Degeneracies with the Q.Q interaction in a single j shell

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1 Abstract

Previously [1] it was shown that for a configuration of 2 protons and 2 neutrons in the g9/2 shell there is a certain degeneracy that occurs when the quadrupole-quadrupole (Q.Q) interaction is used to obtain wave functions. We here show 3 other examples of such degenerate T=0, T=2 pairs. More importantly we note a peculiarity of the T=0 partner in the original example (g9/2, J=4 T=0). Also we point out that degeneracies can be confusing and steps can be taken to remove them.

2 Introduction

In a work for a proceedings honoring Aplodor Raduta [1] Escuderos and Zamick discussed several topics. One of these involved surprising degeneracies that occur in single j shell calculations in which he quadrupole quadrupole (Q.Q) interaction is used to obtain the wave functions of a system of 2 protons and 2 neutrons in a single j shell. It was found that two J=4+ states were degenerate—one with isospin T=0 and the other with isospin T=2. There was no obvious explanation for this. In this work we do not offer an explanation either but we systematically search for other examples. As will soon be shown we find three others also involving one state with T=0 and the other with T=2. This suggests that this behavior is no accident. We will find that in one seniority comes into play in determining the properties of the degenerate pair.

3 List of degenerate states and their wave functions.

In tables I,II,III and IV we show the energies and wave functions of degenerate pairs (T=0 and T=2) for a system of 2 protons and 2 neutrons in a single j shell. The wave functions are represented by column vectors with entries $D_{J_P J_N}^{J_T}(J_P, J_N)$, the latter being the probability amplitude that in the $\alpha$'th state of total angular momentum $J$ the protons couple to $J_P$ and the neutrons to $J_N$. Note that $J_P$ and $J_N$ are both even. For brevity we only list the amplitudes where $J_P \leq J_N$. We can infer the others from the relation $D_{J_P J_N}^{J_T}(J_P, J_N) = (-1)^s D_{J_N J_P}^{J_T}(J_N, J_P)$ with $s = J+T$.

| J_P, J_N | T=0 | T=2 |
|----------|-----|-----|
| 0, 2     | 0.0832 | -0.2877 |
| 2, 2     | -0.2689 | -0.2694 |
| 2, 4     | -0.3510 | 0.3510 |
| 4, 4     | 0.1212 | -0.0431 |
| 4, 6     | -0.2115 | 0.3138 |
| 6, 6     | 0.7507 | 0.5125 |

Table II g9/2 shell, J=4+ E= 3.5284 MeV.
Table III $g_{9/2}$ shell $J=5^+$ $E=4.5890$ MeV.

| $J_P, J_N$ | $T=0$ | $T=2$ |
|-----------|-------|-------|
| 0.4       | 0     | 0     |
| 2.2       | 0.3132| -0.4270|
| 2.4       | 0.2289| -0.2542|
| 2.6       | -0.2076| 0.3107|
| 4.4       | -0.0135| 0.2395|
| 4.6       | 0.1784| -0.1418|
| 4.8       | -0.0888| 0.1567|
| 6.6       | 0.1362| 0.1638|
| 6.8       | 0.1353| 0.0316|
| 8.8       | 0.7549| 0.5665|

We now comment on the zeros that appear in the tables. The ones for odd $J$ i.e. $J=5$ are easy to understand. They follow from Eq 1 above. We have

$$D^5(J_P,J_N) = - D^5(J_N,J_P) (-1)^T.$$ Hence we have the result, $D^{5,0}(J,J)=0$.

We will discuss the zeros for $g_{9/2}$ $J=4^+$ in the next section.

Table IV $h_{11/2}$ shell $J=5^+$ $E=4.1458$ MeV

| $J_P, J_N$ | $T=0$ | $T=2$ |
|-----------|-------|-------|
| 2.4       | 0.0542| 0.3711|
| 2.6       | 0.2983| 0.1707|
| 4.4       | 0     | 0     |
| 4.6       | 0.3133| -0.2679|
| 4.8       | -0.4809| -0.2371|
| 6.6       | 0     | 0     |
| 6.8       | -0.2805| -0.4531|
| 8.8       | 0     | 0     |

In the above the Q
Q interaction was scaled so that the $J=0$ two-body matrix element was set equal to -1.0 MeV.

