On the absence of fifth-order contributions to the nucleon mass in heavy-baryon chiral perturbation theory

Judith A. McGovern and Michael C. Birse

Theoretical Physics Group, Department of Physics and Astronomy
University of Manchester, Manchester, M13 9PL, U.K.

We have calculated the contribution of order $M^2$ in the chiral expansion of the nucleon mass in two-flavour heavy-baryon chiral perturbation theory. Only one irreducible two-loop integral enters, and this vanishes. All other contributions in the heavy-baryon limit can be absorbed in the physical pion-nucleon coupling constant which enters in the $M^2$, term, and so there are no contributions at $M^3$. Including finite nucleon mass corrections, the only contribution agrees with the expansion of the relativistic one-loop graph in powers of $M/r$, and is only 0.3% of the $M^4$ term. This is an encouraging result for the convergence of two-flavour heavy-baryon chiral perturbation theory.

The fundamental degrees of freedom of the strong interaction are quarks and gluons, but in spite of the many successes of QCD in describing high energy phenomenology, a full description of the particles that constitute ordinary matter still eludes us. There is however increasing interest in the interactions of such particles at low energies as new, precise, data on nucleon properties and interactions becomes available. For low enough energy it has long been known that the interaction of pions is governed only by the symmetries of QCD, in particular SU(2) x SU(2) chiral symmetry, and a successful systematic effective field theory of pions, Chiral Perturbation Theory [1], has been developed. In this theory the pionic lagrangian is expanded as a power series, with each sub-

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*Bernard et al. [5] consider the contribution of two-loop diagrams to the imaginary part of the nucleon isoscalar electromagnetic formfactor, but this does not require the evaluation of two-loop integrals.
alently the pion-nucleon sigma commutator, $\sigma_{qN} = M^2 d\sigma_{qN}/dM^2$, estimated from scattering data to be $45 \pm 8$ MeV \cite{1}. To order $q^0$ there are two contributions, with one LEC which has now been estimated independently from pion-nucleon scattering data \cite{8,9}. To order $q^4$ more LEC’s will enter, for which there we as yet have no experimental handle. Borasoy and Meißner have attempted from pion-nucleon scattering data \cite{8,7}. To order $q^4$ more LEC’s will enter, for which there we as yet have no experimental handle.

In order to calculate most quantities to order $q^2$, the expansion of the nucleonic Lagrangian up to $\mathcal{L}^{(5)}_{\chi N}$ would be required. The only contribution to $\Sigma(0)$ from the fifth order term would be a simple counterterm of order $M^6$. However the Lagrangian is analytic in the quark masses, that is in $M_q^2$, so such a term cannot exist. (For the same reason any irreducible two-loop diagrams must give finite contributions to $\Sigma(0)$, since there can be no counterterm to cancel divergences.) Similarly, in the absence of mass insertions of order $M_X$, $\mathcal{L}^{(4)}_{\pi\pi}$ cannot contribute at this order. All the relevant Feynman amplitudes for these calculations can be found in the work of Meißner \textit{et al.} \cite{10} or, using an alternative form of $\mathcal{L}^{(3)}_{\pi N}$, in that of Mojić \textit{et al.} \cite{11} (who also gives the relevant amplitudes from $\mathcal{L}^{(4)}_{\pi\pi}$).

The heavy-baryon propagator is given by

$$S^{-1} = \omega - \Sigma(\omega, k),$$

where the nucleon momentum is written as $p = mv + k$, $m$ is the bare mass, and $\omega = v \cdot k$. The mass shift $\delta m = m_N - m$ is the value of $\omega$ for which the propagator has a pole at zero three-momentum:

$$\delta m - \Sigma(\delta m, 0) = 0.$$  \hspace{1cm} (2)

(Of course the mass shift could be found from the pole of the propagator for any three-momentum, but as HBCPT is constructed to respect Lorentz invariance, the result will not change.)

In order to solve Eq. 2 to a given order in $M_\pi$, both $\delta m$ and $\Sigma(\omega)$ must be expanded in powers of $M_\pi$. To order $M^2_\pi$, $\Sigma(\delta m) = \Sigma(0)$, so

$$\delta m^{(2)} + \delta m^{(3)} = -4c_1 M^2_\pi \frac{3g^2 M^3_\pi}{32\pi F^2_\pi},$$

where the second term comes from the diagram in Fig. 1. Writing as $\Sigma^{(n)}$ the expression for the $O(q^n)$ part of $\Sigma$, which will have an expansion $\Sigma^{(n)}(\omega) = a_1 M^2_\pi + a_2 M_\pi^{3-n-1} \omega + \ldots$, we obtain

$$\delta m^{(5)} = \Sigma^{(5)}(0) + \delta m^{(2)} \Sigma^{(4)'}(0) + \delta m^{(3)} \Sigma^{(3)'}(0) + \frac{1}{2} (\delta m^{(2)})^2 \Sigma^{(3)''}(0)$$  \hspace{1cm} (4)

where primes indicate derivatives with respect to $\omega$. Any calculation in HBCPT yields an answer in terms of the bare Lagrangian parameters, $M, q, F$ and $m$ to lowest order, which are the first terms in an expansion in powers of $M_\pi, g_A, F_\pi$ and $m_N$. It is customary to replace the bare parameters by the physical ones so that the lowest order predictions do not change as higher orders are added. This however gives an extra contribution to the higher order calculations.

