Sharp Nash inequalities on manifolds with boundary in the presence of symmetries

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Abstract
In this paper we establish the best constant $\tilde{A}_{\text{opt}}(M)$ for the Trace Nash inequality on a $n$–dimensional compact Riemannian manifold in the presence of symmetries, which is an improvement over the classical case due to the symmetries which arise and reflect the geometry of manifold. This is particularly true when the data of the problem is invariant under the action of an arbitrary compact subgroup $G$ of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal.

Key words: Manifolds with boundary, Symmetries, Trace Nash inequalities, Best constants.

1 Introduction

We say that the Nash inequality (1) is valid if there exists a constant $A > 0$ such that for all $u \in C^\infty_0(\mathbb{R}^n)$, $n \geq 2$

$$
\left( \int_{\mathbb{R}^n} u^2 \, dx \right)^{1+\frac{2}{n}} \leq A \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \left( \int_{\mathbb{R}^n} |u| \, dx \right)^{\frac{4}{n}} \tag{1}
$$

Such an inequality first appeared in the celebrated paper of Nash [15], where he discussed the H"older regularity of solutions of divergence form in uniformly
elliptic equations. It is a particular case of the Gagliardo-Nirenberg type inequalities $\|u\|_r \leq C\|\nabla u\|_a^a \|u\|_s^{1-a}$ and it is well known that the Nash inequality $\|u\|_r \leq C \|\nabla u\|_a^a \|u\|_s^{1-a}$ and it is well known that the Nash inequality (1) and the Euclidian type Sobolev inequality are equivalent in the sense that if one of them is valid, the other one is also valid (i.e. see [1]). It is, also, well known that with this procedure of passing from the one type of inequalities to the other, is impossible to compare the best constants, since the inequalities under use are not optimal.

As far as the optimal version of Nash inequality (1) is concerned, the best constant $A_0(n)$, that is

$$A_0(n)^{-1} = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \left( \int_{\mathbb{R}^n} |u| \, dx \right)^{\frac{4}{n}}} {\left( \int_{\mathbb{R}^n} u^2 \, dx \right)^{1+\frac{2}{n}}} \mid u \in C_0^\infty (\mathbb{R}^n), u \neq 0 \right\},$$

has been computed by Carlen and Loss in [3], together with the characterization of the extremals for the corresponding optimal inequality, as

$$A_0(n) = \frac{(n + 2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_1^N |B^n|^\frac{2}{n}},$$

where $|B^n|$ denotes the euclidian volume of the unit ball $B^n$ in $\mathbb{R}^n$ and $\lambda_1^N$ is the first Neumann eigenvalue for the Laplacian for radial functions in the unit ball $B^n$.

For an example of application of the Nash inequality with the best constant, we refer to Kato [13] and for a geometric proof with an asymptotically sharp constant, we refer to Beckner [2].

For compact Riemannian manifolds, the Nash inequality still holds with an additional $L^1$-term and that is why we will refer this as the $L^1$-Nash inequality.

Given $(M, g)$ a smooth compact Riemannian $n$-manifold, $n \geq 2$, we get here the existence of real constants $A$ and $B$ such that for any $u \in C^\infty(M)$,

$$\left( \int_M u^2 dV_g \right)^{1+\frac{4}{n}} \leq A \int_M |\nabla u|^2 dV_g \left( \int_M |u| \, dV_g \right)^{\frac{4}{n}} + B \left( \int_M |u| \, dV_g \right)^{2+\frac{4}{n}}$$

(2)

The best constant for this inequality is defined as

$$A_{opt}^1(M) = \inf \{ A > 0 : \exists B > 0 \text{ s.t. } (2) \text{ is true } \forall u \in C^\infty(M) \}$$
This inequality has been studied completely by Druet, Hebey and Vaugon. They proved in [6] that \( A_{\text{opt}}^1(M) = A_0(n) \), and (2) with its optimal constant \( A = A_0(n) \) is sometimes valid and sometimes not, depending on the geometry of \( M \).

Humbert in [11] studied the following \( L^2 \)-Nash inequality

\[
\left( \int_M u^2 \, dV_g \right)^{1+\frac{2}{n}} \leq \left( A \int_M |\nabla u|^2 \, dV_g + B \int_M u^2 \, dV_g \right) \left( \int_M |u| \, dV_g \right)^{\frac{2}{n}},
\]

for all \( u \in C^\infty(M) \), of which the best constant is defined as

\[
A_{\text{opt}}^2(M) = \inf \{ A > 0 : \exists B > 0 \text{ s.t. } (3) \text{ is true } \forall u \in C^\infty(M) \}.
\]

Contrary to the sharp \( L^1 \)-Nash inequality, in this case, he proved that \( B \) always exists and \( A_{\text{opt}}^2(M) = A_0(n) \).

We denote \( \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, +\infty) \) and \( \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\} \). The trace Nash inequality states that a constant \( \tilde{A} > 0 \) exists such that for all \( u \in C^\infty_0(\mathbb{R}^n_+) \), \( n \geq 2 \) with \( \nabla u \in L^2(\mathbb{R}^n) \) and \( u|_{\partial \mathbb{R}^n_+} \in L^1(\partial \mathbb{R}^n_+) \cap L^2(\partial \mathbb{R}^n_+) \)

\[
\left( \int_{\partial \mathbb{R}^n_+} u^2 \, ds \right)^{\frac{n}{n-1}} \leq \tilde{A} \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx \left( \int_{\partial \mathbb{R}^n_+} |u| \, ds \right)^{\frac{2}{n-1}},
\]

where \( ds \) is the standard volume element on \( \mathbb{R}^{n-1} \) and the trace of \( u \) on \( \partial \mathbb{R}^n_+ \) is also denoted by \( u \).

Let \( \tilde{A}_0(n) \) be the best constant in Nash inequality (4). That is

\[
\tilde{A}_0(n)^{-1} = \inf \left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx \left( \int_{\partial \mathbb{R}^n_+} |u| \, ds \right)^{\frac{2}{n-1}}}{\left( \int_{\partial \mathbb{R}^n_+} u^2 \, ds \right)^{\frac{n}{n-1}}} \mid u \in C^\infty_0(\mathbb{R}^n_+), u \neq 0 \right\}
\]

The computation problem of the exact value of \( \tilde{A}_0(n) \) still remains open.

