Existence and uniqueness of $L^p$ solutions to the Boltzmann equation with an angle-potential concentrated collision kernel

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Abstract

We solve the Cauchy problem associated to the space homogeneous Boltzmann equation with an angle-potential singular concentration modeling the collision kernel, proposed in [6]. The potential under consideration ranges from Coulomb to hard spheres cases. However, the motivation of such a collision kernel is to treat the case of Coulomb potentials, on which this particular form of collision operator is well defined. We also show that the scaled angle-potential singular concentration in a grazing collisions limit makes the Boltzmann operator converge in the sense of distributions to the Landau operator acting on the Boltzmann solutions.

Keywords: kinetic theory, soft potentials, Coulomb forces, grazing collisions limit, long range interactions, abstract ODE theory.

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1 Introduction

Aside from being evolution equations in nonlocal (differential-integral) form, the Boltzmann and Landau equations are closely related mathematically in the sense that the Landau equation was formally derived in 1937 (see [15]) from the Boltzmann equation which, on its own, cannot describe plasmas. This is due to the fact that the intermolecular Coulomb forces are so strong, that the singularities they create in the collision integral of the Boltzmann equation (the nonlocal integral term) are of the critical order at which the integral diverges. Hence, the Boltzmann equation in this case is ill-posed. Landau, however, was able to use the structure of the Boltzmann equation’s collision kernel (the weight in the collision integral that models probability rates of two interacting particles transitioning from their pre- to their post-collisional states) heuristically to derive a proper, convergent, collision operator that describes particle interactions in this special case.

1.1 Description of the Boltzmann and Landau equations

The formulation of the problem, in the x-uniform framework (known as the space homogeneous problem), is posed as follows: let $f = f(v, t)$, for $(v, t) \in \mathbb{R}^3 \times (0, \infty)$, be a probability density function describing the probability of finding a particle with velocity $v$ at time $t$. Let $v_*$ denote the velocity of a particle about to collide with the first, and let $v'$ and $v'_*$ denote their respective velocities before or after a reversible (elastic) collision. Also in the elastic case, collisions must conserve momentum and kinetic energy:

$$v' + v'_* = v + v_* \quad \text{and} \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \quad (1.1)$$
Defining the relative velocities \( u := v - v_* \) and \( u' := v' - v'_* \) and letting \( \sigma := \hat{u}' = u' / |u'| \in S^2 \) denote the scattering direction, the post- (or pre-) collisional velocities may be written as

\[
v' = v'(v, v_*, \sigma) = v + \frac{1}{2}(|u|\sigma - u), \quad v'_* = v'_*(v_*, v, \sigma) = v_* - \frac{1}{2}(|u|\sigma - u).
\]

One can represent \( \sigma \in S^2 \) as

\[
\sigma = \hat{u} \cos \theta + \omega \sin \theta,
\]

where \( \omega \in u^\perp, |\omega| = 1 \) is in turn decomposed into

\[
\omega = \hat{j} \cos \phi + \hat{k} \sin \phi.
\]

Here \( j, k \in u^\perp \) are defined as \( j := (1, 0, 0) - \hat{u}\hat{u}_1, k = j \times \hat{u} \).

The Cauchy problem for the Boltzmann equation is written in strong form as

\[
\begin{aligned}
\partial_t f(v) &= Q_B(f, f)(v) := \int_{\mathbb{R}^3 \times S^2} B(|u|^\gamma, \hat{u} \cdot \sigma) \cdot \left( f(v') f(v'_*) - f(v) f(v_*) \right) d\sigma dv_* \quad (1.4)
\end{aligned}
\]

where \( \hat{z} := z/|z| \) for any \( z \in \mathbb{R}^3 \), and we have omitted the \( t \) variable for convenience. The collision kernel, \( B(|u|^\gamma, \hat{u} \cdot \sigma) \), is often modeled as

\[
B(|u|^\gamma, \hat{u} \cdot \sigma) d\sigma := |u|^\gamma b(\hat{u} \cdot \sigma) d\sigma = |u|^\gamma b(\cos \theta) \sin \theta d\theta d\omega = |u|^\gamma \sin^{-m}(\theta/2) \sin \theta d\theta d\omega, \quad (1.5)
\]

and the spaces \( L^p_k \) are defined as

\[
L^p_k(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : \|f\|_{L^p_k(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} f^p(v)(1 + |v|^2)^k/2 \, dv \right)^{1/p} \right\}.
\]

The parameter \( \gamma \in (-3, 0) \) corresponds to soft potentials (repulsive forces), and \( \gamma = -3 \), which is only possible in (1.7), corresponds to Coulomb forces. The case \( \gamma \in (0, 1] \) corresponds to hard potential and was extensively studied in [11, 12, 13], and \( \gamma = 0 \) describes Maxwell molecule interactions (see for example [2, 3, 5, 6, 16]).
The weight \( b(\hat{u} \cdot \sigma) \) d\( \sigma \) is known as the angular cross section, and while it does not need to be defined exactly as it is in (1.5), \( b(\cdot) \) is always an even, nonnegative function that must satisfy
\[
\int_{0}^{\pi/2} b(\cos \theta) \sin^2 \theta \cos(\theta/2) d\theta < \infty \quad (1.6)
\]
in order for \( Q_B \) to be well defined in weak form (see [1, 8] for discussion on the cancellation lemma). In the case of (1.5), this means that \( m < 4 \).

In the case of Coulomb potentials (when \( \gamma = -3 \)), the cross section has been determined to be of the Rutherford type, where \( b(\hat{u} \cdot \sigma) \) d\( \sigma \) = \( b(\cos \theta) \sin \theta d\theta \sim \sin^{-3}(\theta/2)d\theta \) for \( \theta << \pi/2 \). This corresponds to \( m = 4 \) from (1.5), so \( Q_B \) is no longer well defined. Recall that in this case we use the Landau equation to model particle interactions. Nonetheless, an \( \varepsilon \)-truncation of the Boltzmann equation’s \( b(\cos \theta) \) helps us analyze the asymptotics and derive the Landau equation, whose strong form is
\[
\begin{align*}
\partial_t f(v) &= Q_L(f,f)(v) := \nabla_v \cdot \int_{\mathbb{R}^3} |u|^2 \Pi(u) \cdot (f(v) \nabla f(v) - \nabla f(v) f(v)) \, dv, \\
f(v) \bigg|_{t=0} &= f_0(v),
\end{align*}
\]
where \( \Pi(u) := I_{3 \times 3} - \hat{u} \otimes \hat{u} \in \mathbb{R}^{3 \times 3} \) projects onto the space \( u^\perp \).

There are several important similarities between the Boltzmann and Landau equations. For example, one can check, by using (1.1), the symmetry of \( B(|u|, \hat{u} \cdot \sigma) \) and exchanging variables in \( Q_B(f,f) \) and \( Q_L(f,f) \), that solutions of both the Boltzmann and Landau equations conserve mass, momentum and kinetic energy:
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(v,t)(1,v,|v|^2)dv = \int Q_{B,L}(f,f)(v,t)(1,v,|v|^2)dv = 0, \quad (1.8)
\]
that is,
\[
\int f(v,t)(1,v,|v|^2)dv = \int f_0(v)(1,v,|v|^2)dv. \quad (1.9)
\]
Also both equations satisfy the H-Theorem, meaning that
\[
-\frac{d}{dt} \mathcal{H}(t) = -\frac{d}{dt} \int f \log f dv = \int Q_{B,L}(f,f) \log f dv \leq 0, \quad (1.10)
\]
with equality holding if and only if \( f \) is a Gaussian in the velocity variable.

However, in order to see exactly how \( Q_B \) turns into \( Q_L \) when collisions become grazing, we need to carefully study the limiting behavior of a properly \( \varepsilon \)-truncated collision cross section, \( b_\varepsilon \), for which the Boltzmann equation does not yet fall apart.
1.2 The grazing collisions limit

The most common truncation of the Boltzmann collision cross section is

\[ b_\varepsilon(\cos \theta) = \frac{I_{\theta \geq \varepsilon}}{\log \sin(\varepsilon/2)} b(\cos \theta) \]  

(1.11)

(see [15, 9, 16, 14]). In [14], the authors were able to extend this to an even stronger \( \theta \)-singularity in \( b(\cos \theta) \) with a suitable truncation, and still obtain the Landau equation: for \( \delta \in [0, 2) \) they define

\[ b^\delta_\varepsilon(\cos \theta) := \frac{I_{\theta \geq \varepsilon}}{H_\delta(\sin(\varepsilon/2))} \sin^{-\delta}(\theta/2), \]  

(1.12)

where \( H_\delta \) is such that \( H'_\delta(x) = x^{-(\delta+1)} \) (if \( \delta = 0 \) then we recover (1.11)).

Moreover, the rate of convergence of the Boltzmann collision integral \( Q_B \) to the corresponding Landau collision term is much higher for \( \delta > 0 \). In a sense, the angular cross sections \( b_\varepsilon \) approximate a singular point mass distribution as \( \varepsilon \to 0 \), which is a signature of the Landau derivation. This limit is called the grazing collisions limit.

It’s important to note that one does not need to use the exact truncation (or the exact collision kernel) above to get the grazing collisions limit. In fact, according to [16] it suffices for \( B_\varepsilon \) to satisfy the following three conditions, pointwise in \( u \in \mathbb{R}^3 \):

\[ \beta_2[B_\varepsilon](u) := \int_0^{\frac{\pi}{2}} B_\varepsilon(|u|, \cos \theta) \sin^2(\theta/2) \sin \theta d\theta \]

\[ \to \frac{2}{\pi} |u|^{-\gamma} \text{ as } \varepsilon \to 0, \]  

(1.13)

\[ \forall k > 2, \ \beta_k[B_\varepsilon](u) := \int_0^{\frac{\pi}{2}} B_\varepsilon(|u|, \cos \theta) \sin^k(\theta/2) \sin \theta d\theta \]

\[ \to 0 \text{ as } \varepsilon \to 0, \]  

(1.14)

\[ |u|^{-\gamma} B_\varepsilon(|u|, \cos \theta) \to 0 \text{ as } \varepsilon \to 0, \]  

uniformly on \( \{ \theta > \theta_0 \}, \forall \theta_0 \in (0, \pi/2) \).  

