Random symmetric matrices with a constraint: The spectral density of random impedance networks

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We derive the mean eigenvalue density for symmetric Gaussian random $N \times N$ matrices in the limit of large $N$, with a constraint implying that the row sum of matrix elements should vanish. The result is shown to be equivalent to a result found recently for the average density of resonances in random impedance networks [Y.V. Fyodorov, J. Phys. A 32, 7429 (1999)]. In the case of banded matrices, the analytical results are compared with those extracted from the numerical solution of Kirchhoff equations for quasi-one-dimensional random impedance networks.

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I. FORMULATION OF THE PROBLEM

We consider an ensemble of $N \times N$ random matrices $M$ with matrix elements

$$M_{mn} = J_{mn} - \delta_{mn} \sum_{l=1}^{N} J_{ml},$$

(1)

where $J$ is a real symmetric $N \times N$ matrix with random entries. Such ensembles have been considered in Refs. [6,7,10,11]. In Ref. [10], $M$ was used to model a random transition matrix for a model describing glassy relaxation, $\partial_{t}\mu(t) = -M\mu(t)$, with matrix elements $J_{mn}$ distributed independently (subject to the constraint $J_{mn} = J_{nm}$) according to

$$P(J_{mn}) = (p/N) \delta(J_{mn} - 1/p) + (1 - p/N) \delta(J_{mn}).$$

(2)

The form of Eq. (1) yields $\sum_{n} M_{mn} = 0$, implying probability conservation in the problem considered in Ref. [10]. The average density $d(\lambda)$ of eigenvalues $\lambda$,

$$d(\lambda) = N^{-1}(\text{tr} \delta(M - \lambda I)),$$

(3)

was calculated in the limit of large $N$ and $p$, using the method of replicas. $\langle \cdots \rangle$ is an average over the ensemble defined by $P(J_{mn})$. In Ref. [6], eigenvalues of matrices $M$ with elements similar to Eq. (1) were shown to model resonance frequencies in random impedance networks [5], with $J_{mn} = J_{nm}$ and

$$P(J_{mn}) = \frac{1}{2} \delta(J_{mn} - 1) + \frac{1}{2} \delta(J_{mn} + 1).$$

(4)

In Refs. [6,7] the average density of resonance frequencies was calculated, in the limit of large $N$, using a variant of the supersymmetry technique. It was found that the result agrees

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The average density of eigenvalues $N_{\mathbf{G}}$ of Gaussian real variables. The corresponding symmetric random matrix $\mathbf{G}$ coincides with the result derived in Refs. [6,7]. In this limit, correlations between diagonal and off-diagonal matrix elements are found to be irrelevant, and the result can be understood in terms of an averaged Pastur equation [14]. Furthermore, we have also considered the case of banded symmetric random matrices. This case is of interest for random impedance networks with quasi-one-dimensional solutions of the Kirchhoff equations for such networks.

II. Method

The average eigenvalue density may be obtained from the trace of the averaged resolvent $\mathbf{G}=(\langle \mathbf{E} \mathbf{1} - \mathbf{M} \rangle^{-1})$.

$$d(E) = - (N\pi)^{-1} \text{Im} \text{tr} \mathbf{G}.$$  \hspace{1cm} (6)

Expanding the resolvent, Wick’s theorem may be employed for performing the average, using

$$\langle M_{ij}M_{mn} \rangle = \sigma^{2}[(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - \delta_{mn}(\delta_{im}\delta_{ij} + \delta_{in}\delta_{jm})$$

$$- \delta_{ij}(\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}) + \delta_{ij}\delta_{mn}(N\delta_{im} + 1)].$$  \hspace{1cm} (7)

In the limit of large $N$, adopting the scaling $\sigma^{2} = N^{-1}$, Eq. (7) can be simplified to

$$\langle M_{ij}M_{mn} \rangle \approx N^{-1} (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{ij}\delta_{mn}\delta_{im}.$$ \hspace{1cm} (8)

It is convenient to keep track of the contributions with the help of diagrams. To this end, a graphical representation of the contraction (8) is introduced in Fig. 1(a). The averaged resolvent $\mathbf{G}$ turns out to be a diagonal matrix. Figure 1(b) defines a graphical representation for the diagonal elements $G_{jj}$ of $\mathbf{G}$. Terms contributing to $G_{jj}$ are shown in Fig. 1(c–e). One observes that, in the limit of large $N$, only diagrams with no intersections between dashed or between dashed and wavy lines contribute. Thus, to leading order in $N$, the contribution (c) in Fig. 1 must be considered, but not (d) or (e), for example.

All diagrams contributing to the resolvent to leading order in $N^{-1}$ may be summed, resulting in a set of two coupled equations shown in Figs. 2(a) and 2(b). The diagonal elements $G_{jj}$ and $H_{jj}$ are independent of $j$ and denoted by $G$ and $H$ in the following. The first equation [Fig. 2(a)] contains an infinite sum of diagrams. This sum may be performed exactly. It is denoted by $g(H^{-1})$, where

$$g(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{dJ}{z - J} e^{-J^{2}/2}.$$ \hspace{1cm} (9)
The result for the resolvent scales with $\sigma^2$ and $N$ as $(\sigma^2 N)^{-1/2}g[(E - G)(\sigma^2 N)^{1/2}]$. Consequently, the equations shown in Fig. 2(a, b) imply the following self-consistent equation:

$$G = (\sigma^2 N)^{-1/2}g[(E - G)(\sigma^2 N)^{1/2}],$$

(10)
equivalent to Eq. (51) in Ref. [6] for the average density of eigenvalues in an ensemble of highly connected random impedance networks [15].

