Heisenberg-limited metrology with coherent control on the probes’ configuration

Giulio Chiribella\textsuperscript{1,2,3,*} and Xiaobin Zhao\textsuperscript{1}

\textsuperscript{1} QICI Quantum Information and Computation Initiative, Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong 999077, China
\textsuperscript{2} Department of Computer Science, University of Oxford, Parks Road, Oxford OX1 3QD, United Kingdom and
\textsuperscript{3} Perimeter Institute for Theoretical Physics, Caroline Street, Waterloo, Ontario N2L 2Y5, Canada

A central feature of quantum metrology is the possibility of Heisenberg scaling, a quadratic improvement over the limits of classical statistics. This scaling, however, is notoriously fragile to noise. While for some noise types it can be restored through error correction, for other important types, such as dephasing, the Heisenberg scaling appears to be irremediably lost. Here we show that this limitation can sometimes be lifted if the experimenter has the ability to probe physical processes in a coherent superposition of alternative configurations. As a concrete example, we consider the problem of phase estimation in the presence of a random phase kick, which in normal conditions is known to prevent the Heisenberg scaling. We provide a parallel protocol that achieves Heisenberg scaling with respect to the probes’ energy, as well as a sequential protocol that achieves Heisenberg scaling with respect to the total probing time. In addition, we show that Heisenberg scaling can also be achieved for frequency estimation in the presence of continuous-time dephasing noise, by combining the superposition of paths with fast control operations.

Quantum metrology, the precise estimation of physical parameters aided by quantum resources, promises striking enhancements with respect to the limits of classical statistics \cite{1–3}. The best known example is the Heisenberg scaling, corresponding to a root mean square error with inverse linear scaling \(1/N\) when an unknown physical process is probed by \(N\) entangled particles \cite{4–6}, or by a single particle in a sequence of \(N\) time steps \cite{7–9}. The inverse linear scaling \(1/N\) amounts to a quadratic improvement over the standard quantum limit \(1/\sqrt{N}\), corresponding to a classical statistics over \(N\) repeated experiments.

The Heisenberg scaling was originally derived for the estimation of noiseless quantum processes \cite{10–12}. Later, it was found out to be extremely fragile to noise \cite{13–15}. This finding stimulated an extensive search for methods to counteract noise in quantum metrology \cite{16–19}. For certain types of noise, it was found that the Heisenberg limit can be restored by error correction \cite{18}, weak measurements \cite{19}, and fast quantum control \cite{17}. However, some important types of noise have so far resisted all attempts. The prototype of such resistant noise types is dephasing \cite{20}, corresponding to random fluctuations with the same generator as the signal. While the Heisenberg scaling can be restored for some particular models of correlated dephasing \cite{21,22}, all the existing methods give rise to an error scaling worse than the Heisenberg scaling in the standard setting involving uncorrelated dephasing \cite{13,15,17,23,24}. Intermediate scalings between the Heisenberg and standard quantum limit have been achieved using error correction \cite{18,25–27} or nondemolition measurements \cite{19,28}, but as long as all the probes experience uncorrelated dephasing processes, the Heisenberg scaling remained so far unattainable.

In this paper, we show that the Heisenberg scaling can sometimes be achieved even in the presence of uncorrelated dephasing noise, provided that the experimenter has the ability to probe quantum processes in a coherent superposition of multiple configurations \cite{29–37}. We consider the standard phase estimation problem where an unknown phase \(\theta\) is imprinted on the probes by multiple uncorrelated instances of a noisy process \(C_{\theta}\), and we explore setups where each probe is sent through \(M\) alternative trajectories, each leading to an independent instance of the unknown process \(C_{\theta}\), as illustrated in Figure 1. We then use this architecture as a building block for parallel protocols using \(N\) entangled probes, and for sequential protocols using a single probe in \(N\) time steps. Remarkably, we find out that the Heisenberg limit can be restored, both in terms of number of probes/total energy (for parallel protocols) and in terms of the number of time steps (for sequential protocols), when the process \(C_{\theta}\) corresponds to a random phase kick, which shifts the phase either by \(\theta\) or by \(\theta + \delta_{0}\), where \(\delta_{0}\) is a fixed offset. Building on these results, we show that the Heisenberg limit can also be achieved for a physically relevant model of continuous-time Markovian dephasing \cite{20}, allowing an accurate frequency estimation with an error decreasing quadratically with the total probing time.

Our results provide a new application of the technology of coherent control over multiple trajectories, which has recently attracted increasing interest due to its potential for quantum communication \cite{32–34,38,39}. Superpositions of trajectories have been demonstrated experimentally \cite{40,41} (see also \cite{41–45} for related experiments using the superposition of trajectories to investigate indefinite causal order). Applications of the superposition of trajectories to quantum metrology have been considered more recently \cite{46} (see also \cite{47–49} for the study of quantum metrology with indefinite causal order). To the best of our knowledge, however, the fundamental ques-

\* giulio@cs.hku.hk
tion of trajectories.

