Current Statistics for Quantum Transport through Two-Dimensional Open Chaotic Billiards

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The probability current statistics of two-dimensional open chaotic ballistic billiards is studied both analytically and numerically. Assuming that the real and imaginary parts of the scattering wave function are both random Gaussian fields, we find a universal distribution function for the probability current. In by-passing we recover previous analytic forms for calculations of transport through a Bunimovich billiard. The expressions bridge the entire region from GOE to GUE type statistics. Our analytic expressions are verified numerically by explicit quantum-mechanical calculations of transport through a Bunimovich billiard.

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I. INTRODUCTION

For quantum chaotic closed system it is well known that the statistical properties of the energy levels are described by random matrix theory (RMT) 1. They follow the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE), depending on whether time-reversal symmetry (TRS) of a system is preserved or not. In the same way the wave function statistics obeys different laws in the two cases. Let the scaled local density be \( \rho(r) = \mathcal{A} |\psi(r)|^2 \) where \( \psi(r) \) is the normalized wave function and \( \mathcal{A} \) is the area (volume) of the system. As prescribed by GOE the probability distribution is the well-known Porter-Thomas (P-T) distribution \( P(\rho) = (1/\sqrt{2\pi\rho}) \exp(-\rho/2) \) when TRS is present (the Hamiltonian \( H \) is invariant under time reversal \( \hat{T} \)). On the other hand, the distribution takes the exponential Rayleigh form \( P(\rho) = \exp(-\rho) \) as described by GUE when TRS is broken (\( H \) is not invariant on \( \hat{T} \) and \( \psi \) must be complex). It is easy to understand qualitatively why the statistics is so different in the two cases, for example, why small values of \( \rho \) have a much larger weight in GOE than in GUE. In the first case the real wave function vanishes along nodal lines in two-dimensional (2D) systems (surfaces in 3D). On the other hand, \( \psi \) is complex in the second case and vanishes only at nodal points (lines in 3D) resulting in less probability for small \( \rho \).

II. THEORY

In the following derivation of the wave function and probability current statistics we assume that the real and imaginary parts of \( \psi \) can be viewed as two independent isotropic Gaussian fields. An explicit example of such a state is given in ref. 2 in the form of a Berry-type wave-chaotic function. In general the assumption of independent fields can only make sense if we first extract a common phase factor. This feature will turn out to be most useful. Let us introduce the notations

\[
\langle u^2 \rangle = \sigma_u^2, \quad \langle v^2 \rangle = \sigma_v^2, \quad \langle uv \rangle = \gamma, \quad (2)
\]

\[
\sigma^2 = \sigma_u^2 + \sigma_v^2 = \langle |\psi|^2 \rangle, \quad \langle u \rangle = 0, \quad \langle v \rangle = 0. \quad (3)
\]

We define the averages as

\[
\langle u \rangle = 0, \quad \langle v \rangle = 0.
\]
\[ \langle ... \rangle = \frac{1}{A} \int_A d^2r \ldots, \]  

where \( A \) is the area to be sampled. In our case it will be area of the cavity, but in principle it could be any area that one may wish to diagnose. In what follows we assume that wave function \( \psi(r) \) is normalized as

\[ \int_A d^2r |\psi(r)|^2 = 1, \]  

and therefore \( \sigma^2 A = 1 \). To bring \( \psi \) to a "diagonal" form in which the real and imaginary parts are independent Gaussian fields we introduce the new functions \( p(x,y) \) and \( q(x,y) \) by changing the phase as

\[ \psi(x,y) \rightarrow e^{i\alpha} \psi(x,y) = p(x,y) + iq(x,y). \]  

The condition \( pq = 0 \) now allows us to determine \( \alpha \). By this step we will also be able to find the analytic expressions for the wave function and probability current statistics. Straightforward algebra gives

\[ \tan 2\alpha = \frac{2\gamma}{\sigma_u^2 - \sigma_v^2}, \]  

\[ \langle p^2 \rangle = \frac{1}{2} \left[ \sigma^2 + \sqrt{\sigma^4 - 4(\sigma_u^2 \sigma_v^2 - \gamma^2)} \right], \]

\[ \langle q^2 \rangle = \frac{1}{2} \left[ \sigma^2 - \sqrt{\sigma^4 - 4(\sigma_u^2 \sigma_v^2 - \gamma^2)} \right]. \]

Next let us consider the cumulative distribution \( G(\rho) \) for the scaled density \( \rho(r) = A|\psi(r)|^2 \):

\[ G(\rho) = \int_{C(\rho)} f(p,q)dpdq. \]  

The integration is defined by the circle \( C(\rho) \) in the \( (p,q) \)-plane centered at the origin and with radius \( \sqrt{\rho(r)} \), i.e., \( (p^2 + q^2)/\sigma^2 \leq \rho(\mathbf{r}) \) in the integral above. The function \( f(p,q) \) is the joint distribution for the random Gaussian fields \( p \) and \( q \):

\[ f(p,q) = \frac{1}{2\pi \sqrt{(p^2)(q^2)}} e^{-\left(\frac{p^2}{(\sigma_p^2)} + \frac{q^2}{(\sigma_q^2)}\right)/2}. \]

