Random potentials for Markov processes

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ABSTRACT
This paper is devoted to the integral functionals \( \int_0^\infty f(X_t) \, dt \) of Markov processes in \( \mathbb{R}^d \) in the case \( d \geq 3 \). It is established that such functionals can be presented as the integrals \( \int_{\mathbb{R}^d} f(y)G(x, dy, \omega) \) with vector valued random measure \( G(x, dy, \omega) \). Some examples such as compound Poisson processes, Brownian motion and diffusions are considered.

ARTICLE HISTORY
Received 1 August 2020
Accepted 9 July 2022

COMMUNICATED BY
K. Lee

KEYWORDS
Markov processes; green function; random green measure; compound Poisson process; Brownian motion

AMS SUBJECT CLASSIFICATIONS 2010
47D07; 37P30; 60G22; 47A30

1. Introduction
Let \( X = \{X(t), t \geq 0\} \) be a Markov process in \( \mathbb{R}^d \) starting from \( x \in \mathbb{R}^d \). For a function \( f : \mathbb{R}^d \to \mathbb{R} \) the potential of \( f \) is defined as [1]

\[
uf(x) = \int_0^\infty \mathbb{E}^x[f(X(t))] \, dt.
\]

The existence of the potential \( uf(x) \) is, in general, a difficult question and the class of admissible \( f \) shall be investigated for each process separately. An alternative approach is based on the use of the generator \( L \) of the process \( X \). Namely, the potential \( uf \) may be constructed as the solution to the following equation:

\[
-Lu = f.
\]

Of course, there will appear the technical problem of the characterization of the domain of the inverse generator \( L^{-1} \). In analogy with the PDE framework, we would like to have an integral representation

\[
uf(x) = \int_{\mathbb{R}^d} f(y)G(x, dy),
\]

where \( G(x, dy) \) is a measure on \( \mathbb{R}^d \). This measure is nothing but the fundamental solution to the considered equation and traditionally may be called the Green measure for the operator \( L \).
Another possibility to study the potential $u_f$ of $f$ is based on the following observation. A standard way to define a homogeneous Markov process is to give the probability $P_t(x, B)$ of the transition from the point $x \in \mathbb{R}^d$ to the set $B \subset \mathbb{R}^d$ in time $t > 0$. In some cases we have

$$P_t(x, B) = \int_B p_t(x, y) \, dy,$$

where $p_t(x, y)$ is the density of the transition probability. The function

$$g(x, y) = \int_0^\infty p_t(x, y) \, dt$$

is called the Green function. Of course, the existence of the Green function for a given process or for a given transition probability is a non-trivial fact also. Green functions for different classes of Markov processes are traditional objects in probability theory, see, e.g. [2,3] and references therein.

The existence of the transition density $p_t(x, y)$, even for simple classes of Markov processes, is not always guaranteed, see examples below. Hence, we introduce the Green measure

$$G(x, dy) = \int_0^\infty P_t(x, dy) \, dt$$

assuming the existence of this object as a Radon measure on $\mathbb{R}^d$, see [4].

In this paper, we are interested in another object associated with the Markov process $X$. Namely, having in mind that $X$ starts from $x \in \mathbb{R}^d$, for certain class of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we define the random variables $Y^x(f)$ by

$$Y^x(f) := \int_0^\infty f(X(t)) \, dt.$$  

It is reasonable to call these random variables random potentials of $f$. This is an additive functional, but contrary to usual for $d = 1$, we have no normalization. It will be clear from our examples given below that such object exists typically for $d \geq 3$. Then its relation to the Green measure is the following:

$$\mathbb{E}^x[Y^x(f)] = \int_{\mathbb{R}^d} f(y) G(x, dy).$$

The aim of this paper is to show that the random variables $Y^x(f)$, for certain classes of Markov processes $X$, have the representation

$$Y^x(f)(\omega) = \int_{\mathbb{R}^d} f(y) G(x, dy, \omega)$$

with vector-valued random measure $G(x, dy, \omega)$ which we will call random Green measure. This problem was already discussed in [5] in the framework of stochastic analysis. Here we would like to develop purely analytic approach to the construction and analysis of random Green measures for Markov processes. As a result, we are dealing with pairs of related objects: potentials and their representations via Green measures and random potentials and their representations via random Green measures.

