Local subgroupoids II: Examples and properties *

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Abstract

The notion of local subgroupoid as a generalisation of a local equivalence relation was defined in a previous paper by the first two authors. Here we the notion of star path connectivity for a Lie groupoid to give an important new class of examples, generalising the local equivalence relation of a foliation, and develop in this new context basic properties of coherence, due earlier to Rosenthal in the special case. These results are required for further applications to holonomy and monodromy.

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**Introduction**

Any foliation gives rise to a local equivalence relation, defined by the path components of local intersections of small open sets with the leaves. Local equivalence relations were generalised to local subgroupoids in a previous paper by the first two authors [2], referred to hereafter as paper I. In this paper we show that a basic topological groupoid notion, that of **identity star path component**, leads easily to a local subgroupoid of a wide class of Lie groupoids and this generalises the local equivalence relation of a foliation. We define local subgroupoids \( c_1(Q,U) \) for certain open covers \( U \) of the object space of a Lie groupoid \( Q \). Further, we show that the theory of coherence, which is prominent in the papers of Rosenthal [14, 15], generalises nicely to the local subgroupoid case. The application of paper I to the holonomy and monodromy Lie groupoids of local subgroupoids (to be dealt with elsewhere) requires a condition on the local subgroupoid of having a ‘globally adapted atlas’. We develop the ‘coherence’ theory for giving conditions on \( U \) for it to be globally adapted to the local subgroupoid \( c_1(Q,U) \) (Corollary 7.9). This is related to the construction of a locally Lie groupoid from a foliation [3, 12].

We also develop similar theory for Lie groupoids with a ‘path connection’ [4, 13, 16] \( \Gamma \) leading to a local subgroupoid \( c_\Gamma(Q,U) \) for certain open covers \( U \).

This paper is strongly influenced by papers of Rosenthal [14, 15] on local equivalence relations, a concept due originally to Grothendieck and Verdier [8] in a series of exercises presented as open problems concerning the construction of a certain kind of topos. The concept was investigated more recently by Kock and Moerdijk [9, 10]. The main aims of the papers [6, 9, 10, 14, 15] are towards the connections with sheaf theory and topos theory.

The starting point of our investigation was to notice that an equivalence relation on \( X \) is a wide subgroupoid of the groupoid \( X \times X \). However an equivalence relation is just one of the standard examples of a groupoid, and so it natural to consider the corresponding theory for subgroupoids of a given groupoid \( Q \). The expectation is that this will allow applications to combinations of foliation and bundle theory, since a standard example of a Lie groupoid is the Ehresmann symmetry groupoid of a principal bundle [13].

In the case \( Q \) is the indiscrete groupoid \( X \times X \), we recover the well known concept of **local equivalence relation**.

The remarkable fact is that the theory goes very smoothly, and so suggests it is a natural generalisation of the foliation case, and one which illuminates some constructions in that area.

1 **Local subgroupoids: definitions and examples**

We first recall some definitions from [2].

Consider a groupoid \( Q \) on a set \( X \) of objects, and suppose also \( X \) has a topology. For any
open subset $U$ of $X$ we write $Q|U$ for the full subgroupoid of $Q$ on the object set $U$. Let $L_Q(U)$ denote the set of all subgroupoids of $Q|U$ with object set $U$ (these are called \textit{wide} subgroupoids of $Q|U$). For $V \subseteq U$, there is a restriction map $L_{UV}: L_Q(U) \to L_Q(V)$ sending $H$ in $L_Q(U)$ to $H|V$. This gives $L_Q$ the structure of presheaf on $X$.

We interpret the sheaf $p_Q: L_Q \to X$ constructed in the usual way from the presheaf $L_Q$.

For $x \in X$, the stalk $p_Q^{-1}(x)$ of $L_Q$ has elements the germs $[U,H]_x$ where $U$ is open in $X$, $x \in U$, $H_U$ is a wide subgroupoid of $Q|U$, and the equivalence relation $\sim_x$ yielding the germs at $x$ is that $H_U \sim_x K_V$, where $K_V$ is wide subgroupoid of $Q|V$, if and only if there is a neighbourhood $W$ of $x$ such that $W \subseteq U \cap V$ and $H_U|W = K_V|W$. The topology on $L_Q$ is the usual sheaf topology, with a sub-base of sets $\{[U,H]_x : x \in U\}$ for all open $U$ of $X$ and wide subgroupoids $H$ of $G|U$.

**Definition 1.1** A \textit{local subgroupoid} of $Q$ on the topological space $X$ is a continuous global section of the sheaf $p_Q: L_Q \to X$ associated to the presheaf $L_Q$.

Two standard examples of $Q$ are $Q = X$, $Q = X \times X$, where $X \times X$ has the multiplication $(x,y)(y,z) = (x,z)$. In the first case, $L_X$ is already a sheaf and $L_X \to X$ is a bijection. More generally, we have:

**Proposition 1.2** If $Q$ is a bundle of groups, then $L_Q$ is a sheaf.

**Proof:** By our assumption, if $U$ is a subset of $X$ then a wide subgroupoid $H|U$ of $Q|U$ is uniquely defined by the values $H(x)$ for all $x \in U$. This easily implies the usual two compatibility conditions for a sheaf. \hfill $\square$

In the case $Q$ is the indiscrete groupoid $X \times X$ the local subgroupoids of $Q$ are the local equivalence relations on $X$, as mentioned in the Introduction. It is known that $L_{X\times X}$ is in general not a sheaf [14].

**Definition 1.3** If $G$ is a wide subgroupoid of the groupoid $Q$ on $X$, then $\text{loc}(G)$ is the local subgroupoid defined by

$$\text{loc}(G)(x) = [X,G]_x.$$ 

This gives a wide and important class of local subgroupoids, but we are more interested in those which derive from connectivity considerations on a topological groupoid. For this we need to discuss atlases for local subgroupoids, which is the major way of giving a local subgroupoid.

**Definition 1.4** An \textit{atlas} $U_H = \{(U_i,H_i) : i \in I\}$ for a local subgroupoid consists of an open cover $U = \{U_i : i \in I\}$ of $X$, and for each $i \in I$ a wide subgroupoid $H_i$ of $Q|U_i$ such that the following compatibility condition holds:
$\text{Comp}(H)$: for all $i, j \in I, x \in U_i \cap U_j$ there is an open set $W$ such that $x \in W \subseteq U_i \cap U_j$ and $H_i|W = H_j|W$.

