The tensor structure on the representation category of the $\mathcal{W}_p$ triplet algebra

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Abstract

We study the braided monoidal structure that the fusion product induces on the Abelian category $\mathcal{W}_p$-mod, the category of representations of the triplet $\mathcal{W}$-algebra $\mathcal{W}_p$. The $\mathcal{W}_p$-algebras are a family of vertex operator algebras that form the simplest known examples of symmetry algebras of logarithmic conformal field theories. We formalize the methods for computing fusion products, developed by Nahm, Gaberdiel and Kausch, that are widely used in the physics literature and illustrate a systematic approach to calculating fusion products in non-semi-simple representation categories. We apply these methods to the braided monoidal structure of $\mathcal{W}_p$-mod, previously constructed by Huang, Lepowsky and Zhang, to prove that this braided monoidal structure is rigid. The rigidity of $\mathcal{W}_p$-mod allows us to prove explicit formulae for the fusion product on the set of all simple and all projective $\mathcal{W}_p$-modules, which were first conjectured by Fuchs, Hwang, Semikhatov and Tipunin; and Gaberdiel and Runkel.

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1. Introduction

The theory of vertex operator algebras is an algebraic approach to describing the chiral symmetry algebras of conformal field theories, at least when the number of irreducible representations of the symmetry algebra is finite [1–6]. Over the last few years a class of conformal field theories, called logarithmic conformal field theories, has been the subject of a lot of research. Logarithmic conformal field theories appear in the description of critical points of a number of interesting physical systems. Examples are polymers, spin chains, percolation and sand-pile models [7–12]. Logarithmic conformal field theories generalize the conformal field theories most commonly considered, by allowing the singularities encountered in correlation functions, when two field insertion approach each other, to be logarithmic divergences rather than just poles [13]. Two necessary consequences of the logarithmic
divergences are that $L_0$ the generator of scale transformations is no longer diagonalizable and that the representation theory of the symmetry algebra is non-semi-simple. The non-semi-simplicity in particular has made it quite challenging to find rigorous mathematical classifications of the representations of symmetry algebras associated with logarithmic conformal field theories.

Arguably the two best understood families of vertex operator algebras associated with logarithmic conformal field theories are the $\mathcal{W}_p^+$ and the $\mathcal{W}_{p_+,p_-}$-series, where $p \geq 2$ and $p_\pm \geq 2$, with $p_+, p_-$ coprime respectively. The $\mathcal{W}_p^+$-series is by now quite well understood [14–23]. It was shown in [24] that the $\mathcal{W}_2$ triplet algebra satisfies Zhu’s $c_2$-cofiniteness condition. The same result for general $p \geq 2$ was shown in [17, 21]. The representation category $\mathcal{W}_p^+$-mod was completely classified in [25] as a $\mathbb{C}$-linear Abelian category. The representation theory of $\mathcal{W}_{p_+,p_-}$-series is not so well understood yet. There are well supported conjectures for lists of all irreducible and all projective representations, but there is still a lot of work to be done [19, 26–30]. For all odd $p_- \geq 3$ it was shown in [31–33] that the triplet algebra $\mathcal{W}_{2,p_-}$ satisfies Zhu’s $c_2$-cofiniteness condition and additionally the centre of Zhu’s algebra was determined.

It was shown by Huang, Lepowsky and Zhang in the series of papers [16, 34–43] that the fusion product of $\mathcal{W}_p^+$-representations induces a braided monoidal structure on $\mathcal{W}_p^+$-mod, the representation category of the $\mathcal{W}_p^+$ triplet algebra. The purpose of this paper is to analyse the fusion product of $\mathcal{W}_p^+$-representations by making heavy use of all that is known of $\mathcal{W}_p^+$-mod as an Abelian category. The main results can be summarized as follows:

- Section 2.3 in which a systematic description is given on how to define and compute fusion products in a non-semi-simple setting.
- Theorem 40 which states that the braided monoidal structure of $\mathcal{W}_p^+$-mod induced by fusion is rigid.
- Theorem 44 which gives explicit formulae for the fusion products of all simple and all projective $\mathcal{W}_p^+$-modules.

The formulae in theorem 44 were first conjectured in [15, 20].

As a final comment we would like to note that the $\mathcal{W}_p^+$-series is closely related to quantum groups at roots of unity [18, 19]. Indeed it was shown in [25] that the representation categories of $\mathcal{W}_p^+$ and its corresponding quantum group are equivalent as Abelian categories. It was shown in [44] that the standard quantum group tensor product cannot coincide with the $\mathcal{W}_p^+$ fusion product, because it is not braided.

The paper is organized as follows. In section 2 we introduce our notation for vertex operator algebras, give a short definition of monoidal categories and explain how to define and compute the fusion product in the representation category $V$-mod of an arbitrary $c_2$-cofinite vertex operator algebra $V$. In section 3 we introduce the $\mathcal{W}_p^+$ triplet algebra and its representation category $\mathcal{W}_p^+$-mod. Sections 2 and 3 are introductory and serve to familiarize the reader with fusion products, the $\mathcal{W}_p^+$-algebra, representations of the $\mathcal{W}_p^+$-algebra and to introduce our notation.

Section 4 contains a detailed analysis of two simple $\mathcal{W}_p^+$-modules which we call $X_{1^-}$ and $X_{2^+}$. We calculate their fusion products with all simple $\mathcal{W}_p^+$-modules and prove that they are rigid objects in $\mathcal{W}_p^+$-mod. This section relies heavily on the notions discussed in sections 2.2 and 2.3. In section 5 we prove this paper’s two main theorems 40 and 44 by exploiting the rigidity of $X_{1^-}$ and $X_{2^+}$ to compute the fusion product of $X_{1^-}$ and $X_{2^+}$ with all projective modules. This allows us to prove the rigidity of $\mathcal{W}_p^+$-mod and to compute the fusion product on the set of all simple and all projective modules as well as the induced product on the Grothendieck group $K(\mathcal{W}_p^+)$. 

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2. The definition of fusion tensor products

2.1. Vertex operator algebras and current algebras

In this section we will briefly summarize our definitions and notation for vertex operator algebras. For a more detailed discussion see [45, 46].

**Definition 1.** A tuple \((V, \Omega, T, Y)\)—consisting of a complex vector space \(V\), two distinguished non-trivial elements \(\Omega, T \in V\) and a map \(Y\)—is called a vertex operator algebra (VOA for short), if it satisfies the following conditions.

(1) The vector space \(V\) is non-negative integer graded

\[
V = \bigoplus_{n=0}^{\infty} V[n],
\]

such that \(V[0] = \mathbb{C}\Omega, \dim V[n] < \infty \forall n \geq 0\) and \(T \in V[2]\).

(2) For \(A \in V[hA],\) the map \(Y\) is a \(\mathbb{C}\)-linear map

\[
Y : V \to \text{End}_\mathbb{C}(V)[[z, z^{-1}]]
\]

\[
A \mapsto Y(A; z) = A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-h_A},
\]

such that

\[
Y(A; z)\Omega - A \in V[[z]]z
\]

and

\[
Y(\Omega; z) = \text{id}_V.
\]

(3) If we set

\[
Y(T; z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]

then the modes \(L_n\) satisfy the commutation relations of the Virasoro algebra with fixed central charge \(c = c_V\)

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0}.
\]

(4) The zero mode of the Virasoro algebra \(L_0\) acts semi-simply on \(V\) and

\[
V[n] = \{ A \in V | L_0 A = nA \}.
\]

(5) For any element \(A \in V\) we have

\[
\frac{d}{dz} Y(A; z) = Y(L_{-1} A; z).
\]

(6) For any elements \(A, B \in V, Y(A; z)\) and \(Y(B; z)\) are local with respect to each other, that is, there exists an \(N > 0\), such that

\[
(z - w)^N [Y(A; z), Y(B; w)] = 0
\]

as a formal power series in \(\text{End}_\mathbb{C}(V)[[z, z^{-1}, w, w^{-1}]]\).
(7) For any elements $A \in V[h_A]$, $B \in V$, $Y(A; z)$ and $Y(B; z)$ satisfy the operator product expansion

$$Y(A; z)Y(B; w) = \sum_{n \in \mathbb{Z}} Y(A_n B; w)(z - w)^{-n-h_A}. \quad (2.10)$$

When there is no chance of confusion we will refer to a VOA just by its graded vector space $V$.

**Remark 2.** By the above definition it follows that for $A \in V[h_A]$ the Virasoro generators $L_0$ and $L_{-1}$ satisfy

$$[L_{-1}, A_n] = -(n + h_A - 1)A_{n-1}$$

$$[L_0, A_n] = -nA_n. \quad (2.11)$$

Next we introduce a finiteness condition due to Zhu [3, 5].

**Definition 3.** A VOA $V$ is said to be $c_2$-cofinite if

$$\dim \mathbb{C} V/c_2(V) < \infty, \quad (2.12)$$

where $c_2(V)$ the subspace of $V$ defined by

$$c_2(V) = \text{span}[A_n B | A \in V[h_A], B \in V, n \leq -(h_A + 1)]. \quad (2.13)$$

In this paper we will mainly be considering $c_2$-cofinite VOAs. Among many other helpful properties $c_2$-cofiniteness guarantees that the $V$ has only a finite number of irreducible representations.

The algebra of the modes of a VOA $V$ can be understood by using the concepts of the current Lie algebra $g(V)$ and the current algebra $U(V)$ of $V$. The representation theory of a VOA $V$ can be defined by left $U(V)$-modules with some extra properties, which we will explain in the following.

Let $V$ be a VOA. Consider the spaces $V^{(1)} = \bigoplus_{h \geq 0} V[h] \otimes \mathbb{C}[[\xi, \xi^{-1}]](d\xi)^{1-h}$ and $V^{(0)} = \bigoplus_{h \geq 0} V[h] \otimes \mathbb{C}[[\xi, \xi^{-1}]](d\xi)^{-h}$ as well as the $\mathbb{C}$-linear map $\nabla : V^{(0)} \to V^{(1)}$ defined by

$$\nabla(A \otimes f(\xi)(d\xi)^{-h_A}) = L_{-1}A \otimes f(\xi)(d\xi)^{-h_A} + A \otimes \frac{df(\xi)}{d\xi}(d\xi)^{-h_A}. \quad (2.14)$$

**Definition 4.** Let

$$g(V) = V^{(1)}/\nabla V^{(0)}, \quad (2.15)$$

then $g(V)$ has the structure of a Lie algebra given by

$$[A \otimes f(d\xi)^{1-h_A}, B \otimes g(d\xi)^{-h_B}] = \sum_{m=0}^{h_A+h_B-1} \frac{1}{m!} A_{m-h_A+1} B \otimes \frac{d^m f}{d\xi^m} g(d\xi)^{m+2-h_A-h_B}. \quad (2.16)$$

For each element $A \in V[h]$ we denote

$$A_n = [A \otimes \xi^{n-h_A}(d\xi)^{1-h}] \in g(V) \quad (2.17)$$

and

$$L_n = [T \otimes \xi^{n+1} (d\xi)^{-1}] \in g(V). \quad (2.18)$$
The $L_n$ generate the Virasoro Lie algebra as a subalgebra in $\mathfrak{g}(V)$ and the Lie algebra $\mathfrak{g}(V)$ has a $\mathbb{Z}$-graded Lie algebra structure by

$$\mathfrak{g}(V)[n] = \{ g \in \mathfrak{g}(V) | [L_0, g] = ng \}. \quad (2.19)$$

By definition the $A_n$ are the elements in $\mathfrak{g}(V)[-n]$.

We now define the current algebra $\mathcal{U}(V)$ of $V$. We first consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g}(V))$ of $\mathfrak{g}(V)$. Then $\mathcal{U}(\mathfrak{g}(V))$ has the structure of a $\mathbb{Z}$-graded algebra by decomposition into $L_0$ eigenspaces

$$\mathcal{U}(\mathfrak{g}(V))[n] = \{ P \in \mathcal{U}(\mathfrak{g}(V)) | [L_0, P] = nP \}. \quad (2.20)$$

Consider the degreewise completion of $\mathcal{U}(\mathfrak{g}(V))$

$$\mathcal{U}(\mathfrak{g}(V)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{U}(\mathfrak{g}(V))[n] \quad (2.21)$$

and consider the degreewise closed two sided ideal

$$\mathcal{I} = \bigoplus_{n \in \mathbb{Z}} \mathcal{I}[n] \quad (2.22)$$

of $\mathcal{U}(\mathfrak{g}(V))$ generated by the Borcherds relations which arise from the operator product expansion

$$Y(A; z)Y(B; w) = \sum_{n \in \mathbb{Z}} Y(A_nB; w)(z - w)^{-n - h_A}. \quad (2.23)$$

**Definition 5.** The current algebra $\mathcal{U}(V)$ of $V$ is the topological $\mathbb{Z}$-graded algebra

$$\mathcal{U}(V) = \bigoplus_n \mathcal{U}(V)[n] = \bigoplus_n \mathcal{U}(\mathfrak{g}(V))[n]/\mathcal{I}[n]. \quad (2.24)$$

The following proposition is very important in this paper, because it allows us to switch back and forth between calculations in the current Lie algebra and the current algebra.

**Proposition 6.** The canonical $\mathbb{Z}$-graded Lie algebra map

$$\mathfrak{g}(V) = \bigoplus_n \mathfrak{g}(V)[n] \to \mathcal{U}(V) = \bigoplus_n \mathcal{U}(V)[n] \quad (2.25)$$

has a dense image.

We define filtrations of $\mathcal{U}(V)$

$$\mathcal{F}_k(\mathcal{U}) = \bigoplus_{n \geq k} \mathcal{U}(V)[n],$$

$$\mathcal{F}^k(\mathcal{U}) = \bigoplus_{n \leq k} \mathcal{U}(V)[n], \quad (2.26)$$

satisfying

$$\cdots \mathcal{F}_{k-1}(\mathcal{U}) \supset \mathcal{F}_k(\mathcal{U}) \supset \mathcal{F}_{k+1}(\mathcal{U}) \cdots,$$

$$\cdots \mathcal{F}^{k-1}(\mathcal{U}) \subset \mathcal{F}^k(\mathcal{U}) \subset \mathcal{F}^{k+1}(\mathcal{U}) \cdots. \quad (2.27)$$

The category $V$-mod of representations of the VOA $V$ is defined using left $\mathcal{U}(V)$-modules.

**Definition 7.** A representation $M$ of the VOA $V$, also called a $V$-module, is a left $\mathcal{U}(V)$-module containing a finite dimensional $\mathcal{F}_0(\mathcal{U})$ invariant subspace $M_0$ such that $\mathcal{U}(V) \cdot M_0 = M$. We denote the Abelian category of $V$-modules by $V$-mod.
For any $V$-module $M$ we can define the $C$-linear map

$$Y^M : V \to \text{End}_C(M)[[z, z^{-1}]]$$

$$A \mapsto Y^M(A; z) = \sum_{n \in \mathbb{Z}} \rho_M(A_n) z^{-n-h_A},$$  \hspace{1cm} (2.28)

where $\rho_M$ is a representation of the current Lie algebra $\mathfrak{g}(V)$ on $M$ and we have assumed that $A \in V[h_A]$. Then we have the following formulae

$$Y^M(\Omega; z) = \text{id}_M$$

$$\frac{d}{dz} Y^M(A; z) = Y^M(L^{-1}A; z)$$ \hspace{1cm} (2.29)

and for all $A \in V[h_A]$ and $B \in V$, the operators $Y^M(A; z)$ and $Y^M(B; w)$ are local with respect to each other and satisfy the operator product expansion

$$Y^M(A; z) Y^M(B; w) = \sum_{n \in \mathbb{Z}} Y^M(A_nB; w)(z-w)^{-n-h_A}. \hspace{1cm} (2.30)$$

**Definition 8.** The current algebra $\mathcal{U}(V)$ admits the algebra anti-automorphism $\sigma : \mathcal{U}(V)[n] \to \mathcal{U}(V)[-n]$, \hspace{1cm} (2.31)

which for $A \in V[h_A]$ is defined by $\sigma(A_n) = (-1)^n (\mathcal{E}^1 A)_n$.

**Definition 9.** For any $V$-module $M$ the contragredient $V$-module $M^*$ is defined by

$$M^* = \sum_{h \in H} \text{Hom}_C(M[h], C)$$ \hspace{1cm} (2.32)

as a vector space. While the action of the current algebra is given by

$$\langle P\phi, u \rangle = \langle \phi, \sigma(P)u \rangle,$$ \hspace{1cm} (2.33)

for $\phi \in M^*, u \in M, P \in \mathcal{U}(V)$. The contragredient of the contragredient is again the original module $M = (M^*)^*$. 

