problem. Therefore, a good understanding of our geometry problem is central for solving the data generation problem.

1.3 Other Related Work

Loomis and Whitney proved a class of projection inequalities in [7], allowing one to upper bound the volume of a $d$–dimensional object by the volumes of its $(d-1)$-dimensional projection volumes. Their inequalities are special cases of BT inequalities. BT inequalities and their generalizations also play an essential role in the worst-case optimal join problem in databases (we can get an upper bound of the size of the relation $R$ knowing the cardinalities of its projections). See e.g., [8] for some most recent results on this problem.
There is a large body of literature on the constructible region $\Gamma_n$ for joint entropy function over $n$ random variables $X_1, \ldots, X_n$. More specifically, for each joint distribution over $X_1, \ldots, X_n$, there is a point in $\Pi_n$, which is a $2^n - 1$ dimensional vector, with the entry indexed by $S \subseteq [n]$ being $H(\{X_i\}_{i \in S})$. Characterizing $\Gamma_n$ is one major problem in information theory and has been studied extensively. Many entropy inequalities are known, including Shannon-type inequalities and several non-Shannon-type inequalities. For a comprehensive treatment of this topic, we refer interested readers to the book [9]. There are close connections between entropy inequalities and projection inequalities [2–5]. In particular, BT inequalities can be easily derived from the well known Shearer's entropy inequalities [4] (many even regard them as the same).

2 Proof of Theorem 2 and Theorem 3

In this section, we prove Theorem 2 and Theorem 3. We need to introduce a class of special geometric objects, which are crucial to our proofs. We say an $n$-dimensional object $F \subseteq \mathbb{R}^n_+$ is cornered if $x \in T$ implies $y \in F$ for all $y \leq x$ (i.e., $y_i \leq x_i$ for all $i \in [n]$). An object $R \subseteq \mathbb{R}^n_+$ is said to be an open rectangle if $R = (0, a_1] \times (0, a_2] \times \ldots \times (0, a_n]$, or a closed rectangle if $R = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$.

**Definition 2.** We say $F \subseteq \mathbb{R}^n_+$ is a rectangular flower if

1. $F$ is cornered,
2. $F \cap (0, \infty)^S$ is a open rectangle in $(0, \infty)^S$ for any $S \subseteq [n]$.

Figure 1: (i) A 3-dimensional rectangular flower. (ii) The network flow $N(A, B)$. The dashed line represents the minimum $s$-$t$ cut.

See Figure 1 for an example. It is easy to see that a rectangular flower $F \subseteq \mathbb{R}^n_+$ is a union of $2^n - 1$ closed rectangles $\bigcup_{S \subseteq [n], S \neq \emptyset} F_S$, each $F_S$ being a closed rectangle in Span($S$). Moreover, If $S \subseteq S'$, for any $i \in S$, the edge length of $R_S$ along axis $i$ is no shorter than that of $R_{S'}$ (since $F$ is cornered).

We also need to introduce a new class of inequalities, call fractional nonuniform-cover inequalities, which can be seen as the fractional generalization of NC inequalities. We need some notations first. Let $A = \{(A_i, \alpha_i)\}_{i=1}^k$, $B = \{(B_j, \beta_j)\}_{j=1}^m$ be two families of weighted subsets of $[n]$, where $A_i$s and $B_j$s are subsets of $[n]$ and $\alpha_i > 0$ ($\beta_j$ resp.) is the nonnegative weight associated with $A_i$ ($B_j$ resp.). Construct a network flow instance $N(A, B)$ as follows: Let $\Sigma = \{(A_i, x) \mid x \in A_i, A_i \in A\}$
and $\Lambda = \{(B_j, y) \mid y \in B_j, B_j \in \mathcal{B}\}$ be sets of nodes. Let node $s$ be the source and node $t$ be the sink. There is an arc from $s$ to each node $(A_i, x) \in \Sigma$ with capacity $\alpha_i$. There is an arc from each node $(B_j, y) \in \Lambda$ to $t$ with capacity $\beta_j$. For each pair of $(A_i, x)$ and $(B_j, y)$, there is an arc with capacity $+\infty$ from $(A_i, x)$ to $(B_j, y)$ if $A_i \subseteq B_i$ and $x = y$. We say $\mathcal{A}$ saturates $\mathcal{B}$ if the following properties hold:

C1. For any $x \in [n]$, $\sum_{i=1}^{k} \alpha_i 1(x \in A_i) = \sum_{j=1}^{m} \beta_j 1(x \in B_j)$.

C2. The maximum $s$-$t$ flow (or equivalently, the minimum $s$-$t$ cut) of $\mathcal{N}(\mathcal{A}, \mathcal{B})$ is $\sum_j \beta_j$.

