We consider the problem of two interacting particles on a sphere. The potential of the interaction depends on the distance between the particles. The case of Newtonian-type potentials is studied in most detail. We reduce this system to a system with two degrees of freedom and give a number of remarkable periodic orbits. We also discuss integrability and stochastization of the motion.

1. Introduction and historical notes

The origins of the analysis of motion of mass particles and rigid bodies in constant curvature spaces (plane, two- and three-dimensional spheres, and Lobachevsky space) trace back to the classical XIXth century papers by Serret, Killing, Lipschitz, Liebmann, Schering, etc. One of the most comprehensive is the paper by W. Killing [4]. The author studied the motion of a particle in the “central field”, a curved-space analog of the Newtonian field (the Kepler problem). He also presented the generalization of the Euler two-center problem and N-dimensional analogs of the above-mentioned problems. He derived and partially studied the equations of motion of a rigid body in these spaces and discussed some issues of the theory of the Newtonian potential. More than a century after that, the study of dynamics in constant curvature spaces attracted the interest of scientists again. This resulted in appearance of a series of papers, reviewed in [2]. Note, though, that some of the classical results were independently rediscovered in these papers.

One of the major distinctive features of curved spaces is the absence of translational (Galilean) invariance, resulting in the absence of the center-of-inertia integrals and non-existence of the corresponding barycentric frame of reference. As a result, this takes us out of the realm of the classical celestial mechanics.

For example, the two-body problem, when considered on a two-dimensional (or three-dimensional) sphere $S^2$ ($S^3$) or on a Lobachevsky plane (or space) $L^2$ ($L^3$), can no longer be reduced to the corresponding problem of motion in a central potential field (an analog of the Kepler problem) and, generally, is not integrable, as will be shown below. Hereinafter, we will discuss only the case of a two-dimensional sphere $S^2$, though the obtained results can be easily extended to $L^2$ and their many-dimensional generalizations.

First of all, let us note that the Kepler problem of motion in a central potential field on a sphere is integrable, and the orbits are ellipses with one of the foci at the center. An analog of the Newtonian potential on a sphere was first offered, apparently, by Serret in [11] (1860), the same for a Lobachevsky plane was done by Lobachevsky himself and Bolyai. A generalization of the Bertrand theorem to a curved space was performed by Liebmann (1903). Analogs of Kepler’s laws for $S^n$ and $L^n$ can be found in a number of papers, of which the most fundamental is the above-cited paper by W. Killing. Among the modern explorations of the Kepler problem from various perspectives, we recommend [2, 7].
So, an analog of the Newtonian potential on $S^2 (L^2)$ is
\[ U = -\gamma \cot \theta \quad (U = -\gamma \coth \theta), \] (1.1)
where $\theta$ is the longitude measured from the pole, at which the gravitating center is located, and $\gamma$ is the gravitational constant. The potential (1.1) can be obtained as a centrosymmetric solution to the Laplace–Beltrami equations for $S^3 (L^3)$ or obtained from an analog of Bertrand theorem for $S^2 (L^2)$ [6].

In [2], we state the restricted problem of two bodies on a sphere (or on $L^2$), where one of the bodies circumscribes a great circle on the sphere with constant velocity (i.e., moves freely along a geodesic), while the other moves in its field (with the potential (1.1)) and does not affect the first body. In [8] (as well as in [2]), we offer also a numerical analysis of the Poincaré map for this problem, which is not integrable, judging by its stochastic orbits; we also use the averaging method to study the perigee motion caused by the curvature of the space. In [17], it is proved, for the restricted problem, the non-existence of an additional meromorphic analytic integral.

2. Reduction of the two-body problem on $S^2$

The general (not restricted) two-body problem on $S^2 (L^2)$ was studied in respect to reduction in [12], where the quantization problems were also discussed. The general method of reduction, offered in [12], was first suggested by E. Cartan and further developed by J. Marsden and A. Weinstein. This method is based on rather formal argumentation, which, at the same time, has a profound differential geometric interpretation. Unfortunately, the method does not always give and suitable expressions, which have been sought after in the classical celestial mechanics (for example, when reducing the three-body problem).