**Proposition 10.** Let $V$ be a $c_2$ cofinite VOA.

1. The number of isomorphism classes of simple $V$-modules is finite.
2. For any $V$-module $M$ all Jordan–Hölder series

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\},$$ \hspace{1cm} (2.34)

such that $M_i/M_{i+1}$ are simple $V$-modules, are finite.
3. Each $V$-module $M$ decomposes into a direct sum of finite dimensional generalized $L_0$-eigenspaces $M[h]$

$$M = \bigoplus_{h \in H} M[h],$$

$$M[h] = \{ u \in M, (L_0 - h)^N u = 0, \text{ for some } N \geq 1 \},$$ \hspace{1cm} (2.35)

where $H$ is the set of all weights, a discrete subset of $\mathbb{C}$ generated from the finite set $H_0$ of highest weights by adding all non-negative integers.
4. $V$-mod admits a contravariant endofunctor called the contragredient or contragredient dual.

Proposition 10 was shown in [43, 47].
To better analyse $V$-mod we define finite dimensional $\mathbb{C}$-algebras $A_k(V)$, $k = 0, 1, \ldots$ in the following way. Consider the degreewise closed right $\mathcal{U}(V)$-ideal
\[ I_k = \mathcal{F}_{k+1}(\mathcal{U}) \cdot \mathcal{U}(V) \subset \mathcal{U}(V) \] (2.36)
and consider the two sided $\mathcal{F}_0(\mathcal{U})$ ideal
\[ I_k = I_k \cap \mathcal{F}_0(\mathcal{U}). \] (2.37)
Taking the quotient of $\mathcal{F}_0(\mathcal{U})$ by $I_k$ we define a series of $\mathbb{C}$-algebras
\[ A_k(V) = \mathcal{F}_0(\mathcal{U}) / I_k. \] (2.38)
For the remainder of this section we will assume that the VOA $V$ is $c_2$-cofinite.

**Proposition 11.** For $k = 0, 1, 2, \ldots$ the $\mathbb{C}$-algebras $A_k(V)$ are all finite dimensional.

We denote by $A_k(V)$-mod, the Abelian category of finite dimensional left $A_k(V)$-modules. We define the covariant functor
\[ A_k : V\text{-}\mathsf{mod} \to A_k(V)\text{-}\mathsf{mod} \]
\[ M \mapsto A_k(M) = \frac{M}{I_k(M)}. \] (2.39)

**Proposition 12.**

1. A $V$-module $M$ is the zero module if and only if $A_k(M)$ is the zero module.
2. If $M$ is a simple $V$-module, then $A_k(M)$ is simple and the set of isomorphism classes of simple $V$-modules are in one to one correspondence with the isomorphism classes of simple $A_k(V)$-modules.

The $k = 0$ case is most important to our work at hand, we will refer to $A_0(V)$ as the zero mode algebra of the VOA $V$.

To define the notion of the fusion tensor product on $V$-mod, we prepare some additional concepts and definitions. As a first step we define left and right completions of the current algebra $\mathcal{U}(V)$. For each $k \in \mathbb{Z}$ define
\[ \mathcal{F}^k(\mathcal{U}) = \bigcap_{d \leq k} \mathcal{U}(V)[d] \]
\[ \mathcal{F}_k(\mathcal{U}) = \bigcap_{d \geq k} \mathcal{U}(V)[d] \]
\[ \mathcal{U}^L = \bigcup_k \mathcal{F}^k(\mathcal{U}) \]
\[ \mathcal{U}^R = \bigcup_k \mathcal{F}_k(\mathcal{U}). \] (2.40)

Then $\mathcal{U}^L$ and $\mathcal{U}^R$ are topological $\mathbb{C}$-algebras with topologies defined by the filtrations $\mathcal{F}^k(\mathcal{U})$ and $\mathcal{F}_k(\mathcal{U})$ and the canonical inclusions
\[ \mathcal{U}(V) \to \mathcal{U}^L \]
\[ \mathcal{U}(V) \to \mathcal{U}^R \] (2.41)
have dense images.

For any object $M$ of $V$-mod, we define its closure by
\[ \overline{M} = \lim_k \frac{M}{\mathcal{F}_k(\mathcal{U})(M)}. \] (2.42)

Then there is a continuous action of $\mathcal{U}^R$ on $\overline{M}$. Note that $M$ already has the structure of a $\mathcal{U}^L$-module. By the properties of projective limits $\overline{M}$ is equipped with a complete Hausdorff linear topology. Then the action of the current algebra $\mathcal{U}(V)$ is uniquely extended to an
action of $\mathcal{U}^R$ on $\overline{M}$. The two spaces $M$ and $\overline{M}$ share the same generalized $L_0$-eigenspaces

\[ M[h] = \{ m \in M \mid (L_0 - h)^n m = 0, \text{ for some } n \geq 1 \} \]

\[ \overline{M}[h] = \{ m \in \overline{M} \mid (L_0 - h)^n m = 0, \text{ for some } n \geq 1 \}, \]

but unlike $M$, $\overline{M}$ also contains infinite sums of generalized $L_0$-eigenvectors

\[ \overline{M} = \prod_{h \in H} M[h]. \]

The image of the canonical inclusion $M \rightarrow \overline{M}$ is dense.

For any $V$-module $M$ there is a canonical surjective linear map

\[ A_{k+1}(M) \rightarrow A_k(M) \]

and the projective limit

\[ \lim_k A_k(M) \]

has a unique continuous $\mathcal{U}^R$ action. Indeed

\[ \overline{M} = \lim_k A_k(M) \]

as a continuous $\mathcal{U}^R$-module. We also have the canonical homomorphisms

\[ g^\ell(V) \rightarrow \mathcal{U}^\ell \]

\[ g^R(V) \rightarrow \mathcal{U}^R, \]

which both have dense images.

For each $k \in \mathbb{Z}$ we define

\[ g_k(V) = \sum_{d \geq k} g(V)[d]. \]

Then the canonical map

\[ g_k(V) \rightarrow F_k(\mathcal{U}) \rightarrow F_k(\mathcal{U}^R) \]

has dense image. So we have the $\mathbb{C}$-linear isomorphism

\[ \frac{M}{g_{k+1}(V)(M)} \rightarrow \frac{M}{I_k(M)} = A_k(M). \]

Therefore we have

\[ \overline{M} = \lim_k \frac{M}{g_{k+1}(V)(M)} \]

For later use we define for each $V$-module $M$ the quotient

\[ \frac{M}{c_1(M)} = \frac{M}{\text{span}[A_m m \in M, A \in V[h], n \geq h > 0]} \]

Then $M/c_1(M)$ is a finite dimensional complex vector space and there exists a canonical surjective linear map

\[ \frac{M}{c_1(M)} \rightarrow A_0(M). \]

Also for later use we introduce the following notation.
Definition 13. Let $M$ be a $V$-module, then we define $L_0$-graded subspaces $M^0$ and $M^\ast$, such that

$$
M = M^0 \oplus g_1(V) \cdot M, \\
M = M^\ast \oplus c_1(M),
$$

(2.55)

respectively called the zero mode and the special subspace, such that the canonical maps

$$
M^0 \to A_0(M), \\
M^\ast \to \frac{M}{c_1(M)}
$$

(2.56)

are $\mathbb{C}$-linear isomorphisms.

Remark 14. The subspaces $M^0$ and $M^\ast$ are not uniquely defined. We will fix specific choices of subspaces in a later section.

2.2. General properties of braided monoidal categories

In this section we introduce the concepts of monoidal categories, their rigidness and some general properties. We mainly follow the appendices of the seminal papers due to Kazhdan and Lusztig [48, appendix A] as well as the standard reference for monoidal categories [49]. We will only be considering monoidal categories that are also $\mathbb{C}$-linear and Abelian and we assume that the reader is familiar with basic notions of Abelian categories such as exact sequences, projective modules, injective modules, etc.

Definition 15. A monoidal category is a tuple $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$—just $(\mathcal{C}, \otimes, 1)$ for short—where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the tensor product bi-functor, $1 \in \mathcal{C}$ is the tensor unit, $\alpha_{L,M,N} : L \otimes (M \otimes N) \cong (L \otimes M) \otimes N$ is the associator, $\lambda_M : 1 \otimes M \to M$ is the left unit isomorphism, and $\rho_M : M \otimes 1 \to M$ is the right unit isomorphism. These data are subject to conditions, in particular $\alpha$ satisfies the pentagon axiom and $\lambda$, $\rho$, $\alpha$ obey the triangle axiom.

Definition 16. We say that an object $M$ is weakly rigid if the contravariant functor

$$
F_M(-) = \text{Hom}(- \otimes M, 1)
$$

(2.57)

is representable, i.e. for all objects $N$ there exists an object $M^\vee$ called the tensor dual\footnote{Strictly speaking $M^\vee$ is called the right dual. There is a similar notion of a left dual. However if the tensor category is braided then these notions are related. We will therefore only be considering right duals.} such that

$$
\text{Hom}(N \otimes M, 1) \cong \text{Hom}(N, M^\vee).
$$

(2.58)

Therefore if $M$ is weakly rigid there exists a morphism

$$
e_M : M^\vee \otimes M \to 1
$$

(2.59)

that is isomorphic to $\text{id}_{M^\vee} \in \text{Hom}(M^\vee, M^\vee)$ by the equivalence (2.58). A monoidal category is called weakly rigid if all its object are weakly rigid.

Definition 17. An object $M$ is said to be rigid if it is weakly rigid and there exists a morphism

$$
i_M : 1 \to M \otimes M^\vee,
$$

(2.60)

such that

$$
\text{id}_M = \rho_M \circ (\text{id}_M \otimes e_M) \circ \alpha_{M,M^\vee,M}^{-1} \circ (i_M \otimes \text{id}_M) \circ \lambda_M^{-1}
$$

$$
i_M^\vee = \lambda_{M^\vee} \circ (e_M \otimes \text{id}_M^\vee) \circ \alpha_{M^\vee,M,M^\vee} \circ (\text{id}_M^\vee \otimes i_M) \circ \rho_M^\vee.
$$

(2.61)
Definition 18. A braiding $b$ on a monoidal category $(C, \otimes, 1)$ is a natural transformation between the functors $\otimes$ and $\otimes \circ P$, where $P: C \times C \to C \times C$ is the permutation $(M, N) \to (N, M)$. For $M, N$ in $C$, $b$ defines a morphism $b_{M,N} \in \text{Hom}(M \otimes N, N \otimes M)$ satisfying the following.

1. All $b_{M,N}$ are isomorphisms.
2. For any $L, M, N$ in $C$ we have
   
   \begin{align*}
   b_{L \otimes M, N} &= \alpha_{N,L,M}^{-1} \circ (b_{L,N} \circ \text{id}_M) \circ \alpha_{L,N,M} \circ (\text{id}_L \otimes b_{M,N}) \circ \alpha_{L,M,N}^{-1}, \\
   b_{L,M \otimes N} &= \alpha_{M,N,L} \circ (\text{id}_M \otimes b_{L,N}) \circ \alpha_{M,L,N}^{-1} \circ (b_{L,M} \circ \text{id}_N) \circ \alpha_{L,M,N}.
   \end{align*}

3. For all $M$ in $C$
   
   \begin{align*}
   b_{M,1} &= b_{1,M} = \text{id}_M.
   \end{align*}

Proposition 19. Let $(C, \otimes, 1)$ be a monoidal category, then

1. For any rigid object $M$ in $C$ the functors $M \otimes -$, $- \otimes M : C \to C$

   \begin{align*}
   \text{(2.64)}
   \end{align*}

   are exact.

2. For a rigid object $M$ with dual $M^\vee$ and arbitrary objects $N, L$ we have the isomorphism

   \begin{align*}
   \text{Hom}(N, M \otimes L) \cong \text{Hom}(M^\vee \otimes N, L).
   \end{align*}

   \begin{align*}
   \text{(2.65)}
   \end{align*}

3. Let $M$ in $C$ be rigid with dual $M^\vee$. Then for any projective object $P$, $M^\vee \otimes P$ is projective.

   For any injective object $I$, $M \otimes I$ is injective in $C$ if $M^\vee$ is also rigid.

4. Assume that

   (a) The Abelian category $C$ has enough projective and injective objects.

   (b) All projective objects are injective and all injective objects are projective.

   (c) All projective objects are rigid.

   Then if

   \begin{align*}
   0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0
   \end{align*}

   \begin{align*}
   \text{(2.66)}
   \end{align*}

   is an exact sequence in $C$ such that two of $L, M, N$ are rigid, then the third object is also rigid.

5. If $M, N$ in $C$ are rigid objects, then $M \otimes N$ is also rigid and its dual $(M \otimes N)^\vee$ is given by

   \begin{align*}
   (M \otimes N)^\vee = N^\vee \otimes M^\vee.
   \end{align*}

   \begin{align*}
   \text{(2.67)}
   \end{align*}

2.3. Fusion tensor products and their properties

Fusion plays a central role in analysing conformal field theories and is indeed the central theme of this paper. Fusion describes the short distance expansion of two fields on the level of representations. The fusion product $M \otimes N$ of two $V$-modules $M$ and $N$ is the smallest $V$-module in which all the fields appearing in the short distance expansions—of fields transforming in $M$ with fields transforming in $N$—transform.

There is a wealth of literature on fusion tensor products in both mathematics and physics. In the case of rational conformal field theory the representation category is semi-simple and the theory of fusion tensor products is well established [50–52]. If the conformal field theory is logarithmic, the representation category is not semi-simple. Fortunately there are computational methods in physics one can fall back on for defining fusion tensor products.
without assuming semi-simplicity [53, 54]. For VOAs arising from affine Lie algebras, there exists a mathematically rigorous definition of fusion tensor products due to Kazhdan and Lusztig [48] that does not rely on semi-simplicity. In the series of papers [16, 34–43] by Huang, Lepowsky and Zhang a non-meromorphic operator product expansion and braided tensor category structure were constructed for representation categories of VOAs satisfying natural, general hypotheses, without the assumption of semi-simplicity. In this paper we will state our definition of fusion in the spirit of [48, 50, 51, 53, 54] in the case of $\mathfrak{c}_2$-cofinite VOAs without assuming semi-simplicity.

We fix a $\mathfrak{c}_2$-cofinite VOA $V$ and prepare some notation.

**Definition 20.** We define the current Lie algebra on the Riemann sphere with punctures at $0, 1, \infty$ by

$$\mathfrak{g}^p = \bigoplus_{n=0}^{\infty} \mathfrak{g}[h] \otimes \mathbb{C}[z, z^{-1}, (z - 1)^{-1}] dz^{1-h},$$

(2.68)

where $\nabla$ is defined as in (2.14)

$$\nabla(A \otimes f(z) dz^{-h_a}) = L_{-1} A \otimes f(z) dz^{-h_a} + A \otimes \frac{df(z)}{dz} dz^{1-h_a}.$$  

(2.69)

Then $\mathfrak{g}^p$ has the structure of a Lie algebra given by

$$[A \otimes f(z) dz^{1-h_a}, B \otimes g(z) dz^{-h_b}] = \sum_{m=0}^{\infty} \frac{1}{m!} A_{m-h_a+1} B \otimes \frac{d^m f(z)}{dz^m} g(z) dz^{m+h_a-h_b}.$$  

(2.70)

for $A \in \mathfrak{V}[h_A]$ and $B \in \mathfrak{V}[h_B]$.

For $f(z) \in \mathbb{C}[z, z^{-1}, (z - 1)^{-1}]$ we denote the Laurent expansions at $0, 1$ and infinity by $f_0(\xi_0) \in \mathbb{C}(\xi_0)$, $f_1(\xi_1) \in \mathbb{C}(\xi_1)$ and $f_{\infty}(\xi_\infty) \in \mathbb{C}(\xi_\infty)$ respectively. For example for

$$f(z) = z^{-1}$$

(2.71)

the expansions and radii of convergence are given by

$$f_0(\xi_0) = -\sum_{n \geq 0} \frac{\xi_0^n}{n!}, \quad \xi_0 = z, \quad 1 > |z| > 0,$$

$$f_1(\xi_1) = -\frac{1}{\xi_1}, \quad \xi_1 = z - 1, \quad 1 > |z - 1| > 0,$$

$$f_{\infty}(\xi_\infty) = \sum_{n \geq 0} \frac{\xi_\infty^{-n}}{n!}, \quad \xi_\infty = z, \quad |z| > 1.$$  

(2.72)

We define Lie algebra homomorphisms

$$j_a^L : \mathfrak{g}^p \rightarrow \mathfrak{g}^L, \quad a = 0, 1,$$

$$j_{\infty}^R : \mathfrak{g}^p \rightarrow \mathfrak{g}^R,$$

(2.73)

where $\mathfrak{g}^L$ and $\mathfrak{g}^R$ are the left and right completions of $\mathfrak{g}(V)$ defined in (2.48), by

$$j_a^L ([A \otimes f(z) dz^{1-h_a}]) = [A \otimes f_a(\xi_0) dz_a^{1-h_a}],$$

$$j_{\infty}^R ([A \otimes f(z) dz^{1-h_\infty}]) = [A \otimes f_\infty(\xi_\infty) dz_\infty^{1-h_\infty}].$$  

(2.74)

The Lie algebra maps $j_a^L$ and $j_{\infty}^R$ have dense images.

