Floquet-heating-induced non-equilibrium Bose condensation in an open optical lattice

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Periodically driven quantum systems suffer from heating via resonant excitation. While such Floquet heating guides a generic isolated system towards the infinite temperature state, a driven open system, coupled to a thermal bath, will approach a non-equilibrium steady state, which is determined by the interplay of driving and dissipation. Here, we show that this interplay can give rise to the counterintuitive effect that Floquet heating can induce Bose condensation. We consider a one-dimensional Bose gas in an optical lattice of finite extent, which is coupled weakly to a three-dimensional thermal bath given by a second atomic species. The bath temperature T lies well above the crossover temperature, below which the majority of the system’s particles form a (finite-size) Bose condensate in the ground state. However, when a strong local potential modulation is switched on, which resonantly excites the system, a non-equilibrium Bose condensate is formed in a state that decouples from the drive. Our predictions, which are based on a microscopic model that is solved using kinetic equations of motion derived from Floquet-Born-Markov theory, can be probed under realistic experimental conditions.

Introduction.— Floquet engineering is a powerful means for controlling the properties of isolated quantum systems [1–4]. The idea is to employ time-periodic forcing, so that the coherent evolution of the system in stroboscopic steps of the driving period acquires interesting novel properties. Prominent examples include the control of phase transitions [5–9], the engineering of artificial magnetic fields and topological band structures in systems of charge-neutral particles, such as atoms or photons [10–17], as well as the realization of so-called anomalous topological states that cannot exist in undriven systems [18–21]. Nevertheless, periodically driven isolated many-body systems also suffer from heating, as it is caused by unwanted resonant excitation processes [22–28]. This type of Floquet heating will generically guide an isolated system towards an infinite-temperature state, corresponding to eigenstate thermalization without energy conservation [29, 30]. However, when a periodically driven quantum system is coupled to a bath [31–45], it will not heat up to infinite temperature, but approach a non-equilibrium steady state. It was predicted that in this way, for instance, thermal states of the time-averaged Hamiltonian based on Floquet-Born-Markov theory [31, 32] and provide an intuitive explanation of the effect.

The System.—We consider a system of N noninteracting bosonic atoms in a species-selective one-dimensional optical lattice with strong transverse confinement. It is described by the tight-binding Hamiltonian

$$H_0 = -\sum_{i=1}^{M-1} (\hat{a}_{i+1}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+1}) = \sum_k \varepsilon_k \hat{b}_k^\dagger \hat{b}_k,$$  

with tunnelling parameter J, number of lattice sites M (assuming a box-like confinement in lattice direction), and annihilation operator \(\hat{a}_i\) for a boson on lattice site \(i\). The eigenmodes, with annihilation operators \(\hat{b}_k = \sum_i (\psi_k i) \hat{a}_i\), are characterized by wavefunctions \(\langle i | \psi_k \rangle = \sqrt{2/(M+1)} \sin(kai)\) and energies \(\varepsilon_k = -2J \cos(ka)\), with lattice spacing \(a\) and wavenumbers \(k = \nu\pi/[a(M+1)]\) with \(\nu = 1,\ldots,M\). Additionally, the system is subjected to a local time-periodic potential modulation of amplitude A and angular frequency \(\omega\) on lattice site \(\ell\) [cf. Fig. 1(a)], so that the total Hamiltonian reads

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_D(t), \quad \hat{H}_D(t) = A[1 + \cos(\omega t)] \hat{a}_\ell^\dagger \hat{a}_\ell.$$  

Both the longitudinal box confinement as well as the local modulation can be realized experimentally using spatial light modulators. The single-particle Floquet states of the system, \(|\phi_{\alpha}(t)\rangle = e^{-i\varepsilon_\alpha t/\hbar} |u_{\alpha}(t)\rangle\), labeled by \(\alpha = 1,\ldots,M\), are characterized both by their quasienergies \(\varepsilon_{\alpha}\) and by time-periodic Floquet modes \(|u_{\alpha}(t)\rangle = |u_{\alpha}(t+T)\rangle\), where we define the driving period \(T = 2\pi/\omega\). They are the eigenstates with eigenvalues \(\exp(-i\varepsilon_{\alpha} T/\hbar)\) of the corresponding single-particle one-cycle evolution operator from time \(t\) to \(t+T\). We also define the Floquet-mode annihilation operators \(\hat{f}_{\alpha} = \sum_i (\langle u_{\alpha}(t) | i \rangle \hat{a}_i)\).

The Bath.—The system interacts weakly with a three-dimensional Bose-Einstein condensate (BEC) given by
another atomic species at a temperature $T$ well below its critical temperature $T_c$, but well above the crossover temperature of the lattice bosons [cf. Fig. 1(a)]. Thus, for the bath the condensate depletion caused by quantum and thermal fluctuations is small, and at the same time the thermal contact with the lattice system does not lead to trivial condensation in the lattice. Similar scenarios have recently been realized experimentally [47–54]. Approximating the bath by a homogeneous BEC of $N_B$ particles in a volume $V$, so that the number density $n_B = N_B/V$ corresponds to the density of the background gas at the lattice position and applying standard Bogoliubov theory [61], the bath Hamiltonian reads $\hat{H}_B = \sum_q E_B(q) \hat{\beta}_q^\dagger \hat{\beta}_q$. Here $E_B(q) = \sqrt{\varepsilon^2 + 2G E_0(q)}$ is the energy, with $q = |\vec{q}|$, and $\hat{\beta}_q$ the annihilation operator of a Bogoliubov quasiparticle with momentum $h\vec{q}$. Additionally, $E_0(q) = \hbar^2 q^2 / (2m_B)$, with bare mass $m_B$, is the free dispersion relation and $G = gn_B$ with intrabath contact interaction strength $g$, characterizes the interactions between bath particles.

