ON ASYMPTOTIC STABILITY IN ENERGY SPACE OF GROUND STATES FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider nonlinear Schrödinger equations

$$iut + \Delta u + \beta(|u|^2)u = 0, \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $d \geq 3$ and $\beta$ is smooth. We prove that symmetric finite energy solutions close to orbitally stable ground states converge to a sum of a ground state and a dispersive wave as $t \to \infty$ assuming the so called Fermi Golden Rule (FGR) hypothesis. We improve the “sign condition” required in a recent paper by Gang Zhou and I.M.Sigal.

1. Introduction

We consider asymptotic stability of standing wave solutions of nonlinear Schrödinger equations

\begin{equation}
(\text{NLS}) \quad \begin{cases}
  iut + \Delta u + \beta(|u|^2)u = 0, \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
  u(0, x) = u_0(x) \text{ for } x \in \mathbb{R}^d,
\end{cases}
\end{equation}

where $d \geq 3$ and $\beta$ is smooth.

In this paper, we discuss the asymptotic stability of ground states in the energy class. Following Soffer and Weinstein [31], the papers [2, 3, 4, 7, 8, 26, 27, 32, 35, 36, 37] studied the case when the initial data are rapidly decreasing and the linearized operators of (NLS) at the ground states have at most one pair of eigenvalues that lie close to the continuous spectrum. Cases when the linearized operators have many eigenvalues were considered in [34]. One of the difficulties in proving asymptotic stability is the possible existence of invariant tori corresponding to eigenvalues of the linearization. A large amount of effort has been spent to show that “metastable” tori decay like $t^{-1/2}$ as $t \to \infty$ by means of a mechanism called Fermi Golden Rule.

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(FGR) introduced by Sigal [29] and by a normal form expansion. Recently, thanks to a significant improvement of the normal form expansion, Zhou and Sigal [45] were able to prove asymptotic stability of ground states in the case when the linearized operators have two eigenvalues not necessarily close to the continuous spectrum. In a different direction, Gustafson et al. [18] proved that small solitons are asymptotically stable in $H^1(\mathbb{R}^d)$ if $d \geq 3$ and if the linearized operators do not have eigenvalues except for the 0 eigenvalue. Recently, [23, 24] extended [18] to the lower dimensional cases ($d = 1, 2$). The papers [18, 23, 24] utilize the endpoint Strichartz estimate or local smoothing estimates.

In the present paper, we unify the methods in [45] and [18] and show that the result proved by [45] in a weighted space holds also in $H^1(\mathbb{R}^d)$. Furthermore, our assumption on (FGR) is weaker than [45]. [45] assumes a sign hypothesis on a coefficient of the ODE for the discrete mode. See [44] for a conjecture behind this assumption. By exploiting the orbital stability of solitons, we show that it is enough to assume the nondegeneracy of the coefficient, without any need to assume anything about its sign.

To be more precise, let us introduce our assumptions.

\textbf{(H1)} $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$;

\textbf{(H2)} there exists a $p \in (1, \frac{d+2}{d-2})$ such that for every $k = 0, 1$,

$$\frac{d^k}{dv^k} |v|^p \beta(v^2) \lesssim |v|^{p-k-1} \quad \text{if } |v| \geq 1;$$

\textbf{(H3)} there exists an open interval $\mathcal{O}$ such that

\begin{equation}
\Delta u - \omega u + \beta(u^2)u = 0 \quad \text{for } x \in \mathbb{R}^d,
\end{equation}

admits a $C^1$-family of ground states $\phi_\omega(x)$ for $\omega \in \mathcal{O}$.

We also assume the following.

\textbf{(H4)}

\begin{equation}
\frac{d}{d\omega} \|\phi_\omega\|^2_{L^2(\mathbb{R}^d)} > 0 \quad \text{for } \omega \in \mathcal{O},
\end{equation}

\textbf{(H5)} Let $L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2)\phi_\omega^2$ be the operator whose domain is $H^2_{rad}(\mathbb{R}^d)$. Then $L_+$ has exactly one negative eigenvalue and does not have kernel.

\textbf{(H6)} For any $x \in \mathbb{R}^d$, $u_0(x) = u_0(-x)$. That is, the initial data $u_0$ of (NLS) is even.
(H7) Let $H_\omega$ be the linearized operator around $e^{it\omega}\phi_\omega$ (see Section 2 for the precise definition). $H_\omega$ has a positive simple eigenvalue $\lambda(\omega)$ for $\omega \in \mathcal{O}$. There exists an $N \in \mathbb{N}$ such that $N\lambda(\omega) < \omega < (N+1)\lambda(\omega)$.

(H8) (FGR) is nondegenerate (see Hypothesis 3.5 in Section 3).

(H9) The point spectrum of $H_\omega$ consists of 0 and $\pm \lambda(\omega)$. The points $\pm \omega$ are not resonances.

Theorem 1.1. Let $d \geq 3$. Let $\omega_0 \in \mathcal{O}$ and $\phi_{\omega_0}(x)$ be a ground state of (1.1). Let $u(t, x)$ be a solution to (NLS). Assume (H1)–(H9). Then, there exist an $\varepsilon_0 > 0$ and a $C > 0$ such that if $\varepsilon := \inf_{\gamma \in [0, 2\pi]} \|u_0 - e^{i\gamma}\phi_\omega\|_{H^1} < \varepsilon_0$, there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R}; \mathbb{R})$ and $h_\infty \in H^1$ with $\|h_\infty\|_{H^1} + |\omega_+ - \omega_0| \leq C\varepsilon$ such that

$$\lim_{t \to \infty} \|u(t, \cdot) - e^{i\theta(t)}\phi_{\omega_+} - e^{it\Delta} h_\infty\|_{H^1} = 0.$$ 

Remark 1.1. Under the assumption (H1)–(H5), it is well known that the standing waves are stable (see [6, 16, 17, 28, 40] and the references in [5]).

Remark 1.2. Ground states of (1.1) are known to be unique for typical nonlinearities like $\beta(s) = s^{(p-1)/2}$ or $\beta(s) = s^{(p-1)/2} - s^{(q-1)/2}$ (see [14, 21, 22] and [41]). The assumption (H5) is satisfied for those cases (see [19, 22]).

Remark 1.3. Hypothesis (H9) is generic because resonances and embedded eigenvalues can be eliminated by perturbations following the ideas in [11, 12].

Remark 1.4. Hypothesis (H8), that is Hypothesis 3.5 in Section 3, probably holds generically.

Remark 1.5. Hypothesis (H6), that is the symmetry assumption $u_0(x) = u_0(-x)$, can be dropped maintaining the same proof, if we add some inhomogeneity to the equation, for example a linear term $V(x)u$. In particular our result holds in the setting of [45].

Remark 1.6. Theorem 1.1 supports the conjecture by Soffer and Weinstein in [33] about the sign in "dispersive" normal forms for 1 dimensional Hamiltonian systems coupled to dispersive equations, since we prove in our case that the sign is the expected one.

Conclusions similar to Theorem 1.1 can be obtained allowing more eigenvalues for the linearization, replacing (H7)–(H9) with:

(H7') $H_\omega$ has a certain number of simple positive eigenvalues with $0 < N_j\lambda_j(\omega) < \omega < (N_j + 1)\lambda_j(\omega)$ with $N_j \geq 1$. 
The (FGR) Hypothesis 5.2 in Section 5 holds.

$H_\omega$ has no other eigenvalues except for 0 and the $\pm \lambda_j(\omega)$. The points $\pm \omega$ are not resonances.

For a multi indexes $m = (m_1, m_2, \ldots)$ and $n = (n_1, \ldots)$, setting $\lambda(\omega) = (\lambda_1(\omega), \ldots)$ and $(m - n) \cdot \lambda = \sum (m_j - n_j) \lambda_j$, we have the following non resonance hypotheses: $(m - n) \cdot \lambda(\omega) = 0$ implies $m = n$ and $(m - n) \cdot \lambda(\omega) \neq \omega$ for all $(m, n)$

**Theorem 1.2.** The same conclusions of Theorem 1.1 hold assuming (H1)–(H6) and (H7')–(H10').

**Remark 1.7.** The (FGR) Hypothesis 5.2 is an analogue of the (FGR) in [45] and is a sign hypothesis on the coefficients of the equation of the discrete modes. In particular it is stronger than Hypothesis 3.5. In the case $N_j = 1$ for all $j$, one can replace Hypothesis 5.2 with an hypothesis similar to Hypothesis 3.5 in the sense that it is known that if certain coefficients are non zero, then they have a specific sign.

**Remark 1.8.** If we do not assume (H6), the solitary waves can move around. This causes technical difficulties when trying to show asymptotic stability in the energy space. However the results of this paper go through if we break the translation invariance of (NLS) by adding for example a linear term $V(x)u(t, x)$ as in [45] or by replacing the nonlinearity by $V(x)\beta(|u|^2)u$, for appropriate $V(x)$.

**Remark 1.9.** The result in [45] is restricted to initial data satisfying a certain symmetry assumption and to an (NLS) with an additional linear term $V(x)u(t, x)$ with $V(x) = V(|x|)$. The argument of Theorem 1.2 can be used to generalize the result in [45] to generic, not spherically symmetric, $V(x)$ and for initial data in $H^1$ not required to satisfy symmetry assumptions. The case when $V(x)$ is spherically symmetric is untouched by our argument, because in that case the linearization admits a nonzero eigenvalue which is non simple.

**Remark 1.10.** Theorem 1.2 is relevant to a question in [33] on whether in the multi eigenvalues case the interaction of distinct discrete modes causes persistence of some excited states or radiation always wins. Theorem 1.2 suggests that the latter case is the correct one.
Remark 1.1. Theorems [1.1] and [1.2] can be proved also in dimensions 1 and 2 extending to the linearizations the smoothing estimates for Schrödinger operators proved in [23, 24]. See [9, 13].

Remark 1.2. Theorem [1.2] seems also relevant to $L^2$ critical Schrödinger equations with a spatial inhomogeneity in the nonlinearity treated by Fibich and Wang [15], in the sense that if certain spectral assumptions and a (FGR) hold, it should be possible to prove that the ground states which are shown to be stable in [15], are also asymptotically stable, at least in the low dimensions $d = 1, 2$ when the critical nonlinearity is smooth.

Remark 1.3. The ideas in this paper can also be used to give partial proof of the orbital instability of standing waves with nodes, even in the case when these waves are linearly stable, see [10].

Gustafson, Nakanishi and Tsai have announced Theorem 1.1 in the case $N = 1$ for the equation of [35] where some small ground states are obtained by bifurcation. Our proof is valid in their case and has the advantage that can treat large solitons and the case where eigenvalues are not necessarily close to the edge of continuous spectrum.

Our paper is planned as follows. In Section 2, we introduce the ansatz and linear estimates that will be used later. In Section 3, we introduce normal form expansions on dispersive part and discrete modes of solutions. In Section 4, we prove a priori estimates of transformed equations and prove Theorem 1.1. In Section 5 we sketch the proof of Theorem 1.2. In the Appendix, we give the proof of the normal form expansion used in Theorem 1.1 following [3, 4, 45].

Finally, let us introduce several notations. Given an operator $L$, we denote by $N(L)$ the kernel of $L$ and by $N_g(L)$ the generalized kernel of $L$. We denote by $R_L$ the resolvent operator $(L - \lambda)^{-1}$.

