Training Convolutional ReLU Neural Networks in Polynomial Time: Exact Convex Optimization Formulations

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Abstract

We study training of Convolutional Neural Networks (CNNs) with ReLU activations and introduce exact convex optimization formulations with a polynomial complexity with respect to the number of data samples, the number of neurons and data dimension. Particularly, we develop a convex analytic framework utilizing semi-infinite duality to obtain equivalent convex optimization problems for several CNN architectures. We first prove that two-layer CNNs can be globally optimized via an $\ell_2$ norm regularized convex program. We then show that certain three-layer CNN training problems are equivalent to an $\ell_1$ regularized convex program. We also extend these results to multi-layer CNN architectures. Furthermore, we present extensions of our approach to different pooling methods.

1 Introduction

Convolutional Neural Networks (CNNs) have shown a remarkable success across various machine learning problems [1]. However, our theoretical understanding of CNNs still remains restricted, where the main challenge arises from the highly non-convex and nonlinear structure of CNNs with nonlinear activations such as Rectified Linear Units (ReLU). To this end, we study the training problem for various CNN architectures with ReLU activations and introduce equivalent finite dimensional convex formulations that can be used to globally optimize these architectures. Our results characterize the role of network architecture in terms of equivalent convex regularizers. Remarkably, we prove that the proposed methods are polynomial time with respect to all problem parameters.

Convex neural network training was previously considered in [2, 3]. However, these studies are restricted to two-layer fully connected networks with infinite width, thus, the optimization problem involves infinite dimensional variables. Moreover, it has been shown that even adding a single neuron to a neural network leads to a non-convex optimization problem which cannot be solved efficiently [3]. Another line of research [4–11] focuses on the effect of implicit and explicit regularization in neural network training and aims to explain why the resulting network generalizes well. Among these studies, [4–8] proved that the minimum $\ell_2$ norm two-layer network that perfectly fits a one dimensional dataset outputs the linear spline interpolation. Moreover, [7] studied certain linear convolutional networks and showed an implicit non-convex quasi-norm regularization. However, as the number of layers increases, the regularization approaches to $\ell_0$ quasi-norm, which is not computationally tractable. Recently, [8] showed that two-layer CNNs with linear activations can be equivalently optimized as nuclear and $\ell_1$ norm regularized convex problems. Although all the norm characterizations provided by these studies are insightful for future research, existing results are quite restricted due to linear activations, simple settings or intractable optimization
problems.

**Shallow CNNs and their representational power:** As opposed to their relatively simple and shallow architecture, CNNs with two or three layers are very powerful and efficient inference models. In [12], the authors show that greedy training of two or three layer CNNs can achieve comparable performance to deeper models such as VGG-11[13]. However, a full theoretical and interpretable understanding of CNNs even with a single hidden layer is lacking in the literature.

**Our contributions:** Unlike previous studies, we introduce exact and finite dimensional convex programs to globally optimize various CNN architecture through our convex analytic framework utilizing semi-infinite duality. Our contributions can be summarized as follows:

- We develop convex programs that are polynomial time with respect to all input parameters: the number of samples, data dimension, and the number of neurons to globally train CNNs. To the best of our knowledge, this is the first work characterizing polynomial time trainability of non-convex CNN models. More importantly, we achieve this complexity with explicit and interpretable convex optimization problems. Consequently, training CNNs, especially in practice, can be further accelerated by leveraging extensive tools available from convex optimization theory.

- Our work reveals a hidden regularization mechanism behind CNNs and characterizes how the architecture and pooling strategies, e.g., max-pooling, average pooling, and flattening, dramatically alter the regularizer. As we show, ranging from $\ell_1$ and $\ell_2$ norm to nuclear norm (see Table 1 for details), ReLU CNNs exhibit an extremely rich and elegant regularization structure which is implicitly enforced by architectural choices. In convex optimization and signal processing, $\ell_1$, $\ell_2$ and nuclear norm regularizations are well studied, where these structures have been applied in compressed sensing, inverse problems and matrix completion. Our results bring light to unexplored and promising connections of ReLU CNNs with these established disciplines.

**Notation and preliminaries:** We denote the matrices and vectors as uppercase and lowercase bold letters, respectively, for which a subscript indicates a certain element or column. We use $I_k$ to denote the identity matrix of size $k$. We denote the set of integers from 1 to $n$ as $[n]$. The Frobenius norm and nuclear norms are denoted as $\| \cdot \|_F$ and $\| \cdot \|_*$. We use $B_p$ to denote the unit $\ell_p$ ball, i.e., $B_p := \{ u \in \mathbb{C}^d : \|u\|_p \leq 1 \}$. We also use $1[x]$ as a function that outputs 1 if $x$ is true, 0 otherwise.

In order to keep the presentation and notation simple, we will use a regression framework with scalar outputs and squared loss. However, we also note that all of our results can be extended to vector outputs and arbitrary convex regression and classification loss functions. We present these extensions in Appendix. In our regression framework, we denote the input data matrix and the corresponding label vector as $X \in \mathbb{R}^{n \times d}$ and $y \in \mathbb{R}^n$, respectively. Moreover, we represent the patch matrices, i.e., subsets of columns, extracted from $X$ as $X_k \in \mathbb{R}^{n \times h}$, $k \in [K]$, where $h$ denotes the filter size. With this notation, $\{X_k u \}_{k=1}^K$ describes a convolution operation between the filter $u \in \mathbb{R}^h$ and the data matrix $X$. Throughout the paper, we will use the ReLU activation function defined as $\text{ReLU}(x) = \max\{0, x\}$. However, since CNN training problems with ReLUs are not convex in their conventional form, below we introduce an alternative formulation for this activation, which will be crucial for our derivations.

### 1.1 Hyperplane arrangements

Let $H$ be the set of all hyperplane arrangement patterns of $X$, defined as the following set

$$H := \bigcup \{ \{ \text{sign}(Xw) \} : w \in \mathbb{R}^d \},$$

Table 1: CNN architectures and the corresponding norm regularization in our convex programs

| Architecture | 2-layer | 2-layer | 2-layer | 3-layer | $L$-layer |
|-------------|---------|---------|---------|---------|-----------|
| Implicit Regularization | $\sum_{j,k} (X_k u_k, w_j) + \|X_u \|_F$ | $\sum_{j,k} \maxpool (X_k u_k, w_j) + \|X_u \|_F$ | $\sum_{j,k} (X_k u_k, w_j) + \|X_u \|_F$ | $\sum_{j,k} (X_k u_k, w_j) + \|X_u \|_F$ | $\sum_{j,k} (X_k u_k, w_j) + \|X_u \|_F$ |

1The results on two-layer CNNs are presented in Appendix A.4
2This refers to an $L$-layer network with only one ReLU layer. We discuss this limitation in Appendix.
which has finitely many elements, i.e., \(|H| \leq N_H < \infty, N_H \in \mathbb{N}\). We now define a collection of sets that correspond to positive signs for each element in \(H\), by \(S := \{\cup_{h_{i=1}}^{\infty}\} : h \in H\). We extend ReLU to an elementwise function that masks the negative entries of a vector or matrix. Hence, given a set \(S \subseteq \mathbb{R}\), we define a diagonal mask matrix \(D(S) \in \mathbb{R}^{n \times n}\) as follows

\[
D(S)_i := \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{otherwise}
\end{cases}.
\]

Then, we can equivalently represent \((Xw)_+\) as \(D(S)Xw\) provided that \(D(S)Xw \geq 0\) and \((I_n - D(S))Xw \leq 0\). Note that these constraints can be compactly defined as \((2D(S) - I_n)Xw \geq 0\). If we denote the cardinality of \(S\) as \(P\), i.e., the number of regions in a partition of \(\mathbb{R}^d\) by hyperplanes passing through the origin and are perpendicular to the rows of \(X\), then

\[
P \leq 2 \sum_{k=0}^{r-1} \binom{n-1}{k} \leq 2r \left(\frac{e(n-1)}{r}\right)^r
\]

for \(r \leq n\), where \(r := \text{rank}(X) \leq d\) [14-17] (see Appendix A.2 for details).

### 1.2 Convolutional hyperplane arrangements

We now define a notion of hyperplane arrangements for CNNs, where we introduce the patch matrices \(\{X_k\}_{k=1}^K\) instead of directly operating on \(X\). We first construct a new data matrix as \(M = [X_1; X_2; \ldots; X_K] \in \mathbb{R}^{nK \times h}\). We then define **convolutional hyperplane arrangements** as the hyperplane arrangements for \(M\) and denote the cardinality of this set as \(P_{\text{conv}}\). Then, we have

\[
P_{\text{conv}} \leq 2 \sum_{k=0}^{r_e-1} \binom{nK - 1}{k} \leq 2r_e \left(\frac{e(nK - 1)}{r_e}\right)^{r_e}
\]

where \(r_e := \text{rank}(M) \leq h\) and \(K = \left\lfloor \frac{d}{\text{stride}} \right\rfloor + 1\). Note that when the filter size \(h\) is fixed, \(P_{\text{conv}}\) is polynomial in \(n\) and \(d\). Similarly, we consider hyperplane arrangements for circular CNNs followed by a linear pooling layer, i.e., in the form \(XUw\), where \(U \in \mathbb{R}^{h \times d}\) is a circulant matrix generated by the elements \(u \in \mathbb{R}^{h}\). Then, we define **circular convolutional hyperplane arrangements** and denote the cardinality of this set as \(P_{\text{cconv}}\), which is exponential in the rank of the circular patch matrices, i.e., denoted as \(r_{c}\).

