Unidirectional decomposition method for obtaining exact localized waves solutions totally free of backward components.

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Abstract

In this paper we use a unidirectional decomposition capable of furnishing localized wave pulses, with luminal and superluminal peak velocities, in exact form and totally free of backward components, which have been a chronic problem for such wave solutions. This decomposition is powerful enough for yielding not only ideal nondiffracting pulses but also their finite energy versions still in exact analytical closed form. Another advantage of the present approach is that, since the backward spectral components are absent, the frequency spectra of the pulses do not need to possess ultra-widebands, as it is required by the usual localized waves (LWs) solutions obtained by other methods. Finally, the present results bring the LW theory nearer to the real experimental possibilities of usual laboratories.

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1 Introduction

Localized waves (or nondiffracting waves)[1-60] are very special free space solutions of the linear wave equation, \((\nabla^2 - \partial_t^2)\psi = 0\), whose main characteristic is that of resisting
the diffraction effects for long distances. In their pulse versions, the LWs can possess subluminal, luminal or supeluminal peak velocities. As it is well known\(^7, 10, 17, 28, 34\), these impressive features of the nondiffracting waves are due to the special space-time coupling of their spectra.

In the last years, several methods\(^7, 27, 28, 34\) have been developed to yield localized pulses in exact analytical closed form. The most successful approaches are those dealing with bidirectional or unidirectional decomposition\(^7, 28, 34\), in which the time variable \(t\) and the spatial variable \(z\) (considered as the propagation direction) are replaced with other two variables, linear combinations of them.

Examples of bidirectional and unidirectional decompositions are i) \(\zeta = z - ct, \eta = z + ct\), developed by Besieris et.al.\(^7\); ii) \(\zeta = z - Vt, \eta = z - c^2t/V\), also introduced by those authors\(^28\); iii) \(zeta = z - Vt, \eta = z + Vt\), developed by Zamboni-Rached et.al.\(^34\); iv) \(\zeta = z - Vt, \eta = z + ct\), proposed by Besieris et.al.\(^28\).

Even if all those decompositions allow us to get exact analytical LWs solutions, all of them suffer with the same problem, that is, the occurrence of backward travelling components in their spectra.\(^\star\) This drawback makes it necessary the use of ultra-wideband frequency spectra to minimize\(^7, 28, 34\) the contribution of the backward components. This fact can suggest the wrong idea that ultra-wide frequency bands are a characteristic of the LW pulses, which is not true. As a matter of fact, it is quite simple to choose spectra that eliminate completely such "noncausal" components and at the same time have narrow frequency bands. The real problem is that no closed form analytical solution is known for those cases, and one has to make recourse to time consuming numerical simulations. This problem was already solved in the case of localized subluminal waves\(^60\) but still persists in the general cases of luminal and superluminal LW pulses.

We are going to show here that, by a very simple decomposition, a unidirectional one,

\(^\star\)Only the simplest X-wave pulses, which possess the form \(\psi(\rho, \phi, z - Vt)\), does not present this problem.
one can overcome the problems cited above, getting LW pulses with superluminal and
luminal peak velocities, constituted by forward travelling components only, and without
mandatory recourses to ultra wide frequency bands. These results make nondiffracting
waves more easily experimentally realizable and applicable.

2 The method

The first subsection forwards a brief overview of the LWs theory. The rest of this
section is devoted to the new method and to its results.

2.1 A brief overview about LWs

In the case of the linear and homogeneous wave equation in free space, in cylindrical
coordinates \((\rho, \phi, z)\) and using a Fourier-Bessel expansion, we can express a general
solution \(\psi(\rho, \phi, z, t)\) as

\[
\Psi(\rho, \phi, z, t) = \sum_{n=-\infty}^{\infty} \left[ \int_{0}^{\infty} dk_{\rho} \int_{-\infty}^{\infty} dk_{z} \int_{-\infty}^{\infty} d\omega k_{\rho} A'_{n}(k_{\rho}, k_{z}, \omega) J_{n}(k_{\rho} \rho) e^{ik_{z}z} e^{-i\omega t} e^{in\phi} \right] \tag{1}
\]

with

\[
A'_{n}(k_{\rho}, k_{z}, \omega) = A_{n}(k_{z}, \omega) \delta \left( k_{\rho}^{2} - \left( \frac{\omega^{2}}{c^{2}} - k_{z}^{2} \right) \right) , \tag{2}
\]

\(A_{n}(k_{z}, \omega)\) being an arbitrary function and \(\delta(.)\) the Dirac’s delta function.

An ideal nondiffracting wave can be defined as a wave capable of maintaining its
spatial form indefinitely (except for local variations) while propagating. This property
can be expressed in a mathematical way\cite{27} by (when assuming propagation in the \(z\)
direction)

\(^1\)In the following, we shall briefly write "forward components" and even "forward pulses".
\[ \Psi(\rho, \phi, z, t) = \Psi(\rho, \phi, z + \Delta z_0, t + \frac{\Delta z_0}{\mathcal{V}}) \]  

where \(\Delta z_0\) is a certain length and \(\mathcal{V}\) is the pulse-peak velocity, with \(0 \leq \mathcal{V} \leq \infty\).