(1.15)

Indeed, one can show that if (1.13)- (1.15) hold and \( B_\varepsilon(|u|^{\gamma}, \hat{u} \cdot \sigma) = |u|^{\gamma} b_\varepsilon(\hat{u} \cdot \sigma) \), then \( Q_{B_\varepsilon}(f,f) \to Q_L(f,f) \) as \( \varepsilon \to 0 \) in the sense of distributions. A sketch of the following theorem can be found in [16]:

**Proposition 1.1.** Consider a sequence of nonnegative collision kernels, \( B_\varepsilon = B_\varepsilon(|u|^{\gamma}, \hat{u} \cdot \sigma) = |u|^{\gamma} b_\varepsilon(\hat{u} \cdot \sigma), -3 \leq \gamma \leq -1 \), satisfying properties
(1.13) - (1.15), and let \(0 \leq f \in L^1_2 \cap L^p\), where

\[
p \geq \frac{6}{6 - |\gamma + 2|} \quad \text{if } \gamma \leq -2, \\
p > 1 \quad \text{if } \gamma > -2.
\]

Then for any test function \(\varphi \in C_0^\infty(\mathbb{R}^3)\) and for any \(t > 0\),

\[
\lim_{\varepsilon \to 0} \left| \int (Q_{B_\varepsilon}(f,f)(v,t) - Q_L(f,f)(v,t)) \varphi(v) dv \right| = 0. \tag{1.16}
\]

**Proof.** We will first formally split \(\int Q_{B_\varepsilon}(f,f)\varphi dv\) and \(\int Q_L(f,f)\varphi dv\) into several integrals, and justify the splitting at the end. By an exchange of variables, one can check that formally,

\[
\int Q_L(f,f)(v)\varphi(v) dv = \int \int f f_* |u|^\gamma \left( -2(\nabla \varphi - \nabla_\ast \varphi_\ast) \cdot u + \frac{1}{2} |u|^2 (D^2 \varphi + D^2_\ast \varphi_\ast) : \Pi(u) \right) dv_* dv \\
= \frac{1}{2} \int \int f f_* |u|^\gamma G_L(v,v_\ast) dv_* dv, \tag{1.17}
\]

where

\[
G_L(v,v_\ast) := -2(\nabla \varphi - \nabla_\ast \varphi_\ast) \cdot u + \frac{1}{2} |u|^2 (D^2 \varphi + D^2_\ast \varphi_\ast) : \Pi(u). \tag{1.18}
\]

Define

\[
G^1_L(v,v_\ast) = -4(\nabla \varphi - \nabla_\ast \varphi_\ast) \cdot u = -4(\partial_{v_i} \varphi - \partial_{v_\ast i} \varphi_\ast) u_i, \\
G^2_L(v,v_\ast) = |u|^2 (D^2 \varphi + D^2_\ast \varphi_\ast) : \Pi(u) = |u|^2 (\partial_{v_i v_j} \varphi + \partial_{v_\ast i v_\ast j} \varphi_\ast) \Pi(u)_{ij}.
\]

Similarly for the Boltzmann collision term,

\[
\int Q_{B_\varepsilon}(f,f)(v)\varphi(v) dv = \frac{1}{2} \int \int f f_* |u|^\gamma \int_{S^2} b_\varepsilon(\hat{u} \cdot \sigma)(\varphi' + \varphi'_\ast - \varphi - \varphi_\ast) d\sigma dv_* dv \\
= \frac{1}{2} \int \int f f_* G[B_\varepsilon](v,v_\ast) dv_* dv,
\]

where
\[ G[B_\xi(v, v_s)] := |u|^\gamma \int_0^{\pi/2} B_\xi(|u|^\gamma, \cos \theta) \int_{-\pi}^\pi (\varphi' + \varphi_s' - \varphi - \varphi_s) d\phi \sin \theta d\theta \]
\[ = \int_{S^2} B_\xi(|u|^\gamma, \hat{\phi} \cdot \sigma)(\varphi' + \varphi_s' - \varphi - \varphi_s) d\sigma. \]

We begin by taking the second order Taylor expansion of \( \varphi' + \varphi_s' - \varphi - \varphi_s \), keeping in mind that \( v'_s - v_s = -(v' - v) \):

\[
(\varphi' - \varphi) + (\varphi_s' - \varphi_s) = \\
= \nabla \varphi(v) \cdot (v' - v) + \frac{1}{2} \partial_{v_iv_j} \varphi(v)(v'_i - v_i)(v'_j - v_j) \\
+ \frac{1}{6} \partial_{v_iv_j, v_k} \varphi(\xi)(v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \\
+ \nabla \varphi(v_s) \cdot (v'_s - v_s) + \frac{1}{2} \partial_{v_iv_j} \varphi(v_s)(v'_i - v_i)(v'_j - v_j) \\
+ \frac{1}{6} \partial_{v_iv_j, v_k} \varphi(\xi)(v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \\
= (\nabla \varphi(v) - \nabla \varphi_s(v_s)) \cdot (v' - v) \\
+ \frac{1}{2} (\partial_{v_iv_j} \varphi(v) + \partial_{v_s, v_s} \varphi(v_s))(v'_i - v_i)(v'_j - v_j) \\
+ \frac{1}{6} (\partial_{v_iv_j, v_k} \varphi(\xi) - \partial_{v_s, v_s, v_s, v_s} \varphi(\xi))(v'_i - v_i)(v'_j - v_j)(v'_k - v_k), \quad (1.19)
\]

where \( \xi \) and \( \zeta \) are convex combinations of \( v, v', \) and \( v_s, v'_s \) respectively. Next, substitute this expansion into \( G[B_\xi] \):

\[
G[B_\xi(v, v_s)] = (\nabla \varphi(v) - \nabla \varphi_s(v_s)) \cdot \int_0^{\pi/2} B_\xi(|u|^\gamma, \cos \theta) \int_{-\pi}^\pi (v' - v) d\phi \sin \theta d\theta \\
+ \frac{1}{2} (\partial_{v_iv_j} \varphi + \partial_{v_s, v_s} \varphi) \int_0^{\pi/2} B_\xi(|u|^\gamma, \cos \theta) \int_{-\pi}^\pi (v'_i - v_i)(v'_j - v_j) d\phi \sin \theta d\theta \\
+ \frac{1}{6} \int_0^{\pi/2} B_\xi(|u|^\gamma, \hat{\phi} \cdot \sigma) \int_{-\pi}^\pi (\partial_{v_iv_j, v_k} \varphi(\xi) - \partial_{v_s, v_s, v_s, v_s} \varphi(\xi)) \cdot \\
\cdot (v'_i - v_i)(v'_j - v_j) d\phi d\sigma \quad (1.20)
\]

Let

\[
G_1[B_\xi(v, v_s)] := \int_0^{\pi/2} B_\xi(|u|^\gamma, \hat{\phi} \cdot \sigma) \int_{-\pi}^\pi (v' - v) d\phi \sin \theta d\theta, \\
G_2[B_\xi(v, v_s)] := \frac{1}{2} \int_0^{\pi/2} B_\xi(|u|^\gamma, \cos \theta) \int_{-\pi}^\pi (v'_i - v_i)(v'_j - v_j) d\phi \sin \theta d\theta, \\
\]
\[ G_3[B_\varepsilon](v, v_*) := \frac{1}{6} \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos \theta) \int_{-\pi}^{\pi} (\partial_{v_i v_j v_k} \varphi(\xi) - \partial_{v_* v_* v_*} \varphi(\zeta)) \cdot (v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \sin \theta d\theta d\phi \]

\[ \leq \frac{1}{3} \|D^3\|_{L^\infty} \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos \theta) \int_{-\pi}^{\pi} |v' - v|^3 \sin \theta d\phi. \]

Using the representations of the post-collisional velocities and of the scattering direction from Section 1.1, it is not hard to show that

\[ \int_{-\pi}^{\pi} (v' - v) d\phi = \pi u(\cos \theta - 1) = -2\pi u \sin^2(\theta/2), \quad (1.21) \]

\[ \int_{-\pi}^{\pi} (v'_i - v_i)(v'_j - v_j) d\phi = \frac{\pi}{2}(\cos \theta - 1)^2 u_i u_j + \frac{\pi}{4} \Pi(u)_{ij} |u|^2 \sin^2 \theta \]

\[ = \pi \sin^4(\theta/2)(2u_i u_j - |u|^2 \Pi(u)_{ij}) + \pi |u|^2 \Pi(u)_{ij} \sin^2(\theta/2), \quad (1.22) \]

\[ \int_{-\pi}^{\pi} |v' - v|^3 d\phi = |u|^3 \int_{-\pi}^{\pi} \sin^3(\theta/2) d\phi = 2\pi \sin^3(\theta/2). \quad (1.23) \]

Then \( G_1[B_\varepsilon] \) and \( G_2[B_\varepsilon] \) become

\[ G_1[B_\varepsilon](v, v_*) = -2\pi u \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos \theta) \sin^2(\theta/2) \sin \theta d\theta \]

\[ = -2\pi u \beta_2[B_\varepsilon](u) \quad (1.24) \]

and

\[ G_2[B_\varepsilon](v, v_*) = \frac{\pi}{2}(2u_i u_j - |u|^2 \Pi(u)_{ij}) \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos \theta) \sin^4(\theta/2) \sin \theta d\theta \]

\[ + \frac{\pi}{2} |u|^2 \Pi(u)_{ij} \int_{-\pi}^{\pi} B_\varepsilon(|u|^\gamma, \cos \theta) \sin^2(\theta/2) \sin \theta d\theta \]

\[ = \frac{\pi}{2}(2u_i u_j - |u|^2 \Pi(u)_{ij}) \beta_4[B_\varepsilon](u) + \frac{\pi}{2} |u|^2 \Pi(u)_{ij} \beta_2[B_\varepsilon](u), \quad (1.25) \]

and \( G_3[B_\varepsilon] \) is bounded by

\[ G_3[B_\varepsilon](v, v_*) \leq \frac{2\pi}{3} |u|^3 \|D^4\|_{L^\infty} \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos \theta) \sin^3(\theta/2) \sin \theta d\theta \]

\[ = \frac{2\pi}{3} |u|^3 \|D^4\|_{L^\infty} \beta_3[B_\varepsilon](u). \quad (1.26) \]
Together,

\[
\int Q_{B_\epsilon}(f, f)(v)\varphi(v)\,dv = \frac{1}{2} \iint f f_s((\nabla \varphi - \nabla_* \varphi_\ast) \cdot G_1[B_\epsilon](v, v_*)\,dv, dv
\]

\[
+ \frac{1}{2} \iint f f_s(\partial_{v_i} v_j \varphi + \partial v_i v_j \varphi_\ast) G_2[B_\epsilon](v, v_*)\,dv, dv
\]

\[
+ \frac{1}{2} \iint f f_s G_3[B_\epsilon](v, v_*)\,dv, dv
\]

\[= -\pi \int \beta_2[B_\epsilon](u) f f_s(\nabla \varphi - \nabla_* \varphi_\ast) \cdot udv, dv
\]

\[
+ \frac{\pi}{4} \int \beta_4[B_\epsilon](u) f f_s(\partial_{v_i} v_j \varphi + \partial v_i v_j \varphi_\ast)(2u_i u_j - |u|^2 \Pi(u)_{ij})\,dv, dv
\]

\[
+ \frac{\pi}{4} \int \beta_2[B_\epsilon](u) f f_s|u|^2(\partial_{v_i} v_j \varphi + \partial v_i v_j \varphi_\ast) \Pi(u)_{ij}\,dv, dv
\]

\[
+ \frac{1}{2} \iint f f_s G_3[B_\epsilon](v, v_*)\,dv, dv. \tag{1.27}
\]

