Solutions of the Einstein-Dirac Equation on Riemannian 3-Manifolds with Constant Scalar Curvature. *

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Abstract

This paper contains a classification of all 3-dimensional manifolds with constant scalar curvature \( S \neq 0 \) that carry a non-trivial solution of the Einstein-Dirac equation.

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1 Introduction

Consider a Riemannian spin manifold of dimension \( n \geq 3 \) and denote by \( D \) the Dirac operator acting on spinor fields. A solution of the Einstein-Dirac equation is a spinor field \( \psi \) solving the equations

\[
Ric - \frac{1}{2} S \cdot g = \pm \frac{1}{4} T_\psi , \quad D(\psi) = \lambda \psi.
\]

Here \( S \) denotes the scalar curvature of the space, \( \lambda \) is a real constant and \( T_\psi \) is the energy-momentum tensor of the spinor field \( \psi \) defined by the formula

\[
T_\psi(X, Y) = (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi).
\]

The scalar curvature \( S \) is related to the eigenvalue \( \lambda \) and the length of the spinor field \( \psi \) by the formula

\[
S = \pm \frac{\lambda}{n-2} |\psi|^2.
\]

In [KimF] we introduced the weak Killing equation for a spinor field \( \psi^* \):

\[
\nabla_X \psi^* = \frac{n}{2(n-1)} dS(X)\psi^* + \frac{2\lambda}{(n-2)S} Ric(X) \cdot \psi^* - \frac{\lambda}{n-2} X \cdot \psi^* + \frac{1}{2(n-1)S} X \cdot dS \cdot \psi^*
\]

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Any weak Killing spinor $\psi^*$ (WK-spinor) yields a solution $\psi$ of the Einstein-Dirac equation after normalization

$$\psi = \sqrt{(n-2)|S|/|\psi^*|^2} \psi^*. $$

In fact, in dimension $n = 3$ the Einstein-Dirac equation is essentially equivalent to the weak Killing equation (see [KimF]). Up to now the following 3-dimensional Riemannian manifolds admitting WK-spinors are known:

1. the flat torus $T^3$ with a parallel spinor;
2. the sphere $S^3$ with a Killing spinor;
3. two non-Einstein Sasakian metrics on the sphere $S^3$ admitting WK-spinors. The scalar curvature of these two left-invariant metrics equals $S = 1 \pm \sqrt{5}$.

The aim of this paper is to classify all Riemannian 3-manifolds with constant scalar curvature and admitting a solution of the Einstein-Dirac equation. In particular, we will prove the existence of a one-parameter family of left-invariant metrics on $S^3$ with WK-spinors. This family contains the two non-Einstein Sasakian metrics with WK-spinors on $S^3$, but does not contain the standard sphere $S^3$ with Killing spinors. Moreover, any simply-connected, complete Riemannian manifold $N^3 \neq S^3$ with WK-spinors and constant scalar curvature is isometric to a space of this one-parameter family. In order to formulate the result precisely, we fix real parameters $K, L, M \in \mathbb{R}$ and denote by $N^3(K, L, M)$ the 3-dimensional, simply-connected and oriented Riemannian manifold defined by the following structure equations:

$$\omega_{12} = K \sigma^3, \quad \omega_{13} = L \sigma^2, \quad \omega_{23} = M \sigma^1,$$

or, equivalently:

$$d\sigma^1 = (L - K) \sigma^2 \wedge \sigma^3, \quad d\sigma^2 = (M + K) \sigma^1 \wedge \sigma^3, \quad d\sigma^3 = (L - M) \sigma^1 \wedge \sigma^2.$$ 

The 1-forms $\sigma^1, \sigma^2, \sigma^3$ are the dual forms of an orthonormal frame of vector fields. Using this frame the Ricci tensor of $N^3(K, L, M)$ is given by the matrix

$$Ric = \begin{pmatrix}
-2KL & 0 & 0 \\
0 & 2KM & 0 \\
0 & 0 & -2LM
\end{pmatrix}.$$ 

