Exact Noncommutative KP and KdV Multi-solitons

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Abstract

We derive the Kadomtsev-Petviashvili (KP) equation defined over a general associative algebra and construct its \( N \)-soliton solution. For the example of the Moyal algebra, we find multi-soliton solutions for arbitrary space-space noncommutativity. The noncommutativity of coordinates is shown to obstruct the general construction of a \( \tau \) function for these solitons. We investigate the two-soliton solution in detail and show that asymptotic observers of soliton scattering are unable to detect a finite spatial noncommutativity. An explicit example shows that a pair of solitons in a noncommutative background can be interpreted as several pairs of image solitons. Finally, a dimensional reduction gives the general \( N \)-soliton solution for the previously discussed noncommutative KdV equation.
1 Introduction

Noncommutativity of coordinates [1, 2] appears to be an important ingredient of string theory. For example, in Witten’s formulation of open string field theory [3], space-time noncommutativity leads to an elegant formulation of the string field equations of motion. More recently, the calculation [4] and analysis [5, 6] of solitons in field theories simplified by strong noncommutativity has given evidence to support a conjecture of Sen [7] relating nonperturbative objects in open string field theory to (unstable) D-branes. This success has fueled further developments on the subject which are reviewed in [8].

Given the fundamental and calculational importance of noncommutative geometry, gaining a deeper understanding of its physical effects is essential. Here we present a non-trivial, yet completely solvable model of soliton physics in a noncommutative geometry where the parameter controlling the noncommutativity is left arbitrary. The solutions we find are not only of interest in answering questions in string theory but they also present a laboratory for detailed investigations into the impact the noncommutativity of coordinates has on soliton physics.

As our model we derive the analogue of the 2 + 1-dimensional Kadomtsev-Petviashvili (KP) equation for soliton profiles which take values in a general associative algebra. The resulting noncommutative KP (ncKP) equation is an integrodifferential equation. In spite of the complexity of the ncKP equation, we are able to explicitly construct an N-soliton solution via a modification of the trace method [9]. As a particular example we specify the Moyal algebra of the two spatial coordinates to find the ncKP equation and its associated N-soliton solution for noncommutative geometry.

We then proceed to examine the one- and two-soliton solutions in detail. The single soliton solution is unchanged in the deformation to noncommutative space due to the linear structure of the KP soliton. In the general two-soliton solution, the noncommutative nature of the underlying geometry leads to interesting effects. We show that the solution is periodic in a dimensionless parameter which controls the noncommutativity. The asymptotic behaviour of soliton scattering is shown to be independent of finite spatial noncommutativity, making it impossible to detect a noncommutative geometry through the asymptotics. Finally, as an explicit example we show how finite noncommutativity changes the physics of a pair of solitons to resemble the interaction of four different pairs of solitons.

Even though we can construct the N-soliton solution in an explicit manner, space-space noncommutativity obstructs the direct construction of a \( \tau \) (generating) function for these solutions. While it is possible that there is an analogue of a \( \tau \) function to generate soliton
solutions and conserved currents for the noncommutative KP equation, its construction is more involved than in the commutative case. Consequently, our results suggest that the analysis of integrable systems in noncommutative geometry is fundamentally more involved than in the commutative case.

While we concentrate on the ncKP equation defined in 2 + 1 dimensions, it is a simple matter to eliminate a spatial direction and recover the noncommutative version of the Korteweg-de Vries (ncKdV) equation [10]. In this way, all of our calculations and observations for the ncKP equation lead to analogous results for the ncKdV equation, including its general \( N \)-soliton solution. This is an efficient way to optimize our calculational efforts since the ncKP and ncKdV equations with their respective solutions are of similar complexity.

During the completion of this project, we discovered that the KP equation defined over a general finite-dimensional associative (matrix) algebra has been derived previously. These and other results on noncommutative integrable systems may be found in the very recent book [11].

### 2 The noncommutative KP equation for general associative algebras

The (commutative) KP equation is a non-linear equation in two spatial coordinates which admits stable solitary wave (soliton) solutions. Theoretically it is interesting since it yields the simplest extension of the classic KdV equation which is defined in one space dimension. One way to construct the KP equation is through the following process. Consider the differential equation

\[
L\psi = \left( \frac{\partial^2}{\partial x^2} + u(x, y, t) \right) \psi = \lambda \psi \tag{2.1}
\]

It is well-known that for reasonable choices of “potential” \( u(x, y, t) \) and boundary conditions, there is a discrete infinity of eigenvalues \( \lambda \). One can also view the time evolution of the wavefunction \( \psi \)

\[
\partial_t \psi = A\psi \tag{2.2}
\]

where we will choose the operator \( A \) to be linear and third order in derivatives of the coordinate \( x \)

\[
A = \frac{\partial^3}{\partial x^3} + a_2(x, y, t) \frac{\partial^2}{\partial x^2} + a_1(x, y, t) \frac{\partial}{\partial x} + a_0(x, y, t) \tag{2.3}
\]

The Lax method of generating the KP equation in particular, and more generally other integrable non-linear equations, requires the mutual consistency of the two operators \( L \) and
A. Considering the commutator of the Lax pair, $L$ and $A$ and requiring the eigenvalues to be constants of motion, one finds Lax’s equation

$$[L, A] = \partial_t L - \partial_y A$$  \hspace{1cm} (2.4)$$

In the present case, this equation allows one to solve for the coefficients $a_i$ in (2.3) and implies a consistency requirement for the potential $u$. This constraint is known as the KP equation

$$0 = (u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u^2)_x)_x + \frac{3}{4}u_{yy}$$  \hspace{1cm} (2.5)$$

Subsequently we will refer to $u$ as the profile of the solitary wave (soliton) which solves (2.5).

