OBSTRUCTIONS TO CHOOSING DISTINCT POINTS ON CUBIC PLANE CURVES

WEIYAN CHEN

Abstract. Every smooth cubic plane curve has 9 inflection points, 27 sextatic points, and 72 “points of type nine”. Motivated by these classical algebro-geometric constructions, we study the following topological question: Is it possible to continuously choose \( n \) distinct unordered points on each smooth cubic plane curve for a natural number \( n \)? This question is equivalent to asking if certain fiber bundle admits a continuous section or not. We prove that the answer is no when \( n \) is not a multiple of 9. Our result resolves a conjecture of Benson Farb.

1. Introduction

A cubic plane curve in \( \mathbb{CP}^2 \) is the zero locus of a homogeneous polynomial \( f(x, y, z) \) of degree 3. It has been a classical topic to study certain special points on smooth cubic curves, such as the 9 inflection points (dating back at least to Maclaurin; see the Introduction in [1] for an account of the history of this topic), the 27 sextatic points (studied by Cayley [2]), and the 72 “points of type 9” (studied by Gattazzo [5]). Inspired by these classical constructions, Benson Farb (private communication) asked the following question: For what integer \( n \) is it possible to continuously choose \( n \) distinct points on each cubic plane curve as the curve varies in families?

To make the question precise, let \( X \) denote the parameter space of smooth cubic plane curves:

\[
X := \{ f(x, y, z) : f \text{ is a homogeneous polynomial of degree 3 and is smooth} \}/\sim
\]

where \( f \sim \lambda f \) for any \( \lambda \in \mathbb{C}^\times \). There is a fiber bundle whose fiber over \( f \in X \) is \( \text{UConf}^n C_f \), the configuration space of \( n \) distinct unordered points on the cubic plane curve \( C_f \) defined by \( f = 0 \).

\[
\text{UConf}^n C_f \longrightarrow E_n \\
\downarrow \xi_n \\
X
\]

(1.1)

Question 1 (Farb). For which values \( n \) do the bundles \( \xi_n \) admit continuous sections?

The aforementioned algebraic constructions and their generalizations give continuous sections to \( \xi_n \) for various \( n \):

Theorem 1 (Maclaurin, Cayley, Gattazzo). The bundle \( \xi_n \) has a continuous section when \( n = 9 \sum_{k \in S} J_2(k) \) where \( J_2 \) is Jordan’s 2-totient function and \( S \) is an arbitrary finite set of positive integers, for example, when \( n = 9, 27, 36, 72, 81, 99, 108, 117, 135, 144, 180 \ldots \)

See Section 2 for more discussion on Theorem 1 and a definition of Jordan’s 2-totient function.

In contrast to the various algebraic constructions of sections of \( \xi_n \), it was previously unknown whether there is any \( n \) such that \( \xi_n \) does not have any continuous section.

Conjecture 1 (Farb). \( n = 9 \) is the smallest value for \( \xi_n \) to have a continuous section.

Behind this conjecture is the following speculation: the classical algebraic constructions should be the only possible continuous sections. The minimal number comes from the 9 inflection points.
After Conjecture 1, Farb made a much stronger conjecture: the only continuous sections of \( \xi_n \) (up to homotopy) are those given by the algebraic constructions in Theorem 1.

We will prove Conjecture 1 in this paper. In fact, we will prove the following stronger statement.

**Theorem 2.** \( \xi_n \) has no continuous section unless \( n \) is a multiple of 9.

Let us remark on the significance of Theorem 2: it tells us that in those case when \( 9 \nmid n \) and when our current knowledge since Maclaurin had failed to identify any natural structure of \( n \) special points on smooth cubic curves, there is in fact none.

Notice that \( \xi_1 \) is precisely the tautological bundle whose fiber over every curve in \( X \) is the curve itself. Theorem 2 thus implies:

**Corollary 3.** The tautological bundle \( \xi_1 \) does not have any continuous section.

Corollary 3 has the following interpretation: it is not possible to continuously choose an elliptic curve structure for every cubic plane curve \( C_f \) in the parameter space \( X \), because it is not possible to continuously choose a point on \( C_f \) to serve as the identity.

Between the known algebraic constructions (Theorem 1) and the topological obstructions presented here (Theorem 2), the smallest \( n \) for which we don’t know whether a continuous section to \( \xi_n \) exists or not is \( n = 18 \).

