Ultrarelativistic limits of boosted dilaton black holes

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Abstract

We investigate the ultrarelativistic limits of dilaton black holes, black $p$-branes (strings), multi-centered dilaton black hole solutions and black $p$-brane (string) solutions when the boost velocity approaches the speed of light. For dilaton black holes and black $p$-branes (boost is along the transverse directions), the resulting geometries are gravitational shock wave solutions generated by a single particle and membrane. For the multi-centered dilaton black hole solutions and black $p$-brane solutions (boost is along the transverse directions), the limiting geometries are shock wave solutions generated by multiple particles and membranes. When the boost is along the membrane directions, for the black $p$-brane and multi-centered black $p$-brane solution, the resulting geometries describe general plane-fronted waves propagating along the membranes. The effect of the dilaton on the limit is considered.

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I. INTRODUCTION

In recent years, much attention has been focused on the ultrarelativistic limits of diverse black hole spacetimes. There exist some reasons responsible for the interest. For example, people wish to know the spacetime structure of a moving black hole and use it to investigate the emission of gravitational waves when two black holes collide \cite{1}. The ultrarelativistic boost of black holes usually leads to a class of gravitational wave spacetimes, which are exact solutions of Einstein’s field equations. These resulting gravitational shock wave solutions have been used to study the scattering of particles and strings at the energy of Planckian scale \cite{2–5}. At that energy scale, just as pointed out by t’Hooft \cite{2}, the gravitational interaction dominates their collision processes, the picture of particles (strings) propagating in a flat spacetime ceases to be valid. The gravitational field generated by the particles (strings) must be taken into account. In addition, they may play an important role in understanding the back reaction of Hawking radiation \cite{6}, because particles with high velocity moving in curved spacetimes also produce gravitational shock waves.

In 1971, Aichelburg and Sexl (AS) \cite{7} obtained an exact solution of Einstein equation describing a particle moving at the speed of light by boosting the Schwarzschild metric in the limit of the boost velocity $v \to c = 1$, where $c$ is the speed of light. The AS geometry is flat everywhere, except for the location of the particle, where the geometry has a delta-type singularity. The AS geometry has been explained as a shock wave spacetime generated by the null particle by Dray and ’t Hooft \cite{8}, who have rederived the AS solution by using the coordinate shift method. The coordinate shift method can also be used to produce the shock wave in curved spacetimes, such as black hole spacetimes \cite{8–10}. The shock wave solutions have been investigated in the higher derivative gravitational theory \cite{11,12}.

The AS boost method has recently been used to study the ultrarelativistic limit of the Reissner-Nordström \cite{13,14}, Kerr \cite{15} and Kerr-Newman \cite{16} spacetimes, and even the topological defect spacetimes \cite{17} and the Schwarzschild-(anti-)de Sitter spacetimes \cite{18}. There some authors have investigated the propagation and scattering of quantum fields in these ultrarelativistic limiting geometries. Note that the geometric structure of black holes is changed significantly due to the appearance of the dilaton field and the low energy actions of superstring and supergravity theories can be viewed as the modification of the Hilbert-Einstein action in general relativity. In this paper, we would like to investigate the effects of the dilaton field on the resulting ultrarelativistic geometry by boosting the four dimensional and higher dimensional dilaton black holes to the ultrarelativistic limit.

In order to investigate the scattering process of multiple particles \cite{2} and the back reaction of Hawking radiation on the black hole geometry, it is necessary to study the shock wave solution generated by multiple particles. However, except for the shock wave solutions generated by topological defects, those solutions mentioned above are all produced by a single null particle (the shock wave in the de Sitter space is generated by two null particles \cite{18}). By boosting the multi-centered dilaton black holes in the four dimensional and higher dimensional spacetimes, we find the shock wave solutions generated by multiple particles. Note that extremal black strings and black p-branes are precisely solutions of fundamental strings and membranes themselves. We get the ultrarelativistic geometries of black string and p-branes by boosting those solutions along transverse directions and membrane (string) directions. Also we boost the multi-centered black string and p-brane solutions and obtain
gravitational wave spacetimes generated by multiple strings and membranes, respectively.

The organization of this paper is as follows. In the next section we consider boosting a single four dimensional and higher dimensional dilaton black hole. The case of multi-centered dilaton black hole solutions will be discussed in Sec.III. A single black string and p-brane, and multi-centered black string and p-brane solutions are considered in Sec.IV and V, respectively. We present our conclusion with a discussion in Sec.VI.

II. BOOSTING THE DILATON BLACK HOLES

A. Four dimensional dilaton black holes

Consider the following action

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2\alpha \phi} F_2^2 \right], \quad (2.1) \]

where \( F_2 \) is the Maxwell field. The coupling constant \( \alpha \) governs the interaction between the dilaton field and the Maxwell field. For \( \alpha = 1 \), the action is the low energy approximation of superstring action. The charged dilaton black hole solutions are \( [19] \)

\[ F_{tr} = \frac{e^{2\alpha \phi}}{R^2} Q, \quad (2.2) \]

\[ e^{2\alpha \phi} = \left( 1 - \frac{r}{r} \right)^{2a^2/(1+a^2)}, \quad (2.3) \]

\[ ds^2 = -A^2(r) dt^2 + A^{-2}(r) dr^2 + R^2(r) d\Omega_4^2, \quad (2.4) \]

where

\[ A^2(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right)^{(1-a^2)/(1+a^2)}, \quad (2.5) \]

\[ R^2(r) = r^2 \left( 1 - \frac{r_-}{r} \right)^{2a^2/(1+a^2)}. \quad (2.6) \]

The mass and electric charge of the hole are

\[ 2M = r_+ + \frac{1-a^2}{1+a^2} r_-, \quad Q^2 = \frac{r_- r_+}{1+a^2}. \quad (2.7) \]