To obtain a coherent generalization of the KP equation for the case where $u$ takes values in some general associative algebra, we propose to deform the commutator in Lax’s equation (2.4) to the $\star$-commutator. This leads to a deformed version of Lax’s equation

$$[L, A]_\star = L \star A - A \star L = \partial_t L - \partial_y A$$  \hspace{1cm} (2.6)$$

We will leave the explicit definition of the $\star$-product arbitrary at this point and only require it to commute with derivatives.

We follow as in the commutative case and substitute the definition of $L$ (2.1) and the ansatz (2.3) for $A$ into the deformation of Lax’s equation (2.6). Again we arrive at a consistency requirement for the profile $u$ which in this case gives the noncommutative version of the KP equation (ncKP)

$$0 = (u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u \star u)_x + \frac{3}{4}[u, \partial_x^{-1} u_y]_\star)_x + \frac{3}{4}u_{yy}$$  \hspace{1cm} (2.7)$$

While it can be readily checked that this equation reduces to the KP equation (2.5) in the commutative limit, in general it is an integrodifferential equation for the profile $u$. Note that due to the presence of the $\star$-commutator term in (2.7), the noncommutative KP equation is not the commutative version with $\star$-products naively replacing commutative multiplication. This ‘non-covariance’ in moving to the noncommutative equation can only arise when one includes terms nonlinear in derivatives. Additionally, we remark that this equation is identical to the generalized KP one obtains when the profile $u$ is an element of a generic matrix algebra [12, 11].

We can obtain a differential equation equivalent to (2.7) by moving to a potential formulation of the problem. We do this by defining an auxiliary field $\phi$ where

$$u = 2\phi_x$$  \hspace{1cm} (2.8)$$
Substituting into (2.7) transforms the ncKP equation into a convenient form which we will use for subsequent calculations

\[ 0 = (4\phi_t + \phi_{xxx} + 6\phi_x \star \phi_x)_x + 6[\phi_x, \phi_y]_* + 3\phi_{yy} \]  

(2.9)

Once more, the \( \star \)-commutator term in this equation produces a contribution which is not present in the commutative case.

3 Generating the \( N \)-soliton solution

There are many ways to solve the commutative KP equation but for our purposes the trace method \[9\] provides the most convenient starting point for generalizing to associative algebras. The trace method assumes a formal power series for the \( N \)-soliton solution of the commutative KP equation in the potential formulation

\[ \phi = \sum_{n=1}^{\infty} \epsilon^n \phi^{(n)} \]  

(3.1)

In our case, substituting into (2.9) and solving order by order in the formal expansion parameter \( \epsilon \), one can verify the first few terms

\[ \phi^{(1)} = \sum_{l=1}^{N} \phi_l \]  

(3.2)

\[ \phi^{(2)} = -\sum_{l_1, l_2=1}^{N} \frac{\phi_{l_1} \star \phi_{l_2}}{(p_{l_1} + q_{l_2})} \]  

(3.3)

where

\[ \phi_l = c_l e^{(p_l + q_l)x + (p_l^2 - q_l^2)y - (p_l^3 + q_l^3)t} \]  

(3.4)

Here we require \( c_l(p_l + q_l) > 0 \), but otherwise \( c_l \) is an arbitrary constant.

For the higher order terms, we have the recursive equation at \( O(\epsilon^n) \) which gives an equation for \( \phi^{(n)} \) in terms of lower order coefficients

\[ [4\partial_t + \partial_{xxx} + 3\partial_x^{-1}\partial_{yy}] \phi^{(n)} = -6 \sum_{r=1}^{n-1} \left( \partial_x^{-1}[\phi_x^{(r)}, \phi_y^{(n-r)}]_* + \phi_x^{(r)} \star \phi_x^{(n-r)} \right) \]  

(3.5)

A formal solution to this recursion relation, and the ncKP equation, is given by the coefficients

\[ \phi^{(n)} = (-1)^{n-1} \sum_{l_1,...,l_n=1}^{N} \frac{\phi_{l_1} \star \phi_{l_2} \star \cdots \star \phi_{l_n}}{(p_{l_1} + q_{l_2})(p_{l_2} + q_{l_3}) \cdots (p_{l_{n-1}} + q_{l_n})} \]  

(3.6)
This claim can be proved through substitution and by using the following algebraic identity

\[
(p_1 + q_1 + \cdots + p_n + q_n)^4 - 4(p_1 + q_1 + \cdots + p_n + q_n)(p_1^3 + q_1^3 + \cdots + p_n^3 + q_n^3) \\
+ 3(p_1^2 - q_1^2 + \cdots + p_n^2 - q_n^2)^2
\]

\[(3.7)\]

\[
= 6 \sum_{r=1}^{n-1} (p_r + q_{r+1})[(p_1 + q_1 + \cdots + p_r + q_r)(p_{r+1}^2 - q_{r+1}^2 + \cdots + p_n^2 - q_n^2) \\
- (p_{r+1} + q_r + \cdots + p_n + q_n)(p_1^2 - q_1^2 + \cdots + p_r^2 - q_r^2) \\
+ (p_1 + q_1 + \cdots + p_r + q_r)(p_{r+1} + q_r + \cdots + p_n + q_n)(p_1 + q_1 + \cdots + p_n + q_n)]
\]

The identity itself can be proved through a process of matching the coefficients on either side of the equality. This procedure is simplified by noticing that each side is a quartic polynomial in the variables \(p_k\) and \(q_k\). In the limit of a commuting algebra, we can use the symmetry of the summand in (3.6) to recover the known result for the commutative case [13, 9].

