RELATIVE ISOPERIMETRIC INEQUALITY IN THE PLANE: THE ANISOTROPIC CASE

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Abstract. In this paper we prove a relative isoperimetric inequality in the plane, when the perimeter is defined with respect to a convex, positively homogeneous function of degree one $H : \mathbb{R}^2 \to [0, +\infty]$. Under suitable assumptions on $\Omega$ and $H$, we also characterize the minimizers.

1. Introduction

Let $\Omega$ be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. The classical relative isoperimetric inequality states that

$$P^2(E; \Omega) \geq C \min\{|E|, |\Omega \setminus E|\},$$

for any measurable subset $E$ of $\Omega$ (see, for example, [13],[16],[8]). Here $|E|$ is the Lebesgue measure of $E$, and $P(E; \Omega)$ is the usual perimeter in $\Omega$. Being $P(E; \Omega) = P(\Omega \setminus E; \Omega)$, the inequality (1.1) can be written as

$$P^2(E; \Omega) \geq C|E|,$$

for any $E \subset \Omega$ such that $|E| \leq |\Omega|/2$.

Natural questions related to the inequality (1.2) are the following: finding the optimal constant

$$C(\Omega) = \inf \left\{ \frac{P^2(E; \Omega)}{|E|} : 0 < |E| \leq \frac{|\Omega|}{2}, E \subseteq \Omega \right\},$$

proving that it is attained, and characterizing the minimizers.

First results in this direction can be found in [8] or [16], where it is proved that $C(\Omega) = \frac{8}{\pi}$ when $\Omega$ is the unit disk in $\mathbb{R}^2$, and it is attained at a semicircle. More generally, in [10] the author proves that for an open convex set $\Omega$ of the plane, $C(\Omega)$ is actually a minimum. Moreover, there exists a convex minimizer of (1.3) whose measure equals $\frac{|\Omega|}{2}$, and any minimizer $E$ has the following properties:

(a) $\partial E \cap \Omega$ is either a circular arc or a straight segment. Moreover, neither $E$ nor $\Omega \setminus E$ is a circle.

(b) Let $T$ be one of the terminal points of $\partial E \cap \Omega$. Then $T$ is a regular point of $\partial \Omega$ and $\partial E \cap \Omega$ is orthogonal to $\partial \Omega$. As a consequence, either $E$ or $\Omega \setminus E$ is convex.
(c) If \(|E| < \frac{|\Omega|}{2}\), then \(E\) is a circular sector having sides on \(\partial \Omega\). In such a case, there exists another minimizer \(F\) which is a sector with sides on \(\partial \Omega\), having the same vertex as \(E\), such that \(|F| = \frac{|\Omega|}{2}|\).

Furthermore, in \([10]\) \(C(\Omega)\) is explicitly computed under the additional assumption that \(\Omega\) is symmetric about a point and also in special cases of convex domains. If \(r(\Omega)\) is the inradius of \(\Omega\), then
\[
C(\Omega) = \frac{8r^2(\Omega)}{|\Omega|}.
\]

We refer the reader to \([12]\) for some extremal problems involving \(C(\Omega)\).

The purpose of the present paper is to find analogous results when the Euclidean perimeter is replaced by an “anisotropic” perimeter. More precisely, if \(H\) is an arbitrary norm on \(\mathbb{R}^2\), the perimeter with respect to \(H\) for a set \(E \subseteq \mathbb{R}^2\) with sufficiently smooth boundary is given by
\[
P_H(E; \Omega) = \int_{\partial E \cap \Omega} H(\nu_E) \, dH^1,
\]
where \(H^1\) is the 1-dimensional Hausdorff measure and \(\nu_E\) is the unit outer normal to \(E\) (see Section 2 for the precise definition).

We recall that in this setting it is well-known that the following isoperimetric inequality holds for any \(E \subseteq \mathbb{R}^2\)
\[
P_H^2(E; \mathbb{R}^2) \geq 4|W||E|, \tag{1.4}
\]
where \(W = \{(x, y) : H^\circ(x, y) < 1\}\) and \(H^\circ\) is polar to \(H\) (see \([9, 11, 14, 2, 19]\)). Moreover, the equality in (1.4) holds if and only if \(E\) is homothetic to \(W\). We refer to \(W\) as the Wulff shape.

Our results can be summarized as follows. Under suitable assumptions on \(H\), we first show that an anisotropic relative isoperimetric inequality holds. That is: when \(\Omega\) is an open, bounded connected set of \(\mathbb{R}^2\), with Lipschitz boundary, then there exists \(C_H(\Omega) > 0\) such that
\[
C_H(\Omega) = \inf \left\{ \frac{P_H^2(E; \Omega)}{|E|} : 0 < |E| \leq \frac{|\Omega|}{2}, \ E \subseteq \Omega \right\}. \tag{1.5}
\]
Then we prove that, for a convex set \(\Omega\), \(C_H(\Omega)\) is actually a minimum, there exists a convex minimizer of (1.5) whose measure equals \(\frac{|\Omega|}{2}\), and any minimizer \(E\) has the following properties:

\((\alpha)\) \(\partial E \cap \Omega\) is either homothetic to a Wulff arc (that is an arc of \(\partial W\)) or a straight segment. Moreover, neither \(E\) nor \(\Omega \setminus E\) is homothetic to a Wulff shape.

\((\beta)\) Let \(T\) be one of the terminal points of \(\partial E \cap \Omega\). Then \(T\) is a regular point of \(\partial \Omega\) and \(\partial E \cap \Omega\) verifies the following contact angle condition with \(\partial \Omega\):
\[
\langle \nabla H(\nu_E), \nu_\Omega \rangle = 0,
\]
where \(\nu_\Omega\) and \(\nu_E\) are the usual unit outer normal vectors to \(\partial \Omega\) and \(\partial E\) at \(T\) respectively.
(γ) If $|E| < \frac{|\Omega|}{2}$, then $E$ is homothetic to a Wulff sector (see section 2 for the precise definition) having sides on $\partial \Omega$. In such a case, there exists another minimizer $F$ which is a sector with sides on $\partial \Omega$, having the same vertex as $E$, such that $|F| = \frac{|\Omega|}{2}$.

Furthermore, we explicitly compute $C_H(\Omega)$ under the additional assumption that $\Omega$ is symmetric about a point. Indeed,

$$C_H(\Omega) = \frac{8r_H^2(\Omega)}{|\Omega|},$$

where $r_H(\Omega)$ is defined in Theorem 3.6. For example, if $\Omega$ is obtained by a rotation of $\frac{\pi}{2}$ of a level set of $H$, that is $\Omega = \{(x,y) : H(-y,x) < r\}$, then

$$C_H(\Omega) = \frac{8r^2}{|\Omega|} = \frac{8}{\kappa_H},$$

where $\kappa_H = |\{(x,y) : H(x,y) < 1\}|$. We recover immediately the classical result $C_H = \frac{8}{\pi}$ when $H$ is the Euclidean norm.

The paper is organized as follows. In Section 2 we give the precise definitions of anisotropic perimeter and some basic properties. In Section 3 we prove the main result. A fundamental argument is to study problem (1.5) by considering the area $|E|$ fixed.

Finally, we give some examples.

2. Notation and preliminaries

Let $H : \mathbb{R}^2 \to [0, +\infty[$ be a $C^2(\mathbb{R}^2 \setminus \{0\})$ function such that $H^2(\xi)$ is strictly convex and

$$H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^2, \, \forall t \in \mathbb{R}.$$  \hspace{1cm} (2.1)

Moreover, suppose that there exist two positive constants $\alpha \leq \beta$ such that

$$\alpha|\xi| \leq H(\xi) \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^2.$$  \hspace{1cm} (2.2)

We define the polar function $H^o : \mathbb{R}^2 \to [0, +\infty[$ of $H$ as

$$H^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{H(\xi)},$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbb{R}^2$. It is easy to verify that also $H^o$ is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$H(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{H^o(\xi)}.$$

The set

$$W = \{\xi \in \mathbb{R}^2 : H^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin.