For our purposes, we rewrite the metric (2.4) in the isotropic coordinates. Defining

\[ r = \bar{r} \left( 1 + \frac{r_+ + r_-}{2\bar{r}} + \frac{(r_+ - r_-)^2}{16\bar{r}^2} \right), \quad (2.8) \]

we have

\[ ds^2 = -A^2(r) dt^2 + \bar{r}^{-2} R^2(r) [dr^2 + \bar{r}^2 d\Omega_2^2]. \quad (2.9) \]

Here \( \bar{r} \) is asymptotically a Cartesian coordinate and one has \( \bar{r}^2 = x^2 + y^2 + z^2 \). Performing a Lorentz transformation in the x-direction:
We now take the limit $v \to 1$. This limit is a delicate point. Employing the method in Ref. \[7\] (see also Ref. \[13\]), we can obtain the results of physical meaning. This method works as follows. One integrates first the expression with respect to $u$, and then differentiates the corresponding expression with respect to $v'$. In order that the black hole can be boosted to the velocity of light, the mass and charge must go to zero. Following \[7,13,16\], we set

$$
\begin{align*}
    t &= \gamma(t' - vx'), \\
    y &= y', \\
    x &= \gamma(x' - vt'), \\
    z &= z',
\end{align*}
$$

where $\gamma = (1 - v^2)^{-1/2}$ and the constant $v$ is the boost velocity, one has

$$
ds^2 = \gamma^2[(r')^{-2}R^2(r') - A^2(r')]\left(dt' - vdx'\right)^2 + (r')^{-2}R^2(r')(-dt'^2 + dx'^2 + dy'^2 + dz'^2),
$$

(2.11)

where $\bar{r}^2 = \gamma^2(x' - vt')^2 + y'^2 + z'^2$. We expand the components of the metric (2.11) up to the order of $r^2_+, r^2_r$, and $r-r_+$ (higher-order contributions will vanish due to the boost). The results are

$$
A^2(r') = 1 - \left(r_+ + \frac{1 - a^2}{1 + a^2} r_-ight) \frac{1}{r'} + \left[\frac{r_+(r_+ + r_-)}{2} + \frac{1 - a^2}{1 + a^2} r_-(r_+ + r_-) \right] \frac{a^2(1 - a^2)}{(1 + a^2)^2} r_-^2 + \frac{1 - a^2}{1 + a^2} r_+ r_- \right] \frac{1}{r'^2},
$$

(2.12)

$$(r')^{-2}R^2 = 1 + \left(r_+ + \frac{1 - a^2}{1 + a^2} r_-ight) \frac{1}{r'} + \left[\frac{3r_+^2 + 3r_-^2 + 2r_+ r_-}{8} - \frac{a^2 r_+ r_-}{1 + a^2} - \frac{2a^2 r_-^2}{(1 + a^2)^2} \right] \frac{1}{r'^2}.
$$

(2.13)

We now take the limit $v \to 1$. One integrates first the expression with respect to $u' = x' - vt'$, takes the limit $v \to 1$, and then differentiates the corresponding expression with respect to $u'$. In order that the black hole can be boosted to the velocity of light, the mass and charge must go to zero. Following \[7,13,16\], we set

$$
M = \gamma^{-1}p, \quad Q^2 = \gamma^{-1}p_e^2,
$$

(2.14)

where $p$ and $p_e$ are two constants and can be interpreted as the kinematic and the electromagnetic momenta, respectively. Thus we can obtain

$$
ds^2 = \left\{-4p \left[\delta(x' - t') \ln \rho^2 - \frac{1}{|t' - x'|} \right] - \frac{3 - 4a^2}{2(1 - a^2)} \frac{\pi p_e^2 \delta(x' - t')}{\rho} \right\}(dt' - dx')^2 + (-dt'^2 + dx'^2 + dy'^2 + dz'^2),
$$

(2.15)

where $\rho^2 = y'^2 + z'^2$. Performing the following coordinate transformation

$$
\begin{align*}
    dy' &= dy, \quad dz' = dz, \\
    dt' - dx' &= dt - dx, \\
    dt' + dx' &= dt + dx - \frac{4p}{|t' - x'|}(dt - dx),
\end{align*}
$$

(2.16)

we have
\[ ds^2 = dudv + dy^2 + dz^2 - \left\{ 4p \ln \rho^2 + \frac{3 - 4a^2}{2(1 - a^2)} \frac{\pi \rho_e^2}{\rho} \right\} \delta(u) du^2, \]  

(2.17)

where \( u = x - t, \ v = x + t \) are two null coordinates. When \( a = 0 \), the solution (2.17) reduces to the one for the Reissner-Nordström black holes [13]:

\[ ds^2 = dudv + dy^2 + dz^2 - \left[ 4p \ln \rho^2 + \frac{3 \pi \rho_e^2}{2} \right] \delta(u) du^2. \]  

(2.18)

From the result (2.17) it is easy to find that the contribution of the dilaton field enters the expression through the contribution of the electromagnetic field. This is expected because the dilaton field has to be zero when the electromagnetic field disappears. In that case the solutions (2.17) and (2.18) both reduce to

\[ ds^2 = dudv + dy^2 + dz^2 - 4p \ln \rho^2 \delta(u) du^2. \]  

(2.19)

This is the AS solution obtained by boosting the Schwarzschild black hole. From (2.17) we find that when \( a^2 = 3/4 \), the contributions of the dilaton and electric fields are canceled each other, and the resulting limit is the AS solution (2.19). This is a very interesting case. In addition, it is worth noting that if we set \(|Q|\) go to zero as the mass, that is

\[ Q^2 = \gamma^{-2} \rho_e^2, \]  

(2.20)

both the contributions of the electric field and dilaton field vanish, and the resulting metric is just the AS solution.

In eq. (2.14) the first relation has a good motivation [7]; the second relation is to have a distributionally well defined limit of the metric. But, in this case, the electromagnetic field has a vanishing field strength (the gauge potential is a pure gauge [4]) but a nonzero, delta-like energy density in the ultrarelativistic limit [13]. It is easy to prove that the dilaton field also has the similar properties. However, this situation is still mathematically perfectly defined [14].