It is interesting to observe that even though the noncommutative version of the KP equation (2.9) involves certain ‘non-covariant’ terms (the \(\star\)-commutator), its solution (3.6) is just the commutative solution with the \(\star\)-product replacing the usual commutative product.

### 4 The KP equation in noncommutative geometry

We will now specify the associative product of the previous section to be the Moyal product for two noncommuting spatial coordinates. In particular, the \(\star\)-product of two functions is explicitly defined in terms of the non-commutativity parameter \(\theta\) by

\[
f(x, y) \star g(x, y) \equiv \exp \left[ \frac{\theta}{2} (\partial_x \partial_{y'} - \partial_y \partial_{x'}) \right] f(x, y)g(x', y') \bigg|_{x=x', y=y'}
\]

(4.1)

For many calculations the parameterization in terms of \(p_l\) and \(q_l\) as introduced in (3.4) will be most convenient. These parameters can be used to construct physically relevant quantities such as the wavevector \(\vec{k} = (k_x, k_y)\) and the frequency \(\omega\) of the \(l^{th}\) soliton through

\[
(k_x)_l = p_l + q_l \\
(k_y)_l = p_l^2 - q_l^2 \\
\omega_l = p_l^3 + q_l^3
\]

(4.2) - (4.4)

With these definitions we can now introduce the cross-product notation for soliton wavevectors, which will simplify calculations

\[
\vec{k}_n \times \vec{k}_m = (k_x)_n(k_y)_m - (k_x)_m(k_y)_n \\
= (p_n + q_n)(p_m^2 - q_m^2) - (p_m^2 - q_m^2)(p_n + q_n)
\]

(4.5)
Given these definitions, the $N$-soliton solution of the ncKP equation for the case of non-commuting spatial coordinates can be calculated from (3.1) and (3.6)

$$\phi_N = \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{l_1 \ldots l_n = 1}^{N} \frac{e^{\theta_2 \sum_{i<j} \vec{k}_{i} \times \vec{k}_{j} \phi_{l_1} \phi_{l_2} \cdots \phi_{l_n}}}{(p_{l_1} + q_{l_2})(p_{l_2} + q_{l_3}) \cdots (p_{l_{n-1}} + q_{l_n})}$$

(4.6)

The result of noncommutativity in the spatial coordinates is the appearance of an index-dependent factor

$$e^{\frac{\theta_2}{2} \sum_{i<j} \vec{k}_{i} \times \vec{k}_{j}}$$

(4.7)

which is familiar from the calculation of Feynman diagrams in noncommutative field theories [14, 15, 16] or open string amplitudes in a background magnetic field [17]. This noncommutative factor has important implications for the solution. Following as in the commutative case, we can better see the structure of the solution by introducing the matrix

$$B_{lm} = c_m e^{p_{lm} x} e^{\theta_m x + (p_m^2 - q_m^2) y - (p_m^3 + q_m^3) t}$$

(4.8)

where $c_m (p_m + q_m) > 0$ as before. With this definition we can rewrite the $N$-soliton solution (4.6) in a concise manner

$$\phi_N = \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{l_1 \ldots l_n = 1}^{N} e^{\frac{\theta_2}{2} \sum_{i<j} \vec{k}_{i} \times \vec{k}_{j} B_{l_1 l_2} B_{l_2 l_3} \cdots B_{l_{n-1} l_n} \frac{\partial}{\partial x} B_{l_n l_1}}$$

(4.9)

In the commutative case ($\theta = 0$), the sum over indices $l_i$ is a trace over the product of $B$ factors. Moreover, the derivative can be extracted from the sum and the series over $n$ summed

$$\phi_N = \sum_{n=1}^{\infty} (-1)^{n-1} \mathrm{Tr} \left( B^{n-1} \frac{\partial}{\partial x} B \right) = \frac{\partial}{\partial x} \mathrm{Tr} \ln (\mathbb{I} + B) = \frac{\partial}{\partial x} \ln \det (\mathbb{I} + B)$$

(4.10)

Consequently, we obtain a well-known result: the $N$-soliton profile for the KP equation can be recovered from logarithmic derivatives of a generating function

$$u = 2 \frac{\partial^2}{\partial x^2} \ln \det (\mathbb{I} + B)$$

(4.11)

Customarily, the generating function is labeled $\tau$

$$\tau = \det (\mathbb{I} + B)$$

(4.12)

and it plays a fundamental role in the theory of integrable systems (see [18] for example).

In the present noncommutative case, the appearance of noncommutative factors in (4.9) leads to substantial changes in the character of the matrix series. If one considers each term in (4.9) as a matrix chain with periodic boundary conditions, then noncommutativity can
be seen to introduce non-local, anisotropic interactions between the sites \( (l_i) \) on the chain. While long-range interactions complicate the evaluation of (4.9), it is the anisotropy of the interactions which has the most profound effect. For any non-zero \( \theta \), each factor of \( B \) in the series is distinguished, making it impossible to extract the derivative as in the commutative case. This makes the direct identification of a \( \tau \) function for the noncommutative KP equation impossible in the current framework. It would appear that there are significant changes to the theory of integrable systems as soon as spatial noncommutativity is turned on. This same type of obstruction was observed previously in the noncommutative KdV equation \[11\].

