Asymptotic Behavior
in
Polarized $T^2$-symmetric Vacuum Spacetimes

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Abstract

We use the Fuchsian algorithm to study the behavior near the singularity of a class of solutions of Einstein’s vacuum equations. These solutions admit two commuting spacelike Killing fields like the Gowdy spacetimes, but their twist does not vanish. The spacetimes are also polarized in the sense that one of the ‘gravitational degrees of freedom’ is turned off. Examining an analytic family of solutions with the maximum number of arbitrary functions, we find that they are all asymptotically velocity-term dominated as one approaches the singularity.

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I Introduction

There is increasing evidence that cosmological solutions exhibit rather special dynamical behavior in the neighborhood of their singularities. The evidence is still essentially limited to families of solutions with at least one Killing field. However, it is quite striking that although the Hawking-Penrose singularity theorems [1] require nothing more than geodesic incompleteness in generic cosmological solutions, every study to date indicates that the solutions under investigation are either ‘asymptotically velocity-term dominated’ (AVD) or show ‘Mixmaster’ behavior, see [2, 3, 4, 5, 6, 7, 8].

In a space with AVD behavior, the metric tensor $g_{ab}(x, t)$ evolves in such a way that an observer with fixed $x_0$ moving toward the singularity sees the dynamics of $g_{ab}(x_0, t)$ asymptotically approach that of a Kasner spacetime, with there being generally a different Kasner limit for each different $x_0$ (see [9, 3, 5, 7], and references therein). Mixmaster behavior is similar, except that this observer sees $g_{ab}(x_0, t)$ move through an infinite sequence of Kasner epochs, with regular intermittent bounces from one epoch to another (see [10]). Again, different observers generally see different sequences (see for instance [3, 8]). While neither AVD nor Mixmaster behavior as described above is trivial, the Einstein equations, even with the simplification of an assumed symmetry, are sufficiently complicated that the prevalence of these special behaviors is quite remarkable.

The earliest verifications of AVD behavior in a family of inhomogeneous solutions, the polarized Gowdy spacetimes, took the form of a theorem [5, 11]. The techniques developed in proving that result have not, however, been readily extended to more general families. Instead, most of the recent evidence for AVD and Mixmaster behavior in cosmological spacetimes has been based on numerical work: Berger and Moncrief [12] provide strong numerical evidence for AVD behavior in general ($T^3$) Gowdy spacetimes, but find that the Kasner exponents should satisfy some inequalities in generic solutions (the solutions should be ‘low-velocity’); they also have evidence in polarized $U(1)$-symmetric spacetimes [13]. One should note that it is not always easy to be sure, in numerical computations, that the constraint equations do hold, except in the Gowdy class. Note also that Weaver, Berger and Isenberg [8] provide similar evidence that locally $T^2$-symmetric spacetimes with certain magnetic fields have Mixmaster behavior.

This numerical evidence motivates the search for a theoretical explanation for the prevalence of these behaviors and numerical observations such as the distinction between high- and low-velocity solutions, and if possible a means to predict which behavior occurs. The recent work of Kichenassamy and Rendall [7] introduces a new tool for obtaining such information. They use the Fuchsian algorithm to prove that there is a family of general (non-polarized) Gowdy spacetimes parametrized by the maximum number of free functions, namely four, which all exhibit AVD ‘low-velocity’ behavior. If the derivative of one of these functions vanishes, ‘high-velocity’ behavior is allowed. This family of solutions...
includes all of the previously known solutions in this class. The results also shed
new light on other features of the numerical computations.

It is very likely that one can show that these new Gowdy spacetimes are
stable under smooth perturbation of Cauchy data, by adapting the techniques
described in [14, 15]. The general strategy consists in showing, using the Nash-
Moser implicit function theorem, that the free functions which determine the
solutions given by the Fuchsian algorithm can be used to parametrize solutions
much in the same way as one uses Cauchy data on a hypersurface to label regular
solutions. In a sense, one therefore generates systematically an ‘asymptotic phase
space’ for families of solutions, as was called for in [5].

In this work, we show that the Fuchsian algorithm is an effective tool for proving
that AVD behavior occurs in a wider class of spacetimes: those which possess,
like the Gowdy spacetimes, a $T^2$ isometry group with spacelike generators, but
in which, unlike the Gowdy case, the Killing vectors have a non-vanishing twist.
The main new difficulty is that this non-vanishing twist prevents the constraint
equations from decoupling from the evolution equations, resulting in a considerably
more complicated PDE system than what obtains in the case of Gowdy spacetimes [16, 17, 18]. This difficulty is overcome by abandoning the separation
of constraint and evolution equations. It is found that, combining some of the
constraints with some of the ‘evolution’ equations, one can form a system which
is sufficient to determine the metric. One then proves directly that the remaining
constraints hold everywhere if they hold asymptotically at the singularity. This
latter condition can be expressed explicitly in terms of the data which determine
the asymptotics at the singularity, or ‘singularity data’ for short.