The local subgroupoid $s$ of the atlas is then well defined by $s(x) = [U_i, H_i]|x, x \in X$.

The above atlas is compatible with an atlas $U'H' = \{(U'_j, H'_j) : j \in J\}$ if for all $i \in I, j \in J$ and $x \in U_i \cap U'_j$ there is an open set $W$ such that $x \in W \subseteq U_i \cap U'_j$ and $H_i|W = H'_j|W$. Clearly, two compatible atlases define the same local subgroupoid.

It is well known from general sheaf theory that any local subgroupoid has a compatible atlas. Note also that the atlas $\{(X, H)\}$ with a single element determines the local subgroupoid $\text{loc}(H)$. So the atlas is a crucial part of the construction of a local subgroupoid $s$.

2 The star path component of a topological groupoid

A key concept for topological groups is the path component of the identity. The analogue for topological groupoids is the star identity path component.

**Definition 2.1** Let $Q$ be a topological groupoid. If $x \in \text{Ob}(Q)$ we write $\text{St}_Q x$ for the star of $Q$ at $x$, namely the union of all the $Q(x, y)$ for $y \in \text{Ob}(Q)$. The star identity path component $C_1(Q)$ of $Q$ consists of all $g \in Q$ such that if $x = \alpha(g)$ then there is a path in $\text{St}_Q x$ joining $g$ to the identity $1_x$. Such a path is called a star path. We say $Q$ is star path connected if $Q = C_1(Q)$.

**Proposition 2.2** The star identity path component of $Q$ is a subgroupoid of $Q$.

**Proof:** Write $C$ for this star identity component. Let $g \in Q(x, y), h \in Q(y, z)$ and suppose also $g, h \in C$. Then there are paths $l_t$ in $\text{St}_Q x, m_t$ in $\text{St}_Q y$ such that $l_0 = g, l_1 = 1_x, m_0 = h, m_1 = 1_y$. Hence $g.m_t$ is a path in $\text{St}_Q x$ joining $gh$ to $g$, and this composed with $l_t$ joins $gh$ to $1_x$. So $C$ is closed under composition.

If $g \in C(x, y)$ and $l_t$ joins $g$ to $1_x$ then $g^{-1}l_{1-t}$ joins $g^{-1}$ to $1_y$. So $C$ is a subgroupoid of $Q$. \qed

Note that [13, Example II.3.7, p46] gives an example where $C_1(Q)$ is not normal in $Q$.

We will later need the following.

**Proposition 2.3** Let $Q$ be star path connected and let $U$ be an open cover of $X$. Then $Q$ is generated by the subgroupoids $C_1(Q|U)$ for all $U \in U$. 

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Proof: Let \( g \in \text{St}_Q x \). Then there is a path \( \lambda \) in \( \text{St}_Q x \) from \( g \) to \( 1_x \). Let \( \mu = \beta \lambda \). By the Lebesgue covering lemma, we can write \( \mu = \mu_1 + \cdots + \mu_n \) where each \( \mu_r \) lie in an open set \( U_r \) of \( \mathcal{U} \) for \( r = 1, \ldots, n \). Then we can write \( \lambda = \lambda_1 + \cdots + \lambda_n \) where \( \beta \lambda_r = \mu_r, r = 1, \ldots, n \), and \( \lambda_r \) is a path in \( \text{St}_Q x \cap U_r \) for \( r = 1, \ldots, n \). Let \( g_r = -\lambda_r(0) + \lambda_r(1) \). Then \( g_r \in C_1(Q|U_r) \) and \( g = g_1 + \cdots + g_n \).

Now we recall some major examples of Lie groupoids.

Example 2.4 Let \( \mathcal{E} \) be a principal bundle \( p: E \to B \) with group \( \Omega \). Then \( E \times E \) is certainly a topological groupoid, and so also is its quotient \( Q = E \times_G E \) by the diagonal action of the topological group \( \Omega \). If \( b, b' \in B \), we can by choosing a point in \( p^{-1}(b) \), identify \( Q(b, b') \) with the \( \Omega \)-maps \( p^{-1}(b) \to p^{-1}(b') \). For this reason, we also write \( \text{Sym}(\mathcal{E}) \) for \( Q \). In the case \( \mathcal{E} \) is locally trivial, and assuming \( X \) is a manifold, the topology may also be constructed from this alternative description, since the \( \Omega \)-maps \( p^{-1}(b) \to p^{-1}(b') \) may, again by choosing a point in \( p^{-1}(b) \), be identified with the elements of \( \Omega \). It is this description we now use.

Consider in particular the double cover of the circle \( p: S^1 \to S^1 \) given by \( z \mapsto z^2 \). In this case \( \Omega \) is the cyclic group of order 2. Let \( Q = \text{Sym}(p) \).

This groupoid \( Q \) is star path connected. For suppose \( g \in Q(z, w) \). Let \( \lambda \) be a path of shortest length in \( S^1 \) from \( z \) to \( w \) (if \( z = -w \) then there are two such paths). Let \( u \in S^1 \) satisfy \( u^2 = z \). Since \( p \) is a covering map, there are unique paths \( \lambda^+, \lambda^- \) starting at \( u, -u \) and covering \( \lambda \). Let \( v = \lambda^+(1) \). Then \( g \) is a bijection \( \{u, -u\} \to \{v, -v\} \). If \( g(u) = v \), then the pair of paths \( \lambda^+, \lambda^- \) define a star path from the identity on \( \{u, -u\} \) to \( g \). If \( g(u) = -v \), then such a path is determined by \( \lambda' \), the shortest path joining \( z \) to \( w \) in the opposite direction round \( S^1 \), and its corresponding lifts.

However, if \( U = S^1 \) with a single point removed, then \( Q|U \) is not star-connected, since if \( \lambda \) is a path joining \( z \) to \( w \) in \( U \), then \( \lambda' \) is not a path in \( U \).

Example 2.5 Let \( \Omega \) be a Lie group acting smoothly on the right of a \( \mathcal{C}^r \)-manifold \( X \). Form the Lie action groupoid \( Q = X \rtimes \Omega \). Even if \( Q \) is star path connected, this is not necessarily so for \( Q|U \) for all open subsets \( U \) of \( X \).

3 Local subgroupoids and star path connectivity

The previous notions give us our major examples of new and interesting local subgroupoids.

