Hanson-Wright Inequality for Random Tensors under Einstein Product

Shih Yu Chang *

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Abstract

The Hanson-Wright inequality is an upper bound for tails of real quadratic forms in independent subgaussian random variables. In this work, we extend the Hanson-Wright inequality for the maximum eigenvalue of the quadratic sum of random Hermitian tensors under Einstein product. We first prove Weyl inequality for tensors under Einstein product and apply this fact to separate the quadratic form of random Hermitian tensors into diagonal sum and coupling (non-diagonal) sum parts. For the diagonal part, we can apply Bernstein inequality to bound the tail probability of the maximum eigenvalue of the sum of independent random Hermitian tensors directly. For coupling sum part, we have to apply decoupling method first, i.e., decoupling inequality to bound expressions with dependent random Hermitian tensors with independent random Hermitian tensors, before applying Bernstein inequality again to bound the tail probability of the maximum eigenvalue of the coupling sum of independent random Hermitian tensors. Finally, the Hanson-Wright inequality for the maximum eigenvalue of the quadratic sum of random Hermitian tensors under Einstein product can be obtained by the combination of the bound from the diagonal sum part and the bound from the coupling (non-diagonal) sum part. In Appendix of this work, we also include the Hanson-Wright inequality under T-product tensor, which can be derived by the same method of establishing the Hanson-Wright inequality under Einstein product except changing the rule of tensors product operation.

Index terms— Hanson-Wright inequality, Bernstein bound, Courant-Fischer theorem under Einstein product, random tensors, Weyl inequality under Einstein product, decoupling method

1 Introduction

The Hanson-Wright inequality provides us an upper bound for tails of real quadratic forms in independent subgaussian random variables. We define a random variable $X$ is a $\alpha$-subgaussian if for every $\theta > 0$, we have \[ \Pr(|X| \geq \theta) \leq 2 \exp(-\frac{\theta^2}{2\alpha^4}). \] The Hanson-Wright inequality states that for any sequence of independent mean zero $\alpha$-subgaussian random variables $X_1, \cdots, X_n$, and any symmetric matrix $A = (a_{i,j})_{i,j \leq n}$, we have

\[ \Pr \left( \left| \sum_{i,j=1}^{n} a_{i,j} (X_i X_j - \mathbb{E}(X_i X_j)) \right| \geq \theta \right) \leq 2 \exp \left( -\frac{1}{C} \min \left\{ \frac{\theta^2}{\alpha^4 \|A\|_{\text{HS}}}, \frac{\theta}{\alpha^2 \|A\|_{\text{op}}} \right\} \right), \] (1)

where $\|A\|_{\text{HS}}$ is defined as \( \left( \sum_{i,j=1}^{n} |a_{i,j}|^2 \right)^{1/2} \), and $\|A\|_{\text{op}}$ is defined as $\max_{\|x\| \leq 1} \|Ax\|_2$. The bound in Eq. (1) was essentially proved in [1] in the symmetric case and in [2] in the zero mean case. The Hanson-Wright inequality

*Shih Yu Chang is with the Department of Applied Data Science, San Jose State University, San Jose, CA, U. S. A. (e-mail: shihyu.chang@sjsu.edu).
inequality has been applied to numerous applications in high-dimensional probability and statistics, as well as in random matrix theory [3]. For example, the estimation of bound in Eq. (1) is applied to the theory of compressed sensing with circulant type matrices [4]. In [1], they applied Hanson-Wright inequality to study the concentration properties for sample covariance operators corresponding to Banach space-valued Gaussian random variables.

In recent years, tensors have been applied to different applications in science and engineering [5]. In data processing fields, tensor theory applications include unsupervised separation of unknown mixtures of data signals [6,7], signals filtering [8], network signal processing [9–11] and image processing [12,13]. In wireless communication applications, tensors are applied to model high-dimensional communication channels, e.g., MIMO (multi-input multi-output) code-division [14,15], radar communications [16,17]. In numerical multilinear algebra computations, tensors can be applied to solve multilinear system of equations [18], high-dimensional data fitting/regression [19], tensor complementary problem [20], tensor eigenvalue problem [21], etc. In machine learning, tensors are also used to characterize data with coupling effects, for example, tensor decomposition methods have been reported recently to establish the latent-variable models, such as topic models in [22], and the method of moments for undertaking the Latent Dirichlet Allocation (LDA) in [23]. Nevertheless, all these applications assume that systems modelled by tensors are fixed and such assumption is not true and practical in problems involving tensor formulations. In recent years, there are more works beginning to develop theory about random tensors, see [24–31], and references therein. In this work, we extend the Hanson-Wright inequality from random variables to random Hermitian tensors under Einstein product. This is a related work to our another work about Hanson-Wright inequality for symmetric T-product tensor [32].

We first prove Theorem 2 about Weyl inequality for tensors under Einstein product and apply Theorem 2 to separate the quadratic form of random tensors into diagonal sum and coupling (non-diagonal) sum parts. For the diagonal part, we can apply Theorem 3 directly to bound the tail probability of the maximum eigenvalue of the sum of independent random Hermitian tensors. For coupling sum part, we have to upper bound this part by Theorem 4 via decoupling method first, i.e., decoupling inequality to bound expressions with dependent random Hermitian tensors with independent random Hermitian tensors, before applying again Theorem 3 to bound the tail probability of the maximum eigenvalue of the sum of independent random Hermitian tensors. Therefore, the main result of this work is presented by the following Theorem 1.1.

**Theorem 1.1 (Hanson-Wright Inequality for Random Tensors)** We define a vector of random tensors \( \mathcal{X} \in \mathbb{R}^{(n \times I_1 \times \cdots \times I_M) \times (I_1 \times \cdots \times I_M)} \) as:

\[
\mathcal{X} = \begin{bmatrix}
\mathcal{X}_1 \\
\mathcal{X}_2 \\
\vdots \\
\mathcal{X}_n
\end{bmatrix},
\]

(2)

where random Hermitian tensors \( \mathcal{X}_i \in \mathbb{R}^{(I_1 \times \cdots \times I_M) \times (I_1 \times \cdots \times I_M)} \) are independent random Hermitian tensors with \( \mathbb{E} \mathcal{X}_i = O \) for \( 1 \leq i \leq n \). We also require another fixed tensor \( \mathcal{A} \in \mathbb{R}^{(n \times I_1 \times \cdots \times I_M) \times (n \times I_1 \times \cdots \times I_M)} \), which is defined as:

\[
\mathcal{A} = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n,1} & A_{n,2} & \cdots & A_{n,n}
\end{bmatrix},
\]

(3)

where \( A_{i,j} \in \mathbb{R}^{(I_1 \times \cdots \times I_M) \times (I_1 \times \cdots \times I_M)} \) are Hermitian tensors also. We also require following assumptions.
Define random Hermitian tensor $Y_i$ for $i = 1, 2, \cdots, n$ as

$$Y_i \overset{\text{def}}{=} X_i \star_M A_{i,i} \star_M X_i - \mathbb{E} (X_i \star_M A_{i,i} \star_M X_i),$$

we assume that

$$\mathbb{E}Y_i = O$$

and $\lambda_{\max}(Y_i) \leq T_{dg}$ almost surely.

Define the total variance $\sigma_{dg}^2$ as:

$$\sigma_{dg}^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} \mathbb{E} (Y_i^2) \right\|,$$

where $\| \cdot \|$ represents the spectral norm, which equals the largest singular value of a tensor. The term $I_M$ is defined as the product of each dimension size:

$$I_M \overset{\text{def}}{=} \prod_{j=1}^{M} I_j.$$

Moreover, we define random Hermitian tensor $Z_k$ for $k = 1, 2, \cdots, n^2 - n$ as

$$Z_k \overset{\text{def}}{=} X^{(1)}_i \star_M A_{i,j} \star_M X^{(2)}_j \text{ for } 1 \leq i \neq j \leq n,$$

where the tensors $X^{(1)}_i$ are identical distribution copy for the tensors $X_i$, and the tensors $X^{(2)}_j$ are identical distribution copy for the tensors $X_j$, then we assume that

$$\mathbb{E}Z_k = O$$

almost surely.

Given any realization of the random tensor $X_i$, denoted as $\tilde{X}_i$, we assume that

$$\lambda_{\max} \left( \tilde{X}_i \star_M A_{i,j} \star_M X_j \right) \leq T_{cp}$$

almost surely,

where $T_{cp}$ is a positive real number, and all $1 \leq i \neq j \leq n$. The total variance with respect to the tensor $\tilde{X}_i$ is defined as

$$\sigma_{cp}^2(\tilde{X}_i) \overset{\text{def}}{=} \left\| \sum_{j=1, \neq i}^{n} \mathbb{E} \left( \left( \tilde{X}_i \star_M A_{i,j} \star_M X_j \right)^2 \right) \right\|.$$

The function $f(\tilde{X}_i)$ is the probability density function for the realization tensor $\tilde{X}_i$.

Then, we have

$$\Pr \left( \lambda_{\max} \left( \tilde{X}_i^\top \tilde{X}_i - \mathbb{E} \left( \tilde{X}_i^\top \tilde{X}_i \right) \right) \geq \theta \right) \leq \Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X_i \star_M A_{i,j} \star_M X_j \right) \geq \frac{\theta}{2} \right)$$

$$+ \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \left( X_i \star_M A_{i,i} \star_M X_i - \mathbb{E} (X_i \star_M A_{i,i} \star_M X_i) \right) \right) \geq \frac{\theta}{2} \right)$$

$$\leq C_4 I_M^{\text{def}} \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( \frac{-\theta^2}{8n^2 C_4^2 \sigma_{cp}^2(\tilde{X}_i) + 4T_{cp} \theta n C_4 / 3} \right) f(\tilde{X}_i) d\tilde{X}_i$$

$$+ I_M^{\text{def}} \exp \left( \frac{-\theta^2}{8\sigma_{dg}^2 + 4T_{dg} \theta / 3} \right),$$

(10)
where $P_{cp}$ and $P_{dg}$ are probability bounds related to the coupling sum and the diagonal sum parts, respectively, and the term $C_4$ is a positive constant.

