Abstract

This paper considers a stochastic Nash game in which each player $i$ minimizes a composite objective $f_i(x) + r_i(x_i)$, where $f_i$ is an expectation-valued smooth function and $r_i$ is a nonsmooth convex function with an efficient prox-evaluation. In this context, we make the following contributions. (I) Under suitable monotonicity assumptions on the concatenated gradient map of $f_i$, we derive (optimal) rate statements and oracle complexity bounds for the proposed variable sample-size proximal stochastic gradient-response (VS-PGR) scheme when the sample-size increases at a geometric rate. If the sample-size increases at a polynomial rate of $[(k+1)^v]$ with $v > 0$, the mean-squared error of the iterates decays at a corresponding polynomial rate while the iteration and oracle complexities to obtain an $\epsilon$-Nash equilibrium (NE) are $O(1/\epsilon^{1/v})$ and $O(1/\epsilon^{1+1/v})$, respectively. (II) We then overlay (VS-PGR) with a consensus phase with a view towards developing distributed protocols for aggregative stochastic Nash games. In the resulting (d-VS-PGR) scheme, when the sample-size and the number of consensus steps at each iteration grow at a geometric and linear rate respectively while the communication rounds grow at the rate of $k+1$, computing an $\epsilon$-NE requires similar iteration and oracle complexities to (VS-PGR) with a communication complexity of $O(\ln^2(1/\epsilon))$; (III) Under a suitable contractive property associated with the proximal best-response (BR) map, we design a variable sample-size proximal BR (VS-PBR) scheme, where each player solves a sample-average BR problem. When the sample-size increases at a suitable geometric rate, the resulting iterates converge at a geometric rate while the iteration and oracle complexity are respectively $O(\ln(1/\epsilon))$ and $O(1/\epsilon)$; If the sample-size increases at a polynomial rate with degree $v$, the mean-squared error decays at a corresponding polynomial rate while the iteration and oracle complexities are $O(1/\epsilon^{1/v})$ and $O(1/\epsilon^{1+1/v})$, respectively. (IV) Akin to (II), the distributed variant (d-VS-PBR) achieves similar iteration and oracle complexities to the centralized (VS-PBR) with a communication complexity of $O(\ln^2(1/\epsilon))$ when the communication rounds per iteration increase at the rate of $k+1$. Finally, we present some preliminary numerics to provide empirical support for the rate and complexity statements.

1 Introduction

Noncooperative game-theoretic models [5, 11] consider the resolution of conflicts among selfish players, each of which tries to optimize its payoff, given its rival strategies. Nash games, an important subclass of noncooperative games originating from the seminal work by [31], have seen wide applicability in a breadth of engineered systems, such as power grids, communication networks, and transportation networks (see

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e.g. [2,3,35,36,42]). Recently, there has been an interest in “designing” games to effect distributed control; consequently, in networked regimes, the role of distributed protocols for computing equilibria over graphs is of increasing relevance.

In this paper, we consider the Nash equilibrium problem (NEP) with a finite set of $n$ players indexed by $i$ where $i \in \mathcal{N} \triangleq \{1, \ldots, n\}$. For any $i \in \mathcal{N}$, the $i$th player is characterized by a strategy $x_i \in \mathbb{R}^{d_i}$ and a payoff function $F_i(x_i, x_{-i})$ depending on its strategy $x_i$ and the rival strategies $x_{-i} \triangleq \{x_j\}_{j \neq i}$. Let $x$ denote the strategy profile, defined as $x \triangleq (x_1, \ldots, x_n) \in \mathbb{R}^d$ with $d \triangleq \sum_{i=1}^n d_i$. We consider a stochastic Nash game $\mathcal{P}$ where the objective of player $i$, given its rival strategies $x_{-i}$, is to solve the following parametrized stochastic optimization problem:

$$\min_{x_i \in \mathbb{R}^{d_i}} F_i(x_i, x_{-i}) \triangleq f_i(x_i, x_{-i}) + r_i(x_i), \quad (\mathcal{P}_i(x_{-i}))$$

where $f_i(x) \triangleq \mathbb{E}[\psi_i(x; \xi(\omega))]$, the random variable $\xi : \Omega \to \mathbb{R}^m$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\psi_i : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is a scalar-valued function, and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}$. We focus on structured nonsmooth convex Nash games where $f_i(x_i, x_{-i})$ is assumed to be smooth and convex in $x_i$ for any $x_{-i}$ while $r_i(x_i)$ is assumed to be a convex but possibly nonsmooth function with an efficient prox-evaluation. Note that the function $r_i(\cdot)$ allows for incorporating convex constraints by defining $r_i(x_i)$ as the indicator function of a convex set. A Nash equilibrium (NE) of the stochastic Nash game in which the $i$th player solves the parametrized problem $(\mathcal{P}_i(x_{-i}))$ is a tuple $x^* \triangleq \{x_i^*\}_{i=1}^n \in \mathbb{R}^d$ such that for any $i \in \mathcal{N}$,

$$F_i(x_i^*, x_{-i}^*) \leq F_i(x_i, x_{-i}^*) \quad \forall x_i \in \mathbb{R}^{d_i}.$$ 

In other words, $x^*$ is an NE if no player can profit from unilaterally deviating from its equilibrium strategy $x_i^*$.

Our focus is two-fold: (i) Development of variable sample-size stochastic proximal gradient-response (VS-PGR) and proximal best-response (VS-PBR) schemes with optimal (deterministic) rates of convergence as well as iteration and oracle complexities; (ii) Extension to distributed (consensus-based) regimes (referred to as (d-VS-PGR) and (d-VS-PBR)), allowing for resolving aggregative stochastic Nash games where each player’s payoff depends on its strategy and an aggregate of all players’ strategies over a static communication graph, where linear rates of convergence are achieved by combining increasing number of consensus steps with a growing batch-size of sampled gradients or payoffs.

Prior research. We discuss some relevant prior research on gradient-response, best-response, and distributed schemes for continuous-strategy Nash games and variance-reduced schemes for stochastic optimization.

(i) Gradient-response schemes. Early work considered convex Nash games where player problems are convex and that an NE [8 Chapter 1] is equivalent to a solution of the associated variational inequality. Gradient-response schemes have proven useful in flow control and routing games (see, e.g. [1,36,43]) and generally impose a suitable monotonicity property on the concatenated gradient map. Merely monotone gradient-response schemes have proven useful in flow control and routing games (see, e.g. [1,36,43]) and generally impose a suitable monotonicity property on the concatenated gradient map. Merely monotone

(ii) Best-response schemes. By observing that an NE is a fixed point of the BR map for convex Nash games (cf. [5]) we may apply fixed-point or BR approaches, where each player selects the strategy that optimizes its payoff, given rival strategies (cf. [5,10]). There have been efforts to extend such schemes to engineered settings (cf. [39]), where the BR correspondence can be expressed in a closed form. However, BR schemes do not always lead to convergence to Nash equilibria even in potential convex Nash games.
(see [9] for a counterexample). The proximal BR schemes appear to have been first discussed in [7] where it is shown that the set of fixed points of the proximal BR map is equivalent to the set of Nash equilibria. In [9], several regularized Gauss-Seidel BR schemes are suggested for generalized potential games and it is shown that a limit point of the generated sequence is an NE. More recently, sampled BR schemes have been developed [37] to solve risk-averse two-stage noncooperative games while the rate statements and complexity bounds have been provided for a distinctly different class (specifically single-loop) of inexact synchronous, asynchronous, and delay-tolerant stochastic proximal BR schemes in [26, 29, 41]. We emphasize that in this scheme, each strongly convex expectation-valued subproblem is solved by a stochastic gradient scheme and the overall iteration complexity in terms of projected gradient steps is $O(1/\epsilon)$, in contrast with $O(\ln(1/\epsilon))$ complexity obtained in the present work, albeit in terms of sample-average subproblems. Finally, a.s. and mean convergence of sequences produced by proximal BR schemes is shown in [25, 26] for stochastic and misspecified potential games.

(iii) **Consensus-based distributed schemes for Nash games.** Aggregative games [17] are Nash games where player payoffs are coupled through an aggregate of player strategies; however, players might not have access to the aggregate and hence cannot compute payoffs or gradients, precluding the direct use of gradient/BR schemes. Inspired by the consensus-based protocols for distributed optimization [15, 23, 24, 32, 33], Koshal et al. [22] developed distributed synchronous and asynchronous algorithms for such games in which players utilize an estimate of the aggregate and update it by communicating with their neighbors. Deterministic aggregative games subject to coupling constraints are considered in [6, 34], while in [34], an asymmetric projection algorithm is adopted for seeking a variational generalized Nash equilibrium (GNE). In [6], non-differentiable payoff functions are considered and a semi-decentralized algorithm is presented for finding a zero of the associated generalized equation. However, in both [6] and [34], an additional central node is required for updating the Lagrange multiplier associated with the coupling constraints. Distributed primal-dual algorithms were proposed in deterministic regimes [46] while the only known distributed gradient-based scheme for generalized stochastic Nash games was developed in [45], which considers asymptotic behavior under a constant steplength. We focus on stochastic aggregative Nash games but consider both gradient and BR schemes while providing rate and complexity guarantees.

(iv) **Variance reduction schemes.** There has been an effort to utilize variance reduction schemes for solving stochastic programs within stochastic gradient-based schemes, where the true gradient is replaced by the average of an increasing batch of sampled gradients, leading to a progressive reduction of the variance of the sample-average gradient. Thus, such schemes can improve the rates of convergence or even allow for recovering deterministic convergence rates (in an expected value sense) if the batch size grows sufficiently fast, as seen in convex [12, 13, 16, 19, 40] and nonconvex optimization regimes [12, 28, 38]. However, there has been no known effort to apply such avenues for resolving stochastic Nash games, particularly via BR schemes.

**Research Gaps and Novelty of Proposed Research.** Prior algorithmic research on stochastic Nash games has largely resided in standard gradient-based approaches (without utilizing variance reduction) with either little or no available rate and complexity analysis for either best-response schemes or distributed variants for both gradient or best-response schemes. In addition, much of the prior rate statements show distinct gaps with deterministic analogs. In this paper, we address the following gaps: (i) **Best-response schemes.** We provide a novel best-response scheme that can address stochastic Nash games characterized by a suitable contractive property; (ii) **Variance-reduction schemes.** By overlaying a variable sample-size framework, we observe that both gradient and best-response schemes achieve deterministic rates of convergence with optimal or near-optimal oracle complexities; (iii) **Distributed variants.** Finally, we then proceed to extend each scheme to a distributed regime capable of contending with ag-
| Algorithm | $S_k$ | Rate $\mathbb{E}[\|x_k - x^*\|^2]$ | Iter. Comp. | Oracle Comp. | Assump. |
|-----------|-------|-------------------------------|-------------|-------------|---------|
| VS-PGR    | $\rho^{-(k+1)}$ | Linear: $\mathcal{O}(\rho^k)$ | $\mathcal{O}(\ln(1/\epsilon))$ | $\mathcal{O}(1/\epsilon)$ | SM      |
|           | $(k+1)^v$ | $\mathcal{O}(q^k) + \mathcal{O}(k^{-v})$ | $\mathcal{O}((1/\epsilon)^{1/v})$ | $\mathcal{O}(1/\epsilon)^{(1+1/v)}$ | SM      |
| VS-PBR    | $\rho^{-(k+1)}$ | $\mathcal{O}(\rho^k)$ | $\mathcal{O}(\ln(1/\epsilon))$ | $\mathcal{O}(1/\epsilon)$ | CPBRM   |
|           | $(k+1)^v$ | $\mathcal{O}(q^k) + \mathcal{O}(k^{-v})$ | $\mathcal{O}((1/\epsilon)^{1/v})$ | $\mathcal{O}(1/\epsilon)^{(1+1/v)}$ | CPBRM   |

Table 1: (VS-PGR) and (VS-PBR) schemes ($v > 0$, SM: Strongly monotone, CPBRM: Contract. prox. BR Map)

| Algorithm | $S_k$ | Comm. $\tau_k$ | Rate $\mathbb{E}[\|x_k - x^*\|^2]$ | Iter. Comp. | Oracle Comp. | Comm. Comp. | Assump. |
|-----------|-------|----------------|-------------------------------|-------------|-------------|-------------|---------|
| d-VS-PGR  | $\rho^{-(k+1)}$ | $k+1$ | Linear: $\mathcal{O}(\rho^k)$ | $\mathcal{O}(\ln(1/\epsilon))$ | $\mathcal{O}(1/\epsilon)$ | $\mathcal{O}(\ln^2(1/\epsilon))$ | SM      |
|           | $(k+1)^v$ | $[(k+1)^v]$ | $\mathcal{O}((k+1)^{-v})$ | $\mathcal{O}((1/\epsilon)^{1/v})$ | $\mathcal{O}((1/\epsilon)^{(1+1/v)})$ | SM      |
| d-VS-PBR  | $\rho^{-(k+1)}$ | $k+1$ | Linear: $\mathcal{O}(\rho^k)$ | $\mathcal{O}(\ln(1/\epsilon))$ | $\mathcal{O}(1/\epsilon)$ | $\mathcal{O}(1/\epsilon)^{(1+1/v)}$ | CPBRM   |
|           | $(k+1)^v$ | $[(k+1)^v]$ | $\mathcal{O}((k+1)^{-v})$ | $\mathcal{O}((1/\epsilon)^{1/v})$ | $\mathcal{O}((1/\epsilon)^{(1+1/v)})$ | CPBRM   |

Table 2: (d-VS-PGR) and (d-VS-PBR) schemes for Aggregative games ($v > 0$, $u \in (0, 1)$)

Gregarious Nash games and prove that under suitable communication requirements, the aforementioned geometric rates of convergence can be retained.

**Contributions.** We summarize the key aspects of our schemes in Tables 1 and 2 and elaborate on these next.

(i). **VS-PGR.** In Section 2.1, we propose a variable sample-size proximal gradient response (VS-PGR) scheme, where in each iteration an increasing batch of sampled gradients is utilized. Under a strong monotonicity assumption, the mean-squared error admits a linear rate of convergence (Th. 1) when batch-sizes increase geometrically. We further show in Th. 2 that the iteration complexity (no. of proximal evaluations) and oracle complexity (no. of sampled gradients) to achieve an $\epsilon$–NE denoted by $x$ satisfying $\mathbb{E}[\|x - x^*\|^2] \leq \epsilon$ are respectively $\mathcal{O}(\ln(1/\epsilon))$ and $\mathcal{O}((1/\epsilon)^{1+\delta})$ with $\delta \geq 0$. In Cor. 1, with a suitable choice of algorithmic parameters, the iteration and oracle complexity to obtain an $\epsilon$–NE are shown to be optimal and are bounded by $\mathcal{O}(\kappa^2 \ln(1/\epsilon))$ and by $\mathcal{O}(\kappa^2/\epsilon)$, where $\kappa$ denotes the condition number. Finally, under a polynomially increasing sample-size $[(k+1)^v]$, $v > 0$, we show in Lemma 1 that $\mathbb{E}[\|x_k - x^*\|^2] = \mathcal{O}(k^{-v}) + \mathcal{O}(q^k)$ (where $q < 1$), and establish that the iteration and oracle complexity to obtain an $\epsilon$–NE are $\mathcal{O}((1/\epsilon)^{1/v})$ and $\mathcal{O}((1/\epsilon)^{(1+1/v)})$, respectively.

(ii). **Distributed VS-PGR.** In Section 2.2, we design a distributed VS-PGR scheme (see Alg. 1) to compute an NE of an aggregative stochastic Nash game over a static communication graph where players combine variance reduced proximal gradient response with a consensus update for learning the aggregate. By suitably increasing the number of consensus steps and sample-size at each iteration, this scheme is characterized by a linear rate of convergence (Th. 3) while in Th. 4 and Cor. 2 the iteration, oracle, and communication complexity to compute an $\epsilon$–NE are proven to be $\mathcal{O}(\ln(1/\epsilon))$, $\mathcal{O}(1/\epsilon)$, and $\mathcal{O}(\ln^2(1/\epsilon))$, respectively. With polynomially increasing communication rounds $[(k+1)^v]$ and sample-size $[(k+1)^v]$ for $u \in (0, 1)$ and $v > 0$, we can obtain a polynomial convergence rate $\mathbb{E}[\|x_k - x^*\|^2] = \mathcal{O}(k^{-v})$ associated with the iteration, communication, and oracle complexity to obtain an $\epsilon$–NE bounded by $\mathcal{O}((1/\epsilon)^{1/v})$, $\mathcal{O}((1/\epsilon)^{(1+1/v)})$, and $\mathcal{O}((1/\epsilon)^{(1+1/v)})$, respectively.

