ON A TVERBERG GRAPH

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Abstract. For a graph whose vertex set is a finite set of points in \( \mathbb{R}^d \), consider the closed (open) balls with diameters induced by its edges. The graph is called a (an open) Tverberg graph if these closed (open) balls intersect. Using the idea of halving lines, we show that (i) for any finite set of points in the plane, there exists a Hamiltonian cycle that is a Tverberg graph; (ii) for any \( n \) red and \( n \) blue points in the plane, there exists a perfect red-blue matching that is a Tverberg graph. Also, we prove that (iii) for any even set of points in \( \mathbb{R}^d \), there exists a perfect matching that is an open Tverberg graph; (iv) for any \( n \) red and \( n \) blue points in \( \mathbb{R}^d \), there exists a perfect red-blue matching that is a Tverberg graph.

1. Introduction

Tverberg’s Theorem is an essential result of modern Discrete and Convex Geometry proved in 1966 by Helge Tverberg [12]. It claims that for any set of \( (r-1)(d+1)+1 \) points in \( \mathbb{R}^d \), there exists a partition of points into \( r \) parts whose convex hulls intersect.

In the current paper, we consider a variation of Tverberg’s problem introduced recently in [3,8,11]. For two points \( x, y \in \mathbb{R}^d \), we denote by \( D(xy) \) the closed ball for which the segment \( xy \) is its diameter. Let \( G \) be a graph whose vertex set is a finite set of points in \( \mathbb{R}^d \). We say that \( G \) is a Tverberg graph if

\[
\bigcap_{xy \in E(G)} D(xy) \neq \emptyset.
\]

Replacing closed balls by open balls in the definition of Tverberg graph, we define an open Tverberg graph. A graph whose vertices are points in \( \mathbb{R}^d \) is called an open Tverberg graph if the open balls with diameters induced by its edges intersect. For the sake of brevity, a perfect matching for an even set of points in \( \mathbb{R}^d \) is called a (an open) Tverberg matching if it is a (an open) Tverberg graph. Analogously, a Hamiltonian cycle for a set of points in \( \mathbb{R}^d \) is called a Tverberg cycle if it is a Tverberg graph.

In 2019, Huemer, Pérez-Lantero, Seara, and Silveira [8] showed that for any \( n \) red points and any \( n \) blue points in the plane, there is a red-blue Tverberg matching (every edge of this Tverberg matching connects a red vertex with a blue one). This result can be considered as a colorful variation of the problem. Later, Bereg, Chacón-Rivera, Flores-Peñaloza, Huemer, and Pérez-Lantero [3] found a second proof of the monochromatic version of this result, that is, for any \( 2n \) points in the plane, there is a Tverberg matching. Recently, Soberón and Tang [11] showed the existence of a Tverberg cycle for an odd set of points in the plane. It seems that the mentioned authors overlooked Lemma 2 from the paper [6] of Dumitrescu, Pach, and Tóth. This lemma easily implies the following

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strengthening of the result from [3]: For any $2n$ distinct points in the plane, there exists a perfect matching $\mathcal{M}$ and a point $z$ in the plane such that either $\angle xzy \geq 2\pi/3$ or $z \in xy$ for all $xy \in \mathcal{M}$.

The goal of the current paper is to show a variety of methods that can be useful in proving the existence of Tverberg cycles or matchings for point sets.

Using the idea of halving lines [9], we give short proofs of the following results.

**Theorem 1.** For any finite set of points in the plane, there exists a Tverberg cycle.

**Theorem 2.** For any set of $n$ red points and $n$ blue points in the plane, there exists a Tverberg red-blue matching.

Remark that Theorem 1 is a refinement of the main result from [11]: Our approach also works for even sets of points in the plane. Theorem 2 is the main result from [8] and we give a new proof which seems to be simpler than the original one.

Our main results is higher-dimensional generalizations of the main theorem from [3] and [8].

**Theorem 3.** For any even set of distinct points in $\mathbb{R}^d$, there exists an open Tverberg matching.

**Theorem 4.** For $n$ red points and $n$ blue points in $\mathbb{R}^d$, there exists a red-blue Tverberg matching.

Remark that Theorem 2 is a special case of Theorem 4. However, we decided to leave the proofs of both theorems because they are absolutely different. It is worth to mention that Theorems 2 and 4 are in a sense tight: Consider the vertices of a square which are colored alternatively in red and blue; for any red-blue matching the intersection of the induced balls is a point.