**Remark 1.1.** There exist \(P\) hyperplane arrangements of \(X\) where \(P\) is exponential in \(r\). Thus, if \(X\) is full rank, \(r = d\), then \(P\) can be exponentially large in the dimension \(d\). As it will be shown, this makes the training problem for fully connected networks challenging. On the other hand, for CNNs, the number of relevant hyperplane arrangements \(P_{\text{conv}}\) is exponential in \(r_e\). If \(M\) is full rank, then \(r_e = h \ll d\) and accordingly \(P_{\text{conv}} \ll P\). This shows that the parameter sharing structure in CNNs enables a significant reduction in the number of possible hyperplane arrangements. As shown in the sequel and Table 3 our results imply that the complexity of training problem is significantly lower compared to fully connected networks.
2 Two-layer CNNs

In this section, we present exact convex formulation for two-layer CNN architectures.

2.1 Two-layer CNNs with average pooling

We first consider an architecture with \( m \) filters, average pooling\(^3\) and standard weight decay regularization, which can be trained via the following problem

\[
\begin{align*}
    p_1^* &= \min_{\{u_j, w_j\}_{j=1}^m} \frac{1}{2} \left\| \sum_{j=1}^m \sum_{k=1}^K (X_k u_j)_+ \ w_j - y \right\|^2_2 + \frac{\beta}{2} \sum_{j=1}^m (\|u_j\|^2 + w_j^2),
\end{align*}
\]

(1)

where \( u_j \in \mathbb{R}^h \) and \( w \in \mathbb{R}^m \) are the filter and output weights, respectively, and \( \beta > 0 \) is a regularization parameter. After a rescaling (see Appendix A.3), we obtain the following problem

\[
\begin{align*}
    p_1' &= \min_{u_j, w_j} \frac{1}{2} \left\| \sum_{j=1}^m \sum_{k=1}^K (X_k u_j)_+ \ w_j - y \right\|^2_2 + \frac{\beta}{2} \sum_{j=1}^m \|w_j\|_1.
\end{align*}
\]

(2)

Then, taking dual with respect to \( w \) and changing the order of min-max yields the weak dual

\[
\begin{align*}
    p_1^* \geq d_1^* &= \max_v -\frac{1}{2}\|v - y\|_2^2 + \frac{1}{2}\|y\|_2^2 \text{ s.t. } \max_{u_j, w_j} \sum_{k=1}^K v^T (X_k u_j)_+ \leq \beta.
\end{align*}
\]

(3)

which is a semi-infinite optimization problem with \( n \) variables. The dual of (3) can be obtained as a finite dimensional convex program using semi-infinite optimization theory \[18\]. The same dual also corresponds to the bidual of (1). Suprisingly, strong duality holds as soon as the number of neurons exceed a critical value. In the sequel, we use this result to derive an exact convex formulation for (1).

**Theorem 2.1.** Let \( m \) be a number such that \( m \geq m^* \) for some \( m^* \in \mathbb{N} \), \( m^* \leq n + 1 \), then strong duality holds for (3), i.e., \( p_1^* = d_1^* \), and the equivalent convex program for (1) is

\[
\begin{align*}
    \min_{\{w_i, w_j\}_{i,j=1}^{P_{\text{conv}}}} \frac{1}{2} \left\| \sum_{i=1}^{P_{\text{conv}}} \sum_{k=1}^K D(S_k^i) X_k (w_i' - w_i) - y \right\|^2_2 + \frac{\beta}{2} \sum_{i=1}^{P_{\text{conv}}} (\|w_i\|_2 + \|w_i'\|_2) \\
    \text{s.t. } (2D(S_k^i) - I_n) X_k w_i \geq 0, (2D(S_k^i) - I_n) X_k w_i' \geq 0, \forall i, k.
\end{align*}
\]

(4)

Moreover, an optimal solution to (1) with \( m^* \) filters can be constructed from (4) as follows

\[
\begin{align*}
    u_j^* &= \begin{cases} 
    \frac{w_j'}{\sqrt{\|w_j'\|_2}}, & w_j^* = \sqrt{\|w_j'\|_2} \quad \text{if } \|w_j'\|_2 > 0 \\
    \frac{w_j'}{\sqrt{\|w_j'\|_2}}, & w_j^* = -\sqrt{\|w_j'\|_2} \quad \text{otherwise},
    \end{cases}
\end{align*}
\]

where \( \{w_i', w_i^*\}_{i=1}^{P_{\text{conv}}} \) are the optimal solutions to (4), and at most one of \( w_i' \) or \( w_i^* \) is non-zero for all \( i = 1, \ldots, P_{\text{conv}} \). Here, we have \( m^*: = \sum_{i=1}^{P_{\text{conv}}} 1(\|w_i'\|_2 \neq 0 \text{ or } \|w_i^*\|_2 \neq 0) \).\(^4\)

Therefore, we obtain a finite dimensional convex formulation with \( 2hP_{\text{conv}} \) variables and \( 2nP_{\text{conv}}K \) constraints for the non-convex problem in (1). Since \( P_{\text{conv}} \) is polynomial in \( n \) and \( d \) given a fixed \( r_c \leq h \), (4) can be solved by a standard convex optimization solver in polynomial time.

**Remark 2.1.** Table\(^\text{3}4\) shows that for fixed rank \( r_c \), or fixed filter size \( h \), the complexity is polynomial in all problem parameters: \( n \) (number of samples), \( m \) (number of filters, i.e., neurons), and \( d \) (dimension). The filter size \( h \) is typically a small constant, e.g., \( h = 9 \) for \( 3 \times 3 \) filters. We also note that for fixed \( n \) and rank(\( X \)) = \( d \), the complexity of fully connected networks is exponential in \( d \), which cannot be improved unless \( P = NP \) even for \( m = 2 \). However, this result shows that CNNs can be trained to global optimality with polynomial complexity as a convex program.

\(^3\)We define the average pooling operation as \( \sum_{k=1}^K (X_k u_k)_+ \), which is also known as global average pooling.

\(^4\)Since our proof technique is similar for different CNNs, we present only the proof of Theorem 2.1 in Section 5. The rest of the proofs can be found in Appendix.
Interpreting non-convex CNNs as convex variable selection models: Interestingly, we have the sum of the squared $\ell_2$ norms of the weights in the non-convex problem (1) as the regularizer, however, the equivalent convex program in (4) is regularized by the sum of the $\ell_2$ norms of the weights. This particular regularizer is known as group $\ell_1$ norm, and is well-studied in the context of sparse recovery and variable selection [20,21]. Hence, our convex program reveals an implicit variable selection mechanism in the original non-convex problem in (4). More specifically, the original features in $X$ are mapped to higher dimensions via convolutional hyperplane arrangements as $\{D(S^k_i)X_k\}_{k=1}^P_{\text{conv}}$ and followed by a convex variable selection strategy using the group $\ell_1$ norm. In the following sections, we will show that this implicit regularization changes significantly with the CNN architecture and pooling strategies and can range from $\ell_1$ and $\ell_2$ norms to nuclear norm.

2.2 Two-layer CNNs with max pooling

Here, we consider the architecture with max pooling, which is trained via the following optimization problem after a rescaling

$$p^*_i = \min_{\{u_j, w_j\}_{j=1}^m} \frac{1}{2} \left( \sum_{j=1}^m \text{maxpool} \left( \{ (X_k u_j) \}_{k=1}^K \right) w_j - y \right)^2 + \beta \|w\|_1,$$

where maxpool$(\cdot)$ is an elementwise max function over the patch index $k$. Then, taking dual with respect to $w$ and changing the order of min-max yields

$$p^*_i \geq d^*_i = \max_v \left\{ -\frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|y\|_2^2 \right\} \text{ s.t. } \max_{u \in \mathbb{B}_2} \|v^T \text{maxpool} \left( \{ (X_k u) \}_{k=1}^K \right) \| \leq \beta. \quad (6)$$

**Theorem 2.2.** Let $m$ be a number such that $m \geq m^*$ for some $m^* \in \mathbb{N}$, $m^* \leq n + 1$, then strong duality holds for (6), i.e., $p^*_i = d^*_i$, and the equivalent convex program for (5) is

$$\min_{\{w_i, w'_i\}_{i=1}^n} \left( \sum_{i=1}^n \left( \sum_{k=1}^K D(S^k_i)X_k (w'_i - w_i) - y \right)^2 + \beta \sum_{i=1}^n (\|w_i\|_2 + \|w'_i\|_2) \right) \quad (7)$$

s.t. $(2D(S^k_i) - I_n)X_k w_i \geq 0$, $(2D(S^k_i) - I_n)X_k w'_i \geq 0$, $\forall i, k$, $D(S^k_i)X_k w_i \geq D(S^k_i)X_k w'_i \geq 0$, $\forall i, j, k$.