A single quantum particle (in yellow) is used to probe an unknown process depending on a parameter \( \theta \). The particle is sent through \( M \) alternative trajectories, each traversing an independence of the unknown process. In the end the trajectories are recombined and an interferometric measurement is performed. This basic architecture can be used as a building block for protocols using \( N \) entangled probes, and for sequential protocols using a single probe in \( N \) time steps.

mentioning that our protocols do not require entanglement of the N00N type \([50]\); for our parallel protocols, we will only use polarization entangled GHZ states \([51]\), which are comparatively easier to produce experimentally (see e.g. \([52]\) for a recent experiment with \( N = 12 \)).

Phase estimation with superposition of paths. – Consider the estimation of a phase shift \( \theta \) acting on a single photon. In the ideal scenario, the parameter \( \theta \) is imprinted on the system by a unitary gate \( U_\theta := \exp(-i\theta Z/2) \), with \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). In the noisy scenario, the parameter \( \theta \) is affected by random fluctuations by an amount \( \delta \), distributed according to a probability distribution \( \rho(\delta) \). The combined action of the signal and the noise is described by a single qubit dephasing channel

\[
\hat{C}_\theta(\rho) = \int d\delta \rho(\delta) U_{\theta+\delta} \rho U_{\theta+\delta}^\dagger,
\]

where \( \rho \) is the initial density matrix of the system.

Traditionally, phase estimation is modelled as the task of estimating the parameter \( \theta \) with \( N \) independent accesses to channel \( C_\theta \). In this setting, it is known that any finite amount of dephasing noise compromises the Heisenberg scaling \([13–15]\), even if one adopts arbitrary error correction operations and fast quantum control \([17, 24]\). In many relevant scenarios, however, the channel \( C_\theta \) only represents the effective evolution of a two-dimensional subspace of a larger quantum system. For example, a polarization qubit corresponds to the two-dimensional subspace spanned by the states \( |H\rangle := |1\rangle_H \otimes |0\rangle_V \) and \( |V\rangle := |0\rangle_H \otimes |1\rangle_V \), where the subscripts \( H \) and \( V \) refer to two modes of the electromagnetic field with vertical and horizontal polarization, respectively. In this picture, the qubit channel (1) is just a restriction of the overall dephasing channel

\[
\tilde{C}_\theta(\rho) = \int d\delta \rho(\delta) \tilde{U}_{\theta+\delta} \rho \tilde{U}_{\theta+\delta}^\dagger,
\]

where the unitary operator \( \tilde{U}_\theta = \exp[-i\theta(a^\dagger a - b^\dagger b)/2] \) represents the action of the phase shift on the relevant modes of the electromagnetic field, and \( a \ (b) \) is the annihilation operator for the mode with horizontal (vertical) polarisation.

The full description of the dephasing process (2) allows one to analyze the situation where a single photon is sent through multiple paths, each path subject to an independent dephasing process. Each path is associated to two polarization modes, say \( H_j \) and \( V_j \) for the \( j \)-th path, and a single photon in a superposition of multiple paths is associated to the Hilbert space spanned by Fock states with total photon number equal to \( 1 \). A convenient way to represent such states is to introduce a factorization of the Hilbert space in terms of a path degree of freedom and a polarization degree of freedom. A photon with polarization state \( |\psi\rangle = \alpha |H\rangle + \beta |V\rangle \) placed on the \( j \)-th path is represented by the state \( |\psi\rangle \otimes |j\rangle :=

FIG. 1. Single-particle probe in a coherent superposition of trajectories. A single quantum particle (in yellow) is used to probe an unknown process depending on a parameter \( \theta \). The particle is sent through \( M \) alternative trajectories, each traversing an independence of the unknown process. In the end the trajectories are recombined and an interferometric measurement is performed. This basic architecture can be used as a building block for protocols using \( N \) entangled probes, and for sequential protocols using a single probe in \( N \) time steps.
time behavior induced by the master equation \[20\] to a qubit environment. Moreover, the random phase kick can be realized in a collisional model of Hamiltonian $H = \sum \omega_{ij} a_i^\dagger a_j$, where $\omega_{ij}$ are random phase kicks that shift the phase by $\delta \in [0, 2\pi]$.

Appendix A for the exact expression). $\theta$ is a fixed offset, depending on the noise distribution (see Appendix A, we show that this term is proportional to a phase shift by the amount $\theta$). Under this condition, the second term in Eq. (3) converges to $F_\theta \rho F_\theta^\dagger$ in the large $M$ limit. Direct calculation shows that $F_\theta$ is proportional to a unitary operator: explicitly, one has $F_\theta = |f(1)\rangle U_{\theta_0 + \delta}$, where $f(k) := \int_{\theta_0}^{\theta_1} d\delta \ p(\delta) \ e^{-ik\delta/2}$ is the Fourier transform of the noise distribution, and $\theta_0 = -2 \arctan(\text{Im}[f(1)]/\text{Re}[f(1)])$.