We will call Wulff sector with vertex at the origin the set $A \cap W$, where $A$ is an open cone with vertex at $(0,0)$. 
The following properties of $H$ and $H^o$ hold true (see for example [6]):

\begin{align}
(2.3) & \quad H(\nabla H^o(\xi)) = H^o(\nabla H(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \\
(2.4) & \quad H^o(\xi)\nabla H(\nabla H^o(\xi)) = H(\xi)\nabla H^o(\nabla H(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}.
\end{align}

**Definition 2.1** (Anisotropic relative perimeter). Let $\Omega$ be an open bounded set of $\mathbb{R}^2$. In [3], the perimeter of $F \subset \mathbb{R}^2$ in $\Omega$ with respect to $H$ is defined as the quantity

\[ P_H(F; \Omega) = \sup \left\{ \int_F \operatorname{div} \sigma dx : \sigma \in C^1_0(\Omega; \mathbb{R}^2), H^o(\sigma) \leq 1 \right\}. \]

The equality

\[ P_H(F; \Omega) = \int_{\partial^* F} H(\nu_F) dH^1 \]

holds, where $\partial^* F$ is the reduced boundary of $F$ and $\nu_F$ is the unit outer normal to $F$ (see [3]).

The anisotropic perimeter of a set $F$ is finite if and only if the usual Euclidean perimeter $P(F; \Omega)$

\[ P(F; \Omega) = \sup \left\{ \int_F \operatorname{div} \sigma dx : \sigma \in C^1_0(\Omega; \mathbb{R}^N), |\sigma| \leq 1 \right\}. \]

is finite. Indeed, by properties (2.1) and (2.2) we have that

\[ \frac{1}{\beta} |\xi| \leq H^o(\xi) \leq \frac{1}{\alpha} |\xi|, \]

and then

\[ (2.5) \quad \alpha P(E; \Omega) \leq P_H(E; \Omega) \leq \beta P(E; \Omega). \]

**Remark 2.1.** We observe that when $\partial E \cap \Omega$ is the image of a smooth curve $\gamma(t) = (x(t), y(t)), t \in [a, b]$, then $P_H(E; \Omega)$ coincides with the value

\[ (2.6) \quad \mathcal{L}_H(\gamma) = \int_a^b H(-y'(t), x'(t)) dt. \]

By regularity of $H$, the curve joining two points $P_0$ and $P_1$ which minimizes $\mathcal{L}_H$ is the straight segment $P_0P_1$. This can be shown by classical argument of Calculus of Variations. We consider, for sake of simplicity, the curves $\gamma(t) = (t, u(t))$. Denoting by $\mathcal{L}_H(u) = \mathcal{L}_H(\gamma)$, the minimum of the problem

\[ \min \{ \mathcal{L}_H(u), \quad u(a) = u_a, \ u(b) = u_b, \} \]

is the solution to

\[ \begin{cases}
  \frac{d}{dt}H_x(-u'(t), 1) = 0, \\
  u(a) = u_a, \ u(b) = u_b.
\end{cases} \]

Such solution is the linear function passing through $P_0 = (a, u_a)$ and $P_1 = (b, u_b)$. 
**Definition 2.2** (Anisotropic curvature ([1], [5])). Let $F \subset \mathbb{R}^2$ be a bounded open set with smooth boundary, $\nu_F(x, y)$ the unit outer normal at $(x, y) \in \partial F$, in the usual Euclidean sense. Let $u$ be a $C^2$ function such that $F = \{ u > 0 \}$, $\partial F = \{ u = 0 \}$ and $\nabla u \neq (0, 0)$ on $\partial F$. Hence, $\nu_F = -\frac{\nabla u}{|\nabla u|}$ on $\partial F$. The anisotropic curvature at $(x, y) \in \partial F$ is

$$ k_H(x, y) = \frac{\nabla H(\nu_F(x, y))}{|\nabla H(\nu_F(x, y))|}, \quad (x, y) \in \partial F, $$

and, by the properties of $H$,

$$ H^o(n_F) = 1. $$

The anisotropic curvature $k_H$ of $\partial F$ is

$$ k_H(x, y) = \operatorname{div} n_F(x, y) = \operatorname{div} \left[ \nabla H \left( -\frac{\nabla u}{|\nabla u|} \right) \right], \quad (x, y) \in \partial F. $$

Let $(x_0, y_0) \in \partial F$. Without loss of generality, we can locally describe $\partial F$ with a $C^2$ function $v: [x_0 - \delta, x_0 + \delta] \mapsto \mathbb{R}$, that is $F$ is the epigraph of $v$ near $(x_0, y_0) = (x_0, v(x_0))$. By properties of $H$, the anisotropic curvature $k_H(x_0, y_0)$ of $\partial F$ at $(x_0, y_0)$ can be written as

$$ k_H(x_0, y_0) = -\frac{d}{dt} H_x(-v'(t), 1) \bigg|_{t=x_0}. $$

**Remark 2.2.** We stress that if $F$ is homothetic to the Wulff shape $W$ and centered at $(x_0, y_0)$, the anisotropic outer normal at $(x, y) \in \partial F$ has the direction of $(x - x_0, y - y_0)$. Indeed, being $F = \{(x, y): H^o(x-x_0, y-y_0) = \lambda \}$, for some positive $\lambda$, by property (2.4) it follows that

$$ n_F(x, y) = \nabla H(\nabla H^o(x-x_0, y-y_0)) = \frac{1}{\lambda} (x-x_0, y-y_0). $$

See Figure 1 for an example.

**Figure 1.** Here $H(x, y) = (x^2/a^2 + y^2/b^2)^{1/2}$ and $H^o(x, y) = (a^2x^2 + b^2y^2)^{1/2}$. When $a \neq b$, the usual and the anisotropic outer normal are, in general, different.
Remark 2.3. Let be $F = \frac{1}{\lambda} W$, with $\lambda > 0$. It is not difficult to show (see, for instance, [4], [6]) that the anisotropic curvature at $(x, y) \in \partial F$ is
$$k_H(x, y) = \lambda.$$

3. An anisotropic relative isoperimetric inequality

Theorem 3.1. Let $\Omega$ be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Then an anisotropic relative isoperimetric inequality holds. Namely, there exists a constant $C > 0$ such that
\begin{equation}
P_H^2(E; \Omega) \geq C \min\{|E|, |\Omega \setminus E|\},
\end{equation}
for every measurable set $E \subset \Omega$.

Proof. The hypotheses made on $\Omega$ guarantee that a relative isoperimetric inequality holds when we consider the usual perimeter $P(E; \Omega)$ (see [13], [16], [10]). Hence the inequality (3.1) follows immediately from property (2.5). □

Our aim is to study, for $\Omega$ bounded and convex, the best constant in the inequality (3.1), that is to find the infimum
\begin{equation}
C_H = \inf \left\{ Q(E) : 0 < |E| \leq \frac{1}{2} |\Omega|, E \subset \Omega \right\},
\end{equation}
where
$$Q(E) = \frac{P_H^2(E; \Omega)}{|E|},$$
to prove that $C_H$ is actually a minimum, and to characterize the minimizers. Furthermore, we will find the explicit value of $C_H$ in some special case.

If $E$ is a minimizer of (3.2), then $E$ solves also the following problem under volume constraint:
$$\min\{P_H(F; \Omega), F \subset \Omega \text{ and } |F| = |E|\}.$$
The following result characterizes the minimizers of the above problem.

Theorem 3.2. Let $\Omega$ be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Then there exists a minimizer $E$ of the problem
\begin{equation}
\min\{P_H(F; \Omega), F \subset \Omega \text{ and } |F| = k\},
\end{equation}
with $0 < k \leq |\Omega|/2$ fixed. Moreover, $\partial E \cap \Omega$ is either homothetic to an arc of $\partial W$, or a straight segment. Hence a minimizer of (3.2), if exists, has the same characterization.