**System–Bath coupling.**—The system particles interact with the bath particles via contact interactions of strength $\gamma$, described by $\hat{H}_{SB} = \gamma \int d^3r \hat{n}_S(r) \hat{B}(\vec{r})$, with $\hat{B}(\vec{r}) = [\hat{n}_B(\vec{r}) - n_B]$, where the operator $\hat{n}_S(\vec{r}) = \sum_i w_i(\vec{r}) w_i^\dagger(\vec{r}) \hat{a}_i^\dagger \hat{a}_i$ describes the density of the system particles, with $w_i(\vec{r}) = (\vec{r} | i)$ denoting the lowest-band Wannier function of lattice site $i$, and the operator $\hat{n}_B(\vec{r})$ corresponds to the density of the (bare) bath particles. The subtraction of the average bath density $n_B$ only shifts the energies by an irrelevant constant, while it ensures $\text{Tr}_B (\hat{n}_B \hat{H}_{SB}) = 0$, as required by the Born-Markov formalism [62]. The bath state is given by $\hat{\rho}_B = \exp(-\hat{H}_B/T) / Z_B$ ($k_B = 1$ throughout the paper). In leading (linear) order with respect to the Bogoliubov modes, the bath operator is given by $\hat{B}(\vec{r}) \approx \sqrt{n_B/V} \sum_{q \neq \vec{q}} e^{i\vec{q}\cdot\vec{r}} [u_q \hat{\beta}_q^\dagger - v_q \hat{\beta}_q]$, where $u_q$ and $v_q$ are the usual Bogoliubov coefficients connecting quasiparticles with bare particles. They obey $u_q^2 - v_q^2 = 1$ and $2u_q v_q = G/E_B(q)$ (see supplemental material [62]).

We assume ultraweak system-bath coupling $\gamma$, where $\gamma$ is small when compared to all energy splittings in the system (thus, bath-mediated interactions in the system are negligible). Under this assumption, we derive a master equation for the system, using Floquet-Born-Markov theory, in combination with the secular approximation [32–34, 63–65]. Since we treat the bath operator $\hat{B}(\vec{r})$ in linear order of the quasiparticle operators, our analysis is restricted to single-phonon scattering in the bath, which dominates for low temperatures $T$ [55]. For a single particle the master equation is characterized by Golden-rule-type rates for quantum jumps from Floquet state $\beta$ to $\alpha$,

$$R_{\alpha\beta} = \frac{2\pi\gamma^2}{\hbar} \sum_{k \in \mathbb{Z}} \sum_{ij} (v_i u_j)^{(K)} (v_j u_i)^{(K)} W_{ij} (\Delta^{(K)}) ,$$

with quasienergy differences $\Delta^{(K)} = \varepsilon_{\alpha} - \varepsilon_{\beta} + K \omega$, coupling matrix elements $(v_i u_j)^{(K)} = T^{-1} \int_0^T dt e^{-iK\omega t} (u_i(t) | i) (i | u_j(t))$, and the bath-correlation function $W_{ij}(E) = J_{ij}(E)/(e^{E/T} - 1)$. It enters the spectral density $J_{ij}(E) = \text{sgn}(E) n_B q(E)^3 I_{ij,q}(q(E))/((8\pi^2 \sqrt{E^2 + G^2}))$, with the wavenumber $q(E)$ for a bath quasiparticle with energy $E$, and $I_{ij,q}(q) \approx e^{-\frac{1}{2}q^2 + 2\sin[qa(i-j)]}$ (approximating Wannier functions by oscillator ground-states with isotropic oscillator length $d$) [62].

For the case of many particles, one obtains the equation of motion [36, 39, 66],

$$\langle \dot{n}_\alpha \rangle = \sum_\beta [R_{\alpha\beta} \langle \hat{n}_\alpha + 1 | \hat{n}_\beta \rangle - R_{\beta\alpha} \langle \hat{n}_\beta + 1 | \hat{n}_\alpha \rangle] ,$$

for the expectation values of the Floquet number operators $\hat{n}_\alpha = \int_{t_0}^t |i\rangle \langle i|$. These depend on two-body correlations $\langle \hat{n}_\alpha \hat{n}_\beta \rangle$. This reflects that, even though we assume vanishing intra-system interactions, we are dealing with an interacting problem, since the coupling operator $\hat{H}_{SB}$ is
cubic in the system and bath (quasiparticle) operators [39, 62]. In order to get a closed set of kinetic equations for the mean occupations, we additionally make the mean-field approximation $\langle \hat{n}_a \hat{n}_b \rangle \approx \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle$, corresponding to a Gaussian ansatz for the system state, which, in similar situations, was previously found to show excellent agreement with the full master equation [39].

