BORDER RANK OF MONOMIALS VIA ASYMPTOTIC RANK

MATTHIAS CHRISTANDL, FULVIO GESMUNDO, AND ALESSANDRO ONETO

ABSTRACT. We determine the Waring border rank of monomials. The approach is based on recent results on tensor asymptotic rank which allow us to extend the Ranestad-Schreyer lower bound for the Waring rank of a homogeneous polynomial to the level of border rank.

1. Introduction

Let \( f \) be a homogeneous polynomial of degree \( d \) in \( n+1 \) variables over the complex numbers. The Waring problem asks:

What is the minimal number of linear forms \( \ell_i \) needed to write \( f = \sum_{i=1}^r \ell_i^d \)?

This number is called the Waring rank, or simply rank, of \( f \). From a geometric perspective, the Waring rank of \( f \) is the rank with respect to the \( d \)-th Veronese embedding \( \mathcal{V}_{n,d} \) of \( \mathbb{P}C^{n+1} \) in \( \mathbb{P}\text{Sym}^d C^{n+1} \), where \( C^{n+1} \cong \text{Sym}^1 C^{n+1} \) is the vector space of linear forms in \( n+1 \) variables \( x_0, \ldots, x_n \) and \( \text{Sym}^d C^{n+1} \) is the vector space of homogeneous polynomials of degree \( d \) in \( x_0, \ldots, x_n \); write \( R_d(f) \) for the Waring rank of \( f \). This notion generalizes the rank of a matrix, and indeed, if \( f \) is a quadratic form, its Waring rank coincides with the usual rank of the associated symmetric matrix.

The Waring problem has a long history, dating back at least to Sylvester [Syl51] who gave a complete solution in the case of binary forms, i.e., polynomials in two variables. A complete answer has been given for generic forms for every degree and number of variables in the celebrated Alexander-Hirschowitz Theorem [AH95]. As far as specific forms are concerned, only few cases are known: relevant for this work is the answer in the case of monomials, provided by Carlini, Catalisano and Geramita [CCG12]. Their result shows that if \( m = x_0^{a_0} \cdots x_n^{a_n} \) is a monomial with \( a_0 = \min \{ a_i \} \), then

\[
R_d(m) = \frac{1}{a_0 + 1} \prod_{i=0}^n(a_i + 1).
\]

Over other fields the problem remains open. For monomials in two variables, it is known that the real Waring rank always equals the degree [BCG11]. For monomials in more variables, it is known that the real Waring rank and the complex Waring rank coincide if and only if one of the exponents is equal to one [CKOV17]; to the extent of our knowledge, already the real Waring rank of the monomial \( x_0^2 x_1^2 x_2^2 \) is not known. Over other subfields of the complex numbers, we refer to [RT17] for some partial result. We refer to [BCC18] for an extensive survey on the state-of-the-art on the subject from an algebraic and geometric point of view.

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Waring problem received a great deal of attention from a broad community in the last few decades due to its relations with problems on additive decompositions of tensors appearing in several fields of applied mathematics; see e.g. [Lan12]. There is a natural identification between homogeneous polynomials of degree \(d\) and symmetric tensors of order \(d\): via this identification, Waring decompositions for polynomials provide symmetric tensor decompositions for symmetric tensors, which often find a role in applications.

An important property of Waring rank is its failure to be upper semicontinuous, in contrast to matrix rank. The notion of border Waring rank, introduced by Bini [Bin80], but essentially dating back to Terracini [Ter16], is the semicontinuous closure of the Waring rank; for \(f \in \text{Sym}^d \mathbb{C}^{n+1}\), the border Waring rank of \(f\), or simply border rank, denoted \(R_d(f)\), is the minimum \(r\) such that \(f\) can be approximated by forms of rank \(r\); more precisely

\[
R_d(f) = \min \left\{ r : f = \lim_{\varepsilon \to 0} f_{\varepsilon} \quad \text{with} \quad R_d(f_{\varepsilon}) = r \quad \text{for every} \quad \varepsilon \neq 0 \right\}.
\]

