SMOOTHNESS CONDITIONS IN COHOMOGENEITY MANIFOLDS

LUIGI VERDIANI AND WOLFGANG ZILLER

Abstract. We present an efficient method for determining the conditions that a metric on a cohomogeneity one manifold, defined in terms of functions on the regular part, needs to satisfy in order to extend smoothly to the singular orbit.

A group action is called a cohomogeneity one action if its generic orbits are hypersurfaces. Such actions have been used frequently to construct examples of various types: Einstein metrics, soliton metrics, metrics with positive or non-negative curvature and metrics with special holonomy. See [DW, FH, GKS, GVZ, KS] for a selection of such results. The advantage of such metrics is that geometric problems are reduced to studying its behavior along a fixed geodesic $c_\ell$ normal to all orbits. The metric is described by a finite collection of functions of $t$, which for each time specifies the homogeneous metric on the principal orbits. One aspect one needs to understand is what conditions these functions must satisfy if regular orbits collapse to a lower dimensional singular orbit. These smoothness conditions are often crucial ingredients in obstructions, e.g. to non-negative or positive curvature, see e.g. [GVWZ, VZ1, VZ2]. This problem was studied in [EW], but in practice their method is difficult to apply. The goal of this paper is to devise a straightforward procedure in order to derive such conditions explicitly.

The local structure of a cohomogeneity one manifold near a collapsing orbit can be described in terms of Lie subgroups $H \subset K \subset G$ with $K/H = S^\ell$, $\ell > 0$. The action of $K$ on $S^\ell$ extends to a linear action on $D = D^{\ell+1} \subset \mathbb{R}^{\ell+1}$ and thus $M = G \times_K D$ is a homogeneous disc bundle, where $K$ acts as $(g, p) \to (gk^{-1}, kp)$, and with boundary $G \times_K \partial D = G \times_K K/H = G/H$ a principal orbit. The Lie group $G$ acts by cohomogeneity one on $M$ by left multiplication in the first coordinate. A compact (simply connected) cohomogeneity one manifold is the union of two such homogeneous disc bundles. For simplicity we write $M = G \times_K V$ with $V \simeq \mathbb{R}^n$. Given a smooth $G$ invariant metric on the open dense set of regular points, i.e., the complement of the singular orbit, we choose a normal geodesic $c : [0, \infty) \to V$ orthogonal to all orbits, where we have identified $V$ with the slice at $c(0)$. The stabilizer group of the $G$ action at $c(t)$ is $H$ for $t > 0$ and $K$ for $t = 0$. The metric on the regular part is described uniquely by a set of inner products $g_t$, $t > 0$ on $T_{c(t)}M$, invariant under the action of $H$ and smooth in $t$. The problem is when the extension of this metric to the singular points is smooth.

Theorem A. Let $G$ act by cohomogeneity one on $M = G \times_K V$ and $g$ be a smooth cohomogeneity one metric defined on the set of regular points in $M$. Then $g$ has a smooth extension to the singular orbit if and only if it is smooth when restricted to every 2 plane in the slice $V$ containing $c(0)$.

As we will see, it follows from the classification of transitive actions on spheres, that it is sufficient to require the condition only for a finite set of 2-planes $P_i = \{c(0), v_i\}$, in fact at most 4 are necessary. We can furthermore assume that $L = \exp(\theta v_i) \subset K$ is a closed one parameter group, and hence the action of $L$ on $V$ and on a $K$ invariant complement of $\ell$ in $g$ splits into 2 dimensional invariant subspaces $\ell_k$ isomorphic to $\mathbb{C}$, on which $L$ acts by multiplication with $e^{id_k \theta}$.

The first named author was supported by a PRIN grant. The second named author was supported by a grant from the National Science Foundation.
The integers $d_k$ are determined by the weights of the representation of $K$ on $V$ and $\mathfrak{m}$. These integers will determine the smoothness conditions, see Table B, C, D.

To be more explicit, let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ be the Lie algebras of $G, K, H$. Let $\mathfrak{p}$ be an $\text{Ad}_H$ invariant complement of $\mathfrak{h} \subset \mathfrak{k}$, which represents the tangent space of the sphere $K/H$, and $\mathfrak{m}$ an $\text{Ad}_K$ invariant complement of $\mathfrak{k} \subset \mathfrak{g}$, which represents the tangent space of the singular orbit $G/K$. Thus $\mathfrak{p} \oplus \mathfrak{m}$ can be identified with the tangent space to the regular orbits along $c$, and the metric is given by $g = dt^2 + h$, where $h, t > 0$ is a smooth set of inner products on $\mathfrak{p} \oplus \mathfrak{m}$.

The metric is described in terms of the length of Killing vector fields. We choose a basis $X_i$ of $\mathfrak{p} \oplus \mathfrak{m}$ and let $X^*_i$ be the corresponding Killing vector fields such that $X_i$ span $\mathfrak{p}$ for $i = 1, \ldots, r$ and $\mathfrak{m}$ for $i = r + 1, \ldots, s$. Then $X^*_i(c(t))$ is a basis of $\dot{c}^*(t) \subset T_{c(t)} M$ for all $t > 0$ and the metric is determined by the matrix $g_{ij} = g(X^*_i, X^*_j)_{c(t)}$.

Combining the finite set of smoothness conditions obtained from Theorem A, we will see that:

**Theorem B.** Let $g_{ij}(t)$ be a smooth family of matrices describing the cohomogeneity one metric on the regular part. Then there exist integers $d_k \geq 1$ and constants $a_k, b_k, c_k, a_{ij}^k$ such that the metric is smooth if and only if

\[
\sum_{i,j} a_{ij}^k g_{ij}(t) = a_k t^2 + t^k \phi_k(t^2) \quad \text{for } k = 1, \ldots, r,
\]

\[
\sum_{i,j} a_{ij}^k g_{ij}(t) = b_k + c_k t + t^{d_k} \phi_k(t^2) \quad \text{for } k = r + 1, \ldots, s
\]

for some smooth functions $\phi_1, \ldots, \phi_{r+s}$. Here $r + s$ is the number of distinct non-zero functions in the matrix $g_{ij}$.

The constants $b_k$ and $c_k$ represent the metric on $G/K$ and its second fundamental form, whereas $a_k$ describes smoothness on the slice $V$. This system of equations can also be solved for the coefficients $g_{ij}$ of the metric. We will illustrate that it is straightforward to determine the integers $d_k$ and constants $a_k, b_k, c_k, a_{ij}^k$ in specific examples.

In a future paper, we will show that our new description is also useful in proving general theorems about cohomogeneity one manifolds, in particular solving the initial value problem, starting at the singular orbit, for cohomogeneity one Einstein manifolds or prescribed Ricci tensors.

The paper is organized as follows. After discussing some preliminaries in Section 1, we prove Theorem A in Section 2. In Section 3 we describe how the action of the one parameter group $L \subset K$ on $V$ and on $\mathfrak{m}$ is used to derive the smoothness conditions. This is an over determined system of equations, and we will show how it can be reduced to the system in Theorem B. In Section 4 we illustrate the method in some specific examples, and in Section 5 determine the integers $d_k$ for the action of $K$ on $V$.

1. Preliminaries

For a general reference for this Section see, e.g., [AA] [AB]. A noncompact cohomogeneity one manifold is given by a homogeneous vector bundle and a compact one by the union of two homogeneous disc bundles. Since we are only interested in the smoothness conditions near a singular orbit, we restrict ourselves to only one such bundle. Let $H, K, G$ be Lie groups with inclusions $H \subset K \subset G$ such that $H, K$ are compact and $K/H = S^\ell$. The transitive action of $K$ on $S^\ell$ extends (up to conjugacy) to a unique linear action on the disc $V = \mathbb{R}^{\ell+1}$. We can thus define the homogeneous vector bundle $M = G \times_K V$ and $G$ acts on $M$ via left action in the first component. This action has principal isotropy group $H$, and singular isotropy group $K$ at a fixed base point $p_0 \in G/K$ contained in the singular orbit. A disc $\mathbb{D} \subset V$ can be viewed as the
slice of the $G$ action since, via the exponential map, it can be identified $G$ equivariantly with a submanifold of $M$ orthogonal to the singular orbit at $p_0$.

Given a $G$-invariant metric $g$ on the regular part of the $G$ action, i.e. on the complement of $G \cdot p_0$, we want to determine when the metric can be extended smoothly to the singular orbit. We choose a geodesic $c$ parameterized by arc length and normal to all orbits with $c(0) = p_0$. Thus, with the above identification, $c(t) \subset V$. At the regular points $c(t)$, i.e., $t > 0$, the isotropy is constant equal to $H$. We fix an $\text{Ad}_H$ invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ and identify the tangent space $T_{c(t)}G/H = \dot{c}^t \subset T_{c(t)}M$, with $\mathfrak{n}$ via action fields: $X \in \mathfrak{n} \rightarrow X^*(c(t))$. $H$ acts on $\mathfrak{n}$ via the adjoint representation and a $G$ invariant metric on $G/H$ is described by an $\text{Ad}_H$ invariant inner product on $\mathfrak{n}$. For $t > 0$ the metric along $c$ is thus given by $g = dt^2 + h_t$ with $h_t$ a one parameter family of $\text{Ad}_H$ invariant inner products on the vector space $\mathfrak{n}$, depending smoothly on $t$. Conversely, given such a family of inner products $h_t$, we define the metric on the regular part of $M$ by using the action of $G$.

By the slice theorem, for the metric on $M$ to be smooth, it is sufficient that the restriction to the slice $V$ is smooth. This restriction can be regarded as a map $g(t): V \rightarrow S^2(\mathfrak{n})$. The metric is defined and smooth on $V \setminus \{0\}$, and we need to determine when it admits a smooth extension to $V$.

We choose an $\text{Ad}_H$ invariant splitting

$$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r,$$

where $\text{Ad}_H$ acts trivially on $\mathfrak{n}_0$ and irreducibly on $\mathfrak{n}_i$ for $i > 0$. On $\mathfrak{n}_i$, $i > 0$ the inner product $h_t$ is uniquely determined up to a multiple, whereas on $\mathfrak{n}_0$ it is arbitrary. Furthermore, $\mathfrak{n}_i$ and $\mathfrak{n}_j$ are orthogonal if the representations of $\text{Ad}_H$ are inequivalent. If they are equivalent, inner products are described by 1, 2 or 4 functions, depending on whether the equivalent representations are orthogonal, complex or quaternionic.

Next, we choose a basis $X_i$ of $\mathfrak{n}$, adapted to the above decomposition, and thus the metrics $h_t$ are described by a collection of smooth functions $g_{ij}(t) = g(X^*_i(c(t)), X^*_j(c(t)))$, $t > 0$. In order to be able to extend this metric smoothly to the singular orbit, they must satisfy certain smoothness conditions at $t = 0$, which we will discuss in the next two Sections. Notice that in order for the metric to be well defined on $M$, the limit of $h_t$, as $t \rightarrow 0$, must exist and be $\text{Ad}_K$ invariant on $\mathfrak{m}$. But we will see that the smoothness conditions in Section 3 already contain this information.

Choosing an $\text{Ad}_K$ invariant complement to $\mathfrak{k} \subset \mathfrak{g}$, we obtain the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p} \quad \text{and thus} \quad \mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m},$$

where we can also assume that $\mathfrak{n}_i \subset \mathfrak{p}$ or $\mathfrak{n}_i \subset \mathfrak{m}$. Here $\mathfrak{m}$ can be viewed as the tangent space to the singular orbit $G/K$ at $p_0 = c(0)$ and $\mathfrak{p}$ as the tangent space of the sphere $K/H$.

It is important for us to identify $V$ in terms of action fields. For this we send $X \in \mathfrak{p}$ to $\tilde{X} := \lim_{t \rightarrow 0} \frac{X^*(c(t))}{t} \in V$. Since $K$ preserves the slice $V$ and acts linearly on it, we thus have $X^*(c(t)) = t\tilde{X} \in V$. In this language, $V \simeq \dot{c}(0) \oplus \mathfrak{p}$. For simplicity we denote $\tilde{X}$ again by $X$ and, depending on the context, use the same letter if considered as an element of $\mathfrak{p}$ or of $V$.

Notice that since $K$ acts irreducibly on $V$, an invariant inner product on $V$ is determined uniquely up to a multiple. Since for any $G$ invariant metric we fix a geodesic $c$, which we assume is parameterized by arc length, this determines an inner product on $V$, which we denote by $g_0$. Thus $g_0 = g_{\dot{c}(0)}|_V$ for any $G$ invariant metric for which $c$ is a normal geodesic. We point out that in fact any $G$ invariant metric is $G$-equivariantly isometric to one where the given curve $c$ is a normal geodesic.