Our proofs of Theorems 3 and 4 are based on the method of infinite descent. Note that the both proofs of Tverberg’s Theorem by Tverberg and Vrečica [13] and by Roudeff [10] give a good illustration of this method. Recalling that Tverberg’s Theorem has a colorful variation (unfortunately, proved only in the special cases; see [1,4,5]), we can also interpret Theorems 3 and 4 as a Tverberg-type theorem and its colorful version, respectively. At last, remark that the combinatorial and linear-algebraic ingredients of the proofs of Theorem 3 and 4 are different.

Throughout the paper, we use the standard notation of Convex Geometry and Graph Theory; see the books [2] and [14] on Convexity and Graph Theory, respectively.

The paper is organized as follows. Sections 2 and 3 are devoted to the proofs of Theorems 1 and 2, respectively. In Subsection 4.1, we study properties of the global minimum of a function playing the key role in the proofs of Theorems 3 and 4. In Subsection 4.2, we introduce a concept of obtuse graphs and study their properties. Using the lemmas from Subsection 4.1 and 4.2, we prove Theorem 3 in Section 5. Finally, applying the lemma from Subsections 4.1, we give a proof of Theorem 4 in Section 6.

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2. Proof of Theorem 1

Let $S$ be a set of $n$ points in the plane. By a standard compactness argument, it is sufficient to prove the theorem additionally assuming that no three points of $S$ lie on a
Consider a bisecting line $\ell$ for $S$. Recall that for a finite set $P$ of points in the plane, a line is called a bisecting line if there are at most $|P|/2$ points of $P$ in each of its open half-planes. Choose a bisecting line $\ell_{\pi/2}$ for $S$ that is orthogonal to $\ell$. Denote by $o$ the intersection point of $\ell$ and $\ell_{\pi/2}$. Later, we specify the choice of the bisecting lines $\ell$ and $\ell_{\pi/2}$ depending on the parity of $n$.

Notice that the lines $\ell$ and $\ell_{\pi/2}$ determine four closed quadrants in the plane. Let us enumerate them $1, 2, 3, 4$ in the counterclockwise order starting from any quadrant; see Figure 1, where we denote the quadrants as $Q_1, Q_2, Q_3, Q_4$, respectively. Let $s_i$ be the number of points of $S$ in the $i$-th closed quadrant. Remark that a point may belong to two distinct quadrants if and only if it lies on the intersection of their boundaries.

**Figure 1.** Illustrations for Case 1.1, Case 1.2, Case 2.

Our key observation is that the point $o$ lies in any disk with a diameter whose endpoints lie in two opposite quadrants. Hence it remains to find a cycle whose segments satisfy this property.

There are several possible cases.

**Case 1.** Let $n$ be even. Choose a bisecting line $\ell$ passing through exactly two points of $S$. Denote these points by $x$ and $y$. Since no two segments spanned by points of $S$ are orthogonal, one can choose a bisecting line $\ell_{\pi/2}$ containing no points of $S$. There are two possible cases depending on the arrangement of the points $x, y, o$ on the line $\ell$. (By the choice of $\ell_{\pi/2}$, the point $o$ is not from $S$.)

**Case 1.1.** The point $o$ lies between the points $x$ and $y$ on the line $\ell$. Without loss of generality, assume that $x$ belongs to the second and third quadrants and the point $y$ belongs to the first and fourth quadrants. Since $\ell$ and $\ell_{\pi/2}$ are bisecting lines, we obtain that $s_1 = s_3$ and $s_2 = s_4$. Therefore, we easily construct a cycle with edges satisfying the key observation: We run alternatively between the points of the third and first quadrants starting from $x$ and finishing at $y$, and then we continue to run alternatively between the points of the fourth and second quadrants starting from $y$ and finishing at $x$.

