Twistor theory of higher dimensional black holes: II. Examples

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Abstract

We apply the twistor construction for higher dimensional black holes to known examples in five spacetime dimensions. First, the patching matrices are calculated from the explicit metric for these examples. Then, an ansatz is proposed for obtaining the patching matrix instead from the data of a rod structure and angular momenta. The ansatz is tested on examples with up to three nuts, and these are shown to give flat space, the Myers–Perry solution and the black ring, as expected. Rules for the transition between different adaptations of the patching matrix and the elimination of conical singularities are developed and seen to work.

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1. Introduction

This paper follows part I in which basic results were established and we shall freely use those results and the terminology established there. Thus, in five dimensions the twistor construction for black hole spacetimes can be summarized as follows.

**Summary 1.1.** There exists a one-to-one correspondence between five-dimensional stationary and axisymmetric spacetimes and rank-3 bundles $E \rightarrow \mathcal{R}$ over reduced twistor space $\mathcal{R}$, where $\mathcal{R}$ consists of two Riemann spheres identified over a certain region. $E$ can be encoded in a symmetric meromorphic $3 \times 3$ matrix $P(z)$ as follows.

If $J$ is the matrix of inner products of Killing vectors, then we define the Ernst potential adapted to a particular rod as the matrix

$$J' = \frac{1}{\det \tilde{A}} \begin{pmatrix} 1 & -x^i \\ -x^i & \det \tilde{A} \cdot \tilde{A} + x^i \end{pmatrix},$$

where $x^i$ are orbifold weights.
where \( \tilde{A} \) is obtained from \( J \) by cancelling the appropriate row and column, and \( \chi = (\chi_1, \chi_2) \) are the twist potentials.

The bundle \( E \to \mathcal{R} \) is characterized by the twistor data which for an axis-regular Ernst potential consists only of the patching matrix \( P \). The patching matrix is an analytic continuation of the Ernst potential, that is, \( P(z) = J'(0, z) \), where \( J' \) is non-singular for \( r \to 0 \).

Note that the axis-regularity fixes the three integers which are initially part of the twistor data. For the bundle corresponding to \( J \) itself, the integers are \( p_0 = 1, p_1 = p_2 = 0 \) and for the bundle corresponding to the Ernst potential \( J' \), they are \( p_0 = p_1 = p_2 = 0 \).

Moreover, we have seen in part I that the rod structure of the spacetime is coded into the poles and residues of the patching matrix.

**Proposition 1.2.** A patching matrix \( P \) has real singularities, that is, points \( z \in \mathbb{R} \), where an entry of \( P \) has a singularity, at most at the nuts of the rod structure and these real singularities are simple poles of \( P \).

This proposition requires horizons to be nondegenerate. From the definition of \( J' \), it is clear that there are different patching matrices adapted to different rods. In section 2, we calculate the patching matrix on the top-end rod for the Myers–Perry solution and the black ring solution; for flat space and the five-dimensional Schwarzschild spacetime, the patching matrix is easily computed on all rods.

The converse direction will be studied in section 3, that is, we present an ansatz for constructing the patching matrix given the data of angular momenta and the rod structure. By summary 1.1, knowing \( P \) is equivalent to determining the spacetime metric. Not all rod structures lead to solutions. In order to determine the genuinely free parameters in \( P \), it is necessary to understand how \( P \) behaves when changing from the adaptation to one rod to the one for an adjacent rod, a process we call switching. In theorem 6.5 of part I, we have seen how to switch around the nut at infinity and in section 3 we study this for an arbitrary nut. Finally, we show how to eliminate conical singularities and apply this to the black ring.

### 2. Patching matrix for relevant examples

In order to find the patching matrix, we first need to know the metric of our spacetime in the \( \sigma \)-model form, that is, we have to calculate \( J(r, z) \). After that the Ernst potential, respectively, the patching matrix, can be computed which mainly means determining the twist potentials on the axis. The easiest example to start with is flat space in five dimensions.

#### 2.1. Five-dimensional Minkowski space

The real five-dimensional Minkowski space is the manifold \( \mathbb{R}^5 \) with the metric that, in double polar coordinates, takes the form

\[
\mathrm{d}s^2 = -\mathrm{d}z^2 + R_1^2 \, \mathrm{d}\phi^2 + \mathrm{d}R_2^2 + R_2^2 \, \mathrm{d}\psi^2.
\]

The rotational Killing vector fields are \( X_1 = \frac{1}{r} \partial_\phi \) and \( X_2 = \partial_\psi \). To obtain the \( \sigma \)-model coordinates, we introduce \( z \) and \( r \) by

\[
z + ir = \frac{1}{2} \left( R_1 + iR_2 \right)^2.
\]

Then

\[
J(r, z) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & z + \sqrt{r^2 + z^2} & 0 \\
0 & 0 & -z + \sqrt{r^2 + z^2}
\end{pmatrix}
\]  

(2.1)
and
\[ e^{2\nu} = \frac{1}{2\sqrt{r^2 + z^2}}. \] (2.2)

Since dim(\text{ker}(J(0, z))) \geq 1 only for \( z = 0 \), we can read off that the metric admits two semi-infinite rods, namely \((-\infty, 0)\) and \((0, \infty)\). Because \( J \) is diagonal we have for the Killing 1-forms \( \theta_t = g(X_t, \cdot) \) that \( \theta_t \wedge \mathcal{d} \theta_t = 0 \). It follows that the twist potentials are zero without loss of generality and thereby we obtain the patching matrix as
\[ P_\mu(z) = \text{diag} \left( \frac{1}{\pm 2z}, -1, \pm 2z \right), \] (2.3)
where the upper sign combination is for \( P \) adapted to \( z > 0 \), and the lower one for \( z < 0 \).

### 2.2. Twist potentials on the axis

As part of the algorithm for obtaining \( P(z) \) from the metric we need to calculate the twist potentials just on the axis. Explicit expressions for twist potentials have been obtained for example in Tomizawa et al (2004) and Tomizawa et al (2009), but these are not in Weyl coordinates which we need here. Therefore, it is simpler to rederive some results, not only for completeness but also for providing a way of calculating the twist potentials on the axis for other spacetimes where they are not yet in the literature.

First we derive general formulae. Assume that the metric takes the form
\[ ds^2 = J_{00} \, dr^2 + 2J_{01} \, dt \, d\varphi + 2J_{02} \, dt \, d\psi + J_{11} \, d\varphi^2 + 2J_{12} \, d\psi \, d\varphi \]
and rewrite it as
\[ ds^2 = -F^2 (d\varphi + \omega_1 \, d\varphi + \omega_2 \, d\psi)^2 + G^2 (d\psi + \Omega \, d\varphi)^2 + H^2 \, d\varphi^2 + \varepsilon^{2i} (dr^2 + dz^2), \]
with
\[ F^2 = -J_{00}, \quad -F^2 \omega_1 = J_{01}, \quad -F^2 \omega_2 = J_{02}, \]
\[ -F^2 \omega_1^2 + G^2 = J_{12}, \quad -F^2 \omega_1 \omega_2 + G^2 \Omega = J_{11}, \]
\[ -F^2 \omega_2^2 + G^2 \Omega^2 + H^2 = J_{11}. \]
The latter form has been chosen to facilitate calculating \( P \) adapted to part of the axis where \( z \to \infty \) and \( d_r = 0 \). In terms of the orthonormal frame
\[ \theta^0 = F (d\varphi + \omega_1 \, d\varphi + \omega_2 \, d\psi), \quad \theta^1 = G (d\psi + \Omega \, d\varphi), \]
\[ \theta^2 = H \, d\varphi, \quad \theta^3 = e^v \, dr, \quad \theta^4 = e^v \, dz, \]
the Killing 1-forms take the form
\[ \frac{\partial}{\partial t} \to T = -F \theta^0 = -F^2 (d\varphi + \omega_1 \, d\varphi + \omega_2 \, d\psi), \]
\[ \frac{\partial}{\partial \psi} \to \Psi = G \theta^1 - F \omega_2 \theta^0. \]
Using \( d\varphi = H^{-1} \theta^2 \) and \( d\psi = G^{-1} \theta^1 - \Omega H^{-1} \theta^2 \), this yields for the first twist potential
\[ d\chi_1 = *(T \wedge \Psi \wedge dT) \]
\[ = -\frac{F^3 G}{H} * (\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge (d\omega_1 - \Omega \, d\omega_2)). \]
and for the second
\[ d\chi_2 = * (T \wedge \Psi \wedge d\Psi) = \frac{FG}{H} \ast (\theta^0 \wedge \theta^1 \wedge (G^2 d\Omega - F^2 \omega_2 \omega_1 - F^2 \omega_2 \Omega \omega_1)). \]

Since \( J = J(r, z) \), all the functions depend only on \( r \) and \( z \), hence so do \( \chi_1 \) and \( \omega_i \). Then, the total derivatives are \( d\chi_i = \partial_i \chi d\tau + \partial_i \omega_i dz \), and analogous for \( \omega_i \). Furthermore, noting that \( dr = e^{-\nu} \theta^3 \) and \( dz = e^{-\vartheta} \theta^4 \), the above equations read
\[ d\chi_1 = -\epsilon \frac{FG}{H} ((\partial_0 \omega_1 - \Omega \partial_1 \omega_2) dz - (\partial_2 \omega_1 - \Omega \partial_3 \omega_2) d\tau) \]
\[ \Rightarrow \partial_1 \chi_1 = -\epsilon \frac{FG}{H} (\partial_0 \omega_1 - \Omega \partial_1 \omega_2), \]
and
\[ d\chi_2 = \epsilon \frac{FG}{H} (G^2 \partial_1 \Omega - F^2 \omega_2 \partial_1 \omega_1 - F^2 \omega_2 \Omega \partial_1 \omega_2) dz \]
\[ - (G^2 \partial_2 \Omega - F^2 \omega_2 \partial_2 \omega_1 - F^2 \omega_2 \Omega \partial_2 \omega_2) d\tau) \]
\[ \Rightarrow \partial_2 \chi_2 = \epsilon \frac{FG}{H} (G^2 \partial_1 \Omega - F^2 \omega_2 \partial_1 \omega_1 - F^2 \omega_2 \Omega \partial_1 \omega_2), \]
with \( \epsilon \in \{\pm 1\} \) only depending on the chosen orientation of our orthonormal tetrad. To proceed, we need to specify our metric functions in order to calculate the twist potentials. First, we are going to look at the asymptotics since they will give us important information later.

