Resource allocation models at resource quantity dependence on demand

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Abstract. The paper describes the reverse priority mechanism for a resource allocation problem under the condition when the quantity of resource allocated depends on demand, i.e. on the sum of consumers’ requests. A particular case of the problem for two consumers and a general case for more than two consumers with equal and different priorities are discussed. The linear dependence of the quantity of resource allocated on demand is considered for the particular and general cases. It has been obtained that the use of linear dependence is appropriate in the case with two consumers, when a certain predictable strategy of consumer behavior can be found. For the higher number of consumers, it is feasible to use this dependence, when there is a consumer with a significantly higher priority as compared to the others. Otherwise, the consumers’ behavior becomes hard to predict. The piecewise-constant dependence of the quantity of resource allocated on demand has been studied for the particular and general cases. The overall conclusion has been obtained and has stated that the consumers’ behavior becomes hard to predict, including surges of estimates, when the reverse priority mechanism is used in cases of resource quantity dependence on demand.

1. Introduction

The reverse priority mechanism has been studied for a case when the resource quantity is preset [1-6], or the function of resource quantity allocation is preset [7-9]. In a number of cases, the quantity of resource allocated depends on demand, i.e. the sum of requests. For example, if demand \( s = \sum_{i} s_i \) exceeds the quantity of resource \( R \), then the Center can increase this quantity by adding the resource in direct proportion to deficit \( \Delta = s - R \), which means that the allocated quantity of resource is

\[ R(s) = R + k(s - R), \]  

(1)
where $0 < k < 1$

Another example is connected with resource allocation in a three-level resource allocation system (see figure 1).

**Figure 1. Three-Level Resource Allocation System**

Each consumer $I_{ij}$ files its request $s_{ij}$ to its Center $C_i$. Center $C_i$ sums up the requests and files the total request $s_i = \sum_j s_{ij}$ to the Upper-Level Center (ULC). The Upper-Level Center allocates the resource among the Centers by using the direct priority mechanism, i.e. in direct proportion to requests $s_i$, which means that resource quantity $x_i$ received by $C_i$ equals $x_i = \frac{s_i \cdot R}{\sum_j s_j}$, where $R$ – resource of the Upper-Level Center.

When there is a large number of Centers, the weak dependence hypothesis, according to which the Centers do not take into account the influence of their request on parameter $k = \frac{R}{\sum_j s_j}$, is true.

That is the reason why, while allocating the resource among the consumers, each Center assumes that its resource equals

$$R(S) = kS, \; k < 1$$

(2)

Let us analyze the reverse priority mechanism at dependences (1) and (2) [10].

2. Two-Consumers Case

Let the consumers’ priorities be equal $A_i = 1, \; i = 1, n$.

Let the dependence be $R(S) = kS$.

We have

$$x_1 = \min \left( s_1; \frac{k(s_1 + s_2)}{s_1 \left( \frac{1}{s_1} + \frac{1}{s_2} \right)} \right) = \min(s_1; k s_2), \; x_2 = \min \left( s_2; \frac{k(s_1 + s_2)}{s_2 \left( \frac{1}{s_1} + \frac{1}{s_2} \right)} \right) = \min(s_2; k s_1).$$

Let $s_1 \geq s_2$. In this case

$$x_1 = k s_2,$$
Theorem 1. Any situation \((s_1, s_2)\) fulfilling the following conditions
\[
  \begin{align*}
    s_1 &\geq ks_2, \\
    s_2 &\geq ks_1 
  \end{align*}
\]
is a Nash Equilibrium (see figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{nash_equilibrium.png}
\caption{Graphic Representation of Nash Equilibrium.}
\end{figure}

Proof. When conditions (3) are fulfilled, the following occurs
\[
x_1 = ks_2, \quad x_2 = ks_1.
\]

It is obvious that, by changing its strategy, each consumer can increase the quantity of its resource as it is determined by the request of another consumer.

The theorem is proved.

However, it should be noted that if consumer 1 increases its request, then the second consumer can increase the quantity of resource received by increasing its request if necessary. This, in turn, increases the quantity of resource received by the first consumer.

Thus, any Nash Equilibrium makes it possible for the consumers to increase the quantity of resource received by helping each other. In this case, any Nash point can be called the equilibrium with possibilities (Nash Equilibrium B).

