Quantum entropy of a non-extreme stationary axisymmetric black hole due to minimally coupled quantum scalar field

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By using the 't Hooft "brick wall" model and the Pauli-Villars regularization scheme we calculate the statistical-mechanical entropy arising from the minimally coupled scalar fields which rotate with the azimuthal angular velocity $\Omega_0 = \Omega_H$ ($\Omega_H$ is the angular velocity of the black hole horizon) in the general four-dimensional non-extreme stationary axisymmetric black hole space-time. We also show, for the Kerr-Newman and the Einstein-Maxwell dilaton-axion black holes, that the statistical-mechanical entropy obtained from our derivation and the quantum thermodynamical entropy by the conical singularity method are equivalent.

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I. INTRODUCTION

Since Bekenstein and Hawking found that the black hole entropy is proportional to the event horizon area by comparing black hole physics with thermodynamics and from the discovery of the black hole evaporation \cite{1} - \cite{3}, many efforts have been devoted to study the statistical origin of the black hole entropy. Especially, the idea to relate the entropy of the black hole to quantum excitations of the black hole has attracted many attentions \cite{4} - \cite{23}. The thermodynamical entropy of the black hole is related to the covariant Euclidean free energy $F_E[g, \beta] = \beta^{-1} W[g, \beta]$ \cite{25}, where $\beta$ is the inverse temperature. The function $W[g, \beta]$ is given on Euclidean manifolds with the period $\beta$ in the Euclidean time $\tau$. We can calculate the free energy $F_E$ by the conical singularities method. This procedure was consistently carried out for the studies of the static black holes and the rotating charged Kerr black hole \cite{5} \cite{19} \cite{26} \cite{27} \cite{23}. On the other hand, the canonical statistical-mechanical entropy can be derived from the free energy $F_C$ of a system \cite{24}, where $F_C$ can be defined in term of the one particle spectrum. One of the ways to calculate $F_C$ is the "brick wall" model (BWM) proposed by 't Hooft \cite{4}. He argued that the black hole entropy is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the horizon. In this model, in order to eliminate divergence which appears due to the infinite growth of the density of states closed to the horizon, 't Hooft introduces a "brick wall" cutoff: a fixed boundary $\Sigma_h$ near the event horizon within the quantum field does not propagate and the Dirichlet boundary condition was imposed on the boundary, i.e., wave function $\phi = 0$ for $r = r(\Sigma_h)$. Later, J. G. Demers, R, Lafrance and R. C. Myers \cite{28} pointed out that the Dirichlet condition can be removed if we use the Pauli-Villars regulated theory. The BWM has been successfully used in studies of the statistical-mechanical entropy for many black holes \cite{1} \cite{14} \cite{17} \cite{22} \cite{23}.

Recently, Frolov and Fursaev \cite{23} reviewed the studies of the relation between the thermodynamic entropy and the statistical-mechanical entropy of the black holes. They shown that for the general static black holes the covariant Euclidean free energy $F_E$ and the statistical-mechanical free energy $F_C$ are equivalent when ones use the ultraviolet regularization method \cite{23}.

As static case, the quantum entropy for the stationary axisymmetric black holes has also been studied by many authors recently. Mann and Solodukhin \cite{23} investigated the covariant Euclidean formulation for the Kerr-Newman black hole. They showed that an Euclidean manifold which was obtained by Wick rotation of the Kerr-Newman geometry with Killing horizon has a conical singularity similar to the one which appears in the static black holes. The one-loop quantum correction to the entropy of the charged Kerr black hole was calculated by applying the method of the conical singularities. They found an interesting result that the logarithmic term of the quantum entropy for

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II. THE SPACE-TIME OF THE GENERAL NON-EXTREME STATIONARY AXISYMMETRIC BLACK HOLE

