Stabilization of slow-fast control systems: the non-hyperbolic case

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Abstract

In this paper we study the stabilization problem of a general class of slow-fast systems with one fast and arbitrarily many slow states. Moreover, the class of systems under study is \textit{slowly actuated}, meaning that only the slow states are subject to the action of a controller. Furthermore, we are particularly interested in the case where normal hyperbolicity is lost. We show that by using the Geometric Desingularization method, it is possible to design controllers to locally stabilize non-hyperbolic points of any finite degeneracy. The main novelty of this paper is that, unlike previous research on the topic, we make use of more than one chart of the blow up space to enhance the region of attraction of the operating point. A couple of numerical examples highlight our contribution.

Key words: Nonlinear control; Slow-fast systems; singular perturbations.

1 Introduction

Multiple timescales are ubiquitous in mathematical modeling and applications. Examples of real life phenomena with several timescales can be found in non-linear circuits \cite{9,23,24,25,30}, neuron models \cite{27}, biochemical systems \cite{18,25}, etc. There exists a great body of literature dealing with systems for which the timescale separation is global \cite{5,17}. However, in many complex models, the aforementioned global timescale separation does not hold. Usually, from a dynamical systems point of view, a non-global timescale separation is associated with the presence of certain singularities. The correct and thorough analysis of the behavior of a multi-timescale system near such singularities is crucial for the progress of our understanding of phenomena with several timescales.

In the context of control systems, considerable effort has been given to study systems with two timescales and global separation of such timescales \cite{16,17,22,27,31}. However, with the increased interest in shaping the behavior of complex multi-timescale systems, we require control techniques that can tackle problems where the timescale separation is not global. Preliminary steps in this regard have been developed for regulation purposes in \cite{12,14} in the planar case, and \cite{15} for the case of singularities of quadratic degeneracy.

In this article we extend the results of \cite{15} to a broader class of slow-fast control systems, and we propose a geometric way to enlarge the region of stability of an equilibrium point. The idea is to show that Geometric Desingularization \cite{4,21} can be used in combination with control strategies to stabilize degenerate points of slow-fast systems, c.f. \cite{12,14} for the planar case. In few words, the Geometric Desingularization technique is a suitable change of coordinates, well-defined around singularities, that allows a simpler analysis of the involved behavior of slow-fast systems without global timescale separation. In turn, by employing such a technique, the design of controllers for slow-fast systems in the aforementioned scenario becomes simpler.

Briefly speaking, we study slow-fast control systems with one fast and an arbitrary amount of slow states. Moreover, the system is actuated \textit{only} on the slow variables. Our contributions can be summarized as follows: first, we show that by using the Geometric Desingularization technique we can design, in a rather simple way, controllers that locally stabilize a non-hyperbolic point of a slowly actuated slow-fast systems as described above (Theorem 1). Next, we provide a constructive way to design a controller that accomplishes the aforementioned
The rest of this document is organized as follows: in Section 2 we provide the necessary preliminaries and the formal setting of the problem to be studied. Next, in Section 3 we briefly recall the Geometric Desingularization technique, which is essential to prove our results. Afterwards, in Section 4, we present our contributions, followed by two examples in Section 5 highlighting our results. Finally, Section 6 presents some concluding remarks and a couple of open problems for future research.

2 Preliminaries

From now on, we shall confine ourselves to the study of two-timescale systems, also known as slow-fast system (SFS). A SFS is an Ordinary Differential Equation of the form

\[
\begin{align*}
\dot{x} &= f(x, z, \epsilon) \\
\epsilon \dot{z} &= g(x, z, \epsilon),
\end{align*}
\]

where \( x \in \mathbb{R}^{n_x} \) are the slow states, \( z \in \mathbb{R}^{n_z} \) the fast states, \( 0 < \epsilon \ll 1 \) is a small parameter responsible for the timescale separation between \( x \) and \( z \), and \( f \) and \( g \) are smooth functions. For \( \epsilon > 0 \) we can define a new time parameter \( \tau = t/\epsilon \) and obtain an equivalent system to (1) of the form

\[
\begin{align*}
\dot{x}' &= \epsilon f(x, z, \epsilon) \\
\dot{z}' &= g(x, z, \epsilon).
\end{align*}
\]

Usually, in the study of SFSs, we define two reduced subsystems by taking the limit \( \epsilon \to 0 \) of (1) and (2). By doing so we obtain the DAE (Differential Algebraic Equation) and the Layer Equation, which read as

\[
\text{DAE: } \begin{cases}
\dot{x} = f(x, z, 0) \\
0 = g(x, z, 0)
\end{cases} \quad \text{Layer: } \begin{cases}
x' = 0 \\
\dot{z}' = g(x, z, 0)
\end{cases}
\]

The main idea of (Geometric) Singular Perturbation Theory is to draw conclusions (for example: qualitative and quantitative description, stability, etc.) from the reduced systems (i.e. the DAE and the Layer Equation) and then extend them to similar results of the corresponding SFS. In the analysis of SFSs, one of the most important geometric objects to consider is the critical manifold.