Now, we show that the four integrals at the end of (1.27) are bounded. This will justify this splitting of \(\int Q_{B_\epsilon}(f, f)\varphi\) and \(\int Q_L(f, f)\varphi\). First, the structure of \(B_\epsilon(|u|^\gamma, \hat{u} \cdot \sigma) = |u|^\gamma, b_\epsilon(\hat{u} \cdot \sigma)\) and assumptions (1.13), (1.14) imply that \(\beta_k[B_\epsilon](u) \leq A_k|u|^\gamma\) for some \(K > 0\). Using this, and that \(\varphi \in C_0^\infty\),

\[
\int f f_s \beta_2[B_\epsilon](u)(\nabla \varphi - \nabla_* \varphi_\ast) \cdot udv, dv \leq A_2\|D^2 \varphi\|_L \iint f f_s|u|^{\gamma + 2} dv, dv, \tag{1.28}
\]

\[
\int \beta_4[B_\epsilon](u) f f_s(\partial_{v_i} v_j \varphi + \partial v_i v_j \varphi_\ast)(2u_i u_j - |u|^2 \Pi(u)_{ij})\,dv, dv
\]

\[
\leq 6A_4\|D^2 \varphi\|_L \iint f f_s|u|^{\gamma + 2} dv, dv, \tag{1.29}
\]

\[
\int \beta_2[B_\epsilon](u) f f_s|u|^2\,dv, dv(\partial_{v_i} v_j \varphi + \partial v_i v_j \varphi_\ast) \Pi(u)_{ij}\,dv, dv
\]

\[
\leq 2A_2\|D^2 \varphi\|_L \iint f f_s|u|^{\gamma + 2} dv, dv, \tag{1.30}
\]

and

\[
\iint f f_s G_3[B_\epsilon]\,dv, dv \leq \frac{2\pi}{3} A_3 \iint f f_s|u|^{\gamma + 3} dv, dv. \tag{1.31}
\]

It remains to show that \(\iint f f_s|u|^{\gamma + 2}, \iint f f_s|u|^{\gamma + 3}\) are finite. For this, we apply the following lemma, which was inspired by Lemma 4 from [10]:

\[9\]
Lemma 1.2. Let $p > 1$, $k > 0$ and $\alpha \leq k$ such that $\alpha p' > -6$. If $f \in L^1_k \cap L^p(\mathbb{R}^3)$ and $h(v) := |v|^\alpha$, then there exists $C = C(p) > 0$ such that

\[
\iint f(v)f(v_*)|v - v_*|^\alpha dv_*dv \leq 2^k \|f\|_{L^1(\mathbb{R}^3)}\|f\|_{L^k_1(\mathbb{R}^3)} \quad \text{if } \alpha \geq 0, \tag{1.32}
\]
\[
\iint f(v)f(v_*)|v - v_*|^\alpha dv_*dv \leq \|f\|_{L^1(\mathbb{R}^3)}^2 + C\|f\|_{L^p(\mathbb{R}^3)}^2 \quad \text{if } \alpha < 0. \tag{1.33}
\]

Proof. If $\alpha \geq 0$, then we can use the convexity of $x \mapsto x^\alpha$ to get

\[
|v - v_*|^\alpha \leq 2^{\alpha-1}(\|v\|^\alpha + |v_*|^\alpha) \leq 2^{k-1}\left(1 + |v|^2\right)^{\frac{\alpha}{2}} + \left(1 + |v_*|^2\right)^{\frac{\alpha}{2}},
\]
so

\[
\iint f(v)f(v_*)|v - v_*|^\alpha dv_*dv \leq 2^k\|f\|_{L^1} \|f\|_{L^k_1}.
\]
If $\alpha < 0$, let $h_1(v) := |v|^\alpha 1_{|v| \leq 1}$ and $h_2(v) := |v|^\alpha 1_{|v| > 1}$, so that $h_1 + h_2 = h$. Then

\[
\iint f(v)f(v_*)h(v - v_*)dv_*dv = \int f(v)f \ast h_1(v)dv + \int f(v)f \ast h_2(v)dv.
\]
By Holder's and Young's inequalities,

\[
\int ff \ast h_1dv \leq \|f\|_{L^p} \|f \ast h_1\|_{L^{p'}} \leq \|f\|_{L^p}^2 \|h_1\|_{L^{p'/2}}, \tag{1.34}
\]
\[
\int ff \ast h_2dv \leq \|f\|_{L^1} \|f \ast h_2\|_{L^\infty} \leq \|f\|_{L^1}^2 \|h_2\|_{L^\infty} = \|f\|_{L^1}^2, \tag{1.35}
\]
Note that $\|h_1\|_{L^{p'/2}}$ depends only on $|S^2|$ and $p'$. 

Now that our steps until now have been justified, we can take the limit as $\varepsilon \to 0$ in (1.27). (1.13) - (1.15) allow the second and fourth integrals to vanish, leaving us with

\[
\int Q_{B_\varepsilon}(f, f)(v)\varphi(v)dv \to -2 \iint |u|^n ff_*(\nabla \varphi - \nabla_\varepsilon \varphi_\varepsilon) \cdot udv_*dv
\]
\[
+ \frac{1}{2} \iint |u|^n ff_*(D^2 \varphi + D^2_{\varepsilon} \varphi_\varepsilon) : \Pi(u)dv_*dv
\]
\[
= \int Q_{L}(f, f)(v)\varphi(v)dv. \tag{1.36}
\]

\[\square\]
If, additionally, $f_\varepsilon$ is a solution to (1.4) with $B = B_\varepsilon$, then one can go a step further and replace $f$ in Proposition 1.1 with $f_\varepsilon$:

**Theorem 1.3.** Let $f_\varepsilon \in L^p(\mathbb{R}^3) \cap L^1_2(\mathbb{R}^3)$, $p > \frac{6}{5}$, be a weak solution of (1.4) with the collision cross section $b_\varepsilon$ satisfying (1.13) - (1.15), and with $0 \leq f_0 \in L^1_2 \cap L \log L(\mathbb{R}^3)$. Then, for all $t > 0$, $Q_{B_\varepsilon}(f, f) \to Q_L(f, f)$ as distributions. That is, for any $\varphi \in C_0^\infty$,

$$
\lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}^3} (Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v, t) - Q_L(f_\varepsilon, f_\varepsilon)(v, t)) \varphi(v)dv \right| = 0. \quad (1.37)
$$

The proof of this theorem is almost identical to the proof the previous proposition.

1.3 Weak and weak-H formulations.

Let $\varphi = \varphi(v, t) \in C^1(\mathbb{R}^+, C_0^\infty(\mathbb{R}^3))$ and consider the weak form of $Q_B$ for a collision kernel $B(|u|, \hat{u} \cdot \sigma) = |u|^\gamma b(\hat{u} \cdot \sigma)$, with $b$ even, nonnegative and symmetric about $\theta = \pi/2$. Then $B$ is invariant under the change of variables $(v, v_s) \leftrightarrow (v_s, v)$ and $(v, v_s) \leftrightarrow (v', v')$. Furthermore, the Jacobian of these transformations has an absolute value of one, therefore we may write the weak form of $Q_B$ as

$$
\int_0^t \int_{\mathbb{R}^3} Q_B(f, f)(v, s) \varphi(v, s)dvdv = \int_0^t \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u} \cdot \sigma)(f' f'_* - ff_*) \varphi(v, s)d\sigma dv_* dsdvdv = \frac{1}{4} \int_0^t \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u} \cdot \sigma) \cdot (f' f'_* - ff_*) (\varphi + \varphi_* - \varphi' - \varphi'_*) d\sigma dv_* dsdvdv. \quad (1.38)
$$

In [16] the author ensures that the integral on right hand side of (1.38) converges by making the extra assumption that solutions of the Boltzmann equation have finite entropy decay, that is,

$$
0 \leq -\frac{d}{dt} \mathcal{H}(t) = -\frac{d}{dt} \int f(v, t) \log f(v, t)dv
= \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} B(u, \hat{u} \cdot \sigma) \cdot (f' f'_* - ff_*) .
$$
\[
\cdot (\log f'_s - \log f_s) \, d\sigma d\nu_s < \infty. \quad (1.39)
\]

Because of the assumption on the entropy, the variational formulation of the Boltzmann equation (1.4) with (1.38) representing \( Q_B \varphi \) is called the weak-H form, and its solutions are consequently the weak-H, or H solutions.

However, if the singularities of \( B(|u|^{\gamma}, \hat{u} \cdot \sigma) \) are mild enough - that is, if \( \gamma \geq -2 \) and if (1.6) holds - then the right hand side of (1.38) is well defined even without the assumption (1.39). In fact, in this case we can even go further by splitting \( Q_B \) into its gain and loss parts, \( Q_B^+ \) and \( Q_B^- \) (which are still well defined):

\[
Q_B(f, f) = \int_{\mathbb{R}^3 \times S^2} B(|u|^{\gamma}, \hat{u} \cdot \sigma)(f'_s f'_* - f_s f_*) d\sigma d\nu_s
= \int_{\mathbb{R}^3 \times S^2} B(|u|^{\gamma}, \hat{u} \cdot \sigma)f'_s d\sigma d\nu_s - \int_{\mathbb{R}^3 \times S^2} B(|u|^{\gamma}, \hat{u} \cdot \sigma)f_s d\sigma d\nu_s
=: Q_B^+(f, f) - Q_B^-(f, f).
\]

Then we can break up \( \int Q_B \varphi \) into

\[
\int_0^t \int_{\mathbb{R}^3} Q_B(f, f)(v, s) \varphi(v, s) dv ds
= \frac{1}{4} \int_0^t \int_{\mathbb{R}^3} \int_{S^2} B(|u|, \hat{u} \cdot \sigma)f'_s (\varphi + \varphi_* - \varphi' - \varphi'_*) d\sigma d\nu_s ds
- \frac{1}{4} \int_0^t \int_{\mathbb{R}^3} \int_{S^2} B(|u|, \hat{u} \cdot \sigma)f_s (\varphi + \varphi_* - \varphi' - \varphi'_*) d\sigma d\nu_s ds
= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{S^2} f_s f'_s B(|u|, \hat{u} \cdot \sigma)(\varphi' + \varphi'_* - \varphi - \varphi_*) d\sigma d\nu_s ds, \quad (1.40)
\]

and the right hand side of (1.40) is still well defined. Because no additional assumption on the entropy is required in this case, this form of \( \int Q_B \varphi \) is stronger, and is known simply as the weak form. The weak formulation of (1.4) would then have (1.40) on its left hand side, and solutions to this problem are called weak solutions.