A vector or a matrix will be called real when all of their components are real valued. Let $\langle x \rangle = \sqrt{1 + |x|^2}$ and let $\mathcal{H}_a$ be a set of functions defined by $\mathcal{H}_a(\mathbb{R}^d) = \left\{ u \in \mathcal{S}(\mathbb{R}^d) : \|e^{a(x)} u\|_{H^k(\mathbb{R}^d)} < \infty \text{ for every } k \in \mathbb{Z}_{\geq 0} \right\}$. For any Banach spaces $X, Y$, we denote by $B(X, Y)$ the space of bounded linear operators from $X$ to $Y$. Various constants will be simply denoted by $C$ in the course of calculations.
2. Linearization, modulation and set up

Now, we review some well known facts about the linearization at a ground state. We can write the ansatz

\[
(2.1) \quad u(t, x) = e^{i\theta(t)}(\phi_{\omega(t)}(x) + r(t, x)) , \quad \theta(t) = \int_0^t \omega(s)ds + \gamma(t)
\]

Inserting the ansatz into the equation we get

\[
(2.2) \quad i\dot{r}_t = -\Delta r + \omega(t)r - \beta(\phi_{\omega(t)}^2)r - \beta'(\phi_{\omega(t)}^2)\phi_{\omega(t)}^2 r
\]

\[- - \beta'(\phi_{\omega(t)}^2)\phi_{\omega(t)}^2 + \gamma(t)\phi_{\omega(t)} - i\dot{\omega}(t)\partial_\omega\phi_{\omega(t)} + \gamma(t)r + O(r^2).
\]

Because of \(\overline{C} \), we write the above as a system. Let

\[
(2.3) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
\]

\[H_{\omega,0} = \sigma_3(-\Delta + \omega), \quad V_\omega = -\sigma_3 \left[ \beta(\phi_{\omega}^2) + \beta'(\phi_{\omega}^2)\phi_{\omega}^2 \right] + i\beta'(\phi_{\omega}^2)\phi_{\omega}^2 \sigma_2;
\]

\[H(\omega) = H_{\omega,0} + V_\omega, \quad R = \ell(r, \bar{r}), \quad \Phi_\omega = \ell(\phi_\omega, \phi_\omega).
\]

Then \((2.2)\) is rewritten as

\[
(2.4) \quad iR_t = H_{\omega(t)} R + \sigma_3 \dot{\gamma} R + \sigma_3 \dot{\gamma} \Phi_{\omega(t)} - i\dot{\omega}_t \partial_\omega \Phi_{\omega(t)} + N,
\]

where

\[
N = \sigma_3 \left\{ \beta(|\Phi_\omega + R|^2/2)(\Phi_\omega + R) - \beta(|\Phi_\omega|^2/2)\Phi_\omega
\]

\[- - \partial_\epsilon \beta(\Phi_\omega + \epsilon R|^2/2)(\Phi_\omega + \epsilon R) \big|_{\epsilon = 0} \right\} = O(R^2) \quad \text{as} \quad R \to 0.
\]

The essential spectrum of \(H_\omega\) consists of \((-\infty, -\omega] \cup [\omega, +\infty)\). It is well known (see [10]) that under the assumption (H3)–(H6), 0 is an isolated eigenvalue of \(H_\omega\), \(\dim N_g(H_\omega) = 2\) and

\[
H_\omega \sigma_3 \Phi_\omega = 0, \quad H_\omega \partial_\omega \Phi_\omega = -\Phi_\omega.
\]

Since \(H_\omega^* = \sigma_3 H_\omega \sigma_3\), we have \(N_g(H_\omega^*) = \text{span}\{\Phi_\omega, \sigma_3 \partial_\omega \Phi_\omega\}\). Let \(\xi(\omega)\) be a real eigenfunction with eigenvalue \(\lambda(\omega)\). Then we have

\[
H_\omega \xi(\omega) = \lambda(\omega) \xi(\omega), \quad H_\omega \sigma_1 \xi(\omega) = -\lambda(\omega) \sigma_1 \xi(\omega).
\]

Note that \(\langle \xi, \sigma_3 \xi \rangle > 0\) since \(\langle \sigma H_\omega, \cdot \rangle\) is positive definite on \(\frac{1}{2} N_g(H_\omega^*)\).

Both \(\phi_\omega\) and \(\xi(\omega, x)\) are smooth in \(\omega \in \mathcal{O}\) and \(x \in \mathbb{R}^d\) and satisfy

\[
\sup_{\omega \in K, x \in \mathbb{R}^d} e^{a|x|}(\phi_\omega(x) + |\xi(\omega, x)|) < \infty
\]

for every \(a \in (0, \inf_{\omega \in K} \sqrt{\omega - \lambda(\omega)})\) and every compact subset \(K\) of \(\mathcal{O}\).
For $\omega \in \mathcal{O}$, we have the $H_{\omega}$-invariant Jordan block decomposition

\begin{equation}
L^2(\mathbb{R}^d, \mathbb{C}^2) = N_g(H_{\omega}) \oplus (\oplus_+ N(H_{\omega} \leq \lambda(\omega))) \oplus L^2_c(H_{\omega}),
\end{equation}

where $L^2_c(H_{\omega}) := \{ N_g(H^*_{\omega}) \oplus (\oplus_+ N(H^*_{\omega} \leq \lambda(\omega))) \}$. Correspondingly, we set

\begin{align}
R(t) &= z(t)\xi(\omega(t)) + \overline{z(t)}\sigma_1\xi(\omega(t)) + f(t), \\
R(t) &= \overline{N_g(H^*_{\omega(t)})} \quad \text{and} \quad f(t) \in L^2_c(H_{\omega(t)}).
\end{align}

By using the implicit function theorem, we obtain the following (see e.g. [25] for the proof).

**Lemma 2.1.** Let $I$ be a compact subset of $\mathcal{O}$ and let $u(t)$ be a solution to (NLS). Then there exist a $\delta_1 > 0$ and a $C > 0$ satisfying the following. If

\[ \delta := \sup_{0 \leq t \leq T} \| u(t) - e^{i\theta_0} \phi_{\omega_0} \|_{H^1(\mathbb{R}^d)} < \delta_1 \]

holds for a $T \geq 0$, an $\omega_0 \in I$ and a $\theta_0 \in \mathbb{R}$, then there exist $C^1$-functions $z(t)$, $\omega(t)$ and $\theta(t)$ satisfying (2.1), (2.6) and (2.7) for $0 \leq t \leq T$, and

\[ \sup_{0 \leq t \leq T} (|z(t)| + |\omega(t) - \omega_0| + |\theta(t) - \theta_0|) \leq C\delta. \]

**Remark 2.1.** Let $\varepsilon$ and $\varepsilon_0$ be as in Theorem 1.1 and let $\delta$ and $\delta_1$ be as in Lemma 2.1. By (H4) and (H5), we have orbital stability of $e^{i\omega_0 t} \phi_{\omega_0}$ and it follows that

\[ \sup_{t \geq 0} (\| f(t) \|_{H^1} + |z(t)| + |\omega(t) - \omega_0|) \lesssim \varepsilon. \]

(See [39] and also [30].) Thus there exists $\varepsilon_0 > 0$ such that

\[ \inf_{\gamma \in \mathbb{R}} \| u(t) - e^{i\gamma} \phi_{\omega_0} \|_{H^1} < \delta_1 / 2. \]

By continuation argument (see e.g. [25]), we see that there exist $z \in C^1([0, \infty); \mathbb{C})$ and $\omega$, $\theta \in C^1([0, \infty); \mathbb{R})$ such that (2.6) and (2.7) are satisfied for $t \in [0, \infty)$.

Substituting (2.6) into (2.4), we have

\begin{equation}
if_t = (H_{\omega(t)} + \sigma_3 \dot{\gamma}) f + l + N,
\end{equation}

where

\[ l = \sigma_3 \dot{\gamma} \Phi_{\omega(t)} - i\dot{\omega} \partial_\omega \Phi_{\omega(t)} \]

\[ + (z\lambda(\omega(t)) - i\dot{z})\xi(\omega(t)) - (\overline{z}\lambda(\omega(t)) + i\overline{\dot{z}})\sigma_1\xi(\omega(t)) \]

\[ + \sigma_3 \dot{\gamma}(z\xi(\omega(t)) + \overline{\dot{z}}\sigma_1\xi(\omega(t))) - i\dot{\omega}(z\partial_\omega \xi(\omega(t)) + \overline{\dot{z}} \sigma_1 \partial_\omega \xi(\omega(t))). \]
We expand $\mathcal{N}$ in (2.2) as

$$\mathcal{N}(R) = \sum_{2 \leq |m+n| \leq 2N+1} \Lambda_{m,n}(\omega) z^m \bar{z}^n + \sum_{1 \leq |m+n| \leq N} z^m \bar{z}^n A_{m,n}(\omega) f$$

(2.9)

$$+ O(\|f\|^2(\Phi_\omega + R)^{p-3}) + O(|\beta(\|f\|^2)|) + O_{loc}(\|z^{2N+2}\|),$$

where $\Lambda_{m,n}(\omega)$ and $A_{m,n}(\omega)$ are real vectors and matrices which decay like $e^{-a|x|}$ as $|x| \to \infty$, with $\sigma_1 \Lambda_{m,n} = -A_{n,m}$ and $A_{m,n} = -\sigma_1 A_{n,m} \sigma_1$. In the sequel, we denote by $O_{loc}(g)$ terms with $g$ multiplied by a function which decays like $e^{-a|x|}$. By taking the $L^2$-inner product of the equation with generators of $N_g(H^*)$ and $N(H^* - \lambda)$, we obtain a system of ordinary differential equations on modulation and discrete modes.

$$A \begin{pmatrix} i\omega \\ \dot{\gamma} \\ i\dot{z} - \lambda z \end{pmatrix} = \begin{pmatrix} \langle \mathcal{N}, \Phi_\omega \rangle \\ \langle \mathcal{N}, \sigma_3 \partial_\omega \Phi_\omega \rangle \\ \langle \mathcal{N}, \sigma_3 \xi(\omega) \rangle \end{pmatrix},$$

(2.10)

where

$$A = \text{diag} \left( \|\phi_\omega\|_{L^2}^2/d\omega, -d\|\phi_\omega\|_{L^2}^2/d\omega, \langle \xi, \sigma_3 \xi \rangle \right)$$

$$+ O(|z| + \|e^{-a|x|} f\|_{L^2}).$$

Finally, we introduce linear estimates which will be used later. Let $P_c(\omega)$ be the spectral projection from $L^2(\mathbb{R}^d, \mathbb{C}^2)$ onto $L^2_c(H_\omega)$ associated to the splitting (2.3).

**Lemma 2.2** (the Strichartz estimate). Let $d \geq 3$. Assume (H3)–(H9). Let $\omega \in \mathcal{O}$ and $k \in \mathbb{Z}_{\geq 0}$. Then

$$\|\nabla^k e^{itH_\omega} P_c(\omega) \varphi\|_{L^\infty_t L^2_x \cap L^2_t L^{2d/(d-2)}_x} \lesssim \|\nabla^k \varphi\|_{L^2}$$

(2.11)

for any $\varphi \in L^2(\mathbb{R}^d, \mathbb{C}^2)$, and

$$\left\| \nabla^k \int_0^t e^{-isH_\omega} P_c(\omega) g(s) ds \right\|_{L^2_x} \lesssim \|\nabla^k g\|_{L^1_t L^2_x + L^2_t L^{2d/(d+2)}_x},$$

(2.12)

$$\left\| \nabla^k \int_0^t e^{i(t-s)H_\omega} P_c(\omega) g(s) ds \right\|_{L^\infty_t L^2_x \cap L^2_t L^{2d/(d-2)}_x} \lesssim \|\nabla^k g\|_{L^1_t L^2_x + L^2_t L^{2d/(d+2)}_x}$$

(2.13)

for any $g \in L^1_t L^2_x + L^2_t L^{2d/(d+2)}_x$.

**Proof.** As is explained in Yajima [42, 43], Lemma 2.2 follows from the Strichartz estimates in the flat case and $W^{k,p}$-boundedness of wave operators and their inverses. Specifically, let $W(\omega) = \lim_{t \to \infty} e^{-itH_\omega} e^{it\sigma_3(-\Delta + \omega)}.$
By \textbf{[12]},

\[ W(\omega): W^{k,p}(\mathbb{R}^d; \mathbb{C}^2) \to W^{k,p}(\mathbb{R}^d; \mathbb{C}^2) \cap N_g(H^\omega) \]

and its inverse are bounded for \( k \in \mathbb{N} \cup \{0\} \) and \( 1 \leq p \leq \infty \). By \( e^{-itH^\omega}P_c(\omega) = W(\omega)e^{it\sigma_3(\Delta - \omega)}W^{-1}(\omega) \) and by Keel and Tao \textbf{[20]}, we obtain (2.11)–(2.13).

By our hypotheses and by regularity theory, the map \( \omega \to V^\omega \) which associates to \( \omega \) the vector potential in (2.3), is a continuous function with values in the Schwartz space \( S(\mathbb{R}^d; \mathbb{C}^4) \). The following holds also under weaker hypotheses.

\textbf{Lemma 2.3.} Let \( s_1 = s_1(d) > 0 \) be a fixed sufficiently large number. Let \( K \) be a compact subset of \( O \) and let \( I \) be a compact subset of \((\omega, \infty) \cup (-\infty, -\omega)\). Assume that \( \omega \to V^\omega \) is continuous with values in the Schwartz space \( S(\mathbb{R}^d; \mathbb{C}^4) \). Assume furthermore that for any \( \omega \in O \) there are no eigenvalues of \( H^\omega \) in the continuous spectrum and the points \( \pm \omega \) are not resonances. Then there exists a \( C > 0 \) such that

\[ \| (x)^{-s_1} e^{-itH^\omega}R_{H^\omega}(\mu + i0)P_c(\omega)g \|_{L^2(\mathbb{R}^d)} \leq C(t)^{-\frac{d}{2}} \| (x)^{s_1} g \|_{L^2(\mathbb{R}^d)} \]

for every \( t \geq 0 \), \( \mu \in I \), \( \omega \in K \) and \( g \in S(\mathbb{R}^d; \mathbb{C}^2) \).

