Non-Bloch band theory of non-Hermitian Hamiltonians in the symplectic class

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Non-Hermitian Hamiltonians are generally sensitive to boundary conditions, and their spectra and wave functions under open boundary conditions are not necessarily predicted by the Bloch band theory for periodic boundary conditions. To elucidate such a non-Bloch feature, recent works have developed a non-Bloch band theory that works even under arbitrary boundary conditions. Here, it is demonstrated that the standard non-Bloch band theory breaks down in the symplectic class, in which non-Hermitian Hamiltonians exhibit Kramers degeneracy because of reciprocity. Instead, a modified non-Bloch band theory for the symplectic class is developed in a general manner, as well as an illustrative example. This nonstandard non-Bloch band theory underlies the $Z_2$ non-Hermitian skin effect protected by reciprocity.

I. INTRODUCTION

Topology plays a central role in contemporary physics. In particular, it describes a variety of phases of matter that cannot be described by spontaneous symmetry breaking [1, 2]. Such topological phases are ubiquitous in insulators [3, 5] and superconductors [6, 7], as well as semimetals [8], all of which are classified according to symmetry [9–11]. A signature of topology manifests itself as the bulk-boundary correspondence: nontrivial bulk topology of Bloch Hamiltonians results in the emergence of anomalous boundary states. For example, zero modes appear at the two ends of one-dimensional systems [3, 7], of anomalous boundary states. For example, zero modes appear at the two ends of one-dimensional systems [3, 7].

An important consequence of this symmetry is Kramers degeneracy, which ensures the $Z_2$ topological phase and their anomalous boundary modes are protected by symmetry. As a prime example, topological phases in quantum spin Hall insulators [4] are protected by time-reversal symmetry (reciprocity). Time-reversal-invariant (reciprocal) Hamiltonians $H$ respect

$$\mathcal{T} H^\mathcal{T} T^{-1} = H, \quad \mathcal{T} T^* = -1$$

with a unitary matrix $\mathcal{T}$ (i.e., $\mathcal{T} \mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{T} = 1$), and are defined to belong to the symplectic class (class AII). An important consequence of this symmetry is Kramers degeneracy, which ensures the $Z_2$ topological phase and the helical edge states [5].

Despite the enormous success, the existing framework of topological phases was confined to Hermitian systems. Nevertheless, non-Hermiticity appears, for example, in a variety of nonequilibrium open systems as a consequence of nonconservation of energy or particles [12–14]. To understand the role of topology in non-Hermitian systems, topological characterization of non-Hermitian systems [14–15] has recently been developed both in theory [16–58] and experiments [59–70]. On the basis of the 38-fold internal symmetry in non-Hermitian physics [35–71], topological classification of non-Hermitian systems was established [23, 24, 35–39, 89], which predicts a number of non-Hermitian topological phases that have no analog in Hermitian systems.

Furthermore, non-Hermiticity is found to alter the nature of the bulk-boundary correspondence [20]. This breakdown arises from the extreme sensitivity of non-Hermitian systems to boundary conditions, which is called the non-Hermitian skin effect [25–27]. In fact, spectra and wave functions of non-Hermitian systems under open boundary conditions can be strikingly different from those under periodic boundary conditions, only the latter of which are predicted by the Bloch band theory. To elucidate such a non-Bloch feature of non-Hermitian systems, recent works have developed a non-Bloch band theory that works even under arbitrary boundary conditions [25, 27, 30–34, 40, 53, 54]. Reference [20] numerically investigated a non-Hermitian extension of the Su-Schrieffer-Heeger model [3] with asymmetric hopping, which is a prototypical example that exhibits the non-Hermitian skin effect. Providing the exact solution to this model, Ref. [25] showed a clear understanding about the non-Hermitian skin effect and the non-Bloch bulk-boundary correspondence. Reference [40] further generalized this result and gave a non-Bloch band theory in a general manner, which is summarized as follows:

Suppose $H(\beta)$ denotes a bulk Hamiltonian in one dimension with $\beta = e^{ik}$ and complex-valued wavenumbers $k \in \mathbb{C}$. Moreover, $\beta_i$’s ($i = 1, 2, \cdots, 2M$; $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_{2M}|$) denote the solutions to the characteristic equation $\det [H(\beta) - E] = 0$ in terms of $\beta$ for the given eigenenergy $E \in \mathbb{C}$. Then, continuum bands are formed by $H(\beta)$ with the trajectory of $\beta_M$ and $\beta_{M+1}$ satisfying

$$|\beta_M| = |\beta_{M+1}|.$$
In Hermitian systems, we always have $|\beta| = 1$ and all the eigenstates for continuum bands are delocalized through the bulk. In the non-Hermitian case, by contrast, this is not always the case, and an enormous number of localized states can appear, which is a signature of the skin effect. The non-Bloch band theory correctly describes a number of non-Hermitian systems and their topology. Recent experimental observations confirmed it in mechanical metamaterials [55], electrical circuits [59], and quantum walk [70]. It may bring about phenomena and functionalities unique to non-Hermitian systems, some of which were recently explored [24, 29, 43, 48, 55]. However, the validity of the non-Bloch band theory has been unclear in the presence of symmetry.

