Tridiagonal pairs of $q$-Serre type and their linear perturbations

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Abstract

A tridiagonal pair is an ordered pair of diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on the eigenspaces of the other in a block-tridiagonal fashion. We consider a tridiagonal pair $(A, A^*)$ of $q$-Serre type; for such a pair the maps $A$ and $A^*$ satisfy the $q$-Serre relations. There is a linear map $K$ in the literature that is used to describe how $A$ and $A^*$ are related. We investigate a pair of linear maps $B = A$ and $B^* = tA^* + (1 - t)K$, where $t$ is any scalar. Our goal is to find a necessary and sufficient condition on $t$ for the pair $(B, B^*)$ to be a tridiagonal pair. We show that $(B, B^*)$ is a tridiagonal pair if and only if $t \neq 0$ and $P(t(q - q^{-1})^{-2}) \neq 0$, where $P$ is a certain polynomial attached to $(A, A^*)$ called the Drinfel’d polynomial.

Keywords: Tridiagonal pair; Tridiagonal system; Split sequence; Linear perturbation; Drinfel’d polynomial

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1 Introduction

In this paper we consider a linear algebraic object known as a tridiagonal pair \[5\]. Roughly speaking, a tridiagonal pair is an ordered pair of diagonalizable linear maps \((A, A^*)\) on a nonzero finite-dimensional vector space, that each act on the eigenspaces of the other in a block-tridiagonal fashion. We consider a tridiagonal pair \((A, A^*)\) of \(q\)-Serre type; for such a pair the \(A\) and \(A^*\) satisfy the \(q\)-Serre relations as shown in (3) and (4) below. There is a linear map \(K\) in the literature (see [7, Section 1.1] and [2,3,9]) that is used to describe how \(A\) and \(A^*\) are related. We investigate a pair of linear maps \((B, B^*)\) of the form

\[
B = A, \quad B^* = tA^* + (1 - t)K,
\]

where \(t\) is any scalar. Our goal is to find a necessary and sufficient condition on \(t\) such that \((B, B^*)\) is a tridiagonal pair. To reach this goal, we associate with \((A, A^*)\) a polynomial \(P(x)\) called the Drinfel’d polynomial \([4,8,10]\). The degree of \(P\) plus one is equal to the number of eigenspaces of \(A\) and the number of eigenspaces of \(A^*\). We show that the pair \((B, B^*)\) is a tridiagonal pair if and only if both

\[
t \neq 0, \quad P\left(\frac{t}{(q - q^{-1})^2}\right) \neq 0.
\]

Our main result is Theorem 8.9.

The paper is organized as follows. In Section 2 we make some basic definitions and set some notation. In Section 3 we introduce the notion of a parallel system. In Section 4 we recall the notion of a tridiagonal system. In Section 5 we recall the split decomposition. In Section 6 we recall the notion of a mock tridiagonal system. In Section 7 we introduce the notion of a \(t\)-linear perturbation of a tridiagonal pair and prove a number of results about it. In Section 8 we define the Drinfel’d polynomial and use it to prove our main result.

2 Preliminaries

In this section we give some basic definitions and set our notation. Let \(K\) denote an algebraically closed field. Throughout this paper, a scalar will refer to an element of \(K\) and every vector space we mention will be understood to be over \(K\). Let \(V\) denote a nonzero finite-dimensional vector space. For a linear map \(A : V \to V\), an eigenspace refers to a nonzero subspace \(\{v \in V : Av = \lambda v\}\) for some scalar \(\lambda\). This scalar is called the eigenvalue for the given eigenspace. The map \(A\) is said to be diagonalizable whenever \(V\) is spanned by the eigenspaces of \(A\).

Definition 2.1. (See [5 Definition 1.1]) A tridiagonal pair on \(V\) is an ordered pair \((A, A^*)\) of linear maps \(A : V \to V\) and \(A^* : V \to V\) that satisfy the following conditions:

(i) \(A\) and \(A^*\) are diagonalizable.
(ii) There exists an ordering \( \{V_i\}_{i=0}^d \) of the eigenspaces of \( A \) such that
\[
A^* V_i \subseteq V_{i-1} + V_i + V_{i+1},
\]
where \( 0 \leq i \leq d \) and \( V_{-1} = V_{d+1} = 0 \).

(iii) There exists an ordering \( \{V_i^*\}_{i=0}^\delta \) of the eigenspaces of \( A^* \) such that
\[
A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*,
\]
where \( 0 \leq i \leq \delta \) and \( V_{-1}^* = V_{\delta+1}^* = 0 \).

(iv) If a subspace \( W \) of \( V \) is such that \( AW \subseteq W \) and \( A^* W \subseteq W \), then either \( W = 0 \) or \( W = V \).

The above tridiagonal pair is said to be over \( \mathcal{K} \).

With reference to the above definition, it is known that \( d = \delta \) (see [5, Lemma 4.5]), so \( A \) and \( A^* \) have the same number of eigenspaces.

Note that if \( (A, A^*) \) is a tridiagonal pair on \( V \), then so is \( (A^*, A) \).

**Remark 2.2.** For a tridiagonal pair \( (A, A^*) \), call an ordering \( \{V_i\}_{i=0}^d \) of the eigenspaces of \( A \) **standard** if it satisfies the conditions of Definition 2.1(ii). Observe that a standard ordering is not unique, as the ordering \( \{V_{d-i}\}_{i=0}^d \) is also standard. Since \( (A^*, A) \) is also a tridiagonal pair, a similar discussion applies to an ordering \( \{V_i^*\}_{i=0}^d \) of the eigenspaces of \( A^* \).

For a given tridiagonal pair \( (A, A^*) \), observe that a standard ordering of the eigenspaces of \( A \) gives an ordering \( \{\theta_i\}_{i=0}^d \) of the eigenvalues of \( A \) known as an **eigenvalue sequence** of \( (A, A^*) \).

Looking instead at the tridiagonal pair \( (A^*, A) \), we obtain the **dual eigenvalue sequence** \( \{\theta_i^*\}_{i=0}^d \). Note that since a standard ordering is not unique, neither are the eigenvalue and dual eigenvalue sequences. Indeed, \( \{\theta_{d-i}\}_{i=0}^d \) is also an eigenvalue sequence, while \( \{\theta_i^*\}_{i=0}^d \) is also a dual eigenvalue sequence.

The tridiagonal conditions impose a great deal of structure on the linear maps \( A \) and \( A^* \); for instance, the following relations must always be satisfied.

**Theorem 2.3.** (See [5, Theorem 10.1]) Let \( (A, A^*) \) denote a tridiagonal pair over \( \mathcal{K} \). Then there exist scalars \( \beta, \gamma, \gamma^*, \rho, \rho^* \) such that
\[
[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \rho A^*] = 0, \tag{1}
\]
\[
[A^*, A^2 A - \beta^* A A^* A + A A^2 - \gamma^* (A A^* + A^* A) - \rho^* A] = 0 \tag{2}
\]
where \( [X, Y] = XY - YX \).
Remark 2.4. (See [1]) The relations (1) and (2) are relevant in physics, with certain cases appearing in quantum integrable models and exactly solvable systems in statistical mechanics.