**Definition 3.** (Fractional-Nonuniform-Cover (FNC) inequalities) Suppose $T$ is an object in $\mathbb{R}^n$ and $\mathcal{A}$ covers $\mathcal{B}$. A fractional-nonuniform-cover inequality is of the following form:

$$\prod_{(A_i, \alpha_i) \in \mathcal{A}} |T_{A_i}|^{\alpha_i} \geq \prod_{(B_j, \beta_j) \in \mathcal{B}} |T_{B_j}|^{\beta_j}.\tag{1}$$

When the context is clear, we also refer to linear inequalities of the form $\sum_{A_i \in \mathcal{A}} \alpha_i \pi_{A_i} \geq \sum_{B_j \in \mathcal{B}} \beta_j \pi_{B_j}$ as FNC inequalities.

**Lemma 1.** The set of FNC inequalities (the linear form) is exactly the set of all nonnegative linear combinations of NC inequalities.

**Proof.** It is trivial to see that a nonnegative linear combination of NC inequalities is an FNC inequality. Now, we show the other direction. Fix the dimension to be $n$. Consider an arbitrary FNC inequality $cx \leq 0$. If all entries in $c$ are rational number, the FNC inequality itself is an NC inequality by scaling all coefficients by some integer factor (this is because if all capacities of the network are integral, there is an integral maximum flow). So, we only need to handle the case where some entries of $c$ are not rational. Now, we show that every point in $\mathcal{N}C_n$ is satisfied by $cx \leq 0$. Suppose the contrary that there is point $y \in \mathcal{N}C_n$ but $cy = \epsilon > 0$. However, we claim that there is a sequence of FNC inequalities with rational coefficients \{c(i)x \leq 0\}_i such that $\lim_i c(i) = c$. Hence, we have that, for some sufficiently large $i$, $c(i)y \geq \epsilon/2 > 0$, which renders a contradiction.

Now, we briefly argue why the claimed sequence exists. It is not hard to see that the set of coefficient vectors $c$ corresponding to FNC inequalities is a rational polyhedral cone, defined by the linear constraints C1 and the flow constraint C2, which again can be captured by linear constraints (using auxiliary flow variables). So there is a set $V$ of rational generating vectors and $c$ can be written as a nonnegative combination of these vectors. Suppose $c = V\gamma$, $\gamma \geq 0$ (each column of $V$ is a generating vector). Pick an arbitrary rational nonnegative sequence of vectors $\{\gamma(j)\}_j$ that approach to $\gamma$, and $\{V\gamma(j)\}$ would be the desired sequence.

The rest can be seen from Farka’s Lemma: Let $Ax \leq 0$ be a feasible system of inequalities and $cx \leq 0$ be an inequality satisfied by all $x$ with $Ax \leq 0$. Then, By Farka’s Lemma, $cx \leq 0$ is a nonnegative linear combination of the inequalities in $Ax \leq 0$ (see e.g., [6]).

**Proof of Theorem 2.** We only need to show that all non-FNC inequalities are wrong. Suppose $F$ is an object. Consider an arbitrary non-FNC inequality:

$$\prod_{A} |F_A|^{\alpha_i} \geq \prod_{B} |F_B|^{\beta_j},$$

where $\mathcal{A}$ does not saturate $\mathcal{B}$. We show that we can construct a rectangular flower $F$ that this inequality does not hold.
Consider the network flow instance $\mathcal{N}(A, B)$. Suppose $C_1$ does not hold: for some $x \in [n]$, $\sum_{i=1}^{k} \alpha_i(1 \in A_i) \neq \sum_{j=1}^{m} \beta_j(1 \in B_j)$. First, if $\sum_{i=1}^{k} \alpha_i|A_i| \neq \sum_{j=1}^{m} \beta_j|B_j|$, we can easily see that (1) is false by considering $F = [0, 2]^n$ (log $LHS = \sum_{i=1}^{k} \alpha_i|A_i|$ and log $RHS = \sum_{j=1}^{m} \beta_j|B_j|$).

Now, suppose $\sum_{i=1}^{k} \alpha_i|A_i| = \sum_{j=1}^{m} \beta_j|B_j|$ but $\sum_{i=1}^{k} \alpha_i(1 \in A_i) \neq \sum_{j=1}^{m} \beta_j(1 \in B_j)$ for some $x$. W.l.o.g., assume $x = 1$. Let $F = [0, 2] \times [0, 1]$ and ... Again, it is easy to see (1) is false since log $LHS = \sum_{i=1}^{k} \alpha_i(1 \in A_i)$ and log $RHS = \sum_{j=1}^{m} \beta_j(1 \in B_j)$.

Now, suppose $C_2$ is false, that is the value of the min-cut is less than $\sum_{j=1}^{m} \beta_j$. Suppose the minimum $s-t$ cut defines the partition $(S, T)$ of vertices such that $s \in S$ and $t \in T$. Let $\Sigma$ and $\Lambda$ be defined as above, and $\Sigma_S = \Sigma \cap S$, $\Sigma_T = \Sigma \cap T$, $\Lambda_S = \Lambda \cap S$, $\Lambda_T = \Lambda \cap T$. Since the min-cut is less than $\sum_{j=1}^{m} \beta_j$, none of the above four sets are empty. Clearly, there is no edge from $\Sigma_S$ to $\Lambda_T$ since otherwise the value of the cut is infinity. In other words, $\Lambda_S$ absorbs all outgoing edges from $\Sigma_S$. (See Figure 1 ii).

