Ramanujan’s Most Singular Modulus

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Abstract

We present an elementary self-contained detailed computation of Ramanujan’s most famous singular modulus, $k_{210}$, based on the Kronecker Limit Formula.

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1 The Singular Modulus $\alpha$

1.1 Introduction

In his second letter to G. H. Hardy, dated 27 February 1913, the self-taught Indian genius Srinivasa Ramanujan announced the following outrageous theorem (numbered (17) in the letter [17]):

**THEOREM 1 (Ramanujan’s Singular Modulus).** If

$$F(\alpha) := 1 + \left(\frac{1}{2}\right)^2 \alpha + \left(\frac{1\cdot3}{2\cdot4}\right)^2 \alpha^2 + \cdots$$  \hspace{1cm} (1.1)

and

$$F(1 - \alpha) = \sqrt{210} \cdot F(\alpha),$$ \hspace{1cm} (1.2)

then

$$\alpha = (4 - \sqrt{15})^4(8 - 3\sqrt{7})^2(2 - \sqrt{3})^2(6 - \sqrt{35})^2$$
$$\cdot (\sqrt{10} - 3)^4(\sqrt{7} - \sqrt{6})^4(\sqrt{15} - \sqrt{14})^2(\sqrt{2} - 1)^2.$$ \hspace{1cm} (1.3)

(We have written $\alpha$ in place of Ramanujan’s $k$.)

Our paper develops an elementary self-contained presentation of the history, theory and algorithms for computing $\alpha$ as well as the details of its computation. Although (1.1)–(1.3) “...is one of Ramanujan’s most striking results,” (Hardy [13]), there is no published *ab initio* account of it. Watson [18] published the only computation of $\alpha$ in the literature, but it presupposes a considerable background knowledge on the part of the reader.

There always has been and continues to be a large population of students and practicing mathematicians who are intrigued by the folklore surrounding Ramanujan’s origins, who are thrilled by the wondrous classical beauty of his mathematics, but who are put off by the necessity of mastering multiple branches of advanced mathematics in order to be able to read the proofs.

We present for such readers an expository development of one of his most famous formulae, a detailed elementary self-contained connected account which takes an inquiring mind...
with a modest background step-by-step to the full result. Our exposition thus contains historical information, motivational commentary, and offers a “cultural framework” as it were, so as to explain to the reader where the material “fits in” within the greater subject of mathematics in general. It is far more than just a computation. It is a focused introduction to and an exposition of that specific area of mathematics which Ramanujan’s result so strikingly illuminates.

Although our paper is expository, one hopes that the specialist can enjoy the point of view and some novelties of detail. To begin with, our version of Ramanujan’s “very curious algebraical lemma” is new variant, and we use it for the major computations in our paper (see §2.4.2).

Moreover, our proof of Weber’s fundamental formula for $g_{2n}$ avoids Weber’s use of genus theory by appealing to an explicit composition formula for the quadratic form $AX^2 + BY^2$ (see §3.7.1). This new approach simplifies the proof considerably and brings it within the reach of our target reader.

This reader needs to understand elementary analysis and algebra at the level of, say, three years at a university, as well as to be familiar with number theory to the level of quadratic reciprocity and the Jacobi symbol. All of this material is more than sufficiently covered in Niven and Zuckerman [16].

Much of the underlying general theory has been published in David Cox’s beautiful book [6] but he never deals with Ramanujan’s particular result. Finally, much of the material is in scattered references in several languages.

We call special attention to Volume 5 of Bruce Berndt’s edition of Ramanujan’s Notebooks [1], which devotes more than one hundred and fifty pages to the computation of singular moduli, and which is the most exhaustive reference on the subject. Unfortunately, although Berndt presents Chan’s proof of the “very curious algebraical lemma,” he does not discuss the complex computation of $g_{210}$, the most difficult step, and thus is not a source for computing Ramanujan’s specific result. However, he does give some bibliographic references.

1.2 Singular moduli and units in quadratic fields

If we take Ramanujan’s theorem at face value and try to see if it makes sense (!) we can observe from the definition:

$$F(\alpha) := 1 + \left(\frac{1}{2}\right)^2 \alpha + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \alpha^2 + \cdots$$

(1.4)

that $F(\alpha)$ converges for $-1 \leq \alpha < 1$. Moreover, as $\alpha$ increases from 0 to 1, $F(\alpha)$ increases from 1 to infinity and $F(1 - \alpha)$ decreases from infinity to 1, and therefore

$$\frac{F(1 - \alpha)}{F(\alpha)}$$
decreases monotonically from infinity to zero as $\alpha$ increases from 0 to 1. Thus there exists a unique real number $\alpha$ with $0 < \alpha < 1$ such that

$$
\frac{F(1-\alpha)}{F(\alpha)} = \sqrt{210}.
$$

(1.5)

The number

$$
k_{210} := \sqrt{\alpha} \equiv \sqrt{\alpha}
$$

is called the singular modulus (for 210). If the right hand side of (1.2) is $\sqrt{n}$, $n \in \mathbb{R}^+$, the same argument shows that there exists a unique real number $\alpha_n$ with $0 \leq \alpha_n < 1$ which satisfies

$$
\frac{F(1-\alpha_n)}{F(\alpha_n)} = \sqrt{n},
$$

and we write

$$
k_n := \sqrt{\alpha_n}.
$$

and we call $k_n$ the singular modulus (for $n$). We use $\alpha$ and $k$ without the subscript quite frequently for ease of exposition. Its meaning should be clear from the context. Of course, what makes Ramanujan’s result (1.1)–(1.3) so astonishing is the explicit numerical value of $\alpha_{210}$ as a finite product of the differences of quadratic surds. It is all the more amazing and unexpected since there is no evident a priori reason to expect such a value based on the expansion (1.4). Where does $\alpha$ come from?

Let’s examine $\alpha$, itself. In the first place, all the quadratic surds,

$$
\sqrt{15}, \sqrt{7}, \sqrt{3}, \sqrt{35}, \sqrt{10}, \sqrt{6}, \sqrt{14}, \sqrt{2}
$$

which appear in $\alpha$ are the square roots of the divisors of 210. This can hardly be an accident!

In the second place, $\alpha$ is a product

$$
\alpha := (4 - \sqrt{15})^4(8 - 3\sqrt{7})^2(2 - \sqrt{3})^2(6 - \sqrt{35})^2
$$

\begin{equation*}
(\sqrt{10} - 3)^4(\sqrt{7} - \sqrt{6})^4(\sqrt{15} - \sqrt{14})^2(\sqrt{2} - 1)^2
\end{equation*}

of terms of the form $T - U\sqrt{m}$ where

$$
T^2 - mU^2 = 1
$$

(1.6)

which is the famous Pell’s equation. Moreover the numbers $u := T - U\sqrt{m}$, where $T, U$ are rational integers, are integers in the quadratic field $Q(\sqrt{m})$. The norm of the quadratic integer $u$ is defined by the equation $N(u) := T^2 - mU^2$. See [16]. The equation (1.6) shows that $u$ has norm equal to one, that is, $u$ a unit in $Q(\sqrt{m})$. Thus, each factor in $\alpha_{210}$ is a unit in a subfield of $Q(\sqrt{m})$, and therefore $\alpha_{210}$ itself is a unit! An accident?

Not only is this not an accident, but rather is an instance of a general theorem which Chan and Huang [5] proved in 1997:
THEOREM 2 (Chan–Huang Unit Theorem). If \( n \) is of the form \( 4m + 2 \), then \( \alpha_n \) is a unit.

We will call this result the Chan–Huang–UNIT–THEOREM, or “CHUT” for short. In fact, the CHUT deals with \( \alpha_n \) for any \( n \), but this suffices for now.

The fact that \( \alpha_{210} \) (and therefore \( k_{210} \)) is a unit suggests that the explanation of its value lies in the theory of quadratic number fields. But the theory of quadratic fields is a reformulation of the more classical theory of quadratic forms, and so we now turn to discussing \( \alpha \) in terms of this classical theory.

1.3 Binary quadratic forms. The “numeri idonei” of Euler

In 1776, Euler [8] made the following observation:

**THEOREM 3.** Let \( n \) be an odd number not divisible by 3 nor 5 nor 7, but which is properly represented by \( x^2 + 210y^2 \). If the equation

\[ x^2 + 210y^2 = n \]

has only ONE solution \((x, y)\) with \( x, y \geq 0 \), then \( n \) is a prime number.

This property of “210” is shared by 64 other known numbers, each of which was called a “numerus idoneus” (a “convenient” or “useful” number) by Euler. Gauss [11] conjectured that these 65 numbers are all of them, and Weinberger [21] proved that at most there may be one more.

This common “useful” property, in our case, can be formulated as follows in terms of the theory of binary quadratic forms:

**THEOREM 4.** All the binary quadratic forms of determinant 210 belong to 8 genera and each genus contains one class of equivalent forms.

We will define the technical terms later on.

1.4 Elliptic modular functions and abelian extensions

Our formulation of the “useful” property of 210 can be stated in terms of the theory of class fields as follows. Let \( \tau \) be any complex number with positive imaginary part. Define

\[
j(\tau) := \left[ 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)e^{2\pi i n \tau} \right]^3 \prod_{n=1}^{\infty} \left[ 1 - e^{2\pi i n \tau} \right]^{24}
\]

where

\[
\sigma_3(n) := \sum_{d \mid n, \ d \geq 1} d^3
\]
and “$d | n$” means “$d$ divides $n$”.

In the theory of elliptic functions (Borwein and Borwein [3]) one proves

$$j = 64 \frac{(1 - \alpha + \alpha^2)^3}{\alpha^2(1 - \alpha)^2}$$

(1.7)

where $\tau := \sqrt{-n}$, and $\alpha \equiv \alpha_n$. (The reader should take note of the two uses of the letter

$n$, as a summation index and also as the value whose square root appears in the variable). Taking $n = 210$ and using Ramanujan’s value of $\alpha$ (1.3) as well as the equation (1.7), some
tedious algebra leads us to the value

$$j(\sqrt{-210}) = 64 \left\{ 4[\sqrt{3} + \sqrt{2}]^{12}[3\sqrt{14} + 5\sqrt{5}]^{4} \left[ \frac{\sqrt{7} + \sqrt{3}}{2} \right]^{12} \left[ \frac{(\sqrt{5} + 1)}{2} \right]^{12} + 1 \right\}^{3}$$

$$\left( [\sqrt{3} - \sqrt{2}]^{12}[3\sqrt{14} - 5\sqrt{5}]^{4} \left[ \frac{\sqrt{7} - \sqrt{3}}{2} \right]^{12} \left[ \frac{\sqrt{5} - 1}{2} \right]^{12} \right)$$

We collect some facts about $j(\sqrt{-210})$. (See Cox [6].)

1. $j(\sqrt{-210})$ is an algebraic integer.

2. $j(\sqrt{-210})$ is a primitive element of the ring class field of $Q(\sqrt{-210})$.

3. $j(\sqrt{-210})$ satisfies a monic algebraic equation of degree 8 with rational integer coefficients:

$$\alpha^8 + a_1\alpha^7 + \cdots + a_8 = 0.$$  

The coefficients are enormous! For example, the coefficient of the seventh power of the variable is

$$a_1 = -3494487845306481075093315600749304691200$$

while the constant term is

$$a_8 = 7587169380271379738636919142674280077130435043277$$

$$326055125100897851220991378671072700656000000000000.$$  

(We thank Anthony Varilly for performing these computations with the program PARI.)

