TOPOLOGICAL ENTROPY AND RECURRENCE PROPERTIES IN NON-AUTONOMOUS DYNAMICAL SYSTEMS

MEHDI FATEHI NIA
DEPARTMENT OF MATHEMATICS, YAZD UNIVERSITY, 89195-741 YAZD, IRAN
E-MAIL: FATEHINIAM@YAZD.AC.IR

Abstract. In this paper we study topological entropy and recurrence properties of non-autonomous dynamical system generated by a family of continuous self maps on a compact space \( X \). Specially, we introduce the pseudo-entropy and periodic-pseudo-entropy and prove their equivalence with the topological entropy for non-autonomous dynamical systems.

Keywords: Non-autonomous dynamics, non-wandering, topological entropy, pseudo orbits, chain recurrent, chain mixing.

MSC(2010): Primary: 37C50; Secondary: 37C15.

1. Introduction

The notion of topological entropy is a well known tool for measure the complexity of dynamical systems. Topological entropy was introduced by Adler et al ([1]) and later extended by Bowen ([3]). After these works many research articles appeared on different dynamical systems and computation methodology of topological entropy. In [2], the authors prove that the topological entropy of a map is equal to the exponential growth rate of the number of separated (periodic) pseudo orbits. In [11], the authors introduced the notions of chain mixing rate and making use of this notion gave a lower bound for topological entropy.

In the recent years, non-autonomous discrete dynamical systems have been extensively studied by many researchers [4, 5, 7, 9, 14, 17]. The main difference between autonomous and non-autonomous systems is that the basic elements dictating the dynamics are fixed in the former, and change over time in the latter. Specially, Kolyada and Snoha [9] introduced topological entropy for a non-autonomous dynamical system given by a sequence of continuous self-maps of a compact metric space. Recently, Kawan introduced the notion of metric entropy for a non-autonomous dynamical system which is related via a variational inequality to the topological entropy of non-autonomous systems as defined by Kolyada and Snoha. In this way they generalized several properties of the classical metric entropy such as
Rokhlin inequality and power rule to non-autonomous dynamical systems [8]. In [4, 6, 10, 15], the authors have studied main notions in discrete dynamical systems such as; non-wandering sets, shadowing, chain recurrent, topological stability, and expansiveness for non-autonomous discrete dynamical systems induced by a sequence of continuous on a compact metric space. Chain recurrent sets and non-wandering sets have an important role in the study of ergodic properties of dynamical systems. In [14], Thakkar and Das consider the chain recurrent sets of non-autonomous dynamical systems and study chain recurrent sets in a non-autonomous discrete system with the shadowing property. Also, in [16] they defined and studied non-wandering set, $\alpha$–limit set, $\omega$–limit set and recurrent set for non-autonomous discrete dynamical systems.

We use briefly $NDS$ to denote the non-autonomous dynamical systems.

In this paper, first we present some definitions and results that will be used in the sequel. In Section 3, we review the work of Thakkar and Das on non-wandering sets and recurrent points for $NDS$’s and prove that topological entropy of an equi-continuous $NDS$ is equal to the topological entropy of this $NDS$ restricted to its chain recurrent set. In the rest, shadowing property for a $NDS$ considered and this is proved that for an equi-continuous $NDS$ with the shadowing property every recurrent point is a chain recurrent point. Section 4 is the main part of the paper. There, we introduce pseudo-entropy and periodic-pseudo entropy for non-autonomous dynamical systems and prove the equivalence of these notions and topological entropy for $NDS$’s. This section is a generalization of the results presented in [2]. Finally, in the last section, we study chain mixing and topological mixing in $NDS$’s and a lower boundary for topological entropy is evaluated.

2. Preliminaries

In this section, we recall the notion of entropy for a $NDS$, as defined in Kolyada et al. [9]. As in the classical autonomous dynamical systems, the definition of topological entropy using open covers and the definition using separated/spanning sets are coincide.

Let $X$ be a compact topological space and $\mathcal{F} = \{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps from $X$ to $X$. For any positive integers, $i$, $n$; set $\mathcal{F}_{[i,n]} = f_{i+(n-1)} \circ f_{i+(n-2)} \circ ... \circ f_{i+1} \circ f_{i}$ and additionally $\mathcal{F}_{[i,0]} = id$. We also write $\mathcal{F}_{[i,-n]} = (\mathcal{F}_{[i,n]})^{-1}$ [9]. For case $i = 1$ we will use $\mathcal{F}_n$. The pair $(X, \mathcal{F})$ is called a non-autonomous discrete dynamical system. The trajectory of a point $x \in X$ is the sequence $(\mathcal{F}_n(x))_{n \geq 0}$.

Now we consider the topological entropy for a non-autonomous dynamical system.