From (2.17), it is obvious that the solution is invalid as \( a = 1 \). The case \( a = 1 \) is a special one. In this case, we have \( r_+ = 2M, r_- = Q^2/M \), and

\[ A^2 = 1 - \frac{r_+}{\bar{r}'} + \frac{r_+(r_+ + r_-)}{2\bar{r}'^2}, \]  

(2.21)

\[ (\bar{r}')^{-2}R^2 = 1 + \frac{r_+}{\bar{r}'} + \left[ \frac{(r_+ + r_-)^2}{4} - \frac{r_-(r_+ + r_-)}{2} + \frac{(r_+ - r_-)^2}{8} \right] \frac{1}{\bar{r}'^2}. \]  

(2.22)

If one still uses the relations in eq. (2.14), he will find that the contribution of the dilaton field is divergent. To gain a well defined limit of this metric, we have to rescale the mass and charge as

\[ M = \gamma^{-1} \rho, \quad Q^2 = \gamma^{-3/2} \rho_e^2. \]  

(2.23)

Using the same method, we get the ultrarelativistic limit

\[ ds^2 = dudv + dy^2 + dz^2 - \left[ 4p \ln \rho^2 + \frac{\rho_e^2 \pi \rho_e^2}{8\bar{r}'^2} \right] \delta(u) du^2. \]  

(2.24)

This situation happens due to the fact that the energy-momentum tensor of the dilaton field is proportional to \( r_+^2 = Q^4/M^2 \), but the one of the electric field is proportional to \( Q^2 \). The second term in the square brackets in (2.24) comes from the contribution of the dilaton field and the electric field has not any contribution in this case.
B. Higher dimensional dilaton black holes

Now we extend the above discussion to the arbitrary higher dimensional dilaton black holes, which come from the action

\[
S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - \frac{2 e^{\beta \phi} F_{D-2}^2}{(D-2)!} \right],
\]

(2.25)

where \( F_{D-2} \) is a \((D-2)\)-form satisfying \( dF = 0 \) and \( \beta \) is a coupling constant. The black hole solutions are

\[
F_{d+1} = Q \epsilon_{d+1},
\]

(2.26)

\[
e^{\beta \phi} = \left[ 1 - (r_-/r)^d \right]^{\frac{d}{2b}},
\]

(2.27)

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 C(r) d\Omega_{d+1},
\]

(2.28)

where

\[
A(r) = \left[ 1 - (r_+/r)^d \right] \left[ 1 - (r_-/r)^d \right]^{1-db},
\]

\[
B(r) = \left[ 1 - (r_+/r)^d \right]^{-1} \left[ 1 - (r_-/r)^d \right]^{b-1},
\]

\[
C(r) = \left[ 1 - (r_-/r)^d \right]^b,
\]

d = D - 3 and the constant \( b \) is

\[
b = \frac{2(d+1)\beta^2}{d[2d + \beta^2(d+1)]}.
\]

The magnetic charge \( Q \) satisfies

\[
Q^2 = \frac{bd^3 (r_+ r_-)^d}{2b^2}.
\]

(2.29)

The mass of the black hole is

\[
M = \frac{(d+1)A_{d+1}}{16\pi} [r_+^d + (1 - db)r_-^d].
\]

(2.30)

Here \( A_{d+1} = 2\pi^{(d+2)/2}/\Gamma((d+2)/2) \) is the volume of the unit \((d+1)\)-sphere. Rewriting the metric (2.28) in the isotropic coordinates yields

\[
ds^2 = -A(r) dt^2 + \bar{r}^{-2} r^2 C(r) [d\bar{r}^2 + \bar{r}^2 d\Omega_{d+1}^2]
\]

(2.31)

where

\[
r^d = \bar{r}^d \left[ 1 + \frac{r_+^d + r_-^d}{2\bar{r}^d} + \frac{(r_+^d - r_-^d)^2}{16\bar{r}^2d} \right].
\]

(2.32)

Expanding the functions \( A \) and \( r^2(\bar{r})^{-2}C \), up to the order of \( r_+^{2d} \), \( r_-^{2d} \) and \( r_+^{d}r_-^{d} \), we obtain
where $u$ is the line element (2.35) with dimensions, we have to use the scaling relation (2.23) in this case, and then the resulting limit is the solution (2.37) again. In addition, when $b = 1/d$, i.e., $\beta = 2d/(d + 1)$, the metric (2.35) is invalid. Like the case $a = 1$ in four dimensions, we have to use the scaling relation (2.23) in this case, and then the resulting metric is the line element (2.33) with

$$f = \frac{2\pi A_{d+1}}{(16\pi)^2 d^2 p^2} \pi(2d - 3)!!.$$  

(2.38)

III. BOOSTING THE MULTI-CENTERED DILATON BLACK HOLE SOLUTIONS
A. Four dimensional multi-centered dilaton black hole solutions

In the action (2.1), when the extremality or zero-force condition is satisfied, besides the extremal dilaton black hole solution (2.4) with $r_+ = r_-$, one has the so-called multi-centered dilaton black hole solutions [21],

$$ds^2 = -U^{-2}(r)dt^2 + U^2(r)(dx^2 + dy^2 + dz^2),$$

(3.1)

where

$$U(r) = \left(1 + \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|}\right)^{1/(1+a^2)},$$

(3.2)

$r_i = (x_i, y_i, z_i)$ is the location of the $i$th black hole. The potential and the dilaton configuration are

$$A = \pm \left(\frac{1}{1 + a^2}\right)^{1/2} \left(1 + \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|}\right)^{-1} dt,$$

(3.3)

$$e^{-2a\phi} = \left(1 + \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|}\right)^{2a^2/(1+a^2)}.$$  

(3.4)

This solution describes $n$ extremal dilaton black holes in static equilibrium. The constant $\mu_i$ has the relation to the mass of $i$th black hole as $\mu_i = (1 + a^2)m_i$. When $a = 0$, this solution reduces to the Majumdar-Papapetrou (MP) solution [22], which describes $n$ extremal Reissner-Nordström black holes in the equilibrium.

