Electromagnetic vortex lines riding atop null solutions of the Maxwell equations

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Abstract. New method of introducing vortex lines of the electromagnetic field is outlined. The vortex lines arise when a complex Riemann-Silberstein vector $(E + iB)/\sqrt{2}$ is multiplied by a complex scalar function $\phi$. Such a multiplication may lead to new solutions of the Maxwell equations only when the electromagnetic field is null, i.e. when both relativistic invariants vanish. In general, zeroes of the $\phi$ function give rise to electromagnetic vortices. The description of these vortices benefits from the ideas of Penrose, Robinson and Trautman developed in general relativity.

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1. Introduction

Complex scalar fields in a three-dimensional space carry generically a collection of vortex lines. These vortex lines are located at the zeroes of the field. Complex generic vector fields do not have such properties because the requirement that all field components simultaneously vanish leads to an overdetermined set of equations. There are at least two ways to overcome this difficulty and to introduce vortex lines also for vector fields. One may either select one relevant component of the field (as, for example, in the scalar theory of light) or one may build a single field entity from various vector components. The first approach has been used in most papers on phase singularities in wave fields (for a thorough review and list of references see [1]). In our previous publication [2] we have chosen the second method. We have studied phase singularities and vortex lines associated with the zeroes of the square $F \cdot F$ of the Riemann-Silberstein‡ (RS) vector (my units are chosen so that $\epsilon = 1$ and $\mu = 1$)

$$F = (E + iB)/\sqrt{2}.$$  

(1)

The vortex lines of the electromagnetic field defined in this manner are rather elusive objects — they lack clear signature and may be hard to observe. Even though the phase of the field has dislocations, the electromagnetic field does not vanish on these vortex lines, but it is just a null electromagnetic field — the two relativistic field invariants vanish on vortex lines. Nevertheless, the vortex lines built on $F \cdot F$ may still have some role to play in singular optics as discussed in the papers by M. Berry [5] and G. Kaiser [6] appearing in this issue.

In the present paper I explore a different approach. It is based on the observation that when a null solution of the Maxwell equations is taken as the background field, an extra scalar multiplier may imprint on this solution a rich vortex structure. This time all components of the electromagnetic field will vanish on vortex lines. Of course, the vortex lines that are introduced in this manner do not have a generic character since they are found only in special cases — for null fields. However, the class of solutions of the Maxwell equations that carry these vortex lines is, in my opinion, sufficiently broad and interesting to make this approach relevant for singular optics. In addition, the present study touches upon some important concepts discovered before in the general relativistic context.

2. Scalar prefactor as a carrier of phase singularities

The starting point of this investigation is an observation that a complex scalar function $\phi(r, t)$ multiplying the RS vector will control the zeroes of the electromagnetic field. Wherever $\phi(r, t)$ vanishes, the electric and magnetic field vectors vanish. Assuming that the background field $F$, as well as the product $\phi F$, satisfy the Maxwell equations

$$i \partial_t F(r, t) = \nabla \times F(r, t), \quad \nabla \cdot F(r, t) = 0,$$

(2)

$$i \partial_t (\phi(r, t)F(r, t)) = \nabla \times (\phi(r, t)F(r, t)), \quad \nabla \cdot (\phi(r, t)F(r, t)) = 0.$$  

(3)

‡ This name was introduced in a review paper [3] on the photon wave function, where one may also find information on the history of the Riemann-Silberstein vector. The first to make a practical use of the RS vector in the analysis of electromagnetic waves was Bateman [4]. The RS vector offers a very convenient representation of the electromagnetic field, especially in the study of vortex lines, and I shall make an extensive use of it here.
we arrive at the following conditions on the prefactor field $\phi(r,t)$ (to simplify the notation, from now on I omit the dependence of $F$ on space-time coordinates)

$$F_i \partial_\xi \phi(r,t) = -F \times \nabla \phi(r,t),$$

$$F \cdot \nabla \phi(r,t) = 0.$$  \hspace{1cm} (4)

This set of equations possesses nontrivial solutions only when the background field is null, i.e. $F^2 = E^2/2 - B^2/2 + iE \cdot B = 0$. In order to prove this assertion, we can take the scalar product of both sides of Eq. (4) with the vector $F$. Since the scalar product on the right hand side is equal to zero, the left hand side must also vanish and this results in the following alternative: either $\partial_i \phi(r,t)$ or $F \cdot F$ must vanish. In the first case, we end up with a trivial result: the scalar function $\phi(r,t)$ must be a constant. It cannot depend on space and time variables since its gradient is at the same time parallel to $F$, as seen from Eq. (4), and orthogonal to $F$, as seen from Eq. (5). In the second case we may have a nontrivial form of $\phi(r,t)$. Taking the scalar product of Eq. (4) with the complex conjugate RS vector $F^*$ we obtain the condition

$$i F^* \cdot F \partial_\xi \phi(r,t) = -F^* \cdot (F \times \nabla \phi(r,t)) = -(F^* \times F) \cdot \nabla \phi(r,t).$$

(6)

Finally, I obtain the following two partial differential equations for the function $\phi(r,t)$

$$\partial_\xi \phi(r,t) + n \cdot \nabla \phi(r,t) = 0, \quad F \cdot \nabla \phi(r,t) = 0,$$

(7)

where $n$ is the normalized Poynting vector

$$n = \frac{-i F^* \times F}{F^* \cdot F} = \frac{E \times B}{E^2/2 + B^2/2}.$$  \hspace{1cm} (8)

The solutions of Eqs. (7) will in general have zeroes and this will lead to vortex lines of the electromagnetic field riding atop the background solution $F$. On each vortex line the electromagnetic field vanishes. Near the vortex line the electric and magnetic field vectors followed around a closed contour rotate by $2\pi m$, where $m$ is the topological charge of the vortex. In contrast to nonrelativistic wave mechanics, there is no interaction between vortex lines introduced in this manner. All vortices move independently, since the product of solutions of the first order partial differential equations (7) is again a solution. We may always add new vortices without changing the motion of the existing ones. In the following sections I shall give general solutions of Eqs. (7) in two special cases: when the background field is a plane monochromatic wave and when it is the Robinson-Trautman field.