$K$ acts via the isotropy action $\text{Ad}(K)_\mathfrak{m}$ of $G/K$ on $\mathfrak{m}$ and via the slice representation on $V$. The action on $V$ is determined by the fact that $K/H = S^k$, and that the stabilizer group at $c(t), t > 0$, is $H$. The smoothness conditions only depend on the $\text{Id}$ component of $K$ since, as
we will see, they are determined by $L \simeq S^1 \subset K_0$. Furthermore, the action of $K$ on $S^\ell$, and hence on $V$ is often highly ineffective. But there exists a normal subgroup $N \subset K_0$ such that $N$ acts almost effectively and transitively on $S^\ell$ with stabilizer group $N \cap H$. Since $L \subset N$, the smoothness conditions again only depend on the $1d$ component of $N$. We list the almost effective actions by connected Lie groups acting transitively on spheres in Table A.

2. Reduction to a 2-plane

In this Section we show how to reduce the question of smoothness of the metric on $M = G \times_K V$ to a simpler one. At an exceptional point, smoothness (of order $C^k$ or $C^\infty$) of the metric is equivalent to the invariance with respect to the Weyl group since the slice is the normal geodesic. Hence we only need to discuss the conditions at singular points.

At a singular point, recall that the slice theorem for the action of $H$ implies that the metric is smooth if and only if its restriction to a slice $V$, i.e. $g_{|V} : V \to S^2(p \oplus m)$ is smooth. We choose for each Ad$_H$ irreducible summand in $p$ an (arbitrary) vector $v_i \neq 0$. If there exists a 3-dimensional trivial module $p_0 \subset p$, we pick in $p_0$ an arbitrary fixed basis.

**Proposition 2.1.** A cohomogeneity one metric $g$ defined on the set of regular points in $M$ extends smoothly to the singular orbit if and only if it is smooth when restricted to the 2 planes $P_i \subset V$ spanned by $\dot{c}(0)$ and $v_i$.

**Proof.** It is sufficient to show that $g(X,Y)|_V$ is smooth for any non-vanishing vector fields $X,Y$ defined on $V$, i.e. $X,Y : V \to TM$. We can use equivariance of the metric and its derivatives with respect to the action of $K$ on $V$.

We start by proving continuity. Let $p_i \in V \setminus \{0\}$ such that $p_i \to 0$. We want to show that $g(X,Y)(p_i)$ converges to $g(X,Y)(0)$. If not, there exists a subsequence that does not converge to $g(X,Y)(0)$. We will show that for any subsequence there exists a further subsequence whose limit is $g(X,Y)(0)$. This will be a contradiction. Since $K$ acts transitively on any sphere in $V$, there exists $k_i \in K$ such that $k_ip_i$ lies on the normal geodesic $c$. Given a subsequence, compactness of $K$ implies that there exists a further subsequence such that $k_i \to k_0$. But if $k_i \to k_0$, the equivariance of the metric, and continuity of the metric along the normal geodesic, implies that

$$g(X,Y)(p_i) = g(k_iX, k_iY)(k_i \cdot p_i) \to g(k_0X, k_0Y)(0) = g(X,Y)(0)$$

since by assumption the metric at the origin is invariant under $K$. This proves continuity.

Next, we prove the metric is $C^1$. For simplicity we first assume that the action of $H$ on $p$ is irreducible and non-trivial. Using the classification of such actions in Table A, i.e. entries 1,1',7 and 8, one easily sees that $H$ acts transitively on the unit sphere in $p$. By assumption, the metric is smooth when restricted to the 2-plane $P$ spanned by $v \in p$ and $\dot{c}(0)$. Given a vector $w \in V$, possibly $w = \dot{c}(0)$, we need to show that the derivative with respect to $w$ extends continuously across the origin, i.e. that

$$(2.2) \quad \lim_{i \to \infty} \frac{\partial}{\partial w} g(X,Y)(p_i) = \frac{\partial}{\partial w} g(X,Y)(0)$$

for any sequence $p_i \in V$ with $p_i \to 0$. Let us first show that the right hand side derivative in fact exists. For this, since $K$ acts transitively on every sphere in $V$, we can choose $k \in K$ such that $kw \in P$ and hence:

$$\frac{\partial}{\partial w} g(X,Y)(0) = \lim_{h \to 0} \frac{g(X,Y)(h \cdot w) - g(X,Y)(0)}{h} = \lim_{h \to 0} \frac{g(kX, kY)(h \cdot kw) - g(kX, kY)(0)}{h}$$
where we have used $K$ equivariance away from the origin and $K$ invariance of $g$ at the origin. But the right side is the derivative

$$\frac{\partial}{\partial (kw)} g(k_*X,k_*Y)(0)$$

which exists by assumption since $kw \in P$.

Now choose as before $k_i \in K$ such that $k_i p_i$ lies on the geodesic $c$. Since $H$ acts transitively on the unit sphere in $p$, and since $p$ is the orthogonal complement to $\dot{c}(0) \in V$, we can choose $h_i \in H$ such that $h_i k_i w$ lies in $P$. As before, we can assume that $k_i \to k_0$ and $h_i \to h_0$. Equivariance and smoothness of the metric away from the origin implies that for each fixed $i$

$$\frac{\partial}{\partial w} g(X,Y)(p_i) = \frac{\partial}{\partial (h_i k_i w)} g((h_i k_i)_*X,(h_i k_i)_*Y)(h_i k_i p_i)$$

Since $h_i k_i p_i = k_i p_i$ lies on the geodesic, and since $h_i k_i w \in P$, we get

$$\lim_{i \to \infty} \frac{\partial}{\partial w} g(X,Y)(p_i) = \frac{\partial}{\partial (h_0 k_0 w)} g((h_0 k_0)_*X,(h_0 k_0)_*Y)(0)$$

$$= \lim_{h \to 0} \frac{g((h_0 k_0)_*X,(h_0 k_0)_*Y)(h \cdot h_0 k_0 w) - g((h_0 k_0)_*X,(h_0 k_0)_*Y)(0)}{h}$$

$$= \lim_{h \to 0} \frac{g(X,Y)(h \cdot w) - g(X,Y)(0)}{h} = \frac{\partial}{\partial w} g(X,Y)(0)$$

Thus the metric is $C^1$. The proof proceeds by induction. Assume the metric is $C^k$. This means that $T(w_1, \ldots, w_k, X, Y)(p) = \frac{\partial^k}{\partial w_1 \cdots \partial w_k} g(X,Y)(p)$ is a smooth multi linear form on the slice $V$ which is equivariant in all its arguments. We can thus use the same proof as above to show that

$$\frac{\partial}{\partial w} \left( \frac{\partial^k}{\partial w_1 \cdots \partial w_k} g(X,Y) \right)(p)$$

extends continuously across the origin, and hence the metric is $C^{k+1}$.

We now extend the above argument to the case where $p$ is not irreducible. We first observe that in the decomposition of $p$, the group $H$ acts transitively on the unit sphere in each irreducible factor of positive dimension. And in the case where there are 2 irreducible factors, say $p_1, p_2$, (there can be at most two), $H$ acts transitively on the unit sphere in both irreducible factor separately, i.e. it takes any 2 unit vectors $v_i \in p_i$ into any 2 other unit vectors in $p_i$. This is clear in case 5, 5’ and 6,6’ in Table A. In the remaining case of $K = \Spin(9)$ and $H = \Spin(7)$ we have $p_1 \simeq \mathbb{R}^7$ with $\Spin(7)$ acting via the 2-fold cover $\Spin(7) \to \SO(7)$, and $p_2 \simeq \mathbb{R}^8$ on which $\Spin(7)$ acts via its spin representation. The claim now follows since the stabilizer of $H$ at $v_1 \in \mathbb{R}^7$ is $\Spin(6)$, and the restriction of the spin representation of $\Spin(7)$ on $\mathbb{R}^8$ to this stabilizer is the action of $\Spin(6) = \SU(4)$ on $\mathbb{C}^4$, which is again transitive on the unit sphere and hence takes $v_2 \in p_2$ into any other unit vector in $p_2$.

Let $P_i$ be the 2-plane spanned by $v_i$ and $\dot{c}(0)$. By the above, any vector in $p$, can be transformed by the action of $H$ into a linear combination of the vectors $v_i$. Following the strategy in the previous case, we choose $k_i \in K$ such that $k_i p_i$ lies on the geodesic $c$, and $h_i \in H$ such that $h_i k_i w = \sum a_{ij} w_j$ with $w_j \in P_j$. Furthermore, $k_i \to k_0$ and $h_i \to h_0$ with $h_0 k_0 w = \sum a_{ij} w_j$. By
linearity of the derivative, and since the metric is smooth on $P_i$ by assumption, we have

$$\lim_{i \to \infty} \frac{\partial}{\partial(h_i k_i w)} g((h_i k_i)_* X, (h_i k_i)_* Y)(h_i k_i p_i) = \lim_{i \to \infty} \sum_j a_{ij} \frac{\partial}{\partial w_j} g((h_i k_i)_* X, (h_i k_i)_* Y)(h_i k_i p_i)$$

$$= \sum_j \lim_{i \to \infty} a_{ij} \frac{\partial}{\partial w_j} g((h_i k_i)_* X, (h_i k_i)_* Y)(h_i k_i p_i)$$

$$= \sum_j a_{0j} \frac{\partial}{\partial w_j} g((h_0 k_0)_* X, (h_0 k_0)_* Y)(0)$$

$$= \frac{\partial}{\partial(h_0 k_0 w)} g((h_0 k_0)_* X, (h_0 k_0)_* Y)(0).$$

The proof now continues as before. □

**Remark 2.3.** Notice that unless the group $K$ is $\text{Sp}(n)$ or $\text{Sp}(n) \cdot \text{U}(1)$, only one or two 2-planes are required. For the exceptions one needs four resp. three 2-planes. Notice also, that we can choose any vector $v$ in an irreducible submodule in $p$. In fact, the condition is clearly independent of such a choice since $H$ acts transitively on the unit sphere in every irreducible submodule.

We point out that Proposition 2.1 also holds for any tensor on $M$ invariant under the action of $G$, using the same strategy of proof.

### 3. Smoothness on 2-planes

In this section we show that smoothness on 2-planes can be determined explicitly in a simple fashion.

Recall that on $V$ we have the inner product $g_0$ with $g_0 = g_c(0)_V$ for any $G$ invariant metric with normal geodesic $c$. We fix a basis $e_0, e_1, \ldots, e_k$ of $V$, orthonormal in $g_0$, such that $c$ is given by the line $c(t) = te_0 = (t, 0, \ldots, 0)$. The tangent space to $M$ at the points of the normal geodesic can be identified with $c(t) \oplus m \oplus p$ via action fields. The metric $g = dt^2 + h_t$ on the set of regular points in $M$ is determined by a family of $\text{Ad}_H$ invariant inner products $h_t$ on $m \oplus p$, $t > 0$, which depend smoothly on $t$. Furthermore, $m$ and $p$ are orthogonal at $t = 0$, but not necessarily for $t > 0$. The inner products $h_t$ extend in a unique way to $V$, smoothly on $V \setminus \{0\}$. In order to prove smoothness at the origin, it is sufficient to show that $g(X_i, X_j)$ is smooth for vector fields which are a basis at every point in a neighborhood of $c(0)$. For this we use the action fields $X^*_i$ corresponding to an appropriately chosen basis $X_i$ of $m$, restricted to the slice $V$, and the (constant) vector fields $e_i$ on $V$. Recall also that we identify $p$ with a subspace of $V$ by sending $X \in p$ to $\lim_{t \to 0} X^<_t c(t) \in V$ and that $X^*(c(t)) = tX$. Finally, we have the splitting $p = p_1 \oplus \ldots \oplus p_a$ into $\text{Ad}_H$ irreducible subspaces.