For compact Riemannian manifolds with boundary, Humbert, also, studied in [12] the trace Nash inequality.

On smooth compact \( n \)-dimensional, \( n \geq 2 \), Riemannian manifolds with boundary, for all \( u \in C^\infty(M) \), consider the following trace Nash inequality

\[
\left( \int_{\partial M} u^2 \, dS_g \right)^{\frac{n}{n-1}} \leq \left( \tilde{A} \int_M |\nabla u|^2 \, dV_g + \tilde{B} \int_{\partial M} u^2 \, dS_g \right) \left( \int_{\partial M} |u| \, dS_g \right)^{\frac{2}{n-1}}.
\]
The best constant for the above inequality is defined as
\[
\tilde{A}_{\text{opt}}(M) = \inf \left\{ \tilde{A} > 0 : \exists \tilde{B} > 0 \text{ s.t. } (5) \text{ is true } \forall u \in C^\infty(M) \right\}
\]
It was proved in [12] that \( \tilde{A}_{\text{opt}}(M) = \tilde{A}_0(n) \), and (5) with its optimal constant \( \tilde{A} = \tilde{A}_0(n) \) is always valid.

In this paper we prove that, when the functions are invariant under an isometry group, all orbits of which are of infinite cardinal, the Nash inequalities can be improved, in the sense that we can get a higher critical exponent. More precisely we establish:

\textbf{(A)} The best constant for the \textbf{Nash inequality} on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup \( G \) of the isometry group \( \text{Is}(M, g) \), where all the orbits have infinite cardinal, and

\textbf{(B)} The best constant for the \textbf{Trace Nash inequality} on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup \( G \) of the isometry group \( \text{Is}(M, g) \), where all the orbits have infinite cardinal.

These best constants are improvements over the classical cases due to the symmetries which arise and reflect the geometry of the manifold.

2 Results and Examples

2.1 Results

\textbf{Theorem 2.1} Let \((M, g)\) be a smooth, compact \( n \)-dimensional Riemannian manifold, \( n \geq 3 \), with boundary, \( G \)-invariant under the action of a subgroup \( G \) of the isometry group \( \text{Is}(M, g) \). Let \( k \) denotes the minimum orbit dimension of \( G \) and \( V \) denotes the minimum of the volume of the \( k \)-dimension orbits. Then for any \( \varepsilon > 0 \) there exists a constant \( B_\varepsilon \) such that and for all \( u \in H^2_{I_0, G}(M) \) the following inequality

\[
\left( \int_M u^2 dV_g \right)^{\frac{n-k+2}{n-k}} \leq \left( (A_G + \varepsilon)^\frac{n-k}{n-k+2} \int_M |\nabla u|^2_g dV_g + B_\varepsilon \int_M u^2 dV_g \right) \\
\times \left( \int_M |u| dV_g \right)^{\frac{4}{n-k}}
\]
holds, where \( A_G = \frac{A_0(n-k)}{V^{n-k-1}} \).
Moreover the constants \( A_G \) is the best constant for this inequality.

**Theorem 2.2** Let \((M, g)\) be a smooth, compact \( n \)-dimensional Riemannian manifold, \( n \geq 3 \), with boundary, \( G \)-invariant under the action of a subgroup \( G \) of the isometry group \( Is(M, g) \). Let \( k \) denotes the minimum orbit dimension of \( G \) and \( V \) denotes the minimum of the volume of the \( k \)-dimensional orbits. Then for any \( \varepsilon > 0 \) there exists a constant \( \tilde{B}_\varepsilon \) such that and for all \( u \in H^2_{1,G}(M) \) the following inequality holds,

\[
\left( \int_{\partial M} u^2 dS_g \right)^{\frac{n-k}{n-k-1}} \leq \left( (\tilde{A}_G + \varepsilon) \int_M |\nabla u|^2 dV_g + \tilde{B}_\varepsilon \int_{\partial M} u^2 dS_g \right)^{\frac{n-k-1}{2}} \times \left( \int_{\partial M} |u| dS_g \right)^{\frac{n-k-1}{n-k-1}}
\]

holds, where \( \tilde{A}_G = \frac{\tilde{A}_0(n-k)}{V^{n-k-1}} \).
Moreover the constants \( \tilde{A}_G \) is the best constant for this inequality.

**Corollary 2.1** For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that and for all \( u \in H^2_{1,G}(T) \) the following inequality holds,

\[
\left( \int_{\partial T} u^2 dS_T \right)^2 \leq \left( \frac{\tilde{A}_0(2) + \varepsilon}{2\pi (l-r)} \int_T |\nabla u|^2 dV + C_\varepsilon \int_{\partial T} u^2 dS_T \right) \left( \int_{\partial T} |u| dS_T \right)^2.
\]

Moreover the constant \( \tilde{A}_{opt}(T) = \frac{\tilde{A}_0(2)}{2\pi (l-r)} \) is the best constant for this inequality and verifies

\[
\frac{3\sqrt{3}}{4\pi^2(l-r)} \leq \tilde{A}_{opt}(T) \leq \frac{2}{\pi^2(l-r)}.
\]

**2.2 Examples**

**Example 1.** Let \( T \) be the three dimensional solid torus

\[
T = \left\{ (x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - l)^2 + z^2 \leq r^2, l > r > 0 \right\}.
\]


with the metric induced by the \( \mathbb{R}^3 \) metric. Let \( G = O(2) \times I \) be the group of rotations around axis \( z \). All \( G \)-orbits of \( T \) are circles and the orbit of minimum volume is the circle of radius \( l - r \), and of length \( 2\pi(l - r) \). Then \( T \) is a compact 3-dimensional manifold with boundary, invariant under the action of the subgroup \( G \) of the isometry group \( O(3) \).