The Fuchsian algorithm has been extensively studied and takes a variety of
forms (see [15, 19, 20, 7]). In section II, we briefly review the form of the algorithm
we use here, and a few relevant results we will need. Next, we describe in section
III the $T^2$-symmetric spacetimes, noting some of their properties and defining
the polarized sub-family. Then, in section IV, we propose an AVD Ansatz for
the metric coefficients and show that the ‘regular part’ of the field is indeed
negligible in comparison with the leading terms. Finally, we discuss in section V
our conclusions and plans for future work.

II The Fuchsian algorithm

The Fuchsian algorithm was initially developed to understand the behavior of
solutions of differential equations in the neighborhood of a possible singularity of
unknown location. The rationale was that if singularities are to form, it would be
desirable to figure out by what mechanism they form: which components of the
solution becomes singular? do singularities occur only in higher derivatives? is
the locus of the singularity arbitrary? how does it vary with Cauchy data given
on a surface where the solution is smooth?
Existing results prior to Fuchsian techniques gave some information on the
time of the first singularity, but did not shed light on the mechanism of singularity
formation, except for special classes of singularities, such as shock waves in low
dimensions, or caustic formation.

The questions asked above would be answered if it were possible to establish
an expansion of the solution to relatively high order. To achieve this, one needs
to establish a formal solution, and to prove that this formal expansion does
characterize the solution. In practice, one is not primarily interested in the
convergence or divergence of a series representation. Rather, one would like
to know whether the parameters entering in a formal series representation do
determine uniquely the solution, or whether there are infinitely many solutions
differing from each other by, say, exponentially small corrections.

The Fuchsian method tackles this problem by seeking a reduction of the given
system of PDEs to a Fuchsian system; that is, one which has a regular singular
point with respect to one of the variables, which we call $t$. Using a change of
coordinates if necessary, one may assume that the locus of the singularity is
$t = 0$. It is also possible to set things up so that one always deals with first-
order Fuchsian systems, by the introduction of new variables. This is a familiar
procedure for the Cauchy problem: for instance, if $u$ solves the wave equation in
Minkowski space, it is easy to check that the quantities $(u, \partial_t u)$ satisfy a first-
order system.

Let us consider a PDE system which we write symbolically as
\[ F[u] = 0. \]

The exact form of the nonlinearity is not important for what follows. Generally,
$u$ can have any number of components.

Schematically, the Fuchsian algorithm has three parts:

Step 1. Identify the leading part of the desired expansion for $u$. This can
be done in many cases by seeking a leading balance; that is, a leading term $a(t)$
such that, upon substitution into the equation, the most singular terms cancel
each other.

Step 2. Introduce a renormalized unknown. This means that one writes
\[ u = a(t) + t^s v(t), \]
where $v$ is the new unknown. It is generally useful to compute $a$ to relatively
high order if possible, so that any arbitrary functions in the expansion are already
included in $a$. If $a$ is a solution up to order $n$, one may usually take $s = n + \varepsilon$,
where $\varepsilon$ is small.

Step 3. Obtain and solve a Fuchsian system for $v$. Indeed, one finds under
rather general circumstances that the function $v$ solves an equation of a very
particular form, namely
\[ (t\partial_t + A)v = t^\varepsilon f(t, x^\rho, v, \nabla_\rho v), \]
where $\varepsilon$ is small.
where $\rho$ stands for spatial indices in this formula. The matrix $A$ depends at most on the spatial variables, and $f$ is, say, bounded. By taking $t^\varepsilon$ to be a new time variable, one may always assume that $\varepsilon$ is equal to one. Observe how spatial derivatives are effectively switched off from the equation when $t$ goes to zero: the Fuchsian algorithm provides a systematic procedure to guarantee AVD behavior.

There is a variety of existence results for Fuchsian systems [19, 21, 7]. For our purposes, it suffices to note the following (see [7, 19])

**Theorem 1** There is a unique local solution which is continuous in time and analytic in space, and vanishes as $t$ goes to zero, provided that (a) $f$ is continuous in $t$ and analytic in its other arguments and satisfies an estimate of the form

$$|f(t, x^\rho, v, \nabla_\rho v) - f(t, x^\rho, w, \nabla_\rho w)| \leq C(|v - w| + |\nabla_\rho v - \nabla_\rho w|)$$

for some constant $C$ provided $v$ and $w$ are bounded, and (b) the matrix $\sigma^A (= \exp(A \ln \sigma))$ is uniformly bounded for $0 < \sigma < 1$.

Condition (b) is usually most conveniently checked by simply computing the matrix exponential.

We emphasize that we are not allowed to prescribe arbitrarily the initial value of $v$. The free data (usually called ‘singularity data’) which label the solution $u$ are already built into the choice of $a$ in (1), and are subsequently incorporated into the function $f$ in (2). A straightforward extension of the theorem can be made if we assume only that $v(0)$ belongs to the null-space of $A$. By considering the equation satisfied by $v - v(0)$, one can reduce the problem to an equation to which the theorem applies. In such a case, $v(0)$ must be added to the list of singularity data.