**Case 1.2.** The points $x$ and $y$ lie on the same side with respect to $o$ on the line $\ell$. Without loss of generaly assume that $x$ and $y$ belong to the first and fourth quadrants. Using the fact that $\ell$ and $\ell_{\pi/2}$ are bisecting lines, we obtain that $s_1 = s_3 + 1$ and $s_4 = s_2 + 1$. Therefore, we easily construct a cycle with edges satisfying the key observation: We run alternatively between the points of the first and third quadrants starting from $x$ and finishing at $y$, and then we continue to run alternatively between the points of the fourth and second quadrants starting from $y$ and finishing at $x$.
Case 2. Let \( n \) be odd. Since any bisecting line for \( S \) passes through at least one point of \( S \), one can choose bisecting lines \( \ell \) and \( \ell_{\pi/2} \) such that each of them passes through exactly one point of \( S \). We may assume that these two points are distinct, and thus, they are also distinct from \( o \). (Indeed, if the lines \( \ell \) and \( \ell_{\pi/2} \) intersect at a point of \( S \), one can find a proper arrangement of lines rotating them simultaneously.) Denote by \( x \) the point of \( S \) lying on \( \ell \) and by \( y \) the point of \( S \) lying on \( \ell_{\pi/2} \).

Without loss of generality, assume that the point \( x \) belongs to the second and third quadrants and the point \( y \) belongs to the first and second quadrants. Since \( \ell \) and \( \ell_{\pi/2} \) are bisecting lines, we obtain that \( s_1 = s_3 \) and \( s_2 = s_4 + 1 \). Therefore, we easily construct a cycle with edges satisfying the key observation: We run alternatively between the points of the third and first quadrants starting from \( x \) and finishing at \( y \), and then we continue to run alternatively between the points of the second and fourth quadrants starting from \( y \) and finishing at \( x \).

3. Proof of Theorem 2

Denote by \( S \) the set of \( n \) red points and \( n \) blue points. By a standard compactness argument, it is sufficient to prove the theorem additionally assuming that no three points of \( S \) lie on a line, no three determine a right angle, and no two segments spanned by four points of \( S \) are orthogonal.

Choose a unit vector \( v \). Denote by \( v_\alpha \) the unit vector obtained by the counterclockwise rotation of \( v \) by an angle \( \alpha \). Let \( H_\alpha \) be the set of bisecting lines for \( S \) orthogonal to \( v_\alpha \); see the definition of bisecting line in the second paragraph of Section 2. Since \( H_\alpha \) is a closed plank lying between two parallel bisecting lines, the midline of \( H_\alpha \) is well defined. Denote this line by \( \ell_\alpha \). Remark that \( H_\alpha \) coincides with the line \( \ell_\alpha \) if and only if \( \ell_\alpha \) passes through exactly two points of \( S \). Denote by \( o_\alpha \) the intersection point of \( \ell_\alpha \) and \( \ell_{\alpha+\pi/2} \).

Since \( v_\alpha = -v_{\alpha+\pi} \), we have that the pair of lines \( \{\ell_\alpha, \ell_{\alpha+\pi/2}\} \) coincides with the pair of lines \( \{\ell_{\alpha+\pi/2}, \ell_{\alpha+\pi}\} \). Thus, \( o_\alpha = o_{\alpha+\pi/2} \).

Consider a set \( \Omega \) of \( \alpha \in \mathbb{R}/2\pi\mathbb{Z} \) such that none of the lines \( \ell_\alpha \) and \( \ell_{\alpha+\pi/2} \) pass through points of \( S \). Remark that the complementary set \( \overline{\Omega} = (\mathbb{R}/2\pi\mathbb{Z}) \setminus \Omega \) is finite. Moreover, for every \( \alpha \in \overline{\Omega} \), only one of two lines \( \ell_\alpha \) or \( \ell_{\alpha+\pi/2} \) passes through a point of \( S \) (in fact, it passes through exactly two points of \( S \)). Notice that the lines \( \ell_\alpha \) and \( \ell_{\alpha+\pi/2} \) determine in the plane four \textit{open} quadrants. Let us enumerate them in the counterclockwise order by 1, 2, 3, 4 starting from the quadrant bounded by two rays emanating from \( o_\alpha \) in directions \( v_\alpha \) and \( v_{\alpha+\pi/2} \); see Figure 2, where we denote the quadrants as \( Q_1, Q_2, Q_3, Q_4 \), respectively. For \( \alpha \in \Omega \), denote the number of red and blue points in the \( i \)-th quadrant by \( r_i(\alpha) \) and \( b_i(\alpha) \), respectively.