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**2. Background**

2.1. The bundle \( \xi_n \). We will first give more details about the fiber bundle \( \xi_n \). Let \( \text{PConf}^n C_f \) denote the configuration space of \( n \) distinct ordered points on \( C_f \):

\[
\text{PConf}^n C_f := \{(x_1, x_2, \cdots, x_n) \in (C_f)^n : x_i \neq x_j \forall i \neq j\}.
\]

Define

\[
\widetilde{E}_n := \{(f, x_1, x_2, \cdots, x_n) \in X \times (\mathbb{CP}^2)^n : x_i \in C_f \forall i \text{ and } x_i \neq x_j \forall i \neq j\}.
\]

The projection \( \widetilde{E}_n \to X \) gives a fiber bundle:

\[
\text{PConf}^n C_f \longrightarrow \widetilde{E}_n \quad \xrightarrow{\xi_n} \quad X
\]

The symmetric group \( S_n \) acts freely on \( \widetilde{E}_n \) by permuting the \( n \) coordinates. Define \( E_n := \widetilde{E}_n / S_n \). Since the bundle projection \( \xi_n : \widetilde{E}_n \to X \) is invariant of \( S_n \), it descends to another bundle \( \xi_n : E_n \to X \) as in (1.1). The fiber of \( \xi_n \) over \( f \in X \) is precisely the unordered configuration space of \( C_f \)

\[
\text{UConf}^n C_f := \text{PConf}^n C_f / S_n.
\]

**Remark 1.** The Lie group \( \text{PGL}_3(\mathbb{C}) = \text{GL}_n(\mathbb{C}) / \mathbb{C}^* \) acts on \( \mathbb{CP}^2 \) by projective linear maps, and thus acts on \( X \) by projective linear change of coordinates. Hence, \( \text{PGL}_3(\mathbb{C}) \) also acts on \( \widetilde{E}_n \) diagonally; this action projects to a \( \text{PGL}_3(\mathbb{C}) \)-action on \( E_n \). The bundle projection \( \xi_n : E_n \to X \) is \( \text{PGL}_3(\mathbb{C}) \)-equivariant. However, a continuous section \( s \) of the bundle \( \xi_n \) does not have to be \( \text{PGL}_3(\mathbb{C}) \)-equivariant. The \( \text{PGL}_3(\mathbb{C}) \)-action on \( X \) will play a key role in the proof of Theorem 2.
2.2. Algebraic sections of $\xi_n$. Every smooth cubic plane curve $C_f$ has

(a) 9 inflection points where the tangent line intersect $C_f$ with multiplicity 3,
(b) 27 sextatic points where an irreducible conic intersects $C_f$ with multiplicity 6 ([2]),
(c) 72 points of type 9 where an irreducible cubic intersects $C_f$ with multiplicity 9 ([5]).

More generally, Gattazzo [5] defined points of type $3k$ on $C_f$ to be points where an irreducible curve of degree $k$ intersects $C_f$ with multiplicity $3k$. In particular, inflection points and sextatic points are precisely points of type 3 and 6, respectively. A proof of the following fact can be found on page 392 in [3]:

Step 1: $s$ induces a continuous map $\phi : \text{PGL}_3(\mathbb{C}) \rightarrow \text{UConf}^n C_f$

Throughout the proof, we will fix a basepoint $f \in \mathcal{X}$ to be the Fermat cubic:

$$f(x, y, z) = x^3 + y^3 + z^3.$$  

If we choose a different basepoint in $\mathcal{X}$, the argument will go through with only small modification. See Remark [4] for an explanation. Recall from Section [2] that $\text{PGL}_3(\mathbb{C})$ acts on both $E_n$ and $\mathcal{X}$. Define a map

$$\phi : \text{PGL}_3(\mathbb{C}) \rightarrow E_n$$

$$g \mapsto g \cdot s(g^{-1} \cdot f)$$

where again $E_n$ is the total space of the fiber bundle $\xi_n$. Notice that for any $g \in \text{PGL}_3(\mathbb{C})$, we have

$$\xi_n(\phi(g)) = \xi_n(g \cdot s(g^{-1} \cdot f))$$

$$= g \cdot \xi_n(s(g^{-1} \cdot f)) \quad \text{since } \xi_n \text{ is } \text{PGL}_3(\mathbb{C})\text{-equivariant}$$

$$= g \cdot (g^{-1} \cdot f) \quad \text{since } \xi_n \circ s = \text{id}_\mathcal{X}$$

$$= f.$$  

Therefore, the image of $\phi$ is entirely in the fiber of $\xi_n$ over $f$. Hence, we will simply consider $\phi$ as a map $\phi : \text{PGL}_3(\mathbb{C}) \rightarrow \text{UConf}^n C_f$. 

