Global existence and regularity estimate for stochastic Cahn–Hilliard equation with unbounded noise diffusion

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Abstract

In this article, we consider the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise with diffusion coefficient of sublinear growth. By introducing the spectral Galerkin method, we first obtain the well-posedness of the approximated equation in finite dimension. Then with the help of the semigroup theory and the factorization method, the approximation processes is shown to possess many desirable properties. Further, we show that the approximation process is strongly convergent in certain Banach space via the interpolation inequality and variational approach. Finally, the global existence and regularity estimate of the unique solution process are proven by means of the strong convergence of the approximation process.

Keywords: stochastic Cahn–Hilliard equation, multiplicative space-time white noise, spectral Galerkin method, global existence, regularity estimate

2010 MSC: 60H15, 35R60, 60H35

1. Introduction

In this article, we consider the following stochastic Cahn–Hilliard equation with multiplicative space-time white noise

\[ dX(t) + A(AX(t) + F(X(t)))dt = G(X(t))dW(t), \quad t \in (0, T) \]
\[ X(0) = X_0. \]

Here $0 < T < \infty$, $H := L^2(\mathcal{O})$ with the rectangular domain $\mathcal{O} \subset \mathbb{R}^d$, $d \leq 3$, $-A : D(A) \subset H \to H$ is the Laplacian operator under homogenous Dirichlet or

\textsuperscript{*}This work was funded by National Natural Science Foundation of China (No. 91630312, No. 91530118, No.11021101 and No. 11290142).

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Neumann boundary condition, and \( \{W(t)\}_{t \geq 0} \) is a generalized Wiener process on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). The nonlinearity \( F \) is assumed to be the Nemytskii operator of \( f' \), where \( f \) is a polynomial of degree 4, i.e.,

\[
c_4 \xi^4 + c_3 \xi^3 + c_2 \xi^2 + c_1 \xi + c_0 \quad \text{with} \quad c_i \in \mathbb{R}, \ i = 0, \ldots, 4, \ c_4 > 0.
\]

A typical example is the double well potential \( f = \frac{1}{4}(\xi^2 - 1)^2 \). When \( G = I \), Eq. (1.1) corresponds to the stochastic Cahn–Hilliard–Cook equation. The diffusion coefficient \( G \) is assumed to be the Nemytskii operator of \( g \), where \( g \) is a global Lipschitz function with the sublinear growth condition \( |g(\xi)| \leq C(1 + |\xi|^\alpha) \), \( \alpha < 1 \). This equation is used to describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably (see e.g. [CH58, NCS84]).

The existence and uniqueness of the solution to Eq. (1.1) have already been proven by [DPD96] in the case of \( G = I, \ d = 1 \) and if \( G = I, \ d \geq 2 \), the driving noise is more regular than the space-time white noise. When \( G \) is a bounded diffusion coefficient, the authors in [CW01] extend the results on the local existence and uniqueness of the solution in the case that \( |g(\xi)| \leq C(1 + |\xi|^\alpha) \), \( \alpha \in (0, 1], \ d \leq 3 \). Meanwhile, the global existence of the solution is achieved under the restriction that \( \alpha < \frac{1}{d}, \ d = 1 \). However, for the global existence of the solution, it is still unknown whether the sublinear growth condition \( \alpha < \frac{1}{d} \) could be extended to the general sublinear growth condition, i.e., \( |g(\xi)| \leq C(1 + |\xi|^\alpha), \ \alpha \in (0, 1) \), which is one main motivation of this article.

To study such problem, our strategy is different from that in the existing literature (see e.g. [CW01, AKM16]). Instead of introducing an appropriated cut-off SPDE, we firstly use the spectral Galerkin method to discretize Eq. (1.1) and get the spectral Galerkin approximation

\[
dX^N(t) + A(X^N(t))dt + P^N F(X^N(t)))dt = P^N G(X^N(t))dW(t), \quad t \in (0, T] \tag{1.2}
\]

\[X^N(0) = P^N X_0,\]

where \( N \in \mathbb{N}^+ \). Then by making use of the factorization formula and the equivalent random form of the semi-discrete equation, we show the well-posedness of the semi-discrete equation (1.2), as well as its uniform a priori estimate and regularity estimate. Furthermore, we show that the limit of the solution of the spectral Galerkin method exists globally and is the unique mild solution of Eq. (1.1). As a consequence, the exponential integrability property, the optimal temporal and spatial regularity estimates of the exact solution are proven. Meanwhile, with the help of the Sobolev interpolation equality and the smoothy effect of the semigroup \( S(t) := e^{-A^2t} \), the sharp spatial strong convergence rate of the spectral Galerkin method is established under homogenous Dirichlet boundary condition. To the best of our knowledge, this is not only the new result on the global existence and regularity estimate of the solution, but also the first result on the strong convergence rate of numerical approximation for the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise.
The rest of this article is organized as follows. In Section 2 the setting and assumptions used are formulated. In Section 3 we prove several uniform a priori estimates and regularity estimates of the spatial spectral Galerkin method. The strong convergence analysis of the spatial spectral Galerkin method is presented in Section 4. Our main result which states existence, uniqueness and regularity of solutions of Eq. (1.1) with nonlinear multiplicative noise is presented in Section 5.

2. Preliminaries

In this section, we present some preliminaries and notations, as well as the assumptions on Eq. (1.1).

Given two separable Hilbert spaces \((\mathcal{H}, \| \cdot \|_\mathcal{H})\) and \((\tilde{\mathcal{H}}, \| \cdot \|_{\tilde{\mathcal{H}}})\), \(\mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})\) and \(\mathcal{L}_1(\mathcal{H}, \tilde{\mathcal{H}})\) are the Banach spaces of all linear bounded operators and the nuclear operators from \(\mathcal{H}\) to \(\tilde{\mathcal{H}}\), respectively. The trace of an operator \(T \in \mathcal{L}_1(\mathcal{H})\) is \(\text{tr}[T] = \sum_{k \in \mathbb{N}} \langle T f_k, f_k \rangle_\mathcal{H}\), where \(\{f_k\}_{k \in \mathbb{N}} (\mathbb{N} = \{0, 1, 2, \cdots\})\) is any orthonormal basis of \(\mathcal{H}\). In particular, if \(T \geq 0\), \(\text{tr}[T] = \|T\|_{\mathcal{L}_1}\). Denote by \(\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})\) the space of Hilbert–Schmidt operators from \(\mathcal{H}\) into \(\tilde{\mathcal{H}}\), equipped with the usual norm given by \(\| \cdot \|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})} = \langle \sum_{k \in \mathbb{N}} \| f_k \|_{\tilde{\mathcal{H}}}^2 \rangle^{\frac{1}{2}}\). The following useful property and inequality hold

\[
\|ST\|_{\mathcal{L}_2(\tilde{\mathcal{H}}, \hat{\mathcal{H}})} \leq \|S\|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})}\|T\|_{\mathcal{L}_1(\mathcal{H})}, \quad T \in \mathcal{L}(\mathcal{H}), \ S \in \mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}}), \quad (2.1)
\]

\[
\text{tr}[Q] = \|Q\|_{\mathcal{L}_2(\mathcal{H})}^2 = \|T\|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})}^2, \quad Q = TT^*, \quad T \in \mathcal{L}_2(\hat{\mathcal{H}}, \mathcal{H}),
\]

where \(T^*\) is the adjoint operator of \(T\).

Given a Banach space \((\mathcal{E}, \| \cdot \|_\mathcal{E})\), we denote by \(\gamma(\mathcal{H}, \mathcal{E})\) the space of \(\gamma\)-radonifying operators endowed with the norm \(\|T\|_{\gamma(\mathcal{H}, \mathcal{E})} = \mathbb{E}\| \sum_{k \in \mathbb{N}} \gamma_k T f_k \|_\mathcal{E}^2\), where \(\{\gamma_k\}_{k \in \mathbb{N}}\) is a Rademacher sequence on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For convenience, let \(L^q = L^q(\mathcal{O}), 1 \leq q < \infty\) and \(E = \mathcal{L}(\mathcal{O})\) equipped with the usual inner product and norm. We also need the following Burkholder inequality in \(E\),

\[
\left( \sup_{t \in [0, T]} \left\| \int_0^t \phi(r) d\widetilde{W}(r) \right\|_{E} \right)^p L^p(\Omega) \leq C_p \left\| \phi \right\|_{L^p(\mathcal{O}):L^2([0, T]; \gamma(H; E))} \leq C_p \left( \mathbb{E} \left( \int_0^T \left\| \sum_{k \in \mathbb{N}} (\phi(t)e_k)^2 \right\|_{E} dt \right)^{\frac{p}{2}} \right)^{\frac{1}{p}},
\]

(2.2)

where \(\widetilde{W}\) is the \(H\)-valued cylindrical Wiener process and \(\{e_k\}_{k \in \mathbb{N}}\) is any orthonormal basis of \(H\). For \(L^q = L^q(\mathcal{O}), 1 \leq q < \infty\), the Burkholder inequality also holds (see e.g. [vNVW08]).