Example 3.1 Consider an equivalence relation \( E \) on the space \( X \). Then for each open set \( U \) of \( X \) we have an equivalence relation \( E|U \) on \( U \) and we can consider the partition of \( U \) given by the path components of the equivalence classes of \( E|U \). In general, this will not give us a
local equivalence relation. Instead we need to assume given an open cover \( U = \{ U_i : i \in I \} \) of \( X \) satisfying the compatibility condition that for all \( i, j \in I, x \in U_i \cap U_j \) there is an open set \( W \) such that \( x \in W \subseteq U_i \cap U_j \) and the path components of \( E|W \) are the intersections with \( W \) of the path components of the classes of each of \( E|U_i, E|U_j \). The resulting local equivalence relation will be written \( c_1(E, U) \). The compatibility condition is satisfied in for example equivalence relations given by the leaves of a foliation on a manifold, and is the standard example of the local equivalence relation defined by a foliation. \( \square \)

We now consider similar questions for topological groupoids.

Of course if \( G \) is a wide subgroupoid of \( Q \), then so also is \( C_1(G) \) and then \( \text{loc}(C_1(G)) \) is a local subgroupoid of \( Q \).

Suppose \( Q \) is star path connected, that is \( Q = C_1(Q) \). Let \( X = \text{Ob}(Q) \) and let \( U \) be a subset of \( X \). In general \( Q|U \) need not be star path connected, as we show below. Further, while \( C_1(Q|U) \subseteq C_1(Q)|U \), in general we do not have equality here. Such a condition is needed locally to obtain the local subgroupoid \( c_1(Q, U) \) defined below.

**Definition 3.2** An open cover \( U = \{ U_i : i \in I \} \) of \( X \) is said to be **path compatible** with a topological groupoid \( Q \) on \( X \) if for all \( i, j \in I, x \in U_i \cap U_j \) there is an open set \( W \) such that \( x \in W \subseteq U_i \cap U_j \) and

\[
C_1(Q|U_i)|W = C_1(Q|U_j)|W.
\]

In this case, the local subgroupoid \( c_1(Q, U) \) is defined to have value \([U_i, C_1(Q|U_i)]_x\) at \( x \in U_i \).

The next proposition gives useful sufficient conditions for \( c_1(Q, U) \) to be defined.

**Proposition 3.3** Let \( Q \) be a topological groupoid on \( X \) and suppose there is an open cover \( U = \{ U_i : i \in I \} \) of \( X \) such that for all \( i, j \in I \) and \( x \in U_i \cap U_j \) there is an open set \( W_x \) such that \( x \in W_x \subseteq U_i \cap U_j \) and there are groupoid retractions \( r_{i,W_x} : Q|U_i \to Q|W_x, r_{j,W_x} : Q|U_j \to Q|W_x \) over retractions \( U_i \to W_x, U_j \to W_x \). Then a local subgroupoid \( c_1(Q, U) \) is well defined by for \( i \in I, x \in U_i, x \mapsto [U_i, C_1(Q|U_i)]_x \).

**Proof:** The retractions ensure the compatibility condition, since if \( x, y \in W \) and if \( \lambda \) is a path in \( \text{St}_{Q|U_i} \) joining \( 1_x \) to the element \( g : x \to y \) of \( Q|W \), then \( r_{i,W}\lambda \) is a path in \( \text{St}_{Q|W} \) joining \( 1_x \) to \( g \). So \( C_1(Q|U_i)|W = C_1(Q|W) \), and similarly for \( j \). \( \square \)

Let \( Q \) be a topological groupoid on \( X \). Then \( Q \) is called **locally trivial** if for all \( x \in X \) there is an open set \( U \) containing \( x \) and a section \( s : U \to \text{St}_Gx \) of \( \beta \). Thus \( \beta s = 1_U \) and for each \( y \in U, \alpha(s(y)) = x \), i.e. \( s(y) : x \to y \) in \( Q \). We recall the following standard result (see for example [13]).

**Proposition 3.4** Let \( Q \) be a topological groupoid on \( X \) and \( U \) be an open subset of \( X \). If \( s : U \to \text{St}_Qx \) is a continuous section of \( \beta \) for some \( x \in U \), then the topological groupoid \( Q|U \)
is topologically isomorphic to the product groupoid \( Q(x) \times (U \times U) \), and if \( x \in W \subseteq U \), then any retraction \( U \to W \) is covered by a retraction \( Q|U \to Q|W \).

**Proof:** Remark that the groupoid multiplication on \( Q(x) \times (U \times U) \) is defined by

\[
(g, (y, z))(h, (z, w)) = (gh, (y, w)).
\]

Define

\[
\phi: Q|U \to G(x) \times (U \times U), g \mapsto (s(y)gs(z)^{-1}, (y, z))
\]

where \( y = \alpha(g) \) and \( z = \beta(g) \). Since \( s \) is continuous, \( \phi \) is clearly an isomorphism of topological groupoids.

The last part follows easily.

We have emphasised these results, despite their simple proofs, because they have useful applications for example to manifold and bundle theory.

If \( s \) is a local subgroupoid of \( Q \) defined by an atlas \( U = \{(U_i, H_i) : i \in I\} \) and \( U \) is an open subset of \( X \) then \( s|U \) is the local subgroupoid of \( Q|U \) defined by the atlas \( U \cap U = \{(U_i \cap U, H_i|[U_i \cap U]) : i \in I\} \). It is easy to verify this is an atlas, and as a section \( s|U \) is just the restriction of \( s \) to the open subset \( U \).

Suppose now that we have the local subgroupoid \( c_1(Q, U) \) defined by the open cover \( U \), and \( U \) is an open subset of \( X \). We will later need a result which follows easily from compatibility:

**Proposition and Definition 3.5** The equality

\[
c_1(Q, U)|U = c_1(Q|U, U \cap U)
\]

holds if for any \( i, j \in I \) and \( x \in U_i \cap U_j \cap U \) there is an open set \( W \) such that \( x \in W \subseteq U_i \cap U_j \cap U \) and \( C_1(Q|U_i)|W = C_1(Q|U_j \cap U)|W \). If this condition holds for all open sets \( U \) of \( X \), then we say that the cover \( U \) is path local for \( c_1(Q, U) \).

**Remark 3.6** There is a variation of the local subgroupoid \( c_1(Q, U) \) in which the paths in \( Q \) which are used are controlled, for example to belong to a given class, or to derive from the paths in \( X \) in a specified way. We give an example of this in the next section.

