SELF-SIMILAR COLLISIONLESS SHOCKS

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ABSTRACT

Observations of γ-ray burst afterglows suggest that the correlation length of magnetic field fluctuations downstream of relativistic nonmagnetized collisionless shocks grows with distance from the shock to scales much larger than the plasma skin depth. We argue that this indicates that the plasma properties are described by a self-similar solution and derive constraints on the scaling properties of the solution. For example, we find that the scaling of the characteristic magnetic field amplitude with distance from the shock is $B \propto D^{s_B}$, with $-1 < s_B \leq 0$; that the spectrum of accelerated particles is $dn/dE \propto E^{-2/3(s_B+1)}$; and that the scaling of the magnetic correlation function is $\langle B(x)B(x+\Delta x) \rangle \propto (\Delta x)^{2s_B}$ (for $\Delta x \gg D$). We show that the plasma may be approximated as a combination of two self-similar components: a kinetic component of energetic particles and an MHD-like component representing "thermal" particles. We argue that the latter may be considered as infinitely conducting, in which case $s_B = 0$ and the scalings are completely determined (e.g., $dn/dE \propto E^{-2}$ and $B \propto D^0$, with possible logarithmic corrections). Similar claims apply to nonrelativistic shocks such as in supernova remnants, if the upstream magnetic field can be neglected. Self-similarity has important implications for any model of particle acceleration and/or field generation. For example, we show that the diffusion function in the angle $\mu$ of momentum $p$ in diffuse shock acceleration models must satisfy $D_{\mu\mu}(p,D) = D^{-1}D_{pp}(p/D)$ (where $p$ is the particle momentum) and that a previously suggested model for the generation of large-scale magnetic fields through a hierarchical merger of current filaments should be generalized. A numerical experiment testing our analysis is outlined.

Subject headings: acceleration of particles — gamma rays: bursts — magnetic fields — plasmas — shock waves — supernova remnants

Online material: color figure

1. INTRODUCTION

Due to the low densities characteristic of a wide range of astrophysical environments, shocks observed in many astrophysical systems are collisionless, i.e., mediated by collective plasma instabilities rather than by particle-particle collisions. For example, collisionless shocks play an important role in supernova remnants (SNRs) (e.g., Begelman et al. 1994; Maraschi 2003), gamma-ray bursts (GRBs, e.g., Zhang & Mészáros 2004), and the formation of the large-scale structure of the universe (e.g., Loeb & Waxman 2000).

Although collisionless shocks have been studied for several decades, theoretically and experimentally, in space and in the laboratory, a self-consistent theory of collisionless shocks based on first principles has not yet emerged (see, e.g., comments in Krall 1997).

Observations of GRB "afterglows," the delayed low-energy emission following the prompt gamma-ray emission, provide a unique probe of the physics of collisionless shocks. Current understanding suggests that the afterglow radiation observed is the synchrotron emission of energetic, nonthermal electrons in the downstream of a strong collisionless shock driven into the surrounding interstellar medium (ISM) or stellar wind. These collisionless shocks start out highly relativistic, with shock Lorentz factor $\gamma_s \sim 100$ on a timescale of minutes after the GRB, and gradually decelerate to $\gamma_s \sim 10$ on a timescale of 1 day, and $\gamma_s \sim 1$ on a timescale of a month. This allows one to probe the physics of shocks over a wide range of Lorentz factors. Afterglow shocks are highly "nonmagnetized": the ratio of magnetic field to kinetic energy flux ahead of the shock is very small, $U_B/n_1 m_P c^2 \sim 10^{-9}$, where $U_B$ and $n_1$ are the magnetic energy density and particle number density in the upstream rest frame, respectively. This strongly suggests that the shock structure is determined by the upstream density and the shock Lorentz factor alone (e.g., Gruzinov 2001a). We therefore adopt the assumption that the shock structure approaches a well-defined limit as $U_B/n_1 m_P c^2 \to 0$ and that GRB afterglow shocks are approximately described by this limiting solution (see § 2.1 for a detailed discussion).

It may be noted here that the upstream density can be eliminated from the problem by measuring time in units of the (shock-frame proton) plasma time, $\omega_p^{-1} = (4\pi \gamma_s n_1 e^2 m_p \gamma_s)^{-1/2} = (4\pi n_1 e^2 m_p)^{-1/2}$, and by measuring distances in units of the corresponding skin depth, $l_{sd} = c/\omega_p$. The shock is then completely specified by the dimensionless parameter $\gamma_s \omega_p/c$ (and the dimensionless mass ratio $n_1 m_P$; the upstream pressure is assumed negligible). In this sense, GRB shocks may be considered "simple."

The synchrotron model of GRB afterglows requires a strong magnetic field and a population of energetic electrons to be present in the downstream. Optical observations (e.g., Zhang & Mészáros 2004), the clustering of explosion energies (Frail et al. 2001), and the observed X-ray luminosity (Freedman & Waxman 2001; Berger et al. 2003) suggest that the fraction of postshock thermal energy density carried by nonthermal electrons, $\epsilon_e$, is large, $\epsilon_e \approx 0.1$. The fraction of postshock thermal energy carried by the magnetic field, $\epsilon_B$, is less well constrained by observations. However, in cases in which $\epsilon_B$ can be reliably constrained by multi-wave band spectra, values close to equipartition, $\epsilon_B \approx 0.01–0.1$, are inferred (e.g., Frail et al. 2000).3

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3 Eichler & Waxman (2005) have pointed out that observations determine $\epsilon_e$ and $\epsilon_B$ only up to a factor $f$, the fraction of electrons accelerated, where $m_e/m_p < f < 1$. However, it is expected that $f$ is not very small, $f \approx 1/10$ (Eichler & Waxman 2005).
Near-equipartition magnetic fields may conceivably be produced in the collisionless shock driven by the GRB explosion by electromagnetic (EM) (e.g., Weibel-like) instabilities (e.g., Blandford & Eichler 1987; Gruzinov & Waxman 1999; Medvedev & Loeb 1999; Wiersma & Achterberg 2004). The main challenge associated with the downstream magnetic field is related to the fact that in order to account for the observed radiation as synchrotron emission from accelerated electrons, the field amplitude must remain close to equipartition deep into the downstream, over distances \( \sim 10^{10} l_{sd} \); at \( t \sim 1 \) day the magnetic field must be strong throughout the (proper) width \( \Delta \sim 2 \gamma_e c t \sim 10^{17} \) cm, while \( l_{sd} \sim 10^{7} (n_1/1 \text{ cm}^{-3})^{1/2} \) cm. This is a challenge since EM instabilities are believed to generate (near equipartition) magnetic fields with coherence length \( L \sim l_{sd} \), and a field varying on such a scale is expected to decay within a few skin depths downstream (Gruzinov 2001a). This suggests that the correlation length of the magnetic field far downstream must be much larger than the skin depth, \( L \gg l_{sd} \), perhaps even of order the distance from the shock (Gruzinov & Waxman 1999; Gruzinov 2001a).

Growth of the characteristic length scale by many orders of magnitude, from \( L \sim l_{sd} \) to \( L \gg l_{sd} \), is a strong indication of self-similarity. In regions where \( L \gg l_{sd} \), which also implies that \( L \) is much larger than the Larmor radius of thermal protons (\( L \gg R_{L,th} \sim \gamma_e m_p c^2/eb \sim l_{sd} \gamma_e^{1/2} \)), it is reasonable to assume that \( L \) is the only relevant length scale, and self-similarity is expected (see § 3 for a detailed discussion). The main goal of the current paper is to introduce and formulate the assumption of a self-similar collisionless shock structure, and to study some of its consequences.

Although our analysis is motivated by GRB afterglow observations, it may be relevant also for nonrelativistic collisionless shocks, such as shocks in young SNRs. In the past few years, high-resolution X-ray observations have provided indirect evidence for the presence of strong magnetic fields, \( \geq 10 \mu \text{G} \), in the nonrelativistic (\( \gamma_e \sim \text{ few times } 1000 \text{ km s}^{-1} \)) shocks of young SNRs (see Vink & Lamming 2003; Bamba et al. 2003; Völk et al. 2005). These fields extend to distances \( D > 10^{17} \text{ cm} \sim 10^{10} l_{sd} \) downstream, and possibly even \( \geq 10^{16} \text{ cm} \) upstream, of the shock. As in GRBs, such strong magnetic fields cannot result from the shock compression of a typical ISM magnetic field, \( B_1 \sim \gamma_e \mu \text{G} \). In SNRs, the discrepancy is somewhat less severe, \( U_{B1}/n_1 m_p c^2 \sim 10^{-4} \), and the possibility that these magnetic fields are related to the large-scale ISM fields cannot be ruled out. If the ISM magnetic fields can be neglected, this suggests that these shocks too may have a self-similar nature. Henceforth, when discussing nonrelativistic shocks, we assume that this is indeed the case.

The nonthermal energetic electron (and proton) population is believed to be produced by the diffusive (Fermi) shock acceleration (DSA) mechanism (for reviews, see Drury 1983; Blandford & Eichler 1987; Malkov & Drury 2001). Acceleration of charged particles to high, nonthermal energies is a ubiquitous phenomenon in both relativistic and nonrelativistic collisionless shocks. The accelerated particles are estimated to carry a considerable part of the energy: electrons alone carry \( \sim 10\% \) of the thermal energy in GRB external shocks (Eichler & Waxman 2005) and \( \sim 5\% \) of the thermal energy in SNR shocks (Keshet et al. 2004 and references therein), and at least 10\% of the energy in SNR shocks must be converted into relativistic protons if these shocks are responsible for Galactic cosmic rays (Drury et al. 1989). This has several important implications. The accelerated particles are likely to have an important role in generating and maintaining the inferred magnetic fields. This conclusion is supported also by the evidence of strong amplification of the magnetic field in the upstream of GRB afterglow shocks (Li & Waxman 2006), which is most likely due to the streaming of high-energy particles ahead of the shock. Since the high-energy particles are likely to play an important role in the generation of the fields, a theory of collisionless shocks must provide a self-consistent description of particle acceleration, which depends on the scattering of these particles by magnetic fields, and field generation, which is likely driven by the accelerated particles.

The search for a self-consistent theory of collisionless shocks has led to extensive numerical studies. Particle-in-cell (PIC) simulations were performed in one dimension (e.g., Dieckmann et al. 2006), in two dimensions (2D; e.g., Wallace & Epperlein 1991; Kato 2005), in two spatial and three velocity dimensions (2D3V; e.g., Gruzinov 2001a, 2001b; Medvedev et al. 2005), and in the past few years in three dimensions (3D; e.g., Silva et al. 2003; Nishikawa et al. 2003; Frederiksen et al. 2004; Jaroschek et al. 2004; Spitkovsky 2005). Such simulations have provided compelling evidence that transverse, EM (Weibel-like) instabilities generate near-equilibrium magnetic fields in pair \((e^+e^-)\) plasma, and \( \epsilon_B \geq m_e/m_p \) magnetic fields in ion-electron plasma. However, 3D simulations are limited to very small simulation boxes and at present can reliably probe small length scales no larger than \( \sim 100 \text{ electron skin depths and short timescales no longer than } \sim 100 \text{ electron plasma times} \). Hence, there is only preliminary evidence for the existence of collisionless shocks in 3D, and only in a pair-plasma. (Simulations of ion-electron plasma are forced to employ an effective, small proton to electron mass ratio, \( m_p/m_e \sim 20 \), with present computational resources, and the preliminary results thus obtained are not easily extrapolated to more realistic mass ratios.) Obviously, the question of field survival and correlation length evolution on length scales \( > l_{sd} \) are not yet answered. Similarly, highly energetic particles cannot be contained in the small simulation boxes used, so Fermi-like acceleration processes are suppressed. It is important to note, that some published results are based on PIC simulations in stages in which the boundary conditions strongly modify the plasma evolution. For example, claims that the magnetic fields decay slowly or saturate at some finite level remain questionable, until verified by simulations with sufficiently large simulation boxes. A discussion of 3D PIC simulations and their physical implication appears in the Appendix.