Using (3) in (1), and taking into account (2), we can show\(^{27, 1}\) that any localized wave solution, when eliminating evanescent waves and considering only positive angular frequencies, can be written as

\[ \Psi(\rho, \phi, z, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \int_{0}^{\infty} \mathrm{d}\omega \int_{-\omega/c}^{\omega/c} \mathrm{d}k_z A_{nm}(k_z, \omega) \right. \]

\[ \times J_n \left( \rho \sqrt{\frac{\omega^2}{c^2} - k_z^2} \right) e^{ik_z z} e^{-\omega t} e^{im\phi} \]

with

\[ A_{nm}(k_z, \omega) = S_{nm}(\omega) \delta (\omega - (\mathcal{V}k_z + b_m)) \]  

\[ b_m = \frac{2m\pi\mathcal{V}}{\Delta z_0}, \]  

and quantity \(S_{nm}(\omega)\) being an arbitrary frequency spectrum.

We should note that, due to Eq.(5), each term in the double sum (4), namely in the expression within square brackets, is a truly nondiffracting wave (beam or pulse), and their sum (4) is just the most general form representing an ideal nondiffracting wave defined by Eq.(3).

We can also notice that (4) is nothing but a superposition of Bessel beams with a specific space-time coupling in their spectra: more specifically with linear relationships between their angular frequency \(\omega\) and longitudinal wave number \(k_z\).

Concerning such a superposition, the Bessel beams with \(k_z > 0\) \((k_z < 0)\) propagate in the positive (negative) \(z\) direction. As we wish to obtain LWs propagating in the positive \(z\) direction, the presence of ”backward” Bessel beams \((k_z < 0)\), i.e. of ”backward components”, is not desirable. This problem can be overcome, however, by appropriate
choices of the spectrum (5), which can totally eliminate those components, or minimize their contribution, in superposition (4).

Another important point refers to the energy of the LWs[3, 28, 33, 34]. It is well known that any ideal LW, i.e., any field with the spectrum (5), possesses infinite energy. However finite-energy LWs can be constructed by concentrating the spectrum $A_{nm}(k_z, \omega)$ in the surrounding of a straight line of the type $\omega = \mathcal{V}k_z + b_m$ instead of collapsing it exactly over that line. In such a case, the LWs get a finite energy, but are endowed with finite field depths: i.e., they maintain their spatial forms for long (but not infinite) distances.

Despite the fact that expression (4), with $A_{nm}(k_z, \omega)$ given by (5), does represent ideal nondiffracting waves, it is difficult to use it for obtaining analytical solutions, especially when having the task of eliminating the backward components. This difficulty becomes even worse in the case of finite-energy LWs.

As an attempt to bypass these problems, many different bidirectional and unidirectional decomposition methods have been proposed in the last years[7, 28, 34]. Those methods consist essentially in the replacement of the variables $z$ and $t$ in Eq.(4) with new ones $\zeta = z + v_1 t$ and $\zeta = z + v_2 t$, where $v_1$ and $v_2$ are constants possessing a priori any value in the range $[-\infty, \infty]$. The names bidirectional and unidirectional decomposition correspond to $v_1/v_2 > 0$ and $v_1/v_2 < 0$, respectively.

For instance, in [7] Besieris et.al. introduced the bidirectional decomposition $\zeta = z - ct$, $\eta = z + ct$ and obtained interesting ideal and finite energy luminal LWs. They also worked, in [28], with the unidirectional decomposition $\zeta = z - Vt$, $\eta = z - c^2t/V$ (with $V > c$) for obtaining ideal and finite-energy superluminal LWs. In [34] Zamboni-Rached et.al. introduced the bidirectional decomposition $\zeta = z - Vt$, $\eta = z + Vt$ (with $V > c$), being thus able to provide many other ideal and finite-energy superluminal nondiffracting pulses.
Subluminal LWs have been obtained \cite{28,32,60}, for instance, through a decomposition of the type $\zeta = z - vt$, $\eta = z$, with $v < c$.

All such decompositions are very efficient in furnishing LWs in closed forms, but yield solutions that suffer with the problem that backward travelling wave components enter their spectral structure. A way out was found in the case of subluminal waves \cite{60}, but the problem still persists for the luminal and superluminal ones. In the latter cases, they succeeded till now only in minimizing the contribution of the backward Bessel beams in Eq.\((1)\) by choosing ultra-wideband frequency spectra. This is not the best approach because, in general, the solutions found in this way resulted to be far from the experimental possibilities of the usual laboratories.

As it was said before, the backward components can be totally removed by a proper choice of the spectrum: but none of the previous decompositions are then able to yield analytical solutions for the integral \((1)\).

We are going to introduce, therefore, a unidirectional decomposition that allows one to get ideal and finite energy LWs, with superluminal and luminal peak velocities, without any occurrence in their spectral structure of backward components.

