ON QUASI-BAER RINGS OF ORE EXTENSIONS

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ABSTRACT. Let \( R \) be a ring and \( S = R[x; \sigma, \delta] \) its Ore extension. We prove under some conditions that \( R \) is a quasi-Baer ring if and only if the Ore extension \( R[x; \sigma, \delta] \) is a quasi-Baer ring. Examples are provided to illustrate and delimit our results.

1. Introduction

Throughout this paper, \( R \) denotes an associative ring with unity. For a subset \( X \) of \( R \), \( r_R(X) = \{ a \in R | xa = 0 \} \) and \( \ell_R(X) = \{ a \in R | aX = 0 \} \) will stand for the right and the left annihilator of \( X \) in \( R \) respectively. By \cite{9}, a right annihilator of \( X \) is always a right ideal, and if \( X \) is a right ideal then \( r_R(X) \) is a two-sided ideal. An Ore extension of a ring \( R \) is denoted by \( R[x; \sigma, \delta] \), where \( \sigma \) is an endomorphism of \( R \) and \( \delta \) is a \( \sigma \)-derivation, i.e., \( \delta: R \rightarrow R \) is an additive map such that \( \delta(ab) = \sigma(a)\delta(b) + \delta(a)b \) for all \( a, b \in R \). Recall that elements of \( R[x; \sigma, \delta] \) are polynomials in \( x \) with coefficients written on the left. Multiplication in \( R[x; \sigma, \delta] \) is given by the multiplication in \( R \) and the condition \( xa = \sigma(a)x + \delta(a)b \) for all \( a, b \in R \). We say that a subset \( X \) of \( R \) is \((\sigma, \delta)\)-stable if \( \sigma(X) \subseteq X \) and \( \delta(X) \subseteq X \). A ring \( R \) is \((quasi)\)-Baer if the right annihilator of every nonempty subset (every right ideal) of \( R \) is generated by an idempotent. From \cite{1}, an idempotent \( e \in R \) is left (resp. right) semicentral in \( R \) if \( exe = xe \) (resp. \( exe = ex \)), for all \( x \in R \). Equivalently, \( e^2 = e \in R \) is left (resp. right) semicentral if \( eR \) (resp. \( Re \)) is an ideal of \( R \). Since the right annihilator of a right ideal is an ideal, we see that the right annihilator of a right ideal is generated by a left semicentral in a quasi-Baer ring. We use \( S_l(R) \) and \( S_r(R) \) for the sets of all left and right semicentral idempotents, respectively. Also note \( S_l(R) \cap S_r(R) = B(R) \), where \( B(R) \) is the set of all central idempotents of \( R \). If \( R \) is a semiprime ring then \( S_l(R) = S_r(R) = B(R) \). Recall that \( R \) is a reduced ring if it has no nonzero nilpotent elements. A ring \( R \) is abelian if every idempotent of \( R \) is central. We can easily observe that every reduced ring is abelian.

According to \cite{10}, an endomorphism \( \sigma \) of a ring \( R \) is said to be rigid if \( a\sigma(a) = 0 \) implies \( a = 0 \) for all \( a \in R \). We call a ring \( R \) \( \sigma \)-rigid if there exists

