ON A THEOREM OF STAFFORD

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Abstract. In [6] Stafford proved that every left or right ideal of the Weyl algebra \(A_n(K) = K[x_1, ..., x_n][\partial_1, ..., \partial_n]\) (\(K\) a field of characteristic zero) is generated by two elements. Consider the ring \(D_n := K[[x_1, ..., x_n]][\partial_1, ..., \partial_n]\) of differential operators over the ring of formal power series \(K[[x_1, ..., x_n]]\).

In this paper we prove that every left or right ideal of the ring \(E_n := K((x_1, ..., x_n))[\partial_1, ..., \partial_n]\) of differential operators over the field of formal Laurent series \(K((x_1, ..., x_n))\) is generated by two elements. The same is true for the ring of differential operators over the convergent Laurent series \(C\{x_1, ..., x_n\}\). This is in accordance with the conjecture that says that in a (noncommutative) noetherian simple ring, every left or right ideal is generated by two elements.

1. Introduction

In [6] Stafford proved that every left or right ideal of the Weyl algebra \(A_n(K) = K[x_1, ..., x_n][\partial_1, ..., \partial_n]\) (\(K\) a field of characteristic zero) is generated by two elements. It would be interesting to have a similar result for the ring \(D_n := K[[x_1, ..., x_n]][\partial_1, ..., \partial_n]\) of differential operators over the ring of formal power series \(K[[x_1, ..., x_n]]\). In this paper we prove that every left or right ideal of the ring \(E_n := K((x_1, ..., x_n))[\partial_1, ..., \partial_n]\) of differential operators over the field of formal Laurent series \(K((x_1, ..., x_n))\) is generated by two elements. The same is true for the ring of differential operators over the convergent Laurent series \(C\{x_1, ..., x_n\}\) (this is the field of fractions of the domain \(O_n\) of germs of convergent complex power series around the origin of \(\mathbb{C}^n\)). Note that this is in accordance with the conjecture that says that in a (noncommutative) noetherian simple ring, every left or right ideal is generated by two elements.

The main difficulty here is that the symmetry between the \(x\)'s and the \(\partial\)'s, present in the Weyl algebra and which is fundamental in the prove of Stafford’s theorem, is broken in the ring \(D_n\). Since we are working over the base ring of power series, infinite sums in the \(x\)'s are permitted while only finite sums in

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the \( \partial \)'s occur. To deal with this problem we appeal to Weierstrass' preparation theorem to put an element \( v \in D_n \) in an appropriate form (see lemma 4.1). In general, however, we follow the proof of Stafford’s theorem in Björk’s book ([1]) and we do the necessary modifications to deal with the power series case.

2. Basic Notations

Let \( K \) be a field of characteristic zero and \( T \) be a skew field of characteristic zero. By \( T[[x]] \) we denote the ring of power series in one indeterminate with coefficients in \( T \) and by \( T((x)) \) its quotient skew field. In paragraph 3 we will be interested in the ring \( S = T((x))\langle \partial_x \rangle \), the ring of differential operators with coefficients in \( T((x)) \). This paragraph is mostly concerned with free \( S \)-modules of finite rank over \( S \).

In paragraph 4 the following notation will be used. For each \( 0 \leq r \leq n \) let \( D_r = K[[x_1, \ldots, x_r]]\langle \partial_1, \ldots, \partial_r \rangle \) be the ring of differential operators over the ring of formal power series \( K[[x_1, \ldots, x_r]] \) and let \( F_r \) be its quotient ring. \( F_r \) exists because \( D_r \) is an Ore domain. Since \( x_{r+1}, \ldots, x_n \) commute with the elements in \( F_r \) we also get the division ring \( F_r((x_{r+1}, \ldots, x_n)) \) which by definition is the quotient ring of \( D_r((x_{r+1}, \ldots, x_n)) \). Also, for each \( 0 \leq r \leq n \), we put \( R_r = F_r((x_{r+1}, \ldots, x_n))\langle \partial_{r+1}, \ldots, \partial_n \rangle \). Of course if \( r = n \), then \( R_n = F_n \).

3. The ring \( S = T((x))\langle \partial_x \rangle \)

Let \( S = T((x))\langle \partial_x \rangle \), the ring of linear differential operators with coefficients in rational expressions in \( x \) over the skew field \( T \). Recall that \( S \) is a noncommutative noetherian simple ring.