Moreover, an optimal solution to (5) can be constructed from (7) using the method in Theorem 2.1.

3 Three-layer circular CNNs

In this section, we consider three-layer circular CNNs, which can be trained via the following non-convex optimization problem

$$p^*_2 = \min_{u_j \in \mathbb{B}_2, \forall j} \left( \sum_{j=1}^m (XU_j w_{1j}) + w_{2j} - y \right)^2 + \frac{\beta}{2} \sum_{j=1}^m (\|w_{1j}\|_2 + \|w_{2j}\|_2), \quad (8)$$

where $U_j \in \mathbb{R}^{d \times d}$ is a circulant matrix generated by a circular shift modulo $d$ using the elements $u_j \in \mathbb{R}^h$ and we include unit norm constraints for filter weight vectors without loss of generality.

**Theorem 3.1.** Let $m$ be a number such that $m \geq m^*$ for some $m^* \in \mathbb{N}$, $m^* \leq n + 1$, then strong duality holds for (8), i.e., $p^*_2 = d^*_2$, and the equivalent convex program is

$$\min_{\{w_i, w'_i\}_{i=1}^n} \left( \sum_{i=1}^n D(S_i)\tilde{X}(w'_i - w_i) - y \right)^2 + \frac{\beta}{\sqrt{d}} \sum_{i=1}^n (\|w_i\|_1 + \|w'_i\|_1) \quad (9)$$

s.t. $(2D(S_i) - I_n)\tilde{X}w_i \geq 0$, $(2D(S_i) - I_n)\tilde{X}w'_i \geq 0$, $\forall i$, where $\tilde{X} = XF$ and $F \in \mathbb{C}^{d \times d}$ is the DFT matrix. Additionally, as in Theorem 2.1, we can construct an optimal solution to (8) from (9).\footnote{The details are presented in Appendix A.9}
Here, we again have sum of the squared $\ell_2$ norms in the non-convex problem (8) as the regularization, however, the equivalent convex program (9) is regularized by the sum of the $\ell_1$ norms. Thus, even with the same regularization in the non-convex problem, the architectural choice for a CNN determines a different implicit regularization structure revealed by our convex optimization approach.

4 Multi-layer circular CNNs

In this section, we consider the following $L$-layer circular CNN training problem

$$
P_3^* = \min_{\{u_i\}_{i=1}^{L}} \sum_{l=1}^{L} \left( \frac{1}{2} \sum_{j=1}^{m} (x_{l-1} U_{lj} w_{lj})^2 + w_{lj} = y \right) + \frac{\beta}{2} \sum_{j=1}^{m} (\|w_{lj}\|_2 + w_{lj}^2),
$$

(10)

where $U_{lj} \in \mathbb{R}^{d \times d}$ is a circulant matrix generated by a circular shift modulo $d$ using the elements $u_{lj} \in \mathbb{R}^{h_i}$ and $U_L := \{(u_1, \ldots, u_L) : u_i \in \mathbb{R}^{h_i}, \forall i \in [L]; \prod_{l=1}^{L} U_l \|p\|_F \leq 1\}$.

Theorem 4.1. Let $m$ be a number such that $m \geq m^*$ for some $m^* \in \mathbb{N}$, $m^* \leq n + 1$, then strong duality holds for (10), i.e., $p_3^* = d_3^*$, and the equivalent convex problem is

$$
\min_{\{w_i, w_i^\prime\}_{i=1}^{P_{conv}}} \frac{1}{2} \sum_{i=1}^{P_{conv}} D(S_i) \tilde{X} (w_i - w_i^\prime) - y \|2 + \beta \sum_{i=1}^{P_{conv}} (\|w_i\|_1 + \|w_i^\prime\|_1) \|2
$$

s.t. $(2D(S_i) - I_n) \tilde{X} w_i \geq 0$, $(2D(S_i) - I_n) \tilde{X} w_i^\prime \geq 0, \forall i$.

Moreover, as in Theorem 3.1, we can construct an optimal solution to (10) from (11).

5 Proof of the main result (Theorem 2.1)

We start with the following claim.

Proposition 5.1. Given $m \geq m^*$, strong duality holds for (3), i.e., $p_1^* = d_1^*$.

We now focus on the single-sided dual constraint

$$
\max_{u \in B_2} \sum_{k=1}^{K} v^T (X_k u) \leq \beta,
$$

(12)

which can be written as

$$
\max_{S^k \subseteq [n]} \max_{u \in B_2} \sum_{k=1}^{K} v^T D(S^k) X_k u \text{ s.t. } (2D(S^k) - I_n) X_k u \geq 0, \forall k.
$$

(13)

Since the inner maximization is convex and there exists a strictly feasible solution for fixed $D(S^k)$ matrices, (13) can also be written as

$$
\max_{S^k \subseteq [n]} \min_{\alpha_k \geq 0} \sum_{k=1}^{K} (v^T D(S^k) X_k + \alpha_k^T (2D(S^k) - I_n) X_k) u
$$

$$
= \max_{S^k \subseteq [n]} \min_{\alpha_k \geq 0} \sum_{k=1}^{K} v^T D(S^k) X_k + \alpha_k^T (2D(S^k) - I_n) X_k.
$$

The proof is presented in Appendix [A.7] where the definition of $m^*$ is given.
We note that the above problem is convex and strictly feasible for $v$. We now enumerate all hyperplane arrangements and index them in an arbitrary order, i.e., denoted $w$. Then, we apply a change of variables and define $\alpha$. By recalling Sion’s minimax theorem \cite{23} for the inner max-min, we express the strong dual of (16) as

$$\min_{\lambda, \lambda'_{ik} \geq 0} \max_{\alpha_{ik}, \alpha'_{ik} \geq 0} \frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta - \sum_{k=1}^{K} v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right)$$

$$+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_{i} \left( \beta - \sum_{k=1}^{K} -v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right).$$

Next, we introduce new variables $z_i, z'_{i} \in \mathbb{R}^h$ to represent (15) as

$$\min_{\lambda, \lambda'_{ik} \geq 0} \max_{\alpha_{ik}, \alpha'_{ik} \geq 0} \frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta + \sum_{k=1}^{K} v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right) z_i$$

$$+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_{i} \left( \beta + \sum_{k=1}^{K} -v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right) z'_{i},$$

which is concave in $v, \alpha_{ik}, \alpha'_{ik}$ and convex in $z_i$ and $z'_{i}$. Moreover, the set $B_2$ is convex and compact. By recalling Sion’s minimax theorem \cite{23} for the inner max-min, we express the strong dual of (16) as

$$\min_{\lambda, \lambda'_{ik} \geq 0} \max_{\alpha_{ik}, \alpha'_{ik} \geq 0} \frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta + \sum_{k=1}^{K} v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right) z_i$$

$$+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_{i} \left( \beta + \sum_{k=1}^{K} -v^T D(S^k_i) X_k + \alpha_{ik}^T (2D(S^k_i) - I_n) X_k \right) z'_{i}.$$
where we eliminate the variables $\lambda_i$, $\lambda_i'$, since $\lambda_i = \|w_i\|_2$ and $\lambda_i' = \|w_i'\|_2$ are feasible and optimal. We now note that there will be $m^*$ pairs $\{w_i', w_i^*\}, \forall i \in P_{\text{conv}}$, where at most one of $w_i'$ or $w_i^*$ is non-zero since only one side of $|\sum_{k=1}^K v^T(X_k u^*)_+| \leq \beta$ can be active at the optimum in (3).

Then, using the prescribed $\{u_j^*, w_j^*\}_{j=1}^{m^*}$, we evaluate the non-convex objective in (1) as follows

$$p^*_i \leq \frac{1}{2} \left( \sum_{j=1}^{m^*} \sum_{k=1}^K (X_k u_j^*)_+ w_j^* - y \right) \leq \beta 2 \left( \sum_{j=1, w_j^* \neq 0}^{m^*} \left( \frac{w_j^*}{\|w_j^*\|_2} \right)^2 + \left( \frac{\sqrt{\|w_j^*\|_2}}{\|w_j^*\|_2} \right)^2 \right) + \beta 2 \left( \sum_{j=1, w_j^* \neq 0}^{m^*} \left( \frac{w_j^*}{\|w_j^*\|_2} \right)^2 + \left( \frac{\sqrt{\|w_j^*\|_2}}{\|w_j^*\|_2} \right)^2 \right)$$

which has the same objective value with (19). Since strong duality holds for the convex program, we have $p^*_i = d^*_i$, which is equal to the value of (19) achieved by the prescribed parameters above. □

### 6 Numerical experiments

In this section, we present numerical experiments to verify our results in the previous sections. We first consider an experiment with a synthetic dataset, where $(n, d) = (6, 20)$, $X \in \mathbb{R}^{d \times 20}$ is generated using a multivariate normal distribution with zero mean and identity covariance, and $y = (1 - 1, 1 - 1, 1, 1)^T$. We then train the three-layer circular CNN model in (8) on this dataset using SGD and the convex program (9). In Figure 1, we plot the regularized objective value with respect to the computation time. Here, we plot 5 different independent realizations for SGD. We also plot both the non-convex objective in (8) and the convex objective in (9) for our convex program, where we use a specific rescaling to achieve the optimal objective value (see Appendix A.9). In Figure 1a, we use 5 filters with $h = 3$ and stride 1, where only one out of five trials successfully converge to the optimal objective value achieved by both our convex program and feasible network. As we increase $m$, all the trials are able to converge to the optimal objective value in Figure 1b.

