Research on the elastic–plastic external contact mechanical properties of cylinder

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Abstract
Based on Hertz contact theory, an elastic-plastic contact mechanics model of outer cylinder under different contact angles of axis is proposed. The relationship among contact angle, load and contact deformation, contact stiffness and contact area is established. The finite element method is used to simulate the elastic-plastic contact process of the cylinder. The influence of the load and radius of the cylinder model on the contact deformation and the contact stiffness is compared and analyzed under different contact angles. The error of the analysis results of the finite element and the mechanical model is within 9%. On this basis, the influence of contact deformation, contact area and contact angle on the contact stiffness of the outer cylinder in elastic and plastic stage is explored. The results show that in the stage of elastic and plastic deformation, the amount of contact deformation and contact area increase with the increase of load. The contact stiffness decreases with the increase of contact angle and increases with the increase of cylinder radius. The amount of contact deformation decreases with the increase of cylinder radius, and tends to constant gradually. In the elastic stage, the contact stiffness increases with the increase of load. The contact area decreases with the increase of contact angle and increases with the increase of cylinder radius. In the plastic stage, the contact stiffness is constant with the increase of load, and the contact area is independent of contact angle and cylinder radius.

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Keywords
Cylinder contact model, contact finite element, elastic–plastic, contact angles, contact stiffness

Introduction
Hertz contact theory\(^1\) and fractal theory are the main methods to analyze cylinder contact model. In recent years, several typical cylinder contact models have been proposed based on Hertz contact theory.\(^2\) Johnson\(^3\) proposed a model of long cylinder in non-conformal contact, which considered the internal and external contact between two cylinders. Radzimovsky\(^4\) proposed an outer cylinder contact model, which was derived from the stress–strain formula. Goldsmith\(^5\) also proposed an inner cylinder contact model based on Hertz theory, which gave the expression between the load and the contact deformation. However, the above studies are all elastic contact models, so it is impossible to analyze the mechanical properties of the model under the condition of plastic deformation under heavy load. Moreover, the above model formulas include logarithmic function; so when the load is applied to a certain value, the model will fail.\(^2\) Through the research of rolling bearing, Lui and Shao\(^6,7\) put forward a cylinder contact model with lubricating oil film, which analyzed the influence of support stiffness on contact deformation. Sharma and Jackson\(^8\) carried out the finite element analysis of the elastoplastic cylinder contact problem, and compared the finite element results under the elastic and full plastic conditions with the existing Hertz contact and spherical elastoplastic contact models. Vijaywargiya and Green\(^9\) simulated two-dimensional sliding between two interacting elastoplastic cylinders by finite element analysis. Dumas and Baronet\(^10\) used the finite element method to study the elastic–plastic indentation of an infinite rigid cylinder with its axis parallel to the surface of half-space in detail, and gave the calculation results of different strain hardening slopes. Jackson\(^11\) presented a simple solution to a cylindrical rigid frictionless punch indenting a half-space considering only perfectly plastic deformation, which was verified by the finite element calculation. Based on the finite element method, Komvopoulos\(^12\) solved the elastic contact problem of a layered semi-infinite solid with rigid surface compression. Cinar and Sinclair\(^13\) studied the problem of infinite rigid cylinder under the condition of frictionless and complete adhesion release under incremental elastic–plastic condition. Bower and Johnson\(^14\) evaluated the effect of strain hardening on the cumulative plastic deformation of cylinder under repeated rolling and sliding contact by using the non-linear kinematic hardening law. By comparing different contact models, Beheshti and Khonsari\(^15\) found that the use of elastic–plastic micro-contact model can predict lower maximum normal pressure and larger contact width and actual contact area. Based on the fractal theory and Hertz contact theory, Huang et al.\(^16\) established the contact model of two cylinders. Zhao et al.\(^17\) established the fractal model of normal contact stiffness of the joint surface of two cylinders. Based on the theory of classification, Li et al.\(^18\) established a fractal prediction model of the normal stiffness of the joint surface of two cylinders considering friction factors. Based on
the Hertz contact theory and the fractal theory, the deformation properties of the model in elastic, elastic–plastic, and plastic stages are analyzed, but the deformation properties of the cylinder contact model in different contact angles are not considered.

Therefore, based on Hertz contact theory, this article proposes an elastic–plastic contact model of cylinder under different contact angles; establishes the relationship between load and contact deformation, contact stiffness, and contact area; and uses the finite element method to simulate the elastic–plastic contact process for comparison. The first section of the article establishes the contact mechanical model of the cylinder, which mainly derives the deformation formula of the cylinder model in the elastic and plastic phases. The second section conducts simulation analysis and comparison of the cylinder model, which mainly analyzes the effect of the load on the contact deformation and the effect of load on contact stiffness. In the third section, the contact mechanical characteristics of cylinder are studied, which mainly analyzes the effect of the load on the contact area and the influence of the cylinder radius on the model mechanical properties.

