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The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT

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Abstract

We study entanglement Renyi entropies (EREs) of $1+1$ dimensional CFTs with classical gravity duals. Using the replica trick the EREs can be related to a partition function of $n$ copies of the CFT glued together in a particular way along the intervals. In the case of two intervals this procedure defines a genus $n-1$ surface and our goal is to find smooth three dimensional gravitational solutions with this surface living at the boundary. We find two families of handlebody solutions labelled by the replica index $n$. These particular bulk solutions are distinguished by the fact that they do not spontaneously break the replica symmetries of the boundary surface. We show that the regularized classical action of these solutions is given in terms of a simple numerical prescription. If we assume that they give the dominant contribution to the gravity partition function we can relate this classical action to the EREs at leading order in $G_N$. We argue that the prescription can be formulated for non-integer $n$. Upon taking the limit $n \to 1$ the classical action reproduces the predictions of the Ryu-Takayanagi formula for the entanglement entropy.

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Entanglement entropy (EE) is a powerful observable for many-body quantum systems. This is especially so when defined with respect to the reduced density matrix associated to a
spatial subregion \( \mathcal{A} \) of the full system \([1, 2]\). EE then detects spatial quantum correlations in a fixed many-body state. One simple reason for the appeal of EE is the universal nature of its definition allowing for model independent characterizations of many-body phases. To list a few applications: EE has been used as an order parameter to distinguish trivially gapped phases from those with topological degrees of freedom \([3, 4]\), as a c-function on CFTs in two and three dimensions \([5–7]\) and as a measure of thermalization in non-equilibrium situations \([8]\).

Unfortunately EEs are rather hard to compute theoretically even for free theories. Techniques for CFTs are available \([9–11]\) and give results for fairly simple spatial regions \( \mathcal{A} \). However a more general understanding of EE in QFT is lacking.

Surprisingly there is a simple formula for computing EE in AdS/CFT given by a prescription of Ryu and Takayanagi (RT) \([12–14]\) involving the area of minimal surfaces. The formula applies to quantum field theories with dual classical Einstein gravity descriptions. Some higher derivative corrections have been attempted, see for example \([15]\), while bulk quantum corrections are unknown. The status of the formula remains as a further conjecture above and beyond the usual rules of the Maldacena conjecture \([16–18]\). In principle one should be able to derive it using just these rules, however the attempt in \([19]\) failed as was emphasized in \([20]\). In particular a derivation would forge the way to understanding bulk quantum and classical corrections to the formula.

The focus of this paper will be \( 1+1 \) CFTs where the sub-region of interest \( \mathcal{A} \) is the union of a set of intervals along the spatial axis \([10, 21]\) and we consider only the vacuum state of the CFT. The RT prediction for this case was discussed in \([20, 23]\) and involves the lengths of bulk geodesics which we summarize in Figure 1. We will attempt to prove the RT formula for this case using the replica trick. This trick involves calculating the Entanglement Renyi Entropies (ERE) as an intermediate step

\[
S_n = -\frac{1}{n-1} \ln \text{Tr}_A(\rho_A)^n
\]  

(1.1)

where \( \rho_A \) is the reduced density matrix in the vacuum of the CFT for the Hilbert space associated to the intervals \( \mathcal{A} \). The EREs are defined for integer \( n \geq 2 \) and can be calculated by the partition function of the CFT on a surface \( \mathcal{M} \) of genus \( n-1 \). Assuming one can analytically continue the partition function to non-integer \( n \) then the limit \( n \to 1 \) gives the von Neumann entropy expression for the EE.

This paper was inspired by some of the results of Headrick in \([20]\) where the ERE for two intervals and \( n = 2 \) was found for CFTs with gravitational duals. We attempt to generalize Headrick’s results by finding the gravity solutions which are needed to compute the EREs holographically for \( n > 2 \). We seek handlebody solutions whose conformal boundary is the genus \( n-1 \) surface \( \mathcal{M} \). To generate such solutions we need to represent \( \mathcal{M} \) in terms of its so called Schottky uniformization. This representation of \( \mathcal{M} \) can be roughly described as a connected domain in the complex plane with certain identifications on the boundaries of the domain. Schottky uniformization allows us to find the bulk handlebody solution

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FIG. 1: The RT prescription for computing the EE in 1+1 dimensional CFTs for 3 disjoint intervals. The CFT spatial direction is $\sigma$ and $r$ is the radial direction of the dual $AdS_3$. The minimal surfaces are simply geodesics connecting the ends of the intervals. The sum of the regularized lengths of these geodesics computes the EE. There is more than one minimal set of such geodesics and one is instructed to find the global minimum. We have shown only 2 cases out of a total of 5.

by extending the domain boundaries and identifications into the bulk radial direction in a particular way.

Actually there is an infinite set of such gravitational solutions. At finite Newton’s constant $G_N$ one expects all of these to contribute to the partition function as

$$Z_M = \sum_{\gamma} \exp(-S_{gr}^\gamma + O(G^0_N)) \quad (1.2)$$

where $S_{gr}^\gamma \propto G_N^{-1}$ is the gravitational action for the classical solution labelled by $\gamma$. However in the classical limit where $G_N \to 0$ only the least action solution will dominate and we only need to find this one. In this paper we show that one can easily construct a small finite subset of the infinite set of solutions that contribute to (1.2).

Interestingly the solutions we can construct in this way have the property that one can formulate a simple numerical problem which computes their gravitational action. The answer can then be found numerically for integer $n \geq 2$. This formulation can be continued in the replica index $n$ to non-integers. This is true despite the fact that the bulk solutions no longer make any sense. The limit $n \to 1$ can be studied exactly and the actions computed in this way reproduce the Ryu-Takayanagi formula for the EE involving the lengths of bulk geodesics.

Unfortunately since it is more difficult to construct the missing solutions in (1.2) to check that they are all subdominant we are left only with a partial result. The gravitational actions we compute via the numerical prescription can only be related to the EREs if we assume they are in fact the dominant ones. If one could show that this assumption is correct then we could compute the EREs and prove the RT formula. A simple way to characterize the missing handlebody saddles is by the fact that the bulk solution breaks some of the symmetries of the boundary manifold including for example the replica symmetry which
interchanges the different replicas.\footnote{There are also non-handlebody solutions which are usually assumed to be subdominant since they would be pathological from an AdS/CFT point of view. We come back to these as well as the replica breaking saddles in the discussion.} This would be an interesting phenomena if it were to happen, and the investigation of this possibility is left to the future.

Although we will discuss some results for multiple intervals most of the discussion will be for the case of 2 intervals. We expect our results to generalize to multiple intervals.

Our results match the calculations of a complementary paper [22] which takes the CFT perspective to this problem. CFTs with large $c$ and a small number of low dimension primary operators were considered. The arguments in [22] are based on semiclassical conformal blocks. We comment more on this paper in the discussion.

The paper is organized as follows. In Section II we introduce the replica trick which tells us to compute the partition function of a certain genus $n - 1$ Riemann surface $\mathcal{M}$ the properties of which we also discuss here. In Section III we give a numerical prescription for computing EREs in 1 + 1 CFTs with a classical gravity description. This prescription remains a conjecture since we could not rule out the possibility of other saddles being dominant. However since the end result we found is rather simple it is useful to present this before delving into the details of its derivation. We subsequently show that these saddles reproduce the RT prescription and reproduce other known results in the literature. In Section IV we discuss the essential ideas behind the program of Schottky uniformization. In the Section V we gives details of the bulk solutions that we find. In Section VI we compute the bulk action in a few ways and relate the answer to the prescription given earlier on. We end with a discussion. There are several appendices with details.

\section{II. The Replica Trick and the Riemann Surface}

We are interested in computing the ERE for a spatial region $\mathcal{A}$ - the set of $N$ intervals:

$$\mathcal{A} = [z_1, z_2] \cup [z_3, z_4] \ldots \cup [z_{2N-1}, z_{2N}]$$ (2.1)

where the $z_i$ are cyclicly ordered. The Hilbert space factors locally: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\mathcal{A}^c}$ where $\mathcal{A}^c$ is the complement region to the above intervals. The EE in the vacuum state is defined by:

$$\rho_A = \text{Tr}_{\mathcal{A}^c} |0\rangle \langle 0| \rightarrow S_{EE} = -\text{Tr}_A (\rho_A \log \rho_A)$$ (2.2)

and the ERE generalizations were given in (1.1) such that $\lim_{n \to 1} S_{n} = S_{EE}$. The replica trick allows one to formulate $\text{Tr}(\rho_A)^n$ as a partition function of the theory on a particular manifold. The arguments are standard and can be found for example in the review [23]. For each of the $n$ factors of $\rho_A$ one introduces a Euclidean path integral on the complex $z$-plane with certain boundary conditions on the $z$ real axis. The trace and sum over intermediate
states then glues together these $n$ copies of the $z$-plane along the intervals in $\mathcal{A}$ in a particular way. The result is a Euclidean path integral on an $n$-sheeted Riemann surface or branched covering defined by:

$$\mathcal{M} : \quad y^n = \prod_{i=1..N} \frac{(z - z_{2i-1})}{(z - z_{2i})}$$

(2.3)

with the entanglement region $\mathcal{A}$ lying on the real $z$ axis. The genus of this surface is $(N-1)(n-1)$. Beyond this specifying (2.3) only tells us the complex structure of the surface, however to compute the CFT partition function we also need to give a particular metric in the fixed conformal class. We take this to be the original metric that the CFT lives on:

$$ds^2 = dzd\bar{z}$$

(2.4)

On the branched covering this metric necessarily has conical excess singularities at the branch points. These can be resolved by cutting out a region $\epsilon$ from the branch points and replacing the singular metric with a smooth one. The details of this procedure are standard and given in Appendix C. The Euclidean path integral on $\mathcal{M}$ can be used to compute entanglement Renyi entropies:

$$S_n = -\frac{1}{n-1} \left( \ln Z_\mathcal{M}(ds^2) - n \ln Z_1 \right)$$

(2.5)

where $Z_1$ is the partition function of the theory on the flat $z$ plane without any branch points.

The isometries of the surface (2.3) include $\mathbb{Z}_n$ cyclic rotations of the replicas and the anti-holomorphic involution which reflects about the real $z$ axis (the symmetry associated to complex conjugation due to the fact that the $z_i$ all lie on the real axis.) Together these generate the dihedral group $D_n$ and we refer to this as the “replica symmetry”. For more discussions on the relevance of these symmetries to computations of the ERE see [24].

It is common to think of $Z_\mathcal{M}$ as the correlation function of twist operators in the product orbifold theory of $n$ copies of the CFT under consideration:

$$Z_\mathcal{M} \propto \langle \sigma_1(z_1)\sigma_{-1}(z_2)\ldots\sigma_1(z_{2N-1})\sigma_{-1}(z_{2N}) \rangle$$

(2.6)

up to some regulator factors which deal with the divergences associated to the conical singularities. The twist operator $\sigma_1$ enacts the generator of cyclic permutation of the $n$ CFTs upon circling it. And the operator $\sigma_{-1}$ acts inversely to $\sigma_1$. See for example [47] whose results are relevant for computations of EREs for general CFTs. The dimension of these twist operators is fixed by the central charge $c$ of the CFT:

$$h_n = \frac{cn}{12} \left( 1 - \frac{1}{n^2} \right)$$

(2.7)

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2 Some properties of this surface for two intervals $N = 2$ are summarized in Appendix A.
III. PRESCRIPTION

In this section we give a prescription for finding and computing certain saddles of 3 dimensional Einstein gravity that contribute to $Z_M$ by the usual rules of AdS/CFT. Many things will be introduced in an ad-hoc way leaving their justification to later. We also leave discussions of the explicit bulk solution to later sections.

The prescription reproduces several known cases as well as the Ryu-Takayanagi formula. Throughout this section we will assume that one of the saddles we construct is the dominant solution, and thus at leading order in $1/G_N$ computes the ERE. It should be kept in mind that this might not be the case. And so the prescription given here remains to be proven.

We claim that in order to compute $S_n$ holographically one should use the following recipe:

1. Consider the ordinary differential equation (ode) defined on the Euclidean $z$-plane with the points $z_i$ lying on the real axis:

$$\psi''(z) + \frac{1}{2} T_{zz} \psi(z) = 0; \quad T_{zz} = \sum_{i=1,\ldots,2N} \left( \frac{\Delta}{(z - z_i)^2} + \frac{p_i}{z - z_i} \right)$$  \hspace{1cm} (3.1)

where $\Delta = (n^2 - 1)/(2n^2)$. The $p_i$ are called *accessory* parameters.

2. Tune $p_i$ such that the solutions of (3.1) have *trivial* monodromy around a set of $N$ cycles $C_M$ in the $z$-plane with the points $z_i$ removed. We label this set by,

$$\Gamma = \{C_M : M = 1, \ldots, N\}$$  \hspace{1cm} (3.2)

The $C_M$ are defined to be simple non-intersecting (homologically) independent and non-trivial and each encircle an even number of the $z_i$. At fixed $N$ there is some number $N_N$ of independent configurations of cycles $\Gamma_\gamma$ which we label by

$$\mathcal{T}_N = \{\Gamma_\gamma : \gamma = 1, \ldots, N_N\}$$  \hspace{1cm} (3.3)

3. For a fixed configuration $\Gamma_\gamma \in \mathcal{T}_N$ the monodromy conditions determine the $p_i^\gamma$. From these construct the following “saddle” Renyi entropies $S_n^\gamma$ by integrating:

$$\frac{\partial S_n^\gamma}{\partial z_i} = -\frac{cn}{6(n-1)} p_i^\gamma$$  \hspace{1cm} (3.4)

where $c$ is the central charge.

4. The true ERE is claimed to satisfy:

$$S_n = \min_\gamma S_n^\gamma$$  \hspace{1cm} (3.5)

We give some clarifying comments:
• The solution $\psi(z)$ will later be used to construct a bulk gravitational solution.

• Prior to imposing the monodromy conditions the accessory parameters are real and unconstrained except for the three conditions:

$$\sum_i p_i = 0, \quad \sum_i p_i z_i = -2N\Delta, \quad \sum_i p_i z_i^2 = -2\Delta \sum_i z_i$$

such that the point $z = \infty$ is not a singular point of the ode. Thus the point $z = \infty$ has trivial monodromy and one can think of (3.6) as being contained within the monodromy conditions on the cycles in $\Gamma_\gamma$.

• The counting of the number of unique configurations of cycles proceeds recursively. As we add one more interval $N - 1 \to N$ we can use configurations $\mathcal{T}_{N-1}$ to construct those in $\mathcal{T}_N$. This is illustrated in Figure 2.

![Figure 2](image)

**FIG. 2**: A recursive argument for generating configurations of cycles in $\mathcal{T}_N$. The black solid lines are new curves. The other curves are represented by the shaded blob and are taken from a configuration $\Gamma \in \mathcal{T}_{N-1}$ or $\mathcal{T}_{N-2}$ as indicated. The last term subtracts off some over counting of the previous two terms. The answer is $\mathcal{N}_N = 3\mathcal{N}_{N-1} - \mathcal{N}_{N-2}$.