According to Proposition 2.1 it is sufficient to determine smoothness on a finite list of 2-planes. Let $P^* \subset V$ be one of these 2-planes, spanned by $e_0 = \dot{c}(0)$ and $X \in p_i$ for some $i$. We choose $X$ such that $\{\dot{c}(0), X\}$ is a basis of $P^*$, orthonormal in $g_0$, and $L := \{\exp(\theta X) \mid \theta \in \mathbb{R}\}$ is a closed one parameter subgroup of $K$. This is possible, since $\text{Ad}_H$ acts transitively on the unit sphere in $p_i$. The one parameter group $L$ may not act effectively on $P^*$, even if $K$ acts effectively on $V$. Since $L \simeq S^1$, acting via rotation on $P^*$, the ineffective kernel is $L \cap H$. Let $a$ be the order of the finite cyclic group $L \cap H$. Equivalently, $a$ is the largest integer with $\exp(\frac{2\pi a}{a} X)c(0) = c(0)$. Thus $L$ operates on $P^*$ as a rotation $R(a\theta)$ in the basis $c(0), X$. We can also assume $a > 0$ by replacing, if necessary, $X$ by $-X$. This integer $a$ will be a crucial ingredient in the smoothness conditions. Notice that $a$ is the same for any unit vector $X \in p_i$ and we can thus simply denote it
by $a_i$. In the Appendix we will compute the integers $a_i$ for each almost effective transitive action on a sphere.

The action of $L$ on $m$ decomposes $m$:

$$m = \ell_0 \oplus \ell_1, \ldots, \ell_r$$

with $L|_{\ell_0} = \text{Id}$, and $L|_{\ell_i} = R(d_i\theta)$ for some integers $d_i$. Similarly we have a decomposition of $V$:

$$V = \ell'_0 \oplus \ell'_1, \ldots, \ell'_s$$

with $\ell'_{-1} = \text{span}\{c(0), X\}$, $L|_{\ell'_{-1}} = R(a\theta)$, $L|_{\ell'_0} = \text{Id}$ and $L|_{\ell'_s} = R(d_s\theta)$.

We choose the basis $e_i$ of $V$ and $X_i$ of $m$ such that it is adapted to this decomposition and oriented in such a way that $a, d_i$ and $d'_i$ are positive. For simplicity, we denote the basis of $\ell_i$ by $Y_1, Y_2$, the basis of $\ell'_i$ by $Z_1, Z_2$, and reserve the letter $X$ for the one parameter group $L = \exp(\theta X)$. We choose the vectors $Z_i \in p$ such that they correspond to $e_i$ under the identification $p \subset V$ and hence $Z_i^*(c(t)) = t e_i \in V$, as well as $X^*(c(t)) = t e_0$. We determine the smoothness of inner products module by module, and observe that an $L$ invariant function $f$ on $P^*$ extends smoothly to the origin if and only if its restriction to the line $t e_0$ is even, i.e. $f(t e_0) = g(t^2)$ with $g: (-\epsilon, \epsilon) \to \mathbb{R}$ smooth. Furthermore, we use the fact that the metric $V \to S^2(p \oplus m)$ is equivariant with respect to the action of $L$. Once the condition is determined when inner products are smooth when restricted to $P^*$, we restrict to the geodesic $c$ to obtain the smoothness condition for $h_t$.

In the following, $\phi_i(t)$ stands for a generic smooth function defined on an interval $(-\epsilon, \epsilon)$.

We will separate the problem in three parts: smoothness of scalar products of elements in $m$, in $p$ and mixed scalar products between elements of $m$ and $p$. We will start with the easier case of the metric on $p$.

### 3.1 Smoothness on $p$.

Recall that on a 2-plane a metric given in polar coordinates by $dt^2 + f^2(t) d\theta^2$ is smooth if and only if $f$ extends to a smooth odd function with $f(0) = 0$ and $f'(0) = 1$, see, e.g., [KW]. Since $X$ has unit length in $g_0$, we have $X^* = \frac{\partial}{\partial \theta}$ in the two plane spanned by $c(0)$ and $X$. Hence smoothness on $p$ is equivalent to:

$$g_{c(t)}(X^*, X^*) = t^2 + t^4\phi(t^2)$$

for all $X \in p$ with $g_0(X, X) = 1$ for some smooth function $\phi$, defined on an interval $(-\epsilon, \epsilon)$.

Notice that $p_i$ and $p_j$, for $i \neq j$, are orthogonal for any $G$ invariant metric, unless $(K, H) = (Sp(n), Sp(n-1))$, in which case there exists a 3 dimensional module $p_0$ on which $H$ acts as $\text{Id}$. We choose three vectors $X_i \in p_0$, orthonormal in $g_0$. Applying (3.1) to $(X_i^* + X_j^*)/\sqrt{2}$, it follows that the metric is smooth on $p_0$ if and only if

$$g_{c(t)}(X_i^*, X_j^*) = t^2\phi_{ij} + t^4\phi_{ij}(t^2)$$

for some smooth functions $\phi_{ij}$.

We finally point out the following relationship with the integer $a$ from the previous section. It is necessary to normalize the vector $X \in p$ such that it has unit length in $g_0$. This can be determined algebraically as follows: Let $t_0$ be the first value such that $\exp(t_0 X) \in H$. Then, according to the above discussion, $t_0 = \frac{2\pi}{a}$ for $a = |L \cap H|$ and hence $X/a$ has unit length in $g_0$.

See [V] for a more detailed description, including the case when the geodesic is not necessarily parametrized by arc length.

### 3.2 Inner products in $m$.

In the remaining sections $L = \{\exp(\theta X) \mid \theta \in \mathbb{R}\}$ is a one parameter group acting via $R(a\theta)$ on $\ell'_{-1}$. We first describe the inner products in a fixed module $\ell_i$.

**Lemma 3.3.** Let $\ell$ be an irreducible $L$ module in $m$ on which $L$ acts via a rotation $R(d\theta)$ in a basis $Y_1, Y_2$. If the metric on $\ell$ is given by $g_{ij} = g_{c(t)}(Y_i^*, Y_j^*)$, then

$$
\begin{pmatrix}
g_{11} & g_{12} 
g_{12} & g_{22}
\end{pmatrix} =
\begin{pmatrix}
\phi_1(t^2) & 0 
0 & \phi_1(t^2)
\end{pmatrix} + t^2
\begin{pmatrix}
\phi_2(t^2) & \phi_3(t^2) 
\phi_3(t^2) & -\phi_2(t^2)
\end{pmatrix}
$$


for some smooth functions $\phi_k$, $k = 1, 2, 3$.

**Proof.** The metric on $\ell$, restricted to the plane $P^* \subset V$, can be represented by a matrix $G(p)$ whose entries are functions of $p \in P^*$. We identify $\ell \simeq \mathbb{C}$ and $P^* \simeq \mathbb{C}$ such that the action of $L$ is given by multiplication with $e^{i\theta \ell}$ on $\ell$ and $e^{i\theta}$ on $P^*$. The metric $G$ must be $L$ equivariant, i.e.

$$G(p) = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \quad \text{with} \quad G(e^{i\theta} p) = R(d\theta)G(p)R(-d\theta).$$

The right hand side can also be seen as a linear action of $L$ on $S^2\ell \simeq \mathbb{R}^3$ and we may describe it in terms of its (complex) eigenvalues and eigenvectors. We then get:

$$(g_{11} + g_{22})(e^{i\theta} p) = (g_{11} + g_{22})(p)$$

$$(g_{12} + i(g_{11} - g_{22}))(e^{i\theta} p) = e^{2i\theta}(g_{12} + i(g_{11} - g_{22}))(p)$$

$$(g_{12} - i(g_{11} - g_{22}))(e^{i\theta} p) = e^{-2i\theta}(g_{12} - i(g_{11} - g_{22}))(p).$$

The first equality just reflects the fact that the trace is a similarity invariant. Let $w(p) = (g_{12} + i(g_{11} - g_{22}))(p)$. Then the second equality says that $w(e^{i\theta} p) = e^{2i\theta}w(p)$, and the third one is the conjugate of the second. Setting $p = te_0, t \in \mathbb{R}$ and replacing $\theta$ by $\theta/a$, we get

$$w(e^{i\theta} t) = e^{2i\theta}a \cdot w(t) = (te^{i\theta})^{2d/a} t^{-2d/a} w(t).$$

If we let $z = te^{i\theta}$, then

$$w(z) = z^{-2d/a} w(t) \quad \text{or} \quad z^{-2d/a} w(z) = t^{-2d/a} w(t), \quad \text{where} \quad t = |z|.$$  

The first equation says that if $w(z)$ is smooth, then $w(z)$ must have a zero of order $2d/a$ at $z = 0$. If so, the second equation says that the function $z^{-2d/a} w(z)$ is $L$-invariant. This means that $g_{11} + g_{22}$ and $z^{-2d/a} w(z)$ must be smooth functions of $|z|^2$. If we restrict $z^{-2d/a} w(z)$ to the real axis and we separate the real and the imaginary part this is equivalent to the existence of smooth functions $\phi_i$ such that

$$(g_{11} - g_{22})(t) = t^{2d/a} \phi_1(t^2), \quad g_{12}(t) = t^{2d/a} \phi_2(t^2), \quad (g_{11} + g_{22})(t) = \phi_3(t^2).$$

Conversely, given 3 functions $g_{11}, g_{22}, g_{12}$ along the real axis that verify these relations, they admit a (unique) smooth $L$-invariant extension to $\mathbb{C}$. Indeed, the first two equalities guarantee that $z^{-2d/a} w(z)$ and hence $w(z)$ is a smooth function on $P^*$. The third equality guarantees that $g_{11} + g_{22}$, and hence $G(p)$, has a smooth extension to $P^*$.

\[\square\]

**Remark.** If $a$ does not divide $2d$, the proof shows that $w(z)$ is smooth only if $w(t) = 0$ for all $t$. But then $g_{12} = 0$ and $g_{11} = g_{22}$ is an even function. Thus in this Lemma, as well as in all following Lemmas’s, in case of a fractional exponent of $t$, the term should be set to be 0. In practice, this will already follow from $\text{Ad}_H$ invariance.

For inner products between different modules we have:

**LEMMA 3.4.** Let $\ell_1$ and $\ell_2$ be two irreducible $L$ modules in $\mathfrak{m}$ with basis $Y_1, Y_2$ resp. $Z_1, Z_2$ on which $L$ acts via a rotation $R(d_i \theta)$ with $d_i > 0$. If the inner products between $\ell_1$ and $\ell_2$ are given by $h_{ij} = g_{c(t)}(Y_i^*, Z_j^*)$, then

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \frac{1}{t^{1/2d/a}} \begin{pmatrix} \phi_1(t^2) & \phi_2(t^2) \\ -\phi_2(t^2) & \phi_1(t^2) \end{pmatrix} + \frac{1}{t^{1/2d/a}} \begin{pmatrix} \phi_3(t^2) & \phi_4(t^2) \\ -\phi_4(t^2) & \phi_3(t^2) \end{pmatrix}.$$
for some smooth functions $\phi_k$.

Proof. $L$ acts on $\ell_1 \oplus \ell_2$ via conjugation with $\text{diag}(R(d_1 \theta), R(d_2 \theta))$ and hence

\[
\begin{pmatrix}
    h_{11} & h_{12} \\
    h_{21} & h_{22}
\end{pmatrix} \rightarrow R(d_1 \theta) \begin{pmatrix}
    h_{11} & h_{12} \\
    h_{21} & h_{22}
\end{pmatrix} R(-d_2 \theta)
\]

This action has eigenvectors

\[
w_1 = h_{11} + h_{22} + i(h_{12} - h_{21}), \quad w_2 = h_{12} + h_{21} - i(h_{11} - h_{22})
\]

with eigenvalues $e^{(d_1 - d_2) i \theta}$ and $e^{(d_1 + d_2) i \theta}$, and their conjugates. We set

\[
w_1(e^{a i \theta} p) = e^{d_1 - d_2 i \theta} w_1(p), \quad w_2(e^{a i \theta} p) = e^{d_1 + d_2 i \theta} w_2(p),
\]

where we replaced, if necessary, $w_1$ by its conjugate. A computation similar to the previous ones shows that a smooth extension to the origin is equivalent to

\[
(h_{11} + h_{22})(t) = t^{d_1 - d_2} \phi_1(t^2), \quad (h_{11} - h_{22})(t) = t^{d_1 + d_2} \phi_2(t^2)
\]

\[
(h_{12} - h_{21})(t) = t^{d_1 - d_2} \phi_3(t^2), \quad (h_{12} + h_{21})(t) = t^{d_1 + d_2} \phi_4(t^2)
\]

where $\phi_i, i = 1, \ldots, 4,$ are smooth real functions. Conversely, these relationships enable one to extend $h_{11} \pm h_{22}$ and $h_{12} \pm h_{21}$, and hence all inner products, smoothly to $P^*$.

