ON THE END DEPTH AND ENDS OF GROUPS

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Abstract. We prove that any finitely generated one ended group has linear end depth. Moreover, we give alternative proofs to theorems relating the growth of a finitely generated group to the number of its ends.

1. Introduction

The topology of a group at infinity is an important asymptotic invariant of groups (see [3]). In particular the question which groups are simply connected at infinity is relevant to topology ([2], [15]). In order to study finitely generated groups that are simply connected at infinity (sci), Otera in [10], introduces the function $V_1(r)$, measuring, “in which way” a group is sci. The growth of this function, called sci growth and denoted by $V_1$, is a quasi-isometry invariant for finitely generated groups. Expecting the existence of a group with super-linear $V_1$, Otera introduces the end depth function (see Definition 1), $V_0(r)$, a 0 - dimensional analogue of the sci growth for one ended groups. The end depth function measures the “depth” of the bounded components of the complement of the ball $B(r)$ in the Cayley graph of the group. The growth of this function is a quasi-isometry invariant for finitely generated groups and it is called the end depth of the group. Otera [10] shows that given a group in which the growth of $V_0$ is super-linear, we can construct a group where the function $V_1$ has super-linear growth:

Theorem (Otera). If $G = A * B_H$ is the free product with amalgamation over a (sub)group $H$ which is one ended with super-linear end depth and $A,B$ are sci groups, then $G$ is simply connected at infinity with super-linear $V_1$.

One may also remark that a group with non-linear $V_0$ has dead end elements (see [1]). So a group with non-linear $V_0$ has dead end elements with respect to any generating set (to our knowledge there are no such examples in the literature).

Key words and phrases. End depth, Ends, Growth, Virtually cyclic group.
In this paper, we show that the function $V_0$ of any one ended group is linear (see Theorem 2).

In section 4, we give an alternative proof of the following theorem that was first proven by Erschler [4], based on the Varopoulos inequality and answers question VI.19, posed by Pierre de la Harpe in [4]:

**Theorem 3.** Let $G = \langle S \rangle$ be a finitely generated group and $X = \Gamma(G, S)$. If there exists a sequence of positive integers $\{r_i\}_{i \geq 1}$ such that $\lim_{i \to \infty} r_i = \infty$ and $\lim_{i \to \infty} |S(r_i)| < \infty$, then $G$ is virtually cyclic.

By $|S(r_i)|$ we denote the number of vertices in the sphere of radius $r_i$ in the Cayley graph of the group $G$. Another proof, using different methods, was given by Timar [13]. We give a proof, using elementary methods, without using the Varopoulos inequality.

In section 5, we show a stronger result. We relate the number of ends of a finitely generated group with the growth of the spheres in its Cayley graph (see Theorem 4). This Theorem is a weaker version of similar theorems proven by Justin in [8], by Wilkie and Van Den Dries in [14] and by Imrich and Seifter in [6]. Also, in October 2009, Shalom and Tao proved a more general result for groups of polynomial growth in [11], by refining Kleiner’s work in [9].

2. Preliminaries

The definitions and notation introduced in this section will be used throughout this paper.

Let $(X, d)$ be a metric space and $A, B$ non-empty subsets of $X$. The distance of the sets $A$ and $B$, $d(A, B)$ is defined by:

$$d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$$

We denote by $|A|$ the number of elements in the set $A$. For $r > 0$, we define the $r$–neighbourhood of $A$, $N(A, r)$ by:

$$N(A, r) = \{y \in X \mid d(y, A) < r\}$$

For any $x \in X$, we denote by $S(x, r)$ ($B(x, r)$) the sphere (ball) in $X$ of radius $r$, centered at $x$.

We recall the definition of the number of ends of a metric space:

Let $(X, d)$ be a locally compact, connected metric space. For any compact subset, $K$, of $X$ we denote the number of unbounded connected components of $X \setminus K$ by $e(X, K)$. Then, the number of ends of $X$, denoted by $e(X)$, is the supremum, over all compact subsets $K$ of $X$, of $e(X, K)$:

$$e(X) = \sup\{e(X, K) \mid K \subset X\text{compact}\}$$
If \( e(X) = 1 \), we say that \( X \) is a *one ended* metric space.