Proof. The existence of a minimizer of (3.3) follows by the lower semicontinuity of $P_H$ (see [3]) using standard methods of Calculus of Variations.

To prove the result, we proceed by steps.

Step 1. First, we show that a minimizer $E$ is locally homothetic to an arc of $\partial W$, or a straight segment.

Fixed $(x_0, y_0) \in \partial E \cap \Omega$, we can locally describe $\partial E \cap \Omega$ with a $C^2$ function $u$ (see [1], [2], [11], [17]). That is, without loss of generality, there exists a rectangle $R = [x_0 -$
\[ \delta, x_0 + \delta \times I \text{ where } E \cap R \text{ is the epigraph of } u : x_0 - \delta, x_0 + \delta \rightarrow I. \] Moreover, there exists \( \lambda \) such that \( u \) is the minimum of the functional

\[ J(v) = \int_{x_0 - \delta}^{x_0 + \delta} H(-v'(t), 1) dt + \lambda \int_{x_0 - \delta}^{x_0 + \delta} v(t) dt, \]

with boundary conditions \( v(x_0 + \delta) = u(x_0 + \delta) \) and \( v(x_0 - \delta) = u(x_0 - \delta) \). The corresponding Euler equation associated to \( J \) is

\[
\begin{align*}
\frac{d}{dt} H_x(-v'(t), 1) &= \lambda, \quad t \in ]x_0 - \delta, x_0 + \delta[, \\
v(x_0 \pm \delta) &= u(x_0 \pm \delta).
\end{align*}
\]

If \( \lambda = 0 \), there exists a linear function \( u_0 \) which solves \( (3.4) \). If \( \lambda \neq 0 \), by Remark 2.3, the function \( u_\lambda(t) \), which describes \( \frac{1}{\lambda} \partial W \) (up to translation) near \( x_0 \), is a solution of \( (3.4) \).

On the other hand, for any \( \lambda \in I \), the regularity on \( H \) guarantees that the functional \( J \) is strictly convex. Hence, \( u_\lambda = u \) is the unique solution of \( (3.4) \) (see also [5, 17]).

**Step 2.** Now we show that the minimizer has the same anisotropic curvature at any point.

Let us take \( (x_1, y_1) \) and \( (x_2, y_2) \) in \( \partial E \cap \Omega \). As in the step 1, let us consider \( u_1 : B_1 = ]x_1 - \delta_1, x_1 + \delta_1[ \rightarrow I_1 \) and \( u_2 : B_2 = ]x_2 - \delta_2, x_2 + \delta_2[ \rightarrow I_2 \) two functions which locally describe \( \partial E \cap \Omega \). Moreover, there exist \( \lambda_1 \) and \( \lambda_2 \) such that \( u_i \), for \( i = 1, 2 \), minimizes the functional

\[ J_i(v) = \int_{B_i} H(-v'(t), 1) dt + \lambda_i \int_{B_i} v(t) dt, \quad i = 1, 2, \]

with boundary conditions \( v(x_i \pm \delta_i) = u_i(x_i \pm \delta_i) \). We claim that \( \lambda_1 = \lambda_2 \). This can be shown by arguing as in [15], Theorem 2. We briefly describe the idea, and we refer to the quoted paper for the precise details.

We assume that \( 0 \leq \lambda_1 < \lambda_2 \). A similar argument can be repeated in the other cases.

For every \( \lambda \in ]\lambda_1, \lambda_2[ \) there exists a function \( u_{\rho,i} \) which is the unique minimizer to

\[ \int_{B_{\rho,i}} H(-v'(t), 1) dt + \lambda \int_{B_{\rho,i}} v(t) dt, \quad i = 1, 2, \]

where \( 0 < \rho < \min_i \delta_i \) and \( B_{\rho,i} = ]x_i - \rho, x_i + \rho[, \) with boundary conditions \( v(x_i \pm \rho) = u_i(x_i \pm \rho) \).

By convexity of \( H \), a comparison argument shows that \( u_{\rho,1} \leq u_1 \) in \( B_{\rho,1} \), and \( u_{\rho,2} \geq u_2 \) in \( B_{\rho,2} \). Defining

\[ V_{\rho,i} = \int_{B_{\rho,i}} |u_i - u_{\rho,i}| dt, \]

it is possible to prove that there exist two suitable positive numbers \( r_1 \) and \( r_2 \) such that

\[
\begin{align*}
V_{r_1,1} &= V_{r_2,2}.
\end{align*}
\]

This implies that, defining the set \( E^* \) as

\[ E^* = [E \cup (\text{epi } u_{r_1,1} \cap C_1)] \setminus [C_2 \cap (E \setminus \text{epi } u_{r_2,2})], \]

where \( C_i = B_i \times I_i \), we have that \( |E^*| = |E| \).
Finally, we get that $E \Delta E^* \in \Omega$ and
\[ P_H(E; \Omega) - P_H(E^*; \Omega) = \]
\[ = \int_{B_{r,1}} H(-u'_{1,1})dt + \int_{B_{r,2}} H(-u'_{2,1})dt + \]
\[ - \int_{B_{r,1}} H(-u'_{1,1})dt - \int_{B_{r,2}} H(-u'_{r,2,2})dt + \]
\[ + \lambda \int_{B_{r,1}} (u_1 - u_{r,1,1})dt + \lambda \int_{B_{r,2}} (u_2 - u_{r,2,2})dt, \]

where last line in the above equality vanishes, by (3.5).

By minimality of $u_{r,1,1}$ and $u_{r,2,2}$, $P_H(E; \Omega) > P_H(E^*; \Omega)$, and this contradicts the minimality of $E$. Hence, $\lambda_1 = \lambda_2$.

**Step 3.** We point out that the claim of Step 2 assure that $\partial E \cap \Omega$ consists of Wulff arcs, all with the same curvature, or straight segments. To conclude the proof of the Theorem, we have to prove that $E$ and $\partial E \cap \Omega$ are connected. This can be shown by repeating line by line the proof of Theorem 2 in [10].

\[ \square \]

The following property of the minimizers is a direct consequence of Remark 2.1.

**Proposition 3.1.** Let $\Omega$ be an open bounded connected set of $\mathbb{R}^2$, with Lipschitz boundary. Suppose that $E$ is a minimizer of (3.2). If $|E| < |\Omega|/2$ and $\partial E \cap \Omega$ is not a straight segment, $\partial E \cap \Omega$ is concave towards $E$.

**Proof.** If $|E| < |\Omega|/2$ and $\partial E \cap \Omega$ is strictly concave towards $\Omega \setminus E$, we can consider a new set $E^*$ by adding to $E$ the region of $\Omega$ between $\partial E \cap \Omega$ and a straight segment joining two suitable points of $\partial E \cap \Omega$. Choosing the two points sufficiently near, we get that $|E| \leq |E^*| \leq |\Omega|/2$ and, by Remark 2.1, $P_H(E^*; \Omega) < P_H(E; \Omega)$. This contradicts the minimality of $E$. \[ \square \]

**Theorem 3.3.** Let $\Omega$ be an open bounded convex set of $\mathbb{R}^2$. Suppose that $E$ is a minimizer of (3.2), and let $T$ be a terminal point of $\partial E$. Then $\partial \Omega$ at $T$ is $C^1$, and
\[ \langle n_E, \nu_\Omega \rangle = 0 \]

where $n_E$ is the anisotropic outer normal to $\partial E$ and $\nu_\Omega$ is the usual unit outer normal to $\partial \Omega$ at $T$.