General strategies for carrying out the algorithm can be found in [15, 20, 19, 7], with applications to several examples. Let us simply describe here what these steps entail for the first PDE to which these ideas were applied successfully:

$$\eta^{ab} \partial_{ab} u = e^u \quad (3)$$

in Minkowski space, where $u$ is a scalar field (there are similar results for power nonlinearities as well). Let $t = \psi(x)$ be the locus of the (yet unknown) singularity, and let $x$ stand for the spatial variables. For one space dimension, equation (3) has a closed-form solution (“Liouville field theory”); however, we allow here the number of space dimensions to be arbitrary. Let $T = t - \psi(x)$. If we choose the leading part of $u$ so that $\exp(u) \sim \varphi(x) T^s$ where $s$ and $\varphi$ are unknown, one readily finds that, to eliminate the most singular term in the expansion of (3), we need to choose $s = -2$ and $\varphi = 2(1 - |\nabla_x \psi|^2)$ which must therefore be positive. Hence the leading part of $u$ takes the form

$$u \approx \ln \frac{2}{T^2} + \ln(1 - |\nabla_x \psi|^2).$$
This completes the first step.

It is useful to write out the rest of the leading part \( a(t) \) of \( u \) up to order two in \( T \) for two reasons: (a) this reveals that the solution contains logarithmic terms, which disappear in fact only if the scalar curvature of the singularity manifold vanishes identically; (b) this shows that the coefficient of \( T^2 \) in the expansion is arbitrary. We therefore compute, by direct substitution

\[
  u \approx \ln \frac{2}{T^2} + u_0(x) + u_1(x)T + u_{1,1}(x)T^2 \ln T + u_2(x)T^2 + \ldots ,
\]

where \( u_0, u_1 \) and \( u_{1,1} \) are entirely determined by \( \psi \); in particular, \( u_0 = \ln(1 - |\nabla_x \psi|^2) \). However, \( u_2 \) remains arbitrary. One then sets

\[
  u = \ln \frac{2}{T^2} + u_0(x) + u_1(x)T + u_{1,1}(x)T^2 \ln T + vT^2
\]

so that the arbitrary function \( u_2 \) appears as an ‘initial value’ for the renormalized unknown \( v \). This completes the second step.

The singularity data in this case are \( \psi \) and \( u_2 = v(0) \). Once they are known, the formal solution is completely determined.

For the third step, we now substitute expression (4) for \( u \) into (3), and find that \( v \) solves an equation which can be thought of as a non-linear perturbation of the Euler-Poisson-Darboux equation. One then checks that \( v, Tv_T \) and \( T\nabla_\rho v \) solve a Fuchsian system. This has the following consequences:

(a) There is a formal solution to all orders, in powers of \( T \) and \( T \ln T \); for \( T < 0 \), one replaces \( T \ln T \) by \( T \ln |T| \). The series are convergent if \( \psi \) and \( u_2 \) are analytic; otherwise, they are valid as far as the differentiability of the free functions allows. As already mentioned, the logarithmic terms cannot be dispensed with, except for very special solutions. The example of Gowdy spacetimes shows that one should even allow terms such as \( T^k(x) \) for the application to Einstein’s equations.

(b) The solution is uniquely determined by the ‘singularity data’ \( \{ \psi, u_2 \} \). In fact, the correspondence between these data and the Cauchy data on a hypersurface where the solution is regular can be inverted, using the Nash-Moser version of the inverse function theorem, see \[14\]. From a practical viewpoint, this means that (i) the smoother the Cauchy data, the smoother the singularity surface; (ii) there is a recipe for computing how the singularity data vary when the Cauchy data vary. The question in the opposite direction is simpler: the series representation gives explicitly the solution in terms of the singularity data.

All this makes the series representations generated by Fuchsian techniques as reliable as an exact solution in the vicinity of the singularity—a place where at present no other representation is available.

It should be stressed that the Fuchsian method applies without symmetry or integrability restrictions. For this reason, it enables one to study directly the stability of solutions furnished by generation techniques, even under fully...
inhomogeneous’ or ‘asymmetric’ perturbations, and the information it provides yields concrete analytical insight into the properties of solutions.

III Polarized $T^2$-symmetric spacetimes

While the Gowdy $T^3$ spacetimes [22] have been extensively studied over the years [5, 16, 6, 12, 7], and are relatively well-understood, the more general $T^2$-symmetric spacetimes have only recently begun to be considered [16, 17, 18]. The technical condition which distinguishes the Gowdy sub-family is the requirement that the Killing fields $X$ and $Y$ which generate the isometry group have vanishing twist constants $\kappa_x := \varepsilon^{abcd} X_a Y_b \nabla_c X_d$ and $\kappa_y := \varepsilon^{abcd} X_a Y_b \nabla_c Y_d$, where $\varepsilon^{abcd}$ is the Levi-Civita tensor. The essential difference in practice is that, if one chooses the constant orbit area time-foliation (“Gowdy time”) [22], the constraint equations decouple from the evolution equations in the Gowdy case, and can therefore more or less be ignored in the analysis. If however either $\kappa_x$ or $\kappa_y$ is nonzero, then no such decoupling occurs.