In § 2 we lay the basis for our analysis of afterglow (relativistic) and SNR (nonrelativistic) shocks. In § 2.1 we discuss our assumption that these shocks are highly nonmagnetized, i.e., that the shock structure approaches a well-defined limit as \( U_{B1}/n_1 m_p c^2 \rightarrow 0 \) and that the shocks observed are approximately described by this limiting solution. In § 2.2 we present the governing equations and discuss their dependence on dimensional parameters, which is relevant for the discussion of self-similarity. In particular, we demonstrate that when distances are measured in units of \( l_{sd} \), the shock structure depends only on \( \gamma_e c \) and \( m_e/m_p \). In § 2.3 we clarify the notion of “shock structure”: since the EM fields and particle distributions fluctuate with time (at any given point downstream, and perhaps also upstream of the shock), the “stationary shock structure” is given by the correlation functions of the fluctuating quantities (which are expected to depend only on the distance from the shock).
In § 3 we introduce and formulate the assumption of a self-similar structure in the downstream of nonmagnetized collisionless shocks, derive several scaling relations of the physical quantities, and discuss some of the physical implications. One of the conclusions of § 3 is that the plasma may be approximately described as a combination of two self-similar components: a kinetic component of energetic particles and an MHD-like component representing the bulk, “thermal” particles. The MHD-like component is discussed in § 4. We argue that this component may be treated as an infinitely conducting fluid and show that this leads to a complete determination of the scaling laws.

In § 5 we present various extensions of the analysis. Self-similarity is studied in the upstream of nonmagnetized collisionless shocks and in homogenous time-dependent plasmas, which may be more accessible to simulations than (nonhomogeneous) collisionless shocks. In § 6 we discuss some of the implications of the self-similarity assumption to models of diffusive particle acceleration and to the phenomenological model, suggested by Medvedev et al. (2005), of field generation through hierarchical acceleration and to the phenomenological model, suggested by Spitkovsky (2005), of self-similarity assumption to models of diffusive particle acceleration and to the phenomenological model, suggested by Medvedev et al. (2005), of field generation through hierarchical acceleration and to the phenomenological model, suggested by Spitkovsky (2005), of self-similarity assumption.

Henceforth, we assume that strong, nonmagnetized collisionless shocks do indeed exist. We consider a quasi–steady state planar shock. First, we consider the shock frame, these conditions are valid only in the far downstream and for which self-similarity is assumed, are discussed in §§ 3.3 and 4.1.

The flow is governed by Vlasov’s equation,

$$\partial_t f_\alpha + v(p) \cdot \nabla f_\alpha + q_\alpha \left( \frac{E}{c} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_p f_\alpha = 0,$$

(2)

and Maxwell’s equations,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \partial_t \mathbf{E},$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B},$$

(3)

where the electric current is related to $f_\alpha$ through

$$\mathbf{j} = \sum_\alpha q_\alpha \int d^3 p \, v(p) f_\alpha(p).$$

(4)

The velocity of the particles is given in terms of the momentum by

$$v(p) = \frac{p}{\sqrt{m_p c^2 + p^2}}.$$  

(5)

As usual, the equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{E} = 4\pi \rho$ are assumed to hold at some initial time and are therefore preserved by the other equations at all times. The shock is completely defined by the far upstream boundary conditions. Written in the shock frame, these conditions are

$$E(D \rightarrow -\infty) = B(D \rightarrow -\infty) = 0,$$

$$f_\alpha(D \rightarrow -\infty) = \frac{\gamma}{\gamma - 1} \mathbf{E} \cdot \mathbf{v}^2$$

where $D$ is the distance from the shock front in the direction of the downstream ($D$ is negative in the upstream). Note that the equations are independent of the frame of reference.
In order to highlight the dimensional dependencies of the Vlasov-Maxwell equations, we express the physical quantities $A = t, x \ldots$ in terms of dimensionless variables $A$, where

$$t = \frac{l_{ad}}{v_s} \hat{t}, \quad x = l_{ad} \hat{x}, \quad p = \frac{m_a}{m_p} p_{th} \hat{p},$$

$$B = \frac{p_n c}{e l_{ad}} \hat{B}, \quad E = \frac{p_n v_s}{e l_{ad}} \hat{E},$$

$$j = e v_s \gamma_m \hat{j},$$

and

$$f_\alpha = \frac{\gamma_m n_1}{[m_a/m_p] p_{th}} \hat{f}_\alpha,$$

where

$$\omega^2_{pi} = \frac{4 \pi n_1 e^2}{m_p c^2} \sim \frac{4 \pi n_2 e^2}{\gamma_s m_p},$$

$$l_{ad} = \frac{c}{\omega^2_{pi}},$$

$$p_{th} = m_p \gamma_m v_s$$

are the characteristic ion-plasma frequency (squared), the ion skin depth, and the characteristic momentum of thermal particles in the downstream, respectively. Inserting these into equations (2) and (3), we obtain

$$\partial_t \hat{f}_\alpha + \frac{v}{v_s} \cdot \nabla \hat{f}_\alpha + \frac{q_\alpha/e}{m_a/m_p} \left( \hat{E} + \frac{v}{v_s} \times \hat{B} \right) \cdot \nabla \hat{p} \hat{f}_\alpha = 0,$$

$$\nabla \times \hat{B} = \hat{j} + \frac{v^2}{c^2} \partial_t \hat{E},$$

$$\nabla \times \hat{E} = -\partial_t \hat{B}$$

and

$$\hat{j} = \sum \frac{q_\alpha}{e} \int d^3 \hat{p} \frac{v}{v_s} \hat{f}_\alpha,$$

with

$$\frac{v}{v_s} = \frac{\hat{p}}{\sqrt{(1/\gamma_m^2) + (v_s^2/c^2) \hat{p}^2}}.$$ 

The boundary conditions at upstream infinity (written in the shock frame) can similarly be written in dimensionless form,

$$\hat{E}(\hat{D} \to -\infty) = \hat{B}(\hat{D} \to -\infty) = 0$$

and

$$\hat{f}_\alpha(\hat{D} \to -\infty) = \delta^1(\hat{p} - \hat{\zeta}),$$

where $\hat{D} = D/l_{ad}$. Measuring distances in units of $l_{ad}$, we have thus arrived at a set of dimensionless equations, equations (10)–(14), that depend only on $\gamma_m v_s/c$ (and on $m_a/m_p$).

2.3. Stationary Shock Structure

The EM fields and particle distributions at any given point behind (downstream of) the shock fluctuate with time. For each set of values of the shock parameters, $n_1$ and $\gamma_m v_s$, there is a large ensemble of time-dependent “specific solutions,” with specific temporal and spatial dependence of particle distribution functions and EM fields. We assume that the averages and correlation functions of the fluctuating quantities depend only on the distance from the shock and are identical for all specific solutions (in the limit that the size of the shock plane is infinite).

Consider, for example, the particle distribution function $f_\alpha(x = D \hat{z} + x_\perp, p, t)$ or magnetic field $B(x = D \hat{z} + x_\perp, t)$, where we have separated the $x$-dependence to dependence on the distance from the shock front, using $\hat{z}$ as the direction of the shock normal, and on $x_\perp$, a two-dimensional vector perpendicular to $\hat{z}$. While $f_\alpha$ depends on $t$ and $x_\perp$, we assume that an average of $f_\alpha$ over planes perpendicular to the shock normal is independent of $t$ and $x_\perp$:

$$\langle f_\alpha \rangle(p; D) \equiv \lim_{r \to \infty} \frac{1}{2 \pi r^2} \int_{x_{\perp} < r} d^2x_\perp f_\alpha(D \hat{z} + x_\perp + x_\perp', p, t).$$

Similarly, we assume that correlation functions of $f$, $B$, and $E$ may be defined, which are independent of $t$ and $x_\perp$, e.g.,

$$B_{ij}(\Delta x, \Delta t, D) \equiv \lim_{r \to \infty} \frac{1}{2 \pi r^2} \int_{x_{\perp} < r} d^2x_\perp B_{ij}[D \hat{z} + x_\perp + x_\perp', t] \times B_{ij}[D \hat{z} + x_\perp + \Delta x + x_\perp', t + \Delta t].$$

The stationary shock structure is described by $\langle f_\alpha \rangle$ and by the (infinite) set of correlation functions. Note that the Maxwell-Vlasov equations may be converted to an (infinite) hierarchy of equations for the correlation functions.

It is useful for the analysis that follows to define the 2D power spectrum of the magnetic field in a plane parallel to the shock front. We define the 2D power spectrum at distance $D$ as

$$B_{ij}^{(2D)}(\Delta x, D) \equiv (2\pi)^{-1} \int d^2x_\perp \eta_{ij}(x_\perp', D) e^{i \Delta x \cdot x_\perp'},$$

where $\eta_{ij}(x_\perp', D) \equiv B_{ij}(\Delta x = x_\perp', \Delta t = 0, D)$. The plane-averaged magnetic energy density is related to the correlation function by

$$\langle U_B \rangle(D) \equiv (8\pi)^{-1} \int d^2k_{\perp} B_{ij}^{(2D)}(k_{\perp}, D) = (8\pi)^{-1} \sum_B \eta_B(\Delta x = 0, \Delta t = 0, D).$$

Note that the plane-averaged magnetic field vanishes. As a consequence of the planar symmetry, a nonzero averaged magnetic field would be oriented along the z-axis. As its divergence vanishes, the magnitude of $\langle B \rangle$ would be constant and thus equal to its far upstream value, which is zero. The root mean square (rms) magnetic field,

$$\bar{B}(D) = \left[ \sum_B \eta_B(\Delta x = 0, \Delta t = 0, D) \right]^{1/2},$$

serves as a measure of the characteristic magnetic field amplitude at a distance $D$ from the shock (note $\langle U_B \rangle = \bar{B}^2/8\pi$).

3. Self-Similarity in the Downstream

As discussed in §1, afterglow observations suggest that the characteristic length scale $L$ for variations in the magnetic field becomes much larger than $l_{ad}, L \gg l_{ad}$, at distances $D \gg l_{ad}$ downstream of
the shock. We assume here that in the limit $L/l_{sd} \to \infty$, $L$ becomes the only length scale relevant for the evolution of the plasma, which implies self-similarity. There is no proof that self-similarity will be present whenever the characteristic length scale diverges (or becomes infinitesimal). However, the self-similarity assumption is known to be valid for many physical systems in which such divergence occurs (see, e.g., Zel’’ dovich & Raizer [1968, ch. 12] for self-similarity in hydrodynamics and Kadanno et al. [1967] for self-similarity in critical phenomena). In order to clarify the reasoning behind this assumption and its implications, we first briefly describe in § 3.1 an example of a physical system, where the presence of a diverging characteristic length scale leads to self-similar behavior. We then formulate in § 3.2 the self-similarity assumption for the downstream of collisionless shocks. The separation of the plasma into two components is discussed in § 3.3. The scaling relations are derived in § 3.4, and some of their implications are discussed in § 3.5.

A note is in order here regarding nonrelativistic collisionless shocks. As mentioned in the introduction, self-similarity may be expected in the nonrelativistic shocks observed in young SNRs. The analysis is similar for relativistic and nonrelativistic shocks, and in particular leads to the same scaling relations and implications. The main difference is that in the nonrelativistic case there is a physically relevant length scale that is different from the skin depth $l_{sd}$: the Larmor radius of mildly relativistic protons, $R_L \sim m_p c^2/eB \sim R_{L,th}(v/c) \sim (v/c)^{-1}l_{sd}$, where $R_{L,th}$ is the characteristic Larmor radius of thermal protons. In other words, there is a small parameter in the problem, $(v/c)$, that is likely to have a physical implication. Hence, although in this case we also expect self-similarity as $L/l_{sd} \to \infty$, we expect the self-similar solution to provide a good approximation for $L \gg (v/c)^{-1}l_{sd}$, rather than for $L \gg l_{sd}$, and the scaling relations of the particle distribution functions to be valid for momenta $p > m_p c$.