The analogue of $\mathfrak{g}_0(V)$ for the current Lie algebra on the Riemann sphere is given by

$$\mathfrak{g}^p_0 = \text{span} [A \otimes f(z) dz^{1-h_a}] \in \mathfrak{g}^p \otimes_{\mathfrak{g}^p} \mathfrak{g}^L$$

(2.75)

where $\text{ord}_{\infty}(f(z))$ is the order of the pole of $f(z)$ at infinity. The image of the map

$$\mathfrak{g}^p_0 \rightarrow \mathfrak{g}_0^L \rightarrow \mathfrak{F}_k(\mathfrak{U}^R)$$  

(2.76)

is dense.
is dense and therefore the canonical map
\[
\frac{\mathfrak{g}_P(V)}{\mathfrak{g}_k(V)} \to \frac{\mathfrak{g}_R}{\mathfrak{g}_k(V)}
\]
is an isomorphism, hence
\[
\lim_{\leftarrow k} \frac{\mathfrak{g}_P(V)}{\mathfrak{g}_k(V)} \to \lim_{\leftarrow k} \frac{\mathfrak{g}_R}{\mathfrak{g}_k(V)} = \mathfrak{g}_R.
\]
Consider the map
\[
j_{1,0} = j_l^1 \otimes 1 + 1 \otimes j^0_l : \frac{\mathfrak{g}_P(V)}{\mathfrak{g}_k(V)} \to \frac{\mathfrak{g}_R}{\mathfrak{g}_k(V)}.
\]
For any two \(V\)-modules \(M, N\), the vector space \(M \otimes N\) is a left \(\mathfrak{g}_P(V)\)-module by \(j_{1,0}\).

**Proposition 21.** For each \(k = 0, 1, 2, \ldots\)
\begin{enumerate}
  \item \(\dim \mathbb{C} \otimes N / \mathfrak{g}_k(V)(M \otimes N) < \infty\).
  \item The Lie algebra \(\mathfrak{g}_R = \lim_{\leftarrow k} \mathfrak{g}_P(V)/\mathfrak{g}_k(V)(M \otimes N)\) acts continuously on the projective limit
\[
M \otimes N = \lim_{\leftarrow k} \mathfrak{g}_P(V)(M \otimes N)
\]
by \(j_{\infty}^R : \mathfrak{g}_P(V) \to \mathfrak{g}_R\).
\end{enumerate}
Furthermore by \(\mathfrak{g}_R \to \mathcal{U}_R\) the right completion \(\mathcal{U}_R\) of the current algebra acts continuously on \(M \otimes N\).

For each \(h \in \mathbb{C}\), let
\[
M \otimes N[h] = \{m \in M \otimes N \mid \exists n \geq 1 \text{ s.t. } (L_0 - h)^n m = 0\}.
\]
The fusion product of \(M\) and \(N\) is given by
\[
M \otimes N = \bigoplus_{h \in \mathbb{C}} M \otimes N[h].
\]

**Proposition 22.**
\begin{enumerate}
  \item The space \(M \otimes N\) is a \(V\)-module.
  \item For each \(k \geq 0\) we have the \(\mathbb{C}\)-linear isomorphisms
\[
\mathcal{A}_k(M \otimes N) = \frac{M \otimes N}{\mathfrak{g}_k(M \otimes N)} \cong \frac{M \otimes N}{\mathfrak{g}_{k+1}(M \otimes N)}.
\]
Most notably we have the \(A_0(V)\)-module isomorphism
\[
\mathcal{A}_0(M \otimes N) \cong \frac{M \otimes N}{\mathfrak{g}_1(M \otimes N)}.
\]
\end{enumerate}

**Theorem 23.**
\begin{enumerate}
  \item The triplet \((\mathbb{C}-\text{mod}, \otimes, V)\) is a braided monoidal category with unit object \(V = 1\).
  \item For any \(V\)-module \(M\), the contravariant functor from \(\mathbb{C}\)-\text{mod} to \(\mathbb{C}\)-\text{Vec} the category of complex vector spaces
\[
F_M : \mathbb{C}\text{-mod} \to \mathbb{C}\text{-Vec}
\]
\[
N \mapsto F_M(N) = \text{Hom}_{\mathbb{C}\text{-mod}}(N \otimes M, V^*),
\]
where \(V^*\) is the contragredient of \(V\), is represented by \(M^*\) the contragredient of \(M\), i.e.
\[
F_M(N) \cong \text{Hom}_{\mathbb{C}\text{-mod}}(N, M^*).
\]
\end{enumerate}
Theorem 23 was proved in the series of papers [16, 34–43] (originally announced in [16]) by Huang, Lepowsky and Zhang\(^2\). A new proof of theorem 23 for \(c_2\)-cofinite VOAs on Riemann surfaces of arbitrary genus is in preparation [55].

Unfortunately the definition of \(M \otimes N\) is rather difficult to work with, because even though the image of canonical map

\[
M \otimes N \rightarrow M \otimes N
\]

is dense, it generally does not lie in \(M \otimes N\).

However for each \(k \geq 0\) we can make use of the isomorphism

\[
\mathcal{A}_k(M \otimes N) \cong M \otimes N \otimes \mathfrak{g}_{k+1}(V)(M \otimes N).
\]

By analysing these quotients, we can study \(M \otimes N\) level for level. For any given element \(m \otimes n \in M \otimes N\), we will denote the class that it represents in \(\mathcal{A}_k(M \otimes N)\) by \([m \otimes n]\).

As we shall see in the following, for the purposes of this paper it will be sufficient to make statements about \(\mathcal{A}_0(M \otimes N)\) and in one case \(\mathcal{A}_1(M \otimes N)\). As a vector space \(\mathfrak{g}_1^+(V)\) is spanned by elements of the form

\[
[v \otimes z^{-n+h-1}dz^{1-h}], \quad v \in V[h]
\]

for \(n \geq 0\) and

\[
[v \otimes (z-1)^{-m+h-1}dz^{1-h}], \quad v \in V[h]
\]

for \(m \geq h\). From the expansions defined above it therefore follows that in \(\mathcal{A}_0(M \otimes N)\) for \(1 \leq n \leq h-1\) we have the relations

\[
j_{1,0}(v \otimes z^{-n+h-1}dz^{1-h}) = \sum_{k=0}^{h-n} \binom{h-1-n}{k} v_{k-(h-1)} \otimes 1 + 1 \otimes v_{-n} = 0,
\]

and for \(m \geq h\)

\[
j_{1,0}(v \otimes z^{-m+h-1}dz^{1-h}) = \sum_{k=0}^{m-h-1} \binom{m-h+k}{m-h} (-1)^k v_{k-(h-1)} \otimes 1 + 1 \otimes v_{-m} = 0.
\]

\[
j_{1,0}(v \otimes (z-1)^{-m+h-1}dz^{1-h}) = v_{-m} \otimes 1 + \sum_{k=0}^{m-h-1} \binom{m-h+k}{m-h} (-1)^{h-1-m} 1 \otimes v_{k-(h-1)} = 0.
\]

The action of the zero modes is given by

\[
j_{1,0}(v_0) = j_{1,0}(v \otimes z^{h-1}dz^{1-h})
\]

\[
= \sum_{k=0}^{h-1} \binom{h-1}{k} v_{k-(h-1)} \otimes 1 + 1 \otimes v_{-1}.
\]

For the generators of the Virasoro algebra this means

\[
L_{-1} \otimes 1 \cong -1 \otimes L_{-1}
\]

\[
L_{-n} \otimes 1 \cong \sum_{j=0}^{\infty} \binom{n+2+j}{n} (-1)^n 1 \otimes L_{j-1}
\]

\[
1 \otimes L_{-n} \cong -\sum_{j=0}^{\infty} \binom{n+2+j}{n} (-1)^j L_{j-1} \otimes 1,
\]

\(^2\) Theorem 4.13 in [43] states that theorem 23 holds for any \(c_2\)-cofinite VOA \(V\) satisfying \(\dim V[0] = 1, \dim V[n] = 0, n < 0\). Theorem 23 for the case \(V = V_p\) was explicitly spelled out in [42, section 5.2].
for $n \geq 2$ and
\[ j_{1,0}(L_0) = L_{-1} \otimes 1 + L_0 \otimes 1 + 1 \otimes L_0. \]  
(2.95)

To aid us in computing $\mathcal{A}_0(M \otimes N)$, we make use of the special and zero mode subspaces in definition 13 to state the following proposition due to Nahm [53].

**Proposition 24.** Let $M$ and $N$ be $V$-modules. Then the canonical $\mathbb{C}$-linear map
\[ M' \otimes N^0 \rightarrow \mathcal{A}_0(M \otimes N) \rightarrow 0 \]  
(2.96)
is surjective.

**Proof.** Let $m \otimes n \in M \otimes N$, $A \in V[h_A]$, $B \in V[h_B]$, $k \geq h_A$ and $\ell > 0$. We introduce two kinds of manipulations by using the formulae in (2.92)

1. Moving modes to the right
\[ [A_{-k}m \otimes n] = -\sum_{j \geq 0} \binom{k - h_A + j}{k - h_A} [m \otimes A_{j-(h_A-1)}n]. \]  
(2.97)

Note that $A_{-k}$ is replaced by modes with a mode number that is greater than $-k$, i.e. the grading is lowered.

2. Moving modes to the left
\[ [m \otimes B_{-\ell}n] = -\sum_{j > 0} \binom{h_B - 1 - \ell}{j} [B_{j-(h_B-1)}m \otimes n]. \]  
(2.98)

Note that $B_{-\ell}$ is replaced by modes with a mode number that is greater than or equal to $-\ell$, i.e. the grading is lowered or stays the same.

Since the image of the canonical map $\mathcal{G}_k(V) \rightarrow \mathcal{F}_j(V)$ is dense and $\mathcal{F}_j(V)(M) = I_0(M)$, any element $n$ in $N$ is represented by $n = x \cdot n_0$ for some $x \in \mathcal{G}_1(V)$ and $n_0 \in N^0$. Then by using formula (2.98), the class $[m \otimes n]$ can be represented by $m' \otimes n_0$ for some $m' \in M$. Consider $m \in c_1(M)$ and $n_0 \in N^0$, we can assume that $m$ is homogeneous such that $m \in c_1(M)[h]$ for some $h$. By definition we can assume that $m$ has the form
\[ m = A_{-k}m_0 \]  
(2.99)
for some $m_0 \in M'$ and $A \in V[h_A]$, $k \geq h_A$. Then by using formula (2.97)
\[ [m \otimes n_0] = -\sum_{j \geq 0} \binom{k - h_A + j}{k - h_A} [m_0 \otimes A_{j-(h_A-1)}n_0]. \]  
(2.100)

For each summand we use again the fact that the class $[m_0 \otimes A_{j-(h_A-1)}n_0]$ can be represented by an element $m_0' \otimes n_0' \in M' \otimes N^0$. If we decompose $m_0'$ into homogeneous summands, then the weights of the individual summands will all be less then the original weight $h$ of $m$. Because the weights are bounded from below, a finite number of applications of the formulae (2.97) and (2.98) will yield a representative in $M' \otimes N^0$ for any class $[m \otimes n]$. \qed

Before we end this section on the fusion product we consider the relation between the fusion product $\otimes_V$ of a VOA $V$ and the fusion product $\otimes_{V'}$ of a subVOA $V' \subset V$.

**Proposition 25.** Let $V'$ be a $c_2$-cofinite subVOA of the VOA $V$. Let $M$ and $N$ be $V$-modules, then they are also $V'$-modules and there exists a surjective $V'$-module map
\[ M \otimes_{V'} N \rightarrow M \otimes_{V'} N. \]  
(2.101)
Proof. Since
\[ g_k(V') \subset g_k(V) \] (2.102)
there is a canonical surjection of \( A_k(V') \)-modules
\[ A_k(M \otimes_V N) = M \otimes N \] (2.103)
\[ g_{k+1}^p(V') \rightarrow g_{k+1}^p(V) \]
Note that for sufficiently large \( k \), a given generalized \( L_0 \)-eigenspace is stable, i.e.
\[ A_k(M \otimes_V N)[h] = A_{k+1}(M \otimes_V N)[h] = (M \otimes_V N)[h], \quad k \gg 0. \] (2.104)
Therefore the surjection (2.103) implies a surjection
\[ M \otimes_V N[h] \rightarrow M \otimes_V N[h] \] (2.105)
between generalized \( L_0 \)-eigenspaces. This can be repeated for all values of \( h \) and therefore there exists a surjective \( V' \)-module map
\[ M \otimes_V N \rightarrow M \otimes_V N. \] (2.106)
\[ \Box \]
Proposition 26. Let \( M \) be a \( V \)-module, then the covariant functors \( M \otimes - \) and \( - \otimes M \) are right exact.
Proof. We prove the proposition for \( M \otimes - \). The proof for \( - \otimes M \) follows analogously.
Let \( A, B, C \in V \)-mod satisfy the exact sequence
\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0. \] (2.107)
Then the sequence
\[ M \otimes A \rightarrow M \otimes B \rightarrow M \otimes C \rightarrow 0 \] (2.108)
is exact if the restriction to the generalized \( L_0 \)-eigenspaces
\[ M \otimes A[h] \rightarrow M \otimes B[h] \rightarrow M \otimes C[h] \rightarrow 0 \] (2.109)
is exact. Because for sufficiently large \( k \) the generalized \( L_0 \)-eigenspaces for fixed generalized eigenvalue \( h \) are stable under taking \( A_k \) quotients, we consider the sequence
\[ \frac{M \otimes A}{g_k^p(V)(M \otimes A)}[h] \rightarrow \frac{M \otimes B}{g_k^p(V)(M \otimes B)}[h] \rightarrow \frac{M \otimes C}{g_k^p(V)(M \otimes C)}[h] \rightarrow 0, \] (2.110)
which is clearly exact. \[ \Box \]

2.4. Algebra morphisms between products of modules
We have defined the fusion tensor product in \( V \)-mod. Now we introduce the concept of vertex operators in a conformal field theory on \( \mathbb{P} \) associated with \( V \)-mod, by extending the notions in [50].

For a \( V \)-module \( M \) we have defined the topological completion
\[ \overline{M} = \prod_{h \in H} M[h]. \] (2.111)
For any two \( V \)-modules \( M, N \) we denote by \( \text{Hom}_{C}(\overline{M}, \overline{N}) \), the space of continuous \( C \)-linear maps from \( \overline{M} \) to \( \overline{N} \). Then we have a \( C \)-linear isomorphism
\[ \text{Hom}_{C}(M, N) \cong \text{Hom}_{C}(\overline{M}, \overline{N}). \] (2.112)
Now for the \( V \)-modules \( L, M, N \) consider the complex vector spaces
\[ \text{Hom}_{C}(M \otimes N, L) \cong \text{Hom}_{C}(\overline{M} \otimes \overline{N}, \overline{L}). \] (2.113)
We know that
\[ M \otimes N = \lim_{k \to \infty} M \otimes N. \]  
(2.114)

So there exists an injective \( \mathbb{C} \)-linear map
\[ \text{Hom}_{\mathbb{C}(V)}(M \otimes N, L) \to \text{Hom}_{\mathbb{C}}(M \otimes N, L) \to \text{Hom}_{\mathbb{C}}(M \otimes N, L). \]  
(2.115)

The following proposition characterizes the image of this map.

**Proposition 27.** For \( \psi \in \text{Hom}_{\mathbb{C}}(M \otimes N, L) \) the necessary and sufficient condition for \( \psi \) to lie in the image of the map from \( \text{Hom}_{\mathbb{C}(V)}(M \otimes N, L) \) is that for all \( f(z) \in \mathbb{C}[z, z^{-1}, (z-1)^{-1}] \).

\[ A \in V[h_A], \ m \in M \text{ and } n \in N \]
\[ j^R_{\infty}([A \otimes f(z) \, dz^{1-h_A}])n = j^R_{1}([A \otimes f(z) \, dz^{1-h_A}])m \]
\[ + \psi(m)j^R_{1}([A \otimes f(z) \, dz^{1-h_A}])n, \]  
where we denote
\[ \psi : m \otimes n \mapsto \psi(m)n. \]  
(2.117)

Now consider the complex algebras \( \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \) and \( \mathbb{R} = \mathbb{C}[y, y^{-1}] \). An element \( f(z, y) \in \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \) can be thought of as a rational function on \( \mathbb{P} \times \mathbb{P} \).

We consider three domains.

1. \( U_0 = \{(z, y) \in \mathbb{P} \times \mathbb{P} | |y| > |z| > 0\} \),
2. \( U_y = \{(z, y) \in \mathbb{P} \times \mathbb{P} | |z - y| > 0\} \),
3. \( U_\infty = \{(z, y) \in \mathbb{P} \times \mathbb{P} | |z| > |y| > 0\} \),

and define local coordinates \( \xi_a, y \) on \( U_0 \), \( a = 0, y, \infty \) by defining \( \xi_0 = z, \xi_y = y, \text{ and } \xi_\infty = z \).

