Sensitivity of BPA SAR Image Formation to Initial Position, Velocity, and Attitude Navigation Errors

Colton Lindstrom, Dr. Randall Christensen, Dr. Jacob Gunther

Abstract—The Back-Projection Algorithm (BPA) is a time domain matched filtering technique to form synthetic aperture radar (SAR) images. To produce high quality BPA images, precise navigation data for the radar platform must be known. Any error in position, velocity, or attitude results in improperly formed images corrupted by shifting, blurring, and distortion. This paper develops analytical expressions that characterize the relationship between navigation errors and image formation errors. These analytical expressions are verified via simulated image formation and real data image formation.

I. INTRODUCTION

SYNTHETIC aperture radar (SAR) is a class of radar processing that uses the flight path of a spacecraft or aircraft, referred to as a radar platform, to create a synthetic imaging aperture. Through a collection of matched filters, raw radar data is processed into images. Many efficient matched filtering algorithms have been developed that employ the frequency domain, such as the range-Doppler algorithm, the chirp scaling algorithm, the omega-K algorithm, and more [1]. Time domain algorithms also exist, such as the Back-Projection Algorithm (BPA) [2].

This paper explores the sensitivity of BPA images to navigation errors. This is done first analytically using the range equation and back-projection equation, which are both defined in Section II. Secondly the analysis is verified by injecting error into a flight trajectory estimate of an aircraft and using the corrupted trajectory estimate to form BPA images. This process is performed on both simulated and real data.

A. Motivation

The research in this paper is primarily motivated by the field of GPS denied navigation but may be of interest to other fields relating to SAR image quality or image autofocusing. GPS denied navigation is a field of research that involves estimating the state of a vehicle in the absence of Global Navigation Satellite Systems (GNSS) such as GPS. Typical approaches utilize an inertial navigation system (INS) as the core sensor, aided by measurements from auxiliary sensors, in the framework of an extended Kalman filter. Such auxiliary sensors may include cameras, lidar, radar, etc, [3].

When forming a SAR image using back-projection, navigation data and raw radar data are processed to form an image. Obtaining precise navigation data in an ideal application requires the use of GPS. However, in a GPS denied environment, navigation errors may be present, which result in distorted SAR images. This research is motivated by the potential of inferring navigation errors from induced image errors during BPA image formation [4]. This paper works toward building the foundation and intuition needed to achieve such a potential.

B. Literature Review

BPA is more sensitive to navigation errors than other types of SAR image formation techniques. This can be inferred from Duersch and Long who explore some of the sensitivities inherent in forming images using back-projection [2]. This research expands the sensitivity analysis to motion errors as seen from a navigation point of view with a more complete navigation state.

BPA is essentially a matched filter along a hyperbolic curve within a set of range compressed data. Integrating along a curve requires that each data sample be precisely selected in correspondence with the current position of the vehicle. Any error in navigation data results in integrating data on an incorrect curve with an incorrect phase. Very precise navigation data is therefore necessary to form accurate BPA images [2]. Further details on BPA are discussed in Section II

Errors in navigation data manifest themselves in a SAR image as shifts and distortions of a given target. Research performed by Christensen et al explores the effects of navigation errors on fully formed SAR images and hypothesizes that navigation errors can be determined by comparing degraded SAR images to a reference SAR map [4].

Many types of errors can affect the quality of SAR images. As such a comprehensive analysis of image errors is difficult and requires further investigation. Current efforts in analyzing image errors include research performed by Bamler [5] and Farrel et al [6]. They explore image errors caused by servo transients, quantization, range uncertainty, range-rate uncertainty, and focusing algorithm selection. Additionally,
Chen explores image errors caused by moving targets in the illuminated scene [7].

In previous literature, navigation errors have been expressed as range displacement, line of sight displacement, and forward velocity error. Moreira and Xing et al adjust the SAR pulse repetition frequency (PRF) to compensate for forward velocity errors [8], [9]. Moreira further adjusts phase and range delays to compensate for line of sight displacement errors. Velocity errors in particular have been shown to affect the Doppler parameters of the SAR data, which cause target location errors and image degradation in the final image [9], [10].