The precise definitions of weak and weak-H solutions of the Boltzmann equation are defined by Villani in [16] as follows:

**Definition 1.4.** A function \( f(v, t) \) is said to be a weak solution of the Boltzmann equation with initial data \( 0 \leq f_0 \in L^1_2(\mathbb{R}^3) \) if the following conditions are satisfied:
\( f \geq 0, f \in C(\mathbb{R}^+, D'), f \in L^1([0, T], L^1_{2+\gamma}), \forall t \geq 0, f(\cdot, t) \in L^1_2(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3), \)

\( f(v, 0) = f_0(v) \) for a.e. \( v \in \mathbb{R}^3 \)

\( \forall t \geq 0, \int f(v, t) \left( \frac{1}{v} \right) dv = \int f_0(v) \left( \frac{1}{v} \right) dv \)  \hspace{1cm} (1.41)

\( \int f(v, t) \log f(v, t) dv \leq \int f_0(v) \log f_0(v) dv \)  \hspace{1cm} (1.42)

\( \forall \varphi = \varphi(v, t) \in C^1(\mathbb{R}^+, C^\infty_0(\mathbb{R}^3)), \forall t > 0, \)

\( \int f(v, t)\varphi(v, t) dv - \int f_0(v)\varphi(v, 0) dv - \int_0^t \int f(v, s)\partial_s \varphi(v, s) dv ds = \frac{1}{2} \int_0^t \int f(v, s)f(v_*, s)|v|^{\gamma} \cdot \int_{S^2} b(\hat{u} \cdot \sigma) (\varphi' + \varphi'_* - \varphi - \varphi_*) d\sigma dv_* dv. \)  \hspace{1cm} (1.43)

**Definition 1.5.** A function \( f(v, t) \) is an \( H \)-solution of the Boltzmann equation if it satisfies all of the conditions (i)-(iii) above, and item (iv) with the right hand side of (1.49) replaced by (1.38).

**Remark 1.6.** The difference between weak solutions and \( H \) solutions lies only in the interpretation of \( \int Q_B \varphi \). The entropy assumption, (1.39), is only a condition under which (1.38) is well defined. If there is another way to ensure the finiteness of the right hand side of (1.38) (for example if \( B(|u|^{\gamma}, \hat{u} \cdot \sigma) \) has a special, non-traditional structure), then (1.39) is not needed. Whether the entropy assumption is needed or not, solutions to the variational problem are called \( H \)-solutions as long as \( \int Q_B \varphi \) is defined as in (1.38), and they are called weak solutions if \( \int Q_B \varphi \) is defined as in (1.40).

A similar definition of weak and weak-\( H \) solutions can be derived for the Landau equation. One can check in [16] that the weak-\( H \) form of \( \int Q_L \varphi \) is...
\[
\int Q_L(f, f)\varphi(v)dv = -\int \sqrt{ff_s}|u|^{\gamma+2}(\nabla \varphi - \nabla \varphi_s)^T\Pi(u) \cdot \left( (\nabla - \nabla_s)\sqrt{ff_s}|u|^{\gamma+2} \right) dv_s dv, \quad (1.44)
\]
which is well defined given the assumption of finite entropy decay of the solutions:

\[
0 \leq -\frac{d}{dt}\int f(t, v) \log f(t, v)dv = \int ff_s|u|^{\gamma+2} \left( \frac{\nabla f}{f} - \frac{\nabla f_s}{f_s} \right)^T \Pi(u) \left( \frac{\nabla f}{f} - \frac{\nabla f_s}{f_s} \right) dv_s dv < \infty. \quad (1.45)
\]

For \( \gamma \geq -2 \), similar to \( Q_B \) one can split \( Q_L \) into two integrals to further expand (1.44):

\[
\int Q_L(f, f)\varphi dv = -\frac{1}{2} \int \int ff_s|u|^{\gamma}(\nabla \varphi - \nabla \varphi_s) \cdot udv_s dv \\
+ 2 \int \int ff_s|u|^{\gamma+2}\Pi(u) : (D^2\varphi + D^2\varphi_s)dv_s dv \quad (1.46)
\]

Now we define a weak solution for the Landau equation:

**Definition 1.7.** A function \( f(v, t) \) is said to be a weak solution of the Landau equation with initial data \( 0 \leq f_0 \in L^1_2(\mathbb{R}^3) \) if the following conditions are satisfied:

(i)

\[
f \geq 0, f \in C([0, T], L^1_2), f \in L^1([0, T], L^1_{2+\gamma}), \\
\forall t \geq 0, f(\cdot, t) \in L^1_2(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3),
\]

(ii)

\[
f(v, 0) = f_0(v) \text{ for a.e. } v \in \mathbb{R}^3
\]

(iii)

\[
\forall t \geq 0, \int f(v, t) \left( \frac{1}{v} \right) dv = \int f_0(v) \left( \frac{1}{v} \right) dv \quad (1.47)
\]

\[
\int f(v, t) \log f(v, t)dv \leq \int f_0(v) \log f_0(v)dv \quad (1.48)
\]
\[ \forall \varphi = \varphi(v, t) \in C^1(\mathbb{R}^+, C_0^\infty(\mathbb{R}^3)), \forall t > 0, \]
\[ \int f(v, t)\varphi(v, t)dv - \int f_0(v)\varphi(v, 0)dv - \int_0^t \int f(v, s)\partial_s \varphi(v, s)dvds \]
\[ = \frac{1}{2} \int_0^t \iiint f(v, s)f'(v, s)|u|^{-1}(D^2\varphi(v, s) + D_s^2\varphi(v, s)) : \Pi(u)dv dv ds \]
\[ - 2 \int_0^t \iiint f f'_s |u|^{-3}(\nabla \varphi(v, s) - \nabla_s \varphi(v, s)) \cdot udv dv. \quad (1.49) \]

By now, thanks to Desvillettes in [10], we know that H-solutions of the Landau equation for all soft potentials (even if $\gamma = -3$) are weak solutions, so $\int Q_L(f, f)\varphi$ can be defined as in (1.46), and (1.44) is not necessary. Desvillettes shows that not only does (1.45) ensure that the right hand side of (1.38) well defined, but that the right hand side of (1.40) is finite too. However, it is important to note that the weak solutions of Desvillettes for very soft potentials require extra integrability, which is obtained by using (1.45), so the entropy assumption is still needed.

2 An angle-potential concentrated collision kernel

2.1 Description of the kernel

Recall that for soft potentials the traditional Boltzmann collision operator, $B(|u|, \hat{u} \cdot \sigma) = |u| b(\hat{u} \cdot \sigma)$ has two singularities - one when $u = 0$, the other when $\hat{u} \cdot \sigma = 1$. However, classical truncations of the kernel only take care of the singularity in $b(\cos \theta)$, not in $|u|$, so in fact the collision operator $Q_{B_\varepsilon}$ is still a singular integral. This is a problem when looking for weak solutions, partly because it does not allow us to make $L^p$ estimates on $Q^+_B$ and $Q^-_B$. We are therefore tempted to truncate the collision kernel in a way that also controls its singularity at $u = 0$, while still sending $Q_{B_\varepsilon}(f, f)$ to $Q_L(f, f)$ in the grazing collisions limit.

We present here a collision kernel, originally from [6], that links the two singularities existing in $B$:

\[ g_\varepsilon(|u|, \hat{u} \cdot \sigma) = g_\varepsilon(|u|, \mu) = \frac{4}{\pi \varepsilon} \delta_0(1 - \mu - \min\{2, \varepsilon|u|\}), \quad (2.1) \]

where $\mu := \cos \theta$. Notice that, unlike the standard $B(|u|, \hat{u} \cdot \sigma)$, this collision kernel does not separate its two variables, $|u|$ and $\hat{u} \cdot \sigma$. 

We can check that, for fixed \( u \neq 0 \), \( g_\varepsilon \) satisfies properties (1.13) and (1.14) from before. Indeed, letting \( m_\varepsilon(x) := \min\{2, \varepsilon x^2\} \) and \( \mu_\varepsilon(x) := 1 - m_\varepsilon(x) \) we have

\[
\beta_2[g_\varepsilon](u) = \int_0^{\pi/2} g_\varepsilon(|u|^\gamma, \cos \theta) \sin^2(\theta/2) \sin \theta d\theta = \frac{1}{2} \int_0^{1} g_\varepsilon(|u|^\gamma, \mu)(1-\mu) d\mu
\]

\[
= \frac{2}{\varepsilon}(1 - \mu_\varepsilon(|u|))1_{|u|^\gamma \leq 1} = \frac{2}{\pi \varepsilon} m_\varepsilon(|u|)1_{|u|^\gamma \leq 1} \geq \frac{2}{\pi} |u|^\gamma 1_{|u|^\gamma \leq 1}. \quad (2.2)
\]

Similarly, for \( k > 2 \) and a fixed \( u \neq 0 \),

\[
\beta_k[g_\varepsilon](u) = \int_0^{\pi/2} g_\varepsilon(|u|^\gamma, \mu) \sin^k(\theta/2) \sin \theta d\theta
\]

\[
= \int_0^{1} g_\varepsilon(|u|^\gamma, \mu) \left(\frac{1}{2} (1 - \mu)\right)^{\frac{k}{2}} d\mu = 2^{-\frac{k}{2}} \frac{4}{\pi \varepsilon} (1 - \mu_\varepsilon(|u|))^{\frac{k}{2}} 1_{|u|^\gamma \leq 1}
\]

\[
= 2^{-\frac{k}{2}} \frac{4}{\pi \varepsilon} m_\varepsilon(|u|)^{\frac{k}{2}} 1_{|u|^\gamma \leq 1} \leq \frac{2^2 \varepsilon^{\frac{k}{2}}}{\pi} |u|^\gamma 1_{|u|^\gamma \leq 1}
\]

\[
\rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0, \quad (2.3)
\]

thus (1.13) and (1.14) are satisfied. And in some sense, \( g_\varepsilon \) satisfies (1.15) as well, because the mass of \( g_\varepsilon \) is concentrated at just one point which corresponds to \( \theta \) being very small and \( \varepsilon |u|^\gamma \leq 2 \).

The Boltzmann operator with this new cross section is written as

\[
Q_{g_\varepsilon}(f,f)(v,t) = \int_{R^3} \int_{S^2} (f(v', t)f(v'_*, t) - f(v, t)f(v_*, t)) g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) d\sigma dv_*
\]

\[
= \int_{R^3} \int_{S^2} f'f'_* g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) d\sigma dv_* - \frac{8}{\varepsilon} f(v, t), \quad (2.4)
\]

and the corresponding Boltzmann equation is

\[
\begin{aligned}
\partial_t f + \frac{8}{\varepsilon^2} f = \int_{R^3} \int_{S^2} f'f'_* g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) d\sigma dv_* \\
f(v, 0) = f_0(v).
\end{aligned}
\]

(2.5)

In view of Proposition 1.1, it would be reasonable to expect for there to be a grazing collisions limit, and in fact we will show that this is true.

