Monitoring Quantum Oscillations with very small Disturbance

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We present a new scheme to detect and visualize oscillations of a single quantum system in real time. The scheme is based upon a sequence of very weak generalized measurements, distinguished by their low disturbance and low information gain. Accumulating the information from the single measurements by means of an appropriate Bayesian Estimator, the actual oscillations can be monitored nevertheless with high accuracy and low disturbance. For this purpose only the minimum and the maximum expected oscillation frequency need to be known. The accumulation of information is based on a general derivation of the optimal estimator of the expectation value of a hermitian observable for a sequence of measurements. At any time it takes into account all the preceding measurement results.

I. INTRODUCTION

Consider a two-level quantum system where the probability to find the system in a projection measurement on a specific level oscillates due to a periodically time-dependent external potential. We propose a measurement scheme which allows to monitor these oscillations by means of a sequence of consecutive measurements carried out on a single two-level system. As in the case of the detection of gravitational waves we assume that we are dealing with a one shot experiment, i.e. the measurements have to be carried out on a single quantum system and the experiment cannot be repeated in order to acquire more measurement data. The proposed measurement scheme yields a real time record of the actual oscillations, which are sometimes called Rabi oscillations. This is done by appropriately estimating after each measurement in the sequence the actual value of the oscillating probability.

In contrast to measurements in classical physics, quantum measurements have the following prominent feature: The more information they provide, the more they change the state of the measured system. The most precise measurements are von Neumann projection measurements. They project a quantum system in an eigenstate of the measured observable, which represents in general a drastic disturbance of the system’s state.

There is a broader class of measurements called generalized measurements, which can be realized by coupling the system in question to an ancilla system and carrying out a projection measurement on the ancilla. Depending on the kind of coupling and its strength these indirect measurements can exert an influence on the system which ranges from very weak to very strong. They can be elegantly described in the POVM formalism [1]. In order to keep the disturbance caused by a sequence of measurements low, we employ generalized measurements with very weak influence. On the other hand, the weaker the influence of these measurements is, the less information about the measured system they convey. This disadvantage can be compensated by accumulating data from all measurements. A corresponding data processing scheme is proposed below.

The strength of the influence of a sequence of measurements can be determined by considering the case where, apart from the measurements, no other dynamical influence is present. In the example of the oscillating two-level system this corresponds to screening or turning off the time-dependent potential. For an appropriate sequence of measurements a measure for the strength of the influence is then given by the “decoherence time”. That is the period after which the coherences of the systems’ state have decayed to $1/e$ of their initial value (for qubits cp. [2]). The undisturbed dynamics of the system, on the other hand, can be characterized by the time scale $T_R$, which is the period of the oscillations, if no measurements are carried out.

Comparing the decoherence time $T_d$ to the period $T_R$, roughly three modes or regimes of measurement can be distinguished [3]: (i) for $T_d \gg T_R$ the system evolves approximately according to its undisturbed dynamics, i.e. the disturbing influence of the measurements is comparatively small. (ii) $T_d \approx T_R$, both dynamical influences are equally strong. (iii) $T_d \ll T_R$, decoherence induced by the measurements dominates the dynamics of the system.

Looking in mode (iii) at the systems dynamics in the selective regime, i.e. given certain measurement outcomes, one finds quantum jumps or in the limit of a continuous projection measurement the Quantum Zeno effect, where the system freezes in an eigenstate of the measured observable.

In [3] it was shown that a detection of the oscillations of a two-level system under the influence of an external field with reasonable accuracy and disturbance is possible employing mode (ii). A physical realization of this measurement scheme was proposed by probing a photon oscillating between two cavities by a sequence of Rydberg atoms [4], based on an experiment of Haroche et al. [5].