If estimates \(s_1, s_2\) are bounded from above, i.e. \(s_1 \leq D, s_2 \leq D\), then point \(s_1 = D, s_2 = D\) is the only Pareto point.

Indeed, in any other point, at least one consumer receives less than \(kD\).

Let us consider the problem with different priorities. To be specific, let us take \(A_1 > A_2\).

We have
\[
x_1 = \min \left( \frac{A_1 \cdot kS}{S_1 \cdot \left( \frac{A_1 + A_2}{S_1} \right)} \right),
\]
\[
x_2 = \min \left( \frac{A_2 \cdot kS}{S_2 \cdot \left( \frac{A_1 + A_2}{S_1} \right)} \right).
\]

Through simple transformations, we reduce these expressions to
\[
x_1 = \min \left( \frac{A_1 kS_2}{A_2} \left( 1 - \frac{(A_1 - A_2)S_2}{A_1 S_1 + A_2 S_1} \right) \right),
\]
\[
x_2 = \min \left\{ S_2 : \frac{A_2 k S_1}{A_1} \left( 1 - \frac{A_1 - A_2 S_1}{A_1 S_2 + A_2 S_1} \right) \right\}
\]

It is readily seen that \( x_1 \) is the increasing function of \( S_1 \), and \( x_2 \) is the decreasing function of \( S_2 \).

Theorem 2. There is the only equilibrium situation \( S_1^* = D \),
\[
S_2^* = D \frac{(1-k)^2 a^2 + 4ak - a(1-k)}{2}
\]
where \( a = \frac{A_2}{A_1} < 1 \).

It should be noted that at \( a=1 \) we get point B (see figure 2).

Proof. Since \( x_1 \) is the increasing function of \( S_1 \), then \( S_1 = D \). Since \( x_2 \) is the decreasing function of \( S_2 \), then \( x_2 = S_2^* \) in equilibrium. We get the following equation:
\[
S_2^* = \frac{A_2 k (D + S_2^*) D}{A_1 S_2^* + A_2 D}
\]

By solving the equation, we obtain (4). The theorem is proved.

3. General Case
Let us consider a case when the number of consumers is \( n > 2 \).

As before, let us first consider the case with the equal priorities.

In this case, let us write the reverse priority mechanism in the following form
\[
x_i = \min \left\{ S_i : \frac{1}{S_i} \right\}, \quad i = 1 \ldots n,
\]
where parameter \( \gamma \) is determined based on the following condition
\[
\sum_i x_i = k S.
\]

This form of the mechanism assures the complete allocation of resources.

However, it should be noted that if \( x_i = S_i < \frac{k S}{S_j \gamma(S)} \), where \( \gamma(S) = \sum_j \frac{1}{S_j} \), then the \( i \)th consumer will obtain the equality by increasing \( S_j \). That is why, without loss of generality, it can be believed that
\[
x_i = \frac{k S}{S_j \gamma(S)}.
\]

By taking a derivative, we get
\[
\frac{\partial x_i}{\partial S_j} = \frac{k S \gamma(S) - k S \gamma(i)}{S_j^2 \gamma^2(S)}, \quad \text{where} \quad \gamma(i) = \sum_{j \neq i} \frac{1}{S_j}.
\]

Let us show that \( \frac{\partial x_i}{\partial S_j} < 0 \), i.e. \( k S \gamma(S) < k S \gamma(i) \) or \( 1 < S(i) \gamma(i) \).

Let us define the minimum \( s(i) \cdot \gamma(i) \).
For this purpose, let us define the minimum \( y(i) = \frac{1}{\sum_{j=1}^{i-1} S_j} \) under constraint \( \sum_{j \neq i} S_j = S_i \).

By using the Lagrange multipliers method, we get \( S_j = \frac{s(i)}{n-1}, \quad j \neq i, \quad y(i) = \frac{n-1}{\sum_{j \neq i} s(i)} = \frac{(n-1)^2}{s(i)}, \)

\( y(i)s(i) = (n-1)^2. \)

Since, at \( n > 2 \) \( s(i)y(i) > (n-1)^2 > 1 \), then \( \frac{\partial x_i}{\partial S_i} < 0 \) holds good for all \( i \).

Thus, at \( S > 0 \) it is advisable for each consumer to decrease \( S_i \).