2.3. Asymptotic Minkowski spacetimes

For an asymptotically flat stationary and axisymmetric spacetime in five dimensions, we learn from Harmark (2004, section IV.C) the leading terms in the approach to Minkowski space. In \( \sigma \)-model form (for \( \sqrt{r^2 + z^2} \rightarrow \infty \) and \( z/\sqrt{r^2 + z^2} \) finite), the metric coefficients behave as follows:
\[ J_{00} = -1 + \frac{4M}{3\pi \sqrt{r^2 + z^2}} + O((r^2 + z^2)^{-1}), \]
\[ J_{01} = -\frac{L_1}{\pi} \sqrt{r^2 + z^2} - \frac{z}{r^2 + z^2} + O((r^2 + z^2)^{-1}), \]
\[ J_{02} = -\frac{L_2}{\pi} \sqrt{r^2 + z^2} + \frac{z}{r^2 + z^2} + O((r^2 + z^2)^{-1}), \]
\[ J_{11} = (\sqrt{r^2 + z^2} - z) \left[ 1 + \frac{2M + \eta}{3\pi \sqrt{r^2 + z^2}} + O((r^2 + z^2)^{-1}) \right], \]
\[ J_{12} = \frac{\zeta}{r^2 (r^2 + z^2)^{1/2}} + O((r^2 + z^2)^{-1}), \]
\[ J_{22} = (\sqrt{r^2 + z^2} + z) \left[ 1 + \frac{2M - \eta}{3\pi \sqrt{r^2 + z^2}} + O((r^2 + z^2)^{-1}) \right], \]
\[ e^{2\eta} = \frac{1}{2\sqrt{r^2 + z^2}} + O((r^2 + z^2)^{-1}). \]

Here \( M \) is the mass of the spacetime and \( L_1, L_2 \) are the angular momenta; \( \zeta \) and \( \eta \) are constant where \( \eta \) is not gauge-invariant, that is, it changes under \( z \rightarrow z + \text{const.} \), unlike \( \zeta \); the periodicity of \( \varphi \) and \( \psi \) is assumed to be \( 2\pi \) (the case when it is \( 2\pi \epsilon \) is given in Harmark (2004, section IV.C) as well).
Calculating the twist potentials on the top end rod by the method described above, we obtain to leading order in $z$ the expressions
\[ \chi_1|_{r=0} \sim \frac{2\epsilon L_1}{\pi z}, \quad \chi_2|_{r=0} \sim -\frac{4\epsilon \xi}{z}, \]
where additive constants are dropped, so that both potentials go to zero at large $z$. Thus, the patching matrix to leading order in $z$ beyond (2.3) is
\[
P_+ = \begin{pmatrix}
-1 & \frac{M + \eta}{3\pi z^2} & \frac{\epsilon L_1}{\pi z^2} & \frac{2\epsilon \xi}{z^2} \\
\epsilon L_1 & \frac{M + \eta}{3\pi z^2} - 1 & 4M & -\frac{2L_2}{\pi z} \\
\frac{2\epsilon \xi}{z^2} & \frac{2L_2}{\pi z} & 2\zeta & 4(M - \eta)
\end{pmatrix}.
\]

The subscript $+$ indicates that the patching matrix is adapted to the top asymptotic end. The adaptation $P_-$ to the bottom asymptotic end, that is, the one which extends to $z \rightarrow -\infty$, is obtained by swapping $\phi$ and $\psi$ in their roles. This leads to $z \rightarrow -z$ and $L_1 \leftrightarrow L_2$. Furthermore, one has to check what happens with $\zeta$ and $\eta$ in this case. From Harmark (2004, equation (5.18)), we see that $\zeta \rightarrow \zeta$ and $\eta \rightarrow -\eta$ for the Myers–Perry solution. But all asymptotically flat spacetimes have the same fall off up to the order (2.4), so this behaviour must be generic. For the ease of reference later on we will include $P_-$ again explicitly:
\[
P_- = \begin{pmatrix}
\frac{1}{2z} & \frac{M - \eta}{3\pi z^2} & -\frac{\epsilon L_2}{\pi z^2} & -\frac{2\epsilon \xi}{z^2} \\
\frac{\epsilon L_2}{\pi z^2} - 1 & -\frac{4M}{3\pi z} & \frac{2L_1}{\pi z} \\
-\frac{2\epsilon \xi}{z^2} & \frac{2L_1}{\pi z} & 2\zeta & 4(M + \eta)
\end{pmatrix}.
\]

The Myers–Perry solution (Myers and Perry 1986), which we will study next, is the five-dimensional pendant of the Kerr solutions, that is, it describes a five-dimensional spinning black hole with a topologically spherical horizon.

2.4. Five-dimensional Myers–Perry solution

The calculation in the first part of this example up to the expression for $J(r, z)$ is based on Harmark (2004). The Myers–Perry metric is given by
\[
ds^2 = -dt^2 + \rho_0^2 \sum \left( dr - a_1 \sin^2 \theta \, d\varphi - a_2 \cos^2 \theta \, d\psi \right)^2 + (\rho_0^2 + a_1^2) \sin^2 \theta \, d\varphi^2 + (\rho_0^2 + a_2^2) \cos^2 \theta \, d\psi^2 + \frac{\Delta}{\Sigma} \, d\rho^2 + \Sigma \, d\theta^2,
\]
where
\[
\Delta = \rho^2 \left( 1 + \frac{a_1^2}{\rho^2} \right) \left( 1 + \frac{a_2^2}{\rho^2} \right) - \rho_0^2,
\]
\[
\Sigma = \rho^2 + a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta,
\]
and the coordinate ranges are
\[ t \in \mathbb{R}, \quad \varphi, \psi \in [0, 2\pi), \quad \theta \in [0, \pi]. \]
The Weyl coordinates can be taken to be
\[ r = \frac{1}{2}\rho \sqrt{\Delta} \sin 2\theta, \quad z = \frac{1}{2}\rho^2 \left( 1 + \frac{a_1^2 + a_2^2 - \rho_0^2}{2\rho^2} \right) \cos 2\theta. \]
The rod structure consists of three components \((-\infty, \alpha), (-\alpha, \alpha)\) and \((\alpha, \infty)\), where
\[
\alpha = \frac{1}{2} \sqrt{\left(\rho_0^2 - a_1^2 - a_2^2\right)^2 - 4a_1^2a_2^2}.
\]
The rod vectors turn out to be as follows.

1. If \(z\) lies in the semi-infinite spacelike rod \((\alpha, \infty)\), then the rod vector is \(\partial_y\).
2. If \(z\) lies in the finite timelike rod \((-\alpha, \alpha)\), then the kernel of \(J\) is spanned by the vector
\[
(1 \quad \Gamma_1 \quad \Gamma_2)^t,
\]
in the basis \((\partial_r, \partial_y, \partial_y)\), where \(\Gamma_{1,2}\) are the angular velocities
\[
\Gamma_1 = \frac{\rho_0^2 + a_1^2 - a_2^2 - 4\alpha}{2a_1\rho_0^2}, \quad \Gamma_2 = \frac{\rho_0^2 - a_1^2 + a_2^2 - 4\alpha}{2a_2\rho_0^2}.
\]

This rod corresponds to an event horizon with topology \(S^3\) (see Hollands and Yazadjiev (2008), proof of proposition 2 in section 3).
3. If \(z\) lies in the semi-infinite spacelike rod \((-\infty, -\alpha)\), then the rod vector is \(\partial_y\).

The conserved Komar quantities are
\[
M = \frac{3\pi}{8} \rho_0^2, \quad L_1 = \frac{3\pi}{8} a_1 \rho_0^2, \quad L_2 = \frac{3\pi}{8} a_2 \rho_0^2.
\]

Now we can again calculate the twist potentials on the top end rod as shown earlier and obtain
\[
\chi_{1|\theta=0} = -\frac{\epsilon \rho_0^2 a_1}{\rho^2 + a_1^2}, \quad \chi_{2|\theta=0} = \frac{\epsilon a_1 a_2 \rho_0^2}{\rho^2 + a_1^2}.
\]
On \(\theta = 0\), we have
\[
\rho^2 = 2z + \frac{1}{2}(\rho_0^2 - a_1^2 - a_2^2),
\]
so the notation in the calculation of \(P\) can be somewhat streamlined by introducing
\[
\beta = \frac{1}{4}(\rho_0^2 + a_1^2 - a_2^2), \quad \gamma = \frac{1}{4}(\rho_0^2 + a_1^2 - a_2^2).
\]
Then, a straightforward computation shows
\[
P_1 = \begin{pmatrix}
-\frac{z + \gamma}{2(z^2 - \alpha^2)} & -\frac{\rho_0^2 a_1}{4(z^2 - \alpha^2)} & \frac{\rho_0^2 a_1 a_2}{4(z^2 - \alpha^2)} \\
\cdot & -\frac{z^2 + z(\beta - \gamma) + \gamma^2 - \beta\gamma - \alpha^2}{z^2 - \alpha^2} & \frac{a_2 \rho_0^2(z - \gamma)}{2(z^2 - \alpha^2)} \\
\cdot & \cdot & \frac{2(z - \beta) + a_2^2 \rho_0^2(z - \gamma)}{2(z^2 - \alpha^2)}
\end{pmatrix},
\]
where the subscript indicates that it is adapted to rod 1 according to the numbering above.

In the case of \(a_1 = a_2 = 0\), the Myers–Perry metric becomes the five-dimensional Schwarzschild metric
\[
dr^2 = \left(-1 + \frac{\rho_0^2}{\rho^2}\right) \, dt^2 + \rho^2 \sin^2 \theta \, d\Omega^2 + \rho^2 \cos^2 \theta \, d\psi^2 + \left(1 - \frac{\rho_0^2}{\rho^2}\right)^{-1} \, d\rho^2 + \rho^2 \, d\theta^2.
\]
The twist potentials are globally constant and we set them without loss of generality to zero. The adaptations to the three different parts of the axis then take the following form.
1. Spacelike rod \(z \in (\alpha, \infty)\):
\[
P_1(z) = \text{diag} \left( -\frac{1}{2(z-\alpha)}, -\frac{z-\alpha}{z+\alpha}, 2(z+\alpha) \right).
\]
(2) Horizon rod \( z \in (-\alpha, \alpha) \):
\[
P_2(z) = \text{diag}\left( \frac{1}{4(z^2 - \alpha^2)}, -2(z - \alpha), 2(z + \alpha) \right).
\]
(3) Spacelike rod \( z \in (-\infty, -\alpha) \):
\[
P_3(z) = \text{diag}\left( \frac{1}{2(z + \alpha)}, -\frac{z + \alpha}{z - \alpha}, -2(z - \alpha) \right).
\]

2.5. Black ring solutions

The five-dimensional black ring of Emparan and Reall (2002) is a spacetime with a black hole whose horizon has topology \( S^1 \times S^2 \). We shall take formulae and notation from Harmark (2004, section VI).

