Abstract

A broader range of analytical tools can enhance understanding of the unusual mechanical properties of metamaterials and other advanced material systems. Here, we discuss a mechanics analogue of the Parseval’s energy theorem that leads to a density of the strain energy in the reciprocal space. It reflects the ways for an elastic medium to translate static deformation patterns between two points in space. A normalized spectral density also provides an information entropy of deformation at those points. Both differential and discrete (numerical) entropies of the Shannon’s type are discussed. Spectral entropy is a basic measure of information available in the material interior about surface loads, or a measure of disorder introduced into elastic medium by the deformation. An exact analytical entropy function is derived for an isotropic plane solid under Gauss-distributed and point loads. Approaches to numerical calculation of spectral entropy in computational solid mechanics are also discussed. Energy spectral density and spectral entropy of an elastic continuum is shown to translate, logically, in agreement with the Saint-Venant’s principle. However, it also becomes clear that microstructured media may demonstrate anomalous pathways of evolution of the strain energy spectrum, enabling interesting transformation mechanics studies of engineered material systems.

Keywords: transformation mechanics, strain energy spectral density, mechanical metamaterials, information entropy

1. Introduction

Recent advances in mechanical metamaterials create interesting opportunities for the control of strain energy distribution in deformable bodies. Mechanical metamaterials [1-10] are structural composites that manifest behaviors beyond the scope of traditional mechanics of materials, including negative elastic moduli [1-7] and basic symmetries breaking [8-9]. Identification of dangerous stress and strain profiles in autonomous materials systems, and pre-programmed processing and modification of surface loads in the materials interior are some examples of their potentially diverse applications. In a broader sense, control and analysis of spectral content of static deformation is an interesting subject that has been widely overlooked in materials mechanics. In this paper, we demonstrate a mechanics analogue of the Parseval’s theorem from the field of information theory and signal processing. This theorem gives a strain energy spectral density (SESD), whose spatial evolution provides a fundamental insight on how mechanical stresses are transformed and modified in the material interior. The strain energy spectral density also...
enables a straightforward calculation of a spectral entropy of static deformation and its variance with
distance to loads. Spectral entropy is a measure of smoothness of the energy spectrum and therefore can
indicate proximity of defects in the material, or unusual and potentially hazardous load patterns. We also
show that the spectral entropy has a simple relation to the Shannon’s information entropy [11-12], and
therefore, it can serve as a measure of information contained in the material deformation about any
available surface and volume forces.

Spectral energy density and the corresponding spectral entropy of deformation for a Gauss-type pressure
load acting on surface of a plane solid are derived analytically as functions of distance to the load. A
discussion is also provided on approaches to numerical calculation of the spectral entropy in general solid
mechanics problems. The reader will clearly see what information is contained in mechanical deformation
of a homogenous elastic continuum, and it how changes when the strain energy spectrum evolves in the
material interior with distance to the surface load. Also, approaches presented here can be extended to
analyze energy and entropic properties of various classes of advanced microstructured materials, and
even mechanical metamaterials, including those featuring the non-reciprocity of mechanical deformation
[9] and the Saint-Venant’s effect reversal [8]. These material systems may demonstrate anomalous strain
energy transformation capabilities and enable a wide range of interesting practical opportunities.

2. Parseval’s theorem and strain energy spectral density

A mechanics version of the Parseval’s theorem, also known as energy theorem of the Fourier transform,
see references [13-15] and the equation (A.1) in Appendix, can be written in a variety of ways. For
example, we could consider a standard volumetric or spatial density of strain energy for an arbitrary plane
stress problem, as a quadratic form of the strain components,

\[ W(x, y) = \frac{1}{2} \varepsilon^*(x, y) \mathbf{E} \varepsilon(x, y) \quad (1) \]

where \( \mathbf{E} \) is a constitutive matrix (A.15), and \( \varepsilon^* \) is a conjugate transpose of the vector of strain components,

\[ \varepsilon(x, y) = \begin{bmatrix} \varepsilon_x(x, y) \\ \varepsilon_y(x, y) \\ \gamma_{xy}(x, y) \end{bmatrix} \quad (2) \]

and then integrate (1) over the \( y \)-coordinate,

\[ \Pi(x) = \int_{-\infty}^{\infty} W(x, y) dy \quad (3) \]

This integral gives a distribution of the strain energy in the \( x \)-axis direction. We may represent the
vector (2) via the inverse Fourier transform (A.3) of its own Fourier image (A.2),

\[ \bar{\varepsilon}(x, q) = \int_{-\infty}^{\infty} \varepsilon(x, y) e^{-i q y} dy \quad (4) \]

and rewrite (3) as the following:
\[ \Pi(x) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \bar{\varepsilon}'(x, q') e^{-i \gamma y} dq' \right) \mathbf{E} \left( \int_{-\infty}^{\infty} \bar{\varepsilon}(x, q) e^{i \gamma y} dq \right) dy \]  