In contrast to the latter investigations we want to show that the results can be improved by working in mode (i). The advantage of mode (i) obviously is the weak influence and thus the low disturbance inflicted by the measurements. On the other hand mode (i) represents a
challenge because there the measurement results are only poorly correlated to the actual state evolution. In order to overcome this difficulty we study optimal estimates in sections III and IV. The presented approach is rather general. We optimally estimate the expectation value of an arbitrary hermitian observable of a quantum system with finite dimensional Hilbert space, given the result of a generalized measurement (which can also consist of a sequence of consecutive measurements). The result can be applied to our special case to estimate the probability to find a two-level system in a projection measurement on a specific level. This is done in section IV after giving a brief description of this scheme. The results of numerical simulations of our measurement scheme are discussed. An appendix contains a recipe for these simulations with useful formulae to abbreviate the computations.

II. ESTIMATOR FOR MEAN VALUES OF OBSERVABLES

There probably exists considerably more literature on state-estimation than on the estimation of mean values of physical quantities such as energy, position or spin of quantum systems. Nevertheless there might be questions which do not require the maximal knowledge of the statistics of all measurements that can be carried out on a quantum system—as it is represented by the state of the system—but rather the knowledge of a single physical property such as the mean position of a quantum particle. For example, when attempting to detect gravitational waves, only the mean spacial distances between test masses have to be estimated at different times. Of course the calculation of the mean value of an observable as well as state determination is not an issue if a large ensemble of identically prepared systems is available to be measured. But for experiments with restricted resources and especially in one-shot experiments estimation procedures become essential.

Let us consider the following task. Given a quantum system with \(d\)-dimensional Hilbert space \(\mathcal{H}\). After a POVM measurement with result \(m\) the state of the system reads

\[
|\psi_m\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m|\psi)}},
\]

where \(M_m\) is the Kraus operator corresponding to the measurement result \(m\), and

\[
p(m|\psi) = \langle \psi | M_m^\dagger M_m |\psi\rangle
\]

represents the probability to obtain \(m\), provided the state before the measurement was \(|\psi\rangle\). Let us assume that we know the result \(m\) and the respective Kraus operator \(M_m\) but we do not know the initial state \(|\psi\rangle\). What is the best way to estimate the parameter

\[
\theta_m := \langle \psi_m | A |\psi_m\rangle,
\]

which represents the expectation value of the observable \(A\) with respect to the state \(|\psi_m\rangle\)?

It turns out to be rewarding to base the parameter estimation on the least squared error criterion: The value \(g_m\) is an optimal estimate of the parameter \(\theta_m\) if it minimizes the expected square of the error

\[
E \left( (\theta_m - g_m)^2 \right) = \frac{\int (\theta_m - g_m)^2 p(m|\psi)p(\psi)d\psi}{\int p(m|\psi)p(\psi)d\psi}, \quad (4)
\]

For the sake of simplicity we assume no prior knowledge about the initial state \(|\psi\rangle\), i.e. \(p(\psi) = 1\) and \(d\psi\) is a normed measure over the set of all pure states which is invariant under the action of the rotation group \(SU(d)\).

Taking into account the linearity of the expectation value, it is easy to see that in this case the optimal estimator \(g_m\) is equal to the expected value of \(\theta_m\):

\[
E \left( (\theta_m - g_m)^2 \right) = E ((\theta_m)^2) - 2gE(\theta_m) + g_m^2
\]

\[
= (E(\theta_m) - g_m)^2 + \text{Var}(\theta_m), \quad (5)
\]

where

\[
\text{Var}(\theta_m) = E((\theta_m)^2) - E(\theta_m)^2 \quad (6)
\]

represents the variance of \(\theta_m\). The right-hand side of (5) assumes a minimum for \(g_m = E(\theta_m)\). Such a value \(g_m\) is also called the Bayesian estimate (cp. [7]).

Evaluating \(g_m\) we obtain:

\[
g_m = \frac{\int \theta_m p(m|\psi)d\psi}{\int p(m|\psi)d\psi} = \frac{\int \langle \psi | M_m^\dagger A M_m |\psi\rangle d\psi}{\int \langle \psi | M_m^\dagger M_m |\psi\rangle d\psi} = \frac{\text{tr}[M_m^\dagger A M_m]}{\text{tr}[M_m^\dagger M_m]} \quad (7)
\]

This quantity represents the best estimate of the expectation value of observable \(A\) after one single generalized measurement with result \(m\), if the state before the measurement is completely unknown. It is the best estimate in the sense that it leads to the least expected squared error.

