Dust-acoustic solitary waves in a magnetized dusty plasma with nonthermal electrons and trapped ions

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The nonlinear theory of electrostatic dust-acoustic (DA) waves in a magnetized dusty plasma consisting of negatively charged mobile dusts, nonthermal fast electrons and trapped ions with vortex-like distribution is revisited. Previous theory in the literature [Phys. Plasmas 20, 104505 (2013)] is rectified and put forward to include the effects of the external magnetic field, the adiabatic pressure of charged dusts as well as the obliqueness of propagation to the magnetic field. Using the reductive perturbation technique, a Korteweg-de Vries (KdV)-like equation is derived which governs the dynamics of the small-amplitude solitary waves in a magnetized dust nonthermal plasma. It is found that due to the dust thermal pressure, there exists a critical value ($\beta_c$) of the nothermal parameter $\beta$ ($>1$), denoting the percentage of energetic electrons, below which the DA solitary waves cease to propagate. The soliton solution (travelling wave) of the KdV-like equation is obtained, and is shown to be only of the rarefactive type. The properties of the solitons are analyzed numerically with the system parameters. It is also seen that the effect of the static magnetic field (which only modifies the soliton width) becomes significant when the dust gyrofrequency is smaller than one-tenth of the dust plasma frequency. Furthermore, the amplitude of the soliton is found to increase (decrease) when the ratio of the free to trapped ion temperatures ($\sigma$) is positive (negative). The effects of the system parameters including the obliqueness of propagation ($l_z$) and $\sigma$ on the dynamics of the DA solitons are also discussed numerically, and it is found that the soliton structures can withstand perturbations and turbulence during a considerable time. The results should be useful for understanding the nonlinear propagation of DA solitary waves in laboratory and space plasmas (e.g., Earth’s magnetosphere, auroral region, heliospheric environments etc.).

I. INTRODUCTION

Recently, Paul et al.\textsuperscript{3} has investigated the nonlinear propagation of dust-acoustic (DA) solitary waves in an unmagnetized dusty plasma with a population of nonthermal electrons and trapped positive ions (vortex-like distribution). They derived a Korteweg-de Vries (KdV)-like equation which has a nonlinearity proportional to the three-half power of the wave potential, i.e., $\Phi^{3/2}$. Furthermore, they predicted that the DA waves may propagate in the form of stationary solitons with positive amplitude. This soliton solution was obtained in the form of sech$^2(x)$, similar to the Korteweg-de Vries (KdV) soliton. We, however, rectify these results and show that the nonlinearity in the KdV-like equation should be proportional to $(-\Phi)^{3/2}$. Here, the negative sign appears due to the vortex-like distribution of positively charged species, i.e., ions. Thus, a travelling wave solution of the KdV-like equation should be in the form of sech$^4(x)$ with negative amplitude instead of sech$^2(x)$ with positive amplitude.\textsuperscript{1} We also put forward the work of Paul et al.\textsuperscript{3} to include the effects of the external magnetic field, the obliqueness of wave propagation as well as the thermal pressure of charged dusts. These effects are shown to modify the soliton characteristics significantly.

Since the work of Rao et al.\textsuperscript{4}, in which the existence of nonlinear DA waves in unmagnetized dusty plasmas (where the charged dusts provide the inertia and the restoring forces come from the pressures of inertialless electrons and ions) was first predicted, there has been a number of works focusing on the linear or nonlinear properties of low-frequency DA waves in magnetized or unmagnetized plasmas (See, e.g., Refs.\textsuperscript{5,12}). The nonlinear propagation of such waves can give rise to the formation of solitons with negative or positive wave amplitudes, which has potential applications in astrophysical and space environments (such as cometary tails, planetary rings, interstellar clouds and lower parts of Earth’s ionosphere etc.) as well as in laboratory and technological studies.\textsuperscript{13–16} Furthermore, electrostatic solitary waves have been observed in several regions, including the Earths magnetotail, bow shock/solar wind, and polar magnetosphere.\textsuperscript{17–19}

On the other hand, the observations in space plasma environments indicate the presence of electrons and ions which are not in thermodynamic equilibrium\textsuperscript{19,20}. The presence of nonthermal electrons in Earth’s bow-shock region has been confirmed by the Vela satellite.\textsuperscript{18} Also, the ASPERA on the Phobos 2 satellite has recorded the loss of energetic electrons from the upper ionosphere of...
Mars. Furthermore, observations made by the Viking spacecraft and Freja satellite have indicated the presence of electrostatic solitary structures in the magnetosphere with density depression. It has been shown by numerical simulation that the propagation of DA waves can give rise to a significant amount of ion trapping in the wave potential. Malkki et al. had demonstrated that the nonlinear ion-hole instability is the best candidate for the generation mechanism of the negative isolated potentials. Naturally, there is a departure from the Boltzmann distribution of electrons with an excess of energetic particles. The higher-order approximation enhances the amplitude of the DA waves. It was pointed out that the higher-order approximations support only rarefaction solitons. Furthermore, the nonlinear propagation of DA waves in a magnetized dusty plasma consisting of very massive, micrometer-sized, negatively charged inertial dust grains, nonthermally distributed fast electrons, and ions with trapped particles. The external static magnetic field is considered along the z-axis, i.e., \( \mathbf{B} = B_0 \mathbf{z} \). Thus, the nonlinear dynamics of the low-phase velocity (in comparison with the electron and ion thermal velocities, i.e., \( v_{te} \ll \omega/k \ll v_{ij} \)) where \( v_{ij} = \sqrt{k_B T_j/m_j} \) is the thermal velocity of j-species particles with \( j = e, i, d \) for electrons, ions and charged dusts, low-frequency (in comparison with the dust-cyclotron frequency) DA waves is given by the following set of normalized equations:

\[
\frac{\partial n_d}{\partial t} + \nabla \cdot (n_d \mathbf{u}_d) = 0, \quad (1)
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u}_d \cdot \nabla \right) \mathbf{u}_d = \nabla \phi + \omega_c (\mathbf{u}_d \times \mathbf{z}) - \frac{5}{3} T_n^{-1/3} \nabla n_d, \quad (2)
\]

\[
\nabla^2 \phi = \delta n_e - \mu n_i + n_d, \quad (3)
\]

where \( n_d \) and \( \mathbf{u}_d \) are, respectively, the number density and velocity of charged dusts (with mass \( m_d \)) normalized by the unperturbed value \( n_{d0} \) and the DA speed \( c_d = \sqrt{Z_d k_B T_d/m_d} \), with \( T_d \) denoting the electron temperature, \( k_B \) the Boltzmann constant and \( Z_d \) the charged dust state, i.e., the number of electrons/ions residing on the dust-grain surface. Also, \( \phi \) is the electrostatic wave potential normalized by \( k_B T_c/e \), where \( e \) is the magnitude of the electron charge, \( T_c = T_d/T_e \) with \( T_d \) denoting the dust temperature, \( \omega_c = |q_d| B_0 / m_d \omega_{pd} \) is the dust-cyclotron frequency normalized by the dust plasma oscillation frequency \( \omega_{pd} = \sqrt{4 \pi n_{d0} Z_d^2 e^2 / m_d} \). The time and space variables are, respectively, in the units of the dust plasma period \( \omega_{pd}^{-1} \) and the Debye length \( \lambda_D = \sqrt{k_B T_c / 4 \pi n_{d0} Z_d e^2} \). At equilibrium, the charge neutrality condition reads \( \mu \equiv n_{e0}/Z_d n_{d0} = 1 + \delta = 1 + n_{e0}/Z_d n_{d0} \). For the propagation of DA waves since the thermal motion of charged dusts can not keep up with the wave, we consider adiabatic compression of the dust fluid, and use the pressure law: \( P_d/P_{d0} = (n_d/n_{d0})^{3} \), where \( P_{d0} = k_B T_d n_{d0} \) and \( \gamma = 5/3 \) for three-dimensional configuration. In this assumption, the effects of viscosity, thermal conductivity and the energy transfer due to collisions can be neglected. In our fluid model we have also considered the plasma thermal pressure to be smaller than the magnetic pressure and the charging of the dust grains is held constant.

In what follows, the nonthermal distribution of electrons (fast or energetic particles) are given by\( ^{21,20} \):

\[
n_e = (1 - \beta \phi + \beta \phi^2) e^\phi, \quad (4)
\]

where \( \beta = 4 \gamma / (1 + 3 \gamma) \), with \( \gamma > 0 \), denotes the degree of nonthermality of the charged particles (percentage of fast or energetic electrons) in the plasma, and \( \beta \gtrsim 1 \) according to when \( \gamma \gtrsim 1 \). The value \( \gamma = 0 \) corresponds to the case of thermal equilibrium (Boltzmann distribution) of electrons.

II. BASIC EQUATIONS AND THE DERIVATION OF KDV-LIKE EQUATION

We consider the nonlinear propagation of DA waves in a magnetized dusty plasma consisting of very massive, micrometer-sized, negatively charged inertial dust grains, nonthermally distributed fast electrons, and ions with trapped particles. The external static magnetic field is considered along the z-axis, i.e., \( \mathbf{B} = B_0 \mathbf{z} \). Thus, the nonlinear dynamics of the low-phase velocity (in comparison with the electron and ion thermal velocities, i.e., \( v_{te} \ll \omega/k \ll v_{ij} \)) where \( v_{ij} = \sqrt{k_B T_j/m_j} \) is the thermal velocity of j-species particles with \( j = e, i, d \) for electrons, ions and charged dusts, low-frequency (in comparison with the dust-cyclotron frequency) DA waves is given by the following set of normalized equations:

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\left( \frac{\partial}{\partial t} + \mathbf{u}_d \cdot \nabla \right) \mathbf{u}_d = \nabla \phi + \omega_c (\mathbf{u}_d \times \mathbf{z}) - \frac{5}{3} T_n^{-1/3} \nabla n_d, \quad (2)
\]

\[
\nabla^2 \phi = \delta n_e - \mu n_i + n_d, \quad (3)
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where \( n_d \) and \( \mathbf{u}_d \) are, respectively, the number density and velocity of charged dusts (with mass \( m_d \)) normalized by the unperturbed value \( n_{d0} \) and the DA speed \( c_d = \sqrt{Z_d k_B T_d/m_d} \), with \( T_d \) denoting the electron temperature, \( k_B \) the Boltzmann constant and \( Z_d \) the charged dust state, i.e., the number of electrons/ions residing on the dust-grain surface. Also, \( \phi \) is the electrostatic wave potential normalized by \( k_B T_c/e \), where \( e \) is the magnitude of the electron charge, \( T_c = T_d/T_e \) with \( T_d \) denoting the dust temperature, \( \omega_c = |q_d| B_0 / m_d \omega_{pd} \) is the dust-cyclotron frequency normalized by the dust plasma oscillation frequency \( \omega_{pd} = \sqrt{4 \pi n_{d0} Z_d^2 e^2 / m_d} \). The time and space variables are, respectively, in the units of the dust plasma period \( \omega_{pd}^{-1} \) and the Debye length \( \lambda_D = \sqrt{k_B T_c / 4 \pi n_{d0} Z_d e^2} \). At equilibrium, the charge neutrality condition reads \( \mu \equiv n_{e0}/Z_d n_{d0} = 1 + \delta = 1 + n_{e0}/Z_d n_{d0} \). For the propagation of DA waves since the thermal motion of charged dusts can not keep up with the wave, we consider adiabatic compression of the dust fluid, and use the pressure law: \( P_d/P_{d0} = (n_d/n_{d0})^{3} \), where \( P_{d0} = k_B T_d n_{d0} \) and \( \gamma = 5/3 \) for three-dimensional configuration. In this assumption, the effects of viscosity, thermal conductivity and the energy transfer due to collisions can be neglected. In our fluid model we have also considered the plasma thermal pressure to be smaller than the magnetic pressure and the charging of the dust grains is held constant.

In what follows, the nonthermal distribution of electrons (fast or energetic particles) are given by\( ^{21,20} \):

\[
n_e = (1 - \beta \phi + \beta \phi^2) e^\phi, \quad (4)
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where \( \beta = 4 \gamma / (1 + 3 \gamma) \), with \( \gamma > 0 \), denotes the degree of nonthermality of the charged particles (percentage of fast or energetic electrons) in the plasma, and \( \beta \gtrsim 1 \) according to when \( \gamma \gtrsim 1 \). The value \( \gamma = 0 \) corresponds to the case of thermal equilibrium (Boltzmann distribution) of electrons.
On the other hand, the ion distribution with free and trapped particles is modelled for small-amplitude perturbations \(|\phi| \ll 1\) as

\[
n_i \approx 1 - \phi - \frac{4(1 - \sigma)}{3\sqrt{\pi}} (-\phi)^{3/2} + \frac{1}{2} \phi^2,
\]

where \(\sigma = T_i/T_d\) is the ratio of the free to trapped ion temperatures. This temperature ratio \(\sigma\) is allowed to be negative \((\sigma < 0)\) corresponding to a depression in the trapped particle distribution, while \(\sigma = 0\) represents the plateu (flat-topped). Furthermore, in the limit of \(\sigma \to 1\), \(n_i(\phi)\) approaches the Boltzmann distribution. Thus, one can consider either \(\sigma < 0\) or \(0 < \sigma < 1\).

Next, for the dynamical evolution of the small-amplitude electrostatic DA perturbations we follow the same reductive perturbation technique as in Refs.12,23 in which the independent variables are stretched as

\[
\xi = \epsilon^{1/4}(l_xx + l_yy + l_zz - v_0t),
\]
\[
\tau = \epsilon^{3/4}t,
\]

where \(0 < \epsilon < 1\) is a small parameter measuring the weakness of the wave amplitude, \(v_0\) is the nonlinear wave phase velocity normalized by the DA speed \(c_d\). Also, \(l_x\), \(l_y\) and \(l_z\) are the direction cosines of the wave vector along the axes such that \(l_x^2 + l_y^2 + l_z^2 = 1\). The dependent variables, namely \(n_d\), \(u_d\) and \(\phi\) are expanded as

\[
n_d = 1 + \epsilon n_d^{(1)} + \epsilon^{3/2} n_d^{(2)} + \cdots,
\]
\[
\phi = \phi^{(1)} + \epsilon^{3/2} \phi^{(2)} + \cdots,
\]
\[
u_d = \epsilon u_d^{(1)} + \epsilon^{3/2} u_d^{(2)} + \cdots,
\]
\[
u_d(x,y) = \epsilon^{5/4} u_{d,xy}^{(1)} + \epsilon^{3/2} u_{d,xy}^{(2)} + \cdots.
\]

In the above expansions, the first-order perturbations for the transverse velocity components of the dust fluids appear in higher-orders of \(\epsilon\) than that for the parallel component. For the nonlinear DA waves, this anisotropy is introduced due to the fact that the dust gyromotion (perpendicular to the magnetic field) is treated as a higher-order effect than the parallel motion to the magnetic field.20,21