For the rest of the paper, we fix a nonzero scalar $q$ that is not a root of unity. Our primary focus is a special case of (1) and (2) known as the $q$-Serre relations. This special case is described as follows. Setting $\gamma = \gamma^* = \rho = \rho^* = 0$ and $\beta = q^2 + q^{-2}$, the relations of Theorem 2.3 become

\begin{align*}
A_3 A^* &- [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0, \quad (3) \\
A^* A_3 &- [3]_q A^2 A A^* + [3]_q A A^* A^2 - A A^* A^3 = 0, \quad (4)
\end{align*}

where $[i]_q = \frac{q^i - q^{-i}}{q - q^{-1}}$ for any $i \in \mathbb{Z}$.

Definition 2.5. We say a tridiagonal pair $(A, A^*)$ has $q$-Serre type if it satisfies relations (3) and (4).

The next result uses (3) and (4) to describe the eigenvalue sequences and dual eigenvalue sequences of a tridiagonal pair of $q$-Serre type.

Proposition 2.6. (See [14, Lemma 4.8]) Let $(A, A^*)$ be a tridiagonal pair over $K$. Then the following are equivalent:

(i) $(A, A^*)$ satisfies the $q$-Serre relations.

(ii) There exists an eigenvalue sequence $\{\theta_i\}_{i=0}^d$ for $(A, A^*)$ and a dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$ for $(A, A^*)$ such that $\theta_i = q^{2i} \theta_0$ and $\theta_i^* = q^{2i} \theta_0^*$, for $0 \leq i \leq d$.

3 Parallel Systems and Split Sequences

In this section we present some basic linear algebraic constructions and corresponding results. We begin with some notation. Let $x$ denote an indeterminate, and let $K[x]$ denote the algebra of polynomials in $x$ that have all coefficients in $K$.

Definition 3.1. Given scalars $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$, we define some polynomials in $K[x]$ as follows. For $0 \leq i \leq d$,

\begin{align*}
\eta_i &= (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-i+1}), \\
\eta_i^* &= (x - \theta_d^*)(x - \theta_{d-1}^*) \cdots (x - \theta_{d-i+1}^*), \\
\tau_i &= (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1}), \\
\tau_i^* &= (x - \theta_0^*)(x - \theta_1^*) \cdots (x - \theta_{i-1}^*).
\end{align*}

A straightforward algebraic calculation confirms the following.
Lemma 3.2. (See [11, Lemma 5.5]) We have both
\[ \eta_d = \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\tau_i, \quad \eta^*_d = \sum_{i=0}^{d} \eta^*_{d-i}(\theta^*_0)\tau^*_i. \] (9)

Next we have some comments about linear maps. Let \( A : V \to V \) be a diagonalizable linear map with eigenvalues \( \theta_0, \theta_1, \ldots, \theta_d \). For \( 0 \leq i \leq d \), consider the primitive idempotent
\[ E_i = \prod_{j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j}. \] (10)

It is clear from (10) that \( E_i \) acts as the identity on the \( \theta_i \)-eigenspace of \( A \) and acts as the zero map on the other eigenspaces of \( A \). From this observation, \( E_i \) is the projection map onto the \( \theta_i \)-eigenspace so that
\[ E_i V = \{ v \in V : Av = \theta_i v \}. \]

The following relations are immediate:
\[ \sum_{i=0}^{d} E_i = I, \] (11)
\[ E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d), \] (12)
\[ E_i A = AE_i = \theta_i E_i \quad (0 \leq i \leq d). \] (13)

Definition 3.3. A parallel system on \( V \) is a sequence
\[ \Phi = (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E^*_i \}_{i=0}^{d}) \]
satisfying the following conditions:

(i) \( A \) and \( A^* \) are diagonalizable linear maps from \( V \) to itself.

(ii) \( \{ E_i \}_{i=0}^{d} \) is an ordering of the primitive idempotents of \( A \).

(iii) \( \{ E^*_i \}_{i=0}^{d} \) is an ordering of the primitive idempotents of \( A^* \).

We fix a parallel system \( \Phi = (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E^*_i \}_{i=0}^{d}) \) on \( V \) for the remainder of the section. For \( 0 \leq i \leq d \), let \( \theta_i \) (resp. \( \theta^*_i \)) denote the eigenvalue of \( A \) (resp. \( A^* \)) corresponding to \( E_i \) (resp. \( E^*_i \)). From (10), we obtain
\[ E_0 = \frac{\eta_d(A)}{\eta_d(\theta_0)}, \quad E^*_0 = \frac{\eta^*_d(A)}{\eta^*_d(\theta^*_0)}, \] (14)
\[ E_d = \frac{\tau_d(A)}{\tau_d(\theta_d)}, \quad E^*_d = \frac{\tau^*_d(A)}{\tau^*_d(\theta^*_d)}. \] (15)

Definition 3.4. The parallel system \( \Phi \) is said to be sharp whenever \( \dim(E^*_0 V) = 1 \).
For the remainder of this section, assume $\Phi$ is sharp. Fix an integer $i$ with $0 \leq i \leq d$ and consider the map

$$E_0^* \tau_i(A)$$
on $E_0^* V$. Note that this is a linear map from $E_0^* V$ to itself. Since $E_0^* V$ has dimension 1, it follows that this map acts as multiplication by a scalar $\chi_i$.

**Lemma 3.5.** Let $E : V \to V$ and $F : V \to V$ denote two linear maps such that $E^2 = E$ and $\dim(EV) = 1$. Then

(i) $EFE = cE$, where $c = \text{tr}(FE)$.

(ii) $\text{tr}(FE)$ is nonzero if and only if $EFE$ is nonzero.

**Proof.** Consider the restriction of the map $EF$ on $EV$, and observe that it must act by scalar multiplication since $\dim(EV) = 1$. Let this scalar be $c$, so that $EFE = cE$. Using $\dim(EV) = 1$ and linear algebra we obtain $\text{tr}(E) = 1$. Then observe that $\text{tr}(FE) = \text{tr}(FEE) = \text{tr}(EFE) = c \text{tr}(E) = c$ by commutativity of the trace, establishing (i). The proof of (ii) follows immediately from (i). \qed

**Corollary 3.6.** For $0 \leq i \leq d$,

$$\chi_i = \text{tr}(\tau_i(A)E_0^*)$$

(16)

**Proof.** Observe that $E_0^* \tau_i(A)E_0^* = \chi_i E_0^*$. By Lemma 3.5 $E_0^* \tau_i(A)E_0^* = \text{tr}(\tau_i(A)E_0^*)E_0^*$ as well, so it follows that $\chi_i = \text{tr}(\tau_i(A)E_0^*)$. \qed

**Definition 3.7.** For $0 \leq i \leq d$, let

$$\zeta_i = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \chi_i.$$

We call the sequence $\{\zeta_i\}_{i=0}^d$ the split sequence of $\Phi$. Observe that $\zeta_0 = 1$.

