Non-Einstein relative Yamabe metrics

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Abstract

In this paper, we give a sufficient condition for a positive constant scalar curvature metric on a manifold with boundary to be a relative Yamabe metric, which is a natural relative version of the classical Yamabe metric. We also give examples of non-Einstein relative Yamabe metrics with positive scalar curvature.

1 Introduction

Let $M$ be a compact connected smooth manifold of dimension $n \geq 3$. Let $\mathcal{M}$ be the space of all Riemannian metrics on $M$ and $\mathcal{C}(M)$ the set of all conformal classes on $M$. We consider the normalized Einstein-Hilbert functional $\mathcal{E}$ on the space $\mathcal{M}$:

$$
\mathcal{E} : \mathcal{M} \to \mathbb{R}, \quad g \mapsto \mathcal{E}(g) := \frac{\int_M R_g dv_g}{\text{Vol}_g(M)^\frac{n-2}{n}},
$$

where $R_g, dv_g, \text{Vol}_g(M)$ denote respectively the scalar curvature of $g$, the volume measure of $g$ and the volume of $(M,g)$. In the case of $\partial M = \emptyset$, for $C \in \mathcal{C}(M)$, we define a number $Y(M,C) := \inf_{g \in C} \mathcal{E}(g)$ and it is called the Yamabe constant of $C$. And a metric $g \in C$ which achieves this infimum is called a Yamabe metric and has constant scalar curvature. Yamabe, Trudinger, Aubin and Schoen have proved that any conformal class $C$ contains a Yamabe metric. Conversely, a metric $g$ with constant scalar curvature is a Yamabe metric if either $R_g \leq 0$ or $g$ is an Einstein metric ([8], [9]). On the other hand, Kato [10] gave a sufficient condition for a metric to be a Yamabe metric and examples of non-Einstein Yamabe metrics with positive scalar curvature.

For the case of $\partial M \neq \emptyset$, Yamabe metrics under minimal boundary condition have been studied (cf. [1], [3], [7]). For $\bar{C} \in \mathcal{C}(M)$, we set $\bar{C}_0 := \{ g \in \bar{C} \mid H_g = 0 \text{ on } \partial M \}$, that is, the space of all relative metrics in $\bar{C}$, where $H_g$ denotes the mean curvature of $\partial M$ in $(M,g)$. A relative metric $g \in \bar{C}_0$ is called a relative Yamabe metric if $g$ is a minimizer of $\mathcal{E}|_{\bar{C}_0}$. The infimum of $\mathcal{E}|_{\bar{C}_0}$ is called the relative Yamabe constant of $C$, which is denoted by $Y(M,\partial M,C)$. Like the case of $\partial M = \emptyset$, a relative metric $g$ with constant scalar curvature is a

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relative Yamabe metric if \( R_g \leq 0 \). Moreover, it is known that some Obata-type theorems hold under a suitable boundary condition (see [2], [6] and Section 2 in this paper).

Our main result of this paper is a relative version of Kato’s result in [10]. More precisely, we will give the following sufficient condition for a relative metric of constant scalar curvature to be a relative Yamabe metric, and examples of non-Einstein relative Yamabe metrics with positive scalar curvature. Our main result is the following:

**Theorem 1.1.** Let \( g \) be a relative Yamabe metric on a compact connected smooth manifold \( M \) of dimension \( n \geq 3 \) with non-empty smooth boundary \( \partial M \) with \( R_g > 0 \) on \( M \). Assume that \( h \) is a relative metric on \( M \) with constant scalar curvature and that \( \varphi \) is a diffeomorphism of \( M \) such that \( dv_{\varphi^* h} = \gamma dv_g \) for some positive constant \( \gamma \). If

\[
R_h h \leq R_g g, \tag{1}
\]

then \( h \) is also a relative Yamabe metric. Moreover, if

\[
R_h h < R_g g, \tag{2}
\]

then \( h \) is a unique relative Yamabe metric (up to positive constant) in the relative conformal class \([ h ]_0\) of \( h \). Here, \([ h ]_0 := \{ g \in [ h ] \mid H_g = 0 \text{ on } \partial M \} = \{ u \frac{\Delta}{\Delta + 2} h \mid u \in C^\infty(M), \nu_h(u) = 0 \text{ on } \partial M \}\), where \( \nu_h \) denotes the inward unit normal vector field of \( \partial M \) with respect to \( h \) in \( M \).