### 2.2 Connection with \( Q_L \)

**Theorem 2.1.** Let \( f_\varepsilon \in L^p(\mathbb{R}^3), p > \max\{\frac{6}{8 + \gamma}, 1\} \) be a weak solution of (1.4) with the cross section \( g_\varepsilon \) defined as in (2.1). Then for all time, \( |Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon) - \)
\[ Q_L(f_\varepsilon, f_\varepsilon) \rightarrow 0 \text{ in the distributional sense as } \varepsilon \rightarrow 0. \text{ That is, for any } \varphi \in C_0^\infty(\mathbb{R}^3), \]

\[
\lim_{\varepsilon \rightarrow 0} \left| \int_{\mathbb{R}^3} (Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v, t) - Q_L(f_\varepsilon, f_\varepsilon)(v, t)) \varphi(v) dv \right| = 0. \tag{2.6}
\]

**Proof.** We need to compute \( G_1[|u|^{-\gamma}g_\varepsilon], G_2[|u|^{-\gamma}g_\varepsilon] \) and \( G_3[|u|^{-\gamma}g_\varepsilon] \). Let \( m_\varepsilon(|u|) := \min\{\varepsilon |u|/2, 2\} \) and \( \mu_\varepsilon(|u|) := 1 - m_\varepsilon(|u|) \) denote the mass of \( \delta_0(1 - \mu - m_\varepsilon(|u|)). \) Using integration by parts with the Dirac mass \( \delta_0 \) and recalling (1.24), (1.25), we have:

\[
G_1[g_\varepsilon](v, v_\ast) = -2\pi u \int_0^1 g_\varepsilon(|u|^\gamma, \dot{u} \cdot \sigma) \sin^2(\theta/2) d\mu
\]

\[= -2\pi u \beta_2[g_\varepsilon](u), \tag{2.7}\]

\[
G_2[g_\varepsilon](v, v_\ast) = \frac{1}{2} \pi(2u_i u_j - |u|^2 \Pi(u)_{ij}) \int_0^{\pi/2} g_\varepsilon(\cos \theta) \sin^4(\theta/2) \sin \theta d\theta
\]

\[+ \frac{1}{2} \pi |u|^2 \Pi(u)_{ij} \int_0^{\pi/2} g_\varepsilon(|u|^\gamma, \cos \theta) \sin^2(\theta/2) \sin \theta d\theta
\]

\[= \frac{\pi}{2}(2u_i u_j - |u|^2 \Pi(u)_{ij}) \beta_4[g_\varepsilon](u) + \frac{\pi}{2} |u|^2 \Pi(u)_{ij} \beta_2[g_\varepsilon](u) \tag{2.8}\]

and

\[
G_3[g_\varepsilon](v, v_\ast) \leq \frac{1}{3} \|D^3 \varphi\|_{L^3} \int_{-\pi}^{\pi} \int_0^1 \int_0^1 g_\varepsilon(|u|^\gamma, \mu) |v' - v|^3 d\mu d\phi
\]

\[= \frac{2\pi}{3} |u|^3 \|D^3 \varphi\|_{L^3} \int_0^1 g_\varepsilon(|u|^\gamma, \mu) \sin^3(\theta/2) d\mu
\]

\[= \frac{8}{3\varepsilon} |u|^3 \|D^3 \varphi\|_{L^3} \beta_3[g_\varepsilon](u). \tag{2.9}\]

All together,

\[
\int Q_{g_\varepsilon}(f, f) \varphi dv = \frac{1}{2} \int f f_s(\nabla \varphi - \nabla_\ast \varphi_\ast) \cdot G_1[g_\varepsilon](v, v_\ast) dv_s dv
\]

\[+ \frac{1}{2} \int f f_s(\partial_{v_i} \varphi + \partial_{v_i} \varphi_\ast) G_2[g_\varepsilon](v, v_\ast) dv_s dv
\]

\[+ \frac{1}{2} \int f f_s G_3[g_\varepsilon](v, v_\ast) dv_s dv \]
We already showed that this integral converges in Proposition 1.1. For \( \beta \), we already showed this integral is finite in Proposition 1.1. Finally, with \( I_1 - I_4 \) defined accordingly.

Now we show that \( I_1 \leq \pi \). Note also that, by the construction of \( \beta \).

\[
I_1 = -\pi \int ff_{s\beta_2}[s\beta_2](u)(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv
\]

\[
= -\frac{2}{3} \int_{|u| \leq \varepsilon} |u|^2(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv
\]

\[
\rightarrow -\frac{2}{3} \int_{|u| \leq \varepsilon} |u|^2(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv = G^1_{L^1}(v, v_\ast). \quad (2.11)
\]

We already showed that this integral converges in Proposition 1.1. For \( I_2 \), let \( 1 < P < \min \{ \frac{3}{|\gamma|P}, 2 \} \). Then

\[
I_2 = \frac{\pi}{4} \int \beta_2[g_2](u)ff_{s\beta_2}(u)\frac{2}{3} |u|^2(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv
\]

\[
\leq \frac{1}{4} \varepsilon \int |u|^2(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv
\]

\[
\rightarrow 0 \quad (2.12)
\]

as \( \varepsilon \rightarrow 0 \). Note also that, by the construction of \( P \) and by Lemma 1.2, \( I_2 < \infty \).

\[
I_3 = \frac{\pi}{4} \int \beta_2[g_2](u)ff_{s\beta_2}(u)\frac{2}{3} |u|^2(\nabla \varphi - \nabla \varphi_\ast) \cdot udv \, dv
\]

\[
= \frac{1}{2} \int |u|^{\gamma+2} ff_{s\beta_2}(D^2 \varphi + D^2 \varphi_\ast) : \Pi(u) \, dv \, dv
\]

\[
\rightarrow \frac{1}{2} \int |u|^{\gamma+2} ff_{s\beta_2}(D^2 \varphi + D^2 \varphi_\ast) : \Pi(u) \, dv \, dv = G^2_{L^2}(v, v_\ast), \quad (2.13)
\]

and we already showed this integral is finite in Proposition 1.1. Finally,
\[ I_4 = \frac{1}{2} \int f f_s G_3 g \varepsilon(v, v_*) dv_s dv \]
\[ \leq \frac{1}{6} |u|^3 \| D^3 \|_{L^\infty} \beta_3 g \varepsilon(u) \leq \frac{4}{3 \varepsilon} 2^{3+2} \varepsilon \| D^3 \varphi \|_{L^\infty} \int_{|u| \gamma \leq 1} f f_s |u|^3 dv_s dv \]
\[ = \sqrt{\varepsilon} 2^{1+2} \| D^3 \varphi \|_{L^\infty} \int f f_s |u|^3 dv_s dv \rightarrow 0 \quad (2.14) \]
as \varepsilon \rightarrow 0, and the integral is convergent by Lemma 1.2. The result follows.

### 3 Estimates on \( Q_{g \varepsilon} \)

In this section we prove that \( Q_{g \varepsilon} \) maps from \( L^1 \cap L^p \) into itself. This will establish its continuity on these spaces and help us show existence, uniqueness and uniform bounds on solutions to (2.5).

#### 3.1 Auxiliary lemmas

In this section we prove a few lemmas that are necessary to show continuity of \( Q_{g \varepsilon} \). We roughly follow the arguments of Lemmas 3, 4 and Theorem 5 of [2].

First, we introduce some notation. Recall from before that we can split \( Q_{g \varepsilon} \) into its gain and loss parts:

\[ Q_{g \varepsilon}(f, f)(v, t) = \int_{\mathbb{S}^2} f(v', t) f(v'_*, t) g \varepsilon(|u| \gamma, \hat{u} \cdot \sigma) d\sigma d\sigma_\gamma - \frac{8}{\varepsilon} f(v, t) \]
\[ =: Q_{g \varepsilon}^+(f, f)(v, t) = Q_{g \varepsilon}^-(f, f)(v, t). \quad (3.1) \]

Next, for \( \eta, \psi, \in C_B(\mathbb{R}^3) \), define

\[ P_\varepsilon(\eta, \psi)(u) := \int_{\mathbb{S}^2} \eta(u^-) \psi(u^+) g \varepsilon(|u| \gamma, \hat{u} \cdot \sigma) d\sigma \]

\[ u^- := \frac{1}{2}(u - |u| \sigma) \]

\[ u^+ := \frac{1}{2}(u + |u| \sigma). \]

Finally, define the following radially symmetric functions: for any \( f \in L^p \), \( 1 \leq p \leq \infty \), let

\[ f^*_p(u) := \left( \int_{R \in SO(3)} |f(Ru)|^p dR \right)^{\frac{1}{p}} \]
\[
\left( \frac{1}{|S^2|} \int_{\sigma \in S^2} |f(|u|\sigma)|^p d\sigma \right)^\frac{1}{p} \text{ if } 1 \leq p < \infty,
\]

\[f_p(u) := \text{ess sup}_{R \in SO(3)} |f(Ru)| = \text{ess sup}_{\sigma \in S^2} |f(|u|\sigma)|.\]

Such functions satisfy the following properties:

(i) \(f_p^* \) is radial, i.e. \(f_p^*(u) = f_p^*(x)\) whenever \(|u| = |x|\).

(ii) If \(g\) is radial, then \((fg)_p^*(u) = f_p^*(u)\).

(iii) If \(d\nu\) is a rotationally invariant measure on \(\mathbb{R}^3\), then
\[
\int_{\mathbb{R}^3} |f(u)|^p d\nu(u) = \int_{\mathbb{R}^3} |f^*_p(u)|^p d\nu(u),
\]

and in particular \(\|f\|_{L^p(\mathbb{R}^3)} = \|f^*_p\|_{L^p(\mathbb{R}^3)}\).

We are now ready to introduce the auxiliary lemmas:

**Lemma 3.1.** Let \(\eta, \psi, \phi \in C_0(\mathbb{R}^3)\) and \(1/p + 1/q + 1/r = 1\), with \(1 \leq p, q, r \leq \infty\). Then,
\[
\left| \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(u)\phi(u) du \right| \leq \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta^*_p, \psi^*_p)(u)\phi^*_p(u) du. \tag{3.2}
\]

**Proof.** This lemma and its proof are almost identical to Lemma 3 of [2]. For some \(R \in SO(3)\) we begin with the changes of variable \(u \rightarrow Ru\) and then \(\sigma \rightarrow R\sigma\) in the left hand side of (3.2):

\[
\left| \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(u)\phi(u) du \right| = \left| \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(Ru)\phi(Ru) du \right|
\]

\[
= \left| \int_{\mathbb{R}^3} \int_{S^2} \eta \left( \frac{1}{2} (Ru - |u|R\sigma) \right) \psi \left( \frac{1}{2} (Ru + |u|R\sigma) \right) \cdot \cdot \cdot g_\varepsilon(|u|^\gamma, \tilde{R}u \cdot R\sigma) d\sigma \phi(Ru) du \right|
\]

\[
\leq \int_{\mathbb{R}^3} \int_{S^2} |\eta(Ru^-)| |\psi(Ru^+)| g_\varepsilon(|u|^\gamma, \tilde{u} \cdot \sigma) d\sigma |\phi(Ru)| du. \tag{3.3}
\]

We can characterize the rotation \(R = R_{\tilde{\theta}, \tilde{\omega}}\), where \(\tilde{\theta} \in [0, \pi], \tilde{\omega} \in S^1\) are defined such that \(R_{\tilde{\theta}, \tilde{\omega}} \tilde{u} = \tilde{u} \cos \tilde{\theta} + \tilde{\omega} \sin \tilde{\theta}\). Since \(R\) is arbitrary and the left
hand side of (3.2) does not depend on $R$ we can take the average over all possible rotations in (3.3) to get

$$\left| \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(u) \psi(u) \mathrm{d}u \right|$$

$$\leq \int_{\mathbb{R}^3} \int_{S^2} \left( \int_{SO(3)} |\eta(Ru)| |\psi(Ru)^+| |\phi(Ru)| \mathrm{d}R \right) g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) \mathrm{d}\sigma \mathrm{d}u$$

$$\leq \int_{\mathbb{R}^3} \int_{S^2} \left( \int_{SO(3)} |\eta(Ru^-)| \mathrm{d}R \right)^{\frac{1}{p}} \left( \int_{SO(3)} |\psi(Ru^+)| \mathrm{d}R \right)^{\frac{1}{q}} \cdot \left( \int_{SO(3)} |\varphi(Ru^-)| \mathrm{d}R \right)^{\frac{1}{r}} g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) \mathrm{d}\sigma \mathrm{d}u$$

$$= \int_{\mathbb{R}^3} \int_{S^2} (\eta_\varepsilon^+(u) \psi_\varepsilon^+(u) \phi_\varepsilon^+(u)) g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) \mathrm{d}\sigma \mathrm{d}u$$

$$= \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta_\varepsilon^*, \psi_\varepsilon^*)(u) \phi_\varepsilon^*(u) \mathrm{d}u, \quad (3.4)$$

where in the end we used Holder’s inequality with the exponents $p, q, r$. This concludes the proof. \(\square\)