If $\frac{\theta}{2mC_4} \leq \frac{\sigma_{cp}^2(\tilde{X}_i)}{\tau_{cp}}$ with respect to $i = 1, 2, \ldots, n$, and $\theta \leq 2\sigma_{dg}^2/T_{dg}$, we have

$$\Pr \left( \lambda_{\max} \left( X^T A X - \mathbb{E} \left( X^T A X \right) \right) \geq \theta \right) \leq C_4 \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( -\frac{3\theta^2}{32n^2\sigma_{cp}^2(\tilde{X}_i)} \right) f(\tilde{X}_i)d\tilde{X}_i + \frac{1}{M} \exp \left( -\frac{3\theta^2}{32\sigma_{dg}^2} \right). \quad (11)$$

Moreover, if $\frac{\theta}{2nC_4} \geq \frac{\sigma_{cp}^2(\tilde{X}_i)}{\tau_{cp}}$ with respect to $i = 1, 2, \ldots, n$, and $\theta \geq 2\sigma_{dg}^2/T_{dg}$, we have

$$\Pr \left( \lambda_{\max} \left( X^T A X - \mathbb{E} \left( X^T A X \right) \right) \geq \theta \right) \leq nC_4 \sum_{i=1}^{M} \exp \left( -\frac{3\theta^2}{16nC_4T_{cp}} \right) + \frac{1}{M} \exp \left( -\frac{3\theta^2}{16T_{dg}} \right). \quad (12)$$

The rest of this paper is organized as follows. In Section 2, we review tensors under Einstein product and discuss Bernstein bounds for random Hermitian tensors. In Section 3, we will formulate our quadratic form for random Hermitian tensors used for Hanson-Wright inequality under Einstein product. Under such quadratic formulation, we will separate this form into diagonal sum and coupling sum parts. The probability bound for the diagonal sum part will also be discussed. The decoupling technique is presented and applied to bound the coupling sum of random Hermitian tensors in Section 4. Main result of this work: the Hanson-Wright inequality for random Hermitian tensors, is given in Section 5. Finally, concluding remarks are given by Section 6.

## 2 Fundamentals of Tensors and Random Tensors Tail Bounds

In this section, we will provide a brief introduction of tensors and related theorems in Section 2.1. In Section 2.2, we will present extended Bernstein bounds for a sum of zero-mean random tensors proved in [30].

### 2.1 Preliminaries of Tensors

Throughout this work, scalars are represented by lower-case letters (e.g., $d$, $e$, $f$, ...), vectors by boldfaced lower-case letters (e.g., $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$, ...), matrices by boldfaced capitalized letters (e.g., $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, ...), and tensors by calligraphic letters (e.g., $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, ...), respectively. Tensors are multidimensional arrays of values which are higher-dimensional generalizations from vectors and matrices. Given a positive integer $N$, let $[N] \triangleq \{1, 2, \ldots, N\}$. An order-$N$ tensor (or $N$-th order tensor) denoted by $\mathcal{X} \triangleq (a_{i_1, i_2, \ldots, i_N})$, where $1 \leq i_j = 1, 2, \ldots, I_j$ for $j \in [N]$, is a multidimensional array containing $I_1 \times I_2 \times \cdots \times I_N$ entries. Let $\mathbb{C}^{I_1 \times \cdots \times I_N}$ and $\mathbb{R}^{I_1 \times \cdots \times I_N}$ be the sets of the order-$N$ $I_1 \times \cdots \times I_N$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. For example, $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is an order-$N$ multiaarray, where the first, second, ..., and $N$-th dimensions have $I_1$, $I_2$, ..., and $I_N$ entries, respectively. Thus, each entry of $\mathcal{X}$ can be represented by $a_{i_1, \ldots, i_N}$.

Without loss of generality, one can partition the dimensions of a tensor into two groups, say $M$ and $N$ dimensions, separately. Thus, for two order-$(M+N)$ tensors: $\mathcal{X} \triangleq (a_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $\mathcal{Y} \triangleq (b_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, according to [33], the tensor addition $\mathcal{X} + \mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$.
\( \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N \) is given by
\[
(\mathcal{X} + \mathcal{Y})_{i_1,\ldots,i_M,j_1,\ldots,j_N} \overset{\text{def}}{=} a_{i_1,\ldots,i_M,j_1,\ldots,j_N} + b_{i_1,\ldots,i_M,j_1,\ldots,j_N}.
\] (13)

On the other hand, for tensors \( \mathcal{X} \overset{\text{def}}{=} (a_{i_1,\ldots,i_M,j_1,\ldots,j_N}) \in \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N \) and \( \mathcal{Y} \overset{\text{def}}{=} (b_{j_1,\ldots,j_N,k_1,\ldots,k_L}) \in \mathbb{C}^J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L \), according to [33], the Einstein product (or simply referred to as tensor product in this work) \( \mathcal{X} \ast_N \mathcal{Y} \in \mathbb{C}^I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L \) is given by
\[
(\mathcal{X} \ast_N \mathcal{Y})_{i_1,\ldots,i_M,k_1,\ldots,k_L} \overset{\text{def}}{=} \sum_{j_1,\ldots,j_N} a_{i_1,\ldots,i_M,j_1,\ldots,j_N} b_{j_1,\ldots,j_N,k_1,\ldots,k_L}.
\] (14)

Note that we will often abbreviate a tensor product \( \mathcal{X} \ast_N \mathcal{Y} \) to “\( \mathcal{X} \mathcal{Y} \)” for notational simplicity in the rest of the paper. This tensor product will be reduced to the standard matrix multiplication as \( L = M = N = 1 \). Other simplified situations can also be extended as tensor–vector product (\( M > 1, N = 1, \text{and} \ L = 0 \)) and tensor–matrix product (\( M > 1, \text{and} \ N = L = 1 \)). In analogy to matrix analysis, we define some basic tensors and elementary tensor operations as follows.

**Definition 1** A tensor whose entries are all zero is called a zero tensor, denoted by \( O \).

**Definition 2** An identity tensor \( I \in \mathbb{C}^I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N \) is defined by
\[
(I)_{i_1,\ldots,i_N,j_1,\ldots,j_N} \overset{\text{def}}{=} \prod_{k=1}^N \delta_{i_k,j_k},
\] (15)
where \( \delta_{i_k,j_k} \overset{\text{def}}{=} 1 \) if \( i_k = j_k \); otherwise \( \delta_{i_k,j_k} \overset{\text{def}}{=} 0 \).

In order to define Hermitian tensor, the conjugate transpose operation (or Hermitian adjoint) of a tensor is specified as follows.

**Definition 3** Given a tensor \( \mathcal{X} \overset{\text{def}}{=} (a_{i_1,\ldots,i_M,j_1,\ldots,j_N}) \in \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N \), its conjugate transpose, denoted by \( \mathcal{X}^H \), is defined by
\[
(\mathcal{X}^H)_{j_1,\ldots,j_N,i_1,\ldots,i_M} \overset{\text{def}}{=} \overline{a_{i_1,\ldots,i_M,j_1,\ldots,j_N}},
\] (16)
where the overline notion indicates the complex conjugate of the complex number \( a_{i_1,\ldots,i_M,j_1,\ldots,j_N} \). If a tensor \( \mathcal{X} \) satisfies \( \mathcal{X}^H = \mathcal{X} \), then \( \mathcal{X} \) is a Hermitian tensor.

**Definition 4** Given a tensor \( \mathcal{U} \overset{\text{def}}{=} (u_{i_1,\ldots,i_M,j_1,\ldots,j_M}) \in \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M \), if
\[
\mathcal{U}^H \ast_M \mathcal{U} = \mathcal{U} \ast_M \mathcal{U}^H = I \in \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M,
\] (17)
then \( \mathcal{U} \) is a unitary tensor.

In this work, the symbol \( \mathcal{U} \) is reserved for a unitary tensor.

**Definition 5** Given a square tensor \( \mathcal{Y} \overset{\text{def}}{=} (a_{i_1,\ldots,i_M,j_1,\ldots,j_M}) \in \mathbb{C}^I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M \) such that
\[
\mathcal{Y} \ast_M \mathcal{X} = \mathcal{X} \ast_M \mathcal{Y} = I,
\] (18)
then \( \mathcal{X} \) is the inverse of \( \mathcal{Y} \). We usually write \( \mathcal{X} \overset{\text{def}}{=} \mathcal{Y}^{-1} \) thereby.
We also list other crucial tensor operations here. The \textit{trace} of a square tensor is equivalent to the summation of all diagonal entries such that
\[
\text{Tr}(X) \overset{\text{def}}{=} \sum_{1 \leq i, j \leq I, j \in [M]} X_{i, \ldots, i, i, \ldots, i}.
\] (19)

The \textit{inner product} of two tensors $X, Y \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is given by
\[
\langle X, Y \rangle \overset{\text{def}}{=} \text{Tr}(X^H \ast_M Y).
\] (20)

According to Eq. (20), the \textit{Frobenius norm} of a tensor $X$ is defined by
\[
\|X\| \overset{\text{def}}{=} \sqrt{\langle X, X \rangle}.
\] (21)

We use $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ to represent the minimum and the maximum eigenvalues of a Hermitian tensor \cite{34}. The notation $\succeq$ is used to indicate the semidefinite ordering of tensors. If we have $X \succeq Y$, this means that the difference tensor $X - Y$ is a positive semidefinite tensor \cite{34}.

