$k$-nets embedded in a projective plane over a field

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Abstract

We investigate $k$-nets with $k \geq 4$ embedded in the projective plane $PG(2, \mathbb{K})$ defined over a field $\mathbb{K}$; they are line configurations in $PG(2, \mathbb{K})$ consisting of $k$ pairwise disjoint line-sets, called components, such that any two lines from distinct families are concurrent with exactly one line from each component. The size of each component of a $k$-net is the same, the order of the $k$-net. If $\mathbb{K}$ has zero characteristic, no embedded $k$-net for $k \geq 5$ exists; see [10, 13]. Here we prove that this holds true in positive characteristic $p$ as long as $p$ is sufficiently large compared with the order of the $k$-net. Our approach, different from that used in [10, 13], also provides a new proof in characteristic zero.

1 Introduction

An (abstract) $k$-net is a point-line incidence structure whose lines are partitioned in $k$ subsets, called components, such that any two lines from distinct components are concurrent with exactly one line from each component. The components have the same size, called the order of the $k$-net and denoted by

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A $k$-net has $n^2$ points and $kn$ lines. A $k$-net (embedded) in $PG(2, \mathbb{K})$ is a subset of points and lines such that the incidence structure induced by them is a $k$-net.

In the complex plane, there are known plenty of examples and even infinity families of 3-nets but only one 4-net up to projectivity; see [10, 11, 12, 13]. This 4-net, called the classical 4-net, has order 3 and it exists since $PG(2, \mathbb{C})$ contains an affine subplane $AG(2, \mathbb{F}_3)$ of order 3, unique up to projectivity, and the four parallel line classes of $AG(2, \mathbb{F}_3)$ are the components of a 4-net in $PG(2, \mathbb{C})$. By a result of Stipins [10], see also [13], no $k$-net with $k \geq 5$ exists in $PG(2, \mathbb{C})$. Since Stipins’ proof works over any algebraically closed field of characteristic zero, his result holds true in $PG(2, \mathbb{K})$ provided that $\mathbb{K}$ has zero characteristic.

Our present investigation of $k$-nets in $PG(2, \mathbb{K})$ includes groundfields $\mathbb{K}$ of positive characteristic $p$, and as a matter of fact, many more examples. This phenomena is not unexpected since $PG(2, \mathbb{K})$ with $\mathbb{K}$ of characteristic $p > 0$ contains an affine subplane $AG(2, \mathbb{F}_p)$ of order $p$ from which $k$-nets for $3 \leq k \leq p + 1$ arise taking $k$ parallel line classes as components. Similarly, if $PG(2, \mathbb{K})$ also contains an affine subplane $AG(2, \mathbb{F}_{p^h})$, in particular if $\mathbb{K} = \mathbb{F}_q$ with $q = p^r$ and $h|r$, then $k$-nets of order $p^h$ for $3 \leq k \leq p^h + 1$ exist in $PG(2, \mathbb{K})$. Actually, more families of $k$-nets in $PG(2, \mathbb{F}_q)$ when $q = p^r$ with $r \geq 3$ exist; see Example 5.3. On the other hand, no 5-net of order $n$ with $p > n$ is known to exist. This suggests that for sufficiently large $p$ compared with $n$, Stipins’ theorem remains valid in $PG(2, \mathbb{K})$. Our Theorem 5.2 proves it for $p > 3^\varphi(n^2 - n)$ where $\varphi$ is the classical Euler $\varphi$ function, and in particular for $p > 3^{n^2/2}$. Our approach also works in zero characteristic and provides a new proof for Stipins’ result.

