Long range statistical fluctuations of the crossed Josephson current

Régis Mélin∗
Centre de Recherches sur les Très Basses Températures (CRTBT†),
CNRS, BP 166, 38042 Grenoble Cedex 9, France

We investigate the crossed Josephson effect in a geometry consisting of a double ferromagnetic bridge between two superconductors, with tunnel interfaces. The crossed Josephson current vanishes on average because the Andreev reflected hole does not follow the same sequence of impurities as the incoming electron. We show that i) the root mean square of the crossed Josephson current distribution is proportional to the square root of the junction area; and ii) the coherent coupling mediated by fluctuations is “long range” since it decays over the ferromagnet phase coherence length $l_\phi$, larger than the exchange length. We predict a crossed Josephson current due to fluctuations if the length of the ferromagnets is smaller than $l_\phi$ and larger than the exchange length $\xi_h$.

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I. INTRODUCTION

Transport properties of hybrid structures consisting of a superconductor (S) multiply connected to several normal metal (N) or ferromagnetic (F) electrodes has focused an important interest recently1,2,3. In usual Andreev reflection at a single NS interface, a spin-up electron coming from the N side is reflected as a hole in the spin-down band while a Cooper pair is transferred in the superconductor. Multiterminal structures allow “non local” processes, in which a spin-up electron in one electrode is Andreev reflected as a hole in the spin-down band in another electrode, corresponding to non local transmission in the electron-hole channel. Conversely non local transmission in the electron-electron channel corresponds to a process in which a spin-up electron from one electrode is transmitted as a spin-up electron in another electrode. Transport theory of three-terminal FSF junctions including non local transmission in the electron-electron and electron-hole channels has been discussed recently4,5,6,7,8,9,10,11,12,13,14,15,16,17,18, in the tunnel limit4,5 and for highly transparent interfaces6,7, in the framework of the scattering approach11,12,13. The models were also extended to describe disorder14,15,16, non collinear ferromagnets17,18, and the noise17,18. On the experimental side, two experiments probing non local transport were carried out recently19,20, in FSF and NSN three-terminal junctions.

The question arises of whether the phase coherence of crossed Andreev reflection can be probed experimentally. We show here that this is possible with a double ferromagnetic bridge between two superconductors (see Fig. 1). We have already shown that the crossed Josephson current vanishes on average in the diffusive limit9, because the Andreev reflected hole does not follow the same sequence of impurities as the incoming electron since they propagate in different electrodes. However, by evaluating the statistical fluctuations of the dc crossed Josephson current, we show here that the fluctuations of the Josephson current decay over the phase coherence length $l_\phi$ in the ferromagnet, larger than the decay length of the local average Josephson current set by the exchange length $\xi_h$ (see Refs. 21,22,23,24,25,26,27,28,29,30). The fluctuations of the crossed supercurrent do not show $\pi$-shift oscillations and damping as a function of the fermilmagnet length, because the spin-up and spin-down electrons of correlated pairs extracted from one superconductor do not see the same realization of disorder, so that the center of mass momentum of the spatially separated correlated pair averages to zero after propagation over a length comparable to the elastic mean free path. The root mean square of the crossed Josephson current is proportional to the square root of the junction area, because the number of diagrams involved in the supercurrent is equal to the junction area divided by the Fermi wave-length. The crossed Josephson supercurrent due to fluctuations can in principle be detected experimentally, provided the length of the ferromagnets is smaller than $l_\phi$ and larger than $\xi_h$.

The article is organized as follows. Preliminaries regarding Green’s functions are given in section III. The analytical results are presented in section IV for the average local supercurrent, and in section V for the statistical fluctuations of the supercurrent. Concluding remarks are given in section VI. Some details on disorder averaging are provided in the Appendix.
FIG. 1: (Color online.) Schematic 3D representation of the Josephson junction considered in the article.

FIG. 2: Schematic 2D cut of the junction on Fig. 1. We have represented some pairs of sites at the interfaces: \((a_k, \alpha_k), (a'_k, \alpha'_k), (b_m, \beta_m), \) and \((b'_m, \beta'_m)\).

