INFINITELY MANY RADIAL SOLUTIONS FOR A SUPER-CUBIC KIRCHHOFF TYPE PROBLEM IN A BALL

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Abstract. We prove the existence of infinitely many radial solutions to a Kirchhoff type problem in a ball with a super-cubic nonlinearity. Our methods rely on bifurcation analysis and energy estimates.

1. Introduction. We consider the Kirchhoff type problem

\(- \left( a + b \int_{\Omega} |\nabla u|^2 \,dx \right) \Delta u = f(u) \text{ in } \Omega, \]
\(u = 0 \text{ on } \partial \Omega \)

where \(a > 0, b > 0, \Omega \) is the unit ball in \(\mathbb{R}^N, N = 1,2,3\) and

\[ f(u) = \begin{cases} u^p + u & \text{if } u \geq 0, \\ -|u|^q + u & \text{if } u < 0, \end{cases} \]

with

\[ 3 < p, q < \frac{N+2}{N-2} < \infty. \]

That is, \(f\) has super-cubic and subcritical growth. Let \(\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty\) denote the eigenvalues of

\[ u'' + \frac{N-1}{r} u' + \lambda u = 0 \quad r \in (0,1], \]

\[ u'(0) = u(1) = 0. \]

Our main result is the following theorem.

**Theorem 1.1.** If \(f\) satisfies (1.2)-(1.3), then for each \(k\) with \(a\lambda_k > 1\) the equation (1.1) has two radial solutions with \(k\) nodal sets. In particular, (1.1) has infinitely many radial solutions.

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Remark 1.1. It is easily verified that the hypotheses in Theorem 1.1 may be weakened to
\[ f \text{ is increasing, } a\lambda_k > f'(0) > 0, 3 < p, q < \frac{N + 2}{N - 2}, \]
\[ \lim_{u \to \pm \infty} \frac{f(u)}{u^p} > 0, \text{ and } \lim_{u \to \pm \infty} \frac{f(u)}{-|u|^q} > 0. \]

Remark 1.2. Unlike other results on the existence of infinitely many solutions, we do not assume \( f \) to be odd. Our assumption \( f'(0) > 0 \) plays a crucial role in the bifurcation analysis of radial solutions to be used in this paper.

For \( \Omega \) a general smooth bounded domain, the existence and multiplicity of solutions for (1.1) has been extensively studied. In [14], Perera and Zhang proved the existence of a nontrivial solution using the Yang index and critical groups for \( f \) asymptotically cubic and not resonant with the nonlinear spectrum. In [24], they revisited (1.1) via invariant sets of descent flow and found the existence of a positive, a negative solution and a sign changing solution for \( f \) sub-cubic, asymptotically cubic and super-cubic. In [16], Song, Tang and Chen proved the existence of three solutions for \( f \) nearly resonant to the first nonlinear eigenvalue from below based on Ekeland’s variational principle and the mountain pass lemma. In [15] the same authors proved the existence of solutions for \( f \) resonant to higher nonlinear eigenvalues.

The existence of infinitely many solutions for problem (1.1) in general bounded domains and \( f \) odd can be found in [8, 18, 22, 21]. In [21] the existence of infinitely many sign-changing solutions was proved using a combination of invariant sets of descent flow and Ljusternik-Schnirelman type minimax method for \( f(u) = |u|^{p-2}u, p \in (2, 2^*) \). In [8] infinitely many large energy solutions were found via the fountain theorem under Ambrosetti-Rabinowitz’s 4-super quadratic condition or general 4-super quadratic at infinity with the global monotonicity condition:
\[ \frac{f(x,t)}{t^3} \text{ is an increasing function of } t \geq 0 \text{ for every } x \in \Omega. \] (1.5)

These results were extended in [18, 22]. In [18], (1.5) was replaced by the following condition:

There exists \( \theta \geq 1 \) such that \( \theta G(x,t) \geq G(x,st) \) for all \((x,t) \in \Omega \times \mathbb{R} \) and \( s \in [0,1] \),

where \( G(x,t) = f(x,t)t - 4F(x,t) \) and \( F(x,t) = \int_{0}^{1} f(x,s)ds \). In [22], Ye replaced (1.5) with

there exists \( r > 0 \) such that for all \( x \in \Omega \), \( \frac{f(x,t)}{t^3} \) is increasing in \( t \geq r \).

