SEMI-STABLE LOCUS OF A GROUP
COMPACTIFICATION

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Abstract. In this paper, we consider the diagonal action of a
connected semisimple group of adjoint type on its wonderful com-
 pactification. We show that the semi-stable locus is a union of the
$G$-stable pieces and we calculate the geometric quotient.

0.1. Introduction. Let $G$ be a connected, semisimple algebraic group
of adjoint type over an algebraically closed field and $X$ be its wonder-
ful compactification. We will give an explicit description of the semi-stable
locus of $X$ (for the diagonal $G$-action) using Lusztig’s $G$-stable pieces
and calculate the geometric quotient $X//G$. We also deal with the case
where the $G$-action is twisted by a diagram automorphism.

The results will be used by the first author [He] to study character
sheaves on the wonderful compactification.

During the time the article was writing, we learned that De Concini,
Kanna and Maffei [CKM] described the semi-stable locus and geometr ic
quotient for complete symmetric varieties (which includes as a special
case the non-twisted conjugation action of $G$ on its wonderful compac-
tification).

0.2. Geometric invariant theory. The foundations of geometric in-
variant theory are developed in [MFK94]. We quickly review that part
which we use. Let $k$ be a field. The setup for geometric invariant
theory over $k$ consists of $(G, X, \tau, \mathcal{L}, \psi)$ where

(i) $G$ is a reductive algebraic group over $k$,
(ii) $X$ is a separated, finite type $k$-scheme,
(iii) $\tau : G \times X \to X$ is an algebraic action of $G$ on $X$,
(iv) $\mathcal{L}$ is an invertible sheaf on $X$, and
(v) $\psi : \tau^*\mathcal{L} \to \text{pr}_1^*\mathcal{L}$ is a $G$-linearization of $\mathcal{L}$ (where $\text{pr}_1 : G \times X \to
X$ is the projection), i.e., an isomorphism of invertible sheaves
on $G \times X$ which defines a lifting of the action $\tau$ to an action of
$G$ on $\text{Spec}_X \text{Sym}^n(\mathcal{L})$.

The fundamental theorem of geometric invariant theory, [MFK94,
Theorem 1.10, p. 38], associates to this datum a pair $(X^{\text{ss}}(\mathcal{L}), \phi)$. Here
$X^{\text{ss}}(\mathcal{L})$ is the union $X_s$ over all positive integers $n$ and all $G$-invariant

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sections $s$ of $\Gamma(X, \mathcal{L} \otimes^n)$, provided $X_s$ is affine (recall, $X_s$ is defined to be the maximal open subscheme of $X$ on which $s$ is a generator of $\mathcal{L} \otimes^n$). And $\phi$ is a $G$-invariant $k$-morphism

$$\phi : X^{ss}(\mathcal{L}) \to X//_{\mathcal{L}}G$$

which is a uniform categorical quotient of the action of $G$ on $X^{ss}(\mathcal{L})$.

Moreover the following hold.

(i) The morphism $\phi$ is affine and universally submersive.

(ii) For some integer $n > 0$, there exists an ample invertible sheaf $M$ on $X//_{\mathcal{L}}G$ such that $\phi^*M$ is isomorphic to $\mathcal{L} \otimes^n$ as $G$-linearized invertible sheaves (in particular, $X//_{\mathcal{L}}G$ is quasi-projective).

(iii) There exists a unique open subscheme $U$ of $X//_{\mathcal{L}}G$ such that $\phi^{-1}(U)$ is the stable locus. And the induced morphism $\phi : \phi^{-1}(U) \to U$ is a uniform geometric quotient of $\phi^{-1}(U)$.

Since we do not make use of them, we will not make precise the definitions of uniform categorical quotient, stable locus and uniform geometric quotient. But we will use a few other known facts about geometric invariant theory.

**Fact 1.** When $X$ is projective and $\mathcal{L}$ is ample, every open $X_s$ is affine. Thus $X//_{\mathcal{L}}G$ is canonically isomorphic to

$$X//_{\mathcal{L}}G = \text{Proj } \oplus_{n \geq 0} \Gamma(X, \mathcal{L} \otimes^n)^G$$

and $X^{ss}(\mathcal{L})$ is the maximal open subscheme of $X$ on which the natural rational map from $X$ to $X//_{\mathcal{L}}G$ is defined, [Ses77].