**Definition 1** The critical manifold associated to the SFS (1) is defined as

\[
S = \{ (x, z) \in \mathbb{R}^{n_x+n_z} | g(x, z, 0) = 0 \}. \tag{3}
\]

Note that \( S \) is the phase-space of the DAE and the set of equilibrium points of the Layer Equation. We say that \( S \) is Normally Hyperbolic if every point \( s \in S \) is a hyperbolic equilibrium point of the reduced dynamics \( z' = g(x, z, 0) \). The theory regarding SFSs with Normally Hyperbolic critical manifold is nowadays well understood, see e.g. \cite{3} for a general treatment, and \cite{17} for applications in the context of control systems. However, we still find many open problems in the situation where \( S \) has non-hyperbolic points.

2.1 Setting

The class of slow-fast control systems (SFCSs) that we study are defined as

\[
\begin{align*}
\dot{x} &= f(x, z, u, \epsilon) \\
\epsilon \dot{z} &= g(x, z, \epsilon),
\end{align*}
\]

where \( x \in \mathbb{R}^{n_x}, n_x \geq 1 \) is an integer, \( z \in \mathbb{R}, f(x, z, 0, 0) \) is smooth, \( u \in \mathbb{R}^{n_u} \) denotes the control input, and \( g(x, z, \epsilon) \) is a smooth function satisfying

\[
g(x, z, 0) = - \left( z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right), \tag{5}
\]

where \( k \in \mathbb{N}, \) with \( k \geq 2 \).

**Remark 1**

- The control input \( u \) only acts on the slow dynamics, that is, the class of systems (4)-(5) is under-actuated.
- The origin is the most degenerate non-hyperbolic point of the critical manifold \( S = \{ g(x, z, 0) = 0 \} \). This is because the origin is the unique point where \( g(0) = 0 \) and \( \frac{\partial g}{\partial z}(0) = 0 \). Therefore, we are interested in stabilizing the origin of (4).
- The fact that the class of systems (4)-(5) is large is explained by the classification of singularities of smooth functions. Let us give a brief recollection of the relevant arguments, for more details see \cite{4,5}. Let \( V_0(z) \), where \( z \in \mathbb{R} \), be a smooth function with \( V_0(0) = 0 \). Let \( k \geq 0 \) be the minimum integer such that \( \frac{\partial^{k+1} V_0}{\partial z^{k+1}}(0) = 0 \) and \( \frac{\partial^{k+1} V_0}{\partial z^{k+1}}(0) \neq 0 \). Then \( V_0(z) \) is locally equivalent to \( \pm z^{k+1} \) (see e.g. \cite{4} Theorem 1). Next, let \( V_\epsilon(z) = V(x, z) \) be a generic family of functions with \( V_0(z) \) as above. Then, the universal unfolding (containing the minimum number of parameters such that the singularity is generic) of \( V_0(z) \) is...
\(V_z(z) = \pm z^{k+1} + x_{k-1}z^{k-1} + \cdots + x_1z\) (see e.g. [3, Example 14.9]). Thus, the critical manifold \(S\) associated to (4) is equivalently defined as the set of critical points of \(V_z(z)\). The fact that we need at least \(k-1\) parameters to unfold \(V_0\) implies that it suffices to fix \(n_k = k-1\). If \(n_k < k-1\), then the class of SFCSs (4)-(5) is not generic, on the other hand, if \(n_k > k-1\) similar techniques as used here can be employed, compare with [13]. The choice of the negative sign in (5) is just for convenience. It simply means that away from \(S\), the trajectories of the layer equation travel “downwards”. Choosing a positive sign in (5) reverses the aforementioned direction, but a similar analysis as the one performed here can be used in such a case.

3 Geometric Desingularization

The Geometric Desingularization method is used to overcome the difficulties that the presence of non-hyperbolic points pose. Briefly speaking, this method provides a new system, equivalent to (4), but with simpler singularities. In this way, the design of controllers for (4) becomes more accessible. The Geometric Desingularization method is motivated by the regularization of singularities in algebraic varieties [7, 8]. In the context of slow-fast systems it was first introduced in [3], and has been further developed afterwards, see e.g. [10, 11, 19, 21] and [12, 15] for some applications in control systems.

To start the description of the method, let us rewrite a SFCS as

\[
X = \varepsilon f(x, z, u, \varepsilon) \frac{\partial}{\partial x} + g(x, z, \varepsilon) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]

which is a smooth vector field on \(\mathbb{R}^{k+1}\). The Geometric Desingularization method consists of the following steps (see more details in [13, 21]).