The metric is
\[
ds^2 = -\frac{F(v)}{F(u)} \left( dt - C \kappa \frac{1 + v}{F(v)} d\phi \right)^2 + \frac{2\kappa^2 F(u)}{(u-v)^2} \left[ -\frac{G(v)}{F(v)} d\psi^2 + \frac{G(u)}{F(u)} d\psi^2 + \frac{1}{G(u)} du^2 - \frac{1}{G(v)} dv^2 \right],
\]
where \( F(\xi) \) and \( G(\xi) \) are
\[
F(\xi) = 1 + b\xi, \quad G(\xi) = (1 - \xi^2)(1 + c\xi),
\]
and the parameters vary in the ranges
\[
0 < c \leq b < 1.
\]
The parameter \( \kappa \) has the dimension of length, and for thin rings it is roughly the radius of the ring circle. The constant \( C \) is given in terms of \( b \) and \( c \) by
\[
C = \sqrt{2b(b-c)\frac{1+b}{1-b}},
\]
and the coordinate ranges for \( u \) and \( v \) are
\[
-1 \leq u \leq 1, \quad -\infty \leq v \leq -1,
\]
with asymptotic infinity recovered as \( u \to v \to -1 \). For the \( \phi \)-coordinate, the axis of rotation is \( v = -1 \), and for the \( \psi \)-direction the axis is divided into two components. First \( u = 1 \) which is the disc bounded by the ring, and second \( u = -1 \) which is the outside of the ring, that is, up to infinity. The horizon is located at \( v = -\frac{1}{b} \) and outside of it at \( v = -\frac{1}{b} \) lies an ergosurface. As argued in Emparan and Reall (2008, section 5.1.1), three independent parameters \( b, c, \kappa \) are one too many, since for a ring with a certain mass and angular momentum we expect its radius to be dynamically fixed by the balance between centrifugal and tensiinal forces. This is here the case as well, because in general there are conical singularities on the plane containing the ring, \( u = \pm 1 \). In order to cure them, \( \phi \) and \( \psi \) have to be identified with periodicity
\[
\Delta \phi = \Delta \psi = 4\pi \frac{\sqrt{F(-1)} |G(-1)|}{|G'(-1)|} = 2\pi \sqrt{\frac{1-b}{1-c}},
\]
and the two parameters have to satisfy
\[
b = \frac{2c}{1+c^2}.
\]
This leaves effectively a two-parameter family of solutions as expected with the Killing vector fields \( X_0 = \partial_t, X_1 = \partial_X \) and \( X_2 = \partial_y \). For the moment, however, we will keep the conical singularity in and regard the parameter \( b \) as free. It can be eliminated at any time using (2.12).

A straightforward calculation shows

\[
\det J = \frac{4\kappa^4}{(u-v)^4} G(u)G(v);
\]
hence, we define

\[
r = \frac{2\kappa^2}{(u-v)^2} \sqrt{-G(u)G(v)}.
\]

The harmonic conjugate can be calculated in the same way as for the Myers–Perry solution (for details see Harmark (2004, appendix H)) and one obtains

\[
z = \kappa^2 (1-uv)(2+cu+cv)/(u-v)^2.
\]

Using expressions for \( u, v \) in terms of \( r, z \) (see Harmark (2004, appendix H))

\[
u = \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 + 2(1-c^2)\kappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3},
\]

\[
v = \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 - 2(1-c^2)\kappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3},
\]

where

\[
R_1 = \sqrt{r^2 + (z+c\kappa)^2}, \quad R_2 = \sqrt{r^2 + (z-c\kappa)^2}, \quad R_3 = \sqrt{r^2 + (z-\kappa^2)^2},
\]

the \( J \)-matrix can be computed as

\[
J_{00} = -\frac{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 - 2b(1-c^2)\kappa^2}{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\kappa^2},
\]

\[
J_{01} = -\frac{2c(1-c)[R_1 - R_3](1-c)R_1 + (1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\kappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3},
\]

\[
J_{02} = \frac{R_1 - z - c\kappa^2}{J_{00}J_{22} - J_{01}^2},
\]

\[
J_{11} = \frac{-r^2}{J_{00}J_{22} - J_{01}^2},
\]

with the remaining components vanishing, and

\[
e^{2y} = [(1+b)(1-c)R_1 + (1-b)(1+c)R_2 + 2(c-b)R_3 + 2b(1-c^2)\kappa^2]
\]

\[
\times \frac{(1-c)R_1 + (1+c)R_2 + 2cR_3}{8(1-c^2)^2 R_1 R_2 R_3}.
\]

The rod structure consists of four components \((-\infty, -c\kappa^2), (-c\kappa^2, c\kappa^2), (c\kappa^2, \kappa^2)\) and \((\kappa^2, \infty)\).

1. For \( r = 0 \) and \( z \in (\kappa^2, \infty) \), we have \( R_3 - R_1 + (1+c)\kappa^2 = 0 \) which implies \( J_{00} = J_{11} = 0 \). Hence, the interval \((\kappa^2, \infty)\) is a semi-infinite spacelike rod in the direction \( \partial_z \).
2. For \( r = 0 \) and \( z \in (c\kappa^2, \kappa^2) \), we have \( R_2 + R_1 - (1-c)\kappa^2 = 0 \) which implies \( J_{22} = 0 \). Hence, the interval \((c\kappa^2, \kappa^2)\) is a finite spacelike rod in the direction \( \partial_y \).
3. For \( r = 0 \) and \( z \in (-c\kappa^2, c\kappa^2) \), we have \( R_1 + R_2 - 2c\kappa^2 = 0 \) which implies that the kernel of \( J \) in this range is spanned by the vector

\[
(1 \quad \Gamma \quad 0)^t, \quad \text{where} \quad \Gamma = \frac{b-c}{(1-c)\kappa}.
\]

8
θ where we used again the orthonormal basis thus

\[ \partial \phi \]

We see immediately that only one of the twist 1-forms is non-vanishing. On the top end rod

\[ r = 0 \] and \( z \in (\infty, -c \kappa^2) \), we have \( R_1 - R_3 + (1 + c) \kappa^2 = 0 \) which implies \( J_{22} = 0 \). Hence, the interval \((\infty, -c \kappa^2)\) is a semi-infinite spacelike rod in the direction \( \partial \phi \).

As before we compute the patching matrix. However, this time some of the metric components vanish and the metric can be written as

\[ dx^2 = J_{00} dr^2 + 2J_{01} dr d\phi + J_{11} d\phi^2 + J_{22} d\psi^2 + e^{2\nu} (dr^2 + dz^2) \]

\[ = -F^2(dr + \omega d\phi)^2 + G^2 d\phi^2 + H^2 d\psi^2 + e^{2\nu} (dr^2 + dz^2), \]

with

\[ F^2 = -J_{00}, \quad F^2 \omega = -J_{01}, \quad G^2 - F^2 \omega^2 = J_{11}, \quad H^2 = J_{22}. \]

We see immediately that only one of the twist 1-forms is non-vanishing. On the top end rod

\[ \partial \phi = 0, \]

so that we obtain

\[ \frac{\partial}{\partial t} \rightarrow T = -F^2(dr + \omega d\phi) = -F\theta^0 \]

\[ \frac{\partial}{\partial \psi} \rightarrow \Psi = H^2 d\psi = H\theta^3, \]

where we used again the orthonormal basis

\[ \theta^0 = F(dr + \omega d\phi), \quad \theta^1 = G d\phi, \quad \theta^2 = H d\psi, \quad \theta^3 = e^\nu dr, \quad \theta^4 = e^\nu dz. \]

For the twist 1-form, we then obtain

\[ dx = \partial_\chi dr + \partial_\chi dz = *(T \wedge \Psi \wedge dT) \]

\[ = *(F\theta^0 \wedge H\theta^2 \wedge F^2 d\omega \wedge G^{-1}\theta^1) \]

\[ = -\frac{F^3 H}{G} *(\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge d\omega) \]

\[ = -\frac{F^3 H}{G} *(\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge (\partial_\nu dr + \partial_\omega dz)); \]

thus

\[ \partial_\nu \chi = \epsilon \frac{F^3 H}{G} \partial_\omega, \quad \partial_\omega \chi = -\epsilon \frac{F^3 H}{G} \partial_\nu. \]

Note that

\[ \omega = \frac{J_{01}}{J_{00}}, \quad G^2 = -\frac{r^2}{J_{00}J_{22}}, \]

as \(-r^2 = \det J = (J_{00}J_{11} - J_{01}^2)J_{22}\). On \( r = 0 \), we also see that

\[ R_1 = |z + c k^2|, \quad R_2 = |z - c k^2|, \quad R_3 = |z - k^2|, \]

and for \( k^2 < z < \infty \) the moduli signs can be dropped. Then, the metric coefficients behave as

\[ J_{00} = \mathcal{O}(1), \quad J_{01} = \mathcal{O}(r^2), \quad J_{22} = \mathcal{O}(1), \quad \omega^2 = \mathcal{O}(r^2), \]

so that we obtain

\[ \partial_\nu \chi = -\epsilon \frac{(-J_{00})^2 (J_{22})^2}{r} (\partial_\nu \chi) \]

\[ = -\epsilon \frac{J_{00}J_{22}}{r} \partial_\nu \left( \frac{J_{01}}{J_{00}} \right). \]
Now, if $J_{01} = r^2 B(z) + O(r^4)$, then
\[
\lim_{r \to 0} \frac{\partial z}{\partial r} = -\epsilon \lim_{r \to 0} 2J_{00}J_{22}B(z).
\]
In order to determine $B(z)$ we perform some auxiliary calculations. Denote $\alpha = \epsilon \kappa^2$ and $\beta = \kappa^2$. Then, with $z > \beta$ and to leading order in $r$ it is
\[
R_1 = (z + \alpha) \left(1 + \frac{r^2}{2(z + \alpha)^2}\right), \quad R_2 = (z - \alpha) \left(1 + \frac{r^2}{2(z - \alpha)^2}\right),
\]
whence
\[
J_{22} = 2(z - \beta) \frac{2(z + \alpha) - r^2}{r^2} = \frac{2(z - \beta)(z + \alpha)}{z - \alpha}.
\]
Second, we compute
\[
J_{00} = -\frac{z - \alpha}{z + \lambda}, \quad \text{where} \quad \lambda = \kappa^2 \cdot \frac{2b - bc - c}{1 - c}.
\]
Last, we obtain
\[
J_{01} = -\frac{C(1 - c)\kappa^3}{2(1 - b)(z - \beta)(z + \alpha)(z + \lambda)} \cdot r^2.
\]
Using these results, (2.13) can be integrated to
\[
\chi \big|_{r=0} = \frac{2v}{z + \lambda}, \quad v = \frac{\epsilon C(1 - c)\kappa^3}{1 - b}.
\]
Note that this agrees up to a constant with Tomizawa et al (2004, equation (25)). Now we can compute the quantities which go in the patching matrix. The restriction $r = 0$ is not explicitly mentioned, but still assumed in the following:
\[
g\chi = \frac{\chi}{J_{00}J_{22}} = -\frac{v}{(z - \beta)(z + \alpha)}, \quad g = \frac{1}{J_{00}J_{22}} = -\frac{z + \lambda}{2(z + \alpha)(z - \beta)}.
\]
For the last matrix entry, we first calculate some auxiliary quantities. From (2.14), we obtain
\[
b = \frac{\lambda + \alpha}{\lambda + 2\beta - \alpha};
\]
hence,
\[
b - c = \frac{(\beta - \alpha)(\lambda - \alpha)}{\beta(\lambda + 2\beta - \alpha)}, \quad 1 + b = \frac{2(\lambda + \beta)}{\lambda + 2\beta - \alpha}, \quad 1 - b = \frac{2(\beta - \alpha)}{\lambda + 2\beta - \alpha}.
\]
This yields
\[
2v^2 = \frac{4b(b - c)(1 + b)(1 - c)^2 \kappa^6}{(1 - b)^3} = (\lambda + \alpha)(\lambda - \alpha)(\lambda + \beta),
\]
which in turn justifies the following factorization:
\[
(z - \alpha)(z + \alpha)(z - \beta) + 2v^2 = (z + \lambda)(z^2 - (\beta + \lambda)z - \alpha^2 + \beta\lambda + \lambda^2).
\]
and eventually
\[
J_{00} + g\chi^2 = -\frac{z - \alpha}{z + \lambda} - \frac{2v^2}{(z + \lambda)(z + \alpha)(z - \beta)} = -\frac{z^2 - (\beta + \lambda)z - \alpha^2 + \beta\lambda + \lambda^2}{(z + \alpha)(z - \beta)}.
\]
The patching matrix for $z \in (\beta, \infty)$ and $r = 0$ is now
\[
P_1 = \begin{pmatrix}
-\frac{z + \lambda}{2(z + \alpha)(z - \beta)} & \frac{\nu}{(z + \alpha)(z - \beta)} & 0 \\
\cdot & -\frac{z^2 - \gamma z + \delta}{(z + \alpha)(z - \beta)} & 0 \\
0 & 0 & \frac{2(z + \alpha)(z - \beta)}{z - \alpha}
\end{pmatrix}, \quad (2.15)
\]
where the index again only indicates that it is adapted to the part of the axis which extends to $+\infty$ and where
\[
\alpha = c\kappa^2, \quad \beta = \kappa^2, \quad \lambda = \kappa^2 \cdot \frac{2b - bc - c}{1 - b}, \quad \nu = c\kappa^3 \left(1 - c\right), \quad \gamma = \kappa^2 + \lambda, \quad \delta = -c^2\kappa^4 + \kappa^2\lambda + \lambda^2. \quad (2.16)
\]
Note that this is based on the assumption that the periodicity of $\varphi$ and $\psi$ is $2\pi$, otherwise it has to be modified according to Harmark (2004, equation (4.17)).