(iii). **VS-PBR.** In Section 3.2, we develop a variable sample-size proximal best-response (VS-PBR) scheme (see Alg. 2) when the proximal BR map is contractive and requires that each player solves a sample-average BR problem per step. In Th. 5, we show that the generated iterates converge to the NE at a linear rate in the mean-squared sense when the sample-size for computing the sample-average payoff increases geometrically, leading to an iteration (no. of deterministic opt problems solved) and oracle complexity (no. of samples) to achieve an $\epsilon$–NE of $\mathcal{O}(\ln(1/\epsilon))$ and $\mathcal{O}(1/\epsilon)$ respectively. Akin to Section 2.1 we show in Cor. 4 that when the sample-size increases at a polynomial rate of $[(k+1)^v]$,
\[ \mathbb{E}[\|x_k - x^*\|^2] = O(k^{-v}) + O(a^k) \] (where \( a < 1 \)) with the iteration and oracle complexities bounded by \( O(1/\epsilon^{1/v}) \) and \( O(1/\epsilon^{1+1/v}) \), respectively.

(iv. Distributed VS-PBR. In Section 3.3 we design a distributed VS-PBR scheme (see Alg. 3) to compute an NE of an aggregative Nash game with contractive proximal BR maps, akin to (d-VS-PGR) where the aggregate is estimated by taking multiple consensus steps while the proximal BR is approximated by solving a sample-average best-response problem. When the number of consensus steps and sample-sizes are raised suitably fast, the mean-squared error diminishes at a geometric rate (Prop. 4). We further show in Th. 6 that the iteration, oracle, and communication complexity to compute an \( \epsilon \)-NE are \( O(\ln(1/\epsilon)) \), \( O(1/\epsilon) \), and \( O(\ln^2(1/\epsilon)) \), respectively.

**Notation:** A vector \( x \) is assumed to be a column vector while \( x^T \) denotes its transpose. \( \|x\| \) denotes the Euclidean vector norm, i.e., \( \|x\| = \sqrt{x^Tx} \). We write a.s. for “almost surely” and for \( x \in \mathbb{R} \), \( \lceil x \rceil \) denotes the smallest integer greater than \( x \). For a closed convex function \( r(\cdot) \), the prox. operator is defined by \( (1) \) for \( \alpha > 0 \):

\[
\text{prox}_{\alpha r}(x) \triangleq \arg\min_y \left( r(y) + \frac{1}{2\alpha} \|y - x\|^2 \right).
\]  

## 2 VS-PGR Scheme and the Distributed Variant

This section considers the development of a variable sample-size proximal stochastic gradient-response scheme for a class of strongly monotone Nash games. We derive rate and complexity statements when players can observe rival strategies in Section 2.1 and provide analogous statements for a distributed variant in Section 2.2 for an aggregative game where players overlay an additional consensus phase for learning the aggregate.

### 2.1 Variable sample-size proximal stochastic gradient-response scheme

We impose the following conditions on \( \mathcal{P} \).

**Assumption 1** (i) \( r_i \) is lower semicontinuous and convex with effective domain denoted by \( \mathcal{R}_i \triangleq \text{dom}(r_i) \). (ii) For every fixed \( x_{-i} \in \mathcal{R}_{-i} \triangleq \prod_{j \neq i} \mathcal{R}_j \), \( f_i(x_i, x_{-i}) \) is \( C^1 \) and convex in \( x_i \in \mathcal{R}_i \); (iii) For all \( x_{-i} \in \mathcal{R}_{-i} \) and any \( \xi \in \mathbb{R}^n \), \( \psi_i(x_i, x_{-i}; \xi) \) is differentiable in \( x_i \in \mathcal{R}_i \) such that \( \nabla_{x_i} f_i(x_i, x_{-i}) = \mathbb{E} [\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)] \).

Define \( G(x; \xi) \triangleq (\nabla_{x_i} \psi_i(x_i; \xi))_{i=1}^n \) and \( G(x) \triangleq \mathbb{E}[G(x; \xi)] \). Then \( G(x) = (\nabla_{x_i} f_i(x))_{i=1}^n \) by Assumption 1(iii). The following lemma establishes that a tuple \( x^* \) is an NE if and only if it is a fixed point of a suitable map.

**Lemma 1 (Equivalence between NE and fixed point of proximal response map)** Given the stochastic Nash game \( \mathcal{P} \), suppose Assumption 1 holds for each player \( i \in \mathcal{N} \). Let \( r(x) \triangleq (r_i(x_i))_{i=1}^n \). Then \( x^* \in X \) is an NE if and only if \( x^* \) is a fixed point of \( \text{prox}_{\alpha r}(x - \alpha G(x)) \), i.e.,

\[
x^* = \text{prox}_{\alpha r}(x^* - \alpha G(x^*)), \quad \forall \alpha > 0.
\]  

**Proof.** Note that for any \( i \in \mathcal{N} \), \( f_i(x_i, x_{-i}^*) \) and \( r_i(x_i) \) is convex in \( x_i \). Then \( x_i^* \) is an optimal solution of \( f_i(x_i, x_{-i}^*) + r_i(x_i) \) if and only if the following holds for any \( \alpha > 0 \):

\[
x_i^* = \text{prox}_{\alpha r_i}(x_i^* - \alpha \nabla_{x_i} f_i(x^*)).
\]  

Then by concatenating \( 3 \) for \( i = 1, \ldots, n \), we obtain Equation (2). \( \square \)
Suppose the iteration index is given by $k$. Player $i$ at iteration $k$ holds an estimate $x_{i,k} \in \mathbb{R}^{d_i}$ of the equilibrium strategy $x^*$. We consider a variable sample-size generalization of the standard proximal stochastic gradient method, in which $S_k$ number of sampled gradients are utilized at iteration $k$. For any $i \in \mathcal{N}$, given $S_k$ realizations $\nabla x_i \psi_i(x_k; \xi_k^1), \ldots, \nabla x_i \psi_i(x_k; \xi_k^{S_k})$, $x_{i,0} \in \mathcal{R}_i$, player $i$ updates $x_{i,k+1}$ as follows:

$$
x_{i,k+1} = \text{prox}_{\alpha r} \left[ x_{i,k} - \frac{\alpha}{S_k} \sum_{p=1}^{S_k} \nabla x_i \psi_i(x_k; \xi_k^p) \right],
$$

where $\alpha > 0$ is the step size. If $w_k^p \triangleq G(x_k; \xi_k^p) - G(x_k)$ and $\bar{w}_{k,S_k} \triangleq \frac{1}{S_k} \sum_{p=1}^{S_k} w_k^p$, then by concatenating \cite{[1]} for $i = 1, \ldots, n$, we obtain the compact form:

$$
x_{k+1} = \text{prox}_{\alpha r} \left[ x_k - \alpha (G(x_k) + \bar{w}_{k,S_k}) \right].
$$

(\text{VS-PGR})

We impose the following conditions on the gradient mapping $G(x)$ and noise $\bar{w}_{k,S_k}$ and rely on $\mathcal{F}_k$, defined as $\mathcal{F}_k \triangleq \sigma\{x_0, x_1, \ldots, x_k\}$.

**Assumption 2** (i) $G(x)$ is $L$-Lipschitz continuous, i.e., $\|G(x) - G(y)\| \leq L \|x - y\|$, $\forall x, y \in \mathcal{R} \subseteq \prod_{j=1}^{p} \mathcal{R}_j$. (ii) $G(x)$ is $\eta$-strongly monotone, i.e., $(G(x) - G(y))^T (x - y) \geq \eta \|x - y\|^2$, $\forall x, y \in \mathcal{R}$. (iii) There exists a constant $\nu > 0$ such that for any $k \geq 0$, $\mathbb{E}[\bar{w}_{k,S_k} \mid \mathcal{F}_k] = 0$, a.s. and $\mathbb{E}[\|\bar{w}_{k,S_k}\|^2 \mid \mathcal{F}_k] \leq \nu^2 / S_k$, a.s.

We now establish a simple recursion for the conditional mean-squared error in terms of sample size $S_k$, step size $\alpha$, and the problem parameters.

**Lemma 2** Consider (VS-PGR) and suppose Assumptions [1] and [2] hold. Then

$$
\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq (1 - 2\alpha \eta + \alpha^2 L^2) \|x_k - x^*\|^2 + \frac{\alpha^2 \nu^2}{S_k}, \forall k \geq 0, \text{ a.s.}.
$$

(5)

**Proof.** Consider the reformulation (VS-PGR). By using the nonexpansive property of the proximal operator and Eqn. [2], $\|x_{k+1} - x^*\|^2$ can be bounded as follows:

$$
\|x_{k+1} - x^*\|^2 = \|\text{prox}_{\alpha r} [x_k - \alpha (G(x_k) + \bar{w}_{k,S_k})] - \text{prox}_{\alpha r} [x^* - \alpha G(x^*)]\|^2
\leq \|x_k - x^* - \alpha (G(x_k) - G(x^*)) + \bar{w}_{k,S_k}\|^2
\leq \|x_k - x^*\|^2 - 2\alpha (G(x_k) - G(x^*))^T (x_k - x^*) + \alpha^2 \|G(x_k) - G(x^*)\|^2
- 2\alpha (x_k - x^*)^T \bar{w}_{k,S_k} + 2\alpha^2 (G(x_k) - G(x^*))^T \bar{w}_{k,S_k} + \alpha^2 \|\bar{w}_{k,S_k}\|^2.
$$

Then by taking expectations conditioned on $\mathcal{F}_k$ on both sides of the above inequality, recalling that $x_k$ is adapted to $\mathcal{F}_k$, and by invoking Assumption [2] it follows that Eqn. [5] holds.  

\[\square\]

### 2.1.1 Geometrically Increasing Sample Sizes

We begin with a preliminary lemma that will be used in the rate analysis.

**Lemma 3** Let the sequence $\{v_k\}_{k \geq 0}$ with initial value $v_0 \leq c_0$ satisfy the following recursion \cite{[6]} for some $q, \rho \in (0, 1)$:

$$
v_{k+1} \leq q v_k + c_1 \rho^{k+1} \quad \forall k \geq 0.
$$

(6)

Then for any $k \geq 0$, (i) $v_k \leq \left(c_0 + \frac{c_1}{\max(q^{1/\rho}, q^{1/q})^{-1}}\right) \max\{\rho, q\}^k$ when $\rho \neq q$; (ii) $v_k \leq \left(c_0 + \frac{c_1}{\max(q^{1/\rho}, q^{1/q})}\right) \bar{q}^k$ for any $\bar{q} \in (q, 1)$ when $\rho = q$. 
Theorem 2 (Iteration and Oracle Complexity) Let $\epsilon$ be needed to obtain an $\rho$ and the number of sampled gradients required is bounded by $M$. For $\rho \neq q$, when $\rho < q$. Therefore, $\sum_{m=1}^{k} q^{k+1-m} \rho^{m} = \frac{\rho^{m}}{1-\rho/q} q^{k} = \frac{1}{q/\rho - 1} q^{k+1}$.

Similarly, $\sum_{m=1}^{k} q^{k+1-m} \rho^{m} \leq \frac{1}{\rho/q-1} \rho^{k+1}$ when $\rho < q$. Hence by (7) and $v_{0} \leq c_{0}$, (i) follows.

(ii) Consider $\rho = q$. Recall from Lemma 2 that $kq^{k} \leq \bar{q}^{k}/\ln((\bar{q}/q)^{\epsilon})$ for any $\tilde{q} \in (q, 1)$. This together with (7) implies that $v_{0} \leq k^{c_{0}+1} q^{k} \leq c_{0} + c_{1}$.

Based on Lemmas 2 and 3, we proceed to prove the linear convergence of the iterates generated by VS-PGR in a mean-squared sense with geometrically increasing batch-size.

Theorem 1 (Linear convergence rate of VS-PGR) Let VS-PGR be applied to $\mathcal{P}$, where $S_{k} \triangleq \lceil \rho^{-(k+1)} \rceil$ for some $\rho \in (0, 1)$ and $E[\|x_{0} - x^{*}\|^{2}] \leq C$ for some $C > 0$. Suppose Assumptions 2 and 1 hold. Let $C \in (0, 2\eta/L^{2})$ and $q \triangleq 1 - 2\alpha\eta + \alpha^{2}L^{2}$. Then the following hold for any $k \geq 0$.

(i) If $\rho \neq q$, then $E[\|x_{k} - x^{*}\|^{2}] \leq C(\rho, q) \max(\rho, q)^{k}$ where $C(\rho, q) \triangleq C + \frac{\alpha^{2}\nu^{2}}{\max(\rho, q/\rho, q/\rho - 1)}$.

(ii) If $\rho = q$, then for any $\tilde{q} \in (q, 1)$, $E[\|x_{k} - x^{*}\|^{2}] \leq \tilde{D}q^{k}$, where $\tilde{D} \triangleq C + \frac{\alpha^{2}\nu^{2}}{\ln((\tilde{q}/q)^{\epsilon})}$.

Proof. By definition, $q \in (0, 1)$ when $\alpha \in (0, 2\eta/L^{2})$. Then by taking unconditional expectations on both sides of Eqn. (5) and using $S_{k} = \lceil \rho^{-(k+1)} \rceil$, we obtain that for any $k \geq 0$, $E[\|x_{k+1} - x^{*}\|^{2}] \leq qE[\|x_{k} - x^{*}\|^{2}] + \alpha^{2}\nu^{2}\rho^{k+1}$. Then by using Lemma 3 we obtain the results.

Next, we examine the iteration (no. of prox. evals.) and oracle complexity (no. of sampled gradients) of VS-PGR to compute an $\epsilon$-Nash equilibrium. We refer to a random strategy profile $x : \Omega \rightarrow \mathbb{R}^{n}$ as an $\epsilon$-NE if $E[\|x - x^{*}\|^{2}] \leq \epsilon$.

Theorem 2 (Iteration and Oracle Complexity) Let VS-PGR be applied to $\mathcal{P}$, where $S_{k} = \lceil \rho^{-(k+1)} \rceil$ for some $\rho \in (0, 1)$ and $E[\|x_{0} - x^{*}\|^{2}] \leq C$ for some $C > 0$. Suppose Assumptions 2 and 1 hold. Let $\alpha \in (0, 2\eta/L^{2})$ and define $q \triangleq 1 - 2\alpha\eta + \alpha^{2}L^{2}$. Set $\tilde{q} \in (q, 1)$. Then the number of proximal evaluations needed to obtain an $\epsilon$-NE is bounded by $K(\epsilon)$, defined as

$$K(\epsilon) \triangleq \begin{cases} \frac{1}{\ln(1/q)} \ln((C + \frac{\alpha^{2}\nu^{2}}{q-\rho})\epsilon^{-1}) & \text{if } \rho < q < 1, \\ \frac{1}{\ln(1/q)} \ln((C + \frac{\alpha^{2}\nu^{2}}{\ln(q/\rho)})\epsilon^{-1}) & \text{if } q = \rho, \\ \frac{1}{\ln(1/\rho)} \ln((C + \frac{\alpha^{2}\nu^{2}}{\rho-q})\epsilon^{-1}) & \text{if } q < \rho < 1, \end{cases}$$

and the number of sampled gradients required is bounded by $M(\epsilon)$, defined as

$$M(\epsilon) \triangleq \begin{cases} \frac{1}{\rho \ln(1/\rho)} \left((C + \frac{\alpha^{2}\nu^{2}}{q-\rho})\epsilon^{-1}\right) \frac{\ln(1/\rho)}{\ln(q/\rho)} + K(\epsilon) & \text{if } \rho < q < 1, \\ \frac{1}{q \ln(1/q)} \left((C + \frac{\alpha^{2}\nu^{2}}{\ln(q/\rho)})\epsilon^{-1}\right) \frac{\ln(1/q)}{\ln(q/\rho)} + K(\epsilon) & \text{if } q = \rho, \\ \frac{1}{\rho \ln(1/\rho)} \left((C + \frac{\alpha^{2}\nu^{2}}{\rho-q})\epsilon^{-1}\right) + K(\epsilon) & \text{if } q < \rho < 1. \end{cases}$$
Proof. We first consider the case \( \rho \neq q \). From Theorem 1(i) it follows that for any \( k \geq K_1(\epsilon) \) defined in Eqn. (8), we obtain the bound on the iteration complexity defined in Eqn. (8) for cases \( \rho < q < 1 \) and \( q < \rho < 1 \). For any \( \lambda > 1 \) and positive integer \( K \), we have that

\[
\sum_{k=0}^{K} \lambda^k \leq \int_{0}^{K+1} \lambda^x dx \leq \frac{\lambda^{K+1}}{\ln(\lambda)}.
\]

(10)

Therefore, we achieve the following bound on the number of samples utilized:

\[
\sum_{k=0}^{K_1(\epsilon)-1} S_k \leq K_1(\epsilon)-1 \rho^{-(k+1)} + K_1(\epsilon) \leq \frac{1}{\rho \ln(1/\rho)} \rho^{-K_1(\epsilon)} + K_1(\epsilon).
\]

Note that for any \( 0 < \epsilon, p, q < 1 \), the following holds:

\[
\rho^{-\frac{\ln(c_1/\epsilon)}{\ln(1/p)}} = \left( e^{\ln(\rho^{-1})} \right)^{\frac{\ln(c_1/\epsilon)}{\ln(1/p)}} = e^{\ln(c_1/\epsilon) \frac{\ln(1/p)}{\ln(1/\rho)}} = (c_1/\epsilon)^{\frac{\ln(1/p)}{\ln(1/\rho)}}.
\]

(11)

Thus, the number of sampled gradients required to obtain an \( \epsilon \)-NE is bounded by \( \frac{1}{\rho \ln(1/\rho)} \left( \frac{C(\rho,q)}{\epsilon} \right)^{\frac{\ln(1/p)}{\ln(1/\rho)}} \) for cases \( \rho < q < 1 \) and \( q < \rho < 1 \).