In what follows, we substitute Eqs. \(6\) and \(7\) into Eqs. \(1\), \(3\), and equate the coefficients of different powers of \(\epsilon\). Thus, equating the coefficients of \(\epsilon^{5/4}\), \(\epsilon^{3/4}\) and \(\epsilon^{7/4}\), from Eq. \(1\), we successively obtain

\[
-v_0 n_d^{(1)} + l_x u_d^{(1)} = 0,
\]
\[
-l_x u_d^{(1)} + l_y u_d^{(1)} = 0,
\]
\[
-v_0 \frac{\partial n_d^{(2)}}{\partial \xi} + \frac{\partial n_d^{(1)}}{\partial \tau} + l_x \frac{\partial u_d^{(2)}}{\partial \xi} + l_y \frac{\partial u_d^{(2)}}{\partial \xi} + l_z \frac{\partial u_d^{(2)}}{\partial \xi} = 0.
\]

From the \(x\)-component of Eq. \(2\), equating the coefficients of \(\epsilon^{5/4}\) and \(\epsilon^{3/4}\) we obtain

\[
l_x \frac{\partial \phi^{(1)}}{\partial \xi} + \omega_c u_d^{(1)} - \frac{5}{3} T_l z \frac{\partial n_d^{(1)}}{\partial \xi} = 0,
\]

Also, from the \(y\)-component of Eq. \(2\), equating the coefficients of \(\epsilon^{5/4}\) and \(\epsilon^{3/4}\) we obtain

\[
-l_y \frac{\partial \phi^{(1)}}{\partial \xi} - \omega_c u_d^{(1)} - \frac{5}{3} Y T_l z \frac{\partial n_d^{(1)}}{\partial \xi} = 0,
\]

\[
-v_0 u_d^{(1)} = \omega_c u_d^{(2)}.
\]

Similarly, from the \(z\)-component of Eq. \(2\), equating the coefficients of \(\epsilon^{5/4}\) and \(\epsilon^{3/4}\) we successively obtain

\[
v_0 u_d^{(2)} = -l_z \phi^{(1)} + \frac{5}{3} T_l z n_d^{(1)},
\]

\[
-v_0 \frac{\partial n_d^{(2)}}{\partial \xi} + \frac{\partial n_d^{(1)}}{\partial \tau} = l_x \frac{\partial \phi^{(2)}}{\partial \xi} - \frac{5}{3} T_l z \frac{\partial n_d^{(2)}}{\partial \xi}.
\]

Next, using the expressions \(\[1\] and \(\[2\]\) for electron and ion number densities, we successively obtain after equating the coefficients of \(\epsilon^3\) from Eq. \(3\) as

\[
n_d^{(1)} = -[\mu + \delta(1 - \beta)] \phi^{(1)},
\]

\[
n_d^{(2)} = \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - \frac{4 \mu_5(1 - \sigma)}{3 \sqrt{\pi}} \left(- \phi^{(1)}\right)^{3/2} - [\mu + \delta(1 - \beta)] \phi^{(2)}.
\]

Using Eq. \(\[17\]\), we obtain from Eqs. \(\[11\]\), \(\[13\]\) and \(\[15\]\) the first-order components of the dust fluid velocity as

\[
u_x^{(1)} = \frac{l_x}{\omega_c} \left[1 + \frac{5}{3} T(\mu + \delta(1 - \beta)) \right] \frac{\partial \phi^{(1)}}{\partial \xi},
\]

\[
u_y^{(1)} = -\frac{l_x}{\omega_c} \left[1 + \frac{5}{3} T(\mu + \delta(1 - \beta)) \right] \frac{\partial \phi^{(1)}}{\partial \xi},
\]

\[
u_z^{(1)} = -\frac{v_0}{l_z} [\mu + \delta(1 - \beta)] \phi^{(1)}.
\]

Similarly, using Eqs. \(\[19\]\) and \(\[20\]\), we obtain from Eqs. \(\[12\]\) and \(\[14\]\) the second order quantities for the velocity as

\[
u_x^{(2)} = -\frac{v_0 l_x}{\omega_c} \left[1 + \frac{5}{3} T(\mu + \delta(1 - \beta)) \right] \frac{\partial^2 \phi^{(1)}}{\partial \xi^2},
\]

\[
u_y^{(2)} = -\frac{v_0 l_y}{\omega_c} \left[1 + \frac{5}{3} T(\mu + \delta(1 - \beta)) \right] \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}.
\]