We skip the proof. See \textbf{[8]} for \( d = 3 \) and \( I \subset (\omega, \infty) \), see also \textbf{[33]}. The proof for \( d = 3 \) and \( I \subset (-\infty, -\omega) \) is almost the same. Finally for \( d > 3 \) a similar proof to \textbf{[8]} holds, changing the formulas for \( R_{-\Delta}(\mu + i0) \).

\section{Normal form expansions}

In this section, following \textbf{[45]} we introduce normal form expansions on the dispersive part \( f \), the modulation mode \( \omega \) and the discrete mode \( z \).

First, we will expand \( f \) into normal forms isolating the slowly decaying part of solutions that arises from the nonlinear interaction of discrete and continuous modes of the wave.

\textbf{Lemma 3.1.} Assume (H1)–(H9) and that \( \varepsilon_* > 0 \) in Theorem \textbf{[14]} is sufficiently small. Let \( a \in (0, \inf_{\omega \in K} \sqrt{\omega - \lambda(\omega)}) \). Then there exist \( \Psi_{m,n}^{(N)}(\omega) \in H^a(\mathbb{R}^d, \mathbb{R}^2) \cap L^2_c(H^\omega) \) for \((m, n) \in \mathbb{Z}_{\geq 0} \) with \( m + n = N + 1 \) and \( \Psi_{m,n}(\omega) \in H^a(\mathbb{R}^d, \mathbb{R}^2) \cap L^2_c(H^\omega) \) for \((m, n) \in \mathbb{Z}_{\geq 0} \) with \( 2 \leq m + n \leq N \) such that for
\( t \geq 0, \)

\[
f(t) = f_N(t) + \sum_{2 \leq m+n \leq N} \Psi_{m,n}(\omega(t))z(t)^m \bar{z}(t)^n,
\]

\[
i P_e(\omega(t)) \partial_t f_N - (H_\omega(t) + P_e(\omega(t)) \gamma(\sigma)) f_N
\]

\[
= \sum_{m+n=N+1} \Phi_{m,n}^{(N)}(\omega(t))z(t)^m \bar{z}(t)^n + N_N,
\]

where \( N_N \) is the remainder term satisfying

\[
|N_N| \lesssim (|z|^{N+2} + |zf_N| + |f_N|^2) e^{-a|x|} + |f_N|^{1+4/d} + |f_N|^{(d+2)/(d-2)}
\]

\[
+ |z||e^{-a|x|} f_N|_{L^2} + \|e^{-a|x|/2} f_N\|_{H^1}^2 e^{-a|x|}.
\]

Before we start to prove Lemma 3.1 we observe the following.

**Lemma 3.2.** Suppose \((H1)–(H9)\) and that \(\varepsilon_0 > 0\) is a sufficiently small number. Then for \(t \geq 0,\)

\[
\begin{pmatrix}
  i\omega \\
  \gamma \\
  i\bar{z} - \lambda z
\end{pmatrix}
= \begin{pmatrix}
  p(z, \bar{z}) \\
  q(z, \bar{z}) \\
  r(z, \bar{z})
\end{pmatrix} + \sum_{1 \leq m+n \leq N} \begin{pmatrix}
  \langle f, \alpha_{m,n}(\omega) \rangle \\
  \langle f, \beta_{m,n}(\omega) \rangle \\
  \langle f, \gamma_{m,n}(\omega) \rangle
\end{pmatrix} z^m \bar{z}^n
\]

\[
+ O(|z|^{2N+2} + \|e^{-a|x|/2} f\|_{H^1}^2),
\]

where \(p(x, y), q(x, y), r(x, y)\) are real polynomials of degree \((2N + 1)\) satisfying

\[
|p(x, y)| + |q(x, y)| + |r(x, y)| = O(x^2 + y^2)
\]

as \((x, y) \to (0, 0)\) and \(\alpha_{m,n}(\omega), \beta_{m,n}(\omega), \gamma_{m,n}(\omega) \in \mathcal{H}_d(\mathbb{R}^d; \mathbb{R}^2) \cap L^2_c(\mathcal{H}_\omega^*)\) with \(0 < a < \inf_{\omega \in \mathcal{K}} \sqrt{\lambda(\omega)}\).

**Proof.** Let us substitute (2.9) into (2.10). Since \(N(R) = O(R^2)\) as \(R \to 0,\) the resulting equation can be written as (3.3). The components of the matrix \(A\) in (2.10) are given by real linear expressions of \(z, \bar{z} \) and \(\langle f, \Phi_\omega \rangle, \langle f, \sigma \partial_z \Phi_\omega \rangle\) and \(\langle f, \sigma_3 \xi \rangle\). Hence it follows that \(p(x, y), q(x, y), r(x, y)\) are real polynomials and \(\alpha_{m,n}(\omega), \beta_{m,n}(\omega), \gamma_{m,n}(\omega) \in \mathcal{H}_d(\mathbb{R}^d; \mathbb{R}^2).\) Since \(f \in L^2_c(\mathcal{H}_\omega),\) we choose \(\alpha_{m,n}(\omega, x), \beta_{m,n}(\omega, x)\) and \(\gamma_{m,n}(\omega, x)\) in \(L^2_c(\mathcal{H}_\omega^*).\)

**Proof of Lemma 3.1.** We will prove Lemma 3.1 by induction. Let \(f_1 = f\) and let

\[
f_{k+1}(t) = f_k(t) + \sum_{m+n=k+1} z(t)^m \bar{z}(t)^n \Psi_{m,n}^{(k)}(\omega(t)) \quad \text{for } 1 \leq k \leq N - 1,
\]

where \(N_N \) is the remainder term satisfying

\[
|N_N| \lesssim (|z|^{N+2} + |zf_N| + |f_N|^2) e^{-a|x|} + |f_N|^{1+4/d} + |f_N|^{(d+2)/(d-2)}
\]

\[
+ |z||e^{-a|x|} f_N|_{L^2} + \|e^{-a|x|/2} f_N\|_{H^1}^2 e^{-a|x|}.
\]
where \( \mathcal{O} \ni \omega \mapsto \Psi_{m,n}(\omega) \mapsto \mathcal{H}_a(\mathbb{R}^d, \mathbb{R}^2) \cap L_c^2(H_\omega) \) is \( C^1 \) in \( \omega \). We will choose \( \Psi_{m,n}(\omega) \) so that for \( k = 1, \cdots, N \), there exist \( \Phi_{m,n}(\omega) \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap L_c^2(H_\omega) \) \( (m, n \in \mathbb{Z}_{\geq 0}, m + n = k + 1) \) and \( N_k \in L_c^2(H_\omega) \) such that

\[
(3.6) \quad P_c(\omega)i\partial_t f_k - (H_\omega + P_c(\omega)\gamma\sigma_3) f_k = \sum_{m+n=k+1} \Phi_{m,n}(\omega) z^m z^n + N_k,
\]

\[
(3.7) \quad |N_k| \lesssim (|z|^{k+2} + |z f_k| + |f_k|^2 f_k)^p e^{-a|x|} + |\beta(|f_k|^2)f_k| + |z||e^{-a|x|}f_k||L^2 + ||e^{-a|x|/2}f_k||_{H^1}^2 e^{-a|x|}.
\]

By (2.8), (2.9) and Lemma 3.2 there exist \( \Phi_{2,0}(\omega), \Phi_{1,1}(\omega), \Phi_{1,2}(\omega) \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap L_c^2(H_\omega) \) such that

\[
P_c(\omega)(l + N) = \Phi_{2,0}(\omega) z^2 + \Phi_{1,1}(\omega) |z|^2 + \Phi_{1,2}(\omega) z^2 + N_1,
\]

and

\[
|N_1| \lesssim e^{-a|x|}(|z|^3 + |z f| + |f|^2 f)^p e^{-a|x|} + |\beta(|f|^2)f| + e^{-a|x|}||e^{-a|x|}f||L^2 + ||e^{-a|x|/2}f||_{H^1}^2 e^{-a|x|}.
\]

Thus we have (3.6) and (3.7) for \( k = 1 \).

Suppose that there exist \( \Phi_{m,n} \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap L_c^2(H_{\omega(t)}) \) satisfying (3.6) and (3.7). Substituting (3.5) into (3.6), we have

\[
(3.8) \quad iP_c(\omega)\partial_t f_{k+1} = (H_\omega + i\gamma\sigma_3) f_{k+1} = N_k + \sum_{m+n=k+1} P_c(\omega) \left( \gamma\sigma_3 \Psi_{m,n}(\omega) - i\omega \partial_{\omega} \Psi_{m,n}(\omega) \right) z^m z^n + \sum_{m+n=k+1} z^m z^n (H_\omega - (m-n)\lambda) \Phi_{m,n}(\omega) + \sum_{m+n=k+1} z^m z^n \Phi_{m,n}(\omega) - \sum_{m+n=k+1} \left( m z^{m-1} z^n (i \dot{z} - \lambda z) - n z^m z^{n-1} (i \dot{z} - \lambda z) \right) \Sigma_{m,n} \Phi_{m,n}(\omega)
\]

Put

\[
\Sigma_{m,n} = -R_{H_\omega}((m-n)\lambda) \Phi_{m,n}(\omega).
\]

Then by (3.4), the right hand side of (3.8) can be rewritten as

\[
\sum_{m+n=k+1} \Phi_{m,n}(\omega) z^m z^n + N_{k+1}
\]
for some $\Phi^{(k+1)}_{m,n} \in L^2_c(H_w) \cap \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \ (m, \ n \in \mathbb{Z}_{\geq 0} \text{ and } m + n = k + 2)$ and $\mathcal{N}_{k+1}$ satisfying

$$\mathcal{N}_{k+1} \lesssim |z|^{k+3} + |zf_k| + |f_k|^2(f_k)^{p-3}e^{-a|x|} + |\beta(|f_k|^2)f_k| + |z|(|z||e^{-a|x|}\|f_k\|_{L^2} + \|e^{-a|x|}/2f_k\|_{L^1})e^{-a|x|}.$$  

By (H1) and (H2),

$$f \text{ where }$$

Thus we have (3.3).

Let $\tilde{f}_N = P_c(\omega_0)f_N$ and

$$f_{N+1} = \tilde{f}_N + \sum_{m+n=N+1} \Psi^{(N)}_{m,n}(\omega_0)z^mz^n,$$

where

$$\Psi^{(N)}_{m,n}(\omega_0) = -R_{H_{\omega_0}}((m-n)\lambda)\Phi^{(N)}_{m,n}(\omega_0) \text{ for } |m-n| \leq N$$

(3.10)

$$\Psi^{(N)}_{N+1,0}(\omega_0) = -R_{H_{\omega_0}}((N+1)\lambda + i0)\Phi^{(N)}_{N+1,0}(\omega_0),$$

$$\Psi^{(N)}_{0,N+1}(\omega_0) = -R_{H_{\omega_0}}(-(N+1)\lambda + i0)\Phi^{(N)}_{0,N+1}(\omega_0).$$

To simplify (3.1), we will introduce new variables

$$\tilde{\omega} := \omega + P(z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^mz^n(f_N, \tilde{\alpha}_{m,n}(\omega)),$$

$$\tilde{z} := \omega + Q(z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^mz^n(f_N, \tilde{\beta}_{m,n}(\omega)),$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials and $\tilde{\alpha}_{m,n}, \tilde{\beta}_{m,n} \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2)$.

