Abstract

We investigate the Hamiltonian analysis of Nappi-Witten model (WZW action based on non semi simple gauge group) and find a time dependent non-commutativity by canonical quantization. Our procedure is based on constraint analysis of the model in two parts. A first class analysis is used for gauge fixing the original model following by a second class analysis in which the boundary condition are treated as Dirac constraints. We find the reduced phase space by imposing our second class constraints on the variables in an extended Fourier space.

Keywords: Noncommutativity, constraint analysis

1 Introduction

Treating boundary conditions as Dirac constrains has been considered in the recent years by so many authors [1, 2, 3, 4]. This approach has been used first in studying the Polyakov string coupled to a B-field. The common feature of all works is non commutativity of the coordinate fields on the boundaries which may lie on some brains, as first predicted by [5]. However, there are different approaches in defining the constraints and investigating their consistency in time. We have reviewed the whole subject in our previous work [6] and showed if we impose the set of constraints on the Fourier expansions of the fields, the redundant modes will be omitted in a natural way.

For simple physical models obeying linear equations of motion, the ordinary Fourier expansion gives appropriate coordinates to reach the reduced phase space. In other words, the infinite set of second class constraints emerging as the result of boundary conditions, forces us to omit a number of Fourier modes. However, ordinary Fourier transformation is not essential for quantization; it is just one tool that acts well for most physical models at hand. In the general case one should search for "appropriate coordinates", in which imposing the set of second class constraints is equivalent to omitting some canonical pairs from the theory.

In this paper we study the constraint structure of the Nappi-Witten model in the Hamiltonian formalism. This model acquires complicated boundary conditions so that the ordinary Fourier expansion seems inadequate to impose the whole set of constraints which
emerge from the boundary conditions. Nevertheless, the Nappi-Witten model, on its own
grands, is an attractive one since it describes a non semi simple gauge group as well as giving
time dependent non commutativity in some gauges \[7\]. Our next interest is to emphasize
that solving the equations of motion is not necessarily needed for quantizing a theory; the
only necessity is finding the dynamics of the constraints and construct their algebra with
the Hamiltonian such that they remain consistent with time on the constraint surface.

We give a precise Hamiltonian treatment of the model in which the constraint structure
is followed step by step from the initial action to the final reduced phase space. In section 2
we introduce the model and find primary and secondary constraints of the system. Section
3 is devoted to fixing the gauge by introducing appropriate gauge fixing conditions. In
section 4 we follow our strategy of treating the boundary conditions as primary Dirac
constraints and follow their consistencies. The boundary conditions which come from the
original action, in fact, make the system more complicated. So, it is not possible to write
down the solutions in a closed form similar to a simple Fourier expansion (see reference
\[8\]). We try to find a basis which is appropriate for imposing the infinite set of constraints
in section 5. In section 6 we will give our concluding remarks and will compare our results
with parallel approaches.

2 Hamiltonian structure of the model

The Nappi-Witten model describes a 4-component bosonic string \(X_a = (a_1, a_2, u, v)\) living
in the background metric \(G_{ab}(X)\) and coupled to a \(B\)-field. The action is given as:

\[
S = \int d^2\sigma \left[ \sqrt{-g} g^{ij} G_{ab}(X) \partial_i X^a \partial_j X^b + B_{ab} \epsilon^{ij} \partial_i X^a \partial_j X^b \right],
\]

where

\[
G(X) = \begin{pmatrix}
  1 & 0 & \frac{a_2}{2} & 0 \\
  0 & 1 & -\frac{a_1}{2} & b \\
  \frac{a_2}{2} & -\frac{a_1}{2} & 1 & 0 \\
  0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
B(X) = \begin{pmatrix}
  0 & u & 0 & 0 \\
  -u & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
\]

The special form of \(G(X)\) and \(B(X)\) are chosen so that the gauge group of the model is
non semi-simple \[8\]. The metric field can be written in terms of the following variables:

\[
N_1 = \frac{1}{g^{00} \sqrt{-g}}, \quad N_2 = -\frac{g^{01}}{g^{00}}, \quad N_3 = \sqrt{-g} = \frac{1}{\sqrt{(g^{01})^2 - g^{00} g^{11}}}. (3)
\]

In terms of the variables \(X_a\) and \(N_\alpha\) the action becomes:

\[
S = \int d^2\sigma \left[ \frac{1}{N_1} G_{ab}(X) (X^a \dot{X}^b - 2N_2 X^a X^b \dot{u} + (N_2^2 - N_1^2) X^a X^b) + 2B_{ab} \dot{X}^a X^b \right], (4)
\]

where dot and prime means temporal and spatial derivatives, respectively. The canonical
momenta \(\pi^\alpha\) and \(p_a\) conjugate to \(N_\alpha\) and \(X^a\) are:

\[
\pi^\alpha = 0, \quad \alpha = 1, 2, 3
\]

\[
p_i = \frac{1}{N_1} (2\dot{a}_i + \dot{u} \epsilon_{ij} a_j) - \frac{N_2}{N_1} (2a'_i + u' \epsilon_{ij} a_j) + 2u \epsilon_{ij} a'_j,
\]

\[
p_u = \frac{1}{N_1} (2\dot{u} + 2\dot{v} + \epsilon_{ij} \dot{a}_i a_j) - \frac{N_2}{N_1} (2\dot{b} + 2\dot{v} + \epsilon_{ij} \dot{a}_i a_j),
\]

\[
p_v = \frac{2\dot{v}}{N_1} - \frac{N_2}{N_1} 2\dot{u}'.
\]
The Canonical Hamiltonian reads:

$$H = \int d^2\sigma \frac{1}{N_1} G_{ab}(F^a F^b - (N_2^2 - N_1^2)X^{ia} X^{ib}),$$