1. Define the quasi-homogeneous blow up map \(\Phi: S^k \times I \to \mathbb{R}^{k+1}\), where \(I \subseteq [0, \infty)\) is an interval (possibly infinite), and \(S^k\) denotes the \(k\)-sphere, by

\[
\Phi(\bar{x}, \bar{z}, \bar{\varepsilon}, r) = (r^\alpha \bar{x}, r^\beta \bar{z}, r^\gamma \bar{\varepsilon}) = (x, z, \varepsilon),
\]

where \(\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{k-1}), (\bar{x}, \bar{z}, \bar{\varepsilon}) \in S^k\), that is

\[
\sum_{i=1}^{k-1} \bar{x}_i^2 + \bar{z}^2 + \bar{\varepsilon}^2 = 1, r^\alpha \bar{x} = (r^\alpha \bar{x}_1, \ldots, r^\alpha \bar{x}_{k-1}),
\]

\(\alpha = (\alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{N}^{k-1}, \beta, \gamma \in \mathbb{N}, r \in I\). If the weights are all equal to 1, simply refer to \(\Phi\) as blow up.

2. Define the desingularized vector field \(\bar{X}\) as follows: i) Note that the blow up map \(\Phi\) induces a vector field \(\overline{X}\) on \(S^k \times I\) defined as \(\overline{X} = D\Phi^{-1} \circ X \circ \Phi\), where \(D\Phi\) denotes the directional derivative of \(\Phi\). Since \(\Phi\) is a diffeomorphism for \(r > 0\), the vector fields \(X\) and \(\overline{X}\) are conjugate for all \(r > 0\). Moreover, the definition of \(\overline{X}\) extends continuously to \(r = 0\) [21], that is \(\overline{X}\) is well defined on \(S^k \times I\). Note that \(S^k \times \{0\}\) is mapped to the origin \(0 \in \mathbb{R}^{k+1}\) via the blow up map, therefore, since \(X(0) = 0\), \(\overline{X}\) vanishes along \(S^k \times \{0\}\), so ii) Define the desingularized vector field \(\bar{X}\) as \(\bar{X} = \frac{1}{r^\gamma} \overline{X}\), where \(p \in \mathbb{N}\) is suitably chosen so that \(\bar{X}\) does not vanish along \(S^k \times \{0\}\).

Note that the vector fields \(\overline{X}\) and \(\bar{X}\) are smoothly equivalent on \(S^k \times \{r > 0\}\), their only difference is the time-parametrization. Thus, it is qualitatively the same to study \(\bar{X}\) than \(\overline{X}\). This also implies that the qualitative properties of \(\bar{X}\) can be related to similar ones of \(X\), after all, they are also smoothly equivalent for all \(r > 0\). The idea of Geometric Desingularization is to appropriately choose the weights of the blow up map (7) so that the vector field \(\overline{X}\) has simpler singularities (e.g. hyperbolic, or semi-hyperbolic) with respect to those of the SFCS (6), and that the singular dependence of \(X\) on \(\varepsilon\) is overcome in the blow up space. In turn, the design of the controller for \(\bar{X}\) becomes simpler.

Remark 2 Usually, if the vector field \(X\) is quasihomogeneous [14, 24] then the weights of the blow up map \(\Phi\) correspond to the quasihomogeneity type of \(X\), but in general, finding the appropriate values of the weights is a non-trivial task. However, as every singularity of an algebraic variety over a field of characteristic 0 (such as \(\mathbb{R}\)) can be regularized after a finite number of blow ups [21, 22], we conjecture that, for any vector field on \(\mathbb{R}^n\), this methodology also holds.

While performing the computations, it is more convenient to introduce charts rather than working on spherical coordinates. A chart is just a parametrization of a hemisphere of the blown up space. In practice, a chart is obtained by simply setting one of the coordinates \((\bar{x}_1, \ldots, \bar{x}_{k-1} , \bar{z}, \bar{\varepsilon}) \in S^k\) to \(\pm 1\) in the definition of \(\Phi\). In this way we define, for example, the chart \(K_{\bar{\varepsilon}} = (r^\alpha \bar{x}, r^\beta \bar{z}, r^\gamma \bar{\varepsilon})\). Note that each chart parametrizes just a part of \(S^k \times I\); however, all possible charts define an open cover of \(S^k \times I\). To avoid confusion, whenever we work on more than one chart, we shall define local coordinates and distinguish them in each chart. The charts are related to each other via transition maps [13, 21].

Remark 3 The chart \(K_{\bar{\varepsilon}}\) is the most important one in our analysis, and it is called the family chart. All other charts, which we denote by \(K_{\bar{\varepsilon} \bar{z}} = \{ \bar{x}_1 = \pm 1\}\) and \(K_{\bar{\varepsilon} \bar{z} \bar{\varepsilon}} = \{ \bar{z} = \pm 1\}\) are called directional charts. In the chart \(K_{\bar{\varepsilon}}\) the singular dependence of the vector field on \(\varepsilon\) is overcome. It is also the chart where most of the local dynamical properties of \(X\) near the origin can be seen.
\section{Main results}

Here we present our main results: we show that a controller designed for the desingularizable system $\tilde{X}$ gives, after change of coordinates, a controller for $X$.

