Classically, G. Fano proved that the family of (smooth, anticanonically embedded) Fano 3-dimensional varieties is bounded, and moreover provided their classification, later completed by V.A. Iskovskikh, S. Mukai and S. Mori. For singular Fano varieties with log terminal singularities, there are two basic boundedness conjectures: Index Boundedness and the much stronger $\epsilon$-lt Boundedness.

The $\epsilon$-lt Boundedness was known only in two cases: in dimension 2 [Ale94] and for toric varieties [BB93]. In this paper we prove it for a significantly less “elementary” class, that of spherical varieties. In addition to an argument adapted from the toric case, the proof contains quite a few new twists.

In Section 5, we introduce a new invariant of a spherical subgroup $H$ in a reductive group $G$ which measures how nice an equivariant Fano compactification of $G/H$ there exists.

1. Boundedness conjectures

Let $X$ be a log terminal Fano variety defined over an algebraically closed field $k$, i.e.

1. $X$ is normal,
2. the canonical class $K_X$ is $\mathbb{Q}$-Cartier,
3. for a log resolution of singularities $f : Y \to X$, in the formula
   
   $K_Y = f^*K_X + \sum a_i E_i$

   the log discrepancies $b_i = 1 + a_i$ of exceptional divisors $E_i$ are all positive,
4. $-K_X$ is ample.

**Definition 1.1.** The index of $X$ is the minimal positive integer $I$ such that $I \cdot K_X$ is Cartier.

**Definition 1.2.** The minimal log discrepancy of $X$ is mld($X$) = min $b_i$. For $\epsilon > 0$, we will say that $X$ is $\epsilon$-lt if mld($X$) $> \epsilon$.

**Definition 1.3.** One says that a certain set of varieties (or schemes) is bounded if they appear as geometric fibers of a family $X \to S$, with $X$ and $S$ both schemes of finite type over the base field.

The following conjecture appeared independently in [Ale94] and the work of Alexander and Lev Borisovs [BB93].

**Conjecture 1.4** ($\epsilon$-lt Boundedness or BAB Conjecture). The family of $n$-dimensional $\epsilon$-lt Fano varieties is bounded.

The only cases where this conjecture has been proven are:

1. $n = 2$, arbitrary char $k$ [Ale94].
(2) for toric Fano varieties, arbitrary char k $BB93$.
(3) $n = 3$ and $\epsilon = 1$ (i.e. with terminal singularities), char $k = 0$ $Kaw92$.

A weaker conjecture (which follows from the previous one by taking $\epsilon = 1/I$) is due to V. Batyrev:

**Conjecture 1.5 (Index Boundedness).** The family of $n$-dimensional log terminal Fano varieties of index $I$ is bounded.

This conjecture is known in the following cases not covered by the above (char $k = 0$ everywhere):

1. for smooth Fanos $KMM92$.
2. $n = 3$ $Bor01$.

Recently, J. McKernan $McK02$ announced a proof of Index Boundedness Conjecture for any $n$.

For the rest of the paper, we will be working over an algebraically closed field of characteristic 0.

## 2. Recall of spherical varieties

Let $T = \mathbb{G}_m^n$ be a multiplicative torus. Toric varieties are of course normal varieties with an open $T$-orbit which are (partial) compactifications of $T$. They have a familiar combinatorial description. One starts with two free abelian groups of rank $n$, the group of characters $M$ of $T$ and the dual group $N$. Every toric variety $X$ corresponds uniquely to a fan $\Sigma$ in $N_{\mathbb{Q}}$, and $X$ is complete iff $\text{supp} \Sigma = N_{\mathbb{Q}}$.

$T$-invariant Cartier divisors correspond to piecewise linear functions on the fan. Let $v \in \Sigma(1)$ be the integral generators of 1-dimensional cones, they correspond to $T$-invariant divisors $D_v$. Then $-K_X = \sum D_v$. In particular, $K_X$ is $\mathbb{Q}$-Gorenstein iff for any cone $\sigma$ its integral generators $v \in \sigma(1)$ lie on a common hyperplane.