This paper organized as follows. In Section 2, we recall some background materials and prove Theorem 1.1. In Section 3, we give some examples of non-Einstein relative Yamabe metrics.

## 2 Backgrounds and the proof of Theorem 1.1

Let \( M \) be a compact connected smooth manifold of dimension \( n \geq 3 \) with non-empty smooth boundary \( \partial M \). As pointed out in [1, Lemma 2.1], the relative Yamabe constant \( Y(M, \partial M; \bar{C}) \) is characterized as follows:

**Proposition 2.1** ([1, Lemma 2.1]). For any fixed \( g \in \bar{C}_0 \),

\[
Y(M, \partial M; \bar{C}) = \inf_{h \in \bar{C}_0} \mathcal{E}(h) = \inf_{u \in C^\infty_c(M), \nu_g(u)|_{\partial M} = 0} \mathcal{E}(u \frac{\Delta}{\Delta + 2} g)
\]

\[
= \inf_{u \in C^\infty_c(M), \nu_g(u)|_{\partial M} = 0} \frac{\int_M \left( \frac{4(n-1)}{n-2} |du|^2_g + R_g u^2 \right) dv_g}{\left( \int_M u \frac{2n}{n-2} dv_g \right)^{\frac{n-2}{n}}}
\]

\[
= \inf_{f \in L^{1,2}(M), f \not\equiv 0} \frac{\int_M \left( \frac{4(n-1)}{n-2} |df|^2_g + R_g f^2 \right) dv_g}{\left( \int_M |f| \frac{2n}{n-2} dv_g \right)^{\frac{n-2}{n}}},
\]
where $L^{1,2}(M)$ denotes the Sobolev space of square-integrable functions on $M$ up to their first weak derivatives (see [4] for example).

Remark 2.1. From this proposition, the relative Yamabe constant $Y(M, \partial M ; \bar{C})$ coincides with the conformal invariant $Q(M) = Q(M, \bar{C})$ (up to the positive factor $\frac{4(n-1)}{n-2}$) defined by Escobar [7].

Each relative Yamabe metric $g \in \bar{C}_0$ has constant scalar curvature $R_g = Y(M, \partial M ; \bar{C}) \cdot \text{Vol}_{g}(M)^{-2/n}$ with $H_g = 0$ on $\partial M$. Conversely, like the case of closed manifolds, a relative metric $g$ with $R_g = \text{const}$ is a relative Yamabe metric if $R_g \leq 0$. In the case of closed manifolds, it is also known that a metric with constant scalar curvature is a Yamabe metric if it is an Einstein metric (see [8, 9]). On the other hand, in the case of $\partial M \neq \emptyset$, there is an Obata-type theorem for manifolds with totally geodesic boundary by Escobar as follows:

**Theorem 2.1 ([3 Theorems 3.2, 4.1]).** Let $g$ be an Einstein metric of positive scalar curvature on a compact $n$-manifold $M$ with totally geodesic boundary. Then, for any constant scalar curvature relative metric $h \in [g]_0$, the following uniqueness result holds:

1. If $(M, [g])$ is conformally equivalent to $(S^n_+, [g_S])$, then there exist a homothety $\Phi : (S^n_+, g_S) \to (M, [g])$ and a conformal transformation $\varphi \in \text{Conf}(S^n_+, [g_S])$ such that $\Phi^* h = \varphi^* (\Phi^* g)$. Here, $g_S$ denotes the standard metric on $S^n_+$.