For inner products with elements in $\ell_0$ we have:

**Lemma 3.5.** Let $\ell_0 \subset \mathfrak{m}$ be the module on which $L$ acts as $\text{Id}$, and $\ell$ an irreducible $L$ module with basis $Y_1, Y_2$ on which $L$ acts via a rotation $R(d \theta)$.

(a) If $Y \in \ell_0$, then $g_{c(t)}(Y^*, Y^*)$ is an even functions of $t$,

(b) If $Y \in \ell_0$ and $h_i = g_{c(t)}(Y^*, Y_i^*)$, then

\[
h_1(t) = t^{d_1} \phi_1(t^2), \quad h_2(t) = t^{d_2} \phi_2(t^2),
\]

for some smooth functions $\phi_k$.

Proof. If $Y \in \ell_0$, then $g(Y^*, Y^*)$ is invariant under $L$ and hence an even function.

In case (b), we consider the restriction of the metric to the three dimensional space spanned by $\ell$ and $Y$. This can be represented by a matrix $G(p) = \begin{pmatrix} g_{11} & g_{12} & h_1 \\ g_{12} & g_{22} & h_2 \\ hystem{11} & h_2 & h \end{pmatrix}$ whose entries are functions of $p \in P^*$. In particular, $h_i = g(Y_i^*, Y_i^*)$. The action of $L$ on $G(p)$ is given by conjugation with $\text{diag}(R(d \theta), 1)$. Decomposing into eigenvectors, we get, in addition to the eigenvectors already described in Lemma 3.3, the eigenvector $w(z) = h_1(z) + i h_2(z)$ with eigenvalue $e^{d i \theta}$. But $w(e^{a i \theta} p) = e^{a i \theta} w(p)$ implies that $z^{-a} w(z)$ is an invariant function. Thus smoothness for the $h_i$ functions is equivalent to

\[
h_1(t) = t^{d_1} \phi_1(t^2), \quad h_2(t) = t^{d_2} \phi_2(t^2)
\]

for some smooth functions $\phi_1$.

### 3.3. Inner products between $\mathfrak{p}$ and $\mathfrak{m}$.

Recall that for an appropriately chosen basis $e_0, \ldots, e_k$ of $V$, we need to show that the inner products $g(e_i, X_i^*)$, where $X_i$ is a basis of $\mathfrak{m}$, are smooth functions when restricted to the plane $P^* \subset V$. When restricting to the geodesic $c$, we obtain the smoothness conditions on the corresponding entries in the metric.

Recall also that the plane $P^*$ is spanned by $e_0 = c$ and $X \in \mathfrak{p} \subset V$ such that $L = \{ \exp(\theta X) \mid \theta \in \mathbb{R} \}$ is a closed one parameter group in $K$. We also have the decomposition of $V$:

\[
V = \ell_{-1}' \oplus \ell_0' \oplus \ell_1' \oplus \cdots \oplus \ell_s' \quad \text{with} \quad \ell_{-1}' = \text{span}\{c(0), X\}, \quad L|_{\ell_{-1}'} = R(a \theta), \quad L|_{\ell_0'} = \text{Id} \quad \text{and} \quad L|_{\ell_i'} = R(d_i \theta)
\]
which we use in the following. Finally, recall that $X^*(c(t)) = tX \in V$ for $X \in p$ and that $g_{c(t)}(\frac{\partial}{\partial t}, X^*) = 0$ for all $X \in p \oplus m$.

**Lemma 3.6.** Let $X \in \ell'_{-1}$. Then we have:

(a) If $Y \in \ell_0$, then $g_{c(t)}(X^*, Y^*) = t^2\phi(t^2)$,

(b) If $Y_1, Y_2$ a basis of the irreducible module $\ell = \ell_i$, on which $L$ acts as $R(d\theta)$ with $d > 0$, then $g_{c(t)}(X^*, Y_k^*) = t^2t^\frac{d}{a}\phi_k(t^2)$

for some smooth functions $\phi, \phi_k$.

**Proof.** For part (a) the proof is similar to Lemma 3.5. On the 3-space spanned by $e_0 = \hat{c}(0), \ e_1 = X, \ e_2 = Y$, the one parameter group $L$ acts via conjugation with $\text{diag}(R(a\theta), 1)$ and, using that fact that $Y^*$ is orthogonal to $\hat{c}$, the metric is given by $G(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ 0 & h & f \end{pmatrix}$ with $h = g(e_2, Y^*)$ and $f = g(Y^*, Y^*)$. We already saw that $f$ is an even function, and as in the proof of Lemma 3.5, we see, when restricted to the geodesics, $h(t) = t^2\phi(t^2) = t\phi(t^2)$. Hence $g_{c(t)}(X^*, Y^*) = t\phi(t^2) = t^2\phi(t^2)$.

For part (b) the proof is similar to Lemma 3.4. On the 4 dimensional space spanned by $e_1, e_2$ and $Y_1, Y_2$ the group $L$ acts via conjugation with $\text{diag}(R(a\theta), R(d\theta))$ and the metric is given by

$$G(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & h_1 & h_2 \\ 0 & h_1 & g_{11} & g_{12} \\ 0 & h_2 & g_{12} & g_{22} \end{pmatrix}$$

with $h_k = g(e_2, Y_k^*)$ and $g_{kl} = g(Y_k^*, Y_l^*)$. As in the proof of Lemma 3.4 it follows that

$$h_2(t) = \frac{\partial}{\partial t} \phi_1(t^2) \quad \text{and} \quad h_2(t) = \frac{\partial}{\partial t} \phi_2(t^2)$$

and hence $h_2(t) = t^\frac{d}{a}+1\phi(t^2)$, and similarly for $h_1(t)$. Thus $g_{c(t)}(X^*, Y_k^*) = t\phi(t^2) = t^2t^\frac{d}{a}\phi_k(t^2)$.

Next the inner products with $\ell'_0$.

**Lemma 3.7.** For $Z \in \ell'_0$ we have:

(a) If $Y \in \ell_0$, then $g_{c(t)}(Z^*, Y^*) = t^2\phi(t^2)$,

(b) If $Y_1, Y_2$ is a basis of the irreducible module $\ell_i$, then $g_{c(t)}(Z^*, Y_k^*) = t^{\frac{d_i}{a}}\phi_k(t^2)$

for some smooth functions $\phi_i$.

**Proof.** For part (a), let $Z = e_1$. Then $g(e_1, Y^*)$ is $L$ invariant and hence even. Furthermore, it vanishes at $t = 0$ since the slice is orthogonal to the singular orbit at $c(0)$. Hence $g(e_1, Y^*) = t^2\phi(t^2)$, which implies $g_{c(t)}(Z^*, Y^*) = t^2\phi(t^2)$.

Similarly for (b), using the proof of Lemma 3.5 it follows that $g_{c(t)}(e_1, Y_k^*) = t^\frac{d}{a}\phi_k(t^2)$. Since $d_i, a > 0$, this already vanishes as required. The proof now finishes as before.

And finally the remaining inner products:

**Lemma 3.8.** Let $\ell'_i$ and $\ell'_j$ with $i, j > 0$ be two irreducible $L$ modules with basis $Z_1, Z_2$ resp.

$Y_1, Y_2$ on which $L$ acts via a rotation $R(\frac{d_i}{a}\theta)$ resp. $R(\frac{d_j}{a}\theta)$ with $d'_i, d'_j > 0$.

(a) The inner products $h_{ij} = g_{c(t)}(Z_k^*, Y_l^*)$ satisfy

$$h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = t^b t^{\frac{d_i - d_j}{a}} \begin{pmatrix} \phi_1(t^2) & \phi_2(t^2) \\ -\phi_2(t^2) & \phi_1(t^2) \end{pmatrix} + t^c t^{\frac{d_i + d_j}{a}} \begin{pmatrix} \phi_3(t^2) & \phi_4(t^2) \\ \phi_4(t^2) & -\phi_3(t^2) \end{pmatrix}$$
where \( b = 3 \) if \( d_i' = d_j \), and \( b = 1 \) if \( d_i' \neq d_j \),

(b) If \( Y \in \ell_0 \), then \( g_{c(t)}(Y^*, Z_k^*) = tt^k \phi_k(t^2) \)

do some smooth functions \( \phi_i \).

Proof. (a) We repeat the proof of Lemma 3.4 for the basis \( e_1 = Z_1, e_2 = Z_2, e_3 = Y_1, e_4 = Y_2 \) of \( \ell_i' \cup \ell_j \). But if \( d_i' = d_j \), we have to require in addition that the inner products vanish at \( t = 0 \), i.e., \( \phi_1(0) = \phi_2(0) = 0 \), which means the first matrix must be multiplied by \( t^2 \). The proof then proceeds as before.

(b) We proceed as in Lemma 3.5 (b).

This finishes the discussion of all possible inner products in \( n = p \oplus m \).

3.4. Smoothness conditions for symmetric \( 2 \times 2 \) tensors. The above methods can be applied to obtain the smoothness conditions for any \( G \)-invariant tensor, defined along a curve \( c \) transverse to all orbits. One needs to take care though, since for a metric \( g \) the slice and singular orbit are orthogonal at \( t = 0 \), whereas for a general tensor this may not be the case. For the purpose of applying this to the Ricci tensor, we briefly discuss how to derive the smoothness conditions for any symmetric \( 2 \times 2 \) tensor \( T \).

For the functions \( T(p,m) \), the conditions for smoothness of \( T \) and a metric \( g \) are clearly the same. For \( T(p,p) \) the only difference is that now \( T(X_i^*, X_i^*) = t^2 \phi_i(t^2) \) for \( X_i \in p_i \), with \( \phi_i(0) = \phi_0 \in \mathbb{R} \) for all \( i \), where \( X_i \) has unit length in \( g_0 \). For the case of \( T(p,m) \), one needs to examine the proof of the Lemma’s in Section 3.3. In some cases, for a metric tensor, certain components are forced to have a zero of one order higher at \( t = 0 \) than a generic symmetric tensor. We list the results in Table D.

A new feature is that, unlike in the case of a metric, the mixed terms \( T(\dot{c}(t), X^*) \) do not have to vanish if \( X \in p \oplus m \) lies in a module on which \( \text{Ad}_H \) acts trivially. The conditions on \( T(m, c) \) can easily be deduced from the proofs of the Lemma’s in Section 3.3, keeping in mind that \( T \) on the 2 plane spanned by \( e_0 = \dot{c}(0) \), \( e_1 = X \) is now not \( I_d \), but an arbitrary symmetric matrix.

For the remaining values one easily shows, working in Euclidean coordinates, that for \( X \in p_0 \)

\[
T_{c(t)}(\dot{c}, X^*) = tv_1(t^2), \quad T_{c(t)}(\dot{c}, \dot{c}) = \psi_2(t^2) \quad \text{with} \quad \psi_2(0) = \phi_0,
\]

where \( \dot{c}(0) \) and \( X \) have unit length in \( g_0 \). Notice that \( g_0 \) only depends on \( \dot{c}(0) \) and not on the choice of a \( G \)-invariant metric on \( M \). Recall also that, as explained in Section 3.1, the normalization of \( X \) such that it has unit length in \( g_0 \) can in fact be determined algebraically.

Finally, observe that \( T(\dot{c}, p_j) = T(p_i, p_j) = 0 \) for \( 0 < i < j \) since the \( \text{Ad}_H \) representations are inequivalent.

4. Examples

Before we illustrate the method with some examples, let us make some general comments.

If \( G \) is compact, one often starts with a basis of \( n = p \oplus m \) which is orthonormal in a fixed biinvariant metric \( Q \) of \( g \). Thus one needs to determine the real numbers \( r_i > 0 \) such that \( Q_{|p_i} = r_i g_0 \), \( i = 1, \ldots, s \), which needs to be used in order to translate the conditions in (3.1) into a basis orthonormal in \( Q \). We point out that if \( s > 1 \), \( r_i \) depends on \( i \) since in that case the biinvariant metric \( Q_{|K} \) does not restrict to a constant curvature metric on \( K/H \). This issue is studied in [GZ], Table 2.5, which can be useful in determining \( r_i \). Alternatively, one chooses the basis in \( p \) not \( Q \)-orthonormal, but such that their image under the inclusion \( p \subset V \), determined by the action of \( K \) on \( V \), has unit length in \( g_0 \).