4 Path connections

The purpose of this section is to give new examples of local subgroupoids with a possibility of working towards relating the concepts of holonomy in foliation theory and in bundle theory.
Let $\Lambda(X)$ denote the path space of a topological space $X$. Let $Q$ be a topological groupoid over $X$. A path connection $\Gamma$ in $Q$ is a continuous map

$$\Gamma: \Lambda(X) \rightarrow \Lambda(Q), \lambda \mapsto \Gamma(\lambda)$$

satisfying the following conditions

(i) $\alpha(\Gamma(\lambda)(t)) = \lambda(0)$ and $\beta(\Gamma(\lambda)(t)) = \lambda(t)$, $t \in [0, 1]$

(ii) the transport condition: If $\psi: [0, 1] \rightarrow [t_0, t_1] \subseteq [0, 1]$ is a homeomorphism, then

$$\Gamma(\lambda) \circ \psi = \Gamma(\lambda)(\psi(0)) \circ \Gamma(\lambda \psi).$$

The second condition means

$$\Gamma(\lambda)(\psi(t)) = \Gamma(\lambda)(\psi(0)) \circ \Gamma(\lambda \psi(t))$$

for $t \in [0, 1]$.

By taking the homeomorphism $\psi$ to be the identity map $\psi: [0, 1] \rightarrow [0, 1]$ it follows from the condition (ii) that $\Gamma(\lambda)(0) = 1_{\lambda(0)}$. Let $\lambda, \mu \in \Lambda(X)$ and $\lambda(1) = \mu(0)$, that is the composition $\lambda + \mu$ is defined, then we have $\lambda = (\lambda + \mu) \circ \psi_0$ and $\mu = (\lambda + \mu) \circ \psi_1$ where $\psi_0(t) = \frac{1}{2}t$ and $\psi_1(t) = \frac{1}{2}t + \frac{1}{2}$. Moreover applying (ii) to the path $\lambda + \mu$ and $\psi_0$ and then applying to $\lambda + \mu$ and $\psi_1$ we obtain

$$\Gamma(\lambda + \mu)(t) = \begin{cases} 
\Gamma(\lambda)(2t) & 0 \leq t \leq \frac{1}{2} \\
\Gamma(\lambda)(1) \circ \Gamma(\mu)(2t - 1) & \frac{1}{2} \leq t \leq 1.
\end{cases}$$

In particular

$$\Gamma(\lambda + \mu)(1) = \Gamma(\lambda)(1) \circ \Gamma(\mu)(1).$$

Let $Q$ be a topological groupoid on $X$ with a continuous path connection $\Gamma: \Lambda(X) \rightarrow \Lambda(Q)$. Let $C_\Gamma(Q)$ be the set of all $g \in Q$ such that if $\alpha(g) = x$ then there is a path $\lambda$ in $X$ such that $\Gamma(\lambda)$ joins $g$ to the identity $1_x$ at $x$, that is, $\Gamma(\lambda)(0) = 1_x$ and $\Gamma(\lambda)(1) = g$. Then we prove the following proposition.

**Proposition 4.1** $C_\Gamma(Q)$ is a wide subgroupoid of $Q$.  

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Proof: Let \( g, h \in C_\Gamma(Q) \) such that \( gh \) is defined in \( Q \). Then there are paths \( \lambda \) and \( \mu \) joining \( g \) to \( 1_{\alpha(g)} \) and \( h \) to \( 1_{\alpha(h)} \) respectively. Here note that \( \lambda(0) = \alpha(g) \), \( \lambda(1) = \beta(g) \) and \( \mu(0) = \alpha(h) \), \( \mu(1) = \beta(h) \). So the composition \( \lambda + \mu \) of the paths is defined and \( \Gamma(\lambda + \mu)(0) = \Gamma(\lambda)(0) = 1_{\alpha(g)} \) and \( \Gamma(\lambda + \mu)(1) = \Gamma(\lambda)(1) \circ \Gamma(\mu)(1) = gh \). So \( gh \in C_\Gamma(Q) \). That means \( C_\Gamma(Q) \) is closed under the groupoid composition.

If \( g \in C_\Gamma(Q) \) with \( \alpha(g) = x \) then there is a path \( \lambda \) in \( X \) such that \( \Gamma(\lambda)(0) = 1_x \) and \( \Gamma(\lambda)(1) = g \). Define a path \( \bar{\lambda} \) in \( X \) such that \( \bar{\lambda}(t) = \lambda(1 - t) \). Then \( \bar{\lambda}(t) = (\lambda \psi)(t) \) with \( \psi(t) = 1 - t \). By the transport law we have \( \Gamma(\lambda)(\psi(t)) = \Gamma(\lambda)(1) \circ \Gamma(\bar{\lambda})(t) \) where \( \Gamma(\lambda)(1) = g \). So we have

\[
\Gamma(\bar{\lambda})(0) = g^{-1}\Gamma(\lambda)(1) = g^{-1}g = 1_y
\]

and

\[
\Gamma(\bar{\lambda})(1) = g^{-1} \circ (\Gamma(\lambda)(0) = g^{-1} \circ 1_x = g^{-1}
\]

So \( g^{-1} \in C_\Gamma(Q) \). Hence \( C_\Gamma(Q) \) is a wide subgroupoid of \( Q \).

We also need an analogue of Proposition 2.3.

**Proposition 4.2** If \( \Gamma \) is a path connection on the topological groupoid \( Q \) and \( U \) is an open cover of \( X \), then \( C_\Gamma(Q) \) is generated by the family \( C_\Gamma(Q|U) \) for all \( U \in \mathcal{U} \).

**Proof:** If \( g \in C_\Gamma(Q) \) is joined to \( 1_x \) by the path \( \Gamma(\mu) \), then we can write \( \mu = \mu_1 + \cdots + \mu_n \) where each \( \mu_r \) lies in some set \( U_r \) of \( \mathcal{U} \). Let \( g_r = -\Gamma(\mu_r)(0) + \Gamma(\mu_r)(1) \). Then \( g_r \in C_\Gamma(Q|U_r) \) and \( g = g_1 + \cdots + g_n \). \( \square \)

If \( Q \) is a topological groupoid on \( X \) with a path connection \( \Gamma : \Lambda(X) \to \Lambda(Q) \) then of course \( \text{loc}(C_\Gamma(Q)) \) is a local subgroupoid. However we would like an analogue of \( c_1(Q, \mathcal{U}) \) and this needs extra conditions. In fact the existence of a smooth path connection for the groupoid \( \text{Sym}(p) \) of a principal bundle \( p : E \to B \) relies on the existence of an infinitesimal connection (see [12], [13]) which itself requires extra structure on the space involved.