2. If $(M, [g])$ is not conformally equivalent to $(S^n_+, [g_S])$, then, up to homothety, $h = g$.

On the other hand, in [2], Akutagawa gave a different rigidity theorem as follows (which is released from the assumption that the boundary is totally geodesic):

**Theorem 2.2 ([2 Theorem 1.1]).** Let $\bar{g}$ be a relative positive Einstein metric on a compact $n$-manifold $M$ with boundary. Then, for any relative Einstein metric $\bar{g} \in [\bar{g}]_0$, the following holds:

1. Assume that $\bar{g}$ is a metric of positive constant curvature, and set $g := \bar{g}|_{\partial M}$.

   1.1 If $(\partial M, [g])$ is conformally equivalent to $(S^{n-1}_+, [g_{S^{n-1}}])$, then there exist a homothety $\Phi : (S^n_+, g_S) \to (M, [\bar{g}])$ and a conformal transformation $\phi \in \text{Conf}(S^n_+, [g_S])$ such that $\Phi^* \bar{g} = \phi^* (\Phi^* \bar{g})$.

   1.2 If $(\partial M, [g])$ is not conformally equivalent to $(S^{n-1}_+, [g_{S^{n-1}}])$, then up to rescaling, $\bar{g} = \bar{g}$.

2. If $\bar{g}$ is not a metric of positive constant curvature, then, up to rescaling, $\bar{g} = \bar{g}$.

For further details on the relative Yamabe problem, refer to [1], [2], [3], [6] or [7].

In [2], Akutagawa also gave an example of a metric which is not relative Einstein but relative Yamabe (see [2 Countereexample] or Section 3 of this paper). On the other hand, in Section 3, we will also give another examples of such metrics. In the following, we will give the proof of Theorem 1.1 which is similar to the proof of [10 Theorem].
Proof of Theorem 1.1: Since the scalar curvature is preserved under the pull-back action of diffeomorphisms, it is enough to consider the case that \( \varphi = \text{id}_M \).

Let \( \bar{h} = u^{\frac{4}{n-2}} h \in [h] \), \( u \in C^\infty(M) \) with \( \nu_h(u)|_{\partial M} = 0 \). From Proposition 2.1, it is enough to prove for this situation. Then

\[
E(\bar{h}) = \frac{\int_M R_{\bar{h}} dv_{\bar{h}}}{\text{Vol}_{\bar{h}}(M)^{\frac{n-2}{n}}} = \frac{\int_M u^{-\frac{n+2}{n-2}} \left( -\frac{4(n-1)}{n-2} \Delta_h u + R_h u \right) u^{\frac{2n}{n-2}} dv_{\bar{h}}}{\left( \int_M u^{\frac{2n}{n-2}} dv_{\bar{h}} \right)^{\frac{n-2}{n}}},
\]

where \(-\Delta_h\) denotes the non-negative Laplacian with respect to \( h \). Thus, using integration by parts and \( \nu_h(u)|_{\partial M} = 0 \), we obtain

\[
E(\bar{h}) = \frac{\int_M \left( -\frac{4(n-1)}{n-2} u \Delta_h u + R_h u^2 \right) dv_{\bar{h}}}{\left( \int_M u^{\frac{2n}{n-2}} dv_{\bar{h}} \right)^{\frac{n-2}{n}}},
\]

Using \( \frac{R_h}{R_g} |\nabla u|_g^2 \leq |\nabla u|_h^2 \) (from the assumption (1) in Theorem 1.1), we obtain

\[
E(\bar{h}) = \frac{\int_M \left( -\frac{4(n-1)}{n-2} |\nabla u|^2_h + R_h u^2 \right) \gamma dv_g}{\left( \int_M u^{\frac{2n}{n-2}} \gamma dv_g \right)^{\frac{n-2}{n}}},
\]

\[
\geq \gamma^{1-\frac{n-2}{n-2}} \frac{R_h}{R_g} \int_M \left( -\frac{4(n-1)}{n-2} |\nabla u|^2_g + R_g u^2 \right) dv_g \left( \int_M u^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}
\]

\[
= \gamma^{1-\frac{n-2}{n-2}} \frac{R_h}{R_g} E(u^{\frac{4}{n-2}} g).
\]

Since \( g \) is a relative Yamabe metric, \( R_g > 0 \) and from (1),

\[
E(\bar{h}) \geq \gamma^{1-\frac{n-2}{n-2}} \frac{R_h}{R_g} E(g) \tag{3}
\]

\[
= \gamma^{1-\frac{n-2}{n-2}} \frac{R_h}{R_g} \int_M R_g dv_g \left( \int_M dv_g \right)^{\frac{n-2}{n}} = \frac{\int_M R_h dv_g}{\left( \int_M dv_g \right)^{\frac{n-2}{n}}} = \frac{\int_M R_h dv_h}{\left( \int_M dv_h \right)^{\frac{n-2}{n}}}
\]

\[
= E(h).
\]
Therefore, by the definition of the relative Yamabe metrics, $h$ is a relative Yamabe metric on $M$.