A key idea in our proof is to consider the cross-ratio of four concurrent lines from different components of a 4-net. Proposition 5.1 states that the cross-ratio remains constant when the four lines vary without changing component. In other words, every 4-net in $PG(2, \mathbb{K})$ has constant cross-ratio. By Theorem 4.2 in zero characteristic, and by Theorem 4.3 in characteristic $p$ with $p > 3^\varphi(n^2 - n)$, the constant cross-ratio is restricted to two values only, namely to the roots of the polynomial $X^2 - X + 1$. From this, the non-existence of $k$-nets for $k \geq 5$ easily follows both in zero characteristic and in characteristic $p$ with $p > 3^\varphi(n^2 - n)$. It should be noted that without a suitable hypothesis on $n$ with respect to $p$, the constant cross-ratio of a 4-net may assume many different values, even for finite fields, see Example 5.3.

In $PG(2, \mathbb{K})$, $k$-nets naturally arise from pencils of curves, the components
of the $k$-net being the completely reducible curves in the pencil. This has given a motivation for the study of $k$-nets in Algebraic geometry; see [2], and [12]. We discuss this relationship in Section 2 and state an equation that will be useful in Section 3.

2 $k$-nets and completely irreducible curves in a pencil of curves

Let $\lambda_1, \lambda_2, \lambda_3$ be three components of a $k$-net of order $n$ embedded in $PG(2, \mathbb{K})$. Let $r_i = 0, w_i = 0, t_i = 0$ ($i = 1, \ldots, n$) be the equations of the lines in $\lambda_1, \lambda_2, \lambda_3$, respectively. The completely reducible polynomials $R = r_1 \cdots r_n$, $W = w_1 \cdots w_n$ and $T = t_1 \cdots t_n$ define three plane curves of degree $n$, say $\mathcal{R}$, $\mathcal{W}$ and $\mathcal{T}$. Consider the pencil $\Lambda$ generated by $\mathcal{R}$ and $\mathcal{W}$. Since $\lambda_1, \lambda_2, \lambda_3$ are the components of a $3$-net of order $n$, there exist $\alpha, \beta \in \mathbb{K}^*$ such that $\mathcal{T}$ and the curve $\mathcal{H}$ of $\Lambda$ with equation $\alpha R + \beta W = 0$ have $n^2 + 1$ common points but no common components. From Bézout’s theorem, $\mathcal{T} = \mathcal{H}$. Therefore,

$$\alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma t_1 \cdots t_n = 0 \quad (1)$$

holds for a homogeneous triple $(\alpha, \beta, \gamma)$ with coordinates $\mathbb{K}^*$. Changing the projective coordinate system in $PG(2, \mathbb{K})$ the equations of the lines in the components of the $3$-net change but the homogeneous triple $(\alpha, \beta, \gamma)$ remains invariant.

Conversely, assume that an irreducible pencil $\Lambda$ of plane curves of degree $n$ contains $k$ curves each splitting into $n$ distinct lines, that is, $k$ completely reducible curves. Let $\lambda_i$ with $1 \leq i \leq k$ be the set of the $n$ lines which are the factors of a completely reducible curve. Then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the components of a $k$-net embedded in $PG(2, \mathbb{K})$.

3 The invariance of the cross-ratio of a 4-net

Consider a 4-net of order $n$ embedded in $PG(2, \mathbb{K})$ and label their components with $\lambda_i$ for $i = 1, 2, 3, 4$. We say that the 4-net $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ has constant cross-ratio if for every point $P$ of $\lambda$ the cross-ratio $(\ell_1, \ell_2, \ell_3, \ell_4)$ of the four lines $\ell_i \in \lambda_i$ through $P$ is constant.

**Proposition 3.1.** Every 4-net in $PG(2, \mathbb{K})$ has constant cross-ratio.
Proof. In a projective reference system, let \( r_i = 0, w_i = 0, t_i = 0, s_i = 0 \) with \( 1 \leq i \leq n \) be the lines of a 4-net \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) respectively. Then there exist \( \alpha, \beta, \gamma \in \mathbb{K}^* \) such that (1) holds and \( \alpha', \beta', \gamma' \in \mathbb{K} \) such that

\[
\alpha' r_1 r_2 \cdots r_n + \beta' w_1 w_2 \cdots w_n + \gamma' s_1 s_2 \cdots s_n = 0.
\]