4. Combining this monic equation and the equation (1.7), we conclude that $k_{210}$ satisfies

a monic equation of degree 96.

This theory is called the theory of Complex Multiplication and today is in the forefront of
modern mathematical research.

We will expand on these results later.
1.5 Prospectus

Let’s stop to catch our breath. We’ve used several (as of yet) undefined technical terms and notions. We’ve wandered over complex function theory, elementary and algebraic number theory, elliptic and modular functions, the genus theory of binary quadratic forms, and finally, complex multiplication and class field theory (the theory of normal field extensions with abelian Galois group)!

Did Ramanujan know any of this? Hardy [13] suggests that he most certainly did not know the algebraic number theory, much less class field theory.

Then how did he arrive at his results? Nobody knows, really, although Watson has made a clever suggestion which we will not consider in this paper. (See [18] and [4]).

Our paper presents an elementary method of computing $\alpha_{210}$. It is based on the theory of binary quadratic forms and uses the so-called Kronecker Limit Formula. We will develop the details in full since they have never been published. The fundamental sources are Weber’s treatise [19] and his paper [20]. In both references Weber simply states the result of his computation, but suppresses the computation itself. Ramanujan uses the result of Weber’s computation in order to compute $\alpha_{210}$, although he obtained it independently of Weber. The fact is, nobody knows how he did it.

Our plan is the following: we will first develop the theory of the function $F(\alpha)$ and learn the two-step algorithm which Ramanujan used. Then we will develop the theory of quadratic forms and the theory of Kronecker’s Limit Formula so as to be able to carry out the steps in Ramanujan’s algorithm. Finally we will carry out the details of the computation and crown our memoir with an elementary and detailed demonstration of the general formula implied by the computation which will explain a priori the miraculous computations we carry out.

2 Ramanujan’s Function $F(\alpha)$

2.1 Complete elliptic integrals

We began our inquiry of Ramanujan’s theorem (1.1)–(1.3) by looking at the value of $\alpha$. Now we look at the function $F(\alpha)$. In fact, this function is quite well known. Define

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

(2.1)

The method suggested by Watson [18] and Chan [4] is superficially more elementary. It entails “guessing” the fields which contain the singular moduli and then using skillful ingenuity with radicals to obtain the final formulas. Unfortunately, most readers have not mastered class-field theory, the theorems of Chan are not applicable to Ramanujan’s computation in any event, and Watson’s presentation presupposes a profound knowledge, as Ramanujan had, of singular moduli. Indeed, we think that this is the method Ramanujan most likely used for the bulk of his computations of singular moduli, but it is not remotely “elementary” for most readers who don’t know class field theory.
Then $K(k)$, for $0 \leq k < 1$, is called the **complete elliptic integral of the first kind**. If we put

$$
\alpha := k^2,
$$

and then expand the denominator by the binomial theorem and integrate term by term, *we see that Ramanujan’s $F(\alpha)$ is $K(k)$:*

\[ F(\alpha) = K(k) \] (2.2)

The number $k$ is called the **modulus** of $K$. The number $k'$, defined by

$$
k'^2 := 1 - k^2,
$$
is called the **complementary modulus** and

$$
K' := K(k')
$$

is called the **complementary integral**. We also write

$$
K \equiv K(k).
$$

Mathematicians have studied $K$ and $K'$ for (at least) two and a half centuries, and therefore they’ve studied Ramanujan’s $F(\alpha)$ and $F(1 - \alpha)$. See Euler [7] and Fagnano [9].

### 2.2 The modular equation

In particular, in 1771 Landen [15] discovered the following transformation law:

\[ K(k) = \frac{1}{1 + k} K\left(\frac{2\sqrt{k}}{1 + k}\right), \] (2.4)
\[ K'(k) = \frac{2}{1 + k} K'\left(\frac{2\sqrt{k}}{1 + k}\right). \]

Dividing $K'(k)$ by $K(k)$, we obtain

\[ \frac{K'(k)}{K(k)} = 2 \frac{K'\left(\frac{2\sqrt{k}}{1 + k}\right)}{K\left(\frac{2\sqrt{k}}{1 + k}\right)}. \] (2.5)

which is a *multiplication law* for the quotient $\frac{K'}{K}(k) \equiv \frac{K'(k)}{K(k)}$. If we put

$$
l := \frac{2\sqrt{k}}{1 + k},
$$

8
the duplication law (2.5) takes the form
\[ 2 \frac{L'}{L} = \frac{K'}{K}, \tag{2.7} \]
where \( L' \) and \( L \) are the complete elliptic integrals with respective moduli \( l' \) and \( l \). The equation (2.6) can also be written
\[ l^2(1 + k)^2 = 4k. \tag{2.8} \]
Equations (2.6) and (2.8) are both called the modular equation of degree 2. The duplication law (2.7) is one of an infinity of multiplication laws and each such law has a modular equation associated with it. We now make the following general definition.

**DEFINITION 1.** The modular equation of degree \( n \) is the algebraic equation
\[ \Omega_n(k, l) = 0, \]
relating the moduli \( k \) and \( l \) for which the multiplication law
\[ n \frac{L'}{L} = \frac{K'}{K} \tag{2.9} \]
holds. Here \( K, K', L, L' \) are the complete elliptic integrals with moduli \( k \) and \( l \), and \( n > 0 \) is a rational number.

Jacobi [14] proved that for every integer \( n > 0 \) there is a modular equation \( \Omega_n(k, l) = 0 \). We just proved Jacobi’s result, with Landen’s help (!) for \( n = 2 \). We list a few more examples of modular equations (Greenhill [12]):

| Degree | Modular Equation |
|--------|------------------|
| 3      | \( \sqrt{kl} + \sqrt{k'l'} = 1 \) |
| 5      | \( kl + k'l' + \sqrt{32klk'l'} = 1 \) |
| 7      | \( \sqrt{kl} + \sqrt{k'l'} = 1 \) |

### 2.3 Ramanujan’s theorem for \( \alpha_2 \), \( \alpha_3 \), and \( \alpha_7 \)

Why are we interested in the modular equation? Equation (2.9) is
\[ n \frac{K(l')}{K(l)} = \frac{K(k')}{K(k)} \]
Now comes the “brilliancy”, as they say in chess. Choose \( k \) such that
\[ l = k', \quad l' = k. \]
Then (2.9) becomes

\[ \frac{nK}{K'} = \frac{K'}{K}, \]

or

\[ \left( \frac{K'}{K} \right)^2 = n, \]

that is,

\[ \frac{K'}{K}(k) = \sqrt{n}. \tag{2.10} \]

or, by (2.2), (2.3) and (2.10),

\[ \frac{F(1 - \alpha)}{F(\alpha)} = \sqrt{n}, \tag{2.11} \]

and (2.11) is precisely Ramanujan’s equation (1.2) —indeed it is for any \( n \), and not just \( n = 210 \)!

So we now understand the origin of Ramanujan’s equation in general: namely

The modular equation of degree \( n \) takes the form of Ramanujan’s equation (2.11) when in it we choose \( k = l' \) and \( k' = l \).

Let us compute a few examples.

**Example 1.** \( n = 2 \) Suppose \( \frac{F(1 - \alpha)}{F(\alpha)} = \sqrt{2} \). Then the modular equation of degree 2, namely

\[ l^2(1 + k)^2 = 4k, \]

under the substitution \( l = k', l' = k \), becomes

\[ (1 - k^2)(1 + k)^2 = 4k, \]

or

\[ (k^2 + 1)(k^2 + 2k - 1) = 0, \]

or

\[ k = -1 \pm \sqrt{2}. \]

But \( 0 \leq k_2 \leq 1 \), and therefore

\[ k_2 = \sqrt{2} - 1. \]

**THEOREM 5.** If \( F(\alpha) = \sqrt{2} \cdot F(1 - \alpha) \), then

\[ \alpha = (\sqrt{2} - 1)^2. \]

Thus we have found the singular modulus \( k_2 \).
Example 2. \( n = 3 \) Our table of modular equations gives us the following equation of degree 3:

\[
\sqrt{kl} + \sqrt{k'l'} = 1.
\]

Putting \( l = k', l' = k \), we obtain

\[
2\sqrt{kk'} = 1,
\]

or

\[
k^2(1 - k^2) = \frac{1}{16},
\]

that is,

\[
k = \sqrt{\frac{2 \pm \sqrt{3}}{4}}.
\]

To properly choose the sign, we recall that

\[
\alpha = k^2 = \frac{2 \pm \sqrt{3}}{4}
\]

(which makes \( 1 - \alpha = (2 \mp \sqrt{3})/4 \), is the unique solution in \([0, 1]\) to the equation

\[
\sqrt{3} = \frac{F(1 - \alpha)}{F(\alpha)}
\]

\[
= \frac{1 + \left(\frac{1}{2}\right)^2 \left[\frac{2 \mp \sqrt{3}}{4}\right]}{1 + \left(\frac{1}{2}\right)^2 \left[\frac{2 \pm \sqrt{3}}{4}\right]} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left[\frac{2 \mp \sqrt{3}}{4}\right]^2 + \cdots
\]

and comparing both sides numerically, we see that we must choose the negative sign. Therefore

\[
k_3 = \sqrt{\frac{2 - \sqrt{3}}{4}}.
\]

and our Ramanujan type theorem is the following.

**THEOREM 6.** If \( F(\alpha) = \sqrt{3} \cdot F(1 - \alpha) \), then

\[
\alpha = \frac{2 - \sqrt{3}}{4}.
\]

We have therefore found the singular modulus \( k_3 \). We note that the choice

\[
\alpha = \frac{2 + \sqrt{3}}{4}
\]

is the root of the complementary equation

\[
F(1 - \alpha) = \frac{1}{\sqrt{3}} F(\alpha).
\]
Example 3. \[ n = 7 \] The modular equation of degree 7 (by our table) is:

\[ \sqrt{kl} + \sqrt{k'l'} = 1. \]

Putting \( l = k' \), \( l' = k \), and solving for \( k_7 \), we obtain

\[ k_7 = \sqrt{\frac{8 - \sqrt{7}}{16}}. \]

(We leave the details to the reader.)

**THEOREM 7.** If \( F(\alpha) = \sqrt{7} \cdot F(1 - \alpha) \), then

\[ \alpha = \frac{8 - 3\sqrt{7}}{16}. \]

We have therefore found the singular modulus \( k_7 \).

2.4 Ramanujan’s two-step algorithm

The computation of \( \alpha_2, \alpha_3, \) and \( \alpha_7 \) we just carried out would suggest a procedure to calculate Ramanujan’s \( \alpha_{210} \). Namely, find the modular equation of degree 210 and then transform it into an algebraic equation for \( k_{210} \) by means of the substitution \( k = l' \) and \( k' = l \).

Although it is theoretically possible to construct \( \Omega_{210}(k, l) \), the labor is formidable beyond belief and the coefficients are so huge and unwieldy as to make it impossible in practice. We just have to recall the coefficients of the equation for \( j(\sqrt{-210}) \) that we saw in Section 1.4. Ramanujan unquestionably did not follow this procedure.

Then what did he do?

Fortunately, he indicated the general procedure; but unfortunately, he left no clue about how to carry out the hardest part.