In this paper, to reduce the unrestricted two-body problem on a sphere, we use Bour’s reduction [16] of the spatial three-body problem (similarly, Radau’s reduction can be used [16]). In a sense, the reduction we need can be regarded as its special case. Connection between the mentioned problems is based on a famous observation by Jacobi: once a barycentric frame of reference was introduced, which is equivalent to reduction to the center of inertia, the classical three-body problem could be reduced to the problem of two mass points moving in the fixed space $Oxyz$ with a potential, which was a function of the points’ distances from the fixed center $O$ and of the angle between their radius-vectors. Such a reduced system has a (vector) integral of angular momentum, and Bour’s reduction is an effective reduction of the system’s order using this integral (elimination of a node). With that, two more degrees of freedom are eliminated. In the case of a sphere, the distances $r_1, r_2$ between two points and the center are fixed, so in the Bour-reduced system we only have to put $r_1 = r_2 = R$, $\dot{r}_1 = \dot{r}_2 = 0$ ($R$ is the sphere’s radius). The specified reduction is valid for the potential, which arbitrarily depends on the distance between the points (the two-body problem is sometimes referenced as E. I. Kugushev’s problem [1]).

Below we give the calculations in more detail. Let $Oxyz$ be a fixed frame of reference, and $O\xi\eta\zeta$ — a moving frame of reference, such that the plane $O\eta\xi$ contains the mass points $m_1$ and $m_2$. The axis $\eta$, being an intersection of the plane ($\eta\xi$) with ($xy$), is the line of nodes (see Fig. 1). The mass points are described by the coordinates $\theta_1$ and $\theta_2$ (the angular polar coordinates in the plane $O\eta\xi$) and $\varphi, \psi$ (these angles specify the location of the frame $O\xi\eta\zeta$ relative to $Oxyz$). It is easy to see that all these variables are identical with the Euler angles from the rigid body dynamics (this analogy was first noted by J. Silvestr).

In terms of the introduced variables, the Cartesian coordinates read:
\[
\begin{align*}
x_i &= -R(\cos \varphi \cos \psi \sin \theta_i + \sin \psi \cos \theta_i), \\
y_i &= -R(\cos \varphi \sin \psi \sin \theta_i - \cos \psi \cos \theta_i), \\
z_i &= R \sin \theta_i \sin \varphi,
\end{align*}
\] (2.1)
Using (2.1), the Lagrangian function of the system can be written as

\[ L = \frac{R^2}{2} \left( m_1 (\dot{\theta}_1 + \dot{\psi} \cos \varphi)^2 + m_2 (\dot{\theta}_2 + \dot{\psi} \cos \varphi)^2 + J \dot{\varphi}^2 + R \sin^2 \varphi \dot{\psi}^2 - 2L \dot{\varphi} \dot{\psi} \sin \varphi - U(\theta_1 - \theta_2) \right), \]  

(2.2)

where

\[ J = m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2, \quad K = m_1 \cos^2 \theta_1 + m_2 \cos^2 \theta_2, \]

\[ L = m_1 \sin \theta_1 \cos \theta_1 + m_2 \sin \theta_2 \cos \theta_2. \]

Let the axis Oz be directed along the vector of angular momentum \(|M| = c\). The Lagrangian equations with the Lagrangian (2.2) admit the integral

\[ \frac{\partial L}{\partial \dot{\psi}} = M_z = c = \text{const} \]  

(2.3)

and invariant relations

\[ \frac{\partial L}{\partial \dot{\varphi}} = M_\eta = 0, \quad \frac{\partial L}{\partial \varphi} = M_{z \times \eta} \dot{\psi} = 0. \]  

(2.4)

Indeed,

\[ M_\eta = M_{z \times \eta} \dot{\psi}, \quad M_{z \times \eta} = -M_\eta \dot{\psi}, \]

where \(M_z, M_\eta, M_{z \times \eta}\) are the projections of the angular momentum \(M\) onto the given axes (see Fig. 1). The invariant manifold (2.3), (2.4) is explicitly given by:

\[ \begin{align*}
\dot{\psi} &= \frac{cJ}{m_1 m_2 R^2 \sin^2(\theta_1 - \theta_2)}, \\
\dot{\varphi} &= \frac{cL \sin \varphi}{m_1 m_2 R^2 \sin^2(\theta_1 - \theta_2)}, \\
\cos \varphi &= \frac{m_1 \dot{\theta}_1 + m_2 \dot{\theta}_2}{c/R^2 - (m_1 + m_2) \dot{\psi}}.
\end{align*} \]  

(2.5)

It follows from (2.3) that the variable \(\psi\) is cyclic, so we can reduce the system’s order by one degree of freedom. Besides, the variable \(\varphi\) is also cyclic on the invariant manifold (2.4), and here we
again can reduce the number of degrees of freedom by one, using the Routh reduction procedure. The Routh function is

\[ R = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \psi - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \varphi = \frac{R^2}{2} (m_1 \dot{\theta}_1^2 + m_2 \dot{\theta}_2^2) - \frac{R^2 J (m_1 \dot{\theta}_1 + m_2 \dot{\theta}_2)^2}{2 (J (m_1 + m_2) - m_1 m_2 \sin^2(\theta_1 - \theta_2))} - \frac{c^2 J}{2 m_1 m_2 R^2 \sin^2(\theta_1 - \theta_2)} - U(\theta_1 - \theta_2). \]  