After integration of Eq. (8) we obtain

\[ G(\rho) \approx \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp[-\rhoq(\mu + \nu \cos \theta)]}{\mu + \nu \cos \theta} d\theta, \]  

where we introduced the following notations

\[ \mu = \frac{1}{2} \left( \frac{1}{\epsilon} + \epsilon \right), \quad \nu = \frac{1}{2} \left( \frac{1}{\epsilon} - \epsilon \right), \quad \epsilon = \sqrt{\frac{\langle q^2 \rangle}{\langle p^2 \rangle}}. \]

Differentiating (11) with respect to \( \rho \) we find the final expression for the density distribution

\[ P(\rho, \epsilon) = \mu \exp(-\rho^2 \rho) I_0(\mu \rho^2), \]  

where \( I_0(z) \) is the modified Bessel function of zeroth order.

The distribution in Eq. (12) coincides with the results obtained from RMT for closed systems \[ 5 \] and is therefore not new. For weakly open systems with point contacts Šeba et al. \[ 6 \] have related the statistical properties of the scattering matrix elements with the distribution \( P(\rho) \) and have obtained the expression above. Analogous derivation of ours is also found in Ref. \[ 7 \]. Our way of deriving Eq. (12), however, explicitly shows how to identify the two independent random fields in a given wave function. For the random wave function in Eq. (8) the limits \( \langle q^2 \rangle \rightarrow 0, \epsilon \rightarrow 0 \) correspond to a closed system and as a consequence one recovers the P-T distribution and GOE statistics. On the other hand, the case \( \langle p^2 \rangle \rightarrow \langle q^2 \rangle, \epsilon \rightarrow 1 \) corresponds in this context to an open system through which there is a current flow. Consequently one finds the exponential Rayleigh distribution that corresponds to the GUE statistics. In the crossover region, the value of \( \epsilon \) is obtained numerically using Eqs. (4), (6), (8), and (11), i.e., \( \epsilon \) is not merely a fitting parameter. We have recently verified this type of crossover in wave function statistics for a Bunimovich stadium using numerical scattering methods \[ 8 \].

In view of all previous work on the generic form of wave function statistics in chaotic systems it is surprising that no attention has been paid to the corresponding current distributions except Ref. \[ 9 \] where they only relate the average of squared current to \( \langle \rho \rangle \). Since currents may be measured \[ 3 \] it is of interest to establish a form also for currents. Below we will show how to find a useful form that is both simple and universal. Let us limit ourselves to the case of a weak net current between narrow input and output leads. Inside the cavity, however, there will be a rich, whirling flow pattern, which is strongly influenced by the vortical motions around the nodal points associated with the complex form of the wave function. Hence the net current through the through the billiard turns out to be only a tiny fraction of the total internal flow, in particular for asymmetric arrangements of leads and wave lengths that are small compared to the dimensions of the cavity. As a result the corresponding distributions may to a good approximation be chosen to be isotropic. Hence the components of the current effectively average to zero. These assumptions are verified by the numerical calculations to be discussed in the next section.

Our complex wave function (1) carries the probability current density \( \hbar = m = 1 \)

\[ \mathbf{j} = \text{Im}(\psi^* \nabla \psi) = p \nabla q - q \nabla p. \]  

To find the corresponding distribution it is convenient to begin with a characteristic function for the components of the probability current density

\[ \Theta(\mathbf{a}) = \langle e^{i\mathbf{a} \cdot \mathbf{j}} \rangle = \langle \exp[i(p\mathbf{a} \cdot \nabla q - q\mathbf{a} \cdot \nabla p)] \rangle. \]

Since \( \langle p \nabla q \rangle = \langle q \nabla p \rangle = 0 \) for isotropic fields \( \nabla p \) and \( \nabla q \) are statistically independent of \( p \) and \( q \). They have
the same distribution as in Eq. \([13]\) with dispersions
\[
\langle (\nabla p)^2 \rangle = k^2\langle p^2 \rangle \quad \text{and} \quad \langle (\nabla q)^2 \rangle = k^2\langle q^2 \rangle
\]
which follows from the Schrödinger equation. Using the relation
\[
\langle a\nabla p \rangle^2 = a^2 k^2 \langle p^2 \rangle / 2 \quad \text{and similarly for} \quad \nabla q
\]
we obtain
\[
\Theta(a) = \frac{1}{1 + \tau^2 a^2}, \quad (15)
\]
where \(a = |a|\) and
\[
\tau^2 = k^2 \langle p^2 \rangle / 2. \quad (16)
\]
From Eq. \([13]\) it is now easy to calculate the distribution functions. For the components we have
\[
P(j_x) = \langle \delta(j_x - p \frac{\partial}{\partial x} + q \frac{\partial}{\partial y}) \rangle = \\
\frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta(|a_x|) e^{-i a_x j_x} \, da_x = \frac{1}{2\pi} \exp(-|j_x|/\tau)
\]
and the same for \(P(j_y)\). In order to derive the distribution function for the absolute value of the probability current density let us consider the joint distribution function
\[
P(\{j_x, j_y\}) = \frac{1}{2\pi} \int_0^\infty a J_0(a j) \Theta(a) \, da = \frac{1}{2\pi \tau^2} K_0 \left( \frac{j}{\tau} \right),
\]
where \(j = |j|\) and \(K_0(z)\) is the modified Bessel function of the second kind. Since this expression is radially symmetric one can find the probability density function \(P(j)\) for \(j\) by just multiplying Eq. \([15]\) with a factor of \(2\pi j\). This gives us the final expression
\[
P(j) = \frac{j}{\tau^2} K_0 \left( \frac{j}{\tau} \right). \quad (19)
\]