• The condition that each cycle encircles an even number of points $z_i$ is related to the fact that these cycles actually live on the Riemann surface $\mathcal{M}$ and we want them to come back to the same sheet.$^3$

• We will sometimes refer to a given $\gamma$ as a *saddle* since it will ultimately correspond to a particular three dimensional gravitational solution. The monodromy conditions on $C_M \in \Gamma_\gamma$ will tell us which cycles of the manifold $\mathcal{M}$ are contractable inside the bulk three dimensional handlebody solution.

• For a manifold of genus $(N - 1)(n - 1)$ we should pick $(N - 1)(n - 1)$ non-intersecting cycles (out of $2(N - 1)(n - 1)$) to be contractable in order to specify a unique handlebody. We will sometimes refer to these as “A-cycles.” So far we have specified $N - 1$

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$^3$ Note the non-crossing condition on the cycles is still appropriate despite the fact that some of the cycles actually move into the second sheet. This follows from the comment on cyclic symmetry.
of these not including one of the cycles in \( \Gamma_\gamma \) which is not independent due to the monodromy condition at infinity in the \( z \) plane \((3.6)\). As we will see the remaining cycles are related to these by demanding the bulk solution respects the replica symmetry. That is we are also implicitly picking a basis of A-cycles:

\[
\{g^m(C_M) : m = 0, \ldots n - 1; M = 1, \ldots N - 1\}
\]

\( (3.7) \)

where \( g \) enacts the cyclic replica symmetry and moves \( C_M \) to the adjacent sheet of the branched covering. Note that not all of these cycles are independent because \( \sum_{m=0}^{n-1} [g^m(C_M)] = 0 \). This gives the desired \((N - 1)(n - 1)\) counting.

- It is easy to see that the anti-holomorphic involution (symmetry under complex conjugation) is also preserved by this choice of cycles.

- Saddles we are missing include ones where the monodromy condition on the Riemann surface \( \mathcal{M} \) do not obey the replica symmetry. These cannot be constructed by the ode \((3.1)\) which must be generalized in an appropriate way.

- Note that up to some constants \( T_{zz} \) will be the expectation value of the stress tensor for the associated saddle. It is then clear that \( \Delta \) is related to the dimension of twist operators \((2.7)\). Furthermore \((3.4)\) follows from applying the conformal Ward identity to \( T_{zz} \) and comparing to the conformal transformation of the twist operator correlation function \((2.6)\) (albeit on a saddle by saddle basis.) We will derive \((3.4)\) later using the bulk action for the constructed solutions.

- The central charge is related as usual \([25]\) to the bulk Newton’s constant \( c = 3/(2G_N) \). The prescription above is for large central charge, otherwise the different bulk solutions will all contribute to \((1.2)\) including the ones we have not constructed.

- Each saddle will have a counterpart set of geodesics which we can identify with a locally minimal surface of the RT prescription. These geodesics can be constructed by noting that the configuration of cycles \( \Gamma_\gamma \) partitions the \( z_i \) into pairs:

\[
P_\gamma = \{(z_i, z_j)_K; K = 1, \ldots, N\}
\]

\( (3.8) \)

such that \((z_i, z_j) \in P_\gamma \) are either both inside or both outside every cycle \( C_M \in \Gamma_\gamma \). Joining these pairs by geodesics gives the counterpart RT saddle. The homology condition which is part of the RT prescription \([14]\) is satisfied for these geodesics. See Figure 3 for an example of this.

As a zeroth order check we consider \( N = 1 \) where we find that the conditions \((3.6)\) are sufficient to fix the \( p_i \),

\[
p_1 = -p_2 = -\frac{2\Delta}{(z_1 - z_2)}
\]

\( (3.9) \)
FIG. 3: A picture of the correspondence between boundary cycles for a fixed configuration \( \Gamma_\gamma \in \mathcal{T}_3 \) and the bulk geodesics of the RT formula (green curves hanging down from the boundary). The geodesics connect points defined by \( P_\gamma \) in (3.8). Notice that in this picture the cycles in \( \Gamma_\gamma \) are contractable in the bulk without crossing the geodesics.

There is only a single configuration and it is clear that the monodromy is trivially fixed to zero when passing around the points \( z_1 \) and \( z_2 \). Integrating this we find the standard CFT result \[ S_{n}[N=1] = \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln((z_2 - z_1)/\epsilon) + \kappa_{N=1} \] (3.10)

where \( \epsilon \) is a UV cutoff \( \kappa_{N=1} \) is un fixed, but scheme dependent.

In general it is a difficult problem to carry out the above steps for \( N > 1 \). Firstly there is no analytic way to solve for \( p_\gamma^i \) given a monodromy condition \( \gamma \), so one needs to proceed numerically. Secondly one needs to integrate (3.4) to find \( S_\gamma^n \). We assume that the partial derivatives commute so we only have a single integration constant for each saddle. Clearly the issue here is that the relative integration constants for the saddle entropies are not fixed by the above prescription. We will fix these constants by taking limits where the saddle entropies \( S_\gamma^n \) are related to the results for one less interval. This allows us to fix the integration constant recursively.

Note also that we can give an absolute formula for computing \( S_\gamma^n \) which will fix this integration constant, see (6.40) for the two interval case. The formula is written in terms of the solution \( \psi \) to the ode problem but is more complicated than the prescription given above so we leave that till a later section. The prescription we have given is sufficient for our current purposes.

A. Reproducing the Ryu-Takayanagi prediction

We wish to compute the monodromy matrices in the replica limit \( \delta_n = n - 1 \to 0 \). We can do this using perturbation theory. Assume that \( p_\gamma^i \) vanishes linearly in the replica limit:

\[ p_\gamma^i \sim \rho_i \delta_n + \mathcal{O}(\delta_n^2) \]

where \( \rho_i \) are constants which we need to determine. The fact that \( p_\gamma^i \) should vanish in the replica limit follows from the conjectured formula for the entanglement Renyi entropy (3.4) which we expect to be finite in this limit. Also note that \( \Delta = \delta_n + \mathcal{O}(\delta_n^2) \) in this limit. The second order ode (3.1) can be conveniently represented as a first order
\[
\frac{d}{dz} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}T_{zz} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \quad \rightarrow \quad \frac{d}{dz} u(z) = H(z)u(z)
\]
(3.11)

The monodromies are then simply path ordered exponentials:

\[
u(z) = M(C)u(z_0) \quad M(C) = \mathcal{P} \exp \left( \int_C dz H(z) \right)
\]
(3.12)

where \(C\) is a specific path from \(z_0\) to \(z\). Note that \(M(C)\) has unit determinant which follows from the Wronskian condition of two solutions to the ode. Perturbatively we have

\[
H = H_0 + \delta_n H_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{\delta_n}{2} T_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_1 \equiv \sum_i \left( \frac{1}{(z - z_i)^2} + \frac{\rho_i}{(z - z_i)} \right)
\]
(3.13)

So we can then use time dependent perturbation theory methods to solve this problem where \(z\) is thought of as “time”. Firstly move to the interaction picture:

\[
M(C) = M_0(z) M_I(z) \quad \rightarrow \quad \frac{d}{dz} M_I(z) = \delta_n (M_0^{-1}H_1M_0)(z) M_I(z)
\]
(3.14)

where the zeroth order solution to the ode \(\psi_0 = Az + B\) can be used to find the zeroth order monodromy matrix:

\[
M_0(z) = \begin{pmatrix} 1 & (z - z_0) \\ 0 & 1 \end{pmatrix}
\]
(3.15)

The path ordered exponential expression for \(M_I\) can then be computed to first order by simply expanding the exponential:

\[
M_I \approx 1 + \delta_n \int_C dz M_0^{-1}H_1M_0 = 1 + \frac{\delta_n}{2} \int_C dz \begin{pmatrix} (z - z_0) & (z - z_0)^2 \\ 1 & - (z - z_0) \end{pmatrix} T_1
\]
(3.16)

If we close the cycle \(C\) by sending \(z \rightarrow z_0\) we find the monodromy condition requires the vanishing of resulting contour integral in Eq. 3.16. For a cycle \(C = C_M \in \Gamma_\gamma\) we get three independent conditions:

\[
\sum_{z_i \in D_M^n} \rho_i = 0, \quad \sum_{z_i \in D_M^n} (\rho_i z_i + 1) = 0, \quad \sum_{z_i \in D_M^n} (\rho_i z_i^2 + 2z_i) = 0,
\]
(3.17)

where the sum is over points \(z_i\) contained in the interior of \(C_M\) which we have denoted by the domain \(D_M\). Note that it does not matter which “interior” we choose - because of the monodromy condition at \(\infty\) given in (3.6). After some thought it becomes clear that this set of \(N\) equations is solved by the following conditions on the pairs \((z_i, z_j) \in P^\gamma\) into which the cycles \(C_M\) partitioned the \(z_i\).

\[
\rho_i = -\frac{2}{z_i - z_j}, \quad \rho_j = -\frac{2}{z_j - z_i} \quad \forall \ (z_i, z_j) \in P^\gamma
\]
(3.18)
The saddle entanglement entropy can then be found by taking \( \lim_{n \to 1} \) in (3.4) and integrating the result:

\[
S_{\gamma}^{\gamma}_{EE} = \frac{c}{3} \sum_{(z_i, z_j) \in P_{\gamma}} \ln(|z_i - z_j|/\epsilon) + \kappa_{\gamma}^N
\]

(3.19)

This result is exactly \( c/6 \) times the regulated lengths of geodesics in \( AdS_3 \) connecting the points \((z_i, z_j) \in P_{\gamma}\) on the boundary. As we discussed around Figure 3 there is a correspondence between the saddles that we construct (at any integer \( n \geq 2 \)) and the minimal surfaces (geodesics) needed to compute the RT answer. We have shown here that the action of these saddles can be continued to non-integer \( n \) and in the limit \( n \to 1 \) they become the lengths of the corresponding RT geodesics. While other aspects of this section are somewhat conjectural, the last statement is correct and hints at the inner workings of the RT formula.

To completely reproduce the RT prescription we are left to compute the integration constants \( \kappa_{\gamma} \) relative to all the different saddles \( \Gamma_{\gamma} \in T_N \). We give the following argument. Firstly consider an adjacent pair \((z_k, z_{k+1}) \in P_{\gamma}\) which is enclosed by a unique single cycle \( C_L \in \Gamma_{\gamma} \) which does not enclose any other \( z_i \). Note that there must be at least one such pair. Now take the limit \( z_k \to z_{k+1} \) where we expect to reproduce the entanglement entropy for \( N-1 \) intervals and a configuration of cycles given by \((\Gamma_{\gamma} = \Gamma_{\gamma} \setminus C_L) \in T_{N-1}\). At least up to a UV divergence associated with the closing of the interval \([z_k, z_{k+1}]\). For a very small interval \( z_k \approx z_{k+1} \) we can zoom in on this and ignore all the other intervals - allowing us to exactly subtract off the EE associated with this single interval. Note that it might be that \([z_k, z_{k+1}]\) is not an interval in \( \mathcal{A} \) but is an interval in the complement \( \mathcal{A}^c \). In which case we can appeal to approximate purity of the state at small distances so this still contributes the same divergence. Further we require that we are in a regime where \( S_{\gamma}^{\gamma}_{EE} \) is the dominant saddle - this should be possible to arrange for by moving around the other \( z_i \). We find that

\[
\lim_{z_k \to z_{k+1}} \left( S_{\gamma}^{\gamma}_{EE} - \frac{c}{3} \ln((z_{k+1} - z_k)/\epsilon) - \kappa_0 \right) = S_{\gamma}^{\gamma}_{EE}
\]

(3.20)

Note we are assuming that the regulator we use is such that it treats the UV divergences located at the different points \( z_i \) in a uniform way. This way we get exactly \( S_{\gamma}^{\gamma}_{EE} \) on the right hand side of (3.20) and no other ambiguous constants. Assuming that \( \kappa_{\gamma'} = (N - 1)\kappa_1 \) is fixed for all configurations in \( T_{N-1} \) then by induction we find \( \kappa_{\gamma}^N = N\kappa_1 \) which also must hold for all saddle configurations in \( T_N \). The final answer: \( \min_{\gamma} S_{\gamma}^{\gamma}_{EE} \) is then the RT formula for disjoint intervals in a 2d CFT.

**B. Two intervals and the mutual information**

We now specialize to the case of two intervals \( N = 2 \). We think that most of the following results work for \( N > 2 \) but the arguments become cumbersome and we content ourselves to looking in more detail at the first non-trivial case. According to our prescription we have two different configurations of cycles which we label \( \gamma = \alpha, \beta \). These cycles are shown in
Figure 4. For example they correspond to the following partitioning of the $z_i$ into pairs:

$$P_\alpha = \{(z_1, z_2), (z_3, z_4)\}, \quad P_\beta = \{(z_1, z_4), (z_2, z_3)\} \quad (3.21)$$

For ease of notation we will often drop the $\gamma = \alpha, \beta$ subscript when the distinction is not important.

To start we would like to understand more about the dependence of the prescription on the $z_i$. The conformal transformations that leave the $z$ plane vacuum invariant and also leave the $z_i$ on the real axis are given by $SL(2, \mathbb{R})$ transformations:

$$z \rightarrow \frac{Az + B}{Cz + D}, \quad z_i \rightarrow \frac{Az_i + B}{Cz_i + D}, \quad AD - BC = 1 \quad (3.22)$$

where $A, B, C, D$ are all real. These move around the points $z_i$ and do not change the ordering up to cyclic permutations. We would now like to track the transformation property of $p_i$ in (3.1) under $SL(2, \mathbb{R})$. Firstly the conditions (3.6) which the $p_i$ satisfy are left invariant if we transform:

$$p_i \rightarrow (Cz_i + D)^2 \left( p_i + 2\Delta \frac{C}{Cz_i + D} \right) \quad (3.23)$$

This property makes it clear that $p_i$ transforms almost like a differential $\partial z_i$. More precisely according to (3.4) $p_i$ is conjugate to $z_i$ and so in order to reproduce the transformation given in (3.23) we must demand that the entanglement entropy is not $SL(2, \mathbb{R})$ invariant but rather:

$$S_n \rightarrow S_n - \frac{c}{6} \left( 1 + \frac{1}{n} \right) \sum_{i=1}^{4} \ln(Cz_i + D) \quad (3.24)$$

These statements follow trivially from the fact that ERE can be represented as the log of a four point function of twist operators (2.6). Then the $SL(2, \mathbb{R})$ transformations above are simply due to the conformal weights of the twist operators.
In order to soak up this transformation we can define what is known as the Mutual Renyi Information (MRI). The MRI is the following combination of entanglement Renyi entropies:

\[ I_n \equiv I_n([z_1, z_2], [z_3, z_4]) = S_n([z_1, z_2]) + S_n([z_3, z_4]) - S_n([z_1, z_2] \cup [z_3, z_4]) \]  

such that:

\[ I_n = -S_n + \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln \left( \frac{(z_2 - z_1)(z_4 - z_3)}{\epsilon^2} \right) + 2\kappa_1 \]  

It follows that \( I_n \) is \( SL(2, \mathbb{R}) \) invariant and as such can only depend on the cross ratio:

\[ x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_2)(z_3 - z_1)} \]  

From this we can define the following \( SL(2, \mathbb{R}) \) invariant accessory parameter:

\[ \frac{dI_n(x)}{dx} = -\frac{cn}{6(n-1)}p_x \]  

There is one final property we have not exploited. Since we are working in vacuum (to define our density matrix \( \rho_A \)) it follows that the Renyi entropies satisfy \( S_A = S_{A^c} \) where \( A^c \) is the complement of the region \( A \). This implies the following:

\[ S_n([z_1, z_2] \cup [z_3, z_4]) = S_n([z_4, z_1] \cup [z_2, z_3]) \]  

This purity relation corresponds to switching \( z_4 \leftrightarrow z_2 \) and as such can be thought of as a very simple crossing relation for the twist operators. When we plug this into the mutual information we find:

\[ I_n(x) = I_n(1-x) + \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln \left( \frac{x}{1-x} \right) \]  

Actually we can go a little further. If we track what happens to the configurations of cycles in \( T_2 \) as we send \( z_4 \leftrightarrow z_2 \) it is clear that \( \Gamma_\alpha \leftrightarrow \Gamma_\beta \). So in terms of the saddle ERE we can define the saddle MRI which satisfies:

\[ I_\beta_n(x) = I_\alpha_n(1-x) + \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln \left( \frac{x}{1-x} \right) \]  

and similarly for \( \alpha \leftrightarrow \beta \).