However, it should be noted that there is one more strategy for each consumer (let it be consumer \( i \)), the point of which is to increase the estimate up to the maximum value of \( D \). In this context, the resource received by other consumers does not exceed their estimates \( S_j \).

When the value of \( D \) is sufficiently high, all consumers \( j \neq i \) receive the resource in quantity \( S_j \), and the remaining resource in quantity \( x_i = k(s(i) + D) - s(i) = kD - (1 - k)s(i) \) is received by consumer \( i \).

When the value of \( D \) is sufficiently high, this strategy is more profitable for consumer \( i \) as compared to the strategy of estimate decrease.

We will get the profitability conditions of shifting to strategy \( S_i = D \) for the \( i \)th consumer. When the decrease strategy is used, consumer \( i \) receives the resource in quantity \( x_i(S_i) = \frac{k(s_i + s(i))}{1 + s_iy(i)}. \)

To make strategy \( S_i = D \) more profitable, the following condition needs to be fulfilled

\[ k(s(i) + D) - s(i) > x_iS(i) \] or

\[ D \geq \frac{S_i^0 + s(i)}{1 + s_i^0y(i)} + \frac{1 - k}{k} s(i), \] \( (8) \)

where \( S_i^0 \) is defined from equation \( S_i^0 = \frac{(s_i^0 + s(i))k}{1 + s_i^0y(i)}. \)

It should be noted that validation of condition (8) is quite a difficult task for consumer \( i \), as it does not possess the full information on values \( s(i) \) and \( y(i) \). It brings about the unpredictability of consumers’ behavior, surges of estimates, etc.

Let us consider the problem with different priorities. We have

\[ x_i(S_i) = \frac{A_i(s_i + s(i))k}{A_i + s_iy(i)}, \]

\[ \frac{\partial x_i}{\partial S_i} = \frac{A_i(s_i + s_iy(i))k - y(i)A_i(s_i + s(i))k}{(A_i + s_iy(i))^2}. \]

Through simple transformations, we get the condition of derivative negativeness

\( (A_i + s_iy(i)) < y(i)(s_i + s(i)) \) or

\[ A_i < y(i)s(i). \] \( (8) \)
Let us define the minimum 
\[ y(i) = \sum_{j \neq i} \frac{A_j}{s_j} \] 
under constraint \[ \sum_{j \neq i} s_j = s(i). \]

We get \[ s_j = y(A_j) = \frac{\sqrt{\sum_{j \neq i} A_j}}{s_j} \]
\[ s(i)Y(i) = \left( \sum_{j \neq i} \sqrt{A_j} \right)^2. \]

Let us show that there exists no more than one consumer \( i \) such that
\[ A_i > \left( \sum_{j \neq i} \sqrt{A_j} \right)^2, \tag{9} \]
and \[ A_i = \max_j A_j. \]

We have \[ \sqrt{A_i} > \sum_{j \neq i} \sqrt{A_j} > \sqrt{A_j} \text{ or } A_i > A_j, j \neq i. \]

If condition (9) is fulfilled, then consumer \( i \) selects strategy \( s_j = D. \) Other consumers select strategies based on the following condition \[ s_j = \frac{A_j(s_j + D)}{A_j + s_j y(i)}. \]

This is the only equilibrium situation.
If condition (9) is not fulfilled, then a hard-to-predict situation occurs as in case with equal priorities.

The overall conclusion, which can be made based on the analysis carried out, is as follows: it is appropriate to use dependence \( R(S) = kS \) in the two-consumers case, when there exists a certain predictable strategy of consumers’ behavior. When the number of consumers is higher, the use of this strategy is justified if there is a consumer with a significantly higher priority. Otherwise, the consumers’ behavior becomes hard to predict.

4. Case of Linear Dependence of Quantity of Resource Allocated on Demand
Let us consider the linear dependence of quantity of resource allocated on demand 
\[ R(S) = R + k(S - R) \]

As before, let us start with the case of two consumers with equal priorities.
We have
\[ x_1 = \min \left( s_1 : \frac{R + k(S - R)}{s_1 + s_2} \right), \]
\[ x_2 = \min \left( s_2 : \frac{R + k(S - R)}{s_1 + s_2} \right). \]

Through simple transformations
\[ \frac{R + k(S - R)}{s} \cdot \frac{s_2}{s_1 + s_2} = \left( \frac{1 - k}{s} \right) \cdot \frac{R + k(S - R)}{s_1 + s_2}, \]
\[ \frac{R + k(S - R)}{s} \cdot \frac{s_1}{s_1 + s_2} = \left( \frac{1 - k}{s} \right) \cdot \frac{R + k(S - R)}{s_1 + s_2}. \]

Both dependences are the decreasing functions of \( s_1 \) and, consequently, \( s_2 \), and, at the same time, are the increasing function of strategy of the other consumer.

