Quantum Field Theory and the Space of All Lie Algebras

William Gordon Ritter

Harvard University Department of Physics
17 Oxford St., Cambridge, MA 02138
Email: ritter@fas.harvard.edu

Abstract: The space $\mathcal{M}_n$ of all isomorphism classes of $n$-dimensional Lie algebras over a field $k$ has a natural non-Hausdorff topology, induced from the Segal topology by the action of $\text{GL}(n)$. One way of studying this complicated space is by topological invariants. In this article we propose a new class of invariants coming from quantum field theory, valid in any dimension, inspired by Jaffe’s study of generalizations of the Witten index.

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1. The Space of All Lie Algebras

Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra over an arbitrary field \( k \). In what follows we will mostly assume \( k \) is \( \mathbb{R} \) or \( \mathbb{C} \).

**Definition 1.** The *structural tensor* \( f \) is defined to be the element of \( \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g} \) given by considering the bracket as a skew-symmetric bilinear mapping \( \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \) (which is the same as a map \( \wedge^2 \mathfrak{g} \to \mathfrak{g} \)).

In a basis \( \{ E_a \} \) of \( \mathfrak{g} \), the components of the structural tensor satisfy
\[
[E_a, E_b] = f_{ab}^c E_c.
\]
The statement that \( \mathfrak{g} \) forms a Lie algebra is equivalent to the following relations for the structure constants:
\[
f_{ij}^k + f_{ji}^k = 0, \quad f_{ij}^\ell \cdot f_{\ell k}^m + f_{jk}^\ell \cdot f_{\ell i}^m + f_{ki}^\ell \cdot f_{\ell j}^m = 0
\]
These relations are more compactly written as
\[
f_{(ij)}^k = f_{(ij)}^\ell f_{\ell k}^m = 0
\]

The space of all structure constants of \( n \)-dimensional Lie algebras inherits a topology from \( k^{n^3} \). We call this the *Segal topology* because of its relevance to Ref. [3]. The space of all sets \( \{ f_{ij}^k \} \) satisfying (1.1) is a subvariety \( W^n \subset k^{n^3} \) of dimension
\[
\dim W^n \leq n^3 - \frac{n^2(n+1)}{2} = \frac{n^2(n-1)}{2}.
\]
The space $\mathcal{M}_n$ of all isomorphism classes of $n$-dimensional Lie algebras over $k$ has a natural weakly separating (i.e. $T_0$, not $T_1$) non-Hausdorff topology $\kappa^n$, induced from the Segal topology by the action of $\text{GL}(n)$. Basis transformations induce $\text{GL}(n)$ tensor transformations between equivalent structure constants, $C_{ij}^k \sim (A^{-1})_{ij}^k C_{fg}^{hj} A_f^i A_g^j$, $A \in \text{GL}(n)$.

One can either define $\mathcal{M}_n = W^n/\text{GL}(n)$, or avoid the basis-dependent notation entirely, and simply define $\mathcal{M}_n$ as the set of all $n$-dimensional structural tensors in the sense of Definition 1.

For $n = 3$ the structure constants can be written as

$$f_{ij}^k = \epsilon_{ijk}(n^{ij} + \epsilon^{km}a_m),$$

where $n^{ij}$ is symmetric and $\epsilon_{ijk} = \epsilon^{ijk}$ totally antisymmetric with $\epsilon_{123} = 1$. With (1.3) the constraints are equivalent to $n^{lm}a_m = 0$, which are 3 independent relations. There is a classification due to Behr of Lie algebras in $\mathcal{M}_3$ according to possible inequivalent eigenvalues of $n^{lm}$ and values of $a_m$ (see [4]).

Invariants of real Lie algebras have been calculated for $n \leq 5$ by Patera, Sharp, Winternitz and Zassenhaus [5]. In this article we propose a new class of invariants, valid in any dimension, inspired by Jaffe’s study of generalizations of the Witten index [1, 2].

2. Deformation Theory

Definition 2. A Lie algebra deformation (or simply, a deformation) is a continuous curve $f : [0, \epsilon] \to \mathcal{M}_n$, where $\epsilon > 0$. The deformation is said to be trivial if all $\mathfrak{g}(t)$ are isomorphic, where $\mathfrak{g}(t)$ is the Lie algebra with structural tensor $f(t)$.