Next, we introduce some assumptions and spaces associated with \(A\). Let \(\mathcal{O} = [0, L], L > 0\). We denote by \(H^k := H^k(\mathcal{O})\) the standard Sobolev space,
For convenience, we mainly focus on the well-posedness and numerical approximation for Eq. (1.1) under homogenous Dirichlet boundary condition. We would like to mention that the approach for proving the global existence of the unique solution is also available for Eq. (1.1) under homogenous Neumann boundary condition. Denote $A = -\Delta$ the Dirichlet Laplacian operator with

$$D(A) = \{ v \in H^2(\Omega) : v = 0 \text{ on } \partial \Omega \}.$$  

It is known that $A$ is a positive definite, self-adjoint and unbounded linear operator on $H$. Thus there exists an orthonormal eigensystem $\{ (\lambda_j, e_j) \}_{j \in \mathbb{N}}$ such that $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$ with $\lambda_j \sim j^2$ and $\sup_{j \in \mathbb{N}^+} \| e_j \|_E < \infty$. Equipped with the norm $\| \cdot \|_{H^\alpha}$ and corresponding inner product, the Hilbert space $H^\alpha$ equals $D(A^{1/2})$. It is obvious that $H = \mathbb{H}$. Throughout this article, the Wiener process $W$ is assumed to be the $H$-valued cylindrical Wiener process, which implies that for any $\gamma \in (0, 1/2)$, $\| A^{\frac{\gamma-2}{2}}Q^{\gamma/2} \|_{\mathcal{L}_2(\mathbb{H})} < \infty$. We denote by $C$ a generic constant which may depend on several parameters but never on the projection parameter $N$ and may change from occurrence to occurrence. We also remark that the approach for proving the global existence of the unique solution is available for the cases of higher dimension and more regular $Q$-Wiener process.

3. A priori estimate and regularity estimate of the spectral Galerkin method

In this section, we give the a priori estimate and regularity estimate of the solution of Eq. (1.2). Notice that Eq. (1.2) is equivalent to the following random PDE and stochastic linear equation

$$dY^N(t) + A(AY^N(t) + PN F(Y^N(t) + Z^N(T)))dt = 0, \quad Y^N(0) = P^N X_0, \quad (3.1)$$

$$dZ^N(t) + A^2 Z^N(t)dt = P^N G(Y^N(t) + Z^N(t))dW(t), \quad Z^N(0) = 0. \quad (3.2)$$

The above decomposition is inspired by [Cer03] where the authors use similar decomposition to show the well-posedness of stochastic reaction-diffusion systems. In the following, we present the a priori and regularity estimates of $Z^N$ and $Y^N$.

**Lemma 3.1.** Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \geq 1$. There exists a unique solution $X^N$ of Eq. (1.2) satisfying

$$\sup_{t \in [0,T]} \mathbb{E} \left[ \| X^N(t) \|_{H^{-1}}^q \right] \leq C(X_0, T, q), \quad (3.3)$$

where $C(X_0, T, q)$ is a positive constant.

**Proof.** Thanks to the fact all the norms in finite dimensional normed linear spaces are equivalent, the norm $\| \cdot \|$ and $\| \cdot \|_{H^{-1}}$ in $P^N(\mathbb{H})$ are equivalent up to
constants depend on $N$. The local existence of a unique solution for Eq. \cite{1.2} can be obtained by the the arguments in \cite[Chapter 3]{PR07}. Thus to achieve the global existence of a solution in $\mathbb{H}$, we only need to find a proper Lyapunov functional and to derive the a priori estimate of the solution under the norm $\| \cdot \|_{\mathbb{H}^{-1}}$. According to Eq. \cite{3.1}, by using the chain rule and integration by parts, we have for any $t \leq T$,

$$
\| Y^N(t) \|_{\mathbb{H}^{-1}}^2 = \| Y^N(0) \|_{\mathbb{H}^{-1}}^2 - 2 \int_0^t \langle \nabla Y^N(s), \nabla Y^N(s) \rangle ds
$$

$$
- 2 \int_0^t \langle F(Y^N(s) + Z^N(s)), Y^N(s) \rangle ds
$$

$$
= \| Y^N(0) \|_{\mathbb{H}^{-1}}^2 - 2 \int_0^t \| \nabla Y^N(s) \|^2 ds
$$

$$
- 2 \int_0^t \langle F(Y^N(s) + Z^N(s)), Y^N(s) \rangle ds.
$$

The expression of $F$ and Young inequality implies that

$$
\| Y^N(t) \|_{\mathbb{H}^{-1}}^2 + 2 \int_0^t \| \nabla Y^N(s) \|^2 ds + 8 \langle c_4 - \epsilon \rangle \int_0^t \| Y^N(s) \|_{L^4_s}^4 ds
$$

$$
\leq \| Y^N(0) \|_{\mathbb{H}^{-1}}^2 + C(\epsilon) \int_0^t (1 + \| Z^N(s) \|_{L^4_s}) ds.
$$

Thus it suffices to deduce the a priori estimate of $\int_0^t \| Z^N(s) \|^4_{L^4_s} ds$. From the mild form of $Z^N$, the Hölder inequality, the Burkholder inequality and the smooth effect of $S(\cdot)$, it follows that for $p \geq 2$ and $q \geq 4$,

$$
\mathbb{E}[\| Z^N(s) \|_{L^q_s}^q]
$$

$$
= \mathbb{E}\left[ \left\| \int_0^s S(s-r)P^NG(Y^N(r) + Z^N(r))dW(r) \right\|_{L^q_s}^q \right]
$$

$$
\leq \mathbb{E}\left[ \left( \int_0^s \left\| S(s-r)P^NG(Y^N(r) + Z^N(r)) \right\|^2_{\mathbb{H}} dr \right)^{\frac{q}{2}} \right]
$$

$$
\leq \mathbb{E}\left[ \left( \int_0^s \sum_{k=1}^\infty \left\| S(s-r)P^NG(Y^N(r) + Z^N(r)) e_k \right\|^2_{L^p} dr \right)^{\frac{q}{2}} \right]
$$

$$
\leq \mathbb{E}\left[ \left( \int_0^s (s-r)^{-\frac{1}{2} q \left( \frac{3}{2} - \frac{1}{p} \right)} \sum_{k=1}^\infty \left\| S \left( s-r \right)^{\frac{3}{2} - \frac{1}{p}} P^N(G(Y^N(r) + Z^N(r)) e_k \right\|^2_{L^p} dr \right)^{\frac{q}{2}} \right]
$$

The Paserval equality and the sublinear growth of $G$ yield that

$$
\mathbb{E}[\| Z^N(s) \|_{L^q_s}^q]
$$

$$
\leq \mathbb{E}\left[ \left( \int_0^s (s-r)^{-\frac{1}{2} q \left( \frac{3}{2} - \frac{1}{p} \right)} \sum_{j,k=1}^\infty \left\langle G(Y^N(r) + Z^N(r)) e_k, e^{-\frac{q}{2} \lambda^2(s-r) e_j} \right\rangle^2 dr \right)^{\frac{q}{2}} \right].
$$
\[
= CE \left[ \left( \int_0^s (s - r)^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{j=1}^{\infty} e^{-\lambda_j^2 (s-r)} \| G(Y^N(r) + Z^N(r)) e_j \|_2^2 dr \right)^{\frac{q}{2}} \right] 
\leq CE \left[ \left( \int_0^s (s - r)^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{j=1}^{\infty} e^{-\lambda_j^2 (s-r)} \| G(Y^N(r) + Z^N(r)) \|_2^2 dr \right)^{\frac{q}{2}} \right] 
\leq CE \left[ \left( \int_0^s (s - r)^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{j=1}^{\infty} e^{-\lambda_j^2 (s-r)} \| G(Y^N(r) + Z^N(r)) \|_2^2 dr \right)^{\frac{q}{2}} \right] 
\leq CE \left[ \left( \int_0^s (s - r)^{-\frac{1}{p} + \frac{1}{2}} (1 + \| Y^N(r) \|_2^{2\alpha} + \| Z^N(r) \|_2^{2\alpha}) dr \right)^{\frac{q}{2}} \right] 
\leq C\left( \int_0^s (s - r)^{-\frac{1}{p} + \frac{1}{2}} dr \right)^{\frac{q}{2}} E \left[ \left( 1 + \int_0^s \| Y^N(r) \|_{\mathbb{H}^k}^2 + \| Z^N(r) \|_{\mathbb{H}^k}^2 dr \right)^{\frac{q}{2}} \right].
\]

Using the Young inequality, we obtain for \(0 \leq s \leq t\),
\[
E[\| Z^N(s) \|_{\mathbb{L}^p}] \leq Cs^{\frac{q}{2p}} \left[ 1 + E\left( \int_0^s \| Y^N(r) \|_{\mathbb{L}^p}^4 dr \right)^{\frac{q}{2}} \right] + \int_0^s E[\| Z^N(r) \|_p^q] dr 
\leq Cs^{\frac{q}{2p}} \left[ 1 + E\left( \int_0^s \| Y^N(r) \|_{\mathbb{L}^p}^{4\alpha} dr \right)^{\frac{q}{2}} \right] + \int_0^s E[\| Z^N(r) \|_{\mathbb{L}^p}^q] dr.
\]

Gronwall’s inequality yields that for \(0 \leq s \leq T\),
\[
E[\| Z^N(s) \|_{\mathbb{L}^p}^q] \leq C(T) \left( 1 + E\left( \int_0^s \| Y^N(r) \|_{\mathbb{L}^p}^{4\alpha} dr \right)^{\frac{q}{2}} \right).
\]