In all we can now refine our prescription a little more. We can make a conformal transformation to move the points to \( z_1 = 0, z_2 = x, z_3 = 1, z_4 = \infty \). After which our ode looks like:

\[ \psi''(z) + \frac{1}{2} \left( \frac{\Delta}{z^2} + \frac{\Delta}{(z-x)^2} + \frac{\Delta}{(z-1)^2} - \frac{2\Delta}{z(z-x)} - \frac{p_x(x-1)}{z(z-1)(z-x)} \right) \psi(z) = 0 \]  

and the prescription to compute the mutual information is simply to integrate (3.28). We should then find the maximum of the two possible saddle mutual informations (note the sign switch in the definition (3.26) of \( I_n \)):

\[ I_n(x) = \max \{ I_\alpha_n(x), I_\beta_n(x) \} \]
where we remind the reader that it is possible there are some missing saddles which become dominant at some $x$ and thus override this answer.

It is clear that in the limit where $x \to 0$ the dominant configuration is $\Gamma_\alpha$ and in this limit the mutual information vanishes since this limit corresponds to moving the two intervals infinitely far apart. This condition will be used to fix the integration constant in (3.28).

Since when $x = 1/2$ the two different saddle mutual informations agree (by purity of the vacuum state) it must be the case that the $\Gamma_\alpha$ and $\Gamma_\beta$ saddles switch dominances at $x = 1/2$ [20]. This results in a first order phase transition for any $n$.

If we are feeling lazy we can reconstruct the calculated contribution to the mutual information from a single saddle:

$$I_n(x) = \begin{cases} I_\alpha^n(x), & 0 < x < 1/2 \\ I_\alpha^n(1 - x) + \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln \left( \frac{x}{1 - x} \right), & 1/2 < x < 1 \end{cases}$$ (3.34)

C. Reproducing the known answer for $n = 2$

Set $n = 2, \Delta = 3/8$ and it turns out in this limit we can analytically solve the ode. The reason lies in the fact that we are in this case secretly describing a genus one torus. The case for $n = 2$ was already worked out in [20] based on fairly extensive computations given in [47]. We will see that our prescription reproduces their results with relative ease.

The two independent solutions can be written as:

$$\psi(z) = \frac{1}{(t'(z))^{1/2}} \exp(\pm ht(z)), \quad t'(z) = \frac{1}{\sqrt{z(z - 1)(z - x)}}$$ (3.35)

where $h$ is an unfixed constant which is related to the accessory parameter:

$$p_x = \frac{2 - x}{4x(x - 1)} + \frac{h^2}{2x(x - 1)}$$ (3.36)

These solutions can then be used to find the monodromy matrix:

$$M(C) = \Psi(z) \begin{pmatrix} e^{h \int_C t'(z)dz} & 0 \\ 0 & e^{-h \int_C t'(z)dz} \end{pmatrix} \Psi(z_0)^{-1}$$ (3.37)

where $C$ is a path from $z_0$ to $z$ and the non-path dependent factors are:

$$\Psi(z) = \frac{1}{(t')^{1/2}} \begin{pmatrix} -1 & -1 \\ \frac{1}{2} t'' - ht' & \frac{1}{2} t'' + ht' \end{pmatrix}$$ (3.38)

The trivial monodromy condition for the curve $C \in \Gamma_\alpha$ is then simply:

$$2\pi ik = h \int_C t'(z)dz = 2h \int_0^x \frac{dz}{\sqrt{z(z - 1)(z - x)}} = 4hK(x)$$ (3.39)
where \( k \) is an integer and \( K(x) \) is the complete elliptic integral (defined by the integral above.) The integer \( k \) is unfixed so far. The case \( k = 0 \) does not work since one has to be carefully when taking \( h \to 0 \) due to the degeneration of the matrix \( \Psi(z) \) given in \((3.38)\) in this limit. For \(|k| \geq 2\) the solutions one finds involve a multiply wound uniformization coordinates (see Section \([IV]\)) and should not be included in our prescription since they will not correspond to sensible bulk solutions. We are left with \( k = \pm 1 \) of which either gives the same answer. The accessory parameter is:

\[
p_x^\alpha = \frac{2 - x}{4x(1 - x)} - \frac{\pi^2}{8x(1 - x)K(x)^2}
\]

which integrates to:

\[
I_2^\alpha = -\frac{c}{12} \log \left(2^8(1 - x)/x^2\right) + \frac{c\pi}{6}\tau_2
\]

where we have defined the (purely imaginary) modular parameter for the underlying torus:

\[
\tau_2 = \frac{K(1 - x)}{K(x)}
\]

and we have added an integration constant such that \( I_2^\alpha(x = 0) = 0 \). Similarly we can find \( I_2^\beta \) by imposing the different monodromy condition on the cycles \( \Gamma_\beta \). The answer one finds satisfies the expected purity relation \((3.31)\) where under \( x \to 1 - x \) the modular parameter of the torus undergoes the modular \( S \) transformation \( \tau_2 \to 1/\tau_2 \). This makes sense because \( S \) switches the cycles on the torus and thus the two monodromy conditions we are working with.

\[\text{D. Numerics for } n > 2\]

In order to compute the monodromy matrices numerically it is convenient to define connection matrices along the real line between the singular points. These matrices relate canonically chosen linearly independent solutions at adjacent singular point. We relegate the details to Appendix \([B]\). The monodromy condition can easily be read off from these connection matrices and from this we can compute \( p_x \).

The results are shown in Figure \([5]\). Actually there is very little difference between the Mutual Renyi Information for different values of \( n \) and in order to effectively compare them we subtract off a scaled version of the EE \((n = 1)\) which takes into account the scaling of the twist operators with \( n \):

\[
J_n(x) = I_n(x) - \frac{1}{2} \left(1 + \frac{1}{n}\right)I_1(x)
\]

Recall that the RT formula for the MI is \( I_1(x) = \max\{0, (c/3) \ln \frac{x}{1-x}\} \). The function \( J_n(x) \) has the property that it is symmetric about \( x = 1/2 \).
FIG. 5: Calculated contributions to the Mutual Renyi Information (MRI) in holographic CFTs. We show in the left panel a subtracted version of the MRI as defined in (3.43). Both saddles $\Gamma_\alpha$ and $\Gamma_\beta$ are important and dominate for $x < 1/2$ and $x > 1/2$ respectively. In the right we show the dependence of the MRI on $n$ for fixed $x$ (after analytically continuing from integer $n$). The $n = 1$ limit for $x \leq 1/2$ is zero as predicted by the RT formula. Numerically it was more convenient to use (6.40) to find this right plot.

IV. SCHOTTKY UNIFORMIZATION

Having introduced the ode (3.1) as the main crux for constructing certain bulk solutions we should now explain where this came from, and in particular give some pictures of what the bulk solution looks like. Since all solutions of Einsteins equations with a negative cosmological constant in 3 dimensions are globally quotients of $AdS_3$ we simply need to determine the appropriate quotient. The technology we need in order to do this goes under the name of Schottky uniformization. We give here a rough general discussion of this technology. We follow closely the discussion in [27] and [28] see also [26].

Pick the following coordinates on $AdS_3$:

$$ds^2 = \frac{d\xi^2 + dv dw}{\xi^2} \quad (4.1)$$

with conformal boundary at $\xi \to 0$ the complex $w$ plane. The isometry group of $AdS_3$ is $PSL(2, \mathbb{C})$ where the action induces a conformal isometry on the boundary. The action on $AdS_3$ is:

$$w \to \frac{(aw + b)(cw + d) + ac\xi^2}{|cw + d|^2 + |c|^2\xi^2} \quad \xi \to \frac{\xi}{|cw + d|^2 + |c|^2\xi^2} \quad (4.2)$$

where $ad - bc = 1$. As $\xi \to 0$ this action becomes:

$$w \to \frac{aw + b}{cw + d} \equiv L(w) \quad \xi \to \xi|L'(w)| \quad (4.3)$$
In this way the quotient of $AdS_3$ by a discrete subgroup $\Sigma \subset PSL(2, \mathbb{C})$ descends to a quotient of the complex $w$ plane.\footnote{We are being heuristic here - for example we should first remove a certain set of measure zero from the $w$ plane, for which $\Gamma$ acts badly (fixed points of $\Gamma$): $C' = \mathbb{C}/\{\text{bad points}\}$. We can then form the quotient $\mathbb{C}'/\Sigma$. For a proper discussion see. We will continue to be heuristic without making similar admissions.} This quotient is then a way of representing the surface $\mathcal{M} = \mathbb{C}'/\Sigma$. Thus one of the steps we will need to understand is how to map the $w$ complex plane into $\mathcal{M}$:

$$\pi_S : \mathbb{C} \rightarrow \mathcal{M}$$

consistent with the action of the quotient. In fact the ode \footnote{Note that this also implies that the solutions of the ode transform as $-1/2$ differentials:}

\begin{equation}
\left\{ \pi^{-1}_S, z \right\} = \frac{w''}{w'} - \frac{3}{2} \left( \frac{w''}{w'} \right)^2 \equiv T_{zz}(z)
\end{equation}

This equation can be thought of as a differential equation for $\pi^{-1}_S(z) = w(z)$ where $T_{zz}$ is taken as a fixed input. We will construct $T_{zz}$ independently in a moment. Solving equation \footnote{Note that this also implies that the solutions of the ode transform as $-1/2$ differentials:}

\begin{equation}
w(z) = \frac{\psi_1(z)}{\psi_2(z)}, \quad \psi'' + \frac{1}{2}T_{zz} \psi = 0
\end{equation}

where $\psi = \psi_{1,2}$ are two linearly independent solutions of this ode. At this point we have made a connection with the prescription of Section $\text{III}$. However we still need to give an argument that the stress tensor $T_{zz}$ takes the form quoted in (3.1). Since the map $\pi^{-1}_S$ is globally defined (but multivalued) and since the the Schwarzian derivative does not change under the $PSL(2, \mathbb{C})$ action on $w$ the stress tensor is globally defined on $\mathcal{M}$ and not multi-valued. However $T_{zz}$ does not transform homogeneously under general conformal transformations $z \rightarrow z(\tilde{z})$ since the Schwarzian derivative shifts under such coordinate changes: $\{t, \tilde{z}\} = z'(\tilde{z})^2\{t, z\} + \{z, \tilde{z}\}$. The rule on the overlapping patches is: \footnote{Note that this also implies that the solutions of the ode transform as $-1/2$ differentials:}

\begin{equation}
\tilde{T}_{\tilde{z}\tilde{z}} = \left( \frac{\partial z}{\partial \tilde{z}} \right)^2 T_{zz} + \{z, \tilde{z}\}
\end{equation}
Given these properties we claim that the following expression (on the \( z \) coordinate patch) is smooth on \( \mathcal{M} \) and completely general:

\[
T_{zz} = \Delta \left( \sum_{i=1}^{4} \frac{1}{(z - z_i)^2} + \frac{2(-z_3 + z_1 + z_2 + z_4 - 2z)}{(z - z_1)(z - z_2)(z - z_4)} \right) + \sum_{s=1}^{3(n-2)} \hat{p}_s \omega_{zz}^s \quad (4.9)
\]

where \( \Delta = 1/(1 - 1/n^2) \). To argue for this form consider to begin with the the last sum over \( \omega_{zz}^s \) which are holomorphic quadratic differentials on \( \mathcal{M} \). There are \( 3(n-2) \) of these and they are enumerated in Appendix A. Given a fixed \( T_{zz} \) we can always add a linear combination of quadratic differential since they transform homogeneously under conformal transformations thus preserving (4.8).

Finally we have to check that the remaining term multiplying \( \Delta \) in (4.9) is smooth on \( \mathcal{M} \). This term is not a quadratic differential which are smooth by definition. We only need to check the behavior near \( z \to z_i \) and \( z \to \infty \). Close to for example \( z \to z_1 \) we can use the coordinate \( y \) defined by the branched covering \( (A1) \) \( y^n \sim (z - z_1) \) such that the Schwarzian derivative is:

\[
\{z, y\} \approx -\frac{1}{2} \frac{n^2 - 1}{y^2} + O(y^{n-2}) \quad (4.10)
\]

and thus the new stress tensor in the \( y \) coordinate patch is:

\[
\tilde{T}_{yy}(y) = \frac{1}{y^2} \left( \Delta n^2 - \frac{1}{2} (n^2 - 1) \right) + O(y^{n-2}) \quad (4.11)
\]

This is smooth provided \( \Delta = 1/2(1 - 1/n^2) \) and \( n \geq 2 \). The last term in the brackets of (4.9) is then required for smoothness as \( z \to \infty \).

The logic of the preceding discussions is that we have replaced the problem of finding the map \( \pi^{-1} \) with the problem of finding the accessory parameters \( \hat{p}_s \). As was discussed extensively in Section III these should be determined by the monodromy conditions imposed on solutions to the ode. For example if we traverse a closed path \( \mathcal{C} \) on \( \mathcal{M} \) the map \( w = \pi^{-1}_s \) defined by (4.6) is not single valued since the solutions \( (\psi_1, \psi_2) \) undergo a monodromy \( M(\mathcal{C}) \):

\[
(\psi_1, \psi_2) \to (\psi_1, \psi_2) M(\mathcal{C}) \quad \Rightarrow \quad \omega \to \frac{a \omega + b}{c \omega + d} ; \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv M(\mathcal{C}) \quad (4.12)
\]

Thus the monodromies of the ode determine a \( \text{PSL}(2, \mathbb{C}) \) action on \( w \). And in this way they determine the discrete quotient group \( \Sigma \). Note that the monodromies form a representation of the fundamental group of the surface \( \mathcal{M} \) and so does the quotient group \( \Sigma \).