The general form of the metric and the field equations for the $T^2$-symmetric spacetimes is presented in [17], along with a proof that the Gowdy time always exists globally for these spacetimes. To write the metric, we assume that all metric components depend on two coordinates, the Gowdy time $t$ and spatial coordinate $\theta \in S^1$ with $\partial/\partial x$ and $\partial/\partial y$ generating the $T^2$ isometry. By choosing $X$ and $Y$ to be suitable linear combinations of the generators, we may always assume without loss of generality that $\kappa_x = 0$. We then drop the subscript from $\kappa_y$. We now focus our attention on the sub-class of polarized spacetimes, which have $A \equiv 0$ in the notation of [17]. The metric takes the form

$$ds^2 = e^{2(\nu-u)}(-\alpha dt^2 + d\theta^2) + \lambda e^{2u}(dx + G_1 d\theta + M_1 dt)^2 + \lambda e^{-2u} t^2(dy + G_2 d\theta + M_2 dt)^2$$

(5)

where $\lambda$ is a positive constant and the functions $u, \nu, \alpha, G_1, M_1, G_2, M_2$ depend on $t$ and $\theta$. The vacuum field equations take the form, writing $u_t$ for $\frac{\partial u}{\partial t}$ etc.,
\[ D = t \partial_t, \quad D^2 = t^2 \partial_t^2 + t \partial_t, \text{ and } m = \lambda \kappa^2, \]

\[ D^2 u - t^2 \alpha u_{\theta \theta} = \frac{1}{2\alpha} D\alpha Du + \frac{t^2}{2} \alpha \theta u_\theta \quad (6a) \]

\[ D\alpha = -\frac{\alpha^2}{t^2} me^{2\nu} \quad (6b) \]

\[ D\nu = (Du)^2 + t^2 \alpha u_\theta^2 + \frac{\alpha}{4t^2} me^{2\nu} \quad (6c) \]

\[ \partial_\theta \nu = 2u_\theta Du - \frac{\alpha \theta}{2\alpha} \quad (6d) \]

\[ G_{1,t} = M_{1,\theta} \quad (6e) \]

\[ G_{2,t} = M_{2,\theta} + \frac{\kappa \alpha^{1/2}}{t^3} e^{2\nu} \quad (6f) \]

\[ \kappa_t = 0 \quad (6g) \]

\[ \kappa_\theta = 0 \quad (6h) \]

Note that the Gowdy case is recovered if \( \kappa = 0, \alpha = 1, \) and \( G_1 = G_2 = M_1 = M_2 = 0. \) Since \( G_{1,t} = M_{1,\theta}, \) \( G_1 d\theta + M_1 dt \) is locally an exact differential \( d\phi. \) Replacing \( x \) by \( x + \varphi, \) we may assume locally that \( G_1 = M_1 = 0. \) Similarly, one can set \( M_2 = 0 \) by redefining \( y. \) Since these reductions are only local and may be incompatible with global requirements, we do not consider them further, even though they do make the geometric ‘degrees of freedom’ more clear.

Equations (6) constitute an initial-value problem for the polarized spacetimes, in which the equations (6a-d) decouple from the rest. They form an independent system for \( \{u, \alpha, \nu\}. \) Once these three functions are known, the other equations can be solved easily.

We note that equations (6b-d) in particular—three of the four equations which constitute the heart of the Cauchy problem for these spacetimes—actually derive from the constraint equations of Einstein’s theory. Unlike the Gowdy case, the wave equation (6a) does not decouple from the constraints, since it contains the function \( \alpha. \) We therefore take (6a-d) as our basic equations, treating (6a-c) as evolution equations, and (6d) as the only effective constraint.

The local well-posedness of the initial-value problem away from the singularity at \( t = 0 \) is not quite straightforward, for we must prove that equation (6d) propagates. This is not an immediate consequence of standard results because we are not using any of the standard set-ups for the initial-value problem. It nevertheless does hold, and this can be ascertained in two ways.

One approach is as follows (this is basically the argument used by [17, 16]): if we choose \( \{u, u_t, \alpha, \nu, \ldots\} \) at some initial time \( t_0 > 0 \) so that they satisfy the constraint (6d), then we can view these as an initial data set for the Einstein equations without any symmetry and construct a local solution in the standard way. One then uses the results of [16] to introduce coordinates in this region so that the metric takes the form (5).
We can also give a direct argument, which will be useful later. We first deal with the analytic case, which is all we need for the results of section IV. In view of its independent interest, we show in the appendix how to deal with the non-analytic Cauchy problem as well.

Away from $t = 0$, the PDE system (6a-c) is of Cauchy-Kowalewska type. More precisely, we can reduce it to the following first-order system for $(z_0, z_1, z_2, \alpha, \nu) := (u, u_t, u_\theta, \alpha, \nu)$:

\[
\begin{align*}
\partial_t z_0 &= z_1 \\
\partial_t z_1 &= \alpha \partial_\theta z_2 - \frac{z_1}{t} - \frac{m}{2t^3} z_1 e^{2\nu} + \frac{1}{2} z_2 \alpha \theta \\
\partial_t z_2 &= \partial_\theta z_1 \\
\partial_t \alpha &= -\frac{\alpha^2}{t^3} m e^{2\nu} \\
\partial_t \nu &= t z_1^2 + t \alpha z_2^2 + \frac{\alpha}{4t^3} m e^{2\nu}.
\end{align*}
\]