2.4.1 Ramanujan’s invariant \( g_n \)

One of Jacobi’s great discoveries is the infinite product representation of the moduli \( k \) and \( k' \) (see [3]). Let

\[ q := e^{-\pi K'/K}. \]

Then Jacobi’s marvelous formulas are:

\[ k(q) = 4\sqrt{q} \left[ \frac{1 + q^2}{1 + q} \frac{1 + q^4}{1 + q^3} \frac{1 + q^6}{1 + q^5} \frac{1 + q^8}{1 + q^7} \cdots \right]^4 \]

\[ k'(q) = \left[ \frac{1 - q}{1 + q} \frac{1 - q^3}{1 + q^3} \frac{1 - q^5}{1 + q^5} \frac{1 - q^7}{1 + q^7} \cdots \right]^4. \]
Needless to say, Ramanujan was intimately familiar with both of these formulas. Indeed, it is highly probable that he (re)discovered them! If we assume \( K'/K(k) = \sqrt{n} \), then we follow Ramanujan [17] and we define the function \( g_n \) by the equation:

\[
g_n := 2^{-\frac{1}{2}} e^{\frac{\pi \sqrt{n}}{24}} (1 - e^{-\pi \sqrt{n}})(1 - e^{-3\pi \sqrt{n}})(1 - e^{-5\pi \sqrt{n}}) \ldots
\]

(2.13)

where we have used boldface type for emphasis. Then (2.12) and (2.13) and some algebra permit us to conclude that

\[
\frac{k'^2}{k} = 2g_{12}^2,
\]

(2.14)

and solving (2.14) for the singular modulus \( k \equiv k_n \), we obtain the formula

\[
k_n = g_n^6 \left( \sqrt{g_{12}^2 + \frac{1}{g_{12}^2}} - g_n^6 \right).
\]

(2.15)

Watson [18] was the first person to point out Ramanujan’s TWO-STEP ALGORITHM:

- Step 1: compute \( g_n \),
- Step 2: compute \( k_n \).

Because of (2.15), all the difficulty is concentrated in the computation of \( g_n \).

2.4.2 "a very curious algebraical lemma"

If we recall our earlier computation of \( k_2 \), \( k_3 \), and \( k_7 \) in Section 2.3, we note that they were all representable as products of \textit{quadratic units}, i.e., of quantities of the form \( \sqrt{A} - \sqrt{A - 1} \). Similar computations give us the values

\[
\begin{align*}
k_6 & = (2 - \sqrt{3})(\sqrt{3} - \sqrt{2}), \\
k_{10} & = (\sqrt{10} - 3)(3 - 2\sqrt{2}), \\
k_{30} & = (5 - 2\sqrt{6})(4 - \sqrt{15})(\sqrt{6} - \sqrt{5})(2 - \sqrt{3}),
\end{align*}
\]

(2.16)

which are again all products of quadratic units, as is Ramanujan’s value of \( k_{210} \), a fact noted earlier.

Unfortunately, the formula (2.15):

\[
k_n = g_n^6 \left( \sqrt{g_{12}^2 + \frac{1}{g_{12}^2}} - g_n^6 \right)
\]

\footnote{We do not claim that \textsc{Ramanujan} was the first person to use this algorithm; we only claim that this was the one he used.}
does not easily transform itself into a product of quadratic units. Perhaps the reader would like to try to transform (2.15) into (2.16) by using the value

\[ g_{30}^{6} = (3 + \sqrt{10})(2 + \sqrt{5}). \]

Observe that \( g_{30} \) is a product of units. The same is true of all \( g_{n} \) used by Ramanujan. Writing (2.14) in the form

\[ \frac{1}{k} - k = 2g_{12}^{12}, \]

we see that Ramanujan had to solve the quadratic equation (2.17), in which \( g_{n} \) is a product of units, and then express the root \( k_{n} \) again as a product of units.

We now quote Hardy [13]:

"This result Ramanujan [achieved], granted the value of \([g_{n}]\), by a very curious algebraical lemma."

For brevity, we refer to this lemma as the “VCAL.” We now reproduce Watson’s statement [18] of the VCAL, which he, in turn, copied from Ramanujan’s notebook (with a few changes in notation):

**THEOREM 8 (VCAL).** Suppose that

1. \( uv := g_{n}^{6} \),

2. \( 2U := u^{2} + \frac{1}{u^{2}} \); \( 2V := v^{2} + \frac{1}{v^{2}} \),

3. \( W := \sqrt{U^{2} + V^{2} - 1} \), and

4. \( 2S := U + V + W + 1 \);

then

\[ \alpha = (\sqrt{S} - \sqrt{S - 1})^{2}(\sqrt{S - U} - \sqrt{S - U - 1})^{2} \]
\[ \cdot (\sqrt{S - V} - \sqrt{S - V - 1})^{2}(\sqrt{S - W} - \sqrt{S - W - 1})^{2}. \]

The VCAL shows that \( \alpha_{n} \) (and therefore \( k_{n} \)) is a product of units in certain algebraic fields. Watson proved the VCAL by verification without any indication of its origin. A more natural proof may be found in Berndt [1].

We have found the following alternate version of the VCAL, which we believe to be new, though it can be obtained from the work of Chan [4], the “AVCAL”, to be useful.

**THEOREM 9 (AVCAL).** If

\[ \sqrt{\alpha} := \sqrt{ab} + \sqrt{(a + 1)(b - 1)}, \]
\[ \sqrt{\beta} := \sqrt{cd} + \sqrt{(c - 1)(d - 1)}, \]

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then
\[
x_1 := (\sqrt{a+1} - \sqrt{a})(\sqrt{b} - \sqrt{b-1})(\sqrt{c} - \sqrt{c-1})(\sqrt{d} - \sqrt{d-1}),
\]
\[
x_2 := -(\sqrt{a+1} + \sqrt{a})(\sqrt{b} + \sqrt{b-1})(\sqrt{c} + \sqrt{c-1})(\sqrt{d} + \sqrt{d-1})
\]
are the roots of
\[
\frac{1}{x} - x = 2\left[\sqrt{\alpha \beta} + \sqrt{(\alpha + 1)(\beta - 1)}\right].
\]
\[\square\]

The proof is quite simple. We offer the following two examples.

2.4.3 The computation of \(\alpha_{30}\)

From Ramanujan’s paper “Modular equations and approximations to \(\pi\)” [17, p. 26], we find that

\[
g_{30}^{12} = (2 + \sqrt{5})^2(3 + \sqrt{10})^2
\]
\[
= 171 + 54\sqrt{10} + 76\sqrt{5} + 120\sqrt{2}.
\]

Take

\[
\sqrt{\alpha \beta} := 171 + 54\sqrt{10}, \\
\sqrt{(\alpha + 1)(\beta - 1)} := 76\sqrt{5} + 120\sqrt{2}.
\]

Then

\[
\alpha = 759 + 240\sqrt{10} = (8\sqrt{6} + 5\sqrt{15})^2
\]
\[
\implies \sqrt{\alpha} = 8\sqrt{6} + 5\sqrt{15},
\]

and

\[
\beta = 39 + 12\sqrt{10} = (2\sqrt{6} + \sqrt{15})^2
\]
\[
\implies \sqrt{\beta} = 2\sqrt{6} + \sqrt{15}.
\]

Our value of \(\alpha\) and (2.18) suggest that we take

\[
\sqrt{ab} := 8\sqrt{6}
\]
\[
\implies ab = (8\sqrt{6})^2 = 384,
\]

and

\[
\sqrt{(a + 1)(b - 1)} := (5\sqrt{15})
\]
\[
\implies (a + 1)(b - 1) = (5\sqrt{15})^2 = 375.
\]
Solving for $a$ and $b$, we obtain

$$a = 24, \quad b = 16.$$  

Similarly,

$$cd = 24, \quad (c - 1)(d - 1) = 15,$$

and therefore

$$c = 6, \quad d = 4.$$  

Since $0 \leq \alpha < 1$, we conclude from the AVCAL that

$$k_{30} = (5 - 2\sqrt{6})(4 - \sqrt{15})(\sqrt{6} - \sqrt{5})(2 - \sqrt{3})$$

as given in Berndt’s paper “Ramanujan’s singular moduli” [2, Thm. 2.1].

### 2.4.4 The computation of $k_{210}$

This is the computation that inspired this paper. From Weber’s tables [19, 20], or Watson’s paper [18], we learn that

$$g_{210}^{12} = \left(\sqrt{3} + \sqrt{2} \cdot \sqrt{3}\sqrt{14 + 5\sqrt{5}} \cdot \sqrt{7 + \sqrt{3}} \cdot \sqrt{5 + 1}\right)^{12}$$

$$= 120134025 + 53725540\sqrt{5} + 26215380\sqrt{21} + 11723880\sqrt{105} + 49044510\sqrt{6} + 32107152\sqrt{14} + 21933360\sqrt{30} + 14358762\sqrt{70}.$$  

Now take

$$\sqrt{\alpha\beta} := 120134025 + 53725540\sqrt{5} + 26215380\sqrt{21} + 11723880\sqrt{105},$$

$$\sqrt{(\alpha + 1)(\beta - 1)} := 49044510\sqrt{6} + 32107152\sqrt{14} + 21933360\sqrt{30} + 14358762\sqrt{70}.$$  

Solving for $\alpha$ and $\beta$, we obtain

$$\alpha = 120621959 + 53943744\sqrt{5} + 26321856\sqrt{21} + 11676456\sqrt{105}$$

$$= (3168\sqrt{3} + 2076\sqrt{7} + 1419\sqrt{15} + 928\sqrt{35})^2$$

$$\Rightarrow \sqrt{\alpha} = 3168\sqrt{3} + 2076\sqrt{7} + 1419\sqrt{15} + 928\sqrt{35},$$

$$\beta = 119648071 + 53508216\sqrt{5} + 26109336\sqrt{21} + 11676456\sqrt{105}$$

$$= (3156\sqrt{3} + 2068\sqrt{7} + 1413\sqrt{15} + 924\sqrt{35})^2$$

$$\Rightarrow \sqrt{\beta} = 3156\sqrt{3} + 2068\sqrt{7} + 1413\sqrt{15} + 924\sqrt{35}.$$  

Now since

$$\alpha = \sqrt{ab} + \sqrt{(a + 1)(b - 1)}$$

$$= 3168\sqrt{3} + 2076\sqrt{7} + 1419\sqrt{15} + 928\sqrt{35},$$
we take
\[ \sqrt{ab} := 2076\sqrt{7} + 1419\sqrt{15}, \]
\[ \sqrt{(a + 1)(b - 1)} := 3168\sqrt{3} + 928\sqrt{35}. \]

Solving for \(a\) and \(b\), we obtain
\[ a = 121983 + 11904\sqrt{105}, \quad b = 249 + 24\sqrt{105}. \]

Similarly, in
\[ \sqrt{\beta} = \sqrt{cd} + \sqrt{(c - 1)(d - 1)} \]
\[ = 3156\sqrt{3} + 2068\sqrt{7} + 1413\sqrt{15} + 924\sqrt{35}, \]
we take
\[ \sqrt{cd} := 2068\sqrt{7} + 1413\sqrt{15}, \]
\[ \sqrt{(c - 1)(d - 1)} := 3156\sqrt{3} + 924\sqrt{35}. \]

Solving for \(c\) and \(d\), we obtain
\[ c = 121489 + 11856\sqrt{105}, \quad d = 247 + 24\sqrt{105}. \]