In this case, therefore, to the calculation of $\Sigma^{(5)}(0)$ from the diagrams of Fig. 2 must be added a piece from $\Sigma^{(3)}(0)$. Here the relevant parameters are $M_\pi$ and the pseudo-vector $\pi N$ coupling $f_{\pi NN} \equiv g_{\pi NN}/m_N = g/F(1 + O(M^2_\pi))$.

The diagrams which contribute to $\Sigma^{(5)}(0)$ are shown in Fig. 2. They consist of two-loop diagrams, and one-loop diagrams with either one insertion from $\mathcal{L}^{(3)}_{\pi N}$ or $\mathcal{L}^{(4)}_{\pi\pi}$, or two insertions from $\mathcal{L}^{(2)}_\pi$. In all cases only zero external momentum is required. The two-loop diagrams shown in Fig. 2f-i all vanish trivially. The diagram in Fig. 2a can after some algebra be written as $(3M^2/4(2d - 3)F^4)I$, where

$$I = \int \frac{d^d l \, d^d k}{(2\pi)^{2d}} \frac{1}{v \cdot k (M^2 - l^2)(M^2 - (k - l)^2).}$$

This integral can be done by using Feynman parameters, first combining the mesonic propagators in the standard way and integrating over $l$, and then using the identity

$$\frac{1}{P^q Q^r} = \frac{2\pi^{(p + q)}}{(p + q)! (p - q)!} \int_0^\infty \frac{y^{p - 1} dy}{(P + 2yQ)^{p + q}}$$

to include the heavy-baryon propagator. Integration over $y$ and $k$ yields

$$I = -\frac{M^{2d-5} \pi^{(p + q)} (d - 2)}{(4\pi)^d} \int_0^1 (x - x^2)^{(1-d)/2} dx$$

$$= -\frac{M^{2d-5} \pi^{(p + q)} (d - 2)(d - 3)}{(4\pi)^d} \int_0^1 (x - x^2)^{(1-d)/2} dx$$

which tends to zero as $d \to 4$.

The same integral also appears in the evaluation of Fig. 2b and 2d, and it is the only non-separable two-loop
Fig. 2: Contributions to $\Sigma^{(5)}$. Solid dots represent insertions from $\mathcal{L}_{\pi N}^{(3)}$ and $\mathcal{L}_{\pi \pi}^{(4)}$, and crosses from $\mathcal{L}_{\pi N}^{(2)}$ (both include fixed terms from the expansion in $1/m_N$).

Fig. 3: Contributions to $\Sigma^{(4)}$. Integral which does. Since it vanishes, all the two-loop diagrams 2a-e are proportional to $\Delta_0 J_0(0)$, in the notation of Fig. 1, and $\Delta_0 = 2M^2 L(M)$ is just the integral of the meson propagator and diverges as $1/(d-4)$. These divergences are cancelled by the graphs of Fig. 2j-l with counterterm insertions from $\mathcal{L}_{\pi N}^{(3)}$ and $\mathcal{L}_{\pi \pi}^{(4)}$, which however bring in the low energy constants $2d_{16} - d_{18}$ from the baryonic, and $\bar{t}_3$ and $\bar{t}_4$ from the mesonic Lagrangians. (The notation is that of [3], with LEC’s defined to absorb the usual factors of $\log(M/\mu)$; in [8], $\bar{t}_n \rightarrow \bar{t}_{n+1}$ and $\bar{t}_n \rightarrow \bar{t}_n/16\pi^2$.) The graphs with two insertions from $\mathcal{L}_{\pi N}^{(2)}$, Fig. 2o-q, are all finite. (Counterterm graphs 2m-o all give vanishing contributions.) The final contribution of Fig. 2, with the conventions of [7], is

$$
\Sigma^{(5)}_{2\text{-loop+CT}}(0) = \frac{3g^2 M^5}{32\pi F^2} \left( \frac{2\bar{t}_4 - 3\bar{t}_3}{F^2} - \frac{4(2d_{16} - d_{18})}{g} \right) + \frac{g^2}{32\pi^2 F^2} + \frac{1}{8m^2} + \frac{6c_1}{m} + 24c_2^2 \right) .
$$