5 One Soliton

In the case of \( N = 1 \), the single soliton, there is only one wavevector, \( \vec{k} \) and hence the factor (4.7) associated with the spatial noncommutativity vanishes. Consequently, the noncommutative one-soliton solution coincides with the solution in commutative space. From equation (4.11) we find the well-known one-soliton profile

\[
    u_1(x, y, t) = \frac{1}{2} k^2 \text{sech}^2 \left( \frac{1}{2} (k_x x + k_y y - \omega t) + \delta \right)
\]

(5.1)

This is not surprising given the nature of the KP soliton. It is a one-dimensional object which moves in the \( x - y \) plane with constant wavevector \( \vec{k} \). Consequently, the profile (5.1) depends only on a single argument. By changing coordinates to those parallel and perpendicular to the soliton, it is easy to show that \( \ast \)-products involving \( u_1 \) reduce to commutative multiplication. Analogous behaviour is seen for noncommutative versions of the KdV \[11, 10\] and nonlinear Schrödinger equations \[19\].

This argument may be extended to cases with an arbitrary number of parallel solitons, where the cross-products of all wavevectors vanish

\[
    \vec{k}_n \times \vec{k}_m = 0 \quad \forall \; n, m
\]

(5.2)

With this condition satisfied, from (4.9) it is clear that the solution for parallel solitons will reduce to that of the commutative case. Note that even though the effect of spatial noncommutativity is trivial in this case, the dynamics of parallel solitons in the commutative case is non-trivial.
6 Two Solitons

The two-soliton solution of the ncKP equation is much more interesting since the noncommutativity generally affects the physics of the colliding solitons. This minimally complicated case has only one new factor due to noncommutativity which involves the wavevectors of the two solitons

\[ e^{\frac{\theta}{2} \vec{k}_1 \times \vec{k}_2} \]  

(6.1)

The labeling of the solitons is made unique by requiring the cross-product of the wavevectors to be non-negative

\[ \vec{k}_1 \times \vec{k}_2 = (k_x)_1(k_y)_2 - (k_y)_1(k_x)_2 \geq 0 \]  

(6.2)

For further computation it is useful to introduce a dimensionless noncommutativity parameter \( \Theta \)

\[ \Theta = \frac{\theta}{2} \vec{k}_1 \times \vec{k}_2 \]  

(6.3)

As noted above, the effective noncommutativity vanishes for non-zero \( \theta \) when the wavevectors of the solitons are parallel: \( \vec{k}_1 \times \vec{k}_2 = 0 \).

In the case of the two-soliton solution, there is a substantial simplification of the structure of the nonlocality in the index space of the solution as compared to the general noncommutative \( N \)-soliton solution (4.9). For two solitons the nonlocality can be re-interpreted purely as anisotropy in the index space. More precisely, since \( l_i = 1, 2 \) we have from the noncommutative factor (4.7)

\[ \Theta = \frac{\theta}{2} \vec{k}_1 \times \vec{k}_2 \]  

(6.4)

The advantage gained by this observation is that it allows for the two-soliton solution to be written as the trace over a product of matrices. Using (4.9) and (6.4) we find

\[ \phi_2 = \sum_{n=1}^{\infty} \left( -1 \right)^{n-1} \sum_{t_1 \ldots t_n=1}^{2} e^{-\Theta(n-1)l_1} B_{t_1 t_2} e^{-\Theta(n-3)l_2} B_{t_2 t_3} \ldots e^{\Theta(n-3)l_{n-1}} B_{t_{n-1} t_n} e^{\Theta(n-1)l_n} \frac{\partial}{\partial x} B_{t_n t_1} \]  

= \sum_{n=1}^{\infty} \left( -1 \right)^{n-1} \text{Tr} \left( e^{\frac{\Theta}{2} (n-1)\sigma_3} B e^{\frac{\Theta}{2} (n-3)\sigma_3} B \ldots e^{-\frac{\Theta}{2} (n-3)\sigma_3} B e^{-\frac{\Theta}{2} (n-1)\sigma_3} \frac{\partial B}{\partial x} \right) \]  

(6.5)

where the matrices \( B \) are defined in (4.8).

In this form we can already see some of the analytic structure of the two-soliton solution as a function of the effective noncommutativity. Consider the effect of an imaginary shift in \( \Theta \)

\[ \Theta \rightarrow \Theta + 2\pi i m \]  

(6.6)
where \( m \) is some integer. From the form (6.5) we see that each of the factors associated with noncommutativity is changed by a factor

\[
e^{\frac{2}{\Theta} (n+1-2k)\sigma_3} \to (-1)^m (n+1-2k) e^{\frac{2}{\Theta} (n+1-2k)\sigma_3}
\]

(6.7)

It is straightforward to show that the factor \((-1)^m (n+1-2k)\) has no effect on the solution (6.5) as it cancels out in each term of the series. Furthermore, it is not difficult to convince oneself that (6.5) is not periodic under shifts in \( \Theta \) of less than \( 2\pi \). Hence the two-soliton solution of the ncKP equation is a periodic function of \( \Theta \) with period \( 2\pi \). This is a simple prototype of a multi-dimensional periodicity which arises in the \( N \)-soliton solution (4.9).