**Definition 1** (Jordan’s 2-totient function). $J_2(k) = \text{number of elements in } (\mathbb{Z}/k\mathbb{Z})^2 \text{ of order } k.$

Jordan’s 2-totient function can be computed using the following formula:

$$J_2(k) = k^2 \prod_{p|k, \text{prime}} (1 - \frac{1}{p^2}).$$

It follows from the discussion above that

$$9J_2(k) = \text{the number of points of type 3k on a cubic curve}$$

The coordinates of the points of type $3k$ on $C_f$ change continuously as we vary $f$ in the parameter space $\mathcal{X}$. For example, the 9 inflections points are precisely the vanishing locus of the Hessian function, which depends continuously on the defining equation of the cubic curve. Therefore, the points of type $3k$ define a continuous section of $\xi_n$ for $n = 9J_2(k)$. More generally, given a finite set of positive integers $S$, the points of type $3k$ for $k \in S$ define a continuous section of $\xi_n$ for $n = 9 \sum_{k \in S} J_2(k)$, which gives Theorem [4] in the Introduction.

3. Proof of Theorem [2]

Suppose the bundle $\xi_n$ has a continuous section $s : \mathcal{X} \rightarrow E_n$.

If we choose a different basepoint in $\mathcal{X}$, the argument will go through with only small modification.
Both the domain and the codomain of \( \phi \) are acted upon by a finite group \( \Gamma_f \), the group of projective linear automorphisms of the Fermat cubic curve \( C_f \):

\[
\Gamma_f := \{ g \in \text{PGL}_3(\mathbb{C}) : g \cdot f = f \in \mathcal{X} \}.
\]

\( \Gamma_f \) acts on \( \text{UConf}^n C_f \) since it acts on the curve \( C_f \). \( \Gamma_f \) is a subgroup of \( \text{PGL}_3(\mathbb{C}) \) and thus acts on \( \text{PGL}_3(\mathbb{C}) \) via multiplication from the left.

**Lemma 4.** The map \( \phi : \text{PGL}_3(\mathbb{C}) \to \text{UConf}^n C_f \) is \( \Gamma_f \)-equivariant.

**Proof.** For any \( \gamma \in \Gamma_f \) and any \( g \in \text{PGL}_3(\mathbb{C}) \), we have

\[
\phi(\gamma g) = (\gamma g) \cdot s(g^{-1} \gamma^{-1} \cdot f) = \gamma \cdot \left( g \cdot s(g^{-1} \cdot f) \right) \quad \text{since } \gamma^{-1} \in \Gamma_f \text{ fixes } f = \gamma \cdot \phi(g) \]

Observe that \( \pi_1(\text{PGL}_3(\mathbb{C})) \cong \mathbb{Z}/3\mathbb{Z} \) because we have \( \text{PGL}_3(\mathbb{C}) \cong \text{PSL}_3(\mathbb{C}) \) which is \( \text{SL}_3(\mathbb{C}) \) modulo third roots of unity. On the other hand, Fadell-Neuwirth (Corollary 2.2 in [4]) proved that the ordered configuration space \( \text{PConf}^n \) (defined in Section 2) is aspherical. The same is true for \( \text{UConf}^n C_f \). So \( \pi_1(\text{UConf}^n C_f) \) must be torsion free because the Eilenberg–MacLane space of any group with torsion must have infinite dimensions. Therefore, \( \phi : \text{PGL}_3(\mathbb{C}) \to \text{UConf}^n C_f \) must induce a trivial map on fundamental groups. By lifting criterion, the map \( \phi \) can be lifted to a map \( \tilde{\phi} \) making the following diagram commute:

\[
\begin{array}{ccc}
\text{PConf}^n C_f & \xrightarrow{\tilde{\phi}} & S_n \\
\text{PGL}_3(\mathbb{C}) & \xrightarrow{\phi} & \text{UConf}^n C_f
\end{array}
\]

The natural \( \Gamma_f \)-action on \( \text{PConf}^n C_f \) commutes with the \( S_n \) action.

