Gradient Descent Ascent in Min-Max Stackelberg Games

Denizalp Goktas
Brown University
Computer Science
Providence, Rhode Island, USA
denizalp_goktas@brown.edu

Amy Greenwald
Brown University
Computer Science
Providence, Rhode Island, USA
amy_greenwald@brown.edu

ABSTRACT

Min-max optimization problems (i.e., min-max games) have attracted a great deal of attention recently as their applicability to a wide range of machine learning problems has become evident. In this paper, we study min-max games with dependent strategy sets, where the strategy of the first player constrains the behavior of the second. Such games are best understood as sequential, i.e., Stackelberg, games, for which the relevant solution concept is Stackelberg equilibrium, a generalization of Nash. One of the most popular algorithms for solving min-max games is gradient descent ascent (GDA). We present a straightforward generalization of GDA to min-max Stackelberg games with dependent strategy sets, but show that it may not converge to a Stackelberg equilibrium. We then introduce two variants of GDA, which assume access to a solution oracle for the optimal Karush Kuhn Tucker (KKT) multipliers of the games’ constraints. We show that such an oracle exists for a large class of convex-concave min-max Stackelberg games, and provide proof that our GDA variants with such an oracle converge in $O(\varepsilon^2)$ iterations to an $\varepsilon$-Stackelberg equilibrium, improving on the most efficient algorithms currently known which converge in $O(\varepsilon^4)$ iterations. We then show that solving Fisher markets, a canonical example of a min-max Stackelberg game, using our novel algorithm, corresponds to buyers and sellers using myopic best-response dynamics in a repeated market, allowing us to prove the convergence of these dynamics in $O(\varepsilon^2)$ iterations in Fisher markets. We close by describing experiments on Fisher markets which suggest potential ways to extend our theoretical results, by demonstrating how different properties of the objective function can affect the convergence and convergence rate of our algorithms.

KEYWORDS
Equilibrium Computation; Learning in Games; Market Dynamics

1 INTRODUCTION

Min-max optimization problems (i.e., zero-sum games) have attracted a great deal of attention recently as their applicability to a wide range of machine learning problems has become evident. Applications of min-max games include, but are not limited to, generative adversarial networks [57], fairness in machine learning [9, 17, 40, 59], generative adversarial imitation learning [7, 27], reinforcement learning [10], adversarial learning [64], and statistical learning (e.g., learning parameters of exponential families) [9].

These applications often require solving min-max games, which are constrained min-max optimization problems: i.e.,
\[
\min_{x \in X} \max_{y \in Y} f(x, y),
\]
where $f : X \times Y \rightarrow \mathbb{R}$ is continuous, and $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are non-empty and compact. In this paper, we focus on convex-concave min-max games, in which $f$ is convex in $x$ and concave in $y$. In the special case of convex-concave objective functions, the seminal minmax theorem holds:
\[
\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)
\]
This theorem guarantees the existence of a saddle point, i.e., a point that is simultaneously a minimum of $f$ in the $x$-direction and a maximum of $f$ in the $y$-direction, which allows us to interpret the optimization problem as a simultaneous-move, zero-sum game, where $y^* \ (\text{resp. } x^*)$ is a best-response of the outer (resp. inner) player to the other’s action $x^* \ (\text{resp. } y^*)$, in which case a saddle point is also called a minmax point or a Nash equilibrium.

More recently, Goktas and Greenwald [24] have considered the more general problem of solving min-max Stackelberg games, which are constrained min-max optimization problems with dependent feasible sets: i.e.,
\[
\min_{x \in X} \max_{y \in Y} g(x, y) \geq 0 f(x, y),
\]
where $f : X \times Y \rightarrow \mathbb{R}$ is continuous, $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are non-empty and compact, and $g(x, y) = (g_1(x, y), \ldots, g_l(x, y))^T$ with $g_i : X \times Y \rightarrow \mathbb{R}$. These authors observe that the minmax theorem does not necessarily hold assuming dependent feasible sets [24]. As a result, such games are more appropriately viewed as sequential, i.e., Stackelberg, games, where the outer player chooses $x \in X$ before the inner player responds with their choice of $g(x) \in Y$ s.t. $g(x, y(x)) \geq 0$. In these games, the outer player’s value function $V_X : X \rightarrow \mathbb{R}$ is defined as $V_X(x) = \max_{y \in Y} g(x, y) \geq 0 f(x, y)$. This function represents the outer player’s loss, assuming the inner player chooses a feasible best response, so it is the function the outer player seeks to minimize. The inner player’s value function, $V_Y : X \rightarrow \mathbb{R}$, which they seek to maximize, is simply the objective function given the outer player’s action: i.e., $V_Y(y, x) = f(x, y)$.

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1When a Nash equilibrium exists in a min-max Stackelberg game, the game reduces to a simultaneous-move game and the Stackelberg equilibrium coincides with the Nash.

2One could also view such games as pseudo-games (also known as abstract economies) [19, 20], in which the players move simultaneously under the unreasonable assumption that the moves they make will satisfy the game’s dependency constraints. Under this view, the relevant solution concept is generalized Nash equilibrium.
One of the most popular algorithms for solving min-max games is gradient descent ascent (GDA), which at each iteration carries out gradient descent for $x$, the variable being minimized, and gradient ascent for $y$, the variable being maximized. Although Goktas and Greenwald provide first-order methods (FOMs) to solve min-max Stackelberg games with a convex-concave objective function in polynomial time, their methods are nested GDA algorithms, which require two nested gradient update loops. As such, they do not have the flavor of simultaneous GDA, i.e., an algorithm that simultaneously runs a gradient descent step for the outer player and a gradient ascent step for the inner player. In this paper, we investigate the behavior of simultaneous GDA variants in min-max Stackelberg games. The following example—which relies on Goktas and Greenwald’s insight that the direction of steepest descent in Stackelberg games is not given by the gradient of the objective function, but rather by the gradient of the outer player’s value function—demonstrates how traditional simultaneous GDA algorithms can fail to converge to Stackelberg equilibria.

**Example 1.1.** Consider the following min-max Stackelberg game:

$$\min_{x \in [-1,1]} \max_{y \in [-1,1]} (1-x+y) = x^2 + y + 1.$$  

The optimal solution (i.e., the Stackelberg equilibrium) of this game is given by $x = 1/2, y^* = 1/2$. Consider the following simultaneous GDA algorithm, similar to that given by Nedic and Ozdaglar [43] and Goktas and Greenwald [24],

$$\begin{align*}
x(t+1) &= \Pi_{x \in [-1, 1]} \left[ x(t) - \nabla_x f(x(t), y(t)) \right], \\
y(t+1) &= \Pi_{y \in [y(t) + \nabla_y f(x(t), y(t))]} \left[ y(t) - \nabla_y f(x(t), y(t)) \right].
\end{align*}$$

Applied to this game, this algorithm yields the following update rules:

$$\begin{align*}
x(t+1) &= \Pi_{x \in [-1, 1]} \left[ x(t) - 2x(t) \right], \\
y(t+1) &= \Pi_{y \in [-1, 1]} \left[ y(t) + 1 \right].
\end{align*}$$

Starting at $x(0) = 0, y(0) = 0$, it then proceeds as follows: $x(1) = 0, y(1) = 1; x(2) = 0, y(1) = 1$; and so on, thus converging to a point which is not a Stackelberg equilibrium. Indeed, the algorithm is not even stable when initialized at the Stackelberg equilibrium.

In this paper, we introduce two variants of simultaneous GDA, hereafter GDA unless otherwise noted, that converge in polynomial time to Stackelberg equilibria in a large class of min-max Stackelberg games. Our first algorithm is a deterministic algorithm whose average iterate converges in $O(1/\varepsilon^2)$ iterations to an $\varepsilon$-Stackelberg equilibrium in convex-strictly-concave min-max Stackelberg games under standard smoothness assumptions. Our second algorithm is a randomized algorithm whose expected output converges to an $\varepsilon$-Stackelberg equilibrium in $O(1/\varepsilon^2)$ iterations in convex-concave min-max Stackelberg games under the same assumptions. The iteration complexity of these algorithms improve on the nested gradient descent ascent algorithm provided by Goktas and Greenwald [24], which computed an equilibrium in $O(1/\varepsilon^2)$ iterations, and thereby show that the same convergence rate as that of simultaneous GDA assuming independent strategy sets [43] can be achieved.

Having developed GDA algorithms for the dependent strategy set setting, we conclude by using them to compute competitive equilibria in Fisher markets, which have been shown to be instances of min-max Stackelberg games [24], so that their Stackelberg equilibria coincide with the competitive equilibria. Applied to the computation of competitive equilibria in Fisher markets, our GDA algorithms correspond to myopic best-response dynamics. In a related dynamic price-adjustment process called tâtonnement [72], which also converges to competitive equilibria, sellers adjust their prices incrementally, while buyers respond optimally to the sellers’ price adjustments. Our dynamics give rise to a novel tâtonnement process in which both buyers and sellers exhibit bounded rationality [63], with sellers and buyers adjusting their prices and demands incrementally, respectively.