Therefore our AVCAL gives us the following formula for the singular modulus \(k_{210}\):
\[
k_{210} = \left[ \sqrt{121984 + 11904\sqrt{105}} - \sqrt{121983 + 11904\sqrt{105}} \right]
\cdot \left[ \sqrt{249 + 24\sqrt{105}} - \sqrt{248 + 24\sqrt{105}} \right]
\cdot \left[ \sqrt{121489 + 11856\sqrt{105}} - \sqrt{121488 + 11856\sqrt{105}} \right]
\cdot \left[ \sqrt{247 + 24\sqrt{105}} - \sqrt{246 + 24\sqrt{105}} \right].
\]

Now we follow Watson’s computation [18]. The following identities hold:
\[ 121984 + 11904\sqrt{105} = (248 + 24\sqrt{105})^2, \]
\[ 121983 + 11904\sqrt{105} = (93\sqrt{7} + 64\sqrt{15})^2, \]
\[ 249 + 24\sqrt{105} = (12 + \sqrt{105})^2, \]
\[ 248 + 24\sqrt{105} = (6\sqrt{3} + 2\sqrt{35})^2, \]
\[ 121489 + 11856\sqrt{105} = (247 + 24\sqrt{105})^2, \]
\[ 121488 + 11856\sqrt{105} = (78\sqrt{10} + 38\sqrt{42})^2, \]
\[ 247 + 24\sqrt{105} = (4\sqrt{4} + 3\sqrt{15})^2, \]
\[ 246 + 24\sqrt{105} = (3\sqrt{14} + 2\sqrt{30})^2. \]
So also do the identities

\[
248 + 24\sqrt{105} - (93\sqrt{7} + 64\sqrt{15}) = (31 - 8\sqrt{15})(8 - 3\sqrt{7}) \\
= (4 - \sqrt{15})^2(8 - 3\sqrt{7}), \\
12 + \sqrt{105} - (6\sqrt{3} + 2\sqrt{35}) = (6 - \sqrt{35})(2 - \sqrt{3}), \\
247 + 24\sqrt{105} - (78\sqrt{10} + 38\sqrt{42}) = (13 - 2\sqrt{42})(19 - 6\sqrt{10}) \\
= (\sqrt{7} - \sqrt{6})^2(\sqrt{10} - 3)^2, \\
4\sqrt{4} + 3\sqrt{15} - (3\sqrt{14} + 2\sqrt{30}) = (3 - 2\sqrt{2})(\sqrt{15} - \sqrt{14}) \\
= (\sqrt{2} - 1)^2(\sqrt{15} - \sqrt{14}).
\]

Taken together, they transform our formula for \( k_{210} \) into

\[
k_{210} = \frac{(4 - \sqrt{15})^2(8 - 3\sqrt{7})(6 - \sqrt{35})(2 - \sqrt{3})}{(\sqrt{7} - \sqrt{6})^2(\sqrt{10} - 3)^2(\sqrt{2} - 1)^2(\sqrt{15} - \sqrt{14})}
\]

which is the value that Ramanujan announced!

3 Kronecker’s Limit Formula and the Computation of \( g_{210} \)

Let’s review our progress up to now. Our task is to compute \( k_{210} \) by use of Ramanujan’s two-step algorithm. We have just computed it, granted the value of \( g_{210} \). Thus we still have to compute the value of \( g_{210} \).

There are at least four methods available for computing \( g_n \):

1. The Kronecker Limit Formula,
2. Watson’s empirical process,
3. Modular equations,
4. Class field theory.

In this paper we will deal with Kronecker’s limit formula. We begin with an introductory sketch of the theory of quadratic forms, using the form \( x^2 + 210y^2 \), fundamental to the computation of Ramanujan’s result, as the underlying example, then we state the formula itself, and its specialization to the computation of Ramanujan’s function \( g_n \), and finally we carry out the detailed computation of the numerical value of \( g_{210} \). Since the formula is a statement about quadratic forms, we must first introduce the basic concepts and terminology of this beautiful classical theory.
3.1 Binary quadratic forms

DEFINITION 2. A binary quadratic form $F$ (or just a form) is a polynomial in two variables $X$ and $Y$, given by

$$F \equiv F(X, Y) := aX^2 + bXY + cY^2$$

with real coefficients $a$, $b$, and $c$.

1. $F$ is rational iff $a$, $b$, and $c$ are in $\mathbb{Q}$,
2. $F$ is integral iff $a$, $b$, and $c$ are in $\mathbb{N}$,
3. the discriminant of $F$, $\Delta$, is defined by

$$\Delta \equiv \Delta(F) := b^2 - 4ac,$$

4. $F$ is positive definite iff

$$\Delta < 0, \quad a > 0, \quad \text{and} \quad c > 0.$$

DEFINITION 3. An integral form $F(X, Y) := aX^2 + bXY + cY^2$ represents an integer $n$ iff the equation

$$aX^2 + bXY + cY^2 = n$$

has a solution in integers $X$ and $Y$.

Example 4. The polynomial

$$F(X, Y) := X^2 + 210Y^2$$

is a positive definite integral binary quadratic form of discriminant $\Delta = -840$, and any number represented by $F(X, Y)$ is also represented by the form

$$G(X, Y) := 5266X^2 - 8424XY + 3369Y^2,$$

which again has discriminant $\Delta(G) = -840$. However, even though the form

$$H(X, Y) := 14X^2 + 15Y^2$$

also is positive definite, integral, binary, quadratic and has discriminant $\Delta(H) = -840$, it turns out that $H(X, Y)$ represents no number represented by $F$ and $G$, and vice versa.

The explanation of this example is to be found in the notion of equivalence.
**Definition 4.** \( GL_2(\mathbb{Z}) \) is the multiplicative group of \( 2 \times 2 \) matrices

\[
g := \begin{pmatrix} r & s \\ t & u \end{pmatrix}
\]

such that \( r, s, t \) and \( u \) are integers which satisfy \( ru - st = \pm 1 \).

\( SL_2(\mathbb{Z}) \) is the subgroup of matrices in \( GL_2(\mathbb{Z}) \) with \( ru - st = +1 \).

**Example 5.** The matrix \( g = \begin{pmatrix} 4 & -5 \\ -3 & 4 \end{pmatrix} \) lies in \( SL_2(\mathbb{Z}) \).

**Definition 5.** Given a form \( F(X, Y) := aX^2 + bXY + cY^2 \) and a matrix

\[
g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in SL_2(\mathbb{Z}),
\]

we define the form \( gF \) by the formula

\[
gF := a(rX + tY)^2 + b(rX + tY)(sX + uY) + c(sX + uY)^2.
\]

That is, \( gF \) is got from \( F \) by making the substitution

\[
X \mapsto rX + tY, \quad Y \mapsto sX + uY.
\]

**Example 6.** The transform of

\[
F(X, Y) := X^2 + 210Y^2 \quad \text{under} \quad g = \begin{pmatrix} 4 & -5 \\ -3 & 4 \end{pmatrix}
\]

is

\[
gF := (4X - 3Y)^2 + 210(-5X + 4Y)^2
= 5266X^2 - 8424XY + 3369Y^2
\equiv G(X, Y),
\]

and \( G(X, Y) \) is the form introduced in Example 4.

**Definition 6.** Two forms \( F \) and \( F' \) are equivalent iff there exists a \( g \in GL_2(\mathbb{Z}) \) such that \( F' = gF \).

If there exists \( g \in SL_2(\mathbb{Z}) \) such that \( F' = gF \), we say that \( F \) and \( F' \) are properly equivalent.

**Example 7.** The forms \( F(X, Y) \) and \( G(X, Y) \) of Example 4 and Example 6 are properly equivalent.

We assume as known the following facts about quadratic forms. The details may be found in Flath [10].
1. Integral forms that are equivalent represent precisely the same integers.

2. There is only a finite number, \( h \equiv h(\Delta) \), of proper equivalence classes of positive definite integral forms of a given discriminant \( \Delta \). This number \( h \) is called the class number.

3. A positive definite form \( F(X, Y) := aX^2 + bXY + cY^2 \) is called reduced iff
   - \(|b| \leq a \leq c\),
   - \(|b| = a \implies b = a\), and
   - \(a = c \implies b \geq 0\).

4. There is a unique reduced form in every proper equivalence class of positive definite integral forms.

Example 8. The form \( x^2 + 210y^2 \) has discriminant \( \Delta = -840 \). Let us compute all of the reduced forms of discriminant \( \Delta = -840 \).

The following Lemma is useful in numerical work.

**Lemma 10.** If \( F \) is reduced and \( \Delta < 0 \), then

\[
a \leq \sqrt{\frac{|\Delta|}{3}}.
\]

**Proof.**

\[
4a^2 \leq 4ac = b^2 - \Delta \leq a^2 - \Delta
\]

\[
\implies 3a^2 \leq -\Delta = |\Delta|
\]

\[
\implies a \leq \sqrt{\frac{|\Delta|}{3}}.
\]

In our case, \( a \leq \sqrt{840/3} = 16.7 \ldots \) and the candidates for \( a \) are \( a = 1, 2, 3, \ldots, 16 \).

Case \( a = 1 \): Then \(|b| = 0, 1\) since \(|b| \leq a\). If \(|b| = 0\), then

\[
c = \frac{b^2 + 840}{4a} = 210,
\]

and therefore we obtain the reduced form

\[F(X, Y) = X^2 + 210Y^2.\]

If \(|b| = 1\), then \(c = (1 + 840)/4\) is not an integer, and we have exhausted all possibilities with \( a = 1 \).

Case \( a = 2 \): Then \(|b| = 0, 1, 2\). Now \(|b| = 0 \implies c = 840/8 = 105\), and therefore

\[F(X, Y) = 2X^2 + 105Y^2.\]
On the other hand, $|b| = 1 \implies c = 841/8$, not an integer, and $|b| = 2 \implies c = 844/8$, not an integer either.

The remaining values of $a$ are treated in the same way and we simply tabulate the final results.

**THEOREM 11.** The reduced forms with $\Delta = -840$ are:

|   |   |
|---|---|
| 1. | $X^2 + 210Y^2$ |
| 2. | $2X^2 + 105Y^2$ |
| 3. | $3X^2 + 70Y^2$ |
| 4. | $5X^2 + 42Y^2$ |
| 5. | $6X^2 + 35Y^2$ |
| 6. | $7X^2 + 30Y^2$ |
| 7. | $10X^2 + 21Y^2$ |
| 8. | $14X^2 + 15Y^2$ |

Since there are 8 primitive reduced forms,

$$h(-840) = 8,$$

or, simply, *the class number is 8.*

### 3.2 Kronecker’s limit formula

Let $(A,B,C)$ denote the positive definite quadratic form

$$Q(X,Y) := AX^2 + 2BXY + CY^2.$$

We call

$$m := AC - B^2$$
the **determinant** of \( Q(X, Y) \). These so-called *Gauss forms* are a special case of the general binary forms in the first part, but we will only deal with these Gauss forms from now on. We also point out that the **determinant** of a Gauss form plays the role of the **discriminant** for the general binary forms. We could express all of our results in terms of discriminants, instead of determinants, but there is no need to do so.

The sum
\[
S(s) := \sum' \frac{1}{(AX^2 + 2BXY + CY^2)^s},
\]
which is taken over all positive and negative integers \( X \) and \( Y \) with the exception of \( X = Y = 0 \), converges absolutely (and therefore independently of the order of the terms) as long as \( s > 1 \), to a finite value which is a **function** of \( s \).