3.1. An Example of Self-Similarity

Consider a macroscopic system in thermodynamic equilibrium, exhibiting a phase transition at some temperature $T_c$. Near the critical temperature, $T \sim T_c$, the correlation length $L$ of fluctuations within the system diverges, $L(\delta T) \to \infty$ as $\delta T \equiv T - T_c \to 0$. It is commonly assumed that in this limit $L$ becomes the only “relevant” length scale and that the microscopic length scales, e.g., the interparticle distance $d$, become irrelevant.

To clarify this assumption, let us consider the correlation function $f(r)$ of some fluctuating quantity. The correlation function $f$ is a function of $\delta T$ and of the various parameters defining the thermodynamic system (e.g., $d$). Assuming $L$ diverges monotonically with $\delta T$, we may replace the dependence on $\delta T$ with dependence on $L$, $f(r; L, d, c_1)$, where $c_1$ represents the parameters defining the thermodynamic system. We may assume that these parameters contain no parameter with the dimensions of length, since any parameter $c_1$ may be replaced by a dimensionless parameter, $c_1/d$. Using the dimensional parameters, a constant $A_f$ with the same dimensions of $f$ may be constructed, and $f$ may be written as $f = A_f \tilde{f}$, where $\tilde{f}$ is a dimensionless function. Since $f$ is dimensionless, it must be a function of only dimensionless combinations of $(r, L, d, c_1)$. Thus, $f = A_f f(r; L, d, \tilde{c}_1)$, where $\tilde{c}_1$ represent the dimensionless combinations of $c_1$. As $L/d$ diverges, it is assumed that $d$ becomes “irrelevant,” in the sense that $\tilde{f}$ approaches a limit

$$f(r; L, d, c_1) = A_f \tilde{f} \left( \frac{r}{L}, \frac{d}{L}, \tilde{c}_1 \right) \to A_f \left( \frac{d}{L} \right)^{-\eta} g \left( \frac{r}{L}, \tilde{c}_1 \right)$$

for $d/L \to 0$. Note that it is not assumed that, as $d/L \to 0$, $\tilde{f}$ approaches a limit independent of $d/L$, but rather that in this limit it has a scale-free (power law) dependence on $d/L$. In the former case, $f$ depends on the dimensional parameters $A_f$ and $L$ alone, and in the latter on $A_f d^{-\eta}$ and $L$. In both cases, the only length scale that may be extracted from $f$ is $L$. Similar arguments lead to the conclusion that $L$ has a power-law dependence on $\delta T$,

$$\delta T(L, d, c_1) = T_c \tilde{\delta} T \left( \frac{d}{L}, \tilde{c}_1 \right) \to T_c \left( \frac{d}{L} \right)^{-\eta} \tilde{h} \left( \tilde{c}_1 \right),$$

for $d/L \to 0$. In general, the self-similarity assumption may be stated as the assumption that all the properties of the system at some $\delta T = \delta T_1$, corresponding to $L_1 = L(\delta T_1)$, are identical to those of the system at $\delta T = \delta T_2$, corresponding to $L_2 = L(\delta T_2)$, up to a scaling transformation. That is, any function describing some properties of the system, e.g., $f(r, \delta T(L))$, is identical at $L_1$ and $L_2$ up to a scaling of $r$ and $L$,

$$f(r, L_2) = \left( \frac{L_2}{L_1} \right)^{\eta} f \left( \frac{r}{L_2/L_1}, L_1 \right)$$

for some $\eta$. The function $f$ is termed self-similar, as it is similar to itself at different $L$-values. Comparing equations (22) and (20), it is clear that the assumption that $d$ is irrelevant is equivalent to the assumption that $f$ is self-similar. It is clear from the above example that the assumption of self-similarity provides powerful constraints on the properties of the system and that a complete characterization of its properties requires determination of the similarity exponents $(\eta, \eta_d)$.

3.2. The Self-Similarity Assumption

Consider the plasma at a distance $D \gg l_{sd}$ downstream of the shock, where the field correlation length is assumed to satisfy $L \gg l_{sd}$. Assuming $L/l_{sd}$ diverges with $D$, we expect the structure of the shock to become self-similar. That is, we expect the averaged distribution function $(f_a)$, the rms magnetic field $B$, and the (infinite) set of correlation functions to be self-similar:

$$\tilde{B}(L) = \left( \frac{L}{L_0} \right)^{\eta} \tilde{B}(L_0),$$

$$\langle f_a \rangle(p, L) = \left( \frac{L}{L_0} \right)^{\eta} \langle f_a \rangle \left[ \frac{p}{(L/L_0)^{\eta}} L_0 \right],$$

$$B_{ij}(\Delta x, \Delta t, L) = \left( \frac{L}{L_0} \right)^{2\eta} B_{ij} \left[ \frac{\Delta x}{L/L_0} \frac{\Delta t}{(L/L_0)^{\eta}} L_0 \right],$$

and so on, where $L_0$ is a reference length scale. Note that we have replaced the dependence on $D$ with a dependence on $L$, assuming that $L$ diverges monotonically with $D$.

The values of the similarity exponents are not determined by the requirement of self-similarity alone and must be derived based on additional arguments. In what follows, we derive constraints on the similarity exponents $(\xi, \eta, \eta_d, \ldots)$ using the Maxwell-Vlasov equations. Since self-similarity applies to averaged quantities, e.g., $(f_a)$ and $B_{ij}$, rather than to specific solutions, this derivation should be based on the equations for the averaged quantities, which are obtained from the Maxwell-Vlasov equations. Such a derivation is, however, rather cumbersome. Identical constraints on the similarity exponents may be derived in a simpler way by assuming that the specific solutions form a
scalable family in the sense that if \{f_0(x, p, t), B(x, t)\} is a solution of the Maxwell-Vlasov equations, then the scaled functions

\[
B'(x, t) = \xi^{\alpha} B \left( \frac{x}{\xi}, \frac{t}{\xi^2} \right),
\]

\[
f'_0(x, p, t) = \xi^{\alpha} f_0 \left( \frac{x}{\xi}, \frac{p - \xi^2 \mathbf{v} \times \mathbf{B}}{\xi^2 \xi^2} \right),
\]

(26)

also constitute a solution (at least approximately). It is straightforward to verify that the self-similarity requirements, equations (24) and (25), are automatically satisfied under this assumption by substituting \(\xi = L/L_0\) in equation (26) and recalling that the averages and correlation functions are identical for all specific solutions. Under the assumption that the solutions are scalable, we can use the Maxwell-Vlasov equations directly (instead of the equations for the correlation functions) in order to derive the constraints on the similarity exponents. Before doing so, we address (in §3.3) the separation of the plasma into two components.

It is important to emphasize here that, although the constraints derived under the assumption that the solutions are scalable (in the above sense) are identical to those derived from the self-similarity requirements, equations (24) and (25), it is not obvious that the self-similarity requirements indeed imply that the specific solutions are scalable.

### 3.3. Two Plasma Components

As the plasma flows away from the shock, most of the particles remain “thermal,” i.e., most protons carry a momentum \(\sim p_{th}\) and most electrons carry a momentum \((m_e/m_p)^{1/2} p_{th}\), with \(\nu > 0\). This implies the existence of a length scale \(R_{th} = p_{th} c/eB \propto L^{−\alpha} R_{th}/L \propto L^{−(1+\alpha)}\). Obviously, in order for \(L\) to be the only relevant scale, we must have \(s_B > -1\). Combined with the requirement that the field energy density does not diverge, we have

\[
-1 < s_B \leq 0.
\]

Our assumption that \(L\) is the only relevant length scale implies that \(R_{th}\) is irrelevant. However, \(R_{th}\) is obviously relevant for the description of the microscopic motion of individual thermal particles. This apparent contradiction may be resolved only if the microscopic motion of thermal particles, i.e., of particles with momenta \(p\), such that \(pc/eB \ll L\), is unimportant. These particles should therefore be described by effective equations, where the microscopic particle motion is unimportant. We leave the discussion of the thermal particles, to which we refer hereafter as the “fluid” particles, to §4 and note here only several points that are important for the discussion that follows.

First, the scaling of \(f_0\) derived in §3.2 does not apply to all particle momenta, but rather to the large-\(p\), \(p \gg p_{th}(\nu/e)^{-1}\), behavior of \(f_0\). In particular, the Vlasov equation in this region, where \(v(p) \approx c\hat{p}\), can be written as

\[
\partial_t f_0 + c\hat{p} \cdot \nabla f_0 + q_0 (E + c\hat{p} \times B) \cdot \nabla_p f_0 = 0.
\]

Second, the total electric current is the sum of the electric currents carried by the high-energy particles, \(j_h\), and by the fluid, \(j_F\). Hence, \(j = j_h + j_F\). Hereafter, \(h\) and \(F\) subscripts refer to the high-energy and to the fluid particles, respectively. The electric current \(j_h\) is given by

\[
j_h = \sum_{\alpha} q_0 \int_{p > q_0 c/eB/L_c} d^3 p \, c p \hat{p} f_0(p).
\]

(29)

Here, \(\xi_g(L)\) is a dimensionless function that determines the threshold momentum between accelerated and fluid components. It must satisfy \(\xi_g > (v/e)^{-1} p_{th} c/eB \propto (v/c)^{-1} R_{th}/L\) in order to ensure that the accelerated component is not affected by the thermal scale. It must also tend to zero as \(L\) diverges, as self-similarity is expected for all scales \(\gg (v/c)^{-1} R_{th}\).

Solutions of the type we have arrived at, in which the self-similar solution describes the evolution in some part of \((x, p, t)\) phase space while the other part is described by a different solution, are usually called “second-type solutions” (e.g., Zel’dovich & Raizer 1968, ch. 12; Waxman & Shvarts 1993). In second-type solutions, the similarity exponents cannot be determined by global conservation laws. For example, one may have argued that a relation between \(s_p\) and \(s_e\) may be derived by requiring the integral over momenta of \(\langle f_p \rangle\), given by equation (24), to be equal to the (downstream) particle density,

\[
n_e = \int d^3 p \langle f_0 \rangle (p, L) = \left( \frac{L}{L_0} \right)^{3s_p + s_e} \int d^3 y \langle f_0 \rangle (y, L_0),
\]

(30)

which would imply \(3s_p + s_e = 0\). Such a constraint cannot be derived, however, since the scaling relation, equation (24), used for obtaining the second equality of equation (30) does not hold for small values of \(y\). In fact, as we show below, the functional form of \(\langle f \rangle\) derived by the self-similarity arguments is such that the integral on the right-hand side of equation (30) diverges at small \(y\). The number of particles described by the self-similar solution does not diverge, however, since the integration extends only down to \(y = \xi_g(L/L_0)^{−\nu} c/eB L_0/c\) (note that this constrains the dependence of \(\xi_g\) on \(L\)). This is analogous to the divergence of energy in second-type self-similar solutions of hydrodynamic flows (e.g., Waxman & Shvarts 1993).