**Fact 2.** Again when $X$ is proper, every $G$-orbit $O$ in $X^{ss}(\mathcal{L})$ contains a unique closed $G$-orbit in its closure (in $X^{ss}(\mathcal{L})$). And two $G$-orbits $O_1$ and $O_2$ in $X^{ss}(\mathcal{L})$ are in the same fiber of $\phi$ if and only if the associated closed $G$-orbits are equals. In particular, $\phi$ establishes a natural bijection between the points of $X//_{\mathcal{L}}G$ and the closed $G$-orbits in $X^{ss}(\mathcal{L})$, [Ses77].

**Fact 3.** (Matsushima’s criterion) A $G$-orbit $O$ is affine if and only if the stabilizer group of one (and hence every) closed point is itself reductive, [Ric77]. In particular, since the fibers of $\phi$ are affine, every closed $G$-orbit in $X^{ss}(\mathcal{L})$ is affine, and hence has reductive stabilizer group.

**Fact 4.** If $X$ is normal (or if $X^{ss}(\mathcal{L})$ is normal), then $\phi$ factors through the normalization of the target. Thus by the universal property, the target $X//_{\mathcal{L}}G$ is normal.

0.3. **Notations.** Now we fix the notations used in the rest of this article. Let $G$ be a connected semisimple algebraic group of adjoint type over an algebraically closed field $k$. Let $B$ be a Borel subgroup of $G$, $B^-$ be an opposite Borel subgroup and $T = B \cap B^-$. Let $(\alpha_i)_{i \in I}$ be the set of simple roots determined by $(B, T)$. We denote by $W$ the Weyl group $N(T)/T$. For $w \in W$, we choose a representative $\dot{w}$ in
$N(T)$. For $i \in I$, we denote by $\alpha_i$ and $s_i$ the fundamental weight and the simple reflection corresponding to $\alpha_i$.

For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by $J$ and let $P_J^- \supset B^-$ be the parabolic subgroup opposite to $P_J$. Set $L_J = P_J \cap P_J^-$. Then $L_J$ is a Levi subgroup of $P_J$ and $P_J^-$. The semisimple quotient of $L_J$ of adjoint type will be denoted by $G_J$. We denote by $\pi_{P_J^-}$ (resp. $\pi_{P_J}$) the projection of $P_J$ (resp. $P_J^-$) onto $G_J$.

Let $W_J$ be the subgroup of $W$ generated by $\{s_j \mid j \in J\}$ and $W'$ be the set of minimal length coset representatives of $W/W_J$.

0.4. Wonderful compactification of $G$. We consider $G$ as a $G \times G$-variety by left and right translation. Then there exists a canonical $G \times G$-equivariant embedding $X$ of $G$ which is called the wonderful compactification ([DCP83], [Str87]). The variety $X$ is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits $Z_J$ indexed by the subsets $J$ of $I$. The boundary $X - G$ is a union of smooth divisors $Z_{J \setminus \{i\}}$ (for $i \in I$), with normal crossing. The $G \times G$-variety $Z_J$ is isomorphic to the product $(G \times G) \times_{P_J \times P_J} G_J$, where $P_J^- \times P_J$ acts on $G \times G$ by $(q, p) \cdot (g_1, g_2) = (g_1 g q, g_2 p)$ and on $G_J$ by $(q, p) \cdot z = \pi_{P_J^-}(q) \pi_{P_J}(p).$ We denote by $h_J$ the image of $(1, 1, 1)$ in $Z_J$ under this isomorphism.

0.5. Twisted actions. We follow the approach in [HT06, Section 3]. Let $\sigma$ be an automorphism on $G$ such that $\sigma(B) = B$ and $\sigma(T) = T$. We also assume that $\sigma$ is a diagram automorphism, i.e., the order of $\sigma$ coincides with the order of the associated permutation on $I$.

Let $G_\sigma$ (resp. $X_\sigma$) be the $(G \times G)$-variety which as a variety is isomorphic to $G$ (resp. $X$), but the $G \times G$-action is twisted by $(g, g') \mapsto (g, \sigma(g'))$. Then $G_\sigma$ is an open $G \times G$-subvariety of $X_\sigma$ and we call $X_\sigma$ the wonderful compactification of $G_\sigma$.