In this work, although the standard non-Bloch band theory [25, 43] is applicable to generic non-Hermitian systems without symmetry, we demonstrate its breakdown in the symplectic class. For non-Hermitian Hamiltonians, the symplectic class (class AI′) in Ref. [55] is defined by the symmetry given by Eq. (1). Because of this symmetry, Hamiltonians exhibit Kramers degeneracy even in non-Hermitian systems, leading to the breakdown of the standard non-Bloch band theory. Instead, we generally provide a modified condition for continuum bands in the symplectic class, summarized as follows:

When non-Hermitian Hamiltonians respect reciprocity in Eq. (1) and belong to the symplectic class, the solutions to the characteristic equation $\det [H - \beta E] = 0$ are generally denoted as

$$\beta_1, \beta_2, \ldots, \beta_{2M}; \beta_{2M}^{-1}, \beta_{2M-1}, \ldots, \beta_1^{-1}$$  \hspace{1cm} (3)

with $|\beta_1| \leq \cdots \leq |\beta_{2M}| \leq 1 \leq |\beta_{2M}^{-1}| \leq \cdots \leq |\beta_1^{-1}|$. Here, $\beta_1$ and $\beta_1^{-1}$ form a Kramers pair. Then, the condition for continuum bands is given as

$$|\beta_{2M-1}| = |\beta_{2M}|.$$  \hspace{1cm} (4)

Remarkably, the standard non-Bloch band theory [25, 43] predicts Eq. (2), i.e., $|\beta_{2M}| = |\beta_{2M}^{-1}|$, for continuum bands, but this is not the case in the symplectic class. The condition (2) intuitively implies the interference between the non-Bloch waves with $\beta_{2M}$ and $\beta_{2M+1}^{-1}$. In the symplectic class, however, the non-Bloch waves with $\beta_{2M}$ and $\beta_{2M}^{-1}$ cannot interfere with each other since they form a Kramers pair; instead, the non-Bloch waves with $\beta_{2M-1}$ and $\beta_{2M}$ interfere, replacing the condition (2) with the condition (4). This nonstandard non-Bloch band theory underlies a new type of non-Hermitian skin effects protected by reciprocity [25, 43].

More precisely, the conditions (2) and (4) are derived from boundary conditions. In the standard (symplectic) case, boundary conditions impose a constraint on $\beta_i$’s, which forms an $M$-th-order (a $2M$-th-order) algebraic equation in terms of $\beta_1, \beta_2, \ldots, \beta_M$ (and $\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_M^{-1}$) with the system size $L$ [see Eq. (60) for the symplectic case]. Because of the assumption $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_{2M}|$, the leading-order term includes $(\beta_{2M+1} \beta_{2M+2} \cdots \beta_{2M})^L$ and the next-to-leading-order term includes $(\beta_{2M} \beta_{2M+2} \cdots \beta_{2M})^L$, in general. To respect the constraint, these two terms should be comparable to each other for $L \to \infty$, which leads to Eq. (3). In the symplectic class, by contrast, reciprocity forbids the appearance of the term proportional to $(\beta_{2M} \beta_{2M-1} \cdots \beta_1)^L$, which should be dominant in the absence of symmetry (see Sec. III B for details). Consequently, the leading-order-term including $(\beta_{2M} \beta_{2M-1} \cdots \beta_1)^L$ and the next-to-leading-order term including $(\beta_{2M-2} \beta_{2M-3} \cdots \beta_1)^L$ should be comparable, which yields Eq. (4).

This work is organized as follows. In Sec. II we summarize basic properties of reciprocity. In Sec. III A we investigate a symplectic extension of the Hatano-Nelson model [54, 72], which is a prototypical non-Hermitian system in the symplectic class. Then, we generally demonstrate the non-Bloch band theory in the symplectic class in Sec. III B. We conclude this work in Sec. IV.

II. RECIPROCITY

Reciprocity is one of the fundamental internal symmetry [55, 71]. There are two types of reciprocity according to the sign of the symmetry operator $T$, one of which is defined by

$$TH^T T^{-1} = H, \quad TT^* = +1$$  \hspace{1cm} (5)

with a unitary operator $T$, and the other of which is defined by $T = e^{i\pi T}$ in Ref. [55]. This symmetry is called TRS [71] since it is a Hermitian-conjugate counterpart of time-reversal symmetry (TRS), and non-Hermitian Hamiltonians with Eqs. (5) and (1) are defined to belong to classes AI and AI′, respectively. In this work, classes AI and AI′ are also called the orthogonal and symplectic classes, respectively, in a similar manner to the Hermitian case. Reciprocity appears in a variety of non-Hermitian systems. For example, time-reversal-invariant Hermitian Hamiltonians with gain or loss (i.e., complex onsite potential) respect it, and belong to the orthogonal (symplectic) class in the absence (presence) of the spin degree of freedom. In addition, it is relevant to open quantum systems described by the Lindblad master equation [73, 74].

In this section, we describe basic properties of reciprocity relevant to the non-Bloch band theory. We begin with reviewing the non-Bloch band theory in Sec. II A. In Sec. II B we investigate the orthogonal class and show the absence of skin effects due to the symmetry. We next investigate the symplectic class in Sec. II C. In this case, the symmetry brings about Kramers degeneracy and does not necessarily lead to the absence of skin effects.
A. Non-Bloch band theory

In the following, we consider a generic non-Hermitian Hamiltonian in one dimension described by
\[ \hat{H} = \sum_{n} \sum_{j=-l}^{l} \sum_{\mu,\nu=1}^{q} H_{j,\mu\nu} \hat{c}_{n,j,\mu}^{\dagger} \hat{c}_{n,\mu\nu}, \]
where \( \hat{c}_{n,\mu} \) (\( \hat{c}_{n,\mu}^{\dagger} \)) is the annihilation (creation) operator on site \( n \), and \( H_{j,\mu\nu} \) is the single-particle Hamiltonian. Moreover, \( n \) describes the spatial degree of freedom, \( l \) describes the hopping range, and \( \mu, \nu \) describe the internal degree of freedom per unit cell. We assume translation invariance under periodic boundary conditions. Thus, \( H_{j,\mu\nu} \) is independent of the sites \( n \) away from the edges. Because of the noninteracting (quadratic) nature of the Hamiltonian, diagonalization of the many-body Hamiltonian \( H \) reduces to diagonalization of the single-particle Hamiltonian \( H \), whose elements are given by \( H_{j,\mu\nu} \). A complex eigenenergy \( E \) of \( H \) is denoted as \( E \in \mathbb{C} \), and the corresponding right (left) eigenstate is denoted as \( |\phi \rangle \) (|\chi \rangle) [76]:
\[ H |\phi \rangle = E |\phi \rangle, \quad H^{\dagger} |\chi \rangle = E^{*} |\chi \rangle. \quad (7) \]