(2.6)

Using the Legendre transformations

\[ p_i = \frac{\partial R}{\partial \dot{\theta}_i} = m_i R^2 \dot{\theta}_i - \frac{m_i R^2 J (m_1 \dot{\theta}_1 + m_2 \dot{\theta}_2)}{J (m_1 + m_2) - m_1 m_2 \sin^2(\theta_1 - \theta_2)}, \quad i = 1, 2, \]  

(2.7)

we obtain the equations of motion of the Hamiltonian form with the Hamiltonian

\[ \mathcal{H} = \frac{1}{2 R^2} \left( \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + (c^2 - (p_1 + p_2)^2) \frac{J}{m_1 m_2 R^2 \sin^2(\theta_1 - \theta_2)} \right) + U(\theta_1 - \theta_2). \]  

(2.8)

This is the sought-for reduced system with two degrees of freedom.

The absolute motion (i.e. motion in the fixed frame of reference) can be obtained, using the two additional quadratures given by (2.5).

Remark. On the manifold (2.5), the domain of possible values of the new variables \( p_1 \) and \( p_2 \) is bounded because it follows from (2.5) that \( \cos \varphi = \frac{p_1 + p_2}{c} \) and \( |\cos \varphi| \leq 1 \).

So, the phase space of the reduced system is parametrized with the canonical variables \( \theta_1, \theta_2, p_1, p_2 \). The angle variables \( \theta_1, \theta_2 \mod 2\pi \) define a two-dimensional torus (configuration space of the reduced system), while the conjugate momenta \( p_1, p_2 \) form the strip \(|p_1 + p_2| = c \leq |\cos \varphi| \leq |c| \). Besides, it is necessary to identify the points of the phase space with coordinates

\[ (\theta_1, \theta_2, p_1, p_2) \quad \text{and} \quad (\pi - \theta_1, \pi - \theta_2, -p_1, -p_2). \]  

(2.9)

Indeed, according to (2.5), the different sets of coordinates, \( (\theta_1, \theta_2, \varphi, \psi) \) and \( (\pi - \theta_1, \pi - \theta_2, \pi - \varphi, \pi + \psi) \), define the same configuration of the bodies on the sphere. For the velocities we have \((\pi - \theta_1) = -\theta_1, (\pi - \theta_2) = -\theta_2 \) and, using (2.7), obtain (2.6). We believe it would be an interesting to develop a better geometric description of the reduced phase space.

The Hamiltonian (2.8) is a homogeneous, but not positively defined, quadratic function of the momenta. Since the system (2.8) is not natural, well-developed methods of topological and qualitative analysis [5] can hardly be applied to it.

3. Particular solutions and the Smale diagram of the two-body problem on \( S^2 \)

Let us start with the general problem of \( n \) interacting bodies. The potential of interaction \( U \) depends on the distances between the bodies.

A particular solution to the \( n \)-body problem will be called a rigid-body motion if the distances between the bodies remain constant. A particular solution to the \( n \)-body problem will be called a stationary configuration (relative equilibrium) if all the bodies rotate uniformly, with the same angular velocity, about the axis containing the vector \( M \).

It is obvious that any stationary configuration is a rigid-body motion, but the reverse is, generally, not true.
3.1. Stationary configurations and the Smale diagram

The stationary configurations in the general n-body problem on $S^2$ can be obtained as the critical points of the reduced potential.

**Proposition 1.** The stationary configurations (relative equilibria) of the n-body problem correspond to the critical points of the reduced (effective) potential

$$ U_* = U - \frac{1}{2} \omega^2 I = U - \frac{M^2}{2I}, \quad (3.1) $$

where $\omega$ is the angular velocity of the configuration, $I$ is the total moment of inertia of the system with respect to the axis of rotation, $M^2$ is the squared total angular momentum of the system of bodies in a fixed frame of reference.

**Proof.**

Consider in the frame of reference, rotating with the angular velocity $\omega$ about the axis $z$. In this frame, the Lagrangian (in terms of spherical coordinates) reads:

$$ \mathcal{L} = \frac{1}{2} \sum m_i R^2 (\dot{\theta}_i^2 + \sin^2 \theta_i \dot{\varphi}_i^2) + \omega \sum m_i \dot{\varphi}_i R^2 \sin^2 \theta_i - U + \frac{1}{2} \omega^2 I, \quad (3.2) $$

where $I = \sum m_i R^2 \sin^2 \theta_i$ is the total moment of inertia with respect to the $z$-axis ($i$ is the number of a particle).