Suppose that \( F \) is a field contained in \( T \). By \( F[[x]]\langle \partial_x \rangle \) we denote the ring of differential operators with coefficients in the ring of power series \( F[[x]] \).

**Lemma 3.1.** Let \( 0 \neq \alpha \in S \). Then the \( S \)-module \( S/\alpha S \) has finite length.

**Proof.** It follows immediately from the division algorithm on \( S \). See also [1] lemma 8.8, page 27.

**Lemma 3.2.** Let \( \delta_1, \ldots, \delta_m \) be a set of non zero elements in \( F[[x]]\langle \partial_x \rangle \subset S \).

Let \( 0 \neq \alpha \in S \) and \( S^{(m)} = S\varepsilon_1 + \cdots + S\varepsilon_m \) be a free \( S \)-module of rank \( m \) with basis \( \varepsilon_1, \ldots, \varepsilon_m \). Let \( M \) be the \( S \)-submodule of \( S^{(m)} \) generated by the set \( \{ \alpha \delta_1 f \varepsilon_1 + \cdots + \alpha \delta_m f \varepsilon_m | f \in \mathbb{Z}[[x]]\langle \partial_x \rangle \} \).

Then \( M = S^{(m)} \).

**Proof.** To simplify the notation we put \( \partial := \partial_x \).

Let us first observe that both the assumption and the conclusion are unchanged if the \( m \)-tuple \( \delta_1, \ldots, \delta_m \) is replaced by an \( m \)-tuple \( \beta_1, \ldots, \beta_m \), where \( \beta_i = \Sigma a_{ij} \delta_j \) and \( (a_{ij}) \) is an \( m \times m \) invertible matrix with \( a_{ij} \in F \). Of course, while we replace \( \delta_1, \ldots, \delta_m \) by \( \beta_1, \ldots, \beta_m \) under a \( F \)-linear transformation \( (a_{ij}) \) we also replace the free generators \( \varepsilon_1, \ldots, \varepsilon_m \) of \( S^{(m)} \) by \( \zeta_1, \ldots, \zeta_m \) where \( \zeta_i = \Sigma b_{ij} \varepsilon_j \), \( (b_{ij}) = (a_{ij})^{-1} \).
Let $ord(\delta_i)$ be the $\partial-$order of $\delta_i$. We can assume that these $\partial$-orders decrease, i.e., $ord(\delta_1) \geq ord(\delta_2) \geq \cdots \geq ord(\delta_m)$. Hence there exists an integer $\omega$ and some $1 \leq l \leq m$ such that $\omega = ord(\delta_1) = \cdots = ord(\delta_l)$, while $ord(\delta_i) < \omega$ if $i > l$.

If $1 \leq i \leq l$ we can write $\delta_i = r_i + p_i(x)\partial^s$, where $ord(r_i) < \omega$ and $p_i(x) \in F[[x]]$. We can assume that $val(p_1) \leq val(p_2) \leq \cdots \leq val(p_l)$, where $val(p_i)$ is the usual valuation of the power series $p_i$. If $val(p_1) = val(p_2) = \mu$, then there exists some $t \in F$ such that $val(p_2 - tp_1) > \mu$. Replace $\delta_2$ by $\delta_2 - t\delta_1$ while $\delta_1, \delta_3, \ldots, \delta_m$ are unchanged. Then, after some $F-$linear transformations we can assume that $val(p_1) < val(p_2) < \cdots < val(p_l)$. With these normalizations in hand we begin to prove that $\varepsilon_1 \in M$.

Let $k = ord(\alpha)$. Since $S = T((x))\langle \partial \rangle$ we can assume that $\alpha = \alpha_0 + \partial^k$ where $ord(\alpha_0) < k$. If $1 \leq i \leq l$ we have $\alpha\delta_i = p_i(x)\partial^{k+\omega} + \psi_i$ where $ord(\psi_i) < k + \omega$ and if $l < i \leq m$, $ord(\alpha\delta_i) < k + \omega$.

Now if $g \in S$ we put $g_1 = [g, x] = gx - xg$ the commutator of $g$ and $x$. Inductively, let $g_{\nu+1} = [g_\nu, x]$. The element $g_\nu$ is called the $\nu-$fold commutator of $g$ and $x$. The $\nu-$ fold commutator of $\partial^s$ and $x$ is $\nu$! for all positive integers $\nu$, while the $\nu-$ fold commutator of $\partial^s$ and $x$ is zero if $s < \nu$.