**Lemma 3.8.** The split sequence $\{\zeta_i\}_{i=0}^d$ of $\Phi$ satisfies

$$\zeta_d = \eta_d^*(\theta_0^*) \tau_d(\theta_d) \text{tr}(E_dE_0^*),$$

(17)

$$\sum_{i=0}^d \eta_d-i(\theta_0) \eta_d-i(\theta_0^*) \zeta_i = \eta_d^*(\theta_0^*) \eta_d(\theta_0) \text{tr}(E_0E_0^*).$$

(18)

**Proof.** To obtain (17), observe that $\zeta_d = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*) \chi_d = \eta^*(\theta_0^*) \chi_d$. From (16), we have $\chi_d = \text{tr}(\tau_d(A)E_0^*)$, so (15) yields (17), as desired.

To obtain (18), from (14) we have that $\eta_d(A) = \eta_d(\theta_0)E_0$, so

$$\eta_d^*(\theta_0^*) \eta_d(\theta_0) \text{tr}(E_0E_0^*) = \eta_d^*(\theta_0^*) \text{tr}(\eta_d(A)E_0^*).$$
Now, from the equation on the left in (9), we can expand the previous expression into

$$
\sum_{i=0}^{d} \eta_{d}^{k}(\theta_{d}^{*}) \eta_{d-i}(\theta_{0}) \text{tr}(\tau_{i}(A)E_{0}^{d}) = \sum_{i=0}^{d} \eta_{d}^{k}(\theta_{0}) \eta_{d-i}(\theta_{0}) \chi_{i},
$$

using (16) in the last simplification. By splitting \( \eta_{d}^{k}(\theta_{0}^{*}) = \eta_{d-i}(\theta_{0}^{*})(\theta_{0}^{*} - \theta_{i}^{*})(\theta_{0}^{*} - \theta_{i}^{*}) \cdots (\theta_{0}^{*} - \theta_{i}^{*}) \) and using the definition of the split sequence, we obtain (18).

**Definition 3.9.** The parameter array for \( \Phi \) is the sequence \( \{\theta_{i}\}_{i=0}^{d}, \{\theta_{i}^{*}\}_{i=0}^{d}, \{\zeta_{i}\}_{i=0}^{d} \).

### 4 Tridiagonal Systems

We now recall an object known as a tridiagonal system that conveniently packages a tridiagonal pair along with a standard ordering of its eigenspaces.

**Definition 4.1.** (See [5, Definition 2.1]) A tridiagonal system on \( V \) is a sequence

$$\Phi = (A; \{E_{i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d})$$

satisfying the following conditions:

(i) \( A \) and \( A^{*} \) are diagonalizable linear maps from \( V \) to itself.

(ii) \( \{E_{i}\}_{i=0}^{d} \) is an ordering of the primitive idempotents of \( A \).

(iii) \( \{E_{i}^{*}\}_{i=0}^{d} \) is an ordering of the primitive idempotents of \( A^{*} \).

(iv) \( E_{i}A^{*}E_{j} = 0 \) if \( |i - j| > 1 \) and \( 0 \leq i, j \leq d \).

(v) \( E_{i}^{*}AE_{j}^{*} = 0 \) if \( |i - j| > 1 \) and \( 0 \leq i, j \leq d \).

(vi) If a subspace \( W \) of \( V \) is such that \( AW \subseteq W \) and \( A^{*}W \subseteq W \), then either \( W = 0 \) or \( W = V \).

**Remark 4.2.** From the definition above, it is straightforward to observe that if \( (A; \{E_{i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d}) \) is a tridiagonal system, then so are \( (A; \{E_{d-i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d}), (A; \{E_{i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d}), \) and \( (A; \{E_{d-i}\}_{i=0}^{d}; A^{*}; \{E_{d-i}^{*}\}_{i=0}^{d}) \). These are called relatives of the original tridiagonal system.

**Definition 4.3.** Referring to Definition 4.1, we say \( d \) is the diameter of the tridiagonal system \( \Phi \).

The resemblance between Definitions 2.1 and 4.1 indicates the close relation between the notions of tridiagonal pairs and tridiagonal systems, as will be described in the following lemma.

**Lemma 4.4.** (See [5, Lemma 2.2 and Lemma 2.3]) Let \( \Phi = (A; \{E_{i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d}) \) denote a tridiagonal system on \( V \). Then \( (A, A^{*}) \) is a tridiagonal pair on \( V \). Conversely, if \( (A, A^{*}) \) is a tridiagonal pair on \( V \), then \( (A; \{E_{i}\}_{i=0}^{d}; A^{*}; \{E_{i}^{*}\}_{i=0}^{d}) \) is a tridiagonal system, where \( \{E_{i}V\}_{i=0}^{d} \) is a standard ordering of the \( A \)-eigenspaces and \( \{E_{i}^{*}V\}_{i=0}^{d} \) is a standard ordering of the \( A^{*} \)-eigenspaces.
**Definition 4.5.** We say a tridiagonal system $\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$ has $q$-Serre type if the tridiagonal pair $(A, A^*)$ has $q$-Serre type.

The following is a consequence of the assumption that $K$ is algebraically closed.

**Proposition 4.6.** (See [13, Theorem 1.3]) Suppose $(A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$ is a tridiagonal system on $V$. Then $\dim(E_0V) = \dim(E_0^*V) = 1$.

# 5 The Split Decomposition

Throughout this section we fix a tridiagonal system $\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$ on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^{d}$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^{d}$. We shall be discussing decompositions of $V$. By a decomposition of $V$, we mean a sequence of nonzero subspaces whose direct sum is $V$. For example, both $\{E_iV\}_{i=0}^{d}$ and $\{E_i^*V\}_{i=0}^{d}$ are decompositions of $V$. Another decomposition of interest to us is called the split decomposition and is described as follows.

**Definition 5.1.** A decomposition $\{U_i\}_{i=0}^{d}$ of $V$ is said to be split with respect to $\Phi$ whenever

(i) $(A - \theta_i)U_i \subseteq U_{i+1}$ for $0 \leq i \leq d$ and $U_{d+1} = 0$,

(ii) $(A^* - \theta_i^*)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$ and $U_{-1} = 0$.

The existence of a unique split decomposition with respect to $\Phi$ is confirmed by the following proposition.

**Proposition 5.2.** (See [5, Theorem 4.6]) Let $U_0, U_1, ..., U_d$ denote any subspaces of $V$. Then the following are equivalent:

(i) $U_i = (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$ for $0 \leq i \leq d$.