Finally, we run experiments with our randomized GDA algorithm which suggest that our randomized algorithm can be derandomized, because in all our experiments the average iterates converge to a competitive equilibrium. Our experiments suggest avenues for future work, namely investigating how varying the degree of smoothness of the objective function can impact convergence rates.

**Related Work.** Our model of min-max Stackelberg games seems to have first been studied by Wald, under the posthumous name of Wald’s maximin model [71]. A variant of Wald’s maximin model is the main paradigm used in robust optimization, a fundamental framework in operations research for which many methods have been proposed [3, 29, 53]. Shimizu and Aiyoshi [61, 62] proposed the first algorithm to solve min-max Stackelberg games via a relaxation to a constrained optimization problem with infinitely many constraints, which nonetheless seems to perform well in practice. More recently, Segundo et al. [60] proposed an evolutionary algorithm for these games, but they provided no guarantees. As pointed out by Postek and Shtern, all prior methods either require oracles and are stochastic in nature [3], or rely on a binary search for the optimal value, which can be computationally complex [29]. The algorithms we propose in this paper circumvent the aforementioned issues and can be used to solve a large class of convex robust optimization problems in a simple and efficient manner.

Much progress has been made recently in solving min-max games with independent strategy sets, both in the convex-concave case and in the non-convex-non-concave case. For the former case, when $f$ is $\mu_x$-smoothly-convex in $x$ and $\mu_y$-smoothly-convex in $y$, gradient-descent-ascent (GDA)-based methods, compute a solution in $O(\sqrt{\mu_y} + \mu_x)$ iterations [1, 23, 38, 41, 46, 68] and $O(\sqrt{\mu_y})$ iterations [1, 31, 38, 73]. For the special case where $f$ is $\mu_x$-smoothly convex in $x$ and linear in $y$ there exist methods that converge to an $\varepsilon$-approximate solution in $O(\sqrt{\mu_y})$ iterations [26, 34, 55]. When the strong concavity or linearity assumptions of $f$ on $y$ are dropped, and $f$ is assumed to be $\mu_x$-smoothly-convex in $x$ but only concave in $y$, there exist methods that converge to an $\varepsilon$-approximate solution in $O(\mu_y/\varepsilon^2)$ iterations [38, 51, 55] with a lower bound of $O(\sqrt{\mu_y} \varepsilon)$ iterations. When $f$ is simply assumed to be convex-concave, a multitude of well-known first-order methods exist that solve for an $\varepsilon$-approximate solution with $O(1/\varepsilon^2)$ iteration complexity [44, 45, 69]. When $f$ is assumed to be non-convex-$\mu_y$-smoothly-convex, there exist methods that converge to various solution concepts, all in $O(\varepsilon^{-2})$ iteration [33, 37, 39, 54, 58]. When $f$ is non-convex-non-concave various algorithms have been proposed to compute first-order Nash equilibrium [39, 49], with at best an upper bound of

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1 All notation is defined in the next section.
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\(O(e^{-2.9})\) iterations \([38, 50]\). When \(f\) is non-convex-non-concave and the desired solution concept is a "local" Stackelberg equilibrium, there exist many algorithms to compute a solution \([33, 37, 54]\), with the most efficient ones converging to an \(\epsilon\)-approximate solution in \(O(e^{-3})\) iterations \([38, 56, 67]\).

Extensive-form games in which players’ strategy sets can depend on other players’ actions have been studied by Davis et al. \([13]\) assuming payoffs to be bilinear, and by Farina et al. \([21]\) for another specific class of convex-concave payoff functions. Fabiani et al. \([18]\) and Kebrabse and Iannelli \([35]\) study more general settings than ours, namely non-zero-sum Stackelberg games with more than two players. Both sets of authors derive convergence guarantees assuming specific payoff structures, but their algorithms do not converge in polynomial time.

Min-max Stackelberg games naturally model various economic settings. They are related to abstract economies, first studied by Arrow and Debreu \([2]\); however, the solution concept that has been the focus of this literature is generalized Nash equilibrium \([19, 20]\), which, like Stackelberg, is a weaker solution concept than Nash, but which makes the arguably unreasonable assumption that the players move simultaneously and nonetheless satisfy the constraint dependencies on their strategies imposed by one another’s moves.

Optimal auction design problems can be seen as min-max Stackelberg games. Duetting et al. \([16]\) propose a neural network architecture called RegretNet to solve auctions; however, as their objectives can be non-convex-concave, their guarantees do not apply.

In this paper, we observe that solving for the competitive equilibrium of a Fisher market can also be seen as solving a (convex-concave) min-max Stackelberg game. The study of the computation of competitive equilibria in Fisher markets was initiated by Devanur et al. \([14]\), who provided a polynomial-time method for the case of linear utilities. Jain et al. \([32]\) subsequently showed that a large class of Fisher markets could be solved in polynomial-time using interior point methods. Recently, Gao and Kroer \([22]\) studied an alternative family of first-order methods for solving Fisher markets (only; not min-max Stackelberg games more generally), assuming linear, quasilinear, and Leontief utilities, as such methods can be more efficient when markets are large.

2 PRELIMINARIES

**Notation.** We use Roman uppercase letters to denote sets (e.g., \(X\)), bold lowercase letters to denote vectors (e.g., \(p\)), and Roman lowercase letters to denote scalar quantities (e.g., \(c\)). We denote the \(i\)th row vector of a matrix (e.g., \(X\)) by the corresponding bold lowercase letter with subscript \(i\) (e.g., \(x_i\)). Similarly, we denote the \(j\)th entry of a vector (e.g., \(p\) or \(x_j\)) by the corresponding Roman lowercase letter with subscript \(j\) (e.g., \(p_j\) or \(x_{ij}\)). We denote the vector of ones of size \(n\) by \(1_n\). We denote the set of integers \(\{1, \ldots, n\}\) by \([n]\), the set of natural numbers by \(\mathbb{N}\), the set of positive natural numbers by \(\mathbb{N}_+\), the set of real numbers by \(\mathbb{R}\), the set of non-negative real numbers by \(\mathbb{R}_+\), and the set of strictly positive real numbers by \(\mathbb{R}_{++}\). We denote the orthogonal projection operator onto a convex set \(C\) by \(\Pi_C\), i.e., \(\Pi_C(x) = \arg \min_{y \in C} ||x - y||^2\).

**Problem Definition.** A min-max game with dependent strategy sets, denoted \((X, Y, f, g)\), is a two-player, zero-sum game, where one player, who we call the outer, or \(x\), (resp. inner, or \(y\)) player, is trying to minimize their loss (resp. maximize their gain), defined by a continuous objective function \(f : X \times Y \rightarrow \mathbb{R}\), by taking an action from their strategy set \(X \subset \mathbb{R}^n\) (resp. \(Y \subset \mathbb{R}^m\)) s.t. \(g(x, y) \geq 0\), where \(g(x, y) = (g_1(x, y), \ldots, g_d(x, y))^\top\) with \(g_d : X \times Y \rightarrow \mathbb{R}\). A strategy profile \((x, y) \in X \times Y\) is said to be feasible if for all \(k \in [d]\), \(g_k(x, y) \geq 0\). The function \(f\) maps a pair of feasible actions taken by the players \((x, y) \in X \times Y\) to a real value (i.e., a payoff), which represents the loss (resp. the gain) of the outer (resp. inner) player. A min-max game is said to be convex-concave if the objective function \(f\) is convex-concave.

One way to see this game is as a Stackelberg game, i.e., a sequential game with two players, where WLOG, we assume that the minimizing player moves first and the maximizing player moves second. The relevant solution concept for Stackelberg games is the Stackelberg equilibrium: A strategy profile \((x^*, y^*) \in X \times Y\) s.t. \(g(x^*, y^*) \geq 0\) is an \((\epsilon, \delta)\)-Stackelberg equilibrium if

\[
\max_{y \in Y} \min_{x \in X} \left( f(x, y) - \delta \right) \leq \min_{y \in Y} \max_{x \in X} \left( f(x, y) + \epsilon \right).
\]

Intuitively, an \((\epsilon, \delta)\)-Stackelberg equilibrium is a point at which the outer (resp. inner) player’s payoff is no more than \(\epsilon\) (resp. \(\delta\)) away from its optimum. A \((0, 0)\)-Stackelberg equilibrium is guaranteed to exist in min-max Stackelberg games \([24]\). Note that when \(g(x, y) \geq 0\) for all \((x, y) \in X \times Y\), the game reduces to a min-max game, for which, by the min-max theorem, a Nash equilibrium exists \([47]\).