Because of translation invariance of \( H \) away from the edges, the eigenstates are given by a linear combination of fundamental solutions
\[ \sum_{n=1}^{L} \beta_{i}^{n} |n \rangle |\phi_{i} \rangle := \sum_{n=1}^{L} \sum_{\mu=1}^{q} \beta_{i}^{n} \phi_{\mu}^{(i)} |n \rangle |\mu \rangle, \quad \beta_{i}, \phi_{\mu}^{(i)} \in \mathbb{C}, \quad (8) \]
where \( L \) is the number of unit cells, \( |n \rangle \) is a state localized at the site \( n \), and \( |\mu \rangle \) is a state with the internal degree \( \mu \). This wave function is delocalized through the bulk for \( |\beta| = 1 \), while it is localized around the edge \( n = 1 \) (\( n = L \)) for \( |\beta| < 1 \) (\( |\beta| > 1 \)). The corresponding bulk Hamiltonian is described by
\[ H (\beta) := \sum_{j=-l}^{l} H_{j} \beta^{j}, \quad (9) \]
where \( H_{j} \) is a \( q \times q \) matrix defined by \( (H_{j})_{\mu\nu} := H_{j,\mu\nu} \) and satisfies
\[ H (\beta_{i}) |\phi_{i} \rangle = E |\phi_{i} \rangle. \quad (10) \]
In the presence of Hermiticity, Eq. [5] is just a plane wave and \( H (\beta) \) is a conventional Bloch Hamiltonian because of \( |\beta| = 1 \). The possible \( \beta_{i} \)'s for given \( E \) are determined by the characteristic equation
\[ \det [H (\beta) - E] = 0, \quad (11) \]
which is the \( 2q \times 2q \)-th order algebraic equation in terms of \( \beta \). For these \( \beta_{i} \)'s (\( i = 1, 2, \cdots, 2q \)), the right eigenstate \( |\phi \rangle \) in real space can be represented as
\[ |\phi \rangle = \sum_{i=1}^{2q} \sum_{n=1}^{L} \beta_{i}^{n} |n \rangle |\phi_{i} \rangle. \quad (12) \]

Remarkably, if the right eigenstate \( |\phi \rangle \) is localized at one end, the corresponding left eigenstate \( |\chi \rangle \) is localized at the other end. To see this property, we notice that \( |\chi \rangle^{*} \) is, by definition, a right eigenstate of \( H^{T} \) with the eigenenergy \( E \). Then, let us consider transposition \( H \rightarrow H^{T} \), which leads to the transformations
\[ H_{j,\mu\nu} \rightarrow H_{-j,\mu\nu}, \quad (13) \]
and
\[ H (\beta) \rightarrow \sum_{j=-l}^{l} H_{j}^{T} \beta^{-j} = \sum_{j=-l}^{l} H_{j}^{T} \beta^{-j} = H^{T} (\beta^{-1}). \quad (14) \]

This result implies that if \( \beta \) satisfies Eq. [11] for \( H, \beta^{-1} \) satisfies Eq. [11] for \( H^{T} \), and vice versa. Recalling that \( |\chi \rangle^{*} \) is a right eigenstate of \( H^{T} \), we conclude that if \( |\phi \rangle \) is localized at one end, \( |\chi \rangle \) is localized at the other end, and vice versa. This also means that delocalization of \( |\phi \rangle \) occurs simultaneously with delocalization of \( |\chi \rangle \).

B. Orthogonal class (class AIr)

Reciprocity imposes some constraints on the eigenstates \( |\phi \rangle \) and \( |\chi \rangle \). In fact, in the orthogonal class, Eq. [5] yields
\[ H (T |\chi \rangle^{*}) = T H^{T} |\chi \rangle^{*} = E (T |\chi \rangle^{*}), \quad (15) \]
which means that \( T |\chi \rangle^{*} \) is also a right eigenstate with the eigenenergy \( E \). The eigenenergies are, in general, not degenerate solely in the presence of Eq. [5]. Hence, the two right eigenstates are equivalent to each other, i.e.,
\[ |\phi \rangle \propto T |\chi \rangle^{*}. \quad (16) \]

Because of the relationship between \( |\phi \rangle \) and \( |\chi \rangle \) discussed in Sec. II A, they are forbidden to be localized in the orthogonal class. In fact, if \( |\phi \rangle \) were localized at one end, \( T |\chi \rangle^{*} \) would be localized at the other end, which contradicts Eq. [10]. Here, we use the fact that the internal-symmetry operation does not change the place at which eigenstates are localized. Thus, no skin effects appear in the orthogonal class, and this is why we call the symmetry in Eq. [5] reciprocity.

The absence of skin effects in the orthogonal class can be derived also on the basis of the non-Bloch band theory [33]. Since transposition transforms \( H (\beta) \) to \( H^{T} (\beta^{-1}) \) as shown in Eq. [14], reciprocity for \( H \) [i.e., Eq. [5]] imposes
\[ T H^{T} (\beta) T^{-1} = H (\beta^{-1}). \quad (17) \]

Then, when \( \beta \) is a solution to the characteristic equation [11], we have
\[ \det [H (\beta^{-1}) - E] = \det [T H^{T} (\beta) T^{-1} - E] = \det [H (\beta) - E] = 0. \quad (18) \]
which implies that $\beta^{-1}$ is also a solution to Eq. \[11\]. Hence, the solutions to the $2lq$-th order equation \[11\] can be represented as

$$|\beta_1| \leq \cdots \leq |\beta_{lq}| \leq 1 \leq |\beta_{lq}^{-1}| \leq \cdots \leq |\beta_1^{-1}|. \quad (19)$$

Then, using Eq. \[2\], which is the salient result of the non-Bloch band theory, we have $|\beta_{lq}| = |\beta_{lq}^{-1}|$, i.e., $|\beta_{lq}| = 1$. Consequently, bulk eigenstates are delocalized and no skin effects occur.