If we apply this to the elements $\alpha\delta_1, \ldots, \alpha\delta_m$ we see that the $(k + \omega)-$fold commutator of $\alpha\delta_i$ and $x$ is $p_i(x)(k + \omega)!$ for all $1 \leq i \leq l$, while they are zero if $l < i \leq m$.

The definition of $M$ implies that $M$ is stable under the $\nu-$fold commutator with $x$, i.e., if $m \in M$ then the $\nu$-fold commutator of $m$ and $x$ is in $M$ for all positive integers $\nu$. In fact, note that if $f \in \mathbb{Z}[[x]]\langle \partial \rangle$, then $[\alpha\delta f, x] = \alpha\delta(fx) - x(\alpha\delta f)$.

For $f = 1 \in \mathbb{Z}[[x]]\langle \partial \rangle$ we have that $a = \alpha\delta_1 \varepsilon_1 + \cdots + \alpha\delta_m \varepsilon_m \in M$. If $v_1$ is the $(k + \omega)-$fold commutator of $a$ and $x$ divided by $(k + \omega)!$, then $v_1 \in M$ and

$$v_1 = p_1(x)\varepsilon_1 + \cdots + p_l(x)\varepsilon_l.$$  

Now, if $h \in S$, let $h_1 = [h, \partial] = h\partial - \partial h$ be the commutator of $h$ with $\partial$. Inductively, we define $h_{\nu+1}$ by $h_{\nu+1} = [h_{\nu}, \partial]$. The element $h_{\nu}$ is called the $\nu-$fold commutator of $h$ and $\partial$. We observe that $M$ is stable under the $\nu-$fold commutator with $\partial$ for all positive integers $\nu$. In fact, note that if $f \in \mathbb{Z}[[x]]\langle \partial \rangle$, then $[\alpha\delta f, \partial] = \alpha\delta(f\partial) - \partial(\alpha\delta f)$.

Since $v_1 \in M$ we have that if $v_1^{(\mu)}$ is the $\mu-$fold commutator of $v_1$ and $\partial$, where $\mu = val(p_1(x)) < \cdots < val(p_l(x))$, then $v_1^{(\mu)}$ is in $M$ and

$$v_1^{(\mu)} = p_1^{(\mu)}(x)\varepsilon_1 + p_2^{(\mu)}(x)\varepsilon_2 + \cdots + p_l^{(\mu)}(x)\varepsilon_l \in M,$$

where $p^{(\mu)}(x)$ denotes the usual $\mu$-derivative of a power series $p(x)$. Note that $u(x) := p_1^{(\mu)}(x)$ is a unit in $F[[x]]$ and $val(p_j^{(\mu)}(x)) = val(p_j(x)) - \mu > 0$, $\mu = 2, \cdots, l$.

Define $v_2 := v_1 - p_1(x)(u(x))^{-1}v_1^{(\mu)}$. Then $v_2 \in M$ and

$$v_2 = q_2(x)\varepsilon_2 + \cdots + q_l(x)\varepsilon_l,$$
where \( q_j(x) = p_j(x) - p_1(x)u(x)\), \( j = 2, \ldots, l \). A simple calculation shows that \( q_j(x) \) is non-zero with \( \text{val}(q_j(x)) = \text{val}(p_j(x)) \), for all \( j = 2, \ldots, l \). Therefore \( \text{val}(q_2) < \text{val}(q_3) < \cdots < \text{val}(q_l) \).

We now, repeat the previous argument using the commutator of \( v_2 \) and \( \partial \) until we get \( v_3 = r_3(x)\varepsilon_3 + \cdots + r_l(x)\varepsilon_l \in M \). Proceeding in this way we finally get \( v_l(x) = \overline{u}(x)\varepsilon_l \in M \), where \( \overline{u}(x) \) is a unit in \( F[[x]] \). Therefore \( \varepsilon_l \in M \).

If \( l = 1 \), we are done. If \( l > 1 \), since \( v_{l-1} \in M \), we have that \( \varepsilon_{l-1} \in M \). Going backwards we get \( \varepsilon_1 \in M \).