The notion of border Waring rank can be defined in terms of secant varieties of Veronese varieties. The \(r\)-th secant variety of \(V_{n,d}\), denoted \(\sigma_r(V_{n,d})\), is the closure (equivalently in the Zariski or the Euclidean topology) of the union of all linear spaces spanned by \(r\) points of \(V_{n,d}\); hence, we have

\[
\sigma_r(V_{n,d}) = \left\{ [f] \in \mathbb{P} \text{Sym}^d \mathbb{C}^{n+1} : R_d(f) \leq r \right\}.
\]

As briefly mentioned above, Waring rank is not semicontinuous and there are examples of forms for which the rank is strictly larger than the border rank. The very first example, already known to Sylvester [Syl52], is the monomial \(f = x_0^a x_1^n\), with \(R_d(f) = 3\) and \(R_d(f) = 2\). In the case of monomials, a conjecture circulated in the community for almost a decade. In this work, we prove this conjecture as follows.

**Theorem 1.1.** Let \(m = x_0^{a_0} \cdots x_n^{a_n}\), with \(a_n = \max_i \{a_i\}\), then

\[
R_d(m) = \frac{1}{a_n + 1} \prod_{i=0}^{n} (a_i + 1).
\]

The upper bound \(R_d(m) \leq \frac{1}{a_n + 1} \prod_{i=0}^{n} (a_i + 1)\) was proved by Landsberg and Teitler in [LT10, Theorem 11.2] and equality was shown under the assumption \(a_n > a_0 + \ldots + a_{n-1}\). In [Oed16], Oeding proved equality for certain families of monomials. In order to determine the lower bound, these results rely on so-called generalized flattening methods, introduced in [LO13], where one obtains lower bounds for the border rank by reducing the problem to determining the rank of certain matrices; see, e.g., [BBCG19, §7.2] for a more general description of the method. At the UMI-SIMAI-PTM Joint Meeting in Wroclaw (PL) in September 2018, Buczyński announced a proof for a number of other cases, in a restricted range, obtained using a method based on apolarity theory. The problem was still open in general.

Our approach is based on two fundamental building blocks:

- the Ranestad-Schreyer lower bound for Waring rank given in [RS11, Proposition 1] is multiplicative under tensor product, see Proposition 2.5.
• multiplicative lower bounds for rank are lower bounds for the tensor asymptotic rank introduced in [CGJ19], see Lemma 2.7 and in turn lower bound for border rank by [CGJ19, Proposition 6.2] and [CJZ18, Theorem 8].

We establish these results in Section 2.

More generally, Theorem 2.9 and Remark 2.10 will show that the Ranestad-Schreyer lower bound, and in fact even a stronger version for partially symmetric tensors provided in [Tei14], holds for border rank.

2. Ranestad-Schreyer multiplicativity and border rank of monomials

This section is devoted to developing the proof of Theorem 1.1. In particular, first we introduce the partially symmetric versions of rank and border rank, and we discuss apolarity theory, both in the homogeneous and the multihomogeneous setting. Then we recall the Ranestad-Schreyer lower bound from [RS11] and its generalization to the multigraded setting in [Tei14]. In conclusion, we use these results, together with the results of [CGJ19, §6], to obtain the proof of Theorem 1.1.

2.1. Partly symmetric tensors. Let $V_1, \ldots, V_k$ be complex vector spaces of dimension $n_1 + 1, \ldots, n_k + 1$, respectively; we write $\{x_{ij} : j = 0, \ldots, n_i\}$ for a basis of $V_i$. Consider the ring of polynomials in all the variables $x_{ij}$, that is

$$
\mathbb{C}[x_{ij} : i = 1, \ldots, k, j = 0, \ldots, n_i] \simeq \text{Sym}^*(V_1 \oplus \cdots \oplus V_k),
$$

identified with the symmetric algebra of $V_1 \oplus \cdots \oplus V_k$. Then,

$$
\text{Sym}^*(V_1 \oplus \cdots \oplus V_k) \simeq \text{Sym}^* V_1 \otimes \cdots \otimes \text{Sym}^* V_k,
$$

where $\text{Sym}^* V_i$ is the ring of polynomials in the variables $x_{i,0}, \ldots, x_{i,n_i}$.

