Tighter Constraints of Multiqubit Entanglement*

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Abstract Monogamy and polygamy relations characterize the distributions of entanglement in multipartite systems. We provide classes of monogamy and polygamy inequalities of multiqubit entanglement in terms of concurrence, entanglement of formation, negativity, Tsallis-\(q\) entanglement, and Rényi-\(\alpha\) entanglement, respectively. We show that these inequalities are tighter than the existing ones for some classes of quantum states.

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1 Introduction

Quantum entanglement is an essential feature of quantum mechanics, which distinguishes the quantum from the classical world and plays a very important role in quantum information processing.[1−4] One singular property of quantum entanglement is that a quantum system entangled with one of the other subsystems limits its entanglement with the remaining ones, known as the monogamy of entanglement (MoE).[5−6] MoE plays a key role in many quantum information and communication processing tasks such as the security proof in quantum cryptographic scheme[7] and the security analysis of quantum key distribution.[8]

For a tripartite quantum state \(\rho_{ABC}\), MoE can be described as the following inequality

\[E(\rho_{ABC}) \geq E(\rho_{AB}) + E(\rho_{AC}),\]

(1)

where \(\rho_{AB} = \text{tr}_C(\rho_{ABC})\) and \(\rho_{AC} = \text{tr}_B(\rho_{ABC})\) are reduced density matrices, and \(E\) is an entanglement measure. However, it has been shown that not all entanglement measures satisfy such monogamy relations. It has been shown that the squared concurrence \(C^2\)[9−10] the squared entanglement of formation (EoF) \(E^2\)[11] and the squared convex roof extended negativity (CREN) \(N^2\)[12−13] satisfy the monogamy relations for multipartite states.

Another important concept is the assisted entanglement, which is a dual amount to bipartite entanglement measure. It has a dually monogamous property in multipartite quantum systems and gives rise to polygamy relations. For a tripartite state \(\rho_{ABC}\), the usual polygamy relation is of the form,

\[E^\alpha(\rho_{ABC}) \leq E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC}),\]

(2)

where \(E^\alpha\) is the corresponding entanglement measure of assistance associated to \(E\). Such polygamy inequality has been deeply investigated in recent years, and was generalized to multiqubit systems and classes of higher-dimensional quantum systems.[12,14−20]

Recently, generalized classes of monogamy inequalities related to the \(\beta\)-th power of entanglement measures were proposed. In Refs.[21−22], the authors proved that the squared concurrence and CREN satisfy the monogamy inequalities in multiqubit systems for \(\beta \geq 2\). It has also been shown that the EoF satisfies monogamy relations when \(\beta \geq \sqrt{2}\).[21−23] Besides, the Tsallis-\(q\) entanglement and Rényi-\(\alpha\) entanglement satisfy monogamy relations when \(\beta \geq 1\).[14−22] for some cases. Moreover, the corresponding polygamy relations have also been established.[16−18,20,25−26]

In this paper, we investigate monogamy relations and polygamy relations in multiqubit systems. We provide tighter constraints of multiqubit entanglement than all the existing ones, thus give rise to finer characterizations of the entanglement distributions among the multiqubit systems.

2 Tighter Constraints Related to Concurrence

We first consider the monogamy inequalities and polygamy inequalities for concurrence. For a bipartite pure state \(|\psi\rangle_{AB}\) in Hilbert space \(H_A \otimes H_B\), the concurrence is defined as[27−28] \(C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{tr}\rho_B^2)}\) with

\[\rho_B = \frac{1}{2} (\mathbb{1}_B + \text{tr}_A |\psi\rangle\langle \psi|)\]
\[ \rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|. \]  

The concurrence for a bipartite mixed state \( \rho_{AB} \) is defined by the convex roof extension,

\[
C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle \}} \sum_i p_i C(|\psi_i\rangle),
\]

where the minimum is taken over all possible decompositions of \( \rho_{AB} = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) with \( \sum_i p_i = 1 \) and \( p_i \geq 0 \). For an N-qubit state \( \rho_{AB_1 \cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}} \), the concurrence \( C(\rho_{AB_1 \cdots B_{N-1}}) \) of the state \( \rho_{AB_1 \cdots B_{N-1}} \) under bipartite partition \( A \) and \( B_1 \cdots B_{N-1} \) satisfies[21]

\[
C^\beta(\rho_{AB_1 \cdots B_{N-1}}) \geq C^\beta(\rho_{A_1}) + C^\beta(\rho_{AB_2}) + \cdots + C^\beta(\rho_{AB_{N-1}}),
\]

for \( \beta \geq 2 \), where \( \rho_{AB_j} \) denote two-qubit reduced density matrices of subsystems \( AB_j \) for \( j = 1, 2, \ldots, N-1 \). Later, the relation (3) is improved for the case \( \beta \geq 2 \) as

\[
C^\beta(\rho_{AB_1 \cdots B_{N-1}}) \geq C^\beta(\rho_{A_1}) + \frac{\beta}{2} C^\beta(\rho_{AB_2}) + \cdots + \left( \frac{\beta}{2} \right)^{m-1} C^\beta(\rho_{AB_m}) + \cdots + \left( \frac{\beta}{2} \right)^{N-1} C^\beta(\rho_{AB_{N-1}}),
\]

conditioned that \( C(\rho_{A_i}) \geq C(\rho_{AB_1 \cdots B_{N-1}}) \) for \( i = 1, 2, \ldots, m \), and \( C(\rho_{AB_i}) \leq C(\rho_{AB_1 \cdots B_{N-1}}) \) for \( j = m+1, \ldots, N-2 \). The relation (4) is further improved for \( \beta \geq 2 \) as[22]

\[
C^\beta(\rho_{AB_1 \cdots B_{N-1}}) \geq C^\beta(\rho_{A_1}) + \left( 2^{\beta/2} - 1 \right) C^\beta(\rho_{AB_2}) + \cdots + \left( 2^{\beta/2} - 1 \right)^{m-1} C^\beta(\rho_{AB_m}) + \cdots + \left( 2^{\beta/2} - 1 \right)^{N-1} C^\beta(\rho_{AB_{N-1}}),
\]

with the same conditions as in Eq. (4).