**Lemma 3.3.** Assume (H1)–(H9) and that $\varepsilon$ is sufficiently small. Then there exist a polynomial $P(x, y)$ of degree $2N + 1$ satisfying $P(x, y) = O(x^2 + y^2)$ as $(x, y) \to (0, 0)$ and $\tilde{\alpha}_{m,n}(\omega) \in L^2_c(H^*_w) \cap \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2)$ such that for $t \geq 0$,

$$i\tilde{\omega} = O(|z|^{2N+2} + \|e^{-a|x|}/2f_N+1\|_{L^2}^2) \text{ for } t \geq 0.$$  

**Lemma 3.4.** Assume (H1)–(H9) and that $\varepsilon$ is sufficiently small. Then there exists a polynomial $Q(x, y)$ of degree $2N + 1$ satisfying $Q(x, y) = O(x^2 + y^2)$ as $(x, y) \to (0, 0)$, and $\tilde{\beta}_{m,n}(\omega) \in L^2_c(H^*_w) \cap \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2)$ such that for $t \geq 0$,

$$i\tilde{z} - \lambda \tilde{z} = \sum_{1 \leq m \leq N} a_m(\omega, \omega_0)|\tilde{z}|^{2m} \tilde{z} + \tilde{\gamma}^{(N)}_{m,n}(f_N, \tilde{\gamma}^{(N)}_{0,n}(\omega))$$

$$+ O(|\tilde{z}|^{2N+2} + \|e^{-a|x|}/2f_N+1\|_{L^2}^2),$$

(3.12)
where $a_m(\omega, \omega_0)$ ($1 \leq m \leq N - 1$) are real numbers, and $\tilde{\gamma}_{0,N}(\omega) \in \mathcal{H}_d(\mathbb{R}^d; \mathbb{C}^2)$.

Lemmas 3.3 and 3.4 can be obtained in the same way as [35]. See Appendix for the proof.

Now, let us introduce our assumption on (FGR). Let

$$\Gamma(\omega, \omega_0) := \Im a_N(\omega, \omega_0).$$

**Hypothesis 3.5.** There exists a positive constant $\Gamma$ such that

$$\inf_{\omega \in \mathcal{O}} |\Gamma(\omega, \omega_0)| > \frac{1}{2} \Gamma.$$

Under the above assumption, we have the following.

**Lemma 3.6.** Assume (H1)–(H9) and that $\epsilon_0 > 0$ is sufficiently small. Then there exist a positive constant $C$ such that for every $T \geq 0$,

$$\int_0^T |z(t)|^{2N+2} dt \leq C \left( |z(T)|^2 + |z(0)|^2 + \int_0^T \|e^{-a|x|/2} f_{N+1}(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right).$$

**Proof.** Choosing $\epsilon_0$ smaller if necessary, we may assume that $|\Gamma(\omega(t), \omega_0)| > \frac{1}{2} \Gamma$ for every $t \geq 0$. Multiplying (3.12) by $\bar{z}$ and taking the imaginary part of the resulting equation, we have

$$\frac{d}{dt} |\bar{z}|^2 = \Gamma(\omega, \omega_0) |\bar{z}|^{2N+2} + 3 |\bar{z}|^2 f_{N+1} + O(|z|^{2N+3} + |\bar{z}|^2 |e^{-a|x|/2} f_{N+1}|^2_{L^2}).$$

By the Schwarz inequality, we have for a $c > 0$,

$$\left| 3 |\bar{z}|^2 f_{N+1} + O(|z|^{2N+3} + |\bar{z}|^2 |e^{-a|x|/2} f_{N+1}|^2_{L^2}) \right| \leq \frac{1}{4} |z|^{2N+2} + C |\bar{z}|^2 |e^{-a|x|/2} f_{N+1}|^2_{L^2}.$$

Combining (3.13) and (3.14), we obtain Lemma 3.6.

4. **Proof of Theorem 1.1**

To begin with, we restate Theorem 1.1 in a more precise form.

**Theorem 4.1.** Assume (H1)–(H9) and that $d \geq 3$. Let $u$ be a solution of (NLS), $U = \Psi_{0,N}(\omega)$, and let $\Psi_{m,n}(\omega)$ be as in Lemma 3.1. Then if $\epsilon_*$ in Theorem 1.1 is sufficiently small, there exist $C^1$-functions $\omega(t)$ and $\theta(t)$, a
constant $\omega_+ \in \mathcal{O}$ such that $\sup_{t \geq 0} |\omega(t) - \omega_0| = O(\varepsilon)$, $\lim_{t \to +\infty} \omega(t) = \omega_+$ and we can write

$$U(t, x) = e^{i\theta(t)\sigma_3} \left( \Phi_{\omega(t)}(x) + z(t)\xi(\omega(t)) + \overline{z(t)}\sigma_1\xi(\omega(t)) \right) + e^{i\theta(t)\sigma_3} \sum_{2 \leq m+n \leq N} \Psi_{m,n}(\omega(t)) z(t)^m \overline{z(t)}^n + e^{i\theta(t)\sigma_3} f_N(t, x),$$

with

$$\|z(t)\|_{L^{N+1}_t L^{N+2}_x} + \|f_N(t, x)\|_{L^\infty_t H^1_x \cap L^2_t W^{1,2d/(d-2)}_x} \leq C\varepsilon.$$

Furthermore, there exists $f_\infty \in H^1(\mathbb{R}^d, \mathbb{C}^2)$ such that

$$\lim_{t \to \infty} \left\| e^{i\theta(t)\sigma_3} f_N(t) - e^{it\Delta} f_\infty \right\|_{H^1} = 0.$$

Theorem 4.1 shows that a solution to (NLS) around the ground state can be decomposed into a main solitary wave, a well localized slowly decaying part, and a dispersive part that decays like a solution to $iu_t + \Delta u = 0$.

To prove Proposition 3.1, we will apply the endpoint Strichartz estimate. Let $T > 0$ and let

$$X_T = L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; L^{2d/(d-2)}(\mathbb{R}^d)),$$

$$Y_T = L^1(0, T; L^2(\mathbb{R}^d)) + L^2(0, T; L^{2d/(d+2)}(\mathbb{R}^d)),$$

$$Z_T = L^2(0, T; L^2(\mathbb{R}^d; \langle x \rangle^{-2s_1} dx)),$$

where $s_1$ is the positive number given in Lemma 2.3. To prove Theorem 4.1, we need the following.

**Lemma 4.2.** Assume (H1)–(H9) and assume that $\varepsilon_*$ is sufficiently small. Then there exists a $C > 0$ such that for every $T \geq 0$,

$$\begin{aligned}
\|\tilde{f}_N\|_{X_T} + \|\nabla \tilde{f}_N\|_{X_T} & \leq C\varepsilon + C \sup_{0 \leq t \leq T} (1 + |\omega(t) - \omega_0| + |z(t)|) \|z\|_{L^{N+1}_t L^{N+2}_x} \\
& + C \left( \sup_{0 \leq t \leq T} |z(t)| + \|\tilde{f}_N\|_{X_T}^{\min(1, \frac{N}{2d})} \right) \left( \|\tilde{f}_N\|_{X_T} + \|\nabla \tilde{f}_N\|_{X_T} \right).
\end{aligned}$$

**Lemma 4.3.** Assume (H1)–(H9). Let $s_1$ be as in Lemma 2.3 and let $\varepsilon_0 > 0$ be a sufficiently small number. Then there exists a $C > 0$ such that for every
\( T > 0, \)
\[
\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T} \\
\leq C \varepsilon + C \sup_{0 \leq t \leq T} (|\omega(t) - \omega_0| + |\dot{\gamma}(t)| + |z(t)|) \|z\|_{L^2}^{N+2}
\]
(4.2)
\[+ C \left( \sup_{0 \leq t \leq T} |z(t)| + \|\tilde{f}_N\|_{X_T}^{\min(1, \frac{2}{N})} \right) (\|\tilde{f}_N\|_{X_T} + \|\nabla \tilde{f}_N\|_{X_T}) \]
\[+ C \sup_{0 \leq t \leq T} |z(t)|^N (\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T})^2. \]

As in [3, 8], let \( P_+ (\omega) \) and \( P_- (\omega) \) be the spectral projections defined by
\[
P_+ (\omega) f = \frac{1}{2 \pi i} \int_{\lambda \geq \omega} \{ R_{H_\omega} (\lambda + i0) - R_{H_\omega} (\lambda - i0) \} f d\lambda, \]
\[
P_- (\omega) f = \frac{1}{2 \pi i} \int_{\lambda \leq -\omega} \{ R_{H_\omega} (\lambda + i0) - R_{H_\omega} (\lambda - i0) \} f d\lambda. \]

To apply the Strichartz estimate (Lemma 2.2) to (3.2), we will use a gauge transformation introduced by [3] and give a priori estimates for the remainder terms.

**Lemma 4.4.** Assume (H1)–(H9) and that \( \varepsilon_* \) is sufficiently small. For \( t \geq 0, \)
\[
i \partial_t \tilde{f}_N = \left( H_{\omega_0} + (\dot{\theta} - \omega_0)(P_+ (\omega_0) - P_- (\omega_0)) \right) \tilde{f}_N
\]
(4.3)
\[+ \sum_{m+n=N+1} \Phi_{m,n}^{(N)} (\omega_0) z^m \bar{z}^n + \tilde{N}_N, \]
\[
i \partial_t f_{N+1} = \left( H_{\omega_0} + (\dot{\theta} - \omega_0)(P_+ (\omega_0) - P_- (\omega_0)) \right) f_{N+1}
\]
(4.4)
\[+ N_{N+1} + \tilde{N}_{N+1}, \]
where
\[
N_{N+1} = (N + 1) \left\{ z^N (i \dot{z} - \lambda z) \Psi_{N+1,0}^{(N+1)} (\omega) + \bar{z}^N (i \bar{z} - \bar{\lambda} \bar{z}) \overline{\Psi_{0,N+1}^{(N+1)}} (\omega) \right\}
\[+ (\dot{\theta} - \omega_0)(P_+ (\omega) - P_- (\omega)) (\Psi_{N+1,0}^{(N+1)} z^{N+1} + \overline{\Psi_{0,N+1}^{(N+1)}} (\omega) \bar{z}^{N+1}), \]
and
\[
\|\tilde{N}_N\|_{Y_T} + \|\nabla \tilde{N}_N\|_{Y_T} + \|\tilde{N}_{N+1}\|_{Y_T} + \|\nabla \tilde{N}_{N+1}\|_{Y_T}
\]
(4.5)
\[\leq C \sup_{0 \leq t \leq T} (|\omega(t) - \omega_0| + |z(t)|) \|z\|_{L^2}^{N+2}
\[+ \left( \sup_{0 \leq t \leq T} (|\omega(t) - \omega_0| + |z(t)|) + \|f_N\|_{X_T}^{\min(1, \frac{2}{N})} \right) (\|\tilde{f}_N\|_{X_T} + \|\nabla \tilde{f}_N\|_{X_T}). \]
To obtain Lemma 4.4, we need the following, which holds also under weaker hypotheses.

**Lemma 4.5** ([8]). Assume that \( \omega \to V_\omega \) is continuous with values in the Schwartz space \( \mathcal{S}(\mathbb{R}^d; \mathcal{C}^4) \). Assume furthermore that for any \( \omega \in \mathcal{O} \) there are no eigenvalues of \( H_\omega \) in the continuous spectrum and the points \( \pm \omega \) are not resonances. Then

\[
\| P_\omega(\omega) \sigma_3 - (P_+(\omega) - P_-(\omega)) \|_{B(L^p, L^q)} \leq c_{p,q}(\omega) < \infty.
\]

for any \( p \in [1, 2] \) \( q \in [2, \infty) \), where \( c_{p,q}(\omega) \) is a constant upper semicontinuous in \( \omega \).

