Supporting decisions by unleashing multiple mindsets using pairwise comparisons method

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Abstract

Inconsistency in pairwise comparison judgements is often perceived as an unwanted phenomenon and researchers have proposed a number of techniques to either reduce it or to correct it. We take a viewpoint that this inconsistency unleashes different mindsets of the decision maker(s) that should be taken into account when generating recommendations as decision support. With this aim we consider the spanning trees analysis which is a recently emerging idea for use with the pairwise comparison approach that represents the plurality of mindsets (in terms of a plurality of vectors corresponding to different spanning trees). Until now, the multiplicity of the vectors supplied by the spanning trees approach have been amalgamated into a single preference vector, losing the information about the plurality of mindsets. To preserve this information, we propose a novel methodology taking an approach similar to Stochastic Multi-criteria Acceptability Analysis. Considering all the rankings of alternatives corresponding to the different mindsets, our methodology gives the probability that an alternative attains a given ranking position as well as the probability that an alternative is preferred to another one. Since the exponential number of spanning trees makes their enumeration prohibitive, we propose computing approximate probabilities using statistical sampling of the spanning trees. Our approach is also appealing because it can be applied also to incomplete sets of pairwise comparisons. We demonstrate its usefulness with a didactic example as well as with an application to a real-life case of selecting a Telecom backbone infrastructure for rural areas.

Keywords: Multiple Criteria Decision Analysis; Pairwise comparisons; Graph theory; Spanning trees; Stochastic Multi-attribute Acceptability Analysis; Analytical Hierarchy Process
1. Introduction

In any decision of our life we have to compare a plurality of alternatives with respect to several points of views taking into account our preferences. For example, if we have to choose a city car, we have to consider the models available on the market and we have to compare them with respect to their salient characteristics such as maximum speed, acceleration, space, price, fuel consumption. To make a well thought-out decision it is necessary

(1) to assign a priority to each alternative with respect to each criterion in order to evaluate its related performance,

(2) to assign a priority to each criterion in order to define its importance

(3) to aggregate the evaluations of performances on considered criteria obtained in point (1) taking into account the importance of criteria obtained in point (2).

In order to assign priorities to alternatives and criteria, the use of the pairwise comparison approach is quite common due to the fact that it allows decision makers to focus on one comparison at a time. When prioritising alternatives in point (1), the decision maker focuses on two alternatives at a time, deciding which one is more preferred on the given criterion. In most cases, the decision maker is also asked about the strength of his/her preference. These two questions can be combined together as a single question, however, one may argue that establishing the strength of preference can only be established after deciding the direction of preference. The same process can be repeated for prioritising criteria in point (2).

When there are only two options at hand, obviously there will be only one comparison required and therefore the prioritisation problem becomes trivial. However, when there are three or more options to prioritise, the number of comparisons increases significantly. When comparing three options, say A, B and C, we have three possibilities of comparing A to B, then B to C, and finally comparing A to C. One may argue that the last comparison is not required as the comparison of A to C can be mathematically derived from the other two comparisons. However, the counter-argument is that we should still collect this apparently redundant information as the decision maker might have different opinion about options A and C when compared directly without involving option B. If the derived judgement and the directly obtained judgement turn out to be same, then we can easily calculate the underlying preferences of the decision maker. Instead, if these two judgements happen to be different, then we cannot straightforwardly calculate the underlying preferences. Nevertheless, even if these judgements are different, in any case, from
a psychological viewpoint, this gives us an additional piece of information about the problem. In fact, we expose the inconsistency amongst the direct judgements, and there are a number of techniques for eliciting preferences from inconsistent judgements, for example, the row geometric mean (Crawford and Williams 1985), the logarithmic least squares method (de Jong 1984; Rabinowitz 1976), and the eigenvector method (Saaty 1977). The basic idea of all these techniques for preference elicitation is to average and to amalgamate the preference information supplied by the decision maker (DM), trying to discover the "true preferences" behind the DM’s inconsistent judgements. We believe that with the existing methods a relevant wealth of information is lost. In fact, we believe that more than merely identifying internal inconsistency, the judgements provided by the DM reveal the plurality of his/her mindsets. Consequently, unleashing these multiple mindsets is an important area of investigation in an effective decision support process.

In this context, a recent work on the use of the spanning trees approach for analysing pairwise comparison judgements (Tsyganok 2010; Siraj et al. 2012) has gained our attention due to the fact that the generation of spanning trees does not involve any aggregation of underlying preferences. The aim of this paper is to further investigate the use of the spanning trees approach to uncover all possible alternative preference vectors that represent the decision maker in one way or another.

Our aim in this paper is to use stochastic approaches to analyse all the possible preference vectors from inconsistent judgements. In order to appreciate the use of stochastic approaches, one must realise that even a small number of alternatives and criteria can generate a large number of possible combination of priority vectors for the weights of the criteria (we call them weight vectors) and the evaluation of the alternatives on considered criteria (we call them evaluation vectors). For example, a decision problem having only four criteria and four alternatives will still end up in generating over a million possible combinations of weight vectors and evaluation vectors. We discuss this in more detail later in Section 4.

The stochastic approach proposed in this article can be used as a support in practical decision making situations. A typical Multiple Criteria Decision Analysis (MCDA) support system aims at recommending a solution and to assess its robustness. On the contrary, our proposed approach provides insights to the DM for better understanding of his/her preferences, which, in turn, will help the DM to make a well-informed decision.

