Viscous Decretion Discs around Rapidly Rotating Stars

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Abstract
We discuss steady viscous Keplerian decretion discs around rapidly rotating stars. We assume that low-frequency modes, which may be excited by the opacity bump mechanism, convective motion in the core, or tidal force if the star is in a binary system, can transport a sufficient amount of angular momentum to the region close to the stellar surface. Under this assumption, we construct a star-disc system, in which there forms a viscous decretion disc around a rapidly rotating star because of the angular-momentum supply. We find a series of solutions of steady viscous decretion discs around a rapidly rotating star that extend to \( R_{\text{disc}} \gg R_* \), with \( R_* \) being the equatorial radius of the star, depending on the amount of angular momentum supply.

Key words: stars: oscillation — stars: rotation — stars: winds, outflows

1. Introduction

Discs around Be stars are now believed to be viscous Keplerian decretion discs (e.g., Porter & Rivinius 2003; Lee et al. 1991), although their formation mechanism has not yet been identified. There have appeared, however, several promising scenarios concerning the formation mechanism. For example, stellar evolution calculations of rapidly rotating main-sequence stars (e.g., Ekström et al. 2008; Granada et al. 2013) have suggested that the rotation velocity of the surface layers can reach the critical velocity for mass shedding from the equatorial regions as a result of angular-momentum transfer from the inner region to the surface, where the transport of angular momentum in the evolution calculation is implemented as in Meynet and Maeder (2005), employing the theory of the transport mechanisms in rotating stars developed by Zahn (1992) and Maeder and Zahn (1998). Cranmer (2009), on the other hand, has proposed a mechanism of angular-momentum transfer by waves propagating and damping in the evanescent atmosphere, assuming that they are driven by oscillation modes below the photosphere. Cranmer (2009) has shown that as a result of angular-momentum deposition by waves, the rotation velocity in the atmospheric layers is accelerated to the Keplerian velocity, leading to the formation of a decretion disc around the rotating star.

Angular-momentum transfer by non-axisymmetric oscillations in rotating stars has been discussed by various authors. The circularization of a binary orbit and the synchronization between the spin of a massive star and the orbital motion of the companion in a binary system are believed to result from angular-momentum exchange between the orbital motion and the spin of the star, where the angular-momentum redistribution in the star takes place through dissipative low-frequency \( g \)-modes tidally excited by the orbital motion of the companion (e.g., Zahn 1975, 1977; Goldreich & Nicholson 1989; Papaloizou & Savonije 1997; Witte & Savonije 2001; Willems et al. 2003). The problem of angular-momentum transport in the Sun by low-frequency \( g \)-modes has long been a hot topic in the field since the early years of helio-seismology, and has been investigated by many researchers, including Press (1981), Schatzman (1993), Gough (1997), Kumar, Talon, and Zhan (1999), Talon, Kumar, and Zahn (2002), Mathis et al. (2008), where low-frequency modes are assumed to be excited by convection motion in the convective envelope, and suffer radiative damping as they propagate into the radiative core. For Be stars, it has also been suggested that angular-momentum transferred by non-axisymmetric oscillations could play a role in the formation of viscous discs around the stars (e.g., Ando 1983, 1986; Lee & Saio 1993).

As a model of decretion discs around Be stars, Okazaki (2001) has calculated steady and transonic viscous flows around the stars, while taking account of the effects of the radiation pressure on the radial flow (see also a more recent discussion by Krtička et al. 2011). He has shown that the velocity field in the disc is very close to the Keplerian near the star, and tends to angular momentum conserving in the region far from the star. In his numerical analysis, the disc was treated as being mechanically decoupled from the central star, and no angular momentum source to support the disc was specified. These two points are what we are concerned with in this paper.

We construct a disc-star model, consisting of a rapidly rotating star and a viscous decretion disc around it. To support a decretion disc around a rotating star, a sufficient amount of angular momentum must be supplied to the surface layers of the star, and this angular-momentum supply is assumed to result from angular-momentum deposition through non-axisymmetric oscillation modes. The rotation velocity in the surface layers is accelerated while acquiring angular momentum transferred by non-axisymmetric low frequency modes, and a viscous Keplerian decretion disc forms around the star with excessive angular momentum. Since the \( \kappa \)-mechanism associated with the iron opacity bump excites low frequency and high radial order \( g \)-modes as well as \( r \)-modes in slowly pulsating B (SPB) stars and low radial order \( f \) - and \( p \)-modes in \( \beta \) Cephei stars (Dziembowski et al. 1993; Gautschy & Saio 1993), we may use the low-frequency
modes as an agent that causes angular-momentum transfer in SPBe stars. For early Be stars, for which the opacity bump mechanism does not work for driving low-frequency modes, we could use low-frequency modes stochastically excited by convective motion in the core (Neiner et al. 2012). If the Be star is in a binary system, low-frequency modes tidally excited by the orbital motion of the companion star could work for disc formation.

We employ a theory of wave-meanflow interaction to derive a meanflow equation for rotation, which describes the angular-momentum transfer by waves (e.g., Andrews & McIntyre 1987a, 1987b; Dunkerton 1980; Grimshaw 1984). We regard the forcing term in the meanflow equation to be a source term for angular-momentum. In the angular-momentum conservation equation for a disc-star system, therefore, we include both the forcing term in the wave-meanflow equation and the viscous torque term, which is essential for angular-momentum transfer in the disc. In section 2, we derive a set of ordinary differential equations, which we solve for a steady disc-star system. Section 3 is devoted to numerical results obtained for steady disc-star systems, and we conclude in section 4. In the Appendix, we derive the meanflow equation that we use in this paper.

2. Equations for Viscous Decretion Discs around Rotating Stars

To discuss angular-momentum transfer by non-axisymmetric oscillations in a rotating star, we use a theory of wave-meanflow interaction, in which fluid motions are separated into waves and a meanflow and dissipative processes of the waves propagating in the mean flow have an essential role for the forcing on the meanflow (e.g., Andrews & McIntyre 1978a, 1978b; Dunkerton 1980; Grimshaw 1984; Ando 1983; Goldreich & Nicholson 1989). For rotating stars, we regard the rotation velocity field as the meanflow, and global oscillations as waves. In the Cowling approximation (Cowling 1941), in which the Euler perturbation of the gravitational potential is neglected, the meanflow equation may be given by (see the Appendix)

\[ \frac{d}{dt} \frac{\rho}{\tilde{\xi}} = \frac{m}{2} \text{Im} \left[ \nabla \cdot (\tilde{\xi}^* p') \right], \]  

(1)

where \( \tilde{\xi}^* \) is the complex conjugate of the displacement vector, \( \tilde{\xi} \) of an oscillation mode; \( p' \) is the Eulerian pressure perturbation, \( m \) is the azimuthal wave number, \( \rho \) is the mass density in the equilibrium state used for linear treatment of stellar pulsations, and \( \tilde{\ell} \) is the specific angular momentum of rotation. In the wave-meanflow interaction theory, the Eulerian coordinates, \( \tilde{x}_0 \), are separated into two parts, such that \( \tilde{x}_0 = x_0 + \xi_0(t, x) \), where \( x_0 \) are the Lagrangian mean coordinates in the sense that \( \langle \xi_0 \rangle = 0 \) with \( \xi_0 \) being the displacement vector component associated with the wave, and \( \langle f \rangle \equiv \int_0^{2\pi} f e^{i\phi} d\phi \) is the zonal average of a physical quantity, \( f \), with \( \phi \) being the azimuthal angle around the rotation axis (see the Appendix). The right-hand-side of equation (1) represents forcing effects caused by the oscillation mode; an example of the forcing term calculated for low-frequency modes in a slowly pulsating B star is given in the Appendix.