Since the action of \( L_i = \{ \exp(\theta v_i) \mid 0 \leq \theta \leq 2\pi \} \) on \( m \) is given by the restriction of \( \text{Ad}_K \), the exponents \( d_i \) can be determined in terms of Lie brackets, i.e. on \( \ell_i \) we have \([v_i, Y_1] = d_i Y_2\).
and \([v_i, Y_2] = -d_i Y_2\), where \(Y_1, Y_2 \in \ell_i\) are \(Q\) orthogonal vectors of the same length. This also determines the orientation of the basis so that \(d_i > 0\). The values of \(d_i\) that will arise are already determined by the weights \(\alpha_i\) of the irreducible subrepresentations of \(K\) on \(m\). The decomposition under \(L = \{\exp(\theta v) \mid 0 \leq \theta \leq 2\pi\}\) can be considered as the weight space decomposition of the action of \(K\) on \(m\) with respect to a maximal abelian subalgebra containing \(v\). Thus \(d_i = \alpha_i(v)\), for all weights \(\alpha_i\), and hence the largest integer is \(\lambda(v)\) where \(\lambda\) is the dominant weight.

The slopes \(d_i^e\) are not determined by Lie brackets. Instead, one needs to use the knowledge of \(K\) and \(H\) to determine the action of \(K\), and hence \(L\), on \(V\). For the almost effective actions of \(K\) on spheres, a choice of the vectors \(Y\) equivalent to equivariance of the second fundamental form into \(\text{Ad}_H\) direction.

We thus in general get a highly over determined system of equations, whose coefficients do not depend on the metric, but only on the Lie groups involved. On the other hand, we know that smooth metrics exist on \(M\). This implies that this over determined system has solutions. Row reducing the coefficient matrix, we obtain relationships between the even functions. Substituting, we obtain \(k\) equations in \(k\) unknown even functions, where \(2k\) is the number of nonzero entries in \(g_{ij}\). These are the equations described in Theorem B. The coefficient matrix must be invertible, again since we know smooth metrics exist on \(M\). Thus we can also write \(g_{ij}\) in terms of the unknown even functions.

The conditions of order \(0\) are equivalent to \(K\) invariance. The conditions of order \(1\) are equivalent to equivariance of the second fundamental form \(B: S^2T \to T^\perp = V\) of the singular orbit \(G/K\) with tangent space \(T = T_{p_0}K/H\) under the action of \(K\). Recall also that one has the Weyl group element \(w \in K\) with \(w(\hat{c}(0)) = -\hat{c}(0)\), uniquely determined mod \(H\). Clearly \(w \in L_i\) for all \(i\), up to a change by \(\text{Ad}_H\).

Example 1

A simple example is given by the groups \(G = Sp(1) \times S^1, K = \{(e^{i\theta}, 1) \mid 0 \leq \theta \leq 2\pi\} \cdot H\) and \(H \cong \mathbb{Z}_4\) with generator \((i, i)\). There exists an infinite family of inequivalent cohomogeneity one actions on \(S^5\) as a special case of the Kervaire sphere examples, see [GVWZ], the simplest one being the tensor product action of \(SO(3)SO(2)\) on \(S^5\). For all of them one half of the group diagram is given by the above groups.

If we let \(X_1 = (i, 0), X_2 = (j, 0), X_3 = (k, 0)\) and \(Y = (0, i)\) then we have the \(\text{Ad}_H\) invariant decomposition \(p = t = \mathbb{R} \cdot X_2\) and \(m = m_0 \oplus m_1\) with \(m_0 = \text{span}\{X_1, Y\}, m_1 = \mathbb{R} \cdot X_3\). Since \(\text{Ad}_H\) acts as \(\text{Id}\) on \(m_0\) and as \(-\text{Id}\) on \(p \oplus m_1\) the nonvanishing inner products are given by

\[
 f_1 = (X_i, X_i), i = 1, 2, 3, \quad g = \langle Y, Y \rangle, \quad h_1 = \langle X_1, Y \rangle, \quad h_2 = \langle X_2, X_3 \rangle.
\]

There is only the one parameter group \(L = \{\exp(\theta v) \mid 0 \leq \theta \leq 2\pi\}\) to be considered. \(L\) acts via \(R(\theta)\) on \(\ell_{-1} = \text{span}\{X_2, \hat{c}(0)\}\) trivially on \(\ell_0 = \mathbb{R} \cdot Y\), and by \(R(\theta)\) on \(\ell_1 = \text{span}\{X_1, X_3\}\). Thus \(a = 1\) and \(d_1 = 2\). According to Section 3.1, Lemma 3.5 and Lemma 3.6 we have

\[
 f_2 = 2t^2 + t^4 \phi_1(t^2), \quad h_1 = t^2 \phi_2(t^2), \quad h_2 = t^4 \phi_3(t^2)
\]

and according to Lemma 3.3

\[
 f_1 = \phi_4(t^2) + t^4 \phi_5(t^2), \quad f_3 = \phi_4(t^2) - t^4 \phi_5(t^2).
\]

See also [GVZ] Appendix 1 for a further class of examples with \(K/H \cong S^1\).
**Example 2**

Let $H \subset K \subset G$ be given by $\text{SO}(2) \subset \text{SO}(3) \subset \text{SO}(5)$, where the embedding of $\text{SO}(3)$ in $\text{SO}(5)$ is given by the unique irreducible representation of $\text{SO}(3)$ on $\mathbb{R}^5$. The singular orbit $G/K$ is the Berger space (which is positively curved in a biinvariant metric).

We consider the following basis of $\mathfrak{g} = \mathfrak{so}(5)$:

$$
\begin{align*}
K_1 &= 2E_{12} + E_{34}, & K_2 &= E_{23} - E_{14} + \sqrt{3}E_{45}, & K_3 &= E_{13} + E_{24} + \sqrt{3}E_{35} \\
V_1 &= \frac{1}{\sqrt{5}}E_{12} - \frac{2}{\sqrt{5}}E_{34}, & V_2 &= \frac{\sqrt{2}}{\sqrt{5}}E_{45} - \frac{\sqrt{3}}{\sqrt{10}}(E_{23} - E_{14}), & V_3 &= \frac{\sqrt{2}}{\sqrt{5}}E_{35} - \frac{\sqrt{3}}{\sqrt{10}}(E_{13} + E_{24}) \\
V_4 &= E_{25}, & V_5 &= E_{15}, & V_6 &= \frac{1}{\sqrt{2}}(E_{24} - E_{13}), & V_7 &= -\frac{1}{\sqrt{2}}(E_{23} + E_{14}).
\end{align*}
$$

Then $K_1, K_2, K_3$ span the subalgebra $\mathfrak{k} \simeq \mathfrak{so}(3)$ with $[K_1, K_2] = K_3$ and cyclic permutations. Thus $K_i$ is orthonormal with respect to the biinvariant metric $Q_{\mathfrak{k} \mathfrak{o}(5)}(A, B) = -\frac{1}{2} \text{tr}(AB)$ which induces the metric of constant curvature 1 on $\text{SO}(3)/\text{SO}(2) = S^2$. We choose the base point such that the Lie algebra of its stabilizer group $H$ is spanned by $K_1$. Hence $c(0)$, $K_2$, $K_3$ is an orthonormal basis in the standard inner product on $V = \mathbb{R}^3$. Notice that for the biinvariant metric $Q_{\mathfrak{so}(5)}(A, B) = -\frac{1}{2} \text{tr}(AB)$ we have $Q_{\mathfrak{so}(5)}(A, B) = 5Q_{\mathfrak{so}(3)}(A, B)$ for $A, B \in \mathfrak{so}(3)$. Thus, if we abbreviate $Q = Q_{\mathfrak{so}(5)}$, we have $Q(K_i, K_j) = 5\delta_{ij}$. On the other hand, $V_i$ are orthonormal unit vectors in $Q$.

We have the following decomposition of $\mathfrak{p} \oplus \mathfrak{m}$ as sum of irreducible $H$-modules:

$$
\text{Ad}_H \text{ acts trivially on } \mathfrak{m}_0, \text{ speed on } \mathfrak{p} \text{ and } \mathfrak{m}_1, \text{ and with speed 2 and } 3 \text{ on } \mathfrak{m}_2 \text{ resp. } \mathfrak{m}_3. \text{ E.g., since } H = \{ \exp(tk_1) \mid 0 \leq t \leq 2\pi \}, \text{ one needs to check that } [K_1, V_4] = 2V_5 \text{ and } [K_1, V_5] = -2V_4. \text{ Thus } \mathfrak{p} \text{ and } \mathfrak{m}_1 \text{ are equivalent as } H\text{-modules while all the other modules are inequivalent. An } \text{Ad}_H \text{ invariant metric } g \text{ along } c(t) \text{ is thus defined by the following functions:}
$$

$$
f = \langle K_2, K_2 \rangle = \langle K_3, K_3 \rangle, \quad g_1 = \langle V_1, V_1 \rangle, \quad g_2 = \langle V_2, V_2 \rangle = \langle V_3, V_3 \rangle, \quad g_3 = \langle V_4, V_4 \rangle = \langle V_5, V_5 \rangle, \quad g_4 = \langle V_6, V_6 \rangle = \langle V_7, V_7 \rangle, \quad h_{11} = \langle K_2, V_2 \rangle = \langle K_3, V_3 \rangle, \quad h_{12} = \langle K_2, V_3 \rangle = \langle K_3, V_2 \rangle.
$$

and all other scalar products are zero. The action of $\text{SO}(3)$ on $\mathfrak{m}$ is irreducible and hence $K$-invariance of the metric at the singular orbit implies $g_1(0) = g_2(0) = g_3(0) = g_4(0)$.

For the other smoothness conditions, since $\text{Ad}_H$ acts irreducibly on $\mathfrak{p}$, we need to choose only one vector and set $X = K_2$ with $L = \exp(tk_2) \subset \text{SO}(3)$. One easily sees that $L \cap H = \{e\}$ and hence $a = 1$. Furthermore, $V = \ell_{-1} \oplus \ell_0$ with $\ell_{-1} = \text{span}\{c(0), K_2\}$ and $\ell_0 = \text{span}\{K_3\}$ since $L$ acts via rotations in the $c(0), K_2$ plane, and hence trivially on $c_3 = K_3(0)$.

Under the action of $L$, one easily sees that $\mathfrak{m}$ decomposes as the sum of the following irreducible modules:

$$
\ell_0 = \text{span}(\sqrt{6}V_2 + \sqrt{10}V_7), \quad \ell_1 = \text{span}(V_3 + \sqrt{15}V_6, -\sqrt{6}V_1 - \sqrt{10}V_4), \quad \ell_2 = \text{span}(\sqrt{10}V_2 - \sqrt{6}V_7, 4V_5), \quad \ell_3 = \text{span}(-\sqrt{15}V_3 + V_6, \sqrt{10}V_1 - \sqrt{6}V_4).
$$

and a Lie bracket computation shows that under the action of $L$ we have $d_i = i$ for $i = 1, 2, 3$. E.g. $[K_2, -\sqrt{15}V_3 + V_6] = 3(\sqrt{10}V_1 - \sqrt{6}V_4)$ and $[K_2, \sqrt{10}V_1 - \sqrt{6}V_4] = -3(\sqrt{15}V_3 + V_6)$.

1) **Irreducible modules in $\mathfrak{m}$**. We have three irreducible $L$-modules in $\mathfrak{m}$ and for each of them we apply Lemma [Lemma 3.3] and use the notation $g_{ij}$ therein. Notice that due to $\text{Ad}_H$ invariance, all vectors $V_i$ are orthogonal to each other.
For $\ell_1$ we have:

\[
g_{11} = \langle V_3 + \sqrt{15}V_6, V_3 + \sqrt{15}V_6 \rangle = g_2 + 15g_4
\]
\[
g_{22} = \langle -\sqrt{6}V_1 - \sqrt{10}V_4, -\sqrt{6}V_1 - \sqrt{10}V_4 \rangle = 6g_1 + 10g_3, \quad g_{12} = 0.
\]

Since $d_1 = 1$ and $a = 1$, we need

\[
(2 + 15g_4) + (6g_1 + 10g_3) = \phi_1(t^2), \quad (2 + 15g_4) - (6g_1 + 10g_3) = t^2 \phi_2(t^2)
\]

For $\ell_2$ we have:

\[
g_{11} = \langle \sqrt{10}V_2 - \sqrt{6}V_7, \sqrt{10}V_2 - \sqrt{6}V_7 \rangle = 10g_2 + 6g_4
\]
\[
g_{22} = \langle 4V_5, 4V_5 \rangle = 16g_3, \quad g_{12} = 0
\]

Since $d_2 = 2$, smoothness requires that

\[
(10g_2 + 6g_4) + 16g_3 = \phi_3(t^2), \quad (10g_2 + 6g_4) - 16g_3 = t^2 \phi_4(t^2)
\]

For $\ell_3$ we have:

\[
g_{11} = \langle -\sqrt{15}V_3 + V_6, -\sqrt{15}V_3 + V_6 \rangle = 15g_2 + g_4
\]
\[
g_{22} = \langle \sqrt{10}V_1 - \sqrt{6}V_4, \sqrt{10}V_1 - \sqrt{6}V_4 \rangle = 10g_1 + 6g_3, \quad g_{12} = 0
\]

Since $d_3 = 3$, we need

\[
(15g_2 + g_4) + (10g_1 + 6g_3) = \phi_5(t^2), \quad (15g_2 + g_4) - (10g_1 + 6g_3) = t^6 \phi_6(t^2).
\]

In particular, all functions $g_1, g_2, g_3, g_4$ are even, a fact that one could have already obtained from invariance of the metric under the Weyl group element.

