Electromagnetic fields of slowly rotating magnetized compact stars in conformal gravity

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The exact analytical solutions for vacuum electromagnetic fields of slowly rotating magnetized compact stars in conformal gravity have been studied. Taking the realistic dipolar magnetic field configuration for the star, analytical solutions of the Maxwell equations for the near zone magnetic and electric fields exterior to a slowly rotating magnetized relativistic star in conformal gravity are obtained. In addition, the dipolar electromagnetic radiation and energy losses from the rotating magnetized compact star in conformal gravity have been studied. With the aim to find observational constraints on the L parameter of conformal gravity, the theoretical results for the electromagnetic radiation from the rotating magnetized relativistic star in conformal gravity have been combined with the precise observational data on the radio pulsars periods slow down and it is estimated that the upper limit i.e. the maximum value of the parameter of conformal gravity is less than $L \lesssim 9.5 \times 10^5 \text{cm} \left( L/M \lesssim 5 \right)$.

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I. INTRODUCTION

The end state of the life-cycle evolution of massive stars from several to ~ hundred Solar masses through a supernova explosion may form either a neutron star or a black hole. Collapsed black hole according to the no-hair theorem does not have intrinsic magnetic field (see e.g. [1–5]). On the contrary the formed neutron stars are highly magnetized objects and one of the main aims of the modern astrophysics of compact relativistic stars is to get a clear understanding of the configuration, structure and evolution of the stellar magnetic field. The precise measurements of electromagnetic signals from the radio-pulsars show that the magnetic fields of compact relativistic stars decrease in strength with the stellar age and the recycled old neutron stars have weaker magnetic fields. The strong electromagnetic field affects the observational data on the high energetic processes in the vicinity of the compact stars in all electromagnetic radiation spectra. Observational data of radio-pulsars and soft Gamma ray repeaters (SGR) have shown that the surface magnetic field of a typical neutron star is about $10^{12}$ G, while for magnetars observed as SGRs and anomalous X-ray pulsars (AXP) it may reach the extreme values as $10^{15}$ G [6, 7]. Therefore, the comparison of the evolution of magnetic fields and of the rotation spin down observed in neutron stars with those modeled and theoretically predicted provide a great challenge and powerful tool to get the constraints on the neutron star properties in the extreme physics regime and conditions. Consequently, the continued analysis of the evolution of the stellar magnetic fields together with the precise measurement of the spin of relativistic stars at the various evolutionary stages provides an opportunity to get the constraints on the alternate theories gravity (ATG) [8] in the strong field regime. The electromagnetic signal detected from radio pulsars is mainly due to the magneto-dipolar radiation from the rotating magnetized compact star. The energy loss due to the electromagnetic radiation causes the spin-down of the rotating relativistic star [9–17]. The structure of the pulsar magnetosphere and related astrophysical processes have been widely studied in literature, see e.g. [18–26].

Thus the strong gravitational field regime near relativistic compact stars can play a role of laboratory to test general relativity versus other modified or alternative theories of gravity. Testing gravity theories using the strong field regime has been performed for X-ray sources from the stellar black hole candidates [27–38]. The comparison of the electromagnetic field and radiation of the compact star with the pulsar spin down can also be used to constrain the alternative theories of gravity [38, 39].

The impact of strong electromagnetic fields can be observed by other astrophysical processes such as gravitational lensing, motion of test particles, and the electromagnetic spectrum of accretion discs [40–45]. An analytical solution of the exterior electromagnetic field of a rotating magnetized star in the Newtonian limit has been found in [46]. Interior solutions for the electromagnetic fields of a constant magnetic density star are studied by
many authors, see, for example, [47]. General relativistic corrections to the electric and magnetic field structure outside magnetized compact gravitational objects have been studied in [1] and have been further extended by a number of authors [2, 11, 12, 25, 26, 48–71]. Magnetic fields of spherical compact stars in a braneworld have been studied in [72].