In Boyer-Lindquist coordinates the most general line element for a stationary axisymmetric black hole in four-dimensional space-time can be expressed as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{t\phi}dtd\phi + g_{\theta\phi}d\theta^2 + g_{\varphi\varphi}d\varphi^2,$$  \hspace{1cm} (1)

where $g_{tt}$, $g_{rr}$, $g_{t\phi}$, $g_{\theta\phi}$ and $g_{\varphi\varphi}$ are functions of the coordinates $r$ and $\theta$ only. Because the space-time (1) is stationary and axisymmetric one it exists a stationary Killing vector field $\xi^\mu = (1, 0, 0, 0)$ and an axial Killing field $\Psi^\mu = (0, 0, 0, 1)$ \cite{24}. By taking a liner combination of $\xi^\mu$ and $\Psi^\mu$ we obtain a new Killing field $l^\mu = \xi^\mu + \Omega_H \Psi^\mu$, \hspace{1cm} (2)

which is normal to the horizon of the black hole. In Eq. (2) the constant $\Omega_H$ is called the angular velocity of the event horizon. An interesting feature of the black hole worthy of note is that the norm of the Killing field $l^\mu$ is zero on the horizon since the horizon is a null surface and vector $l^\mu$ is normal to the horizon. That is to say, the black hole horizon is a surface where the Killing field $l^\mu$ is null. Substituting $l^\mu$ into the formula of the surface gravity \cite{30} $\kappa^2 = -\frac{1}{2}l_{\mu\nu}l^{\mu\nu}$, we obtain

$$\kappa = \frac{-1}{2} \lim_{r \to r_H} \left( \frac{-1}{g_{tt}} \frac{d}{dr} \left( g_{tt} - \frac{g_{t\phi}^2}{g_{\varphi\varphi}} \right) \right) = \frac{2\pi}{\beta_H},$$  \hspace{1cm} (3)
where \( r_H \) represents the outermost event horizon, \( 1/\beta_H \) is the Hawking temperature, and here and hereafter the metric signature is taken as \((-++,++)\). We know that the event horizon is a null hypersurface determined by
\[
g^\mu_\nu \frac{\partial H}{\partial x^\mu} \frac{\partial H}{\partial x^\nu} = 0. \tag{4}
\]

For the stationary axisymmetric black hole [3] the function \( H \) is a function of \( r \) and \( \theta \) only. Substituting the metric (1) into Eq. (4) and discussing carefully we find
\[
\frac{1}{g^{tt}(r_H)} = \left( g_{tt} - \frac{g^{2}_{t\varphi}}{g_{\varphi\varphi}} \right)_{r_H} = 0. \tag{5}
\]

Solutions of which determine the location of the event horizons. From Eq. (5) we know that for a non-extreme stationary axisymmetric black hole \( 1/g^{tt} \) can be expressed as
\[
\left( g_{tt} - \frac{g^{2}_{t\varphi}}{g_{\varphi\varphi}} \right) \equiv G_1(r, \theta)(r - r_H), \tag{6}
\]
where \( G_1(r, \theta) \) is a regular function in the region \( \infty > r \geq r_H \) and its value is nonzero on the outermost event horizon \( r_H \). On the other hand, since \( \kappa = constant \) and \( 1/g^{rr} = 0 \) on the event horizon \( r = r_H \), we find from Eq.(3) that \( g^{rr} \) must take the following form
\[
g^{rr} \equiv G_2(r, \theta)(r - r_H), \tag{7}
\]
where \( G_2(r, \theta) \) is a well-defined function in the region \( \infty > r \geq r_H \) and is nonzero on the horizon \( r_H \) too. Making use of Eqs. (6) and (7), we obtain
\[
g_{rr} \left( g_{tt} - \frac{g^{2}_{t\varphi}}{g_{\varphi\varphi}} \right) = \frac{G_1(r, \theta)}{G_2(r, \theta)} \equiv -f(r, \theta), \tag{8}
\]
where the \( f(r, \theta) \) is a constant or a regular function on the outermost event horizon and outside the horizon.

III. THE STATISTICAL-MECHANICAL ENTROPY OF GENERAL NON-EXTREME STATIONARY AXISYMMETRIC BLACK HOLE

We now try to find an expression of the statistical-mechanical entropy due to the minimally coupled quantum scalar fields in a general four-dimensional stationary axisymmetric black hole. We first seek the total number of modes with energy less than \( E \) by using Klein-Gordon equation, and then calculate a free energy. The statistical-mechanical entropy of the black hole is obtained by the variation of the free energy with respect to inverse temperature and setting \( \beta = \beta_H \).