(5)

Next, rearrange this expression by changing the integration order,

\[ \Pi(x) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-i \gamma y} e^{i \gamma y} dq' \right) \bar{\varepsilon}'(x, q') \mathbf{E} \bar{\varepsilon}(x, q) d\gamma dq' \]

\[ = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(q - q') \bar{\varepsilon}'(x, q') \mathbf{E} \bar{\varepsilon}(x, q) d\gamma dq' \]

(6)

Here, \( \delta \) is the Dirac’s delta function (A.8-9) which removes the integral over \( q' \) and gives finally

\[ \Pi(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \bar{\varepsilon}'(x, q) \mathbf{E} \bar{\varepsilon}(x, q) dq \]

(7)

A comparison (3) and (7) proves the **strain energy spectral theorem**: An integral of strain energy volumetric density \( W \) is equal to a Fourier integral of strain energy spectral density \( \tilde{W} \). Equivalently, an integral over a quadratic form of strain components is equal to an integral over the quadratic form of Fourier transforms of these strain components:

\[ \int_{-\infty}^{\infty} W(x, y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{W}(x, q) dq \]

(8)

\[ W(x, y) = \frac{1}{2} \bar{\varepsilon}'(x, y) \mathbf{E} \varepsilon(x, y) \]

(9)

\[ \tilde{W}(x, q) = \frac{1}{2} \bar{\varepsilon}'(x, q) \mathbf{E} \varepsilon(x, q) \]

(10)

The strain energy spectral density (SESD) function, \( \tilde{W}(x, q) \), is a very interesting characteristic of a state of deformation. Dependence of this function on \( q \) shows a spectral density (spectral distribution) of the strain energy, and it tells us how much strain energy is contained in the wavenumber interval from \( q \) to \( q + dq \). Furthermore, its dependence on \( x \) determines how this spectral density transforms from one point in space to another.

Considering a general three-dimensional energy density \( W(x, y, z) \) and applying single or multiple integrals similar to (3), we may also get functions of the type \( \tilde{W}(x, y, q) \), as well as \( \tilde{W}(x, q_1, q_2) \) and a full transform \( \tilde{W}(q_1, q_2, q_3) \) under a double or triple Fourier integral in (8). Examples are shown in Appendix (A.10-12). The function \( \tilde{W}(x, q_1, q_2) \) will show a two-dimensional spectral distribution of the strain energy contained in the layer \( (x, x + dx) \) of the material. However, in the analysis of material responses to surface loads, strain energy spectral density of the type (10) or (A.11) depending on a single coordinate, as a distance to the loads, will be most interesting in the practice of transformation mechanics studies.

For the discussion to follow, we also introduce the normalized spatial (\( w \)) and spectral (\( \tilde{w} \)) densities,

\[ w(x, y) = \frac{W(x, y)}{\Pi(x)} \]

\[ \tilde{w}(x, q) = \frac{1}{2\pi} \frac{\tilde{W}(x, q)}{\Pi(x)} \]

(11)

(12)
such that

\[ \int_{-\infty}^{\infty} w(x, y) dy = \int_{-\infty}^{\infty} \tilde{w}(x, q) dq = 1 \quad (13) \]

Transformation or evolution of the strain energy spectral density in space (with the coordinate \( x \)) will be specific to a given material, and a recent study [8] shows that microstructured materials may be highly selective to certain spectral components of their deformation patterns. In particular, some Fourier modes may dissipate unexpectedly fast in the material volume or even get localized on the surface. The present formalism can help to understand these interesting behaviors, and in Section 6 below, we initiate the discussion by considering an isotropic continuum as a reference case.

3. Differential spectral entropy

Knowledge of a normalized spectral density of strain energy (12) enables us to determine the corresponding spectral entropy. **Strain energy spectral entropy (SESE),** as a function of the coordinate \( x \), can be written using the Shannon’s definition of differential entropy [11-12],

\[ S(x) = -\int_{-\infty}^{\infty} \tilde{w}(x, q) \ln \tilde{w}(x, q) \ dq \quad (14) \]

The functional \( S \) is a measure of complexity of the strain field, or a measure of disorder introduced to the elastic continuum by mechanical forces.

