INVIARNTS IN SEPARATED VARIABLES: YANG-BAXTER, ENTWINING AND TRANSFER MAPS

PAVLOS KASSOTAKIS

Abstract. We present the explicit form of a family of Liouville integrable maps in 3 variables, the so-called triad family of maps and we propose a multi-field generalisation of the later. We show that by imposing separability of variables to the invariants of this family of maps, the $H_I$, $H_{II}$ and $H_{III}$ Yang-Baxter maps in general position of singularities emerge. Two different methods to obtain entwining Yang-Baxter maps are also presented. The outcomes of the first method are entwining maps associated with the $H_I$, $H_{II}$ and $H_{III}$ Yang-Baxter maps, whereas by the second method we obtain non-periodic entwining maps associated with the whole $F$ and $H$-list of quadrirational Yang-Baxter maps. Finally, we show how the transfer maps associated with the $H$-list of Yang-Baxter maps can be considered as the $(k-1)$-iteration of some maps of simpler form. We refer to these maps as extended transfer maps and in turn they lead to $k$-point alternating recurrences which can be considered as alternating versions of some hierarchies of discrete Painlevé equations.

1. Introduction

The quantum Yang-Baxter equation originates from the theory of exactly solvable models in statistical mechanics [73] [11]. It reads:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

(1)

where $R : V \otimes V \rightarrow V \otimes V$ a linear operator and $R_{lm}$, $l \neq m \in \{1, 2, 3\}$ the operators that acts as $R$ on the $l$-th and $m$-th factors of the tensor product $V \otimes V \otimes V$. For the history of the later and for the early developments on the theory see [37]. Replacing the vector space $V$ with any set $X$ and the tensor product with the cartesian product, Drinfeld [21] introduced the set theoretical version of (1). Solutions of the later appeared under the name of set theoretical solutions of the quantum Yang-Baxter equation. The first instance of such solutions, appeared in [65] [23]. The term Yang-Baxter maps was proposed by Veselov [71] as an alternative name to the Drinfeld’s one. Early results on the context of Yang-Baxter maps were provided in [25] [24]. In the recent years, many results arose in the interplay between studies on Yang-Baxter maps and the theory of discrete integrable systems [8] [12] [10] [18] [19] [31] [20] [9].

In [23] it was considered a special type of set theoretical solutions of the quantum Yang-Baxter equation, the so called non degenerate rational maps. Nowadays, this type of solutions is referred to as quadrirational Yang-Baxter maps. Note that the notion of quadrirational maps, was extended in [40] to the notion of $2^n$-rational maps, where highly symmetric higher dimensional maps were considered. Under the assumption of quadrirationality and modulo conjugation (see Definition 3.1), in [6] [59] a list of ten families of maps was obtained. Five of them were given in [6], which constitute the so-called $F$-list of quadrirational Yang-Baxter maps and five more in [59], which constitute the so-called $H$-list of quadrirational Yang-Baxter maps. For their explicit form see Appendix A The Yang-Baxter maps of the $F$-list and the $H$-list can also be obtained from some of the integrable lattice equations in the classification scheme of [5], by using the invariants of the generators of the Lie point symmetry group of the later [50]. In the series of papers [45] [44] [50], from the Yang-Baxter maps of the $F$-list and of the $H$-list, integrable lattice equations and correspondences (relations) were systematically constructed. Invariant, under the maps, functions where the variables appeared in separated form, played an important role to this construction. The cornerstone of this manuscript are invariant functions where the variables appear in separated form.

In [4], it was introduced a family rational of maps in 3 variables that preserves two rational functions the so-called the triad map. The triad map serves as a generalisation of the QRT map [61] (cf. [22]). In Section 2 we present an explicit formula for Adler’s triad map as well as we prove the Liouville integrability of the later. We also propose an extension of the triad map in $k \geq 3$ number of variables. If one imposes separability to the variables of the invariants of the triad map, the $H_I$, the $H_{II}$ and the $H_{III}$ Yang-Baxter maps in general
positions of singularities, emerge. This is presented in Section 3 together with the explicit formulae for these maps.

In Section 4, we develop two methods to obtain non-equivalent entwining maps \cite{52} i.e. maps \( R, S, T \) that satisfy the relation

\[
R_{12} S_{13} T_{23} = T_{23} S_{13} R_{12}.
\]

The first method gives us entwining maps associated with the \( H_1, H_{11} \) and the \( H_{111}^A \) members of the \( H \)-list of Yang-Baxter maps. The second one produces entwining maps for the whole \( F \)-list and the \( H \)-list. In this manuscript we present the entwining maps associated with the \( H \)-list of quadrirational Yang-Baxter maps only.

In Section 5, we re-factorise the transfer maps \cite{71} associated with the \( H \)-list of Yang-Baxter maps. We show that the transfer maps coincide with the \((k-1)\)-iteration of some maps of simpler form that we refer to as extended transfer maps. Moreover, we show that the extended transfer maps, after a change of variables followed by an integration, are written as \( k \)-point recurrences, which some of them can be considered as alternating versions of discrete Painlevé hierarchies \cite{57,18,33}. In Section 6 we end this manuscript with some conclusions and perspectives.

2. The Adler’s Triad family of maps

In \cite{4}, Adler proposed the so-called triad family of maps. The triad map is a family of maps in 3 variables that consists of the composition of involutions which preserve two rational invariants of a specific form. In what follows we present the explicit form of the later in terms of its invariants.

Consider the polynomials

\[
n^i = \sum_{j,k,l=0}^{1} \alpha_{j,k,l}^i x_1^{1-j} x_2^{1-k} x_3^{1-l}, \quad d^i = \sum_{j,k,l=0}^{1} \beta_{j,k,l}^i x_1^{1-j} x_2^{1-k} x_3^{1-l}, \quad i = 1, 2.
\]

Where \( x_1, x_2, x_3 \) are considered as variables and \( \alpha_{j,k,l}^i, \beta_{j,k,l}^i \) as parameters. We consider also 3 maps \( R_{ij}, i < j \), \( i, j \in \{1, 2, 3\} \). These maps can be build out of the polynomials \( n^i, d^i \) and they read: \( R_{ij} : (x_1, x_2, x_3) \mapsto (X_1(x_1, x_2, x_3), X_2(x_1, x_2, x_3), X_3(x_1, x_2, x_3)) \) where

\[
X_i = x_i - 2 \begin{vmatrix}
\partial_{x_i} D_{x_i} n^1 \cdot d^1 & \partial_{x_i} D_{x_i} n^2 \cdot d^2 \\
D_{x_i} n^1 \cdot d^1 & D_{x_i} n^2 \cdot d^2
\end{vmatrix},
\]

\[
X_j = x_j + 2 \begin{vmatrix}
\partial_{x_j} D_{x_j} n^1 \cdot d^1 & \partial_{x_j} D_{x_j} n^2 \cdot d^2 \\
D_{x_j} n^1 \cdot d^1 & D_{x_j} n^2 \cdot d^2
\end{vmatrix},
\]

\[
X_k = x_k \text{ for } k \neq i,j.
\]

Here with \( \partial_z \) we denote the partial derivative operator wrt. to \( z \) i.e. \( \partial_z h = \frac{\partial h}{\partial z} \). \( D_z \) is the Hirota’s bilinear operator i.e. \( D_z h \cdot k = (\partial_z h) k - h \partial_z k \).

**Proposition 2.1.** The following holds:

1. Mappings \( R_{ij} \) depend on 32 parameters \( \alpha_{j,k,l}^i, \beta_{j,k,l}^i \), \( i = 1, 2, j, k, l \in \{0, 1\} \). Only 15 of them are essential.
2. The functions \( H_1 = n^1/d^1 \), \( H_2 = n^2/d^2 \) are invariant under the action of \( R_{ij} \) i.e. \( H_i R_{ij} = H_i \), \( i = 1, 2 \).
3. Mappings \( R_{ij} \) are involutions i.e. \( R_{ij}^2 = \text{id} \).
4. Mappings \( R_{ij} \) are anti-measure preserving with respect to the measures \( m_1 = n^1 d^1 \), \( m_2 = n^2 d^1 \).
5. It holds \( R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \).

**Proof.**

1. The invariants \( H_1, H_2 \) depend on 3 variables and they include 32 parameters. Acting with a different Möbius transformation to each of the variables, 9 parameters can be removed. A Möbius transformation of an invariant remains an invariant, since we have 2 invariants, 6 more parameters can be removed. Finally, since any multiple of an invariant remains an invariant, 2 more parameters can be removed. That leaves us with 32 - 9 - 6 - 2 = 15 essential parameters for the invariants \( H_1, H_2 \) and hence for the maps \( R_{ij} \).
(2) The functions $H_1 = n^1/d^1$, $H_2 = n^2/d^2$, reads
\[ H_1(x_1, x_2, x_3) = \frac{a_1 x_1 x_2 + b_1 x_1 + c_1 x_2 + d_1}{a_1 x_1 x_2 + b_1 x_1 + c_1 x_2 + d_1}, \quad H_2(x_1, x_2, x_3) = \frac{k x_1 x_2 + l x_1 + m x_2 + n}{k x_1 x_2 + l x_1 + m x_2 + n}, \]
where $a, a_1, b, b_1, k, k_1, \ldots$ are linear functions of $x_3$ (note we have suppressed the dependency on $x_3$ of the functions $H_1, H_2$). From the set of equations
\[ H_1(X_1, X_2, x_3) = H_1(x_1, x_2, x_3), \quad H_2(X_1, X_2, x_3) = H_2(x_1, x_2, x_3), \] (3)
by eliminating $X_2$ or by eliminating $X_1$ the resulting equations respectively factorize as:
\[ (X_1 - x_1)A = 0, \quad (X_2 - x_2)B = 0. \]
The factor $A$ is linear in $X_1$ and the factor $B$ is linear in $X_2$. By solving these equations (we omit the trivial solution $X_1 = x_1$, $X_2 = x_2$) we obtain
\[ X_1 = \frac{\gamma_{11}^1 x_2^2 + (\gamma_{21}^{11} + \gamma_{31}^{11}) \gamma_{11}^2 x_2 + \gamma_{21}^{12} + (\gamma_{11}^{12} + \gamma_{31}^{12}) x_2 + \gamma_{21}^{22} x_2}{\gamma_{11}^2 x_2 + (\gamma_{11}^{11} + \gamma_{31}^{11}) x_2 + \gamma_{21}^{11} + (\gamma_{11}^{12} + \gamma_{31}^{12}) x_2 + \gamma_{21}^{21} x_2}, \]
\[ X_2 = \frac{\gamma_{11}^2 x_1^2 + (\gamma_{21}^{11} + \gamma_{31}^{11}) x_1 + \gamma_{21}^{12} + (\gamma_{11}^{12} + \gamma_{31}^{12}) x_1 + \gamma_{21}^{22} x_1}{\gamma_{11}^2 x_1^2 + (\gamma_{11}^{11} + \gamma_{31}^{11}) x_1 + \gamma_{21}^{11} + (\gamma_{11}^{12} + \gamma_{31}^{12}) x_1 + \gamma_{21}^{21} x_2}. \] (4)
where $\gamma_{ij}^{kl} := \begin{vmatrix} u_{ij} & u_{kl} \\ v_{ij} & v_{kl} \end{vmatrix}$, with $u_{ij}$ the determinants of a matrix generated by the $i$th and $j$th column of the matrix
\[ u = \begin{pmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \end{pmatrix} \]
and $v_{kl}$ the determinants of a matrix generated by the $k$th and $l$th column of the matrix
\[ v = \begin{pmatrix} k & l & m & n \\ k_1 & l_1 & m_1 & n_1 \end{pmatrix} \]
Now it is a matter of long and tedious calculation to prove that the map $\phi : (x_1, x_2, x_3) \mapsto (X_1, X_2, x_3)$, where $X_1, X_2$ are given by \(4\) coincides with the map \(R_{12}\) of \(2\). Similarly we can work on \(R_{13}\) and \(R_{23}\).

(3) Since the map \(R_{12} : (x_1, x_2, x_3) \mapsto (X_1, X_2, x_3)\) satisfies \(3\), the proof of involutivity follows.

(4) It is enough to prove that the map \(R_{12}\) anti-preserves the measure $m_1 = n^1 d^2$ i.e. the Jacobian determinant
\[ \frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} := \begin{vmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} \end{vmatrix} \]
equals:
\[ \frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} = -\frac{n^1(X_1, X_2, x_3) d^2(X_1, X_2, x_3)}{n^1(x_1, x_2, x_3) d^2(x_1, x_2, x_3)}. \] (5)
Since the functions $H_i = n^i/d^i$, $i=1,2$ are invariant under the action of the map \(R_{12}\), it holds:
\[ n^1(X_1, X_2, x_3) = \kappa(x_1, x_2, x_3)n^1(x_1, x_2, x_3), \quad d^1(X_1, X_2, x_3) = \kappa(x_1, x_2, x_3)d^1(x_1, x_2, x_3), \]
\[ n^2(X_1, X_2, x_3) = \lambda(x_1, x_2, x_3)n^2(x_1, x_2, x_3), \quad d^2(X_1, X_2, x_3) = \lambda(x_1, x_2, x_3)d^2(x_1, x_2, x_3), \] (6)
where $\kappa, \lambda$ are rational functions of $x_1, x_2, x_3$. So,
\[ \frac{n^1(X_1, X_2, x_3) d^2(X_1, X_2, x_3)}{n^1(x_1, x_2, x_3) d^2(x_1, x_2, x_3)} = \kappa(x_1, x_2, x_3) \lambda(x_1, x_2, x_3). \] (7)
We differentiate equations \(6\) with respect to $x_1$ and we eliminate $\frac{\partial \kappa(x_1, x_2, x_3)}{\partial x_1}$ and $\frac{\partial \lambda(x_1, x_2, x_3)}{\partial x_1}$ to obtain
\[ \frac{1}{n^1} \left( \frac{\partial n^1}{\partial x_1} - \kappa \frac{\partial n^1}{\partial x_1} \right) = \frac{1}{d^1} \left( \frac{\partial d^1}{\partial x_1} - \kappa \frac{\partial d^1}{\partial x_1} \right), \quad \frac{1}{n^2} \left( \frac{\partial n^2}{\partial x_1} - \lambda \frac{\partial n^2}{\partial x_1} \right) = \frac{1}{d^2} \left( \frac{\partial d^2}{\partial x_1} - \lambda \frac{\partial d^2}{\partial x_1} \right). \] (8)
Remark 2.2. Any map that can be build out of the involutions $R_{12}, R_{13}, R_{23}$ can be considered as an Adler’s triad map. Hence here we have provided the explicit form of Adler’s Triad family of maps.