**Step 2:** \( \phi \) induces a group homomorphism \( \rho : \Gamma_f \to S_n \).

The lift \( \tilde{\phi} \) may not be \( \Gamma_f \)-equivariant, but it projects down to a \( \Gamma_f \)-equivariant map \( \phi \) as in Lemma 3. Therefore, for any \( \gamma \in \Gamma_f \), there exists a unique permutation \( \sigma_\gamma \in S_n \) such that

\[
\tilde{\phi}(\gamma) = \sigma_\gamma (\gamma \cdot \tilde{\phi}(1)).
\]

Let \( \rho : \Gamma_f \to S_n \) denote the function \( \gamma \mapsto \sigma_\gamma \).

**Lemma 5.** (1) For any \( \gamma \in \Gamma_f \) and any \( h \in \text{PGL}_3(\mathbb{C}) \), we have

\[
\tilde{\phi}(\gamma h) = \sigma_\gamma (\gamma \cdot \tilde{\phi}(h)). \tag{3.1}
\]

(2) The function \( \rho : \Gamma_f \to S_n \) is a group homomorphism.

**Proof.** (1) Since \( \text{PGL}_3(\mathbb{C}) \) is connected, we can take a path \( p : [0, 1] \to \text{PGL}_3(\mathbb{C}) \) such that \( p(0) = 1 \) and \( p(1) = h \). Now \( \tilde{\phi}(\gamma \cdot p(t)) \) and \( \sigma_\gamma (\gamma \cdot \tilde{\phi}(p(t))) \) are paths in \( \text{PConf}^n C_f \) that both lift the path \( \phi(\gamma \cdot p(t)) \) in \( \text{UConf}^n C_f \), with the same starting point \( \tilde{\phi}(\gamma) = \sigma_\gamma (\gamma \cdot \tilde{\phi}(1)) \) when \( t = 0 \). Thus they must end at the same point by the uniqueness of path lifting.

(2) Take any \( \beta, \gamma \in \Gamma_f \). On one hand,

\[
\tilde{\phi}(\beta \gamma) = \sigma_{\beta \gamma} (\beta \gamma \cdot \tilde{\phi}(1)).
\]

On the other hand, by (3.1)

\[
\tilde{\phi}(\beta \gamma) = \sigma_{\beta} (\beta \cdot \tilde{\phi}(\gamma)) = \sigma_{\beta} (\beta \cdot \sigma_\gamma (\gamma \cdot \tilde{\phi}(1))) = \sigma_{\beta \sigma_\gamma} (\beta \gamma \cdot \tilde{\phi}(1))
\]

where the last equality follows because the \( S_n \)-action commutes with the \( \Gamma_f \)-action on \( \text{PConf}^n C_f \). Therefore, \( \sigma_{\beta \sigma_\gamma} = \sigma_{\beta \gamma} \).  \[\square\]
It follows from Lemma 5 that the homomorphism \( \rho \) is trivial if and only if the lift \( \tilde{\phi} \) is \( \Gamma_f \)-equivariant. Therefore, the group homomorphism \( \rho : \Gamma_f \to S_n \) measures the failure of the lift \( \phi \) to be \( \Gamma_f \)-equivariant.

**Lemma 6.** Let \( \text{Proj}_i : \text{PConf}^n C_f \to C_f \) denote the projection onto the \( i \)-th coordinate. Consider the action of \( \Gamma_f \) on \( \{1, 2, \ldots, n\} \) given by \( \rho : \Gamma_f \to S_n \). For any \( \gamma \in \Gamma_f \) and any \( i \in \{1, 2, \ldots, n\} \), if \( \sigma_\gamma \) fixes \( i \), then the composition \( \text{Proj}_i \circ \tilde{\phi} \)

\[
\text{PGL}_3(\mathbb{C}) \xrightarrow{\tilde{\phi}} \text{PConf}^n C_f \xrightarrow{\text{Proj}_i} C_f
\]

is equivariant with respect to \( \gamma \).