Let $(X, d)$ be a compact metric space. For each $n \geq 1$ the function $d_n(x,y) = \max \{d(\mathcal{F}_j(x), \mathcal{F}_j(y)) : 0 \leq j \leq n - 1\}$ is a metric on $X$ and equivalent to $d$. A subset $E$ of $X$ is called $(n, \mathcal{F}, \epsilon)$–separated if for any two distinct point
$x, y \in E, d_n(x, y) > \epsilon$. A subset $F$ of $X$ is called $(n, \mathcal{F}, \epsilon)$-spanning if for each $x \in X$ there is $y \in F$ for which $d_n(x, y) \leq \epsilon$ [9].

We define $s_n(\mathcal{F}, \epsilon)$ as the maximal cardinality of a $(n, \mathcal{F}, \epsilon)$-seperated set and $r_n(\mathcal{F}, \epsilon)$ as the minimal cardinality of a set which is a $(n, \mathcal{F}, \epsilon)$-spanning set. The topological entropy $h(\mathcal{F})$ of the system $(X, \mathcal{F})$ is defined by

$$h(\mathcal{F}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log s_n(\mathcal{F}, \epsilon) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_n(\mathcal{F}, \epsilon).$$

Now, we consider the other definition for topological entropy of non-autonomous dynamical systems which is based on open covers. [9]

For open covers $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ we denote:

$$\bigvee_{i=1}^{n} \mathcal{A}_i = \mathcal{A}_1 \lor \mathcal{A}_2 \lor \cdots \lor \mathcal{A}_n = \{A_1 \cap A_2 \cap \cdots \cap A_n : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, \cdots A_n \in \mathcal{A}_n\}.$$ 

This is clear that $\bigvee_{i=1}^{n} \mathcal{A}_i$ is also an open cover for $X$. For an open cover $\mathcal{A}$, let $\mathcal{F}_{[i,-n]}(\mathcal{A}) = \{\mathcal{F}_{[i,-n]}(A) : A \in \mathcal{A}\}$ and $\mathcal{A}_i^n = \bigvee_{j=0}^{n-1} \mathcal{F}_{[i-j]}(\mathcal{A})$. Let $\mathcal{N}(\mathcal{A})$ denote the minimal possible cardinality of a subcover chosen from $\mathcal{A}$. Then

$$h^*(\mathcal{F}, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}_i^n)$$

is said to be the topological entropy of $\mathcal{F}$ on the open cover $\mathcal{A}$.

Let

$$h^*(\mathcal{F}) = \sup\{h^*(\mathcal{F}, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } X\}.$$

By Lemma 3.1 of [9], $h^*(\mathcal{F}) = h(\mathcal{F}).$

This is well known that entropy is an important invariant of topological conjugate. In [8] the authors introduce conjugated non-autonomous systems and prove that topological entropy for non-autonomous dynamical systems is an invariant of topological conjugacy.

**Definition 2.1.** [8] Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two non-autonomous system. We say that $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are conjugated, if for every $n \in \mathbb{N}$ there exists a homeomorphism map $\pi_n : X \to Y$ such that $\pi_{n+1} \circ f_n = g_n \circ \pi_n$.

If the maps $\pi_n$ are only continuous then we say that the systems $(X, \mathcal{F})$ and $(Y, g_{1,\infty})$ are semi-conjugate.

**Proposition 2.2.** [8] Let $(X, \mathcal{F})$ be semi-conjugated to $(Y, \mathcal{G})$. Then $h(\mathcal{F}) \leq h(\mathcal{G})$.

### 3. Non-wandering and chain recurrent sets

Dynamical systems with some kind of orbit's recurrence have been always attractive to researchers and various notions of recurrence have been studied in last years. In this section we consider the main ones: non-wandering sets,
periodic points, chain recurrent sets and recurrent sets for non-autonomous dynamical systems.

**Definition 3.1.** [16] Let \((X, d)\) be a metric space and \(f_k : X \rightarrow X\) be a sequence of homeomorphisms, \(k = 1, 2, \cdots\). A point \(x \in X\) is said to be a non-wandering point for \((X, F)\) if for any neighborhood \(U\) of \(x\) and for any \(n \geq 0\) there exists \(m \geq n\) and \(r \geq 0\) such that \(F_{[m, r]}(U) \cap U \neq \emptyset\). The set of all non-wandering points is denoted by \(\Omega(F)\).

Theorems 2.1 and 2.2 in [16] show that if \(X\) be compact then \(\Omega(F)\) is a nonempty and closed set.

**Definition 3.2.** [16] Let \((X, d)\) be a metric space and \(f_n : X \rightarrow X\) be a sequence of homeomorphisms, \(n = 0, 1, 2, \cdots\). A point \(x_0 \in X\) is said to be periodic point of \(NDS (X, F)\) if the orbit of \(x_0\) is periodic, i.e. there exists an integer \(k > 0\) such that \(F_{ik+j}(x_0) = F_j(x_0)\), for every \(i \in \mathbb{N}\) and \(0 \leq j < k\). The set of all periodic points of \(F\) is denoted by \(\text{Per}(F)\).

