Remarks on a recent result by Paul Skoufranis

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Abstract

We give a different proof of Paul Skoufranis’s very recent result showing that the strong convergence of possibly non-commutative random variables $X^{(k)} \to X$ is stable under reduced free product with a fixed non-commutative random variable $Y$. In fact we obtain a more general fact: assuming that the families $X^{(k)} = \{X^{(k)}_i\}$ and $Y^{(k)} = \{Y^{(k)}_j\}$ are $\ast$-free as well as their limits (in moments) $X = \{X_i\}$ and $Y = \{Y_j\}$, the strong convergences $X^{(k)} \to X$ and $Y^{(k)} \to Y$ imply that of $\{X^{(k)}, Y^{(k)}\}$ to $\{X, Y\}$. Phrased in more striking language: the reduced free product is “continuous” with respect to strong convergence. The analogue for weak convergence (i.e. convergence of all moments) is obvious.

By a $C^\ast$-probability space, we mean a unital $C^\ast$-algebra equipped with a state. Let $(A^{(k)}, \phi^{(k)})$ and $(A, \phi)$ be $C^\ast$-probability spaces. Let $\{X^{(k)}_m \mid m \in M\} \subset A^{(k)}$, $\{X_m \mid m \in M\} \subset A$ be families of non-commutative random variables. We say that $\{X^{(k)}_m \mid m \in M\}$ tends strongly to $\{X_m \mid m \in M\}$ if for any polynomial $P$ in the non-commutative variables $\{x_m \mid m \in M\}$ we have $\phi^{(k)}(P(X^{(k)}_m)) \to \phi(P(X_m))$ (this is the convergence of moments) and moreover $\|P(X^{(k)}_m)\| \to \|P(X_m)\|$. The notion of strong convergence that was introduced in [5], was inspired by Haagerup and Thorbjørnsen’s paper [3]. More recent examples of strong convergence are obtained in [2]. The results below are motivated by work by Camille Male [5], who first considered the question of the stability of strong convergence, and by D. Shlyakhtenko’s proof (see the appendix of [3]) that the reduced free product with the $C^\ast$-algebra generated by free creators on the Fock space satisfies the desired stability property. Very recently, this was generalized by P. Skoufranis [7] to essentially all reduced free products. In this note we give a different more direct proof based on an inequality due to É. Ricard and Q. Xu that is a generalization to arbitrary reduced free products of results proved previously by Voiculescu, Haagerup and Buchholz (see [6]) for free products of groups. Our proof yields actually a stronger stability than the one appearing in [7], as described in the abstract, but P. Skoufranis informed us that the original proof of [7] also yields that improvement.

1. A rough outline

The main point to prove the result stated in the abstract is this: if we are dealing with a polynomial $P$ that is a polynomial in $X$’s and $Y$’s that are $\ast$-free and we want to compute its norm, we observe that if $P$ is of (joint) degree at most $d$ then $Q = (P^*P)^m$ will be of degree at most $2md$.

The Ricard-Xu non-commutative Khintchine inequality ([6]) gives

$$(8md)^{-1}kh(Q) \leq \|Q\| \leq (4md + 1)^2 \kh(Q)$$

where $\kh(Q)$ is a certain expression (actually a norm depending on $m$) that we will need to analyse below.
Fix $\varepsilon > 0$. The last inequality gives us that if $m = m(d, \varepsilon)$ is fixed but chosen large enough so that $(\max\{8md, (4md + 1)^2\})^{1/2m} < 1 + \varepsilon$ then we have

$$(1 + \varepsilon)^{-1}[kh((P^*P)^m)]^{1/2m} \leq \|P\| \leq (1 + \varepsilon)[kh((P^*P)^m)]^{1/2m}. $$

Thus to show the strong convergence of $X^{(k)}, Y^{(k)}$ to $X, Y$ it suffices to show that for $m$ fixed and $Q = (P^*P)^m$ we have

$$[kh(Q(X^{(k)}, Y^{(k)}))]^{1/2m} \to [kh(Q(X, Y))]^{1/2m},$$

or merely

$$\text{(1.1) } kh(Q(X^{(k)}, Y^{(k)})) \to kh(Q(X, Y)).$$

But now a closer look at $kh(Q(X, Y))$ in (1.1) will show that this holds.

2. Background on ultraproducts

It will be convenient to use ultraproducts, but we only need very basic and elementary facts that are recalled below.

2.1. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$. Given a sequence $(X^{(k)})$ of Banach spaces, we will usually denote by $X^\mathcal{U}$ their ultraproduct (see e.g. [4] for more background information). The elements $x \in X^\mathcal{U}$ are equivalence classes of bounded sequences $(x^{(k)})$ with $x^{(k)} \in X^{(k)}$ for all $k$. By definition, two such sequences $(x^{(k)})$, $(y^{(k)})$ are equivalent if $\lim_{\mathcal{U}} \|x^{(k)} - y^{(k)}\| = 0$. We will sometimes write $x = [x^{(k)}]_{\mathcal{U}}$ to denote that $(x^{(k)})$ is a representative of $x$. Whenever this holds we have $\|x\| = \lim_{\mathcal{U}} \|x^{(k)}\|$.

2.2. As is well known, when all the spaces in $(X^{(k)})$ are Hilbert spaces $X^\mathcal{U}$ is also a Hilbert space. It may be worthwhile to remind the reader that if $(Y^{(k)})$ is another family of Hilbert spaces, we have a canonical isometric embedding of $X^\mathcal{U} \otimes_2 Y^\mathcal{U}$ into the ultraproduct of the family $(X^{(k)} \otimes_2 Y^{(k)})$. Of course this extends to an arbitrary finite number of factors. Moreover, if $\sup_h \dim(X^{(k)}) < \infty$, then this embedding is an isomorphism.

