Generalized Amitsur–Levitski theorem and equations for Sheets
in a Reductive complex Lie algebra

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0. Introduction

0.1. The Amitsur-Levitski theorem is a famous result stating that matrix algebras satisfy a certain identity often referred to as the standard identity. This result can be generalized when viewed as a statement in Lie theory. If $\mathfrak{g}$ is a complex reductive Lie algebra the generalization involves the nilcone of $\mathfrak{g}$.

If $U$ is the universal enveloping algebra then in [2] we have shown

$$U = Z \otimes E$$

(0.1)

where $E$ is the graded $\mathfrak{g}$–module (under the adjoint action) spanned all powers of nilpotent elements in $\mathfrak{g}$ and $Z = \text{Cent } U$. Since $U$ is a quotient of the tensor algebra of $\mathfrak{g}$ one has a $\mathfrak{g}$-map

$$\Gamma_T : A(\mathfrak{g}) \to U$$

(0.2)

where $A(\mathfrak{g})$ is the graded $\mathfrak{g}$-module of alternating tensors. We have proved in (4) that

$$\Gamma_T(A^{2k}(\mathfrak{g})) \subset E^k$$

(0.3)

for any $k \in \mathbb{Z}$. If $\pi$ is a representation of $U$ on a vector space $V$ this obviously implies

$$\text{Ker } \pi|\Gamma_T(A^{2k}(\mathfrak{g})) \subset \text{Ker } \pi|E^k$$

(0.4)

This as one easily notes generalizes the Amitsur–Levitski theorem.

0.2. Since the exterior algebra $\wedge \mathfrak{g}$ is also a quotient of the tensor algebra the quotient map defines an isomorphism

$$A(\mathfrak{g}) \to \wedge \mathfrak{g}$$
of graded $\mathfrak{g}$-modules. In particular by restriction an isomorphism

$$A^{even}(\mathfrak{g}) \rightarrow \wedge^{even} \mathfrak{g}$$  

(0.5)

noting the right side of (0.5) is a commutative algebra.

Identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the Killing form. We now consider the “commutative” analogue of the statements in §0.1. Let $P(\mathfrak{g})$ be the ring of polynomial functions on $\mathfrak{g}$. Analogous to (0.1) one has

$$P(\mathfrak{g}) = J \otimes H$$  

(0.6)

where $J$ is the algebra $P(\mathfrak{g})^{\mathfrak{g}}$ of polynomial invariants on $\mathfrak{g}$ and $H$ is the graded $\mathfrak{g}$-module of harmonic polynomials on $\mathfrak{g}$. (See Theorem 11 in [2]). Poincaré–Birkhoff–Witt symmetrization induces a $\mathfrak{g}$-module isomorphism

$$U \rightarrow P(\mathfrak{g})$$  

(0.7)

which restricts to a $\mathfrak{g}$-module isomorphisms

$$Z \rightarrow J$$  

$$E \rightarrow H$$  

(0.8)

Using these isomorphisms (0.2) and (0.3) now define $\mathfrak{g}$-module maps

$$\Gamma : \wedge^{even} \mathfrak{g} \rightarrow P(\mathfrak{g})$$  

(0.9)

and for $k \in \mathbb{Z}$,

$$\Gamma : \wedge^{2k} \mathfrak{g} \rightarrow H^k$$  

(0.10)

Let $R^k(\mathfrak{g})$ be the image of (0.10) so that $R^k(\mathfrak{g})$ is a $\mathfrak{g}$-module of harmonic polynomials on $\mathfrak{g}$ of degree $k$. The significance of $R^k(\mathfrak{g})$ has to do with the dimensions of $Ad \mathfrak{g}$ adjoint (=coadjoint) orbits. Any such orbit is symplectic and hence is even dimensional. For $j \in \mathbb{Z}$ let $\mathfrak{g}^{(2j)} = \{x \in \mathfrak{g} \mid \dim [\mathfrak{g}, x] = 2j\}$. We recall that a $2j \mathfrak{g}$-sheet is an irreducible component
of \( g^{(2j)} \). Let \( \text{Var} R^k(g) = \{ x \in g \mid p(x) = 0, \forall p \in R^k(g) \} \). In [4] (see Proposition 3.2 in [4]) we prove

**Theorem 0.1.** One has

\[
\text{Var} R^k(g) = \bigcup_{2j < 2k} g^{(2j)}
\]

or that \( \text{Var} R^k(g) \) is the set of all \( 2j \, g \)-sheets for \( j < k \).