To treat both a rotating star and a geometrically thin viscous disc around it as one system, we work in cylindrical coordinates \((R, \varphi, z)\), where we have omitted the hat, \( ^\hat{\cdot} \), from the Eulerian coordinates, \( \tilde{x}_0 \), for simplicity. The displacement vector of an oscillation mode may be given in cylindrical coordinates by \( \tilde{\xi} = \xi_R e_R + \xi_\varphi e_\varphi + \xi_z e_z \), where \( e_R, e_\varphi, \) and \( e_z \) are orthonormal vectors in the \( R, \varphi, \) and \( z \) directions, respectively. If we regard equation (1) as an angular-momentum conservation equation, its right-hand-side can be considered to be a source term for angular momentum attributable to the oscillations. Including both the viscous torque term for decretion discs and the angular momentum source term due to the oscillations, we may write the angular-momentum conservation equation as

\[ \frac{\rho}{\tilde{\xi}} = \frac{1}{R} \frac{\partial}{\partial R} \left( R^2 \sigma_{R\varphi} \right) + \frac{m}{2} \text{Im} \left[ \frac{1}{R} \frac{\partial}{\partial R} \left( R \xi_R^* p' + \frac{\partial (\xi_z^* p')}{\partial z} \right) \right], \]  

(2)

where \( \frac{d}{dt} \) denotes the substantial time derivative; \( \rho \) is the mass density, \( v_\varphi \) is the azimuthal velocity, \( \sigma_{R\varphi} \) is the \( R \varphi \) component of the viscous tensor, and is assumed to dominate other components of the tensor for a geometrically thin disc. We have assumed the system is axisymmetric.

We also assume that the oscillation amplitudes are saturated, for example, by non-linear couplings between the oscillation modes (e.g., Lee 2012), so that the source term in equation (2) to be time-independent. Assuming further that the disc-star system is in a steady state, and integrating vertically equation (2), we obtain

\[ \frac{\partial}{\partial R} \left( R^2 \Omega^2 \right) = \frac{\partial}{\partial R} \left( 2\pi R^3 \Omega^2 \int_{z_0}^{z} \eta dz \right) + \frac{\pi}{R} \frac{\partial}{\partial R} \int_{z_0}^{z} dz \text{Im} (\xi_R^* p'), \]  

(3)

where we have used \( \sigma_{R\varphi} = \eta R (\partial \Omega / \partial R) \) with \( \eta \) being the shear viscosity coefficient, \( v_\varphi = R \Omega \), and we have assumed \( p' = 0 \) at the surface given by \( z = z_0(R) \) (see below). Here, \( M \equiv 2\pi R \Omega \Sigma_{vR} \) is the mass decretion rate, which is a constant for steady flows, and \( \Sigma = \int_{z_0}^{z} \rho dz \) is the surface density. Integrating equation (3) with respect to the coordinate \( R \), we obtain

\[ q(R) \equiv 2\pi R^3 \int_{z_0}^{z} \eta dz = M j(R) - f(R) - J, \]  

(4)

where

\[ j(R) = R^2 \Omega, \]  

(5)

\[ f(R) = m \pi R \int_{z_0}^{z} dz \text{Im} (\xi_R^* p'), \]  

(6)

and

\[ J \equiv M j_0 - q_0 - f_0 = M j - q - f. \]  

(7)

Here, \( j_0, q_0, \) and \( f_0 \) denote the quantities evaluated at some arbitrary point \( R_0 \), which could be, for example, \( R_{\text{ts}} \) (see below for the definition of \( R_{\text{ts}} \)).
Following Paczyński (1991), we assume the disc-star system is in hydrostatic balance:

\[
\begin{align*}
\frac{1}{\rho} \frac{\partial p}{\partial R} + \frac{\partial \Phi}{\partial R} &= R \Omega^2, \\
\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial \Phi}{\partial z} &= 0,
\end{align*}
\]

where the gravitational potential is given by \( \Phi = -GM_*/\sqrt{R^2 + z^2} \) with \( M_* \) being the mass of the star. To determine the surface shape of the disc-star system, we consider two neighboring points \((R, z_0)\) and \((R + \delta R, z_0 + \delta z_0)\) on the surface such that \( p(R, z_0) = 0 = p(R + \delta R, z_0 + \delta z_0) \); we then obtain, using equations (8) and (9) for hydrostatic balance,

\[
\frac{dz_0}{dR} = \frac{R}{z_0} \left( \frac{r_0^2 \Omega^2}{GM_*} - 1 \right),
\]

where \( r_0 = \sqrt{R^2 + z_0^2} \).

Equations (4) and (10) are two ordinary differential equations that we solve with appropriate boundary conditions for disc-star systems. To treat a disc-star system, we divide the system into two parts, that is, the inner part \((R \leq R_{tr})\) and the outer part \((R \geq R_{tr})\). The inner part is assumed to be uniformly rotating at a constant rate, \( \Omega_e \), and we have no need to integrate equation (4) for the inner part. The surface shape of the inner part is obtained by integrating equation (10) for the constant rate \( \Omega = \Omega_e \). On the other hand, the outer part of the system is composed of the outer part of the rotating star and a decretion disc, and is allowed to rotate differentially. We assume that the outer disc part extends to the radius \( R_{out} \gg R_{tr} \). We therefore have to integrate both equations (4) and (10) to determine the rotation rate, \( \Omega(R) \), and the shape, \( z_0(R) \).

To determine the inner part of the system, let us consider a star uniformly rotating at a rate \( \Omega_e \) with no decretion discs around it. Rewriting equation (10) as \( dr_0^2/dR^2 = r_0^2 \Omega_e^2/GM_* \), we may integrate this equation, since \( \Omega_e^2/GM_* \) is a constant for uniform rotation, to obtain

\[
z_0^*(R) = \left[ \frac{1}{R_p - \frac{\Omega_e^2 R^2}{2GM_*}} - R^2 \right]^{1/2},
\]

where \( z_0^*(R) \) defines the surface of the uniformly rotating star (without decretion discs), and \( R_p = z_0^*(0) \) is its polar radius. Using the condition \( z_0^*(R_e) = 0 \), we may define, as a function of \( R_p \) and \( \Omega_e \), the equatorial radius, \( R_e \), of a star that is uniformly rotating at a rate \( \Omega_e \). With the condition \( z_0^*(R_{tr}) = 0 \), we can also define the critical equatorial radius, \( R_{tr} \), of a star uniformly rotating at the critical angular velocity \( \Omega_{tr} = (GM_*/R_e^3)^{1/2} \). For the critical radius, \( R_{tr} \), and the rotation rate, \( \Omega_{tr} \), we have \( R_{tr}/R_e = 1.5 \). Using this critical radius, we rewrite (11) as

\[
y_0(x) = \left[ \frac{1}{y_p - \frac{\Omega_e^2 x^2}{2}} - x^2 \right]^{1/2},
\]

where \( x = R/R_{tr} \), \( y_0(x) = z_0^*(R)/R_{tr} \), \( y_p = R_p/R_{tr} = 2/3 \), and \( \Omega_e = \Omega_e/\Omega_{tr} \). Note that \( y_0(x) = 0 \) gives the critical radius \( x = 1 \) at \( \Omega_e = 1 \) for a uniformly rotating star. The rotation rate, \( \Omega_e \), may also be normalized by using \( \Omega_e \equiv (GM_*/R_e^3)^{1/2} \), which is a critical angular velocity for the actual stellar equatorial radius, \( R_e \), and in this case we have \( \Omega_e/\Omega_{tr} = x_e^{3/2} \). With \( x_e = R_e/R_{tr} \), we therefore have to determine both equations (4) and (10) to determine the rotation rate, \( \Omega_e(R) \), and the shape, \( z_0(R) \).