Spherical varieties generalize this picture to the non-commutative case. Let $G$ be a connected reductive group with a Borel subgroup $B$. A variety $X$ with $G$-action is called spherical if it is normal and has an open $B$-orbit. If $H$ is the stabilizer of a point $x$ in the open orbit then $X$ is a partial compactification of $G/H$. Luna and Vust proved the structure theory of spherical varieties which was later translated into the language of colored fans. The basic references for this theory are $Kno91$ and $Bri97b$. Note that the class of spherical varieties is much richer than that of toric varieties, and they may have fairly complicated singularities (for example $\mathbb{Q}$-factorial spherical singularities need not be quotient).

**Remark 2.1.** A major caveat is that the whole theory is relative to $G/H$. Once the homogeneous variety $G/H$ is fixed, spherical compactifications correspond uniquely to colored fans. The question of which subgroups $H \subset G$ are spherical is open, with complete classification available only for groups of type $A$ $Lun01$. In any case, for any noncommutative group $G$ there exist infinitely many non-isomorphic spherical subgroups, and infinitely many non-isomorphic colored lattices corresponding to them.

Let us summarize all the combinatorial facts that we need and fix the notation. Let $X$ be a spherical embedding of $G/H$. Then we have:

1. a free abelian group $\Lambda$ of rank $r \leq n$ (analog of $M$ for toric varieties), it is defined as the multiplicative group of $B$-eigenvectors of $k(G/H)$ up to scalars, $\Lambda = k(G/H)^{(B)}/k^*$;
(2) its dual $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z})$, which we will denote by $N$, by analogy with the toric case;

(3) the vector space $N_\mathbb{Q} = \Lambda_\mathbb{Q}^*$.

The new elements are:

(4) a valuation cone $V$; this is a co-simplicial cone spanning $N_\mathbb{Q}$;

(5) a finite set $D$ of colors (these are the irreducible $B$-stable divisors in $G/H$) and a map $\rho : D \to N$.

A simple embedding, i.e. one with a unique closed $G$-orbit $Y$, defines two finite sets $V_Y \subseteq V$ and $D_Y \subseteq D$. The corresponding colored cone is the pair $(\sigma_Y, D_Y)$ where the cone $\sigma_Y$ is generated by $V_Y$ and $\rho(D_Y)$. It is known that $\sigma_Y$ is strictly convex and that $\sigma_Y \cap V \neq \emptyset$. Variety $X$ is uniquely described by a colored fan $\Sigma$, i.e. a compatible collection of colored cones. $X$ is complete iff the cones cover $V$; in this case the collection $\{\sigma \cap V\}$ is an ordinary fan with support $V$. In particular, the embeddings without colors correspond to partial subdivisions of $V$. Such embeddings are called toroidal.

An element of $V_X = \bigcup V_Y$, resp. $D_X = \bigcup D_Y$, where the sum goes over the finite set of $G$-orbits $Y$, defines a $G$-invariant, resp. $B$-invariant but not $G$-invariant, divisor on $X$. Every $B$-invariant divisor can be written as

$$-K_X = \sum_{v \in V_X} D_v + \sum_{D \in D_X} a_D D + \sum_{D \in D \setminus D_X} n_D D.$$  

This divisor is Cartier iff the coefficients in the first half of this expression are values of a piecewise linear function $l_\sigma \in \Lambda, \sigma \in \Sigma$ at the points $v, \rho(D)$. The divisors $D \in D \setminus D_X$ are always Cartier. By [Bri97a], the anticanonical divisor on a spherical variety can be written as

$$-K_X = \sum_{v \in V_X} D_v + \sum_{D \in D} a_D D$$

for unique, explicitly computable positive integers $a_D$. Variety $X$ is $\mathbb{Q}$-Gorenstein iff for every colored cone $(\sigma_Y, D_Y)$ the points $v \in \sigma_Y(1)$ and $\rho(D)/a_D, D \in D_Y$, lie on a common hyperplane.