Using WKB approximation with
\[
\phi = \exp[-iEt + im\varphi + iW(r, \theta)], \tag{9}
\]
and substituting the metric (1) into the Klein-Gordon equation of the scalar field \( \phi \) with mass \( \mu \) and nonminimal \( \xi R\phi^2 \) (\( R \) is the scalar curvature of the spacetime) coupling
\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - (\mu^2 + \xi R)\phi = 0, \tag{10}
\]
we find [3]
\[
p_r^2 = \frac{1}{g^{rr}} \left[ -g^{tt} E^2 + 2g^{t\varphi} Em - g^{\varphi\varphi} m^2 - g^{\theta\theta} p_\theta^2 - (\mu^2 + \xi R) \right], \tag{11}
\]
where \( p_r = \partial_r W(r, \theta) \) and \( p_\theta = \partial_\theta W(r, \theta) \). If the scalar curvature \( R \) takes a nonzero value at the horizon then this region can be approximated by some sort of constant curvature space. However, the exact results for such a black
hole showed that the mass parameter in the solution enters only in the combination \((\mu^2 - R/6)\) \[21\] \[32\]. Therefore, inserting the covariant metric into Eq.(11) we arrive at

\[
p_{r}^{2} = -\frac{g_{tt}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} \left[ (E - \Omega m)^2 + \left( g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right) \left( \frac{m^2}{g_{\varphi\varphi}} + \frac{p_{\varphi}^2}{g_{\theta\theta}} + M^2(r, \theta) \right) \right],
\]

(12)

where \(\Omega \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\) and \(M^2(r, \theta) \equiv \mu^2 - (\frac{1}{2} - \xi)R\). In this paper our discussion is restricted to study minimally coupled \((\xi = 0)\) scalar fields. We know from Eq. (12) that \(W(r, \theta)\) can be expressed as

\[
W(r, \theta) = \pm \int r^2 \sqrt{-\frac{g_{tt}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}} K(r, \theta) dr + c(\theta),
\]

(13)

where

\[
K(r, \theta) = \sqrt{(E - \Omega m)^2 + \left( g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right) \left( \frac{m^2}{g_{\varphi\varphi}} + \frac{p_{\varphi}^2}{g_{\theta\theta}} + M^2(r, \theta) \right)}.
\]

(14)

Therefore, in phase space the number of the modes with \(E, m\) and \(p_{\varphi}\) is shown by \[33\]

\[
n(E, m, p_{\varphi}) = \frac{1}{\pi} \int d\theta \int_{r_H + h}^{r_{\infty}} \frac{-g_{tt}g_{\varphi\varphi}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} K(r, \theta) dr.
\]

(15)

Zhao and Gui \[34\] pointed out that “a physical space” must be dragged by the gravitational field with a azimuth angular velocity \(\Omega_{H}\) in the stationary axisymmetric space-time \([1]\). Apparently, a quantum scalar field in thermal equilibrium at temperature \(1/\beta\) in the stationary axisymmetric black hole must be dragged too. Therefore, it is rational to assume that the scalar field is rotating with angular velocity \(\Omega_{0} = \Omega_{H}\). For such an equilibrium ensemble of states of the scalar field, the free energy is given by

\[
\beta F = \int dm \int dp_{\varphi} \int dn(E, m, p_{\varphi}) ln \left[ 1 - e^{-\beta (E - \Omega_{0} m)} \right]
\]

\[
= \int dm \int dp_{\varphi} \int dn(E + \Omega_{0} m, m, p_{\varphi}) ln \left[ 1 - e^{-\beta E} \right]
\]

\[
= -\beta \int dm \int dp_{\varphi} \int \frac{n(E + \Omega_{0} m, m, p_{\varphi})}{e^{\beta E} - 1} dE
\]

\[
= -\beta \int \frac{n(E)}{e^{\beta E} - 1} dE,
\]