We give some conditions which are sufficient for \( c_r(Q, \mathcal{U}) \) to be well defined.

We suppose given an open cover \( \mathcal{U} = \{ U_i : i \in I \} \) for \( X \) and for each \( i \in I \) a collection \( \text{geod}(U_i) \) of paths in \( U_i \) — an element \( \lambda \in \text{geod}(U_i) \) with \( \lambda(0) = x, \lambda(1) = y \) is called a “geodesic path” from \( x \) to \( y \). We suppose

(i) if \( x, y \in U_i \), then there is a unique geodesic path \( \text{geod}_i(x, y) \) from \( x \) to \( y \).

(ii) if \( x, y \in U_i \cap U_j \) then \( \text{geod}_i(x, y) = \text{geod}_j(x, y) \).
We also need the connection to be “flat” for this structure in the sense that if \( \lambda: x \to y \) is any path in \( U_i \) then \( \Gamma(\lambda)(1) = \Gamma(\text{geod}_i(x,y))(1) \).

**Proposition 4.3** Under the above atlas assumptions, there is a local subgroupoid \( c_r(Q, \mathcal{U}) \) defined by

\[
c_r(Q, \mathcal{U})(x) = [U_i, C \Gamma(Q|U_i)]_x.
\]
**Proof:** We have to prove that if \( x \in U_i \cap U_j \) then
\[ [U_i, C_T(Q|U_i)]_x = [U_j, C_T(Q|U_j)]_x. \]
This means there is an open neighbourhood \( W \) of \( x \) in \( U_i \cap U_j \) such that
\[ C_T(Q|U_i)|W = C_T(Q|U_j)|W. \]
Let \( W \) be an open neighbourhood of \( x \) in \( U_i \cap U_j \). Let \( g \in C_T(Q|U_i)|W \) with \( \alpha(g) = x \) and \( \beta(g) = y \). So there is a path \( \lambda : x \to y \) in \( U_i \) such that \( \Gamma(\lambda)(1) = g \). Let \( \lambda_i : x \to y \) be the geodesic path. So \( \Gamma(\lambda_i)(1) = g \), by the flat condition, and so \( g \in C_T(Q|U_j)|W \). Hence \( C_T(Q|U_i)|W \leq C_T(Q|U_j)|W \). Since the converse proof is similar we have \( C_T(Q|U_i)|W = C_T(Q|U_j)|W \). \( \square \)

Suppose now that we have the local subgroupoid \( c_t(Q, \mathcal{U}) \) defined by the open cover \( \mathcal{U} \), and \( U \) is an open subset of \( X \). We will later need a result which follows easily from compatibility:

**Proposition and Definition 4.4** The equality
\[ c_t(Q, \mathcal{U})|U = c_t(Q|U, \mathcal{U} \cap U) \]
holds if for any \( i, j \in I \) and \( x \in U_i \cap U_j \cap U \) there is an open set \( W \) such that \( x \in W \subseteq U_i \cap U_j \cap U \) and \( C_T(Q|U_j)|W = C_T(Q|U_j \cap U)|W \). If this condition holds for all open sets \( U \) of \( X \), then we say that the cover \( \mathcal{U} \) is \( \Gamma \) path local for \( c_t(Q, \mathcal{U}) \).

## 5 Partial orders and induced morphisms

We first establish some elementary but essential basic theory.

The set \( L_Q(X) \) of wide subgroupoids of \( Q \) is a poset under inclusion. We write \( \leq \) for this partial order. This poset has a top element namely \( Q \) and a bottom element namely the discrete groupoid \( X \).

Let \( \text{Loc}(Q) \) be the set of local subgroupoids of \( Q \). Let \( x \in X \). We define a partial order on the stalks \( p_Q^{-1}(x) = (\mathcal{L}_Q)_x \) by \( [U', H']_x \leq [U, H]_x \) if there is an open neighbourhood \( W \) of \( x \) such that \( W \subseteq U \cap U' \) and \( H'|W \) is a subgroupoid of \( H|W \). Clearly this partial order is well defined. Its bottom element is of the form \([U, H]_x\) where \( H \) is discrete, and its top element is of the form \([U, Q|U]_x\). This partial order induces a partial order on \( \text{Loc}(Q) \) by \( s \leq t \) if and only if \( s(x) \leq t(x) \) for all \( x \in X \).

The major purpose of the next topic is to relate local subgroupoids of \( Q \) and local equivalence relations on \( \text{Ob}(Q) \). This seems an area requiring much more development, and we hope will be the start of new ways of relating bundle and foliation theory.
Suppose given two groupoids $Q, H$ and a groupoid morphism $\phi: Q \to H$ on $X$, which is the identity on objects. Then we obtain morphisms of presheaves $\phi^*: L_Q \to L_H, \phi_*: L_H \to L_Q$ as follows.

Let $U$ be open in $X$. Then $\phi^*: L_Q(U) \to L_H(U)$ is given by $\phi(K)$ is the image of $K \in L_Q(U)$ by $\phi$. Here $K$ is a wide subgroupoid of $Q|U$, and so its image $\phi(K)$ is a subgroupoid of $H|U$, since $Ob(\phi)$ is injective and is clearly wide.

Further $\phi^*: L_H(U) \to L_Q(U)$ is given by $\phi^*(K') = \phi^{-1}(K')$, for $K' \in L_H(U)$.

Hence we get induced morphism of sheaves $\phi_*: L_Q \to L_H, \phi^*: L_H \to L_Q$.

In particular, we get for a groupoid $Q$ an ‘anchor’ morphism of groupoids $A: Q \to X \times X$ and so sheaf morphisms

$$A_*: L_Q \to L_{X \times X}, \quad A^*: L_{X \times X} \to L_Q.$$  

Hence a local subgroupoid $s$ of $Q$ yields a local equivalence relation $A_*(s)$ on $X$, and a local equivalence relation $r$ on $X$ yields a local subgroupoid $A^*(r)$ of $Q$. This gives further examples of local subgroupoids.

Clearly also $\phi_*, \phi^*$ are order preserving on stalks for any morphism $\phi: Q \to H$ of groupoids over $X$. Hence they induce morphism of posets

$$\phi_*: \text{Loc}(Q) \to \text{Loc}(H), \quad \phi^*: \text{Loc}(H) \to \text{Loc}(Q).$$

Further, $s \leq \phi^*r$ if and only if $\phi_*s \leq r$. This can be expressed by saying that $\phi_*$ is left adjoint to $\phi^*$.