Let us consider the second term in Eq. (3). In Appendix A, we show that this term is proportional to a unitary phase shift if and only if the Fourier transform of the noise distribution satisfies the condition

$$|f(2) - f^2(1)| = 1 - |f^2(1)| , \quad (4)$$

Under this condition, the second term in Eq. (3) is proportional to a phase shift by the amount $\theta + \theta_1$, where $\theta_1$ is a fixed offset, depending on the noise distribution (see Appendix A for the exact expression).

An example of noisy process that satisfies condition (4) is a random phase kick that shifts the phase by $\delta_0$, where $\delta_0 \in [0, 2\pi]$ is a fixed (but otherwise arbitrary) offset. In this model, the photon has probability $p$ to get a phase shift $\theta$ and probability $1-p$ to get a phase shift $\theta + \delta_0$, and $p$ can have any value between 0 and 1. Physically, the random phase kick can be realized in a collisional model for dephasing [54], where it corresponds to the case of a qubit environment. Moreover, the random phase kick model is important in that it corresponds to the short-time behavior induced by the master equation [20]

$$\frac{d C_{\omega,t}(\rho)}{dt} = -i \frac{\omega}{2} [Z, C_{\omega,t}(\rho)] + \frac{\gamma}{2} [Z C_{\omega,t}(\rho) Z - C_{\omega,t}(\rho)] , \quad (5)$$

where $t$ is the time parameter, $\omega$ is the frequency, and $\gamma$ is the dephasing rate.

When the condition (4) is satisfied, one can remove the offsets $\theta_0$ and $\theta_1$ in the two terms of Eq. (3), obtaining a noiseless channel in the limit $M \to \infty$. In the following, we analyze the finite $M$ scenario, showing that Heisenberg limit can be achieved whenever $M$ grows linearly with $N$, where $N$ is the number of probes (for parallel protocols) or the number of time steps (for sequential protocols).

**Parallel phase estimation protocol.** Let us start by considering the scenario where the probe consists of $N$ entangled photons sent through parallel uses of the same dephasing process, as shown in Fig. 2.

For simplicity, we will illustrate the ideas in the basic setting involving the preparation of the $N$ photons in a GHZ state [51]. In the noiseless case, this state allow one to estimate small phase shifts in the interval $[0, 2\pi/N]$, and a setup using multiple GHZ states with different values of $N$ can achieve Heisenberg scaling of the error with respect to the total number of photons [53]. We now provide a protocol that restores this ideal scaling in the presence of dephasing noise satisfying condition (4). The steps of protocol are the following:

1. Prepare $N$ photons in the polarization entangled GHZ state $|\Psi_N\rangle = (|H\rangle^\otimes N + |V\rangle^\otimes N) / \sqrt{2}$.
2. Put each photon in a uniform superposition of $M$
paths, initializing the path degree of freedom in the maximally coherent state $|e_0\rangle$.

3. Let the noisy process $\widetilde{C}_\theta$ act on each of the paths.

4. Perform a Fourier measurement on each path, getting outcome $m$. If the outcome is $m = 0$, perform a phase shift of $-\theta_0$, otherwise perform a phase shift of $-\theta_1$.

5. Measure the polarization of the $N$ photons with a nonorthogonal measurement including the four operators $P_{\pm} = |\Psi_{\pm}\rangle\langle\Psi_{\pm}|/2$ and $Q_{\pm} = |\Phi_{\pm}\rangle\langle\Phi_{\pm}|/2$, with $|\Psi_{\pm}\rangle = (|H\rangle^{\otimes N} \pm |V\rangle^{\otimes N})/\sqrt{2}$ and $|\Phi_{\pm}\rangle = (|H\rangle^{\otimes N} \pm i|V\rangle^{\otimes N})/\sqrt{2}$.

6. Repeat the above procedure for $\nu$ times, and output the maximum likelihood estimate $\hat{\theta} = \arg\max_{\theta} \log p(x_1, \ldots, x_\nu|\theta)$, where $\{x_1, \ldots, x_\nu\}$ are the outcomes of the $\nu$ measurements on the photons’ polarization, and $p(x_1, \ldots, x_\nu|\theta)$ is the probability of obtaining such outcomes when the true phase shift is $\theta$.

It is important to note that Step 4 can also be postponed to the end, and that the conditional phase shifts do not need to be implemented actively, as they can be included in the data processing stage. However, our description of the protocol includes these operations because they simplify the presentation and analysis of the results.

In Appendix B, we compute the Fisher information for the outcomes of the measurement at step (5), under the assumption that Eq. (4) is satisfied. Denoting the Fisher information by $F_\theta$, we prove the bound

$$F_\theta \geq \frac{N^2}{2} \left|1 - \frac{e^{\phi_0} - 1}{M}\right|^{2N}$$

This bound guarantees Heisenberg scaling whenever $M$ grows linearly with $N$.