3. Boundedness of toric Fanos

Let us first recall the proof of boundedness in the toric case. The anticanonical divisor on a $\mathbb{Q}$-Gorenstein toric variety corresponds to a piecewise linear function $l$ taking value 1 at every integral generator $v \in \Sigma(1)$. This divisor is ample iff the function is strictly convex, i.e. if $v$'s are vertices of a convex lattice polytope, call it $Q$. The log discrepancies at $T$-invariant exceptional divisors are precisely the values of $l$ at integral point $u \in N$. Since every toric variety has a $T$-equivariant resolution, $X$ is $\epsilon$-lt if and only if the following condition is satisfied (where $Q^0$ denotes the interior):

$$\epsilon Q^0 \cap N = \{0\}$$

Hence, the boundedness follows from the following theorem, [Hen83]:

**Theorem 3.1** (Hensley). Up to $\text{Aut}(N) = \text{GL}(n, \mathbb{Z})$, there are only finitely many lattice polytopes in $N_\mathbb{Q}$ satisfying the condition (2).

Borisovs gave a different proof of Theorem 3.1 in [BB93].
4. Boundedness of spherical Fanos

**Notation 4.1.** Let $X$ be a spherical $G$-variety with open orbit $G/H$. Denote by $C = Z(G)^0$ the connected component of the center, and by $G^{ss} = [G, G]$ the derived subgroup, the semisimple part of $G$. Then $G$ is the quotient of $G^{ss} \times C$ by a finite central subgroup. We will also denote by $\tilde{H} = N_G(H)$ the normalizer of $H$ in $G$. Then $\tilde{H}$ contains $C$, and $\tilde{H}/H = \text{Aut}^G(G/H)$, the automorphism group of the $G$-variety $G/H$. In fact, any element of $\tilde{H}/H$ extends to an automorphism of $X$, i.e., $\text{Aut}^G(X) = \tilde{H}/H$. Further, this group is diagonalizable.

Here is our main theorem:

**Theorem 4.2.** For any $\epsilon > 0$ and $n \in \mathbb{N}$, the set of $\epsilon$-lt $n$-dimensional spherical Fano varieties is finite.

As a first reduction, we can assume that the action is faithful (just divide by the kernel). However, it will be convenient to work with almost faithful actions, in the following sense:

**Definition 4.3.** A $G$-action on $X$ is called almost faithful if its kernel is finite, and $C$ acts faithfully.

As a second reduction, we can assume that $G = G^{ss} \times C$ with simply connected $G^{ss}$ (go to a finite cover); then the Picard group of $G$ is trivial. We can also assume that the action is almost faithful (divide by a finite subgroup of $C$, if necessary).

**Proof of Theorem 4.2.** By Theorem 4.7 below, it suffices to bound the dimension of $G$ in terms of $n = \dim(X)$, where $X$ is $G$-spherical. This will imply that only finitely many isomorphism classes of connected reductive groups $G$ occur, and our theorem will follow.

The dimension of $C$ is bounded by $n$, of course. To bound the dimension of $G^{ss}$, consider a toroidal resolution $\tilde{X}$ of $X$. All closed orbits in $\tilde{X}$ are isomorphic to the same flag variety $G/P = G^{ss}/(P \cap G^{ss})$, on which $G^{ss}$ acts with a finite kernel. And, by a result of Akhiezer [Akh95], the dimension of $\text{Aut} G/P$ is bounded by a function of $\dim G/P \leq n$ only. □

**Definition 4.4.** A $G$-action on $X$ is called smart if

1. the action is almost faithful, and
2. the natural homomorphism $C \to \text{Aut}^G(X)^0$ is an isomorphism.

**Lemma 4.5.** Any almost faithful $G$-action can be extended to a smart action of a bigger connected reductive group.

**Proof.** Consider the map

$$\varphi : G = G^{ss} \times C \to \tilde{G} = G^{ss} \times (\tilde{H}/H)^0, \quad (g, c) \mapsto (g, c\tilde{H}).$$

Then $\varphi$ is a group homomorphism, injective since $C \cap H$ is trivial (as $C$ acts faithfully). Further, $\tilde{G}$ acts on $G/H$ by $(g, \gamma) \cdot xH = gx\gamma H$, this action extends to $X$, and $\varphi$ is equivariant. We check that the $\tilde{G}$-action is smart.