We now prove the results for the case \( \rho = q \). From Theorem 1(ii) it follows that for any \( \tilde{q} \in (q,1) \) and \( k \geq K_2(\epsilon) \) defined in Eqn. (8) for the case \( \rho = q \). Therefore, we may bound the number of sampled gradients for obtaining an \( \epsilon \)-NE by

\[
\sum_{k=0}^{K_2(\epsilon)-1} S_k \leq \frac{\epsilon^{-K_2(\epsilon)}}{q \ln(1/\tilde{q})} + K_2(\epsilon) = \frac{1}{q \ln(1/\tilde{q})} \left( \tilde{D}/\epsilon \right)^{\frac{\ln(1/\tilde{q})}{\ln(1/\rho)}} + K_2(\epsilon). \]

This is the bound given in Eqn. (9) for the case \( \rho = q \).

The above theorem establishes that the iteration and oracle complexity to achieve an \( \epsilon \)-NE are \( O(\ln(1/\epsilon)) \) and \( O((1/\epsilon)^{1+\delta}) \), where \( \delta = 0 \) when \( \rho \in (q,1) \), \( \delta = \frac{\ln(q/\rho)}{\ln(1/\rho)} \) when \( q < \rho < 1 \), and \( \delta = \frac{\ln(\tilde{q}/q)}{\ln(1/\rho)} \) when \( \rho = q \). In the following, we further examine the influence of the condition number on the iteration and oracle complexity.

**Corollary 1** Let the scheme \( \text{VS-PGR} \) be applied to \( \mathcal{P} \), where \( \mathbb{E}||x_k - x^*||^2 \leq C \). Suppose Assumptions 1 and 2 hold. Define the condition number \( \kappa \triangleq \frac{L}{\eta} \). Set \( \alpha = \frac{\eta}{L^2} \) and \( S_k \geq \lceil \rho^{-(k+1)} \rceil \) with \( \rho = 1 - \frac{1}{2\kappa^2} \).

Then iteration and oracle complexity to obtain an \( \epsilon \)-NE are bounded by \( O(\kappa^2 \ln(1/\epsilon)) \) and \( O(\kappa^2/\epsilon) \), respectively.

Proof. By \( \alpha = \frac{\eta}{L^2} \) and \( \kappa = \frac{L}{\eta} \), we obtain that \( q = 1 - 2\eta\alpha + \alpha^2 L^2 = 1 - \frac{\eta^2}{L^2} = 1 - \frac{1}{\kappa^2} \). Note that \( \rho > q \) by \( \rho = 1 - \frac{1}{2\kappa^2} \). Then \( \frac{\alpha^2 \rho^2 q}{\rho - q} \leq 2 \left( \frac{1}{L^2} \right) \kappa^2 = 2L^2 \). Since \( 1/\rho = 1 + \frac{1}{2\kappa^2x} \), \( \ln(1/\rho) \geq \frac{1}{2\kappa^2x}/(1 + \frac{1}{2\kappa^2x}) = \frac{1}{2\kappa^2} \) by \( \ln(1+x) \geq x/(x+1) \) for any \( x \geq 0 \). Thus, by Eqns. (8) and (9) for the case \( q < \rho < 1 \), the results hold by the following:

\[
K(\epsilon) = \frac{\ln \left( C + \alpha^2 \nu^2 q/\rho - q \right) + \ln(1/\epsilon)}{\ln(1/\rho)} \leq 2 \left( \ln \left( C + 2\nu^2/L^2 \right) + \ln(1/\epsilon) \right) \kappa^2 = O(\kappa^2 \ln(1/\epsilon)),
\]

\[
M(\epsilon) = \frac{C + \alpha^2 \nu^2 q/\rho - q}{\rho \ln(1/\rho)} \epsilon^{-1} + K(\epsilon) \leq \frac{C + 2\nu^2/L^2}{\epsilon} 2\kappa^2 (1 + 1/(2\kappa^2 - 1)) + K(\epsilon) = O(\kappa^2)(1/\epsilon). \]

□
2.1.2 Polynomially Increasing Sample-Size

We now investigate the convergence properties of the scheme \textbf{VS-PGR} with polynomially increasing sample size. We first prove a preliminary result.

**Lemma 4** Consider the function \( d(x) = q^u x^v \) where \( q \in (0, 1) \), \( x > 0 \), \( u \in (0, 1] \), and \( v > 0 \). Then \( d(x) \) is unimodal on \( \mathbb{R}_+ \) with a unique maximizer given by \( x^* = \frac{v}{\ln(1/q)} \). Furthermore, \( q^x \leq c_{q,v} x^{-v} \) for all \( x \in \mathbb{R}_+ \) where \( c_{q,v} \triangleq e^{-v/u} \left( \frac{v}{u \ln(1/q)} \right)^{v/u} \).

**Proof.** We begin by noting that 
\[
d'(x) = \ln(q) u q^u x^{u-1} x^v + v q^u x^{v-1} = q^u x^v - (v - u \ln(1/q)) x^v \quad \text{and} \quad d''(x) = 0 \text{ if } x^* = \left( \frac{v}{u \ln(1/q)} \right)^{1/u}.
\]

Unimodality follows by noting that \( d(0) = 0 \), \( d'(x) > 0 \) if \( x \in (0, x^*) \), and \( d'(x) < 0 \) when \( x > x^* \). It follows that
\[
c_{q,v} \triangleq \max_{x \geq 0} d(x) = q^{(x^*)^u} (x^*)^v = q^{\ln(1/q)} \left( \frac{v}{u \ln(1/q)} \right)^{v/u} = e^{-v/u} \left( \frac{v}{u \ln(1/q)} \right)^{v/u}. \quad \Box
\]

**Proposition 1** Let \textbf{VS-PGR} be applied to \( P \), where \( S_k \triangleq \lceil (k+1)^v \rceil \) for some \( v > 0 \) and \( \mathbb{E}[\|x_0 - x^*\|^2] \leq C \). Suppose Assumptions 1 and 2 hold. Let \( \alpha \in (0, 2\eta/L^2) \) and \( q \triangleq 1 - 2\alpha \eta + \alpha^2 L^2 \). Then the following holds:

\[
\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq q^{k+1} \left( C + \alpha^2 \nu^2 \ln(1/q)^{-1} \right) + \frac{2\alpha^2 \nu^2 q^{-1}}{\ln(1/q)} (k+1)^{-v}, \quad \forall k \geq 0.
\]

In addition, the iteration and oracle complexity to obtain an \( \epsilon \)-NE are \( \mathcal{O}(v(1/\epsilon)^{1/v}) \) and \( \mathcal{O} \left( e^v v^v (1/\epsilon)^{1+1/v} \right) \), respectively.

**Proof.** By taking unconditional expectations on both sides of Eqn. (5), using \( S_k = \lceil (k+1)^v \rceil \) we obtain that
\[
\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq q \mathbb{E}[\|x_k - x^*\|^2] + \alpha^2 \nu^2 (k+1)^{-v}.
\]

Hence
\[
\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq q^{k+1} \mathbb{E}[\|x_0 - x^*\|^2] + \alpha^2 \nu^2 \sum_{m=1}^{k+1} q^{k+1-m} m^{-v}. \quad (13)
\]

Since \( q \in (0, 1) \) and \( v > 0 \), by \( q^{-m} m^{-v} \leq \int_{m}^{m+1} q^{-t} (t-1)^{-v} dt \forall m \geq 2 \), we have
\[
\sum_{m=1}^{k+1} q^{k+1-m} m^{-v} - q^{k+1} \sum_{m=1}^{\lceil 2v/\ln(1/q) \rceil} q^{-m} m^{-v} + \frac{q^{k+1}}{\ln(1/q)} + \int_{\lceil 2v/\ln(1/q) \rceil}^{k+1} \frac{(q^{-1})^t}{(t-1)^v} dt \leq q^{k+1} \sum_{m=1}^{\lceil 2v/\ln(1/q) \rceil} q^{-m} m^{-v} + \frac{q^{k+1}}{\ln(1/q)})
\]

Integrating by parts, we obtain that
\[
\int_{a}^{b} \frac{(q^{-1})^t}{t^v} dt = \int_{a}^{b} \frac{1}{t^v} \left( \frac{(q^{-1})^t}{\ln(q^{-1})} \right)' dt = \left( \frac{(q^{-1})^t}{t^v \ln(q^{-1})} \right)_{a}^{b} + \int_{a}^{b} \frac{v}{t^v \ln(q^{-1})} \frac{(q^{-1})^t}{t^v} dt. \quad (15)
\]
Note that \( \frac{v}{\ln(1/q)} \leq \frac{1}{2} \) when \( t \geq [2v/\ln(1/q)] \). Therefore, by setting \( a = [2v/\ln(1/q)] \), \( b = k + 1 \) in the above equation, the following holds:

\[
\int_{[2v/\ln(1/q)]}^{k+1} \frac{(q-1)^t}{t^v} dt \leq \frac{(q-1)^t}{t^v} \Big|_{[2v/\ln(1/q)]}^{k+1} + \frac{1}{2} \int_{[2v/\ln(1/q)]}^{k+1} \frac{(q-1)^t}{t^v} dt 
\]

which yields (16). Therefore, the oracle complexity is \( O(v/v) \). Then by substituting (16) into (14), we have that

\[
\sum_{m=1}^{k+1} q^{k+1-m} m^{-v} \leq q^{k+1} \frac{e^{2q-1}}{1-q} + \frac{2q^{-1}(k+1)^{-v}}{\ln(1/q)}.
\]

This incorporated with (13) and \( \mathbb{E}[\|x_0 - x^*\|^2] \leq C \) produces (12).

Since \( q \in (0, 1) \) and \( v > 0 \), by Lemma 4 with \( u = 1 \), \( q^k \leq c_{q,v} k^{-v} \) with \( c_{q,v} = e^{-v} \left( \frac{v}{\ln(1/q)} \right)^v \). Then by (12), we conclude that for any \( k \geq 1 \),

\[
\mathbb{E}[\|x_k - x^*\|^2] \leq \left( C_{q,v} + \alpha^2 \nu^2 c_{q,v} \frac{e^{2q-1}}{1-q} + \frac{2\alpha^2 \nu^2 q^{-1}}{\ln(1/q)} \right) k^{-v} \leq C_v k^{-v}.
\]

Then for any \( k \geq K(\epsilon) = \left( \frac{C_v}{\epsilon} \right)^{1/v} \), \( \mathbb{E}[\|x_k - x^*\|^2] \leq \epsilon \). By noting that \( C_v = \mathcal{O}(e^v v^v) \), the iteration complexity is \( \mathcal{O}(v(1/e)^{1/v}) \). Therefore, the number of sampled gradients required to obtain an \( \epsilon \)-NE is bounded by

\[
\sum_{k=0}^{K(\epsilon)-1} (k+1)^v = (K(\epsilon))^v + \sum_{k=1}^{K(\epsilon)-1} k^v \leq (K(\epsilon))^v + \int_1^{K(\epsilon)} t^v dt 
\]

\[
= \frac{C_v}{\epsilon} + \frac{t^{v+1}}{v+1} \Big|_1^{K(\epsilon)} = \frac{C_v}{\epsilon} + (v+1)^{-1} \left( \frac{C_v}{\epsilon} \right)^{1+\frac{1}{v}}.
\]

Therefore, the oracle complexity is \( \mathcal{O} \left( e^v v^v (1/e)^{1+1/v} \right) \). □

**Remark 1** It is worth emphasizing that Prop. 4 implies that as \( v \) is increased, the constant in both the rate and oracle complexity grows at an exponential rate. Choosing an appropriate \( v \) requires assessing both available computational resources and the impact of generating either sample-average gradients or solving sample-average problems with large sample-sizes and remains a focus of ongoing research.

### 2.2 Distributed VS-PGR for Aggregative Games

Next, we consider a structured nonsmooth stochastic aggregative game \( \mathcal{P}^{agg} \), where player \( i \in \mathcal{N} \) solves the following parametrized problem:

\[
\min_{x_i \in \mathbb{R}^d} F_i^{agg}(x_i, x_{-i}) \triangleq f_i(x_i, \bar{x}) + r_i(x_i), \quad (\mathcal{P}_i^{agg}(x_{-i}))
\]

where \( \bar{x} \triangleq \sum_{i=1}^n x_i \) denotes the aggregate of all players’ strategies and \( f_i(x_i, \bar{x}) \triangleq \mathbb{E} [\psi_i(x_i, x_i + \bar{x}_{-i}; \xi)] \) is expectation-valued with \( \bar{x}_{-i} \triangleq \sum_{j=1,j\neq i}^n x_j \) and the random variable \( \xi : \Omega \to \mathbb{R}^m \). We impose the following assumptions on the stochastic aggregative game \( \mathcal{P}^{agg} \).

**Assumption 3** (i) Assumption [5](i); (ii) For any \( y \in \mathbb{R}^d \), \( f_i(x_i, y) \) is \( C^1 \) and convex in \( x_i \in \mathcal{R}_i \); (iii) For all \( y \in \mathbb{R}^d \) and any \( \xi \in \mathbb{R}^m \), \( \psi_i(x_i, y; \xi) \) is differentiable in \( x_i \in \mathcal{R}_i \) such that \( \nabla_{x_i} f_i(x_i, y) = \mathbb{E} [\nabla_{x_i} \psi_i(x_i, y; \xi)] \).
2.2.1 Algorithm Design

In this section, we design a distributed algorithm to compute an NE of $P_{agg}$, where each player may exchange information with its local neighbors, and subsequently update its estimate of the aggregate and the equilibrium strategy. The interaction among players is defined by an undirected graph $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} \triangleq \{1, \ldots, n\}$ is the set of players and $\mathcal{E}$ is the set of undirected edges between players. The set of neighbors of player $i$, denoted by $\mathcal{N}_i$, is defined as $\mathcal{N}_i = \{ j \in \mathcal{N} : (i, j) \in \mathcal{E} \}$, and player $i$ is assumed to be a neighbor of itself. Define the adjacency matrix $A = [a_{ij}]_{i,j=1}^n$, where $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$, otherwise.

A path in $G$ with length $p$ from $i_1$ to $i_{p+1}$ is a sequence of distinct nodes, $i_1 i_2 \ldots i_{p+1}$, such that $(i_m, i_{m+1}) \in \mathcal{E}$, for all $m = 1, \ldots, p$. The graph $G$ is termed connected if there is a path between any two distinct players $i, j \in \mathcal{N}$. Though each player does not have access to all players’ strategies, it may estimate the aggregate $\bar{\xi}$ by communicating with its neighbors. Player $i$ at time $k$ holds an estimate $x_{i,k}$ for its equilibrium strategy and an estimate $v_{i,k}$ for the average of the aggregate. To overcome the fact that the communication network is sparse, we assume that to compute $v_{i,k+1}$, players communicate $\tau_k$ rounds rather than once at major iteration $k + 1$. The strategy of each player is updated by a variable sample-size proximal stochastic gradient scheme characterized by (18) dependent on the constant step size $\alpha > 0$ and $S_k$, the number of sampled gradient are used at time $k$. We specify the scheme in Algorithm 1.