Now we can take advantage of the fact that $\eta_\varepsilon^*, \psi_\varepsilon^*$ are radial to simplify the expression $\mathcal{P}_\varepsilon(\eta_\varepsilon^*, \psi_\varepsilon^*)$. For any function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, let $\tilde{f} : \mathbb{R}^+ \rightarrow \mathbb{R}$ be such that $f(x) = \tilde{f}(|x|)$ for all $x \in \mathbb{R}^3$. Then,

$$\mathcal{P}_\varepsilon(\eta_\varepsilon^*, \psi_\varepsilon^*)(u) = \int_{S^2} \bar{\eta}_\varepsilon^+(|u|^\gamma, \mu) \bar{\psi}_\varepsilon^+(|u|^\gamma, \mu) g_\varepsilon(|u|^\gamma, \mu) \mathrm{d}\mu,$$

$$= 2\pi \int_0^1 \bar{\eta}_\varepsilon^+(a_1(|u|^\gamma, \mu)) \bar{\psi}_\varepsilon^+(a_2(|u|^\gamma, \mu)) g_\varepsilon(|u|^\gamma, \mu) \mathrm{d}\mu, \quad (3.5)$$

where

$$a_1(|u|^\gamma, \mu) := |u| \sqrt{\frac{1}{2}(1 + \mu)} = |u^+|,$$

$$a_2(|u|^\gamma, \mu) := |u| \sqrt{\frac{1}{2}(1 - \mu)} = |u^-|.$$

This motivates the introduction of a new, simpler bilinear operator defined over bounded, continuous functions of one variable: for $\eta, \psi \in C_B(\mathbb{R}^+)$ define

$$\mathcal{B}_\varepsilon(\eta, \psi)(x) := \int_0^1 \eta(a_1(x, \mu)) \psi(a_2(x, \mu)) g_\varepsilon(x, \mu) \mathrm{d}\mu$$

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We prove the following lemma, which is the equivalent of Lemma 4 from [2]:

**Lemma 3.2.** Let $1 \leq p \leq \infty$. Then for any $\eta \in L^p(\mathbb{R}^+, x^2 dx)$ and $\psi \in L^\infty(\mathbb{R}^+)$,

$$
\|B_\varepsilon(\eta, \psi)\|_{L^p(\mathbb{R}^+, x^2 dx)} \leq \frac{8}{\pi \varepsilon} \|\psi\|_{L^\infty(\mathbb{R}^+)} \|\eta\|_{L^p(\mathbb{R}^+, x^2 dx)}.
$$

(3.6)

**Proof.** Let $C_\infty = 8$ and for $p < \infty$ let $C_p := 2^{3+\frac{4}{p}}$. Then $C_p \leq 16$ for all $p \in [1, \infty]$. If we let $\mu_\varepsilon(x) := 1 - m_\varepsilon(x) \in [-1, 1)$ be the dirac mass of $g_\varepsilon$, then by definition,

$$
B_\varepsilon(\eta, \psi)(x) = \frac{4}{\pi \varepsilon} \eta(1 - \varepsilon a_1(x)) \psi(a_2(x)) \mathds{1}_{0 \leq \mu_\varepsilon(x) \leq 1}
$$

$$
= \frac{4}{\pi \varepsilon} \eta(a_3(x)) \psi(a_2(x)) \mathds{1}_{0 \leq \mu_\varepsilon(x) \leq 1}
$$

$$
= \frac{4}{\pi \varepsilon} \eta(a_1(x)) \psi(a_2(x)) \mathds{1}_{x \geq \varepsilon^2}. 
$$

(3.7)

The case $p = \infty$ is trivial, so we assume that $p \neq \infty$. Then

$$
\|B_\varepsilon(\eta, \psi)\|^p_{L^p(\mathbb{R}^+, x^2 dx)} = \left(\frac{4}{\pi \varepsilon}\right)^{p} \int_{\varepsilon^2}^\infty \eta(1 - \varepsilon a_1(x)) \psi(a_2(x)) \mathds{1}_{0 \leq \mu_\varepsilon(x) \leq 1} x^2 dx
$$

$$
\leq \left(\frac{4}{\pi \varepsilon}\right)^{p} \|\psi\|^p_{L^\infty(\mathbb{R}^+)} J_{p, \varepsilon}(\eta),
$$

(3.8)

where

$$
J_{p, \varepsilon}(\eta) := \int_{\varepsilon^2}^\infty \eta(a_1(x)) \mathds{1}_{x \geq \varepsilon^2} x^2 dx.
$$

We estimate the integral $J_{p, \varepsilon}(\eta)$ by performing the change of variable $a_1(x, \varepsilon) = a_1(x, \varepsilon x^{-3}) \rightarrow x :$

$$
a_1(x, \varepsilon) = x \sqrt{\frac{1}{2} \left(1 + \mu_\varepsilon(x)\right)} = x \sqrt{\frac{1 - \varepsilon}{2x^3} + \frac{\varepsilon}{2x^3}} = \sqrt{x^2 - \frac{\varepsilon}{2x}},
$$

$$
a'_1(x, \varepsilon(x)) = \frac{1}{2 a_1(x, \varepsilon(x))} \left(2x + \frac{\varepsilon}{2x^2}\right) = \frac{4x^3 + \varepsilon}{4x^2 a_1(x, \varepsilon(x))} \cdot a_1(x, \varepsilon(x)),
$$

$$
x^2 dx = \frac{x^2}{a'_1(x, \varepsilon(x)))} da_1(x, \varepsilon(x)) = \frac{4x^4}{4x^3 + \varepsilon} \frac{a_1(x, \varepsilon(x))^2}{x \sqrt{1 - \frac{\varepsilon}{2x^3}}} da_1(x, \varepsilon(x)) \leq \frac{a_1^2}{\sqrt{1 - \frac{\varepsilon}{2x^3}}} da_1 \leq \sqrt{2} a_1^2 da_1.
$$

(3.9)
so

\[ J_{p,\varepsilon}(\eta) = \int_{\mathbb{R}^3} \eta(a_1(x, \mu_\varepsilon(x))) p \cdot x^2 \, dx \]

\[ \leq \sqrt{2} \int_0^\infty \eta^p(a_1) a_1^2 \, da_1 = \sqrt{2} \| \eta^p \|_{L^p(\mathbb{R}^+, x^2 \, dx)}, \quad (3.10) \]

as was to be shown. \hfill \Box

**Lemma 3.3.** Let \( 1 \leq p \leq \infty \). The bilinear operator \( \mathcal{P}_\varepsilon \) extends to a bounded operator from \( L^p(\mathbb{R}^3) \times L^\infty(\mathbb{R}^3) \) to \( L^p(\mathbb{R}^3) \), and

\[ \| \mathcal{P}_\varepsilon(\eta, \psi) \|_{L^p(\mathbb{R}^3)} \leq \frac{16}{\varepsilon} \| \psi \|_{L^\infty(\mathbb{R}^3)} \| \eta \|_{L^p(\mathbb{R}^3)}. \quad (3.11) \]

**Proof.** Let \( \eta \in L^p(\mathbb{R}^3), \psi \in L^\infty(\mathbb{R}^3) \) and \( \phi \in L^{p'}(\mathbb{R}^3) \). By Lemma 3.1 combined with a density argument,

\[ \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(u) \phi(u) \, du \leq \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta^*_p, \psi^*_\infty)(u) \phi^*_p(u) \, du. \]

Since the functions \( \eta^*_p, \psi^*_\infty, \phi^*_p \) are radial in \( u \), let \( \bar{\eta}_p, \bar{\psi}_\infty, \bar{\phi}^*_p : \mathbb{R}^+ \mapsto \mathbb{R} \) such that for any \( u \in \mathbb{R}^3 \), \( \eta^*_p(u) = \bar{\eta}_p(|u|), \psi^*_\infty(u) = \bar{\psi}_\infty(|u|), \phi^*_p(u) = \bar{\phi}^*_p(|u|) \). Then for \( p \neq \infty \) \( \bar{\eta}_p \in L^p(\mathbb{R}^+, x^2 \, dx), \bar{\psi}_\infty \in L^2(\mathbb{R}^3, x^2 \, dx) \) and \( \bar{\phi}^*_p \in L^r(\mathbb{R}^3, x^2 \, dx) \) and

\[ \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta^*_p, \psi^*_\infty)(u) \phi^*_p(u) \, du = 2\pi \int_0^\infty \mathcal{B}_\varepsilon(\bar{\eta}_p, \bar{\psi}_\infty)(x) \bar{\phi}^*_p(x) x^2 \, dx \]

\[ \leq 2\pi \| \mathcal{B}_\varepsilon(\bar{\eta}_p, \bar{\psi}_\infty) \|_{L^p(\mathbb{R}^+, x^2 \, dx)} \| \phi^*_p \|_{L^{p'}(\mathbb{R}^3)} \]

\[ \leq 2\pi \frac{8}{\pi \varepsilon} \| \bar{\psi}_\infty \|_{L^2(\mathbb{R}^3)} \| \bar{\eta}_p \|_{L^p(\mathbb{R}^+, x^2 \, dx)} \| \bar{\psi} \|_{L^{p'}(\mathbb{R}^3)} \]

\[ = \frac{16}{\varepsilon} \| \bar{\psi} \|_{L^2(\mathbb{R}^3)} \| \eta \|_{L^p(\mathbb{R}^3)} \| \phi \|_{L^{p'}(\mathbb{R}^3)} \]

For \( p = \infty \) the estimate above is almost identical (replace \( \| \bar{\eta}_p \|_{L^p(\mathbb{R}^+, x^2 \, dx)} \) with \( \| \bar{\eta}_\infty \|_{L^\infty(\mathbb{R}^+)} \)). This shows us that \( \mathcal{P}_\varepsilon(\eta, \psi) \) is a bounded, real-valued linear operator acting on \( L^{p'} \), and therefore belongs to \( L^p \) with norm bounded by

\[ \| \mathcal{P}_\varepsilon(\eta, \psi) \|_{L^p(\mathbb{R}^3)} \leq \frac{16}{\varepsilon} \| \psi \|_{L^\infty(\mathbb{R}^3)} \| \eta \|_{L^p(\mathbb{R}^3)}, \quad (3.12) \]

as was to be shown. \hfill \Box
3.2 Continuity of $Q_{g_e}$

Our first step here is to prove boundedness of $Q_{g_e}^+$:

**Theorem 3.4.** Let $1 \leq p, q, r \leq \infty$, $1/p + 1/q = 1 + 1/r$. Then the bilinear operator $Q_{g_e}^+$ extends to a bounded operator from $L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$ to $L^r(\mathbb{R}^3)$, and

$$\|Q_{g_e}^+(f, h)\|_{L^r(\mathbb{R}^3)} \leq \frac{16}{\varepsilon} \|f\|_{L^p(\mathbb{R}^3)} \|h\|_{L^q(\mathbb{R}^3)}$$

(3.13)

with $C_p$ defined as before.

