A Practical Coding Scheme for the BSC with Feedback

Ke Wu* and Aaron B. Wagner†

*Computer Science Department, Carnegie Mellon University, Pittsburgh, PA 15213 USA. kew2@andrew.cmu.edu.
†School of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14850 USA. wagner@cornell.edu.

Abstract—We provide a practical implementation of the rubber method of Ahlswede et al. for binary alphabets. The idea is to create the “skeleton” sequence therein via an arithmetic decoder is nearly optimal in a strong sense for certain parameters.

We consider the binary symmetric channel with ideal feedback, both in its stochastic- and adversarial-noise forms. In the former, each bit is flipped independently with some probability p. In the latter, an omniscient adversary can flip up to a fraction ℓ of the bits in order to disrupt the communication.

The information-theoretic limits for both forms of the channel, assuming perfect feedback, are well-known. In the adversarial case, the capacity as a function of f was determined by Zigangirov [1], building on earlier results of Berlekamp [2]. For the stochastic version, the capacity equals that of the non-feedback version (e.g., [3], [4]) and likewise the high-rate error exponent, normal approximation, and moderate deviations performance are all unimproved by feedback. In fact, the third-order coding rate is unimproved by feedback [5], as is the order of the optimal “pre-factor” in front of the error exponent at high rates. Thus, at least for the stochastic version of the channel, feedback offers very little improvement in coding performance.

In general, feedback is known to simplify the coding problem even if it does not provide for improved performance. The erasure (e.g., [6] Section 17.1), and Gaussian channels [7], [8] provide striking examples of this phenomenon. For the BSC, see [9], [10] for classical and [11], [12] for recent work on devising implementable schemes using feedback.

For the adversarial symmetric channel with feedback (and arbitrary, finite alphabet size), Ahlswede et al. [13] proposed an explicit scheme called the rubber method. In the binary case, for a fixed ℓ > 2, the message is encoded as a “skeleton” string containing no substring of ℓ consecutive zeros. The encoder then transmits this string, sending ℓ consecutive zeros to indicate that an error has occurred. For each ℓ, this scheme achieves the capacity of the adversarial channel for a certain choice of f. This scheme simplifies significantly the original achievability argument of Berlekamp [2]. For ternary and larger alphabets, the scheme is even simpler. Rubber method has since been generalized [14]–[16].

We only consider the binary case in this paper, and we make two contributions. The first is to propose the use of arithmetic coding applied to a particular Markov chain in order to efficiently encode the message sequence into the corresponding skeleton string. This results in a practically-implementable end-to-end scheme, with only a negligible rate penalty. The second contribution is showing that, for each ℓ, there is a special rate R∗ ℓ and crossover probability pℓ such that the resulting scheme is optimal with respect to the second-order coding rate and moderate deviations performance for the channel with crossover probability p ℓ and error-exponent optimal at rate R∗ ℓ for all channels with crossover probability less than p ℓ. We also consider the third-order coding rate and the “pre-factor” of the error exponent of the scheme. These turn out to be nearly, but not exactly optimal. See Section V.

In Section II we introduce our notation and provide various preliminaries. In Section III and IV we describe our coding scheme. In Section V we present our main results.

II. Notation and Preliminaries

Capital letters such as X or Y denote random variables. We use x^n to denote the first n bits of the sequence x₁, x₂, ..., xₙ, and we use z∥z′ to denote the concatenation of two strings z and z′. In addition, [x^n]_L denotes the truncation of x^n to the first L bits.

We use Bin(n, p) to denote the binomial distribution with size n and success probability p and N(µ, σ²) to denote the normal distribution with mean µ and variance σ². Moreover, B(p) denotes the Bernoulli distribution with success probability p. We use D(P||Q) to denote the Kullback-Leibler divergence between distribution P and Q.

A. The Channel Model

Let BSC(p) denote a binary symmetric channel with crossover probability p ∈ (0, 1/2) without feedback. That is, BSC(p) has input alphabet $\mathcal{X} = \{0, 1\}$ and output alphabet $\mathcal{Y} = \{0, 1\}$, and probability transition matrix

$$p(y|x) = \begin{pmatrix} 1 - p & p \\ p & 1 - p \end{pmatrix}.$$ 

Suppose that an encoder wishes to send a message m in a message space M through BSC(p). It first encodes the message m using an encoding function f, and sends $x^n = f(m)$ through the channel. The decoder, upon receiving $y^N$ from the channel, runs a decoding function g on $y^N$ to obtain $m'$. The pair $(f, g)$ is called a code $C_{N, R}$ with block length N and rate
For the adversarial feedback BSC channel BSC capacity: \( m \times f \). He also gives a lower bound that coincides with the upper bound on the capacity was first shown by Berlekamp where \( h(p) = -p \log p - (1-p) \log (1-p) \) is the binary entropy function, and the log is base-2 throughout.

We will also consider the adversarial binary symmetric channel BSC adv \( f \) in which at most \( f \) fraction of transmitted bits can be adversarially flipped.

Feedback allows the encoder to see exactly what the decoder receives after each transmission and update its next transmission accordingly. In the BSC with feedback, which we denote \( \text{BSC}^{fb} \), the encoding function \( f \) consists of a sequence \( \{ f_i \}^N_{i=1} \). Each \( f_i \) takes as input \( m, y_1, \ldots, y_{i-1} \), and outputs \( x_i \), the next bit to send. The decoder then runs \( g(y^N) \) to obtain \( m' \).