For \( \Omega = \mathbb{R}^N \) and \( f \) odd, the problem
\[- \left( a + b \int_{\Omega} |
abla u|^2 dx \right) \Delta u = V(x)u + f(u) \quad \text{in } \mathbb{R}^N, \]
\[ u \to 0 \quad \text{as } x \to \infty. \]

has been extensively studied. See, [2, 6, 9, 10, 13, 1, 5, 11, 17, 19, 23, 7, 12, 20, 25].

Since the Sobolev embedding \( H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)(2 \leq s \leq 2^*) \) is not compact, it is usually difficult to prove the Palais-Smale condition for the problem in \( \mathbb{R}^N \). In order to overcome this difficulty, some conditions have been imposed on the potential function \( V \). For \( V \) constant or radial, see [2, 6, 9, 10, 13]; for \( V \) bounded
from below, see [2, 1, 5, 11, 17, 19, 23] and for V’s such that Palais-Smale sequences converge while the the corresponding Sobolev embedding may not be compact, see [7, 12, 20, 25]. It is worth pointing out that assuming $f$ to be odd is a key ingredient in the aforementioned references on the existence of infinitely many solutions for (1.1) for both $\Omega$ bounded and $\mathbb{R}^N$.

We base our arguments on the fact that if $u$ is a solution to the singular ordinary differential equation

$$u_{rr} + \frac{N-1}{r} u_r + \lambda f(u(r)) = 0 \quad r \in (0,1]$$

$$u'(0) = u(1) = 0$$

with

$$\lambda = \frac{1}{a + b \int_0^1 r^{N-1}(u'(r))^2 dr},$$

i.e., $\lambda(a + b \int_0^1 r^{N-1}(u'(r))^2 dr) = 1,$

then $u$ is a solution to (1.1). We investigate the solutions to (1.6) by considering the initial value problem

$$u_{rr} + \frac{N-1}{r} u_r + \lambda f(u(r)) = 0 \quad r \in (0,1]$$

$$u(0) = d, \; u'(0) = 0$$

and the bifurcation properties of (1.6).

2. Bifurcation analysis of radial solutions. Since each eigenvalue $\lambda_k$ of (1.4) is simple, $f(0) = 0, f'(0) = 1$, by Theorem 1.7 of [4], for each positive integer $k$ there exists a continuum of solutions to (1.8) bifurcating from $(\lambda_k,0)$ with $u(0) > 0$. Let such a continuum be $\Gamma_k$. By uniqueness of solutions to initial value problems, if $(\lambda, u) \in \Gamma_k$ then $u'(x) \neq 0$ for $u(x) = 0$. This and the connectedness of $\Gamma_k$ imply that if $(\lambda, u) \in \Gamma_k$ then $u$ has exactly $k$ zeros in $(0,1]$. Hence $\Gamma_k \cap \Gamma_j$ is empty for $k \neq j$. Thus, by global bifurcation theory (see Theorem 8.2, [3]), $\Gamma_k$ is unbounded. Since $p < (N+2)/(N-2)$, a priori estimates for elliptic equations imply that if $\{(\lambda_j, u_j)\}_j$ is a sequence in $\Gamma_k$ and $\{\|u_j\|\}_j$ converges to $+\infty$ then $\lambda_j$ converges to zero. Similarly, for each positive integer $k$, there exists an unbounded continuum $\tilde{\Gamma}_k$ of solutions to (1.8) bifurcating from $(\lambda_k,0)$ with $u(0) < 0$. Figure 1 below provides a sketch of the above analysis.

3. Proof of main result. Let $(\lambda, u)$ be a solution to (1.8) and

$$E(r) = \frac{(u'(r))^2}{2} + \lambda F(u(r))$$

be the energy function associated with problem (1.8). Multiplying (1.8) by $r^{N-1} u$ and integrating on $[s, t]$, then multiplying the same equation by $r^N u'$ and integrating also on $[s, t]$, one has

$$t^{N-1} H(t) - s^{N-1} H(s) = \int_s^t \lambda r^{N-1} \left( NF(u(r)) - \frac{N-2}{2} u(r)f(u(r)) \right) dr$$

where $F(u) = \int_0^u f(s) ds$ and $H(x) = x E(x) + \frac{N-2}{2} u^2(x) u(x)$. Identity (3.1) is known as a Pohozaev’s identity. Since $E''(r) = -\frac{N-1}{r} (u'(r))^2 \leq 0$, one has