The paper is organised as follows. The next section provides background on the pairwise comparisons and the potential of the spanning trees approach; Section 3 then introduces the enumeration of all possible combinations of weight vectors and evaluation vectors, and the use of a stochastic approach for eliciting preferences; then the statistical sample selection of a well distributed family of spanning trees based on a
random walk procedure is discussed in Section 4. Section 5 discusses the Telecom backbone case study to demonstrate the use of random spanning trees, and then finally conclusions and future work are discussed in Section 6.

2. Background

Pairwise comparisons are used to obtain DM's preferences using a "one-at-a-time" approach. For example, in a car selection problem, we can either ask the DM to directly assign weights to the three criteria of Price, Speed, and Looks; or we can alternatively ask him/her to compare two criteria at a time and make these comparisons for all three possible pairs, that is, Price versus Speed, Speed versus Looks, and then Price versus Looks. These comparative judgements can then be used to calculate the relative weights in the form of a weight vector (sometimes referred as preference vector).

Whilst comparing the three criteria, one would expect that if Price is considered more important than Speed and Speed more important than Looks, then Price must be more important than Looks. This is often referred as the ordinal consistency requirement. A much stronger requirement would be that if Price is preferred to Speed $p$ times, and Speed is preferred to Looks $q$ times, then Price should be preferred to Looks $pq$ times. This is often termed as cardinal consistency. Although these are rational and justified requirements from mathematical perspective, DMs often break these rules. Obviously, if the DM is cardinally consistent, then the ordinal consistency will already be achieved. However, the opposite is not true, that is, ordinal consistency does not ensure that the judgements are cardinally consistent.

This is demonstrated through an example shown in Fig. 1 where Price is considered twice as important as Speed, Speed is considered three times as important as Looks - which might suggest that that Price is 6 times as important as Looks. If this is indeed the case then we can easily calculate the preference vector shown on the right side of this figure. However, if instead of choosing 6, the DM suggests Price is 3 times as important as Looks, then this makes the three judgements cardinally inconsistent - though note that the order of preference is still preserved and so the judgements are not ordinally inconsistent. In the case of cardinal inconsistency, eliciting the preference vector is not a straight-forward task. There are numerous algorithms proposed to estimate the preference vector from an inconsistent set of judgements, however, it remains a widely debated issue as there is no single method that can be justified as the most appropriate method for eliciting preferences from inconsistent judgements.

So far, we have assumed that the DMs provide a complete set of pairwise comparison judgements. However, in practice, it is quite common to confront situations where DMs provide an incomplete set of
judgements, and where it is not always possible to ask DMs to provide the missing data. This leaves us in situations where preferences should be elicited from inconsistent, as well as, incomplete sets of judgements.

The most widely-used technique involving pairwise comparison judgements is the Analytic Hierarchy Process (AHP) (Saaty, 1980) where DMs structure their criteria into a hierarchy and then evaluate alternatives with respect to each of these criteria. Fig. 2 demonstrates this technique with the help of the car selection problem discussed earlier. In this figure, the DM has evaluated four alternatives with respect to Price, Speed and Looks (therefore, providing three sets of pairwise comparison judgements). Note that these judgements are shown in the form of a 4-by-4 matrix, which is a standard representation used to show pairwise comparisons. On the top-right of this figure, the DM has also provided the relative importance of the three criteria, again by using pairwise comparison judgements.

These pairwise comparison judgements are then used to elicit preference vectors with the help of some elicitation technique. Although there are many techniques for elicitation, the most widely used techniques are Right Eigenvector (REV) (Saaty 1980) and Row Geometric Mean (RGM) (Crawford 1987). These elicited vectors are then used to construct a decision table as shown at the bottom right of this figure (Fig. 2). At the bottom of this table, note that the criteria weights can also be elicited using the same REV or RGM method. Finally, all these evaluation vectors and criteria weights can be compiled to generate some aggregated scores and/or to produce some form of recommendations to the decision maker(s).

Historically, REV method has been widely used for eliciting preference vectors for both consistent and inconsistent judgements. The inconsistency is usually measured in terms of the Consistency Ratio (CR) which is an Eigenvalue-based measure. According to this measure, the PC matrix is usually considered
acceptable when the CR value remains below a value of 0.1. However, the REV method has been criticised due to its left-right eigenvector asymmetry, the use of arbitrary thresholds for inconsistency acceptability, as well as a few other further issues (Barzilai 1997; Brunelli 2018). Due to these shortcomings, several other methods have been proposed in the literature. Choo and Wedley (2004) analysed and numerically compared a variety of these prioritisation methods and concluded that there is no single best method that outperforms the others in every situation. Although REV is the most commonly used method, the RGM approach has gained popularity due to its mathematical properties, and due to its ease of implementation (Kulakowski 2020).

