PSPACE hardness of approximating the capacity of Markoff channels with noiseless feedback *

MUKUL AGARWAL†‡,

It will be proved that approximating the capacity of a Markoff channel with noiseless feedback is PSPACE hard.

AMS 2000 subject classifications: Primary 00K00 fill, 00K01; secondary 00K02 fill.

Keywords and phrases:

1. Introduction

It will be proved that approximating the capacity of a Markoff channel with noiseless feedback is PSPACE-hard. By 'approximating,' we mean, computing the capacity within a certain given additive error $e$. A class of channels will be constructed for which it will be proved that approximating the capacity to within 0.1 bits of the correct capacity is PSPACE-hard. To the best of the knowledge of the author, this is the first result in literature on hardness of computing the capacity or approximately computing the capacity of a channel. The observation that computing the capacity of a Markoff channel with noiseless feedback can be formulated as a stochastic dynamic programming problem with partial observations as formulated by Tatikonda and Mitter [1] and thereby connecting it to the work of Tsitsiklis and Papadimitriou on the complexity of Partially Observed Markoff Decision Processes [2] is what is novel here.

For the relevant background on complexity theory, see [2] and references therein. An understanding of Section 4 in [2], in particular, the statement of Theorem 6 and its proof is needed to understand the proof here. Once this is understood, and the reader has an understanding of information theory and basic probability theory, the proofs presented here can be understood.

*Footnote to the title with the ‘thankstext’ command.
†Some comment.
‡First supporter of the project.
§Second supporter of the project.
For literature on Markoff channels with feedback, the reader is referred to [1] and references therein.

2. Literature survey

In [2], the complexity of Markoff Decision Processes and Partially Observed Markoff Decision Processes has been considered and in [3], the Witsenhausen and team decision problems have been considered. In both these papers, it is proved that there are problems in each category which are hard. See further references therein for problems which have been proved to be hard, especially in a control setting. In [4], it has been proved that the generalized Lloyd-Max algorithm is NP-complete. In [5], it has been proved that general decoding problem for linear codes and the general problem of finding the weights of a linear code are both NP-complete.

3. The result

This section contains the channel construction, lemmas and the theorem.

3.1. Channel construction and reliable communication

Let a quantified formula \( Q = \exists x_1 \exists x_2 \ldots \forall x_n F(x_1, x_2, \ldots, x_n) \) with \( n \) variables and \( m \) clauses in conjunctive normal form be given.

Construct a channel corresponding to this Boolean formula as follows:

Channel states \( s_0, A_{ij}, A'_{ij}, T_{ij}, T'_{ij}, F_{ij}, F'_{ij} \); sets \( A_j, A'_j, T_j, T'_j, F_j, F'_j \) exactly as [2]. The only difference is that there is no terminating state, and from \( A_{i,n+1}, A'_{i,n+1} \), the channel state goes back to itself or comes back to state \( s_0 \) and the process continues.

Channel transitions are exactly as in [2], except that at state \( A_{i,n+1} \), the channel stays at state \( A_{i,n+1} \) with probability \( 1 - p \) and transitions to state \( s_0 \) with probability \( p \). Similarly, from state \( A'_{i,n+1} \), the channel either stays at the state \( A'_{i,n+1} \) with probability \( 1 - q \) or transitions to the state \( s_0 \) with probability \( q \).

The states \( A_{i,n+1} \) will be called good states and the rest of the states will be called ‘bad’ states. The reason for this will become clear below.
The input to the channel is what is called ‘decision’ in [2] along with a bit (called the information bit) which the encoder ‘feeds’ directly into the channel if the channel state is ‘good’. The ‘decision’ variable, at each point, takes a maximum of two possible values. Mathematically, the channel input space is \( \{D_1, D_2\} \times \{0, 1\}\).

The output of the channel is the partial state information (this is the partial knowledge in [2], that is, one of \(s_0, A_i, A'_i, T_i, T'_i, F_i, F'_i\) along with an output bit (corresponding to the input information bit). Mathematically, the output space of the channel is \(S \times \{0, 1\}\) where we will describe \(S\) shortly after we have commented on the channel evolution. In states \(A_{i,n+1}\), the channel will output the information bit perfectly, and in other states, the the channel will output a 0 for the information bit. It is for this reason, that the states \(A_{i,n+1}\) are called good states and the rest of the states are called bad states. The output of the channel is ‘fed back’ immediately to the encoder without delay.