In this work we investigate the vacuum electromagnetic fields of slowly rotating magnetized compact stars in conformal gravity proposed in [73]. An example of Lagrangian in this large class of conformally invariant theories of gravity is

\[ \mathcal{L} = \phi^2 R + 6g^{\mu\nu} \partial^\mu \phi \partial_\nu \phi , \]

(1)

where the scalar field \( \phi \) (dilaton) is combined with the Ricci scalar. This Lagrangian is invariant under conformal transformations

\[ g_{\mu\nu} \rightarrow g^{*}_{\mu\nu} = S g_{\mu\nu} , \]

\[ \phi \rightarrow \phi^* = S^{-1/2} \phi , \]

(2)

where \( S = S(x) \) is a function of the spacetime coordinates.

Since the world around us is not conformally invariant, conformal symmetry must be broken, and one of the possibilities is that it is spontaneously broken. In such a case, Nature must select one of the vacua, namely a certain gauge corresponding a specific choice of the conformal factor \( S \). In the symmetric phase, the theory is invariant under conformal transformations, i.e. the physics is independent of the conformal factor \( S \). In the broken phase, the choice of the conformal factor \( S \) does lead to observational effects. Such a choice may look arbitrary, but this is a fundamental feature of any spontaneously broken symmetry, not just of conformal gravity. In what follows, we will consider the infinite family of conformal factors found in [73] because they have the property to solve the singularity problem in the Kerr metric.

In the paper [74] the quasinormal modes of the scalar fields of a black hole in conformal gravity have been studied. The energy conditions of a black hole in conformal gravity have been studied in [75]. Conformal invariance preservation at the quantum level has been discussed in [76].

The present paper is organized as follows. Sect. II is devoted to the vacuum electromagnetic fields of a rotating magnetized compact star in conformal gravity and we derive an exact analytical solutions of the Maxwell equations for the magnetic and the electric fields of a slowly rotating neutron star in conformal gravity. In the next Sect III, we calculate the energy losses from a slowly rotating neutron star in conformal gravity. In Sect. IV, we obtain astrophysical constraints on the value of the parameter of conformal gravity, \( L \), from the comparison of theoretical results on spin down of the rotating star in conformal gravity with the current observational data from radio pulsars period evolution. Finally, in Sect. V we summarize our obtained results. Throughout the paper, all physical quantities are denoted with an asterisk. We use a space-like signature \((-+,+,+,-\)) , a system of units in which \( G = c = 1 \) and, we restore them when we need to compare our results with observational data. Greek indices run from 0 to 3, Latin indices from 1 to 3.

II. THE VACUUM ELECTROMAGNETIC FIELDS OF A Rotating Magnetized COMPACT STAR IN CONFORMAL GRAVITY

In this section we briefly discuss the electromagnetic fields in the spacetime of a magnetized compact star in conformal gravity. One of the most difficult mathematical problems is to solve the combined Einstein-Maxwell equations, which are coupled nonlinear differential equations, but one can solve them in the realistic approximation when the electromagnetic field plays no role into the space-time geometry in the vicinity of the compact star (see, for example, [11, 12]). Assuming that the energy of the electromagnetic field is too weak to modify the space-time geometry around the compact star, one can consider the electromagnetic field in the fixed background spacetime and investigate the effects of the background gravitational field on the electromagnetic field of the slowly rotating relativistic magnetized star in conformal gravity.

Even the spacetime of the most rapidly rotating compact (neutron) stars observed as millisecond pulsars can be approximately described within the slow rotation limit [77]. In Boyer-Lindquist coordinates \((t,r,\theta,\phi)\) the space-time outside the slowly rotating magnetized star in conformal gravity can be expressed through the following line element [73]

\[ ds^2 = S(r) \left[ -N^2 dt^2 + \frac{1}{N^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] , \]

(3)

where \( M \) is the total mass and \( R \) is the radius of the compact star, \( \omega(r) = 2aM/r^3 \) is the angular velocity of the dragging of inertial frames and \( a \) is the specific angular momentum of the star, which is defined as \( a = J/M \), and \( J = I \Omega \) is the total angular momentum, with the moment of inertia \( I \) and the angular velocity \( \Omega \) (or the period \( P = 2\pi/\Omega \) of rotation of the star) which are very important and precisely measurable quantities/parameters in observation of pulsars.

