THE GAUSS IMAGE OF \( \lambda \)-HYPERSURFACES AND A BERNSTEIN TYPE PROBLEM

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Abstract. In this paper, our purpose is to study rigidity theorems for \( \lambda \)-hypersurfaces in Euclidean space under Gauss map. As a Bernstein type problem for \( \lambda \)-hypersurfaces, we prove that an entirely graphic \( \lambda \)-hypersurface in Euclidean space is a hyperplane.

1. Introduction

Let \( X : M \to \mathbb{R}^{n+1} \) be a smooth \( n \)-dimensional immersed hypersurface in the \( (n + 1) \)-dimensional Euclidean space \( \mathbb{R}^{n+1} \). In [1], Cheng and Wei have introduced the notation of the weighted volume-preserving mean curvature flow, which is defined as the following: a family \( X(\cdot, t) \) of smooth immersions

\[
X(\cdot, t) : M \to \mathbb{R}^{n+1}
\]

with \( X(\cdot, 0) = X(\cdot) \) is called a weighted volume-preserving mean curvature flow if

\[
\frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + H(t)
\]

holds, where

\[
\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu},
\]

\( H(t) = H(\cdot, t) \) and \( N(t) \) denote the mean curvature vector and the unit normal vector of hypersurface \( M_t = X(M^n, t) \) at point \( X(\cdot, t) \), respectively and \( N \) is the unit normal vector of \( X : M \to \mathbb{R}^{n+1} \). One can prove that the flow (1.1) preserves the weighted volume \( V(t) \) defined by

\[
V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu.
\]

The weighted area functional \( A : (-\varepsilon, \varepsilon) \to \mathbb{R} \) is defined by

\[
A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,
\]

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where $d\mu_t$ is the area element of $M$ in the metric induced by $X(t)$. Let $X(t) : M \to \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of $X$. If $V(t)$ is constant for any $t$, we call $X(t) : M \to \mathbb{R}^{n+1}$ a weighted volume-preserving variation of $X$. Cheng and Wei [1] have proved that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations if and only if there exists constant $\lambda$ such that

\[
\langle X, N \rangle + H = \lambda.
\]

An immersed hypersurface $X(t) : M \to \mathbb{R}^{n+1}$ is called a $\lambda$-hypersurface if the equation (1.2) is satisfied.

**Remark 1.1.** When $\lambda = 0$, the $\lambda$-hypersurface becomes a self-shrinker of the mean curvature flow.

**Example 1.1.** The $n$-dimensional Euclidean space $\mathbb{R}^n$ is a complete and non-compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = 0$.

**Example 1.2.** The $n$-dimensional sphere $S^n(r)$ with radius $r > 0$ is a compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$.

**Example 1.3.** For $1 \leq k \leq n-1$, the $n$-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with radius $r > 0$ is a complete and non-compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$.

It is well known that the Gauss map of hypersurfaces in $\mathbb{R}^{n+1}$ plays a very important role in study of hypersurfaces. For constant mean curvature surfaces in $\mathbb{R}^3$, a beautiful result of Hoffman, Osserman and Schoen [5] shows that the plane and the right circular cylinder are the only complete surfaces with constant mean curvature in $\mathbb{R}^3$, of which the image under Gauss map lies a closed hemisphere.

Our purpose in this paper is to study $\lambda$-hypersurfaces by Gauss map. We want to attack the following problem:

**Problem.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the image under the Gauss map is contained in an open hemisphere, then is $X : M \to \mathbb{R}^{n+1}$ a hyperplane? If the image under the Gauss map is contained in a closed hemisphere, then is $X : M \to \mathbb{R}^{n+1}$ a hyperplane or a cylinder whose cross section is an $(n-1)$-dimensional $\lambda$-hypersurface in $\mathbb{R}^n$?

For the above problem, we solve it under the assumption of proper.

**Theorem 1.1.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete properly $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the image under the Gauss map is contained in an open hemisphere, then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane. If the image under the Gauss map is contained in a closed hemisphere, then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane or a cylinder whose cross section is an $(n-1)$-dimensional $\lambda$-hypersurface in $\mathbb{R}^n$.

**Theorem 1.2.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete properly $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the image under the Gauss map is contained in $S^n \setminus S^{n-1}_+$, then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane.