C. Contributions

A comprehensive study of BPA SAR image errors in the context of the full navigation state has not been performed to date. The research seeks to fill this void by developing relationships between image shifts, blurs, and distortions and all components of navigation state, specifically position, velocity, and attitude errors.

Section [11] begins by providing necessary background knowledge concerning inertial navigation and BPA processing. Section [13] develops the math necessary to predict how navigation errors affect the final SAR image. Sections [12] and [14] demonstrate the application of the error analysis to simulated and real data, respectively. Section [16] provides concluding discussion and summary.

II. BACKGROUND

A. Inertial Navigation

The purpose of this section is to define an inertial navigation framework applicable to the short data collection times typical of SAR imagery. The framework is then used to develop analytical expressions of position estimation error growth.

Inertial navigation is a large field with an equally-large body of literature dating back to the 1930’s. An excellent overview of the history and motivating factors behind the development of this field is provided in [11]. The navigation framework developed in this section utilizes concepts discussed in [11], [12], [13]. The developed framework is most directly related to the so-called “Tangent Frame” kinematic model [12], with the assumptions of constant gravity and a non-rotating earth, both of which are applicable over the short time frame typical of an airborne SAR data collection. In the development that follows, the truth and navigation states are defined, with the associated differential equations. Consistent with an extended Kalman filter framework, the truth state differential equations are linearized about the estimated navigation state to derive the differential equations of the estimation errors, or error states.

The truth state vector comprises the true position (p^n), velocity (v^n), and attitude quaternion (q^n_b) of the vehicle

\[ x = [ p^n \ v^n \ q^n_b ]^T \]  

where n and b refer to the navigation and body frame, respectively. The body frame origin is coincident with the navigation center of the inertial measurement unit (IMU), with the axes aligned and rotating with the vehicle body axes. Out of convenience for the subsequent analysis, and without loss of generality, the navigation frame is defined with the x-axis parallel to the velocity of the vehicle, the z-axis in the direction of gravity, and the y-axis defined by the right-hand-rule. The x, y, and z axes, therefore, correspond to the along-track, cross-track, and down directions typical of radar imaging conventions. Consistent with Ferell’s definition of the “Tangent Frame” [12], the position and velocity are defined relative to a fixed origin, whose location is the position of the vehicle at the beginning of the SAR data collection. The differential equations of the truth states are defined as follows [7]

\[
\begin{align*}
\dot{\hat{p}}^n &= T_{b}^{n}v^n + g^n \\
\dot{\hat{v}}^n &= \frac{1}{2} q^n_b \otimes \left[ \omega^b \right] \\
\dot{\hat{q}}^n_b &= 0 \\
\end{align*}
\]  

The strapdown inertial navigation system comprises a three-axis accelerometer and gyro, which provide measurements of specific force (\(\nu^b\)) and angular rate (\(\omega^b\)) in the body frame, corrupted by noise

\[
\begin{align*}
\nu^b &= \nu^b + n_{v} \\
\omega^b &= \omega^b + n_{\omega} \\
\end{align*}
\]  

The navigation states are defined identical to the truth states but are propagated using noisy accelerometer and gyro measurements

\[
\begin{align*}
\dot{\hat{p}}^n &= T_{b}^{n}\nu^n + g^n \\
\dot{\hat{v}}^n &= \frac{1}{2} q^n_b \otimes \left[ \omega^b \right] \\
\dot{\hat{q}}^n_b &= 0 \\
\end{align*}
\]  

The estimation error, or error state vector,

\[
\delta x = [ \delta p^n \ \delta v^n \ \delta \theta^n ]^T
\]  

is defined as the difference between the truth states and the navigation states. For all but the attitude states, the difference is defined by a simple subtraction. For the attitude quaternion the difference is defined by a quaternion product.

\[
\delta p^n = p^n - \hat{p}^n
\]  

\[
\delta v^n = v^n - \hat{v}^n
\]  

\[
\frac{1}{2} \delta \theta^n = q^n_b \otimes (\delta \hat{q}^n_b)^* \]

It is also convenient to define the attitude errors in terms of the true and estimated transformation matrices

\[
[I - (\delta \theta^n \times)] = T_{b}^{n} \left( \hat{T}_{b}^{n} \right)^T
\]  