In [5] we found the best constant in inequality (5) in the 3-dimensional solid torus, which is \( G \)-invariant under the action of a subgroup \( G = O(2) \times I \) of the isometry group \( O(3) \).

**Example 2.** Let \( \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m, k \geq 2, m \geq 1 \) and \( \Omega \subset (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^m \). Let \( G_{k,m} = O(k) \times I_d \), the subgroup of the isometry group \( O(n) \) of the type \( (x_1, x_2) \rightarrow (\sigma(x_1), x_2), \sigma \in O(k), x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^m \). Suppose that \( \Omega \) is invariant under the action of \( G_{k,m} (\tau(\Omega) = \Omega, \forall \tau \in G_{k,m}) \). Then \( \Omega \) is a compact \( n \)-dimensional manifold with boundary, invariant under the action of the subgroup \( G_{k,m} \) of the isometry group \( O(n) \).

### 3 Notations and preliminary results

Let \((M, g)\) be a compact \( n \)-dimensional, \( n \geq 3 \), Riemannian manifold with boundary \( G \)-invariant under the action of a subgroup \( G \) of the isometry group \( I(M, g) \). We assume that \((M, g)\) is a smooth bounded open subset of a slightly larger Riemannian manifold \((\widetilde{M}, g)\) (i.e. see [14]), invariant under the action of a subgroup \( G \) of the isometry group of \((\widetilde{M}, g)\).

Consider the spaces of all \( G \)-invariant functions under the action of the group \( G \)

\[
C^\infty_G(M) = \{ u \in C^\infty(M) : u \circ \tau = u, \forall \tau \in G \}
\]

\[
C^\infty_{0,G}(M) = \{ u \in C^\infty_0(M) : u \circ \tau = u, \forall \tau \in G \}
\]

Denote \( H^p_1(M) \) the completion of \( C^\infty(M) \) with respect to the norm

\[
\| u \|_{H^p_1(M)} = \left( \| \nabla u \|_{L^p(M)}^p + \| u \|_{L^p(M)}^p \right)^{1/p},
\]

and \( H^p_{1,G}(M) \) the space of all \( G \)-invariant functions of \( H^p_1(M) \).

For completeness we cite some background material and results from [4].

Given \((\widetilde{M}, g)\) a Riemannian manifold (complete or not, but connected), we denote by \( I(\widetilde{M}, g) \) its group of isometries.

Let \( P \in M \) and \( O_P = \{ \tau(P) : \tau \in G \} \) be its orbit of dimension \( k \), \( 0 \leq k < n \). According to ([14] § 9, [7]) the map \( \Phi : G \rightarrow O_P \), defined by \( \Phi(\tau) = \tau(P) \),
is of rank $k$ and there exists a submanifold $H$ of $G$ of dimension $k$ with \( \text{Id} \in H \), such that $\Phi$ restricted to $H$ is a diffeomorphism from $H$ onto its image denoted $V_P$.

Let $N$ be a submanifold of $M$ of dimension \((n-k)\), such that $T_P \Phi (H) \oplus T_P N = T_PM$. Using the exponential map at $P$, we build a \((n-k)\)-dimensional submanifold $W_P$ of $N$, orthogonal to $O_P$ at $P$ and such that for any $Q \in W_P$, the minimizing geodesics of $(M, g)$ joining $P$ and $Q$ are all contained in $W_P$.

Let $\Psi : H \times W_P \to M,$ be the map defined by $\Psi (\tau, Q) = \tau (Q)$. According to the local inverse theorem, there exists a neighborhood $\mathcal{V}_{(\text{Id},P)} \subset H \times W_P$ of $(\text{Id}, P)$ and a neighborhood $\mathcal{M}_P \subset M$ such that $\Psi^{-1} = (\Psi_1 \times \Psi_2)$, from $\mathcal{M}_P$ onto $\mathcal{V}_{(\text{Id},P)}$ is a diffeomorphism.

Up to restricting $V_P$, we choose a normal chart $(\mathcal{V}_P, \varphi_1)$ around $P$ for the metric $\tilde{g}$ induced on $O_P$, with $\varphi_1 (\mathcal{V}_P) = U \subset \mathbb{R}^k$. In the same way, we choose a geodesic normal chart $(\mathcal{W}_P, \varphi_2)$ around $P$ for the metric $\tilde{g}$ induced on $W_P$, with $\varphi_2 (\mathcal{W}_P) = W \subset \mathbb{R}^{n-k}$.

We denote by $\xi_1 = \varphi_1 \circ \Phi \circ \Psi_1, \xi_2 = \varphi_2 \circ \Psi_2, \xi = (\xi_1, \xi_2)$ and $\Omega = \mathcal{M}_P$.

From the above and the Lemmas 1 and 2 in [10] the following lemma holds:

\begin{lemma}
Let $(M, g)$ be a compact Riemannian $n$–manifold with boundary, $G$ a compact subgroup of $I(M, g)$, $P \in M$ with orbit of dimension $k$, $0 \leq k < n$. Then there exists a chart $(\Omega, \xi)$ around $P$ such that the following properties are valid:

1. $\xi (\Omega) = U \times W$, where $U \subset \mathbb{R}^k$ and $W \subset \mathbb{R}^{n-k}$.

2. $U, W$ are bounded, and $W$ has smooth boundary.

3. $(\Omega, \xi)$ is a normal chart of $M$ around of $P$, $(\mathcal{V}_P, \varphi_1)$ is a normal chart around of $P$ of submanifold $O_P$ and $(\mathcal{W}_P, \varphi_2)$ is a normal geodesic chart around of $P$ of submanifold $W_P$.

4. For any $\varepsilon > 0$, $(\Omega, \xi)$ can be chosen such that:

$$1 - \varepsilon \leq \sqrt{\det (g_{ij})} \leq 1 + \varepsilon \text{ on } \Omega, \text{ for } 1 \leq i, j \leq n$$

$$1 - \varepsilon \leq \sqrt{\det (\tilde{g}_{ij})} \leq 1 + \varepsilon \text{ on } \mathcal{V}_P, \text{ for } 1 \leq i, j \leq k.$$ 
\end{lemma}
5. For any \( u \in C^\infty_G(M) \), \( u \circ \xi^{-1} \) depends only on \( W \) variables.