2.3. Let $(X^{(k)}), (Y^{(k)})$ be sequences of Banach spaces (resp. unital $C^*$-algebras). Let $T^{(k)} : X^{(k)} \to Y^{(k)}$ be a bounded sequence of linear mappings (resp. unital $*$-homomorphisms) then the mapping $T^\mathcal{U} : X^\mathcal{U} \to Y^\mathcal{U}$ defined whenever $x = [x^{(k)}]_{\mathcal{U}}$ by $T^\mathcal{U}(x) = [T^{(k)}(x^{(k)})]_{\mathcal{U}}$ is bounded (resp. a unital $*$-homomorphism) with $\|T^\mathcal{U}\| = \lim_{\mathcal{U}} \|T^{(k)}\|$.

Given two Banach spaces $X, Y$, we will view a finite sum $t = \sum_\alpha x^{(k)}(\alpha) \otimes y^{(k)}(\alpha) \in X \otimes Y$ as an operator from $Y^\ast$ to $X$, and we denote by $\|t\|$ the associated operator norm.

Consider now a sequence $t^{(k)} \in X^{(k)} \otimes Y^{(k)}$ in the algebraic tensor product with uniformly bounded rank $R$. Let us assume more precisely that there are, for each $1 \leq \alpha \leq R$, bounded sequences $(x^{(k)}(\alpha))$ and $(y^{(k)}(\alpha))$ such that $t^{(k)} = \sum_1^R x^{(k)}(\alpha) \otimes y^{(k)}(\alpha) \in X^{(k)} \otimes Y^{(k)}$. Let $x^{(k)}(\alpha) = [x^{(k)}(\alpha)]_{\mathcal{U}}$ and $y^{(k)}(\alpha) = [y^{(k)}(\alpha)]_{\mathcal{U}}$, and let $t^{\mathcal{U}} = \sum_1^R x^{\mathcal{U}}(\alpha) \otimes y^{\mathcal{U}}(\alpha) \in X^\mathcal{U} \otimes Y^\mathcal{U}$. Then

$$\|t^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|t^{(k)}\|.$$

Incidentally, this says that the injective tensor product $X^\mathcal{U} \otimes Y^\mathcal{U}$ embeds canonically isometrically into the ultraproduct of the sequence $(X^{(k)} \otimes Y^{(k)})$. 


2.4. Let $S^{(k)} \in B(H^{(k)})$ be a bounded sequence. Let $S^U \in B(H^U)$ be the associated operator. Then it is easy to see that 
\[ \|S^U\| = \lim_{U} \|S^{(k)}\|. \]
More generally, let $n$ be a fixed integer and $H$ a Hilbert space. We denote by $M_n(B(H))$ the space of $n \times n$ matrices with entries in $B(H)$ with the usual norm. Let $S^{(k)} \in M_n(B(H^{(k)}))$ be a bounded sequence. Then clearly \[
\|(S^{(k)})_{ij}\|_{M_n(B(H^U))} = \lim_{U} \|S^{(k)}\|_{M_n(B(H^{(k)}))}.
\]

2.5. Let $C$ be a unital $*$-algebra. By a state on $C$ we mean a linear functional such that $\phi(1) = 1$ and $\phi(x^*x) \geq 0$ for all $x \in C$. Given this, the classical GNS construction produces a Hilbert space denoted by $L_2(\phi)$ and a $*$-homomorphism $\pi : C \to B(L_2(\phi))$ equipped with a distinguished cyclic unit vector $\xi = \xi_\phi \in L_2(\phi)$, such that $\phi(c) = \langle c\xi, \xi \rangle$ for any $c \in C$. Let $A = \pi(C) \subset B(L_2(\phi))$. Then $A$ is a unital $C^*$-algebra. Let $\phi(a) = \langle a\xi, \xi \rangle$ for any $a \in A$. Then $\phi$ is a state on $A$, $L_2(\phi) \simeq L_2(\phi)$ and the representation $A \subset B(L_2(\phi))$ can be identified with the result of the GNS construction applied to $(A, \phi)$.

2.6. Given a sequence of states $\phi^{(k)}$ on a unital $*$-algebra $B$, let $(\pi^{(k)}, H^{(k)}, \xi^{(k)})$ be the associated GNS construction and let $A^{(k)} = \pi^{(k)}(B) \subset B(H^{(k)})$ be the associated $C^*$-algebra. Let $\pi^U : B \to B(H^U)$ be the representation defined for any $z = [z^{(k)}]_U \in H^U$ and $b \in B$ by 
\[
\pi^U(b)([z^{(k)}]_U) = [\pi^{(k)}(b)(z^{(k)})]_U.
\]
Obviously, \[
\|\pi^U(b)\| = \lim_{U} \|\pi^{(k)}(b)\|.
\]
Let $\phi = \lim_U \phi^{(k)}$ relative to pointwise convergence on $B$, let $\pi : B \to B(L_2(\phi))$ be the associated GNS representation and let $\xi = \xi_\phi$. Let also $\xi^U = [\xi^{(k)}]_U$. Then 
\[
\|\pi^U(b)\xi^U\|^2 = \lim_{U} \|\pi^{(k)}(b)\xi^{(k)}\|^2 = \lim_{U} \|\phi^{(k)}(b^*b)\| = \phi(b^*b) = \|\pi(b)\xi\|^2.
\]
Therefore, the correspondence $\pi(b)\xi \mapsto \pi^U(b)\xi^U$ extends to an isometric isomorphism from $L_2(\phi)$ onto the subspace $K^U \subset H^U$ that is the closure of $\{\pi^U(b)\xi \mid b \in B\}$. More precisely, the restriction of $\pi^U$ to $K^U$, i.e. $b \mapsto \pi^U(b)|_{K^U} \in B(K^U)$, is unitarily equivalent to the representation $\pi$.