Clearly, the derived subgroup $G^{ss}$ acts with finite kernel. Thus, the connected kernel of the $\tilde{G}$-action is contained in the connected center $(\tilde{H}/H)^0$. But the latter acts faithfully, so that (1) holds. Finally, since $G$ embeds into $\tilde{G}$, we have $\text{Aut}^\tilde{G}(X)^0 \subseteq \text{Aut}^G(X)^0 = (H/H)^0$ which implies (2). □
Definition 4.6. Two $G$-actions are called equivalent if they differ by an automorphism of $G$.

If $G = G^{ss} \times C$ then Aut $G$ contains as a subgroup of finite index the product of the group of inner automorphisms of $G^{ss}$ and of Aut $C \cong \text{GL}(\dim C, \mathbb{Z})$.

Theorem 4.7. For a fixed connected reductive group $G = G^{ss} \times C$ and $\epsilon > 0$, there exist only finitely many spherical $G$-varieties with smart action which are $\epsilon$-lt Fano varieties, up to choosing an equivalent action.

To prepare for the proof, let us first translate in terms of colored fans the conditions that $X$ is Fano and $\epsilon$-lt. Let $X$ be our spherical $G$-variety. Therefore, we have a lattice $N$, valuation cone $V$ and colors $\rho : D \to N$. This time, the anti-canonical divisor corresponds to a piecewise linear function $l = (l_\sigma, \sigma \in \Sigma)$ which takes values 1 at the points $v \in V_X$ and $\rho(D)/a_D$. Ampleness translates into two conditions (see [Bri97b, Ch.5.2]):

1. $l$ is strictly convex; in other words $l_\sigma < l_{\sigma'}$ on $\sigma' \setminus \sigma \cap \sigma$, and
2. $l_\sigma(\rho(D)/a_D) = 1$ for every $D \in D_\sigma$, and $< 1$ for every $D \in D \setminus D_\sigma$.

This means that the linear inequalities $l_\sigma \leq 1$ define a possibly unbounded polyhedral body, call it $P$, such that

1. the points $v \in V_X$ and $\rho(D)/a_D$, $D \in D_X$, are vertices of $P \cap |\Sigma|$, and
2. the points $\rho(D)/a_D$, $D \in D \setminus D_X$, are in the interior $P^0$.

Every spherical variety has a toroidal $G$-equivariant resolution, and the log discrepancies are precisely the values of $l$ at integral points $u \in N \cap V$. Therefore, the $\epsilon$-lt condition is equivalent to

$$\epsilon P^0 \cap N \cap V = \{0\}$$

Now consider the convex hull $Q$ of all points $v \in V_X$ and $\rho(D)/a_D$. It is known that the cone $V$ and the set $\{\rho(D)/a_D\}$ do not lie in a common half-space of $N_\mathbb{Q}$, see [Bri97b, Rem.3.4(3)]. This implies that $Q$ is a polytope of maximal dimension whose interior contains the origin. Clearly, $Q$ also satisfies the condition (3) above.

The role of Hensley’s theorem will be played by the following combinatorial statement:

Lemma 4.8. Suppose we have fixed the following data:

1. a surjective homomorphism of lattices $\pi : N \to \tilde{N}_1$,
2. a strictly convex rational polyhedral cone $\tilde{V}$ generating $\tilde{N}_1 \otimes \mathbb{Q}$ and a cone $V = \pi^{-1}(\tilde{V})$,
3. a positive integer $A$,
4. finitely many points $\tilde{d}_i = \pi(d_i) \in \tilde{N}_1/A$.

Then for every $\epsilon > 0$ there exist only finitely many maximal-dimensional polytopes $Q$ in $N_\mathbb{Q}$ satisfying the following condition:

1. $Q$ is the convex hull of some $d_i \in N/A$ with $\pi(d_i) = \tilde{d}_i$ and finitely many elements of $V \cap N$,
2. $\epsilon Q^0 \cap N \cap V = \{0\}$

up to the action of the group Aut($N, \pi) \subseteq \text{Aut}(N)$ leaving ker $\pi$ invariant and inducing the identity on $\tilde{N}_1$. 