It is well-known that feedback does not improve the channel capacity:

\[
C(\text{BSC}^{fb}(p)) = 1 - h(p).
\]

For the adversarial feedback BSC channel BSC adv \( f \), an upper bound on the capacity was first shown by Berlekamp \( [2] \). He also gives a lower bound that coincides with the upper bound when \( f \geq \frac{3-\sqrt{5}}{4} \). A lower bound that coincides with the upper bound for \( f < \frac{3-\sqrt{5}}{4} \) was given by Zigangirov \( [1] \), thus determining the capacity for BSC adv \( f \):

\[
C(\text{BSC}^{fb}(p)) = \begin{cases} 1 - h(f) & \text{if } 0 \leq f \leq \frac{3-\sqrt{5}}{4}, \\ (1 - 3f) \log \frac{1+\sqrt{5}}{2} & \text{if } \frac{3-\sqrt{5}}{4} < f \leq 1. \end{cases}
\]

We say that a code \( C \) for the BSC adv \( f \) is admissible if \( C \) can correct any error pattern with error fraction at most \( f \).

We say that a sequence of codes \( \{ C_{N,R} \}_N \) for BSC adv \( f \) is admissible if the error probability \( P_e(C_{N,R}) \) tends to 0 as \( N \) goes to infinity.

**Proposition 2.** For skeleton sequence set \( A^N \) and block length \( N \), a code constructed using the rubber method is admissible for BSC adv \( f \) if

\[
N' + (\ell + 1)f N \leq N.
\]

**Proof.** See Section 2.2 of \( [17] \).

**Example 3.** Suppose the encoder chooses \( x = 011010 \in A^6_2 \) and the maximum fraction of adversarial errors is \( f = 1/3 \). Suppose the first three bits the decoder receives are 010, which is not a prefix of \( x \). The encoder then sends 0 and suppose decoder sees 0100. The decoder then erases the last three bits (the continuous zeros and the one before them) and its stack becomes 0. This is now a prefix of \( x \) and the encoder would thus resend the second bit in \( x \), which is 1. See Figure 2.
D. Shannon–Fano–Elias Code and Arithmetic Coding

The Shannon–Fano–Elias code compresses a source sequence with known distribution to near-optimal length. It uses the cumulative distribution function \( F(x) \) to allot codewords. For a random variable \( X \in \{1, 2, \ldots, M\} \) with distribution \( p \), the codeword is \( \lfloor F(x) \rfloor \) where

\[
F(x) = \sum_{a<x} p(a) + \frac{1}{2} p(x),
\]

lies between \( F(x) \) and \( F(x+1) \) and \( l(x) = \lceil \log \frac{1}{x(x+1)} \rceil + 1 \). In addition, the Shannon-Fano-Elias code is prefix-free. That is, no codeword is a prefix of any other.

Arithmetic coding is an algorithm for efficiently computing the Shannon–Fano–Elias codeword for sequences given a method for computing the probability of the next symbol given the past (e.g., [18, Ch. 4]).

E. Constant Recursive Sequences and the Perron–Frobenius Theorem

**Lemma 4** (Theorem 2.3.6, [19]). A sequence \( A(n) \) is an order-\( d \) constant-recursive sequence if for all \( n \geq d + 1 \),

\[
A(n) = c_1 A(n-1) + c_2 A(n-2) + \cdots + c_d A(n-d).
\]

The \( n \)-th term \( A(n) \) in the sequence must be of the form

\[
A(n) = k_1(n) \lambda_1^n + k_2(n) \lambda_2^n + \cdots + k_d(n) \lambda_d^n,
\]

where \( \lambda_i \) is a root with multiplicity \( d_i \) of the polynomial

\[
\lambda^d - c_1 \lambda^{d-1} - \cdots - c_d,
\]

and \( k_i(n) \) is a polynomial with degree \( d_i - 1 \).

**Definition 5** ([9, 8.3.16], [20]). A matrix \( M \) is a positive (non-negative) matrix if every entry of \( M \) is positive (non-negative).

A non-negative square matrix \( M \) is primitive if its \( k \)-th power is positive for some natural number \( k \).

**Lemma 6** (Perron–Frobenius Theorem, Page 674, [20]). If \( M \) is a primitive matrix, then \( M \) has a positive real eigenvalue \( \lambda^* \) such that all other eigenvalues \( \lambda_i \) have absolute value \( |\lambda_i| < |\lambda^*| \). Moreover, \( \lambda^* \) is a simple eigenvalue and its corresponding column and row eigenvectors are positive.

See [20, Ch. 8] for further detail about the Perron-Frobenius Theorem.

III. A Key Markov Chain

In this section we show that we can efficiently compute the distribution of a Markov Chain that is uniformly distributed over \( A_N^\ell \).

Recall that \( A_N^\ell \) denotes the set of binary sequences of length \( N \) with no consecutive \( \ell \) zeros.

**Lemma 7.** Let \( \lambda^\ell \) be the unique real solution that lies in \((1,2)\) of

\[
\lambda = \lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + 1.
\]

Then \( \lim_{N\to\infty} \frac{|A_N^\ell|}{\lambda^\ell N} \) exists and is positive and finite.

**Proof.** We first compute the cardinality of \( A_N^\ell \). Let \( A_\ell(N) \) denote \( |A_N^\ell| \).

Consider all allowable sequences in \( A_N^\ell \). The number of sequences in \( A_N^\ell \) that begin with 1 is \( A_\ell(N-1) \). The number of sequences in \( A_N^\ell \) that begin with 01 is \( A_\ell(N-2) \), and so on. Continuing recursively we have that

\[
A_\ell(N) = A_\ell(N-1) + A_\ell(N-2) + \cdots + A_\ell(N-\ell).
\]

Let \( \lambda_1, \ldots, \lambda_{\ell} \) be the roots of equation (2), where \( \lambda_i \) has multiplicity \( d_i \). Note that \( \lambda^\ell = \lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + 1 \) is also the characteristic polynomial for the following \( \ell \times \ell \) non-negative matrix:

\[
M^\ell_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.
\]