Moreover, we can see the value of the min-cut is $\sum_{(A_i, x)}(A_i, x) \in \Sigma_S \alpha_i + \sum_{(B_j, y)}(B_j, y) \in \Lambda_T \beta_j$. Since this value is less than $\sum_{(B_j, y)}(B_j, y) \in \Lambda_T \beta_j$, we have that $\sum_{(A_i, x)}(A_i, x) \in \Sigma_S \alpha_i < \sum_{(B_j, y)}(B_j, y) \in \Lambda_T \beta_j$ due to $C_1$. Now, we construct the rectangular flower $F$. Suppose $F = \bigcup_{S \subseteq [n], S \neq \emptyset} F_S$ and we use $F_{S, x}$ to denote the edge length of the close rectangle $F_S$ along axis $x \in S$. We only need to specify all $F_{S, x}$ as follows:

$$F_{S, x} = \begin{cases} t & S \subseteq B_j, \text{ for some } (B_j, y) \in \Lambda_S \\ 1 & \text{otherwise.} \end{cases}$$

Now, we verify that the above rectangular flow $F$ violates the given non-FNC inequality. In fact, we can easily see that for any node $(A_i, x) \in \Sigma_S$, there is a node $(B_j, x) \in \Lambda_S$ and we have that $F_{A_i, x} = t$. Hence,

$$\log \prod_{(A_i, x) \in A} |A_i|^\alpha_i = \log t \sum_{(A_i, x) \in \Sigma_S} \alpha_i.$$ 

On the other hand, we have that

$$\log \prod_{B \in B} |B|^\beta_j \geq \log t \sum_{(B_j, y) \in \Lambda_S} \beta_j,$$

which implies that the given inequality is false. This proves Theorem 2. \hfill \Box

We denote the set of log-projection vectors generated by rectangular flowers to be

$$\mathcal{R}F_n = \{ \pi \in \mathbb{R}^{|2^{n} - 1|} | \pi \text{ is the log-projection vector of some rectangular flower } F \}.$$ 

Now, we prove Theorem 3.

Proof of Theorem 3. Clearly, $\mathcal{R}F_n \subseteq \Psi_n$. We only need to show that $\mathcal{R}F_n = \mathcal{NC}_n$.

We can see that a given vector $\pi$ is the log-projection vector of some rectangular flower in $\mathbb{R}^n$ if the following linear program, denoted as $\text{LP}(\pi)$, is feasible (treating $f_{S,i}$ as variables):

$$\sum_{i \in S} f_{S,i} = \pi_S, \quad \text{for all } S \subseteq [n],$$ 

$$f_{S,i} \geq f_{S',i}, \quad \text{for all } S \subset S' \subseteq [n].$$

Hence, $\mathcal{R}F_n = \{ \pi \in \mathbb{R}^{|2^{n} - 1|} | \text{LP}(\pi) \text{ is feasible} \}$. It is easy to check that $\mathcal{R}F_n$ is a convex cone (i.e., if $\pi_1, \pi_2 \in \mathcal{R}F_n$, $a\pi_1 + b\pi_2 \in \mathcal{R}F_n$ for any $a, b > 0$). In fact, from basic linear programming
fact, $\mathcal{RF}_n$ is a polyhedron cone. In fact, this can be easily seen as follows: We can write LP($\pi$) as the standard matrix form $\{Ax = (\pi, 0), x \geq 0\}$. Obviously, $\{Ax = (y_1, y_2) | x \geq 0\}$ is a finitely generated cone (generated by columns of $A$), thus a polyhedral cone. $\mathcal{RF}_n$ is the intersection of the above cone with the subspace $\{(y_1, y_2) | x_2 = 0\}$, which is again a polyhedral cone.

It is straightforward to verify that each point in $\mathcal{RF}_n$ satisfies all NC inequalities (we leave the verification to the reader). So $\mathcal{RF}_n \subseteq NC_n$. Suppose for contradiction that there is a point $\pi \in NC_n$ but $\pi \not\in \mathcal{RF}_n$. So there is a hyperplane $\sum_{S \subseteq [n]} \alpha_S x_S = 0$ separating $\mathcal{RF}_n$ and $\pi$ (with $\sum_{S \subseteq [n]} \alpha_S \pi_S > 0$). So $\sum_{S \subseteq [n]} \alpha_S x_S \leq 0$ is not an FNC inequality (since $\pi \in NC_n$ should satisfy all FNC inequalities). From the proof Theorem 2 we have shown that for any non-FNC inequality, we can construct a rectangular flower that violates the inequality. This contradicts that $\sum_{S \subseteq [n]} \alpha_S x_S \leq 0$ for all $x \in \mathcal{RF}_n$. Hence, NC$_n \subseteq \mathcal{RF}_n$. This concludes the proof of the theorem.