We denote by $\text{Der} \mathfrak{g}$ the set of all linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that $D[a, b] = [Da, b] + [a, Db]$.

Let $G$ and $H$ be Lie groups and let $B$ be an action of $G$ on $H$ by group homomorphisms with $B : G \times H \to H$ smooth. Then the semidirect product group $G \ltimes H$ is a Lie group with group operation: $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1B(g_1)h_2)$. We denote by $\beta : G \to \text{Aut} \mathfrak{h}$ the map defined by $\beta(g) = T_e B(g)$ and $b : \mathfrak{g} \to \text{Der} \mathfrak{h}$ the differential of $\beta$ at the identity. Then $b$ defines a semidirect product of Lie algebras $\mathfrak{g} \ltimes_b \mathfrak{h}$ by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2] + b(X_1)Y_2 - b(X_2)Y_1)$$

and this is the Lie algebra of $G \ltimes H$.

Example 1. An $\mathbb{R}$-linear map $\varphi : \mathbb{R} \to \text{gl}_2(\mathbb{R})$ is determined by its value at 1. Let $\mathfrak{a}(t) = \mathbb{R} \ltimes_{\varphi_t} \mathbb{R}^2$, where $\varphi_t(1) = \text{diag} \left( (1 + t)^{\lambda}, \ (1 + t + \alpha t^2)^{\mu} \right)$ for $\lambda, \mu \neq 0$ and $\alpha > 0$. Then $t \to \mathfrak{a}(t)$ is continuous and the Lie algebras $\mathfrak{a}(t)$ are pairwise nonisomorphic.
As a vector space, \( \mathfrak{a}(t) \cong \mathbb{R}^3 \), so we can choose the standard basis \( e_1, e_2, e_3 \). With this choice of basis, the commutation relations of Example 1 are

\[
[e_1, e_2] = (1 + t) \lambda e_2, \quad [e_1, e_3] = (1 + t + \alpha t^2) \mu e_3, \quad [e_2, e_3] = 0 \quad (2.1)
\]

In terms of the structure constants, \( f_{ij}^k = a(j) \delta_j^k \) for \( a(2) = (1 + t) \lambda, a(3) = (1 + t + \alpha t^2) \mu \), and \( f_{2j}^1 = -\delta_j^1 \delta_j^2 a(2) \).

For any finite-dimensional Lie algebra \( \mathfrak{g} \), there exists a unique (up to isomorphism) connected, simply-connected Lie group \( G \) having \( \mathfrak{g} \) as its tangent algebra. It follows that the classification of simply-connected Lie groups reduces to the classification of the corresponding Lie algebras. In particular, Example 1 proves the existence of pairwise nonisomorphic continuous deformations of Lie groups.

3. Lie Algebra Cohomology and BRST Theory

We review the connection between Lie algebra cohomology and quantization of gauge theories, setting notation for later sections.

Let \( \{e_\alpha\} \) be a basis of \( V \) and \( \{T_i\} \) of \( \mathfrak{g} \), with \( f_{ij}^k \) the structure constants, so that \( [T_i, T_j] = f_{ij}^k T_k \). Let \( t_i \) be the generators in the representation \( \rho \), i.e. \( t_i = \rho(T_i) \). A \( V \)-valued \( n \)-cochain is an antisymmetric linear map \( u : \mathfrak{g}^n \rightarrow V \) specified by

\[
u(T_{i_1}, \ldots, T_{i_n}) = u_{\alpha_{i_1} \ldots i_n}^\alpha e_\alpha, \quad (n \geq 0)
\]

with coefficients \( u_{i_1 \ldots j}^\alpha \) being totally antisymmetric in lower indices. The coboundary operator \( \delta \) increases the valence of arbitrary \( n \)-cochain \( u \) by one, sending it to \( (n+1) \)-cochain \( \delta u \). Explicitly for \( x_k \in \mathfrak{g} \), we have

\[
\delta u(x_1, \ldots, x_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} \rho(x_k)u(x_1, \ldots, \hat{x}_k, \ldots, x_{n+1}) + \sum_{j<k} (-1)^{j+k} u([x_j, x_k], x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}). \quad (3.2)
\]