Eliminating \(u_x^{(1)}\) and \(n_d^{(1)}\) from Eqs. \(\[15\]\), \(\[17\]\) and \(\[21\]\) we obtain the following dispersion relation

\[
v_0 = l_z \left[\frac{1}{\mu + \delta(1 - \beta)} + \frac{5}{3} \frac{T}{T} \right]^{1/2}.
\]
This represents the phase velocity of the obliquely propagating DA waves in the moving \( \xi - \tau \) frame of reference. It can be shown that this phase velocity corresponds to the longitudinal propagation of low-frequency, long-wavelength slow-wave eigen modes that can be obtained by Fourier analyzing the set of basic equations. Furthermore, for the DA waves to propagate with the velocity \( v_0 \), the expression in the square brackets of Eq. (24) must be positive, which gives \( \beta > \beta_c = 1 + (\mu + 3/5T)/\delta > 1 \). This implies that in hot dusty plasmas, the small-amplitude DA wave propagation is possible when the phase velocity \( v_0 \) is not only influenced by the effects of the nonlinearity and dispersion of charged dusts, but also greatly modified by the effects of the number density of charged particles and the nonthermality of electrons, but also greatly modified by the effects of the number density of charged particles and the nonthermality of electrons, but also greatly modified by the effects of the dust temperature \( (T_d) \) as well as the obliqueness of propagation \( (\lambda) \).

Typical variations of the phase velocity \( v_0 \) with respect to the density ratio \( \delta \) are shown in Fig. 1 (See the left panel) for different values of \( \lambda \) and \( T \). We find that the phase velocity \( v_0 \) is not only influenced by the effects of the number density of charged particles and the nonthermality of electrons, but also greatly modified by the effects of the dust temperature \( (T_d) \) as well as the obliqueness of propagation \( (\lambda) \).

Next, from Eqs. (10) and (16) we obtain

\[
\left( v_0 - \frac{5}{3} \frac{l^2}{v_0} T \right) \frac{\partial \phi_1}{\partial \xi} = \frac{\partial u_1}{\partial \tau} - \frac{l^2}{v_0} \frac{\partial \phi_2}{\partial \xi} + l \frac{\partial u_2}{\partial \xi} + l_y \frac{\partial u_y}{\partial \xi} + l_z \frac{\partial u_1}{\partial \xi}.
\]

Substituting the expressions for \( u_1^{(2)}, u_2^{(2)} \) and \( u_y^{(2)} \) from Eqs. (18), (22) and (23) into Eq. (25), and noting that the coefficient of \( \phi^{(2)} \) vanishes by the dispersion relation \( (24) \), we obtain, after a few steps, the following KdV-like equation

\[
\frac{\partial \phi^{(1)}}{\partial \tau} + A \sqrt{-\phi^{(1)}} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0.
\]

where the coefficients of nonlinearity and dispersion are

\[
A = \frac{1 - \sigma}{\sqrt{\sigma}} \frac{\mu c^2}{v_0^2 (\mu + \delta (1 - \beta))},
\]

\[
B = \frac{1}{6} \frac{3 + [\mu + \delta (1 - \beta)] (5T - 3v_0^2)}{\omega_c^2 (\mu + \delta (1 - \beta))^2} \left( \frac{l^2}{2v_0 (\mu + \delta (1 - \beta))^2} \right),
\]

in which \( \mu = 1 \pm \delta \) for negatively/positively charged dusts.

We note that in contrast to the usual KdV equation appeared in various contexts of plasmas (See, e.g., Ref. 11), Eq. (26) has a nonlinear term proportional to the threehalf power of the wave potential. This modification of the nonlinear term is, however, due to the consideration of vortex-like ion distribution in the plasma. The similar form of the KdV equation has also been appeared in unmagnetized dusty plasma. We note that both the coefficients \( A \) and \( B \) are modified by the parameters \( l_1, T, \delta \) and \( \beta \), while the effects of the trapped ions characterized by \( \sigma \) and the static magnetic field characterized by \( \omega_c \) are, respectively, entered in \( A \) and \( B \) only. It should be mentioned that the negative sign in \( -\phi^{(1)} \) in the nonlinear term of Eq. (26), which appears due to the vortex-like distribution of the positively charged particles, i.e., ion, does not appear in a recent work on DA waves in an unmagnetized plasma with mobile dusts, nonthermal electrons and trapped positive ions. Furthermore, in Ref. 11 it has been shown that the DA soliton exists only of the compressive type with a form of \( \text{sech}^4(\eta) \) as the KdV soliton. We, however, rectify these results in our present work and show that the evolution equation for the propagation of DA solitary waves in dusty plasmas with trapped ions can be described by a KdV-like equation (26), not in the form as derived in Ref. 11. Also, as in the previous study (26), Eq. (26) must have a stationary soliton in the form of \( \text{sech}^4(\eta) \) (to be shown in the next section) not the \( \text{sech}^2 \)-type. We also show that unlike compressive solitons to exist in Ref. 11, the dusty plasmas with nonthermal fast electrons and trapped ions support only rarefactive-type solitons.