At the end of this section, we briefly mention projection inequalities with nonzero constant terms ($\sum_S \alpha_S x_S \leq \beta$, for $\beta \neq 0$). If $\beta < 0$, none such inequality is true by just considering the hypercube. Obviously, if $\sum_S \alpha_S x_S \leq 0$ is true for all $x \in \Psi_n$, $\sum_S \alpha_S x_S = \beta$ for all $\beta > 0$ also. Moreover, if $\sum_S \alpha_S x_S \leq 0$ is not an FNC inequality, $\sum_S \alpha_S x_S \leq \beta$ cannot be true for any $\beta > 0$, since we can make $t$ large enough in the proof of Theorem 2. Conversely, if $\sum_S \alpha_S x_S \leq \beta$ for some $\beta > 0$ is true for all $x \in \Psi_n$, it must hold that $\sum_S \alpha_S x_S \leq 0$ for all $x \in \Psi_n$. This is because if $x \in \Psi_n$, $ax \in \Psi_n$ for any $a > 0$. Therefore, it suffices to consider only those inequalities with zero constant term.

### 3 Proof of Theorem 4: Non-Convexity of $\Psi_n$

In this section, we will prove Theorem 4: the non-convexity of constructible region $\Psi_n$ for $n \geq 4$. We suppose the converse that $\Psi_n$ is convex. First, we can see that if $\Psi_n$ is convex, it must be a convex cone (this is because if $x \in \Psi_n$, $ax \in \Psi_n$ for $a > 0$). Hence, each supporting hyperplane of $\Psi_n$ must correspond to an FNC inequality.

Consider

$$
\Pi_0 = \{(\pi(T)_{1,2}, \pi(T)_{1,3}, \pi(T)_{1,3}, \pi(T)_{2,3}, \pi(T)_{2,4}, \pi(T)_{3,4}, \pi(T)_{1,2,3}, \pi(T)_{2,3,4}) | T \text{ is an object in } \mathbb{R}^n_+\},
$$

which is the projection of $\Psi_n$ onto the subspace spanned by

$$
\{v_{\{1,2\}}, v_{\{1,3\}}, v_{\{2,3\}}, v_{\{2,4\}}, v_{\{3,4\}}, v_{\{1,2,3\}}, v_{\{2,3,4\}}\}
$$

where $v_S$ is the axis indexed by $S \subseteq [n]$. Since $\Psi_n$ is a convex cone, $\Pi_0$ must also be a convex cone. Hence, each linear inequality that defines $\Pi_0$ must be some FNC inequality with terms $T_{\{1,2\}}, T_{\{1,3\}}, T_{\{2,3\}}, T_{\{2,4\}}, T_{\{3,4\}}, T_{\{1,2,3\}}, T_{\{2,3,4\}}$.

Now, we prove that any FNC inequality involving only the above terms is a nonnegative linear combination of the following two BT inequalities:

$$
|T_{\{1,2\}}| \cdot |T_{\{1,3\}}| \cdot |T_{\{2,3\}}| \geq |T_{\{1,2,3\}}|^2, \quad (2)
$$

$$
|T_{\{2,3\}}| \cdot |T_{\{2,4\}}| \cdot |T_{\{3,4\}}| \geq |T_{\{2,3,4\}}|^2. \quad (3)
$$

By Lemma 1, it suffices to consider only NC inequalities. In fact, according to the definition of NC, the right hand side can only contain the terms $T_{\{1,2,3\}}$ and $T_{\{2,3,4\}}$. Apply Corollary 2 in
Single Cover Theorem (which is discussed in next section) on this inequality, we can see that it must be a combination of (2) and (3). In other words, \( \Pi_0 \) is defined by (2) and (3).

Now, we consider the vector \( \phi_0^{(t)} \), \( t > 0, t \neq 1 \),
\[
\phi_0^{(t)} = (0, 2 \ln t, 0, 2 \ln t, 0, \ln t, \ln t)
\]
The example is essentially adopted from the example in [3]. It is easy to see that \( \phi_0^{(t)} \) satisfies (2) and (3). Now, we briefly show \( \phi_0^{(t)} \notin \Pi_0 \). Suppose there exists an object \( T \) with the log-projection vector consistent with \( \phi_0^{(t)} \). In other words, \(|T_{\{1,2\}}| = |T_{\{2,3\}}| = |T_{\{3,4\}}| = 1, |T_{\{1,3\}}| = |T_{\{2,4\}}| = t^2, |T_{\{1,2,3\}}| = |T_{\{2,3,4\}}| = t \). Note that \(|T_{\{1,2\}}| \cdot |T_{\{1,3\}}| \cdot |T_{\{2,3\}}| = |T_{\{1,2,3\}}|^2 \).