**Proof of Lemma 4.4.** By a simple computation, we have (4.3) and (4.4) with \( \omega \in N \) not resonances. Then
\[
\tilde{N}_N = P_\omega(\omega_0) N_0 + \delta N_N, \quad \tilde{N}_{N+1} = \tilde{N}_N + \tilde{N}_{N+1}, \quad \text{where}
\]

\[
\delta N_N = P_\omega(\omega_0) \left\{ i \dot{\omega} \partial_\omega P_\omega(\omega) + (\dot{\theta} - \omega_0) (P_\omega(\omega) \sigma_3 - P_+(\omega) + P_-(\omega)) \right\} f_N
\]

\[+ \sum_{m+n=N+1} P_\omega(\omega_0) \left( \Phi^{(N)}_{m,n}(\omega) - \Phi^{(N)}_{m,n}(\omega_0) \right) z^m \bar{z}^n,
\]

and

\[
\tilde{N}_{N+1} = \sum_{m,n \in \mathbb{N}} \left( m z^{m-1} \overline{z}(i \dot{z} - \lambda z) - n z^m \overline{z}(i \dot{z} - \lambda z) \right) \Psi^{(N+1)}_{m,n}(\omega_0)
\]

\[\qquad \quad - (\dot{\theta} - \omega_0) \sum_{m,n \in \mathbb{N}} (P_+(\omega_0) - P_-(\omega_0)) \Psi^{(N)}_{m,n}(\omega_0) z^m \bar{z}^n.
\]

Applying Hölder’s inequality to (3.3), we have

\[
\| N_N \|_{Y_T} \lesssim \sup_{0 \leq t \leq T} |z(t)| \left( \| z \|_{L^{N+1}_2(N+1)}(0,T) + \| f_N \|_{X_T} \right)
\]

\[\quad \quad + \| f_N \|_{\dot{X}_T}^2 + \| f_N \|_{\dot{X}_T}^{\frac{4+4}{2}} + \| f_N \|_{X_T} \| f_N \|_{L^\infty(0,T;L^\frac{2d}{d-2})} \cdot
\]

Similarly, we have

\[
\| \nabla N_N \|_{Y_T} \lesssim \sup_{0 \leq t \leq T} |z(t)| \left( \| z \|_{L^{N+1}_2(N+1)}(0,T) + \| f_N \|_{X_T} + \| \nabla f_N \|_{X_T} \right)
\]

\[\quad \quad + \| \nabla f_N \|_{X_T} \left( \| f_N \|_{X_T} + \| f_N \|_{\dot{X}_T}^4 + \| f_N \|_{L^\infty(0,T;L^\frac{2d}{d-2})} \right).
\]

See [18] for the details. By (2.10), we have

\[
|\dot{\omega}| + |\dot{\theta} - \omega| + |i \dot{z} - \lambda z| \lesssim |z|^2 + \| f_N \|_{L^\frac{2d}{d-2}}^2.
\]
From the definition, it is obvious that $\partial_\omega P_c(\omega) \in B(L^{2d/4}, L^{2d/4})$. Thus by Lemma 4.5, it follows that

$$
\|\delta_N\|_{Y_T} + \|\nabla \delta_N\|_{Y_T} \lesssim \sup_{0 \leq t \leq T} \left( |z(t)|^2 + \|f(t)\|_{H^1} \right) \|f_N\|_{X_T}
$$

$$
+ \sup_{0 \leq t \leq T} |\omega(t) - \omega_0| \left( \|z\|_{L^2(N+1)(0,T)} + \|f_N\|_{X_T} \right).
$$

Similarly, we have

$$
\|\delta_{N+1}\|_{Y_T} + \|\nabla \delta_{N+1}\|_{Y_T} \lesssim \|z\|_{N+2}^{N+2} \|f_N\|_{X_T}.
$$

Combining the above, we obtain (4.5). Thus we complete the proof. \qed

**Proof of Lemma 4.2.** Let $f_\pm = P_\pm(\omega_0)\tilde{f}_N$ and

$$
U_\pm(t, s) = e^{\pm i \int_s^t (\omega_0 - \theta) ds} P_\pm(\omega_0) e^{-i(t-s)H_{\omega_0}} P_\pm(\omega_0).
$$

It follows from Lemma 2.2 that there exists a $C > 0$ such that

$$
\|U_\pm(\cdot, s)\|_{X_T} \leq C \|\varphi\|_{L^2}
$$

for every $T \geq 0$, $s \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^d)$, and

$$
\left\|\int_0^t U_\pm(t, s) g(s) ds\right\|_{X_T} \leq C \|g\|_{Y_T}
$$

for every $T \geq 0$ and $g \in \mathcal{S}(\mathbb{R}^{d+1})$.

By Lemma 4.4

$$
f_\pm(t) = U_\pm(t, 0) f_\pm(0) - i \int_0^t U_\pm(t, s) \left\{ \sum_{m+n=N+1} \Phi_{m,n}^{(N)}(\omega_0) z^m \bar{z}^n + \tilde{N}_N \right\} ds.
$$

In view of Lemma 2.1 and the definition of $f_\pm(t)$, we have $\|f_\pm(0)\|_{H^1} \lesssim \varepsilon$.

Applying (4.6) and (4.7) to (4.8), we have

$$
\|f_\pm(t)\|_{X_T} + \|\nabla f_\pm(t)\|_{X_T}
$$

$$
\lesssim \|f_\pm(0)\|_{H^1} + \|z\|_{L^2(N+1)(0,T)} + \|\tilde{N}_N\|_{Y_T} + \|\nabla \tilde{N}_N\|_{Y_T}
$$

$$
\lesssim \varepsilon + \sup_{0 \leq t \leq T} (1 + |\omega(t) - \omega_0| + |z(t)|) \|z\|_{L^2(N+2)}^{N+1}
$$

$$
+ \left( \sup_{0 \leq t \leq T} |z(t)| + \|f_N\|_{X_T}^{\min(1,2)} \right) (\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T}).
$$

By the definition of $P_c(\omega)$,

$$
\|f_N - \tilde{f}_N\|_{H^1} \lesssim |\omega - \omega_0| e^{-a|x|} \|f_N\|_{L^2}.
$$
Substituting (4.10) into (4.9), we obtain (4.1). Thus we complete the proof of Lemma 4.2.

Proof of Lemma 4.3. Let \( h_\pm(t) = P_\pm(\omega_0)f_{N+1} \). Using the variation of constants formula, we have

\[
h_\pm(t) = U_\pm(t,0)h_\pm(0) - i \int_0^t U_\pm(t,s)(N_{N+1} + \tilde{N}_{N+1})ds.
\]

Put \( h_\pm(0) = h_{0,1,\pm} + h_{0,2,\pm} \), where

\[
h_{0,2,\pm} = f_\pm(0) + \sum_{m+n=N+1} \Psi_{m,n}^{(N)}(\omega_0)z(0)^m \overline{z(0)^n}.
\]

Note that \( \Psi_{m,n}^{(N)}(0) \in H^1 \) if \( m, n \geq 1 \), whereas \( \Psi_{N+1,0}^{(N)}(0) \) and \( \Psi_{0,N+1}^{(N)}(0) \) may not belong to \( L^2 \).

Since \( s_1 > 0 \), we have \( \|f\|_{Z_T} \lesssim \|f\|_{X_T} \). Applying (4.6) and (4.7), we have

\[
\|U_\pm(t,0)h_{0,2,\pm}\|_{Z_T} + \|\nabla U_\pm(t,0)h_{0,2,\pm}\|_{Z_T} \\
\lesssim \|U_\pm(t,0)h_{0,2,\pm}\|_{X_T} + \|\nabla U_\pm(t,0)h_{0,2,\pm}\|_{X_T} \lesssim \varepsilon,
\]

and

\[
\left\| \int_0^t U_\pm(t,s)\tilde{N}_{N+1}ds \right\|_{Z_T} + \left\| \nabla \int_0^t U_\pm(t,s)\tilde{N}_{N+1}ds \right\|_{Z_T} \\
\lesssim \left\| \int_0^t U_\pm(t,s)\tilde{N}_{N+1}ds \right\|_{X_T} + \left\| \nabla \int_0^t U_\pm(t,s)\tilde{N}_{N+1}ds \right\|_{X_T} \\
\lesssim \sup_{0 \leq t \leq T} (|\omega(t) - \omega_0| + |z(t)|) \|z\|_{L^2_{2N+2}}^{N+1} \\
+ \left( \sup_{0 \leq t \leq T} |z(t)| + \|\tilde{f}_N\|_{X_T}^\min(1,\frac{1}{4})^2 \right) (\|\tilde{f}_N\|_{X_T} + \|\nabla \tilde{f}_N\|_{X_T})
\]

in the same way as the proof of Lemma 4.2.

By Lemma 2.3 and the definition of \( \Psi_{N+1,0}^{(N)}(0) \) and \( \Psi_{0,N+1}^{(N)}(0) \), we have

\[
\|U_\pm(t,0)h_{0,1,\pm}\|_{Z_T} + \|\nabla U_\pm(t,0)h_{0,2,\pm}\|_{Z_T} \\
\lesssim \|t\|_{d/2} \left( \|\langle x \rangle^{s_1} \Phi_{N+1,0}^{(N)}(0)\|_{H^1} + \|\langle x \rangle^{s_1} \Phi_{0,N+1}^{(N)}(0)\|_{H^1} \right) \|\tilde{f}_N\|_{L^2(0,T)} \\
\lesssim \varepsilon.
\]

It follows from Lemma 3.2 that

\[
|iz\dot{z} - \dot{\lambda}z| \lesssim |z|^2 + \|e^{-\alpha |x|/2}f_{N+1}\|_{H^1}^2, \\
|\dot{\theta} - \omega_0| \leq |\dot{\theta} - \omega| + |\omega - \omega_0| \lesssim |\omega - \omega_0| + |z|^2 + \|e^{-\alpha |x|/2}f_{N+1}\|_{H^1}^2.
\]
Thus by Lemma 2.3
\[
\sum_{i=0}^{1} \left\| \int_0^t U_{\pm}(t,s)N_{N+1} ds \right\|_{Z_T} + \left\| \nabla \int_0^t U_{\pm}(t,s)N_{N+1} ds \right\|_{Z_T} \\
\lesssim \left\| \int_0^t (t-s)^{-d/2}(\varepsilon |z(s)|^{N+1} + |z(s)|^{N+2} e^{-a|x|^2/2} f_{N+1}(s))^{2}\|_{H_1} ds \right\|_{L^2(0,T)} \\
\lesssim \varepsilon \|z\|_{L^2N+2(0,T)}^N + \sup_{0 \leq t \leq T} |z(t)|^N (\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T})^2.
\]
Combining the above, we obtain (4.12).

Now, we are in position to prove Theorem 1.1 and 4.1.

Proof of Theorems 1.1 and 4.1 Since \( e^{i\omega_0 t} \phi_{\omega_0} \) is orbitally stable, Lemma 3.2 and Remark 2.1 imply that
\[
\sup_{t \geq 0} (|z(t)| + |\omega(t) - \omega_0| + |\gamma(t)|) \lesssim \varepsilon.
\]
We have
\[
(4.11) \quad \|f_N\|_{W^{k,p}} \lesssim \|\tilde{f}_N\|_{W^{k,p}}
\]
for every \( k \in \mathbb{Z}_{\geq 0} \) and \( 1 \leq p \leq \infty \) because
\[
\|f_N - \tilde{f}_N\|_{W^{k,p}} = \|(P_c(\omega) - P_c(\omega_0))f_N\|_{W^{k,p}} \lesssim |\omega - \omega_0|\|f_N\|_{W^{k,p}}.
\]
Thus by Lemmas 3.64.2 and 4.3 it holds that for every \( T \geq 0 \),
\[
(4.12) \quad \|\tilde{z}\|_{L^2N+2(0,T)} \lesssim \varepsilon + \|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T},
\]
\[
\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T}
\]
\[
(4.13) \quad \lesssim \varepsilon + \|\tilde{z}\|_{L^2N+2(0,T)}^{N+1} + \left( \varepsilon + \|f_N\|_{X_T}^{\min(\frac{1}{4}, \frac{1}{2})} \right) (\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T}),
\]
\[
\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T}
\]
\[
(4.14) \quad \lesssim \varepsilon + \varepsilon \|\tilde{z}\|_{L^2N+2(0,T)}^{N+1} + \left( \varepsilon + \|f_N\|_{X_T}^{\min(\frac{1}{4}, \frac{1}{2})} \right) (\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T})
\]
\[
+ \varepsilon^N (\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T})^2.
\]
Let \( A > 0 \) be a sufficiently large number. Adding (4.13) to (4.14) multiplied by \( A \) and substituting (4.12) into the resulting equation, we have
\[
\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T} + \frac{A}{2} (\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T})
\]
\[
\lesssim \varepsilon + \|f_N\|_{X_T}^{\min(\frac{1}{4}, \frac{1}{2})} (\|f_N\|_{X_T} + \|\nabla f_N\|_{X_T}) + \varepsilon^N (\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T})^2.
\]
Letting $T \to \infty$, we obtain
\begin{align}
\|f_N\|_{L^\infty_t H^1_0 \cap L^2_t W^{1,2}_x} + \|\langle x \rangle^{-s_i} f_{N+1}\|_{L^2_t H^1_0} & \lesssim \varepsilon, \\
\int_0^\infty |z(t)|^{2N+2} dt & \lesssim \varepsilon.
\end{align}
\begin{equation}
(4.15)
\end{equation}
\begin{equation}
(4.16)
\end{equation}
Since $\dot{z}$ is bounded from (2.10), it follows from (4.16) that $\lim_{t \to \infty} z(t) = 0$. Furthermore, Lemma 3.3, (4.15) and (4.16) imply that there exists an $\omega_+ \in \mathcal{O}$ such that
\begin{equation}
\lim_{t \to \infty} \omega(t) = \lim_{t \to \infty} \tilde{\omega}(t) = \omega_+.
\end{equation}
Thus we prove Theorem 1.1.