Therefore \( \lambda_1, \ldots, \lambda_{\ell} \) are also the eigenvalues of \( M^\ell_N \). It is easy to see that equation (2) has exactly one positive real root that lies inside \((1,2)\) and no real root in \([2, +\infty)\). Without loss of generality, we assume that \( \lambda_1 \) is this root. Moreover, \( M^\ell_N \) is primitive since \( (M^\ell_N)^\ell \) is a positive matrix. According to Perron–Frobenius theorem, \( \lambda_1 \) is a simple root with multiplicity 1 of equation (2) and \( |\lambda_1| < |\lambda^\ell| \) for \( i = 2, \ldots, \ell \). Therefore,

\[
A_\ell(N) = k_1(n) \lambda_1^n + k_2(n) \lambda_2^n + \cdots + k_{\ell}(n) \lambda_{\ell}^n,
\]

where \( k_i(\cdot) \) is a polynomial with degree \( d_i - 1 \). Since \( \lambda_1 \) is a simple dominating root and its corresponding column and row eigenvectors are positive, according to Theorem 2.4.2 in [19] and Lemma 6,

\[
\lim_{N\to\infty} \frac{|A_N^\ell|}{\lambda_1^\ell N} = k_1 > 0.
\]

Note that Lemma 7 implies that

\[
\lim_{N\to\infty} \frac{1}{N} \log |A_N^\ell| = \log \lambda_1^\ell.
\]

**Lemma 8.** The stochastic process that is uniformly distributed over \( A_N^\ell \) is an \((\ell - 1)\)-th order Markov Chain.

**Proof.** Let \( z \) be any binary sequence. We abuse the notation slightly by defining \( A_\ell(z) \) to be the number of allowable sequences in \( A_N^\ell \) that begin with \( z \).

Suppose \( \{X_i\}_{i=1}^N \) is a stochastic process that is uniformly distributed over \( A_N^\ell \). Then we have

- \( \Pr[X_1 = 1] = \frac{\text{number of sequences begin with 1}}{|A_N^\ell|} = \frac{A_\ell(N-1)}{A_\ell(N)} \);
- \( \Pr[X_0] = 1 - \frac{A_\ell(N-1)}{A_\ell(N)} \);
- \( \Pr[X_i = 1 | X_1, \ldots, X_{i-1} = z] = \frac{A_\ell(N - i)}{A_\ell(z)} \);
- \( \Pr[X_i = 0 | X_1, \ldots, X_{i-1} = z] = 1 - \frac{A_\ell(N - i)}{A_\ell(z)} \).
To see that \( \{X_i\} \) is an \((\ell - 1)\)-th order Markov Chain, we only need to show that for any \( i \geq \ell, z \in \{0, 1\}^{i-1} \)
\[
\Pr[X_i = 1|X_1, \ldots, X_{i-1} = z] = \Pr[X_i = 1|X_{i-\ell+1}, \ldots, X_{i-1} = z[i-\ell+1, i-1]].
\]

Fix a \( z \in \{0, 1\}^{i-1} \) for \( i \geq \ell \). Suppose \( z \) ends with \( \alpha \) zeros. Then \( 0 \leq \alpha \leq \ell - 1 \) since the sequence is in \( A_\alpha^N \). We have that
\[
A_\ell(z) = A_\ell(N - i + \alpha + 1) - \alpha \sum_{k=0}^{\alpha-1} A_\ell(N - i + k + 1). \tag{4}
\]

When \( \alpha = 0 \), equation \((4)\) becomes \( A_\ell(z) = A_\ell(N - i + \alpha + 1) \). This indicates that for any \( z \) and \( z' \) that have the same last \( \ell - 1 \) bits, \( A_\ell(z) = A_\ell(z') \) and that \( \Pr[X_i = 1|X_1, \ldots, X_{i-1} = z] = \Pr[X_i = 1|X_1, \ldots, X_{i-1} = z'] \).

Then for any \( x \in \{0, 1\}^{\ell-1} \) and any \( x' \in \{0, 1\}^{i-\ell} \), we have
\[
\Pr[X_i = 1|X_{i-\ell+1}, \ldots, X_{i-1} = x] = \sum_{x'' \in \{0, 1\}^{i-\ell}} \{\Pr[X_i = 1|X_1, \ldots, X_{i-1} = x''|x]\}
\]
\[= \sum_{x'' \in \{0, 1\}^{i-\ell}} \{\Pr[X_i = 1|X_1, \ldots, X_{i-1} = x''] | x\}
\]
\[= \sum_{x'' \in \{0, 1\}^{i-\ell}} \{\Pr[X_i = 1|X_1, \ldots, X_{i-1} = x''] | x\}
\]
\[= \Pr[X_i = 1|X_1, \ldots, X_{i-1} = x''] | x],
\]
where the second equation comes from the fact that for any two sequences with the same last \( \ell - 1 \) bits, we have
\[
\Pr[X_i = 1|X_1, \ldots, X_{i-1} = x'' | x] = \Pr[X_i = 1|X_1, \ldots, X_{i-1} = x''] | x].
\]

The proof shows that to compute the probability of the next symbol in the string given the past, we only need to compute \( A_\ell(N) \) for various values of \( N \). This can be computed using
\[
A_\ell(N) = k_1 \lambda_1^N + k_2 \lambda_2^N + \cdots + k_\ell \lambda_\ell^N \text{ where } \lambda_i \text{ are the roots of equation } \frac{d}{dx} \sum_{i=1}^{\ell} c_i k_i(N) = 0 \text{ and } c_1, k_1(N), \ldots, k_\ell(N) \text{ can be determined by the initial conditions } A_\ell(1) = 2, \ldots, A_\ell(\ell - 1) = 2^{\ell-1}, A_\ell(\ell) = 2^\ell - 1.
\]

Note that when \( N \) is large, \( A_\ell(N) \) is well-approximated as \( A_\ell(N) \approx k_1 \lambda_1^N \). Under this approximation the Markov Chain becomes time-invariant.