(6)

where

$$F^a = \dot{X}^a = N_1(G^{-1})^{ab}(p_b - B_{bc}X^c) + N_2B_{ab}X^b,$$

(7)

In terms of component fields $a_i$, $u$ and $v$ we have

$$H = \int d^2\sigma(N_1\Psi^1 + N_2\Psi^2),$$

(8)

where

$$\Psi^1 = \frac{1}{4}p_i^2 + \frac{1}{4}\epsilon_{ij}pvp_j + \frac{1}{2}pap - \frac{1}{4}bp_i^2 + \frac{1}{16}a_i^2p_v^2 + u\epsilon_{ij}a_i^2a_j + \epsilon_{ij}au^2p_j + \frac{1}{2}vpv_i^2a_i + (1 + u^2)a_i^2 + bu^2 + 2u''v' \Psi^2 = a_i^2p_i + u'p_u + v'p_v,$$

(9)

As can be seen from Eqs. (8) the momenta $\pi^\alpha$ are primary constraints. The dynamics of the system is achieved by the total Hamiltonian:

$$H_T = H + \int d\sigma\lambda_\alpha\pi^\alpha(\sigma, \tau),$$

(10)

in which $\lambda_\alpha$ are Lagrange multipliers. As usual we should impose the consistency conditions on the constraints so that they remain valid during the time. For this reason we demand $\dot{\pi}^\alpha \approx 0$, where $\approx$ means weak equality i.e. equality on the constraint surface. Using Eqs. (10) and (6) we have:

$$\dot{\pi}^1 = \{\pi^1, H_T\} = -\Psi^1$$

$$\dot{\pi}^2 = \{\pi^2, H_T\} = -\Psi^2$$

$$\dot{\pi}^3 = \{\pi^3, H_T\} = 0,$$

(11)

Therefore, the consistency of three primary constraints $\pi^\alpha$ gives two second level constraints $\Psi^1$ and $\Psi^2$. In this way we have so far two levels of constraints as

$$\begin{pmatrix}
\pi^1 \\
\Psi^1 \\
\Psi^2 \\
\pi^3
\end{pmatrix} \approx \begin{pmatrix}
\pi^1 \\
\Psi^1 \\
\Psi^2 \\
\pi^3
\end{pmatrix}.$$

(12)

In order to investigate the consistency of second level constraints, we need to calculate the Poisson brackets of $\Psi^1(\sigma, \tau)$ and $\Psi^2(\sigma, \tau)$ at different points. Direct calculation, using the fundamental Poisson brackets among the four conjugate pairs $(u, p_u)$, $(v, p_v)$ and $(a_i, p_i)$ gives:

$$\{\Psi^1(\sigma, \tau), \Psi^1(\sigma', \tau)\} = \frac{1}{2}(\Psi^2(\sigma, \tau)\partial_\sigma - \Psi^2(\sigma', \tau)\partial_\sigma')\delta(\sigma - \sigma')$$

$$\{\Psi^1(\sigma, \tau), \Psi^2(\sigma', \tau)\} = \Psi^2(\sigma, \tau)\partial_\sigma\delta(\sigma - \sigma')$$

$$\{\Psi^2(\sigma, \tau), \Psi^2(\sigma', \tau)\} = \frac{1}{2}(\Psi^2(\sigma, \tau)\partial_\sigma - \Psi^2(\sigma', \tau)\partial_\sigma')\delta(\sigma - \sigma'),$$

(13)

where $\delta'(\sigma - \sigma') \equiv \frac{\partial}{\partial \sigma}\delta(\sigma - \sigma')$. It should be noted that each of the above Poisson brackets leads to a set of terms at different points $\sigma$ and $\sigma'$ multiplied by $\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma')$ or $\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma')$ which equals to $-\frac{\partial}{\partial \sigma}\delta(\sigma - \sigma')$. However, since these terms have only non vanishing value when $\sigma'$ approaches to $\sigma$, one can consider all of them at the same point. Then they add up to give the above results. The algebra (13) shows that $\Psi^1(\sigma, \tau)$ and $\Psi^2(\sigma, \tau)$ are first class constraints. Moreover, from (8) we see that:

$$\{\Psi^1, H\} = N_2\Psi^1 + N_1\Psi^2 \approx 0$$

$$\{\Psi^2, H\} = N_2\Psi^1 + N_1\Psi^2 \approx 0$$

(14)
This shows that the consistency of $\Psi^1(\sigma, \tau)$ and $\Psi^2(\sigma, \tau)$ does not give any new constraint, and we are left with the five first class constraints given in (12).

In this way we have derived three constraint chains $\left( \frac{\pi^1}{\Psi^1} \right)$, $\left( \frac{\pi^2}{\Psi^2} \right)$ and $\left( \pi^3 \right)$ in the terminology of reference [9]. In fact, the chain relation $\{\pi^\alpha, H\} = \Psi^\alpha$ holds for all of the chains. However the first two chains are correlated, since the Poisson bracket of the last element of each chain with the Hamiltonian contains the other constraint. This means that it is not possible to construct closed algebra within each chain. The last chain contains just one element and is not correlated to other chains, since it commutes with all of them as well as with Hamiltonian.

As in ordinary Polyakov string one can show that $\pi^3$ generates the Weyl symmetry of the model which affects only the components of the world-sheet metric. In terms of the variables $N_{\alpha}$ we have $N_3 \rightarrow N_3 + \epsilon$ under Weyl transformation. On the other hand the constraint chains $\left( \frac{\pi^1}{\Psi^1} \right)$, $\left( \frac{\pi^2}{\Psi^2} \right)$ can be shown that generate the effect of reparametrization invariance on the metric variables $N_1$ and $N_2$ as well as the variables $X_a$.