In component form,

\[
(\delta u)^\alpha_{i_1 \ldots i_{n+1}} = \sum_{k=1}^{n+1} (-1)^{k+1} (t_{ik})^\alpha_{\beta_{i_1 \ldots \hat{i}_k \ldots i_{n+1}}} + \sum_{j<k} (-1)^{j+k} f_{ij}^m \alpha u_{mi_1 \ldots \hat{i}_j \ldots \hat{i}_k \ldots i_{n+1}}. \quad (3.3)
\]

The coboundary operator is nilpotent, \( \delta \circ \delta = 0 \). The corresponding cohomology groups are denoted by \( H^n(\mathfrak{g}, V) \). The exterior derivative provides one example of such a coboundary operator (with \( \mathfrak{g} \) being the Lie algebra \( \mathcal{X}(M) \) of vector fields, and \( V \) the algebra of functions on a manifold \( M \)).

Due to the antisymmetry of \( (3.3) \), \( n \)-cochains may be considered as elements of \( \wedge^n \mathfrak{g}^* \otimes V \). A particularly convenient notation for cochains is to view the basis
elements \( \{ c^i \} \) of \( g^* \) as odd (anticommuting) objects: \( c^i \wedge c^j = c^j c^i = -c^i c^j \), etc. The elements \( c^i \) act on the basis vectors according to:
\[
c^i \ldots c^m (T_{j_1}, \ldots, T_{j_m}) = \det (\delta^i_{j_j}),
\]
Any product of \( n + 1 \) such elements must vanish by symmetry, so the number of independent products of Grassman generators is given by
\[
1 + n + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n-1} + \binom{n}{n} = (1+1)^n
\]
We denote the algebra generated by these \( 2^n \) elements by \( \mathcal{F} \), and we remark that \( \mathcal{F} \) has the structure of a fermionic Fock space, with multiplication by the various \( c^i \)'s playing the role of creation operators for different species of fermions.

In this notation, a general \( n \)-cochain takes the form
\[
u = \frac{1}{n!} \nu_{\alpha_1 \ldots \alpha_n} c^{i_1} \ldots c^{i_n} e_{\alpha},
\]
and the coboundary operator \( \delta \) assumes the BRST-like form
\[
\delta = c^i t_i - \frac{1}{2} f^i_{jk} c^j c^k \frac{\partial}{\partial c^k}.
\]
As an odd differential operator, \( \delta \) acts upon each \( c \) and \( e \) in (3.5) by
\[
\delta e_\alpha = c^i (t_i)_\alpha e_\beta, \quad \delta c^i = -\frac{1}{2} f^i_{jk} c^j c^k, \quad \delta \circ c^i = \delta c^i - c^j \circ \delta.
\]
Viewing \( c^i \in g^* \) as left invariant forms subject to the Maurer-Cartan equation, Eq. (3.6) becomes
\[
\delta = c^i t_i \quad \text{and} \quad \delta c^i = -\frac{1}{2} f^i_{jk} c^j c^k,
\]
and we deduce \( \delta^2 = 0 \) very easily:
\[
(c^i t_i + d)^2 = d^2 + t_i (d \circ c^i + c^j d) + c^i c^k t_j t_k = t_i d c^i + \frac{1}{2} c^i c^k [t_j, t_k] = 0.
\]
The relations (3.8) show that the BRST differential is a homological perturbation of the differential of the Koszul-Tate complex.

4. Faddeev-Popov Quantization

Consider a Yang-Mills theory, take \( g \) to be the Lie algebra of the gauge group, and recall that \( t_i \) denote generators of \( g \). In this context, the operator (3.7) is known as the minimal nilpotent extension of \( c^i t_i \). Other nilpotent extensions are possible; a conventional form of the BRST operator in gauge theory is:
\[
\delta = c^i t_i - \frac{1}{2} f^i_{jk} c^j c^k \frac{\partial}{\partial c^k} + b_i \frac{\partial}{\partial \bar{c}_i}.
\]
where $t_i$ are generators of gauge transformations of original (matter and gauge) fields $\varphi$ present in the classical theory, $c^i$ and $\bar{c}_i$ are ghosts and antighosts, and $b_i$ are auxiliary boson fields. We treat $c^i$ and $\bar{c}_i$ as independent fermionic fields, of ghost number +1 and −1 respectively. The operator (4.1) is nilpotent, odd, of ghost number +1, and acts on the fields giving their variations under BRST transformations:

\[
\begin{align*}
\delta \varphi &= c^i t_i \varphi \\
\delta c^i &= -\frac{1}{2} f_{jk}^i c^j c^k \\
\delta \bar{c}_i &= b_i \\
\delta b_i &= 0
\end{align*}
\]