We say that we choose a neighborhood of \( O_P \) when we choose \( \delta > 0 \) and we consider
\[
O_{P, \delta} = \{ Q \in \tilde{M} : d(Q, O_P) < \delta \}.
\]
Such a neighborhood of \( O_P \) is called a tubular neighborhood.

Let \( P \in M \) and \( O_P \) be its orbit of dimension \( k \). Since the manifold \( M \) is
included in \( \tilde{M} \), we can choose a normal chart \((\Omega_P, \xi_P)\) around \( P \) such that
Lemma 3.1 holds for some \( \varepsilon_0 > 0 \). For any \( Q = \tau(P) \in O_P \), where \( \tau \in G \),
we build a chart around \( Q \), denoted by \( (\tau(\Omega_P), \xi_P \circ \tau^{-1}) \) and “isometric” to
\((\Omega_P, \xi_P)\). \( O_P \) is then covered by such charts. We denote by \((\Omega_{P, m})_{m=1,...,M}\)
a finite extract covering. Then we can choose \( \delta > 0 \) small enough, depending
on \( P \) and \( \varepsilon_0 \) such that the tubular neighborhood \( O_{P, \delta} \), (where \( d(\cdot, O_P) \) is the
distance to the orbit) has the following properties:
(i) \( O_{P, \delta} \) is a submanifold of \( \tilde{M} \) with boundary,
(ii) \( d^2(\cdot, O_P) \) is a \( C^\infty \) function on \( O_{P, \delta} \) and
(iii) \( O_{P, \delta} \) is covered by \((\Omega_{m})_{m=1,...,M}\).

Clearly, \( M \) is covered by \( \cup_{P \in M} O_{P, \delta} \). We denote by \((O_{j, \delta})_{j=1,...,J}\) a finite extract covering of \( M \), where all \( O_{j, \delta} \)'s are covered by \((\Omega_{jm})_{m=1,...,M_j}\). Then
we will have
\[
M \subset \bigcup_{j=1}^J \bigcup_{m=1}^{M_j} \Omega_{jm} = \bigcup_{m=1}^{\sum_{j=1}^J M_j} \Omega_i
\]
So we obtain a finite covering of \( M \) consisting of \( \Omega_i \)'s, \( i = 1,...,\sum_{j=1}^J M_j \). We
choose such a covering in the following way:

(i) If \( P \) lies in the interior of \( M \), then there exist \( j, 1 \leq j \leq J \) and
\( m, 1 \leq m \leq M_j \) such that the tubular neighborhood \( O_{j, \delta} \) and \( \Omega_{jm} \), with
\( P \in \Omega_{jm} \), lie entirely in \( M \)'s interior, (that is, if \( P \in M \setminus \partial M \), then \( O_{j, \delta} \subset
M \setminus \partial M \) and \( \Omega_{jm} \subset M \setminus \partial M \)).

(ii) If \( P \) lies on the boundary \( \partial M \) of \( M \), then a \( j, 1 \leq j \leq J \) exists,
such that the tubular neighborhood \( O_{j, \delta} \) intersects the boundary \( \partial M \) and an
\( m, 1 \leq m \leq M_j \) exists, such that \( \Omega_{jm} \), with \( P \in \Omega_{jm} \), cuts a part of the
boundary \( \partial M \). Then the \( \Omega_{jm} \) covers a patch of the boundary of \( M \), and the
whole of the boundary is covered by charts around \( P \in \partial M \).

We denote \( N \) the projection of the image of \( M \), through the charts
\((\Omega_{jm}, \xi_{jm})\), \( j = 1,...,J, m = 1,...,M_j \), on \( \mathbb{R}^{n-k} \). Then \((N, \tilde{g}) \) is a \( (n-k) \)-
dimensional compact submanifold of \( \mathbb{R}^{n-k} \) with boundary and \( N \) is covered
by \((W_i), i = 1, \ldots, \sum_{j=1}^{J} M_j\), where \(W_i\) is the component of \(\xi_i(\Omega_i)\) on \(\mathbb{R}^{n-k}\) for all \(i = 1, \ldots, \sum_{j=1}^{J} M_j\).

Let \(p\) be the projection of \(\xi_i(P), P \in M\) on \(\mathbb{R}^{n-k}\). Thus one of the following holds:

(i) If \(p \in N \setminus \partial N\), then \(W_i \subset N \setminus \partial N\) and \(W_i\) is a normal geodesic neighborhood with normal geodesic coordinates \((y_1, \ldots, y_{n-k})\).

(ii) If \(p \in \partial N\), then \(W_i\) is a Fermi neighborhood with Fermi coordinates \((y_1, \ldots, y_{n-k-1}, t)\).

In these neighborhoods we have

\[
1 - \varepsilon_0 \leq \sqrt{\det(\bar{g}_{ij})} \leq 1 + \varepsilon_0 \quad \text{on} \quad N, \quad \text{for} \quad 1 \leq i, j \leq n - k,
\]

where \(\varepsilon_0\) can be as small as we want, depending on the chosen covering.

For convenience in the following we set

\[
O_j = O_{j, \delta} = \{ Q \in \widetilde{M} : d(Q, O_{P_j}) < \delta\}
\]

We still need the following lemma:

**Lemma 3.2 (a)** For any \(v \in H^p_{1,G}(O_j \cap M), v \geq 0\) the following properties are valid:

1. 
\[
(1 - c\varepsilon_0) V_j \int_N v_2^p d\bar{v}_g \leq \int_M v^p dV_g \leq (1 + c\varepsilon_0) V_j \int_N v_2^p d\bar{v}_g
\]

2. 
\[
(1 - c\varepsilon_0) V_j \int_N |\nabla_{\bar{g}} v_2|^p d\bar{v}_g \leq \int_M |\nabla_{g} v|^p dV_g \leq (1 + c\varepsilon_0) V_j \int_N |\nabla_{\bar{g}} v_2|^p d\bar{v}_g
\]

**(b)** For any \(v \in H^p_{1,G}(O_j \cap \partial M), v \geq 0\) the following property is valid:

\[
(1 - c\varepsilon_0) V_j \int_{\partial N} v_2 ds_g \leq \int_{\partial M} v ds_g \leq (1 + c\varepsilon_0) V_j \int_{\partial N} v_2 ds_g
\]

where \(V_j = Vol(O_j), v_2 = v \circ \xi^{-1}\) and \(c\) is a positive constant.