Among all the maps that can be constructed by the involutions $R_{ij}$, the following maps

$$T_i = R_{i+1}R_{i-1}, \quad T_1 = R_{13}R_{12}, \quad T_2 = R_{12}R_{23}, \quad T_3 = R_{23}R_{13}$$

(10)

are of special interest since they are not periodic and moreover they satisfy [14]

$$(T_iT_j)^2 = T_1T_2T_3 = id.$$

Proposition 2.3. For the maps $T_i$, $i = 1, 2, 3$ it holds:

1. they preserve the functions $H_1, H_2$.
2. they are measure-preserving with respect to the measures $m_1, m_2$
3. they preserve the following degenerate Poisson tensors,

$$\Omega_i^j = m_j \left( \frac{\partial H_i}{\partial x_1} \right) \left( \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right) \left( \frac{\partial H_i}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right) \left( \frac{\partial H_i}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right), \quad i, j \in \{1, 2\},$$

where it holds

$$0 = \Omega_1^j \nabla H_1, \quad \Omega_1^j \nabla H_2 = -\Omega_2^j \nabla H_1, \quad \Omega_2^j \nabla H_2 = 0, \quad j = 1, 2$$

4. they are Liouville integrable maps

Proof. The statements (1), (2) follows from Proposition 2.1. To prove the statement (3), (4), first note that since the maps $T_i$ are measure preserving, they preserve the following volume forms

$$V^j = m_j \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3},$$

Hence, the contractions $V^j|dH_i$, $i, j \in \{1, 2\}$, (see [22] [29]) are degenerate Poisson tensors. Namely:

$$\Omega_i^j = \left( m_j \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right)[dH_i] = m_j \left( \frac{\partial H_i}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right) \left( \frac{\partial H_i}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} \right) + \frac{\partial H_i}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3},$$

where $i, j \in \{1, 2\}$.

(5) The maps $T_i$ preserve the Poisson tensors $\Omega_i^j$ and the 2 invariants $H_1, H_2$, so they are Liouville integrable maps [33] [14] [70].
Note that on the level surfaces $H_2(x_1, x_2, x_3) = c$, maps $T_1, T_2, T_3$ reduce to pair-wise commuting maps on the plane which preserve the function $H_1(x_1, x_2; c)$. One of these reduced maps is the associated with the invariant $H_1(x_1, x_2; c)$ QRT map. Examples of commuting maps with specific members of the QRT family of maps were also constructed in [30].

The involution $R_{12}$ under the reduction $x_2 = x_1$, $H_2 = H_1 = H$, so $H = \frac{ax_1^2 + bx_1 + c}{kx_1^2 + lx_1 + m}$, reads:

$$R_{12} : (x_1, x_3) \mapsto \left(x_1 - 2 \frac{D_{x_1} n \cdot d}{D_{x_1} n \cdot d}, x_3\right),$$

that coincides with the QRT involution $i_2$ that preserves the invariant $H$. This formulae for the QRT involution $i_2$ was firstly given in [33], where an elegant presentation of the QRT map was considered.

### 2.1. A generalisation of the triad family of maps.

Following the same generalisation procedures introduced for the QRT family of maps [16] [56] [67] [52] [29], the triad family of maps can be generalised in similar manners. Here, in order to generalise the triad family of maps, we mimic the generalisation of the QRT family of maps introduced in [67].

Consider the following polynomials

$$n^i = \sum_{j_1, j_2, \ldots, j_k = 0}^{1} \alpha^i_{j_1, j_2, \ldots, j_k} x_1^{1-j_1} x_2^{1-j_2} \cdots x_k^{1-j_k},$$

$$d^i = \sum_{j_1, j_2, \ldots, j_k = 0}^{1} \beta^i_{j_1, j_2, \ldots, j_k} x_1^{1-j_1} x_2^{1-j_2} \cdots x_k^{1-j_k},$$

$i = 1, 2, k \geq 3$. (11)

Where $x_1, x_2, \ldots, x_k$ are considered as variables and $\alpha^i_{j_1, j_2, \ldots, j_k}, \beta^i_{j_1, j_2, \ldots, j_k}$ as parameters. We consider the $\binom{k}{2}$ maps $R_{ij}$, $i < j$, $i, j \in \{1, 2, \ldots, k\}$. These maps can be build out of the polynomials $n^i, d^i$ and they read:

$R_{ij} : (x_1, x_2, \ldots, x_k) \mapsto (X_1, X_2, \ldots, X_k)$, where $X_1 = x_1 \forall l \neq i, j$ and $X_i, X_j$ are given by the formulae (2), where $n^i, d^i$, $i = 1, 2$ are given by (11).

Proposition 2.1 is straight forward extended to the $k$-variables case. Namely for the mappings $R_{ij}$ it holds:

- Mappings $R_{ij}$ depend on $4 \cdot 2^n$ parameters $\alpha^i_{j_1, j_2, \ldots, j_k}, \beta^i_{j_1, j_2, \ldots, j_k}, i = 1, 2, j_1, j_2, \ldots, j_k \in \{0, 1\}$. Only $4 \cdot 2^n - 3n - 8$ of them are essential.
- The functions $H_1 = n^1 / d^1, H_2 = n^2 / d^2$ are invariant under the action of $R_{ij}$ i.e. $H_i \circ R_{ij} = H_i$, $i = 1, 2$.
- Mappings $R_{ij}$ are involutions i.e. $R^2_{ij} = id$.
- Mappings $R_{ij}$ are anti-measure preserving with respect to the measures $m_1 = n^1 d^2, m_2 = n^2 d^1$.
- It holds $R_{ij} R_{kl} R_{ij} = R_{kj} R_{il} R_{ij}$.

We take a stand here to comment that for $k = 3$ the construction above coincides with the Adler’s triad family of maps hence we have Liouville integrability. For $k > 3$ we have a generalisation of the later and since always we will have maps in $k$ variables with $2$ invariants, Liouville integrability is not expected for generic choice of the parameters $\alpha^i_{j_1, j_2, \ldots, j_k}, \beta^i_{j_1, j_2, \ldots, j_k}$. For a specific but quite general choice of the parameters though, one can associate a Lax pair to these maps and recover the additional integrals which are required for the Liouville integrability to emerge.

We also have to note that the case $k = 4$ was firstly introduced in [48]. Although for $k = 4$ we have mappings in $4$ variables with $2$ invariants, Liouville integrability is not apparent unless we specify the parameters. A specific choice of the parameters which leads to integrability is presented to the following example.

### Example 2.4 (The Adler-Yamilov map [7]).

Consider the following special form of the functions $n^i, d^i$

$$d^1 = d^2 = 1, \quad n^1 = x_1 x_2 + x_3 x_4, \quad n^2 = x_1 x_2 x_3 x_4 + x_1 x_4 + x_2 x_3 + ax_1 x_2 + bx_3 x_4.$$

Then the functions $H_1 = n^1 / d^1, H_2 = n^2 / d^2$ are preserved by construction by the maps $R_{ij}$ as well as by the following elementary involutions:

$i : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3), \quad \phi : (x_1, x_2, x_3, x_4) \mapsto (x_1 x_2 / x_3, x_3, x_2, x_3 x_4 / x_2)$.

The Adler-Yamilov map $(\xi)$ is considered by the following composition:

$$\xi := R_{14} \phi \iota : (x_1, x_2, x_3, x_4) \mapsto \left(x_3 - \frac{(a-b)x_1}{1 + x_1 x_4}, x_4, x_1, x_2 + \frac{(a-b)x_1}{1 + x_1 x_4}\right).$$
The Adler-Yamilov map is Liouville integrable since it preserves, and the invariants $H_1, H_2$ are in involution with respect to the canonical Poisson Bracket. For further discussions on the Adler-Yamilov map see [49, 30].

3. INVARIANTS IN SEPARATED VARIABLES AND YANG-BAXTER MAPS

Mappings $R_{mn}$, $m < n \in \{1, 2, \ldots, k\}$, presented in subsection 2.1 satisfy the identities $R_{ij}R_{ik}R_{jl} = R_{jl}R_{il}R_{lj}$, nevertheless as they stand they are not Yang-Baxter. Take for example the map $R_{12} : (x_1, x_2, x_3, \ldots, x_k) \mapsto (X_1, X_2, x_3, \ldots, x_k)$. The formulae for $X_1$ is fraction linear in $x_1$ with coefficients that depend on all the remaining variables and $X_2$ is fraction linear in $x_2$ with coefficients that depend on all the remaining variables. In order for $R_{12}$ to be a Yang-Baxter map the coefficients of $x_1$ in the formulae of $X_1$ should depend only on $x_2$ and the coefficients of $x_2$ in the formulae of $X_2$ should depend only on $x_1$. This “separability” requirement can be easily achieved by requiring separability of variables on the level of the invariants of the map $R_{12}$. We have two invariants $H_1 = \frac{n^3}{d^3}$, $H_2 = \frac{n^2}{d^2}$, so we can have three different kinds of separability. (I) Both $H_1$ and $H_2$ to be multiplicative separable on the variables $x_1$ and $x_2$. (II) $H_1$ to be multiplicative and $H_2$ to be additive separable and finally (III) both $H_1$ and $H_2$ to be additive separable on the variables $x_1$ and $x_2$. In what follows we explicitely present these three different kinds of separability in all variables of the invariants $H_1$ and $H_2$.

(I) Multiplicative/multiplicative separability of variables.

$$H_1 = \prod_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad H_2 = \prod_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i}.$$  \hspace{1cm} (12)

(II) Multiplicative/additive separability of variables.

$$H_1 = \prod_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad H_2 = \sum_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i},$$  \hspace{1cm} (13)

(III) Additive/additive separability of variables.

$$H_1 = \sum_{i=1}^{k} \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad H_2 = \sum_{i=1}^{k} \frac{A_i - B_i x_i}{C_i - D_i x_i}.$$  \hspace{1cm} (14)

Where $a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i, i = 1, \ldots k$, parameters, $8k$ in total. In all three cases above, the number of essential parameters is $3k - 6$. This argument can be proven by the following reasoning. Since the invariants $H_1, H_2$ depends on $k$ variables, by a M"obius transformation on each of the $k$ variables $3k$ parameters can be removed. Also any M"obius transformation of an invariant remains an invariant so since we have two invariants $2 \times 3$ more parameters can be removed. Finally, for each one of the $2k$ in number functions $\frac{a_i - b_i x_i}{c_i - d_i x_i}, \frac{A_i - B_i x_i}{C_i - D_i x_i}, i = 1, \ldots k$, one non-zero parameter can be absorbed simply by dividing with it (and reparametrise), so $2k$ in number more parameters can be removed. In total we have $8k - 3k - 2 \times 3 - 2k = 3k - 6$ essential parameters.

3.0.1. Multiplicative/multiplicative separability of variables. Let us first introduce some definitions.

**Definition 3.1.** The maps $R, \tilde{R} : \mathbb{CP}^1 \times \mathbb{CP}^1 \mapsto \mathbb{CP}^1 \times \mathbb{CP}^1$ are $(\text{Mob})^2$ equivalent if there exists bijections $\phi, \psi : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$ such that the following conjugation relation holds

$$\tilde{R} = \phi^{-1} \times \psi^{-1} R \phi \times \psi.$$

**Definition 3.2.** The map $R : \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u, v) \mapsto (U, V) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, where

$$U = \frac{a_1 + a_2 u}{a_3 + a_4 u}, \quad V = \frac{b_1 + b_2 v}{b_3 + b_4 v}$$

with $a_i, b_i, i = 1, \ldots, 4$ known polynomials of $v$ and $u$ respectively, will be said to be of subclass $[\gamma : \delta]$, if the highest degree that appears in the polynomials $a_i$ is $\gamma$ and the higher degree that appears in the polynomials $b_i$ is $\delta$.

Clearly, maps that belong to different subclasses are not $(\text{Mob})^2$ equivalent.
Proposition 3.3. Consider the multiplicative/multiplicative separability of variables of the invariants $H_1$ and $H_2$ (see (12)). Consider also the following sets of parameters

$$\mathbf{p}_{ij} := \mathbf{p}_i \cup \mathbf{p}_j$$

where $\mathbf{p}_i := \{a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i\}$, $i < j \in \{1, 2, \ldots, k\}$

and the functions

$$f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \quad i = 1, \ldots, k.$$

The following holds:

(1) The invariants $H_1 = \prod_{i=1}^k f_i$, $H_2 = \prod_{i=1}^k g_i$ depend on $8k$ parameters. Only $3k - 6$ of them are essential.

(2) Mappings $R_{ij}$ explicitly read:

$$R_{ij} : (x_1, x_2, \ldots, x_k) \mapsto (X_1, X_2, \ldots, X_k),$$

where $X_i = x_i \forall i \neq i, j$ and $X_i, X_j$ are given by the formulae

$$X_i = x_i - 2 \left( f'_i f_j f'_j f_i f'_i f'_j f_i g'_i g_j g'_j g_i + \frac{f'_i g'_i}{g_i g'_i} \right)$$

$$X_j = x_j + 2 \left( f'_j g'_i f'_i g'_j f'_j f'_i f'_j f'_i g'_i g_j g'_j g_i + \frac{f'_i g'_i}{g_i g'_i} \right)$$

where $f'_i \equiv \frac{\partial f_i}{\partial x_i}, g'_i \equiv \frac{\partial g_i}{\partial x_i}, g''_i \equiv \frac{\partial^2 g_i}{\partial x_i^2}$, etc. Note that in the expressions of $X_i, X_j$ appears only the parameters $\mathbf{p}_{ij}$. From further on we denote the maps $R_{ij}$ as $R_{ij}^{P_{ij}}$, in order to stress this separability feature.

(3) Mappings $R_{ij}$ are anti-measure preserving with respect to the measures $m_1 = n^2 d^2$, $m_2 = n^2 d^3$, where $n^i, d^i$ the numerators and the denominators respectively, of the invariants $H_i$, $i = 1, 2$.

(4) Mappings $R_{ij}^{P_{ij}}$ satisfy the Yang-Baxter identity

$$R_{ij}^{P_{ij}} R_{jk}^{P_{jk}} R_{kj}^{P_{kj}} = R_{kj}^{P_{kj}} R_{ij}^{P_{ij}} R_{jk}^{P_{jk}}.$$

(5) Mappings $R_{ij}^{P_{ij}}$ are involutions with the sets of singularities

$$\Sigma_{ij} = \{P_{ij}^1, P_{ij}^2, P_{ij}^3, P_{ij}^4\} = \left\{ \left( \frac{a_i}{b_i}, \frac{c_i}{d_i} \right), \left( \frac{a_j}{b_j}, \frac{c_j}{d_j} \right), \left( \frac{A_i}{B_i}, \frac{C_i}{D_i} \right), \left( \frac{A_j}{B_j}, \frac{C_j}{D_j} \right) \right\},$$

and the sets of fixed points

$$\Phi_{ij} = \{Q_{ij}^1, Q_{ij}^2, Q_{ij}^3, Q_{ij}^4\} = \left\{ \left( \frac{a_i}{b_i}, \frac{c_i}{d_i} \right), \left( \frac{a_j}{b_j}, \frac{c_j}{d_j} \right), \left( \frac{A_i}{B_i}, \frac{C_i}{D_i} \right), \left( \frac{A_j}{B_j}, \frac{C_j}{D_j} \right) \right\}$$

where in the formulae for $P_{ij}^m$ and $Q_{ij}^m$, $m = 1, \ldots, 4$, we have suppressed the dependency on the remaining variables. For example, with $P_{ij}^1 = \left( \frac{a_i}{b_i}, \frac{c_i}{d_i} \right)$ we denote  

$$(x_1, \ldots, x_{i-1}, \frac{a_i}{b_i}, x_{i+1}, \ldots, x_{j-1}, \frac{c_j}{d_j}, x_{j+1}, \ldots, x_k)$$

and similarly for the remaining $P_{ij}^m$ and $Q_{ij}^m$.

(6) Each one of the maps $R_{ij}^{P_{ij}}$ is (Möbius$)^2$ equivalent to the $H_1$ Yang-Baxter map.

Proof. (1) See at the end of the previous subsection.

(2) Mappings written in terms of the functions $f_i, g_i$ get exactly the desired form.