For $\ell_6$, Lemma 3.5 says that $\langle \sqrt{6}V_2 + \sqrt{10}V_7, \sqrt{6}V_2 + \sqrt{10}V_7 \rangle = 6g_2 + 10g_4$ is even, a condition that is already implied by the previous ones.

2) Products between modules in $m$. Inner products between $\ell_0$ and $\ell_2$, and between $\ell_1$ and $\ell_3$ are not necessarily 0. For the first one, Lemma 3.3 implies that

\[
\langle \sqrt{6}V_2 + \sqrt{10}V_7, \sqrt{6}V_2 + \sqrt{10}V_7 \rangle = \sqrt{60}(g_2 - g_4)
\]

and hence $g_2 - g_4 = t^2 \phi_7(t^2)$, a condition already implied by $K$ invariance at $t = 0$.

For the second one, Lemma 3.4 and

\[
(\sqrt{15}V_6, -\sqrt{15}V_3 + V_6) = \sqrt{15}(g_4 - g_2), \quad \langle -\sqrt{6}V_1 - \sqrt{10}V_4, \sqrt{10}V_1 - \sqrt{6}V_4 \rangle = \sqrt{60}(g_3 - g_1)
\]

as well as

\[
\langle \sqrt{15}V_6, \sqrt{10}V_1 - \sqrt{6}V_4 \rangle = \langle -\sqrt{6}V_1 - \sqrt{10}V_4, -\sqrt{15}V_3 + V_6 \rangle = 0
\]

implies that

\[
(g_4 - g_2) - 2(g_3 - g_1) = t^4 \phi_8(t^2), \quad (g_4 - g_2) + 2(g_3 - g_1) = t^2 \phi_8(t^2).
\]

Notice though that the second condition is already implied by $K$ invariance at $t = 0$.

3) Smoothness on the slice. Section 3.1 implies that $f = 2t^2 + t^4 \phi(t^2)$ since $a = 1$.

4) Products between $m$ and the slice $V$. All of the modules $\ell_i$ have nontrivial inner products with the slice. For the 4 inner products between $\ell_{-1}$, i.e. $K_2$, and $\ell_i$ we get from Lemma 3.6

\[
h_{11} = t^2 \phi(t^2), \quad h_{12} = t^3 \phi(t^2), \quad h_{11} = t^4 \phi(t^2), \quad h_{12} = t^5 \phi(t^2).
\]

On the other hand, for the 4 inner products between $\ell_0$, i.e. $K_3$, and $\ell_i$ we get from Lemma 3.7

\[
h_{12} = t^3 \phi(t^2), \quad h_{11} = t^2 \phi(t^2), \quad h_{12} = t^3 \phi(t^2), \quad h_{11} = t^4 \phi(t^2).
\]

Thus we need:

\[
h_{11} = t^4 \phi(t^2), \quad h_{12} = t^5 \phi(t^2).
\]
5) Combining all conditions. Summarizing the conditions in 1) and 2), we have for the inner products in $m$: 

\[
\begin{align*}
(g_2 + 15g_4) + (6g_1 + 10g_3) &= \phi_1(t^2) \\
(g_2 - 15g_4) - (6g_1 + 10g_3) &= t^4 \phi_2(t^2) \\
(10g_2 + 6g_4) + (16g_3) &= \phi_3(t^2) \\
(10g_2 - 6g_4) - (16g_3) &= t^4 \phi_4(t^2) \\
(15g_2 + g_4) + (10g_1 + 6g_3) &= \phi_5(t^2) \\
(15g_2 - g_4) - (10g_1 + 6g_3) &= t^6 \phi_6(t^2) \\
(g_1 - g_2) - 2(g_3 - g_1) &= t^4 \phi_7(t^2) \\
g_2 - g_4 &= t^2 \phi_8(t^2), \quad g_2 - g_3 = t^2 \phi_9(t^2)
\end{align*}
\]

Notice that the last two conditions imply that $(g_3 - g_1) = t^2 \phi(t^2)$ and hence $K$ invariance at $t = 0$ is encoded in the above equations.

This is an over determined linear system of equations in the metric functions. Since we know there always exist solutions, we can row reduce in order to get the following relationships between the smooth functions:

\[
\begin{align*}
\phi_1 &= \phi_5 - 16t^2 \phi_7 - 2t^4 \phi_8 \\
\phi_2 &= t^4 \phi_6 - 12 \phi_7 + 2t^2 \phi_8 \\
\phi_3 &= \phi_5 - 10t^2 \phi_7 - 5t^4 \phi_8 \\
\phi_4 &= t^2 \phi_6 + 5 \phi_8
\end{align*}
\]

Thus necessary and sufficient conditions for smoothness in $m$ are:

\[
\begin{align*}
15g_2 + 6g_3 + g_4 + 10g_1 &= \phi_5 \\
15g_2 - 6g_3 + g_4 - 10g_1 &= t^6 \phi_6 \\
-2g_3 + 2g_1 &= t^2 \phi_7 \\
-g_2 - 2g_3 + g_4 + 2g_1 &= t^4 \phi_8
\end{align*}
\]

which we can also solve for the metric and obtain (after renaming the even functions):

\[
\begin{align*}
g_1 &= \frac{1}{32} \phi_1 + \frac{1}{16} t^2 \phi_2 + \frac{3}{16} t^4 \phi_3 - \frac{1}{32} t^6 \phi_4 \\
g_2 &= \frac{1}{16} \phi_1 + \frac{1}{16} t^2 \phi_2 + \frac{3}{16} t^4 \phi_3 \\
g_3 &= \frac{1}{32} \phi_1 - \frac{5}{16} t^2 \phi_2 - \frac{5}{16} t^4 \phi_3 - \frac{1}{32} t^6 \phi_4 \\
g_4 &= \frac{1}{32} \phi_1 - \frac{15}{16} t^2 \phi_2 + \frac{3}{32} t^6 \phi_4
\end{align*}
\]

for some smooth functions $\phi_1, \phi_2, \phi_3, \phi_4$ of $t^2$. Furthermore,

\[
f = t^2 + t^4 \phi_5(t^2), \quad h_{11} = t^4 \phi_6(t^2), \quad h_{12} = t^5 \phi_7(t^2)
\]

Example 3

This example shows how to predict the exponents $d_k$ in terms of representation theory. Let $\phi_n$ be the complex $n$-dimensional irreducible representation of $SU(2)$. Choose $K = SU(2) \subset G = SU(2n)$ given by the embedding $\phi_{2n}$, and $H = SO(2) = \text{diag}(e^{i\theta}, e^{-i\theta}) \subset SU(2)$. Thus $K/H = S^2$ with slice representation $\phi_3$ and hence $a = 2$. By Clebsch-Gordan, the isotropy representation of $G/K$ is $\phi_{4n-2} \oplus \phi_{4n-4} \oplus \cdots \oplus \phi_2$. Thus the isotropy representation $G/H$ is the sum of 2 dimensional representations $\mathfrak{n}_i$ with multiplicity $i$ and weight $4n-2i$ for $i = 1, \cdots, 2n-2$ and $\mathfrak{n}_{2n-1}$ and $\mathfrak{n}_{2n}$ with multiplicity $2n-2$ and weight 2 resp. 0, as well as $\mathfrak{n}_{2n+1}$ with weight 2 coming from the isotropy representation of $K/H$. 
We only need to consider the one parameter group \( L = \exp(tA) \) with \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Since \( A \) is conjugate to \( \text{diag}(i, -i) \), the decomposition under \( L \) has the same weights and multiplicity. Thus in the description of the metric, we have exponents \( t^k \) for \( k = 1, \ldots, 4n - 2 \).

5. Actions on Spheres

In order to facilitate the applications of determining the smoothness conditions in examples, we discuss here the decomposition of the action by \( L \) on the slice and the integers \( a, d'_i \). Since \( L \subset K_0 \), we can assume that \( K \) is connected. Although the action of \( K \) on \( V \) can be highly ineffective, there exists a normal subgroup containing \( L \), acting almost effectively and transitively on \( V \). In Table A we list the almost effective transitive actions by connected Lie groups on spheres. The effective actions and the decomposition of \( p \) into irreducibles one can e.g. find in [Z]. In order to add the ineffective ones we make the following remarks. We can make an effective action of \( K \) ineffective, only if \( K \) is not simply connected since it acts as an irreducible representation on \( V \). Indeed, representations are described in terms of representations of the Lie algebra \( \mathfrak{k} \), and those are in one to one correspondence with the representation of the universal cover of \( K \). This explains the entries where \( K \) is semisimple. If not, since \( K \) is compact, \( K = K' \cdot T^k \) with \( K' \) semisimple and \( T^k \) a normal subgroup of \( K \). Since \( K \) acts irreducibly, \( k = 1 \). We can write \( K \) as \( K = (K' \times S^1)/\Gamma \), where the finite central subgroup \( \Gamma \) is determined by the embedding \( K \subset G \). We can thus describe all slice representations as (possibly ineffective) actions by \( K' \times S^1 \). The circle \( S^1 \) can act on \( V \) with certain slopes, but must commute with the irreducible action of \( K' \) on \( V \).

Recall that the inclusion \( \mathfrak{p} \subset V \) is determined by the action fields of the action of \( K \) on \( V \). For each irreducible module we choose a vector \( X \in \mathfrak{p} \), of unit length in \( g_0 \) and \( L = \exp(tX) \subset K \) a closed one parameter group. Furthermore, the integer \( a = |L \cap H| \) is the ineffective kernel of the action of \( L \) on \( V \) and \( V \) is the sum of two dimensional \( L \) invariant modules:

\[
V = \ell'_{-1} \oplus \ell'_0 \oplus \ell'_1, \ldots, \ell'_s \quad \text{with} \quad \ell'_{-1} = \text{span}\{\dot{c}(0), X\}
\]

and

\[
L|_{\ell'_{-1}} = R(a\theta), \quad L|_{\ell'_0} = \text{Id} \quad \text{and} \quad L|_{\ell'_i} = R(d'_i\theta).
\]

where \( a, d'_i \) are integers, which we can assume to be positive.

We now discuss each transitive action, one at a time, using the numbering in Table A.

1) \( K/H = \text{SO}(n+1)/\text{SO}(n) = S^n \)

\( K \) acts by matrix multiplication \( x \rightarrow Ax \) on \( V = \mathbb{R}^{n+1} \) with orthonormal basis \( e_1, e_2, \ldots, e_{n+1} \). We choose the geodesic such that \( c(t) = te_1 \) and let \( H \) be the stabilizer group of \( e_1 \), i.e. \( H = \{ \text{diag}(1, A) \mid A \in \text{SO}(n) \} \).

As usual, we use the notation \( E_{ij} \) for the skew symmetric matrix with non-zero entries in the \( (i, j) \) and \( (j, i) \) spot and biinvariant inner product \( Q(A, B) = -\frac{1}{2} \text{tr}(AB) \). Then \( \mathfrak{p} = \text{span}\{E_{12}, \ldots, E_{1(n+1)}\} \) and for the action fields we get \( E_{12}^* = e_1 \).