As observed in Section 2, the coefficients \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) remain invariant when the reference system is changed. Take a point \( P \) of \( \lambda \) and relabel the lines of \( \lambda \) such that \( r_1 = 0, w_1 = 0, t_1 = 0 \) and \( s_1 = 0 \) are the four lines of \( \lambda \) passing through \( P \). We temporarily introduce the notation \((x_1, x_2, x_3)\) for the homogeneous coordinates of a point, and we arrange the reference system in such a way that \( P \) coincides with the point \((0, 0, 1)\), the line \( x_3 = 0 \) contains no point from \( \lambda_1 \) or \( \lambda_2 \) while \( r_1 = x_1 \) and \( w_1 = x_2 \). Also, non-homogeneous coordinates \( x = x_1/x_3 \) and \( y = x_2/x_3 \) can be used so that \( r_1 = x \) and \( w_1 = y \). Note that we have arranged the coordinates so that \( r_i, w_i, t_i, s_i \) have a zero constant term if and only if \( i = 1 \). Let

\[
\rho = \prod_{i=2}^{n} r_i(0, 0), \quad \omega = \prod_{i=2}^{n} w_i(0, 0), \quad \tau = \prod_{i=2}^{n} t_i(0, 0), \quad \sigma = \prod_{i=2}^{n} s_i(0, 0).
\]

Observe that

\[
0 = \alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma t_1 \cdots t_n = \alpha \rho x + \beta \omega y + \gamma \tau t_1 + [\cdots],
\]

where \([\cdots]\) stands for the sum of terms of degree at least 2. From (1),

\[
\frac{\alpha \rho}{\gamma \tau} x + \frac{\beta \omega}{\gamma \tau} y + t_1 = 0.
\]

Similarly,

\[
\frac{\alpha' \rho}{\gamma' \sigma} x + \frac{\beta' \omega}{\gamma' \sigma} y + s_1 = 0.
\]

Therefore, the cross-ratio of the lines of \( \lambda \) passing through \( P \) is equal to

\[
\kappa = \frac{\alpha \beta'}{\alpha' \beta}
\]

and hence it is independent of the choice of the point \( P \). ■

As an illustration of Proposition 3.1 we compute the constant cross-ratio of the known 4-net embedded in the complex plane.
Example 3.2. Let $n = 3$, and take a primitive third root of unity $\xi$. In homogeneous coordinates $(x, y, z)$ of $PG(2, \mathbb{K})$, let

\[
\begin{align*}
    r_1 &:= x, & r_2 &:= y, & r_3 &:= z, \\
    w_1 &:= x + y + z, & w_2 &:= x + \xi y + \xi^2 z, & w_3 &:= x + \xi^2 y + \xi z, \\
    t_1 &:= \xi x + y + z, & t_2 &:= x + \xi y + z, & t_3 &:= x + y + \xi z, \\
    s_1 &:= \xi^2 x + y + z, & s_2 &:= x + \xi^2 y + z, & s_3 &:= x + y + \xi^2 z.
\end{align*}
\]

Then these lines form a 4-net $\lambda$ order 3. Moreover,

\[
\begin{align*}
    t_1 t_2 t_3 &= 3(2\xi + 1)r_1 r_2 r_3 + \xi w_1 w_2 w_3, \\
    s_1 s_2 s_3 &= -3(2\xi + 1)r_1 r_2 r_3 + \xi^2 w_1 w_2 w_3.
\end{align*}
\]

Hence, the constant cross-ratio of $\lambda$ is $\kappa = -1/\xi$.