Now let us assume moreover that $\phi = \lim_U \phi^{(k)}$ strongly. This means (see below) that $\|\pi(b)\| = \lim_{U} \|\pi^{(k)}(b)\| = \|\pi^U(b)\|$ for any $b \in B$. Then the mapping $\pi(b) \mapsto \pi^U(b)$ defines an isometric embedding \[
\psi : A = \overline{\pi(B)} \to B(H^U).
\]

3. Main result

We now turn to a more formal description of our main result. A more abstract (but equivalent) version of the statement in the abstract can be given in terms of convergence of states. We use the notation in [2.5]

Let $\phi$ (resp. $\phi^{(k)}$, $(k \in \mathbb{N})$) be states on a unital $*$-algebra $C$ with associated GNS Hilbert spaces denoted by $L_2(\phi)$ (resp. $L_2(\phi^{(k)})$). Let $\pi$ (resp. $\pi^{(k)}$) be the associated GNS representations of $C$ on these Hilbert spaces. We say that $\phi^{(k)}$ tends to $\phi$ strongly and we write $\phi^{(k)} \overset{\pi}{\to} \phi$ if $\phi^{(k)}$ tends to $\phi$ pointwise on $C$ and moreover if $\|\pi^{(k)}(c)\| \to \|\pi(c)\|$ for any $c \in C$.  

3
In [8] the notion of free product of a family of states is defined. It can be described as follows. Consider a family of states \{\phi_i \mid i \in I\} with GNS Hilbert space \(H_i = L_2(\phi_i)\), GNS representation \(\pi_i : C_i \to B(L_2(\phi_i))\) and distinguished unit vector \(\xi_i\). Let \(A_i \subset B(L_2(\phi_i))\) be the associated C*-algebra. We denote by \(\hat{\pi}_i : A_i \to B(L_2(\phi_i))\) the inclusion map. Let \(B = \ast_{i \in I} C_i\) be the (algebraic) free product of unital *-algebras. Following Voiculescu (see [8]) one defines a Hilbert space free product \((H, \xi) = \ast_{i \in I} (H_i, \xi_i)\) and a representation \(\pi\) of \(B\) acting on \((H, \xi)\). Let \(\hat{\phi}_i\) (resp. \(\phi\)) be the vector state on \(A_i\) (resp. \(\pi(B)\)) associated to \(\xi_i\) (resp. \(\xi\)). The unital C*-subalgebra \(A = \pi(B) \subset B(H)\), equipped with \(\hat{\phi}\), is called the reduced free product of \((A_i, \hat{\phi}_i)_{i \in I}\). We will denote by \(\phi = \ast_{i \in I} \phi_i\) the vector state on \(B\) defined by \(\phi(b) = \langle \pi(b) \xi, \xi \rangle\). We call it the free product of the states \(\{\phi_i \mid i \in I\}\). Then we can reformulate the main result like this:

**Theorem 3.1.** Let \(C_i \ (i \in I)\) be a family of unital *-algebras. Let \(\{\phi_i^{(k)} \mid i \in I\} \ (k \in \mathbb{N})\) be a sequence of families of states, each \(\phi_i^{(k)}\) being a state on \(C_i\). Assume that we have states \(\phi_i\) on \(C_i\) such that, for each \(i \in I\), when \(k \to \infty\) we have

\[
\phi_i^{(k)} \xrightarrow{s} \phi_i.
\]

Then

\[
\ast_{i \in I} \phi_i^{(k)} \xrightarrow{s} \ast_{i \in I} \phi_i.
\]

3.1. Let \(A_i\) be associated to \((C_i, \phi_i)\) as above. We view each \(A_i\) as a subalgebra of \(A = \ast_{i \in I} A_i\). By a monomial of length \(d\) we mean a product of the form \(x_1 \cdots x_d\) with \(x_j \in A_{i_j}\) such that \(\phi_{i_j}(x_j) = 0\) and such that \(i_1 \neq i_2 \neq \cdots \neq i_d\). By a homogeneous element of length \(d\) in \(A = \ast_{i \in I} A_i\) we mean a finite sum of monomials of length \(d\). An element is called of length \(\leq d\) if it is a sum of homogeneous elements each of length \(\leq d\). Note that the elements of finite length are dense in \(A\).

Let us denote by \(W_d\) (resp. \(W_{\leq d}\)) the space of homogeneous elements of length \(d\) (resp. \(\leq d\)) in the preceding sense. We also set \(W_0 = \mathbb{C}1\) and denote by \(\overline{W_d}\) the closure of \(W_d\) in \(A\). Then the Ricard-Xu inequality we will use is this (note that the assumption in [6] that all the GNS constructions are faithful is here automatic): There are constants \(c' > 0\) and \(\beta > 0\) such that

\[
\forall d \ \forall x \in W_d \quad (c' d^3)^{-1} k_h(x) \leq \|x\| \leq c' d^3 k_h(x),
\]

where we set

\[
k_h(x) = \max \{ \max_{0 \leq r \leq d} \|t_r(x)\|, \max_{1 \leq r \leq d} \|s_r(x)\| \},
\]

and where \(t_r(x), s_r(x)\) are defined as follows: Consider an element \(x \in W_d\), of the form \(x = \sum_{\alpha} x(\alpha)\) with \(x(\alpha)\) monomial of degree \(d\) as described above. Let us denote by \(\xi\) the distinguished unit vector in \(H = \ast_{i \in I} H_i\). We define when \(1 \leq r \leq d - 1\)