Let us consider different priorities, \( A_1 > A_2 \).
We have
\[
x_1 = \min \left( S_1: \frac{\left[ (1-k)R + k (S_1 + S_2) \right] S_2}{A_1 S_2 + A_2 S_1} \right) =
\]

\[
x_1 = \min \left( S_2: \frac{\left( 1-k \right) RA_2 - k \left( A_1 - A_2 \right) S_2}{A_1 S_2 + A_2 S_1} \right).
\]

\[
x_2 = \min \left( S_1: \frac{\left( 1-k \right) RA_1 + k \left( A_1 - A_2 \right) S_1}{A_1 S_2 + A_2 S_1} \right) .
\]

It should be noted that \( x_2 \) is the decreasing function of \( S_2 \). That is why, \( x_2 \) will be decreasing at

\[
S_2 < \frac{\left( 1-k \right) RA_2}{k (A_1 - A_2)} .
\]

\( x_1 \) will also be the decreasing function of \( S_1 \). The situation is similar to the case of resource growth in proportion to demand growth. Each consumer has two options: either to decrease the request or to increase it to the maximum \( D \). The game becomes hard to predict.

Let us consider the case of \( n > 2 \).

We have

\[
x_i(S) = \frac{\left[ (1-k)R + k (S_i + S(i)) \right] A_i}{A_i + S_i Y(i)} .
\]

As it has been noted above, we consider only the situations when

\[
S_i = \frac{\left[ (1-k)R + k (S_i + S(i)) \right] A_i}{A_i + S_i Y(i)} .
\]

Let us fulfill

\[
\frac{\partial x_i}{\partial S_i} = A_i \left[ k A_i + S_i Y(i) \right] - Y(i) \left[ (1-k)R + k (S_i + S(i)) \right] \over (1 + S_i Y(i))^2 .
\]

The conditions of derivative negativeness are as follows

\[
\frac{1-k}{k} R Y(i) + Y(i) S(i) > A_i .
\]

If this condition is violated, then the consumer with the maximum priority announces the maximum request, and the remaining consumers announce the requests based on condition \( S_j = x_j(S) \), \( j \neq i \).

This is the only Nash Equilibrium situation.

If conditions are fulfilled, then, as with the equal priorities case, the game becomes hard to predict.

5. Case of Piecewise-Constant Dependence of Resource Quantity on Demand

Let us consider the piecewise-constant dependence of resource quantity on demand (see figure 3), \( (R_1 < R_2) \).

\[
R(S) = \begin{cases} R_1, & S < Q, \\ R_2, & S \geq Q. \end{cases}
\]
As before, let us first consider the case of two consumers with equal priorities.

It should be noted that if \( R_2 = R_1 \), then there is the only equilibrium \( S_1 = S_2 = \frac{R_1}{2} \).

If \( Q \leq R_1 \), then situation \( S_1 = S_2 = \frac{R_2}{2} \) is the only equilibrium, which is quite obvious.

Let \( R_2 \geq Q > R_1 \). Let us define the condition, under which it will be profitable for any consumer (for example, for the first one) to announce

\[
S_1 \geq Q - \frac{R_1}{2} = a
\]  

in order to assure the level of resource \( R_2 \). By announcing \( S_1 \) equal to (11), the first consumer obtains \( x_1 = R_2 - \frac{R_1}{2} > \frac{R_1}{2} \), which means that it wins.

However, in the next iteration, the second consumer announces \( S_2 \) based on condition \( S_2 = \frac{R_2 a}{a + S_2} \).

By solving this equation, we get \( x_2 = S_2 = \frac{a}{2} \left( \sqrt{1+ \frac{4R_2}{a}} - 1 \right) \), and the first \( x_1 = R_2 - x_2 \).

The total quantity of resource of the first consumer in two iterations amounts to

\[
R_2 - \frac{R_1}{2} + R_2 - \frac{a}{2} \left( \sqrt{1+ \frac{4R_2}{a}} - 1 \right).
\]  

If this value exceeds \( R_1 \), then strategy (12) is profitable, otherwise, it is not.