We need to understand which groups \( \Sigma \) produce the desired handlebody solutions when acting on \( \text{AdS}_3 \). These groups are called Schottky groups and we simply quote some results. They have the property that \( \Sigma \) is freely generated by half of the \( 2(n - 1) \) generators in the fundamental group. We define these generators through their \( \text{PSL}(2, \mathbb{C}) \) representative:

\[
\{L_m : m = 1 \ldots n - 1\}. \quad \text{Upon traversing around the other half of the generators of the}
\]
fundamental group one finds trivial monodromy and trivial action in the quotient group. These generators correspond to a basis of non-intersection cycles in the homology of $\mathcal{M}$, a basis of “A cycles”. It turns out that these are the cycles which are contractable in the bulk of the corresponding $AdS_3$ handlebody. The “B cycles” then correspond to the generators which have nontrivial action $L_m$ and are not contractable.

We can get a rough picture of the quotient looking more carefully at the fundamental domain of $C'/\Sigma$ and $AdS'_3/\Sigma$. See Fig. 6 for a picture. The fundamental domain is given by specifying $(n-1)$ pairs of non-intersecting circles in the $w$ plane: \{$C_m, \tilde{C}_m : m = 1 \ldots n-1$\} and then identifying the circles $C_m$ and $\tilde{C}_m$ via the non-trivial generators $L_m(C_m) = \tilde{C}_m$. Note that the fundamental domain is not unique. The generators $L_m$ map the outside of the circle $C_m$ into the inside of the circle $\tilde{C}_m$. So for example we can shrink $C_m$ while making $\tilde{C}_m$ larger and still have a fundamental domain for the quotient.

The fundamental domain of the quotient of $AdS_3$ is simply found by extending the circles $C_m, \tilde{C}_m$ living on the boundary to hemispheres in the bulk of (4.1). These are two dimensional minimal surfaces in $AdS_3$ ending on the circles. Note that on $\mathcal{M}$ the cycles which encircle $C_m, \tilde{C}_m$ are contractable within the three dimensional bulk solution. These correspond to the cycles of $\mathcal{M}$ with trivial monodromies (the A-cycles.) The B-cycles are paths in the fundamental domain which connect the identified circles.

Finally we come to the accessory parameters. In (4.9) we have $3(n-2)$ of these, however we claimed in Section III that there was only a single independent accessory parameter for two intervals $p_\pi$. Note that of all $3(n-2)$ quadratic differentials enumerated in Appendix A
only one of them $\omega^1_{zz}$ does not change under the actions of the replica symmetry. This means that in order to preserve this symmetry $\hat{p}_s = 0$ for $s \neq 1$. If we include the anti-holomorphic involution (complex conjugation on the $z$-plane) in the replica symmetry we find that the remaining accessory parameter $\hat{p}_1 \equiv p_x$ should be real. Fixing most of the accessory parameters to zero is only possible if the monodromy conditions respect the replica symmetry otherwise these should be turned on and the resulting bulk solution will also not be symmetric.

V. BULK SOLUTION

We turn now to a detailed description of the bulk solution, from which the final goal is to compute the bulk action which we get to in the next section. We start with the details of the quotient $\mathcal{C}'/\Sigma$.

Assume the Schottky monodromy problem has been solved. As discussed in the previous sections for two intervals and for bulk solutions which are replica symmetric there are two different monodromy conditions that we can impose that we labelled $\Gamma_\alpha, \Gamma_\beta \in T_2$ (see Figure 4). For arguments sake pick $\Gamma_\beta$ which involves imposing trivial monodromy around a cycle $C_\beta$ which encircles the pair $(z_2, z_3)$. The other case can be worked out in an analogous manner. We would like to work out the identification circles $C_m, \tilde{C}_m$ for the Schottky fundamental domain as well as the generators $L_m$ linking them. The Schottky group $\Sigma$ is only defined up to common conjugation by $PSL(2, \mathbb{C})$ and thus we can choose two independent solutions of the ode at will in order to produce the map $w(z)$. We pick the solutions,

$$\psi_{\pm} = (z - z_1)^{1/2 \pm 1/2/n}(1 + \mathcal{O}(z - z_1))$$

which diagonalize the monodromy around the point $z_1$. Then define:

$$w = \lambda \frac{\psi_+}{\psi_-}$$

such that $w(z_1) = 0$. Note that under this choice the $\mathbb{Z}_n$ replica symmetry is generated by rotations of the $w$ plane by an angle $2\pi/n$. A nice way to get a concrete picture of the map generated by (5.2) is to consider the images under $w(z)$ of the 4 real axis segments in the $z$-plane between the points $z_i$. Segments slightly above and slightly below the real $z$-axis $\pm i\eta$ should both be considered since these will map to different curves in the $w$ plane. We should also consider the segments on all the $n$ replicas. The images of these segments will then trace out a particular fundamental domain in $w$. The identifications $L_m$ can be worked out by appropriately glueing the real line segments together amongst the different replicas.

See Figure 7 for the resulting picture. This can be confirmed numerically by plotting (5.2) after one has imposed the monodromy condition. We will not give the full detailed argument that leads to this picture. However we summarize some of the more important aspects:
FIG. 7: A picture of the map from the branched covering $z$ (left) to the Schottky domain $w$ (right) for $n = 3$. We only show the image of the first replica. The solid line tracks the real line segments of the $z$ plane which are slightly above the real axis and the dashed line tracks the segments just below the real axis. The blue fuzzy lines represent the branch cuts. The point $z_4$ maps to $w(z_4) = \infty$. Note that we have imposed the monodromy condition on the cycles in $\Gamma_\beta$ (Figure 4) so that the map $w(z)$ jumps discontinuously across the segment $[z_2, z_3]$ via the generator $L_1$.

- Since the ode (3.1) is real along the real $z$-axis all the segment images must be circular arcs or straight lines. That is there is always a basis of solutions to the ode $\psi_I, \psi_{II}$ which is real along a given segment. Then:

$$w_{\text{seg}} = \frac{a\ell + b}{c\ell + d}, \quad \ell = \frac{\psi_I(z)}{\psi_{II}(z)} \in \mathbb{R}$$

for some $a, b, c, d \in \mathbb{C}$. This describes a circle or line in the complex $w$-plane with affine parameter $\ell$.

- Most of the real axis segments map to straight lines along rays emanating from the origin in the $w$-plane $w = \ell \exp(i2m\pi/n)$ and $w = \ell \exp(i(2m + 1)\pi/n)$ where $m = 1, \ldots n$. In particular $w(z_4) = \infty$. This behavior for (5.2) only follows once the monodromy condition is imposed.

- Images of the segments $z \in [z_2, z_3] \pm i\eta$ on all the different replicas map to circular arcs which meet the above straight lines at an angle of $\pi/n$. These arcs can be described as:

$$w(\sigma + i\eta) = e^{i2\pi \frac{(m-1)}{n}} \left( \frac{x_S - \ell e^{-i\frac{\pi}{n}}}{1 - \ell e^{-i\frac{\pi}{n}}} \right), \quad 0 < \ell < \infty, \quad z_2 < \sigma < z_3$$

$$w(\sigma - i\eta) = e^{i2\pi \frac{m}{n}} \left( \frac{x_S - \ell e^{i\frac{\pi}{n}}}{1 - \ell e^{i\frac{\pi}{n}}} \right), \quad 0 < \ell < \infty, \quad z_2 < \sigma < z_3$$

where $m$ labels the images generated from the different replicas $m = 1, \ldots n$. 

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We have chosen the magnitude of $\lambda$ in (5.2) so that $|w(z_3)| = 1$. Then $|w(z_2)| = x_S$ is the single remaining parameter which can be computed numerical in terms of the cross ratio $x$. It satisfies $x_S < 1$.

Note that (5.2) jumps discontinuously across $[z_2, z_3]$ on the $z$-plane since the arcs (5.4) are different from (5.5). This jump is encoding a non-trivial monodromy element and occurs on one of the B-cycles. Note in particular this jump does not occur across a branch cut on the $z$-plane.

We can compute the associated generator in $\Sigma$ by finding the $\text{PSL}(2, \mathbb{C})$ transformation which identifies an arc in (5.4) with the corresponding arc in (5.5) (with the same $m$.) By symmetry the two affine parameters $\ell$ map onto each other. One finds:

$$L_m = \frac{1}{1 - x_S} \begin{pmatrix} x_S - e^{i\frac{2\pi}{n}} & 2ix_S e^{i\frac{(2m-1)}{n}} \sin(\pi/n) \\ -2ie^{-i\frac{2\pi}{n}} \sin(\pi/n) & x_S - e^{-i\frac{2\pi}{n}} \end{pmatrix}$$

Gluing the replicas together we get the global picture on the left side of Figure 8. Note that the circular arcs are the boundaries of the fundamental domain and they are identified pairwise as in the Figure.

Note that the arcs start to cross unless we demand $\cos^2(\pi/n) < x_S$. This actually corresponds to the boundary of moduli space $x_S = 0$. See Figure 10 for numerically calculated plots of $x_S(x)$. The generators (5.6) satisfy

$$\text{Tr} L_m = 2 \frac{(x_S - \cos(2\pi/n))}{1 - x_S}$$

and the condition that the arcs do not cross also corresponds to the requirement that the elements are loxodromix ($|\text{Tr} L| > 2$.) If we parameterize $L_m$ as:

$$L_m(w) - a_m = q \left( \frac{w - a_m}{w - r_m} \right)$$

then

$$q = \frac{x_S - \cos(2\pi/n) - \sin(2\pi/n)\sqrt{(x_S/\cos^2(\pi/n)) - 1}}{x_S - \cos(2\pi/n) + \sin(2\pi/n)\sqrt{(x_S/\cos^2(\pi/n)) - 1}}$$

Note the parameter $q$ does not depend on $m$. The attractive and repulsive fixed points $a_m, r_m$ are:

$$|a_m| = |r_m| = \sqrt{x_S}, \quad \text{arg}(a_m, r_m) = \pi \frac{(2m-1)}{n} \pm \tan^{-1} \left( \sqrt{(x_S/\cos^2(\pi/n)) - 1} \right)$$

The two fixed points come together at the boundary of moduli space when $\cos^2(\pi/n) = x_S$. The generators are not all independent: $L_n \ldots L_2 L_1 = (-1)^n$ where $-1$ acts trivially as a fractional linear transformation. This leaves $n - 1$ generators for the group $\Sigma$. Which is
FIG. 8: Pictures of the fundamental domain of the Schottky quotient. We have drawn the case $n = 3$ for some fixed $x_S(x)$. The left plot shows a non-standard domain $D_s$ which is however clearly $\mathbb{Z}_n$ symmetric. There are three (generally $n$) generators $L_1, L_2, L_3$ for this case which are shown as green arrows. These generators are not all independent. In the right picture we have deformed some of the circles in the left picture to form a more standard fundamental domain $D_d$ where there are now only two generators $L_1, L_2$. Note that the images of the points $w(z_i)$ only appear once within this fundamental domain. In the right figure we label the blue circles $O_1, \tilde{O}_1$ and the red ones $O_2, \tilde{O}_2$.

FIG. 9: The symmetric fundamental domain $D_s$ for $n = 6$. 
FIG. 10: The single parameter $x_S$ which goes into the Schottky group for the surface $\mathcal{M}$ with monodromy conditions $\Gamma_\beta$. For $x \to 0$ (where this saddle is subdominant to $\Gamma_\alpha$) the limit is $x_S = \cos^2(\pi/n)$ - although seeing this numerically requires high precision.

the number that we expected. The fundamental domain is pictured in the left of Figure 8 for $n = 3$ and Figure 9 for $n = 6$ and uses the real $z$-axis segments that we found above. This domain also does not conform to the standards of the usual fundamental domain of a Schottky group. This fact goes hand in hand with the over counting of generators. It is easy to see how to fix this. By deforming the circular arcs in an appropriate way the last generator $L_n$ becomes superfluous and one is left with $2(n-1)$ identified closed circles rather than arcs. The argument is sketched in the right of Figure 8. The existence of this deformed fundamental domain for the group generated by $L_1, \ldots, L_{n-1}$ means that this group is by definition a classical Schottky group [26]. In what follows we will go back and forth from considering these two different fundamental domains. We will refer to the first symmetric domain as $D_s$ and the later deformed domain as $D_d$.

The boundary of the fundamental domain is defined as:

$$\partial D_s = \bigcup_{m=1}^n \left( U_m \cup \tilde{U}_m \right) \quad \partial D_d = \bigcup_{m=1}^{n-1} \left( O_m \cup \tilde{O}_m \right)$$

(5.11)

where $U_m, \tilde{U}_m$ are the arcs given in (5.4), (5.5) respectively. While the circles $O_m, \tilde{O}_m$ are not uniquely defined; an example is given in the right side of Figure 8.

The attractive and repulsive fixed points all lie along the same circle $|w| = \sqrt{x_S}$ in the complex $w$ plane. This demonstrates a fact that was speculated upon in [24]. The authors showed that if the Schottky parameters $a_m, r_m, q_m$ can be chosen to be real then there is a so called real duality between the compact boson CFT at the self dual radius and free fermions (with a fixed spin structure) and this implied the EREs of these two theories were the same for two intervals. They demonstrated the real duality using a different method and speculated that this meant one could choose the Schottky parameters to be real. Indeed we
see here explicitly that this is the case - by making an \( PSL(2, \mathbb{C}) \) transformation to send the circle \( |w| = \sqrt{x_S} \) to the real axis then \( a_m, r_m \) will all become real. \( q \) remains fixed under this transformation and is real as above. \(^6\)

Note that this reality argument applies to the Schottky parameters for our specific choice of A-cycles (cycles with trivial monodromy). It probably does not apply for other replica symmetry breaking choice of A-cycles, although for the arguments in [24] one only needs the reality condition for one such choice of cycles.

Finally a picture of the bulk solution can be drawn by extending the circles into hemispheres in \( AdS_3 \) \(^{[4.1]}\). We depict in Figure [11] the symmetric case where the fundamental domain on the boundary is \( \mathcal{D}_s \). In this picture two bulk hemispheres intersect over a geodesic. We speculate that one can identify this with a generalized version of geodesic in the RT prescription.

![FIG. 11: The gravity solution found by extending the arcs \( U_m, \tilde{U}_m \) which live on the boundary of \( AdS_3 \) into hemispheres inside the bulk and identifying these hemispheres. We show again the case \( n = 3 \). The picture also represents a (non standard) fundamental domain for the quotient of \( AdS_3 \) by the action of \( \Sigma \) which acts as \( (4.2) \). The hemispheres intersect over the green curves which are bulk geodesics.](image)

\(^6\) This also means that the Schottky group is actually a Fuchsian group acting nicely on the disk \( |w| < \sqrt{x_S} \). Interestingly this allows us to find a real-time three dimensional black hole based upon \( \mathcal{M} \), see [29, 30] for details. These are generalizations of the usual BTZ black holes [31]. It would be interesting to understand what such a solution means for the EREs.
VI. BULK ACTION

We will consider two ways to calculate the regularized on shell Einstein action for the bulk solution. Certain results in the literature will be used heavily. To begin with we work on the deformed domain $\mathcal{D}_d$ defined in the previous subsection. It was shown in [27] and further in [32] that the on-shell action can be written in terms of a certain two dimensional Liouville action living on the boundary. The action is defined on the domain $\mathcal{D}_d$. It was first written down in [28] by Zograf and Takhtajan (ZT) and we will refer to it as the ZT action. It was further shown in [28] that the variation of the ZT action with respect to the moduli of $\mathcal{M}$ behaves nicely and we will use these results to prove the assertion of the prescription that the variation of the EREs gives the accessory parameters (3.4).