Let \( G = \langle S \rangle \) be a finitely generated group. For any \( A \subset G \) and \( x \in G \) we denote by \( xA \) the set \( \{ xa \mid a \in A \} \). Also, we denote by \( \Gamma(G, S) \) the Cayley graph of \( G \) with respect to the finite generating set \( S \) and \( d_S \) the word metric in \( \Gamma(G, S) \). If \( e \) is the identity element of \( G \), for any positive integer \( r \), we write \( S(e, r) \) (ball \( B(e, r) \)) in \( \Gamma(G, S) \). The *size of a sphere* \( S(g, r) \) in \( X \) is the number of vertices (elements of \( G \)) in that sphere and we denote it by \( |S(g, r)| \).

We remark that \( (\Gamma(G, S), d_S) \) is a locally compact, connected metric space, thus \( e(\Gamma(G, S)) \) is defined. It is a well known fact that this is independent of the finite generating set chosen. Thus, the *number of ends*, \( e(G) \), of \( G \) is defined to be the number of ends of its Cayley graph, \( \Gamma(G, S) \), with respect to a finite generating set \( S \). We say that a finitely generated group is *one ended* if its Cayley graph, with respect to a finite generating set, is a one ended metric space. Note that the number of ends is a quasi-isometry invariant of finitely generated groups.

Regarding the number of ends of a finitely generated group, we recall the following important theorem of Hopf [5]:

**Theorem 1.** A finitely generated group \( G \) has either 0,1,2 or infinitely many ends.

It is clear that a finitely generated group \( G \) is finite if and only if \( e(G) = 0 \).

On the other hand, from Stallings’ classification Theorem [12] we have that a finitely generated group \( G \) has exactly two ends if and only if \( G \) has an infinite cyclic, finite index subgroup. Therefore, we have the following equivalences:

\[
e(G) = 2 \iff G \text{ is virtually } \mathbb{Z} \iff G \text{ is quasi isometric to } \mathbb{Z}
\]

Finally we define the growth of a function, which we will need in section 3.

Let \( f, g : \mathbb{R}_+ \to \mathbb{R}_+ \). We say that the *growth of the function* \( f \) is at most the growth of the function \( g \) and we write \( f \prec g \), if there exist real constants \( a_1 > 0, a_2 > 0, a_3 \) such that, for any \( x \in \mathbb{R}_+ \), the following inequality holds:

\[
f(x) \leq a_1 g(a_2 x) + a_3
\]

The functions \( f \) and \( g \) have the *same growth*, denoted by \( f \sim g \), if \( f \prec g \) and \( g \prec f \).

Note that the relation \( f \sim g \) is an equivalence relation. The *growth rate* of a function \( f \) is defined as the corresponding equivalence class of the function \( f \). Lastly, we say that \( f \) has *linear growth* if \( f(x) \sim x \).
3. The End Depth Function

In this section we examine the growth of the end depth function of a one ended group. We remark that this notion is a 0-dimensional analogue of the sci growth for one ended groups and it was introduced by Otera [10]. We start by giving the definition of the end depth function that is due to Otera.

**Definition 1.** Let \( G = \langle S \rangle \) be a finitely generated one ended group and \( X = \Gamma(G,S) \). For any \( r > 0 \), we denote by \( N(r) \) the set of all \( k \in \mathbb{R} \) such that any two points in \( X \setminus B(k) \) can be joined by a path outside \( B(r) \). The function \( V^X_0(r) = \inf N(r) \) is called the end depth function of \( X \).

The idea of the end depth function can be grasped more easily if we consider the bounded connected components of \( X \setminus B(r) \):

**Remark 1.** Let \( G = \langle S \rangle \) be a finitely generated group, \( X = \Gamma(G,S) \), \( d = d_S \) and \( e \) the identity element of \( G \). For any \( r > 0 \) the set \( X \setminus B(r) \) has finitely many connected components. We denote by \( U_r \) the unique unbounded connected component and by \( B_r \) the union of the bounded components of \( X \setminus B(r) \).

\[ \text{Figure 1.} \]

Then, we have the following:

1. Clearly, \( B_r = \emptyset \) if and only if \( V^X_0(r) = r \).
2. Suppose that \( B_r \neq \emptyset \), i.e. \( X \setminus B(r) \) has at least one bounded connected component. Then, for any \( x \in B_r \) any path that joins \( x \) to an element \( y \in U_r \) must pass through \( B(r) \). Thus, for any \( x \in B_r \), \( V^X_0(r) \geq d(e,x) \), so:

\[ V^X_0(r) \geq \max\{d(e,x) \mid x \in B_r\} \]