Cylinder contact mechanics model

The contact state of cylinder model can be divided into three stages: elastic stage, elastic–plastic stage, and plastic stage. In this model, the deformation formulas of cylinder in elastic and plastic stage are derived.

**Full elastic contact**

Let the radius of both cylinders be $R$. When there is no load, the two cylinders only contact at one point. As shown in Figure 1, the original gap between the two contact points $M_1$ and $M_2$ is $h = z_1 - z_2$. Under the pressure, the deformation displacement of the upper cylinder at $M_1$ is $\omega_1$, and that of the lower cylinder at $M_2$ is $\omega_2$. From the geometric relationship, the distance $\alpha$ of the two contact points $M_1$ and $M_2$ approaching each other can be expressed as

$$\alpha = h + \omega_1 + \omega_2$$

$$= \frac{\sin^2\theta}{2R}x^2 + \frac{1 + \cos^2\theta}{2R}y^2 + \omega_1 + \omega_2$$

From equation (1) and the displacement formula of the midpoint of the object under the distributed pressure on the semi-infinite boundary plane

$$\left(k_1 + k_2\right) \iint \frac{F'}{R_1} dx'dy' = \alpha - \frac{\sin^2\theta}{2R}x^2 - \frac{1 + \cos^2\theta}{2R}y^2$$

where $k_1 = \frac{(1 - v_1^2)}{(\pi E_1)}$ and $k_2 = \frac{(1 - v_2^2)}{(\pi E_2)}$; $F'dx'dy'$ is the pressure acting on element $dx'dy'$ of the contact part; $R_1$ is the distance from any point on the
contact surface to the specified point on the contact surface; \( \theta \) is the contact angle; \( \nu_{1,2} \) is Poisson’s ratio; and \( E_{1,2} \) is Young’s moduli.

From equation (2), the area of the contact ellipse and the relationship between the load and the deformation can be obtained as

\[
S = \pi \left( \frac{1 - \nu^2}{2} \right)^{\frac{1}{2}} \left[ \frac{3}{2} (k_1 + k_2) PRE_{(e)} \right]^{\frac{1}{2}}
\]

\[
P_1 = \left[ \frac{(2/3)^{\frac{1}{2}}}{E_{(e)} R^{\frac{1}{2}}} \left( 1 - \nu^2 \right) \frac{1}{K_{(e)} (k_1 + k_2)^{\frac{1}{2}}} \right] \alpha^{\frac{1}{2}}
\]

where \( K_{(e)} = \int_0^{\pi/2} d\varphi / \sqrt{1 - \nu^2 \sin^2 \varphi} \), \( E_{(e)} = \int_0^{\pi/2} \sqrt{1 - \nu^2 \sin^2 \varphi} d\varphi \); \( S \) is the contact area; \( P_1 \) is the load; and \( e \) is the radius of curvature of the contact ellipse.

**Elastic–plastic contact**

When the two cylinders in contact appear initial yield, the average contact pressure between the contact surfaces can be expressed as\(^{20}\)

\[
P_e = \sigma_e C^{-1}
\]

where \( P_e \) is the average contact pressure, \( \sigma_e \) is the maximum von Mises stress, and \( C^{-1} \) is a constant.

Green\(^{20}\) gave the calculation formula of constant \( C \)

\[
C = \begin{cases} 
\frac{1}{\sqrt{1 + 4(v-1)v}} & v \leq 0.1938 \\
1.164 + 2.975v - 2.906v^2 & v > 0.1938 
\end{cases}
\]
where \( v \) is Poisson’s ratio.

From equation (4)

\[
P_e = \left[ \left( \frac{2}{3} \right)^{\frac{1}{5}} \left( \frac{E(e)R}{1 - e^2} \right)^{\frac{1}{5}} \frac{1}{K(e)(k_1 + k_2)^{\frac{2}{5}}} \right] \alpha_{1}^{\frac{2}{5}}
\]

(7)

According to equations (5) and (7), the displacement distance of two cylinders at initial yield is

\[
\alpha_1 = (\sigma_e C^{-1})^{\frac{1}{5}} \left( \frac{9(1 - e^2)K^3(e)(k_1 + k_2)^2}{4E(e)R} \right)^{\frac{2}{5}}
\]

(8)

where \( \alpha_1 \) is the displacement distance at initial yield.