4 Some constraints on the constant cross-ratio of a 4-net

It is well known that the cross-ratio of four distinct concurrent lines can take six possible different values depending on the order in which the lines are given. If $\kappa$ is one of them then $\kappa \neq 0, 1$ and these six cross-ratios are

\[
\kappa, \quad \frac{1}{\kappa}, \quad 1 - \kappa, \quad \frac{1}{1 - \kappa}, \quad \frac{\kappa}{\kappa - 1}, \quad 1 - \frac{1}{\kappa}.
\]

It may happen, however, that some of these values coincide, and this is the case if and only if either $\kappa \in \{-1, 1/2, 2\}$, or

\[
\kappa^2 - \kappa + 1 = 0. \tag{4}
\]

Proposition 3.1 says that the cross-ratio of four concurrent lines of a 4-net takes the above six values for a given $\kappa \neq 0, 1$, and each of these values can be considered as the constant cross-ratio of the 4-net. Now, the problem consists in computing $\kappa$. We are able to do it in zero characteristic showing that $\kappa$ satisfies Equation (4). In positive characteristic there are more possibilities. This will be discussed after proving the following result.

**Proposition 4.1.** Let $\lambda$ be a 4-net of order $n$ embedded in $PG(2, \mathbb{K})$. Then the cross-ratio $\kappa$ of $\lambda$ is an $N$–th root of unity of $\mathbb{K}$ such that $N = n(n - 1)$ and

\[
(\kappa - 1)^N = 1. \tag{5}
\]
Proof. We prove first that $\kappa^N = 1$. Let $P_{ij}$ be the common point of the lines $r_i$ and $w_j$ with $1 \leq i, j \leq n$. Then the unique line from $\lambda_3$ through $P_{ij}$ has equation $\sigma_{ij}r_i + \tau_{ij}w_j$ with $\sigma_{ij}, \tau_{ij} \in \mathbb{K}^*$. Moreover, for any $k = 1, \ldots, n$ there is a unique index $j$ such that $t_k = \sigma_{ij}r_i + \tau_{ij}w_j$. For every $i = 1, \ldots, n$,

$$\alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma[(\sigma_{i1}r_i + \tau_{i1}w_1) \cdots (\sigma_{in}r_i + \tau_{in}w_n)] = 0. \quad (6)$$

Take a point $Q$ on the line $r_i = 0$ such that $w_j(Q) \neq 0$ for every $1 \leq j \leq n$. Then

$$w_1(Q) \cdots w_n(Q)(\beta + \gamma \prod_{j=1}^n \tau_{ij}) = 0$$

yields

$$-\frac{\beta}{\gamma} = \prod_{j=1}^n \tau_{ij} \quad (7)$$

for any fixed index $i$. The above argument applies to any line $w_j$ and gives

$$-\frac{\alpha}{\gamma} = \prod_{i=1}^n \sigma_{ij} \quad (8)$$

for any fixed index $j$. Therefore,

$$\left(\frac{\beta}{\alpha}\right)^n = \prod_{i=1}^n \prod_{j=1}^n \frac{\tau_{ij}}{\sigma_{ij}}. \quad (9)$$

A similar argument can be carried out for $\lambda_4$. The unique line from $\lambda_4$ through $P_{ij}$ has equation $\delta_{ij}r_i + \omega_{ij}w_j$ with $\delta_{ij}, \omega_{ij} \in \mathbb{K}^*$. Then

$$\left(\frac{\beta'}{\alpha'}\right)^n = \prod_{i=1}^n \prod_{j=1}^n \frac{\omega_{ij}}{\delta_{ij}}. \quad (10)$$

From Lemma 3.1,

$$\frac{\tau_{ij}}{\sigma_{ij}} \cdot \frac{\delta_{ij}}{\omega_{ij}} = \kappa$$

for every $1 \leq i, j \leq n$. Then Equations (9) and (10) yield $\kappa^n = \kappa^{n^2}$ whence

$$\kappa^N = 1. \quad (11)$$

From the discussion at the beginning of this section, Equation (11) holds true when $\kappa$ is replaced with any of the other five cross-ratio values. Therefore, (5) also holds. \qed
In the complex plane, the cross-ratio equation has only two solutions, namely the roots of (4). In fact, let \( \kappa = x + yi \) with \( x, y \in \mathbb{R} \). Then with respect to the complex norm, (11) and (5) imply \( |x + yi| = x^2 + y^2 = 1 \) and \( |x - 1 + iy| = (x - 1)^2 + y^2 = 1 \). It hence follows that \( \kappa = \frac{1}{2}(1 \pm \sqrt{3}i) \), or equivalently (4).

