A new classification scheme for Random Matrix Theories

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Abstract

In the last few years several new Random Matrix Models have been proposed and studied. They have found application in various different contexts, ranging from the physics of mesoscopic systems to the chiral transition in lattice gauge theory. These new ensembles can be classified in terms of the same Dynkin diagrams and root lattices which are used in the classification of the Lie algebras.
In the last few years several new Random Matrix Theories (RMT) have been discussed in the literature. They can be considered as generalizations of the six ensembles which were originally proposed by Wigner and Dyson [1], and share with them several features. At the same time they are also characterized by some new properties, which make them particularly apt to describe universal phenomena in a wide set of new physical contexts ranging from the physics of mesoscopic systems [2] (where most of them were originally introduced and discussed) to lattice gauge theory [3].

It is the aim of this letter to organize these new ensembles within a compact classification scheme. As we shall see, several entries in our classification will be left empty. They represent new RMT’s which can be easily constructed by using the techniques which will be discussed below but which have not found yet application in any physical context.

At the same time much progress has been done in the physics of quantum electronic transport in disordered wires [4]. Several features of disordered wires can be described by studying suitable ensembles of random transfer matrices. These ensembles are completely different from the RMT’s discussed above. The main difference is that these transfer matrix ensembles are characterized by an additional “time” coordinate and one is usually interested in the time evolution of the eigenvalues distribution, which is described by a suitable Fokker-Planck (FP) equation. In the application to the disordered wires the role of “time” coordinate is played by the length of the wire, the Fokker-Planck equation is known as Dorokhov Mello Pereyra Kumar (DMPK) equation [4] and describes the evolution of the transmission eigenvalues (and hence of the conductance) as the length of the wire increases.

A major result of our analysis is that we can insert also these, apparently unrelated, transfer matrix ensembles in our classification scheme.

Our classification is strongly related to that of the irreducible Symmetric Spaces (SS) and makes use of the same Dynkin diagrams and root lattices which are used in the classification of the Lie algebras. Some of these results were anticipated in [5]. A similar approach was also recently discussed in [6].

Let us start by listing some known results about symmetric spaces. A more detailed account can be found for instance in [7].

Let us take two Lie groups $G$ and $K$ such that $K \subset G$ and let $G$ and $K$ be the corresponding Lie algebras. Decompose $G$ as: $G = K + L$, then the coset manifold $X \equiv G/K$ is a symmetric space if the following commutation relations hold:

\[
[K, K] \subset K, \quad [K, L] \subset L, \quad [L, L] \subset K.
\] (1)

The exponential map from $G$ to $G$ maps $L$ into $X$.

There are three possibilities:

$X^+$] $G$ is semisimple and compact, then $X$ has everywhere non-negative curvature.
$X^-$] $G$ is semisimple and non-compact, then $X$ has everywhere non-positive curvature.

$X^0$] $G$ is non-semisimple and $[L, L] = 0$, then $X$ has everywhere zero curvature.

The following four properties of SS are particularly relevant for our analysis:

1] **Triality**.

SS always appear in triplets: $\{X^+, X^-, X^0\}$ of non-negative, non-positive and zero curvature, with the same subgroup $K$. If $L$ is the algebra which generates $X^-$, then $X^+$ is generated by $iL$ and $X^0$ coincides with $L$.

2] **Spherical coordinate system**.

Each SS admits a spherical coordinate systems, whose radial coordinates can be obtained as the exponential map of the Cartan subalgebra of $L$. Notice however that other choices of coordinates on the SS are also possible. These “non-Cartan” basis become important if some extra symmetry is imposed to the system[8].

3] **Complete classification**.

Irreducible SS can be classified completely with techniques similar to those used to classify the Lie algebras. A triplet of irreducible SS is identified by its particular root lattice and by a set of positive integers $m_\alpha$ which are called “root multiplicities”. A “root lattice” is a lattice generated by a finite set of non-zero vectors $R \equiv \{\alpha\} \subset V$, where $V$ is a $n$ dimensional vector space. These vectors are called roots and must fulfill a set of very stringent properties, which we shall not discuss here. The main consequence of these constraints is that the root lattices can be classified completely. They fall into five infinite classes which are conventionally denoted as $A_n, B_n, C_n, D_n$ and $BC_n$. The index $n$ denotes the rank of lattice which coincides with the dimensions of $V$. Exceptional isolated solution also exist for low values of $n$, but they are not relevant for the present analysis. The five infinite classes are defined as follows:

$A_n$] As space $V$ let us take the hyperplane in $\mathbb{R}^{n+1}$ for which $v_1 + v_2 + \cdots + v_{n+1} = 0$, with $v_i \in \mathbb{R}^{n+1}$. Let us take a canonical basis in $\mathbb{R}^{n+1}$: $\{e_1, e_2, \cdots, e_{n+1}\}$. Then $R = \{e_i - e_j, \ i \neq j\}$

