MILNOR’S ISOSPECTRAL TORI AND HARMONIC MAPS

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ABSTRACT. A well-known question asks whether the spectrum of the Laplacian on a Riemannian manifold \((M, g)\) determines the Riemannian metric \(g\) up to isometry. A similar question is whether the energy spectrum of all harmonic maps from a given Riemannian manifold \((\Sigma, h)\) to \(M\) determines the Riemannian metric on the target space. We consider this question in the case of harmonic maps between flat tori. In particular, we show that the two isospectral, non-isometric 16-dimensional flat tori found by Milnor cannot be distinguished by the energy spectrum of harmonic maps from \(d\)-dimensional flat tori for \(d \leq 3\), but can be distinguished by certain flat tori for \(d \geq 4\). This is related to a property of the Siegel theta series in degree \(d\) associated to the 16-dimensional lattices in Milnor’s example.

1. INTRODUCTION

Let \((\Sigma, h)\) and \((M, g)\) be smooth, connected Riemannian manifolds, where \(\Sigma\) is closed and oriented (in the following all manifolds are assumed smooth and connected). The Dirichlet energy of a smooth map \(f: \Sigma \to M\) is defined by

\[
E[f] = \frac{1}{2} \int_{\Sigma} |df|^2 \text{d}vol_h,
\]

where \(df: T\Sigma \to TM\) is the differential and \(|df|^2\) is the length-squared determined by the Riemannian metrics \(h\) and \(g\). Stationary points of the functional \(E[f]\) under variations of \(f\) are called harmonic maps \([6]\).

The energy spectrum of harmonic maps \(f: \Sigma \to M\), for fixed Riemannian metrics on both manifolds, is the set of critical values of the energy functional:

\[
\{E[f] \mid f: \Sigma \to M \text{ harmonic}\} \subset \mathbb{R}_{\geq 0}.
\]

The energy spectrum of harmonic maps for specific source and target spaces has been studied in \([1, 3, 22]\). In general the energy spectrum does not have to be a discrete subset of \(\mathbb{R}_{\geq 0}\) \([7, 17, 18]\). Depending on the situation one would also like to include the multiplicities of values in the energy spectrum, i.e. how often an energy value is attained by several (in a suitable sense) different harmonic maps.

Remark 1.1. In the case of \(\Sigma = S^1\) harmonic maps \(\gamma: S^1 \to M\) are precisely the closed geodesics in \(M\). The energy of a closed geodesic is related to its length \(L[\gamma]\) by \(E[\gamma] = \frac{1}{4\pi R} L[\gamma]^2\), where \(R\) is the radius of \(S^1\), because \(|\dot{\gamma}|\) is constant. The set

\[
\{L[\gamma] \mid \gamma: S^1 \to M \text{ closed geodesic}\} \subset \mathbb{R}_{\geq 0}
\]

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together with multiplicities is known as the length spectrum of $(M, g)$. Here the multiplicity of a length $l$ is defined as the number of free homotopy classes of closed loops $\gamma: S^1 \rightarrow M$ that contain a closed geodesic of length $l$ (see e.g. [10]).

If we assume that a precise notion of energy spectrum for harmonic maps is given, we can ask the following question.

**Question.** Let $(M, g)$ and $(M', g')$ be Riemannian manifolds which are not isometric. Does there exist a closed, oriented Riemannian manifold $(\Sigma, h)$, such that the energy spectrum of harmonic maps $\Sigma \rightarrow M$ and $\Sigma \rightarrow M'$ is different?

This question is similar to the question whether "one can hear the shape of a drum", i.e. whether the spectrum of eigenvalues of the Laplacian on functions on a Riemannian manifold determines the metric up to isometry [14, 11].

In the particular case where both the source and target manifold are flat tori, the notion of energy spectrum can be given a precise meaning. We consider flat tori $(T^m = \mathbb{R}^m/\Lambda_m, g_m)$ and $(T^n = \mathbb{R}^n/\Lambda_n, g_n)$ for lattices $\Lambda_m$ and $\Lambda_n$ (without loss of generality we always use the Riemannian metric induced from the standard Euclidean scalar product on $\mathbb{R}^m$ and $\mathbb{R}^n$). It is well-known that a map $f: T^m \rightarrow T^n$ is harmonic if and only if it is affine (see Section 2 for more details). Affine maps are of the form

$$f_{C,s}: T^m \rightarrow T^n$$

$$[x] \mapsto f([x]) = [Cx + s]$$

where $C \in \mathbb{R}^{n \times m}$ with $C\Lambda_m \subset \Lambda_n$ and $s \in \mathbb{R}^n$. Since the differential acts by multiplication with the constant matrix $C$, the energy of the harmonic map $f = f_{C,s}$ is equal to

$$E[f] = \frac{1}{2}||C||^2 \text{vol}_{g_m}(T^m),$$

(1.2)

where $||C||^2 = \text{Tr}(C^tC)$ and $\text{Tr}$ denotes the trace, $t$ the transpose.