Moreover, we need the following propositions:
Proposition 3.1 For any $\varepsilon > 0$ and for all $u \in C_0^\infty(G)$ the following inequality holds

$$\left( \int_{\partial M} (\eta_j u)^2 \, dS_g \right)^{\frac{n-k}{n-k-1}} \leq \frac{\tilde{A}_0 (n-k) + \varepsilon}{V_j^{\frac{1}{n-k-1}}} \int_M |\nabla (\eta_j u)|^2 \, dV_g$$

$$\times \left( \int_{\partial M} \eta_j |u| \, dS_g \right)^{\frac{2}{n-k-1}}$$

(6)

where $(\eta_j)$ is a partition of unity associating to $(O_j)$.

Proof. By Lemma 3.2 (b) with $\nu = \eta_j u$ and $p = 2$ we obtain

$$\int_{\partial M} (\eta_j u)^2 \, dS_g \leq (1 + c\varepsilon_0) V_j \int_{\partial N} (\eta_j u)^2 \, ds_{\bar{g}}$$

(7)

Let $\varepsilon_0 > 0$. Then there exist $\delta > 0$ such that for any $Q = \xi(P) \in \partial N, P \in \partial M$ and for all $\phi \in C_0^\infty(B_Q(\delta))$, $(B_Q(\delta)$ is the $(n-k)$–dimensional ball of radius $\delta$ centered on $Q$), according to Theorem 2 of [12] the following inequality holds

$$\left( \int_{\partial N} \phi^2 \, ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \left( \tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} \right) \int_N |\nabla_{\bar{g}} \phi|^2 \, dV_{\bar{g}} \left( \int_{\partial N} |\phi| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}$$

(8)

where $\tilde{A}_0(n-k)$ is the best constant of the trace Nash inequality

$$\left( \int_{\partial N} \phi^2 \, ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \left( \tilde{A} \int_N |\nabla_{\bar{g}} \phi|^2 \, dV_{\bar{g}} + \tilde{B} \int_{\partial N} \phi^2 \, ds_{\bar{g}} \right) \left( \int_{\partial N} |\phi| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}$$

By (7) and (8) we have

$$\left( \int_{\partial M} (\eta_j u)^2 \, dS_g \right)^{\frac{n-k}{n-k-1}} \leq \left[ (1 + c\varepsilon_0) V_j \right]^{\frac{n-k}{n-k-1}} \left( \tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} \right)$$

$$\times \int_N \left| \nabla_{\bar{g}} (\eta_j u) \right|^2 \, dV_{\bar{g}} \left( \int_{\partial N} |(\eta_j u)| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}$$

(9)

From (9) and Lemma 3.2 arises

$$\left( \int_{\partial M} (\eta_j u)^2 \, dS_g \right)^{\frac{n-k}{n-k-1}} \leq \frac{\left[ (1 + c\varepsilon_0) V_j \right]^{\frac{n-k}{n-k-1}}}{\left[ (1 - c\varepsilon_0) V_j \right]^{\frac{n-k}{n-k-1}}}$$

$$\times \frac{\left[ (1 - c\varepsilon_0) V_j \right]^{\frac{1}{n-k-1}} (1 - c\varepsilon_0) V_j}{\left[ (1 - c\varepsilon_0) V_j \right]^{\frac{1}{n-k-1}}}$$

(10)
\[
\times \left( \tilde{A}_0 (n - k) + \frac{\varepsilon_0}{2} \right) \int_M |\nabla_g (\eta_j u)|^2 dV_g \\
\times \left( \int_{\partial M} |(\eta_j u)| dS_g \right)
\]

or

\[
\left( \int_{\partial M} (\eta_j u)^2 dS_g \right)^{\frac{n-k}{n-k-1}} \leq \frac{1 + c \varepsilon_0}{1 - c \varepsilon_0} \left( \tilde{A}_0 (n - k) + \frac{\varepsilon_0}{2} \right) \frac{1}{V^{n-k-1}} \\
\times \int_M |\nabla_g (\eta_j u)|^2 dV_g \left( \int_{\partial M} |(\eta_j u)| dS_g \right)^{\frac{2}{n-k-1}}
\]

(10)

Given \( \varepsilon > 0 \) we can choose \( \varepsilon_0 > 0 \) small enough such that

\[
\frac{1 + c \varepsilon_0}{1 - c \varepsilon_0} \left( \tilde{A}_0 (n - k) + \frac{\varepsilon_0}{2} \right) \leq \tilde{A}_0 (n - k) + \varepsilon
\]

and then by (10) we obtain (6).

**Proposition 3.2** For any \( \varepsilon > 0 \) and for all \( u \in C^\infty_G(M) \) the following inequality holds

\[
\left( \int_{\partial M} u^2 dS_g \right)^{\frac{n-k}{n-k-1}} \leq \frac{\tilde{A}_0 (n - k) + \varepsilon}{V^{n-k-1}} \left( \int_M |\nabla u|^2 dV_g + C \int_M u^2 dV_g \right) \\
\times \left( \int_{\partial M} |u| dS_g \right)^{\frac{2}{n-k-1}}
\]

(11)

**Proof.** We set \( \alpha_j = \frac{\eta_j^2}{\sum_{m=1}^J \eta_m^2}, j = 1, 2, \ldots, J \) and so \( \{\alpha_j\} \) is a partition of unity for \( M \) subordinated in the covering \((O_j)_{j=1,2,\ldots,J}\) and functions \( \sqrt{\alpha_j} \) are smooth, \( G \)-invariants and there exist a constant \( H \) such that for any \( j = 1, \ldots, J \) holds

\[
|\nabla \sqrt{\alpha_j}| \leq H
\]