$B_n$] In this case let us take $V = \mathbb{R}^n$ Then $R = \{\pm e_i, \ \pm e_i \pm e_j, \ i \neq j\}$

$C_n$] $V = \mathbb{R}^n$ and $R = \{\pm 2e_i, \ \pm e_i \pm e_j, \ i \neq j\}$

$D_n$] $V = \mathbb{R}^n$ and $R = \{\pm e_i \pm e_j, \ i \neq j\}$

$BC_n$] $V = \mathbb{R}^n$ and $R = \{\pm e_i, \pm 2e_i, \ \pm e_i \pm e_j, \ i \neq j\}$
The roots of type \{±e_i ± e_j, \ i \neq j\} are called ordinary roots while \{±2e_i\} and \{±e_i\} are called, respectively, long and short roots. In the following we shall denote with \(m_0\), \(m_s\) and \(m_l\) the multiplicities of the ordinary, short and long roots respectively. For each class of root lattices there are only few sets of \(m_\alpha\) values which are compatible with the constraints which characterize the symmetric spaces. Altogether they generate 11 different infinite series of triplets, which, apart from few exceptional solutions, exhaust the set of all possible irreducible symmetric spaces. They are listed in tab.1 below. To make more readable the table, in the first column only the coset description of the \(X^-\) element of each triplet is reported and the \(SO(2p)/U(p)\) class has been splitted into two separate entries.

4] Laplace Beltrami operator

The informations on the root lattice are enough to construct, for each SS, the explicit form of the radial part of the Laplace-Beltrami operator \(B\) in terms of the radial coordinates \(\{q_\alpha\}\):

\[
B = \frac{1}{J^s(q)} \sum_{k=1}^{n} \frac{\partial}{\partial q_k} J^s(q) \frac{\partial}{\partial q_k} \tag{2}
\]

where \(n\) is the rank of the SS, \(s \in \{0, +, -\}\) and \(J^s(q)\) is the Jacobian of the transformation to the spherical coordinates and is defined as

\[
J^0(q) = \prod_{\alpha \in R^+} q_\alpha^{m_\alpha} \quad \text{(zero curvature)} \tag{3}
\]

\[
J^+(q) = \prod_{\alpha \in R^+} [\sin(q_\alpha)]^{m_\alpha} \quad \text{(positive curvature)} \tag{4}
\]

\[
J^-(q) = \prod_{\alpha \in R^+} [\sh(q_\alpha)]^{m_\alpha} \quad \text{(negative curvature)} \tag{5}
\]

where \(R^+\) is the subset of the positive roots of the lattice, \(m_\alpha\) is the multiplicity of the root \(\alpha\) and \(q_\alpha\) denotes the projection in \(\{q\}\) of the root \(\alpha\). E.g. if \(\alpha = e_i - e_j\) then \(q_\alpha = q_i - q_j\).

We are now in the position to list our main results. Some of them are already well known in the literature, even if they are rephrased here in a different language. They are collected here only for completeness. Some of them however are new and require some further comment.

a1] To each symmetric space it can be associated a suitable RMT by mapping the radial coordinates of the SS in the eigenvalues of the RMT. The root multiplicities \(m_\alpha\) correspond to the critical indices of the RMT. Like the symmetric spaces, also the RMT’s can be organized in triplets: circular, gaussian and transfer matrix ensembles with the same eigenvalue content, the same symmetries and the same critical indices. Then the following relations hold:
SS of \textbf{positive} curvature correspond to \textbf{circular} ensembles

those of \textbf{negative} curvature to \textbf{transfer matrix} ensembles

and those of \textbf{zero} curvature to \textbf{gaussian} ensembles.

These RMT’s are defined, in the case of the ensembles of gaussian or circular type, by the following joint probability density for the eigenvalues:

\begin{align*}
\text{gaussian} & \quad P(x_i) = J^0(x) e^{\sum_{j=1}^n x_j^2} \\
\text{circular} & \quad P(x_i) = J^+(x)
\end{align*}

and in the case of the transfer matrix ensembles by the following Fokker-Planck equation for the time dependent probability density \(P(x_i, t)\):

\[ f \frac{\partial P}{\partial t} = \sum_{k=1}^n \frac{\partial}{\partial x_k} [J^- \frac{\partial}{\partial x_k} J^-] P \equiv D P \tag{8} \]

where the constant \(f\) sets the units in which the time \(t\) is measured. Also for the gaussian and circular RMT it is possible to write FP equations which describe the approach to the equilibrium configuration of the ensembles. They have the same form of eq.(8), but with \(J^0\) or \(J^+\) instead of \(J^-\). One can easily see that the differential operator \(D\) which appears in the Fokker-Planck eq. of the RMT is simply related to the radial part \(B\) of the Laplace-Beltrami operator on the SS:

\[ D = J^s B [J^s]^{-1} \tag{9} \]

As a consequence, several properties of these FP equations and in some cases also asymptotic solutions can be obtained by only resorting to group theoretical arguments.