In particular, $\mathbb{C}[x_{ij} : i = 1, \ldots, k, j = 0, \ldots, n_i]$ inherits the natural multigrading given by the tensor products of the symmetric algebras

$$
\text{Sym}^*(V_1 \oplus \cdots \oplus V_k) \simeq \bigoplus_{d_1, \ldots, d_k \geq 0} \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k;
$$

a partly symmetric tensor of multidegree $d = (d_1, \ldots, d_k)$ is an element of the multigraded component $\text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k$.

If $t \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k$ is a partly symmetric tensor, the partly symmetric rank of $t$ is defined as

$$
R_d(t) = \min \left\{ r : t = \sum_{i,j} \rho_i \otimes \cdots \otimes \rho_j, \text{ for some } \rho_i \in \text{Sym}^1 V_i \right\}.
$$

This is the rank with respect to the Segre-Veronese variety $V_{d_1,n_1} \times \cdots \times V_{d_k,n_k}$ obtained as the embedding of multidegree $(d_1, \ldots, d_k)$ of $V_1 \times \cdots \times V_k$ in the projective space $\mathbb{P}(\text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k)$. In the case $k = 1$, this coincides with the Waring rank. In the case $k = 2, d_1 = d_2 = 1$, the tensor $t$ is a bilinear form and the partly symmetric rank coincides with the rank of the associated matrix.

Similarly to the homogeneous setting, the partly symmetric border rank of a tensor $t \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k$ is the minimum $r$ such that $t$ can be approximated by
partially symmetric tensors of rank \( r \), namely
\[
\mathcal{R}_d(t) = \min \left\{ r : t = \lim_{\varepsilon \to 0} t_\varepsilon \text{ with } \mathcal{R}_d(t_\varepsilon) = r \text{ for } \varepsilon \neq 0 \right\}.
\]
The number \( \mathcal{R}_d(t) \) is the smallest \( r \) such that \( \{t\} \in \mathbb{P}(\text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k) \) belongs to the \( r \)-th secant variety \( \sigma_r(V_{d_1,n_1} \times \cdots \times V_{d_k,n_k}) \) of the Segre-Veronese variety.

It is straightforward to verify that partially symmetric rank and border rank are submultiplicative under tensor product in the following sense. If \( t, s \) are partially symmetric tensors of \( t \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k \) and \( s \in \text{Sym}^{e_1} V_{k+1} \otimes \cdots \otimes \text{Sym}^{e_\ell} V_{k+\ell} \), then
\[
\mathcal{R}_{d|e}(t \otimes s) \leq \mathcal{R}_d(t) \cdot \mathcal{R}_e(s) \quad \text{and} \quad \mathcal{R}_{d|e}(t \otimes s) \leq \mathcal{R}_d(t) \cdot \mathcal{R}_e(s),
\]
where \( d|e \) denotes the concatenation of \( d = (d_1, \ldots, d_k) \) and \( e = (e_1, \ldots, e_\ell) \).

Both inequalities can be strict, as observed in [ČJZ18, CGJ19, BBC19].

2.2. Apolarity theory. Apolarity is a classical approach to the Waring problem: it dates back to Sylvester [Syl52] and it has been used, directly or indirectly, to achieve most of the known results for Waring rank of specific forms.

We briefly present the subject and we refer the reader to [IK99, Ger96, CGO14, BCC18, Gal16, GRV16] for a complete explanation of this material in the homogeneous setting and to [Tei14] for the multigraded version.

Let \( V = \mathbb{C}^{n+1} \) be the vector space spanned by a set of variables \( x_0, \ldots, x_n \) and let \( \text{Sym}^* V = \mathbb{C}[x_0, \ldots, x_n] \) be the symmetric algebra on \( V \), identified with the standard graded ring of polynomials in \( x_0, \ldots, x_n \). The symmetric algebra \( \text{Sym}^* V^* \) has a natural action on \( \text{Sym}^* V \) via differentiation:
\[
\circ : \text{Sym}^* V^* \times \text{Sym}^* V \longrightarrow \text{Sym}^* V,
\]
\[
(D, f) \quad \mapsto \quad D(f),
\]
where \( D \in \text{Sym}^* V^* \) is regarded as a differential operator with constant coefficients, that is a polynomial in \( \partial_0, \ldots, \partial_n \), with \( \partial_i = \frac{\partial}{\partial x_i} \) and \( D(f) \) is the result of the differentiation of the polynomial \( f \) by \( D \).