Notably, Refs. \[53, 54\] showed that the skin effects originate from nontrivial topology that cannot be continuously deformed to any Hermitian systems. Consistently, such intrinsic non-Hermitian topology is absent in the one-dimensional orthogonal class; see class AI in Table V of Ref. \[35\].

\section*{C. Symplectic class (class AII’)}

In the symplectic class, in which Eq. \[1\] is respected, we still have Eq. \[15\]. A crucial distinction is Kramers degeneracy due to $T^T = -T$. Such generic degeneracy is absent in the orthogonal class. In fact, because of $T^T = -T$, we have

$$\langle \chi | T \chi \rangle^* = \langle \chi | T^T \chi \rangle^* = -\langle \chi | T \chi \rangle^*, \quad (20)$$

which leads to $\langle \chi | T \chi \rangle^* = 0$. This indicates that $|\phi\rangle$ and $T |\chi\rangle^*$, which belong to the same eigenenergy, are biorthogonal \[76\] to each other and linearly independent of each other. Thus, all the eigenenergies are at least twofold degenerate.

Similarly, we have Eq. \[18\] even in the symplectic class. In terms of $H (\beta)$, the non-Bloch waves $|\phi_i\rangle$ and $T |\chi_i\rangle^*$ form a Kramers pair; the former satisfies Eq. \[10\], while the latter satisfies

$$H (\beta_i^{-1}) (T |\chi_i\rangle^*) = E (T |\chi_i\rangle^*). \quad (21)$$

Because of this Kramers degeneracy, the characteristic equation has the $4lq$-th order and its solutions are generally represented as

$$|\beta_1| \leq \cdots \leq |\beta_{2lq}| \leq 1 \leq |\beta_{2lq}^{-1}| \leq \cdots \leq |\beta_1^{-1}|. \quad (22)$$

If the standard non-Bloch band theory is applicable, we have $|\beta_{2lq}| = |\beta_{2lq}^{-1}|$, and hence no skin effects appear in a similar manner to the orthogonal class. However, a reciprocal skin effect is feasible in the symplectic class, as shown in Ref. \[51\] and the next section. This fact implies modification of the standard non-Bloch band theory, as demonstrated in the following.

\section*{III. NON-BLOCH BAND THEORY IN THE SYMPLECTIC CLASS}

We establish the non-Bloch band theory in the symplectic class. In Sec. IIIA we begin with exactly solving a symplectic extension of the Hatano-Nelson model \[54, 72\] and confirming the skin effect even in the presence of reciprocity. On the basis of this prototypical model, we generally demonstrate our nonstandard non-Bloch band theory in Sec. III B. There, Kramers degeneracy in Sec. IIC plays a key role.

\subsection*{A. Symplectic Hatano-Nelson model}

The Hatano-Nelson model \[72\] is a prototypical non-Hermitian model that exhibits the skin effect, which is given by

$$\hat{H} = \sum_n \left[ (t + g) \hat{c}_{n+1}^\dagger \hat{c}_n + (t - g) \hat{c}_n^\dagger \hat{c}_{n+1} \right]. \quad (23)$$

Here, $t \in \mathbb{R}$ is the Hermitian part of the hopping, and $g \in \mathbb{R}$ describes the asymmetry of the hopping as the degree of non-Hermiticity. The corresponding bulk Hamiltonian defined as Eq. \[9\] is

$$H (\beta) = (t + g) \beta^{-1} + (t - g) \beta. \quad (24)$$

The exact solution to the Hatano-Nelson model is provided, for example, in the Supplemental Material of Ref. \[40\]. Under open boundary conditions, we have

$$|\beta| = \sqrt{\frac{t + g}{t - g}}, \quad (25)$$

and all the eigenstates are localized at the right (left) edge for $g/t > 0$ ($g/t < 0$). We note in passing that the original works \[72\] introduced onsite random potential and revealed delocalization transitions even in one dimension due to the interplay between non-Hermiticity and disorder.

Combining a reciprocal pair of the Hatano-Nelson models, we below investigate the following symplectic generalization \[51\]:

$$\hat{H} = \sum_n \left[ \hat{c}_{n+1}^\dagger (t - i \Delta \sigma_x + g \sigma_z) \hat{c}_n + \hat{c}_n^\dagger (t + i \Delta \sigma_x - g \sigma_z) \hat{c}_{n+1} \right], \quad (26)$$

and

$$H (\beta) = (t - i \Delta \sigma_x + g \sigma_z) \beta^{-1} + (t + i \Delta \sigma_x - g \sigma_z) \beta. \quad (27)$$

Here, $\hat{c}_n (\hat{c}_n^\dagger)$ annihilates (creates) a spinful particle with two components, $\sigma_i$’s are Pauli matrices that describe the spin degree of freedom, and $\Delta$ is spin-orbit interaction. The bulk Hamiltonian respects

$$\sigma_y H^T (\beta) \sigma_y^{-1} = H (\beta^{-1}), \quad (28)$$

and indeed belongs to the symplectic class.

Under periodic boundary conditions, $\beta$ satisfies $|\beta| = 1$ and hence is given by $\beta := e^{ik}$ with real wavenumbers $k \in [0, 2\pi]$. Then, the Bloch Hamiltonian is

$$H (k) = 2t \cos k - 2 (\Delta \sigma_x + ig \sigma_z) \sin k. \quad (29)$$
The spectrum of $H(k)$ is given as

$$E(k) = 2t \cos k \pm 2i\sqrt{g^2 - \Delta^2} \sin k,$$  

(30)

which is entirely real for $|g| \leq |\Delta|$ and form a loop in the complex plane for $|g| > |\Delta|$ (Fig. [1]).