While focusing on this "single solution" aspect, it can be argued that an in-depth analysis of the inconsistency gets neglected. We contend that a prioritisation method must have the capabilities to focus on both aspects of the problem i.e. production of a "good quality" preference vector and also facilitation of an in-depth analysis.
2.1. Spanning tree approach

In this context, a graph-theoretic approach was proposed to generate a set of all possible preference vectors through enumeration (Tsyganok 2010, see also Siraj et al. 2012). This approach is briefly summarised in Fig. 3 where the DM provides a set of PC judgements (shown on the top-left) which is then translated into a fully connected graph. This graph is then analysed using spanning tree analysis to identify all possible combination of judgements, and eventually, generating all possible preference vectors. Each of these alternative preference vectors essentially represents a mindset of the DM.

| DM provides pairwise comparison judgments | Analyst translates DM’s judgments into mathematical representations, for example, a matrix or a graph |
|----------------------------------------|------------------------------------------------------------------------------------------------|
|                                        | Each vector represents a mindset of the DM                                                                 |
|                                        | Elicit a vector from each of these combinations of judgments                                               |
|                                        | Spanning tree analysis unleashes all possible combinations                                                 |

The proposed method was shown to have a number of desirable properties, however, since the original method used the arithmetic mean to calculate the average of all these vectors, it was again focusing on the "single solution" aspect. Also, one may argue that this computationally expensive enumeration will be overkill when the judgements are (fully) consistent, as all vectors will have identical values. On the contrary, we argue that this rarely happens in real life; in the majority of cases, human judgements are found to be inconsistent, and therefore, unleashing these multiple mindsets can provide useful insights in practical
decision making problems.

More recently, Lundy et al. (2017) proposed a geometric version of this spanning tree approach and established the mathematical equivalence of this approach with RGM. They also proposed the possibility of using statistical techniques to gain insights into inconsistency. In this paper, we develop this idea further and demonstrate its usefulness when we apply this technique at a meta-level, that is, applying it to the whole problem involving multiple criteria, instead of processing just one set of judgements.

3. Stochastic approach to spanning trees

We have seen that until now the multiplicity of prioritisation vectors supplied by spanning trees approach have been amalgamated into a single priority vector either using the arithmetic mean or geometric mean. However, with this approach the information about the plurality of mindsets is lost. This is the price to pay to get a single overall ranking of alternatives. A different way of thinking could be to consider the plurality of ranking supplied by the plurality of priority vectors corresponding to each combination of (evaluation and weighting) spanning trees, that in turn corresponds to a mindset.

3.1. Problem formulation

Consider a situation where the DM has \( n \) alternative options to choose from, and \( m \) criteria to consider whilst making this decision. These alternatives and criteria can be denoted as:

- Set of alternatives: \( A = \{a_1, \ldots, a_i, \ldots, a_n\} \),
- Set of criteria: \( G = \{g_1, \ldots, g_j, \ldots, g_m\} \)

In order to assess these alternatives with respect to each criterion, \( m \) sets of pairwise comparison judgements will be collected. Let us denote each of these pairwise comparison by

\[
M^j = [c^j_{i_1, i_2}], g_j \in G
\]

with \( c^j_{i_1, i_2} \) being the pairwise comparison judgement of alternative \( a_{i_1} \) with alternative \( a_{i_2} \) with respect to criterion \( g_j \).

The DM also needs to assess the relative importance of the criteria, and therefore another pairwise comparison matrix will be required for elicitation, say:

\[
M^G = [c^G_{j_1, j_2}]
\]
with $c^G_{j_1,j_2}$ being the pairwise comparison judgement of criterion $g_{j_1}$ with criterion $g_{j_2}$.

### 3.1.1. Spanning trees

Considering a generic pairwise comparison matrix $M = [c_{rs}]$, a spanning tree $\tau_k$ consists of $(n - 1)$ mutually independent judgements out of the total judgements, that is

$$\tau_k = \{c^k_{r_1,s_1}, \ldots, c^k_{r_{n-1},s_{n-1}}\}$$

with $c^k_{r_1,s_1}, \ldots, c^k_{r_{n-1},s_{n-1}}$ independent judgements. Each of these spanning trees can be used to calculate a preference vector, and hence, we will interchangeably use the term "spanning tree" and "spanning tree vector".

For the matrices $M^j$ assessing alternatives with respect to considered criteria $g_j, j = 1, \ldots, m$, the set of spanning trees is denoted as:

$$T^j = \{\tau^j_k\},$$

For the matrix $M^G$ assessing the importance of criteria, the set of spanning trees is denoted as:

$$T^G = \{\tau^G_k\},$$

This gives us a total of $m + 1$ sets of spanning trees (spanning tree vectors), and therefore, $m + 1$ sets of preference vectors.

As shown in Fig. 4, we can pick one tree from each of these sets and construct a decision table that is traditionally used to calculate overall preferences. These combinations of spanning trees can be formulated as

$$(\tau^1_{k_1}, \ldots, \tau^m_{k_m}, \tau^G_k) \in T^1 \times \ldots T^m \times T^G,$$

### 3.1.2. Evaluating the combinations of spanning trees

The evaluation vector of alternatives $a_i \in A$ corresponding to the spanning tree $\tau^j_{k_j} \in T^j$ can thus be represented as:

$$(u^j_k(a_1), \ldots, u^j_k(a_n))$$
while the weight vector corresponding to the spanning tree $\tau_k^G \in T^G$ is denoted as:

$$(w_1^k, \ldots, w_m^k)$$

Now the overall priority for alternatives $a_i \in A$ given by the combination of spanning trees $\tau = (\tau_1^1, \ldots, \tau_m^m, \tau^G) \in T^1 \times \ldots T^m \times T^G$ will be:

$$u_\tau(a_i) = w_1^{k_1} u_1^{k_1}(a_i) + \ldots + w_m^{k_m} u_m^{k_m}(a_i)$$

From this formulation, we can deduce the set of combinations of spanning trees for which any $a_{i_1}, a_{i_2} \in A$