The set of affine maps between the tori can be identified with

$$\text{Hom}_\mathbb{Z}(\Lambda_m, \Lambda_n) \times T^n \cong \mathbb{Z}^{n \times m} \times T^n.$$

There is a unique affine and hence harmonic map in each homotopy class of maps from $T^m$ to $T^n$, up to translations (a map to $T^n$ is determined up to homotopy by the map $f_*$ on first integral homology, corresponding to an integral matrix of winding numbers). The translations on $T^n$ are isometries and do not change the energy of the map. We can thus define

**Definition 1.2.** The energy spectrum

$$\{ E[f] \mid f: T^m \rightarrow T^n \text{ harmonic} \}$$

of harmonic maps between flat tori is a countable subset of $\mathbb{R}_{\geq 0}$. The multiplicity of a value $E$ in the energy spectrum is defined as the number of homotopy classes of harmonic maps $f: T^m \rightarrow T^n$ with $E[f] = E$. This includes the case of multiplicity 0 if $E$ does not occur in the energy spectrum.

1In [19] the corresponding question for a spectrum of immersed totally geodesic surfaces in Riemannian manifolds has been studied.
Remark 1.3. This can be generalized to harmonic maps $f: \Sigma \rightarrow T^n$ from any closed, oriented Riemannian manifold $(\Sigma, h)$; cf. Section 2.

As an explicit example we consider the 16-dimensional isospectral tori constructed by Milnor [20]. The tori are given by

$$T_{8,8} = \mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8$$
$$T_{16} = \mathbb{R}^{16}/\Gamma_{16}$$

where $\Gamma_8 \oplus \Gamma_8$ and $\Gamma_{16}$ are certain integral even unimodular lattices in $\mathbb{R}^{16}$ (with the restriction of the standard Euclidean scalar product). These lattices are also denoted by $E_8 \oplus E_8$ and $D_{16}^+$. The flat tori $T_{8,8}$ and $T_{16}$ are non-isometric, but the spectra of the Laplacian on functions and differential forms (with multiplicities) are the same.

Let $(\Lambda, Q)$ be an even, positive definite, unimodular lattice of rank $m$ with integral quadratic form $Q$. The Siegel upper half space of degree $d$ is defined by

$$H_d = \left\{ Z = X + iY \mid X, Y \in \mathbb{R}^{d\times d} \text{ symmetric}, Y \text{ positive definite} \right\} \subset \mathbb{C}^{d\times d}.$$ 

For a $d$-tuple of elements $x = (x_1, \ldots, x_d) \in \Lambda$ define the matrix

$$Q(x) = (Q(x_i, x_j))_{i,j=1,\ldots,d} \in \mathbb{Z}^{d\times d}.$$ 

The theta series [8, 15] of degree $d$ associated to the lattice $\Lambda$ is

$$\Theta^{(d)}_\Lambda(Z) = \sum_{x \in \Lambda^d} e^{\pi i \text{Tr}(Q(x)Z)}$$

where $Z \in H_d$. The theta series is a Siegel modular form of degree $d$ and weight $\frac{m}{2}$. It has a Fourier expansion

$$\Theta^{(d)}_\Lambda(Z) = \sum_{T \in P_d} r_\Lambda(T) e^{\pi i \text{Tr}(TZ)},$$

with

$$P_d = \left\{ T \in \mathbb{Z}^{d\times d} \mid T \text{ symmetric, positive semi-definite, even} \right\},$$

where an integral matrix is called even if all of its diagonal elements are even. The Fourier coefficients are the representation numbers

$$r_\Lambda(T) = \# \left\{ x \in \Lambda^d \mid Q(x) = T \right\}.$$ 

It turns out that the representation numbers and thus the theta series of degree $d$ for the lattices $\Gamma_8 \oplus \Gamma_8$ and $\Gamma_{16}$ are closely related to the energy spectrum of harmonic maps from flat tori $T^d$ to $T_{8,8}$ and $T_{16}$, respectively.