**Remark 3.1.** The angle condition is justified by the following natural geometric argument.

Let $s: \alpha_s x + \beta_s y + q_s = 0$ be a straight line, $P_0 = (x_0, y_0) \in \mathbb{R}^2 \setminus s$. By an immediate calculation, the straight segment which minimizes $L_H$ between $P_0$ and $s$ is parallel to the straight line $r: \alpha_r x + \beta_r y = 0$ which has to satisfy the following orthogonality condition:
\[ \langle \nabla H(\beta_r, \alpha_r), (\beta_s, \alpha_s) \rangle = 0. \]

Using the notation of Theorem 3.3, if we consider as $r$ the tangent line to $\partial \Omega$ at a terminal point $T$ of $\partial E \cap \Omega$, and as $s$ the tangent straight line to $\partial E$ at $T$, then (3.6) and (3.7) coincide.
Proof of Theorem 3.3. We first assume that \( \partial \Omega \) is \( C^1 \) at \( T \).

Let us suppose, by contradiction, that (3.6) is not verified. The idea is to construct a new set \( E^* \) such that \( Q(E^*) < Q(E) \). This will contradict the minimality of \( E \). To do that, we need to distinguish three cases.

First of all, we denote with \( s \) the tangent line to \( \partial \Omega \) at \( T \), and with \( t \) and \( r \) two half line with vertex at \( T \) and towards \( \Omega \) such that \( t \) is tangent to \( \partial E \) at \( T \) and \( r \) satisfies the angle condition (3.7) with respect to \( s \).

Case 1. We first assume that \( |E| < |\Omega|/2 \) and the angle between \( s \) and \( t \) towards \( E \) is greater than the one between \( s \) and \( r \) towards \( E \). We construct the straight segment \( QQ_0 \) parallel to \( r \) joining a suitable point \( Q \in \partial E \cap \Omega \) and \( Q_0 \in s \). Being \( \Omega \) convex, we can consider the point \( \bar{Q} = QQ_0 \cap \partial \Omega \). Denoted by \( D \) the closed region delimited by \( QQ \), the arc of \( \partial E \) joining \( Q \) and \( T \) and the arc of \( \partial \Omega \) between \( T \) and \( Q \), let be \( E^* = E \cup D \) (see figure 3).

We choose \( Q \) sufficiently near to \( T \) such that \( |E^*| < |\Omega|/2 \). Hence \( E^* \) has larger area than \( E \) and, by Remark 3.1 and Lemma 2.1 also smaller anisotropic perimeter.

Case 2. Now we still suppose that \( |E| < |\Omega|/2 \), and the angle between \( s \) and \( t \) towards \( E \) is smaller that the one between \( s \) and \( r \) towards \( E \). We construct the straight segment \( QQ_0 \) parallel to \( r \) joining a suitable point \( Q \in t \) and \( Q_0 \in s \), the point \( \bar{Q} = QQ_0 \cap \partial \Omega \) and the set \( D \) as the intersection between the triangle \( QTQ_0 \) and \( E \) (see figure 4). We define \( E^* = E \setminus D \).

We show that, for a suitable choice of \( Q \),

\[
\frac{P^2_H(E; \Omega)}{|E|} > \frac{P^2_H(E^*; \Omega)}{|E^*|}.
\]

Differently from the case 1, inequality (3.8) is not obvious because \( E^* \) has both smaller perimeter and area. Hence, we explicitly calculate the right-hand side in (3.8). Denoted by \( A = |E|, P = P_H(E; \Omega), \delta A = |D| = |E| - |E^*|, \delta P = P_H(E; \Omega) - P_H(E^*; \Omega) \), the

Figure 2. Contact angle condition, with \( H(x, y) = (x^4 + y^4)^{\frac{1}{4}} \) and \( E \) is homothetic to the Wulff shape \( W \) and centered at \((x_0, y_0)\). The tangent lines to \( \partial \Omega \) at the contact points have the same direction of the anisotropic normal to \( \partial E \) at the same points.
inequality (3.8) becomes

\[
(3.9) \quad \frac{P^2}{A} > \frac{(P - \delta P)^2}{A - \delta A}.
\]

Denoting by \( l_{1,H} = \mathcal{L}_H(\gamma_1) \) and \( l_{2,H} = \mathcal{L}_H(\gamma_2) \), where \( \gamma_1 \) and \( \gamma_2 \) are the curves which represent \( TQ \) and \( QQ_0 \) respectively, it is easy to prove that

\[
l_{1,H} = l_1 \cdot H(-\beta, \alpha) = l_1 C_1,
\]

where \( l_1 \) and \((\alpha, \beta)\) are respectively the usual length and the direction of \( TQ \), and

\[
l_{2,H} = l_2 \cdot H(-\beta_r, \alpha_r) = l_2 C_2,
\]

where \( l_2 \) and \((\alpha_r, \beta_r)\) are respectively the usual length and the direction of \( QQ_0 \). Observe that by construction, \( l_{1,H} > l_{2,H} \).
We first show (3.9) replacing \( \delta P \) with \( \delta \tilde{P} = l_{1,H} - l_{2,H} \) and \( \delta A \) with \( \delta \tilde{A} = \delta A + A_1 + A_2 \), where \( A_1 \) and \( A_2 \) are the measures of the sets as in Figure 5.

By elementary properties of triangles,

\[
\frac{(P - \delta \tilde{P})^2}{A - \delta \tilde{A}} = \frac{(P - l_1C_1 + l_2C_2)^2}{A - l_1l_2 \sin(\gamma + \vartheta)} = \frac{(P - l_1 (C_1 - \frac{\sin \gamma}{\sin \theta} C_2))^2}{A - l_1^2 \frac{\sin \gamma}{\sin \theta} \sin(\gamma + \vartheta)} = f(l_1)
\]

The function \( f \) is strictly decreasing in the interval \([0, \bar{C}]\), with

\[
\bar{C} = \frac{A}{P} \frac{C_1 - C_2 \frac{\sin \gamma}{\sin \theta}}{\sin \gamma + \theta}
\]

which is strictly positive, being \( l_{1,H} > l_{2,H} \). This implies that, for \( l_1 < \bar{C} \),

\[
(3.10) \quad \frac{P^2}{A} > \frac{(P - \delta \tilde{P})^2}{A - \delta \tilde{A}}.
\]

On the other hand, by Remark 2.1 we get

\[
\delta P \geq \delta \tilde{P}.
\]

Hence, being obviously \( \delta \tilde{A} \geq \delta A \), by (3.10), it follows (3.9) for a suitable choice of \( Q \).

**Case 3.** Finally, if \(|E| = |\Omega|/2\), we can both consider, as minimum sets, \( E \) and \( \Omega \setminus E \). Hence, if the angle condition is not verified, we can suppose, without loss of generality, that the lines \( r, s \) and \( t \) verify the hypotheses of case 2.

If \( \partial E \cap \Omega \) is a straight segment, or it is strictly concave towards \( E \), we can repeat line by line the same argument of case 2. Otherwise, if \( \partial E \cap \Omega \) is strictly concave towards \( \Omega \setminus E \), proceeding as in case 1 we construct the straight segment \( QQ_0 \), and another straight segment \( BC \) joining two suitable points of \( \partial E \cap \Omega \). Let \( D_1 \) and \( D_2 \) be as in Figure 6 and define \( E^* = (E \setminus D_1) \cup D_2 \). Choosing \( B, C \) and \( Q \) in such a way that \(|E| = |E^*|\), since \( P_H(E^*; \Omega) < P_H(E; \Omega) \) we obtain a contradiction, and the proof of the Theorem is
completed when $T$ is a regular point of $\partial \Omega$. Finally, we show that $\partial E \cap \Omega$ cannot join $\partial \Omega$

at a non regular point.

By contradiction, suppose that $\partial \Omega$ is not regular at $T$. By convexity it has different right and left tangent straight lines, that we denote by $s_1$ and $s_2$ respectively.