It is easy to show that (12) is the increasing function of \( R_2 \), and at large \( R_2 \) this expression is larger than \( R_1 \), and at small – it is smaller. That is why, there is \( R_2^* \) such that (12) equals \( R_1 \). If \( R_2 < R_2^* \), then strategy (11) is not profitable for the first consumer. In this case, we get the equilibrium situation, which is called the “equilibrium in safe strategies”. If \( R_2 > R_2^* \), then strategy (11) is profitable for the first consumer.

Let \( Q > R_2 \) (see figure 3). In this case, to make the resource be equal to \( R_2 \) it is necessary to have \( S_1 + S_2 = Q \) or \( S_1 = Q - S_2 \).
We have $x_1 = \frac{R_2}{Q} - \frac{R_1}{2} = \frac{R_1 R_2 - R_1}{2Q} < \frac{R_1}{2}$.

Thus, strategy $s_1 = Q - S_2$ is not profitable for the consumer, and situation $s_1 = S_2 = \frac{R_1}{2}$ is equilibrium.

Let us consider the situation when $s_1 = s_2 = \frac{Q}{2}$, $x_1 = x_2 = \frac{R}{2}$. Can the first consumer receive more than $\frac{R_2}{2}$ by announcing $s_1 < \frac{Q}{2}$.

We have $x_1 = s_1 = \frac{R_1}{2} - \frac{Q}{2s_1 + Q}$,

by solving this equation, we get

$$x_1 = s_1 = \frac{1}{2} \left( \sqrt{Q^2 + 8R_1Q - Q} \right).$$

We obtain the conditions, under which $s_1 < \frac{R_2}{2}$ or

$$Q^2 + 8R_1Q < (R_2 + Q)^2.$$

We have

$$R_2^2 + 2R_2Q = 8R_1Q,$$

$$R_2^* = \sqrt{Q^2 + 8R_1Q - Q}.$$

Thus, at $R_2 > R_2^*$ situation $s_1 = s_2 = \frac{Q}{2}$ is the Nash Equilibrium.

Let us consider the case with more than two consumers with equal priorities.

If $Q \leq R_1$, then, as before, $s_i = \frac{R_2}{n}$, $i = 1, n$ is the only equilibrium situation.

If $R_1^* < Q < R_2$, then situation $s_i = \frac{R_1}{n}$, $i = 1, n$ is not an equilibrium, as any consumer $i$ by announcing estimate $s_i = Q - \left( \frac{n-1}{n} \right) R_1$ obtains the following quantity of resource

$$x_i = R_2 - \frac{(n-1)}{n} R_1 > \frac{R_1}{n}.$$

We get the conditions, under which situation $s_i = \frac{R_1}{n}$, $i = 1, n$ is an equilibrium in safe strategies, when if the $i^{th}$ consumer announces estimate $Q - \left( \frac{R_1}{n} \right) R_1$, each $j^{th}$ consumer announces estimate $s_j$ satisfying the following equation $s_j = \frac{R_2}{1 + s_j y(i)}$. 


By carrying out the analysis as in the previous case, we get the similar results. And namely, there is level \( R_2^* \), above which we get the Nash Equilibrium situation, at which either \( S = R_2 \) (if \( R_2 \geq Q \)), or \( S = Q \) (if \( R_2 \leq Q \)).

Finally, let us consider the general case with n consumers and different priorities. We omit the lengthy calculations, but would like to mention the most important things.

There are three types of equilibrium.
1. Nash Equilibrium at \( S = R_2 \).
2. Equilibrium in safe strategies \( S = R_1 \).
3. Nash Equilibrium at \( S = Q \).

The type is defined by \( R_1 \cdot R_2 \cdot Q \) ratio, and in some case – by priorities \( A_i \), \( i = 1, n \).

Overall conclusion: the consumers’ behavior can become hard to predict, including surges of estimates, if the reverse priority mechanism is used in cases when the resource quantity depends on demand.

Therefore, in such cases it is advisable to use the absolute priority mechanism

\[
x_i = \min \left( \frac{S_i}{\sqrt{A_i}} \right),
\]

where \( \gamma \) is defined based on condition \( \sum x_i = R(S) \).

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