From Theorem 4 in [3], we know that the projection of \( T_{\{2,3\}} \) must be a rectangle \( B(\frac{1}{t}, t) \). However, since \(|T_{\{2,3\}}| \cdot |T_{\{2,4\}}| \cdot |T_{\{3,4\}}| = |T_{\{2,3,4\}}|^2 \), the projection of \( T_{\{2,3\}} \) must be a rectangle \( B(t, \frac{1}{t}) \). Since \( t \neq 1 \), the two boxes are not the same and we arrive at a contradiction. This shows that \( \Psi_n \) is not convex and thus completes the proof of Theorem 4.

### 4 Counterexample Construction for NC\BT Inequalities

We have shown that the constructible region cannot be fully characterized by a set of linear inequalities as it is not convex. However, it is still interesting to see what are all correct linear inequalities. Equivalently, we want to figure out the set of linear inequalities that define \( \text{Conv}(\Psi_n) \), the convex hull of \( \Psi_n \).

In this section, we construct counterexamples for several NC but non-BT (denoted as NC\BT) inequalities. Note that a compact object can be approximated by the union of small cubes, our counterexamples are also unions of cubes.

#### 4.1 Skeleton

In this subsection, we use an \( n \)-tuple \((t_1, t_2, \cdots, t_n)\) where \( t_i \)s are non-negative integers to represent the \( n \)-dimensional unit hypercube: \( \{(x_1, \cdots, x_n) \mid \forall i; t_i \leq x_i \leq t_i + 1\} \), i.e., \( \prod_{i=1}^n [t_i, t_i + 1] \). Denote the sum of two sets by their Minkovski sum, namely \( A + B = \{a + b \mid a \in A, b \in B\} \). We need to the notion of skeleton, which is important for our construction.

**Definition 4.** (Connection Graph) In \( \mathbb{R}^d \), consider an FNC inequality \( \prod_{i=1}^k |T_{A_i}|^{\alpha_i} \geq \prod_{j=1}^m |T_{B_j}|^{\beta_j} \) \((\alpha_i, \beta_j > 0)\). The connection graph \( G_C \) for the above inequality is an undirected graph \( G_C = (V, E) \), where \( V = \{v_1, \cdots, v_n\} \), representing \( n \) dimensions. The edge \((v_i, v_j) \in E\) if and only if both \( i \) and \( j \) appear in some \( B_j \) but not in any \( A_i \).

**Definition 5.** Let \( C_1, C_2, \cdots, C_s \) be all cliques (complete subgraphs) in \( G_C \). \( M \) is a large positive integer. For every \( C_r \), we define
\[
\text{SK}_{C_r}(M) = \{ t \mid 0 \leq t_i \leq M - 1, \forall i \in C_r; \ t_i = 0, \forall i \notin C_r \}.
\]
The skeleton for the given NC inequality is defined as
\[
\text{SK}_{G_C}(M) = \bigcup_{r=1}^{s} \text{SK}_{C_r}(M)
\]
See Figure 2 for an example.
In the figure above, the connection graph is the right one and the corresponding skeleton is the object on the left.

For \( S \subset [n] \), let \( \Delta(S) \) be the size of maximum clique in \( G_C[S] \), the subgraph induced by vertices in \( S \). For sufficiently large \( M \), we have the following asymptotic estimations:

\[
\prod_{i=1}^{k} |T_{A_i}|^{\alpha_i} \approx M^{\sum_{i=1}^{k} \alpha_i},
\]

\[
\prod_{j=1}^{m} |T_{B_j}|^{\beta_j} \approx M^{\sum_{j=1}^{m} \beta_j \Delta(B_j)}.
\]

The following lemma is a direct consequence of the above estimate.

**Lemma 2.** If the NC inequality \( \prod_{i=1}^{k} |T_{A_i}|^{\alpha_i} \geq \prod_{j=1}^{m} |T_{B_j}|^{\beta_j} \) satisfies that

\[
\sum_{i=1}^{k} \alpha_i < \sum_{j=1}^{m} \beta_j \Delta(B_j)
\]

then it is incorrect, i.e., there exists a counterexample for it.

**Example 3.** Consider the NC inequality \( |T_{12}| \cdot |T_{23}| \cdot |T_{34}| \geq |T_{123}| \cdot |T_{234}| \). The connection graph \( G_C \) contains two edges (1, 3) and (2, 4). We have \( \sum_{i} \alpha_i = 3 \) and \( \sum_{j} \beta_j \Delta(B_j) = 4 \). Hence, the inequality is not true in general.