Note again, that as stated above, the channel transitions happen in exactly the same way as in [2].

It will be assumed that the channel starts in state \(s_0\). Note the following: the functioning of channel state is sequential in the sense that it starts at \(s_0\), at time 1, the channel is in some state \(X_{i1}\) for some \(i\), where \(X\) stands for one of \(A, A', T, T', F, F'\). At the next time point, the channel state is in some set \(X_{i2}\). At time \(2n+1\), the channel is in state \(X_{i,n+1}\) where \(X = A\) or \(A'\), where the channel stays with a certain probability (\(p\) or \(q\)) or jumps to \(s_0\) and the process continues. Note that each time, the set the channel state belongs to is one of two possible sets, and the receiver and the encoder, already know, which these two sets are. Thus, we can let the set \(S\) be \(\{S_1, S_2\}\) where \(S_1\) and \(S_2\) will be used to distinguish between these two sets.

For the purpose of understanding, it is best to think of the problem as a sequential problem where a set of bits enter the encoder at a certain rate \(R\) which causes an encoding and the channel produces the outputs from which a decoding needs to happen. A rate \(R\) is achievable if, for every \(\delta, \exists t_\delta\) such that for \(t > t_\delta\), \(Rt\) bits can be communicated and the average error, that is, \(\Pr(\hat{B}_{tR} \neq B_{tR})\) is less than \(\delta\), where \(B_{tR}\) denotes the bit input upto time \(t\) and \(\hat{B}_{tR}\) is the corresponding decoding.

Further, assume that \(p = 2^{-(mn)^{100}}\) and \(q = 2^{-(mn)^{200}}\). The way these are chosen satisfy at least the following conditions: \(p, q \rightarrow 0\) as \(mn \rightarrow \infty\) and \(q/p \rightarrow 0\) at a sufficiently fast rate as \(mn \rightarrow \infty\).
3.2. Lemmas, theorem and proof

As has been stated previously, assume, in what follows, that the channel starts in state $s_0$.

**Lemma 1.** Given $\epsilon > 0$. Then, $\exists m_0, n_0$, depending only on $\epsilon$ such that for $m > m_0, n > n_0$, if $Q$ is true, capacity of the channel corresponding to $Q$ is larger than $1 - \epsilon$.

**Proof.** By [2], $Q$ is true implies we can choose the channel input (decisions in [2]) so that we always end up in some $A_{i,n+1}$, not $A'_{i,n+1}$. The transitions from $s_0$ to $A_{i,n+1}$ takes $2n + 1$ units of time where at worst, no bits can be communicated, and the channel stays in state $A_{i,n+1}$ for an average of order of magnitude $2^{(mn)^{100}}$ number of transitions. By these considerations, it follows that given any $\kappa > 0, \exists m_1, n_1$ such that for $m > m_1, n > n_1$, the stationary distribution of the set of states $A_{i,n+1}$ of the Markoff chain with the above chosen channel inputs is $> 1 - \epsilon$, and in states $A_{i,n+1}$, noiseless communication of one bit per channel use occurs. By use of the ergodic theory for Markoff chains, the lemma follows. \hfill $\square$

**Lemma 2.** Given $\alpha > 0$. Then, $\exists m_0, n_0$ sufficiently large, depending only on $\alpha$ such that for $m > m_0, n > n_0$, the capacity of the channel corresponding to the formula $Q$ is larger than $\alpha$ implies that $Q$ is true.