But, when \( s \to 1^+ \), the sum \( S(s) \) **diverges** to infinity, and the Kronecker "**Grenzformel**" gives us the first two terms in the Laurent expansion of \( S(s) \) around \( s = 1 \):
\[
S(s) = \frac{a-1}{s-1} + a_0 + a_1(s - 1) + \cdots
\]

**THEOREM 12 (Kronecker’s Limit Formula).**

\[
\lim_{s \to 1^+} \left\{ \sum' \frac{1}{(AX^2 + 2BXY + CY^2)^s} - \frac{\pi}{\sqrt{m}} \cdot \frac{1}{s-1} \right\} = -\frac{2\pi\gamma}{\sqrt{m}} + \frac{\pi}{\sqrt{m}} \ln \left( \frac{A}{4m} \right) - \frac{2\pi}{\sqrt{m}} \ln \eta(\omega_1)\eta(\omega_2),
\]

where
\[
\gamma := \text{Euler’s constant} = 0.5772 \ldots
\]
\[
\omega_1 := \frac{B + i\sqrt{m}}{A}, \quad \omega_2 := \frac{-B + i\sqrt{m}}{A},
\]
\[
\eta(\omega) := e^{\frac{2\pi i}{\omega}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \omega}).
\]

This statement of the Kronecker **Grenzformel** is taken from Weber [19]. Kronecker uses his formula to prove the following result.

**THEOREM 13 (Fundamental Lemma).** If \((A, B, C)\) and \((A_1, B_1, C_1)\) are two forms with determinant \( m \), then

\[
\lim_{s \to 1^+} \left\{ \sum' \frac{1}{(AX^2 + 2BXY + CY^2)^s} - \sum' \frac{1}{(A_1X^2 + 2B_1XY + C_1Y^2)^s} \right\} = \frac{2\pi}{\sqrt{m}} \ln \left\{ \left( \frac{A}{A_1} \right)^{\frac{1}{2}} \cdot \frac{\eta(\Omega_1)}{\eta(\omega_1)} \cdot \frac{\eta(\Omega_2)}{\eta(\omega_2)} \right\}
\]

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where
\[
\begin{align*}
\omega_1 &= \frac{B + i\sqrt{m}}{A}, \quad \omega_2 = -\frac{B + i\sqrt{m}}{A}, \\
\Omega_1 &= \frac{B_1 + i\sqrt{m}}{A_1}, \quad \Omega_2 = -\frac{B_1 + i\sqrt{m}}{A_1}.
\end{align*}
\]

**Proof.** Apply the Grenzformel to \((A, B, C)\) and \((A_1, B_1, C_1)\) separately and subtract. The left hand side of the difference is the left hand side of the stated proposition, while the right hand side equals
\[
\frac{\pi}{\sqrt{m}} \left\{ \ln \left( \frac{A}{4m} \right) - \ln \left( \frac{A_1}{4m} \right) - 2 \left[ \ln \eta(\omega_1) \eta(\omega_2) - \ln \eta(\Omega_1) \eta(\Omega_2) \right] \right\}
\]
\[
= \frac{\pi}{\sqrt{m}} \left\{ \ln \left( \frac{A}{A_1} \right)^{\frac{1}{2}} \left[ \frac{\eta(\Omega_1)}{\eta(\omega_1)} \right]^{\frac{1}{2}} \left[ \frac{\eta(\Omega_2)}{\eta(\omega_2)} \right]^{\frac{1}{2}} \right\}
\]
\[
= \frac{2\pi}{\sqrt{m}} \left\{ \ln \left( \frac{A}{A_1} \right)^{\frac{1}{2}} \left[ \frac{\eta(\Omega_1)}{\eta(\omega_1)} \right] \left[ \frac{\eta(\Omega_2)}{\eta(\omega_2)} \right] \right\}. \tag*{\Box}
\]

**Note:** \(\omega_1\) and \(\omega_2\) are called the roots of the form \((A, B, C)\).

### 3.3 The relation with Ramanujan’s function \(g_n\)

The key observation which permits us to apply Kronecker’s Fundamental Lemma to the computation of \(g_{210}\) is that the eight reduced forms of determinant 210 (or of discriminant \(-840\)) can be paired off as follows:

\[
\begin{align*}
X^2 + 210Y^2 &\quad \Leftrightarrow \quad 2X^2 + 105Y^2, \\
3X^2 + 70Y^2 &\quad \Leftrightarrow \quad 6X^2 + 35Y^2, \\
5X^2 + 42Y^2 &\quad \Leftrightarrow \quad 10X^2 + 21Y^2, \\
7X^2 + 30Y^2 &\quad \Leftrightarrow \quad 14X^2 + 15Y^2.
\end{align*}
\]

The “pairing” is of the forms

\[
[AX^2 + 2CY^2] \Leftrightarrow [2AX^2 + CY^2]
\]

and the “roots” of the forms on each side are respectively \(\omega\) and \(\Omega\), i.e.,

\[
\omega = \omega_1 = \omega_2 = \frac{\sqrt{-m}}{A} \quad \Leftrightarrow \quad \Omega = \Omega_1 = \Omega_2 = \frac{\sqrt{-m}}{2A} = \frac{\omega}{2}
\]
that is, the root $\omega$ is paired off with the root $\omega/2$. Applying the Fundamental Lemma to this pair of forms $(A, 0, 2C)$ and $(2A, 0, C)$, we obtain

$$\lim_{s \to 1^+} \left\{ \sum' \frac{1}{(AX^2 + 2CY^2)^s} - \sum' \frac{1}{(2AX^2 + CY^2)^s} \right\}$$

$$= \frac{2\pi}{\sqrt{m}} \ln \left( \left( \frac{A}{2A} \right)^{\frac{1}{2}} \frac{\eta(\omega/2)}{\eta(\omega)} \frac{\eta(\omega/2)}{\eta(\omega)} \right)$$

$$= \frac{2\pi}{\sqrt{m}} \ln \left( \left( \frac{1}{2} \right)^{\frac{1}{2}} \left[ \frac{\eta(\omega/2)}{\eta(\omega)} \right]^2 \right).$$

But, if $q := e^{\pi i \omega}$, then

$$\eta(\omega) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n}),$$

and so

$$\eta(\omega/2) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Therefore,

$$\frac{\eta(\omega/2)}{\eta(\omega)} = \frac{q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)}{q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n})}$$

$$= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})$$

$$= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n-1}).$$

Since $B = 0$ for both of our forms, we conclude that $\omega = \frac{i\sqrt{m}}{A}$, and so the right-hand side becomes

$$e^{-\frac{\pi i \sqrt{m}}{24A}} \prod_{n=1}^{\infty} [1 - e^{-(2n-1)\pi \sqrt{m}/A}] = 2^\frac{1}{4} g_{m/A^2}$$

as we see from the definition of $g_n$, given in equation (2.13). Therefore

$$\lim_{s \to 1^+} \left\{ \sum' \frac{1}{(AX^2 + 2CY^2)^s} - \sum' \frac{1}{(2AX^2 + CY^2)^s} \right\} = \frac{2\pi}{\sqrt{m}} \ln \left( \left( \frac{1}{2} \right)^{\frac{1}{2}} \left\{ 2^\frac{1}{4} g_{m/A^2} \right\}^2 \right)$$

$$= \frac{4\pi}{\sqrt{m}} \ln g_{m/A^2}.$$
We have therefore obtained the following formula.

**THEOREM 14 (Kronecker’s Formula for $g_n$).**

$$\lim_{s \to 1^+} \left\{ \sum' \frac{1}{(AX^2 + 2CY^2)^s} - \sum' \frac{1}{(2AX^2 + CY^2)^s} \right\} = \frac{4\pi}{\sqrt{m}} \ln \frac{g_m}{A^2}. \quad (G)$$

This is the formula which we will use to compute $g_{210}$.

Kronecker, himself, expressed the right hand side of his Grenzformel in terms of the theta functions, while we have chosen Ramanujan’s notation. This statement of the Kronecker formula in terms of the Ramanujan function $g_n$ has not been explicitly stated in the literature (although it is clearly implicitly contained in Weber’s work).

Following earlier work of Dirichlet, Kronecker and Weber [19, 20] summed both sides of \((G)\) over all reduced forms of determinant $m$. However, and here comes another “brilliancy,” they weighted the sum by multiplying each of the $h(m) \equiv h(\Delta)$ summands by the Jacobi symbol $\left( \frac{\delta}{A + 2C} \right)$ (see Niven and Zuckerman [16]) where $\delta$ is a fundamental discriminant. We will define this term later on.

Since the weights are numerically equal to $+1$ or $-1$, one hopes for useful cancellations. Miraculously everything cancels (!) and one obtains a sum computable from the Dirichlet class number formulas.

### 3.4 The Kronecker–Weber computation of the weighted sum for $m = 210$

The sum which we wish to compute is
\[ R := \lim_{s \to 1^+} \left\{ \left( \frac{\delta}{1} \right) \left[ \sum' \frac{1}{(X^2 + 210Y^2)^s} - \sum' \frac{1}{(2X^2 + 105Y^2)^s} \right] + \left( \frac{\delta}{107} \right) \left[ \sum' \frac{1}{(2X^2 + 105Y^2)^s} - \sum' \frac{1}{(X^2 + 210Y^2)^s} \right] \right. \\
\left. + \left( \frac{\delta}{73} \right) \left[ \sum' \frac{1}{(3X^2 + 70Y^2)^s} - \sum' \frac{1}{(6X^2 + 35Y^2)^s} \right] + \left( \frac{\delta}{47} \right) \left[ \sum' \frac{1}{(5X^2 + 42Y^2)^s} - \sum' \frac{1}{(10X^2 + 21Y^2)^s} \right] \right. \\
\left. + \left( \frac{\delta}{41} \right) \left[ \sum' \frac{1}{(6X^2 + 35Y^2)^s} - \sum' \frac{1}{(3X^2 + 70Y^2)^s} \right] + \left( \frac{\delta}{37} \right) \left[ \sum' \frac{1}{(7X^2 + 30Y^2)^s} - \sum' \frac{1}{(14X^2 + 15Y^2)^s} \right] \right. \\
\left. + \left( \frac{\delta}{31} \right) \left[ \sum' \frac{1}{(10X^2 + 21Y^2)^s} - \sum' \frac{1}{(5X^2 + 42Y^2)^s} \right] + \left( \frac{\delta}{29} \right) \left[ \sum' \frac{1}{(14X^2 + 15Y^2)^s} - \sum' \frac{1}{(7X^2 + 30Y^2)^s} \right] \right\}, \]

where \( \delta \) is a fundamental discriminant. By definition, this means that

1. \( \delta \) divides 210, and
2. \( \delta \equiv 1 \pmod{4} \),

which implies that

\( \delta = 1, -3, 5, -7, -15, 21, -35, 105. \)

Kronecker and Weber compute this sum in two different ways. Following their lead, first we will use Kronecker’s formula \((G)\) and obtain linear combinations of Ramanujan’s function \(g_n\). Second, we will use the Dirichlet class number formulas.