Proof. Of course, by replacing lattices $N, \tilde{N}_1$ by $N/A, \tilde{N}_1/A$ we can assume from the start that $Q$ is a lattice polytope. Polyhedral cone $V$ is defined by several linear inequalities $f_i(x) \geq 0$ in $N_Q$, where $f_i = \tilde{f}_i \circ \pi$ for some linear integral-valued functions on $\tilde{N}_1$. Let $-F = \min \tilde{f}_j(d_i)$. For every point $x$ of the polytope $Q$ one has $f_j(x) \geq -F$. Therefore, if $\epsilon < 1/F$ then 
$$\epsilon Q^0 \cap N \cap V = \epsilon Q^0 \cap N = \{0\}.$$ 

Therefore, by Hensley's theorem there exist only finitely many polytopes satisfying our condition, up to $\text{Aut}(N)$. Now let us fix one of those polytopes $Q$ and ask: for which $g \in \text{Aut}(N)$ does the polytope $gQ$ has the same shape – it is the convex hull of some points $d_i$ with $\pi(d_i) = \tilde{d}_i$ and some elements in $V \cap N$?

The vertices of $Q$ split into two sets: vertices $v_j$ that are in $V$, and vertices lying outside of $V$. The latter are necessarily some of $d_i$’s, say $d_1 \ldots d_s$. Up to a permutation of vertices (finitely many choices), we can assume that $\pi(gd_i) = \pi(d_i), i = 1 \ldots s$, and that $gv_j \in V$. Since $0 \in Q$, there exist some positive integers $n_i, m_j$ such that

$$\sum m_j v_j + \sum_{i=1}^s n_i d_i = 0.$$ 

Therefore, for some positive combination of $v_j$, one has 
$$\bar{v} = \sum m_j \pi(v_j) = \sum m_j \pi(gv_j).$$ 

Consequently, each of $gv_j$ belongs to $V + (\pi^{-1}(\bar{v}) - V)$, and since $\overline{V}$ is strictly convex, this is a finite union of cosets of the lattice ker $\pi$. Hence, up to finitely many choices again, we can assume that $gv_j \in v_j + \text{ker} \pi$. Together with $gd_i \in d_i + \text{ker} \pi, i = 1 \ldots s$, this implies $g \in \text{Aut}(N, \pi)$. 

We will also use the following basic boundedness result from [AB03] which follows from Knop’s theorem about local rigidity of wonderful compactifications:
**Theorem 4.9.** For any connected reductive group $G$, up to conjugation, there exist only finitely many spherical subgroups $H$ (i.e. $G/H$ is spherical) which coincide with their normalizer.

We are now ready for our theorem.

**Proof of Thm. 4.7.** Let $X$ be an embedding of $G/H$. By Theorem 4.9, we can fix the normalizer $\tilde{H}$ from now on. For the connected components of identity we also have $H_0 \subseteq \tilde{H}_0$. We now have four lattices, $N^* = k(G/H)^{(B)/k^*}$ etc., of rank bounded by the rank of $G$, and a diagram

$$
\begin{array}{ccc}
N_0 & \longrightarrow & N \\
\downarrow \pi_0 & & \downarrow \pi \\
\tilde{N}_0 & \longrightarrow & \text{im } N & \longrightarrow & \tilde{N}.
\end{array}
$$

In this diagram, horizontal arrows are injective with finite cokernels, and vertical arrows are surjective ($\pi_0$ is surjective because $H_0, \tilde{H}_0$ are connected). The map $N \to \tilde{N}$ need not be surjective. However, there are only finitely many possibilities for intermediate lattice $\tilde{N}_1 = \text{im } N$, so we can fix one of them.