(3) See Proposition 2.4

(4) See Proposition 2.4

(5) Because mappings $R_{ij}^{P_{ij}}$, for generic parameter sets $\mathbf{p}_{ij}$, belong to the $[2 : 2]$ subclass, we expect at most 8 singular points, 4 singular points from the first fraction of the map and 4 from the second. By direct calculation we show that the singular points of the first and the second fraction of $R_{ij}^{P_{ij}}$ coincide. Moreover, $P_{ij}^m$, $m = 1, \ldots, 4$ are the singular points of the maps $R_{ij}^{P_{ij}}$ i.e.

$$R_{ij}^{P_{ij}} : P_{ij}^m \mapsto \left( x_1, \ldots, x_{i-1}, \frac{0}{0}, x_{i+1}, \ldots, x_{j-1}, \frac{0}{0}, x_{j+1}, \ldots, x_k \right).$$
Note that the values of the invariants $H_i$ at the singular points $P_i^{m}$ are undetermined i.e. $H_1(P_1^{m}) = 0$, $m = 1, 2$, $H_2(P_2^{m}) = 0$, $m = 3, 4$. For the fixed points $Q_{ij}^{m}$, $m = 1, \ldots, 4$ it holds $R_{ij}^{P_0} : Q_{ij}^{m} \rightarrow Q_{ij}^{m}$. Note also that $H_1(Q_{ij}^{1}) = 0$, $H_1(Q_{ij}^{2}) = \infty$, $H_2(Q_{ij}^{3}) = 0$, $H_2(Q_{ij}^{4}) = \infty$.

6) Introducing the new variables $y_i$, $i \neq j = 1, \ldots k$ though

$$CR[x_i, a_i/b_i, c_i/d_i, A_i/B_i] = CR[y_i, 0, 1, \infty], \quad CR[x_j, c_j/d_j, a_j/b_j, C_j/D_j] = CR[y_j, \infty, 1, 0],$$

after a re-parametrization mappings $R_{ij}$ gets exactly the form of the $H_1$ map. Here, with $CR[a, b, c, d]$ we denote the cross-ratio of 4 points, namely

$$CR[a, b, c, d] := \frac{(a - c)(b - d)}{(a - d)(b - c)}.$$

Each one of the maps $R_{ij}$ has a set of singularities which consists of 4 distinct points. With appropriate limits we are allowed to merge some of the singularities and obtain Yang-Baxter maps which are not $(\text{Möb})^2$ equivalent with the original one.

By setting $C_i = cA_i$, $D_i = cB_i$, $A_j = cC_j$, $B_j = cD_j$ and letting $c \rightarrow 0$ the singular points $P_i^{4}$ and $P_i^{3}$ merge.

The resulting maps, under a re-parametrization, coincide with the ones obtained in the multiplicative/additive case (see subsection 3.0.2), hence are $(\text{Möb})^2$ equivalent with the $H_{11}$ Yang-Baxter map. The same result can be obtained by merging $P_i^{3}$ and $P_i^{2}$.

Note that merging $P_i^{1}$ or $P_i^{3}$ or $P_i^{2}$ with $P_i^{4}$ is not of interest since the resulting maps are trivial.

By further setting $c_i = c_1$, $d_i = c_b$, $a_j = c_c$, $b_j = c_d$ and letting $c \rightarrow 0$ the singular points $P_i^{2}$ and $P_i^{4}$ merge as well. The resulting maps, under a re-parametrization, coincide with the ones obtained in the additive/additive case (see subsection 4.0.3), hence are $(\text{Möb})^2$ equivalent with the $H_{11}^A$ Yang-Baxter map. Any further merging of singularities leads to trivial maps.

Remark 3.4. An interesting observation is that the reduction $Q_{ij}^{4} = P_{ij}^{2}$ or $Q_{ij}^{4} = P_{ij}^{1}$, (equivalently $Q_{ij}^{3} = P_{ij}^{3}$ or $Q_{ij}^{2} = P_{ij}^{1}$) that merges singular with fixed points, leads to maps that belong to the $[1 : 1]$ subclass of maps.

Remark 3.5. For generic sets of parameters $p_{ij}$, each one of the $k$ in number maps $R_{ij}^{P_0}$, is $(\text{Möb})^2$ equivalent to the $H_1$ Yang-Baxter map. For degenerate choices of the sets $p_{ij}$, this is no longer the case. Hence, in that respect, mappings $R_{ij}^{P_0}$ are more general than the $H_1$ map since they include degenerate cases as well. In the same respect $Q_{ij}$ [72], the rational version of the discrete Krichever-Novikov equation $Q_{ij}$ [9], is more general.

Example 3.6 ($k = 3$). For $k = 3$, the invariants $H_1 = f_1f_2f_3$, $H_2 = g_1g_2g_3$ are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants $H_1, H_2$ can be cast into the form:

$$H_1 = x_1x_2x_3, \quad H_2 = \frac{x_1 - p_1}{x_1 - 1}, \frac{x_2 - p_2}{x_2 - 1}, \frac{x_3 - p_3}{x_3 - 1}.$$ 

Then each of the mappings $R_{ij}, i \neq j \in \{1, 2, 3\}$ is exactly the $H_1$ Yang-Baxter map. The $H_1$ Yang-Baxter map explicitly reads: $H_1 : (u, v) \mapsto (U, V)$ where

$$U = vQ, \quad V = uQ^{-1}, \quad Q = \frac{(\alpha - 1)uv + (\beta - \alpha)u + \alpha(1 - \beta)}{(\beta - 1)uv + (\alpha - \beta)v + \beta(1 - \alpha)}.$$ (15)

By the identifications $u \equiv x_1, u \equiv x_3$, $\alpha \equiv p_1$ and $\beta \equiv p_3$, from [15] we recover the maps $R_{ij}$.

The maps $\phi_i : (x_1, x_2, x_3) \mapsto (X_1, X_2, X_3)$ where $X_i = x_i \forall \neq i$ and $X_i = \frac{x_i}{x_i - 1}, \ i = 1, 2, 3$ and the maps $\psi_i : (x_1, x_2, x_3) \mapsto (X_1, X_2, X_3)$ where $X_i = x_i \forall \neq i$ and $X_i = \frac{x_i - p_i}{x_i - 1}, \ i = 1, 2, 3$ satisfy

$$H_1\phi_1\phi_2\phi_3 = \frac{p_1p_2p_3}{H_1}, \quad H_2\phi_1\phi_2\phi_3 = \frac{p_1p_2p_3}{H_1}, \quad H_1\psi_1\psi_2\psi_3 = \frac{p_1p_2p_3}{H_2}.$$ 

The maps $\phi_i$ and $\psi_i$ have a special role in [59] since though them the $H_1$ map was derived out of the $F_1$ Yang-Baxter map. We will discuss more about these maps in the next Section. We just quickly recall that $\phi_1R_{12}\phi_2$ is exactly the $F_1$ Yang-Baxter map.
Remark 3.7. We have to remark that with loss of generality, mappings $R_{ij}$ can belong on a different subclasses than the $[2 : 2]$ subclass of maps that the $H_1$ map belongs to. For example, for
\begin{equation}
H_1 = (x_1 - p_1)(x_2 - p_2)(x_3 - p_3), \quad H_2 = \frac{x_1 - p_1}{x_1} \frac{x_2}{x_2 - p_2} \frac{x_3}{x_3 - p_3}.
\end{equation}
$R_{12}$ is the Hirota’s KdV map (see [14]) that belongs on the subclass $[1 : 1]$ and $R_{13}, R_{23}$ are maps which belong to the subclass $[2 : 1]$. Explicitly the maps read
\begin{align*}
R_{12} : (x_1, x_2, x_3) &\mapsto \left( \frac{p_1(p_2x_1 + p_1x_2 - x_1x_2)}{p_2x_1}, \frac{p_2(p_2x_1 + p_1x_2 - x_1x_2)}{p_1x_2}, x_3 \right), \\
R_{13} \equiv S_{13} : (x_1, x_2, x_3) &\mapsto \left( \frac{p_1(-1 + x_3)(p_3x_1 + p_1x_3 - x_1x_3)}{-p_3x_1 - p_1x_3 + p_1p_3x_3 + x_3x_3}, x_2, \frac{p_1x_1 + p_1x_3 - x_1x_3}{p_1x_3} \right), \\
R_{23} \equiv T_{23} : (x_1, x_2, x_3) &\mapsto \left( x_1, \frac{p_2p_3x_3 - p_2x_2 + p_2x_3 - 2x_3x_3}{-p_2p_3 + p_3x_2 + 2p_3x_3 - 2x_3}, x_2(-p_3 + x_3) \right).
\end{align*}
The Hirota’s KdV map entwines with $S_{13}$ and $T_{23}$, since $R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}$ holds.

Example 3.8 ($k \geq 4$). For $k = 4$ the invariants depend on 32 parameters and only 6 of them are essential. Without loss of generality they can be cast into the form
\begin{equation}
H_1 = x_1x_2x_3x_4, \quad H_2 = \frac{x_1 - p_1}{x_1} \frac{x_2 - p_2}{x_2} \frac{x_3 - p_3}{x_3} \frac{x_4 - p_4}{x_4}.
\end{equation}
For $k > 4$ the invariants depend on $8k$ parameters and only $3k - 6$ of them are essential. Without loss of generality they can be cast into the form
\begin{equation}
H_1 = \prod_{i=1}^{k} x_i, \quad H_2 = \prod_{i=4}^{k} x_i - \prod_{i=1}^{k} \frac{x_1 - p_1}{x_1} \frac{x_2 - p_2}{x_2} \frac{x_3 - p_3}{x_3} \frac{x_4 - p_4}{x_4}.
\end{equation}

3.0.2. Multiplicative/additive separability of variables.

Proposition 3.9. Consider the multiplicative/additive separability of variables of the invariants $H_1$ and $H_2$ (see [13]). Consider also the following sets of parameters
\begin{equation}
p_{ij} := p_i \cup p_j \text{ where } p_i := \{a_i, b_i, c_i, d_i, A_i, B_i, C_i, D_i\}, \quad i < j \in \{1, 2, \ldots, k\}
\end{equation}
and the functions
\begin{equation}
f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \quad i = 1, \ldots, k.
\end{equation}
The following holds:
\begin{enumerate}
\item The invariants $H_1 = \prod_{i=1}^{k} f_i, H_2 = \sum_{i=1}^{k} g_i$ depend on $8k$ parameters. Only $3k - 6$ of them are essential.
\item Mappings $R_{ij}$ explicitly read:
\begin{equation}
R_{ij} : (x_1, x_2, \ldots, x_k) \mapsto (X_1, X_2, \ldots, X_k),
\end{equation}
where $X_l = x_l \forall l \neq i, j$ and $X_i, X_j$ are given by the formulae
\begin{align*}
X_i = x_i - 2 \begin{vmatrix}
{f_i}' & {f_j}' & \gamma_i \cr
f_i' & f_j' & \gamma_j \cr
\end{vmatrix} + \begin{vmatrix}
{f_j}' & \gamma_j \cr
f_i' & f_j' \cr
\end{vmatrix} + \frac{\partial f_i}{\partial x_i} \frac{\partial f_j}{\partial x_j} + \frac{\partial g_i}{\partial x_i} \frac{\partial g_j}{\partial x_j},
\end{align*}
\begin{align*}
X_j = x_j + 2 \begin{vmatrix}
{f_i}' & {f_j}' & \gamma_i \cr
f_i' & f_j' & \gamma_j \cr
\end{vmatrix} + \begin{vmatrix}
{f_j}' & \gamma_j \cr
f_i' & f_j' \cr
\end{vmatrix} + \frac{\partial f_i}{\partial x_i} \frac{\partial f_j}{\partial x_j} + \frac{\partial g_i}{\partial x_i} \frac{\partial g_j}{\partial x_j},
\end{align*}
where $f_i' \equiv \frac{\partial f_i}{\partial y_i}, \quad g_i' \equiv \frac{\partial g_i}{\partial y_i}, \quad \gamma_i' \equiv \frac{\partial \gamma_i}{\partial y_i},$ etc. Note that in the expressions of $X_i, X_j$ appears only the coordinates $x_i, x_j$ and the parameters $p_{ij}$. From further on we denote the maps $R_{ij}$ as $R_{ij}^{p_{ij}}$, in order to stress this separability feature.
(3) Mappings $R_{ij}$ are anti-measure preserving with respect to the measures $m_1 = n^1d^2$, $m_2 = n^2d^1$, where $n^1,d^2$ the numerators and the denominators respectively, of the invariants $H_i$, $i = 1,2$.

(4) Mappings $R_{ij}^{P_0}$ satisfy the Yang-Baxter identity

$$R_{ij}^{P_0} R_{jk}^{P_0} R_{kj}^{P_0} = R_{jkl}^{P_0} R_{ijk}^{P_0} R_{ikl}^{P_0}.$$  

(5) Mappings $R_{ij}^{P_0}$ are involutions with the sets of singularities

$$R_{ij}^{P_0} = \{(x_1, x_2, x_3, x_4) \mid x_i \neq 0, i = 1,2,3,4\}$$

where the superscript 2 in $P_0^{ij}$ denotes that these singular points appears with multiplicity 2. In the formulae for $P_m^{ij}$, $m = 1, \ldots, 3$, we have suppressed the dependency on the remaining variables. For example, with $P_1^{ij} = (\frac{a_i}{b_i}, \frac{a_j}{b_j})$ we denote $(x_1, \ldots, x_{i-1}, \frac{a_i}{b_i}, x_{i+1}, \ldots, x_{j-1}, \frac{a_j}{b_j}, x_{j+1}, \ldots, x_k)$ and similarly for the remaining $P_m^{ij}$.

(6) Each one of the maps $R_{ij}^{P_0}$ is (Möbi)² equivalent to the $H_{11}$ Yang-Baxter map.

Proof. The proof follows similarly to the proof of proposition 3.5.

Example 3.10 ($k \geq 3$). For $k = 3$, the invariants $H_1 = f_1 f_2 f_3$, $H_2 = g_1 + g_2 + g_3$ are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants $H_1, H_2$ can be cast into the form:

$$H_1 = \frac{x_1 - p_1 x_2 - p_2 x_3 - p_3}{x_1}, \quad H_2 = x_1 + x_2 + x_3.$$  

Then each of the mappings $R_{ij}, i \neq j \in \{1,2,3\}$ is exactly the $H_{11}$ Yang-Baxter map.

For $k > 3$ the invariants depend on 8k parameters and only $3k - 6$ of them are essential. Without loss of generality they can be cast into the form:

$$H_1 = \frac{x_1 - p_1 x_2 - p_2 x_3 - p_3}{x_1}, \quad H_2 = \frac{\sum_{i=1}^{k} x_i}{x_i}.$$  

3.0.3. Additive/additive separability of variables.

Proposition 3.11. Consider the additive/additive separability of variables of the invariants $H_1$ and $H_2$ (see [4]). Consider also the following sets of parameters

$$\mathbf{p}_{ij} := \mathbf{p}_i \cup \mathbf{p}_j$$

and the functions

$$f_i := \frac{a_i - b_i x_i}{c_i - d_i x_i}, \quad g_i := \frac{A_i - B_i x_i}{C_i - D_i x_i}, \quad i = 1, \ldots, k.$$  

The following holds:

(1) The invariants $H_1 = \prod_{i=1}^{k} f_i$, $H_2 = \sum_{i=1}^{k} g_i$ depend on 8k parameters. Only $3k - 6$ of them are essential.