**Main Theorem:** Let $N^3 \neq S^3$ be a complete, simply-connected Riemannian manifold with constant scalar curvature $S \neq 0$. If $N^3$ admits a WK-spinor, then $N^3$ is isometric to $N^3(K, L, M)$ and the parameters are a solution of the equation

$$-K^2L(L - M)^2 + L^3M^3 + KL^2M^2(M - L) + K^3(L - M)(L + M)^2 = 0 \quad (*)$$

Conversely, any space $N^3(K, L, M)$ such that $(K, L, M) \neq (0, 0, 0)$ is a solution of $(*)$ admits two WK-spinors for one and only one WK-number $\lambda$. With respect to the fixed orientation of $N^3(K, L, M)$ we have the two cases:

$$\lambda = \frac{S}{2\sqrt{2}} \sqrt{\frac{S}{S^2 - 2|Ric|^2}} \quad \text{if} \quad -K < M.$$
\[ \lambda = -\frac{S}{2\sqrt{2}} \sqrt{\frac{S}{S^2 - 2|Ric|^2}} \quad \text{if} \quad M < -K. \]

The spaces \( N^3(K, L, M) \) are isometric to \( S^3 \) equipped with a left-invariant metric.

**Remark:** If the parameters \( K = M \) coincide, the solution of the equation \((*)\) is given by

\[ L = \frac{1}{4} K(1 - \sqrt{5}) \quad , \quad L = \frac{1}{4} K(1 + \sqrt{5}) \]

and we obtain the Ricci tensors

\[ Ric = \begin{pmatrix} \frac{1}{2} K^2(\sqrt{5} - 1) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & \frac{1}{2} K^2(\sqrt{5} - 1) \end{pmatrix} \]

or

\[ Ric = \begin{pmatrix} -\frac{1}{2} K^2(1 + \sqrt{5}) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & -\frac{1}{2} K^2(1 + \sqrt{5}) \end{pmatrix}. \]

The non-Einstein-Sasakian metrics on \( S^3 \) occur for the parameter \( K = 1 \) (see [KimF]).

**Remark:** Using the standard basis of the Lie algebra \( so(3) \) we can write the left-invariant metric of the space \( N^3(K, L, M) \) in the following way:

\[ \begin{pmatrix} \frac{1}{|M-L||K+M|} & 0 & 0 \\ 0 & \frac{1}{|K-L||M-L|} & 0 \\ 0 & 0 & \frac{1}{|K-L||K+M|} \end{pmatrix}. \]

The equation \((*)\) is a homogeneous equation of order six. The transformation \((K, L, M) \rightarrow (\mu K, \mu L, \mu M)\) corresponds to a homothety of the metric. Therefore - up to a homothety - the moduli space of solutions is a subset of the real projective space \( \mathbb{P}^2(\mathbb{R}) \) given by the equation \((*)\). This subset is a configuration of six curves in \( \mathbb{P}^2(\mathbb{R}) \) connecting the three points \([K : L : M] = [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\) corresponding to flat metrics.
In particular, we have constructed two paths of solutions of the Einstein-Dirac equation deforming the non-Einstein Sasakian metrics on $S^3$.

2 The integrability condition for the Einstein-Dirac equation in dimension $n = 3$.

The spinor bundle of a 3-dimensional Riemannian manifold is a complex vector bundle of dimension two. Moreover, there exists a quaternionic structure commuting with the Clifford multiplication by real vectors (see [F]). Consequently, in case of a real WK-number $\lambda$, the corresponding space of WK-spinors is a quaternionic vector space. In the spinor bundle let us introduce the metric connection $\nabla^\lambda$ given by the formula

$$\nabla^\lambda_X \psi := \nabla_X \psi - \frac{3}{4} dS(X) \psi - \lambda \left\{ \frac{2}{S} Ric(X) - X \right\} \cdot \psi - \frac{1}{4S} X \cdot dS \cdot \psi$$

and denote by $\Omega^\lambda$ its curvature form. Then we obtain the following

**Proposition 1:** Let $N^3$ be a simply-connected 3-dimensional Riemannian manifold and suppose that the scalar curvature $S \neq 0$ does not vanish. Then the following conditions are equivalent:

1. $N^3$ is a non-trivial solution of the Einstein-Dirac equation with real eigenvalue $\lambda$;
2. $N^3$ admits a WK-spinor with real WK-number $\lambda$;
3. $N^3$ admits two WK-spinors with real WK-number $\lambda$;
4. The curvature $\Omega^\lambda \equiv 0$ vanishes identically.