For a tripartite state \( |\psi\rangle_{ABC} \), the concurrence of assistance (CoA) is defined by[29–30]

\[
C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle \}} \sum_i p_i C(|\psi_i\rangle),
\]

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} \), and \( C(|\psi\rangle_{AB}) = C_a(|\psi\rangle_{AB}) \). The generalized polygamy relation based on the concurrence of assistance was established in Refs. [16–17]

\[
C^2(|\psi\rangle_{AB_1 \cdots B_{N-1}}) = C_a^2(|\psi\rangle_{AB_1 \cdots B_{N-1}}) \leq C_a^2(\rho_{AB_1}) + C_a^2(\rho_{AB_2}) + \cdots + C_a^2(\rho_{AB_{N-1}}).
\]

These monogamy and polygamy relations for concurrence can be further tightened under some conditions. To this end, we first introduce the following lemma.

**Lemma 1** Suppose that \( k \) is a real number satisfying \( 0 < k \leq 1 \), then for any \( 0 \leq t \leq k \) and non-negative real numbers \( m, n \), we have

\[
(1 + t)^n \geq 1 + \frac{(1 + k)^n - 1}{k^n} t^n,
\]

for \( m \geq 1 \), and

\[
(1 + t)^n \leq 1 + \frac{(1 + k)^n - 1}{k^n} t^n,
\]

for \( 0 \leq n \leq 1 \).

**Proof** We first consider the function \( f(m, x) = (1 + x)^m - x^m \) with \( x \geq 1/k \) and \( m \geq 1 \). Then \( f(m, x) \) is an increasing function of \( x \), since

\[
\frac{\partial f(m, x)}{\partial x} = m[(1 + x)^{m-1} - x^{m-1}] \geq 0.
\]

Thus,

\[
f(m, x) \geq f\left(m, \frac{1}{k}\right) = \left(1 + \frac{1}{k}\right)^m - \left(\frac{1}{k}\right)^m = \frac{(k + 1)^m - 1}{k^m}.
\]

Set \( x = 1/t \) in Eq. (10), we get the inequality (8).

Similar to the proof of inequality (8), we can obtain the inequality (9), since in this case \( f(n, x) \) is a decreasing function of \( x \) for \( x \geq 1/k \) and \( 0 \leq n \leq 1 \).

In the next, we denote \( C_{ABC} = C(\rho_{ABC}) \) the concurrence of \( \rho_{ABC} \) and \( C_{A|B_1 \cdots B_{N-1}} = C(\rho_{A|B_1 \cdots B_{N-1}}) \) for convenience.

**Lemma 2** Suppose that \( k \) is a real number satisfying \( 0 < k \leq 1 \). Then for any \( 2 \otimes 2 \otimes 2^n-2 \) mixed state \( \rho \in H_A \otimes H_B \otimes H_C \), if \( C_{ABC}^2 \leq k C_{AB}^2 \), we have

\[
C_{A|BC}^\beta \geq C_{A|B}^\beta + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} C_{AC}^\beta,
\]

for all \( \beta \geq 2 \).

**Proof** Since \( C_{ABC}^2 \leq k C_{AB}^2 \) and \( C_{ABC} > 0 \), we obtain

\[
C_{A|BC}^\beta \geq C_{A|B}^\beta + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} C_{AC}^\beta
\]

\[
C_{A|B}^\beta \left[ 1 + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} \left( C_{AC}^\beta \right)^{\beta/2} \right] \geq C_{A|B}^\beta + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} C_{AC}^\beta,
\]

where the first inequality is due to the fact, \( C_{ABC}^2 \geq C_{AB}^2 + C_{AC}^2 \) for arbitrary \( 2 \otimes 2 \otimes 2^n-2 \) tripartite state \( \rho_{ABC} \) and the second is due to Lemma 1. We can also see that if \( C_{AB} = 0 \), then \( C_{AC} = 0 \), and the lower bound becomes trivially zero.

For multiqubit systems, we have the following Theorems.

**Theorem 1** Suppose \( k \) is a real number satisfying \( 0 < k \leq 1 \). For an N-qubit mixed state \( \rho_{AB_1 \cdots B_{N-1}} \), if \( k C_{AB}^2 \geq k C_{A|B_1 \cdots B_{N-1}} \) for \( i = 1, 2, \ldots, m \), and \( C_{A|B}^2 \leq k C_{A|B_1 \cdots B_{N-1}} \) for \( j = m + 1, \ldots, N - 2 \), \( \forall 1 \leq m \leq N - 3 \), \( N > 4 \), then we have

\[
C_{A|B_1 \cdots B_{N-1}}^\beta \geq C_{A|B_1}^\beta + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} C_{AB}^\beta + \cdots + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} C_{AB_{m+1}}^\beta + \cdots + C_{AB_{N-2}}^\beta
\]
for all $\beta \geq 2$.

**Proof** From the inequality (11), we have

$$C_{AB_{1}} + (1 + k)\beta/2 - \frac{1}{k^{3/2}} C_{AB_{2}} + \ldots + (1 + k)\beta/2 - \frac{1}{k^{3/2}} C_{AB_{m}} + \ldots \geq \ldots \geq C_{AB_{m+1} \ldots B_{N-1}} \geq (1 + k)\beta/2 - \frac{1}{k^{3/2}} C_{AB_{m+1}} + \ldots + C_{AB_{m}} + \ldots + C_{AB_{N-1}}.$$ (15)

Combining Eqs. (14) and (15), we get the inequality (13).