Now we boost the multi-centered dilaton black holes to the ultrarelativistic limit. Using the Lorentz boost (2.10), we have

$$ds^2 = \gamma^2[U^2(r') - U^{-2}(r')]dt' - vdx'^2 + U^{-2}(r')(-dt'^2 + dx'^2 + dy'^2 + dz'^2),$$

(3.5)

where $r'^2 = \gamma^2(x' - vt')^2 + y'^2 + z'^2$. Taking the limit $v \to 1$, in order to get a solution of physical meaning, one has to make $m_i$ change as

$$m_i = p_i \gamma^{-1},$$

(3.6)

where $p_i$ remains unchanged in the process of the boost. Expanding the components of metric (3.5) up to the order $m_i$, yields

$$U^2(r) = 1 + \frac{2}{1 + a^2} \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|},$$

(3.7)

$$U^{-2}(r) = 1 - \frac{2}{1 + a^2} \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|}.$$  

(3.8)

Taking the limit $v \to 1$, we obtain

$$ds^2 = dudv + dy^2 + dz^2 - \sum_{i=1}^{n} 4\rho_i \left(\delta(u - u_i) \ln \rho_i^2 + \frac{1}{|u - u_i|}\right) du^2,$$

(3.9)
where we have dropped the prime, and

\[ u = x - t, \quad v = x + t, \quad \rho_i^2 = (y - y_i)^2 + (z - z_i)^2. \]  

(3.10)

(3.11)

Performing the following coordinate transformation

\[ du \rightarrow dv, \quad dv \rightarrow dv - \sum_{i=1}^{n} \frac{4p_i}{|u - u_i|} du, \]  

(3.12)

yields the result we expect

\[ ds^2 = dudv + dy^2 + dz^2 - \sum_{i=1}^{n} 4p_i \ln \rho_i^2 \delta(u - u_i) du^2. \]  

(3.13)

This solution is independent of the coupling parameter \( a \). So the ultrarelasitic limit of the MP solution is also the equation (3.13). From the solution, the nonvanishing component of the stress-energy tensor is

\[ T_{uu} = \sum_{i=1}^{n} p_i \delta(u - u_i) \delta(y - y_i) \delta(z - z_i), \]  

(3.14)

this is the sum of tensors of the \( n \) independently null particles. Therefore the shock wave spacetime (3.13) is generated by \( n \) particles moving along a same direction at the speed of light. No interaction between \( n \) particles is related to the fact that the multi-centered solution describes the spacetime where \( n \) extremal black holes are in the static equilibrium. Therefore it is an extension of AS geometry to the multiple particles. When \( n = 1 \) and \( x_1 = y_1 = z_1 = 0 \), obviously, the solution (3.13) goes back to the AS solution. Similar to the AS geometry, our solution (3.13) is flat except for those surfaces where particles live.

**B. Higher dimensional multi-centered dilaton black hole solutions**

We now generalize the above solution to the higher dimensional spacetimes. Consider the action [21]

\[ S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left[ R - \frac{4}{D - 2} (\nabla \phi)^2 - e^{-4a\phi/(D-2)} F_2^2 \right], \]  

(3.15)

where \( F_2 \) is also the Maxwell field. In this case, the multi-centered solution is

\[ ds^2 = -U^{-2}(r) dt^2 + U^{2/d}(r) \delta_{ab} dx^a dx^b, \]  

(3.16)

\[ A = \pm \left( \frac{d + 1}{2(d + a^2)} \right)^{1/2} F^{-1}(r) dt, \]  

(3.17)

\[ e^{-4a\phi/(D-2)} = [F(r)]^{2a^2/(d+a^2)}. \]  

(3.18)

The function \( U(r) \) is
\[ U(r) = [F(r)]^{d/(d+a^2)}, \]  
with 
\[ F(r) = 1 + \frac{1}{d} \sum_{i=1}^{n} \frac{\mu_i}{|r - r_i|^d}. \]

Here \( d = D - 3 \), \( a, b = 1, 2, \cdots, d + 2 \), and 
\[ r^2 = (x^1)^2 + (x^2)^2 + \cdots + (x^{d+2})^2, \quad r_i^2 = (x_i^1)^2 + (x_i^2)^2 + \cdots + (x_i^{d+2})^2. \]

The constant \( \mu_i \) is related to the mass of the \( i \)th hole as 
\[ \mu_i = \frac{8\pi (d + a^2)}{(d + 1)A_{d+1}} m_i. \] (3.20)

When \( a = 0 \), the solution (3.16) is just the higher dimensional generalization of the MP solution given by Myers [23]. Now we boost the multi-centered solution along the \( x^1 \) direction and use the relation (3.6). The resulting metric is 
\[ ds^2 = dudv + (dx^2)^2 + (dx^3)^2 + \cdots + (dx^{d+2})^2 + \frac{16\pi p_i}{(d - 1)A_d} \rho_{i-1}^2 \delta(u - u_i) du^2, \] (3.21)

with 
\[ \rho_i^2 = (x^2 - x_i^2)^2 + (x^3 - x_i^3)^2 + \cdots + (x^{d+2} - x_i^{d+2})^2. \] (3.22)

Again, the solution is independent of the coupling constant \( a \). Therefore this is also the ultra-relativistic limit of the Myers’ solution. This metric is the higher dimensional generalization of the solution (3.13).