Formula (7) can in particular be applied to estimate, after a generalized measurement, the probability to find a system with two levels 0 and 1 on level 1. In this case \(A\) should be chosen to be the projector on level 1 since the expectation value of this projector is equal to the desired probability.

III. ESTIMATOR FOR SEQUENTIAL MEASUREMENTS

In this section we derive an optimal estimator for a sequence of measurements. For the sake of broad applicability we consider the general case of a sequence of
generalized measurements carried out on a single $d$-level system with unknown Hamiltonian.

The single measurements with Kraus operators $N_{m_n}$ are carried out consecutively on a single quantum system at times $t = n \tau$, where $n$ is an integer. Here the number $m_n$ represents the result of the $n$-th measurement. The single measurements are of duration $\delta \tau$ and during this time the motion due to the system’s Hamiltonian $H$ can be neglected (impulsive measurement approximation). Between two consecutive measurements, the system evolves according to the unitary operator

$$U = \exp \frac{i}{\hbar} H \tau.$$  \hspace{1cm} (8)

For later convenience we express the unitary evolution by means of a unit vector $k \in \mathbb{R}^{d^2 - 1}$ with components $k_j$ and an angle $0 \leq \phi < 2 \pi$:

$$U = \exp \frac{i}{\hbar} H \tau = \exp ik \cdot \text{e}^{\phi},$$ \hspace{1cm} (9)

where $k \cdot \text{e} = \sum_j k_j e_j$, and the $e_j$ form a complete set of generators of $SU(d)$. The state $|v_m\rangle$ of the system after $n$ measurements with results $(m_1, \ldots, m_n) = m$ is then given by equation (11) with Kraus operator

$$M_m = N_{m_1} U N_{m_2} U \ldots N_{m_n} U$$ \hspace{1cm} (10)

instead of $M_m$.

As above we assume that we know the results of the $n$ measurements and, consequently, the corresponding Kraus operators $N_{m_1}, \ldots, N_{m_n}$, but we know neither the unitary evolution $U$ between the measurements nor the initial state $|\psi\rangle$ of the system. Our ignorance about $U$ has to be incorporated into the optimal estimate of the observable $A$ after the $n$-th measurement. Instead of only averaging over all possible initial states as in (3) we also have to average over all possible unit vectors $k$ and possible angles $\phi$ weighted by the corresponding probabilities $p(k)$ and $p(\phi)$:

$$g_m = \frac{\int \theta_m p(m|\psi, k, \phi)p(k)p(\phi)d\psi d^2-1 k d\phi}{\int p(m|\psi, k, \phi)p(k)p(\phi)d\psi d^2-1 k d\phi}.$$ \hspace{1cm} (11)

We assume that the direction of $k$ and the angle $\phi$ are equally distributed, i.e. $p(k)=$ const and $p(\phi)=$ const. The optimal estimator $g_m$ is then given by

$$g_m = \frac{\int \theta_m p(m|\psi, k, \phi)d\psi d^2-1 k d\phi}{\int p(m|\psi, k, \phi)d\psi d^2-1 k d\phi} = \frac{\text{tr}[M_m^\dagger A M_m]d^2-1 k d\phi}{\text{tr}[M_m^\dagger M_m]d^2-1 k d\phi}.$$ \hspace{1cm} (12)

For unknown unitary evolution, $g_m$ represents the estimate of the expectation value of observable $A$ after $n$ measurements with results $m = (m_1, \ldots, m_n)$. It minimizes the expected squared error. Calculating $g_m$ after each measurement in a sequence of measurements yields an optimally updated estimate of the current mean value of observable $A$.