**Remark 1.2.** The set $(S^n \setminus S^{n-1}_+) \cup \{p\}$ contains a great circle, here $p \in S^{n-1}_+$, and the nontrivial $\lambda$-hypersurface $S^1 \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ whose Gauss image is a great circle. Hence the Gauss image restriction in the theorem (1.2) is optimal.
Remark 1.3. Since a $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is a self-shrinker when $\lambda = 0$, we should remark that Ding, Xin and Yang [3] have proved the same results for complete proper self-shrinkers in $\mathbb{R}^{n+1}$.

Furthermore, since the image of an entire graphic $\lambda$-hypersurface under Gauss map is contained in an open hemisphere, we prove that the assertion in the problem is true for entire graphic $\lambda$-hypersurfaces.

Theorem 1.3. Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional entire graphic $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane $\mathbb{R}^{n}$.

Remark 1.4. In the case of $\lambda = 0$, that is, in the case of self-shrinkers, Ecker and Huisken [4] proved that $X : M \to \mathbb{R}^{n+1}$ is a hyperplane if it is an entire graphic self-shrinker with polynomial area growth in $\mathbb{R}^{n+1}$. Recently, Wang [6] removed the assumption of polynomial area growth (cf. Ding and Wang [2]).

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2. Proof of Theorem 1.3

Letting $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional entire graphic hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, then we can write

$$X = (x_1, \cdots, x_n, f) = (x, f),$$

where $x = (x_1, \cdots, x_n), f = f(x_1, \cdots, x_n)$. Denote the orthonormal basis of $\mathbb{R}^{n+1}$ by $\{E_1, E_2, \cdots, E_{n+1}\}$. We know that tangent vectors of $X : M \to \mathbb{R}^{n+1}$ are given by

$$e_i = E_i + f_i E_{n+1},$$

for $i = 1, 2, \cdots, n$, where $f_i = \frac{\partial f}{\partial x_i}$. The induced metric on $M$ is given by

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} + f_i f_j,$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^{n+1}$. The unit normal vector $N$ is given by

$$N = \frac{1}{\sqrt{1 + |Df|^2}} \left( - \sum_i f_i E_i + E_{n+1} \right),$$

where $Df = (f_1, \cdots, f_n)$ and $|Df|^2 = f_1^2 + \cdots + f_n^2$. The mean curvature $H$ of $M$ is given by

$$H = \sum_{i,j} g^{ij} \langle f_i E_{n+1}, N \rangle = \sum_{i,j} \frac{g^{ij} f_{ij}}{\sqrt{1 + |Df|^2}},$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. On the other hand,

$$\langle X, N \rangle = \frac{f - \sum_i x_i f_i}{\sqrt{1 + |Df|^2}}.$$
From (1.2), (2.4) and (2.5), we get

\[(2.6)\]
\[
\sum_{i,j} g_{ij} f_{ij} + \frac{f - \sum_i x_i f_i}{\sqrt{1 + |Df|^2}} = \lambda,
\]

that is,

\[(2.7)\]
\[
\sum_{i,j} g_{ij} f_{ij} = -f + \sum_i x_i f_i + \lambda \sqrt{1 + |Df|^2}.
\]

We shall consider a differential operator \(L(\lambda,f)\) defined by

\[(2.8)\]
\[
L(\lambda,f) \psi = \sum_{i,j} a_{ij} \psi_{ij} - \langle x, D\psi \rangle - \lambda \langle Df, D\psi \rangle \sqrt{1 + |Df|^2},
\]

where \(\lambda\) is constant, \(f\) and \(\psi\) are functions on \(\mathbb{R}^n\), \((a_{ij})\) is the inverse of a positive definite matrix, \(x = (x_1, \cdots, x_n) \in \mathbb{R}^n\), \(\psi_i = \frac{\partial \psi}{\partial x_i}\), \(D\psi = (\psi_1, \cdots, \psi_n)\), \(\psi_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}\). The following proposition is very important in order to prove the theorem

**Proposition 2.1.** Assume the minimum eigenvalue \(\mu(x)\) of the matrix \((a_{ij})\) at \(x \in M\) satisfies

\[(2.9)\]
\[
\lim \inf_{|x| \to +\infty} \mu(x)(|x|^2 - |\lambda||x|) > n.
\]

For a \(C^2\)-function \(\psi\), if there is a positive constant \(\varepsilon\) such that

\[(2.10)\]
\[
L(\lambda,f) \psi \geq \varepsilon \sum_{i,j} a_{ij} \psi_i \psi_j,
\]

then \(\psi\) is constant.