Following theorem is the min-max theorem for a Hermitian tensor under Einstein product.

\textbf{Theorem 1} Given a Hermitian tensor $C \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $k$ positive integers between 1 and $I_1^M$. Then we have
\[
\lambda_k = 1 \max_{S \in \mathbb{R}^{I_1 \times \cdots \times I_M}} \min_{\dim(S) = k} \frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle},
\]
\[
= 2 \min_{T \in \mathbb{R}^{I_1 \times \cdots \times I_M}} \max_{\dim(T) = I_1^M - k + 1} \frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle},
\] (22)

where $\lambda_k$ is the $k$-th largest eigenvalue of the tensor $C$.

\textbf{Proof:} We will just verify the first characterization of $\lambda_k$ by $=1$ in Eq. (22). The other characterization by $=2$ in Eq. (22) can be proved similar. For every $X \in S$ with $\dim(S) = k$ spanned by $V_1, V_2, \cdots, V_k$ (unitary orthogonal tensors), we can write $X = \sum_{j=1}^k c_j V_j$. To show that the value $\lambda_k$ is achievable, note that
\[
\frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle} = \sum_{j=1}^k \lambda_j c_j^* c_j \geq \sum_{j=1}^k \lambda_k c_j^* c_j = \lambda_k.
\] (23)

To verify that this is the maximum, since $T$ has dimension $\mathbb{I}_1^{M} - k + 1$ spanned by $V_k, V_{k+1}, \cdots, V_{I_1^M}$, we have
\[
\min_{X \in S} \frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle} \leq \min_{X \in S \cap T} \frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle}.
\] (24)

Any such $X \in S \cap T$ can be expressed as $X = \sum_{j=k}^{I_1^M} c_j V_j$. Then, we have
\[
\frac{\langle X, C \ast_M X \rangle}{\langle X, X \rangle} = \frac{\sum_{j=k}^{I_1^M} \lambda_j c_j^* c_j}{\sum_{j=k}^{I_1^M} c_j^* c_j} \leq \frac{\sum_{j=k}^{I_1^M} \lambda_k c_j^* c_j}{\sum_{j=k}^{I_1^M} c_j^* c_j} = \lambda_k.
\] (25)
inequality ≥

Proof: Due to Theorem 1, we will prove the inequality

Therefore, for all subspaces \( S \) of dimensions \( k \), we have \( \min_{X \in S} \frac{\langle X, \lambda X \rangle}{\langle X, X \rangle} \leq \lambda_k. \)

We then can apply Theorem [1] to prove Weyl inequality for Hermitian tensors under Einstein product.

**Theorem 2** Suppose \( A, B \in \mathbb{C}^{I_1 \times \cdots \times I_M} \) are Hermitian tensors with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2^M} \) and \( \epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{2^M} \), respectively. Let \( \mathcal{C} = A + B \) with eigenvalues \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{2^M} \). We then have:

\[
\lambda_k + \epsilon_1 \geq \mu_k \geq \lambda_k + \epsilon_{2^M}.
\]

**Proof:** Due to Theorem [1], we will prove the inequality \( \geq 1 \) by \( \min_{T \in \mathbb{R}^{I_1 \times \cdots \times I_M}} \max_{\lambda \in T} \frac{\langle X, (A + B) * M \lambda \rangle}{\langle X, X \rangle} \) only since the inequality \( \geq 2 \) can be proved similarly from \( \max_{S \subseteq \mathbb{R}^{I_1 \times \cdots \times I_M}} \min_{\lambda \in S} \frac{\langle X, C * M \lambda \rangle}{\langle X, X \rangle} \).

Because we have

\[
\mu_k = \min_{T \in \mathbb{R}^{I_1 \times \cdots \times I_M}} \max_{\lambda \in T} \frac{\langle X, (A + B) * M \lambda \rangle}{\langle X, X \rangle}
\]

Then, this theorem is proved. □

### 2.2 Tensor Bernstein Bounds

Tensor Bernstein inequality is an important inequality to bound the sum of independent, bounded random tensors by restricting the range of the maximum eigenvalue of each random tensor. Following theorem is proved at Theorem 6.2 in [30].

**Theorem 3 (Bounded \( \lambda_{\max} \) Tensor Bernstein Bounds)** Given a finite sequence of independent Hermitian tensors \( \{\mathcal{X}_i \in \mathbb{C}^{I_1 \times \cdots \times I_M} \mid i = 1, \ldots, N\} \) that satisfy

\[
\mathbb{E} \mathcal{X}_i = 0 \text{ and } \lambda_{\max}(\mathcal{X}_i) \leq T \text{ almost surely.}
\]

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^N \mathbb{E} \left( \mathcal{X}_i^2 \right) \right\|. \) Then, we have following inequalities:

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^N \mathcal{X}_i \right) \geq \theta \right) \leq \Pi_i^M \exp \left( \frac{-\theta^2/2}{\sigma^2 + T\theta/3} \right); \tag{29}
\]

and

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^N \mathcal{X}_i \right) \geq \theta \right) \leq \Pi_i^M \exp \left( \frac{-3\theta^2}{8\sigma^2} \right) \text{ for } \theta \leq \sigma^2/T; \tag{30}
\]
and
\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq \frac{1}{\sqrt{2\pi T}} \exp \left( -\frac{3\theta}{8T} \right) \text{ for } \theta \geq \sigma^2 / T. 
\]

3 Quadratic Form for Random Hermitian Tensors and Its Diagonal Sum

In Section 3.1, we will formulate our quadratic form for random tensors used for Hanson-Wright inequality under Einstein product. Under such quadratic formulation, we will separate this form into the diagonal sum and the coupling sum parts. The probability bound for the diagonal sum part will be presented by Section 3.2. The coupling sum part will be discussed at next Section 4.

3.1 Quadratic Form for Random Tensors

We define a vector of random tensors \( \Xi \in \mathbb{R}^{(n \times I_1 \times \cdots \times I_M) \times (I_1 \times \cdots \times I_M)} \) as:
\[
\Xi = \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix},
\]
(32)
where random Hermitian tensors \( X_i \) are independent random tensors with \( \mathbb{E} X_i = O \) for \( 1 \leq i \leq n \). We also require another fixed tensor \( \Xi \in \mathbb{R}^{(n \times I_1 \times \cdots \times I_M) \times (n \times I_1 \times \cdots \times I_M)} \), which is defined as:
\[
\Xi = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n,1} & A_{n,2} & \cdots & A_{n,n}
\end{bmatrix},
\]
(33)
where \( A_{i,j} \in \mathbb{R}^{(I_1 \times \cdots \times I_M) \times (I_1 \times \cdots \times I_M)} \) are Hermitian tensors also.

By independence and zero mean of \( X_i \), we can represent \( \Xi^T \Xi - \mathbb{E} \left( \Xi^T \Xi \right) \) as
\[
\Xi^T \Xi - \mathbb{E} \left( \Xi^T \Xi \right) = \sum_{i=1,j=1}^{n} \lambda_i \mathcal{M} A_{i,j} \mathcal{M} X_j - \sum_{i=1}^{n} \mathbb{E} \left( \lambda_i \mathcal{M} A_{i,i} \mathcal{M} X_i \right)
\]
\[
= \sum_{i=1}^{n} \left( \lambda_i \mathcal{M} A_{i,i} \mathcal{M} X_i - \mathbb{E} \left( \lambda_i \mathcal{M} A_{i,i} \mathcal{M} X_i \right) \right)
+ \sum_{1 \leq i \neq j \leq n} A_{i,j} \mathcal{M} X_i \mathcal{M} X_j
\]
(34)

From Theorem 2 and Eq. (34), we have
\[
\lambda_{\max} \left( \Xi^T \Xi - \mathbb{E} \left( \Xi^T \Xi \right) \right)
\leq \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} \lambda_i \mathcal{M} A_{i,j} \mathcal{M} X_j \right) + \lambda_{\max} \left( \sum_{i=1}^{n} \left( \lambda_i \mathcal{M} A_{i,i} \mathcal{M} X_i - \mathbb{E} \left( \lambda_i \mathcal{M} A_{i,i} \mathcal{M} X_i \right) \right) \right)
\]
(35)
Therefore, we have

\[
\Pr \left( \lambda_{\text{max}} \left( \frac{\mathbf{X}^\top \mathbf{X}}{n} - \mathbb{E} \left( \frac{\mathbf{X}^\top \mathbf{X}}{n} \right) \right) \geq \theta \right) \leq \Pr \left( \lambda_{\text{max}} \left( \sum_{1 \leq i \neq j \leq n} \mathbf{X}_i \star_M \mathbf{A}_{i,j} \star_M \mathbf{X}_j \right) \geq \frac{\theta}{2} \right) + \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} (\mathbf{X}_i \star_M \mathbf{A}_{i,i} \star_M \mathbf{X}_i - \mathbb{E} (\mathbf{X}_i \star_M \mathbf{A}_{i,i} \star_M \mathbf{X}_i)) \right) \geq \frac{\theta}{2} \right),
\]

(36)

where \( P_{\text{cp}} \) and \( P_{\text{dg}} \) are probability bounds related to the coupling sum and the diagonal sum parts, respectively.