Sequential phase estimation protocol. – In this protocol, a single photon is prepared in the polarization state $|+\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$. Then, the photon is sent in a uniform superposition of paths through $M$ independent dephasing channels. After the action of the channels, the paths are recombined, and a Fourier measurement is performed on the paths, followed by phase shifts that remove the offsets $\theta_0$ and $\theta_1$, in the same way as in the parallel protocol. This procedure is repeated for $N$ steps, and a polarization measurement is finally performed after the $N$-th step, using a measurement with operators $F_{\pm} := |\pm\rangle\langle\pm|/2$ and $Q_{\pm} = |\pm i\rangle\langle\pm i|/2$, with $|\pm\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$ and $|\pm i\rangle = (|H\rangle \pm i|V\rangle)/\sqrt{2}$. Also in this case, the intermediate measurements can in principle be postponed to the last step, and the conditional phase shifts of $\theta_0$ and $\theta_1$ can be absorbed into the data processing.

In Appendix B, we show that this sequential protocol is mathematically equivalent to the parallel one, and that the Fisher information is still given by Eq. (6). Hence, Heisenberg limit with respect to the number of time steps can be obtained with a single particle in a superposition of $M = O(N)$ paths per step.

The sequential protocol can also be applied to the estimation of the frequency $\omega$ in the master equation (5). We consider the scenario where a single photon undergoes the evolution for a total time $T$, and fast control operations are applied at short intervals of time $t$ [17]. The control operations are measurements on the path degree of freedom, followed by appropriate shifts in the photon’s polarization, as in the discrete-time sequential protocol illustrated above. In Appendix C, we show that the protocol achieves a Fisher information $F_\omega$ satisfying the bound $F_\omega \geq T^2/2$ in the limit $t \to 0$ using a number of paths growing as $T/t$. The benefit of the superposition of paths carries over also to finite values of $t$, as illustrated in Figure (3).

Conclusions. – In this paper we explored the precision scaling achieved by probing quantum processes in a coherent superposition of configurations. We showed that routing each probe on a superposition of trajectories can unlock Heisenberg scaling (with respect to number of probes, total energy, or total probing time) in the presence of dephasing noise induced by a random phase kick. Our findings are in stark contrast with the scenario where the probes are sent on definite trajectories, in which case the random phase kick is known to prevent the Heisenberg scaling.

The key resource exploited by our protocols is quan-
beamsplitters used to route photons on different paths. Notably, this resource is different from other resources in quantum metrology, such as the total number/energy of the probes, or the total time. The fact that the number of trajectories does not affect the total energy suggests that the superposition of trajectories could be used to achieve higher levels of precision in scenarios where the energy is bounded, such as e.g. in biological probes [56].

Our results have only scratched the surface of the potential benefits of the superposition of configurations in quantum metrology. An interesting direction for future research is to consider superpositions of more complex configurations, for example including tree-like structures in which the basic setups of this paper are used as building blocks. In this context, two key open problems arise. The first is to characterize what is the most general class of noise models for which the Heisenberg scaling can be restored through a superposition of configurations. Recently developed frameworks for quantum circuits with quantum control [36, 57] are a valuable tool to address this problem. The second problem is to extend the analysis to scenarios where the path degree of freedom is also subject to noise, e.g. due to imperfections of the beamsplitters used to route photons on different paths. An interesting approach here is to consider concatenated schemes, similar to those considered in quantum error correction and fault tolerance [58–60].

Finally, an appealing direction is the experimental demonstration of quantum metrology boosted by coherent control on the probes’ trajectories. For moderate values of $N$, the proof-of-principle demonstration of the protocols proposed appears to be within reach with current technologies of photonic quantum metrology [61, 62], especially in the sequential setting, which does not require multiphoton entanglement.

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Eq. (4) is satisfied. By explicit evaluation, we have

$$\theta_{ij} \in \{0, 1\}$$

Combining these two relations, we obtain

$$A = \left( \begin{array}{cc}
\frac{1+\text{Re}[f(2)]-2\text{Re}[f(1)]^2}{2} + i \frac{\text{Im}[f(2)]-2\text{Re}[f(1)]\text{Im}[f(1)]}{2} & \frac{1-\text{Re}[f(2)]-2\text{Re}[f(1)]\text{Im}[f(1)]}{2} \\
-i \frac{\text{Im}[f(2)]-2\text{Re}[f(1)]\text{Im}[f(1)]}{2} & 1-\text{Re}[f(2)]-2\text{Re}[f(1)]^2
\end{array} \right).$$

Appendix A: Derivation of Eq. (4)