**Theorem 1** Consider the SFCS $X$ defined by (6) and suppose that $X$ can be desingularized as described in Section 3. Denote the desingularization of $X$ by $\tilde{X}$. If $\tilde{u}$ is a controller that renders $S^k \times \{0\}$ asymptotically stable under the flow of $\tilde{X}$, then $u(x, z, \varepsilon) = \Phi \circ \tilde{u} \circ \Phi^{-1}(x, z, \varepsilon)$ renders the origin $0 \in \mathbb{R}^{k+1}$ asymptotically stable under the flow of $X$.

**Proof.** Let us rewrite the closed-loop blown-up system as $\tilde{Y}$, that is $\tilde{Y}(\tilde{x}, \tilde{z}, \tilde{e}, r) = \bar{X}(\tilde{x}, \tilde{z}, \tilde{e}, r, \tilde{u})$, where $\tilde{u} = \bar{u}(\tilde{x}, \tilde{z}, \tilde{e}, r)$ is a feedback controller that renders $B_0 = S^k \times \{0\}$ asymptotically stable. Similarly we denote by $Y$ the closed-loop system $X$ with $u$ induced by $\Phi$, that is $u(x, z, \varepsilon) = \Phi \circ \tilde{u} \circ \Phi^{-1}(x, z, \varepsilon)$. Therefore $\tilde{Y}$ is the desingularization of $Y$. Recall that $\Phi$ maps $S^k \times \{0\}$ to $0 \in \mathbb{R}^k$ and that $\tilde{Y}$ and $Y$ are equivalent. This means that trajectories of $\tilde{Y}$ are mapped, via $\Phi$, to trajectories of $Y$ preserving direction. Therefore, trajectories of $\tilde{Y}$ approaching $S^k \times \{0\}$ are mapped to trajectories of $Y$ that approach $0 \in \mathbb{R}^k$. The asymptotic convergence is preserved since $\Phi$ is a diffeomorphism. Note that by taking $r \in I$ with $I$ arbitrarily large, we can find an equivalence between the trajectories of $Y$ and of $\tilde{Y}$ within an arbitrarily large compact subset of $\mathbb{R}^k$. \hfill $\Box$

The main idea of Theorem 1 is that we can design controllers for a SFCS $X$ in the blow up space. Due to the fact that the blow up simplifies the singularities of a vector field, the design of controllers in the blow up space is expected to be simpler than in the original scenario, i.e., without blow up. Note, however, that Theorem 1 requires an analysis within the whole blow up space. This may be computationally tedious, so we also present a more relaxed result and with a particular choice of controller, which for certain applications may be sufficient. The main idea of the following result is to design a feedback controller in the blow up space that linearizes the slow dynamics.

**Theorem 2** Consider a SFCS given by

\begin{align}
\dot{x} &= f(x, z, \varepsilon) + u \\
\varepsilon \dot{z} &= -\left( z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right),
\end{align}

where $x = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$, $z \in \mathbb{R}$ and $u \in \mathbb{R}^{k-1}$.

Then, for $\varepsilon > 0$ sufficiently small, the controller

\begin{align}
u = -C + b \varepsilon^{\frac{1}{k-1}} \bar{z} \varepsilon_1 - \varepsilon^{\frac{2}{k-1}} A \bar{x},
\end{align}

where $C = \text{diag}\{f_i(0,0,0)\}$, $b > 0$, $A > 0$ diagonal, and $\varepsilon_1 \in [0, 0, \ldots, 0]^T \in \mathbb{R}^{k-1}$, renders the origin $(x, z) = (0,0) \in \mathbb{R}^k$ locally asymptotically stable.

**Proof.** The proof consists of designing the controller in the chart $K_\varepsilon$. Therefore, the very first step on this proof is to find appropriate weights for the blow up map $\Phi$. In what follows, we show one possible way to choose such weights. Let us first rewrite (8) as the vector field

\begin{align}
X = \varepsilon(f(x, z, \varepsilon) + u) \frac{\partial}{\partial x} + g(x, z) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\end{align}

Note that $g(x, z) = -\left( z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right)$ is a quasi-homogeneous polynomial of type $(k, k-1, \ldots, 1)$. Thus, let us propose the coordinate transformation

\begin{align}
(x_1, \ldots, x_{k-1}, z, \varepsilon) = (\tilde{r}^k \tilde{x}_1, \ldots, \tilde{r}^2 \tilde{x}_{k-1}, \tilde{r} \tilde{z}, \tilde{r}^{k-1} \gamma),
\end{align}

where $\gamma \in \mathbb{N}$ shall be appropriately chosen below. Next, we obtain the blown up vector field. First, from $\varepsilon^\prime = 0$ and $\varepsilon = \tilde{r}^{k-1}$, it follows that $\tilde{r}^\prime = 0$. Similarly we obtain