The next contribution to $\delta m^{(5)}$ is from the last three terms of Eq. (7), the relevant graphs are shown in Fig. 3 and Fig. 1, and the integrals are finite. The result is

$$
\delta m^{(3)}\Sigma^{(3)}(0) + \frac{1}{2}(\delta m^{(2)})^2\Sigma''(0) + \frac{1}{2}(\delta m^{(2)})^2\Sigma''(0)
$$

$$
= \frac{3g^2 M^5}{32\pi F^2} \left( \frac{3g^2}{32\pi^2 F^2} - 4c_1 \left( \frac{3}{2m} + 12c_1 \right) + 24c_2^2 \right) .
$$

It may be seen that all terms involving the LEC $c_1$ vanish from the sum of Eqs (7,8).

Finally we need the contribution to $\Sigma^{(5)}(0)$ from replacing the bare constants in $\Sigma^{(3)}(0)$ with their physical values. (The expression for $\Sigma^{(3)}(0)$ may be found from that for $\delta m^{(3)}$ in Eq. (3) by reinstating the bare coupling constants.) The diagrams which contribute to the renormalised $\pi N$ coupling $f_{\pi NN}(= g_{\pi NN}/m_N)$ are given in Fig. 4.

The last diagram of Fig. 4 indicates the contribution from the expansion of the pion and nucleon wavefunction renormalisation to order $M^2$. Some comment is required about $Z_N$, which is defined as the residue of the nucleon propagator at the pole. This has recently been the subject of a paper by Ecker and Možiš [12], who point out a correction which is necessary in order to reproduce the results of a relativistic calculation in HBCPT. It arises because a nucleon can also be created by the eliminated
“small” component of the relativistic nucleon field, and so the normalisations of the relativistic and heavy baryons do not match. The same correction was included through the spinor normalisation by Fearing et al. [13]. The net effect of including the small-component sources, to order \( q^3 \), is to give
\[
Z_N = (1 + k^2/4m_N^2)Z_N^{\text{HB}},
\]
where \( Z_N^{\text{HB}} \) is calculated purely from the HBCPT Lagrangian. (Whereas in the relativistic theory \( Z_N \) is a constant, in HBCPT it may depend on the on-shell three-momentum \( k \).) In the framework of Ref. [4], which we have been using here,
\[
Z_N^{\text{HB}} = 1 - k^2/4m_N^2 + \ldots,
\]
where other terms of order \( q^2 \) have been suppressed. Thus in this framework, the dependence on \( k \) cancels to this order. With this small-component-source correction, we find that there are no \( O(1/m_N^2) \) corrections to \( f_{\pi NN} \). (In [4] such terms are also shown to be absent from \( g_{\pi N} \), so the usual expression for the Goldberger-Treiman discrepancy [13], proportional only to \( \sigma_{16} \), holds.) Thus we obtain for the physical pion mass and \( \pi N \) coupling constant,
\[
M_{\pi}^2 = M^2(1 + 2t_3 M^2/F^2)
\]
\[
f_{\pi NN} = \frac{g}{F} \left[ 1 - M^2 \left( \frac{g^2}{16\pi^2 F^2} + \frac{t_4}{F^2} - \frac{4d_{16} - 2d_{18}}{g} \right) \right]
\]
and substituting in \( \delta m^{(3)}(0) \) to obtain the final contribution to \( \delta m^{(5)} \) gives
\[
\Sigma_{1-\text{loop}}^{(5)}(0) = -\frac{3g^2 M_\pi^2}{32\pi F^2} \left[ \frac{g^2}{8\pi^2 F^2} - \frac{2t_4 - 3t_3}{F^2} + \frac{4(2d_{16} - d_{18})}{g} \right].
\]
Collecting all contributions, Eq. (5,8,12), we obtain our final result,
\[
\delta m^{(3)} + \delta m^{(5)} = -\frac{3g^2 M_\pi M_N^2}{32\pi m_N^2} \left( 1 - \frac{M_\pi^2}{8m_N^2} \right).
\]
Thus in the heavy-baryon limit the order \( M_N^2 \) contribution vanishes, with all corrections being absorbed in the physical pion mass and pion-nucleon coupling constant in the \( M_\pi^3 \) contribution. For finite nucleon mass, the correction is just that obtained obtained if the relativistic one-loop contribution is expanded in powers of \( M_\pi/m_N \).

1In [12], \( Z_N \) in this framework is given wrongly, since the \( O(1/m_N^2) \) kinetic energy insertion, Eq. C1 of [13], has been missed. We have repeated our calculations in Ecker and Mojžíš’s framework, and obtained the same final result.
nately the analyses are not sensitive to $c_1$, though the preferred value would leave a substantial $M_4^2$ contribution. None-the-less it would appear at least that the odd and even power series are converging separately.

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