**Algorithm 1** Distrib. VS-PGR for Agg. Stoch. Nash Games

Initialize: Set $k = 0$, and $v_{i,0} = x_{i,0} \in \mathcal{R}_i$ for any $i \in \mathcal{N}$. Let $\alpha > 0$ and $\{\tau_k, S_k\}$ be deterministic sequences.

Iterate until $k \geq K$.

**Consensus.** $\dot{v}_{i,k} := v_{i,k}$ $\forall i \in \mathcal{N}$ and repeat $\tau_k$ times, $\dot{v}_{i,k} := \sum_{j \in \mathcal{N}_i} a_{ij} \dot{v}_{j,k}$ $\forall i \in \mathcal{N}$.

**Strategy Update.** for every $i \in \mathcal{N}$ $x_{i,k+1} := \text{prox}_{\alpha r_i} \left[ x_{i,k} - \frac{\alpha}{S_k} \sum_{p=1}^{S_k} \nabla x_i \psi_i \left( x_{i,k}, n \hat{v}_{i,k}; \xi^p_k \right) \right]$, \hspace{1cm} (18)

$v_{i,k+1} := v_{i,k} + x_{i,k+1} - x_{i,k}$. \hspace{1cm} (19)

If $e_{i,k} \triangleq \frac{1}{S_k} \sum_{p=1}^{S_k} \left( \nabla x_i \psi_i (x_{i,k}, n \hat{v}_{i,k}; \xi^p_k) - \nabla x_i f_i (x_{i,k}, n \hat{v}_{i,k}) \right)$, then (18) can be rewritten as:

$x_{i,k+1} = \text{prox}_{\alpha r_i} \left[ x_{i,k} - \alpha \left( \nabla x_i f_i (x_{i,k}, n \hat{v}_{i,k}) + e_{i,k} \right) \right]$. \hspace{1cm} (20)

We impose the following conditions on the communication graph, gradient mapping, and observation noises.

**Assumption 4** (i) The undirected graph $G$ is connected and the adjacency matrix $A$ is symmetric with row sums equal to one. (ii) $\phi(x) \triangleq (\nabla x_i f_i (x_i, \sum_{i=1}^n x_i))_{i=1}^n$ is $\eta_\phi$-strongly monotone, i.e., $\langle \phi(x) - \phi(y), (x - y) \rangle \geq \eta_\phi \|x - y\|^2$ $\forall x, y \in \mathcal{R}$. (iii) The mapping $\phi(x)$ is $L_\phi$-Lipschitz continuous over $\mathcal{R}$, i.e., $\|\phi(x) - \phi(y)\| \leq L_\phi \|x - y\|$ $\forall x, y \in \mathcal{R}$. (iv) For any $i \in \mathcal{N}$ and every fixed $x_i \in \mathcal{R}_i$, $\nabla x_i f_i (x_i, y)$ is $L_i$-Lipschitz continuous in $y$, namely, $\|\nabla x_i f_i (x_i, y_1) - \nabla x_i f_i (x_i, y_2)\| \leq L_i \|y_1 - y_2\|$ $\forall y_1, y_2 \in \mathbb{R}^d$. (v) For any $i \in \mathcal{N}$, there exists a constant $\nu_i > 0$ such that for any $k \geq 0$, $\mathbb{E}[e_{i,k} | F_k] = 0$ and $\mathbb{E}[\|e_{i,k}\|^2 | F_k] \leq \nu_i^2 / S_k$, a.s..
2.2.2 Preliminary Results

Define $A(k) \triangleq A^k$. Then by Assumption 4(i), $A(k)$ is symmetric with the sum of each row equaling one. We now recall some results from relevant prior research.

Lemma 5  (i) [33, Proposition 1] Suppose Assumption 4(i) holds. Then there exists a constant $\theta > 0$ and $\beta \in (0, 1)$ such that for any $i, j \in \mathcal{N}$, $|A^k|_{ij} - \frac{1}{n} | \leq \theta \beta^k$, $\forall k \geq 1$.

(ii) [22, Lemma 2] If $y_k \triangleq \sum_{i=1}^{n} v_{i,k}/n$, then $y_k = \sum_{i=1}^{n} x_{i,k}/n$.

We now introduce the transition matrices $\Phi(k, s)$ from time instance $s$ to $k \geq s$, defined as $\Phi(k, s) = A(k), \Phi(k, s) = A(k)A(k-1) \cdots A(s) \forall 0 \leq s < k$. We may then establish an upper bound on the consensus error.

Lemma 6  Suppose Assumption 4(i) holds. Let Algorithm 1 be applied to $\mathcal{P}^{agg}$. Then

$$\|y_k - \hat{v}_{i,k}\| \leq \theta D_R \beta^{\sum_{p=0}^{k} \tau_p} + 2\theta D_R \sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \forall k \geq 0, \quad (21)$$

where $D_R \triangleq \sum_{j=1}^{n} \max_{x_j \in X_j} \|x_j\|$, and the constants $\theta$ and $\beta$ are defined in Lemma 5(i).

Proof. By the consensus step in Algorithm 1, we note that $\hat{v}_{i,k} = \sum_{j=1}^{n} [A(k)]_{ij} v_{j,k}$. Then by [22] (16) in Lemma 4, we obtain the following bound on $\|y_k - \hat{v}_{i,k}\|$:

$$\|y_k - \hat{v}_{i,k}\| \leq \sum_{j=1}^{n} \left| \frac{1}{n} - [\Phi(k, 0)]_{ij} \right| \|v_{j,0}\| + \sum_{s=1}^{k} \sum_{j=1}^{n} \left| \frac{1}{n} - [\Phi(k, s)]_{ij} \right| \|x_{j,s} - x_{j,s-1}\|. \quad (22)$$

By definition of $\Phi(k, s)$ and $A(k)$, we have that $\Phi(k, s) = A^{\sum_{p=s}^{k} \tau_p}$. Then by Lemma 5(i) it follows that for any $i, j \in \mathcal{N}$, $\left| \frac{1}{n} - [\Phi(k, s)]_{ij} \right| \leq \theta \beta^{\sum_{p=s}^{k} \tau_p}$, $\forall k \geq s$. Then by substituting this bound into (22) we obtain that

$$\|y_k - \hat{v}_{i,k}\| \leq \theta \beta^{\sum_{p=0}^{k} \tau_p} \sum_{j=1}^{n} \|v_{j,0}\| + \theta \sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \sum_{j=1}^{n} (\|x_{j,s}\| + \|x_{j,s-1}\|),$$

and hence by defining $D_R \triangleq \sum_{j=1}^{n} \max_{x_j \in X_j} \|x_j\|$, we obtain (21).

Prior to the main results, we provide a supporting lemma.

Lemma 7  Let $\beta \in (0, 1)$. Define $\tau_k \triangleq [(k + 1)^u]$, for some $u \in (0, 1]$. Then the following holds for any $k \geq 1$:

$$\sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \leq e(\ln(\beta^{-1/(u+1)}))^{\frac{1}{u+1}} \left( \frac{(k+1)^{u+1} - k u + 1}{u+1} \right) \left( 1 + \frac{u + 1}{k n \ln(1/\beta)} \right). \quad (23)$$

Proof. Since $\tau_k = [(k + 1)^u]$ and $u > 0$, we obtain that

$$\sum_{p=s}^{k} \tau_p \geq \sum_{p=s+1}^{k+1} p^u \geq \int_{s}^{k+1} \tau \sum_{s=0}^{k+1} \tau^u dt = \left[ \frac{\tau^{u+1}}{u+1} \right]_{s}^{k+1} = \frac{k+1}{u+1} s^{u+1} - \frac{k u + 1}{u+1}.$$
Then by \( \beta \in (0, 1) \), the following holds with \( b \equiv \beta^{-1/(u+1)} \):

\[
\sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \leq \beta^{\frac{(k+1)u+1}{u+1}} \sum_{s=1}^{k} b^{s+1} + \beta^{\frac{(k+1)u+1}{u+1}} \left( b^{k+1} + \sum_{s=1}^{k-1} b^{s+1} \right).
\]

(25)

By defining \( t = s^{u+1} \), implying that \( s = t^{1/(u+1)} \) and \( ds = \frac{1}{u+1} t^{-u/(u+1)} dt \). Then from \( b > 1 \) it follows that

\[
\sum_{s=p}^{k-1} b^{s+1} \leq \int_{p}^{k} b^{s+1} ds = \frac{1}{u+1} \int_{p^{u+1}}^{k^{u+1}} \frac{b^t}{t^{u/(u+1)}} dt.
\]

(26)

Using (15) with \( v = u/(u+1) \) and \( q = b^{-1} \), we obtain that

\[
\int \frac{b^t}{t^{u/(u+1)}} dt = \frac{b^t}{t^{u/(u+1)}} + \frac{u}{(u+1)} \int t \ln(t) \frac{b^t}{t^{u/(u+1)}} dt.
\]

Define \( k_0 \equiv \lfloor \ln(b)^{-1/(u+1)} \rfloor \). Then \( (k_0 + 1)^{u+1} \geq 1/\ln(b) \) and \( \frac{1}{\ln(b)} \leq 1 \) if \( t \geq (k_0 + 1)^{u+1} \). Thus,

\[
\int_{(k_0+1)^{u+1}}^{k^{u+1}} \frac{b^t}{t^{u/(u+1)}} dt \leq \frac{b^{(k_0+1)^{u+1}}}{t^{u/(u+1)} \ln(b)^{(k_0+1)^{u+1}}} + \frac{u}{u+1} \int_{(k_0+1)^{u+1}}^{k^{u+1}} \frac{b^t}{t^{u/(u+1)}} dt
\]

\[
\Rightarrow \frac{1}{u+1} \int_{(k_0+1)^{u+1}}^{k^{u+1}} \frac{b^t}{t^{u/(u+1)}} dt \leq \frac{b^{(k_0+1)^{u+1}}}{t^{u/(u+1)} \ln(b)^{(k_0+1)^{u+1}}} \left( 1 + \frac{1}{k^{u+1} \ln(b)} \right).
\]

This together with (25), (26), and \( b = \beta^{-1/(u+1)} > 1 \) implies that

\[
\sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \leq \beta^{\frac{(k+1)u+1}{u+1}} \sum_{s=1}^{k_0} b^{s+1} + \beta^{\frac{(k+1)u+1}{u+1}} \left( \sum_{s=1}^{k_0} b^{s+1} + b^{k+1} \right)
\]

\[
\leq \beta^{\frac{u+1}{u+1}} k_0 b^{k+1} + \beta^{\frac{u+1}{u+1}} b^{k+1} \left( 1 + \frac{1}{k^{u+1} \ln(b)} \right).
\]

Then by the fact that \( b^{k+1} \leq b^{1/\ln(b)} = e \) since and \( k^{u+1} \leq 1/\ln(b) \), using \( b = \beta^{-1/(u+1)} \) we obtain (23).

\[ \Box \]

2.2.3 Convergence Analysis

Proposition 2 Suppose Assumptions 3 and 4 hold. Let Algorithm 7 be applied to \( P_{\text{avg}} \), where \( \tau_k \equiv k+1 \) and \( S_k \equiv [\rho^{-(k+1)}] \) for some \( \rho \in (0, 1) \). Suppose \( \gamma \equiv \max\{\rho, \beta\} \), \( \varrho \equiv 1 - 2\alpha \eta_0 + 2\alpha^2 L_0^2 \), \( C_1 \equiv kD_{\mathcal{R}} \) and \( C_2 \equiv 2\theta D_{\mathcal{R}} \left( e^{\sqrt{1/\ln(\beta^{-1/2})}} + \frac{2 + \ln(1/\beta)}{\beta^{1/2} \ln(1/\beta)} \right) \) with \( D_{\mathcal{R}} \equiv \sum_{j=1}^{n} \max_{x_j \in \mathcal{R}_j} \|x_j\| \), where the constants \( \theta \) and \( \beta \) are given in Lemma 3(i). Then the following holds for any \( k \geq 0 \):

\[
\mathbb{E} \left[ \|x_{k+1} - x^*\|^2 \right] \leq \varrho \mathbb{E} [\|x_k - x^*\|^2] + C_3 \gamma^{k+1},
\]

(27)

where \( C_3 \equiv \alpha^2 \sum_{i=1}^{n} \nu_i^2 + 4\alpha n D_{\mathcal{R}} (C_1 + C_2) \sum_{i=1}^{n} L_i + 4\alpha^2 n^2 \beta (C_1^2 + C_2^2) \sum_{i=1}^{n} L_i^2. \)

(28)
Proof. Similar to Lemma 1, $x^* \in X$ is an NE if and only if $x^*$ is a fixed point of $\text{prox}_{\alpha}(x - \alpha \phi(x))$ $\forall \alpha > 0$. Then by using (20) and the non-expansive property of the proximal operator, we have that

$$
\|x_{i,k+1} - x^*_i\|^2 \leq \|x_{i,k} - x^*_i\|^2 - \alpha \|\nabla_{x_i} f_i(x_{i,k}, \hat{v}_{i,k}) - \nabla_{x_i} f_i(x^*_i, \bar{x}^*)\|^2 + \alpha \|\nabla_{x_i} f_i(x_{i,k}, \hat{v}_{i,k}) - \nabla_{x_i} f_i(x^*_i, \bar{x}^*)\|^2
$$

(32)

The above inequality over $i = 1, \cdots, n$ produces

$$
\mathbb{E}[\|x_{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \|x_k - x^*\|^2 + 2\alpha^2 \sum_{i=1}^{n} \|\nabla_{x_i} f_i(x_{i,k}, \hat{v}_{i,k}) - \nabla_{x_i} f_i(x^*_i, \bar{x}^*)\|^2 + \frac{\alpha^2 \nu_i^2}{S_k}
$$

(31)

Then using $\mathbb{E}[\|x_{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \|x_k - x^*\|^2 + 2\alpha^2 \sum_{i=1}^{n} \|\nabla_{x_i} f_i(x_{i,k}, \hat{v}_{i,k}) - \nabla_{x_i} f_i(x^*_i, \bar{x}^*)\|^2 + \frac{\alpha^2 \nu_i^2}{S_k}$

(32)
Then using Assumptions \([4] (\text{ii})\) and \([4] (\text{iii})\), we obtain that

\[
\mathbb{E}[\|x_{k+1} - x^*\|^2 | F_k] \leq (1 - 2\alpha\eta + 2\alpha^2 L^2_\phi) \|x_k - x^*\|^2 + 4\alpha nD_R \sum_{i=1}^{n} L_i \|\hat{v}_{i,k} - y_k\|
+ 2\alpha^2 n^2 \sum_{i=1}^{n} L_i^2 \|\hat{v}_{i,k} - y_k\|^2 + \alpha^2 \sum_{i=1}^{n} \nu_i^2 / S_k.
\] (33)

We now estimate the bound for \(\|\hat{v}_{i,k} - y_k\|\) based on Lemmas \([6], [7]\) By using \([21]\) and \([23]\) with \(a = 1\), from \(\beta \in (0,1)\) and \(\sum_{p=0}^{k} \tau_p = (k+1)(k+2)/2\) it follows that

\[
\|y_k - \hat{v}_{i,k}\| \leq 2\theta D_R \left( e^{\sqrt{1/\ln(\beta^{-1/2})} \beta^{\frac{1+k}{2}} + \beta^2 k \ln(1/\beta)} + 1 \right) + \theta D_R \beta^{(k+1)(k+2)/2}
\leq 2\theta D_R \left( e^{\sqrt{1/\ln(\beta^{-1/2})} \beta^{\frac{1+k}{2}} + \beta^2 k \ln(1/\beta)} + \theta D_R \beta^{(k+1)(k+2)/2} \right) = C_1 \beta^{(k+1)(k+2)/2} + C_2 \beta^{k+1}, \quad \forall k \geq 1.
\]

By \([21]\) it is seen that the above inequality also holds for \(k = 0\), and hence

\[
\|y_k - \hat{v}_{i,k}\| \leq C_1 \beta^{(k+1)(k+2)/2} + C_2 \beta^{k+1} \quad \forall k \geq 0.
\] (34)

Then by using \((a+b)^2 \leq 2(a^2 + b^2)\), \(S_k \geq \rho^{-k+1}\), \(g_\phi = 1 - 2\alpha\eta + 2\alpha^2 L^2_\phi\), and by taking the unconditional expectations on both sides of \([33]\), we obtain that

\[
\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq g_\phi \mathbb{E}[\|x_k - x^*\|^2] + 4\alpha nD_R \left( C_1 \beta^{(k+1)(k+2)/2} + C_2 \beta^{k+1} \right) \sum_{i=1}^{n} L_i
+ 4\alpha^2 n^2 \left( C_1^2 \beta^{(k+1)(k+2)} + C_2^2 \beta^{2(k+1)} \right) \sum_{i=1}^{n} L_i^2 + \alpha^2 \sum_{i=1}^{n} \nu_i^2 \quad \forall k \geq 0.
\] (35)

This implies \([27]\) by the definition of \(C_3\) in \([28]\) and \(\beta \in (0,1)\).