Similarly, a point \(x \in X\) is said to be a fixed point of \((X, F)\) if \(f_n(x) = x\) for all \(n \geq 0\). In [16], the authors prove that \(\text{Per}(F) \subset \Omega(F)\).

**Definition 3.3.** [16] Let \((X, F)\) be a NDS. By \(\alpha\)–limit set of a point \(x \in X\), we mean the set

\[\alpha(x) = \{y \in X | \lim_{k \rightarrow \infty} d(F_{n_k}(x), y) = 0\},\]

where \(\{n_k\}\) is some strictly decreasing sequence of negative integers. Similarly, by \(\omega\)–limit set of a point \(x \in X\), we mean the set

\[\omega(x) = \{y \in X | \lim_{k \rightarrow \infty} d(F_{n_k}(x), y) = 0\},\]

where \(\{n_k\}\) is some strictly increasing sequence of positive integers.

A point \(x \in X\) is said to be recurrent if \(x \in \alpha(x) \cap \omega(x)\). We denote the set of all recurrent points of \(F\) by \(R(F)\) and the closure of it by \(C(F)\) [16].

**Remark 3.4.** By Theorem 2.5 of [16], if \(X\) is compact then for any \(x \in X\), \(\alpha(x) \subseteq \Omega(F)\) and \(\omega(x) \subseteq \Omega(F)\). Then we have \(\text{Per}(F) \subset R(F) \subseteq \Omega(F)\) and \(C(F) \subseteq \Omega(F)\).

**Definition 3.5.** [14] Let \((X, F)\) be a NDS. A point \(x \in X\) is said to be a chain recurrent point for \(F\) if for any \(\delta > 0\) and any \(n \geq 0\), there exist \(m \geq n\) and a finite sequence \(\{x_i\}_{i=0}^{k}\) of points of \(X\) with \(x_0 = x_k = x\) such that \(d(f_{m+i}(x_i), x_{i+1}) < \delta\) or \(d(f_{m+i}(x_i), x_{i+1}) < \delta\) for all \(i = 0, 1, \cdots, k - 1\). The sequence \(\{x_i\}_{i=0}^{k}\) is said to be a \(\delta\)–chain for \(x\) with action starting at \(m\). The set of all chain recurrent points of \(F\) is denoted by \(CR(F)\).

Recall that the sequence \(\{h_n\}_{n \geq 1}\) of homeomorphisms on \(X\) is said to be equi-continuous, if for any \(\epsilon > 0\) there exists a constant \(\delta > 0\) such that such that \(d(h_n(x), h_n(y)) < \epsilon\) for all \(x, y \in X\) with \(d(x, y) < \delta\) and for all \(n \geq 1\)
implies that $\implies$ $\implies$

\[ 3.4 \]

$2$

subset $\subset$

topological entropy on non-autonomous dynamical systems investigated.

$\text{autonomous dynamical systems introduced and the equivalence of that with }$

Theorem 3.8.

Theorem 3.6. Let $F$ be an equi-continuous NDS then $h(F) = h(F, CR(F))$.

Proof. By Theorems 3.3 and 3.4 of \[14\], if the family of homeomorphisms \{f_n, f_n^{-1}\}_{n \geq 0} is equi-continuous on $X$, then $CR(F)$ is a closed set and $\Omega(F) \subset CR(F)$. In \[9\] the authors introduce the entropy $h(F, Y)$ with respect to any subset $Y$ of $X$ and also prove that $h(F) = h(F, \Omega(F))$. So $h(F) = h(F, \Omega(F)) \leq h(F, CR(F)) \leq h(F)$, which complete the proof. $\square$

Definition 3.7. \[15\] Let $F = (X, F)$ be a NDS. For $\delta > 0$, the sequence $\{x_n\}_{n=-\infty}^{\infty}$ in $X$ is said to be a $\delta-$pseudo orbit of $F$ if $d(f_n(x_n), x_{n+1}) < \delta$ for $n \geq 0$ and $d(f_n^{-1}(x_{n+1}), x_n) < \delta$ for $n \leq -1$. For given $\epsilon > 0$, a $\delta-$pseudo orbit $\{x_n\}_{n=-\infty}^{\infty}$ is said to be $\epsilon-$traced by $y \in X$ if $d(F_n(y), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. The NDS $(X, F)$ is said to have shadowing property or pseudo orbit tracing property (P.O.T.P.) if, for every $\epsilon > 0$, there exists $\delta > 0$ such that every $\delta-$pseudo orbit is $\epsilon-$traced by some points of $X$.