6 Coherence for wide subgroupoids and local subgroupoids

We now fix a groupoid $Q$ on $X$, so that $L_Q(X)$ is the set of wide subgroupoids of $Q$, with its inclusion partial order, which we shall write $\leq$.

Clearly $\text{loc}_Q$ as defined in Definition 1.3 gives a poset morphism

$$\text{loc}_Q: L_Q(X) \to \text{Loc}(Q).$$

**Definition 6.1** Let $s$ be a local subgroupoid of $Q$. Then $\text{glob}(s)$ is the wide subgroupoid of $Q$ which is the intersection of all wide subgroupoids $H$ of $Q$ such that $s \leq \text{loc}(H)$.

We think of $\text{glob}(s)$ as an approximation to $s$ by a global subgroupoid.

**Proposition 6.2** (i) $\text{loc}$ and $\text{glob}$ are morphisms of posets.
(ii) For any wide subgroupoid $H$ of $Q$, $\text{glob}(\text{loc}(H)) \subseteq H$. 

The proofs are clear.

However, $s \leq \text{loc}(\text{glob}(s))$ need not hold. Rosenthal in [14] gives the example of the local equivalence relation $s = \text{loc}(E)$ where $E$ is the equivalence relation $aEb$ if and only if $a = \pm b$. Here is a similar example.

**Example 6.3** Let $Q$ be a groupoid on $\mathbb{R}$ such that all $x \in \mathbb{R}$ with $x \neq 0$ there is a neighbourhood $U$ of $x$ such that $Q|U$ is a bundle of groups, while no such neighbourhood of 0 exists. Let $s = \text{loc}(Q)$. Then $H = \text{glob}(s)$ on $\mathbb{R} \setminus \{0\}$ coincides with $Q$ on this set, and in fact $H$ is the bundle of groups $Q(x)$ for all $x \in \mathbb{R}$. It follows that $s(0) > \text{loc}(H)(0)$. 

We therefore adapt from [14, 15] some notions of coherence.

**Definition 6.4** Let $s$ be a local subgroupoid of $Q$ on $X$.

(i) $s$ is called **coherent** if $s \leq \text{loc}(\text{glob}(s))$.

(ii) $s$ is called **globally coherent** if $s = \text{loc}(\text{glob}(s))$.

(iii) $s$ is called **totally coherent** if for every open set $U$ of $X$, $s|U$ is coherent.

**Definition 6.5** Let $H \in L_Q(X)$, so that $H$ is a wide subgroupoid of $Q$.

(i) $H$ is called **locally coherent** if $\text{loc}(H)$ is coherent.

(ii) $H$ is called **coherent** if $H = \text{glob}(\text{loc}(H))$. 

**Example 6.6** Let $Q$ be a groupoid on the discrete space $X$. Then $\text{glob}(\text{loc}(Q)) = \text{Inn}(Q)$, the groupoid of vertex groups of $Q$. Thus in general, $Q$ is not coherent.

At another extreme we have:

**Proposition 6.7** Let $Q$ be a bundle of groups. Then any local subgroupoid of $Q$ is globally coherent, and any wide subgroupoid of $Q$ is coherent.

**Proof:** Let $s$ be a local subgroupoid of $Q$ and let $\{(U_i, H_i) : i \in I\}$ be an atlas for $s$. Then if $x \in U_i$, we have $s(x) = [U_i, H_i]_x$. Let $H(x) = H_i(x)$. If $x \in U_i \cap U_j$, there is a neighbourhood $W$ of $x$ such that $W \subseteq U_i \cap U_j$ and $H_i|W = H_j|W$, and hence $H_i(x) = H_j(x)$. Thus $H$ is independent of the choices. Also $H|U_i = H_i$. Hence $\text{loc}(H)(x) = [U_i, H_i]_x$, and so $\text{loc}(H) = s$, $H = \text{glob}(s)$. 

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Coherence of \( s \) says that in passing between local and global information nothing is lost due to collapsing. Notice also that these definitions depend on the groupoid \( Q \).

**Proposition 6.8**  
loc and glob induce morphisms of posets from the locally coherent subgroupoids of \( Q \) to the coherent local subgroupoids of \( Q \), and on these posets glob is left adjoint to loc. Further, glob and loc give inverse isomorphisms between the posets of coherent subgroupoids and of globally coherent local subgroupoids.

**Proof:** Let \( H \) be a locally coherent subgroupoid of \( Q \) and let \( s \) be a coherent local subgroupoid of \( Q \). By the definition of locally coherent subgroupoid \( \text{loc}(H) \) is a coherent local subgroupoid of \( Q \).

Conversely, let \( K = \text{glob}(s) \). Since \( s \) is coherent, \( s \leq \text{loc}(K) \). Since glob is a poset morphism,  
\[ \text{glob}(s) \leq \text{glob}(\text{loc}(K)), \]
i.e., \( K \leq \text{glob}(\text{loc}(K)) \). Since loc is also a poset morphism,  
\[ \text{loc}(K) \leq \text{loc}(\text{glob}(\text{loc}(K))). \]
So \( \text{loc}(K) \) is a coherent local subgroupoid, and \( K \) is locally coherent.

The adjointness relation is that \( \text{glob}(s) \leq H \iff s \leq \text{loc}(H) \). The implication \( \Leftarrow \) follows from the fact that for all \( H \) we have \( \text{glob}(\text{loc}(H)) \leq H \). The implication \( \Rightarrow \) follows from the coherence of \( s \).

The final statement is obvious. \( \square \)

Note in particular that coherence of \( H \) implies local coherence of \( H \).

**Proposition 6.9**  
Let \( Q \) be a topological groupoid on \( X \) and \( G \) a star path connected wide subgroupoid of \( Q \). Then \( G \) is coherent and \( \text{loc}(G) \) is globally coherent.

**Proof:** We prove that \( \text{glob}(\text{loc}(G)) = G \). By Proposition 6.2 \( \text{glob}(\text{loc}(G)) \leq G \). To prove that \( G \leq \text{glob}(\text{loc}(G)) \) let \( H \) be a wide subgroupoid of \( Q \) such that \( \text{loc}(G) \leq \text{loc}(H) \). Then for \( x \in X \),
\[ [X, G]_x \leq [X, H]_x \]
and so for some open neighbourhood \( U_x \) of \( x \), \( G|U_x \leq H|U_x \). These sets \( U_x \) form a cover \( U \) of \( X \). By Proposition 2.3, \( G \) is generated by the \( G|U \) for \( U \in U \). It follows that \( G \leq H \). \( \square \)

**Corollary 6.10**  
If \( X \) is a topological space then its fundamental groupoid \( \pi_1 X \) is coherent and globally coherent.
Corollary 6.11 Let $Q$ be a topological groupoid on $X$. Then the star identity component $C_1(Q)$ is coherent.