Recall our key observation from Section 2 that \( o_\alpha \) lies in any disk with a diameter whose endpoints lie in two opposite quadrants. Thus, it is enough to choose a red-blue matching such that all segments have this property. For this, we need to find \( \alpha \in \Omega \) such that the number of red points in any of the quadrants is equal to the number of blue points in the opposite one, that is,

\[
0 = r_1(\alpha) - b_2(\alpha) = r_2(\alpha) - b_3(\alpha) = r_3(\alpha) - b_4(\alpha) = r_4(\alpha) - b_1(\alpha).
\]

Next, we show that the equality \( r_1(\alpha) - b_2(\alpha) = 0 \) implies the other three. Suppose \( r_1(\alpha) = b_2(\alpha) \). Since the lines \( \ell_\alpha \) and \( \ell_{\alpha+\pi/2} \) are bisecting, we have \( r_1(\alpha) + b_1(\alpha) = r_3(\alpha) + b_3(\alpha) \). Consequently, \( r_3(\alpha) - b_1(\alpha) = r_1(\alpha) - b_3(\alpha) = 0 \). Next, from the equality of the numbers of red and blue points we have

\[
r_1(\alpha) + r_2(\alpha) + r_3(\alpha) + r_4(\alpha) = |S|/2 = n = b_1(\alpha) + b_2(\alpha) + b_3(\alpha) + b_4(\alpha).
\]
Excluding the known equal parts, we obtain
\[ r_2(\alpha) + r_4(\alpha) = b_2(\alpha) + b_4(\alpha). \]
Using this and the equality of the numbers of points of \( S \) in the second and fourth quadrants, that is, \( r_2(\alpha) + b_2(\alpha) = r_4(\alpha) + b_4(\alpha) \), we easily obtain the remaining required equalities. Thus, it remains to show that \( r_1(\alpha) = b_3(\alpha) \) for some \( \alpha \in \Omega \).

Finally, consider the function \( F : \Omega \to \mathbb{Z} \) defined by the equality \( F(\alpha) = r_1(\alpha) - b_3(\alpha) \). Since the line \( \ell_\alpha \) changes continuously for \( \alpha \in \mathbb{R}/2\pi\mathbb{Z} \), we get that \( F \) is a piecewise constant function. Moreover, it is constant between any two consecutive points of \( \Omega \). We claim that \( F \) changes its value at points of \( \Omega \) by 1, 0, or -1. Indeed, consider \( \alpha_0 \in \Omega \). Let us change \( \alpha \) continuously from \( \alpha_0 - \varepsilon \) to \( \alpha_0 + \varepsilon \), where \( \varepsilon > 0 \) is chosen in a way that \( \alpha_0 \) is the only point of \( \Omega \) lying between \( \alpha_0 - \varepsilon \) and \( \alpha_0 + \varepsilon \). We know that one of the lines \( \ell_{\alpha_0} \) or \( \ell_{\alpha_0 + \pi/2} \) contains exactly two points of \( S \) and another does not pass through any point of \( S \). Therefore, \( |F(\alpha_0 + \varepsilon) - F(\alpha_0 - \varepsilon)| \leq 2 \). Notice that \( |F(\alpha_0 + \varepsilon) - F(\alpha_0 - \varepsilon)| \) can be equal to 2 only if one of the following properties holds for \( \alpha = \alpha_0 \): (i) the boundary of the first quadrant contains two red points; (ii) the boundary of the third quadrant contains two blue points; (iii) the boundary of the first quadrant contains one red point and the boundary of the third quadrant contains one blue point. There are 6 possible arrangements of these two points of \( S \) and the point \( \alpha_0 \) satisfying one of these properties. We leave to the reader as a simple exercise the exhaustion verification of the fact that \( F \) changes by 1, 0 or -1 at the point \( \alpha_0 \); see Figure 2.

Without loss of generality, assume that \( 0 \in \Omega \). Since \( \ell_\alpha = \ell_{\alpha + \pi} \) and \( \ell_{\alpha + \pi/2} = \ell_{\alpha + 3\pi/2} \), we have
\[
F(0) = r_1(0) - b_3(0), \quad F(\pi/2) = r_1(\pi/2) - b_3(\pi/2) = r_2(0) - b_4(0), \\
F(\pi) = r_1(\pi) - b_3(\pi) = r_2(0) - b_4(0), \quad F(3\pi/2) = r_1(3\pi/2) - b_3(3\pi/2) = r_2(0) - b_4(0).
\]
From the equality of the numbers of red and blue points, we get that the equality $F(0) + F(\pi/2) + F(\pi) + F(3\pi/2) = 0$ holds, and thus, the function $F$ takes nonnegative and nonpositive values. Since the function $F$ takes only integer values and changes its value at a finite number of points by at most 1, there is a value of $\alpha \in \Omega$, where $F$ vanishes. Using the corresponding pairs of orthogonal lines $\ell_\alpha$ and $\ell_{\alpha+\pi/2}$, we construct the desired matching.

