Exponential Stability for Linear Evolutionary Equations.

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Abstract. We give an approach to exponential stability within the framework of evolutionary equations due to [R. Picard. A structural observation for linear material laws in classical mathematical physics. Math. Methods Appl. Sci., 32(14):1768–1803, 2009]. We derive sufficient conditions for exponential stability in terms of the material law operator which is defined via an analytic and bounded operator-valued function and give an estimate for the expected decay rate. The results are illustrated by three examples: differential-algebraic equations, partial differential equations with finite delay and parabolic integro-differential equations.

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1 Introduction

The article gives an approach to the exponential stability of equations of the form

$$\partial_0 V + AU = F,$$  \hspace{1cm} (1)

where we denote the derivative with respect to the temporal variable by $\partial_0$. The (unbounded) linear operator $A$, acting on some Hilbert space, is assumed to be maximal monotone and we refer to the monographs \[4, 15\] for the topic of monotone operators on Hilbert spaces. Equation (1) is completed by a constitutive relation of the form

$$V = MU,$$

where $M$ is a bounded linear operator acting in time and space. Thus, we end up with an equation of the form

$$(\partial_0 M + A) U = F,$$

which we refer to as an evolutionary equation. This class of problems was introduced by Picard in \[16\], where $A$ was assumed to be skew-selfadjoint, and it was illustrated that many equations of classical mathematical physics are covered by this abstract class. The well-posedness for such problems is proved, by showing that the operator $\partial_0 M + A$ is boundedly invertible in a suitable Hilbert space. More precisely, the derivative $\partial_0$ is established as a normal, continuously invertible operator in an exponentially weighted $L_2$-space and $M$ is defined as a function of $\partial_0^{-1}$ (see Section 2) and the well-posedness is shown under a positive definiteness constraint on the operator $M$ (see \[16\] Solution Theory and Theorem 2.4 of this article). Moreover, the question of causality, which can be seen as a characterizing property for evolutionary processes, was addressed which leads to additional constraints on the operator $M$, namely that $M$ is defined via the Fourier-Laplace transformation of an analytic and bounded function $M : B_C(r,r) \rightarrow L(H)$ (for more details see Section 2). Especially the analyticity of $M$ is crucial for the causality, due to the correlation of supports of $L_2$-functions and the analyticity of their Laplace transforms by the Paley-Wiener Theorem (see \[21\] Theorem 19.2). Later on these results were generalized to the case of $A$ being a maximal monotone relation in \[22, 24\].

In this work we give sufficient criteria for the exponential stability of the evolutionary problem in terms of the function $M$. The study of stability issues for differential equations, which goes back to Lyapunov (see \[14\] for a survey), has become a very active field of research for many decades and there exist numerous works dealing with this topic. We just like to mention some standard approaches to exponential stability. The first strategy goes back to Lyapunov. The aim is to find a suitable function (a so-called Lyapunov function) yielding a certain differential inequality which allows to derive statements about the asymptotic behavior of solutions of the differential equation. A second approach, which applies to linear differential equations, is based on the theory of semigroups. In this framework different criteria for exponential stability were derived in terms of the semigroup or its generator, e.g. Datko’s Lemma (\[8\] or \[9\] p. 300), Gearharts Theorem (\[11\] or \[9\] p. 302) or the Spectral Mapping Theorem (see \[9\] p. 302, Theorem 1.10). A third approach uses the Fourier or the Laplace transform of a solution to derive statements of their asymptotics. These methods are sometimes referred to as Frequency Domain Methods. In our framework it seems to be appropriate to employ the last approach,
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since, by the definition of $M$, methods of vector-valued complex analysis are at hand through the Fourier-Laplace transformation. Note that, due to the general structure of evolutionary equations, the results apply to a broad class of differential equations, such as differential-algebraic equations, equations with memory effects or integro-differential equations, where semigroup methods may be difficult to apply.

The article is structured as follows. In Section 2 we recall the framework of evolutionary equations and its solution theory (Theorem 2.4). Section 3 provides an abstract condition for exponential stability for evolutionary equations in terms of the function $M$ and the proof of the main stability result (Theorem 3.7). To illustrate the versatility of the previous results, we discuss several examples in Section 4. We begin with studying differential-algebraic equations of mixed type and derive a condition for their exponential stability. Moreover, we give a concrete example for such an equation, which seems hard to be tackled by other approaches, since the type of the differential equation switches on different parts of the underlying domain. Furthermore, we give a possible approach of how to deal with initial value problems. In Subsection 4.2 we consider an example of a partial differential equation with memory effect. A similar problem was also treated by Batkai and Piazzera in [2], using a semigroup approach. In contrast to their result our approach directly extends to the case of differential-algebraic equations with delay. We conclude the article by discussing a parabolic integro-differential equation with an operator-valued kernel, where we adopt the ideas of [23] to derive the exponential stability.

Throughout let $H$ be a complex Hilbert space. We denote its inner product by $\langle \cdot | \cdot \rangle$ which is assumed to be linear in the second and conjugate linear in the first argument. We denote its induced norm by $| \cdot |$.

2 The framework of evolutionary equations

We recall some basic notions and results on linear evolutionary equations, i.e. equations of the form

$$\left( \partial_{0,\varrho} M(\partial_{0,\varrho}^{-1}) + A \right) u = f, \quad (2)$$

where $A : D(A) \subseteq H \to H$ is a linear, maximal monotone operator, $\partial_{0,\varrho}$ denotes the time-derivative, established in a suitable Hilbert space and $M(\partial_{0,\varrho}^{-1})$ is a bounded linear operator in time and space, a so-called linear material law. We refer to [16, 17, 13, 22, 24] for more details and proofs of the following statements. First we begin by introducing the Hilbert space setting, where we want to consider equation (2).

For $\varrho \in \mathbb{R}$ we define the space $H_{\varrho,0}(\mathbb{R}; H)$ as the space of (equivalence classes of) measurable function $f : \mathbb{R} \to H$ which are square-integrable with respect to the exponentially weighted Lebesgue measure $e^{-2\varrho t} \, dt$, equipped with the inner product

$$\langle f | g \rangle_{H_{\varrho,0}(\mathbb{R}; H)} := \int_{\mathbb{R}} \langle f(t) | g(t) \rangle e^{-2\varrho t} \, dt.$$ 

Note that $H_{0,0}(\mathbb{R}; H)$ is just the space $L_2(\mathbb{R}; H)$. We define the derivative $\partial_{0,\varrho}$ as the closure
of the operator

\[ \partial_{0,\varrho}|_{C_\infty^c(\mathbb{R}; H)} : C_\infty^c(\mathbb{R}; H) \subseteq H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H) \]

\[ \phi \mapsto \phi', \]

where we denote by \( C_\infty^c(\mathbb{R}; H) \) the space of function \( \phi : \mathbb{R} \to H \) which are arbitrarily often differentiable and have compact support.

**Remark 2.1.**

(a) For each \( \varrho \in \mathbb{R} \) the operator \( \partial_{0,\varrho} \) is normal with \( \mathfrak{R} \partial_{0,\varrho} = \frac{1}{2} \left( \partial_{0,\varrho} + \partial_{0,\varrho}^* \right) = \varrho \). Moreover we obtain that \( \partial_{0,\varrho}^* = -\partial_{0,\varrho} + 2 \varrho \). In particular, for \( \varrho = 0 \) the operator \( \partial_{0,0} \) is skew-selfadjoint and coincides with the usual weak derivative on \( L_2(\mathbb{R}; H) \) with domain \( H_{0,1}(\mathbb{R}; H) = H^1(\mathbb{R}; H) = W^1_2(\mathbb{R}; H) \).

(b) For \( \varrho \neq 0 \) the operators \( \partial_{0,\varrho} \) on \( H_{\varrho,0}(\mathbb{R}; H) \) and \( \partial_{0,0} + \varrho \) on \( L_2(\mathbb{R}; H) \) are unitarily equivalent and \( \partial_{0,\varrho} \) is boundedly invertible with \( \| \partial_{0,\varrho}^{-1} \| = \frac{1}{|\varrho|} \). For \( \varrho = 0 \) the operator \( \partial_{0,0} \) is not boundedly invertible, since 0 lies in its continuous spectrum.