$$E(r) \geq E(1) \quad \text{for all} \quad 0 < r \leq 1.$$
For $r \in [\frac{1}{2}, 1]$, we have

$$E'(r) \geq -\frac{N - 1}{2}(u'(r))^2 \geq -4(N - 1)E(r),$$

that is

$$\frac{E'(r)}{E(r)} \geq -4(N - 1).$$

Integrating (3.2) on $[r, 1]$ we have

$$\ln \frac{E(1)}{E(r)} \geq -4(N - 1)(1 - r) \geq -2(N - 1),$$

which implies

$$E(r) \leq E(1)e^{2(N - 1)}.$$  \hspace{1cm} (3.3)

Therefore,

$$F(u(r)) \leq \lambda^{-1}E(r) \leq \lambda^{-1}e^{2(N - 1)}\left(\frac{u'(1))^2}{2}\right).$$

Let $\rho = \max_{r \in [\frac{1}{2}, 1]} u(r)$. From (3.4) we infer

$$\rho \leq \left[\frac{p + 1}{2}e^{2(N - 1)}\right]^{\frac{1}{p+1}} \frac{|u'(1)|}{\lambda^{\frac{1}{p+1}}}. \hspace{1cm} (3.5)$$

**Lemma 3.1.** Let $k$ be a positive integer, $\lambda \in (0, 1)$ and $|\sigma| > 2$ be such that

$$\lambda|\sigma|^{p-1} \geq (2^{\frac{p+10}{2}}k)^{p+1}(p + 1)e^{(p+1)(N - 1)}.$$  \hspace{1cm} (3.6)
Let \((\lambda, u)\) satisfy
\[
\frac{N-1}{r} u_r + N u_{rr} + \lambda f(u(r)) = 0 \quad r \in (0, 1)
\]
\[
u(1) = 0, u'(1) = \sigma.
\]
If \(u(t_0) = 0\) for some \(t_0 \in \left[\frac{1}{2}, \frac{1}{3} \cdot 1\right]\) and \(u'(t_0) < 0\) then there exists \(t_1 \in [t_0 - \frac{1}{3\varepsilon}, t_0]\) such that \(u(t_1) = 0\) and \(u > 0\) on \([t_1, t_0]\).

Proof. Let
\[
r_1 := \inf \{r > 0; |u'(s)| \geq \frac{|u'(t_0)|}{2} \text{ for all } s \in [r, t_0]\},
\]
and
\[
s_1 := \inf \{r > 0; u'(s) \leq 0 \text{ for all } s \in [r, s_1]\}.
\]
Without loss of generality we may assume that \(\sigma < 0\). Since \(E(t_0) \geq E(1), |u'(t_0)| \geq |\sigma|\). From the definition of \(r_1\), \(-u'(r) \geq \frac{2}{\sigma}\) for all \(r \in [r_1, t_0]\). Integrating on \([r_1, t_0]\), we obtain
\[
-u(r_1) = \int_{r_1}^{t_0} u'(s)ds \leq \int_{r_1}^{t_0} \frac{\sigma}{2} ds = \frac{\sigma}{2}(t_0 - r_1).
\]
Therefore, combining (3.5), (3.6) and (3.7), one has
\[
t_0 - r_1 \leq \frac{2u(r_1)}{-\sigma} \leq \frac{2}{\left[p + \frac{1}{2} e^{2(N-1)}\right]^{\frac{2}{p+1}}} \left(\frac{|\sigma|^{1-p}}{\lambda}\right)^{\frac{1}{p+1}} \leq \frac{1}{16\lambda}.
\]
Since \(E(t_0) \leq E(r_1), u'(r_1) = \frac{u'(t_0)}{2}, u(r_1) > 0, \) and \(|\sigma| > 2\), we have
\[
\frac{|u'(t_0)|^2}{2} \leq \left(\frac{u'(r_1)}{2}\right)^2 + \lambda F(u(r_1))
\]
\[
\leq \frac{|u'(t_0)|^2}{8} + \lambda \left[\frac{|u(r_1)|^{p+1}}{p+1} + \frac{|u(r_1)|^2}{2}\right]
\]
\[
\leq \frac{|u'(t_0)|^2}{8} + \lambda \left[\frac{1}{p+1} + \frac{1}{2}\right] |u(r_1)|^{p+1}
\]
\[
\leq \frac{|u'(t_0)|^2}{8} + \frac{3}{4} \lambda |u(r_1)|^{p+1}.
\]
Hence
\[
u(r_1) \geq \left(\frac{\sigma^2}{2\lambda}\right)^{\frac{1}{p+1}}.
\]
Let \(r \in [s_1, r_1]\). It follows from (3.9) that \(|u'(t_0)| \leq |\sigma|^{N-1}\) and then \(u'(r_1) = \frac{u'(t_0)}{2} \geq \frac{\sigma^{N-1}}{2}\). Multiplying (1.6) by \(r^{N-1}\) and integrating on \([r, r_1]\), one has
\[
0 \geq r^{N-1} u'(r) = r^{N-1} u'(r) + \int_r^{r_1} \lambda s^{N-1} f(u(s))ds
\]
\[
\geq \frac{1}{2} r_1^{N-1} \sigma e^{N-1} + \lambda \int_r^{r_1} s^{N-1} u^p(r_1)ds
\]
\[
\geq \frac{1}{2} r_1^{N-1} \sigma e^{N-1} + \lambda \left(r_1^{N} - r^{N}\right) u^p(r_1)
\]
\[
\geq \frac{1}{2} r_1^{N-1} \sigma e^{N-1} + \lambda \left(r_1^{N-2} r + \cdots + r^{N-1}\right) u^p(r_1).
\]
Thus, by inequality above and (3.9) we have
\[ r_1 - r \leq \frac{1}{2} \sigma e^{N-1} \frac{r_1^{N-1}}{r_1^{N-2} + \ldots + r^{N-1}} \frac{N}{\lambda} u^{-p}(r_1) \]
\[ \leq N e^{N-1} \left( \frac{|\sigma|^{1-p}}{\lambda} \right)^{\frac{1}{p+p-1}} \leq N e^{N-1} \frac{1}{2^{\frac{p+10}{p}} k} \left[ \frac{1}{(p+1)e^{(p+1)(N-1)}} \right]^{\frac{1}{p+p-1}} \]
\[ \leq \frac{1}{16k}. \]