Then we can define the algebra homomorphisms
\[ j^L_{\infty} : \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \to \mathbb{R}((\xi_a)), \quad a = 0, y, \]  
(2.118)

and
\[ j^R_{\infty} : \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}] \to \mathbb{R}((\xi_\infty^{-1})) \]  
(2.119)

by expanding \( f(z) \) on the open sets \( U_0 \) by the local coordinates \( (\xi_a, y) \). We denote \( j^L_{\infty}(f) = f_a(\xi_a; y) \) for \( a = 0, y \) and \( j^R_{\infty}(f) = f_\infty(\xi_\infty; y) \).

Then we can define the current Lie algebra \( g^R_{\infty}(V) \) over \( \mathbb{R} \) by
\[ g^R_{\infty}(V) = \frac{\bigoplus_{h=0}^{\infty} V[h] \otimes \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}, dz^{1-h}]}{\nabla \bigoplus_{h=0}^{\infty} V[h] \otimes \mathbb{C}[z, z^{-1}, y, y^{-1}, (z-y)^{-1}, dz^{1-h}]}, \]  
(2.120)

where \( \nabla \) is defined as in (2.14).

\[ \nabla(A \otimes f(z, y) \, dz^{-h}) = L_{-1}A \otimes f(z, y) \, dz^{-h} + A \otimes \frac{df(z, y)}{dz} \, dz^{1-h}. \]  
(2.121)

Then \( g^R_{\infty}(V) \) has the structure of a Lie algebra given by
\[ [A \otimes f(z, y) \, dz^{1-h}, B \otimes g(z, y) \, dz^{1-h}] = \sum_{m=0}^{\infty} \frac{1}{m!}A_{-m-h+1}B \otimes \frac{d^m f(z, y)}{dz^m} g(z, y) \, dz^{m+2-h-s}. \]  
(2.122)

Let
\[ j^L_{\infty} : g^R_{\infty}(V) \to \mathbb{R} \otimes g^L, \quad a = 0, y, \]  
(2.123)

\[ j^R_{\infty} : g^R_{\infty}(V) \to \mathbb{R} \otimes g^R, \]  
be the Lie algebra homomorphisms over \( \mathbb{R} \) defined in the same way as in (2.74).
Definition 28. For \( V \)-modules \( L, M, N \), a \( \text{Hom}_\mathbb{C}(M \otimes N, \overline{L}) \)-valued holomorphic function \( \psi(y) \) on \( \mathbb{C}^* \)—which may be multi-valued—is called a vertex operator of type \( \left( \frac{M}{L,N} \right) \) if it satisfies the following two conditions.

1. For \( f(z) \in \mathbb{C}[z, z^{-1}, y, y^{-1}, (z - y)^{-1}] \), \( A \in V[h_A] \), \( m \in M \) and \( n \in N \) we have
   \[
   f^R_\infty([A \otimes f(z, y) dz^{-1-h_A}]) \psi(m; y)n = \psi(j^L_\infty([A \otimes f(z, y) dz^{-1-h_A}])m; y)n + \psi(m; y) j^R_0([A \otimes f(z, y) dz^{-1-h_A}])n. \tag{2.124}
   \]

2. For \( m \in M \) and \( n \in N \)
   \[
   \frac{d}{dy} \psi(m; y)n = \psi(L^{-1}m; y)n. \tag{2.125}
   \]

We denote by \( \mathfrak{I}_{L,N}^M \) the complex vector space of vertex operators of type \( \left( \frac{M}{L,N} \right) \). By taking \( y = 1 \) we can define linear maps

\[
\mathfrak{I}_{L,N}^M \to \text{Hom}_\mathbb{C}(M \otimes N, \overline{L})
\]

\[
\psi(-; y) \mapsto \psi(-; 1). \tag{2.126}
\]

Then we have the following theorem.

Theorem 29. The image of the map \( \psi(-; y) \mapsto \psi(-; 1) \) is contained in the image of the injection

\[
\text{Hom}_{\mathcal{U}(V)}(M \otimes N, L) \to \text{Hom}_\mathbb{C}(M \otimes N, \overline{L}), \tag{2.127}
\]

and the two images are equal.

By the above vertex operator one can define \( N \)-point conformal blocks and prove the validity of the associativity and braiding constraints as was shown in [16].

We will revisit these concepts for one special case when proving the rigidity of the \( \mathcal{W}_p \)-module \( X_i^+ \). We also make note of a slight abuse of notation we will be using. When considering an element \( \psi \in \text{Hom}_{\mathcal{U}(V)}(M \otimes N, L) \) and \( m \in M, n \in N \) we will identify \( m \otimes n \) with \( \psi(m)n \), when there is no chance of confusion.

3. The Abelian category \( \mathcal{W}_p \)-mod

In this section we briefly review the structure of \( \mathcal{W}_p \)-mod as an Abelian category following [25].

3.1. General properties of \( \mathcal{W}_p \)-mod

For \( p \geq 2 \) the VOA \( \mathcal{W}_p \) is generated by the identity \( \mathbb{I} \), the energy momentum tensor \( T(z) \) and three weight \( 2p - 1 \) primary fields \( W_\epsilon(z) \), where \( \epsilon = \pm, 0 \) labels \( \mathfrak{sl}_2 \)-charges. The central charge of the theory is given by

\[
c_p = 1 - \frac{6}{p} (p - 1)^2. \tag{3.1}
\]

It has been shown in [21] that \( \mathcal{W}_p \) is \( c_2 \)-cofinite.

As an Abelian category \( \mathcal{W}_p \)-mod decomposes into a \( \mathbb{C} \)-linear sum of Abelian subcategories

\[
\mathcal{W}_p \text{-mod} = \bigoplus_{\epsilon=0}^{p} \mathcal{C}_\epsilon, \tag{3.2}
\]
where for \( 0 \leq s \leq p \) the \( C_s \) are full Abelian subcategories of \( \mathcal{W}_p \)-mod. For \( s \neq s' \) and \( M \in C_s \), \( M' \in C_{s'} \) the spaces \( \text{Ext}^{i}_{\mathcal{W}_p}(M, M') = 0 \) for any \( i \in \mathbb{Z} \), in particular \( \text{Hom}_{\mathcal{W}_p}(M, M') = \text{Ext}^0_{\mathcal{W}_p}(M, M') = 0 \). The two subcategories \( C_0 \) and \( C_p \) are semi-simple and contain one simple object each

\[
X^+_p \in \text{obj}(C_p), \quad X^-_p \in \text{obj}(C_0),
\]

which are projective in \( C_p \) and \( C_0 \) respectively as well as in \( \mathcal{W}_p \)-mod. We will therefore occasionally also denote these modules by \( P^+_p = X^+_p \). For \( 1 \leq s \leq p - 1 \) the subcategories \( C_s \) are not semi-simple. They contain two simple objects each \( M \) and \( \dot{M} \). They are projective in \( \text{Hom}_{\mathcal{W}_p}(M, M') = \text{Ext}^0_{\mathcal{W}_p}(M, M') = 0 \). The two subcategories \( C_0 \) and \( C_p \) are semi-simple and contain one simple object each

\[
X^+_s, \quad X^-_{p-s} \in \text{obj}(C_s).
\]

We denote the projective covers of \( X^+_s \) and \( X^-_{p-s} \) by \( P^+_s \) and \( P^-_{p-s} \) respectively. They are characterized by socle series\(^3\) of length 3

\[
X^+_s = S_0(P^+_s) \subset S_1(P^+_s) \subset S_2(P^+_s) = P^+_s,
\]

\[
X^-_{p-s} = S_0(P^-_{p-s}) \subset S_1(P^-_{p-s}) \subset S_2(P^-_{p-s}) = P^-_{p-s},
\]

such that

\[
S_1(P^+_s)/S_0(P^+_s) = 2X^-_{p-s}, \quad S_2(P^+_s)/S_1(P^+_s) = X^+_s,
\]

\[
S_1(P^-_{p-s})/S_0(P^-_{p-s}) = 2X^+_s, \quad S_2(P^-_{p-s})/S_1(P^-_{p-s}) = X^-_{p-s}.
\]

Both \( P^+_s \) and \( P^-_{p-s} \) each have two occurrences of \( X^+_s \) and \( X^-_{p-s} \) as subquotients and therefore they have identical characters.

The simple and the projective modules of \( \mathcal{W}_p \)-mod are all self-contragredient, i.e. \( X^*_s = X^+_s \) and \( P^*_s = P^+_s \) for \( 1 \leq s \leq p \), \( s = \pm \). In particular the vacuum representation, \( X^+_1 \), which is the tensor unit, is self-contragredient. Therefore by proposition 19 it follows that \( (\mathcal{W}_p \text{-mod}, \otimes, X^+_1) \) is weakly rigid and that for each \( \mathcal{W}_p \)-module \( M \) the weakly rigid dual \( M^* \) coincides with the contragredient \( M^* \).

For \( 1 \leq s \leq p - 1 \) the subcategories \( C_s \) also contain six families of indecomposable modules characterized by socle series of length 2. For \( d \geq 1 \) these are summarized in the table below.

| \( S_1/S_0 \) \( G^+_{d/2} \) | \( G^-_{p-s,d} \) | \( H^+_{d/2} \) | \( H^-_{p-s,d} \) | \( I^+_{d/2}(\lambda) \) | \( I^-_{p-s,d}(\lambda) \) |
|---|---|---|---|---|---|
| \( (d+1)X^+_s \) | \( (d+1)X^-_{p-s} \) | \( dX^+_s \) | \( dX^-_{p-s} \) | \( dX^+_s \) | \( dX^-_{p-s} \) |
| \( S_2/S_1 \) \( dX^-_{p-s} \) | \( dX^+_s \) | \( (d+1)X^-_{p-s} \) | \( (d+1)X^+_p \) | \( dX^-_{p-s} \) | \( dX^+_s \) |

Note that \( I^+_{d/2}(\lambda) \) and \( I^-_{p-s,d}(\lambda) \) are not uniquely characterized by their socle series alone. To each of the two series there corresponds a continuous family of inequivalent indecomposable modules parametrized by \( \lambda \in \mathbb{P} \).

The simple modules \( X^*_s \) can be decomposed into direct sums of simple Virasoro modules

\[
X^+_s = \bigoplus_{m=1}^{\infty} (2m-1) \mathcal{L}_{2m-1},
\]

\[
X^-_s = \bigoplus_{m=1}^{\infty} 2m \mathcal{L}_{2m},
\]

\(^3\) A socle series of a module \( M \) is a filtration of submodules \( S_1(M) \subseteq \cdots \subseteq S_n(M) = M \) such that \( S_1(M) \) is the maximal semi-simple submodule of \( M \) and \( S_n(M)/S_{n-1}(M) \) is the maximal semi-simple submodule of \( S_{n+1}/S_{n-1}(M) \).
where \( \mathcal{L}_{h_{1,s}} \) is the highest weight irreducible Virasoro module of weight

\[
    h_{1,s} = \frac{1}{4p} \left( (rp - s)^2 - (p - 1)^2 \right),
\]

therefore the weights of \( X_s^+ \) and \( X_s^- \), which we will denote by \( h_s^+ \) and \( h_s^- \), are \( h_{1,s} \) and \( h_{2,s} \) respectively.

The dimension of the highest weight spaces \( X_s^+[h_s^+] \) is 1 and the dimension of the highest weight spaces \( X_s^-[h_s^-] \) is 2. We fix non-zero vectors \( u_s \in X_s^+[h_s^+] \) and we fix a basis \( v_s^+, v_s^- \) of \( X_s^-[h_s^-] \) which satisfies the following conditions

\[
    W_0^+ v_s^+ = 0, \quad W_0^- v_s^- = 0.
\]

Then \( v_s^+ \) and \( v_s^- \) are universally determined up to constants.

We have the following results.

1. The Virasoro submodules \( \mathcal{U}(\mathcal{L})u_s \) are isomorphic to the irreducible Virasoro modules \( \mathcal{L}_{h_{1,s}} \) and the Virasoro submodules \( \mathcal{U}(\mathcal{L})v_s^+ \) are isomorphic to the irreducible Virasoro modules \( \mathcal{L}_{h_{2,s}} \).

2. The action of the first few modes of the \( W \)-fields on the highest weight vector \( u_s \) of \( X_s^+ \), \( 1 \leq s \leq p \) is given by

\[
    W_s^\varepsilon v_s^\mu \begin{cases} \varepsilon = 0, & k < 2p - s. \\ \in \mathcal{U}(\mathcal{L})v_s^\mu, & \varepsilon = \pm, 0 < k < 3p - s. \end{cases}
\]

3. The action of the first few modes of the \( W \)-fields on the highest weight vectors \( v_s^\mu \) of \( X_s^- \), \( 1 \leq s \leq p \), \( \mu = \pm \) is then given by

\[
    W_s^\varepsilon v_s^\mu \begin{cases} \varepsilon = 0, & \mu = \varepsilon = \pm, k < 2p - s. \\ \in \mathcal{U}(\mathcal{L})v_s^\mu, & \varepsilon = \mp, 0 < k < 3p - s. \end{cases}
\]

After defining \( \mathcal{W}_p \)-mod, we now turn to \( A_0(\mathcal{W}_p \text{-mod}) \)—the category of zero mode quotients of \( \mathcal{W}_p \)-modules. We define elements of \( A_0(\mathcal{W}_p \text{-mod}) \), in the following way

\[
    \mathcal{X}_s^\varepsilon = A_0(X_s^\varepsilon), \quad 1 \leq s \leq p, \quad \varepsilon = \pm.
\]

Then the \( \mathcal{X}_s^\varepsilon \) are simple objects in \( A_0(\mathcal{W}_p \text{-mod}) \) and any simple object of \( A_0(\mathcal{W}_p \text{-mod}) \) is isomorphic to one of the objects \( \mathcal{X}_s^\varepsilon \).

Just like \( \mathcal{W}_p \)-mod, \( A_0(\mathcal{W}_p \text{-mod}) \) also decomposes into a \( \mathbb{C} \)-linear direct sum of Abelian subcategories

\[
    A_0(\mathcal{W}_p \text{-mod}) = \bigoplus_{s=1,\varepsilon=\pm}^p \mathcal{X}_s^\varepsilon.
\]

The subcategories \( \mathcal{X}_s^+ \) and \( \mathcal{X}_s^- \), \( 1 \leq s \leq p \) are semi-simple and each contains one simple module

\[
    \mathcal{X}_s^+ \in \text{obj}(\mathcal{X}_s^+), \quad \mathcal{X}_s^- \in \text{obj}(\mathcal{X}_s^-).
\]

For \( 1 \leq s \leq p - 1 \) the subcategories \( \mathcal{X}_s^+ \) are not semi-simple. In addition to one simple module

\[
    \mathcal{X}_s^\varepsilon \in \text{obj}(\mathcal{X}_s^\varepsilon),
\]

they also contain one reducible but indecomposable module \( \widetilde{\mathcal{X}}_s^+ \) that is the projective cover of \( \mathcal{X}_s^+ \) and satisfies the exact sequence

\[
    0 \longrightarrow \mathcal{X}_s^+ \longrightarrow \widetilde{\mathcal{X}}_s^+ \longrightarrow \mathcal{X}_s^+ \longrightarrow 0.
\]
The image of the indecomposable $W_p$-modules in $A_0(W_p)$-mod is

\[
\begin{align*}
A_0(X^\pm_s) &= X^\pm_s, & A_0(I^\pm_s(\lambda)) &= dX^\pm_s, & A_0(G^s_{\pm,d}) &= (d+1)X^\pm_s \\
A_0(G^s_{+a}) &= (d+1)X^+_s, & A_0(P^+_s) &= X^+_s, & A_0(H^+_s) &= dX^+_s \\
A_0(H^s_+) &= dX^+_s, & A_0(P^+_{-s}) &= X^+_s \\
\end{align*}
\]

As one can see from the above table, the indecomposable structure of $A_0(W_p)$ is much simpler than that of $W_p$-mod as only the images of $P^+_s$ are non-semi-simple.

The detailed $W_p$-module structure of $X^+_1$ and $X^+_2$ is crucial to calculations. We have the following results.

**Proposition 30.** As complex vector spaces the $A_0$ quotients of simple $W_p$-modules satisfy:

1. For $1 \leq s \leq p$, $\dim A_0(X^+_s) = 1$ and the space is spanned by the equivalence class represented by $u_s$ and therefore has conformal weight $h^+_s$.
2. For $1 \leq s \leq p$, we fix the zero mode subspace $(X^+_s)^0$ to be spanned by the highest weight vector $u_s$.
3. For $1 \leq s \leq p$, $\dim A_0(X^+_s) = 2$ and the space is spanned by the equivalence classes represented by the two highest weight vectors $v^+_s$, $\varepsilon = \pm$ and therefore has conformal weight $h^+_s$.
4. For $1 \leq s \leq p$, we fix the zero mode subspace $(X^+)^0$ to be spanned by the two highest weight vectors $v^+_s$, $\varepsilon = \pm$.