(c) For \( \varrho \neq 0 \) the inverse operator \( \partial_{0,\varrho}^{-1} \) is given by

\[ \left( \partial_{0,\varrho}^{-1} u \right)(t) = \begin{cases} \int_{-\infty}^{t} u(s) \, ds & \text{if } \varrho > 0, \\ -\int_{t}^{\infty} u(s) \, ds & \text{if } \varrho < 0 \end{cases} \]

for \( u \in H_{\varrho,0}(\mathbb{R}; H) \) and almost every \( t \in \mathbb{R} \). This representation yields that for \( \varrho > 0 \) the operator \( \partial_{0,\varrho}^{-1} \) is forward causal whereas it is backward causal\(^1\) for \( \varrho < 0 \).

(d) Let \( \varrho \in \mathbb{R} \) and denote by \( \mathcal{L}_\varrho : H_{\varrho,0}(\mathbb{R}; H) \to L_2(\mathbb{R}; H) \) the so-called Fourier-Laplace transformation, defined as the unitary extension of the operator given by

\[ (\mathcal{L}_\varrho \phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} e^{-\varrho t} \phi(t) \, dt \]

for \( \phi \in C_\infty^c(\mathbb{R}; H) \) and \( x \in \mathbb{R} \). Then

\[ \partial_{0,\varrho} = \mathcal{L}_\varrho^*(im + \varrho) \mathcal{L}_\varrho, \quad (3) \]

where by \( m : D(m) \subseteq L_2(\mathbb{R}; H) \to L_2(\mathbb{R}; H) \) we denote the multiplication-by-the-argument operator with maximal domain, i.e., \( (mf)(t) = tf(t) \) for almost every \( t \in \mathbb{R} \) and every \( f \in D(m) := \{ f \in L_2(\mathbb{R}; H) \mid (t \mapsto tf(t)) \in L_2(\mathbb{R}; H) \} \). Note that in the case \( \varrho = 0 \), \( (3) \) is just the usual spectral representation for the weak derivative on \( L_2(\mathbb{R}; H) \) via the Fourier transformation (see [11] p. 112).

We consider \( (2) \) as an equation in the Hilbert space \( H_{\varrho,0}(\mathbb{R}; H) \). As a matter of physical relevance, we require that the corresponding solution operator \( \left( \partial_{0,\varrho}M(\partial_{0,\varrho}^{-1} + A) \right)^{-1} \), if it

\(^1\) A mapping \( F : D(F) \subseteq H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H) \) is called forward causal if for each \( f, g \in D(F) \) with \( f = g \) on some interval \( ] - \infty, a[ \) for \( a \in \mathbb{R} \) the functions \( F(f) \) and \( F(g) \) coincide on the same interval \( ] - \infty, a[ \). Analogously \( F \) is called backward causal if for each \( f, g \in D(F) \) with \( f = g \) on some interval \( ]a, \infty[ \) for \( a \in \mathbb{R} \) the functions \( F(f) \) and \( F(g) \) coincide on the same interval \( ]a, \infty[ \).
We now define the operator $M$ on a linear material law (see Remark 2.5). Thus, it is natural that $\rho > 0$.

**Definition 2.2.** Let $r > 0$ and $M : B_C(r, r) \to L(H)$\(^2\). For $\rho > \frac{1}{2r}$ we define

$$M(\partial_{0,\rho}^{-1}) := \mathcal{L}_e M \left( \frac{1}{im + \rho} \right) \mathcal{L}_e,$$

where $M \left( \frac{1}{im + \rho} \right)$ is defined as the multiplication operator $\left( M \left( \frac{1}{im + \rho} \right) f \right)(t) := M \left( \frac{1}{im + \rho} \right) f(t)$ on $L_2(\mathbb{R}; H)$ with domain $\left\{ f \in L_2(\mathbb{R}; H) \mid \left( t \mapsto M \left( \frac{1}{im + \rho} \right) f(t) \right) \in L_2(\mathbb{R}; H) \right\}$. We call $M(\partial_{0,\rho}^{-1})$ a linear material law if the function $M$ belongs to $\mathcal{H}^\infty(B_C(r, r); L(H))$, i.e., $M$ is bounded and analytic.

**Remark 2.3.** The notion material law is motivated by several examples of mathematical physics, since it turns out that all material parameters, such as mass density, conductivity, permeability etc. can be incorporated into the operator $M(\partial_{0,\rho}^{-1})$ (see [16] for several examples). Thus, it is natural that $M(\partial_{0,\rho}^{-1})$ is forward causal. Since the operator $M(\partial_{0,\rho}^{-1})$ is linear and it commutes with the translation operators $\tau_h$, mapping $u \in H_{e,0}(\mathbb{R}; H)$ to $t \mapsto u(t + h)$ for $h \in \mathbb{R}$, causality can be characterized via the requirement that $\text{spt} \ M(\partial_{0,\rho}^{-1})u \subseteq [0, \rho]$ if $\text{spt} \ u \subseteq [0, \infty]$, where by $\text{spt} g$ we denote the support of a function $g \in L_2_{\text{loc}}(\mathbb{R}; H)$. This, however, can be characterized by the analyticity and boundedness of the mapping $M$, employing a Paley-Wiener-type result (see e.g. [21] Theorem 19.2). Moreover, note that due to the boundedness of $M$, the operator $M \left( \partial_{0,\rho}^{-1} \right)$ becomes a bounded operator on $H_{e,0}(\mathbb{R}; H)$.

We are now able to state the solution theory for evolutionary equations. For the proof we refer to [16] [24].

**Theorem 2.4 (Solution Theory).** Let $A : D(A) \subseteq H \to H$ be a maximal monotone linear operator. Moreover, let $r > 0$ and $M \in \mathcal{H}^\infty(B_C(r, r); L(H))$ and assume that the solvability condition is satisfied:

$$\exists c > 0 \forall z \in B_C(r, r), x \in H : \Re\{z^{-1}M(z)x|x\} \geq c|x|^2. \quad (4)$$

Then for each $\rho > \frac{1}{2r}$ the problem [3] is well-posed in $H_{e,0}(\mathbb{R}; H)$, i.e. for each $\rho > \frac{1}{2r}$ the operator $\partial_{0,\rho}M(\partial_{0,\rho}^{-1}) + A$ is boundedly invertible and has a dense range. Moreover, the solution operator $(\partial_{0,\rho}M(\partial_{0,\rho}^{-1}) + A)^{-1}$ is forward causal.

**Remark 2.5.** Note that the solution operator $(\partial_{0,\rho}M(\partial_{0,\rho}^{-1}) + A)^{-1}$ commutes with the time-derivative $\partial_{0,\rho}$. This yields, that for right hand sides $f \in D(\partial_{0,\rho}^k)$ for $k \in \mathbb{N}$ in (2) the corresponding solution $u$ also lies in $D(\partial_{0,\rho}^k)$, i.e. the solution operator preserves “temporal” regularity\(^3\).

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\(^2\)We denote by $B_C(x, s)$ the open ball in $\mathbb{C}$ with center $x \in \mathbb{C}$ and radius $s > 0$.

\(^3\)Indeed, one can show that the solution operator $(\partial_{0,\rho}M(\partial_{0,\rho}^{-1}) + A)^{-1}$ extends to the whole Sobolev-chain $(H_k(\partial_{0,\rho}))_{k \in \mathbb{Z}}$ associated with the derivative $\partial_{0,\rho}$, see [17] which yields a solution theory for distributional right-hand sides.
3 An abstract condition for exponential stability

In this section we show that under certain constraints on the function $M$, the corresponding evolutionary problem is exponentially stable. Although in the literature, exponential stability usually means that the solution of an initial value problem decays with an exponential rate as time tends to infinity, we like to introduce a slightly weaker notion within our framework which, however, yields the desired decay if the source term $f$ is regular enough (see Remark 3.2 (a)). Moreover, since the framework of evolutionary equations covers different types of equations, where initial values do not make sense, such as differential-algebraic or even pure algebraic equations or equations with memory effect, where a given pre-history would be more appropriate then an initial value, we cannot treat initial value problems within this general approach. However, in concrete examples we can reformulate initial value problems as evolutionary equations with a modified right hand side (see Subsection 4.1) such that the following results still apply.