The strain energy spectral entropy (14) will generally be higher near stress concentrators, localized surface loads or a physical inhomogeneity in the form of voids, cracks and secondary phase inclusions. Indeed, stress concentrators give volumetric strain energy density functions \( W \) highly localized in space, whose reciprocal space counterpart \( \tilde{W} \), on the contrary, will be a smooth function of the wavenumber \( q \). Although \( \tilde{W} \) is not a Fourier transform of \( W \), this example is insightful: Fourier transform of a Gaussian function is also a Gaussian function, but with an inverse width,

\[ \tilde{w}_G(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{a^2 q^2}{2}} e^{-i q y} dy = \frac{1}{a\sqrt{2\pi}} e^{-\frac{q^2}{2a^2}} \quad (15) \]

In simple terms, a narrow function in the coordinate space generally corresponds to a wide function in the reciprocal space, and vice versa. We further note that differential entropy of the normal distribution (15) is a logarithmic function of its width \( a \),

\[ S_G = -\int_{-\infty}^{\infty} \tilde{w}_G(q) \ln \tilde{w}_G(q) \ dq = \ln \left( \frac{a}{\sqrt{2\pi}} \right) \quad (16) \]

Therefore, theoretical range of the differential entropy integral (14) is from \(-\infty\) to \( \infty \), when the width \( a \) of the normal distribution varies from 0 to \( \infty \). Moreover, if the material coordinates are not dimensionless originally, the function \( \tilde{w}(x, q) \) will have a physical dimensionality of length. Then, a dimensionless coordinate \( y/\Lambda \) should be introduced leading to a scaling factor \( 1/\Lambda \) for \( \tilde{w}(x, q) \), and because of the negative logarithm in (14) and property (13), this will produce a shift of the entire entropy function (14)
by a constant proportional to \( \ln \Lambda \). Thus, only a relative change of the differential entropy (14) between two points in a coordinate space can have a physical meaning, while its absolute value will generally depend on a physical scaling of that coordinate space.

4. Numerical entropy calculation

Fortunately, the unlimited range of the differential entropy of mechanical deformation (14), varying from \(-\infty\) to \(\infty\), and uncertainty of its absolute value are only “theoretical artifacts”. Its unbound character is a formal mathematical consequence of an absolute accuracy of the analytical integration, which is only possible for a few simple problems (some are discussed Section 6). In below, we explain how practical numerical calculations will produce a limited positive entropy, independent of the coordinate scaling, and how it can be related to the exact integral (14).

In practical solid mechanics problems, solved by the finite element or other numerical methods, entropy calculation will begin with evaluation of the strain components (2) on a discrete set of material points \((x, y_m), m = 0, \pm 1, ..., \pm M/2\). These points should be equidistant from each other along the \(y\)-axis on a sufficiently large interval of length \(L\), so that \(y_0 = 0\), and their spacing along the \(x\)-axis can be arbitrary.

Then, discrete Fourier transform of the strain components are calculated,

\[
\tilde{\varepsilon}_x(x, \mu) = \sum_{m=-M/2}^{M/2-1} \varepsilon_x(x, y_m) e^{-i q \mu y_m}, \quad q \mu = \frac{2\pi \mu}{L}, \quad \mu = 0, \pm 1, ..., \pm M/2
\]

and similar for \(\tilde{\varepsilon}_y\) and \(\tilde{\gamma}_{xy}\), leading to the discrete spectral values \(\tilde{W}(x, q\mu)\) by the equation (10),

\[
\tilde{W}(x, \mu) = \frac{1}{2} \tilde{\varepsilon}^*(x, \mu) \tilde{\varepsilon}(x, \mu)
\]

The entropy integral (14) is then replaced (at each \(x\)) with a finite sum,

\[
S(x) \approx -\sum_{\mu=-M/2}^{M/2-1} p_\mu(x) \ln \frac{p_\mu(x)}{\Delta q}, \quad \Delta q = q_{\mu+1} - q_\mu = \frac{2\pi}{L}
\]

Since \(\sum p_\mu(x) = 1\) for any \(x\), we may also write a relationship between the differential entropy \(S\) and a discrete spectral entropy \(H\):

\[
S(x) \approx H(x) + \ln \frac{2\pi}{L}
\]

\[
H(x) = -\sum_{\mu=-M/2}^{M/2-1} p_\mu(x) \ln p_\mu(x)
\]

As can be seen, the discrete entropy \(H\) is limited in range from 0 to \(\ln M\). The zero value occurs for a Kronecker delta (A.19) distribution, \(p_\mu = \delta_{\mu,0}\), because of the property \(0 \ln 0 = 0\), and the maximal value (\(\ln M\)) may happen for a uniform distribution, where \(p_\mu = 1/M\) for all \(\mu\). Since an exact continuous
spectral density function (10,12) can be represented by an infinite number of Fourier harmonics ($M \to \infty$), the unbound nature of the differential entropy (14) is now more intuitive.