**Proof.** Given any \( h \in \text{PGL}_3(\mathbb{C}) \), Lemma 5 gives that \( \tilde{\phi}(\gamma \cdot h) = \sigma_\gamma (\gamma \cdot \tilde{\phi}(h)) \). Thus, whenever \( \sigma_\gamma (i) = i \), we have

\[
\text{Proj}_i \circ \tilde{\phi}(\gamma \cdot h) = \text{Proj}_i \left( \sigma_\gamma (\gamma \cdot \tilde{\phi}(h)) \right) = \text{Proj}_i (\gamma \cdot \tilde{\phi}(h)) = \gamma \cdot \text{Proj}_i (\tilde{\phi}(h)). \quad \square
\]

**Remark 2** (Interpretation of the group homomorphism \( \rho \) as monodromy). The above construction of the group homomorphism \( \rho \) from a continuous section \( s \) appears to be ad hoc. We now briefly sketch a more conceptual but less direct construction of \( \rho \) to illustrate that \( \rho \) is a natural object to consider for finding obstructions to sections. We do not need this equivalent construction of \( \rho \) anywhere in the proof.

A continuous section \( s \) produces a cover of \( \mathcal{X} \) of degree \( n \), where the fiber over each curve is the set of \( n \) points on the curve chosen by \( s \). The monodromy representation of this cover gives a group homomorphism

\[
\pi_1(\mathcal{X}) \to S_n.
\]

By restricting the monodromy representation to the \( \text{PGL}_3(\mathbb{C}) \)-orbit of the Fermat curve

\[
\text{PGL}_3(\mathbb{C})/\Gamma_f \approx \text{PGL}_3(\mathbb{C}) \cdot f \leftrightarrow \mathcal{X}
\]

we obtain a group homomorphism

\[
\pi_1(\text{PGL}_3(\mathbb{C})/\Gamma_f) \to S_n.
\]

This homomorphism factors through the quotient \( \pi_1(\text{PGL}_3(\mathbb{C})/\Gamma_f) \to \Gamma_f \), making the following diagram commute:

\[
\begin{array}{ccc}
\pi_1(\text{PGL}_3(\mathbb{C})/\Gamma_f) & \to & S_n \\
\downarrow & & \\
\Gamma_f & \to &
\end{array}
\]

The induced homomorphism \( \Gamma_f \to S_n \) is conjugate to \( \rho \).

**Step 3: \( n \) must be a multiple of 9.** The homomorphism \( \rho : \Gamma_f \to S_n \) gives an action of \( \Gamma_f \) on \( \{1, 2, \ldots, n\} \). In the final step, we will show that \( \Gamma_f \) contains a subgroup \( K \) of order 9 that acts freely. Thus, \( \{1, 2, \ldots, n\} \) is a disjoint union of orbits of \( K \), each of size 9. In particular, \( n \) must be a multiple of 9.

**Lemma 7.** There are two commuting elements \( a \) and \( b \) in \( \Gamma_f \) such that

1. the subgroup \( K := \langle a, b \rangle \leq \Gamma_f \) is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \).
2. \( a \) and \( b \) act on \( C_f \) as translations of order 3.

**Remark 3.** Lemma 7 was mentioned without proof in [11], first paragraph of Section 4. We include a brief proof here to make this paper complete and self-contained.
Proof. Let $a$ and $b$ be two elements in $\text{PGL}_3(\mathbb{C})$ that are represented by the following two matrices:

\[
 a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{bmatrix}.
\]

The commutator of the two matrices is in the center of $\text{GL}_3(\mathbb{C})$. Thus, $a$ and $b$ commute in $\text{PGL}_3(\mathbb{C})$. Both elements are of order 3 and therefore must be the identity map.

We first claim that $a$ acts on $C_f$ via translation. In general, every morphism of an elliptic curve is a composition of a translation and a group homomorphism. Thus, it suffices to show that the induced map $a_\ast \in \text{Aut}(H_1(C_f, \mathbb{Z})) = \text{SL}_2(\mathbb{Z})$ is identity. A straightforward computation shows that $a_\ast$ in $C_f$ does not have any fixed point. Thus, by the Lefschetz fixed point theorem, $a_\ast$ must have trace 2. Now $a_\ast$ is a finite order element in $\text{SL}_2(\mathbb{Z})$ of trace 2, and therefore must be identity.