If we replace the conditions $kC_{AB_{i}} \geq C_{A|B_{i+1} \ldots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $C_{AB_{j}} \leq k^{2}C_{A|B_{j+1} \ldots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, we have the following theorem.

**Theorem 2** Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an $N$-qubit mixed state $\rho_{AB_{1} \ldots B_{N-1}}$, if $k^{2}C_{AB_{i}} \geq C_{A|B_{i+1} \ldots B_{N-1}}$ for $i = 1, 2, \ldots, N - 2$, we then have

$$C_{A|B_{1} \ldots B_{N-1}} \geq C_{AB_{1}} + \frac{(1 + k)\beta/2 - 1}{k^{3/2}} C_{AB_{2}} + \ldots \geq \frac{(1 + k)\beta/2 - 1}{k^{3/2}} C_{AB_{N-1}},$$ (16)

for $\beta \geq 2$.

It can be seen that the inequalities (13) and (16) are tighter than the ones given in Ref. [22], since

$$\frac{(1 + k)\beta/2 - 1}{k^{3/2}} \geq 2^{\beta/2} - 1,$$

for $\beta \geq 2$ and $0 < k \leq 1$. The equality holds when $k = 1$. Namely, the result (5) given in Ref. [22] are just special cases of ours for $k = 1$. As $\{(1 + k)\beta/2 - 1\}/k^{3/2}$ is a decreasing function with respect to $k$ for $0 < k \leq 1$ and $\beta \geq 2$, we find that the smaller $k$ is, the tighter the inequalities (11), (13), and (16) are.

**Example 1** Consider the three-qubit state $|\psi\rangle_{ABC}$ in generalized Schmidt decomposition form,[32–33]

$$|\psi\rangle_{ABC} = \lambda_{0} |000\rangle + \lambda_{1} e^{i\varphi} |100\rangle + \lambda_{2} |101\rangle + \lambda_{3} |110\rangle + \lambda_{4} |111\rangle,$$ (17)

where $\lambda_{i} \geq 0$, $i = 1, 2, \ldots, 4$, and $\sum_{i=0}^{4} \lambda_{i}^{2} = 1$. Then we get $C_{A|BC} = 2\lambda_{0}\sqrt{\lambda_{1}^{2} + \lambda_{2}^{2} + \lambda_{3}^{2}}$, $C_{AB} = 2\lambda_{0}\lambda_{2}$, and $C_{AC} = 2\lambda_{0}\lambda_{3}$. Set $\lambda_{0} = \lambda_{3} = 1/2$, $\lambda_{2} = \sqrt{2}/2$, and $\lambda_{1} = \lambda_{4} = 0$. We have $C_{A|BC} = \sqrt{3}/2$, $C_{AB} = \sqrt{2}/2$, and $C_{AC} = 1/2$. Then

$$C_{AB}^{\beta} + (2\beta/2 - 1)C_{AC}^{\beta} = \left(\frac{\sqrt{2}}{2}\right)^{\beta} + (2\beta/2 - 1)\left(\frac{1}{2}\right)^{\beta},$$

$$C_{AB}^{\beta} + \left(\frac{1 + k)\beta/2 - 1}{k^{3/2}} C_{AC}^{\beta} = \left(\frac{\sqrt{2}}{2}\right)^{\beta} + \left(\frac{1 + k)\beta/2 - 1}{k^{3/2}} \right)\left(\frac{1}{2}\right)^{\beta}.$$ (18)

One can see that our result is better than the result (5) in Ref. [22] for $\beta \geq 2$, hence better than Eqs. (3) and (4) given in Refs. [21, 23], see Fig. 1.

![Fig. 1](image-url) (Color online) The y axis is the lower bound of the concurrence $C_{A|BC}$. The red (green) line represents the lower bound from our result for $k = 0.6 (k = 0.8)$, and the blue line represents the lower bound of Eq. (5) from Ref. [22].
we have
\begin{equation}
+ \frac{(1+k)^\beta/2 - 1}{k^{\beta/2}} \right)^m \mathcal{C}_{AB}\mathcal{N}_{-1}, 
\end{equation}
for all $0 \leq \beta \leq 2$.

**Proof** The proof is similar to the proof of Theorem 1 by using inequality (9).

**Theorem 4** Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an $N$-qubit pure state $|\psi\rangle_{AB_1\ldots B_{N-1}}$, if $k^2a_{AB} \geq C^2_{AB_1\ldots B_{N-1}}$ for all $i = 1, 2, \ldots, N - 2, \ldots$ we have
\begin{equation}
\mathcal{C}_{a|\psi\rangle_{AB_1\ldots B_{N-1}}} \leq C_{AB_1} + \frac{(1+k)^{\beta/2} - 1}{k^{\beta/2}} C_{AB_2} \ldots + \frac{(1+k)^{\beta/2} - 1}{k^{\beta/2}} \right)^N - 2 \mathcal{C}_{AB}\mathcal{N}_{-1},
\end{equation}
for all $0 \leq \beta \leq 2$.

The inequalities (18) and (19) are also upper bounds of $C(\psi|\psi\rangle_{AB_1\ldots B_{N-1}})$ for pure state $|\psi\rangle_{AB_1\ldots B_{N-1}}$ since $C(\psi|\psi\rangle_{AB_1\ldots B_{N-1}}) = C(\psi|\psi\rangle_{AB_1\ldots B_{N-1}})$.