IV. APPLICATION; TRACKING AN OSCILLATING QUBIT

We will now apply the estimator given in equation (12) to the sequential measurement of an oscillating qubit. Oscillating qubits are realized for example by two-level systems such as coupled quantum dots, trapped atoms in an external field or photons oscillating between two microwave cavities [8]. For the sake of concreteness, we consider a two-level atom under the influence of a resonant laser field. The Hamiltonian of such an atom can be approximated by

$$H = E_0 |0\rangle\langle 0| + E_1 |1\rangle\langle 1|$$ \hspace{1cm} (13)

$$+ \frac{\hbar \Omega_R}{2} (|1\rangle\langle 0| \exp(-i\omega t) + |0\rangle\langle 1| \exp(i\omega t)),$$

where $\omega = (E_1 - E_0)/\hbar$, and the Rabi frequency $\Omega_R$ represents the strength of the coupling between the atom and the electromagnetic field. The resulting motion of a state that is not subjected to measurement is represented by

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$ \hspace{1cm} (14)

with $|c_1|^2 = \frac{1}{2}(1 + a \cos(\Omega_R t + \varphi))$ where the constants $a$ and $\varphi$ depend on the initial state of the qubit.

In order to observe the actual behaviour of the system in real time, we have to measure and estimate the expectation value of $|1\rangle\langle 1|$, which is equal to $|c_1(t)|^2$. In the following we take the viewpoint that we already know the Hamiltonian (13) apart from the precise value of the coupling strength $\Omega_R$. This is quite a natural assumption, since the form of $H$ has to be known in order to design measurements, which requires the knowledge of how to couple a meter to the system.

In a first step we choose the Kraus operators. To obtain information about the observable $|1\rangle\langle 1|$, it seems natural to employ measurements the effects $(N_0^m N_1^m)$ of which commute with $|1\rangle\langle 1|$ (cp. [8]). Because such effects are diagonal with respect to the basis states $|0\rangle$ and $|1\rangle$, i.e.

$$N_0^m N_1^m = p_0^m |0\rangle\langle 0| + p_1^m |1\rangle\langle 1|,$$ \hspace{1cm} (15)

the probability to obtain the result $m$ depends directly on $|c_1(t)|^2$:

$$p(m|\psi(t)) = \langle \psi(t)|N_0^m N_1^m |\psi(t)\rangle,$$

$$= p_0^m + \Delta p^m |c_1|^2,$$ \hspace{1cm} (16)

where $\Delta p^m := p_1^m - p_0^m$. For the sake of simplicity we consider measurements with two possible results, + and −, and Kraus operators $N_\pm = U_\pm \sqrt{N_0^m N_1^m}$ with trivial unitary part, i.e. $U_\pm = 1$. The Kraus operators of each single measurement thus read

$$N_+ := \sqrt{p_0^m} |0\rangle\langle 0| + \sqrt{p_1^m} |1\rangle\langle 1|,$$ \hspace{1cm} (17)

$$N_- := \sqrt{p_0^m} |0\rangle\langle 0| + \sqrt{p_1^m} |1\rangle\langle 1|.$$

(18)
with positive numbers $p_j^\pm$ which satisfy $p_j^+ + p_j^- = 1$. A detailed analysis of optimal Kraus operators from the viewpoint of Bayesian estimates will be presented elsewhere.

The change of state caused by consecutive measurements with Kraus operators $N_\pm$ as given in Eqs. (17) can be quantified by the decoherence time $T_d$. That is the average period after which the off-diagonal elements $\langle i|\rho(t)|j \rangle$ with $i \neq j \in \{0, 1\}$ of the systems density operator $\rho(t)$ are decayed to $1/e$ of their original value, if the only dynamical influence is given by the measurements. $T_d$ is related to the decoherence rate $\gamma$ by

$$T_d = \frac{2}{\gamma} = \frac{8\pi \bar{p}(1 - \bar{p})}{(\Delta p)^2}$$

with $\bar{p} := (p_0^+ + p_1^+)/2$ and $\Delta p := p_1^+ - p_0^+$. $\tau$ is the time which passes between two consecutive measurements. If unitary dynamics generated by the Hamiltonian $H$ are present, they can create coherences representing a counter weight to the decoherence caused by the measurements. For

$$T_d \gg T_R,$$  \hspace{1cm} (20)

where $T_R = 2\pi/\Omega_R$ is the period of the Rabi oscillation, the influence of the single measurements on the state of the system becomes negligibly small as compared to the influence of the unitary dynamics. In this mode, which was called mode (i) in the introduction, we run our sequential measurement.