\[
t_r(x) = \sum_{\alpha} x_1(\alpha) \cdots x_r(\alpha) \xi \otimes x_{r+1}(\alpha) \cdots x_d(\alpha) \xi \in H \otimes H,
\]

and also \(t_0(x) = \sum_{\alpha} \xi \otimes x_1(\alpha) \cdots x_d(\alpha) \xi\) and \(t_d(x) = \sum_{\alpha} x_1(\alpha) \cdots x_d(\alpha) \xi\otimes \xi\). We will identify an element \(a \otimes b \in H \otimes H\) with the linear map \(T \in B(H^*, H)\) defined by \(T(z) = z(b)a\). In this way we will view \(t_r(x)\) as an element of \(B(H^*, H)\), and we denote by \(\|t_r(x)\|\) its norm.

Assuming \(I = \{1, 2\}\) for simplicity. Let us denote by \(J_i : A_i \to A_1 \oplus A_2\) (\(i \in I\)) the canonical embedding. Assuming that \(x_r(\alpha) \in A_i\) for \(i = i(\alpha, r)\), we set

\[
s_r(x) = \sum_{\alpha} x_1(\alpha) \cdots x_{r-1}(\alpha) \xi \otimes x_{r+1}(\alpha) \cdots x_d(\alpha) \xi \otimes [J_{i(\alpha, r)}(x_r(\alpha))] \in B(H^*, H) \otimes [A_1 \oplus A_2].
\]
We then denote by \( \|s_i(x)\| \) its norm in the minimal tensor product \( B(H^*, H) \otimes_{\min} [A_1 \oplus A_2] \). Equivalently this is the maximum of two norms, one in \( B(H^*, H) \otimes_{\min} A_1 \) (induced by \( B(H^* \otimes H_1, H \otimes H_1) \)) and one in \( B(H^*, H) \otimes_{\min} A_2 \) (induced by \( B(H^* \otimes H_2, H \otimes H_2) \)).

3.2. By [6] there is a (completely) bounded projection \( Q_d: *_{i \in I} A_i \to W_d \) with \( \|Q_d\| \leq \max\{1, 4d\} \), defined so that if \( x = x_0 + \cdots + x_D \) with \( x_d \in W_d \) for all \( 0 \leq d \leq D \), then \( Q_d(x) = x_d \).

Obviously this implies

\[
(\text{max}\{1, 4d\})^{-1} \max_{0 \leq d \leq D} \{\|Q_d(x)\|\} \leq \|x\| \leq (d + 1) \max_{0 \leq d \leq D} \{\|Q_d(x)\|\}.
\]

It will convenient for us to extend the above definition of \( kh \) as follows: for any \( D \) and any \( x \in W_{\leq D} \) we define

\[
kh(x) = \sup_{0 \leq d \leq D} kh(Q_d(x)).
\]

Combining (3.2) with (3.1) we now obtain that there are constants \( c > 0 \) and \( \alpha > 0 \) such that

\[
\forall d \forall x \in W_{\leq d} \quad (cd^n)^{-1} kh(x) \leq \|x\| \leq cd^\alpha kh(x).
\]

3.3. For any \( i \in I \) we set \( C_i = \{x \in C_i \mid \phi(x) = 0\} \). The elements of \( C_i \) will be sometimes called “centered” (with respect to \( \phi \)). Let \( B_0 = C_1 \). By elementary (free) algebra, one can show that the algebraic free product \( B \) is linearly isomorphic to the direct sum

\[
B_0 \oplus \oplus_{d \geq 1, i_1 \neq \cdots \neq i_d} B(i_1, \cdots, i_d)
\]

where the subspaces \( B(i_1, \cdots, i_d) \) \( (d \geq 1, i_1 \neq i_2 \neq \cdots) \) are formed of all products of the form

\[
x_{i_1} \cdots x_{i_d} \quad \text{with} \quad x_{i_j} \in C_{i_j} \quad \forall 1 \leq j \leq d.
\]

Recall that, for each \( k \), we are given a state \( \phi_i^{(k)} \) on \( C_i \). We will denote by \( v_i^{(k)}: C_i \to C_i \) the linear mapping that transforms centering with respect to \( \phi \) into centering with respect to \( \phi_i^{(k)} \).

More precisely, \( v_i^{(k)}(1) = 1 \) and \( \forall a \in C_i \quad v_i^{(k)}(a) = a - \phi_i^{(k)}(a)1 \). Using the above direct sum decomposition of \( B \), and viewing \( C_i \subset B \) we can extend the mappings \( v_i^{(k)} \) to a single mapping on \( B \). More precisely, using the freeness of the product, there is a unique linear map \( v^{(k)}: B \to B \) that coincides with \( v_i^{(k)} \) on \( C_i \) for each \( i \in I \) and is such that for any element of the form (3.6) we have

\[
v^{(k)}(x_{i_1} \cdots x_{i_d}) = v^{(k)}_{i_1}(x_{i_1}) \cdots v^{(k)}_{i_d}(x_{i_d}).
\]

We note that if we equip \( B \) with the maximal \( C^* \)-norm then we clearly have

\[
\forall b \in B \quad \|v^{(k)}(b) - b\| \to 0.
\]

Let us denote by \( P_d \) the linear projection (relative to (3.5)) from \( B \) to the subspace \( W_d \subset B \) defined by

\[
W_d = \oplus_{i_1 \neq \cdots \neq i_d} B(i_1, \cdots, i_d).
\]

Fix \( k \). Again let \( W_0 = C_1 \) and \( W_{\leq d} = W_0 + \cdots + W_d \).