In particular, taking \( r = s_1 \) in inequality above we have
\[ r_1 - s_1 \leq \frac{1}{16k}. \quad (3.10) \]

Let
\[ \hat{r}_1 := \inf \{ r > 0 ; u(s) \geq \frac{u(s_1)}{2} \text{ for all } s \in [r, s_1] \} \]
and
\[ t_1 := \inf \{ r > 0 ; u(s) \geq 0 \text{ for all } s \in [r, \hat{r}_1] \}. \]

Let \( r \in [\hat{r}_1, s_1] \). Multiplying (1.6) by \( r^{N-1} \) and integrating on \([r, s_1]\), one has
\[ r^{N-1} u'(r) = \int_r^{s_1} \lambda s^{N-1} f(u(s)) ds \geq \frac{\lambda}{N} (s_1^{N-1} - r^{N-1}) u^p(r), \]
which yields
\[ s_1 - r \leq \frac{1}{\lambda} u'(r) \frac{u'(r)}{u^p(r)}. \]

Then, integrating on \([\hat{r}_1, s_1]\), one has
\[ \frac{(s_1 - \hat{r}_1)^2}{2} \leq \frac{1}{\lambda(1-p)} (u_1^{1-p}(s_1) - u_1^{1-p}(\hat{r}_1)) = \frac{(2^{p-1} - 1)}{\lambda(p-1)} u_1^{1-p}(s_1). \]

Combining the inequality above, (3.6), (3.9) and the fact \( u(s_1) \geq u(r_1) \), we have
\[ s_1 - \hat{r}_1 \leq \left[ \frac{2(2^{p-1} - 1)}{p-1} \right]^{\frac{1}{2}} \left( \frac{1}{\lambda} \right)^{\frac{1}{2}} \left( \frac{\sigma^2}{2\lambda} \right)^{\frac{1}{p+p-1}} \]
\[ \leq \left[ \frac{2(2^{p-1} - 1)}{p-1} \right]^{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{p+p-1}} \left( \frac{|\sigma|^{1-p}}{\lambda} \right)^{\frac{1}{p+p-1}} \]
\[ \leq \frac{2^{\frac{5}{2}}}{2^{\frac{p+10}{p}} k} \left[ \frac{1}{(p+1)e^{(p+1)(N-1)}} \right]^{\frac{1}{p+p-1}} \leq \frac{1}{16k}. \quad (3.11) \]