Clearly, the tangent line $t$ does not satisfy the contact angle condition with both $s_1$ and $s_2$. So we can repeat the arguments just considered by replacing the straight line $s$ with $s_1$ or $s_2$, and obtaining a contradiction with the minimality of $E$. \hfill \Box

**Proposition 3.2.** Let $\Omega$ be an open bounded convex set of $\mathbb{R}^2$, $0 < k \leq |\Omega|/2$, and set $E_k$ be a minimizer of problem

$$
\min \{ P_H(F; \Omega), \ F \subset \Omega \text{ and } |F| = k \}.
$$

We have the following properties:

1. neither $E_k$ nor $\Omega \setminus E_k$ is homothetic to a Wulff shape;
2. if $k < |\Omega|/2$, and $T_1$ and $T_2$ are the terminal points of $\partial E_k \cap \Omega$ on $\partial \Omega$, then the left and right tangent straight lines at $T_1$ to $\partial \Omega$ do not make a cone towards $\Omega \setminus E_k$ with the analogous lines at $T_2$.
3. if $k < |\Omega|/2$ and $\partial E_k \cap \Omega$ is not a straight segment, $\partial E_k \cap \Omega$ is concave towards $E$.

*Proof.* We prove the three properties by contradiction with the minimality of $E_k$, finding a set with same area and smaller perimeter.

Let $E_k$ or $\Omega \setminus E_k$ be homothetic to a Wulff shape. Since the perimeter $P_H(E_k; \Omega)$ is invariant up to translations in $\Omega$, we can suppose that $\partial E_k$ touches at least at one (regular) point $P \in \partial \Omega$, and there exists a small ball $B_P$ centered at $P$ such that $B_P \cap \partial E_k \not\subset \partial \Omega$.

We stress that in $P$ the contact angle condition cannot hold. Indeed $\nu_{E_k}(P) = \nu_{\Omega}(P)$, and by (3.6) and the homogeneity of $H$ we should have that

$$
0 = \langle n_{E_k}(P), \nu_{\Omega}(P) \rangle = \langle \nabla H(\nu_{\Omega}(P)), \nu_{\Omega}(P) \rangle = H(\nu_{\Omega}(P)),
$$

![Figure 6.](image-url)
so \( \nu_\Omega = 0 \) and this is absurd. Then arguing as in case 3 of the proof of Theorem 3.3, being \( E_k \) (or \( \Omega \setminus E_k \)) strictly convex we can add and subtract two small regions in order to get a new set with the same area and smaller perimeter (see Figure 6). This proves (1).

Property (2) easily follows by the convexity of \( \Omega \). Indeed, if \( E_k \) has measure smaller than \( |\Omega|/2 \) and does not verify (2), we can do a suitable translation \( \partial E_k \) of \( \partial E_k \) towards the vertex \( V \) of the cone in \( \mathbb{R}^2 \), in such a way that the set \( \tilde{E} \) bounded by \( \partial E_k \cap \Omega \) towards \( V \) and \( \partial \Omega \), has measure \( k \) and smaller perimeter than \( E_k \) in \( \Omega \) (see figure 7). This contradicts the minimality of \( E_k \).

![Figure 7](image)

Finally, suppose that \( \partial E_k \) is concave towards \( \Omega \setminus E_k \). By property (2), the tangent straight lines at terminal points of \( \partial E_k \cap \Omega \) either make a cone towards \( E_k \) or are parallel. As in Proposition 3.1 in both cases we can add a small region to \( E_k \) in order to decrease the perimeter and, similarly as in the proof of property (2), with a suitable translation of \( \partial E_k \cap \Omega \) towards the vertex of the cone, keep fixed the area \( |E_k| \). This proves property (3).

In order to prove the existence of a minimizer of (3.2), we need the following technical lemma.

**Lemma 3.1.** Let \( \mu : [0, +\infty[ \to \mathbb{R} \) be a lower semicontinuous function. Suppose that for any \( k > 0 \) there exists \( \delta_k > 0 \) such that

\[
(3.11) \quad \mu(k + \delta) \leq \mu(k), \quad \text{for any } \delta \in [0, \delta_k].
\]

Then \( \mu \) is decreasing in \( ]0, +\infty[ \).

**Proof.** By contradiction, suppose that there exist \( k_1 < k_2 \) such that

\[
(3.12) \quad \mu(k_1) < \mu(k_2).
\]

Define \( \varphi(k) \) as

\[
\varphi(k) = \begin{cases} 
\mu(k_1) & \text{if } k \leq k_1, \\
\mu(k) & \text{if } k_1 < k < k_2, \\
\mu(k_2) & \text{if } k \geq k_2.
\end{cases}
\]
The function \( \varphi \) is lower semicontinuous, and for any \( k \) there exists \( \delta_k > 0 \) such that \( \varphi(k + \delta) \leq \varphi(k) \), for any \( \delta \in [0, \delta_k] \). Hence, we can define \( \bar{\delta} > 0 \) as
\[
\bar{\delta} = \sup\{\delta > 0 : \varphi(k_1 + \delta) \leq \varphi(k_1)\}.
\]
If \( \bar{\delta} = +\infty \), then \( \varphi(k_2) \leq \varphi(k_1) \), and this contradicts (3.12). Hence, suppose that \( \bar{\delta} < +\infty \). Being \( \varphi \) lower semicontinuous, \( \bar{\delta} \) is actually a maximum:
\[
\varphi(k_1 + \bar{\delta}) \leq \liminf_{\delta \to \bar{\delta}} \varphi(k_1 + \delta) \leq \varphi(k_1).
\]
But this contradicts the definition of \( \bar{\delta} \). Indeed, by the property of \( \varphi \) we can take \( \tilde{\delta} > \bar{\delta} \) such that \( \varphi(k_1 + \tilde{\delta}) \leq \varphi(k_1 + \bar{\delta}) \leq \varphi(k_1) \). Hence, necessarily \( \mu(k_1) \geq \mu(k_2) \), and the proof is concluded.

**Theorem 3.4.** Let \( \Omega \) be an open bounded convex set of \( \mathbb{R}^2 \). Let \( \mu(k) \) be the function defined in \( ]0, |\Omega|/2] \) as
\[
(3.13) \quad \mu(k) = \min \left\{ \frac{P_H^2(F; \Omega)}{k}, F \subset \Omega \text{ and } |F| = k \right\}.
\]
Then, we have the following results hold:

1. \( \mu(k) \) is a decreasing lower semicontinuous function in \( ]0, |\Omega|/2] \),
2. the sets which minimize (3.13) verify the contact angle condition. More precisely, they verify the thesis of Theorem 3.3.

**Proof.** We first prove that the function \( \mu \) is lower semicontinuous in \( ]0, |\Omega|/2] \).

Let be \( k \in ]0, |\Omega|/2] \), and take a positive sequence \( k_n \) such that \( k_n \to k \). Consider \( E_n \subset \Omega \) such that \( |E_n| = k_n \) and \( \mu(k_n) = Q(E_n) = k_n^{-1}P_H^2(E_n; \Omega) \). By Proposition 3.2, \( E_n \) is convex. Hence, by the Blaschke selection Theorem (see [18], page 50) \( E_n \) converges (up to a subsequence) to a set \( E \) in the Hausdorff metric. Being \( E_n \) convex and bounded, then \( \chi_{E_n} \to \chi_E \) in \( L^1(\Omega) \) strongly, and \( |E| = k \). Using the lower semicontinuity of \( P_H(\cdot; \Omega) \) (see [3]) we get
\[
\mu(k) \leq Q(E) \leq \liminf \frac{P_H^2(E_n; \Omega)}{k_n} = \liminf \mu(k_n).
\]
In order to prove that \( \mu \) is decreasing, let be \( k \in ]0, |\Omega|/2] \) fixed and consider \( E_k, |E_k| = k \) such that \( \mu(k) = Q(E_k) \).

We claim that there exists a positive number \( \delta_k \) and a family of sets \( E_k(\delta), 0 < \delta \leq \delta_k \) with continuously increasing area and \( Q(E_k(\delta)) \leq Q(E_k) \). Then
\[
(3.14) \quad \mu(|E_k(\delta)|) \leq Q(E_k(\delta)) \leq \mu(k), \quad \delta \in ]0, \delta_k].
\]
Being \( \mu \) lower semicontinuous in \( ]0, |\Omega|/2] \), by Lemma 3.1 this is sufficient to show that \( \mu \) is decreasing.