**Parameters.**— We assume $N = 200$ bosonic atoms on $M = 49$ lattice sites and consider realistic parameters, inspired by the several experimental systems [47–53] that have been reported. We here consider the case of K atoms in a Rb bath. An alternative combination of atomic species, Cs in Rb, is discussed in the supplemen- tal material [62]. We choose a lattice depth of $V_0 = 6E_R$, with recoil energy $E_R = \hbar^2 k_f^2/(2m_R)$, Potassium mass $m_R$, lattice momentum $k_L = \pi/a$ and lattice spacing $a = 395.01 \text{ nm}$ corresponding to the Rb tune-out wave-length of 790.01 nm [67, 68]. For convenience, the lattice minima are considered to be isotropic, which slightly underestimates the transverse confinement. Moreover, the bath particles have Rubidium mass and interaction parameter $G = g n_B$ with $g = 2 \pi \hbar^2 a_{Rb}/m_B$ and $a_{Rb} = 104 a_0$ and a number density of $n_B = 6.29/a^3$ (giv- ing $G = 0.05 E_R$). Within our theoretical framework, the system–bath coupling $\gamma$ enters through the time scale $t_{\text{ref}} = 16\pi \hbar^2/(m_R k_L n_B \gamma^2)$ of the relaxation dynamics. In an experiment $\gamma = 2 \pi \hbar^2 a_{Rb}/m$ is given by the K-Rb scattering length $a_{Rb}$ and reduced mass $m$. While the bath possesses the temperature $T = 2.38 J$, corre- sponding to 15 percent of its estimated critical temperature $T_{\text{ref}}^{\text{Bath}} = 15.9 J$, the system is prepared in a thermal undriven state of temperature 6.67 T.

**Relaxation dynamics of the undriven system.**— Let us first discuss the equilibration dynamics of the undriven system, $A = 0$. Here the Floquet modes and quasienergies equal the single-particle eigenstates of $\hat{H}_0$ and their energies. In Fig. 1(b) we show the time- evolution of their mean occupations $\langle \hat{n}_k \rangle = \langle \hat{b}_k^\dagger \hat{b}_k \rangle$. Since the system is one-dimensional, equilibrium Bose condensation is a finite-size effect. The corresponding crossover temperature, below which the majority of bosons occupies the single-particle ground state and the coherence length exceeds the system extent, can be estimated as $T_c^{\text{eq}} \approx 8.3 J/N^2 \approx 0.69 k_B$ [62, 69]. As the bath temperature $T = 2.38 J$ lies above this crossover temperature, at equilibrium, Bose condensation is not expected. This is confirmed in Fig. 1(b), where the dashed line indicates the thermal occupation of the ground state, which is ap- proached in the long-time limit.

We, moreover, observe that a rather fast dynamics for $t \lesssim 0.1 t_{\text{ref}}$ is followed by very slow relaxation taking much longer than $t_{\text{ref}}$. The reason for this separation of time scales becomes apparent from Fig. 2(a) where we depict snapshots of the distribution $\langle \hat{n}_k \rangle$ at intermediate times $t$ (solid lines) and compare them to the equilibrium distribution at the bath temperature (dashed line). Below (above) the critical wave numbers $k_{\text{crit}} (k_L - k_{\text{crit}})$ the absorption or emission of bath excitations is strongly sup- pressed, since in an infinite system it would not be possible to conserve both energy and momentum in such a process. Namely, for a transition $k \rightarrow k'$, the absolute value of the quasi-particle momentum obeys $q > |q_x| = |k - k'|$ (cf. Fig. 2(c)) [70], corresponding to quasi-particle energies $E_B(q) \geq E_B(|k - k'|)$ that have to match $|\varepsilon_k - \varepsilon_{k'}|$, which is impossible for too small or too large $k'$. This is illustrated in Fig. 2(c): Only after shifting the Bogoliubov dispersion $E_B(q)$ (green line) from the origin to the point $(k_{\text{crit}}, \varepsilon_{k_{\text{crit}}})$ (dashed-dotted line) or further to $k' \geq k_{\text{crit}}$, there is more than one intersection with the lattice dispersion of the system $\epsilon_k$ (red line) where for the corresponding $k'$, energy and momentum conservation can be fulfilled. Similar behavior has been found for a free impurity immersed in a superfluid [55] (where such a critical momentum only exists for $m_B > m_{Rb}$). Since for the finite lattice of $M$ sites momentum is conserved only approximately, ultimately, for $t \gg t_{\text{ref}}$, the system thermalizes with temperature $T$.

**Driven-dissipative system.**— We now turn to the dynamics of the driven system with driving amplitude $A = J$, frequency $\omega = 1.5 J$, and position $\ell = 20$. Unlike in the typical regime of Floquet engineering, the driving fre-
condensate mode $k_c$ does not correspond to the undriven ground state, emphasizing further the non-equilibrium nature of the effect.

The intriguing observation of driving-induced non-equilibrium Bose condensation appears counterintuitive at first glance. However, noting that the condensate occurs in the mode $k_c = 2\pi/(\ell a)$, which possesses a node approximately at the driven lattice site $\ell$, it becomes apparent that the condensation is a “strategy” of the system to minimize the coupling to the drive. By condensing into $k_c$, a steady state is maintained, in which the otherwise huge energy inflow resulting from Floquet heating is balanced by the outflow of energy into the environment, which is limited by the weak system-bath coupling. A similar phenomenon has recently been discussed in a related scenario, where non-equilibrium Bose condensation was not induced by local driving, but rather by the local coupling to a very hot bath [69] (note that such a local coupling to a second bath would be difficult to implement experimentally in an optical lattice system).

Thinking of the Floquet-heating as somewhat equivalent to the coupling to a very hot bath, suggests that the non-equilibrium condensation can also be understood as a thermoelectric effect. Namely, we can roughly divide the system into two parts, a cool part governed by the temperature $T$ of the environment, which is given by those few modes having wavenumbers $k = \nu 2\pi/(\ell a)$ with integer $\nu$ that decouple from the drive, and a hot part, given by the remaining state space, which suffers from Floquet heating. As a result of the Floquet heating the hot part is depleted, so that the average mode occupation dramatically increases in the cool part. This in turn leads to Bose-Einstein condensation in the mode $k_c$ possessing the lowest energy in the cool part. This division into cool and hot part is supported also by Fig. 2(d), showing for each undriven eigenstate $|\psi_k\rangle$ its maximum overlap with all Floquet states $|u_\alpha(0)\rangle$ for $A = J$. Decoupled eigenstates with $k = \nu 2\pi/(\ell a)$, which form the cold part, possess overlaps close to one with corresponding Floquet states, indicating that they are protected from the influence of the drive.