The particles with \(p \leq \xi_g(L)c/eB/L_c\) are described by a solution different than the self-similar solution describing the higher momenta, “accelerated,” particles. As in any second-type self-similar solution, it is necessary that the non-self-similar part of the solution does not affect the self-similar part (this requirement often determines the similarity exponents of second-type solutions, e.g., Zel’dovich & Raizer 1968, ch. 12; Waxman & Shvarts 1993). In our case, the particles at \(p \leq \xi_g(L)c/eB/L_c\) may affect the higher momenta particles only through their contribution to the electric current. This implies that the current contributed by \(p \leq \xi_g(L)c/eB/L_c\) particles must either be negligible or scale with \(L\) in a similar way as the current of the higher momenta particles does. For the fluid current, this requirement implies that either \(j_F \ll j_h\) or \(j_F(L)/j_h(L) \propto L^{0}\). In addition, this requirement implies that the integral on the right-hand side of equation (29) converges [as otherwise the current would be dominated by \(p \leq \xi_g(L)c/eB/L_c\) particles].

The following point should be emphasized here. We have implicitly assumed above that the high-momenta, accelerated particles are dynamically important in the sense that their electric current makes a considerable contribution to the total electric current, \(j_h \sim j\). However, self-similar solutions in which the accelerated particles do not contribute to the current, \(j_h \ll j\), are possible in principle. In this case, the growth of the magnetic field fluctuation correlation length should be driven by the fluid, and the accelerated particles can be treated as “test particles,” which
do not affect the flow. As mentioned in § 1, such a picture is unlikely, due to the fact that the accelerated particles carry a significant fraction of the energy and due to the evidence that they play a role in generating magnetic fields in the upstream plasma.

3.4. Scaling Relations

Let us consider first the scaling of \( L \) with \( D \). Since the characteristic length scale for changes in \( L \) is \( L \), we must have \( L \propto D \): requiring the scale for changes in \( L \) to be proportional to \( L \) may be written as \( d(\log L)/dD = 1/\alpha \), which implies \( L \propto D \). It should be noted that this assumption is not equivalent to assuming that the only length scale relevant for the evolution is \( D \). Under such an assumption, \( d(\log L)/dD = s/D \), implying \( L = D^s \), which allows the possibility \( L/D \rightarrow 0 \) as \( D \rightarrow \infty \).

Next, we consider the scaling of time, i.e., the value of \( s_t \). Substituting the scaled solutions, equation (26), into the Vlasov equation, equation (28), one finds that the various terms in the equation scale differently with \( \xi \). In order for the scaling of the temporal and the spatial derivative terms to be similar, one must require \( s_t = 1 \). However, one cannot conclude from this that \( s_t = 1 \) is the only value allowed, since it is possible that one of the first two terms, \( \partial_t f \) or \( v \cdot \nabla f \), becomes negligible as \( L \rightarrow \infty \) and may be neglected altogether in the Vlasov equation. Nevertheless, we conclude that \( s_t = 1 \) is required based on the following arguments. The case \( s_t < 1 \), implying that \( cT/L \rightarrow 0 \) as \( L \rightarrow \infty \), where \( \Gamma \) is the characteristic time for variations in the physical quantities, is ruled out since it would imply that at large distances from the shock the electric field is much larger than the magnetic field \( (E/B) \rightarrow \infty \) as \( L \rightarrow \infty \), which is inconsistent with the synchrotron model of afterglow emission. Assuming \( s_t > 1 \), \( cT/L \rightarrow \infty \) as \( L \rightarrow \infty \), would imply that at large distances from the shock the magnetic fields are approximately static (velocities of the field lines of order \( L/T \)) in the shock frame. Once the condition \( L/T \ll v_s \) would be reached, the thermal protons would gyrate around the static field lines (the gyration radius, \( R_{\text{L,th}} \), is much smaller than \( L \), and thus also the gyration time, \( T_{\text{L,th}} \approx R_{\text{L,th}}/v_s \), is much shorter than \( T \)), in contradiction with the fact that the bulk downstream fluid moves with velocity \( v_s \approx v_L \).

[Note that \( \gamma_d = (1 - v_s^2/c^2)^{-1/2} \approx 1 \).] We conclude that we must have \( s_t = 1 \).

The scaling of the momentum, i.e., the value of \( s_p \), is determined by comparing the momentum derivative term with the spatial (or temporal) derivative term in the Vlasov equation. Substituting the scaled solutions, equation (26), into the Vlasov equation, equation (28), and requiring all terms to scale similarly implies \( s_p = 1 + s_B \). One implication of this scaling is that the Larmor radius scales as \( L \) (in fact, the entire trajectory of each particle scales with \( L \)).

Finally, we derive the relation between \( s_f \) and \( s_B \). The current provided by the accelerated particles is

\[
\dot{j}_h(x, t) = \sum \alpha \int d^3p \phi f_0 \left( x, \frac{p}{\xi}, \frac{t}{\xi} \right)
\]

\[
= \xi^{s_f+3(s_B+1)} \sum \alpha \int d^3p \phi f_0 \left( x, \frac{p}{\xi}, \frac{t}{\xi} \right)
\]

\[
= \xi^{s_f+3(s_B+1)} j_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
\]

We have not shown here explicitly the lower limit of the integration over \( p \), since, as discussed in § 3.3, the current integral must converge at small \( p \). Substituting the scaled current and the scaled magnetic field into Maxwell’s equation, \( \nabla \times \mathbf{B} = 4\pi c^2 \dot{j} + \mathbf{E} \):

\[
4\pi c^2 \dot{j}_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
\]

\[
\frac{\partial}{\partial \xi} \mathbf{B} = 4\pi c^2 \dot{j}_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
\]

We have not shown here explicitly the lower limit of the integration over \( p \), since, as discussed in § 3.3, the current integral must converge at small \( p \). Substituting the scaled current and the scaled magnetic field into Maxwell’s equation, \( \nabla \times \mathbf{B} = 4\pi c^2 \dot{j} + \mathbf{E} \):

\[
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\]

\[
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\[
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\]

\[
\frac{\partial}{\partial \xi} \mathbf{B} = 4\pi c^2 \dot{j}_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
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\[
4\pi c^2 \dot{j}_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
\]

\[
\frac{\partial}{\partial \xi} \mathbf{B} = 4\pi c^2 \dot{j}_h \left( \frac{x}{\xi}, \frac{t}{\xi} \right). \tag{31}
\]
A similar constraint may be obtained by taking the limit $p \to 0$, which yields

$$
\langle f_{\alpha}(p, L) \rangle = \left( \frac{p}{p_0} \right)^{\eta_\beta/p} \left[ \hat{p}_{p_0} \hat{L} \left( \frac{p_0}{p} \right)^{1/p} \right] e^{-\frac{p}{p_0}} \langle f_\alpha \rangle,$$

for $p \to 0$. This implies that $\langle f_{\alpha}(p, L) \rangle$ approaches a power-law distribution at small $p$, $\langle f_{\alpha}(p, L) \rangle \propto p^{\eta_\beta/p}$, provided that $\langle f_{\alpha}(p, L) \rangle$ approaches a finite limit for some $p$ as $L \to \infty$. The condition that $\langle f_{\alpha}(p, L) \rangle \to \eta \neq 0$ as $L \to \infty$ is equivalent to the requirement that accelerated particles reach downstream infinity. This is physically reasonable since cosmic rays are expected to be advected with the flow to the downstream (e.g., in diffusive Fermi acceleration). Moreover, the presence of high-energy electrons in the far downstream ($\sim 10^{10} \text{erg}$) is required to account for afterglow observations. Thus, we conclude that the self-similar distribution function approach a power-law dependence on $p$ at both high and low momenta,

$$
\frac{dn_{\alpha}}{dp} \propto p^2 \int d^2 \hat{p} f_{\alpha}(p \hat{p}) e^{-p^{2/\eta_\beta+1}}.
$$

Let us consider the magnetic field correlation function. Using $L/L_0 = \Delta x/\Delta x_0$ in equation (25), we obtain

$$
B_{ij}(\Delta x, \Delta t, L) = \left( \frac{\Delta x}{\Delta x_0} \right)^{2s_0} B_{ij} \left[ \frac{\Delta t}{\Delta x_0} \frac{\Delta x_0}{\Delta x} L \right],
$$

for $\Delta x \to 0$, where $\hat{x}$ is a unit vector in the direction of $\Delta x$. This implies that $B_{ij}(\Delta x, \Delta t, L)$ approaches a power-law behavior at large $\Delta x$, $B_{ij}(\Delta x, \Delta t, L) \propto \Delta x^{2s_0}$, provided that $B_{ij}(\Delta x, \Delta t, L)$ approaches a finite limit for some $\Delta x_0$ as $L, \Delta t \to 0$. This is expected to be the case under the assumption that the accelerated particles reach the shock front, since in this case the number of particles with Larmor radius greater than (some given) $\Delta x$ approaches a constant as $L \to 0$, and such particles are expected to generate a correlation between the magnetic fields at points separated by $\Delta x$.

Equation (37) implies that the magnetic power spectrum has a component that scales as $k^{-2-2s}$. Assuming that the result $B_{ij}(\Delta x, 0, L) \approx \left( \Delta x/\eta \Delta x_0 \right)^{2s_0} B_{ij}(\Delta x \hat{x}, 0, 0)$ holds for $\Delta x > \eta \Delta x_0$, where $\eta$ is some dimensionless constant, the power spectrum may be written as (see §2.3)

$$
2\pi B_{ij}^{(2D)}(k_L, L) \approx \int_{k_L < \eta \Delta x_0} d^2 x' B_{ij}(x', L)e^{-ik_L \cdot x'},
$$

for $\Delta x \to 0$, where $\eta \Delta x_0$ is the exponential factor in $f_1$ is approximately constant, such that $f_1$ is nearly independent of $k_L$. In addition, the lower limit in the second integral, $k_L \eta \Delta x_0$, may be taken to be zero (recall that $s_B > -1$). Thus, $2\pi B_{ij}^{(2D)}(k_L, L) \approx f_1(\eta \Delta x_0, L + k_L^{-2-2s} k_L^{-2} \hat{k}_L \Delta x_0),$ where we assumed that the integral in the second term converges. As $s_B > -1$, for $k_L \to 0$, we can neglect $f_1$ so that

$$
B_{ij}^{(2D)}(k_L, L) \propto k_L^{-2-2s}.
$$

The scaling of various other correlation functions, such as $\langle f_{\alpha} f_{\beta} \rangle$, may be deduced in the method shown above. For example, the arguments leading to the conclusion $B_{ij}(\Delta x, \Delta t, L) \propto \Delta x^{2s_0}$ for $\Delta x \to 0$ also imply that $B_{ij}(\Delta x, \Delta t, L) \propto \Delta t^{2s_0}$ for $\Delta t \to 0$.

4. THE FLUID COMPONENT, DOWNSTREAM

As explained in §3.3, the assumption that $L$ is the only relevant scale implies that the microscopic motion of the thermal particles (referred to as the fluid particles) is unimportant for the global solution. One option is that the current carried by the fluid is negligible, $j_F \ll j_B$, in which case these particles do not affect the EM fields. The second option is that the current carried by this population is considerable $j_F \sim j_B \sim j$ (i.e., that it scales as the electric current carried by the accelerated component), and that the thermal fluid is described by effective equations independent of the microscopic scales $\lambda_B$ and $R_{l,th} = \rho_0/eB$. The first option is unlikely since, as we show in §4.2, the fluid is expected to be highly conducting and thus probably carries negligible currents. In this section we study the implications of the second option, $j_F \sim j_B$. We assume that the appropriate effective degrees of freedom are those of a single fluid, since the skin depth $\lambda_B$ and thermal Larmor radius $R_{l,th}$ have an important role already in two-fluid equations.