We choose the closed one-parameter group \( L = \{\exp(\theta E_{12}) \mid 0 \leq \theta \leq 2\pi\} \) which induces a rotation \( R(\theta) \) in the \( e_1, e_2 \) plane. Thus \( X = E_{12} \) and

\[
\ell'_{-1} = \{\dot{c}(0), E_{12}\} \quad \text{with} \quad a = 1, \quad \text{and} \quad \ell'_0 = \{E_{13}, \ldots, E_{1(n+1)}\}
\]

1) \( K/H = \text{Spin}(n+1)/\text{Spin}(n) = S^n \)
Spin(n + 1) acts via the two fold cover Spin(n + 1) → SO(n + 1) ineffectively on V. Since 
L ⊂ SO(n + 1) is a generator in π1(SO(n + 1)) ≅ Z2, the lift of L ⊂ SO(n + 1) to Spin(n + 1) has 
twice its length. Thus, if E12 is the lift of E12, the one parameter group L = {exp(θE12) | θ ∈ R} 
induces a rotation R(2θ) in the e1, e2 plane. Hence ℓ′−1 = {c(0), E12} with a = 2 and ℓ′0 as before.

2) K/H = SU(n + 1)/SU(n) = S2n+1
K acts by matrix multiplication x → Ax on V = Cn+1 with orthonormal basis e1, ie1, ..., en+1, 
iseen+1. H is the stabilizer of e1, i.e. H = SU(n) = {diag(1, A) | A ∈ SU(n)}. Besides E12, 
we have the skew hermitian matrix iE12 (by abuse of notation). We use the inner product 
Q(A, B) = −1/2Re(tr(AB)), and hence p = p0 ⊕ p1 with p0 = R · F with F = diag(ni, −i, ..., −i) 
and p1 = span{E12, iE12, .. E1(n+1), iE1(n+1)}. For the action fields we have F* = nie1 and 
E*12 = ie1, iE12 = ie1, i = 2, ..., n + 1.

We need to choose two closed one parameter subgroups, L1 = {exp(θE12) | 0 ≤ θ ≤ 2π} and 
L2 = exp(θF) with 0 ≤ θ ≤ 2π. L1 induces a rotation R(θ) in the e1, e2 plane, and in the ie1, ise2 plane as well. Thus

ℓ′−1 = {c(0), E12}, with a = 1, ℓ′1 = {1/nF, iE12}, with d′1 = 1 and ℓ′0 = {E1k, iE1k, k ≥ 3}

Next, L2 = {exp(θF) | 0 ≤ θ ≤ 2π} induces a rotation R(nθ) in the e1, ie1 plane, and R(−θ) 
in the e1, ie1 plane, k ≥ 2. Thus X = 1/nF and

ℓ′−1 = {c(0), 1/nF}, with a = n, and ℓ′r = {E1r, iE1r}, r ≥ 2, with d′r = −1

3) K/H = U(n + 1)/U(n) = S2n+1
Same action and basis as in the previous case, but now F = diag(i, 0, ..., 0) and hence F* = 
ie1. Thus the result for L1 = {exp(θE12) | 0 ≤ θ ≤ 2π} is as before, except that ℓ′1 = {F, iE12}.

But now L2 = {exp(θF) acts as R(θ) in the e1, ie1 plane, and as Id on the rest. Hence

ℓ′−1 = {c(0), F}, with a = 1, and ℓ′0 = {E1k, iE1k, k ≥ 2}

3') K/H = U(n + 1)/U(n)k = S2n+1
In this case U(n+1) acts as v → (det A)^k Av for some integer k ≥ 1, and hence the stabilizer 
group of e1 is H = SU(n) · S_k with S_k = diag(z^21, z^2k+1, ..., z^2k). Thus we have p = p0 ⊕ p1 as 
in case 3), but now p0 = R · F with F = diag((k + 1)i, ki, ..., ki) and hence F* = (k + 1)ie1.

The case of L1 = {exp(θE12) is as in the previous 2 cases, except that ℓ′1 = {1/(k+1)F, iE12}.
Now L2 = {exp(θF) | 0 ≤ θ ≤ 2π} acts as R((k + 1)θ) in the e1, ie1 plane, and R(θ) in the 
e1, ie1 plane, r ≥ 2. Hence

ℓ′−1 = {c(0), 1/(k+1)F}, with a = k + 1, and ℓ′r = {E1r, iE1r}, r ≥ 2, with d′r = k.

3') K/H = U(1)/Z_k = S2n+1
We list here separately the common case of K = U(1) acting on C as w → z^kw with stabilizer 
group Z_k the k-th roots of unity. Here p = p0 spanned by F = i with F* = kie1. Thus

ℓ′−1 = {c(0), 1/kF} with a = k.

4) K/H = Sp(n + 1)/Sp(n) = S4n+3
K acts by matrix multiplication x → Ax on V = H^{n+1}, with orthonormal basis e1, ie1, ike1, ..., 
H is the stabilizer of e1 i.e., H = {diag(1, A) | A ∈ Sp(n)} acting on p = Im H ⊕ H^n as 
(s, x) → (s, Ax). We also have the basis of t given by Eij, iEij, jEij, kEij, where, by abuse of 
notation, the last three are skew hermitian, and F1 = diag(i, 0, ..., 0), F2 = diag(j, 0, ..., 0), F3 =
diag(k, 0, · · · , 0). As before, Q(A, B) = −\frac{1}{2} Re(\text{tr}(AB)) and \( p = p_0 \oplus p_1 \) with \( p_0 = \text{span}(F_1, F_2, F_3) \) and \( p_1 = \text{span}\{E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r = 2, \cdots n + 1\} \). For the action fields we have \( F^*_1 = ie_1, F^*_2 = je_1, F^*_3 = ke_1 \) and \( E^*_s = es, iE^*_s = ie_s, jE^*_s(c1) = je_s, kE^*_s = ke_s, s = 2, \cdots n + 1 \).

We need to consider four 1-parameter groups \( L_1 = \{\exp(\theta E_{12}) \mid 0 \leq \theta \leq 2\pi\} \), \( L_2 = \exp(\theta F_1) \), \( L_3 = \exp(\theta F_2) \) and \( L_4 = \exp(\theta F_3) \) with \( 0 \leq \theta \leq 2\pi \).

For \( L_1 \), acting on \( V \), we get:

\[ \ell'_{-1} = \{c(0), E_{12}\} \quad \text{with} \quad a = 1, \quad \ell'_1 = \{F_1, iE_{12}\}, \quad \ell'_2 = \{F_2, jE_{12}\}, \quad \ell'_3 = \{F_3, kE_{12}\} \quad \text{with} \quad d'_r = 1 \]

and \( \ell'_0 = \{E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 3\} \).

The one parameter group \( L_2 = \exp(\theta F_1) \) rotates the planes \( e_1, ie_1 \) and \( je_1, ke_1 \) by \( R(\theta) \) and fixes all remaining vectors. Thus

\[ \ell'_{-1} = \{c(0), F_1\}, \quad \text{with} \quad a = 1, \quad \ell'_1 = \{F_2, F_3\} \quad \text{with} \quad d'_r = 1 \]

and \( \ell'_0 = \{E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 2\} \), and similarly for \( L_3, L_4 \).

5) \( K/H = \text{Sp}(n + 1) \times \text{Sp}(1)/\text{Sp}(n) \cdot \Delta \text{Sp}(1) = S^{4n+3} \)

The slice is \( V = H = \mathbb{H}^{n+1} \) with basis \( e_1, ie_1, je_1, ke_1, \cdots \) and \( (A, q) \in K \) acting as \( v \to Avq^{-1} \). Here we are considering the effective action and thus \( K = \text{Sp}(n + 1) \times \text{Sp}(1)/\mathbb{Z}_2 \) with \( \mathbb{Z}_2 = (\text{Id}, -1) \). The stabilizer group of \( e_1 \) is \( H = \text{Sp}(n)\Delta \text{Sp}(1) = \{((\text{diag}(A, q)), q) \mid A \in \text{Sp}(n), q \in \text{Sp}(1)\} \approx \text{Sp}(n) \times \text{Sp}(1)/\mathbb{Z}_2 \) acting on \( p = \text{Im} H \oplus \mathbb{H}^n \) as \( (s, x) \to (qs^{-1}, Ax^{-1}) \). Again, \( p = p_0 \oplus p_1 \) with \( p_0 = \text{span}(F_1, F_2, F_3) \) and \( p_1 = \text{span}\{E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r = 2, \cdots n + 1\} \), but now \( F_1 = (\text{diag}(i, 0, \cdots , 0), -i) \), \( F_2 = (\text{diag}(j, 0, \cdots , 0), -j) \), \( F_3 = (\text{diag}(k, 0, \cdots , 0), -k) \) with \( F^*_1 = 2i e_1, F^*_2 = 2j e_1, F^*_3 = 2k e_1 \).

We need to consider only two 1-parameter groups \( L_1 = \{\exp(\theta E_{12}), 1\} \mid 0 \leq \theta \leq 2\pi \), \( L_2 = \{\exp(\theta F_1) \mid 0 \leq \theta \leq 2\pi \} \).

For \( L_1 = \exp(\theta E_{12}) \) we get:

\[ \ell'_{-1} = \{c(0), E_{12}\}, \quad \text{with} \quad a = 1, \quad \ell'_1 = \{\frac{1}{2} F_1, iE_{12}\}, \quad \ell'_2 = \{\frac{1}{2} F_2, jE_{12}\}, \quad \ell'_3 = \{\frac{1}{2} F_3, kE_{12}\} \quad \text{with} \quad d'_r = 1 \]

and \( \ell'_0 = \{E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 3\} \).

The one parameter group \( L_2 \) rotates the planes \( e_1, ie_1 \) by \( R(\theta) \) and fixes all remaining vectors, including \( F_2, F_3 \). Thus

\[ \ell'_{-1} = \{c(0), \frac{1}{2} F_1\}, \quad \text{with} \quad a = 2, \quad \ell'_0 = \{\frac{1}{2} F_2, \frac{1}{2} F_3, E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 2\} \]
and \( \ell'_0 = \{ E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 3 \} \).

For \( L_2 = \exp(\theta F_1) \) on the other hand, we have

\[
\ell'_{-1} = \{ \dot{c}(0), \frac{1}{2} F_1 \}, \quad \text{with } a = k + 1, \ell'_{1} = \{ F_2, F_3 \} \text{ with } d'_1 = 1 - k
\]

and \( \ell'_0 = \{ E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 2 \} \).

For \( L_3 = \exp(\theta F_2) \) we have:

\[
\ell'_{-1} = \{ \dot{c}(0), F_2 \}, \quad \text{with } a = 1, \ell'_{1} = \{ F_1, F_3 \} \text{ with } d'_1 = -1
\]

and \( \ell'_0 = \{ E_{1r}, iE_{1r}, jE_{1r}, kE_{1r}, r \geq 2 \} \).

7) \( \mathbf{K}/\mathbf{H} = \mathbf{G}_2/\mathbf{SU}(3) = \mathbb{S}^6 \)

We regard \( \mathbf{G}_2 \) as the automorphism group of the Cayley numbers with basis \( 1, i, j, k, \ell, i\ell, j\ell, k\ell. \) This embeds \( \mathbf{G}_2 \) naturally into \( \mathbf{SO}(7) \) and its action is transitive on \( \mathbb{S}^6 \). On the Lie algebra level, a skew symmetric matrix \( (a_{ij}) \in \mathfrak{so}(7) \) belongs to \( \mathfrak{g}_2 \) iff

\[
a_{23} + 4a_{45} + a_{76} = 0, \quad a_{12} + 4a_{47} + a_{65} = 0, \quad a_{13} + a_{64} + a_{75} = 0
\]

\[
a_{14} + a_{72} + a_{36} = 0, \quad a_{15} + a_{26} + a_{37} = 0, \quad a_{16} + a_{52} + a_{43} = 0, \quad a_{17} + a_{24} + a_{53} = 0.
\]

Thus a basis for the Lie algebra \( \mathfrak{g}_2 \subset \mathfrak{so}(7) \) is given by

\[
\begin{pmatrix}
0 & x_1 + x_2 & y_1 + y_2 & x_3 + x_4 & y_3 + y_4 & x_5 + x_6 & y_5 + y_6 \\
-(x_1 + x_2) & 0 & \alpha_1 & -y_5 & x_5 & -y_3 & x_3 \\
-(y_1 + y_2) & -\alpha_1 & 0 & x_6 & y_6 & -x_4 & -y_4 \\
-(x_3 + x_4) & y_5 & -x_6 & 0 & \alpha_2 & y_1 & -x_1 \\
-(x_3 + y_4) & -x_5 & -x_6 & -\alpha_2 & 0 & x_2 & y_2 \\
-(x_5 + x_6) & y_3 & x_4 & -y_1 & -x_2 & 0 & \alpha_1 + \alpha_2 \\
-(y_5 + y_6) & -x_3 & y_4 & x_1 & -y_2 & -(\alpha_1 + \alpha_2) & 0
\end{pmatrix}
\]