For a homogeneous polynomial \( f \in \text{Sym}^* V \), the apolar ideal of \( f \) is defined by
\[
f^\perp = \{ D \in \text{Sym}^* V^* : D \circ f = 0 \};
\]
it is clear that \( f^\perp \) is a homogeneous ideal in \( \text{Sym}^* V^* \).

Given \( f \in \text{Sym}^d V \), the \( e \)-th catalecticant of \( f \) is the linear map
\[
\text{cat}_e(f) : \text{Sym}^e V^* \longrightarrow \text{Sym}^{d-e} V,
\]
\[
D \quad \mapsto \quad D \circ f.
\]
It is immediate that the homogeneous component of degree \( e \) in \( f^\perp \) coincides with the kernel of the \( e \)-th catalecticant map, namely \( (f^\perp)_e = \ker(\text{cat}_e(f)) \) and, in particular, we have \( (f^\perp)_e = \text{Sym}^e V^* \) for \( e > \deg(f) \).

Example 2.1. Let \( m = x_0^{a_0} \cdots x_n^{a_n} \) be a monomial; then \( m^\perp = (\partial_0^{a_0+1}, \ldots, \partial_n^{a_n+1}) \), the ideal generated by \( \partial_0^{a_0+1}, \ldots, \partial_n^{a_n+1} \).
The apolarity action and the notion of apolar ideal extend to the multigraded setting (and in fact even more in general, see \cite{Gal16}) and to tensor products. Indeed, the polynomial ring $\text{Sym}^*(V^* \oplus \cdots \oplus V^*_k)$ acts via apolarity on the polynomial ring $\text{Sym}^*(V^1 \oplus \cdots \oplus V^*_k)$; if $t \in \text{Sym}^{d_1} V^1 \oplus \cdots \oplus \text{Sym}^{d_k} V^*_k$ is a partially symmetric tensor, then $t^\perp$ is a multihomogeneous ideal in $\text{Sym}^*(V^1 \oplus \cdots \oplus V^*_k)$.

Now, for every $e = (e_1, \ldots, e_k)$, define a multigraded catalecticant map:

$$\text{cat}_e(t) : \text{Sym}^{e_1} V^1 \oplus \cdots \oplus \text{Sym}^{e_k} V^*_k \to \text{Sym}^{d_1 - e_1} V^1 \oplus \cdots \oplus \text{Sym}^{d_k - e_k} V^*_k,$$

where $D$ and $t$ are regarded as elements of $\text{Sym}^*(V^1 \oplus \cdots \oplus V^*_k)$ and $\text{Sym}^*(V^1 \oplus \cdots \oplus V^*_k)$ respectively, and $D(t)$ is the result of the natural differentiation.

For $t \in \text{Sym}^{d_1} V^1 \oplus \cdots \oplus \text{Sym}^{d_k} V^*_k$, the multihomogeneous components of the apolar ideal $t^\perp$ coincide with the kernels of the catalecticant maps, i.e., $(t^\perp)_e = \ker(\text{cat}_e(t))$. Similarly to the homogeneous case, we have that $(t^\perp)_e = \text{Sym}^{e_1} V^1 \oplus \cdots \oplus \text{Sym}^{e_k} V^*_k$ if $e_j > d_j$ for at least one $j$.

In the special case where $t = f_1 \otimes \cdots \otimes f_k \in \text{Sym}^{d_1} V^1 \oplus \cdots \oplus \text{Sym}^{d_k} V^*_k$ there is a direct characterization of the apolar ideal of $t$ in terms of the apolar ideals of the $f_i$'s.

Let us first recall a basic lemma from linear algebra that we will use in the proof.

Lemma 2.2. Let $A_1, \ldots, A_k$ be linear maps, with $A_i : \mathbb{C}^{n_i} \to \mathbb{C}^{m_i}$. Let $A_1 \otimes \cdots \otimes A_k$ be the Kronecker product of the $A_i$'s, i.e., the linear map defined by

$$A_1 \otimes \cdots \otimes A_k : \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k} \to \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_k}$$

$$v_1 \otimes \cdots \otimes v_k \mapsto A_1(v_1) \otimes \cdots \otimes A_k(v_k),$$

and extended linearly.