Thus, we may write a final **normalized form** of the strain energy spectral entropy that can be used in computational solid mechanics:

$$h(x) = \frac{1}{\ln M} H(x) = -\frac{1}{\ln M} \sum_{\mu=-M/2}^{M/2-1} p_\mu(x) \ln p_\mu(x) \quad (23)$$

The range of this entropy measure is limited to [0, 1]. A Kronecker delta distribution $p_\mu = \delta_{\mu 0}$, leading to $h = 0$, is the case of a uniform mechanical deformation. A uniform distribution $p_\mu = 1/M$, corresponding to $h \to 1$, may be seen in vicinity of sharp stress concentrators, because of the obvious property (A.18). As can be seen, the values $p_\mu$ are dimensionless for any scaling of the material coordinates, and therefore the discrete spectral entropy (23) is an invariant measure of a state of deformation, similar strain.

When an exact differential entropy $S(x)$ is available by equation (14) for a simple benchmark problem (see Section 6), its modified form $s(x)$ can be used to validate the normalized numerical entropy $h(x)$:

$$s(x) = \frac{1}{\ln M} \left( S(x) + \ln \frac{L}{2\pi} \right) \approx h(x) \quad (24)$$

Fig.1: A Gauss-type load component acting on a plane solid at $x = 0$, and evolution of the normal stress $\sigma_x$ by equations (35-36).

5. **Information content of mechanical deformation**

Mathematical form of spectral entropy of mechanical deformation (22) is identical to the Shannon’s information entropy in the field of data science and signal processing, for example [11-12]. The Shannon’s entropy represents a mean information per one observed event in a random process, where $p_\mu$ is a probability of each possible outcome. This gives a hint that mechanical deformation can be interpreted as information as well.
For example, assume that the strain energy spectrum (10) represents an average spectrum of many random static loads applied to an elastic body at different times. Then the value $p_{\mu}$ in (20) gives a probability for these loads to produce a Fourier harmonic $q_{\mu}$ in the strain energy spectrum of the materials at the position $x$. Therefore, the entropy (22) is a mean information contained in the strain energy spectrum per one Fourier harmonic about any unknown or random loading conditions applied to the elastic body. In the area of signal processing, a unit entropy is attributed to a white noise signal, comprised of all possible Fourier harmonics with an equal probability. It corresponds to a complete disorder in the system sourcing such a signal. A state of perfect order is assigned to a constant or purely harmonic signal with a spectral density in the form of a Dirac delta function, whose entropy is equal to zero. Logically, such a perfect signal cannot transmit information, and a practical data transmission requires an amplitude or frequency modulated signal.

Thus, spectral entropy of mechanical deformation, in homogeneous materials, will always decrease from a loaded boundary toward material’s interior (examples are shown in Section 6). This entropy behavior will represent a loss of information about details of surface loads or tractions acting on that boundary, in accordance with the Saint-Venant’s principle. When the strain components become zero or constant in regions very far away from the loads, the entropy (22,23) will approach zero. A perfectly uniform deformation contains no information about the pattern and location of the loads that produce this deformation. The material in those regions is “unaware” of how exactly it is loaded, because a great number of possible loading scenarios can produce the same uniform deformation pattern in the material interior. When the entropy is nonzero, at least some amount of information about the loads is available. As a measure of information about loading patterns, spectral entropy could be useful for inverse problems of solid mechanics, helping to determine existence and uncertainty of the inverse solutions.

6. Example: Gauss-type distributed and point loads

Simple analytical forms of the strain energy density (10,12) and differential entropy (14) can be obtained for a state of plane stress in an elastic continuum loaded at $x = 0$ with a Gauss-type pressure force (see Figure 1) and governed by the homogenous Navier’s equations [16] for the displacement field at $x > 0$:

$$
2u''_{xx}(x,y) + (1 + \nu)v''_{xy}(x,y) + (1 - \nu)u''_{yy}(x,y) = 0
$$

$$
(1 - \nu)v''_{xx}(x,y) + (1 + \nu)v''_{xy}(x,y) + 2v''_{yy}(x,y) = 0
$$

(25)

We can utilize a fundamental solution to these equations in the form of static Raleigh waves [8],

$$
C(q)e^{-\eta x}e^{iqy}
$$

(26)

The shape of these waves is preserved in the material interior, but the amplitude decays at an exponential rate $\eta$. An integral superposition of these waves gives

$$
u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_2(q)e^{-\eta x}e^{iqy} dq
$$

$$
\nu(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_2(q)e^{-\eta x}e^{iqy} dq
$$

(27)
Such a solution can satisfy (25), but only if $C_2 = -i \text{sgn}(q)C_1$, and

$$\eta = |q|$$

(28)

Thus, a general static Raleigh wave solution for a plane stress problem reads

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(q)e^{-|q|x}e^{iqy} dq$$

$$v(x, y) = -i \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} \text{sgn}(q)C(q)e^{-|q|x}e^{iqy} dq$$

(29)

Here, $\text{sgn}$ is the sign function (A.13), and $C(q)$ is any arbitrary complex-valued function of the wave number $q$ that can be selected to satisfy boundary conditions, $u(0, y)$ and $v(0, y)$. Both real and imaginary parts of the solution (29) will separately satisfy the governing equation (25). We also note that (29) is valid for a state of plane strain as well.