### 2.2 Totally “forward” LW pulses

Let us start with eqs.\((1,2)\), which describe a general free-space solution (without evanescent waves) of the homogeneous wave equation, and consider in Eq.\((2)\) a spectrum $A_n(k_z, \omega)$ of the type

$$A_n(k_z, \omega) = \delta_{n0}H(\omega)H(k_z)A(k_z, \omega)\delta(k^2 - (\omega^2/c^2 - k_z^2))$$  \(\text{(6)}\)

where $\delta_{n0}$ is the Kronecker delta function, $H(\cdot)$ the Heaviside function and $\delta(\cdot)$ the Dirac delta function, quantity $A(k_z, \omega)$ being an arbitrary function. Spectra of the type \((6)\) restrict the solutions to the axially symmetric case, with only positive values to the
angular frequencies and longitudinal wave numbers. With this, the solutions proposed by us get the integral form

$$\psi(\rho,z,t) = \int_0^\infty d\omega \int_0^{\omega/c} dk_z A(k_z,\omega) J_0(\rho\sqrt{\omega^2/c^2 - k_z^2}) e^{ik_z} e^{-i\omega t}$$ (7)

i.e., result to be general superpositions of zero-order Bessel beams propagating in the positive z direction only. Therefore, any solution obtained from (7), be they nondiffracting or not, are completely free from backward components.

At this point, we introduce the unidirectional decomposition

$$\begin{cases} 
\zeta = z - Vt \\
\eta = z - ct 
\end{cases}$$ (8)

with $V > c$.

A decomposition of this type was used till now in the context of paraxial approximation only[54]; but we shall show below that it can be much more effective, giving important results in the exact context and in situations that cannot be analyzed in the paraxial approach.

With (8), we can write the integral solution (7) as

$$\psi(\rho,\zeta,\eta) = (V-c) \int_0^\infty d\sigma \int_{-\infty}^\sigma d\alpha A(\alpha,\sigma)J_0(\rho\sqrt{\gamma^2\sigma^2 - 2(\beta - 1)\sigma\alpha}) e^{-i\eta} e^{i\alpha\zeta}$$ (9)

where $\gamma = (\beta^2 - 1)^{-1/2}$, $\beta = V/c$ and where

$$\begin{cases} 
\alpha = \frac{1}{V-c} (\omega - Vk_z) \\
\sigma = \frac{1}{V-c} (\omega - ck_z) 
\end{cases}$$ (10)

are the new spectral parameters.

It should be stressed that superposition (9) is not restricted to LWs: It is the choice of the spectrum $A(\alpha,\sigma)$ that will determine the resulting LWs.
2.2.1 Totally “forward” ideal superluminal LW pulses

The X-type waves:

The most trivial LW solutions are those called X-type waves\[^{13, 14}\]. They are constructed by frequency superpositions of Bessel beams with the same phase velocity \( V > c \) and till now constitute the only known ideal LW pulses free of backward components. Obviously, it is not necessary to use the approach developed here to obtain such X-type waves, since they can be obtained by using directly the integral representation in the parameters \( (k_z, \omega) \), i.e., by using Eq.\(^{(7)}\). Even so, just as an exercise, let us use the present approach to construct the ordinary X wave.

Consider the spectral function \( A(\alpha, \sigma) \) given by

\[
A(\alpha, \sigma) = \frac{1}{V - c} \delta(\alpha) e^{-s\sigma}
\]  

(11)

One can note that the delta function in (11) implies that \( \alpha = 0 \rightarrow \omega = V k_z \), which is just the spectral characteristic of the X-type waves. In this way, the exponential function \( \exp(-s\sigma) \) represents a frequency spectrum starting at \( \omega = 0 \), with an exponential decay and frequency bandwidth \( \Delta \omega = V/s \).

Using (11) in (9), we get

\[
\psi(\rho, \zeta) = \frac{1}{\sqrt{(s - i\zeta)^2 + \gamma^{-2}\rho^2}} \equiv X
\]

(12)

which is the well known ordinary X wave.

Totally “forward” Superluminal Focus Wave Modes:

Focus wave modes (FWMs)\[^{7, 28, 34}\] are ideal nondiffracting pulses possessing spectra with a constraint of the type \( \omega = V k_z + b \) (with \( b \neq 0 \)), which links the angular frequency
and the longitudinal wave number, and are known for their strong field concentrations.

Till now, all the known FWM solutions possess, however, backward spectral components, a fact that, as we know, forces one to consider large frequency bandwidths to minimize their contribution. However we are going to obtain solutions of this type completely free of backward components, and able to possess also very narrow frequency bandwidths.