Then:

(i) $\text{rk}(A_1 \otimes \cdots \otimes A_k) = \text{rk}(A_1) \cdots \text{rk}(A_k)$;

(ii) $\ker(A_1 \otimes \cdots \otimes A_k) = \ker(A_1) \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_k} + \cdots + \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_{k-1}} \otimes \ker(A_k)$.

Proof. We prove it in the case $k = 2$. The general result follows by induction.

(i) It is enough to show that $\text{Im}(A_1 \otimes A_2) = \text{Im}(A_1) \otimes \text{Im}(A_2)$. This is immediate from the fact that tensor products are generated by product elements.

(ii) The inclusion of the right-hand side in the left-hand side is immediate. The other inclusion follows by a dimension argument. Let $r_i := \text{rk}(A_i)$. From part (i),

$$\dim \ker(A_1 \otimes A_2) = m_1 m_2 - \text{rk}(A_1 \otimes A_2) = m_1 m_2 - r_1 r_2.$$ 

On the other hand, by Grassmann's formula,

$$\dim(\ker(A_1) \otimes \mathbb{C}^{m_2} + \mathbb{C}^{m_1} \otimes \ker(A_2)) =$$

$$= (m_1 - r_1) m_2 + m_1 (m_2 - r_2) - (m_1 - r_1) (m_2 - r_2) = m_1 m_2 - r_1 r_2.$$ 

□
Now, notice that for every $j = 1, \ldots, k$, the polynomial ring $\text{Sym}^* V_j^*$ can be regarded as a subring of $\text{Sym}^*(V_1^* \oplus \cdots \oplus V_k^*)$; if $I \subseteq \text{Sym}^* V_j^*$ is an ideal for some $j = 1, \ldots, k$, write $I^{ext}$ for the ideal generated by $I$ in $\text{Sym}^*(V_1^* \oplus \cdots \oplus V_k^*)$.

**Lemma 2.3.** Let $t = f_1 \otimes \cdots \otimes f_k \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k$. Then,

$$t^\perp = (f_1^\perp)^{ext} + \cdots + (f_k^\perp)^{ext}.$$

**Proof.** We show that the two ideals coincide in every multidegree $e = (e_1, \ldots, e_k)$.

Recall that $(t^\perp)_e = \ker(\text{cat}_e(t))$. Moreover, we have

$$(f_i^\perp)_e = \text{Sym}^{e_1} V_i^* \otimes \cdots \otimes \text{Sym}^{e_{i-1}} V_{i-1}^* \otimes (f_i^\perp)_{e_i} \otimes \text{Sym}^{e_{i+1}} V_{i+1}^* \otimes \cdots \otimes \text{Sym}^{e_k} V_k^*$$

and recall that $(f_i^\perp)_{e_i} = \ker(\text{cat}_{e_i}(f_i))$.

Therefore, by Lemma 2.2(ii), it is enough to prove the following Claim.

**Claim.** $\text{cat}_e(t) = \text{cat}_{e_1}(f_1) \boxtimes \cdots \boxtimes \text{cat}_{e_k}(f_k)$.

**Proof of Claim.** It suffices to observe that both sides coincide on product elements $D = D_1 \otimes \cdots \otimes D_k \in \text{Sym}^{e_1} V_1^* \otimes \cdots \otimes \text{Sym}^{e_k} V_k^*$. Regard $D \in \text{Sym}^{e_1} V_1^* \otimes \cdots \otimes \text{Sym}^{e_k} V_k^*$ as the differential operator $D = D_1 \cdots D_k \in \text{Sym}^* (V_1^* \oplus \cdots \oplus V_k^*)$ and $t = f_1 \cdots f_k \in \text{Sym}^* (V_1 \oplus \cdots \oplus V_k)$, where the factors of $t$ involve disjoint sets of variables and the factors of $D$ act on disjoint sets of variables. Then, we obtain

$$\text{cat}_e(t)(D_1 \otimes \cdots \otimes D_k) = (D_1 \cdots D_k)(f_1 \cdots f_k)
= D_1(f_1) \cdots D_k(f_k)
= D_1(f_1) \otimes \cdots \otimes D_k(f_k)
= \text{cat}_{e_1}(f_1)(D_1) \otimes \cdots \otimes \text{cat}_{e_k}(f_k)(D_k)
= [\text{cat}_{e_1}(f_1) \boxtimes \cdots \boxtimes \text{cat}_{e_k}(f_k)](D_1 \otimes \cdots \otimes D_k),$$

and therefore $\text{cat}_e(t) = \text{cat}_{e_1}(f_1) \boxtimes \cdots \boxtimes \text{cat}_{e_k}(f_k)$.