![Figure 1](image1.png)

(a) Independent realizations with $m = 5$

![Figure 1](image2.png)

(b) Independent realizations with $m = 15$

Figure 1: Training cost of the three-layer circular CNN trained with SGD (5 initialization trials) on a synthetic dataset $(n = 6, d = 20, h = 3, \text{stride} = 1)$, where the green and red line with a marker represent the objective value obtained by the proposed convex program in (9) and the non-convex objective value in (8) of a feasible network with the weights found by the convex program, respectively. We use markers to denote the total computation time of the convex optimization solver.

We also evaluate the same model on a small subset of MNIST dataset [28] for a binary classification task. Here, we first select two classes and then randomly undersample to obtain a subset of the dataset. Particularly, we select $(n, d, m, h, \text{stride}) = (99, 50, 20, 3, 1)$ and use a batch size of 10 for SGD. In Figure 2, we plot both the regularized objective values in (8) and (9), and the corresponding test accuracies with respect to the computation time. Here, since the number of filters is large enough, all the SGD trials are able converge the optimal value provided by our convex program.

---

1. Additional numerical results can be found in Appendix A.1
2. We use CVX [24] and CVXPY [25, 26] with the SDPT3 solver [27] to solve convex optimization problems.
7 Concluding remarks

We studied various non-convex CNN training problems and introduced exact finite dimensional convex programs. Particularly, we provide equivalent convex characterizations for ReLU CNN architectures in a higher dimensional space. Unlike the previous studies, we prove that these equivalent characterizations have polynomial complexity in all input parameters and can be globally optimized via convex optimization solvers. Furthermore, we show that depending on the type of a CNN architecture, equivalent convex programs might exhibit different norm regularization structure, e.g., $\ell_1$, $\ell_2$, and nuclear norm. Thus, we claim that the implicit regularization effect in neural networks architectures can be also viewed as an architectural bias. More importantly, in the light of our results, efficient optimization algorithms can be developed to exactly (or approximately) optimize deep CNN architectures for large scale experiments in practice, which is left for future research.

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A Appendix

In this section, we present additional materials and proofs of the main results that are not included in the main paper due to the page limit.

A.1 Additional numerical results

Here, we present additional numerical experiments to further verify our theory. We first perform an experiment with another synthetic dataset, where \( X \in \mathbb{R}^{6 \times 15} \) is generated using a multivariate normal distribution with zero mean and identity covariance, and \( y = [1 \ 1 \ 1 \ 1 \ 1 \ 1] \). In this case, we use the two-layer CNN model in (1) and the corresponding convex program in (4). In Figure 3, we perform the experiment using \( m = 5, 8, 15 \) filters of size \( h = 10 \) and stride 5, where we observe that as the number of filters increases, the ratio of the trials converging to the optimal objective value increases as well.

We also perform an experiment for the three-layer circular CNN model on a small subset of CIFAR-10 dataset [29] for a binary classification task. In this experiment, we first select two classes and then randomly under-sample to obtain a subset of the original dataset. Particularly, we select \((n, d, m, h, \text{stride}) = (99, 50, 40, 3, 1)\) and use a batch size of 10 for SGD. In Figure 4, we plot both the regularized objective values in (8) and (9), and the corresponding test accuracies with respect to the computation time. Here, since the number of filters is large enough, all the SGD trials are able to converge the optimal value provided by our convex program.

A.2 Constructing hyperplane arrangements in polynomial time

In this section, we discuss the number of distinct hyperplane arrangements, i.e., \( P \), and present algorithm that enumerates all the distinct arrangements in polynomial time.

We first consider the number of all distinct sign patterns \( \text{sign}(Xw) \) for all \( w \in \mathbb{R}^d \). This number corresponds to the number of regions in a partition of \( \mathbb{R}^d \) by hyperplanes passing through the origin, and are perpendicular to the rows of \( X \). Here, one can replace the dimensionality \( d \) with the rank of the data matrix \( X \), i.e., denoted as \( r \), without loss of generality. Let us first introduce the Singular Value Decomposition of \( X \) in a compact form as \( X = U \Sigma V^T \), where \( U \in \mathbb{R}^{n \times r} \), \( \Sigma \in \mathbb{R}^{r \times r} \), and \( V \in \mathbb{R}^{r \times d} \). Then, for a given vector \( w \in \mathbb{R}^d \), \( Xw = Uw' \), where \( w' = \Sigma V^T w \), \( w' \in \mathbb{R}^r \). Hence, the number of distinct sign patterns \( \text{sign}(Xw) \) for all possible \( w \in \mathbb{R}^d \) is equal to the number of sign patterns \( \text{sign}(Uw') \) for all possible \( w' \in \mathbb{R}^r \).
Consider an arrangement of \( n \) hyperplanes in \( \mathbb{R}^r \), where \( n \geq r \). Let us denote the number of regions in this arrangement by \( P_{n,r} \). In \([14, 17]\), it is shown that this number satisfies

\[
P_{n,r} \leq 2 \sum_{k=0}^{r-1} \binom{n-1}{k}.
\]

For hyperplanes in general position, the above inequality is in fact an equality. In \([30]\), the authors present an algorithm that enumerates all possible hyperplane arrangements \( O(n^r) \) time, which can be used to construct the data for the convex programs we present throughout the paper.

### A.3 Equivalence of the \( \ell_1 \) penalized objectives

In this section, we prove the equivalence between the original problems with \( \ell_2 \) regularization and their \( \ell_1 \) penalized versions. We also note that similar equivalence results were also presented in \([5, 6, 31, 32]\). We start with the equivalence between \( (1) \) and \( (2) \).
Lemma A.1. The following two problems are equivalent:

\[
\min_{\{u_j, w_j\}_{j=1}^m} \frac{1}{2} \bigg\| \sum_{j=1}^m \sum_{k=1}^K (X_k u_j) + w_j - y \bigg\|^2 + \frac{\beta}{2} \sum_{j=1}^m \left( \|w_j\|^2 + w_j^2 \right)
\]

\[
= \min_{\{u_j, w_j\}_{j=1}^m, u_j \in \mathbb{R}_2, w_j} \frac{1}{2} \bigg\| \sum_{j=1}^m \sum_{k=1}^K (X_k u_j) + w_j - y \bigg\|^2 + \frac{\beta}{2} \sum_{j=1}^m \|w\|_1.
\]

Proof of Lemma A.1. We rescale the parameters as \(\tilde{u}_j = \gamma_j u_j\) and \(\tilde{w}_j = w_j / \gamma_j\), for any \(\gamma_j > 0\). Then, the output becomes

\[
\sum_{j=1}^m \sum_{k=1}^K (X_k \tilde{u}_j) + \tilde{w}_j = \sum_{j=1}^m \sum_{k=1}^K (X_k \gamma_j u_j) + \frac{w_j}{\gamma_j} = \sum_{j=1}^m \sum_{k=1}^K (X_k u_j) + w_j,
\]

which proves that the scaling does not change the network output. In addition to this, we have the following basic inequality

\[
\frac{1}{2} \sum_{j=1}^m \left( \|u_j\|^2 + w_j^2 \right) \geq \sum_{j=1}^m \left( |w_j| \|u_j\|_2 \right),
\]

where the equality is achieved with the scaling choice \(\gamma_j = \left( \frac{|w_j|}{\|u_j\|_2} \right)^\frac{1}{2}\) is used. Since the scaling operation does not change the right-hand side of the inequality, we can set \(\|u_j\|_2 = 1\), \(\forall j\). Therefore, the right-hand side becomes \(\|w\|_1\).

Now, let us consider a modified version of the problem, where the unit norm equality constraint is relaxed as \(\|u_j\|_2 \leq 1\). Let us also assume that for a certain index \(j\), we obtain \(\|u_j\|_2 < 1\) with \(w_j \neq 0\) as an optimal solution. This shows that the unit norm inequality constraint is not active for \(u_j\), and hence removing the constraint for \(u_j\) will not change the optimal solution. However, when we remove the constraint, \(\|u_j\|_2 \to \infty\) reduces the objective value since it yields \(w_j = 0\). Therefore, we have a contradiction, which proves that all the constraints that correspond to a nonzero \(w_j\) must be active for an optimal solution. This also shows that replacing \(\|u_j\|_2 = 1\) with \(\|u_j\|_2 \leq 1\) does not change the solution to the problem.

Next, we prove the equivalence between (8) and (29).