From (2.6), we read off the conserved Komar quantities as
\[
M = \frac{3\pi}{4}(\lambda + c^2\kappa^4), \quad L_1 = \frac{\pi C(1 - c)\kappa^3}{1 - b}, \quad L_2 = 0.
\]

3. The converse

As already mentioned in the introduction, the following is an immediate consequence of the twistor construction that we described in part I.

**Corollary 3.1.** The patching matrix $P$ (adapted to any portion of the axis $r = 0$) determines the metric and conversely.

**Sketch of Proof.** $J'(r,z)$ is obtained from $P(w)$ by the splitting procedure, see section 3 in Part I, and conversely $P(w)$ is the analytic continuation of $J'(r=0,z)$. □

It is known that the classification of black holes in four dimensions does not straightforwardly generalize to five dimensions. The Myers–Perry solution and the black ring are spacetimes whose range of parameters (mass and angular momenta) do have a non-empty intersection, but their horizon topology is different, which means they cannot be isometric. In order to address this issue, the rod structure is introduced to supplement the set of parameters.

Using this extended set of parameters, the following theorem from Hollands and Yazadjiev (2008) is the first step towards a classification.

**Theorem 3.2.** Two five-dimensional, asymptotically flat vacuum spacetimes with a connected horizon where each of the spacetimes admits three commuting Killing vector fields, one time translation and two axial Killing vector fields are isometric if they have the same mass and two angular momenta, and their rod structures coincide.

Note, however, that Chruściel and Nguyen (2011, proposition 3.1) suggest that by adding the rod structure to the list of parameters the mass becomes redundant, at least for the connected horizon.

**Theorem 3.2** answers the question about the uniqueness of five-dimensional black holes, but not existence. In other words, we do not yet know which combinations of the rod structure
and angular momenta are permitted, and how they determine the twistor data, that is essentially \( P \), and thereby the metric. It is natural to conjecture:

**Conjecture 3.3.** Rod structure and angular momenta determine \( P \) (even for a disconnected horizon).

### 3.1. From rod structure to patching matrix—an ansatz

In the following, we will present an ansatz for this reconstruction of the patching matrix from the given data, exemplified in cases where the rod structure has up to three nuts.

Given a rod structure with nuts at \( \{ a_i | 1 \leq i \leq N \} \), we know that \( P \) can at most have single poles at these nuts, see corollary 6.3 and proposition 6.4 in part I. We shall see that this fact can also be derived from the switching procedure (theorem 3.12) and thus we make the ansatz

\[
P(z) = \frac{1}{\Delta} P'(z),
\]

where \( \Delta = \prod_{i=1}^{N} (z - a_i) \) and the entries of \( P'(z) \) are polynomials in \( z \). If we now choose \( P \) to be adapted to the top outermost rod \((a_\gamma, \infty)\), then section 2.3 tells us its asymptotic behaviour as \( z \to \infty \), that is, \( P \) asymptotes \( P_\gamma \) given in (2.5). This implies that the entries of \( P'(z) \) are in fact polynomials of the following degrees:

\[
P'(z) = \begin{pmatrix}
q_{\gamma-2}(z) & q_{\gamma-1}(z) & q_{\gamma-1}(z) \\
\cdot & q_{\gamma}(z) & q_{\gamma-1}(z) \\
\cdot & \cdot & q_{\gamma+1}(z)
\end{pmatrix},
\]

where \( q_k \) is a polynomial of degree \( k \). (Here the notation shall just indicate the degree of the polynomials, that is, two appearances of \( q_{\gamma-1} \) or \( q_{\gamma-2} \) in different entries of the matrix can still be different polynomials, and if \( N - 2 < 0 \) then it shall be the zero-polynomial.) In fact, from (2.5) we can not only deduce the degree of the polynomials but also their leading coefficients. The diagonal entries will have leading coefficients \( -\frac{1}{2}, -1, \) and \( 2 \), respectively, and the leading coefficients on the superdiagonal will be proportional to the angular momenta. Similarly, one can use (2.6) for \( P \) adapted to the bottom outermost rod \((-\infty, a_1)\). Note that this does not impose any further restrictions on the coefficients of the spacetime metric apart from being analytic.

The number of free parameters in \( P \) equals the number of independent coefficients in the polynomials. Our aim must be to tie down our spacetime metric by fixing all those parameters in terms of the \( a_i \) and the angular momenta \( L_1 \) and \( L_2 \). Any free parameter left in \( P \) is then a free parameter in our (family of) solutions.

**Example 3.4** (One-nut rod structure), Consider the case where the rod structure has one nut, which is without loss of generality at the origin (remember that a shifted rod structure corresponds to a diffeomorphic spacetime), see figure 1. We do not make assumptions about
the angular momenta \( L_1 \) and \( L_2 \). According to our ansatz, we have for the patching matrix on the top part of the axis

\[
P_1 = \frac{1}{z} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ z+c_1 & c_2 & 2z^2 + c_3 z + c_4 \\ 0 & c_2 & 0 \end{pmatrix},
\]

which implies \( L_1 = \zeta = 0 \). On the other hand, for the bottom part it is

\[
P_2 = \frac{1}{z} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ z+c_1 & c_2 & 2z^2 + c_3 z + c_4 \\ 0 & c_2 & 0 \end{pmatrix},
\]

and therefore necessarily \( L_2 = \zeta = 0 \); thus \( c_2 = 0 \). This forces the patching matrix to be diagonal and since it has to have unit determinant,

\[
\det P_1 = \frac{1}{z^3} \left( -\frac{1}{2} \right) (z+c_1) (2z^2 + c_3 z + c_4) = 1,
\]

we obtain \( c_1 = c_3 = c_4 = 0 \). But this is the patching matrix for flat space, see (2.3).

Hence, we have shown that for a rod structure with one nut not all values for the conserved quantities are allowed, in fact they all (including mass) have to vanish, which in turn uniquely determines the spacetime as Minkowski space.

Attempting the same for a rod structure with two nuts one will quickly note that more tools are necessary in order to fix all the parameters. Here corollary 6.7 in part I is useful.

**Corollary 3.5.** In five spacetime dimensions, if \( P_+ \) is the patching matrix adapted to \((a_n, \infty)\), then \( \Delta := \prod_{i=1}^{N} (z-a_i) \) divides all \( 2 \times 2 \)-minors of \( \Delta \cdot P_+ = P'_+ \).

In part I, we have also seen that this guarantees the metric coefficients on \((a_n, \infty)\) to be bounded for \( z \downarrow a_n \). However, despite the regularity of the metric, this does not have to hold for the other nuts, as we have seen for example for the black ring.

**Example 3.6** (Two-nut rod structure). Consider the rod structure as in figure 2. In line with the above ansatz, we start off from

\[
P = \frac{1}{z^2 - \alpha^2} \begin{pmatrix} -\frac{1}{2} z + c_1 & L_1 & c_2 \\ \frac{L_1}{\pi} & -z^2 + c_3 z + c_4 & \frac{2 L_3}{\pi} z + c_5 \\ \frac{L_1}{\pi} & \frac{2 L_3}{\pi} z + c_5 & 2 z^3 + c_6 z^2 + c_7 z + c_8 \end{pmatrix},
\]

which we assume to be adapted to the top section of the axis \((\alpha, \infty)\) and where the orientation of the basis is without loss of generality chosen such that \( \epsilon = 1 \) in (2.5). Nondegeneracy requires \( \alpha > 0 \).

One restriction on the constants is immediate from (3.1): the top-left entry must not change sign on the top rod so that

\[
\alpha > 2c_1.
\]
We now make use of corollary 3.5 which for the minor obtained by cancelling the third row and first column yields

\[ c_2 z^2 + \left( -c_2 c_3 - \frac{2L_1 L_2}{\pi^2} \right) z - c_2 c_4 + \frac{L_1 c_5}{\pi} \sim z^2 - a^2, \]

where \( \sim \) means that the left-hand side has a factor \( z^2 - a^2 \). Assume that \( L_1, L_2 \neq 0 \); then comparing the (ratio of) coefficients gives

\[ c_2 c_3 = -\frac{2L_1 L_2}{\pi^2}, \quad \text{(3.3)} \]
\[ c_4 = \alpha^2 + \frac{L_1 c_5}{\pi c_2}. \quad \text{(3.4)} \]

Choosing the minor obtained from cancelling the second row and third column we obtain

\[ \frac{L_2}{\pi} z^2 - \left( \frac{1}{2} c_5 + \frac{2L_2 c_1}{\pi} \right) z + c_1 c_5 - \frac{L_1 c_2}{\pi} \sim z^2 - a^2; \]

thus

\[ c_5 = -\frac{4L_2}{\pi} c_1, \quad \text{(3.5)} \]
\[ 4c_1^2 = \alpha^2 - \frac{L_1}{L_2} c_2. \quad \text{(3.6)} \]

These four equations allow us to express \( c_1, c_2, c_4 \) and \( c_5 \) in terms of \( c_3 \) (the sign of \( c_1 \) is fixed by (3.10)).