(ii) $\{U_i\}_{i=0}^{d}$ is a decomposition of $V$ that is split with respect to $\Phi$.

(iii) For $0 \leq i \leq d$,

\[
U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV,
\]

\[
U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V.
\]

**Proposition 5.3.** (See [5, Corollary 5.7]) Let $\{U_i\}_{i=0}^{d}$ be the decomposition of $V$ that is split with respect to $\Phi$. Then

\[
\dim(E_iV) = \dim(E_i^*V) = \dim(U_i)
\]

for $0 \leq i \leq d$. 

Lemma 5.4. Let \( \{U_i\}_{i=0}^d \) be the decomposition of \( V \) split with respect to \( \Phi \). Then for \( 0 \leq i \leq d \), we have

\[
(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I)U_0 \subseteq U_i
\]

(19)

and

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)U_i \subseteq U_0.
\]

(20)

Proof. \( (19) \) and \( (20) \) follow after repeated application of inclusions (i) and (ii) respectively from Definition 5.1.

Let \( \{U_i\}_{i=0}^d \) denote the decomposition of \( V \) that is split with respect to \( \Phi \). By definition of a tridiagonal system and Proposition 4.6, \( \Phi \) is a sharp parallel system, so we may refer to its associated split sequence. We now use the decomposition \( \{U_i\}_{i=0}^d \) to interpret this split sequence. By Proposition 5.2, we have \( U_0 = E_0^* V \). From \( (19) \) and \( (20) \), the subspace \( U_0 \) is invariant under the map

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_1 I) \cdots (A - \theta_0 I).
\]

(21)

Since \( \dim(U_0) = 1 \), it follows that \( (21) \) acts on \( U_0 \) as a scalar multiple of the identity. Next, we determine this scalar.

Lemma 5.5. Let \( \{\zeta_i\}_{i=0}^d \) denote the split sequence of \( \Phi \). Then for \( 0 \leq i \leq d \), \( \zeta_i \) is the eigenvalue for the map

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_1 I) \cdots (A - \theta_0 I)
\]

(22)

acting on \( U_0 \).

Proof. We must show that the map \( (22) \) and the map

\[
(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)E_0^* (A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I)
\]

(23)

have the same action on \( U_0 = E_0^* V \). However, since \( A^*E_0^* = \theta_0^*E_0^* \), we have

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)E_0^* = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)E_0^*.
\]

so replacing the right hand side of this equality with the left hand side in \( (23) \), we can rewrite \( (23) \) as

\[
(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)E_0^*(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I).
\]

At the same time, \( (13) \) tells us that \( E_0^* \) commutes with \( A^* \), so we may shift \( E_0^* \) to the beginning of the product to obtain

\[
E_0^*(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I).
\]

(24)
But $U_0 = E_0^*V$ is invariant under (22) while $E_0^*$ is the identity on $E_0^*V$, so (24) has the same action as (22) on $U_0$, as desired.

We mention some inequalities involving the split sequence of $\Phi$.

**Proposition 5.6.** (See [12, Lemma 6.1]) We have both
\[
\text{tr}(E_d E_0^*) \neq 0, \tag{25}
\]
\[
\text{tr}(E_0 E_0^*) \neq 0. \tag{26}
\]

**Corollary 5.7.** (See [12, Corollary 8.3]) The split sequence $\{\zeta_i\}_{i=0}^d$ of $\Phi$ satisfies
\[
\zeta_d \neq 0, \tag{27}
\]
\[
\sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i \neq 0. \tag{28}
\]

**Proof.** Observe that line (27) follows from (17) and (25), while line (28) follows from (18) and (26).

As described in the following proposition, $\Phi$ is determined up to isomorphism by its parameter array. We refer the reader to [13, Definition 5.1] for the definition of an isomorphism of tridiagonal systems.

**Proposition 5.8.** (See [13, Theorem 1.6]) Two tridiagonal systems over $\mathbb{K}$ are isomorphic if and only if they have the same parameter array.

## 6 Mock Tridiagonal Systems

To complete our preliminary discussion, we will need the notion of a mock tridiagonal system, which is obtained from the notion of a tridiagonal system by weakening the conditions in a mild way.

**Definition 6.1.** (See [6, Definition 1.4]) A mock tridiagonal system on $V$ is a sequence
\[
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
\]
satisfying the following conditions:

(i) $A$ and $A^*$ are diagonalizable linear maps from $V$ to itself.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $A$.

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i A^* E_j = 0$ if $|i - j| > 1$ and $0 \leq i, j \leq d$. 

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(v) \( E_i^* AE_j^* = 0 \) if \(|i - j| > 1 \) and \( 0 \leq i, j \leq d \).

(vi) The maps \( E_0^* E_0 E_0^* \) and \( E_0^* E_d E_0^* \) are nonzero on \( V \).

**Proposition 6.2.** (See [6, Lemma 1.5]) If \( \Phi \) is a tridiagonal system on \( V \), then \( \Phi \) is a mock tridiagonal system on \( V \).

**Definition 6.3.** Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a mock tridiagonal system on \( V \). Then \( \Phi \) is said to be **sharp** if \( \dim(E_0^* V) = 1 \).

Let \( \Phi \) denote a mock tridiagonal system that is sharp. Then \( \Phi \) is also a sharp parallel system, so we may construct the associated parameter array.

**Proposition 6.4.** (See [6, Theorem 2.7 and Proposition 3.7]) Let \( \Phi \) denote a mock tridiagonal system on \( V \) that is sharp. Then there exists a vector space \( V^\dagger \) such that \( \dim(V^\dagger) \leq \dim(V) \) and a sharp tridiagonal system \( \Phi^\dagger \) on \( V^\dagger \) such that \( \Phi \) and \( \Phi^\dagger \) share the same parameter array. Moreover, if \( \dim(V) = \dim(V^\dagger) \), then \( \Phi \) is a tridiagonal system isomorphic to \( \Phi^\dagger \).

**7 Linear Perturbations of Tridiagonal Pairs**

We now specialize to the setting of tridiagonal pairs of \( q \)-Serre type. Let \( (A, A^*) \) be such a pair. From Proposition 2.4 there exists an eigenvalue sequence \( \{q^{2i} \theta_0\}_{i=0}^d \) and a dual eigenvalue sequence \( \{q^{2i} \theta_0^*\}_{i=0}^d \). By factoring out the constants \( q^{d} \theta_0 \) and \( q^{d} \theta_0^* \) respectively and utilizing Remark 4.2, without loss of generality we may assume the eigenvalue sequence \( \{q^{2i-d} \theta_0\}_{i=0}^d \) and the dual eigenvalue sequence \( \{q^{d-2i} \theta_0^*\}_{i=0}^d \). From Lemma 4.4 there exists an associated tridiagonal system \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) such that \( E_i V \) is the \( A \)-eigenspace corresponding to the eigenvalue \( \theta_i = q^{2i-d} \) and \( E_i^* V \) is the \( A^* \)-eigenspace corresponding to the eigenvalue \( \theta_i^* = q^{d-2i} \).