The sets of colors $\mathcal{D}, \mathcal{D}$ for groups $H, \tilde{H}$ are nearly the same. More precisely, let $B$ be a Borel subgroup such that $BH$ is open in $G$, then the colors of $G/H$ are the irreducible components of $(G \setminus BH)/H$. But $BH = B\tilde{H}$, so that the set $\mathcal{D}$ of irreducible components of $G \setminus BH = G \setminus B\tilde{H}$ is the same. The sets of colors are obtained from $\mathcal{D}$ by dividing by $H$, resp. $\tilde{H}$, which identifies some of the divisors into one color. We have a commutative diagram

$$
\begin{array}{ccc}
\hat{D} & \twoheadrightarrow & D \\
\downarrow & & \downarrow \pi \\
\hat{\mathcal{D}} & \twoheadrightarrow & \mathcal{D} \\
\end{array}
$$

We will denote by $d_i$ (resp. $\tilde{d}_i$) the points $\rho(D)/a_D$, $D \in \mathcal{D}$, (resp. $\tilde{\rho}(\mathcal{D})/a_D$, $\mathcal{D} \in \mathcal{D}$) and by $A$ the LCM$(a_D)$. By Lemma 4.8 there are only finitely many possibilities for the polytope $Q$ in $N_0$, and hence also for the set $\rho(D)$, up to the action of group $\text{Aut}(N, \pi)$.

Consider the multiplicative group $k(G)^{(B \times H)}$ of those rational functions on $G$ that are eigenvectors of $B$ (for left multiplication) and of $H$ (for right multiplication). Any such function may have zeroes and poles along elements of $\mathcal{D}$ only; conversely, any divisor in $\mathcal{D}$ has a global equation (since $G$ has trivial Picard group) which is a $B \times H$-eigenvector. Moreover, the functions in $k(G)^{(B \times H)}$ having no zero or poles are the regular invertible functions on $G$, i.e., the scalar multiples of characters. This yields an exact sequence

$$
0 \to \chi(G) \to k(G)^{(B \times H)}/k^* \to \mathbb{Z}\hat{D} \to 0,
$$

where $\chi(G) = \chi(C)$ denotes the character group, and $\mathbb{Z}\hat{D}$ the free abelian group on $\mathcal{D}$. This sequence splits by sending each $f \in k(G)^{(B \times H)}$ to its restriction to $C$. Hence, the group in the middle is canonically $\chi(G) \oplus \mathbb{Z}\hat{D} = \chi(C) \oplus \mathbb{Z}\hat{D}$.

On the other hand, assigning to each $f \in k(G)^{(B \times H)}$ its $H$-weight yields another exact sequence

$$
0 \to N^* = k(G/H)^{(B)}/k^* \to \chi(C) \oplus \mathbb{Z}\hat{D} = k(G)^{(B \times H)}/k^* \to \chi(H) \to 0.
$$
Dually, we have a surjective homomorphism $N_C \oplus \mathbb{Z}\hat{D} \to N$. Its restriction to $N_C$ is injective, since $C$ acts faithfully. In the analogous sequences for the group $\tilde{H}$, the map $N_C \to \tilde{N}$ is zero (since $\tilde{H}$ contains $C$), and $\tilde{N}_0$ is generated by the images of colors $\mathbb{Z}\hat{D}$.

Denote ker$(\pi : N \to \tilde{N})$ by $N_{\tilde{H}/H}$. Since the action is assumed to be smart, the map $N_C \to N$ is injective, since $C$ acts faithfully. Changing the action to an equivalent action composes this isomorphism with an automorphism of $N_C$.

Now, fix one polytope $Q$ in one of the finitely many equivalence classes modulo Aut$(N, \pi)$ that we obtained above. This also fixes a rational polytope $Q' = Q \cap N_{\tilde{H}/H} \otimes \mathbb{Q}$. The images of colors belong to $Q$, hence there are only finitely many choices for the map $\hat{D} \to Q$. Pick and fix a basis $e_1 \ldots e_s$ of $N_C$ and integral points $f_1 \ldots f_s$ in a fixed multiple $mQ'$ giving a basis of $N_{\tilde{H}/H}$. By switching to an equivalent action, we can assume that $N_C \to N$ sends $e_i$ to $f_i$. Importantly, if we replace the polytope $Q$ by another polytope $gQ$, $g \in$ Aut$(N, \pi)$ in the same equivalence class, then the kernel of $N_C \oplus \mathbb{Z}\hat{D} \to N$ will not change. Hence, for each of the finitely many equivalence classes of polytopes we obtained one surjection ker$N_C \oplus \mathbb{Z}\hat{D} \to N$ and a single polytope $Q$ in $N$. Now, consider the diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & \tilde{N}^* & \rightarrow & k(G)^{(B \times H)}/k^* & \rightarrow & \chi(\tilde{H}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N^* & \rightarrow & k(G)^{(B \times \tilde{H})}/k^* & \rightarrow & \chi(H) & \rightarrow & 0 \\
\end{array}
$$