Here we show that the second term in Eq. (3) is proportional to a unitary channel if and only if the condition in Eq. (4) is satisfied. By explicit evaluation, we have

$$U_{\theta}^{\dagger} C_{\theta}(\rho) U_{\theta} = \int d\delta \rho(\delta) \left\{ \frac{1+\cos \delta}{2} \rho + \frac{1-\cos \delta}{2} Z \rho Z - i \frac{\sin \delta}{2} Z \rho + i \frac{\sin \delta}{2} \rho Z \right\}$$

$$= \frac{1+\text{Re}[f(2)]}{2} \rho + \frac{1-\text{Re}[f(2)]}{2} Z \rho Z + i \frac{\text{Im}[f(2)]}{2} Z \rho - i \frac{\text{Im}[f(2)]}{2} \rho Z$$

(A1)

and

$$U_{\theta}^{\dagger} F_{\theta} = \int d\delta \rho(\delta) \left( \cos \frac{\delta}{2} I - i \frac{\sin \delta}{2} Z \right)$$

$$= \text{Re}[f(1)] I + i \text{Im}[f(1)] Z$$

(A2)

Combining these two relations, we obtain

$$U_{\theta}^{\dagger} \left( C_{\theta}(\rho) - F_{\theta} \rho F_{\theta}^{\dagger} \right) U_{\theta} = A_{00} \rho + A_{11} Z \rho Z + A_{01} Z \rho + A_{10} \rho Z ,$$

(A3)

where $A_{ij}, i,j \in \{0,1\}$ are the entries of the matrix.
The map on the right hand side of Eq. (A3) is proportional to a unitary channel if and only if the matrix $A$ has not full rank, that is, if and only if $\det(A) = 0$. Explicit calculation of the determinant yields

$$\det(A) = \frac{1 - |f(1)^2|}{4}.$$  \hfill (A5)

Hence, we have $\det(A) = 0$ if and only if $1 - |f(1)^2| = |f(1)^2 - f(2)|$. Using the inequality $|f(1)| \leq 1$, this condition can be rewritten as

$$1 - |f(1)^2| = |f(1)^2 - f(2)|.$$  \hfill (A6)

This is the condition given in Eq. (4) of the main text.

When this condition is satisfied, one has

$$U_{\theta}^\dagger \left( C_\theta - F_{\theta} \rho F_{\theta}^\dagger \right) U_{\theta} = \left( 1 - |f(1)^2| \right) U_{\theta_1},$$  \hfill (A7)

with $\theta_1 = \arctan \{-i A_{10}/(A_{00} - (1 - |f(1)^2|)/2\} = \arctan \frac{2 \Re[f(1)] |\text{Im}[f(1)] - |\text{Im}[f(2)]|}{\text{Re}[f(2)] - \text{Re}[f(1)] + |\text{Im}[f(1)]|^2}$.  

**Appendix B: Derivation of Eq. (5) in the main text**

1. **Expression for the effective channel acting on the probe**

In our protocol, the experimenter measures the path degree of freedom on the Fourier basis $\{|e_m\}, m = 0, \cdots, M - 1\}$, obtaining outcome $m$. The experimenter performs a phase shift $-\theta_0$ if $m = 0$, or a phase shift $-\theta_1$ if $m \neq 0$. The effective evolution of the probe is given by the channel

$$E_{\theta}(\rho) = \frac{U_{\theta_0}^\dagger}{M} \left[ C_\theta + (M - 1) F_{\theta} \rho F_{\theta}^\dagger \right] U_{\theta_0} + \frac{M - 1}{M} \ U_{\theta_1} \left( C_\theta - F_{\theta} \rho F_{\theta}^\dagger \right) U_{\theta_1}.$$  \hfill (B1)

Recall the relation $U_{\theta_0}^\dagger F_{\theta} = |f(1)| U_{\theta_0}$, given in the main text, and the relation $U_{\theta_1} \left( C_\theta - F_{\theta} \rho F_{\theta}^\dagger \right) U_{\theta_1} = \left( 1 - |f(1)^2| \right) U_{\theta_0}^\dagger$, valid when the condition $1 - |f(1)^2| = |f(1)^2 - f(2)|$ is satisfied [cf. Eq. (A7) of this Appendix]. Using these two relations, we obtain

$$E_{\theta}(\rho) = \frac{1}{M} U_{-\theta_0} C_{\theta}(\rho) U_{-\theta_0}^\dagger + \frac{M - 1}{M} U_{\theta_0} \rho U_{\theta_0}^\dagger.$$  \hfill (B2)

Now, Eq. (A1) yields the relation $C_{\theta}(\rho) = \left( \begin{array}{cc} \rho_{HH} & \rho_{HV} f(2) e^{-i\theta} \\ \rho_{VH}^* f(2)^* e^{i\theta} & \rho_{VV} \end{array} \right)$, where $\rho_{ij} := \langle i | \rho | j \rangle, i, j \in \{H, V\}$ are the matrix elements of $\rho$. Using this relation, Eq. (B2) becomes

$$E_{\theta}(\rho) = \left( \begin{array}{cc} \frac{e^{-i\theta_0 f(2)}}{M} + \frac{M - 1}{M} e^{i\theta} \rho_{VH} & \frac{e^{i\theta_0 f(2)}}{M} + \frac{M - 1}{M} e^{-i\theta} \rho_{VV} \\ \frac{\lambda e^{-i(\theta + \theta_2)}}{\rho_{VH}} & \rho_{VV} \end{array} \right) \lambda,$$

the second equality following from the definitions $\lambda := \left| \frac{e^{-i\theta_0 f(2)}}{M} + \frac{M - 1}{M} \right|$ and $e^{-i\theta_2} := \left( \frac{e^{i\theta_0 f(2)}}{M} + \frac{M - 1}{M} \right)/\lambda$.