Under open boundary conditions, an eigenenergy is denoted as $E \in \mathbb{C}$, and the corresponding right eigenstate is denoted as $|\phi\rangle = \sum_{n=1}^{L} \phi_{n,s} |n\rangle |s\rangle$. The Schrödinger equation in real space reads

$$(t - i\Delta \sigma_x + g \sigma_z) \left( \frac{\phi_{n+1,\uparrow}}{\phi_{n,\downarrow}} \right) + (t + i\Delta \sigma_x - g \sigma_z) \left( \frac{\phi_{n+1,\downarrow}}{\phi_{n,\uparrow}} \right) = E \left( \frac{\phi_{n,\uparrow}}{\phi_{n,\downarrow}} \right),$$

(31)

in the bulk ($n = 2, 3, \cdots, L - 1$), and

$$(t + i\Delta \sigma_x - g \sigma_z) \left( \frac{\phi_{1,\uparrow}}{1 \phi_{1,\downarrow}} \right) = E \left( \frac{\phi_{1,\uparrow}}{1 \phi_{1,\downarrow}} \right),$$

(32)

$$(t - i\Delta \sigma_x + g \sigma_z) \left( \frac{\phi_{L+1,\downarrow}}{\phi_{L+1,\uparrow}} \right) = E \left( \frac{\phi_{L+1,\downarrow}}{\phi_{L+1,\uparrow}} \right),$$

(33)

at the edges. Defining $\phi_{0,s}$ and $\phi_{L+1,s}$ by the bulk equation (31), the boundary equations (32) and (33) reduce to

$$\begin{align*}
\left( \frac{\phi_0,\uparrow}{\phi_{0,\downarrow}} \right) &= \left( \frac{\phi_{L+1,\uparrow}}{\phi_{L+1,\downarrow}} \right) = 0. 
\end{align*}$$

(34)

Suppose a fundamental solution is given as $\phi_{n,s} \propto \beta^n \phi_s$. From the bulk equation (31), we have

$$[H(\beta) - E] \left( \frac{\phi_{1,\uparrow}}{\phi_{1,\downarrow}} \right) = 0. $$

(35)

To have a nontrivial solution $(\phi_{1,\uparrow} \phi_{1,\downarrow})^T \neq 0$, the coefficient matrix $H(\beta) - E$ should not be invertible, leading to the characteristic equation

$$\det [H(\beta) - E] = 0,$$

(36)

i.e.,

$$E = t (\beta + \beta^{-1}) \mp \sqrt{g^2 - \Delta^2} (\beta - \beta^{-1}).$$

(37)

This is a quartic equation in terms of $\beta$ for given $E$. In fact, it decomposes into a pair of quadratic equations

$$
\begin{align*}
(t + \sqrt{g^2 - \Delta^2}) \beta^2 - E \beta + t - \sqrt{g^2 - \Delta^2} &= 0, \\
(t - \sqrt{g^2 - \Delta^2}) \beta^2 - E \beta + t + \sqrt{g^2 - \Delta^2} &= 0.
\end{align*}
$$

(38)

Remarkably, when $\beta$ satisfies this characteristic equation, $\beta^{-1}$ also satisfies it; in particular, when $\beta$ satisfies Eq. (38), $\beta^{-1}$ satisfies Eq. (39), and vice versa. This is a direct consequence of reciprocity, as discussed in Sec. [II]. Furthermore, a fundamental solution with $\beta$ and another fundamental solution with $\beta^{-1}$ are linearly independent of each other and form a Kramers pair. Now, we define

The solutions to Eq. (38) as $\beta_1$ and $\beta_2$ ($|\beta_1| \leq |\beta_2|$), which satisfy

$$\beta_1 \beta_2 = \frac{t - \sqrt{g^2 - \Delta^2}}{t + \sqrt{g^2 - \Delta^2}},$$

(40)

The solutions to Eq. (39) are given as $\beta_1^{-1}$ and $\beta_2^{-1}$. Since the solutions $\beta_1, \beta_2, \beta_1^{-1}, \beta_2^{-1}$ to the characteristic equation are defined to respect $|\beta_1| \leq |\beta_2|$, the standard non-Bloch band theory [25, 40] predicts $|\beta_2| = |\beta_2^{-1}|$ for continuum bands. However, this is not the case in the symplectic class; instead, we have $|\beta_1| = |\beta_2|$, as shown below.

Now, the eigenstate $|\phi\rangle = \sum_{n=1}^{L} \sum_{s \in \{\uparrow, \downarrow\}} \phi_{n,s} |n\rangle |s\rangle$ can be obtained as a linear combination of the above fundamental solutions:

$$
\begin{align*}
\left( \phi_{n,\uparrow} \phi_{n,\downarrow} \right) &= \beta_1^n \phi_{n,1+} + \beta_2^n \phi_{n,1-} + \beta_1^{-n} \phi_{n,2+} + \beta_2^{-n} \phi_{n,2-},
\end{align*}
$$

(41)

for $n = 1, 2, \cdots, L$. Here, since $(\phi_{1,\uparrow} \phi_{1,\downarrow})^T$ satisfies Eq. (35), we have

$$
\begin{align*}
\left( \pm \sqrt{g^2 - \Delta^2} + g \right) \left( \frac{-i\Delta}{\sqrt{g^2 - \Delta^2} - g} \right) \phi_{1,\uparrow} &= 0.
\end{align*}
$$

(42)

Remarkably, $(\phi_{1,\uparrow} \phi_{1,\downarrow})^T$ does not depend on $i$. Hence, Eq. (41) further simplifies to

$$
\begin{align*}
\left( \phi_{n,\uparrow} \phi_{n,\downarrow} \right) &= (\beta_1 \phi_{n,1+} + \beta_2 \phi_{n,1-}) \left( \frac{1}{c_1} \right) + (\beta_1^{-1} \phi_{n,2+} + \beta_2^{-1} \phi_{n,2-}) \left( \frac{1}{c_2} \right),
\end{align*}
$$

(43)