The coefficients \( c_7 \) and \( c_8 \) can be fixed by the minor which results from cancelling the second row and the first column:

\[ \frac{2L_1}{\pi} z^3 + \frac{c_6 L_1}{\pi} z^2 + \left( \frac{c_7 L_1}{\pi} + \frac{2c_2 L_2}{\pi} \right) z + \frac{L_1 c_8}{\pi} - c_2 c_5 \sim z^3 + b z^2 - a^2 z - b a^2, \]

where \( b \) is some constant. Again the ratios of the coefficients for the linear over the cubic and the constant over the quadratic term give

\[ c_7 = -2\alpha^2 - \frac{2L_2}{L_1} c_2, \quad \text{(3.7)} \]
\[ c_8 = -\alpha^2 c_6 + \frac{\pi}{L_1} c_2 c_5. \quad \text{(3.8)} \]

The last coefficient that remains undetermined is \( c_6 \), but the determinant is going to help us for this. The requirement \( \text{det } P = 1 \) implies

\[ (z^2 - a^2)^3 = z^6 + \left( \frac{1}{2} c_6 - 2c_1 - c_3 \right) z^5 + \left( 2c_1 c_3 - c_1 c_6 - c_4 - \frac{1}{2} c_3 c_6 + \frac{1}{2} c_7 \right) z^4 + \cdots. \]

The quintic term immediately gives the desired expression

\[ c_6 = 4c_1 + 2c_3. \quad \text{(3.9)} \]

Exploiting furthermore the quartic term we obtain

\[ -3\alpha^2 = 2c_1 c_3 - c_1 c_6 - c_4 - \frac{1}{2} c_3 c_6 + \frac{1}{2} c_7, \]

which, by using the above obtained relations, is equivalent to

\[ \alpha^2 = 4c_1^2 + 4c_1 c_3 + c_3^2 + \frac{L_2}{L_1} c_2. \quad \text{(3.10)} \]
Let us relabel the parameters in accordance with Harmark (2004) as follows:

\[ c_3 = \frac{1}{2} \rho_0^2, \quad L_1 = \frac{\pi}{4} a_1 \rho_0^2, \quad L_2 = \frac{\pi}{4} a_2 \rho_0^2. \]

Note that from the asymptotic patching matrix we see that \( c_3 \) is proportional to the mass which justifies the implicit assumption about its positiveness in the above definition. However, the parameters \( (\rho_0, a_1, a_2) \) are not unconstrained as we will see soon.

By (3.3) we have

\[ c_2 = -\frac{1}{4} a_1 a_2 \rho_0^2. \quad (3.11) \]

Equations (3.6), (3.10) and (3.11) imply

\[ \frac{L_1}{L_2} c_2 = 4c_1 c_3 + c_3^2 + \frac{L_2}{L_1} c_2 \quad \Rightarrow \quad c_1 = -\frac{1}{8} \left( \rho_0^2 + a_1^2 - a_2^2 \right). \]

Moreover, from (3.5) and (3.9) we obtain

\[ c_5 = \frac{1}{2} a_2 \rho_0^2 \left( \rho_0^2 + a_1^2 - a_2^2 \right) \quad \text{and} \quad c_6 = \frac{1}{4} \left( \rho_0^2 - a_1^2 + a_2^2 \right). \]

Continuing with (3.4) yields

\[ c_4 = \alpha^2 - \frac{1}{4} \rho_0^2 \left( \rho_0^2 + a_1^2 - a_2^2 \right), \]

and (3.7) and (3.8) give

\[ c_7 = -2 \alpha^2 + \frac{1}{4} a_2^2 \rho_0^2, \]

\[ c_8 = \frac{1}{4} \alpha^2 \left( -\rho_0^2 + a_1^2 - a_2^2 \right) - \frac{1}{4} a_2^2 \rho_0^2 \left( \rho_0^2 + a_1^2 - a_2^2 \right). \]

With these parameters being determined and with the help of (3.10), we can write \( \alpha \) explicitly as

\[ \alpha^2 = \frac{1}{4} \left( \rho_0^2 - a_1^2 - a_2^2 \right)^2 - \frac{1}{4} a_1^2 a_2^2. \quad (3.12) \]

Comparing those expressions with (2.10) one will find that they coincide. However, note that

\[ 16 \alpha^2 = \rho_0^4 - 2 \rho_0^2 \left( a_1^2 + a_2^2 \right) + (a_1^2 - a_2^2)^2, \]

which implies that for real non-zero \( \alpha \) we need the left-hand side to be positive and therefore we need either \( \rho_0^2 > (\|a_1\| + |a_2|)^2 \), a condition on the asymptotic quantities familiar from the discussion of the Myers–Perry solution in Emparan and Reall (2008) and Myers (2011), or \( 0 < \rho_0^2 < (\|a_1\| - |a_2|)^2 \). This latter possibility is ruled out by (3.2): we want \( \alpha > 2c_1 \) while (3.6) gives \( \alpha^2 \leq 4c_1^2 \), so we must have \( c_1 < 0 \) or

\[ \rho_0^2 \geq a_1^2 - a_2^2. \]

From the bottom rod we must obtain this condition with \( a_1 \) and \( a_2 \) interchanged, so that we require

\[ \rho_0^2 \geq |a_1^2 - a_2^2|. \quad (3.13) \]

and this is incompatible with \( 0 < \rho_0^2 < (\|a_1\| - |a_2|)^2 \) and nondegeneracy.

Mass and angular momenta form a set of three parameters and the position of the nuts can be expressed in terms of these three parameters. This is more than one would have expected just from theorem 3.2. However, we stated already that by Chruściel and Nguyen (2011, proposition 3.1) the mass is redundant in the set of parameters. Here we did not eliminate the mass, but rather the rod length. If one instead replaces \( M \) by \( \alpha \) in the set of parameters, then one obtains an equation for \( M \): by rearranging (3.12) one seeks positive \( c_3 \) which satisfy a sixth-order polynomial. With no further conditions on \( (\alpha > 0, L_1, L_2) \), there are again two positive solutions for \( c_3 \) (unless \( L_1^2 = L_2^2 \) when there is only one), but once again one branch is ruled out by (3.3).
Some of the steps above, when we determined all the parameters in the patching matrix, required \( L_1 L_2 \neq 0 \). Assuming that one of the angular momenta vanishes leads to dichotomies at certain steps when solving for the \( c_i \). Some of the branches in this tree of possibilities lead to contradictions, while others lead to valid solutions such as the Myers–Perry solution with one vanishing angular momentum or an ultrastatic solution, that is, where \( g_{tt} = 1, g_{ti} = 0 \) (which must violate one of the global conditions as the mass is zero). On the other hand, at no point did we use the fact that the middle rod is a horizon.

Note also that issues of conicality cannot arise here as the periodicities of \( \phi \) and \( \psi \) are chosen to be \( 2\pi \) on the outer parts of the axis and no further spatial rods are left. When we turn to a larger numbers of nuts there could be conical singularities.

Moving on to a rod structure with three nuts, we will consider the simpler case where one of the Killing vectors is hypersurface-orthogonal.

**Example 3.7** (Three-nut rod structure with one hypersurface-orthogonal Killing vector). We consider the rod structure as in figure 3. Together with \( L_1 = L \neq 0, L_2 = 0 \) this comprises our twistor data. In order to simplify the calculations, we would like to make assumptions such that the two non-diagonal entries in the third row and column of the patching matrix vanish (when adapted to \( \beta, \infty \)). One therefore needs \( g_{\phi \psi} = g_{\psi \psi} = 0 \). This cannot be concluded from \( L_2 = 0 \), as the Black Saturn shows (see Elvang and Figueras (2007)). We thus make the assumption that \( \partial \psi \) is hypersurface-orthogonal, that is, \( \Psi \) and \( d\Psi = 0 \), so that \( g_{t\phi} = g_{\psi \psi} = 0 \), and \( \chi_2 = 0 \).

These assumptions turn our ansatz into

\[
P = \frac{1}{\Delta} \begin{pmatrix} q(z) & l(z) & 0 \\ l(z) & c(z) & 0 \\ 0 & 0 & Q(z) \end{pmatrix},
\]

where

\[
\Delta(z) = (z + \alpha)(z - \alpha)(z - \beta),
\]
\[
q(z) = \frac{1}{2} z^2 + c_1 z + c_2,
\]
\[
l(z) = \frac{L}{\pi} z + c_3,
\]
\[
c(z) = -z^3 + c_4 z^2 + c_5 z + c_6,
\]
\[
Q(z) = 2z^4 + c_7 z^3 + c_8 z^2 + c_9 z + c_{10}.
\]

Corollary 3.5 gives the following conditions:

\[
qc - l^2 = \tilde{q}_1 \Delta, \quad \tilde{q}_1 \text{ quadratic},
\]
\[
Qq = \tilde{c}_1 \Delta, \quad \tilde{c}_1 \text{ cubic},
\]
\[
QL = \tilde{q}_2 \Delta, \quad \tilde{q}_2 \text{ quadratic},
\]
\[
Qc = \tilde{Q}_1 \Delta, \quad \tilde{Q}_1 \text{ quartic}.
\]

(3.14)
The condition for the patching matrix to have unit determinant then implies
\[ \Delta^3 = Q(qc - l^2) = Q\tilde{q}_1 \Delta \iff \Delta^2 = Q\tilde{q}_1. \]  
(3.15)

Now, as \( \tilde{q}_1 \) is a quadratic, there are six possibilities for it to be a product of \((z + \alpha), (z - \alpha)\) and \((z - \beta)\). But \( \partial_{\psi} \) is 0 on \((\alpha, \beta)\); thus \( Q/\Delta \to 0 \) for \( z \downarrow \beta \). To guarantee this \((z - \beta)^3 \) has to divide \( Q \), which rules out three of those six possibilities. Furthermore, by corollary 3.5 we have
\[ \frac{\tilde{q}_1}{\Delta} = \frac{1}{\det \tilde{A}_4} \text{ on } (-\infty, -\alpha), \]
where \( \tilde{A}_4 \) is obtained from \( J \) by cancelling the rows and columns containing inner products with \( \partial_{\psi} \). But from the general theory we know that the entry of \( P \) with the inverse determinant contains a simple pole when approaching the nut, that is, \( z \downarrow -\alpha \), so that \( \tilde{q}_1(-\alpha) \neq 0 \). This immediately yields
\[ \tilde{q}_1 = \frac{1}{2}(z - \alpha)^2 \text{ and by (3.15) also } Q = 2(z + \alpha)^2(z - \beta)^2. \]

Now observe that there is a factor of \((z - \alpha)\) in \( \Delta \) but not in \( Q \), so that by (3.14) the monic \((z - \alpha)\) has to divide \( l, q \) and \( c \). We write this as
\[ l = \frac{L}{\pi}(z - \alpha), \quad q = -\frac{1}{2}(z - \alpha)\tilde{l}, \quad c = -(z - \alpha)\tilde{q}_3, \]
where
\[ \tilde{l}_1 = z + A, \quad \tilde{q}_3 = z^2 + Bz + C \quad \text{ for } A, B, C = \text{const.} \]

The first equation in (3.14) then turns into
\[ \tilde{l}_1\tilde{q}_3 - \frac{2L^2}{\pi^2} = \Delta \]
\[ \iff z^3 + (A + B)z^2 + (C + AB)z + AC - \frac{2L^2}{\pi^2} = z^3 - \beta z^2 - \alpha^2 z + \alpha^2 \beta. \]