**Plastic contact**

When the material enters the complete plastic deformation stage, the average contact pressure is close to the yield strength\(^{21}\)

\[
P_e = \sigma_s
\]

(9)

where \( \sigma_s \) is the yield strength.

According to the theory of ABBOTT, E.J.\(^{22}\), the contact area can be expressed as

\[
A_p = 2S
\]

(10)

where \( A_p \) represents the contact area in plastic stage.

From equations (3) and (4)

\[
S = \pi \left[ \frac{3(k_1 + k_2)(E(e)R)^4}{2(1 - e^2)^{\frac{2}{5}}K^3(e)} \right]^{\frac{1}{5}} \alpha
\]

(11)

According to equation (10), the contact area can be obtained as

\[
A_p = 2\pi \left[ \frac{3(k_1 + k_2)(E(e)R)^4}{2(1 - e^2)^{\frac{2}{5}}K^3(e)} \right]^{\frac{1}{5}} \alpha
\]

(12)

Then the total load can be expressed as

\[
P = P_eA_p
\]

(13)

where \( P \) is the total load.

Namely
\[ P = 2\pi\sigma_s \left( \frac{3(k_1 + k_2)(E_c R)}{2(1 - e^2)^{\frac{1}{2}}K_{(e)}^3} \right)^{\frac{1}{2}} \]  

(14)

According to equation (14), the contact stiffness is

\[ K = \frac{dP}{d\alpha} \]  

(15)

where \( K \) represents the contact stiffness.

Namely

\[ K = 2\pi\sigma_s \left( \frac{3(k_1 + k_2)(E_c R)}{2(1 - e^2)^{\frac{1}{2}}K_{(e)}^3} \right)^{\frac{1}{2}} \]  

(16)

**Finite element method analysis and comparison**

**Modeling and analysis**

In this article, the upper and lower cylinder models are made of carbon structural steel of the same material. The following material properties are selected: Young’s moduli \( E = 2.1 \times 10^5 \) MPa, \( \nu = 0.3 \), yield strength \( H = 380 \) MPa, and \( R = 2.5 \) mm. This article mainly analyzes the mechanical characteristics of the cylinder model when it is in contact with 45°, 60°, and 90°. As shown in Figure 2, the cylinder mesh model adopts the face-to-face contact mode and refines the mesh around the contact surface. The minimum mesh size is \( 4 \times 10^{-4} \) mm. The boundary condition is that the bottom degree of freedom of the lower cylinder model is fully fixed. And the top of the upper cylinder model is fixed except for the free end in the vertical direction. The vertical load is applied to the upper surface of the upper cylinder model, and the simulation calculation is carried out.

![Figure 2. Cylinder mesh model: (a) 45°, (b) 60°, and (c) 90°.](image)
The simulation results of strain pattern can be obtained after solution. Figure 3(a) is the integral deformation cloud of the cylinder model, the contact angle is 90°, and the load is 20 KN, and Figure 3(b) is the vertical partial deformation cloud diagram of the lower cylinder, which clearly reflects the distribution and size of deformation. It can be seen that the maximum deformation of the contact surface is symmetrically distributed around the middle point, rather than appearing at the middle point. This is because the overall deformation of the cylinder model is characterized by point-to-surface contact. At the beginning of the contact, there is no contact but a certain distance around the middle point of the contact surface between the upper and lower cylinders, so the amount of deformation around the contact will be greater than the middle position during the contact process. And finally the contact surface will be formed.

**Table 1.** Comparison between simulation value and theoretical value of contact deformation.

| Contact angle/° | P/KN | $\alpha$ (simulation value)/μm | $\alpha$ (theoretical value)/μm | Deformation error/% |
|-----------------|------|-------------------------------|-------------------------------|---------------------|
| 45              | 14   | 1.86543                       | 1.71227                       | 8.95                |
|                 | 20   | 2.61457                       | 2.44610                       | 6.89                |
| 60              | 14   | 1.97147                       | 1.98737                       | 0.81                |
|                 | 20   | 2.65624                       | 2.81639                       | 5.96                |
| 90              | 14   | 2.15773                       | 1.98988                       | 8.43                |
|                 | 20   | 3.08247                       | 3.30868                       | 7.34                |

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**Contact deformation comparison**

The amount of deformation obtained by the theoretical derivation formula of the elastic–plastic deformation of the cylinder model is compared with the result obtained by the finite element simulation, and the data are fitted by the image.