In the above argument, if we assume that $\bar{h}$ is also a relative Yamabe metric, then it must be $\mathcal{E}(\bar{h}) = \mathcal{E}(h)$. In particular, the inequality in (3) must be an equality. Hence, if we assume (2) in Theorem 1.1 in addition, then $\nabla u \equiv 0$ on $M$. Therefore, $u \equiv \text{const}$ on $M$ since $M$ is connected. This means that $h$ is a unique relative Yamabe metric up to positive constant in $[h]_0$. \qed

We have a corollary for Theorem 1.1.

**Corollary 2.1.** Let $M$ be the one as in Theorem 1.1 and $\{g_t \mid T \leq t \leq T'\}$ ($T < T'$) a smooth variation in $\mathcal{M}$ satisfying the following conditions:

1. $R_{g_t} = \text{const}$ on $M$ and $H_{g_t} = 0$ on $\partial M$ for all $t \in [T, T']$,
2. $g_T$ is a relative Yamabe metric,
3. $R_{g_t} > 0$ on $M$ for all $t \in (T, T')$,
4. $R_{g_{T'}} = 0$ on $M$.

Then there exists a positive constant $\delta > 0$ such that $g_t$ is also a relative Yamabe metric for every $t \in [T' - \delta, T')$.

**Proof.** From [5, THÉORÈME], there exists a family $\{\varphi_t \mid T \leq t \leq T'\}$ of diffeomorphisms of $M$ with $\varphi_t(\partial M) = \partial M$ such that $dv_{g_t}^\ast = \gamma_t dv_{g_T}$ for some $\gamma_t \in \mathbb{R}$ which is smooth with respect to $t$. By the assumptions of the Corollary, $g_t$ satisfies the condition (1) in the Theorem 1.1 when $t$ is sufficiently close to $T'$. Hence, we can apply Theorem 1.1 to such $g_t$ and then obtain Corollary 2.1. \qed

### 3 Examples of non-Einstein relative Yamabe metrics

In this section, we shall give some examples of non-Einstein relative Yamabe metrics.

(1)(cf. [10, Example 2]) We consider the Berger sphere:

$$S^3(1) \cong SU(2) = \left\{ A \in M(2; \mathbb{C}) \mid \det A = 1, \ A^* = A^{-1} \right\} = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \mid (z, w) \in \mathbb{C}^2, |z|^2 + |w|^2 = 1 \right\},$$

and set

$$X_1 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, X_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$ 

Then, this is a left invariant orthonormal frame with respect to the standard metric on $SU(2)$. And we define a left invariant metric $g_{s,t}$ ($1 \leq s \leq t$) on $SU(2)$ so that

$$g_{s,t}(X_1, X_1) = 1, \ g_{s,t}(X_2, X_2) = s, \ g_{s,t}(X_3, X_3) = t, \ g_{s,t}(X_i, X_j) = 0 \ (i \neq j).$$
Then, \( \{ X_1, \frac{1}{\sqrt{s}} X_2, \frac{1}{\sqrt{t}} X_3 \} \) is an orthonormal basis of \( SU(2) \) with respect to \( g_{s,t} \).
Using this basis, we can derive the scalar curvature of \( g_{s,t} \) as
\[
R_{g_{s,t}} = \frac{2}{st} \{ 2(s + t + st) - (1 + s^2 + t^2) \}.
\]
And we define a subspace \( SU(2)_+ \) of \( SU(2) \) so that
\[
SU(2)_+ := \left\{ \left( \frac{z}{w}, -\frac{z}{\bar{w}} \right) \mid \text{Im} z \geq 0 \right\}.
\]
Then, \( (SU(2)_+, g_{s,t} |_{SU(2)_+}) \) is a Riemannian manifold with boundary \( \partial SU(2)_+ \):
\[
\partial SU(2)_+ = \left\{ \left( \frac{z}{w}, -\frac{z}{\bar{w}} \right) \mid \text{Im} z = 0 \right\}.
\]
Then, \( X_1 \) forms a left invariant unit normal vector field of the boundary \( \partial SU(2)_+ \) and it is minimal with respect to \( g_{s,t} \) (that is, \( H_{g_{s,t}} = 0 \) on \( \partial SU(2)_+ \)). Hence, from Theorem 1.1, \( g_{s,t} \) is a relative Yamabe metric with constant positive scalar curvature, if \( t \geq s + \sqrt{s + 1} \) (since \( g_{1,1} \) is the standard metric, therefore it is a relative Yamabe metric).