To determine the outer part of the system by integrating equations (4) and (10), we have to give a prescription for the viscous angular-momentum transport. For a thin disc in \( R \geq R_{tr} \), employing the so-called \( \alpha \)-regulation, we may give the shear viscosity coefficient, \( \eta \), as (Shakura & Sunyaev 1973; see also Frank et al. 2002)

\[
\eta = \alpha \rho_0 v_{tr} z_0,
\]

where \( \alpha \) is a dimensionless constant parameter, such that \( 0 < \alpha < 1 \), and \( \rho_0 \) and \( v_{tr} = (\partial p_0/\partial \rho_0)^{1/2} \) are the density and the sound speed evaluated at the equatorial plane. We note that there exist other ways of prescribing the \( \alpha \)-viscosity; for example, the \( R \varphi \) component of the viscous tensor is given by \( \sigma_{R \varphi} = -\alpha \int_{-z_0}^{z_0} p dz \). This prescription was employed to calculate transonic viscous decretion flows, for example, by Okazaki (2001). Although this prescription has an advantage that the rank of differential equations can be reduced by one, we use the prescription (13) to calculate expected steep changes in \( \Omega_e \) in the boundary layers between the star and disc. Assuming a polytropic relation, \( p = K \rho^{1+1/n} \), and \( z_0 \ll R \), we integrate equation (9) to obtain (Paczyński 1991).
\[ \rho = \rho_0 \left( 1 - \frac{z^2}{z_0^2} \right)^n \quad \text{with} \quad \rho_0 = \left[ \frac{GM_*}{2(n+1)KR^3} \right]^n z_0^{2n}, \]  

and hence

\[ v_{r,0} = \left( \frac{GM_*}{2n} \right)^{1/2} \frac{z_0}{R^{1/2}} \]  

and

\[ \eta = \alpha c_0 \left( \frac{GM_*}{K^n} \right)^{n+0.5} \frac{z_0^{2n+2}}{R^{3n+1.5}}, \]  

where \( c_0 = (2n)^{-1/2}(2n+2)^{-m}. \) Equation (4) is now given by

\[ 4\pi c_0 \left( \frac{GM_*}{K^n} \right)^{n+0.5} \frac{z_0^{2n+2}}{R^{3n+1.5}} \frac{d\Omega}{dR} = M R^2 \Omega - f - J. \]  

Equations (10) and (17) now make a set of ordinary differential equations which we have to solve with boundary conditions imposed at \( R = R_t \) and \( R = R_{\text{out}}. \)

In this paper, we assume \( n = 1.5 \) for the polytropic index (Paczynski 1991). Using non-dimensional variables, we rewrite equations (10) and (17) as

\[ \frac{dy}{dx} = \frac{x}{y} \left[ \Omega^2 (x^2 + y^2)^{3/2} - 1 \right], \]  

\[ \frac{d\Omega}{dx} = \frac{1}{a} \frac{x^3}{y^6} \left[ x^2 \Omega - f \hat{x} (x) - j \right], \]  

where

\[ y = \frac{z_0(R)}{R_t}, \quad \Omega = \frac{\Omega_{t}}{\Omega_{c}}, \quad a = 4\pi c_0 \frac{G^2 M_*^2}{MK^{1.5}}, \]  

\[ b = \frac{3}{4} \frac{M_* \Omega_{t}}{R_t}, \quad j = \frac{j}{M \sqrt{GM_* R_t}}, \]  

and

\[ \hat{f} (x) = x \int_{z_0}^{z} \frac{dz}{R_t} \frac{\rho g r}{\rho_* GM_* / R_t} \sum_k m \Im \left( \frac{\xi_k^*}{r} \frac{p'}{\rho g r} \right). \]  

Here, \( \bar{\rho}_* = M_* / (4\pi R_*^3 / 3), \) \( g = GM_* / r^2 \) with \( M_* \) being the mass within the sphere of radius \( r = \sqrt{R^2 + z^2}, \) and the summation in equation (21) is over the oscillation modes that contribute to the forcing on the meanflow, where \( k \) denotes a collective mode index. For the polytropic index, \( n = 1.5, \) we use \( K = 0.4242 GM_*^{1/3} R_* \) with \( R_* \) being the radius of the star (Chandrasekhar 1939) to obtain

\[ a \approx 4\pi c_0 \left( \frac{M_*}{M} \right) \left( \frac{GM_*}{R_t^2} \right)^{1/2}, \]  

and, from equation (14),

\[ \rho_0 \sim 1.5 \bar{\rho}_* y^3 / x^{9/2}, \]  

where \( c_0' = c_0 / (0.4242)^{1.5} \simeq 0.2 \) for \( n = 1.5, \) where we have replaced \( R_* \) by \( R_t, \) which may be in between \( R_* \) and \( 1.5 R_* \) for rapidly rotating stars. Although the constant \( a \) depends on the viscosity parameter, \( \alpha, \) the magnitude of \( a / c_0' \) is almost the same as that of the constant \( b, \) that is, \( a / c_0' \sim b. \) As suggested by Paczyński (1991), the constant \( a \) can be as large as \( a \sim 10^{12} \alpha, \) depending on quantities such as \( M, M_* \), and \( R_* \) (or \( R_t \)). As \( a \) increases, however, it becomes difficult to numerically find solutions to the set of differential equations. In this paper, we employ \( a = 10^5, \) which leads to the disc thickness, \( z_0 / R \sim 0.1 \) (see below).

For rapidly rotating SPB stars, for example, numerous \( r \)-modes and prograde sectoral \( g \)-modes are destabilized by the opacity bump mechanism (e.g., Aprilia et al. 2012). To determine the forcing function, \( b(\hat{f}(x)), \) for SPB stars, we need to know their amplitudes and to sum up all of the accelerating and decelerating contributions to the forcing. With a linear theory of oscillations, however, we have no means to determine the amplitudes, and hence the forcing function, \( b \hat{f}(x). \)