Let us choose a spectral function $A(\alpha, \sigma)$ like

$$A(\alpha, \sigma) = \frac{1}{V - c} \delta(\alpha + \alpha_0)e^{-s\sigma}$$

with $\alpha_0 > 0$ a constant. This choice confines the spectral parameters $\omega, k_z$ of the Bessel beams to the straight line $\omega = Vk_z - (V - c)\alpha_0$, as it is shown in the figure below

![Figure 1: The Dirac delta function in (13) confines the spectral parameters $\omega, k_z$ of the Bessel beams to the straight line $\omega = Vk_z - (V - c)\alpha_0$, with $\alpha_0 > 0$.](image)

Substituting (13) in (9), we have

$$\psi(\rho, \zeta, \eta) = \int_0^\infty d\sigma \int_0^\sigma d\alpha \delta(\alpha + \alpha_0)e^{-s\sigma} J_0(\rho\sqrt{\gamma^2\sigma^2 - 2(\beta - 1)\sigma\alpha})e^{-i\alpha\eta}e^{i\sigma\zeta} ,$$

which, on using identity 6.616 in Ref.[61], results in
\[ \psi(\rho, \zeta, \eta) = X e^{i\alpha_0 \eta} \exp \left[ \frac{i\alpha_0}{2} \left( -i\zeta - \frac{1}{X} \right) \right] \]  

(15)

where \( X \) is the ordinary X-wave given by Eq.(12).

Solution (15) represents an ideal superluminal LW of the type FWM, but totally free from backward components.

As we already said, the Bessel beams constituting this solution have their spectral parameters linked by the relation \( \omega = V k_z - (V - c) \alpha_0 \); thus, by using (13) and (10), it is easy to see that the frequency spectrum of those Bessel beams starts at \( \omega_{\text{min}} = c \alpha_0 \) with an exponential decay \( \exp(-s\omega/V) \), and so possesses the bandwidth \( \Delta \omega = V/s \). It is clear that \( \omega_{\text{min}} \) and \( \Delta \omega \) can assume any values, so that the resulting FWM, eq. (15), can range from a quasi-monochromatic to an ultrashort pulse. This is a great advantage with respect to the old FWM solutions.

As an example, we plot two situations related with the LW pulse given by Eq.(15).

\[(a)\text{ and } (b)\text{ show, respectively, the intensity of the complex and real part of a quasi-monochromatic, totally "forward", superluminal FWM optical pulse, with } V = 1.5c, \alpha_0 = 1.256 \times 10^7 \text{ m}^{-1} \text{ and } s = 1.194 \times 10^{-4} \text{ m}, \text{ which correspond to } \omega_{\text{min}} = 3.77 \times 10^{15} \text{ Hz}, \text{ and } \Delta \omega = 3.77 \times 10^{12} \text{ Hz}, \text{ i.e., to a picosecond pulse with } \lambda_0 = 0.5 \mu\text{m}.\]

The first, in Fig.(2), is a quasi-monochromatic optical FWM pulse, with \( V = 1.5c \),
\[ \alpha_0 = 1.256 \times 10^7 \text{ m}^{-1} \text{ and } s = 1.194 \times 10^{-4} \text{ m}, \text{ which correspond to } \omega_{\text{min}} = 3.77 \times 10^{15} \text{Hz}, \text{ and } \Delta \omega = 3.77 \times 10^{12} \text{Hz}, \text{ i.e., to a picosecond pulse with } \lambda_0 = 0.5 \mu\text{m}. \] Figure (2a) shows the intensity of the complex LW field, while Fig. (2b) shows the intensity of its real part. Moreover, in Fig. (2b), in the right upper corner, it is shown a zoom of this LW, on the z axis and around the pulse’s peak, where the carrier wave of this quasi-monochromatic pulse shows up.

The second example, in Fig. (3), corresponds to an ultrashort optical FWM pulse with \( V = 1.5 \, c \), \( \alpha_0 = 1.256 \times 10^7 \text{ m}^{-1} \) and \( s = 2.3873 \times 10^{-7} \text{ m}, \) which correspond to \( \omega_{\text{min}} = 3.77 \times 10^{15} \text{Hz}, \) and \( \Delta \omega = 1.88 \times 10^{15} \text{Hz}, \) i.e., to a fentosecond optical pulse. Figures (3a,3b) show the intensity of the complex and real part of this LW field, respectively.

Figure 3: (a) and (b) show, respectively, the intensity of the complex and real part of an ultrashort, totally “forward”, superluminal FWM optical pulse, with \( V = 1.5 \, c \), \( \alpha_0 = 1.256 \times 10^7 \text{ m}^{-1} \) and \( s = 2.3873 \times 10^{-7} \text{ m}, \) which correspond to \( \omega_{\text{min}} = 3.77 \times 10^{15} \text{Hz}, \) and \( \Delta \omega = 1.88 \times 10^{15} \text{Hz}, \) i.e., to a fentosecond optical pulse.

Now, we apply the present approach to obtain totally “forward” finite-energy LW pulses.
2.2.2 Totally “forward”, finite-energy LW pulses

Finite-energy LW pulses are almost nondiffracting, in the sense that they can retain their spatial forms, resisting to the diffraction effects, for long (but not infinite) distances.