FIG. 1. Complex spectra of the symplectic Hatano-Nelson model. The black dashed curves denote the spectra under periodic boundary conditions, and the red dots denote the spectra under open boundary conditions ($L = 100$). (a) The periodic-boundary spectrum and the open-boundary spectrum coincide with each other, and no skin effect occurs ($t = 1.0$, $\Delta = 0.3$, $g = 0.2$). (b) The periodic-boundary spectrum forms a loop in the complex plane, but the open-boundary spectrum lies on the real axis, which is a signature of the skin effect ($t = 1.0$, $\Delta = 0.3$, $g = 0.4$).
with some constants $\tilde{\phi}_{1\pm}, \tilde{\phi}_{2\pm} \in \mathbb{C}$ and $c_{\pm} := -i(\pm \sqrt{g^2 - \Delta^2} + g)/\Delta$. Then, the boundary condition (44) reduces to

$$
(\tilde{\phi}_{1+} + \tilde{\phi}_{2+}) \left( \frac{1}{c_+} \right) + (\tilde{\phi}_{1-} + \tilde{\phi}_{2-}) \left( \frac{1}{c_-} \right) = 0, \quad (44)
$$

$$(\beta_1^{L+1} \phi_{1+} + \beta_2^{L+1} \phi_{2+}) \left( \frac{1}{c_+} \right) + (\beta_1^{L+1} \phi_{1-} + \beta_2^{L+1} \phi_{2-}) \left( \frac{1}{c_-} \right) = 0. \quad (45)
$$

The vectors $(1 c_+)^T$ and $(1 c_-)^T$ form a Kramers pair and linearly independent of each other. In particular, they are biorthogonal to each other [76], i.e., the left counterpart of $(1 c_-)^T$ is orthogonal to $(1 c_+)^T$. As a result, we have

$$
\tilde{\phi}_{1\pm} + \tilde{\phi}_{2\pm} = \beta_1^{\pm (L+1)} \phi_{1\pm} + \beta_2^{\pm (L+1)} \phi_{2\pm} = 0, \quad (46)
$$

leading to

$$
\beta_1^{L+1} = \beta_2^{L+1} \quad (47)
$$

for a nontrivial solution $(\tilde{\phi}_{1\pm}, \tilde{\phi}_{2\pm}) \neq 0$. This equation means that the absolute values of $\beta_1$ and $\beta_2$ coincide with each other and are given, from Eq. (40), as

$$
|\beta_1| = |\beta_2| = \sqrt{\frac{t - \sqrt{g^2 - \Delta^2}}{t + \sqrt{g^2 - \Delta^2}}}. \quad (48)
$$

The relative phase between $\beta_1$ and $\beta_2$ can be different, resulting in the formation of continuum bands.

Equation (48) provides the localization length of eigenstates and the criteria of the skin effect. For $|g| \leq |\Delta|$, we have $|\beta_1| = |\beta_2| = 1$ and hence eigenstates are delocalized. For $|g| > |\Delta|$, on the other hand, we have $|\beta_1| = |\beta_2| \neq 1$ and hence eigenstates are localized at the edges. In contrast to the conventional skin effect, skin modes appear at both edges; when an eigenstate is localized at one edge, the Kramers partner is localized at the other edge. The numerical calculations shown in Fig. 1 confirm this result. For $|g| > |\Delta|$, the spectrum under periodic boundary conditions forms a loop in the complex plane, but the spectrum under open boundary conditions lies on the real axis, which is a signature of the non-Hermitian skin effect.

In the above calculations, an important distinction from the standard case is the equivalence between $(\phi_1^{(1\pm)} \phi_2^{(1\pm)})^T$ and $(\phi_1^{(2\pm)} \phi_2^{(2\pm)})^T$. In fact, if they were linearly independent, we have $|\beta_2| = |\beta_2^{-1}|$ instead of $|\beta_1| = |\beta_2|$ in a similar manner to the standard case [25, 40]. However, symplectic reciprocity makes $(\phi_1^{(1\pm)} \phi_2^{(1\pm)})^T$ and $(\phi_1^{(2\pm)} \phi_2^{(2\pm)})^T$ linearly dependent on each other and changes the condition for continuum bands, as demonstrated below.

### B. General condition

Now, we demonstrate the non-Bloch band theory in the symplectic class in a general manner. We consider a generic non-Hermitian Hamiltonian described by

$$
\hat{H} = \sum_{n,j=-L}^{L} \sum_{\mu=1}^{q} \sum_{s,t \in \{\uparrow, \downarrow\}} H_{j,\mu;st} c_{n+j,\mu,s}^\dagger c_{n,\mu,t}. \quad (49)
$$

In comparison with the standard class discussed in Sec. II A, the indices $s, t \in \{\uparrow, \downarrow\}$ are added to Eq. (6) to account for the internal degree of freedom arising from Eq. (1). A prime example of such an internal degree of freedom is the spin degree of freedom. Correspondingly, the eigenstates are given by a linear combination of fundamental solutions

$$
\sum_{n=1}^{L} |m\rangle (\beta^n_+ |\phi_{1+}^m\rangle + \beta^{-n}_- |\phi_{1-}^m\rangle), \quad (50)
$$

where $|\phi_{1\pm}\rangle$ can be expanded as

$$
|\phi_{1\pm}\rangle := \sum_{\mu=1}^{q} \sum_{s \in \{\uparrow, \downarrow\}} |\phi_{1\pm}^{(\mu)s}\rangle |\mu\rangle |s\rangle, \quad (51)
$$

and is a right eigenstate of $H (\beta_1^{\pm})$:

$$
H (\beta_1^{\pm}) |\phi_{1\pm}\rangle = E |\phi_{1\pm}\rangle. \quad (52)
$$

The corresponding left eigenstate $|\chi_{1\pm}\rangle$ of $H (\beta_1^{\pm})$ is defined by

$$
H^\dagger (\beta_1^{\pm}) |\chi_{1\pm}\rangle = E^* |\chi_{1\pm}\rangle. \quad (53)
$$