(2) Mappings $R_{ij}$ explicitly read:

$$R_{ij} : (x_1, x_2, \ldots, x_k) \mapsto (X_1, X_2, \ldots, X_k),$$

where $X_l = x_l$ $\forall l \neq i, j$ and $X_i, X_j$ are given by the formulae

$$X_i = x_i - 2 \left| \begin{array}{cc} f_i' & g_i' \\ g_i' & f_i' \\ \frac{f_i''}{f_i'} + \frac{f_j''}{f_j'} & \frac{g_i''}{g_i'} + \frac{g_j''}{g_j'} \\ \frac{g_i''}{g_i'} + \frac{g_j''}{g_j'} & \frac{f_i''}{f_i'} + \frac{f_j''}{f_j'} \end{array} \right|,$$

$$X_j = x_j + 2 \left| \begin{array}{cc} f_j' & g_j' \\ g_j' & f_j' \\ \frac{f_i''}{f_i'} + \frac{f_j''}{f_j'} & \frac{g_i''}{g_i'} + \frac{g_j''}{g_j'} \\ \frac{g_i''}{g_i'} + \frac{g_j''}{g_j'} & \frac{f_i''}{f_i'} + \frac{f_j''}{f_j'} \end{array} \right|,$$

where $f_i' \equiv \frac{\partial f_i}{\partial x_i}$, $g_i' \equiv \frac{\partial g_i}{\partial x_i}$, $g_i'' \equiv \frac{\partial^2 g_i}{\partial x_i^2}$, etc. Note that in the expressions of $X_i, X_j$ appears only the coordinates $x_i, x_j$ and the parameters $\mathbf{p}_{ij}$. From further on we denote the maps $R_{ij}$ as $R_{ij}^{P_0}$, in order to stress this separability feature.
(3) Mappings \( R_{ij} \) are anti-measure preserving with respect to the measures \( m_1 = n^1 d^2 \), \( m_2 = n^2 d^1 \), where \( n^1, d^1 \) the numerators and the denominators respectively, of the invariants \( H_i \), \( i = 1, 2 \).

(4) Mappings \( R_{ij}^{P_0} \) satisfy the Yang-Baxter identity

\[
R_{ij}^{P_0} R_{jk}^{P_0} R_{kj}^{P_0} = R_{kj}^{P_0} R_{jk}^{P_0} R_{ij}^{P_0}.
\]

(5) Mappings \( R_{ij}^{P_0} \) are involutions with the sets of singularities

\[
\Sigma_{ij} = \{ P_{ij}^1, P_{ij}^2 \} = \left\{ \left( \frac{c_1}{d_1}, \frac{c_2}{d_2} \right), \left( \frac{c_3}{d_3}, \frac{c_4}{d_4} \right) \right\},
\]

where the superscript 2 in \( P_{ij}^1 \) and \( P_{ij}^2 \) denotes that these singular points appear with multiplicity 2. In the formulae for \( P_{ij}^m \), \( m = 1, \ldots, 2 \), we have suppressed the dependency on the remaining variables. For example, with \( P_{ij}^1 = \left( \frac{c_1}{d_1}, \frac{c_2}{d_2} \right) \) we denote \( \left( x_1, \ldots, x_{i-1}, \frac{c_1}{d_1}, x_{i+1}, \ldots, x_{j-1}, \frac{c_2}{d_2}, x_{j+1}, \ldots, x_k \right) \) and similarly for \( P_{ij}^2 \).

(6) Each one of the maps \( R_{ij}^{P_0} \) is \((\mathcal{M} \mathcal{O} \mathcal{B})^2\) equivalent to the \( H_{111}^A \) Yang-Baxter map.

Proof. The proof follows similarly to the proof of proposition 3.3.

Example 3.12 \((k \geq 3)\). For \( k = 3 \), the invariants \( H_1 = f_1 + f_2 + f_3 \), \( H_2 = g_1 + g_2 + g_3 \) are functions of 3 variables with 24 parameters, 3 of them are essential. Without loss of generality, after removing the redundancy of the parameters, the invariants \( H_1, H_2 \) can be cast into the form:

\[
H_1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \quad H_2 = p_1 x_1 + p_2 x_2 + p_3 x_3.
\]

Then each of the mappings \( R_{ij} \), \( i \neq j \in \{1, 2, 3\} \) is exactly the \( H_{111}^A \) Yang-Baxter map.

For \( k > 3 \) the invariants depend on \( 8k \) parameters and only \( 3k - 6 \) of them are essential. Without loss of generality they can be cast into the form

\[
H_1 = \sum_{i=1}^{k} \frac{1}{x_i}, \quad H_2 = p_1 x_1 + p_2 x_2 + p_3 x_3 + \sum_{i=4}^{k} \alpha_i - \beta_i x_i,
\]

4. Entwining Yang-Baxter maps

Following \cite{15}, three different maps \( S, T, U \) are called entwining Yang-Baxter maps if they satisfy:

\[
S_{12} T_{13} U_{23} = U_{23} T_{13} S_{12}.
\]

We consider two maps to be different if they are not \((\mathcal{M} \mathcal{O} \mathcal{B})^2\) equivalent. Hence, in order to ensure that we have different maps we require that at least one of the maps \( S, T, U \) either belongs to a different subclass than the remaining ones or it has different singularity pattern (even if it belongs to the same subclass with the remaining ones) or it has different periodicity. In what follows we present two methods to obtain entwining maps. The first one is based on degeneracy i.e. we construct maps which belong to different subclasses and we obtain entwining maps associated with the \( H_1, H_{111} \) and \( H_{111}^A \) families of maps. The second one is based on the symmetries of the \( H \)-list of Yang-Baxter maps and we obtain entwining maps for all members of the \( H \)-list.

4.1. Degeneracy and entwining Yang-Baxter maps. In subsection 3.0.1 it was shown that for \( k = 3 \) and for the multiplicative/multiplicative case, the invariants \( H_1, H_2 \) depend on 3 essential parameters. Without loss of generality they read

\[
H_1 = x_1 x_2 x_3, \quad H_2 = \frac{x_1 - p_1 x_1 - p_2 x_2 - p_3}{x_1 - 1} \frac{x_2 - 1}{x_2 - 1} \frac{x_3 - 1}{x_3 - 1}.
\]

The associated maps \( R_{12}, R_{13} \) and \( R_{23} \) which preserve the invariants have exactly the form of the \( H_I \) map. In order to obtain entwining maps associated with the \( H_I \) map, we consider:

\[
H_1 = x_1 x_2 x_3, \quad H_2 = \frac{x_1 - p_1 x_2 - p_2 x_3}{x_1 - 1} \frac{x_2 - 1}{x_2 - 1} \frac{x_3 - 1}{x_3 - 1}.
\]

For these invariants, \( R_{12} \) is exactly the \( H_I \) map and for generic \( \alpha_3, \beta_3, \gamma_3 \) mappings \( R_{13} \) and \( R_{23} \) are \((\mathcal{M} \mathcal{O} \mathcal{B})^2\) equivalent to the \( H_I \). In order to obtain entwining maps we need to violate this \((\mathcal{M} \mathcal{O} \mathcal{B})^2\) equivalence of the maps \( R_{13} \) and \( R_{23} \) with the \( H_I \) map. This is achieved by violating the generality, i.e. setting \( \alpha_3 = 0 \) or \( \beta_3 = 0 \), the maps \( R_{13} \) and \( R_{23} \), belongs to different subclasses than the \( H_I \) map does. Working similarly for the \( H_{11} \) map.
we find 1 family of maps which entwine with the later without being \((M\bar{b})^2\) equivalent. Finally, for \(H^4_{14}\) we find also 1 family of entwining maps which are not \((M\bar{b})^2\) equivalent with the later. Our results are presented in Propositions 4.1–4.3.

**Proposition 4.1.** The \(H_1\) Yang-Baxter map entwines with the maps \(e^a H_1\) and \(e^b H_1\) of Table 1 according to

| Map            | \((u, v) \mapsto (U, V)\)       | subclass |
|----------------|---------------------------------|----------|
| \(e^a H_1\)    | \(U = \frac{\alpha(1-u) + \beta(\alpha-1)uv}{\alpha-u}, \quad V = \frac{uv(\alpha-u)}{\alpha(1-u) + \beta(\alpha-1)uv}\) | \(1:2\)  |
| \(e^b H_1\)    | \(U = \frac{u - \alpha}{u - 1}, \quad V = \frac{uv(u-1)}{u - \alpha}\)        | \(0:2\)  |

the entwining relation

\[S_{12} T_{13} T_{23} = T_{23} T_{13} S_{12},\]

where \(S_{12}\) is the \(H_1\) map acting on the \((1,2)\)–coordinates, \(T_{13}\) and \(T_{23}\) are \(e^a H_1\) acting on \((1,3)\) and \((2,3)\) coordinates respectively, or \(e^b H_1\) acting on \((1,3)\) and \((2,3)\) coordinates respectively.

**Proof.** Starting with the invariants

\[H_1 = x_1 x_2 x_3, \quad H_2 = \frac{x_1 - p_1 x_2 - p_2 a - bx_3}{x_1 - 1 x_2 - 1 b - cx_3},\]

the map \(R_{12}\) is exactly the \(H_1\) map. By setting \(a = 0\), \(R_{13}\) and \(R_{23}\) takes the form of \(e^a H_1\) of Table 1 (where \(\beta \equiv c/b\)). The map \(e^a H_1\) is of subclass \([1:2]\) so clearly non-(\(M\bar{b})^2\) equivalent to \(H_1\).

By setting \(b = 0\), \(R_{13}\) and \(R_{23}\) takes the form of \(e^b H_1\) of Table 1 (where \(\beta \equiv a/c\)). The map \(e^b H_1\) is of subclass \([0:1]\) so clearly non-(\(M\bar{b})^2\) equivalent to \(H_1\) or to \(e^a H_1\).

Finally, by setting \(c = 0\), mappings \(R_{13}\) and \(R_{23}\) are \((M\bar{b})^2\) equivalent to \(e^a H_1\).

\(\square\)

**Proposition 4.2.** The \(H_{11}\) Yang-Baxter map entwines with the map of Table 2

| Map            | \((u, v) \mapsto (U, V)\)       | subclass |
|----------------|---------------------------------|----------|
| \(e^b H_{11}\) | \(U = \frac{uv}{\alpha-u}, \quad V = \frac{u \alpha - u - v}{\alpha-u}\)        | \([1:1]\) |

according to the entwining relation

\[S_{12} T_{13} T_{23} = T_{23} T_{13} S_{12},\]

where \(S_{12}\) is the \(H_{11}\) map acting on the \((1,2)\)–coordinates, \(T_{13}\) and \(T_{23}\) are \(e^b H_{11}\) acting on \((1,3)\) and \((2,3)\) coordinates respectively.

**Proof.** Starting with the invariants

\[H_1 = x_1 + x_2 + x_3, \quad H_2 = \frac{x_1 - p_1 x_2 - p_2 a - bx_3}{x_1 x_2 - 1 b - cx_3},\]

the map \(R_{12}\) is exactly the \(H_{11}\) map. By setting \(a = 0\), \(R_{13}\) and \(R_{23}\) are \((M\bar{b})^2\) equivalent to the \(H_{11}\) map.

By setting \(b = 0\), \(R_{13}\) and \(R_{23}\) takes the form of \(e^b H_{11}\) of Table 2. The map \(e^b H_{11}\) is of subclass \([1:1]\) so clearly non-(\(M\bar{b})^2\) equivalent to the \(H_{11}\) map.

Finally, by setting \(c = 0\), mappings \(R_{13}\) and \(R_{23}\) are \((M\bar{b})^2\) equivalent to \(e^b H_{11}\).

\(\square\)
Table 3. Entwining maps associated with the $H_{111}^4$ Yang-Baxter map though degeneracy

| Map          | $(u, v) \mapsto (U, V)$ | subclass   |
|--------------|-------------------------|------------|
| $e^bH_{111}^4$ | $U = \frac{\beta}{\alpha} u, \ V = \frac{\beta uv}{\beta(u + v) - \alpha u^2 v}$ | $[0 : 2]$ |

Proposition 4.3. The $H_{111}^4$ Yang-Baxter map entwines with the map of Table 3 according to the entwining relation

$$S_{12}T_{13}T_{23} = T_{23}T_{13}S_{12},$$

where $S_{12}$ is the $H_{111}^4$ map acting on the $(1, 2)$–coordinates, $T_{13}$ and $T_{23}$ are $e^bH_{111}^4$ acting on $(1, 3)$ and $(2, 3)$ coordinates respectively.

Proof. Starting with the invariants

$$H_1 = x_1 + x_2 + x_3, \quad H_2 = p_1x_1 + p_2x_2 + \frac{a - bx_3}{b - cx_3},$$

the map $R_{12}$ is exactly the $H_{111}^4$ map. By setting $a = 0$, $R_{13}$ and $R_{23}$ are $(M\phi)^2$ equivalent to the $H_{111}^4$ map.

By setting $b = 0$ and $R_{13}$ and $R_{23}$ take the form of $e^bH_{111}^4$ of Table 3 (where $\beta = a/c$). The map $e^bH_{111}^4$ is of subclass $[0 : 2]$ so clearly non-$(M\phi)^2$ equivalent to the $H_{111}^4$ map.

Finally, by setting $c = 0$, mappings $R_{13}$ and $R_{23}$ are $(M\phi)^2$ equivalent to the $H_{111}^4$ map.

In the following section, we are using the notion of symmetry of Yang-Baxter maps in order to generate entwining maps.

### 4.2. Symmetries of Yang-Baxter maps and the entwining property.

The notion of symmetry in the context of Yang-Baxter maps was introduced in [59].

Definition 4.4. An involution $\phi : \mathbb{C}P^1 \mapsto \mathbb{C}P^1$ is a symmetry of the Yang-Baxter map $R : \mathbb{C}P^1 \times \mathbb{C}P^1 \mapsto \mathbb{C}P^1 \times \mathbb{C}P^1$ if it holds

$$\phi_1\phi_2R_{12} = R_{12}\phi_1\phi_2,$$

where $\phi_1$ is the involution that acts as $\phi$ to the first factor of the cartesian product $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\phi_2$ is the involution that acts as $\phi$ to the second factor of the cartesian product.

Let $m < n \in \{1, \ldots, k\}$, $k \geq 3$ fixed. A direct consequence of the previous definition is that if $\phi$ is a symmetry of the Yang-Baxter map $R$, then the map $\phi_nR_{mn}\phi_n$ is a new Yang-Baxter map since it is not $(M\phi)^2$ equivalent with $R_{mm}$. By finding the symmetries of the $F$–list of Yang-Baxter maps, the authors of [59] derived the $H$–list of Yang-Baxter maps. Clearly the symmetries of the $F$–list are symmetries of the $H$–list and vice versa.

Theorem 4.5. Let $\phi$ a symmetry of a Yang-Baxter map $R$ and let $\phi_0$ the identity map i.e. $\phi_0 : (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$. Out of the possible $4^3$ entwining relations of the form

$$R_{12}\phi_1R_{13}\phi_2R_{23}\phi_3R_{34}\phi_4 = R_{23}\phi_3R_{13}\phi_2R_{12}\phi_1, \quad i, j, k \in \{0, 1, 2, 3\},$$

apart the Yang-Baxter relation that holds, only the following three entwining relations holds

$$R_{12}R_{13}\phi_3R_{23}\phi_2 = R_{23}\phi_2R_{13}\phi_1R_{12}, \quad R_{12}\phi_2R_{13}\phi_3R_{23}\phi_2 = R_{23}\phi_2R_{13}\phi_3R_{12}\phi_2, \quad R_{12}\phi_2R_{13}\phi_3R_{23}\phi_2 = R_{23}\phi_2R_{13}\phi_3R_{12}\phi_2.$$

Proof. To show that only the entwining relations (19), (20), (21) holds, we start with

$$R_{12}\phi_1R_{13}\phi_2R_{23}\phi_3 = R_{23}\phi_3R_{13}\phi_2R_{12}\phi_1, \quad i, j, k \in \{0, 1, 2, 3\},$$

By direct calculations, we prove that if the Yang-Baxter relation holds out of the $4^3$ different relations (17), only (19), (20), (21) holds.