1In this work, the quaternion is interpreted as a “left-handed” quaternion, and the \(\otimes\) operator is the Hamiltonian quaternion product [14].
Linearization of (2) about the estimated state results in the error state differential equation

$$\delta \dot{x} = F \delta x + B w$$  \hfill (10)$$

where the state dynamics matrix, $F$, and noise coupling matrix, $B$, are defined respectively as

$$F = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & [T^*_b P^b] \end{bmatrix}$$  \hfill (11)$$

$$B = \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ -T^*_b & 0_{3 \times 3} & T^*_b \end{bmatrix}$$  \hfill (12)$$

and where the white noise ($w$) consists of the accelerometer and gyro measurement noise

$$w = [n_v \ n_w]^T$$  \hfill (13)$$

The focus of this paper is to analyze the sensitivity of the BPA image to errors in position, velocity, and attitude at the beginning of the synthetic aperture. The effect of $w$ is therefore ignored, and the analysis is facilitated by determining the homogenous solution to (19)

$$\delta x_k = \Phi(t_k, t_{k-1}) \delta x_{k-1}$$  \hfill (14)$$

where $\Phi(t_k, t_{k-1})$ is the state transition matrix (STM) from the $t_{k-1}$ to $t_k$. The STM is defined as the matrix which satisfies the differential equation and initial condition

$$\dot{\Phi}(t_k, t_{k+1}) = F(t) \Phi(t, t_k), \quad \Phi(t_k, t_k) = I_{n \times n}$$  \hfill (15)$$

For the case of straight-and-level flight, the dynamic coupling matrix of (11) is constant,

$$F = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & (\nu^n)^x \end{bmatrix} \times$$  \hfill (17)$$

Where the accelerometer measurements are expressed in the $n$ frame as

$$\nu^n = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$$  \hfill (18)$$

Since $F$ is constant, the STM is derived using the matrix exponential ([15], page 42)

$$\Phi(t_{k+1}, t_k) = \begin{bmatrix} I_{3 \times 3} & T^*_b \Delta t \nu^{nx} \Delta t \nu^n \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}$$  \hfill (19)$$

The desired analytical expression for position errors is obtained from the first row of the STM to yield

$$\delta p^n(t) = \delta p^n_0 + \delta v^n_0 \Delta t + \nu^n \times \delta \theta^n_0 + \frac{\Delta t^2}{2}$$  \hfill (20)$$

In all subsequent sections, the variables representing positions, velocities, and attitudes are all assumed to be in the $n$ frame. As such, the $n$ superscript on all navigation states is omitted for notational brevity.

B. Back-Projection Algorithm

Forming images using SAR is a process of matched filtering that transformed raw returned radar signals into focused pixels. A raw SAR signal is typically a linear frequency modulated (FM), or “chirp”, signal. Chirp signals are a sinusoid-like signal with an instantaneous frequency that is linear with time. A transmitted chirp signal is denoted $s_{tx}(t)$ and is equal to

$$s_{tx}(t) = \begin{cases} \exp(j2\pi f_0 t + j\pi K t^2), & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$  \hfill (21)$$

where $f_0$ is the initial frequency, $K$ is the linear FM rate in hertz per second, and $T$ is the pulse duration.

The chirp signal is transmitted several times along the trajectory, and return signals are collected for each transmitted signal. Using a “stop and hop” approximation, the return radar signal is a time shifted, attenuated version of the transmitted signal given by

$$s_{rx}(t) = A s_{tx}(t - \tau)$$  \hfill (22)$$

The return signal is fed through a matched filter in a process called “range compression”. The matched filter is a time reversed, conjugate version of the transmitted signal $s_{tx}(t)$. The output of the matched filter is denoted $s_{out}(t)$ and is equal to the convolution of $s_{rx}(t)$ with $s_{tx}(T - t)$.

$$s_{out}(t) = s_{rx}(t) * s_{tx}(T - t)$$  \hfill (23)$$

Evaluating the convolution results in

$$s_{out}(t) = e^{-j\rho(2\pi f_0 t + \pi KT)} \begin{cases} \sin(\pi K \rho t), & 0 \leq t < T \\ \frac{T}{\pi K \rho}, & T = T \leq 2T \end{cases}$$  \hfill (24)$$

where $\rho = T - t$. This expression can be written in a closed form using the sinc function $\text{sinc}(x) = \sin(\pi x)/\pi x$.