The driving induced non-equilibrium condensation is a robust effect, not relying on fine tuning of parameters. We find it equally for model parameters describing Cs atoms in a Rb bath (see supplemental material [62]). In Fig. 3 we investigate its dependence on both the driving strength (a) and the bath temperature (b), when plotting the mean occupations $\langle\hat{n}_k\rangle$ both at time $t/t_{\text{ref}} = 1$ (solid line) and in the steady state (dashed lines) versus $A/J$ and $T/T_{\text{bath}}$ (the dotted lines are the ground-state occupation in equilibrium for comparison).

More than half of the particles occupy a single mode in a large parameter regime for $A/J \gtrsim 0.5$ (a) and $T \lesssim T_{\text{c eq}} = JN/M \approx 4J \gg T_{\text{c eq}} \approx 0.69J$ (b). Here $T_{\text{c eq}}$ denotes a rough estimate of the non-equilibrium condensation temperature below which half the particles occupy...
It is obtained by assuming that within the cold part of the system, due to the large level spacing between the few decoupled modes, the bath transports essentially all $N - N'$ particles to the “cold mode” of lowest energy mode, $k_c$, while the remaining $N'$ in the hot part are equally distributed over the “hot modes” as a result of Floquet heating. $T_{\text{crit}}^\text{ne}$ then results from setting $N' = N/2$ and approximating the coupling between hot and cold modes by non-resonant ($K = 0$) bath-induced transitions [62].

**Transient condensation.**— Also for the driven system the relaxation is somewhat slower for $k < k_{\text{crit}}$ as compared to intermediate $k$. This effect starts to play a dominant role, when $k_{\text{crit}}$ exceeds the condensate mode $k_0$, which can be achieved by increasing $G/E_R$ to 0.1 (e.g. by doubling the bath density $n_B$). Now, the next mode with a node close to the driving position $\ell$, $k'_c = 2k_c$, starts acquiring large occupations during the relaxation dynamics [see Figs. 3(c) and (d)]. However, the corresponding state is only metastable and ultimately, on a much longer time scale, the system relaxes to its true steady state with a condensate at $k = k_c$ (dashed lines). Thus via both the driving position $\ell$ and the bath density one can control the state of the system.

**Conclusions.**— We have shown that the interplay between driving-induced heating and dissipation in open Floquet systems can be exploited for engineering interesting non-equilibrium steady states. In particular, Bose condensation can be achieved in excited states and at bath temperatures well above the condensation temperature at equilibrium. Our results, moreover, predict that these phenomena can be observed with ultracold atoms. It will be interesting to study also interacting systems as well as the regime of stronger system-bath coupling in future work.

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Supplemental material for “Floquet-heating-induced non-equilibrium Bose condensation in an open optical lattice”

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BATH HAMILTONIAN AND SYSTEM-BATH COUPLING

The bath is given by a system of weakly interacting bosonic atoms described by the Hamiltonian

$$\hat{H}_B = \int_r \left\{ \hat{\chi}^\dagger(\vec{r}) \left[ -\frac{\hbar^2}{2m_B} \nabla^2 + G \right] \hat{\chi}(\vec{r}) + \frac{g}{2} \hat{\chi}^\dagger(\vec{r})\hat{\chi}(\vec{r})\hat{\chi}(\vec{r}) \right\},$$

(A.1)

where we used the convention $\int_r = \int d^3r$. Moreover, $\hat{\chi}(\vec{r})$ denotes the bosonic field operator, $m_B$ the mass, and $g$ the contact interaction strength of the bath particles. To find an effective low-energy and low-temperature description of the bath Hamiltonian, we perform the usual Bogoliubov approximation. We assume that the extent of the bath is large compared to the system, so that the bath’s density $n_B$ is approximately homogeneous, where it overlaps with the system. Its bulk properties are thus approximated by assuming a homogeneous system of $N_B$ particles in a volume $V$ with periodic boundary conditions, so that $n_B = N_B/V$. After defining the annihilation operators for a bath particle of momentum $\vec{q}$ (we reserve the symbol $\vec{q}$ for bath momenta as a convention), $\hat{c}_q = \frac{1}{\sqrt{V}} \int_r e^{-i\vec{q}\cdot\vec{r}} \hat{\chi}(\vec{r})$, for temperatures $T$ well below the critical bath temperature $T_{\text{Bath}} = 2\pi\hbar^2(n_B/\zeta(3/2))^{2/3}/m_B$ (we set $k_B = 1$ throughout the manuscript and neglect the small change of $T_{\text{Bath}}$ due to the presence of interactions) and weak interactions $g \ll 4\pi\hbar^2(V/N_B)^{1/3}/m_B$ one may represent the field operator $\hat{\chi}$ as $\hat{\chi}(\vec{r}) = \chi_0 + \delta\hat{\chi}(\vec{r})$ with c-number field $\chi_0 = \sqrt{n_B/V}$ (with ground state occupation $N_0 = \langle \hat{c}_0^\dagger\hat{c}_0 \rangle$) describing the condensate, and small operator-valued fluctuations

$$\delta\hat{\chi}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{q} \neq 0} e^{i\vec{q}\cdot\vec{r}} \hat{c}_q.$$  \hfill (A.2)