On the other hand, for any \( y,z \in X \) with \( d(e,y), d(e,z) > \max\{d(e,x) \mid x \in B_r\} \) we have that \( y,z \in U_r \). This implies
that $y$ and $z$ can be joined by a path outside $B(r)$, so $y, z \in X \setminus B(V^X_0(r))$. It follows that:

$$V^X_0(r) = \max\{d(e, x) \mid x \in B_r\}$$

From the latter equality we see how, in a sense, $V^X_0$ measures the depth of the bounded connected components of $X \setminus B(r)$. Furthermore, there exists a bounded connected component, $A_r$, of $X \setminus B(r)$ and an element $a \in A_r$ such that

$$V^X_0(r) = d(e, a) = \max\{d(e, x) \mid x \in B_r\}$$

Figure 2.

The end depth function depends on the choice of the generating set, but its growth rate does not. Actually, it is a quasi-isometry invariant for finitely generated groups [10]. Therefore, we recall the following definition that is due to Otera [10].

**Definition 2.** Let $G = \langle S \rangle$ be a one ended group and $X = \Gamma(G, S)$. The end depth of $G$ is the growth rate of the function $V^X_0$.

**Theorem 2.** The end depth of a one ended group is linear.

**Proof.** Let $G = \langle S \rangle$ be a one ended group, $X = \Gamma(G, S)$ and $d = d_s$. We argue by contradiction that, for any integer $r \geq 2$, $V^X_0(r) \leq 4r$.

Suppose that there is a positive integer $r \geq 2$, such that $V^X_0(r) > 4r$. Then, as stated in Remark[1] there exists a bounded connected component, $A$, of $X \setminus B(r)$ and an element $a \in A$, such that $V^X_0(r) = d(e, a)$.

Note that $d(a, B(r)) > 3r$, therefore $d(B(a, r), B(r)) > 2r$.

We consider the left action of $a$ on $X$. Then $aB(r) = B(a, r) \subset A$ and $aA \cap A \neq \emptyset$. Moreover, since $aA \cap aB(r) = \emptyset$ and $|A| = |aA|$, we have that $aA \setminus A \neq \emptyset$. Therefore $aA \cap B(r) \neq \emptyset$. 

\[\text{Figure 2.}\]
Recall that $G$ is one ended, so there exists a unique unbounded connected component, $U$, of $X \setminus B(r)$ and an infinite geodesic path $\gamma = (\gamma_0, \gamma_1, \ldots)$ of vertices in $U$, such that $d(\gamma_0, B(r)) = 1$. Clearly, $d(a\gamma_0, a) = r + 1$ and since $d(a, X \setminus A) > 3r$, it follows that $a\gamma_0 \in A$. On the other hand, since $a\gamma = (a\gamma_0, a\gamma_1, \ldots)$ is an infinite path while $B(r) \cup B_r$, where $B_r$ is the union of the connected components of $X \setminus B(r)$, is finite, there exists $n > 0$, such that $a\gamma_n \in U$. Therefore, the path $\gamma' = (a\gamma_0, a\gamma_1, a\gamma_n)$ joins an element of $A$ to an element of $U$. But $A$ and $U$ are connected components of $X \setminus B(r)$, so $\gamma'$ passes through $B(r)$. Thus, there exists $m \in \{0, 1, \ldots, n\}$, such that $y = a\gamma_m \in B(r)$. Let $x \in aA \cap B(r)$. Then $x = az$, for some $z \in A$. The elements $x$ and $y$ are joined by a path $\varepsilon = (\varepsilon_0 = x, \varepsilon_1, \ldots, \varepsilon_k = y)$ in $B(r)$, for some $k \in \mathbb{N}$. The sequence:

$$\varepsilon' = a^{-1}\varepsilon = (a^{-1}\varepsilon_0, a^{-1}\varepsilon_1, \ldots, a^{-1}\varepsilon_k)$$

is a path that joins $a^{-1}x = z \in A$ to $a^{-1}y = \gamma_m \in U$. Therefore, $\varepsilon'$ passes through $B(r)$. Thus, there exists $j \in \{1, 2, \ldots, k\}$ such that $a^{-1}\varepsilon_j \in B(r)$. But then, $\varepsilon_j \in B(r) \cap aB(r) \subset B(r) \cap A$, which is a contradiction since $B(r) \cap A = \emptyset$. 
In conclusion, for any integer \( r \geq 2 \), we have that \( V_0^X(r) \leq 4r \). Hence, \( V_0^X \) has linear growth.

\[ \square \]

4. On Ends of Groups

The main objective of this section is to present an alternative approach to question VI.19, posed by Pierre de la Harpe in [4]. Theorem 3 answers this question and it was first proven by Erschler [4], based on the Varopoulos inequality, and later by Timar [13], using different methods. We give a geometric proof, without using the Varopoulos inequality.