One can explicitly describe the \( g \)-module \( R^k(g) \). (See §1.2 in [4]).

One notes that that there exists a subset \( \Pi(k) \) of the symmetric group \( \text{sym}(k) \), having the cardinality of is \((2k - 1)(2k - 3) \cdots 1\), such that the correspondence

\[
\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, (\nu((2k - 1)), \nu(2k)))
\]

sets up a bijection of \( \Pi(k) \) with the set of all partitions of \( 1, 2, \ldots, 2k \) into a union of subsets each of which has two elements. We also observe that \( \Pi(k) \) can and will be chosen such that \( \text{sg} \nu = 1 \) for all \( \nu \in \Pi(k) \).

The following is a restatement of the results in §3.2 of [4] (see especially (3.25) and (3.29) in [4])

**Theorem 0.2.** For any \( k \in \mathbb{Z} \) there exists a nonzero scalar \( c_k \) such that for any \( x_i \, i = 1, \ldots, 2k, \) in \( g \)

\[
\Gamma(x_1 \wedge \cdots \wedge x_{2k}) = c_k \sum_{\nu \in \Pi(k)} [x_{\nu(1)}, x_{\nu(2)}] \cdots [x_{\nu(2k-1)}, x_{\nu(2k)}]
\]

Furthermore the homogeneous polynomial of degree \( k \) on the right side of (0.12) is harmonic and \( R^k(g) \) is the span of all such polynomials for an arbitrary choice of the \( x_i \).

0. Introduction to results of our joint paper, [5], with Nolan Wallach
0.2. Henceforth assume that $\mathfrak{g}$ is simple so that the adjoint representation is irreducible. Let $n = \text{dim} \mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ so that $\ell = \text{dim} \mathfrak{h}$ is the rank of $\mathfrak{g}$. Let $\Delta_+$ be a choice of positive roots for $(\mathfrak{g}, \mathfrak{h})$ so that $n = \ell + 2r$ where $r = \text{card} \Delta_+$. One readily has that $R^k(\mathfrak{g}) = 0$ if $k > r$ and $k = r$ is the maximal value of $k$ for which $R^k(\mathfrak{g})$ is interesting. In fact $R^r(\mathfrak{g})$ is the variety of singular elements of $\mathfrak{g}$. The paper [5] is devoted to a study of $R^r(\mathfrak{g})$ and in particular to its remarkable $\mathfrak{g}$-module structure. Of course Theorem 0.2 may be applied to determine $R^r(\mathfrak{g})$. However in [5] we establish a different determination of $R^r(\mathfrak{g})$. The results of [2] imply that the adjoint representation, has multiplicity $\ell$ in $H$. Choosing a basis of $\mathfrak{g}$ and suitable generators of $J$ this means there exists a $\ell \times n$ matrix $Q$ whose rows are the $\ell$ occurrences of the adjoint representation in $H$. One notes that this matrix has $\binom{n}{\ell}$ minors of size $\ell \times \ell$. We prove

**Theorem 0.3.** The determinant of any $\ell \times \ell$ minor of $Q$ is an element of $R^r(\mathfrak{g})$ and indeed $R^r(\mathfrak{g})$ is the span of the determinants of all these minors.

If $\varphi \in \Delta_+$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let $\mathfrak{n}$ be the Lie algebra spanned by $e_\varphi$ for $\varphi \in \Delta_+$ and let $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ defined by putting $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Now a subset $\Phi \subset \Delta_+$ will be called an ideal in $\Delta_+$ if the span, $\mathfrak{n}_\Phi$, of $e_\varphi$, for $\varphi \in \Phi$, is an ideal of $\mathfrak{b}$. In such case let

$$\langle \Phi \rangle = \sum_{i=1}^{k} \varphi_i \quad (0.13)$$

It follows easily that $\langle \Phi \rangle$ is a dominant weight. Let $V_\Phi$ be an irreducible $\mathfrak{g}$-module having $\langle \Phi \rangle$ as highest weight. In [4] we have shown if $\Phi, \Psi$ are distinct ideals of $\Delta_+$ then

$$V_\Phi \text{ and } V_\Psi, \text{ define inequivalent representations of } \mathfrak{g} \quad (0.14)$$

Let $\mathcal{I}(\ell)$ be the set all ideals $\Phi$ in $\Delta_+$ such that $\text{dim} \mathfrak{n}_\Phi = \ell$.