By Theorem 3.2, \( \partial E_k \cap \Omega \) is a straight segment or a Wulff arc, and by property (1) of Proposition 3.2, it has two terminal points \( T_i \) on \( \partial \Omega \). We suppose that such points are regular for \( \partial \Omega \), so that by property (2) Proposition 3.2 the tangent lines to \( \partial \Omega \), \( s_i \) at \( T_i \) either are parallel or make a cone \( A \) towards \( E_k \). In the first case, the claim follows immediately by the convexity of \( \Omega \) and making a suitable translation of \( \partial E_k \). Hence, we
consider the second case, and suppose without loss of generality that \( s_1 \cap s_2 = (0, 0) \).

Moreover, by property 3 of Proposition 3.2, \( \partial E_k \cap \Omega \) is a straight segment, or concave towards \( E_k \).

We need to distinguish two cases for the shape of \( \Omega \).

**Case 1.** \( \partial E_k \cap \partial \Omega \) is not contained in \( \partial A \).

We set \( C(\delta), \delta \geq 0 \), the region bounded by \((1 + \delta)\partial E_k \) and \( \partial A \), and \( E_k(\delta) = C(\delta) \cap \Omega \).

For sake of simplicity, we define \( C(0) = C \).

Let \( A_i(\delta) \) be the boundary point of \( \partial C(\delta) \cap A \) on \( s_i \). Moreover, let be \( B_i(\delta) = \partial \Omega \cap w_i \), where \( w_i \) is the tangent line to \( \partial C(\delta) \) at \( A_i(\delta) \). (see figure 8).

\[
\begin{figure}
\centering
\includegraphics{figure8}
\caption{Figure 8.}
\end{figure}
\]

Now we compute area and relative perimeter of \( E_k(\delta) \). Observe that the triangles \( D_i \) of vertex \( A_i(\delta) \), \( B_i(\delta) \) and \( T_i \) have area \( |D_i| = o(\delta) \). We have:

\[
|E_k(\delta)| \geq |E_k| + (|C(\delta)| - |C|) + o(\delta) = |E_k| + 2\delta|C| + o(\delta)
\]

and

\[
P_H(E_k(\delta); \Omega) \leq P_H(C(\delta); A) = (1 + \delta)P_H(C; A) = (1 + \delta)P_H(E_k; \Omega).
\]

It follows that

\[
(3.15) \quad \frac{1}{\delta} [Q(E_k(\delta)) - Q(E_k)] \leq
\]

\[
\leq \frac{1}{\delta} Q(E_k) \left[ \frac{(1 + \delta)^2}{1 + 2\delta \frac{|C|}{|E_k|} + o(\delta)} - 1 \right] =
\]

\[
= Q(E_k) \left[ \frac{2 \left( 1 - \frac{|C|}{|E_k|} \right) + o(1)}{1 + o(1)} \right].
\]
Since $|E_k| < |C|$, then for $\delta$ sufficiently small we obtain that the left-hand side of (3.15) is negative. This proves (3.14), and hence (1), if $T_i$ are regular points of $\partial \Omega$. If, for example, $T_1$ is not a regular point, we can repeat the arguments just considered by replacing $s_1$ with the left or right tangent straight line.

Now we prove (2). In order to fix the ideas, we consider the regular point $T_1$ and the straight line $r$ which verifies the contact angle condition with $s_1$. Let $\alpha_{opt}$ be the angle between $s_1$ and $r$ towards $E_k$, and $\alpha$ the corresponding angle between $\partial E_k \cap \Omega$ and $s_1$ towards $E_k$. Suppose by contradiction that $\alpha \neq \alpha_{opt}$.

If $\alpha < \alpha_{opt}$, then the construction made in the proof of case 2 of Theorem 3.3 allows to take $E^*$ such that $|E^*| < |E_k|$ and $Q(E^*) < Q(E_k)$, and this contradicts the monotonicity of $\mu$. If $\alpha > \alpha_{opt}$, and $\partial E_k \cap \Omega$ is a Wulff arc, as in case 3 of Theorem 3.3 we can add and subtract two sets in order to decrease the perimeter and to preserve the area, contradicting the minimality of $E_k$. In the case that $\partial E_k \cap \Omega$ is a straight segment, we can add a small region to $E_k$ in order to decrease the perimeter and with a suitable translation, keep fixed the area $|E_k|$.

Finally, $T_1$ cannot be a singular point for $\partial \Omega$. Otherwise, similarly as observed at the end of the proof of Theorem 3.3 and proceeding as above, we get a contradiction with the minimality of the minimizer.

**Case 2.** $\partial E_k \cap \partial \Omega$ is contained in $\partial A$, that is $E_k = C$.

Define $E_k(\lambda) = \lambda E_k$, $\lambda \geq 0$, and $r \geq 0$ such that

$$\lambda_{max} = \max\{\lambda \geq 0 : E_k(\lambda) \cap \partial \Omega \subset \partial A\}.$$

First, we prove that at the terminal points of $\partial E_k \cap \Omega$ it holds the contact angle condition (3.6).

In order to fix the ideas, we consider the regular point $T_1 \in \partial \Omega$ and the straight line $r$ which verifies the contact angle condition with $s_1$. Let $\alpha_{opt}$ be the angle between $s_1$ and $r$ towards $E_k$, and $\alpha$ the corresponding angle between $\partial E_k \cap \Omega$ and $s_1$ towards $E_k$. Suppose by contradiction that $\alpha \neq \alpha_{opt}$.

**Case 2-a** Let be $\lambda_{max} > 1$. Reasoning as in the proof of Theorem 3.3 we find $E^*$ such that $Q(E^*) < Q(E_k)$, with $|E_k| - |E^*|$ sufficiently small. Then there exists $\rho > 0$ such that $|\rho E^*| = \rho |E^*| = |E_k|$, and $Q(\rho E^*) = Q(E^*) < Q(E_k)$. This contradicts the minimality of $E_k$.

Repeating the same argument for $T_2$, we have that the terminal points of $\partial E_k \cap \Omega$ have to verify the angle condition, that is $E_k$ is homothetic to a Wulff sector $W \cap A$.

**Case 2-b** Let be $\lambda_{max} = 1$. Then, as $0 < \lambda < \lambda_{max}$, the set $\lambda E_k$ is such that $Q(\lambda E_k) = Q(E_k)$. Thanks to case 2-a, we have that $\mu(\lambda E_k)$ is attained at a Wulff sector, namely the set $(\tilde{\lambda} W) \cap A = (\tilde{\lambda} W) \cap \Omega$, for $\lambda > 0$ such that $|\lambda E_k| = |(\tilde{\lambda} W) \cap A|$. Hence $\mu(|E_k|) = Q((\tilde{\lambda} W) \cap \Omega) < Q(E_k)$. Define

$$\gamma_{max} = \max\{\gamma \geq 0 : (\gamma W) \cap \Omega \text{ is homothetic to a Wulff sector}\} \tag{3.16}$$

We have that $\gamma_{max}$ is finite and $|\gamma_{max} W \cap \Omega| < |E_k|$. Otherwise, there exists $\gamma \leq \gamma_{max}$ such that $|\gamma W \cap \Omega| = |E_k|$ and $Q(\gamma W \cap \Omega) = Q(\tilde{\lambda} W \cap \Omega) < Q(E_k)$, and this is a contradiction.
As matter of fact, the homogeneity of \( H \) and (2.4) imply, for \( \xi \in \partial W \), that \( H(\nu_W(\xi)) = \langle \nu_W(\xi), \xi \rangle \). Moreover, for \( \xi \in \partial A \), \( \langle \nu_A(\xi), \xi \rangle = 0 \). Hence by the divergence Theorem we get that, for \( \gamma > 0 \),

\[
\text{(3.17)} \quad P_H(\gamma W; A) = 2\gamma |W \cap A|.
\]

Define \( E(\delta) = \Omega \cap [\gamma_{\text{max}} + \delta)W] \), and \( A_\delta \) the cone made by the two half-straight lines \( s_i\), \( i = 1, 2 \) with origin at \((0,0)\) and passing through one of the two terminal points of \( \partial[E(\delta) \cap \Omega] \).