**IV. BOOSTING THE DILATON BLACK STRINGS AND P-BRANES**

Now let us consider the low energy action of supergravity theory 
\[ S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - \frac{e^{-\alpha(d)\phi}}{2(d+1)!} F_{d+1}^2 \right], \] (4.1)

where \( F_{d+1} \) denotes the \((d+1)\)-form tensor field, and 
\[ \alpha^2(d) = 4 - \frac{2dd}{d + d}, \quad \tilde{d} = D - d - 2. \]

In the action (4.1) one has the black \((p = \tilde{d} - 1)\)-brane (black string for \( p = 1 \)) solution with “magnetic” charge [24]
\[ F = Q_\epsilon d+1, \] (4.2)
\[ e^{-2\phi} = \left[ 1 - (r_\epsilon/r)^d \right]^{\gamma \phi}, \] (4.3)
\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 C(r)d\Omega_{d+1} + D(r)\delta_{ab}dy^a dy^b, \] (4.4)
where

\[ A(r) = \left[ 1 - (r_+/r)^d \right] \left[ 1 - (r_-/r)^d \right] ^{\gamma_y-1}, \]
\[ B(r) = \left[ 1 - (r_+/r)^d \right] ^{-1} \left[ 1 - (r_-/r)^d \right] ^{\gamma_2-1}, \]
\[ C(r) = \left[ 1 - (r_-/r)^d \right] ^{\gamma_2}, \]
\[ D(r) = \left[ 1 - (r_-/r)^d \right] ^{\gamma_y}, \]

and

\[ \gamma_y = \frac{d}{d+d}, \quad \gamma_2 = \frac{\alpha^2(d)}{2d}, \quad \gamma_\phi = \alpha(d). \]

The black \((d - 1)\)-brane solution with “electric” charge in the action (4.1) can be obtained by using the duality transformation \([25]\). In the solution (4.4) the charge \(Q\) and the ADM mass \(M\) per unit \(p\)-brane are

\[ Q = d(r_- r_+)^{d/2}, \quad \text{(4.5)} \]
\[ M = \frac{A_{d+1}}{16\pi} [(d+1)r_+^d - r_-^d]. \quad \text{(4.6)} \]

There exist two different kinds of directions in the \(p\)-brane solutions (4.4): transverse direction and membrane direction. We will boost the solution along the two kinds of directions, respectively.

**A. Boosting in the transverse directions**

Rewriting the black \(p\)-brane (4.4) in the isotropic coordinates yields

\[ ds^2 = -A(r)dt^2 + r^2\bar{r}^{-2}C(r)[dr^2 + r^2\Omega_{d+1}^2] + D(r)\delta_{ab}dy^a dy^b, \quad \text{(4.7)} \]

where the relation between \(r\) and \(\bar{r}\) is the same as the one in (2.32). Expanding the functions \(A(r)\) and \(r^2\bar{r}^{-2}C(r)\) up to the order of \(r_+^{2d}, r_-^{2d}\) and \(r_+^d r_-^d\), one has

\[ A(r) = 1 - \left[ r_+^d + (\gamma_y - 1)r_-^d \right] \frac{1}{r^d} + \left[ \frac{r_+^d (r_+^d + r_-^d)}{2} + \frac{(\gamma_y - 1)r_-^d (r_+^d + r_-^d)}{2} \right] \frac{1}{r^{2d}}, \quad \text{(4.8)} \]
\[ r^2\bar{r}^{-2}C(r) = 1 + \left[ \frac{r_+^d + r_-^d}{d} - \gamma_2 r_-^d \right] \frac{1}{\bar{r}^d} + \left[ \frac{(r_+^d - r_-^d)^2}{8d} - \frac{\gamma_2 r_-^d (r_+^d + r_-^d)}{d} \right] \frac{1}{\bar{r}^{2d}} + \frac{(2-d)(r_+^d + r_-^d)^2}{4d^2} + \frac{\gamma_2 r_-^d (r_+^d + r_-^d)^2}{2} + \frac{\gamma_2 (\gamma_y - 1)(r_+^d + r_-^d)^2}{2} \right] \frac{1}{\bar{r}^{2d}}, \quad \text{(4.9)} \]

Using the relation (2.13) and taking the limit \(v \to 1\), we obtain the resulting solution due to the boost along a direction, say \(x^1\), of the transverse directions
\[ ds^2 = du dv + (dx^2)^2 + (dx^3)^2 + \cdots + (dx^{d+2})^2 + \delta_{ab} dy^a dy^b \]
\[ + \left[ \frac{16\pi p}{(d-1)A_d \rho^{d-1}} + \frac{g p_c^2}{\rho^{2d-1}} \right] \delta(u) du^2, \]  
(4.10)

where \( u = x^1 - t, v = x^1 + t \), and the constant \( g \) is
\[ g = \left[ \frac{1}{8d(d+1)} + \frac{(d^2 - 4)\gamma_\Omega}{2d^3} - \frac{(d + 2)(4 - 3d^2)}{4d^3(d + 1)} \right] \frac{\pi(2d - 3)!!}{2^{d-1}(d - 1)!}. \]

The solution (4.10) is the gravitational shock wave spacetime generated by the membrane moving at the speed of light along the direction perpendicular to the membrane itself. If we use the relation (2.20) or let \( Q = 0 \), the result will be
\[ ds^2 = du dv + (dx^2)^2 + (dx^3)^2 + \cdots + (dx^{d+2})^2 + \delta_{ab} dy^a dy^b \]
\[ + \left[ \frac{16\pi p}{(d-1)A_d \rho^{d-1}} \right] \delta(u) du^2. \]  
(4.11)

Note that the ultrarelativistic limit for extremal black \( p \)-branes (\( r_+ = r_- \)) is also the metric (4.11). Therefore, the metric (4.11) is a gravitational shock wave solution generated by a fundamental membrane (string) travels at the speed of light. Here the difference should be noticed between the solutions (4.11) and (2.37). The former is generated by a membrane (string) and the latter by a point-like particle. For the solution (4.11), the internal space \( (\delta_{ab} dy^a dy^b) \) is regular and does not relate to the delta-type singularity.