$$s_{out}(t) = e^{-j\rho(2\pi f_0 t + \pi KT)} \xi \text{sinc}(K \rho \xi)$$  \hfill (25)$$

After range compression, the sequential returns from a single target form a hyperbolic curve in the range compressed data. This hyperbola is quantified via the range equation denoted $R(p_t, \eta)$ and defined as

$$R(p_t, \eta) = \|p_t - p(\eta)\|$$  \hfill (26)$$

where $p_t$ is the position of a target of the ground and $p$ is the true time-varying position of the aircraft from ([1]). The aircraft position varies with azimuth time (or slow time), $\eta$.

To form an image using BPA, a second matched filter is applied to the range compressed data in the azimuth direction.
This is called “azimuth compression”. Azimuth compression using BPA is performed in the time domain and is dependent on the range equation, which is dependent on the position of the radar vehicle. For a particular pixel location, \( p_{\text{pix}} \), azimuth compression is defined by the summation,

\[
A(p_{\text{pix}}) = \sum_k s_{\text{out}}(t_{\text{pix},k}) \exp \left\{ j4\pi \frac{R_k(p_{\text{pix}}, \eta)}{\lambda} \right\} \tag{27}
\]

where \( k \) is used to denote the \( k^{\text{th}} \) range compressed signal, \( \lambda \) is the wavelength at the center frequency of the chirp signal, and \( t_{\text{pix},k} \) is the time during the \( k^{\text{th}} \) range compressed signal at which \( R_k(p, \eta) = R_k(p_{\text{pix}}, \eta) \). This time can be calculated via the conversion,

\[
t_{\text{pix}} = 2R_k(p_{\text{pix}}, \eta)/c \tag{28}
\]

where \( c \) is the speed of light. To go from \( t_{\text{pix}} \) to \( t_{\text{pix},k} \), an index in the \( k^{\text{th}} \) range compressed pulse must be found that corresponds with time \( t_{\text{pix}} \). Forming a BPA image is a matter of performing azimuth compression for a collection of pixels within some chosen geographical region.

### III. Analysis

Errors in the estimated trajectory cause errors in the range equation \((26)\), which in turn cause errors in the back-projection equation \((27)\). The range equation appears in two places in the back-projection equation, namely the index of the range compressed data and the phase of the matched filter. As such, an error in the range equation causes two types of errors.

First, an error in the index of the range compressed data appears as a change in the hyperbolic curve that \((27)\) uses to perform azimuth compression. Changes in the chosen curve relative to the correct curve manifest as shifts, eccentricity changes, and distortions. These errors are referred to as “curve errors”. Second, an error appears in the phase of the matched filter. This affects the focus of a target in the final image through a phase mismatch. Phase mismatches lead to target blurring. These errors are referred to as “phase errors”.

Curve errors and phase errors are explored individually for position, velocity, and attitude navigation errors. Intuition for each type of error is aided by first expanding \((26)\) using a Taylor series approximation. For conciseness, the notation for \((26)\) is abbreviated to \( \tilde{R}(\eta) \). According to \(1\), the Taylor approximation for the range equation, denoted \( \tilde{R}(\eta) \), is approximated as

\[
\tilde{R}(\eta) \approx R_0 + \frac{d^2 \|R(\eta)\|^2}{d\eta^2} \bigg|_{\eta = \eta_0} \frac{1}{2R_0} (\eta - \eta_0)^2 \tag{29}
\]

where \( \eta_0 \) is the time of closest approach and \( R_0 \) is the range of closest approach, which is also equal to \( \tilde{R}(\eta_0) \). As a common practice in literature, this approximation is expanded about the time of closest approach \( \eta_0 \). By doing so, the first order term of the expansion equals zero.