Finally, we will prove that $f_N(t)$ is asymptotically free as $t \to \infty$. Let $U(t, s) = U_+(t, s) + U_-(t, s)$ and $t_2 \geq t_1 \geq 0$. Lemma 2.2 and (4.3) yield that as $t_1 \to \infty$,
\begin{align}
\left\| U(0, t_2) \tilde{f}_N(t_2) - U(0, t_1) \tilde{f}_N(t_1) \right\|_{H^1} & = \left\| \int_{t_1}^{t_2} U(0, s) \left\{ \sum_{m+n=N+1} \Phi_{m,n}(\omega) z^m \bar{z}^n + \tilde{N}_N \right\} ds \right\|_{H^1} \\
& \lesssim \|z\|_{L^{2N+2}(t_1, t_2)} + \|\tilde{N}_{N+1}\|_{L^1(1; H^1(\mathbb{R}^d)) + L^2(1; W^{1,2}_x(\mathbb{R}^d))} \to 0,
\end{align}
Hence there exists $\tilde{f}_\infty \in H^1(\mathbb{R}^d)$ such that
\begin{equation}
\lim_{t \to \infty} \left\| \tilde{f}_N(t) - U(t, 0) \tilde{f}_\infty \right\|_{H^1} = 0.
\end{equation}
For $q \in (2, \frac{2d}{d-2})$, we have
\begin{equation}
\lim_{t \to \infty} \|\tilde{f}_N(t)\|_{L^q} = \lim_{t \to \infty} \|U(t, 0) \tilde{f}_\infty\|_{L^q} = 0.
\end{equation}
By the definition of $f_N$ and $\tilde{f}_N$ and (1.11),
\begin{align}
\|\tilde{f}_N(t) - f_N(t)\|_{H^1} & = \|(P_+(\omega) - P_-(\omega)) f_N\|_{H^1} \\
& \lesssim \|\omega - \omega_0\| \|f_N\|_{L^q} \lesssim \|\tilde{f}_N\|_{L^q} \to 0,
\end{align}
as $t \to \infty$. Combining the above, we have by the definition of $U(t, 0)$
\begin{equation}
\lim_{t \to +\infty} \|f_N(t) - e^{i[(t\omega_0 - \theta(t) + \theta(0))(P_+(\omega_0) - P_-(\omega_0))] e^{-i\Delta t} \omega_0} \tilde{f}_\infty\|_{H^1} = 0.
\end{equation}
Consider the strong limit $W(\omega_0) = \lim_{t \to \infty} e^{i\Delta t} e^{it\Delta} (\Delta - \omega_0) \sigma_3$ and set
\begin{equation}
f_{\infty} = W(\omega_0)^{-1} e^{i\theta(0)(P_+(\omega_0) - P_-(\omega_0))} \tilde{f}_\infty.
\end{equation}
Notice that since $e^{it\omega_0\sigma_3}$ is a unitary matrix periodic in $t$ and $e^{it\omega_0\sigma_3}f_\infty$ describes circle in $L^2$, we have

$$\lim_{t \to +\infty} \left( W(\omega_0)e^{i t \omega_0 \sigma_3} f_\infty - e^{i t H_{\omega_0}} e^{i t (\Delta - \omega_0) \sigma_3} e^{i t \omega_0 \sigma_3} f_\infty \right) = 0.$$ 

Since $\|e^{it H_{\omega_0}}\|_{L^\infty B(L^2(\mathcal{H}_{\omega_0}), L^2(\mathcal{H}_{\omega_0}))} < 1$, Lemma 2.2 implies

$$\|e^{-it H_{\omega_0}} W(\omega_0)e^{it \omega_0 \sigma_3} f_\infty - e^{it (\Delta - \omega_0) \sigma_3} e^{it \omega_0 \sigma_3} f_\infty\|_{H^1} \approx \|W(\omega_0)e^{it \omega_0 \sigma_3} f_\infty - e^{it H_{\omega_0}} e^{it (\Delta - \omega_0) \sigma_3} e^{it \omega_0 \sigma_3} f_\infty\|_{H^1},$$

the above 0 limit implies

$$\lim_{t \to +\infty} \|e^{-it H_{\omega_0}} W(\omega_0)e^{it \omega_0 \sigma_3} f_\infty - e^{it (\Delta - \omega_0) \sigma_3} e^{it \omega_0 \sigma_3} f_\infty\|_{H^1} = 0.$$ 

Since $W(\omega_0)$ conjugates $H_{\omega_0}$ into $\sigma_3(\Delta + \omega_0)$, we get

$$e^{i t \omega_0 + i \theta(0)}(P_+ (\omega_0) - P_-(\omega_0)) e^{-it H_{\omega_0}} f_\infty = e^{-it H_{\omega_0}} W(\omega_0)e^{it \omega_0 \sigma_3} f_\infty.$$ 

Thus we get the following, completing the proof of Theorem 4.1:

$$\lim_{t \to +\infty} \|e^{i \theta(t) \sigma_3} f_N(t) - e^{i t \Delta \sigma_3} f_\infty\|_{H^1} = 0.$$ 

\[\square\]

**Corollary 4.6.** If Hypothesis 3.5 holds, then $\Gamma(\omega, \omega) > \Gamma$ holds.

Suppose we have $\Gamma(\omega, \omega_0) < -\Gamma/2$. We can pick initial datum so that $f_{N+1}(0) = 0$ and $z(0) \approx \epsilon$. Then from Lemma 4.3 we get $\|f_{N+1}\|_{Z_T} + \|\nabla f_{N+1}\|_{Z_T} \leq C \epsilon^2$ for any $T$ for fixed $C > 0$. Then integrating (3.13) we get

$$|\tilde{z}(t)|^2 - |\tilde{z}(0)|^2 \geq \frac{\Gamma}{2} \int_0^t |\tilde{z}|^{2N+2} + o(\epsilon) \left( \int_0^t |\tilde{z}|^{2N+2} \right)^{\frac{1}{2}} + o(\epsilon^2).$$

For large $t$ $|\tilde{z}(t)| < |\tilde{z}(0)|$ since $z(t) \to 0$, so for large $t$ we get $\int_0^t |\tilde{z}|^{2N+2} = o(\epsilon^2)$. In particular for $t \to \infty$ we get $\epsilon^2 \leq o(\epsilon^2)$ which is absurd for $\epsilon \to 0$.

5. **Proof of Theorem 1.2**

We will provide only a sketch of the proof. The argument is essentially the same of Theorem 1.1. However, when we select the main terms of the equations of the discrete modes we have more than just one dominating term. Since these dominating terms could cancel with each others, the situation is harder than the one in (3.13). We resolve all problems by assuming Hypothesis 5.2 which is very close in spirit to the (FGR) hypothesis in [15].
The eigenvectors \( \lambda_j(\omega) \) have corresponding real eigenvectors \( \xi_j(\omega) \), normalized so that \( \langle \xi_j, \sigma \xi_j \rangle = \delta_j \). \( \sigma_1(\omega) \) generates \( N(H_\omega + \lambda(\omega)) \). The \( \xi_j(\omega) \) can be chosen real because \( H_\omega \) has real coefficients. The functions \( (\omega, x) \in \mathcal{O} \times \mathbb{R}^d \to \xi_j(\omega, x) \) are \( C^2 \); \( |\xi_j(\omega, x)| < ce^{-a|x|} \) for fixed \( c > 0 \) and \( a > 0 \) if \( \omega \in K \subset \mathcal{O} \), \( K \) compact. \( \xi_j(\omega, x) \) is even in \( x \) since by assumption we are restricting ourselves in the category of such functions. We order the indexes so that \( N_1 \leq N_2 \leq \cdots \). We set

\[
R(t) = (z \cdot \xi + \bar{z} \cdot \sigma_1 \xi) + f(t) \in \left[ \sum_{j, \pm} N(H(t) \mp \lambda_j(t)) \right] + L^2(H(t))
\]

where \( z \cdot \xi = \sum z_j \xi_j \). In the sequel we use the multi index notation \( z^m = \prod z_j^m \). Denote by \( N \) the largest of the \( N_j \). Given two vectors we will write \( \bar{a} \leq \bar{b} \) if \( a_j \leq b_j \) for all components. If this happens we write \( \bar{a} < \bar{b} \) if we have \( a_j < b_j \) for at least one \( j \). We will set \( (m-n) \cdot \lambda = \sum_j (m-n)_j \lambda_j \). We will denote by \( \text{Res} \) the set of vectors \( \bar{M} \geq 0 \), with integer entries, with the property that \( \bar{M} \cdot \lambda > \omega \) and if \( \bar{M} < \bar{M} \) then \( \bar{M} \cdot \lambda < \omega \). Then we have:

**Theorem 5.1.** Assume (H1)–(H6), (H7’)–(H10’) (in particular Hypothesis 5.2 below) and that \( d \geq 3 \). Let \( u \) be a solution of \( \text{NLS} \), \( U = (u, \bar{u}) \). Let \( \Psi_{m,n}(\omega) \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^2) \) be the vectors rapidly decreasing for \( |x| \to \infty \), with real entries, and with continuous dependence on \( \omega \). Then if \( \varepsilon_* \) is sufficiently small, there exist \( C^1 \)-functions \( \omega(t) \) and \( \theta(t) \), a constant \( \omega_+ \in \mathcal{O} \) such that \( \sup_{t \geq 0} |\omega(t) - \omega_0| = O(\varepsilon) \), \( \lim_{t \to +\infty} \omega(t) = \omega_+ \) and we can write

\[
U(t, x) = e^{it(0)\alpha_1} \left( \Phi_{\omega(t)}(x) + \zeta(t) \cdot \xi(\omega(t)) + \xi_1(\omega(t)) \right) + e^{it(\alpha_1)} \sum_{2 \leq |m+n| \leq N} \Psi_{m,n}(\omega(t)) \zeta^{m}(t) + e^{it(\alpha_2)} f_N(t, x),
\]

with for a fixed \( C > 0 \)

\[
\sum_{M \in \text{Res}} \|\zeta^{M}(t)\|_{L^2}^2 + \|f_N(t, x)\|_{L^2 H_{\chi}^1 \cap L^2 W^{2d/(d-2)}} \leq C\varepsilon.
\]

Furthermore, there exists \( f_+ \in H^1(\mathbb{R}^d, \mathbb{C}^2) \) such that

\[
\lim_{t \to \infty} \|f_N(t) - e^{-i\theta(t)\alpha_3} e^{it\alpha_3} f_+\|_{H^1} = 0.
\]

We consider \( k = 1, 2, \ldots, N \) and set \( f = f_k \) and \( z_{(k),j} = z_j \) for \( k = 1 \). The other \( f_k \) and \( z_{(k),j} \) are defined below by induction.

\[
E_{ODE}(k) = \sum_{M \in \text{Res}} \left\{ O(|z_{(k)}^M|^2) + O(z_{(k)}^M f_k) \right\} + O(f_k^2) + O(\beta(|f_k|^2 f_k))
\]
In the PDE’s there will be error terms of the form
\[
E_{PDE}(k) = \sum_{M \in \text{Res}} O_{loc}(|z(k)|^M |\dot{z}(k)|) + O_{loc}(\dot{z}(k)f_k) + O(f_k^2) + O(\beta(|f_k|^2 f_k)).
\]