Example 9. Consider the case \( \ell = 2 \). That is, we forbid two consecutive zeros in the skeleton sequence. Then the characteristic polynomial is \( x^2 - \lambda - 1 = 0 \). The two roots are \( \lambda_1 = 1 + \sqrt{5} \) and \( \lambda_2 = 1 - \sqrt{5} \) respectively. The initial condition is \( A_2(1) = 2, A_2(2) = 3 \). Therefore \( A_2(N) = k_1 \lambda_1^N + k_2 \lambda_2^N \) where \( k_1 = \frac{3 + \sqrt{5}}{2\sqrt{5}}, k_2 = \frac{3 - \sqrt{5}}{2\sqrt{5}} \). See also Ex. 4.7.

\[\]

IV. A PRACTICAL CODING SCHEME

In this section we combine arithmetic coding and the rubber method to give an efficient feedback code for BSC\(_{adv}(f)\) and BSC\(_{dB}(p)\). First we describe a modified version of arithmetic coding that will be used in our scheme. Consider the following pair of algorithms (Decom\(_\ell\), Com\(_\ell\)):

**Algorithm 10. (Decom\(_\ell\), Com\(_\ell\))**

Let \( L = \lceil \log |A_\ell^N| \rceil \). Let \( \{X_i\}_{i=1}^N \) be a stochastic process that is uniformly distributed over \( A_\ell^N \). Let \( (A_C, A_D) \) where \( A_C : A_\ell^N \rightarrow \{0, 1\}^{L+1} \) and \( A_D : \{0, 1\}^{L+1} \rightarrow A_\ell^N \) be the compression and decompression algorithms for arithmetic coding applied to \( \{X_i\}_{i=1}^N \), where the decompressor outputs \( \perp \) if its input is not a valid codeword. Let \( L' \) be any integer such that \( L' \leq L - 3 \).

\[
\text{Decom}_\ell(m) : \{0, 1\}^{L'} \rightarrow A_\ell^N
\]

1. Run the decompress algorithm \( A_D(m || m') \) for all possible \( m' \in \{0, 1\}^{L+1-L'} \). Let the first non-\( \perp \) output be \( A_D(m || m') = x_N \). If there’s no such \( x_N \), set \( x_N \) to be a random sequence in \( A_\ell^N \).

2. Output \( x_N \).

\[
\text{Com}_\ell(x_N) : A_\ell^N \rightarrow \{0, 1\}^{L'}:\]

1. Output \( |C(x_N) \| L' \).

**Lemma 11. The pair of algorithms (Decom\(_\ell\), Com\(_\ell\)) described in Algorithm 10 satisfies**

\[
\text{Com}_\ell(\text{Decom}_\ell(m)) = m, \forall m \in \{0, 1\}^{L'}.
\]

**Proof.** Suppose that all sequences in \( A_\ell^N \) are lexicographically sorted and \( x_N + 1 \) is the sequence following \( x_N \). Note that for some binary sequences of length \( L + 1 \), \( A_D \) might output \( \perp \) if the binary sequence is not a Shannon-Fano-Elias codeword for any \( x_N \in A_\ell^N \).

As long as there exists an \( m' \) such that \( A_D(m || m') \neq \perp \), \( \text{Com}_\ell(\text{Decom}_\ell(m)) = m \) due to the correctness of arithmetic coding. Therefore we only need to show that for any \( m \in \{0, 1\}^{L'} \), there exist \( m' \in \{0, 1\}^{L+1-L'} \) such that \( A_D(m || m') \neq \perp \).

We will prove that for any \( m \in \{0, 1\}^{L'} \), there must exist an \( x_N \in A_\ell^N \) such that \( m \) is a prefix of \( A_C(x_N) \).

To see this, let each sequence \( m \in \{0, 1\}^{L'} \) represent an interval of length \( \frac{1}{2^L} \) in \([0, 1]\) such that all of the real numbers inside the interval represented by \( m \) have prefix \( m \).

Note that \( A_C(x_N) \) falls between \( F(x_N) \) and \( F(x_N + 1) \), where \( F(\cdot) \) is the cumulative distribution function of \( \{X_i\}_{i=1}^N \).

As \( X_N \) is uniformly distributed over \( A_\ell^N \), for any \( x_N \), \( F(x_N) - F(x_N + 1) = \frac{1}{2^\ell} \leq \frac{1}{2^L} \). Therefore, for any \( m \), the interval represented by \( m \) with length \( \frac{1}{2^L} \) must contain both \( F(x_N) \) and \( F(x_N + 1) \) for at least one \( x_N \). This indicates that \( A_C(x_N) \in (F(x_N), F(x_N + 1)) \) must fall inside the interval represented by \( m \). That is, \( m \) must be a prefix of \( A_C(x_N) \).

Now we describe the construction of our overall scheme:
Construction 12. The encoding and decoding of $C_{\ell,N,R}$ are as follows:

**Encoding:**
- Let $m^{NR}$ be a message source of length $NR$. Find the minimum natural number $N'$ such that $\lfloor \log |A_{\ell}^{N'}| \rfloor \geq NR + 3$.
- Run $\text{Decom}_i(m)$ and denote the output as $x^{N'}$. Let $x^{N'}$ be the skeleton sequence and send it through the feedback channel using the rubber method.

**Decoding:**
- Let $y^N$ be the sequence received from the feedback channel. Run the decoding algorithm of the rubber method on $y$ to get $\tilde{x}^{N'}$. If $\tilde{x}^{N'} \not\in A_{\ell}^{N'}$, set $\tilde{x}^{N'}$ to be a random skeleton sequence in $A_{\ell}^{N'}$.
- Otherwise, output $m' = \text{Com}_i(\tilde{x}^{N'})$.