Suppose now that we replace \( \phi \) by \( \phi_i^{(k)} \) so that we have a direct sum decomposition as above but
now associated to \( \phi^{(k)}_i \). This leads to subspaces \( \mathcal{W}^{(k)}_d \subset B \) defined exactly like \( \mathcal{W}_d \) but with respect to \( \phi^{(k)}_i \). Let \( P^{(k)}_d \) denote the linear projection from \( B \) to the subspace \( \mathcal{W}^{(k)}_d \subset B \) in the said direct sum decomposition relative to \( \phi^{(k)}_i \). It is easy to check that we have for any \( k \)

\[
v^{(k)} P_d = P^{(k)}_d v^{(k)}.
\]

3.4. Returning to the situation of Theorem 3.1, let \( H^{(k)}_i = L_2(\phi^{(k)}_i) \) with distinguished vector \( \xi^{(k)}_i \), GNS representation \( \pi^{(k)}_i : C_i \to A^{(k)}_i \subset B(\mathcal{H}^{(k)}_i) \) with \( A^{(k)}_i = \pi^{(k)}_i(C_i) \). We define \( A^{(k)} = *_{i \in I} A^{(k)}_i \), \( H^{(k)} = *_{i \in I} H^{(k)}_i \) and let \( \pi^{(k)} : B \to *_{i \in I} A^{(k)}_i \subset B(H^{(k)}) \) be the corresponding representation. Let \( H^\U \) (resp. \( H^\U_i \)) denote the ultraproduct of \( (H^{(k)}_i) \) (resp. \( H^{(k)}_i \)).

We will use (2.6) when \( \phi = *_{i \in I} \phi_i \) and \( \phi^{(k)} = *_{i \in I} \phi^{(k)}_i \). Let \( H = L_2(\phi) \) with \( \xi = \xi_\phi \). We denote by \( \pi : B = *_{i \in I} C_i \to B(H) \) the GNS representation relative to \( \phi \). We have natural identifications

\[
(H, \xi) = *_{i \in I} (H_i, \xi_i) \quad \text{and} \quad (H^{(k)}, \xi^{(k)}) = *_{i \in I} (H^{(k)}_i, \xi^{(k)}_i).
\]

In this case, the correspondence \( \pi(b)\xi \mapsto \pi^{(k)}(b)\xi^{(k)} = [\pi^{(k)}(b)\xi^{(k)}]_U \) defines a unitary operator from \( H \) to \( K^\U \). We will denote by

\[
V : H \to H^\U
\]

the corresponding isometry.

Assume now that \( \phi^{(k)}_i \to \phi_i \). Then (see (2.6)) we have an isometric embedding

\[
\psi_i : A_i \to B(H^\U_i)
\]

such that \( \psi_i(\pi_i(b)) = \pi^{(k)}_i(b) \) for any \( b \in C_i \).

3.5. Consider now a monomial of length \( d \) in \( \pi(B) \subset *_{i \in I} A_i \) as defined in 3.1 above, of the form \( x = x_1 \cdots x_d \) with \( x_j \in A_{i_j} \) such that \( \phi^{(k)}_{i_j}(x_j) = 0 \) and such that \( i_1 \neq i_2 \neq \cdots \neq i_d \). We can choose \( b_j \in C_{i_j} \) such that \( x_j = \pi_i(b_j) \). We may also view \( b_j \) as an element of \( B = *_{i \in I} C_i \), then with this abuse of notation we may write also \( x_j = \pi(b_j) \).

Let \( x^{(k)}_j = \pi^{(k)}(b_j) \) and \( y^{(k)}_j = \pi^{(k)}(v^{(k)}(b_j)) \). Then by (3.7) \( \lim_U \|x^{(k)}_j - y^{(k)}_j\| = 0 \).

It will be useful to record here also, that a fortiori, for any \( 1 \leq r \leq d \) we have by (3.7)

\[
V(x_1 \cdots x_r) = [y^{(k)}_1 \cdots y^{(k)}_r]_U \quad \text{and} \quad V(x_{r+1} \cdots x_d) = [y^{(k)}_{r+1} \cdots y^{(k)}_d]_U.
\]

**Proof of Theorem 3.4** The pointwise convergence of \( *_{i \in I} \phi^{(k)}_i \) to \( *_{i \in I} \phi_i \) is obvious by definition of the free product of states. To show the strong convergence it suffices to show that \( \lim_U \|\pi^{(k)}(b)\| = \|\pi(b)\| \) for any \( b \in B \) and any non trivial ultrafilter \( U \) on \( \mathbb{N} \). Recall \( B = *_{i \in I} C_i \) (algebraic free product). Note that \( B = \cup_b \mathcal{W}_d \). The main point is that, by the Ricard-Xu inequality, there are constants \( c > 0 \) and \( \alpha > 0 \) such that

\[
\forall b \in \mathcal{W}_d \quad (cd^\alpha)^{-1} \lim_U \|\pi^{(k)}(b)\| \leq \|\pi(b)\| \leq cd^\alpha \lim_U \|\pi^{(k)}(b)\|.
\]

If we accept this result, the proof is immediate: we just note that \( (b^*b)^m \) is of degree at most \( 2md \), therefore \( (c(2md)^\alpha)^{-1} \lim_U \|\pi^{(k)}((b^*b)^m)\| \leq \|\pi(b^*b)^m\| = \|\pi(b)\|^{2m} \leq c(2md)^\alpha \lim_U \|\pi^{(k)}((b^*b)^m)\| \) and \( \|\pi^{(k)}((b^*b)^m)\| = \|\pi^{(k)}(b^*b)^m\| \). So we find

\[
(c(2md)^\alpha)^{-1/2m} \lim_U \|\pi^{(k)}(b)\| \leq \|\pi(b)\| \leq (c(2md)^\alpha)^{1/2m} \lim_U \|\pi^{(k)}(b)\|.
\]
and letting \( m \to \infty \) yields the equality \( \lim_{\mathcal{U}} \| \pi^{(k)}(b) \| = \| \pi(b) \| \).