For the rest of the paper, we shall fix \( \Phi \) to this tridiagonal system. To avoid trivialities, we assume that \( d \geq 1 \).

Now consider the decomposition \( \{U_i\}_{i=0}^d \) of \( V \) that is split with respect to \( \Phi \). Let \( K : V \to V \) be a linear map such that \( U_i \) is the eigenspace of \( K \) corresponding to the eigenvalue \( q^{d-2i} \) for \( 0 \leq i \leq d \).

**Lemma 7.1.** We have the relations

\[
\frac{qKA - q^{-1}AK}{q - q^{-1}} = I, \quad \frac{qK^{-1}A^* - q^{-1}A^*K^{-1}}{q - q^{-1}} = I.
\]

**(29)**

**Proof.** It suffices to show that these relations hold on each \( U_i \) for \( 0 \leq i \leq d \). Let \( i \) be given. Observe that

\[
\frac{qKA - q^{-1}AK}{q - q^{-1}} = \frac{qK(A - \theta_i I) - q^{-1}(A - \theta_i I)K}{q - q^{-1}} + \theta_i K.
\]

**(30)**

Apply each side of (30) to \( U_i \). Using the fact that on \( U_i \), we have \( K(A - \theta_i I) = q^{d-2i-2}(A - \theta_i I) \) and \( K = q^{d-2i} I \), we find that the right hand side of (30) is equal to \( I \), establishing the relation on
the left of (29). Concerning the relation on the right of (29), note that
\[
qK^{-1}A^* - q^{-1}A^*K^{-1} = \frac{qK^{-1}(A^* - \theta_i^*I) - q^{-1}(A^* - \theta_i^*I)K^{-1}}{q - q^{-1}} + \theta_i^*K^{-1}. \tag{31}
\]

Applying each side of (31) to $U_i$ and using the fact that on $U_i$, we have $K^{-1}(A^* - \theta_i^*I) = q^{2i - 2 - d}(A^* - \theta_i^*I)$ and $K^{-1} = q^{2i - d}I$, we find that the right hand side of (31) is equal to $I$. This establishes the relation on the right of (29).

**Definition 7.2.** For a scalar $t$, the \textit{$t$-linear perturbation} of the tridiagonal pair $(A, A^*)$ with respect to $\Phi$ is the ordered pair $(B, B^*)$ such that
\
B = A, \quad B^* = tA^* + (1 - t)K. \tag{32}
\

Our goal in this paper is to determine a necessary and sufficient condition on $t$ such that the $t$-linear perturbation of $(A, A^*)$ with respect to $\Phi$ is a tridiagonal pair.

**Lemma 7.3.** Referring to Definition 7.2, for $0 \leq i \leq d$ the following equation holds on $U_i$:
\[
B^* - \theta_i^*I = t(A^* - \theta_i^*I). \tag{33}
\]

Proof. Since $K = \theta_i^*I$ on $U_i$, by the equality on the right of (32) we find that on $U_i$,
\[
B^* - \theta_i^*I = tA^* + (1 - t)\theta_i^*I - \theta_i^*I = t(A^* - \theta_i^*I),
\]
as desired. \hfill \square

**Lemma 7.4.** Referring to Definition 7.2, for $0 \leq i \leq d$ we have
\[
(B^* - \theta_i^*I)U_i \subseteq U_{i-1}. \tag{34}
\]

Proof. Use condition (ii) from Definition 5.1 along with Lemma 7.3 \hfill \square

**Lemma 7.5.** Referring to Definition 7.2, for $0 \leq i \leq d$ the following equation holds on $U_i$:
\[
(B^* - \theta_1^*I) \cdots (B^* - \theta_i^*I) = t^i(A^* - \theta_i^*I) \cdots (A^* - \theta_i^*I). \tag{35}
\]
Moreover,
\[
(B^* - \theta_1^*I) \cdots (B^* - \theta_i^*I)U_i \subseteq U_0. \tag{36}
\]

Proof. The equality (34) follows from repeated use of Lemmas 7.3 and 7.4. The inclusion (35) follows from repeated use of Lemma 7.4 \hfill \square

**Lemma 7.6.** Referring to Definition 7.2, the map $B^*$ is diagonalizable with eigenvalues $\{\theta_i^*\}_{i=0}^d$. Moreover, for $0 \leq i \leq d$ the dimension of the $\theta_i^*$-eigenspace of $B^*$ is $\dim U_i$.\hfill 11
Proof. First, we show that the map
\[ \prod_{i=0}^{d} (B^* - \theta_i^* I) \] is zero on \( V \). To this end, it suffices to show (36) is the zero on \( U_i \) for \( 0 \leq i \leq d \). Let \( i \) be given. Using Lemma 7.5, we obtain that
\[ (B^* - \theta_0^* I)(B^* - \theta_1^* I) \cdots (B^* - \theta_{i-1}^* I)(B^* - \theta_i^* I) \] is zero on \( U_i \).

The map (37) is a factor of (36), so (36) is zero on \( U_i \).

We have shown that the map (36) is zero on \( V \). Let \( m(x) \) denote the minimal polynomial of \( B^* \). By our above comments, \( m(x) \) divides (36). In particular, \( m(x) \) has no repeated roots, so \( B^* \) is diagonalizable. It remains to show that for \( 0 \leq i \leq d \), the dimension of the \( \theta_i^* \)-eigenspace of \( B^* \) is \( \dim U_i \). We establish this as follows. For \( 0 \leq i \leq d \) choose a basis for \( U_i \), so that the union of all these bases gives a basis of \( V \). Observe that under this basis, by (33) the matrix representation of \( B^* \) is upper triangular, with the scalar \( \theta_i^* \) appearing on the diagonal with multiplicity \( \dim U_i \) for \( 0 \leq i \leq d \). Then \( \theta_i^* \) has multiplicity \( \dim U_i \) as a root of the characteristic polynomial of \( B^* \). The result follows.

For the rest of the paper, let \( E'_i \) denote the primitive idempotent of \( B^* \) associated with the eigenvalue \( \theta_i^* \) for \( 0 \leq i \leq d \).

Lemma 7.7. We have the equality \( U_0 = E'_0 V \).

Proof. On one hand, since \( (B^* - \theta_0^* I)U_0 = 0 \), we have the containment \( U_0 \subseteq E'_0 V \). On the other hand, Lemma 7.6 gives us that \( \dim U_0 = \dim E'_0 V \), so \( U_0 = E'_0 V \).

Lemma 7.8. The sequence \( \Phi' = (B; \{E_i\}_{i=0}^{d}; B^*; \{E'_i\}_{i=0}^{d}) \) is a sharp parallel system.