Based on the recursion \([27]\) in Prop. 2 and by using Lemma 3 we may obtain the linear convergence of Alg. 1 with geometrically increasing sample-size and communication rounds increasing at a linear rate given by \(\tau_k = k + 1\).

**Theorem 3 (Linear rate of convergence)** Suppose Assumptions 3 and 4 hold. Let Algorithm \([4]\) be applied to \(P^{agg}\), where \(\tau_k \triangleq k + 1\), \(S_k \triangleq \lfloor \rho^{-(k+1)} \rfloor\) for some \(\rho \in (0,1)\), and \(\mathbb{E}[\|x_0 - x^*\|^2] \leq C\). Suppose \(\alpha \in \left( 0, \eta_\phi / L^2_\phi \right)\), define \(g_\phi \triangleq 1 - 2\alpha\eta + 2\alpha^2 L^2_\phi\) and \(\gamma \triangleq \max\{\rho, \beta\}\). Then \(g_\phi \in (0,1)\) and the following hold for any \(k \geq 0\):

(i) If \(\gamma \neq g_\phi\), then \(\mathbb{E}[\|x_k - x^*\|^2] \leq \left( C + \frac{C_1}{\max\{g_\phi, \gamma\}} \cdot \frac{1}{\beta\gamma / \rho^k} \right) g_\phi^k\).

(ii) If \(\gamma = g_\phi\), then for any \(d_\phi \in (g_\phi, 1)\), \(\mathbb{E}[\|x_k - x^*\|^2] \leq \left( C + \frac{C_1}{\max\{g_\phi, \gamma\}} \cdot \frac{1}{\beta\gamma / \rho^k} \right) d_\phi^k\).

Similarly to Theorem 2 we may derive bounds on the iteration and oracle complexity as well as the communication complexity to compute an \(\epsilon\)-NE.

**Theorem 4** Suppose Assumptions 3 and 4 hold. Let Alg. 1 be applied to \(P^{agg}\), where \(\tau_k = k + 1\), \(S_k = \lfloor \rho^{-(k+1)} \rfloor\) for some \(\rho \in (0,1)\), and \(\mathbb{E}[\|x_0 - x^*\|^2] \leq C\). Let \(g_\phi \triangleq 1 - 2\alpha\eta + 2\alpha^2 L^2_\phi\), \(\gamma \triangleq \max\{\rho, \beta\}\), \(\alpha \in \left( 0, \eta_\phi / L^2_\phi \right)\), \(\tilde{g}_\phi \in (g_\phi, 1)\), and \(C_3\) be defined by \([28]\). Then the iteration, communication, and oracle complexity to obtain an \(\epsilon\)-NE are respectively bounded by \(K(\epsilon), \frac{K(\epsilon)(K(\epsilon)+1)}{2}, \) and \(M(\epsilon)\), where \(K(\epsilon)\) and
obtain that Assumptions 3 and 4 hold. Set \( \alpha \) by the fact that it is not the first to utilize increasing communication rounds. Recall that in \([15]\), a distributed accelerated gradient algorithm is employed to solve a distributed convex optimization problem,\( \frac{1}{\rho} \) by the fact that \( O(\ln(1/\rho)) \) in the same way as of Theorem 2. Since \( \tau_k = k + 1 \), the communication complexity (no. of communication rounds) required to obtain an \( \epsilon \)-NE is bounded by \( \sum_{k=0}^{K(\epsilon)-1} \tau_k = \sum_{k=1}^{K(\epsilon)} k = \frac{K(\epsilon)(K(\epsilon)+1)}{2} \). Thus, the theorem is proved.

We now prove that the optimal oracle complexity \( O(1/\epsilon) \) is obtainable under suitable algorithm parameters.

\[ K(\epsilon) \triangleq \begin{cases} \frac{1}{\ln(1/\rho)} \left( (C + \frac{C_3}{\kappa/\rho-1})\epsilon^{-1} \right) & \text{if } \gamma < \phi < 1, \\ \frac{1}{\ln(1/\rho)} \left( (C + \frac{C_3}{\ln(\rho/\phi)\gamma})\epsilon^{-1} \right) & \text{if } \gamma = \phi, \\ \frac{1}{\ln(1/\rho)} \left( (C + \frac{C_3}{\gamma/\rho-1})\epsilon^{-1} \right) & \text{if } \phi < \gamma < 1, \end{cases} \] (36)

\[ M(\epsilon) \triangleq \begin{cases} \frac{1}{\rho \ln(1/\rho)} \left( (C + \frac{C_3}{\kappa/\rho-1})\epsilon^{-1} \right) \ln(1/\rho) + K(\epsilon) & \text{if } \gamma < \phi < 1, \\ \frac{1}{\rho \ln(1/\rho)} \left( (C + \frac{C_3}{\ln(\rho/\phi)\gamma})\epsilon^{-1} \right) \ln(1/\rho) + K(\epsilon) & \text{if } \gamma = \phi, \\ \frac{1}{\rho \ln(1/\rho)} \left( (C + \frac{C_3}{\gamma/\rho-1})\epsilon^{-1} \right) + K(\epsilon) & \text{if } \phi < \gamma < 1. \end{cases} \] (37)

Proof. Based on the geometric rate of convergence established in Theorem 3, we can establish the iteration complexity \( (K(\epsilon)) \) defined in (36) and oracle complexity \( (M(\epsilon)) \) defined in (37) in the same way as of Theorem 2. Since \( \tau_k = k + 1 \), the communication complexity (no. of communication rounds) required to obtain an \( \epsilon \)-NE is bounded by \( \sum_{k=0}^{K(\epsilon)-1} \tau_k = \sum_{k=1}^{K(\epsilon)} k = \frac{K(\epsilon)(K(\epsilon)+1)}{2} \). Thus, the theorem is proved.

Corollary 2 Let Algorithm 1 be applied to \( \mathcal{D}^{agg} \), where \( E[\|x_0 - x^*\|^2] \leq C \) and \( \tau_k = k + 1 \). Suppose Assumptions \( \beta \) and \( \lambda \) hold. Set \( \alpha = \frac{n_{\rho/\phi}}{2L_{\phi}} \) and \( S_k = \lceil \rho^k \beta \rceil \) with \( \rho \) defined by \( \max \left\{ 1 - \frac{\eta^2}{2L_{\phi}}, \beta \right\} \). Then the iteration, communication, and oracle complexity to obtain an \( \epsilon \)-NE are \( \mathcal{O}(\ln(1/\epsilon)) \), \( \mathcal{O}(\ln^2(1/\epsilon)) \), and \( \mathcal{O}(1/\epsilon) \), respectively.

Proof. By \( \alpha = \frac{n_{\rho/\phi}}{2L_{\phi}} \), we obtain that \( \phi = \phi_0 + 2\alpha^2L_{\phi} = 1 - \frac{\eta^2}{2L_{\phi}} \). Note that \( \gamma = \max \{ \rho, \beta \} = \rho > \phi_0 \) by the fact that \( \rho \geq 1 - \frac{\eta^2}{2L_{\phi}} > 1 - \frac{\eta^2}{2L_{\phi}} \). Thus, by using (36) and (37) for the case \( \gamma > \phi_0, \gamma = \rho \), we obtain that \( K(\epsilon) = \frac{1}{\ln(1/\rho)} \left( (C + \frac{C_3}{\kappa/\rho-1})\epsilon^{-1} \right) \ln(1/\rho) + K(\epsilon) \) and \( M(\epsilon) = \frac{1}{\rho \ln(1/\rho)} \left( (C + \frac{C_3}{\kappa/\rho-1})\epsilon^{-1} \right) + K(\epsilon) \).

Remark 2 Our work is not the first to utilize increasing communication rounds. Recall that in \([15]\), a distributed accelerated gradient algorithm is employed to solve a distributed convex optimization problem, where at each step, \( \mathcal{O}(\ln(k)) \) consensus steps are taken. The authors show that in convex settings, the rate is \( \mathcal{O}(1/k^2) \) (optimal) while the total number of communications rounds is \( \mathcal{O}(k \ln(k)) \) up to time \( k \). The recent paper \([14]\) considers the multi-agent constrained optimization of a strongly convex function and the authors design a distributed primal-dual algorithm requiring \( \mathcal{O}(\ln(k)) \) communication rounds at iteration \( k \). The proposed method \([14]\) has a non-asymptotic convergence rate \( \mathcal{O}(1/k^2) \) and requires \( \mathcal{O}(k \ln(k)) \) local communications for all \( k \) iterations. Our scheme (Alg. 1) requires \( k + 1 \) communication rounds at iteration \( k \) and a total \( \mathcal{O}(k^2) \) up to time instance \( k \) to recover the optimal geometric convergence rate but does so in a stochastic game-theoretic regime. To the best of our knowledge, the optimal communication complexity for aggregative game in deterministic regimes is still an open question and this paper is amongst the first one to establish the communication complexity in stochastic Nash games via stochastic gradient-based techniques.

In the following, we will explore the performance of Algorithm 1 with polynomially increasing sample sizes and communication rounds.
Corollary 3 Suppose Assumptions 3 and 4 hold. Let Alg. 1 be applied to $P^{agg}$, where $E\|x_0 - x^*\|^2 \leq C$, $\tau_k \triangleq \left\lceil (k + 1)^u \right\rceil$ and $S_k \triangleq \left\lceil (k + 1)^v \right\rceil$ for some $u \in (0, 1)$ and $v > 0$. Let $\alpha \in (0, \eta_\phi / L^2_\phi)$ and $g_\phi \triangleq 1 - 2\alpha \eta_\phi + 2\alpha^2 L^2_\phi$. Then for all $k \geq 1$, $E\|x_k - x^*\|^2 = O(k^{-v})$, and the iteration, communication, and oracle complexity to obtain an $\epsilon$–NE are $O((1/\epsilon)^{1/v})$, $O((1/\epsilon)^{(u+1)/v})$, and $O((1/\epsilon)^{(1+1)/v})$, respectively.

Proof. We first estimate the bound for $\|\hat{v}_{i,k} - y_k\|$ based on Lemmas 6 and 7. Since $t_{u+1}, u > 0$ is convex in $x > 0$, we have $(k+1)^u+1 - k^{u+1} \geq \nabla x^u_{u+1}|_{x = k} = (u+1)k^u$. Thus, $(k+1)^u+1/u+1 \geq k^{u+1}/u+1 + k^u$ and $(k+1)^u+1 - k^{u+1}/u+1 \geq k^u$. Then by using (21), (23), and (24), from $\beta \in (0, 1)$ it follows that for any $k \geq 1$:

$$\|y_k - \hat{v}_{i,k}\| \leq \theta D_R \left( 1 + 2e \left( \ln(\beta^{-1/(u+1)}) \right)^{\frac{1}{u+1}} \right) \beta^{\frac{k+1}{u+1} + k^u} + 2\theta D_R \beta^{k^u} \left( 1 + \frac{u+1}{k^u \ln(1/\beta)} \right) \leq \tilde{C}_\beta \beta^{k^u} \text{where } \tilde{C}_\beta \triangleq \theta D_R \beta^{\frac{1}{u+1}} \left( 1 + 2e \left( \ln(\beta^{-1/(u+1)}) \right)^{\frac{1}{u+1}} \right) + 2\theta D_R \frac{u+1 + \ln(1/\beta)}{\ln(1/\beta)}.$$ By (21), the above inequality also holds for $k = 0$. Hence $\|y_k - \hat{v}_{i,k}\| \leq \tilde{C}_\beta \beta^{k^u}$, $\forall k \geq 0$. Then using (33), $S_k = \left\lceil (k + 1)^v \right\rceil$, and $g_\phi = 1 - 2\alpha \eta_\phi + 2\alpha^2 L^2_\phi$, we obtain that

$$E[\|x_{k+1} - x^*\|^2] \leq g_\phi E[\|x_k - x^*\|^2] + \alpha^2 (k+1)^{-v} \sum_{i=1}^{n} \nu_i^2 + 4\alpha n D_R \tilde{C}_\beta \beta^{k^u} \sum_{i=1}^{n} L_i + 2\alpha^2 n^2 \tilde{C}_\beta^2 \beta^{2k^u} \sum_{i=1}^{n} L_i^2 \leq g_\phi E[\|x_k - x^*\|^2] + \tilde{C}_\beta^2 \beta^{k^u} + \alpha^2 (k+1)^{-v} \sum_{i=1}^{n} \nu_i^2 = g_\phi^{k+1} E[\|x_0 - x^*\|^2] + \left( \alpha^2 \sum_{i=1}^{n} \nu_i^2 \right) \sum_{m=0}^{k-1} g_\phi^{k-1-m} m^{-v} + \tilde{C}_\beta \sum_{m=0}^{k} g_\phi^{k-m} \beta^m u, \quad (38)$$

where $\tilde{C}_\beta \triangleq 4\alpha n D_R \tilde{C}_\beta \sum_{i=1}^{n} L_i + 2\alpha^2 n^2 \tilde{C}_\beta^2 \sum_{i=1}^{n} L_i^2$. We now estimate the upper bound of the last term in the above inequality. Since $g_\phi \in (0, 1)$ and $\beta \in (0, 1)$, $\sum_{m=0}^{k} g_\phi^{k-m} \beta^m u \leq \int_{0}^{k+1} g_\phi^{-u} \beta^u dt$. Integrating by parts,

$$\int g_\phi^{-u} \beta^u dt = \int \beta^u \left( \frac{g_\phi^{-u}}{\ln(g_\phi^{-1})} \right)\prime dt = \frac{\beta^u g_\phi^{-u}}{\ln(g_\phi^{-1})} - \int \frac{g_\phi^{-u}}{\ln(g_\phi^{-1})} (\beta^u)\prime dt = \frac{\beta^u g_\phi^{-u}}{\ln(g_\phi^{-1})} + \ln(1/\beta) u \int \frac{g_\phi^{-u}}{\ln(g_\phi^{-1})} \beta^u t^{-1} dt = \frac{\beta^u g_\phi^{-u}}{\ln(g_\phi^{-1})} + \int \frac{\ln(1/\beta) u}{\ln(g_\phi^{-1})} e^{1-u} \beta^u t^{-1} dt \leq t \leq t_0 \frac{2 \alpha \eta_\phi (\ln(1/\beta) / \ln(1/g_\phi))^{1/(1-u)} \eta_\phi}{\ln(g_\phi^{-1})}. \quad (16)$$

Note that $\frac{\ln(1/\beta) u}{\ln(g_\phi^{-1})} \eta_\phi \leq \frac{1}{2}$ when $t \geq t_0 \triangleq \left\lceil (2u \ln(1/\beta) / \ln(1/g_\phi))^{1/(1-u)} \right\rceil$. Similarly to (16), we obtain

$$\int_{t_0}^{k+1} g_\phi^{-u} \beta^u dt \leq \frac{2 \alpha \eta_\phi (\ln(1/\beta) / \ln(1/g_\phi))^{1/(1-u)} \eta_\phi}{\ln(g_\phi^{-1})}.$$ This incorporated with (38) and (17) implies that

$$E[\|x_{k+1} - x^*\|^2] = O \left( (k + 1)^{-v} + e^{2u} g_\phi^{k+1} + \beta (k+1)^u + g_\phi^{-(2 u \ln(1/\beta) / \ln(1/g_\phi))^{1/(1-u)}} g_\phi^{k+1} \right).$$

Thus, by using Lemma 4, we obtain that

$$E[\|x_{k+1} - x^*\|^2] = O \left( \left( e^v v^v + e^{-v/u} \left( w / \ln(1/\eta) \right)^{v/u} + v^v e^{-v/u} g_\phi^{-(2 u \ln(1/\beta) / \ln(1/g_\phi))^{1/(1-u)}} \right) (k + 1)^{-v} \right).$$
Then the iteration and oracle complexity to obtain an $\epsilon$-NE are $O\left(\left(\frac{u}{n}\right)^{1/u} \frac{1}{\theta_\phi} \frac{(2u \ln(1/\beta) / \ln(1/\theta_\phi))^{1/(1-\nu)} - 1}{(1/\epsilon)^{1/u}}\right)$ and $O\left(e^u \left(\frac{u}{n}\right)^{(v+1)/u} \frac{1}{\theta_\phi} \frac{(2u \ln(1/\beta) / \ln(1/\theta_\phi))^{(v+1)/(1-\nu)} - 1}{(1/\epsilon)^{1/u}}\right)$, respectively while the communication complexity $\sum_{k=0}^{K(\epsilon)-1} \tau_k \leq (K(\epsilon))^u + \int_0^{K(\epsilon)} t^n dt = (K(\epsilon))^u + \frac{u^u}{u+1} \left(\frac{1}{\epsilon}\right)^{(u+1)/u}$ is bounded by

$$O\left(e^u \left(\frac{u}{n}\right)^{(v+1)/u} \frac{1}{\theta_\phi} \frac{(2u \ln(1/\beta) / \ln(1/\theta_\phi))^{(v+1)/(1-\nu)} - 1}{(1/\epsilon)^{1/u}}\right).$$ □

### 3 VS-PBR Scheme and the distributed variant

In this section, after providing some background in Section 3.1, we consider a class of stochastic Nash games in which the proximal BR map is contractive [7]. In Section 3.2, we conduct a rate and complexity analysis for a variable sample size proximal BR scheme for computing an NE, where in each iteration, each player solves a sample-average BR problem. Distributed variants of (VS-PBR) are examined in Section 3.3 where analogous rate and complexity statements are provided.