**Mathematical Preliminaries.** Given \(A \subset \mathbb{R}^n\), the function \(f : A \rightarrow \mathbb{R}\) is \(f_l\)-Lipschitz-continuous if \(\forall x_1, x_2 \in A, \|f(x_1) - f(x_2)\| \leq f_l \|x_1 - x_2\|\). If the gradient of \(f\), \(\nabla f\), is \(f_l\)-Lipschitz-continuous, we refer to \(f\) as \(f_l\)-Lipschitz-smooth. A function \(f : A \rightarrow \mathbb{R}\) is \(\mu\)-strongly convex if \(f(x_1) \geq f(x_2) + \langle \nabla f(x_2), x_1 - x_2 \rangle + \mu/2 \|x_1 - x_2\|^2\), and \(\mu\)-strongly concave if \(-f\) is \(\mu\)-strongly convex.

3 (SIMULTANEOUS) GDA

In this section, we explore GDA algorithms for min-max Stackelberg games. We prove that a min-max Stackelberg game with dependent strategy sets is equivalent to a three-player game with independent strategy sets, with the Lagrangian as the payoff function and a third player who chooses the optimal Karush Kuhn Tucker (KKT) multipliers \([36]\). This theorem provides a straightforward path to generalizing GDA to the dependent strategy sets setting. All the results in this paper are based on the following assumptions:

**Assumption 3.1.** 1. (Slater’s condition \([65, 66]\)) \(\forall x \in X, \exists \bar{y} \in Y\) s.t. \(g_k(x, \bar{y}) > 0\) for all \(k \in [d]\); 2. \(f, g_1, \ldots, g_d\) are continuous and convex-concave; and 3. \(\nabla f, \nabla g_1, \ldots, \nabla g_d\) are continuous in \((x, y)\).

We note that these assumptions are in line with previous work geared towards solving min-max Stackelberg games with first-order methods (FOMs) \([24]\). Part I of Assumption 3.1, Slater’s condition, is a constraint qualification condition which is necessary to derive the optimality conditions for the inner player’s payoff maximization problem. This condition is a standard constraint qualification in the convex programming literature \([5]\); without it the problem

\[\text{We use the terminology "min-max game" to mean a min-max game with independent strategy sets, and "min-max Stackelberg game" to mean a min-max game with dependent strategy sets.}\]
becomes analytically intractable. Part 2 of Assumption 3.1 is a standard assumption in exploratory studies of min-max games (e.g., [49]). Finally, we note that Part 3 of Assumption 3.1 can be replaced by a subgradient boundedness assumption; however, for simplicity, we assume this stronger condition.

Building on ideas developed by Nouiehed et al. [49] and Jin et al. [33] for min-max games, Goktas and Greenwald [24] recently developed a two-player non-convex-concave min-max game with independent (G2DA) algorithm with polynomial-time guarantees. Example 1.1 demonstrates that the GDA algorithms used to solve min-max games [11, 37, 43] do not solve min-max Stackelberg games. Nonetheless, we resolve this open question by developing a new algorithm that approximates Stackelberg equilibria.

We define \(L_x(y, \lambda) = f(x, y) + \sum_{k=1}^{d} \lambda_k g_k(x, y)\) to be the Lagrangian associated with the outer player’s value function, or equivalently, the inner player’s payoff maximization problem, given the outer player’s strategy \(x \in X\). Our algorithms rely heavily on the next theorem, proven in Appendix B, which states that any min-max Stackelberg game with independent strategy sets is equivalent to a three-player min-max game with independent strategy sets and payoff function \(L_x(y, \lambda)\), where the inner two players, \(y\) and \(\lambda\), can move simultaneously (i.e., they play a Nash equilibrium), but the outer player, \(x\), must move first.

**Theorem 3.2.** Under Assumption 3.1, any min-max Stackelberg game with independent strategy sets \((X, Y, f, g)\) can be viewed as a min-max game with independent strategy sets: i.e.,

\[
\min_{x \in X} \max_{y \in Y} f(x, y) = \min_{x \in X} \max_{y \in Y} L_x(y, \lambda) \quad (\text{1})
\]

\[
= \min_{x \in X, \lambda \geq 0} L_x(y, \lambda). \quad (\text{2})
\]

Note, that \(\min_{x \in X, \lambda \geq 0} \max_{y \in Y} L_x(y, \lambda)\) is a non-convex-concave min-max game, since \(L_x(y, \lambda)\) is non-convex-concave, due to the term \(\sum_{k=1}^{d} \lambda_k g_k(x, y)\), which is not guaranteed to be convex in \(\lambda\) and \(x\) jointly.\(^5\) This reduction of a min-max game with independent strategy sets to one with independent strategy sets does not imply that the game is solvable in polynomial time, since for non-convex-concave min-max games the computation of (global) min-max points is NP-Hard [12]. Nonetheless, Theorem 3.2 is suggestive of a new algorithm, which we call gradient descent ascent (Algorithm 2), which is essentially GDA, but run on the two-player non-convex-concave min-max game with independent strategy sets \(\min_{x \in X, \lambda \geq 0} \max_{y \in Y} L(x, y)\).

Unfortunately, as shown next, G2DA does not solve Stackelberg equilibria in general.

**Example 3.3.** Recall Example 1.1. The Lagrangian associated with the outer player’s value function is \(L(x, y, \lambda) = x^2 + y + \lambda (1 - (x + y))\). The optimal solution to this game is \(x^* = 1, y^* = 1/2\). G2DA applied to this game with learning rates \(\eta_x = \eta_y = 1\) for all \(t \in \mathbb{N}\) yields: \(\lambda^{(t+1)} = \Pi_{\mathbb{R}^+} \left[ \lambda^{(t)} - \eta_x \nabla_x L(x^{(t)}, y^{(t)}, \lambda^{(t)}) \right] = \Pi_{[-1,1]} [-x^{(t)} + \lambda^{(t)}], \quad \text{and} \quad y^{(t+1)} = \Pi_{[-1,1]} [y^{(t)} + \eta_y \nabla_y L(x^{(t)}, y^{(t)}, \lambda^{(t)})] = \Pi_{[-1,1]} [y^{(t)} + \lambda^{(t)}]
\]

\(^5\)We recall that a function \(f : A \times B \rightarrow \mathbb{R}\) is jointly convex in \(a\) and \(b\) if it is convex in \((a, b)\), and that the product of two convex functions need not give rise to a convex function. In particular, \(\lambda \cdot g(x, y)\) is not necessarily jointly convex in \((\lambda, x)\).

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**Algorithm 1 Gradient Descent Descent Ascent (G2DA)**

**Inputs:** \(\Lambda, X, Y, f, g, \eta_x, \eta_y, \eta_x, T, \lambda^{(0)}, x^{(0)}, y^{(0)}\)

**Output:** \(x^{(T)}, y^{(T)}\)

1. for \(t = 1, \ldots, T - 1\) do
2. Set \(\lambda^{(t+1)} = \Pi_{\mathbb{R}^+} \left[ \lambda^{(t)} - \eta_x \nabla_x L(x^{(t)}, y^{(t)}, \lambda^{(t)}) \right] = \Pi_{[-1,1]} [-x^{(t)} + \lambda^{(t)}], \quad \text{and} \quad y^{(t+1)} = \Pi_{[-1,1]} [y^{(t)} + \lambda^{(t)}] = \Pi_{[-1,1]} [y^{(t)} + \lambda^{(t)}]
3. end for
4. return \((x^{(T)}, y^{(T)})\)

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at \(\lambda^{(0)} = 0, x^{(0)} = 0, y^{(0)} = 0\), the algorithm proceeds as follows:

\(\lambda^{(1)} = 0, x^{(1)} = 0, y^{(1)} = 1, \lambda^{(2)} = 0, x^{(2)} = 0, y^{(2)} = 1\), and so on, and thus does not converge to a Stackelberg equilibrium of the game.

This result is slightly discouraging, however, not entirely surprising, since one timescale GDA, i.e., GDA run with the same step sizes while descending and ascending, with fixed step sizes does not necessarily converge to stationary points of the outer player’s value function in non-convex-concave min-max games with independent strategy sets [11]. Recently, it has been shown that two timescale GDA [37], i.e., GDA run with different step sizes while descending and ascending, converges to stationary points of the outer player’s value function in these games. Hence, one might wonder whether a two timescale variant of Algorithm 1 would converge to a Stackelberg equilibrium. Even more discouraging, however, is that the above example converges to a solution that does not correspond to a Stackelberg equilibrium for any learning rate \(\eta_x, \eta_y > 0\). But all hope is not lost; in what follows, we rely on an oracle, inspired by the algorithms in Goktas and Greenwald [24].

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**4 GDA WITH OPTIMAL KKT MULTIPLIERS**

We prove the main theoretical result of this paper in this section. For a large class of min-max Stackelberg games, we provide first-order methods with polynomial-time guarantees which, in contrast to Goktas and Greenwald [24], who developed a nested GDA algorithm (i.e., one with a nested loop), require only a single loop of gradient evaluations. To do so, we rely on a Lagrangian oracle, which outputs the optimal KKT multipliers of the Lagrangian associated with the inner player’s payoff maximization problem. We also provide sufficient conditions for this oracle to exist.