Interestingly, the displacement solution (29) represents an inversion (A.3) of its own Fourier transform over the coordinate $y$, parallel to the solid boundary,

$$\tilde{u}(x, q) = C(q)e^{-|q|x}$$

$$\tilde{v}(x, q) = -i \text{sgn}(q)C(q)e^{-|q|x}$$

(30)

For any $q$, it will satisfy a Fourier image of the governing equations, according to (A.4),
\[
2\dddot{u}_{xx}(x,q) + iq(1 + \nu)\dddot{v}_{x}(x,q) - q^2(1 - \nu)\dddot{u}(x,q) = 0
\]
\[
(1 - \nu)\dddot{v}_{xx}(x,q) + iq(1 + \nu)\dddot{u}_{x}(x,q) - 2q^2\dddot{v}(x,q) = 0
\]

It is interesting to consider the following case of the Fourier domain solution (30), up to a constant factor,

\[
\dddot{u}(x,q) = \frac{1}{a|q|} e^{-|q|x - \frac{q^2}{2a^2}}
\]
\[
\dddot{v}(x,q) = -\frac{i}{aq} e^{-|q|x - \frac{q^2}{2a^2}}
\]

where \(a\) is a spectral width of deformation. This leads to the strain components, due to (A.4),

\[
\dddot{\varepsilon}_x(x,q) = -\dddot{\varepsilon}_y(x,q) = -\frac{1}{a} e^{-|q|x - \frac{q^2}{2a^2}}
\]
\[
\dddot{\gamma}_{xy}(x,q) = \frac{2i \text{sgn}(q)}{a} e^{-|q|x - \frac{q^2}{2a^2}}
\]

whose originals, as functions of the coordinate \(y\), can be written in terms of the complementary error function \(\text{erfc}\) of (A.14),

\[
\dddot{\varepsilon}_x(x,y) = -\dddot{\varepsilon}_y(x,y) = -\frac{1}{\sqrt{8\pi}} (\beta + \beta^*)
\]
\[
\dddot{\gamma}_{xy}(x,y) = \frac{i}{\sqrt{2\pi}} (\beta - \beta^*)
\]
\[
\beta = \beta(x,y) = e\frac{a^2(x+iy)^2}{2} \text{erfc}\frac{a(x+iy)}{\sqrt{2}}
\]

Because of (A.15), the corresponding stress components are

\[
\sigma_x(x,y) = -\sigma_y(x,y) = E\frac{1}{1+\nu}\dddot{\varepsilon}_x(x,y)
\]
\[
\tau_{xy}(x,y) = E\frac{2+2\nu}{2+2\nu}\dddot{\gamma}_{xy}(x,y)
\]

and therefore, the horizontal traction force at \(x = 0\) is simply a Gaussian in space of width \(1/a\), see Figure 1,

\[
T_x(y) = -\sigma_x(0,y) = E\frac{a^2y^2}{\sqrt{2\pi(1+\nu)}} e^{-\frac{a^2y^2}{2}}
\]

Now, we can write analytically the strain energy spatial density,

\[
W(x,y) = \frac{1}{2} \dddot{\varepsilon}_x(x,y) E\dddot{\varepsilon}_x(x,y) = \frac{Ee a^2(x^2-y^2)}{2\pi(1+\nu)} \text{erfc}\frac{a(x+iy)}{\sqrt{2}} \text{erfc}\frac{a(x-iy)}{\sqrt{2}}
\]

as well as the strain energy spectral density,

\[
\bar{W}(x,q) = \frac{1}{2} \dddot{\varepsilon}_x(x,q) E\dddot{\varepsilon}_x(x,q) = \frac{E(1+\text{sgn}^2q)}{a^2(1+\nu)} e^{-2|q|x - \frac{q^2}{a^2}}
\]

We note that these energy densities are valid for both plain stress and plain strain constitutive matrices (A.15). Their evolution in the material interior with \(x\) is shown in Figure 2.
Fig. 3: (top) Evolution of the effective width of the strain energy density distributions for Fig. 1 problem, and (bottom) the corresponding differential spectral entropy plotted from the analytical form (42). According to (45), $S(x) \approx 1 - \ln x$, when $x \gg 1/a$.