To extend this result to any field of characteristic zero, and discuss the positive characteristic case, look at

\[
 f(X) = \frac{X^N - 1}{X - 1} \text{ and } g(X) = \frac{(X - 1)^N - 1}{X}
\]
as polynomials in \( \mathbb{Z}[X] \). From the preceding discussion on the complex case, their maximum common divisor is either \( X^2 - X + 1 \), or 1 according as 6 divides \( N \) or does not. In the former case, divide both by \( X^2 - X + 1 \) and then replace \( f(X) \) and \( g(X) \) by them accordingly. Now, \( f(X) \) and \( g(X) \) are coprime, and hence their resultant is a non-zero integer \( R \). Using a basic formula on resultants, see [4, Lemma 2.3], \( R \) may be computed in terms of a primitive \( N \)-th root of unity \( \xi \), namely

\[
 R = \prod_{1 \leq i,j \leq N-1} (1 + \xi^i - \xi^j), \text{ when } 6 \nmid N,
\]
and

\[
 R = \prod_{\substack{1 \leq i,j \leq N-1 \\text{ s.t. } i,j \neq N/6,5N/6}} (1 + \xi^i - \xi^j), \text{ when } 6 \mid N,
\]
hold in the \( N \)-th cyclotomic field \( \mathbb{Q}(\xi) \). Therefore, \( R \neq 0 \) provided that \( \mathbb{K} \) has zero characteristic.

**Theorem 4.2.** Let \( \mathbb{K} \) be a field of characteristic 0. If a 4-net \( \lambda \) is embedded in \( PG(2, \mathbb{K}) \) then \(-3\) is a square in \( \mathbb{K} \) and the constant cross-ratio \( \kappa \) of \( \lambda \) satisfies (4).

To investigate the positive characteristic case, we will use the well known result that \( \mathbb{Q}(\xi) \) is a cyclic Galois extension of \( \mathbb{Q} \) of degree \( \varphi(N) \) where \( \varphi \) is the classical Euler function. Let \( \alpha \) be a generator of the Galois group. Then \( \alpha(\xi) = \xi^m \) for a positive integer \( m \) prime to \( N \). Therefore, \( \alpha \) permutes the factors in the right hand side. Given such a factor \( 1 + \xi^i - \xi^j \), its cyclotomic norm

\[
 \| 1 + \xi^i - \xi^j \| = (1 + \xi^i - \xi^j) \cdot (1 + \xi^i - \xi^j)^{\alpha} \cdot \ldots \cdot (1 + \xi^i - \xi^j)^{\alpha^{N-1}}
\]
is in \( \mathbb{Q} \). Actually, it is an integer since the factors are algebraic integers. Hence, the prime divisors of \( R \) come from the prime divisors of the norms \( \| 1 + \xi^i - \xi^j \| \). Therefore, to find an upper bound on the largest prime divisor of \( R \) it is enough to find an upper bound on these norms. Obviously,

\[
\| 1 + \xi^i - \xi^j \| \leq \| 1 + \xi^i - \xi^j \| \cdot \| 1 + \xi^{im} - \xi^{jm} \| \cdots \| 1 + \xi^{i(\varphi(N)-1)} - \xi^{j(\varphi(N)-1)} \|.
\]

Since \( | 1 - \xi^i + \xi^j | \leq 3 \), this shows that \( \| 1 + \xi^i - \xi^j \| \leq 3^{\varphi(N)} \). Hence the largest prime divisor of \( R \) is at most \( 3^{\varphi(N)} \). Therefore, the following result is proven.

**Theorem 4.3.** Let \( \mathbb{K} \) be a field of characteristic \( p > 0 \). If \( p > 3^{\varphi(n^2-n)} \) then Theorem 4.2 holds.

For planes over finite fields, Equations (11) and (5) may provide further non-existence results on embedded 4-nets.