### 3 Tighter Constraints Relate to EoF

Let $H_A$ and $H_B$ be two Hilbert spaces with dimension $m$ and $n$ ($m \leq n$), respectively. Then the entanglement of formation (EoF) [34–35] is defined as follows: for a pure state $\psi_{AB} \in H_A \otimes H_B$, the EoF is given by
\begin{equation}
E(\psi_{AB}) = S(\rho_A),
\end{equation}
where $\rho_A = \text{tr}_B(\psi_{AB})$ and $S(\rho) = -\text{tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the EoF is given by
\begin{equation}
E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(\psi_i),
\end{equation}
with the minimum taken over all possible pure state decomposition of $\rho_{AB}$.

In Ref. [36], Wootters showed that
\begin{equation}
E(|\psi\rangle) = f(C^2(|\psi\rangle))
\end{equation}
for $2 \otimes m$ ($m \geq 2$) pure state $|\psi\rangle$, and $E(\rho) = f(C^2(\rho))$ for two-qubit mixed states $\rho_A \otimes \rho_B$. Suppose $H(x) = H([1+\sqrt{1-x}]/2)$ and $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$, $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$, and satisfies the following relations:
\begin{equation}
f'[(x^2 + y^2)] \geq f'[(x^2)] + f'[(y^2)]
\end{equation}
where $f'[(x^2 + y^2)] = \left[f(x^2) + y^2\right]^{1/2}$.

Although EoF does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{AB}\mid C$ [37], the authors in Ref. [38] showed that EoF is a monotonic function satisfying
\begin{equation}
E^2(\rho_{AB_1\ldots B_{N-1}}) \geq \sum_{i=1}^{N-1} E^2(\rho_{AB_i}).
\end{equation}

For $N$-qubit systems, one has [31]
\begin{equation}
E^3_{AB_1\ldots B_{N-1}} \geq E^3_{AB_1} + E^3_{AB_2} \ldots + E^3_{AB_{N-1}}.
\end{equation}
for $\beta \geq \sqrt{2}$, where $E_{AB_1\ldots B_{N-1}}$ is the EoF of $\rho$ under bipartite partition $A|B_1 B_2 \ldots B_{N-1}$, and $E_{AB_i}$ is the EoF of the mixed state $\rho_{AB_i} = \text{tr}_{B_1\ldots B_{i-1} B_{i+1} \ldots B_{N-1}}(\rho)$ for $i = 1, 2, \ldots, N - 1$. Recently, the authors in Ref. [23] proposed a monogamy relation that is tighter than the inequality (23),
\begin{equation}
E^3_{AB_1 B_2 \ldots B_{N-1}} \geq E^3_{AB_1} + \frac{\beta}{\sqrt{2}} E_{AB_2} + \cdots + \left(E^3_{AB_{N-1}} \right)
\end{equation}
for $\beta \geq \sqrt{2}$. The inequality (24) is also improved to
\begin{equation}
E^3_{AB_1 B_2 \ldots B_{N-1}} \geq E^3_{AB_1} + (2^{3/\sqrt{2}} - 1) E_{AB_2} \ldots
\end{equation}
\begin{equation}
+ (2^{3/\sqrt{2}} - 1)^{m-1} E_{AB_m} + (2^{3/\sqrt{2}} - 1)^{m+1} \times (E^3_{AB_{m+1}} + \cdots + E^3_{AB_{N-2}}) + (2^{3/\sqrt{2}} - 1)^m E^3_{AB_{N-1}},
\end{equation}
under the same conditions as that of inequality (24).

In fact, these inequalities can be further improved to even tighter monogamy relations.

**Theorem 5** Suppose $k$ is a real number satisfying $0 < k \leq 1$. For any $N$-qubit mixed state $\rho_{AB_1 \ldots B_{N-1}}$, if $k E_{AB_1}^2 \geq E_{AB_1^2 \ldots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $E_{AB_1}^2 \geq k E_{AB_1^2 \ldots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, $\forall 1 \leq m \leq N - 3, N \geq 4$, the entanglement of formation $E(\rho)$ satisfies
\begin{equation}
E_{AB_1^2 \ldots B_{N-1}} \geq E_{AB_1} + \frac{(1+k)^t - 1}{k^t} E_{AB_2} + \cdots
\end{equation}
\begin{equation}
+ \frac{(1+k)^t - 1}{k^t} \right)^{m-1} E_{AB_m} + \frac{(1+k)^t - 1}{k^t} \right)^{m+1} \times (E^3_{AB_{m+1}} + \cdots + E^3_{AB_{N-2}}) + (1+k)^t - 1 \right)^m } E^3_{AB_{N-1}},
\end{equation}
for $\beta \geq \sqrt{2}$, where $t = \beta/\sqrt{2}$.

**Proof** For $\beta \geq \sqrt{2}$ and $k f'[(x^2)] \geq f'[(y^2)]$, we find
\begin{equation}
f''(x^2 + y^2) \geq f''(x^2) + f''(y^2),
\end{equation}
where $f''(x^2 + y^2) = [f(x^2) + y^2]^{1/2}$. The first inequality is due to the inequality (22), and the second inequality can be obtained from inequality (8).

Let $\rho = \sum_i p_i |\psi_i\rangle |\psi_i\rangle \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ be the optimal decomposition of $E_{AB_1 B_2 \ldots B_{N-1}}(\rho)$ for the $N$-qubit mixed state $\rho$. Then [22]
\begin{equation}
E_{AB_1 B_2 \ldots B_{N-1}} \geq f(C^2_{A|B_1 B_2 \ldots B_{N-1}}).
\end{equation}
Thus,
\begin{equation}
E^3_{AB_1 B_2 \ldots B_{N-1}} \geq f^3(C^2_{A|B_1 B_2 \ldots B_{N-1}}).
\end{equation}
where the first inequality holds due to Eq. (28), the second inequality is similar to the proof of Theorem 1 by using inequality (27), and the last equality holds since for any $2 \otimes 2$ quantum state $\rho_{AB}$, $E(\rho_{AB}) = f[C^2(\rho_{AB})]$. □

Similar to the case of concurrence, we have also the following tighter monogamy relation for EoF.