**Proposition 31.** The two copies of $L_{h^+_1}$ in the simple module $X^+_1$ each contains a null vector at level 2

\[
(L^2_{-1} - mL_{-2}) v^\mu_1 = 0, \quad \mu = \pm.
\]  

and a well defined choice for the special subspace $(X^+_1)^\ell$ is given by

\[
(X^+_1)^\ell = \bigoplus_{j=0}^1 \bigoplus_{\mu=\pm} C L^j_{-1} v^\mu_1.
\]

**Proposition 32.** The Virasoro submodule $L_{h^+_2}$ of the simple module $X^+_2$ contains a null vector at level 2

\[
(L^2_{-1} - \frac{1}{p} L_{-2}) u_2 = 0
\]

and a well defined choice for the special subspace $(X^+_2)^\ell$ is given by

\[
(X^+_2)^\ell = \bigoplus_{j=0}^1 C L^j_{-1} u_2 = \bigoplus_{\ell=0,\pm} C W_{-2p+2} \ u_2.
\]

**3.2. The free field realization of $W_p$**

One can explicitly construct $W_p$ as a subVOA of a free field VOA $V_L$ on a lattice by the method of screening operators. The free field VOA is constructed by means of the Heisenberg algebra

\[
a = \mathbb{C} \mathbb{I} \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C} a_n.
\]

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as well as an operator $\hat{a}$, satisfying the commutation relations
\[ [a_m, \hat{a}] = \delta_{m,0}, \quad [a_m, a_n] = m\delta_{m,-n}. \]  
(3.22)

The Heisenberg algebra acts on Fock spaces $\mathcal{F}^\lambda$ generated by a state $|\lambda\rangle$, $\lambda \in \mathbb{C}$
\[ a_m|\lambda\rangle = \lambda \delta_{m,0}|\lambda\rangle, \quad m \geq 0. \]  
(3.23)

For the free field VOA $V_L$ we restrict the charges $\lambda$ of $\mathcal{F}^\lambda$ to a rescaled $A_1$ root lattice $L$ and its dual $L^\vee$:
\[ L = \mathbb{Z}\alpha_+, \quad L^\vee = \mathbb{Z}\frac{\alpha_-}{2}, \]  
(3.24)

where $\alpha_+ = \sqrt{2}p$ and $\alpha_- = -\sqrt{2}p$. The theory contains a single free bosonic field
\[ \varphi(z) = \hat{a} + a_0 \log z + \sum_{n \neq 0} \frac{a_n}{-n}z^{-n}. \]  
(3.25)

that satisfies the OPE
\[ \varphi(z)\varphi(w) \sim \log(z - w). \]  
(3.26)

The energy momentum tensor is given by
\[ T(z) = \frac{1}{2} : (\partial \varphi(z))^2 : + \frac{\alpha_+ + \alpha_-}{2}\partial \varphi(z), \]  
(3.27)

where $: :$ indicates normal ordering, i.e. arranging the Heisenberg operators in ascending order from left to right according to their index with $\hat{a}$ on the very left. Calculating the OPE of $T$ with itself, one reproduces the central charge
\[ c_p = 1 - 6\frac{(p-1)^2}{p}. \]  
(3.28)

of $W_p$. The primary fields are given by
\[ V_\mu(z) = : e^{\mu \varphi(z)} :, \]  
(3.29)

where $\mu \in L^\vee$ and the weight of $V_\mu(z)$ is
\[ h_\mu = \frac{1}{2}\mu - (\alpha_+ + \alpha_-). \]  
(3.30)

The OPE of two primary fields is given by
\[ V_\mu(z)V_\nu(w) = (z - w)^{\mu - \nu} : V_\mu(z)V_\nu(w) :. \]  
(3.31)

The VOA $V_L$ contains the fields $\mathbb{1}$, $T(z)$ and $V_\mu(z)$ for $\mu \in L$ but not $V_\nu(z)$ for $\nu \in L^\vee \setminus L$. The representation category $V_L$-mod is semi-simple with $2p$ simple modules $V_{[\lambda]}$, $[\lambda] \in L^\vee/L$. For later calculations it will prove useful to parametrize the classes $[\lambda] \in L^\vee/L$ by
\[ [r, s] = [\alpha_{r,s}] = \left[ \frac{1 - r}{2}\alpha_+ + \frac{1 - s}{2}\alpha_- \right], \quad r, s \in \mathbb{Z}, \]  
(3.32)

where
\[ [r + 1, s + p] = [r, s]. \]  
(3.33)

The $V_L$-modules decompose into infinite sums of Fock spaces
\[ V_{[\alpha_{r,s}]} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{\alpha_{r,s} + 2rn}. \]  
(3.34)
The \( V_L \)-theory contains two weight 1 primary fields that can be used as screening operators

\[
Q_+ (z) = V_{a_+} (z), \quad Q_- (z) = V_{a_-} (z). \tag{3.35}
\]

The \( \mathcal{W}_p \) VOA is realized by screening with \( Q_- (z) \)

\[
\mathcal{W}_p = \ker \left( \oint dz Q_- (z) : V_{[1,1]} \to V_{[1,-1]} \right). \tag{3.36}
\]

As \( \mathcal{W}_p \)-modules the simple \( V_L \)-modules \( \mathcal{V}_{[1,p]} \) and \( \mathcal{V}_{[2,p]} \) are isomorphic to \( X_p^+ \) and \( X_p^- \) respectively. The remaining \( 2p - 2 \) simple \( V_L \)-modules are reducible as \( \mathcal{W}_p \)-modules and form Felder complexes [56]

\[
\cdots \to \mathcal{V}_{[01],s} \xrightarrow{Q_-^{(s)}} \mathcal{V}_{[02,p-s]} \xrightarrow{Q^{(p-s)}} \mathcal{V}_{[01],r} \to \cdots \tag{3.37}
\]

where \( Q^{(a)} \) is the zero mode of a rather complicated \( a \)-fold product of \( Q_- (z) \) whose details need not concern us here [57]. The simple \( \mathcal{W}_p \)-modules \( X_s^\pm \) for \( 1 \leq s \leq p - 1 \) are equivalent to the kernels and images of \( Q^{(a)} \)

\[
X_s^+ = \ker \left( Q^{(a)} : \mathcal{V}_{[1,1]} \to \mathcal{V}_{[2,p-s]} \right) = \text{im} \left( Q^{(p-s)} : \mathcal{V}_{[2,p-s]} \to \mathcal{V}_{[1,1]} \right)
\]

\[
X_s^- = \ker \left( Q^{(p-s)} : \mathcal{V}_{[2,p-s]} \to \mathcal{V}_{[1,1]} \right) = \text{im} \left( Q^{(a)} : \mathcal{V}_{[1,1]} \to \mathcal{V}_{[2,p-s]} \right). \tag{3.38}
\]

**Proposition 33.** The screening operator maps \( Q^{(a)} \) induce surjective \( \mathcal{W}_p \)-module maps

\[
\mathcal{V}_{[1,1]} \to X_{p-s}^-, \quad \mathcal{V}_{[2,1]} \to X_{p-s}^+
\]

\[
|\alpha_{-s},s\rangle \mapsto u_{p-s}, \quad |\alpha_{s,s}\rangle \mapsto v_{p-s}, \tag{3.39}
\]

for \( 1 \leq s \leq p - 1 \) as well as injective \( \mathcal{W}_p \)-module maps

\[
X_s^+ \to \mathcal{V}_{[1,1]}, \quad X_s^- \to \mathcal{V}_{[2,1]}\]

\[
u_s \mapsto |\alpha_{1,s}\rangle, \quad v_s \mapsto |\alpha_{2,s}\rangle, \tag{3.40}
\]

for \( 1 \leq s \leq p \).

### 3.3. The \( V_L \) fusion product

We recall some well known facts about the \( V_L \)-mod fusion product that will be relevant to our calculations below. The tuple \((V_L\text{-mod}, \otimes, \mathcal{V}_{[1,1]}\)) defines a braided monoidal category, i.e. there is a well defined fusion product of \( V_L \)-modules

\[
\mathcal{V}_{[r_1,r_2]} \otimes \mathcal{V}_{[s_1,s_2]} = \mathcal{V}_{[r_1+s_2-1,n+r_2-1]}. \tag{3.41}
\]

The free field VOA \( V_L \) contains another subVOA \( (\mathcal{F}^\alpha, \mathcal{O}, \mathcal{T}, Y) \) other than \( \mathcal{W}_p \) called the Heisenberg VOA\(^4\). The current algebra \( \mathcal{U}(\mathcal{F}^\alpha) \) is given by the universal enveloping algebra \( \mathcal{U}(\alpha) \) of the Heisenberg algebra, while the simple objects of \( \mathcal{F}^{\alpha \lambda} \) are given by \( \mathcal{F}^\lambda \), \( \lambda \in \mathbb{C} \)

**Proposition 34.** For \( \lambda_1, \lambda_2 \in \mathbb{C} \) the fusion product in \( \mathcal{F}^{\alpha \lambda_1 \lambda_2} \)-mod is given by

\[
\mathcal{F}^{\lambda_1} \otimes \mathcal{F}^{\lambda_2} \sim \mathcal{F}^{\lambda_1 + \lambda_2}
\]

\[
|\lambda_1\rangle \otimes |\lambda_2\rangle \mapsto |\lambda_1 + \lambda_2\rangle \tag{3.42}
\]

\(^4\) This is the only appearance of a non-c\(_2\)-cofinite VOA in this paper.
and for \((r_1, s_1), (r_2, s_2) \in \mathbb{Z}^2\) the following diagram commutes
\[
\begin{array}{cc}
\mathcal{F}^{\alpha_{r_1+1}} \otimes \mathcal{F}^{\alpha_{s_2}} & \mathcal{F}^{\alpha_{r_1+1}+r_2-1,s_1+s_2-1} \\
\mathcal{V}_{[r_1, s_1]} \otimes_{V_{L}} \mathcal{V}_{[r_2, s_2]} & \mathcal{V}_{[r_1+r_2-1,s_1+s_2-1]}
\end{array}
\]
where the vertical arrows are injective \(\mathcal{F}^{\alpha_{1,1}}\)-module maps and the horizontal arrows are a \(\mathcal{F}^{\alpha_{1,1}}\) and a \(V_{L}\)-isomorphism respectively.

4. Computing certain fusion products in \(\mathcal{W}_p\)-mod

In this section we apply the methods explained above to analyse the monoidal structure of \(\mathcal{W}_p\)-mod.

4.1. The fusion rules and rigidity of \(X^-_1\)

In this section we analyse the fusion products of \(X^-_1\) with simple modules and prove the rigidity of \(X^-_1\).

Theorem 35.

(1) The fusion square of \(X^-_1\) is
\[
X^-_1 \otimes X^-_1 = X^+_1.
\]
(2) \(X^-_1\) is rigid and self-dual.

Sketch of proof. We prove the theorem in three steps.

(1) We prove that there exists a surjection of \(A_0(\mathcal{W}_p)\)-modules
\[
A_0(X^+_1) \to A_0(X^-_1 \otimes X^-_1).
\]
(2) We prove that \(\dim A_1(X^-_1 \otimes X^-_1)[1] = 0\).
(3) We prove the existence of a non-trivial \(\mathcal{W}_p\)-module map
\[
X^-_1 \otimes X^-_1 \to \mathcal{V}_{[2,p-1]}.
\]

Step 1 implies that \(X^-_1 \otimes X^-_1\) is a (possibly trivial) highest weight module generated by a state of conformal weight 0. Since \(h_{p-1} = 1\) step 2 excludes the possibility of \(X^-_{p-1}\) being a submodule of \(X^-_1 \otimes X^-_1\). Step 3 implies that \(X^-_1 \otimes X^-_1\) is non-trivial and since the only non-trivial submodule of \(\mathcal{V}_{[2,p-1]}\), generated by a state of conformal weight 0 is \(X^+_1\), it follows that \(X^-_1 \otimes X^-_1 = X^+_1\).

The rigidity of \(X^-_1\) follows by choosing
\[
\text{id}_{X^-_1} : X^-_1 \otimes X^-_1 \to X^+_1
\]
\[
\text{id}_{X^+_1} : X^+_1 \otimes X^-_1 \to X^-_1
\]
and the fact that therefore all the maps appearing in definition 17 are isomorphisms. \(\square\)

Proof of step 1. As in proposition 31 we choose
\[
(X^-_1)^\prime = \mathop{\oplus}_{j=0}^1 \mathop{\oplus}_{\epsilon=\pm} C L^j_{L-1} v^\epsilon_1
\]
and as in proposition 30 we choose
\[ (X^{-})^{0} = \bigoplus_{\epsilon = \pm} \mathbb{C} \nu^{\pm}_{1}. \] (4.6)

Using the formulae (2.94) as well as the null vector in proposition 31 we can compute the action of \( L_{0} \) on the classes represented by the elements of \( (X^{-})^{0} \cap (X^{-})^{0} \):
\[
\begin{align*}
[v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] & \mapsto 2h_{1}^{\pm}[v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] + [L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] \\
[L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] & \mapsto (2h_{1}^{\pm} + 1)[v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] + [L_{-1}^{2} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}]
\end{align*}
\] (4.7)

Thus for each pair \( \epsilon_{1}, \epsilon_{2} \) we can represent \( L_{0} \) by
\[ L_{0} \equiv \begin{pmatrix} 2h_{1}^{\pm} & ph_{1}^{\pm} \\ 1 & 2h_{1}^{\pm} + 1 - p \end{pmatrix} \] (4.8)
on the basis \( L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1} \), \( j = 0, 1 \). The eigenvalues of this matrix are \( h_{1}^{\pm} = 0 \) and \( 2p - 1 \).

Next we will determine a lower bound on the dimension of the kernel of the surjection
\[ (X^{-})^{y} \otimes (X^{-})^{0} \to \mathcal{A}_{0}(X_{I}^{-} \otimes X_{I}^{-}). \] (4.9)

Because \( \mathcal{A}_{0}(W_{p}) \)-mod does not contain any module with eigenvalue \( 2p - 1 \) eigenvectors, the eigenvectors
\[ v^{\epsilon_{1}}_{1} \otimes v^{\epsilon_{2}}_{1} + \frac{2}{3p - 2} L_{-1} v^{\epsilon_{1}}_{1} \otimes v^{\epsilon_{2}}_{1}, \] (4.10)
corresponding to the eigenvalue \( 2p - 1 \), must lie in the kernel of the surjection (4.9).

From formula (3.11) illustrating the action of \( W \)-field modes on \( X_{I}^{-} \) we know
\[ L_{-1} v_{1}^{\epsilon} = C_{\epsilon} \cdot W^{e} v_{1}^{\epsilon}, \quad \epsilon = \pm \] (4.11)
for some constant \( C_{\epsilon} \). This implies that \( L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}, \int = \pm \) lies in the kernel of (4.9), because
\[ [L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] = C_{\epsilon} \cdot [W^{e} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] \]
\[ = C_{\epsilon} \cdot \sum_{j=0}^{2p-3} \binom{2p-3}{j} (-1)^{j} [v^{\epsilon}_{1} \otimes W^{e} v_{(2p-2)}^{\epsilon}] = 0. \] (4.12)

Finally
\[ [L_{-1} v^{\epsilon}_{1} \otimes v^{-\epsilon}_{1}] = C_{\epsilon} \cdot [W^{e} v^{\epsilon}_{1} \otimes v^{-\epsilon}_{1}] \]
\[ = C_{\epsilon} \cdot \sum_{j=0}^{2p-3} \binom{2p-3}{j} (-1)^{j} [v^{-\epsilon}_{1} \otimes W^{e} v_{(2p-2)}^{-\epsilon}] \]
\[ = A_{1} [L_{-1} v^{\epsilon}_{1} \otimes v^{-\epsilon}_{1}] + B_{1} [v^{\epsilon}_{1} \otimes v^{-\epsilon}_{1}], \] (4.13)
for some constants \( A_{z} \) and \( B_{z} \), since by the action of the \( W \)-modes (3.11) \( W^{e} v_{(2p-2)}^{\epsilon} \in L^{e}_{h_{1}^{\pm}}. \)

This implies that some non-trivial linear combination of \( [L_{-1} v^{\epsilon}_{1} \otimes v^{\epsilon}_{1}] \) and \( [L_{-1} v^{\epsilon}_{1} \otimes v^{-\epsilon}_{1}] \) lies in the kernel of (4.9). Therefore the kernel of (4.9) is at least seven dimensional and \( \mathcal{A}_{0}(X_{I}^{-} \otimes X_{I}^{-}) \) is at most one dimensional. If \( \mathcal{A}_{0}(X_{I}^{-} \otimes X_{I}^{-}) \) is indeed non-trivial, then the eigenvalue of \( L_{0} \) is 0. \( \square \)