Then, using equations of motion with the Lagrangian (3.2) and bearing in mind that $\dot{\theta}_i = \dot{\varphi}_i = 0$ (stationary solution), we obtain the following conditions of the relative equilibrium:

$$ \frac{\partial}{\partial \theta_i} \left( U - \frac{1}{2} \omega^2 I \right) = 0, \quad \frac{\partial}{\partial \varphi_i} \left( U - \frac{1}{2} \omega^2 I \right) = 0, \quad i = 1, \ldots, n. \quad (3.3) $$

To finish the proof, we have yet to express the angular momentum of the stationary configuration in the fixed frame of reference:

$$ M_x = M_y = 0, \quad M_z = \sum m_i R^2 \sin^2 \theta_i \dot{\varphi}_i = \omega \sum m_i R^2 \sin^2 \theta_i = I \omega. $$

Now we consider the case of the generalized Newtonian potential and give a complete description of the relative equilibria of the two-body problem on $S^2$. According to Proposition 1 it is necessary to find the critical points of the reduced potential

$$ U_* = -\varepsilon R^2 m_1 m_2 \cot \theta_{12} - \frac{1}{2} \omega^2 R^2 \left( m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2 \right), $$

where $\theta_{12}$ is the angular distance the particles on the sphere, and $\varepsilon$ is the gravitational constant.

It can be shown that for relative equilibria of two particles on $S^2$, the following two observations are true:

1. While moving, the particles never leave the great circle that contains the axis of rotation.

2. When $\varepsilon > 0$ (the case of attraction) the particles stay on the great circle, with the axis of rotation between them (see Fig. 2).

Indeed, the cosine of the angle between the mass points can be rewritten as

$$ \xi = \cos \theta_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2), $$

and we find that

$$ \frac{\partial U_*}{\partial \varphi_1} = -\frac{\partial U_*}{\partial \varphi_2} = \frac{\varepsilon R^2 m_1 m_2}{(1 - \xi^2)^{3/2}} \sin \theta_1 \sin \theta_2 \sin(\varphi_1 - \varphi_2) = 0. $$
Hence, either $\varphi_1 = \varphi_2$, or $\varphi_2 = \varphi_1 + \pi$. If $\varphi_1 = \varphi_2$ the equations $\frac{\partial U}{\partial \theta_i} = 0$ do not have a common solution.

So, to find stationary configurations, it is necessary to find the critical points of the function

$$U_* = -\omega R^2 m_1 m_2 \cotg(\theta_1 + \theta_2) - \frac{1}{2}\omega^2 R^2 (m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2)$$

where the angles $\theta_1$ and $\theta_2$ are measured from the axis of rotation (Fig. 2). These angles are given by

$$\omega^2 \sin \theta_1 \cos \theta_1 - \frac{\omega m_2}{\sin^2(\theta_1 + \theta_2)} = 0, \quad \omega^2 \sin \theta_2 \cos \theta_2 - \frac{\omega m_1}{\sin^2(\theta_1 + \theta_2)} = 0. \quad (3.4)$$

Fig. 2.

From this, we obtain the equation that generalizes the planar “law of the lever” and relates the angles $\theta_1$ and $\theta_2$

$$m_1 \sin 2\theta_1 = m_2 \sin 2\theta_2. \quad (3.5)$$

Using (3.4) and (3.5), one can show that for $\omega > 0$, either $\theta_1, \theta_2 < \pi/2$, or $\theta_1, \theta_2 > \pi/2$. These configurations are identical up to the reflection about the equatorial plane. Therefore, we will assume that

$$0 < \theta_1, \theta_2 < \frac{\pi}{2}. \quad (3.6)$$

If we put $m_1 < m_2$, then the equation has two roots on the interval (3.6)

$$\theta_2^{(1)} = \frac{1}{2} \arcsin \left( \frac{m_1}{m_2} \sin 2\theta_1 \right), \quad \theta_2^{(2)} = \frac{\pi}{2} - \frac{1}{2} \arcsin \left( \frac{m_1}{m_2} \sin 2\theta_1 \right); \quad (3.7)$$

here $\theta_2^{(1)} < \theta_1$, and $\theta_2^{(2)} > \theta_1$.