**Theorem 6** Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an $N$-qubit mixed state $\rho_{AB_1\ldots B_{N-1}}$, if $kE_{AB_i}^2 \geq E_{A|B_{i+1}\ldots B_{N-1}}^2$ for all $i = 1, 2, \ldots, N-2$, we have

$$E_{A|B_1B_2\ldots B_{N-1}}^\beta \geq E_{A|B_1B_2B_{N-1}}^\beta + \left(\frac{1+k}{k^t}\right)^{N-2}E_{A|B_2B_{N-1}}^\beta,$$

for $\beta \geq \sqrt{2}$ and $t = \beta/\sqrt{2}$.

As $\beta^2 \geq \beta - 1$, our new monogamy relations (29) and (30) are tighter than the ones given in Refs. [21–23]. Also, for $0 < k \leq 1$ and $\beta \geq 2$, the smaller $k$ is, the tighter inequalities (26) and (30) are.

**Example 2** Let us again consider the three-qubit state $|\psi\rangle_{ABC}$ defined in Eq. (17) with $\lambda_0 = \lambda_3 = 1/2$, $\lambda_2 = \sqrt{2}/2$ and $\lambda_1 = \lambda_4 = 0$. Then $E_{ABC} = 2 - \log_2 3 \approx 0.811278$,

$$E_{AB} = -\frac{2 + \sqrt{2}}{4} \log_2 \frac{2 + \sqrt{2}}{4} - \frac{2 - \sqrt{2}}{4} \log_2 \frac{2 - \sqrt{2}}{4} \approx 0.600876,$$

$$E_{AB} = -\frac{2 + \sqrt{3}}{4} \log_2 \frac{2 + \sqrt{3}}{4} - \frac{2 - \sqrt{3}}{4} \log_2 \frac{2 - \sqrt{3}}{4} \approx 0.354579.$$

Thus,

$$E_{AB}^\beta + (2^{\beta/2} - 1)E_{AC}^\beta = (0.600876)^\beta + (2^{\beta/2} - 1)0.354579^\beta,$$

$$E_{AB}^\beta + \frac{1.5^{\beta/2} - 1}{0.5^{\beta/2}} E_{AC}^\beta = (0.600876)^\beta + \frac{1.5^{\beta/2} - 1}{0.5^{\beta/2}} 0.354579^\beta,$$

$$E_{AB}^\beta + \frac{1.7^{\beta/2} - 1}{0.7^{\beta/2}} E_{AC}^\beta = (0.600876)^\beta + \frac{1.7^{\beta/2} - 1}{0.7^{\beta/2}} 0.354579^\beta,$$

$$E_{AB}^\beta + \frac{1.9^{\beta/2} - 1}{0.9^{\beta/2}} E_{AC}^\beta = (0.600876)^\beta + \frac{1.9^{\beta/2} - 1}{0.9^{\beta/2}} 0.354579^\beta.$$

One can see that our result is better than the one in Ref. [22] for $\beta \geq \sqrt{2}$, hence better than the ones in Refs. [21, 23], see Fig. 2.

![Fig. 2](image)

Fig. 2 (Color online) The y axis is the lower bound of the EoF $E_{ABC}^\beta$. The red (green resp. blue) line represents the lower bound from our result for $k = 0.5$ ($k = 0.7$ resp. $k = 0.9$), and the yellow line represents the lower bound from the result in Ref. [22].

We can also provide tighter polygamy relations for the entanglement of assistance. The entanglement of assistance (EoA) of $\rho_{AB}$ is defined as [39]

$$E_{\rho_{AB}}^a = \max_{\{|\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$

with the maximization taking over all possible pure decompositions of $\rho_{AB}$. For any dimensional multipartite quantum state $\rho_{AB_1\ldots B_{N-1}}$, a general polygamy inequality of multipartite quantum entanglement was established in [18],

$$E_a^a(\rho_{AB_1\ldots B_{N-1}}) \leq \sum_{i=1}^{N-1} E_a^a(\rho_{A|B_iB_{i+1}\ldots B_{N-1}}).$$

Using the same approach as for concurrence, we have the following Theorems.

**Theorem 7** Suppose $k$ is a real number satisfying $0 < k \leq 1$. For any $N$-qubit mixed state $\rho_{AB_1\ldots B_{N-1}}$, if $kE_{AB_i}^2 \geq E_{A|B_{i+1}\ldots B_{N-1}}^2$ for $i = 1, 2, \ldots, m$, and $E_{AB_i}^2 \leq kE_{A|B_{i+1}\ldots B_{N-1}}^2$ for $j = m + 1, \ldots, N - 2$, $\forall 1 \leq m \leq N - 3,$
\[ N \geq 4, \text{ we have} \]
\[
(E_{AB|B_1\cdots B_{N-1}}^a)^\beta \leq (E_{AB})^\beta + \left(\frac{1 + k}{k^\beta} - 1\right)(E_{AB_m}^a)^\beta + \cdots + \left(\frac{1 + k}{k^\beta} - 1\right)^{m-1}(E_{AB_{m+1}}^a)^\beta + \cdots + \left(\frac{1 + k}{k^\beta} - 1\right)^m(E_{AB_{N-1}}^a)^\beta,
\]
for \(0 \leq \beta \leq 1\).

**Theorem 8** Suppose \(k\) is a real number satisfying \(0 < k \leq 1\). For any \(N\)-qubit mixed state \(\rho_{AB_1\cdots A_{N-1}}\), if \(kE_{AB_i}^a \geq E_{AB|B_1\cdots B_{N-1}}^a\) for all \(i = 1, 2, \ldots, N - 2\), we have
\[
(E_{AB|B_1\cdots B_{N-1}}^a)^\beta \leq (E_{AB})^\beta + \left(\frac{1 + k}{k^\beta} - 1\right)(E_{AB_2}^a)^\beta + \cdots + \left(\frac{1 + k}{k^\beta} - 1\right)^{N-2}(E_{AB_{N-1}}^a)^\beta,
\]
for \(0 \leq \beta \leq 1\).