Proof of step 2. We know that the image of the action of the \( W \)-field modes \( W^{e}_{k} \) on the highest weight vectors \( v^{\epsilon}_{1} \) lies in the Virasoro submodules generated by \( v^{\epsilon}_{1} \) and \( v^{-\epsilon}_{1} \) for \( k < 3p - 1 \). Therefore a spanning set of representatives for \( \mathcal{A}_{1}(X_{I}^{-} \otimes X_{I}^{-}) \) can be chosen from Virasoro descendants of \( v^{e}_{1} \otimes v^{\epsilon}_{1}, \epsilon_{1} = \pm, \epsilon_{2} = \pm \). Also since the relations (2.94) for Virasoro modes still hold for \( n \geq 2 \), we can restrict the spanning set of representatives for \( \mathcal{A}_{1}(X_{I}^{-} \otimes X_{I}^{-}) \) to
L_{-1}^-\text{-descendants of } v_1^{i_1} \otimes v_1^{i_2}. Finally because of the null vector (3.17) at level 2 of X^- we have the following surjection of complex vector spaces

\[
\bigoplus_{i,j=0}^{\neq} \mathbb{C}[L_{-1}^-v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}] \to \mathcal{A}_1(X^-_1 \otimes X^-_1). \tag{4.14}
\]

Because the image of the canonical Lie algebra homomorphism

\[
\mathfrak{g}(\mathcal{W}_p) \to \mathcal{U}(\mathcal{W}_p) \tag{4.15}
\]

is dense, we know that the image of \( \mathbb{C}^2 \) lies in \( g_2(\mathcal{W}_p)(X^-_1 \otimes X^-_1) \) and that \( \mathbb{C}^2 \) therefore acts trivially on \( \mathcal{A}_1(X^-_1 \otimes X^-_1) \) even if \( L_{-1}^- \) does not. This implies the relation

\[
(f_{1,0}(T \otimes 1 \cdot dz^{-1}))^2 [v_1^{i_1} \otimes v_1^{i_2}]
\]

\[
= [L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}] + 2[L_{-1}^-v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}] + [v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}] = 0 \tag{4.16}
\]

We therefore take \([v_1^{i_1} \otimes v_1^{i_2}], [L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}] \) and \([v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}] \) as a spanning set for \( \mathcal{A}_1(X^-_1 \otimes X^-_1) \) and compute the action of \( L_0 \)

\[
[v_1^{i_1} \otimes v_1^{i_2}] \mapsto \left( \frac{3}{2} p - 1 \right) [v_1^{i_1} \otimes v_1^{i_2}] + [L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}]
\]

\[
[L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}] \mapsto \left( \frac{3}{2} p - 1 \right) \frac{p}{2} [v_1^{i_1} \otimes v_1^{i_2}] + \frac{3}{2} p [L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}] + \frac{p}{2} [v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}]
\]

\[
[v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}] \mapsto - \left( \frac{3}{2} p - 1 \right) \frac{p}{2} [v_1^{i_1} \otimes v_1^{i_2}] + \frac{p}{2} [L_{-1}^-v_1^{i_1} \otimes v_1^{i_2}] + \frac{p}{2} [v_1^{i_1} \otimes L_{-1}^-v_1^{i_2}]. \tag{4.17}
\]

As a matrix \( L_0 \) is represented by

\[
\begin{pmatrix}
\frac{3}{2} p - 1 & \frac{3}{2} p - 1 & \frac{3}{2} p - 1 \\
1 & \frac{3}{2} p & \frac{3}{2} p \\
0 & \frac{p}{2} & \frac{p}{2}
\end{pmatrix}
\]

on the basis \( v_1^{i_1} \otimes v_1^{i_2}, L_{-1}^-v_1^{i_1} \otimes v_1^{i_2} \) and \( v_1^{i_1} \otimes L_{-1}^-v_1^{i_2} \) and the eigenvalues of this matrix are 0, \( 2p - 1 \) and \( 2p \), none of which are \( h_{p-1} = 1 \). \( \Box \)

**Proof of step 3.** According to proposition 33 there exists an injective \( \mathcal{W}_p \)-module map

\[
X^-_1 \to \mathcal{V}_{[2,1]}
\]

\[
v^-_1 \mapsto [\omega_{2,1}]. \tag{4.19}
\]

By this map and proposition 25 there exists a non-trivial \( \mathcal{W}_p \)-module map

\[
X^-_1 \otimes X^-_1 \to \mathcal{V}_{[2,1]} \otimes \mathcal{V}_{[2,1]}
\]

\[
v^-_1 \otimes v^-_1 \mapsto [\omega_{2,1}] \otimes [\omega_{2,1}]. \tag{4.20}
\]

By proposition 34 there exits a \( \mathcal{V}_2 \)-module isomorphism

\[
\mathcal{V}_{[2,1]} \otimes \mathcal{V}_{[2,1]} \cong \mathcal{V}_{[3,1]} \equiv \mathcal{V}_{[1,1]}
\]

\[
[\omega_{2,1}] \otimes [\omega_{2,1}] \mapsto [\omega_{3,1}]. \tag{4.21}
\]

By composing these two maps we have constructed a non-trivial \( \mathcal{W}_p \)-module map

\[
X^-_1 \otimes X^-_1 \to \mathcal{V}_{[1,1]}. \tag{4.22}
\]
Theorem 36. The fusion rules of $X_i^-$ with simple modules is given by

$$X_i^- \otimes X_i^s = X_i^{-\varepsilon} \quad 1 \leq s \leq p, \; \varepsilon = \pm.$$  \hfill (4.23)

Sketch of proof. We prove the theorem in two steps

(1) Let $M$ be a simple module, then $X_i^- \otimes M$ is also simple.

(2) We prove the existence of a non-trivial $W_p$-module map

$$X_i^- \otimes X_i^s \to V_{[2,s]}.$$  \hfill (4.24)

The simplicity of $X_i^- \otimes X_i^s$ implied by step 1 and the non-triviality of the map in step 2 implies that $X_i^- \otimes X_i^s$ is a simple submodule of $V_{[2,s]}$. Therefore $X_i^- \otimes X_i^s = X_i^-$. The theorem then follows by $X_i^- \otimes X_i^- = X_i^1$. \hfill \Box

Proof of step 1. Proof by contradiction. Assume $X_i^- \otimes M$ is not simple, then there exists an exact sequence

$$0 \to A \to X_i^- \otimes M \to B \to 0,$$  \hfill (4.25)

for some non-trivial $W_p$-modules $A$ and $B$. Because $X_i^-$ is rigid, the sequence

$$0 \to X_i^- \otimes A \to M \to X_i^- \otimes B \to 0,$$  \hfill (4.26)

must also be exact which is in contradiction to $M$ being simple. \hfill \Box

Proof of step 2. According to proposition 33 there exist injective $W_p$-module maps

$$X_i^- \to V_{[2,1]},$$

$$v_i^- \mapsto |x_{2,1}|$$  \hfill (4.27)

and for $1 \leq s \leq p$

$$X_i^s \to V_{[1,s]},$$

$$u_s \mapsto |x_{1,s}|.$$  \hfill (4.28)

By the above injective $W_p$-module maps and proposition 25 there exists a non-trivial $W_p$-module map

$$X_i^- \otimes X_i^s \to V_{[2,1]} \otimes V_{[1,s]}$$

$$v_i^- \otimes u_s \mapsto |x_{2,1}| \otimes |x_{1,s}|.$$  \hfill (4.29)

By proposition 34 there exists a $V_2$-module isomorphism

$$V_{[2,1]} \otimes V_{[1,s]} \to V_{[2,s]}$$

$$|x_{2,1}| \otimes |x_{1,s}| \mapsto |x_{2,s}|.$$  \hfill (4.30)

By composing these two maps we have constructed a non-trivial $W_p$-module map

$$X_i^- \otimes X_i^s \to V_{[2,s]}.$$  \hfill (4.31)

\hfill \Box
4.2. The fusion rules and rigidity of $X_2^+$

In this section we analyse the fusion products of $X_2^+$ with simple modules and prove the rigidity of $X_2^+$.

**Theorem 37.** The $W_p$-module $X_2^+$ is rigid and the fusion rules of $X_2^+$ with simple modules are given by

$$X_2^+ \hat{\otimes} X_2^s = \begin{cases} X_2^s & s = 1 \\ X_2^{s-1} \oplus X_2^{s+1} & 2 \leq s \leq p - 1 \\ P_{p-1} & s = p. \end{cases}$$ (4.32)

**Sketch of proof.** We prove the theorem in a number of steps

1. We prove the existence of surjections of $A_0$-modules

   $\begin{align*}
   &s = 1 & A_0(X_2^-) \\
   &1 < s < p & A_0(X_2^{s-1}) \oplus A_0(X_2^{s+1}) \\
   &s = p & A_0(P_{p-1})
   \end{align*}$

   $\rightarrow A_0(X_2^+ \otimes X_2^-) \rightarrow 0.$ (4.33)

2. We prove the existence of non-trivial $W_p$-module maps

   $X_2^+ \otimes X_2^s \rightarrow V_{[2,s+1]},$ (4.34)

   for $1 \leq s \leq p - 1.$

3. We prove the existence of non-trivial $W_p$-module maps

   $X_2^+ \otimes V_{[1,p-s]} \rightarrow X_2^{-s-1},$ (4.35)

   for $2 \leq s \leq p - 1.$

4. We prove the existence of a surjective $W_p$-module map

   $X_2^+ \otimes X_2^{-p} \rightarrow X_2^{-p-1}.$ (4.36)

5. We use the formalism outlined in section 2.4 to prove that $X_2^+$ is rigid and that therefore $(X_2^+)^\vee = X_2^+.$

Steps 1 through 3 prove

$$X_2^+ \otimes X_2^s = \begin{cases} X_2^s & s = 1 \\ X_2^{s-1} \oplus X_2^{s+1} & 1 < s < p. \end{cases}$$ (4.37)

Step 5 implies that $X_2^+ \otimes X_2^s$ is projective, since the product of the dual of a rigid module and a projective module is again projective. The only projective module compatible with steps 1 and 4 is $P_{p-1},$ therefore

$$X_2^+ \otimes X_2^{-p} = P_{p-1}.$$ (4.38)

Finally the fusion products of the theorem follow by multiplying with $X_2^-$ and the associativity of the fusion product. □

**Proof of step 1.** We choose the special subspace of $X_2^+$ to be given by

$$(X_2^+)^\vee = \bigoplus_{j=0}^{1} \mathbb{C}L_j u_2 \oplus \bigoplus_{s=0,\pm} \mathbb{C}W_{2s+3} u_2.$$ (4.39)
as in proposition 32 and we choose the zero mode subspace of \( X^+ \) to be given by
\[
(X^-)^0 = \bigoplus_{\epsilon = \pm} C v^\epsilon_r.
\] (4.40)
as in proposition 30. By proposition 24 there is a canonical surjection
\[
(X^+)^0 \otimes (X^-)^0 \to \mathcal{A}_0(\mathfrak{h}^+ \otimes \mathfrak{h}^-).
\] (4.41)

We first show that the spanning set
\[
\text{span}([L^j_{-1} u_2 \otimes v^\pm], [W^\mu_{2p+2} u_2 \otimes v^\pm], \ j = 0, 1, \ \epsilon = \pm, \ \mu = \pm, 0)
\] (4.42)
is redundant. Consider
\[
[W^\mu_{2p+2} u_2 \otimes v^\pm] = -[u_2 \otimes W^\mu_{2p+2} v^\pm].
\] (4.43)
The difference in conformal highest weights between the Virasoro representations \( \mathcal{L}_{h_{1, \epsilon}} \) and \( \mathcal{L}_{h_{2, \delta}} \) in the decomposition (3.7) of \( X^- \), is \( h_{1, \epsilon} - h_{2, \delta} = 3p - s \). Therefore
\[
W^\mu_{2p+2} v^\pm_s \in \bigoplus_{\epsilon = \pm} \mathcal{U}(\mathcal{L}) v^\pm
\] (4.44)
which implies that \([W^\mu_{2p+2} u_2 \otimes v^\pm]\) depends linearly on the \([L^j_{-1} u_2 \otimes v^\pm]\), \( j = 0, 1, \ \delta = \pm \).

We therefore have a four dimensional spanning set for \( \mathcal{A}_0(\mathfrak{h}^+ \otimes \mathfrak{h}^-) \) on which we can compute the action of \( L_0 \):

\[
\begin{align*}
[u_2 \otimes v^\pm_s] &\mapsto (h^{+}_s + h^{-}_s)[u_2 \otimes v^\pm_s] + [L^0_{-1}u_2 \otimes v^\pm_s] \\
[L^j_{-1} u_2 \otimes v^\pm_s] &\mapsto \frac{h^{+}_s}{p}[u_2 \otimes v^\pm_s] + \left(h^{+}_s + h^{-}_s + 1 - \frac{1}{p}\right)[L^j_{-1}u_2 \otimes v^\pm_s].
\end{align*}
\] (4.45)

We can therefore represent \( L_0 \) by the matrix
\[
\begin{pmatrix}
\frac{h^{+}_s}{p} & h^{-}_s - 1 \\
1 & h^{+}_s + h^{-}_s + 1 - \frac{1}{p}
\end{pmatrix}
\] (4.46)
on the basis \( L^j_{-1} u_2 \otimes v^\pm_s, j = 0, 1, \ \epsilon = \pm \). For \( 2 \leq s \leq p - 1 \) the eigenvalues of this matrix are \( h^{+}_{s-1} \) and \( h^{-}_{s+1} \) and for \( s = p \) the eigenvalues of the above matrix are \( h^{+}_{p-1} \) and \( h^{-}_1 \). \( \square \)

**Proof of step 2.** According to proposition 33 there exist injective \( \mathcal{W}_p \)-module maps
\[
X^+_s \to \mathcal{V}_{[2, s]} \\
u_2 \mapsto [\alpha_1, s] \tag{4.47}
\]
and for \( 1 \leq s < p \)
\[
X^-_s \to \mathcal{V}_{[2, s]} \\
v^+_s \mapsto [\alpha_2, s]. \tag{4.48}
\]
By these maps and proposition 25 there exists a non-trivial \( \mathcal{W}_p \)-module map
\[
X^+_s \otimes X^-_s \to \mathcal{V}_{[1, 2]} \otimes \mathcal{V}_{[2, s]} \\
u_2 \otimes v^-_s \mapsto [\alpha_1, 2] \otimes [\alpha_2, s]. \tag{4.49}
\]
By proposition 34 there exists a \( \mathcal{V}_p \)-module isomorphism
\[
\mathcal{V}_{[1, 2]} \otimes \mathcal{V}_{[2, s]} \to \mathcal{V}_{[2, s+1]} \\
[\alpha_1, 2] \otimes [\alpha_2, s] \mapsto [\alpha_1, 2] \otimes [\alpha_2, s+1]. \tag{4.50}
\]
By composing these two maps we have constructed a non-trivial $W_p$-module map
\[ X_s^+ \otimes X_s^- \to V_{[2,s+1]} \]  
(4.51)

**Proof of step 3.** Before we begin with the proof we note that the proof of step 1 implies the existence of a surjective $W_p$-module map
\[ X_{s-1}^- \oplus X_{s+1}^- \to X_s^+ \otimes X_s^- \]  
(4.52)
for $1 < s < p$, i.e. the results for $A_0(X_s^+ \otimes X_s^-)$ allow for no modules larger than $X_{s-1}^-$ or $X_{s+1}^-$. Therefore because $X_1^-$ is rigid and the functor $X_i^- \otimes -$ is exact, there exists a surjective $\mathcal{W}_p$-module map
\[ X_{s-1}^+ \oplus X_{s+1}^+ \to X_s^+ \otimes X_s^+ \]  
(4.53)
According to proposition 33 there exists an injective $\mathcal{W}_p$-module map
\[ X_2^+ \to V_{[2,1]} \]  
\[ u_2 \mapsto [\alpha_{1,2}] \]  
(4.54)
By this map and proposition 25 there exists a non-trivial $\mathcal{W}_p$-module map
\[ X_2^+ \otimes V_{[1,p-s]} \to V_{[1,2]} \otimes V_{[1,p-(s-1)]} \]  
\[ u_2 \otimes [\alpha_{1,2-p,s}] \mapsto [\alpha_{1,2}] \otimes [\alpha_{1,2-p,s}] \]  
(4.55)
By proposition 34 there exists a $V_\ell$-module isomorphism
\[ V_{[1,2]} \otimes V_{[1,p-s]} \to V_{[1,p-(s-1)]} \]  
\[ [\alpha_{1,2}] \otimes [\alpha_{1,2-p,s}] \mapsto [\alpha_{1,2-p,s}] \]  
(4.56)
By composing these two maps we have constructed a non-trivial $\mathcal{W}_p$-module map
\[ \psi : X_2^+ \otimes V_{[1,p-s]} \to V_{[1,p-(s-1)]} \]  
(4.57)
Also according to proposition 33 there exists a surjective $\mathcal{W}_p$-module map
\[ \pi : V_{[1,p-(s-1)]} \to X_1^+ \]  
\[ [\alpha_{1,p-(s-1)}] \mapsto \psi_{s-1}^+ \]  
(4.58)
The composition $\pi \circ \psi$ is therefore a non-trivial $\mathcal{W}_p$-module map
\[ \pi \circ \psi : X_2^+ \otimes V_{[1,p-s]} \to X_1^+ \]  
(4.59)
By the surjection (4.53) $X_2 \otimes X_{p-s}^-$ must lie in the kernel of $\pi \circ \psi$. Therefore there exists a non-trivial $\mathcal{W}_p$-module map
\[ X_2^+ \otimes X_1^- \to X_{s-1}^- \]  
(4.60)

**Proof of step 4.** According to proposition 33 there exists an injective $\mathcal{W}_p$-module map
\[ X_2^+ \to V_{[1,2]} \]  
\[ u_2 \mapsto [\alpha_{1,2}] \]  
(4.61)
and a $\mathcal{W}_p$-module isomorphism $X_p^- \to V_{[2,p]}$. By the above maps and proposition 25 there exists a non-trivial $\mathcal{W}_p$-module map
\[ X_2^+ \otimes V_{[2,p]} \to V_{[1,2]} \otimes V_{[2,p]} \]  
\[ u_2 \otimes [\alpha_{0,p}] \mapsto [\alpha_{1,2}] \otimes [\alpha_{0,p}] \]  
(4.62)
By proposition 34 there exists a $V_2$-module isomorphism
\[
V_{[1,1]} \otimes v_2 V_{[2,p]} \rightarrow V_{[2,p+1]} = V_{[1,1]}
\]
\[|\alpha_{1,2} \otimes |\alpha|_{0,p} \mapsto |\alpha|_{0,p+1} = |\alpha|_{-1,1}]. \tag{4.63}
\]
By composing these two maps we have constructed a non-trivial $W_p$-module map
\[
X_2^+ \otimes X_p^+ \rightarrow V_{[1,1]}. \tag{4.64}
\]
Also according to proposition 33 there exists a surjective $W_p$-module map
\[
V_{[1,1]} \rightarrow X_{p-1}^-
\]
\[|\alpha_{-1,1} \mapsto v_{p-1}^+]. \tag{4.65}
\]
Therefore there exists a non-trivial $W_p$-module map
\[
X_2^+ \otimes X_p^- \rightarrow X_{p-1}^-. \tag{4.66}
\]

**Proof of step 5.** In order to prove the rigidity of $X_2^+$, we need to consider three fold products of $X_2^+$, which at this stage we can only compute for $p \geq 4$. We will explain how the proof of rigidity can be reduced to analysing formal solutions of hypergeometric equations for $p \geq 4$. The advantage of this analysis is that it does not require us to explicitly know the three fold fusion product of $X_2^+$ and we can therefore also apply it to $p = 2, 3$ once we have discussed $p \geq 4$.