(16)

with

\[
n(E) = \int dm \int dp_{\varphi} \int n(E + \Omega_{0} m, m, p_{\varphi})
\]

\[
= \frac{1}{3\pi} \int d\theta \int_{r_H + h}^{r_{\infty}} \frac{dr \sqrt{g_{\varphi\varphi}}}{\left( g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right) \left( 1 + \frac{g_{\varphi\varphi}^2(\Omega - \Omega_{0})^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} \right) \left( \frac{m^2}{g_{\varphi\varphi}} + \frac{p_{\varphi}^2}{g_{\theta\theta}} + M^2(r, \theta) \right) ^{\frac{1}{2}}}
\]

\[
+ \left( g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right) \left( 1 + \frac{g_{\varphi\varphi}^2(\Omega - \Omega_{0})^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} \right) M^2(r, \theta) \right] ^{\frac{1}{2}}.
\]

(17)

where the function \(n(E)\) presents the total number of the modes with energy less than \(E\). The integrations of the \(m\) and \(p_{\varphi}\) in the Eq.(16) are taken only over the value for which the square root in Eq.(14) exists.

Taking the integration of the \(r\) in the Eq.(17) for the case \(\Omega_{0} = \Omega_{H}\) we have

\[
n(E) = -\frac{1}{2\pi} \int d\theta \sqrt{g_{\theta\theta}g_{\varphi\varphi}} \left[ \frac{2}{3} \frac{E_{3} H}{4\pi} \right] \left[ C(r, \theta) + M^2(r, \theta) \left( \frac{E_{3} H}{4\pi} \right) ^{3} \right] \ln \left( \frac{E^2}{E_{\min}^2} \right) \biggr|_{r_H}
\]

\[
- \frac{1}{3\pi} \frac{E_{3} H}{4\pi} \int d\theta \sqrt{g_{\theta\theta}g_{\varphi\varphi}} M^2(r, \theta) \left( E - \frac{E_{3}^2}{E_{\min}^2} \right) \biggr|_{r_H},
\]

(18)
where

\[ C(r, \theta) = \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2} \frac{\partial g^{rr}}{\partial r} \frac{\partial \ln f}{\partial r} - \frac{2 \pi}{\beta_H \sqrt{f}} \left( \frac{1}{g_{\theta \theta}} \frac{\partial g_{\theta \theta}}{\partial r} + \frac{1}{g_{\varphi \varphi}} \frac{\partial g_{\varphi \varphi}}{\partial r} \right) - \frac{2 g_{\varphi \varphi}}{f} \left( \frac{\partial}{\partial r} \left( \frac{g_{\varphi \varphi}}{g_{\varphi \varphi}} \right) \right)^2, \]

\[ E_{min}^2 = -M^2(r_H, \theta) \left( g_{tt} - \frac{g_{t \varphi}}{g_{\varphi \varphi}} \right) \left( g_{tt} - \frac{g_{t \varphi}}{g_{\varphi \varphi}} \right), \]

\[ \tilde{K}^2 = E^2 + \left( g_{tt} - \frac{g_{t \varphi}}{g_{\varphi \varphi}} \right) M^2(r_H, \theta), \]

(19)

here and hereafter \( f \equiv f(r, \theta) \) which is defined by Eq. (8).

Now let us use the Pauli-Villars regularization scheme [28] by introducing five regulator fields \( \{ \phi_i, i = 1, ..., 5 \} \) of different statistics with masses \( \{ \mu_i, i = 1, ..., 5 \} \) dependent on the UV cutoff [29]. If we rewrite the original scalar field \( \phi = \phi_0 + \mu = \mu_0, \) then these fields satisfy \( \Sigma_{i=0}^5 \Delta_i = 0 \) and \( \Sigma_{i=0}^5 \Delta_i \mu_i^2 = 0, \) where \( \Delta_0 = \Delta_3 = \Delta_4 = 1 \) for the commuting fields and \( \Delta_1 = \Delta_2 = \Delta_5 = -1 \) for the anticommuting fields. Since each of the fields makes a contribution to the free energy of Eq. (14), the total free energy can be expressed as

\[ \beta F = \sum_{i=0}^{5} \Delta_i \beta F_i. \]