**Proposition 13.** The code $C_{\ell,N,R}$ in Construction 12 is admissible for the $\text{BSC}^{fb}(p)$ if $N' \leq (1 - (\ell + 1)p) \log \lambda_{\ell}^*$.

**Proof.** Follows directly from Proposition 2 and Lemma 11.

Note that in the first step of encoding, we can find $N'$ simply by computing $|A_{\ell}^{N'}|$ for $N = NR + 3, \ldots, 2NR + 6$ since $2^{\frac{N}{\ell}} \leq |A_{\ell}^{N'}| \leq 2^{N}$. See Lemma 25 in Appendix.

We further note that the above coding scheme also works for stochastic feedback channel $\text{BSC}^{fb}(p)$:

**Proposition 14.** The sequence of codes $\{C_{\ell,N,R}\}_N$ of which is constructed as in Construction 12 is admissible for the $\text{BSC}^{fb}(p)$ if $R < R_{\ell}(p) = (1 - (\ell + 1)p) \log \lambda_{\ell}^*$.

**Proof.** The fraction of errors that can be corrected by $C_{\ell,N,R}$ is

$$f_N = \frac{1}{\ell + 1} \left( 1 - \frac{N'}{N} \right).$$

When $N$ tends to infinity,

$$\lim_{N \to \infty} f_N = f^* = \frac{1}{\ell + 1} \left( 1 - \frac{R}{\log \lambda_{\ell}^*} \right).$$

according to Lemma 7.

If the fraction of errors is less than $f_N$, then $C_{\ell,N,R}$ can decode correctly. Let $E_i$ be the indicator of whether the $i$-th transmitted bit is flipped. The error probability of $C_{\ell,N,R}$ is thus

$$P_e(C_{\ell,N,R}) = \Pr \left[ \frac{1}{N} \sum_{i=1}^{N} E_i \geq f_N \right].$$

The result then follows by the law of large numbers.

**V. MAIN RESULTS**

We now show that, for certain parameters, our codes achieve the capacity and the optimal error-exponent, second-order rate, and moderate deviations constant for certain parameters.

**TABLE I:** Numerical results of $\log \lambda_{\ell}^*$, tangent points $p_{\ell}$ and tangent rates $R_{\ell}^*$

| $\ell$ | $\log \lambda_{\ell}^*$ | $p_{\ell}$ | $R_{\ell}^*$ |
|-------|-----------------|----------|-----------|
| 2     | 0.6942          | 0.1910   | 0.2965    |
| 3     | 0.8791          | 0.0804   | 0.5065    |
| 4     | 0.9468          | 0.0362   | 0.7754    |

**A. Capacity**

**Theorem 15.** For any integer $\ell \geq 2$, $R_{\ell}(p)$ is tangent to $C(\text{BSC}^{fb}(p))$. For $p_{\ell} = \frac{1}{1+2(\ell+1)\log \lambda_{\ell}^*}$,

$$R_{\ell}(p_{\ell}) = C(\text{BSC}^{fb}(p_{\ell})).$$

That is, for any $\epsilon > 0$, the sequence of codes $\{C_{\ell,N,R}\}_N$ as constructed in Construction 12 is admissible for $\text{BSC}^{fb}(p_{\ell})$ with $R = C(\text{BSC}^{fb}(p_{\ell})) - \epsilon$.

**Proof.** Note that according to Theorem 2 of [17], $R_{\ell}(p)$ is tangent to $C(\text{BSC}^{fb}(p))$. Moreover according to Section 3.6 of [17], when $p = p_{\ell}$, $R_{\ell}(p_{\ell}) = C(\text{BSC}^{fb}(p_{\ell}))$. The result then follows from Proposition 14.

We call $p_{\ell}$ the tangent points and $R_{\ell}^* = R_{\ell}(p_{\ell})$ the tangent rates. The tangent points $p_{\ell}$, tangent rates $R_{\ell}^*$, and log $\lambda_{\ell}^*$ values for different $\ell$ are listed in Table I.

**B. Error-exponent**

**Lemma 16.** (Sphere-packing bound with pre-factor [5], [21]). Let $\{C_{N,R}\}_N$ be a sequence of codes for the $\text{BSC}^{fb}(p)$, each with rate $R < C(\text{BSC}^{fb}(p))$. Let $q \in (0, \frac{1}{2})$ such that $R = 1 - h(q)$. Let $E_{sp}(R) = D(B(q)|B(p))$ and $E_{sp}'(R)$ be the slope of

**Fig. 3.** $R_{\ell}(p)$ for different $\ell$.

**A. Capacity**

**Theorem 15.** For any integer $\ell \geq 2$, $R_{\ell}(p)$ is tangent to $C(\text{BSC}^{fb}(p))$. For $p_{\ell} = \frac{1}{1+2(\ell+1)\log \lambda_{\ell}^*}$,

$$R_{\ell}(p_{\ell}) = C(\text{BSC}^{fb}(p_{\ell})).$$

That is, for any $\epsilon > 0$, the sequence of codes $\{C_{\ell,N,R}\}_N$ as constructed in Construction 12 is admissible for $\text{BSC}^{fb}(p_{\ell})$ with $R = C(\text{BSC}^{fb}(p_{\ell})) - \epsilon$.

**Proof.** Note that according to Theorem 2 of [17], $R_{\ell}(p)$ is tangent to $C(\text{BSC}^{fb}(p))$. Moreover according to Section 3.6 of [17], when $p = p_{\ell}$, $R_{\ell}(p_{\ell}) = C(\text{BSC}^{fb}(p_{\ell}))$. The result then follows from Proposition 14.