4. Preliminaries for high-dimensional results

Throughout the next sections, we denote by $o$ the origin of $\mathbb{R}^d$.

4.1. Extreme point of the maximum of dot products. For a finite set $\mathcal{M}$ of pairs of points in $\mathbb{R}^d$, consider the function $H_\mathcal{M} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$ H_\mathcal{M}(x) = \max \{ \langle a - x, b - x \rangle : ab \in \mathcal{M} \}. $$

Lemma 5. Let $\mathcal{M}$ be a finite set of pairs of points in $\mathbb{R}^d$. Then the function $H_\mathcal{M}(x)$ attains its strict global minimum at a unique point $x_\mathcal{M}$. Moreover, if $\mathcal{M}_0$ is the subset of $\mathcal{M}$ consisting of pairs $ab$ such that $\langle a - x_\mathcal{M}, b - x_\mathcal{M} \rangle = H_\mathcal{M}(x_\mathcal{M})$ and $K$ is the set of the midpoints of pairs in $\mathcal{M}_0$, then $x_\mathcal{M} \in \text{conv } K$.

Proof. Since $\langle a - x, b - x \rangle = (\frac{a+b}{2} - x)^2 - (\frac{a-b}{2})^2$, the function $H_\mathcal{M}$ is strictly convex and bounded from below, and thus, it attains its strict global minimum at a unique point $x_\mathcal{M}$. Without loss of generality we may assume that $x_\mathcal{M}$ coincides with the origin $o$. Suppose to the contrary $o \notin \text{conv } K$. By the Separation Theorem [2], there exists a non-zero vector $t$ such that $\langle c, t \rangle > 0$ for any $c \in \text{conv } K$. Hence, for sufficiently small $\varepsilon > 0$, the point $\varepsilon t$ is closer to all points of $K$, and thus, the dot product $\langle a - \varepsilon t, b - \varepsilon t \rangle$ for any pair $ab \in \mathcal{M}_0$ is strictly less than $H_\mathcal{M}(o)$. Also, for sufficiently small $\varepsilon > 0$, the value of $\langle a - \varepsilon t, b - \varepsilon t \rangle$ for $ab \in \mathcal{M} \setminus \mathcal{M}_0$ does not exceed $H_\mathcal{M}(o)$. By a standard compactness argument, we find a proper point $\varepsilon t$ such that $H_\mathcal{M}(\varepsilon t) < H_\mathcal{M}(o)$, a contradiction. \(\square\)

4.2. Properties of an obtuse graph. A finite set $V \subset \mathbb{R}^d$ is called dependent if there are positive coefficients $\lambda_v > 0$ for $v \in V$ such that

$$ \sum_{v \in V} \lambda_v v = o. $$

A graph $G$ is called obtuse if its vertex set is a finite dependent set and its vertices $a, b$ are adjacent if and only if $\langle a, b \rangle < 0$. Let us show some properties of an obtuse graph.

Lemma 6. A vertex of an obtuse graph $G$ is isolated if and only if it coincides with the origin $o$.

Proof. If the origin is a vertex of the obtuse graph, then it is isolated because of $\langle v, o \rangle = 0$ for any $v \in V(G)$. Hence, it remains to show that any non-zero vertex $v$ is not isolated. Indeed, there are positive $\lambda_u$ for $u \in V(G)$ such that

$$ \sum_{u \in V(G)} \lambda_u u = o. $$

Considering the dot product of this vector and $v$, we obtain

$$ \lambda_v \langle v, v \rangle + \sum_{u \in V(G) \setminus \{v\}} \lambda_u \langle u, v \rangle = 0. $$

Since $\lambda_w > 0$ for $w \in V(G)$, there is a vertex $u \in V(G) \setminus \{v\}$ such that $\langle u, v \rangle < 0$. This finishes the proof of the lemma. \(\square\)
Lemma 7. Any two vertices from different connected components of an obtuse graph $G$ are orthogonal.