### III. SOLUTION OF THE KDV-LIKE EQUATION AND DISCUSSION

The travelling wave solution of Eq. (26) can be obtained by transforming the independent variables \( \xi \) and \( \tau \) to \( \eta = \xi - u_0 \tau \) and \( \tau = \tau \) where \( u_0 \) is a constant velocity normalized by \( c_d \), and imposing the appropriate boundary conditions, namely, \( \phi \to 0 \), \( d\phi^{(1)}/d\eta \to 0 \), \( d^2\phi^{(1)}/d\eta^2 \to 0 \) as \( \eta \to \pm \infty \). Thus, one can find the soliton solution as (See for details, Appendix A)

\[
\Phi = \phi^{(1)} = -\phi_m^{(1)} \text{sech}^4 \left( \frac{\xi - u_0 \tau}{\Delta} \right),
\]
where the amplitude \( \phi_m^{(1)} \) and the width \( \Delta \) (normalized by \( \lambda_D \)) are given by

\[
\phi_m^{(1)} = \left( \frac{15u_0}{8A} \right)^2,
\]

\[
\Delta = \left( \frac{16B}{u_0} \right)^{1/2}.
\]

This soliton solution corresponds to the rarefaction or compression of the dust number density, i.e., the dust density holes or humps [See Eq. 17] according to when \( \beta \gtrsim 2 + 1/\delta \). From Eq. (29) we find that in contrast to the usual KdV soliton (See, e.g., Ref. 11), the amplitude of the soliton [Eq. (26)] does not depend on the sign of the nonlinear coefficient \( A \). However, for the existence of such solitons, \( B \) must be positive. It can be shown that \( B > 0 \) for a wide range of values of the parameters, namely \( 0 < l_z, T \lesssim 1, \delta > 1 \) and \( \beta > \beta_c \).

We point out that the perturbation method, which is valid for the small but finite amplitude perturbations may not be valid for large \( \Phi \), which makes the wave amplitude large enough to break the condition \( \epsilon |\phi^{(1)}| < 1 \). We numerically analyze the properties of the stationary soliton [Eq. (29)] by the effects of the parameters \( l_z, \delta, T, \beta, \sigma \) and \( \omega_c \). We consider those plasma parameters for which \( \beta > \beta_c \) and \( B > 0 \) are satisfied. Different characteristic features of the DA soliton are exhibited in Fig. 2. Inspecting on the coefficients \( A \) and \( B \), we find that the external magnetic field characterized by \( \omega_c \) (the parameter \( \sigma \) for trapped ions) does not have any effect on the amplitude (width), but on the width (amplitude) of the soliton. Next, we study the effects of the plasma parameters as follows:

(i) **Effects of the obliqueness of propagation \( (l_z) \) and the plasma density \( (\delta) \):** The effects of the obliqueness parameter \( l_z \) and the electron to dust density ratio \( \delta \) on the amplitude and width of the soliton are shown in Fig. 2(a) (See the solid and dashed lines for the variation with \( l_z \), and dashed and dotted lines for \( \delta \)). It is found that, in each case, increasing the values of \( l_z \) (which means that the cosine of the angle of propagation decreases) and \( \delta \) (i.e., one enters into a relatively dense plasma regime in which both electron and ion number densities increase in order to maintain the charge neutrality), keeping others constant (i.e., \( T = 0.1, \beta = 0.1 + \beta_c \) and \( u_0 = 0.03 \)), lead to a decrease in the wave amplitude (Hereafter, the wave amplitude means its magnitude \( |\Phi| \)), and a small increase in the width.

(ii) **Effects of the dust temperature \( (T) \) and the presence of fast electrons \( (\beta) \):** The effects of the thermal pressure of charged dusts (solid and dashed lines) as well as the presence of fast electrons (See the dashed and dotted lines) in the plasma (Keeping others constant, i.e., \( l_z = \delta = 0.1, \sigma = -2.5, \omega_c = 0.05, \beta = 0.1 + \beta_c \) and \( u_0 = 0.03 \)) on the soliton characteristics are exhibited in Fig. 2(b). It is seen that the temperature ratio \( T \) of dust to electron temperature has stronger influence with decreasing the amplitude and a small increase in the width of the soliton. The effect of \( \beta \) is quite noticeable, i.e., a small increase in its value \( (> \beta_c) \) leads to a significant increase of both the amplitude and width of the soliton. Thus, in magnetized warm dusty nonthermal plasmas, the percentage of fast particles has to be larger than the slow counterparts in order to sustain the DA solitary waves, and higher the percentage of fast particles the larger are the amplitude and width of the
solitons.

(iii) Effects of the free to trapped ion temperature ($\sigma$): Figure 2(c) shows the effects of the presence of trapped ions characterized by $\sigma$ in the plasma. We find that when $\sigma < 0$, which corresponds to a depression in the trapped particle distribution, its higher values (in magnitude) lead to a significant decrease of the soliton amplitude. Since $B$ is independent of $\sigma$, the soliton width remains unchanged with the variation of $\sigma$. On the other hand, when $\sigma (> 0)$ tends to 1, i.e., as one approaches thermal equilibrium of ions in the plasma, the soliton amplitude is found to increase (not shown in the figure).

(iv) Effects of the static magnetic field ($\omega_c$): The influence of the magnetic field strength characterized by the nondimensional dust gyrofrequency $\omega_c$ on the properties of solitons is shown in Fig. 2(d). As mentioned before, the magnetic field does not have any effect on the amplitude of the soliton, however, it causes the solitons to become narrower with higher values of $\omega_c < 0.1$. For $\omega_c \gtrsim 0.1$, the effect of the magnetic field on the soliton width is no longer appreciable. The effect of $\omega_c$ is significant at its lower values, and thus, lower the values of $\omega_c$, the higher is the soliton width.