In particular, ignoring the constraint (6d), we obtain a unique solution of the remaining equations by prescribing the data \{u, u_t, \alpha, \nu, \ldots\} for $t = t_0$. Now let us set

\[ N := \nu_\theta - 2u_\theta Du + \frac{\alpha_\theta}{2\alpha}. \]  

(7)

Calculating

\[ 0 = D\nu_\theta - \partial_\theta D\nu = DN + D(2u_\theta Du - \frac{\alpha_\theta}{2\alpha}) - \partial_\theta D\nu, \]

we find, using (6a-c),

\[ DN - \frac{1}{2\alpha} ND\alpha = 0. \]  

(8)

This is a linear ordinary differential equation for $N$ (there are no $\theta$-derivatives). Hence if we choose data \{u, u_t, \alpha, \nu, \ldots\} for $t = t_0$ so that $N(t_0) = 0$, the uniqueness theorem for ODEs guarantees that $N$ is identically zero for all time.

We therefore have proved the well-posedness of the initial-value problem. The results of [17] ensure that the solution remains bounded for $t > \rho$, where $\rho \geq 0$ is independent of $\theta$. It is expected that $\rho > 0$ in special cases only, such as exact Kasner spacetimes [24]. We are interested in asymptotics near $t = 0$. Note that Fuchsian techniques may be useful for analyzing singularities for $t$ near $\rho > 0$; however, if these solutions are non-generic in some reasonable sense, they should not contain the full number of free parameters, and they may be non-polarized as well. It does appear that there are consistent asymptotics of the form $u \approx u_0$, $\nu \approx \frac{1}{2} \ln(t - \rho) + \nu_0(\theta)$ and $\alpha \approx \alpha_0(\theta)(t - \rho)^{-2}$. 

9
As far as the number of free functions in the metric is concerned, one might expect that there will only be two, since one of the gravitational degrees of freedom has been turned off. Indeed, while the initial data for (6a–c) consist of four functions \{u, u_t, \alpha, \nu, \ldots\}, they are constrained by one relation, \textit{viz.} (6d), and, if we set aside the choice of the initial value for the lapse function \(\alpha\), we obtain two arbitrary functions in the solution.

Similarly, we will obtain a family of \textit{singular} solutions of (6a–c) depending on four arbitrary functions occurring in its singular expansion, and will show that if these ‘singularity data’ are constrained by one relation, the constraint (6d) holds for all time as well.

**IV Application of the Fuchsian algorithm**

We are interested in generating solutions to (6) which have controlled asymptotics near \(t = 0\) and which are parametrized by as many arbitrary singularity data as possible. We achieve this by following the program outlined in section II.

**Step 1. Leading-order asymptotics.**

Since we expect Kasner-like behavior at the singularity, and since \(u\) and \(\nu\) appear in the metric exponentially, we choose logarithmic leading terms for \(u\) and \(\nu\):

\[
\begin{align*}
  u &\approx k(\theta) \ln t + u_0(\theta) + \ldots; \\
  \nu &\approx (1 + \sigma(\theta)) \ln t + \nu_0(\theta) + \ldots; \\
  \alpha &\approx \alpha_0(\theta) + \ldots.
\end{align*}
\]

For equation (6b) to hold at leading order, it is sufficient that \(\sigma > 0\). For (6c) to hold at leading order, one needs \(D\nu\) and \((Du)^2\) to balance each other, which requires that

\[
k^2 = 1 + \sigma
\]

which we assume from now on. The function \(\alpha_0\) should be taken to be positive, to ensure the metric has the correct signature.

Note that there are four free functions, namely \((k, u_0, \alpha_0, \nu_0)\), in these leading term expansions, just as there were four Cauchy data in the discussion of section III. These four free functions are the singularity data for this system. They are \(2\pi\)-periodic; furthermore, \(\alpha_0\) and \(\sigma = k^2 - 1\) are positive.

These asymptotics may be compared with those of the solutions obtained in the Gowdy case in [7]. If \(k_G\) denotes the parameter called \(k\) in [7], the correspondence is: \(\pm k_G = 2k - 1\). This means that the solutions we obtain here, with \(k^2 > 1\), are similar to the “high-velocity” Gowdy solutions, for which \(k_G > 1\).

The asymptotics (9a–c) are not compatible with equations (6) if \(0 < k < 1\), unless \(m = 0\), which is the Gowdy case. Indeed, (6b) implies that \(\alpha\) is of the order \(t^{2\sigma}\),
which is singular if $\sigma = k^2 - 1$ is negative. This makes the term $D\alpha Du/(2\alpha)$ in (6a) more singular than all the other terms in this equation, so that (6a) cannot hold. There are two ways to circumvent this: (1) take $k = 0$, so that $Du$ vanishes to leading order, giving a consistent balance, at the expense of losing the freedom to vary $k$; (2) add terms to the field equations which would compensate the most singular term in (6a)—which is possible by going over to the non-polarized field equations. These possibilities will be addressed when we deal with non-polarized spacetimes, in a forthcoming paper.