**Definition 3.1.** Let $A : D(A) \subseteq H \rightarrow H$ be a maximal monotone linear operator and $M \in \mathcal{H}^{\infty}(B_{\mathbb{C}}(r,r); L(H))$ for some $r > 0$ which satisfies the solvability condition (4). Let $\rho > \frac{1}{2}$ and $\left( \partial_{\rho}M(\partial_{\rho}^{-1}) + A \right)^{-1}$ exponentially stable with stability rate $\nu_0 > 0$ if for each $0 \leq \nu < \nu_0$ and $f \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)$ the solution $u$ of (2) satisfies

$$u = \left( \partial_{\rho}M(\partial_{\rho}^{-1}) + A \right)^{-1} f \in \bigcap_{-\nu < \mu \leq \rho} H_{\mu,0}(\mathbb{R}; H),$$

which especially implies $\int_{\mathbb{R}} e^{2\mu t} |u(t)|^2 \, dt < \infty$ for all $0 \leq \mu < \nu$.

**Remark 3.2.**

(a) We show that our notion of exponential stability indeed yields an exponential decay of the solution if the given right hand side is regular enough. For doing so, let $\rho > \frac{1}{2}$ and $\left( \partial_{\rho}M(\partial_{\rho}^{-1}) + A \right)^{-1}$ exponentially stable with stability rate $\nu_0 > 0$. Moreover, assume that $f \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)$ and $f \in D(\partial_{\rho}u)$ such that $\partial_{\rho}u \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)$ for some $0 < \nu < \nu_0$. Then $u = \left( \partial_{\rho}M(\partial_{\rho}^{-1}) + A \right)^{-1} f$ also lies in $D(\partial_{\rho}u)$ and

$$\partial_{\rho}u = \left( \partial_{\rho}M(\partial_{\rho}^{-1}) + A \right)^{-1} \partial_{\rho}u,$$

(2.5). By the assumed exponential stability, we get that $e^{\mu \nu} u \in L_2(\mathbb{R}; H)$ and $e^{\mu \nu} \partial_{\rho}u \in L_2(\mathbb{R}; H)$ for each $0 \leq \mu < \nu$. The latter yields $e^{\mu \nu} u \in W^{1,2}_2(\mathbb{R}; H)$. Indeed, for $\phi \in C_c^\infty(\mathbb{R}; H)$ we compute

$$\int_{\mathbb{R}} \langle u(t) | e^{2\mu \nu} \phi(t) \rangle_{L^2(\mathbb{R}; H)} \, dt = \int_{\mathbb{R}} \langle u(t) \partial_{\rho}u(t) | e^{2\mu \nu} \phi(t) e^{-2\mu \nu} \rangle_{L^2(\mathbb{R}; H)} \, dt$$

$$= \int_{\mathbb{R}} \langle u(t) | e^{2\mu \nu} \phi(t) \rangle_{L^2(\mathbb{R}; H)} \, dt + \int_{\mathbb{R}} \langle u(t) | e^{2\mu \nu} \phi(t) e^{-2\mu \nu} \rangle_{L^2(\mathbb{R}; H)} \, dt$$

$$= \int_{\mathbb{R}} \langle u(t) | e^{2\mu \nu} \phi(t) \rangle_{L^2(\mathbb{R}; H)} \, dt + \langle u | \partial_{\rho}u \rangle_{H^{1,0}(\mathbb{R}; H)}$$

$$= \int_{\mathbb{R}} \langle u(t) | e^{2\mu \nu} \phi(t) \rangle_{L^2(\mathbb{R}; H)} \, dt + \langle u | -\partial_{\rho} \left( e^{2\mu \nu} \phi \right) \rangle_{H^{1,0}(\mathbb{R}; H)}$$
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\[ + \left\langle u, 2\varrho \left( e^{(2\varrho + \mu) t} \phi \right) \right\rangle_{H_{\varrho,0}(\mathbb{R};H)} \]

\[ = \int_{\mathbb{R}} \langle u(t) | e^{(2\varrho + \mu) t} \phi(t) \rangle e^{-2\varrho t} dt \]

\[ - \int_{\mathbb{R}} \langle u(t) | e^{(2\varrho + \mu) t} \phi'(t) \rangle e^{-2\varrho t} dt + \int_{\mathbb{R}} 2\varrho \langle u(t) | e^{(2\varrho + \mu) t} \phi(t) \rangle e^{-2\varrho t} dt \]

\[ = - \int_{\mathbb{R}} \langle u(t) | e^{(2\varrho + \mu) t} \phi'(t) \rangle e^{-2\varrho t} dt \]

\[ = - (e^{\mu t} u) \frac{d}{dt} e^{\varrho t} \] \(L_2(\mathbb{R};H)\).

Thus, we obtain \(e^{\mu t} |u(t)| \rightarrow 0\) as \(t\) tends to infinity for each \(0 \leq \mu < \nu\) due to Sobolev’s embedding theorem (see e.g. [9, p. 408]), i.e. \(u\) decays exponentially with a decay rate less than \(\nu\).

(b) If \(f \in H_{\mu,0}(\mathbb{R};H) \cap H_{\varrho,0}(\mathbb{R};H)\) for some \(\mu, \varrho \in \mathbb{R}\) with \(\mu > \varrho\), then \(f \in H_{\nu,0}(\mathbb{R};H)\) for all \(\nu \in [\varrho, \mu]\). Indeed, we estimate

\[ \int_{\mathbb{R}} |f(t)|^2 e^{-2\nu t} dt = \int_{-\infty}^{0} |f(t)|^2 e^{-2\nu t} dt + \int_{0}^{\infty} |f(t)|^2 e^{-2\nu t} dt \]

\[ \leq \int_{-\infty}^{0} |f(t)|^2 e^{-2\mu t} dt + \int_{0}^{\infty} |f(t)|^2 e^{-2\nu t} dt \]

\[ \leq |f|^2_{H_{\mu,0}(\mathbb{R};H)} + |f|^2_{H_{\varrho,0}(\mathbb{R};H)}. \]

We now give conditions for the function \(M\) and show that they yield the well-posedness and exponential stability for the corresponding evolutionary problem.

**Hypotheses.** Let \(\nu_0 > 0\). We assume that

(a) \(M : \mathbb{C} \setminus B_C \left[ -\frac{1}{2\nu_0}, \frac{1}{2\nu_0} \right] \rightarrow L(H)\) is analytic\(^4\),

(b) for each \(r > 0\) and \(0 \leq \nu < \nu_0\) the function\(^5\)

\[ B_C(r, r) \setminus \{\nu^{-1}\} \ni z \mapsto (1 - \nu z) M(z(1 - \nu z)^{-1}) \]

has a bounded and analytic extension to \(B_C(r, r)\),

(c) for every \(0 < \nu < \nu_0\) there exists \(c > 0\) such that for all \(z \in \mathbb{C} \setminus B_C \left[ -\frac{1}{2\nu}, \frac{1}{2\nu} \right]\)

\[ \Re z^{-1} M(z) \geq c. \]

**Remark 3.3.** Let \(M\) satisfy the assumptions above and let \(r > 0\). Then the restriction of \(M\) to \(B_C(r, r)\) is an element of \(H^\infty(B_C(r, r); L(H))\). Indeed, the analyticity is clear from (a) and

\(^4\)We denote by \(B_C[x, s]\) the closed ball in \(\mathbb{C}\) with center \(x \in \mathbb{C}\) and radius \(s \geq 0\).

\(^5\)Here and further on we set \(B_C(r, r) \setminus \{0^{-1}\} := B_C(r, r)\).
the boundedness follows from (b). Hence, together with (c), it follows that the problem (2) is well-posed for every \( q > 0 \) according to Theorem 2.4.