- $a_{i_1}$ is preferred to $a_{i_2}$:

$$B(a_{i_1} > a_{i_2}) = \{\tau \in T^1 \times \ldots T^m \times T^G : u_\tau(a_{i_1}) > u_\tau(a_{i_2})\}$$
\( a_{i1} \) is indifferent with \( a_{i2} \):

\[
B(a_{i1} \sim a_{i2}) = \{ \tau \in \mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G : u_\tau(a_{i1}) = u_\tau(a_{i2}) \}
\]

The set of combinations of spanning trees \( \tau \in \mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G \) for which alternative \( a_i \in A \) attains the \( p \)-th position with respect to overall priority \( u_\tau(a_i) \) can be formulated as:

\[
R(a_i, p) = \{ \tau \in \mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G : \text{rank}(a_i, \tau) = p \}
\]

with

\[
\text{rank}(a_i, \tau) = 1 + \sum_{i' \neq i} \rho(u_\tau(a_{i'}) > u_\tau(a_i))
\]

where \( \rho(\text{false}) = 0 \) and \( \rho(\text{true}) = 1 \).

Inspired by the Stochastic Multicriteria Acceptability Approach or SMMA [Lahdelma et al. 1998, Lahdelma and Salminen 2001], the probability that alternative \( a_{i1} \) is preferred to alternative \( a_{i2} \), with \( a_{i1}, a_{i2} \in A \) can be represented as:

\[
P(a_{i1} > a_{i2}) = \frac{\text{card}(B(a_{i1} > a_{i2}))}{\text{card}(\mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G)}
\]

that, remembering that the total number of spanning trees in a graph of \( k \) nodes is \( k^{k-2} \) [Cayley 1889], and, consequently \( \text{card}(\mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G) = m^{m-2}n^{m(n-2)} \), becomes

\[
P(a_{i1} > a_{i2}) = \frac{\text{card}(B(a_{i1} > a_{i2}))}{m^{m-2}n^{m(n-2)}}
\]

the probability that alternative \( a_i \in A \) is attaining the \( p \)-th ranking position will be:

\[
P(\text{rank}(a_i, \tau) = p) = \frac{R(a_i, p)}{\text{card}(\mathcal{T}^1 \times \ldots \mathcal{T}^m \times \mathcal{T}^G)} = \frac{R(a_i, p)}{m^{m-2}n^{m(n-2)}}
\]

### 3.2. Spanning trees for incomplete pairwise comparison judgements

Harker [1987] investigated incomplete sets of judgements where the DMs are allowed to respond with "do not know" or "not sure" to some judgements. This is an important issue to investigate as the probability of acquiring an incomplete set of PC judgements increases with an increase in the total number of items for comparison [Fedrizzi and Giove 2007]. Both the REV and the RGM methods are inappropriate in such
cases due to the fact that the PC matrix cannot be constructed without estimating/imputing the missing judgements.

The spanning trees can be applied also to partial comparison matrix with the help of the Kirchoff formula (See details in (Siraj et al., 2012)). This implies that generating combinations of spanning trees is not restricted to only complete sets of judgements, and therefore, generating preference frequencies and rank-order frequencies are possible from incomplete pairwise comparisons, without any modification. The implication of having an incomplete set of judgements is that the total number of combinations will be smaller than the number possible from a complete sets of judgements. This permits the DM to only express values for those PC judgements for which he/she is sufficiently confident, avoiding forcing the DM to provide judgements when he/she is not sufficiently confident - in so doing maintaining the reliability of the preference information considered in the decision support process.

3.3. Didactic example

Let us show how our approach works in practice by using the classic example of school selection proposed in [Saaty (1980)]. The parents have to decide the high school for their son. They consider six criteria being the following

1. \( g_1 \): Learning,
2. \( g_2 \): Friends,
3. \( g_3 \): School life,
4. \( g_4 \): Vocational training,
5. \( g_5 \): College preparation,
6. \( g_6 \): Music classes.

There are three alternatives corresponding to three schools denoted \( A \), \( B \) and \( C \). For each criterion, the parents compared in pairs the schools as shown in the following pairwise comparison matrices.

\[
M_{g_1} = \begin{pmatrix}
3 & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{3} & 1 & 3 \\
\frac{1}{2} & \frac{1}{3} & 1
\end{pmatrix}
\] (1)

\[
M_{g_2} = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\] (2)
\[
M_{g_3} = \begin{pmatrix}
1 & 5 & 1 \\
\frac{1}{5} & 1 & \frac{1}{5} \\
1 & 5 & 1
\end{pmatrix} 
\]

(3)

\[
M_{g_4} = \begin{pmatrix}
1 & 9 & 7 \\
\frac{1}{5} & 1 & \frac{1}{5} \\
\frac{1}{5} & 5 & 1
\end{pmatrix} 
\]

(4)

\[
M_{g_5} = \begin{pmatrix}
1 & \frac{1}{2} & 1 \\
2 & 1 & 2 \\
1 & \frac{1}{2} & 1
\end{pmatrix} 
\]