The radial function \( S(r) \) in Eq.(3) is the scaling factor and, in the slowly rotating limit, has the form [73]

\[ S(r) = \left( 1 + \frac{L^2}{r^2} \right)^{2(n+1)} , \quad n = 1, 2, 3, \ldots , \]

(4)

where \( L \) is a parameter with dimensions of a length and \( n \) is an integer positive number. The theory does not
provide any prediction for the value of $L$, so we can expect that $L$ is either of the order of the Planck length, $L \sim L_P$, or of the order of the gravitational radius of the object, $L \sim M$, as these are the only two length scales of the system [73]. The first option is realized with the scale already present in the action, while the latter is with the scale that breaks conformal symmetry on-shell. A priori, both scenarios are possible and natural. In the present paper, we will consider the second option with $L \sim M$, as it is the only one with potential astrophysical implications in compact objects. If $L \sim L_P$, modifications of Einstein’s gravity would only show up in high energy/high curvature regimes. The choice of $n$ is related to the symmetry breaking. As in any spontaneously broken symmetry, we cannot say why Nature selects a particular vacuum in the class of good vacua. In our work, we consider the simplest case $n = 1$ and we briefly describe how our results change for larger values of $n$. In order to study the electromagnetic properties of slowly rotating magnetized compact stars in conformal gravity, one has to find the solutions of the Maxwell equations in conformal gravity which can be written in similar way as as it has been done in [11, 78].

**Stellar Model:**

Before doing any calculation, we list here the realistic stellar model assumptions.

- The magnetic moment of the relativistic star does not have strong dependence on time due to the high electrical conductivity of the stellar medium $\sigma \to \infty$, see e.g., [78].

- In the slowly rotation limit, the linear approximation of the angular velocities of rotation is very good one that is $O(\omega)$ and $O(\Omega)$, respectively.

- It is assumed that the rotating star has a spherical shape in the slowly rotation approximation and the deformation due to the stellar rotation is negligible.

- The medium is vacuum outside the star.

- Using the above assumptions we look for the stationary solutions of the Maxwell equations for the components of the magnetic field of the rotating star in conformal gravity in the following form [11, 78]

\[
B^i(r, \theta, \phi, t) = F^i(r) \times [\cos \chi \cos \theta + \sin \chi \sin \theta \cos \lambda], \quad (5)
\]

\[
B^\theta(r, \theta, \phi, t) = G^\theta(r) \times [\cos \chi \sin \theta - \sin \chi \cos \theta \cos \lambda], \quad (6)
\]

\[
B^\phi(r, \theta, \phi, t) = H^\phi(r) \times \sin \chi \sin \lambda, \quad \lambda = \phi - \Omega t, \quad (7)
\]

where the unknown radial functions $F^i(r)$, $G^\theta(r)$, and $H^\phi(r)$ are responsible for the corrections to the magnetic field due to conformal gravity parameters and neutron star’s mass and $\chi$ is the inclination angle of the magnetic field with respect to the stellar rotation axis.

In the paper [11], such a consideration has already been performed in the general relativistic case and the expressions for the stationary vacuum electromagnetic fields of a slowly rotating relativistic star have been clearly shown. Following to the techniques used in the paper [11] we look for the solutions and relations for the electromagnetic fields of a slowly rotating compact star in conformal gravity that are distinguished by the scaling factor $S(r)$ in comparison with the general relativistic ones. Then one can simply write them in the following form

\[
\left( B^i, E^i \right)_{CG} = \frac{1}{S} \left( B^i, E^i \right)_{GR}, \quad (i = 1, 2, 3), \quad (8)
\]

or

\[
\left( B, E \right)_{CG} = \frac{1}{S} \left( B, E \right)_{GR}, \quad (9)
\]

where the quantities $B$ and $E$ are the magnetic and the electric fields, respectively.