To carry out the first computation, we compute the following table of Jacobi symbols.
Applying the Kronecker formula (G) to the “pairs”, we find:

\[
\begin{align*}
\lim_{s \to 1^+} \left[ \sum' \frac{1}{(X^2 + 210Y^2)s} - \sum' \frac{1}{(2X^2 + 105Y^2)s} \right] &= \frac{4\pi}{\sqrt{210}} \ln g_{210}, \\
\lim_{s \to 1^+} \left[ \sum' \frac{1}{(3X^2 + 70Y^2)s} - \sum' \frac{1}{(6X^2 + 35Y^2)s} \right] &= \frac{4\pi}{\sqrt{210}} \ln g_{210/3^2}, \\
\lim_{s \to 1^+} \left[ \sum' \frac{1}{(5X^2 + 42Y^2)s} - \sum' \frac{1}{(10X^2 + 21Y^2)s} \right] &= \frac{4\pi}{\sqrt{210}} \ln g_{210/5^2}, \\
\lim_{s \to 1^+} \left[ \sum' \frac{1}{(14X^2 + 15Y^2)s} - \sum' \frac{1}{(7X^2 + 30Y^2)s} \right] &= \frac{4\pi}{\sqrt{210}} \ln g_{210/7^2}.
\end{align*}
\]

Therefore, the sum \( R \) becomes

\[
R = \left[ \left( \delta \right)_1 - \left( \delta \right)_{107} \right] \frac{4\pi}{\sqrt{210}} \ln g_{210} + \left[ \left( \delta \right)_{73} - \left( \delta \right)_{41} \right] \frac{4\pi}{\sqrt{210}} \ln g_{210/3^2} + \left[ \left( \delta \right)_{47} - \left( \delta \right)_{31} \right] \frac{4\pi}{\sqrt{210}} \ln g_{210/5^2} + \left[ \left( \delta \right)_{37} - \left( \delta \right)_{29} \right] \frac{4\pi}{\sqrt{210}} \ln g_{210/7^2}.
\]

(3.1)

From the table, we see that

| \( \delta \) | \( \left( \delta \right)_{1} - \left( \delta \right)_{107} \) | \( \left( \delta \right)_{73} - \left( \delta \right)_{41} \) | \( \left( \delta \right)_{47} - \left( \delta \right)_{31} \) | \( \left( \delta \right)_{37} - \left( \delta \right)_{29} \) |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| -3 | 2 | 2 | -2 | 2 |
| 5 | 2 | -2 | -2 | -2 |
| -7 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 |
| 21 | 2 | -2 | 2 | 2 |
| -35 | 2 | 2 | 2 | -2 |
| 105 | 0 | 0 | 0 | 0 |
and four sums survive:

| $\delta$ | $\text{SUM}$ |
|---------|-------------|
| $-3$    | $\frac{8\pi}{\sqrt{210}} \{\ln g_{210} + \ln g_{210/3^2} - \ln g_{210/5^2} + \ln g_{210/7^2}\}$ |
| $5$     | $\frac{8\pi}{\sqrt{210}} \{\ln g_{210} - \ln g_{210/3^2} - \ln g_{210/5^2} - \ln g_{210/7^2}\}$ |
| $21$    | $\frac{8\pi}{\sqrt{210}} \{\ln g_{210} - \ln g_{210/3^2} + \ln g_{210/5^2} + \ln g_{210/7^2}\}$ |
| $-35$   | $\frac{8\pi}{\sqrt{210}} \{\ln g_{210} + \ln g_{210/3^2} + \ln g_{210/5^2} - \ln g_{210/7^2}\}$ |

If we sum these four survivors, the \textit{miraculous cancellation takes place} and we obtain:

$$R = \frac{8\pi}{\sqrt{210}} \cdot 4 \cdot \ln g_{210} = \frac{32\pi}{\sqrt{210}} \ln g_{210} \hspace{1cm} (3.2)$$

a truly astonishing result!

This method of summing the series $R$ to obtain the four series in (3.1) to obtain the final sum (3.2) taken over all values of the remaining $\delta$'s is Kronecker’s novel contribution. Weber [19, 20] refined it into a broadly applicable tool. We have presented the simplest version possible so as to recapture the classical beauty and elegance of Kronecker’s original version.

### 3.5 The Dirichlet computation of the weighted sum

We group the coefficients of the \textit{same} summand in $R$ so that $R$ becomes:
\[ R = \lim_{s \to 1} \left\{ \left[ \left( \frac{\delta}{41} \right) - \left( \frac{\delta}{73} \right) \right] \sum' \frac{1}{(X^2 + 35Y^2)^s} \right. \]

\[ + \left[ \left( \frac{\delta}{107} \right) - \left( \frac{\delta}{1} \right) \right] \sum' \frac{1}{(2X^2 + 105Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{7} \right) - \left( \frac{\delta}{41} \right) \right] \sum' \frac{1}{(3X^2 + 70Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{47} \right) - \left( \frac{\delta}{31} \right) \right] \sum' \frac{1}{(5X^2 + 42Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{41} \right) - \left( \frac{\delta}{73} \right) \right] \sum' \frac{1}{(6X^2 + 35Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{37} \right) - \left( \frac{\delta}{29} \right) \right] \sum' \frac{1}{(7X^2 + 42Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{31} \right) - \left( \frac{\delta}{47} \right) \right] \sum' \frac{1}{(10X^2 + 21Y^2)^s} \]

\[ + \left[ \left( \frac{\delta}{29} \right) - \left( \frac{\delta}{37} \right) \right] \sum' \frac{1}{(14X^2 + 15Y^2)^s} \}

To see what’s going on, we examine a particular sum, say

\[ \left[ \left( \frac{\delta}{41} \right) - \left( \frac{\delta}{73} \right) \right] \sum' \frac{1}{(6X^2 + 35Y^2)^s}. \]  

(3.3)

First we observe that

\[ \left( \frac{\delta}{41} \right) = - \left( \frac{\delta}{73} \right), \]

so that (3.3) can be written

\[ 2 \left( \frac{\delta}{41} \right) \sum' \frac{1}{(6X^2 + 35Y^2)^s} \].

Let’s look at a particular summand. For example, take

\[ X = 3Y = 7. \]

Then \( 6X^2 + 35Y^2 = 1769 \). The summand \( \frac{1}{(1769)^s} \) appears once for every pair of integers \((X, Y)\) which satisfies

\[ 6X^2 + 35Y^2 = 1769. \]

Since, as it is easy to verify (see below), the set of solutions is

\[ (X, Y) = (\pm 3, \pm 7) \text{ and } (\pm 17, \pm 1), \]
each of the possible eight pairs contributes 1 to the coefficient of \( \frac{1}{(1769)^s} \). We therefore conjecture that the total contribution of the denominator \((1769)^s\) to the sum \(R\) is equal to

\[
\frac{8}{(1769)^s}.
\]

How do we know that the eight listed pairs \((X, Y)\) exhaust all solutions of \(6X^2 + 35Y^2 = 1769\), or that there are no other summands from the other reduced forms of determinant 210?

Dirichlet, himself, answered this question when he found the formula for the total number of proper representations of an integer \(N\) by the forms of determinant \(m\): see Cox [6]. If \(f(N)\) denotes this number of representations, then

\[
f(N) = 2 \sum_{d | N, d \geq 1} \left( \frac{-m}{d} \right)
\]

where \(\left( \frac{-m}{d} \right)\) is the Jacobi symbol. In our case

\[
f(1729) = 2 \left\{ \left( \frac{-210}{1} \right) + \left( \frac{-210}{29} \right) + \left( \frac{-210}{61} \right) + \left( \frac{-210}{61} \right) \right\} = 8,
\]

which verifies that the eight pairs we displayed are all of them.

Thus, the total contribution to the sum \(R\) is

\[
4 \left( \frac{\delta}{41} \right) \left\{ \left( \frac{-210}{1} \right) + \left( \frac{-210}{29} \right) + \left( \frac{-210}{61} \right) + \left( \frac{-210}{1769} \right) \right\} \frac{1}{(1769)^s}.
\]

Now we use some properties of the Jacobi symbol:

\[
\left( \frac{m}{p} \right) \left( \frac{m}{p} \right) = \left( \frac{mn}{p} \right) \quad \text{and} \quad \left( \frac{m}{pq} \right) \left( \frac{m}{q} \right) = \left( \frac{m}{p} \right),
\]

as well as

\[
\left( \frac{\delta}{41} \right) = \left( \frac{\delta}{1769} \right).
\]
Therefore, if $\delta \delta' = 210$, the expression in (3.4) equals

$$4 \left( \frac{\delta}{41} \right) \left\{ \left( -\frac{210}{1} \right) + \left( -\frac{210}{29} \right) + \left( -\frac{210}{61} \right) + \left( -\frac{210}{1769} \right) \right\}$$

(1769)$s$

$$= \frac{4 \left( \frac{\delta}{1769} \right) \left\{ \left( -\frac{210}{1} \right) + \left( -\frac{210}{29} \right) + \left( -\frac{210}{61} \right) + \left( -\frac{210}{1769} \right) \right\}}{(1769)^s}$$

$$= \frac{4 \left\{ \left( -\frac{210}{1} \right) + \left( -\frac{210}{29} \right) + \left( -\frac{210}{61} \right) + \left( -\frac{210}{1769} \right) \right\}}{(1769)^s} .$$

But this is the coefficient of $\frac{1}{(1769)^s}$ in the expansion of the product series:

$$4 \left\{ \sum_{n=1}^{\infty} \frac{(-3/2) \ 1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \frac{70 \ 1}{m^s} \right\} ,$$

(3.5)

and we conclude that

**The sum $R$ is equal to the product (3.5).**

We boxed this statement since it is, in fact, a fundamental theorem in the theory of these series.

Therefore, the first line in our table of surviving sums is

$$\frac{8\pi}{\sqrt{210}} \left\{ \ln g_{210} + \ln \frac{g_{210}}{3^2} - \ln \frac{g_{210}}{5^2} + \ln g_{210}/7^2 \right\}$$

$$= 4 \left\{ \sum_{n=1}^{\infty} \frac{(-3/2) \ 1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \frac{70 \ 1}{m^s} \right\}$$

$$= 4 \left\{ \sum_{n=1}^{\infty} \frac{(-3/2) \ 1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \frac{280 \ 1}{m^s} \right\} .$$

If we carry out the same computation for the other three lines of surviving sums we
obtain the following four equations:

\[
\frac{8\pi}{\sqrt{210}} \left\{ \ln g_{210} + \ln g_{210/3^2} - \ln g_{210/5^2} + \ln g_{210/7^2} \right\} = 4 \left\{ \sum_{n=1}^{\infty} \left( -\frac{3}{n} \right) \frac{1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \left( \frac{280}{m} \right) \frac{1}{m^s} \right\},
\]

\[
\frac{8\pi}{\sqrt{210}} \left\{ \ln g_{210} - \ln g_{210/3^2} - \ln g_{210/5^2} - \ln g_{210/7^2} \right\} = 4 \left\{ \sum_{n=1}^{\infty} \left( \frac{5}{n} \right) \frac{1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \left( -\frac{168}{m} \right) \frac{1}{m^s} \right\},
\]

\[
\frac{8\pi}{\sqrt{210}} \left\{ \ln g_{210} - \ln g_{210/3^2} + \ln g_{210/5^2} + \ln g_{210/7^2} \right\} = 4 \left\{ \sum_{n=1}^{\infty} \left( \frac{21}{n} \right) \frac{1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \left( -\frac{40}{m} \right) \frac{1}{m^s} \right\},
\]

\[
\frac{8\pi}{\sqrt{210}} \left\{ \ln g_{210} + \ln g_{210/3^2} + \ln g_{210/5^2} - \ln g_{210/7^2} \right\} = 4 \left\{ \sum_{n=1}^{\infty} \left( -\frac{35}{n} \right) \frac{1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \left( \frac{24}{m} \right) \frac{1}{m^s} \right\}.
\]  

(3.6)

3.5.1 The Dirichlet class number formulas

One of the glories of 19th century mathematics is Dirichlet’s formulas for the limit of the sums of the series appearing in (3.6) as \( s \to 1^+ \). Namely,

\[
\sum_{n=1}^{\infty} \left( \frac{\delta}{n} \right) \frac{1}{n} = \frac{\pi}{\sqrt{-\delta}} \cdot K(\delta) \quad \text{if} \quad \delta < 0, \tag{3.7}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{\delta}{n} \right) \frac{1}{n} = \frac{\ln \varepsilon}{\sqrt{\delta}} \cdot K(\delta) \quad \text{if} \quad \delta > 0, \tag{3.8}
\]

where \( K(\delta) \) is the number of properly primitive classes of discriminant \( \delta \) and

\[
\varepsilon := \frac{T + U\sqrt{\delta}}{2}
\]

is the minimal solution of the “even” Pell equation:

\[
T^2 - \delta U^2 = 4.
\]

These formulas are developed in numerous books including Weber [19].