In this paper, therefore, we just assume a simple form for the function \( b \hat{f}(x). \) Assuming that the contributions to the acceleration of the surface layers are dominant over those to deceleration, we employ for the forcing function, \( \hat{f}(x), \) a form given by

\[ b \hat{f}(x) = b_0 \left( \frac{1}{\exp[-b_1(x - x_0 \epsilon)]} + 1 \right) \equiv b_0 \bar{f}(x), \]  

where \( b_0, b_1, \) and \( x_0 \) are parameters, and \( x_0 = R_c / R_t. \) The parameter \( b_0 \) corresponds to the square of the oscillation amplitudes; we have chosen the functional form for \( \hat{f}(x) \), so that \( d \hat{f}(x) / dx \) roughly reproduces the \( x \)-dependence of \( 1 / t^{AM} \) shown in figure 6. In the surface region of the star where \( x \simeq x_0 \leq 1, \) we may approximate

\[ | \hat{f}(x_0) = x_0 \epsilon_0 | \sim x_0^2 \gamma_0 (x_0) \left( \rho_0 / \bar{\rho}_* \right) \left| \sum_k m \Im \left( \frac{\xi_k^*}{r} \frac{p'}{\rho g r} \right) \right| \]  

\[ \sim x_0^{-2.5} \gamma_0^4 (x_0) \left| \sum_k m f_k^2 \sin \delta_k \right| \]  

where \( f_k \sim | \xi_k / R_t | \) is the normalized oscillation amplitude and \( \delta_k \) is the phase difference between \( \xi_k \) and \( p' \) near the surface; we have approximated \( (GR_* / R_t) \sim 1. \) Since \( | \hat{f}(x_0)| = 0.5, \) we may have \( b_0 \sim b \times x_0^{-2.5} \gamma_0^4 (x_0) \left| \sum_k m f_k^2 \sin \delta_k \right|. \) Although it is difficult to correctly estimate the magnitude of the quantities, such as \( | \sum_k m f_k^2 \sin \delta_k |, \) assuming \( | \sum_k m \sin \delta_k | = 1 \) and \( x_0^{-2.5} \gamma_0^4 (x_0) \sim 10^{-3}, \) we have \( b_0 \sim 1 \) for \( f_k \sim 10^{-2} \) and \( b \sim a / c_0' \sim 10^8 \) for \( a = 10^5. \) Note that for \( x \gg 1, \) we should have \( b \hat{f}(x) = 0, \) expecting that there occurs no forcing on the meanflow in the disc.

The set of differential equations (18) and (19) for the outer part are integrated with three boundary conditions, one given by \( d\Omega / dx = d\Omega_K / dx \) at \( x_{\text{out}} = R_t / R_{\text{out}} \) and other two conditions given by \( \Omega(x_\text{tr}) = \Omega_t \) and \( y(x_\text{tr}) = y_0(x_\text{tr}) \) at \( x_{\text{tr}} \equiv R_t / R_{\text{tr}}, \) where \( \Omega_K = x^{-3/2} \) is the Keplerian angular velocity. The first two boundary conditions are used to integrate the set of coupled two first-order, ordinary differential equations, (18) and (19), for a given \( j, \) which may be regarded as an eigenvalue of the system of differential equations, and is determined by using the third condition. The third condition, \( y(x_\text{tr}) = y_0(x_\text{tr}), \) ensures the physical continuity of the
inner part and the outer part of the system at \( x_{\text{tr}} \). In this paper, we use \( x_{\text{tr}} = 0.8 \).

As indicated by equation (19), \( d\Omega/dx \) changes sign at \( x_j \), at which point \( x_j^2\Omega(x_j) = 0 \), and if we assume \( b f(x) = 0 \) for \( x > 1 \), we obtain \( x_j \approx x_j^2 \) if we substitute \( \Omega_K \) for \( \Omega \). To understand a rough property of the solution in the region \( 1 < x < x_j \), in which \( x^2\Omega < j \), we assume functional forms given by \( y = c_1 x^s \) and \( \Omega = c_2 \sqrt{x} \Omega_K \), where the parameters \( c_1, c_2, \) and \( s \) are assumed to be only weakly dependent on \( x \). Substituting the forms into equations (18) and (19), we obtain \( s = \frac{11}{12} \) and \( c_1 \sim \left( \frac{j}{ac_2} \right)^{1/6} \), \( c_2 \sim \frac{1}{\left(1 + c_1^2 x^{2-2}\right)^{1/2}} \left(1 + c_1^2 x^{2-2}\right)^{3/4} \sim \frac{1}{\left(1 + c_1^2 x^{2-6}\right)^{1/4}}, \) \( x \approx x_j \).

Here, we approximated \( (2/3)^{1/6} \approx 1 \) for the second equation, and we set the factor \( s \), on the left hand of \( c_1^2 \) in the numerator of the third equation, to be equal to 1. These relations are consistently satisfied if \( c_1 \approx 0.1 \) and \( c_2 \approx 1 \) for \( j \approx 10 \) and \( a \approx 10^2 \), which indicates that the disc flows may undergo angular momentum conservation, so that \( x^2\Omega \approx j \) must be a constant (see Okazaki 2001). If this is the case, the outer boundary condition must be modified, and an appropriate treatment of solutions around the point \( x_j \) will be required.

3. Numerical Results

Let us give a brief description of the procedure that we employ to obtain solutions to the set of differential equations (18) and (19) for a given \( \Omega_s \leq 1 \). Since it is difficult to solve the differential equations for the entire region from \( x_{\text{tr}} \) to \( x_{\text{out}} \) by using a Runge-Kutta method (or a relaxation method), we divide the interval \( (x_{\text{tr}}, x_{\text{out}}) \) into two intervals, that is, \( (x_{\text{tr}}, x_m) \) and \( (x_m, x_{\text{out}}) \) with \( x_m \approx 1 \), and for integration we use an implicit Runge-Kutta method for the former and a Henyey type relaxation method for the latter. Here, for \( x_m \) we choose a point \( x_m \approx 1 \) that satisfies \( d\Omega/dx = 0 \). For a given value of the parameter \( j \), we integrate the differential equations (18) and (19) from \( x = x_{\text{tr}} \), with starting values of \( y(x_{\text{tr}}) = y_0(x_{\text{tr}}) \) and \( \Omega(x_{\text{tr}}) = \Omega_s \) to the point \( x_m \). This integration gives \( y_m, y_m \), and \( \Omega_m = \Omega(x_m) \) as a function of \( j \), or equivalently, \( y_m, \Omega_m, \) and \( j = \Omega(z) \) as a function of \( x_m \). For the interval \( (x_m, x_{\text{out}}) \), we then solve equations (18) and (19) using the relaxation method with the initial guesses given by \( y = x \left( \frac{j x^{-1/2} - 1}{1/2} \right) a^1/2 \) and \( \Omega = x^{-3/2} \) to find the value of \( x_m \), such that the boundary conditions \( y(x_m) = y_m \) and \( \Omega(x_m) = \Omega_m \) at \( x = x_m \) and \( d\Omega/dx = d\Omega_K/dx \) at \( x = x_{\text{out}} \) are satisfied. This procedure gives us a complete solution, \( y(x) \) and \( \Omega(x) \), for the region from \( x_{\text{tr}} \) to \( x_{\text{out}} \), which corresponds to the outer part of a disc-star system. The inner part of the system is the part of a star uniformly rotating at the rate of \( \Omega_s \), and its surface shape, \( y_0(x) \), is given by equation (12). The inner part and the outer part of the system are connected at \( x_{\text{tr}} \), and continuous connection is ensured by the boundary conditions given by \( y(x_{\text{tr}}) = y_0(x_{\text{tr}}) \) and \( \Omega(x_{\text{tr}}) = \Omega_s \).