**Remark 0.1** One notes that if $\mathfrak{g}$ is of type $A_\ell$ then $\text{card} \mathcal{I}(\ell)$ is $P(\ell)$ where $P$ here is the classical partition function

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Let $Cas \in Z$ be the quadratic Casimir element normalized by the condition that the eigenvalue of $Cas$ in the adjoint representation is 1. The following theorem, giving the rather striking $g$-module structure of $R^r(g)$ is one of the main result of [5]. The theory of abelian ideals of $b$ is an area of considerable current research activity. The proof of theorem makes direct contact with this theory.

**Theorem 0.4.** For any $\Phi \in \mathcal{I}(\ell)$ the $b$–ideal $n_\Phi$ is abelian. Furthermore as $g$–modules one has the equivalence

\[ R^r(g) \cong \sum_{\Phi \in \mathcal{I}(\ell)} V_\Phi \]  

so that $R^r(g)$ is a multiplicity one $g$–module with $\text{card} \mathcal{I}(\ell)$ irreducible components. In addition $Cas$ has the eigenvalue $\ell$ on each and every such component.

**Results**

1. Let $R$ be an associative ring and for any $k \in \mathbb{Z}$ and $x_1, \ldots, x_k$, in $R$ one defines an alternating sum of products

\[ [[x_1, \ldots, x_k]] = \sum_{\sigma \in \text{Sym } k} sg(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)} \]  

One says that $R$ satisfies the standard identity of degree $k$ if $[[x_1, \ldots, x_k]] = 0$ for any choice of the $x_i \in R$. Of course $R$ is commutative if and only if it satisfies the standard identity of degree 2.

Now for any $n \in \mathbb{Z}$ and field $F$, let $M(n, F)$ be the algebra of $n \times n$ matrices over $F$. The following is the famous Amitsur–Levitski theorem.

**Theorem 1.** $M(n, F)$ satisfies the standard identity of degree $2n$.

**Remark 1.** By restricting to matrix units , for a proof, it suffices to take $F = \mathbb{C}$.  

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Without any knowledge that it was a known theorem we came upon Theorem 1 in [1], a long time ago, from the point of Lie algebra cohomology. In fact the result follows from the fact that if $\mathfrak{g} = M(n, \mathbb{C})$, then the restriction to $\mathfrak{g}$ of the primitive cohomology class of degree $2n + 1$ of $M(n + 1, \mathbb{C})$ to $\mathfrak{g}$ vanishes.

Of course $\mathfrak{g}_1 \subset \mathfrak{g}$ where $\mathfrak{g}_1 = \text{Lie } SO(n, \mathbb{C})$. Assume $n$ is even. One proves that the restriction to $\mathfrak{g}_1$ of the primitive class of degree $2n - 1$ (highest primitive class) of $\mathfrak{g}$ vanishes on $\mathfrak{g}_1$. This leads to a new standard identity, namely

**Theorem 2.**

$$[[x_1, \ldots, x_{2n-2}]] = 0$$  \hspace{1cm} (2)

for any choice of $x_i \in \mathfrak{g}_1$. That is any choice of skew-symmetric matrices.

**Remark 2.** Theorem 2 is immediately evident when $n = 2$.

Theorems 1 and 2 suggest that standard identities can be viewed as a subject in Lie theory. Theorem 3 below offers support for this idea. Let $\mathfrak{r}$ be a complex reductive Lie algebra and let

$$\pi : \mathfrak{r} \to \text{End } V$$  \hspace{1cm} (3)

be a finite-dimensional complex completely reducible representation. If $w \in \mathfrak{r}$ is nilpotent then $\pi(w)^k = 0$ for some $k \in \mathbb{Z}$. Let $\varepsilon(\pi)$ be the minimal integer $k$ such that $\pi(w)^k = 0$ for all nilpotent $w \in \mathfrak{r}$. In case $\pi$ is irreducible one can easily give a formula for $\varepsilon(\pi)$ in terms of the highest weight. If $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) is given as above and $\pi$ (resp. $\pi_1$) is the defining representation then $\varepsilon(\pi) = n$ and $\varepsilon(\pi_1) = n - 1$. Consequently the following theorem (see [4]) generalizes Theorems 1 and 2.