3.5.2 The final computation of \( g_{210} \)

We make a table:
The values $K(\delta)$ in the table can be found in Flath [10]. Substituting these numbers in (3.7) and (3.8) we obtain, for example,

$$\sum_{n=1}^{\infty} \left( \frac{-3}{n} \right)^{1/n} = \frac{\pi}{\sqrt[3]{3}} \cdot K(-3) = \frac{\pi}{3\sqrt[3]{3}},$$

and

$$\sum_{n=1}^{\infty} \left( \frac{280}{n} \right)^{1/n} = \frac{K(280)}{\sqrt[8]{280}} \cdot \ln \left( \frac{T + U\sqrt{280}}{2} \right)$$

$$= \frac{4}{\sqrt[8]{280}} \cdot \ln \left( \frac{502 + 30\sqrt{280}}{2} \right)$$

$$= \frac{4}{\sqrt[8]{280}} \cdot \ln \left( 5\sqrt{5} + 3\sqrt{14} \right)^2$$

$$= \frac{8}{\sqrt[8]{280}} \cdot \ln \left( 5\sqrt{5} + 3\sqrt{14} \right).$$

Both of these summations have independent interest.

Therefore,

$$\frac{2\pi}{\sqrt[3]{210}} \left\{ \ln g_{210} + \ln g_{210/3^2} - \ln g_{210/5^2} + \ln g_{210/7^2} \right\} = \frac{\pi}{3\sqrt[3]{3}} \cdot \frac{8}{\sqrt[8]{280}} \cdot \ln \left( 5\sqrt{5} + 3\sqrt{14} \right),$$

or

$$\ln g_{210} + \ln g_{210/3^2} - \ln g_{210/5^2} + \ln g_{210/7^2} = \frac{1}{3} \ln \left( 5\sqrt{5} + 3\sqrt{14} \right)^2. \quad (3.9)$$
The Dirichlet class number formulas applied to the three remaining product series give us

$$\frac{4}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{5}{n} \right\} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \right\} = \frac{1}{\sqrt{210}} \frac{\ln \left( \frac{3 + \sqrt{5}}{2} \right)}{2} \cdot 4,$$

$$\frac{4}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{21}{n} \right\} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \right\} = \frac{1}{\sqrt{210}} \frac{\ln \left( \frac{5 + \sqrt{21}}{2} \right)}{2} \cdot 4,$$

$$\frac{4}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{35}{n} \right\} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \right\} = \frac{1}{\sqrt{210}} \frac{\ln \left( \frac{10 + 2\sqrt{24}}{2} \right)}{2} \cdot 4.$$

Thus the last three lines of (3.6) permit us to conclude

$$\ln g_{210} - \ln g_{210/3^2} - \ln g_{210/5^2} - \ln g_{210/7^2} = \ln \left( \frac{3 + \sqrt{5}}{2} \right),$$

$$\ln g_{210} - \ln g_{210/3^2} + \ln g_{210/5^2} + \ln g_{210/7^2} = \ln \left( \frac{5 + \sqrt{21}}{2} \right),$$

$$\ln g_{210} + \ln g_{210/3^2} + \ln g_{210/5^2} - \ln g_{210/7^2} = \ln \left( \frac{10 + 2\sqrt{24}}{2} \right). \tag{3.10}$$

Summing (3.9) and (3.10), we obtain

$$4 \ln g_{210} = \frac{2}{3} \ln \left( 5\sqrt{5} + 3\sqrt{14} \right) + \ln \left( \frac{3 + \sqrt{5}}{2} \right) + \ln \left( \frac{5 + \sqrt{21}}{2} \right) + \ln \left( \frac{10 + 2\sqrt{24}}{2} \right)$$

$$= \ln \left[ \left( 5\sqrt{5} + 3\sqrt{14} \right)^{\frac{2}{3}} \cdot \left( \frac{3 + \sqrt{5}}{2} \right)^{\frac{1}{3}} \cdot \left( \frac{5 + \sqrt{21}}{2} \right)^{\frac{1}{3}} \cdot \left( \frac{10 + 2\sqrt{24}}{2} \right)^{\frac{1}{3}} \right]$$

$$\Rightarrow g_{210} = \left( 5\sqrt{5} + 3\sqrt{14} \right)^{\frac{1}{3}} \cdot \left( \frac{3 + \sqrt{5}}{2} \right)^{\frac{1}{3}} \cdot \left( \frac{5 + \sqrt{21}}{2} \right)^{\frac{1}{3}} \cdot \left( \frac{10 + 2\sqrt{24}}{2} \right)^{\frac{1}{3}}$$

$$= \left( 5\sqrt{5} + 3\sqrt{14} \right)^{\frac{1}{3}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{\frac{1}{3}} \cdot \left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^{\frac{1}{3}} \cdot \left( \sqrt{2} + \sqrt{3} \right)^{\frac{1}{3}}$$

That is,

$$g_{210} = \sqrt{2 + \sqrt{3}} \cdot \sqrt[6]{5\sqrt{5} + 3\sqrt{14}} \cdot \sqrt[3]{\frac{\sqrt{3} + \sqrt{7}}{2}} \cdot \sqrt[6]{\frac{\sqrt{5} + 1}{2}}$$

This is the value of \( g_{210} \) which Watson copied from Ramanujan’s first draft of his notebook!
3.6 Final comments

Weber showed that these beautiful and miraculous computations are instances of general theorems, and *always occur* for each of the 65 idoneal numbers, and not just for \( m = 210 \). His classic exposition, however, never deals with the case of *even* \( m \) which (as we saw) converts the Kronecker Limit Formula into a statement about the elliptic modular function \( \frac{\eta(\omega/2)}{\eta(\omega)} \). Instead, he develops the entire theory for the related function \( e^{-\pi i/24} \frac{\eta((\omega + 1)/2)}{\eta(\omega)} \).

Since the discussion of the former function is nowhere to be found, we will carry it out in detail.

3.7 The miracle explained: Kronecker’s theory

Weber [19, 20] showed that the “miraculous cancellation” is no accident but rather is explainable *a priori*, and we will now develop his theory. Our proofs are modelled on Weber’s, but since we consider a very special case, we achieve considerable simplifications.

We make two simplifying assumptions:

1. *All reduced forms of determinant \( m \) are of the form*

   \[ AX^2 + CY^2. \]

   *This means that the determinant \( m = AC \) is a convenient number* [6, Thm. 3.22(ii)].

2. \( m = 2P \) where \( P \) is a product of *distinct odd primes*.

Therefore (as is easy to see directly, or also is proven in Theorem 3.22(v) of Cox [6]),

\[ h(m) = 2^t, \]

and the reduced forms break up into \( 2^t - 1 \) forms

\[ Q = AX^2 + 2CY^2 \]

and \( 2^t - 1 \) homologues

\[ Q = 2AX^2 + CY^2. \]

Accordingly, we can *pair them off*:

\[
\begin{align*}
Q_1 &:= A_1X^2 + 2C_1Y^2 &\quad& \Rightarrow &Q'_1 &:= 2A_1X^2 + C_1Y^2, \\
Q_2 &:= A_2X^2 + 2C_2Y^2 &\quad& \Rightarrow &Q'_2 &:= 2A_2X^2 + C_2Y^2, \\
&\vdots & & \quad & \vdots & \\
Q_T &:= A_TX^2 + 2C_TY^2 &\quad& \Rightarrow &Q'_T &:= 2A_TX^2 + C_TY^2,
\end{align*}
\]

where

\[ T := 2^{t-1}. \]
DEFINITION 7. The number $\delta$ is a fundamental discriminant if and only if

1. $\delta \equiv 1 \pmod{4}$ and $\delta$ is square-free; or
2. $\delta = 4\delta_1$, where $\delta_1 \not\equiv 1 \pmod{4}$ and $\delta_1$ is square-free.

Thus, every square-free odd number, taken with appropriate sign, is a fundamental discriminant.

The number of fundamental discriminants which divide our discriminant $D = -4m$ is therefore the number of odd divisors of $m$. But the number of odd divisors of $m$ is equal to

$$2^t = h(m).$$

We associate the Jacobi symbol

$$\chi(\delta; A, C) := \left(\frac{\delta}{A+C}\right) \equiv \chi(\delta; Q)$$

to the form $Q := AX^2 + BY^2$.

Then the sum, $R$, which we wish to compute is

$$R(\delta) := \lim_{s \to 1^+} \left\{ \chi(\delta; Q_1) \left[ \sum' \frac{1}{Q_1^s} - \sum' \frac{1}{Q_1'^s} \right] + \chi(\delta; Q'_1) \left[ \sum' \frac{1}{Q'_1^s} - \sum' \frac{1}{Q_1^s} \right] + \chi(\delta; Q_2) \left[ \sum' \frac{1}{Q_2^s} - \sum' \frac{1}{Q_2'^s} \right] + \chi(\delta; Q'_2) \left[ \sum' \frac{1}{Q_2'^s} - \sum' \frac{1}{Q_2^s} \right] + \cdots 
+ \chi(\delta; Q_h) \left[ \sum' \frac{1}{Q_h^s} - \sum' \frac{1}{Q_h'^s} \right] + \chi(\delta; Q'_h) \left[ \sum' \frac{1}{Q_h'^s} - \sum' \frac{1}{Q_h^s} \right] \right\},$$

where $\delta$ is any one of the $2^t$ fundamental discriminants which divide $D = -4m$.

Our final result comes from computing

$$H := \sum_\delta R(\delta),$$

where $\delta$ runs over the fundamental discriminants which divide $-4m$.

3.7.1 The first cancellation: Kronecker’s computation

The key step in Kronecker’s computation is the following Lemma which shows what happens when we compute the contribution, $R_1$, of the two “paired off” forms, $Q$ and $Q'$, to the sum $R$. 