After this we will go through a re-derivation of the results in [27] for the bulk action using the symmetric fundamental domain $\mathcal{D}_s$ which is somewhat more convenient for our purposes. This will allow us to give an absolute expression for the EREs not involving derivatives with respect to $z_i$.

A. Zograf-Takhtajan Action

Firstly we introduce the notion of Fuchsian uniformization. We only need it as an intermediate step so we will be brief. This is another kind of uniformization compared to the Schottky variety which aims to place a constant negative (for genus $(n-1)>1$) curvature metric on $\mathcal{M}$. The method is very similar to the Schottky case. Consider the Poincaré disc $D$ with metric:

$$d\hat{s}^2 = \frac{dtd\bar{t}}{(1-|t|^2)^2}$$  \hspace{1cm} (6.1)

where $|t|<1$. Fuchsian uniformization represents $\mathcal{M}$ as a quotient of $D$ by a discrete group which acts nicely on it. That is a discrete subgroup $\Sigma_F$ of $SL(2,\mathbb{C})$ which leaves the metric (6.1) invariant.

Several results in the literature are available for computing the gravity partition function when the metric is taken to be $d\hat{s}^2$. Because of the conformal anomaly the result does depend on which metric we use within a fixed conformal class. We actually want the partition function on $ds^2$ given in (2.4) and the difference between these two is given by the Liouville action:

$$Z_{\mathcal{M}}[ds^2] = e^{S_L} Z_{\mathcal{M}}[d\hat{s}^2], \quad S_L = \frac{c}{96\pi} \int_{\mathcal{M}} d^2t \left( (\partial_t \phi_F)^2 - \frac{16\phi_F}{(1-|t|^2)^2} \right)$$  \hspace{1cm} (6.2)

where $\phi_F$ is the conformal factor which relates the two metrics:

$$ds^2 = e^{-\phi_F} d\hat{s}^2 = e^{-\phi_F(z)}dzd\bar{z}$$  \hspace{1cm} (6.3)

The metric $ds^2$ has conical singularities which means $\phi_F$ is singular at these points. We deal with this by cutting out holes around these points which is a standard [47] procedure.
Details are given in Appendix C. To find $\phi_F$ we need to map the branched covering to this representation of $\mathcal{M}$. Once again the ode (3.1) allows us to construct the analytic map and the quotient group $\Sigma_F$ as the monodromy group:
\[ t(z) = \frac{\psi_1(z)}{\psi_2(z)}, \quad \psi'' + \frac{1}{2} T_{zz}^F \psi = 0 \]
where the stress tensor $T_{zz}^F$ has the same form as the Schottky case (3.1) however now the accessory parameters $p_i \rightarrow p_i^F$ will be different. In order to fix the $p_i^F$ we must impose the condition that all monodromy elements generated by the fundamental group leave the metric (6.1) invariant. This is called the Fuchsian monodromy condition. Note there is a unique condition here, we do not need to pick different “A cycles” and "B cycles". We emphasize that this is a different monodromy problem to the one given in Section III and that we expect to find a different accessory parameter. The field $\phi_F$ is then:
\[ \phi_F = - \ln |t'(z)|^2 + 2 \ln (1 - |t(z)|^2) \]
and we are now in a position to compute $S_L$.

To calculate $Z_M(d\bar{s}^2)$ we introduce the ZT action which also happens to be a Liouville type action however now living on the Schottky $w$ space. Firstly one introduces a new Liouville field $\phi_S$ which is the conformal factor on the Schottky $w$-space which uniformizes that space, placing on it a constant negative curvature metric:
\[ ds^2 \equiv e^{-\phi_S} dw d\bar{w} \]
\[ \phi_S = - \ln |t'(w)|^2 + 2 \ln (1 - |t(w)|^2) \]

A major difference between the two Liouville fields is that $\phi_S$ is not single valued on $\mathcal{M}$ where as $\phi_F$ is. As we move around on $\mathcal{M}$ the $w$ coordinate undergoes $PSL(2,\mathbb{C})$ transformations along the B-cycles: $\tilde{w} = L_m(w)$. In order that $\phi_S$ is consistent with these jumps in $w$ (the action of $\Sigma$) we must also require that $\phi_S$ jumps:
\[ \phi_S(\tilde{w}) = \phi_S(w) + \log |L'_m(w)|^2 \]
under which the metric (6.6) is preserved:
\[ e^{-\phi_S(\tilde{w})} d\tilde{w} d\bar{\tilde{w}} = e^{-\phi_S(w)} dw d\bar{w} \]
The field $\phi_S$ is uniquely specified by the identifications (or boundary conditions) and the requirement that it satisfies the Liouville equation:
\[ \partial_w \bar{\partial}_w \phi_S = -2e^{-\phi_S} \]
This equation follows from the requirement that (6.6) has constant negative curvature. More succinctly $\phi_S$ is the solution of the equations of motion which follow from varying the
The following ZT action defined in [28]:

\[ S_{ZT} = \frac{c}{96\pi} \int_{D_d} d^2w \left( (\partial \phi_S)^2 + 16e^{-\phi_S} \right) + S^{bd}_{ZT} \]  

(6.11)

\[ \frac{96\pi}{c} S_{ZT}^{bd} = \sum_{m=1}^{n-1} \int_{O_m} (2\phi_S + \log |L'_m|^2 + 2\log |c_m|^2) \left( idw \frac{L''_m}{L'_m} - id\bar{w} \frac{\bar{L}''_m}{\bar{L}'_m} \right) \]  

(6.12)

where the boundary terms are designed to impose (6.8). Note that \( c_m \) is the lower left component of the \( L_m \) matrix defined in (5.6). Recall that \( \partial \mathcal{D}_d = \cup_m (O_m \cup \tilde{O}_m) \) where \( \tilde{O}_m = L_m(O_m) \). Also note that the addition of the boundary term at \( |w| = R_w \to \infty \) is to deal with an IR divergence due to the \( w \) plane being infinite. Typically one picks the generators \( L_m \) using the freedom to conjugate by \( PSL(2, \mathbb{C}) \) such that one of the fixed points is at \( \infty \). Then this IR divergence is absent since \( w = \infty \) does not appear in the fundamental domain. This will not be convenient for us and we choose instead to directly deal with the IR divergence.

In [27] it was shown that the on shell value of the action (6.11) for \( \phi_S \) gives the regularized action of the bulk gravity solution (up to some minor additions, see (6.13) below.) The ZT action captures the conformal anomaly and depends on the choice of metric in a fixed conformal class. The appropriate metric here is (6.6) and by the usual dictionary of AdS/CFT the action \( S_{ZT} \) gives a contribution to the partition function \( Z_M(d\hat{s}^2) \) of the CFT defined on this metric:

\[ \hat{S}_\gamma^{\text{gr}} = -S_{ZT} + \frac{c}{2}(n-2) - \frac{c}{3}(n-2) \log \Lambda \]  

(6.13)

where \( \gamma \) labels the particular gravitational saddle. We have also included a UV cutoff factor \( \propto \log \Lambda \) which cannot be removed in the limit \( \Lambda \to 0 \) due to the conformal anomaly. It is proportional to the Euler character of \( M \) which is \(-2(n-2)\).

Putting everything together the partition function we seek (1.2) is the sum over the different saddles \( \gamma \) (including the ones we do not construct):

\[ Z_M(d\hat{s}^2) = \sum_\gamma \exp(-\hat{S}_\gamma^{\text{gr}} + S_L + O(c^0)) \]  

(6.14)

Rather than work directly with \( S_{ZT} \) and \( S_L \) we would like to compute their on-shell variation with respect to the \( z_i \). We can use several results in the literature. These results can be understood as essentially arising from conformal ward identities.

For \( S_L \) the variation was given originally by Polyakov using the Liouville theory path integral. A proof using just the classical Liouville action was given in [33][34]. The variation gives the Fuchsian accessory parameters defined in terms of the Fuchsian monodromy.
\[ \frac{\partial S_L}{\partial z_i} = \frac{cn}{6} p_i^F \] (6.15)

For \( S_{ZT} \) the results in [28] and [36] can be applied. Here a mathematically rigorous procedure for varying the action with respect to the moduli was used and goes under the name of quasiconformal transformations. From the results in these papers we can derive:

\[ \frac{\partial S_{ZT}}{\partial z_i} = \frac{cn}{6} (p_i^\gamma - p_i^F) \] (6.16)

the difference in Fuchsian and Schottky accessory parameters. Details are given in Appendix E. The reason the difference in accessory parameters appears is that the stress tensor of \( S_{ZT} \) which appear when varying the action takes the form:

\[ \hat{T}_{ww} = \partial_w \phi_S + \frac{1}{2} (\partial_w \phi_S)^2 = w'(z)^{-2} (T_{zz} - T_{zz}^F) = w'(z)^{-2} \left( \sum_i \frac{(p_i^\gamma - p_i^F)}{z - z_i} \right) \] (6.17)

This stress tensor is the one associated to the ground state of the CFT living on \( \hat{d}s^2 \).

Adding (6.15) to (6.16) explains equation (3.4) given in the prescription of Section III which was the main goal of this current subsection. Note that \( p_i^F \) cancels between (6.15) and (6.16) so the final result does not depend on the Fuchsian uniformization. This is expected since we used the uniform metric \( \hat{d}s^2 \) only as an intermediary.

**B. Regularization Surface**

The goal of the next two subsections is to find an expression for \( S_{gr} \) without resorting to taking derivatives thereof. To do this we go through the derivation in [27] using a slightly different regularization procedure.

Following [27] we need to pick a regularization surface in order to define the bulk action. This surface should be consistent with the \( L_m \) identifications. Also the desired boundary metric should be induced on this cutoff surface. We use the field \( \phi_S \) to define our cutoff surface:

\[ \xi \approx \Lambda e^{\phi_S/2} \quad ds^2|_\Lambda \approx \Lambda^{-2} e^{-\phi_S} dwd\bar{w} = \Lambda^{-2} \hat{d}s^2 \] (6.18)

with UV cutoff \( \Lambda \to 0 \). Note we could not simply cutoff at fixed \( \xi \) since under the \( PSL(2,\mathbb{C}) \) isometries (4.2) the coordinate \( \xi \) changes. For small \( \xi \) this transformation is consistent with

\[ ^7 \text{In the literature on Liouville theory one considers a slightly different form of the Liouville action } S_L \text{ from (6.2). Firstly the action is defined on the } z \text{ plane with reference metric } ds^2. \text{ We can get to this form by integrating by parts on (6.2) and we go through this in Appendix C. Secondly the } z \text{-plane is not multi-sheeted, rather the points } z_i \text{ are conical deficit singularities for the uniform metric } \hat{d}s^2. \text{ One can think of these as arising from a quotient of our surface } M \text{ by } \mathbb{Z}_n. \text{ This explains the extra factor of } n \text{ in (6.15) compared to for example [36].} \]
the transformation of $e^{\phi S/2}$. Hence this choice. Using this surface which we call $D_\Lambda$ we define the regularized action in the usual way:

$$16\pi G_N \hat{S}_{gr} = \int_{Q_\Lambda} d^3x \sqrt{g}(R-2) + 2 \int_{D_\Lambda} d^2x \sqrt{h}(1 - K)$$

(6.19)

where $Q_\Lambda$ is the regularized portion of the fundamental domain for the quotient $AdS_3/\Sigma$ with boundary $D_\Lambda$. $K$ is the trace of the extrinsic curvature and $h$ is the induced metric. Note that depending on the choice of fundamental domain for the quotient $Q_\Lambda$ the boundary will be conformally equivalent to either $D_s$ or $D_d$ as $\Lambda \to 0$. The choice of $D_d$ was worked out in [27] so here we pick $D_s$.

We can be a little more precise and pick coordinates $\{\Lambda, u, \bar{u}\}$ in order to write the original $AdS_3$ metric in the Fefferman-Graham expansion with induced metric $d\hat{s}^2$. Following [37] write

$$\xi = \frac{4\Lambda e^{\phi S/2}}{4 + \Lambda^2 e^{\phi S}|\partial_u \phi S|^2} \quad w = u - \partial_u \phi S \frac{2\Lambda^2 e^{\phi S}}{4 + \Lambda^2 e^{\phi S}|\partial_u \phi S|^2}$$

(6.20)

where now $\phi S$ is considered a function of $\phi S(u, \bar{u})$ which anyway approaches $(w, \bar{w})$ at the boundary. The bulk metric is:

$$ds^2 = \frac{d\Lambda^2}{\Lambda^2} + \frac{1}{\Lambda^2} \left| \left( \frac{1 + \Lambda^2}{1 - |t|^2} \right) dt + \frac{1}{2} \Lambda^2 (1 - |t|^2) \hat{T}_{tt} dt \right|^2$$

(6.21)

where we have written the answer in terms of the Poincaré disk coordinate $t$.

The answer has a particularly simple form since the FG expansion terminates in three bulk dimensions [38, 39]. The stress tensor $\hat{T}_{tt}$ of the field theory living on $d\hat{s}^2$ and in the state defined by the saddle at hand appears as a sub leading term in the FG expansion [40]. In these coordinates we can write the stress tensor as:

$$\hat{T}_{tt} = (t'(z))^{-2} \left( \{w, z\} - \{t, z\} \right) = (t'(z))^{-2} \left( T_{zz} - T_{zz}^F \right)$$

(6.22)

where $t, w$ were given in terms of solutions to the appropriate Fuchsian [6.4] or Schottky [4.6] odes. We use the notation $\hat{T}$ to denote the stress tensor for the vacuum state of the theory living on the uniform metric $d\hat{s}^2$. While $T$ is reserved for the stress tensor of the theory defined on the singular metric $ds^2$.

Fefferman-Graham coordinates typically develop a coordinate singularity away from the boundary. This is indeed the case, the metric becomes degenerate when:

$$\Lambda^2_c = \frac{4}{(r_n^2 - |t|^2)|\hat{T}_{tt}| - 4r_n^2}$$

(6.23)

Actually we could attempt to bypass altogether the Fuchsian uniformization - and never even mention $d\hat{s}^2$ or the Poincaré disk coordinates $t$. In this case we should pick our regularization surface such that the induced metric is directly $ds^2$ which is anyhow the desired

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$^8$ The normalized stress tensor of the state is related to this by a factor $\hat{T}_{CFT} = \frac{c}{2\pi} \hat{T}$. 

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metric. The fact that the partition function depends on this regularization surface in the limit $\Lambda \to 0$ is a manifestation of the Weyl anomaly in holography [41]. This would leave the introduction of the Liouville action $S_L$ unnecessary. We can do this by setting

$$\phi \equiv \phi_S - \phi_F = \ln w'(z) + \ln \bar{w}'(\bar{z})$$

(6.24)

and using $\phi$ to define a new regularization surface. Note that $\phi$ is a locally harmonic function on the $z$ plane. Since $\phi_F$ is single valued on $\mathcal{M}$ the new field transforms in the same way as $\phi_S$ around the non-trivial B-cycles (6.8).