**Proof.** From the condition (2.9), there is a real number \(R_1\) such that if \(x \in (\mathbb{R}^n \setminus B_{R_1})\), then

\[(2.11)\]
\[
\sum_i a_{ii} < |x|^2 - |\lambda||x|,
\]

where \(B_{R_1}\) is an open ball of radius \(R_1\) centered at origin. We consider a function \(h(x)\) on \(\mathbb{R}^n\) defined by

\[(2.12)\]
\[
h(x) = \begin{cases} 
1, & |x| \leq R_0 \\
-t(|x|^2 - R_0^2) + 1, & |x| > R_0,
\end{cases}
\]

where \(R_0 \geq R_1\) and \(t, 0 < t < 1\), are constant. We know that the function \(he^C\psi\) attains its maximum at a point \(p \in \{x \in \mathbb{R}^n : h > 0\}\), where \(C < \varepsilon\) is a positive constant. If \(\psi\) is not constant in \(\mathbb{R}^n\), then there is a ball \(B_{R_0}\) with \(R_0 \geq R_1\) such that \(\psi\) is not constant in \(B_{R_0}\). Suppose the function \(\psi\) attains its maximum at a point \(q \in B_{R_0}\). We obtain from the strong maximum principle that \(\psi\) is constant since \(L(\lambda,f)\) is a linear elliptic operator and \(L(\lambda,f) \psi \geq 0\). It is a contradiction. Hence, \(\psi\) attains its maximum only on the boundary \(\partial B_{R_0}\). By the same assertion, in \(B_{\sqrt{R_0+1}}\), \(\psi\) attains its maximum only on the boundary \(\partial B_{\sqrt{R_0+1}}\). We can assume
max \( \psi = \psi(p_1) \) and \( \max_{\mathcal{B} \cap \overline{B_{R_0}}} \psi = \psi(p_2) \) where \( p_1 \in \partial B_{R_0} \) and \( p_2 \in \partial B_{\sqrt{R_0^2 + 1}} \). Then we have \( \psi(p_1) < \psi(p_2) \). Therefore, as long as we choose \( t \) sufficiently small, we obtain
\[
(2.13) \quad (he^{C\psi})(p_1) = (e^{C\psi})(p_1) < ((1-t)e^{C\psi})(p_2) = (he^{C\psi})(p_2).
\]
This means that the maximum of \( he^{C\psi} \) can only be attained in the set \( \{ x \in \mathbb{R}^n : |x| > R_0 \geq R_1 \} \).