3 Gauge fixing

We began the theory with 14 field variables in the phase space, i.e. $X^a$, $N_{\alpha}$ and their corresponding momentum fields $p_a$ and $\pi^\alpha$. Then we derived 5 first class constraints given in (12). As is well known from Dirac theory the first class constraints are generators of gauge transformations [10]. One needs to consider additional conditions to fix the gauges. These ”gauge fixing conditions” are functions of phase space variables which should vanish to fix the gauges. The gauge fixing conditions should fulfill two conditions. First, they should constitute a system of second class constraints when added to the original first class constraints of the system. This condition is necessary to fix the values of variables which vary under the action of gauge generators [12]. Second, they should have a closed algebra under the consistency conditions, i.e. under the successive Poisson brackets with the Hamiltonian.

For a ”complete gauge fixing” the number of independent gauge fixing conditions should be equal to the number of first class constraints [13]. In this way, we should suggest 5 gauge fixing conditions to fix the gauges generated by the constraints given in (12), and reach a ”reduced phase space” of 4 field variables. Since the momenta $\pi^\alpha$ are generators of transformations in $N_{\alpha}$, we fix the corresponding gauge by choosing the values of $N_{\alpha}$ as $N_1 \approx 1$, $N_2 \approx 0$ and $N_3 \approx 1$. These values are chosen such that $g_{ij} = \eta_{ij}$. In this way we have so far introduced three gauge fixing conditions

$$\begin{align*}
\Omega_1 &\equiv N_1 - 1, \\
\Omega_2 &\equiv N_2, \\
\Omega_3 &\equiv N_3 - 1.
\end{align*}$$

It can easily seen that the system of 6 constraints $\pi^\alpha$ and $\Omega_\alpha$ are second class. The consistency of $\Omega_\alpha$’s by the use of total Hamiltonian (10) determines the lagrange multipliers $\lambda_\alpha$ to be zero and does not give any new constraint. This makes us sure that the two criterions of a good gauge mentioned above are satisfied. In fact, by the above gauge fixing three degrees of freedom $N_{\alpha}$ are removed completely from the theory. This gauge has fixed the
Weyl symmetry as well as the effect of the reparametrization on the metric variables \(N_1\) and \(N_2\). On the other hand, we are still left with the remaining gauges generated by \(\Psi^1\) and \(\Psi^2\) which generate the effect of reparametrization on the variables \(X_a\). In fact, since we have fixed the gauge from the middle of the constraint chains, the gauge is fixed partially in the language of reference [13]. In partial gauge fixing the Lagrange multipliers are determined while the variations generated by some of the gauge generators are not fixed.

To fix the effect of the parametrization of the world-sheet on \(X_a\)'s, as in so many models in string theory we need to determine some definite combinations of fields as the time variable in target space. Taking a look on the form of the constraints \(\Psi^1\) and \(\Psi^2\) in (9) shows that the choice \(u = \mu \tau\) is more economical in the sense that simplifies the constraints better. Here \(\mu\) is a parameter with dimension of \((\text{length})^{-1}\). We recall that all of the dynamical variables in the action are dimensionless. Hence, we consider the gauge fixing condition

\[
\Omega_4 = u - \mu \tau. \tag{16}
\]

To fulfill the second criterion of a good gauge we choose the last gauge fixing condition as

\[
\Omega_5 \equiv \dot{\Omega}_4 = \{\Omega_1, H_T\} + \frac{\partial \Omega_1}{\partial \tau} \approx p_v - 2\mu \tag{17}
\]

This new constraint should also be valid during the time. Since

\[
\dot{\Omega}_5 = 2\mu(-\frac{N_2}{N_1} + N_2') \approx 0, \tag{18}
\]

the chosen gauges are consistent and make a closed algebra with the Hamiltonian. It is also clear that \(\Omega_4\) and \(\Omega_5\) make a second class system with \(\Psi^1\) and \(\Psi^2\). Imposing strongly the constraints (16) and (17) on the system, simplifies the constraints \(\Psi^1\) and \(\Psi^2\) as

\[
\Psi^1 \rightarrow \bar{\Psi}^1 = \frac{1}{4}p_i^2 + \frac{1}{2}\epsilon_{ij}\mu a_ip_j + \epsilon_{ij}\mu \tau a'_ip_j + (1 + \mu^2\tau^2)a_i^2 + \mu p_a - b\mu^2 + \frac{1}{2}\mu^2 a_i^2 + \mu^2 \tau a_i a_i',
\]

\[
\Psi^2 \rightarrow \bar{\Psi}^2 = a'_ip_i + 2\mu v', \tag{19}
\]

This shows that \(p_u\) and \(v\) can be derived on the constraint surface, i.e. from identities \(\bar{\Psi}^1 = 0\) and \(\bar{\Psi}^2 = 0\), in terms of the physical variables \(a_i\) and \(p_i\). In this way the reduced phase space is just the four dimensional space of \((a_i, p_i)\) whose original Poisson brackets serve as the Dirac brackets in the remaining physical space. The terms \(\mu p_a\) and \(\mu^2 b\) in the expressions of \(\bar{\Psi}^1\) have nothing to do with the dynamics of \((a_i, p_i)\) and can be dropped. The parameter \(b\) has in fact no important role in the theory and only shifts the spectrum of the energy with a constant value.