**Proposition 1.4.** If $\Theta^{(d)}_{\Gamma_8 \oplus \Gamma_8} = \Theta^{(d)}_{\Gamma_{16}}$ for an integer $d \in \mathbb{N}$, then the energy spectrum (including multiplicities) of harmonic maps from any given flat torus $T^d$ to the tori $T_{8,8}$ and $T_{16}$ are the same.
It is known that
\[ \Theta^{(d)}_{\Gamma_8 \oplus \Gamma_8} = \Theta^{(d)}_{\Gamma_{16}}, \quad d = 1, 2, 3 \]
\[ \Theta^{(d)}_{\Gamma_8 \oplus \Gamma_8} \neq \Theta^{(d)}_{\Gamma_{16}}, \quad d \geq 4. \]

For \( d = 1 \) this follows from the fact that the dimension of the space \( M_8 \) of modular forms of degree 1 and weight 8 is equal to 1. The case \( d = 2 \) was proved by Witt \([23]\) in 1941 (this is the article that Milnor referred to in \([20]\)) and Witt also claimed the result for \( d = 4 \) and conjectured the case \( d = 3 \) (known as ”problem of Witt”). The case \( d \geq 3 \) was proved independently by Igusa \([13]\) and Kneser \([16]\) in 1967.

Using these results we show:

**Theorem 1.5.** Consider the energy spectrum of harmonic maps from flat tori \( T^d \) to \( T_{8,8} \) and \( T_{16} \).

1. For any given flat torus \( T^d \) of dimension \( d = 1, 2, 3 \) the energy spectrum (including multiplicities) for \( T_{8,8} \) and \( T_{16} \) are the same.
2. In every dimension \( d \geq 4 \) there exists a flat torus \( T^d \) and an energy \( E \in \mathbb{R}_{\geq 0} \) whose multiplicities in the energy spectrum for \( T_{8,8} \) and \( T_{16} \) are different.

The case \( d = 1 \) (the length spectrum of \( T_{8,8} \) and \( T_{16} \) are the same) was known before and implies that \( T_{8,8} \) and \( T_{16} \) are isospectral for the Laplacian.

### 2. Harmonic Maps Between Flat Tori

Let \((T^m = \mathbb{R}^m / \Lambda_m, g_m)\) and \((T^n = \mathbb{R}^n / \Lambda_n, g_n)\) be flat tori. The following is well-known (see \([6\), p. 129], \([7]\), \([21]\)).

**Proposition 2.1.** A map \( f : T^m \rightarrow T^n \) is harmonic if and only if it is affine.

For completeness we give a proof.

**Proof.** The proof is almost verbatim to the the case of holomorphic maps between complex tori \([12\), p. 325]. Any harmonic map \( f : T^m \rightarrow T^n \) lifts to a harmonic map \( \tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n \) with
\[
\tilde{f}(x + \lambda) - \tilde{f}(x) \in \Lambda_n \quad \forall x \in \mathbb{R}^m, \lambda \in \Lambda_m.
\]

For a given vector \( \lambda \in \Lambda_m \), consider the map
\[
\tilde{f}_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n
\]
\[
x \mapsto \tilde{f}(x + \lambda) - \tilde{f}(x).
\]

Then \( \tilde{f}_\lambda \) is a smooth map with image in the lattice \( \Lambda_n \), hence constant. It follows that
\[
\partial_{x_k} \tilde{f}(x + \lambda) = \partial_{x_k} \tilde{f}(x) \quad \forall x \in \mathbb{R}^m, \lambda \in \Lambda_m,
\]
for each \( k = 1, \ldots, m \), hence \( \partial_{x_k} \tilde{f} \) descends to a harmonic map \( T^m \rightarrow \mathbb{R}^n \). Since \( T^m \) is compact, this map has to be constant by the maximum principle, which implies that \( \tilde{f} \) is affine. \( \square \)
Remark 2.2. This result can be generalized: Let $(\Sigma, h)$ be a closed, oriented Riemannian manifold and

$$a: \Sigma \to A(\Sigma) = H_1(\Sigma; \mathbb{R})/H'_1(\Sigma; \mathbb{Z}) \cong T^{h_1(\Sigma)}$$

the Albanese map (the prime indicates the image of integral homology in real homology). In [21] it is shown that every harmonic map $f: \Sigma \to T^n$ factors through $a$:

$$\Sigma \xrightarrow{a} A(\Sigma) \xrightarrow{\phi} T^n$$

where $\phi$ is a uniquely determined affine map. Conversely, for every affine map $\phi: A(\Sigma) \to T^n$ the map $f = \phi \circ a$ is harmonic.