Since the Lagrangian is convex-concave in \(x, y\), if the optimal KKT multipliers \(\lambda^* \in \mathbb{R}^d\) were known for the problem \(\min_{x \in X} \max_{y \in Y} g(x, y) \geq 0 \quad f(x, y) = \min_{x \in X} \max_{y \in Y} L(x, y, \lambda^*)\), then one could plug them back into the Lagrangian to obtain a convex-concave saddle point problem given by \(\min_{x \in X} \max_{y \in Y} L(x, y, \lambda^*)\). This might lead one to think that they can use GDA [43] to solve the Lagrangian for \(x, y\) giving rise to Lagrangian Gradient Descent Ascent (LGDA, Algorithm 2) for any convex-concave min-max Stackelberg game.

**Example 4.1.** Consider the following this min-max Stackelberg game:

\[\min_{x \in [-1,1]} \max_{y \in [-1,1]} (1 - (x + y)) x^2 - y^2 + 1\]

The Stackelberg equilibrium of this game is \(x^* = 0, y^* = 0\). The Lagrangian is given by \(L(x, y, \lambda) = x^2 - y^2 + \lambda (1 - (x + y))\). When we plug the optimal KKT multiplier \(\lambda^* = 0\) into the Lagrangian, we obtain \(L(x, y, \lambda) = x^2 - y^2\).
Algorithm 2 Lagrangian Gradient Descent Ascent (GDALO)

**Inputs:** $\lambda^s, X, Y, f, g, \eta^x, \eta^y, T, (\lambda^{(0)}, x^{(0)}, y^{(0)})$

**Output:** $x^*, y^*$

1. for $t = 1, \ldots, T - 1$ do
2. \quad Set $x^{(t+1)} = \Pi_X (x^{(t)} - \eta^x \nabla_x L(x^{(t)}, y^{(t)}, \lambda^{(t)}) + y^{(t)})$
3. \quad Set $y^{(t+1)} = \Pi_Y (y^{(t)} + \eta^y \nabla_y L(x^{(t)}, y^{(t)}, \lambda^{(t)}) + x^{(t)})$
4. end for
5. return $\{(x^{(t)}, y^{(t)})\}_{t=1}^T$

$x^2 - y^2$. Thus, Algorithm 2 yields the update rules $x^{(t+1)} = x^{(t)} - 2x^{(t)}$ and $y^{(t+1)} = y^{(t)} - 2y^{(t)}$. The sequence of iterates starting at $x^{(0)} = 1, y^{(0)} = 1$, when $\eta^x = \eta^y = 1$ for all $t \in \mathbb{N}$, thus cycles as follows: $x^{(t)} = -1, y^{(t)} = -1; x^{(2)} = 1, y^{(2)} = 1; \text{and so on}$. Nonetheless, the average of the iterates corresponds to the Stackelberg equilibrium.

Unfortunately, LGDA does not converge in general, as once $\lambda^*$ is plugged back into the Lagrangian, the Lagrangian might become degenerate in $y$, i.e., the dependence of the Lagrangian on $y$ is lost, in which case GDCA can converge to the wrong solution.

**Example 4.2.** Recall Example 1.1. When we plug the optimal KKT multiplier $\lambda^* = 1$ into the Lagrangian associated with the outer player’s value function, we obtain $L^*(y, \lambda) = x^2 + y + 1 - (x + y) = x^2 - x + 1$, with $\partial L/\partial x = 2x - 1$ and $\partial L/\partial y = 0$. It follows that the $x$ iterate converges to $1/2$, but the $y$ iterate will never get updated, and hence unless $y$ is initialized at its Stackelberg equilibrium value, LGDA will not converge to a Stackelberg equilibrium.

This degeneracy issue arises when $\nabla_y f(x, y) = -\sum_{i=1}^d \lambda^*_i \nabla_y g_i(x, y)$, $\forall x \in X$, and can be side-stepped if we restrict attention to min-max Stackelberg games with convex-strictly-concave payoff functions, in which case the Lagrangian is guaranteed not to be degenerate in $y$, and convergence in average iterates is guaranteed. The proof of the following theorem is relegated to Appendix C.

**Theorem 4.3.** Let $\{(x^{(t)}, y^{(t)})\}_{t=1}^T$ be the sequence of iterates generated by LGDA run on the convex-strictly-concave min-max Stackelberg game $(X, Y, f, g)$. Let $(x^*, y^*) \in X \times Y$ be a Stackelberg equilibrium of $(X, Y, f, g)$. Suppose that Assumption 3.1 holds, and for all $t \in [T]$, $\eta^x = \eta^y = 1/\sqrt{T}$. Let $\ell_L = \max_{(x,y) \in X \times Y} \|\nabla L(x, y, \lambda^*)\|$, and $\ell_T = \frac{1}{T} \sum_{t=1}^T x^{(t)}$ and $\ell_T^* = \frac{1}{T} \sum_{t=1}^T y^{(t)}$. We then have:

$$
\frac{\|y^{(t)} - y^*\|^2}{2} + \frac{\ell^2_L}{2} \leq f\left(\frac{x^{(t)}, y^{(t)}}{T}\right) - f(x^*, y^*) \leq \frac{\|y^{(t)} - y^*\|^2}{2} + \frac{\ell_L^2}{2} \frac{\|x^{(t)} - x^*\|^2}{2} + \frac{\ell_T^2}{2}
$$

(3)

Because of the potential degeneracy of the Lagrangian in $y$, we propose running the gradient ascent step for the inner player on the objective function, rather than the Lagrangian. Algorithm 3, which we dub GDALO, once again assumes access to the optimal KKT multipliers $\lambda^*$, with which it runs gradient descent on the Lagrangian for the outer player and gradient ascent on the objective function for inner player.

Algorithm 3 Gradient Descent Ascent with a Lagrangian Oracle

**Inputs:** $\lambda^*, X, Y, f, g, \eta^x, \eta^y, T, (x^{(0)}, y^{(0)})$

**Output:** $x^*, y^*$

1. for $t = 1, \ldots, T - 1$ do
2. \quad Set $x^{(t+1)} = \Pi_X (x^{(t)} - \eta^x \nabla_x L(x^{(t)}, y^{(t)}, \lambda^{(t)}) + y^{(t)})$
3. \quad Set $y^{(t+1)} = \Pi_Y (y^{(t)} + \eta^y \nabla_y L(x^{(t)}, y^{(t)}, \lambda^{(t)}) + x^{(t)})$
4. end for
5. Draw $(\tilde{x}, \tilde{y})$ uniformly at random from $\{(x^{(t)}, y^{(t)})\}_{t=1}^T$
6. return $(\tilde{x}, \tilde{y})$

We are not able to prove that GDALO converges in average iterates to a Stackelberg equilibrium. We do show, however, that in expectation any iterate selected uniformly at random corresponds to a Stackelberg equilibrium. In particular, the following convergence rate holds for Algorithm 3. We refer the reader to Appendix C for the proofs, which we note do not follow from known results.

**Theorem 4.4.** Let $(\tilde{x}, \tilde{y})$ be the output generated by Algorithm 3 run on the min-max Stackelberg game $(X, Y, f, g)$. Let $(x^*, y^*) \in X \times Y$ be a Stackelberg equilibrium of $(X, Y, f, g)$. Suppose that Assumption 3.1 holds and that for all $t \in [T]$, $\eta^x = \eta^y = 1/\sqrt{T}$. Let $\ell_L = \max_{(x,y) \in X \times Y} \|\nabla L(x, y, \lambda^*)\|$, and $\ell_T = \max_{(x,y) \in X \times Y} \|\nabla f(x, y)\|$. We then have:

$$
\frac{\|y^{(t)} - y^*\|^2}{2} + \ell^2_L \leq E\left[\frac{1}{T} f\left(\frac{x^{(t)}, y^{(t)}}{T}\right)\right] - f(x^*, y^*) \leq \frac{\|y^{(t)} - y^*\|^2}{2} + \ell_L^2 + \frac{\ell^2_T}{2}
$$

(4)

We conclude this section by providing an explicit closed-form solution for the optimal KKT multipliers for a large class of min-max Stackelberg games. The existence of a Lagrangian oracle is guaranteed in these games.

**Theorem 4.5.** Consider a min-max Stackelberg game of the following form:

$$
\min_{x \in X} \max_{y \in Y} \{\nu_i(x, y), (y, x) \leq c_i\} = f_1(x) + \sum_{i=1}^n a_i \log(f_2(x, y)) + \sum_{i=1}^n b_i \log(f_3(y))
$$

(5)

where $f_1 : X \rightarrow \mathbb{R}, f_2 : Y \rightarrow \mathbb{R}, f_3 : X \rightarrow \mathbb{R}, f : X \times X \rightarrow \mathbb{R}$, and $X, Y \subseteq \mathbb{R}^m$ are compact-convex. Suppose that 1. $f_2, f_3, g_1, \ldots, g_n$ are concave in $y$, for all $x \in X$, 2. $f_2, f_3$ are homogeneous in $y$, for all $x \in X$, i.e., $\forall k \in \mathbb{R}, f_2(x, ky) = k f_2(x, y), f_3(ky) = k f_3(y)$, and continuous. Then, the optimal KKT multipliers $\lambda^* \in \mathbb{R}^n$ $\lambda_i^* = \frac{a_i b_i}{c_i}$, for all $i \in [n]$.