The first solution $\theta_2^{(1)}$ describes a configuration, where the heavier particle moves along the smaller circle. As $R \to \infty$, this solution tends to ordinary stationary configuration of the two-body problem on a plane.

The second solution $\theta_2^{(2)}$ describes a configuration, where the heavier particle moves along the larger circle; this solution does not have an analog in the two-body problem on a plane.

Let us plot a bifurcation diagram (see [14]) for the two-body problem on $S^2$ on the integral plane ($c^2 = M^2/R^4$, $E = h/R^2$), where $h$ is the energy integral (Fig. 3). When $m_1 \neq m_2$, we obtain
two curves, corresponding to the solutions of (3.7) (see Fig. 3b). When $m_1 = m_2$, the curves merge at a point $P$ (Fig. 3b); in this case, also according to (3.7), we have either $\theta_1 = \theta_2$, or $\theta_1 + \theta_2 = \pi/2$.

Here we also give a bifurcation diagram for the two-body problem on a sphere in the presence of a potential that, unlike the gravitational potential, has no singularities (see Fig. 4). For that potential, in the paper [1] a geometrical (topological) analysis of the integral manifolds $M^2 = \text{const}$, $h = \text{const}$ and of the level surface of the reduced system’s energy integral is performed. Note that, due to singularities at the poles in the case of the gravitational potential (1.1), it is impossible to directly apply the results of [1] to this case. So, the complete topological analysis of three-dimensional isoenergetic manifolds does not seem to have been done.

Fig. 3. Bifurcation diagram for the two-body problem with the gravitational potential $U = -\varepsilon m_1 m_2 \cot \theta$.

Fig. 4. Bifurcation diagram for the two-body problem on a sphere with the potential $U = \varepsilon m_1 m_2 (1 - \cos \theta)$.

3.2. A particular solution in the case of equal masses

The $n$-body problem on a plane has a particular solution (collinear configuration), in which all the particles move along ellipses, while staying at all times on the same straight line. This solution was first offered by Euler (for the three-body problem) and Moulton (for the $n$-body case) [9]. In [9], Moulton also proved that for the case when all masses are different the number of such collinear configurations (for different permutations of the particles) is equal to $\frac{n!}{2}$. Let us show that analogous (pulsating) solutions on a sphere do not, generally, exist.

Proposition 2. For the two-body ($n$-body) problem on $S^2$, when $m_i \neq m_j$, there are no solutions other than a stationary configuration, in which all the particles stay on the same great circle that contains the axis of rotation.
Proof.

Let \( Oz \) be the axis of rotation. The equations of motion in spherical coordinates are

\[
m_i \ddot{\theta}_i = m_i \sin \theta_i \cos \theta_i \phi_i^2 - \frac{\partial U}{\partial \theta_i},
\]

\[
\frac{d}{dt} (m_i \sin^2 \theta_i \dot{\phi}_i) = -\frac{\partial U}{\partial \phi_i} = \sum_{j \neq i, j \neq 1} U'_{ij} \sin \theta_i \sin \theta_j \sin (\phi_i - \phi_j),
\]

(3.8)

where \( \xi_{ij} = \cos \theta_i \cos \theta_j + \cos \theta_i \sin \theta_j \cos (\varphi_i - \varphi_j) \).

If the particles always stay on a great circles that contains the axis of rotation, then either \( \varphi_i = \varphi_j \), or \( \varphi_j = \varphi_i + \pi \) and, besides,

\[
\dot{\phi}_i = \dot{\phi}_j = \dot{\psi}, \quad i, j = 1, \ldots, n.
\]

Therefore, according to (3.5), the following must hold:

\[
m_i \sin^2 \theta_i \dot{\psi} = m_i c_i^2, \quad c_i = \text{const}
\]

Hence, all the angles \( \theta_i \) depend only on the constants \( c_i \) and the function \( \lambda^2(t) = 1/\dot{\psi}(t) \)

\[
\zeta_i = \sin \theta_i = c_i \lambda(t).
\]

(3.10)

Differentiating this and substituting into the first equation of (3.5), we obtain

\[
\frac{m_i c_i}{\sqrt{1 - c_i^2 \lambda^2}} \left( \ddot{\lambda} + \frac{c_i^2 \lambda \ddot{\lambda}}{1 - c_i^2 \lambda^2} \right) = m_i c_i \lambda^{-3} \sqrt{1 - c_i^2 \lambda^2} - \frac{\partial U}{\partial \theta_i},
\]