An object is called BRST invariant when it is annihilated by $\delta$. In the BRST gauge fixing procedure, the gauge-fixed Lagrangian has to be chosen in the form

\[ L = L_{\text{inv}} + \delta \Lambda \]

with $L_{\text{inv}}(\varphi)$ invariant (and $\Lambda(\varphi, c, \bar{c}, b)$ non-invariant) under (4.2), i.e., under local gauge transformations with ghosts $c^i$ playing the role of gauge parameters. Also, the gauge fixing term $\Lambda$ should be of ghost number -1. Due to $\delta^2 = 0$, gauge invariance of $L_{\text{inv}}$ means BRST invariance of the total Lagrangian $L$, and vice versa. For instance, the Faddeev-Popov Ansatz is generally recovered by the choice

\[ \Lambda = \Phi^i \bar{c}_i \quad \Rightarrow \quad L = L_{\text{inv}} + c^j t_j \Phi^i \bar{c}_i + \Phi^i b_i \]

where $\Phi^i(\varphi)$ are gauge-fixing conditions. Subsequent functional integration over $b$ yields the delta-function of $\Phi$ (fixing the gauge), and functional integration over $c, \bar{c}$ yields the determinant $\det(t_j \Phi^j)$ of variations under the gauge transformations, introduced by Faddeev and Popov.

We now discuss the relationship of physical operators and states to the BRST invariant objects. First, consider a functional integral of the type

\[ Z_\delta = \int D\psi Y \delta X e^{iL}, \]

where $\psi$ indexes all fields in the theory (including ghosts), $Y$ is BRST invariant and $X$ arbitrary. Then, assuming BRST invariance of functional measure (no anomalies), we see that

\[ Z_\delta = \int \delta(D\psi Y X e^{iL}) = 0, \]

as a (zero) result of a mere change of integration variables. But this simple fact immediately implies that amplitudes which we would like to call physical,

\[ Z_{\text{ph}} = \int D\psi Y e^{i(L_{\text{inv}} + \delta \Lambda)}, \]
do not depend on the choice of gauge fixing term $\Lambda$, because any change in $\Lambda$ produces additional contributions of the form (4.8).

Thus physical operators (or states) are classes of objects, of the form $Y + \delta X$, with $Y$ being BRST invariant, $\delta Y = 0$. Physical objects are associated with cohomology classes of the BRST operator $\delta$. Objects of the type $\delta X$ (those in zero cohomology) are called spurious. As a rule, physical field $Y$ is also required to be of definite ghost number (usually, 0).

5. Lie Algebra Deformations and the BRST Operator

The operator

$$Q = c^i t_i - \frac{1}{2} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k}, \quad (5.1)$$

depends on the Lie algebra $\mathfrak{g}$ through its structure constants $f_{ij}^k$, and on a $\mathfrak{g}$-module $V$ in which the elements of some basis for $\mathfrak{g}$ act via representation matrices $t_i$. Thus $Q$ is an algebraic object which is completely determined by an ordered pair $(\mathfrak{g}, V)$ where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $V$ is a $\mathfrak{g}$-module.

Let $\mathfrak{g}(t)$ denote a deformation of the type considered in Section 1. We propose that invariants of deformations $\mathfrak{g}(t)$ can be constructed using the operators (5.1). Obvious invariants such as the trace do not yield interesting results, since the BRST operator (5.1) is traceless in a large class of examples. However, more sophisticated invariants inspired by quantum field theory, such as heat kernel-regularized traces of combinations of $Q$ with other operators, are known to exist. A detailed study of one class of such invariants is due to Jaffe [1, 2], and we consider the application of Jaffe’s theory to Lie algebras in Section 6.