#### 3.1 Background on proximal best-response maps

For any tuple $y \in \mathbb{R}^d$, let the proximal BR map $\hat{x}(y)$ be defined as follows:

$$\hat{x}(y) \triangleq \arg\min_{x \in \mathbb{R}^d} \left[ \sum_{i=1}^{n} F_i(x_i, y_{-i}) + \frac{\mu}{2} \|x - y\|^2 \right] \quad \text{for some } \mu > 0. \quad (39)$$

It is clear that the objective function is separable in $x_i$ and (39) reduces to a set of player-specific proximal BR problems, where player $i$ solves the following problem:

$$\hat{x}_i(y) \triangleq \arg\min_{x_i \in \mathbb{R}^{d_i}} \left[ \mathbb{E}[\psi_i(x_i, y_{-i}; \xi)] + r_i(x_i) + \frac{\mu}{2} \|x_i - y_i\|^2 \right]. \quad (40)$$

We impose the following assumption on problem ($P_i(x_{-i})$).

**Assumption 5** (i) The function $r_i$ is lower semicontinuous and convex with effective domain denoted by $\mathcal{R}_i$, which is required to be compact. (ii) For every fixed $x_{-i} \in \mathcal{R}_{-i}$, $f_i(x_i, x_{-i})$ is $C^1$ and convex in $x_i \in \mathcal{R}_i$, and $\nabla x_i f_i(x_i, x_{-i})$ is $L_i$-Lipschitz continuous in $x_i$, i.e., $\|\nabla x_i f_i(x_i, x_{-i}) - \nabla x_j f_i(x_j, x_{-i})\| \leq L_i \|x_i - x_j\|$ $\forall x_i, x_j \in \mathcal{R}_i$. (iii) For any $i \in \mathcal{N}$, all $x_{-i} \in \mathcal{R}_{-i}$ and any $\xi \in \mathbb{R}^{m}$, $\psi_i(x_i, x_{-i}; \xi)$ is differentiable in $x_i$ such that for some $\nu_i > 0$, $\mathbb{E}[\|\nabla x_i \psi_i(x_i, x_{-i}; \xi)\|^2] \leq \nu_i^2 \quad \forall x \in \mathcal{R}$.

Then by [7, Proposition 12.5], $x^*$ is an NE of the game $\mathcal{P}$ if and only if $x^*$ is a fixed point of the proximal best-response map $\hat{x}(\bullet)$, that is, if and only if $x^* = \hat{x}(x^*)$. By Assumption 5, the second derivatives of the functions $f_i, \forall i \in \mathcal{N}$ on $\mathcal{R}$ are bounded. Analogous to the avenue adopted in [7], we may define

$$\Gamma \triangleq \begin{pmatrix}
\mu & \mu + \xi_{1, \min} & \xi_{1, \max} \\
\mu + \xi_{2, \min} & \mu & \xi_{2, \max} \\
\vdots & \vdots & \vdots \\
\mu + \xi_{n, \min} & \mu + \xi_{n, \max} & \mu
\end{pmatrix}, \quad (41)$$

where $\mu > 0$ is a fixed stepsize.
where $\zeta_{i,\text{min}} \triangleq \inf_{x \in \mathcal{R}} \lambda_{\text{min}}(\nabla^2_{x,x} f_i(x))$ and $\zeta_{i,\text{max}} \triangleq \sup_{x \in \mathcal{R}} \|\nabla^2_{x,x} f_i(x)\| \forall j \neq i$. Then by [37] Theorem 4, we may obtain the following relation:

$$\left( \begin{array}{c} \|\hat{x}_1(y') - \hat{x}_1(y)\| \\
\vdots \\
\|\hat{x}_n(y') - \hat{x}_n(y)\| \end{array} \right) \leq \Gamma \left( \begin{array}{c} \|y'_1 - y_1\| \\
\vdots \\
\|y'_n - y_n\| \end{array} \right).$$

(42)

If the spectral radius $\rho(\Gamma) < 1$, then the proximal best-response map is contractive w.r.t. some monotonic norm. Sufficient conditions for the contractive property of the proximal BR map $\hat{x}(\bullet)$ can be found in [7,37].

### 3.2 Variable sample-size proximal BR schemes

Suppose at iteration $k$, we have $S_k$ realizations $\xi_{k,1}^1, \ldots, \xi_{k}^{S_k}$ of the random vector $\xi$. For any $x_i \in \mathcal{R}_i$, we approximate the payoff $f_i(x_i, y_{-i,k})$ by its sample-average $\frac{1}{S_k} \sum_{p=1}^{S_k} \psi_i(x_i, y_{-i,k}; \xi_{k}^p)$ and solve the sample-average best-response problem (43). We then obtain the variable-size proximal BR scheme (Algorithm 2).

**Algorithm 2** Variable-size proximal best-response scheme

Set $k := 0$. Given $K > 0$, let $y_{i,0} = x_{i,0} \in X_i$ for $i = 1, \ldots, n$.

1. For $i = 1, \ldots, n$, player $i$ updates estimate $x_{i,k+1}$ as

$$x_{i,k+1} = \arg\min_{x_i \in \mathcal{R}_i} \left[ \frac{1}{S_k} \sum_{p=1}^{S_k} \psi_i(x_i, y_{-i,k}; \xi_{k}^p) + r_i(x_i) + \frac{\mu}{2} \|x_i - y_{i,k}\|^2 \right].$$

(43)

2. For $i = 1, \ldots, n$, $y_{i,k+1} := x_{i,k+1}$; (3) Set $k := k + 1$ and return to (1) if $k < K$.

Denote by $\varepsilon_{i,k+1} \triangleq x_{i,k+1} - \hat{x}_i(y_k)$ the inexactness associated with the approximate proximal BR solution. We now give the bound of $\mathbb{E} \left[ \|\varepsilon_{i,k+1}\|^2 \right]$ regarding the inexactness sequence in the following lemma.

**Lemma 8** Suppose Assumption [3] holds. Let Algorithm 2 be applied to $\mathcal{P}$. Let $C_{i,b} \triangleq \frac{\mu}{\mu^2 + L_i^2}(1 - L_i/\sqrt{\mu^2 + L_i^2})^{-1}$. Then for any $i = 1, \ldots, n$, $\mathbb{E} \left[ \|x_{i,k+1} - \hat{x}_i(y_k)\|^2 \right] \leq \frac{\nu^2 C_{i,b}^2}{S_k}, \forall k \geq 0$.

**Proof.** Define $\bar{w}_{i,k}(x_i) \triangleq \frac{1}{S_k} \sum_{p=1}^{S_k} \nabla y_{i,p}(x_i, y_{-i,k}; \xi_{k}^p) - \nabla x_i f_i(x_i, y_{-i,k})$. By the optimality condition of (43), $x_{i,k+1}$ is a fixed point of $\text{prox}_{\alpha\bar{w}_{i,k}} \left[ x_i - \alpha \left( \nabla x_i f_i(x_i, y_{-i,k}) + \mu(x_i - y_{i,k}) + \bar{w}_{i,k}(x_i) \right) \right]$ for any $\alpha > 0$. By applying the optimality condition on $\text{prox}_{\alpha\bar{w}_{i,k}} \left[ x_i - \alpha \left( \nabla x_i f_i(x_i, y_{-i,k}) + \mu(x_i - y_{i,k}) \right) \right]$ for any $\alpha > 0$. Then by the nonexpansive property of the proximal operator, we have the following:

$$\|x_{i,k+1} - \hat{x}_i(y_k)\| \leq \left\| (1 - \alpha \mu)(x_{i,k+1} - \hat{x}_i(y_k)) - \alpha \left( \nabla x_i f_i(x_{i,k+1}, y_{-i,k}) - \nabla x_i f_i(\hat{x}_i(y_k), y_{-i,k}) \right) \right\|$$

$$+ \alpha \|\bar{w}_{i,k}(x_{i,k+1})\|. \quad (44)$$

Note by Ass. [5] (ii) that $f_i(x_i, y_{-i})$ is convex in $x_i \in \mathcal{R}_i$ for every fixed $x_{-i} \in \mathcal{R}_{-i}$. Then

$$\left\| (1 - \alpha \mu)(x_{i,k+1} - \hat{x}_i(y_k)) - \alpha \left( \nabla x_i f_i(x_{i,k+1}, y_{-i,k}) - \nabla x_i f_i(\hat{x}_i(y_k), y_{-i,k}) \right) \right\|^2$$

$$= (1 - \alpha \mu)^2 \|x_{i,k+1} - \hat{x}_i(y_k)\|^2 + \alpha^2 \|\nabla x_i f_i(x_{i,k+1}, y_{-i,k}) - \nabla x_i f_i(\hat{x}_i(y_k), y_{-i,k})\|^2$$

$$- 2\alpha(1 - \alpha \mu) \left( \nabla x_i f_i(x_{i,k+1}, y_{-i,k}) - \nabla x_i f_i(\hat{x}_i(y_k), y_{-i,k}) \right)^T (x_{i,k+1} - \hat{x}_i(y_k))$$

$$\leq (1 - \alpha \mu)^2 + \alpha^2 L_i^2 \|x_{i,k+1} - \hat{x}_i(y_k)\|^2,$$
which incorporated with (44) implies that for any $\alpha > 0$,
\[
\|x_{i,k+1} - \tilde{x}_i(y_k)\| \leq \sqrt{(1 - \alpha \mu)^2 + \alpha^2 L_i^2 \|x_{i,k+1} - \tilde{x}_i(y_k)\| + \alpha \|\tilde{w}_i,x_i(x_{i,k+1})\|}.
\]
In the above inequality, by setting $\alpha = \frac{\mu}{\mu^2 + L_i^2}$, we obtain that $\|x_{i,k+1} - \tilde{x}_i(y_k)\| \leq C_{i,b} \|\tilde{w}_i,x_i(x_{i,k+1})\|$. By Ass. 5(iii), there holds $E[\|\tilde{w}_i,x_i(x_{i,k+1})\|^2] \leq \frac{\nu^2}{S_k}$ and $E[\|x_{i,k+1} - \tilde{x}_i(y_k)\|^2] \leq C_{i,b}^2 E[\|\tilde{w}_i,x_i(x_{i,k})\|^2] \leq \frac{\nu^2 C_{i,b}^2}{S_k}$.

Based on this lemma, we obtain a linear rate of convergence with a suitably selected sample size $S_k$. 

**Proposition 3 (Linear rate of convergence)** Suppose Assumption 3 holds and $a \triangleq \|\Gamma\| < 1$, where $\Gamma$ is defined in (41). Define $C_{ns} = \max_i \nu_i^2 C_{i,b}^2$ with $C_{i,b} \triangleq \frac{\mu}{\mu^2 + L_i^2} \left(1 - \frac{1}{\sqrt{\mu^2 + L_i^2}}\right)^{-1}$. Let Algorithm 2 be applied to $P$, where $E[\|x_0 - x^*\|^2] \leq C$ and $S_k = \left\lceil \frac{C_{ns}}{\eta^2} \right\rceil$ for some $\eta \in (0, 1)$. Then the following holds for any $k \geq 0$.

(i) If $a \neq \eta$, then $E[\|x_k - x^*\|^2] \leq \left(\sqrt{C} + \frac{\eta}{\max\{a, \eta, n/a\}}\right)^2 \max\{a, \eta\} 2^k$.

(ii) If $\eta = a$, then for any $\tilde{a} \in (a, 1)$, $E[\|x_k - x^*\|^2] \leq \left(\sqrt{C} + \frac{\eta}{\ln(\tilde{a}/a)}\right)^2 \tilde{a} 2^k$.

**Proof.** By $x_i^* = \tilde{x}_i(x^*)$, using the triangle inequality and $y_k = x_k$ we obtain that:
\[
\|x_{i,k+1} - x_i^*\| \leq \|x_{i,k+1} - \tilde{x}_i(x_k)\| + \|\tilde{x}_i(x_k) - \tilde{x}_i(x^*)\|.
\]
Then by the triangle inequality, (42), and $a \triangleq \|\Gamma\| < 1$, we have the following bound:
\[
v_{k+1} \triangleq \left\| \begin{array}{c}
\|x_{1,k+1} - x_1^*\| \\
\vdots \\
\|x_{n,k+1} - x_n^*\|
\end{array} \right\| \leq a \left\| \begin{array}{c}
\|x_{1,k} - x_1^*\| \\
\vdots \\
\|x_{n,k} - x_n^*\|
\end{array} \right\| + \left\| \begin{array}{c}
\|x_{1,k+1} - \tilde{x}_1(x_k)\| \\
\vdots \\
\|x_{n,k+1} - \tilde{x}_n(x_k)\|
\end{array} \right\|. \tag{46}
\]
Therefore, the following holds:
\[
v_{k+1}^2 \leq a^2 v_k^2 + 2av_k \left( \begin{array}{c}
\|x_{1,k+1} - \tilde{x}_1(x_k)\| \\
\vdots \\
\|x_{n,k+1} - \tilde{x}_n(x_k)\|
\end{array} \right)^2 \tag{47}
\]
Since $S_k = \left\lceil \frac{\max_i \nu_i^2 C_{i,b}^2}{\eta^2} \right\rceil \geq \frac{\nu^2 C_{i,b}^2}{\eta^2} \left(\frac{1}{\eta}ight)$, by Lemma 2, $E[\|x_{i,k+1} - \tilde{x}_i(y_k)\|^2] \leq \eta^2(k+1)$. Then by taking expectations of the inequality (47), using the Hölder’s inequality $E[\|XY\|] \leq (E[\|X\|^2])^{\frac{1}{2}} (E[\|Y\|^2])^{\frac{1}{2}}$, we obtain that
\[
E[\|x_{k+1} - x^*\|^2] \leq a^2 E[\|x_k - x^*\|^2] + 2a \sqrt{m} k^{k+1} \sqrt{E[\|x_k - x^*\|^2]} + m \eta^2(k+1)
\]
\[
= (a \sqrt{E[\|x_k - x^*\|^2]} + \sqrt{m} k^{k+1})^2 \Rightarrow \sqrt{E[\|x_{k+1} - x^*\|^2]} \leq a \sqrt{E[\|x_k - x^*\|^2]} + \sqrt{m} k^{k+1}. \tag{48}
\]
Based on the resursion (48), by using Lemma 3 we obtain the results. □