**Step 2. Renormalized unknown.**

We now introduce new unknowns which will provide an exact form for the remainders indicated with ‘...’ in (9a-c). Because of the $e^{2\nu}$ term, we see that it is not possible to assume that the remainder terms are of order $t$. We do expect them to be of order $t^{\varepsilon}$ if $\varepsilon$ is small compared to the minimum of $\sigma$. We therefore define the renormalized unknowns $(v, \mu, \beta)$ by

\begin{align*}
u(\theta, t) &= k^2(\theta) \ln t + \nu_0(\theta) + t^{\varepsilon} \mu(\theta, t); \\
\alpha(\theta, t) &= \alpha_0 + t^{\varepsilon} \beta(\theta, t).
\end{align*}

**Step 3. Fuchsian system.**

We shall now show that the renormalized field variables solve a Fuchsian problem. Consequently, once the functions $(k, u_0, \alpha_0, \nu_0)$ have been specified, and $\varepsilon$ has been chosen small enough, the unknowns $v$, $\mu$ and $\beta$ are uniquely determined via theorem 1.

To achieve this, let us first, since we are looking for a first-order system, introduce first-order derivatives of $v$ as new unknowns. This suggests letting

\[
\bar{v} = (v_1, v_2, v_3, v_4, v_5) := (v, Dv, t^{\varepsilon} v_\theta, \beta, \mu).
\]

Let us also introduce the abbreviation $E = m \exp(2\nu_0 + 2t^{\varepsilon} \mu)$. It is helpful to remove the $t$-derivatives of $\alpha$ in the right-hand side of (6a) by using:

\[
\frac{D\alpha}{\alpha} = -at^{2\sigma(\theta)} E,
\]

which follows from (6b) and (11c). We then find the following evolution equations
for $\vec{v}$:

\[
Dv_1 = v_2; \\
Dv_2 + 2\varepsilon v_2 + \varepsilon^2 v_1 = t^{-\varepsilon} (\alpha_0 + t^\varepsilon \beta)(k_{\theta\theta} \ln t + u_{0,\theta\theta} + v_{3,\theta}) - \frac{1}{2} E t^{2\varepsilon - \varepsilon}(k + t^\varepsilon (v_2 + \varepsilon v_1)) + \frac{1}{2} t^{2\varepsilon - \varepsilon} (\alpha_0 + t^\varepsilon \beta)(k_{\theta} \ln t + u_{0,\theta} + v_3); \\
Dv_3 = t^\varepsilon \partial_\theta (\varepsilon v_1 + v_2); \\
(D + \varepsilon)v_4 = -t^{2\varepsilon - \varepsilon}(\alpha_0 + t^\varepsilon \beta)^2 E; \\
(D + \varepsilon)v_5 = 2k(v_2 + \varepsilon v_1) + t^\varepsilon (v_2 + \varepsilon v_1)^2 + \frac{1}{2} E t^{2\varepsilon - \varepsilon}(\alpha_0 + t^\varepsilon \beta) + \alpha t^{2\varepsilon - \varepsilon}(k_{\theta} \ln t + u_{0,\theta} + v_3)^2.
\]

This system has the general form

\[
(D + A)\vec{v} = t^\varepsilon \vec{f}(t, x, \vec{v}, \partial_\theta \vec{v}),
\]

where

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
\varepsilon^2 & 2\varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \varepsilon \\
-2k\varepsilon & -2k & 0 & 0 & \varepsilon
\end{pmatrix},
\]

and $\vec{f}$ is a five-component object containing all the terms in the system that are not already included in the right-hand side.

By taking $\varepsilon$ small (less than the smaller of $1$ and any possible value of $\sigma$), we can ensure that $\vec{f}$ is continuous in $t$ and analytic in all the remaining variables. Since the eigenvalues of $A$ are $\varepsilon$ and $0$, of multiplicities four and one respectively, we conclude that the boundedness condition of Theorem 1 holds. Explicitly, we have $P^{-1}AP = A_0$, hence $\sigma^A = P\sigma^{A_0}P^{-1}$, where

\[
A_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & 1 & 0 \\
0 & 0 & \varepsilon & 0 & 0 \\
0 & 0 & 0 & \varepsilon & 2k \\
0 & 0 & 0 & 0 & \varepsilon
\end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & -\varepsilon & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

so that

\[
\sigma^{A_0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \sigma^\varepsilon & 0 & 0 & \sigma^\varepsilon \ln \sigma \\
0 & 0 & \sigma^\varepsilon & 0 & 0 \\
0 & 0 & 0 & \sigma^\varepsilon & 2k\sigma^\varepsilon \ln \sigma \\
0 & 0 & 0 & 0 & \sigma^\varepsilon
\end{pmatrix}.
\]
We conclude from Theorem 1 that there is a unique solution of the Fuchsian system (13) which vanishes as \( t \) tends to zero, and which is analytic in \( \theta \) and continuous in time. We note in particular that if we construct \( u, \nu \) and \( \alpha \) from (11a-c) with \( v = v_1, \mu = v_5, \) and \( \beta = v_4, \) then \( (u, \nu, \alpha) \) is a solution of equations (6a-c). To verify this, we note that equations (13a-c) imply that

\[
D(v_3 - t^\varepsilon v_{1,\theta}) = 0,
\]

so that any solution which tends to zero with \( t \) has also the property that \( v_2 = tv_{1,t} \) and \( v_3 = t^\varepsilon v_{1,\theta}. \)

We now wish to show that, by imposing a constraint on the singularity data \( (k, u_0, \alpha_0, \nu_0) \), we can guarantee that the solution \( (u, \nu, \alpha) \) of (6a-c) obtained by solving the Fuchsian system (13) will satisfy the constraint (6d) as well, in order to obtain genuine solutions of Einstein’s vacuum equations. We achieve this using (8), which in turn has been derived using only (6a-c).