3. Milnor’s Example of Two Isospectral 16-Dimensional Tori

Let $L_n \subset \mathbb{Z}^n$ be the lattice defined by

$$L_n = \{ z \in \mathbb{Z}^n \mid \sum_{i=1}^n z_i \text{ is even} \}$$

and $\Gamma_n \subset \mathbb{R}^n$ the lattice generated by $L_n$ and the vector

$$\left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \in \frac{1}{2}\mathbb{Z}^n.$$

We consider the 16-dimensional lattices (see [20] and the discussion in [2, 4])

$$\Gamma_8 \oplus \Gamma_8 \subset \mathbb{R}^8 \oplus \mathbb{R}^8 = \mathbb{R}^{16}$$
$$\Gamma_{16} \subset \mathbb{R}^{16}.$$

With the bilinear form $Q$ induced from the standard Euclidean scalar product $\langle , \rangle$ on $\mathbb{R}^{16}$ both lattices are integral and even, i.e.

$$\langle w, z \rangle \in \mathbb{Z}$$
$$|w|^2 = \langle w, w \rangle \in 2\mathbb{N} \ (w \neq 0)$$

for all vectors $w, z$ in one of the lattices.

For a lattice $\Gamma$ equal to either $\Gamma_8 \oplus \Gamma_8$ or $\Gamma_{16}$ and $T$ given by the associated flat torus

$$T_{8,8} = \mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8$$
$$T_{16} = \mathbb{R}^{16}/\Gamma_{16}$$

consider a harmonic map

$$f_{C,s}: T^d \to T$$

$$[x] \mapsto f([x]) = [Cx + s]$$

from a flat torus $T^d = \mathbb{R}^d/\Lambda_d$, where $C \in \mathbb{R}^{16\times d}$ with $C\Lambda_d \subset \Gamma$ and $s \in \mathbb{R}^{16}$. Let $(b_1, \ldots, b_d)$ be a fixed basis of $\Lambda_d$ and $\gamma_k = Cb_k$ the images in $\Gamma$. Writing

$$b = (b_1, \ldots, b_d) \in \text{GL}(d, \mathbb{R})$$
$$\gamma = (\gamma_1, \ldots, \gamma_d) \in \Gamma^d \subset \mathbb{R}^{16\times d}$$
for the matrices of column vectors we have \( \gamma =Cb \) and 
\[ C^tC = (b^t)^{-1} \gamma b^{-1} = (b^t)^{-1} Q(\gamma) b^{-1}. \]
By equation (1.2) the energy of the harmonic map \( f = f_{C,a} \) is given by
\[ E[f] = \frac{1}{2} \text{Tr} \left( Q(\gamma) (b^t b)^{-1} \right) \det(b). \]
Since \( C \) determines \( \gamma \) and vice versa, the multiplicity of harmonic maps \( T^d \to T \) of energy \( E \) is equal to
\[ \# \left\{ \gamma \in \Gamma^d \mid E = \frac{1}{2} \text{Tr} \left( Q(\gamma) (b^t b)^{-1} \right) \det(b) \right\}. \] (3.1)
We deduce the following:

**Lemma 3.1.** For any given flat torus \( T^d = \mathbb{R}^d/\Lambda_d \), with basis \( b \) of \( \Lambda_d \), and any energy \( E \in \mathbb{R}_{\geq 0} \) the multiplicity of harmonic maps \( T^d \to T \) with energy \( E \) is equal to
\[ \# \left\{ \gamma \in \Gamma^d \mid E = \frac{1}{2} \text{Tr} \left( Q(\gamma) (b^t b)^{-1} \right) \det(b) \right\} = \sum_{S \in Q(E)} r_{\Gamma}(S), \]
where
\[ Q(E) = \{ S \in \mathcal{P}_d \mid E = \frac{1}{2} \text{Tr}(S(b^t b)^{-1}) \det(b) \}. \]

The set \( Q(E) \) depends only on the lattice \( \Lambda_d \) and not on \( \Gamma \), hence Proposition 1.4 follows.

**Proof of Theorem 1.5.** Claim (1) follows from Proposition 1.4 and the results of Witt, Igusa and Kneser for \( d \leq 3 \) mentioned above.