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a rigid endomorphism $\sigma$ of $R$. Following Hashemi and Moussavi [4], a ring $R$ is $\sigma$-compatible if for each $a, b \in R$, $a\sigma(b) = 0 \Longleftrightarrow ab = 0$. Moreover, $R$ is said to be $\delta$-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If $R$ is both $\sigma$-compatible and $\delta$-compatible, we say that $R$ is $(\sigma, \delta)$-compatible. A ring $R$ is $\sigma$-rigid if and only if $R$ is $(\sigma, \delta)$-compatible and reduced [4 Lemma 2.2]. Also, if $R$ is $\sigma$-rigid then $R[x; \sigma, \delta]$ is reduced [10 Theorem 3.3]. From [8], a ring $R$ is said to be a $\sigma$-skew Armendariz ring if for $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x; \sigma]$, $pq = 0$ implies $a_i \sigma^i(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. From [3], a ring $R$ is called an $(\sigma, \delta)$-skew Armendariz ring if for $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x; \sigma, \delta]$, $pq = 0$ implies $a_i x^i b_j x^j = 0$ for each $i, j$. Note that $(\sigma, \delta)$-skew Armendariz rings are generalization of $\sigma$-skew Armendariz rings, $\sigma$-rigid rings and Armendariz rings, see [8], for more details. It was proved in [7 Corollary 12], that if $R$ is a $\sigma$-rigid ring then $R[x; \sigma, \delta]$ is a quasi-Baer ring if and only if $R$ is quasi-Baer. Also in [4 Corollary 2.8], it was shown that, if $R$ is $(\sigma, \delta)$-compatible, then $R[x; \sigma, \delta]$ is a quasi-Baer ring if and only if $R$ is quasi-Baer.

The aim of this paper is to show that if $R$ is an $(\sigma, \delta)$-skew Armendariz ring with $\sigma$ an automorphism such that $Re$ is $(\sigma, \delta)$-stable for all $e \in S_l(R)$, then $R$ is a quasi-Baer ring if and only if $R[x; \sigma, \delta]$ is a quasi-Baer ring. Many examples are provided to illustrate and delimit results and to show that they are not consequences of [4 Corollary 2.8]. Moreover, we obtain a partial generalization of [7 Corollary 12].

2. Preliminaries and Examples

For any $0 \leq i \leq j$ ($i, j \in \mathbb{N}$), $f^j_i \in \text{End}(R, +)$ will denote the map which is the sum of all possible words in $\sigma, \delta$ built with $i$ letters $\sigma$ and $j - i$ letters $\delta$ (e.g., $f^3_2 = \sigma^2$ and $f^n_0 = \delta^n, n \in \mathbb{N}$). The next lemma appears in [11 Lemma 4.1].

**Lemma 2.1.** For any $n \in \mathbb{N}$ and $r \in R$ we have $x^n r = \sum_{i=0}^{n} f^n_i(r)x^i$ in the ring $R[x; \sigma, \delta]$.

**Lemma 2.2.** [5 Lemma 5]. Let $R$ be an $(\sigma, \delta)$-skew Armendariz ring. If $e^2 = e \in R[x; \sigma, \delta]$ where $e = e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n$, then $e = e_0$.

**Lemma 2.3.** Let $R$ be a ring, $\sigma$ an endomorphism and $\delta$ be a $\sigma$-derivation of $R$. Then $\sigma(Re) \subseteq Re$ implies $\delta(Re) \subseteq Re$ for all $e \in B(R)$.

**Proof.** Let $e \in B(R)$ and $r \in R$. Then $\delta(re) = \delta(ere) = \sigma(ere)\delta(e) + \delta(ere)e = \sigma(ere)\delta(e) + \delta(ere)e = se\delta(e) + \delta(ere)e$, for some $s \in R$, but $e \in B(R)$, then $e\delta(e) = e\delta(e)e$, so $\delta(re) = (se\delta(e) + \delta(ere))e$. Therefore $\delta(Re) \subseteq Re$. \hfill $\Box$

**Lemma 2.4.** Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. If $R$ is $(\sigma, \delta)$-compatible. Then for $a, b \in R$, $ab = 0 \Rightarrow a f^j_i(b) = 0$ for all $j \geq i \geq 0$.
Proof. If \( ab = 0 \), then \( a\sigma^i(b) = a\delta^j(b) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \), because \( R \) is \((\sigma, \delta)\)-compatible. Then \( a\sigma^i_j(b) = 0 \) for all \( i, j \).

\( \square \)

**Lemma 2.5.** Let \( R \) be a ring, \( \sigma \) an endomorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \). If \( R \) is \( \sigma \)-rigid then \( R \) is \((\sigma, \delta)\)-skew Armendariz.