The scheme $\mathcal{M}_n$ discussed in Section 1 is, as one would expect, quite a complicated object; by definition it contains all Lie algebras of dimension $n$. One way of elucidating some of its structure is to identify path-components using invariants from quantum field theory.

6. Quantum Invariants

Let $\mathcal{H}$ be a Hilbert space, with an operator $\gamma$ on $\mathcal{H}$ that is both self-adjoint and unitary. Such an operator is called a $\mathbb{Z}_2$-grading, because $\mathcal{H}$ splits into $\pm 1$ eigenspaces for $\gamma$, i.e. $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Let $X = \{a_0, \ldots, a_n\}$ be a set of operators on $\mathcal{H}$ which we will call vertices. The heat-kernel regularization density of this set of vertices is defined to be

$$X(s) = \begin{cases} 
  b_0 e^{-s_0 Q^2} b_1 e^{-s_1 Q^2} \ldots b_n e^{-s_n Q^2}, & \text{when every } s_j > 0 \\
  0, & \text{otherwise}
\end{cases}$$
and the Radon transform is \( \hat{X}(\beta) = \int X(s)[d^n s]_\beta \), where \([d^n s]_\beta\) is Lebesgue measure on the hyperplane \( \sum s_j = \beta \). Let \( U(g) \) be a continuous unitary representation of a compact Lie group \( G \) on \( \mathcal{H} \). The expectation of a heat-kernel regularization \( \hat{X} \) is defined by

\[
\big\langle \hat{X}; g \big\rangle = \text{Tr}(\gamma U(g)\hat{X})
\]

\( Q \) is a self-adjoint operator commuting with \( U(g) \), called the supercharge, which is odd with respect to \( \gamma \) and we assume that \( e^{-\beta Q^2} \) is trace class \( \forall \beta > 0 \).

**Definition 3.** The JLO cochain is the expectation whose \( n \)th component is defined by

\[
\tau_{\text{JLO}}^n(a_0, \ldots, a_n; g) = \langle a_0, da_1, \ldots, da_n; g \rangle
\]

The most general structure necessary to define a JLO cocycle is a \( \Theta \)-summable fractionally differentiable structure, which is defined as a sextuple

\[
\{ \mathcal{H}, Q, \gamma, G, U(g), \mathfrak{A} \},
\]

where \( \mathcal{H}, Q, \gamma, G, U(g) \) are all as defined above, and \( \mathfrak{A} \) is an algebra of operators (actually a subalgebra of an interpolation space; see [1]) which is pointwise invariant under \( \gamma \) and for which \( U(g)\mathfrak{A}U(g)^* \subset \mathfrak{A} \). Further references to “cyclic cohomology” refer to cohomology of the algebra \( \mathfrak{A} \).

The JLO expectation is a cocycle in the sense of entire cyclic cohomology theory, its application to physics is that pairing an operator with a family of JLO cocycles, each coming from a fractionally-differentiable structure on \( \mathcal{H} \), gives a natural generalization of a well-known equivariant index (6.1) in supersymmetric physics. Moreover, algebraic properties of cyclic cohomology imply that the equivariant index is a homotopy invariant. The equivariant index is defined by

\[
\mathfrak{Z}^{Q(\lambda)}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \text{Tr}(\gamma U(g)a e^{-Q(\lambda)^2 + it da}) dt
\]

(6.1)

and it is shown in [1] that

\[
\mathfrak{Z}^{Q(\lambda)}(a; g) = \langle \tau^{\text{JLO}}(\lambda), a \rangle
\]

and

\[
d\lambda \langle \tau^{\text{JLO}}(\lambda), a \rangle = 0
\]

All known cocycles in entire cyclic cohomology are of the form \( \tau^{\text{JLO}} \) for some choice of a \( \Theta \)-summable fractionally differentiable structure.

A simple proof that \( \mathfrak{Z}^{Q(\lambda)}(a; g) \) is a geometric invariant is given in [2], however, the proof relies on analytic hypotheses which have to be checked in each case of interest. Following [2], we note that the following regularity hypotheses are sufficient to establish \( \mathfrak{Z}^{Q(\lambda)}(a; g) \) as an invariant.
1. The operator \( Q \) is self-adjoint operator on \( \mathcal{H} \), odd with respect to \( \gamma \), and \( e^{-\beta Q^2} \) is trace class for all \( \beta > 0 \).