**Theorem 3.** Let $\mathfrak{r}$ be a complex reductive Lie algebra and let $\pi$ be as above. Then for any $x_i \in \mathfrak{r}$, $i = 1, \ldots, 2\varepsilon(\pi)$, one has

$$[[\hat{x}_1, \ldots, \hat{x}_{2\varepsilon(\pi)}]] = 0$$  \hspace{1cm} (4)
where $\hat{x}_i = \pi(x_i)$.

2. Henceforth $\mathfrak{g}$, until mentioned otherwise, will be an arbitrary reductive complex finite dimensional Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$ and let $S(\mathfrak{g}) \subset T(\mathfrak{g})$ (resp. $A(\mathfrak{g}) \subset T(\mathfrak{g})$) be the subspace of symmetric (resp. alternating) tensors in $T(\mathfrak{g})$. The natural grading on $T(\mathfrak{g})$ restricts to a grading on $S(\mathfrak{g})$ and $A(\mathfrak{g})$. In particular, where multiplication is tensor product one notes

**Proposition 1.** $A^j(\mathfrak{g})$ is the span of $[[x_1, \ldots, x_j]]$ over all choices of $x_i$, $i = 1, \ldots, j$, in $\mathfrak{g}$.

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is the quotient algebra of $T(\mathfrak{g})$ so that there is an algebra epimorphism

$$\tau : T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

Let $Z = Cent U(\mathfrak{g})$ and let $E \subset U(\mathfrak{g})$ be the graded subspace spanned by all powers $e^j$, $j = 1, \ldots$ where $e \in \mathfrak{g}$ is nilpotent. In [2] (see Theorem 21 in [2]) we proved (where tensor product identifies with multiplication)

$$U(\mathfrak{g}) = Z \otimes E$$

In [4] (see Theorem 3.4 in [4]) we proved

**Theorem 4.** For any $k \in \mathbb{Z}$ one has

$$\tau(A^{2k}(\mathfrak{g})) \subset E^k$$

Theorem 3 is then an immediate consequence of Theorem 4. Indeed, using the notation of Theorem 3, let $\pi_U : U(\mathfrak{g}) \rightarrow End V$ be the algebra extension of $\pi$ to $U(\mathfrak{g})$. One then has
Theorem 5. If $E^k \subset \text{Ker } \pi_U$ then

$$[[\hat{x}_1, \ldots, \hat{x}_{2k}]] = 0 \quad (7)$$

for any $x_1, \ldots, x_{2k}$ in $\mathfrak{g}$.

3. The Poincaré–Birkhoff–Witt theorem says that the restriction $\tau : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a linear isomorphism. Consequently, given any $t \in T(\mathfrak{g})$ there exists a unique element $\tilde{t}$ in $S(\mathfrak{g})$ such that

$$\tau(t) = \tau(\tilde{t}) \quad (8)$$

Let $A_{\text{even}}(\mathfrak{g})$ be the span of alternating tensors of even degree. Restricting to $A_{\text{even}}(\mathfrak{g})$ one has a $\mathfrak{g}$-module map

$$\Gamma_T : A_{\text{even}}(\mathfrak{g}) \to S(\mathfrak{g})$$

defined so that if $a \in A_{\text{even}}(\mathfrak{g})$ then

$$\tau(a) = \tau(\Gamma_T(a)) \quad (9)$$

Now the (commutative) symmetric algebra $P(\mathfrak{g})$ over $\mathfrak{g}$ and exterior algebra $\wedge \mathfrak{g}$ are quotient algebras of $T(\mathfrak{g})$. The restriction of the quotient map clearly induces $\mathfrak{g}$-module isomorphisms

$$\tau_S : S(\mathfrak{g}) \to P(\mathfrak{g})$$
$$\tau_A : A_{\text{even}}(\mathfrak{g}) \to \wedge_{\text{even}} \mathfrak{g} \quad (10)$$

where $\wedge_{\text{even}} \mathfrak{g}$ is the commutative subalgebra of $\wedge \mathfrak{g}$ spanned by elements of even degree. We may complete the commutative diagram defining

$$\Gamma : \wedge_{\text{even}} \mathfrak{g} \to P(\mathfrak{g}) \quad (11)$$

so that on $A_{\text{even}}(\mathfrak{g})$ one has

$$\tau_S \circ \Gamma_T = \Gamma \circ \tau_A \quad (12)$$
By (6) one notes that for $k \in \mathbb{Z}$ one has

$$\Gamma : \wedge^{2k} \mathfrak{g} \rightarrow P^k(\mathfrak{g})$$  \hspace{1cm} (13)