As in reference [8] we consider the dimensionless quantity \(\mu l\) as a small parameter which should be considered only in the first order. Therefore, in all of the foregoing calculations we keep only linear terms with respect to \(\mu\), assuming that \(l\) is finite. Therefore, the Hamiltonian (8) in the reduced phase space can be written in terms of the Hamiltonian density:

\[
H_{GF} = \frac{1}{4}p_i^2 + \frac{1}{2}\epsilon_{ij}\mu a_ip_j + \epsilon_{ij}\mu \tau a'_ip_j + a_i^2. \tag{20}
\]

Since \(B(X)\) in (2) is linear with respect to \(u\) one may think of \(\mu\) as the order of magnitude of the \(B\)-field. This assumption is equivalent to considering the effect of the \(B\)-field only up to the first order.
4 Boundary conditions as constraints

From now on we forget about the original theory and suppose we are given a theory with two degrees of freedom $a_i$ and the corresponding momenta $p_i$ whose dynamics is given by the final Hamiltonian (20). We make a change of variables from $(a_i, p_i)$ to $(A_i = \epsilon_{ij}a_j, P_i = p_i)$. Then the the fundamental Poisson brackets which is the same as the final Dirac bracket of the original theory read

\[
\{A_i(\sigma, \tau), P_j(\sigma', \tau)\} = \epsilon_{ij}\delta(\sigma - \sigma'),
\]

\[
\{A_i(\sigma, \tau), A_j(\sigma', \tau)\} = \{P_i(\sigma, \tau), P_j(\sigma', \tau)\} = 0
\]

(21)

The Hamiltonian equation of motion for the remaining fields, can be written as

\[
\dot{A}_i = \frac{1}{2}\epsilon_{ij}(P_j - 2\mu\tau A_j' - \mu A_j)
\]

\[
\dot{P}_i = -\epsilon_{ij}(\frac{1}{2}\mu P_j - \mu\tau P_j'' + 2A_j'')
\]

(22)

The only things that should be brought from the original theory are the boundary conditions. Using the original action (1) the boundary condition after gauge fixing emerge in terms of phase space variables as:

\[
\Phi_i^{(1)} = \mu\tau P_i - 2A_i' = 0 \quad \text{at} \quad \sigma = 0, l
\]

(23)

We have shown in the appendix that the boundary condition (23) can also be derived from the parallel approach as the equations of motion of the end points in the discretized version.

As mentioned in the introduction we do not want to find the general solution of the dynamical equations of motion. On the other hand, we are interested to follow the dynamics of the boundary conditions which means investigating the consistency of primary constraints $\Phi_i^{(1)}(\sigma)|_{\sigma=0}$ and $\Phi_i^{(1)}(\sigma)|_{\sigma=l}$. Using the gauge fixed Hamiltonian of the previous section (20) the total Hamiltonian at this stage is

\[
\bar{H}_T = \int_0^l d\sigma \left[ \frac{1}{4}P_i P_i - \frac{1}{2}\mu A_i P_i - \mu\tau A_i' P_i + A_i' A_i' + \Lambda_i^i \Phi_i^{(1)}(\sigma)|_{\sigma=0} + \Lambda_i^2 \Phi_i^{(1)}(\sigma)|_{\sigma=l} \right].
\]

(24)

The consistency of primary constraints for instance at $\sigma = 0$ gives

\[
0 = \left[ \mu P_i - \epsilon_{ij} P_j' + \mu\epsilon_{ij} A_j' \right]_{\sigma=0} + \Lambda_i^i \left\{ \Phi_i^{(1)}|_{\sigma=0}, \Phi_j^{(1)}|_{\sigma=0} \right\}
\]

(25)

Similar equations should be written at the end-point $\sigma = l$. As discussed in details in [14] the first term in the LHS of Eq. (25) has not the same order as the coefficient of $\Lambda_i^1$ (and $\Lambda_i^2$) in the second term when regularizing the Dirac delta function. Therefore this condition can be fulfilled identically only if $\Lambda_i^1, 2$ as well as the first term vanish simultaneously. In this way we have used the consistency conditions of the constraints for simultaneously determining the undetermined Lagrange multiplier and finding the next level of constraints as $\Phi_i^{(2)}(0)$ and $\Phi_i^{(2)}(l)$ where

\[
\Phi_i^{(2)}(\sigma) = P_i - \epsilon_{ij} P_j' + \mu\epsilon_{ij} A_j'.
\]

(26)

Then we should consider the consistency of second level constraints by using the Hamiltonian

\[
\bar{H} = \int_0^l d\sigma \left[ \frac{1}{4}P_i P_i - \frac{1}{2}\mu A_i P_i - \mu\tau A_i' P_i + A_i' A_i' \right]
\]

(27)
which is the same as the total Hamiltonian (24) after imposing $A^i_{1,2} = 0$. This gives the third level of constraints. Subsequent levels of constraints can be derived in the same way. Using the relations:

$$\{ A^{(n)}_i, H \} = \frac{1}{2} \epsilon_{ij} (P^{(n)}_j - \mu A^{(n)}_j - 2 \mu \tau A^{(n+1)}_j) + O(\mu^2)$$

$$\{ P^{(n)}_i, H \} = -\epsilon_{ij} \left( \frac{1}{2} \mu P^{(n)}_j - \mu \tau P^{(n+1)}_j + 2 A^{(n+2)}_j \right) + O(\mu^2),$$

(28)

where $A^{(n)}_i = \partial_\sigma A_i$ and $P^{(n)}_i = \partial_\sigma P_i$ one can inductively show that the full set of constraints are $\Phi^{(N)}_i(0) \approx 0$ and $\Phi^{(N)}_i(l) \approx 0$ where

$$\Phi^{(2n+1)}_i = -n \mu P^{(2n-1)}_i + \mu \tau P^{(2n)}_i - 2 n \mu \epsilon_{ij} A^{(2n)}_j - 2 A^{(2n+1)}_i + O(\mu^2),$$

$$\Phi^{(2n+2)}_i = (n + 1) \mu P^{(2n)}_i - \epsilon_{ij} P^{(2n+1)}_j + (2n + 1) \mu \epsilon_{ij} A^{(2n+1)}_j + O(\mu^2) \quad n = 0, 1, 2, \ldots$$

(29)

For practical calculations we write the constraints as ordinary functions in the bulk of the string and then integrate them with the use of $\delta(\sigma)$ and $\delta(\sigma - l)$ respectively.