**B. Boosting in the membrane directions**

Now we boost the black \( p \)-brane along a direction, say \( y^1 \), of the membrane directions \( y \). That is, we do the following Lorentz transformation
\[ t = \gamma(t' - vy^1), \quad y^1 = \gamma(y^1' - vt'), \]  
(4.12)

and other coordinates keep unchanged. The solution (4.4) becomes
\[ ds^2 = \gamma^2[D(r) - A(r)](dt' - vy^1)^2 + B(r) dr^2 + r^2 C(r)d\Omega_{d+1}^2 \]
\[ + D(r)[-dt'^2 + (dy^1)^2 + \delta_{ab} dy^a dy^b], \]  
(4.13)

where \( a', b' = 2, 3, \cdots, d - 1 \). Expanding the function \( D(r) - A(r) \), up to the order of \( r_+^{2d}, r_-^{2d}, \) and \( r_+^d r_-^d \), yields
\[ D(r) - A(r) = (r_+^d - r_-^d) \frac{1}{r^d} + (\gamma_y - 1)(r_+^{2d} - r_-^{2d}) \frac{1}{r^{2d}}. \]  
(4.14)

From (4.14) it is easy to find that the extremal black \( p \)-brane (\( r_+ = r_- \)) is invariant under the boost along the membrane directions. This can be understood in terms of the fact that the extremal black \( p \)-branes (black string for \( p = 1 \)) is the precisely the solutions of the
fundamental membranes (strings) \[20\]. For nonextremal black \(p\)-branes \((r_+ \neq r_-)\), in order to extract results of physical meaning from (4.13), we assume that the mass and charge go to zero in the following way:

\[
M = \gamma^{-2}P, \quad Q = \gamma^{-2}P_e, \quad (4.15)
\]

where \(P\) and \(P_e\) are two constants. Thus the resulting solution is

\[
ds^2 = du dv + dr^2 + r^2d\Omega_{d+1}^2 + \delta_{\alpha \beta}dy^\alpha dy^\beta + \frac{h}{r^d}du^2, \quad (4.16)
\]

where \(u = y^1 - t,\ v = y^1 + t\), and the constant \(h\) is

\[
h = \frac{8\pi (d+2)P}{(d+1)A_{d+1}} - d\sqrt{\left(\frac{8\pi P}{(d+1)A_{d+1}}\right)^2 + \frac{P_e^2}{d^2(d+1)}}.
\]

The solution (4.16) is a plane-fronted gravitational wave propagating along the direction \(v\). When \(Q^2 = 0\), the constant \(h\) becomes

\[
h = \frac{16\pi P}{(d+1)A_{d+1}}.
\]

In this case, the solution (4.16) is the ultrarelativistic boost limit of the neutral black membrane along the membrane directions. For the neutral black string case, the solution (4.16) reduces to the result given in \[26\].

V. BOOSTING THE MULTI-CENTERED BLACK STRINGS AND \(P\)-BRANES

In this section we will discuss boosting the multi-centered black strings and \(p\)-branes. Let us consider the most generic multi-centered black \(p\)-brane solution in the following action \[27\]

\[
S = \frac{1}{16\pi} \int d^{D+p+q}x\sqrt{-d}\left[R - \frac{1}{2}\lambda(\nabla \phi)^2 - \frac{1}{2(p+2)!}e^{a_2\lambda \phi}F_{p+2}^2\right], \quad (5.1)
\]

where \(F_{p+2}\) is the \((p+2)\)-form tensor field, \(\lambda = 2/(D+p+q-2)\), and

\[
a_2^2 = \frac{D + p + q - 2}{D - 2}a^2 - \frac{(D - 2)pq + (D - 3)^2p + q}{D - 2}.
\]

The multi-centered \((p, q)\)-brane solution is \[27\]

\[
ds^2 = U^{-2}(r)(-dt^2 + \delta_{\alpha \beta}dy^\alpha dy^\beta) + U^{2(p+1)/(d+q)}(r)(\delta_{ab}dx^adx^b + \delta_{\mu \nu}dz^\mu dz^\nu), \quad (5.2)
\]

with

\[
U(r) = H(r)^{(d+q)/[(p+1)(d+q)+a_2^2]},
\]

\[
H(r) = 1 + \frac{1}{d} \sum_{i=1}^n \frac{\mu_i}{|r - r_i|^{d}}. \quad (5.3)
\]
Here $\alpha, \beta = 1, 2, \ldots, p$, $\mu, \nu = 1, 2, \ldots, q$, $a, b = 1, 2, \ldots, d + 2$ ($d = D - 3$), and $r_i = (x_1^i, x_2^i, \ldots, x_d^i)$ is the location of the $i$th membrane. The vector potential and dilaton field are

$$A = \pm \sqrt{2a_1 \over H(r)} dt dy^1 \cdots dy^p, \quad (5.4)$$

$$e^{-\phi/a_2} = H(r) a_1, \quad (5.5)$$

where $a_1 = (d + p + q + 1) / [(p + 1)(d + q) + a_2^2]$. The mass density of the $i$th membrane is

$$m_i = {A_{d+1} a_1 \over 8\pi} \mu_i. \quad (5.6)$$

The multi-centered black $(p, q)$-brane solution (5.2) is obviously boost invariant in the $y^\alpha$ directions. We boost therefore the solution in the $x^1$ and $z^1$ directions, respectively.

(i) In the transverse directions. Performing the Lorentz transformation in the $x^1$ direction, we have

$$ds^2 = \gamma^2 [U^{2(p+1)/(d+q)}(r') - U^{-2}(r')] (dt' - v dx^1')^2 + U^{-2}(r') \delta_{\alpha\beta} dy^\alpha dy^\beta$$

$$+ U^{2(p+1)/(d+q)}(r') [-dt'^2 + (dx'^1)^2 + \delta_{a'b'} dx^a dx^b + \delta_{\mu\nu} dz^\mu dz^\nu], \quad (5.7)$$

where $a', b' = 2, 3, \ldots, d + 2$. Taking the limit $v \to 1$ and using the relation (3.6), we obtain

$$ds^2 = du dv + \delta_{a'b'} dx^a dx^b + \delta_{\alpha\beta} dy^\alpha dy^\beta + \sum_{i=1}^{n} {d \rho_i \delta(u - u_i) du^2 \over (d - 1) A_{d-1}^2}, \quad (5.8)$$

where $u = x^1 - t$, $v = x^1 + t$, and the definition of $\rho^2_i$ is the same as the one in (3.22). Obviously, this solution is the generalization of the one for a single membrane (4.10) and describes the gravitational shock waves generated by $n$ membranes, which move parallel to each other at the speed of light. The solution (5.8) is invalid when $d = 1$, i.e., $D = 4$. As $d = 1$, it should be replaced by

$$ds^2 = du dv + \delta_{a'b'} dx^a dx^b + \delta_{\alpha\beta} dy^\alpha dy^\beta - \sum_{i=1}^{n} 4 \rho_i^2 \delta(u - u_i) du^2. \quad (5.9)$$