Theorem 3.8. Let $(X, F)$ has the shadowing property and the family homeomorphisms \{f_n, f_n^{-1}\}_{n \geq 0} is equi-continuous on $X$, then every recurrent point for $F$ is a chain recurrent point for $F$.

Proof. In \[14\], the authors prove that if $(X, F)$ has the shadowing property then $CR(F) \subset \Omega(F)$, and consequently, if $(X, F)$ has the shadowing property and the family homeomorphisms \{f_n, f_n^{-1}\}_{n \geq 0} is equi-continuous on $X$, then $CR(F) = \Omega(F)$ and hence Remark 3.4 implies that $C(F) \subset CR(F)$. $\square$

4. Pseudo orbits and entropy

In \[2\] Barge and Swanson use of pseudo-orbits and periodic-pseudo-orbits instead of orbits in their definition of topological entropy which is one of the equivalent definitions of topological entropy. In this section, as the main part of the paper, pseudo-entropy and periodic-pseudo-entropy for non-autonomous dynamical systems introduced and the equivalence of that with topological entropy on non-autonomous dynamical systems investigated.

A subset $E$ of $(\alpha, F)$-pseudo orbits is $(n, \epsilon)-$separated if, for each distinct sequences $x = \{x_n\}_{n=-\infty}^{\infty}$, $y = \{y_n\}_{n=-\infty}^{\infty}$ in $E$, there is a $0 \leq k < n$, for which $d(x_k, y_k) > \epsilon$. Let $c_n(\epsilon, \alpha, F)$ denote the maximal cardinality of a $(n, \epsilon)-$separated set of $(\alpha, F)$-pseudo orbits. Since $X$ is compact then $c_n(\epsilon, \alpha, F)$ is finite.

Definition 4.1. The number $h_p(F) = \lim_{\epsilon \to 0} \lim_{\alpha \to 0} \lim_{n \to \infty} \frac{1}{n} \log c_n(\epsilon, \alpha, F)$ will be called the pseudo-entropy of $F$.

Now, we are going to show that $h_p(F) = h(F)$. For this purpose we need some notions and lemmas. Our proof of Lemmas and Theorems in this
section is based upon ideas found in [2]. Suppose that $X_\alpha$ is the set of all $(\alpha, F)$-pseudo orbits and define the metric $\rho$ on $X_\alpha$ by $\rho(x, y) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{||i||}}$. By definition, $X_\alpha$ is a closed subset of $X^Z$ and hence is compact. Let $\sigma_\alpha : X_\alpha \to X_\alpha$ denote the shift $\sigma_\alpha(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, x_{-1}, x_0, x_1, \cdots)$.

For each $\varepsilon > 0$, let $A_{1, \varepsilon}$ be a finite cover of $X_1$ by $\varepsilon$-balls and form the restricted covers $A_{\alpha, \varepsilon} = \{ A \cap X_\alpha / A \in A_{1, \varepsilon} \}$, $0 \leq \alpha \leq 1$ [2].

**Lemma 4.2.** $h(\sigma_0) = h(F)$.

**Proof.** Consider the NDS $G = (X_0, \{ g_i \})$, where $g_i = \sigma_0$, for all $i \geq 0$. Let $\pi_k : X_0 \to X$ be given by $\pi_k(x) = x_k$, $k \in Z$. So $f_k \circ \pi_k = \pi_{k+1} \circ \sigma_0$, where for $k < 0$ we define $f_k(x) = f^{-1}_k(x)$. Then $F$ is semi conjugate to $G$ and hence by Proposition 2.2, $h(F) \leq h(G) = h(\sigma_0)$. On the other hand by Lemma 1 of [2] and Proposition 3.2 of [8], for any finite cover $A$ of $X_0$ there is a finite cover $B$ of $X$ such that $h(\sigma_0, A) \leq h(F, B)$. So that $h(\sigma_0) \leq h(F)$.

**Lemma 4.3.** [2] $h(\sigma_0, A_{0, \varepsilon}) \geq \inf_{0 < \alpha \leq 1} h(\sigma_\alpha, A_{\alpha, \varepsilon})$

**Lemma 4.4.** $\lim_{n \to \infty} \frac{1}{n} \log c_n(2\varepsilon, \alpha, F) \leq h(\sigma_\alpha, A_{\alpha, \varepsilon})$.

**Proof.** This is clear that for any pair $(n, \varepsilon)$-separated, $x = \{ x_i \}_{i=-\infty}^{\infty}$ and $y = \{ y_i \}_{i=-\infty}^{\infty}$ of $(\alpha, F)$-pseudo orbits in $E$, $x$ and $y$ are $(n, \varepsilon)$-separated orbits of $\sigma_\alpha : X_\alpha \to X_\alpha$. Then, by proof of Lemma 3 of [2], the maximal cardinality of a $(n, \varepsilon)$-separated set for $F$ is not greater than $\text{card}(\text{V}_{i=0}^{n-1} \sigma^{-1}_\alpha(A_{\alpha, \varepsilon}))$.