We now turn to the proof of \((3.10)\). By the Ricard-Xu inequality \((3.4)\), we have
\[
(3.11) \quad \forall b \in \mathcal{W}_{\leq d} \quad (cd^\alpha)^{-1} k h(\pi(b)) \leq \| \pi(b) \| \leq cd^\alpha k h(\pi(b)).
\]
and
\[(cd^\alpha)^{-1} \lim_{\mathcal{U}} k h(\pi^{(k)}(b)) \leq \lim_{\mathcal{U}} \| \pi^{(k)}(b) \| \leq cd^\alpha \lim_{\mathcal{U}} k h(\pi^{(k)}(b)).\]
Thus to conclude, it suffices to show that \( \lim_{\mathcal{U}} k h(\pi^{(k)}(b)) = k h(b) \).
We claim that it suffices to prove this for \( b \in \cup_d \mathcal{W}_d \), i.e. such that \( P_d(b) = b \) for some \( d \).
Let \( Q^{(k)}_d : A^{(k)} \to A^{(k)} \) be the analogue of \( Q_d \) with respect to \( A^{(k)} \). By the very definition of \( k h \) in \((3.3)\) it suffices to show \( \lim_{\mathcal{U}} k h(Q^{(k)}_d(\pi^{(k)}(b))) = k h(Q_d\pi(b)) \) for any \( d \).
At this point it is worthwhile to observe that \( Q_d\pi = \pi P_d \) and \( Q^{(k)}_d(\pi^{(k)}) = \pi^{(k)} P_d^{(k)} \).
Thus it suffices to show equivalently that for any \( d \) we have
\[
(3.12) \quad \lim_{\mathcal{U}} k h(\pi^{(k)} P_d^{(k)}(b)) = k h(\pi P_d(b)).
\]
Note that by \((3.8)\), since \( P_d^{(k)}(b) \) is idempotent, we have \( P_d^{(k)} v^{(k)} = P_d^{(k)} v^{(k)} P_d \) and by \((3.7)\) this implies \( \| P_d^{(k)}(b) - P_d^{(k)} P_d(b) \| \to 0 \). Then the equivalence \((3.4)\), applied to each \( \pi^{(k)} \), implies \( k h(\pi^{(k)} P_d^{(k)}(b) - \pi^{(k)} P_d^{(k)} P_d(b)) \to 0 \), and hence \( \lim_{\mathcal{U}} k h(\pi^{(k)} P_d^{(k)}(b)) = \lim_{\mathcal{U}} k h(\pi^{(k)} P_d^{(k)} P_d(b)) \).
Therefore, to prove \((3.12)\) we may assume that \( b = P_d(b) \) for some \( d \), proving our claim.

To conclude it suffices to show, when \( b = P_d(b) \), that we have for any \( r \):
\[
\lim_{\mathcal{U}} t_r(\pi^{(k)} P_d^{(k)}(b)) = t_r(\pi(b)) \quad \text{and} \quad \lim_{\mathcal{U}} s_r(\pi^{(k)} P_d^{(k)}(b)) = s_r(\pi(b)).
\]
In order to verify this, let \( x = \pi(b) \), and let \( \bar{x}^{(k)} = \pi^{(k)} v^{(k)}(b) = \pi^{(k)} P_d^{(k)}(b) \).

We may assume that \( b \) is a finite sum of the form \( b = \sum_{\alpha=1}^{N} b(\alpha) = \sum_{\alpha=1}^{N} b(\alpha) \in \mathcal{W}_d \), and \( b(\alpha) = b_1(\alpha) \cdots b_d(\alpha) \) with \( b_1(\alpha) \in C_{i_1}, \cdots, b_d(\alpha) \in C_{i_d} \) and \( i_1 \neq i_2 \neq \cdots \neq i_d \).
Then \( x = \pi(b) = \sum_{\alpha=1}^{N} x(\alpha) \) with \( x(\alpha) = \pi(b(\alpha)) = x_1(\alpha) \cdots x_d(\alpha) \), with \( x_j(\alpha) = \pi(b_j(\alpha)) \), and we have \( \bar{x}^{(k)} = \sum_{\alpha} \bar{x}^{(k)}(\alpha) \) with \( \bar{x}^{(k)}(\alpha) = y_1^{(k)}(\alpha) \cdots y_d^{(k)}(\alpha) \) with \( y_j^{(k)}(\alpha) = \pi^{(k)} v^{(k)}(b_j(\alpha)) \).