\[ \square \]

### 2.3. Multiplicativity of Ranestad-Schreyer lower bound

For any homogeneous polynomial $f \in \text{Sym}^d V$, denote by $A_f$ the quotient algebra $\text{Sym}^* V^*/f^\perp$. Since we have that $(f^\perp)_e = \text{Sym}^e V$ for $e > d$, we deduce that $A_f$ is a finite dimensional vector space, and, since $f^\perp$ is a homogeneous ideal, $A_f = \bigoplus_{e=0}^d (A_f)_e$.

Similarly, if $t \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k$, we define $A_t = \text{Sym}^* (V_1^* \oplus \cdots \oplus V_k^*)/t^\perp$, which is again finite dimensional and multigraded because $t^\perp$ is a multihomogeneous ideal and $(t^\perp)_e = 0$ if $e_j > d_j$ for at least one $j$; in particular, $A_t = \bigoplus_{e_i=0, \ldots, d_i} (A_t)_e$.

The Ranestad-Schreyer lower bound for homogeneous forms given in [RS11, Proposition 1] was later generalized to the multihomogeneous case by Teitler in [Tei13].
Theorem 2.4 ([14, Theorem 5.13]). Let \( t \in \text{Sym}^{d_1} V_1 \otimes \cdots \otimes \text{Sym}^{d_k} V_k \). Let \( \delta = (\delta_1, \ldots, \delta_k) \) such that \( t^\perp \) is generated in multidegrees smaller or equal to \( \delta \) (in the sense of the standard partial ordering of the degrees). Then,

\[
R_d(t) \geq \frac{\dim A_t}{\delta_1 \cdots \delta_k}.
\]

In the particular case where \( t = f_1 \otimes \cdots \otimes f_k \), we obtain the following.

Proposition 2.5. Let \( t = f_1 \otimes \cdots \otimes f_k \), with \( f_i \in \text{Sym}^{d_i} V_i \). Then \( A_t \cong A_{f_1} \otimes \cdots \otimes A_{f_k} \) as multigraded algebras. In particular,

\[
\dim A_t = (\dim A_{f_1}) \cdots (\dim A_{f_k}).
\]

Moreover, if \( f_i^\perp \) is generated in degree at most \( \delta_i \), for \( i = 1, \ldots, k \), then \( t^\perp \) is generated in degree at most \( \delta \).

Proof. For every \( i = 1, \ldots, k \), since \( (f_i^\perp)^{\text{ext}} \subseteq t^\perp \), the natural inclusion \( \text{Sym}^* V_i^* \to \text{Sym}^* (V_i^* \oplus \cdots \oplus V_k^*) \) descends to the quotients giving a graded algebra homomorphism \( \varphi_i : A_{f_i} \to A_t \). Notice that elements of degree \( e \) are mapped to elements of multidegree \( (0, \ldots, 0, e, 0, \ldots, 0) \), where \( e \) is at the \( i \)-th entry.

By the universal property of tensor products, the \( \varphi_i \)'s lift to a homomorphism of graded algebras

\[
\varphi : A_{f_1} \otimes \cdots \otimes A_{f_k} \to A_t
\]

defined by \( \varphi(g_1 \otimes \cdots \otimes g_k) = \varphi_1(g_1) \cdots \varphi_k(g_k) = g_1 \cdots g_k \) on product elements and extended linearly. Notice that \( \varphi \) is surjective, because the algebra \( A_t \) is generated by elements of multidegree \( (0, \ldots, 0, 1, 0, \ldots, 0) \), that are the images of the variables \( x_{i,j} \), and those are in the image of \( \varphi \).