The Killing form extends to a nonsingular symmetric bilinear form on $P(\mathfrak{g})$ and $\wedge \mathfrak{g}$. This enables us to identify $P(\mathfrak{g})$ with the algebra of polynomial functions on $\mathfrak{g}$ and to identify $\wedge \mathfrak{g}$ with its dual space $\wedge \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual space to $\mathfrak{g}$. Let $R^k(\mathfrak{g})$ be the image (13) so that $R^k(\mathfrak{g})$ is a $\mathfrak{g}$-module of homogeneous polynomial functions of degree $k$ on $\mathfrak{g}$. The significance of $R^k(\mathfrak{g})$ has to do with the dimensions of $Ad \mathfrak{g}$ adjoint (=coadjoint) orbits. Any such orbit is symplectic and hence is even dimensional. For $j \in \mathbb{Z}$ let $\mathfrak{g}^{(2j)} = \{ x \in \mathfrak{g} | \dim [\mathfrak{g}, x] = 2j \}$. We recall that a $2j \mathfrak{g}$-sheet is an irreducible component of $\mathfrak{g}^{(2j)}$. Let $Var R^k(\mathfrak{g}) = \{ x \in \mathfrak{g} | p(x) = 0, \forall p \in R^k(\mathfrak{g}) \}$. In [4] (see Proposition 3.2 in [4]) we prove

**Theorem 6.** One has

$$Var R^k(\mathfrak{g}) = \bigcup_{2j < 2k} \mathfrak{g}^{(2j)}$$  \hspace{1cm} (14)

or that $Var R^k(\mathfrak{g})$ is the set of all $2j \mathfrak{g}$-sheets for $j < k$.

Let $\gamma$ be the transpose of $\Gamma$. Thus

$$\gamma : P(\mathfrak{g}) \rightarrow \wedge^{\text{even}} \mathfrak{g}$$  \hspace{1cm} (15)

and one has for $p \in P(\mathfrak{g})$ and $u \in \wedge \mathfrak{g}$,

$$(\gamma(p), u) = (p, \Gamma(u))$$  \hspace{1cm} (16)

One also notes

$$\gamma : P^k(\mathfrak{g}) \rightarrow \wedge^{2k} \mathfrak{g}$$  \hspace{1cm} (17)

A proof of Theorem 6 depends upon establishing some nice algebraic properties of $\gamma$. Since we have, via the Killing form, identified $\mathfrak{g}$ with its dual, $\wedge \mathfrak{g}$ is the underlying space for a
standard cochain complex \((\wedge g, d)\) where \(d\) is the coboundary operator of degree +1. In particular if \(x \in g\) then \(dx \in \wedge^2 g\). Identifying \(g\) here with \(P^1(g)\) one has a map

\[ P^1(g) \to \wedge^2 g \]

\(18\)

**Theorem 7.** The map (15) is the homomorphism of commutative algebras extending (18). In particular for any \(x \in g\)

\[ \gamma(x^k) = (-dx)^k \]

(19)

The connection with Theorem 6 follows from

**Proposition 2.** Let \(x \in g\). Then \(x \in g^{(2k)}\) if and only if \(k\) is maximal such that \((dx)^k \neq 0\), in which case there is a scalar \(c \in \mathbb{C}^\times\) such that

\[ (dx)^k = c \ w_1 \wedge \cdots \wedge w_{2k} \]  

(20)

where \(w_i, i = 1, \ldots, 2k\), is a basis of \([x, g]\).

For a proof of Theorem 7 and Proposition 2 see Theorem 1.4 and Proposition 1.3 in [4].