(ii) In the membrane directions. Let us boost the solution in the $z^1$ direction. Performing the Lorentz transformation in the $z^1$ direction, we have

$$ds^2 = \gamma^2 [U^{2(p+1)/(d+q)}(r) - U^{-2}(r)] (dt' - v dz^1')^2 + U^{-2}(r) \delta_{\alpha\beta} dy^\alpha dy^\beta$$

$$+ U^{2(p+1)/(D+q-3)}(r) [-dt'^2 + (dz'^1)^2 + \delta_{a'b'} dx^a dx^b + \delta_{\mu\nu} dz^\mu dz^\nu]. \quad (5.10)$$

We set that the mass density goes to zero as $m_i = \gamma^{-2} P_i$. Thus we can arrive at

$$ds^2 = du dv + \delta_{ab} dx^a dx^b + \delta_{\alpha\beta} dy^\alpha dy^\beta + \delta_{\mu\nu} dz^\mu dz^\nu + \sum_{i=1}^{n} {16\pi \over dA_{d+1}} {P_i \over |r - r_i|} du^2, \quad (5.11)$$

where $\mu', \nu' = 2, 3, \ldots, q$, $u = z^1 - t$, and $v = z^1 + t$. This gravitational wave solution is the extension of a single membrane solution (4.10) and describes $n$ gravitational waves propagating along the $v$ direction.
VI. CONCLUSION AND DISCUSSION

By boosting the four dimensional and higher dimensional dilaton black hole, black p-brane (black string for \( p = 1 \)), multi-centered dilaton black hole and multi-centered black p-brane solutions, we have investigated in some detail the ultrarelativistic limit of these solutions. For the single dilaton black hole solution and black p-brane solution for which the boost is done along the transverse directions, the resulting solutions are the gravitational shock wave solutions generated by a single particle and membrane (string) moving at the speed of light, respectively. When the charge disappears, the contribution of the dilaton field vanishes as well. This point can be seen from the energy-momentum tensor of the dilaton field. For the multi-centered dilaton black hole solution and multi-centered black p-brane solution for which the boost is done along the transverse directions, the resulting spacetimes are also shock wave solutions, but produced by multiple particles and membranes (strings). These particles or membranes (strings) move parallel to each other. When the boost is made along the membrane directions for the signal p-brane and multi-centered p-brane solutions, the ultrarelativistic limits are the general plane-fronted wave solutions propagating along the membrane directions. Therefore, the ultrarelativistic limits for black p-brane solutions are different due to the different boost directions. The effect of the dilaton field on the ultrarelativistic limit has been considered. Some peculiar cases appear. For example, in the \( a^2 = 3/4 \) four dimensional and \( \beta^2 = d/(d+1) \) higher dimensional dilaton black holes the contribution of the dilaton field cancels just the one of the tensor fields. And for \( a = 1 \) four dimensional and \( \beta^2 = 2d/(d+1) \) higher dimensional dilaton black holes, the rescaling relation (2.14) ceases to be valid and one has to use the relation (2.23) in order to get a distributionally well defined limit.

It should be pointed out that, to get well defined ultrarelativistic limits of dilaton black holes, the mass and charge must go to zero in an appropriate way. In this paper we have used the relations (2.14), (2.23), (3.6) and (4.15), respectively. As for this point, it is worth adding some remarks here. The rescaling of the mass in (2.14) is well motivated and saves the energy of the particle from diverging due to its finite rest mass by keeping the total energy \( p \) fixed and letting \( M \) approach zero in the ultrarelativistic limit [7,14]. The rescaling of charge in (2.14) is a unique manner to get a distributionally well defined result and a finite correction due to the charge. This rescaling of charge is valid for the Reissner-Nordström black hole [13] and \( a \neq 1 \) four dimensional dilaton black hole (2.4) and \( \beta^2 \neq 2d/(d+1) \) higher dimensional dilaton black hole (2.28). For \( a = 1 \) four dimensional and \( \beta^2 = 2d/(d+1) \) higher dimensional dilaton black holes, we have to rescale the charge in the manner (2.23). In this case, the contribution of the electromagnetic field vanishes, only the dilaton field makes its contribution through the charge. This can be seen from the energy-momentum tensor of the dilaton, whose first order correction is proportional to \( r^{-2d} \). However, the resulting fields in the rescaling relations (2.23) and (2.14) have physically highly unsatisfactory property of a vanishing field but a nonzero, delta-type energy density, although it is mathematically perfectly defined [14]. A physically intuitive rescaling of charge is eq. (2.20). That is the charge goes to zero in the same way as that of the mass. In this case, all contributions coming from the tensor field and dilaton field vanish as the ultrarelativistic limit is approached, and resulting geometries are gravitational shock wave solutions generated by neutral particles or membranes (strings) moving at the speed of light. In fact, the rescaling relation (3.6) in the
multi-centered solutions implies that the charge and mass have same manners going to zero. Therefore the resulting solution is independent of the charge and dilaton. This rescaling relation may be the basis of the argument by 't Hooft that the electric charge shifts only the pole points in the S-matrix of the scattering process \[4\]. However, it should be noticed that, when the boost is along the membrane (string) direction, the charge has the correction to the resulting solution \[1\], although the charge has the same rescaling relation as the mass \[1\]. At the same time, it should also be stressed that the boost is just a method to produce a new exact solution of Einstein’s field equations. It changes the type of solutions, from the type D of the original solutions to the type N of the new solutions in this paper. From the mathematical point of view, therefore, the different rescaling relations of mass and charge are admissible. But the physical relevances need to be studied further.