We remark that since the optimal KKT multipliers of the min-max Stackelberg games of the form of Theorem 4.5 are given in closed form, such games can be converted into min-max games with independent strategy sets. That said, Algorithm 3 is still necessary, because as we have shown, the Lagrangian can become degenerate in which case LGDA can converge to the wrong solution.

注：Note that convergence in expectation is weaker than convergence in average iterates. Intuitively, convergence in expectation means that as the number of iterations for which the algorithm is run increases, in expectation the output of the algorithm becomes a better and better approximation of a Stackelberg equilibrium.
5 APPLICATION TO FISHER MARKETS

The Fisher market model, attributed to Irving Fisher [6], has received a great deal of attention recently, in particular by computer scientists, as its applications to fair division and mechanism design have proven useful for the design of automated markets in many online marketplaces. In this section, we use our algorithms to compute competitive equilibria in Fisher markets, which have been shown to be instances of min-max Stackelberg games [24].

A Fisher market consists of \( n \) buyers and \( m \) divisible goods [6]. Each buyer \( i \in [n] \) has a budget \( b_i \in \mathbb{R}_+ \) and a utility function \( u_i : \mathbb{R}^m_+ \rightarrow \mathbb{R} \). As is standard in the literature, we assume that there is one divisible unit of each good available in the market [48]. An instance of a Fisher market is given by a tuple \((n, m, U, b, \bar{p})\), where \( U = \{u_1, \ldots, u_n\} \) is a set of utility functions, one per buyer, and \( b \in \mathbb{R}^m_+ \) is the vector of buyer budgets. We abbreviate as \((U, b)\) when \( n \) and \( m \) are clear from context.

An allocation \( X = (x_1, \ldots, x_n)^T \in \mathbb{R}^{n \times m}_+ \) is a map from goods to buyers, represented as a matrix, s.t. \( x_{ij} \geq 0 \) denotes the amount of good \( j \in [m] \) allocated to buyer \( i \in [n] \). Goods are assigned prices \( p = (p_1, \ldots, p_m)^T \in \mathbb{R}^m_+ \). A tuple \((p^*, X^*)\) is said to be a competitive (or Walrasian) equilibrium of Fisher market \((U, b)\) if 1. buyers are utility maximizing, constrained by their budget, i.e., \( \forall i \in [n], x_{ij}^* = \arg\max_{x \in \mathbb{R}^m_+} x_j p_j \leq b_i u_i(x); \) and 2. the market clears, i.e., \( \forall j \in [m], p_j^* > 0 \Rightarrow \sum_{i \in [n]} x_{ij}^* = 1 \) and \( p_j^* = 0 \Rightarrow \sum_{i \in [n]} x_{ij}^* \leq 1 \).

Goktas and Greenwald [24] observe that any competitive equilibrium \((p^*, X^*)\) of a Fisher market \((U, b)\) corresponds to a Stackelberg equilibrium of the following min-max Stackelberg game:

\[
\min_{p^* \in \mathbb{R}^m_+} \max_{X^* \in \mathbb{R}^{n \times m}_+} \sum_{j \in [m]} p_j^* \sum_{i \in [n]} b_i \log \left( u_i(x_i^*) \right). \tag{6}
\]

By Theorem 4.5, we can obtain a Lagrangian solution oracle for Equation (6), given by the following corollary. We note that as for all buyers \( i \in [n], b_i > 0 \), Slater’s condition is satisfied.

**Corollary 5.1.** Consider the min-max Stackelberg game described by Equation (6). The optimal KKT multipliers are given by \( \lambda^* = 1_n \).

Let \( \mathcal{L} \) be the Lagrangian of the outer player’s value function in Equation (6), i.e., \( \mathcal{L}(X, \lambda) = \sum_{j \in [m]} p_j^* + \sum_{i \in [n]} b_i \log \left( u_i(x_i) \right) + \sum_{i \in [n]} \lambda_i (b_i - x_i^* \cdot p) \). Using Corollary 5.1, we can define myopic best-response dynamics (Algorithm 4; MBRD) [8, 42] in Fisher markets as GDALO run on Equation (6), by noting that \( \nabla_p \mathcal{L}(p, X, 1_n) = 1_m - \sum_{i \in [n]} x_i^* \) (Goktas et al. [25], Theorem 3).

In words, under myopic best-response dynamics, at each time step the (fictional Walrasian) auctioneer takes a gradient descent step, and then all the buyers take a gradient ascent step to maximize their utility. We note that myopic best-response dynamics can also be interpreted as a tâtonnement process run with boundedly rational buyers who take a step in the direction of their optimal bundle, but do not actually compute their optimal bundle at each time step. We thus have the following corollary of Theorem 4.4:

**Corollary 5.2.** Let \((U, b)\) be a Fisher market with equilibrium price vector \( p^* \), where \( U \) is a set of continuous, concave, homogeneous, and continuously differentiable utility functions. Running myopic best-response dynamics (Algorithm 4) on the Fisher market \((U, b)\) yields an output which is in expectation an \( \varepsilon \)-competitive equilibrium with \( \varepsilon \)-utility maximizing allocations in \( O(1/\varepsilon^2) \) iterations.\(^7\)

Experiments. As we are not able to prove average-iterate convergence for GDALO, we ran a series of experiments on Fisher markets in which we track whether or not the sequence of average iterates produced by MBRD converges to a competitive equilibrium.\(^8\) We consider three buyer utility structures, each of which endows Equation (6) with different smoothness and convexity properties, thus allowing us to compare the efficiency of the algorithms under these varying conditions. We summarize the properties of Equation (6) in these three Fisher markets in Table 1, and include a more detailed description of our experimental setup in Appendix A.

| Buyer utilities | Linear-concave | Linear-strictly concave | Assumption 3.1 holds |
|-----------------|----------------|-------------------------|---------------------|
| Linear          | ✓              | ✓                       | ✓                   |
| Cobb-Douglas    | ×              | ✓                       | ✓                   |
| Leontief        | ×              | ✓                       | ✓                   |

Let \( (x_1^{(t)}, \ldots, x_n^{(t)})^T \) be the sequence of iterates generated by MBRD and let \( p(t) = \frac{1}{T} \sum_{t=1}^{T} p(t) \). Figure 1 then depicts average exploitability, i.e. \( \forall t \in \mathbb{N}_+ \),

\[
\max_{X \in \mathbb{R}^{n \times m}_+} \sum_{j \in [m]} p_j (x_j - x_j^*) + \sum_{i \in [n]} b_i \log \left( u_i(x_i) \right) - \min_{p^* \in \mathbb{R}^m_+} \sum_{j \in [m]} \sum_{i \in [n]} p_j x_{ij} - \sum_{i \in [n]} b_i \log \left( u_i(x_i) \right),
\]

across all runs, divided by \( 1/\sqrt{T} \). Note that if this quantity is constant then the (average) iterates converge empirically, at a rate of \( O(1/\sqrt{T}) \), while if it is an increasing (decreasing) function then the iterates converge empirically at a rate slower (faster) than \( O(1/\sqrt{T}) \).

\(^7\)We note that one can ensure that the derivatives of \( \sum_{i \in [n]} b_i \log u_i(x_i) \) w.r.t. \( x_{ij} \) in Equation (6) are bounded at \( x_{ij} = 0 \) for all \( i \in [n] \) and \( j \in [m] \) by reparametrizing the program as \( \sum_{i \in [n]} p_j + \sum_{i \in [n]} \log u_i(x_i) - \delta \) for \( \delta > 0 \).

\(^8\)Our code can be found at https://github.com/denizalp/min-max-gda-fisher.
Convergence is fastest in Fisher markets with Cobb-Douglas utilities, followed by linear, and then Leontief. Linear and Cobb-Douglas Fisher markets appear to converge at a rate faster than $O(1/\sqrt{T})$ in both Like for linear utilities, the objective function is twice differentiable for Cobb-Douglas utilities, and for Cobb-Douglas utilities the objective is also linear-strictly-concave, which could explain the faster convergence rate. Fisher markets with Leontief utilities, in which the objective function is not differentiable, are the hardest markets of the three for our algorithms to solve; we seem to obtain a convergence rate slower than $O(1/\sqrt{T})$. Our experiments suggest that the convergence of MBRD (Algorithm 4), and more generally, GDALO (Algorithm 3) can be improved to convergence in average iterates at a $O(1/\sqrt{T})$ rate, when Assumption 3.1 holds.