We can pick Fefferman-Graham coordinates with respect to the Liouville field $\phi$ by replacing $\phi_S \to \phi$ in (6.20). The bulk metric is then:

$$ds^2 = \frac{d\Lambda^2}{\Lambda^2} + \frac{1}{\Lambda^2} \left| d\bar{z} + \frac{\Lambda^2}{2} T_{zz} dz \right|^2$$

(6.25)

From this we see that $T_{zz}$ is the stress tensor of the theory living on $ds^2$. This is stress tensor that appears in the original ode.

Compared to the Fefferman-Graham coordinates for the metric $d\hat{s}^2$ those for $ds^2$ are rather singular. Here we find a breakdown of the FG coordinates arbitrarily close to the points on the boundary ($z \to z_i, \Lambda \to 0$). This breakdown was discussed and confronted in [42] in a similar computation of EREs. They break down at $\Lambda^2 = 2/|T_{zz}| \sim |z - z_i|^2$. This is because in addition to UV regulating the theory using the cutoff surface $\Lambda = \text{const}$ we need to regulate the divergences associated with the conical singularities in $ds^2$. We achieved this previously by using the singular Liouville field $\phi_F$ to transform to the non-singular metric $d\hat{s}^2$. Then the Liouville action for $\phi_F$ contained the divergences associated to these conical singularities. So the field $\phi$ has to take into account both the divergences associated to the conical singularities as well as the $L_m$ identifications. We found it convenient to deal with these issues separately by splitting this into two steps.

C. Action from the Symmetric Domain

We are now ready to calculate (6.19). Firstly let us compute the bulk integral using the $(\xi, w, \bar{w})$ coordinates. We use the FG coordinate (6.20) to define the regulating surface at $\Lambda = \text{const}$. The fundamental domain was depicted in Figure 11 consisting of removing hemispheres from $AdS_3$. Define $V$ the volume of this domain:

$$\int d^3x \sqrt{g}(R - 2) = 4V = 4 \int d^2w \int_{\xi_{\text{min}}}^{\infty} \frac{d\xi}{\xi^3} = 2 \int d^2w \frac{1}{\xi_{\text{min}}^2(w, \bar{w})}$$

(6.26)

Away from the hemispheres the radial integral is cutoff at $\xi_{\text{min}}^{-2} = \Lambda^{-2} e^{-\phi_S(w, \bar{w})} - |\partial_w \phi_S|^2/2 + \ldots$ where we should emphasize that we are working with $w, \bar{w}$ coordinates at the boundary and not $u, \bar{u}$. We define $V_m, \tilde{V}_m$ the volumes of the chunks of $AdS_3$ below the hemispheres.
segments (below in the sense of Figure 11). There are $2n$ of these but by the replica symmetry they are all the same:

$$V = \frac{1}{4} \int_{D_s} d^2 w \left( 2\Lambda^{-2} e^{-\phi_S} - |\partial w \phi_S|^2 \right) + 2nV_1$$  \hspace{1cm} (6.27)$$

Where $V_1$ corresponds to the “first” hemisphere segment - on the boundary it becomes the $m = 1$ segment of (5.4). The volume is:

$$V_1 = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{r_{m(\theta)}} dr \frac{1}{\rho^2 - r^2}, \quad \rho = \frac{(1 - x_S)}{2\sin(\pi/n)}$$  \hspace{1cm} (6.28)$$

where we do the integral using cylindrical coordinates about the center of the hemisphere. We have given the radius of the hemispheres $\rho$ in terms of the Schottky parameter $x_S$ defined in Section V.

Examining the geometry of the hemispheres shown in Figure 11 we see that we get two terms, one from where the $r$ integral is cutoff by the intersection of the hemisphere with the regularization surface and the other from the remaining triangular shaped region. That is where the radial integral is cutoff by the intersection with another hemisphere. These two terms are:  

$$V_1 = -\frac{1}{4} \int_{U_1} d\theta (\phi_S + 2 \ln(\Lambda/\rho)) + N_1, \quad N_1 = -\frac{1}{2} \int_0^{\pi/n} d\theta \ln \left( 1 - \frac{\cos^2(\pi/n)}{\cos^2(\theta)} \right)$$  \hspace{1cm} (6.30)$$

The first term is an integral on the $AdS_3$ boundary along the segment in the $w$-plane which can be described as (see (5.4)):

$$U_1 = \left\{ w = w_1 + re^{i\theta}; \frac{\pi}{2} + \frac{\pi}{n} < \theta < \frac{3\pi}{2} - \frac{\pi}{n} \right\}, \quad w_1 = \left( \frac{x_S - e^{-i2\pi/n}}{1 - e^{-i2\pi/n}} \right)$$  \hspace{1cm} (6.31)$$

where $w_1$ is the center of the circular arc in the complex plane. Note that along this arc $\phi_S$ is identified under $L_1$ with the $\phi_S$ at the next arc moving in an anti-clockwise direction on Figure 11. We can write this identification (6.8) simply as:

$$\phi_S \left( \tilde{w}_1 + r^{-1} \exp(-i\theta + i2\pi/n) \right) = \phi_S (w_1 + r \exp(i\theta)) - 4\ln(r/\rho)$$  \hspace{1cm} (6.32)$$

where $\tilde{w}_1$ is the center of this adjacent arc ($\tilde{w}_1 = e^{i2\pi/n} \tilde{w}_1$).

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9 The expression for $N_1$ is only a function of $n$ and does not analytical continue well to $n < 2$. We guess an expression that has a better continuation in $n$

$$N_1 = -\frac{1}{2} \text{sign}(n-2) \int_0^{\min(\pi/n,\pi-\pi/n)} d\theta \ln \left( 1 - \frac{\cos^2(\pi/n)}{\cos^2(\theta)} \right)$$  \hspace{1cm} (6.29)$$

For $n < 2$ this expression subtracts the volume of a triangular shaped region, since now the other term in $V_1$ over count the volumes of the hemisphere segments. This is a guess since the bulk solution does not make any sense for $n < 2$. This guess seems to yield the correct answer.
An issue we have ignored so far is related to the IR divergence associated with working in the Poincare patch. To fix this we momentarily move to global coordinates where the metric is:

$$ds^2 = \frac{d\xi^2}{\xi^2} + \left(\frac{R_w}{\xi} - \frac{\xi}{R_w}\right)^2 R^2_w \frac{dwd\bar{w}}{R^2_w(\xi^2 + |w|^2)^2} \quad (6.33)$$

and where the radial coordinate ranges over $0 < \xi < R_w$. The limit $R_w \to \infty$ returns us to Poincare coordinates. However before we take this limit we get an extra log contribution to the bulk Einstein action:

$$V = \int d^2 w \frac{R^2_w}{(R^2_w + |w|^2)^2} \left(\frac{R^2_w}{2\xi^2_{\min}} + 2\log(\xi_{\min}/R_w) + \ldots\right) \quad (6.34)$$

The log term encodes the coupling of $\phi_S$ to the curvature of the $w$ sphere, which we have hidden at $|w| \to \infty$ by working on the plane. We must keep this term which in the limit $R_w \to \infty$ gives us the addition:

$$V \to V + \frac{1}{2} \int_{|w|=R_w} d\theta \phi_S - 2\pi \ln(R_w/\Lambda) \quad (6.35)$$

The extrinsic curvature part of the gravitational action (6.19) is most conveniently evaluated in FG coordinates $(\Lambda, u, \bar{u})$ - which can be related to $(\xi, w, \bar{w})$ coordinates close to the boundary (note if we were using $\phi$ and not $\phi_S$ as our Liouville field there would be some extra complications to deal with here.) That is:

$$2 \int d^2 x \sqrt{h}(1 - K) \approx -\int d^2 w \left(2e^{-\phi_S(w, \bar{w})} \Lambda^{-2} + 4\partial_w \partial_{\bar{w}} \phi_S\right) \quad (6.36)$$

Adding everything together the quadratic UV divergence associated to $\Lambda$ vanishes leaving:

$$\frac{96\pi}{c} \hat{S}_{gr} = -\int_{D_s} d^2 w \left((\partial \phi_S)^2 + 4\partial^2 \phi_S\right) + 32n V_1 + 8 \int_{|w|=R_w} d\theta \phi_S - 32\pi \ln(R_w/\Lambda) \quad (6.37)$$

where one should remove $|w| > R_w$ to define the symmetric domain $D_s$. The $\partial^2 \phi_S$ term evaluates to something proportional to the Euler character of $M$:

$$\int d^2 w \partial^2 \phi_S = -8\pi(n - 2) \quad (6.38)$$

Combining the gravitational action with the Liouville action (6.2) as in (6.14) we find after integrating by parts:

$$\frac{96\pi}{c} S_{gr} = -\int_{D_s} d^2 w (\partial \phi)^2 - 8n \int_{U_1} d\theta (\phi - 2 \ln \rho) + 8 \int_{|w|=R_w} d\theta \phi - 32\pi \ln R_w \left(32\pi(n - 2) \ln \Lambda + 32\pi(n - 2) + 32n N_1\right) \quad (6.39)$$

where everything is written in terms of $\phi \equiv \phi_S - \phi_F$. Recall that this is a harmonic field that can be defined solely in terms of the Schottky uniformization coordinates $w(z)$ see
The new domain \( \tilde{D}_s \) is a regularized version of \( D_s \) defined by cutting out various holes where \( \phi \) diverges. The justification for this cutting procedure is the same as for the Liouville action that we went through in Appendix D. Some details have been swept under the rug in arriving at (6.39). For example we need to disentangle the conical singularity at \( z_4 \) from the curvature singularity of the infinite \( w \) plane at \( w = \infty \). Recall that for our choice (5.2) these points were the same \( w(z_4) = \infty \). The quickest way to deal with this is to deform the point \( w(z_4) \to w_4 \gg 1 \) such that \( w(z_\infty) = \infty \) for \( z_\infty \approx z_4 \) on only one of the replicas. Taking \( w_4 \to \infty \) of this procedure defines the action (6.39).

We can now evaluate (6.39) since \( \phi \) is harmonic. After some work (the details of which are given in Appendix D) one finds an answer which can be succinctly written in terms of \( \psi_- \) the particular solution to the ode appearing in the denominator of the Schottky coordinate \( w = \lambda \psi_+ / \psi_- \) (see the discussion around (5.2).) We send \( z_1 = 0, z_2 = x, z_3 = 1 \) and \( z_4 \to \infty \). By defining the Mutual Information (for this particular saddle \( \gamma = \beta \)) as in (3.20) we can take the limit \( z_4 \to \infty \) without the associated IR divergence. We find:

\[
c^{-1} f_\beta^\alpha = \frac{n}{12\pi(n-1)} \int_x^1 dz \text{Im} \left( \frac{\psi'_-}{\psi_-} - \frac{\tilde{\psi}'_-}{\tilde{\psi}_-} \right) \ln \left| \frac{\psi'_-}{\psi_-} - \frac{\tilde{\psi}'_-}{\tilde{\psi}_-} \right| - \frac{n N_1}{3\pi(n-1)}
\]

\[
+ \frac{1}{12} \ln \left| \frac{\mu_1 \mu_2 \mu_3}{\mu_4 \rho^2} \right| + \frac{1}{6} \left( \frac{1}{n} + 1 \right) \ln(x) - \frac{(n+1) \ln(n)}{6(n-1)}
\]

(6.40)

where \( \psi_- \approx z^{1/2-1/(2n)} \) at the origin of the \( z \) plane. The integral is along the real axis with \( \psi_- = \psi_-(z + i\eta) \) evaluated just above the real axis and we have defined \( \psi_- \) evaluated just below the real axis as \( \tilde{\psi}_- = \psi_-(z - i\eta) \). Note that \( \tilde{\psi}_- \) is related to \( \psi_- \) by a monodromy around the loop on \( \mathcal{M} \) which connects the top of the real line segment \( z \in [x, 1] + i\eta \) with the bottom \( z \in [x, 1] - i\eta \). See the left panel in Figure 7. This is the monodromy loop that defined the matrix \( L_1 \) and we can write \( \tilde{\psi}_- = c_1 \psi_+ + d_1 \psi_- \). Recall that we are studying the saddle associated to the monodromy conditions in \( \Gamma_\beta \). A similar expression to (6.40) exists for \( \Gamma_\alpha \). Finally \( \rho \) and \( \mu_i \) are also extracted from \( \psi_- \) simply as:

\[
\rho = \left| \frac{\lambda n^{-1}}{W[\psi_-, \tilde{\psi}_-]} \right| \\
\mu_1 = \lim_{z \to z_i} \frac{\lambda}{\psi_-^2 (z - z_i)^{1-1/n}} \\
\hat{\mu}_4 = -\lim_{z \to \infty} \frac{\lambda}{\psi_-^2 z^{1-1/n}}
\]

(6.41)

where \( W \) is the Wronskian of two solutions to the ode. Note that when we plug these constants into (6.40) the factor \( \lambda \) drops out so we can effectively ignore it.

We have confirmed numerically that the expression (6.40) gives the same answer as the one obtained by integrating the accessory parameter. We think formula like (6.40) should generalize for other non-replica symmetric saddles which would be useful in checking the assumption that replica symmetry remains unbroken. The details of course will be slightly different and we leave this to future work.
VII. DISCUSSION

To summarize we have calculated some contributions to EREs in holographic CFTs by constructing higher genus gravitational handlebody solutions. We only found a subset of all possible classical solutions. These are the solutions which were highly symmetric - respecting the symmetries of the boundary surface. We found that the bulk actions of this set of solutions continued nicely under $n \rightarrow 1$ to the Ryu-Takayanagi formula involving lengths of bulk geodesics. We did this for arbitrary numbers of intervals, however the bulk solutions were only described in detail for $N = 2$.

If we could show that the symmetric solutions dominate in the sum over saddles at large central charge for all $0 < x < 1$ then we would have found the Mutual Renyi Information for all 1 + 1 CFTs with an Einstein gravity dual description and we would have proven the RT formula in this case. We have not managed to come up with a proof necessary for this purpose and in fact after some thought we are not sure it is true.

More conservatively for two intervals one should be able to show that our prescription for computing the EREs is correct for any $n$ in a perturbative expansion about $x = 0$. We can argue pictorially that the saddle which we called $\Gamma_\alpha$ dominates over all other gravitational saddles in the limit $x \rightarrow 0$. In particular the minimal length of a curve living on $M$ (in terms of either metric $ds^2$ or $d\hat{s}^2$) which is homologous to the cycle $C_\alpha$ (see Figure 4) becomes parametrically small compared to the minimal length curve homologous to $C_\beta$. This is true on all replicas. The bulk solution which has the least volume and hence least action will be the one where all the short cycles $C_\alpha$ are contractable. If any of the other longer $C_\beta$ cycles were contractible then we will clearly get a larger volume subdominant solution.