There exist many analytical solutions representing finite-energy LWs\cite{7, 28, 34}, but, once more, all the known solutions suffer from the presence of backward components. We can overcome this limitation.

Superluminal finite-energy LW pulses, with peak velocity $V > c$, can be got by choosing spectral functions in (7) which are concentrated in the vicinity of the straight line $\omega = V k_z + b$ instead of lying on it. Similarly, in the case of luminal finite-energy LW pulses the spectral functions in (7) have to be concentrated in the vicinity of the straight line $\omega = c k_z + b$ (note that in the luminal case, one must have $b \geq 0$).

Indeed, from Eq.(10) it is easy to see that, by our approach, finite-energy superluminal LWs can be actually obtained by concentrating the spectral function $A(\alpha, \sigma)$ entering in (9), in the vicinity of $\alpha = -\alpha_0$, with $\alpha_0$ a positive constant. And, analogously, the finite-energy luminal case can be obtained with a spectrum $A(\alpha, \sigma)$ concentrated in the vicinity of $\sigma = \sigma_0$, with $\sigma_0 \geq 0$.

To see this, let us consider the spectrum

$$A(\alpha, \sigma) = \frac{1}{V - c} H(-\alpha - \alpha_0) e^{\alpha \alpha} e^{-\sigma \sigma}$$

where $\alpha_0 > 0$, $a > 0$ and $s > 0$ are constants, and $H(\cdot)$ is the Heaviside function.

Due to the presence of the Heaviside function, the spectrum (16), when written in terms of the spectral parameters $\omega$ and $k_z$, has its domain in the region shown below:

We can see that the spectrum $A(\alpha, \sigma)$ given by Eq.(16) is more concentrated on the line $\alpha = \alpha_0$, i.e, around $\omega = V k_z - (V - c)\alpha_0$, or on $\sigma = 0$ (i.e., around $\omega = c k_z$), depending on the values of $a$ and $s$: More specifically, the resulting solution will be a superluminal finite-energy LW pulse, with peak velocity $V > c$, if $a >> s$; or a luminal
finite-energy LW pulse if $s \gg a$.

Inserting the spectrum (16) into (9), we have

$$
\psi(\rho, \zeta, \eta) = \int_0^\infty d\sigma \int_{-\infty}^{-\alpha_0} d\alpha e^{a\alpha} e^{-s\sigma} J_0(\rho \sqrt{\gamma^2 \sigma^2 - 2(\beta - 1)\sigma}) e^{-i\alpha \eta} e^{i\sigma \zeta},
$$

and, by using identity 6.6.16 in Ref.[61], we get

$$
\psi(\rho, \zeta, \eta) = X \int_{-\infty}^{-\alpha_0} d\alpha e^{a\alpha} e^{-i\alpha \eta} \exp \left[ -\frac{\alpha}{\beta + 1} (s - i\zeta - X^{-1}) \right],
$$

which can be directly integrated to furnish

$$
\psi(\rho, \zeta, \eta) = \frac{X \exp \left\{ -\alpha_0 \left[ (a - i\eta) - \frac{1}{\beta + 1} \left( s - i\zeta - X^{-1} \right) \right] \right\}}{(a - i\eta) - \frac{1}{\beta + 1} \left( s - i\zeta - X^{-1} \right)},
$$

As far as we know, the new solution (19) is the first one to represent finite-energy LWs completely free of backward components.

\textit{Superluminal or luminal peak velocities:}
The finite-energy LW (19) can be superluminal (peak-velocity $V > c$) or luminal (peak-velocity $c$) depending on the relative values of the constants $a$ and $s$. To see this in a rigorous way, in connection with solution (19), we should calculate how its global maximum of intensity (i.e., its peak), which is located on $\rho = 0$, develops in time. One can obtain the peak’s motion by considering the field intensity of (19) on the $z$ axis, i.e., $|\psi(0, \zeta, \eta)|^2$, at a given time $t$, and finding out the value of $z$ at which the pulse presents a global maximum: We shall call $z_p(t)$ (the peak’s position) this value of $z$. Obviously the peak velocity will be $dz_p(t)/dt$.

The on-axis field intensity of (19) is

$$|\psi(0, \zeta, \eta)|^2 = e^{-2a\alpha_0} \left( \frac{(as - \eta\zeta)^2 + (s\eta + a\zeta)^2}{(as - \eta\zeta)^2 + (s\eta + a\zeta)^2} \right)$$

(20)

For a given time $t$, we can find out the $z$ position of the peak by setting $\partial|\psi(0, \zeta, \eta)|^2/\partial z = 0$, that is,

$$\frac{e^{-2a\alpha_0}[2(as - \eta\zeta)(\zeta + \eta) - 2(s\eta + a\zeta)(a + s)]}{[(as - \eta\zeta)^2 + (s\eta + a\zeta)^2]^2} = 0$$

(21)

where we have used $\partial/\partial z = \partial/\partial \zeta + \partial/\partial \eta$.