By (3.17) and the convexity of \( \Omega \), we get, for an appropriate \( \delta \), \( |E(\delta)| = k \) and

\[
Q(E(\delta)) \leq 4|W \cap A_\delta| < 4|W \cap A| = Q(\gamma_{\text{max}} W) < Q(E_k).
\]

Then \( \partial E_k \) must verify the contact angle condition at each \( T_i \), and this concludes the case 2-b, and (2) is proved.

In order to prove (3.14), and hence (1), we observe that from (2), \( E_k = (\lambda W) \cap \Omega \), for some \( \lambda > 0 \). Let \( \gamma_{\text{max}} \) as in (3.16), and suppose that \( \gamma_{\text{max}} = \lambda \), otherwise (3.14) is immediate, being \( Q(E_k) = Q(\gamma W \cap \Omega) \), for any \( 0 < \gamma < \gamma_{\text{max}} \). Defining \( E(\delta) = \Omega \cap [\gamma_{\text{max}} + \delta)W] \) and reasoning as in case 2-b, we get (3.14).

Finally, the regularity of \( T_i \) on \( \partial \Omega \) follows exactly as in the case 1, and the proof is completed. \( \square \)

**Remark 3.2.** We observe that if \( E \) is a minimizer of (3.2), and \( |E| < |\Omega|/2 \), then \( E \) is homothetic to a Wulff sector with sides on \( \partial \Omega \). Otherwise, arguing as in case 1 of the proof of Theorem 3.4, we construct a new set \( E^* \) with \( Q(E^*) < Q(E) \). Hence, \( E = E(\lambda) = A \cap (\lambda W) \) with sides on \( \partial \Omega \). Being

\[
Q(E(\rho)) = 4|W \cap A|, \quad \forall \rho: |E(\rho)| \leq \frac{|\Omega|}{2},
\]

where \( E(\rho) = A \cap (\rho W) \), there exists another minimizer \( F \) which is a Wulff sector with sides on \( \partial \Omega \) and \( |F| = |\Omega|/2 \).

Now we are able to prove the main result.

**Theorem 3.5.** Let \( \Omega \) be an open bounded convex set of \( \mathbb{R}^2 \). Then there exists a convex minimizer of problem (3.2) whose measure is equal to \( |\Omega|/2 \). More precisely, either a minimizer \( E \) of (3.2) has measure \( |\Omega|/2 \), or \( E \) is homothetic to a Wulff sector with sides on \( \partial \Omega \). Finally, it verifies the contact angle condition.

**Proof.** Let \( \mu \) defined as in the above theorem and, being \( \mu \) decreasing in \( |0, |\Omega|/2| \), it attains its minimum at \( k = |\Omega|/2 \).

Now we are able to prove that (3.2) has a minimum. Let \( \tilde{E} \) be such that \( |\tilde{E}| = |\Omega|/2 \) and \( \mu(|\Omega|/2) = Q(\tilde{E}) \). Let \( E_n, n \in \mathbb{N} \) be a minimizing sequence of problem (3.2), that is

\[
\lim \frac{Q(E_n)}{n} = C_H, \quad 0 < |E_n| \leq |\Omega|/2.
\]

Without loss of generality, we may suppose that, for any \( n \in \mathbb{N} \), \( Q(E_n) = \mu(|E_n|) \). Otherwise, we replace \( E_n \) with the minimizer of problem (3.3) with volume constraint \( k = |E_n| \). Then

\[
C_H \leq Q(\tilde{E}) = \mu(|\Omega|/2) \leq \mu(|E_n|) = Q(E_n).
\]
Passing to the limit, 
\[ C_H = \mu \left( \frac{|\Omega|}{2} \right), \]
and \( \tilde{E} \) is a minimizer of (3.2), whose boundary in \( \Omega \) is a straight segment or a Wulff arc. From the proof of Theorem 3.4 it follows that if \( E \) is another minimizer of (3.2) with \( |E| < |\Omega|/2 \), then it is a Wulff sector with sides on \( \partial \Omega \). Recalling Theorem 3.3 the result is completely proved. \( \square \)

In the following theorem, we characterize the minimizers for centrosymmetric sets, and find the constant \( C_H \) in (3.2).

For sake of simplicity, if \( T \) is a point in \( \mathbb{R}^2 \), we put \( L_H(T) = \mathcal{L}_H(\gamma) \), where \( \mathcal{L}_H \) is defined in (2.6), and \( \gamma \) is a curve which represent the straight segment \( OT \) joining \( T \) with the origin \( O \). We observe that if \( T = (x, y) \), then \( L_H(T) = H(-y, x) \).

**Theorem 3.6.** Let \( \Omega \subset \mathbb{R}^2 \) be a convex bounded set, symmetric about the origin \( O \). Then a minimizer of (3.2) is a set \( E \) whose boundary in \( \Omega \) is a straight segment passing through the origin \( O \). We observe that if \( T = (x, y) \), then \( L_H(T) = H(-y, x) \).

Proof. The first step is to prove the existence of a set \( E \) enjoying the properties of the statement. Let us consider the set 
\[ B(r_H) = \{(x, y) \in \mathbb{R}^2 : L_H(x, y) < r_H \}. \]
Then \( \partial B(r_H) \) meets \( \partial \Omega \) at least at two symmetric regular points \( T_1, T_2 \). We observe that in \( T_i \) the contact angle condition is satisfied. Indeed, the anisotropic outer normal to the straight segment \( OT_i \) is \( n_E(T_i) = \nabla H(-y_i, x_i) \), where \( T_i = (x_i, y_i), i = 1, 2 \). Denoted by \( \nu_{\Omega}(T_i) \) the unit outer normal to \( \partial \Omega \) at \( T_i \), being \( \nu_{\Omega}(T_i) = (H_y(-y_i, x_i), -H_x(-y_i, x_i)) \), we have \( \langle n_E(T_i), \nu_{\Omega}(T_i) \rangle = 0 \).

We show that \( T_1T_2 \) is the boundary in \( \Omega \) of the required set \( E \), and \( Q(E) = \frac{8 r_H^2}{|\Omega|} \).

By Theorem 3.5 there exists a convex minimizer of (3.2) whose measure is \( |\Omega|/2 \), which is a straight segment or a Wulff arc. If we show that \( P_H(E; \Omega) \leq P_H(F; \Omega) \), where \( F \) is an open convex subset of \( \Omega \) such that \( |F| = |\Omega|/2 \) and \( \partial F \cap \Omega \) is a straight segment or a Wulff arc, we have done.

Clearly, any straight segment passing through the origin bounds in \( \Omega \) a set with greater perimeter than \( E \) and with same area \( |\Omega|/2 \). We do not consider the straight segments which not contain the origin, because they bounds in \( \Omega \) sets with measure different from \( |\Omega|/2 \). Hence we can suppose that \( \partial F \cap \Omega \) is a Wulff arc.