Once we have shown $\Gamma_\alpha$ is the dominant saddle as $x \rightarrow 0$ then at infinite central charge the other saddles cannot be seen to all orders in an expansion about $x = 0$. This leaves open the possibility that another non-symmetric solution becomes dominant at some finite value of $0 < x < 1/2$. The danger region is $x \approx 1/2$ since then the minimal length curves of $C_\alpha$ and $C_\beta$ approach the same length. Similar arguments can be made about the point $x = 1$ as well as for more than two intervals.

The paper [22] found the same monodromy prescription that we gave in a completely different manner. They studied semi-classical (in the sense of large central charge) Virasoro conformal blocks for low dimension operators which can be computed in terms of this monodromy condition [49]. The ERE thought of as a 4-point function of twist operators was then argued to receive its dominant contribution from the conformal block for exchange of the unit operator (including the stress tensor and its descendants.) These conformal blocks can be identified with the bulk solutions we constructed and since they contain the unit operator in the s-channel (t-channel) exchange as $x \rightarrow 0$ ($x \rightarrow 1$) must give the dominant contribution to the 4-point function about $x = 0$ ($x = 1$). The question of intervening saddles in the danger region was also unresolved in [22] and would involve the conformal block of some other heavy operator (with conformal dimension $\mathcal{O}(c)$) potentially becoming
We now review some material that may help for trying to construct these missing saddles in this danger region.

## A. The missing saddles

Summing over saddles in $AdS_3$ holography is a well studied subject at genus one (see for example [43, 44]) but is less explored at higher genus (however see [45, 46].) The basic idea can be understood in terms of the moduli space of Riemann surfaces $M_n$. At genus $(n-1)$ this is a $3(n-2)$ complex dimensional space. The covering space of $M_n$ is called Teichmüller space $T_n$ and this space distinguishes Riemann surfaces related by large coordinate transformations - sometimes referred to as Modular transformations or elements in the Mapping Class Group (MCG). $M_n$ is then just the quotient of $T_n$ by the MCG. If one studies a modular invariant CFT on a Riemann surface then the partition function is a function on $T_n$ which is consistent with the action of the MCG - thus it is also well defined on $M_n$.

The handlebody solutions to $AdS_3$ gravity however are not invariant under the action of the MCG. This is because we had to choose a set of $(n-1)$ A-cycles which were contractable in the bulk and these cycles change under the MCG action. This means that the gravitational action thought of as a function on $T_n$ becomes multivalued on the quotient $M_n$. To find a modular invariant partition function we need to sum over all images of the MCG. Schematically,  

$$Z_M = \sum_{g \in MCG} \exp (-S^{g(\alpha)}_{gr} + O(c^0))$$  \hspace{1cm} (7.1)$$

where we continue to use $\alpha$ to denote the replica symmetric solution corresponding to the monodromy conditions specified by $\Gamma_\alpha$. The action $g$ on this solution which we denote $g(\alpha)$ then scans through all relevant handlebody solutions. After the sum $Z_M$ is modular invariant and well defined on $M_n$.

For example there is one element in the MCG which sends $\Gamma_\alpha \to \Gamma_\beta$. The phase transition between these two saddles at $x = 1/2$ is a fixed point of this action. The other solutions $g(\alpha)$ have not yet been constructed. At finite central charge $c$ we would need all of them to find a consistent partition function, however at large $c$ we can ignore all but the dominant ones. One then just needs to find which is the dominant as a function of $x$.

We have found $S_{gr}$ along a one dimensional slice in $T_n$ which is the special slice distinguished by the replica symmetry. We just need to construct the other one dimensional slices of $T_n$ related by elements of the MCG to the replica symmetric slice. Since the replica symmetry acts non-trivially on these solutions there will be more than one slice of $T_n$ space (related by the replica symmetry) with the same action. If one of these is dominant then

---

10 not all!, a subgroup leaves the bulk solution untouched, but we do not dwell here on such details.
it will clearly give rise to the phenomena that might be called replica symmetry breaking. Actually we require infinite central charge otherwise the usual arguments about the lack of symmetry breaking in a finite system apply. It would be interesting to understand this phenomenon (if it were to occur) in more detail. Some questions that come to mind: What is the order parameter? What does this imply about the spectrum of the reduced density matrix? What are the implications for the product orbifold theory and EREs written as twist operator correlation functions?\footnote{We give an answer to the first question in this footnote: the non-symmetric accessory parameters $p^s$ (for $s \neq 1$) are good order parameters, and can be extracted from integrals of stress tensor on $\mathcal{M}$, see \cite{44}. Formally one would need to slightly break the replica symmetry in order to get a non-zero expectation value for this order parameter. Then at large central charge one can remove the symmetry breaking term and still potentially arrive at a non-zero answer for $\langle p^s \rangle$, indicating symmetry breaking has occurred.}

Unfortunately finding the other solutions is easier said than done. We sketch how one might construct these missing saddles using the methods of this paper. Pick a non-symmetric set of $(n - 1)$ A-cycles and demand that they are contractible in the bulk. The ode \eqref{3.1} can be used to construct the bulk solution - however as we alluded to throughout the paper we need to turn on the other accessory parameter \eqref{4.9}. Which ones we have to turn on depends on the symmetry breaking pattern. Already we see the problem becomes intractable for large $n$ - we have to search in the $3(n - 2)$ dimensional space of accessory parameters in order to satisfy the monodromy conditions. However it is feasible to attempt this for small values of $n$. One then needs to compute $S_{gr}^{\alpha}$. For this we can no longer resort to integrating the accessory parameters and instead have to work with an absolute expression for $S_{gr}$. This is where a suitably generalized version of \eqref{6.40} will come in handy.

To further complicate things there are nonhandlebody solutions when $n > 2$. See \cite{48} for a discussion in the context of partition functions of $AdS_3$ gravity. See also \cite{32} where these solutions are constructed using Quasi-Fuchsian Kleinian groups. Note that they exist if and only if the boundary surface has some discrete symmetry, and indeed we sit on a point in moduli space with lots of symmetry. We have access to their actions since they are simply related to Fuchsian uniformization and the Liouville action $S_L$ \cite{48}. It is believed that these solutions cannot be dominant because they would lead to certain pathologies from the dual CFT perspective - relating to studying the CFT on more than one disconnected surface. But it remains an open question to show this.\footnote{We thank Alex Maloney for discussion on this.} We are actually in a position to study this question however for now we leave this to future work.

For the case $n = 2$ one can construct all the saddles. This is the case where $\mathcal{M}$ is a torus which is then the same surface which is used to compute the thermal partition function of the CFT on a circle. As discussed in \cite{20} the transition between the $\Gamma_\alpha$ and $\Gamma_\beta$ saddles at $x = 1/2$ is related to the black hole Hawking-Page transition. There are also an infinite set of saddles \cite{43} which turn out not to contribute at large central charge for a purely imaginary
torus modulus $\tau = i\tau_2$. These come from $SL(2, \mathbb{Z})$ modular transformations of the torus and lead to the following saddle contributions to the Mutual Information:

$$I_{2}^{A,B}(x) = \frac{c\pi}{6} \frac{\tau_2}{A^2 + B^2 \tau_2^2} - \frac{c}{12} \ln \left(\frac{2^8 (1 - x)}{x^2}\right)$$

(7.2)

where $A, B \in \mathbb{Z}$ and are co-prime. The cycle that is contractable in the bulk is $AC_{\alpha} + BC_{\beta}$ where $C_{\alpha,\beta}$ were defined in Figure 4. Note that under the action of complex conjugation $(C_{\alpha}, C_{\beta}) \rightarrow (-C_{\alpha}, C_{\beta})$ so these bulk solutions break this symmetry when both $A$ and $B$ are not zero (in which case there are multiple solutions with the same action.) However as is clear from (7.2) the dominant solutions are either $(A = 1, B = 0)$ or $(B = 1, A = 0)$. We take this as evidence in favor of the absence of replica symmetry breaking.

Finally note that understanding how the actions of these missing saddles can be continued to $n$ non-integer so we can take the limit $n \rightarrow 1$ is also important. This seems rather tricky when the replica symmetry is broken since we have to continue the symmetry braking pattern to non-integer $n$ whatever that means.

**B. Further work**

Aside from studying the possibility of replica symmetry breaking we see several avenues for how to extend this work. Firstly it is important to understand quantum corrections to the RT formula. This involves calculating one loop determinants for fluctuations about the saddles that we constructed.

Secondly one could try to generalize our setup in several ways. For example one could try to work with different states in the CFT like finite temperature on a circle. Moving to higher dimensions would be difficult because we no longer have the power of $AdS_3$ holography. The power stems from the fact that gravity has no propagating modes in three dimensions. Of course one could still try to work numerically in higher dimensions, and as we have learned it may not be necessary to solve the full numerical problem at integer $n$ in order to reproduce RT. It might be possible to proceed by simply setting up the problem well enough so that in principle the ERE could be computed. Then if the bulk action can be read off without reference to the specifics of the bulk solution it may be possible to continue the answer to $n \rightarrow 1$ without doing the hard numerical work.

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Appendix A: The Riemann surface

Specializing to two intervals $N = 2$ the Riemann surface is defined by:

$$y^n = \frac{(z - z_1)(z - z_3)}{(z - z_2)(z - z_4)}$$  \hspace{1cm} (A1)

- $z$ and $y$ can be thought of meromorphic functions on the surface $M$
- $z$ defines a holomorphic map from $M$ to $\mathbb{CP}^1$ branched at the points $z_i$. The degree of the covering map is $n$. And the branching order is:

$$B_z(z_i) = n - 1$$  \hspace{1cm} (A2)

so that the Riemann-Hurwitz relation tells us that the genus of $M$ is $g = n(-1) + 1 + 2(n - 1) = n - 1$.

- The surface has a cyclic $\mathbb{Z}_n$ symmetry (automorphisms of $M$) defined by cyclic permutations of the $n$ different sheets. It is generated by $(y \rightarrow ye^{i\pi/n}, z \rightarrow z)$.

- The space of complex structure deformations of such a surface is $3(n - 2)$ complex dimensional (for $n > 2$), however we are only interested in a one real dimension slice of this space. These will be the deformations which preserve the $\mathbb{Z}_n$ rotations discussed above and leave the surface in the form described by (A1).

- A basis for the holomorphic differential forms is:

$$\nu^t = \frac{dz}{y^t(z - z_2)(z - z_4)} \hspace{1cm} t = 1, \ldots, n - 1$$  \hspace{1cm} (A3)

- A linearly independent basis for the quadratic holomorphic differential forms is:\(^{13}\)

$$\omega^{s+1} = \frac{dz^2}{y^s(z - z_1)(z - z_3)(z - z_2)(z - z_4)} \hspace{1cm} s = 0, \ldots, n - 2$$  \hspace{1cm} (A4)

$$\omega^{s+n-2} = \frac{dz^2}{y^s(z - z_1)(z - z_3)(z - z_2)^2} \hspace{1cm} s = 2, \ldots, n - 2$$  \hspace{1cm} (A5)

$$\omega^{s+2n-5} = \frac{dz^2}{y^s(z - z_1)(z - z_3)(z - z_4)^2} \hspace{1cm} s = 2, \ldots, n - 2$$  \hspace{1cm} (A6)

$$\omega^{3(n-2)} = \frac{dz^2}{y^{n-1}(z - z_2)^2(z - z_4)^2}$$  \hspace{1cm} (A7)

There are $3(n-2)$ of these, and they label the space of complex structure deformations.

\(^{13}\) In order to find these one notes that the point defined by $z \rightarrow \infty$ is smooth and thus $\omega$ takes the form $P(z)dz^2y^{-s}$ where $P(z)$ is meromorphic function with poles at the branch point and net pole order $-4$. There are linear relations that need to be taken into account between these meromorphic functions. Then the bounds on $s$ can be found by examining the behavior close to each branch point.
Only one of the quadratic differentials is invariant under the $Z_n$ symmetry:

$$\omega^1 \equiv \frac{dz^2}{(z - z_1)(z - z_3)(z - z_2)(z - z_4)} \quad (A8)$$

As usual we can use an $SL(2, \mathbb{R})$ transformation on $z$ to move the points $z_1, z_2, z_3, z_4$ to $0, x, 1, \infty$ respectively. After scaling $y$ by a constant the surface can be described by:

$$y^n = \frac{z(z - 1)}{(z - x)} \quad (A9)$$

### Appendix B: The connection matrices

In this section we setup the connection matrices which allow us to study the monodromy problems numerically. For each point $z_i$ consider the two linearly independent solutions:

$$u^{(z_i)}(z) = \begin{pmatrix} \psi_+^{(z_i)}(z) \\ \psi_-^{(z_i)}(z) \end{pmatrix} \approx \begin{pmatrix} (z - z_i)^{1 + \frac{1}{2n}} \\ (z - z_i)^{1 - \frac{1}{2n}} \end{pmatrix} \quad (B1)$$

where $z$ is taken slightly to the right of $z_i$ on the real axis. Since we know the ode is real along the $z$ axis it is useful to construct the following real connection matrices:

$$u^{(z_i)}(z) = R_{i,i+1}Tu^{(z_{i+1})}(z) \quad (B2)$$

where $T$ enacts the monodromy of moving $z$ from slightly to the right of $z_{i+1}$ to slightly to the left of $z_{i+1}$

$$T = \begin{pmatrix} -ie^{-i\pi/(2n)} & 0 \\ 0 & -ie^{i\pi/(2n)} \end{pmatrix} \quad (B3)$$

From this it is clear that the remaining matrix $R_{i,i+1}$ is real. Further the Wronskian condition on two linearly independent solutions tells us that $\det R_{i,i+1} = -1$.

To summarize we have defined the following set of $2N$ real connection problems:

$$\psi \approx \alpha_i(z - z_i)^{1 + \frac{1}{2n}} + \beta_i(z - z_i)^{1 - \frac{1}{2n}}, \quad z \to z_i^+$$

$$\psi \approx \alpha_{i+1}(z_{i+1} - z)^{1 + \frac{1}{2n}} + \beta_{i+1}(z_{i+1} - z)^{1 - \frac{1}{2n}}, \quad z \to z_{i+1}^-$$

which are amenable to numerical work. From these $R$'s and the matrix $T$ we can construct the full set of monodromies in a straightforward manner. For example the trivial monodromy condition on a path encircling two adjacent points $(z_k, z_{k+1})$ is:

$$1 = T^2R_{k,k+1}T^2R_{k,k+1}^{-1} \quad (B5)$$
Appendix C: Regulating the twist operators

The Liouville action (6.2) needs to be regulated by smoothing out the singularities of \( \phi_F \) close to the singular points \( z_i \). Following \[47\] we cut out infinitesimal holes of \( \mathcal{M} \) around these points \( |z - z_i| < \epsilon \) and “insert the unit operator” by picking a different metric within these holes:

\[
ds^2 = \frac{1}{|z|^4} \propto dt d\bar{t} \quad |z| > R_z \quad (|t - t^\infty_m| < \# / R_z) \tag{C2}
\]

and we have matched onto the metric \( ds^2 = dz d\bar{z} \) at the boundary of the hole. Doing this allows us to extend the definition of \( \phi_F \) to the closed surface \( \mathcal{M} \). Note that for this choice of metric \( \phi_F \) is essentially constant in this hole (up to corrections due to the curvature of the hyperbolic space.) There are 4 of these holes.