Plugging the decomposition into the bath Hamiltonian, Eq. (A.1) and keeping fluctuations only up to quadratic order in $\delta\hat{\chi}(\vec{r})$, we obtain

$$\hat{H}_B \approx \int_r \delta\hat{\chi}^\dagger(\vec{r}) \left[ -\frac{\hbar^2}{2m_B} \nabla^2 + G \right] \delta\hat{\chi}(\vec{r}) + \frac{Gn_B}{2}$$

$$+ \frac{G}{2} \int_r \delta\hat{\chi}(\vec{r})\delta\hat{\chi}(\vec{r}) + \delta\hat{\chi}^\dagger(\vec{r})\delta\hat{\chi}^\dagger(\vec{r}) \right].$$

(A.3)

Here we have used $N_B = N_0 + \int \delta\hat{\chi}^\dagger(\vec{r})\delta\hat{\chi}(\vec{r})$ and introduced $G = gn_B$. We then use Eq. (A.2) and perform the standard Bogoliubov transformation

$$\hat{\beta}_q = u_q \hat{c}_q + v_q \hat{c}_q^\dagger$$

(A.4)

to bring the Hamiltonian to the form

$$\hat{H}_B = \sum_{\vec{q}} E_B(q) \hat{\beta}_q^\dagger \hat{\beta}_q,$$

(A.5)

with Bogoliubov dispersion

$$E_B(q) = \sqrt{E_0(q)^2 + 2GE_0(q),}$$

(A.6)

where $E_0(q) = \hbar^2 q^2/(2m_B)$, and the transformation follows from $u_q^2 - v_q^2 = 1$ and $u_qv_q = G/(2E_B(q))$.

The system-bath coupling Hamiltonian

$$\hat{H}_{SB}(t) = \gamma \int_r \hat{\Psi}^\dagger(\vec{r},t) \hat{\Psi}(\vec{r},t) \hat{B}(\vec{r})$$

(A.7)

can be expressed in terms of Bogoliubov quasiparticles

$$\hat{B}(\vec{r}) = \hat{\chi}^\dagger(\vec{r})\hat{\chi}(\vec{r}) - n_B$$

(A.8)

$$= \sqrt{n_B} \left[ \delta\hat{\chi}(\vec{r}) + \delta\hat{\chi}^\dagger(\vec{r}) \right] + \mathcal{O}(\delta\hat{\chi}^2)$$

(A.9)

$$= \sqrt{n_B} \sum_{\vec{q} \neq 0} (u_q - v_q) \left[ e^{i\vec{q}\cdot\vec{r}} \hat{\beta}_q^\dagger + e^{-i\vec{q}\cdot\vec{r}} \hat{\beta}_q \right] + \mathcal{O}(\delta\hat{\chi}^2).$$

(A.10)

In the last step we have employed Eq. (A.2) as well as the inverse Bogoliubov transformation $\hat{c}_q = u_q \hat{\beta}_q - v_q \hat{\beta}_q^\dagger$.

Within the tight-binding approximation, the field operator of the system is expanded in terms of the lowest-band Wannier states $|i\rangle$, $\hat{\Psi}(\vec{r},t) = \sum_{i=1}^M w_i(\vec{r}) \hat{a}_i$, with Wannier functions $w_i(\vec{r}) = |i\rangle \langle i|$ and corresponding annihilation operators $\hat{a}_i$. Thus, in leading order in $\delta\hat{\chi}$ we have

$$\hat{H}_{SB}(t) = \gamma \sum_{i,j,\vec{q} \neq 0} \hat{a}_i^\dagger \hat{a}_j \left[ \kappa_{ij}(q) \hat{\beta}_q + \kappa_{ji}(q) \hat{\beta}_q^\dagger \right],$$

(A.11)

with coefficients

$$\kappa_{ij}(q) = \sqrt{\frac{n_B E_0(q)}{VE_B(q)}} \int_r w_i(\vec{r})^* w_j(\vec{r}) e^{i\vec{q}\cdot\vec{r}} \approx \delta_{ij} \kappa_i(q).$$

(A.12)
In the last step we neglect all contributions from off-site Wannier orbitals \( w_i(\vec{r})^* w_j(\vec{r}) \approx \delta_{ij}|w_i(\vec{r})|^2 \). Thus, the system–bath coupling operator, is brought to the standard form of a Hubbard–Holstein model \([1, 2]\)

\[
\hat{H}_{\text{SB}}(t) = \gamma \sum_i \hat{n}_i \sum_{\vec{q} \neq 0} \left[ \kappa_i(q) \hat{b}_{\vec{q}} \hat{b}^\dagger_{\vec{q}} + \kappa_i(q)^* \hat{b}^\dagger_{\vec{q}} \hat{b}_{\vec{q}} \right] \equiv \gamma \sum_i \hat{n}_i \hat{B}_i. \tag{A.13}
\]

The system coupling operators \( \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \) couple to the phononic Bogoliubov modes through bath operators \( \hat{B}_i \) at each individual site \( i \).