In agreement with the Parseval’s theorem (8), derived in the previous section, an integral of the spatial density (37) over the coordinate $y$ is indeed equal to the integral of the spectral density (38) over the wavenumber $q$:

$$\Pi(x) = \int_{-\infty}^{\infty} W(x, y) \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{W}(x, q) \, dq = \frac{E_0 a^2 x^2}{a\sqrt{\pi(1+\nu)}} \text{erfc} ax$$

(39)

Next, we can write the normalized energy density, both spatial and spectral,

$$w(x, y) = \frac{W(x, y)}{\Pi(x)} = \frac{ae^{-a^2 y^2} \text{erfc} \frac{a(x+y)}{\sqrt{2}} - \text{erfc} \frac{a(x-y)}{\sqrt{2}}}{\sqrt{4\pi} \text{erfc} ax}$$

(40)

$$\tilde{w}(x, q) = \frac{1}{2\pi} \frac{\tilde{W}(x, q)}{\Pi(x)} = \frac{(1+\text{sgn}^2 q) \, e^{-\frac{q^2}{a^2} - \frac{a^2 q^2}{2(1+\nu)}}}{2a\sqrt{\pi} \text{erfc} ax}$$

(41)
The spectral density function (41) evolves in space narrowing with \( ax \), as shown in Figures 2 and 3, indicating that the elastic continuum works as a low-pass filter for the static Raleigh waves (26) filtering out higher Fourier harmonics from the strain energy spectrum. This transformation of the energy spectrum may continue further in the elastic continuum with distance to loads, until harmonics with \( q \approx 0 \) begin to dominate the spectrum and the material deformation becomes uniform.

The spectral density function (41) allows solving the differential entropy integral (14) analytically:

\[
S(x) = -\int_{-\infty}^{\infty} \tilde{w}(x, q) \ln \tilde{w}(x, q) \, dq = \frac{ax}{\sqrt{\pi} \text{erfc} ax} + \ln(a\sqrt{\pi e} \text{erfc} ax)
\]

(42)

This entropy function decreases steadily with \( x \) at any \( a \), and it is independent of the continuum material properties, \( E \) and \( \nu \). A general reason for this decreasing behavior is the narrowing of the spectral density (41) with \( x \). We also see a logarithmic increase of \( S(x) \) with the spectral width \( a \), as expected from the example (16). The initial value of (42) at the material boundary is \( S(0) = \ln a\sqrt{\pi e} \), which is similar to the differential entropy of the normal distribution (16).

The normalized spatial (40) and spectral (41) energy densities, and the entropy of deformation (42) have simple asymptotic forms at large distances \( x \), compared to the load width \( x \gg 1/a \), i.e. at \( a \to \infty \),

\[
w_0(x, y) = \lim_{a \to \infty} w(x, y) = \frac{x}{\pi(x^2+y^2)}
\]

(43)

\[
\tilde{w}_0(x, q) = \lim_{a \to \infty} \tilde{w}(x, q) = xe^{-2|q|y}
\]

(44)

\[
S_0(x) = \lim_{a \to \infty} S(x) = 1 - \ln x
\]

(45)

Since function \( w_0(x, y) \) approaches a Dirac delta function \( \delta(y) \), when \( x \to 0 \), the results (43-45) also describe the limit case of a localized (point) load of a unit intensity, acting at the coordinate origin normal to the material surface \( x = 0 \). In case of multiple points loads spaced periodically from each other on a distance \( L \) along the line \( x = 0 \), the following \( L \)-periodic energy densities and entropy function can be obtained with a Fourier series decomposition of the displacement field in (25):

\[
w_L(x, y) = \frac{(e^{4\pi y/L}-1)/L}{1+e^{4\pi y/L}-2e^{2\pi y/L}\cos 2\pi y/L}
\]

(46)

\[
\tilde{w}_L(x, \mu) = \frac{(2-\delta_{\mu 0})e^{-4\pi|\mu|y/L}}{2 \coth 2\pi y/L - 1}
\]

(47)

\[
S_L(x) = -\frac{4\ln 2}{3+e^{4\pi y/L}} + \ln(2 \coth 2\pi y/L - 1) + \frac{8\pi y/L}{1-\cosh 4\pi y/L + 2 \sinh 4\pi y/L}
\]

(48)

These formulas are also valid for the case of a finite domain with periodic boundary conditions applied along the lines \( y = \pm L/2 \) and a single point load acting at the origin normal to the material surface. Here, \( \lim_{L \to \infty} w_L(x, y) = w_0(x, y) \), \( \lim_{L \to \infty} L \cdot \tilde{w}_L(x, qL/2\pi) = 2\pi \tilde{w}_0(x, q) \) and \( \lim_{L \to \infty} (S_L(x) - \ln L/2\pi) = S_0(x) \), and the integer wavenumber index \( \mu \) is identical to one defined in (17).
Fig. 4: Comparisons of the modified analytical (24,42) and numerical (23) spectral entropies as functions of the material coordinate for Fig.1 problem. An accurate match is achieved with sufficiently large values of the model parameters, $La$ and $M/La$.