Proof. If \( R \) is \( \sigma \)-rigid then \( R \) is \((\sigma, \delta)\)-compatible by \[4\] Lemma 2.2. Let \( f = \sum_{i=0}^{m} a_i x^i \), \( g = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma, \delta] \) such that \( fg = 0 \), then \( a_i b_j = 0 \) for all \( i, j \), by \[7\] Proposition 6. So \( a_i \sigma^j(b_j) = 0 \), for all \( 0 \leq \ell \leq i \leq n, \ 0 \leq j \leq m \), by Lemma 2.4. Hence \( a_i x^i b_j x^j = \sum_{\ell=0}^{i} a_i \sigma^j(b_j) x^{i+j} = 0 \). Therefore \( R \) is \((\sigma, \delta)\)-skew Armendariz.

The next example illustrates that there exists a ring \( R \) and an endomorphism \( \sigma \) of \( R \) such that \( Re \) is \( \sigma \)-stable for all \( e \in S_t(R) \), but \( R \) is not \( \sigma \)-rigid.

**Example 2.6.** \[8\] Example 1. Consider the ring
\[
R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\},
\]
where \( \mathbb{Z} \) and \( \mathbb{Q} \) are the set of all integers and all rational numbers, respectively. The ring \( R \) is commutative, let \( \sigma : R \rightarrow R \) be an automorphism defined by \( \sigma \left( \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix} \).

(1) \( R \) is not \( \sigma \)-rigid.

\[
\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \sigma \left( \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right) = 0, \text{ but } \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \neq 0, \text{ if } t \neq 0.
\]

(2) \( \sigma(Re) \subseteq Re \) for all \( e \in S_t(R) \). \( R \) has only two idempotents:

\[
e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ end } e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ let } r = \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \in R, \text{ we have } \sigma(re_0) \in Re_0 \text{ and } \sigma(re_1) \in Re_1.
\]

Also we have an example of an endomorphism \( \sigma \) of a ring \( R \) such that \( Re \) is \( \sigma \)-stable for all \( e \in S_t(R) \) and \( R \) is not \( \sigma \)-compatible.

**Example 2.7.** Let \( \mathbb{K} \) be a field and \( R = \mathbb{K}[t] \) a polynomial ring over \( \mathbb{K} \) with the endomorphism \( \sigma \) given by \( \sigma(f(t)) = f(0) \) for all \( f(t) \in R \).

(1) \( R \) is not \( \sigma \)-compatible (so not \( \sigma \)-rigid). Take \( f = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \) and \( g = b_1 t + b_2 t^2 + \cdots + b_m t^m \), since \( g(0) = 0 \) so, \( f \sigma(g) = 0 \), but \( fg \neq 0 \).

(2) \( R \) has only two idempotents 0 and 1 so \( Re \) is \( \sigma \)-stable for all \( e \in S_t(R) \).

There is an example of a ring \( R \) and an endomorphism \( \sigma \) of \( R \) such that \( R \) is \( \sigma \)-skew Armendariz and \( R \) is not \( \sigma \)-compatible.

**Example 2.8.** Consider a ring of polynomials over \( \mathbb{Z}_2 \), \( R = \mathbb{Z}_2[x] \). Let \( \sigma : R \rightarrow R \) be an endomorphism defined by \( \sigma(f(x)) = f(0) \). Then:

(i) \( R \) is not \( \sigma \)-compatible. Let \( f = \overline{1} + x, \ g = x \in R, \) we have \( fg = (\overline{1} + x)x \neq 0, \text{ however } f \sigma(g) = (\overline{1} + x)\sigma(x) = 0. \)

(ii) \( R \) is \( \sigma \)-skew Armendariz \[8\] Example 5].
In the next example, $S = R/I$ is a ring and $\sigma$ an endomorphism of $S$ such that $S$ is $\sigma$-compatible and not $\sigma$-skew Armendariz.