![Figure 1: Average exploitability divided by $1/\sqrt{T}$ after running MBRD (Algorithm 4) on randomly initialized linear, Cobb-Douglas, and Leontief Fisher markets.]

### 6 CONCLUSION

We began this paper by observing that a straightforward generalization of GDA to min-max Stackelberg games does not converge to a Stackelberg equilibrium in convex-concave min-max Stackelberg games. We then introduced two variants of GDA that do converge in polynomial time to Stackelberg equilibrium in a large class of min-max Stackelberg games. Both of our algorithms, LGDA and GDALO, converge in $O(1/\varepsilon^2)$ iterations to an $\varepsilon$-Stackelberg equilibrium. While LGDA converges in averages iterates only in convex-strictly-concave min-max Stackelberg games under standard smoothness assumptions, GDALO converges in expected iterates in all convex-concave min-max Stackelberg games. The iteration complexity of these algorithms improve on the state-of-the-art nested GDA algorithm proposed by Goktas and Greenwald [24], which computes an equilibrium in $O(1/\varepsilon^2)$ iterations.

We then applied GDALO the computation of competitive equilibria in Fisher markets, which yielded myopic best-response dynamics for Fisher markets—a new form of tâtonnement in which both the buyers and sellers are boundedly rational. Our experiments suggest that GDALO’s expected iterate convergence can be improved to average iterate convergence, since in all experiments, the average iterates do indeed converge to competitive equilibria. Additionally, our experiments suggest avenues for future work, namely investigating how varying the degree of smoothness of the objective function can impact convergence rates.

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A AN ECONOMIC APPLICATION: DETAILS

Our experimental goal was to understand if GDALO (Algorithm 3) converges in average iterates and if so how the rate of convergences changes under different utility structures, i.e. different smoothness and convexity properties of the objective function.

To achieve this goal, we ran multiple experiments, each time recording the prices and allocations during each iteration $t$ of the main (outer) loop. We checked for each experiment if the average iterates converged. We have found no experiment for which there was no average iterate convergence. Let $(x^{(t)}, y^{(t)})_{t=1}^T$ be the sequence of iterates generated by MBRD and let $p^t = \frac{1}{t} \sum_{t=1}^T p^{(t)}$. Figure 1 then depicts average exploitability, i.e. $\forall t \in \mathbb{N}_*$,

$$\max_{x \in \mathbb{R}^{n \times m}} x \in \mathbb{R}^{n \times m} : x \in \mathbb{R}^{n \times m}$$

$$\min_{\bar{y} \in \mathbb{R}^m} \max_{\bar{y} \in \mathbb{R}^m} \sum_{j \in [m]} b_j \log (u_i(x_i)) - \sum_{i \in [n]} b_i \log (u_i(y_i)) \ ,$$

across all runs, divided by $1/\sqrt{T}$.

Hyperparameters. We randomly initialized 500 different linear, Cobb-Douglas, Leontief Fisher markets, each with 5 players and 8 goods. Buyer $i$'s budget $b_i$ was drawn randomly from a uniform distribution ranging from 10 to 20 (i.e., $U[10, 20]$), while each buyer $i$'s valuation for good $j$, $u_{ij}$, was drawn randomly from $U[5, 15]$. We ran MBRD (Algorithm 4) for 1000, 500, and 500 iterations for linear, Cobb-Douglas, and Leontief Fisher markets, respectively. We started the algorithm with initial prices drawn randomly from $U[5, 15]$. After manual hyperparameter tuning, we opted for fixed learning rates of $\forall t \in \mathbb{N}_*, n_i^L = 3$, $n_i^U = 1$ for Cobb-Douglas and Leontief Fisher markets, while we picked fixed learning rates of $\forall t \in \mathbb{N}_*$, $n_i^L = 3$, $n_i^U = 0.1$ for linear Fisher markets.

Programming Languages, Packages, and Licensing. We ran our experiments in Python 3.7 [70], using NumPy [28], Pandas [52], and CVXPY [15]. Figure 1 was graphed using Matplotlib [30].

Python software and documentation are licensed under the PSF License Agreement. Numpy is distributed under a liberal BSD license. Pandas is distributed under a new BSD license. Matplotlib only uses BSD compatible code, and its license is based on the PSF license. CVXPY is licensed under an Apache license.

Implementation Details. In order to project each allocation computed onto the budget set of the consumers, i.e., $\{X \in \mathbb{R}^{n \times m} | Xp \leq b\}$, we used the alternating projection algorithm for convex sets, and alternatively projected onto the sets $\mathbb{R}^{n \times m}$ and $\{X \in \mathbb{R}^{n \times m} | Xp \leq b\}$.

Computational Resources. Our experiments were run on MacOS machine with 8GB RAM and an Apple M1 chip, and took about 2 hours to run. Only CPU resources were used.

Code Repository. The data our experiments generated, as well as the code used to produce our visualizations and run the statistical tests, can be found in our code repository (https://github.com/denizalp/min-max-gda-fisher).

B OMITTED PROOFS SECTION 3

Proof of Theorem 3.2. By the definition of the Lagrangian, $\max_{y \in \mathcal{Y}, g(x,y) \geq 0} f(x,y) = \max_{y \in \mathcal{Y}} \inf_{\lambda \geq 0} \mathcal{L}_x(y, \lambda)$. Slater’s condition guarantees that the infimum exists, so that it can be replaced by a minimum. Thus, $\max_{y \in \mathcal{Y}, g(x,y) \geq 0} f(x,y) = \max_{y \in \mathcal{Y}} \min_{\lambda \geq 0} \mathcal{L}_x(y, \lambda)$. In other words, we can re-express the inner player’s maximization problem as a convex-concave min-max game via the Lagrangian. Slater’s condition also implies strong duality, meaning $\max_{y \in \mathcal{Y}} \min_{\lambda \geq 0} \mathcal{L}_x(y, \lambda) = \min_{\lambda \geq 0} \max_{y \in \mathcal{Y}} \mathcal{L}_x(y, \lambda)$. It follows that we can re-express any min-max Stackelberg game with dependent strategy sets as a three-player game with payoff function $\mathcal{L}_x(y, \lambda)$, where the outer player, $x$, moves first and the inner two players, $y$ and $\lambda$, play a Nash equilibrium.

C OMITTED PROOFS SECTION 4

Proof Sketch of Theorem 4.3. Note that when $f$ is convex-strictly-concave in $x$ and $y$, then $\mathcal{L}_x(y, \lambda^*) = f(x,y) + \sum_{k=1}^d \lambda_k^* g_k(x,y)$ is also convex-strictly-concave, since $\sum_{k=1}^d \lambda_k^* g_k(x,y)$ is convex-concave, and the sum of a convex-concave function and a convex-strictly-concave function is convex-strictly-concave. This implies that for all $x \in X$, there exists a unique $y^*(x)$ that solves $\mathcal{L}_x(y, \lambda^*)$. Thus, the Lagrangian is well-defined, i.e., non-null, and we can recover the optimal primal variables from the Lagrangian [5]. Convergence of GDA follows as a direct extension of Nedic and Ozdaglar’s Theorem 3.1 [43].

Lemma C.1. Let the sequences $\left\{x^{(t)}\right\}$ and $\left\{y^{(t)}\right\}$ be generated by Algorithm 3. Suppose that assumption 3.1 holds. Let $\ell_f = \max_{(x,y) \in X \times Y} \|f(x,y)\|_f$ and $\ell_x = \max_{(x,y) \in X \times Y} \|f(x,y)\|_f$, we then have:

(a) For any $x \in X$ and for all $t \in \mathbb{N}_*$,

$$\|x^{(t+1)} - x\|^2 \leq \|x^{(t)} - x\|^2 - 2\eta \left(\mathcal{L}_x(y^{(t)}, \lambda^*) - \mathcal{L}_x(y^{(t)}, \lambda^*)\right) + \eta^2 \ell_f^2 \ .$$

(7)
(b) For any $y \in Y$ and for all $t \in \mathbb{N}_+$,
\[
\left\| y^{(t+1)} - y_t \right\|^2 \leq \left\| y^{(t)} - y_t \right\|^2 + 2\eta \left( f \left( x^{(t)}, y^{(t)} \right) - f \left( x^{(t)}, y \right) \right) + \eta^2 \tau_f^2
\]  
(8)

**Proof.** (a) By the non-expansivity of the projection operation and the definition of algorithm 3 we obtain for any $x \in X$ and all $t \in \mathbb{N}_+$,
\[
\left\| x^{(t+1)} - x \right\|^2 \leq \left\| x^{(t)} - x \right\|^2 - 2\eta \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_{x^{(t)}, y} \left( y \right) \right) + \eta^2 \left\| \nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) \right\|^2
\]  
(9)

or equivalently
\[
\left\| x^{(t+1)} - x \right\|^2 \leq \left\| x^{(t)} - x \right\|^2 - 2\eta \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_{x^{(t)}, y} \left( y \right) \right) + \eta^2 \left\| \nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) \right\|^2
\]  
(10)