3. Simple example

Let us consider the solution of the Maxwell equations described by the following RS vector

$$F(r,t) = (\hat{x} + i \hat{y}) \exp(ikz - i\omega t).$$

(9)

This solution describes the left-handed circularly polarized wave propagating in the $z$ direction. It is a null field, since $(\hat{x} + i \hat{y}) \cdot (\hat{x} + i \hat{y}) = 0$. In this case, Eqs. (7) take on the form

$$(\partial_t + \partial_z) \phi(r,t) = 0, \quad (\partial_x + i\partial_y) \phi(r,t) = 0.$$  \hspace{1cm} (10)

General solution of these equations is an arbitrary function of two variables: the real variable $z-t$ and the complex variable $x + iy$. Thus, we arrive at the family of solutions of the Maxwell equations of the form

$$F(r,t) = f(z-t, x + iy)(\hat{x} + i \hat{y}) \exp(ik(z-t)).$$

(11)
These solutions may have a rich vortex structure. Assuming, for simplicity, a polynomial dependence on $x + iy \equiv w$, we may write down the function $f$ in the factorized form

$$f(z - t, x + iy) = (w - a_1(z - t))(w - a_2(z - t)) \ldots (w - a_n(z - t)),$$

(12)

where $a_k(z - t)$ are some functions of $z - t$. Each term in this product gives rise to an infinite vortex line whose shape is determined by the equation $x + iy = a_k(z - t)$. Each vortex line moves independently. In general, each vortex line flies with the speed of light in the $z$-direction. However, when the function $a_k(z - t)$ is just a constant, the corresponding vortex line is a stationary straight line in $z$-direction. A simple example of the $\phi$ function of the type (12) is

$$\phi(r, t) = (x + iy)^m.$$

(13)

In this case the RS vector $\phi F$ carries a straight vortex line along the $z$-axis with the topological charge $m$.

4. Trautman-Robinson fields

Null solutions of the Maxwell equations were extensively studied by Robinson [9] in connection with his work in general relativity. In particular, these solutions played an important role in the discovery of new solutions of Einstein field equations [10] and also they appear in Penrose twistor theory [11]. A family of null solutions of the Maxwell equations that featured prominently in these studies was described in [12] — I shall call them the Robinson-Trautman (RT) fields. They are usually expressed in a special coordinate system but I shall use here more intuitive Cartesian coordinates.

In these coordinates the RS vectors representing the RT solutions can be written in the form

$$F_x = f(\alpha, \beta) \frac{\beta^2 - 1}{t - ia - z},$$

(14a)

$$F_y = if(\alpha, \beta) \frac{\beta^2 + 1}{t - ia - z},$$

(14b)

$$F_z = -f(\alpha, \beta) \frac{2\beta}{t - ia - z},$$

(14c)

where $f(\alpha, \beta)$ is an arbitrary function of the following complex combinations of the Cartesian coordinates

$$\alpha = t - ia + z - \frac{x^2 + y^2}{t - ia - z}, \quad \beta = \frac{x - iy}{t - ia - z}.$$ 

(15)

The RT Maxwell fields are related to the simple plane wave field discussed in the previous section. Taking the $f$ function in the form $f(\alpha, \beta) = \alpha^{-\beta}$, we obtain the following localized null solution of Maxwell equations

$$F_x = \left((x - iy)^2 - (t - ia - z)^2\right)/d^3,$$

(16a)

$$F_y = i \left((x - iy)^2 + (t - ia - z)^2\right)/d^3,$$

(16b)

$$F_z = -2(x - iy)(t - ia - z)/d^3,$$

(16c)

where $d = ((t - ia)^2 - x^2 - y^2 - z^2)$. This field may be obtained from the plane wave solution of the previous section by the coordinate transformation (conformal

§ Topical review has been recently published by Trautman [13]
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reflected) $x^\mu \rightarrow x^\mu/x^2$ accompanied by the shift in time by an imaginary constant $-ia$ and evaluated in the limit of infinite wavelength, when $k \rightarrow 0$. The energy density of this field is equal to

$$F^* \cdot F = 2\frac{(a^2 + (t-z)^2 + x^2 + y^2)^2}{((a^2-t^2 + x^2 + y^2 + z^2)^2 + 4a^2t^2)^{3/2}}. \quad (17)$$

From this formula we see that the electromagnetic field (16 can never vanish. Note, that the presence of $ia$ eliminates the singularity on the light cone and makes the energy of the field finite,

$$\int d^3r F^* \cdot F = \frac{\pi}{4a^5}. \quad (18)$$

There are some simple solutions with a single vortex line built on this background function. The simplest ones are obtained by taking the scalar multiplier function $\phi$ equal to $\beta^n$. These solutions have a stationary vortex line along the $z$-direction with the topological charge equal to $-n$. A bit more complex vortex line is obtained when $\phi = \alpha + 4a\beta$. The vanishing of $\phi$ leads to the following two equations for the vortex coordinates

$$2y = t, \quad (x + 2a)^2 + z^2 = 3a^2 + \frac{3}{4}t^2. \quad (19)$$

Thus, at each time the vortex line forms a circle lying on the uniformly moving $y = t/2$ plane. The radius of the circle is shrinking until $t = 0$ and then it starts expanding. Possibilities for constructing solutions with more intricate vortex lines are unlimited.

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