The navigation frame is chosen such that the initial position of the radar platform is the origin. This origin can be interpreted globally as the point of GPS denial or locally as the beginning of the synthetic aperture. The platform is assumed to be flying at a constant velocity. For ease of visualization in subsequent figures, the radar platform is assumed to be flying northward. In this scenario, the true time-varying position of the platform is expressed simply as

\[
p(\eta) = v_0 \eta \tag{30}
\]

where \( v_0 \) is the true initial velocity. In \((26)\), the time-varying range is expressed in terms of the truth state. Error analysis is performed by replacing the truth state with the estimated navigation state. Then \((6)\) is used to write the navigation state as the difference between the truth state and error state. This is expressed as

\[
\tilde{R}(\eta) = \left\| p_t - (p(\eta) - \delta p) \right\| \tag{31}
\]

where the hat on \( \tilde{R}(\eta) \) distinguishes this value as an estimate rather than the true value. This construction allows for an intuitive analysis of the back-projection equation with the help of the Taylor approximation from \((29)\), for the cases of position, velocity, attitude errors at the beginning of the synthetic aperture.

#### A. Position Errors

Using \((20)\), an initial position estimation error, denoted \( \delta p_0 \), is introduced into the estimated range equation.

\[
\tilde{R}(\eta) = \left\| p_t - v_0 \eta_0 + \delta p_0 \right\| \tag{32}
\]

This equation is then expanded using the Taylor approximation, again denoted with a tilde.

\[
\hat{\tilde{R}}(\eta) = \left\| p_t - v_0 \eta_0 + \delta p_0 \right\| + \frac{(v_0)^T v_0}{2 \left\| p_t - v_0 \eta_0 + \delta p_0 \right\|^2} (\eta - \eta_0)^2 \tag{33}
\]

In the first term of the expansion, \( \delta p_0 \) causes a constant shift of the hyperbola used for azimuth compression. In the second term, \( \delta p_0 \) in the denominator is typically small compared \( p_t - v_0 \eta_0 \). As such, its contribution to the overall error is very small and can be ignored. In terms of curve errors, the estimated hyperbola is shifted in the direction of \( \delta p_0 \) due to the first term of the expansion. For phase errors, constant offsets do not affect the overall focus of any imaged target \(1\). Phase offsets only affect knowledge of absolute phase.

The notional effects of position errors are illustrated in Figures \(1\) and \(2\). Each figure is split into three subfigures showing how a position error propagates through different stages of radar processing. The first subfigure shows the error’s effect on the flight trajectory. The second subfigure shows the error’s effect on the range compressed data. The third subfigure shows the error’s effect on the final image.

In each figure, light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent actual data given no estimation errors. These figures primarily provide intuition primarily on curve errors but can be useful in visualizing phase errors as well.
B. Velocity Errors

From (20), an initial velocity estimation error is introduced into the estimated range equation as

$$\hat{R}(\eta) = \|p_t - (v_0 - \delta v_0)\eta\|$$  \hspace{1cm} (34)

Again, the Taylor expansion is taken and results in

$$\hat{R}(\eta) = \|p_t - (v_0 - \delta v_0)\eta\| + \frac{(v_0)^T v_0 - 2(v_0)^T \delta v_0 + (\delta v_0)^T \delta v_0 (\eta - \eta_0)^2}{\|p_t - (v_0 - \delta v_0)\eta\|}$$  \hspace{1cm} (35)

In the first term of the expansion, there is again a constant offset due to $\delta v_0$. As is the case for initial position errors, this constant offset causes a shifted curve error and a negligible offset phase error. In the second term of the expansion, $(\delta v_0)^T \delta v_0$ in the numerator is a quadratic error term and contribute little overall error. Similar to the case of position errors, the $\delta v_0\eta$ term in the denominator contributes negligible overall error due to its relative size compared to the rest of the denominator. In the numerator of the second term, $2(v_0)^T \delta v_0$ causes a time-varying error. In terms of curve errors, this changes the eccentricity of the expected hyperbola in the range compressed data. For phase errors, this term can be thought of a linearly changing frequency error or azimuth FM rate error.

An azimuth FM rate error is characterized by a phase that changes quadratically in time. A quadratically varying phase yields a linearly changing instantaneous frequency. This is similar to the linear FM signal modeled by equation (21). For both curve errors and phase errors, the second numerator term results in blurring of the imaged target in the azimuth dimension. This blur is only present in along track errors, as cross track and elevation errors result in a $\delta v_0$ that is orthogonal to $v_0$.

Again, the notional effects are illustrated in Figures 4, 5, and 6 for various stages of SAR processing. Each figure is again split into three subfigures with identical interpretations as Figures 1, 2, and 3.