(5)

\[
M_{g_6} = \begin{pmatrix}
1 & 6 & 4 \\
\frac{1}{5} & 1 & \frac{1}{5} \\
\frac{1}{3} & 3 & 1
\end{pmatrix} 
\]

(6)

After comparing the alternatives with respect to these criteria, the parents also compared the criteria in terms of their importance as shown below.

\[
M_{\text{criteria}} = \begin{pmatrix}
1 & 4 & 3 & 1 & 3 & 4 \\
\frac{1}{4} & 1 & 7 & 3 & \frac{1}{5} & 1 \\
\frac{1}{3} & \frac{1}{7} & 1 & \frac{1}{9} & \frac{1}{5} & \frac{1}{6} \\
1 & \frac{1}{3} & 5 & 1 & 1 & \frac{1}{3} \\
\frac{1}{3} & 5 & 5 & 1 & 1 & 3 \\
\frac{1}{4} & 1 & 6 & 3 & \frac{1}{3} & 1
\end{pmatrix} 
\]

(7)

The consistency ratios for the above pairwise comparison matrices are as follows:

- CR\((M_{g_1}) = 0.04\),
- CR\((M_{g_2}) = 0\),
- CR\((M_{g_3}) = 0\),
- CR\((M_{g_4}) = 0.18\),
- CR\((M_{g_5}) = 0\),
• $\text{CR}(M_{g4}) = 0.04$,

• $\text{CR}(M_{\text{criteria}}) = 0.24$

As we can see, there are four matrices which are not completely consistent and two of these, namely $M_{g4}$ and $M_{\text{criteria}}$ have unacceptable levels of inconsistency as measured by the usual threshold of 0.1 for the CR value [Saaty (1980)].

The priorities $u_j(X)$, $X = A, B, C, j = 1, \ldots, 6$, obtained from pairwise comparisons matrices $M_{g1} - M_{g6}$ representing the evaluations of schools with respect to the considered criteria are shown in Table 1 while the weights of the considered criteria, according to the priority obtained from pairwise comparison’s matrix $M_{\text{criteria}}$, are the following:

$$w_1 = 0.32, w_2 = 0.14, w_3 = 0.03, w_4 = 0.13, w_5 = 0.24, w_6 = 0.14.$$ 

| Alternatives | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|--------------|-------|-------|-------|-------|-------|-------|
| School A     | 0.16  | 0.33  | 0.45  | 0.77  | 0.25  | 0.69  |
| School B     | 0.59  | 0.33  | 0.09  | 0.05  | 0.50  | 0.09  |
| School C     | 0.25  | 0.33  | 0.46  | 0.17  | 0.25  | 0.22  |

Using priorities $w_j$ and $u_j(X)$, $X = A, B, C, j = 1, \ldots, 6$, we can compute the overall priority $u(X)$ of each school, with

$$u(X) = \sum_{j=1}^{6} w_j u_j(X)$$

obtaining the following results:

$$u(A) = 0.37, u(B) = 0.38, u(C) = 0.25.$$ 

Let us now handle the same problem with the spanning tree approach. With this aim, we have to consider all the combinations of spanning trees from the pairwise comparison matrix of criteria $M_{\text{criteria}}$ and from the pairwise comparison matrices of alternatives with respect to the considered criteria $M_{gj}, j = 1, \ldots, m$. Remembering that with $m$ criteria and $n$ alternatives, we have $n^{(n-2)m(n-m-2)n}$ combinations of spanning trees, for the decision problem at hand, we have $6^4 \cdot 3^6 = 944784$ combinations of spanning trees. We compute weights of criteria

$$u_1^k, \ldots, u_6^k$$

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and evaluations of alternatives with respect to considered criteria

\[ w_j^k(A), w_j^k(B), w_j^k(C), j = 1, \ldots, 6, \]

for each combination of spanning trees \( \tau = (\tau_{k_1}, \ldots, \tau_{k_6}, \tau^G_G) \in \mathcal{T}^1 \times \ldots \mathcal{T}^6 \times \mathcal{T}^G \).

Comparing alternatives
Comparing criteria

The combination shown above has favoured School B over other two schools while the one shown below is favouring School A.

| Rank | 1st | 2nd | 3rd |
|------|-----|-----|-----|
| Score | .304 | .433 | .263 |

The judgments included in the combination are highlighted using rounded rectangles.

| SCHOOL | A | B | C |
|--------|---|---|---|
| Rank   | 1st | 2nd | 3rd |
| Score  | .451 | .333 | .216 |

Figure 5: Demonstrating the two different combinations of spanning trees, and the difference in their rankings and scores.