We now state some auxiliary results which in particular imply that the solution operator (for each \( k \))

\[
\left( \partial_{0,t}M \left( \partial_{0,t}^{-1} + A \right) \right)^{-1}
\]

for an evolutionary problem does not depend on the particular choice of \( q \). These results can also be found in [17, p. 429 f.]. However, for sake of completeness we present them again with a slightly modified proof.

**Lemma 3.4.** Let \( q, \mu \in \mathbb{R} \) with \( \mu > q \) and set \( U := \{ z \in \mathbb{C} \mid \Re z \in [q, \mu] \} \). Moreover, let \( f : U \to H \) be continuous on \( U \) and analytic in the interior of \( U \), such that \( f^\mu f(iR + s) \, ds \to 0 \) as \( R \to \pm \infty \). Then

\[
\limsup_{R \to \infty} \left| \int_{-R}^{R} f(it + \mu) \, dt - \int_{-R}^{R} f(it + q) \, dt \right| = 0.
\]

**Proof.** According to Cauchy’s integral theorem we have

\[
i \int_{-R}^{R} f(it + \mu) \, dt + \int_{\mathbb{R}} f(iR + s) \, ds - \left( i \int_{-R}^{R} f(it + \mu) \, dt + \int_{\mathbb{R}} f(-iR + s) \, ds \right) = 0
\]

for each \( R > 0 \). The assertion now follows by taking the limes superior as \( R \) tend to \( \infty \). \( \Box \)

The next lemma shows that we can approximate a function which belongs to two different exponentially weighted \( L_2 \)-spaces by the same sequence of test functions with respect to both topologies.

**Lemma 3.5.** Let \( q, \mu \in \mathbb{R} \) and \( f \in H_{q,0}(\mathbb{R}; H) \cap H_{\mu,0}(\mathbb{R}; H) \). Then, for each \( \varepsilon > 0 \) there exists \( \phi \in C_c^\infty(\mathbb{R}; H) \) such that

\[
\max \{ |f - \phi|_{H_{q,0}(\mathbb{R}; H)}, |f - \phi|_{H_{\mu,0}(\mathbb{R}; H)} \} \leq \varepsilon.
\]

**Proof.** Let \( \varepsilon > 0 \). Then we choose \( N \in \mathbb{N} \) such that \( f_N := f \chi_{[-N,N]} \) satisfies

\[
\max \{ |f - f_N|_{H_{q,0}(\mathbb{R}; H)}, |f - f_N|_{H_{\mu,0}(\mathbb{R}; H)} \} \leq \frac{\varepsilon}{2}.
\]

We denote by \( (\psi_k)_{k \in \mathbb{N}} \in C_c^\infty(\mathbb{R})^N \) the Friedrichs mollifier (see e.g. [10 Chapter C.4]). Then, for each \( k \in \mathbb{N} \) we have that \( \text{spt} \psi_k * f_N \subseteq [-N-1,N+1] \). Now, we choose \( k_0 \) large enough such that

\[
|\psi_{k_0} * f_N - f_N|_{L_2([-N-1,N+1]; H)} \leq e^{-2\max\{|\varepsilon|,\mu\}(N+1)} \frac{\varepsilon}{2}
\]

Then the function \( \psi_{k_0} * f_N \in C_c^\infty(\mathbb{R}; H) \) has the desired property. \( \Box \)

**Lemma 3.6.** Let \( q, \mu \in \mathbb{R} \) with \( \mu > q \) and set \( U := \{ z \in \mathbb{C} \mid \Re z \in [q, \mu] \} \). Moreover, let \( f \in H_{q,0}(\mathbb{R}; H) \cap H_{\mu,0}(\mathbb{R}; H) \) and \( T \in \mathcal{H}^\infty(U; L(H)) \cap C_b(U; L(H)) \) (i.e. \( T \) is bounded and continuous on \( U \) and analytic in the interior of \( U \)). Then

\[
(L^*_q T(i \mu + \phi) L_q f) (t) = (L^*_\mu T(i \kappa + \mu) L_\mu f) (t)
\]
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for almost every $t \in \mathbb{R}$.

Proof. According to Lemma 3.5 it suffices to prove the assertion for test functions. So let $\phi \in C_c^\infty(\mathbb{R}; H)$. We show that the function

$$U \ni z \mapsto e^{zt} T(z) ((L_{Re z} \phi)(\text{Im } z)) \in H$$

satisfies the assumptions of Lemma 3.4. Indeed it is continuous in $U$ and analytic in the interior of $U$ as a composition of analytic functions. Furthermore, we estimate for $s \in [\varrho, \mu], \xi \in \mathbb{R}$:

$$|L_s \phi(\xi)| \leq \frac{1}{\sqrt{2\pi}} \int e^{-rs}|\phi(r)| \, dr \leq \frac{C}{\sqrt{2\pi}} \int |\phi(r)| \, dr,$$

where $C := \sup \{e^{-rs} | s \in [\varrho, \mu], r \in \text{spt } \phi \}$, which shows that the function

$$U \ni z \mapsto (L_{Re z} \phi)(\text{Im } z)$$

is bounded. Moreover, due to the Riemann-Lebesgue lemma, we get that $(L_s \phi)(R) \to 0$ as $R \to \pm \infty$ for every $s \in \mathbb{R}$. Therefore, according to Lebesgue’s dominated convergence theorem, we deduce that

$$\int_\varrho^\mu e^{i(R+s)t} T(iR+s)(L_s \phi)(R) \, ds \leq \max \{e^{it}, e^{it}\}|T|_\infty \int_\varrho^\mu (|L_s \phi|(R)) \, ds \to 0 \quad (R \to \pm \infty).$$

Thus, by Lemma 3.4 we get that

$$\int_\mathbb{R} e^{i(s+\varrho)t} T(is+\varrho)(L_\varrho \phi)(s) \, ds = \int_\mathbb{R} e^{i(s+\mu)t} T(is+\mu)L_\mu \phi(s) \, ds,$$

which yields the assertion. 

We are now able to prove our main theorem.

**Theorem 3.7.** Let $A : D(A) \subseteq H \to H$ be a maximal monotone linear operator and $M$ a mapping satisfying the hypotheses above for some $\nu_0 > 0$. Then for each $\varrho > 0$ the solution operator $\left(\partial_{\varrho, \varrho} M \left(\partial_{\varrho, \varrho}^{-1} + A \right)^{-1} \right)$ is exponentially stable with stability rate $\nu_0$.

Proof. Let $\varrho > 0$, $0 \leq \nu < \nu_0$ and take $f \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\varrho,0}(\mathbb{R}; H)$. We set

$$u := \left(\partial_{\varrho, \varrho} M \left(\partial_{\varrho, \varrho}^{-1} + A \right)^{-1} f \right)$$

and we have to show that $u \in H_{\mu,0}(\mathbb{R}; H)$ for each $\mu \in [-\nu, \varrho]$. Let $0 < \eta < \varrho + \nu$. We define

$$\tilde{N}(z) = (1 - \nu z) M \left(z (1 - \nu z)^{-1} \right)$$
for \( z \in B_C \left( \frac{1}{2\eta}, \frac{1}{2\eta} \right) \setminus \{ \nu^{-1} \} \). Note that according to hypotheses (b), \( \tilde{N} \) has an extension \( N \in \mathcal{H}^\infty \left( B_C \left( \frac{1}{2\eta}, \frac{1}{2\eta} \right); L(H) \right) \). Moreover,

\[
\Re z^{-1} N(z) = \Re z^{-1} \tilde{N}(z) = \Re \left( \frac{1 - \nu z}{z} \right) M \left( \frac{z}{1 - \nu z} \right) \geq c \tag{5}
\]

for every \( z \in B_C \left( \frac{1}{2\eta}, \frac{1}{2\eta} \right) \setminus \{ \nu^{-1} \} \) and some suitable \( c > 0 \), since \( z (1 - \nu z)^{-1} \notin B_C \left[ -\frac{1}{2\nu}, \frac{1}{2\nu} \right] \).