Proof. Denote by $U \subset V(G)$ the set of vertices of some connected component. Put $W = V(G) \setminus U$. We show that any vector from $U$ is orthogonal to any vector of $W$. Since $V(G)$ is a dependent set, there are positive $\lambda_u$ for all $v \in V(G)$ such that
$$\sum_{u \in U} \lambda_u u = - \sum_{w \in W} \lambda_w w.$$ Therefore, we have
$$\langle \sum_{u \in U} \lambda_u u, \sum_{w \in W} \lambda_w w \rangle \leq 0.$$ Since $\lambda_v > 0$ for all $v \in V(G)$ and there are no edges between $U$ and $W$, that is, $\langle u, w \rangle \geq 0$ for all $u \in U$ and $w \in W$, we obtain $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$. $\square$

5. Proof of Theorem 3

Let $S$ be an even set of distinct points in $\mathbb{R}^d$. For any perfect matching $\mathcal{M}$ on $S$, we consider a function $H_\mathcal{M}$ from Subsection 4.1. Recall that
$$H_\mathcal{M}(x) = \max \{ \langle a - x, b - x \rangle : ab \in \mathcal{M} \}.$$ If $H_\mathcal{M}(x) < 0$ for some matching $\mathcal{M}$ and some $x \in \mathbb{R}^d$, then we are done. Suppose to the contrary that for any perfect matching $\mathcal{M}$ and any $x \in \mathbb{R}^d$, we have $H_\mathcal{M}(x) \geq 0$. Consider the function $P$ depending on a perfect matching $\mathcal{M}$ defined by
$$P(\mathcal{M}) := \inf_{x \in \mathbb{R}^d} H_\mathcal{M}(x) = \min_{x \in \mathbb{R}^d} H_\mathcal{M}(x).$$ For a matching $\mathcal{M}$, denote by $x_\mathcal{M}$ the unique point such that $H_\mathcal{M}(x_\mathcal{M}) = P(\mathcal{M})$.

Among all perfect matchings $\mathcal{M}$ for $S$, choose a matching $\mathcal{M}'$ such that
$$P(\mathcal{M}') = \min_{\mathcal{M}} P(\mathcal{M}) \geq 0.$$ For the sake of brevity, put $m := P(\mathcal{M}')$. Additionally assume that among all perfect matchings $\mathcal{M}$ with $P(\mathcal{M}) = m$, the matching $\mathcal{M}'$ contains the minimum number of pairs $ab$ such that $\langle a - x_\mathcal{M}, b - x_\mathcal{M} \rangle = m$. Without loss of generality, we assume that $x_{\mathcal{M}'}$ coincides with the origin $o$. Consider the matching $\mathcal{M}'' \subseteq \mathcal{M}'$ consisting of pairs $ab$ with $\langle a, b \rangle = m$. Applying Lemma 5 to $H_\mathcal{M}$, we obtain that the midpoints of some submatching $\mathcal{M}''' \subseteq \mathcal{M}''$ form a dependent set. Let $S_1 \subseteq S$ be the points of the pairs from $\mathcal{M}'''$. Hence the set $S_1$ is dependent as well.

Consider an obtuse graph $G$ with the vertex set $S_1$. Let $G_1, \ldots, G_r$ be its connected components. Since we assume that all points of $S$ are distinct, Lemma 6 yeilds that there is at most one isolated vertex. If some vertex of $G$ coincides with the origin $o$, then we set $v_1 = o$, otherwise let $v_1$ be any vertex of $G$. Without loss of generality, assume that $v_1$ is a vertex of $G_1$. By Lemma 6, the connected components $G_i$ for $2 \leq i \leq r$ contains at least two vertices, and thus, we choose a vertex $v_i$ from $G_i$ for $2 \leq i \leq r$ such that $v_1v_i \not\in \mathcal{M}'''$. By Lemma 7, we have $\langle v_1, v_i \rangle = 0$ for $2 \leq i \leq r$. Put
$$E_{\pi/2} = \{ v_1v_i : 2 \leq i \leq r \}.$$ Consider the graph $G'$ on the vertex set $S_1$ and with the edge set
$$E(G) \cup E_{\pi/2} \cup \mathcal{M}'''.$$
Remark that the sets $E(G) \cup E_{n/2}$ and $\mathcal{M}''$ do not intersect. Let us color the edges in $E(G) \cup E_{n/2}$ in red and the edges in $\mathcal{M}''$ in blue.

Recall that an edge $e$ of a connected graph $H$ is called a cut edge, if $H - e$ is disconnected graph. The key combinatorial ingredient of our proof is the following result of J.W. Grossman and R. Häggkvist; see Corollary 1 in [7].