**Example 2.9.** Let $Z$ be the ring of integers and $\mathbb{Z}_2$ be the ring of integers modulo 4. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z}_4 \right\}.$$ 

Let $\sigma : R \to R$ be an endomorphism defined by $\sigma \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$.

Take the ideal $I = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in 4\mathbb{Z} \right\}$ of $R$. Consider the factor ring $R/I \cong \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in 4\mathbb{Z} \right\}$.

(1) $R/I$ is not $\sigma$-skew Armendariz. In fact, $\left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right)^2 x = 0 \in (R/I)[x; \sigma]$, but $\left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \sigma \left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right)^2 \neq 0$.

(2) $R/I$ is $\sigma$-compatible. Let $A = \begin{pmatrix} \sigma & \beta \\ 0 & \alpha \end{pmatrix}, B = \begin{pmatrix} a^t & \beta \\ 0 & \alpha \end{pmatrix} \in R/I$. If $AB = 0$ then $\alpha a^t = 0$ and $\alpha b = \beta a^t = 0$, so that $A \sigma(B) = 0$. The same for the converse. Therefore $R/I$ is $\sigma$-compatible.

### 3. Ore Extensions over quasi-Baer rings

It was proved in [1, Theorem 1.2], that if $R$ is a quasi-Baer ring and $\sigma$ an automorphism of $R$ then $R[x; \sigma]$ is a quasi-Baer ring. The following example shows that “$\sigma$ is an automorphism” is not a superfluous condition in Proposition 3.2.

**Example 3.1.** [6, Example 2.8]. There is an example of a quasi-Baer ring $R$ and an endomorphism $\sigma$ of $R$ such that $R[x; \sigma]$ is not a quasi-Baer ring. In fact, let $R = \mathbb{K}[t]$ be the polynomial ring over a field $\mathbb{K}$ and $\sigma$ be the endomorphism given by $\sigma(f(t)) = f(0)$. Then the ring $R[x; \sigma]$ is not a quasi-Baer ring.

**Proposition 3.2.** Let $R$ be a ring, $\sigma$ an automorphism and $\delta$ be a $\sigma$-derivation of $R$. Suppose that $Re$ is $(\sigma, \delta)$-stable for all $e \in S_e(R)$. If $R$ is quasi-Baer then the Ore extension $R[x; \sigma, \delta]$ is quasi-Baer.

**Proof.** Let $S = R[x; \sigma, \delta]$ and $I$ be an ideal of $S$. We claim that $rS(I) = eS$, for some idempotent $e \in R$. We can suppose that $I \neq 0$, we set $I_0 = \{ 0 \} \cup \{ a \in R \mid \exists a_0, a_1, \cdots, a_{n - 1} \in R \text{ such that } ax^n + \sum_{i=0}^{n-1} a_ix^i \in I, n \in \mathbb{N} \}$. It is clear that $I_0$ is a nonzero left ideal of $R$. Given $a \in I_0$ and $r \in R$,
there is an element in $I$ of the form \( ax^n + \sum_{i=0}^{n-1} a_ix^i \). Multiplying on the right by \( \sigma^{-n}(r) \) gives an element of the form \( arx^n + \sum_{i=0}^{n-1} b_ix^i \), for some elements \( b_0, b_1, \cdots, b_{n-1} \in R \), and so \( ar \in I_0 \), thus \( I_0 \) is a two-sided ideal. So there exists an idempotent \( e \in R \) such that \( rr_R(I_0) = eR \). We have \( eS \subseteq r_S(I) \).