By what we just have shown, for the cokernel $\chi(H)$ we have only finitely many choices. For each of them, the diagram tells us which characters of $\tilde{H}$ vanish on $H \subseteq \tilde{H}$, and this determines $H$ since $\tilde{H}/H$ is diagonalizable. For each $N^*$, we have finitely many polytopes in $N$, so finally only finitely many Fano varieties. The proof of Theorem 4.7 is now complete.

5. **Fano invariant of a spherical subgroup**

By [Bri93], every spherical $G/H$ has an equivariant Fano compactification, which is automatically a spherical variety. It is obtained by running the "anti Minimal Model Program", as follows. Choose an arbitrary projective compactification $X$ of $G/H$. Then by [Bri93, Cor.4.7], there exists a finite sequence $X \leftarrow X'$ of $G$-equivariant divisorial contractions in rays $[C]$ with $K_X \cdot C > 0$ and antiflips such that $-K_{X'}$ is nef; and that there exists a contraction $X' \rightarrow X''$ of a face of $NE(X')$ such that $X''$ is Fano.

**Definition 5.1.** The Fano invariant of a spherical subgroup $H \subset G$ is

$$F_G(H) = \max \{ \text{mld}(X) \mid X \text{ is an equivariant Fano compactification of } G/H \}$$

As an immediate consequence of Theorem 4.7 we have

□
Corollary 5.2. \( F_G(H) \) is well-defined. For any \( \epsilon > 0 \), there exist only finitely many \( H \) with \( F_G(H) > \epsilon \), up to an equivalence defined by automorphisms of a smart action of a bigger group \( G' \).

By definition, \( 0 < F_G(H) \leq 2 \), and if there exists a smooth Fano compactification then \( F_G(H) = 2 \); that is the best case.

Example 5.3. If \( G \) is a torus then for any closed subgroup \( H \), \( F_G(H) = 2 \). Indeed, in this case \( G/H \) is a torus, and can be equivariantly compactified to a smooth Fano variety, for example a projective space.

On the other hand, we have

Lemma 5.4. If \( G \) is a non-commutative reductive group then \( \inf_H F_G(H) = 0 \).

Proof. We can assume the action is smart. Let \( U \) be the unipotent radical of a Borel subgroup \( B \), and \( S \) be a finite subgroup of \( T = B \cap G^{ss} \). By taking \( S = T_n \), the subgroup of \( n \)-torsions, we obtain infinitely many spherical subgroups \( H = US \), and none of them are equivalent. But by Corollary 5.2, for any \( \epsilon > 0 \) there are only finitely many equivalence classes of \( H \) with \( F_G(H) > \epsilon \). \( \Box \)

Example 5.5. If \( G \) is a semisimple adjoint group then \( F_{G \times G}(\text{diag } G) = 2 \). Indeed, in this case one has the wonderful compactification of De Concini and Procesi [CP83], which is a smooth Fano.

The proof of the above Lemma may give an impression that perhaps Fano invariant merely counts the number of connected components of \( H \) for a faithful action of \( G \). But this is not the case. Indeed, for any spherical subgroup that equals its normalizer \( H = \bar{H} \), there exists a wonderful compactification: in this case the valuation cone \( V \) is strictly convex, and one simply takes the toroidal compactification for the colored fan consisting of \( V \) and its faces. The wonderful compactifications are always smooth, and they “tend to be” Fanos. However, \( H \) may have many connected components.

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