C. Attitude Errors

Again using (20), initial attitude errors are injected into the range equation to yield

$$\hat{R}(\eta) = \|p_t - v_0\eta + \nu^\eta \times \delta \theta_0 \frac{\eta^2}{2}\|$$  \hspace{1cm} (36)

Errors in attitude manifest as errors in acceleration. Specific effects from attitude errors are apparent after computing the cross product. Using constant accelerometer measurements,

$$\nu^\eta \times \delta \theta_0 = \begin{bmatrix} 0 & 0 & -g \\ 0 & \delta \theta_{z,0} & \delta \theta_{y,0} \\ -g & \delta \theta_{y,0} & \delta \theta_{x,0} \end{bmatrix}$$  \hspace{1cm} (37)

This equation illustrates how attitude errors only cause acceleration errors in the along track and cross track directions. Specifically, errors in roll, $\delta \theta_{x,0}$, cause cross track acceleration errors. Errors in pitch, $\delta \theta_{y,0}$, cause along track acceleration errors. Errors in yaw do not cause any errors in acceleration.

For conciseness, acceleration errors resulting from (37) are collectively referred to as $\delta \dot{v}_0$. As such, the estimated range equation takes the form

$$\hat{R}(\eta) = \|p_t - v_0\eta + \frac{1}{2} \delta \dot{v}_0 \eta^2\|$$  \hspace{1cm} (38)

The effects of acceleration errors are again explored using the Taylor approximation of the estimated range equation.

$$\hat{R}(\eta) = \|p_t - v_0\eta + \frac{1}{2} \delta \dot{v}_0 \eta_0^2\|$$  \hspace{1cm} (39)

$$\hat{R}(\eta) = \|p_t - v_0\eta + \frac{1}{2} \delta \dot{v}_0 \eta_0^2\| + \frac{Q}{\|p_t - v_0\eta + \frac{1}{2} \delta \dot{v}_0 \eta_0^2\|} (\eta - \eta_0)^2$$

where

$$Q = 1.5(\delta \dot{v}_0)^T \delta \dot{v}_0 \eta_0^2 - 3(\delta \dot{v}_0)^T v_0\eta_0$$  \hspace{1cm} (40)

In the first term of the expansion, $\frac{1}{2} \delta \dot{v}_0 \eta_0^2$ causes a constant offset. For curve errors, this term causes a small shift in the imaged target. In practice, this shift isn’t strongly apparent, because the image degrades due to other terms before the shifting becomes strong. For phase errors, this constant offset doesn’t affect the focus of the image.

In the second term of the expansion, $1.5(\delta \dot{v}_0)^T \delta \dot{v}_0 \eta_0^2$ is a quadratic error term and is considered very small. The $\frac{1}{2} \delta \dot{v}_0 \eta_0^2$ term in the denominator is small compared to other terms in the denominator and causes negligible overall error. The $p_t^T \delta \dot{v}_0$ in the numerator causes a time-varying error. Interestingly, this error is in terms of the target location implying that the location of the target affects the severity of attitude imaging error. For both curve and phase errors, this term causes blurring similar to the along track velocity errors. The $3(\delta \dot{v}_0)^T v_0\eta_0$ term also causes a time-varying error. For both curve and phase errors, this again results in blurring; however, this term only becomes significant for along track acceleration errors. This term is negligible for cross track and elevation errors due to orthogonality. Note that this term is dependent on the time of closest approach and therefore changes depending on the location of the target in azimuth.

Notional illustrations of how attitude errors propagate through the SAR processing steps are shown in Figures 7 and 8. It was shown that yaw errors do not affect SAR images, therefore these figures depict only roll and pitch errors. Each figure is split into subfigures with interpretations identical to those of the position and velocity error figures.

IV. SIMULATED DATA

The analysis presented in the previous section is now verified via simulation. SAR images are first formed using the true trajectory. Initial errors are then injected and propagated to
Fig. 1. Illustration of how cross track position errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 2. Illustration of how along track position errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 3. Illustration of how elevation position errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 4. Illustration of how cross track velocity errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.
Fig. 5. Illustration of how along track velocity errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 6. Illustration of how elevation velocity errors affect the various stages of radar processing. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 7. Progression of roll errors through the SAR data. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.