**Proposition 1.** Let \( G = \langle S \rangle \) be a finitely generated group and \( X = \Gamma(G,S) \). Suppose that there is a positive integer \( n \) and a sequence of positive integers \( \{r_i\} \) so that, for any \( i \in \mathbb{N} \), there exists a compact subset \( K_i \) of \( X \), with the following properties:

1. \( \text{diam}(K_i) < n \)
2. \( N(K_i, r_i) \setminus K_i \) has at least two connected components, \( A_i \) and \( B_i \)
3. \( \lim_{i \to \infty} r_i = \lim_{i \to \infty} \text{diam}(A_i) = \lim_{i \to \infty} \text{diam}(B_i) = \infty \)

Then \( e(G) > 1 \).

\[ \text{Figure 5.} \]

**Proof.** We may assume that, for all \( i \geq 1 \), \( K_i \) is a graph.

\( G \) is finitely generated and for any \( i \), the set \( K_i \) has diameter less than \( n \), so the number of edges in \( K_i \) is less than \( |S|^{2n} \). Therefore, there exists a subsequence, \( \{K_{i_j}\} \), such that any two sets of this subsequence are isometric. We re-index this subsequence to avoid double indices, so we
write $K_j$ for $K_{ij}$. The action of $G$ on $X$ is by isometries, so there exists a subsequence of $\{K_j\}_j$, that we still denote by $\{K_j\}_j$ for convenience, so that for any $j > 1$ there is $g_j \in G$ such that $g_jK_j = K_1$. Again, as $G$ is finitely generated and for any $j > 1$, $\text{diam}(K_j) < n$, we conclude that the number of connected components of $X \setminus K_j$ is uniformly bounded. Therefore, there exists yet another subsequence of $\{K_j\}_j$, denoted also for convenience by $\{K_j\}_j$, so that

$$\bigcap_{j>1} g_j A_j \cap A_1 \neq \emptyset \quad \text{and} \quad \bigcap_{j>1} g_j B_j \cap B_1 \neq \emptyset$$

Now, let $A$ and $B$ be connected components of $X \setminus K_1$ such that $A_1 \subset A$ and $B_1 \subset B$. Then, for all $j$, we have that $g_j A_j \subset A$ and $g_j B_j \subset B$, so $\text{diam}(A) \geq \text{diam}(A_j)$ and $\text{diam}(B) \geq \text{diam}(B_j)$. This implies that $\text{diam}(A) = \infty$ and $\text{diam}(B) = \infty$.

Finally, we will argue by contradiction that $A$ and $B$ are different connected components of $X \setminus K_1$. So, suppose that $A$ and $B$ are connected in $X \setminus K_1$. Let $x \in \bigcap_{j>1} g_j A_j \cap A_1$ and $y \in \bigcap_{j>1} g_j B_j \cap B_1$, so that $d_S(x, K_1) = d_S(y, K_1) = 1$. Then, there exists a finite path, $\gamma$, of length $l \in \mathbb{N}$ in $X \setminus K_1$ that joins $x$ to $y$. Clearly, for any $j > 1$, $\gamma_j = g_j^{-1}\gamma$ is a finite path of length $l$, that joins $x_j = g_j^{-1}x \in A_j$ to $y_j = g_j^{-1}y \in B_j$. Thus there exists $m \in \mathbb{N}$ so that, for any $j > m$, the path $\gamma_j$ is contained in $N(K_j, r_j)$. Note that, for any $j > m$, the elements $x_j$ and $y_j$ are connected outside $N(K_j, r_j)$ and that their distance from $X \setminus N(K_j, r_j)$ is greater than $r_j - 2$. Therefore, we reach to the conclusion that, for any $j > m$, $l > r_j - 2$. This however contradicts our hypothesis that $\lim_{i \to \infty} r_i = \infty$.