Now taking \(k\)th moment, \(k \in \mathbb{N}^+\) on and letting \(p = 4, q = 4k\), we have
\[
E \left[ \left( \int_0^t \| Y^N(s) \|_{L^4}^4 ds \right)^k \right] 
\leq C\| Y^N(0) \|_{L^2}^{2k} + C(\epsilon) \int_0^t (1 + E[\| Z^N(s) \|_{L^2}^4]) ds 
\leq C\| Y^N(0) \|_{L^2}^{2k} + C(\epsilon, T) \left( C(\epsilon_1) + \epsilon_1 \int_0^t E[\left( \int_0^s \| Y^N(r) \|_{L^4}^4 dr \right)^k] ds \right),
\]
where \(\epsilon_1 > 0\) is a small number such that \(C(\epsilon, T)\epsilon_1 T < \frac{1}{4}\). The above estimation leads to
\[
E \left[ \left( \int_0^t \| Y^N(s) \|_{L^4}^4 ds \right)^k \right] \leq C\| Y^N(0) \|_{L^2}^{2k} + C(k, \epsilon, \epsilon_1, T),
\]
which in turns yields that for \(k \in \mathbb{N}^+\),
\[
E \left[ \| Y^N(t) \|_{L^2}^{2k} \right] + E \left[ \left( \int_0^t \| \nabla Y^N(s) \|_{L^2}^2 ds \right)^k \right] + E \left[ \left( \int_0^t \| Y^N(s) \|_{L^4}^4 ds \right)^k \right] 
\leq C(X_0, T, k).
\]

Based on the a priori estimates of \(Z^N\) and \(Y^N\) in \(L^p\) and \(H^{-1}\), respectively, we complete the proof via the Hölder inequality.
Lemma 3.2. Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \geq 1$. There exists a positive constant $C(X_0, T, q)$ such that

$$
E \left[ \sup_{t \in [0,T]} \left\| Z^N(t) \right\|_E^q \right] \leq C(X_0, T, q). \tag{3.7}
$$

**Proof.** By using the factorization formula in [DPZ92, Chapter 5], we have that for $\alpha > \frac{1}{p} + \gamma$, $p > 1$, $\gamma = \frac{1}{8}$,

$$
E \left[ \sup_{s \in [0,T]} \left\| Z^N(s) \right\|_E^q \right] \leq C(q, T) E \left[ \left\| Y_{\alpha,N} \right\|_{L^p(0,T;H)}^q \right],
$$
where

$$
Y_{\alpha,N}(s) = \int_0^s (s - r)^{-\alpha_1} S(s - r) P^N G(Y^N(s) + Z^N(s)) dW(s).
$$

Thus it suffices to estimate $E \left[ \left\| Y_{\alpha,N} \right\|_{L^p(0,T;H)}^q \right]$. From the Hölder and Burkholder inequalities, it follows that for $q \geq \max(p, 2)$,

$$
E \left[ \left\| Y_{\alpha,N} \right\|_{L^p(0,T;H)}^q \right] = E \left[ \left( \int_0^T \left\| \int_0^s (s - r)^{-\alpha_1} S(t - s) P^N G(Y^N(s) + Z^N(s)) dW(r) \right\|_p^p ds \right)^{\frac{q}{p}} \right]
$$

$$
\leq C(T,q) \int_0^T E \left[ \left\| \int_0^s (s - r)^{-\alpha_1} S(t - s) P^N G(Y^N(s) + Z^N(s)) dW(r) \right\|^q \right] ds
$$

$$
\leq C(T,q) \int_0^T E \left[ \left( \int_0^s (s - r)^{-2\alpha_1} \sum_{i \in \mathbb{N}^+} \left\| S(t - s) P^N G(Y^N(s) + Z^N(s)) e_i \right\|_p^{2\alpha_1} dr \right)^{\frac{q}{2\alpha_1}} ds \right]
$$

$$
\leq C(T,q) \int_0^T E \left[ \left( \int_0^s (s - r)^{-\alpha_1 + \frac{1}{2}} (1 + \left\| Y^N(r) \right\|_p^{2\alpha_1} + \left\| Z^N(r) \right\|_p^{2\alpha_1} ds \right)^{\frac{q}{2\alpha_1}} ds \right].
$$

Since $\alpha < 1$, one can choose a positive number $l > 2$ and a large enough number $p$ such that $2\alpha l < 4$ and $(2\alpha_1 + \frac{1}{2}) \frac{q}{2\alpha_1} < 1$. Then by using a priori estimates (3.5) and (3.6), we obtain

$$
E \left[ \left\| Y_{\alpha,N} \right\|_{L^p(0,T;H)}^q \right] \leq C(T,q,\alpha) \int_0^T \left( \int_0^s (s - r)^{-\alpha_1 + \frac{1}{2}} \frac{q}{2\alpha_1} dr \right)^{\frac{q}{2\alpha_1}}
$$

$$
\times E \left[ \left( \int_0^s (1 + \left\| Y^N(r) \right\|_p^{2\alpha_1} + \left\| Z^N(r) \right\|_p^{2\alpha_1} dr \right)^{\frac{q}{2\alpha_1}} ds \right].
$$

which implies that

$$
E \left[ \sup_{s \in [0,T]} \left\| Z^N(s) \right\|_E^q \right] \leq C(T,q,\alpha, X_0).
$$

**Corollary 3.1.** Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \geq 1$. Then the solution $X^N$ of Eq. (1.2) satisfies

$$
E \left[ \sup_{t \in [0,T]} \left\| X^N(t) \right\|_{\mathbb{H}^{-1}}^q \right] \leq C(X_0, T, q), \tag{3.8}
$$

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where \( C(X_0, T, q) \) is a positive constant.

**Proof.** Similar arguments in the proof of (3.6) yield that for any \( k \geq 1 \),

\[
E \left[ \sup_{t \in [0,T]} \| Y^N(t) \|_{H^{k+1}}^{2k} \right] \leq C(X_0, T, k).
\]

Combining this estimate with Lemma 3.2, we complete the proof. \( \square \)

Thanks to the above a priori estimates of \( Y^N \) and \( Z^N \), we are now in a position to deduce the a priori estimate of \( X^N \) in \( \mathbb{H} \).

**Lemma 3.3.** Let \( X_0 \in \mathbb{H}, T > 0 \) and \( q \geq 1 \). There exists a positive constant \( C(X_0, T, q) \) such that

\[
E \left[ \sup_{t \in [0,T]} \| X^N(t) \|_{H^q}^q \right] \leq C(X_0, T, q). \tag{3.9}
\]

**Proof.** By applying the integration by parts and the dissipativity of \(-F\), we obtain

\[
\| Y^N(t) \|^2 + (2 - \epsilon) \int_0^t \| (-A)Y^N(s) \|^2 \, ds \\
\leq \| X_0 \|^2 + C \int_0^t (\| Z^N(s) \|_{L^2}^2 + 1) \| Y^N(s) \|_{L^4}^4 \, ds \\
+ C \int_0^t (1 + \| \nabla Y^N(s) \|^2 + \| Z^N(s) \|_{L^8}^8) \, ds.
\]

Taking the \( p \)th moment and using the a priori estimates (3.6) and (3.7), we have that for \( p \geq 1 \),

\[
E \left[ \sup_{t \in [0,T]} \| Y^N(t) \|^{2p} \right] + E \left[ \int_0^T \| (-A)Y^N(s) \|^{2p} \, ds \right] \\
\leq C(p, T) \left( E \left[ \| X_0 \|^{2p} \right] + E \left[ (1 + \sup_{s \in [0,T]} \| Z^N(s) \|_{L^2}^{2p}) \int_0^T \| Y^N(s) \|_{L^4}^{4p} \, ds \right] \right) \\
+ E \left[ \left( \int_0^T (1 + \| \nabla Y^N(s) \|^2 + \| Z^N(s) \|_{L^8}^8) \, ds \right)^p \right] \\
\leq C(T, X_0^N, p),
\]

which, together with (3.7) and the Hölder inequality, completes the proof. \( \square \)

Based on the a priori estimate of \( \| X^N \| \), we are in a position to deduce the regularity estimate of \( X^N \). Before that, we first give the regularity estimate of \( Z^N \).
Lemma 3.4. Let $X_0 \in \mathbb{H}$, $q \geq 1$ and $\gamma \in (0, \frac{3}{2})$. Then the approximated stochastic convolution $Z^N$ satisfies

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| Z^N(t) \|_{\mathbb{H}^q}^q \right] \leq C(X_0, T, q, \gamma)$$

(3.10)

for a positive constant $C(X_0, T, q, \gamma)$.