Under the natural bijection between $X$ and $X_\sigma$, we may identify the $G \times G$-orbits on $X$ with the $G \times G$-orbits on $X_\sigma$. We denote by $Z_{J,\sigma}$ the $G \times G$-orbit on $X_\sigma$ that corresponds to $Z_{\sigma(J)} \subset X$. Accordingly, we denote by $h_{J,\sigma}$ the base point in $Z_{J,\sigma}$ which corresponds to the base point $h_{\sigma(J)}$ of $Z_{\sigma(J)}$.

0.6. $\sigma$-semisimple elements in $G_\sigma$. We follow the notation of [Spr06]. An element $g \in G_\sigma$ is called $\sigma$-semisimple if it is conjugated to an element in $T$. We have the following result.

**Theorem 0.1.** Let $g \in G_\sigma$. Then the following conditions are equivalent:

1. The element $g$ is $\sigma$-semisimple.
2. The $G$-orbit of $g$ is closed in $G_\sigma$.

In this case, the isotropy subgroup of $g$ in $G$ is reductive.
The equivalence of (1) and (2) can be found in [Lus03, 1.4 (e)] (in terms of disconnected groups instead of twisted conjugation action). In the case of simply connected group, the equivalence is also proved in [Spr06, Proposition 3]. By Fact 3, Matsushima’s criterion, the $G$-orbit of $g$ is closed implies that the isotropy subgroup of $g$ is reductive.

0.7. $G$-stable-piece decomposition. Let $G_{\Delta}$ be the diagonal image of $G$ in $G \times G$. The classification of the $G_{\Delta}$-orbits on $X$ was obtained by Lusztig [Lus04] in terms of $G$-stable pieces. A similar result also occurs in [EL06]. We list some known results which will be used later.

For $J \subset I$ and $w \in W^\sigma(J)$, set
\[ Z_{J,\sigma;w} = G_{\Delta}(B\dot{w}, B) \cdot h_{J,\sigma}. \]
We call $Z_{J,\sigma;w}$ a $G$-stable piece of $X_\sigma$. By [Lus04, 12.3] and [He06, Proposition 2.6], $X_\sigma$ is a disjoint union of the $G$-stable pieces.

Fact 5. $X_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^\sigma(J)} Z_{J,\sigma;w}$.

Set $I(J, \sigma; w) = \max\{K \subset J \mid w\sigma(K) = K\}$. Then the subvariety $L_{I(J, \sigma; w)}\dot{w}$ of $G_\sigma$ is stable under the action of $L_{I(J, \sigma; w)} \times L_{I(J, \sigma; w)}$ and in particular, is stable under the conjugation action of $L_{I(J, \sigma; w)}$. Moreover, by [Lus04, 12.3(a)] and [He07, Lemma 1.4],
\[ Z_{J,\sigma;w} = G_{\Delta}(L_{I(J,\sigma;w)}\dot{w}, 1) \cdot h_{J,\sigma} \]
and there exists a natural bijection between the $G_{\Delta}$-orbits on $Z_{J,\sigma;w}$ and the $L_{I(J, \sigma; w)}$-orbits on $L_{I(J, \sigma; w)}\dot{w}/Z^0(L_J) \subset G_\sigma/Z^0(L_J)$ (for the conjugation action of $L_{I(J, \sigma; w)}$).

For any point $z$ in $Z_{J,\sigma;w}$, the isotropy subgroup
\[ G_z = \{g \in G \mid (g, g) \cdot z = z\} \]
was described explicitly in [EL06, Theorem 3.13]. We only need the following special case in our paper.

Fact 6. Let $z = (gl\dot{w}, g) \cdot h_{J,\sigma}$ for $g \in G$ and $l \in L_{I(J, \sigma; w)}$. Then $G_z$ is reductive if and only if $w = 1$ and $l$ is a $\sigma$-semisimple element in $L_{I(J, \sigma; 1)}$.

By [He07, Theorem 4.5], the closure of each $G$-stable piece is a union of $G$-stable pieces and the closure relation can be described explicitly. More precisely, for $J \subset I$, $w \in W^\sigma(J)$ and $w' \in W$, we write $w' \leq J,\sigma w$ if there exists $u \in W_J$ such that $w' \geq uw\sigma(u)$. Then
\[ \overline{Z_{J,\sigma;w}} = \bigsqcup_{J' \subset J} \bigsqcup_{w' \in W^\sigma(J')} Z_{J',\sigma;w'}. \]
Notice that if $1 \leq J,\sigma w$, then we must have $w = 1$. Therefore,

Fact 7. $\bigsqcup_{J \subset I} Z_{J,\sigma;1}$ is open in $X_\sigma$. 