### 3.2 Diagonal Sum of Random Tensors

The purpose of this subsection is to determine the bound for the probability \( P_{\text{dg}} \). If we define the following relation:

\[
\mathcal{Y}_i \overset{\text{def}}{=} \mathbf{X}_i \star_M \mathbf{A}_{i,i} \star_M \mathbf{X}_i - \mathbb{E} (\mathbf{X}_i \star_M \mathbf{A}_{i,i} \star_M \mathbf{X}_i),
\]

(37)

then, we have \( P_{\text{dg}} \) expressed as:

\[
P_{\text{dg}} = \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{Y}_i \right) \geq \frac{\theta}{2} \right).
\]

(38)

From Theorem 3, we will have following lemma about the bound for \( P_{\text{dg}} \).

**Lemma 1 (Bound for \( P_{\text{dg}} \))** Suppose a finite sequence of independent Hermitian tensors \( \{\mathcal{Y}_i \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \} \), which are defined by Eq. (37), that satisfy

\[
\mathbb{E} \mathcal{Y}_i = 0 \text{ and } \lambda_{\text{max}} (\mathcal{Y}_i) \leq T_{dg} \text{ almost surely.}
\]

(39)

Define the total variance \( \sigma_{\text{dg}}^2 \) as: \( \sigma_{\text{dg}}^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} \mathbb{E} (\mathcal{Y}_i^2) \right\| \). Then, we have following inequalities:

\[
P_{\text{dg}} = \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{Y}_i \right) \geq \frac{\theta}{2} \right) \leq I_1^M \exp \left( \frac{-\theta^2_{\text{dg}}}{8 \sigma_{\text{dg}}^2 + 4 T_{dg} \theta / 3} \right); \]

(40)

and

\[
P_{\text{dg}} = \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{Y}_i \right) \geq \frac{\theta}{2} \right) \leq I_1^M \exp \left( \frac{-3 \theta^2}{32 \sigma_{\text{dg}}^2} \right) \text{ for } \theta \leq 2 \sigma_{\text{dg}}^2 / T_{dg};
\]

(41)

and

\[
P_{\text{dg}} = \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{Y}_i \right) \geq \frac{\theta}{2} \right) \leq I_1^M \exp \left( \frac{-3 \theta}{16 T_{dg}} \right) \text{ for } \theta \geq 2 \sigma_{\text{dg}}^2 / T_{dg}.
\]

(42)
4 Coupling Sum of Random Hermitian Tensors

Our next goal is to bound the probability $P_{cp}$, which is

$$P_{cp} = \Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X_i \ast_M A_{i,j} \ast_M X_j \right) \geq \frac{\theta}{2} \right).$$  \hspace{1cm} (43)

However, it is not independent among each summand $X_i \ast_M A_{i,j} \ast_M X_j$. In order to apply Bernstein bounds in Theorem 3, we are interested in decoupling the following expression

$$\sum_{1 \leq i \neq j \leq n} X_i \ast_M A_{i,j} \ast_M X_j \geq \theta$$

by the following expression

$$\sum_{1 \leq i \neq j \leq n} X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(2)},$$  \hspace{1cm} (44)

where $\{X_i^{(1)}\}, \{X_i^{(2)}\}$ are two independent copies of random tensors $\{X_i\}$. We will prove the following relation:

$$\Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X_i \ast_M A_{i,j} \ast_M X_j \right) \geq \theta \right)$$

$$\leq C_4 \cdot \Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(2)} \right) \geq \frac{\theta}{C_4} \right),$$  \hspace{1cm} (45)

where $C_4$ is a positive constant. Our decoupling method discussed in this section is based on the work in [35], but we extend their approach to the setting of tensors hereof.

We will present several lemmas before proving the main result of this section.

**Lemma 2** Let $S_n$ to be

$$S_n = \sum_{1 \leq i \neq j \leq n} \left( X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(1)} + X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(2)} \right)$$

$$+ X_i^{(2)} \ast_M A_{i,j} \ast_M X_j^{(1)} + X_i^{(2)} \ast_M A_{i,j} \ast_M X_j^{(2)}$$

we will have

$$\Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} \left( X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(1)} + X_i^{(2)} \ast_M A_{i,j} \ast_M X_j^{(2)} \right) \right) \geq \theta \right)$$

$$\leq \Pr \left( \lambda_{\max} (S_n) \geq \frac{\theta}{3} \right) + 2\Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} \left( X_i^{(1)} \ast_M A_{i,j} \ast_M X_j^{(2)} \right) \geq \frac{\theta}{3} \right) \right)$$  \hspace{1cm} (47)

**Proof:** By Theorem 2 and the triangle inequality of $\lambda_{\max}$.

**Lemma 3** Let $X, Y$ be two independent and identically distributed random Hermitian tensors with $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Then

$$\Pr (\lambda_{\max} (X) \geq \theta) \leq 3\Pr \left( \lambda_{\max} (X + Y) \geq \frac{2\theta}{3} \right),$$  \hspace{1cm} (48)

where $\theta > 0$.  

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Proof: Let \( Z \) be another independent and identically distributed random Hermitian tensors compared to random Hermitian tensors \( X,Y \) with \( \mathbb{E}(Z) = \mathcal{O} \). Then, we have

\[
\Pr (\lambda_{\text{max}} (X) \geq \theta) = \Pr (\lambda_{\text{max}} ((X + Y) + (X + Z) - (Y + Z)) \geq 2\theta) \\
\leq \Pr (\lambda_{\text{max}}(X + Y) \geq \frac{2\theta}{3}) + \Pr (\lambda_{\text{max}}(X + Z) \geq \frac{2\theta}{3}) \\
+ \Pr (\lambda_{\text{max}}(Y + Z) \geq \frac{2\theta}{3}) \\
= 3\Pr (\lambda_{\text{max}}(Y + Z) \geq \frac{2\theta}{3})
\]

(49)

From Lemma 2, the proof for the relation provided by Eq. (45) can be reduced as the proof of following two bounds

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{1 \leq i \neq j \leq n} X^{(2)}_{i} \odot_{M} A_{i,j} \odot_{M} X^{(2)}_{j} \right) \geq \theta \right) \\
\leq 3\Pr \left( \lambda_{\text{max}} \left( \sum_{1 \leq i \neq j \leq n} (X^{(1)}_{i} \odot_{M} A_{i,j} \odot_{M} X^{(1)}_{j} + X^{(2)}_{i} \odot_{M} A_{i,j} \odot_{M} X^{(2)}_{j}) \right) \geq \frac{2\theta}{3} \right);
\]

(50)

and, there exists a positive constant \( C_2 \) to have

\[
\Pr (\lambda_{\text{max}} (S_n) \geq \theta) \leq C_2 \Pr \left( C_2 \lambda_{\text{max}} \left( \sum_{1 \leq i \neq j \leq n} X^{(1)}_{i} \odot_{M} A_{i,j} \odot_{M} X^{(2)}_{j} \right) \geq \theta \right),
\]

(51)

where the bound provided by Eq. (50) is obtained by Lemma 3.

We still require two more lemmas before presenting the main result of this section.

Lemma 4 Let \( X \in \mathcal{B} \), where \( \mathcal{B} \) is the Banach space with spectral norm, be any zero mean random Hermitian tensor. Then for all non-random Hermitian tensor \( A \) same dimensions with \( X \) and \( \lambda_{\text{max}} (A) > 0 \), we have

\[
\Pr (\lambda_{\text{max}} (A + X) \geq \lambda_{\text{max}} (A)) \geq \frac{1}{4} \inf_{f \in F} \left( \frac{\mathbb{E}(|f(X)|)^2}{\mathbb{E}(f^2(X))} \right)
\]

(52)

where \( F \) is the family of linear functionals on \( \mathcal{B} \).

Proof: Note that if \( x \) is a random variable with \( \mathbb{E}x = 0 \), then we have \( \Pr(x \geq 0) \geq \frac{1}{4} \frac{(\mathbb{E}|x|^2)}{\mathbb{E}(x^2)} \). From this fact, we have

\[
\Pr (f(X) \geq 0) \geq \frac{1}{4} \frac{(\mathbb{E}(|f(X)|)^2)}{\mathbb{E}(f^2(X))},
\]

(53)

since if \( f \in F \) is such that \( f(A) = \lambda_{\text{max}}(A) \), then \( \{\lambda_{\text{max}} (A + X) \geq \lambda_{\text{max}} (X)\} \) contains \( \{f (A + X) \geq f (A)\} = \{f (X) \geq 0\} \). □
Lemma 5 Let \( A_{i,j} \) for \( 1 \leq i, j \leq n \), and \( \mathcal{B} \) are non-random Hermitian tensors, where \( \lambda_{\text{max}}(\mathcal{B}) > 0 \). Also let \( \{ \beta_i \} \) be a sequence of independent and symmetric Bernoulli random variables, that is \( \Pr(\beta_i = 1) = \Pr(\beta_i = -1) = \frac{1}{2} \). Then, we have

\[
\Pr\left( \lambda_{\text{max}}\left( \mathcal{B} + \sum_{i=1}^{n} A_{i,i} \beta_i + \sum_{1 \leq i \neq j \leq n} A_{i,j} \beta_i \beta_j \right) \geq \lambda_{\text{max}}(\mathcal{B}) \right) \geq C_3
\]

where \( C_3 \) is a constant depend on \( \left( \sum_{i=1}^{n} A_{i,i} \beta_i + \sum_{1 \leq i \neq j \leq n} A_{i,j} \beta_i \beta_j \right) \), but independent of \( \mathcal{B} \).