We remind the reader that \( \xi \) denotes the distinguished unit vector in \( H = *_{i \in I} H_1^{(k)} \), and
\[
t_r(x) = \sum_{\alpha} x_1(\alpha) \cdots x_r(\alpha) \xi \otimes x_{r+1}(\alpha) \cdots x_d(\alpha) \xi \in H \otimes H
\]
is viewed as an element of \( B(H^*, H) \). Moreover, in the case \( I = \{1, 2\} \), \( s_r(x) \in B(H^*, H) \otimes [A_1 \oplus A_2] \).
Recall also that \( H^{(k)} \) denotes the Hilbert space ultraproduct of the free products defined by \( (H^{(k)}), \xi^{(k)} \) = \( *_{i \in I} (H_1^{(k)}, \xi^{(k)}_i) \).
We will compare \( t_r(x) \) with
\[
t_r(\bar{x}^{(k)}) = \sum_{\alpha} y_1^{(k)}(\alpha) \cdots y_r^{(k)}(\alpha) \xi^{(k)} \otimes y_{r+1}^{(k)}(\alpha) \cdots y_d^{(k)}(\alpha) \xi^{(k)} \in H^{(k)} \otimes H^{(k)}.
\]
We can naturally associate to the sequence \((t_r(\bar{x}^{(k)}))\) an element \( T_r \in H^{(k)} \otimes H^{(k)} \). Moreover, since the sum over \( \alpha \) is a finite sum, the ranks of the tensors \( t_r(\bar{x}^{(k)}) \) are uniformly bounded, and hence it is easy to check (see \((2.3)\) that
\[
(3.13) \quad \| T_r \| = \lim_{\mathcal{U}} \| t_r(\bar{x}^{(k)}) \|.
\]
Consider now (see \((3.4)\) the isometry \( V : H \to H^{(k)} \) defined by \( V(\pi(b)\xi) = [\pi^{(k)}(b)\xi^{(k)}]_H \). We have then, by \((3.3)\)
\[
(V \otimes V)(t_r(x)) = T_r.
\]
from which \(\|T_r\| = \|t_r(x)\|\) follows.

Similarly, assuming \(I = \{1, 2\}\) for simplicity, we will compare \(s_r(x)\) with

\[
s_r(x^{(k)}) = \sum_{\alpha} y^{(k)}_{1} (\alpha) \cdots y^{(k)}_{r-1} (\alpha) \xi^{(k)} \otimes y^{(k)}_{r+1} (\alpha) \cdots y^{(k)}_{d} (\alpha) \xi^{(k)} \otimes [J_{i(\alpha,r)}(y^{(k)}_{r}(\alpha))].
\]

Now again, since \(\alpha\) runs over a fixed finite set, the sequence \((s_r(x^{(k)}))\) defines an element \(S_r\) in \((H^d \otimes H^d) \otimes_{\min} [B(H^d_1) \oplus B(H^d_2)]\), where \(H^d \otimes H^d\) is viewed as embedded in \(B(H^d, H^d)\) and the norm is the minimal norm (see 3.1).

Using 2.4 and the uniform bound on the ranks of the tensors involved, it follows that (see 2.4)

\[
||s_r|| = \lim_{U} ||s_r(x^{(k)})||.
\]

We now again use 3.4. We have then, by (3.9)

\[
S_r = (V \otimes V) \otimes [\psi_1 \otimes \psi_2](s_r(x))
\]

from which \(||S_r|| = ||s_r(x)||\) follows. Thus we obtain by (3.13) and (3.14) that

\[
||t_r(x)|| = \lim_{U} ||t_r(x^{(k)})|| \quad \text{and} \quad ||s_r(x)|| = \lim_{U} ||s_r(x^{(k)})||,
\]

and this concludes the proof of (3.12). \(\square\)

We now turn to the situation considered in the abstract. Given a family \(X = \{X_m \mid m \in M\}\) in a \(C^*\)-probability space \((A, \phi)\), let \(C\) be the unital \(*\)-algebra of all polynomials in the non-commutative variables \(\{x_m, x_m^\ast \mid m \in M\}\). We will say that \((A, \phi)\) is adapted to \(X\) if \(C\) is dense in \(A\) and the GNS-representation associated to \(\phi\) (or equivalently to the restriction of \(\phi\) to \(C\)) is isometric (it suffices for this that \(\phi\) be faithful on \(A\)).

**Corollary 3.2.** Let \(\{X^{(k)}_m \mid m \in M\}\), \(\{X^{(k)}_n \mid m \in M\}\) and \(\{Y^{(k)}_n \mid n \in N\}\), \(\{Y^{(k)}_n \mid n \in N\}\) be sequences of families of non-commutative random variables in some adapted \(C^*\)-probability spaces. Assume that \(\{X^{(k)}_m \mid m \in M\}\) tends strongly to \(\{X_m \mid m \in M\}\) and that \(\{Y^{(k)}_n \mid n \in N\}\) tends strongly to \(\{Y_n \mid n \in N\}\) when \(k \to \infty\). Assume moreover that for each \(k\) \(\{X^{(k)}_m \mid m \in M\}\) and \(\{Y^{(k)}_n \mid n \in N\}\) are \(*\)-free and also (a fortiori) that \(\{X_m \mid m \in M\}\) and \(\{Y_n \mid n \in N\}\) are \(*\)-free. Then the joint family \(\{X^{(k)}_m, Y^{(k)}_n \mid m \in M, n \in N\}\) tends strongly to \(\{X_m, Y_n \mid m \in M, n \in N\}\).