For example let us show that (19) holds. We have:

$$R_{12}R_{13}\phi_1R_{23}\phi_2 = R_{12}R_{13}R_{23}\phi_1\phi_2 = R_{23}R_{13}R_{12}\phi_1\phi_2$$

(21)
since φ₁ commutes with R₂₃ and the Yang-Baxter relation R₁₂R₁₃R₂₃ = R₂₃R₁₃R₁₂ holds. But due to the 
symmetry we have R₁₂φ₁φ₂ = φ₁φ₂R₁₂ so (21) reads:
\[ R₂₃R₁₃R₁₂φ₁φ₂ = R₂₃R₁₃φ₁φ₂R₁₂ = R₂₃φ₂R₁₃φ₁R₁₂, \]
and that completes the proof that (18) holds. For the remaining relations we work similarly for their proof. □

Note that any of the entwining relations (18), (19) and (20) is uniquely described by the symmetries φ₁, φ₂, φ₃ that take part in this relation. For example in (18) the symmetries φ₀, φ₁, φ₂ appear in this order, hence we refer to (18) as relation of entwining type (φ₀, φ₁, φ₂) or by using just the subscripts, relation of entwining type (0, 1, 2).

In Table 4 we present the entwining maps S, T, U that correspond to the entwining relations (18) − (20), where R is any Yang-Baxter map. In what follows, we specify R to be any member of the H−list\(^{1}\) of quadrirational Yang-Baxter maps.

| Entwining type | S₁₂ | T₁₃ | U₂₃ |
|---------------|-----|-----|-----|
| (0, 1, 2)     | R₁₂ | R₁₃φ₁ | R₂₃φ₂ |
| (2, 3, 0)     | R₁₂φ₂ | R₁₃φ₃ | R₂₃ |
| (2, 2, 2)     | R₁₂φ₃ | R₁₃φ₂ | φ₂R₂₃φ₂ |

4.2.1. Entwining maps associated with the H₁ Yang-Baxter map. The involutions φ, ψ

\[ \phi : u \mapsto \frac{\alpha}{u}, \quad \psi : u \mapsto \frac{u - \alpha}{u - 1}, \]

where α a complex parameter, are symmetries for the H₁ map (see [59]), since it holds
\[ φ₁φ₂R₁₂ = R₁₂φ₁φ₂, \quad ψ₁ψ₂R₁₂ = R₁₂ψ₁ψ₂, \]
where R₁₂ is the H₁ map acting on the 12−coordinates and
\[ φ₁ : (x₁, x₂) \mapsto (p₁/x₁, x₂), \quad φ₂ : (x₁, x₂) \mapsto (x₁, p₂/x₂), \]
\[ ψ₁ : (x₁, x₂) \mapsto ((x₁ − p₁)/(x₁ − 1), x₂), \quad ψ₂ : (x₁, x₂) \mapsto (x₁, (x₂ − p₂)/(x₂ − 1)). \]

Note that the symmetries φ and τ can be derived from our considerations (see example 3.0) since for k = 3 it holds
\[ H₁φ₁φ₂φ₃ = \frac{p₁p₂p₃}{H₁}, \quad H₂φ₁φ₂φ₃ = \frac{1}{H₂}, \]
\[ H₁ψ₁ψ₂ψ₃ = H₂, \quad H₂ψ₁ψ₂ψ₃ = H₁. \]

Remark 4.6. By using similar arguments as in the proof of the Theorem 4.5, entwining relations where the symmetries φ and ψ of the H₁ map interlace do not exist i.e. it does not exists for example any relation of entwining type (φ₁, φ₂, ψ₃).

In Table 5 we present the entwining maps associated with the H₁ map which are generated by using the symmetries φ and ψ. In Table 5 it appears the H₁ map, the companion of the H₁ map that is denoted as cH₁, as well as cF₁ which is the companion map of the map F₁ that was derived in [59]. We also have four novel maps which are not (Möb)² equivalent to H₁, which we refer to as Φ₁, Φ₂, Ψ₁ and Ψ₂. In the proposition that follows we present their explicit form.

\(^{1}\)It is easy to show that the entwining maps associated with the F−list of quadrirational Yang-Baxter maps are (Möb)² equivalent to the corresponding to the H−list entwining maps. This is the reason that we present the entwining maps associated with the H−list only.
Table 5. Left Table: Entwining maps $S, T, U$ associated with $H_I$ Yang-Baxter map using the symmetry $\phi$. Right Table: Entwining maps $S, T, U$ associated with $H_I$ Yang-Baxter map using the symmetry $\psi$

| Entwining type | $S_{12}$ | $T_{13}$ | $U_{23}$ |
|---------------|---------|---------|---------|
| $(0, 1, 2)$   | $H_I$   | $\Phi_I^\alpha$ | $\Phi_I^\alpha$ |
| $(2, 3, 0)$   | $\Phi_I^\alpha$ | $\Phi_I^\alpha$ | $H_I$ |
| $(2, 2, 2)$   | $\Phi_I^\alpha$ | $H_I$ | $cH_I$ |

| Entwining type | $S_{12}$ | $T_{13}$ | $U_{23}$ |
|---------------|---------|---------|---------|
| $(0, 1, 2)$   | $H_I$   | $\Psi_I^\alpha$ | $\Psi_I^\beta$ |
| $(2, 3, 0)$   | $\Psi_I^\alpha$ | $\Psi_I^\beta$ | $H_I$ |
| $(2, 2, 2)$   | $\Psi_I^\alpha$ | $H_I$ | $cF_I$ |

Proposition 4.7. The following non-periodic\footnote{A non-periodic map cannot be equivalent by conjugation ((Mob)\textsuperscript{2} equivalent) to a periodic map. Since the $H_I$ map is involutive, the maps presented in this proposition are not (Mob)\textsuperscript{2} to the $H_I$ map.} maps $(u, v) \mapsto (U, V)$, where

\begin{align*}
U &= \alpha vQ, \\
V &= \frac{1}{u}Q^{-1}, \\
Q &= \frac{\beta - \alpha + u(1 - \beta) + v(a - 1)}{\beta(1 - \alpha)u - \alpha(1 - \beta)v + (a - \beta)uv}, \quad (\Phi_I^\alpha) \\
U &= \frac{1}{v}Q^{-1}, \\
V &= \beta uQ, \\
Q &= \frac{\beta - \alpha + u(1 - \beta) + v(a - 1)}{\beta(1 - \alpha)u - \alpha(1 - \beta)v + (a - \beta)uv}, \quad (\Phi_I^\beta) \\
U &= vQ, \\
V &= \frac{u - \alpha}{u - 1}Q^{-1}, \\
Q &= \frac{\alpha(1 - v) - \beta u + uv}{\beta(1 - u) - \beta v + uv}, \quad (\Phi_I^\gamma) \\
U &= \frac{v - \beta}{v - 1}Q, \\
V &= uQ^{-1}, \\
Q &= \frac{\alpha(1 - u - v) + uv}{\beta(1 - u) - \alpha v + uv}, \quad (\Psi_I^\delta)
\end{align*}

entwine with the $H_I$ Yang-Baxter map according to the entwining relations of Table 5.

4.2.2. Entwining maps associated with the $H_{11}$ Yang-Baxter map. The invariants

\begin{align*}
H_1 &= x_1 + x_2 + x_3, \\
H_2 &= \frac{x_1 - p_1}{x_1}, \quad \frac{x_2 - p_2}{x_2}, \quad \frac{x_3 - p_3}{x_3},
\end{align*}

generate the maps $R_{ij}$, $i < j \in \{1, 2, 3\}$ which are exactly the $H_{11}$ map acting on the $(ij)$–coordinates. Explicitly the $H_{11}$ map reads

\begin{align*}
U &= v + \frac{(\alpha - \beta)uv}{\beta u + \alpha v - \alpha \beta}, \quad V = u - \frac{(\alpha - \beta)uv}{\beta u + \alpha v - \alpha \beta}, \quad (H_{11})
\end{align*}

A symmetry of the $H_{11}$ map is $\phi : u \mapsto \alpha - u$, since it holds $\phi_1 \phi_2 R_{12} = R_{12} \phi_1 \phi_2$, where $R_{12}$ is the $H_{11}$ map acting on the $(12)$–coordinates and

$$
\phi_1 : (x_1, x_2) \mapsto (p_1 - x_1, x_2), \quad \phi_2 : (x_1, x_2) \mapsto (x_1, p_2 - x_2),
$$

Table 6. Entwining maps $S, T, U$ associated with $H_{11}$ Yang-Baxter map using the symmetry $\phi$.

| Entwining type | $S_{12}$ | $T_{13}$ | $U_{23}$ |
|---------------|---------|---------|---------|
| $(0, 1, 2)$   | $H_{11}$ | $\Phi_{11}^\alpha$ | $\Phi_{11}^\alpha$ |
| $(2, 3, 0)$   | $\Phi_{11}^\alpha$ | $\Phi_{11}^\alpha$ | $H_{11}$ |
| $(2, 2, 2)$   | $\Phi_{11}^\alpha$ | $H_{11}$ | $cH_{11}$ |
Proposition 4.8. The following non-periodic maps \((u, v) \mapsto (U, V)\), where
\[
U = \frac{u - v + \beta - \alpha}{\beta u - \alpha v}, \quad V = \frac{(\alpha - u)(u - v)}{\beta u - \alpha v}, \quad (\Phi_H^a)
\]
\[
U = \frac{\alpha(\beta - v)(u - v)}{\beta u - \alpha v}, \quad V = \frac{u - v + \beta - \alpha}{\beta u - \alpha v}, \quad (\Phi_H^b)
\]
entwine with the \(H_{11}\) Yang-Baxter map according to the entwining relations of Table 7.

The map \(cH_{11}\) denotes the companion map of the \(H_{11}\) map.

4.2.3. Entwining maps associated with the \(H_{11}^A\) Yang-Baxter map. The invariants
\[
H_1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \quad H_2 = p_1x_1 + p_2x_2 + p_3x_3.
\]
generate the maps \(R_{ij}, \quad i < j \in \{1,2,3\}\) which are exactly the \(H_{11}^A\) map acting on the \((ij)\)–coordinates. Explicitly the \(H_{11}^A\) map reads
\[
U = \frac{v \alpha u + \beta v}{\alpha u + v}, \quad V = \frac{u \alpha u + \beta v}{\beta u + v}, \quad (H_{11}^A)
\]

Two symmetries of the \(H_{11}^A\) map are
\[
\phi : u \mapsto \frac{1}{\alpha u}, \quad \psi : u \mapsto -u
\]
since it holds
\[
\phi \phi_2 R_{12} = R_{12} \phi \phi_2, \quad \psi \psi_2 R_{12} = R_{12} \psi \psi_2,
\]
where \(R_{12}\) is the \(H_{11}^A\) map acting on the \((12)\)–coordinates and
\[
\phi_1 : (x_1, x_2) \mapsto \left(\frac{1}{\alpha u}, x_2\right), \quad \phi_2 : (x_1, x_2) \mapsto (x_1, \frac{1}{\beta u}),
\]
\[
\psi_1 : (x_1, x_2) \mapsto (-x_1, x_2), \quad \psi_2(x_1, x_2) \mapsto (x_1, -x_2).
\]

Note that the map \(\phi R_{12} \psi_2\) is exactly the \(H_{11}^B\) Yang-Baxter map.

Proposition 4.9. The following non-periodic maps \((u, v) \mapsto (U, V)\) where
\[
U = v \frac{1 + \beta uv}{1 + \alpha uv}, \quad V = \frac{1}{\beta u} \frac{1 + \beta uv}{1 + \alpha uv}, \quad (\Phi_{11}^{aA})
\]
\[
U = \frac{1}{\alpha u} \frac{1 + \alpha uv}{1 + \beta uv}, \quad V = \frac{1}{\beta u} \frac{1 + \alpha uv}{1 + \beta uv}, \quad (\Phi_{11}^{bA})
\]
\[
U = \frac{v \alpha u - \beta v}{\alpha u - v}, \quad V = \frac{u \alpha u - \beta v}{\beta u - v}, \quad (\Psi_{11}^{aA})
\]
\[
U = \frac{v \alpha u - \beta v}{\alpha u - v}, \quad V = \frac{u \alpha u - \beta v}{\beta u - v}, \quad (\Psi_{11}^{bA})
\]
entwine with the \(H_{11}^A\) Yang-Baxter map according to the entwining relations of Table 7.

Table 7. Left Table: Entwining maps \(S, T, U\) associated with \(H_{11}^A\) Yang-Baxter map using the symmetry \(\phi\). Right Table: Entwining maps \(S, T, U\) associated with \(H_{11}^A\) Yang-Baxter map using the symmetry \(\psi\).

| Entwining type | \(S_{12}\) | \(T_{13}\) | \(U_{23}\) | Entwining type | \(S_{12}\) | \(T_{13}\) | \(U_{23}\) |
|---------------|-----------|-----------|-----------|---------------|-----------|-----------|-----------|
| \((0,1,2)\)   | \(H_{11}^A\) | \(\Phi_{11}^{aA}\) | \(\Phi_{11}^{bA}\) | \((0,1,2)\)   | \(H_{11}^A\) | \(\Psi_{11}^{aA}\) | \(\Psi_{11}^{bA}\) |
| \((2,3,0)\)   | \(\Phi_{11}^{aA}\) | \(\Phi_{11}^{bA}\) | \(H_{11}^A\)   | \((2,3,0)\)   | \(\Psi_{11}^{aA}\) | \(\Psi_{11}^{bA}\) | \(H_{11}^A\)   |
| \((2,2,2)\)   | \(\Phi_{11}^{bA}\) | \(H_{11}^A\)   | \(H_{11}^A\)   | \((2,2,2)\)   | \(\Psi_{11}^{bA}\) | \(H_{11}^A\)   | \(cH_{11}^A\) |

The map \(cH_{11}^A\) denotes the companion map of the \(H_{11}^A\) map and with \(\hat{H}_{11}^A\) we denote a \((M\delta)^2\) equivalent map to the \(H_{11}^A\).
4.2.4. Entwining maps associated with the $H_{II}^B$ Yang-Baxter map. The invariants that were derived in \cite{45,47,56,47},

$$H_1 = x_1 x_2 x_3, \quad H_2 = p_1 x_1 + p_2 x_2 + p_3 x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3},$$

generate the maps $R_{ij}$, $i < j \in \{1, 2, 3\}$ which are exactly the $H_{II}^B$ map acting on the $(ij)$-coordinates. Explicitly the $H_{II}^B$ map reads

$$U = v \frac{1 + \beta uv}{1 + \alpha uv}, \quad V = u \frac{1 + \alpha uv}{1 + \beta uv}, \quad (H_{II}^B)$$

The symmetries $\phi, \psi$ of the $H_{II}^A$ map are symmetries of $H_{II}^B$ as well.