If the scalar curvature $S \neq 0$ is constant, the condition $\Omega^\lambda \equiv 0$ has been investigated and yields algebraic equations involving the Ricci tensor and its covariant derivative (see [KimF], Theorem 8.3.). In order to formulate the integrability condition, we denote
by $X \times Y$ the vector cross product of two vectors $X, Y \in T(N^3)$. For brevity, let us introduce the endomorphism $T : T(N^3) \to T(N^3)$ given by the formula

$$T(X) = \sum_{i=1}^{3} e_i \times (\nabla_{e_i} Ric)(X),$$

which will be used in the proof of the main Theorem.

**Theorem 1 (see [KimF]):** Let $N^3$ be a simply-connected 3-dimensional Riemannian manifold with constant scalar curvature $S \neq 0$. $N^3$ admits a solution of the Einstein-Dirac equation with real eigenvalue $\lambda$ if and only if the following three conditions are satisfied:

1. $8\lambda^2\{S^2 - 2|Ric|^2\} = S^3$;
2. $8\lambda^2\{SRic(X) - 2Ric \circ Ric(X)\} - 4\lambda ST(X) - S^2 Ric(X) = 0$;
3. $8\lambda^2\{2Ric(X) - SX\} \times \{2Ric(Y) - SY\} + 8\lambda S\{(\nabla_X Ric)(Y) - (\nabla_Y Ric)(X)\} + S^3 X \times Y = 2S^2 \sum_{i<j}\{R_{iY}\delta_{iX} + R_{iX}\delta_{iY}\} e_i \times e_j$.

### 3 Proof of the Main Theorem

We fix an orthonormal frame $e_1, e_2, e_3$ of vector fields on $N^3$ consisting of eigenvectors of the Ricci tensor:

$$Ric = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$  

Denote by $\sigma^1, \sigma^2, \sigma^3$ the dual frame and consider the connection forms $\omega_{ij} = \langle \nabla e_i, e_j \rangle$ of the Levi-Civita connection. The structure equations of the Riemannian manifold $N^3$ are

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} + \frac{C - A - B}{2} \sigma^1 \wedge \sigma^2$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23} + \frac{B - A - C}{2} \sigma^1 \wedge \sigma^3$$

$$d\omega_{23} = \omega_{21} \wedge \omega_{13} + \frac{A - B - C}{2} \sigma^2 \wedge \sigma^3$$

and the covariant derivative $\nabla Ric$ is given by the matrix of 1-forms

$$\nabla Ric = \begin{pmatrix} dA & (A - B)\omega_{12} & (A - C)\omega_{13} \\ (A - B)\omega_{12} & dB & (B - C)\omega_{23} \\ (A - C)\omega_{13} & (B - C)\omega_{23} & dC \end{pmatrix}.$$  

Using the third equation of Theorem 1 we obtain

$$\langle (\nabla_{e_i} Ric)(e_j) - (\nabla_{e_j} Ric)(e_i), e_i \rangle = 0$$

and, consequently,

$$dA(e_1) = dA(e_2) = dB(e_1) = dB(e_3) = dC(e_2) = dC(e_3) = 0.$$
Since $A + B + C = S$ is constant, we conclude that any eigenvalue $A, B, C$ of the Ricci tensor is constant, too. The second equation of Theorem 1 yields the condition that all elements outside the diagonal of the $(1,1)$-tensor $T$ are zero:

\[
(A - B)\omega_{12}(e_1) = 0 = (A - B)\omega_{12}(e_2)
\]
\[
(C - A)\omega_{13}(e_1) = 0 = (C - A)\omega_{13}(e_3)
\]
\[
(B - C)\omega_{23}(e_2) = 0 = (B - C)\omega_{23}(e_3).
\]

First, we discuss the generic case that $A, B, C$ are pairwise different. Then there exist numbers $K, L, M$ such that

\[
\omega_{12} = K\sigma^3, \quad \omega_{13} = L\sigma^2, \quad \omega_{23} = M\sigma^1.
\]

The parameter triples $\{A, B, C\}$ and $\{K, L, M\}$ are related via the structure equations by the formulas

\[
A = -2KL, \quad B = 2KM, \quad C = -2LM.
\]