The stabilizer group at \( i \) is given by the complex linear automorphisms, which is equal to \( \mathbf{SU}(3) \). Thus its Lie algebra \( \mathfrak{h} \) is given by the constraints \( x_1 + x_{i+1} = y_1 + y_{i+1} = 0 \) for \( i = 1, 3, 5 \), and the complement \( \mathfrak{p} \) by

\[
\begin{pmatrix}
0 & 2x_1 & 2y_1 & 2x_3 & 2y_3 & 2x_5 & 2y_5 \\
-2x_1 & 0 & 0 & -y_5 & x_5 & -y_3 & x_3 \\
-2y_1 & 0 & 0 & x_5 & y_5 & -x_3 & -y_3 \\
-2x_3 & y_5 & -x_5 & 0 & 0 & y_1 & -x_1 \\
-2y_3 & -x_5 & -x_5 & 0 & 0 & x_1 & y_1 \\
-2x_5 & y_3 & x_3 & -y_1 & -x_1 & 0 & 0 \\
-2y_5 & -x_3 & y_3 & x_1 & -y_1 & 0 & 0
\end{pmatrix}
\]

Since the action of \( \text{Ad}_H \) on \( \mathfrak{p} \) is irreducible, it is sufficient to consider only one one-parameter group, and we choose \( F = 2E_{12} - E_{47} + E_{56} \in \mathfrak{p} \) with \( L = \{ \exp(\theta F) \mid 0 \leq \theta \leq 2\pi \} \). It acts as a rotation in the \( e_4, e_7 \) plane and \( e_5, e_6 \) plane at speed 1, and in the \( e_1, e_2 \) plane at speed 2, and as \( \text{Id} \) on \( e_3 \).

Thus

\[
\ell'_{-1} = \{ \dot{c}(0), F \} \text{ with } a = 2, \quad \ell'_{1} = \{ 2E_{14} + E_{27} - E_{36}, 2E_{17} + E_{35} - E_{24} \} \text{ with } d'_1 = 1
\]

\[
\ell'_{2} = \{ 2E_{15} - E_{26} - E_{37}, 2E_{16} + E_{25} + E_{34} \} \text{ with } d'_2 = -1, \text{ and } \ell'_{0} = \{ 2E_{13} + E_{57} + E_{46} \}
\]

8) \( \mathbf{K}/\mathbf{H} = \mathbf{Spin}(7)/\mathbf{G}_2 = \mathbb{S}^7 \)
The embedding $\text{Spin}(7) \subset \text{SO}(8)$, and hence the action of $K$ on the slice, is given by the spin representation. On the Lie algebra level we can describe this as follows. A basis of $\mathfrak{g}_2 \subset \mathfrak{so}(8)$ is given by the span of

$$E_{24} + E_{68}, \ E_{28} + E_{46}, \ E_{26} - E_{48}E_{23} + E_{67}, \ E_{27} + E_{36}, \ E_{34} + E_{78}, \ E_{38} + E_{47}, \ E_{37} - E_{48}$$

and the complement $\mathfrak{p}$ by the span of

$$E_{12} + E_{56}, \ E_{13} + E_{57}, \ E_{14} + E_{58}, \ E_{15} - E_{48}, \ E_{16} + E_{25}, \ E_{17} + E_{35}, \ E_{18} + E_{45}.$$  

Since the action of $\text{Ad}_H$ on $\mathfrak{p}$ is irreducible, we need to consider only one one-parameter group and we choose $L = \{ \exp(\theta F) \mid 0 \leq \theta \leq 2\pi \}$ with $F = E_{12} + E_{56}$. It acts as a rotation in the $e_1, e_2$ plane and $e_5, e_6$ plane at speed 1, and as Id on $e_3, e_4, e_7, e_8$.

Thus

$$\ell'_{-1} = \{\dot{c}(0), F\} \text{ with } a = 1, \ \ell'_1 = \{E_{15} - E_{48}, \ E_{16} + E_{25}\} \text{ with } d'_1 = 1$$

and $\ell'_0 = \{E_{13} + E_{57}, \ E_{14} + E_{58}, \ E_{17} + E_{35}, \ E_{18} + E_{45}\}$.

9) $K/H = \text{Spin}(9)/\text{Spin}(7) = S^{15}$

The embedding of $H$ in $K$ is given by the spin representation of $\text{Spin}(7)$ in $\text{Spin}(8)$ followed by the (lift of) the standard block embedding of $\text{Spin}(8)$ in $\text{Spin}(9)$. Let $S_{ij}$ be the standard basis of $\text{spin}(9)$ under the isomorphism $\text{spin}(9) \cong \text{spin}(9)$ and denote by $E_{ij}$ the standard basis of $\mathfrak{so}(16)$. Furthermore, $\text{Spin}(9)$ acts on the slice $V \simeq \mathbb{R}^{16}$ via the spin representation and one easily computes the image of $S_{ij}$ in $\mathfrak{so}(16)$. We only need the basis of $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The irreducible 7-dimensional module $\mathfrak{p}_1$ is spanned by

$$Z_2 : = \begin{array}{l} -S_{78} + S_{12} + S_{34} + S_{56} = 2E_{12} + E_{9,10} + E_{11,12} + E_{13,14} - E_{15,16} \\ Z_3 : = S_{68} + S_{13} - S_{24} + S_{57} = 2E_{13} + E_{9,11} - E_{10,12} + E_{13,15} + E_{14,16} \\ Z_4 : = S_{58} + S_{14} + S_{23} - S_{67} = 2E_{14} + E_{9,12} + E_{10,11} + E_{13,16} - E_{14,15} \\ Z_5 : = -S_{48} + S_{15} - S_{26} - S_{37} = 2E_{15} + E_{9,13} - E_{10,14} - E_{11,15} - E_{12,16} \\ Z_6 : = -S_{38} + S_{16} + S_{25} + S_{47} = 2E_{16} + E_{9,14} + E_{10,13} - E_{11,16} + E_{12,15} \\ Z_7 : = S_{28} + S_{17} + S_{35} - S_{46} = 2E_{17} + E_{9,15} + E_{10,16} + E_{11,13} - E_{12,14} \\ Z_8 : = S_{18} - S_{27} + S_{36} + S_{45} = 2E_{18} + E_{9,16} - E_{10,15} + E_{11,14} + E_{12,13} \end{array}$$

and the irreducible 8-dimensional module $\mathfrak{p}_2$ is spanned by $S_{i9}$

$$S_{19} = \frac{1}{7}(E_{1,9} + E_{2,10} + E_{3,11} + E_{4,12} + E_{5,13} + E_{6,14} + E_{7,15} + E_{8,16})$$

If $e_1, \ldots, e_{16}$ is a basis of the slice, then $Z^*_i = e_i, \ i = 2, \ldots, 8$ and $S^*_{i9} = e_{i+8}, \ i = 1, \ldots, 8$.

For the smoothness conditions we need to choose two one parameter groups. For $L_1 = \{\exp(\theta Z_2) \mid 0 \leq \theta \leq 2\pi\}$ we obtain

$$\ell'_{-1} = \{\dot{c}(0), Z_2\} \text{ with } a = 2, \ \ell'_i = \{S_{i9}, \ S_{i+1,9}\}, \ i = 1, 3, 5, 7 \text{ with } d'_i = 1 \text{ for } i = 1, 3, 5, \ d'_7 = -1$$

and $\ell'_0 = \{Z_3, \ldots, Z_8\}$.  

and for $L_2 = \{ \exp(\theta S_{19}) \mid 0 \leq \theta \leq 2\pi \}$

$$\ell'_{-1} = \{ \dot{c}(0), S_{19} \} \text{ with } a = 1, \ \ell'_i = \{ Z_i, S_{i,9} \}, \ i = 2, \cdots 8 \text{ with } d'_i = 1$$

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University of Firenze
E-mail address: verdiani@math.unifi.it

University of Pennsylvania
E-mail address: wziller@math.upenn.edu
| K                  | H                  | p_i    | dim p_i |
|--------------------|--------------------|--------|---------|
| 1                  | SO(n+1)            | p_1    | n       |
| 1'                 | Spin(n+1)          | p_1    | n       |
| 2                  | SU(n+1)            | p_0 + p_1 | 1, 2n  |
| 3                  | U(n+1)             | p_0 + p_1 | 1, 2n  |
| 3'                 | U(n+1) k           | p_0 + p_1 | 1, 2n  |
| 4                  | Sp(n+1)            | p_0 + p_1 | 3, 4n  |
| 5                  | Sp(n+1) ◯ Sp(1)    | p_1 + p_2 | 3, 4n  |
| 5'                 | Sp(n+1) × Sp(1)    | p_1 + p_2 | 3, 4n  |
| 6                  | Sp(n+1) ◯ U(1)     | p_0 + p_1 + p_2 | 1, 2, 4n |
| 6'                 | Sp(n+1) × U(1)     | p_0 + p_1 + p_2 | 1, 2, 4n |
| 7                  | G_2                | p_1    | 6       |
| 8                  | Spin(7)            | p_1    | 7       |
| 9                  | Spin(9)            | p_1 + p_2 | 8, 7   |

Table A. Almost effective transitive actions on spheres

| ⟨m, m⟩ | ℓ_0  | ℓ_i   | ℓ_j  |
|--------|------|-------|------|
| ℓ_0   | φ(t^2) | t^2\phi(t^2) | t^2\phi(t^2) |
| ℓ_i   | \frac{d_j}{t^n} \phi(t^2) | \frac{d_j}{t^n} \phi(t^2) | \frac{d_j}{t^n} \phi(t^2) |
| g_{11} + g_{22} = \phi_1(t^2) | h_{11} + h_{22} = t^{\frac{|d_j - d_i|}{a}} \phi_1(t^2), h_{11} - h_{22} = t^{\frac{|d_j + d_i|}{a}} \phi_1(t^2) |
| g_{11} - g_{22} = t^{\frac{2d_j}{a}} \phi_2(t^2) | h_{12} - h_{21} = t^{\frac{|d_j - d_i|}{a}} \phi_1(t^2), h_{12} + h_{22} = t^{\frac{|d_j + d_i|}{a}} \phi_1(t^2) |
| g_{12} = t^{\frac{2d_j}{a}} \phi_3(t^2) | |

Table B. Smoothness Conditions I for G invariant metrics or symmetric 2 × 2 tensors

| ⟨p, m⟩ | ℓ_0 | ℓ_j |
|--------|-----|-----|
| ℓ_{-1} | t^2\phi(t^2) | t^2\phi(t^2) |
| ℓ_0   | t^2\phi(t^2) | t^2\phi(t^2) |
| ℓ_i   | t^\frac{d_j}{a} \phi(t^2) | t^\frac{d_j}{a} \phi(t^2) |
| h_{11} + h_{22} = t^{\frac{d_j - d_i}{a}} \phi_1(t^2), h_{11} - h_{22} = t^{\frac{d_j + d_i}{a}} \phi_2(t^2) |
| h_{12} - h_{21} = t^{\frac{d_j - d_i}{a}} \phi_3(t^2), h_{12} + h_{22} = t^{\frac{d_j + d_i}{a}} \phi_4(t^2) |
| b = 2 if d'_i = d_j |

Table C. Smoothness Conditions II for G invariant metrics
| $\langle p, m \rangle$ | $\ell_0$ | $\ell_j$ |
|-----------------|--------|--------|
| $\ell_{-1}$    | $t^2 \phi(t^2)$ | $t^2 t^\frac{d_j}{a} \phi(t^2)$ |
| $\ell_0^j$     | $t \phi(t^2)$ | $t t^\frac{d_j}{a} \phi(t^2)$ |
| $\ell_i^j$     | $t t^\frac{d_j}{a} \phi(t^2)$ | $T_{11} + T_{22} = t^\frac{|d_i-d_j|}{a} \phi_1(t^2), \ T_{11} - T_{22} = t^\frac{|d_i+d_j|}{a} \phi_2(t^2)$ |
|                |        | $T_{12} - T_{21} = t^\frac{|d_i-d_j|}{a} \phi_3(t^2), \ T_{12} + T_{22} = t^\frac{|d_i+d_j|}{a} \phi_4(t^2)$ |

Table D. Smoothness Conditions II for a $G$ invariant symmetric $2 \times 2$ tensor $T$