(3.11)

where \( \frac{\partial U}{\partial \theta_i} = \frac{\partial U}{\partial \zeta_i} \frac{\partial \zeta_i}{\partial \theta_i} = \sqrt{1 - c_i^2 \lambda^2} \frac{\partial U}{\partial \zeta_i} \). The required solution exists if (3.11) can be made independent on the particle’s number \( i \). Provided that \( \dot{\lambda} \neq 0 \), this can be done if \( \theta_i \) are assumed to be the critical points of the reduced potential (3.8) and all \( c_i \) are assumed to be equal, i.e.

\[
\frac{\partial U}{\partial \theta_i} = \omega^2 m_i \sin \theta_i \cos \theta_i, \quad c_i = c = \text{const}.
\]

(3.12)

Then, taking into account (3.10) and (3.11), incompatible systems are obtained

\[
m_i \sin 2\theta_i = \pm m_j \sin 2\theta_j \quad \text{and} \quad \sin \theta_i = \pm \sin \theta_j,
\]

which does not have a solution \( m_i \neq m_j \).

Let us consider more closely the case of equal masses in the two-body problem, \( m_1 = m_2 = 1 \) (with \( \varphi_2 = \varphi_1 + \pi \) and \( \theta_1 = \theta_2 = \theta \)). Using directly (3.8) and (3.9), we obtain

\[
\dot{\psi} = \dot{\varphi}_1 = \dot{\varphi}_2 = \frac{c}{\sin^2 \theta} \quad \text{and} \quad \ddot{\theta} = \sin \theta \cos \theta \dot{\psi}^2 - \frac{\omega^2}{\sin^2 2\theta}, \quad c = \text{const}.
\]

(3.13)

These equations describe the motion of a particle on a sphere in the field of an attracting center with the potential

\[
U = \frac{1}{2} \omega^2 \cot 2\theta.
\]

For this potential, the orbits are unclosed, which shows another difference between curved and flat spaces. It is known that, due to uniformity condition, the particles’ orbits in flat spaces are ellipses with one of the foci at the center of mass.

Nevertheless, orbits on a sphere are closed in some rotating frame of reference. Moreover, one can choose the velocity of rotation so that both particles will move along one and the same curve (such motion is called a relative choreography [13]).
3.3. Rigid-body motions in the two-body problem

Consider the motion of particles in a flat space \( \mathbb{R}^3 \) in a fixed frame of reference. In this case, the relative equilibria of the two-body (\( n \)-body) problem in the center-of-mass reference frame, generally (i.e. when the velocity of the center of mass is not zero), correspond to the simplest rigid-body solutions. The mutual distances between the particles are always the same. We show that in a curved space (on \( S^2 \)), lacking the notion of the center of mass, all the rigid-body solutions of the two-body problem are stationary configurations discussed above (3.4), (3.7). Therefore, the corresponding family of configurations has less free parameters than such a family in a flat space.

**Theorem.** When the interaction potential depends only on the mutual distances, all rigid-body solutions of the two-body problem on \( S^2 \), are stationary configurations.

**Proof.** The proof is based on successive differentiation of the invariant relation

\[
(x, y) = \cos \theta_0 = \text{const},
\]

where \( x, y \) are the radius-vectors of the particles on \( S^2 \) in \( \mathbb{R}^3 \), and finding the maximal invariant manifold that contains these motions.

Now let us perform an algebraic reduction of our system by restricting it onto a level surface of \( M \). For the variables

\[
M = m_1 \dot{x} \times x + m_2 \dot{y} \times y, \quad L = \mu(\dot{x} \times x - \dot{y} \times y),
\]

where \( \mu = \frac{m_1 m_2}{m_1 + m_2} \), we have nine equations:

\[
\begin{align*}
\dot{L} &= U' (x \times y) \\
\dot{x} &= \frac{1}{m_1 + m_2} x \times M + \frac{1}{m_1} x \times L, \\
\dot{y} &= \frac{1}{m_1 + m_2} y \times M - \frac{1}{m_1} y \times L.
\end{align*}
\]

Here \( U(\xi) \) is the potential energy of the interaction, expressed in terms of the cosine of the angle between the particles \( \xi = (x, y) \), and vector \( M \) is a parameter. Due to the condition \((\dot{x}, x) = (\dot{y}, y) = 0\) and definition of \( x, y \), the following relations hold:

\[
\begin{align*}
x^2 &= y^2 = 1; \\
\frac{m_1}{m_1 + m_2} (x, M) + (x, L) &= 0, \quad \frac{m_2}{m_1 + m_2} (y, M) - (y, L) = 0.
\end{align*}
\]

The energy integral in this case reads

\[
E = \frac{1}{2\mu} L^2 + U(\xi) = \text{const}.
\]