We wish to explicitly describe the \(g\)-module \(R^k(g)\). (See §1.2 in [4]). Let \(J = P(g)^g\) so that \(J\) is the ring of \(Ad g\) polynomial invariants. Let \(Diff P(g)\) be the algebra of differential operators on \(P(g)\) with constant coefficients. One then has an algebra isomorphism

\[ P(g) \to Diff P(g), \ q \mapsto \partial_q \]

where for \(p, q, f \in P(g)\) one has

\[ (\partial_q p, f) = (p, qf) \]

(21)

and \(\partial_x\), for \(x \in g\), is the partial derivative defined by \(x\).
Let $J_+ \subset J$ be the $J$-ideal of all $p \in J$ with zero constant term and let

$$H = \{ q \in P(g) \mid \partial_p q = 0 \ \forall p \in J_+ \}$$

$H$ is a graded $g$-module whose elements are called harmonic polynomials. Then one knows (see Theorem 11 in [2]) that, where tensor product is realized by polynomial multiplication,

$$P(g) = J \otimes H \quad (22)$$

It is immediate from (21) that $H$ is the orthocomplement of the ideal $J_+ P(g)$ in $P(g)$. However since $\gamma$ is an algebra homomorphism one has

$$J_+ P(g) \subset \text{Ker} \gamma \quad (23)$$

since one easily has that $J_+ \subset \text{Ker} \gamma$. Indeed this is clear since

$$\gamma(J_+) \subset d(\land g) \cap (\land g)^g = 0$$

But then (16) implies

**Theorem 8.** For any $k \in \mathbb{Z}$ one has

$$R^k(g) \subset H$$

Let $\text{Sym}(2k, 2)$ be the subgroup of the symmetric group $\text{Sym}(2k)$ defined by

$$\text{Sym}(2k, 2) = \{ \sigma \in \text{Sym}(2k) \mid \sigma \text{ permutes the set of unordered pairs } \{(1, 2), (3, 4), \ldots, ((2k - 1), 2k)\} \}$$

That is if $\sigma \in \text{Sym}(2k, 2)$ and $1 \leq i \leq k$ there exists $1 \leq j \leq k$ such that as unordered sets

$$(\sigma(2i - 1), \sigma(2i)) = ((2j - 1), 2j)$$
It is clear that Sym$(2k, 2)$ is a subgroup of order $2^k \cdot k!$. Let $\Pi(k)$ be a cross-section of the set of left cosets of Sym$(2k, 2)$ in Sym$(2k)$ so that one has a disjoint union

$$Sym(2k) = \bigcup \nu Sym(2k, 2)$$

indexed by $\nu \in \Pi(k)$.

**Remark 3.** One notes that the cardinality of $\Pi(k)$ is $(2k - 1)(2k - 3)\cdots 1$ and the correspondence

$$\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, (\nu((2k - 1)), \nu(2k)))$$

sets up a bijection of $\Pi(k)$ with the set of all partitions of $(1, 2, \ldots, 2k)$ into a union of subsets each of which has two elements. We also observe that $\Pi(k)$ may be chosen - and will be chosen - such that $sg \nu = 1$ for all $\nu \in \Pi(k)$. This is clear since the $sg$ character is not trivial on Sym$(k, 2)$ for $k \geq 1$. The following is a restatement of the results in §3.2 of [4] (see especially (3.25) and (3.29) in [4])

**Theorem 9.** For any $k \in \mathbb{Z}$ there exists a nonzero scalar $c_k$ such that for any $x_i \ i = 1, \ldots, 2k$, in $\mathfrak{g}$

$$\Gamma(x_1 \wedge \cdots \wedge x_{2k}) = c_k \sum_{\nu \in \Pi(k)} [x_{\nu(1)}, x_{\nu(2)}] \cdots [x_{\nu(2k-1)}, x_{\nu(2k)}]$$

(25)

Furthermore the homogeneous polynomial of degree $k$ on the right side of (25) is harmonic and $R^k(\mathfrak{g})$ is the span of all such polynomials for an arbitrary choice of the $x_i$.