First of all, since \( \vec{f} \) is bounded, we know that \( (D + A)\vec{v} \) is actually \( O(t^\varepsilon) \), which implies in particular that \( \alpha \) and \( D\alpha \) are of order one and \( t^\varepsilon \) respectively. In particular, \( D\alpha/\alpha = t\alpha_t/\alpha = O(t^\varepsilon) \). This means, using (8), that

\[
\frac{\partial_t N}{N} = \frac{\alpha_t}{2\alpha} = O(t^{\varepsilon - 1})
\]

which is integrable up to \( t = 0 \). (One could also have estimated \( D\alpha/\alpha \) directly from (12).) Letting \( z(t, \theta) \) be the integral of this function from 0 to \( t \), we find that

\[
N(t, \theta) \propto \exp z(t, \theta).
\]

Thus, if we can choose the data so that \( N \to 0 \) as \( t \to 0 \) for fixed \( \theta \), we will know that \( N \) is in fact identically zero, and therefore that the constraint is satisfied. Now

\[
N = \nu_\theta - 2u_\theta Du + \frac{\alpha_\theta}{2\alpha} = \nu_{0,\theta} - 2ku_{0,\theta} + \frac{\alpha_{0,\theta}}{2\alpha_0} + o(1),
\]

where \( o(1) \) is some expression which tends to zero with \( t \). We conclude that the constraint is satisfied if and only if the singularity data satisfy:

\[
\nu_{0,\theta} - 2ku_{0,\theta} + \frac{\alpha_{0,\theta}}{2\alpha_0} = 0. \tag{14}
\]

Note also that all the considerations in this paper are in fact local in \( \theta \), and therefore allow in principle for other spatial topologies.

To summarize, we have proved the following result:
Theorem 2 For any choice of the singularity data \(k(\theta), u_0(\theta), v_0(\theta)\) and \(\alpha_0(\theta)\), subject to condition (14), the \(T^2\)-symmetric vacuum Einstein equations have a solution of the form (11) where \(\beta, v\) and \(\nu\) are bounded near \(t = 0\). It is unique once the twist constant \(\kappa\) has been fixed, except for the freedom in the functions \(G_1, G_2, M_1\) and \(M_2\). Each of these solutions generates spacetimes with AVD asymptotics.

V Concluding remarks

We have therefore obtained a family of singular \(T^2\)-symmetric spacetimes with precise asymptotics at the singularity, which is of AVD type, and which depends on the maximum number of singularity data, that is, as many singularity data as there are Cauchy data for solutions away from the singularity. Fuchsian techniques therefore apply even if the constraints do not decouple from the ‘evolution’ equations as in the Gowdy case.

We may also note the following.

First, it is likely that, as in the case of scalar fields, these singular solutions are stable in a Sobolev topology, by application of the Nash-Moser theorem, in which case these solutions form an open set in the space of all solutions. This means that this type of AVD behavior is **stable** in this class, and is therefore not a special feature of some closed-form solution.

Second, the polarized U(1)-symmetric solutions are believed to be AVD as well [13], and work is underway to address this class by Fuchsian methods.

Third, it appears that the general (nonpolarized) \(T^2\)-symmetric spacetimes may show Mixmaster behavior [23]. Numerical and analytical work to explore this possibility is being carried out.

Appendix

In this appendix, we consider the non-analytic version of the initial-value problem for \(T^2\)-symmetric spacetimes. The strategy is as follows: We first promote \(\alpha_\theta\) to a new field variable \(\zeta := \alpha_\theta\), and produce an evolution equation for \(\zeta\) by differentiating (6b) with respect to \(\theta\). We then use (6b) to eliminate \(D\alpha\) from (6a), and equation (6d) to express \(\partial_\theta \nu\) in terms of the other field variables. This gives us a symmetric-hyperbolic system (15) for \((z_0, z_1, z_2, \alpha, \zeta, \nu)\). Standard theorems then ensure that (15) admits a unique solution, defined in a small time interval, for non-analytic, but sufficiently smooth, initial data. We then show that the constraints \(\zeta = \alpha_\theta\) and \(N = 0\) do propagate, by a variant of the argument used for the propagation of the constraint \(N = 0\). This will establish that we do obtain solutions to (6a-d) non-analytic initial data.