To see this, let \( 0 \neq f(x) = \sum_{k=0}^{n} a_kx^k \in I \), then \( f(x)e = \sum_{k=0}^{n} (\sum_{i=k}^{n} a_kf_i^k(e))x^k \), where \( f_i^i \) are sums of all possible words in \( \sigma, \delta \) built with \( k \) letters \( \sigma \) and \( i-k \) letters \( \delta \). \( Re \) is \( f_i^i \)-stable \( (0 \leq k \leq i) \), so there exists \( u_k \in R \) such that \( f_i^i(e) = u_k e \). \( \lambda \) Therefore \( f(x)e = \sum_{k=0}^{n} (\sum_{i=k}^{n} a_ku_i^k)ex^k \), if we set \( \alpha_k = \sum_{i=k}^{n} a_ku_i^k \), then \( f(x)e = \sum_{k=0}^{n} \alpha_kx^k \). If \( \alpha_n \neq 0 \), then \( \alpha_n \in I_0 \) and so, \( \alpha_ne = \alpha_n = 0 \) (because \( rr_R(I_0) = eR \)). Contradiction, hence \( \alpha_n = 0 \).

Now suppose that \( \alpha_j = 0 \) for \( j = n, n-1, \cdots, k+1 \) with \( k \in \mathbb{N} \). But \( f(x)e = \alpha_kx^k + \sum_{\ell=0}^{k-1} \alpha_\ell x^\ell \), with the same manner as above we have \( \alpha_k = 0 \). So we can get \( \alpha_n = \alpha_{n-1} = \cdots = \alpha_0 = 0 \). Consequently \( eS \subseteq r_S(I) \).

Conversely, we can claim that \( r_S(I) \subseteq eS \). Let \( 0 \neq f(x) = \sum_{k=0}^{n} a_kx^k \in I \) and \( \lambda(x) = \sum_{j=0}^{m} b_jx^j \in S \), such that \( f(x)\lambda(x) = 0 \), we shall show that \( \lambda(x) = \sigma^{-n}(e)\lambda(x) \). If we set \( \xi(x) = \lambda(x) - \sigma^{-n}(e)\lambda(x) = \sum_{j=0}^{m} (b_j - \sigma^{-n}(e)b_j)x^j \), we have \( f(x)\xi(x) = \sum_{i=0}^{n} (\sum_{j=0}^{m} (b_j - \sigma^{-n}(e)b_j)x^j) = a_n\sigma^n(b_m - \sigma^{-n}(e)b_m)x^{n+m+Q} = 0 \), where \( Q \) is a polynomial with \( \deg(Q) < n + m \). Thus \( a_n\sigma^n(b_m - \sigma^{-n}(e)b_m) = 0 \), since \( a_n \neq 0 \), then \( a_n \in I_0 \). Hence \( \sigma^n(b_m - \sigma^{-n}(e)b_m) \in r_R(I_0) = eR \). So \( \sigma^n(b_m - \sigma^{-n}(e)b_m) = \sigma^n(b_m - \sigma^{-n}(e)b_m) \), then \( b_m - \sigma^{-n}(e)b_m = \sigma^{-n}(e)(b_m - \sigma^{-n}(e)b_m) = 0 \) (because \( \sigma^{-n}(e) \) is idempotent), hence \( b_m - \sigma^{-n}(e)b_m = 0 \). Now, suppose that \( b_j - \sigma^{-n}(e)b_j = 0 \) for \( j = m, m-1, \cdots, k+1 \) with \( k \in \mathbb{N} \) and showing that \( b_k - \sigma^{-n}(e)b_k = 0 \). Effectively, \( f(x)\xi(x) = a_n\sigma^n(b_k - \sigma^{-n}(e)b_k)x^{n+k} + Q' = 0 \), where \( Q' \) is a polynomial with \( \deg(Q') < n + k \), then \( a_n\sigma^n(b_k - \sigma^{-n}(e)b_k) = 0 \), with the same manner as below, we obtain \( b_k - \sigma^{-n}(e)b_k = 0 \). Therefore \( b_j - \sigma^{-n}(e)b_j = 0 \) for all \( 0 \leq j \leq m \), then \( \xi(x) = 0 \). But \( \lambda(x) = \sigma^n(e)\lambda(x) \) or \( \sigma^n(e) = ue \) for some \( u \in R \), but \( e \) is left semicentral then \( \lambda(x) = eue\lambda(x) \). Hence \( r_S(I) \subseteq eS \). So \( R[x; \sigma, \delta] \) is a quasi-Baer ring. \( \square \)