In order to obtain equations that are independent of the downstream bulk velocity of the fluid, which cannot scale with $L$, we consider the flow in the downstream frame (where the bulk velocity vanishes). Note that since $s_i = 1$, the self-similarity relations, e.g., equation (26), hold both in the shock frame and in the downstream frame (and in any other frame, as long as we set $t = 0$ as the time when the shock was at $z = 0$). As the current carried by the fluid is not negligible, $j_F \sim j$, it must have a self-similar structure. We thus assume that all the dynamically important fluid quantities have a self-similar structure. In particular, the fluid velocity field $u$ (in the downstream frame) scales as

$$
u(x,t) = (x,t) \frac{e^{-i\hat{c} \cdot x}}{\hat{c}}.
$$

In §4.2 we show that for $L > R_{l,th}$ the fluid can be considered as infinitely conducting, i.e.,

$$
\frac{\epsilon}{c} = \frac{1}{\varepsilon} u \times B.
$$

This, together with Maxwell’s equation $\nabla \times E = -\varepsilon^{-1} \partial_t B$, implies that $L/T \sim u$ and thus $s_i = 1 - s_i = 0$. In other words, the velocity fluctuations scale trivially. In order to relate this to the magnetic field, we consider the force density applied to the fluid, $\sim c^{-1} n_i p_i u/T$. The force has a contribution from the pressure, $\sim PL$, where $P$ is the fluctuating part of the pressure, and from the EM fields, $\sim c^{-1} j_B \sim U_B/L$, where $U_B$ is the magnetic field energy density. If $U_B$ is not negligible compared to $P$, requiring
that the two force densities, \( \gamma^2 n_1 m_pc^2 \) and \( c^{-1} j_F B \sim U_B/L \), scale similarly, we find, using \( u \sim L \) and \( T \sim L^3 \), that \( s_B = 0 \). If \( U_B \) is negligible, \( P \gg U_B \), the force density is dominated by \( P/L \), and \( s_B \) cannot be constrained by considering the force density scaling. However, \( P \gg U_B \) is unlikely. Our basic assumption that the accelerated component plays an important role in the solution suggests that the fluctuations in the fluid are driven by it. As the interaction between the accelerated component and the fluid is mediated through the EM fields, this implies that they have a considerable contribution to the force density applied to the fluid. Moreover, the observational evidence that \( \epsilon_B \) is not very small, \( \epsilon_B \sim 10^{-2} \) to \( 10^{-1} \), does not leave much room for \( P \) (which is smaller than \( U_B/\epsilon_B \)) to be much larger than \( U_B \). We conclude therefore that

\[
s_u = s_B = 0, \tag{42}
\]

which, using equation (32), implies

\[
s_p = 1, \quad s_f = -4. \tag{43}
\]

Equations (42) and (43) imply that \( U_u \propto U_B \propto U_h \), where \( U_u, U_B, \) and \( U_h \) are the energy densities of the fluid fluctuations, magnetic fields, and accelerated particles, respectively. Implications regarding the correlation functions of fluid quantities such as \( \mathbf{u} \) can be drawn in analogy to \( \xi_B \) and \( \gamma \).

In § 4.1 below we study the scaling properties of the fluid equations of motion, under the simplifying assumptions of an ideal fluid and that corrections of order \( \epsilon_B \) can be neglected. A closed set of equations is derived for the fluid component, and it is demonstrated that self-similar (scalable) solutions of the equations exist only for \( s_u = s_B = 0 \). In § 4.2 we argue that under the assumption \( R_{1,\text{sh}} \ll L \), the fluid can be treated as infinitely conducting (as we show, the exact requirement is \( R_{1,\text{sh}}/L \ll \{e/T, L/e, T\} \)).

One consequence of the scaling indices derived above is a divergence of the energy associated with the self-similar components. A flat power-law energy spectrum of accelerated particles, \( dn/dE \propto E^{-\gamma} \) (eq. [36] with \( s_B = 0 \)), which is a natural consequence of the model and agrees with observations and with linear DSA theory, implies that the energy density diverges at large momenta. Similarly, a flat magnetic power spectrum of the form \( B_{\text{rms}}^2(k) \propto k_{\perp}^{-2} \) (eq. [39] with \( s_B = 0 \)) implies that the magnetic energy density diverges at large wavelengths (the same applies to the kinetic energy associated with the velocity fluctuations). Possible remedies for the divergence problem are outlined in § 4.3, but resolving the problem is beyond the scope of this paper.

### 4.1. Fluid Equations

For simplicity, we focus below on relativistic shocks. We comment at the end of this subsection on the application of the analysis to nonrelativistic shocks. As mentioned above, we consider the flow in the downstream frame.

The total electric current is \( \mathbf{j} = \mathbf{j}_F + \mathbf{j}_B \). The fluid current may be written as

\[
\mathbf{j}_F = \frac{c}{4\pi} \mathbf{\nabla} \times \mathbf{B} - \mathbf{j}_B - \frac{1}{4\pi} \partial_\| \mathbf{E}. \tag{44}
\]

In order to guarantee conservation of charge, we must include

\[
\mathbf{\nabla} \cdot \mathbf{E} = 4\pi (\rho_B + \rho_F) \tag{45}
\]

as an independent equation, where \( \rho_B \) is the electric charge density carried by the accelerated component,

\[
\rho_B = \sum_\alpha q_\alpha \int_{p > \epsilon B \mathbf{e} \mathbf{B}} d^3p f_\alpha, \tag{46}
\]

and \( \rho_F \) is the electric charge density carried by the fluid. Conservation of fluid energy and momentum is described by

\[
\partial_\| \mathbf{T}^{\mu\nu} = F^{\mu\nu} j_\|, \tag{47}
\]

where \( T^{\mu\nu} \) is the fluid energy momentum tensor, \( j_\| \) is the fluid four-current, and \( F^{\mu\nu} \) is the EM tensor. The fluid velocity \( \mathbf{u} \) is defined so that \( \mathbf{T}^{00} = 0 \) in a local frame moving with velocity \( \mathbf{u} \). These equations can be closed with the addition of an “equation of state” relating the different components of \( T^{\mu\nu} \) in the fluid rest frame.

As the energy density of the fluid must scale as \( L^\gamma \), the above equations are consistent only with a scaling in which the EM fields and \( T^{\mu\nu} \) both scale as \( L^\gamma \), implying \( s_u = s_B = 0 \). However, since the equations may contain terms that can be neglected, and the scaling laws are likely to be relevant to fluctuations in \( T^{\mu\nu} \) (imposed on a constant background) rather than to its average value, different scaling exponents may also be allowed. In order to examine this in more detail, we make the simplifying assumption of an ideal fluid, i.e., \( T^{00} \propto \delta^0 \) in the fluid rest frame. This is the case if the distribution of the fluid particles is approximately isotropic in the fluid’s local rest frame, which is reasonable as their Larmor radius is much smaller than the length scale for variations in the magnetic field. Under this assumption, equation (47) can be written as

\[
\partial_\| \left[ \gamma^2 (w_0 + w) \right] + \mathbf{\nabla} \cdot \left[ \gamma^2 (w_0 + w) \mathbf{u} \right] = \mathbf{E} \cdot j_\| + \partial_\| \mathbf{P} \tag{48}
\]

and

\[
\frac{\gamma^2 (w_0 + w)}{c^2} \left( \partial_\| + \mathbf{u} \cdot \mathbf{\nabla} \right) \mathbf{u} + \frac{\mathbf{E} \cdot j_\| + \partial_\| \mathbf{P}}{c^2} \mathbf{u} = -\mathbf{\nabla} \mathbf{P} + \rho_F \mathbf{E} + \frac{1}{c} j_\| \times \mathbf{B}, \tag{49}
\]

where \( P_0 + P \) and \( w_0 + w \) are the pressure and rest frame enthalpy per unit volume, respectively, assumed to consist of a constant part, \( P_0 \sim w_0 \sim \gamma^2 n_1 m_pc^2 \), and a fluctuating part, \( P, w \). Here, \( \gamma \equiv (1 - u^2/c^2)^{-1/2} \) is the Lorentz factor associated with the fluid’s velocity fluctuations.

The fluid equations must be simplified by identifying terms that are negligible, in order to determine the correct scaling laws. As explained above, we assume that the magnetic force density, \( c^{-1} |j_F \times \mathbf{B}| \sim L^{-2} B^2 \sim L^{-2} \epsilon_B n_1 m_pc^2 \), makes a considerable contribution to the momentum balance (eq. [49]). This implies \( P \lesssim \epsilon_B \gamma^2 n_1 m_pc^2 \approx \epsilon_B w_0 \). We assume that \( w/w_0 \sim P/P_0 \), which holds, for example, for any equation of state in which the enthalpy is a function of the pressure alone; in particular this is true for a relativistic equation of state, \( w_0 + w = 4(P_0 + P) \). We find \( w \lesssim \epsilon_B \gamma^2 n_1 m_pc^2 \approx \epsilon_B w_0 \). Comparing the first and the last terms in equation (49), combined with the above result, \( L/T \sim u \), we find that \( \gamma^2 u^2/c^2 \approx \epsilon_B \); thus, \( \gamma \approx 1 \), and the fluid fluctuations are nonrelativistic.

We thus find that the thermal energy \( P \) and the kinetic energy \( U_u \sim w_0 u^2 \) of the fluid fluctuations are related to the magnetic
energy by \( P, U_u \leq U_B \). Note that \( \mathbf{E} \cdot \mathbf{j}_F \sim uB^2/L \) [since \( \mathbf{E}j_F \sim (u/cB)(cB/L) \)]. Hence, equations (48) and (49) can be approximately written as

\[
\frac{w_0}{c^2} \nabla \cdot u = 0 \quad (50)
\]

and

\[
\frac{w_0}{c^2} \left( \partial_t u + \nabla \cdot \mathbf{u} \right) = -\nabla P + \rho_F \mathbf{E} + \frac{1}{c} \mathbf{j}_F \times \mathbf{B}. \quad (51)
\]

In this case, we may eliminate the pressure from the equations by taking the curl of equation (51), yielding

\[
\frac{w_0}{c^2} \left\{ \partial_t \nabla \cdot \mathbf{u} - \nabla \times \left[ \mathbf{u} \times (\nabla \times \mathbf{u}) \right] \right\} = \nabla \times \left( \rho_F \mathbf{E} + \frac{1}{c} \mathbf{j}_F \times \mathbf{B} \right). \quad (52)
\]

Equations (3), (28), (29), (41), (50), and (52) constitute a closed set of equations for the unknowns \( \mathbf{u}, f_p, \) (for momentum \( p > \xi eBL/c \), and \( \mathbf{B} \). As can be readily seen, these equations imply (and with a suitable \( s_B = s_u = 0 \)).

The above analysis is also applicable to nonrelativistic shocks. A few comments should, however, be added. In the nonrelativistic case, the electric field is much weaker than the magnetic field, and the forces acting on the proton component approximately cancel each other and thus lead to equation (41). Consider the proton fluid in a box of size \( x < L \) during a timescale \( T \), in which the magnetic and electric fields are approximately constant in space and time. The proton momentum inside the box is \( \sim \gamma_u m_p uL^2 \). The time derivative of the momentum in the box consists of the electric force, \( f_E \sim enEL^3 \sim en(L/cT)BL^3 \), the magnetic force, \( f_B \sim emu/cBL^3 \), and the flow of momentum into the box, \( f_F \sim L^2 \gamma_u m_p uLv_T \). The ratio of the momentum flow to the electric force is of order

\[
\frac{f_F}{f_E} \leq \frac{\gamma_u m_p uL^2 cT}{eBL} \sim \frac{R_{L_{th}} v_T}{L} \frac{L}{L} \ll 1. \quad (53)
\]

Hence, the only force that can balance \( f_E \) is \( f_B \). The forces are balanced only if \( \mathbf{E} = -c^{-1} \mathbf{u} \times \mathbf{B} \). If the forces are not balanced, the proton fluid is accelerated. The time it takes the velocity \( u \) to reach the scale \( L/T \) (for which \( \mathbf{E} \sim -c^{-1} \mathbf{u} \times \mathbf{B} \) is very short, of order \( \gamma_u m_p n(L/T) e^{-3/enEL^3} \sim (R_{L_{th}}/v_T)T \ll T \). Therefore, we may assume that the forces are approximately balanced at any given time and that the fluid is nearly infinitely conducting.