Proof. By Definition 7.2 the map \( B \) is diagonalizable with primitive idempotents \( \{E_i\}_{i=0}^{d} \). By Lemma 7.6 the map \( B^* \) is diagonalizable with primitive idempotents \( \{E'_i\}_{i=0}^{d} \). Thus, by Definition 3.3 \( \Phi' \) is a parallel system. Moreover, this parallel system is sharp because \( \dim(E'_0 V) = 1 \) by Lemma 7.7.

Definition 7.9. Referring to Lemma 7.8 we call \( \Phi' \) the \( t \)-linear perturbation of \( \Phi \).

Since \( \Phi' \) is a sharp parallel system, by our discussion in Section 3 we may refer to the split sequence \( \{\zeta_i\}_{i=0}^{d} \) associated with \( \Phi' \).

Lemma 7.10. The split sequences \( \{\zeta_i\}_{i=0}^{d} \) of \( \Phi \) and \( \{\zeta'_i\}_{i=0}^{d} \) of \( \Phi' \) are related by \( \zeta'_i = t^i \zeta_i \) for \( 0 \leq i \leq d \).

Proof. Recall from the discussion following Definition 3.4 that for \( 0 \leq i \leq d \), there exists a scalar \( \chi'_i \) such that the map
\[ E'_0 \tau_i(B) \]
acts on $E^1_0V$ as $\chi_i I$. From Definition 3.7 we have
\[ \zeta_i' = (\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i)\chi_i'. \] (38)

Applying formula (10) to $E^1_0$ we find that $E^1_0$ is equal to
\[ \prod_{j=i+1}^d \frac{B^* - \theta^*_j I}{\theta^*_0 - \theta^*_j}, \] (39)
times
\[ \prod_{j=1}^i \frac{B^* - \theta^*_j I}{\theta^*_0 - \theta^*_j}. \] (40)

Recall $E^1_0V = U_0$ from Lemma 7.7. By (19), we have that $\tau_i(B)U_0 \subseteq U_i$. By Lemma 7.5, the map (40) sends $U_i$ into $U_0$, and the map (39) acts on $U_0$ as the identity. Combining these observations with Lemma 7.5, we find that the following equation holds on $U_0$:
\[ E^1_0\tau_i(B) = \frac{t^i(A^* - \theta^*_i I) \cdots (A^* - \theta^*_1 I)(B - \theta_{i-1} I) \cdots (B - \theta_1 I)(B - \theta_0 I)}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i)}. \] (41)

Consider the map which is the common value in (41). By Lemma 5.5, this map acts on $U_0$ as
\[ \frac{t^i \zeta_i}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i)} \]
times the identity. By this and (38) we find that $\zeta_i' = t^i \zeta_i$ for $0 \leq i \leq d$. \[ \square \]

**Corollary 7.11.** The parameter array of $\Phi'$ is the sequence $(\{\theta_i\}^d_{i=0}, \{\theta^*_i\}^d_{i=0}, \{t^i \zeta_i\}^d_{i=0})$, where $(\{\theta_i\}^d_{i=0}, \{\theta^*_i\}^d_{i=0}, \{\zeta_i\}^d_{i=0})$ is the parameter array of $\Phi$.

**Proof.** The result follows from Definition 7.2, Lemma 7.6 and Lemma 7.10. \[ \square \]

**Lemma 7.12.** The maps $B$ and $B^*$ from Definition 7.3 satisfy the $q$-Serre relations.

**Proof.** We first show that
\[ B^3B^* - [3]_q B^2B^*B + [3]_q B B^*B^2 - B^*B^3 = 0. \] (42)

Expanding the left hand side of (42) using (32), we obtain $t$ times
\[ A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 \] (43)
plus $1 - t$ times
\[ A^3K - [3]_q A^2KA + [3]_q AKA^2 - KA^3. \] (44)

The expression (43) is zero since $A$ and $A^*$ satisfy the $q$-Serre relations. We can show (44) is zero by
using the relation on the left of (29) to pull all the $K$’s to the right, whence the resulting expression will be zero. We have thus verified (42).

Next, we show that

$$B^3 B - [3]_q B^2 B^* + [3]_q B^* B B^2 - B B^* = 0.$$  \hfill (45)

Expanding the left hand side of (45) using (32), we find that the expression is equal to $t^3$ times

$$A^3 A - [3]_q A^2 A^* + [3]_q A A^* A - A A^* A^3$$  \hfill (46)

plus $t^2(1 - t)$ times

$$\left(A^2 K A + A^* K A^* A + KA^2 A\right) - [3]_q \left(A^2 K A + A^* K A^* A + KA^2 A^*\right)$$

$$ + [3]_q \left(A^* A^* A K + A^* A K A^* + KA A^* A\right) - \left(A A^2 K + A A^* K A^* + AK A^* A\right)$$  \hfill (47)

plus $t(1 - t)^2$ times

$$\left(A^* K^2 A + KA^* K A + K^2 A^* A\right) - [3]_q \left(A^* K^2 A + KA^* K A + K^2 A A^*\right)$$

$$ + [3]_q \left(A^* A K^2 + K A A^* K + K A K A^*\right) - \left(A A^* K + A K A^* K + AK^2 A^*\right)$$  \hfill (48)

plus $(1 - t)^3$ times

$$K^3 A - [3]_q K^2 A K + [3]_q K A K^2 - AK^3.$$  \hfill (49)

We now show that each of (46), (47), (48), and (49) is zero. First, observe that (46) is zero since $A$ and $A^*$ satisfy the $q$-Serre relations. For (47), we may use both relations in (29) to pull all instances of $K$ to the right, upon which the resulting expression will be zero. A similar calculation establishes that (48) is zero. Finally, for (49), we may once more use the relation on the left of (29) to pull all the $K$’s to the right, whence the resulting expression is zero.

Having established (46), (47), (48), and (49) are all zero, we obtain the relation (45). It follows that $B$ and $B^*$ satisfy the $q$-Serre relations. \hfill \Box

**Lemma 7.13.** We have that

$$B^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V$$  \hfill (50)

for all $0 \leq i \leq d$, where $E_{-1} = E_{d+1} = 0$, and

$$B E'_i V \subseteq E'_{i-1} V + E'_i V + E'_{i+1} V$$  \hfill (51)

for all $0 \leq i \leq d$, where $E'_{-1} = E'_{d+1} = 0$.