If \( p \in \{ x \in \mathbb{R}^n : |x| > R_0 \} \), then, at point \( p \), we have
\[
(2.14) \quad h_i + Ch\psi_i = 0,
\]
and
\[
(2.15) \quad 0 \geq \sum_{i,j} a^{ij}(he^{C\psi})_{ij}
\]
since \( (a^{ij}) \) is positive definite. By a direct calculation, we have, from the definition of \( h \) and \( (2.14) \),
\[
e^{-C\psi} \sum_{i,j} a^{ij}(he^{C\psi})_{ij}
\]
\[
= \sum_{i,j} a^{ij}h_{ij} + 2C \sum_{i,j} a^{ij}h_i\psi_j + Ch \sum_{i,j} a^{ij}\psi_i\psi_j + C^2h \sum_{i,j} a^{ij}\psi_i\psi_j
\]
\[
\geq -2t \sum_{i} a^{ii} + 2C \sum_{i,j} a^{ij}(-C\psi_i)\psi_j + C^2h \sum_{i,j} a^{ij}\psi_i\psi_j
\]
\[
+ Ch \left( (x, D\psi) + \varepsilon \sum_{i,j} a^{ij}\psi_i\psi_j + \lambda \frac{\langle Df, D\psi \rangle}{\sqrt{1 + |Df|^2}} \right)
\]
\[
(2.16) \quad = -(x, Dh) - 2t \sum_{i} a^{ii} + C(\varepsilon - C)h \sum_{i,j} a^{ij}\psi_i\psi_j + Ch\lambda \frac{\langle Df, D\psi \rangle}{\sqrt{1 + |Df|^2}}
\]
\[
= 2t(|x|^2 - \sum_{i} a^{ii}) + C(\varepsilon - C)h \sum_{i,j} a^{ij}\psi_i\psi_j - \lambda \sum_{i} \frac{f_i h_i}{\sqrt{1 + |Df|^2}}
\]
\[
= 2t \left( |x|^2 - \sum_{i} a^{ii} + \lambda \sum_{i} \frac{f_i x_i}{\sqrt{1 + |Df|^2}} \right) + C(\varepsilon - C)h \sum_{i,j} a^{ij}\psi_i\psi_j
\]
\[
\geq 2t \left( |x|^2 - \sum_{i} a^{ii} - |\lambda||x| \right) + C(\varepsilon - C)h \sum_{i,j} a^{ij}\psi_i\psi_j.
\]
Since \( p \in \{ x \in \mathbb{R}^n : |x| > R_0 \} \), \( R_0 \geq R_1 \) and \( C < \varepsilon \), then we obtain from \( (2.15) \) and \( (2.16) \)
\[
0 \geq e^{-C\psi} \sum_{i,j} a^{ij}(he^{C\psi})_{ij}
\]
\[
(2.17) \quad \geq 2t \left( |x|^2 - \sum_{i} a^{ii} - |\lambda||x| \right) + C(\varepsilon - C)h \sum_{i,j} a^{ij}\psi_i\psi_j > 0.
\]
This is a contradiction. Hence, \( he^{C\psi} \) does not attain its maximum in \( \{ x \in \mathbb{R}^n : |x| > R_0 \geq R_1 \} \). Thus, \( \psi \) must be constant. \( \square \)

**Proof of Theorem 1.3.** Since \( (g_{ij}) = (\delta_{ij} + f_if_j) \) is the induced metric, we know that \( (g_{ij}) \) a positive definite matrix. Taking \( (a_{ij}) = (g_{ij}) = (\delta_{ij} + f_if_j) \) in the proposition 2.1 we know \( (g_{ij}) \) satisfies the condition (2.9). Putting \( \psi = \log \det(g_{ij}) \), we have

\[
\psi_i = \sum_{p,q} g^{pq}_{ij} \frac{\partial g_{pq}}{\partial x_i}, \quad \psi_{ij} = \sum_{p,q} \left( \frac{\partial g_{pq}}{\partial x_j} \frac{\partial g_{pq}}{\partial x_i} + g^{pq} \frac{\partial^2 g_{pq}}{\partial x_i \partial x_j} \right).
\]

By a direct calculation, we obtain

\[
\frac{\partial g_{ip}}{\partial x_l} = -\sum_{j,k} g^{ij}_{jk} \frac{\partial g_{jk}}{\partial x_l},
\]

\[
\sum_{i,j} g^{ij} \psi_{ij} = -2 \sum_{i,j,p,k,q,l} g^{ij}_{pk} g^{ql} f_{pi} f_q f_{kj} f_l - 2 \sum_{i,j,p,k,q,l} g^{ij}_{pk} g^{ql} f_{qi} f_p f_{kj} f_l
\]
\[
+ 2 \sum_{i,j,p,q} g^{ij}_{pq} f_{pi} f_q + 2 \sum_{i,j,p,q} g^{ij}_{pq} f_{pi} f_q.
\]

From (2.21), we get

\[
\frac{\partial (\sum_{i,j} g^{ij} f_{ij})}{\partial x_p} = \langle x, Df_p \rangle + \lambda \sum_i \frac{f_i f_{ip}}{\sqrt{1 + |Df|^2}}.
\]

Hence, we have

\[
\sum_{i,j} g^{ij} f_{ijp} = \langle x, Df_p \rangle + \lambda \sum_i \frac{f_i f_{ip}}{\sqrt{1 + |Df|^2}} + 2 \sum_{i,j,k,l} g^{ik} g^{jl} f_{kp} f_{lj} f_{ij}.
\]