So $\lim_{n \to \infty} \frac{1}{n} \log c_n(\varepsilon, \alpha, F) \leq h(\sigma_\alpha, A_{\alpha, \varepsilon})$.

The following theorem is one the main result of this paper.

**Theorem 4.5.** Let $X$ be a compact metric space and $(X, \mathcal{F})$ be a non-autonomous dynamical systems on $X$. Then $h_p(F) = h(F)$.

**Proof.** Since every orbit of $F$ is an $(\alpha, F)$-pseudo orbit, for all $\alpha > 0$, this is clear that $h(F) \leq h_p(F)$. On the other hand, Lemma 4.4 implies that $\lim_{\alpha \to 0} \frac{1}{n} \log c_n(2\varepsilon, \alpha, F) \leq \inf_{0 < \alpha \leq 1} h(\sigma_\alpha, A_{\alpha, \varepsilon})$. Then by Lemma 4.3

$\lim_{\alpha \to 0} \frac{1}{n} \log c_n(2\varepsilon, \alpha, F) \leq h(\sigma_0, A_{0, \varepsilon})$. So, Letting $\varepsilon \to 0$ and use of Lemma 4.2, we have $h_p(F) \leq h(F)$.

In [9] the authors consider the relation between the topological entropy of a NDS, as a uniformly convergent sequence of maps and the classical topological entropy of its limit and gave the following lemma.

**Lemma 4.6.** [9] Let $\mathcal{F}$ be a sequence of continuous self-maps of a compact metric space $X$ converging uniformly to $f$. Then $h(F) \leq h(f)$. 
By Lemma 4.6 and Theorem 4.5 we have the following result:

**Corollary 4.7.** Let $X$ be a compact metric space and $(X, \mathcal{F})$ be a NDS contains a sequence of continuous functions converging uniformly to $f$. Then $h_p(\mathcal{F}) \leq h(f)$.

### 4.1. Chain transitivity and periodic-pseudo entropy.

**Definition 4.8.** We say that the NDS $(X, \mathcal{F})$ is $\alpha$-chain transitive if for every $x, y \in X$ there is an $(\alpha, \mathcal{F})$-chain from $x$ to $y$ and an $(\alpha, \mathcal{F})$-chain from $y$ to $x$. The NDS $(X, \mathcal{F})$ is chain transitive if for every $\alpha > 0$, is an $\alpha$-chain transitive.

**Lemma 4.9.** Let the NDS $(X, \mathcal{F})$ be chain transitive, then there is a positive number $K$ such that for every pair $x, y \in X$ there is a $(2\alpha, \mathcal{F})$-chain from $x$ to $y$ of length less than or equal to $K$.

*Proof.* For every $(a, b) \in X \times X$, choose the number $k(a, b)$ such that there is a $(2\alpha, F)$-chain from $a$ to $b$. So, we can find open sets $V_a, V_b$ contains $a$ and $b$, respectively, such that if $a_1 \in V_a$ and $b_1 \in V_b$ then there is a $(2\alpha, \mathcal{F})$-chain of length $k(a, b)$ from $a_1$ to $b_1$. The collection $\{V_x \times V_y\}_{a,b \in X}$ is an open cover for $X \times X$. Compactness of $X$ implies that the open cover $\{V_a, V_b\}_{a,b \in X}$ for $X \times X$ has a finite subcover $\{V_{a_i} \times V_{b_i}\}$. Let $K = \max\{k(a_i, b_i)\}$. □

A subset $S$ of $(\alpha, \mathcal{F})$-pseudo orbits of period $n$ is $(n, \epsilon)$—separated if, for each $x = \{x_i\}_{0 \leq i < n}, y = \{y_i\}_{0 \leq i < n} \subset S, x \neq y$, there is a $i$, $0 \leq i < n$, for which $d(x_i, y_i) > \epsilon$. Let $p_n(\epsilon, \alpha, \mathcal{F})$ denote the maximal cardinality of a $(n, \epsilon)$—separated set of $(\alpha, \mathcal{F})$-pseudo orbits. Since $X$ is compact then $p_n(\epsilon, \alpha, \mathcal{F})$ is finite.

**Remark 4.10.** Let $\alpha' < \alpha$, since every $(\alpha', \mathcal{F})$-pseudo orbit is an $(\alpha, \mathcal{F})$-pseudo orbit, then $p_n(\epsilon, \alpha, \mathcal{F}) \leq p_n(\epsilon, \alpha', \mathcal{F})$.

The number $H_p(\mathcal{F}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \log \frac{1}{n} \log p_n(\epsilon, \alpha, \mathcal{F})$ will be called the periodic-pseudo-entropy of $\mathcal{F}$. The following theorem shows that $H_p(\mathcal{F}) = h(\mathcal{F})$.