Due to the continuity of \( N \), estimate (5) also holds for \( z = \frac{1}{\nu} \). Thus, according to Theorem 2.4, we obtain a solution

\[
v := \left( \partial_{0,\eta} N(\partial_{0,\eta}^{-1}) + A \right)^{-1} e^{\nu m} f \in H_{\eta,0}(\mathbb{R}; H),
\]

where we have used that \( e^{\nu m} f \in H_{0,0}(\mathbb{R}; H) \cap H_{\eta+\nu,0}(\mathbb{R}; H) \subseteq H_{\eta,0}(\mathbb{R}; H) \) (see Remark 3.2 (b)). We apply Lemma 3.6 to \( T(z) = (zN(z^{-1}) + A)^{-1} \) for \( z \in \mathbb{C} \) with \( \Re z \geq \eta \) and get that

\[
v = \mathcal{L}_\eta^* \left( (im + \eta) N \left( \frac{1}{im + \eta} + A \right) \right)^{-1} \mathcal{L}_\eta e^{\nu m} f
\]

\[
= \mathcal{L}_{\eta+\nu}^* \left( (im + \eta + \nu) N \left( \frac{1}{im + \eta + \nu} + A \right) \right)^{-1} \mathcal{L}_{\eta+\nu} e^{\nu m} f.
\]

Using \((it + \eta + \nu) N \left( \frac{1}{it + \eta + \nu} \right) = (it + \eta) M \left( \frac{1}{it + \eta} \right) \) for \( t \in \mathbb{R} \), we derive

\[
e^{-\nu m} v = \mathcal{L}_\eta^* \left( (im + \eta) M \left( \frac{1}{im + \eta} + A \right) \right)^{-1} \mathcal{L}_\eta f.
\]

Again, applying Lemma 3.6 to \( T(z) = (zM(z^{-1}) + A)^{-1} \) for \( z \in \mathbb{C} \) with \( \Re z \geq \min \{ \eta, \eta \} \) we get that

\[
e^{-\nu m} v = \mathcal{L}_\eta^* \left( (im + \eta) M \left( \frac{1}{im + \eta} \right) + A \right)^{-1} \mathcal{L}_\eta f
\]

\[
= \mathcal{L}_\theta^* \left( (im + \theta) M \left( \frac{1}{im + \theta} \right) + A \right)^{-1} \mathcal{L}_\theta f
\]

\[
= u,
\]

which gives \( u \in H_{\eta-\nu}(\mathbb{R}; H) \). Since \( \eta \in ]0, \eta+\nu[ \) was chosen arbitrarily, we get the assertion. \( \square \)

## 4 Examples

In this section we illustrate our results of the previous section by three different types of differential equations which, however, are all covered by the abstract notion of evolutionary equations. We emphasize that we do not claim that in the forthcoming examples the stability
4 Examples

rates are optimal under the given constraints nor that an exponential decay could not be obtained under lesser constraints. But we emphasize that our approach provides a unified way to study exponential stability of a broad class of differential equations.

We begin to study a class of differential-algebraic equations, where the material law is of the simplest form. Moreover, we provide a strategy of how to deal with initial values problems for this class. As a second example we treat a partial differential-algebraic equation with finite delay. We conclude this section with an example of a parabolic integro-differential equation with an operator-valued kernel.

4.1 Differential-algebraic equations of mixed type

It turns out that in applications, the material law is often of the form 

\[ M(\partial_{0,\varrho}^{-1}) = M_0 + \partial_{0,\varrho}^{-1}M_1 \]

(see for instance [16, 17, 22]), where \( M_0, M_1 \in L(H) \). The corresponding evolutionary equation is then of the form

\[ (\partial_{0,\varrho}M_0 + M_1 + A)u = f, \]  

(6)

where \( A : D(A) \subseteq H \to H \) is a maximal monotone linear operator. In order to obtain the well-posedness of this evolutionary equation we require that \( M_0 \) is selfadjoint and strictly positive definite on its range, while \( \Re M_1 := \frac{1}{2}(M_1 + M_1^*) \) is strictly positive definite on the kernel of \( M_0 \) (see [16, 22, 24] for the proof of well-posedness). In order to obtain exponential stability for this problem, we require that \( \Re M_1 \) is strictly positive definite on the whole space \( H \).

**Theorem 4.1.** Let \( A : D(A) \subseteq H \to H \) be a maximal monotone linear operator and \( M_0, M_1 \in L(H) \) such that \( M_0 \) is selfadjoint and strictly positive definite on its range, and \( \Re M_1 \geq c > 0 \). Then for each \( \varrho > 0 \) the solution operator \( (\partial_{0,\varrho}M_0 + M_1 + A)^{-1} \) is exponentially stable with stability rate \( c \|M_0\| \).

**Proof.** We have to verify the hypotheses (a)-(c) for the function

\[ M(z) = M_0 + zM_1 \quad (z \in \mathbb{C}). \]

Obviously the assumption (a) holds. Let now \( r > 0 \) and \( \nu \geq 0 \). Then we compute

\[ \quad (1 - \nu z)M \left( (1 - \nu z)^{-1} \right) = (1 - \nu z)M_0 + zM_1 \]

for \( z \in B_{\mathbb{C}}(r, r) \setminus \{\nu^{-1}\} \), which shows (b). Let now \( \nu \in ]0, \frac{c}{\|M_0\|}[ \). In order to show (c) we note, that for \( z \in \mathbb{C} \setminus B_{\mathbb{C}} \left[ -\frac{1}{2
u}, \frac{1}{2
u} \right] \) there exists \( t \in \mathbb{R} \) and \( \varrho > -\nu \) such that \( z^{-1} = it + \varrho \). Thus, for \( \varrho \geq 0 \) we can estimate

\[ \Re z^{-1}M(z) = \varrho M_0 + \Re M_1 \geq c, \]

while for \( \varrho \in ]-\nu, 0[ \) we estimate

\[ \Re z^{-1}M(z) = \varrho M_0 + \Re M_1 \geq -\nu\|M_0\| + c > 0. \]

Thus, the assertion follows by Theorem 3.7. \( \square \)
4.1 Differential-algebraic equations of mixed type

To illustrate the versatility of our approach, we discuss the following simple example of an evolutionary equation, whose type (elliptic, parabolic or hyperbolic) changes on different parts of the underlying domain. A similar example was also discussed in [18, 19].

Example 4.2. Let $\Omega \subseteq \mathbb{R}^n$ and $\Omega_0, \Omega_1 \subseteq \Omega$ be measurable disjoint subsets with positive Lebesgue measure. We consider the evolutionary equation

$$\left( \partial_{0,\varepsilon} \left( \begin{array}{ccc} \chi_{\Omega_0} + \chi_{\Omega_1} & 0 & 0 \\ 0 & \chi_{\Omega_0} & 0 \\ 0 & 0 & \chi_{\Omega_0} \end{array} \right) + \left( \begin{array}{ccc} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{array} \right) + \left( \begin{array}{ccc} 0 & \text{div}_c \\ \text{grad} & 0 \\ 0 & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (7)$$

where $c > 0$. The differential operator $\text{div}_c$ is defined as the closure of the operator

$$\text{div} |_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n \subseteq L_2(\Omega)^n \to L_2(\Omega)$$

$$(\phi_i)_{i \in \{1, \ldots, n\}} \mapsto \sum_{i=1}^{n} \partial_i \phi_i,$$

where we denote by $\partial_i$ the partial derivative with respect to the $i$-th coordinate. The operator $\text{grad}$ is defined as the negative adjoint of $\text{div}_c$, i.e.

$$\text{grad} := - (\text{div}_c)^*.$$

This operator is just the usual weak gradient on $L_2(\Omega)$ with domain $H^1(\Omega)$. Note that then the operator matrix $\left( \begin{array}{ccc} 0 & \text{div}_c \\ \text{grad} & 0 \end{array} \right)$ is a skew-selfadjoint operator (and hence maximal monotone) on $H := L_2(\Omega) \oplus L_2(\Omega)^n$. Moreover, the operators

$$M_0 = \begin{pmatrix} \chi_{\Omega_0} + \chi_{\Omega_1} & 0 \\ 0 & \chi_{\Omega_0} \end{pmatrix}, \quad M_1 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

satisfy the assumptions of Theorem 4.1 and hence, the solution operator is exponentially stable with stability rate $c$.