We call $p_{\ell}$ the tangent points and $R_{\ell}^* = R_{\ell}(p_{\ell})$ the tangent rates. The tangent points $p_{\ell}$, tangent rates $R_{\ell}^*$, and log $\lambda_{\ell}^*$ values for different $\ell$ are listed in Table I.

The function $R_{\ell}(p)$ for different $\ell$ is plotted in Figure 3. That the rubber method would achieve the capacity of the $\text{BSC}^{fb}(p_{\ell})$ is implicit in [17]. We consider three more-refined performance measures.
the error exponent at \( R \). Then the error probability \( P_e(C_{N,R}) \) satisfies
\[
P_e(C_{N,R}) \geq K_1 \frac{1}{N^{\frac{1}{2} (1 + E_{sp}(R))}} e^{-NE_{sp}(R)},
\]
where \( K_1 \) is a positive constant depending on \( R \).

**Theorem 17.** For any fixed \( \ell \geq 2 \), consider the sequence of codes \( \{C_{\ell,N,R}\}_N \) at the tangent rate \( R_\ell^* \). That is, \( R_\ell = R_\ell(p_\ell) = 1 - h(p_\ell) \). Then for the BSC\(^B\)(\( p \)) with \( p < p_\ell \), \( \{C_{\ell,N,R}\}_N \) at rate \( R_\ell^* \) achieves optimal error exponent
\[
P_e(C_{\ell,N,R}) \leq O \left( \frac{1}{\sqrt{N}} \right) e^{-NE_{sp}(R)}.
\]

In particular,
\[
\lim_{N \to \infty} -\frac{1}{N} \log P_e(C_{\ell,N,R}) = E_{sp}(R).
\]

**Remark 18.** The “pre-factor” order achieved by our scheme is \( O(1/\sqrt{N}) \), which is worse than the optimal order of \( O(1/\sqrt{\log N}) \) in Theorem 16. Interestingly, for the binary erasure channel (BEC), both with and without feedback, the optimal pre-factor is \( O(1/\sqrt{N}) \) [5, Theorem 2]. Rubber coding attempts to emulate a BEC using the BSC, which might explain this connection. A similar gap from strict optimality occurs in the second-order coding rate results to follow. Making the connection between rubber coding and the BEC more precise is an interesting topic for future study.

**Proof.** Let \( R_0 = \log \lambda_\ell^* \). Let \( f_N = \frac{1}{\ell+1} \left( 1 - \frac{N'}{N} \right) \) be the fraction of errors \( C_{\ell,N,R} \) can correct. Since \( |\log |A_{N'}^{\ell+1}| | \geq NR_N + 3 \), and \( |\log |A_{N'}^{\ell}| | < NR_N + 3 \), we have
\[
N' \leq \frac{R_\ell^*}{\log \lambda_\ell^*} + O \left( \frac{1}{N} \right).
\]

This indicates that
\[
f_N = \frac{1}{\ell+1} \left( 1 - \frac{N'}{N} \right) = p_\ell + O \left( \frac{1}{N} \right).
\]

If the number of errors is less than \( N f_N \), \( C_{\ell,N,R} \) can correctly decode the message. Define \( r_N = \frac{p_\ell}{f_N} \). Let \( E_i \) be the indicator random variable of whether the \( i \)-th bit is flipped. By Lemma 24, when \( N \) is large, the error probability \( P_e(C_{\ell,N,R}) \) satisfies
\[
P_e(C_{\ell,N,R}) = \Pr \left[ \sum_{i=1}^{N} E_i \geq N f_N \right]
\leq e^{-ND(B(f_N))} \left( a_N + o \left( \frac{1}{N} \right) \right),
\]
where
\[
a_N = \frac{1 - r_N^{(1-f_N)N+1}}{1 - r_N} \exp \left( \frac{(1-f_N)N}{2f_N(1-f_N)N} \right).
\]

Since \( D(B(\cdot))B(p(\cdot)) \) is continuous,
\[
D(B(f_N)) = D(B(p_\ell)) + O \left( \frac{1}{N} \right).
\]

Therefore,
\[
P_e(C_{\ell,N,R}) \leq O \left( \frac{1}{\sqrt{N}} \right) e^{-NE_{sp}(R)}. \]

\[
C - \frac{1}{\sqrt{N}} \sqrt{\frac{p(1-p)\log 2}{p}} - \frac{1}{\sqrt{N}} \sqrt{\frac{p(1-p)\log 2}{p}} \Phi^{-1}(1-\epsilon) - O \left( \frac{1}{N} \right).
\]

**C. Second-order Rate**

**Lemma 19** (Second-order coding rate: Theorem 15, [22]). Given a block length \( N \) and an \( \epsilon \) such that \( 0 < \epsilon < 1 \), the largest possible rate of a code for the BSC\(^B\)(\( p \)) with error probability less than or equal to \( \epsilon \) is
\[
C - \frac{1}{\sqrt{N}} \sqrt{\frac{p(1-p)\log 2}{p}} \Phi^{-1}(1-\epsilon) - O \left( \frac{1}{N} \right).
\]

**Remark 21.** Note the log \( N/N \) term is “missing” from the expansion in Theorem 20 [22]. See Remark 18.

**Proof.** Let
\[
R_N = C - \frac{1}{\sqrt{N}} \sqrt{\frac{p(1-p)\log 2}{p}} \Phi^{-1}(1-\epsilon) - \frac{c_0}{N},
\]
where \( c_0 \) is a positive constant which we will specify later. We now show that for sufficiently large \( N \), the error probability of \( C_{\ell,N,R} \) satisfies \( P_e(C_{\ell,N,R}) < \epsilon \).