Equivalently, we have $E_{\theta}(\rho) = \frac{1 + \lambda}{2} U_{\theta + \theta_2} \rho U_{\theta + \theta_2}^\dagger + \frac{1 - \lambda}{2} U_{\theta + \theta_2 + \pi} \rho U_{\theta + \theta_2 + \pi}^\dagger$.

2. **Achievable Fisher information in the parallel protocol**

In our first protocol, $N$ probes are initialized in the entangled state $|\Psi_+\rangle = (|H\rangle \otimes N + |V\rangle \otimes N) / \sqrt{2}$, and undergo $N$ independent applications of the channel $E_{\theta}$ in Eq. (B3). The resulting state is

$$E_{\theta}^{\otimes N} (|\Psi_N^+\rangle < \Psi_N^+\rangle) = \frac{1}{2} \left[ |H\rangle \langle H| \otimes N + |V\rangle \langle V| \otimes N + \lambda^N e^{-N(\theta + \theta_2)} |H\rangle \langle V| \otimes N + \lambda^N e^{-N(\theta + \theta_2)} |V\rangle \langle H| \otimes N \right].$$  \hfill (B4)
At this point, suppose that the experimenter implement a measurement containing the operators $P_{\pm} = |\Psi_{\pm}\rangle \langle \Psi_{\pm}|/2$ and $Q_{\pm} = |\Phi_{\pm}\rangle \langle \Phi_{\pm}|/2$, with $|\Psi_{\pm}\rangle = (|H\rangle \otimes N \pm |V\rangle \otimes N)/\sqrt{2}$ and $|\Phi_{\pm}\rangle = (|H\rangle \otimes N \pm i|V\rangle \otimes N)/\sqrt{2}$. The probabilities of the corresponding outcomes are

$$p(P_{\pm}|\theta) = \frac{1}{4} \left[ 1 \pm \lambda^N \cos(N(\theta + \theta_2)) \right] \quad \text{and} \quad p(Q_{\pm}|\theta) = \frac{1}{4} \left[ 1 \pm \lambda^N \sin(N(\theta + \theta_2)) \right] \quad (B5)$$

The corresponding classical Fisher information is:

$$F_{\theta} = \sum_{j \in \{P_{\pm}, P_{\mp}, Q_{\pm}, Q_{\mp}\}} p(j|\theta) \left( \frac{d\ln p(j|\theta)}{d\theta} \right)^2$$

$$= \frac{N^2 \lambda^{2N}}{2} \frac{1 - \lambda^{2N} + \lambda^{2N} \sin^2(2N(\theta + \theta_2))}{1 - \lambda^{2N} + \lambda^{4N} \sin^2(2N(\theta + \theta_2))}$$

$$= \frac{N^2 \lambda^{2N}}{2} \left[ 1 + \frac{\lambda^{2N} \sin^2(2N(\theta + \theta_2))}{1 - \lambda^{2N} + \lambda^{4N} \sin^2(2N(\theta + \theta_2))} \right]. \quad (B6)$$

From this exact expression, we now derive a $\theta$-independent bound. Note that $\lambda \leq 1$: indeed, if we set $x := \Re[e^{i\theta_0} f(2)]$ and $y = \Im[e^{i\theta_0} f(2)]$, we obtain

$$\lambda = \left| \frac{e^{i\theta_0} f(2)}{M} + \frac{M - 1}{M} \right| \leq \left| \frac{e^{i\theta_0} f(2)}{M} \right| + \left| \frac{M - 1}{M} \right| \leq 1, \quad (B7)$$

where we used the triangle inequality for the modulus, and the relation $|f(2)| \leq 1$.

Since $\lambda \leq 1$, the second summand in Eq. (B6) is nonnegative, and we have the bound $F_{\theta} \geq \frac{N^2 \lambda^{2N}}{2}$, which coincides with Eq. (5) in the main text.