The plan of this paper is as follows. We will analyze two classes of stochastic processes in $\mathbb{R}^d$, $d \geq 3$: continuous time translation invariant random walks and diffusions, in particular, the Brownian...
motion (Bm for short) and fractional Brownian motion (fBm). For each such class, we will show the existence of the random Green measures and establish some of their properties. In particular, we will describe a natural class of admissible function \( f \) integrable w.r.t. the Green measures.

Note that even for such simple Markov processes as random walks the transition probability may have a complicated behavior in time–space variables, see [6]. But the Green measures may have nice visible properties. This is related with the averaging of the transition probabilities in the definition of Green measures. The last effect is well known in different models of dynamics.

2. Random walks

2.1. Jump generators and Green measures

Let us describe briefly certain results from [4] which we need for our considerations.

Let us fix a density kernel \( a : \mathbb{R}^d \rightarrow \mathbb{R} \) with the following properties:

\[
a(-x) = a(x), \quad a \geq 0, \quad a \in C_b(\mathbb{R}^d), \\
\int_{\mathbb{R}^d} a(y) \, dy = 1.
\]

Consider the generator

\[
L f(x) = \int_{\mathbb{R}^d} a(x - y) [f(y) - f(x)] \, dy.
\]

This operator can be defined in a proper function space \( E \). As \( E \), we may consider the space of bounded measurable functions \( B(\mathbb{R}^d) \), the Banach space of bounded continuous functions \( C_b(\mathbb{R}^d) \) or the Lebesgue spaces \( L^p(\mathbb{R}^d) \), \( p \geq 1 \), depending on the case.

In particular, \( L^* = L \) in \( L^2(\mathbb{R}^d) \) and \( L \) is a bounded linear operator in all \( L^p(\mathbb{R}^d) \). We call this operator a jump generator with the jump kernel \( a \). The corresponding Markov process is of a pure jump type and is known in stochastic as a compound Poisson process [7].

Several analytic properties of jump generators were studied recently in [6,8,9]. We will formulate certain necessary facts concerning these operators.

Because \( L \) is a convolution operator, it is natural to apply Fourier transform to study it. Consider the Fourier image of the jump kernel

\[
\hat{a}(k) = \int_{\mathbb{R}^d} e^{-i(k,y)} a(y) \, dy.
\]

Then

\[
\hat{a}(0) = 1, \quad |\hat{a}(k)| \leq 1, \quad k \neq 0, \\
\hat{a}(k) \rightarrow 0, \quad k \rightarrow \infty.
\]

In the Fourier image \( L \) is the operator of multiplication by the function

\[
\hat{L}(k) = \hat{a}(k) - 1,
\]

that is, the symbol of \( L \).
In the following, we will always assume that \( \hat{a} \in L^1(\mathbb{R}^d) \) and \( a \) has finite second moment, that is,

\[
\int_{\mathbb{R}^d} |x|^2 a(x) \, dx < \infty.
\]

The resolvent \( R_\lambda(L) = (\lambda - L)^{-1} \) for \( \lambda > 0 \) has a kernel

\[
G_\lambda(x, y) = \frac{1}{1 + \lambda} \left( \delta(x - y) + G_\lambda(x - y) \right)
\]

with

\[
G_\lambda(x) = \sum_{k=1}^{\infty} \frac{a_k(x)}{(1 + \lambda)^k},
\]

where

\[
a_k(x) = a^{*k}(x)
\]

is the \( k \)-fold convolution of the kernel \( a \). As any Radon measure, the Green measure may be considered as a (translation invariant) generalized function of the form

\[
G_0(x) = \delta(x) + G_0(x).
\]

The transition probability density \( p(t, x) \) in terms of Fourier transform has representation

\[
p(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(k, x) + t(\hat{a}(k) - 1)} \, dk,
\]

and for the resolvent kernel

\[
G_\lambda(x) = -(L - \lambda)^{-1}(x),
\]

it holds that

\[
G_\lambda(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i(k, x - y)}}{1 - \hat{a}(k) + \lambda} \, dk.
\]