(12)

Let \( u \in C^\infty_G(M) \). Then we have

\[
\int_{\partial M} u^2 dS_g = \int_{\partial M} \left( \sum_{j=1}^J \alpha_j u^2 \right) dS_g = \sum_{j=1}^J \int_{\partial M} (\sqrt{\alpha_j} u)^2 dS_g
\]

(13)
By (13) and Proposition 3.1 arises

\[
\int_{\partial M} u^2 dS_g \leq \sum_{j=1}^{J} \left( \frac{\tilde{A}_0 (n - k) + \varepsilon}{V_{\frac{n-k-1}{n-k}}} \right)^{\frac{n-k-1}{n-k}} \left( \int_{M} |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}}
\]

\[
\times \left( \int_{\partial M} \sqrt{\alpha_j} |u| dS_g \right)^{\frac{2}{n-k}}
\]

and since \( \min V_j = V \) we obtain

\[
\int_{\partial M} u^2 dS_g \leq \left( \frac{\tilde{A}_0 (n - k) + \varepsilon}{V_{\frac{n-k-1}{n-k}}} \right)^{\frac{n-k-1}{n-k}} \sum_{j=1}^{J} \left( \int_{M} |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}}
\]

\[
\times \left( \int_{\partial M} \sqrt{\alpha_j} |u| dS_g \right)^{\frac{2}{n-k}}
\]

(14)

By Hölder’s inequality we have

\[
\left( \int_{\partial M} \sqrt{\alpha_j} |u| dS_g \right)^{\frac{2}{n-k}} = \left[ \int_{\partial M} \left( \sqrt{\alpha_j} |u| \sqrt{|u|} \right) dS_g \right]^{\frac{2}{n-k}}
\]

\[
\leq \left( \int_{\partial M} \alpha_j |u| dS_g \right)^{\frac{1}{n-k}} \left( \int_{\partial M} |u| dS_g \right)^{\frac{1}{n-k}}
\]

and by (13) we obtain

\[
\int_{\partial M} u^2 dS_g \leq \left( \frac{\tilde{A}_0 (n - k) + \varepsilon}{V_{\frac{n-k-1}{n-k}}} \right)^{\frac{n-k-1}{n-k}} \left( \int_{\partial M} |u| dS_g \right)^{\frac{1}{n-k}}
\]

\[
\times \sum_{j=1}^{J} \left( \int_{M} |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}} \left( \int_{\partial M} \alpha_j |u| dS_g \right)^{\frac{1}{n-k}}
\]

(15)

Moreover the following Hölder’s inequality

\[
\sum_{j=1}^{J} a_j b_j \leq \left( \sum_{j=1}^{J} a_j^{p} \right)^{1/p} \left( \sum_{j=1}^{J} b_j^{q} \right)^{1/q}
\]

(16)
holds for any $a_j, b_j$ nonnegative and for all $p \geq 1, q \geq 1$ with $(1/p) + (1/q) = 1$. Setting in (16)

$$a_j = \left( \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}}, \quad b_j = \left( \int_{\partial M} \alpha_j |u| dS_g \right)^{\frac{1}{n-k}}$$

$$p = \frac{n-k}{n-k-1}, \quad q = n-k$$

we obtain

$$\sum_{j=1}^J \left( \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}} \left( \int_{\partial M} \alpha_j |u| dS_g \right)^{\frac{1}{n-k}} \leq \left( \sum_{j=1}^J \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}} \left( \int_{\partial M} \left( \sum_{j=1}^J \alpha_j \right) |u| dS_g \right)^{\frac{1}{n-k}}$$

$$= \left( \int_{\partial M} |u| dS_g \right)^{\frac{1}{n-k}} \left( \sum_{j=1}^J \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}} \left( \sum_{j=1}^J \int_{\partial M} \alpha_j |u| dS_g \right)^{\frac{1}{n-k}}$$

(17)

By (15) and (17) arises

$$\int_{\partial M} u^2 dS_g \leq \left( \frac{A_0 (n-k) + \varepsilon}{V^{n-k-1}} \right)^{\frac{n-k-1}{n-k}} \left( \int_{\partial M} |u| dS_g \right)^{\frac{2}{n-k}} \times \left( \sum_{j=1}^J \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \right)^{\frac{n-k-1}{n-k}}$$

(18)

Further more since

$$|\nabla (\sqrt{\alpha_j} u)|^2 = \alpha_j |\nabla u|^2 + u^2 |\nabla \sqrt{\alpha_j}|^2 + 2 \langle \nabla u, \nabla \sqrt{\alpha_j} \rangle u \sqrt{\alpha_j}$$

and since (12) holds, after some computations we obtain

$$\sum_{j=1}^J \int_M |\nabla (\sqrt{\alpha_j} u)|^2 dV_g \leq \int_M |\nabla u|^2 dV_g + HJ \int_M u^2 dV_g$$

$$+ 2 \int_M \sum_{j=1}^J \langle \nabla \sqrt{\alpha_j}, \nabla u \rangle u \sqrt{\alpha_j} dV_g$$

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But

\[ 0 = \nabla (\sum_{j=1}^{J} \alpha_j) = \sum_{j=1}^{J} (\nabla \alpha_j) \]
\[ = \sum_{j=1}^{J} 2\sqrt{\alpha_j} \nabla (\sqrt{\alpha_j}) = 2 \sum_{j=1}^{J} \sqrt{\alpha_j} \nabla (\sqrt{\alpha_j}) \]

Thus the following inequality holds

\[ \sum_{j=1}^{J} \int_M |\nabla (\sqrt{\alpha_j}u)|^2 dV_g \leq \int_M |\nabla u|^2 dV_g + C \int_M u^2 dV_g \quad (19) \]

Finally by (18) and (19) we have

\[ \int_{\partial M} u^2 dS_g \leq \left( \frac{A_0 (n-k) + \varepsilon_0}{V} \right)^{\frac{n-k+1}{n-k+1}} \left( \int_{\partial M} |u| dS_g \right)^{\frac{2}{n-k}} \]
\[ \times \left( \int_M |\nabla u|^2 dV_g + C \int_M u^2 dV_g \right)^{\frac{n-k+1}{n-k}} \]

and the proposition is proved.