For \( r \in [t_1, \hat{r}_1] \). It follows from \( E(r) \geq E(s_1) \) that
\[ \frac{[u'(r)]^2}{2} + \frac{\lambda}{p+1} u^{p+1}(r) + \frac{\lambda}{2} u^2(r) \geq \frac{\lambda}{p+1} u^{p+1}(s_1) + \frac{\lambda}{2} u^2(s_1), \]
which yields
\[ \frac{[u'(r)]^2}{2} \geq \left( \frac{\lambda}{p+1} u^{p+1}(s_1) - \frac{\lambda}{p+1} u^{p+1}(\hat{r}_1) \right) + \left( \frac{\lambda}{2} u^2(s_1) - \frac{\lambda}{2} u^2(\hat{r}_1) \right) \]
\[ \geq \frac{\lambda}{p+1} \left( u^{p+1}(s_1) - \left( \frac{u(s_1)}{2} \right)^{p+1} \right). \]
Hence,
\[ u'(r) \geq \left( \frac{2}{p+1} \left( 1 - \frac{1}{2^p+1} \right) \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}} u^{\frac{p+1}{2}}(s_1). \]

It follows from the inequality above that
\[ u(\hat{r}_1) \geq \int_{r}^{\hat{r}_1} u'(s) ds \geq \left( \frac{2}{p+1} \left( 1 - \frac{1}{2^p+1} \right) \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}} u^{\frac{p+1}{2}}(s_1)(\hat{r}_1 - r). \tag{3.12} \]

It follows from \( u(s_1) \geq u(r_1) \) and (3.9) that
\[ u(s_1) \geq \left( \frac{\sigma^2}{2\lambda} \right)^{\frac{1}{p+1}}. \tag{3.13} \]

Combining (3.6), (3.12) and (3.13), one has
\[ \hat{r}_1 - r \leq \frac{1}{2} \left( \frac{2}{p+1} \left( 1 - \frac{1}{2^p+1} \right) \right)^{-\frac{1}{2}} \left( \frac{1}{\lambda} \right)^{\frac{1}{2}} u^{\frac{1-p}{2}}(s_1) \]
\[ \leq \left( \frac{2}{p+1} \left( 1 - \frac{1}{2^p+1} \right) \right)^{-\frac{1}{2}} \left( \frac{|\sigma|^{1-p}}{\lambda} \right)^{\frac{1}{p+1}} \leq \frac{1}{16k} \leq \frac{1}{16k}. \tag{3.14} \]

Thus, taking \( r = t_1 \) in inequality above we have
\[ \hat{r}_1 - t_1 \leq \frac{1}{16k}. \]

Thus, from (3.8), (3.10), (3.11), (3.14), we infer that \( t_0 - t_1 \leq \frac{1}{4k}. \)

For \( u(t_0) = 0 \) with \( u'(t_0) > 0 \), imitating the proof of Lemma 3.1, we have the following estimate for \( t_1 \) with \( u(t_1) = 0 \) and \( u < 0 \) on \([t_1, t_0]\).

**Lemma 3.2.** Let \( k \) be a positive integer, \( \lambda \in (0, 1) \) and \( |\sigma| > 2 \) be such that
\[ \lambda |\sigma|^{q-1} \geq (2^{q-1+1} k)^{q+1}(q+1)e^{(q+1)(N-1)}. \tag{3.15} \]

Let \((\lambda, u)\) satisfy
\[ u_{rr} + \frac{N-1}{r} u_r + \lambda f(u(r)) = 0 \quad r \in (0, 1) \]
\[ u(1) = 0, u'(1) = \sigma. \]

If \( u(t_0) = 0 \) for some \( t_0 \in [\frac{1}{2} + \frac{1}{4k}, 1] \) and \( u'(t_0) > 0 \) then there exists \( t_1 \in [t_0 - \frac{1}{4k}, t_0] \) such that \( u(t_1) = 0 \) and \( u < 0 \) on \([t_1, t_0]\).