From (21), we have that

$$\eta\zeta^2 + \eta^2\zeta + s^2\eta + a^2\zeta = 0$$

(22)

We are interested in the cases where $a >> s$ or $s >> a$.

For the case $a >> s$, we have for the two last terms in the l.h.s. of (22): $s^2\eta + a^2\zeta = s^2(z - ct) + a^2(z - Vt) = (s^2 + a^2)z - (s^2c + a^2V)t \approx a^2(z - Vt)$, and so we can approximate (22) with

$$\eta\zeta^2 + \eta^2\zeta + a^2\zeta \approx 0$$

(23)
which yields three values of $z$. It is not difficult to show that one of them, $\zeta = 0$, i.e., $z = Vt$, furnishes the global maximum of the intensity \((20)\), and therefore also of \((19)\). We can also show that the other two roots have real values only for $t \geq \sqrt{8} a/(V - c)$, and in any case furnish values of the intensity of \((19)\) much smaller than the global maximum, already found at $z = Vt \equiv z_p(t)$. So we can conclude that for $a >> s$ the peak velocity is $V > c$.

For the case $s^2c >> a^2V \rightarrow s >> a$, the two last terms in the l.h.s. of \((22)\) are:

$$s^2\eta + a^2\zeta = s^2(z - ct) + a^2(z - Vt) = (s^2 + a^2)z - (s^2c + a^2V)t \approx s^2(z - ct),$$

and so we can approximate \((22)\) with

$$\eta\zeta^2 + \eta^2\zeta + s^2\eta \approx 0 \quad (24)$$

and we can show that the root $\eta = 0$, i.e. $z = ct \equiv z_p(t)$, furnishes the global maximum of the intensity \((20)\), and therefore also of \((19)\). The other two roots have real values only for $t \geq \sqrt{8} s/(V - c)$, and furnish, once more, much smaller values of the intensity of \((19)\). In this way, for $s^2c >> a^2V \rightarrow s >> a$, the peak velocity is $c$.

Now, let us analyze, in details, examples of both cases.

**Totally “forward”, finite-energy superluminal LW pulses:**

As we have seen above, superluminal finite-energy LW pulses can be obtained from \((19)\) by putting $a >> s$. In this case, the spectrum $A(\alpha, \sigma)$ is well concentrated around the line $\alpha = \alpha_0$, and therefore in the plane $(k_z, \omega)$ this spectrum starts at $\omega_{\text{min}} \approx c\alpha_0$ with an exponential decay, and the bandwidth $\Delta \omega \approx V/s$.

The field depth of the superluminal LW pulse obtained from \((19)\), it can be calculated in a simple way. Let us examine the evolution of its peak intensity by putting $\zeta = 0 \rightarrow \eta = (1 - \beta^{-1})z$ in eq. \((20)\):
\[ |\psi(0, \zeta = 0, \eta)|^2 = \frac{e^{-2\alpha_0}}{a^2 s^2 + s^2 (1 - \beta^{-1})^2 z^2} \]  

where, of course, the coordinate \( z \) above is the position of the peak intensity, i.e., \( z = Vt \equiv z_p \). It is easy to see that the pulse presents its maximum intensity at \( z = 0 \) \( (t = 0) \), and maintains it for \( z \ll a/(1 - \beta^{-1}) \). Defining the field depth \( Z \) as the distance over which the pulse’s peak intensity remains at least 25% of its initial value, we can obtain from (25) the depth of field

\[ Z = \frac{\sqrt{3} a}{1 - \beta^{-1}} \]  

which depends on \( a \) and \( \beta = V/c \): Thus, the pulse can get large field depths by suitably adjusting the value of parameter \( a \).

Figure 5 shows the space-time evolution, from the pulse’s peak at \( z_p = 0 \) to \( z_p = Z \), of a finite-energy superluminal LW pulse represented by eq. (19) with the following parameter values: \( a = 20 \text{ m} \), \( s = 3.99 \times 10^{-6} \text{ m} \) (note that \( a >> s \) ), \( V = 1.005 c \) and \( \alpha_0 = 1.26 \times 10^7 \text{ m}^{-1} \). For such a pulse, we have a frequency spectrum starting at \( \omega_{\text{min}} \approx \omega_{\text{min}} \approx 3.77 \times 10^{15} \text{ Hz} \) (with an exponential decay) and the bandwidth \( \Delta \omega \approx 7.54 \times 10^{13} \text{ Hz} \). From these values and since \( \Delta \omega/\omega_{\text{min}} = 0.02 \), it is a optical pulse with \( \lambda_0 = 0.5 \mu\text{m} \) and time width of 13 fs. At the distance given by the field depth \( Z = \sqrt{3} a/(1 - \beta^{-1}) = 6.96 \text{ km} \) the peak intensity is a fourth of its initial value. Moreover, it is interesting to note that, in spite of the intensity decrease, the pulse’s spot size \( \Delta \rho_0 = 7.5 \mu\text{m} \) remains constant during the propagation.

\[^1\text{We can expect that, while the pulse peak intensity is maintained, the same happens for its spatial form.}\]
Figure 5: The space-time evolution, from the pulse’s peak at $z_p = 0$ to $z_p = Z$, of a totally “forward”, finite-energy, superluminal LW optical pulse represented by eq. (19), with the following parameter values: $a = 20 \text{ m}$, $s = 3.99 \times 10^{-6} \text{ m}$ (note that $a \gg s$), $V = 1.005 c$ and $\alpha_0 = 1.26 \times 10^7 \text{ m}^{-1}$.