Fig. 8. Progression of pitch error through the SAR data. Left: flight trajectory with error. Center: range compressed data with error. Right: final image with error. Light colored or dotted illustrations represent expected data given estimation errors. Solid black illustrations represent truth data.
yield a corrupted estimate of the trajectory. Images are formed with estimation errors and are compared to the truth reference image. The presence and extent of shifting and blurring, as predicted by the development of section III, is also verified.

For each navigation error, a figure is presented with a simulated SAR image superimposed with the predicted target shift. The reference image to which each SAR chip should be compared is provided in Figure 9.

For each navigation error, a figure is presented with a simulated SAR image superimposed with the predicted target shift. The reference image to which each SAR chip should be compared is provided in Figure 9.

The SAR images formed given estimation errors are provided in Figures 11, 12, and 13. For each image, a superimposed “X” shows the location of the reference target in relation to the current image. A superimposed “O” shows the predicted location of the target given the injected estimation errors. The “X” and “O” are generated using the true range equation, (26), and the estimated range equation, (31), respectively. Equation (26) is used to find the true range of closest approach and time of closest approach, denoted $R_0$ and $\eta_0$. Equation (31) is used to find the estimated range of closest approach and time of closest approach, denoted $\hat{R}_0$ and $\hat{\eta}_0$.

\[
R_0 = \min_{\eta} R(\eta), \quad \hat{R}_0 = \min_{\eta} \hat{R}(\eta) \quad (41)
\]
\[
\eta_0 = \arg \min_{\eta} R(\eta), \quad \hat{\eta}_0 = \arg \min_{\eta} \hat{R}(\eta)
\]

The range of closest approach and time of closest approach are used as coordinates to overlay “X” and “O” onto each image.

Figures 11, 12, and 13 illustrate that in all cases, the direction of shifts and blurs is consistent with the development of section III. Furthermore, in cases where blur is negligible, the amount of shift is accurately predicted utilizing the method described previously. It is important to highlight the ambiguity that exists in relating the SAR image error with the attributing navigation error. From a single image, for example, it is impossible isolate the effects of cross track position and elevation errors, since both cause shifts in the cross track position of the target. Similar difficulties existing in attributing along-track shifts and blurs to the corresponding navigation errors.

V. REAL DATA

The analysis from Section III is be further verified using real SAR data. Radar data was collected in Logan, Utah. SAR images are formed using a post-processed, high fidelity navigation solution, which is considered truth for the purposes of this research. The reference image in Figure 10 is formed using the truth trajectory. Each type of navigation error is then injected into the truth trajectory, from which the distorted SAR images of Figures 14, 15, and 16 are formed.

Fig. 10. Reference image for real SAR data.

The results on real SAR data mirror the trends observed in section IV. The type and direction of the shifts and blurs are consistent with the predictions of section III. In the case of negligible blurs, the shifts on real data are accurately predicted using the method described in section IV for all cases except yaw error, where a very small prediction error is observed. Finally, ambiguity in the attribution of error sources to image errors is observed in the real SAR data. Despite the small discrepancy in yaw, these results serve to further validate the relationships developed in section III.

VI. CONCLUSION

This paper analyzes errors in the formation of SAR images using the Back-Projection Algorithm from a navigation perspective, for the case of straight-and-level flight. Relationships are developed between the position, velocity, and attitude estimation errors at the beginning of the synthetic aperture and the observed shifts and blurs of the corrupted BPA SAR image.

The developed relationships were observed and validated on both simulated and real SAR data. In the case of negligible blurring, the location of the target in the corrupted SAR image is accurately predicted given knowledge of the attributing navigation error. These results suggest that errors in BPA SAR images could potentially be used in reverse; i.e. image errors could be characterized and exploited as a navigation aid in GPS-denied applications. For a single image, however, it was
observed that the shifts/blurs are not unique to an individual navigation error. The presence of the target location in the developed relationships suggests that the effect of navigation errors can be modified by the selection of the target location. One obvious extension of this work is the consideration of multiple targets with large geometric diversity, to resolve the ambiguity in attributing error sources. Furthermore, methods which characterize the amount and direction of image blurring must be developed to exploit the information contained therein.