Let \( \epsilon_N = N \frac{1}{\ell+1} \left( 1 - \frac{N'}{N} \right) \). Define the number of errors that \( C_{\ell,N,R} \) is capable of correcting. According to our construction, \( \log |A_{N'}^{\ell}| \geq NR_N + 3 \), and \( \log |A_{N'}^{\ell-1}| < NR_N + 3 \), we have
\[
N' \leq \frac{R_N}{\log \lambda_\ell^*} + \frac{c_1}{N} + O \left( \frac{1}{N} \right),
\]
where \( c_1 = 3 - \frac{2k_1}{\log \lambda_\ell^*} \). Therefore
\[
\epsilon_N \geq \frac{N}{\ell+1} \left( 1 - \frac{R}{\log \lambda_\ell^*} \right) - \frac{c_1}{\ell+1} - o(1).
\]

Let \( E_i \) be the random variable such that \( E_i = 1 \) if the \( i \)-th bit is flipped. Let \( \Psi_N \) be the c.d.f. of the binomial distribution \( \text{Bin}(N,p) \). According to Berry–Esseen theorem (Section 5, [23]), for any \( N \), for any \( x \),
where \( \Phi \) is the c.d.f. of standard Gaussian and \( \sigma = \sqrt{p(1-p)} \),
\[ c_2 = \frac{0.56p}{\sigma^2}. \]
Therefore
\[
P_e(C_{\ell,N,R_N}) \leq 1 - \Psi_N \left( \frac{N}{\ell + 1} \left( 1 - \frac{R_N}{\log \lambda^*_R} \right) - \frac{c_1}{\ell + 1} - o(1) \right)
\leq 1 - \Phi \left( \frac{1}{\sqrt{N}} \left( \frac{N}{\ell + 1} \left( 1 - \frac{R_N}{\log \lambda^*_R} \right) - \frac{c_1}{\ell + 1} - o(1) \right) + \frac{c_2}{\sqrt{N}} \right).
\]

Let \( R_0 = \log \lambda^*_R \). Note that
\[
\frac{1}{\sqrt{N}} \left( \frac{N}{\ell + 1} \left( 1 - \frac{R_N}{\log \lambda^*_R} \right) - \frac{c_1}{\ell + 1} - o(1) - Np \right)
= \frac{1}{\sqrt{N}} \frac{N}{\ell + 1} \left[ 1 - (\ell + 1)p - \frac{C}{R_0} \right]
+ \frac{\sigma}{\sqrt{N}} (\ell + 1)\Phi^{-1}(1 - \epsilon)
+ \frac{1}{\sqrt{N}} \frac{c_0}{(\ell + 1)R_0} - \frac{c_1}{(\ell + 1)} - o(1)
= \Phi^{-1}(1 - \epsilon) + \frac{1}{\sqrt{N}} \left( \frac{c_0}{(\ell + 1)R_0} - \frac{c_1}{(\ell + 1)} - o(1) \right),
\]
where the first equality comes from the fact that when \( p = p_\ell \),
\[ \log \frac{1-p_\ell}{p_\ell} = R_0(\ell + 1). \]
See Section 3.6 in [2]. The second equality comes from the fact that \( C = (1 - (\ell + 1)\ell)R_0 \).
Therefore
\[
P_e(C_{\ell,N,R_N}) \leq 1 - \Phi \left( \Phi^{-1}(1 - \epsilon) \right)
+ \frac{1}{\sqrt{N}} \left( \frac{c_0}{(\ell + 1)R_0} - \frac{c_1}{(\ell + 1)} - o(1) \right) + \frac{c_2}{\sqrt{N}}
= 1 - \left\{ \Phi(\Phi^{-1}(1 - \epsilon)) + \frac{1}{\sqrt{N}} \left( \frac{c_0}{(\ell + 1)R_0} - \frac{c_1}{(\ell + 1)} - o(1) \right) \right\}
+ \frac{c_2}{\sqrt{N}} + o(\frac{1}{\sqrt{N}}).
\]
For \( N \) large enough, \( o(\frac{1}{\sqrt{N}}) < \frac{1}{\sqrt{N}} \). By picking
\[ c_0 \geq \left( \frac{c_0}{\Phi(\Phi^{-1}(1 - \epsilon))} + \frac{c_1}{\ell + 1} \right)(\ell + 1)R_0, \]
we have that \( \Phi(\Phi^{-1}(1 - \epsilon)) \frac{1}{\sqrt{N}} \left( \frac{c_0}{(\ell + 1)R_0} - \frac{c_1}{(\ell + 1)} - o(\frac{1}{\sqrt{N}}) \right) \)
is positive eventually, which implies \( P_e(C_{\ell,N,R_N}) < \epsilon \).

\[ D. \text{ Moderate Deviations} \]

**Lemma 22** (Moderate deviations, Corollary 1, [24]). For any sequence of real numbers \( \varepsilon_N \), \( s.t. \varepsilon_N \to 0 \) as \( N \to \infty \) and \( \varepsilon_N \sqrt{N} \to \infty \) as \( N \to \infty \), for any sequence of codes \( \{C_{\ell,N,R_N} \}_{N} \) for the BSC\( f(\ell) \) such that \( R_N \geq C(\text{BSC} f(\ell)) - \varepsilon_N \), we have
\[
\liminf_{N \to \infty} \frac{1}{N \varepsilon_N^2} \log P_e(C_{\ell,N,R_N}) \geq - \frac{1}{2p(1-p) \log^2 \frac{1-p}{p}}.
\]

**Theorem 23.** Fix any \( \ell \geq 2 \). Let \( C \) be the capacity of the BSC\( f(\ell) \). For any sequence of real numbers \( \varepsilon_N \), \( s.t. \varepsilon_N \to 0 \) as \( N \to \infty \) and \( \varepsilon_N \sqrt{N} \to \infty \) as \( N \to \infty \), consider the sequence of codes \( \{C_{\ell,N,R_N} \}_{N} \) such that \( R_N = C - \varepsilon_N \). Let \( P_e(C_{\ell,N,R_N}) \) denote the average error probability of \( C_{\ell,N,R_N} \) over the BSC\( f(\ell) \). Then
\[
\lim_{N \to \infty} \frac{1}{N \varepsilon_N^2} \log P_e(C_{\ell,N,R_N}) = - \frac{1}{2p(1-p) \log^2 \frac{1-p}{p}}.
\]