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LEMMA 15.

\[ R_1 := \lim_{s \to 1^+} \left\{ \chi(\delta; Q) \left[ \sum' \frac{1}{Q^s} - \sum' \frac{1}{Q'^s} \right] + \chi(\delta; Q') \left[ \sum' \frac{1}{Q'^s} - \sum' \frac{1}{Q^s} \right] \right\} \]

\[ = \lim_{s \to 1^+} \left\{ \left[ 1 - \left( \frac{2}{\delta} \right) \right] \left[ \chi(\delta; Q) \sum' \frac{1}{Q^s} - \chi(\delta; Q') \sum' \frac{1}{Q'^s} \right] \right\}. \]

Proof. We begin with the “composition identity”:

\[ 2AX^2 + CY^2 \equiv (Ax_1^2 + 2Cy_1^2) (2x_2^2 + ACy_2^2), \]

where

\[ X := x_1x_2 + Ay_1y_2, \quad Y := 2x_1x_2 - Cy_1y_2. \]

This identity reveals the relation which every form \( Q \) has with its homologue.

We associate the Jacobi symbol

\[ \left( \frac{\delta}{2 + AC} \right) \]

to the form

\[ 2x_2^2 + ACy_2^2. \]

But, by the generalized law of quadratic reciprocity for Jacobi symbols:

\[ \left( \frac{\delta}{2 + AC} \right) = (-1)^{\frac{\delta - 1}{2}} \left( \frac{2 + AC}{\delta} \right) \]

and, since

\[ \delta \equiv 1 \pmod{4} \quad \text{and} \quad \delta \mid AC, \]

we conclude that

\[ (-1)^{\frac{\delta - 1}{2}} = 1 \]

and

\[ 2 + AC \equiv 2 \pmod{\delta}, \]

which mean

\[ \left( \frac{\delta}{2 + AC} \right) = \left( \frac{2}{\delta} \right), \]

and therefore

\[ \left( \frac{\delta}{2A + C} \right) = \left( \frac{2}{\delta} \right) \left( \frac{\delta}{A + 2C} \right), \quad \left( \frac{\delta}{A + 2C} \right) = \left( \frac{2}{\delta} \right) \left( \frac{\delta}{2A + C} \right), \]

and we obtain

\[ R_1 = \lim_{s \to 1^+} \left\{ \left[ \chi(\delta; Q) - \left( \frac{2}{\delta} \right) \chi(\delta; Q) \right] \sum' \frac{1}{Q^s} + \left[ \chi(\delta; Q') - \left( \frac{2}{\delta} \right) \chi(\delta; Q') \right] \sum' \frac{1}{Q'^s} \right\} \]

\[ = \lim_{s \to 1^+} \left\{ \left[ 1 - \left( \frac{2}{\delta} \right) \right] \left[ \chi(\delta; Q) \sum' \frac{1}{Q^s} - \chi(\delta; Q') \sum' \frac{1}{Q'^s} \right] \right\}. \]
Now we compute $R_1$ according as $\left(\frac{2}{\delta}\right)$ is $+1$ or $-1$, respectively.

**Lemma 16.** $\left(\frac{2}{\delta}\right) = +1 \implies R_1 = 0$, and

$$\left(\frac{2}{\delta}\right) = -1 \implies R_1 = \frac{2}{\lim_{s \to 1^+}} \left\{ \chi(\delta; Q) \sum' \frac{1}{Q^s} - \chi(\delta; Q') \sum' \frac{1}{Q'^s} \right\}. \quad \square$$

Now we used the value of $R_1$ to compute the full weighted sum $R$.

**Lemma 17.** $\left(\frac{2}{\delta}\right) = +1 \implies R = 0$, and

$$\left(\frac{2}{\delta}\right) = -1 \implies R = \frac{4\pi}{\sqrt{m}} \sum_{k=1}^{h} \chi(\delta, Q_k) \ln g_{m/A_k^2}.$$

where the sum is over all $h(-4m)$ reduced forms $Q_k \equiv A_k X^2 + C_k Y^2$ of determinant $m$.

**Note:** Case 1 of Proposition 2 explains the first “miraculous cancellation” in the computation of $R$.

**Proof.** By Kronecker’s formula for $g_n$, we conclude

$$R = \sum_{k=1}^{h} \chi(\delta, Q_k) \frac{4\pi}{\sqrt{m}} \ln g_{m/A_k^2}.$$ \[39\]

But, by Lemma 15,

$$R = \left[ 1 - \left(\frac{\delta}{2}\right) \right] \lim_{s \to 1^+} \left\{ \sum_{k=1}^{h} \chi(\delta, Q_k) \cdot \sum' \frac{1}{Q_k^s} \right\}.$$ \[39\]

If $\left(\frac{\delta}{2}\right) = 1$, then $R = 0$, as claimed in the statement. If $\left(\frac{\delta}{2}\right) = -1$, then

$$R = 2 \cdot \lim_{s \to 1^+} \left\{ \sum_{k=1}^{h} \chi(\delta, Q_k) \cdot \sum' \frac{1}{Q_k^s} \right\} = \frac{4\pi}{\sqrt{m}} \sum_{k=1}^{h} \chi(\delta, Q_k) \ln g_{m/A_k^2}. \quad \square$$
3.7.2 The second cancellation: Dirichlet’s computation

We propose to compute the sum

\[ F := \sum_{\delta} \chi(\delta, Q), \]

where \( \delta \) runs over all \( 2^t \) fundamental discriminants which divide \( D = -4m \) and \( Q \equiv AX^2 + BY^2 \) is any one of the \( 2^t \) reduced forms of determinant \( m \). We will consider two cases: when \( Q \) is the so-called *principal* form

\[ X^2 + mY^2, \]

and the case when \( Q \) is a *non-principal* form and therefore

\[ Q \neq X^2 + mY^2. \]

**Case 1: \( Q \equiv X^2 + mY^2 \)**

Then

\[
\chi(\delta, Q) = \left( \frac{\delta}{m+1} \right) \\
= (-1)^{\left(\frac{\delta - 1}{2}\right) \left(\frac{m+1-1}{2}\right)} \left( \frac{m+1}{\delta} \right) \\
= \left( \frac{1}{\delta} \right) \text{ since } m + 1 \equiv 1 \pmod{\delta}, \\
= 1.
\]

Therefore,

\[ F = \sum_{\delta} \chi(\delta, Q) = 2^t = h(m). \]

**Case 2: \( Q \neq X^2 + mY^2 \)**

Then there exists at least one fundamental discriminant, \( \delta_1 \), for which

\[ \chi(\delta_1, Q) \neq 1 \]

(why?) and thus

\[
\chi(\delta_1, Q) \cdot F = \chi(\delta_1, Q) \sum_{\delta} \chi(\delta, Q) \\
= \sum_{\delta} \chi(\delta_1, Q) \chi(\delta, Q) \\
= \sum_{\delta} \chi(\delta_1 \delta, Q),
\]

40
where we have used the multiplicativity of the Jacobi symbol. Now comes the fundamental observation (no pun intended!): As $\delta_1$ runs over the complete set of fundamental discriminants dividing $-4m$, so does $\delta_1 \delta$, and therefore the set of Jacobi symbols $\{\chi(\delta_1, Q)\}$ 

**coincides** with the set of Jacobi symbols $\{\chi(\delta_1 \delta, Q)\}$. Therefore,

$$\chi(\delta_1, Q) \cdot F = F.$$ 

But

$$\chi(\delta_1, Q) = -1,$$

or $-F = F$, i.e.,

$$F = 0.$$ 

Written out,

$$F := \sum_{\delta} \chi(\delta, Q) = 0 \quad \text{for any} \quad Q \neq X^2 + mY^2.$$ 

**Note:** Case 2 explains the second “miraculous cancellation” in the computation of $R$.

### 3.7.3 The general formula for $g_m$ where $m$ is even

If we conjoin the pair of “miraculous cancellations” into a single generic computation we obtain a general formula for Ramanujan’s function $g_m$.

Our starting point, then, is to compute:

$$H := \sum_{\delta} R(\delta)$$

$$= \sum_{\delta} \sum_{k=1}^{h} \chi(\delta, Q_k) \frac{4\pi}{\sqrt{m}} \ln g_m/A_k^2$$

$$= \frac{4\pi}{\sqrt{m}} \sum_{k=1}^{h} \left\{ \sum_{\delta} \chi(\delta, Q_k) \right\} \ln g_m/A_k^2$$

$$= \begin{cases} 
0, & \text{if } \left( \frac{2}{\delta} \right) = +1, \\
\frac{4\pi}{\sqrt{m}} \cdot h \cdot \ln g_m, & \text{if } \left( \frac{2}{\delta} \right) = -1.
\end{cases}$$

But, as we saw in (3.6),

$$R = 2 \cdot \lim_{s \to 1^+} \left\{ \sum_{k=1}^{h} \chi(\delta, Q_k) \cdot \sum' \frac{1}{Q_k^s} \right\}$$

$$= 2 \cdot \left\{ \sum_{n=1}^{\infty} \left( \frac{\delta}{n} \right) \frac{1}{n^s} \right\} \left\{ \sum_{m=1}^{\infty} \left( \frac{\delta}{m} \right) \frac{1}{m^s} \right\}.$$
Now we assume that \( \delta \delta' = -4m \),

which means that \( \delta' \) is even and of opposite sign to \( \delta \), and we conclude from the Dirichlet class number formulas (3.7) and (3.8) that

\[
\frac{4\pi}{\sqrt{m}} \sum_{k=1}^{h} \chi(\delta, Q_k) \ln g_{m/A_k^2} = \frac{2\pi}{\sqrt{m}} \cdot K(\delta) \cdot K(\delta') \cdot \ln \varepsilon
\]

\[
\Rightarrow 2 \cdot \sum_{k=1}^{h} \chi(\delta, Q_k) \ln g_{m/A_k^2} = K(\delta) \cdot K(\delta') \cdot \ln \varepsilon
\]

\[
\Rightarrow 2 \cdot h \cdot \ln g_m = \sum_{\delta} K(\delta) \cdot K(\delta') \cdot \ln \varepsilon
\]

\[
\Rightarrow (g_m)^{2h} = \prod_{\delta} e^{K(\delta) \cdot K(\delta')},
\]

or, putting

\[
m = 2n
\]

and

\[
h(m) \equiv K(-4m) = K(-8n),
\]

we obtain our crowning result:

**THEOREM 18.** Let \( \delta_1, \delta_2, \cdots, \delta_{K(-8n)} \) the distinct odd divisors of \(-8n\), and suppose the complimentary divisor \( \delta_k' \) to \( \delta_k \) is defined by

\[
\delta_k \cdot \delta_k' := -8n.
\]

Then

\[
g_{2n} = \prod_{k=1}^{K(-8n)} \left( \frac{T_k + U_k \sqrt{\delta_k}}{2} \right)^{K(\delta_k) \cdot K(\delta_k')} \frac{K(\delta_k) \cdot K(\delta_k')}{K(-8n)},
\]

a formula which can only be called glorious!

**Note:** This formula has been proved by Weber [19, 20], and by Berndt [1]. Weber’s development is based on quadratic forms, but invokes genus theory, which we wished to avoid. Berndt’s proof uses ideal theory and appeals to results in various journals, although both he
and Weber obtain a more general result. Our development reveals the fundamental structure of all of these proofs without the extra baggage of (mathematically useful, but pedagogically obfuscatory) generalization.

This same formula can be used to compute $g_{2n}$ for all even convenient numbers which are of the form $2P$ where $P$ is a product of distinct odd primes. In principle, this is just what Weber did.

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