Now we want to investigate whether the constraints are first or second class. For this reason one should calculate the Poisson brackets of the constraints. Since the constraints contain different orders of derivatives of $A_i(\sigma, \tau)$ and $P_i(\sigma, \tau)$, the Poisson brackets $C^{i,k'}_{ij} \equiv \{ \Phi^k_i, \Phi^{k'}_j \}$ contain derivatives of orders $k + k'$, $k + k' - 1$, etc, of the Dirac delta function, which are highly divergent and independent of each other. One way of treating the matrix of Poisson brackets is regularizing the delta functions as gaussian functions of width $\varepsilon$ and let $\varepsilon \to 0$ after all. A tedious calculation gives

$$C^{2m+1, 2n+1}_{ij} = \frac{-2 \mu \epsilon_{ij}}{\sqrt{\pi}} \varepsilon^{-2(m+n+1)} (\varepsilon(m + n) H_{2m+2n}(0) - 2 \tau H_{2m+2n+1}(0)) + O(\mu^2)$$

$$C^{2m+2, 2n+1}_{ij} = \frac{-2 \mu \epsilon_{ij}}{\sqrt{\pi}} \varepsilon^{-2(m+n+1)-1} (\mu \epsilon_{ij} H_{2m+2n+2}(0) + \delta_{ij} H_{2m+2n+2}(0)) + O(\mu^2),$$

$$C^{2m+2, 2n+2}_{ij} = \frac{2 \mu \epsilon_{ij}}{\sqrt{\pi}} \varepsilon^{-2(m+n+1)-1} H_{2m+2n+2}(0) + O(\mu^2)$$

(30)

where $H_n(x)$ are Hermite polynomials. Similar expressions should be considered with $H_n(1)$ at the end-point $\sigma = l$. The non vanishing elements on each row are located such that no vanishing linear combination of rows may be found. This means that the infinite dimensional matrix $C^{i,k'}_{ij}$ is not singular and can in principle be inverted. Therefore, all of the constraints are second class. However, it is not practically possible to find the inverse of $C^{i,k'}_{ij}$. The problem is how we can find the Dirac brackets of the fields which need to have $C^{-1}$.

5 Reduced phase space

As stated before, we seek for appropriate coordinates in which imposing the constraints (24) lead to omitting a set of canonical pairs. Here we have a difficult problem in which the ordinary Fourier expansion does not do this job. However, in the limit $\mu \to 0$ the boundary condition (23) is the ordinary Neumann one and the Hamiltonian (27) has a simple quadratic form in terms of coordinates and momenta. Hence, we need to write extended Fourier transformations for the fields $A_i$ and $P_i$ that include at most linear corrections with respect to the parameter $\mu$ and go to the ordinary Fourier transformation in the limit $\mu \to 0$. Since $\mu \tau$ and $\mu \sigma$ are the only dimensionless quantities that can be used for this correction, what can we do is correcting the Fourier coefficients by correction terms linear in $\tau$ or $\sigma$. The
linear term in \( \tau \), however, is not needed at this stage, since it can be considered as part of
the solution of the equations of motion. Adding all these points up together we suggest the
following extended Fourier transformations for the fields

\[
A_i(\sigma, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[ (A_i(k, \tau) + \mu \sigma \alpha_i(k, \tau)) \cos k \sigma + (B_i(k, \tau) + \mu \sigma \beta_i(k, \tau)) \sin k \sigma \right],
\]

\[
P_i(\sigma, \tau) = \frac{-\epsilon_{ij}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[ (C_j(k, \tau) + \mu \sigma \gamma_j(k, \tau)) \cos k \sigma + (D_j(k, \tau) + \mu \sigma \delta_j(k, \tau)) \sin k \sigma \right].
\]

In ordinary Fourier expansions the coefficients \( A_i(k, \tau) \), \( B_i(k, \tau) \), \( C_i(k, \tau) \) and \( D_i(k, \tau) \) contain
the same amount of data as the original fields \( A_i(\sigma, \tau) \) and \( P_i(\sigma, \tau) \). Comparing the
expansions (31) and (32) with ordinary Fourier expansions shows that we have used a dupli-
cated basis including sin’s, cos’s, \( \sigma \) times sin’s and \( \sigma \) times cos’s for expanding our fields.
This basis is complete but its elements are not independent. Mathematically it is allowed
to use a basis which is "larger than necessary". However, the essential point is that one
should assume appropriate Poisson brackets among the extended Fourier modes such that
the desired fundamental Poisson brackets (21) remain valid. In other words, we should
tune their brackets in such a way that our physical phase space variables, which are half of
the extended phase space variables, do obey the right Poisson brackets. Direct calculation
shows that the following Poisson brackets lead to the standard Poisson algebra (21) for the
physical fields,

\[
\{ A_i(k, \tau), C_j(k', \tau) \} = \{ B_i(k, \tau), D_j(k', \tau) \} = \delta_{ij} \delta(k-k'), \\
\{ \alpha_i(k, \tau), D_j(k', \tau) \} = \{ \gamma_i(k, \tau), B_j(k', \tau) \} = \delta_{ij} \delta(k-k').
\]

All other Poisson brackets are assumed to vanish. Specially the modes \( \beta_i \) and \( \delta_i \) have
vanishing Poisson brackets with all other variables in the extended Fourier space and so
decouple from the theory. This means that we can put them away and write down the expansions
only with linear terms in the cosine modes. We will see on the other hand
that omitting the modes \( \beta_i \) and \( \delta_i \) does not disturb our analysis of imposing the boundary
conditions. We have, up to this point, 6 sets of real variables in the extended Fourier space
which depend on real, continues and positive variable \( k \).