4.3. Diverging Energy

As explained in the beginning of §4, the vanishing of the magnetic scaling index, \( s_B \), implies logarithmic energy divergences. The divergence may be prevented if the scaling relations given in equations (26) and (40) are approximately, rather than accurate. For example, high-order terms, \( xL^2, 1/L^2, \ldots \), that were neglected in our analysis may introduce logarithmic corrections to the scaling relations. Alternatively, physical processes that were not included in the analysis, such as cooling, may limit the parameter range over which self-similarity is applicable, e.g., self-similarity may hold up to some cutoff momentum \( p_{max} \) or cutoff distance \( D_{max} \).

Modeling GRB external shocks (Waxman 1995) and SNR shocks (Zhang 1993) implies that in both systems the maximal energy of accelerated ions is limited by the age (size) of the system, whereas the maximal energy of the accelerated electrons is limited by energy losses. If the accelerated electrons play an important role in the evolution, self-similarity may break down at some scale and associated with electron cooling. Otherwise, the accelerated proton configuration may remain self-similar beyond the electron-cooling scale. In this case, the accelerated protons have a time-dependent energy cutoff and may (if their spectrum is sufficiently flat) modify the structure of the shock with time, rendering our assumption of a steady state inaccurate (self-similarity may still exist). We are not aware of observational constraints (on the shock thickness, say) that rule out this possibility.

5. Extensions of the Model

In §§1–4 we have motivated the self-similarity assumption based on observational evidence that is relevant directly to the downstream of strong, nonmagnetized shocks. It is possible, however, that in other related circumstances, in which power-law distributions of energetic particles are interacting with nonmagnetized MHD plasmas, similar scalable solutions arise. In §5.1 we examine the possibility that the hydromagnetic structure of the upstream is also self-similar. In §5.2 we discuss time-dependent, homogenized toy models, in which energetic particles with a power-law distribution in momentum are added to a plasma.

5.1. Upstream

It is expected that the accelerated component precursor generates turbulence in the upstream (Blandford & Eichler 1987). In the context of GRBs, it was recently claimed that the magnetic field energy density in the upstream is larger than that of the typical ISM by at least 3 orders of magnitude (Li & Waxman 2006), which suggests that high-energy particles generate waves upstream of the shock. If this is indeed the case, it is likely that higher momentum particles are important at larger distances from the shock, which suggests that the characteristic length scale for variations in the fields grows with the distance from the shock. Therefore, the EM structure in the upstream may also have a self-similar character, although we do not have as strong observational evidence for self-similarity in the upstream as we have for such behavior in the downstream.

Consider the conditions many skin depths upstream of the shock. The equations governing the accelerated component’s distribution function and the EM fields are the same as for the
downstream. In particular we expect the relation between the scaling indices to be

\[ s_{f_1} = s_{g_1} + 1, \quad s_{y_1} = -2s_{g_1} - 4, \]  

(54)

where the subscript 1 denotes the upstream. As explained in § 3.5, assuming that the accelerated component reaches the shock front, we expect the distribution function to have a power-law momentum dependence with a power-law index \( l_p = s_{f_1}/s_{g_1} \). From the same considerations, we must have in the upstream \( l_p = s_{y_1}/s_{y_1} \). We thus find that the scaling indices of the upstream and downstream are equal. In particular, for the expected value \( s_y = 0 \), the energy in the magnetic field does not decay as we go farther into the upstream. This result, which cannot be true for arbitrary distances in the upstream, is related to the diverging energy in accelerated particles. Note also that the value of \( \bar{B}_t \), which is constant according to our analysis in the upstream and far downstream regions, is expected to change considerably in the shock vicinity (distance of order \( l_{ad} \) where the self-similarity breaks down) and thus is probably different in the upstream and downstream.

Although the scaling relations are the same for the upstream and the downstream, the accelerated distribution function can have a qualitatively different spatial dependence. Note that for sufficiently high momentum the distribution is continuous across the shock. In particular, for a given momentum, the accelerated particle distribution function in the upstream is expected to decay exponentially (if the acceleration process is DSA; see, e.g., Kirk et al. 2000), while in the downstream it is expected to remain approximately constant. The analysis of the fluid component in the upstream is more complicated, as the magnitude of the (velocity, pressure) fluctuations and the amount of fluid heating are unknown.

### 5.2. Homogenous Configurations

In this subsection we consider the development of magnetic fields in homogenous configurations as a consequence of the interaction of the thermal fluid with the accelerated particles (for related simulations, see, e.g., Lucek & Bell 2000; Bell 2004). Such configurations are much simpler to simulate and analyze than space-dependent configurations and may give insights into dynamically important mechanisms.

If spatial inhomogeneity has an important role in the acceleration mechanism in collisionless shocks (as it does in the case of Fermi acceleration), it is likely that no acceleration takes place in homogenous configurations. In order to see the effects of an accelerated component on the dynamics, it would be interesting to perform homogenous simulations with an accelerated component (with a power-law distribution function) included in the initial conditions.

In the following, we assume a homogenous configuration with a relativistic accelerated component with some anisotropic distribution function \( f_a(\mathbf{p}) = g_a(\mu)p^p \), \( p > p_{\text{min}} \), included in the initial conditions, where \( \mu = \cos(\mathbf{p} \cdot \hat{z}) \) (the \( \hat{z} \)-axis is chosen to represent the direction of the shock normal; the cylindrical symmetry around this axis is present in planar nonmagnetized shocks, and there is no reason not to include it in homogenous simulations). We assume that the fluid component may be described by single fluid equations for which there is no inherent physical length scale. We assume initial neutrality, in the sense that any charge density (electric current) implied by \( f_a \) is compensated by an opposite charge density (current) carried by the fluid.

Quite generally, a nonisotropic distribution is unstable. The distribution function is expected to evolve from the initial unstable configuration to a stable (probably isotropic) configuration by the development of instabilities followed by dissipation. Particles with larger momentum will respond more slowly to the generated EM fields (which are limited in strength by the initial energy) and on longer length scales (perhaps of order their Larmor radius). Fluctuations of the EM fields on small scales are expected to decay with time (Gruzinov 2001a), while EM field generation at later times due to instabilities in larger momentum regimes is expected to occur on longer length scales. It is thus likely that the magnetic field length scale grows with time. Self-similarity is to be expected once the length scale of the magnetic field is much larger than the Larmor radius of particles having \( p_{\text{min}} \).

The time development may be affected by the choice of \( g_a \). In particular, the value of the bulk electric current carried by the initial population \( \int_0^1 \mu(g_a(\mu) - g_c(\mu)) \) and bulk velocity \( \int_0^1 \mu g_a(\mu) + g_c(\mu) \) may affect the nature of the instabilities involved. As we show in this subsection, the self-similarity assumption (as long as it is valid) allows determination of some physically interesting quantities without dependence on the precise instability mechanism.

In this case, we cannot assume a steady state. However, there is a full (3D) translational symmetry (homogeneity). We consider averaged quantities at fixed times. In particular, equations (15) and (16) are replaced by

\[
\langle f_a(\mathbf{p}, t) \rangle \equiv \lim_{r \to -\infty} \frac{1}{(4/3)\pi r^3} \int_{|x'| < r} d^3x' f_a(x + x', \mathbf{p}, t) 
\]

and

\[
B_{ij}(\Delta x, \Delta t, t) \equiv \lim_{r \to -\infty} \frac{1}{(4/3)\pi r^3} \int_{|x'| < r} d^3x' B_{ij}(x + x', t) \delta(x + \Delta x + x', t + \Delta t) 
\]

(56)

where these functions do not depend on \( x \) due to the homogeneity \( \bar{B} \) defined similarly.

We assume that a scaling property, following equations (23)–(25), is obtained at late times, i.e.,

\[
\bar{B}(L) = \left( \frac{L}{L_0} \right)^{s_B} \bar{B}(L_0),
\]

(57)

\[
\langle f_a(\mathbf{p}, L) \rangle = \left( \frac{L}{L_0} \right)^{s_f} \langle f_a(\mathbf{p}, L_0) \rangle \left( \frac{L}{L_0} \right)^{s_y},
\]

(58)

and

\[
B_{ij}(\Delta x, \Delta t, L) = \left( \frac{L}{L_0} \right)^{2s_B} B_{ij} \left( \frac{\Delta x}{L/L_0}, \frac{\Delta t}{(L/L_0)^2}, L_0 \right),
\]

(59)

with \( L \propto t^{1/s_t}, \quad T \propto t (s_t > 0) \).

An important difference with respect to the shock case arises from the absence of a finite bulk velocity, and therefore the assumption that time scales as distance no longer applies in general. We study cases in which the magnetic fields are stronger than electric fields and thus restrict \( s_t \geq 1 \). A value of \( s_t > 1 \) is consistent with Vlasov’s equation, equation (28), if we neglect the time derivative term and the electric field term, which is justified for \( s_t > 1 \).

The distribution function of the accelerated component is by definition finite at \( t \to 0 \) (also at \( L \to 0 \)). In analogy to equation (34),
we therefore have \( \langle f_n \rangle (p, t) \propto p^{n/\nu} \) for \( p \to \infty \). On the other hand, \( \langle f_n \rangle (p, t = 0) \propto p^{2/\nu} \), which implies that \( s_f / s_p = l_p \) (this also shows that a finite distribution must be a power law in order to achieve self-similarity). From Vlasov’s equation (and the expression for the electric current) we find, as for the shock case, that

\[
s_p = s_B + 1, \quad s_f = -2s_B - 4. \tag{60}
\]

Here it is useful to solve for the scaling indices in terms of the initial power-law index \( l_p \),

\[
s_B = -\frac{l_p + 4}{l_p + 2}, \quad s_p = -\frac{2}{l_p + 2}, \quad s_f = -\frac{2l_p}{l_p + 2}. \tag{61}
\]

Assuming that the fluid can be considered as infinitely conducting (see § 4.2), we have \( u \sim L/T \) so \( s_B = 1 - s \). Assuming also that the magnetic force is not negligible in the fluid momentum equation (51), we find \( s_p = s_B \) (this is reasonable as the energy source is the energetic component, and energy is transferred to the fluid through the magnetic fields). Together with equation (61), we have

\[
s_f = 1 - s_B = \frac{2(l_p + 3)}{l_p + 2}. \tag{62}
\]

In particular, this implies \( B \propto L^{x\nu} \propto t^{s_B/s_p} = t^{(l_p+4)/(2l_p+3)} \).

6. SOME GENERAL IMPLICATIONS

If self-similarity holds, it has important implications for any model of particle acceleration and/or field generation. In § 6.1 we show that a previously suggested model, which describes the flow in terms of merging current filaments (Medvedev et al. 2005) in a self-similar manner, must be generalized in order to be a self-consistent model. We show that after the appropriate generalization the model follows the scaling relations derived in § 5.2 and that the generalization significantly modifies the model’s predictions. In § 6.2 we discuss the relevance of our analysis to DSA.

6.1. Current Merger Model

Medvedev et al. (2005) suggest a model of coalescing electric current filaments in a quasi-2D configuration for describing the magnetic field evolution in the downstream of collisionless shocks. In this model there is no distinction between accelerated and thermal particles, so it does not hold information regarding the energy dependence of the particle distribution function. As the model is essentially homogenous and self-similar, it is interesting to compare it to the scalings derived in § 5.2.