In order to obtain a simple analytical expression for the coefficients \( \kappa_i(q) \), we approximate the Wannier functions by harmonic oscillator ground states

\[
w_i(\vec{r}) \approx \varphi^{\text{HO}}_L(x - x_i) \varphi^{\text{HO}}_T(y) \varphi^{\text{HO}}_T(z) \tag{A.14}
\]

with site position \( x_i = ia \). With the oscillator frequency in the lattice minimum \( \Omega_L = 2\sqrt{\epsilon_0 E_R/\hbar} \), one has \( \varphi^{\text{HO}}_L(x) = (d_L/\sqrt{\pi})^{-0.5} e^{-x^2/2d_L^2}/\sqrt{\pi} \) with oscillator length \( d_L = \sqrt{\hbar/m\Omega_L} \). In transverse direction \( \varphi^{\text{HO}}_T \) is defined equivalently, with oscillator length \( d_T \). This yields

\[
\kappa_i(q) = \sqrt{\frac{n_B E_0(q)}{\sqrt{E_B(q)}}} e^{iq_x x_i} e^{-\frac{1}{4}(d_T^2 q_y^2 + d_T^2 (q_x^2 + q_z^2))}. \tag{A.15}
\]

**FLOQUET-BORN-MARKOV-SECULAR APPROXIMATION**

In the system-bath coupling Hamiltonian, we have omitted contributions beyond the linear order \( \delta \hat{X}(\vec{r}) \). This means that we restrict ourselves to one-phonon scattering in the bath, which largely dominates over higher-order phonon scattering for low temperatures \( T \) \([3]\). As a consequence \( \hat{H}_{\text{SB}} \) is already in the form \( \hat{H}_{\text{SB}} = \sum_i \hat{v}_i \otimes \hat{B}_i \) required for the Floquet-Born-Markov formalism \([4–8]\). Here the part of the coupling operator acting in the system’s state space is given by \( \hat{v}_i = \hat{n}_i \). Since we are dealing with non-interacting particles in the system, we can obtain the many-particle master equation from the single-particle one. For the single-particle problem, we have to replace \( \hat{n}_i \) by \( |i\rangle \langle i| \), giving \( \hat{v}_i = |i\rangle \langle i| \). In the limit of weak system–bath coupling, where the rotating wave (or secular) approximation is valid, one finds golden rule-type rates \([4–9]\)

\[
R_{\alpha\beta} = \frac{2\pi \gamma^2}{\hbar} \text{Re} \sum_{\vec{q} \neq 0} \sum_{ij} (v_{ij}(\vec{q}))^{(K)} \kappa_i^{(K)}(q) W_{ij}(\Delta_{\alpha\beta}^{(K)}), \tag{A.16}
\]

for a bath-induced quantum jump of a single particle from Floquet state \( \beta \) to Floquet state \( \alpha \). Here we have defined the quasienergy difference \( \Delta_{\alpha\beta}^{(K)} = \varepsilon_\alpha - \varepsilon_\beta + K \hbar \omega \), and the Fourier components of the coupling matrix elements

\[
(v_{ij})^{(K)}_{\alpha\beta} = \frac{1}{T} \int_0^T dt e^{-iK\omega t} \langle u_\alpha(t) | i \rangle \langle i | u_\beta(t) \rangle, \tag{A.17}
\]

\[
= \sum_r u_{\alpha,i}^{(r)} u_{\beta,i}^{(r + K)}. \tag{A.18}
\]

Here \( T = 2\pi/\omega \) is the driving period and \( |u_\alpha(t)\rangle \) a Floquet mode, the \( r \)-th Fourier component of which is denoted by \( u_{\alpha,i}^{(r)} = \langle i | u_\alpha^{(r)} \rangle \). We have also employed the half-sided Fourier transform

\[
W_{ij}(E) = \frac{1}{\pi \hbar} \int_{0}^\infty d\tau e^{-\frac{\tau}{2\hbar}} \langle \hat{B}_i(\tau) \hat{B}_j^\dagger \rangle \tag{A.19}
\]

of the bath correlation function. Here \( \langle \cdot \rangle_B = \text{Tr}_B(\hat{n}_B \cdot) \) and where \( \hat{O}(\tau) \) indicates the operator \( \hat{O} \) in the interaction picture,

\[
\hat{O}(\tau) = e^{i(\hat{H}_B + \hat{H}_B^\dagger)} \hat{O} e^{-i(\hat{H}_B + \hat{H}_B^\dagger)} \tau. \tag{A.20}
\]

We use that the bath is in a thermal state \( \hat{n}_B = \frac{1}{Z} \exp(-\hat{H}_B/T) \), to evaluate

\[
\langle \hat{B}_i(\tau) \hat{B}_j^\dagger \rangle_B
\]

\[
= \sum_{\vec{q} \neq 0} \left( \sum_{\vec{q} \neq 0} \left[ \kappa_i(q) \kappa_j(q)^* e^{\frac{i}{\hbar} E_B(q) \tau} + \kappa_i(q) \kappa_j(q)^* \kappa_i^\dagger(q) \kappa_j^\dagger(q^*) e^{\frac{i}{\hbar} E_B(q) \tau} \right] \right)_B \tag{A.21}
\]

\[
= \int_{-\infty}^{\infty} dE J_{ij}(E) e^{\frac{i}{\hbar} E\tau} n(E), \tag{A.22}
\]

with Bose-Einstein occupation function

\[
n(E) = \frac{1}{e^{E/T} - 1}. \tag{A.23}
\]

and spectral density

\[
J_{ij}(E) = \sum_{\vec{q} \neq 0} |\kappa_i(q)\kappa_j(q)^*\delta(E - E_B(q))
\]

\[
-\kappa_i(q)\kappa_j(q)^*\delta(E - E_B(q))| \tag{A.25}
\]

Therefore, using the Sokhotski–Plemelj formula and neglecting the imaginary part of \( W_{ij} \) (giving rise to a Lamb shift, which in the secular coupling limit becomes negligible), we have

\[
W_{ij}(E) = J_{ij}(E) n(E). \tag{A.26}
\]