Hence, $K_1$ is a compact subset of $X$ with, at least, two unbounded connected components. Thus $e(G) > 1$. \hfill \Box

Remark 2. It is worth mentioning that Proposition\[7\] does not hold for arbitrary metric spaces. For example, consider the space $X = [0, \infty)$ with the usual metric. Then, $X$ is a one ended metric space and it is easy to see that all the conditions of Proposition \[7\] hold for $X$: For any $r \in \mathbb{N}$, we set $K_r = [r + 1, r + 2]$. Then, for any $r \in \mathbb{N}$, $K_r$ is a compact subset of $X$ with $\text{diam}(K_r) < 2$. Moreover, the connected components of $X \setminus K_r$ are the sets $A_r = [0, r + 1)$ and $B_r = (r + 2, \infty)$, with $\text{diam}(A_r) > r$ and $\text{diam}(B_r) = \infty$.

In the following theorem we will use Proposition\[7\] to give, as mentioned in the introduction, an alternative approach to question VI.19 posed by Pierre de la Harpe in \[4\].
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Figure 6.

**Theorem 3.** Let $G = \langle S \rangle$ be a finitely generated group and $X = \Gamma(G, S)$. If there exists a sequence of positive integers $\{r_i\}_{i \geq 1}$ such that $\lim_{i \to \infty} r_i = \infty$ and $\lim_{i \to \infty} |S(r_i)| < \infty$, then $G$ is virtually cyclic.

In the case that $G$ is infinite, we will, upon passing to a subsequence, split, for any $t$, the set $S(r_t)$ into 2 subsets, $K_t$ and $F_t$, whose distance tends to infinity and so that $\{\text{diam}(K_t)\}_t$ is bounded. Finally, we show that we can apply Proposition 1 for the sets $K_t$.

**Proof.** This is trivial if $G$ is finite, so suppose that $G$ is infinite, thus $e(G) \geq 1$. For simplicity, let $d = d_S$. There exists a bi-infinite geodesic path, $\gamma = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots)$, of vertices in $X$, where $\gamma_0$ is the identity element of $G$. For all $i \geq 1$, $X \setminus S(r_t)$ has an unbounded connected component $U_i$, such that, for any $j \geq r_t + 1$ $\gamma_j \in U_i$

Obviously, then, for all $i \geq 1$

$U_{i+1} \subset U_i$

On the other hand, $\lim_{i \to \infty} |S(r_i)| < \infty$, so the sequence $\{|S(r_i)|\}_i$ is bounded. Hence $e(G) < \infty$ and there exist a positive integer $m$ and a subsequence $\{r_{i_l}\}_l$, such that $\lim_{l \to \infty} r_{i_l} = \infty$ and, for all $l > 0$, $|S(r_{i_l})| = m$. As usual, we re-index this subsequence to avoid double indices, so we write $r_t$ for $r_{i_l}$. For any $l > 0$, let $S(r_l) = \{x_1(l), x_2(l), \ldots, x_m(l)\}$

We may assume that, for all $l > 0$, $x_1(l) = \gamma_{r_t}$ and $x_m(l) = \gamma_{-r_t}$. For any $l > 0$, we consider, for all $j, k \in \{1, \ldots, m\}$, the distances $d(x_j(l), x_k(l))$. Using a diagonal argument, we can extract a subsequence $\{r_t\}_t$, so that the distances of any two elements of the sphere $S(r_t)$ converge in $\mathbb{R} \cup \{\infty\}$. Again, to keep notation simple we denote this sequence by $\{r_t\}_t$. Therefore, we have that, for any $j, k \in$
{1, \ldots, m}:

\[
\lim_{t \to \infty} d(x_j(t), x_k(t)) = a_{jk} \in \mathbb{R} \cup \{\infty\}
\]

This implies that there exists a partition $\mathcal{P}$ of the set $\{1, \ldots, m\}$ so that any $j, k \in \{1, \ldots, m\}$ belong to the same set of the partition if and only if $a_{jk} < \infty$.

We note that:

\[
a_{1m} = \lim_{t \to \infty} d(\gamma_{rt}, \gamma_{-rt}) = \infty
\]

Therefore, the partition $\mathcal{P}$ is not a singleton.