The exact same argument applies to $b$. $\square$

Lemma 8. For any $\gamma \in K = \langle a, b \rangle$, if there is a continuous $\gamma$-equivariant map $\psi : \text{PGL}_3(\mathbb{C}) \to C_f$, then $\gamma$ must be the identity.

Proof. Note that $\psi : \text{PGL}_3(\mathbb{C}) \to C_f$ must induce a trivial map on fundamental groups since $\pi_1(\text{PGL}_3(\mathbb{C})) \cong \mathbb{Z}/3\mathbb{Z}$ and $\pi_1(C_f) \cong \mathbb{Z}/2$. Thus we can lift $\psi$ to $\tilde{\psi}$ making the following diagram commute:

\[
\begin{array}{ccc}
\text{PGL}_3(\mathbb{C}) & \xrightarrow{\psi} & C_f \\
\downarrow{\tilde{\psi}} & & \\
\mathbb{R}^2 & \xrightarrow{p} & \mathbb{R}^2
\end{array}
\]

$\gamma \in K$ is a translation of the torus $C_f$. Let $\tilde{\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$ be the unique translation of $\mathbb{R}^2$ such that $\tilde{\gamma}(\tilde{\psi}(1)) = \tilde{\psi}(\gamma)$. We claim that for any $h \in \text{PGL}_3(\mathbb{C})$,

\[
\tilde{\gamma}(\tilde{\psi}(h)) = \tilde{\psi}(\gamma h).
\tag{3.2}
\]

Take a path $\mu(t)$ in $\text{PGL}_3(\mathbb{C})$ such that $\mu(0) = 1$ and $\mu(1) = h$. The two paths $\tilde{\gamma}(\tilde{\psi}(\mu(t)))$ and $\tilde{\psi}(\gamma \cdot \mu(t))$ are both lifts of the path $\gamma(\psi(\mu(t))) = \psi(\gamma \cdot \mu(t))$ starting at the same point $\tilde{\gamma}(\tilde{\psi}(1)) = \tilde{\psi}(\gamma)$. Thus they must end at the same point by the uniqueness of path lifting, giving that $\tilde{\gamma}(\tilde{\psi}(h)) = \tilde{\psi}(\gamma h)$.

Applying (3.2) three times, we have

\[
\tilde{\gamma}^3(\tilde{\psi}(1)) = \tilde{\psi}(\gamma^3) = \tilde{\psi}(1).
\]

Thus, $\tilde{\gamma}^3$ must be the trivial translation of $\mathbb{R}^2$, and so is $\tilde{\gamma}$. In this case (3.2) gives that $\tilde{\psi}(1) = \tilde{\psi}(\gamma)$, and thus $\psi(1) = \psi(\gamma) = \gamma \cdot \psi(1)$. $\gamma$ is a translation of the torus $C_f$ that has a fixed point $\psi(1)$, and therefore must be the identity map. $\square$

Recall that the group homomorphism $\rho : \Gamma_f \to S_n$ gives an action of $\Gamma_f$ on $\{1, 2, \cdots, n\}$. Now Lemma 4 and Lemma 5 together imply that the subgroup $K \leq \Gamma_f$ acts freely on $\{1, 2, \cdots, n\}$. Thus, $\{1, 2, \cdots, n\}$ decomposes to a disjoint union of $K$-orbits of size 9. In particular, $n$ must be a multiple of 9. This concludes the proof of Theorem 2.

Remark 4 (What if we choose a different basepoint?). The choice of the Fermat cubic curve $f(x, y, z) = x^3 + y^3 + z^3$ as a basepoint of $X$ allows us to explicitly write down matrices for $a$ and $b$ in Lemma 4. If a different basepoint $h \in X$ is chosen, the argument will go through
with only minor modification. There is an element \( g \in \text{PGL}_3(\mathbb{C}) \) that brings \( h \) to be in the \textit{Hesse form}: \( x^3 + y^3 + z^3 + \lambda xyz \) for some \( \lambda \in \mathbb{C} \) (see e.g. Lemma 1 in [1]). One can check that \( a \) and \( b \) constructed in Lemma 7 act as translations on any smooth cubic curve in the Hesse form. Thus \( gag^{-1} \) and \( gbg^{-1} \) act on \( C_h \) as translations. Now the argument goes through by replacing \( a \) and \( b \) by their \( g \)-conjugates.

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