**Proof.** Let \( R_0 = \log \lambda^*_R \). Note that \( C_{\ell,N,R_N} \) has rate \( R_N = C - \varepsilon_N \). The maximum fraction of errors it can correct is thus
\[
f_N = \frac{1}{\ell + 1} \left( 1 - \frac{N'}{N} \right) = \frac{1}{\ell + 1} \left( 1 - \frac{C - \varepsilon_N}{R_0} \right) + O \left( \frac{1}{N} \right)
= p_\ell + \varepsilon_N \frac{1}{(\ell + 1)R_0} + O \left( \frac{1}{N} \right).
\]
Let \( E_i \) be the indicator random variable of whether the \( i \)-th bit is flipped. Then the error probability \( P_e(C_{\ell,N,R_N}) \leq \Pr[\sum_{i=1}^N E_i \geq N f_N] \) and \( P_e(C_{\ell,N,R_N}) \geq \frac{1}{2} \Pr[\sum_{i=1}^N E_i \geq N f_N] \). Let \( \varepsilon'_N = (f_N - p_\ell)(\ell + 1)R_0 \). Then
\[
\lim_{N \to \infty} \frac{\varepsilon_N}{\varepsilon'_N} = \lim_{N \to \infty} \frac{(f_N - p_\ell - O(\frac{1}{N}))(\ell + 1)R_0}{(f_N - p_\ell)(\ell + 1)R_0} = 1,
\]
where the last step comes from the fact that \( \varepsilon_N = O(\frac{1}{\sqrt{N}}) \), \( f_N - p_\ell = \Omega(\frac{1}{\sqrt{N}}) \). Define \( Z_N = \frac{1}{N \varepsilon'_N^2} \sum_{i=1}^N (E_i - p) \). Then we have
\[
\lim_{N \to \infty} \frac{1}{N \varepsilon'_N^2} \log P_e(C_{\ell,N,R_N})
= \lim_{N \to \infty} \frac{1}{N \varepsilon'_N} \log \Pr \left[ \sum_{i=1}^N E_i \geq N f_N \right]
= \lim_{N \to \infty} \frac{1}{N \varepsilon'_N^2} \log \Pr \left[ Z_N \geq \frac{f_N - p_\ell}{\varepsilon'_N} \right]
= \lim_{N \to \infty} \frac{1}{N \varepsilon'_N^2} \log \Pr \left[ Z_N \geq \frac{1}{(\ell + 1)R_0} \right]
= \lim_{N \to \infty} \frac{\varepsilon'_N^2}{N \varepsilon'_N} \log \Pr \left[ Z_N \geq \frac{1}{(\ell + 1)R_0} \right]
= - \frac{1}{2p(1-p) \log^2 \frac{1-p}{p}}.
\]
where the last equation comes from Theorem 3.7.1 in [25] and the fact that when \( p = p_\ell \), \( (\ell + 1)R_0 = \log \frac{1-p}{p} \).
Therefore for any integer $f$, we have

$$\Pr \left( \sum_{i=1}^{N} Y_i = \left\lceil fN \right\rceil \right) \leq \Pr \left( \sum_{i=1}^{N} Y_i = \left\lceil fN \right\rceil + j \right) \left( \frac{p}{Nf} \right)^{\left\lceil fN \right\rceil + j} \left( 1 - \frac{p}{Nf} \right)^{N - \left\lceil fN \right\rceil - 1} \leq \Pr \left( \sum_{i=1}^{N} Y_i = \left\lceil fN \right\rceil + j \right) \left( \frac{p}{Nf} \right)^{\left\lceil fN \right\rceil} \left( 1 - \frac{p}{Nf} \right)^{N - \left\lceil fN \right\rceil - 1} = \frac{e^{-\frac{(\left\lceil fN \right\rceil + j)(\left\lfloor fN \right\rfloor - j)}{2Nf}}}{\sqrt{2\pi Nf}}.$$ 

Thus the last inequality comes from the fact that $\left\lceil fN \right\rceil$ is large. 

**Proof.** We follow Theorem 2 in [21]. For any fixed $N$, let $Y_1, \ldots, Y_N$ be $i.i.d.$ random variables with $Y_i \sim B(Y)$. For any integer $S \in \{0, N\}$, we have that

$$\Pr \left( \sum_{i=1}^{N} Y_i = S \right) = \sum_{i=1}^{N} \left( \begin{array}{c} N \\end{array} \right) p^i (1 - p)^{N-i} = \sum_{i=1}^{S} \left( \begin{array}{c} N \\end{array} \right) p^i (1 - p)^{N-i} + \sum_{i=1}^{N-S} \left( \begin{array}{c} N \\end{array} \right) p^i (1 - p)^{N-i}.$$

Plugging back we have

$$\Pr \left( \sum_{i=1}^{N} Y_i = \left\lceil fN \right\rceil \right) \leq \Pr \left( \sum_{i=1}^{N} Y_i = \left\lceil fN \right\rceil + j \right) \left( \frac{p}{Nf} \right)^{\left\lceil fN \right\rceil + j} \left( 1 - \frac{p}{Nf} \right)^{N - \left\lceil fN \right\rceil - 1} \leq \frac{e^{-\frac{(\left\lceil fN \right\rceil + j)(\left\lfloor fN \right\rceil - j)}{2Nf}}}{\sqrt{2\pi Nf}}.$$ 

where

$$\left\lceil fN \right\rceil = \text{the least integer } \geq fN.$$
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