Until explicitly stated otherwise we will therefore assume that $p \geq 4$. Then we have proven that
\[
X_2^+ \otimes (X_2^+ \otimes X_2^+) \cong (X_2^+ \otimes X_2^+) \otimes X_2^+ = 2 \cdot X_2^+ \oplus X_2^+. \tag{4.67}
\]
The rigidity of $X_2^+$ and the self-duality $X_2^{+\vee} = X_2^+$ require the existence of $W_p$-module maps $i : X_1^+ \rightarrow X_2^+ \otimes X_2^+$ and $e : X_2^+ \otimes X_2^+ \rightarrow X_1^+$, such that
\[
X_2^+ \cong X_2^+ \oplus X_2^+ \xrightarrow{id \otimes i} X_2^+ \otimes (X_2^+ \otimes X_2^+)
\]
\[
f \xrightarrow{\alpha_{X_2^+,X_2^+,X_2^+}}
\]
\[
X_2^+ \cong X_2^+ \otimes X_2^+ \xrightarrow{e \otimes id} (X_2^+ \otimes X_2^+) \otimes X_2^+
\]
\[
(4.68)
\]
and
\[
X_2^+ \cong X_2^+ \otimes X_2^+ \xrightarrow{i \otimes id} (X_2^+ \otimes X_2^+) \otimes X_2^+
\]
\[
g \xrightarrow{\alpha_{X_2^+,X_2^+,X_2^+}^{-1}}
\]
\[
X_2^+ \cong X_2^+ \otimes X_2^+ \xrightarrow{id \otimes e} X_2^+ \otimes (X_2^+ \otimes X_2^+)
\]
\[
(4.69)
\]
commute, where $f = \mu \cdot \text{id}_{X_2^+}$, $g = \nu \cdot \text{id}_{X_2^+}$ for two non-zero constants $\mu$ and $\nu$. We show that $\mu \neq 0$, the case of $\nu$ is similar so we omit the proof.

We fix highest weight vectors $u_i$ of $X_2^+$ for $s = 1, 2, 3$, such that $X_2^+[h_s^+] = Cu_i$ and $u_1 = \Omega$ as in previous calculations. By the fusion products we have computed so far we know that the spaces of vertex operators
\[
\left( X_2^+, X_2^+ \right), \left( X_2^+, X_2^+ \right), \left( X_2^+, X_2^+ \right), \left( X_2^+, X_2^+ \right), \left( X_2^+, X_2^+ \right), \left( X_2^+, X_2^+ \right).
\]
(4.70)
are all one dimensional for \( s = 1, 3 \). We therefore fix non-trivial vertex operators
\[
\Psi^2 \in \left( \frac{X^+_2}{X^+_2 \times X^+_2} \right), \quad \Psi^2 \in \left( \frac{X^+_4}{X^+_2 \times X^+_2} \right), \quad \Psi^4 \in \left( \frac{X^+_4}{X^+_2 \times X^+_2} \right).
\] (4.71)

These vertex operators can be formally expanded as
\[
b \Psi^2_s(z; z) = \sum_{n \in \mathbb{Z}} b \Psi^2_{c,n}(z) z^{-n-\left(h^+_2 + h^+_4 - h^+_6\right)},
\] (4.72)
for appropriate choices of \( a, b \) and \( c \), where
\[
b \Psi^2_{c,n} \in \bigoplus_{k, \ell \geq 0} \text{Hom}_\mathbb{C}(X^+_n [h^+_2 + k] \otimes X^+_n [h^+_2 + \ell], X^+_n [h^+_4 + k + \ell - n]).
\] (4.73)

This allows us to define four power series
\[
\Phi^{(1)}_s(z_4, z_3, z_2, z_1) = (\Omega)_1 \Psi^2_1(u_2; z_4) \Psi^2_2(u_2; z_3) \Psi^2_3(u_2; z_2) \Psi^2_4(u_2; z_1) \Omega
\]
\[
\Phi^{(2)}_s(z_4, z_3, z_2, z_1) = (\Omega)_1 \Psi^2_1(u_2; z_4) \Psi^2_2(u_2; z_3) \Psi^2_3(u_2; z_2 - z_3) \Psi^2_4(u_2; z_1) \Omega,
\] (4.74)
for \( s = 1, 3 \). The power series \( \Phi^{(1)}_s \) and \( \Phi^{(2)}_s \) converge absolutely on the domains
\[
U^{(1)} = \{ (z_4, z_3, z_2, z_1) \in (\mathbb{C}^*)^4 | |z_4| > |z_3| > |z_2| > |z_1| > 0 \},
\]
\[
U^{(2)} = \{ (z_4, z_3, z_2, z_1) \in (\mathbb{C}^*)^4 | |z_4| > |z_3| > |z_2 - z_3| > 0, |z_3| > |z_2 - z_3| > 0 \},
\] (4.75)
respectively and satisfy the partial differential equations:

(1) For \( n = -1, 0, 1 \)
\[
\sum_{a=1}^{4} z_a^c \frac{\partial}{\partial z_a} + (n + 1) h^+_2 \Phi = 0.
\] (4.76)

(2) For \( a = 1, 2, 3, 4 \)
\[
\frac{\partial^2}{\partial z_a^2} - \frac{1}{p} \sum_{b=1}^{4} \left( \frac{h^+_2}{(z_b - z_a)^2} - \frac{1}{z_b - z_a} \frac{\partial}{\partial z_b} \right) \Phi = 0.
\] (4.77)

The first set of differential equations follows from Möbius covariance and the second set from the null vector at level 2 in \( X^+_2 \).

The solution space of these two sets of differential equations is two dimensional and the solutions define multivalued holomorphic functions on \( (\mathbb{P})^4 \setminus \) diagonals. Therefore \( \Phi^{(1)}_s \) and \( \Phi^{(2)}_s \) define bases of the solution space of the above differential equations on the two domains \( U^{(1)} \) and \( U^{(2)} \) and it is possible to analytically continue \( \Phi^{(1)}_s \) to \( U^{(2)} \) and vice versa. For a given path \( \gamma \) from \( U^{(1)} \) to \( U^{(2)} \), \( \Phi^{(1)}_s \) can be written as a linear combination of \( \Phi^{(2)}_s \) and \( \Phi^{(2)}_{1-s} \). This defines a connection matrix
\[
\begin{pmatrix}
\Phi^{(1)}_s \\
\Phi^{(2)}_s
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\Phi^{(2)}_s \\
\Phi^{(2)}_{1-s}
\end{pmatrix}.
\] (4.78)

Going along the path \( \gamma \) in the opposite direction one can express \( \Phi^{(2)}_s \) as a linear combination of \( \Phi^{(1)}_s \) and \( \Phi^{(1)}_{1-s} \) with the inverse of the connection matrix above
\[
\begin{pmatrix}
\Phi^{(1)}_s \\
\Phi^{(2)}_s
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1}
\begin{pmatrix}
\Phi^{(1)}_s \\
\Phi^{(2)}_{1-s}
\end{pmatrix}.
\] (4.79)
The constant $\mu$, in $f = \mu \cdot \text{id}_{L}^{2}$ of diagram (4.68), being non-zero is equivalent to $\Phi_{1}^{(1)}$ having non-vanishing contributions from $\Phi_{1}^{(2)}$, i.e. $a$ being non-zero. Similarly $v$ is non-vanishing if $\Phi_{1}^{(2)}$ has non-trivial contributions from $\Phi_{1}^{(1)}$, which is the case when $d$ is non-zero.

The first set of differential equations (4.76) guarantees the covariance of $\Phi_{1}^{(2)}$ with respect to Möbius transformations. Since Möbius transformations act transitively on triples of pair wise distinct elements of $\mathbb{P}$, we can fix three of the arguments of $\Phi_{1}^{(2)}$ to uniquely determine

$$
\Phi_{s}^{(2)} = \prod_{1 \leq i < j \leq 4} (z_{i} - z_{j})^{\frac{2s+1}{p} + \frac{1}{2}} (1 - x)^{\frac{si}{2}} H_{s}^{(a)}(x) \quad (4.80)
$$

up to a function $H_{s}^{(a)}(x)$ of the Möbius invariant cross ratio

$$
x = \frac{z_{4} - z_{3}}{z_{4} - z_{2}} \frac{z_{1} - z_{2}}{z_{1} - z_{3}}. \quad (4.81)
$$

See [51] for a wealth of examples regarding such computations. The functions $H_{s}^{(1)}(x)$ and $H_{s}^{(2)}(x)$ are absolutely convergent on $1 > |x| > 0$ and $1 > |1 - x| > 0$ respectively. The second set of differential equations (4.77) arise from the fact that the vertex operators above vanish upon inserting the null vector

$$
\left( L_{-1}^{2} - \frac{1}{p} L_{-2}^{2} \right) u_{2}. \quad (4.82)
$$

The prefactors of $H_{s}^{(a)}(x)$ in equation (4.80) have been chosen such that the differential equation for $H_{s}^{(a)}(x)$ induced by (4.76) is particularly simple. Namely, the well known hypergeometric equations

$$
x(1 - x) \frac{d^2}{dx^2} H_{s}^{(a)}(x) + \frac{2}{p} (1 - 2x) \frac{d}{dx} H_{s}^{(a)}(x) - \frac{3 - p}{p^2} H_{s}^{(a)}(x) = 0. \quad (4.83)
$$

For a detailed list of solutions and all formulae we will be using see [58]. For $\Phi_{1}^{(1)}$ which converges on $U^{(1)}$, $H_{1}^{(1)}(x)$ is a power series in $x$, while for $\Phi_{1}^{(2)}$ which converges on $U^{(2)}$, $H_{1}^{(2)}(x)$ is a power series in $1 - x$

$$
\begin{align*}
H_{1}^{(1)}(x) &= {}_{2}F_{1} \left( \frac{1}{p}, \frac{3 - p}{p}; \frac{2}{p}; x \right), \\
H_{1}^{(1)}(x) &= x^{\frac{3 - p}{p}} {}_{2}F_{1} \left( \frac{p - 1}{p}, \frac{2p - 2}{p}; x \right), \\
H_{1}^{(2)}(x) &= {}_{2}F_{1} \left( \frac{1}{p}, \frac{3 - p}{p}; \frac{2}{p}; 1 - x \right), \\
H_{1}^{(2)}(x) &= (1 - x)^{\frac{3 - p}{p}} {}_{2}F_{1} \left( \frac{1}{p}, \frac{p - 1}{p}; \frac{2p - 2}{p}; 1 - x \right).
\end{align*} \quad (4.84)
$$

To prove that $\Phi_{1}^{(1)}$ has non-vanishing contributions from $\Phi_{1}^{(2)}$ we continue $H_{1}^{(1)}(x)$ along the path from 0 to 1 on the real line. The well known connection formula for hypergeometric functions then yields

$$
\begin{align*}
H_{1}^{(1)}(x) &= \frac{1}{2 \cos \frac{\pi}{p}} H_{1}^{(2)}(x) + \frac{3 - p}{2 - p \Gamma(\frac{3}{p}) \Gamma(\frac{1}{p})} \Gamma(\frac{2}{p})^{2} H_{3}^{(2)}(x), \\
H_{1}^{(2)}(x) &= \frac{1}{2 \cos \frac{\pi}{p}} H_{1}^{(1)}(x) + \frac{3 - p}{2 - p \Gamma(\frac{3}{p}) \Gamma(\frac{1}{p})} \Gamma(\frac{2}{p})^{2} H_{3}^{(1)}(x).
\end{align*} \quad (4.85)
$$

And thus the rigidity of $X_{a}^{+}$ for $p \geq 4$ follows.
For $p = 2, 3$ the analysis is exactly the same. Specifying the domains and codomains of the vertex operators is just a bit trickier. The resulting differential equations are analogous however. For $p = 3$ we have shown so far that

$$X_2^+ \otimes X_2^+ \otimes X_2^+ = X_2^+ \oplus (X_2^+ \otimes X_2^+)$$

and that there exists a surjective $W_3$-module map

$$X_2^+ \otimes X_1^+ \rightarrow X_1^+.$$  

(4.87)

Therefore the right exactness of the fusion product implies the existence of a surjective $W_3$-module map

$$X_2^+ \otimes (X_2^+ \otimes X_1^+) \rightarrow X_1^+ \oplus X_1^+.$$  

(4.88)

The analysis above can therefore be repeated for $p = 3$ without any modifications. We consider the differential equations (4.76) and (4.77), which can again be simplified to the hypergeometric equation (4.83). Analysing the connection formulae for $p = 3$ yields

$$H_1^{(1)}(x) = H_1^{(2)}(x),$$  

(4.89)

thus proving the rigidity of $X_2^+$ for $p = 3$.