**Lemma 3.3.** Let \( k \) be a positive integer. If \( \{(\lambda_j, u_j)\} \) is a sequence in \( \Gamma_k \) and \( \lim_{j \to +\infty} \lambda_j = 0 \), then,
\[ \lim_{j \to +\infty} \lambda_j^2 \int_0^1 r^{N-1} u_j(r)f(u_j(r))dr = 0. \]

**Proof.** We argue by contradiction. Suppose that there exists \( M > 0 \) such that
\[ \lambda_j^2 \int_0^1 r^{N-1} u_j(r)f(u_j(r))dr \geq M \tag{3.16} \]
for any \( j = 1, 2, \ldots \). From (1.2)-(1.3) there exists \( M_1 > 0 \) such that \( 2NF(x) \geq (N-2+M_1)xf(x) \) for all \( x \in \mathbb{R} \). Hence, from (3.1) and (3.16), we have
\[
\frac{|\sigma_j|^2}{2} = \frac{[u_j'(1)]^2}{2} = \lambda_j \int_0^1 r^{N-1} \left( NF(u_j(r)) - \frac{N-2}{2} u_j(r)f(u_j(r)) \right) \, dr \\
\geq \lambda_j \frac{M_1}{2} \int_0^1 r^{N-1} f(u_j(r))u_j(r)dr \geq \frac{MM_1}{2\lambda_j}.
\]
Since \( p, q > 3 \),
\[
\lambda_j |\sigma_j|^{p-1} \geq \lambda_j \left( \frac{MM_1}{\lambda_j} \right)^{\frac{p-1}{p}} \geq (MM_j)^{\frac{p-1}{p}} \lambda_j^{-\frac{2p}{p}} \rightarrow +\infty
\]
and
\[
\lambda_j |\sigma_j|^{q-1} \geq \lambda_j \left( \frac{MM_1}{\lambda_j} \right)^{\frac{q-1}{q}} \geq (MM_j)^{\frac{q-1}{q}} \lambda_j^{-\frac{2q}{q}} \rightarrow +\infty
\]
as \( \lambda_j \to 0 \). Hence, for \( j \) large enough, (3.6) and (3.15) are satisfied. This, Lemma 3.1, and Lemma 3.2 imply that \( u_j \) has at least \( 2k \) zeroes, which contradicts that \( u \) has exactly \( k \) zeroes because of \((\lambda_j, u_j) \in \Gamma_k\).

**Proof of Theorem 1.1.** Let \( k_0 \) be such that \( a\lambda_k > 1 \) for all \( k \geq k_0 \). For \((\lambda, u) \in \Gamma_k\) as shown in Figure 1, \( \lim_{\|u\|_{\infty} \to 0} \lambda = \lambda_k = \frac{\lambda_k}{f(0)} \). Therefore, there exists \((\hat{\lambda}, \hat{u}) \in \Gamma_k\) such that \( a\hat{\lambda} > 1 \) and then
\[
\hat{\lambda} \left( a + b \int_0^1 r^{N-1}(\hat{u}'(r))^2dr \right) > 1. \quad (3.17)
\]
On the other hand, for \((\lambda, u) \in \Gamma_k\), from Lemma 3.3,
\[
\lambda^2 \int_0^1 r^{N-1}uf(u)dr \rightarrow 0 \quad \text{as} \ \lambda \to 0. \quad (3.18)
\]
Since \( \int_0^1 r^{N-1}(u'(r))^2dr = \lambda \int_0^1 r^{N-1}uf(u)dr \), one has, from (3.18), that
\[
\lambda \int_0^1 r^{N-1}(u'(r))^2dr \rightarrow 0 \quad \text{as} \ \lambda \to 0,
\]
which yields
\[
\lambda \left( a + b \int_0^1 r^{N-1}(u'(r))^2dr \right) \rightarrow 0 \quad \text{as} \ \lambda \to 0.
\]
Thus, there exists \((\hat{\lambda}, \hat{u}) \in \Gamma_k\) such that
\[
\hat{\lambda} \left( a + b \int_0^1 r^{N-1}(\hat{u}'(r))^2dr \right) < 1. \quad (3.19)
\]
From (3.17), (3.19) and the intermediate value theorem, there exists some \((\lambda, u) \in \Gamma_k\) such that
\[
\lambda \left( a + b \int_0^1 r^{N-1}(u'(r))^2dr \right) = 1.
\]
This proves (1.7) which shows (1.1) has a radial solution with \( k \) nodal sets. By analogy, we can find a second radial solution in \( \hat{\Gamma}_k\) with \( k \) nodal solution. Thus, Theorem 1.1 is proved. \( \square \)
REFERENCES

[1] P. Chen and X. H. Tang, Existence and multiplicity results for infinitely many solutions for Kirchhoff-type problems in $\mathbb{R}^N$, Math. Methods Appl. Sci., 37 (2014), 1828–1837.