In Example 2.7, \( Re \) is \((\sigma, \delta)\)-stable for all \( e \in S(I) \) but \( R \) is not \((\sigma, \delta)\)-compatible. Thus, Proposition 3.2 is not a consequence of [4, Corollary 2.8].
There is a quasi-Baer ring $R$, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $Re$ is $(\sigma, \delta)$-stable for all $e \in S_l(R)$.

**Example 3.3.** Consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the set of all integers numbers. By [2, Example 1.3(ii)], $R$ is a quasi-Baer ring. Define $\sigma: R \to R$ and $\delta: R \to R$ by
\[
\sigma \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & \sigma(b) \\ 0 & c \end{pmatrix}, \quad \delta \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}
\]
for all $a, b, c \in \mathbb{Z}$.

Clearly, $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation. The nonzero idempotents of $R$ are of the form
\[
e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix},
\]
where $t \in \mathbb{Z}$. $e_2$ is right semicentral not left semicentral and $e_1$ is left semicentral not right semicentral, so the only left semicentral nonzero idempotents of $R$ are $e_0$ and $e_1$. $Re_0$ is $(\sigma, \delta)$-stable. Let $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$, since $\sigma(re_1) = \begin{pmatrix} x & -xt \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}$, then $Re_1$ is $\sigma$-stable, also $Re_1$ is $\delta$-stable. Therefore $Re$ is $(\sigma, \delta)$-stable for all $e \in S_l(R)$.

**Example 3.4.** Consider the ring $S = \begin{pmatrix} D & D \oplus D \\ 0 & D \end{pmatrix}$, where $D$ is a simple domain which is not a division ring. By [3, Example 4.11], $R$ is a quasi-Baer ring and has nonzero idempotents of the form
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & (b, d) \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & (b, d) \\ 0 & 1 \end{pmatrix},
\]
where $b, d \in D$, with $\sigma$ and $\delta$ as in Example 3.3. $Re$ is $(\sigma, \delta)$-stable for all $e \in S_l(R)$.

**Corollary 3.5.** Let $R$ be an abelian or a semiprime ring, $\sigma$ an automorphism and $\delta$ be a $\sigma$-derivation of $R$, such that $\sigma(Re) \subseteq Re$ for all $e \in B(R)$. If $R$ is quasi-Baer then $R[x; \sigma, \delta]$ is quasi-Baer.

**Proof.** By Lemma 2.3 and Proposition 3.2.

In the remainder of this section we focus on the converse of Proposition 3.2. We begin with the next example which shows that there exists a ring $R$ and a derivation $\delta$ of $R$ such that $R[x; \delta]$ is quasi-Baer but $R$ is not quasi-Baer.

**Example 3.6.** [1, Example 1.6]. There is a ring $R$ and a derivation $\delta$ of $R$ such that $R[x; \delta]$ is a Baer ring. But $R$ is not quasi-Baer. Let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation $\delta$ such that $\delta(t) = 1$ where $t = t + (t^2)$ in $R$ and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $R[x; \delta]$. If we set $e_{11} = t$, $e_{12} = t$, $e_{21} = t^2 + x$ and
Theorem 3.10. Let $R[x;\delta]$ be not a superfluous condition in the next theorem.

**Proposition 3.7.** Let $R$ be an $(\sigma,\delta)$-skew Armendariz ring. If $R[x;\sigma,\delta]$ is quasi-Baer then $R$ is quasi-Baer.