Here, $\{\pm\}$ is equivalent to $\{\uparrow, \downarrow\}$ in the absence of perturbations that mix $\uparrow$ and $\downarrow$, including spin-orbit interaction; but this is not necessarily true, in general. In addition, $\beta_1^{\pm}$’s and $\beta_1^{-1}$’s ($i = 1, 2, \cdots, 2q$) are the solutions to the characteristic equation $\det [H (\beta) - E] = 0$. Without loss of generality, they can be chosen so that Eq. (22) will be satisfied. Importantly, as described in Sec. II C, $(\phi_{1+})$ and $(\phi_{1-})$ are biorthogonal to each other and form a Kramers pair. Specifically, we have

$$
|\phi_{1-}\rangle = T |\chi_{1+}\rangle^*, \quad |\phi_{1+}\rangle = -T |\chi_{1-}\rangle^*. \quad (54)
$$

under the appropriate choice of the gauges. Generally, the left eigenstates $|\chi_{i\pm}\rangle$’s are determined when the right eigenstates $|\phi_{i\pm}\rangle$’s are given, except for the arbitrariness of normalization [76]. In this respect, Eq. (54) provides normalization conditions of $|\chi_{i\pm}\rangle$’s. From these fundamental solutions, the right eigenstate $|\phi\rangle$ and the left eigenstate $|\chi\rangle$ in real space can be given as

$$
|\phi\rangle = \sum_{i=1}^{2q} \sum_{n=1}^{L} |n\rangle (\beta_i^n |\phi_{1+}\rangle + \beta_i^{-n} |\phi_{1-}\rangle), \quad (55)
$$

$$
|\chi\rangle = \sum_{i=1}^{2q} \sum_{n=1}^{L} |n\rangle (\beta_i^{-n} |\chi_{1+}\rangle + (\beta_i^n)^* |\chi_{1-}\rangle). \quad (56)
$$
under the appropriate choice of the gauges and normalization. In addition, we have from Eq. (54)

\[ \mathcal{T} |\chi|^* = \sum_{i=1}^{2lq} \sum_{n=1}^{L} |n \rangle \left( -\beta_i^n |\phi_{i+} \rangle + \beta_i^{-n} |\phi_{i-} \rangle \right). \]  (57)

which is the Kramers partner of |\phi\rangle satisfying \langle \chi |\mathcal{T} |\chi|^* = 0.

Generic eigenstates |\phi\rangle and \mathcal{T} |\chi|^* in Eqs. (55) and (57) include 2\(lq\) \(\times\) 2 \(\times\) 2 unknown variables \(\phi_{\mu s}^{(i)}\) \((i = 1, 2, \ldots, 2lq; \mu = 1, 2, \ldots, q; s = \uparrow, \downarrow)\) in Eq. (51). They reduce to 2\(lq\) \(\times\) 2 unknown variables, for example, \(\phi_{\uparrow \downarrow}^{(i)}\), because of the Schrödinger equation (52) for the bulk Hamiltonian \(H(\beta)\). Here, the rank of \(H(\beta)\) is assumed appropriately in a similar manner to Ref. [40].

The 2\(lq\) \(\times\) 2 unknown variables \(\ddot{\phi}_{i\pm} := \phi_{i\uparrow}^{(\pm)}\) \((i = 1, \ldots, 2lq)\) are determined by boundary conditions. In general, the boundary conditions are given by details about the \(lq\) sites around each end. Hence, the boundary conditions for |\phi\rangle can be represented by

\[ \sum_{i=1}^{2lq} \left( f_j (\beta_i) \ddot{\phi}_{i+} + f_j (\beta_i^{-1}) \ddot{\phi}_{i-} \right) = 0, \]  (58)

\[ \sum_{i=1}^{2lq} \left( \beta_i^L g_j (\beta_i) \ddot{\phi}_{i+} + \beta_i^{-L} g_j (\beta_i^{-1}) \ddot{\phi}_{i-} \right) = 0, \]  (59)

where \(f_j (\beta_i)\) and \(g_j (\beta_i)\) \((j = 1, 2, \ldots, 2lq)\) are functions of \(\beta_i\) that are independent of \(\beta_i\). Importantly, the boundary conditions for the Kramers partner \(\mathcal{T} |\chi|^*\) should also be respected:

\[ \sum_{i=1}^{2lq} \left( -f_j (\beta_i) \ddot{\phi}_{i+} + f_j (\beta_i^{-1}) \ddot{\phi}_{i-} \right) = 0, \]  (60)

\[ \sum_{i=1}^{2lq} \left( -\beta_i^L g_j (\beta_i) \ddot{\phi}_{i+} + \beta_i^{-L} g_j (\beta_i^{-1}) \ddot{\phi}_{i-} \right) = 0. \]  (61)

For example, when the eigenstates vanish at \(n = 0\) and \(n = L+1\) in a similar manner to the example in Sec. III A, we have

\[ \sum_{i=1}^{2lq} (|\phi_{i+} \rangle + |\phi_{i-} \rangle) = \sum_{i=1}^{2lq} (\beta_i^{L+1} |\phi_{i+} \rangle + \beta_i^{-L} |\phi_{i-} \rangle) = 0 \] for |\phi\rangle, and \(f_j (\beta_i)\) and \(g_j (\beta_i)\) are given by \(|\phi_{\pm} \rangle\). Indeed, the boundary condition (60) is described by these equations.

Whereas we have 4\(lq\) unknown variables \(\ddot{\phi}_{i\pm}\), the boundary conditions (58)-(61) provide 8\(lq\) linear equations. This implies that Eqs. (58)-(61) are not linearly independent of each other because of some constraints on \(f_j (\beta_i)\) and \(g_j (\beta_i)\). To have such constraints, we notice from Eqs. (58) and (59)

\[ \sum_{i=1}^{2lq} f_j (\beta_i) \ddot{\phi}_{i+} = \sum_{i=1}^{2lq} f_j (\beta_i^{-1}) \ddot{\phi}_{i-} = 0. \]  (62)