Restricting the attention to the \((m-1)\)-tuple \( \delta_2, \ldots, \delta_m \) and the \( S \)-module \( S^{m-1} = S\varepsilon_2 + \cdots + S\varepsilon_m \), the lemma follows by induction over \( m \).

\[ \square \]

**Lemma 3.3.** Let \( \delta_1, \ldots, \delta_m \) be a set of non zero elements in \( F[[x]]\langle \partial_x \rangle \subset S \) and let \( M \) be a \( S \)-submodule of \( S^m \) such that the \( S \)-module \( S^m/M \) has finite length. If \( 0 \neq \alpha \in S \), there exists some \( f \in \mathbb{Z}[[x]]\langle \partial_x \rangle \) such that

\[ S^m = M + S(\alpha\delta_1 f\varepsilon_1 + \cdots + \alpha\delta_m f\varepsilon_m). \]

**Proof.** See [1], Lemma 8.10, page 28.

\[ \square \]

**Corollary 3.4.** Let \( 0 \neq \rho \in S \) and let \( \delta_1, \ldots, \delta_m \) be a set of non zero elements in \( F[[x]]\langle \partial_x \rangle \subset S \). Then there exists some \( f \in \mathbb{Z}[[x]]\langle \partial_x \rangle \) such that

\[ S^m = S^m \rho + S(\rho\delta_1 f\varepsilon_1 + \cdots + \rho\delta_m f\varepsilon_m) \]

**Proof.** See [1], Corollary 8.11, page 29.

\[ \square \]

**Lemma 3.5.** Let \( \delta_1, \ldots, \delta_m \) be a set of non zero elements in \( F[[x]]\langle \partial_x \rangle \subset S \) and let \( 0 \neq \rho \in S \). Consider the free \( S \)-module \( S^{m+1} = S\varepsilon_0 + S\varepsilon_1 + \cdots + S\varepsilon_m \) with basis \( \varepsilon_0, \ldots, \varepsilon_m \). Then there exists some \( f \in \mathbb{Z}[[x]]\langle \partial_x \rangle \) such that

\[ S^{m+1} = S^{m+1} \rho + S(\varepsilon_0 + \delta_1 f\varepsilon_1 + \cdots + \delta_1 f\varepsilon_m) \]

**Proof.** See [1], Lemma 8.13, page 29.

\[ \square \]

4. **LEMMAS FOR \( D_r \) AND \( R_r \)**

In lemma 4.4 we will need to apply Weierstrass’ preparation theorem to an non-zero element \( v \in D_n = K[[x_1, \ldots, x_n]]\langle \partial_1, \ldots, \partial_n \rangle \). We will then state a separate lemma to prepare this element.

**Lemma 4.1.** Let \( v \in D_n \) be a non zero element. For any \( r, 0 \leq r \leq n-1 \), \( v \) can be written in the following form:

\[ v = \omega_1 \beta_1 G_1 + \cdots + \omega_m \beta_m G_m, \]

where \( \omega_1, \ldots, \omega_m \in K[[x_1, \ldots, x_n]] \) are units , \( \beta_1, \ldots, \beta_m \in K[[x_{r+1}]]\langle \partial_{r+1} \rangle \) and \( G_1, \ldots, G_m \in D(r+1) := K[[x_1, \ldots, x_{r+1}, \ldots, x_n]]\langle \partial_1, \ldots, \partial_{r+1}, \ldots, \partial_n \rangle \).
Proof. Since \( v \in D_n \) is a non zero element, we can write \( v \) as a finite sum 
\[
\sum_{\alpha} p_\alpha(x_1, \ldots, x_n) \partial^\alpha,
\]
where each \( p_\alpha(x_1, \ldots, x_n) \in K[[x_1, \ldots, x_n]] \) and \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \).

Let \( k_\alpha \) be the order of the series \( p_\alpha \). After a suitable linear change of variables we can assume that \( p_\alpha(0, \ldots, 0, x_{r+1}, 0, \ldots, 0) \in K[[x_{r+1}]] \) is non zero for all \( \alpha \) and has valuation \( k_\alpha \) as a series in the variable \( x_{r+1} \). In fact, since \( K \) is an infinite field and the \( p_\alpha \) are in a finite number there is a single linear change of variables that works for all \( p_\alpha \). (See [2], lemma 2, page 17 and the remark that follows this lemma.)