Corollary 6.12 Let $Q$ be a topological groupoid with a path connection $\Gamma : \Lambda(X) \to \Lambda(Q)$. Then the wide subgroupoid $C_\Gamma(Q)$ is coherent.

Proposition 6.13 Let $Q$ be a topological groupoid on $X$. Suppose that the local subgroupoid $c_1(Q, U)$ of $Q$ is well defined by an open cover $U$. Then

(i) $\text{glob}(c_1(Q, U)) = C_1(Q)$.

(ii) $c_1(Q, U)$ is coherent.

(iii) If $U$ is path local for $Q$, then $c_1(Q, U)$ is totally coherent.

Proof: (i) Certainly $c_1(Q, U) \leq \text{loc}(C_1(Q))$ since for all open $U$ in $X$, $C_1(Q|U) \leq C_1(Q)|U$ and so $[U_i, C_1(Q|U_i)]_x \leq [X, C_1(Q)]_x$ for all $x \in U$.

Now suppose $H$ is a wide subgroupoid of $Q$ and $c_1(Q, U) \leq \text{loc}(H)$. We have to prove $C_1(Q) \leq H$.

Let $i \in I$ and $x \in U_i$. We have $[U_i, C_1(Q|U_i)]_x \leq [X, H]_x$. Hence there is an open neighbourhood $W_x$ of $x$ contained in $U_i$ and such that $C_1(Q|U_i)|W_x \leq H$.

By Proposition 2.3, $C_1(Q|U_i)$ is generated by the $C_1(Q|U_i)|W_x$ for all $x \in U_i$ and by the same Proposition, $C_1(Q)$ is generated by the $C_1(Q|U_i)$ for all $i \in I$. Hence $C_1(Q) \leq H$.

Coherence of $c_1(Q, U)$ follows from (i) and $C_1(Q|U_i) \leq C_1(Q)|U_i$. Total coherence in the path local case follows by applying (ii) to $Q|U$, using Proposition and Definition 3.5. $\square$

Proposition 6.14 Let $Q$ be a topological groupoid on $X$ such that the local subgroupoid $c_\Gamma(Q, U)$ is well defined by the open cover $U$. Then:

(i) $\text{glob}(c_\Gamma(Q, U)) = C_\Gamma(Q)$.

(ii) $c_\Gamma(Q, U)$ is coherent.

(iii) If $U$ is $\Gamma$ path local for $Q$, then $c_\Gamma(Q, U)$ is totally coherent.
Proof: (i) Note that \( c_*(Q, \mathcal{U}) \leq \text{loc}(C_*(Q)) \) since for all \( U \) in \( X \), \( C_*(Q|U) \leq C_*(Q)|U \). So \( \text{glob}(c_*(Q)) \leq C_*(Q) \). To prove that \( C_*(Q) \leq \text{glob}(c_*(Q, \mathcal{U})) \) suppose that \( H \) is a wide subgroupoid of \( Q \) such that \( c_*(Q, \mathcal{U}) \leq \text{loc}(H) \). We have to prove that \( C_*(Q) \leq H \). If \( U \in \mathcal{U} \) and \( x \in U \) then

\[
[U, C_*(Q|U)]_x \leq [X, H]_x.
\]

Hence \( U \) has a covering by open sets \( W_x \) such that \( [U, C_*(Q|U)]|_{W_x} \leq H \). By Proposition 2.3, \( C_*(Q|U) \) is generated by the groupoids \( C_*(Q|U)|W_x \) and by Proposition 1.2 \( C_*(Q) \) is generated by the \( C_*(Q|U) \) for \( U \in \mathcal{U} \). Hence \( C_*(Q) \leq H \).

The proofs of (ii), (iii) are analogous to those in the previous proposition.

\[\square\]

7 Coherence and atlases

We lead up to conditions for an atlas for a local subgroupoid \( s \) to be \textit{globally adapted} to \( s \). This notion is important for considerations of holonomy (see [2]), and the applications will be developed elsewhere.

The next proposition gives an alternative description of \text{glob}.

Let \( \mathcal{U}_s = \{(U_i, H_i) : i \in I\} \) be an atlas for the local subgroupoid \( s \). Then \( \text{glob}(\mathcal{U}_s) \) is defined to be the subgroupoid of \( Q \) generated by all the \( H_i, i \in I \).

An atlas \( \mathcal{V}_s = \{(V_j, s_j) : j \in J\} \) for \( s \) is said to refine \( \mathcal{U}_s \) if for each index \( j \in J \) there exists an index \( i(j) \in I \) such that \( V_j \subseteq U_{i(j)} \) and \( s_{i(j)}|V_j = s_j \).

**Proposition 7.1** Let \( s \) be a local subgroupoid of \( Q \) given by the atlas \( \mathcal{U}_s = \{(U_i, H_i) : i \in I\} \). Then \( \text{glob}(s) \) is the intersection of the subgroupoids \( \text{glob}(\mathcal{V}_s) \) of \( Q \) for all refinements \( \mathcal{V}_s \) of \( \mathcal{U}_s \).

**Proof:** Let \( K \) be the intersection given in the proposition.

Let \( Q \) be a subgroupoid of \( Q \) on \( X \) such that \( s \leq \text{loc}(Q) \). Then for all \( x \in X \) there is a neighbourhood \( V \) of \( x \) and \( i_x \in I \) such that \( x \in U_{i_x} \) and \( H_{i_x}|V \cap U_{i_x} \leq Q \). Then \( \mathcal{W} = \{(V_x \cap U_{i_x}, H_{i_x}|V_x \cap U_{i_x}) : x \in X \} \) refines \( \mathcal{U}_s \) and \( \text{glob}(\mathcal{W}) \leq Q \). Hence \( K \leq Q \), and so \( K \leq \text{glob}(s) \).