Now we want to impose the full set of constraints (29) on the fields. Using the expansions
(31) and (32) the constraints at the end-point \( \sigma = 0 \) lead to

\[
\int_{-\infty}^{\infty} dk k^{2n} \left[ \mu \tau \epsilon_{ij} C_j + 2n \epsilon_{ij} A_j + (4n + 2) \alpha_i + 2k \tilde{B}_i \right] + O(\mu^2) = 0
\]

\[
\int_{-\infty}^{\infty} dk k^{2n-1} \left[ (n + 1) \epsilon_{ij} C_j + (2n + 1) \gamma_i + k \tilde{D}_i \right] + O(\mu^2) = 0
\]

where \( B_i = \mu \tilde{B}_i \) and \( D_i = \mu \tilde{D}_i \). Since these conditions should be satisfied for arbitrary
values of \( n \) we have

\[
\mu \tau \epsilon_{ij} C_j + 2n \epsilon_{ij} A_j + (4n + 2) \alpha_i + 2k \tilde{B}_i = 0,
\]

\[
(n + 1) \epsilon_{ij} C_j + (2n + 1) \mu \gamma_i + k \tilde{D}_i = 0.
\]

The difficulty arises here since the integer \( n \), which shows the level of constraints, has
appeared in the form of relations among the Fourier modes. This means that it is not
possible to satisfy the constraints of all levels just by considering simple linear relations
among the Fourier modes of a given $k$ as can be done in ordinary Dirichlet, Neumann, or even mixed boundary conditions [3]. In fact, this phenomenon is the reason which makes the ordinary Fourier expansion inadequate for realizing the constraints. However, we have the opportunity of existence of extra variables in the extended phase space, which provides us additional tools for satisfying the constraints. In this way we are allowed to assume that the coefficients of $n$ besides the terms independent of $n$ in (35) vanish. This gives

$$\alpha_i = -\frac{1}{2} \epsilon_{ij} A_j + \mathcal{O}(\mu^2) \quad \tilde{B}_i = \frac{1}{2k} \epsilon_{ij} (A_j - \tau C_j) + \mathcal{O}(\mu^2)$$
$$\gamma_i = -\frac{1}{2} \epsilon_{ij} C_j + \mathcal{O}(\mu^2) \quad \tilde{D}_i = -\frac{1}{2k} \epsilon_{ij} C_j + \mathcal{O}(\mu^2) \quad (36)$$

Hence the main fields $A_i(\sigma, \tau)$ and $P_i(\sigma, \tau)$ can be written in terms of two remaining sets of Fourier modes $A_i(k, \tau)$ and $C_i(k, \tau)$ as

$$A_i(\sigma, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[ (\delta_{ij} - \frac{1}{2} \mu \sigma \epsilon_{ij}) A_j \cos k\sigma + \frac{\mu}{2k} \epsilon_{ij} (A_j - \tau C_j) \sin k\sigma \right], \quad (37)$$
$$P_i(\sigma, \tau) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[ (\epsilon_{ij} + \frac{1}{2} \mu \sigma \delta_{ij}) C_j \cos k\sigma + \frac{\mu}{2k} C_j \sin k\sigma \right]. \quad (38)$$

As expected, the zeroth order (with respect to $\mu$) of the Eqs. (37) and (38) is the expansion of a simple bosonic string with Neumann boundary condition at the end point $\sigma = 0$. The linear term with respect to $\sigma$ in cosine modes as well as the sin term itself are appeared as the first order corrections.

Next we should impose the constraints (29) at the end-point $\sigma = l$ on the fields derived recently in Eqs. (37) and (38). Hence we find

$$\int_{-\infty}^{\infty} dk k^{2n-1} (-1)^n [n \mu \epsilon_{ij} C_j + 2k^2 (A_i - \frac{1}{2} \mu \sigma A_j)] \sin kl + \mathcal{O}(\mu^2) = 0,$$
$$\int_{-\infty}^{\infty} dk k^{2n+1} (-1)^n [(\delta_{ij} - \frac{1}{2} \mu \sigma \epsilon_{ij}) C_j - (2n + 1) \mu \epsilon_{ij} A_j] \sin kl + \mathcal{O}(\mu^2) = 0. \quad (39)$$

The above constraints are satisfied identically for $kl = m\pi$. However, for $k \neq \frac{m\pi}{l}$ there is no way for satisfying the constraints for arbitrary $n$ except assuming that

$$A_i(k, \tau) = C_i(k, \tau) = 0 \quad \text{for} \quad k \neq \frac{m\pi}{l} \quad (40)$$

This leads to descrizing the Fourier modes.

Before writing the final form of the fields in terms of the set of enumerable Fourier modes, care is needed to write the zero modes. The contributions due to cosine modes come out automatically by letting $k = 0$. However, contributions to zero modes originating from sine terms should be derived by taking the following limits:

$$\lim_{k \to 0} \tilde{B}_i \sin k\sigma = \frac{1}{2} \sigma \epsilon_{ij} (A_j(0, \tau) - \tau C_j(0, \tau)), \quad \lim_{k \to 0} \tilde{D}_i \sin k\sigma = -\frac{1}{2} \sigma \epsilon_{ij} C_j(0, \tau), \quad (41)$$

which follow from Eqs. (36). Adding these two contributions the zero mode part of the fields are so far as follows

$$A_i^0(\sigma, \tau) = A_i^0(\tau) - \frac{1}{2} \mu \sigma \tau \epsilon_{ij} C_j^0(\tau) \quad (42)$$