**Proof.** If there was some way for the channel to enter the state $A'_{i,n+1}$ for some $i$ irrespective of the decisions, this would happen with probability at least $\frac{2^{-n}}{m}$ (see [2]), and then, the channel stays in this state for an average of an order of magnitude of $2^{(mn)^{200}}$ amount of time (1 unit of time refers to one state transition). Note that there is no transmission of information possible in state $A'_{i,n+1}$. Even if the channel ended in some $A_{i,n+1}$ with the rest of the probability $1 - \frac{2^{-n}}{m}$, the average order of magnitude amount of time the channel stays in state $A_{i,n+1}$ is $2^{(mn)^{100}}$ which is ‘much less’ than $2^{(mn)^{200} \frac{2^{-n}}{m}}$. It follows that given any $\lambda > 0, \exists m_2, n_2$ such that for $m > m_2, n > n_2$, the fraction of time the channel spends in states $A'_{i,n+1}$ is $> (1 - \lambda$ with high probability. Finally, note that the amount of transmission of information during the $(2n + 1)$ units of time when the channel transitions from $s_0$ to $A_{i,n+1}$ or $A'_{i,n+1}$ is at most 1 bit per channel use. This 1 bit comes because the decision can come from 2 possible choices which can be used to transmit a bit. It follows, then, that if the channel could enter $A'_{i,n+1}$, there exist $m_0, n_0$ sufficiently large such that the capacity of the channel will be less
than $\alpha$ which will contradict the assumption on the channel. It follows, then, that there is a set of decisions (channel inputs) for which the channel never enters the state $A'_{i,n+1}$ which implies, by [2], that the formula is true.

**Theorem 1.** Computing the capacity of this set of Markoff channels (the set of channels formed by taking a channel corresponding to each formula $Q$ where $m, n$ and the particular formula can be any positive integers) when noiseless feedback is available, to within an accuracy of 0.1 bits per channel use is PSPACE hard.

**Proof.** If the capacity of this set of Markoff channels with feedback could be computed to within accuracy 0.1, it would be known whether the capacity of the channel is less than 0.2 or larger than 0.8. This would imply, from the previous lemmas, that we would know, for sufficiently large $m, n$, whether $Q$ is true or not, and this problem is PSPACE hard. This finishes the proof of the theorem. $\square$

4. Discussion

4.1. Comments

**Comment 1.** It has been assumed that the channel starts in state $s_0$. This is only for simplicity of presentation. If the channel does not start at state $s_0$, the set $S$ can be expanded to $(S_0, S_1, S_2)$ where the partial state information part of the output of the channel is $S_0$ as soon as the channel reaches the state $s_0$. Thus, the encoder and receiver will know when the channel has reached state $s_0$ and the previous argument holds.

**Comment 2.** In addition, a transition from state $A_{i,n+1}$ to $A'_{i,n+1}$ with a probability $2^{-(mn)^{500}}$ could be added, and thus, from state $A_{i,n+1}$ to state $s_0$ with probability $1 - 2^{-(mn)^{500}} - 2^{-(mn)^{100}}$ to make the picture a little more realistic. By a similar argument, the lemmas and the theorem would follow.

**Comment 3.** There is nothing special about the number 0.1 in Theorem 1.

**Comment 4.** PSPACE hardness implies NP hardness and thus, the problem dealt with in this paper is also NP hard.

**Comment 5.** By unfolding the POMDP in $?$ or by unfolding the above constructed channel, we get a network and with proper formulation, we can get a hardness result on this network. Details are omitted.
5. Recapitulation and research direction

A class of Markoff channels with noiseless feedback was constructed for which it was proved that approximating the capacity to within 0.1 bit is PSPACE hard.

It would be worthwhile exploring the application of this proof idea to channels with noisy feedback and networks in general.

References

[1] S. Tatikonda and S. K. Mitter, “Capacity of channels with feedback,” *IEEE Transactions on Information Theory*, Vol. 55, Number 11, January 2009.

[2] C.H. Papadimitriou and J. N. Tsitsiklis, “The complexity of Markov Decision Processes,” *Mathematics of Operations Research*, Vol. 12, Number 3, August 1987.

[3] C. H. Papadimitriou and J. N. Tsitsiklis, “Intractable problems in control theory,” *SIAM Journal of Control and Optimization*, Vol. 24, Number 4, July 1986.

[4] M. R. Garey, D. S. Johnson and H. S. Witsenhausen, “The complexity of the generalized Lloyd-Max problem,” *IEEE Transactions on Information Theory*, Vol. IT-28, Number 2, March 1982.

[5] E. Berlekamp, R. McEliece and H. van Tilborg, “On the inherent intractability of certain coding problems,” *IEEE Transactions on Information Theory*, Vol. 24, Issue 3, May 1978.

Mukul Agarwal
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
BOSTON UNIVERSITY
E-mail address: magar@alum.mit.edu

Received August 7, 2015