Hence, for any $x \in X$ and all $t \in \mathbb{N}_+$,
\[
\left\| x^{(t+1)} - x \right\|^2 \leq \left\| x^{(t)} - x \right\|^2 - 2\eta \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_{x^{(t)}, y} \left( y \right) \right) + \eta^2 \left\| \nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) \right\|^2
\]  
(11)

Since the Lagrangian function $L_x(y, \lambda^*)$ is convex in $x$ for each $y \in Y$, and since $\nabla_x f \left( x^{(t)}, y^{(t)} \right)$ is a subgradient of $f \left( x, y^{(t)} \right)$ with respect to $x$ at $x = x^{(t)}$, we obtain for any $x \in X$,
\[
\nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)}, \lambda^* \right) \left( x - x^{(t)} \right) \leq L_x \left( y^{(t)}, \lambda^* \right) - L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right)
\]  
(12)

or equivalently
\[
-\nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)}, \lambda^* \right) \left( x - x^{(t)} \right) \leq - \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_x \left( y^{(t)}, \lambda^* \right) \right)
\]  
(13)

Hence, for any $x \in X$ and all $t \in \mathbb{N}_+$,
\[
\left\| x^{(t+1)} - x \right\|^2 \leq \left\| x^{(t)} - x \right\|^2 - 2\eta \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_x \left( y^{(t)}, \lambda^* \right) \right) + \eta^2 \left\| \nabla_x L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) \right\|^2
\]  
(14)

Since $L$ is continuously differentiable, it is $\ell_L$-Lipschitz continuous with $\ell_L = \max_{(x,y) \in X \times Y} \left\| \nabla L_x(y, \lambda^*) \right\|$. Hence, we have for any $x \in X$ and $t \in \mathbb{N}_+$:
\[
\left\| x^{(t+1)} - x \right\|^2 \leq \left\| x^{(t)} - x \right\|^2 - 2\eta \left( L_{x^{(t)}, y^{(t)}} \left( y^{(t)} \right) - L_x \left( y^{(t)}, \lambda^* \right) \right) + \eta^2 \ell_L^2
\]  
(15)

(b) Similarly, by using the non-expansivity of the projection operation and the definition of algorithm 3 we obtain for any $y \in Y$ and for all $t \in \mathbb{N}_+$:
\[
\left\| y^{(t+1)} - y \right\|^2 \leq \left\| y^{(t)} - y \right\|^2 + 2\eta \left( f \left( x^{(t)}, y^{(t)} \right) - f \left( x^{(t)}, y \right) \right) + \eta^2 \ell_f^2
\]  
(17)

or equivalently
\[
\left\| y^{(t+1)} - y \right\|^2 \leq \left\| y^{(t)} - y \right\|^2 + 2\eta \left( f \left( x^{(t)}, y^{(t)} \right) - f \left( x^{(t)}, y \right) \right) + \eta^2 \ell_f^2 \left( y^{(t)} \right) \right\|^2
\]  
(18)

Since $\nabla_y f \left( x^{(t)}, y^{(t)} \right)$ is a subgradient of the concave function $f \left( x^{(t)}, y \right)$ at $y = y^{(t)}$, we have for all $y \in Y$,
\[
\nabla_y f \left( x^{(t)}, y^{(t)} \right) \left( y - y^{(t)} \right) \leq f \left( x^{(t)}, y \right) - f \left( x^{(t)}, y^{(t)} \right)
\]  
(19)

Hence, for any $y \in Y$ and all $t \in \mathbb{N}_+$,
\[
\left\| y^{(t+1)} - y \right\|^2 \leq \left\| y^{(t)} - y \right\|^2 + 2\eta \left( f \left( x^{(t)}, y^{(t)} \right) - f \left( x^{(t)}, y \right) \right) + \eta^2 \left\| \nabla_y f \left( x^{(t)}, y^{(t)} \right) \right\|^2
\]  
(20)

Since $f$ is continuously differentiable, it is $\ell_f$-Lipschitz continuous with $\ell_f = \max_{(x,y) \in X \times Y} \left\| \nabla f(x,y) \right\|$. Hence, we have for any $y \in Y$ and $t \in \mathbb{N}_+$:
\[
\left\| y^{(t+1)} - y \right\|^2 \leq \left\| y^{(t)} - y \right\|^2 + 2\eta \left( f \left( x^{(t)}, y^{(t)} \right) - f \left( x^{(t)}, y \right) \right) + \eta^2 \ell_f^2
\]  
(21)

**Lemma C.2.** Let the sequences $\{x^{(t)}\}$ and $\{y^{(t)}\}$ be generated by Algorithm 3. Suppose that assumption 3.1 holds. Let $\ell_L = \max_{(x,y) \in X \times Y} \left\| \nabla L_x(y, \lambda^*) \right\|$ and $\ell_f = \max_{(x,y) \in X \times Y} \left\| \nabla f(x,y) \right\|$, and $\bar{x}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} x^{(t)}$ and $\bar{y}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} y^{(t)}$.

(a) For any $x \in X$ and for all $t \in \mathbb{N}_+$:
\[
\frac{1}{T} \sum_{t=1}^{T} L_{x^{(t)}, y^{(t)}} \left( y^{(t)}, \lambda^* \right) \leq L_x \left( \bar{x}^{(T)}, \lambda^* \right) \leq \frac{1}{2 \eta} \frac{\eta \ell_f^2}{2} \text{ for any } x \in X
\]  
(23)
Therefore, establishing the relation in (b). 

\[ f \] implies that \( L \) \( T \) 

Combining the preceding two relations, we obtain for all \( t \in \mathbb{N}_+ \):

\[
\frac{1}{2\eta}(\|\mathbf{x}(t) - \mathbf{x}\|^2 - \|\mathbf{x}(T) - \mathbf{x}\|^2) + \frac{\eta T \ell^2}{2}.
\]

(b) For any \( \mathbf{y} \in Y \) and for all \( t \in \mathbb{N}_+ \):

\[
\frac{1}{2\eta}(\|\mathbf{y}(t) - \mathbf{y}\|^2 - \|\mathbf{y}(T) - \mathbf{y}\|^2) + \frac{\eta T \ell^2}{2}.
\]

PROOF. We first show the relation in (a). From the previous lemma, we have for any \( \mathbf{x} \in X \) and \( t \in \mathbb{N}_+ \),

\[
\|\mathbf{x}(t+1) - \mathbf{x}\|^2 \leq \|\mathbf{x}(t) - \mathbf{x}\|^2 - 2\eta \left( L_{x(0)} \left( \mathbf{x}(t), \mathbf{y}^* \right) - L_x \left( \mathbf{y}(t), \mathbf{y}^* \right) \right) + \eta^2 \ell^2 T.
\]

Therefore,

\[
L_{x(0)} \left( \mathbf{y}(t), \mathbf{y}^* \right) - L_x \left( \mathbf{y}(t), \mathbf{y}^* \right) \leq \frac{1}{2\eta} \left( \|\mathbf{x}(t) - \mathbf{x}\|^2 - \|\mathbf{x}(t+1) - \mathbf{x}\|^2 \right) + \frac{\eta T \ell^2}{2}.
\]

By adding these relations over \( t = 1, \ldots, T \), we obtain for any \( \mathbf{x} \in X \) and \( T \in \mathbb{N}_+ \),

\[
\sum_{t=1}^{T} L_{x(0)} \left( \mathbf{y}(t), \mathbf{y}^* \right) - L_x \left( \mathbf{y}(t), \mathbf{y}^* \right) \leq \frac{1}{2\eta} \left( \|\mathbf{x}(0) - \mathbf{x}\|^2 - \|\mathbf{x}(T) - \mathbf{x}\|^2 \right) + \frac{\eta T \ell^2}{2},
\]

dividing by \( T \):

\[
\frac{1}{T} \sum_{t=1}^{T} L_{x(0)} \left( \mathbf{y}(t), \mathbf{y}^* \right) - \frac{1}{T} \sum_{t=1}^{T} L_x \left( \mathbf{y}(t), \mathbf{y}^* \right) \leq \frac{\|\mathbf{x}(0) - \mathbf{x}\|^2}{2\eta T} + \frac{\eta T \ell^2}{2}.
\]

Since the function \( L_x(y, y^* \rangle \) is concave in \( y \) for any fixed \( \mathbf{x} \in X \), we have

\[
L_x \left( \mathbf{y}(T), \mathbf{y}^* \right) \geq \frac{1}{T} \sum_{t=1}^{T} L_x \left( \mathbf{y}(t), \mathbf{y}^* \right) \quad \text{where} \quad \mathbf{y}(T) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}(t).
\]