Collecting all the statements which are introduced here, one can easily find the profile radial functions $F^i(r)$, $G^\theta(r)$, and $H^\phi(r)$ in the expressions (5)-(7) for the components of the magnetic field in the following form (see e.g., [11])

\[
F^i(r) = - \frac{3\mu}{4M^3S} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right], \quad (10)
\]

\[
G^\theta(r) = H^\phi(r) = \frac{3\mu N}{4\pi M^2 S} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right], \quad (11)
\]

where $\mu$ is the magnetic dipole moment of the magnetized slowly rotating compact star. From the astrophysical point of view, the electric field of compact stars (pulsars and magnetars) is at least $V/c$ times weaker than the stellar magnetic field, where $V$ is the nonrelativistic linear velocity of the neutron star surface. Analytical expressions for the electric field are given in Appendix A.

Hereafter, introducing the normalized dimensionless radial coordinate $\eta = r/R$ and assuming zero inclination angle $\chi = 0$, one can write the exact solutions for the components of the magnetic field (5)-(7) in the following form

\[
B^\phi(\eta, \theta) = - \frac{3B_0}{\epsilon^3 S} \left[ \ln N^2 + \frac{\epsilon}{\eta} \left( 1 + \frac{\epsilon}{2\eta} \right) \right] \cos \theta, \quad (12)
\]

\[
B^\theta(\eta, \theta) = \frac{3B_0N}{\eta^2} \left[ \frac{2\eta}{\epsilon} \ln N^2 + \frac{1}{N^2} + 1 \right] \sin \theta, \quad (13)
\]

\[
B^\phi(\eta, \theta) = 0, \quad (14)
\]

where $B_0 = 2\mu/R^3$ is the Newtonian value of the surface magnetic field at the polar cap of the star, $\epsilon = 2M/R$ is the compactness of the star and $N^2(\eta) = 1 - \epsilon/\eta$ is the lapse function. The scaling factor can be rewritten in
terms of the normalized dimensionless radial coordinate in the following form
\[ S(\eta) = \left[ 1 + \left( \frac{L}{R} \right)^2 \frac{1}{\eta^2} \right]^{2(n+1)}, \quad n = 1, 2, 3, \ldots \] (15)

and in what follows we will focus on the scenario in which \( L \) can be of the order of the gravitational radius of the system, hence also of the order of the stellar radius \( R \). Note that the scaling factor is always greater than 1 \((S \geq 1)\). This means, without doing any calculations, one can conclude that the magnetic field of the compact star decreases in the conformal gravity. More precisely, Fig. 1 and Fig. 2 show the normalized radial dependence of the radial and the tangential components of the magnetic field described by the Eqs. (12) and (13) for a relativistic star in conformal gravity when \( n = 1 \). One can easily see that for both components the magnetic field strength are lowered by increasing the dimensional parameter \( L \), which means the magnetic field of a relativistic star decreases in the spacetime of rotating relativistic star in conformal gravity.

III. ASTROPHYSICAL APPLICATION

In this section, we will briefly study the electromagnetic dipole radiation from a rotating magnetized neutron star in conformal gravity. Note that such a phenomenon is at the basis of the observational evidence of radio pulsars identified with the rotating magnetized neutron stars. In case of pure electromagnetic radiation, the luminosity of the rotating magnetized neutron star in conformal gravity is increased due to the decrease of the magnetic field strength and by the gravitational redshift of the effective rotational angular velocity \( \Omega_R^* \).