**Lemma 8.** Let $M$ be a perfect matching in a graph $H$. If no edge of $M$ is a cut edge of $H$, then $H$ has a cycle whose edges are taken alternately from $M$ and $H - M$.

In the graph $G'$, the blue edges form a perfect matching and no blue edge is a cut edge because $v_1$ is connected to all components of $G$. Hence, the graph $G'$ satisfies the conditions of the Lemma 8, and thus, there is an alternating cycle with blue and red edges. Denote by $R$ and $B$ respectively the set of red and blue edges of this cycle.

Consider the following matching

$$\mathcal{M}^o = (\mathcal{M}' \setminus B) \cup R.$$ 

Since $(a, b) \leq 0$ for any $ab \in R$, we have $H_{\mathcal{M}'}(o) \leq H_{\mathcal{M}'}(o) = m$. Therefore, $H_{\mathcal{M}'}(o) = m$ and the origin $o$ coincides with the point $x_{\mathcal{M}'}$. Recall that $B \subseteq \mathcal{M}'' \subseteq \mathcal{M}''$ and the set $R$ contains at most one edge from $E_{n/2}$ because all edges of $E_{n/2}$ are incident to $v_1$, and thus, the set $R$ contains at least one pair $ab$ with $(a, b) < 0 \leq m$. Therefore, the number of pairs $ab$ from $\mathcal{M}^o$ with $(a, b) = m$ is strictly less than the similar number for $\mathcal{M}'$. This contradicts our choice of $\mathcal{M}'$.

6. Proof of Theorem 4

Let $R$ and $B$ be the sets of $n$ red points and $n$ blue points in $\mathbb{R}^d$, respectively. Consider the function $Q$ depending on a red-blue perfect matching $\mathcal{M}$ for $R \cup B$ defined by

$$Q(\mathcal{M}) = \sum_{rb \in \mathcal{M}} (r - b)^2 = \sum_{r \in R} r^2 + \sum_{b \in B} b^2 - 2 \sum_{rb \in \mathcal{M}} \langle r, b \rangle,$$

Let $\mathcal{M}'$ be a matching for which the function $Q$ attains its maximum. Next, we consider the function $H_{\mathcal{M}'} : \mathbb{R}^d \to \mathbb{R}$ from Subsection 4.1. Recall that

$$H_{\mathcal{M}'}(x) = \max \{ \langle r - x, b - x \rangle : rb \in \mathcal{M}' \}.$$ 

If $H_{\mathcal{M}'}(x) \leq 0$ for some $x \in \mathbb{R}^d$, then we are done. Thus, without loss of generality, we assume that $H_{\mathcal{M}'}$ attains its minimum at the point $o$ and $H_{\mathcal{M}'}(o) > 0$. Applying Lemma 5 to the matching $\mathcal{M}'$, we have that $o$ lies in the convex hull of the midpoints of $rb \in \mathcal{M}'$ such that $(r, b) = H_{\mathcal{M}'}(o)$. Let $\mathcal{M}_0 = \{r_1b_1, \ldots, r_mb_m\}$, where $r_i \in R$ and $b_i \in B$, we have

$$o = \sum_{i=1}^m \lambda_i(r_i + b_i),$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$, and so,

$$\sum_{i=1}^m \lambda_i r_i = -\sum_{i=1}^m \lambda_i b_i.$$

This equality yields

$$0 \geq \left( \sum_{i=1}^m \lambda_i r_i, \sum_{i=1}^m \lambda_i b_i \right) = \sum_{i=1}^m \lambda_i^2 \langle r_i, b_i \rangle + \sum_{1 \leq i < j \leq m} \lambda_i \lambda_j \langle r_i, b_j \rangle + \langle r_j, b_i \rangle.$$
Using $\langle r_i, b_i \rangle = H_M(o)$ and supposing $\langle r_i, b_j \rangle + \langle r_j, b_i \rangle \geq 2H_M(o)$ for all distinct $i$ and $j$, we get

$$0 \geq H_M(o) \cdot \left( \sum_{i=1}^{m} \lambda_i \right)^2 = H_M(o) > 0,$$

a contradiction. Therefore, $\langle r_i, b_j \rangle + \langle r_j, b_i \rangle < 2H_M(o) = \langle r_i, b_i \rangle + \langle r_j, b_j \rangle$ for some distinct $i$ and $j$. Next, consider the new perfect red-blue matching

$$M^o = M' \setminus \{r_ib_i, rjb_j\} \cup \{r_ib_j, rjb_i\}$$

satisfying the inequality

$$Q(M^o) = \sum_{r \in R} r^2 + \sum_{b \in B} b^2 - 2 \sum_{rb \in M^o} \langle r, b \rangle > \sum_{r \in R} r^2 + \sum_{b \in B} b^2 - 2 \sum_{rb \in M'} \langle r, b \rangle = Q(M').$$