Note that $x_{i,k+1}$ defined by (43) requires solving a deterministic optimization. In the following, we establish the iteration complexity (no. of deterministic optimization problems solved) and oracle complexity to obtain an $\epsilon$-NE.
Theorem 5 Suppose Assumption 5 holds, \(a \triangleq ||\Gamma|| < 1\), \(C_{i,b} \triangleq \frac{\mu_{i,i}^2 + L_i^2}{2} 1 - L_i / \sqrt{\mu^2 + L^2} \), and \(C_{ns} \triangleq \max_i \nu_i^2 C_{i,b}^2\). Let Algorithm 2 be applied to \(P\), where \(E[\|x_0 - x^*\|^2] \leq C\) and \(S_k = \left[ \frac{C_{ns}}{\eta^2 (k + 1)} \right] \) for some \(\eta \in (0, 1)\). Let \(\tilde{a} \in (a, 1)\), and \(D = 1 / \ln((\tilde{a} / a)^\delta)\). Then the iteration and oracle complexity to obtain an \(\epsilon\)-NE are bounded by \(K_b(\epsilon)\) and \(M_b(\epsilon)\) respectively, each of which is defined as follows.

\[
K_b(\epsilon) \triangleq \begin{cases} 
\frac{1}{\ln(1/a)} \ln \left( \frac{\sqrt{C + \eta \nu_i / (a - \eta)} \sqrt{\epsilon}}{\sqrt{C + \gamma D}} \right) & \text{if } \eta < a, \\
\frac{1}{\ln(1/a)} \ln \left( \frac{\sqrt{C + \eta \nu_i / (\gamma - \eta)} \sqrt{\epsilon}}{\sqrt{C + \gamma D}} \right) & \text{if } \eta = a, \\
\frac{1}{\ln(1/a)} \ln \left( \frac{\sqrt{C + \eta \nu_i / (\eta - \gamma)} \sqrt{\epsilon}}{\sqrt{C + \gamma D}} \right) & \text{if } \eta > a,
\end{cases}
\]

(49)

\[
M_b(\epsilon) \triangleq \begin{cases} 
\frac{C_{ns}}{\eta^2 \ln(1/\eta^2)} \left( \frac{\ln(1/\eta)}{\ln(1/\eta^2)} \right) + K(\epsilon) & \text{if } \eta < a, \\
\frac{C_{ns}}{\eta^2 \ln(1/\eta^2)} \left( \frac{\ln(1/\eta)}{\ln(1/\eta^2)} \right) + K(\epsilon) & \text{if } \eta = a, \\
\frac{C_{ns}}{\eta^2 \ln(1/\eta^2)} \left( \frac{\ln(1/\eta)}{\ln(1/\eta^2)} \right) + K(\epsilon) & \text{if } \eta > a.
\end{cases}
\]

(50)

Proof. We first validate the case when \(\eta = a\). By Proposition 3(ii), we obtain that for any \(k \geq K_b(\epsilon) = \ln((\sqrt{C + \eta \nu_i / (a - \eta)} \sqrt{\epsilon}) / \sqrt{C + \gamma D})\), there hold \(\tilde{a}^k \leq \frac{a^k}{\sqrt{C + \gamma D}}\) and \(E[\|x_k - x^*\|^2] \leq \epsilon\). Then the bound given by (49) for \(\eta = a\) holds. By using (10) and (11), we may bound the number of sampled gradients by \(\sum_{k=0}^{K_b(\epsilon)-1} S_k \leq C_{ns} \frac{\eta^2 \ln(1/\eta^2)}{\eta^2 \ln(1/\eta^2)} + K_b(\epsilon)\), giving us the required result.

The above theorem establishes that when the number of scenarios increases at a geometric rate, the iteration and oracle complexity to achieve an \(\epsilon\)-NE are respectively \(O((1/\epsilon)^{1+\delta})\) and \(O((1/\epsilon)^{1+\delta})\), where \(\delta = 0\) when \(\eta \in (a, 1)\), \(\delta = \ln(a/\eta) / \ln(1/a)\) when \(\eta < a < 1\), and \(\delta = \ln(\tilde{a}/a) / \ln(1/a)\) when \(\eta = a\). Similar to the discussions in Section 2.1.2, we now establish the rate and complexity properties of Algorithm 2 with polynomially increasing sample-sizes.

Corollary 4 Let Algorithm 2 be applied to \(P\), where \(E[\|x_0 - x^*\|^2] \leq C\) and \(S_k = \left[ (k + 1)^\nu \right] \) for some \(v > 0\). Suppose Assumption 5 holds and \(a \triangleq ||\Gamma|| < 1\), where \(\Gamma\) is defined in (41). Then we obtain the polynomial rate of convergence \(E[\|x_{k+1} - x^*\|^2] = O((k + 1)^{-v})\) and establish that the iteration and oracle complexity bounds to obtain an \(\epsilon\)-NE are \(O(v(1/\epsilon)^{1/v})\) and \(O(e^{v^2(1/\epsilon)^{1+1/v}})\), respectively.

Proof. Define \(C_{mc} = \sqrt{\sum_{i=1}^{n} \nu_i^2 C_{i,b}^2}\), \(C_{i,b} = \frac{\mu_{i,i}^2 + L_i^2}{2} 1 - L_i / \sqrt{\mu^2 + L^2} \), similar to Eqn. (48), by using \(S_k = \left[ (k + 1)^v \right]\) and Lemma 3, we obtain that

\[
\sqrt{E[\|x_{k+1} - x^*\|^2]} \leq a \sqrt{E[\|x_k - x^*\|^2]} + C_{mc}(k + 1)^{-v/2} \leq a^{k+1} \sqrt{E[\|x_0 - x^*\|^2]} + C_{mc} \sum_{m=1}^{k+1} a^{k+1-m} m^{-v/2}.\]
Then by (17), we have that for any \( k \geq 1 \),
\[
    \sqrt{\mathbb{E}[\|x_k - x^*\|^2]} \leq a^k \left( \sqrt{C + C_{mc} e^{v-a-1}} \right) + \frac{C_{mc} e^{v-1}}{ln(1/a)} k^{-v/2}.
\]
Since \( a \in (0,1) \) and \( v > 0 \), by Lemma 4 with \( u = 1 \),
\[
a^k \leq e^{-v/2} \left( \frac{v}{ln(1/a)} \right)^{v/2} k^{-v/2} \quad \forall k \geq 1.
\]
Then by (17), we have that for any \( y \in R^d \),
\[
    \sqrt{\mathbb{E}[\|x_k - x^*\|^2]} = O \left( e^{v/2} v^{v/2} k^{-v/2} \right) \quad \forall k \geq 1.
\]

3.3 Distributed VS-PBR for Aggregative Games

We propose a distributed VS-PBR scheme (Algorithm 3) to solve the aggregative game \( P^{agg} \) formulated in Section 2.2, where the gradient-response update (18) in Algorithm 1 is replaced by the inexact BR update (52).

3.3.1 Algorithm Design

For any \( y_i \in R_i \) and any \( \tilde{y}_i \in R^d \), we define the proximal BR map as follows:
\[
    T_i(y_i, \tilde{y}_i) \triangleq \text{argmin}_{x_i \in R^d} \left[ f_i(x_i, \tilde{y}_i) + r_i(x_i) + \frac{\mu}{2} ||x_i - y_i||^2 \right] \quad \mu > 0. \quad (51)
\]

Then \( T_i(y_i, \tilde{y}_i) \) is uniquely defined by Ass. 3(i) and 3(ii). Suppose at iteration \( k \), each player updates its belief of the aggregate by multiple consensus steps, utilizes \( S_k \) realizations to approximate the payoff \( f_i(x_i, n\tilde{v}_{i,k}) \) at the estimated aggregate \( n\tilde{v}_{i,k} \), then solve the sample-average proximal BR problem (52). We then obtain Alg. 3.

Algorithm 3 Distrib. VS-PBR for Agg. Stoch. Nash Games

Initialize: Set \( k = 0 \), and \( v_{i,0} = x_{i,0} \in R_i \) for any \( i \in \mathcal{N} \). Let \( \alpha > 0 \) and \( \{ \tau_k \} \) be a deterministic sequence.
Iterate until \( k > K \)

Consensus. \( \hat{v}_{i,k} := v_{i,k} \quad \forall i \in \mathcal{N} \) and repeat \( \tau_k \) times: \( \hat{v}_{i,k} := \sum_{j \in \mathcal{N}_i} a_{ij} \hat{v}_{j,k} \quad \forall i \in \mathcal{N} \).

Strategy Update. for every \( i \in \mathcal{N} \)
\[
    x_{i,k+1} = \text{argmin}_{x_i \in R^d} \left[ \frac{1}{S_k} \sum_{p=1}^{S_k} \psi_i(x_i, n\tilde{v}_{i,k}; \xi_k^p) + r_i(x_i) + \frac{\mu}{2} ||x_i - x_{i,k}||^2 \right], \quad (52)
\]
\[
    v_{i,k+1} := v_{i,k} + x_{i,k+1} - x_{i,k}. \quad (53)
\]

3.3.2 Rate Analysis

We impose additional assumptions on the \( P^{agg} \).

Assumption 6 (i) There exists a constant \( L_a > 0 \) such that for any \( i \in \mathcal{N} \) and \( y \in R^d \):
\[
    ||\nabla_x f_i(x_i, y) - \nabla_x f_i(x_i', y) || \leq L_a ||x_i - x_i'|| \quad \forall x_i, x_i' \in R_i. \quad (54)
\]

(ii) For any \( i \in \mathcal{N} \) and any \( y \in R^m \), \( \psi_i(x_i, y; \xi) \) is differentiable in \( x_i \in R_i \) such that for some \( \nu_i > 0 \),
\[
    \mathbb{E}[||\nabla_x f_i(x_i, y) - \nabla_x \psi_i(x_i, y; \xi)||^2] \leq \nu_i^2 \quad \forall x_i \in R_i, y \in R^m.
\]

We may claim the Lipschitz continuity of \( T_i(y_i, \tilde{y}_i) \) in the following Lemma.
Lemma 9 Suppose Assumptions 3(i), 3(ii), 3(iv), and 6(i) hold. If $L_t \triangleq \frac{\mu}{\mu^2 + L_a^2} \left( 1 - L_a / \sqrt{\mu^2 + L_a^2} \right)^{-1}$, where $\mu$ is the parameter used in (51). Then for any $i \in N$ and any $y_i \in \mathcal{R}_i$, the following holds:

$$
\|T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})\| \leq L \|\bar{y}_i - \bar{y}\| \quad \forall \bar{y}_i, \bar{y} \in \mathbb{R}^d. \tag{55}
$$

Proof. By the optimality condition of (51), $T_i(y_i, \bar{y}_i)$ is a fixed point of the map $\text{prox}_{\alpha \mu,\gamma} \left[ x_i - \alpha \left( \nabla f_i(x_i, \bar{y}_i) + \mu (x_i - y_i) \right) \right]$ for any $\alpha > 0$. Then by using the triangle inequality and the nonexpansive property of the proximal operator, we have the following for any $\bar{y}_i, \bar{y} \in \mathbb{R}^d$:

$$
\begin{align*}
&\|T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})\| \\
&\quad \leq \left\| T_i(y_i, \bar{y}_i) - \alpha \left( \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) + \mu (T_i(y_i, \bar{y}_i) - y_i) \right) \right\| \\
&\quad - \left( T_i(y_i, \bar{y}) - \alpha \left( \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) + \mu (T_i(y_i, \bar{y}) - y_i) \right) \right) \\
&\quad = \left\| (1 - \alpha \mu) (T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})) - \alpha \left( \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right) \right\| \\
&\quad \leq \left\| (1 - \alpha \mu) (T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})) - \alpha \left( \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right) \right\| \\
&\quad + \alpha \left\| \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right\|.
\end{align*}
$$

Using Eqn. (54), the convexity of $f_i(x_i, \bar{y}_i)$ in $x_i \in \mathcal{R}_i$ for any $\bar{y}_i \in \mathbb{R}^d$ by Assumption 3(ii), we obtain that

$$
\begin{align*}
&\left\| (1 - \alpha \mu) (T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})) - \alpha \left( \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right) \right\|^2 \\
&\quad \leq (1 - \alpha \mu)^2 \left\| T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y}) \right\|^2 + 2 \alpha^2 \left\| \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right\|^2 \\
&\quad - 2 \alpha (1 - \alpha \mu) \left( T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y}) \right) \left( \nabla f_i(T_i(y_i, \bar{y}_i), \bar{y}_i) - \nabla f_i(T_i(y_i, \bar{y}), \bar{y}) \right) \\
&\quad \leq (1 - 2 \alpha \mu + \alpha^2 (\mu^2 + L_a^2)) \left\| T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y}) \right\|^2 \quad \forall \alpha \in (0, 1/\mu).
\end{align*}
$$

By combining (57) and (56) and invoking Ass. 4(iv) produces

$$
\|T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})\| \leq (1 - 2 \alpha \mu + \alpha^2 (\mu^2 + L_a^2))^{1/2} \|T_i(y_i, \bar{y}_i) - T_i(y_i, \bar{y})\| + \alpha L_i \|\bar{y}_i - \bar{y}\|.
$$

Then by setting $\alpha = \frac{\mu}{\mu^2 + L_a^2}$, we obtain (55). $\square$

Similar to Lemma 8, we can obtain the following result.

Lemma 10 Suppose Assumptions 3(i), 3(ii), 3(iv), and 6 hold. Let Algorithm 3 be applied to $\mathcal{P}^{\text{agg}}$. Define $\varepsilon_{i,k+1} \triangleq x_{i,k+1} - T_i(x_{i,k}, n\bar{y}_i)$ and $C_r \triangleq \frac{\mu}{\mu^2 + L_a^2} \left( 1 - L_a / \sqrt{\mu^2 + L_a^2} \right)^{-1}$. Then for any $i = 1, \cdots, n$, $\mathbb{E}[\|\varepsilon_{i,k,1}\|^2] \leq \frac{\mu^2 + L_a^2}{\mu^2}$.

Proposition 4 Let Assumptions 3(i), 3(ii), 3(iv), and 6 hold. Suppose $a = \|\Gamma\| < 1$ with $\Gamma$ defined by (41), and $\mathbb{E}[\|x_0 - x^*\|^2] \leq C$. Let Algorithm 3 be applied to $\mathcal{P}^{\text{agg}}$, where $\tau_k = k + 1$, $S_k = \left[ \frac{C_2^2 \max \{ \eta, \beta \}}{\sqrt{\eta \tau_k}} \right]$ with $C_r \triangleq \frac{\mu}{\mu^2 + L_a^2} \left( 1 - L_a / \sqrt{\mu^2 + L_a^2} \right)^{-1}$ for some $\eta \in (0, 1)$. Define $C_4 \triangleq \sqrt{n} + n^2 L_a (C_1 + C_2)$, and $\gamma \triangleq \max \{ \eta, \beta \}$, where $C_1$ and $C_2$ are defined in Proposition 3 and $\beta$ and $L_t$ be given in Lemma 3(i) and Lemma 8, respectively. Then the following holds for any $k \geq 0$:

(i) If $a \neq \gamma$, then $\mathbb{E}[\|x_k - x^*\|^2] \leq Q^2 \max \{ a, \gamma \}^{2k}$, where $Q \triangleq \sqrt{C} + \frac{C_4}{\max \{ \alpha \gamma, \alpha \}^2 - 1}$.

(ii) If $\gamma = a$, then for any $\tilde{a} \in (a, 1)$, $\mathbb{E}[\|x_k - x^*\|^2] \leq \left( \frac{Q}{\tilde{a}} \right)^2 \tilde{a}^{2k}$, where $\tilde{Q} \triangleq C + C_4 / \ln ((\tilde{a}/a)^{\varepsilon})$. 