\textit{a2}] A classification similar to that described above for the SS also holds for triplets of RMT’s. However if some extra symmetry imposes the choice of a non-Cartan basis then, following the same steps of point \([a1]\) above one can construct new joint probability densities for the eigenvalues, whose critical indices could well lie outside the Cartan classification of Tab.I \cite{8}. We shall see below some realizations of this type which have relevant physical applications. According to this observation we shall keep in the following the values of the root multiplicities unconstrained. All the following results hold for generic values of \(m_\alpha\).

\textit{a3}] As we mentioned above, the root multiplicities \(m_\alpha\) correspond to the critical indices of the RMT. In particular the multiplicities of the long and short roots correspond to the boundary critical indices while that of the ordinary roots corresponds to the “bulk” critical index of the RMT, the one which is usually denoted with \(\beta\).
Some relevant symmetries of the RMT’s can be understood as symmetries of the associated root lattice. In particular RMT’s of $A_n$ type are characterized by the translational invariance of the eigenvalues, while for all the other lattices ($B_n$, $C_n$, $D_n$ and $BC_n$) this invariance is broken, but a new $Z_2$ symmetry appears. The $A_n$ ensembles of positive or zero curvature (namely of circular or gaussian type) exactly coincide with the original Wigner-Dyson (WD) ensembles which, as it is well known, are the only ones which fulfil such a translational invariance. All the other ensembles are characterized by the presence of a boundary (even if sometimes it is not immediately evident). They are not translational invariant, but are characterized by a new symmetry, the reflection with respect to the boundary, which is exactly the $Z_2$ symmetry mentioned above. In the following we shall refer to these ensembles, in opposition to the WD ensembles, as boundary random matrix theories (BRMT).

To each RMT of the circular or gaussian type, it is possible to associate a set of Classical Orthogonal Polynomials, which can be used to construct correlation functions. As it is well known, the Hermite polynomials are related to the Wigner-Dyson ensembles ($A_n$ root lattices). Simple changes of variables allows to show that the Laguerre polynomials are related to the gaussian BRMT and that the Jacobi ones to the circular BRMT. In these last two cases the boundary critical indices of the BRMT are related to the parameter which define the polynomials as follows:

**Laguerre:**

$$L^{(\lambda)} \equiv \frac{x^{-\lambda} e^x}{n!} \frac{d^n}{dx^n} (x^{n+\lambda} e^{-x}), \quad (x \geq 0) \quad (10)$$

$$\lambda = \frac{m_s + m_l - 1}{2} \quad (11)$$

**Jacobi:**

$$P^{(\mu, \nu)} \equiv \frac{(-1)^n (1-x)^{-\nu}}{2^n n!} \frac{d^n}{dx^n} \left\{ \frac{(1+x)^{n+\mu}}{(1-x)^{n-\nu}} \right\} \quad (12)$$

$$\mu = \frac{m_s + m_l - 1}{2}, \quad \nu = \frac{m_l - 1}{2} \quad (13)$$

We see that $\lambda$ and $\mu$ have the same expression in terms of $m_s$ and $m_l$. Thus the BRMT’s corresponding to Laguerre and Jacobi polynomials with the same $\lambda = \mu$ indices belong to the same triplet in the above classification. They are respectively the zero curvature (Laguerre) and positive curvature (Jacobi) elements of the triplet. This explains the “weak universality” recently observed for the boundary critical indices of these ensembles [9].
Table I: Irreducible Symmetric Spaces and some of their RMT realizations.