Analytical entropy derivation of the type (42) or (45,48) is limited to a few simple problems, where an exact strain field solution is known. Most practical calculations will employ the discrete numerical entropy measure (23) discussed in Section 4. Therefore, it is interesting to note overall behavior of the numerical entropy (23) and its convergence with the exact differential entropy (24,42) in the same solid mechanics problem. For this purpose, imagine that accurate values of the strain components (2) are only known at a discrete set of material points $(x, y_m)$, $y_m = mL/M, m = 0, \pm 1, ..., \pm M/2$. Spacing of these points along the $x$-axis is arbitrary, and we will assume 24 equidistant positions of $x$ in all calculations discussed below. In practice, the strain values $\varepsilon(x, y_m)$ will come from a finite element procedure, but here we calculate them using the formulas (34) $M$ times for each $x$. Next, we select a spectral width parameter $a$, assume a value $x$, and calculate discrete Fourier transforms of the strain components in the $y$-axis direction according to (17). They lead to a set of $M$ values of $\tilde{W}(x, q_\mu)$ for every $x$ by equation (18), and this set represents the discrete (numerical) strain energy spectrum at the position $x$. Then, the normalized set $p_\mu$ and the numerical entropy are calculated by equations (20) and (23).
Comparisons of the analytical, \( s(x) \), and numerical, \( h(x) \), entropies showed importance of only two dimensionless parameters: (1) ratio of the domain size \( L \) and the load width \( 1/a \), and (2) ratio of the load width \( 1/a \) and the finite element size \( L/M \). Naturally, a boundless continuum corresponds to the limits \( La \to \infty \) and \( M/La \to \infty \). Figure 4 data shows that a good match is possible when \( La > 360 \) and \( M/La > 1.5 \). When the \( M/La \) value is large enough, an accurate match exists at small distances to the surface or near the load. If the load contains a feature that is small compared to the finite element size (\( M/La \) is small), the numerical entropy is still limited by the unit value near the surface meaning an equal presence of all computationally resolved Fourier modes, while the differential entropy may take the higher values. Observing a numerical value \( h(x) \approx 1 \) in a practical solid mechanics model can serve as a universal indicator of insufficient accuracy of selected finite element mesh to resolve a small physical feature or a stress concentration present on the position \( x \). Referring again to Figure 4, when the \( La \) value is large enough, influence of the finite boundaries in numerical calculations is insignificant, and an accurate match of \( h(x) \) with \( s(x) \) continues at larger distances \( ax \) in the material interior. However, when the distance \( ax \) is increasingly large the numerical entropy (23) asymptotically approaches a zero, meaning presence of a uniform deformation pattern, while the differential entropy (24, 42) may continue to the negative zone, Figure 4 inset. We conclude that the strain energy spectral entropy in the discrete form (23) is a physically meaningful measure of complexity of mechanical deformation at given numerical accuracy of the model, and it is good for usage in computational solid mechanics.

7. Conclusions

In this paper, we showed a mechanics analogue of the Parseval’s theorem in application to the strain energy density of materials deformation: a volume integral of the strain energy density is equal to a Fourier integral over the strain energy spectral density (SESD) in the reciprocal space. The SESD can be readily written with the same quadratic form as for the usual volumetric strain energy density, \( \frac{1}{2} \varepsilon^\top \varepsilon \), where the strain components are replaced with their Fourier transforms. Dependence of the SESD on a coordinate, when a partial Fourier transform is used, represents evolution of the strain energy spectrum in the material interior.

A normalized strain energy spectrum leads directly to the strain energy spectral entropy. We discussed an exact differential entropy and a discrete (numerical) entropy, similar to the Shannon’s information entropy. Spectral entropy provides a basic quantitative measure of available information about surface loads in the material interior. It can also be interpreted as a measure of nonuniformity or disorder introduced in the elastic continuum by mechanical loads or physical inhomogeneities, such as voids, cracks and secondary phase inclusions. A physically meaningful numerical form (23) of the spectral entropy is suggested, which can be used in computational solid mechanics, and it has been benchmarked by available exact results.

Exact analytical forms of both strain energy spectral density and spectral entropy, as functions of distance \( x \) to the surface are obtained for Gauss-type and point loads acting on a flat boundary of a plane solid. Overall behavior of these functions provides a basic insight into the Saint-Venant’s effect. Surface
Load smoothening in the material interior is explained by narrowing of the strain energy spectral density with $x$. This narrowing of the energy spectrum with distance to surface loads occurs in an elastic continuum, because higher Fourier modes in the energy spectrum decays faster in the material interior than the lower ones, according to the equations (26) and (28). This spectrum evolution continues far into the material interior until the zero Fourier mode, which corresponds to a uniform mechanical deformation, begins to dominate the energy spectrum. Numerical entropy (23) of such a degenerate energy spectrum approaches zero, meaning a loss of information about all peculiarities of loading conditions at boundaries that produced this deformation.