II. TECHNICAL PRELIMINARIES

A. The models

The superconductor is described by the BCS Hamiltonian

\[
\mathcal{H}_{\text{BCS}} = \sum_{(\alpha,\beta),\sigma} -t \left( c_{\alpha,\sigma}^+ c_{\beta,\sigma} + c_{\beta,\sigma}^+ c_{\alpha,\sigma} \right) + \Delta \sum_{\alpha} \left( c_{\alpha,\uparrow}^+ c_{\alpha,\downarrow}^+ + c_{\alpha,\downarrow} c_{\alpha,\uparrow} \right),
\]

(1)

where \(t\) is the hopping amplitude, \(\Delta\) is the superconducting gap, and \(\alpha\) and \(\beta\) correspond to neighboring sites on a cubic lattice with a parameter \(a_0\). The lattice parameter \(a_0\) is taken equal to the Fermi wave-length \(\lambda_F\). The
ferromagnetic electrodes are described by the Stoner model

\[ H_{\text{Stoner}} = \sum_{(\alpha,\beta),\sigma} -t \left( c_{\alpha,\sigma}^+ c_{\beta,\sigma} + c_{\beta,\sigma}^+ c_{\alpha,\sigma} \right) - \hbar^2 \sum_{\alpha} \left( c_{\alpha,\uparrow}^+ c_{\alpha,\uparrow} - c_{\alpha,\downarrow}^+ c_{\alpha,\downarrow} \right), \]  

(2)

where \( h_{\text{ex}} \) is the exchange field. The exchange fields in the two ferromagnets are equal in the parallel alignment, and opposite in the antiparallel alignment. We incorporate also disorder scattering, described by the Hamiltonian

\[ H_{\text{dis}} = \sum_{\alpha,\sigma} V_{\alpha,\sigma} c_{\alpha,\sigma,\sigma}^+ c_{\alpha,\sigma,\sigma}, \]

(3)

where the impurities are located at the sites \( \alpha_n \). The impurity scattering potentials \( V_{\alpha_n} \) are random variables. The site \( \alpha_k \) is on the ferromagnetic side of the interface, and the site \( \alpha_k \) on the superconducting side.

The couplings between the ferromagnets and the superconductors are described by the tunnel Hamiltonian. The tunnel Hamiltonian at the interface \((\alpha, \alpha)\) takes the form

\[ W_{\alpha,\alpha} = \sum_{\kappa,\sigma} \left\{ -t_{\alpha,\alpha} c_{\alpha,\sigma,\kappa}^+ c_{\alpha,\sigma,\kappa} + t_{\alpha,\alpha} c_{\alpha,\sigma,\alpha}^+ c_{\alpha,\sigma,\alpha} S \right\}, \]

(4)

where the summation runs over all sites at the interface (see Fig. 2), and where \( t_{\alpha,\alpha} = t_{\alpha,\alpha} \) is the hopping amplitude connecting the sites \( \alpha_k \) and \( \alpha_k \).

**B. Green’s functions of a ferromagnet and a superconductor**

The starting point is the ballistic Green’s function \( \hat{g}_{i,j}(\omega) \) of the isolated ferromagnetic and superconducting electrodes in the Nambu representation. The ballistic Green’s functions of a ferromagnet take the form

\[ \hat{g}_{a,b}^{1,1}(\omega) = \frac{\pi \rho_F}{k_F d_{a,b}} \exp \left( -i \left( k_F^+ \frac{\omega}{\hbar v_F} d_{a,b} \right) \right) \exp \left( -d_{a,b}/l_{\text{ex}}^{(\text{ball})} \right), \]

(5)

\[ \hat{g}_{a,b}^{2,2}(\omega) = \frac{\pi \rho_F}{k_F d_{a,b}} \exp \left[ i \left( k_F^+ - \frac{\omega}{\hbar v_F} d_{a,b} \right) \right] \exp \left( -d_{a,b}/l_{\text{ex}}^{(\text{ball})} \right), \]