Considering all the combinations of spanning trees from \( \mathcal{T}^1 \times \ldots \mathcal{T}^6 \times \mathcal{T}^G \) we can compute

- Table 2 that shows the probability that each alternative attains a given rank (within parenthesis there is the total number of combinations of spanning trees for which there is the preference of the alternative on the row over the alternative on column),

- Table 3 that for each pair of alternatives shows the probability that the alternative in the row is preferred to the alternative in the column (within parenthesis the total number of combinations of spanning trees for which the alternatives attain the considered ranking position).
Table 2: Preference frequency over 944784 combinations of spanning trees

|          | School A | School B | School C |
|----------|----------|----------|----------|
| School A | 0.51 (483246) | 0.91 (855063) | X |
| School B | 0.49 (461538) | X        | 0.89 (842130) |
| School C | 0.09 (89721)  | 0.11 (102654) | X |

Table 3: Rank order frequency over 944784 combinations of spanning trees

|          | School A | School B | School C |
|----------|----------|----------|----------|
| 1st      | 0.51 (483084) | 0.49 (461268) | 0.00 (432) |
| 2nd      | 0.39 (372141) | 0.40 (381132) | 0.11 (191511) |
| 3rd      | 0.09 (89559)  | 0.11 (102384) | 0.80 (752841) |

4. Random spanning trees

We have seen that, for a problem with $n$ alternatives evaluated across $m$ criteria, the total number of combinations of spanning trees increases exponentially with the problem size - making it impractical to calculate the exact probabilities of $P(rank(a_i, \tau) = p)$ and of $P(a_{i1} \succ a_{i2})$ for large problems.

We therefore propose using an approach which, combined with statistical sampling theory, allows us to provide estimates of the required probabilities of $P(rank(a_i, \tau) = p)$ and of $P(a_{i1} \succ a_{i2})$ to within any user defined degree of accuracy according to any user defined level of confidence.

Of course we cannot physically select a statistical random sample from the population of spanning tree combinations since this would require the generation of the population itself and this is exactly what we aim to avoid. We can however use the ‘random walk’ procedure to generate a tree and hence a sample of trees (see Aldous 1990; Broder 1989). Indeed, the ‘random walk’ procedure has been proven to generate a true statistical random sample in the sense that the generated sample is equivalent to selection of a statistical random sample from the population, that is, where each tree has the same uniform probability of being selected. See (Broder 1989; Aldous 1990) for more details.

As we discussed the possibility of having incomplete sets of pairwise comparison judgements will arise in practice and so it is important to note that the concept of random walks is equally applicable to these incomplete sets without modification.

Since each iteration of the procedure generates a new member of the random sample, the total number of iterations used in the procedure is equivalent to sample size and we can use statistical large sample theory to determine the number of iterations required to generate sample parameters (including the probabilities of $P(rank(a_i, \tau) = p)$ and of $P(a_{i1} \succ a_{i2})$) to any specified degree of accuracy and with any specified level of confidence.
Pairwise comparisons by DM
Assessment for Price
Assessment for Speed
Assessment for Looks
Prioritising the criteria

Generate a decision table from these random-walk trees, and then keep generating tables iteratively.

By generating high number of decision tables using this random-walk approach, we will be able to capture different mind sets.

That is, if we want the estimated probability to have an accuracy within \( \lambda \) and with a \( C \) percent level of confidence, then the required number of iterations would be (Tervonen and Lahdelma, 2007):

\[
It(\lambda, C) = \frac{Z_C^2}{4\lambda^2}
\]

where \( Z_C \) is the z-score calculated from the standardised normal distribution curve.

For example, if we want to achieve an accuracy within 0.01 and with a 99% level of confidence, then the required number of iterations would be

\[
It(0.01, 99\%) = \frac{Z_{99}^2}{4 \times 0.01^2} = \frac{(2.58)^2}{4 \times 0.01^2} = \frac{6.6564}{0.0004} = 16641
\]

As we know that the Z value for 99% confidence interval is equal to 2.58, therefore

\[
It(0.01, 99\%) = \frac{(2.58)^2}{4 \times 0.01^2} = 16,641
\]
This suggests that we need to perform more than 16,641 iterations if we want to achieve an accuracy within ±0.01 with 99% confidence.

4.1. Applying random walk procedure to the school example

In the example below, we illustrate the use of random walks by repeating an experiment multiple times where each experiment itself uses the required number of iterations $It$ determined by the formula above. We do this on the previously discussed School example as this is a tractable problem where the whole population of solutions is available. Later, we will also extend this idea to a bigger problem where generating the whole population is impractical (see Section 5).

Table 4 shows the preference frequencies obtained through random walks, as well as the preference frequencies obtained by taking a sample from the whole population of solutions. Recall that the whole population was generated by enumerating all possible combinations of spanning trees that was shown in Table 2 earlier. Here we can see that the estimated values from random walks and from the sample are both lying within 1% of the population values that were previously shown in Table 2.

| School     | From random walks | Sampling from population |
|------------|-------------------|--------------------------|
| School A   | School B | School C   | School A   | School B | School C   |
| School A   | -       | 51.3% 90.5% | School A   | -       | 50.2% 90.1% |
| School B   | 48.7%   | -       | 89.2%   | School B | 49.8% - 88.6% |
| School C   | 9.5%    | 10.8% | -       | School C | 9.6% 11.4% |

Table 4: Preference frequencies of the school example

Table 5: Rank order profiles generated for the school example

| School     | From random walks | Sampling from population|
|------------|-------------------|--------------------------|
| School A   | School B | School C | School A | School B | School C |
| 1st        | 51.2% 48.7% 0.0% | 1st | 50.2% 49.8% 0.0% |
| 2nd        | 39.3% 40.5% 20.2% | 2nd | 39.9% 38.8% 21.3% |
| 3rd        | 9.5% 10.8% 79.7% | 3rd | 9.9% 11.4% 78.7% |