Lemma A.2. The following two problems are equivalent:

\[
\min_{\{u_j, w_{1j}, w_{2j}\}_{j=1}^m} \frac{1}{2} \bigg\| \sum_{j=1}^m (XU_j w_{1j}) + w_{2j} - y \bigg\|^2 + \frac{\beta}{2} \sum_{j=1}^m \left( \|w_{1j}\|^2 + w_{2j}^2 \right)
\]

\[
= \min_{\{u_j, w_{1j}, w_{2j}\}_{j=1}^m, u_j \in \mathbb{R}_2, \forall j} \frac{1}{2} \bigg\| \sum_{j=1}^m (XU_j w_{1j}) + w_{2j} - y \bigg\|^2 + \frac{\beta}{2} \|w_{2j}\|_1.
\]

Proof of Lemma A.2. We rescale the parameters as \(\tilde{w}_{1j} = \gamma_j w_{1j}\) and \(\tilde{w}_{2j} = w_{2j} / \gamma_j\), for any \(\gamma_j > 0\). Then, the output becomes

\[
\sum_{j=1}^m (XU_j \tilde{w}_{1j}) + \tilde{w}_{2j} = \sum_{j=1}^m (XU_j \gamma_j w_{1j}) + \frac{w_{2j}}{\gamma_j} = \sum_{j=1}^m (XU_j w_{1j}) + w_{2j},
\]

which proves that the scaling does not change the network output. In addition to this, we have the following basic inequality

\[
\frac{1}{2} \sum_{j=1}^m \left( \|w_{1j}\|^2 + w_{2j}^2 \right) \geq \sum_{j=1}^m \left( |w_{1j}| \|w_{2j}\| \right),
\]

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where the equality is achieved with the scaling choice \( \gamma_j = \left( \frac{|w_j|}{\|w_j\|_2} \right)^{\frac{1}{2}} \) is used. Since the scaling operation does not change the right-hand side of the inequality, we can set \( \|w_j\|_2 = 1, \forall j \). Therefore, the right-hand side becomes \( \|w_2\|_1 \). The rest of the proof directly follows from the proof of Lemma [A.1] \( \square \)

### A.4 Two-layer linear CNNs

We now consider two-layer linear CNNs, for which the training problem is

\[
\min_{\{u_j, w_j\}_{j=1}^m} \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^N X_k u_j y_{jk} - y_j \|2 + \frac{\beta}{2} \sum_{j=1}^m (\|u_j\|_2^2 + \|w_j\|_2^2). \tag{20}
\]

**Theorem A.1.** [8] The equivalent convex program for (20) is

\[
\min_{\{z_k\}_{k=1}^K} \frac{1}{2} \sum_{k=1}^K X_k z_k - y \|2 + \beta \|z_1, \ldots, z_K\|_\ast. \tag{21}
\]

**Proof of Theorem A.1.** Let us first reparameterize the primal problem as follows

\[
\max_{M, v} \frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|v\|_2^2 \text{ s.t. } \sigma_{\text{max}}(M) \leq \beta, \ M = [X_1^T v \ldots X_K^T v],
\]

where \( \sigma_{\text{max}}(M) \) represent the maximum singular value of \( M \). Then the Lagrangian is as follows

\[
L(\lambda, Z, M, v) = -\frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|v\|_2^2 + \lambda (\beta - \sigma_{\text{max}}(M)) + \text{trace}(Z^T M) - \text{trace}(Z^T [X_1^T v \ldots X_K^T v])
\]

\[
= -\frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|v\|_2^2 + \lambda (\beta - \sigma_{\text{max}}(M)) + \text{trace}(Z^T M) - v^T \sum_{k=1}^K X_k z_k
\]

where \( \lambda \geq 0 \). Then maximizing over \( M \) and \( v \) yields the following dual form

\[
\min_{\{z_k\}_{k=1}^K} \frac{1}{2} \sum_{k=1}^K X_k z_k - y \|2 + \beta \|z_1, \ldots, z_K\|_\ast,
\]

where \( \|z_1, \ldots, z_K\|_\ast = \|Z\|_\ast = \sum_i \sigma_i(Z) \) is the \( \ell_1 \) norm of singular values, i.e., nuclear norm [33]. \( \square \)

The regularized training problem for two-layer circular CNNs as follows

\[
\min_{\{u_j, w_j\}_{j=1}^m} \frac{1}{2} \sum_{j=1}^m \|X U_j w_j - y\|_2^2 + \frac{\beta}{2} \sum_{j=1}^m (\|u_j\|_2^2 + \|w_j\|_2^2) \tag{22}
\]

where \( U_j \in \mathbb{R}^{d \times d} \) is a circulant matrix generated by a circular shift modulo \( d \) using \( u_j \in \mathbb{R}^h \).

**Theorem A.2.** [8] The equivalent convex program for (22) is

\[
\min_{z \in \mathbb{C}^d} \frac{1}{2} \|\hat{X} z - y\|_2^2 + \beta \|z\|_1, \tag{23}
\]

where \( \hat{X} = X F \) and \( F \in \mathbb{C}^{d \times d} \) is the DFT matrix.

**Proof of Theorem A.2.** Applying a rescaling and then taking the dual yields

\[
\max_{v} -\frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|y\|_2^2 \text{ s.t. } \max_{D \in \mathcal{D}} \|v^T X F D F^H\|_2 \leq \beta,
\]

which can be equivalently written as

\[
\max_{v} -\frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|y\|_2^2 \text{ s.t. } \max_{D \in \mathcal{D}} \|v^T X D\|_2 \leq \beta.
\]
Therefore, the problem above can be further simplified as
\[
\max_w -\frac{1}{2}\|v - y\|_2^2 + \frac{1}{2}\|y\|_2^2 \quad \text{s.t.} \quad \|v^T\tilde{X}\|_\infty \leq \frac{\beta}{\sqrt{d}}.
\]

Then, taking the dual of this problem gives the following
\[
\min_{z\in\mathbb{C}^d} \frac{1}{2}\|\tilde{X}z - y\|_2^2 + \frac{\beta}{\sqrt{d}}\|z\|_1.
\]

\[\square\]

### A.5 Extensions to vector outputs

Here, we present the extensions of our approach to vector output. To keep the notation and presentation simple, we consider the vector output version of the two-layer linear CNN model in Section A.4. The training problem is as follows

\[
\min_{\{u_i, (w_j)_{k=1}^K\}_{i=1}^m} \frac{1}{2}\|\sum_{k=1}^K \sum_{j=1}^m X_ku_jw_j^T - y\|_F^2 + \frac{\beta}{2}\|u_j\|_2^2 + \frac{\beta}{2}\|\sum_{k=1}^K \|w_j\|_2^2\| - \frac{1}{2}\|y\|_F^2.
\]

The corresponding dual problem is given by

\[
\max_v -\frac{1}{2}\|y - v\|_F^2 + \frac{1}{2}\|y\|_F^2 \quad \text{s.t.} \quad \max_{u \in B_d} \sum_{k=1}^K \|v^T X_ku\|_2^2 \leq 1.
\]

The maximizers of the dual are the maximal eigenvectors of \(\sum_{k=1}^K X_k^T Vv^T X_k\), which are optimal filters. The rest of the derivations directly follow Section A.4.

### A.6 Extensions to arbitrary convex loss functions

In this section, we first show the procedure to create an optimal standard CNN architecture using the optimal weights provided by the convex program in (1). Then, we extend our derivations to arbitrary convex loss functions.

In order to keep our derivations simple and clear, we use the regularized two-layer architecture in (1). For a given convex loss function \(\ell(\cdot, y)\), the regularized training problem can be stated as follows

\[
p_1^* = \min_{\{u_i, w_i\}_{i=1}^m} \ell \left( \sum_{i=1}^m \sum_{k=1}^K X_ku_jw_j^T, y \right) + \frac{\beta}{2}\sum_{j=1}^m (\|u_j\|_2^2 + \|w_j\|_2^2) \tag{24}
\]

Then, the corresponding finite dimensional convex equivalent is

\[
\min_{\{w_i, w_i^*, \}^m_{i=1} \subseteq \mathbb{R}^P_{\text{conv}} \mid \forall i, k} \ell \left( \sum_{i=1}^m X_kw_i^T, y \right) + \frac{\beta}{2}\sum_{i=1}^m (\|w_i\|_2 + \|w_i^*\|_2) \tag{25}
\]

s.t. \((2D(S_i^k) - I_n)X_kw_i \geq 0, (2D(S_i^k) - I_n)X_kw_i \geq 0, \forall i, k.

We now define \(m^* := \sum_{i=1}^m 1[\|w_i\|_2 \neq 0 \& \|w_i\|_2 \neq 0]\), where \(\{w_i^*, w_i^*\}^m_{i=1} \subseteq \mathbb{R}^P_{\text{conv}}\) are the optimal weights in (25).