The first and second equation of Theorem 1 become equivalent to the following system of algebraic equations:

1. \[
\lambda = \pm \frac{S}{2A} \sqrt{\frac{S}{S^2 - 2|Ric|^2}};
\]

2. \[
2S(S^2 - 2|Ric|^2)(A - C)L + (B - A)K)^2 = S(SA - 2A^2) - A(S^2 - 2|Ric|^2)
\]
\[
2S(S^2 - 2|Ric|^2)(C - B)M + (A - B)K)^2 = S(SB - 2B^2) - B(S^2 - 2|Ric|^2)
\]
\[
2S(S^2 - 2|Ric|^2)(B - C)M + (C - A)L)^2 = S(SC - 2C^2) - C(S^2 - 2|Ric|^2).
\]

We solve this system of algebraic equations with respect to the parameters $K, L, M$. It turns out that the equations 2’ can be written in the form

\[
P_i(K, L, M) \cdot Q(K, L, M) = 0,
\]

$(1 \leq i \leq 3)$, where the polynomials $P_1, P_2, P_3$ and $Q$ are given by the formulas

\[
P_1(K, L, M) = (-KL^2 + L^2M + K^2(L + M))^2
\]
\[
P_2(K, L, M) = (KM^2 + LM^2 + K^2(L + M))^2
\]
\[
P_3(K, L, M) = (LM(-L + M) + K(L^2 + M^2))^2
\]
\[
Q(K, L, M) = -K^2L(L - M)^2M + L^3M^3 + KL^2M^2(M - L) + K^3(L - M)(L + M)^2.
\]

The real solutions of $P_1 = P_2 = P_3 = 0$ are the pairs $\{K = 0, L = 0\}$ (the flat metric) and $\{K = M, L = -M\}$ (the space of positive constant curvature). Therefore, we proved that a 3-dimensional complete, simply-connected manifold $N^3$ with constant scalar curvature $S \neq 0$ and different eigenvalues of the Ricci tensor is isometric to one of the spaces $N^3(K, L, M)$, where the parameters $K, L, M$ are solutions of the equation $Q(K, L, M) = 0$. These spaces satisfy the conditions 1. and 2. of Theorem 1 and, moreover, a simple computation yields the result that condition 3. of Theorem 1 is
satisfied, too. We next discuss the case that two of the eigenvalues $A, B, C$ coincide, for example, $A = C \neq B$. Then we obtain again
\[ \omega_{12} = K\sigma^3, \quad \omega_{23} = M\sigma^1, \]
but there is no condition for the connection form $\omega_{13}$. We compute the matrix of the $(1,1)$-tensor $T$:
\[
T = \begin{pmatrix}
(B - C)K & 0 & 0 \\
0 & (C - B)(K + M) & 0 \\
0 & 0 & (B - C)M
\end{pmatrix}.
\]

Since the scalar curvature $S$ as well as the eigenvalues $A = C, B$ of the Ricci tensor are constant, the second equation of Theorem 1 yields that $K$ and $M$ are constant and, moreover, coincide:
\[ K = M = \text{const}. \]

In case of $K = M = 0$ we have $\omega_{12} = \omega_{23} = 0$ and $A = C$. In particular, the Ricci tensor is parallel, $\nabla \text{Ric} = 0$. Therefore, in this case $N^3$ is a Ricci-parallel 3-dimensional manifold admitting a WK-spinor. Then $N^3$ is either flat or a space of constant positive curvature (see [KimF], Theorem 8.2.). Finally, we consider that the case of $K = M = 1$, i.e., $\omega_{12} = \sigma^3$ and $\omega_{23} = \sigma^1$. Differentiating the equation $\omega_{12} = \sigma^3$, we obtain
\[ \omega_{13} \wedge \omega_{32} - \frac{B}{2}\sigma^1 \wedge \sigma^2 = d\omega_{12} = d\omega^3 = \omega_{31} \wedge \sigma^1 + \omega_{32} \wedge \sigma^2 - \frac{B}{2}\sigma^1 \wedge \sigma^2 = -\sigma^1 \wedge \sigma^2. \]