In the limit of \( \Theta \) vanishing, the derivative can be extracted and the series (6.5) summed leading to the appearance of the \( \tau \) function, as seen previously. In addition, it is straightforward to calculate the first correction to the two-soliton solution for non-vanishing \( \Theta \). Introducing the phases of each of the two solitons

\[
\eta_1 = \frac{1}{2} \left( (k_x)_{1x} + (k_y)_{1y} - \omega_1 t + \ln \left( \frac{c_1}{p_1 + q_1} \right) \right)
\]

(6.8)

\[
\eta_2 = \frac{1}{2} \left( (k_x)_{2x} + (k_y)_{2y} - \omega_2 t + \ln \left( \frac{c_2}{p_2 + q_2} \right) \right)
\]

(6.9)

and defining the real quantity \( \Delta \) by

\[
e^{-2\Delta} = 1 - \frac{(p_1 + q_1)(p_2 + q_2)}{(p_1 + q_2)(p_2 + q_1)} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 + q_2)(p_2 + q_1)} \geq 0
\]

(6.10)

we can express the leading terms of a perturbative expansion in the dimensionless noncommutativity parameter

\[
u_2 = 2 \frac{\partial^2}{\partial x^2} \ln \left( 1 + e^{2\eta_1} + e^{2\eta_2} + e^{2(\eta_1 + \eta_2 - \Delta)} \right)
\]

\[
+ 2 \frac{\partial}{\partial x} \left( \Theta \frac{(p_1 + q_2 - p_2 - q_1)(1 - e^{-2\Delta}) e^{2(\eta_1 + \eta_2)}}{(1 + e^{2\eta_1} + e^{2\eta_2} + e^{2(\eta_1 + \eta_2 - \Delta)})^2} \right) + O(\Theta^2)
\]

(6.11)

This calculation tells us several things. To first order, the noncommutative two-soliton solution is a smooth deformation of the commutative solution and reduces to it in the limit of vanishing \( \Theta \). It is worth noting the structure of the first correction in \( \Theta \). It is antisymmetric under exchange of soliton label and it vanishes for \( \Delta = 0 \). The antisymmetry is a natural effect of the manner in which the spatial noncommutativity arises in the solution. The vanishing of the correction with \( \Delta \) is true to all orders in \( \Theta \) as we shall soon see.

In order to perform further calculations in the noncommutative case, it is convenient to conjugate the matrices in each term of the series (6.5) to obtain the form

\[
\phi_2 = \sum_{n=1}^{\infty} (-1)^{n-1} \text{Tr} \left( \prod_{k=1}^{n-1} C_{k,n} D \right)
\]

(6.12)
Here we have defined the matrices $C_{k,n}$ and $D$ by

$$C_{k,n} = \left( \begin{array}{c} e^{2\eta_1} \sqrt{1 - e^{-2\Delta} e^{\eta_1 + \eta_2 - \Theta k(n-k)}} \\ e^{2\eta_2} \sqrt{1 - e^{-2\Delta} e^{\eta_1 + \eta_2 - \Theta k(n-k)}} \end{array} \right)$$  \hspace{1cm} \text{(6.13)}$$

and

$$D = \left( \begin{array}{c} (p_1 + q_1) e^{2\eta_1} (p_1 + q_2) \sqrt{1 - e^{-2\Delta} e^{(\eta_1 + \eta_2)}} \\ (p_2 + q_1) \sqrt{1 - e^{-2\Delta} e^{(\eta_1 + \eta_2)}} (p_2 + q_2) e^{2\eta_2} \end{array} \right)$$  \hspace{1cm} \text{(6.14)}$$

Generally speaking, (6.12) is a deformation of the geometric series by spatial noncommutativity. Summing this matrix series exactly appears to be a difficult task. There are however several important observations to be made about noncommutative soliton physics from the form of the matrices $C_{k,n}$.

First of all, we see that the effect of the noncommutative factors lies solely in the off-diagonal components of $C_{k,n}$ and that these components vanish for $\Delta = 0$. The limit of vanishing $\Delta$ is known from the commutative case to lead to the reduction of the two-soliton problem to that of two, non-interacting single solitons. In this way we see another manifestation of the observation made for parallel solitons: spatial noncommutativity is irrelevant without interaction.

Physically, the condition $\Delta = 0$ can be realized in several ways. First, if the x-component of the wavevector for one, or both of the solitons is zero

$$(k_x)_1 = p_1 + q_1 = 0 \text{ \ and, or \ } (k_x)_2 = p_2 + q_2 = 0 \hspace{1cm} \text{(6.15)}$$

Since $k_x$ also gives the amplitude of each of the single solitons, we see that in this limit one, or both of the solitons disappear. In addition to this somewhat degenerate case, $\Delta = 0$ if

$$p_1 + q_2 \to \infty \text{ \ and, or \ } p_2 + q_1 \to \infty \hspace{1cm} \text{(6.16)}$$

In the first case, the first soliton is made infinitely narrow \[9\]. Likewise for the second case and the second soliton.