$$P_i^0(\sigma, \tau) = -\epsilon_{ij} + \mu \sigma \delta_{ij} C_j^0(\tau)$$

At this point we want to notice the reader to a global symmetry of the gauged fixed Lagrangian. If we turn off the B-field we would have an ordinary bosonic string in which
only the derivatives of the A-fields are present in the Lagrangian. This allows one to shift the fields by a constant amount without any change in the Lagrangian. When the B-field is on, Eq. (20) shows that the A-field itself is present in the gauged fixed Hamiltonian. However, the relevant term, i.e. the second term in Eq. (20), is proportional to $\mu$. This shows that the theory is symmetric, up to second order terms with respect to $\mu$, under the following transformation

$$A_i(\sigma, \tau) \rightarrow A_i(\sigma, \tau) + \mu f(\tau)$$

(43)

where $f(\tau)$ is an arbitrary function of time. This symmetry leads to an ambiguity in the zero mode of the A-field. Hence we should correct the first row of Eq. (42) in the most general case as follows

$$A_i^0(\sigma, \tau) = A_i^0(\tau) - \frac{1}{2} \mu \sigma \tau \epsilon_{ij} C_j^0(\tau) + \mu l [a_{ij} A_j^0(\tau) + b_{ij} C_j^0(\tau)]$$

(44)

Note that $\mu l$ is the only relevant dimensionless quantity which is first order in $\mu$. The unknown coefficients $a_{ij}$ and $b_{ij}$ should be determined upon suitable assumptions about the algebra of the fields. The best assumption seems to be keeping the standard algebra in the bulk of the string and letting all changes in the algebra of the fields lay on the boundaries. If we make this choice the final form of the physical fields in terms of the set of discrete Fourier modes $A_i^m(\tau) \equiv A_i(\frac{m\pi}{l}, \tau)$ and $C_i^m(\tau) \equiv C_i(\frac{m\pi}{l}, \tau)$ are as follows

$$A_i(\sigma, \tau) = \frac{1}{\sqrt{7}} \left[ A_i^0(\tau) - \frac{1}{7} \mu \sigma \tau (\sigma - \frac{1}{7}) \epsilon_{ij} C_j^0(\tau) - \frac{1}{2} \mu l \epsilon_{ij} A_j^0(\tau) \right]$$

$$+ \sqrt{\frac{2}{7}} \sum_{m=1}^\infty \left[ (A_i^m(\tau) - \frac{1}{7} \mu \sigma \epsilon_{ij} A_j^m(\tau)) \cos \frac{m\pi \sigma}{l} + \frac{\mu l}{2m\pi} \epsilon_{ij} (A_j^m(\tau) - \tau C_j^m(\tau)) \sin \frac{m\pi \sigma}{l} \right]$$

$$P_i(\sigma, \tau) = -\frac{1}{\sqrt{7}} \left[ \epsilon_{ij} C_j^0(\tau) + \mu \sigma C_i^0(\tau) \right]$$

$$- \sqrt{\frac{2}{7}} \sum_{m=1}^\infty \left[ (\epsilon_{ij} C_j^m(\tau) + \frac{1}{7} \mu \sigma C_i^m(\tau)) \cos \frac{m\pi \sigma}{l} + \frac{\mu l}{2m\pi} C_i^m(\tau) \sin \frac{m\pi \sigma}{l} \right]$$

(45)

The normalization factor $\frac{1}{\sqrt{7}}$ is replaced by $\sqrt{\frac{2}{7}}$ for oscillatory modes and $\frac{1}{\sqrt{7}}$ for zero mode upon going from the continues parameter $k$ to the discrete parameter $m$. With this normalization the brackets of the discrete modes should also be given in terms of Kronecker delta as

$$\{ A_i^m, C_j^m \} = \delta_{ij} \delta_{mm'},$$

$$\{ A_i^m, A_j^{m'} \} = \{ C_i^m, C_j^{m'} \} = 0.$$  

(47)

(48)

In fact, the remaining canonical pairs $A_i^m$ and $C_i^m$ as a small part of the original phase space are natural coordinates of the reduced phase space. On the other hand, a great part of the initial phase space variables are omitted due to the constraints.

Remember that if one is able to omit the redundant variables due to all kinds of constraints and write down the relevant fields in terms of final canonical coordinates of the reduced phase space, then there is no need to find the Dirac brackets. In other words, we pay the expense of using the Dirac brackets whenever it is not possible to find a canonical

\[\text{footnote}{Since another length scale, i.e. } \mu^{-1}, \text{ exists in the model, one may suppose that the normalization factors should differ from the ordinary Fourier series. However, it can be shown that such corrections only changes the observables by amounts of } \mathcal{O}(\mu^2) \text{ which is not important.}\]
basis to describe the reduced phase space. Hence, we will find the Dirac brackets of the original fields \( A_i(\sigma, \tau) \) and \( P_i(\sigma, \tau) \) if we calculate their brackets by using the brackets (47) and (48).

Eq. (46) shows that the momentum-fields \( P_i(\sigma, \tau) \) just include the variables \( C_i^m \) and have vanishing brackets:

\[
\{ P_i(\sigma, \tau), P_j(\sigma', \tau) \} = 0.
\] (49)

Straightforward calculations gives the brackets of coordinate and momentum fields as

\[
\{ A_i(\sigma, \tau), P_j(\sigma', \tau) \} = \epsilon_{ij} \delta_N(\sigma, \sigma'),
\] (50)

where

\[
\delta_N(\sigma, \sigma') \equiv \delta(\sigma - \sigma') + \delta(\sigma + \sigma').
\]

Since both \( \sigma \) and \( \sigma' \) lie in the interval \([0, l]\) their sum never vanishes. So the second delta function does not have any role and Eq. (50) reduces to the usual form of Eq. (21). However, since in the expansion of \( A \)-fields both variables \( A_m^i \) and \( C_m^j \) are present, the interesting phenomenon appears in the bracket of coordinate fields at different points. Direct calculation gives

\[
\{ A_i(\sigma, \tau), A_j(\sigma', \tau) \} = \frac{1}{2} \mu \tau \epsilon_{ij} \left( \frac{\sigma + \sigma'}{l} - 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n \pi}{l} (\sigma + \sigma') \right).
\] (51)