For a regularization of the last expression, we write

\[
\frac{1}{1 - \hat{a}(k) + \lambda} = \frac{1}{1 + \lambda} + \frac{\hat{a}(k)}{(1 + \lambda)(1 - \hat{a}(k) + \lambda)}.
\]

Then for operators we have

\[
G_\lambda = \frac{1}{1 + \lambda} + G_\lambda,
\]

or in the terms of kernels

\[
G_\lambda(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{a}(k)e^{i(k, x - y)}}{1 + \lambda 1 - \hat{a}(k) + \lambda} \, dk.
\]

To summarize our considerations, we note that the study of Green kernels is reduced to the analysis
The following objects. The regular part of the Green kernel is

\[ G_0(x) = \sum_{k=1}^{\infty} a_k(x), \]

\[ a_k(x) = a^{*k}(x) \]

\( k \)-fold convolution.

The Fourier representation of \( G_0 \) has the form:

\[ G_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{a}(k)e^{i(k,x)}}{1 - \hat{a}(k)} \, dk. \]

For \( d \geq 3 \) this integral exists for all \( x \in \mathbb{R}^d \). To see this, it is sufficient to prove that the function \( \frac{\hat{a}(k)}{1 - \hat{a}(k)} \) is integrable. As the second moment of the jump kernel \( a \) is finite, then there exists a positive constant \( C > 0 \) such that, in a neighbourhood of zero, we have \( |1 - \hat{a}(k)| \geq C|k|^2 \). The integrability of \( \frac{\hat{a}(k)}{1 - \hat{a}(k)} \) follows from the boundedness of \( \hat{a} \). The integrability outside the neighbourhood follows from the integrability of \( \hat{a} \).

Denote by \( X(t) \) our compound Poisson process or continuous time random walk. Then [4]

\[
\mathbb{E}^x \left[ \int_0^\infty f(X(t)) \, dt \right] = \int_0^\infty T(t)f(x) \, dt
\]

\[
= \int \int_{\mathbb{R}^d} f(y)G(x,dy)
\]

\[
= -(L^{-1}f)(x).
\]

Here \( G(x,dy) \) (if it exists) is called the Green measure for the process \( X \). The class of admissible \( f \) shall be discussed separately.

For the Green measure \( G(x,dy) \), we can write the representation

\[ G(x,dy) = \mathcal{G}(x,y) \, dy, \]

where \( \mathcal{G}(x,y) \) is a positive generalized function.

We know that for our processes hold

\[ \mathcal{G}(x,y) = \delta(x-y) + G(x-y), \]

where \( G(x) \) is the regular part of the Green density. More precisely, we have

\[
G(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{a}(k)e^{i(k,x)}}{1 - \hat{a}(k)} \, dk.
\]

If \( a \) has finite second moment, for \( d \geq 3 \) this integral exists for all \( x \in \mathbb{R}^d \) and is a uniformly bounded function. Then for continuous \( f \in L^1(\mathbb{R}^d) \) the expression

\[
\int_{\mathbb{R}^d} f(y)\mathcal{G}(x,dy)
\]

is well defined [4].
Consider, for $T > 0$, the random variables
\[ Y(f, T) = \int_0^T f(X(t)) \, dt. \]

Then in the sense of $L^1(P) := L^1(\Omega, P)$ (in expectation) $\lim_{T \to \infty} Y(f, T)$ exists and defines a random variable $Y(f)$. Actually, it will be reasonable to denote $Y(f)$ but in our considerations the starting point $x \in \mathbb{R}^d$ is fixed. The proof for $f \geq 0$ uses monotonicity arguments. For a general $f$ the result follows from the standard techniques.