Proof: By setting \( \mathcal{X} = \sum_{i=1}^{n} A_{i,i} \beta_i + \sum_{1 \leq i,j \leq n} A_{i,j} \beta_i \beta_j \) and \( \mathcal{A} = \mathcal{B} \) in Lemma 4 this lemma is proved. \( \square \)

We are ready to present the main Theorem in this section about the bounds on the tail probability by the decoupling inequality.

Theorem 4 Given determinstic Hermitian tensors \( A_{i,j} \) for \( 1 \leq i, j \leq n \), and random Hermitian tensors \( \mathcal{X}_i \) for \( 1 \leq i \leq n \). Then, there is a positive contant \( C_4 \) such that for all \( n \geq 2 \), we have

\[
\Pr\left( \lambda_{\text{max}}\left( \sum_{1 \leq i \neq j \leq n} (\mathcal{X}_i \star_M A_{i,j} \star_M \mathcal{X}_j) \right) \geq \theta \right) \leq C_4 \Pr\left( \lambda_{\text{max}}\left( \sum_{1 \leq i \neq j \leq n} (\mathcal{X}_i \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)}) \right) \geq \frac{\theta}{C_4} \right),
\]

where \( \theta > 0 \).

Proof: From Lemma 2 this theorem can be proved by proving following two bounds:

\[
\Pr\left( \lambda_{\text{max}}\left( \sum_{1 \leq i \neq j \leq n} (\mathcal{X}_i^{(2)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)}) \right) \geq \theta \right) \leq 3 \Pr\left( \lambda_{\text{max}}\left( \sum_{1 \leq i \neq j \leq n} (\mathcal{X}_i^{(1)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(1)} + \mathcal{X}_i^{(2)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)}) \right) \geq \frac{2\theta}{3} \right),
\]

and

\[
\Pr(\lambda_{\text{max}}(S_n) \geq \theta) \leq C_2 \Pr\left( C_2 \lambda_{\text{max}}\left( \sum_{1 \leq i \neq j \leq n} (\mathcal{X}_i^{(1)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)}) \right) \geq \theta \right),
\]

where \( C_2 \) is a constant.

To prove Eq. (57), we first transform the problem of proving Eq. (57) into a problem conditionally with a non-homogeneous binomial in Bernoulli random variables. Let \( \{ \rho_i \} \) be a sequence of independent and symmetric Bernoulli random variables independent of random Hermitian tensors \( \{ \mathcal{X}_i^{(1)} \}, \{ \mathcal{X}_i^{(2)} \} \). Let \( (\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}) = (\mathcal{X}^{(1)}, \mathcal{X}^{(2)}) \) if \( \rho_i = 1 \), and \( (\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}) = (\mathcal{X}^{(2)}, \mathcal{X}^{(1)}) \) if \( \rho_i = -1 \). Then, we have

\[
4 \mathcal{X}_i^{(1)} \star_M A_{i,j} \star_M \mathcal{Z}_j^{(2)} =
(1 + \rho_i)(1 - \rho_j) \mathcal{X}_i^{(1)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(1)} + (1 + \rho_i)(1 + \rho_j) \mathcal{X}_i^{(1)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)} + (1 - \rho_i)(1 - \rho_j) \mathcal{X}_i^{(2)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(1)} + (1 - \rho_i)(1 + \rho_j) \mathcal{X}_i^{(2)} \star_M A_{i,j} \star_M \mathcal{X}_j^{(2)}. \]

(58)
If we define $\mathcal{P}$ as a realization of $\rho_i$ for $1 \leq i \leq n$, we have

$$4\mathbb{E}(Z^{(1)}_i \ast_M A_{i,j} \ast_M Z^{(2)}_j | \mathcal{P}) = \{X^{(1)}_i \ast_M A_{i,j} \ast_M X^{(1)}_j + X^{(1)}_i \ast_M A_{i,j} \ast_M X^{(2)}_j + X^{(2)}_i \ast_M A_{i,j} \ast_M X^{(1)}_j + X^{(2)}_i \ast_M A_{i,j} \ast_M X^{(2)}_j\}. \quad (59)$$

By setting $B = S_n$ in Lemma 5 and Eqs. (46), (58), (59), we have

$$\Pr \left( 4\lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} (Z^{(1)}_i \ast_M A_{i,j} \ast_M Z^{(2)}_j) \right) \geq \lambda_{\max}(S_n) | \mathcal{P} \right) \geq C_4. \quad (60)$$

By integrating over the set $\{\lambda_{\max}(S_n) \geq \theta\}$, we obtain

$$\frac{1}{C_4} \Pr (\lambda_{\max}(S_n) \geq \theta) \leq \Pr \left( 4\lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} (Z^{(1)}_i \ast_M A_{i,j} \ast_M Z^{(2)}_j) \right) \geq \theta \right) = \Pr \left( 4\lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} (X^{(1)}_i \ast_M A_{i,j} \ast_M X^{(2)}_j) \right) \geq \theta \right), \quad (61)$$

because the sequence $\{X^{(1)}_i, X^{(2)}_i\}$ for $1 \leq i \leq n$ has the same distribution as $\{Z^{(1)}_i, Z^{(2)}_i\}$ for $1 \leq i \leq n$. The proof is completed by using inequality in Eq. (61) with inequalities in Lemma 2 and Eq. (56).

We are ready to determine the bound for the probability $P_{cp}$. If we define the following relation:

$$Z_k \equiv X^{(1)}_i \ast_M A_{i,j} \ast_M X^{(2)}_j, \quad (62)$$

then, we have $P_{cp}$ expressed as:

$$P_{cp} = \Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X^{(1)}_i \ast_M A_{i,j} \ast_M X^{(2)}_j \right) \geq \frac{\theta}{2} \right)$$

$$\leq 1 \cdot C_4 \Pr \left( \lambda_{\max} \left( \sum_{k=1}^{n^2-n} Z_k \right) \geq \frac{\theta}{2C_4} \right), \quad (63)$$

where $\leq 1$ is due to Theorem 4.

From Theorem 4, we will have following lemma about the bound for $P_{cp}$.

**Lemma 6 (Bound for $P_{cp}$)** Given any realization of the random tensor $X_i$, denoted as $\tilde{X}_i$, we assume that

$$\lambda_{\max} (\tilde{X}_i \ast_M A_{i,j} \ast_M X_j) \leq T_{cp} \text{ almost surely,} \quad (64)$$

where $T_{cp}$ is a positive real number, and all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. The total variance with respect to the tensor $\tilde{X}_i$ is defined as

$$\sigma^2_{cp}(\tilde{X}_i) \equiv \left\| \sum_{j=1, j \neq i}^{n} \mathbb{E} \left( (\tilde{X}_i \ast_M A_{i,j} \ast_M X_j)^2 \right) \right\|. \quad (65)$$

The function $f(\tilde{X}_i)$ is the probability density function for the realization tensor $\tilde{X}_i$. We also assume that $X_i \sum_{j=1, j \neq i}^{n} A_{i,j} X_j$ are Hermitian tensors for $i = 1, 2, \ldots, n$. 

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Then, we have following inequalities:[1]

\[ P_{cp} \leq C_4 \Pr \left( \lambda_{\text{max}} \left( \sum_{k=1}^{n^2-n} Z_k \right) \geq \frac{\theta}{2C_4} \right) \leq C_4^M \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( \frac{-\theta^2}{8n^2C_4^2\sigma_{cp}^2(X_i) + 4T_{cp}\theta nC_4/3} \right) f(\tilde{X}_i)d\tilde{X}_i; \] (66)

and

\[ P_{cp} \leq C_4 \Pr \left( \lambda_{\text{max}} \left( \sum_{k=1}^{n^2-n} Z_k \right) \geq \frac{\theta}{2C_4} \right) \leq C_4^M \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( \frac{-3\theta^2}{32n^2C_4^2\sigma_{cp}^2(X_i)} \right) f(\tilde{X}_i)d\tilde{X}_i \]

for \( \frac{\theta}{2nC_4} \leq \frac{\sigma_{cp}^2(X_i)}{T_{cp}} \) with respect to \( i = 1, \ldots, n; \) (67)

and

\[ P_{cp} \leq \Pr \left( \lambda_{\text{max}} \left( \sum_{k=1}^{n^2-n} Z_k \right) \geq \frac{\theta}{2C_4} \right) \leq nC_4^M \exp \left( \frac{-3\theta}{16nC_4^2T_{cp}} \right) \]

for \( \frac{\theta}{2nC_4} \geq \frac{\sigma_{cp}^2(X_i)}{T_{cp}} \) with respect to \( i = 1, 2, \ldots, n. \) (68)

**Proof:** Since all Einstein product are same in this proof, we will remove *M* for space saving in this proof. From Eq. (63), we have

\[ C_4 \Pr \left( \lambda_{\text{max}} \left( \sum_{k=1}^{n^2-n} Z_k \right) \geq \frac{\theta}{2C_4} \right) = C_4 \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \sum_{j=1,\neq i}^{n} A_{i,j}X_j^{(2)} \right) \geq \frac{\theta}{2C_4} \right) \]

\[ \leq C_4 \sum_{i=1}^{n} \Pr \left( \lambda_{\text{max}} \left( X_i^{(1)} \left( \sum_{j=1,\neq i}^{n} A_{i,j}X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right), \] (69)

where we apply Theorem[2] in \( \leq 1. \) By conditional probability with respect to \( X_i, \) each term in Eq. (69) can be expressed as

\[ \Pr \left( \lambda_{\text{max}} \left( X_i^{(1)} \left( \sum_{j=1,\neq i}^{n} A_{i,j}X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right) = \]

\[ \int_{\tilde{X}_i^{(1)}} \Pr \left( \lambda_{\text{max}} \left( \tilde{X}_i^{(1)} \left( \sum_{j=1,\neq i}^{n} A_{i,j}X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right) f(\tilde{X}_i^{(1)})d\tilde{X}_i^{(1)}. \] (70)