Conversely, let \( \mathcal{V}_s = \{(V_j, H'_j) : j \in J\} \) be an atlas for \( s \) which refines \( \mathcal{U}_s \). Then for each \( j \in J \) there is an \( i(j) \in I \) such that \( V_j \subseteq U_{i(j)} \), \( H'_j = H_{i(j)}|V_j \). Then \( s \leq \text{loc}(\text{glob}(\mathcal{V}_s)) \). Hence \( \text{glob}(s) \leq \text{glob}(\mathcal{V}_s) \) and so \( \text{glob}(s) \leq K \).\[\square\]
Corollary 7.2  A wide subgroupoid $H$ of $Q$ is coherent if and only if for every open cover $\mathcal{V}$ of $X$, $H$ is generated by the subgroupoids $H|V, V \in \mathcal{V}$.

Proof: Note that $\{(X, H)\}$ is an atlas for $\text{loc}(H)$, which is refined by $\mathcal{V}_H = \{(V, H|V) : V \in \mathcal{V}\}$ for any open cover $\mathcal{V}$.

Suppose the latter condition holds. Then Proposition 7.1 implies that $H = \text{glob}(\text{loc}(H))$, i.e. $H$ is coherent. The converse holds since $\text{glob}(\mathcal{V}_H) \leq H$. $\square$

Let $U$ be an open subset of $X$. Then we have notions of local subgroupoids of $Q|U$ and also of the restriction $s|U$ of a local subgroupoid $s$ of $Q$. Clearly if $H$ is a wide subgroupoid of $Q$ then $\text{loc}(H|U) = (\text{loc}(H))|U$.

Proposition 7.3  Let $s$ be a local subgroupoid of $Q$ and let $U$ be open in $X$. Then $\text{glob}(s|U) \leq \text{glob}(s)|U$.

Proof: Let $H$ be a wide subgroupoid of $Q$ such that $s \leq \text{loc}(H)$. Then $s|U \leq \text{loc}(H|U)$. So $\text{glob}(s|U) \leq H|U$. The result follows. $\square$

Proposition 7.4  Let $s$ be a local subgroupoid of $Q$. Then

(i) If $s$ is globally coherent, $U$ is open in $X$, and $s|U$ is coherent, then $s|U$ is globally coherent.

(ii) If there is an open cover $\mathcal{V}$ of $X$ such that $s|V$ is coherent for all $V \in \mathcal{V}$, then $s$ is coherent.

(iii) If $s$ is globally coherent then for any open cover $\mathcal{V}$ of $X$, $\text{glob}(s)$ is generated by the groupoids $\text{glob}(s)|V$ for all $V \in \mathcal{V}$.

(iv) If there is open cover $\mathcal{V}$ of $X$ such that $s|V$ is globally and totally coherent for $V \in \mathcal{V}$, then $s$ is totally coherent.

Proof: i) We are given $s = \text{loc}(\text{glob}(s))$. By Proposition 7.3

$$\text{loc}(\text{glob}(s)|U) \leq \text{loc}(\text{glob}(s)|U) = \text{loc}(\text{glob}(s))|U = s|U.$$  

Since $s|U$ is coherent, we have $s|U \leq \text{loc}(\text{glob}(s|U))$. So $s|U = \text{loc}(\text{glob}(s|U))$, i.e. $s|U$ is globally coherent.

ii) We have

$$s|V \leq \text{loc}(\text{glob}(s|V)) \leq \text{loc}(\text{glob}(s)|V) \leq (\text{loc}(\text{glob}(s)))|V.$$  

Since this holds for all $V$ of an open cover, we have $s \leq \text{loc}(\text{glob}(s))$. 17
iii) This follows from Corollary \ref{cor:7.2}.

iv) Let $U$ be open in $X$. Let $V \in \mathcal{V}$. Since $s|V$ is globally and totally coherent, then $s|V \cap U$ is globally coherent. Hence by (ii) $s|U$ is coherent, since the $V \cap U, V \in \mathcal{V}$, cover $U$. \hfill \Box

**Proposition 7.5** Let $\mathcal{U}_s = \{(U_i, H_i) : i \in I\}$ be an atlas for the local subgroupoid $s$. Then:

(i) $s|U_i = loc(H_i)$ for all $i \in I$;

(ii) $loc(glob(s|U_i)) \leq s|U_i$ for all $i \in I$;

(iii) if $s|U_i$ is coherent for all $i \in I$ then $s$ is globally coherent;

(iv) if $s|U_i$ is coherent for all $i \in I$ then $glob(s) = glob(\mathcal{U}_s)$.

**Proof:** i) This is clear.

ii) We have by Proposition \ref{prop:5.2}

$$loc(glob(s|U_i)) = loc(glob(loc(H_i))) \leq loc(H_i) = s|U_i.$$  

iii) This is immediate from the definition of coherence and ii).

iv) Let $H = glob(\mathcal{U}_s)$, i.e. $H$ is the subgroupoid of $Q$ generated by the $H_i, i \in I$. Then $glob(s) \leq H$. Let $K$ be a wide subgroupoid of $Q$ such that $s \leq loc(K)$. Then for all $i \in I$ and $x \in U_i$ there is a neighbourhood $V_x^i$ of $x$ such that $V_x^i \subseteq U_i$ and $H_i|V_x^i \leq K|V_x^i$. By global coherence of $s|U_i$ and Proposition \ref{prop:7.4} i,iii), $H_i$ is generated by the $H_i|V_x^i$ for all $x \in U_i$. Hence $H_i \leq K|U_i \leq K$. Hence $H \leq glob(s)$. \hfill \Box

**Definition 7.6** Let $s$ be a local subgroupoid of the groupoid $Q$ on $X$. An atlas $\mathcal{U}_s$ for $s$ is called **globally adapted** if $glob(s) = glob(\mathcal{U}_s)$.

**Remark 7.7** This is a variation on the notion of an $r$-adaptable family defined by Rosenthal in \cite{15, Definition 4.4} for the case of a local equivalence relation $r$. He also imposes a connectivity condition on the local equivalence classes.

The construction of the holonomy groupoid of a local subgroupoid requires a globally adapted atlas (see Theorems 3.7 and 3.8 \cite{2}). The following proposition is very useful to this end.

**Proposition 7.8** Let $s$ be a totally coherent local subgroupoid of the groupoid $Q$ on $X$. Then any atlas for $s$ is globally adapted.
Proof: This is immediate from the previous Proposition, since total coherence implies that each $s|U_i$ is coherent.

Corollary 7.9 Any path local atlas $\mathcal{U}$ of the local subgroupoid $c_1(Q,\mathcal{U})$ is globally adapted.

Corollary 7.10 Any $\Gamma$ path local atlas of the local subgroupoid $c_1(Q,\mathcal{U})$ is globally adapted.

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