Figure 2 shows \( y \) and \( \Omega \) as a function of \( x \) for \( \Omega_s = 0.98, 0.99, \) and 1.00, where we have assumed \( b_0 = b = 50, x_{\text{out}} = 10, \) and \( x_0 = 0.95 \). In table 1, we tabulate several characteristic quantities, such as \( x_m, j \), and \( -b_0 f(x_m) \), as a function of \( \Omega_s \). As shown by the left panel of the figure, there appears a sharp dip in \( y \) at the boundary between the star and disc, and this dip becomes deeper for smaller values of \( \Omega_s \). Table 1 indicates that this star-disc boundary is located at a radius near \( x_s \). If we go outwards from \( x_s \), \( \Omega \) starts at a point near \( x_s \) to steeply increase to attain a super-Keplerian rate (\( \Omega > \Omega_K \)) at \( x_m \), and then decreases to the Keplerian velocity. Note that \( \Omega \) is slightly sub-Keplerian in the region of \( x > 1 \). For a given \( b_0 \), there exists the lower limit of \( \Omega_s \), below which no solutions to the differential equations are found. As \( \Omega_s \) decreases from unity, the amount of angular-momentum deposition required to accelerate the sub-Keplerian rotation velocity to a super-Keplerian one is increased, and hence the derivative \( d\Omega/dx \) inevitably becomes steeper in the region where the acceleration takes place. The lower limit of \( \Omega_s \) is reached when

| \( \Omega_s/\Omega_s \) | \( x_e \) | \( x_m \) | \( j \) | \( -b_0 f(x_m) \) |
|-----------------|-------|-------|-------|------------------|
| 0.9750 | 0.890 | 0.893 | 5.211 | 4.255 |
| 0.9800 | 0.900 | 0.905 | 4.859 | 3.793 |
| 0.9850 | 0.912 | 0.919 | 4.405 | 3.362 |
| 0.9900 | 0.926 | 0.936 | 3.837 | 2.826 |
| 0.9950 | 0.946 | 0.957 | 3.603 | 2.571 |
| 0.9980 | 0.965 | 0.963 | 5.420 | 4.505 |
| 0.9990 | 0.975 | 0.964 | 7.629 | 6.576 |
| 1.0000 | 1.000 | 0.965 | 16.81 | 16.04 |


| \( b_0 = 10 \) | \( x_0 = 0.95 \) | \( x_{\text{out}} = 10 \) |
|-----------------|-------|-------|
| 0.9935 | 0.940 | 0.945 | 1.688 | 0.691 |
| 0.9940 | 0.942 | 0.948 | 1.666 | 0.657 |
| 0.9950 | 0.946 | 0.953 | 1.619 | 0.621 |
| 0.9960 | 0.952 | 0.960 | 1.578 | 0.584 |
| 0.9970 | 0.958 | 0.966 | 1.552 | 0.576 |
| 0.9980 | 0.965 | 0.972 | 1.577 | 0.594 |
| 0.9990 | 0.975 | 0.976 | 1.761 | 0.767 |
| 1.0000 | 1.000 | 0.976 | 3.160 | 2.142 |


| \( b_0 = 50 \) | \( x_0 = 0.99 \) | \( x_{\text{out}} = 10 \) |
|-----------------|-------|-------|
| 0.9500 | 0.854 | 0.856 | 19.41 | 18.56 |
| 0.9600 | 0.867 | 0.870 | 18.77 | 17.91 |
| 0.9700 | 0.881 | 0.886 | 17.82 | 16.70 |
| 0.9800 | 0.900 | 0.907 | 16.39 | 15.50 |
| 0.9900 | 0.926 | 0.936 | 15.10 | 13.81 |
| 0.9950 | 0.946 | 0.952 | 16.86 | 15.79 |
| 0.9980 | 0.965 | 0.962 | 21.97 | 20.88 |
| 0.9990 | 0.975 | 0.966 | 25.70 | 24.53 |
| 1.0000 | 1.000 | 0.973 | 36.16 | 35.03 |
the point of \( y = 0 \) or \( d\tilde{\Omega}/dx = \infty \) appears in the solution \( y \) or \( \tilde{\Omega} \). For \( b_0 = 50 \) and \( x_{\text{out}} = 10 \), the lower limit of \( \tilde{\Omega}_s \) is \( \approx 0.975 \).

Figure 3 shows that the ratio \( y/x \) is less than \( \sim 0.1 \) in the disc, indicating that the disc is geometrically thin. Since the ratio is approximately proportional to \( j^{1/6} \), as suggested by equation (26), and the value of \( j \) for \( \Omega_s = 0.98 \) is larger than that for \( \Omega_s = 0.99 \) (see table 1), the ratio \( y/x \) for the former is larger than that for the latter. This figure also shows that the ratio gradually decreases as \( x \) increases from \( x \sim 1 \).

If we employ \( b_0 = 10 \) instead of \( b_0 = 50 \), we can obtain solutions for \( x_{\text{out}} = 2 \), but no solutions for \( x_{\text{out}} = 10 \), and the parameter value of \( j \) that we obtain for \( b_0 = 10 \) is \( \sim 1.5 \) for \( \Omega_s < 1 \). This suggests that proper solutions to the differential equations can be obtained only when the outer boundary condition is imposed at \( x_{\text{out}} < x_j \). We also find that the properties of the solutions for a given \( j \) do not strongly depend on \( x_{\text{out}} \) so long as \( x_m < x_{\text{out}} < x_j \). (These properties of disc solutions are also confirmed for the case of \( b_0 = 50 \).) For \( b_0 = 10 \), the lower limit of \( \tilde{\Omega}_s \) is \( \approx 0.9935 \), which is much closer to unity than the lower limit \( \approx 0.975 \) for the case of \( b_0 = 50 \).

Figure 4 plots \( y \) and \( \tilde{\Omega} \) as a function of \( x \) for \( b_0 = 10 \). The discs for \( b_0 = 10 \) are thinner than for \( b_0 = 50 \), and the peak value \( \tilde{\Omega}(x_m) \) attained for the lower limit of \( \tilde{\Omega}_s \) is smaller than that for \( b_0 = 50 \).

To examine the case in which the acceleration takes place in a region much closer to the stellar surface, that is, in the region of much lower density, we carried out similar calculations assuming \( x_0 = 0.99 \) for \( b_0 = b_1 = 50 \) and \( x_{\text{out}} = 10 \). As shown by figure 5 and table 1, we again obtain a series of deceleration disc solutions, the properties of which are quite similar to those for \( x_0 = 0.95 \), except that the values of \( j \) for \( x_0 = 0.99 \) are much larger than those for \( x_0 = 0.95 \). Because of the large values of the parameter \( j \), the discs can have larger radii for \( x_0 = 0.99 \) than for \( x_0 = 0.95 \). The lower limit of \( \tilde{\Omega}_s \) is \( \approx 0.95 \), which is smaller than \( \approx 0.975 \) for the case of \( x_0 = 0.95 \).

Let us discuss the physical meaning of \( j \). With the substitution of \( R_m \) for \( R_0 \), equation (7) becomes \( \dot{J} = M_j m - f_m = M_j (x) - q(x) - f(x) \), where \( f_m = f(x_m) \) and \( j_m = j(x_m) \). The quantity \( j = x_m^2 \tilde{\Omega}(x_m) - b_0 f(x_m) \) is now composed of the angular momentum of rotation and the excessive angular momentum due to forcing by the waves at \( x_m \). Since \( q(x_m) \approx 0 \), we have \( M_j (x_m - j(x_m)) \approx f_m - f(x_m) \), which suggests that the acceleration from \( j(x_{\text{tr}}) \) to \( j_m \) is caused by angular-momentum deposition equal to \( f_m - f(x_{\text{tr}}) \). We also note that the excessive angular momentum, \( -b_0 f(x_m) \), is used to extend the disc outward from \( x_m \), where \( -b_0 f(x) \) tends to zero.
as $x$ increases from $x_m$. Since $x_m^2\tilde{\Omega}(x_m) \sim 1$, as suggested by table 1, the value of $\tilde{j}$, and hence the possible extension of the disc, is determined by the amount of this excessive angular momentum, $-b_0\tilde{f}(x_m)$. If $-b_0\tilde{f}(x_m)$ is large, the possible extension of viscous Keplerian decretion discs becomes large.