**Theorem 4.4.** Let \( \mathbb{K} = \mathbb{F}_q \) be a finite field of order \( q = p^h \) with \( p \) prime. If \( p \neq 3 \), then there exists no 4-net of order \( n \) embedded in \( \text{PG}(2, \mathbb{F}_q) \) for

\[
\gcd(n(n-1), q-1) \leq 2.
\]

**Proof.** From Equation (11), either \( \kappa = 1 \) and \( p = 2 \) or \( \kappa^2 = 1 \) and \( p > 2 \). On the other hand, \( \kappa \neq 1 \). Hence \( \kappa = -1 \) and \( p > 2 \). Now, Equation (5) yields \( p = 3 \), a contradiction. \( \square \)

The following example shows that the hypothesis \( p \neq 3 \) in Theorem 4.4 is essential.

**Example 4.5.** Let \( q = 3^r \), and regard \( \text{PG}(2, \mathbb{F}_q) \) as the projective closure of the affine plane \( \text{AG}(2, \mathbb{F}_q) \). The four line sets \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) form a 4-net of order \( q \) embedded in \( \text{AG}(2, \mathbb{F}_q) \) where \( \lambda_1 \) and \( \lambda_2 \) consist of all horizontal and vertical lines respectively, while \( \lambda_3 \) and \( \lambda_4 \) consist of all lines with slope 1 or \(-1\), respectively. The constant cross-ratio of this 4-net equals \(-1\).

5 Nets with more than four components

We prove the non-existence of 5-nets embedded in \( \text{PG}(2, \mathbb{K}) \) over a field \( \mathbb{K} \) of characteristic 0. This result was previously proved by Stipins [10]; see also...
Those authors used results and techniques from Algebraic geometry. Here, we present a simple combinatorial proof depending on Theorem 4.2. Our proof also works in positive characteristic $p$ whenever $p$ is big enough compared to the order $n$ of 4-net; for example, when $p > 3^{3(n^2-n)}$ so that Theorem 4.3 holds. However, the non-existence result fails in general. This will be illustrated by means of some examples.

We begin with a technical lemma.

**Lemma 5.1.** Let $A, B, C, D, D'$ be collinear points in $\mathbb{P}G(2, \mathbb{K})$ with cross-ratios $\kappa = (ABCD)$ and $1 - \kappa = (ABCD')$. If (4) holds then $(ABDD') = -\kappa$.  

**Proof.** Without loss of generality, $A = (1,0,0)$, $B = (0,1,0)$, $C = (1,1,0)$. Then $D = (\kappa,1,0)$, $D' = (1-\kappa,1,0)$, and the result follows by a direct computation. $lacksquare$

**Theorem 5.2.** If the characteristic of the field $\mathbb{K}$ is either 0 or greater than $3^{3(n^2-n)}$, then there exists no 5-net of order $n$ embedded in $\mathbb{P}G(2, \mathbb{K})$.

**Proof.** Let $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a 5-net of order $n$ embedded in $\mathbb{P}G(2, \mathbb{K})$. Then $\Lambda_5 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\Lambda_4 = (\lambda_1, \lambda_2, \lambda_3, \lambda_5)$, and $\Lambda_{45} = (\lambda_1, \lambda_2, \lambda_4, \lambda_5)$ are three different 4-nets and so we can compare their cross-ratios, say $\kappa_5 = (l_1, l_2, l_3, l_4)$, $\kappa_4 = (l_1, l_2, l_3, l_5)$, $\kappa_{45} = (l_1, l_2, l_4, l_5)$, for five lines from different components and concurrent at a point of $\Lambda$. From Proposition 4.1 each of them is a root of the polynomial $X^2 - X + 1$. Since $\Lambda_5$ and $\Lambda_4$ only differ in the last component, $\kappa_5 \neq \kappa_4$. Therefore, $\kappa_4 = 1 - \kappa_5$. From Lemma 5.1 $\kappa_{45} = -\kappa_5$. This shows that $\kappa_{45}$ is not a root of $X^2 - X + 1$ contradicting Proposition 4.1. $lacksquare$