For $p = 2$ the space of solutions for the hypergeometric equation

$$x(1 - x) \frac{d^2}{dx^2} H_i^{(a)}(x) + (1 - 2x) \frac{d}{dx} H_i^{(a)}(x) - \frac{1}{2} H_i^{(a)}(x) = 0$$

(4.90)

is slightly more complicated than in the previous examples, because the poles encountered at $x = 0$ and $x = 1$ are logarithmic. This implies that vertex operators involved also contain logarithms. We will omit the details however since they are not important for solving the above differential equation. The solutions $H_i^{(a)}$ are given by

$$H_1^{(1)}(x) = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right),$$

$$H_1^{(1)}(x) = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \log(x) + G(x),$$

$$H_1^{(2)}(x) = 2F_1\left(1, \frac{1}{2}; 1; 1 - x\right),$$

$$H_3^{(2)}(x) = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x\right) \log(1 - x) + G(1 - x),$$

(4.91)

where $G(x)$ is a power series with vanishing constant term that converges for $1 > |x|$. The connection formulae for $p = 2$ yield

$$H_1^{(1)} = -\frac{\log(4)}{\pi} H_1^{(2)} - \frac{1}{\pi} H_3^{(2)},$$

(4.92)

thus proving the rigidity of $X_2^+$ for $p = 2$.}

5. The rigidity of $(W_{p^{\otimes}}^{\text{mod}}, \hat{\otimes})$

In the previous sections we proved that $X_1^-$ and $X_2^+$ are rigid self-dual objects in $W_{p^{\otimes}}^{\text{mod}}$. In this section we will exploit this fact to compute the fusion product of $X_1^-$ and $X_2^+$ with the projective modules $P_s^\varepsilon$, $1 \leq s < p$, $\varepsilon = \pm$, allowing us to prove the rigidity of $W_{p^{\otimes}}^{\text{mod}}$ and ultimately compute the fusion product on the set of all simple and all projective modules.
5.1. Fusion products between \( X_1^- \) and \( X_2^+ \) and projective modules

First we prepare some more notation. For any object \( Z \) in \( \mathcal{W}_p - \text{mod} \) we denote by \([Z : X_1^]\) the multiplicity of \( X_1^\) in quotients \( M_{i+1}(Z)/M_i(Z) \) of the Jordan–Hölder series (2.34) of \( Z \), that is,

\[
[Z : X_1^] = \dim \hom(P^\epsilon, Z).
\] (5.1)

We have established that

\[
\begin{align*}
X_2^+ \otimes X_2^+ &= X_2^+ \otimes X_2^+ = \begin{cases} 
X_2^+, & s = 1 \\
X_{p-1}^+ \otimes X_{p+1}^+, & 2 \leq s \leq p - 1 \\
P_{p-1}^\epsilon, & s = p 
\end{cases} \\
X_1^- \otimes X_1^+ &= X_1^- \otimes X_1^+, & 1 \leq s \leq p.
\end{align*}
\] (5.2)

and that \( X_1^- \) and \( X_2^+ \) are self-dual rigid objects. From the Jordan–Hölder series of the projective modules we also know that

\[
\begin{align*}
[P_1^\epsilon : X_1^] &= 2\delta_{(s, t), (t, \sigma)} + 2\delta_{(s, t), (p-t, -\sigma)}, & 1 \leq s < p \\
[P_1^\epsilon : X_1^] &= \delta_{(p, t), (t, \sigma)}.
\end{align*}
\] (5.3)

**Proposition 38.** The fusion rules of \( X_2^+ \) and \( X_1^- \) with projective modules are given by

\[
\begin{align*}
X_2^+ \otimes P_1^\epsilon &= \begin{cases} 
P_2^\mu \oplus 2 \cdot P_{p-1}^\epsilon, & s = 1 \\
P_{s-1}^\mu \oplus P_{s+1}^\mu, & 1 < s < p - 1 \\
P_{p-2}^\mu \oplus 2 \cdot P_p^\mu, & s = p - 1 
\end{cases} \\
X_1^- \otimes P_1^\epsilon &= P_{s-1}^\mu, & 1 \leq s \leq p.
\end{align*}
\] (5.4)

**Proof.** Because \( X_1^- \) and \( X_2^+ \) are rigid, their product with \( P_1^\epsilon \) is projective. The most general ansatz for such a product is therefore

\[
X \otimes P_1^\epsilon = \bigoplus_{m=1}^p \bigoplus_{\mu, \nu} N_{m, \mu} \cdot P_m^\nu,
\] (5.5)

where \( X \) is either \( X_1^- \) or \( X_2^+ \) and \( N_{m, \mu} \in \mathbb{Z} \) is the multiplicity of \( P_m^\nu \) in \( X \otimes P_1^\epsilon \). We can determine \( N_{m, \mu} \) by recalling that a rigid object \( X \) and two arbitrary objects \( A \) and \( B \) satisfy the relation

\[
\hom(A, X \otimes B) \cong \hom(X \otimes A, B).
\] (5.6)

Setting \( A \) to \( P_1^\epsilon \) and \( C \) to \( X_1^\) and calculating the dimensions of the spaces of \( \mathcal{W}_p - \text{mod} \)-module maps in the equation above, we lead to

\[
N_{m, \mu} = \dim \hom(P_1^\epsilon, X \otimes X_1^) = [X \otimes X_1^ : X_1^].
\] (5.7)

We can easily calculate the multiplicities \([X \otimes X_1^ : X_1^]\) for \( X = X_1^- \), \( X_2^+ \) by considering the fusion products 36 and 37

\[
[X_1^- \otimes X_1^ : X_1^] = \delta_{(t, \delta), (m, -\mu)}
\] (5.8)

\[
[X_2^+ \otimes X_1^ : X_1^] = \begin{cases} 
\delta_{(2, +), (m, \mu)} + 2\delta_{(p, \delta), (m, -\mu)} & t = 1 \\
\delta_{(t-1), (m, \mu)} + \delta_{(t+1), (m, -\mu)} & 2 \leq t \leq p - 2 \\
\delta_{(p-2), (m, \mu)} + 2\delta_{(p, \delta), (m, -\mu)} & t = p - 1 \\
\delta_{(p-1), (m, -\mu)} & t = p.
\end{cases}
\]

The proposition then follows directly by plugging in the multiplicities. \( \square \)
5.2. Proving rigidity

We apply point 4 of proposition 19 to $\mathcal{W}_p$-mod. Since all simple and all projective $\mathcal{W}_p$-modules appear in the repeated fusion products of $X_{1}^{-}$ and $X_{2}^{+}$ we have the following proposition.

**Proposition 39.** For $1 \leq s \leq p$, $\varepsilon = \pm$ the simple modules $X_{s}^\varepsilon$ and the projective modules $P_{s}^\varepsilon$ are self-dual rigid objects in $\mathcal{W}_p$-mod.

In $\mathcal{W}_p$-mod all indecomposable objects $M$ except the simple objects and the projective objects satisfy exact sequences

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

such that $L$ and $N$ are direct sums of simple objects. So finally we obtain the rigidity of $\mathcal{W}_p$-mod by applying point 5 of proposition 19.

**Theorem 40.** The weakly rigid monoidal category $(\mathcal{W}_p, \otimes, X_{2}^{+})$ is rigid. For any object $M$ in $\mathcal{W}_p$-mod the dual $M'$ is given by the contragredient $M^*$, i.e. $M' = M^*$.

5.3. The ring structure of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$

We see in theorems 36 and 37, that the fusion products of $X_{1}^{-}$ and $X_{2}^{+}$ with simple modules are direct sums of simple and projective modules. Because all simple modules appear as direct summands of products of $X_{1}^{-}$ and $X_{2}^{+}$, the product of any two simple modules must also be a product of simple and projective modules. Therefore because all the simple modules are rigid, the fusion product closes on the set of all simple and all projective modules. In this section we will compute the fusion product on this set.

We introduce the free Abelian group $P(\mathcal{W}_p)$ of rank $4p - 2$ generated by all projective and all simple modules

$$P(\mathcal{W}_p) = \bigoplus_{s=1}^{p} \bigoplus_{\varepsilon = \pm} \mathbb{Z}[X_{s}^\varepsilon]_p \oplus \bigoplus_{s=1}^{p-1} \bigoplus_{\varepsilon = \pm} \mathbb{Z}[P_{s}^\varepsilon]_p$$

and the rank $2p$ Grothendieck group

$$K(\mathcal{W}_p) = \bigoplus_{s=1}^{p} \mathbb{Z}[X_{s}^\varepsilon]_K.$$

By the rigidity of $\mathcal{W}_p$-mod and the closure of the fusion product on simple and projective modules:

1. $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ have the structure of commutative rings.
2. The canonical projection $\pi : P(\mathcal{W}_p) \rightarrow K(\mathcal{W}_p)$ is a ring homomorphism.

By the above arguments the two operators

$$X = X_{2}^{+} \otimes - \quad Y = X_{1}^{-} \otimes -,$$

define $\mathbb{Z}$-linear endomorphisms of $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$. Because the fusion product is commutative, the two operators $X$ and $Y$ must also commute. Thus by the two operators $X$ and $Y$ the polynomial ring $\mathbb{Z}[X, Y]$ acts on $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$, i.e. $P(\mathcal{W}_p)$ and $K(\mathcal{W}_p)$ are modules over $\mathbb{Z}[X, Y]$ and the canonical projection $\pi$ is a $\mathbb{Z}[X, Y]$-module map.

Before we begin analysing the action of $\mathbb{Z}[X, Y]$ on $P(\mathcal{W}_p)$ we recall some elementary facts about Chebyshev polynomials that will prove helpful.

---

5 The Grothendieck group $K(\mathcal{C})$ can be defined for any Abelian category $\mathcal{C}$. It is given by free Abelian group generated by all objects of $\mathcal{C}$ modulo the subgroup generated by all formal differences $M - L - N$ where $L, M, N$ satisfy an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$. If the number of simple objects in the Abelian category $\mathcal{C}$ is finite, then $K(\mathcal{C})$ is just the finite rank free Abelian group generated by all simple objects.
**Definition 41.** We define elements \( U_n(X) \), \( n = 0, 1, \ldots \) in \( \mathbb{Z}[X] \) recursively
\[
U_0(X) = 1, \quad U_1(X) = X,
U_{n+1}(X) = XU_n(X) - U_{n-1}(X).
\] (5.13)

**Remark 42.**

1. The coefficient of the leading order of \( U_n(X) \) is 1, i.e.
\[
U_n(X) = X^n + \cdots \in \mathbb{Z}[X], \quad m = 0, 1, 2, \ldots
\] (5.14)
so we have
\[
\mathbb{Z}[X] = \bigoplus_{n=0}^{\infty} \mathbb{Z} U_n(X).
\] (5.15)

2. The initial conditions and recursion relations of the polynomials \( U_n(X) \) are those of the Chebyshev polynomials of the second kind, though with a non-standard choice of normalization.

We define the \( \mathbb{Z}[X,Y] \)-module maps
\[
\psi : \mathbb{Z}[X,Y] \rightarrow P(V_p),
\]
\[
f(X,Y) \mapsto f(X,Y) \cdot [X^+]_p,
\]
\[
\varphi : \mathbb{Z}[X,Y] \rightarrow K(V_p),
\]
\[
f(X,Y) \mapsto f(X,Y) \cdot [X^+_1]_K.
\] (5.16)

**Theorem 43.** The maps \( \psi \) and \( \varphi \) are surjective homomorphisms of commutative rings and the kernels are given by the ideals
\[
\ker \psi = (Y^2 - 1, U_{2p-1}(X) - 2YU_{p-1}(X)),
\]
\[
\ker \varphi = (Y^2 - 1, U_p(X) - U_{p-2}(X) - 2Y).
\] (5.17)

**Proof.** Consider the fusion products
\[
X_2^+ \otimes X_1^+ = X_2^+,
X_2^+ \otimes X_1^- = X_{p-1}^- \oplus X_{p+1}^-,
\] (5.18)
for \( 1 < s < p \). Formally this looks exactly like the recursion relations and initial conditions (5.13) if one were to substitute \( X_1^+ \) with \( X \) and \( X_1^- \) with \( U_{s-1}(X) \). We can therefore write the generators of \( P(V_p) \) and \( K(V_p) \) corresponding to the simple modules \( X_s^+ \), \( 1 \leq s \leq p \) as
\[
[X_1^+]_p = U_{s-1}(X)[X_s^+]_p, \quad [X_1^+]_K = U_{s-1}(X)[X_s^+]_K.
\] (5.19)
Since the remaining simple modules \( X_s^- \) can be written as \( X_1^- \otimes X_s^+ \) their corresponding generators in \( P(V_p) \) and \( K(V_p) \) can be written as
\[
[X_1^-]_p = YU_{s-1}(X)[X_s^+]_p, \quad [X_1^-]_K = YU_{s-1}(X)[X_s^+]_K.
\] (5.20)
Thus as a module over \( \mathbb{Z}[X,Y] \), \( K(V_p) \) is generated by \( [X_1^+]_K \) and \( \varphi \) is therefore a surjective \( \mathbb{Z}[X,Y] \)-homomorphism. Next we consider the fusion products
\[
X_2^+ \otimes X_p^+ = P_{p-1}^+, \quad X_2^+ \otimes P_p^+ = P_{s-1}^+ \oplus P_{s+1}^+.
\] (5.21)
for \( 1 < s < p \). These imply that the generators of \( P(W_p) \) corresponding to the projective modules \( P_s^+ \), \( 1 \leq s < p \) can be written as

\[
[P_s^+]_p = (U_{2p-s-1}(X) + U_{s-1}(X))[X_1^+]_p.
\]

(5.22)

Since the remaining projective modules \( P_s^- \) can be written as \( X_1^- \otimes P_s^+ \) their corresponding generators in \( P(W_p) \) can be written as

\[
[P_s^-]_p = Y (U_{2p-s-1}(X) + U_{s-1}(X))[X_1^+]_p.
\]

(5.23)

Thus as a module over \( \mathbb{Z}[X, Y] \), \( P(W_p) \) is generated by \([X_1^+]_p\) and \( \psi \) is therefore a surjective \( \mathbb{Z}[X, Y] \)-homomorphism. We verify that the two ideals

\[
I = (Y^2 - 1, U_{2p-1}(X) - 2YU_{p-1}(X))
\]

\[
J = (Y^2 - 1, U_p(X) - U_{p-2}(X) - 2Y)
\]

(5.24)

are indeed the kernels of \( \psi \) and \( \varphi \) by showing that \( I \) and \( J \) lie in the kernels and that the ranks of \( \mathbb{Z}(X, Y)/I \) and \( \mathbb{Z}(X, Y)/J \) are equal to the ranks of \( P(W_p) \) and \( K(W_p) \). From the fusion product \( X_1^- \otimes X_1^- = X_1^+ \) it follows that

\[
(Y^2 - 1)[X_1^+]_p = 0, \quad (Y^2 - 1)[X_1^+]_K = 0.
\]

(5.25)

The left and right hand sides of \( X_2^+ \otimes X_2^+ = P_{p-1}^+ \) are given by the left and right hand sides of

\[
XU_{p-1}(X)[X_1^+]_K = 2(U_p(X) + Y)[X_1^+]_K
\]

(5.26)

respectively in \( K(W_p) \). By the recursion relations for Chebyshev polynomials it therefore follows that

\[
(U_p(X) - U_{p-2}(X) - 2Y)[X_1^+]_K = 0.
\]

(5.27)

Lastly by the left and right hand sides of the product \( X_2^+ \otimes P_1^+ = P_2^+ \oplus 2X_2^- \) are given by the left and right hand sides of

\[
X(U_{p-2}(X) + U_0(X))[X_1^+]_p = (U_{2p-3}(X) + U_1(X) + 2YU_{p-1}(X))[X_1^+]_p
\]

(5.28)

respectively in \( P(W_p) \). By the recursion relations for Chebyshev polynomials it therefore follows that

\[
(U_{2p-1}(X) - 2YU_{p-1}(X))[X_1^+]_p = 0.
\]

(5.29)

We decompose \( \mathbb{Z}[X, Y]/I \) and \( \mathbb{Z}[X, Y]/J \) as free Abelian groups to compute their rank

\[
\frac{\mathbb{Z}[X, Y]}{I} = \mathbb{Z}[X] \oplus \mathbb{Z}[X] Y = \bigoplus_{i=0}^{2p-2} \mathbb{Z}X^i \oplus \bigoplus_{i=0}^{2p-2} \mathbb{Z}X^i Y
\]

\[
\frac{\mathbb{Z}[X, Y]}{J} = \mathbb{Z}[X] \oplus \mathbb{Z}[X] Y = \bigoplus_{i=0}^{p-1} \mathbb{Z}X^i \oplus \bigoplus_{i=0}^{p-1} \mathbb{Z}X^i Y
\]

(5.30)

and see that the ranks are \( 4p - 2 \) and \( 2p \) respectively.

\[\square\]

**Theorem 44.** The fusion products for all simple and all projective \( W_p \)-modules are given by

\[
X_s^\mu \otimes X_t^\nu = \bigoplus_{i=|s-t|/2}^m X_i^{\mu\nu} \oplus \bigoplus_{i=2p+1-s-t}^{m} P_i^{\mu\nu}
\]

\[
X_s^\mu \otimes P_t^s = \bigoplus_{i=|s-t|/2}^m P_i^{\mu\nu} \oplus \bigoplus_{i=2p+1-(s+t)}^{2m} 2 \cdot P_i^{\mu\nu}
\]

\[
P_i^{\mu} \otimes P_t^s = 2 \cdot X_i^+ \otimes P_t^s \oplus 2 \cdot X_{p-t}^- \otimes P_t^s,
\]

(5.31)
where ‘;2’ indicates that the summation variable is incremented in steps of 2 and
\[
m = \begin{cases} 
p & \text{if } p - i \text{ is even } \\
p - 1 & \text{if } p - i \text{ is odd. }
\end{cases}
\] (5.32)

The product on the Grothendieck group induced by the fusion product is given by
\[
[X^s]_K \cdot [X^t]_K = \sum_{i=|s-t|+1;2}^{s+t-1-p} a(s+i) [X^{s+i}]_K + \sum_{i=b(s+i);2}^{2|s-t|+1} 2[X^{s-i}]_K + [X^{s-i}]_K, \] (5.33)

where
\[
a(s) = \begin{cases} 
s - 1 & \text{if } s - 1 - p \leq 0 \\
p & \text{if } s - 1 - p > 0 \text{ is even } \\
p - 1 & \text{if } s - 1 - p > 0 \text{ is odd, }
\end{cases}
\] (5.34)

\[
b(s) = \begin{cases} 
1 & \text{if } s - 1 - p \text{ is odd } \\
2 & \text{if } s - 1 - p \text{ is even. }
\end{cases}
\] (5.34)

**Proof.** The above fusion rules can be computed directly in \(\mathbb{Z}[X, Y]\) by using multiplication formula for Chebyshev polynomials
\[
U_k(x)U_j(x) = \sum_{i=|k-j|+1;2}^{k+j} U_i(x) \] (5.35)
and subsequently projecting onto \(P(W_p)\) or \(K(W_p)\). Note that the ‘;2’ in the subscript of the sum indicates that the summation variable \(k\) is incremented in steps of 2. \(\square\)

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