**Proposition 4.10.** The following non-periodic maps $(u, v) \mapsto (U, V)$, where

$$U = \frac{v \alpha u + \beta v}{\alpha u + \beta v}, \quad V = \frac{1}{1} \frac{u + v}{u \alpha u + \beta v}, \quad (\Phi_{II}^{Ib})$$

$$U = \frac{1}{v \alpha u + \beta v}, \quad V = \frac{1}{\beta u + v}, \quad (\Phi_{II}^{Ib})$$

$$U = \frac{1 - \beta uv}{1 - \alpha uv}, \quad V = \frac{1 - \alpha uv}{1 - \beta uv}, \quad (\Psi_{II}^{Ib})$$

entwine with the $H_{II}^B$ Yang-Baxter map according to the entwining relations of Table 8.

**Table 8.** Left Table: Entwining maps $S, T, U$ associated with $H_{II}^B$ Yang-Baxter map using the symmetry $\phi$. Right Table: Entwining maps $S, T, U$ associated with $H_{II}^B$ Yang-Baxter map using the symmetry $\psi$

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Entwining type} & S_{12} & T_{13} & U_{23} \\
\hline
(0, 1, 2) & H_{II}^B & \Phi_{II}^{Ib} & \Phi_{II}^{Ib} \\
\hline
(2, 3, 0) & \Phi_{II}^{Ib} & \Phi_{II}^{Ib} & H_{II}^B \\
\hline
(2, 2, 2) & \Phi_{II}^{Ib} & H_{II}^B & H_{II}^B \\
\hline
\end{array}
\]

The maps $\tilde{H}_{II}^B, \tilde{H}_{II}^B$ that appear in Table 8 are $(\text{Möbi})^2$ equivalent to the map $H_{II}^B$. The map $cH_{II}^B$ denotes the companion map of the $H_{II}^B$ map.

4.2.5. Entwining maps associated with the $H_V$ Yang-Baxter map. The invariants that were derived in \cite{45,47,56,47},

$$H_1 = x_1 + x_2 + x_3, \quad H_2 = x_1^3 + 3p_1 x_1 + x_2^3 + 3p_2 x_2 + x_3^3 + 3p_3 x_3$$

generate the maps $R_{ij}$, $i < j \in \{1, 2, 3\}$ which are exactly the $H_V$ map acting on the $(ij)$-coordinates. Explicitly the $H_V$ map reads

$$U = v - \frac{\alpha - \beta}{u + v}, \quad V = u + \frac{\alpha - \beta}{u + v}, \quad (H_V)$$

The involution $\psi : u \mapsto -u$ is a symmetry of the $H_V$ map.

**Proposition 4.11.** The following non-periodic maps $(u, v) \mapsto (U, V)$, where

$$U = v + \frac{\alpha - \beta}{u - v}, \quad V = -u - \frac{\alpha - \beta}{u - v}, \quad (\Psi_V^a)$$

$$U = -v - \frac{\alpha - \beta}{u - v}, \quad V = u + \frac{\alpha - \beta}{u - v}, \quad (\Psi_V^b)$$

twine with the $H_V$ Yang-Baxter map according to the entwining relations of Table 9.
Table 9. Entwining maps $S, T, U$ associated with $H_V$ Yang-Baxter map using the symmetry $\psi$

| Entwining type | $S_{12}$ | $T_{13}$ | $U_{23}$ |
|----------------|---------|---------|---------|
| $(0, 1, 2)$    | $H_V$   | $\Psi^a_V$ | $\Psi^b_V$ |
| $(2, 3, 0)$    | $\Psi^a_V$ | $\Psi^b_V$ | $H_V$ |
| $(2, 2, 2)$    | $\Psi^a_V$ | $H_V$ | $cH_V$ |

The map $cH_V$ denotes the companion map of the $H_V$ map.

5. Transfer maps

The notion of transfer maps associated with Yang-Baxter maps was introduced by Veselov in [71]. In [68] dynamical aspects of the later were discussed. The transfer maps associated with any reversible Yang-Baxter map are defined as

$$T^{(k)}_i = R_{i+1}R_{i+k-2}R_{i+k-1} \ldots R_i, \quad i \in \{1, \ldots, k\},$$

where the indices are considered modulo $k$. There is:

$$T^{(k)}_i T^{(k)}_j = T^{(k)}_j T^{(k)}_i, \quad T^{(k)}_1 T^{(k)}_2 \ldots T^{(k)}_k = Id.$$

For example for $k = 4$ we have $T^{(4)}_1 = R_{14}R_{13}R_{12}, T^{(4)}_2 = R_{12}R_{24}R_{23}, T^{(4)}_3 = R_{23}R_{13}R_{14}$ and $T^{(4)}_4 = R_{34}R_{32}R_{14}$.

**Proposition 5.1.** For the transfer maps $T^{(k)}_i$ associated with the maps $R^{(p)}_{ij}$ of the propositions 3.5, 3.9, 3.11 it holds:

1. They preserve the invariants $H_1, H_2$, presented in the propositions 3.3, 3.9, 3.11.
2. For $k = 2n + 1$ they preserve the measures given in the propositions 3.5, 3.9, 3.11.
3. For $k = 2n$ they anti-preserve the measures given in the propositions 3.5, 3.9, 3.11.
4. They possess Lax pairs.
5. For generic values of the parameter sets $p_{ij}$, are equivalent by conjugation to the transfer maps associated with $H_1, H_{11}$ and $H_{111}$ Yang-Baxter maps respectively.
6. For non-generic values of the parameter sets $p_{ij}$, we have novel transfer maps.

**Proof.** The statements (1) – (3) have already been proven (see propositions 2.1, 3.9, 3.11). As for the statement (4), one can construct a Lax matrix for the Yang-Baxter map $R$ following [66]. Then the Lax equations associated with the transfer maps $T^{(k)}_i$, correspond to certain factorizations of the monodromy matrix (see [71]).

We will show the statement (5) for the transfer maps associated with $R^{(p)}_{ij}$ of proposition 3.9 and for $k = 4$. The proof for arbitrary $k$ follows by induction. In proposition 3.3 it was shown that these maps are $(M \otimes b)^2$ equivalent to the $H_1$ map. Let us denote as $\nu_i$ the maps defined by the cross-ratios

$$CR[x_i, a_i/b_i, c_i/d_i, A_i/B_i] = CR[y_i, 0, 1, \infty], \quad l = 1, \ldots, 4$$

and as $\mu_i$ the maps defined by:

$$CR[x_i, c_i/d_i, a_i/b_i, C_i/D_i] = CR[y_i, \infty, 1, 0], \quad l = 1, \ldots, 4.$$ 

Then the maps $\tilde{R}_{ij}^{(p)}$, where $\tilde{R}_{ij}^{(p)} = \mu^{-1}_j \mu^{-1}_i R^{(p)}_{ij} \mu_i \mu_j$, are exactly the $H_1$ map acting on the $(ij)$-coordinates (see proposition 3.3). For the transfer map $\tilde{T}^{(4)}_1$ associated with $\tilde{R}_{ij}^{(p)}$, there is

$$\tilde{T}^{(4)}_1 = \tilde{R}_{14}\tilde{R}_{13}\tilde{R}_{12} = (\nu^{-1}_1 \mu^{-1}_4 R_{14} \nu_1 \mu_4)(\nu^{-1}_4 \mu^{-1}_3 R_{13} \nu_1 \mu_3)(\nu^{-1}_3 \mu^{-1}_2 R_{12} \nu_1 \mu_2)$$

$$= \mu^{-1}_4 \mu^{-1}_3 \mu^{-1}_2 \nu^{-1}_1 \mu_4 R_{14} R_{13} R_{12} \nu_1 \mu_3 \mu_2 = \mu^{-1}_4 \mu^{-1}_3 \mu^{-1}_2 \nu^{-1}_1 \tilde{T}^{(4)}_1 \nu_1 \mu_3 \mu_2 \nu_4.$$ 

(22)

Note that we have omitted the parameter sets $p_{ij}$ that the maps depends on for simplicity.

(6) For non-generic choice of the parameter sets $p_{ij}$, the conjugation equivalence (22) does not holds. □
5.1. On a re-factorisation of the transfer maps. First, let us introduce some maps. With $\pi_{ij}$ we denote the transpositions:

$$\pi_{ij} : (x_1, \ldots, x_k; p_1, \ldots, p_k) \mapsto (X_1, \ldots, X_k; P_1, \ldots, P_k),$$

$$X_i = x_i, \quad P_i = p_i \quad \forall i \neq i, j, \quad \text{and} \quad X_j = x_i, \quad P_j = p_j,$$

and with $\pi_0$ we denote the following $k$–periodic map

$$\pi_0 : (x_1, \ldots, x_k; p_1, \ldots, p_k) \mapsto (X_1, \ldots, X_k; P_1, \ldots, P_k),$$

$$X_i = x_{i+1}, \quad P_i = p_{i+1}, \quad \forall i \in \{1, \ldots, k\} \quad \text{modulo} \ k.$$ 

We refer to the maps $\pi_i$ as the extended transfer maps associated with the Yang-Baxter map $R$. For small values of $k$, this can be proven by direct calculation. In order to complete the proof, it is enough to show that for arbitrary $k$ the maps $T_i$ and $(T_i)^{-1}$ share the same Lax equation.

Let $L(x, p; \lambda)$ the Lax matrix associated with the Yang-Baxter map $R$. The Lax equation associated with the transfer map $T_i = R^{p_i-1} \cdots R^{p_k} R^{p_1-1} \cdots R^{p_k}$ reads:

$$L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda) = L(x_1, p_1; \lambda)L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda).$$

(23)

5.2. Proposition 5.3. The transfer maps $T_i^{(k)}$ of a Yang-Baxter map $R$, coincide with the $(k-1)$–iteration of the maps:

$$t_i^{(k)} := \pi_{i+1} R^{p_{i+1}-1} = \pi_0 S_i.$$ 

We refer to the maps $t_i^{(k)}$ as the extended transfer maps associated with the Yang-Baxter map $R$.

Proof. It is enough to show that the $(k-1)$–iteration of the map $t_i^{(k)}$ coincides with $T_i^{(k)}$. For small values of $k$, this can be proven by direct calculation. In order to complete the proof, it is enough to show that for arbitrary $k$ the maps $T_i^{(k)}$ and $(T_i^{(k)})^{-1}$ share the same Lax equation.

Let $L(x, p; \lambda)$ the Lax matrix associated with the Yang-Baxter map $R$. The Lax equation associated with the transfer map $T_i^{(k)} = R^{p_i-1} \cdots R^{p_k} R^{p_1-1} \cdots R^{p_k}$ reads:

Since

$$\pi_{12} T_i^{(k)} = L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda) \mapsto$$

$$L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda),$$

and

$$\pi_0 : L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda) \mapsto$$

$$L(x_1, p_1; \lambda)L(x_k, p_k; \lambda) \cdots L(x_2, p_2; \lambda)$$

there is

$$t_i^{(k)} : L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda) \mapsto$$

$$L(x_1, p_1; \lambda)L(x_k, p_k; \lambda) \cdots L(x_2, p_2; \lambda).$$

So the map $t_i^{(k)}$ has the following Lax equation:

$$L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda)L(x_1, p_1; \lambda) =$$

$$L(x_1, p_1; \lambda)L(x_k, p_k; \lambda)L(x_{k-1}, p_{k-1}; \lambda) \cdots L(x_2, p_2; \lambda).$$

But the map $t_i^{(k)}$ acts on the parameter sets $p_i$ as follows

$$t_i^{(k)} : (p_1, \ldots, p_k) \mapsto (P_1, \ldots, P_k),$$

where $P_1 = p_1, \ P_k = p_2$ and $\forall i \neq 1, k \ P_1 = p_{i+1},$

that is periodic with period $k - 1$, so the Lax equation of the map $(t_i^{(k)})^{k-1}$ is exactly (23) i.e. the Lax equation of $T_i^{(k)}$. □
Theorem 5.4. The maps $t_i^{(k)}$ satisfy the relations

\[(t_i^{(k)} t_{i+1}^{(k)})^{k/2} = id, \quad t_i^{(k)} t_{i+1}^{(k)} \cdots t_k^{(k)} = id, \quad k \text{ even},\]
\[t_i^{(k)} t_{i+1}^{(k)} = id, \quad (t_i^{(k)} t_{i+1}^{(k)})^2 = id, \quad k \text{ odd}.\]

Proof. Let us first prove that $t_1^{(k)} t_2^{(k)} \cdots t_k^{(k)} = id$ for $k = 2m$ even. There is

\[t_1^{(2m)} t_2^{(2m)} \cdots t_{2m}^{(2m)} = \pi_0 S_1 \pi_0 S_2 \cdots \pi_0 S_{2m},\]

where we have the composition of $m$ in number expressions of the form $\pi_0 S_1 \pi_0 S_{i+1}$, and for each one of them (using Remark 5.2) it holds $\pi_0 S_1 \pi_0 S_{i+1} = \pi_0 S_{i}^2 \pi_0 = \pi_0$. So

\[t_1^{(2m)} t_2^{(2m)} \cdots t_{2m}^{(2m)} = \underbrace{\pi_0 \pi_0 \cdots \pi_0}_{m\text{-times}} = \pi_0^m = id.\]

Let us now prove that $(t_i^{(k)} t_{i+1}^{(k)})^{k/2} = id$. We have:

\[(t_i^{(k)} t_{i+1}^{(k)})^{k/2} = (t_i^{(k)} t_{i+1}^{(k)})^m = (\pi_0 S_1 \pi_0 S_{i+1})^m = (\pi_0 S_{i+1}^2)^m = \pi_0^2 = id.\]

For $k = 2m + 1$ odd, we have

\[(t_i^{(k)} t_{i+1}^{(k)})^k = (t_i^{(k)} t_{i+1}^{(k)})^{2m+1} = (\pi_0 S_1^{2m+1}) = \pi_0^{2m+1} = id.\]

Also,

\[(t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m+1}^{(2m+1)})^2 = (t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m}^{(2m+1)} \pi_0 S_{2m+1})^2 = \pi_0^{2m+1} S_{2m+1}^2 = S_{2m+1}^2 = id,
\]

where we have used the fact that

\[t_1^{(2m+1)} t_2^{(2m+1)} \cdots t_{2m+1}^{(2m+1)} = \underbrace{\pi_0 \pi_0 \cdots \pi_0}_{m\text{-times}} = \pi_0^m.\]

\[\square\]

Remark 5.5. Note that for $k$ odd, it holds the more general condition

\[(t_i^{(k)} t_j^{(k)})^k = id \quad (i \neq j).\]

5.2. $k$ - point recurrences associated with the transfer maps of the $H$ - list of quadrirational Yang-Baxter maps. We refer to the extended transfer maps $t_i^{(k)}$ that correspond to the $H_I$, $H_{II}$, $H_{III}$ and $H_V$ Yang-Baxter maps respectively as $t_i^{H_I(k)}$, $t_i^{H_{II}(k)}$, $t_i^{H_{III}(k)}$ and $t_i^{H_V(k)}$.