Example 4.5 can be generalized for finite fields $\mathbb{F}_q$ with $q = p^r$ showing that $k$-nets arise from affine subplanes of $\mathbb{P}G(2, \mathbb{K})$. Such $k$-nets have order $p^h$ with $h|r$. Here, we give further $k$-nets of $p$-power order. The construction relies on an idea of G. Lunardon [7]. For the sake of simplicity, we describe the construction in terms of a dual $k$-net, that is, the components are sets of points such that a line connecting two points of different components hits any third component in precisely one point.
Example 5.3. Let $K = \mathbb{F}_q$ such that $q = r^s$ with $s \geq 3$. Take elements $u, v \in \mathbb{F}_q$ such that $1, u, v$ are linearly independent over the subfield $\mathbb{F}_r$. Take a basis $b_1, b_2$ of $\mathbb{F}_q^2$ and put $b_0 = ub_1 + vb_2$. For any $\alpha \in \mathbb{F}_r$, we define the points sets

$$A_\alpha = \{\alpha b_0 + \lambda b_1 + \mu b_2 \mid \lambda, \mu \in \mathbb{F}_r\}$$

in $AG(2, q)$. Then the $A_\alpha$'s ($\alpha \in \mathbb{F}_r$) are components of a dual $r$-net of order $r^2$. In order to see this, take the points

$$P_i = \alpha_i b_0 + \lambda_i b_1 + \mu_i b_2, \quad i = 1, 2, 3.$$  

$P_1, P_2, P_3$ are collinear in $AG(2, q)$ if and only if the vectors

$$(\alpha_1 - \alpha_2)b_0 + (\lambda_1 - \lambda_2)b_1 + (\mu_1 - \mu_2)b_2$$

$(\alpha_1 - \alpha_3)b_0 + (\lambda_1 - \lambda_3)b_1 + (\mu_1 - \mu_3)b_2$  

are linearly dependent over $\mathbb{F}_q$. By the definition of $b_0$ and the independence of $b_1, b_2$, (12) is equivalent with

$$((\alpha_1 - \alpha_2)u + \lambda_1 - \lambda_2)((\alpha_1 - \alpha_3)v + \mu_1 - \mu_3)-$$

$$((\alpha_1 - \alpha_3)u + \lambda_1 - \lambda_3)((\alpha_1 - \alpha_2)v + \mu_1 - \mu_2) = 0. \quad (13)$$

Sorting by $u$ and $v$, we obtain

$$0 = u[(\alpha_1 - \alpha_2)(\mu_1 - \mu_3) - (\alpha_1 - \alpha_3)(\mu_1 - \mu_2)]$$

$$+ v[(\alpha_1 - \alpha_3)(\lambda_1 - \lambda_2) - (\alpha_1 - \alpha_2)(\lambda_1 - \lambda_3)]$$

$$+ (\lambda_1 - \lambda_2)(\mu_1 - \mu_3) - (\mu_1 - \mu_2)(\lambda_1 - \lambda_3).$$

The independence of $1, u, v$ over $\mathbb{F}_r$ implies the system of equations

$$0 = (\alpha_1 - \alpha_2)(\mu_1 - \mu_3) - (\alpha_1 - \alpha_3)(\mu_1 - \mu_2), \quad (14)$$

$$0 = (\alpha_1 - \alpha_3)(\lambda_1 - \lambda_2) - (\alpha_1 - \alpha_2)(\lambda_1 - \lambda_3), \quad (15)$$

$$0 = (\lambda_1 - \lambda_2)(\mu_1 - \mu_3) - (\mu_1 - \mu_2)(\lambda_1 - \lambda_3). \quad (16)$$

With given points $P_1, P_2, \alpha_1 \neq \alpha_2$, (14) and (15) has the unique solution

$$\lambda_3 = \frac{\lambda_1(\alpha_3 - \alpha_2) + \lambda_2(\alpha_1 - \alpha_3)}{\alpha_1 - \alpha_2},$$

$$\mu_3 = \frac{\mu_1(\alpha_3 - \alpha_2) + \mu_2(\alpha_1 - \alpha_3)}{\alpha_1 - \alpha_2},$$