### 4.2 Union of Boxes

By a box we mean a hypercube \( B(b) = \{ x \mid 0 \leq x_i \leq b_i \} \) or a translation of it, i.e., \( B + v \) for some positive vector \( v \), here the sum is the Minkowski sum. The examples in this subsection are the disjoint union of two boxes \( B_1 \) and \( B_2 \). Here we require not only \( B_1 \) and \( B_2 \) are disjoint in \( \mathbb{R}^n_+ \), but their projections onto any subspace \( \mathbb{R}^S \) are disjoint as well for any \( S \subseteq [n] \). In particular, we use the following two boxes:

\[
B_1 = B(1); B_2 = B(M^{t_1}, M^{t_2}, \ldots, M^{t_n}) + 1
\]
As before, for $M$ sufficiently large and $t_i$s to be determined later, the following asymptotic equations hold:

$$
\prod_{i=1}^{k} |T_{A_i}|^{\alpha_i} \approx M^{\sum_{i=1}^{k} \alpha_i \max\{0, \sum_{s \in A_i} t_s\}},
$$

$$
\prod_{j=1}^{m} |T_{B_j}|^{\beta_j} \approx M^{\sum_{j=1}^{m} \beta_j \max\{0, \sum_{s \in B_j} t_s\}}.
$$

Note that we can use absolute value to replace the maximum function, $\max\{0, a\} = \frac{1}{2}(a + |a|)$, we obtain the following lemma.

**Lemma 3.** If there exists $t$ such that the following inequality is true:

$$
\sum_{i=1}^{k} \alpha_i |t| \sum_{s \in A_i} t_s < \sum_{j=1}^{m} \beta_j |t| \sum_{s \in B_j} t_s,
$$

then the corresponding NC inequality is incorrect.

**Proof.** Our counterexample is the union of two boxes $B_1 = B(1), B_2 = B(M^{t_1}, M^{t_2}, \ldots, M^{t_n}) + 1$ where $t$ is the counterexample for the absolute value inequality. By the above asymptotics and $\sum_{i=1}^{k} \sum_{s \in A_i} \alpha_i t_s = \sum_{j=1}^{m} \sum_{s \in B_j} \beta_j t_s$, we conclude that

$$
\sum_{i=1}^{k} \alpha_i \max\{0, \sum_{s \in A_i} t_s\} < \sum_{j=1}^{m} \beta_j \max\{0, \sum_{s \in B_j} t_s\}.
$$

Hence, the object is a counterexample. \qed

**Example 4.** Again, consider the NC inequality in Example 3. Let $t = (1, -1, 1, -1)$. We can see the condition of Lemma 3 is met and the inequality is incorrect.

**Example 5.** Consider the NC inequality $|T_{13}| \cdot |T_{23}| \cdot |T_{124}| \geq |T_{123}| \cdot |T_{1234}|$ and $t = (-1, -1, 1, 2)$. So, this inequality is incorrect.

### 4.3 Exact Single Cover Theorem

Using the union of boxes method we can also obtain the following theorem which is a necessary condition for an inequality to be true. Let $a_i$ be the 0/1 indicator vector for set $A_i$ and $b_j$ for $B_j$, i.e., $a_{ij} = 1$ if and only if $j \in A_i$.

**Theorem 5.** (Exact Single Cover Theorem) If the FNC inequality $\sum_{i=1}^{k} \alpha_i x_{A_i} \geq \sum_{j=1}^{m} \beta_j x_{B_j}$ holds for every $x \in \Psi_n$, then for all $j \in [m]$, there exist nonnegative $c_1, c_2, \ldots, c_k$ such that $c_i \leq \alpha_i$ for all $i$ and

$$
\sum_{i=1}^{k} c_i a_i = \beta_j b_j.
$$
Proof. Let \( K = \{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{k} c_i \mathbf{a}_i, \ 0 \leq c_i \leq \alpha_i, i = 1, 2, \cdots, k \} \). It is immediate that \( K \) is a convex subset of \( \mathbb{R}^n \). If \( K \) does not include \( \beta_j \mathbf{b}_j \), by separating hyperplane theorem, there exists a vector \( \mathbf{t} = (t_1, t_2, \ldots, t_n) \) and real number \( a \) such that

\[
\mathbf{t} \cdot \mathbf{x} < a, \forall \mathbf{x} \in K, \quad \text{but} \quad \beta_j \mathbf{t} \cdot \mathbf{b}_j > a.
\]

We still use a union of two boxes to be the counterexample:

\[
B_1 = B(1); \quad B_2 = B(M^{t_1}, M^{t_2}, \cdots, M^{t_n}) + 1.
\]

It can be seen that

\[
\prod_{j=1}^{m} |T_{B_j}|^{\beta_j} \geq M^{\beta_j \mathbf{t} \cdot \mathbf{b}_j} > M^a.
\]

Now, it suffices to show that \( \prod_{j=1}^{k} |T_{A_i}|^{\alpha_i} < M^a \). In the asymptotic showed before, we have that

\[
\prod_{j=1}^{k} |T_{A_i}|^{\alpha_i} \leq M^{\sum_{i=1}^{k} \alpha_i \max\{0, \sum_{s \in A_i} t_s\}} \leq M^{\sum_{i: \mathbf{a}_i \cdot \mathbf{t} \geq 0} \alpha_i \mathbf{a}_i \cdot \mathbf{t}} < M^a.
\]

The last inequality holds since \( \sum_{i: \mathbf{a}_i \cdot \mathbf{t} \geq 0} \alpha_i \mathbf{a}_i \) is in \( K \). This completes the proof. \( \square \)

Now, we show two simple corollaries.