Finally, we take the continuum limit for the bath sum over \( \vec{q} \), \( \frac{(2\pi)^3}{V} \int d^3q \),

\[
J_{ij}(E) \overset{E \to 0}{=} \frac{n_B}{(2\pi)^3} \int d^3q \frac{E_0(q)}{E_B(q)}
\]

\[
\times e^{-\frac{1}{4}(d_T^2 q_x^2 + d_T^2 (q_x^2 + q_z^2))} \delta(E - E_B(q)), \tag{A.27}
\]
and \( J_{ij}(-E) = -J_{ij}(E^*) \). We introduce spherical coordinates with the \( x \)-axis being the polar axis and set \( \zeta = \cos \theta \), to find

\[
J_{ij}(E) = \frac{n_B}{(2\pi)^2} \int_0^\infty dq q^2 \frac{E_0(q)}{E_B(q)} I_{ij}(q) \delta(E - E_B(q)),
\]

(A.28)

with function

\[
I_{ij}(q) = \int_{-1}^1 d\zeta e^{-\frac{q^2}{2} d_L^2 (1 - \zeta^2)} e^{iq(\zeta x_i - x_j)} = I_{ij}(q^*).
\]

(A.29)

We solve Eq. (A.6) for the momentum

\[
q(E) = \frac{\sqrt{2m_B}}{\hbar} \left( \sqrt{E^2 + G^2} - G \right)^{1/2}
\]

(A.30)

of a Bogoliubov quasiparticle at Energy \( E \). This allows us to transform the differential

\[
2 q dq = \frac{2m_B}{\hbar^2} \frac{E_B}{\sqrt{E_B^2 + G^2}} dE_B.
\]

(A.31)

After transforming the \( q \)-integral into an integral over \( E_B \), we can directly evaluate the delta distribution and find

\[
J_{ij}(E) = \text{sgn}(E) \frac{n_B}{(2\pi)^2} \frac{q(E)^3}{\sqrt{E^2 + G^2}} I_{ij}(q(E)).
\]

(A.32)

Note that for small energies \( E \ll G \) we find super-ohmic behavior \( J(E) \propto E^3 \), while for \( E \gg G \) the spectral density decays again, due to the exponential decay of \( I_{ij}(q(E)) \).

It is left to evaluate the function \( I_{ij}(q) \). Since the integral over \( \zeta \) is hard to evaluate in general, we restrict us for practical reasons to the case where \( d_L = d_T \), so that we find

\[
I_{ij}(q) = e^{-\frac{1}{2} q^2 (d_L^2 - d_T^2)} \text{sinc}[q(x_i - x_j)]
\]

(A.33)

with \( \text{sinc}(x) = \sin(x)/x \). Note that we expect similar results for the dynamics also in the general case where \( d_L \neq d_T \) as long as all \( q \) fulfill \( q \ll (d_L^2 - d_T^2)^{-1/2} \). This can be seen from approximating \( e^{-\frac{1}{2} q^2 (d_L^2 - d_T^2)} \approx 1 \) in the integral of Eq. (A.29), which is a good approximation for such values \( q \).

**CONDENSATION TEMPERATURE IN EQUILIBRIUM**

These results have already been presented in the supplemental material of Ref. [10], but are included here again for completeness. Under equilibrium conditions, when the system is coupled only to the bath of temperature \( T \) (i.e. for \( A = 0 \)), Equation (4) of the main text is solved by the grand-canonical mean occupations

\[
\langle \hat{n}_k \rangle = \frac{1}{e^{(\varepsilon_k - \mu)/T} - 1}
\]

(A.34)

with chemical potential \( \mu \). When (finite-size) Bose condensation sets in, \( \mu \) approaches \( \varepsilon_{k_0} \) from below, so that the occupations of the low-energy modes with \( k \ll 1 \) can be approximated by

\[
\langle \hat{n}_k \rangle \simeq \frac{T}{\varepsilon_k - \mu} \simeq \frac{T}{Jk^2a^2 - 2J - \mu}.
\]

(A.35)

where we have used \( \varepsilon_k = -2J \cos(ka) \approx -2J + Jk^2a^2 \). The chemical potential can be expressed in terms of the occupation \( N_c = \langle \hat{n}_{k_0} \rangle \) of the ground state with wave number \( k_0 = \pi/|a(M + 1)| \),

\[
\mu = -2J + Jk_0^2a^2 - T/N_c.
\]

(A.36)

For low temperatures, the number \( N' \) of particles occupying excited states, with \( k = \nu \pi/|a(M + 1)| \), is dominated by the long-wavelength modes, so that we can approximate

\[
N' = \sum_{k \neq k_0} \langle \hat{n}_k \rangle \simeq \sum_{\nu = 2}^{\infty} \frac{1}{J\pi^2 T M^2 (\nu^2 - 1) + 1}.
\]

(A.37)

For a finite system, we define the characteristic temperature \( T_c \), where Bose condensation sets in, as the temperature for which half of the particles occupy the single-particle ground state, \( N' = N_c = N/2 \). It is given by

\[
T_c \approx \frac{c\pi^2 nJ}{2 M} \approx 8.3 \frac{nJ}{M},
\]

(A.38)

where \( n = N/M \), and \( c \approx 1.68 \) solves \( 1 = c \sum_{\nu = 2}^{\infty} 1/(\nu^2 + c - 1) \). In Fig. 1 we plot the ground-state occupation (i.e. the condensate fraction) of the tight binding chain together with the estimate (A.38) for the condensation temperature \( T_c \). The inverse dependence of \( T_c \) on the system size \( M \) reflects the well-known result that in one spatial dimension, in the thermodynamic limit Bose-Einstein condensation is suppressed by thermal long-wavelength fluctuations.