We conclude that \( \varphi \) is an isomorphism by showing that the multigraded components of \( A_{f_1} \otimes \cdots \otimes A_{f_k} \) and \( A_t \) have the same dimension. Indeed, in multidegree \( e \), we have

\[
\dim(A_{f_1} \otimes \cdots \otimes A_{f_k})_e = (\dim(A_{f_1})_{e_1}) \cdots (\dim(A_{f_k})_{e_k}) = \text{rk}(\text{cat}_{e_1}(f_1)) \cdots \text{rk}(\text{cat}_{e_k}(f_k))
\]

and

\[
\dim(A_t)_e = \text{rk}(\text{cat}_e(f_1 \otimes \cdots \otimes f_k));
\]

hence it is enough to apply Lemma 2.2(i) to the equality \( \text{cat}_{e_1}(f_1)) \times \cdots \times \text{cat}_{e_k}(f_k) = \text{cat}_e(f_1 \otimes \cdots \otimes f_k) \) that we proved in the proof of Lemma 2.3.

The second part of the statement is immediate from Lemma 2.3 as well, since \( t^\perp = (f_1^\perp)^{\text{ext}} + \cdots + (f_k^\perp)^{\text{ext}} \) is generated in the multidegrees of the form \( (0, \ldots, 0, \delta_i, 0, \ldots, 0) \), for all \( i = 1, \ldots, k \).

In particular, Proposition 2.5 implies that the Ranestad-Schreyer bound from Theorem 2.4 is multiplicative under the tensor product.

Corollary 2.6. Let \( f_i \in \text{Sym}^{d_i} V_i \), for \( i = 1, \ldots, k \), and assume that \( f_i^\perp \) is generated in degree at most \( \delta_i \). Then,

\[
R_d(f_1 \otimes \cdots \otimes f_k) \geq \frac{\dim(A_{f_1}) \cdots \dim(A_{f_k})}{\delta_1 \cdots \delta_k};
\]
in particular, if \( f \in \text{Sym}^d V \) and \( f^\perp \) is generated in degree \( \delta \), then, \( R_d(f^{\otimes k}) \geq \left( \frac{\text{dim} A_f}{\delta} \right)^k \), for every \( k \geq 1 \).

2.4. Tensor asymptotic rank and lower bounds for border rank. In [CGJ19], the notion of tensor asymptotic rank was introduced. We recall the definition in the case of Waring rank. Let \( f \in \text{Sym}^d V \). The tensor asymptotic rank of \( f \) is

\[
R_d^\otimes (f) = \lim_{k \to \infty} \left[ R_d(f^{\otimes k}) \right]^{1/k}.
\]

The limit in (1) exists by Fekete’s Lemma (see, e.g., [PS97, pg. 189]) via the submultiplicativity properties of Waring rank under tensor product and in fact it is an infimum over \( k \). As a consequence, multiplicative lower bounds for \( R_d(f) \) are lower bounds for \( R_d^\otimes (f) \); see Lemma 2.7 below. This was observed in [CJZ18] for (generalized) flattening lower bounds, but the general result is immediate for any multiplicative lower bound.

**Lemma 2.7.** Let \( f \in \text{Sym}^d V \). Assume that \( M \) is a multiplicative lower bound for the rank of \( f \), i.e., \( R_d(f^{\otimes k}) \geq M^k \), for every \( k \geq 1 \). Then, \( R_d^\otimes (f) \geq M \).

**Proof.** Consider the inequality \( R_d(f^{\otimes k}) \geq M^k \), and raise both sides to the \( 1/k \), so that \( [R_d(f^{\otimes k})]^{1/k} \geq M \); passing to the limit as \( k \to \infty \), we conclude. \( \square \)

In particular, if \( M \) is the Ranestad-Schreyer lower bound on \( R_d(f) \) obtained from Theorem 2.4 we deduce that \( R_d^\otimes (f) \geq M \).

The definition of the tensor asymptotic rank was inspired by the similar definition of asymptotic rank given by Strassen in the setting of tensors [Str86], in terms of tensor Kronecker (or flattened) product. In [Bin80], Bini proved that the growth of the tensor rank under Kronecker powers is essentially the same as the growth of the border rank under Kronecker powers, so that the definition of asymptotic rank of a tensor, in the sense of Strassen, can equivalently be given in terms of rank or border rank. The analogous result holds for the tensor asymptotic rank of (1), as proved in [CJZ18, Theorem 8] in the case of tensors and in [CGJ19, Proposition 6.2] in full generality. In particular the border rank of a homogeneous form is an upper bound for its tensor asymptotic rank.