Note that for topological defects as well as point-like sources, the ultrarelativistic limit and the weak-field limit (large-distance behavior) of the original geometries gives the same scattering matrices \[3,13,17\]. One may expect that it also holds for multi-centered black hole or \(p\)-brane solutions. Compared with computing the scattering matrices in the these solutions, it is easier to calculate the same quantity in the ultrarelativistic limit of these solutions. For example, let us consider the scalar field propagating in the spacetime \(3.13\). According to \[3\], for an incoming plane wave with ingoing momentum \((k_\perp, \omega)\),

\[
\Psi_<(u, v, x_\perp) = \exp[-i\omega v/4] \cdot \exp[i(k_\perp \cdot x_\perp - iu(k_\perp^2 + m^2)/\omega)],
\]

where \(x_\perp = (y, z)\) and \(m\) is the mass of the scalar field, the outgoing solution with momentum \((p_\perp, \omega)\) is given by

\[
\Psi_>(u, v, x_\perp) = \int d^2 p_\perp \exp[i(p_\perp \cdot x_\perp - iu(p_\perp^2 + m^2)/\omega)] S(k_\perp, p_\perp, \omega).
\]

The scattering matrix \(S\) is

\[
S(k_\perp, p_\perp, \omega) = \int \frac{d^2 x_\perp}{(2\pi)^2} \exp[i(k_\perp - p_\perp) \cdot x_\perp + (i\omega/4) \sum_{i=1}^{n} f(\rho_i)],
\]

where \(f(\rho_i) = 4\rho_i \ln \rho_i^2\). The scattering matrix describes the scattering process of multiple particles at the energy of the Planckian scale, just as discussed in \[2 – 4\]. At the same time, it also describes the scattering of scalar field off multi-centered black hole solution under the weak-field approximation. For the scattering of string-string, one may study the scattering of string in the gravitational wave spacetime generated by string moving at the speed of light. It would be interesting to further study the change of poles in these \(S\) matrices. As an application of shock wave solutions generated by multiple particles and membranes (strings), one may use these geometries to investigate the scattering processes of multiple particles and membranes (strings) at the energies of Planckian scale. Also it should be of interest to study the geodesics and geodesic deviation in the shock wave spacetimes generated by multiple particles and membranes (strings).

As an interesting extension, it is of certain significance to consider the gravitational shock waves generated by multiple particles and membranes (strings) in curved spacetimes.
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REFERENCES

[1] P. D. D’Eath, Phys. Rev. D 18 (1978) 990;  
P. D. D’Eath and P. N. Payne, Phys. Rev. D 46 (1992) 658; 675; 694.  
[2] G. ’t Hooft, Phys. Lett. B 198 (1987) 61; Nucl. Phys.B 304 (1988) 867.  
[3] H. J. de Vega and N. Sanchez, Nucl. Phys. B 317 (1989) 706; 731.  
[4] H. Verlinde and E. Verlinde, Nucl. Phys. B 371 (1992) 246.  
[5] R. Jackiw, D. Kabat, and M. Ortiz, Phys. Lett. B 277 (1992) 148.  
[6] G. ’t Hooft, Nucl. Phys. B 335 (1990) 138.  
[7] P. C. Aichelburg and R. U. Sexl, Gen. Rel. Grav. 2 (1971) 303.  
[8] T. Dray and G. ’t Hooft, Nucl. Phys. B 253 (1985) 173.  
[9] C. O. Lousto and N. Sanchez, Phys. Lett. B 220 (1989) 55;  
C. O. Lousto, Phys. Rev.D 51 (1995) 1733.  
[10] K. Sfetsos, Nucl. Phys. B 436 (1995) 725.  
[11] H. A. Buchdahl, J. Phys. A 16 (1983) 1441.  
[12] M. Campanelli and C. O. Lousto, Phys. Rev. D 54 (1996) 3854;  
C. O. Lousto and F. D. Mazzitelli, Phys. Rev. D 56 (1997) 3471.  
[13] C. O. Lousto and N. Sanchez, Int. J. Mod. Phys. A 5 (1990) 915.  
[14] R. Steinbauer, J. Math. Phys. 38 (1997) 1614.  
[15] V. Ferrari and P. Pendenza, Gen. Rel. Grav. 22 (1990) 1105;  
H. Balasin and H. Nachbagauer, Class. Quantum Grav. 12 (1995) 707; 13 (1996) 731;  
K. Hayashi and T. Samura, Phys. Rev. D 50 (1994) 3666;  
C. Lousto and N. Schanez, Phys. Lett. B 232 (1989) 462.  
[16] C. Lousto and N. Sanchez, Nucl. Phys. B 383 (1992) 395.  
[17] C. Lousto and N. Sanchez, Nucl. Phys. B 355 (1991) 231.  
[18] M. Hotta and M. Tanaka, Class. Quantum Grav. 10 (1993) 307;  
J. Podolsky and J. Griffiths, Phys. Rev. D 56 (1997) 4756.  
[19] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D 43 (1991) 3140.  
[20] G. T. Horowitz and A. Strominger, Nucl. Phys. B 360, (1991) 197.  
[21] K. Shiraishi, J. Math. Phys. 34 (1993) 1480.  
[22] S. D. Majumdar, Phys. Rev. 72 (1947) 390;  
A. Papapetrou, Proc. R. Irish Acad. A 51 (1947) 191.  
[23] R. C. Myers, Phys. Rev. D 35 (1987) 455.  
[24] M. J. Duff, R. R. Khuri, and J. X. Lu, Phys. Rep. 259 (1995) 213.  
[25] R. G. Cai and Y. S. Myung, Nucl. Phys. B 495 (1997) 339.  
[26] J. H. Horne, G. T. Horowitz, and A. R. Steif, Phys. Rev. Lett. 68 (1992) 568.  
[27] R. R. Khuri and R. C. Myers, Nucl. Phys. B 466 (1996) 60.