\begin{align}
\tilde{x}_i^\prime &= \tilde{r}^{\gamma - k + i - 1}(\tilde{f}_i + \tilde{a}_1), \quad i = 1, 2, \ldots, k-1 \\
\tilde{z}^\prime &= -\tilde{r}^{k-1}\left( \tilde{z}^k + \sum_{i=1}^{k-1} \tilde{x}_i \tilde{z}^{i-1} \right), \quad (12)
\end{align}

where $\tilde{f}_i = \tilde{f}_i(\tilde{r}, \tilde{x}, \tilde{z}) = f(\tilde{r}^k \tilde{x}_1, \ldots, \tilde{r}^2 \tilde{x}_{k-1}, \tilde{r} \tilde{z}, \tilde{r}^{k-1})$, and similar notation is used for $\tilde{a}_i$, $i = 1, \ldots, k-1$. Then, to desingularize (12), we need to divide the right hand side by $\tilde{r}^{k-1}$. To obtain a well-defined desingularized vector field when $\tilde{r} = 0$, it is convenient to choose $\gamma = 2k - 1$. A choice such that $\gamma < 2k - 1$ does not provide a well defined vector field for the restriction $r = 0$. On the other hand, the choice $\gamma > 2k - 1$ would imply $\tilde{x}_i^\prime \in O(\tilde{r})$ for all $i = 1, \ldots, k-1$. With this we obtain the desingularized system

\begin{align}
\hat{x}_i^\prime &= \tilde{r}^{\gamma - k + i - 1}(\tilde{f}_i + \tilde{a}_1), \quad i = 1, 2, \ldots, k-1 \\
\hat{z}^\prime &= -\left( \tilde{z}^k + \sum_{i=1}^{k-1} \tilde{x}_i \tilde{z}^{i-1} \right). \quad (13)
\end{align}

We say that the set $B_0 = S^k \times \{0\}$ is asymptotically stable if there is a neighborhood $N$ of $B_0$ such that every trajectory $\gamma(t)$ of $\tilde{X}$ with initial condition in $N$ satisfies $d_H(\gamma(t), B_0) \to 0$ as $t \to \infty$, where $d_H$ denotes Hausdorff distance.
Next, since we can pick \( \bar{u}_i \) arbitrarily, let us propose that each \( \bar{u}_i \) is of the form \( \bar{u}_i = -c_1 - a_i \bar{x}_1 + b \bar{z} \), and \( \bar{u}_i = -c_i - \bar{r}^{-1} a_i \bar{x}_1 \) for \( i = 2, \ldots, k \) and with \( c_i = f_i(0, 0, 0) = c_i > 0 \) and \( b > 0 \). Then (13) is rewritten as

\[
\begin{align*}
\dot{\bar{x}}_1 &= \bar{r}_1 - c_1 - a_1 \bar{x}_1 + b \bar{z} \\
\dot{\bar{x}}_2 &= \bar{r}(\bar{r}_2 - c_2) - a_2 \bar{x}_1 \\
&
\vdots \\
\dot{\bar{x}}_{k-1} &= \bar{r}^{k-2}(\bar{r}_{k-1} - c_{k-1}) - a_{k-1} \bar{x}_{k-1} \\
\bar{z}' &= -\left( \bar{z}^k + \sum_{i=1}^{k-1} \bar{x}_i \bar{z}^{i-1} \right).
\end{align*}
\]

(14)

The role of \( c_i \) is to eliminate the constant values of \( f_i(0, 0, 0) \), while \( a_i \) and \( b \) are chosen so that the origin is locally asymptotically stable. Note that (14) depends regularly on \( \bar{r} \). Therefore, it is convenient to study the stability of (14) for \( \bar{r} = 0 \). Then, the same stability properties hold for \( \bar{r} > 0 \) sufficiently small. We remark at this point that from the relation \( \bar{r} = \bar{r}^{2k-1} \), the arguments regarding the stability of (14) for \( \bar{r} > 0 \) sufficiently small are equivalent to similar ones for (8) with \( \bar{r} = \bar{r}^{1/(2k-1)} \) sufficiently small. Note that \( f_i(0, \bar{x}, \bar{z}) = f_i(0, 0, 0) = c_i \). Therefore, system (14) restricted to \( \bar{r} = 0 \) reads as

\[
\begin{align*}
\dot{x}_1 &= -a_1 \bar{x}_1 + b \bar{z} \\
\dot{x}_2 &= -a_2 \bar{x}_1 \\
&
\vdots \\
\dot{x}_{k-1} &= -a_{k-1} \bar{x}_{k-1} \\
\dot{z}' &= -\left( \bar{z}^k + \sum_{i=1}^{k-1} \bar{x}_i \bar{z}^{i-1} \right).
\end{align*}
\]

(15)

The Jacobian of (15) evaluated at the origin is of the form

\[
J = \begin{bmatrix}
-A & b \hat{e}_1 \\
-\hat{e}_1 & 0
\end{bmatrix},
\]