4. On the variety of singular elements – joint work with Nolan Wallach

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\ell = dim \mathfrak{h}$ so $\ell = rank \mathfrak{g}$. Let $\Delta$ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and let $\Delta_+ \subset \Delta$ be a choice of positive roots. Let $r = card \Delta_+$ so that $n = \ell + 2r$ where we fix $n = dim \mathfrak{g}$. We assume a well ordering is defined on $\Delta_+$. For any
φ ∈ ∆ let \( e_φ \) be a corresponding root vector. The choices will be normalized only insofar as \((e_φ, e_{-φ}) = 1\) for all \( φ \in ∆ \). From Proposition 2 one recovers the well known fact that \( \mathfrak{g}^{(2k)} = 0 \) for \( k > r \) and \( \mathfrak{g}^{(2r)} \) is the set of all regular elements in \( \mathfrak{g} \). One also notes then that (16) implies \( \text{Var } R^r(\mathfrak{g}) \) reduces to 0 if \( k > r \) whereas Theorem 6 implies

\[
\text{Var } R^r(\mathfrak{g}) \text{ is the set of all singular elements in } \mathfrak{g}
\] (26)

The paper [5] is mainly devoted to a study of a special construction of \( R^r(\mathfrak{g}) \) and a determination of its remarkable \( \mathfrak{g} \)-module structure.

It is a classic theorem of C. Chevalley that \( J \) is a polynomial ring in \( ℓ \) homogeneous generators \( p_i \) so that we can write

\[ J = \mathbb{C}[p_1, \ldots, p_ℓ] \]

Let \( d_i = \deg p_i \). Then if we put \( m_i = d_i - 1 \) the \( m_i \) are referred to as the exponents of \( \mathfrak{g} \) and one knows

\[ \sum_{i=1}^{ℓ} m_i = r \] (27)

Henceforth assume \( \mathfrak{g} \) is simple so that the adjoint representation is irreducible. Let \( y_j, j = 1, \ldots, n, \) be basis of \( \mathfrak{g} \). One defines an \( ℓ \times n \) matrix \( Q = Q_{ij}, i = 1, \ldots, ℓ, j = 1, \ldots, n \) by putting

\[ Q_{ij} = \partial_y p_i \] (28)

Let \( S_i, i = 1, \ldots, ℓ, \) be the span of the entries of \( Q \) in the \( i^{th} \) row. The following is immediate

**Proposition 3.** \( S_i \subseteq P^{m_i}(\mathfrak{g}) \). Furthermore \( S_i \) is stable under the action of \( \mathfrak{g} \) and as a \( \mathfrak{g} \)-module \( S_i \) transforms according to the adjoint representation.

If \( V \) is a \( \mathfrak{g} \)-module let \( V_{ad} \) be the set of all of vectors in \( V \) which transform according to the adjoint representation. The equality (24) readily implies \( P(\mathfrak{g})_{ad} = J \otimes H_{ad} \). I proved
the following result some time ago (See §5.4 in [2]. Especially see (5.4.6) and (5.4.7) in §5.4 of [2])

**Theorem 10.** The multiplicity of the adjoint representation in \( H_{ad} \) is \( \ell \). Furthermore the invariants \( p_i \) can be chosen so that \( S_i \subset H_{ad} \) for all \( i \) and the \( S_i, i = 1, \ldots, \ell \), are indeed the \( \ell \) occurrences of the adjoint representation in \( H_{ad} \).

Clearly there are \( \binom{n}{\ell} \ell \times \ell \) minors in the matrix \( Q \). The determinant of any of these minors is an element of \( P^r(\mathfrak{g}) \) by (27). In [5] we offer a different formulation of \( R^r(\mathfrak{g}) \) by proving

**Theorem 11.** The determinant of any \( \ell \times \ell \) minor of \( Q \) is an element of \( R^r(\mathfrak{g}) \) and indeed \( R^r(\mathfrak{g}) \) is the span of the determinants of all these minors.

5. **The \( \mathfrak{g} \) module structure of \( R^r(\mathfrak{g}) \)**

The adjoint action of \( \mathfrak{g} \) on \( \wedge \mathfrak{g} \) extends to \( U(\mathfrak{g}) \) so that \( \wedge \mathfrak{g} \) is a \( U(\mathfrak{g}) \)-module. If \( \mathfrak{s} \subset \mathfrak{g} \) is any subspace and \( k = \text{dim} \mathfrak{s} \) let \( [\mathfrak{s}] = \wedge^k \mathfrak{s} \) so that \( [\mathfrak{s}] \) is a 1-dimensional subspace of \( \wedge^k \mathfrak{g} \). Let \( M_k \subset \wedge^k \mathfrak{g} \) be the span of all \( [\mathfrak{s}] \) where \( \mathfrak{s} \) is any \( k \) dimensional commutative Lie subalgebra of \( \mathfrak{g} \). If no such subalgebra exists put \( M_k = 0 \). It is clear that \( M_k \) is a \( \mathfrak{g} \)-submodule of \( \wedge^k \mathfrak{g} \). Let \( \text{Cas} \in \mathbb{Z} \) be the Casimir element corresponding to the Killing form. The following theorem was proved as Theorem (5) in [3].