**Theorem 4.11.** Let the NDS $(X, \mathcal{F})$ be chain transitive. The topological entropy $h(\mathcal{F})$ is equal to $H_p(\mathcal{F})$.

*Proof.* Let $E$ be a $(n, \epsilon)$—separated set of $(\alpha, \mathcal{F})$-pseudo orbits. For every $x = \{x_i\}$ in $E$ we have an $\alpha$—pseudo orbit from $x_1$ to $x_n$. By Lemma 4.9 there exists a $2\alpha$-periodic pseudo orbit of the length at most $K$ from $x_n$ to $x_1$. So there exists a $2\alpha$-periodic pseudo orbit of the length at most $K + n$ from $x_1$ to $x_1$. This implies that $c_n(\epsilon, \alpha, \mathcal{F}) \leq \Sigma_{i=1}^{K+n} p_i(\epsilon, \alpha, \mathcal{F})$. For each $m \geq 1$, let $i_m, 1 \leq i_m \leq m$, be such that $p_i(\epsilon, 2\alpha, \mathcal{F}) \geq \frac{1}{m}$.
p_i(\epsilon, 2\alpha, \mathcal{F}), \text{ for all } 1 \leq i \leq m. \text{ Then,}
\begin{align*}
h(\mathcal{F}) &= h_p(\mathcal{F}) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log c_n(2\epsilon, \alpha, \mathcal{F}) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^{K+n} p_i(\epsilon, 2\alpha, \mathcal{F}) \\
&\leq \limsup_{n \to \infty} \frac{1}{K+n} \log (K+n)p_{K+n}(\epsilon, 2\alpha, \mathcal{F}) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log p_n(\epsilon, 2\alpha, \mathcal{F}).
\end{align*}

So \( h(\mathcal{F}) \leq H_p(\mathcal{F}) \). On the other hand, by definitions this is clear that \( H_p(\mathcal{F}) \leq h_p(\mathcal{F}) \) and hence by Theorem 4.5 \( H_p(\mathcal{F}) \leq h(\mathcal{F}) \).

By Lemma 4.6 and Theorem 4.11 we have the following result:

**Corollary 4.12.** Let \( X \) be a compact metric space and \((X, \mathcal{F})\) be a chain transitive NDS contains a sequence of continuous functions converging uniformly to \( f \). Then \( H_p(\mathcal{F}) \leq h(f) \).

**Example 4.13.** Let \( I \) be the unit interval and let \( g, h \) be defined as
\[
g(x) = \begin{cases} 
2x + \frac{1}{2} & \text{ for } x \in [0, \frac{1}{4}], \\
-2x + \frac{3}{2} & \text{ for } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\
2x - \frac{3}{2} & \text{ for } x \in \left[\frac{3}{4}, 1\right].
\end{cases}
\]
\[
h(x) = \begin{cases} 
x + \frac{1}{2} & \text{ for } x \in [0, \frac{1}{4}], \\
-4x + 3 & \text{ for } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\
2x - \frac{3}{2} & \text{ for } x \in \left[\frac{3}{4}, 1\right].
\end{cases}
\]
Consider the NDS \((I, \mathcal{F})\), where \( \mathcal{F} = \{g, h, g, h, g, \cdots\} \). In [12], the authors prove \((I, \mathcal{F})\) is transitive and hence by Theorem 4.11 the topological entropy \( h(\mathcal{F}) \) is equal to \( H_p(\mathcal{F}) \).

**Definition 4.14.** [15] Let \((X, d)\) be a metric space and \( f_n : X \to X \) a sequence of homeomorphisms, \( n = 0, 1, 2, \cdots \). The NDS \((X, \mathcal{F})\) is said to be expansive if there exists a constant \( e > 0 \) (called an expansive constant) such that, for any \( x, y \in X \), \( x \neq y \), \( d(F_n(x), F_n(y)) > e \) for some \( n \in \mathbb{N} \).

Fix \( n \geq 1 \). Let \( Fix(F^n) \) be the set of all point \( x \) that \( F_{[k,n]}(x) = x \) and \( N(Fix(F^n)) \) denote its cardinality.