From Theorem[3] and Eq. (70) with conditions given by Eqs. (64) and (65), we have

\[ \Pr \left( \lambda_{\text{max}} \left( X_i^{(1)} \left( \sum_{j=1,\neq i}^{n} A_{i,j}X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right) \leq \]

\[ \sum_{i=1}^{M} \int_{\tilde{X}_i^{(1)}} \exp \left( \frac{-\theta^2}{8n^2C_4^2\sigma_{cp}^2(X_i^{(1)}) + 4T_{cp}\theta nC_4/3} \right) f(\tilde{X}_i^{(1)})d\tilde{X}_i^{(1)}. \] (71)

[1] Note that all superscripts \( X_i^{(1)} \) are removed since they are random copies of \( X_i. \)
If $\frac{\theta}{2nC_4} \leq \frac{\sigma_{cp}(\tilde{X}_i)}{\sigma_{cp}}$ for all $i = 1, 2, \ldots, n$, we can have the following bound

\[
\Pr \left( \lambda_{\text{max}} \left( X_i^{(1)} \left( \sum_{j=1, i \neq j}^n A_{i,j} X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right) \leq \mathbb{I}_M \int_{\tilde{X}_i^{(1)}} \exp \left( \frac{-3\theta^2}{32n^2C_4^2\sigma_{cp}^2(\tilde{X}_i^{(1)})} \right) f(\tilde{X}_i^{(1)}) d\tilde{X}_i^{(1)}. \tag{72}
\]

On the other hand, if $\frac{\theta}{2nC_4} \geq \frac{\sigma_{cp}(\tilde{X}_i)}{\sigma_{cp}}$ for all $i = 1, 2, \ldots, n$, we can have the following bound

\[
\Pr \left( \lambda_{\text{max}} \left( X_i^{(1)} \left( \sum_{j=1, i \neq j}^n A_{i,j} X_j^{(2)} \right) \right) \geq \frac{\theta}{2nC_4} \right) \leq \mathbb{I}_M \int_{\tilde{X}_i^{(1)}} \exp \left( \frac{-3\theta}{16nC_4 T_{cp}} \right) f(\tilde{X}_i^{(1)}) d\tilde{X}_i^{(1)} = \mathbb{I}_M \exp \left( \frac{-3\theta}{16nC_4 T_{cp}} \right). \tag{73}
\]

This Lemma is proved by combining Eq. (69) with Eqs. (71) (72) and (73), respectively.

5 Hanson-Wright Inequality for Random Tensors

In this section, we will present the proof for the main result of this paper, the Hanson-Wright inequality for random Hermitian tensors.

Proof: By combining Lemma 1 and Lemma 6 with Eq. (36), this theorem is proved.

6 Conclusion

In this work, we generalize the Hanson-Wright inequality from the quadratic forms in independent subgaussian random variables to the random Hermitian tensors. First, we apply Weyl inequality for tensors under the Einstein product and apply this fact to separate the quadratic form of random Hermitian tensors into the diagonal sum and the coupling (non-diagonal) sum parts. Second, we apply decoupling inequality to bound expressions with dependent random Hermitian tensors with independent random Hermitian tensors. Finally, the Hanson-Wright inequality can be obtained by utilizing Bernstein inequality to the diagonal sum part and the coupling sum part, respectively.

Appendix: Hanson-Wright inequality for T-product tensors

The T-product operation between two three order tensors was introduced by Kilmer and her collaborators in [36]. In this Appendix, we will apply the same technique used in the previous sections to first establish Courant-Fischer theorem for a T-product tensor in Appendix A and use this fact to build Weyl inequality for symmetric T-product tensors in Appendix B. Finally, we have the Hanson-Wright inequality for random symmetric T-product tensors presented by Appendix C.
Appendix A  Courant-Fischer Theorem for T-product tensor

If a T-product tensor $\mathcal{C} \in \mathbb{R}^{m \times m \times p}$ can be diagonalized as

$$\text{bcirc}(\mathcal{C}) = (F_m^H \otimes I_m) \text{Diag}(C_i : i \in \{1, \cdots, m\}) (F_m \otimes I_m),$$  \hspace{1cm} (74)

the $j$-th eigenvalue of the matrix $C_i$ is called a T-eigenvalue [37], denoted by $\lambda_{i,j}$. If a symmetric T-product tensor $\mathcal{C} \in \mathbb{R}^{m \times m \times p}$ can be expressed as the format shown by Eq. (74), the T-eigenvalues of $\mathcal{C}$ with respect to the matrix $C_i$ are denoted as $\lambda_{i,k_i}$, where $1 \leq k_i \leq m$, and we assume that $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,m}$ (including multiplicities). Then, $\lambda_{i,k_i}$ is the $k_i$-th largest T-eigenvalue associated to the matrix $C_i$. If we sort all T-eigenvalues of $\mathcal{C}$ from the largest one to the smallest one, we use $\tilde{k}$, a smallest integer between $1$ to $m \times p$ (inclusive) associated with $p$ given non-negative integers $k_1, k_2, \cdots, k_p$ such that there are $k_i$ T-eigenvalues greater or equal than $\lambda_{\tilde{k}}$ for the matrix $C_i$. We set $\tilde{i}$ from $\lambda_{\tilde{k}}$ as

$$\tilde{i} = \arg \min \{ \lambda_{\tilde{k}} = \lambda_{i,k_i} | k_i > 0 \}$$ \hspace{1cm} (75)

Then, we will have the following Courant-Fischer theorem for T-product tensors.

**Theorem 5** Given a symmetric T-product tensor $\mathcal{C} \in \mathbb{R}^{m \times m \times p}$ and $p$ non-negative integers $k_1, k_2, \cdots, k_p$ with $0 \leq k_i \leq m$, then we have

$$\lambda_{\tilde{k}} = \max_{S \in \mathbb{R}^{m \times 1 \times p}} \min_{\mathcal{X} \in S} \frac{\langle \mathcal{X}, \mathcal{C} \ast \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}$$

$$= \min_{\dim(T) = \{m-k_1, \cdots, m-k_i, m-k_{i+1}, \cdots, m-k_p\}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{C} \ast \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}$$ \hspace{1cm} (76)

where $\tilde{i}$ is defined by Eq. (75).

**Proof:**

First, we have to express $\langle \mathcal{X}, \mathcal{C} \ast \mathcal{X} \rangle$ by matrices of $C_i$ and $X_i$ through the representation shown by Eq. (74). It is

$$\langle \mathcal{X}, \mathcal{C} \ast \mathcal{X} \rangle = \frac{1}{p} \langle \text{bcirc}(\mathcal{X}), \text{bcirc}(\mathcal{C}) \text{bcirc}(\mathcal{X}) \rangle$$

$$= \frac{1}{p} \text{Tr} (\text{bcirc}(\mathcal{X})^H \text{bcirc}(\mathcal{C}) \text{bcirc}(\mathcal{X}))$$

$$= \frac{1}{p} \text{Tr} (F_p^H \text{Diag} (x_i^H A_i x_i : i \in \{1, \cdots, p\}) F_p)$$

$$= \frac{1}{p} \text{Tr} (\text{Diag} (x_i^H A_i x_i : i \in \{1, \cdots, p\})) = \frac{1}{p} \sum_{i=1}^{p} x_i^H A_i x_i$$ \hspace{1cm} (77)

Without loss of generality, we can assume that all $k_i$ is positive since if any of these $k_i$ is zero, the term of $\langle \mathcal{X}, \mathcal{C} \ast \mathcal{X} \rangle$ is reduced as $\frac{1}{p} \sum_{i'} x_{i'}^H A_{i'} x_{i'}$, where $k_{i'} > 0$. We will just verify the first characterization of $\lambda_{\tilde{k}}$.