**Proposition 2.8 ([CGJ19, Proposition 6.2]).** Let \( f \in \text{Sym}^d V \). Then,

\[
R_d^\otimes (f) \leq R_d(f).
\]

In summary, the results of this section allow us to upgrade the lower bound by Ranestad-Schreyer to the level of border rank:

**Theorem 2.9.** Let \( f \in \text{Sym}^d V \) be a form such that the apolar ideal \( f^\perp \) is generated in degree at most \( \delta \). Then,

\[
R_d(f) \geq \frac{\text{dim} A_f}{\delta}.
\]

**Proof.** Corollary 2.6 provides \( R_d(f^{\otimes k}) \geq \left( \frac{\text{dim} A_f}{\delta} \right)^k \). Passing to the tensor asymptotic rank, as in Lemma 2.7 we obtain \( R_d^\otimes (f) \geq \frac{\text{dim} A_f}{\delta} \) and finally Proposition 2.8 implies \( R_d(f) \geq \frac{\text{dim} A_f}{\delta} \). \( \square \)
We conclude this section with the following observation.

**Remark 2.10.** The same argument as Theorem 2.9 provides a full generalization of Theorem 2.4 to border rank as follows. Let \( t \in \text{Sym}^d V_1 \otimes \ldots \otimes \text{Sym}^d V_k \). Let \( \delta = (\delta_1, \ldots, \delta_k) \) such that \( t^\perp \) is generated in multidegrees smaller or equal to \( \delta \) (in the sense of the standard partial ordering of the degrees). Then,

\[
R_d(t) \geq \frac{\dim A_t}{\delta_1 \cdots \delta_k}.
\]

The proof is the same as the one of Theorem 2.9. Indeed Proposition 2.5 and Corollary 2.6 hold if the factors \( f_i \) are themselves partially symmetric tensors rather than homogeneous forms; moreover, one can define a *partially symmetric tensor asymptotic rank* for which the partially symmetric analog of Proposition 2.8 holds. These results, applied to tensor powers of \( t \), provide the extension of Theorem 2.4 to border rank.

2.5. **Border rank of monomials: proof of Theorem 1.1**

We are now ready to conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1.** The upper bound

\[
R_d(m) \leq \frac{1}{a_n + 1} \prod_{i=0}^{n}(a_i + 1)
\]

holds by [LT10, Theorem 11.2].

As observed in [RS11, Corollary 2], since \( m^\perp = (\partial_a a_0+1, \ldots, \partial_a a_n+1) \), it is immediate to see that the algebra \( A_m \) satisfies \( \dim A_m = \prod_{i=0}^{n}(a_i + 1) \). Therefore, the Ranestad-Schreyer lower bound for \( m \) is exactly \( \frac{1}{a_n + 1} \prod_{i=0}^{n}(a_i + 1) \).

By Theorem 2.9 we conclude that

\[
\frac{1}{a_n + 1} \prod_{i=0}^{n}(a_i + 1) \leq R_d^\otimes(f) \leq R_d(f).
\]

\[\square\]

We conclude with the following observation, that will be developed in subsequent work.

**Remark 2.11.** The cactus rank is a scheme-theoretic version of the rank: more precisely, the cactus rank of a form \( f \in \text{Sym}^d V \) is the smallest length of a 0-dimensional scheme contained in the Veronese variety \( V_{n,d} \) whose linear span contains the point \([f] \in \mathbb{P} \text{Sym}^d V \). The cactus border rank of \( f \) is the smallest \( r \) such that \( f \) can be approximated by forms of cactus rank \( r \). The results of this paper can be extended to the level of cactus rank: indeed, the Ranestad-Schreyer lower bound from [RS11] holds for cactus rank; moreover, [CGJ19, Proposition 6.2] can be extended, with essentially the same proof to the level of cactus rank. In particular, the cactus border rank of monomials equals their cactus rank and their border rank.

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(M. Christandl, F. Gesmundo) QMATH, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen O., Denmark

(A. Oneto) Otto-von-Guericke Universität Magdeburg, Germany

*E-mail address*, M. Christandl: christandl@math.ku.dk

*E-mail address*, F. Gesmundo: fulges@math.ku.dk

*E-mail address*, A. Oneto: alessandro.oneto@ovgu.de, aless.oneto@gmail.com