It can be seen from Figure 4 that there are some errors between the simulation value and the theoretical value of contact deformation, but the trend is consistent.
When the load is about 6 KN, the model begins to plastic deformation, in which the elastic stage is exponential and the plastic stage is linear, and the contact deformation increases with the increase of load. Figure 4 shows clearly that the cylindrical model has a higher data consistency in the elastic stage than in the plastic stage. Therefore, the errors between the simulated and theoretical values of plastic contact deformation at 14 and 20 KN are further compared in Table 1. The maximum deformation error is 8.95% and the minimum deformation error is 0.81%. Under the same load, the contact deformation increases with the increase of contact angle.

From the angle of error generation, it is mainly divided into two aspects. On one hand, it is analyzed from the material itself that the material has not reached the stage of complete plastic deformation during loading, and there is certain elastic deformation which results in deviation of the deformation quantity; on the other hand, it is analyzed from the model itself. The main sources of errors include partition density of grid unit, calculation method of model, setting of boundary conditions, and geometric dimensions. For the first error, the contact deformation can...
be removed from the elastic region by increasing the applied load; for the second error, it can be solved by further improving the accuracy of the grid element, selecting a more appropriate unit type and contact type, and a more accurate geometric model.

**Contact stiffness comparison**

It can be seen from the comparison of theoretical value and simulation value of contact stiffness of cylindrical model that the change trend of theoretical value and simulation value of the model is basically the same as that shown in Figure 5. In the elastic stage, the contact stiffness increases exponentially with increasing load, and in the plastic stage, the contact stiffness remains unchanged regardless of the load.

The simulation value and theoretical value rigidity of $P = 14$ and $20$ KN under different contact angles are also compared here. As shown in Table 2, the maximum rigidity error is $8.44\%$, and the minimum rigidity error is only $0.80\%$. The reason for the error is basically the same as the above statement, and under the same load, the contact rigidity decreases with the increase of contact angle.
Contact analysis

Load influence

Figure 6 shows the influence of load on the contact area. The following material properties were selected: $E = 2.1 \times 10^5$ MPa, $v = 0.3$, $H = 380$ MPa, and $R = 2.5$ m. It can be seen from Figure 6 that the contact area increases with the increase of load, in which the elastic stage is exponential and the plastic stage is linear. Under the same load, the contact area of the elastic stage decreases with the increase of contact angle, and the contact area of the plastic stage is independent of contact angle.

Cylinder radius influence

Figures 7–9 show the influence of cylinder radius on contact deformation, contact stiffness, and contact area, respectively. The following material properties were selected: $E = 2.1 \times 10^5$ MPa, $v = 0.3$, $H = 380$ MPa, and $\theta = 90^\circ$. It can be seen from Figure 7 that the contact deformation decreases with the increase of the cylinder radius. As the contact deformation tends to be constant, the contact can be considered as the contact between two planes. From Figure 8, it can be seen that the contact stiffness increases with the increase of the cylinder radius. Figure 9 shows that in the elastic stage, the contact area increases with the increase of the cylinder radius; in the plastic stage, the contact area is independent of the cylinder radius.

Conclusion

Based on Hertz contact theory, this article presents an elastic–plastic contact mechanics model of cylinder under different contact angles of axis, including the relationship between contact angle, load and contact deformation, contact stiffness, and contact area, which is verified by finite element method. Based on this, the contact mechanical properties of cylinder are explored.

Table 2. Comparison between simulation value and theoretical value of contact stiffness.

| Contact angle/° | P/KN | $K$ (simulation value)/(KN/mm) | $K$ (theoretical value)/(KN/mm) | Stiffness error/% |
|----------------|------|-------------------------------|-------------------------------|-----------------|
| 45             | 14   | 7504.95                       | 8176.27                       | 8.21            |
|                | 20   | 7649.44                       | 8176.27                       | 6.44            |
| 60             | 14   | 7044.50                       | 7101.29                       | 0.80            |
|                | 20   | 7529.44                       | 7101.29                       | 6.03            |
| 90             | 14   | 7035.62                       | 6488.31                       | 8.44            |
|                | 20   | 6044.71                       | 6488.31                       | 6.84            |
1. The contact deformation increases with the increase of load and contact angle.
2. In the elastic stage, the contact stiffness increases with the increase of load, and in the plastic stage, the contact stiffness does not change with the load.

**Figure 6.** Influence of load on contact area.

**Figure 7.** Influence of radius on contact deformation.
1. The increase of load. The contact stiffness decreases with the increase of contact angle.

2. The contact area increases with the increase of load. In the elastic stage, the contact area decreases with the increase of contact angle, and in the plastic stage, the contact area is independent of contact angle.

Figure 8. Influence of radius on contact stiffness.

Figure 9. Influence of radius on contact area.
4. In the elastic stage, the contact area increases with the increase of the cylinder radius, and in the plastic stage, the contact area is independent of the cylinder radius.

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