The other is similar. Let $S_i$ be the projection of $S$ to the space with dimension $k_i$ spanned by $v_{i,1}, \cdots, v_{i,k_i}$,
for every $x_i \in S_i$, we can write $x_i = \sum_{j=1}^{k_i} c_{i,j} v_{i,j}$. To show that the value $\lambda_{\tilde{k}}$ is achievable, note that

$$\frac{\langle \mathcal{X}, \mathcal{C} \ast T \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} = \frac{\frac{1}{p} \sum_{i=1}^{p} x_i^H A_i x_i}{\frac{1}{p} \sum_{i=1}^{p} x_i^H x_i} = \frac{\sum_{i=1}^{p} \sum_{j=1}^{k_i} \lambda_{i,j} c_{i,j}^t c_{i,j}}{\sum_{i=1}^{p} k_i c_{i,j}^t c_{i,j}} \geq \frac{\sum_{i=1}^{p} k_i \lambda_{i,j} c_{i,j}^t c_{i,j}}{\sum_{i=1}^{p} k_i c_{i,j}^t c_{i,j}} = \lambda_{\tilde{k}} \quad (78)$$

To verify that this is the maximum, let $T_i$ be the projection of $T$ to the space with dimension $k_i$ with dimension $n - k_i + 1$, then the intersection of $S$ and $T_i$ is not empty. We have

$$\min_{S} \frac{\langle \mathcal{X}, \mathcal{C} \ast T \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} \leq \min_{S \cap T} \frac{\langle \mathcal{X}, \mathcal{C} \ast T \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}. \quad (79)$$

Any such $x_i \in S \cap T_i$ can be expressed as $x_i = \sum_{j=k_i}^{m} c_{i,j} v_{i,j}$, and any $i$ for $i \neq \tilde{i}$, we have $x_i \in S \cap T_i$ expressed as $x_i = \sum_{j=k_i+1}^{m} c_{i,j} v_{i,j}$. Then, we have

$$\frac{\langle \mathcal{X}, \mathcal{C} \ast T \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} = \frac{\frac{1}{p} \sum_{i=1}^{p} x_i^H A_i x_i}{\frac{1}{p} \sum_{i=1}^{p} x_i^H x_i} = \frac{\sum_{i=1}^{p} \sum_{j=k_i+1}^{m} \lambda_{i,j} c_{i,j}^t c_{i,j}}{\sum_{i=1}^{p} \sum_{j=k_i+1}^{m} c_{i,j}^t c_{i,j}} \leq \frac{\sum_{i=1}^{p} \sum_{j=k_i+1}^{m} \lambda_{i,j} c_{i,j}^t c_{i,j}}{\sum_{i=1}^{p} \sum_{j=k_i+1}^{m} c_{i,j}^t c_{i,j}} = \lambda_{\tilde{k}}. \quad (80)$$

Therefore, for all subspaces $S$ of dimensions $\{k_1, \ldots, k_p\}$, we have $\min_{S} \langle \mathcal{X}, \mathcal{C} \ast T \rangle/\langle \mathcal{X}, \mathcal{X} \rangle \leq \lambda_{\tilde{k}}$. \hfill $\square$

### Appendix B  Weyl Inequality for Symmetric T-product Tensors

We then can apply Theorem 5 to prove Weyl inequality for symmetric T-product tensors.

**Theorem 6** Suppose $A, B \in \mathbb{R}^{m \times m \times p}$ are symmetric tensors with T-eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{mp}$ and $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{mp}$, respectively. Let $C = A + B$ with T-eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{mp}$. We then have:

$$\lambda_{\tilde{k}} + \epsilon_1 \geq \mu_{\tilde{k}} \geq \lambda_{\tilde{k}} + \epsilon_{mp}, \quad (81)$$

where $1 \leq \tilde{k} \leq mp$, where $\tilde{k}$ is associated to $p$ non-negative integers $k_1, \ldots, k_p$ between 0 and $m$ inclusive.
Proof: Due to Theorem 5, we will prove the inequality $\geq 1$ only based on

$$
\lambda_{\hat{k}} = \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{C} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}, \quad (82)
$$

since the inequality $\geq 2$ can be proved similarly from $\lambda_{\hat{k}} = \max_{S \in \mathbb{R}^{m \times 1 \times p}} \min_{\mathcal{X} \in S} \frac{\langle \mathcal{X}, \mathcal{C} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}$.

Because we have

$$
\lambda_{\hat{k}} = \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{A} \times \mathcal{X} \rangle + \langle \mathcal{X}, \mathcal{B} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} \\
\leq \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{A} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} + \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{B} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} \\
\leq \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{A} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle} + \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{B} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}
$$

$$
+ \min_{T \in \mathbb{R}^{m \times 1 \times p}} \max_{\mathcal{X} \in T} \frac{\langle \mathcal{X}, \mathcal{B} \times \mathcal{X} \rangle}{\langle \mathcal{X}, \mathcal{X} \rangle}
$$

$$
= \lambda_{\hat{k}} + \epsilon_1. \quad (83)
$$

Then, this theorem is proved. \qed

Appendix C  Hanson-Wright Inequality for Random Symmetric T-product Tensors

For random variables, Bernstein inequalities give the upper tail of a sum of independent, zero-mean random variables that are either bounded or subexponential. In Theorem 1.7 at [38], we proved Bernstein bounds for a sum of zero-mean random T-product tensors.

Theorem 7 (T-product Tensor Bernstein Bounds with Bounded $\lambda_{\max}$) Given a finite sequence of independent Hermitian T-product tensors $\{\mathcal{X}_i \in \mathbb{C}^{m \times m \times p}\}$ that satisfy

$$
\mathbb{E} \mathcal{X}_i = 0 \quad \text{and} \quad \lambda_{\max}(\mathcal{X}_i) \leq T \quad \text{almost surely.} \quad (84)
$$

Define the total variance $\sigma^2$ as: $\sigma^2 \equiv \left\| \sum_{i=1}^{n} \mathbb{E}(\mathcal{X}_i^2) \right\|$. Then, we have following inequalities:

$$
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \geq \theta \right) \leq m p \exp \left( \frac{-\theta^2 / 2}{\sigma^2 + T \theta / 3} \right); \quad (85)
$$

and

$$
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \geq \theta \right) \leq m p \exp \left( \frac{-3 \theta^2}{8 \sigma^2} \right) \quad \text{for} \ \theta \leq \sigma^2 / T; \quad (86)
$$

Appendix C  Hanson-Wright Inequality for Random Symmetric T-product Tensors
and

$$\Pr\left(\lambda_{\max}\left(\sum_{i=1}^{n} X_i^2\right) \geq \theta\right) \leq m p \exp\left(\frac{-3\theta}{8T}\right) \text{ for } \theta \geq \sigma^2 / T. \quad (87)$$

Finally, we present the Hanson-Wright Inequality for random symmetric T-product tensors.

**Theorem 8 (Hanson-Wright Inequality for Random Symmetric T-product Tensors)**\(\text{thmHWThm}\) We define a vector of random T-product tensors \(\overline{X} \in \mathbb{R}^{(n \times m) \times m \times p}\) as:

$$\overline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad (88)$$

where random symmetric T-product tensors \(X_i \in \mathbb{R}^{m \times m \times p}\) are independent random symmetric T-product tensors with \(\mathbb{E}X_i = O\) for \(1 \leq i \leq n\). We also require another fixed tensor \(\overline{A} \in \mathbb{R}^{(n \times m) \times (n \times m) \times p}\), which is defined as:

$$\overline{A} = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{bmatrix}, \quad (89)$$

where \(A_{i,j} \in \mathbb{R}^{m \times m \times p}\) are symmetric T-product tensors also. We also require the following assumptions.

Define random Hermitian tensor \(Y_i\) as

$$Y_i \overset{\text{def}}{=} X_i \star A_{i,i} \star X_i - \mathbb{E}(X_i \star A_{i,i} \star X_i), \quad \text{for } 1 \leq i \leq n; \quad (90)$$

we assume that

$$\mathbb{E}Y_i = O \quad \text{and} \quad \lambda_{\max}(Y_i) \leq T_{dg} \text{ almost surely.} \quad (91)$$

Define the total variance \(\sigma^2_{dg}\) as:

$$\sigma^2_{dg} \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} \mathbb{E}(Y_i^2) \right\|, \quad \text{where } \| \cdot \| \text{ represents the spectral norm, which equals the largest singular value of a T-product tensor.} \quad (92)$$

Moreover, we define random Hermitian tensor \(Z_k\) for \(k = 1, 2, \cdots, n^2 - n\) as

$$Z_k \overset{\text{def}}{=} X_i^{(1)} \star A_{i,j} \star X_j^{(2)} \quad \text{for } 1 \leq i \neq j \leq n; \quad (93)$$

where the tensors \(X_i^{(1)}\) are identical distribution copy for the tensors \(X_i\), and the tensors \(X_j^{(2)}\) are identical distribution copy for the tensors \(X_j\), then we assume that

$$\mathbb{E}Z_k = O \text{ almost surely.} \quad (94)$$

Given any realization of the random tensor \(X_i\), denoted as \(\tilde{X}_i\), we assume that

$$\lambda_{\max}\left(\tilde{X}_i \star M A_{i,j} \star M \tilde{X}_j\right) \leq T_{cp} \text{ almost surely,} \quad (95)$$

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where $T_{cp}$ is a positive real number, and all $1 \leq i \neq j \leq n$. The total variance with respect to the tensor $\tilde{X}_i$ is defined as

$$\sigma_{cp}^2(\tilde{X}_i) \overset{\text{def}}{=} \left\| \sum_{j=1, j\neq i}^{n} \mathbb{E} \left( (\tilde{X}_i \ast M_{i,j} \ast M_{j,i})^2 \right) \right\|.$$  (95)

The function $f(\tilde{X}_i)$ is the probability density function for the realization tensor $\tilde{X}_i$.

Then, we have

$$\Pr \left( \lambda_{\max} \left( \tilde{X}_i^T A \tilde{X}_i - \mathbb{E} \left( \tilde{X}_i^T A \tilde{X}_i \right) \right) \geq \theta \right) \leq \Pr \left( \lambda_{\max} \left( \sum_{1 \leq i \neq j \leq n} X_i \ast A_{i,j} \ast X_j \right) \geq \frac{\theta}{2} \right) \overset{\text{def}}{=} P_{cp}$$

$$+ \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} (X_i \ast A_{i,i} \ast X_i - \mathbb{E} (X_i \ast A_{i,i} \ast X_i)) \right) \geq \frac{\theta}{2} \right)$$

$$\leq C_4 mp \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( \frac{-\theta^2}{8n^2 C_4^2 \sigma_{cp}^2(\tilde{X}_i) + 4T_{cp} \theta n C_4 / 3} \right) f(\tilde{X}_i) d\tilde{X}_i$$

$$+ mp \exp \left( \frac{-\theta^2}{8\sigma_{dg}^2 + 4T_{dg} \theta / 3} \right),$$  (96)

where $P_{cp}$ and $P_{dg}$ are probability bounds related to the coupling sum and the diagonal sum parts, respectively, and the term $C_4$ is a positive constant.