Although this example seems to be quite easy, it seems hard to attack the problem of solving (7) by using semigroup techniques. The reason for that is that $\Omega$ changes its type on different parts of the domain $\Omega$. Indeed, on $\Omega_0$ we obtain a hyperbolic problem of the form

$$\left( \partial_{0,\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \left( \begin{array}{ccc} 0 & \text{div}_c \\ \text{grad} & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

while on $\Omega_1$ the problem becomes parabolic, namely

$$\left( \partial_{0,\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \left( \begin{array}{ccc} 0 & \text{div}_c \\ \text{grad} & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

which yields, in case of $g = 0$ the parabolic differential equation

$$\partial_{0,\varepsilon}v + cv - c^{-1} \text{div}_c \text{grad} v = f.$$
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On the remaining part $\Omega \setminus (\Omega_0 \cup \Omega_1)$ the problem is elliptic

$$
\left( \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} + \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \right) \begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.
$$

Note that we can treat this problem, without requiring any explicit transmission conditions on the interfaces $\partial \Omega_0$ and $\partial \Omega_1$ and without imposing regularity assumptions on these boundaries.

**Initial value problems**

Now, we present a possible way to tackle initial value problems for equations of the form (6). Consider the following initial value problem:

$$(\partial_{0,\phi} M_0 + M_1 + A) u = f \quad \text{on } [0, \infty[ \quad (8)$$

for $M_0, M_1, A$ as before, $f \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)$ for some $\nu, \varrho > 0$ with $\text{spt} f \subseteq [0, \infty[$ and $u_0 \in D(A)$. One way to deal with such problems is to consider the evolutionary equation

$$(\partial_{0,\phi} M_0 + M_1 + A) \tilde{v} = f - \chi_{[0,\infty)}(m) M_1 u_0 - \chi_{[0,\infty)}(m) A u_0 \quad \text{on } \mathbb{R},$$

for the new unknown $\tilde{v} := u - \chi_{[0,\infty)}(m) u_0$ given by $\tilde{v}(t) = u(t) - \chi_{[0,\infty)}(t) u_0$ for almost every $t \in \mathbb{R}$. Then the right-hand side belongs to $H_{\nu,0}(\mathbb{R}; H)$ for positive $\varrho$, but does not decay if $f$ decays. Hence, this approach can be used for the issue of well-posedness but it is not appropriate for exponential stability.

Instead, we consider an alternative problem for the unknown

$$v := u - \phi(m) u_0,$$

where $\phi$ is given by

$$\phi(t) := \begin{cases} 1 & \text{if } t \in [0, 1], \\ 2 - t & \text{if } t \in [1, 2], \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that if $u$ satisfies (8), then $v$ satisfies the equation

$$(\partial_{0,\phi} M_0 + M_1 + A) v = f + \chi_{[1,2]}(m) M_0 u_0 - \phi(m) M_1 u_0 - \phi(m) A u_0 =: g \quad (9)$$

and vice versa. Since the function $\chi_{[1,2]}(m) M_0 u_0 - \phi(m) M_1 u_0 - \phi(m) A u_0$ belongs to $H_{\mu,0}(\mathbb{R}; H)$ for every $\mu \in \mathbb{R}$, we obtain $g \in H_{\nu,0}(\mathbb{R}; H) \cap H_{-\nu,0}(\mathbb{R}; H)$. Thus, Theorem 4.1 applies to (9) and we get that (9) is well-posed and $v \in H_{\nu,0}(\mathbb{R}; H)$ for each $\mu \in [-\nu, \varrho]$. This gives that $u \in H_{\mu,0}(\mathbb{R}; H)$ for each $\mu \in [-\nu, \varrho]$, since $\phi(m) u_0 \in \bigcap_{\mu \in \mathbb{R}} H_{\mu,0}(\mathbb{R}; H)$. It is left to show in which sense $M_0 u$ attains the initial value $M_0 u_0$. By (8) we get that

$$\partial_{0,\phi} M_0 v = g - M_1 v - A v$$

\footnote{Note that it only makes sense to prescribe an initial value for $M_0 u$ yielding an initial value for the part of $u$ lying in the range of $M_0$.}
4.2 Linear partial differential equations with finite delay

As a second example we study a differential equation with finite delay of the form

\[(\partial_{0,t} M_0 + \tau_h + M_1 + A) u = f,\]

where \(M_0, M_1 \in L(H)\) such that \(M_0\) is selfadjoint and non-negative, \(\Re M_1 \geq c > 1\), \(A : D(A) \subseteq H \to H\) is linear and maximal monotone and \(\tau_h\) is the translation operator with respect to time, i.e. \((\tau_h u) (t) = u(t + h)\) for \(t \in \mathbb{R}\) and some \(h \leq 0\). We will prove that under these assumptions, the corresponding solution operator is exponentially stable and we give an estimate for the stability rate. A similar problem is treated in [24, Example 4.14] for a particular operator \(A\), where the well-posedness is shown via semigroups and a criterion for the exponential stability is given, using the Spectral Mapping Theorem for eventually norm continuous semigroups (cf. [9, p. 280]).

Before we state our stability result for (10), we need to inspect the operator \(\tau_h\) a bit closer.

**Lemma 4.3.** Let \(\varrho, h \in \mathbb{R}\). We define the operator

\[\tau_h : H_{\varrho, 0}(\mathbb{R}; H) \to H_{\varrho, 0}(\mathbb{R}; H),\]

\[u \mapsto (t \mapsto u(t + h)).\]

Then \(\tau_h \in L(H_{\varrho, 0}(\mathbb{R}; H))\) with \(\|\tau_h\| = e^{\varrho h}\) and \(\tau_h = L_{\varrho}^* e^{(im+\varrho)h} L_{\varrho}\).

**Proof.** Obviously, \(\tau_h\) defines a bounded linear operator on \(H_{\varrho, 0}(\mathbb{R}; H)\). For \(\phi \in C_c^\infty(\mathbb{R}; H)\) we compute

\[L_{\varrho} (\tau_h \phi) (t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} e^{-\varrho s} \phi(s + h) \, ds\]

\[= e^{(it+\varrho)h} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} e^{-\varrho s} \phi(s) \, ds\]

equals \(e^{(it+\varrho)h} (L_{\varrho} \phi) (t)\)

for each \(t \in \mathbb{R}\), which gives \(\tau_h = L_{\varrho}^* e^{(im+\varrho)h} L_{\varrho}\). Moreover, by this unitary equivalence we get that

\[\|\tau_h\| = \|e^{(im+\varrho)h}\| = e^{\varrho h}.\]

\[\square\]

For a boundedly invertible linear operator \(C : D(C) \subseteq H \to H\) the extrapolation space \(H_{-1}(C)\) is given as the completion of \(H\) with respect to the norm \(| \cdot |_{H_{-1}(C)}\) defined as \(|x|_{H_{-1}(C)} := |C^{-1} x|_H\) for \(x \in H\) (see [17, Section 2.1]).
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Using this lemma, we are able to write (10) as an evolutionary equation with

\[ M(z) = M_0 + z e^{-\nu h} + z M_1, \]

which is clearly analytic and bounded on balls of the form \( B_C(r, r) \) for \( r > 0 \) if we require that \( h \leq 0 \). This restriction is natural, since for \( h \leq 0 \) the operator \( \tau_h \) is forward causal, while for \( h > 0 \) it is backward causal.