**Corollary 1.** Suppose the following FNC inequality \( \sum_{i=1}^{k} \alpha_i x_{A_i} \geq \sum_{j=1}^{m} \beta_j x_{B_j} \) holds for all \( x \in \Psi_n \), and the set indicator vectors \( \mathbf{a}_i \) are linearly independent. Then this inequality can be written as a nonnegative combination of \( m \) BT inequalities.

**Proof.** Let \( \mathbf{A} \) (\( \mathbf{B} \) resp.) be the matrix with \( \mathbf{a}_i \) being the \( i \)th column (\( \mathbf{b}_j \) the \( j \)th column). Let \( \alpha = \{\alpha_1, \ldots, \alpha_k\}^T \) and \( \beta = \{\beta_1, \ldots, \beta_m\}^T \). By the definition of FNC, we know that \( \mathbf{A} \alpha = \mathbf{B} \beta \). For each \( j \), we know \( \beta_j \mathbf{b}_j = \sum_{i=1}^{k} c_{ji} \mathbf{a}_i \) for some \( 0 \leq c_{ji} \leq \alpha_i \). So \( A \alpha = A(\sum_{j} \mathbf{c}_j) \), where \( \mathbf{c}_j = (c_{j1}, \ldots, c_{jk}) \). Since \( A \) has full column rank, it must be the case that \( \alpha = \sum_{j} \mathbf{c}_j \). \( \square \)

**Corollary 2.** Suppose the following FNC inequality \( \sum_{i=1}^{k} \alpha_i x_{A_i} \geq \sum_{j=1}^{m} \beta_j x_{B_j} \) holds for all \( x \in \Psi_n \), and \( m = 1 \) or 2. Then this inequality can be written as a nonnegative combination of \( m \) BT inequalities.

**Proof.** We only need to consider the case \( m = 2 \). From Theorem 3, \( \beta_1 \mathbf{b}_1 = \sum_{i=1}^{k} \alpha_i \mathbf{a}_i \) for some \( 0 \leq c_i \leq \alpha_i \). Since \( \sum_{i=1}^{k} \alpha_i \mathbf{a}_i = \beta_1 \mathbf{b}_1 + \beta_1 \mathbf{b}_2 \), we have that \( \beta_2 \mathbf{b}_2 = \sum_{i=1}^{k} (\alpha_i - c_i) \mathbf{a}_i \). \( \square \)

**Example 6.** Consider the NC inequality \( |T_{12}| \cdot |T_{23}| \cdot |T_{34}| \geq |T_{123}| \cdot |T_{234}| \) in Example 3. From either of the above corollary, if it is true, it can be decomposed into two BT inequalities. However, it is clear such a decomposition does not exist. So it is not true in general. Similarly, we can also see the inequality in Example 5 is not true.
4.4 A Hybrid Approach

In fact, neither of the above methods are sufficient to disprove all NC\BT inequalities. In this section, we demonstrate an application of the combination of the above approaches.

Example 7. One interesting example is the following NC inequality:

\[ |T_1| \cdot |T_{12}| \cdot |T_{23}| \cdot |T_{34}| \cdot |T_{24}| \geq |T_{123}| \cdot |T_{234}| \cdot |T_{124}|. \]

The example satisfies the statement of Theorem 5, however, we can show it is also not correct. Our counterexample utilizes a combination of skeleton and union-box methods. We observe that the given inequality is a combination of

\[ |T_1| \cdot |T_{12}| \cdot |T_{23}| \cdot |T_{34}| \geq |T_{123}| \cdot |T_{234}|, \quad \text{and} \quad |T_1| \cdot |T_{24}| \approx |T_{124}|. \]

We already have a skeleton counterexample for the former. Our idea is to take the union of the skeleton and a disjoint box \( B \) so that the values of \( |T_1|, |T_{12}|, |T_{23}|, |T_{34}|, |T_{123}|, |T_{234}|, |T_{124}| \) remain (approximately) the same, but \( |T_1| \cdot |T_{24}| \approx |T_{124}| \). Since the skeleton construction allows the left hand side to be arbitrarily larger than the right hand side, we can see that Example 7 is also incorrect.

We can let \( B = B(R^3, R^{-4}, R^{-6}, R^5) \) with \( R > 0 \) large enough (larger than the constant \( M \) in the skeleton construction). Hence, \( |T_1| \cdot |T_{24}| \approx |T_{124}| \approx R^4 \) but \( |T_{12}| \approx M + R^{-1}, |T_{23}| \approx M + R^{-10}, |T_{34}| \approx M + R^{-1}, |T_{123}| \approx M^2 + R^{-7}, |T_{234}| \approx M^2 + R^{-5}. \)

We have shown that some NC\BT inequalities are not correct. It remains to ask whether there is an NC\BT inequality is correct. We have been unable to discover one such inequality. We have checked (in an exhaustive manner) all inequalities in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \), and found out that all NC\BT inequalities are not true. Hence, we propose Conjecture 1 mentioned in the introduction.