4. Conclusion

We have calculated steady viscous Keplerian decretion discs around a rapidly rotating star, assuming the existence of angular-momentum supply to the region close to the surface of the star. The angular-momentum supply may be provided by angular-momentum deposition that takes place through the wave-meanflow interaction, where the waves are low-frequency global oscillations excited by the opacity bump mechanism for SPB stars, or by a stochastic mechanism for early Be stars, or by the tidal force if the star is in a binary system. We may conclude that the angular-momentum supply to the surface layers by the waves can be a mechanism for disc formation around rapidly rotating Be stars. In the sense that angular-momentum supply to the surface layers plays an essential role for disc formation, our calculation may be thought...
to be complementary to recent stellar evolution calculations of rotating main-sequence stars by Granada et al. (2013), who suggested that in the course of evolution the surface layers of the rotating stars reach the critical rotation velocity, leading to mass shedding from the equatorial regions, where the transport of angular momentum inside a star is implemented, following the prescription of Zahn (1992) for the horizontal diffusion coefficient and that of Maeder (1997) for the shear diffusion coefficient.

If the amount of angular-momentum supply, which is represented by the parameter $b_0$ in this paper, is large enough, viscous decretion discs can extend to a distance as far as $R_{\text{out}} \gtrsim 10 R_{\text{cr}}$; and if the acceleration takes place in the region very close to the stellar surface, the possible extension a decretion disc attain can be as large as $\gtrsim 100 R_{\text{cr}}$. If $b_0$ is small, however, disc solutions are found only when $\Omega_\ast$ is very close to unity, and the possible extension of the discs is comparable to the stellar radius, $R_{\ast}$, itself.

If the angular-momentum supply is provided by global oscillations, the parameter $b_0$ represents the square of the oscillation amplitudes. We have argued that amplitudes on the order of $\xi R/R_c \sim 0.01-0.1$ can lead to reasonable values of $b_0$. It is also important to note that for given values of the parameters $\alpha$, $M_\ast$, $R_\ast$ (or $R_c$), assigning a value to the parameter $a$ is almost equivalent to assuming a similar power for $M$.

The discussions made in this paper, therefore, are those for a single value of $M_\ast$, which is determined from the value $a = 10^7$ for given $M_\ast$, $R_\ast$, and $\alpha$.

The density at the mid-plane of the disc may be estimated by using equation (23), which leads to $\rho_0 \propto x^{-1.5}$ if the ratio $y/x$ is assumed to be almost constant. Since $\rho_0 \propto x^{-3.5}$ has been suggested observationally, the $x$ dependence of $\rho_0$ in our model is in serious conflict with the observational estimation (e.g., Porter & Rivinius 2003). For B-type stars we have a typical mean density of $\bar{\rho}_s \sim 10^{-2}$ g cm$^{-3}$, and since $y/x \sim 0.01-0.1$, we obtain $\rho_0 \sim 10^{-9}-10^{-8}$ g cm$^{-3}$ at $x \sim 10$, the value of which is much higher than that observationally estimated (e.g., Waters 1986). We could use much larger (smaller) values for the parameter $a (M)$ to reduce the ratio $y/x$, and hence $\rho_0$, but for the value of $a$ much larger than $10^7$, we find it difficult to numerically obtain solutions to the differential equations. Note that decretion discs calculated for $a$ much larger than $10^7$ (i.e., for $M$ much smaller than that for $a = 10^7$) would have large extensions even for small values of $b_0$, although we cannot prove this because of numerical difficulties.

In our steady disc-star systems discussed in this paper, the extension of the discs is limited by $x_j \approx j^2$. At large radii, $x \gg 1$, the discs flows possibly tend to angular momentum conserving ones (e.g., Okazaki 2001), or the discs would suffer radiative ablation to be truncated at finite radii (e.g., Krtička et al. 2011). To obtain steady and angular-momentum conserving disc solutions at large radii, we need to calculate transonic flows extending indefinitely; however the set of differential equations we have solved in this paper do not provide such transonic solutions. We think this is a reason for the differences in the properties, such as the $x$ dependence of $\rho_0$ discussed in the previous paragraph, of viscous disc solutions at large radii between Okazaki (2001) and the present paper.

It is important to note that the decretion-disc solutions in our model are obtained only for $\Omega_\ast$ that is close to 1 (see table 1), and that since $\Omega(x)/\Omega_{\text{cr}} \gtrsim 1$ in the boundary layers between the disc and the star, as indicated by figures 2, 4, and 5, the actual observed values of $V_{\text{cm}}/V_{\text{cr}}$ will be close to 1 even if $V_c/\Omega_{\ast} \sim 0.8-0.9$ (see figure 1), where we may define $V_{\text{cm}} = R_m \Omega(x_m)$ with $R_m = x_m R_{\text{cr}}$. Although various attempts (e.g., Townsend et al. 2004; Cranmer 2005; Frémat et al. 2005; Rivinius et al. 2006; Delaa et al. 2011) have been made to estimate the ratio $\Omega_\ast/\Omega_{\text{cr}}$ (or $V_c/\Omega_{\ast}$) for Be stars in order to judge whether Be stars are rotating at rates very close to the critical ones, or at rates substantially lower than the critical rates, it may be fair to say that no firm conclusions concerning the ratio have been obtained. For Be stars, for example, Townsend, Owocki, and Howarth (2004) argued for rotation rates very close to the critical rates; however, Frémat et al. (2005) estimated the average rate of rotation as being $\Omega/\Omega_{\text{cr}} \approx 0.8$, which may be considered as substantially subcritical rotation rates. More interestingly, Cranmer (2005) suggested that the lower limits of the rotation rates for early type Be stars are as low as 40%-60% of the critical rates, but those for late-type Be stars could be very close to the critical ones. Since the model discussed in this paper becomes viable only for stars rotating at a rate close to the critical rate, this model will be ruled out if it is proved that most of Be stars are rotating at rates much lower than the critical rates.

As indicated by plots of $\Omega(x)$, there occurs a strong differential rotation in the region close to the surface, particularly for lower values of $\Omega_\ast$. The strong differential rotation could modify the modal properties of oscillations, and hence the accelerating and decelerating contributions to the forcing on the velocity field. Stability analysis of low-frequency modes in differentially rotating stars, which will be one of our future studies, is necessary if we use for the forcing mechanism the oscillation modes that are excited by the opacity bump mechanism.