Consequently, $B = 2$ and the tensors $T$ and $\text{Ric}$ are given by the matrices
\[
T = \begin{pmatrix}
2 - C & 0 & 0 \\
0 & 2(C - 2) & 0 \\
0 & 0 & 2 - C
\end{pmatrix}, \quad \text{Ric} = \begin{pmatrix}
C & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & C
\end{pmatrix}.
\]

The second condition of Theorem 1 yields the equations ($S = 2 + 2C$):
\[
8\lambda^2(SC - 2C^2) - 4\lambda S(2 - C) - S^2C = 0
\quad 8\lambda^2(2S - 8) + 8\lambda S(2 - C) - 2S^2 = 0.
\]

Solving these equations with respect to $\lambda$ and $C$ we obtain the three solutions:

1. $C = 2$ and $\lambda = \pm \frac{3}{2}$. Then $N^3$ is isometric to $S^3$.

2. $C = -1$ and $\lambda = 0$. Then the scalar curvature $S = 0$ is zero.

3. $C = \frac{1}{2}(-1 \pm \sqrt{5})$ and $\lambda = 1 \pm \frac{\sqrt{5}}{2}$. These metrics are the non-Einstein Sasakian metrics on $S^3$ admitting WK-spinors (see [KimF]). The corresponding space is contained in the family $N^3(K, L, M)$.

We have discussed all possibilities and, therefore, we have finished the proof of the main Theorem.

\[\blacksquare\]
4 The moduli space of solutions

The moduli space of all 3-dimensional Riemannian manifolds with constant scalar curvature $S \neq 0$ and WK-spinors is given by the triples \( \{K, L, M\} \) of real numbers satisfying the equation of order six \( Q(K, L, M) = 0 \). The polynomial \( Q \) is symmetric in \( \{K, -L, M\} \). Denote by
\[
\gamma_1 = K - L + M, \quad \gamma_2 = -KL + KM - LM, \quad \gamma_3 = -KLM
\]
the elementary symmetric functions of these variables. Then we have
\[
Q = 4\gamma_1\gamma_2\gamma_3 - \gamma_2^3 - 4\gamma_3^2.
\]
Consider the projective variety \( V_C \subset \mathbb{P}^2(\mathbb{C}) \) defined by the homogeneous polynomial \( Q \):
\[
V_C = \left\{ [K : L : M] \in \mathbb{P}^2(\mathbb{C}) : Q(K, L, M) = 0 \right\}.
\]
\( V_C \) has three singular points:
\[
V_C^{\text{sing}} = \{ [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1] \}
\]
and these points correspond to the flat metric. We will now parametrize the variety \( V_C \) by two meromorphic functions defined on a smooth Riemann surface. \( V_C \) is given by the equation \( (K = 1) \):
\[
Q(1, L, M) = L^3(M - 1)^2(M + 1) + L^2M(1 + M)^2 - LM^2(1 + M) - M^3 = 0.
\]
Let us introduce the variables
\[
a = M - L - LM, \quad b = (L - M)LM.
\]
Then we obtain \( Q(1, L, M) = -a^3 + 4b(1 + a) \) and the equation defining the variety \( V_C \) becomes much simpler:
\[
b = \frac{1}{4} \frac{a^3}{1 + a}.
\]
Next we consider a square root of \( a + 1 \) and we solve the equations
\[
z^2 - 1 = a = M - L - LM, \quad \frac{1}{4} \frac{(z^2 - 1)^3}{z^2} = b = (L - M)LM
\]
with respect to \( L \) and \( M \). Then we obtain four solution pairs \( \{L, M\} \) depending on the variable \( z \). For example,
\[
L(z) = \frac{-(1 + z)(1 - 2z + z^2 + \sqrt{(1 + z)(1 + 3z - 5z^2 + z^3)})}{4z},
\]
\[
M(z) = \frac{(1 + z)(1 - 2z + z^2 + \sqrt{(1 + z)(1 + 3z - 5z^2 + z^3)})}{4z}.
\]
The polynomial
\[
(z + 1)(1 + 3z - 5z^2 + z^3) = (z + 1)(z - 1)(z + (2 + \sqrt{5}))(z + (2 - \sqrt{5}))
\]
has four different zeros. The square root $\sqrt{(1 + z)(1 + 3z^2 - 5z^2 + z^3)}$ is a meromorphic function on the compact Riemann surface of genus $g = 1$. Consequently, there exists a torus $\mathbb{C}/\Gamma$ and elliptic functions $L, M : \mathbb{C}/\Gamma \to \mathbb{P}^1(\mathbb{C})$ such that the components of the variety $V_C \setminus V_C^{\text{sing}}$ are parametrized by $L$ and $M$:

$$V_C = \left\{ [1 : L(z) : M(z)] : z \in \mathbb{C}/\Gamma \right\}.$$ 

The functions $L - M$ and $L \cdot M$ are given by the formulas:

$$L - M = -\frac{(1 + z)(z - 1)^2}{2z}, \quad L \cdot M = -\frac{(1 + z)^2(z - 1)}{2z}.$$ 

The moduli space we are interested in coincides with the real points of the projective variety $V_C$. If $K = 0$, the only solutions of the equation $Q(0,L,M) = 0$ are $L = 0$ or $M = 0$, i.e., the points $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Therefore we can parametrize the moduli space by the parameter $M \in \mathbb{R}$ solving the equation $Q(1,L,M) = 0$ with respect to $L = L(M)$. In this way we obtain a configuration of six curves in $\mathbb{P}^2(\mathbb{R})$ connecting the three singular points of $V_C$ (see the figure in the Introduction). However, we obtain geometrically different metrics on $S^3$ only for two curves parametrized by the real parameter $0 \leq M \leq \infty$. The graphs of the function $L_{\pm}(M)$ are given in the following figure:

(Figure 1)

The functions $L_{\pm}(M)$ are monotone and tend to $\pm 1$ in case that $M$ tends to infinity. Let us discuss the geometric invariants of these metrics. The graph of the scalar curvatures $S_{\pm}(M)$ depending on $M$ is given by the next figure:

(Figure 2: The scalar curvatures)
Next we plot the eigenvalues $A_{\pm}(M)$, $B_{\pm}(M)$, $C_{\pm}(M)$ of the Ricci tensor for both families of metrics:

(Figure 3: The eigenvalues of the Ricci tensor for $L_+(M)$)

(Figure 4: The eigenvalues of the Ricci tensor for $L_-(M)$)

In dimension $n = 3$ the number

$$\lambda^2(D) \cdot [\text{vol}(N^3)]^\frac{1}{8}$$

is a homothety invariant, where $\lambda(D)$ is an eigenvalue of the Dirac operator. In case of a WK-spinor we have

$$\lambda^2 = \frac{1}{8} \frac{S^3}{S^2 - 2|\text{Ric}|^2}$$

and, therefore, we obtain the formula

$$\lambda^2 \cdot \text{vol}^\frac{1}{8} = \frac{1}{8} (2\pi^2)^\frac{3}{8} \frac{S^3}{S^2 - 2|\text{Ric}|^2} \left\{ |K - L||M - L||K + M| \right\}^\frac{1}{8}$$.

The next figures contain the graph of $\lambda^2 \cdot \text{vol}^\frac{1}{8}(M)$ depending on the parameter $M$ for both families of metrics.
Finally, let us discuss the behaviour of the rational function

\[ \Psi = \frac{L^2}{KM} \]

on the variety \( V_C \subset \mathbb{P}^2(\mathbb{C}) \). It turns out that \( \Psi \) has simple zeros at the singular points \([1 : 0 : 0]\) and \([0 : 0 : 1]\). Indeed, solving the equation defining \( V_C \) with respect to \( L = L(M) (K = 1) \) we obtain

\[ \lim_{M \to 0} \frac{L^2(M)}{M} = 0, \quad \lim_{M \to 0} \frac{d}{dM} \left( \frac{L^2(M)}{M} \right) = 1. \]

The third singular point \([0 : 1 : 0]\) is a pole of order two. In the regular part of \( V_C \) the function \( \Psi \) has 12 ramification points. Among them 10 points are first order ramification points. The ramification points of order two are the points

\[ [K : L : M] = [1 : \frac{1}{4}(1 \pm \sqrt{5}) : 1]. \]

These parameters correspond precisely to the non-Einstein Sasakian metrics on \( S^3 \) admitting solutions of the Einstein-Dirac equation.
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