2. For \( \lambda \in J \), where \( J \) is an open interval on the real line, the operator \( Q(\lambda) \) can be expressed as a perturbation of \( Q \) in the form

\[
Q(\lambda) = Q + W(\lambda)
\]

Each \( W(\lambda) \) is a symmetric operator on the domain \( \mathcal{D} = C^\infty(Q) \).

3. Let \( \lambda \) lie in any compact subinterval \( J' \subset J \). The inequality

\[
W(\lambda)^2 \leq aQ^2 + bI ,
\]

holds as an inequality for forms on \( \mathcal{D} \times \mathcal{D} \). The constants \( a < 1 \) and \( b < \infty \) are independent of \( \lambda \) in the compact set \( J' \).

4. Let \( R = (Q^2 + I)^{-1/2} \). The operator \( Z(\lambda) = RW(\lambda)R \) is bounded uniformly for \( \lambda \in J' \), and the difference quotient

\[
\frac{Z(\lambda) - Z(\lambda')}{\lambda - \lambda'}
\]

converges in norm to a limit as \( \lambda' \to \lambda \in J' \).

5. The bilinear form \( d_\lambda a \) satisfies the bound

\[
\| R^\alpha d_\lambda a R^\beta \| < M ,
\]

with a constant \( M \) independent of \( \lambda \) for \( \lambda \in J' \). Here \( \alpha, \beta \) are non-negative constants and \( \alpha + \beta < 1 \).

**Corollary 1 (Ref. [2]).** If a family \( Q(\lambda) \) satisfies conditions 1-5, then the numerical quantity

\[
Z^{Q(\lambda)}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \text{Tr}(\gamma U(g)a e^{-Q(\lambda)^2 + it \alpha d \alpha}) dt
\]

does not depend upon \( \lambda \), and hence is constant on path-components.

In the following theorem, we check the hypotheses 1-5 for the BRST operator.

**Theorem 1.** A family \( Q(\lambda) \) of BRST operators formed by a continuous deformation of Lie group structures satisfies hypotheses 1-5. The numerical quantity

\[
Z^{Q(\lambda)}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \text{Tr}(\gamma U(g)a e^{-Q(\lambda)^2 + it \alpha d \alpha}) dt
\]

is an invariant.
Proof of Theorem 1. Since $Q(\lambda)^2 = 0$, $e^{-\beta Q^2}$ must be trace class. If the grading $\gamma$ is by ghost number, then to see that $Q$ is $\gamma$-odd, note that each term in $Q$ increases total ghost number by one. This proves 1.

2 is also easy, setting $W(\lambda) = Q(\lambda) - Q$ and $Q = Q(0)$. Each $Q(\lambda)$ is symmetric, thus $W$ is also.

To prove 3, we wish to show that there exist $\lambda$-independent constant $b < \infty$ such that $bI + \{Q(\lambda), Q\}$ is a positive operator for all $\lambda \in J'$. Let $E_{\min}(\cdot)$ denote the lowest eigenvalue of a finite-dimensional matrix. $E_{\min}(\{Q(\lambda), Q\})$ is a continuous function of $\lambda$ over the compact interval $J'$, so it is bounded below and above. We choose

$$b > \left| \min_{\lambda} E_{\min}(\{Q(\lambda), Q\}) \right|$$

In property 4, nilpotency requires $R = I$, and $Z(\lambda) = W(\lambda)$. We then wish to show that

$$(\lambda - \lambda')^{-1}(Q(\lambda) - Q(\lambda'))$$

converges in norm to a limit as $\lambda' \to \lambda$ in $J'$.

Therefore the difference quotient (6.4) certainly converges if the structure constants $f(\lambda)_{ij}^k$ are differentiable in $\lambda$.

The same assumption implies that $\lambda \to W(\lambda)$ is a differentiable map, therefore $\lambda \to \|W(\lambda)\|$ is uniformly bounded. The proof of 5 is similar. □

To simplify calculations, for the moment we take $U(g)$ to be a trivial representation. 1 Since $Q$ is nilpotent, the invariant is given by the integral of

$$\pi(\lambda, t) \equiv \text{Tr}(\gamma ae^{-it[Q(\lambda), a]})$$

with respect to a Gaussian measure in the variable $t$.