What we know is
\[ \mathbb{E}^x[Y(f)] = f(x) + \int_{\mathbb{R}^d} f(y) G(x - y) \, dy. \] (2)

In the following, we will discuss vector-valued Radon measure, see, e.g. [10]. These are $L^1(P)$-valued measures which are finite on all bounded Borel sets $A \in \mathcal{B}_b(\mathbb{R}^d)$. To this end, we introduce the Banach space $CL(\mathbb{R}^d) := C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with the norm
\[ \|f\|_{CL} := \|f\|_\infty + \|f\|_1 := \sup_{x \in \mathbb{R}^d} |f(x)| + \int_{\mathbb{R}^d} |f(x)| \, dx. \]

**Theorem 2.1:** For each $x \in \mathbb{R}^d$ the operator
\[ Y : CL(\mathbb{R}^d) \to L^1(P) \]
has a unique representation given, for all $f \in CL(\mathbb{R}^d)$ and every $\omega \in \Omega$, by
\[ Y(f)(\omega) = \int_{\mathbb{R}^d} f(y) \mu^x(dy, \omega) \] (3)
with a vector-valued $\sigma$-additive (in the strong topology of $L^1(P)$) Radon measure $\mu^x(dy, \omega)$ on $\mathcal{B}_b(\mathbb{R}^d)$.

**Proof:** First of all we note that the map
\[ CL(\mathbb{R}^d) \ni f \mapsto Y(f) \in L^1(P) \]
is a linear continuous operator which follows from (2) taking into account the boundedness of $G$. As the space $L^1(P)$ is weakly complete (see Theorem VI.8.6 in [11]), then the map $Y$ is weakly compact, see Theorem 3.5 in [12]. Now we would like to apply the representation theorems from [10,13]. The technical difficulty here concerns the standard framework to which these representation theorems apply. The known approaches consider mappings on spaces of continuous functions (on locally compact spaces) which vanish at infinity. Our space $CL(\mathbb{R}^d)$ is a new type of Banach space of continuous functions which did not appear before in the general theory.

To obtain the desired representation we proceed as follows. At first we take the closed ball $B_N(0) \subset \mathbb{R}^d$ with radius $N \in \mathbb{N}$ centred at zero. Consider our operator $Y$ on the space $C(B_N(0))$. Then we may apply the mentioned results to obtain the representation
\[ Y(f)(\omega) = \int_{B_N(0)} f(x) \mu^x_N(dy, \omega), \quad f \in C(B_N(0)), \quad \omega \in \Omega. \]

Here $\mu^x_N(dy, \omega)$ is a vector-valued Radon measure on $\mathcal{B}(B_N(0))$ with values in $L^1(P)$. The family of measures $\mu^x_N(dy, \omega), N \in \mathbb{N}$, is consistent and defines the limit measure by $\mu^x(dy, \omega)$ on
The next interesting question is to study the random Green measure $\mu^x(dy, \cdot)$, namely if it is really a random variable or may be degenerated in a constant. To this end, we need to compute the variance of the random variable $Y^x(f)$ for $x \in \mathbb{R}^d$ and $f \in CL(\mathbb{R}^d)$. Without lost of generality and for simplicity of calculations, we take $x = 0$ and

$$f(y) = e^{-\sum_{k=1}^d |y_k|}, \quad y \in \mathbb{R}^d. \quad (4)$$

From $\mathbb{E}[\mu^0(dy, \cdot)] = \delta(y) + G(y) dy$ it follows that

$$\int_{0}^{\infty} \mathbb{E}[f(X(t))] \, dt = f(0) + \int_{\mathbb{R}^d} f(y)G(y) \, dy.$$ 

Introduce

$$V(f) = \mathbb{E} \left[ \left( \int_{0}^{\infty} f(X(t)) \, dt - \mathbb{E} \left[ \int_{0}^{\infty} f(X(t)) \, dt \right] \right)^2 \right]$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)f(y+z)G(dy)G(dz) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)f(z)G(dy)G(dz)$$

$$= f^2(0) + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)f(y+z)G(y)G(z) \, dy \, dz - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)f(z)G(y)G(z) \, dy \, dz.$$
Proposition 2.3: Assume that \( a(x) \) is even in each variable:

\[
a(x_1, \ldots, -x_k, \ldots, x_d) = a(x_1, \ldots, x_k, \ldots, x_d), \quad k = 1, \ldots, d.
\]

Then for the function \( f \) given in (4) holds \( V(f) > 0 \).