III. NUMERICAL RESULTS

As a numerical verification of the analytic expressions for the probability current distribution, we consider an open 2D Bunimovich hard-wall stadium (see inset in Fig.\([1]\)). It is characterized by the radius of a semicircle \(a\) and the half-length of a straight section \(l\), and coupled to a pair of leads with a common width \(d\). Here we choose \((a = l)\) and \((d/\sqrt{\lambda} = 0.0935)\) for which the billiard is maximally chaotic and weakly open, respectively. To find the scattering wave function for particles entering and leaving the cavity via the leads we solve the time-independent Schrödinger equation for \(\psi\) under Dirichlet boundary conditions using a plane-wave-expansion method \([3]\), which gives reflection and transmission amplitudes for a given energy. The wave functions are used to compute the different parameters entering the statistics using the explicit expressions stated above.

Figure \([1]\) shows the transmission probability \(T\) as a function of Fermi wave number \(k\) for an incoming wave with transverse mode \(n\) in the leads. There is a sequence of overlapping resonances which become broader in the high energy region shown in the lower section of Fig.\([1]\).

For the statistical analysis of the scattering wave functions we select two typical cases: (A) A low energy with only one fully open channel \((n = 1)\) for which \(T\) reaches unity; (B) A high energy with \(n = 4\) and an intermediate value for \(T\). For the statistics the spatial average is taken over the billiard region corresponding to the closed stadium. For convenience this area is set equal to unity.

FIG. 1. Transmission probability \(T\) as a function of Fermi wave number \(k\) for the open stadium billiard: (A) A low energy case with one open channel in the leads \((n = 1)\), (B) A high energy case with \(n = 4\). The inset shows the hard-wall Bunimovich stadium and the positions of the leads.

Figure \([2]\) shows the numerical results for \(P(j), P(j_x)\) and \(P(j_y)\) together with the analytical predictions in Eqs. \([17]\) and \([19]\). In the case A there is almost no reflection and hence the system is completely coupled to the open channel. The current statistics shows, however, that \(\epsilon = 0.32\), \(i.e.,\) intermediate between a closed and fully open cases. Nevertheless, the numerical results show good agreement with the theory.

Also in the high energy region B in Fig.\([1]\) the probability current distributions are well described by the theory as shown in Fig.\([3]\). Here \(\epsilon = 0.86\) which is close to the exponential Rayleigh (GUE) case \(\epsilon = 1\). The transmis-
sion is, however, lower than in the previous example. As it appears from these there is no simple relation between $T$ and $\epsilon$.

![Graphs of probability current density](image1)

**FIG. 2.** Distribution of probability current density $P(j)$ (top) and its components $P(j_x)$ and $P(j_y)$ (middle and bottom) in the open stadium billiard for the case A in Fig. 1. Solid curves show the analytical predictions for $\epsilon = 0.32$. (For convenience $\hbar = 1, m = 1$.)

**IV. CONCLUDING REMARKS**

We have derived the statistical distributions for wave functions, probability current densities and corresponding components for 2D open quantum systems with classically chaotic dynamics. The expressions for the probability currents are universal in the sense that the shape of the distributions is independent of the mixing parameter $\epsilon$, i.e., only the width changes with $\epsilon$. This is in contrast to the wave function statistics that transforms gradually from GOE to GUE type with increasing $\epsilon$. Obviously these ideas carry over into 3D.

The statistics for mesoscopic transport through a chaotic open billiard was also studied numerically with enough statistical resolution to compare with the analytical predictions. The discussed results give a numerical proof of the predictions for the probability current density developed here. It also appears that experimental verifications are possible. For example, images of the coherent electron flow through a quantum point contact have been observed in recent experiments [8]. There is also the case of thin microwave resonators [9,10] and applying the present theory to the Poynting vector.

![Graphs of probability current density](image2)

**FIG. 3.** Same as in Fig. 2 but for the case B for which $\epsilon = 0.86$.

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