Proof. By the factorization method, we have for $\alpha_1 > \frac{1}{p} + \beta$, $p > 1$, $\beta = \frac{2}{4}$,

$$\mathbb{E} \left[ \sup_{s \in [0,T]} \| Z^N(s) \|_{\mathbb{H}^q}^q \right] \leq C(T, q) \mathbb{E} \left[ \| Y_{\alpha_1,N} \|_{\mathbb{L}^p(0,T;\mathbb{H})}^{q_p} \right],$$

where $Y_{\alpha_1,N}(s) = \int_0^s (s-r)^{-\alpha_1} S(s-r) P^N G(Y^N(r) + Z^N(r)) dW(r)$. From the Hölder and Burkholder inequalities, the estimates (3.7) and (3.9), it follows that for $q \geq \max(p, 2)$,

$$\mathbb{E} \left[ \| Y_{\alpha_1,N} \|_{\mathbb{L}^p(0,T;\mathbb{H})}^{q_p} \right] \leq C(T, q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s-r)^{-2\alpha_1} \sum_{i \in \mathbb{N}^+} \| S(t-s) P^N G(Y^N(s) + Z^N(s)) e_i \|^2 dr \right)^{\frac{q}{2}} ds \right]$$

$$\leq C(T, q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s-r)^{-(2\alpha_1 + \frac{1}{2})} (1 + \| Y^N(r) \|^{2\alpha} + \| Z^N(r) \|^{2\alpha}) dr \right)^{\frac{q}{2}} ds \right]$$

$$\leq C(T, q) \left( 1 + \mathbb{E} \left[ \sup_{r \in [0,T]} \| Y^N(r) \|^{2\alpha q} \right] + \mathbb{E} \left[ \sup_{r \in [0,T]} \| Z^N(r) \|^{2\alpha q} \right] \right)$$

$$\times \left( \int_0^s (s-r)^{-\frac{1}{2}} dr \right)^{\frac{q}{2}} ds$$

$$\leq C(X_0, T, q, \alpha) \int_0^T \left( \int_0^s (s-r)^{-(2\alpha_1 + \frac{1}{2})} dr \right)^{\frac{q}{2}} ds.$$

Since $\gamma < \frac{3}{2}$, one can choose a positive a large enough number $p$ such that $\frac{2}{p} + \frac{2}{2} + \frac{1}{4} < 1$ and $2\alpha_1 + \frac{1}{4} < 1$. Thus we obtain

$$\mathbb{E} \left[ \| Y_{\alpha_1,N} \|_{\mathbb{L}^p(0,T;\mathbb{H})}^{q_p} \right] \leq C(X_0, T, q, \gamma),$$

which implies that for any $q \in \mathbb{N}^+$,

$$\mathbb{E} \left[ \sup_{s \in [0,T]} \| Z^N(s) \|_{\mathbb{H}^q}^q \right] \leq C(X_0, T, q, \gamma).$$

$\square$

Next, we deduce the following uniform regularity estimate of $X^N$.

Proposition 3.1. Let $X_0 \in \mathbb{H}$, $\gamma \in [1, \frac{5}{2})$, $T > 0$, $q \geq 1$ and $N \in \mathbb{N}^+$. Then the unique mild solution $X^N$ of Eq. (1.2) satisfies

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| X^N(t) \|_{\mathbb{H}^q}^q \right] \leq C(X_0, T, q, \gamma)$$

(3.11)

for a positive constant $C(X_0, T, q, \gamma)$.  

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Proof. Due to (3.10), it suffices to give the regularity estimate for $Y^N$. Before that, we give the following estimate of $\|Y^N(t)\|_{L^6}$. The Sobolev embedding theorem, the smoothy effect of $S(t)$ and the Gagliardo–Nirenberg inequality yield that

$$\|Y^N(t)\|_{L^6} \leq \|S(t)X_0^N\|_{L^6} + \int_0^t \|S(t-s)P^N AF(Y^N(s) + Z^N(s))\|_{L^6} ds$$

$$\leq C \|X_0^N\|_{L^6} + C \int_0^t (t-s)^{-\frac{7}{6}} \left(1 + \|Z^N(s)\|^3_{L^6} + \|Y^N(s)\|^3_{L^6}\right) ds$$

$$\leq C \|X_0^N\|_{H^1} + C \int_0^t (t-s)^{-\frac{7}{6}} \left(1 + \|Z^N(s)\|^2_{L^6} + \|AY^N(s)\|^2_{L^6} + \|Y^N(s)\|^2_{L^6}\right) ds.$$

From the Hölder inequality, the estimates (3.6), (3.9) and (3.10), it follows that for any $q \geq 1$,

$$E \left[ \sup_{t \in [0,T]} \|Y^N(t)\|^q_{L^6} \right] \leq C(p) \|X_0^N\|^q_{H^1} + C(p) E \left[ \left( \int_0^T \|AY^N(s)\|^2 ds \right)^{\frac{q}{2}} \right]$$

$$+ C(p) \left( \int_0^T (t-s)^{-\frac{7}{6}} ds \right)^{\frac{q}{2}} E \left[ \sup_{s \in [0,T]} \|Y^N(s)\|^{10q}_{L^6} \right]$$

$$+ C(p) \left( \int_0^T (t-s)^{-\frac{7}{6}} ds \right)^q \left(1 + E \left[ \sup_{s \in [0,T]} \|Z^N(s)\|^{3q}_{L^6} \right] \right)$$

$$\leq C(X_0, T, q).$$

From the mild form of $Y^N(t)$, it follows that

$$\|Y^N(t)\|_{H^7} \leq \|e^{-A^2t}X_0^N\|_{H^7} + \int_0^t \|e^{-A^2(t-s)} AF(Y^N(s) + Z^N(s))\|_{H^7} ds$$

$$\leq C \|X_0^N\|_{H^7} + C \int_0^t (t-s)^{-\frac{7}{6}} \|e^{-\frac{7}{6}A^2(t-s)} F(Y^N(s) + Z^N(s))\|_{L^6} ds$$

$$\leq C \|X_0\|_{H^7} + C \int_0^t (t-s)^{-\frac{7}{6}} \left(1 + \|Y^N(s)\|^3_{L^6} + \|Z^N(s)\|^3_{L^6}\right) ds.$$

By taking $q$th moment and making use of the a priori estimates of $\|Y^N\|_{L^6}$ and $\|Z^N\|_{H^7}$, we finish the proof. \qed

**Remark 3.1.** If $X_0 \in H^\gamma$, $\gamma \in (0, 1)$, the estimate (3.11) still holds for $\gamma \in (0, 1)$. The key ingredient of the proof is use of the ultra-contractivity of $S(t)$ to deal with the term $e^{-A^2t} X_0^N|_{L^6}$. Indeed, ultra-contractivity of $S(t)$ yields that

$$\|e^{-A^2t} X_0^N\|_{L^6} \leq Ct^{-\frac{7}{6}} \|X_0\|.$$

From the Hölder inequality, it follows that there exist $p_1, q_1$ satisfying $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $(\frac{1}{2} + \frac{7}{6})p_1 < 1$ and $q_1 < 4$, such that

$$\int_0^t (t-s)^{-\frac{7}{6}} \|Y^N(s)\|^3_{L^6} ds \leq \left( \int_0^t (t-s)^{-(\frac{1}{2} + \frac{7}{6})p_1} ds \right)^{\frac{1}{p_1}} \left( \int_0^t \|Y^N(s)\|^{3q_1}_{L^6} ds \right)^{\frac{1}{q_1}}.$$
Based on the above estimate and similar arguments in the proof of Proposition 3.7, we obtain the desired result.

After these preparations, we are able to answer the well-posedness problem of Eq. (1.1). Before that, we give the useful lemma whose proof is similar to that of [CHSb, Lemma 4.3].

**Lemma 3.5.** Let \( g : L^4 \rightarrow H \) be the Nemytskii operator of a polynomial of second degree. Then for any \( \beta \in (0,1) \), it holds that

\[
\| g(x)y \|_{H^{-1}} \leq C \left( 1 + \| x \|_E^2 + \| x \|_{H^4}^2 \right) \| y \|_{H^{-\beta}},
\]

where \( x \in E, x \in H^\beta \) and \( y \in H \).

**Proposition 3.2.** Let \( \sup_{N \in \mathbb{N}^+} \| X_0^N \|_E \leq C(X_0), T > 0 \) and \( q \geq 1 \). Then the unique solution \( X^N \) of Eq. (1.2) satisfies

\[
E \left[ \sup_{t \in [0,T]} \| X^N(t) \|_E^q \right] \leq C(X_0, T, q) \tag{3.12}
\]

for a positive constant \( C(X_0, T, q) \).

**Proof.** Due to Corollary 3.2, it remains to bound \( E \left[ \sup_{t \in [0,T]} \| Y^N(t) \|_E^q \right] \). The mild form of \( Y^N \), combined with the estimation of \( \| Y^N \|_{L^6} \) and Corollary 3.2, yields that

\[
E \left[ \sup_{t \in [0,T]} \| Y^N(t) \|_E^q \right] \leq E \left[ \sup_{t \in [0,T]} \| S(t)X_0^N \|_E^q \right] + CE \left[ \left( \int_0^T (t-s)^{-\frac{q}{2}} \| F(Y^N + Z^N) \| ds \right)^q \right] \\
\leq C(X_0, T, q),
\]

which completes the proof.

**Remark 3.2.** If \( X_0 \in H^\gamma, \gamma > \frac{1}{2} \), then \( \| X_0^N \|_E \leq C(X_0) \) holds for every \( N \in \mathbb{N}^+ \). If the bound of \( \| X_0^N \|_E \) is not uniform, then we have that

\[
E \left[ \| X^N(t) \|_E^q \right] \leq C(X_0, T, q)(1 + t^{-\frac{q}{2}}).
\]

In this case, Proposition 4.1, Theorem 4.1 and Proposition 5.1 still hold with \( p = 1 \).