**Theorem A.3.** The convex program (25) and the non-convex problem (24), where \(m \geq m^*\) has identical optimal values. Moreover, an optimal solution to (24) can be constructed from an optimal solution to (25) as follows

\[
u_i^* = \begin{cases} w_i^* \sqrt{\|w_i^*\|_2} & \text{if } \|w_i^*\|_2 \neq 0 \\
\frac{w_i^*}{\sqrt{\|w_i^*\|_2}} & \text{if } \|w_i^*\|_2 = 0 \end{cases}
\]

where \(\{w_i^*, w_i^*\}^m_{i=1} \subseteq \mathbb{R}^P_{\text{conv}}\) are the optimal solutions to (25), and at most one of \(w_i^*\) or \(w_i^*\) is non-zero for all \(i = 1, \ldots, P_{\text{conv}}\).
We first review the basic properties of infinite size neural networks and introduce technical details.

We also show that our dual characterization holds for arbitrary convex loss functions.

We can use \( \mu \) which is identical to the objective value of the convex program (25). Since the value of the convex program is equal to the value of it’s dual \( d^*_1 \) in the dual, we conclude that \( p^*_1 = d^*_1 \), which is equal to the value of the convex program (25) achieved by the prescribed parameters.

We also show that our dual characterization holds for arbitrary convex loss functions:

\[
\min_{\{u_j, w_j\}^m_{j=1} \in B_2, \forall j} \ell \left( \sum_{j=1}^m \sum_{k=1}^K (X_k u_j^*) + w_j, y \right) + \beta \|w\|_1, \tag{26}
\]

where \( \ell(\cdot, y) \) is a convex loss function.

**Theorem A.4.** The dual of (26) is given by

\[
\max -\ell^*(v) \text{ s.t. } \sum_{k=1}^K v^T (X_k u) \leq \beta, \forall u \in B_2,
\]

where \( \ell^* \) is the Fenchel conjugate function defined as

\[
\ell^*(v) = \max_z z^T v - \ell(z, y).
\]

**Proof of Theorem A.4.** The proof follows from classical Fenchel duality (22). We first describe (26) in an equivalent form as follows:

\[
\min_{\{u_j, w_j\}^m_{j=1} \in B_2, \forall j} \ell(z, y) + \beta \|w\|_1 \text{ s.t. } z = \sum_{j=1}^m \sum_{k=1}^K (X_k u_j^*) + w_j,
\]

Then the dual function is

\[
g(v) = \min_{\{u_j, w_j\}^m_{j=1} \in B_2, \forall j} \ell(z, y) - v^T z + v^T \sum_{j=1}^m \sum_{k=1}^K (X_k u_j^*) + w_j + \beta \|w\|_1.
\]

Therefore, using the classical Fenchel duality (22) yields the claimed dual form.

**A.7 Proof of Proposition 5.1**

We first review the basic properties of infinite size neural networks and introduce technical details to derive the dual of (4). We refer the reader to [3, 34] for further details. Let us first consider a measurable input space \( \mathcal{X} \) with a set of continuous basis functions (i.e., neurons or filters in our context) \( \psi_u : \mathcal{X} \to \mathcal{R} \), which are parameterized by \( u \in B_2 \). Next, we use real-valued Radon measures with the uniform norms (35). Let us consider a signed Radon measure denoted as \( \mu \). Now, we can use \( \mu \) to formulate an infinite size neural network as \( f(x) = \int_{u \in B_2} \psi_u(x) d\mu(u) \), where \( x \in \mathcal{X} \) is the input. The norm for \( \mu \) is usually defined as its total variation norm, which is the supremum of \( \int_{u \in B_2} g(u) d\mu(u) \) over all continuous functions \( g(u) \) that satisfy \( |g(u)| \leq 1 \). Now, we
We now focus on the single-sided dual constraint. We first note that the semi-infinite problem (3) is convex. The optimal value, although (27) involves an infinite dimensional integral for \( m \), by Caratheodory’s theorem, we know that the integral can be represented as a finite summation, to be more precise, a summation of at most \( n + 1 \) Dirac delta functions [34]. If we denote the number of Dirac delta functions as \( m^* \), then we have

\[
\mu = \sum_{j=1}^{m^*} \psi_{u_j} w_j
\]

which can be obtained by selecting \( \mu \) as a weighted sum of Dirac delta functions, i.e., \( \mu = \sum_{j=1}^{m^*} w_j \delta(u - u_j) \). In this case, the total variation norm, denoted as \( \| \mu \|_{TV} \), corresponds to the \( \ell_1 \) norm \( \| w \|_1 \).

Now, we ready to derive the dual of (3), which can be stated as follows (see Section 8.6 of [18] and Section 2 of [36] for further details)

\[
d_1^* \leq p_{1,\infty} = \min_{\mu} \frac{1}{2} \left\| \int_{u \in B_2} \sum_{k=1}^K (X_k u)_+ d\mu(u) - y \right\|_2^2 + \beta \| \mu \|_{TV}.
\]  

(27)

Although (27) involves an infinite dimensional integral form, by Caratheodory’s theorem, we know that the integral can be represented as a finite summation, to be more precise, a summation of at most \( n + 1 \) Dirac delta functions [34]. If we denote the number of Dirac delta functions as \( m^* \), where \( m^* \leq n + 1 \), then we have

\[
p_{1,\infty} = \min_{\{u_j, w_j \}_{j=1}^{m^*}} \frac{1}{2} \left\| \sum_{j=1}^{m^*} \sum_{k=1}^K (X_k u_j)_+ w_j - y \right\|_2^2 + \beta \| w \|_1
\]

provided that \( m \geq m^* \). We now need to show that strong duality holds, i.e., \( p_1^* = d_1^* \).

We first note that the semi-infinite problem (3) is convex. Then, we prove that the optimal value is finite. Since \( \beta > 0 \), we know that \( v = 0 \) is strictly feasible, and achieves 0 objective value. Moreover, since \( -\|y - v\|_2^2 \leq 0 \), the optimal objective value \( p_1^* \) is finite. Therefore, by Theorem 2.2 of [36], strong duality holds, i.e., \( p_{1,\infty} = d_1^* \) provided that the solution set of (3) is nonempty and bounded. We also note that the solution set of (3) is the Euclidean projection of \( y \) onto a convex, closed and bounded set since \( (X_k u)_+ \) can be expressed as the union of finitely many convex closed and bounded sets.

### A.8 Proof of Theorem 2.2

The proof follows the proof of Proposition 5.1. The dual of (5) is as follows

\[
d_1^* \leq p_{1,\infty} = \min_{\mu} \frac{1}{2} \int_{u \in B_2} \maxpool \{ (X_k u)_+ \}_{k=1}^K d\mu(u) - y \right\|_2^2 + \beta \| \mu \|_{TV},
\]

which has the following finite equivalent

\[
p_{1,\infty} = \min_{\{u_j, w_j \}_{j=1}^{m^*}} \frac{1}{2} \left\| \sum_{j=1}^{m^*} \maxpool \{ (X_k u_j)_+ \}_{k=1}^K w_j - y \right\|_2^2 + \beta \| w \|_1
\]

provided that \( m \geq m^* \). We now need to show that strong duality holds, i.e., \( p_1^* = d_1^* \).

Since \( \maxpool(\cdot) \) can be expressed as the union of finitely many convex, closed and bounded sets, the rest of the strong duality results directly follow from the proof of Proposition 5.1.

We now focus on the single-sided dual constraint

\[
\max_{u \in B_2} v^T \maxpool \{ (X_k u)_+ \}_{k=1}^K \leq \beta,
\]
which can be written as

\[
\max_{S^k \subseteq [n]} \max_{w \in B_2} \sum_{k=1}^{K} v^T D(S^k) X_k u \quad \text{s.t.} \quad (2D(S^k) - I_n) X_k u \geq 0, \forall k,
\]

\[D(S^k) X_k u \geq D(S^k) X_j u, \forall j, k \in [K], \sum_{k=1}^{K} D(S^k) = I_n.\]

We again enumerate all hyperplane arrangements and index them in an arbitrary order, where we define the overall set as \( S^K \) := \{ (S^1, \ldots, S^K) : S^k \subseteq S, \forall k, i; \sum_{k=1}^{K} D(S^k_i) = I_n, \forall i \} and \( P_{\text{conv}} = |S^K| \). Then, following the same steps in (13)–(18) gives the following convex problem

\[
\min_{w_i, w_j^i \in \mathbb{R}^k} \frac{1}{2} \left| \sum_{i=1}^{K} \sum_{k=1}^{K} D(S^k_i) X_k (w_i^j - w_i) - y \right|_2^2 + \beta \left( \| w_i \|_2 + \| w_i^j \|_2 \right) \quad \text{s.t.} \quad (2D(S^k_i) - I_n) X_i w_i \geq 0, (2D(S^k_i) - I_n) X_i w_i^j \geq 0, \forall i, k,
\]

\[D(S^k_i) X_i w_i \geq D(S^k_i) X_j w_j, D(S^k_i) X_i w_i^j \geq D(S^k_i) X_j w_j^i, \forall j, i, j, k.
\]