We proceed with the details of this argument. The symmetric-hyperbolic
system is:

\[
\begin{align*}
\partial_t z_0 &= z_1 \\
\partial_t z_1 &= \alpha \partial_\theta z_2 - \frac{z_1}{t} - \frac{m}{2t^3}z_1 e^{2\nu} + \frac{1}{2} z_2 \zeta \\
\alpha \partial_t z_2 &= \alpha \partial_\theta z_1 \\
\partial_t \alpha &= -\frac{\alpha^2}{t^3} me^{2\nu} \\
\partial_t \nu &= t z_1^2 + t \alpha z_2^2 + \frac{\alpha}{4t^3} me^{2\nu} \\
\partial_t \zeta &= -\frac{2m\alpha}{t^3} e^{2\nu} [\zeta + \alpha (2t_z z_2 - \frac{\zeta}{2\alpha})].
\end{align*}
\] (15a-f)

One verifies by inspection that this system is symmetric-hyperbolic, so that if we prescribe sufficiently smooth initial data \(\{u, u_t, \alpha, \zeta, \nu\}\) for \(t = t_0\), we obtain a unique solution. The first and third equations ensure respectively that \(z_1 = \partial_t z_0\) and \(\partial_t (z_2 - \partial_\theta z_0) = 0\); we may thus set \(z_0 = u, z_1 = u_t\) and \(z_2 = u_\theta\). Equations (6a-c) therefore hold, with \(\alpha \theta\) replaced by \(\zeta\) in (6a).

Now, let us set

\[
R := \zeta - \alpha \theta \quad \text{and} \quad N' := \nu - 2u_\theta Du + \frac{\zeta}{2\alpha}.
\] (16)

We proceed to derive a first-order system of ODEs for \(R\) and \(N'\). For the rest of this section, we write \(N\) for \(N'\), for convenience.

First of all, using equations (15d) and (15f),

\[
DR = D(\zeta - \alpha \theta)
\]

\[
= -\frac{2m\alpha}{t^2} e^{2\nu} [\zeta + \alpha (2u_\theta Du - \frac{\zeta}{2\alpha})] - \partial_\theta (-\frac{\alpha^2}{t^2} me^{2\nu})
\]

\[
= -\frac{2m\alpha}{t^2} e^{2\nu} [R - \alpha N]
\]

\[
= 2 \frac{D\alpha}{\alpha} [R - \alpha N].
\]

(17)

Using the expression for \(N\) from (16), taking the relation \(\zeta = \alpha \theta + R\) into account, we have

\[
DN = (D\nu)_\theta - 2Du D u_\theta - 2u_\theta D^2 u + D(\frac{\alpha \theta + R}{2\alpha}),
\]

or

\[
DN - D(\frac{R}{2\alpha}) = (D\nu)_\theta - 2Du D u_\theta - 2u_\theta D^2 u + D(\frac{\alpha \theta}{2\alpha}).
\]

Then, from (15a,b,d) and the definition of \(R\), we find

\[
DN - D(\frac{R}{2\alpha}) = \partial_\theta (-\frac{D\alpha}{4\alpha}) - \frac{D\alpha}{\alpha} u_\theta Du - t^2 u_\theta^2 R + D(\frac{\alpha \theta}{2\alpha})
\]

\[
= -t^2 u_\theta^2 R + \partial_\theta (\frac{D\alpha}{4\alpha}) - \frac{D\alpha}{2\alpha} (2u_\theta Du).
\]
Since, from (6b), one has

\[ \frac{D\alpha}{4\alpha} = \frac{D\alpha}{2\alpha}(\nu_\theta + \frac{\alpha_\theta}{2\alpha}), \]

it follows that

\[ DN - D\left(\frac{R}{2\alpha}\right) + t^2u_\theta^2R = \frac{D\alpha}{2\alpha}(N - \frac{R}{2\alpha}). \] (18)

Thus, combining (17) and (18), we have

\[ DN = N\frac{D\alpha}{2\alpha} + R(D\left(\frac{1}{2\alpha}\right) - t^2u_\theta^2) - \frac{RD\alpha}{4\alpha^2} + \frac{D\alpha}{\alpha^2}[R - \alpha N] \]

\[ = R\left[\frac{D\alpha}{\alpha^2}(\frac{1}{2} - \frac{1}{4} + 1) - t^2u_\theta^2\right] - N\frac{D\alpha}{2\alpha} \]

\[ = R\left[\frac{D\alpha}{4\alpha^2} - t^2u_\theta^2\right] - N\frac{D\alpha}{2\alpha}. \] (19)

Equations (17) and (19) constitute a linear, homogeneous system of ODEs for \( R \) and \( N \). Therefore, if the initial data are such that these quantities are zero for \( t = t_0 \), they remain so for all time, QED.

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[24] A Kasner spacetime has a three-dimensional isometry group $T^3$. If we wish to study a Kasner solution as a $T^2$-symmetric spacetime, we must choose a two-dimensional subgroup of $T^3$. For certain choices of this subgroup, the spatial volume of the $T^3$ orbits may go to zero before the spatial area of the chosen subgroup orbit does. Hence, evolving backward in time, the spacetime stops before the Gowdy time $t$, which is proportional to the area of the $T^2$-orbits, hits zero.