(16)

where \( e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{k-1} \) and \( A \) is a \((k-1) \times (k-1)\) diagonal matrix of the form \( A = \text{diag}\{a_i\} \). It is straightforward to show that the eigenvalues of \( J \) are

\[
\left\{ \frac{-a_1 \pm \sqrt{a_1^2 - 4b}}{2}, -a_2, \ldots, -a_{k-1} \right\}.
\]

(17)

Thus, we conclude that the origin of (15) is locally asymptotically stable. To obtain the controller \( u \), we just perform the blow down (inverse of the blow up) resulting in \( u = (u_1, \ldots, u_{k-1}) \) with

\[
\begin{align*}
u_1 &= -c_1 - a_1 \bar{r}^{k-1} \bar{x}_1 - b \bar{r}^{k-2} \bar{z} \\
u_j &= -c_j - a_j \bar{r}^{k-1} \bar{x}_j, \quad j = 2, \ldots, k - 1,
\end{align*}
\]

(18)

which in compact form can be presented as in (9). □

The proof of Theorem 2 consists of designing a controller in the family chart \( K_\varepsilon \). A limitation of Theorem 2 is that, upon choosing \( A \) and \( b \), the region of attraction of the origin shrinks as \( \varepsilon \to 0 \). This can be resolved by studying the system in the directional charts to upgrade and improve the controller, as shown in the following section.

4.1 On the region of attraction

We now discuss a way to extend the region of attraction of the origin via further control actions. We remark that we do this without any Lyapunov analysis but rather by studying the dynamics of the blown up system in an appropriate chart. Increasing the region of attraction is particularly important for SFCS (6) with \( k \) even. This is because, near the origin, the critical manifold is "U-shaped" when \( k \) is even while it is "S-shaped" when \( k \) is odd. This means that, for \( k \) even, trajectories that do not converge to the origin quickly diverge. See Figure 1 for a schematic impression of the previous description.

![Schematic impression of the previous description](image)

Fig. 1. Left: Example of a critical manifold for the case \( k \) even, given by \( S = \{(x, z) \in \mathbb{R}^2 \mid z^2 + x = 0\} \). The origin is a fold point. Right: Example of a critical manifold for the case \( k \) odd, given by \( S = \{(x_1, x_2, z) \in \mathbb{R}^3 \mid z^2 + x_2 z + x_1 = 0\} \). The origin is a cusp point, while the dashed line represents fold points. Note that the trajectory jumps when it passes through a fold point and not through a cusp point.

Theorem 3 Consider the SFCS (8). Suppose that the controller is given by

\[
u = -C + b \varepsilon \frac{1}{\bar{r}} \bar{x}_1 - \varepsilon \frac{1}{\bar{r}} A x + w,
\]

(19)

where \( C \), \( b \), and \( A \) are as in Theorem 2, and \( w = (w_1, \ldots, w_{k-1}) \in \mathbb{R}^{k-1} \) is given by

\[
w_i = K_i (x_i + z^{k-i+2} \chi_i),
\]

(20)

where \( K_i \geq 0 \) and \( \chi_i \in \mathbb{R} \) with \( i = 1, \ldots, k - 1 \). Then, we can choose gains \( K_i \geq 0 \) and constants \( \chi_i \leq -1 \). Therefore, \( j = 2, \ldots, k - 1 \), such that for \( \varepsilon > 0 \) sufficiently small,
the origin is rendered locally asymptotically stable, but its region of attraction is larger compared to the choice $K = 0$.

**PROOF.** We shall prove the result for the case $k$ even. The proof for $k$ odd follows from the fact that if $k$ is odd, trajectories escape from a small neighborhood of the critical manifold when passing through singularities of even degeneracy. By following similar arguments as in the proof of Theorem 2, we can show that the origin of the blown up vector field in the chart $K_2$ is still locally asymptotically stable for $K \geq 0$. This is due to the fact that $w$ is of order $O(\bar{r})$ as $\bar{r} \to 0$. Next, note in (8) that for trajectories that do not converge to the origin, the term $-\frac{k}{2}z^k$ dominates the vector field $\dot{z}$. This means that such trajectories diverge from a small neighborhood of the origin with $z \to -\infty$. So, we look at the chart $K_1$.

Thus, let us define the chart-coordinates

$$(x_1, \ldots, x_{k-1}, z, \varepsilon) = (\rho^k \chi_1, \ldots, \rho^2 \chi_{k-1}, -\rho, \rho^{2k-1} \mu),$$

for $z < 0$. Then, the corresponding blown up vector field reads as

$$\begin{align*}
\rho' &= \rho F(\chi) \\
\chi_i' &= H_i + G_i - K \rho^{-1} \mu \bar{w}_i \\
\mu' &= -(2k-1) \mu F(\chi)
\end{align*}$$

(22)

where $\bar{w}_i = \bar{w}_i(\rho, \chi, \mu)$ is induced by the blow up map (21), that is $\bar{w}_i = \bar{w}_i(\rho^k \chi_1, \ldots, \rho^2 \chi_{k-1}, -\rho, \rho^{2k-1} \mu)$, and

$$\begin{align*}
H_i &= \rho^{i-1} \mu (\dot{f}_i - C - b \mu \sqrt{k+1} \delta_{i1} - \rho^0 \mu \frac{1}{\sqrt{k}} a_i \chi_i) \\
G_i &= -(k - i + 1) F(\chi) \chi_i \\
F(\chi) &= 1 - \sum_{i=1}^{2k-1} (-1)^i \chi_i,
\end{align*}$$

(23)

with $\delta_{i1} = 1$ if $i = 1$ and $\delta_{i1} = 0$ otherwise. Now we have the following crucial observation.