This model assumes that EM instabilities in the shock lead to the formation of current filaments, which may be approximated as infinite in length and having equal electric currents oriented parallel to the flow, half of the currents positive (oriented along the flow) and half negative. The filaments are assumed to be distributed randomly in space. This simple model assumes that current filament coalescence evolves in discrete steps. In each step the filaments carrying similarly oriented currents are divided into neighboring pairs, which attract each other and merge. Two filaments with diameter \( D \), inertial mass per unit length \( \mu \), and current \( I \) unite to form a new filament of diameter \( d' = \sqrt{2D} \) (conserving area), mass per unit length \( \mu' = 2\mu \), and current \( I' = 2I \). In each step, the typical distance \( d \) between neighbor filaments with similarly oriented currents grows by a factor of \( \sqrt{2} \), \( d' = \sqrt{2d} \), as a consequence of the reduced filament number density. It is assumed that \( d \approx 2D \), which implies that the filaments roughly fill the space with their area. Initially, all the filaments are identical (except for the sign of the current) and are hence identical at any given time. The coalescence time, here denoted by \( \tau_{\text{coa}} \), is estimated by analyzing two isolated, parallel, infinite-current filaments attracting each other.

The fact that \( D \) and \( d \) change in each merger by the same factor implies that the configuration following each merger is identical to that before the merger, with rescaled parameters. In particular, the filament coalescence time, \( \tau_{\text{coa}} \propto \mu^{1/2} D/I \), and the maximal velocity during coalescence, \( v \propto I\mu/c \), scale as \( \tau_{\text{coa}} \propto \tau_{\text{coa}} \) and \( v' = \sqrt{2v} \), respectively. The fact that the filament velocity grows with time led the authors to the conclusion that velocities of order the speed of light will be reached. Once the filaments move with \( v \sim c \), the coalescence temporal behavior changes considerably.

This suggested merger model is problematic, as it implies that the magnetic field grows by a factor of \( \sqrt{2} \) in each step. For example, the magnetic field produced by a single filament at its edge, \( B = 4I/eD \), scales as \( B' = \sqrt{2}B \). More generally, the coalescence suggested is self-similar, with \( j'(x) = j(x/\sqrt{2}) \), implying \( B'(x) = \sqrt{2}B(x/\sqrt{2}) \). To see this, note that the current density carried by each filament, \( j = I/D^2 \), does not change in the coalescence, while the filament diameter and distance between filaments grow by \( \sqrt{2} \). The fact that the magnetic field grows by a constant factor in each step implies that after a few steps the magnetic energy density grows beyond equipartition, which is impossible. This can be corrected by changing appropriately the coalescence conditions.

Since the physical process of the merger is complicated, there is no simple way to determine the postmerger current directly. In general, one should therefore set the current of the merged filament to \( (\sqrt{2})^I I \), where \( \zeta \) is a parameter of the model (the mass per unit length is set to 2\( \mu \) since mass is conserved). In this case, the coalescence is self-similar, with \( j'(x) = (\sqrt{2})^{-1}j(x/\sqrt{2}) \), implying \( B'(x) = (\sqrt{2})^{-1}B(x/\sqrt{2}) \) and the magnetic field amplitude does not grow for \( \zeta \leq 1 \). The scaling of the filament coalescence time and maximal velocity changes to \( \tau_{\text{coa}} = (\sqrt{2})^{-1}\tau_{\text{coa}} \) and \( v' = (\sqrt{2})^{-1}v \), respectively.

The (necessary) generalization of the current merger model strongly influences the conclusions that can be drawn based on this model. The growth in length scale is only determined up to a free parameter \( \zeta \). In order for the magnetic field not to diverge with time, we must have \( \zeta \leq 1 \) (if the magnetic field amplitude is postulated to be constant in time, \( \zeta = 1 \)). This implies that the velocities do not grow (remain constant for \( \zeta = 1 \)) and therefore do not reach the speed of light.

The (corrected) scalings are compatible with the scalings for the fluid derived in § 5.2, with \( s_B = \zeta - 1, s_f = 2 - \zeta, s_p = \zeta - 1 \). Note that the self-similar analysis allows us to reach most of the physically interesting conclusions without making oversimplifying and model-specific assumptions, such as the calculation of the length of the time step using two isolated filaments in the current merger model.

It has been claimed (Milosavljevic & Nakar 2006a) that current filaments are unstable to kinklike modes and that the 2D configuration is therefore disrupted. We note that even if the instability is efficient, this does not rule out the model. Since the instabilities are derived within the framework of MHD, for which the above scalings hold, the perturbation’s growth rate scales as all other timescales, i.e., \( \tau_{\text{ins}} = (\sqrt{2})^{2-\zeta}\tau_{\text{ins}} \), where \( \tau_{\text{ins}} \) is the inverse growth rate. If initially \( \tau_{\text{ins}} > \tau_{\text{coa}} \) (which is not ruled out), the filaments merge before they are destroyed by the
instabilities, and this holds, i.e., \( \tau_{\text{ins}} > \tau_{\text{cos}} \), for all subsequent mergers.

### 6.2. Diffusive Shock Acceleration

DSA is the mechanism believed to be responsible for the production of nonthermal populations of charged particles in collisionless shocks. In this first-order Fermi acceleration process, particles gain energy by repeatedly bouncing between the converging upstream and downstream flows. In the lack of a self-consistent theory for the interaction between EM waves and accelerated particles in collisionless shocks, most progress was made under the “test particle” approximation. In this approach, the effects of the waves are modeled by some particle-scattering Ansatz, while the influence of the particles on the waves and on the shock structure are neglected (for reviews, see Drury 1983; Blandford & Eichler 1987; Malkov & Drury 2001).

In the case of nonrelativistic shocks, DSA has been successful in reproducing the power-law spectra observed in strong shocks, under very general assumptions regarding the scattering mechanism (Krymskii 1977; Axford et al. 1977; Bell 1978; Blandford & Ostriker 1978). The analysis of relativistic shocks is more complicated, mainly because the nonthermal particle distribution is not isotropic. The particle spectrum was calculated in relativistic shocks under various assumptions regarding the scattering mechanism, using Monte Carlo simulations (e.g., Ellison et al. 1990; Ostrowski 1991), and by numerical (Kirk & Schneider 1987; Gallant & Achterberg 1999; Vietri 2003; Blasi & Vietri 2005) or analytical (Keshet & Waxman 2005) study of the transport equations. In general, a power-law distribution function of accelerated particles is found, with indices that depend on details of the model and usually satisfy \( J_p < -4 \). In the case of ultra-relativistic shocks with isotropic, small-angle scattering, numerical studies have converged on a spectral index \( J_p \approx -4.22 \) (Bednarcz & Ostrowski 1998; Kirk et al. 2000; Achterberg et al. 2001), in agreement with GRB afterglow observations and with the analytic result \( J_p = 38/9 \) (Keshet & Waxman 2005). However, this value was demonstrated numerically (Ballard & Heavens 1992; Ostrowski 1993; Ellison & Double 2002, 2004; Meli & Quenby 2003b, 2003a; Bednarcz 2004; Niemiec & Ostrowski 2004; Lemoine & Pelletier 2003; Lemoine & Revenu 2006) and analytically (Keshet & Waxman 2005) to be sensitive to the scattering mechanism, which is poorly constrained.

In our analysis, the fluctuations in the fluid provide a natural scattering mechanism for the accelerated particles. Since the analysis reflects near equipartition between fluid fluctuations, accelerated particles, and magnetic fields, it does not naturally evoke (but does not rule out) a test particle approach. A spectral index \( J_p = -4 \) can be reconciled with our analysis only if we relax some of our assumptions. For example, if we assume that the magnetic field energy is constant with distance from the shock, \( J_p < -4 \) would imply that the currents carried by the accelerated component \( j_\parallel \) decrease faster than the total current \( j = \nabla \times B \), and thus the effect of the accelerated particles on the magnetic field is negligible at large distances (in this case, the test particle assumption is self-consistent). If, on the other hand, we assume that the magnetic field decreases with distance as \( s_B = -(L_p + 4)/(L_p + 2) \) (see § 3.5), a value \( J_p < -4 \) would contradict our assumptions regarding the self-similarity of the fluid component (for example, the magnetic force would become negligible in the momentum conservation equation, or the fluid would no longer be highly conductive and its currents would become negligible).

The assumption of self-similarity places constraints on some properties of DSA. As an illustration, consider the case of small-angle scattering, parameterized by a propagation-angle diffusion coefficient \( D_{\mu \nu} \). The stationary transport equation can be written as (Kirk & Schneider 1987)

\[
\gamma_i (v_i/c + \mu_i) \frac{\partial}{\partial z} \langle \mathbf{f}_i \rangle (p_i, \mu_i; z) = \frac{\partial}{\partial \mu_i} \left[ D_{\mu \nu}(\mu_i, \nu_i, z) \frac{\partial}{\partial \nu_i} \langle \mathbf{f}_i \rangle \right].
\]

(63)

where subscript \( i = 1, 2 \) denotes upstream or downstream parameters, respectively; \( \gamma_i = (1 - v_i^2/c^2)^{-1/2} \); and \( \mu = \cos(p_i \cdot \hat{z}) \). The momentum \( p_i \) (and therefore also \( \mu \)) is measured in the rest frame, whereas \( z \) is measured in the shock frame. The boundary conditions are \( \langle \mathbf{f}_i \rangle (p_i, \mu_i, -\infty) = 0 \), \( \langle \mathbf{f}_i \rangle (p_i, \mu_i, \infty) = \langle \mathbf{f}_i \rangle_{\infty} (p_i) \), and \( \langle \mathbf{f}_i \rangle (p_1, \mu_1, 0) = \langle \mathbf{f}_i \rangle (p_2, \mu_2, 0) \), where \( (p_1, \mu_1) \) and \( (p_2, \mu_2) \) are related through an appropriate Lorentz boost. The solution is unique up to the normalization of \( \langle \mathbf{f}_i \rangle \) (Kirk et al. 2000).

The small-angle scattering assumption requires that the Larmor radius of the particles \( R_p \) is much larger than the magnetic field length scale \( L \). In our framework, for any given momentum \( p \) this is true up to a limited distance from the shock (where \( L < pc/eB \)). Consistency of DSA theory thus requires that \( \langle \mathbf{f}_i \rangle (p_1, \mu_1, z) \) converge to its limiting value \( \langle \mathbf{f}_i \rangle_{\infty} (p_1) \) within this range.

In order to comply with the self-similar scaling of equation (24), we must require that

\[
\langle \mathbf{f}_i \rangle (p_i, \mu_i, z) = \left( \frac{z}{z_0} \right)^{J_p} \langle \mathbf{f}_i \rangle \left[ \frac{p_i}{(z/z_0)^{J_p}}, \mu_i, z_0 \right].
\]

(64)

This scaling can be reconciled with the transport equation (eq. [63]) if and only if the diffusion function scales as

\[
D_{\mu \nu}(p_i, \mu_i, z) = \frac{z_0}{z} D_{\mu \nu} \left[ \frac{p_i}{(z/z_0)^{J_p}}, \mu_i, z_0 \right].
\]

(65)

The boundary conditions are invariant to the above scaling (at the shock front \( p_1 \approx p_2 \)). Under this scaling, \( \langle \mathbf{f}_i \rangle (p_i, \mu, z) \propto p_i^{J_p} \) at \( z = \pm \infty \), where \( L_p = s_f / s_p \). The values of \( s_f \) and \( L_p \) are determined in a nontrivial way by the function \( D_{\mu \nu}(p_i, \mu_i, z) \). If \( s_f = 1 \), as is expected when the plasma is highly conductive, we may write equation (65) as

\[
D_{\mu \nu}(p_i, \mu_i, z) = z^{-1} D_{\mu \nu}(\mu_i, p_i/z).
\]

(66)

As an example of the above scaling, consider a highly relativistic particle scattered by weak magnetic fluctuations of correlation length \( L \ll R_i \). The particle trajectory may then be described as a random walk in \( \mu \), and the resulting diffusion function is \( D_{\mu \nu}(p_i, \mu, z) = D(\mu)LR_i^{-L} \). For \( L \ll z \) (and \( s_B = 0 \)), we obtain \( D_{\mu \nu}(p_i, \mu, z) \approx z^{-L} (p/z)^{-L} \).