Now let us fixed \( \alpha \). Then the series \( p_\alpha(x_1, \ldots, x_n) \) can be written as 
\[
\sum_{q=0}^\infty q(x_1, \ldots, x_{r+1}^*, \ldots, x_n) x_{r+1}^q,
\]
where \( q(x_1, \ldots, x_{r+1}^*, \ldots, x_n) \in K[[x_1, \ldots, x_{r+1}, \ldots, x_n]] \).

By our assumptions, \( q_k(x_1, \ldots, x_{r+1}, \ldots, x_n) \) is a unit in \( K[[x_1, \ldots, x_{r+1}, \ldots, x_n]] \). By Weierstrass’ preparation theorem (see [4], page 208) we can write
\[
p_\alpha = u(b_0 + b_1 x_{r+1} + \cdots + x_{r+1}^{k_\alpha}),
\]
where \( u \) is a unit in \( K[[x_1, \ldots, x_n]] \) and \( b_j \in K[[x_1, \ldots, x_{r+1}, \ldots, x_n]] \). Therefore we have
\[
p_\alpha \partial^\alpha = u(b_0 + b_1 x_{r+1} + \cdots + x_{r+1}^{k_\alpha}) \partial_1^{\alpha_1} \cdots \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n} = u(\partial_{r+1}^{\alpha_{r+1}})(b_0 \partial_1^{\alpha_1} \cdots \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n}) + u(x_{r+1}^{k_\alpha} \partial_{r+1}^{\alpha_{r+1}})(\partial_1^{\alpha_1} \cdots \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n}) + \cdots + u(x_{r+1}^{k_\alpha} \partial_{r+1}^{\alpha_{r+1}})(\partial_1^{\alpha_1} \cdots \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n}),
\]
which has the desired form. Since \( v = \sum_\alpha p_\alpha \partial^\alpha \) we are done.

The following lemmas 4.3 and 4.4 are equivalent. We will then prove just the second. To prove it we will need the following result.

**Lemma 4.2.** Let \( 0 \leq r \leq n-1 \) and let \( q \) be a non zero element of \( D_r[[x_{r+1}, \ldots, x_n]] \) and let \( a_1, \ldots, a_t \) be a finite set in \( D_n \). Then there exists some \( \rho \in D_r[[x_{r+1}, \ldots, x_n]] \), \( \rho \neq 0 \), such that \( \rho a_j \in D_n q \), for each \( j = 1, \ldots, t \).

**Proof.** See [1], lemma 8.5, page 26.

**Lemma 4.3.** Let \( 0 \leq r \leq n-1 \) and let \( 0 \neq q \in D_{r+1}[x_{r+2}, \ldots, x_n] \). If \( u \) and \( v \) are two elements in \( D_n \) with \( v \neq 0 \), then there exists \( f \in D_n \) and \( Q_r \in D_r[[x_{r+1}, \ldots, x_n]] \) such that
\[
Q_r \in D_n q + D_n(u + v f).
\]

Recall that \( R_r = F_r((x_{r+1}, \ldots, x_n)) \langle \partial_{r+1}, \ldots, \partial_n \rangle \), where \( F_r \) is the quotient ring of \( D_r = K[[x_1, \ldots, x_r]] \langle \partial_1, \ldots, \partial_r \rangle \).

**Lemma 4.4.** Let \( 0 \leq r \leq n-1 \) and let \( 0 \neq q \in D_{r+1}[[x_{r+2}, \ldots, x_n]] \) and let \( u \) and \( v \in D_n \) with \( v \neq 0 \). Then there exist some \( f \in D_n \) such that
\[
R_r = R_rq + R_r(u + v f).
\]

**Proof.** Let us fix \( r \) such that \( 0 \leq r \leq n-1 \). Since \( v \neq 0 \), by lemma 4.1 we can write
\[
v = \omega_1 \beta_1 G_1 + \cdots + \omega_m \beta_m G_m,
\]
where \( \omega_1, \ldots, \omega_m \in K[[x_1, \ldots, x_n]] \) are units, \( \beta_1, \ldots, \beta_m \in K[[x_{r+1}]] \langle \partial_{r+1} \rangle \) and \( G_1, \ldots, G_m \in D(r + 1) = K[[x_1, \ldots, x_{r+1}, \ldots, x_n]] \langle \partial_1, \ldots, \partial_{r+1}, \ldots, \partial_n \rangle \).
Let $\delta_i = \omega_i \beta_i$, for each $1 \leq i \leq m$. Then $\delta_i \in K[[x_1, \ldots, x_n]](\partial_{r+1})$. To apply lemma 3.5, we observe that $K[[x_1, \ldots, x_n]] = (K[[x_1, \ldots, x_{r+1}, \ldots, x_n]])[[x_{r+1}]] \subset F[[x_{r+1}]]$, where $F$ is the quotient field of $K[[x_1, \ldots, x_{r+1}, \ldots, x_n]]$. Therefore $\delta_i \in F[[x_{r+1}]](\partial_{r+}), F$ a field of characteristic zero. If $T = F_r((x_{r+1}, \ldots, x_n))$, then $T$ is a skew field and $F \subset T$.