If $\frac{\theta^2}{2nC_4} \leq \frac{\sigma_{cp}^2(\tilde{X}_i)}{T_{cp}}$ with respect to $i = 1, 2, \ldots, n$ and $\theta \leq 2\sigma_{dg}^2 / T_{dg}$, we have

$$\Pr \left( \lambda_{\max} \left( \tilde{X}_i^T A \tilde{X}_i - \mathbb{E} \left( \tilde{X}_i^T A \tilde{X}_i \right) \right) \geq \theta \right) \leq mpC_4 \sum_{i=1}^{n} \int_{\tilde{X}_i} \exp \left( \frac{-3\theta^2}{32n^2 C_4^2 \sigma_{cp}^2(\tilde{X}_i)} \right) f(\tilde{X}_i) d\tilde{X}_i$$

$$+ mp \exp \left( \frac{-3\theta^2}{32\sigma_{dg}^2} \right).$$  (97)

Moreover, if $\frac{\theta^2}{2nC_4} \geq \frac{\sigma_{cp}^2(\tilde{X}_i)}{T_{cp}}$ with respect to $i = 1, 2, \ldots, n$, and $\theta \geq 2\sigma_{dg}^2 / T_{dg}$, we have

$$\Pr \left( \lambda_{\max} \left( \tilde{X}_i^T A \tilde{X}_i - \mathbb{E} \left( \tilde{X}_i^T A \tilde{X}_i \right) \right) \geq \theta \right) \leq mnpC_4 \exp \left( \frac{-3\theta}{16nC_4 T_{cp}} \right) + mp \exp \left( \frac{-3\theta}{16T_{dg}} \right).$$  (98)

**Proof:** Since the proof arguments of Theorem 1.1 can still be valid for T-product tensors by applying Theorem 6, this theorem is proved by using Theorem 7 to modify Lemma 1 and Lemma 6 for tensors under T-product.

**References**

[1] R. Adamczak, “A note on the hanson-wright inequality for random vectors with dependencies,” *Electronic Communications in Probability*, vol. 20, pp. 1–13, 2015.
[2] D. L. Hanson and F. T. Wright, “A bound on tail probabilities for quadratic forms in independent random variables,” *The Annals of Mathematical Statistics*, vol. 42, no. 3, pp. 1079–1083, 1971.

[3] R. Vershynin, *High-dimensional probability: An introduction with applications in data science*. Cambridge university press, 2018, vol. 47.

[4] F. Krahmer, S. Mendelson, and H. Rauhut, “Suprema of chaos processes and the restricted isometry property,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 11, pp. 1877–1904, 2014.

[5] L. Qi and Z. Luo, *Tensor analysis: spectral theory and special tensors*. SIAM, 2017.

[6] Q. Wu, L. Zhang, and G. Shi, “Robust multifactor speech feature extraction based on gabor analysis,” *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 19, no. 4, pp. 927–936, Aug. 2010.

[7] S. Mirsamadi and J. H. Hansen, “A generalized nonnegative tensor factorization approach for distant speech recognition with distributed microphones,” *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 24, no. 10, pp. 1721–1731, Jun. 2016.

[8] D. Muti and S. Bourennane, “Survey on tensor signal algebraic filtering,” *Signal Processing*, vol. 87, no. 2, pp. 237–249, Feb. 2007.

[9] Y. Shen, X. Fu, G. B. Giannakis, and N. D. Sidiropoulos, “Topology identification of directed graphs via joint diagonalization of correlation matrices,” *IEEE Transactions on Signal and Information Processing over Networks*, vol. 6, pp. 271–283, Apr. 2020.

[10] Y. Shen, B. Baingana, and G. B. Giannakis, “Tensor Decompositions for Identifying Directed graph Topologies and Tracking Dynamic Networks,” *IEEE Transactions on Signal Processing*, vol. 65, no. 14, pp. 3675–3687, Apr. 2017.

[11] X. Fu, K. Huang, W.-K. Ma, N. D. Sidiropoulos, and R. Bro, “Joint tensor factorization and outlying slab suppression with applications,” *IEEE Transactions on Signal Processing*, vol. 63, no. 23, pp. 6315–6328, Aug. 2015.

[12] C.-Y. Ko, K. Batselier, L. Daniel, W. Yu, and N. Wong, “Fast and accurate tensor completion with total variation regularized tensor trains,” *IEEE Transactions on Image Processing*, May 2020.

[13] T.-X. Jiang, M. K. Ng, X.-L. Zhao, and T.-Z. Huang, “Framelet representation of tensor nuclear norm for third-order tensor completion,” *IEEE Transactions on Image Processing*, vol. 29, pp. 7233–7244, Jun. 2020.

[14] A. L. de Almeida, G. Favier, and J. C. M. Mota, “Constrained tensor modeling approach to blind multiple-antenna CDMA schemes,” *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2417–2428, May 2008.

[15] Z. Zhijin, Y. Hui, and J. S. Fangfang QIANG, “Blind estimation of spreading codes for multi-antenna lc-ds-cdma signals based on tensor decomposition,” *Journal on Communications*, vol. 39, no. 10, p. 52, 2018.

[16] D. Nion and N. D. Sidiropoulos, “Tensor algebra and multidimensional harmonic retrieval in signal processing for MIMO radar,” *IEEE Transactions on Signal Processing*, vol. 58, no. 11, pp. 5693–5705, Jul. 2010.
[17] N. D. Sidiropoulos, R. Bro, and G. B. Giannakis, “Parallel factor analysis in sensor array processing,” IEEE Transactions on Signal Processing, vol. 48, no. 8, pp. 2377–2388, Aug. 2000.

[18] X. Wang, M. Che, and Y. Wei, “Neural networks based approach solving multi-linear systems with m-tensors,” Neurocomputing, vol. 351, pp. 33–42, 2019.

[19] W. Ding, L. Qi, and Y. Wei, “Fast Hankel tensor-vector product and its application to exponential data fitting,” Numer. Linear Algebra Appl., vol. 22, no. 5, pp. 814–832, 2015. [Online]. Available: https://doi.org/10.1002/nla.1970

[20] H.-R. Xu, D.-H. Li, and S.-L. Xie, “An equivalent tensor equation to the tensor complementarity problem with positive semi-definite Z-tensor,” Optim. Lett., vol. 13, no. 4, pp. 685–694, 2019. [Online]. Available: https://doi.org/10.1007/s11590-018-1268-4

[21] L.-B. Cui, C. Chen, W. Li, and M. K. Ng, “An eigenvalue problem for even order tensors with its applications,” Linear Multilinear Algebra, vol. 64, no. 4, pp. 602–621, 2016. [Online]. Available: https://doi.org/10.1080/03081087.2015.1071311

[22] A. Anandkumar, R. Ge, D. Hsu, S. M. Kakade, and M. Telgarsky, “Tensor decompositions for learning latent variable models (A survey for ALT),” in Proceedings of International Conference on Algorithmic Learning Theory. Springer, Oct. 2015, pp. 19–38.

[23] N. D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E. E. Papalexakis, and C. Faloutsos, “Tensor decomposition for signal processing and machine learning,” IEEE Transactions on Signal Processing, vol. 65, no. 13, pp. 3551–3582, Jul. 2017.

[24] R. Gurau, Random tensors. Oxford University Press, Oxford, 2017.

[25] I. R. Klebanov and G. Tarnopolsky, “Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models,” Phys. Rev. D, vol. 95, no. 4, pp. 046004, 13, 2017. [Online]. Available: https://doi.org/10.1103/physrevd.95.046004

[26] R. Vershynin, “Concentration inequalities for random tensors,” Bernoulli, vol. 26, no. 4, pp. 3139–3162, 2020. [Online]. Available: https://doi.org/10.3150/20-BEJ1218

[27] S. Y. Chang, “Tensor expander chernoff bounds,” 2021.

[28] ——, “T product tensors part i: Inequalities,” arXiv preprint arXiv:2107.06285, 2021.

[29] ——, “T product tensors part ii: Tail bounds for sums of random t product tensors,” arXiv preprint arXiv:2107.06224, 2021.

[30] ——, “Convenient tail bounds for sums of random tensors,” 2020.

[31] ——, “General tail bounds for random tensors summation: Majorization approach,” 2021.

[32] ——, “Hanson-Wright inequality for random tensors under T-product,” 2021.

[33] M. Liang and B. Zheng, “Further results on Moore-Penrose inverses of tensors with application to tensor nearness problems,” Comput. Math. Appl., vol. 77, no. 5, pp. 1282–1293, 2019. [Online]. Available: https://doi.org/10.1016/j.camwa.2018.11.001

[34] G. Ni, “Hermitian tensor and quantum mixed state,” arXiv preprint arXiv:1902.02640, 2019.
[35] V. H. de la Peña and S. J. Montgomery-Smith, “Bounds on the tail probability of U-statistics and quadratic forms,” *arXiv preprint math/9309210*, 1993.

[36] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover, “Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging,” *SIAM Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 148–172, 2013.

[37] Y. Miao, L. Qi, and Y. Wei, “T-Jordan canonical form and T-Drazin inverse based on the T-product,” *Communications on Applied Mathematics and Computation*, vol. 3, no. 2, pp. 201–220, 2021.

[38] S. Y. Chang, “T product tensors part II: Tail bounds for sums of random T-product tensors,” 2021.