### 4 Tighter Constraints Related to Negativity

The negativity, a well-known quantifier of bipartite entanglement, is defined as \(N(\rho_{AB}) = (\|\rho_{AB}^T\| - 1)/2\) \([40]\) where \(\rho_{AB}^T\) is the partial transposed matrix of \(\rho_{AB}\) with respect to the subsystem \(A\), and \(\|X\|\) denotes the trace norm of \(X\), i.e., \(\|X\| = \text{tr}\sqrt{XX^T}\). For convenience, we use the definition of negativity as \(\|\rho_{AB}^T\| - 1\). Particularly, for any bipartite pure state \(|\psi\rangle_{AB}\), \(N(\langle\psi|AB) = 2\sum_i c_{ij}\sqrt{\lambda_j} = (\text{tr}\sqrt{|\rho_A|})^2 - 1\), where \(\lambda\)s are the eigenvalues of the reduced density matrix \(\rho_A = \text{tr}_B|\psiangle_{AB}\langle\psi|\). The convex-roof extended negativity (CREN) of a mixed state \(\rho_{AB}\) is defined by
\[
N_c(\rho_{AB}) = \min_{(p_i,|\psi_i\rangle)} \sum_i p_i N(|\psi_i\rangle),
\]
where the minimum is taken over all possible pure state decomposition of \(\rho_{AB}\). Thus \(N_c(\rho_{AB}) = C(\rho_{AB})\) for any two-qubit mixed state \(\rho_{AB}\). The dual to the CREN of a mixed state \(\rho_{AB}\) is defined as
\[
N^\circ_c(\rho_{AB}) = \max_{(p_i,|\psi_i\rangle)} \sum_i p_i N(|\psi_i\rangle),
\]
with the maximum taking over all possible pure state decomposition of \(\rho_{AB}\). Furthermore, \(N^\circ_c(\rho_{AB}) = C^\circ(\rho_{AB})\) for any two-qubit mixed state \(\rho_{AB}\) \([12]\).

Similar to the concurrence and EoF, we have the following Theorems.

**Theorem 9** Suppose \(k\) is a real number satisfying \(0 < k \leq 1\). For any \(N\)-qubit mixed state \(\rho_{AB_1\cdots A_{N-1}}\), if \(kN_{cAB_i}^\circ \geq N_{cAB|B_1\cdots B_{N-1}}^\circ\) for \(i = 1, 2, \ldots, m\), and \(N_{cAB_j}^\circ \leq kN_{cAB|B_1\cdots B_{N-1}}^\circ\) for \(j = m + 1, \ldots, N - 2\), \(\forall 1 \leq m \leq N - 3, N \geq 4\), then we have
\[
N_{cAB|B_1\cdots B_{N-1}}^\circ \geq N_{cAB_1}^\circ + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)(N_{cAB_2}^\circ - 1) + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)^{m-1}(N_{cAB_m}^\circ - 1) + \cdots + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)^{m-1}(N_{cAB_{N-1}}^\circ - 1),\]
for all \(\beta \geq 2\).

**Theorem 10** Suppose \(k\) is a real number satisfying \(0 < k \leq 1\). For any \(N\)-qubit mixed state \(\rho_{AB_1\cdots A_{N-1}}\), if \(kN_{cAB_i}^\circ \geq N_{cAB|B_1\cdots B_{N-1}}^\circ\) for all \(i = 1, 2, \ldots, N - 2\), then
\[
N_{cAB|B_1\cdots B_{N-1}}^\circ \geq N_{cAB_1}^\circ + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)(N_{cAB_2}^\circ - 1) + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)^{m-1}(N_{cAB_m}^\circ - 1) + \cdots + \left(\frac{1 + k}{k^{\beta/2}} - 1\right)^{m-1}(N_{cAB_{N-1}}^\circ - 1),\]
for all \(\beta \geq 2\).

**Example 3** Consider the state in Example 1 with \(\lambda_0 = \lambda_3 = 1/2, \lambda_2 = \sqrt{2}/2, \) and \(\lambda_1 = \lambda_4 = 0\). We have \(N_{cABC} = 3/2, N_{cAB} = \sqrt{2}/2,\) and \(c_{ABC} = 1/2\). Then
\[
N_{cAB}^\circ + (2^{\beta/2} - 1)N_{cAC}^\circ = \left(\frac{\sqrt{2}}{2}\right)\beta + (2^{\beta/2} - 1)\left(\frac{1}{2}\right)\beta,
\]
\[
N_{cAC}^\circ + (1 + k)^{\beta/2} - 1 \frac{k^{\beta/2}}{N_{cAC}^\circ}
\]
One can see that our result is better than the one in Ref. \[22\] for \(\beta \geq 2\), thus also better than the ones in Refs. \[21, 23\], see Fig. 3.

![Fig. 3](image-url) (Color online) The \(y\) axis is the lower bound of the negativity \(N_c(|\psi\rangle_{AB|BC})\), which are functions of \(\beta\). The red (green) line represents the lower bound from our result for \(k = 0.6 (k = 0.8)\), and the blue line represents the lower bound from the result in Ref. \[22\].