In matrix representation, we have

\[
\begin{pmatrix}
    f_1 (\beta_1^\uparrow) & \cdots & f_1 (\beta_{2lq}^\uparrow) & f_1 (\beta_1^{-1}) & \cdots & f_1 (\beta_{2lq}^{-1}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    f_{2lq} (\beta_1) & \cdots & f_{2lq} (\beta_{2lq}) & f_{2lq} (\beta_1^{-1}) & \cdots & f_{2lq} (\beta_{2lq}^{-1}) \\
    \beta_1^L g_1 (\beta_1) & \cdots & \beta_2^L g_1 (\beta_{2lq}) & \beta_1^{-L} g_1 (\beta_1^{-1}) & \cdots & \beta_{2lq}^{-L} g_1 (\beta_{2lq}^{-1}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    \beta_1^L g_{2lq} (\beta_1) & \cdots & \beta_{2lq}^L g_{2lq} (\beta_{2lq}) & \beta_1^{-L} g_{2lq} (\beta_1^{-1}) & \cdots & \beta_{2lq}^{-L} g_{2lq} (\beta_{2lq}^{-1})
\end{pmatrix}
\begin{pmatrix}
    \dddot{\phi}_{\uparrow} \\
    \dddot{\phi}_{\downarrow} \\
    \dddot{\phi}_{2lq+} \\
    \dddot{\phi}_{2lq-} \\
\end{pmatrix} = 0.
\]  (65)

To have a nontrivial solution, the 4lq \(\times\) 4lq coefficient matrix \(F_{\pm}\) should not be invertible, i.e.,

\[ \det F_{\mp} := \det \begin{pmatrix}
    f_1 (\beta_1^\uparrow) & \cdots & f_1 (\beta_{2lq}^\uparrow) & f_1 (\beta_1^{-1}) & \cdots & f_1 (\beta_{2lq}^{-1}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    f_{2lq} (\beta_1) & \cdots & f_{2lq} (\beta_{2lq}) & f_{2lq} (\beta_1^{-1}) & \cdots & f_{2lq} (\beta_{2lq}^{-1}) \\
    \beta_1^L g_1 (\beta_1) & \cdots & \beta_2^L g_1 (\beta_{2lq}) & \beta_1^{-L} g_1 (\beta_1^{-1}) & \cdots & \beta_{2lq}^{-L} g_1 (\beta_{2lq}^{-1}) \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    \beta_1^L g_{2lq} (\beta_1) & \cdots & \beta_{2lq}^L g_{2lq} (\beta_{2lq}) & \beta_1^{-L} g_{2lq} (\beta_1^{-1}) & \cdots & \beta_{2lq}^{-L} g_{2lq} (\beta_{2lq}^{-1})
\end{pmatrix} = 0. \]  (66)

Consistently, these constraints are respected in the specific example in Sec. III A since \((\phi_{\uparrow}^{(1\pm)} \phi_{\downarrow}^{(1\pm)})^T\) and \((\phi_{\uparrow}^{(2\pm)} \phi_{\downarrow}^{(2\pm)})^T\) are linearly dependent on each other. The combination of Eqs. (59) and (61) yields similar constraints on \(g_j (\beta_i)\).

Then, from Eqs. (58) and (59), we have
The determinant on the left-hand side is a $2q$-th-order polynomial in terms of $\beta^2, \cdots, \beta^{2q-2}, \beta^{2q-1}$. Because of Eq. (22), its leading-order term includes $(\beta^{2q-1})^L$ and the next-to-leading-order term includes $(\beta^{2q-2})^L$, in general. To satisfy Eq. (66) for $L \to \infty$, the absolute values of these terms need to coincide with each other, which leads to the condition 2 for the standard case 25.40. However, this is not the case in the symplectic class. In fact, the term including $(\beta^{2q-1})^L$ does not appear since it is proportional to $\det F_+$, which vanishes as shown in Eq. (64). As a result, the leading-order term includes $(\beta^{2q-1})^L$ and the next-to-leading-order term includes $(\beta^{2q-2})^L$, both of which should be comparable to each other for $L \to \infty$. Therefore, it is necessary to have

$$|\beta^{2q-2}^{2q-3} \cdots \beta^{-1}| = \left|\frac{\beta^{2q-2}^{2q-3} \cdots \beta^{-1}}{\beta^{2q-1}}\right| \quad (67)$$

leading to $|\beta^{2q-1}| = |\beta^{2q}|$, i.e., Eq. (1) with $M := q$.

IV. CONCLUSION

In this work, we have established the non-Bloch band theory of non-Hermitian Hamiltonians in the symplectic class. In contrast to the standard non-Bloch band theory, which describes generic non-Hermitian systems without symmetry, reciprocity in Eq. (1) leads to Kramers degeneracy and the modification of the condition for continuum bands. As a consequence of this nonstandard non-Bloch band theory, non-Hermitian skin effects are allowed to occur even in the presence of reciprocity. This contrasts with the orthogonal class, in which reciprocity in Eq. (1) forbids the skin effect.

In closing, we note that similar modification of the non-Bloch band theory can arise because of other symmetry. For example, when unitary symmetry that commutes with the Hamiltonian is present, the Hamiltonian is block diagonal in the eigenbasis of the symmetry. In this case, the non-Bloch band theory should be applied not to the original Hamiltonian but to each subspace of the Hamiltonian. This point and the consequent infinitesimal instability of non-Hermitian systems were discussed also in Ref. 12. Moreover, Ref. 47 found a reciprocal skin effect in the presence of reflection symmetry. We point out that this reflection-symmetry-protected skin effect should be accompanied by the modification of the standard non-Bloch band theory in a similar manner to the modification due to symplectic reciprocity discussed in this work. Our nonstandard non-Bloch band theory in the symplectic class can further be modified in the presence of such additional symmetry.

Remarkably, Refs. 53, 54 identified the origin of the skin effects as non-Hermitian topology that has no counterparts in Hermitian systems. There, a correspondence between a $Z$ topological invariant and the non-Bloch band theory was shown. On the basis of this understanding, Ref. 54 further revealed the reciprocity-protected skin effect that is ensured by a $Z_2$ topological invariant. It merits further study to investigate a similar correspondence between this $Z_2$ topological invariant and the nonstandard non-Bloch band theory developed in the present work.

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Note added. — Recently, we became aware of a related work 17.
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