| $G/K$                                  | $R$     | $m_\alpha$ | $X^+$  | $X^0$  | $X^-$  |
|----------------------------------------|---------|------------|--------|--------|--------|
| $SL(n, \mathbb{R})/SO(n)$              | $A_{n-1}$ | $m_o = 1$  | COE    | GOE    |        |
| $SL(n, \mathbb{C})/SU(n)$              | $A_{n-1}$ | $m_o = 2$  | CUE    | GUE    |        |
| $SU^*(2n)/Sp(2n)$                      | $A_{n-1}$ | $m_o = 4$  | CSE    | GSE    |        |
| $SO(2n+1)/SO(2n)$                      | $B_n$    | $m_o = m_s = 2$ |        |        |        |
| $SO(2n, \mathbb{C})/SO(2n)$           | $D_n$    | $m_o = 2$  |        |        | NS-D   |
| $Sp(2n, \mathbb{C})/Sp(2n)$            | $C_n$    | $m_o = m_l = 2$ |        |        | NS-C   |
| $SO^*(4n)/U(2n)$                       | $C_n$    | $m_o = m_l = 1$ |        |        | NS-CI  |
| $SO^*(4n+2)/U(2n+1)$                   | $BC_n$   | $m_o = m_s = 4, m_l = 1$ |        |        |        |
| $SO(p,q)/SO(p)\otimes SO(q)$, $(p \geq q)$ | $B_{q},D_{q}$ | $m_o = 1$  | $m_s = (p-q)$ |        | chGOE  |
| $SU(p,q)/SU(p)\otimes SU(q)$, $(p \geq q)$ | $BC_{q}$ | $m_o = 2$  | $m_s = 2(p-q)$ | $(p=q)$ SUE | chGUE  |
| $Sp(2p,2q)/Sp(2p)\otimes Sp(2q)$, $(p \geq q)$ | $BC_{q}$ | $m_o = 4$  | $m_s = 4(p-q)$ | $(p=q)$ TUE | chGSE  |

Let us conclude by listing some known examples of RMT’s and giving their collocation in the classification scheme.

Wigner-Dyson ensembles

These ensembles correspond to the $A_n$ type SS discussed above. In this case we have only ordinary roots, and $J^0$ and $J^+$ become:

$$J^0(\{x_n\}) = \prod_{i<j} |x_j - x_i|^{m_0}$$  \hspace{1cm} (14)

$$J^+(\{\theta_n\}) = \prod_{i<j} |e^{i\theta_j} - e^{i\theta_j}|^{m_0}$$  \hspace{1cm} (15)

taking into account that for the three SS of the $A_n$ type we have $m_o = 1, 2, 4$, we immediately recognize the joint probability distribution of the three circular and gaussian WD ensembles.

Chiral RMT’s

These ensembles correspond to the three last rows of table I. They are of gaussian type and their joint probability distribution is given by:

$$J^0(\{x_n\}) = \prod_{i<j} |x_j - x_i|^{m_0} \prod_j x_j^{m_o + m_l - 1}.$$  \hspace{1cm} (16)

They are also known as Laguerre ensembles and find applications in several physical contexts, ranging from quantum wires \cite{11} to lattice QCD \cite{3}.
Transfer matrix ensembles: Disordered wires

These ensembles are related to symmetric spaces of negative curvature. Thus they are better defined in terms of their Fokker-Planck evolution equation. Simple algebraic manipulations show that in the three cases corresponding to the TOE, TUE and TSE entries of tab.I , \( J^- \) is given by:

\[
J^-(\{x_n\}) = \prod_{i<j} |sh^2 x_j - sh^2 x_i|^\frac{m_o}{2} \prod_j |sh 2x_j|^\frac{m_l}{2}
\]

with \( m_o = 1, 2, 4 \) and always \( m_l = 1 \). If inserted in eq.(8) this expression leads to the well known DMPK equation \([4]\). Recently, a generalization with \( m_l = 3 \) has also been proposed \([11]\).

NS ensembles

These are four new RMT’s of gaussian and circular type which have been recently discussed in \([6]\). They describe the behaviour of hybrid microstructure composed of normal metallic conductors in contact with superconducting regions. They are reported in tab.1 keeping the notation \{C, CI, D, DIII\} used in \([6]\).

Ballistic chaotic quantum dots

These are three ensembles of the circular type, which have been introduced in \([12]\) to describe the conductance of ballistic quantum dots, (characterized by a chaotic classical dynamics). We shall denote them in the following as S matrix ensembles. They are described by root lattices of the \( BC_n \) type with the following root multiplicities: \( m_o = \beta, \ m_s = \beta - 2, \ m_l = 1 \), where \( \beta \) can take, as usual the three values \( \beta = 1, 2, 4 \). These three values exhaust the three possible ensembles of this type which are denoted, with abuse of language, as orthogonal, unitary and symplectic: SOE, SUE and SSE in short.

This last example is probably the most interesting, because only one of the three ensembles corresponds to the Cartan decomposition of a symmetric space. It is the unitary (\( \beta = 2 \)) case (which is actually described by a \( C_n \) lattice, since for \( \beta = 2, \ m_s = 0 \)). The other two cases correspond to basis of non-Cartan type (they can be realized as circular \( A_{2n} \) ensembles with the imposition of the extra symmetry \( U(n) \times U(n) \) \([8, 12]\), accordingly the root multiplicities lie outside the classification of tab.I.

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