**Theorem 12.** For any \( k \in \mathbb{Z} \) let \( m_k \) be the maximal eigenvalue of \( \text{Cas} \) on \( \wedge^k \mathfrak{g} \). Then \( m_k \leq k \). Moreover \( m_k = k \) if and only if \( M_k \neq 0 \) in which case \( M_k \) is the eigenspace for the maximal eigenvalue \( k \).

Let \( \Phi \) be a subset of \( \Delta \). Let \( k = \text{card} \Phi \) and write, in increasing order,

\[
\Phi = \{ \varphi_1, \ldots, \varphi_k \} \tag{29}
\]
Let
\[ e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k} \]
so that \( e_\Phi \in \wedge^k g \) is an \((\mathfrak{h})\) weight vector with weight
\[ \langle \Phi \rangle = \sum_{i=1}^{k} \varphi_i \]

Let \( n \) be the Lie algebra spanned by \( e_\varphi \) for \( \varphi \in \Delta_+ \) and let \( b \) the Borel subalgebra of \( g \) defined by putting \( b = \mathfrak{h} + n \). Now a subset \( \Phi \subset \Delta_+ \) will be called an ideal in \( \Delta_+ \) if the span, \( n_\Phi \), of \( e_\varphi \), for \( \varphi \in \Phi \), is an ideal of \( b \). In such a case \( \mathbb{C}e_\Phi \) is stable under the action of \( b \) and hence if \( V_\Phi = U(g) \cdot e_\Phi \) then, where \( k = card \Phi \),
\[ V_\Phi \subset \wedge^k g \]
is an irreducible \( g \)-module of highest weight \( \langle \Phi \rangle \) having \( \mathbb{C}e_\Phi \) as the highest weight space. We will say \( \Phi \) is abelian if \( n_\Phi \) is an abelian ideal of \( b \). Let
\[ \mathcal{A}(k) = \{ \Phi \mid \Phi \text{ is an abelian ideal of cardinality } k \text{ in } \Delta_+ \} \]

The following theorem was established [3]. (See especially Theorems (7) and (8) in [3].)

**Theorem 13.** If \( \Phi, \Psi \) are distinct ideals in \( \Delta_+ \) then \( V_\Phi \) and \( V_\Psi \) are inequivalent (i.e. \( \langle \Phi \rangle \neq \langle \Psi \rangle \)). Furthermore if \( M_k \neq 0 \) then
\[ M_k = \bigoplus_{\Phi \in \mathcal{A}(k)} V_\Phi \quad (30) \]
so that, in particular, \( M_k \) is a multiplicity 1 \( g \)-module.

We now focus on the case where \( k = \ell \). Clearly \( M_\ell \neq 0 \) since \( g^x \) is an abelian subalgebra of dimension \( \ell \) for any regular \( x \in g \). Let \( \mathcal{I}(\ell) \) be the set of all ideals of
cardinality \( \ell \). The following theorem giving the remarkable structure of \( R^r(g) \) as a \( g \)-module is one of the main results in [4].

**Theorem 14.** One has \( \mathcal{I}(\ell) = \mathcal{A}(\ell) \) so that

\[
M_\ell = \bigoplus_{\Phi \in \mathcal{I}(\ell)} V_\Phi
\]

(31)

Moreover as \( g \)-modules one has the equivalence

\[
R^r(g) \cong M_\ell
\]

(32)

so that \( R^r(g) \) is a multiplicity 1 \( g \)-module with \( \text{card} \mathcal{I}(\ell) \) irreducible components and \( \text{Cas} \) takes the value \( \ell \) on each and every one of the \( \mathcal{I}(\ell) \) distinct components.

**Example.** If \( g \) is of type \( A_\ell \) then the elements of \( \mathcal{I}(\ell) \) can identified with Young diagrams of size \( \ell \). In this case therefore the number of irreducible components in \( R^r(g) \) is \( P(\ell) \) where \( P \) here is the classical partition function.

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