### 6.1 Soliton scattering in noncommutative space

In the commutative case, the interaction of two KP solitons takes place where the one-dimensional solitons cross in the $x$-$y$ plane. Far away from this interaction region, the solitons are isolated from each other and we expect that locally, solutions are essentially the one-soliton solution. In the commutative case, direct calculation \[9, 13\] shows this to be the case, up to
an important shift. The asymptotic two-soliton profile is

\[
\begin{align*}
  u_2(x, y, t) &= \begin{cases} 
    \left( \frac{(k_1)^2}{2} \right) \sech^2 \eta_1 + \left( \frac{(k_2)^2}{2} \right) \sech^2 (\eta_2 - \Delta) & \text{as } y \to \infty \\
    \left( \frac{(k_3)^2}{2} \right) \sech^2 (\eta_1 - \Delta) + \left( \frac{(k_4)^2}{2} \right) \sech^2 \eta_2 & \text{as } y \to -\infty
  \end{cases}
\end{align*}
\] (6.17)

The interpretation of this result reveals that the size of the region where the two solitons interact is proportional to \( \Delta \). This produces an offset of the same size between pre- and post-interacting branches of each soliton, as seen in the asymptotic solutions.

In the case of noncommutative geometry, we have already seen that the one-soliton solution coincides with the commutative solution so that the asymptotic solution (6.17) is possible. Here we show indeed that (6.17) gives the asymptotic behaviour in the noncommutative geometry.

The matrix \( C_{k,n} \) which appears in the solution (6.12) can be diagonalized since for imaginary \( \Theta \) it is Hermitian.

\[
C_{k,n} = \Omega_{k,n} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \Omega_{k,n}^{-1}
\] (6.18)

Here the eigenvalues, \( \lambda_{\pm} \), are independent of the indices \( k \) and \( n \)

\[
\lambda_{\pm} = \frac{1}{2} \left[ e^{2\eta_1} + e^{2\eta_2} \pm \sqrt{(e^{2\eta_1} + e^{2\eta_2})^2 - 4e^{2(\eta_1+\eta_2-\Delta)}} \right]
\] (6.19)

The unit-determinant matrix \( \Omega_{k,n} \)

\[
\Omega_{k,n} = \begin{pmatrix} \cos \psi & e^{\Theta(n-k)} \sin \psi \\ -e^{-\Theta(n-k)} \sin \psi & \cos \psi \end{pmatrix}
\] (6.20)

depends on the angle \( \psi \) defined by

\[
\cos 2\psi = \sqrt{\frac{(e^{2\eta_1} - e^{2\eta_2})^2}{(e^{2\eta_1} + e^{2\eta_2})^2 - 4e^{2(\eta_1+\eta_2-\Delta)}}}
\] (6.21)

In the commutative case, this diagonalization is independent of \( k \) and \( n \) and the solution depends only on the eigenvalues \( \lambda_{\pm} \). This is not true in general in the noncommutative case and we see that the noncommutative soliton has more degrees of freedom. In the limits of \( \eta_i \to \pm \infty \), which are correlated with the asymptotic limits of \( y \to \pm \infty \) [13], we see a significant simplification from (6.21) as \( \cos 2\psi \to 1 \). Consequently, to leading order, (6.18) is independent of \( \Theta \) and so the evaluation of the solution in (6.12) agrees with the commutative case and gives the asymptotic behaviour (6.17).

Since the asymptotic form of the two-soliton solution is unchanged for finite spatial noncommutativity, it is impossible for observers far from the interaction region to use soliton
scattering to detect spatial noncommutativity. Even though the interaction of the two solitons can be detected asymptotically due to the shift $\Delta$, the fact that this interaction took place in a noncommutative background can not. While the effects of noncommutative geometry are not visible at long distances, they are certainly not weak, as we shall see in an explicit solution in the next section.

6.2 Closed form solutions for rational imaginary $\Theta$

An extension of the logic we used to show that the two-soliton solution for the ncKP equation is periodic in imaginary $\Theta$ can also lead us to closed form exact solutions when $\Theta$ is a rational multiple of $i\pi$

$$\Theta = i\pi \frac{p}{q}$$

(6.22)

where $p$ and $q$ are relatively prime integers. This simplification can be seen by considering the general noncommutative factor from the solution (6.3). In this case

$$e^{i\pi \frac{p}{2q}(n+1-2k)} = \cos \left( \frac{\pi p}{2q}(n+1-2k) \right) \mathbb{1} + i \sin \left( \frac{\pi p}{2q}(n+1-2k) \right) \sigma_3$$

(6.23)

The presence of rational multiples of $\pi$ here guarantees that there is only a finite number of unique factors which arise in the evaluation of (6.3). Consequently, the infinite series in (6.3) can be split into distinct sums over $2q$ equivalence classes. Each of these sums can be evaluated in terms of elementary functions. The simplest example of this is when $p = q = 1$ and the identity (6.23) allows one to write (6.3) as the sum of contributions from odd $n$ and even $n$

$$\phi_2(\Theta = i\pi) = \frac{\partial}{\partial x} \left[ \sum_{n \text{ odd}} \frac{(-1)^n}{n} \text{Tr} B^n + \sum_{n \text{ even}} \frac{(-1)^{n-1}}{n} \text{Tr}(\sigma_3 B)^n \right]$$

(6.24)