(6)

where \( \hat{g}_{a,b}^{1,1} \) and \( \hat{g}_{a,b}^{2,2} \) are the Green’s functions of a spin-up electron and a hole in the spin-down band respectively, both having \( S_z = \frac{1}{2} \), \( d_{a,b} \) is the distance between the sites \( a \) and \( b \), \( \omega \) the energy with respect to the chemical potential, \( \rho_F \) the density of states, \( k_F^+ \) and \( k_F \) the spin-up and spin-down Fermi wave-vectors, \( v_F^+ \) and \( v_F \) the spin-up and spin-down Fermi velocities. The ballistic ferromagnet Green’s functions given by Eqs. (5) and (6) decay exponentially over the phase coherence length \( l_{\text{ex}}^{(\text{ball})} \), introduced phenomenologically through an imaginary part \( h v_F/l_{\text{ex}}^{(\text{ball})} \) to the energy \( \omega \). We note \( k_F \) and \( v_F \) the Fermi wave-vector and the Fermi velocity in the absence of spin polarization. We neglect in the following the energy dependence of the ferromagnet propagators in Eqs. (5) and (6) since we suppose that the length \( R \) of the ferromagnets is small compared \( h v_F/\Delta \) and \( h v_F^+/\Delta \), both length scales being comparable to the ballistic BCS coherence length \( h v_F/\Delta \).

The Nambu Green’s function of a ballistic isolated superconductor in the sector \( S_z = \frac{1}{2} \) takes the form

\[ \hat{g}_{\alpha,\beta}(\omega) = \frac{\pi \rho_S}{k_F d_{\alpha,\beta}} \exp \left( -\frac{d_{\alpha,\beta}}{\xi_{\text{BCS}}(\omega)} \right) \sin \left( k_F d_{\alpha,\beta} \right) \right] \frac{\sin (k_F d_{\alpha,\beta})}{\sqrt{\Delta^2 - \omega^2}} \left[ \begin{array}{c} -\omega \\ \Delta \end{array} \right] \cos (k_F d_{\alpha,\beta}) \left[ \begin{array}{c} -1 \\ 0 \end{array} \right] + \cos (k_F d_{\alpha,\beta}) \left[ \begin{array}{c} -1 \\ 0 \end{array} \right], \]

(7)

where \( \rho_S \) is the normal state density of states of the superconductor, \( d_{\alpha,\beta} \) the distance between the sites \( \alpha \) and \( \beta \), and \( \xi_{\text{BCS}}(\omega) = h v_F/\sqrt{\Delta^2 - \omega^2} \) the BCS coherence length at a finite energy. The information about propagation in the superconductor in the non local Josephson effect is contained in \( f_{\alpha,\beta}(\omega) \equiv g_{\alpha,\beta}(\omega) \), where “1” and “2” refer to the electron and hole Nambu labels respectively. The statistical fluctuations of the Josephson current involve \( \langle f_{\alpha,\beta}(\omega) \rangle \), where the overline is an average over disorder and over the different conduction channels. We have\[22,33\]

\[ \langle f_{\alpha,\beta}(\omega) \rangle = \frac{\pi \rho_S}{k_F^+ d_{\alpha,\beta}} \frac{\Delta^2}{\Delta^2 - \omega^2} \left( \frac{1}{\xi_{\text{BCS}}(\omega)} \right), \]

(8)

where \( \xi_{\text{BCS}}(\omega) \) is the diffusive limit superconducting coherence length.
C. Supercurrent

The fully dressed Green’s functions $\hat{G}_{i,j}(\omega)$ are obtained from the Dyson equation $\hat{G}(\omega) = \hat{g}(\omega) + \hat{g}(\omega) \otimes \hat{\Sigma} \otimes \hat{G}(\omega)$, where $\otimes$ corresponds to a summation over all the sites in the tunnel Hamiltonian $H$. The self-energy is provided by the couplings of the tunnel Hamiltonian, that, in the Nambu representation, take the form

$$t_{a,\alpha} = \begin{bmatrix} t_a \exp{(i\varphi/4)} & 0 \\ 0 & -t_a \exp{(-i\varphi/4)} \end{bmatrix},$$

where $\varphi$ is the phase difference between the two superconductors and $t_a$ is a real number. The phase does not evolve in time since we restrict here to the dc-Josephson effect. The second order diagrams on Fig. 3 contributing the supercurrent acquire a phase $\exp{(\pm i\varphi)}$, giving rise to a supercurrent proportional to $\sin \varphi$. The equilibrium supercurrent through electrode “a” is given by