**Totally “forward”, finite-energy luminal LW pulses:**
luminal finite energy LW pulses can be obtained from eq. (19) by making $s \gg a$ (more rigorously for $s^2 c \gg a^2 V$). In this case, the spectrum $A(\alpha, \sigma)$ is well concentrated around the line $\sigma = 0$, and therefore in the plane $(k_z, \omega)$ it starts at $\omega_{\text{min}} \approx c\alpha_0$ with an exponential decay and the bandwidth $\Delta \omega \approx c/a$.

The field depth of the luminal LW pulse obtained from (19) can be calculated in a simple way. Let us examine its peak intensity evolution by putting $\eta = 0 \rightarrow \zeta = (1 - \beta)z$ in eq. (20):
\[ |\psi(0, \zeta = 0, \eta)|^2 = \frac{e^{-2a\alpha_0}}{a^2 s^2 + a^2 (1 - \beta)^2 z^2} \]  \hspace{1cm} (27)

where, of course, the coordinate \( z \) above is the position of the peak intensity, i.e., \( z = ct \equiv z_p \). It is easy to see that the pulse has its maximum intensity at \( z = 0 \ (t = 0) \), and maintains it for \( z \ll s/(\beta - 1) \). Defining the field depth \( Z \) as the distance over which the pulse’s peak intensity remains at least 25\% of its initial value, we obtain from (27) the depth of field

\[ Z = \frac{\sqrt{3} s}{\beta - 1} \]  \hspace{1cm} (28)

which depends on \( s \) and \( \beta = V/c \).

Here, we should note that the bigger the value of \( s \), the smaller the transverse field concentration of the luminal pulse. This occurs because for big values of \( s \) the spectrum becomes strongly concentrated around \( \omega = c k_z \) and, as one knows, in this case, the solution tends to become a plane wave pulse.\footnote{\textsuperscript{1}A possible solution for this limitation it would be the use of spectra concentrated around the line \( \sigma = \sigma_0 > 0 \).}

Let us consider, for instance, a finite-energy luminal LW pulse represented by eq.(19) with \( a = 1.59 \times 10^{-6} \text{m} \), \( s = 1 \times 10^4 \text{m} \) (note that \( s >> a \)), \( V = 1.5 \text{c} \), \( \alpha_0 = 1.26 \times 10^7 \text{m}^{-1} \). For such a pulse, which has its peak travelling with the light velocity \( c \), the frequency spectrum starts at \( \omega_{\text{min}} \approx 3.77 \times 10^{15} \text{Hz} \) with a bandwidth \( \Delta \omega \approx 1.88 \times 10^{14} \text{Hz} \). Thus, it is a optical pulse with time width of 5.3 fs.

The space-time evolution of this pulse, from \( z_p = 0 \) to \( z_p = Z \), is shown in Figure (6). At the distance given by the field depth \( Z = \sqrt{3} s/(\beta - 1) = 23.1 \text{km} \) the peak intensity is a fourth of its initial value.

We can see from these two examples, and it can also be shown in a rigorous way, that the superluminal LW pulses obtained from solution (19) are superior than the luminal
Figure 6: The space-time evolution, from the pulse’s peak at \( z_p = 0 \) to \( z_p = Z \), of a totally “forward”, finite-energy, luminal LW optical pulse represented by eq.(19), with \( a = 1.59 \times 10^{-6} \text{m}, \ s = 1 \times 10^4 \text{m} \) (note that \( s \gg a \)), \( V = 1.5 \, c, \ \alpha_0 = 1.26 \times 10^7 \text{m}^{-1} \).

ones obtained from the same solution, in the sense that the former can possess large field depths and, at same time, present strong transverse field concentrations. To obtain more interesting and efficient luminal LW pulses we should use spectra concentrated around the line \( \sigma = \sigma_0 > 0 \).

2.2.3 Nonaxially symmetric, totally “forward”, LW pulses

So far, we have applied the present method to axially symmetric solutions only. A simple way for obtaining the nonaxially symmetric versions of the previous LW
pulses is through the following superposition

\[
\psi(\rho, \phi, \zeta, \eta) = (V - c) \int_0^\infty d\sigma \int_{-\infty}^\sigma d\alpha \, A'(\alpha, \sigma) J_\nu (\sqrt{\gamma^2 - 2(\beta - 1)\sigma \alpha}) \, e^{i\nu\phi} e^{-i\alpha\eta} e^{i\sigma\zeta}
\]

(29)

where \( \phi \) is the azimuth angle, \( \nu \) is an integer and

\[
A'(\alpha, \sigma) = \left( \frac{\sigma}{\sigma - 2\alpha/(\beta + 1)} \right)^{\nu/2} A(\alpha, \sigma)
\]

(30)

with \( A(\alpha, \sigma) \) being the respective spectra of the axially symmetric LW solutions of the previous subsections, which can be recover from Eq.(29) by making \( \nu = 0 \). Thus, the fundamental superposition (29) is more general than (9), in the sense that the former can yield both, axially symmetric and nonaxially symmetric LWs totally free of backward components.