We now note that there will be \( m^* \) pairs \{ \( w_i^j, w_i^j \) \} for \( i = 1, \ldots, P_{\text{conv}} \), where at most one of \( w_i^j \) and \( w_i^j \) is non-zero since either the constraint \( \sum_{k=1}^{K} \sum_{j=1}^{m^*} v^T (X_k u^j) \geq \beta \), or \( \sum_{k=1}^{K} v^T (X_k u^j) \leq \beta \) can be active at the optimum in (4). Then, we can construct a set of weights \{ \( u_i^j, w_i^j \) \}_{i,j}^{m^*} \} as defined in the theorem and evaluate the non-convex objective in (5) using these weights as follows

\[
p_1^* \leq \frac{1}{2} \left( \sum_{j=1}^{m^*} \max \left( \left( \left( X_k u_i^j \right) \right)_{+} \right) \right) \left\| w_i^j - y \right\|_2^2 + \beta \frac{1}{2} \left( \sum_{j=1}^{m^*} \max \left( \left( \left( X_k u_i^j \right) \right)_{+} \right) \right) \left\| w_i^j \right\|_2^2 \left\| w_i^j \right\|_2^2
\]

\[+ \beta \frac{1}{2} \left( \sum_{j=1}^{m^*} \max \left( \left( \left( X_k u_i^j \right) \right)_{+} \right) \right) \left\| w_i^j \right\|_2^2 \left\| w_i^j \right\|_2^2\]

which has the same objective value with (28). Since strong duality holds for the convex program, we have \( p_1^* = d_1^* \), which is equal to the value of the convex program (28) achieved by the prescribed parameters above. \( \square \)

A.9 Proof of Theorem 3.1

By using a rescaling as in the previous section (see Appendix A.3 for details), (8) can be equivalently stated as

\[
p_2^* = \min_{u_j, w_j \in B_2 Y_j} \frac{1}{2} \left( \sum_{j=1}^{m^*} \left( X U_j w_{1j} \right) \right) \left\| w_{2j} - y \right\|_2^2 + \beta \| w_2 \|_1.
\]

Let us denote the eigenvalue decomposition of \( U_j \) as \( U_j = F D_j F^H \), where \( F \in \mathbb{C}^{d \times d} \) is the DFT matrix, \( D_j \in \mathbb{C}^{d \times d} \) is a diagonal matrix, and the superscript \( H \) denotes the Hermitian (conjugate transpose) operation. Then, taking dual with respect to the output layer weights \( w_2 \) and interchanging the order of min-max yields

\[
p_2^* \geq d_2^* = \max_{v} -\frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2 \quad \text{s.t.} \quad \max_{\mathbf{D} \in D} \| v^T \left( X F D_j F^H w_{1j} \right) \|_2 \leq \beta, \forall j,
\]

where \( D := \{ D \in \mathbb{C}^{d \times d} : \| D \|_F \leq d \} \). We note that the maximization in the constraint in (30) is identical for each \( j \). Therefore, the dual constraint is identical to having a single \( D \) as follows

\[
\max_{\mathbf{D} \in D} \| v^T \left( X F D_j F^H w_{1j} \right) \|_2 \leq \beta.
\]
In order to obtain the dual of the semi-infinite problem in \( (30) \), we first substitute \((31)\) into \((30)\) and then take dual with respect to \(v\) (see Appendix A.7 and \([18, 56]\) for further details), which yields

\[
d^2_2 \leq p_{2, \infty} = \min_{\mu} \frac{1}{2} \int_{\theta \in \Theta} \left( X F D F^H w_1 \right) + d \mu(\theta) - y_2 + \beta \| \mu \|_{TV}
\]

where \( \Theta := \{(D, w_1) : D' \in D, w_1 \in B_2 \} \). Then, selecting \( \mu = \sum_{j=1}^{m^*} w_{2j} \delta(\theta - \theta_j) \), where \( m^* \leq n + 1 \), gives

\[
p_{2, \infty} = \min_{\{D_j, w_{1j}, w_{2j}\}_{j=1}^{m^*}} \frac{1}{2} \sum_{j=1}^{m^*} (X F D_j F^H w_{1j}) + w_{2j} - y_2 + \beta \| w_2 \|_1
\]

provided that \( m \geq m^* \) holds. Then, the rest of the strong duality proof directly follows from Proof of Proposition 5.1.

Now, let us first define \( \tilde{X} = X F \) and \( \tilde{w}_1 = F^H w_1 \). Then, we focus on a single-sided dual constraint

\[
\max_{D \in D} \max_{w_1 \in B_2} v^T \left( \tilde{X} D \tilde{w}_1 \right) \leq \beta,
\]

which can be written as

\[
\max_{S \subseteq [n]} \max_{\tilde{D} \in D} \max_{\tilde{w}_1 \in B_2} v^T \tilde{D}(S) \tilde{X} \tilde{D} \tilde{w}_1 \text{ s.t. } (2D(S) - I_n) \tilde{X} \tilde{D} \tilde{w}_1 \geq 0.
\]

Since the inner maximization is convex (after a variable change as \( q = D \tilde{w}_1 \)) and there exists a strictly feasible solution for a fixed \( D(S) \) matrix, strong duality holds and consequently \((33)\) can also be written as

\[
\max_{S \subseteq [n]} \min_{\tilde{D} \in D} \max_{\tilde{w}_1 \in B_2} v^T \tilde{D}(S) \tilde{X} \tilde{D} \tilde{w}_1 + \alpha^T (2D(S) - I_n) \tilde{X} \tilde{D} \tilde{w}_1
\]

\[
= \max_{S \subseteq [n]} \min_{\tilde{D} \in D} \max_{\tilde{w}_1 \in B_2} \| v^T \tilde{D}(S) \tilde{X} + \alpha^T (2D(S) - I_n) \tilde{X} \|_\infty \sqrt{d}.
\]

We now enumerate all hyperplane arrangements and index them in an arbitrary order, which are denoted as \( D(S_i) \), where \( i \in [P_{\text{conv}}] \). Then, we have

\[
(32) \iff \forall i \in [P_{\text{conv}}], \min_{\alpha \geq 0} \| v^T D(S_i) \tilde{X} + \alpha^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \leq \beta
\]

\[
\iff \forall i \in [P_{\text{conv}}], \exists \alpha_i \geq 0 \text{ s.t. } \| v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \leq \beta.
\]

We now use the same approach for the two-sided constraint in \((30)\) to represent \((30)\) as a finite dimensional convex problem as follows

\[
\max_{\alpha, \alpha' \geq 0} \frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2
\]

s.t. \( \| v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \leq \beta, \| - v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \leq \beta \), \( \forall i \).

We note that the above problem is convex and strictly feasible for \( v = \alpha_i = \alpha'_i = 0 \). Therefore, \((34)\) can be written as

\[
\min_{\lambda_i, \lambda_i' \geq 0} \max_{\alpha, \alpha' \geq 0} \frac{1}{2} \| v - y \|_2^2 + \frac{1}{2} \| y \|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta - \| v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \right)
\]

\[
+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_i \left( \beta - \| - v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \|_\infty \sqrt{d} \right).
\]

(35)
Next, we introduce new variables $z_i, z'_i \in \mathbb{C}^d$ to represent (35) as

$$\min_{\lambda_i, \lambda'_i \geq 0} \max_{\alpha_i, \alpha'_i \geq 0} \min_{\substack{z_i \in \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3, z'_i \in \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3}} \frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|y\|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta + \sqrt{d} \left( v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \right) z_i \right)$$

$$+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_i \left( \beta + \sqrt{d} \left( -v^T D(S_i) \tilde{X} + \alpha'_i^T (2D(S_i) - I_n) \tilde{X} \right) z'_i \right). \quad (36)$$

We note that the objective is concave in $v, \alpha_i, \alpha'_i$ and convex in $z_i, z'_i$. Moreover the set $\mathbb{B}_1$ is convex and compact. We recall Sion’s minimax theorem [23] for the inner max-min problem and express the strong dual of the problem (36) as

$$\min_{\lambda_i, \lambda'_i \geq 0} \max_{\alpha_i, \alpha'_i \geq 0} \min_{\substack{z_i \in \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3, z'_i \in \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3}} \frac{1}{2} \|v - y\|_2^2 + \frac{1}{2} \|y\|_2^2 + \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta + \sqrt{d} \left( v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \right) z_i \right)$$

$$+ \sum_{i=1}^{P_{\text{conv}}} \lambda'_i \left( \beta + \sqrt{d} \left( -v^T D(S_i) \tilde{X} + \alpha'_i^T (2D(S_i) - I_n) \tilde{X} \right) z'_i \right). \quad (37)$$

Now, we can compute the maximum with respect to $v, \alpha_i, \alpha'_i$ analytically to obtain the following problem

$$\min_{\lambda_i, \lambda'_i \geq 0} \min_{z_i \in \mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3} \left\{ \sum_{i=1}^{P_{\text{conv}}} \lambda_i \left( \beta + \sqrt{d} \left( v^T D(S_i) \tilde{X} + \alpha_i^T (2D(S_i) - I_n) \tilde{X} \right) z_i \right) \right\}^2$$

$$\quad + \sum_{i=1}^{P_{\text{conv}}} \lambda'_i \left( \beta + \sqrt{d} \left( -v^T D(S_i) \tilde{X} + \alpha'_i^T (2D(S_i) - I_n) \tilde{X} \right) z'_i \right) \quad (38)$$

s.t. $(2D(S_i) - I_n) \tilde{X} z_i \geq 0, (2D(S_i) - I_n) \tilde{X} z'_i \geq 0, \forall i$.