0.8. Nilpotent Cone of $X$. For any dominant weight $l$, let $H(l)$ be the dual Weyl module for $G_{sc}$ with lowest weight $-l$. Let $^\sigma H(l)$ be the $G_{sc}$-module which as a vector space is $H(l)$, but the $G_{sc}$-action is twisted by the automorphism $\sigma$ on $G_{sc}$. Then there exists (up to a nonzero constant) a unique $G_{sc}$ isomorphism $^\sigma H(l) \to H(\sigma(l))$. In particular, if $l = \sigma(l)$, then we have an isomorphism $f_l: ^\sigma H(l) \to H(l)$.

By [DCS99, 3.9], there exists a $G \times G$-equivariant morphism

$$\rho_l : X \to \mathbb{P}(\text{End}(H(l)))$$

which extends the morphism $G_{\sigma} \to \mathbb{P}(\text{End}(H(l)))$ defined by $g \mapsto g[\text{Id}_l]$, where $[\text{Id}_l]$ denotes the class representing the identity map on $H(l)$ and $g$ acts by the left action. We denote by $L_X(\lambda)$ the $G_{sc} \times G_{sc}$-linearized invertible sheaf on $X$ which is the pullback under $\rho_l$ of $\mathcal{O}(1)$ with its canonical linearization. This is the “usual” linearized invertible sheaf on $X$ associated to the weight $\lambda$, e.g., as defined in [BP00, p. 100].

For sufficiently divisible and positive $n$, the $G_{sc} \times G_{sc}$-linearization of $L_X(\lambda)^{\otimes n} = L_X(n \cdot \lambda)$ factors through a $G \times G$-linearization. This induces a $G_{\Delta}$-linearization of $L_X(\lambda)^{\otimes n}$. If moreover, $\lambda$ is regular, then $L_X(\lambda)$ is ample (see [Str87, section 2]).

The morphism $\rho_l$ induces a $G \times G$-equivariant morphism $X_{\sigma} \to \mathbb{P}(\text{Hom}(^\sigma H(l), H(l)))$. When $l = \sigma(l)$, we may apply the isomorphism $f_l : ^\sigma H(l) \to H(l)$ to obtain the $G \times G$-equivariant morphism

$$\rho_{l,\sigma} : X_\sigma \to \mathbb{P}(\text{End}(H(l)))$$

As above, $L_{X_\sigma}(\lambda, \sigma)$ denotes the $G_{sc} \times G_{sc}$-linearized invertible sheaf on $X_\sigma$ which is the pullback under $\rho_{\lambda, \sigma}$ of $\mathcal{O}(1)$ with its canonical linearization. Of course $X_\sigma$ equals $X$ as varieties, and $L_X(\lambda)$ equals $L_{X_\sigma}(\lambda, \sigma)$ as invertible sheaves on this variety. But the $G \times G$-actions are not the same, and thus the $G \times G$-linearized invertible sheaves are not the same.

For $l = \sigma(l)$, let $N(l)_{\sigma}$ be the subvariety of $X_\sigma$ consisting of elements that may be represented by a nilpotent endomorphism of $H(l)$. We call $N(l)_{\sigma}$ the nilpotent cone of $X_l$ associated to the dominant weight $l$. We have an explicit description of $N(l)$ which was obtained in [HT06, Proposition 4.4]

$$N(l)_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^{s(J)}} Z_{J, \sigma; w}$$

where $I(l) = \{i \in I \mid a_i \neq 0\}$ of $I$ for $l = \sum_{i \in I} a_i \varphi_i$ and $\text{supp}(w) \subset I$ is the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of $w$.

Two subvarieties of $X$ related to the nilpotent cones of $X$ are of special interest. One is

$$\cap_l \text{ is dominant } N(l)_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^{s(J)}, \text{supp}(w) = I} Z_{J, \sigma; w}.$$
This subvariety is actually the boundary of the closure in $X_\sigma$ of unipotent subvariety of $G_\sigma$ in the case where $G$ is simple (See [He06, Theorem 4.3] and [HT06, Theorem 7.3]).