Appendix. Mean Flow Equation in the Lagrangian Mean Formalism

Following Grashow (1984), in this Appendix we derive a meanflow equation for zonal flows around the rotation axis of stars, using the Lagrangian mean formalism (see also Andrews & McIntyre 1978b). In a frame rotating with angular velocity $\Omega_\ast$, the $\phi$ component of the momentum conservation equation in spherical polar Eulerian coordinates $\mathbf{x} = (x_1, x_2, x_3) = (r, \theta, \phi)$ is given by

$$\dot{\rho} \left[ \frac{d \dot{v}_\phi}{dt} + \frac{\dot{v}_\theta \dot{v}_\phi}{r} + \frac{\dot{v}_\phi \dot{v}_\phi \cot \theta}{r} \right] + 2\Omega_\ast \left[ \sin \theta \dot{v}_r + \cos \theta \dot{v}_\theta \right] = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} - \frac{1}{r \sin \theta} \frac{\partial \hat{P}}{\partial \phi}, \quad (A1)$$

where $\dot{\rho}$, $\dot{P}$, and $\hat{P}$ are respectively the density, the pressure, and the gravitational potential of the fluid,

$$\dot{v}_r = \frac{d \hat{r}}{dt}, \quad \dot{v}_\theta = \frac{d \hat{\theta}}{dt}, \quad \dot{v}_\phi = \frac{d \hat{\phi}}{dt}, \quad (A2)$$

and
To discuss wave-mean flow interactions, we introduce the Lagrangian mean coordinates, \( x \equiv (x_1, x_2, x_3) = (r, \theta, \phi) \), and the displacement vector, \( \xi_\alpha \), for \( \alpha = 1, 2, 3 \), such that

\[
\hat{x}_\alpha = x_\alpha + \xi_\alpha (t, x) .
\]

(A4)

We assume that for any given \( \hat{u}_\theta \), there is a unique “reference” velocity, \( \bar{v}_\theta \), such that when the point \( x_\alpha \) moves with velocity \( \hat{v}_\theta \), the point \( \hat{x}_\alpha \) moves with velocity \( \bar{v}_\theta \) (e.g., Grimshaw 1984).

Using the velocity \( \bar{v}_\theta \) as the mean velocity associated with the coordinates \( x_\alpha \), and that the displacement, \( \xi_\theta \), represents the waves.

Using the Jacobian, \( J \) for the coordinate transformation between \( (\hat{x}_\alpha) \) and \( (x_\alpha) \) given by

\[
J = \text{det} \left( \frac{\partial \hat{x}_\alpha}{\partial x_\beta} \right) = \text{det} \left( \delta_{\alpha\beta} + \frac{\partial \xi_\alpha}{\partial x_\beta} \right) ,
\]

(A8)

where \( \delta_{\alpha\beta} \) denotes the Kronecker delta, we define the mean density, \( \bar{\rho} \), associated with the coordinate \( x \) as

\[
\bar{\rho} r^2 \sin\theta J = \bar{\rho} \bar{v}_\theta^2 \sin\theta .
\]

(A9)

It is convenient to introduce (e.g., Andrews & McIntyre 1978b)

\[
K_{\alpha\beta} = \frac{\partial J}{\partial (\partial \hat{x}_\alpha / \partial x_\beta)} = \frac{1}{2} \epsilon_{\alpha\beta\gamma} \epsilon_{\beta\gamma\tau} \frac{\partial \hat{x}_\alpha}{\partial x_\epsilon} \frac{\partial \hat{x}_\beta}{\partial x_\tau} ,
\]

(A10)

for which we have

\[
\frac{\partial K_{\alpha\beta}}{\partial x_\gamma} = 0 ,
\]

(A11)

and

\[
K_{\alpha\beta} \frac{\partial \hat{x}_\gamma}{\partial x_\gamma} = \delta_{\gamma\tau} J = K_{\gamma\alpha} \frac{\partial \hat{x}_\gamma}{\partial x_\alpha} ,
\]

(A12)

where we have employed the dual summation convention that Greek indices are summed over the range 1 to 3, and \( \epsilon_{123} = \epsilon_{312} = 1 = -\epsilon_{132} = -\epsilon_{213} = -\epsilon_{321} \) and \( \epsilon_{\alpha\beta\gamma} = 0 \) otherwise.

We rewrite equation (A1) as

\[
\hat{\rho} \frac{d \hat{\dot{L}}}{dt} = -\frac{\partial \hat{\dot{L}}}{\partial \phi} \hat{\dot{\phi}} - \hat{\rho} \frac{\partial \hat{\dot{\phi}}}{\partial \phi} ,
\]

(A13)

where \( \hat{L} \) is the specific angular momentum, in an inertial frame, around the rotation axis defined by

\[
\hat{L} = \hat{r} \sin \hat{\theta} \hat{v}_\theta + \Omega_c r^2 \sin^2 \hat{\theta} .
\]

(A14)

Multiplying equation (A13) by \( \hat{r}^2 \sin \hat{\theta} J \), we obtain

\[
\bar{\rho} \frac{d \hat{L}}{dt} + \frac{\partial \bar{p}}{\partial \phi} = -\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial x_\beta} R_{3\beta} - \hat{\rho} \frac{\partial \hat{\dot{\phi}}}{\partial \phi} ,
\]

(A15)

where

\[
R_{3\beta} = \delta_{3\beta} \left( J \hat{r}^2 \sin \hat{\theta} \hat{p} - \hat{r}^2 \sin \hat{\theta} \hat{p} \right) - \hat{r}^2 \sin \hat{\theta} \hat{p} \frac{\partial \hat{\xi}_\gamma}{\partial x_3} K_{\gamma\beta} .
\]

(A16)

and \( \hat{\rho} = \bar{\rho}(\hat{\rho}, \hat{\phi}) \) with \( \hat{\delta} \) being the specific entropy; we have used the identity

\[
K_{3\beta} \hat{r}^2 \sin \hat{\theta} \hat{\dot{p}} \hat{\phi} = \delta_{3\beta} J \hat{r}^2 \sin \hat{\theta} \hat{\dot{p}} \hat{\phi} - \hat{r}^2 \sin \hat{\theta} \hat{p} \frac{\partial \hat{\xi}_\gamma}{\partial x_3} K_{\gamma\beta} .
\]

(A17)

Applying the averaging procedure (A6) to equation (A15), we obtain

\[
\hat{\rho} \frac{d \hat{L}}{dt} = -\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial x_\beta} \left( R_{3\beta} \hat{\phi} - \hat{\rho} \frac{\partial \hat{\dot{\phi}}}{\partial \phi} \right) ,
\]

(A18)

where we have used \( (\partial \hat{\rho} / \partial \phi) = 0 \). In general,

\[
\hat{\rho} \frac{d \hat{L}}{dt} = \delta_{\alpha\beta} \left( J \hat{r}^2 \sin \hat{\theta} \hat{p} - \hat{r}^2 \sin \hat{\theta} \hat{p} \right) - \hat{r}^2 \sin \hat{\theta} \hat{p} \frac{\partial \hat{\xi}_\gamma}{\partial x_3} K_{\gamma\beta} .
\]

(A19)

is called the radiation stress tensor (e.g., Grimshaw 1984).

Equation (A18) may be regarded as the \( \phi \) component of the meanflow equation, the left-hand-side of which may represent the time evolution of the meanflow and the right-hand-side the forcing by the waves represented by \( \xi_\alpha \). So far we have not assumed that the amplitudes of the waves, \( \xi_\alpha \), are infinitesimally small, and in principle we can formulate the wave-meanflow interaction as a nonlinear theory, which includes the equations of motion for both the meanflows and the waves. To avoid solving such a difficult non-linear problem, we use a linear theory to describe waves, \( \xi_\alpha \), and we are satisfied with calculating the forcing terms in the meanflow equation using the linear waves \( \xi_\alpha \).