For the negativity of assistance \(N_{c}^\circ\), we have the following results.
Theorem 11 Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an $N$-qubit pure state $|\psi\rangle_{AB_1\cdots B_{N-1}}$, if $\langle N_{cA|B_i}\rangle^2 \geq (\langle N_{cA|B_1\cdots B_{N-1}}\rangle^2)^2$ for $i = 1, 2, \ldots, m$, and $\langle N_{cAB}^a\rangle^2 \leq k \langle N_{cAB_1\cdots B_{N-1}}\rangle^2$ for $j = m + 1, \ldots, N - 2$, then
\[
\langle N_{cA|B_1\cdots B_{N-1}}\rangle^2 \leq \langle N_{cAB_1\cdots B_{N-1}}\rangle^2 + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} \langle N_{cA|B} \rangle^2 + \ldots
\]
with the minimum taken over all possible pure state decompositions of $\rho_{AB}$. For $5 - \sqrt{13}/2 \leq q \leq (5 + \sqrt{13})/2$, Yuan et al. proposed an analytic relationship between the Tsallis-$q$ entanglement and concurrence,
\[
T_q(\langle \psi\rangle_{AB}) = g_q(C^2(\langle \psi\rangle_{AB})),
\]
where
\[
g_q(x) = \frac{1}{q - 1} \left[ 1 - \left( \frac{1 - x}{2} \right)^q - \left( \frac{1 - x}{2} \right)^q \right],
\]
with $0 \leq x \leq 1$. It has also been proved that $T_q(|\psi\rangle) = g_q(C^2(|\psi\rangle))$ if $|\psi\rangle$ is a $2 \otimes m$ pure state, and $T_q(\rho) = g_q(C^2(\rho))$ if $\rho$ is a two-qubit mixed state. Hence, Eq. (42) holds for any $q$ such that $g_q(x)$ in Eq. (43) is monotonically increasing and convex. Particularly, one has that
\[
g_q(x^2 + y^2) \geq g_q(x^2) + g_q(y^2),
\]
for $2 \leq q \leq 3$. In Ref. [14], Kim provided a monogamy relation for the Tsallis-$q$ entanglement,
\[
T_q|_{AB_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T_q|_{AB_i},
\]
where $i = 1, 2, \ldots, N - 1$ and $2 \leq q \leq 3$. Later, this relation was improved as follows: If $C_{AB_i} \geq C_{AB_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $C_{AB_1} \leq C_{AB_{1+1}\cdots B_{N-1}}$ for $j = m + 1, \ldots, N$, then
\[
T^\beta_{q|_{AB_1B_2\cdots B_{N-1}}} \geq T^\beta_{q|_{AB_1}} + (2^\beta - 1)T^\beta_{q|_{AB_2}} + \ldots + (2^\beta - 1)^m T^\beta_{q|_{AB_{m+1}}} + (2^\beta - 1)^m T^\beta_{q|_{AB_{N-1}}},
\]
for $\beta \geq 1$ and $T^\beta_{q|_{AB_1B_2\cdots B_{N-1}}}$ quantifies the Tsallis-$q$ entanglement under partition $AB_1B_2\cdots B_{N-1}$, and $T^\beta_{q|_{AB_i}}$ quantifies that of the two-qubit subsystem $AB_i$ if $2 \leq q \leq 3$. Moreover, for $(5 - \sqrt{13})/2 \leq q \leq (5 + \sqrt{13})/2$, one has
\[
T^2_{q|_{AB_1B_2\cdots B_{N-1}}} \geq \sum_{i=1}^{N-1} T^2_{q|_{AB_i}}.
\]
We now provide monogamy relations which are tighter than Eqs. (45) and (46).

Theorem 13 Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an arbitrary $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $kT_q\rho_{AB_i} \geq T_q|_{AB_iB_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $T_q\rho_{AB_1} \leq kT_q|_{AB_1B_{i+1}\cdots B_{N-1}}$ for $j = m + 1, \ldots, N$, then
\[
T^\beta_{q|_{AB_1B_2\cdots B_{N-1}}} \geq T^\beta_{q|_{AB_1}} + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} T^\beta_{q|_{AB_2}} + \ldots + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} T^\beta_{q|_{AB_{m+1}}} + \ldots + \frac{(1 + k)^{\beta/2} - 1}{k^{\beta/2}} T^\beta_{q|_{AB_{N-1}}},
\]
for $\beta \geq 1$ and $2 \leq q \leq 3$.

Theorem 14 Suppose $k$ is a real number satisfying $0 < k \leq 1$. For any $N$-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if all
For Refs. [21, 23], see Fig. 4.

Table 1 with Example 4

Consider the quantum state given in Example 4. As a dual quantity to Tsallis-q function defined in Eq. (43) satisfies

\[ T_{qAB}^\beta = \max_{\langle \psi_i | \psi \rangle} \sum_i p_i T_q(\langle \psi_i | \psi \rangle), \]

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} \). If \( 1 \leq q \leq 2 \) or \( 3 \leq q \leq 4 \), the function \( g_q \) defined in Eq. (43) satisfies

\[ g_q(\sqrt{x^2 + y^2}) \leq g_q(x) + g_q(y), \]

which leads to the Tsallis polygamy inequality

\[ T_{qAB}^\beta = \max_{\langle \psi_i | \psi \rangle} \sum_i p_i T_q(\langle \psi_i | \psi \rangle), \]

for any multi-qubit state \( \rho_{AB|B_1\cdots B_{N-1}} \). Here we provide tighter polygamy relations related to Tsallis-q entanglement. We have the following results.

**Theorem 15** Suppose \( k \) is a real number satisfying \( 0 < k \leq 1 \). For any \( N \)-qubit mixed state \( \rho_{AB_1\cdots B_{N-1}} \), if \( k T_{qAB}^\beta \geq T_{qAB}^\beta \) for \( i = 1, 2, \ldots, m \), and \( T_{qAB}^\beta \leq k T_{qAB}^\beta \) for \( j = m + 1, \ldots, N - 2 \), \( \forall i \leq m \leq N - 3, N \geq 4 \), then

\[ (T_{qAB}^\beta)^\beta \leq (T_{qAB}^\beta)^\beta + \left( \frac{1 + k}{k^\beta} - 1 \right) \frac{T_{qAB}^\beta}{k^\beta} \]

for \( 0 \leq \beta \leq 1 \) with \( 1 \leq q \leq 2 \) or \( 3 \leq q \leq 4 \).