This contradicts our choice of $M'$. Therefore, $H_{M'}(x) \leq 0$ for some $x \in \mathbb{R}^d$.

7. DISCUSSION: PROVING TVERBERG-TYPE THEOREMS USING THE METHOD OF INFINITE DESCENT

First, we sketch the proof of Tverberg’s Theorem by Roudneff [10]. For an $r$-partition $P$ of a set of $(r - 1)(d + 1) + 1$ points in $\mathbb{R}^d$, consider the function $h_P : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$h_P(y) = \sum_{X \in P} \text{dist}^2(y, \text{conv} X),$$

where dist$(A, B)$ is the distance between sets $A, B \subset \mathbb{R}^d$. Since the function $h_P$ is convex, it attains its minimum. Choose a partition $P_0$ for which this minimum is the smallest possible. If this minimum is 0, then we are done. Therefore, we suppose that it is positive and attained at $y = y_{P_0}$. Then, analyzing the arrangement of the point $y_{P_0}$ and the convex hulls of $X \in P_0$, we find a partition $P'_0$ such that

$$h_{P'_0}(y_{P_0}) < h_{P_0}(y_{P_0}),$$

a contradiction. Moreover, the partition $P'_0$ slightly differs from $P_0$ — they share all but two common sets. Also, remark that Tverberg and Vrečica [13] used a similar approach but they consider instead of $h_P$ a different function $g_P : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$g_P(y) = \max_{X \in P} \text{dist}(y, \text{conv} X).$$

The proofs of our Tverberg-type results, Theorems 3 and 4, are based on the method of infinite descent but involve novel ideas. At the last step in the proof of Theorem 3, we consider a new perfect matching $M^o$ that can differ from $M'$ more than in two edges, that is, the matchings can be very different (recall that in Roudneff’s proof, the partitions $P_0$ and $P'_0$ differs only in two sets). In the proof of Theorem 4, we consider two functions instead of one as in Roudneff’s proof: the function $Q$ depending only on matchings and the function $H_{M'} : \mathbb{R}^d \rightarrow \mathbb{R}$ depending on a point in $\mathbb{R}^d$. Then there are two possibilities: Either the minimum point $o$ of the function $H_{M'}$ is the desired intersection point of balls or this point allows to find a new matching $M^o$ increasing the value of the function $Q$.

Interestingly, a standard argument applied only to the function $H_M$ (see Subsection 4.1) does not allow us to finish the proof. To illustrate this, consider the example of two red points $r_1, r_2$ and two blue points $b_1, b_2$ drawn on Figure 3. Clearly, $M^o = \{r_1b_2, r_2b_1\}$ is

1The reader may assume that the the points $r_1$ and $b_2$ are symmetric with respect to the line $\ell$ and the points $r_2$ and $b_1$ are symmetric with respect to $\ell$ as well. Moreover, additionally assume that $\|r_1 - b_1\| = \|r_2 - b_2\| = 1$, $\angle r_1ob_2 = \pi/4$, and the distance between the non-intersecting disks $D(r_1b_1)$ and $D(r_2b_2)$ is close to 0.
Figure 3. Example, where only possible switching increases value of $H_M$

the only desired matching. Choosing the second matching $M' = \{r_1b_1, r_2b_2\}$ as a starting matching, we expect to show the inequality $H_{M'}(o) < H_{M}(o)$, where $o$ is the minimum point of $H_M$. However, it does not hold because

$$H_{M'}(o) = \langle r_1, b_2 \rangle > \langle r_1, b_1 \rangle = \langle r_2, b_2 \rangle = H_{M}(o).$$

This obstacle shows that an extra argument is needed to show that the minimum of the function $H_M$ is less than $H_{M'}(o)$. To finish the proof of Theorem 4, we introduce a new function $Q$ depending only on matching. Unfortunately, we did not find a more direct approach to complete the argument.

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