Here, we associate $k$ - point recurrences with the maps $t_i^{H_I(k)}$, $t_i^{H_{II}(k)}$, $t_i^{H_{III}(k)}$ and $t_i^{H_V(k)}$. Let us first introduce the shift operator $T$ as follows:

\[T^0 : x(n) \to x(n), \quad T^1 : x(n) \to x(n + 1), \quad T^l : x(n) \to x(n + l), \quad T^{-l} : x(n) \to x(n - l), \quad n, l \in \mathbb{Z}.\]

The maps $t_2^{H_I(k)}$, $t_2^{H_{II}(k)}$, $t_2^{H_{III}(k)}$, $t_2^{H_{IV}(k)}$ and $t_2^{H_V(k)}$, explicitly read:

\[(x_1, \ldots, x_k; p_1, \ldots, p_k) \to (Tx_1, \ldots, Tx_k; Tp_1, \ldots, Tp_k).\]
where

\[ Tx_1 = x_2 \frac{p_2(1 - p_2) + (p_2 - p_3)x_3 + (p_3 - 1)x_2x_3}{p_2(1 - p_3) + (p_3 - p_2)x_2 + (p_2 - 1)x_2x_3}, \quad Tp_1 = p_3, \quad Tx_1 = x_{i+1}, \]

\[ Tx_2 = x_3 \frac{p_2(1 - p_3) + (p_3 - p_2)x_2 + (p_2 - 1)x_2x_3}{p_2(1 - p_3) + (p_3 - p_2)x_2 + (p_2 - 1)x_2x_3}, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \]

\[ Tx_1 = p_3 x_2 \frac{x_2 + x_3 - p_2}{p_3 x_2 + p_2 x_3 - p_2 p_3}, \quad Tp_1 = p_3, \quad Tx_i = x_{i+1}, \]

\[ Tx_2 = p_2 x_3 \frac{x_2 + x_3 - p_3}{p_1 x_2 + p_2 x_3 - p_2 p_3}, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \]

\[ Tx_1 = \frac{x_3 p_2 x_2 + p_3 x_3}{p_3 x_2 + x_3}, \quad Tp_1 = p_3, \quad Tx_i = x_{i+1}, \]

\[ Tx_2 = \frac{x_3 p_2 x_2 + p_3 x_3}{p_2 x_2 + x_3}, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \]

\[ Tx_1 = x_2 + 1 + x_2 x_3 \frac{1 + p_2 x_2 x_3}{1 + p_3 x_2 x_3}, \quad Tp_1 = p_3, \quad Tx_i = x_{i+1}, \]

\[ Tx_2 = x_3 + 1 + x_2 x_3 \frac{1 + p_3 x_2 x_3}{1 + p_2 x_2 x_3}, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \]

\[ Tx_1 = \frac{x_2 - p_3 - p_2}{x_2 + x_3}, \quad Tp_1 = p_1, \quad Tx_i = x_{i+1}, \]

\[ Tx_2 = x_3 + \frac{p_1 - p_2}{x_2 + x_3}, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \]

with \( i = 3, 4, \ldots, k \) and \( Tx_k = x_1, Tp_k = p_1 \). Moreover, not just \( t_2^{(k)} \), but all the \( (k \text{ in number}) \) maps \( t_i^{(k)} \) preserve the invariants in separated variables (see Table 10) and they anti-preserve the measures \( m_i = n^i d^k \) where \( n^i, d^k \) the numerator and the denominator respectively of the invariants \( H_i, i = 1, 2 \). Additional invariant can be constructed though the Lax formulation (see the proof of proposition 5.3).

Table 10. Invariants in separated variables for the maps \( t_i^{H_1^{(k)}} \), \( t_i^{H_1^{(k)}} \), \( t_i^{H_1^{(k)}} \), \( t_i^{H_1^{(k)}} \), \( t_i^{H_1^{(k)}} \) and \( t_i^{H_1^{(k)}} \).

| Map       | \( H_1 \)                                                                 | \( H_2 \)                                                                 |
|-----------|--------------------------------------------------------------------------|--------------------------------------------------------------------------|
| \( t_i^{H_1^{(k)}} \) | \( \prod_{i=1}^{k} p_i x_i \) | \( \prod_{i=1}^{k} x_i - p_i + \frac{1}{x_i - 1} \) |
| \( t_i^{H_1^{(k)}} \) | \( \sum_{i=1}^{k} x_i + p_i \) | \( \prod_{i=1}^{k} x_i - p_i \) |
| \( t_i^{H_1^{(k)}} \) | \( \sum_{i=1}^{k} \frac{1}{x_i} + \frac{1}{p_i} \) | \( \sum_{i=1}^{k} p_i x_i \) |
| \( t_i^{H_1^{(k)}} \) | \( \prod_{i=1}^{k} p_i x_i \) | \( \sum_{i=1}^{k} \frac{1}{x_i} + p_i x_i + \frac{1}{p_i} \) |
| \( t_i^{H_1^{(k)}} \) | \( \sum_{i=1}^{k} x_i + p_i \) | \( \sum_{i=1}^{k} x_i^3 + 3 p_i x_i + p_i^3 \) |

Now we show how a \( k \)-point recurrence can be associated with the map \( t_2^{H_1^{(k)}} \). Recall that the map \( t_2^{H_1^{(k)}} \) reads

\[ t_2^{H_1^{(k)}} : (x_1, \ldots, x_k; p_1, \ldots, p_k) \mapsto (Tx_1, \ldots, Tx_k; Tp_1, \ldots, Tp_k), \]

where

\[ Tx_1 = x_2 - \frac{p_3 - p_2}{x_2 + x_3}, \quad Tx_2 = x_3 + \frac{p_3 - p_2}{x_2 + x_3}, \quad Tp_1 = p_3, \quad Tp_2 = p_2, \quad Tp_i = p_{i+1}, \quad i = 3, \ldots, k \]

\(^3\)The invariants in separated variables that appear in the table[11] were firstly introduced, in a different context, in [15][11][50][47].

Note that the invariants \( H_1, H_2 \) for \( t_i^{H_1^{(k)}} \) were also given in [61].
and the indices are considered modulo $k$. Clearly we have, $x_3 = T^{2-k}x_1, p_3 = T^{2-k}p_1$, So we obtain:

$$Tx_1 = x_2 - \frac{T^{2-k}p_1 - p_2}{x_2 + T^{2-k}x_1}, \quad Tx_2 = T^{2-k}x_1 + \frac{T^{2-k}p_1 - p_2}{x_2 + T^{2-k}p_1}, \quad T^{k-1}p_1 = p_1, \quad Tp_2 = p_2.$$  \hfill (24)

Adding the first two equations from above we get the following invariance condition\footnote{This condition is a consequence of the fact that the $t^{H_V(k)}_2$ preserves the invariant $H_1 = \sum_{i=1}^{k} x_i$. Such a condition exists for the remaining extended transfer maps associated with the Yang-Baxter maps of the $H$–list. The later enable us to write $t^{(k)}_2$ maps as $k$–point recurrences.}

$$(T^1 - T^{2-k})x_1 = (T^0 - T^1)x_2.$$ \hfill (25)

So it is guaranteed the existence of a potential function $f$ such that

$$x_1 = c + (T^0 - T^1)f, \quad x_2 = c + (T^1 - T^{2-k})f, \quad \text{where } c \text{ constant.}$$

In terms of $f$, (23) becomes the following $(k + 1)$–point recurrence

$$(T^2 - T^{2-k})f = \frac{-p_2 + T^{2-k}p_1}{2c + (T^3 - T^0)f}, \quad T^{k-1}p_1 = p_1, \quad Tp_2 = p_2.$$ \hfill (26)

In terms of a new variable $h$ defined as $h := \lambda + (T^1 - T^0)f$, there is,

$$(T^2 - T^{2-k})f = -\lambda k + \sum_{i=2-k}^{1} T^i h, \quad (T^3 - T^0)f = \lambda (2-k) + \sum_{i=3-k}^{0} T^i h,$$

so (26) becomes the $k$–point recurrence

$$\frac{2ck}{2-k} + \sum_{i=2-k}^{1} T^i h = \frac{-p_2 + T^{2-k}p_1}{\sum_{i=3-k}^{0} T^i h}, \quad T^{k-1}p_1 = p_1, \quad Tp_2 = p_2,$$ \hfill (27)

where we chose $\lambda = \frac{2c}{k-2}$ to simplify the formulae.

\begin{table}[h]
\centering
\caption{The invariance conditions (25) and the potential functions $f$ for the maps $t^{H_1(k)}_2, t^{H_{11}(k)}_2, t^{H_{111}(k)}_2, t^{H_{1111}(k)}_2$ and $t^{H_V(k)}_2$.}
\begin{tabular}{|c|c|c|}
\hline
Map & Invariance condition & Potential function $f$ \\
\hline
$t^{H_1(k)}_2$ & $\frac{T^0 x_2}{T^2 - k x_1}$ & $x_1 = c \frac{T^0 f}{T f}, \quad x_2 = c \frac{T^0 f}{T^{2-k} f}$ \\
\hline
$t^{H_{11}(k)}_2$ & $(T^0 - T^2) x_1 = (T^0 - T)x_2$ & $x_1 = c + (T^0 - T)f, \quad x_2 = c + (T^1 - T^{2-k})f$ \\
\hline
$t^{H_{111}(k)}_2$ & $(T^2 - T^{2-k}) \frac{1}{x_1} = (T^0 - T) \frac{1}{x_2}$ & $x_1 = \frac{1}{c} + (T^0 - T)f, \quad x_2 = c + (T^1 - T^{2-k})f$ \\
\hline
$t^{H_{1111}(k)}_2$ & $\frac{T^0 x_2}{T^2 - k x_1}$ & $x_1 = c \frac{T^0 f}{T f}, \quad x_2 = c \frac{T^0 f}{T^{2-k} f}$ \\
\hline
$t^{H_V(k)}_2$ & $(T^0 - T^2) x_1 = (T^0 - T)x_2$ & $x_1 = c + (T^0 - T)f, \quad x_2 = c + (T^1 - T^{2-k})f$ \\
\hline
\end{tabular}
\end{table}

\textbf{Proposition 5.6.} The following $(k + 1)$–point recurrences corresponds to the extended transfer map $t^{(k)}_2$ associated with $H_1, H_{11}, H_{111}, H_{1111}$ and $H_V$ Yang-Baxter maps respectively. We refer to these $(k + 1)$–point
For each recurrence presented above we have that the parameters vary as follows: $T p_2 = p_2$, $T^{k-1} p_1 = p_1$. So $p_2$ is constant and $p_1$ is periodic with period $k - 1$.

Note that the recurrences $r_{l_2}^{H_i(k)}$ and $r_{l_2}^{H_{III}(k)}$ are bilinear. Some members of $r_{l_2}^{H_i(k)}$ and $r_{l_2}^{H_{III}(k)}$, for specific choices of the parameters $c, p_2$ and of the function $p_1$, are expected to exhibit the Laurent property \cite{26, 27, 28}.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
recurrence & Variable $h$ & a choice for $\lambda$ \\
\hline
$r_{l_2}^{H_i(k)}$ & $h := \frac{T}{T - f}$ & $\lambda = \frac{1}{T}$ \\
$r_{l_2}^{H_{III}(k)}$ & $h := \lambda + (T - T^0) f$ & $\lambda = \frac{2c}{k - 2}$ \\
$r_{l_2}^{H_{III}(k)}$ & $h := \lambda + (T - T^0) f$ & $\lambda = \frac{2c}{k - 2}$ \\
$r_{l_2}^{H_{III}(k)}$ & $h := \lambda + (T - T^0) f$ & $\lambda = \frac{1}{c}$ \\
$r_{l_2}^{H_{III}(k)}$ & $h := \lambda + (T - T^0) f$ & $\lambda = \frac{2c}{k - 2}$ \\
\hline
\end{tabular}
\caption{Definition of the variables $h$ associated with the recurrences of proposition \cite{26, 27, 28}}
\end{table}
Corollary 5.7. The \((k+1)\)-point recurrences \(rt_2^{H_1(k)}\), \(rt_2^{H_{11}(k)}\), \(rt_2^{H_{111}(k)}\), \(rt_2^{H_{1111}(k)}\) and \(rt_2^{H_{V}(k)}\), in terms of the corresponding variables \(h\) defined in Table [22] get the form of the following \(k\)-point recurrences

\[
\prod_{i=3-k}^{1} T_i^h = \frac{c^{-k}p_2(T^{2-k}p_1 - 1) + (p_2 - T^{2-k}p_1) \prod_{i=2-k}^{0} T_i^h + (1 - p_2) \prod_{i=3-k}^{0} T_i^h}{T^{2-k}p_1 - p_2 + T^{2-k}p_1(p_2 - 1)T^{2-k}h + c^k(1 - T^{2-k}p_1) \prod_{i=3-k}^{0} T_i^h},
\]

\((rt_2^{H_1(k)})\)

and for each recurrence presented above we have that the parameters vary as follows: \(T_{p_2} = p_2\), \(T^{k-1}p_1 = p_1\).

Note that the \((k+1)\)-point recurrences of proposition 5.6 as well as the corresponding \(k\)-point ones introduced in corollary 5.7 are non-autonomous. This is due to the fact that \(p_1\) varies periodically \((T^{k-1}p_1 = p_1)\).

The non-autonomous terms that will be introduced by integrating the relation \(T^{k-1}p_1 = p_1\) are periodic through.

Proper de-autonomisation for the recurrences \(rt_2^{H_{V}(k)}\) and \(rt_2^{H_{1111}(k)}\) will be introduced in what follows.

5.2.1. The recurrences \(rt_i^{H_{V}(k)}\) and discrete Painlevé equations. The dressing chain for the KdV equation [69], reads:

\[
(g_{i+1} + g_i)h = g_{i+1}^2 + g_i^2 + p_{i+1} - p_i.
\]

The recurrences \(rt_i^{H_{V}(k)}\), serve as its discretisations. Actually they are exactly the \((k-1)\)-roots of the discretisations presented in [2]. So, \(rt_i^{H_{V}(k)}\) corresponds to Liouville integrable maps.

Since the dressing chain [29] leads to Painlevé equations \(P_{IV}\) and \(P_V\) and their higher order analogues [69], the recurrences \(rt_i^{H_{V}(k)}\) (after proper de-autonomisation) can be considered as their discrete counter-parts and/or the Bäcklund transformations of the higher order \(P_{IV}\) and \(P_V\) Painlevé equations.