From (2.20) and (2.22), we obtain

\[
\sum_{i,j} g^{ij} \psi_{ij} = -2 \sum_{i,j,p,k,q,l} g^{ij}_{pk} g^{ql} f_{pi} f_q f_{kj} f_l + 2 \sum_{i,j,p,k,q,l} g^{ij}_{pk} g^{ql} f_{qi} f_p f_{kj} f_l
\]
\[
+ 2 \sum_{i,j,p,q} g^{ij}_{pq} f_{pi} f_q + 2 \sum_{p,q} g^{pq} f_q \langle x, Df_p \rangle
\]
\[
+ 2\lambda \sum_{i,p,q} g^{pq} f_q \frac{f_i f_{ip}}{\sqrt{1 + |Df|^2}}.
\]

On the other hand, because of

\[
- \lambda \frac{\langle Df, D\psi \rangle}{\sqrt{1 + |Df|^2}} = -2\lambda \sum_{i,p,q} g^{pq} f_i f_j f_{pi}.
\]
we have

\[ L(\lambda, f)\psi = \sum_{i,j} g^{ij} \psi_{ij} - \langle x, D\psi \rangle - \lambda \frac{\langle Df, D\psi \rangle}{\sqrt{1 + |Df|^2}} \]

(2.25)

\[ = -2 \sum_{i,j,p,k,l} g^{ij} g^{pk} g^{ql} f_{pi} f_{qj} f_{kj} f_{li} + 2 \sum_{i,j,p,k,l} g^{ij} g^{pk} g^{ql} f_{qi} f_{pj} f_{kj} f_{li} + 2 \sum_{i,j,p,q} g^{ij} g^{pq} f_{pi} f_{qj}. \]

At any fixed point, we can choose a coordinate system \( \{x_1, \ldots, x_n\} \) such that

(2.26)

\[ Df = (f_1, 0, \ldots, 0). \]

Then we have from (2.25) and (2.26)

\[ L(\lambda, f)\psi = -2 \sum_{i,p,q} \frac{f_i^2 (f_{pi})^2}{(1 + f_i^2)(1 + f_p^2)(1 + f_q^2)} + 2 \sum_{i,p} \frac{(f_{pi})^2}{(1 + f_i^2)(1 + f_p^2)} + 2 \sum_{i,p,q} \frac{f_{pi} f_{qi} f_{pi} f_{qi}}{(1 + f_i^2)(1 + f_p^2)(1 + f_q^2)} + 2 \sum_{i,p} \frac{(f_{pi})^2}{(1 + f_i^2)(1 + f_p^2)(1 + f_i^2)} \]

\[ = \frac{1}{2} \sum_{i,j} g^{ij} \psi_i \psi_j + 2 \sum_{i,p} \frac{(f_{pi})^2}{(1 + f_i^2)(1 + f_p^2)(1 + f_i^2)} \]

\[ = \frac{1}{2} \sum_{i,j} g^{ij} \psi_i \psi_j. \]

Thus, at any point, we have

(2.27)

\[ L(\lambda, f)\psi \geq \frac{1}{2} \sum_{i,j} g^{ij} \psi_i \psi_j. \]

From the proposition 2.1 we have \( \psi \) is constant. Therefore, \( f_{pi} = 0 \) for any \( p \) and \( i \) from (2.27). Hence \( f \) is a linear function, that is, \( X : M \to \mathbb{R}^{n+1} \) is a hyperplane.

\[ \square \]

### 3. Proofs of Theorem 1.1 and Theorem 1.2

In order to prove the theorem 1.1 and the theorem 1.2 we need the following lemma due to Cheng and Wei \([1]\).

**Lemma 3.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be a complete and non-compact properly immersed \( \lambda \)-hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). Then, there is a positive constant \( C \) such that for \( r \geq 1 \),

(3.1) \[ \text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{2} - 2\beta - \inf H^2}. \]
where $B_r(0)$ is a round ball in $\mathbb{R}^{n+1}$ with radius $r$ and centered at the origin, $\beta = \frac{1}{4} \inf(\lambda - H)^2$.

The next lemma is essentially due to Ding, Xin and Yang [3].

**Lemma 3.2.** Let $X : M \rightarrow \mathbb{R}^{n+1}$ be a complete immersed $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then, its Gauss map is a $e^{-\frac{|X|^2}{2}}$-weighted harmonic map.

By using the above two lemmas and the same assertion as that of [3], we can give the proofs of the theorem 1.1 and the theorem 1.2.

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