Since $v \neq 0$, we have that some $G_i \neq 0$. The ring $D(r + 1)$ is simple, which implies the 2-sided ideal generated by $G_1, \ldots, G_m$ is the whole ring $D(r + 1)$. This gives finite sets $a_1, \ldots, a_l$ and $b_1, \ldots, b_l$ in $D(r + 1)$ such that

$$1 = \sum_{j=1}^{m} \sum_{\nu=1}^{l} b_{\nu} G_j a_{\nu}$$

and hence $D(r + 1) = \sum \sum D(r + 1) G_j a_{\nu}$. Identifying $D(r + 1)$ with a subring of $R_r$ we conclude that $R_r^* = \sum \sum R_r G_j a_{\nu}$.

At this stage we need the following

Claim: To each $m$-tuple $B_1, \ldots, B_m$ in $D(r + 1)$ there exists some $f \in Z[[x_{r+1}]](\partial_{r+1})$ such that

$$R_r q + R_r u + R_r B_1 + \cdots + R_r B_m = R_r q + R_r (u + \delta_1 f B_1 + \cdots + \delta_m f B_m).$$

In fact, since $0 \neq q \in D_{r+1}[[x_{r+2}, \ldots, x_n]]$, it follows from lemma 4.2 that there exists some $0 \neq q \in D_{r+1}[[x_{r+2}, \ldots, x_n]]$ such that $\rho B_j \in D_n q$ for all $j = 1, \ldots, m$ and also $\rho u \in D_n q$.

Let $S = T((x_{r+1}))(\partial_{r+1})$, then $0 \neq q \in S$. Using the lemma 3.5 we get some $f \in Z[[x_{r+1}]](\partial_{r+1})$ such that $S^{(m+1)} = S^{(m+1)} \rho + S(\varepsilon_0 + \delta_1 f \varepsilon_1 + \cdots + \delta_m f \varepsilon_m)$. Since $S$ is a subring of $R_r$, we have that $R_r^{(m+1)} = R_r^{(m+1)} \rho + R_r(\varepsilon_0 + \delta_1 f \varepsilon_1 + \cdots + \delta_m f \varepsilon_m)$.

Considerer the $R_r$-linear application $\pi : R_r^{(m+1)} \rightarrow R_r$ defined by $\pi(\varepsilon_0) = u$ and $\pi(\varepsilon_j) = B_j$ for each $1 \leq j \leq m$. Then the image of $\pi$ is $R_r u + R_r B_1 + \cdots + R_r B_m \subseteq R_r$; but $\rho u \in D_n q$. Then we have that $\pi(\rho \varepsilon_0) = \rho \pi(\varepsilon_0) = \rho u \in R_r q$ and $\pi(\rho \varepsilon_j) = \rho \pi(\varepsilon_j) = \rho B_j \in R_r q$ for each $1 \leq j \leq m$. Therefore $\pi(R_r^{(m+1)} \rho) \subseteq R_r q$. Then

$$R_r q + R_r u + R_r B_1 + \cdots + R_r B_m = R_r q + \pi(R_r^{(m+1)} \rho) \subseteq R_r q + \pi(R_r^{(m+1)} \rho) \rho + R_r(\varepsilon_0 + \delta_1 f \varepsilon_1 + \cdots + \delta_m f \varepsilon_m) \subseteq R_r q + \pi(R_r^{(m+1)} \rho) \rho + R_r(u + \delta_1 f B_1 + \cdots + \delta_m f B_m) \subseteq R_r q + R_r(u + \delta_1 f B_1 + \cdots + \delta_m f b_m)$$

and this proves the claim because the opposite inclusion is clear.