In the case of pure dipolar electromagnetic radiation, the Newtonian value of the luminosity has the following form [79]
\[ L_{0\text{em}} = \frac{\Omega^4 R^6}{6c^3} B_0^2 \sin^2 \chi \] (20)

In order to calculate the rate of the energy loss from the radio pulsar through dipolar electromagnetic radiation in conformal gravity, one has to consider the ratio of the luminosity in the Newtonian and in conformal gravity [12]
\[ \frac{L_{\text{em}}^*}{L_{0\text{em}}} = \left( \frac{f}{1 - \epsilon} \right)^2 \left[ 1 + \left( \frac{L}{R} \right)^2 \right]^{-8(n+1)} \] (21)

The dependence of the rate of energy loss from the compactness of the magnetized neutron star in conformal gravity for the different values of the parameter \( L/M \) is illustrated in Fig. 3. The plot shows the increase of the...
FIG. 3. Dependence of the energy losses $L_{\text{em}}^*/L_{\text{0em}}$ from the compactness $\epsilon$ of the star for the different values of the parameter $L/M$.

FIG. 4. Dependence of the energy losses $L_{\text{em}}^*/L_{\text{0em}}$ from the parameter $L/M$ for the different values of the compactness $\epsilon$ of the star.

rate of energy loss with the increase of the compactness of the star.

In Fig. 4 the dependence of the rate of energy loss of a magnetized neutron star in conformal from the mod-
ule of the parameter $L/M$ for the different values of the compactness $\epsilon$ of the star is shown.

IV. RESULTS AND DISCUSSIONS

Now one can get constraints on the conformal param-
eter $L$ by comparing the obtained theoretical results on electromagnetic radiation from the rotating magnetized star in conformal gravity with the observational data on spin down for the well known rotating magnetized compact stars and magnetars observed as radio pulsars and SGRs/AXPs. In order to get upper limit for the parameter $L$, one can consider the $P - \dot{P}$ diagram for the typical pulsars [10, 80–84]. From the observational data [85] shown in Fig. 5, one can see that the average value of the magnetic field strength for a typical radio pulsar is about $B_{\text{AV}} = B_0 \simeq 10^{12}$ G, its period is $P \simeq 1$ s, the period derivative is about $\dot{P} \simeq 10^{-15}$ s s$^{-1}$, and the lowest value of the magnetic field strength in observation is around $B_{\text{R}}^* = B_{\text{Low}} \simeq 10^{11}$ G (with $P \simeq 1$ s and $\dot{P} \simeq 10^{-17}$ s s$^{-1}$). Using these observational values and the magnetodipolar formula (18) one can find the upper limit for the value of the parameter as $L \lesssim 9.5 \times 10^9$ cm ($L/M \lesssim 5$) for $n = 1$. This statement is in agreement with the Fig. 6 and Fig. 7 on the dependence of the magnetic field at the surface of the neutron star from the parameter $L/M$ for the different values of the compact-
ness of the star.

In the table I, dependence of the model parameters $n$ and $L/M$ is obtained on comparison of the the mag-
etodipolar formula (18) with the observational data on
spin down of the radio pulsars.

V. SUMMARY

In the present work we have investigated the modifi-
cations of the electromagnetic fields of a rotating mag-
etized compact star arising from the parameters of the conformal gravity and their astrophysical implications to
the neutron stars observed as pulsars. We have studied
the relativistic Maxwell equations for the dipolar electro-
magnetic fields of a slowly rotating magnetized compact
star in terms of the parameters of the conformal gravity
and then obtained the analytical solutions for the dipolar
magnetic field in terms of the parameter $L$. Along

| $n$ | 1 | 2 | 3 | 5 | 10 | 20 | 100 |
|-----|---|---|---|---|----|----|-----|
| $L/M$ | 4.87 | 3.74 | 3.15 | 2.49 | 1.79 | 1.28 | 0.58 |

TABLE I. Dependence of the parameters $n$ and $L/M$ from comparison of the magnetodipolar formula (18) with the observational data for the fixed value of the stellar compactness $\epsilon \simeq 0.4$. 

FIG. 5. $P - \dot{P}$ diagram for the observable pulsars and magnetars from the paper [85].
expressions for the electric field of a rotating magnetized star in conformal gravity.