Comparing the coefficients, one sees
\[ B = -A - \beta, \quad C + AB = -\alpha^2, \quad AC - \frac{2L^2}{\pi^2} = \alpha^2 \beta, \]
and therefore \( A \) satisfies
\[ \frac{1}{A} \left( \alpha^2 \beta + \frac{2L^2}{\pi^2} \right) - A(A + \beta) = -\alpha^2 \]
\[ \iff A^3 + \beta A^2 - \alpha^2 A - \alpha^2 \beta - \frac{2L^2}{\pi^2} = 0. \]

Writing \( F(\alpha) := \alpha^3 + \beta \alpha^2 - \alpha^2 \alpha - \alpha^2 \beta - \frac{2L^2}{\pi^2} \), we see that since \( F(0) < 0 \), this last polynomial has to have at least one (positive) real root which we will call \( A \) (see figure 4). Now from \( F'(A) = 3A^2 + 2\beta A - \alpha^2 \) one concludes that the local maximum of \( F \) is at
\[ a_{\max} = -\frac{1}{3}(\beta + \sqrt{\beta^2 + 3\alpha^2}). \]

Furthermore, note that since \( \alpha \leq \beta \), we have
\[ a_{\max} \leq -\frac{1}{3}(\alpha + \sqrt{\alpha^2 + 3\alpha^2}) = -\alpha \quad \text{and} \]
\[ F(-\alpha) = -\frac{2L^2}{\pi^2} < 0, \]
Figure 4. The cubic $F(a)$.

which implies that if $F$ has two more real roots, they will both be smaller than $-\alpha$. On the other hand, there is a constraint on $A$ obtained from the asymptotics. In our patching matrix, the central entry is

$$\frac{c}{\Delta} = -(z - \alpha)(z^2 + Bz + C) = -1 + \frac{1}{\Delta} \left( (\alpha - B)z^{-1} + \cdots \right).$$

Using (2.5) and the relation between $A$ and $B$, this gives

$$A + \alpha = \frac{4M}{3\pi}.$$

Positivity of $M$ thus implies $A > -\alpha$ and we therefore have shown that there is a unique positive $A \in \mathbb{R}$ which satisfies all the constraints.

Consequently, by our ansatz we are able to fix all the parameters in terms of $\alpha$, $\beta$ and $L$, that is in terms of the given rod and asymptotic data, and the patching matrix is

$$P_1 = \begin{pmatrix}
-z + A & \frac{L}{\pi (z + \alpha)(z - \beta)} & 0 \\
\frac{z^2 - \tilde{\gamma} z + \tilde{\delta}}{(z + \alpha)(z - \beta)} & 0 & \frac{2(z + \alpha)(z - \beta)}{z - \alpha} \\
0 & 0 & 1
\end{pmatrix},$$

where

$$\tilde{\gamma} = \beta + A, \quad \tilde{\delta} = -\alpha^2 + \beta A + A^2.$$  

Now compare this with (2.15): since $\lambda$ and $A$ are zeros of the same polynomial and are restricted by the same inequality involving the mass, they are equal and we have derived the patching matrix for the black ring with the conical singularity not yet removed. (We are grateful to Harvey Reall for suggesting this possibility.)

For the regular black ring, removing the conical singularity gives the angular momentum $L$ in terms of $\alpha$ and $\beta$. In this formalism, removing the conical singularity requires more work which we turn to next.
\[
\frac{\partial}{\partial \psi} = 0 \quad \frac{\partial}{\partial \phi} = 0
\]

\[
v = 0 \quad u = 0
\]

**Figure 5.** Two spatial rods with their rod vectors meeting at a nut.

### 3.2. Local behaviour of \( J \) around a nut

For the case of a rod structure with three nuts and \( L_1 L_2 \neq 0 \), and generally as the number of nuts gets higher, one needs more constraints and these will come from the inner rods. It is therefore important to have an understanding of how the patching matrices with adaptations to adjacent rods are related to each other. We have seen an example in theorem 6.5 in part I, which can be considered as such a switch at the nut at infinity. The proof gives an idea of what is happening when changing the adaptation, yet it will be more difficult for interior nuts, that is, nuts for which \( |a_i| \) is finite.

A strategy of how to achieve this is described in Fletcher (1990, chapter 3). There the essence is that ‘... redefining the spheres \( S_0 \) and \( S_1 \) by interchanging double points alters the part of the real axis to which the bundle is adapted.’ (Fletcher 1990, section 3.2). However, as the example in Fletcher (1990, section 5.1) shows, this comes down to a Riemann–Hilbert problem which will be rather hard and impractical to solve in five or even higher dimensions. Thus, we will approach this task in a different way. The idea is that we start off as above on the outermost rods where \( |z| \to \infty \), determine as many free parameters as possible by the constraints which we have got on these rods, then take the resulting \( P \)-matrix (still having free parameters in it which we would like to pin down), calculate its adaptation to the next neighbouring rod and apply analogous constraints there. But before looking at the patching matrix itself let us first study how \( J \) behaves locally around a nut.

Consider first a nut where two spatial rods meet, that is, like in figure 5.

Without loss of generality assume that the nut is at \( z = 0 \). In this case, a suitable choice of coordinates is the \((u, v)\)-coordinates defined as

\[
r = uv, \quad z = \frac{1}{2} (v^2 - u^2) \leftrightarrow u^2 = -z \pm \sqrt{r^2 + z^2}, \quad v^2 = z \pm \sqrt{r^2 + z^2},
\]

where the signs on the right-hand side are either both plus or both minus. If we choose both signs to be plus, then the rod \( \partial \phi = 0 \) corresponds to \( u = 0 \) and \( \partial \psi = 0 \) to \( v = 0 \). The metric in the most general case has the form

\[
d s^2 = X \, d t^2 + 2Y \, d t \, d \phi + 2Z \, d t \, d \psi + U \, d \phi^2 + V \, d \psi^2 + W \, d \phi \, d \psi + e^{2\nu} (u^2 + v^2) (du^2 + dv^2),
\]

or equivalently

\[
J(u, v) = \begin{pmatrix} X & Y & Z \\ U & V & W \end{pmatrix}.
\]

We assume that \( \phi, \psi \) have period \( 2\pi \).

**Theorem 3.8.** For a spacetime regular on the axis, the generic form of \( J \) in \((u, v)\)-coordinates around a nut, where two spacelike rods meet, is

\[
J = \begin{pmatrix} X_0 & u^2Y_0 & v^2Z_0 \\ u^2U_0 & v^2V_0 & u^2v^2W_0 \end{pmatrix},
\]

and, furthermore, one needs
If one of the rods is the horizon instead of a spacelike rod, then corresponding statements hold.

The second part of the theorem is closely tied to the problem of conicality, which we will investigate shortly.

**Proof.** Introduce Cartesian coordinates

\[
x = u \cos \phi, \quad y = u \sin \phi, \quad z = v \cos \psi, \quad w = v \sin \psi,
\]

then the metric becomes in these coordinates

\[
d s^2 = X \, dt^2 + 2 \frac{Y}{u^2} \, dx \, dy + 2 \frac{Z}{v^2} \, dw \, dz + \frac{U}{u^4} \, (dx - y \, dy)^2 \\
+ 2 \frac{V}{u^4} \, (dx - y \, dx)(z \, dw - w \, dz) + \frac{W}{v^4} \, (z \, dw - w \, dz)^2 \\
+ e^{2v} (u^2 + v^2) \left( \frac{1}{u^2} \, (dx + y \, dy)^2 + \frac{1}{v^2} \, (z \, dz + w \, dw)^2 \right).
\]

The \( x, y, z \) and \( w \) are not to be confused with the earlier use of the same symbols. Set \( X_0 = X \). Now as \( u \to 0 \) for constant \( v \) we immediately see that in order for \( g_{ty} \) and \( g_{xw} \) to be bounded we need \( Y = u^2 Y_0 \) and \( V = u^2 V_1 \) for bounded \( Y_0 \) and \( V_1 \). The remaining singular terms are

\[
\frac{U}{u^4} \, (dx - y \, dx)^2 + e^{2v} (u^2 + v^2) \frac{1}{u^2} \, (dx + y \, dy)^2.
\]

For the fourth-order pole not to be dominant, we need \( U = u^2 U_0 \) for bounded \( U_0 \); then it is required that

\[
\frac{U_0}{v^2 e^{2v}} = 1 \quad \text{as a function of } v \text{ on } u = 0
\]

(3.20)

to remove the remaining second-order pole.

Repeating this for \( v \to 0 \) with fixed \( u \) yields \( Z = v^2 Z_0, V_1 = v^2 V_0, W = v^2 W_0 \) and

\[
\frac{W_0}{u^2 e^{2v}} = 1 \quad \text{as a function of } u \text{ on } v = 0.
\]

This is the minimum that we can demand in terms of regularity of \( J \) on the axis and near the nuts.

Assuming now without loss of generality that in figure 5 the axis segment where \( v = 0 \) is the horizon, we have seen in section 5 in part I that then the first row and first column degenerate. So, we substitute

\[
z = v \cosh(\omega t), \quad w = v \sinh(\omega t),
\]

where \( \omega \) is a constant with no further restriction. The coordinates \( x \) and \( y \) choose as in (3.18). Now the above argument works analogously with all results equivalent, but

\[
\frac{X_0}{v^2 e^{2v}} = -\omega^2 \quad \text{as a function of } v \text{ on } u = 0.
\]
3.3. Conicality and the conformal factor

Returning to the case as depicted in figure 5, we saw in (3.20) that regularity at an axis segment where $\partial \varphi$ vanishes forces a relation between $\varphi_{\theta\phi}$ and the conformal factor $e^{2\varphi}$ of the $(r, z)$-metric. In this section, we first establish the following.

**Proposition 3.9.** On a segment of the axis where $u = 0$ we have $\frac{U_0}{v^2e^{2\varphi}} = \text{constant}.$

**Proof.** To prove this we need to consider how the conformal factor varies on the axis and this is obtained from the second part of the Einstein field equations

$$\partial_\xi (\log(r e^{2\varphi})) = \frac{ir}{2} \text{tr}(J^{-1} J^{-1} J^{-1}).$$

(3.21)

It will be convenient to work with $\chi = u + iv$, where $\xi = z + ir = \frac{1}{2} \chi^2$ and concentrate on the conformal factor of the $(u, v)$-metric which is $(u^2 + v^2) e^{2\varphi}$ by (3.19). Then,

$$\partial_\xi (\log((u^2 + v^2) e^{2\varphi})) = \partial_\chi (\log((u^2 + v^2)(uv) e^{2\varphi}))$$

$$= \frac{1}{\chi} \left( \frac{1}{2u} + \frac{i}{2v} + \frac{1 + iuv}{2(u + iv)} \text{tr}(J^{-1} J^{-1} J^{-1}) \right).$$

Close to the axis segment $u = 0$ we substitute from (3.17) and expand in powers of $u$ to find

$$\partial_\xi (\log((u^2 + v^2) e^{2\varphi})) = \frac{1}{\chi} \left( \frac{1}{2u} + \frac{i}{2v} + \frac{1 + iuv}{2(u + iv)} \left( \frac{K_1}{u^2} + \frac{K_2}{u} + O(1) \right) \right),$$

(3.22)

where

$$K_1 = (U_0 X_0 W_0 - v^2 Z_0^2)^2 = 1 + \frac{1}{v^2} O(u^2),$$

$$K_2 = (U_0 X_0 W_0 - v^2 Z_0^2)^2 \frac{\partial U_0}{U_0}.$$ 

The right-hand side of the first equation follows from the determinant

$$u^2 v^2 = \det J = u^2 v^2 X_0 U_0 W_0 - u^2 v^4 U_0 Z_0^2 + O(u^4).$$

Taking in (3.22) the limit on to $u = 0$ we obtain just

$$\partial_\chi (\log(v e^{2\varphi})) = \partial_\chi \log(U_0),$$

so that

$$\frac{U_0}{v^2 e^{2\varphi}} = \text{constant} \quad \text{on } u = 0.$$ 

Thus, (3.20) will hold at all points of the axis segment if it holds at one. The following proposition is an analysis similar to Harmark (2004, appendix H), but it is simpler and more self-contained to rederive it than translate it.