4. Strong convergence analysis of the spectral Galerkin method

The main idea of our approach to proving the global existence of the solution is to show the uniform convergence of the sequence \( \{(Y^N, Z^N)\}_{N \in \mathbb{N}^+} \) and then
to prove the limit process is the unique mild solution of Eq. (1.1). In the following, we first present the strong convergence analysis of the spectral Galerkin approximation in $H^{-1}$. We would like to mention that there already exists some convergence result of finite dimensional approximation for Eq. (1.1) driven by space-time white noise (see e.g. [CHS]). Different from the additive case, the convergence analysis of finite dimensional approximation for Eq. (1.1) driven by multiplicative space-time noise is more involved and has not been studied yet.

**Proposition 4.1.** Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{3}{2})$, $T > 0$, $p \geq 1$ and $\sup_{N \in \mathbb{N}^+} \|X_0^N\|_E \leq C(X_0)$. Assume that $X_N$ and $X_M$ are the spectral Galerkin approximations with different parameters $N, M \in \mathbb{N}^+, N < M$. Then it holds that

$$
\sup_{t \in [0,T]} \mathbb{E} \left[ \|X_N(t) - X_M(t)\|_{H^{-1}}^{2p} \right] \leq C(T, X_0, p, \gamma) \lambda_N^{-\gamma p}, \quad (4.1)
$$

where $C(T, X_0, p, \gamma)$ is a positive constant.

**Proof.** Due to Proposition 4.1 we obtain that for $t \in [0, T]$, $p \geq 1$ and $\gamma \in (0, \frac{3}{2})$,

$$
\mathbb{E} \left[ \| (I - P^N) X_M(t) \|_{H^{-1}}^p \right] \leq \mathbb{E} \left[ \| (I - P^N) A^\frac{1}{2} - \frac{1}{2} A^\frac{1}{2} X_M(t) \|_p \right]
\leq C(X_0, T, p, \gamma) \lambda_N^{-\frac{p}{2}}.
$$

Thus it remains to estimate $\|X_N^N - P^N X_M\|_{H^{-1}}$. From the Taylor expansion and Itô formula, it follows that for $p \geq 2$,

$$
\|X_N(t) - P^N X_M(t)\|_{H^{-1}}^{2p}
= -2p \int_0^t \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-2} \left\| X_N(s) - P^N X_M(s) \right\|_{H^2}^2 ds
- 2p \int_0^t \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-2} \left\langle \int_0^1 F'(\theta X_N(s) + (1 - \theta)X_M(s)) d\theta \right\rangle
\left( X_N(s) - P^N X_M(s) - (I - P^N) X_M(s), X_N(s) - P^N X_M(s) \right) ds
+ 2p \int_0^t \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-2} \left\langle X_N(s) - P^N X_M(s), (G(X_N(s)) - G(X_M(s))) dW(s) \right\rangle_{H^{-1}}
+ 2p \int_0^t \sum_{i \in \mathbb{N}^+} \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-2} \|P^N(G(X_N(s)) - G(X_M(s))) e_i\|_{H^{-1}}^2 ds
+ 2p(2p - 2) \int_0^t \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-4} \sum_{i \in \mathbb{N}^+} \|X_N(s) - P^N X_M(s), P^N(G(X_N(s)) - G(X_M(s))) e_i\|_{H^{-1}}^2 ds
= -2p \int_0^t \|X_N(s) - P^N X_M(s)\|_{H^{-1}}^{2p-2} \|X_N(s) - P^N X_M(s)\|_{H^2}^2 ds
$$
\[ + I_1(t) + I_2(t) + I_3(t) + I_4(t). \]

The monotonicity of \(-F\), the Young inequality and Lemma 3.5 yield that for \(\beta \in (0, 1)\) and small \(\epsilon > 0\),

\[
I_1 \leq C \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \|X^N(s) - P^N X^M(s)\|^2 ds \\
+ 2 \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \left( A^{-\frac{1}{2}} \int_0^1 F'(\theta X^N(s) + (1 - \theta) X^M(s)) d\theta \right) \\
(I - P^N) X^M(s), A^{\frac{1}{2}} (X^N(s) - P^N X^M(s)) ds \\
\leq \epsilon \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \|X^N(s) - P^N X^M(s)\|^2 ds \\
+ C(\epsilon) \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \|A^{-\frac{1}{2}} \int_0^1 F'(\theta X^N(s) + (1 - \theta) X^M(s)) d\theta (I - P^N) X^M(s)\|^2 ds \\
\leq \epsilon \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \|X^N(s) - P^N X^M(s)\|^2 ds \\
+ C(\epsilon) \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} ds \\
+ C(\epsilon) \lambda_N^{\gamma - \beta} \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \left( 1 + \|X^N\|_{H^2}\right)^4 \\
+ \|X^M\|_{H^2} + \|X^N\|_E + \|X^M\|_E \|X^M(s)\|^2_{H^2} ds.
\]

The uniform boundedness of \(\{e_j\}_{j \in \mathbb{N}^+}\) and the Young inequality yield that for small \(\epsilon > 0\),

\[
E[I_3 + I_4] \leq C E \left[ \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \sum_{i \in \mathbb{N}^+} \|P^N((G(X^N(s)) - G(X^M(s)))e_i)\|^2_{H^{-1}} ds \right] \\
\leq C E \left[ \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} \sum_{j \in \mathbb{N}^+} \|(G(X^N(s)) - G(X^M(s)))e_j)\|^2 \lambda_j^{-1} ds \right] \\
\leq C(\epsilon) E \left[ \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{\|\cdot\|_{H^{-1}}} ds \right] \\
+ \epsilon E \left[ \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{H^{-1}} \|X^N(s) - P^N X^M(s)\|^2_{H^2} ds \right] \\
+ C E \left[ \int_0^t \|X^N(s) - P^N X^M(s)\|^2_{H^{-1}} \|(I - P^N) X^M(s)\|^2 ds \right].
\]
Theorem 4.1. Let $\gamma$ be a positive constant and $\beta \in (0, 1)$ and small $\epsilon > 0$,

$$
\mathbb{E}\left[\|X^N(t) - P^N X^M(t)\|_{L^2}^{2p}\right] \\
\leq -2p \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p-2} \|X^N(s) - P^N X^M(s)\|_{L^2}^2\right] ds \\
+ \mathbb{E}\left[I_1(t) + I_2(t) + I_3(t) + I_4(t)\right] \\
\leq C(\epsilon) \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p}\right] ds \\
+ C(\epsilon) \lambda_{X}^{-\gamma - \beta} \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p-2} \left(1 + \|X^N\|_{H^\beta}^4\right) \|X^M(s)\|_{L^2}^2\right] ds \\
+ C \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p-2} \|I - P^N\| X^M(s)\|^2\right] ds \\
\leq C(\epsilon) \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p}\right] ds + C(\epsilon) \lambda_{X}^{-\gamma - \beta - \beta p} \int_0^t \mathbb{E}\left[\left(1 + \|X^N\|_{H^\beta}^4\right) \|X^M(s)\|_{L^2}^{2p}\right] ds \\
+ C\lambda_{X}^{-\gamma p} \int_0^t \mathbb{E}\left[\|X^M(s)\|_{L^2}^{2p}\right] ds.
$$

Combining the regularity estimates of $X^N$ and $X^M$ in Proposition 3.1, we complete the proof by using the Gronwall inequality.

Now, we are in the position to deduce the error estimate in $\mathbb{H}$, which implies that $\{X^N\}_{N \in \mathbb{N}^+}$ is a Cauchy sequence in $L^p(\Omega; C([0, T]; \mathbb{H}))$.

**Theorem 4.1.** Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{1}{2})$, $T > 0$, $p \geq 1$ and $\sup_{N \in \mathbb{N}^+} \|X_0^N\|_E \leq C(X_0)$. Assume that $X^N$ and $X^M$ are the spectral Galerkin approximations with different parameters $N, M \in \mathbb{N}^+, N < M$. Then for $\alpha \in (0, \gamma)$, it holds that

$$
\mathbb{E}\left[\sup_{t \in [0, T]} \|X^N(t) - X^M(t)\|_{L^2}^{2p}\right] \leq C(T, X_0, p, \gamma) \lambda_{X}^{-\alpha p}.
$$

(4.2)

for a positive constant $C(T, X_0, p, \gamma)$.