**Remark 4** Suppose $\phi(t) = (\rho(t), \chi_1(t), \ldots, \chi_{k-1}(t), \mu(t))$ is a trajectory of (22) satisfying $\rho(t) \to 0$ and $\chi_i(t) \to \chi_i^*$ with $|\chi_i^*| < \infty$ as $\tau \to \infty$ (here $\tau$ denotes the rescaled time of (22)). Then, due to the blow up (21), the trajectory $\phi$ is equivalent to a trajectory of the SFCS (8) that converges to the origin as $t \to \infty$.

It is straightforward to show that for $K = 0$, there is indeed an non-empty set of initial conditions such that the corresponding trajectory is as described in Remark 4 (after all, for $K = 0$ we have an equivalent system to the one in the chart $K_2$). So the task of $\bar{w}$ is to enlarge such a set. To design $\bar{w}$, we shall use ”high-gain” arguments. Although more complicated controllers can be designed, we want to keep the arguments as simple as possible to showcase the technique rather than the design itself. Thus, note that the arguments of Remark 4 are satisfied if, for example, $\chi_j^* < -1$ and $\chi_j^* = 0$ for all $j = 2, \ldots, k-1$. With this idea in mind, we propose $\bar{w} = -K \rho^0 (\chi - \chi^*)$, where $K > 0$ is diagonal, $-\infty < \chi_j^* < -1$ and $\chi_j^* = 0$ for all $j = 2, \ldots, k-1$, and the integer $q > 1$ shall be set later. Note then that for $\rho > 0, \mu > 0$ and some large $K > 0$, the point $\chi^*$ is attracting and $\rho \to 0$. To obtain the expression of $w$ in (20) we just blow down using the expressions $\rho = -z$ and $\chi_i = x_i (z)^{-k+i-1}$. Therefore

$$w_i = -K_i (-z)^q (x_i (z)^{-k+i-1} - \chi_i^*).$$

(24)

In principle, any integer $q$ such that $q - k + i - 1 > 0$ provides a smooth expression for the controller $w$. For simplicity we choose $q = k - i + 2$ so that the final expression of the controller is

$$w_i = K_i (x_i z + (-z)^{k-i+2} \chi_i^*),$$

(25)

which is as stated in (20).

□

**Remark 5** To design the “compensation” $w$ in Theorem 3 we chose a high-gain controller. This was done to keep the arguments as simple as possible. Naturally, more elaborate and precise controllers may be designed, but the general idea stays the same: “the design of controllers in the blow up space is rather simple”. We remark, however, that the high-gain nature of $w$ is to be expected. In some sense, the role of $w$ is to capture those trajectories that are quickly diverging and force them to return to the origin.

5 Examples

5.1 Example 1

We now exemplify the result of Theorem 2 with an electric circuit having a tunnel diode [23, 29] as shown in Figure 2a.

In [23] the diode’s constitutive relation is given by $I_D = V_D^3 - 9V_D^2 + 24V_D$, see Figure 2c. A parasitic capacitance is added to regularize the circuit, as shown in Figure 2b, see the justification in [3, 28]. It is assumed that the parasitic capacitance $\varepsilon$ is much smaller than any other parameter of the circuit. The equations describing the regularized circuit are

$$\begin{align*}
\dot{V}_C &= \frac{1}{C} I_L \\
\dot{I}_L &= \frac{1}{L} (V_C + V_D) \\
\varepsilon \dot{V}_D &= -(V_D^3 - 9V_D^2 + 24V_D - I_L),
\end{align*}$$

(26)
where $V_D$ denotes the voltage across the tunnel diode, $I_L$ is the current through the inductor, and $V_C$ the voltage across the capacitor. It is straightforward to show that (26) has a unique equilibrium point at $p = (0, 0, 0)$, which is asymptotically stable. The critical manifold of (26) is precisely given by the constitutive relation

$$S = \{ I_L = I_D = V_D^3 - 9V_D^2 + 24V_D \}.$$ 

Note that $S$ has two fold points $p_1 = (V_D, I_D) = (2, 20)$ and $p_2 = (V_D, I_D) = (4, 16)$. The goal is to design a controller that stabilizes the operating point $(V_C, I_L, V_D)$ at one of the fold points, say $p_2$. The desired value of $V_C$ can be chosen arbitrarily but for simplicity we set it to $V_C = 0$. The controls are given by a voltage source ($u_1$) and a current source ($u_2$) as shown in Figure 2d. Accordingly, the controlled system is described by

$$\dot{V}_C = \frac{1}{C} I_L + \frac{1}{C} u_2$$
$$\dot{I}_L = -\frac{1}{L} (V_C + V_D) + \frac{1}{L} u_1$$
$$\varepsilon \dot{V}_D = -(V_D^3 - 9V_D^2 + 24V_D - I_L).$$