Let us compute $Q$ on a basis element of $F \otimes V$. The Maurer-Cartan equation for ghosts gives

$$Qc^k = -\frac{1}{2} f_{ij}^k c^i c^j$$

The product rule for the Grassmann derivative is

$$\frac{\partial}{\partial c^k}(c_1 c_2 \ldots c_r) = \delta_{ki_1} c_{i_2} \ldots c_{i_r} - \delta_{ki_2} c_{i_1} c_{i_3} \ldots c_{i_r} + \ldots + \frac{(-1)^{r-1}}{r!} \delta_{ki_r} c_{i_1} \ldots c_{i_{r-1}}$$

$$= \sum_p (-1)^{p-1} \delta_{ki_p} c_{i_1} \ldots \hat{c}_{i_p} \ldots c_{i_r}$$

1 It would be interesting to incorporate $U(g)$-dependent terms, since the Lie group $G$ which is the domain of the unitary representation $U$ can be different from the simply connected group(s) $G(\lambda)$ which generate the BRST transformation.
We consider a homogeneous element of $F \otimes V$, which takes the form $X = c^1 \ldots c^n \otimes v$ where $n$ is arbitrary. We compute

$$QX = (-1)^n c^1 \ldots c^n \otimes t_j[v] + \left(f^k_{ij} c^i c^j \sum_p (-1)^{p-1} \delta_{k,p} c^1 \ldots \hat{c}^i \ldots c^n \otimes v\right).$$

This shows how to find the matrix elements of $Q$ in any basis of $F \otimes V$.

Although finite-dimensional, from a computational standpoint the vector space $F \otimes V$ is usually rather large. In the case of dimension 3 algebras (c.f. Example [1]), we need three $c$’s, and the exterior algebra $F$ will be $2^3 = 8$ dimensional. If $V$ is the adjoint representation, $\dim(F \otimes V) = 24$. In practice, one may choose the “standard” basis of $F$ given by lexicographically ordered products of the $c_i$’s. On this space, left multiplication by $c^j$ is a linear operation and its matrix is identical to a permutation matrix up to factors of $\pm 1$ in the various matrix elements. Similarly, $\partial/\partial c^j$ is representable by a permutation matrix up to signs.

**Definition 4.** A monoid is a semigroup with identity. Define $G_N$ to be the set of all $N \times N$ matrices in which each row and each column has at most one nonzero entry, and that entry is $\pm 1$. Then $G$ is closed under products, but contains non-invertible elements (those with a zero row or column), and so it is a monoid.

Left multiplication by $c^j$ is a linear operation on $F$ with non-trivial kernel. Its matrix is given by a permutation matrix, except that there can be factors of $\pm 1$ in the various matrix elements, and there can be rows or columns with all zeros; $\partial/\partial c^j$ is representable by another such matrix.

**Lemma 1.** Let $c^j \ (j = 1, \ldots, n)$ be “Faddeev Popov Ghosts” in the sense considered above, let $A$ denote the $N = 2^n$ dimensional algebra generated by $\{c^j\}$ over the complex numbers, and let $C^j$ denote the operator of left multiplication by $c^j$ on $A$. Then the natural representation

$$\rho : R \longrightarrow M_{N \times N}(\mathbb{C}),$$

takes values in the monoid $G_N$, where

$$R = \mathbb{C} \left[ C^1, \ldots, C^n, \frac{\partial}{\partial c^1}, \ldots, \frac{\partial}{\partial c^n} \right].$$

Determination of the $\dim(F \otimes V)^2$-dimensional matrix of $Q$ is approachable by computer algebra methods. A number of commonly available computer algebra systems contain facilities for working with finitely generated algebras defined by generators and relations, so the system facilitates the creation of subroutines which represent the ghost algebra $A$ and the operators in the ring $R$. In writing such a program, it is extremely useful to keep Lemma [1] in mind. In performing these
calculations for a 24-dimensional example, we discovered that all of the matrices $Q(\lambda)$ for different values of $\lambda$ have the same Jordan canonical form. We have not yet determined whether this must always happen for differentiable families of BRST operators.

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