**Proof.** By Lemma 7.12 $B$ and $B^*$ satisfy the $q$-Serre relations. In particular we have

$$B^3 B^* - [3]_q B^2 B^* B + [3]_q B B^* B^2 - B^* B^3 = 0.$$
Observe that for $0 \leq i \leq d$,

$$
0 = (B^3B^* - [3]_q B^2B^* B + [3]_q BB^* B^2 - B^* B^3) E_i = B^3B^* E_i - [3]_q \theta_i B^* E_i + [3]_q \theta_i^2 B^* E_i - \theta_i^3 B^* E_i = (B - q^2 \theta_i I)(B - \theta_i I)(B - q^{-2} \theta_i I) B^* E_i.
$$

Therefore

$$
B^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V
$$

for all $0 \leq i \leq d$. We have shown the inclusion (50). The inclusion (51) is similarly shown. □

Let us summarize our results so far. For a given scalar $t$, consider the elements $B$ and $B^*$ as in Definition 7. We wish to determine when $(B, B^*)$ is a tridiagonal pair. Lemma 7.12 showed $B$ and $B^*$ satisfy the $q$-Serre relations, while Lemmas 7.16 and 7.13 showed they satisfy conditions (i)–(iii) of Definition 2.1. It remains to determine when $B$ and $B^*$ satisfy condition (iv) of Definition 2.1. This will be done in the next section.

## 8 The Drinfel’d Polynomial

Recall the tridiagonal system $\Phi$ from above Lemma 7.1. The elements $A$ and $A^*$ from $\Phi$ form a tridiagonal pair on $V$ that has $q$-Serre type. For a given scalar $t$ consider the $t$-linear perturbation $(B, B^*)$ of $(A, A^*)$ as shown in (32). Our aim is to find a necessary and sufficient condition on $t$ for $(B, B^*)$ to be a tridiagonal pair. To this end, we define the following polynomial. From now on let $x$ denote an indeterminate. Recall the scalar $q$ from below Remark 2.4.

**Definition 8.1.** (See [4, Section 3.4] and [10, Definition 4.2]) Given a sequence of scalars $\{\zeta_i\}_{i=0}^d$, the corresponding Drinfel’d polynomial is given by

$$
P(x) = \sum_{i=0}^d \frac{(-1)^i \zeta_i x^i}{([i]_q)!^2}, \quad (52)
$$

where

$$
[i]_q = \prod_{n=1}^i [n]_q.
$$

The Drinfel’d polynomial provides a concise interpretation of (28) for $\Phi$ that will be useful in proving our main theorem.

**Lemma 8.2.** For a sequence of scalars $\{\zeta_i\}_{i=0}^d$ the corresponding Drinfel’d polynomial $P$ satisfies

$$
\sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}^*(\theta_0^*) \zeta_i = (-1)^d ([d]_q)!^2 (q - q^{-1})^{2d} P \left( \frac{1}{(q - q^{-1})^2} \right),
$$

where $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{d-2i}$ for $0 \leq i \leq d$. We are using the notation (5)–(8).
Proof. We first compute \( \eta_{d-i}(\theta_0) \) and \( \eta_{d-i}^*(\theta_0^*) \) for \( 0 \leq i \leq d \). Let \( i \) be given. Expanding \( \eta_{d-i}(\theta_0) \) in terms of \( q \) using (5) yields

\[
\eta_{d-i}(\theta_0) = (q^{-d} - q^d)(q^{-d} - q^{d-2}) \ldots (q^{-d} - q^{2i+2-d}).
\]

Similarly, expanding \( \eta_{d-i}^*(\theta_0^*) \) in terms of \( q \) using (6) yields

\[
\eta_{d-i}^*(\theta_0^*) = (q^d - q^{-d})(q^d - q^{-d+2}) \ldots (q^d - q^{d-2i-2}).
\]

Using the algebraic identity

\[
(q^{-d} - q^{d-2j})(q^d - q^{2j-d}) = -(q^{d-j} - q^{-j-d})^2, \quad j \in \mathbb{Z},
\]

we obtain

\[
\eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*) = (-1)^{d-i} \left( \frac{[q]^i_d}{[i]^i_d} \right)^2 (q - q^{-1})^{2d-2i}.
\]

The result follows in view of (52).

Corollary 8.3. Let \( P \) denote the Drinfel’d polynomial associated with the split sequence \( \{\zeta_i\}_{i=0}^d \) of \( \Phi \). Then \( P \) satisfies

\[
P \left( \frac{1}{(q - q^{-1})^2} \right) \neq 0.
\]

Proof. The result follows from Corollary 5.7 and Lemma 8.2.

Lemma 8.4. Let \( P \) and \( P' \) denote the Drinfel’d polynomials associated with the split sequences \( \{\zeta_i\}_{i=0}^d \) of \( \Phi \) and \( \{\zeta'_i\}_{i=0}^d \) of \( \Phi' \) respectively. Then

\[
P'(x) = P(tx).
\]

Proof. The result follows from Lemma 7.10 and (52).

Lemma 8.5. Assume that \( \Phi' \) is a tridiagonal system. Then both

\[
t \neq 0, \quad (53)
\]

\[
P \left( \frac{t}{(q - q^{-1})^2} \right) \neq 0, \quad (54)
\]

where \( P \) refers to the Drinfel’d polynomial associated with the split sequence \( \{\zeta_i\}_{i=0}^d \) of \( \Phi \).

Proof. First we show (53). Since \( \Phi' \) is a tridiagonal system, from (27) we have that \( \zeta'_d \neq 0 \), where \( \{\zeta'_i\}_{i=0}^d \) is the split sequence of \( \Phi' \). By Lemma 7.10, \( \zeta'_d = t^d \zeta_d \), so it follows that \( t \neq 0 \). We have shown (53).
Next we show (54). By Corollary 8.3 we have

\[ P'(\frac{1}{(q^{q-1})^2}) \neq 0, \quad (55) \]

where \( P' \) refers to the Drinfel’d polynomial associated with the split sequence \( \{\zeta'_{i}\}_{i=0}^{d} \). By Lemma 8.4, we have \( P'(x) = P(tx) \). By this and (55) we obtain (54). We have shown (53) and (54), as desired.

Referring to Lemma 8.5, let us now reverse the logical direction.

**Lemma 8.6.** Assume that both

\[ t \neq 0, \quad (56) \]

\[ P(\frac{t}{(q^{q-1})^2}) \neq 0, \quad (57) \]

where \( P \) denotes the Drinfel’d polynomial associated with the split sequence \( \{\zeta_{i}\}_{i=0}^{d} \) of \( \Phi \). Then \( \Phi' \) is a sharp mock tridiagonal system.

**Proof.** Recall \( \Phi' \) is a sharp parallel system from Lemma 7.8 so it suffices to verify that \( \Phi' \) satisfies conditions (iv)-(vi) of Definition 6.4. Lemma 7.13 implies that \( \Phi' \) satisfies conditions (iv) and (v). We now show that \( \Phi' \) satisfies condition (vi).