The other one is $X_\sigma - \cup_l \text{dominant } N(l)_\sigma = \sqcup_{J \subset I} Z_{J,\sigma;1}$, which is the complement of $N(l)_\sigma$ for any $\sigma$-stable dominant regular weight. By the next theorem, this subvariety is actually the semi-stable locus for the $G_\Delta$-action.

**Theorem 0.2.** For $\lambda$ as above, i.e., $\sigma$-stable, dominant and regular, the semistable locus $(X_\sigma)^{ss}(\mathcal{L}_X(\lambda)^{\otimes n})$ equals $\sqcup_{J \subset I} Z_{J,\sigma;1}$. In particular, the semistable locus is independent of the choice of weight $\lambda$.

**Proof.** We simply write the semistable locus $(X_\sigma)^{ss}(\mathcal{L}_X(\lambda)^{\otimes n})$ as $X_\sigma^{ss}$.

On $\operatorname{End}(H(\lambda))$ the characteristic polynomial map

$$\chi : \operatorname{End}(H(\lambda)) \to k[t], \quad (f : H(\lambda) \to H(\lambda)) \mapsto \chi_f(t)$$

is a morphism which is invariant under the conjugation action. The coefficients of the characteristic polynomial define homogeneous polynomials on $\operatorname{End}(H(\lambda))$ which are invariant under the conjugation action. Also the degree is positive except for the leading coefficient (which is 1). Thus each non-leading coefficient defines a $G_\Delta$-invariant sections of positive power $\mathcal{O}(n)$ on $\mathbb{P}(\operatorname{End}(H(\lambda)))$. The pullbacks of these sections are $G_\Delta$-invariant sections of positive powers $\mathcal{L}^{\otimes n}$. By Fact 1, the nonvanishing locus of each of these sections is in the semistable locus. EQUIVALENTLY, the non-semistable locus is contained in the common zero locus of all of these sections. But the common zero locus of these pullback sections on $X_\sigma$ equals the inverse image of the common zero locus of the original sections on $\mathbb{P}(\operatorname{End}(H(\lambda)))$. And this common zero locus is precisely the nilpotent cone in $\mathbb{P}(\operatorname{End}(H(\lambda)))$. Thus the non-semistable locus is contained in $N(\lambda)_\sigma$. So $X_\sigma^{ss}$ contains $X_\sigma - N(\lambda)_\sigma$, i.e., $X_\sigma^{ss}$ contains $\sqcup_{J \subset I} Z_{J,\sigma;1}$.

Also, by Fact 7, $X_\sigma^{ss} - \sqcup_{J \subset I} Z_{J,\sigma;1}$ is closed in $X_\sigma^{ss}$. If $X_\sigma^{ss}$ strictly contains $\sqcup_{J \subset I} Z_{J,\sigma;1}$, then there exists a closed $G_\Delta$-orbit in $X_\sigma^{ss}$ that is not contained in $\sqcup_{J \subset I} Z_{J,\sigma;1}$. Let $z$ be an element in that orbit. By Fact 3 above, the isotropy subgroup of $z$, $\{g \in G \mid (g, g) \cdot z = z\}$, is reductive. By Fact 5 above, $z$ is in $Z_{J,\sigma;w}$ for some $J \subset I$ and $w \in W^{\sigma(J)}$ with $w \neq 1$. But this contradicts Fact 6 above. Therefore $X_\sigma^{ss}$ equals $\sqcup_{J \subset I} Z_{J,\sigma;1}$. $\square$

In the rest of this paper, we will describe the geometric quotient $X//G$. The description of the geometric quotient for the twisted action is based on some detailed analysis on the structure of the semi-stable locus and will be included in a future article.

**Lemma 0.3.** A $G_\Delta$-orbit $O$ in $X^{ss}$ is closed in $X^{ss}$ if and only if it intersects $T$. 
Proof. By \[\text{Lemma 6.1.6 (ii)}\], \(\bar{T} = \bigcup_{J \in I} N(T)_{\Delta} \cdot (T, 1) \cdot h_J\). Let \(z \in X^{ss}\) such that \(G_{\Delta} \cdot z\) is closed in \(X^{ss}\). Then \(G_{\Delta} \cdot z\) is affine and \(G_z\) is reductive. By Fact 6, \(z\) is of the form \(\{(g, g) \cdot h_J\mid g \in G\}\) for some semisimple element \(l \in L_J\). Hence \(l\) is conjugate to an element in \(T\). Therefore \(G_{\Delta} \cdot z \cap \bar{T} \neq \emptyset\).