5 Final Remarks and Acknowledgements

All of our counterexamples in Section 4 are essentially combinatorial, and the constructions allow one side of the inequality to be arbitrarily larger than the other side. We suspect that all incorrect projection inequalities can be refuted in a similar fashion. In other words, we may not need to construct very delicate, twisted geometric objects, but instead just a union of a small number of boxes (the number related to \( n \)), to refute any incorrect linear projection inequality.

We have developed a few other techniques to disprove some of NC inequalities. For example fitting boxes model is the combination of the two models we introduced. It consists of many boxes, each constructed according to the connection graph. Fitting box model can be used to handle all 4-dimensional inequality. However, it is hard to analyze and generalize to higher dimensions, and we decide not to introduce it here.

In 2010, the third author JL proposed the notion of rectangular flowers and suspected that \( \mathcal{RF}_n = \Psi_n \), which, if true, is a natural extension of the box theorem in \([3]\). In fact, JL “verified” the above claim empirically using hundreds of thousands datasets (synthetically generated from different distributions with different dimensions and parameters). Now, we know that \( \mathcal{RF}_n \subseteq \Psi_n \).

But it is still an interesting fact that all NC inequalities are true for many “random-like” data and there may be good mathematical reasons for it. Moreover, our counter-examples, which appear to

\[ \text{Let } K \text{ be a body in } \mathbb{R}^n. \text{ The box theorem states that there is a rectangle } B \text{ with } \text{vol}(B) = \text{vol}(K) \text{ and } \text{vol}(\pi_S(B)) \leq \text{vol}(\pi_S(K)) \text{ for every } S \subseteq [m]. \]
be quite simple in retrospect, may not be totally obvious without realizing the equivalence between rectangular flowers and the NC inequalities.

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A Appendix 1 ($BT_n = \Psi_n$ for $n \leq 3$)

In this section, we prove $BT_3 = \Psi_3$ in $\mathbb{R}^3$. This appears to be a folklore result, and we provide a proof for completeness. Since BT inequalities are correct for every projection vector, it suffices to prove that for any vector $\pi$, there exist an object $T$ such that $\pi(T) = \pi$ if $\pi$ satisfies all BT inequalities in 3-dimension. In fact, we show $BT_n = NC_n$ for $n = 3$. Since $\Psi_n$ is sandwiched between them, all three of them are the same. Hence, it suffices to show that any NC inequality in $\mathbb{R}^3$ is the combination of some BT inequalities.
Suppose an NC inequality in $\mathbb{R}^3$ has the following form:

$$\prod_{S \subseteq \{1, 2, 3\}} |T_S|^\alpha_S \geq 1$$

where $\alpha_S \in \mathbb{Z}$. According to the definition of NC inequalities, it is not hard to verify that $\alpha_S \geq 0$ for all $|S| = 1$. If $\alpha_S \geq 0$ for all $|S| = 2$, the inequality is indeed a BT inequalities.

Now, suppose $\alpha_S < 0$ for some $|S| = 2$. Without lose of generality, we assume that $\alpha_{\{1,2\}} < 0$. By definition of NC, we obtain the following inequalities:

$$\alpha_{\{1\}} \geq -\alpha_{\{1,2\}}; \quad \alpha_{\{2\}} \geq -\alpha_{\{1,2\}}$$

and

$$\prod_{S \subseteq \{1, 2, 3\}} |T_S|^\alpha_S \cdot \left(\frac{|T_{\{1\}}| \cdot |T_{\{2\}}|}{|T_{\{1,2\}}|}\right)^{\alpha_{\{1,2\}}} \geq 1,$$

which is still an NC inequality without the term $|T_{\{1,2\}}|^\alpha_{\{1,2\}}$.

Rewrite it as $\prod_{S \subseteq \{1, 2, 3\}} |T'_S|^\alpha'_S \geq 1$ with $\alpha'_{\{1,2\}} = 0$. If the above embedding inequality is still not an BT inequality yet, then there exist some $|S'| = 2$ such that $\alpha_{S'} < 0$. Without lose of generality, we may assume that $\alpha'_{\{1,3\}} < 0$. Repeating the above operation for $\alpha_{\{1,2\}}$, we can eliminate the term $|T'_{\{1,3\}}|^{\alpha'_{\{1,3\}}}$ as well as $|T'_{\{2,3\}}|^{\alpha'_{\{2,3\}}}$ (if necessary) in the same way. The remaining part must be a BT inequality since the only negative $\alpha_S$ is $\alpha_{\{1,2,3\}}$. Thus, we prove that $\mathcal{BT}_3 = \mathcal{NC}_3 = \Psi_3$. 

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