REFERENCES

[1] I. G. Cumming and F. H. Wong, *Digital Processing of Synthetic Aperture Radar Data*.

[2] M. I. Duerch and D. G. Long, “Analysis of time-domain back-projection for stripmap SAR,” *International Journal of Remote Sensing*, vol. 36, pp. 2010–2036, Apr. 2015. Publisher: Taylor & Francis _eprint: https://doi.org/10.1080/01431161.2015.1030044.

[3] G. Balamurugan, J. Valarmathi, and V. P. S. Naidu, “Survey on uav navigation in gps denied environments,” in *2016 International Conference on Signal Processing, Communication, Power and Embedded System (SCOPES)*, pp. 198–204, 2016.

[4] R. Christensen, J. Gunther, and D. Long, “Radar-Aided, GPS-Denied Navigation.”

[5] R. Bamler, “A comparison of range-Doppler and wavenumber domain SAR focusing algorithms,” *IEEE Transactions on Geoscience and Remote Sensing*, vol. 30, pp. 706–713, July 1992.

[6] J. L. Farrell, J. H. Mims, and A. Sorrell, “Effects of Navigation Errors in Maneuvering SAR,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-9, pp. 758–776, Sept. 1973.

[7] V. C. Chen, “Time-frequency analysis of SAR images with ground moving targets,” in *Wavelet Applications V*, vol. 3391, pp. 295–302, International Society for Optics and Photonics, Mar. 1998.

[8] J. Moreira, “Estimating the residual error of the reflectivity displacement method for aircraft motion error extraction from SAR raw data,” in *IEEE International Conference on Radar*, pp. 70–75, May 1990.

[9] M. Xing, X. Jiang, R. Wu, F. Zhou, and Z. Bao, “Motion Compensation for UAV SAR Based on Raw Radar Data,” *IEEE Transactions on Geoscience and Remote Sensing*, vol. 47, pp. 2870–2883, Aug. 2009. Conference Name: IEEE Transactions on Geoscience and Remote Sensing.

[10] S. Bing, W. Ye, C. Jie, W. Yan, and L. Bing, “Image position analysis of motion errors for missile-borne SAR based on diving model,” in *2013 IEEE International Conference on Imaging Systems and Techniques (IST)*, pp. 206–209, Oct. 2013. ISSN: 1558-2809.

[11] M. S. Grewal, A. P. Andrews, and C. Bartone, *Global navigation satellite systems, inertial navigation, and integration*. Hoboken: John Wiley & Sons, third edition ed., 2013.

[12] J. Farrell, *Aided navigation: GPS with high rate sensors*. Electronic engineering, New York: McGraw-Hill, 2008. OCLC: ocn212908814.

[13] P. G. Savage, *Strapdown analytics*. Maple Plain, Minn.: Strapdown Associates, 2000.

[14] R. Zanetti, “Rotations, Transformations, Left Quaternions, Right Quaternions?,” *The Journal of the Astronautical Sciences*, vol. 66, pp. 361–381, Sept. 2019.

[15] P. S. Maybeck, *Stochastic Models, Estimation, and Control*, vol. 1. New York: Navtech Book and Software Store, 1994.
Fig. 11. Position errors in simulated data: Left, along track position error (3 m). Middle, cross track position error (3 m). Right, elevation position error (3 m). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.

Fig. 12. Velocity errors in simulated data: Left, along track velocity error (0.1 m/s). Middle, cross track velocity error (0.05 m/s). Right, elevation velocity error (0.05 m/s). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.

Fig. 13. Attitude errors in simulated data: Left, roll error (0.001 rad). Middle, pitch error (0.02 rad). Right, yaw error (0.1 rad). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.
Fig. 14. Position errors in real data: Left, along track position error (3 m). Middle, cross track position error (3 m). Right, elevation position error (3 m). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.

Fig. 15. Velocity errors in real data: Left, along track velocity error (1 m/s). Middle, cross track velocity error (0.2 m/s). Right, elevation velocity error (0.2 m/s). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.

Fig. 16. Attitude errors in real data: Left, roll error (0.01 rad). Middle, pitch error (0.5 rad). Right, yaw error (0.1 rad). Each figure is superimposed with a reference target location, “X”, and a predicted target shift, “O”.