On the other hand, if \(z \in \bar{T}\), then the isotropy subgroup of \(z\) in \(G\) contains \(T\). Let \(O\) be the unique closed \(G\)-orbit that is contained in the closure of \(G_{\Delta} \cdot z\). Then \(G_{\Delta} \cdot z\) and \(O\) lie in the same fiber of \(\phi\), which is an affine variety. By \[\text{Ste74, Page 70, Corollary 1}\], \(G_{\Delta} \cdot z\) is closed in that fiber. Hence \(\Gamma_{\Delta} \cdot z = O\). \(\square\)

**Lemma 0.4.** For every element \(z\) in \(\bar{T}\), the intersection \(G_{\Delta} \cdot z \cap \bar{T}\) of the \(G\)-orbit with \(\bar{T}\) equals the \(N(T)\)-orbit \(N(T)_{\Delta} \cdot z\).

**Proof.** Obviously \(N(T)_{\Delta} \cdot z\) is contained in \(G_{\Delta} \cdot z \cap \bar{T}\). The content of the lemma is the opposite inclusion.

We may assume without loss of generality that \(z\) has the form \(z = (t, 1) \cdot h_J\). Suppose that \((g, g) \cdot z\) equals \((t', 1) \cdot h_J\) for some \(t' \in T\), i.e., \((g, g) \cdot z\) is a point of \(G_{\Delta} \cdot z \cap (T, 1) \cdot h_J\).

Let \(F_J = (P_J, P_J) \cdot h_J\), then by \[\text{He07, Proposition 1.10}\], the action of \(G\) on \(X\) induces an isomorphism of \(Z_{J,1}\) with \(G \times_{P_J} F_J\). Thus \(g\) is in \(P_J\). Also both \(t\) and \(t'\) are contained in the same \(P_J\)-orbit in \(L_J/Z^0(L_J)\), i.e. in the same \(G_J\)-conjugacy class. Hence there exists an element \(n\) in \(N(T)\) such that \(t'\) equals \(ntn^{-1}\). Therefore \(G_{\Delta} \cdot z \cap (T, 1) \cdot h_J\) is a subset of \(N(T)_{\Delta} \cdot z\). Now also

\[
G_{\Delta} \cdot z \cap \bar{T} = G_{\Delta} \cdot z \cap N(T)_{\Delta} (T, 1) \cdot h_J = N(T)_{\Delta} \cdot (G_{\Delta} \cdot z \cap (T, 1) \cdot h_J)
\]

\[
\subset N(T)_{\Delta} \cdot N(T)_{\Delta} (T, 1) \cdot h_J = N(T)_{\Delta} \cdot (N(T)_{\Delta} \cdot z) = N(T)_{\Delta} \cdot z
\]

proving the lemma. \(\square\)

**Corollary 0.5.** The embedding \(\bar{T} \to X\) induces an isomorphism

\[
i : \bar{T}/W \to X/G.
\]

**Proof.** Notice that \(\bar{T}/N(T)\) equals \(\bar{T}/W\) for the natural \(W\)-action on \(\bar{T}\) which extends the \(W\)-action on \(T\).

The morphism \(\bar{T} \to X \to X/G\) is \(N(T)\)-invariant, and hence factors through a morphism

\[
i : \bar{T}/W \to X/G.
\]

By **Lemma 0.3**, every closed \(G\)-orbit in \(X^{ss}\) intersects \(\bar{T}\). Thus \(i\) is surjective. For every element \(z\) in \(\bar{T}\), the \(G\)-orbit of \(z\) is closed in \(X^{ss}\). Thus two elements \(z, z'\) in \(\bar{T}\) have the same image under \(i\) if and only if they lie in the same \(G\)-orbit. On the other hand, by **Lemma 0.4**, two elements \(z, z'\) in \(\bar{T}\) lie in the same \(G\)-orbit if and only if they lie in the same \(N(T)\)-orbit. Hence \(i\) is a bijection on points.

By \[\text{Ste65, section 6}\], the restriction of \(i\) to the open subvariety \(T/W\) of \(\bar{T}/W\) gives an isomorphism \(T/W \cong G/G\). Hence, as above, \(i\) is
a bijective, birational morphism of varieties whose target is a normal variety. So again by Zariski’s Main Theorem, $i$ is an isomorphism. □

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