**Proof.** First suppose that $(p, q, r) \neq (1, 1, 1), (1, \infty, \infty), (\infty, 1, \infty)$. Let $\psi \in L^r(\mathbb{R}^3)$ be a test function and define

$$K_\psi := \int_{\mathbb{R}^3} Q_{g_e}^+(f, h)(v)\psi(v)dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)h(v-u)\mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)dudv$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( f(v)^\frac{p}{q}h(v-u)^\frac{q}{p} \right) \left( f(v)^\frac{q}{p} \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)^\frac{p}{q} \right) \cdot \left( g(v-u)^\frac{p}{q} \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)^\frac{q}{p} \right) dudv$$

$$\leq K_1^1 K_2^2 K_3^3 (3.14)$$

by Holder’s inequality with the exponents $p', q', r$, where

$$K_1^1 := \left( \left( \left\| \int f(v)^p h(v-u)^q dudv \right\|_p \right)^\frac{1}{p} = \|f\|_{L^p} \|h\|_{L^q}$$

$$K_2^2 := \left( \left( \left\| \int f(v)^p \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)^{q'} dudv \right\|_p \right)^{\frac{1}{q'}} = \|f\|_{L^p} \|\mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)\|_{L^{q'}}$$

$$K_3^3 := \left( \left( \left\| \int h(v-u)^q \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)^{q'} dudv \right\|_p \right)^{\frac{1}{q'}} = \|h\|_{L^q} \|\mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)\|_{L^{q'}}$$

Then by Lemma (3.3),

$$K_\psi = \langle Q_{g_e}^+(f, h), \psi \rangle_{L^r(\mathbb{R}^3)} \leq \|f\|_{L^p(\mathbb{R}^3)} \|h\|_{L^q(\mathbb{R}^3)} \|\mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)\|_{L^r(\mathbb{R}^3)}$$

$$\leq \frac{16}{\varepsilon} \|f\|_{L^p(\mathbb{R}^3)} \|h\|_{L^q(\mathbb{R}^3)} \|\psi\|_{L^r(\mathbb{R}^3)}$$

(3.15)

$Q_{g_e}^+(f, h)$ is therefore a bounded linear operator defined on $L^r(\mathbb{R}^3)$, that is, it lies in the dual space $L^r(\mathbb{R}^3)$ and in particular $\psi = (Q_{g_e}^+(f, h))^{r-1} \in L^r(\mathbb{R}^3)$. Substituting this choice of $\psi$ into (3.15) we obtain
Lemma 3.5. For any $p \in [1, \infty)$, $K > 1$ and $0 \leq f_\varepsilon \in L_0^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3)$ a weak solution to the Boltzmann equation, the following holds: if $p < \infty$,

$$
\|Q_{g\varepsilon}^+(f, f_\varepsilon)\|_{L^p(\mathbb{R}^3)} \leq K \frac{16}{\varepsilon} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \left( \log K \right) + \frac{16}{\varepsilon} \|f_0\|_{L \log L(\mathbb{R}^3)} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \text{ for } p < \infty, \tag{3.17}
$$

and if $p = \infty$,

$$
\|Q_{g\varepsilon}^+(f, f_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{16K}{\varepsilon} + \frac{16}{\varepsilon \log K} \|f_0\|_{L \log L(\mathbb{R}^3)} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \tag{3.18}
$$

Proof. For $K > 1$ we can write $Q_{g\varepsilon}^+(f, f_\varepsilon)$ as

$$
Q_{g\varepsilon}^+(f, f_\varepsilon) = A + B := Q_{g\varepsilon}^+(f, f_\varepsilon \mathbf{1}_{f_\varepsilon < K}) + Q_{g\varepsilon}^+(f, f_\varepsilon \mathbf{1}_{f_\varepsilon > K}) \leq K \frac{1}{2^p} Q_{g\varepsilon}^+(f, f_\varepsilon^{2p}) + \frac{1}{\log K} Q_{g\varepsilon}^+(f, f_\varepsilon \log f_\varepsilon). \tag{3.19}
$$

First, let $p \neq \infty$.

$$
\|A\|_{L^p(\mathbb{R}^3)} \leq K \frac{1}{2^p} \frac{16}{\varepsilon} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \|f_\varepsilon^{2p}\|_{L^p(\mathbb{R}^3)} \leq K \frac{1}{2^p} \frac{16}{\varepsilon} \|f_\varepsilon\|_{L^1(\mathbb{R}^3)} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \leq K \frac{1}{2^p} \frac{16}{\varepsilon} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \tag{3.20}
$$

where in the end we used the Riesz-Thorin Interpolation Theorem to get

$$
\|f_\varepsilon\|_{L^\frac{2p}{p+1}} \leq \|f_\varepsilon\|_{L^2}^\frac{2}{p+1} \|f_\varepsilon\|_{L^p} = \|f_\varepsilon\|_{L^p}^{\frac{2}{p}}.
$$

For $B$, we simply use Theorem 3.4 with the coefficients $(p, 1, p)$ :}

$$
\|B\|_{L^p(\mathbb{R}^3)} \leq \frac{16}{\varepsilon \log K} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \|f_\varepsilon\|_{L^1(\mathbb{R}^3)}
$$
Now, let $p = \infty$. The proof is almost the same, but this time we bound $A$ by

$$A = Q^+_g(f_\varepsilon, f_\varepsilon 1_{f_\varepsilon \leq K}) \leq K Q^+_g(f_\varepsilon, 1),$$

so that

$$\|A\|_{L^\infty} \leq K \frac{16}{\varepsilon} \|Q^+_g(f_\varepsilon, f_\varepsilon 1_{f_\varepsilon \leq K})\|_{L^\infty} \leq K \frac{16}{\varepsilon} \|f_\varepsilon\|_{L^1} \|1\|_{L^\infty} = \frac{16K}{\varepsilon}.$$ 

For $B$, we follow the same steps as before:

$$\|B\|_{L^\infty} \leq \frac{1}{\log K} \|Q^+_g(f_\varepsilon, f_\varepsilon \log f_\varepsilon)\|_{L^\infty} \leq \frac{16}{\varepsilon \log K} \|f_\varepsilon\|_{L^1} \|f_\varepsilon \log f_\varepsilon\|_{L^1} \leq \frac{16}{\varepsilon \log K} \|f_\varepsilon\|_{L^\infty} \|f_0\|_{L^\log L},$$

as was to be shown.

\[\square\]

4 Existence and uniqueness of $f_\varepsilon$

The proof of existence and uniqueness is inspired by an existence proof from [4], in which the following theorem from [7] is applied:

**Theorem 4.1.** Let $E$ be a Banach space, $F$ a bounded, convex and closed subset of $E$, and $Q : F \rightarrow E$ an operator such that the following holds:

(i) Holder continuity: for all $f, h \in F$,

$$\|Q[f] - Q[h]\|_E \leq C \|f - h\|_E^\beta$$

for some $\beta \in (0, 1)$ \hspace{1cm} (4.1)

(ii) the subtangent condition: for all $f \in F$,

$$\liminf_{\delta \rightarrow 0^+} \frac{1}{\delta} \text{dist}_E(f + \delta Q[f], F) = 0$$

(4.2)

(iii) the one-sided Lipschitz condition: for all $f, h \in F$,

$$[Q[f] - Q[h], f - h] \leq C \|f - h\|_E$$

for all $f, h \in F$ \hspace{1cm} (4.3)

where $[\phi, \psi] := \lim_{\delta \rightarrow 0^+} \delta^{-1}(\|\psi + \delta \phi\|_E - \|\psi\|_E)$. 

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Then, the equation

\[
\begin{align*}
\partial_t f &= Q[f] \text{ on } [0, \infty) \times E \\
f(0) &= f_0 \geq 0 \in F \text{ on } \{0\} \times E
\end{align*}
\]

has a unique solution, \( f \), which lies in \( C^1((0, \infty), E) \cap C([0, \infty), F) \).

The proof of this theorem can be found in [7] or [4]. A direct application will allow us to prove the existence and uniqueness of \( f_\varepsilon \) by choosing an appropriate space \( E \) and set \( F \).

**Theorem 4.2.** For \( 1 \leq p \leq \infty \), let \( E := L^1_2 \cap L^p(\mathbb{R}^3) \) be a Banach space with the norm \( \|f\|_E := \|f\|_{L^1} + \|f\|_{L^p} \), and let \( 0 \leq f_0 \in F \cap L^1_2 \cap L \log L \) with \( \|f_0\|_{L^1} = 1 \), where

\[
F := \{ f \in E : f \geq 0, \|f\|_{L^1} = \|f_0\|_{L^1} = 1, \|f\|_{L^p(\mathbb{R}^3)} \leq C \}
\]

for some \( C > 0 \). Then, there exists a unique solution, \( f_\varepsilon \), to the Boltzmann equation (2.5) which lies in \( C^1((0, \infty), E) \cap C([0, \infty), F) \) that preserves mass, momentum, energy, and whose entropy is bounded by the initial entropy.

**Proof.** Let \( \varepsilon > 0 \). Fix \( p \), and for any \( f \in F \), define \( Q[f] := Q_{g_\varepsilon}(f, f) \). One can easily check that \( F \) is bounded, convex and closed in \( E \), so it remains to show that (4.1), (4.2) and (4.3) hold.

Using the bilinearity of \( Q_{g_\varepsilon} \) and Theorem 3.4, for any \( f, h \in F \),

\[
\begin{align*}
\|Q_{g_\varepsilon}(f, f) - Q_{g_\varepsilon}(f, h)\|_E &\leq \|Q_{g_\varepsilon}(f, f - h)\|_E + \|Q_{g_\varepsilon}(f - h, h)\|_E \\
&= \|Q_{g_\varepsilon}(f, f - h)\|_{L^1} + \|Q_{g_\varepsilon}(f, f - h)\|_{L^p} \\
&\quad + \|Q_{g_\varepsilon}(f - h, h)\|_{L^1} + \|Q_{g_\varepsilon}(f - h, h)\|_{L^p} \\
&\leq \frac{16}{\varepsilon} \|f\|_{L^1} \|f - h\|_{L^1} + \frac{16}{\varepsilon} \|f\|_{L^p} \|f - h\|_{L^1} + \frac{16}{\varepsilon} \|f - h\|_{L^1} \|h\|_{L^1} \\
&\quad + \frac{16}{\varepsilon} \|f - h\|_{L^p} \|h\|_{L^1} = \frac{16}{\varepsilon} \|f - h\|_{L^1} (2 + \|f\|_{L^p}) + \frac{16}{\varepsilon} \|f - h\|_{L^p} \\
&\leq \frac{16}{\varepsilon} (2 + C) \|f - h\|_E. \quad (4.4)
\end{align*}
\]

We have shown that \( Q_{g_\varepsilon} \) is continuous; in particular, it is Holder continuous for any \( \beta \in (0, 1) \), therefore (4.1) holds.