For the analysis, let us perform the following change of coordinates $(x_1, x_2, z) = (V_C, -I_L + 16, V_D - 4)$. Thus, the operating point $p_2 = (V_C, V_D, I_L) = (0, 16, 0)$ is translated to $(x_1, x_2, z) = (0, 0, 0)$. One then obtains

$$\dot{x}_1 = \frac{1}{L} (x_2 + z + 4) + \frac{1}{L} u_1$$
$$\dot{x}_2 = \frac{1}{C} (16 - x_1) - \frac{1}{C} u_2$$
$$\varepsilon \dot{z} = -(3z^2 + x_1 + z^3),$$

which is locally, near the origin, of the form studied in this paper. Using the results of Theorem 2, let us choose the controllers $u_1$ and $u_2$ as

$$u_1 = -4 - \varepsilon - \frac{3}{2} a_1 x_1 + b \varepsilon - \frac{1}{2} z, \quad u_2 = 16 + \varepsilon - \frac{1}{2} a_2 x_2.$$ 

As a benchmark, we compare the performance of (29) with high-gain controllers of the form

$$\varepsilon v_1 = -A_1 x_1 + B z, \quad \varepsilon v_2 = -A_2 x_2.$$ 

For the simulation shown in Figure 3 we have chosen the parameters: $L = C = a_1 = a_2 = A_1 = A_2 = 1$, $b = B = 10$, $\varepsilon = 0.01$. We show trajectories for two initial conditions: $(x_1, x_2, z) = (-10, 10, 10)$ in blue and $(x_1, x_2, z) = (50, -30, -6)$ in green. We let the system run in open-loop for the first 10 seconds. Then, at $t = 10$ we “turn on” the controllers. We observe that both controllers ($u$ and $w$) provide a similar performance, however, the gains of $u_1$ and $u_2$ are approximately 10 times smaller than those of $v_1$ and $v_2$.

### 5.2 Example 2

To highlight the results of Theorem 3, let us consider the planar SFCS

$$\dot{x} = 1 + x + z + u$$
$$\varepsilon \dot{z} = -(z^2 + x)$$

In Figure 4, we compare the performance of the controller (19)-(20) with $K = 0$ and $K > 0$. For this simulation we set the constants $A = 1$, $b = 3$ and $\chi = -2$. The first row of Figure 4 shows the open-loop dynamics, and we observe that trajectories are quickly unbounded after crossing $z = 0$. Next, in the second row of Figure 4 we show the dynamics of the closed-loop system with controller proposed in Theorem 3 with $K = 0$. Note that for $\varepsilon = 0.05$, both trajectories converge to the origin, however, when we decrease $\varepsilon$ to $\varepsilon = 0.01$, one trajectory diverges. Finally in the third row of Figure 4 we show the effect of the compensation proposed in Theorem 3, and note that the origin is asymptotically stable for both values of $\varepsilon$. In all these simulations we show trajectories with initial conditions $(x, z) = (-2, 2)$ in blue and $(x, z) = (0.1, 1)$ in green.
Fig. 3. Top: simulation of (28) with the controllers (29). Bottom: simulation of (28) with the high-gain controllers (30). We show trajectories with initial conditions \((x_1, x_2, z) = (-10, 10, 10)\) in blue and \((x_1, x_2, z) = (50, -30, -6)\) in green. Note that, with a similar performance, \(\max \{|u_i|\} < 0.15 \max \{|v_i|\}\), showing that the proposed controller in Theorem 2 is considerably more convenient for implementation than a high-gain controller.

Fig. 4. Simulation of the results of Theorem 3 for \(k = 2\), and for initial conditions \((x, z) = (-2, 2)\) (blue) and \((x, z) = (0.1, 1)\) (green). The first row shows the open-loop dynamics. The second row shows the action of the controller of Theorem 3 with \(K = 0\) (or equivalently of Theorem 2). The third row shows that the region of attraction of the origin has been enlarged due to the addition of \(w\) as in (20).

6 Conclusions

In this paper we have shown a novel method to design controllers for slow-fast control systems that render a non-hyperbolic point asymptotically stable. To this end we have used a technique called Geometric Desingularization together with simple control techniques. In particular, we have provided a new control methodology for slow-fast control systems that classical techniques \([17]\) do not cover. As future research we propose the extension of the presented methodology to trajectory and path following control problems. Special difficulties arise when the trajectory or path to be followed passes through non-hyperbolic points. Another complicated problem is to study slow-fast control systems near non-hyperbolic points with more than one fast direction.
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