**Proof.** From the mild form of $X^N$ and $P^N X^M$, the smoothy effect of $S(\cdot)$, Lemma 3.5, Proposition 4.1, the interpolation inequality and the Burkholder inequality, it follows that for $p \geq \max\left(\frac{1}{2}, 2\right)$, $l > 2$ and $\beta \in (0, 1)$,

$$
\mathbb{E}\left[\|X^N(t) - P^N X^M(t)\|_{L^2}^{2p}\right] \\
\leq C(\epsilon) \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p}\right] ds \\
+ C(\epsilon) \lambda_{X}^{-\gamma - \beta} \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p-2} \left(1 + \|X^N\|_{H^\beta}^4\right) \|X^M(s)\|_{L^2}^2\right] ds \\
+ C \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p-2} \|I - P^N\| X^M(s)\|^2\right] ds \\
\leq C(\epsilon) \int_0^t \mathbb{E}\left[\|X^N(s) - P^N X^M(s)\|_{L^2}^{2p}\right] ds + C(\epsilon) \lambda_{X}^{-\gamma - \beta - \beta p} \int_0^t \mathbb{E}\left[\left(1 + \|X^N\|_{H^\beta}^4\right) \|X^M(s)\|_{L^2}^{2p}\right] ds \\
+ C\lambda_{X}^{-\gamma p} \int_0^t \mathbb{E}\left[\|X^M(s)\|_{L^2}^{2p}\right] ds.
$$

Combining the regularity estimates of $X^N$ and $X^M$ in Proposition 3.1, we complete the proof by using the Gronwall inequality.\qed
From the Gronwall inequality, it follows that

\[ E \left[ \left( \int_0^T (t - s)^{-\alpha_1} ds \right)^{\frac{2p}{2p - 1}} \right] E \left[ \left( \int_0^T \left( 1 + \|X^N\|_{L^2}^2 + \|X^M\|_{L^2}^2 + \|X^N\|_{H^3}^2 \right) \right] \]

\[ + \|X^M\|_{H^3}^2 \|X^N(t) - X^M(t)\|_{H^1}^2 \|X^N(s) - X^M(s)\|_{L^1} \|X^N(t) - X^M(t)\|_{L^1} (1 - \beta^2) ds \right) \]

\[ + C(p, T) E \left[ \left( \int_0^T S(t-s)(G(X^N(s)) - G(X^M(s)))dW(s)\right)^{2p} \right] \]

\[ \leq C(p, T, X_0, \beta) \lambda_N^{-\beta p} + C(p, T) E \left[ \left( \int_0^T \sum_{i \in N^+} \|S(t-s)(G(X^N(s)) - G(X^M(s))) e_i \|^2 ds \right)^{\frac{q}{2}} \right] \]

\[ \leq C(p, T, X_0, \beta) \lambda_N^{-\beta p} + C(p, T) E \left[ \left( \int_0^T \|X^N(s) - P^N X^M(s)\|^4 ds \right)^{\frac{q}{4}} \right] \]

\[ \leq C(p, T, X_0, \beta) \lambda_N^{-\beta p} + C(p, T) \int_0^T E \left[ \|X^N(s) - P^N X^M(s)\|^{2p} \right] ds. \]

From the Gronwall inequality, it follows that

\[ \sup_{t \in [0,T]} E \left[ \left\| X^N(t) - P^N X^M(t) \right\|^{2p} \right] \leq C(X_0, T, p, \gamma) \lambda_N^{-\beta p}. \] (4.3)

Furthermore, taking supreme over \( t \in [0,T] \), similar arguments yield that for \( p \geq \max(\frac{T}{q}, 2) \), \( l > 2 \) and \( \beta \in (0, 1) \),

\[ E \left[ \sup_{t \in [0,T]} \|X^N(t) - P^N X^M(t)\|^{2p} \right] \]

\[ \leq C(p, T, X_0, \beta) \lambda_N^{-\beta p} + C(p) E \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)(G(X^N(s)) - G(X^M(s)))dW(s) \right\|^{2p} \right] . \]

The factorization method yields that for \( \alpha_1 > \frac{1}{q} \), \( q > 1 \),

\[ E \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)(G(X^N(s)) - G(X^M(s)))dW(s) \right\|^{2p} \right] \]

\[ \leq C(p, q, T) E \left[ \left\| Z_{\alpha_1, N, M} \right\|_{L^q([0,T];H)}^{2p} \right] . \]

where \( Z_{\alpha_1, N, M}(s) = \int_0^s (s-r)^{-\alpha_1} S(s-r) P^N(G(X^N(s)) - G(X^M(s)))dW(s) \).

Thus it suffices to estimate \( E \left[ \left\| Z_{\alpha_1, N, M} \right\|_{L^q([0,T];H)}^{2p} \right] \). From the Hölder and Burkholder inequalities, the estimates (3.1) and (4.1), it follows that for \( 2p \geq q \),

\[ E \left[ \left\| Z_{\alpha_1, N, M} \right\|_{L^q([0,T];H)}^{2p} \right] \]

\[ = E \left[ \left( \int_0^T \left\| \int_0^s (s-r)^{-\alpha_1} S(s-r) P^N(G(X^N(s)) - G(X^M(s)))dW(r) \right\|^{q ds} \right)^{\frac{2p}{q}} \right] \]

\[ \leq C(T, p) \int_0^T E \left[ \left\| \int_0^s (s-r)^{-\alpha_1} S(s-r) P^N(G(X^N(s)) - G(X^M(s)))dW(r) \right\|^{2p} \right] ds \]
More precisely, assume that we have two different mild solutions due to the Lipschitz continuity of \(G\) we complete the proof.

We first show the local uniqueness of the mild solution for Eq. (1.1). Let

\[
\text{Proposition 5.1.} \quad \text{existence of the solution based on this approach.}
\]

use other approximation process (see e.g. [MDPZ13, ASS19]) to show global by multiplicative space-time white noise. We would like to point that once may position to show the global existence of the unique solution for Eq. (1.1) driven

\[
\text{Eq. (1.1) with the same initial datum} \quad X,
\]

\[
\text{Combining the above estimates and } E\left[\sup_{t \in [0,T]} (I-P^N)(t)\right] \leq C(T, X_0, p, \gamma)\lambda_N^{-\gamma p},
\]

we complete the proof.

\[\square\]

5. Global existence and regularity estimate

Based on the convergence of the approximate process \(X^N\), we are in the position to show the global existence of the unique solution for Eq. (1.1) driven by multiplicative space-time white noise. We would like to point that once may use other approximation process (see e.g. [MDPZ13, ASS19]) to show global existence of the solution based on this approach.

\[
\text{Proposition 5.1. Let } T > 0, X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{1}{2}), p \geq 1 \text{ and } \sup_{N \in \mathbb{N}^+} \|X_0^N\| \leq C(X_0). \text{ Then Eq. (1.1) possesses a unique mild solution } X \text{ in } L^{2p}(\Omega; C(0, T; \mathbb{H})).
\]

\[
\text{Proof. We first show the local uniqueness of the mild solution for Eq. (1.1). Let} \quad \tau_R := \inf\{t \in [0, T] : \|X(t)\| > R\}. \text{ Then the uniqueness in } [0, \tau_R] \text{ is obtained due to the Lipschitz continuity of } G \text{ and the local Lipschitz continuity of } F. \text{ More precisely, assume that we have two different mild solutions } X_1 \text{ and } X_2 \text{ for Eq. (1.1) with the same initial datum } X_0. \text{ Next, we prove the local uniqueness, i.e., in each } [0, \tau_R], X_1 = X_2, \text{ a.s. Since the decompositions } X_1 = Y_1 + Z_1 \text{ and } X_2 = Y_2 + Z_2, \text{ we have for } t \in [0, \tau_R],
\]

\[
d(Y_1(t) - Y_2(t)) = -A^2(Y_1(t) - Y_2(t))dt - A(F(X_1(t)) - F(X_2(t)))dt,
\]

\[
Y_1(0) - Y_2(0) = 0,
\]

and

\[
d(Z_1(t) - Z_2(t)) = -A^2(Z_1(t) - Z_2(t))dt + (G(X_1(t)) - G(X_2(t)))dW(t),
\]

\[
Z_1(0) - Z_2(0) = 0.
\]

From the mild form of \(Z_1 - Z_2\) and the factorization method, it follows that

\[
E\left[\sup_{t \in [0, \tau_R]} \|Z_1(t) - Z_2(t)\|^p_E\right]
\]

\[
= E\left[\sup_{t \in [0, \tau_R]} \left\| \int_0^t S(t-s)(G(X_1(s)) - G(X_2(s)))dW(s) \right\|^p\right]
\]

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\[
\alpha_1 = \frac{1}{p} + \frac{1}{8}
\]
for large enough \( p > 1 \). Similar arguments, together with the Young and Gagliardo–Nirenberg inequality, yield that for \( t \in [0, \tau_R] \) and for some \( \epsilon < 1 \),
\[
\|Y_1(t) - Y_2(t)\|^2
\leq - \int_0^t 2\|A(Y_1(s) - Y_2(s))\|^2 ds
+ 2 \int_0^t \langle -A(F(X_1(s)) - F(X_2(s))), Y_1(t) - Y_2(t) \rangle ds
\leq C(\epsilon) \int_0^t \|F(X_1(s)) - F(X_2(s))\|^2 ds
\leq C(\epsilon) \int_0^t \|X_1(s) - X_2(s)\|^2 (1 + \|X_1(s)\|^\frac{1}{p} + \|X_2(s)\|^\frac{1}{p}) ds
\leq C(\epsilon) \int_0^t \|X_1(s) - X_2(s)\|^2 (1 + \|AY_1(s)\|^2 + \|AY_2(s)\|^2 + \|Y_1(s)\|^6
+ \|Y_2(s)\|^6 + \|Z_1(s)\|^\frac{4}{p} + \|Z_2(s)\|^\frac{4}{p}) ds.
\]
Notice that for \( i = 1, 2 \) and \( t \in [0, \tau_R] \),
\[
\int_0^t \|AY_i(s)\|^2 ds \leq C\|Y_0\|^2 + C \int_0^t (\|Z_i(s)\|^\frac{2}{p} + 1)\|Y_i(s)\|^\frac{4}{1} ds
+ C \int_0^t (1 + \|\nabla Y_i(s)\|^2 + \|Z_i(s)\|^\frac{8}{p}) ds
\leq C(R, T, Y_0) < \infty.
\]
By Gronwall’s inequality, we get
\[
\|Y_1(t) - Y_2(t)\|^2 \leq \exp(C(R, T, Y_0)) \int_0^t \|X_1(s) - X_2(s)\|^2 ds.
\]
From the previous estimates, we conclude that for \( 0 \leq s \leq t \leq \tau_R \) and \( p \geq 1 \),
\[
\|X_1(s) - X_2(s)\|^{2p} \leq C_p\|Y_1(s) - Y_2(s)\|^{2p} + C_p\|Z_1(s) - Z_2(s)\|^{2p}
\leq C_p \exp(C(R, T, Y_0)) \int_0^s \|X_1(s) - X_2(s)\|^{2p} ds + C_p\|Z_1(s) - Z_2(s)\|^{2p},
\]
which, together with Gronwall’s inequality, yields that
\[
\|X_1(s) - X_2(s)\|^{2p} \leq \exp(C(p, T) \exp(C(R, T, Y_0)))\|Z_1(s) - Z_2(s)\|^{2p}.
\]
Thus it suffices to prove that
\[ X \parallel \]
which implies that for a subsequence \( N \) and \( \lim_{s \to \infty} \)