**Proof.** Let $I$ be an ideal of $R$ and $S = R[x;\sigma,\delta]$, then since $S$ is quasi-Baer, there exists an idempotent $e \in S$ such that $r_S(IS) = eS$ with $e = e_0 + e_1 x + \cdots + e_n x^n$ ($n \in \mathbb{N}$). By Lemma 2.2 we have $e_0 \in r_R(I)$. Thus $e_0 R \subseteq r_R(I)$.

Conversely, let $a \in r_R(I)$ then $a \in r_S(IS) \cap R = e_0 S \cap R$, so $a = e_0 f$ for some $f = f_0 + f_1 x + \cdots + f_m x^m \in S$. Then $a = e_0 f_0$ and so $a \in e_0 R$. Therefore $r_R(I) \subseteq e_0 R$. Consequently, $R$ is a quasi-Baer ring. $\square$

By Example 2.8 there is a ring $R$ and $\sigma$ an endomorphism of $R$ such that $R$ is $\sigma$-skew Armendariz and $R$ is not $\sigma$-compatible. So that, Proposition 3.7 is not a consequence of [11 Corollary 2.8]. By the next result, we see that Proposition 3.7 is a partial generalization of [7 Corollary 12].

**Corollary 3.8.** Let $R$ be an $\sigma$-rigid ring. If $R[x;\sigma,\delta]$ is quasi-Baer then $R$ is quasi-Baer.

**Proof.** It follows from Lemma 2.5 and Proposition 3.7 $\square$

One might expect the converse of Proposition 3.2 to hold when $R$ is a $(\sigma,\delta)$-skew Armendariz ring. However [8 Example 5] and [6 Example 2.8], shows that this converse does not hold in general.

**Example 3.9.** We consider a commutative polynomial ring over $\mathbb{Z}_2$. $R = \mathbb{Z}_2[x]$, let $\sigma: R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. By [6 Example 2.8], $R[x;\sigma]$ is not quasi-Baer and $R$ is quasi-Baer. But, by [8 Example 5], $R$ is $\sigma$-skew Armendariz. Note that $R$ has only two idempotents $0$ and $1$, so $\sigma(Re) \subseteq Re$ for all $e \in S_1(R)$. Thus $\sigma$ is an automorphism is not a superfluous condition in the next theorem.

**Theorem 3.10.** Let $R$ be a $(\sigma,\delta)$-skew Armendariz ring with $\sigma$ an automorphism such that $Re$ is $(\sigma,\delta)$-stable for all $e \in S_1(R)$. Then $R$ is a quasi-Baer ring if and only if $R[x;\sigma,\delta]$ is a quasi-Baer ring.

**Proof.** It follows immediately from Proposition 3.2 and Proposition 3.7 $\square$

**Example 3.11.** Let $R = \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. Then $R$ is a Baer (so quasi-Baer) reduced ring. Define $\sigma: R \to R$ and $\delta: R \to R$ by $\sigma(z) = \overline{z}$ and $\delta(z) = z - \overline{z}$, where $\overline{z}$ is the conjugate of $z$. $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation. $R$ has only two idempotents $0$ and $1$, so we have the stability indicated in Theorem 3.10.

We claim that $R$ is a $(\sigma,\delta)$-skew Armendariz ring. Consider $R[x;\sigma,\delta]$. Let $p = a_0 + a_1 x + \cdots + a_n x^n$ and $q = b_0 + b_1 x + \cdots + b_m x^m \in R[x;\sigma,\delta]$. 

\[ e_{22} = 1 + 7x \text{ in } R[x;\delta], \text{ then they form a system of matrix units in } R[x;\delta]. \]

Now the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $R[x;\delta]$ is a Baer ring, but $R$ is not quasi-Baer.
Assume that \( pq = 0 \). Since \( R \) is \( \sigma \)-rigid, we have \( a_i b_j = 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \), by [7, Proposition 6]. Thus \( a_i x^i b_j x^j = 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \), because \( R[x; \sigma, \delta] \) is reduced, by [10, Theorem 3.3].

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