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Table V Unique $J=4^+ v=2$ wave function.
4 Partial dynamical symmetry—a new feature

The zeros for $g_{9/2} J=4$ were recognized in ref[1] to be connected to a partial dynamical symmetry. If we focus on $T=2$ states, we note that they are double analogs of states of 4 identical particles—say 4 neutrons. For $J=4^+$ in $g_{9/2}$ there are two seniority 4 states and one with $v=2$. In general though, seniority is not a good quantum number in the $g_{9/2}$ shell (it is in the $f_{7/2}$ shell) and so we expect the resulting eigenstates to have mixed seniority. But Escuderos and Zamick [2] noticed that with any isospin conserving interaction one eigenstate emerged independent of the interaction. This was a seniority $v=4$ state that did not mix with the $v=2$ state or indeed with the other $v=4$ state. There is also such a special $T=2$ state for $J=6^+$. This has lead to many theoretical works which discuss this and prove it [3-8]. We cite in particular the nice work by Qi [6].

This special unique $v=4$ state is usually discussed in the channel of 4 identical particles in terms of fractional parentage coefficients. It is displayed as such in refs [1] and [2]. But now we see the double analog of this $T=2$ state in Table II in the 2-proton-2 neutron channel. We can now understand why $D_{4^-,2}(0,4)=0$. We emphasize again that this unique $T=2 v=4$ state emerges for any interaction, not just Q.Q.

But there is a new feature, as the subheading implies. Note that for the $T=0$ member of the pair the amplitude $D_{4^-,0}(0,4)$ also vanishes. This suggests that the $T=0$ degenerate partner of the unique $J=4 T=2$ state might also have seniority $v=4$. In general states with both valence neutrons and protons do not have good seniority. However the $J=0 T=1$ pairing interaction of Flowers and Edmonds [9,10] does yield states of good seniority. We perform a matrix diagonalization for $J=4^+$ states in the $g_{9/2}$ shell consisting of 2 protons and 2 neutrons. The two-body matrix elements for $J=0$ to $J=9$ are, in MeV), $\{ -1,0,0,0,0,0,0,0,0,0 \}$. We can identify the quantum numbers of the states by the energies. The formula of Flowers and Edmonds [9,10] is

$$E = -C_0 (n-v)/2 (4j+8-n-v) \cdot T(T+1) + t(t+1)$$

where $n$ is the number of nucleons (4 in this case), $T$ is the total isospin (zero in this case), $t$ is the reduced isospin, and $v$ is the seniority. The scale facto $C_0 = -1/(2j+1)$ (-1/10 in this case). The excitation energy relative to the $J=0 , T=0, t=0$ ground state is $E^* = E + 2.2$ MeV. This method of obtaining quantum numbers from energies was previously used by Neergaard [11] and by Harper and Zamick [12].

In this model space there are 16 $J=4^+$ states: 7 $T=0$, 6 $T=1$ and 3 $T=2$ states. We can identify the isospins in several ways, one of which is to change the two-particle matrix element set to $\{ -1,-5,0,-5,0,-5,5,0,-5 \}$. That is, we add -5 MeV to the odd $J (T=0)$ 2-body matrix elements. This will not affect the wave functions but will shift the states upward by an amount 5($T^*(T+1)$). The unshifted states have $T=0$. This is of course also the method to separate the originally degenerate $T=0$, $T=2$ pairs.