4 Proofs

**Proof of Theorem 2.1** The proof is based on [6]. Let as sketch the proof. Following the same steps as in Proposition 3.1 as well as by Lemma 3.2 and Theorem 1.1 in [6] we obtain that for any \( \varepsilon_0 > 0 \) and for all \( u \in C^\infty_G (M) \) the following inequality holds

\[ \left( \int_M (\eta_j u)^2 dV_g \right)^{\frac{n-k+2}{n-k}} \leq \frac{A_0 (n-k) + \frac{\varepsilon_0}{2}}{V_j^{\frac{2}{n-k}}} \int_M |\nabla (\eta_j u)|^2 dV_g \]
\[ \times \left( \int_M \eta_j |u| dV_g \right)^{\frac{4}{n-k}}, \quad (20) \]

where \( (\eta_j) \) is a partition of unity associating to \( (O_j) \).

Let \( \{\alpha_j\}_{j=1, \ldots, J} \) a partition of unity for \( M \) as in Proposition 3.2. Since for any \( u \in C^\infty (M) \) the following holds

\[ \|u\|_2^2 = \|u^2\|_1 = \|\Sigma_{j=1}^{J} \alpha_j u^2\|_1 \leq \Sigma_{j=1}^{J} \|\alpha_j u^2\|_1 = \Sigma_{j=1}^{J} \|\sqrt{\alpha_j} u\|_2^2 \]
where \( \| \cdot \| \) stands for the \( L^p \)-norm.

By Hölder’s inequality \( \| \sqrt{\alpha_j} u \|_1 \leq \| \alpha_j u \|_1^{1/2} \| u \|_1^{1/2} \) from (20) we obtain

\[
\int_{M} u^2 \, dV_g \leq \left( \frac{A_0 (n - k) + \frac{\varepsilon_0}{2}}{V_{n-k}^{\frac{2}{n-k+2}}} \right)^{\frac{n-k+2}{n-k}} \left( \int_{M} |u| \, dV_g \right)^{\frac{n-k}{n-k+2}} \times \sum_{j=1}^{J} \left( \int_{M} \left| \nabla (\alpha_j u) \right|^2 \, dV_g \right)^{\frac{n-k}{n-k+2}} \left( \int_{M} \alpha_j |u| \, dV_g \right)^{\frac{2}{n-k+2}}
\]

By (22) because of (12) and (16) with

\[
a_j = \left( \int_{M} |\nabla (\sqrt{\alpha_j} u)|^2 \, dV_g \right)^{\frac{n-k}{n-k+2}}, \quad b_j = \left( \int_{\partial M} \alpha_j |u| \, dV_g \right)^{\frac{2}{n-k+2}}
\]

\[
p = \frac{n - k + 2}{n - k}, \quad q = \frac{n - k + 2}{2}
\]

arises

\[
\left( \int_{M} u^2 \, dV_g \right)^{\frac{n-k+2}{n-k}} \leq \left( \frac{A_0 (n - k) + \frac{\varepsilon_0}{2}}{V_{n-k}^{\frac{2}{n-k}}} \right)^{\frac{n-k}{n-k+2}} \times \left( \int_{M} \left| \nabla u \right|^2 \, dV_g + H^2 J \int_{M} u^2 \, dV_g \right)^{\frac{4}{n-k}} \times \left( \int_{M} |u| \, dV_g \right)^{\frac{4}{n-k}}
\]

(22)

Given \( \varepsilon > 0 \) we can choose \( \varepsilon_0 > 0 \) such that

\[
\frac{A_0 (n - k) + \frac{\varepsilon_0}{2}}{V_{n-k}^{\frac{2}{n-k}}} \leq \frac{A_0 (n - k)}{V_{n-k}^{\frac{2}{n-k}}} + \varepsilon
\]

and the theorem is proved.

**Proof of Theorem 2.2** According to [8] (Lemma 4) there exists an orbit of minimum dimension \( k \) and of minimum volume. Let \( \mathcal{O} \) be the orbit of dimension \( k \) and of minimum volume, that is \( \text{Vol}(\mathcal{O}) = \min \text{Vol}(\mathcal{O})_{j=1,...,J} = V \). Let also the set \( \mathcal{O}_\delta = \{ Q \in \tilde{M} : d(Q, \mathcal{O}) < \delta \} \), where \( d(\cdot, \mathcal{O}) \) is the distance
to the orbit. For \( u \in C^\infty_0 (\mathcal{O}_\delta \cap M) \) by Proposition 3.2 because of Lemma 3.2 we have sequentially

\[
\left( 1 - c\varepsilon_0 \right) V \int_{\partial N} u_2^2 ds_{\bar{g}} \lesssim \left( \int_M u^2 dV_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \left( \int_{\partial M} u^2 dS_{\bar{g}} \right)^{\frac{n-k}{n-k-1}}
\]

\[
\leq \frac{\tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} (1 + c\varepsilon_0)}{V \frac{n-k}{n-k-1}} \left( \int_M |\nabla u|^2 dV_{\bar{g}} + C \int_M u^2 dV_{\bar{g}} \right) \left( \int_{\partial M} |u| dS_{\bar{g}} \right)^{\frac{2}{n-k-1}}
\]

\[
\leq \frac{\tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} (1 + c\varepsilon_0)}{V \frac{n-k}{n-k-1}} \left( 1 + c\varepsilon_0 \right) V \int_N |\nabla u_2|^2 dv_{\bar{g}} + C \int_N u_2^2 dv_{\bar{g}} \right) \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}
\]

or

\[
\left( \int_{\partial N} u_2^2 ds_{\bar{g}} \right)^{\frac{2}{n-k-1}} \leq \frac{\tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} (1 + c\varepsilon_0)}{V \frac{n-k}{n-k-1}} \left( 1 + c\varepsilon_0 \right) V \int_N |\nabla u_2|^2 dv_{\bar{g}} + C \int_N u_2^2 dv_{\bar{g}} \right) \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}
\]