We performed the same analysis for rank-order frequencies as well, as summarised in Table 5. The table on the left side shows that the rank-order frequencies obtained through random walks are also lying within 1% of the population values shown in Table 3. The rank-order frequencies obtained by taking a sample are shown on the right side of this table. This also illustrates that the accuracy and confidence level results for the generated random sample echo the same results had we been able to physically select a statistical random sample from the population of spanning tree combinations.
5. Telecom backbone example

In order to demonstrate the use of spanning trees to explore multiple mindsets, we consider the practical data acquired in a recent study: the selection of a backbone infrastructure for telecommunication in rural areas (Gasiea, 2010). This study focused on the use of a structured decision making approach towards selecting the telecommunication infrastructure for the rural areas of developing countries. The lack of adequate telecommunications infrastructure in these parts of the world remains a major obstacle for providing affordable services.

The study considered four options of Fiber-optic cable (G1), Power-line communication (G2), Microwave link (G3) and Satellite communication (G4). The authors of this study used AHP and Analytic Network Process (ANP) to structure the problem and acquired the necessary data on preferences and assessments from key stakeholders. The criteria used to compare these alternatives were grouped into six major categories including technical, infrastructural, economic, social, regulatory and environmental factors. These categories and their constituent criteria are presented in Fig. 7.

![Figure 7: Criteria to compare the available backbone infrastructures](image)

The PC matrix, $A_{top}$, acquired for prioritising these six categories (top-level criteria) is shown in Fig. 8. Although $A_{top}$ is a transitive PC matrix, the estimated vectors produced by the widely used EV method does not preserve the original order of preferences in these judgements.

The final weights calculated using the EV and GM methods are found to be almost identical, as given in Table 6 in normalised form. Satellite communication (G4) is considered the most preferred alternative with
a weight of 29.95% (using EV), followed by Microwave (G3) with a weight around 28.34% (using EV).

|       | Fiber   | Powerline | Microwave | Satellite |
|-------|---------|-----------|-----------|-----------|
| Weight| \(w_{G1}\) | \(w_{G2}\) | \(w_{G3}\) | \(w_{G4}\) |
| EV    | 21.7%   | 20.1%     | 28.3%     | 29.9%     |
| GM    | 21.7%   | 20.1%     | 28.4%     | 29.8%     |

Table 6: Estimated weights for the available backbone infrastructure options

Most criteria lie under the *Technical* and *Infrastructure* categories. The *Technical* category includes nine criteria whilst the *Infrastructure* category has eight criteria used to compare the alternatives. The two PC matrices for the *Technical* and *Infrastructure* categories, \(A_{tech}\) and \(A_{infra}\), have been found to be intransitive and should be investigated along with \(A_{top}\) for their impact on the final result.

5.1. Spanning trees analysis using Random walks

In this case study, there are four alternatives and six top-level criteria, therefore, the number of “spanning-trees” solutions for criteria matrix is \(n^{(n-2)} = 6^4 = 1296\), and the number of “spanning-trees” solutions for each alternative matrix is \(n^{(n-2)} = 4^2 = 16\). As we pick one “spanning-trees” solution from each matrix,
there are $m^{(m-2)}n^{(n-2)} = 1296 \times 16^6 = 21,743,271,936$ combinations possible. Generating these billions of combinations is computationally expensive, so sampling is the logical way to estimate probabilities. Therefore we generated a sample of 20,000 solutions with the use of random walks. We generated 20 such samples in order to check whether all these samples produce similar results. In practise, only one sample should suffice but we took this approach for investigation only.

The average scores for the four alternatives are given in Table 7. Each row in this table represents one iteration of generating a sample of 20,000 solutions.

| Iteration | Fiber | Powerline | Microwave | Satellite |
|-----------|-------|-----------|-----------|-----------|
| 1         | 0.218295 | 0.193467 | 0.292819 | 0.29542   |
| 2         | 0.218428 | 0.193078 | 0.292313 | 0.29618   |
| 3         | 0.218372 | 0.192959 | 0.293072 | 0.295597  |
| 4         | 0.21819  | 0.193914 | 0.292752 | 0.295143  |
| 5         | 0.21898  | 0.19307  | 0.291989 | 0.295961  |
| 6         | 0.219305 | 0.193273 | 0.292496 | 0.294927  |
| 7         | 0.218776 | 0.192261 | 0.29268  | 0.296283  |
| 8         | 0.218472 | 0.194001 | 0.292665 | 0.294861  |
| 9         | 0.218751 | 0.193282 | 0.2931   | 0.294867  |
| 10        | 0.218287 | 0.193674 | 0.29279  | 0.295249  |
| 11        | 0.21883  | 0.193428 | 0.293161 | 0.294581  |
| 12        | 0.218442 | 0.193236 | 0.292886 | 0.295436  |
| 13        | 0.218468 | 0.193785 | 0.292581 | 0.295165  |
| 14        | 0.219373 | 0.192499 | 0.292469 | 0.295659  |
| 15        | 0.219038 | 0.193186 | 0.292143 | 0.295633  |
| 16        | 0.218294 | 0.194367 | 0.29286  | 0.29448   |
| 17        | 0.218691 | 0.193434 | 0.292697 | 0.295179  |
| 18        | 0.218749 | 0.193219 | 0.29286  | 0.295171  |
| 19        | 0.218288 | 0.194101 | 0.292734 | 0.294876  |
| 20        | 0.218517 | 0.192994 | 0.292892 | 0.295597  |

Table 7: Scores calculated using random walks; each iteration shows an average of 20k solutions.