Limiting ourselves to $J=4$ $T=0$ states we find the lowest excitation energy is 1.0 MeV and the other 6 states are degenerate at 2.2 MeV. From the quantum numbers $(T,t,v)$ we find that the lowest state has quantum numbers $(0,1,2)$ and the other six have $(0,0,4)$. The first thing that comes to mind is to see if the $T=0$ wave function in Table II matches any of the $v=4$ states. However this will not work because there is a six fold degeneracy here. Any linear combination of these degenerate states is also a $v=4$ state. However

| $J_P, J_N$ | $D(J_P, J_N)$ |
|-----------|-------------|
| 0,4       | 0.6325      |
| 2,2       | 0.1130      |
| 2,4       | -0.0518     |
| 2,6       | -0.1129     |
| 4,4       | -0.0419     |
| 4,6       | 0.0970      |
| 4,8       | 0.0680      |
| 6,6       | -0.1725     |
| 6,8       | 0.2189      |
| 8,8       | -0.0311     |
life is made simple by the fact that there is only one v=2 state. We give the explicit wave function in Table V. We can now take the overlap of the T=0 wave function in Table II with the v=2 wave function in Table V. We find this overlap is zero. This clinches the fact that the T=0 wave function in Table II has seniority v=4 (as does its T=2 partner).

The Q.Q interaction in general does not yield states of good seniority not only for mixed systems but even for identical particles. Here however we have one rare exception—the T=0 partner of the T=2 unique state. We do not get analogous degenerate pairs for J=6+. We did not find any degenerate pairs for 2 protons and 2 neutrons in the 1f_{7/2} shell or the j_{15/2} shell.

We note that most interactions which conserve seniority for particles of one kind do not conserve seniority for configurations containing both valence protons and valence neutrons e.g. the delta interaction. Whereas the delta interaction conserves seniority for say neutrons in the g_{9/2} shell and higher it will not conserve seniority for the 2 proton-2 neutron configurations considered here. In contrast the J=0+ T=1 pairing interaction of Flowers and Edmonds [9,10] will yield states of good seniority for mixed systems.

In summary, with the Q.Q interaction for a configuration of 2 protons and 2 neutrons we find selected degenerate pairs in which one member has isospin T=0 and the other T=2. We have 4 such examples—one in the f_{7/2} shell, two in g_{9/2} and one in h_{11/2}. The most intriguing case involves a J=4+ pair in the g_{9/2} shell in which both members are pure seniority v=4 states. This was anticipated for the T=2 member but is a surprise for the T=0 member.

The important message in this work is that degeneracies can lead to confusions. When a T=0 and T=2 state are degenerate any linear combination of the 2 states is also an eigenstate. The wave functions that appear are each mixtures of T=0 and T=2 and so the isospin information is lost. Another example concerns the unique T=2 v=4 state for J=4+ which was found by Escuderos and Zamick[1]. The reason it took so long to find that such a state existed was that fractional parenage coefficients were generally obtained using the J=0 T=1 pairing interaction of Edmund and Flowers [9,10]. With such an interaction the two T=2 v=4 states are degenerate. Thus an arbitrary linear combination of the states can appear in a given calculation none of which looks like the later to be discovered unique state. Also one needs a seniority violating interaction to find that there is a special state that nevertheless maintains its pure (v=4) seniority. Yes, degeneracies can be confusing and we have here shown what steps have to be taken to handle them.

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5 Appendix

We show in Table VI the two-body matrix elements of Q.Q used in this work. We list them successively from J=0 to J=2j. The even J matrix elements have isospin T=1 whilst the odd ones have T=0.

| J   | f_{7/2} | g_{9/2} | h_{11/2} |
|-----|---------|---------|-----------|
| 0   | -1      | -1      | -1        |
| 1   | -0.8095 | -0.8788 | -0.9161   |
| 2   | -0.4667 | -0.6515 | -0.7554   |
| 3   | -0.0476 | -0.3485 | -0.5325   |
| 4   | 0.3333  | -0.0152 | -0.2687   |
| 5   | 0.5238  | 0.2879  | 0.0070    |
| 6   | 0.3333  | 0.4848  | 0.2587    |
| 7   | -0.4667 | 0.4848  | 0.4434    |
| 8   | 0.1818  | 0.5150  |           |
| 9   | -0.5455 | 0.4026  |           |
| 10  |         | 0.0549  |           |
| 11  |         | -0.0044 |           |
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