The series which appear can be summed exactly in terms of logarithms. Using the representation of the components of $B$ in terms of $\eta_1$, $\eta_2$ and $\Delta$, we have the explicit form for the two-soliton profile for this case

$$u_2(\Theta = i\pi) = \frac{\partial^2}{\partial x^2} \left( \ln (1 + e^{2\eta_1} + e^{2\eta_2} + e^{2(\eta_1+\eta_2-\Delta)}) - \ln (1 - e^{2\eta_1} - e^{2\eta_2} + e^{2(\eta_1+\eta_2-\Delta)}) \right)$$

$$+ \ln (1 + e^{2\eta_1} - e^{2\eta_2} - e^{2(\eta_1+\eta_2-\Delta)}) + \ln (1 - e^{2\eta_1} + e^{2\eta_2} - e^{2(\eta_1+\eta_2-\Delta)})$$

(6.25)

While the solution (6.25) contains singularities in the x-y plane and hence is not physically acceptable, it does give us a glimpse of the effect of finite spatial noncommutativity in an exactly solvable model. With the form of the commutative solution (Eqn. 6.11 with $\Theta = 0$) in mind, we see that (6.25) describes not a single pair of solitons, but rather four pairs of
solitons; one pair for each logarithm. Also it is a simple matter to check that the asymptotic
($\eta_i \rightarrow \pm \infty$) behaviour of the solution agrees with that of the commutative case, as expected
from the general arguments above. As well we note that (6.25) is symmetric under the
exchange of soliton label; a feature which is only present for $\Theta = 0$ and $\pi$.

We see explicitly that the two-soliton solution for $\Theta = i\pi$ can be written in terms of a
generating $\tau$ function

$$
\tau^2 = (1 + e^{2n_1} + e^{2n_2} + e^{2(n_1+n_2-\Delta)})^2
$$

$$
+ 4(1 - e^{-2\Delta}) e^{2(n_1+n_2)} \frac{1 + e^{2n_1} + e^{2n_2} + e^{2(n_1+n_2-\Delta)}}{1 - e^{2n_1} - e^{2n_2} + e^{2(n_1+n_2-\Delta)}}
$$

In this form it is clear that with $\Delta = 0$ the solution reduces to the commutative case. This
example shows that it is possible to have a $\tau$-function for non-vanishing spatial noncommuta-
tivity but we expect this to be a unique feature for $\Theta = i\pi$ and directly tied to the symmetry
of the solution under the exchange of soliton labels.

7 Reduction to noncommutative KdV

As a byproduct of our solution of the ncKP equation, we can find the N-soliton solution for the
previously discussed [10, 20] noncommutative KdV equation (ncKdV). In the commutative
case one can recover the KdV equation from the KP equation through a simple dimensional
reduction. In the present case, assuming a soliton profile independent of the $y$-coordinate,
$u(x,t)$, the ncKP equation (2.7) can be seen to reduce to the ncKdV equation as found in [10]

$$
0 = u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u \ast u)_x
$$

At this point we have not yet defined the $\ast$-product and this equation holds for any associative
product. In order to obtain the KdV equation in noncommutative geometry we specify the
space-time $\ast$-product

$$
f(x,t) \ast g(x,t) \equiv \exp \left[ \frac{\theta}{2} (\partial_x \partial_{t'} - \partial_t \partial_{x'}) \right] f(x,t)g(x',t') \bigg|_{x=x',t=t'}
$$

We can easily produce an $N$-soliton solution to (7.1) in noncommutative space-time by re-
stricting the $y$-component of each soliton wavevector to vanish in the ncKP $N$-soliton solution

$$
(k_y)_l = p_{l}^2 - q_{l}^2 = 0
$$
Choosing \( p_l = q_l \), we can eliminate all \( y \)-dependence and obtain a solution which only depends on \( x \) and \( t \)

\[
u(x, y) = 2 \frac{\partial}{\partial x} \left( \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{l_1 \ldots l_n = 1}^N e^{\theta \sum_{i<j} \vec{k}_l \times \vec{k}_j} \frac{\chi_{l_1} \chi_{l_2} \cdots \chi_{l_n}}{(p_{l_1} + p_{l_2})(p_{l_2} + p_{l_3}) \cdots (p_{l_{n-1}} + p_{l_n})} \right) \tag{7.4}
\]

with

\[
\chi_l = c_l e^{2p_l x - 2p_l^2 t} \tag{7.5}
\]

and

\[
\vec{k}_l = (2p_l, -2p_l^2) \tag{7.6}
\]

An analogous solution appeared in [11] in the context of the matrix KdV equation.

From the explicit solution (7.4) of the ncKdV equation we can immediately recover an important result of [10]: the solution is a symmetric function of the noncommutativity parameter \( \theta \). Here we see that a change in sign of \( \theta \) can be absorbed in the exponential factor

\[
e^{\theta \sum_{i<j} \vec{k}_l \times \vec{k}_j} \tag{7.7}
\]

by reversing the order of the summation indices \( l_j \)

\[
l_1, l_2, \ldots, l_n \rightarrow l_n, l_{n-1}, \ldots, l_1 \tag{7.8}
\]

By inspection, the remainder of the summand in (7.4) is invariant under such a reordering. Hence the \( N \)-soliton solution of the ncKdV equation is an even function of \( \theta \). Note that this property is not shared by the ncKP equation discussed previously.

### 8 Future directions

While we have not been able to find a closed form for the matrix series (6.5) which gives the general two-soliton solution, we expect progress is possible. Given the symmetry of the series, it should be possible to at least obtain an equivalent one-dimensional series representation. It would be desirable to have a more tractable form of the solution in order to answer a number of interesting questions.