A recent study [8] show that discontinuous periodic lattice structures may demonstrate nonlinear and even non-monotonous Raleigh wave decay parameter’s dependence on the wavenumber, in contrast to the simple linear relationship (28). Therefore, advanced material systems, mechanical metamaterials [1-10], microstructured and lattice materials [17-18] could demonstrate more complex pathways of the strain energy and information entropy evolution, compared to the elastic continuum considered here. This could enable many interesting applications of the future transformation mechanics studies.

Applications to periodic microstructured materials is a future task, where a discrete strain distribution over lattice degrees of freedom can be introduced followed by similar numerical procedures as described in Section 5. Three-dimensional problems can be treated with relationships (A.10-12), and in transient problems, entropy of the energy spectrum can be related to transfer properties of mechanical waves.

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Appendix

A common form of the Parseval’s theorem with a scalar function \( f \),

\[
\int_{-\infty}^{\infty} |f(y)|^2 \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(q)|^2 \, dq \tag{A.1}
\]

meaning that information is not lost in the Fourier transform of the function \( f \):

\[
\hat{f}(q) = \int_{-\infty}^{\infty} f(y) e^{-i q y} \, dy \tag{A.2}
\]

Fourier transform properties:

\[
f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(q) e^{i q y} \, dq \tag{A.3}
\]

\[
\int_{-\infty}^{\infty} \partial_y f(y) e^{-i q y} \, dy = i q \hat{f}(q) \tag{A.4}
\]

Partial and full Fourier transforms of a function of several variables:

\[
\hat{f}(x, q, z) = \int_{-\infty}^{\infty} f(x, y, z) e^{-i q y} \, dy \tag{A.5}
\]

\[
\int_{-\infty}^{\infty} \partial_x f(x, y, z) e^{-i q y} \, dy = \partial_x \hat{f}(x, q, z) \tag{A.6}
\]

\[
\hat{f}(q_1, q_2, q_3) = \int_V f(x, y, z) e^{-i(q_1 x + q_2 y + q_3 z)} \, dV \tag{A.7}
\]

Properties of the Dirac delta function \( \delta \) (\( y \), \( q \) and \( q_0 \) are real):

\[
\int_{-\infty}^{\infty} e^{i q y} e^{-i q' y} \, dy = 2\pi \delta(q - q') \tag{A.8}
\]

\[
\int_{-\infty}^{\infty} \delta(q - q') f(q') \, dq' = f(q) \tag{A.9}
\]

Variants of the Parseval’s theorem of mechanics (8) with a fully three-dimensional energy density,

\[
\int_{-\infty}^{\infty} W(x, y, z) \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{W}(x, q, z) \, dq \tag{A.10}
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, y, z) \, dy \, dz = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{W}(x, q_1, q_2) \, dq_1 \, dq_2 \tag{A.11}
\]

\[
\int_V W(x, y, z) \, dV = \frac{1}{8\pi^3} \int_Q \tilde{W}(q_1, q_2, q_3) \, dQ \tag{A.12}
\]

Sign function:

\[
\text{sgn}(q) = \begin{cases} 
1, & q > 0 \\
0, & q = 0 \\
-1, & q < 0 
\end{cases} \tag{A.13}
\]

Complementary error function:
erfc \( z \) = \( \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\alpha^2} d\alpha \) \hspace{1cm} (A.14)

Plane stress (\( \sigma \)) and plane strain (\( \varepsilon \)) constitutive matrices with Young’s modulus \( E \) and Poison’s ratio \( \nu \):

\[
E_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2} (1 - \nu) \end{bmatrix} \quad E_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & \frac{1}{2} - \nu \end{bmatrix}
\] \hspace{1cm} (A.15)

Discrete Fourier transform of a discrete \( M \)-periodic sequence \( g_m \), such that \( g_{m+M} = g_m \):

\[
\tilde{g}_{\mu} = \sum_{m=-M/2}^{M/2-1} g_m e^{-i2\pi \mu m/M}, \quad \mu = 0, \pm 1, \ldots, \pm M/2
\] \hspace{1cm} (A.16)

\[
g_m = \frac{1}{M} \sum_{\mu=-M/2}^{M/2-1} \tilde{g}_{\mu} e^{i2\pi \mu m/M}, \quad m = 0, \pm 1, \ldots, \pm M/2
\] \hspace{1cm} (A.17)

Discrete Fourier transform of a Kronecker delta:

\[
\sum_{m=-M/2}^{M/2-1} \delta_{m0} e^{-i2\pi \mu m/M} = 1
\] \hspace{1cm} (A.18)

where

\[
\delta_{mm'} = \begin{cases} 1, & m = m' \\ 0, & m \neq m' \end{cases}
\] \hspace{1cm} (A.19)