**Theorem 4.4.** Let \( A : D(A) \subseteq H \to H \) be a maximal monotone linear operator, \( M_0, M_1 \in L(H) \) such that \( M_0 \) is selfadjoint and non-negative and \( \Re M_1 \geq c > 1 \). Moreover let \( h < 0 \). Then for each \( \varrho > 0 \) the solution operator \( (\partial_{t, \varrho} M_0 + \tau_h + M_1 + A)^{-1} \) is exponentially stable with stability rate \( \nu_0 > 0 \) satisfying

\[ \nu_0\|M_0\| + e^{-\nu_0 h} = c. \]

**Proof.** We have to show that \( M \) given by (11) satisfies the hypotheses (a)-(c) of Section 3. Obviously \( M \) is analytic on \( \mathbb{C} \setminus \{0\} \), which shows (a). Let now \( r > 0 \) and \( \nu \geq 0 \). As

\[ (1 - \nu z)M \left( z(1 - \nu z)^{-1}\right) = (1 - \nu z)M_0 + z e^{z(1 - \nu)h} + z M_1 \]

for \( z \in B_C(r, r) \setminus \{\nu^{-1}\} \), we see that \( z \mapsto (1 - \nu z)M \left( z(1 - \nu z)^{-1}\right) \) has an analytic extension to \( B_C(r, r) \). Moreover we estimate

\[ \sup_{z \in B_C(r, r)} |e^{(z-1 - \nu)h}| = \sup_{\varrho > \frac{1}{2\nu}} e^{(\nu - \nu)h}, \]

which is finite, since \( h < 0 \). Thus, hypothesis (b) is also satisfied. For showing (c), let \( \nu \in [0, \nu_0[ \) and \( z \in \mathbb{C} \setminus B_C \left[ -\frac{1}{2\nu}, \frac{1}{2\nu}\right] \). Then there exists \( \varrho > -\nu \) and \( t \in \mathbb{R} \) such that \( z^{-1} = it + \varrho \). If \( \varrho \geq 0 \) we estimate

\[ \Re z^{-1} M(z) = g M_0 + e^{\varrho h} \cos(\varrho t) + \Re M_1 \geq c - 1 > 0, \]

and for the case \( \varrho \in ]-\nu, 0[ \) we get

\[ \Re z^{-1} M(z) = g M_0 + e^{\varrho h} \cos(\varrho t) + \Re M_1 \geq -\nu\|M_0\| - e^{-\nu h} + c > 0, \]

since \( \nu < \nu_0 \). This proves (c) and thus, the assertion follows by Theorem 3.7. \( \square \)

**Remark 4.5.** Note that \( M_0 \) in (11) is allowed to have a non-trivial kernel which could also depend on the spatial variable (compare Subsection 4.1). Thus, Theorem 4.4 also covers a certain class of differential-algebraic equations with delay.

4.3 Parabolic integro-differential equations

We consider the following parabolic integro-differential equation

\[ \partial_{t, \varrho} u + B u - C * B u = f, \]

where \( B : D(B) \subseteq H \to H \) is linear such that \( A := B - c \) is maximal monotone for some \( c > 0 \), \( C : [0, \infty[ \to L(H) \) is a weakly measurable function such that \( t \mapsto \|C(t)\| \) is measurable.
and there exists $\nu_0 > 0$ with $\int_0^\infty \|C(t)\| e^{\nu_0 t} \, dt < 1$. We set $U := \{z \in \mathbb{C} | \Im z \leq \nu_0\}$ and define the complex Fourier transform of $C$ by

$$\hat{C}(z) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-itz} C(t) \, dt \quad (z \in U),$$

where the integral is meant in the weak sense. Note that $\hat{C} : U \to \mathcal{L}(H)$ is continuous and bounded on $U$ and analytic in the interior of $U$. The well-posedness and asymptotic behavior for equations of the form (12), including non-linear perturbations, were discussed in several works (e.g. [5, 25, 23] and [6, 20, 3, 7, 23] for a hyperbolic version of the problem), imposing additional constraints on the kernel $C$. Following [23] we are led to assume that $C$ satisfies the following conditions:

1. $C(t)$ is selfadjoint for almost every $t \in \mathbb{R}$,
2. $C(t)$ and $C(s)$ commute for almost every $t, s \in \mathbb{R}$,
3. for all $t \in \mathbb{R}$ we have $t \Im \hat{C}(t+i\nu_0) \leq 0$. (13)

**Remark 4.6.**

(a) Note that (13) is equivalent to

$$\Im \hat{C}(t+i\nu_0) \leq 0 \quad (t \in ]0, \infty[).$$

(b) A typical example for a kernel satisfying the conditions above is a real-valued, differentiable function $k : [0, \infty[ \rightarrow [0, \infty[ \quad \text{with} \quad \int_0^\infty k(t)e^{\nu_0 t} \, dt < 1 \quad \text{and} \quad k'(t) \leq -k(t)v_0 \quad \text{for every} \quad t \geq 0.$

Similar kernels were considered by Prï¿½ss [20] under a weaker constraint on $k'$. Indeed, the conditions 1. and 2. are trivially satisfied, since $k$ is real-valued. For showing condition 3. we note that

$$e^{\nu_0 t}k(t) - e^{\nu_0 s}k(s) \leq \sup_{\xi \in [s,t]} e^{\nu_0 \xi}(\nu_0 k(\xi) + k'(\xi)) \leq 0,$$

for every $t \geq s \geq 0$. Thus, the function $t \mapsto e^{\nu_0 t}k(t)$ is non-increasing and we estimate

$$\Im \hat{k}(t+i\nu_0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\nu_0 s} \sin(-ts)k(s) \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \left( \int_{k\pi/\nu_0}^{(2k+1)\pi/\nu_0} e^{\nu_0 s} \sin(-ts)k(s) \, ds + \int_{(2k+1)\pi/\nu_0}^{2(k+1)\pi/\nu_0} e^{\nu_0 s} \sin(-ts)k(s) \, ds \right)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_{k\pi/\nu_0}^{(2k+1)\pi/\nu_0} \sin(-ts) \left( e^{\nu_0 s}k(s) - e^{\nu_0 (s+\pi/\nu_0)}k \left( s + \pi/\nu_0 \right) \right) \, ds$$

\[8\text{Here we use the fact that scalar analyticity and local boundedness on a norming set yields analyticity (see [12, Theorem 3.10.1]).}\]
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\[ \leq 0 \]

for every \( t \in ]0, \infty[ \) (compare [23, Remark 3.6 (b)]).

(c) In [6] the authors considered real-valued kernels \( k : [0, \infty[ \to \mathbb{R} \) such that \( \int_{0}^{\infty} k(s)e^\nu s \, ds < 1 \) and the integrated kernel \( [0, \infty[ \ni t \mapsto \int_{t}^{\infty} k(s)e^\nu s \, ds \) gives rise to a positive definite convolution operator on \( L^2([0, \infty[) \). Again, the conditions 1. and 2. are satisfied, since \( k \) is real-valued and condition 3. holds according to [6, Proposition 2.2 (a)].

Before we can state a stability result for problems of the form (12), we recall some properties of the convolution operator \( C^* \).

**Lemma 4.7.** We denote by \( S(\mathbb{R}; H) \) the space of simple \( H \)-valued functions. Then for \( \varrho \geq -\nu_0 \) the operator

\[ C^* : S(\mathbb{R}; H) \subseteq H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H) \]

\[ u \mapsto \left(t \mapsto \int_{\mathbb{R}} C(t-s)u(s) \, ds\right) \]

is bounded and linear with \( \| C^* \|_{L(H_{\varrho,0}(\mathbb{R}; H))} \leq \int_{0}^{\infty} \| C(t) \| e^{\nu_0 t} \, dt \) and can therefore be extended to \( H_{\varrho,0}(\mathbb{R}; H) \). Moreover, for \( u \in H_{\varrho,0}(\mathbb{R}; H) \), \( \varrho \geq -\nu_0 \) we have

\[ (L_{\varrho}(C^* u))(t) = \sqrt{2\pi} \hat{C}(t - i\varrho) (L_{\varrho}u(t)) \] (14)

for almost every \( t \in \mathbb{R} \).

**Proof.** A proof for the first assertion can be found in [23, Lemma 3.1]. The proof of formula (14) is straight forward and we omit it.