3. Achievable Fisher information in the sequential protocol

The sequential protocol amounts to $N$ repeated applications of the effective channel $\mathcal{E}_{\theta}$ on the photon’s polarization, initially in the state $|+\rangle$. The output state after the $N$-th application is

$$\mathcal{E}_{\theta}^N(|+\rangle \langle +|) = \frac{1}{2} \left( |H\rangle \langle H| + |V\rangle \langle V| + \lambda^{2N} e^{-iN(\theta + \theta_2)} |H\rangle \langle V| + \lambda^{2N} e^{iN(\theta + \theta_2)} |V\rangle \langle H| \right). \quad (B8)$$

This equation is formally identical to Eq. (B4). At this point, the polarization undergoes a measurement with operators $P_{\pm} := |\pm\rangle \langle \pm|/2$ and $Q_{\pm} = |\pm i\rangle \langle \pm i|/2$, with $|\pm\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$ and $|\pm i\rangle = (|H\rangle \pm i|V\rangle)/\sqrt{2}$. The outcome probabilities of this measurement coincide with the outcome probabilities in Eq. (B5), and therefore the Fisher information is still given by (B6).

Appendix C: Continuous-time dephasing

The single-qubit dynamics of a continuous-time Markovian dephasing is characterized by the Lindblad master equation

$$\frac{dC_{\omega,t}(\rho)}{dt} = -i \frac{\omega}{2} [Z, C_{\omega,t}(\rho)] + \frac{\gamma}{2} [U_{\omega,t} C_{\omega,t}(\rho) U_{\omega,t}^\dagger - C_{\omega,t}(\rho)] \quad \text{(C1)}$$

where $\omega$ is the frequency of the oscillations, and $\gamma$ is the dephasing rate. The solution of this equation is the quantum channel $C_{\omega,t}(\rho)$ of the channel $C_{\omega,t}$ defined by

$$C_{\omega,t}(\rho) = \frac{1 + e^{-\gamma t}}{2} U_{\omega,t} \rho U_{\omega,t}^\dagger + \frac{1 - e^{-\gamma t}}{2} U_{\omega,t+\pi} \rho U_{\omega,t+\pi}^\dagger \quad \text{(C2)}$$

When the qubit is the polarization of a single photon, the single-qubit dynamics (C2) can be obtained from an extended dynamics of the electromagnetic field, according to the master equation

$$\frac{d\bar{C}_{\omega,t}(\rho)}{dt} = -i \frac{\omega}{2} \left[ a^\dagger a - b^\dagger b, \bar{C}_{\omega,t}(\rho) \right] + \frac{\gamma}{2} \left[ \bar{U}_\pi \bar{C}_{\omega,t}(\rho) \bar{U}_\pi^\dagger - \bar{C}_{\omega,t}(\rho) \right], \quad \text{(C3)}$$
where $a$ and $b$ are the annihilation operators for two modes of horizontal and vertical polarization, respectively, and $U_\delta := \exp[-i(a^\dagger a - b^\dagger b)\delta/2]$ for arbitrary $\delta \in [0, 2\pi]$.

When the evolution is restricted to the subspace containing single-photon states and the vacuum, the master equation (C3) has the following solution:

$$\tilde{C}_{\omega,t}(\rho) = \begin{bmatrix} \rho_{HH} & e^{-\gamma t - i \omega t} \rho_{HV} & e^{-\frac{1}{2} i \gamma t - i \frac{2}{2} \lambda_t} \rho_{H\text{vac}} \\ e^{-\gamma t + i \omega t} \rho_{VH} & \rho_{VV} & e^{-\frac{1}{2} i \gamma t + i \frac{2}{2} \lambda_t} \rho_{V\text{vac}} \\ e^{-\frac{1}{2} i \gamma t + i \frac{2}{2} \lambda_t} \rho_{V\text{vac}} & e^{-\frac{1}{2} i \gamma t - i \frac{2}{2} \lambda_t} \rho_{V\text{vac}} & \rho_{\text{vac vac}} \end{bmatrix}.$$  \hspace{1cm} (C4)

where $\tilde{\rho} \in \text{St}(\mathcal{H} \oplus \mathcal{H}_{\text{vac}})$ is an arbitrary state in the space spanned by states $|H\rangle := |1\rangle_H \otimes |0\rangle_V$, $|V\rangle := |0\rangle_H \otimes |1\rangle_V$, and $|\text{vac}\rangle := |0\rangle_H \otimes |0\rangle_V$, and $\rho_{ij} := \langle i|\tilde{\rho}|j\rangle$, $i,j \in \{H, V, \text{vac}\}$ are the matrix elements of $\tilde{\rho}$. This evolution can be rewritten in a compact way, as follows

$$\tilde{C}_{\omega,t}(\rho) = C_{\omega,t}(P\rho P) + F_{\omega,t}(\rho) \langle \text{vac}| + |\text{vac}\rangle \langle \text{vac}| \rho F_{\omega,t}^\dagger + |\text{vac}\rangle \langle \text{vac}| \rho |\text{vac}\rangle \langle \text{vac}|,$$  \hspace{1cm} (C5)

where $P$ is the projector on the original space $\mathcal{H} = \text{Span}\{|H\rangle, |V\rangle\}$, and $F_{\omega,t} := e^{-\frac{1}{2} t (U_{(\omega+\gamma)t})}$.