Now, let $Y \in \mathcal{P}$. For any positive integer $t$, we define the $t-$corresponding set of $Y$ in $X$ as follows:

\[
Y_t = \{x_j(t) \mid j \in Y\}
\]

Let $K \in \mathcal{P}$, so that $1 \in K$. We will show that we can apply Proposition 1 for the $t-$corresponding sets $K_t$. For any $t > 0$, $x_1(t) = \gamma_{rt} \in K_t$, thus $d(K_t, U_t) = 1$. Furthermore:

\[
\lim_{t \to \infty} \text{diam}(K_t) = \lim_{t \to \infty} \sup\{d(x_j(t), x_k(t)) \mid j, k \in K\} < \infty
\]

So, there exists $M > 0$ such that, for all $t > 0$,

\[
\text{diam}(K_t) < M
\]

We denote the set $\{1, \ldots, m\} \setminus K$ by $F$. Considering that $\mathcal{P}$ is not a singleton, we have that $F \neq \emptyset$. The $t-$corresponding set of $F$ in $X$ is the set $F_t = S(r_t) \setminus K_t$. For any $t > 0$, we set

\[
D_t = \frac{1}{2} d(F_t, K_t)
\]

Note that, since $K$ and $F$ are distinct, non empty sets of $\mathcal{P}$, we have that:

\[
\lim_{t \to \infty} D_t = \frac{1}{2} \lim_{t \to \infty} \inf\{d(x_j(t), x_k(t)) \mid j \in F, k \in K\} = \infty
\]

Without loss of generality we assume that, for all $t > 0$, $D_t > 2$. For any $t > 0$, we let

\[
N_t = N(K_t, D_t)
\]

Then, for any $t > 0$, we have that $d(N_t, F_t) > 1$, thus $N_t \cap F_t = \emptyset$.

Finally, for any $t > 0$, let

\[
A_t = N_t \cap U_t \quad \text{and} \quad B_t = N_t \cap (B(r_t) \setminus K_t)
\]

Then, for all $t > 0$, $\gamma_{rt+1} \in A_t$ and $\gamma_{rt-1} \in B_t$, so the sets $A_t$ and $B_t$ are non-empty. Moreover, it is immediate from the definitions that the sets $A_t$ and $B_t$ are different connected components of $N_t \setminus K_t$. 

Finally, for any \( t > 0 \), we have that \( \gamma_{r_t + D_t - 1} \in A_t \) and \( \gamma_{r_t - D_t + 1} \in B_t \). Hence:

\[
diam(A_t) \geq d(\gamma_{r_t + 1}, \gamma_{r_t + D_t - 1}) = D_t - 2
\]

and

\[
diam(B_t) \geq d(\gamma_{r_t - 1}, \gamma_{r_t - D_t + 1}) = D_t - 2
\]

therefore

\[
\lim_{t \to \infty} diam(A_t) = \lim_{t \to \infty} diam(B_t) = \infty
\]

From Proposition \[1\] it follows that \( e(G) > 1 \). We recall that \( e(G) < \infty \), so from Hopf’s Theorem (Theorem \[1\]) we derive that \( e(G) = 2 \), thus \( G \) is virtually \( \mathbb{Z} \). \( \square \)

5. Linear Growth

The main objective of this section is to give a characterization for groups that are virtually cyclic. More specifically, in Theorem \[4\] we give a condition for the growth of spheres in \( G \) that results to \( G \) being virtually \( \mathbb{Z} \). This theorem is, as mentioned in the introduction, a weaker version of theorems proven by Justin \[5\], Wilkie and Van Den Dries \[14\], Imrich and Seifter \[6\], Shalom and Tao \[11\]. The techniques used in this paper are quite elementary and the proof we give has a strong geometric flavour.

We start by giving some definitions.

**Definition 3.** Let \((Y, d)\) be a metric space, \( a, m \) positive integers, with \( m \geq 2 \), and \( A_1, \ldots, A_m \) non-empty subsets of \( Y \). We say that \( \mathcal{A} = (\{A_i\}_i, m, a) \) is a gl-partition of the space \( Y \), if

1. for any \( i, j \in \{1, \ldots, m\} \), either \( A_i = A_j \) or \( A_i \cap A_j = \emptyset \)

\}
(2) \( Y = \bigsqcup_{i=1}^{m} A_i \)

(3) for any \( i \in \{1, \ldots, m\} \), \( d(A_i, Y \setminus A_i) > a \cdot \max\{\text{diam}(A_j), 1 \mid j = 1, \ldots, m\} \)