Differentiating the relation \((x, y) = \text{const}\) by time and using (3.15) and (3.16), we get

\[
\begin{align*}
(x, y \times L) &= 0, \\
\frac{1}{\mu} L(x, y) + U'(\xi)(x \times y)^2 + \frac{2}{m} (x, M)(y, M) &= 0, \\
(x, M \times L) &= 0.
\end{align*}
\]
It follows from the second and the fourth equations of (3.17) that the vectors $x$, $y$, $M$, and $L$ belong to the same plane. Express $M$, $L$ in terms of $x$, $y$:

$$
M = \frac{(M, x) - \cos \theta_0 (M, y)}{\sin^2 \theta_0} x + \frac{(M, y) - \cos \theta_0 (M, x)}{\sin^2 \theta_0} y, \\
L = \frac{(L, x) - \cos \theta_0 (L, y)}{\sin^2 \theta_0} x + \frac{(L, y) - \cos \theta_0 (L, x)}{\sin^2 \theta_0} y;
$$

(3.18)

Substituting this into (3.15) and (3.17), we have $(M, x) = \text{const}$, and $(M, y) = \text{const}$. Thus, the particles move in circles about some axis defined by the vector $M$, and during the motion the particles and the axis always lie in the same plane.

3.4. The case of $M = 0$ and the two-body problem on a circle. Collisions

In the two-body problem on a sphere (with an arbitrary potential, depending only on the inter-particle distance), the motion with zero total momentum ($M = 0$) is a special case. We will show that the particles in this case move along a great circle, which holds its position in a fixed frame of reference.

Indeed, for $M = 0$, equations (3.15) read

$$
\dot{x} = \frac{1}{m_1} x \times L, \quad \dot{y} = -\frac{1}{m_2} y \times L, \quad \dot{L} = \mu' \frac{x \times y}{|x \times y|},
$$

(3.19)

while the relations (3.16) take the form

$$(x, L) = (y, L) = 0.$$

(3.20)

Calculating the orbital derivative of the normal to the particles’ plane $n = \frac{x \times y}{|x \times y|}$ under the flow of (3.19) and using (3.20), we get $\dot{n} \equiv 0$. Thus, the plane that contains the radius-vectors of the particles is fixed, and the particles move along a great circle.

Remark. In the two-body problem on a plane, when the momentum is zero, the particles move along the same straight line, fixed in the center-of-mass reference frame. This frame moves uniformly and in a straight line.

Since for zero momentum $M = 0$, the two-body problem on a sphere is reduced to a system on a circle, we will consider the latter problem in more detail (for arbitrary $M$). We will also show that in this case there is a center-of-mass reference frame, in which the behavior of the particles does not depend on the value of $M$. The new variables are

$$
\psi = \frac{m_1 \theta_1 + m_2 \theta_2}{m_1 + m_2}, \quad \theta = \theta_1 - \theta_2,
$$

(3.21)

where $\theta_1$ and $\theta_2$ are the angles that define the positions of the particles on the circle. So, the Lagrangian reads

$$
\mathcal{L} = \frac{1}{2} (m_1 + m_2) \dot{\psi}^2 + \frac{1}{2} \mu \dot{\theta}^2 - U(\theta), \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.
$$

This implies that the “center-of-mass angle” $\psi$ varies uniformly with time:

$$
\dot{\psi} = \frac{|M|}{R^2 (m_1 + m_2)} = \text{const}.
$$

(3.22)

The time dependency of $\theta$ is given in terms of quadratures

$$
\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2\mu^{-1}(h_1 - U(\theta))}} = t - t_0,
$$

(3.23)

where $\theta_0$, $t_0$, and $h_1$ are some constants.
For the generalized Newtonian potential $U(\theta) = -\alpha m_1 m_2 \cot \theta$, we find that the particles will collide in finite time. If $t_0 = 0$ at the time of collision, the time dependency of the angle can be written in the form of the Puiseux series

$$
\theta(t) = t^{2/3} \sum_{n=0}^{\infty} c_n t^{n/3}.
$$

(3.24)

Here $c_0 \neq 0$, while the coefficients $c_n$ with odd indices are zero ($c_{2n+1} = 0$). In other words, $\theta(t)$ is an even function of time. Consequently, the particles under go perfectly elastic collision (for a suitable regularization).