$$I_S = \frac{e}{\hbar} \int_0^{+\infty} \text{Tr} \left\{ \hat{\sigma}^z \left[ \hat{t}_{a,\alpha} \left( \hat{G}_{a,\alpha}^A(\omega) - \hat{G}_{a,\alpha}^R(\omega) \right) - \hat{t}_{a,\alpha} \left( \hat{G}_{a,\alpha}^A(\omega) - \hat{G}_{a,\alpha}^R(\omega) \right) \right] \right\} d\omega + (h_{ex} \rightarrow -h_{ex}),$$

where the trace is a summation over the Nambu labels and the different conduction channels. The term $(h_{ex} \rightarrow -h_{ex})$ corresponds to the contribution in the sector $S_z = -1/2$. Eq. (10) can be demonstrated from the expression of the current in terms of the Keldysh Green’s function. The transparency of a single junction in the normal state is proportional to $(t/\epsilon_F)^2$, where $\epsilon_F$ is the Fermi energy. We suppose here that $t \ll \epsilon_F$, so that the supercurrent is expanded to order $t^4\rho_0^2\rho_\Delta^2$. The tunnel supercurrent coupling coherently the two ferromagnets is provided by the emission of a correlated pair of electrons from the left superconductor by Andreev reflection, followed by the absorption of the correlated pair by an Andreev reflection at the right superconductor. These two Andreev reflections can be “local” or “non local” in the sense that the incoming electron and outgoing hole can propagate in identical or in different electrodes.

The local term $I_S^{(loc)}$ in the supercurrent involves a diagram with propagation in a single ferromagnet (see Fig. 3(a)), such that the incoming electron and the Andreev reflected hole are scattered by the same sequence of impurities:

$$I_S^{(loc)} = 4\pi \frac{e}{\hbar} \Delta |t_{a,\alpha}|^2 |t_{b,\beta}|^2 (\pi \rho_S)^2 \sin \varphi \Re \sum_{a,b} g_{a,b}^{1,1,A}(\Delta) g_{b,a}^{1,2,A}(\Delta) + (h_{ex} \rightarrow -h_{ex}),$$

where the overline is a disorder averaging, and where the sites $a$ and $b$ belong to the left and right interfaces respectively (see Fig. 3(a)). To obtain Eq. (11), we start from Eq. (10), use several times the Dyson equation, and replace the Green’s functions of the isolated ferromagnets and superconductors by Eqs. (5), (6), and (7). The integral over energy is then calculated by contour integration. The poles of the products of the anomalous Green’s functions are at $\omega = \Delta$, so that the ferromagnet Green’s functions are evaluated at $\omega = \Delta$ in Eq. (11).

The spectral “non local” supercurrent involves a diagram with propagation in both ferromagnets (see Fig. 3(b)):

$$I_S^{(nonloc)}(\omega) = 2\pi \frac{e}{\hbar} \Delta |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \sin \varphi \Re \sum_{a,b,a',b'} \{ f_{a,\beta}(\omega) f_{a',\beta'}(\omega) \}.$$
\[
\begin{align*}
&\times \left[ g_{a,b}^{\uparrow,1,A}(\omega)g_{b',a'}^{\uparrow,2,A}(\omega) + g_{a,b}^{\uparrow,2,A}(\omega)g_{b',a'}^{\uparrow,1,A}(\omega) \right] \right) \right) \\
\end{align*}
\]
(12)

\[ (h_{ex} \rightarrow -h_{ex}), \]  
(13)

where the sites \( a, b, a', \) and \( b' \) belong to different interfaces (see Fig. 3(b)). The variance of the nonlocal supercurrent is obtained by integrating the square of the spectral supercurrent given by Eq. (13) over energy and averaging over disorder:

\[
\left( I_s^{\text{(nonloc)}} \right)^2 = \int d\omega \left( I_s^{\text{(nonloc)}}(\omega) \right)^2.
\]  
(14)