This result is similar to what derived in [6] for a string coupled to constant background B-field. The right hand side of Eq. (51) vanishes in the bulk of the string, i.e. when \( \sigma \) or \( \sigma' \) does not lie on the end points. It gives \((-2)\) when \( \sigma = \sigma' = 0 \) and \((+2)\) when \( \sigma = \sigma' = l \). However, as the B-field itself, the amount of non commutativity grows linearly with time. Our result here defers from reference [7] with a term proportional to \( \mu \tau^2 \) which is the same on both boundaries as well as in the bulk of the string. If, however, we add a term \( -\frac{1}{2} \mu \tau^2 \epsilon_{ij} C_j^0(\tau) \) to the zero mode part of the field \( A_i(\sigma, \tau) \) in Eq. (15), our result will coincide with reference [7]. This correction is allowed according to the global symmetry of Eq. (13). This means that we have forgiven our previous assumption that the components of the A-field commute in the bulk of the string. With this assumption the resulted brackets can be summarized as follows

\[
\begin{align*}
\{ A_i(\sigma, \tau), P_j(\sigma', \tau) \} &= \epsilon_{ij} \delta(\sigma, \sigma'), \\
\{ P_i(\sigma, \tau), P_j(\sigma', \tau) \} &= 0 \\
\{ A_i(\sigma, \tau), A_j(\sigma', \tau) \} &= \left\{ \begin{array}{ll} \\
\frac{\mu \tau \epsilon_{ij}}{2l} & \sigma \neq 0, l \text{ or } \sigma' \neq 0, l \\
\mu \tau \epsilon_{ij}(1 + \frac{\tau}{2l}) & \sigma = \sigma' = 0 \\
\mu \tau \epsilon_{ij}(-1 + \frac{\tau}{2l}) & \sigma = \sigma' = l
\end{array} \right. 
\end{align*}
\] (52)

This shows that the fundamental characters of the \( A \)-fields and \( P \)-fields as coordinate and momentum fields are remained almost as before and the time dependent B-field leads to a time dependent non commutativity in the coordinate fields all over the string.

6 Concluding remarks

In this paper we gave a complete Hamiltonian treatment of the Nappi-Witten model (WZW model based on non semi simple gauge group) as an interesting and non trivial system in
which complicated boundary conditions make the physical subset of variables far from reaching. The initial dynamical variables in this model are 4 components of a bosonic string, \( X_a = (a_1, a_2, u, v) \), and the components of world-sheet metric. We used appropriate variables to find 3 primary and 2 secondary first class constraints. It can be shown that these constraints are generators of reparametrizations as well as Weyl transformations. Then we fixed the gauge such that the world-sheet metric is flat and \( u = \mu \tau \) where the small parameter \( \mu \) is proportional to the strength of the B-field. In this way the components of the world-sheet metric and the variables \( u \) and \( v \) disappeared as the result of constraints and gauge fixing conditions. Hence, we derived a smaller theory with two coordinate fields \( a_1 \) and \( a_2 \) and their corresponding momentum fields.

The most important part of the problem seems to be the boundary conditions which should be brought from the original theory. Considering the boundary condition as Dirac constraints and following their consistency, we found two infinite chains of second class constraints at the end-points which restricted the space of physical variables to a much smaller set. Due to complicated form of the boundary conditions, it is not an easy task to impose them on the space of the physical variables. In fact, with an ordinary Fourier expansion the constraints do not lead simply to omitting some Fourier modes as in Dirichlet or Neumann boundary conditions.

To overcome this difficulty we extended the phase space to a larger one which is given by an extended Fourier expansion in which the Fourier modes are replaced by linear functions of the variables. In this basis the infinite set of constraints can be imposed more easily by using the arbitrariness due to extra variables. This results to disappearing of so many canonical pairs among the used extended Fourier basis and finally a set of discrete modes remain which act as the canonical coordinates of the reduced phase space. Then all physical objects including the original coordinate and momentum fields can be expanded in terms of these modes.

Using these expansions we found that the commutation relations of the coordinate and momentum fields are almost as usual, except that the coordinate fields do not commute at the boundaries, with an amount proportional to time and/or B-field but with opposite signs at two boundaries. We showed that it is allowed to insert a term which gives non commutativity proportional to \( \tau^2 \) throughout the string. This correction may make our results consistent with those of reference \([7]\) in which the authors have given iterative solutions for the equations of motion.

We think that our method here has two main advantages in two different areas. First, we do not solve the equation of motion. Therefore, in our final result the time dependence of remaining modes are not specified. However, this time dependence is not essential for quantization of the model. If needed, one can use the Hamiltonian written in terms of the final modes and then derive their time dependence. In fact, our main objective is that for quantizing a theory, i.e. investigating the algebraic structure of the observables, it is not needed to follow the full dynamics of the system; it is just enough to study the dynamics of constraints. As a matter of fact, for simple models it may seem more simple and economic to solve the equations of motion and then quantize the theory, since this procedure contains the dynamics of the constraints within itself. But this may not be the case for a complicated model such as the model considered in this paper.

The next advantage is in the context of constraint systems. As we see in the literature \([1, 14]\) the main difficulty in considering the infinite set of constraints due to boundary conditions is deriving the Dirac brackets. In this paper, as in our previous work \([6]\) we
showed that if one is able to find a set of canonical variables describing the reduced phase space, then there is naturally no need to calculate the Dirac brackets. In fact, this was the main brilliant idea of Dirac [11], who gave his famous formula of Dirac brackets in such a way that it is equivalent to calculating the Poisson brackets only in the space of canonical variables describing the reduced phase space.

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