Let \( \{\zeta'_{i}\}_{i=0}^{d} \) denote the split sequence for \( \Phi' \), and let \( P' \) denote the Drinfel’d polynomial associated with \( \{\zeta'_{i}\}_{i=0}^{d} \). From Lemma 7.10 we have that \( \zeta'_i = t^i \zeta_i \) for \( 0 \leq i \leq d \). From \( \zeta'_d = t^d \zeta_d \) and (56), we have that \( \zeta'_d \neq 0 \). Using (57) and Lemma 8.3, we obtain

\[ P'(\frac{1}{(q^{q-1})^2}) \neq 0. \]

Moreover, Lemmas 3.8 and 8.2 gives us that \( \text{tr}(E_dE_0') \) and \( \text{tr}(E_0E'_d) \) are nonzero. By Lemma 3.5, the maps \( E'_0E_0E_0' \) and \( E'_0E_dE_0' \) are nonzero. Thus, \( \Phi' \) satisfies condition (vi) of Definition 6.4. We have shown that \( \Phi' \) is a sharp mock tridiagonal system, as desired.

**Proposition 8.7.** Assume that there exists a tridiagonal system \( \Phi' \) that has the same diameter, eigenvalue sequence, and dual eigenvalue sequence as \( \Phi \). Further assume that there exists a nonzero scalar \( t \) such that \( \zeta'_i = t^i \zeta_i \) for \( 0 \leq i \leq d \), where \( \{\zeta_i\}_{i=0}^{d} \) and \( \{\zeta'_i\}_{i=0}^{d} \) are the split sequences of \( \Phi \) and \( \Phi' \) respectively. Then the underlying vector spaces of \( \Phi \) and \( \Phi' \) have the same dimension. Moreover, the \( t \)-linear perturbation of \( \Phi \) is a tridiagonal system isomorphic to \( \Phi' \), and the \( t^{-1} \)-linear perturbation of \( \Phi' \) is a tridiagonal system isomorphic to \( \Phi \).

**Proof.** Recall the underlying vector space \( V \) for \( \Phi \). Let \( V' \) denote the underlying vector space for \( \Phi' \). Let \( P \) and \( P' \) denote the Drinfel’d polynomials associated with \( \{\zeta_{i}\}_{i=0}^{d} \) and \( \{\zeta'_{i}\}_{i=0}^{d} \) respectively. Since \( \Phi' \) is a tridiagonal system, Corollary 8.3 implies that

\[ P'(\frac{1}{(q^{q-1})^2}) \neq 0. \]
Recall from Definition 7.9 that $\Phi'$ is the $t$-linear perturbation of $\Phi$. By Corollary 7.11 $\Phi'$ and $\Phi^\vee$ share the same parameter array and hence the same Drinfel’d polynomial associated with the common split sequence, so by Lemma 8.4 we obtain (57). Furthermore, (56) holds by assumption, so from Lemma 8.6 we have that $\Phi'$ is a sharp mock tridiagonal system. By Proposition 6.4 there exists a vector space $V^\dagger$ with $\dim V^\dagger \leq \dim V$ and a sharp tridiagonal system $\Phi^\dagger$ on $V^\dagger$ that shares the same parameter array as $\Phi'$. The sharp tridiagonal systems $\Phi^\dagger$ and $\Phi^\vee$ have the same parameter array, so they are isomorphic by Proposition 5.8. Consequently $\dim V^\dagger = \dim V^\vee$, so $\dim V \leq \dim V^\vee$. By these comments $\dim V = \dim V^\vee$. We have shown that $\dim V = \dim V^\dagger$. By this and Proposition 6.4 we find that $\Phi'$ is a tridiagonal system isomorphic to $\Phi^\dagger$. We mentioned earlier that $\Phi^\dagger$ is isomorphic to $\Phi^\vee$, so $\Phi'$ is isomorphic to $\Phi^\vee$. Interchanging the roles of $\Phi$ and $\Phi^\vee$, we find that the $t^{-1}$-linear perturbation of $\Phi^\vee$ is a tridiagonal system isomorphic to $\Phi$.

In the statement of Proposition 8.7 we assumed that $\Phi^\vee$ exists. In the next result we find a necessary and sufficient condition for $\Phi^\vee$ to exist.

**Proposition 8.8.** For a nonzero scalar $t$, there exists a tridiagonal system $\Phi^\vee$ that satisfies the assumptions of Proposition 8.7 if and only if

$$P\left(\frac{t}{(q - q^{-1})^2}\right) \neq 0.$$  

(58)

**Proof.** Suppose (58) holds. By Lemma 8.6 we have that $\Phi'$ is a sharp mock tridiagonal system. By Corollary 7.11 $\Phi'$ has parameter array $(\{\theta_i^d\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\tau_i\}_{i=0}^d)$, where $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\tau_i\}_{i=0}^d)$ is the parameter array of $\Phi$. By Proposition 6.4 there exists a tridiagonal system $\Phi^\dagger$ sharing the same parameter array as $\Phi'$. Define the tridiagonal system $\Phi^\vee = \Phi^\dagger$ and note that $\Phi^\vee$ satisfies the assumptions in Proposition 8.7. The result is now proved in one direction.

We now consider the opposite direction. Suppose that there exists a tridiagonal system $\Phi^\vee$ that satisfies the assumptions of Proposition 8.7. Then by Proposition 8.7 $\Phi'$ is a tridiagonal system isomorphic to $\Phi^\vee$. Applying Lemma 8.5 it follows that (58) holds.

The following is our main result.

**Theorem 8.9.** Let $\Phi = (A;\{E_i\}_{i=0}^d; A^*;\{E_i^*\}_{i=0}^d)$ denote a tridiagonal system on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d = \{q^{2i-d}\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d = \{q^{d-2i}\}_{i=0}^d$. For a given scalar $t$ consider the maps $B$ and $B^*$ from (32) and the parallel system $\Phi'$ from Lemma 7.8. Then the following are equivalent:

(i) the pair $(B, B^*)$ is a tridiagonal pair on $V$;

(ii) $\Phi'$ is a tridiagonal system;

(iii) both

$$t \neq 0, \quad P\left(\frac{t}{(q - q^{-1})^2}\right) \neq 0,$$

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where \( P \) refers to the Drinfel’d polynomial associated with the split sequence \( \{ \zeta_i \}_{i=0}^d \) of \( \Phi \).

Moreover, assume that (i)–(iii) hold. Then \( \Phi' \) has eigenvalue sequence \( \{ \theta_i \}_{i=0}^d \), dual eigenvalue sequence \( \{ \theta^*_i \}_{i=0}^d \), and split sequence \( \{ t^i \zeta_i \}_{i=0}^d \). In particular, \( \Phi' \) and \((B, B^*)\) have \( q \)-Serre type.

**Proof.** (i) \( \Leftrightarrow \) (ii) By Lemmas 4.4 and 7.13
(ii) \( \Rightarrow \) (iii) By Lemma 8.5
(iii) \( \Rightarrow \) (ii) By Propositions 8.7 and 8.8

Now assume that (i)–(iii) hold. Then the final assertions of the theorem statement follow from Proposition 2.6 and Corollary 7.11 \( \square \)

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