Next, we prove subtangency. Fix \( f \in F \). For any \( \beta > 0 \), it suffices to find \( \delta_0 = \delta_0(f, \beta) > 0 \) and \( \omega \in F \) such that for any \( \delta \in (0, \delta_0) \),

\[
\frac{1}{\delta} \|f + \delta Q_{g_\varepsilon}(f, f) - \omega\|_E < \beta. \quad (4.5)
\]
For $R, \delta > 0$ we define $f_R$ and $\omega = \omega(f, \delta, R)$ in the same way as it is in Proposition 5.1 in [4]: $f_R := f \mathbb{1}_{B_R}$, $\omega := f + \delta Q_{g_e}(f_R, f)$. We will find suitable values for $R_0$ and $\delta_0$ and use them to define an $\omega_0 = \omega(f, \delta_0, R_0)$ for which (4.5) will hold.

First, we show that $\omega \in F$. Indeed, for any $f \in F$, $\delta, R > 0$, $\omega = (1 - 8\epsilon^{-1}\delta)f + \delta Q_{g_e}(f_R, f) \geq (1 - 8\epsilon^{-1}\delta)f \geq 0$ whenever $\delta \leq \epsilon/8$. Then $\|\omega\|_L = \int \omega = \int f + \delta \int Q_{g_e}(f_R, f) = \int f = \int f_0 = 1$ because $\int Q_{g_e}(h, h) = 0$ for all $h \in L$. Lastly, thanks to Theorem 3.4,

$$\|\omega\|_L \leq \left(1 - \frac{8\epsilon}{\delta}\right)\|f\|_L + \frac{16}{\epsilon}\|f_R\|_L \leq 2C$$

whenever $\delta < \epsilon/8$. This shows that $\omega \in F$ for small enough $\delta$. Now,

$$\|f + \delta Q_{g_e}(f, f) - \omega\|_E = \delta\|(Q_{g_e}(f, f) - Q_{g_e}(f_R, f))\|_E \leq \frac{16}{\epsilon}(2 + C)\|f - f_R\|_E$$

by (4.4). Let $R = R_0$ be large enough so that $\|f - f_{R_0}\|_E \leq \delta \beta \frac{\epsilon}{\delta} (2 + C)^{-1}$. Then if $0 < \delta < \delta_0 := \min\{\epsilon/8, \beta\}$,

$$\frac{1}{\delta}\|f + \delta Q_{g_e}(f, f) - \omega_0\|_E \leq \delta < \beta,$$

so we have (4.5).

Finally, we prove the one sided Lipschitz condition. For $f, h \in F$, let $\phi := Q_{g_e}(f, f) - Q_{g_e}(h, h)$ and $\psi := f - h$. For $\delta > 0$ and $\omega_f, \omega_h \in F$,

$$\|\psi + \delta \phi\|_E - \|\psi\|_E = \|f + \delta Q_{g_e}(f, f) - h - \delta Q_{g_e}(h, h)\|_E - \|f - h\|_E \leq \|f + \delta Q_{g_e}(f, f) - \omega_f\|_E + \|h + \delta Q_{g_e}(h, h) - \omega_h\|_E + \|\omega_f - \omega_h\|_E - \|f - h\|_E.$$

Now, fix $\beta > 0$. By the subtangency condition, there exists $\delta_0 > 0$ and $\omega_f^0, \omega_h^0 \in F$ such that for any $0 < \delta < \delta_0$,

$$\|f + \delta Q_{g_e}(f, f) - \omega_f^0\|_E \leq \frac{\delta \beta}{4},$$

$$\|h + \delta Q_{g_e}(h, h) - \omega_h^0\|_E \leq \frac{\delta \beta}{4}.$$

By the construction of $\omega_f^0, \omega_h^0$, for a large enough $R_0$ and $R > R_0$,
\[ \| \omega_0^f - \omega_0^h \|_E - \| f - h \|_E \]
\[ \leq \| f - h \|_E + \delta \| Q_{g\epsilon}(f_R, f_R) - Q_{g\epsilon}(h_R, h_R) \|_E - \| f - h \|_E \]
\[ \leq \delta \frac{16}{\epsilon} (2 + C) \| f_R - h_R \|_E. \]

If we choose \( R_0 \) large enough so that for \( R > 0 \),
\[ \| f_R - h_R \|_E \leq \frac{\beta \epsilon}{2(2 + C)}, \]
then the Lipschitz condition follows.

\( \square \)

**Remark 4.3.** It is not hard to check that the \( H \)- theorem still holds for \( Q_{g\epsilon} \).
This means that any nonnegative \( f \) that solves the Boltzmann equation (2.5) preserves, mass, momentum and kinetic energy, and has decreasing entropy.
Therefore, these properties did not need to be included in our choice of \( F \).

### 5 A uniform bound on \( f_\epsilon \)

**Theorem 5.1.** Let \( f_\epsilon = f_\epsilon(v, t) \geq 0 \in L^1(\mathbb{R}^3) \) be a weak solution to the Boltzmann equation with nonnegative initial data \( 0 \leq f_\epsilon(v, 0) := f_0 \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3), 1 \leq p \leq \infty. \) Then \( f_\epsilon \) remains in \( L^p(\mathbb{R}^3) \), uniformly in \( \epsilon \) and time. More specifically,
\[ \| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)} \leq \max \left\{ 16e^8\| f_0 \|_{L^\log L}, \| f_0 \|_{L^p} \right\} \text{ for all } t > 0. \tag{5.1} \]

**Proof.** Without loss of generality, \( \| f_\epsilon \|_{L^1} = 1. \) Fix \( \epsilon > 0 \) and let \( K := e^{4\| f_0 \|_{L \log L}} > 1 \) (provided \( f_0 \neq 0 \)). We begin with the case \( p \neq \infty. \) By (3.17),
\[ \epsilon \| Q_{g\epsilon}(f_\epsilon, f_\epsilon)(\cdot, t) \|_{L^p(\mathbb{R}^3)} \]
\[ \leq 16K^{\frac{1}{3p'}} \| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)}^{\frac{1}{3}} + 8\| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)} \left( \frac{2}{\log K} \| f_0 \|_{L \log L(\mathbb{R}^3)} - 1 \right) \]
\[ = 16K^{\frac{1}{3p'}} \| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)}^{\frac{7}{3}} - 4\| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)}. \tag{5.2} \]

Then
\[ \epsilon^p \| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)} \]
\[ \leq \frac{d}{dt} \| f_\epsilon(\cdot, t) \|_{L^p(\mathbb{R}^3)} \]

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\begin{align*}
&= \varepsilon \frac{d}{dt} \| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)}^p = \varepsilon \int \frac{d}{dt} f_\varepsilon (v, t) f_\varepsilon^{p-1} (v, t) dv \\
&= \varepsilon \int_{\mathbb{R}^3} Q_{g_\varepsilon}^+ (f_\varepsilon, f_\varepsilon) (v, t) f_\varepsilon^{p-1} (v, t) dv - 8 \int f_\varepsilon^p (v, t) dv \\
&\leq \varepsilon \| Q_{g_\varepsilon}^+ (f_\varepsilon, f_\varepsilon) (\cdot, t) \|_{L^p(\mathbb{R}^3)} \| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)}^{p-1} - 8 \| f_\varepsilon \|_{L^p}^p \\
&\leq 16 K \frac{1}{2p} \| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)}^{p-\frac{1}{2}} - 4 \| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)} \| f_\varepsilon \|_{L^p(\mathbb{R}^3)}^p. \quad (5.3)
\end{align*}

Multiplying both sides of (5.3) by \( \frac{1}{2} \| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)}^{\frac{1}{2}} \),

\begin{equation}
\varepsilon p \frac{d}{dt} \| f_\varepsilon (\cdot, t) \|_{L^p}^{\frac{1}{2}} \leq 8K - 2 \| f_\varepsilon (\cdot, t) \|_{L^p}^{\frac{1}{2}}. \quad (5.4)
\end{equation}

Then if \( u(t) := \| f_\varepsilon (\cdot, t) \|_{L^p}^{\frac{1}{2}} \), by (5.4),

\begin{align*}
\begin{cases}
  u'(t) \leq \frac{8K}{\varepsilon p} - \frac{2}{\varepsilon p} u(t) \\
  u(0) = \| f_0 \|_{L^p}^{\frac{1}{2}}.
\end{cases} \quad (5.5)
\end{align*}

A maximum principle then shows us that

\begin{align*}
\| f_\varepsilon (\cdot, t) \|_{L^p(\mathbb{R}^3)} = u^2 (t) &\leq \left( 4K + (u(0) - 4K) e^{-\frac{2K}{\varepsilon p}} \right)^2 \\
&\leq \max \left\{ 4K, \| f_0 \|_{L^p(\mathbb{R}^3)}^{\frac{1}{2}} \right\} \leq \max \left\{ 16 \| f_0 \|_{L^p(\mathbb{R}^3)}, \| f_0 \|_{L^p(\mathbb{R}^3)} \right\}.
\end{align*}

Now, suppose that \( p = \infty \). By (3.18),

\[ \varepsilon \| Q_{g_\varepsilon}^+ (f_\varepsilon, f_\varepsilon)(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} \leq 16K + \frac{16 \| f_0 \|_{L^\log L(\mathbb{R}^3)}}{\log K} \| f_\varepsilon (\cdot, t) \|_{L^\infty(\mathbb{R}^3)}. \]

Then

\begin{align*}
\varepsilon \frac{d}{dt} f_\varepsilon (v, t) &\leq \varepsilon \| Q_{g_\varepsilon}^+ (f_\varepsilon, f_\varepsilon)(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} - 8f_\varepsilon (v, t) \\
&\leq 16K + \frac{16 \| f_0 \|_{L^\log L(\mathbb{R}^3)}}{\log K} \| f_\varepsilon (\cdot, t) \|_{L^\infty} - 8f_\varepsilon (v, t).
\end{align*}

In particular, by definition of supremum, this means that for all \( v \in \mathbb{R}^3 \),

\[ \varepsilon \frac{d}{dt} f_\varepsilon (v, t) \leq 16K \]

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The maximum principle then tells us that
\[ f_\varepsilon(v,t) \leq (f_0 - 4K)e^{-\frac{4t}{\epsilon}} + 4K, \]
so that, finally,
\[
\|f_\varepsilon\|_{L^\infty} \leq (\|f_0\|_{L^\infty} - 4K)e^{-\frac{4t}{\epsilon}} + 4K \leq \max \left\{ 4e^{4\|f_0\|_{L^{\log L}}}, \|f_0\|_{L^\infty} \right\} \\
\leq \max \left\{ 16e^{8\|f_0\|_{L^{\log L}}}, \|f_0\|_{L^\infty} \right\}. \tag{5.7}
\]

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