**Proof.** The assumption that \(\{X^{(k)}_m \mid m \in M\}\) tends strongly to \(\{X_m \mid m \in M\}\) means that for any polynomial \(P\) in the non-commutative variables \(\{x_m \mid m \in M\}\) we have \(\phi_1(P(X^{(k)}_m)) \to \phi_1(P(X_m))\) (this is the convergence of moments) and moreover \(\|P(X^{(k)}_m)\| \to \|P(X_m)\|\). We also assume this with \(Y\) in place of \(X\). Let \(C_1\) (resp. \(C_2\)) be the unital \(*\)-algebra generated by non-commutative variables \(\{x_m, x_m^\ast \mid m \in M\}\) (resp. \(\{y_n, y_n^\ast \mid n \in N\}\)). If \(\{X_m \mid m \in M\}\) is given as realized inside a \(C^*\)-algebra equipped with a state \(f\), we define the associated state \(\phi_1\) on \(C_1\) by \(\phi_1(f) = f(P(X_m, X_m^\ast))\) and similarly for \(\phi_2\) on \(C_2\). Repeating this for each \(k\), this leads to states \(\phi_i^{(k)}\) on \(C_i\) for \(i = 1, 2\). We may then identify the free product \(C_1 \ast C_2\) with the unital \(*\)-algebra generated by non-commutative variables \(\{x_m, x_m^\ast, y_n, y_n^\ast \mid m \in M, n \in N\}\). The Corollary then appears as a particular case of the preceding Theorem. \(\square\)

### 4. Supplementary Remarks
4.1. Recall \( (H^{(k)}, \xi^{(k)}) = \ast_{i \in I}(H_{i}^{(k)}, \xi_{i}^{(k)}) \). Going back to the definition of the free product, a moment of thought (recall 2.2) shows that we have a canonical isometric embedding

\[
(4.1) \quad h : \ast_{i \in I}H_{i}^{(k)} \subset H^{(k)}
\]

that respects the distinguished vectors.

Assuming \( I = \{1, 2\} \), the mapping \( h \) can be described like this: First we have \( h(\xi) = \xi^{U} \), then whenever we consider an element \( x_j \) in \( H_{1}^{(k)} \cap \{\xi_{1}^{(k)}\}^\perp \) (resp. \( H_{2}^{(k)} \cap \{\xi_{2}^{(k)}\}^\perp \)) we can choose representatives \( (x_{j}^{(k)}) \) of \( x_j \) with \( x_{j}^{(k)} \) in \( H_{1}^{(k)} \cap \{\xi_{1}^{(k)}\}^\perp \) (resp. \( H_{2}^{(k)} \cap \{\xi_{2}^{(k)}\}^\perp \)), so that given an element \( x = x_{1} \otimes \cdots \otimes x_{d} \) of degree \( d \) in \( \ast_{i \in I}H_{i}^{(k)} \), with alternating factors in \( H_{1}^{(k)} \cap \{\xi_{1}^{(k)}\}^\perp \) and \( H_{2}^{(k)} \cap \{\xi_{2}^{(k)}\}^\perp \), we then define \( h(x) \) as the element of \( H^{(k)} \) admitting as representative the sequence \( (x_{j}^{(k)}) \) with \( x_{j}^{(k)} = x_{1}^{(k)} \otimes \cdots \otimes x_{d}^{(k)} \).

4.2. With the notation of the Corollary, whenever \( X^{(k)} = \{X_{m}^{(k)}, X_{m}^{(k)}^* \mid m \in M\} \) on \((A^{(k)}, \phi^{(k)})\) converges in moments to \( \{X_{m}, X_{m}^* \mid m \in M\} \) on \((A, \phi)\), it is well known that for any non-trivial ultrafilter \( U \) and any polynomial \( P \), we have

\[
(4.2) \quad \|P(X)\| \leq \lim_{U} \|P(X^{(k)})\|.
\]

Indeed, for any \( \varepsilon > 0 \) there are polynomials \( Q, R \) with unit norm in \( L_{2}(\phi) \) so that

\[
\|P(X)\| - \varepsilon < |\phi(P(X)Q(X)R(X))| = \lim_{U} \|\phi^{(k)}(P(X^{(k)})Q(X^{(k)})R(X^{(k)}))\| \leq \lim_{U} \|P(X^{(k)})\|,
\]

from which \((4.2)\) follows. The converse inequality is the essence of strong convergence.

4.3. With the same notation as \((4.2)\) if the variables \( X = \{X_{m}, X_{m}^* \mid m \in M\} \) all commute (i.e. form a family of commuting normal operators) then the (commutative) \( C^{*} \)-algebra \( A \) to which \( X \) is adapted is isometric to the \( C^{*} \)-algebra \( C(K) \) of all continuous functions on a compact set \( K \subset \mathbb{C} \). Similarly, if all \( \{X_{m}^{(k)}, X_{m}^{(k)}^* \mid m \in M\} \) commute, the \( C^{*} \)-algebra \( A^{(k)} \) to which \( X^{(k)} \) is adapted is isometric to the (commutative) \( C^{*} \)-algebra \( C(K^{(k)}) \) for some subset \( K^{(k)} \subset \mathbb{C} \). When the family \( X \) (resp. \( X^{(k)} \)) is reduced to a single normal operator, we can adjust so that this operator corresponds to the function \( P(\lambda) = \lambda \) on \( K \) (resp. \( K^{(k)} \)). In that case, \( K \) (resp. \( K^{(k)} \)) is the spectrum of the operator \( X \) (resp. \( X^{(k)} \)) and it is an easy exercise to check that \( X^{(k)} \xrightarrow{\ast} X \) iff it converges in moments and moreover

\[
\lim_{k \to \infty} \sup_{\lambda \in K^{(k)}} d(\lambda, K) = 0.
\]

Wigner’s classical theorem about the convergence of the eigenvalues of Gaussian random matrices was strengthened in \([3]\) as follows: let \( X^{(k)}(\omega) \) be a random \( k \times k \)-matrix the entries of which are independent complex Gaussian \( N(0, k^{-1}) \), then for almost all \( \omega \), \( X^{(k)}(\omega) \) tends strongly to a circular random variable. A similar result is valid for the classical Gaussian Wigner (Hermitian) matrices (model for the so-called GUE) now with a semi-circular limit. See \([3, 2]\) for more general results.

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