[2] B. T. Cheng and X. H. Tang, Infinitely many large energy solutions for Schrödinger-Kirchhoff type problem in $\mathbb{R}^N$, J. Nonlinear Sci. Appl., 9 (2016), 652–660.

[3] S. N. Chow and J. K. Hale, Methods of Bifurcation Theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 251. Springer-Verlag, New York-Berlin, 1982.

[4] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Functional Analysis, 8 (1971), 321–340.

[5] L. Duan and L. H. Huang, Infinitely many solutions for Schrödinger-Kirchhoff-type equations with general potentials, Results Math., 66 (2014), 181–197.

[6] W. J. Feng and X. J. Feng, Multiple solutions for Kirchhoff equations under the partially sublinear case, J. Funct. Spaces, (2015), Art. ID 610858, 4 pp.

[7] Y. X. Guo and J. J. Nie, Existence and multiplicity of nontrivial solutions for p-Laplacian Schrödinger-Kirchhoff-type equations, J. Math. Anal. Appl., 428 (2016), 1054–1069.

[8] X.-M. He and W.-M. Zou, Existence and multiplicity for a class of Kirchhoff type problems, Acta Math. Sin. Engl. Ser., 26 (2010), 387–394.

[9] J. H. Jin and X. Wu, Infinitely many radial solutions for Kirchhoff-type problems in $\mathbb{R}^N$, J. Math. Anal. Appl., 369 (2010), 564–574.

[10] A. Li and J. B. Su, Existence and multiplicity of solutions for Kirchhoff-type equation with radial potentials in $\mathbb{R}^3$, Z. Angew. Math. Phys., 66 (2015), 3147–3158.

[11] L. Li and X. Zhong, Infinitely many small solutions for the Kirchhoff equation with local sublinear nonlinearities, J. Math. Anal. Appl., 435 (2016), 955–967.

[12] J. J. Nie, Existence and multiplicity of nontrivial solutions for a class of Schrödinger-Kirchhoff-type equations, J. Math. Anal. Appl., 417 (2014), 65–79.

[13] J. J. Nie and X. Wu, Existence and multiplicity of non-trivial solutions for Schrödinger-Kirchhoff-type equations with radial potential, Nonlinear Anal., 75 (2012), 3470–3479.

[14] K. Perera and Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, J. Differential Equations, 221 (2006), 246–255.

[15] S. Z. Song, S. J. Chen and C. L. Tang, Existence of solutions for Kirchhoff type problems with resonance at higher eigenvalues, Discrete Contin. Dyn. Syst., 36 (2016), 6453–6473.

[16] S.-Z. Song, C. L. Tang and S.-J. Chen, Multiple solutions for Kirchhoff type problem near resonance, Electron. J. Differential Equations, 2015, (2015), 7 pp.

[17] J. J. Sun, L. Li, M. Cencelj and B. Gabrovšek, Infinitely many sign-changing solutions for Kirchhoff problems in $\mathbb{R}^3$, Nonlinear Analysis, 186 (2019), 33–54.

[18] J.-J. Sun and C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type equations, Nonlinear Anal., 74 (2011), 1212–1222.

[19] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$, Nonlinear Anal. Real World Appl., 12 (2011), 1278–1287.

[20] Q. L. Xie, S. W. Ma and X. Zhang, Infinitely many bounded state solutions of Kirchhoff problem in $\mathbb{R}^3$, Nonlinear Anal. Real World Appl., 29 (2016), 80–97.

[21] X. Z. Yao and C. L. Mu, Infinitely many sign-changing solutions for Kirchhoff type equations with power nonlinearity, Electron. J. Differential Equations, 2016 (2016), 7 pp.

[22] Y. W. Ye, Infinitely many solutions for Kirchhoff type problems, Differ. Equ. Appl., 5 (2013), 83–92.

[23] Y. W. Ye and C. L. Tang, Multiple solutions for Kirchhoff-type equations in $\mathbb{R}^N$, J. Math. Phys., 54 (2013), 081508, 16 pp.

[24] Z. T. Zhang and K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl., 317 (2006), 456–463.

[25] Q. Y. Zhang and B. Xu, Infinitely many solutions for Schrödinger-Kirchhoff-type equations involving indefinite potential, Electron. J. Qual. Theory Differ. Equ., 2017, (2017), 17 pp.

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