A proper de-autonomisation of \(rt_i^{H_{V}(k)}\) is achieved by breaking the periodicity of the \(p_1\) assuming that \(T^{k-1}p_1 = p_1 + (k-1)a\), where \(a\) constant. This de-autonomisation is proper since the resulting non-autonomous discrete system preserves the same Poisson structure\(^5\) as the autonomous one. So we obtain the following hierarchy of discrete Painlevé equations

\[
- \frac{2ck}{k-2} + \sum_{i=2-k}^{1} T_i^h = \frac{-p_2 + T^{2-k}p_1}{\sum_{i=3-k}^{0} T_i^h}, \quad T_{p_2} = p_2, \quad T^{k-1}p_1 = p_1 + (k-1)a.
\]

For \(k = 3\), [29] reads

\[
-6c + Th + h + T^{-1}h = \frac{-p_2 + T^{-1}p_1}{h}, \quad T_{p_2} = p_2, \quad T^2p_1 = p_1 + 2a.
\]

\(^5\)The Poisson structures associated with the dressing chain for the KdV equation were first derived in [69], see also [29]
So $p_2$ is constant and $p_1 = b_0 + b_1(-1)^n + an$, with $b_0, b_1, a$ constants. We can choose $-p_2 + b_3 = b$ constant, hence we obtain the following discrete Painlevé equation which serves as Bäcklund transformation of $P_{IV}$

$$-6c + T h + h + T^{-1} h = \frac{b + b_1(-1)^n + an}{h}, \quad n \in \mathbb{Z}. \quad (30)$$

For $k = 4$, (29) reads

$$-4c + T^{-2} h + h + T^{-1} h + h + T h = \frac{-p_2 + T^{-2} p_1}{h + T^{-1} h}, \quad T p_2 = p_2, \quad T^3 p_1 = p_1 + 3a.$$

If we define a new variable $w$ as $w := h + T^{-1} h$, then we obtain the following discrete Painlevé equation which serves as Bäcklund transformation of $P_{IV}$

$$-4c + T^{-1} w + T w = \frac{-p_2 + T^{-2} p_1}{w}, \quad T p_2 = p_2, \quad T^3 p_1 = p_1 + 3a.$$

So for $k$ odd, (29) serves as as Bäcklund transformation for the higher order analogues of $P_{IV}$ and for $k$ even (29) serves as as Bäcklund transformation for the higher order analogues of $P_{V}$. Note that in [57], Bäcklund transformation for the higher order analogues of $P_{IV}$ and $P_{V}$ were given in terms of continued fractions. We can recover the form of discrete Painlevé equations introduced in [57] by making use of the alternating terms that appear in (29). For example for $k = 3$, the term $1)^n$ that appears in (30), suggests the introduction of the variables $y(m) := h(2n)$, $z(m) := h(2n + 1)$. Then (30) takes to form of the second discrete Painlevé equation $dP_{I}$

$$y + z + T^{-1} z = \frac{b_0 + b_1 + am}{y}, \quad T y + y + z = \frac{b_0 - b_1 + am}{z}, \quad m \in \mathbb{Z}.$$

5.2.2. The recurrences $\hat{r}^{H P_{I\mathbb{Z}}(k)}_{H}$ and discrete Painlevé equations. As we plan to show in our future work, the recurrences $\hat{r}^{H P_{I\mathbb{Z}}(k)}_{H}$ serves as Liouville integrable discretisations of the following chain introduced in [11]

$$(g_i + g_{i+1}) t = 2(p_i \cosh g_i - p_{i+1} \cosh g_{i+1}).$$

A proper de-autonomisation of the $\hat{r}^{H P_{I\mathbb{Z}}(k)}_{H}$ is achieved by breaking the periodicity of the $p_1$ in a way that the non-autonomous system preserves the same Poisson structure as the autonomous one. This is achieved by imposing that $T^{k-1} p_1 = p_1 a^{k-1}$, where $a$ constant. So we obtain the following hierarchy of discrete Painlevé equations

$$\prod_{i=2-k}^{1} T^i h = \frac{e^{-k} + T^{-k} p_1}{1 + e^{k} p_2} \prod_{i=3-k}^{0} T^i h, \quad T p_2 = p_2, \quad T^{k-1} p_1 = p_1 a^{k-1}. \quad (31)$$

For $k = 3$, (31) reads

$$T h T^{-1} h = \frac{1}{h + 1 + e^{p_2} h}, \quad T p_2 = p_2, \quad T^3 p_1 = p_1 a^2.$$

So $p_2$ is constant and $p_1 = b_0 a^n + b_1(-a)^n$, with $b_0, b_1, a$ constants. Hence we obtain the $q - P_1(A^{(1)}_0)$ discrete Painlevé equation (see [63]). For $k = 4$, (31) reads

$$T h T^{0} h T^{-2} h = \frac{e^{-4} + h T^{-1} h T^{-2} p_1}{1 + e^{4} p_2 h T^{-1} h}, \quad T p_2 = p_2, \quad T^3 p_1 = p_1 a^3.$$

If we define a new variable $w$ as $w := h T^{-1} h$, then we obtain the $q - P_{IV}(A^{(1)}_0)$ discrete Painlevé equation (see [63])

$$T w T^{-1} w = \frac{e^{-4} + w T^{-2} p_1}{1 + e^{4} p_2 w}, \quad T p_2 = p_2, \quad T^3 p_1 = p_1 a^3.$$

The Lax pair associated with the hierarchy (31) first appeared in [63].
Remark 5.8. As for the recurrences $\hat{r}_i^{H_{11\ell}(k)}, \hat{r}_i^{H_{11}(k)}$, one could consider $T^{k-1}p_i = p_i + (k-1)a$ and for $\hat{r}_i^{H_{1}(k)}$ $T^{k-1}p_i = p_i a^{k-1}$, in order to de-autonomise them. We anticipate that this is a proper de-autonomisation, although we have no proof yet. The finding of the Poisson structures that the later recurrences we anticipate that that preserve, will sort this issue out.

Remark 5.9. As a final remark, we note that the $k$--point recurrences associated with the extended transfer maps of the Yang-Baxter map $F_V$, are exactly the same as the $k$--point recurrences associated with the extended transfer maps of the Yang-Baxter map $H_V$ which (one of them) were presented in Corollary 5.7. Since the $(k-1)$--iteration of the extended transfer maps of any Yang-Baxter map coincides with its transfer maps, we conclude that the dynamics of the transfer maps of the Yang-Baxter maps $F_V$ and $H_V$, are the same. The same holds true for the transfer maps associated with the Yang-Baxter maps $F_{111}$ and $H_{111}^A$. As for the remaining members of the $F$ and the $H$ lists of Yang-Baxter maps, further investigation is required in order to prove the equivalence of their transfer dynamics.

6. Conclusions

In Section 2 we have presented a family of maps in $k$ variables which preserve 2 rational invariants of a specific form. One could mimic the procedures introduced in [29] to obtain rational maps in $k$ variables which preserve $m$ rational invariants where $m < k$. For example, there are $\binom{2k}{k}$ rational maps $(x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (X_1, \ldots, X_k, Y_1, \ldots, Y_k)$ which preserve $k$ invariants of the form:

$$H_i = \frac{\alpha_i x_i x_{i+1} + \beta_i x_i + \gamma_i x_{i+1} + \delta_i}{\kappa_i x_i x_{i+1} + \lambda_i x_i + \mu_i x_{i+1} + \nu_i}, \quad i = 1, 2, \ldots, k,$$

where the indices are considered modulo $k$ and $\alpha_i, \beta_i, \kappa_i, \lambda_i, \gamma_i, \delta_i, \mu_i, \nu_i$ are given functions of the variables $y_i, y_{i+1}$.

If separability of variables on the invariants is imposed, then higher rank analogues of the Yang-Baxter maps of propositions 3.5, 3.9 and 3.11 are accepted. Moreover, solutions of the functional tetrahedron equation [50, 42, 43, 64], or even of higher simplex equations [54, 55, 17] are anticipated. For example if we consider the following, different than the [29], choice of invariants:

$$H_1 = \sum_{i=1}^{6} x_i, \quad H_2 = \frac{x_1 x_2 x_3}{x_3}, \quad H_3 = x_2 x_3 x_4 x_5,$$

then the involutions $R_{123}, R_{145}, R_{246}$, and $R_{356}$, preserve $H_i$, $i = 1, 2, 3$ and satisfy the functional tetrahedron equation

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}.$$ 

They are exactly the Hirota’s map [42, 43, 64], i.e. the map $R : (u, v, w) \mapsto (U, V, W)$, where

$$U = \frac{uv}{u + w}, \quad V = u + w, \quad W = \frac{uv}{u + w},$$

acting on (123), (145), (246) and (356) coordinates respectively. For the involution $\phi : u \mapsto -u$, it holds $\phi_1 \phi_2 \phi_3 R_{123} = R_{123} \phi_1 \phi_2 \phi_3$. So $\phi$ is a symmetry of the Hirota’s map $R$ and it can be easily proven that the following entwining relation holds:

$$R_{123} \phi_1 \phi_4 \phi_5 \phi_6 R_{246} \phi_6 R_{356} = R_{356} R_{246} \phi_6 \phi_4 \phi_5 \phi_6 R_{123} \phi_3.$$

Hence we have obtained a solution of the following entwining functional tetrahedron relation

$$S_{123} S_{145} S_{246} T_{356} = T_{356} S_{246} S_{145} S_{123},$$

where $T$ is the Hirota’s map acting on the (356) coordinates and $S : (u, v, w) \mapsto (U, V, W)$ a non-periodic map where

$$U = \frac{uv}{u - w}, \quad V = u - w, \quad W = -\frac{uv}{u - w}.$$

The possible set of entwining relations and maps associated with the Hirota’s map as well as with the Hirota-Miwa’s map, will be considered elsewhere.

In Section 3 we considered two methods to obtain entwining maps. The first method uses degeneracy arguments and produces entwining maps associated with the $H_I, H_{II}$ and $H_{111}^A$ Yang-Baxter maps. The entwining maps of this method belongs to different subclasses than the [2 : 2] subclass of maps that the $H_I, H_{II}$ and $H_{111}^A$ Yang-Baxter maps belongs to so they are not (Möbius) equivalent to the later. The outcomes of the second
method are non-periodic entwining maps of subclass [2 : 2] associated with the whole $H$–list. The fact that the entwining maps which were presented in this Section preserve two invariants in separated variables, enable us to introduce appropriate potentials (as shown in [45, 44, 56]) to obtain integrable lattice equations. Actually we obtain integrable triplets of lattice equations (in some cases even correspondences). Note that integrable triplets of lattice equations were systematically derived in [13] and more recently in [34]. We plan to consider the integrable triplets of lattice equations derived from entwining maps, elsewhere.

In Section 6 we have proved that the transfer maps associated with the $H$ list of Yang-Baxter maps can be considered as the $(k−1)$-iteration of some maps of simpler form. As a consequence of this re-factorisation we have obtained $(k+1)$–point (see proposition 5.7) and $k$–point (see corollary 5.7) alternating recurrences which can be considered as alternating versions of some hierarchies of discrete Painlevé equations. Moreover, the autonomous versions of some of the $k$–point recurrences presented in corollary 5.7 can be obtained by periodic reductions (58, c.f. 35) of integrable lattice equations. Here we have obtained alternating $k$–point recurrences from Yang-Baxter maps without performing periodic reductions. Hence, our results might be compared/extended to the novel and independent frameworks introduced in [8] and 39, 40, where by using symmetry arguments, integrable lattice equations and discrete Painlevé equations of 2nd order were linked.

Acknowledgements

P.K is grateful to Aristophonis Dimakis, Vassilios Papageorgiou and Anastasios Tongas, the organizers of the 4th “Workshop on Mathematical Physics-Integrable Systems, 30 November-1 December 2018, Department of Mathematics, University of Patras, Patras, Greece,” where this work was finalized. Also, P.K. is grateful to James Atkinson, Allan Fordy, Nalini Joshi and to Pol Vanhaecke for very fruitful discussions on the subject, as well as to Maciej Nieszporski for the endless discussions towards the answer to the great question of integrable systems, Yang-Baxter and everything.

References

[1] V.E. Adler and A.B. Shabat. Dressing chain for the acoustic spectral problem. Theoret. and Math. Phys., 149(1):1324–1337, 2006.
[2] V.E. Adler. Recutting of polygons. Funct. Anal. Appl., 27(2):79–80, 1993.
[3] V.E. Adler. Bäcklund transformation for the Krichever-Novikov equation. Intern. Math. Research Notices, 1:1–4, 1998.
[4] V.E. Adler. On a class of third order mappings with two rational invariants. arXiv:nlin/0606052, 2006.
[5] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. Comm. Math. Phys., 233(3):513–543, 2003.
[6] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. Comm. Anal. Geom., 12(5):967–1007, 2004.
[7] V.E. Adler and R.I. Yamilov. Explicit auto-transformations of integrable chains. J. Phys. A: Math. Gen., 27(2):477, 1994.
[8] J. Atkinson. Idempotent biquadratics, Yang-Baxter maps and birational representations of Coxeter groups. arXiv:nlin/1301.4619, 2013.
[9] J. Atkinson and M. Nieszporski. Multi-quadratic quad equations: integrable cases from a factorised-discriminant hypothesis. Int. Math. Res. Not., 2220(15):4215–4240, 2013.
[10] J. Atkinson and Y. Yamada. Quadrirational Yang-Baxter maps and the elliptic Cremona system. arXiv:nlin/1804.01794, 2018.
[11] R.J. Baxter. Exactly solved models in statistical mechanics. Academic Press, London, 1982.
[12] V.V. Bazhanov and S.M. Sergeev. Yang–Baxter maps, discrete integrable equations and quantum groups. Nuclear Physics B, 926:509 – 543, 2018.
[13] R. Boll. Classification of 3d consistent quad-equations. J. Nonlinear Math. Phys., 18(3):337–365, 2011.
[14] M. Bruschi, O. Ragnisco, P.M Santini, and T. Gui-Zhang. Integrable symplectic maps. Physica D, 49:273–294, 1991.
[15] Cresswell C. and Joshi N. The discrete first, second and thirty-fourth Painlevé hierarchies. J. Phys. A: Math. Gen., 32(4):655, 1999.
[16] H.W. Capel and R. Sahadevan. A new family of four-dimensional symplectic and integrable mappings. Physica A, 289:86–106, 2001.
[17] A. Dimakis and F. Muller-Hoissen. Simplex and integrable systems. SIGMA 11, 042:49pp, 2015.
[18] A. Dimakis and F. Müller-Hoissen. Matrix Kadomtsev-Petviashvili equation: Tropical limit, Yang-Baxter and pentagon maps. Theoret. and Math. Phys., 196(2):1164–1173, 2018.
[19] A. Dimakis and F. Müller-Hoissen. Matrix KP: tropical limit and Yang-Baxter maps. Lett. Math. Phys., 2018.
[20] A. Doliwa. Non-commutative rational Yang–Baxter maps. Lett. Math. Phys., 104(3):299–309, 2014.
[21] V. G. Drinfeld. On some unsolved problems in quantum group theory, quantum groups. Lecture Notes in Math., 1510:1–8, 1992.
[22] J. J. Duistermaat. Discrete integrable systems. QRT maps and elliptic surfaces. Springer Monographs in Mathematics. Springer, New York, 2010.

6The non-periodicity assures that these entwining maps are not $(Möbi)^2$ equivalent with the corresponding maps of the $H$–list.
The Yang-Baxter maps $R$ of the $F$ and the $H$ list, explicitly reads:

$$R : \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u, v) \mapsto (U, V) \in \mathbb{CP}^1 \times \mathbb{CP}^1$$

where:

\begin{align*}
U &= \alpha P, \\
V &= \beta P, \\
U &= \alpha P, \\
V &= \beta P, \\
U &= \frac{\alpha P}{\beta}, \\
V &= \frac{\beta P}{\alpha}, \\
U &= \frac{\alpha P}{\beta}, \\
V &= \frac{\beta P}{\alpha}, \\
U &= \frac{\alpha P}{\beta}, \\
V &= \frac{\beta P}{\alpha}, \\
U &= \frac{\alpha P}{\beta}, \\
V &= \frac{\beta P}{\alpha}, \\
U &= \frac{\alpha P}{\beta}, \\
V &= \frac{\beta P}{\alpha},
\end{align*}

\begin{align*}
P &= \frac{(1 - \beta)u + \beta - \alpha + (\alpha - 1)v}{\beta(1 - \alpha)u + (\alpha - \beta)uv + \alpha(\beta - 1)v}, \\
Q &= \frac{\alpha - \beta}{\beta u + \alpha v - \alpha^2}, \\
U &= v Q, \\
V &= u Q, \\
U &= v Q, \\
V &= u Q, \\
U &= v Q, \\
V &= u Q,
\end{align*}

The maps above are depending on 2 complex parameters $\alpha, \beta$. The parameter $\alpha$ is associated with the first factor of the cartesian product $\mathbb{CP}^1 \times \mathbb{CP}^1$, whereas the parameter $\beta$ with the second factor.

Department of Mathematics and Statistics University of Cyprus, P.O Box: 20537, 1678 Nicosia, Cyprus

E-mail address: pavlos1978@gmail.com, pkasso01@ucy.ac.cy