First of all, having a closed form would certainly answer the question of whether the formal solution we have found is well-defined for non-zero \( \Theta \). While we have seen perturbatively and for \( \Theta = i\pi \) that the two-soliton solution is reasonably well-behaved, it is not clear whether this is true in general. Furthermore, with a better understanding of the two-soliton solution for finite \( \Theta \), we would be in a position to investigate the limit of large noncommutativity.
Finding this limit would allow one to make comparisons to recent multi-soliton solutions \cite{21,22,23,24} found at infinite noncommutativity.

A closed form for the two-soliton solution (6.5) would also help to answer the question of whether a $\tau$ function can be found for the ncKP solitons. We are in the unusual position of having the $N$-soliton solution for a non-linear equation without even knowing if a $\tau$-function exists. Clearly, if a generating function exists it is not simply calculable using the methods we have used here. Presumably, knowledge of the $\tau$-function would allow for a complete understanding of the integrable structure of the ncKP (and ncKdV) equation as well as the calculation of conserved quantities.

An interesting application of our results would be to further investigate non-perturbative objects in open $N = 2$ string theory with a $B$-field \cite{25,26}. Using the relationship between this string theory and $2 + 2$-dimensional noncommutative self-dual Yang-Mills theory \cite{25}, one can dimensionally reduce the gauge theory to recover \cite{20} a matrix generalization of the noncommutative KdV equation we have solved here. As shown in \cite{26}, soliton solutions in the lower dimensional theory correspond in the string theory to D-branes with a background magnetic field restricted to their world-volume.

**Acknowledgments**

We thank A.W. Peet for her comments and hospitality at the University of Toronto where part of this work was completed. We have benefited from numerous discussions with V.N. Muthukumar and comments by P. Charland, F. Ferrari, O. Lechtenfeld, J. Liu and E. Verlinde. This work was supported in part by the US Department of Energy grant DE-FG02-91ER40671.

**References**

[1] A. Connes, Noncommutative geometry, San Diego, Academic Press (1994).

[2] A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori,” JHEP\textbf{9802}, 003 (1998) [hep-th/9711162].

[3] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. \textbf{B268}, 253 (1986).
[4] R. Gopakumar, S. Minwalla and A. Strominger, “Noncommutative solitons,” JHEP 0005, 020 (2000) [hep-th/0003160].

[5] K. Dasgupta, S. Mukhi and G. Rajesh, “Noncommutative tachyons,” JHEP 0006, 022 (2000) [hep-th/0005006].

[6] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, “D-branes and strings as noncommutative solitons,” JHEP 0007, 042 (2000) [hep-th/0005031].

[7] A. Sen, “Descent relations among bosonic D-branes,” Int. J. Mod. Phys. A14, 4061 (1999) [hep-th/9902105].

[8] J. A. Harvey, “Komaba lectures on noncommutative solitons and D-branes,” hep-th/0102076.

[9] K. Okhuma and M. Wadati, “The Kadomtsev-Petviashvili equation: the trace Method and the soliton resonances,” J. Phys. Soc. Jpn. 52, 749 (1983).

[10] A. Dimakis and F. Muller-Hoissen, “Noncommutative Korteweg-de-Vries equation,” hep-th/0007074.

[11] B.A. Kupershmidt, KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems, Mathematical Surveys and Monographs V. 78, American Mathematical Society, 2000.

[12] I.Y. Dorfman and A.S. Fokas, “Hamiltonian theory over noncommutative rings and integrability in multidimensions,” J. Math. Phys. 33, 2504 (1992).

[13] E. Infeld and G. Rowlands, Nonlinear waves, solitons and chaos, Cambridge, Cambridge University Press (1990).

[14] T. Filk, “Divergencies in a field theory on quantum space,” Phys. Lett. B376, 53 (1996).

[15] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, “Wilson loops in noncommutative Yang-Mills,” Nucl. Phys. B573, 573 (2000) [hep-th/9910004].

[16] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” hep-th/9912072.

[17] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909, 032 (1999) [hep-th/9908142].

[18] A.C. Newell, Solitons in Mathematics and Physics, Philadelphia, SIAM, 1985.
[19] A. Dimakis and F. Muller-Hoissen, “A noncommutative version of the nonlinear
Schroedinger equation,” [hep-th/0007015]

[20] M. Legare, “Noncommutative generalized NS and super matrix KdV systems from a
noncommutative version of (anti-)self-dual Yang-Mills equations,” [hep-th/0012077]

[21] R. Gopakumar, M. Headrick and M. Spradlin, “On noncommutative multi-solitons,”
[hep-th/0103256]

[22] L. Hadasz, U. Lindstrom, M. Rocek and R. von Unge, “Noncommutative multisolitons:
Moduli spaces, quantization, finite Theta effects and stability,” [hep-th/0104017]

[23] T. Krajewski and M. Schnabl, “Exact solitons on noncommutative tori,” [hep-th/0104090]

[24] T. Araki and K. Ito, “Scattering of noncommutative (n,1) solitons,” [hep-th/0105012]

[25] O. Lechtenfeld, A. D. Popov and B. Spendig, “Open N = 2 strings in a B-field background
and noncommutative self-dual Yang-Mills,” [hep-th/0012200]

[26] O. Lechtenfeld, A. D. Popov and B. Spendig, “Noncommutative solitons in open N = 2
string theory,” [hep-th/0103190].