Proof: For the regular part \( G(x) \) of the Green density we have as above

\[
G(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{a}(k)e^{i(k,x)}}{1 - \hat{a}(k)} \, dk.
\]

Because \( a(x) \) is symmetric in each variable by assumption, the function \( G(x) \) has the same symmetry. Using this symmetry, we reduce the equality for \( V(f) \) to the integration over positive octant:

\[
V(f) = f^2(0) + 2 \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} f(y)f(y+z)G(y)G(z) \, dy \, dz - \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} f(y)f(z)G(y)G(z) \, dy \, dz.
\]

Now after a change of variables the integral part of this formula becomes

\[
\varepsilon^d \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} e^{-\varepsilon \sum \gamma_k - \sum \zeta_k} [2e^{-\varepsilon \sum \gamma_k} - 1] G(\varepsilon y)G(z) \, dy \, dz.
\]

In the last expression \( G(y) \) is continuous at 0 and the integrand monotonically grows, as \( \varepsilon \to 0 \), to a point-wise limiting function with infinite integral. Then, for some \( \varepsilon \) this expression is positive. Note that this expression, in fact, does not depend on \( \varepsilon \). \( \Box \)

2.2. Particular models

The main technical question is a bound for \( a_k(x) \) in \( k \) and \( x \) together for the analysis of the properties of \( G(x) \). From the stochastic point of view, \( a_k(x) \) is the density of sum of \( k \) i.i.d. random variables with distribution density \( a(x) \). Unfortunately, we could not find in the literature any general result in this direction. There are several particular classes of jump kernels for which we shall expect such kind of results. We will consider two examples, see [4] for details.

Example 2.4 (Gauss kernels): Assume that the jump kernels has the following form:

\[
a(x) = C \exp \left( -\frac{b|x|^2}{2} \right), \quad x \in \mathbb{R}^d.
\]

Proposition 2.5 (cf. Section 4.1 in [4]): For the kernel (5) and \( d \geq 3 \) holds

\[
G_0(x) \leq C_1 \exp \left( -\frac{b|x|^2}{2} \right).
\]

Example 2.6 (Exponential tails): Assume

\[
a(x) \leq C \exp(-\delta|x|), \quad x \in \mathbb{R}^d, \quad \delta > 0.
\]

Proposition 2.7 (cf. Section 4.2 in [4]): For the kernel (6) and \( d \geq 3 \) holds

\[
G_0(x) \leq A \exp(-B|x|)
\]

with certain \( A, B > 0 \).

These examples show that for concrete Markov processes the regular component of the random Green measure may have quick decay in the space variable. This gives us the possibility to use a larger class of admissible functions \( f \).
3. Gaussian processes

In this section, we provide two examples of Gaussian processes, namely Bm and fBm, and show the existence of their random Green measure. Although fBm is not a Markov process in general, it shows that our method is general enough to include these processes and also non-Gaussian processes.

3.1. Brownian motion

Let us consider a Markov process the Bm. Namely, denote $B(t), \ t \geq 0$ the Bm in $\mathbb{R}^d$ starting from the point $x \in \mathbb{R}^d$. The generator of this process is the Laplace operator $\Delta_1$ considered in a proper Banach space $E$. A sabo we are interested in studying the random variable

$$Y(f) = \int_0^\infty f(B(t)) \, dt$$

for certain class of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

**Theorem 3.1:** Let $d \geq 3$ and $x \in \mathbb{R}^d$ be given. The map

$$Y : CL(\mathbb{R}^d) \rightarrow L^1(P)$$

is a linear continuous operator and for all $f \in CL(\mathbb{R}^d)$ and every $\omega \in \Omega$ it has a unique representation

$$Y(f)(\omega) = \int_{\mathbb{R}^d} f(y) \mu^x(dy, \omega) \quad (7)$$

with a vector valued $\sigma$-additive (in the strong topology of $L^1(P)$) Radon measure $\mu^x(dy, \omega)$ on $B_0(\mathbb{R}^d)$.