Now we apply a change of variables and define $w_i = \sqrt{d} \lambda_i z_i$ and $w'_i = \sqrt{d} \lambda'_i z'_i$. Thus, we obtain

$$\min_{w_i, w'_i} \frac{1}{2} \left\{ \sum_{i=1}^{P_{\text{conv}}} D(S_i) \tilde{X} (w'_i - w_i) - y \right\}^2 + \beta \sum_{i=1}^{P_{\text{conv}}} (\|w_i\|_1 + \|w'_i\|_1) \quad (39)$$

s.t. $(2D(S_i) - I_n) \tilde{X} w_i \geq 0, (2D(S_i) - I_n) \tilde{X} w'_i \geq 0, \forall i$,

where we eliminate the variables $\lambda_i, \lambda'_i$, since $\lambda_i = \|w_i\|_1/\sqrt{d}$ and $\lambda'_i = \|w'_i\|_1/\sqrt{d}$ are feasible and optimal.

**Optimal weight construction for (5):**

Given the optimal weights for the convex program in (39), i.e., denoted as $\{w^*_i, w'^*_i\}_{i=1}^{P_{\text{conv}}}$, we use the following relation

$$\tilde{X} w^*_i = \tilde{X} \text{diag} \left( \sqrt{d} \left| \frac{w^*_i}{\|w^*_i\|_1} \right| \right) \text{diag} \left( \sqrt{d} \left| \frac{w'^*_i}{\|w'^*_i\|_1} \right| \right) e^{j\phi_i} \sqrt{\frac{\|w^*_i\|_1}{\sqrt{d}}}$$

where $\phi_i$ is defined such that $w^*_i = \text{diag} (\|w^*_i\|) e^{j\phi_i}$. Thus, we can directly set the parameters as follows

$$D^*_i = \text{diag} \left( \sqrt{d} \left| \frac{w^*_i}{\|w^*_i\|_1} \right| \right), \quad \tilde{w}^*_i = \text{diag} \left( \sqrt{d} \left| \frac{w'^*_i}{\|w'^*_i\|_1} \right| \right) e^{j\phi_i}, \quad w^*_i = \sqrt{\frac{\|w^*_i\|_1}{\sqrt{d}}}$$

which can be equivalently written as

$$U^*_i = F \text{diag} \left( \sqrt{d} \left| \frac{w^*_i}{\|w^*_i\|_1} \right| \right) F^H, \quad w^*_i = \sqrt{\frac{\|w^*_i\|_1}{\sqrt{d}}}$$

where $\phi_i$ is defined such that $w^*_i = \text{diag} (\|w^*_i\|) e^{j\phi_i}$. Thus, we can directly set the parameters as follows

$$U^*_i = F \text{diag} \left( \sqrt{d} \left| \frac{w^*_i}{\|w^*_i\|_1} \right| \right) F^H, \quad w^*_i = \sqrt{\frac{\|w^*_i\|_1}{\sqrt{d}}}$$

to exactly match with the problem formulation in (38). We first note that $\|U^*_i\|_F^2 = \|D^*_i\|_F^2 = d, \forall i$, therefore, this set of parameters is feasible for (38). Now, we prove the optimality by showing that these parameters have the same regularization cost with the convex program in (39) as follows

$$\frac{\beta}{2} \sum_{i=1}^{P_{\text{conv}}} \left( \|w^*_i\|_2^2 + \|w'^*_i\|_2^2 \right) = \frac{\beta}{\sqrt{d}} \sum_{i=1}^{P_{\text{conv}}} \|w^*_i\|_1.$$
We now focus on the single-sided dual constraint \( D \). The same steps can also be applied to \( \Theta \).

In order to obtain the dual of the semi-infinite problem in (41), we again take dual with respect to \( U \).

Let us denote the eigenvalue decomposition of \( U \) as \( \sum_{i=1}^{m} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \).

Proof of Theorem 4.1

By using a rescaling for each \( w_{1,j} \) and \( w_{2,j} \), (10) can be equivalently stated as

\[
    p_3^* = \min_{\{\{u_{ij}\}_{i=1}^n, w_{1,j}, w_{2,j}\}_{j=1}^m} \frac{1}{2} \sum_{j=1}^{m} \left[ X \prod_{l=1}^{L} U_{lj} w_{1,j} \right] w_{2,j} - y^T \mathbf{v} + \beta \|w_2\|_1. \tag{40}
\]

Let us denote the eigenvalue decomposition of \( U_{lj} \) as \( \prod_{i=1}^{L} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \).

Then, selecting \( \mu = \sum_{j=1}^{m^*} w_{2,j} \mathbf{v} \), where \( m^* \leq n + 1 \), gives

\[
p_{3,\infty} = \min_{\{\{D_{lj}\}_{i=1}^n, w_{1,j}, w_{2,j}\}_{j=1}^{m^*}} \frac{1}{2} \sum_{j=1}^{m^*} \left[ X \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{H} w_{1,j} \right] w_{2,j} - y^T \mathbf{v} + \beta \|w_2\|_1
\]

provided that \( m \geq m^* \) holds. Then, the rest of the strong duality proof directly follows from Proof of Proposition 5.1.

We now focus on the single-sided dual constraint

\[
    \max_{D_{lj} \in \mathcal{D}_{L}} \mathbf{v}^T \left( X \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{v}_1 \right) \leq \beta,
\]

which can be written as

\[
    \max_{\mathcal{S} \subseteq \mathcal{S}\cap \mathcal{B}_2} \max_{\mathcal{D}_{lj} \in \mathcal{D}_L} \mathbf{v}^T \mathbf{D}(S) \tilde{X} \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{v}_1 \text{ s.t. } (2\mathbf{D}(S) - \mathbf{I}_n) \tilde{X} \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{v}_1 \geq 0. \tag{42}
\]

Since the inner maximization is convex (after a variable change as \( q = \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{v}_1 \)) and there exists a strictly feasible solution for a fixed \( \mathbf{D}(S) \) matrix, (42) can also be written as

\[
    \max_{\mathcal{S} \subseteq \mathcal{S}} \min_{\alpha \geq 0} \max_{\mathcal{D}_{lj} \in \mathcal{D}_L} \mathbf{v}^T \mathbf{D}(S) \tilde{X} z + \alpha^T (2\mathbf{D}(S) - \mathbf{I}_n) \tilde{X} \prod_{l=1}^{L} \mathbf{D}_{lj} \mathbf{v}_1
\]

\[
    = \max_{\mathcal{S} \subseteq \mathcal{S}} \min_{\alpha \geq 0} \|\mathbf{v}^T \mathbf{D}(S) \tilde{X} + \alpha^T (2\mathbf{D}(S) - \mathbf{I}_n) \tilde{X}\|_{\infty}.
\]
Then, the rest of the derivations directly follow (34)–(38) and yield the following optimization problem

\[
\min_{w_i, w'_i} \frac{1}{2} \left\| \sum_{i=1}^{P_{\text{conv}}} D(S_i) \hat{X} (w'_i - w_i) - y \right\|_2^2 + \beta \sum_{i=1}^{P_{\text{conv}}} (\|w_i\|_1 + \|w'_i\|_1)
\]

(43)

s.t. \((2D(S_i) - I_n) \hat{X} w_i \geq 0, (2D(S_i) - I_n) \hat{X} w'_i \geq 0, \forall i.\)

Hence, we obtain a polynomial time convex formulation for the non-convex problem in (10). Then, we can construct an optimal network from the solutions to the convex program as follows.

**Optimal weight construction for (10):**

Given the optimal weights for the convex program in (43), i.e., denoted as \(\{w^*_i, w'^*_i\}_{i=1}^{P_{\text{conv}}},\) we use the following relation

\[
\hat{X} w^*_i = \hat{X} \text{diag} \left( \left( \frac{|w^*_i|}{\|w^*_i\|_1} \right)^{\frac{1}{2}} \right) \text{diag} \left( \sqrt{|w^*_i|} \right) e^{j \phi_i} \sqrt{\|w^*_i\|_1}
\]

\[
= \hat{X} \prod_{l=1}^{L} \text{diag} \left( \left( \frac{|w^*_i|}{\|w^*_i\|_1} \right)^{\frac{1}{2}} \right) \text{diag} \left( \sqrt{|w^*_i|} \right) e^{j \phi_i} \sqrt{\|w^*_i\|_1}
\]

Therefore, we can directly set the parameters as follows

\[
D^*_{li} = \text{diag} \left( \left( \frac{|w^*_i|}{\|w^*_i\|_1} \right)^{\frac{1}{2}} \right), \quad \hat{w}^*_i = \text{diag} \left( \sqrt{|w^*_i|} \right) e^{j \phi_i}, \quad w^*_{1i} = \sqrt{\|w^*_i\|_1}.
\]

Then, the rest of the proof directly follows the proof of Theorem 3.1.