### III. EVALUATION OF THE AVERAGE LOCAL SUPERCURRENT

The average over disorder of the product \( g_{a,b}^{\uparrow,1,A} g_{b,a}^{\uparrow,2,A} \) in Eq. (11), is evaluated in the ladder approximation in the Appendix. We find

\[
\begin{align*}
\langle g_{a,b}^{\uparrow,1,A} g_{b,a}^{\uparrow,2,A} \rangle = & -\left( \frac{\pi \rho_F}{k_F^2 l_d a_{d,a,b}} \right)^2 \exp (iK_h d_{a,b}) \exp (-d_{a,b}/\xi_h), \\
\end{align*}
\]  
(15)

where \( d_{a,b} \) is the distance between the sites “a” and “b”. The decay length of the supercurrent of a single SFS junction in the diffusive limit is given by

\[
\begin{align*}
\xi_h = \sqrt{\frac{3}{2l_d}} \sqrt{\left( \frac{2}{l_F^{\text{(ball)}}} \right)^2 + (\Delta k)^2 + \frac{2}{l_F^{\text{(ball)}}}},
\end{align*}
\]  
(16)

where the exchange field enters through \( \Delta k = k_F^+ - k_F^- \), equal to the difference between the spin-up and spin-down Fermi wave vectors, with \( \Delta k = 2h/v_F \). The wave vector of the supercurrent oscillations is given by

\[
\begin{align*}
K_h = \sqrt{\frac{3}{2l_d}} \sqrt{\left( \frac{2}{l_F^{\text{(ball)}}} \right)^2 + (\Delta k)^2 - \frac{2}{l_F^{\text{(ball)}}}},
\end{align*}
\]  
(17)

where \( l_d \) is the elastic mean free path. Due to the exchange field, the decay length \( \xi_h \) given by Eq. (16) is smaller than the phase coherence length

\[
\begin{align*}
l_F = \sqrt{\frac{l_d l_F^{\text{(ball)}}}{3}}.
\end{align*}
\]  
(18)

Eqs. (16) and (17) follow from the identity

\[
\begin{align*}
\frac{1}{\xi_h} + iK_h = \frac{3}{l_d} \left( \frac{2}{l_F^{\text{(ball)}}} + i\Delta k \right).
\end{align*}
\]  
(19)

demonstrated in the Appendix.

After summing over all pairs of sites \((a, b)\) at the interface, as detailed in the Appendix, we obtain

\[
I_s^{\text{(loc)}} = 8\pi e \frac{N_{ch}}{h} \Delta |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \frac{\pi^2 \rho^2_F}{k_F^2 l_d a_0^2} \frac{1}{(1/\xi_h)^2 + K_h^2} \exp (-R/\xi_h) \cos (K_h R - \theta),
\]  
(20)

with \( \tan \theta = K_h/\xi_h \). The local supercurrent is then proportional to the number of channels \( N_{ch} \) at a single interface, \( N_{ch} \) being itself proportional to the area of a single junction.
IV. STATISTICAL FLUCTUATIONS OF THE NON LOCAL SUPERCURRENT

The average of the non local supercurrent vanishes because of disorder averaging in the diffusive system, or because of averaging over the Fermi oscillations in the ballistic system. The disorder average of the square of the non local supercurrent \((I_S^{(nonloc)})^2\) involves the diagrams on Fig. 3(c), the average over disorder of which does not decay exponentially over the elastic mean free path \(l_d\).

The non local supercurrent \((I_S^{(nonloc)})\) can be recast in the form

\[
I_S^{(nonloc)} = A \sum_{a,b,a',b'} \sum_{\sigma,\tau} \left(X^{(\sigma,\tau),A}_{a,b,a',b'} + X^{(\sigma,\tau),R}_{a,b,a',b'}\right),
\]

where \(A = \pi(e/h)\Delta|t_{a,\alpha}|^2|t_{b,\beta}|^2\), and \(X^{(\sigma,\tau)}_{a,b,a',b'} = f_{a,\alpha}f_{a',\beta}g_{b,\beta}g_{b',\alpha}\), and where \(\sigma = \uparrow, \downarrow\) is the projection of the spin along the \(z\) axis, and \(\tau = 1,2\) is the Nambu index. The notation \(\bar{\tau}\) in the definition of \(X^{(\sigma,\tau)}_{a,b,a',b'}\) corresponds to \(\tau = 1\) if \(\tau = 2\), and \(\bar{\tau} = 2\) if \(\tau = 1\). We deduce from Eq. (21)