\[
\text{According to Theorem 4.1, we have that } \{X^N\}_{N \in \mathbb{N}^+} \text{ is a Cauchy sequence in } L^{2p}(\Omega; C([0, T]; \mathbb{F})). \text{ Then we denote } X \text{ the limit of } X^N \text{ in } L^{2p}(\Omega; C([0, T]; \mathbb{F})). \text{ From } \lim_{N \to \infty} \|X^N - X\|_{L^{2p}(\Omega; C([0, T]; \mathbb{F}))} = 0, \text{ it follows that for each } i \in \mathbb{N}^+,
\]

\[
\lim_{N \to \infty} \|\langle X^N - X, e_i \rangle\|_{L^{2p}(\Omega; C([0, T]; \mathbb{R}))} = 0,
\]

which implies that for a subsequence \( \{X^{N_k}\}_{k \in \mathbb{N}^+} \),

\[
\lim_{k \to \infty} \sup_{t \in [0, T]} |\langle X^{N_k}(t), e_i \rangle|^2 \Lambda_i^2 = \sup_{t \in [0, T]} |\langle X(t), e_i \rangle|^2 \Lambda_i^2, \text{ a.s.,}
\]

and \( \lim_{N \to \infty} X^{N_k} = X \) in \( C([0, T]; \mathbb{H}^\gamma), \text{ a.s.} \) The uniform boundedness of \( \|X^N\|_{L^{2p}(\Omega; C([0, T]; \mathbb{H}^\gamma))}, \text{ together with Fatou’s lemma, yields that}
\]

\[
\|X\|_{L^{2p}(\Omega; C([0, T]; \mathbb{H}^\gamma))}^2 \leq \liminf_{N \to \infty} \|X^{N_k}\|_{L^{2p}(\Omega; C([0, T]; \mathbb{H}^\gamma))}^2 \leq C(T, X_0, p).
\]

Thus it suffices to prove that \( X \) is the mild solution of Eq. (1.1), i.e.,

\[
X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s))dW(s), \text{ a.s.}
\]
Due to the mild form of $X^N$, the regularity of $X_0$, the factorization method, the Burkholder inequality the smoothy effect of $S(\cdot)$ and the error estimate \(\text{(4.3)}\), we have that for $2p \geq q$, $\alpha_1 \geq \frac{1}{q}$, sufficient large $q > 1$ and $\beta \in (0, 1)$,

\[
\|S(t)(I - P^N)X_0\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} + \|\int_0^t S(t-s)A(F(X(s)) - P^N F(X^N(s)))ds\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
+ \|\int_0^t S(t-s)(G(X(s)) - P^N G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
\leq C(T, X_0)\lambda_N^{-\frac{2}{p}} + \|\int_0^t S(t-s)A(I - P^N)F(X^N(s))ds\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
+ \|\int_0^t S(t-s)AP^N(F(X(s) - F(X^N(s)))ds\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
+ \|\int_0^t S(t-s)(I - P^N)G(X(s))dW(s)\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
+ \|\int_0^t S(t-s)P^N(G(X(s)) - G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
\leq C(T, X_0, p)\lambda_N^{-\frac{2}{p}} + C(T, p)\lambda_N^{-\frac{2}{p}}\|\int_0^t (t-s)^{-\frac{1}{4}}(1 + \|X(s)\|^2_{\mathbb{H}} + \|X^N(s)\|^2_{\mathbb{H}})\|X(s) - X^N(s)\|ds\|_{L^{2p}(\Omega; C([0,T]; \mathbb{H}))} \\
+ C(T, p)\|\int_0^t (t-s)^{-\alpha_1}S(t-s)(I - P^N)G(X(s))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
+ C(T, p)\|\int_0^t (t-s)^{-\alpha_1}S(t-s)P^N(G(X(s)) - G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
\leq C(T, X_0, p)\lambda_N^{-\frac{2}{p}} + C(T, p)\lambda_N^{-\frac{2}{p}}\|\int_0^t (t-s)^{-\alpha_1}S(t-s)(I - P^N)G(X(s))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
+ C(T, p)\|\int_0^t (t-s)^{-\alpha_1}S(t-s)P^N(G(X(s)) - G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
\leq C(T, X_0, p)\lambda_N^{-\frac{2}{p}} + C(T, p)\lambda_N^{-\frac{2}{p}}\|\int_0^t (t-s)^{-\alpha_1}S(t-s)P^N(G(X(s)) - G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
+ C(T, p)\|\int_0^t (t-s)^{-\alpha_1}S(t-s)P^N(G(X(s)) - G(X^N(s)))dW(s)\|_{L^{2p}(\Omega; L^4([0,T]; \mathbb{H}))} \\
\leq C(T, X_0, p)\lambda_N^{-\frac{2}{p}}.
\]

The above estimation implies that $S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s))dW(s)$ is the limit of $X^N$ in $L^{2p}(\Omega; C([0,T]; \mathbb{H}))$. By the uniqueness of the limit in $L^{2p}(\Omega; C([0,T]; \mathbb{H}))$, we conclude that $X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s))dW(s)$, a.s. \qed
From the arguments in the above proof, we immediately get that the following well-posedness result under mild assumptions.

**Theorem 5.1.** Let $T > 0$, $X_0 \in H^\gamma$, $\gamma > 0$, $p \geq 1$. Then Eq. (1.1) possesses a unique mild solution $X$ in $L^{2p}(\Omega; C([0, T]; H))$.

**Proof.** Since the strong convergence in Theorem 4.1 holds with $p = 1$ (see Remark 3.2), we have that $\{X_N\}_{N \in \mathbb{N}^+}$ is a Cauchy sequence in $L^{2p}(\Omega; C([0, T]; H))$, which implies that there exists a subsequence $\{X_{N_k}\}_{k \in \mathbb{N}^+}$ converging to $X$ in $C([0, T]; H)$ a.s. Notice that Lemma 3.3 implies that $X_N \in L^{2p}(\Omega; C([0, T]; H))$ for any $p \geq 1$. By using the Hölder inequality and Fatou’s lemma, we obtain

$$
\|X - X^N\|_{L^{2p}(\Omega; C([0, T]; H))} \\
\leq \|X - X^N\|_{L^{2p-\frac{2}{p}}(\Omega; C([0, T]; H))} \|X - X^N\|_{L^2(\Omega; C([0, T]; H))} \\
\leq \|X - X^N\|_{L^2(\Omega; C([0, T]; H))} \left(C(X_0, T, p) + \lim_{k \to \infty} \|X_{N_k}\|_{L^{2p-\frac{2}{p}}(\Omega; C([0, T]; H))}\right)^{\frac{2p-1}{2p}} \\
\leq C(X_0, T, p) \|X - X^N\|_{L^2(\Omega; C([0, T]; H))},
$$

which implies that $\{X_N\}_{N \in \mathbb{N}^+}$ is also a Cauchy sequence in $L^{2p}(\Omega; C([0, T]; H))$.

**Remark 5.1.** Let $T > 0$, $X_0 \in H$, $p \geq 1$. By the similar arguments in the proof of Proposition 5.3 and Theorem 5.1, one can prove that Eq. (1.1) possesses a unique mild solution $X$ in $C([0, T]; L^{2p}(\Omega; H))$. On the other hand, combining with Kolmogorov’s continuity theorem, we get that $X$ has almost surely continuous trajectories in $H$. 

After establishing the well-posedness of Eq. (1.1), we turn to giving the following properties of the exact solution $X$.