**Definition 4.** Let \( (Y, d) \), \( (Z, d') \) be metric spaces with gl-partitions \( A = (\{A_i\}_i, k_1, a) \), \( B = (\{B_i\}_i, k_2, b) \) respectively. We say that \( A \) and \( B \) are similar gl-partitions, if:

(1) \( a = b \)

(2) \( k_1 = k_2 \)

(3) After some rearrangement if necessary, for all \( i = 1, \ldots, k_1 \), \( A_i \) is isometric to \( B_i \)

**Remark 3.** It is an immediate consequence of the definitions that if \( (Y, d) \) and \( (Z, d') \) are isometric metric spaces and \( A \) is a gl-partition of \( Y \), then there exists a gl-partition of \( Z \), similar to \( A \).

We state now a lemma that gives an insight to the structure of a finite metric space that has big diameter compared to the number of its elements.

**Lemma 1** (Distant Galaxies Lemma). Let \( (Y, d) \) be a finite metric space and \( a \in \mathbb{Z} \) greater than 2. Suppose that \( Y \) has \( n \) elements and diameter greater than \((2a + 1)n^2 \). Then there exists a gl-partition \( A = (\{A_i\}_i, n, a) \) of \( Y \).

What we state in this lemma is intuitively obvious, since one expects that if we have very few points to distribute on a great distance, then distant groups of points will be formed, forming in a way distant galaxies in \( Y \).

**Proof.** Suppose that \( Y = \{y_1, \ldots, y_n\} \). For any \( i = 1, \ldots, n \), we set

\[
A_0(i) = \{y_i\} \quad \text{and} \quad d_0 = 1
\]

and we define inductively, for any positive integer \( m \):

\[
A_m(i) = \{y \in Y \mid d(y, A_{m-1}(i)) \leq a \cdot d_{m-1}\}
\]

\[
d_m = \max_{1 \leq j \leq n} \{\text{diam}(A_m(j)), 1\}
\]

Then, for any \( m > 0 \) and \( i = 1, \ldots, n \), we have that

\[
d_m \leq \text{diam}(A_{m-1}(i)) + 2ad_{m-1}
\]

thus,

\[
d_m \leq (2a + 1)d_{m-1}
\]
and finally

\[ d_m \leq (2a + 1)^m \]

Since \( Y \) has \( n \) elements, for any \( i = 1, \ldots, n \), the sequence \( \{A_m(i)\}_m \) is finally constant. So, let \( k \) be the minimum positive integer such that, for all \( i = 1, \ldots, n \), we have that \( A_k(i) = A_{k+1}(i) \). We then denote the set \( A_k(i) \) by \( A_i \).

Note that, from the construction of the sets \( \{A_i\}_i \), we have that, for any \( i = 1, \ldots, n \),

\[ d(A_i, Y \setminus A_i) > a \cdot \max\{diam(A_i), 1 \mid i = 1, \ldots, n\} \]

We will show that, for any \( i \neq j \in \{1, \ldots, n\} \), either \( A_i \cap A_j = \emptyset \) or \( A_i = A_j \).

Let \( i \neq j \in \{1, \ldots, n\} \) such that \( A_i \cap A_j \neq \emptyset \). Then, for any \( y \in A_i \) we have that \( d(y, A_j) \leq d_k \), so \( y \in A_{k+1}(j) = A_j \). Therefore \( A_i \subset A_j \). Similarly, get that \( A_j \subset A_i \), hence \( A_i = A_j \).

In order to proceed we have to show that the steps needed to define the sets \( A_i \) are at most \( n + 1 \). In each step prior to the \( k^{th} \), at least two of the sets \( \{A_m(i)\}_{i=1, \ldots, n} \) have a non-empty intersection. So, eventually these two sets get identified. An example is illustrated in figure 8.

\[ \text{Figure 8.} \]

Therefore, we need at most \( n + 1 \) steps to define the sets \( A_i \), so \( k \leq n + 1 \). Finally, we will show that, for any \( i = 1, \ldots, n \), \( A_i \neq Y \):

Suppose, to the contrary, that there exists \( i \in \{1, \ldots, n\} \), such that \( A_i = Y \). Then:

\[ diam(Y) = diam(A_i) \leq d_k \leq (2a + 1)^k \Rightarrow \]

\[ (2a + 1)^{n+2} \leq (2a + 1)^k \Rightarrow k \geq n + 2 \]

But this contradicts the fact that \( k \) is at most \( n + 1 \).