**Totally “forward”, nonaxially LW of the type FWM** can be reached from Eq.(29) by using (13) into (30). After integrating directly over \( \alpha \), the integration over \( \sigma \) can be made by using the identity 3.12.5.6 in Ref.[62], furnishing:

\[
\psi(\rho, \phi, \zeta, \eta) = e^{i\nu\phi} \left( \frac{\gamma^{-1}\rho}{s - i\zeta + X^{-1}} \right)^\nu X e^{i\alpha_0 \eta} \exp \left[ \frac{\alpha_0}{\beta + 1} (s - i\zeta - X^{-1}) \right]
\]

(31)

**Totally “forward”, finite-energy, nonaxially LW** is obtained from Eq.(29) by using (16) into (30). By integrating first over \( \sigma \) (with the identity 3.12.5.6 in Ref.[62]), the integration over \( \alpha \) can be made directly, and Eq.(29) yields

\[
\psi(\rho, \phi, \zeta, \eta) = e^{i\nu\phi} \left( \frac{\gamma^{-1}\rho}{s - i\zeta + X^{-1}} \right)^\nu X \exp \left\{ -\alpha_0 \left[ \frac{(a - i\eta) - \frac{1}{\beta + 1} (s - i\zeta - X^{-1})}{(a - i\eta) - \frac{1}{\beta + 1} (s - i\zeta - X^{-1})} \right] \right\}
\]

(32)
The solution above can possess superluminal \((a >> s)\) or luminal \((s >> a)\) peak-velocity.

### 2.2.4 A functional expression for totally “forward” LW pulses

In the literature concerning the LWs\(^{36}\) some interesting approaches appear, capable of yielding functional expressions which describe LWs in closed form. Although interesting, even the LWs obtained from those approaches also possess backward components in their spectral structure.

Now, however, we are able to obtain a kind of functional expression capable of furnishing totally “forward” LW pulses.

Let us consider, in eq.(29), spectral functions \(A'(\alpha, \sigma)\) of the type

\[
A'(\alpha, \sigma) = (V - c) H(-\alpha) \left( \frac{\sigma}{\sigma - 2\alpha/(\beta + 1)} \right)^{\nu/2} \Lambda'(\alpha) e^{-s\sigma} \tag{33}
\]

where \(H(.)\) is, as before, the Heaviside function, and \(\Lambda'(\alpha)\) a general function of \(\alpha\).

Using (33) in (29), performing the integration over \(\sigma\) and making the variable change \(\alpha = -u\), we get

\[
\psi(\rho, \phi, \zeta, \eta) = e^{i\nu \phi} \left( \frac{\gamma^{-1} \rho}{s - i\zeta X^{-1}} \right)^{\nu} X \int_{0}^{\infty} du \Lambda(u) e^{-u S} \tag{34}
\]

quantity \(X\) being the ordinary X-wave \(^{12}\) and \(S\) being given by

\[
S = -i\eta - \frac{1}{\beta + 1} (s - i\zeta - X^{-1}) \tag{35}
\]

The integral in (34) is nothing but the Laplace transform of \(\Lambda(u)\).

In this way, we have that...
\[
\psi(\rho, \phi, \zeta, \eta) = e^{i\nu\phi} \left( \frac{\gamma^{-1} \rho}{s - i\zeta + X^{-1}} \right)^{\nu} X F \left( -i\eta - \frac{1}{\beta + 1} (s - i\zeta - X^{-1}) \right),
\]  
(36)

with \( F(\cdot) \) an arbitrary function, is an exact solution to the wave equation that can yield ideal, and also finite-energy LW pulses, with superluminal or luminal peak velocities. Besides this, if the chosen function \( F(S) \) in Eq.(36) is regular and free of singularities at all space-time points \((\rho, \phi, z, t)\), we can show that the LW solutions obtained from Eq.(36) will be totally free of backward components.

3 Conclusions

In conclusion, by using a unidirectional decomposition we were able to get totally "forward", ideal and finite-energy LW pulses. These new solutions are superior than the known LWs already existing in the literature, since the old solutions suffer from the undesirable presence of backward components in their spectra.

By overcoming the problem of these noncausal components, the new LWs here obtained are not obliged to have physical sense only in the cases of ultra-wideband frequency spectra; actually, our new LWs can also be quasi monochromatic pulses, and in such a way get closer to a practicable experimental realization.

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