If we employ a linear theory to describe waves represented by the displacement \( \xi_\alpha \), we may write

\[
\hat{\rho} = \bar{\rho} + \delta \rho = \bar{\rho} + \xi_\alpha \frac{\partial \bar{\rho}}{\partial x_\alpha} ,
\]

(A20)

where \( \bar{\rho} \) is the pressure in the equilibrium state, \( \delta \rho \) and \( \rho' \) denote the Lagrangian and Eulerian perturbations, respectively.

Applying the averaging procedure (A6) to second order of the perturbations we obtain after some manipulations for the second term on the right-hand-side of equation (A16)

\[
\left( \hat{r}^2 \sin \hat{\theta} \hat{p} \frac{\partial \hat{\xi}_\gamma}{\partial x_3} K_{\gamma\phi} \right) \left( \hat{r}^2 \sin \hat{\theta} \hat{p} \frac{\partial \hat{\xi}_\gamma}{\partial x_3} \hat{\phi} \right) + \left( \frac{\partial}{\partial x_3} \left( r^2 \sin \theta \frac{\partial \hat{\xi}_\gamma}{\partial x_3} \hat{\phi} \right) \right) ,
\]

(A21)
where we have used \( \partial f / \partial \phi = \partial (f) / \partial \phi = 0 \) and \( \partial \rho / \partial \phi = 0 \) for the equilibrium pressure \( \rho \), and \( x_3 = \phi \). Because

\[
\frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3} \left( \rho r^2 \sin \theta \frac{\partial \xi_\phi}{\partial \phi} \right) = \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3} \left( \rho r^2 \sin \theta \frac{\partial \xi_\phi}{\partial \phi} \right)
\]

we find

\[
\left( \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3} \left( \rho r^2 \sin \theta \frac{\partial \xi_\phi}{\partial \phi} \right) \right) = \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3} \left( \rho r^2 \sin \theta \frac{\partial \xi_\phi}{\partial \phi} \right)
\]

and hence we omit the second term on the right-hand-side of equation (A21) to obtain

\[
\frac{r^2 \sin \theta \partial \xi_\phi}{\partial x_3} K_{r\theta} = \left( r^2 \sin \theta \frac{\partial \xi_\phi}{\partial x_3} \right),
\]

and hence

\[
\langle R_{3\alpha} \rangle = \delta_{3\alpha} \left( J r^2 \sin \theta \rho - r^2 \sin \theta \rho \right) - \left( r^2 \sin \theta \frac{\partial \xi_\phi}{\partial x_3} \right).
\]

The meanflow equation (A18) is then reduced to

\[
\rho \frac{d (\hat{\xi} k)}{dt} = -\nabla \cdot \left( \xi \frac{\partial \phi}{\partial \phi} \right) - \left( \hat{\rho} \frac{\partial \phi}{\partial \phi} \right),
\]

where \( \xi = \xi_r e_r + \xi_\theta e_\theta + \xi_\phi e_\phi \) with \( e_r, e_\theta, \text{ and } e_\phi \) being the orthonormal vectors in the \( r, \theta, \text{ and } \phi \) directions, and \( \xi_r = \xi_1, \xi_\theta = \xi_2, \text{ and } \xi_\phi = r \sin \theta \xi_3, \rho = \rho \). Since

\[
\left( \frac{\partial}{\partial \phi} \frac{\partial \phi}{\partial \phi} \right) = \nabla \cdot \left( \rho \frac{\partial \phi}{\partial \phi} \right) + \left( \rho \frac{\partial \phi}{\partial \phi} \right),
\]

which is correct to the second order of perturbations, we obtain

\[
\rho \frac{d (\hat{\xi} k)}{dt} = -\nabla \cdot \left( \xi \frac{\partial \phi}{\partial \phi} + \rho \frac{\partial \phi}{\partial \phi} + \nabla \phi \cdot \nabla \phi \right)
\]

where we have used \( \partial \phi / \partial \phi = 0 \), \( \nabla^2 \phi = 4 \pi G \rho \) and \( \langle \nabla \phi \cdot \nabla \phi / \partial \phi \rangle = 0 \). Since the azimuthal and temporal dependence of the perturbations are assumed to be given by the factor \( \exp (i m \phi + i \omega t) \) with \( m \) and \( \omega \) being the azimuthal wavenumber and oscillation frequency, using, for example,

\[
\langle \xi_\phi \xi_\phi \rangle = \frac{1}{2} \mathbb{R} \langle \xi_\phi \xi_\phi \rangle = \frac{1}{2} \mathbb{R} \langle \rho \phi \xi_\phi \rangle.
\]

we can rewrite equation (A28) as

\[
\rho \frac{d (\hat{\xi} k)}{dt} = -\frac{1}{2} \text{Re} \left[ \nabla \cdot \left( \xi_\phi \frac{\partial \phi}{\partial \phi} + \rho \phi \frac{\partial \phi}{\partial \phi} + \nabla \phi^* \nabla \phi \right) \right]
\]

\[
= \frac{m}{2} \text{Re} \left[ \nabla \cdot \left( \xi^* \frac{\partial \phi}{\partial \phi} + \rho \phi^* \frac{\partial \phi}{\partial \phi} + \nabla \phi^* \nabla \phi \right) \right],
\]

where the asterisk indicates the complex conjugation. Integrating over a spherical surface, we obtain

\[
\rho \frac{d (\hat{\xi} k)}{dt} = \frac{d}{dr} \left( \frac{\partial}{\partial r} \hat{\rho} r^2 \sin \theta \left( \xi^* \frac{\partial \phi}{\partial \phi} + \rho \phi^* \frac{\partial \phi}{\partial \phi} + \nabla \phi^* \nabla \phi \right) \right)
\]

\[
= \frac{m}{2} \mathbb{R} \left[ \nabla \cdot \left( \xi^* \frac{\partial \phi}{\partial \phi} + \rho \phi^* \frac{\partial \phi}{\partial \phi} + \nabla \phi^* \nabla \phi \right) \right],
\]

where \( \mathbb{R} \) stands for the real part of a complex number.
Fig. 6. $1/\tau^{AM}$ for unstable $l = |m| = 1$ g$_{22}$ modes for $\Omega_{c}/(GM_{*}/R_{*}^{3})^{1/2} = 0.1$ (left panel) and an unstable $l = |m| = 1 = r_{30}$ mode for $\Omega_{c}/(GM_{*}/R_{*}^{3})^{1/2} = 0.4$ (right panel) of a $4~M_{\odot}$ main-sequence model with $X = 0.7$, $Z = 0.02$, and the central hydrogen content $X_{c} = 0.3646$, where $M_{*}$ and $R_{*}$ denote the mass and radius of the model. The solid and dotted lines indicate retrograde and prograde modes, respectively, and $\tau^{AM}$ is given in seconds. The amplitude normalization is given by setting the radial displacement associated with the spherical harmonic function, $Y_{l}^{m}(\theta, \phi)$, equal to $R_{*}$ at the surface. Note that uniform rotation is assumed.

An example of $1/\tau^{AM}$ is given for low-frequency modes of a $4~M_{\odot}$ main-sequence star with $X = 0.7$ and $Z = 0.02$ in figure 6, where the model has been calculated with a standard stellar evolution code with the OPAL opacity (Iglesias & Rogers 1996), and we have used the method of calculation given by Lee and Saio (1993) for non-adiabatic oscillation modes of a uniformly rotating star. As the figure shows, there occurs to be a strong acceleration by retrograde $g$- and $r$-modes in the layers at $r/R_{*} \sim 0.95$, although the prograde $g$-mode contributes to deceleration of the rotation.

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