\[
\left(I_S^{(nonloc)}\right)^2 = 2A^2 \sum_{a,b,a',b'} \sum_{\sigma,\tau} \sum_{\sigma',\tau'} \text{Re} \left[X^{(\sigma,\tau),A}_{a,b,a',b'} X^{(\sigma',\tau'),A}_{a,b,a',b'} + X^{(\sigma,\tau),A}_{a,b,a',b'} X^{(\sigma',\tau'),R}_{a,b,a',b'}\right],
\]

corresponding to the diagrams on Fig. 3(c). The disorder averages in each electrode are then carried out, following the Appendix. Factoring out the propagators in the superconductor, and carrying out the summation over the conduction channels, leads to

\[
\left(I_S^{(nonloc)}\right)^2 = 4\pi^2(e/h)^2\Delta^2|t_{a,\alpha}|^4|t_{b,\beta}|^4N_{ch}^2 \sum_{\sigma,\tau} \sum_{\sigma',\tau'} \left[\frac{(\pi\rho_d)^2l_{\varphi}}{(k_Fa_0)^2l_d}\right]^2 \times \left\{\exp\left(-\frac{2R}{l_{\varphi}}\right) + \frac{1}{\sqrt{1 + (l_{\varphi}(k^- - k^+))^2}} \exp\left(-\frac{2R}{\xi_h}\right)\right\} \sin^2 \varphi,
\]

where we discarded the terms decaying exponentially over the Fermi wave-length. The product \(\sum_{\sigma,\tau} \sum_{\sigma',\tau'} \left[\frac{(\pi\rho_d)^2l_{\varphi}}{(k_Fa_0)^2l_d}\right]^2\) is proportional to \(1/D^2\), with \(D^2\) of order \(N_{ch}a_0^2(D/\tau)^2\), so that \(\left(I_S^{(nonloc)}\right)^2\) scales like \(N_{ch}\) (see Fig. 2 for the notations \(D\) and \(\tau\)). The variance of the supercurrent given by Eq. (23) involves a “long range” contribution decaying over \(l_{\varphi}\), and a short range contribution decaying over \(\xi_h\). The former propagates over a much larger distance than the latter. Both contributions are identical for normal metals, but the long range contribution dominates for ferromagnets with \(\xi_h \lesssim R \lesssim l_{\varphi}\).

V. CONCLUSIONS

To conclude, we have investigated the possibility of coupling coherently two superconductors by two spatially separated ferromagnets. The statistical fluctuations of the Josephson current are proportional to the square root of the surface of the junctions. The fluctuating part of the Josephson current is “long range” in the sense that it does not decay over the exchange length \(\xi_h\), but decays over the phase coherence length \(l_{\varphi}\). We predict a Josephson current mediated by fluctuations if the length of the ferromagnets is larger than \(\xi_h\), and smaller than \(l_{\varphi}\). The supercurrent is expected to fluctuate as a function of the relative spin orientation of the ferromagnets. This effect can be used as a test of the phase coherence of crossed Andreev reflection, without the competition between the crossed Andreev reflection and elastic cotunneling channels since the Josephson effect probes solely the anomalous propagator in the superconductor.

Another proposal has been made recently to probe a long range Josephson effect in ferromagnets with non collinear magnetizations in a single SFS junction, generating triplet correlations that can also propagate up to \(l_{\varphi}\). This effect is not equivalent to the one considered here since it involves propagation in a single electrode. The fluctuations of the Josephson current discussed here are also not equivalent to universal conductance fluctuations since the root mean square of the supercurrent distribution is proportional to the square root of the junction area.

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APPENDIX A: DETAILS OF DISORDER AVERAGING

1. Elastic scattering time

The Green’s function \( \hat{G}^{(0)}(k, \omega) \) of a disordered isolated ferromagnet, diagonal in the Nambu representation, is given by \( \hat{G}^{(0)}(k, \omega) = \hat{g}(k, \omega) \left[ \hat{I} + \hat{\Sigma}_v \hat{G}^{(0)}(k, \omega) \right] \), where \( \hat{g}(k, \omega) \) is the Green’s function of a ballistic isolated ferromagnet, and

\[
\hat{G}^{(0)}(k, \omega) = \hat{g}(k, \omega) \left[ \hat{I} + \hat{\Sigma}_v \hat{g}(k, \omega) \right],
\]