We numerically investigate the time evolution as well as the stability/instability of the DA solitary waves given by Eq. (29). For the numerical integration of Eq. (29), we use the centered second-order difference approximations for the spatial derivatives with periodic boundary conditions and the standard fourth-order Runge-Kutta scheme for the time stepping. The simulation results are exhibited in Fig. 3. The effects of the oblique wave propagation ($l_z$) on the stability of the DA solitary waves are shown in Figs. 3(a) and 3(b) for $l_z = 0.05$ and $l_z = 0.12$ respectively. The other fixed parameters are $\sigma = -2.5$, $\beta = 0.2 + \beta_c$, $\delta = 1.5$, $T = 0.1$, and $\omega_c = 0.2$. It is found that the solitary waves are more stable for larger $l_z < 1$, i.e., as the angle of propagation to the static magnetic field decreases, the stability of the DA perturbations increases. Figures 3(c) and 3(d) show the similar effects of the trapped ion temperature (characterized by $\sigma$) on the solitary waves for $\sigma = -2.0$, $l_z = 0.05$ and $\sigma = -0.5$ $l_z = 0.12$ respectively. The other fixed parameters are the same as those used for Figs. 3(a) and 3(b). Comparing Figs. 3(a) and 3(c) and Figs. 3(b) and 3(d) we find that as $|\sigma|$ ($\sigma < 0$) decreases, i.e., when the magnitude of the trapped ion temperature is higher than that of the free ions, the instability of the waves increases. The initial conditions used for the plots of Figs. 3(a) and 3(d) respectively, are (i) $\phi^{(1)} = -0.065 \text{sech}^2(\xi/2.002)$, (ii) $\phi^{(1)} = -0.113 \text{sech}^4(\xi/3.099)$, (iii) $\phi^{(1)} = -0.884 \text{sech}^4(\xi/2.002)$, and (iv) $\phi^{(1)} = -0.614 \text{sech}^4(\xi/3.099)$. These are the steady-state solutions of Eq. (26) at $\tau = 0$. From Fig. 3 we also find that the solitons propagate in the negative $\xi$-direction with a profile (almost unchanged) for $\tau < 20$. However, as time goes on, an oscillatory structure is seen to grow in front of the soliton.

It is to be noted that the other parameters, namely $\delta$, $T$, and $\omega_c$ can also have the similar influence on the stability/instability of the solitons. It is found that an increase of $T$ improves the stability of the soliton, while an increase of $\delta$ enhances the instability of the soliton. However, the instability of the soliton is not greatly affected by the variation of $\omega_c$. Since the values of the nonthermal parameter $\beta (= 0.2 + \beta_c)$ depends on $\beta_c = 1 + (1 + \delta + 3/5T)/\delta$, one can also see the effects of $\beta$ with the variations of $T$ and $\delta$ as above.

The dynamical evolution of the soliton profiles is exhibited in Fig. 3 at different times: (a) $\tau = 0$, (b) $\tau = 20$, (c) $\tau = 30$, (d) $\tau = 40$, and (e) $\tau = 50$. The parameters and the initial condition are the same as those in Fig. 3(c). It is seen that in contrast to the KdV soliton (in which the wave steepening occurs[22]), the leading part of the initial pulse flattens due to the modified nonlinearity (proportional to the three-half power of the wave potential). As the time progresses, the pulse separates into solitons that are down-shifted and a residue due to the wave dispersion. It is clear from Fig. 4(d) that once the solitons are formed and get separated, they propagate almost without changing their shape due to the balance of the nonlinearity and the wave dispersion. From Figs. 3 and 4 one can conclude that the DA solitons can withstand perturbations and turbulence in a finite interval of time.

IV. CONCLUSION

We have investigated the propagation characteristics of electrostatic dust-acoustic waves in a magnetized dusty plasma with the combined effects of the nonthermal electrons and trapped ions with vortex-like distribution. Previous theory in the literature[2] is rectified and put forward to include the effects of the external magnetic field, the adiabatic pressure of charged dusts as well as the obliqueness of propagation to the magnetic field. By using the reductive perturbation technique, a KdV-like equation with a nonlinearity proportional to three-half power of the wave potential is derived to investigate the nonlinear propagation of DA solitary waves. It is found that the DA wave propagation is possible when the percentage of energetic electrons remains higher than the slow counterparts, i.e., the degree of nonthermality $\beta > 1$ and exceeds its critical value $\beta_c$. The latter changes with the number densities of the particles as well as the thermal pressure of charged dusts. In absence of the dust thermal pressure, i.e., in cold dusty magnetoplasmas, DA solitary wave can propagate with $\beta < 1$.