For claim (2) it is known [23, p. 325], [13, p. 854], [5] that in the case \( d = 4 \)
\[ r_{\Gamma_8 \oplus \Gamma_8}(S) \neq r_{\Gamma_16}(S) \] (3.2)
for the diagonal matrix
\[ S = \text{diag}(2, 2, 2, 2). \]
This then also holds for all \( d > 4 \) for
\[ S = \text{diag}(2, 2, 2, 0, \ldots, 0). \]
We consider the case \( d = 4 \). Let
\[ M = \begin{pmatrix}
1 & \frac{1}{\pi} & \frac{1}{\pi^2} & \frac{1}{\pi^3} \\
\frac{1}{\pi} & 1 & \frac{1}{\pi} & \frac{1}{\pi^2} \\
\frac{1}{\pi^2} & \frac{1}{\pi} & 1 & \frac{1}{\pi} \\
\frac{1}{\pi^3} & \frac{1}{\pi^2} & \frac{1}{\pi^2} & 1
\end{pmatrix}. \]
It is easy to check that \( M \) is positive definite. According to the Cholesky decomposition there exists a unique upper triangular matrix \( b \in \text{GL}(4, \mathbb{R}) \) such that
\[ M^{-1} = b^t b. \]
The matrix \( b \) defines a basis \( b_1, b_2, b_3, b_4 \) of \( \mathbb{R}^4 \). Let \( \Lambda_4 \) be the integral lattice spanned by these vectors and \( T^4 = \mathbb{R}^4/\Lambda_4 \) the associated flat torus.
Denote by \( \Gamma \) the lattices \( \Gamma_8 \oplus \Gamma_8 \) and \( \Gamma_{16} \). For a 4-tuple \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma^4 \) we have

\[
\text{Tr}(Q(\gamma)(b^t b)^{-1}) = \text{Tr}(Q(\gamma)M)
\]

\[
= |\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 + |\gamma_4|^2
\]

\[
+ 2 \left( \frac{1}{\pi} \langle \gamma_1, \gamma_2 \rangle + \frac{1}{\pi} \langle \gamma_1, \gamma_3 \rangle + \frac{1}{\pi} \langle \gamma_1, \gamma_4 \rangle \right)
\]

\[
+ \frac{1}{\pi} \langle \gamma_2, \gamma_3 \rangle + \frac{1}{\pi} \langle \gamma_2, \gamma_4 \rangle + \frac{1}{\pi} \langle \gamma_3, \gamma_4 \rangle .
\]

Consider the energy

\[
E = \frac{1}{2} \cdot 8 \cdot \det(b).
\]

A harmonic map \( f_{C, \gamma} : T^1 \to T \), determined by \( C \) or equivalently \( \gamma \), has energy \( E \) if any only if

\[
8 = \text{Tr}(Q(\gamma)(b^t b)^{-1}).
\]

For the chosen basis this is happens if and only if

\[
|\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 + |\gamma_4|^2 = 8
\]

\[
\langle \gamma_i, \gamma_j \rangle = 0 \quad \forall i \neq j,
\]

because \( \langle \cdot, \cdot \rangle \) is integral on \( \Gamma \). The second equation means that \( Q(\gamma) \) has to be diagonal. If all lattice vectors \( \gamma_i \) are non-zero, equation (3.3) is equivalent to

\[
|\gamma_1|^2 = |\gamma_2|^2 = |\gamma_3|^2 = |\gamma_4|^2 = 2,
\]

because the lattice is even and positive definite. By the result mentioned at the beginning of the proof, the number of 4-tuples \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \) that satisfy equation (3.5) and (3.4) is different for \( \Gamma_8 \oplus \Gamma_8 \) and \( \Gamma_{16} \).

If some of the \( \gamma_i \) are zero, there are several possibilities, for example,

\[
|\gamma_1|^2 = 4, \quad |\gamma_2|^2 = |\gamma_3|^2 = |\gamma_4|^2 = 0.
\]

However, the number of 4-tuples of this type is the same for both lattices, because \( \Theta_{(d)}^{(d)} = \Theta_{(d)}^{(d)} \) for \( d \leq 3 \).

It follows that the total number of 4-tuples \( \gamma \) that satisfy (3.3) and (3.4) is different for \( \Gamma_8 \oplus \Gamma_8 \) and \( \Gamma_{16} \). This proves the claim with (3.1). \( \square \)

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