is the disorder self-energy, where \( n \) is the concentration of impurities. Evaluating the integral in Eq. (A1) by contour integration leads to

\[
\hat{G}^{(0), \uparrow, 1, A}(k, \omega) = \frac{1}{\omega - h - \xi_k - i/\tau_{1,1}},
\]

with

\[
\tau_{1,1} = \frac{\tau}{1 + \omega/2\varepsilon_F - h/2\varepsilon_F},
\]

and

\[
\hat{G}^{(0), \uparrow, 2, A}(k, \omega) = \frac{1}{\omega - h + \xi_k - i/\tau_{2,2}},
\]

with

\[
\tau_{2,2} = \frac{\tau}{1 - \omega/2\varepsilon_F + h/2\varepsilon_F},
\]

with \( \tau = 4\pi\varepsilon_F/(k_F^2 v^2) \) the elastic scattering time with \( h = \omega = 0 \).

2. Disorder averaging of the supercurrent

The supercurrent involves the disorder average \( g_{a,b}^{\uparrow, 1, A}(\omega) g_{b,a}^{\uparrow, 2, A}(\omega) \), evaluated in Fourier space in the ladder approximation. Using contour integration, we find

\[
\int \frac{d\mathbf{k}}{(2\pi)^3} \hat{G}_{\text{diff}}^{\uparrow, 1, A}(k, \omega) \hat{G}_{\text{diff}}^{\uparrow, 2, A}(k + \mathbf{q}, \omega) \simeq -1 + i\tau(\omega - h + iv_F/\xi_{\text{ball}}) + \frac{q^2 \tau^2 \varepsilon_F^2}{3k_F^2}. \tag{A6}
\]

After the summing the ladder diagrams, the poles are found at wave-vector

\[
\mathbf{q}^{\text{(diff)}} = \frac{6}{l_d} \left( \frac{1}{l_{\text{ball}}} \frac{1}{v_F} + \frac{i}{\xi_{\text{ball}}} \right), \tag{A7}
\]

where we replaced \( \omega \) by \( \Delta \), as obtained in the energy integration of the local supercurrent [see Eq. (11)]. Eq. (A7) leads directly to Eq. (19) if one assumes the ballistic superconducting coherence length \( \xi_{\text{ball}} \) of the order of 1 \( \mu \)m to be large compared to the ballistic exchange length \( v_F/l_{\varphi} \) and to the ferromagnet ballistic phase coherence length \( l_{\varphi}^{\text{ball}} \). The geometrical prefactor \( (\pi\rho_F)^2/k_F^2 l_{\varphi} d_{a,b} \) is then obtained by evaluating the residue in the integral over \( \mathbf{q} \).

3. Summation over the conduction channels

The supercurrent given by Eq. (11) involves a summation over the conduction channels. This summation is evaluated through

\[
\sum_{a,b} \frac{a_0}{d_{a,b}} \exp \left[ -\left( \frac{1}{\xi} - iK_h \right) d_{a,b} \right] \simeq N_{\text{ch}} \int \frac{2\pi y dy}{a_0\sqrt{R^2 + y^2}} \exp \left[ -\left( \frac{1}{\xi} - iK_h \right) \sqrt{R^2 + y^2} \right]
\]

\[
= N_{\text{ch}} \frac{\xi/a_0}{1 - iK_h \xi} \exp \left[ -\left( \frac{1}{\xi} - iK_h \right) R \right], \tag{A8}
\]

where \( d_{a,b} = \sqrt{R^2 + y^2} \) is the distance between the sites \( a \) and \( b \).
4. Disorder averaging from real space Green’s functions

The product of the ballistic Green’s functions $g_{a,b}^{+,1,A}(\omega)g_{b,a}^{+,2,A}(\omega)$ is proportional to $\exp[-q^{(\text{ball})}d_{a,b}]$, with $q^{(\text{ball})} = i\Delta k + 2\omega/v_F + 2/l_{F}^{(\text{ball})}$. The wave-vector $q^{(\text{diff})}$ given by Eq. (A7) is related to $q^{(\text{ball})}$ according to

$$q^{(\text{diff})} = \sqrt{\frac{3q^{(\text{ball})}}{l_d}} \quad \text{(A9)}$$

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