Now we apply the claim to $B_j = G_j a_{j}, j = 1, \ldots, m$. Then, there exists $f_1 \in Z[[x_{r+1}]](\partial_{r+1})$ such that

$$R_r q + R_r u + R_r G_1 a_1 + \cdots + R_r G_m a_1 = R_r q + R_r (u + \delta_1 f_1 G_1 a_1 + \cdots + \delta_m f_1 G_m a_1).$$

Since $f_1 \in Z[[x_{r+1}]](\partial_{r+1})$ and $G_j \in D(r + 1)$ commute, we have that

$$\delta_1 f_1 G_1 a_1 + \cdots + \delta_m f_1 G_m a_1 = \delta_1 G_1 f_1 a_1 + \cdots + \delta_m G_m f_1 a_1 = \omega_1 \beta_1 G_1 f_1 a_1 + \cdots + \omega_m \beta_m G_m f_1 a_1 = v f_1 a_1,$$

since $v = \omega_1 \beta_1 G_1 + \cdots + \omega_m \beta_m G_m$. Then,

$$R_r q + R_r u + R_r G_1 a_1 + \cdots + R_r G_m a_1 = R_r q + R_r (u + v f_1 a_1).$$
Now we apply the claim again with $u$ replaced by $u + vf_1a_1$ and $B_j = G_ja_2, j = 1, \ldots, m$. There exists $f_2 \in \mathbb{Z}[[x_{r+1}]] \langle \partial_{r+1} \rangle$ such that $R_\tau q + R_\tau (u + vf_1a_1) + \sum R_\tau G_ja_2 = R_\tau q + R_\tau (u + vf_1a_1 + vf_2a_2)$. Using the previous equation we have

$$R_\tau q + R_\tau u + \sum R_\tau G_ja_1 + \sum R_\tau G_ja_2 = R_\tau q + R_\tau (u + vf_1a_1 + vf_2a_2).$$

In the next step we apply the claim with $B_j = G_ja_3, j = 1, \ldots, m$ and $u$ replaced by $u + vf_1a_1 + vf_2a_2$. After $l$ steps we have

$$R_\tau q + R_\tau (u + vf_1a_1 + \cdots + vf_ia_1) = R_\tau q + R_\tau u + \sum \sum R_\tau G_ja_r = R_\tau.$$

Hence the lemma follows with $f = f_1a_1 + \cdots + f_ia_1$.

5. The Principal Result

**Lemma 5.1.** Let $a, b$ and $c$ non zero elements of $D_n$. For each $0 \leq r \leq n$ there exist $q_r \in D_r[[x_{r+1}, \ldots, x_n]], q_r \neq 0$ and $d_r, e_r \in D_n$ such that

$$q_r c \in D_n(a + d_r c) + D_n(b + e_r c).$$

**Proof.** With $d_n = 0$ and $e_n = 0$ we see that the statement is true for $r = n$, since $D_n$ is a left Ore ideal and $D_n a \subset (D_n a + D_n b)$.

For $0 \leq r \leq (n - 1)$ the proof is by induction from $r + 1$ to $r$. For this see the prove that Proposition 7.3(r+1) ⇒ Proposition 7.3(r) in the book of Björk [1, page 22], in which the lemma 7.5 should be replaced by our lemma 4.3. □

**Theorem 5.2.** Any left or right ideal in $K((x_1, \ldots, x_n)) \langle \partial_1, \ldots, \partial_n \rangle$ can be generated by two elements. The same is true for the ring $\mathbb{C}\{\{x_1, \ldots, x_n\}\} \langle \partial_1, \ldots, \partial_n \rangle$.

**Proof.** The ring $E_n = K((x_1, \ldots, x_n)) \langle \partial_1, \ldots, \partial_n \rangle$ is a noetherian ring. Therefore, it is enough to show that given $a, b, c \in E_n$ there exists $d, e \in E_n$ such that $c \in E_n(a + dc) + E_n(b + ec)$.

Take $n = 0$ in the previous lemma. Then, there exists $q_0 \in K[[x_1, \ldots, x_n]], q_0 \neq 0$ and $d, e \in D_n$ such that $q_0 c \in D_n(a + dc) + D_n(b + ec)$. Since $D_n = K[[x_1, \ldots, x_n]] \langle \partial_1, \ldots, \partial_n \rangle$, then $c \in E_n(a + dc) + E_n(b + ec)$.

□

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