**Proof:** The proof is essentially similar to the proof of Theorem 2.1. Note that due to (1) we have

$$\mathbb{E}^x[Y(f)] = -\Delta^{-1} f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-2}} \, dy.$$

Then

$$\left| \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-2}} \, dy \right| \leq \int_{|y-x| \leq 1} \frac{f(y)}{|x - y|^{d-2}} \, dy + \int_{|y-x| > 1} \frac{f(y)}{|x - y|^{d-2}} \, dy$$

$$\leq C_1 \|f\|_\infty + C_2 \|f\|_1 \leq C\|f\|_{CL}.$$

In the last but one inequality, we used the local integrability of $|x - y|^{2-d}$ in $y$, hence,

$$\|Y(f)\|_{L^1(P)} \leq C\|f\|_{CL}, \quad \forall \ f \in CL(\mathbb{R}^d).$$

This give us the possibility to apply the same arguments as in Theorem 2.1 and the representation (7) follows.
3.2. Fractional Brownian motion

Consider a $d$-dimensional fBm with Hurst parameter $H \in (0, 1)$, denoted by $B^H := \{B^H(t), \ t \geq 0\}$. Here $B^H(t) := (B^H_1(t), \ldots, B^H_d(t)), \ t \geq 0$, with all coordinates $B^H_i$ being independent $1$-dimensional fBm with the same index $H$. That is, $B^H_i$ are Gaussian processes with zero mean and covariance function

$$
E[B^H_i(t)B^H_i(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \ i = 1, \ldots, d.
$$

We would like to investigate the random variable

$$
Y^H(f) = \int_0^\infty f(x + B^H(t)) \ dt, \ f \in CL(\mathbb{R}^d), \ x \in \mathbb{R}^d.
$$

**Theorem 3.2:** Let $d \in \mathbb{N}$ and $H \in (0, 1)$ be such that $d > 1/H$. The map

$$
Y^H : CL(\mathbb{R}^d) \longrightarrow L^1(P)
$$

is a linear continuous operator and for all $f \in CL(\mathbb{R}^d)$ and every $\omega \in \Omega$ it has a unique representation

$$
Y^H(f)(\omega) = \int_{\mathbb{R}^d} f(y) \mu^{H,x}(dy, \omega)
$$

(8)

with a vector valued $\sigma$-additive (in the strong topology of $L^1(P)$ ) Radon measure $\mu^{H,x}(dy, \omega)$ on $B_b(\mathbb{R}^d)$.

**Proof:** Without lost of generality we may assume that $f \in CL(\mathbb{R}^d)$ is positive such that applying the Fubini theorem we obtain

$$
E \left[ \int_0^\infty f(x + B^H(t)) \ dt \right] = \int_0^\infty \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi t^{2H})^{d/2}} \exp \left(-\frac{|x - y|^2}{2t^{2H}}\right) \ dy \ dt
$$

$$
= \int_{\mathbb{R}^d} f(y) \left( \int_0^\infty \frac{1}{(2\pi t^{2H})^{d/2}} \exp \left(-\frac{|x - y|^2}{2t^{2H}}\right) \ dt \right) \ dx.
$$

The inner integral may be explicitly computed as

$$
\int_0^\infty (2\pi t^{2H})^{-d/2} \exp \left(-\frac{|x - y|^2}{2t^{2H}}\right) \ dt = \frac{\Gamma\left(\frac{d}{2} - \frac{1}{2H}\right)}{H^{1/2H + 1} \pi^{d/2} |x - y|^{d-1/H}}.
$$

(9)

The remaining integral is estimated as follows:

$$
\int_{\mathbb{R}^d} f(y) \frac{f(y)}{|x - y|^{d-1/H}} \ dy \leq \int_{|y| \leq 1} f(y) \frac{f(y)}{|x - y|^{d-1/H}} \ dy + \int_{|y| > 1} f(y) \frac{f(y)}{|x - y|^{d-1/H}} \ dy
$$

$$
\leq C_1 \|f\|_\infty + C_2 \|f\|_1 \leq C \|f\|_{CL},
$$

(10)

which shows that $\|Y^{H,x}(f)\|_1 \leq C \|f\|_{CL}$ for every $f \in CL(\mathbb{R}^d)$. The representation (8) follows by applying the same arguments as in Theorem 2.1.
4. Markov processes