The stationary soliton solution of the KdV-like equation, in the form of $\text{sech}^4(\eta)$ (Quite distinctive from the KdV soliton), is obtained and its properties are studied with the system parameters. It is shown that the external magnetic field characterized by $\omega_c$ (the parameter $\sigma$ for trapped ions) does not have any influence on the amplitude (width), but on the width (amplitude) of the soliton. Thus, the external magnetic field causes the soli-
FIG. 2. Plot of the soliton solution [Eq. (29)] with \( \xi \) at \( \tau = 0 \) for different values of the parameters as in the figure. The effects of the static magnetic field on the width of the soliton remain unaltered for \( \omega_c \gtrsim 0.1 \).

tary structures to become more spiky, while the soliton amplitude decreases (increases) with \( \sigma < 0 \) (0 < \( \sigma < 1 \)). Furthermore, the dynamical evolution of the DA solitary waves shows that the DA soliton can withstand perturbations and turbulence during a considerable time. Our numerical results show that a decrease of the obliqueness parameter \( l_z \), the ratio of dust to electron temperature \( T_e \), as well as the ratio of the free to trapped ion temperature \( |\sigma| \) (\( \sigma < 0 \)) favors the instability of the DA solitons, while the similar instability can be achieved by an increase of the ratio of electron to dust charge density (\( \delta \)). The theoretical results should be useful for understanding the nonlinear features of electrostatic dust-acoustic solitary waves that propagate obliquely to the magnetic field in nonthermal plasmas with higher percentage of energetic (fast) electrons and trapped ions in laboratories as well as space plasmas (e.g., Earth's magnetosphere, auroral region, heliospheric environments etc.).

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**APPENDIX A: STATIONARY SOLITON SOLUTION OF THE KDV-LIKE EQUATION**

Equation (26) can be rewritten as

\[
\frac{\partial \Phi}{\partial \tau} + iA\sqrt{\Phi} \frac{\partial \Phi}{\partial \xi} + B \frac{\partial^3 \Phi}{\partial \xi^3} = 0.
\]  

(A.31)

From Eq. (A.31), using the transformation \( \eta = \xi - u_0 \tau \), we get

\[
\frac{d}{d\eta} \left( B \dot{\Phi} - u_0 \Phi + i \frac{2}{3} A \Phi^{3/2} \right) = 0,
\]  

(A.32)

where the dot denotes differentiation with respect to \( \eta \). Integrating Eq. (A.32) and using the boundary conditions \( \Phi, \Phi \rightarrow 0 \) as \( \xi \rightarrow \pm \infty \) we get

\[
B \dot{\Phi} - u_0 \Phi + i \frac{2}{3} A \Phi^{3/2} = 0.
\]  

(A.33)

Multiplying Eq. (A.33) by \( 2\dot{\Phi} \) and integrating with respect to \( \eta \), we obtain

\[
B \dot{\Phi}^2 - u_0 \Phi^2 + i \frac{8}{15} A \Phi^{5/2} = 0,
\]  

(A.34)

where we have used the boundary conditions \( \Phi, \dot{\Phi} \rightarrow 0 \). From Eq. (A.34) we have

\[
\dot{\Phi} = \pm \Phi \sqrt{\frac{u_0}{B} - i \frac{8A}{15B} \sqrt{\Phi}},
\]  

(A.35)

or,

\[
\int \frac{d\Phi}{\Phi \sqrt{u_0/B - i (8A/15B) \sqrt{\Phi}}} = \pm \int d\eta,
\]  

(A.36)
FIG. 3. The stability and instability of the DA solitons given by Eq. (26) are shown with the variations of the obliqueness parameter ($l_z$) [See subplots (a) for $l_z = 0.05$ and (b) for $l_z = 0.12$ with a fixed $\sigma = -2.5$, $\beta = 0.2 + \beta_c$, $\delta = 1.5$, $T = 0.1$ and $\omega_c = 0.2$] and the free to trapped ion temperature ($\sigma$) [See plots (c) for $\sigma = -2.0$, the other parameters being the same as in the subplot (a), and (d) for $\sigma = -0.5$, the other parameters being the same as in the subplot (b)].

FIG. 4. Numerical solution of Eq. (26) exhibiting soliton profiles at different times: (a) $\tau = 0$, (b) $\tau = 20$, (c) $\tau = 30$, (d) $\tau = 40$, and $\tau = 50$. The parameters values are the same as those in Fig. 3(c).

which gives ($a = u_0/B$ and $b = i8A/15B$)

$$-rac{4}{\sqrt{a}} \tanh^{-1} \sqrt{\frac{a - b\sqrt{\Phi}}{a}} = \pm \eta,$$  \hspace{1cm} \text{(A.37)}

or,

$$\sqrt{a - b\sqrt{\Phi}} = \mp \tanh \left( \frac{\sqrt{a}}{4} \eta \right).$$  \hspace{1cm} \text{(A.38)}

Thus, we obtain the soliton solution as

$$1 - \tanh^2 \left( \frac{\sqrt{a}}{4} \eta \right) = \frac{b}{a} \sqrt{\Phi},$$  \hspace{1cm} \text{(A.39)}

or,

$$\Phi = -\left( \frac{15u_0}{8A} \right)^2 \sech^4 \left( \sqrt{\frac{u_0}{16B}} \eta \right).$$  \hspace{1cm} \text{(A.40)}

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