Apart from condition (20) there is another requirement for the sequence of measurements: a reasonably high number of measurements should take place on the time scale $T_R$ of the unitary dynamics in order to resolve these dynamics. Hence,

$$T_R \gg \tau.$$  \hspace{1cm} (21)

According to our experience based on numerical simulations $T_R/\tau \approx O(10)$ is sufficient to obtain good results (see below). Note that in order to meet conditions (20) and (21) only a vague knowledge about the order of magnitude of the time scale $T_R$ and accordingly of $\Omega_R$ is necessary. Knowing an upper bound $T_R^\geq$ and a lower bound $T_R^\leq$ for $T_R$, both conditions can always be satisfied by first inserting the lower bound $T_R^\leq$ into (21) and choosing $\tau$ accordingly. Having fixed the value of $\tau$, the parameters $p_0$ and $p_1$ can be tuned such that

$$8\bar{p}(1 - \bar{p})/(\Delta p)^2 \gg T_R^\geq/\tau.$$  \hspace{1cm} (22)

In other words: the sequential measurement has to consist of frequent measurements which are sufficiently weak.

Since the influence of the measurements obeying (20) is very weak on the time scale $T_R$ they convey only very little information about the system over the period $T_R$. This is where the accumulation of information by means of the Bayesian estimate (12) comes into play. The estimate after the $n$-th measurement makes the best possible use of the data collected in the previous $n-1$ measurements according to the least square criterion.

In our special case the Bayesian estimate given in (12) reduces to

$$g = \frac{\int \text{tr}[M_m^\dagger AM_m]d\phi}{\int \text{tr}[M_m^\dagger M_m]d\phi}$$

with Kraus operators $M_m$ given by Eq. (16) and (17). The unitary evolution $U$ in $M_m$ (cp. Eq. (16)) is most easily represented in the interaction picture, where Hamiltonian (13) reads $H_I = \hbar \Omega R \sigma_x/2$ with the Pauli spin operator $\sigma_x := |1\rangle \langle 0| + |0\rangle \langle 1|:

$$U = \exp\left(-i \frac{\sigma_x \phi}{2}\right).$$

Here the angle $\phi$ of rotation on the Bloch sphere is given by

$$\phi = \Omega_R \tau = \frac{2\pi \tau}{T_R}.$$  \hspace{1cm} (25)

Because of condition (21), which guarantees the temporal resolution of the Rabi oscillations by the sequence of measurements, the angle $\phi$ is in fact very small. We did not include this a priori information into the estimate $g_m$ used in our numerical simulations. Instead, we let $\phi$ run from $0$ to $2\pi$ in the integral in equation (12). This corresponds to a completely unknown Rabi period $T_R$. An appropriate change of the range of integration might lead to an improvement of the estimate $g_m$.

Note that the representation of the Kraus operators $N_\pm$ in the interaction picture is the same as in the Schrödinger picture used so far:

$$N_{\pm}^{(1)}(t) = e^{\mp \frac{\pi}{2} H_0(t-t_0)} N_\pm e^{-\pm \frac{\pi}{2} H_0(t-t_0)} = N_\pm$$

with $H_0 := H_0|0\rangle \langle 0| + H_1|1\rangle \langle 1|$.

V. RESULTS AND CONCLUSION

In Fig.4 the result from a numerical simulation of a sequential measurement as specified above is plotted. The simulation started with the initial state $|\psi(t_0)\rangle = |1\rangle$. The solid curve represents the evolution of the estimate $g$ and the dashed curve corresponds to the dynamics of $|c_1|^2$, taking into account the influence of the measurements. This represents what really happens. The dotted curve displays how $|c_1|^2$ would have evolved without measurements.