Now, suppose that a single photon is sent on a superposition of $M$ paths, passing through $M$ independent instances of the channel $\tilde{C}_{\omega,t}$. After the action of the channels, the paths are recombined and undergo a measurement on the Fourier basis. Depending on the outcome of the measurement, the photon’s polarization is shifted either by 0 (for outcome 0) or $-\theta_1$ (for outcomes other than 0), where $\theta_1$ will be defined later. The effective channel resulting from these operations is:

$$E_{\omega,t}(\rho) = \frac{1}{M} \left\{ \tilde{C}_{\omega,t}(\rho) + (M-1)F_{\omega,t}\rho F_{\omega,t}^\dagger + (M-1)U_{-\theta_1} \left[ \tilde{C}_{\omega,t}(\rho) - F_{\omega,t}\rho F_{\omega,t}^\dagger \right] U_{-\theta_1}^\dagger \right\}$$

with

$$\lambda_t = \frac{1}{M} e^{-\gamma t} + \frac{M-1}{M} e^{-\gamma(1+i)t} + \frac{M-1}{M} e^{-\gamma t + i \theta t} - \frac{M-1}{M} e^{-\gamma(1+i)t + i \theta t}.$$  \hspace{1cm} (C7)

By applying the channel $E_{\omega,t}$ sequentially for $N = T/t$ times, we then obtain the channel

$$E_{\omega,t}^{(N)} = \begin{bmatrix} \rho_{HH} & \lambda_t e^{-\omega t} \rho_{HV} & \rho_{H\text{vac}} \\ \lambda_t^* e^{i \omega T} \rho_{VH} & \rho_{VV} & \rho_{V\text{vac}} \\ \rho_{V\text{vac}} & \rho_{V\text{vac}} & \rho_{\text{vac vac}} \end{bmatrix}.$$  \hspace{1cm} (C8)

We now consider the problem of estimating the frequency $\omega$ for a given total time $T$ and a given dephasing rate $\gamma$. For this purpose, we initialize the qubit in the state $|+\rangle$ and we perform a measurement with POVM operators $P_{\pm} := |\pm\rangle\langle\pm|/2$ and $Q_{\pm} = |\pm\rangle\langle\pm|/2$. The (classical) Fisher information achieved by this measurement is

$$F_{\omega} = \sum_{j \in \{P_+, P_-, Q_+, Q_-\}} p(j \omega) \left( \frac{d \ln p(j \omega)}{d \omega} \right)^2$$

$$= \frac{T^2 |\lambda_t|^{2T/t}}{2} \left[ 1 + \frac{|\lambda_t|^{2T/t} \sin^2(2T(\omega + \theta_1/2))}{1 - |\lambda_t|^{2T/t} + |\lambda_t|^{2T/t} \sin^2(2T(\omega + \theta_1/2))} \right],$$  \hspace{1cm} (C9)

where the outcome probabilities are $p(P_+ \omega) = |1 + |\lambda_t|^{2T/t} \cos(\omega T + \theta_1 T/t)|/4$ and $p(Q_+ \omega) = |1 - |\lambda_t|^{2T/t} \sin(\omega T + \theta_1 T/t)|/4$ with $\theta_1$ being the phase of the complex number $\lambda_t$.

We now show that the Fisher information has Heisenberg scaling in the limit $t \to 0$, corresponding to fast operations on the photon’s path. For the limit, we set the number of paths $M$ to grow linearly with $T/t$, and optimize the choice of $\theta_1$. From Eq. (C7) one can see that the maximum of $|\lambda_t|$ is obtained for

$$\theta_1 = \arg \left[ \frac{1}{M} e^{-\gamma t} + \frac{M-1}{M} e^{-\gamma(1+i)t} \right]$$

$$= \frac{1}{M} e^{-\gamma t} + \frac{M-1}{M} e^{-\gamma(1+i)t}.$$  \hspace{1cm} (C10)
With this choice, we obtain

\[
\max_{\delta_i} |\lambda_t| = e^{-\gamma t} \left( \left| \frac{1}{M} + \frac{M-1}{M} e^{-i\gamma t} \right| + \left| \frac{M-1}{M} (1 - e^{-i\gamma t}) \right| \right)
\]

\[
e^{-\gamma t} \left( |1 - i\gamma t + O(t^2)| + |i\gamma t + O(t^2)| \right)
\]

\[
e^{-\gamma t} \left[ 1 + \gamma t + O(t^2) \right]
\]

\[
= 1 - O(t^2).
\] (C11)

To conclude, we note that Eq. (C9) implies the bound

\[
F_\omega \geq \frac{1}{2} T^2 |\lambda_t|^{2T/t}
\] (C12)

for sufficiently small \(t\). Using the relation \(\lim_{t \to 0} |\lambda_t|^{T/t} = 1\), we then obtain the asymptotic bound

\[
F_\omega \Big|_{t \to 0} \geq \frac{1}{2} T^2.
\] (C13)