### 7. DISCUSSION

We have studied the consequences of the assumption that the downstream flow of nonmagnetized, collisionless shocks is self-similar. This assumption is motivated by the existence of a strong magnetic field many skin depths downstream of the shock front, as inferred from observations of GRB afterglows and young SNRs. This suggests that the correlation length of magnetic field fluctuations, \( L \), diverges with the distance \( D \) from the shock front. Although our analysis was motivated by evidence for self-similarity in GRB afterglows, and possibly also in SNRs, it may apply to other systems with similar characteristics, such as the shocks involved in the large-scale structure of the universe.

As the EM fields and particle distributions at any given point fluctuate with time, a stationary shock structure may be described
by the averages (over planes parallel to the shock front) and correlation functions of the fluctuating quantities, which depend only on the distance from the shock (see § 2.3). The self-similarity assumption implies that $L \propto D$ and that the averages and correlation functions at different distances $D$ from the shock, corresponding to different values of $L$, are related to each other by simple scaling transformations (see § 3.2), e.g.,

$$\tilde{B}(L) = \left( \frac{L}{L_0} \right)^{s_B} \tilde{B}(L_0),$$

$$\langle f_\alpha \rangle(p, L) = \left( \frac{L}{L_0} \right)^{s_f} \left[ \frac{p}{(L/L_0)^{s_t}} \right] \langle f_\alpha \rangle(p, L_0),$$

$$B_{ij}(\Delta x, \Delta t, L) = \left( \frac{L}{L_0} \right)^{2s_B} B_{ij} \left[ \frac{\Delta x}{L/L_0} \right] \left[ \frac{\Delta t}{(L/L_0)^{s_t}} \right] L_0,$$

where the scaling exponent of the magnetic field must satisfy

$$-1 < s_B \leq 0.$$  

A schematic illustration of a self similar downstream configuration is presented in Figure 1.

We have argued (see § 3.3) that the similarity assumption suggests that the plasma may be approximately described as a combination of two self-similar components: a kinetic component of energetic particles and an MHD-like component representing thermal particles. We have argued that the energetic particles are likely to carry a significant fraction of the current and derived (using the Maxwell-Vlasov equations) the scaling of the characteristic time for variations in the physical quantities, the scaling of the characteristic particle momentum, and the scaling of the particle distribution function normalization:

$$s_t = 1, \quad s_p = s_B + 1, \quad s_f = -4 - 2s_B.$$  

These relations imply that the characteristic Larmor radius of energetic particles scales as $L$ and that the energy density of energetic particles in any momentum interval, with the interval scaling as $L^s$, scales as the magnetic field energy density $\propto L^{2s_f}$. We have then shown (see § 3.5) that (under the assumption that accelerated particles reach the shock front and/or are advected to the downstream) the spectrum of accelerated particles is

$$dn/dE \propto E^{-2/(s_p+1)}$$

and that the scaling of the magnetic correlation function (for $\Delta x \to \infty$) is

$$\langle B_i(x)B_j(x + \Delta x) \rangle \propto \Delta x^{2s_B}.$$  

Similar conclusions can be drawn regarding various other correlation functions.

The thermal particles were discussed in § 4. In § 4.2 we have argued that the thermal component may be considered as an infinitely conducting fluid. We have shown that in this case $s_B = 0$ and the scalings are completely determined, e.g., $dn/dE \propto E^{-2}$ and $B \propto D^{s_f}$, with possible logarithmic corrections. We have derived in § 4.1 a closed set of equations for the fluid component, under the simplifying assumptions of an ideal fluid and neglecting corrections of order $\varepsilon_B$.

The self-similarity assumption and its implications do not hold for arbitrarily large distances and high particle momenta for which new physical processes that were not taken into account in our analysis come into play. For example, our assumption of an infinite planar shock is invalid at distance scales of order the blast wave radius, and the use of the Vlasov equation is invalid for high momenta for which radiative effects cannot be ignored. Upper cutoffs to the distance scale and the momentum range described by the self-similar solution, or logarithmic corrections to this solution, are possible remedies of the energy divergence when $s_B = 0$, as discussed in § 4.3.

If self-similarity holds, it has important implications for any model of particle acceleration and/or field generation. In § 4.2 we have shown that the velocity-angle diffusion coefficient for small-angle scattering in diffusive shock acceleration models must satisfy $D_{\mu\nu}(p, D) = D^{-2} \tilde{D}_{\mu\nu}(p/D)$ (where $p$ is the particle momentum).

In § 4.1 we have discussed the model suggested by Medvedev et al. (2005) for the generation of a large-scale magnetic field through hierarchical merger of current filaments. We have shown that in order to avoid a diverging magnetic field, the model must be generalized by allowing a more general scaling of the electric current of merged filaments. The generalized model follows the scaling laws we have derived in § 5.2. This implies that the self-similar analysis allows us to reach most of the physically interesting conclusions without making oversimplifying and model-specific assumptions (such as those related to the calculation of the merger time). The predictions of the generalized model differ substantially from those of the original model. It predicts, e.g., a scale-independent merging velocity, rather than an increasing velocity approaching the speed of light (this is valid for a nondecaying magnetic field; for a decaying field, the velocity decreases with scale). Finally, we have shown that the instability of current filaments (which was pointed out by Milosavljevic & Nakar 2006a) does not necessarily imply that the current merger model is not viable.

In § 5.2 we have pointed out that the self-similarity assumptions may be tested through their predictions for the evolution of homogeneous (time dependent) plasmas, which may be accessible to direct numerical simulations. An inclusion (at the initial conditions) of an anisotropic, power-law spectrum of high-energy particles, $dn/dE \propto E^{2-s_f}$, in homogeneous simulations may lead to a self-similar evolution in time, described by the scaling exponents given in equation (61). Assuming that the fluid can be considered as infinitely conductive, we predict that the magnetic field evolution follows $B \propto \tau^{-(s_p-4)/2(s_p+3)}$.

Our self-similar model predicts the scaling of all physical quantities related to the accelerated particles, EM fields, and fluid fluctuations. In particular, the particle spectrum is related to the magnetic field scaling: an accelerated particle distribution with a spectral index $l_p < -4$ indicates a decay of the magnetic field amplitude with distance from the shock according to $D^{-4(s_p+4)/2(s_p+3)}$. Such a decay of the magnetic field may be detectable, for example, through the spatial dependence of synchrotron emission from nonthermal electrons gyrating in the downstream magnetic fields.

Finally, we should emphasize two major open questions related to our analysis that need to be resolved. First, the validity of the description of the thermal component in terms of single-fluid MHD equations requires verification. Second, the diverging energy in accelerated particles that results from the distribution $n(E) \propto E^{-2}$ indicates that our assumptions cannot be valid for arbitrarily high momentum (see discussion and possible remedies at the end of § 3.5). A similar divergence is identified in the energy of magnetic fields and fluid fluctuations.

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APPENDIX

COLLISIONLESS SHOCKS IN 3D PARTICLE-IN-CELL SIMULATIONS

In recent years, particle-in-cell (PIC) simulations have provided important preliminary clues to the nature of collisionless shocks. Fully 3D PIC simulations have begun to explore aspects of such shocks in nonhomogeneous (Nishikawa et al. 2003; Frederiksen et al. 2004; Spitkovsky 2005) and homogeneous (Silva et al. 2003; Jaroschek et al. 2004; Romanov et al. 2004) flows. The main conclusions relevant for collisionless shocks are

1. Confirmation that shocks in pair plasma (with no highly accelerated particles) are mediated by EM, two-stream (Weibel-like) instabilities if the upstream is sufficiently nonmagnetized (magnetic energy less than ~1% of the kinetic energy; Spitkovsky 2005) and cold. In interpenetrating shells, this instability produces filaments in density (if $\Gamma \gtrsim 5$) and in electric current (Jaroschek et al. 2005), initially with less than or approximately the skin depth thickness, and transverse (perpendicular to the flow) magnetic fields. Electrons in ion-electron plasma behave similarly (Frederiksen et al. 2004).

2. Preliminary evidence for the existence of relativistic, collisionless shocks in a nonmagnetized pair plasma. Spitkovsky (2005) resolves a shock-like behavior in a 3D simulation of interpenetrating pair-plasma shells, each with Lorentz factor $\Gamma = 15$. A density jump in approximate agreement with the Rankine-Hugoniot adiabat extends over $\sim 70l_{sd}$ and coincides with a region of near-equipartition magnetic fields. Here, $l_{sd} = c/\omega_p$ and $\omega_p = (4\pi n e^2/m_p)^{1/2}$ are the skin depth and plasma frequency, respectively, of simulated particles of mass $m_p$, number density $n$, and Lorentz factor $\Gamma$.

3. Possible indications for self-similarity. In interpenetrating plasma shells, the filaments merge after saturation and grow in a hierarchical process, maintaining a configuration of similar features on gradually larger scales, with magnetic fields decaying slowly and possibly saturating at $\epsilon_B \lesssim 1\%$ (in pair plasmas; Silva et al. 2003; Medvedev et al. 2005). Frederiksen et al. (2004, Fig. 3) provide evidence that the power spectrum of transverse magnetic fields is roughly a power law, $P_k \propto k^{-\alpha}$ with $\alpha \approx 2.3-3.0$, extending to gradually larger scales.

Present 3D PIC simulations are limited to small simulation box volumes, $V < 10^3 l_{sd}^3$, and to short simulation durations, $T < 10^3 \omega_p^{-1}$. Here, $\omega_p$ and $l_{sd}$ corresponds to the species with fastest response; simulating an ion-electron plasma implies much smaller volumes and durations when written in terms of the ion parameters. In order to resolve any effects associated with the ions, an effective, small proton to electron mass ratio, $m_p/m_e \lesssim 20$ (with present computational resources) must be used, and the preliminary results thus obtained are not easily extrapolated to more realistic mass ratios. These constraints limit the relevance of any conclusion drawn from such simulations and may even undermine their reliability:

1. The length of the simulation box parallel to the flow is too small in most cases to resolve a shock, even in pair plasma (with the possible exception of Spitkovsky 2005). The configuration found in most simulations thus represents merely a transient stage in the formation of a shock, which may bare little relevance to its steady state. Moreover, the short length of the box suppresses longitudinal modes of long wavelengths, thus distorting the plasma evolution by effectively reducing it to 2D. For example, an exponential decay of the magnetic energy (after growth saturation) abruptly stops in simulations (Silva et al. 2003; Jaroschek et al. 2005), possibly due to this effect.

2. The small size of the simulation box (approximately tens of skin depths) perpendicular to the flow places an artificial cutoff on transverse modes with long wavelengths. Simulations indicate that the transverse scale of the most energetic modes grows rapidly, reaching the box-size cutoff at early stages of the simulation (Silva et al. 2003, Fig. 3; Frederiksen et al. 2004, Figs 3 and 4). After this occurs, the simulations are strongly affected by the boundary conditions, and the simulated evolution is probably highly distorted.

3. The Larmor radius of particles of Lorentz factor $\gamma$ is $R_l = l_{sd}(\gamma^2 - 1)/2(\Gamma - 1)^{1/2}$, where $\Gamma$ is the average Lorentz factor. A simulation box tens of skin depths long is therefore not sufficiently large to simulate particle magnetization when the ratio between magnetic and thermal energy densities is $\epsilon_B \lesssim 1\%$. Moreover, the box is much too small to resolve acceleration of particles to high energies, so any Fermi-like acceleration process is suppressed. Electrons are observed to be accelerated in ion-electron simulations (Nishikawa et al. 2003; Hededal et al. 2004), but to energies smaller than $\Gamma m_p c^2$.

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