Combining the preceding two relations, we obtain for any \( \mathbf{x} \in X \) and \( t \in \mathbb{N}_+ \),

\[
\frac{1}{T} \sum_{t=1}^{T} L_{x(0)} \left( \mathbf{y}(t), \mathbf{y}^* \right) - L_x \left( \mathbf{y}(T), \mathbf{y}^* \right) \leq \frac{\|\mathbf{x}(0) - \mathbf{x}\|^2}{2\eta T} + \frac{\eta T \ell^2}{2},
\]

thus establishing the relation in (a). Similarly, for (b), from the previous lemma, we have for any \( \mathbf{y} \in Y \) and \( t \in \mathbb{N}_+ \),

\[
\|\mathbf{y}(t+1) - \mathbf{y}\|^2 \leq \|\mathbf{y}(t) - \mathbf{y}\|^2 + 2\eta \left( f \left( \mathbf{x}(t), \mathbf{y}(t) \right) - f \left( \mathbf{x}(t), \mathbf{y} \right) \right) + \eta^2 \ell^2 T.
\]

Hence,

\[
\frac{1}{2\eta} \left( \|\mathbf{y}(t+1) - \mathbf{y}\|^2 - \|\mathbf{y}(t) - \mathbf{y}\|^2 \right) - \frac{\eta T \ell^2}{2} \leq f \left( \mathbf{x}(t), \mathbf{y}(t) \right) - f \left( \mathbf{x}(t), \mathbf{y} \right)
\]

By adding these relations over \( t = 1, \ldots, T \), we obtain for all \( \mathbf{y} \in Y \) and \( T \in \mathbb{N}_+ \),

\[
\frac{1}{2\eta} \left( \|\mathbf{y}(T) - \mathbf{y}\|^2 - \|\mathbf{y}(0) - \mathbf{y}\|^2 \right) - \frac{T \eta T \ell^2}{2} \leq \sum_{t=1}^{T} \left( f \left( \mathbf{x}(t), \mathbf{y}(t) \right) - f \left( \mathbf{x}(t), \mathbf{y} \right) \right)
\]

implying that

\[
\frac{\|\mathbf{y}(0) - \mathbf{y}\|^2}{2\eta T} - \frac{\eta T \ell^2}{2} \leq \frac{1}{T} \sum_{t=1}^{T} f \left( \mathbf{x}(t), \mathbf{y}(t) \right) - \frac{1}{T} \sum_{t=1}^{T} f \left( \mathbf{x}(t), \mathbf{y} \right)
\]

Because the function \( f(x, y) \) is convex in \( x \) for any fixed \( y \in Y \), we have

\[
\frac{1}{T} \sum_{t=1}^{T} f \left( \mathbf{x}(t), \mathbf{y} \right) \geq f \left( \mathbf{x}(T), \mathbf{y} \right) \quad \text{where} \quad \mathbf{x}(T) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t).
\]

Combining the preceding two relations, we obtain for all \( \mathbf{y} \in Y \) and \( t \in \mathbb{N}_+ \),

\[
\frac{\|\mathbf{y}(0) - \mathbf{y}\|^2}{2\eta T} - \frac{\eta T \ell^2}{2} \leq \frac{1}{T} \sum_{t=1}^{T} f \left( \mathbf{x}(t), \mathbf{y}(t) \right) - f \left( \mathbf{x}(T), \mathbf{y} \right)
\]

establishing the relation in (b).
Proof of Theorem 4.4. From relation (a) given in the previous lemma, we have:

\[
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \mathcal{L}_{x} (y(T), \lambda^*) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(38)

\[
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \max_{y \in Y} \min_{x \in X} \mathcal{L}_{x} (y, \lambda^*) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(39)

\[
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \max_{y \in Y} \mathcal{L}_{x} (y, \lambda^*) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(40)

\[
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \max_{y \in Y} \lambda \in \mathbb{R} \mathcal{L}_{x} (y, \lambda) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(41)

where the penultimate line follows from the max-min inequality [5] and the last line from the definition of \( \lambda^* \). Using theorem 3.2, we then get:

\[
\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \max_{y \in Y} \mathcal{L}_{x} (y, \lambda^*) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(42)

Note that for all \( t \in \mathbb{N}^* \), we have \( \mathcal{L}(x(t), y(t), \lambda^*) = f(x(t), y(t)) + \sum_{i=1}^{d} \lambda_i g_i(x(t), y(t)) \geq f(x(t), y(t)), \) since \( \lambda^* \in \mathbb{R}^d \) and for all \( t \in \mathbb{N}^* \), \( g(x(t), y(t)) \geq 0 \). Hence, we have:

\[
\frac{1}{T} \sum_{t=1}^{T} f(x(t), y(t)) - \min_{x \in X} \max_{y \in Y} g(x,y) f(x, y) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(43)

Additionally, using the previous lemma, from relation (b) we have:

\[
-\frac{\|y(0) - y\|^2}{2 \eta T} - \frac{\eta T^2}{2} \leq \frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \max_{y \in Y} g(x,y) f(x, y)
\]

(44)

\[
-\frac{\|y(0) - y\|^2}{2 \eta T} - \frac{\eta T^2}{2} \leq \frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_{x(t)}(y(t), \lambda^*) - \min_{x \in X} \max_{y \in Y} g(x,y) f(x, y)
\]

(45)

Combining equations (46) and (48), we obtain:

\[
-\frac{\|y(0) - y\|^2}{2 \eta T} - \frac{\eta T^2}{2} \leq \frac{1}{T} \sum_{t=1}^{T} f(x(t), y(t)) - \min_{x \in X} \max_{y \in Y} g(x,y) f(x, y) \leq \frac{\|x(0) - x\|^2}{2 \eta T} + \frac{\eta T^2}{2}
\]

(46)

The result of the theorem follows directly.

\[ \square \]

Proof of Theorem 4.5. The Lagrangian, \( \mathcal{L} : X \times Y \times \mathbb{R} \to \mathbb{R} \), associated with the inner player’s payoff maximization problem, \( \max_{Y \in Y} \min_{x \in X} \mathcal{L}(X, Y, \lambda) \), is given by:

\[
\mathcal{L}(X, Y, \lambda) = f_1(x) + \sum_{i=1}^{n} a_i \log(f_2(x, y_i)) + \sum_{j=1}^{m} b_i \log(f_3(y_j(i))) + \sum_{i=1}^{n} \lambda_i (c_i - g_i(x, y_i))
\]

(47)

Let \( Y^* \in \arg \max_{Y \in Y} \min_{x \in X} \mathcal{L}(X, Y, \lambda) \). From the first order KKT optimality conditions [36], for all \( x \in X, i \in [n], j \in [m] \) we have:

\[
\frac{\partial \mathcal{L}}{\partial y_{ij}} = \frac{a_i}{f_2(x, y_i)} \left[ \frac{\partial f_2}{\partial y_{ij}} \right]_{y=y_i} + b_i \left[ \frac{\partial f_3}{\partial y_{ij}} \right]_{y=y_j(i)} \lambda_i \leq 0
\]

(48)

Multiplying both sides by \( y_{ij}^* \) and summing up across the \( j \)'s, for all \( x \in X, i \in [n] \), we get:

\[
\sum_{j\in[m]} \frac{a_i}{f_2(x, y_i)} \left[ \frac{\partial f_2}{\partial y_{ij}} \right] y_{ij}^* + b_i \sum_{j\in[m]} \left[ \frac{\partial f_3}{\partial y_{ij}} \right] y_{ij}^* \lambda_i - \sum_{j\in[m]} \lambda_i \left[ \frac{\partial g_i}{\partial y_{ij}} \right] y_{ij}^* = 0
\]

(49)

\[
\sum_{j\in[m]} \frac{a_i}{f_2(x, y_i)} \left[ \frac{\partial f_2}{\partial y_{ij}} \right] y_{ij}^* + b_i \sum_{j\in[m]} \left[ \frac{\partial f_3}{\partial y_{ij}} \right] y_{ij}^* \lambda_i - \sum_{j\in[m]} \lambda_i \left[ \frac{\partial g_i}{\partial y_{ij}} \right] y_{ij}^* = 0
\]

(50)
Recall that by Euler’s theorem for homogeneous functions [4], we have for any homogeneous function $f : X \rightarrow \mathbb{R}$, $\sum_i \frac{\partial f}{\partial y_i} y_i = f(y)$. Hence, for all $x \in X, i \in [n]$, we have:

$$\frac{a_i}{f_2(x, y_i^*)} f_2(x, y_i^*) + \frac{b_i}{f_3(y_i^*)} f_3(y_i^*) - \lambda_i^* g_i(x, y_i^*) = 0$$  \hspace{1cm} (51)

$$a_i + b_i - \lambda_i^* g_i(x, y_i^*) = 0$$  \hspace{1cm} (52)

From the KKT complementarity conditions, we have $\lambda_i^* g_i(x, y_i^*) = c_i$, which gives us, for all $i \in [n]$:  \hspace{1cm} (53)

$$a_i + b_i - \lambda_i^* c_i = 0$$

$$\lambda_i^* = \frac{a_i + b_i}{c_i}$$  \hspace{1cm} (54)

□