**Theorem 4.15.** Let \((X, \mathcal{F})\) be an expansive chain transitive NDS. If \((X, \mathcal{F})\) has the shadowing property then \( h(\mathcal{F}) = \limsup_{n \to \infty} \frac{1}{n} \log [N(Fix(F^n))] \).
Proof. Consider $\epsilon > 0$ as the expansivity constant and let $\epsilon < \epsilon$. Since $\mathcal{F}$ has the shadowing property then there exists $\alpha < \epsilon$ such that every $\alpha$--pseudo periodic orbit is $\epsilon$--traced by some point $y$ in $X$. Up to this point and expansivity of $\mathcal{F}$ for each $\alpha$--pseudo periodic there exists a corresponding periodic orbit. Then by Theorem 4.11 \( h(\mathcal{F}) = \limsup_{n \to \infty} \frac{1}{n} \log [N(Fix(\mathcal{F}^n))] \)

\[ \square \]

5. Topological entropy and chain recurrent

In this section we investigate the structure of topological mixing NDS’s and define chain mixing time for NDS’s. Theorem 5.7 gives lower bound of the topological entropy in non-autonomous theory by using the chain mixing times.

Definition 5.1. [12] Let $X$ be a compact metric space and $(X, \mathcal{F})$ be a NDS. The system $\mathcal{F}$ is said to be topological mixing if for every non-empty open sets $U, V$ there exists a natural number $K$ such that $f_n \circ f_{n-1} \circ \ldots \circ f_1(U) \cap V \neq \emptyset$, for all $n \geq K$.

Lemma 5.2. [12] The DNS $\mathcal{F} = (X, \mathcal{F})$ is topologically mixing if and only if for each non-empty open set $U$, $\lim_{n \to \infty} f_n \circ f_{n-1} \circ \ldots \circ f_1(U) = X$.

Corollary 5.3. The DNS $\mathcal{F} = (X, \mathcal{F})$ is topologically mixing if and only if for each $\epsilon > 0$ there exists $N_{\epsilon} > 0$ such that for any $x, y \in X$ and any $n \geq N_{\epsilon} > 0$ there is an $\epsilon$--pseudo-orbit from $x$ to $y$ of length exactly $n$.

Definition 5.4. If $0 < \epsilon < \delta$ and $x \in X$, define the chain mixing time $m_{\epsilon}(x, \delta, \mathcal{F})$ to be the smallest $N$ such that for any $n \geq N$ and any $y \in X$, there is an $\epsilon$--chain of length exactly $n$ from some point in $B_\delta(x)$ to $y$. We define $m_{\epsilon}(\delta, \mathcal{F})$ to be the maximum over all $x$ of $m_{\epsilon}(x, \delta, \mathcal{F})$.

The compactness of $X$ implies the existence of the number $m_{\epsilon}(\delta, \mathcal{F})$.

Remark 5.5. Let $\epsilon > 0$ and $x \in X$. We define $(B_\epsilon \circ f_n)(a) = \{ x \in X : d(f_n(a), x) < \epsilon \}$ and for $U \subset X$, $(B_\epsilon \circ f_n)(U) = \bigcup_{a \in U} (B_\epsilon \circ f_n)(a)$.

Put

\[ (B_\epsilon \circ F_2)(a) = (B_\epsilon \circ f_2)((B_\epsilon \circ f_1)(a)) \]

and

\[ (B_\epsilon \circ F_n)(a) = (B_\epsilon \circ f_n)((B_\epsilon \circ F_{n-1})(a)) \]

for all $n \geq 1$.

So, $m_{\epsilon}(x, \delta, \mathcal{F})$ is the smallest $N$ such that $(B_\epsilon \circ F_N)(B_\delta(x)) = X$.

Definition 5.6. We say that the NDS, $\mathcal{F}$ is Lipschitz with Lipschitz constant $c$, if $d(f_n(x), f_n(y)) \leq c d(x, y)$, for all $n \geq 1$ and $x, y \in X$.

Theorem 5.7. Let $\mathcal{F}$ be chain mixing and have Lipschitz constant $c$. Let $D$ be the diameter of $X$. Then for $\delta$ sufficiently small, $m_{\epsilon}(\delta, \mathcal{F}) \geq \log c(D/c-1+2\epsilon)$ if $c > 1$ and $m_{\epsilon}(\delta, \mathcal{F}) \geq \frac{D-2\delta}{2\epsilon}$ if $c = 1$. 

Proof. By definition of \( B_oF_n \) this is clear that 
\[
\text{diam}((B_o f_1)(B_\delta(x))) \leq c(2\delta) + 2\epsilon. \text{ So, diam}((B_o F_2)(B_\delta(x))) \leq c(c(2\delta) + 2\epsilon) + 2\epsilon. \text{ Then by induction diam}((B_o F_n)(B_\delta(x))) \leq c^n(2\delta) + c^{n-1}(2\epsilon) + c^{n-2}(2\epsilon) + \ldots + 2\epsilon \leq c^n(2\delta) + \frac{c^{n-1}}{c-1}(2\epsilon).
\]
Since \( m_\epsilon(x, \delta, F) \) is the smallest \( N \) such that \( (B_o F_n)(B_\delta(x)) = X \). Then \( N \) is at least \( \log_c \frac{D(c-1)+2\epsilon}{2\delta(c-1)+2\epsilon} \).