In the beginning of the sequential measurement (upper picture in Fig.4) the curves of $|c_1|^2$ with and without measurements are close; thus the disturbance due to the measurements is small. A small phase shift of the oscillations however is recognizable. The values of the estimate $g$ are not well correlated to the values of $|c_1|^2$. After many measurements (middle picture in Fig.4) the estimate $g$ starts to approximate the disturbed $|c_1|^2$ values, while
the phase shift between the latter and the values of $|c_1|^2$ without measurements has increased. The amplitude of the oscillation of $|c_1|^2$ is not changed by the measurements if—as seen here—it equals one in the absence of measurements. Eventually (lower picture of Fig. 1) estimate $g$ and the curve of actual values of $|c_1|^2$ nearly coincide.

The main results of the simulations are: i) With increasing time the estimate $g$ reflects the actual oscillations with growing fidelity. After approximately hundred Rabi cycles these oscillations are monitored by $g$ with high accuracy. ii) In the presence of the weak measurements the sinusoidal shape and the period of the Rabi oscillation is almost the same as in the absence of measurements. The measurements however cause a phase shift of the oscillation. Therefore the estimate $g$ also reflects shape and period of the undisturbed Rabi oscillation.

This demonstrates that our approach allows to monitor the periodic evolution of an expectation value with high fidelity. The key to the monitoring are measurements with very low disturbance combined with an estimator which accumulates at any time the information gained in the sequence of previous measurements.

VI. APPENDIX

Simulations of the tracking procedure explained in section IV were performed with a program [9] that is based on the following algorithm.

1. Initialize the qubit’s state vector $|\psi\rangle$ and the number of measurements, $n_{\text{max}}$. Set $n = 1$.
2. Evolve $|\psi\rangle$ in time: $|\psi\rangle \rightarrow e^{-iH\tau/\hbar}|\psi\rangle$.
3. Perform measurement:
   a) Generate a (pseudo) random number $m_n$ whose value is either 0 or 1, depending on the probability $p_{m_n} = \langle \psi|N^*_m N_m|\psi\rangle$.
   b) Update $|\psi\rangle$: $|\psi\rangle \rightarrow (N_{m_n}/\sqrt{p_{m_n}})|\psi\rangle$.
4. Calculate the estimator according to formula (26).
5. If $n < n_{\text{max}}$, then continue at step 2 and increment $n$ by 1.

In the program, the estimator is calculated using a variation of formula (26) that does not contain any integrals. Before specifying this variation, let us introduce the abbreviation $N_{jk} = \langle j|N_{m_n}|k\rangle$ and the coefficient $\delta_{mk}$ which is 1 for $l \leq k \leq m$ and 0 otherwise. Then, the following expression is equal to (26) for all Hamiltonians and all sets of measurement operators (proof is omitted):

$$g = \frac{F_{11}}{F_{00} + F_{11}}; F_{jj} = \sum_{k=0}^{n-1} a_{jj(2k)}^{(n)} b_k(n-1-k)$$

with

$$b_{kl} = \frac{2}{(k+l)!} \Gamma(k + \frac{1}{2}) \Gamma(l + \frac{1}{2})$$

and the recursive relation

$$a_{jk}^{(n=0)} = 0, a_{jk}^{(n=1)} = \sum_{p=0}^{1} N_{jp} N_{lp}^*, a_{jk}^{(n=2)} = \sum_{p=0}^{1} \sum_{q=0}^{(n-1)} (\delta_{2(2n-2)} N_{jp} N_{lp}^* a_{pq(k-2)}^{(n-1)} + \delta_{0(2n-4)} N_{j(1-p)} N_{l(1-q)}^* a_{pq(k-1)}^{(n-1)} + \delta_{0(2n-4)} N_{j(1-p)} N_{l(1-q)}^* a_{pq(k-1)}^{(n-1)})$$

Note, that in order to avoid convergence problems caused by low precision floating point data types, the program uses data types provided by the GMP library [10] for the calculation of the estimator.

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9] The program’s source code is available from the authors upon request.

10] The GNU Multiple Precision Arithmetic Library is available from http://www.swox.com/gmp/.