For \( k = 1 \), \( f_1 = f \) and \( z(k)_{,j} = z_j \) thanks to (2.9) we have
\[
\begin{align*}
&\mathcal{L} \langle \Phi, \partial_x \Phi \rangle = \langle \sum_{2 \leq |m+n| \leq 2N+1} \Lambda^{(k)}_{m,n}(\omega)z^m_{(k)} z^n_{\lambda(k)} + \sum_{1 \leq |m+n| \leq N} z^m_{(k)} z^n_{\lambda(k)} A^{(k)}_{m,n}(\omega) f_k + E_{ODE}(k), \Phi \rangle \\
\end{align*}
\]
(5.1)
\[
\begin{align*}
&i \dot{z}_{j,(k)} - \lambda_j z_{j,(k)} = \sum_{|m|=1}^N a^{(k)}_{j,m}(\omega) |z^m_{(k)}|^2 z_{j,(k)} + \langle \sum_{k+1 \leq |m+n| \leq 2N+1} \Lambda^{(k)}_{m,n}(\omega) z^m_{\lambda(k)} z^n_{\lambda(k)} \\
&+ \sum_{1 \leq |m+n| \leq N} z^m_{(k)} z^n_{\lambda(k)} A^{(k)}_{m,n}(\omega) f_k + E_{ODE}(k), \sigma_3 \xi_j \\
&i \partial_t f_k = (H_\omega + \sigma_3 \gamma) f_k + E_{PDE}(k) + \\
&+ \sum_{k+1 \leq |m+n| \leq 2N+1} R^{(k)}_{m,n}(\omega) z^m_{\lambda(k)} z^n_{\lambda(k)} \text{ (sum over pairs with } |(m - n) \cdot \lambda| < \omega) \\
&+ \sum_{2 \leq |m+n| \leq N+1} R^{(k)}_{m,n}(\omega) z^m_{\lambda(k)} z^n_{\lambda(k)} \text{ (sum over pairs with } |(m - n) \cdot \lambda| > \omega) \\
\end{align*}
\]
with \( \Re \left[ a^{(k)}_{j,m} \right] = 0 \) and
\[
A^{(k)}_{m,n}, R^{(k)}_{m,n} \text{ and } \Lambda^{(k)}_{m,n} \text{ real, rapidly decreasing in } x, \\
\text{continuous in } (\omega, x), \text{ with } \sigma_1 R^{(k)}_{m,n} = -R^{(k)}_{n,m}. 
\]
(5.2)
We set \( f_1 = f \) and, summing only over \((m, n)\) with \( |(m - n) \cdot \lambda| < \omega\), we define inductively \( f_k \) with \( k \leq N \) by
\[
f_k = f_{k-1} + \sum_{|m+n|=k} R_{H_\omega}((m - n) \cdot \lambda) P_c(H_\omega) R_{(m-1)}^{(k-1)}(\omega) z^m_{(k-1)} z^n_{(k-1)}. 
\]
By \( \sigma_1 R^{(k-1)}_{m,n} = -R^{(k-1)}_{n,m} \), by \( [\sigma_1, P_c(H_\omega)] = 0 \), by the fact that \( R^{(k-1)}_{m,n} \) is real
and by \( \sigma_1 H_\omega = -H_\omega \sigma_1 \), we get \( \sigma_1 f_k = \mathcal{I}_k \). Summing only over \((m, n)\) with \( \lambda_j(\omega) \neq (m - n) \cdot \lambda(\omega) \), we set
\[
z_{(k),j} = z_{(k-1),j} + \sum_{|m+n|=k} \frac{z^m_{(k-1)} z^n_{(k-1)}}{\lambda_j - (m - n) \cdot \lambda} \Lambda^{(k-1)}_{m,n} \sigma_3 \xi_j. 
\]
By induction $f_k$ and $z_{(k)}$ solve (5.1) and (5.2). At the step $k = N$, we can define

$$\zeta_j = z_{(N),j} + p_j(z_{(N)}, \overline{z}_{(N)}) \sum_{1 \leq |m+n| \leq N} z_{(N),j}^m \overline{w}_n^0 (f_N, \alpha_{jmn})$$

(5.3)

$$\bar{\omega} = \omega + q(\zeta, \tilde{\zeta}) + \sum_{1 \leq |m+n| \leq N} \zeta^m \bar{\zeta}^n (f_N, \beta_{mn}),$$

with: $\alpha_{jmn}$ and $\beta_{mn}$ vectors with entries which are real valued exponentially decreasing functions; $p_j$ polynomials in $(z_{(N)}, \overline{z}_{(N)})$ with real coefficients and whose monomials have degree not smaller than $N + 1$; $q$ a polynomial in $(\zeta, \bar{\zeta})$ with real coefficients and monomials at least quadratic. The above transformation can be chosen so that with $a_{j,m}(\omega)$ real we have

$$i \dot{\bar{\omega}} = \langle E_{PDE}(N), \Phi \rangle$$

$$i \dot{\zeta}_j - \lambda_j(\omega)\zeta_j = \sum_{1 \leq |m| \leq N} a_{j,m}(\omega) |\zeta|^2 \zeta_j + \langle E_{ODE}(N), \sigma_3 \xi_j \rangle + \sum_{n+\delta j \in Res} \zeta^n \langle A_{0,n}^{(N)}(\omega) f_N, \sigma_3 \xi_j \rangle.$$

(5.4)

Now we fix $\omega_0 = \omega(0)$, set $H = H(\omega(0))$ and rewrite the equation for $f_N$,

$$i \partial_t P_c(\omega_0)f_N = \left\{ H + (\dot{\theta} - \omega_0)(P_-(\omega_0) - P_-(\omega_0)) \right\} P_c(\omega_0)f_N +$$

$$+ P_c(\omega_0)E_{PDE}(N) + \sum_{2 \leq |m+n| \leq N+1} P_c(\omega_0) R_{m,n}^{(N)}(\omega_0) \zeta^m \bar{\zeta}^n$$

(5.5)

where in the summation $|m+n| \leq N$ implies $|(m-n) \cdot \lambda| > \omega$ and with

$$E_{PDE}(N) = E_{PDE}(N) + \sum_{2 \leq |m+n| \leq N+1} P_c(\omega_0) \left( R_{m,n}^{(N)}(\omega) - R_{m,n}^{(N)}(\omega_0) \right) \zeta^m \bar{\zeta}^n +$$

$$+ (\dot{\theta} - \omega_0)(P_c(\omega_0) \sigma_3 - (P_+(\omega_0) - P_-(\omega_0))) f_N + (V(\omega) - V(\omega_0)) f_N$$

$$+ (\dot{\theta} - \omega_0)(P_c(\omega) - P_c(\omega_0)) \sigma_3 f_N.$$  

Next, recall $H = H(\omega(0))$, we set

$$f_N = - \sum_{2 \leq |m+n| \leq N+1} R_H((m-n) \cdot \lambda(\omega_0) + i0) P_c(\omega) R_{m,n}^{(N)}(\omega_0) \zeta^m \bar{\zeta}^n + f_{N+1}$$

(5.7)
where in the summation \(|m+n| \leq N\) implies \(|(m-n) \cdot \lambda| > \omega\). Substituting in (5.4), we get

\[i \dot{\zeta}_j - \lambda_j(\omega)\zeta_j = \sum_{1 \leq |m| \leq N} a_{j,m}(\omega)|\zeta^m|^2 \zeta_j - \sum_{n+\delta_j \in \text{Res}, |m+n| \geq 2} \zeta^m \zeta^{n+\bar{n}} \times (5.8)\]

\[\langle A_{0,n}^{(N)}(\omega)R_H((m-\bar{n}) \cdot \lambda(\omega_0) + i0)P_c(\omega)R_{m,\bar{n}}^{(N)}(\omega_0), \sigma_3 \xi_j \rangle + \langle E_{ODE}(N), \sigma_3 \xi_j \rangle.\]

Substituting in (5.5), where \(k = N\), and writing as in (5.6) we get

\[i \partial_t P_c(\omega_0)f_{N+1} = \left( H + (\dot{\theta} - \omega_0)(P_+(\omega_0) - P_-(\omega_0)) \right) P_c(\omega_0)f_{N+1} + \sum_{2 \leq |m+n| \leq N+1} O(|\zeta^{|m+n|+1}|)R_H((m-n) \cdot \lambda(\omega_0) + i0)R_{m,\bar{n}}^{(N)}(\omega)\]

\[+ \sum_{m+\delta_j \in \text{Res}} O(|\zeta^M(t)|_{L^2}) \approx \sum_{M \in \text{Res}} || \zeta^M(t) ||_{L^2}^2. \]

In the new variables

\[i \dot{\zeta}_j - \lambda_j(\omega)\zeta_j = \sum_{1 \leq |m| \leq N} \tilde{a}_{j,m}(\omega)|\zeta^m|^2 \zeta_j - \sum_{m+\delta_j \in \text{Res}} |\zeta^m|^2 \zeta_j \times (5.10)\]

\[\langle A_{0,n}^{(N)}(\omega)R_H(m \cdot \lambda(\omega_0) + \lambda_j(\omega_0) + i0)R_{m,\bar{n}}^{(N)}(\omega_0), \sigma_3 \xi_j(\omega) \rangle + \langle E_{ODE}(N), \sigma_3 \xi_j \rangle.\]

with \(\tilde{a}_{j,m}, A_{0,m}^{(N)}\) and \(R_{m,\bar{n}}^{(N)}\) real and with all the \(m\) such that \(m+\delta_j \in \text{Res}\).

We can denote by \(\Gamma_{m+\delta_j}(\omega, \omega_0)\) the quantity

\[\Gamma_{m+\delta_j}(\omega, \omega_0) = \Im \left( \langle A_{0,m}^{(N)}(\omega)R_H(m \cdot \lambda(\omega_0) + \lambda_j(\omega_0) + i0)R_{m+\delta_j,0}^{(N)}(\omega_0), \sigma_3 \xi_j(\omega) \rangle \right)\]

\[= \pi \langle A_{0,m}^{(N)}(\omega)\delta(H - m \cdot \lambda(\omega) - \lambda_j(\omega))P_c(\omega_0)R_{m+\delta,j}^{(N)}(\omega), \sigma_3 \xi_j(\omega) \rangle.\]
Then
\[
\frac{d}{dt} |\hat{\zeta}_j|^2 = - \sum_{m+\delta_j \in \text{Res}} \Gamma_{m+\delta,j}(\omega,\omega_0) |\hat{\zeta}_m \hat{\zeta}_j|^2 + + \Im \left( \sum_{m+\delta_j \in \text{Res}} \langle A_{0,m}^{(N)}(\omega) f_{N+1}, \sigma_3 \xi_j(\omega) \rangle \hat{\zeta}_m \hat{\zeta}_j + \langle E_{ODE}(N), \sigma_3 \xi_j(\omega) \rangle \hat{\zeta}_j \right).
\]
(5.12)

Notice that (5.12) contains more terms than (3.13) and that the signs of \(\Gamma_{m+\delta,j}\) now matter. Denote by \(\text{Res}_j\) the subset of \(\text{Res}\) which have at least 1 in the \(j\)th component. We assume the following hypothesis:

**Hypothesis 5.2.** For \(m \in \text{Res}\) let 
\[J(m) = \{j : m \in \text{Res}_j\}\]. There is a fixed \(C_0 > 0\) such that for \(|z| < \epsilon\)
\[
\sum_{m \in \text{Res}} |z_m|^2 \sum_{j \in J(m)} \Gamma_{m,j}(\omega,\omega) \geq C_0 \sum_{m \in \text{Res}} |z_m|^2.
\]

Assuming Hypothesis 5.2, we obtain Theorem 5.1 proceeding along the lines of the proof of Theorem 1.1.

**Remark 5.1.** It is possible that a formula of the following form might be true
\[
\sum_{j \in J(m)} \langle A_{0,m-\delta_j}^{(N)}(\omega) R_{H,\omega}(m \cdot \lambda(\omega) + i0) P_c(\omega) R_{m,0}^{(N)}(\omega_0), \sigma_3 \xi_j(\omega) \rangle = C_m \langle \delta(H_\omega - m \cdot \lambda(\omega)) R_{m,0}^{(N)}(\omega), \sigma_3 R_{m,0}^{(N)}(\omega) \rangle
\]
(5.13)

for some constant \(C_m > 0\). It is elementary to show (5.13) if we replace \(A_{0,m-\delta_j}^{(N)}\) with \(A_{0,\delta_j}\) and \(R_{m,0}^{(N)}\) with \(R_{\delta_j,0}\), from the Taylor expansion in (2.9). For \(N = 1\) this yields Theorem 1.2 substituting the Hypothesis 5.2 with a generic hypothesis similar to Hypothesis 3.5. Indeed if \(N = 1\) it is easy to see that \(A_{0,\delta_j}^{(N)} = A_{0,\delta_j}\) and \(R_{\delta_j,0}^{(N)} = R_{\delta_j,0}\). To get (5.13) in the general case, one should exploit the Hamiltonian nature of (NLS) which has been lost in our proof.