Unlike the School example where we calculated frequencies by generating all the combinations of spanning trees, here we will calculate the frequencies from the samples generated by random walks. It can be argued that these frequencies may vary as they are based on randomly generated set of solutions. However, the aim here is to gain statistical insights, and therefore, these findings should remain useful as long as they produce reliable statistical findings. We investigated this reliability by repeating the same experiment twenty times and comparing the preference frequencies and rank-order frequencies. The standard deviations of these values do not exceed 0.005, which supports the argument that these stochastic results are quite reliable.

Table 8 shows the probability that one alternative is preferred to the other (for all possible pairs). Looking at the first row, we can say that Fiber is preferred over Powerline in 63% (see $0.63 \pm 0.004$ under
the Powerline column) of the random walk solutions, however, it is seldom preferred over Microwave and Satellite. Microwave is clearly preferred over Fiber (with a probability of 87%) and Powerline (with a probability of 96%), however, it has been marginally preferred over Satellite (with a probability of 51%). We can argue that the Fiber and Powerline are the two clearly dominated alternatives, while the Microwave and Satellite are the two dominating alternative. However, when comparing Microwave and Satellite with each other, Microwave is slightly more preferred but there is no clear winner. We are providing the standard deviations for each of these values to show that these scores didn’t deviate much, as discussed earlier.

|          | Fiber   | Powerline | Microwave | Satellite |
|----------|---------|-----------|-----------|-----------|
| Fiber:   | –       | 0.63 ± 0.004 | 0.13 ± 0.002 | 0.09 ± 0.003 |
| Powerline: | 0.37 ± 0.004 | – | 0.04 ± 0.002 | 0.09 ± 0.002 |
| Microwave: | **0.87 ± 0.002** | **0.96 ± 0.002** | – | 0.51 ± 0.004 |
| Satellite: | **0.91 ± 0.003** | **0.91 ± 0.002** | 0.49 ± 0.004 | – |

Table 8: Preference frequencies for Telecom backbone selection

|          | Fiber   | Powerline | Microwave | Satellite |
|----------|---------|-----------|-----------|-----------|
| 1st:     | 4.9% ± 0.2% | 2.5% ± 0.2% | **47.3% ± 0.3%** | **45.3% ± 0.4%** |
| 2nd:     | 11.3% ± 0.3% | 6.8% ± 0.2% | **40.2% ± 0.3%** | **41.7% ± 0.4%** |
| 3rd:     | **47.4% ± 0.4%** | 28.5% ± 0.4% | 11.9% ± 0.2% | 12.3% ± 0.2% |
| 4th:     | 36.5% ± 0.4% | **62.2% ± 0.5%** | 0.6% ± 0.1% | 0.7% ± 0.1% |

Table 9: Rank-order frequencies for Telecom backbone selection

Another interesting way to assess these options is to estimate the ranking of each alternative. Table 9 shows the probability that each alternative attains a given rank. In this table, The probability of Fiber and Powerline taking the first rank is too low (only 4.9% and 2.5% respectively). However, Microwave and Satellite have high probabilities of taking the first rank. On the other end, although Fiber is more likely to attain the third rank, both Fiber and Powerline have high probabilities to attain the lowest rank (36.5% and 62.2% respectively). The two alternatives of Microwave and Satellite are least likely to be placed on the lowest rank, as the percentage of combinations placing them at the 4th position were only 0.6% and 0.7%, respectively.

6. Conclusions

The spanning trees analysis can help understand the plurality of mindsets in terms of a plurality of prioritisation vectors originating from a plurality of spanning trees of pairwise comparison matrices. Considering all the rankings of alternatives corresponding to the different mindsets, we propose to estimate the
probability that an alternative attains a given ranking position, and the probability that an alternative is preferred to another one. Moreover, the proposed approach can be applied to incomplete sets of judgements without any modification.

Since the exponential number of spanning trees makes their enumeration prohibitive, we propose computing approximate probabilities using statistical sampling of spanning trees. The usefulness of statistical sampling is further demonstrated with the case study of Telecom infrastructure selection for rural areas.

An interesting area of future research is to do performance analysis using the sigma-mu approach proposed by Greco et al. (2019). For each alternative, we compute the mean, $\mu$, and the standard deviation, $\sigma$, of all the solutions generated from combinations of spanning trees. The objective would be to maximise the mean while minimising the standard deviation.

The use of interval judgements is also an important area to investigate as we know that decision makers often provide their judgements in intervals (for example, stating that $X$ is about 3 to 5 times better than $Y$). Therefore, the idea of interval judgements need to be combined with the use of spanning trees combinations. Considering the lower and upper limit of intervals, the number of combinations will further increase exponentially, therefore the use of a stochastic approach might be the only practical approach for this purpose.

There is a further potential in the use of spanning trees analysis at group-level decision making where multiple judgements are collected for comparing the same pair of alternatives or weighing the same pair of criteria. The concept of "aggregated individual preferences" and "aggregated individual judgements" has been widely debated in the context of group decision making, and we believe that the use of spanning tree analysis can offer better insights into these types of problems.

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