According to Lemma 4.7, the operator \( (1 - C^*) \) is boundedly invertible in \( H_{\varrho,0}(\mathbb{R}; H) \) for each \( \varrho \geq -\nu_0 \). Therefore, instead of considering (12) we can study

\[ (\partial_{\varrho,0}(1 - C^*)^{-1} + B) u = (1 - C^*)^{-1} f, \] (15)

or equivalently

\[ (\partial_{\varrho,0}(1 - C^*)^{-1} + c + A) u = (1 - C^*)^{-1} f. \]

Note, that this as an evolutionary equation of the form (2) where \( M \) is defined by

\[ M(z) = (1 - \sqrt{2\pi} \hat{C}(-iz^{-1}))^{-1} + cz, \quad z \in \mathbb{C} \setminus B_{\mathbb{C}} \left(-\frac{1}{2\nu_0}, \frac{1}{2\nu_0}\right) \] (16)

where we have used Lemma 4.7. The next lemma shows that (13) already implies that the same condition holds if one replaces \(-\nu_0\) by \( \varrho \) for arbitrary \( \varrho \geq -\nu_0 \).

**Lemma 4.8.** Assume that \( C \) satisfies the conditions 1., 2. and 3. above. Then for every \( \varrho \geq -\nu_0 \) we have

\[ t \Im \hat{C}(t - i\varrho) \leq 0. \]

**Proof.** The proof can be done analogously to the one of [23, Lemma 3.7].
We now state our stability result for \((15)\).

**Theorem 4.9.** Let \(A : D(A) \subseteq H \to H\) be maximal monotone and linear and let \(c > 0\). Moreover, let \(C : [0, \infty) \to L(H)\) be weakly measurable, such that \(t \mapsto \|C(t)\|\) is measurable and there exists \(\nu_0 > 0\) such that \(\int_0^\infty \|C(t)\|e^{\nu t} < 1\) and \(C\) satisfies the conditions 1., 2. and 3. from above. Then for each \(\nu > 0\) the solution operator \((\partial_{\nu 0}(1 - C^*)^{-1} + c + A)^{-1}\) exists and is exponentially stable with a stability rate \(\nu_1 \in [0, \nu_0]\) satisfying

\[
\nu_1 \left(1 - \int_0^\infty \|C(s)\|e^{\nu_1 s} \, ds\right)^{-1} \leq c.
\]

**Proof.** Let \(\nu_1 \in [0, \nu_0]\) such that \(\nu_1 \left(1 - \int_0^\infty \|C(s)\|e^{\nu_1 s} \, ds\right)^{-1} \leq c\). We prove that \(M\) given by \((16)\) satisfies the hypotheses of Section 3. The assumption (a) is clear, since \(\|\sqrt{2\pi \hat{C}}(-iz^{-1})\| < 1\) for each \(z \in C \setminus B_C \left[\frac{1}{2\nu_1}, \frac{1}{\nu_1}\right]\). Let now \(r > 0\) and \(0 \leq \nu < \nu_1\). Then for \(z \in B_C(r, r) \setminus \{\nu^{-1}\}\) we compute

\[
(1 - \nu z)M(z(1 - \nu z)^{-1}) = (1 - \nu z) \left(1 - \sqrt{2\pi \hat{C}}(-iz^{-1}(1 - \nu z)^{-1})\right)^{-1} c(z(1 - \nu z)^{-1})
\]

\[
= (1 - \nu z) \left(1 - \sqrt{2\pi \hat{C}}(-i(z^{-1} - \nu))\right)^{-1} c(z).
\]

which has a holomorphic extension in \(\nu^{-1}\). Noting that for each \(z \in B_C(r, r)\) we have that \(z^{-1} = it + \varrho\) for some \(\varrho > \frac{1}{2\nu}, t \in \mathbb{R}\), we estimate

\[
\|\sqrt{2\pi \hat{C}}(-i(z^{-1} - \nu))\| \leq \int_0^\infty e^{-(\varrho - \nu)s}\|C(s)\| \, ds
\]

\[
\leq \int_0^\infty e^{\nu_0 s}\|C(s)\| \, ds < 1.
\]

Hence, the extension of \((1 - \nu z)M(z(1 - \nu z)^{-1})\) to \(B_C(r, r)\) is indeed bounded for each \(r > 0, \nu \in [0, \nu_1]\). We now show that \(M\) satisfies the assumption (c) on \(C \setminus B_C \left[\frac{1}{2\nu_1}, \frac{1}{\nu_1}\right]\). We follow the strategy of the proof of [23, Lemma 3.8]. Let \(z \in C \setminus B_C \left[\frac{1}{2\nu_1}, \frac{1}{\nu_1}\right]\). Note that then there exists \(\varrho > -\nu_1\) and \(t \in \mathbb{R}\) such that \(z^{-1} = it + \varrho\). We set \(D := |1 - \sqrt{2\pi \hat{C}}(t - i\varrho)|^{-1}\), which is well-defined according to Lemma 4.8. Note that

\[
(1 - \sqrt{2\pi \hat{C}}(t - i\varrho))^{-1} = D^2(1 - \sqrt{2\pi \hat{C}}(-t - i\varrho)),
\]

where we have used assumption 1. Moreover, due to assumption 2., we get that

\[
D^2(1 - \sqrt{2\pi \hat{C}}(-t - i\varrho)) = D(1 - \sqrt{2\pi \hat{C}}(-t - i\varrho))D.
\]
Thus, we obtain for $x \in H$
\[
\Re(z^{-1} M(z)x) = \Re \left( z^{-1} \left( (1 - \sqrt{2\pi} \hat{C}(-iz^{-1}))^{-1} + cz \right) x \right) \\
= \Re \left( \left( \Re \left( 1 - \sqrt{2\pi} \hat{C}(-t - i\theta) \right) + \sqrt{2\pi t} \Im \hat{C}(-t - i\theta) \right) Dx \right) + c|x|^2 \\
\geq \Re \left( D \left( 1 - \sqrt{2\pi} \hat{C}(-t - i\theta) \right) Dx \right) + c|x|^2.
\]

If $\varrho$ is non-negative, the latter term can be estimated by $c$. For negative $\varrho$ we observe that
\[
\left\| D \Re \left( 1 - \sqrt{2\pi} \hat{C}(-t - i\theta) \right) D \right\| \leq \left\| \left( 1 - \sqrt{2\pi} \hat{C}(-t - i\theta) \right)^{-1} \right\| \\
\leq \frac{1}{1 - \|\sqrt{2\pi} \hat{C}(-t - i\theta)\|} \\
\leq \left( 1 - \int_0^\infty e^{\nu_1 s} \|C(s)\| \, ds \right)^{-1}
\]
and hence,
\[
\varrho \Re \left( D \left( 1 - \sqrt{2\pi} \hat{C}(-t - i\theta) \right) Dx \right) + c|x|^2 \geq \left( \varrho \left( 1 - \int_0^\infty e^{\nu_1 s} \|C(s)\| \, ds \right)^{-1} + c \right) |x|^2 \\
> \left( -\nu_1 \left( 1 - \int_0^\infty e^{\nu_1 s} \|C(s)\| \, ds \right)^{-1} + c \right) |x|^2,
\]
which shows that $M$ satisfies hypothesis (c), according to the choice of $\nu_1$. Thus, the assertion follows by Theorem 3.7.

**Remark 4.10.** Theorem 4.9 gives the exponential stability for equation (15). This also yields the exponential stability of the original problem (12), since the operator $(1 - C^*)^{-1}$ leaves the space $H_{-\nu,0}(\mathbb{R}; H)$ for all $\nu \leq \nu_0$ invariant. Indeed, observing that
\[
e^{\nu m}(1 - C^*)^{-1} = (1 - (e^{\nu m}C)^*)^{-1} e^{\nu m}
\]
we obtain for $f \in H_{-\nu,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)$
\[
e^{\nu m}(1 - C^*)^{-1} f = (1 - (e^{\nu m}C)^*)^{-1} e^{\nu m} f \in L_2(\mathbb{R}; H).
\]

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