Proposition 5.8. Let \( F = (X, F) \) be a topological mixing NDS. Then the topological entropy \( h(F) \), satisfies

\[
h(F) \geq d' \cdot \limsup_{\delta \to 0} \frac{\log(\frac{1}{\delta})}{\limsup_{\epsilon \to 0} m_\epsilon(\delta, F)}
\]

where \( d' \) is the lower box dimension of \( X \).

Proof. Consider \( c_n(\epsilon, \alpha, F) \) as the maximal cardinality of a \((n, \epsilon)-\) separated set of \((\alpha, F)\)-pseudo orbits. For \( \alpha > 0 \), let \( N(\alpha) = c_0(0, \alpha, F) \).

Let \( x_1, \cdots, x_{N(3\delta)} \) be a \( 3\delta \)- separated set of points. Fixed \( k \geq 0 \), for each sequence \((i_0, \cdots, i_k)\), where \( 1 \geq i_j \leq N(3\delta) \), there is a sequence of points \( y_{i_1}, \cdots, y_{i_k} \) such that for each \( j, y_{i_j} \in B_\delta(x_i) \) and there is an \( \epsilon \)-pseudo-orbit of length \( m_\epsilon(\delta) \) from \( y_{i_j} \) to \( y_{i_{j+1}} \). Since the points \( x_i \) are \( 3\delta \)- separated, the sequences \((y_{i_1}, \cdots, y_{i_k})\) are \( \delta \)- separated. Then \( c_{km_\epsilon(\delta)}(\epsilon, \delta, F) \geq (N(3\delta))^{k+1} \).

By the proof of Theorem 28 in [11], for small enough \( \alpha \) there exists a positive constant \( C \) such that \( N(\alpha) \geq C(\frac{1}{\alpha})^{d'} \). Then

\[
h(F) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log c_n(\epsilon, \alpha, F)}{n} \quad \text{Theorem 4.5}
\]

\[
\geq \lim_{n \to \infty} \frac{1}{km_\epsilon(\delta)} \log(N(3\delta))^{k+1}
\]

\[
= \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log N(3\delta)}{m_\epsilon(\delta, F)}
\]

\[
\geq \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log \frac{C}{(3\delta)^d}}{m_\epsilon(\delta, F)}
\]

\[
= d' \cdot \limsup_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log \frac{1}{\delta}}{m_\epsilon(\delta, F)}
\]

□

References

[1] R. Adler, A. Konheim and J.M. Andrew, Topological entropy, Trans. Amer. Math. Soc 114 (1965), 309–319.
[2] M. Barge and R. Swanson, Pseudo-orbits and topological entropy, Proc. Amer. Math. Soc 109 (1990), 559–566.
[3] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc 153 (1971) 401–414.
[4] J.S Canovas, Recent results on non-autonomous discrete systems, Bol. Soc. Esp. Mat. Apl 51 (2010), 33–40.
[5] M. Fateh Nia, Parameterized IFS with the asymptotic average shadowing property, Qual. Theory Dyn. Syst (accepted), doi:10.1007/s12346-015-0184-6.
[6] M. Fateh Nia, Iterated function systems with the average shadowing property, Topology Proc 48 (2016), 261–275.
[7] M. Fateh Nia, Iterated function systems with the shadowing property, J. Adv. Res. Pure Math 7 (2015), 83–91.
[8] C. Kawan, Metric entropy of nonautonomous dynamical systems, Nonauton. Dyn. Syst 1 (2014), 26–52.
[9] S. Kolyada and L. Snoha, Topological entropy of nonautonomous dynamical systems, Random Comput. Dynam 4 (1996), 205–233.
[10] T.K.S. Moothathu and P. Oprocha, Shadowing, entropy and minimal subsystems, Monatsh. Math 172 (2013), 357–378.
[11] D. Richeson and j. Wiseman, Chain recurrence rates and topological entropy, Topology Appl 156 (2008), 251–261.
[12] P. Sharma and M. Raghav, Dynamics of non-autonomous discrete dynamical systems, arXiv:1512.08868.
[13] Y. Shi, Chaos in nonautonomous discrete dynamical systems approached by their induced systems, Int. J. Bifurcat. Chaos 22 (2012), no. 11, 1250284, 12 pages.
[14] D. Thakkar and R. Das, Some properties of chain recurrent sets in a nonautonomous discrete dynamical system, Adv. Pure Appl. Math 6 (2015), 173–178.
[15] D. Thakkar and R. Das, On nonautonomous discrete dynamical systems, Int. J. Anal (2014), Article ID 538691.
[16] D. Thakkar and R. Das, A note on non-wandering set of a nonautonomous discrete dynamical system, Appl. Math. Sci 138 (2013), 6849–6854.
[17] K. Yokoi, Recurrence properties of a class of nonautonomous discrete systems, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 689–705.