![FIG. 1. Condensate fraction \( N_c/N \) for a tight binding chain of \( M \) sites at temperature \( T \) (shading). The blue-white dotted line gives the analytical estimate, Eq. (A.38), for the condensation temperature, where half of the particles occupy the single-particle ground state.](image-url)
NONEQUILIBRIUM CONDENSATION TEMPERATURE

Here we present a very rough estimate for the characteristic temperature $T_{\text{ne}}$ for nonequilibrium condensation. As discussed in the main text, we can distinguish modes that decouple from the drive, which form the cold part of the system, and the remaining hot part of the system, which is subjected to strong resonant driving. Since the cold modes are few and equally spaced in momentum, they are well separated in energy. As a result, the bath transfers essentially all particles within the cold part to the cold mode of lowest energy, $k_0$. Within the hot part of the system, the driving mixes states of different energy, so that roughly all Floquet modes acquire the same occupation. This suggests the following approximation for the occupation numbers:

$$\langle \hat{n}_\alpha \rangle = \begin{cases} N_c & \text{for } \alpha = \alpha_c \\ N'/M & \text{else}, \end{cases}$$

(A.39)

where $N = N_c + N'$ (cf. Fig. 2(b) in the main text). Here we have neglected the excited cold modes, which are few and whose occupations are small. In the condensate regime, $N_c \sim N$, one has $N_c \gg N'/M$, and hence the dominating terms $\propto N_c$ in Eq. (4) (of the main text) read for $\alpha = \alpha_c$

$$0 = N_c \sum_{\beta \neq \alpha_c} \left( \frac{N'}{M} - R_{\beta \alpha_c} \right) + O(N_c^0),$$

(A.40)

with $A_{\alpha \beta} = R_{\alpha \beta} - R_{\beta \alpha}$. Since the factor $I_{ij}(q)$, which enters in the rates $R_{\beta \alpha_c}$, has its dominating contribution for $i = j$, we may approximate

$$R_{\alpha \beta} \approx \frac{2\pi \gamma^2}{\hbar} \sum_{K \in \mathbb{Z}} \sum_{i} \{|v_i|^{(K)}_{\alpha \beta}|^2 J(\Delta^{(K)}_{\beta \alpha_c})/e^{\Delta^{(K)}_{\beta \alpha_c}/T - 1}.$$

(A.41)

with $J(E) = J_{ii}(E)$ which is independent of $i$. This gives

$$A_{\alpha \beta} \approx \frac{2\pi \gamma^2}{\hbar} \sum_{K \in \mathbb{Z}} \sum_{i} \{|v_i|^{(K)}_{\alpha \beta}|^2 - J(\Delta^{(K)}_{\beta \alpha_c})|,$$

(A.42)

In Eq. (A.40) we divide by $(2\pi \gamma^2 N_c)/\hbar$ and have

$$0 = \sum_{\beta \neq \alpha_c} \sum_{K \in \mathbb{Z}} \sum_{i} \{|v_i|^{(K)}_{\beta \alpha_c}|^2 J(\Delta^{(K)}_{\beta \alpha_c})\left( \frac{N'}{M} \frac{1}{e^{\Delta^{(K)}_{\beta \alpha_c}/T - 1}.$$

(A.43)

Here, due to the strong driving $A$ and the low frequency $\omega$, the matrix elements $|v_i|^{(K)}_{\beta \alpha_c}$ and quasienergy differences $\Delta^{(K)}_{\beta \alpha_c}$ typically can only be determined numerically.

Nevertheless a very rough estimate can be found by requiring that the term in the brackets is zero after averaging over all states, and that the most prominent contribution stems from photon index $K = 0$,

$$0 = \left( \frac{N'}{M} \frac{1}{e^{\Delta^{(0)}_{\beta \alpha_c}/T - 1} \right) \approx \frac{N'}{M} \frac{1}{e^{\Delta^{(0)}_{\beta \alpha_c}/T - 1},$$

(A.44)

We use the convention to choose the quasienergies (which are defined modulo $\hbar \omega$ only) so that they approach the energy eigenvalues in the limit of vanishing drive ($A = 0$). Assuming that $\langle \Delta^{(0)}_{\beta \alpha_c} \rangle \approx 2J$, which corresponds to the value without the driving $A = 0$, and defining the condensation temperature $T_{\text{ne}}$ by the condition that $N_c = N/2$ (and thus $N' = N/2$), we find

$$T_{\text{ne}} = \frac{2J}{\log(2/\eta_n + 1)} \approx J_n,$$

(A.45)

with filling factor $n = N/M$.

CAESIUM IN RUBIDIUM

When considering Caesium 133 atoms immersed in Rubidium 87, as in the Kaiserslautern experiment [11–14], we find the same behaviour as discussed for the case of Potassium 39 atoms. This can be seen in Fig. 2. However, one difference is that now the time scale for the relaxation now becomes about 10 times larger, since the condensate mode lies below $k_{\text{crit}}$. 

FIG. 2. Similar plots as in the main text, but for Caesium in a Rubidium bath, $m_{133}/m_{87} = 87/137$. Other parameters: $T = 0.17T_{\text{Bath}}, n_B = 1/a^3$, $\alpha = 1000\hbar_B$, $\hbar \omega = J$, other parameters as above. $A = 0.3J$ in (a) and (d). Note that we have to choose a reference time scale that is a factor 10 larger than in the main text, $\ell_{\text{ref}} = 160\hbar \omega^2/(m_B k_L n_B \gamma^2)$, because the condensate lies below $k_{\text{crit}}$, and we have to wait long enough such that momenta below it relax.
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