**Proposition 3.10.** As a function on the axis $\{u = 0\} \cup \{v = 0\}$, that is, as a function of one variable, the factor $(u^2 + v^2) e^{2\varphi}$ is continuous at the nut $u = v = 0$.

**Proof.** Near the nut introduce polar coordinates

$$u = R \cos \Theta, \quad v = R \sin \Theta,$$

so that from (3.21) we obtain

$$\partial_\Theta (\log((u^2 + v^2) e^{2\varphi})) = (u \partial_u - v \partial_v) (\log((u^2 + v^2) e^{2\varphi}))$$

$$= -\frac{u}{v} + \frac{v}{u} - \frac{iuv}{4} \text{tr}(J^{-1} J^{-1} J^{-1} J^{-1} - J^{-1} J^{-1} J^{-1}).$$
Again we expand this using (3.17) to find
\[
\partial_\Theta (\log ((u^2 + v^2) e^{2v})) = -\frac{\mu}{v} + \frac{v}{u} - \frac{\mu}{v} \left( U_0 (X_0 W_0 - u^2 Z_0^2) \right)^2 + \frac{\mu}{v} \left( W_0 (X_0 U_0 - u^2 Y_0^2) \right)^2 + O(u) + O(v)
\]
\[
= O(u) + O(v) = O(R).
\]
Now the jump in \(\log((u^2 + v^2) e^{2v})\) round the nut is
\[
\Delta (\log((u^2 + v^2) e^{2v})) = \lim_{R \to 0} \int_{0}^{2\pi} \partial_\Theta (\log((u^2 + v^2) e^{2v})) \, d\Theta = 0,
\]
and \((u^2 + v^2) e^{2v}\) does not jump either.

On \(u = 0\), \(U_0\) is continuous and by proposition 3.9 \(\frac{U_0}{\nu e^{2\nu}}\) is constant, so \(v^2 e^{2v}\) must be bounded there. Similarly, on \(v = 0\) for \(W_0\) and \(\frac{W_0}{\nu e^{2\nu}}\). Thus, \((u^2 + v^2) e^{2v}\) is continuous on the two rods and has no jump across the nut, so it is continuous on the axis. □

The strategy for removing conical singularities is now clear: we start by assuming that \(\phi\) and \(\psi\) both have period \(2\pi\). On the part of the axis extending to \(z = +\infty\), where the Killing vector \(\partial_\phi\) vanishes, we have \(\frac{U_0}{\nu e^{2\nu}}\) = constant by proposition 3.9 and the asymptotic conditions we are imposing make this constant one. The corresponding statement holds on the part of the axis extending to \(z = -\infty\) for the same reason. When passing by a nut between two spacelike rods we may suppose, by choosing the basis of Killing vectors appropriately, that \(\partial_\phi\) vanishes above the nut and \(\partial_\phi\) below and we know by proposition 3.10 that \((u^2 + v^2) e^{2v}\) is continuous at the nut. If there is no conical singularity above the nut, then we have \(\frac{U_0}{\nu e^{2\nu}}\) = 1 there and we want \(\frac{W_0}{\nu e^{2\nu}}\) = 1 below the nut. Therefore, we require the limits of \(U_0\) from above and \(W_0\) from below to be equal.

**Corollary 3.11.** With the conventions leading to (3.17), the absence of conical singularities requires
\[
\lim_{v \to 0} U_0 = \lim_{u \to 0} W_0.
\]

This is what we have just shown. At a nut where one rod is the horizon we do not obtain further conditions as we have no reason to favour a particular value of \(\omega\). To see how this is applied to the case of the black ring we need a better understanding of going past a nut.

### 3.4. Local behaviour of \(P\) around a nut: switching

In this section, we establish a prescription for obtaining the matrix \(P_\ast\) adapted to the segment of the axis below a nut from the matrix \(P_\ast\) adapted to the segment above. We call this process ‘switching’. Once we have the prescription we can impose the condition of non-conicality found in corollary 3.11. We then apply this to the black ring, but it is clear that with this prescription we have an algorithm for working systematically down the axis given any rod structure so that we obtain all the matrices \(P_\ast\) adapted to the different rods labelled by \(i\). The result is the following.

**Theorem 3.12.** Let \(z = a\) be a nut where two spacelike rods meet, as in figure 5, and assume that we have chosen a gauge where the twist potentials vanish when approaching the nut. Then
\[
\begin{pmatrix}
0 & 0 & \frac{1}{2(z-a)} \\
0 & 1 & 0 \\
\frac{1}{2(z-a)} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 2(z-a) \\
0 & 1 & 0 \\
\frac{1}{2(z-a)} & 0 & 0
\end{pmatrix}
\]
where \(P_\ast\) is adapted to \(u = 0\) and \(P_\ast\) is adapted to \(v = 0\).
We begin by motivating this prescription from a consideration of (3.16). First, calculate the twist potentials in the same way as in section 2. The metric (3.16) can be rearranged in orthonormal form
\[ ds^2 = X (dr + \omega_1 d\phi + \omega_2 d\psi)^2 + \tilde{U} (d\phi + \Omega d\psi)^2 + \tilde{W} d\psi^2 - e^{2\nu} (dr^2 + dz^2). \]

The orthonormal frame is again
\[ \theta^0 = X^{1/2} (dr + \omega_1 d\phi + \omega_2 d\psi), \quad \theta^1 = \tilde{U}^{1/2} (d\phi + \Omega d\psi), \]
\[ \theta^2 = \tilde{W}^{1/2} d\psi, \quad \theta^3 = e^\nu dr, \quad \theta^4 = e^\nu dz, \]
so
\[ X \omega_1 = Y, \quad X \omega_2 = ZX \omega_1 \omega_2 + \tilde{U} \Omega = V, \]
\[ \tilde{U} + X \omega_1^2 = U, \quad \tilde{W} + \tilde{U} \Omega^2 + X \omega_2^2 = W. \]

Adapted to \( \partial \psi = 0 \), then for small \( r \) it is \( Z, V, W \in \mathcal{O}(r^2) \); hence \( \omega_2, \Omega, \tilde{W} \in \mathcal{O}(r^2) \) (in order to see that \( \Omega \in \mathcal{O}(r^2) \), derive from \( X \in \mathcal{O}(1) \) and \( \tilde{U} X = UX - Y^2 \) that \( \tilde{U} \in \mathcal{O}(1) \)) and the other terms \( \mathcal{O}(1) \). This implies \( \frac{W}{W} \rightarrow 1 \) as \( r \rightarrow 0 \).

Now the 1-forms are
\[ \partial_r \rightarrow T = X^{1/2} \theta^0, \quad \partial_\psi \rightarrow \Phi = \omega_1 X^{1/2} \theta^0 + \tilde{U}^{1/2} \theta^1; \]
hence
\[ d\chi_1 = * (T \wedge \Phi \wedge dT) = * (X^{1/2} \theta^0 \wedge \tilde{U}^{1/2} \theta^1 \wedge X d\omega_2 \wedge d\psi) \]
\[ d\chi_2 = * (T \wedge \Phi \wedge d\Phi) = * (X^{1/2} \theta^0 \wedge \tilde{U}^{1/2} \theta^1 \wedge (\omega_1 X d\omega_2 + \tilde{U} d\Omega) \wedge d\psi), \]
which with \( * (\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3) = \epsilon \theta^4 \) leads to
\[ \partial_\psi \chi_1 = \epsilon (X \tilde{U})^{1/2} X \lim_{r \rightarrow 0} \left( \frac{\partial_\psi \omega_2}{\tilde{W}^{1/2}} \right) \]
\[ \partial_\psi \chi_2 = \epsilon (X \tilde{U})^{1/2} \lim_{r \rightarrow 0} \left( \frac{X \omega_1 \partial_r \omega_2 + \tilde{U} \partial_r \Omega}{\tilde{W}^{1/2}} \right). \]

We switch again to \((u, v)\)-coordinates; then on \( v = 0 \) it is
\[ \frac{\partial \chi_1}{\partial u} = u \frac{\partial \psi_1}{\partial z} = u \epsilon (X \tilde{U})^{1/2} X \lim_{v \rightarrow 0} \left( \frac{1}{u \tilde{W}^{1/2}} \frac{\partial \omega_2}{\partial v} \right). \]

Now use
\[ \omega_2 = \frac{Z}{X} = \frac{v^2 Z_0}{X_0} \quad \text{and} \quad \tilde{W} = v^2 W_0 + \mathcal{O}(v^4), \]
to obtain
\[ \frac{\partial \chi_1}{\partial u} = 2 \epsilon Z_0 \left( \frac{U_0 X_0}{W_0} \right)^{1/2} u + \mathcal{O}(u^2) \]
\[ \Rightarrow \chi_1 = \chi_1^0 + \chi_1^1 u^2 + \text{h.o.} \]

Analogous steps lead to
\[ \frac{\partial \chi_2}{\partial u} = 2 \epsilon V_0 \left( \frac{U_0 X_0}{W_0} \right)^{1/2} u^3 + \text{h.o.} \]
\[ \Rightarrow \chi_2 = \chi_2^0 + \chi_2^1 u^4 + \text{h.o.} \]
For $u = 0$ we only have to swap $Y \leftrightarrow Z, U \leftrightarrow W$. With $u^2 \sim 2z$, the above can be summarized as

$$P_-(r = 0, z) = \begin{pmatrix}
g_0 + O(1) & -g_0 z \frac{1}{2} + O(z) & -g_0 z^2 + O(z^2) \\
. & X_0 + O(z) & 2zY_0 + O(z^2) \\
. & . & 2zU_0 + O(z^3)
\end{pmatrix},$$

where $g_0 = (X_0 U_0 - 2zY_0^2)^{-1}$. Note that here we dropped without loss of generality the constant terms of the twist potentials $\chi_i^0$. This can be done just by a gauge transformation to $P$ of the form $P \rightarrow APB$ with constant matrices $A$ and $B$, namely

$$P \rightarrow \begin{pmatrix}1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{pmatrix} P \begin{pmatrix}1 & -c_1 & -c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

For $P$ in standard form this results in $\chi_i \rightarrow \chi_i + c_i$. Removing the constant term in the twist potentials allows us to assume that without loss of generality the entries which become zero or blow up towards a nut are only on the diagonal. The off-diagonal entries are bounded towards the nut.