The form of the simplified contraction (8) implies an interpretation of the result (10) in terms of an averaged Pastur equation: consider a random matrix $\mathbf{M} = \mathbf{J} + \mathbf{V}$, where $\mathbf{J}$ is distributed according to Eq. (5), and $\mathbf{V}$ is a diagonal matrix with Gaussian random entries $v_{ik}$ with zero mean and unit variance, independent of $J_{mn}$. For a given realization of $\mathbf{V}$, Pastur’s equation [14] is $G = (E1 - \mathbf{V} - \mathbf{G})^{-1}$ (see also Ref. [16], and references therein). In this equation $G$ is the $\mathbf{J}$-averaged resolvent, keeping $\mathbf{V}$ fixed. One obtains Eq. (10) after observing that $\mathbf{G}$ is diagonal, by averaging over the matrix elements of $\mathbf{V}$. This interpretation implies that, in the limit of large $N$, the correlations between diagonal and off-diagonal matrix elements of $\mathbf{M}$ [as seen in Eqs. (1) and (7)] are irrelevant.

The above procedure is easily extended to the case of banded matrices, also of interest in random impedance networks [5, 11]. In the banded case, $\langle J_{mn}J_{kl} \rangle = \sigma^2(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})$ [which follows from Eq. (5)] is replaced by

$$\langle J_{mn}J_{kl} \rangle = \sigma^2(|m-n|)(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}).$$

(11)
The function $\sigma^2(x)$ is given by

$$\sigma^2(x) = \begin{cases} \sigma^2 & \text{for } 0 \leq x < b/2 \\ 0 & \text{otherwise.} \end{cases}$$

(12)
The bandwidth of $\mathbf{J}$ is thus $b$. In the limit of large $N$ and large $b$, the spectral density is given by a slight modification of Eq. (10),

$$G = (\sigma^2 b)^{-1/2}g[(E - G)(\sigma^2 b)^{1/2}].$$

(13)
Diagrammatically, the necessary changes are most easily derived by letting $\sigma^2 = N^{-1}$ and assuming that $b = BN$ (with fixed $B$). Then the wavy line in Fig. 1(a) acquires a factor of $B$. Furthermore, the dashed line in Eq. (8) also acquires a factor of $B$. Hence, the self-consistency equation in the banded case becomes (13). Let us also note that the same formula is actually valid not only for $b - N$, but more generally for $1 < b < N$.

This result implies that densities for different values of $b$ can be scaled on one single curve by plotting $\sqrt{b}d(E/\sqrt{b})$. In Fig. 3, solutions of Eq. (13) for $\sigma = 1$ are compared with results of exact diagonalizations of random matrices with $N =$ 500,1000, and $b =$ 50,100. We observe a very good agreement. The results confirm that, for large $N$ and $b$, the average density of eigenvalues is independent of $N$, and that it scales with $b$ as expected.

In the following, we show that Eq. (13) also describes the density of resonances for certain random impedance networks with a large, but finite connectivity.

III. RANDOM IMPEDANCE NETWORKS

Random networks of complex impedances are currently used to model electrical and optical properties of disordered inhomogeneous media [8]. The most common situation is that of a binary composite medium, modeled by attributing a random conductance to each bond $(x, y)$ of a lattice, according to the binary law:

$$\sigma_{x,y} = \begin{cases} \sigma_0 & \text{with probability } p \\ \sigma_1 & \text{with probability } q = 1 - p. \end{cases}$$

(14)
The homogeneous Kirchhoff equations for the electric potentials,

$$\sum_y \sigma_{x,y}(V_y - V_x) = 0,$$

(15)
can be recast as

$$\sum_{y \in Q(x)} (V_y - V_x),$$

(16)
with $\lambda = \sigma_0/(\sigma_0 - \sigma_1)$, and

$$\Delta V_x = \sum_{y \in Q(x)} (V_y - V_x), \quad (\Delta \sigma V)_x = \sum_{y \in P(x)} (V_y - V_x),$$

(17)
where $y(x)$ are all the sites connected to site $x$, whereas $y \in P(x)$ ($y \in Q(x)$) are those connected by a conductance $\sigma_0$ ($\sigma_1$), so that $\Delta = \Delta_0 + \Delta_1$. Resonances appear as nontrivial solutions to Eq. (16), for $0 < \lambda < 1$.

An efficient algorithm allowing for an exact determination of all the resonances of a finite sample has been devel-
oped in Ref. [5]. We have adapted this algorithm to the simplest geometry allowing for long-ranged bonds. Each site of a very long chain is connected to its $b=4$ neighbors.

Our numerical results are shown in Fig. 5, for very long periodic chains with ranges $b=60, 100$, and $120$. For each value of $b$, we have accumulated a number of resonances of order $10^7$. After rescaling the resonances according to $\lambda = (1+E)/2$ the density is given by Eq. (13) with $\sigma=1$. A very satisfactory quantitative agreement with the theoretical prediction is observed.

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