Remark 1. The proof of (3.24) is based on the analysis of (3.23) near $\theta = 0$. Introducing a new variable $x = \tan \theta$, we express (3.23) in the following form (for $t_0 = 0$):

$$
\int_{0}^{x} \frac{\sqrt{x} \, dx}{(1 + x^2) \sqrt{1 + ax}} = \zeta, \quad \zeta = \sqrt{2\alpha (m_1 + m_2)} t, \quad a = \frac{h_1}{\alpha m_1 m_2}.
$$

Using the Taylor expansion of the integrand, we obtain the following series equations defining $x(t)$:

$$
u^3 \sum_{n=0}^{\infty} g_n u^{2n} = \zeta, \quad u^2 = x,
$$

where $g_n$ are constants that depend on $a$. After eliminating $u$, we have

$$
x(\zeta) = \zeta^{2/3} \sum_{n=0}^{\infty} b_n \zeta^{2n/3}.
$$

Remark 2. We believe it would be interesting to generalize the classical results obtained by Zigel, concerning zero measure of collision orbits, to this problem (i.e. to specify the collision manifold).

Let us also mention a long-discussed question [2, 7] about a possibility of extending Sundman’s results (i.e. solutions in power series) to the two- and three-body problems in curved spaces. Unfortunately, the answer to this question has not yet been found and is, most likely, negative.

4. The Poincaré section. Numerical analysis

4.1. The surface of section. Chaos

To study the system’s behavior numerically, we plot the Poincaré section of the reduced system (2.8) in the following way. The two-dimensional surface of section of the three-dimensional energy level $H = E$ is given by $p_2 = \text{const}$. Unlike one-and-a-half degree of freedom systems (or trivial two-degree systems), the surfaces of section in our case might have a complicated structure. Moreover, the projection of such a surface on a plane (for example, ($p_1$, $\theta_1$)) has singularities. To avoid them, we will plot these surfaces in the three-dimensional space of variables ($\theta_1$, $p_1$, $\theta_2$). Some examples of surfaces of section $p_2 = 0$ are given in Fig. 5. The section’s type depends on the location of the point ($E$, $c$) on the bifurcation diagram (Fig. 3). The phase flow of the system (2.8) generates a point Poincaré map on these surfaces. The Poincaré maps for the case of equal masses (for some values of $c$ and $E$) are shown in Fig. 5. According to (2.9) the points with coordinates ($\theta_1$, $\theta_2$, $p_1$) and ($\pi - \theta_1$, $\pi - \theta_2$, $-p_1$) should, generally, be identified. For large energies, there are chaotic motions on the sections (Fig. 5, f, l), which prevent the existence of an additional analytic integral of the system (2.8).

Remark. Non-existence of an additional analytic integral for (2.8) can also be proved using Poincaré’s methods [10], i.e. analyzing the secular terms of the perturbed system and the corresponding number of non-degenerate periodic orbits. In this connection, the expansions of the disturbing function, given in [10], may be useful. These expansions were used in the proof of non-integrability of the classical three-body problem.
4.2. Periodic solutions and choreographies

An important part in the classical celestial mechanics is played by periodic orbits in a center-of-mass reference frame. Such orbits were studied in many papers [15]. Recently, a new class of periodic solutions in the problem of \( n \) equal-mass bodies has been discovered. These are so-called (relative)
choreographies — solutions, when the particles follow each other (in a rotating frame of reference) along the same curve, with time shift \( T/n \) (see also [3]).

A similar part in the two-body problem on a sphere is played by periodic solutions of the reduced system (2.8). The simplest periodic solution that can be expressed analytically is (3.13). There are other similar solutions, which, however, cannot be presented by analytical formulas. A suitable tool to analyze these solutions is the Poincaré map constructed above. Fixed points of the least order correspond to the simplest periodic solutions (see Fig. 5). These solutions define choreographies of particles (the particles move along the same curve) in some rotating frame of reference [3, 13]. For the least-period periodic solutions, the projections of such choreographies on a plane, perpendicular to the axis of rotation, are given in Fig. 6. Figures \( A_1 \)–\( A_6 \) and \( B_1 \)–\( B_3 \) show two different stable solutions, while figures \( C_1 \)–\( C_2 \) show the unstable solution of the reduced system for different energies. Figures \( B_1 \), \( B_2 \), and \( B_3 \) show the analytic solution (3.13).

Remark. It can be shown, using the methods of [3], that for a fixed \( c \), there exists a countable set of values of \( E \), for which the particles form an absolute choreography, i.e. move along a closed curve in a fixed space.

Remark. When \( m_1 \neq m_2 \), choreographies are broken — each particle moves along its own closed curve (in a rotating frame of reference).

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Fig. 6. Relative choreographies for the problem of two equal-mass bodies \((m_1 = m_2 = 1)\) on a sphere, \(c = 1\), \(R = 1\).

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