**Corollary 5.1.** Let $X_0 \in H^\gamma$, $\gamma \in \left(0, \frac{3}{2}\right)$, $T > 0$ and $p \geq 1$. The unique mild solution $X$ of Eq. (1.1) satisfies

$$
\mathbb{E}\left[\sup_{t \in [0, T]} \|X(t)\|^p_{L^{2p-\frac{2}{p}}(H^\gamma)}\right] \leq C(X_0, T, p). \tag{5.1}
$$

**Proof.** By Proposition 3.1, Theorem 4.1 and Fatou’s Lemma, we completes the proof.

**Proposition 5.2.** Let $X_0 \in E$, $T > 0$ and $p \geq 1$. The unique mild solution $X$ of Eq. (1.1) satisfies

$$
\sup_{t \in [0, T]} \mathbb{E}\left[\|X(t)\|^p_E\right] \leq C(X_0, T, p).
$$

**Proof.** The proof is similar to that of Proposition 3.2.

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Remark 5.2. Under the condition of Proposition 5.2, one can prove that the solution $X$ has almost surely continuous trajectories in $E$. In addition we assume that $X_0$ is $\beta$-Hölder continuous with $\beta \in (0, 1)$. By using the fact that $S$ is an analytical semigroup in $E$ and similar arguments in the proof of Proposition 5.3, we have that $X$ is almost surely $\beta$-continuous in space and $\frac{\beta}{2}$-continuous in time.

Proposition 5.3. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{\beta}{2})$, $p \geq 1$. Then the unique mild solution $X$ of Eq. (1.1) satisfies

$$
\|X(t) - X(s)\|_{L^p(E)} \leq C(X_0, T, p, \gamma)(t - s)\frac{\gamma}{2}
$$

(5.2)

for a positive constant $C(X_0, T, p, \gamma)$ and $0 \leq s \leq t \leq T$.

Proof. From the mild form of $X$, it follows that

$$
\|X(t) - X(s)\| \leq \|(S(t) - S(s))X_0\|
+ \int_0^t \|(S(t - r) - S(s - r))AF(X(r))\| \, dr
+ \int_s^t \|S(t - r)AF(X(r))\| \, dr
+ \int_0^s \|S(t - r) - S(s - r)G(X(r))dW(r)\|
+ \int_s^t \|S(t - r)G(X(r))dW(r)\|.
$$

By taking $p$th moment, using (5.1) and the ultra-contractivity of $S(t)$, we get

$$
\mathbb{E}\left[\|(S(t) - S(s))X_0\|^p\right] \leq C(T, X_0, p, \gamma)(t - s)\frac{\gamma}{2}
$$

and

$$
\mathbb{E}\left[\int_0^t \|(S(t - r) - S(s - r))AF(X(r))\|^p \, dr\right]
\leq C(T, p)\mathbb{E}\left[\left(\int_0^s (s - r)^{-\frac{\beta}{2} - \frac{\gamma}{2}}\|(S(t - s) - I)A^{-\frac{\beta}{2}}\|F(X(r))\|dr\right)^p\right]
\leq C(T, X_0, p, \gamma)(t - s)\frac{\gamma}{2},
$$

and

$$
\mathbb{E}\left[\left(\int_s^t \|S(t - r)AF(X(r))\| \, dr\right)^p\right] \leq C(T, p)\mathbb{E}\left[\left(\int_s^t (t - r)^{-\frac{\beta}{2}}\|F(X(r))\|dr\right)^p\right]
\leq C(T, X_0, p)(t - s)^{\frac{\beta}{2}}.
$$

The Burkholder inequality and (5.1) yield that

$$
\mathbb{E}\left[\left\|\int_0^s (S(t - r) - S(s - r))G(X(r))dW(r)\right\|^p\right]
$$
\[ \leq C(T, p) \mathbb{E} \left[ \left( \int_{0}^{t} \sum_{i \in \mathbb{N}^{+}} \| S(s - r)(S(t - s) - I)G(X(r))e_i \|^2 ds \right)^{\frac{p}{2}} \right] \]

\[ \leq C(T, X_0, p, \gamma)(t - s)^{\frac{p}{2}} \]

and

\[ \mathbb{E} \left[ \left\| \int_{s}^{t} S(t - r)G(X(r))dW(r) \right\|^p \right] \]

\[ \leq C(T, p) \mathbb{E} \left[ \left( \int_{s}^{t} \sum_{i \in \mathbb{N}^{+}} \| S(t - r)G(X(r))e_i \|^2 ds \right)^{\frac{p}{2}} \right] \leq C(T, p, X_0)(t - s)^{\frac{p}{4}}. \]

Combining all the above estimates, we complete the proof. \( \square \)

**Remark 5.3.** Under the same condition as in Proposition 5.3, the solution of the spectral Galerkin method \( X^N \) satisfies

\[ \| X^N(t) - X^N(s) \|_{L^p(\Omega; \mathbb{H})} \leq C(X_0, T, p)(t - s)^{\frac{p}{2}}, \]

where \( C(X_0, T, p) > 0 \) and \( 0 \leq s \leq t \leq T \).

As a result of Proposition 5.3, we have the following strong convergence rate of the spectral Galerkin method.

**Corollary 5.2.** Let \( X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}], T > 0, \ p \geq 1 \) and \( \sup_{N \in \mathbb{N}^{+}} \| X^N_0 \|_E \leq C(X_0) \). Then for \( \alpha \in (0, \gamma) \), there exists \( C(X_0, T, p, \alpha) > 0 \) such that

\[ \| X^N - X \|_{L^p(\Omega; C([0, T]; \mathbb{H}))} \leq C(X_0, T, p, \alpha)\lambda_N^{-\frac{\alpha}{2}}. \] (5.3)

As a consequence of the strong convergence of the spectral Galerkin method, the following exponential integrability property of the mild solution holds. We would like to mention that the exponential integrability property has many applications in non-global SPDE and its numerical approximation (see e.g. [CHJ, CHLZ19, CHSa]).

**Corollary 5.3.** Let \( X_0 \in \mathbb{H} \). There exist \( \beta > 0, \ c > 0 \) such that for \( t \in [0, T] \),

\[ \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{0}^{t} e^{-\beta s} \| X(s) \|_{\mathbb{H}^{\beta - 1}}^2 ds + \frac{1}{2} \int_{0}^{t} e^{-\beta s} \| \nabla X(s) \|^2 ds \right) \right] \]

\[ \leq C(X_0, T). \]

**Proof.** From the Gagliardo–Nirenberg and Young inequalities, it follows that

\[ \int_{0}^{t} \| X^N - X \|_{L^4 ds} \leq C \int_{0}^{t} \| \nabla (X^N - X) \|^2 ds + C \int_{0}^{t} \| X^N - X \|^6 ds. \]

The similar arguments in Proposition 4.1 yield that for \( t \in [0, T] \),

\[ \lim_{N \to \infty} \| X^N - X \|_{L^2(\Omega; L^2([0, t]; \mathbb{H}))} = 0, \]
which together with the strong convergence of $X^N$ in $C([0, t]; L^6(\Omega; \mathbb{H}))$ implies that $\lim_{N \to \infty} \|X^N - X\|_{L^6(\Omega; L^4([0, t]; \mathbb{H}^1))} = 0$. Thus by Fatou’s lemma, it suffices to show the uniform boundedness of the exponential moment for $X^N$.

Denote $\mu(x) = -A^2 x - AP^N F(x)$ and $\sigma(x) = P^N G(x)I_\mathbb{H}$ and $U(x) = \frac{1}{2}\|x\|_{H_{-1}}^2$, where $x \in P^N(\mathbb{H})$. By direct calculations and the interpolation inequality, we get

$$\langle DU(x), \mu(x) \rangle + \frac{1}{2} \text{tr}[D^2 U(x) \sigma(x) \sigma^*(x)] + \frac{1}{2} \|\sigma(x)^* DU(x)\|^2$$

$$= \langle x, -A^2 x + AF(x) \rangle_{H_{-1}} + \frac{1}{2} \sum_{i \in \mathbb{N}^+} \|P^N(G(x)e_i)\|_{H_{-1}}^2 + \frac{1}{2} \sum_{i \in \mathbb{N}^+} \langle x, G(x)e_i \rangle_{H_{-1}}^2,$$

$$\leq -(1 - \epsilon)\|\nabla x\|^2 - (4c_4 - \epsilon)\|x\|_{L^4}^4 + \epsilon\|x\|_{H_{-1}}^2 + C(\epsilon).$$

Using the exponential integrability lemma in [CHL17] and taking $\beta = \epsilon$, we have

$$E\left[ \exp\left( e^{-\beta t} \frac{1}{2}\|X^N(t)\|_{H_{-1}}^2 + (4c_4 - \epsilon) \int_0^t e^{-\beta s}\|X^N(s)\|_{L^4}^4 ds + (1 - \epsilon) \int_0^t e^{-\beta s}\|\nabla X^N(s)\|^2 ds \right) \right]$$

$$\leq C(X_0, T, \epsilon),$$

which, combined with Fatou’s lemma, completes the proof.

### 6. Conclusion

In this paper, we introduce a new approach to studying the global existence and regularity estimate of the solution process for stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise. Compared to the existing work, we use the spectral Galerkin method, instead of the cut-off equation, to approximate the original equation. Then by proving the well-posedness and a priori estimates of the approximated equation, we show that the solution $\{X^N\}_{N \in \mathbb{N}^+}$ possesses the sharp strong convergence rate and thus is a Cauchy sequence in certain Banach space. As a consequence, the limit process of $X^N$ is shown to be the global solution of stochastic Cahn–Hilliard equation and to possess the optimal regularity estimates.

### 7. Acknowledgement

The authors are very grateful to Professor Yaozhong Hu (University of Alberta) for his helpful discussions and suggestions.

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