Therefore, \( A_i \neq Y \), for any \( i \in \{1, \ldots, n\} \). We conclude that \( \mathcal{A} = (\{A_j\}, n, a) \) is a gl-partition of the metric space \( Y \). \( \square \)
Theorem 4. Let $G = \langle S \rangle$ be a finitely generated group and $X = \Gamma(G, S)$. If there are $a, n \in \mathbb{N}$ with $a \geq 100$ and $n \geq 2$, such that a sphere of radius $(2a + 1)^{n+2}$ in $X$ has at most $n$ elements, then $G$ is virtually cyclic.

Proof. This is trivial if $G$ is finite, so suppose that $G$ is infinite, thus $e(G) \geq 1$.

For simplicity, let $d = d_S$. There exists a bi-infinite geodesic path, $\gamma = \ldots, \gamma_1, \gamma_0, \gamma_1, \ldots$, of vertices in $X$, where $\gamma_0$ is the identity element of $G$. Let $r = (2a + 1)^{n+2}$ and for any $i \in \mathbb{Z}$ denote the sphere $S(\gamma_i, r)$ by $S_i$.

Since $\gamma_{i+r}, \gamma_{i-r} \in S_i$, we get that $\text{diam}(S_i) > r$. Therefore, from the Distant Galaxies Lemma, it follows that for any $i \in \mathbb{Z}$, there exists a gl-partition, of the set $S_i$. On the other hand, for any $i, j \in \mathbb{Z}$, the sets $S_i$ and $S_j$ are isometric, so there exist similar gl-partitions of these sets. Thus, for any $i \in \mathbb{Z}$, let $A_i = \{A_l(i)\}_{l=1}^k$ be a gl-partition of $S_i$, such that for any $j \in \mathbb{Z}$, $A_i$ and $A_j$ are similar. Let

$$D_i = \max\{\text{diam}(A_l(i)), 1 \mid l = 1, \ldots, k\}$$

For any $i, j \in \mathbb{Z}$, since $A_i$ and $A_j$ are similar gl-partitions, $D_i = D_j$, so we denote $D_i$ by $D$. Also, for any $i \in \mathbb{Z}$, we denote by $A_i$ the set of the gl-partition $A_i$, that $\gamma_{i+r}$ belongs to.

![Figure 9.](image)

Then, for any $i \in \mathbb{Z}$, we have that

$$\text{diam}(A_i) \leq D \quad \text{and} \quad d(A_i, S_i \setminus A_i) > aD$$

We note that $r > aD \geq 100D$.

Let $B = B(\gamma_0, 39D)$ and $x \in B$, then:

(a) $|d(x, \gamma_i) - d(x, \gamma_{i+1})| \leq d(\gamma_i, \gamma_{i+1}) = 1$, for any $i \in \mathbb{Z}$
(b) $d(x, \gamma_{40D-r}) \leq d(x, \gamma_0) + d(\gamma_0, \gamma_{40D-r}) \leq 39D + r - 40D < r$
(c) $d(x, \gamma_{-40D-r}) \geq d(\gamma_0, \gamma_{-40D-r}) - d(\gamma_0, x) \geq 40D + r - 39D > r$
Therefore, there exists \( m \in \{-40D - r, \ldots, 40D - r\} \), so that \( d(x, \gamma_m) = r \), thus \( x \in S_m \).

Furthermore:

\[
d(x, A_m) \leq d(x, \gamma_{r+m}) \leq d(x, \gamma_0) + d(\gamma_0, \gamma_{r+m}) \leq 79D < aD
\]

Thus, \( x \in A_m \) and \( d(x, \gamma_{r+m}) \leq D \), where \( r + m \in \{-40D, \ldots, 40D\} \).

We have shown therefore that

\[
B \subset \bigcup_{i=-40D}^{40D} S(\gamma_i, D)
\]

Figure 10.

Now, since balls of the same radius in \( X \) are isometric, by moving along the path \( \gamma \) we can easily see that

\[
X = \bigcup_{i \in \mathbb{Z}} S(\gamma_i, D)
\]

Therefore \( X \) is quasi-isometric to \( \mathbb{Z} \), so \( e(G) = 2 \). \hfill \Box

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