On the other hand, the Ricci curvature of \( g_{s,t} \) can be calculated as follows:
\[
\begin{align*}
\text{Ric}_{g_{s,t}} (X_1, X_1) &= -\frac{1}{st} (-2 + 2t^2 + 2s^2 - 4st), \\
\text{Ric}_{g_{s,t}} \left( \frac{X_2}{\sqrt{s}}, \frac{X_3}{\sqrt{s}} \right) &= -\frac{1}{st} (2 + 2t^2 - 2s^2 - 4t), \\
\text{Ric}_{g_{s,t}} \left( \frac{X_3}{\sqrt{t}}, \frac{X_3}{\sqrt{t}} \right) &= -\frac{1}{st} (2 - 2t^2 + 2s^2 - 4s), \\
\text{Ric}_{g_{s,t}} (X_i, X_j) &= 0 \ (i \neq j).
\end{align*}
\]
Hence, \( g_{s,t} \) is an Einstein metric if and only if \( s = t = 1 \). Consequently, \( g_{s,t} \) is a non-Einstein relative Yamabe metric with positive scalar curvature on \( SU(2)_+ \) if \( t \geq s + \sqrt{s + 1} \).

(cf. [2] Counterexample]) Consider the Clifford torus \( \Phi(T^2) \) :
\[
\Phi : T^2 = S^1 \times S^1 \to S^3(1) \subset \mathbb{C}^2, \quad (\theta, \phi) \mapsto \frac{1}{\sqrt{2}} (e^{i\sqrt{-1} \theta}, e^{i\sqrt{-1} \phi}) \ (0 \leq \theta, \phi \leq 2\pi).
\]
Set
\[
V_1 \sqcup V_2 = S^3(1) - \Phi(T^2),
\]
and let \( g_S \) be the round metric of constant curvature one on the 3-sphere \( S^3 \). Let \( \tilde{V}_i \ (i = 1, 2) \) be a solid torus with minimal boundary \( \partial \tilde{V}_i = \Phi(T^2) \). Then
$\bar{g} := g_s|_{\bar{V}_1}$ is a metric of constant curvature one on $\bar{V}_1$, and thus it is a relative Einstein metric. And, there exists a relative Yamabe metric $\hat{g} \in [\bar{g}]_0$ such that

$$\mathcal{E}(\hat{g}) = Y(\bar{V}_1, \partial \bar{V}_1, [\hat{g}]) < Y(S^3_+, S^2, [g_s]) = \mathcal{E}(\bar{g})$$

(see [2, Counterexample] for more detail). Hence we have $\hat{g} \neq \bar{g}$, and from Theorem 2.2 (1.2), such $\hat{g}$ is not an Einstein metric.

**Remark 3.1.** The boundary $\partial SU(2)_+$ in the above example (1) is not totally geodesic with respect to $g_{s,t}$. On the other hand, Escobar shown the rigidity Theorem 2.1 ([6, Theorems 3.2, 4.1]) as mentioned above. But, the same statement does not hold in general for Riemannian manifolds with minimal boundary. In fact, there are some examples of metrics which are not relative Einstein, not conformally equivalent to the standard hemisphere but has constant scalar curvature (see [2, Counterexample], [6, p. 875]). These are counterexamples to (2) in Theorem 2.1.

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