Additionally we also have to chop out \( z > R_z \) which contributes an IR divergence to the path integral. There are \( n \) images of these \( z \to \infty \) holes in the \( t \) plane at \( t = t^\infty_m \). To do this we work with the following metric for large \( z \):

\[
ds^2 = R_z^4 \frac{dz d\bar{z}}{|z|^4} \propto dt d\bar{t} \quad (|t - t^\infty_m| < \# / R_z) \tag{C2}\]

which again matches to \( ds^2 = dz d\bar{z} \) at the boundary and sets \( \phi_F \) to approximately a constant in this region. In this way the action \( S_L \) in (6.2) is well defined. It is easy to show that the contributions to \( S_F \) from the \( n + 4 \) holes discussed above is zero essentially because \( \phi_F \) is a constant and the volume of the holes is going to zero in the limit \( \epsilon \to 0, R_z \to \infty \). We can then restrict ourselves to performing the integral in the Liouville action over the region outside the holes: \( \hat{\mathcal{M}} = \mathcal{M} \setminus \{ |z - z_i| < \epsilon \} \cup \{ |z| > R_z \} \).

We now write this integral in terms of the original \( z \) coordinates:

\[
S_L = \frac{c}{96\pi} \int_{\hat{\mathcal{M}}} d^2 z \left( (\partial \phi_F)^2 + 2\phi_F (\partial^2 \phi_F) \right) = \frac{c}{96\pi} \int_{\hat{\mathcal{M}}} d^2 z \left( (\partial \phi_F)^2 + 16e^{-\phi_F} \right) + S_{\text{bdry}} - \frac{c}{3} (n - 2) \tag{C3}
\]

where we have integrated by parts and used the fact that the Euler character of the surface \( \mathcal{M} \) is \( 2(2 - n) \) to introduce the Liouville term \( \propto e^{-\phi_F} \) into the action. Up to this constant the final result is that of the standard Liouville action for the field \( \phi_F \) living on the \( z \)-plane. The boundary term also taking the usual form:

\[
\left( \frac{96\pi}{c} \right) S_{\text{bdry}} = -4 \left( 1 - \frac{1}{n} \right) \int_{|z - z_i| = \epsilon} d\theta \phi_F + 8 \sum_{m=1}^{n} \int_{|z| = R_z} d\theta \phi_F \tag{C4}
\]

where the last sum is over the different sheets of the branched covering. Also the first integral above is from \( \theta = 0, 2\pi n \). We have used the behavior of \( \phi_F \) close to \( z_i \) and \( z = \infty \):

\[
\phi_F \approx \left( 1 - \frac{1}{n} \right) \ln |z - z_i|^2 + \ldots, \quad z \to z_i; \quad \phi_F \approx 2 \ln |z|^2 + \ldots, \quad z \to \infty \tag{C5}
\]
This action actually defines the problem that we have already solved. That is if we vary the action we find the Liouville equations of motion:

$$\partial^2 \phi_F = -8e^{-\phi_F}$$  \hspace{1cm} (C6)

now subject to the boundary conditions (C5). There is a unique solution to this problem. That is the solution which defines a constant negative curvature on the space $M$ and which is equivalent to the solution found using the ode and Fuchsian monodromy condition.

**Appendix D: More on the symmetric domain action**

We continue the calculation of the bulk gravity action starting from (6.39). This action is integrated over the regulated domain $\hat{D}_s$ defined analogously to $\hat{M}$ in (C3). That is $\hat{D}_s = \hat{D}_s \setminus \{(|z - z_i| < \epsilon) \cup (|z| > R_z) \cup (|w| > R_w)\}$.

The field appearing in (6.39) $\phi$ is the Weyl factor for the the conformal transformation from the $w$ to $z$ planes:

$$ds^2 = dzd\bar{z} = e^{-\phi}dwd\bar{w}$$  \hspace{1cm} (D1)

Thus it will have singularities whenever either of these metrics is singular (unless the two metrics have the same singularity and they cancels out.) It is clear that $\phi$ is harmonic so we can evaluate the bulk term of (6.39) by integrating by parts. This then reveals the singularities located at $z_i$ and as well those at $z \rightarrow \infty$ which we remind the reader are actually $n$ points one from each replica - they are located on the $w$-plane at $w^m_\infty$. We also still have the $w$ plane singularity at $w \rightarrow \infty$. Altogether we have:

$$c^{-1}S_{gr} = \frac{1}{12}\phi(|w| = R_w) - \frac{1}{3}\ln R_w + \frac{1}{12}\sum_{m=1}^{n}\phi(|z| = R_z[w = w^m_\infty])$$

$$-n - \frac{1}{24}\sum_{i=1}^{4}\phi(|z - z_i| = \epsilon) - \frac{n}{24\pi}(I_1 - 8N_1) + \frac{(n - 2)}{3}\ln A + \frac{(n - 2)}{3}$$  \hspace{1cm} (D2)

where we have defined the integral:

$$I_1 = \int_{\Omega_1} d\theta (\phi - 4\ln \rho)$$  \hspace{1cm} (D3)

and the notation is for this integral the same as in (6.30).

In order to make sense of (D2) we have temporarily deformed the space $w$ such that $z_4$ no longer maps to $w = \infty$ but instead maps to $w \gg 1$ and also such that there is a point $z_\infty \approx z_4 (on one of the replicas)$ which does map to to $w = \infty$. This will disentangle the $w$-plane curvature which is hidden at $w \rightarrow \infty$ from the twist operator at $z = z_4$ which previously mapped to $w = \infty$ (see Figure 7). The behavior of $\phi$ can then be read of from the behavior of the Schottky uniformization coordinate $w(z)$:
• Around the twist operators \( z \to z_i \):

\[
w(z) \approx w_i + \mu_i (z - z_i)^{1/n} \to \phi \approx 2 \left( \frac{1}{n} - 1 \right) \ln |z - z_i| + 2 \ln |\mu_i/n| \tag{D4}
\]

• Around the \( w \)-plane curvature singularity at \( w \to \infty \) (recall that \( z_\infty \) is close to \( z_4 \) and only on one of the replicas):

\[
w(z) \approx \frac{\nu_\infty}{z - z_\infty} \to \phi \approx 4 \ln |w| - 2 \ln |\nu_\infty| \tag{D5}
\]

• Around the \( z \)-plane curvature singularities at \( z \to \infty \)

\[
w(z) \approx w_\infty^m + \frac{\mu_\infty^m}{z} \to \phi \approx -4 \ln |z| + 2 \ln |\mu_\infty| \quad (m = 1, \ldots n) \tag{D6}
\]

where we have defined many new constants \( \mu_i, \mu_\infty^m, \nu_\infty \) which can be extracted from \( w(z) \) numerical. Plugging this into (D2) we find:

\[
c^{-1} S_{gr} = -\frac{(n - 1)}{12} \sum_{i=1}^{4} \ln |\mu_i/n| - \frac{1}{6} \ln |\nu_\infty| + \frac{1}{6} \sum_{m=0}^{n-1} \ln |\mu_\infty^m|
\]

\[
+ \frac{(n - 2)}{3} - \frac{n}{24\pi}(I_1 - 8N_1) - \frac{n}{3} \ln R_4 - \frac{(n - 2)}{3} \ln \Lambda + \frac{(n - 1)^2}{3n} \ln \epsilon
\]

Note that \( |\mu_\infty^m| \) will actually all be the same by the replica symmetry. We now want to return to the setup of interest (after the aforementioned deformation of the \( w \) plane) by taking the limit \( w(z_4) \to \infty \) and thus \( z_\infty \to z_4 \). We also want to eventually take the limit \( z_4 \to \infty \) (and set \( z_1 = 0, z_2 = x, z_3 = 1 \)) since this is most convenient for numerical work. We take these limits sequentially:

• In the coincidence limit close to \( w \approx w_4 \approx \infty \) one can argue that the analytic map has the form:

\[
z \approx z_4 + w_4^{2n} \mu_4^{-n} (w_4^{-1} - w^{-1})^n \tag{D8}
\]

which reproduces the behavior about \( w = w_4 \) and \( w = \infty \) if we additionally have \( \nu_\infty = -nw_4^{n+1}/\mu_4^n \). Expanding (D8) for large \( w_4 \) we have:

\[
w \approx \tilde{\mu}_4 (z - z_4)^{-1/n}, \quad \tilde{\mu}_4 = -w_4^2/\mu_4 \tag{D9}
\]

where we should fix \( \tilde{\mu}_4 \) in this limit. The we find:

\[
\lim_{w_4 \to \infty} |\nu_\infty^{2n} \mu_4^{-n+1}| = |\tilde{\mu}_4|^{(n+1)/n^{2n-2}} \tag{D10}
\]

• Similarly in the limit \( z_4 \to \infty \) we have:

\[
\lim_{z_4 \to \infty} |(\mu_\infty^1)^{2n} \mu_4^{-n-1}| = |\tilde{\mu}_4^{-1} z_4^{2(n-1/n)/n} - 2/n \tag{D11}
\]

where we have defined as \( z \to \infty \):

\[
w \approx \tilde{\mu}_4 z^{1/n} \tag{D12}
\]
Taking these limit in (D8) gives:

\[
\begin{align*}
   c^{-1}S_{gr} &= -\left(\frac{n-1}{12}\right) \ln \left| \frac{\mu_1 \mu_2 \mu_3}{\tilde{\mu}_4} \right| - \frac{n}{24\pi}(I_1 - 8N_1) + \frac{(n-3)}{6} \ln(n) + \frac{(n-2)}{3} \\
   &\quad + \left(\frac{n^2-1}{6n}\right) \ln z_4 + \frac{(n-1)^2}{3n} \ln \epsilon - \frac{n}{3} \ln R_z - \frac{(n-2)}{3} \ln \Lambda
\end{align*}
\] (D13)

To deal with all the regulator factors we simply compute the mutual information. We need the ERE for a single interval using the same technique as above, so that the non-universal pieces cancel. One finds (the uniformization map can be found in this case analytically, we do not go through the details which can be for example found in [47]):

\[
\begin{align*}
   c^{-1}S_{gr}^{N=1} &= \frac{1}{6} \left( n - \frac{1}{n} \right) \ln |z_1 - z_2| - \frac{1}{3} \ln n - \frac{n}{3} \ln R_z + \frac{1}{3} \ln \Lambda
\end{align*}
\] (D14)

where \(z_1, z_2\) are the end points of the single interval. We also need the partition function \(Z_1\) for the theory on a single replica in order to relate \(Z_M\) to the EE. See (2.5). This can be computed as above to find \(\ln Z_1 = c/3(\ln(R_z/\Lambda) + 1)\). Putting everything together we find:

\[
\begin{align*}
   c^{-1}I_n^\beta &= \frac{1}{12} \ln \left| \frac{\mu_1 \mu_2 \mu_3}{\tilde{\mu}_4} \right| + \frac{n}{24\pi(n-1)}(I_1 - 8N_1) + \frac{1}{6} \left( n + 1 \right) \ln(x) + \frac{(n+1)\ln(n)}{6(n-1)}
\end{align*}
\] (D15)

where we remind the reader this computation was for the \(\Gamma_\beta\) saddle. After mapping the integral \(I_1\) in (D3) to the \(z\) plane we get the expression quoted in (6.40).

**Appendix E: Quasi-conformal transformations**

A variation of the modular parameter of the surface \(M\) can be thought of as a quasi-conformal transformation. A transformation which deforms infinitesimally the complex structure of the manifold \(M\). It was shown in [28] that the ZT action behaves nicely under quasi-conformal transformations. The results make physical sense since the variation gives an integral of the stress tensor associated with the Schottky uniformization:

\[
\begin{align*}
   \delta S_{ZT} &= \frac{c}{24\pi} \int_M d^2w \left( \delta N^w \tilde{T}_{ww} + \delta \tilde{N}^w \tilde{T}_{\tilde{w}\tilde{w}} \right) \\
   \tilde{T}_{ww} &= \frac{1}{2}(\partial_w \phi_S)^2 + \partial_w \partial_{\tilde{w}} \phi_S
\end{align*}
\] (E1)

The quasi-conformal variation is defined through a Beltrami differential \(\delta N^w_{\tilde{w}}\). To describe this consider varying the complex structure of \(M\) by defining new holomorphic coordinates on the different coordinate patches. For example working on the branched covering:

\[
\begin{align*}
   z' &= z + \delta z(z, \bar{z}) \\
   y' &= y + \delta y(y, \bar{y})
\end{align*}
\] (E2)

and equivalently for the antiholomorphic coordinates. Demanding that the new transition functions (defined by the complex curve (A1)) are holomorphic with respect to the coordinates \((z', y', \ldots)\) defines a Beltrami differential:

\[
\delta N^w_{\bar{z}} = \partial_{\bar{z}} \delta z(z, \bar{z})
\] (E3)
which transforms as the placement of the indices suggests. It is this type of differential that appears in (E1). The integral in (E1) is simply over the entire surface $\mathcal{M}$ without boundary (compared to the integrals over the fundamental $\mathcal{D}_d$) since the action of $\Sigma$ does not change the stress tensor $\hat{T}$ and $\delta N$ is well defined on the whole surface.

Let us transform this result to the $z$ plane:

$$\delta S_{ZT} = \frac{c}{24\pi} \int_{\mathcal{M}} d^2z \left( \hat{T}_{zz} \partial_z \delta z + \bar{\hat{T}}_{zz} \partial_{\bar{z}} \delta \bar{z} \right)$$

$$= \frac{c}{24\pi} \sum_i \int_{\mathcal{M}} d^2z \left( \partial_z \delta z(z, \bar{z}) \frac{(p_i - p_i^F)}{(z - z_i)} + \partial_{\bar{z}} \delta \bar{z}(\bar{z}, z) \frac{(\bar{p}_i - \bar{p}_i^F)}{(|z - \bar{z}_i|)} \right)$$

where we have replaced the the Schwarzian derivatives with the their expansion in terms of accessory parameters (4.9). In particular we have set all replica symmetry breaking accessory parameters to zero. We have picked $\delta z$ to be the same on each replica and in this way the quasiconformal transformations leaves us on the moduli space of Riemann surfaces defined by (A1).

Note that what we have arrived at in (E5) looks like a Ward identity for conformal invariance. Integrating we find,

$$\delta S_{ZT} = \frac{cn}{12} \sum_i \left( (p_i - p_i^F) \delta z_i + c.c. \right)$$

where $\delta z_i \equiv \delta z(z_i, \bar{z}_i)$. We have integrated over all the replicas explaining the factor of $n$. A certain amount of regularity in $\delta z(z, \bar{z})$ was assumed in order to be able to invert the operator $\partial_z$ in (E5) and this can be justified by giving an explicit expression for $\delta z$ in terms of $\delta z_i$

$$\delta z(z, \bar{z}) = -\frac{1}{2} e^{\phi_F} \sum_i \delta z_i \partial_{z_i} \partial_{\bar{z}_i} \phi_F$$

where $\phi_F$ is the Fuchsian Liouville field and this equation is the same on each replica. See [34] for the complete discussion.

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