(20)

Substituting Eq. (16) into (18) and then taking the integration over \( E \) we find

\[ \tilde{F} = -\frac{1}{48} \frac{\beta H}{\beta^2} \int d\theta \left\{ \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right\}_{r_H} \sum_{i=0}^{5} \Delta_i M_i^2(r_H, \theta) \ln M_i^2(r_H, \theta) - \frac{1}{2880} \frac{\beta^3}{\beta H^4} \int d\theta \left\{ \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right\}_{r_H} \]

\[ \times \left[ \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2} \frac{\partial g^{rr}}{\partial r} \frac{\partial \ln f}{\partial r} - \frac{2 \pi}{\beta_H \sqrt{f}} \left( \frac{1}{g_{\theta \theta}} \frac{\partial g_{\theta \theta}}{\partial r} + \frac{1}{g_{\varphi \varphi}} \frac{\partial g_{\varphi \varphi}}{\partial r} \right) - \frac{2 g_{\varphi \varphi}}{f} \left( \frac{\partial}{\partial r} \left( \frac{g_{\varphi \varphi}}{g_{\varphi \varphi}} \right) \right)^2 \right] \]

\[ \times \sum_{i=0}^{5} \Delta_i \ln M_i^2(r_H, \theta), \]

(21)

where \( M_i^2(r_H, \theta) = \mu_i^2 - \frac{1}{6} R. \) Then the total statistical-mechanical entropy at the Hawking temperature \( \frac{1}{\beta} = \frac{1}{\beta_H} \) is given by

\[ S = \frac{\beta^2}{\beta H} \frac{\partial \tilde{F}}{\partial \beta} \bigg|_{\beta = \beta_H} \]

\[ = \frac{1}{48 \pi} \int d\theta d\phi \left\{ \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right\}_{r_H} \sum_{i=0}^{5} \Delta_i M_i^2(r_H, \theta) \ln M_i^2(r_H, \theta) + \left\{ \frac{1}{32 \times 45 \pi} \int d\theta d\phi \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right. \]

\[ \times \left[ \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2} \frac{\partial g^{rr}}{\partial r} \frac{\partial \ln f}{\partial r} - \frac{2 \pi}{\beta_H \sqrt{f}} \left( \frac{1}{g_{\theta \theta}} \frac{\partial g_{\theta \theta}}{\partial r} + \frac{1}{g_{\varphi \varphi}} \frac{\partial g_{\varphi \varphi}}{\partial r} \right) - \frac{2 g_{\varphi \varphi}}{f} \left( \frac{\partial}{\partial r} \left( \frac{g_{\varphi \varphi}}{g_{\varphi \varphi}} \right) \right)^2 \right] \]

\[ \times \sum_{i=0}^{5} \Delta_i \ln M_i^2(r_H, \theta), \]

(22)

Using the assumption that the scalar curvature \( R \) at the horizon is much smaller than each \( \mu_i \) and noting that the area of the event horizon is given by \( A_{\Sigma} = \int d\varphi \int d\theta \left\{ \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right\}_{r_H}, \) we obtain at last the following expression of the statistical-mechanical entropy

\[ S = \frac{A_{\Sigma}}{48 \pi} \sum_{i=0}^{5} \Delta_i M_i^2 \ln \mu_i \]

\[ + \left\{ \frac{1}{6 \times 48 \pi} \int d\theta d\phi \left( R \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \right)_{r_H} + \frac{1}{32 \times 45 \pi} \int d\theta d\phi \right. \]

\[ \times \left. \left[ \sqrt{g_{\theta \theta} g_{\varphi \varphi}} \left( \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2} \frac{\partial g^{rr}}{\partial r} \frac{\partial \ln f}{\partial r} - \frac{2 \pi}{\beta_H \sqrt{f}} \left( \frac{1}{g_{\theta \theta}} \frac{\partial g_{\theta \theta}}{\partial r} + \frac{1}{g_{\varphi \varphi}} \frac{\partial g_{\varphi \varphi}}{\partial r} \right) - \frac{2 g_{\varphi \varphi}}{f} \right. \]

\[ \times \left. \left( \frac{\partial}{\partial r} \left( \frac{g_{\varphi \varphi}}{g_{\varphi \varphi}} \right) \right)^2 \right] \right\}_{r_H} \sum_{i=0}^{5} \Delta_i \ln \mu_i^2. \]