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Proof. By noting that \( x_i^* = \hat{x}_i(x^*) \), using the triangle inequality we obtain that
\[
||x_{i,k+1}^* - x^*|| \leq ||x_{i,k+1} - T_i(x_{i,k}, n\hat{v}_i,k) || + ||T_i(x_{i,k}, n\hat{v}_i,k) - \hat{x}_i(x^*)|| + ||\hat{x}_i(x^*) - \hat{x}_i(x^*)||,
\]
where \( \hat{x}_i(\bullet) \) is defined by (39). By the definition of \( T_i(\cdot, \cdot) \) in (51) and \( y_k = \sum_{i=1}^{n} x_{i,k}/n \) from Lemma 5(ii), we have that for any \( i \in \mathcal{N} \), \( \hat{x}_i(x_{i,k}) = T_i(x_{i,k}, n\hat{v}_i,k) = T_i(x_{i,k}, n y_k) \), and hence \( ||T_i(x_{i,k}, n\hat{v}_i,k) - \hat{x}_i(x_{i,k})|| \leq n L_t ||\hat{v}_i,k - y_k|| \) by (55). Then by using (58), there holds \( ||x_{i,k+1}^* - x^*|| \leq \epsilon_{i,k+1} || + n L_t ||\hat{v}_i,k - y_k|| + ||\hat{x}_i(x_{i,k}) - \hat{x}_i(x^*)|| \). Similarly to (46), we may obtain the following recursion:
\[
v_{k+1} \leq a v_k + \left( \begin{array}{c}
||\hat{v}_{1,k} - y_k|| \\
\vdots \\
||\hat{v}_{n,k} - y_k||
\end{array} \right) + n L_t \left( \begin{array}{c}
||\hat{v}_{1,k} - y_k|| \\
\vdots \\
||\hat{v}_{n,k} - y_k||
\end{array} \right).
\]
From (34) and \( \beta \in (0, 1) \) it follows that for any \( k \geq 0 \),
\[
\left( \begin{array}{c}
||\hat{v}_{1,k} - y_k|| \\
\vdots \\
||\hat{v}_{n,k} - y_k||
\end{array} \right) \leq \sqrt{n} \left( C_1 \beta^{(k+1)(k+2)/2} + C_2 \beta^{k+1} \right) \leq \sqrt{n} (C_1 + C_2) \beta^{k+1}.
\]
Note that \( \mathbb{E}[||\epsilon_{i,k+1}||^2] \leq \eta^{2k} \forall i \in \mathcal{N} \) by \( S_k = \left[ \frac{C_2 \max_{i} \nu_i^2}{\eta^{(k+1)}} \right] \) and Lemma 10. Then using (59) and (60), similar to the derivation of (48) we obtain that
\[
\sqrt{\mathbb{E}[||x_{k+1} - x^*||^2]} \leq a \sqrt{\mathbb{E}[||x_k - x^*||^2]} + \sqrt{n} \eta^{k+1} + n^\frac{3}{2} L_t (C_1 + C_2) \beta^{k+1}.
\]
Thus, by the definitions \( \gamma = \max\{\eta, \beta\} \) and \( C_4 = \sqrt{n} + n^\frac{3}{2} L_t (C_1 + C_2) \), we obtain that for any \( k \geq 0 \), \( \sqrt{\mathbb{E}[||x_{k+1} - x^*||^2]} \leq a \sqrt{\mathbb{E}[||x_k - x^*||^2]} + C_4 \gamma^k \). Based on which, by using Lemma 3 we obtain the results. \( \square \)

3.3.3 Iteration and Oracle Complexity

We now establish the iteration, oracle and communication complexity of Algorithm 3 to achieve an \( \epsilon - \text{NE} \).

The derivation of the following theorem is almost the same as that of Theorem 5. So we just state the results while omit the proof.

**Theorem 6** Let \( \mathbb{E}[||x_0 - x^*||^2] \leq C \), Assumptions 3, 4(i), 4(iv), and 5 hold. Suppose \( a = ||\Gamma|| < 1 \) with \( \Gamma \) defined by (41). Let Algorithm 3 be applied to \( \mathcal{P}^{\text{agg}} \), where \( \tau_k = k + 1 \) and \( S_k = \left[ \frac{C_2 \max_{i} \nu_i^2}{\eta^{(k+1)}} \right] \) with \( C_r \triangleq \frac{\mu^2 + L_a^2}{\mu^2} \left( 1 - L_a / \sqrt{\mu^2 + L_a^2} \right)^{- \eta} \) for some \( \eta \in (0, 1) \). Let \( \gamma = \max\{\eta, \beta\} \), \( C_4 \) and \( Q \) be defined in Proposition 4. Suppose \( a \neq \gamma \). Then the number of deterministic optimization solver required by player i to obtain an \( \epsilon - \text{NE} \) is bounded by \( K(\epsilon) \triangleq \left[ \frac{\ln(Q/\sqrt{\gamma})}{\ln(1/\max\{a, \gamma\})} \right] \), while the communication and oracle complexity to obtain an \( \epsilon - \text{NE} \) are \( \frac{K(\epsilon)(K(\epsilon)+1)}{2} \) and \( O \left( \frac{1}{\epsilon} \frac{\ln(1/\max\{a, \gamma\})}{\ln(1/\max\{a, \gamma\})} \right) \).

Remark 3 (i) If the algorithm parameter \( \eta \in (\max\{a, \sqrt{3}\}, 1) \), we obtain the optimal oracle complexity \( O(1/\epsilon) \).

(ii) From Theorem 6 we conclude that the iteration complexity in terms of the deterministic optimization solver, oracle complexity in terms of sampled gradient, and communication complexity to compute an \( \epsilon - \text{NE} \) are \( O(\ln(1/\epsilon)) \), \( O \left( (1/\epsilon)^{1+\delta} \right) \) with \( \delta \geq 0 \), and \( O \left( \ln^2(1/\epsilon) \right) \), respectively.

(iii) Similarly to Corollary 3 when \( \tau_k = \lceil (k+1)^u \rceil \) and \( S_k = \lceil (k+1)^u \rceil \) for \( u \in (0, 1) \) and \( v > 0 \), we can obtain the convergence rate \( \mathbb{E}[||x_{k+1} - x^*||^2] = O((k+1)^{-v}) \) and establish that the iteration, communication, and oracle complexity to obtain an \( \epsilon - \text{NE} \) are \( O((1/\epsilon)^{1/v}) \), \( O((1/\epsilon)^{(u+1)/v}) \), and \( O \left( (1/\epsilon)^{1+1/v} \right) \), respectively. In fact, as done earlier, we may clarify the dependence of the constants on \( u \) and \( v \).
4 Numerical Simulations

In this section, we empirically validate the performance of the proposed algorithms on the networked Nash-Cournot game \[20, 22, 15\], where firms compete in quantity produced. It is a classical example of an aggregative game where the inverse-demand function depends on the sum of production by all firms. We assume that there are \( n \) firms, regarded as the set of players \( \mathcal{N} = \{1, \ldots, n\} \) competing over \( L \) spatially distributed markets (nodes) denoted by \( \mathcal{L} = \{1, \ldots, L\} \). For any \( i \), the \( i \)th firm needs to determine a continuous-valued nonnegative quantity of products to be produced and delivered to the markets, which is defined as \( x_i = (x_i^1, \ldots, x_i^L) \in \mathbb{R}^L \), where \( x_i^l \) denotes the sales of firm \( i \) at the market \( l \). Furthermore, the \( i \)th firm is characterized by a random linear production cost function \( c_i(x_i; \xi_i) = (c_i + \xi_i)\sum_{l=1}^L x_i^l \) for some parameter \( c_i > 0 \) where \( \xi_i \) is a mean-zero random variable. We further assume that the price \( p_l \) of products sold in market \( l \in \mathcal{L} \) is determined by the linear inverse demand (or price) function corrupted by noise \( p_l(\bar{x}_i; \zeta_l) = a_l + \zeta_l - b_l\bar{x}_i \), where \( \bar{x}_i = \sum_{l=1}^n x_i^l \) is the total sales of products at the market \( l \), the positive parameter \( a_l \) indicates the price when the production of the good is zero, the positive parameter \( b_l \) represents the slope of the inverse demand function, and the random disturbance \( \zeta_l \) is zero-mean. Consequently, firm \( i \) has an expectation-valued payoff function defined as \( f_i(x) = \mathbb{E}[c_i(x_i; \xi_i) - \sum_{l=1}^L p_l(\bar{x}_i; \zeta_l)x_i^l] \). Suppose firm \( i \in \mathcal{N} \) has finite production capacity \( X_i = \{x_i \in \mathbb{R}^L : x_i \geq 0, x_i^l \leq \text{cap}_{il}\} \). Then the objective of firm \( i \) is to find a feasible strategy that optimizes its payoff, i.e., \( \min_{x_i \in X_i} f_i(x_i, x_{-i}) \).

Numerical settings. In the numerical study, we consider a network with \( n \) firms and \( L = 10 \) markets with the parameters in the payoffs set as \( a_l \sim U(40, 50), b_l \sim U(1, 2), c_l \sim U(3, 5) \) for all \( i \in \mathcal{N} \) and \( l \in \mathcal{L} \), where \( U(\bar{u}, \bar{u}) \) denotes the uniform distribution over an interval \([\bar{u}, \bar{u}]\) with \( \bar{u} < \bar{u} \). In the stochastic settings, the random variables are assumed to be \( \xi_i \sim U(-c_i/5, c_i/5), \zeta_l \sim U(-a_l/5, a_l/5) \), respectively. We further set \( \text{cap}_{il} = 2 \forall i \in \mathcal{N}, l \in \mathcal{L} \).

We first validate the performance of the distributed VS-PGR (Alg. 1) scheme. We consider four kinds of undirected connected graphs: (i) Cycle graph, which consists of a single cycle and every node has exactly two edges incident with it; (ii) Star graph, where there is a center node connecting to all every other node; (iii) Erdős–Rényi graph, which is constructed by connecting nodes randomly and each edge is included in the graph with probability \( 2/n \) independent from every other edge; (iv) Complete graph, each node has an edge connecting it to every other node. We define a doubly stochastic matrix \( A = [a_{ij}]_{i,j=1}^n \) with \( a_{ii} = 1 - \frac{d(i)-1}{\max_1} \) if \( (j, i) \in \mathcal{E} \) and \( j \neq i, a_{ij} = 0, \) otherwise, where \( d(i) = |\mathcal{N}_i| \) denotes the number of neighbors of player \( i \) and \( \max_1 = \max_{i \in \mathcal{N}} d(i) \). We implement Algorithm 1 with \( \tau_k = [\log(k)] \) and \( S_k = [\rho^{-(k+1)}] \), and terminate it when the total number of samples utilized reached \( 10^6 \) and report the empirical error of \( \frac{\mathbb{E}[\|x_k - x^*\|]}{\|x^*\|} \) by averaging across 50 sample paths. The simulation results are demonstrated in Table 3. As expected, Algorithm 1 with complete graph (namely centralized VS-PGR) has fastest convergence rate and the empirical error at the termination increases with the size of the network. It also indicates that the constant step-size should not be taken too large, otherwise it might lead to non-convergence, see e.g. \( \alpha = 0.02 \) in the case \( n = 50 \). We further display trajectories of the iterates generated by the centralized VS-PGR scheme and its distributed variant over the ER graph. Though the centralized has faster convergence rate than its distributed variant, it requires much more rival information (or communications).
We now investigate how the network structure influences the convergence properties. Set $n = 20$ and run Algorithm 1 over the cycle, star, and Erdős–Rényi graphs with $\tau_k = k + 1$, $\alpha = 0.01$, and $S_k = \lceil \beta^{-(k+1)} \rceil$, where the network connectivity parameter $\beta$ are respectively 0.967, 0.95, 0.986 for cycle, star and ER graphs. The simulation results are demonstrated in Figure 2 with the left and right figure respectively displaying the rate of convergence and oracle complexity. It is shown that the star graph with $\beta = 0.95$ has the fastest convergence rate while the ER graph with $\beta = 0.986$ has the slowest convergence rate, this is consistent with Theorem 3 that smaller $\beta$ may lead to faster rate of convergence (since $\gamma = \beta$). It is also worth noting that for obtaining an $\epsilon$-NE, the ER graph requires the smallest number of samples, while the star graph has the worst oracle complexity. These findings support the theoretic results in Theorem 4 that larger $\beta$ may lead to better oracle complexity.

We then run the VS-PGR algorithm (namely, Algorithm 1 over a complete graph) with geometrically and polynomially increasing sample-sizes and demonstrate the results in Figure 3. The results in Fig. 3(a) show the rate of convergence, implying that with a small number of proximal evaluations there is no big difference on convergence rate, while the algorithm with geometrically increasing sample-size will outperform the algorithm with polynomially increasing sample-size if more proximal evaluations are available. Fig. 3(b) and Fig. 3(c) demonstrate the total number of samples required to obtain an $\epsilon$-NE, where it is shown in Fig. 3(b) that with low accuracy $\epsilon$ the polynomial sample-size with smaller degree $v$ appears to have better oracle complexity, while for a high accuracy $\epsilon$, the geometrically and polynomially increasing (with larger $v$) sample-size may have better oracle complexity. The numerical results are consistent with the discussions in Remark 1.

Comparison with stochastic gradient descent (SGD): We set $n = 20$ and compare Algorithm 1 and SGD by running both schemes over the Erdős–Rényi graph up to $10^6$ samples. We show the results in Table 4 and Figure 4, where SGD-$t$ denotes the minibatch SGD algorithm that utilizes $t$ samples at each iteration while in Algorithm 1 we set $S_k = \lceil \beta^{-(k+1)/2} \rceil$ and $\tau_k = k + 1$. Though it is seen from Table 4 that SGD can obtain slightly better empirical error, Algorithm 1 can significantly reduce the
computation time and the rounds of communication. We can also observe from the iteration complexity bounds demonstrated in Figure 4 that Algorithm 1 requires fewer proximal evaluations than SGD for approximating an NE with the same accuracy.

| Alg. | SGD-16 $\tau_k = \ln(k)$ | SGD-8 $\tau_k = \ln(k)$ |
|------|-------------------------|-------------------------|
| emp.err | 5.74e-04 | 2.58e-04 | 2.48e-4 |
| prox.eval | 469 | 6.25e+4 | 1.25e+5 |
| comm. | 1.11e+5 | 6.55e+5 | 1.41e+6 |
| CPU(s) | 1.8 | 14.67 | 28.76 |

Table 4: Comparison of Alg. 1 and SGD

We now validate the performance of the distributed VS-PBR scheme. Suppose that for each firm $i$, there exists a random quadratic production cost function $c_i(x_i; \xi_i) = (c_i + \xi_i)\sum_{l=1}^{L} x_i^l + \frac{\rho}{2}x_i^T x_i$ for some $c_i > 0$ and random disturbance $\xi_i$ with mean zero. Assume that $\lambda_{\min}(\rho I_L + 2\text{diag}(b)) > (n - 1)\lambda_{\max}(\text{diag}(b)) \; \forall i \in \mathcal{N}$. Then by the definition of $\Gamma$ in (41), $||\Gamma||_{\infty} < 1$ and thus the proximal BR map is contractive. Set $n = 13$, $L = 6$, and $\cap_{d_l} = 2 \; \forall i \in \mathcal{N}, l \in \mathcal{L}$. We then run Algorithm 3 with $\tau_k = \lceil \log(k) \rceil$, $\mu = 20$, and $S_k = \lceil 0.98^{-(k+1)} \rceil$, and demonstrate the convergence rate in Fig. 5, showing that a better network connectivity may lead to a faster rate.

Comparison of distributed VS-PBR and VS-PGR. Let the network be randomly generated by the Erdős–Rényi graph. We run Alg. 1 with $\alpha = 0.04$ and Alg. 3 with $\mu = 30$, where $\tau_k = k + 1$ and $S_k = \lceil 0.98^{-(k+1)} \rceil$. The numerical results for both schemes are shown in Figure 6 from which it is seen that distributed VS-PBR has faster convergence rate since it requires solving a deterministic optimization problem per iteration while distributed VS-PGR merely takes a proximal gradient step. Furthermore, the demonstrated oracle and communication complexity show that distributed VS-PBR necessitates less samples and communication rounds than distributed VS-PGR to obtain an $\epsilon$-NE.
5 Concluding Remarks

Stochastic Nash games and their networked variants represent an important generalization of deterministic Nash games. While there have been recent advances in the computation of equilibria in both distributed and stochastic regimes, at least three gaps currently exist in the available rate statements: (i) Gap between rate statements for deterministic and stochastic Nash games; (ii) Little available by way of implementable best-response schemes in stochastic regimes; (iii) Lack of computational, oracle, and communication complexity statements for distributed gradient and best-response schemes. Motivated by these gaps, we consider four distinct schemes for the resolution of a class of stochastic convex Nash games where each player-specific objective is a sum of an expectation-valued smooth function and a convex nonsmooth function: (i) VS-PGR scheme for strongly monotone stochastic Nash games; (ii) VS-PBR for stochastic Nash games with contractive proximal BR maps, (iii) Distributed VS-PGR for strongly monotone aggregative Nash games, and (iv) Distributed VS-PBR for stochastic aggregative games with contractive proximal BR maps. Under suitable geometrically increasing sample-size and linearly increasing consensus steps for the distributed variants, we show that all schemes generate sequences that converge at the (optimal) geometric rate and derive bounds on the computational, oracle, and communication complexity. We further quantify the rate and complexity bounds of the schemes when the sample-sizes and the rounds of communications increase at prescribed polynomial rates.

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