**APPENDIX A. APPENDIX**

**Proof of Lemma 3.3** Following the idea of [4, Proposition 4.1], we will transform (3.4) into (3.11) and (3.12) by induction. Let \(\omega_1 = \omega\) and let
\[
\omega_{k+1} = \omega_k + \sum_{m,n \geq 0} \langle f_N, \hat{\alpha}_{m,n}^{(k)}(\omega) \rangle z_m z_n.
\]
(A.1)
We will determine $\alpha^{(k)}_{m,n}(\omega) \in \mathcal{H}_a(\mathbb{R}^d, \mathbb{R}^2)$ so that

$$i\hat{\omega}_k = \sum_{2 \leq m+n \leq 2N+1} b^{(k)}_{m,n}(\omega) z^m \bar{z}^n + \sum_{k+1 \leq m+n \leq N} \langle f_N, \alpha^{(k)}_{m,n} \rangle z^m \bar{z}^n + O(|z|^{2N+2} + \|e^{-a|x|/2} f_{N+1}\|_{H^1}^2) \quad (A.2)$$

for $k = 1, \cdots N$. For $k = 1$, Eq. (A.2) follows from Lemma 3.2. Furthermore, we have $b^{(1)}_{m,n}(\omega) = -b^{(1)}_{n,m}(\omega)$, $\alpha^{(1)}_{m,n}(\omega) = \alpha_{n,m}(\omega)$ and $\sigma_1 \alpha^{(1)}_{m,n}(\omega) = -\alpha^{(1)}_{n,m}(\omega)$ because $\omega$ is a real number and

$$\overline{f_N} = \sigma_1 f_N. \quad (A.3)$$

Suppose that (A.2), that $\omega_k$ is a real number, and that

$$(A.4) \quad b^{(k)}_{m,n}(\omega) \text{ are real numbers with } b^{(k)}_{m,n}(\omega) = -b^{(k)}_{n,m}(\omega),$$

$$(A.5) \quad \alpha^{(k)}_{m,n}(\omega) \in \mathcal{H}_a(\mathbb{R}^d, \mathbb{R}^2), \quad \sigma_1 \alpha^{(k)}_{m,n}(\omega) = -\alpha^{(k)}_{n,m}(\omega)$$

are true for $k = l$ with $l \leq N$.

Differentiating (A.1) with respect to $t$ and substituting (3.3), (3.6) and (A.2) with $k = l$ into the resulting equation, we obtain

$$i\hat{\omega}_{l+1} = i\hat{\omega}_l + \sum_{m+n=l} \langle i \partial_t f_N, \alpha^{(l)}_{m,n}(\omega) \rangle z^m \bar{z}^n + \sum_{m+n=l} \left\{ i \langle f_N, \alpha^{(l)}_{m,n}(\omega) \rangle \frac{d}{dt} (z^m \bar{z}^n) + i \hat{\omega}_l \langle f_N, \partial_\omega \alpha^{(l)}_{m,n}(\omega) \rangle z^m \bar{z}^n \right\}$$

$$= \sum_{2 \leq m+n \leq 2N+1} b^{(l)}_{m,n} z^m \bar{z}^n + \sum_{m+n=l} \langle f_N, \alpha^{(l)}_{m,n} \rangle \left( (m-n)\lambda \alpha^{(l)}_{m,n} + H_\omega + (m-n)\lambda \rangle \alpha^{(l)}_{m,n} \right)$$

$$+ \sum_{m+n=l} \left\{ \sum_{p+l=1} \Phi_{p,q}^{(N)}(\omega) z^p \bar{z}^q + \mathcal{N}_N, \alpha^{(l)}_{m,n} \right\} z^m \bar{z}^n$$

$$+ \left( \hat{\gamma} \langle P_c(\omega) \sigma_1 f_N, \alpha^{(l)}_{m,n}(\omega) \rangle + i \hat{\omega}_l \langle f_N, \partial_\omega \alpha^{(l)}_{m,n}(\omega) \rangle \right) z^m \bar{z}^n$$

$$+ \sum_{m+n=l} \langle f_N, \alpha^{(l+1)}_{m,n}(\omega) \rangle \left\{ m \bar{z}^{m-1} (i \bar{z} - \lambda \bar{z}) - n \bar{z}^{m-1}(i \bar{z} - \lambda \bar{z}) \right\}$$

$$+ O(|z|^{2N+2} + \|e^{-a|x|/2} f_{N+1}\|_{H^1}^2).$$

Put $\alpha^{(l)}_{m,n}(\omega) = R_{H_\omega}((n-m)\lambda) \alpha^{(l)}_{m,n}(\omega)$. Then by Lemma 3.2, the definition of $\mathcal{N}_N$ and

$$|\hat{\omega}| + |\hat{\gamma}| + |i \bar{z} - \lambda \bar{z}| + \|e^{-a|x|/2} \mathcal{N}_N\|_{H^1} \lesssim |z|^2 + \|e^{-a|x|/2} f_{N+1}\|_{H^1}^2,$$

it holds that (A.2) with $k = l+1$ is true for some $b^{(l+1)}_{m,n}(\omega) \in \mathbb{R} \ (2 \leq m+n \leq 2N+1)$ and $\alpha^{(l+1)}_{m,n}(\omega) \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap \mathcal{L}^2_c(H_\omega^* \ (m+n = l+1). Note that $\mathcal{N}_N$
can be expanded into a formal power series of \( z, \bar{z} \) and \( f_N \) whose coefficients are real.

By the definition of \( \tilde{\alpha}_{m,n}^{(l)} \), (A.5) with \( k = l \) and the fact that \( \sigma_1 H_\omega \sigma_1 = -H_\omega \),
(A.6) \[ \sigma_1 \tilde{\alpha}_{m,n}^{(l)}(\omega) = \tilde{\alpha}_{m,n}^{(l)}(\omega). \]

From (A.3), (A.6) and (A.2) for \( k \) and that (A.4) and (A.5) are true for \( k = l + 1 \). Thus we prove
\[
i \dot{\omega}_{N+1} = \sum_{2 \leq m+n \leq 2N+1} b_{m,n}^{(N+1)}(\omega) z^m \bar{z}^n + O(\|z\|^{2N+2} + \|e^{-a|x|/2 f_N}\|_H^1),
\]
where \( b_{m,n}^{(N+1)}(\omega) \) are real numbers satisfying \( b_{m,n}^{(N+1)}(\omega) = -b_{m,n}^{(N+1)}(\omega) \). In particular, we have \( b_{n,n}^{(N+1)} = 0 \) for \( n = 1, \cdots, N \).

Using
\[
\frac{d}{dt}(z^m \bar{z}^n) = z^m \bar{z}^n \left\{-i\lambda(m-n) + O(\|z\|^2 + \|e^{-a|x|/2 f_N}\|_H^2)\right\},
\]
we can find a real polynomial \( \tilde{p}(x, y) \) of degree \( 2N+1 \) such that
\[
\dot{\tilde{\omega}} = \omega_{N+1} + \tilde{p}(z, \bar{z}),
\]
\[
\dot{\omega} = O(\|z\|^{2N+2} + \|e^{-a|x|/2 f_N}\|_H^1).
\]
Thus we complete the proof. \( \square \)

Proof of Lemma 3.4. Let \( z_1 = z \) and
\[ z_{k+1} = z_k + \sum_{m+n=k \atop n \neq N} \langle f_N, \tilde{z}_{m,n}^{(k)}(\omega) \rangle z^m \bar{z}^n \quad \text{for} \quad k = 1, \cdots, N. \]

For \( k = 1, \cdots, N + 1 \), we will choose \( \tilde{z}_{m,n}^{(k)} \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap L^2_c(H_\omega^*) \) such that
\[ i \dot{z}_k - \lambda z_k = r_k(z_k, \bar{z}_k) + \langle f_N, \gamma^{(k)}(z) \rangle + O(\|z_k\|^{2N+2} + \|e^{-a|x|/2 f_N}\|_H^2), \]
where \( r_k \) is a real polynomials of degree \( 2N+1 \) with \( r_k(x, y) = O(x^2 + y^2) \) as \( (x, y) \to (0, 0) \),
\[
\gamma^{(k)}(z) = \begin{cases} 
\sum_{m+n \leq N} \gamma_{m,n}^{(k)} z^m \bar{z}^n & \text{for} \quad k = 1, \cdots, N, \\
\gamma_{0,N}^{(N)} \bar{z}^N & \text{for} \quad k = N + 1,
\end{cases}
\]
and \( \gamma_{m,n}(\omega) \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \cap L^2_a(H^\omega). \) This is true for \( k = 1. \) Assume (A.9) for \( k = l \leq N \) and substitute (A.8) into (A.9). Then

(A.10)

\[
i\dot{z}_{l+1} - \lambda z_{l+1} = i\dot{z}_l - \lambda z_l + \sum_{m+n=l, n \neq N} \langle (H_\omega - \lambda(m - n - 1)) f_N, \tilde{\gamma}_{m,n}^{(l)} \rangle z^m \bar{z}^n + \sum_{m+n=l, n \neq N} \left\{ \left( i P_c(\omega) \partial_t f_{N+1} - H_\omega f_N, \tilde{\gamma}_{m,n}^{(l)} \right) + i \dot{\omega} \langle f_N, \partial_\omega \tilde{\gamma}_{m,n}^{(l)} \rangle \right\} z^m \bar{z}^n + \sum_{m+n=l, n \neq N} \left( m z^{m-1} \bar{z}^n (i \dot{z} - \lambda z) - n z^m \bar{z}^{n-1} (i \dot{z} - \lambda z) \right) \langle f_N, \gamma_{m,n}^{(l)} \rangle z^m \bar{z}^n.
\]

Substituting (3.4) into (A.10) and letting

\[
\dot{\tilde{\gamma}}_{m,n}^{(l)}(\omega) = RH_\omega((m - n - 1)\lambda) \gamma_{m,n}^{(l)}(\omega),
\]

we see that (A.9) is true for \( k = l + 1. \) Thus we complete the induction. By (3.9), (A.9) with \( k = N + 1 \) and the fact that

\[
|z_{N+1} - z| = O(z_{N+1}^2),
\]

\[
\|f_N - \tilde{f}_N\|_{H^1} \lesssim |\omega - \omega_0| (\|e^{-|x|} f_{N+1}\|_{H^1} + |z|^{N+1}),
\]

we have

(A.11)

\[
i\dot{z}_{N+1} - \lambda z_{N+1} = r_{N+1}(z_{N+1}, \bar{z}_{N+1}) + \sum_{m+n=N+1} \langle \Psi_{m,n}^{(N+1)}(\omega_0), \gamma_{0,N}^{(N)}(\omega) \rangle z^m \bar{z}^{n+N+1} + z_{N+1}^{N+1} \langle f_{N+1}, \gamma_{0,N}^{(N)}(\omega) \rangle + O\left( \|z\|^{2N+2} + \|e^{-|x|} f_{N+1}\|_{H^1}^2 \right) + O\left( |\omega - \omega_0| (\|z\|^N \|e^{-|x|} f_{N+1}\|_{H^1} + |z|^{2N+1}) \right).
\]

The standard theory of normal forms (see [1]) tells us that by introducing a new variable

\[
\tilde{z} = z_{N+1} + \sum_{2 \leq m+n \leq 2N+1, m,n \geq 0, m-n \neq 1} \tilde{c}_{m,n}(\omega) z^m \bar{z}^n,
\]

we can transform (A.11) into (3.12). Since \( r_{N+1} \) is a real polynomial and \( \Psi_{m,n}^{(N+1)}(\omega) \in \mathcal{H}_a(\mathbb{R}^d; \mathbb{R}^2) \) for \( m, n \in \mathbb{N} \) with \( m + n = N + 1, \) it follows that
\( \tilde{c}_{m,n}(\omega) \in \mathbb{R} \) for \( n \leq 2N \) and \( a_n(\omega, \omega_0) \in \mathbb{R} \) for \( 1 \leq n \leq N-1 \) and by (3.10) with
\[
\Im a_N(\omega, \omega_0) = \Im \langle R_{H_{\omega_0}} ((N+1)\lambda + i0) \Phi_{N+1,0}^{(N)}(\omega_0), \gamma_{0,N}^{(N)}(\omega) \rangle.
\]

\begin{proof}
\end{proof}

**Remark A.1.** By \( \frac{1}{x-i0} = PV \frac{1}{x} + i \pi \delta_0(x) \), by [8] and by the fact that \( \Phi_{N+1,0}^{(N)}(\omega_0) \) and \( \gamma_{0,N}^{(N)}(\omega) \) have real entries, we have
\[
\Im \langle R_{H_{\omega_0}} ((N+1)\lambda(\omega_0) + i0) \Phi_{N+1,0}^{(N)}(\omega_0), \gamma_{0,N}^{(N)}(\omega) \rangle
= \pi \langle \delta_0 \left( H_{\omega_0} - (N+1)\lambda(\omega_0) \right) \Phi_{N+1,0}^{(N)}(\omega_0), \gamma_{0,N}^{(N)}(\omega) \rangle
\]

If Hypothesis 3.5 fails because
\[
\delta \left( H_{\omega} - (N+1)\lambda(\omega) \right) \Phi_{N+1,0}^{(N)}(\omega) = 0
\]
identically in \( \omega \), then by [12] the vector \( \Psi_{N+1,0}^{(N)}(\omega) \) is real and rapidly decreasing to 0 as \( |x| \to \infty \). This suggests that we can continue the normal form expansion one more step.

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