(23)
which is valid for the general non-extreme stationary axisymmetric black holes which the metric can be expressed as \([1]\) in the Boyer-Lindquist coordinates and their signature is \((-\text{,}+\text{,}+\text{,}+)\). For the black holes with signature \((+\text{-},-\text{-},-)\) a corresponding formula can be obtained by replaced the \(\beta_H\) with \(-\beta_H\) in the Eq.\((23)\).

IV. DISCUSSION AND SUMMARY

In this section, let begin discussion with the study of the statistical-mechanical entropy of the Kerr-Newman black hole and EMDA black hole by using the formula \((23)\).

(A) The entropy of the Kerr-Newman black hole

In Boyer-Lindquist coordinates, the metric of the Kerr-Newman black hole \([35]\) \([36]\) takes the form

\[
g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}, \quad g_{\varphi \varphi} = -\frac{\sin^2 \theta (r^2 + a^2 - \Delta)}{\rho^2},
\]

\[
g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta \theta} = \rho^2, \quad g_{\varphi \varphi} = \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2}\right) \sin^2 \theta,
\]

with

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = (r - r_+)(r - r_-),
\]

where \(r_+ = r_H = M + \sqrt{M^2 - Q^2 - a^2}, \ r_- = M - \sqrt{M^2 - Q^2 - a^2}\), and \(M, Q\) represent the mass and charge of the black hole, respectively. Using the metric \((24)\) we get

\[
S_{KN} = \frac{A_\Sigma}{48\pi} \sum_{i=0}^{5} \Delta \mu_i^2 ln \mu_i^2 - \frac{1}{90} \left[ 1 + \frac{3(a^2 - r_+ r_-)}{4r_+^2} \right] \left[ 1 + \frac{a^2}{r_+ Arctan \left( \frac{a}{r_+} \right)} \right] \sum_{i=0}^{5} \Delta \mu_i^2,
\]

where \(A_\Sigma = 4\pi(r_+^2 + a^2)\). Noting \(r_+ r_- - a^2 = Q^2\) and the Pauli-Villars regularization scheme caused a factor \(-\frac{1}{3}\) for the second part in the Eq.\((27)\), we know that the statistical-mechanical entropy \((27)\) coincides with the Mann-Solodukhin’s result \((23)\) which obtained by using the conical singularity method.

(B) The entropy of the Stationary axisymmetric EMDA black hole

The stationary axisymmetric EMDA black hole metric (we take the solution \(b=0\) in Eq.\((14)\) in Ref.\((2)\). The reason we use this solution is that the solution \(b \neq 0\) cannot be interpreted properly as a black hole) is described by \([37]\)

\[
g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}, \quad g_{\varphi \varphi} = -\frac{\sin^2 \theta (r^2 + a^2 - 2dr) - \Delta}{\Sigma},
\]

\[
g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{\theta \theta} = \Sigma, \quad g_{\varphi \varphi} = \left(\frac{(r^2 + a^2 - 2dr)^2 - \Delta a^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta,
\]

with
\[ \Sigma = r^2 - 2dr + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2mr + a^2 = (r - r_+)(r - r_-), \]

where \( r_+ = m + \sqrt{m^2 - a^2}, \) \( r_- = m - \sqrt{m^2 - a^2} \). The mass \( M \), the angular momentum \( J \), the electric charge \( Q \), and the magnetic charge \( P \) of the black hole are respectively given by

\[ M = m - d, \quad J = a(m - d), \quad Q = \sqrt{2\omega d(d - m)}, \quad P = g. \]