As an important application of the obtained results, we have calculated energy losses of slowly rotating magnetized neutron star in conformal gravity through magnetodipolar radiation and found that the rotating star with non-zero $L$ parameter will lose less energy when compared to a rotating neutron star in general relativity. This permits us to check the effects of the scaling factor arising from the conformal gravity in the vicinity of a rotating magnetized star, especially, when one calculates the electromagnetic luminosity from the star. The latter is a very important measurable quantity in pulsar astrophysics.

The obtained dependence from the scaling factor has been combined with the astrophysical data on precise measurement of pulsar period slowdown in order to constrain the $L$ parameter. We have found the upper limit for the parameter of conformal gravity as $L \lesssim 9.5 \times 10^5$ cm.

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Appendix A: Electric field of a rotating compact magnetized star in conformal gravity

The components of the electric field of slowly rotating magnetized neutron star in conformal gravity can be chosen in the following form [11]

$$E^\rho(r, \theta, \phi, t) = [f^\rho(r) + f_3^\rho(r)] \cos \chi (3 \cos^2 \theta - 1) + 3 [g_1^\rho(r) + g_3^\rho(r)] \sin \chi \sin \theta \cos \theta \cos \lambda,$$

(A1) $$E^\theta(r, \theta, \phi, t) = [f_2^\theta(r) + f_4^\theta(r)] \cos \chi \sin \theta \cos \theta + [g_2^\theta(r) + g_4^\theta(r)] \sin \chi \cos \lambda - [g_1^\theta(r) + g_3^\theta(r)] \cos 2\theta \sin \chi \cos \lambda,$$

(A2) $$E^\phi(r, \theta, \phi, t) = [g_5^\phi(r) + g_6^\phi(r)] \sin \chi \cos \theta \sin \lambda - [g_2^\phi(r) + g_4^\phi(r)] \sin \chi \cos \theta \sin \lambda,$$

(A3)

where $\{f^\rho(r)\}$ and $\{g^\rho(r)\}$ are the functions of the radial coordinate $r$ to be found as solutions of the Maxwell equations in spacetime of slowly rotating magnetized neutron star in conformal gravity.

The explicit form of the profile radial functions is given as solutions of the Maxwell equations in spacetime of slowly
where the constants of integration $C^*$, $C_1^*$, and $C_3^*$ can be found from the boundary conditions for the continuity of the tangential components of the electric field and for the jump of the normal component of the electric field due to the surface electric charges at stellar surface.

The other radial functions have the following form as in [11]

\[ g_1^* = f_1^*, \quad g_3^* = f_3^*, \quad g_5^* = \frac{1}{2} f_2^*, \quad g_6^* = \frac{1}{2} f_4^*. \]

\[ f_1^*(r) = \frac{\mu \Omega C^* C_1^*}{6c^2 S^2} \left[ \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 + \left( 3 - \frac{2r}{M} \right) \ln N^2 \right], \]  
\[ f_2^*(r) = -\frac{\mu \Omega C^* C_1^*}{c^2 S^2} N \left[ \left( 1 - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right], \]  
\[ f_3^*(r) = \frac{15\mu \omega r^3}{16c^2 M^2 S^2} \left( C_3^* \left( \frac{2M^2}{3r^2} + \frac{2M}{r} - 4 + \left( 3 - \frac{2r}{M} \right) \ln N^2 \right) + \frac{2M^2}{5r^2} \ln N^2 + \frac{2M^3}{5r^3} \right), \]  
\[ f_4^*(r) = -\frac{45\mu \omega r^3}{8c^2 M^2 S^2} \left( C_3^* \left( 1 - \frac{r}{M} \right) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right) + \frac{M^4}{15r^4 N^2}, \]  
\[ g_2^*(r) = \frac{3\mu \Omega r}{8c^2 M^2 S^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right], \]  
\[ g_4^*(r) = -\frac{3\mu \omega r}{8c^2 M^2 S^2} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right], \]  
\[ g_5^* = \frac{1}{2} f_2^*, \quad g_6^* = \frac{1}{2} f_4^*. \]
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