**Theorem 16** Suppose \( k \) is a real number satisfying \( 0 < k \leq 1 \). For any \( N \)-qubit mixed state \( \rho_{AB_1\cdots B_{N-1}} \), if \( k T_{qAB}^\beta \geq T_{qAB}^\beta \) for all \( i = 1, 2, \ldots, N - 2 \), we have

\[ T_{qAB}^\beta \leq T_{qAB}^\beta + \left( \frac{1 + k}{k^\beta} - 1 \right) \frac{T_{qAB}^\beta}{k^\beta} \]

for \( 0 \leq \beta \leq 1 \) with \( 1 \leq q \leq 2 \) or \( 3 \leq q \leq 4 \).

**5.2 Tighter Monogamy and Polygamy Relations for Rényi-α Entanglement**

For a bipartite pure state \( |\psi\rangle_{AB} \), the Rényi-\( \alpha \) entanglement is defined as[42]

\[ E(\rho_{AB}) = S_\alpha(\rho_A), \]

where \( S_\alpha(\rho) = [1/(1-\alpha)] \log_2 \text{tr} \rho^\alpha \) for any \( \alpha > 0 \) and \( \alpha \neq 1 \), and \( \lim \alpha \to 1 S_\alpha(\rho) = S(\rho) = -\text{tr} \log_2 \rho \). For a bipartite mixed state \( \rho_{AB} \), the Rényi-\( \alpha \) entanglement is given by

\[ E_\alpha(\rho_{AB}) = \min_{\langle \psi_i| \psi \rangle} \sum_i p_i E_\alpha(\langle \psi_i| \psi \rangle), \]

where the minimum is taken over all possible pure-state decompositions of \( \rho_{AB} \). For each \( \alpha > 0 \), one has

\[ E_\alpha(\rho_{AB}) = f_\alpha(C(\rho_{AB})), \]

where

\[ f_\alpha(x) = \frac{1}{1 - \alpha} \log \left( \frac{(1 - \sqrt{1 - x^2})^2 + (1 + \sqrt{1 - x^2})^2}{2} \right) \]

is a monotonically increasing and convex function.\[24\] For \( \alpha \geq 2 \) and any \( n \)-qubit state \( \rho_{AB_1\cdots B_{N-1}} \), one has\[14\]

\[ E_{\alpha|A|B_1B_2\cdots B_{N-1}} \geq E_{\alpha|A|B_1} + E_{\alpha|A|B_2} + \cdots + E_{\alpha|A|B_{N-1}}. \]

We propose the following two monogamy relations for the Rényi-\( \alpha \) entanglement, which are tighter than the previous results.

**Theorem 17** Suppose \( k \) is a real number satisfying \( 0 < k \leq 1 \). For an arbitrary \( N \)-qubit mixed state \( \rho_{AB_1\cdots B_{N-1}} \), if \( k E(\rho_{AB}_i) \geq E(\rho_{AB_1\cdots B_{N-1}}) \) for \( i = 1, 2, \ldots, m \), and \( k E(\rho_{AB_1\cdots B_{N-1}}) \leq k E(\rho_{AB_1\cdots B_{N-1}}) \) for \( j = m + 1, \ldots, N - 2 \), \( \forall i \leq m \leq N - 3, N \geq 4 \), then

\[ (E(\rho_{AB_1\cdots B_{N-1}}))^\beta \]

for \( 0 \leq \beta \leq 1 \) with \( 1 \leq q \leq 2 \) or \( 3 \leq q \leq 4 \).
Ref. [14], and the smaller

One can see that our result is better than the result in

Consider again the state given in Example

Theorem 18 Suppose $k$ is a real number satisfying $0 < k \leq 1$. For an arbitrary $N$-qubit mixed state $\rho_{ABN}$, if $kE_{AB} \geq E_{AB|B_{j} \cdots B_{N-1}}$ for all $i = 1, 2, \ldots, N - 2$, then

$$\left( E_{AB|B_{j} \cdots B_{N-1}} \right)^{\beta} \geq \left( E_{AB} \right)^{\beta} + \left( \frac{(1+k)^{\alpha} - 1}{k^{\alpha}} \right) \left( E_{AB_{m+1}} \right)^{\beta} + \ldots$$

$$+ \left( \frac{(1+k)^{\alpha} - 1}{k^{\alpha}} \right)^{N-2} \left( E_{AB_{N-1}} \right)^{\beta},$$

for $\beta \geq 1$ and $\alpha \geq 2$.

Example 5 Consider again the state given in Example 1 with $\lambda_0 = \lambda_3 = 1/2$, $\lambda_2 = \sqrt{2}/2$, and $\lambda_1 = \lambda_4 = 0$. For $\alpha = 2$, we find $E_{2AC} = \log_{2}(8/5) \approx 0.67807$, $E_{2AB} = \log_{2}(8/7) \approx 0.415037$, and $E_{2AC} = \log_{2}(4/3) \approx 0.192645$. Then

$$E_{2AB} + E_{2AC} = 0.415037^{\alpha} + 0.192645^{\alpha},$$

$$E_{2AB} + \left( \frac{(1+k)^{\alpha} - 1}{k^{\alpha}} \right) E_{2AC}$$

$$= 0.415037^{\alpha} + \left( \frac{(1+k)^{\alpha} - 1}{k^{\alpha}} \right) 0.192645^{\alpha}.$$
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