Let \( X(t), t \geq 0 \), be a homogeneous Markov process in \( \mathbb{R}^d \) with probability \( P_t(x, B) \) of the transition from the point \( x \in \mathbb{R}^d \) to the set \( B \subset \mathbb{R}^d \) in time \( t > 0 \). In some cases we have

\[
P_t(x, B) = \int_B p_t(x, y) \, dy,
\]

where \( p_t(x, y) \) is the density of the transition probability. Formally, applying the Fubini theorem, we obtain

\[
\mathbb{E} \left[ \int_0^\infty f(X(t)) \, dt \right] = \int_0^\infty \mathbb{E}[f(X(t))] \, dt = \int_0^\infty (T_t f)(x) \, dt = \int_0^\infty \int_{\mathbb{R}^d} f(y) P_t(x, dy) \, dt
\]

\[
= \int_{\mathbb{R}^d} f(y) G(x, dy), \tag{11}
\]

where \( G(x, A) = \int_0^\infty P_t(x, A) \, dt \) is a Green measure of the process \( X \), see [4]. If the density of the transition probability exists, then we can consider the Green function

\[
g(x, y) = \int_0^\infty p_t(x, y) \, dt,
\]

and in this case formally

\[
\mathbb{E} \left[ \int_0^\infty f(X(t)) \, dt \right] = \int_{\mathbb{R}^d} f(y) g(x, y) \, dy. \tag{12}
\]

Under certain conditions on \( g(x, y) \) we can check the existence of the right-hand side of (12) and consequently, the perpetual functional or the random potential \( \int_0^\infty f(X(t)) \, dt \). Note that the examples considered before as Bm and compound Poisson process, being Markov processes, provide such examples.

Consider an example of a Markov process without independent increment. Let

\[
Lu(x) = \sum_{k,j=1}^d \partial_{x_k} a_{k,j}(x) \partial_{x_j} u(x)
\]

be a uniformly elliptic differential operator with a symmetric matrix \( (a_{k,j}) \) in the divergent form. By Aronson’s theorem [14], its heat kernel \( p_t(x, y) \) (equivalently, the transition density of the diffusion process \( X(t) \) generated by \( L \)) satisfies the two-side Gaussian bound:

\[
\frac{C_-}{t^{d/2}} \exp \left( -c_- \frac{|x - y|^2}{t} \right) \leq p_t(x, y) \leq \frac{C_+}{t^{d/2}} \exp \left( -c_+ \frac{|x - y|^2}{t} \right). \tag{13}
\]

Using this fact and similarly to the case of the Bm we may show the existence of the random Green measure for the random potential

\[
Y(f) = \int_0^\infty f(X(t)) \, dt
\]

for all \( f \in C_L(\mathbb{R}^d) \). We state this results in the following theorem.
Theorem 4.1: Let \( d \geq 3 \) and \( x \in \mathbb{R}^d \) be given. In addition, let \( X(t), t \geq 0 \), be a Markov process such that the transition density satisfies (13). The map

\[
Y : CL(\mathbb{R}^d) \longrightarrow L^1(P)
\]

is a linear continuous operator and for all \( f \in CL(\mathbb{R}^d) \) and every \( \omega \in \Omega \) it has a unique representation

\[
Y(f)(\omega) = \int_{\mathbb{R}^d} f(y) \mu^x(dy, \omega)
\]

(14)

with a vector valued \( \sigma \)-additive (in the strong topology of \( L^1(P) \)) Radon measure \( \mu^x(dy, \omega) \) on \( B_b(\mathbb{R}^d) \).

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work has been partially supported by Center for Research in Mathematics and Applications (CIMA) related with the Statistics, Stochastic Processes and Applications (SSPA) group, through the grant [UIDB/MAT/04674/2020] of FCT-Fundação para a Ciência e a Tecnologia, Portugal. The financial support by the Ministry for Science and Education of Ukraine through Project [0119U002583] is gratefully acknowledged.

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