Obviously, \( O \notin \partial F \), otherwise \( |F| \neq |\Omega|/2 \). More precisely, denoted by \( P_1 \) and \( P_2 \) the terminal points of \( \partial F \cap \Omega \), we get that \( O \in F \setminus G \), where \( G \subset F \) is bounded by \( \partial \Omega \) and \( P_1P_2 \), otherwise \( |\Omega|/2 \leq |G| < |F| \), and this is impossible. Hence we can consider the straight segments in \( F \), \( OP_1 \) and \( OP_2 \), and it is not difficult to show that 
\[ P_H(F; \Omega) > L_H(P_1) + L_H(P_2) \geq 2r_H = P_H(E; \Omega), \]
and this concludes the proof. \( \square \)
Remark 3.3. If \( \Omega = \{(x, y): H(-y, x) < r\} \), i.e. \( \Omega \) is obtained by a rotation of \( \pi \) the \( r \)-level set of \( H \), then Theorem 3.6 gives
\[
C_H = \frac{8r^2}{|\Omega|} = \frac{8}{\kappa_H},
\]
where \( \kappa_H = |\{(x, y): H(x, y) < 1\}|. \) Observe that any straight segment passing through the origin and joining the boundary of \( \Omega \) bounds a minimizer.

In particular, if \( H(x, y) = H^*_a(x, y) = (x^2 + y^2)^{1/2} \), we recover the classical result \( C_H = \frac{8}{\pi} \) (see for instance [16], [10]).

4. Some examples

Here we apply the results just obtained to some particular function \( H \).

Example 4.1. Let \( H(x, y) \) defined as
\[
H(x, y) = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{1/2}.
\]
An immediate calculation gives that
\[
H^a(x, y) = \left( a^2x^2 + b^2y^2 \right)^{1/2}. \]
If \( \Omega \) is the ellipse \( \Omega = \{(x, y): H^a(x, y) < r\} \), then \( \Omega = \{(x, y): H(-y, x) < \frac{r}{ab}\} \), and \( |\Omega| = \frac{\pi r^2}{ab} \). By Theorem 3.6 and Remark 3.3 we have
\[
(4.1) \qquad P_H^2(E; \Omega) \geq \frac{8}{\pi ab} |E|, \quad \forall E \subset \Omega: |E| \leq \frac{\pi r^2}{2ab}.
\]
Moreover, the equality in (4.1) holds if and only if \( \partial E \cap \Omega \) is any straight segment passing through the origin (see Figure 9).

We observe that if we compute \( C_H \) for the ellipse \( \Omega_1 = \{(x, y): H(x, y) < r\} \), with for example, \( a > b \), then the smaller axis of the ellipse (in the usual sense) is the boundary of the only minimizer of (3.2) (see Figure 9), and the constant \( C_H \) is
\[
C_H = \frac{8}{\pi ab} \frac{b^2}{a^2}.
\]
We point out that the above result for \( \Omega \) can be obtained directly by the classical relative isoperimetric inequality for the Euclidean perimeter. Indeed, the anisotropic relative perimeter of a smooth set \( E \), whose boundary is described by \( (u(t), v(t)) \), with \( t \in [\alpha, \beta] \), is
\[
(4.2) \qquad P_H(E; \Omega) = \int_{\alpha}^{\beta} H(-v', u') \, dt = \int_{\alpha}^{\beta} \left( \frac{(v')^2}{a^2} + \frac{(u')^2}{b^2} \right)^{1/2} \, dt.
\]
Defining \( w = au \) and \( z = bv \), the curve \( (w(t), z(t)) \) describe the boundary of the unit Euclidean disk \( B_r \) with radius \( r \) and centered at the origin. By changing the variables in
Figure 9. In the first figure, \( \Omega_1 \) is a level set of \( H \), and the straight segment is the boundary of the only minimizer of (3.2). In the second figure, \( \Omega \) is a level set of \( H_0 \), and any straight segment passing through the origin is the boundary of a minimizer.

Using (4.2), we get

\[
\int_{\alpha}^{\beta} \left( \frac{(z')^2}{a^2b^2} + \frac{(w')^2}{a^2b^2} \right)^{\frac{1}{2}} dt = \frac{1}{ab} P(\tilde{E}, B_1) \geq 1 \frac{1}{ab} \sqrt{\frac{8}{\pi}} |\tilde{E}|^{\frac{1}{2}} = \sqrt{\frac{8}{\pi ab}} |E|^{\frac{1}{2}},
\]

where \( \tilde{E} \) is the set obtained by \( E \) after the change of variables. Being \( |\tilde{E}| = ab|E| \), we get (4.1).

Finally, the characterization of the minimizers is a direct consequence of the fact that in the classical relative isoperimetric inequality, the minimizers are the diameters. Hence in this case we get the relative anisotropic isoperimetric inequality by a linear transformation, as a consequence of the classical relative isoperimetric inequality.

**Example 4.2.** Now suppose that

\[
H(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}}.
\]

where \( 2 \leq p < +\infty \) and \( p' = \frac{p}{p-1} \). Hence, we have \( H_0(x, y) = (|x|^p + |y|^p)^{\frac{1}{p'}} \).

Let us consider \( \Omega = \{(x, y): |x|^p + |y|^p < r^p\} \). Being \( \Omega \) invariant by \( \frac{\pi}{2} \)-rotations, by Theorem 3.6 and Remark 3.3 we have

\[
P_H^2(E; \Omega) \geq \frac{8}{\kappa_H} |E|, \quad \forall E \subset \Omega: |E| \leq \frac{r^2}{2} \kappa_H,
\]

where \( \kappa_H = |\{(x, y): H(x, y) < 1\}| \), and any straight segment passing through the origin bounds a minimizer.

**Example 4.3.** Let \( H \) be defined as follows:

\[
H(x, y) = \begin{cases} 
(|x|^p + |y|^p)^{1/p} & \text{if } xy \geq 0, \\
(|x|^q + |y|^q)^{1/q} & \text{if } xy \leq 0,
\end{cases}
\]
with \( p > 2, q > 2 \) and \( p > q \). Let us consider \( \Omega = \{(x, y): H(−y, x) < r\} \). Then

\[
C_H = C_H(\Omega) = \frac{8}{\kappa_H}.
\]

We stress that if \( \Omega_1 = r\{(x, y): H(x, y) < r\} \), then easy computations give that

\[
C_H = C_H(\Omega_1) = \frac{8}{\kappa_H} 4^{\frac{1}{p}−\frac{1}{q}}.
\]

Observe that \( C_H(\Omega) > C_H(\Omega_1) \) (compare Figure 10).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{Example 4.3. The solid line represents a level set of \( H \), while the straight segment is the boundary of the only minimizer of (3.2).}
\end{figure}

**Example 4.4 (A non-regular case).** Let us consider \( H(x, y) = \max\{|x|, |y|\} \). The singular behavior of \( H \) does not allow to apply the previous results. Then, in order to prove the anisotropic isoperimetric inequality relative to \( \Omega \) with respect to \( H \), we argue by approximation.

Let be \( \Omega = \{(x, y): \max\{|x|, |y|\} < r\} \), and \( H_p(x, y) = (|x|^p + |y|^p)^{1/p} \). For any set \( E \subset \Omega \) such that \( |E| \leq 2r^2 \), we have

\[
P_{H_p}^2(E; \Omega) \geq 2|E|,
\]

and the best constant is reached by a rectangle whose boundary in \( \Omega \) is the straight segment joining \((-r, 0)\) and \((r, 0)\) (or \((0, −r)\) and \((0, r)\)). We can pass to the limit as \( p \to +\infty \) in (4.3), obtaining

\[
P_H^2(E; \Omega) \geq 2|E|, \quad \forall E \subset \Omega: |E| \leq 2r^2.
\]

Any straight segment passing through the origin and joining the boundary of \( \Omega \) bounds a minimizer. Unlike the case of \( H \) smooth (compare Remark 3.3), such sets are not the only minimizers.

For example, in Figure 12 some minimizer is represented. Indeed, if \( \partial E \) is described by a Lipschitz function \( u(t), t \in [a, b] \), the perimeter is

\[
P_H(E; \Omega) = \int_a^b \max\{1, |−u'(t)|\} dt.
\]
Then in the picture on the left-hand side of Figure 12, the perimeter of $E$ is $2r$ and $|E| = 2r^2$. Moreover, in the other picture any triangle $E$ such that $\partial E \cap \Omega$ is a straight segment parallel to a diagonal is a minimizer.
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