By using the metric (28) we obtain

\[ \begin{align*}
&\left\{ \frac{\partial^2 g^{rr}}{\partial r^2} + \frac{3}{2} \frac{\partial g^{r\theta}}{\partial r} \frac{\partial \ln f}{\partial r} - \frac{2\pi}{\beta_H \sqrt{\Delta}} \left( \frac{1}{g^{r\theta}} \frac{\partial g^{\theta\theta}}{\partial r} + \frac{1}{g^{\varphi\varphi}} \frac{\partial g^{\varphi\varphi}}{\partial r} \right) - \frac{2g^{\varphi\varphi}}{f} \left( \frac{\partial g^{g\varphi}}{\partial r} \right) \right\}_{r_+}^r \\
&= \frac{16r_+^2[(2d - r_+ - r_-)r_+] + 4d(8r_+ - 3d)(r_+^2 - 2dr_+ + a^2)}{\Sigma^3} \\
&+ \frac{4[(3r_+ - 2d)(r_+ + r_+ - 2d) - d]}{\Sigma^2} + \frac{2d(r_+ + r_+ - 2d)(\Sigma - 2dr_+)}{\Sigma^3} \\
&+ \frac{2a^2(1 + \cos^2 \theta)\Sigma - 8a^2(r_+^2 + a^2 - 2dr_+ \cos^2 \theta)}{\Sigma^3}, \\
R &= \frac{2a^2d^2 \sin^2 \theta}{\Sigma^3}. \end{align*} \]

Substituting Eq. (31) into Eq. (23) and then taking the integration of the \( \theta \) and \( \varphi \) we find that the statistical-mechanical entropy of the EMDA black hole is

\[ S = \frac{A_S}{48\pi} \sum_{i=0}^{5} \Sigma \mu_i^2 \ln \mu_i^2 - \frac{1}{90} \left\{ 1 + \frac{9d^2}{8r_+^2 - 16dr_+} + \frac{9d(3a^2d + (r_+^2 - 2dr_+)(r_+ + r_+ - d))}{16(r_+^2 - 2dr_+)^2} \right\} \times \left[ 1 + \frac{r_+^2 + a^2 - 2dr_+}{a\sqrt{r_+^2 - 2dr_+}} \text{Arctan} \left( \frac{a}{\sqrt{r_+^2 - 2dr_+}} \right) \right] \sum_{i=0}^{5} \Sigma \mu_i^2, \]

where \( A_S = 4\pi(r_+^2 + a^2 - 2dr_+) \). In order to compare the entropy (32) with the thermodynamical entropy obtained by the covariant Euclidean formulation, we calculated the thermodynamical entropy of the EMDA black hole by using the conical singularity method of the Ref. [23]. We also find that the results obtained by the two methods are equivalent.

From Eq. (27) or Eq. (32) we find the same result for the Kerr black hole (setting \( Q = 0 \) in Eq. (27) or \( d = 0 \) in Eq. (32)) as that Mann and Solodukhin found in Ref. [24], i.e., the quantum entropy does not depend on the rotation parameter \( a \) and coincides with the quantum entropy of the Schwarzschild black hole. We think the reason is that the quantum entropy is mainly caused by quantum scalar fields near the event horizon and in the region the scalar fields are co-rotating with the black hole.

To summarize, by using BWM and with Pauli-Villars regularization scheme, we investigate the statistical-mechanical entropy arising from the minimally coupled quantum scalar fields rotating with the angular velocity \( \Omega_0 \) in the general four-dimensional non-extreme stationary axisymmetric black hole space-time. An expression of the statistical-mechanical entropy is obtained for the case \( \Omega_0 = \Omega_H \). The Kerr-Newman black hole and the EMDA black hole are studied. It is shown that the statistical-mechanical entropy obtained by using the formula (23) and the quantum thermodynamical entropy derived from the covariant Euclidean formulation (by using the conical singularity method) are equivalent for the Kerr-Newman and the EMDA black holes. The result fills in the gaps mentioned in Ref. [24] that the relation between the canonical and covariant Euclidean formulations in the rotating black hole has not been investigated. The study may provide us with a better understanding of the relationship between the different entropies to the black holes.

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