\[
\left( \int_{\partial N} u_2^2 ds_{\bar{g}} \right)^{\frac{2}{n-k-1}} \leq \frac{(1 + c\varepsilon_0) \frac{n-k+1}{n-k} \tilde{A}_0 (n-k) + \frac{\varepsilon_0}{2} V \frac{n-k+1}{n-k}}{(1 - c\varepsilon_0) \frac{n-k}{n-k-1}} \left( 1 + c\varepsilon_0 \right) V \int_N |\nabla u_2|^2 dv_{\bar{g}} + C \int_N u_2^2 dv_{\bar{g}} \right) \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right)^{\frac{2}{n-k-1}},
\]
\[
\left( \int_{\partial N} u_2^2 ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \frac{(1 + c\varepsilon_0) \frac{n-k+1}{n-k-1}}{(1 - c\varepsilon_0) \frac{n-k}{n-k-1}} \left( \tilde{A}_0 (n - k) + \frac{\varepsilon_0}{2} \right) \\
\times \left( \int_{N} |\nabla u_2|^2 dv_{\bar{g}} + C \int_{N} u_2^2 dv_{\bar{g}} \right) \\
\times \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right) 
\]

hence, given \( \varepsilon > 0 \), we choose \( \varepsilon_0 > 0 \) small enough to imply

\[
\left( \int_{\partial N} u_2^2 ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \left( \tilde{A}_0 (n - k) + \varepsilon \right) \left( \int_{N} |\nabla u_2|^2 dv_{\bar{g}} + C \int_{N} u_2^2 dv_{\bar{g}} \right) \\
\times \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right) \tag{23}
\]

By (23) and Theorem 2 in [12] arises

\[
\left( \int_{\partial N} u_2^2 ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} \leq \left( \tilde{A}_0 (n - k) + \varepsilon \right) \int_{N} |\nabla u_2|^2 dv_{\bar{g}} + \bar{B}_\varepsilon \int_{\partial N} u_2^2 ds_{\bar{g}} \right) \\
\times \left( \int_{\partial N} |u_2| ds_{\bar{g}} \right) \tag{24}
\]

Since the best constant of the trace Nash inequality in \( M \) has the same value with the best constant of the trace Nash inequality of the manifold \( O_\delta \), suppose that for any \( \alpha > 0 \) there exists \( \theta > 0 \) such that

\[
\lambda_\alpha = \inf_A I_\alpha \leq \frac{V \frac{1}{n-k-1}}{A_0(n - k) + \varepsilon} - \theta \tag{25}
\]

where

\[
I_\alpha = \frac{\left( \int_M |\nabla u_\alpha|^2 dV_{\bar{g}} + \alpha \int_{\partial M} u_\alpha^2 dS_{\bar{g}} \right) \left( \int_{\partial M} |u_\alpha| dS_{\bar{g}} \right)^{\frac{2}{n-k-1}}}{\left( \int_{\partial M} u_\alpha^2 dS_{\bar{g}} \right)^{\frac{n-k}{n-k-1}}}
\]

and

\[
A = \{ u \in H^2_{1,G}(M) : u|_{\partial M} \neq 0 \}
\]

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Thus there exists \( u_\alpha \in H^2_{1,\Omega}(\Omega_\delta) \) such that

\[
I_\alpha \leq \frac{V_{n-k-1}}{A_0(n-k) + \varepsilon} - \theta 
\]  

By (26) and Proposition 3.2, we have

\[
\left[ (1 - c\varepsilon_0) V \right]^{\frac{n-k+1}{n-k-1}} \left( \int_N |\nabla g u_\alpha|^2 \, dv_\bar{g} + \alpha \int_{\partial N} u_\alpha^2 \, ds_{\bar{g}} \right) \left( \int_{\partial N} |u_\alpha| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}} 
\]

By (26) and Proposition 3.2, we have

\[
\frac{V_{n-k-1}}{A_0(n-k) + \varepsilon} - \theta
\]

or

\[
\frac{\left( \int_N |\nabla g u_\alpha|^2 \, dv_\bar{g} + \alpha \int_{\partial N} u_\alpha^2 \, ds_{\bar{g}} \right) \left( \int_{\partial N} |u_\alpha| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}}}{\left( \int_{\partial N} u_\alpha^2 \, ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}}}
\]

thus for \( \varepsilon_0 \) small enough we have

\[
\left( \int_N |\nabla g u_\alpha|^2 \, dv_\bar{g} + \alpha \int_{\partial N} u_\alpha^2 \, ds_{\bar{g}} \right) 
\left( \int_{\partial N} |u_\alpha| \, ds_{\bar{g}} \right)^{\frac{2}{n-k-1}} < 
\left( \int_{\partial N} u_\alpha^2 \, ds_{\bar{g}} \right)^{\frac{n-k}{n-k-1}} 
\]

(1 + \varepsilon) \left( \frac{1}{A_0(n-k) - \theta} \right)

\[
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\]
The latter inequality for $\varepsilon$ small enough yields
\[
\frac{\left(\int_{\partial N} |\nabla \bar{g} u_{\alpha}|^2 d\nu_{\bar{g}} + \alpha \int_{\partial N} u_{\alpha}^2 ds_{\bar{g}} \right) \left(\int_{\partial N} |u_{\alpha}| ds_{\bar{g}}\right)^{\frac{2}{n-k-1}}}{\left(\int_{\partial N} u_{\alpha}^2 ds_{\bar{g}}\right)^{\frac{n-k}{n-k-1}}} \leq \frac{1}{A_0 (n-k)}
\] (27)

Because of (24), inequality (27) is false and the proof is complete.

Proof of Corollary 2.1 The proof of the first part of Corollary 2.1 arises immediately by Theorem 2.2. (An another proof is presented in [5]). For the second part we use Theorem 1 in [12], since we have calculated the value of the first Neumann eigenvalue for the Laplacian on radial functions on the interval $[-1,1]$.

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