10
which is a solution for (16), as well. This means that the line \( P_1P_2 \) hits \( A_{\alpha_3} \) in the unique point
\[
P_3 = \frac{\alpha_3 - \alpha_2}{\alpha_1 - \alpha_2} P_1 + \frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_2} P_2.
\]
This formula further shows that the constant cross-ratio can take any value in \( \mathbb{F}_r \setminus \{0, 1\} \).

We are able to describe the geometric structure of \( k \)-nets \( (k \geq 4) \) where one component is contained in a line pencil.

**Theorem 5.4.** Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \), \( k \geq 4 \), be a \( k \)-net of order \( n \) embedded in \( PG(2, \mathbb{K}) \). Assume that the component \( \lambda_1 \) is contained in a line pencil. Then the following hold.

1. The order of \( \lambda \) is \( n = p^e \) where \( p > 0 \) is the characteristic of \( \mathbb{K} \).
2. For each component \( \lambda_i \), \( i > 1 \), there is an elementary Abelian \( p \)-group of collineations acting regularly on the lines of \( \lambda_i \).
3. The components \( \lambda_2, \ldots, \lambda_k \) are projectively equivalent.
4. If any other component is contained in a line pencil then all components are, and the base points of the pencils are collinear.

**Proof.** It suffices to prove the theorem for \( k = 4 \). We give the proof for the dual \( k \)-net by assuming that the component \( \lambda_1 \) is contained in the line \( \ell \). Let \( \kappa \) be the constant cross-ratio of \( (\lambda_2, \lambda_3, \lambda_4, \lambda_1) \) and for any point \( S \not\in \ell \) denote by \( u_S \) the \((S, \ell)\)-perspectivity such that for any point \( P \) and its image \( P' = u_S(P) \), the cross-ratio of \( S, P, P' \) and \( PP' \cap \ell \) is \( \kappa \). Then, for any \( S \in \lambda_2 \), \( u_S \) induces a bijection between \( \lambda_3 \) and \( \lambda_4 \). In particular, \( \lambda_3 \) and \( \lambda_4 \) are projectively equivalent. Let \( S, T \in \lambda_2 \), \( S \neq T \), and assume that \( u_S^{-1}u_T \) has a fixed point \( R \not\in \ell \), that is, \( u_S(R) = u_T(R) = R' \). Then, \( S, T \in RR' \) and with \( R'' = RR' \cap \ell \) the cross-ratios \( (S, R, R', R'') \) and \( (T, R, R', R'') \) are equal to \( \kappa \). This implies \( S = T \), a contradiction. This means that for all \( S, T \in \lambda_2 \), \( S \neq T \), the collineation \( u_S^{-1}u_T \) is an elation with axis \( \ell \), and \( \{u_S^{-1}u_T \mid S, T \in \lambda_2 \} \) generate an elementary Abelian \( p \)-group \( U \) of collineations, leaving \( \lambda_3 \) invariant. Moreover, \( U \) acts transitively, hence regularly on \( \lambda_3 \). This finishes the proof. \( \blacksquare \)
Example 5.5. In Example 5.3 we constructed a dual $r$-net of order $r^2$ in $AG(2, r^s)$, $s \geq 3$. For $P_1 \in A_{\alpha_1}$, $P_2 \in A_{\alpha_2}$, the line $P_1P_2$ has direction vectors

$$(u + \lambda)b_1 + (v + \mu)b_2.$$ 

These are linearly independent for different choices of $\lambda, \mu \in F_r$, hence they determine $r^2$ points at infinity. Let $\lambda_0$ be the set of corresponding infinite points. Then, $(\lambda_0, \lambda_1, \ldots, \lambda_r)$ is a dual $(r + 1)$-net with component $\lambda_0$ contained in a line.

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