Without loss of generality assume that the nut is at $a = 0$. Then, the calculations above show that to leading order in $z$ the patching matrices below and above the nut are (chosen the right orientation for the basis such that the signs which are recorded by $\epsilon$ work out)

$$P_- = \begin{pmatrix}1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{pmatrix} P \begin{pmatrix}1 & -c_1 & -c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(3.23)$$P_+ = \begin{pmatrix}1 & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{pmatrix} P \begin{pmatrix}1 & -c_1 & -c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(3.24)Using that $\det J = -u^2 v^2$ in (3.17) and thus $X_0 U_0 W_0 = -1$ to leading order in $z$ we see that the switching is correct to leading order in $z$. (This is consistent with the different adaptations we calculated for example for the Schwarzschild spacetime or flat space, see section 2.4.)

**Proof of theorem 3.12.** To prove theorem 3.12, the strategy is to follow the splitting procedure outlined in Metzner (2012, section 8.4).

We first observe that splitting $P_+$ as in (3.23) will lead not to $J(r, z)$ as desired, but to $J(r, z)$ with its rows and columns permuted. This can be seen by looking at the diagonal case. To obtain $J(r, z)$ with the rows and columns in the order $(t, \phi, \psi)$, we need to permute

$$P_+ \rightarrow \tilde{P}_+ = E_1 P_+ E_1$$

with $E_1 = \begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Similarly for $P_-$ by (3.24), we permute

$$P_- \rightarrow \tilde{P}_- = E_2 P_- E_2$$

with $E_2 = \begin{pmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.
Note that now the prescription in theorem 3.12 translates to

\[
\widehat{P}^- = D\widehat{P}^+ D \quad \text{with} \quad D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2\zeta & 0 \\
0 & 0 & 1/2\zeta
\end{pmatrix};
\]

(3.25)

Recall that we have set \( a = 0 \). Following the splitting procedure, to obtain \( J \) we split the matrices

\[
\widehat{P}_+ = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/\zeta & 0 \\
0 & 0 & 1
\end{pmatrix}\widehat{P}_+^0 \quad \text{and} \quad \widehat{P}^- = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -r/\zeta
\end{pmatrix}\widehat{P}^0 - \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -r/\zeta
\end{pmatrix};
\]

(3.26)

The location of the diagonal entries which are not 1 is dictated by the position of the Killing vector which vanishes on the section of axis under consideration within the basis of Killing vectors \( (\partial_t, \partial_\theta, \partial_\varphi) \). In the language of section 5.3 in part I, the integers \( (p_0, p_1, p_2) \) are, as we know, a permutation of \( (0, 0, 1) \) and the location of the 1 is determined by the prescription just given.

Assembling (3.25) and (3.26) to

\[
\widehat{P}^- = A\widehat{P}_+ B;
\]

where

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2\zeta/\zeta & 0 \\
0 & 0 & 1/2\zeta
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 \\
0 & -2\zeta/\zeta & 0 \\
0 & 0 & -r/2\zeta
\end{pmatrix},
\]

all that is needed for completing the proof is to show that splitting the left- and right-hand sides of this last equation yield the same \( J \)-matrix. To perform the splitting, we replace all appearances of \( z \) by \( w \) and make the substitution as in equation (3.4) in part I. Note that

\[
w = z + r/2(\zeta^{-1} - \zeta) = \frac{1}{2}(u^2 - v^2 + uv(\zeta^{-1} - \zeta)) = \frac{1}{2\zeta}(u^2 + v)(u - v\zeta),
\]

so that

\[
\frac{2wz}{r} = \frac{(u\zeta + v)(u - v\zeta)}{uv} = 1 + O(\zeta),
\]

\[
\frac{2w}{r\zeta} = \frac{1}{\zeta^2} \frac{(u\zeta + v)(u - v\zeta)}{uv} = 1 + O(\zeta^{-1}).
\]

Thus, \( A(z, r, \zeta) \) is holomorphic and nonsingular in the neighbourhood of \( \zeta = 0 \) with \( A(z, r, 0) = \text{id} \), and \( B(z, r, \zeta^{-1}) \) is holomorphic and nonsingular in the neighbourhood of \( \zeta^{-1} = 0 \) with \( B(z, r, 0) = \text{id} \). Consequently, if \( \widehat{P}_+ \) splits as

\[
\widehat{P}_+ = K^0_+ (r, z, \zeta) \left( K^\infty_+ (r, z, \zeta^{-1}) \right)^{-1},
\]

with \( K^0_+ \) holomorphic and nonsingular in the neighbourhood of \( \zeta = 0 \) and \( K^\infty_+ \) holomorphic and nonsingular in the neighbourhood of \( \zeta^{-1} = 0 \), then a splitting of \( \widehat{P}_- \) is given by taking

\[
\widehat{P}_- = K^0_- (K^\infty_-)^{-1} \quad \text{with} \quad K^0_- = AK^0_+, \quad K^\infty_- = B^* K^\infty_+.
\]

The corresponding expressions for \( J \) are

\[
J = J_+ (r, z) = K^0_+ (0)(K^\infty_+ (0))^{-1}
\]

25
\[ J = J_-(r, z) = K^\alpha_0(0)(K^\infty_0(0))^{-1} = A(r, z, 0)J_+(r, z)B(r, z, 0) = J_+(r, z). \]

These are the same. □

3.5. Application to the black ring

Now we see how to apply the prescription for switching and the discussion of conicality to \( P(z) \) for the black ring as in (2.15). We are interested in the nut with largest \( z \)-value, which is the one at \( z = \beta \). The first step is to make an additive shift to the twist potential \( \chi \) to ensure that the term \( P_{12} \) in (2.15) is finite at \( z = \beta \). This needs

\[ \chi \rightarrow \chi + C, \quad C = -\frac{2\nu}{\beta + \lambda}, \]

when

\[ P_{12} \rightarrow P_{12} - CP_{11} = P_{12} - \frac{\nu(z + \lambda)}{(\beta + \lambda)(z + \alpha)(z - \beta)} = \frac{\nu}{(z + \alpha)(\beta + \lambda)}, \]

which is indeed finite at \( z = \beta \), and

\[ P_{22} \rightarrow P_{22} - 2CP_{12} + C^2P_{11} = -\left(\frac{z + \mu}{(z + \alpha)}\right), \quad \text{where} \quad \mu = \frac{\kappa^2(2b - c + bc)}{(1 + b)}, \]

which is also finite at \( z = \beta \). We are in position to make the switch as at theorem 3.12 with \( \beta \) in place of \( a \) and the result is

\[
\begin{pmatrix}
\frac{(z + \alpha)}{2(z - \alpha)(z - \beta)} & 0 & 0 \\
\cdot & -\frac{(z + \mu)}{(z + \alpha)} & \frac{2\nu(z - \beta)}{\gamma(z + \alpha)} \\
\cdot & \cdot & -2\frac{(z + \lambda)(z - \beta)}{(z + \alpha)}
\end{pmatrix}
\]

We have completed the switching and obtained \( P_2(z) \), the transition matrix adapted to the section of axis \( \alpha < z < \beta \). We could continue to find the transition matrix adapted to the other segments but that is straightforward and we do not need it. Instead, we shall return to the question of conicality addressed in corollary 3.11. Compare with theorem 3.8 to find from \( P_1 \) that

\[ v^2W_0 = \frac{2(z + \alpha)(z - \beta)}{(z - \alpha)}, \]

where now \( v^2 = -2(z - \beta) \) and from \( P_2 \) that

\[ u^2U_0 = -\frac{2(z + \lambda)(z - \beta)}{(z + \alpha)}, \]

where now \( u^2 = 2(z - \beta) \). Corollary 3.11 implies that there is no conical singularity on the axis section \( \alpha < z < \beta \) provided

\[ \lim_{u \to 0} W_0 = \lim_{v \to 0} U_0, \]

which here requires

\[ \frac{\beta + \lambda}{\beta + \alpha} = \frac{\beta + \alpha}{\beta - \alpha}. \]

Using (2.16) this condition can be solved for \( b \) as

\[ b = \frac{2c}{1 + c^2} \]

which is known to be the right condition ((Emparan and Reall 2002) or (Harmark 2004, equation (6.20))).
4. Summary and outlook

In this work, we have presented a possible way for the reconstruction of five- or higher-dimensional black hole spacetimes from what are at the moment believed to be the classifying parameters, namely the rod structure and angular momenta. The method is based on a twistor construction which in turn relies on the Penrose–Ward transform.

Our idea assigns a patching matrix to every rod structure where, apart from the possible poles at the nuts, the entries of the patching matrix have to be rational functions with the same denominator $\Delta_1$—section 3. By imposing boundary conditions the aim is to determine all the coefficients of the polynomials in the numerator of these rational functions in terms of the nuts, rods and angular momenta.

However, with an increasing number of nuts one needs increasingly sophisticated tools and it is of particular importance to gain a detailed understanding of how the patching matrices, adapted to two neighbouring rods, are related. In theorem 3.12, we show how to do this and Metzner (2012, theorem 6.5) provides this statement for the nut at infinity, that is, it relates the patching matrices which are adapted to the outer rods. By means of that we are able to reconstruct the patching matrix for a general two-nut rod structure and we can show that a three-nut rod structure with one Killing vector hypersurface-orthogonal, together with a given angular momentum, fixes the spacetime to be the black ring.

Also, in section 3 we discuss conical singularities on the axis and show how to obtain necessary and sufficient conditions for their removal. Applying this to the black ring we obtain the known relation between the parameters. In particular, this implies a relation between the rod structure and the asymptotic quantities for a non-singular solution known to exist.

Further questions which are interesting to pursue in this context are for example as follows.

Which rod structures are admissible? In other words, are there restrictions on the rod structures arising in nonsingular solutions? The example of flat space treated here shows that there are restrictions.

Can we construct a Lens spacetime this way, that is, a spacetime whose horizon is connected and has the topology of a Lens space (Hollands and Yazadjiev 2008, proposition 2)? We know what the corresponding rod structure looks like, but are we able to fix enough parameters and can we see whether the resulting patching matrix does give rise to a spacetime without singularities? The latter question seems to be difficult to address as by the analytic continuation one can guarantee the existence of the solution with all its nice regularity properties only in a neighbourhood of the axis, but further away from the axis there might be the so-called jumping lines, where the mentioned triviality assumption of the bundle does not hold.

How many dimensions does the moduli space for an $n$-nut rod structure have? Can we find upper and lower bounds on that depending on the imposed boundary conditions? This also does not seem to be an easy questions as most of the conditions we impose on the patching matrix are highly nonlinear, for example, the determinant condition.

Which parts of the theory extend to yet higher dimensions? We have already pointed out along the way that some statements straightforwardly generalize to more than five dimensions as well, but some others do not. A closer look at those points would certainly be interesting.

Also stepping down a dimension leads to a question for which this set of tools might be appropriate. Are we able to disprove the existence of a regular double-Kerr solution in four dimensions by these methods? It is conceivable that for example the imposed compatibility requirements as one switches at the nuts lead finally to an overdetermined system of conditions and thereby a contradiction.
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