Relative Spectral Invariants of Elliptic Operators on Manifolds

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We introduce and study new relative spectral invariants of two elliptic partial differential operators of Laplace and Dirac type on compact smooth manifolds without boundary that depend on both the eigenvalues and the eigensections of and contain much more information about geometry. We prove the existence of the homogeneous short time asymptotics of the new invariants with the coefficients of the asymptotic expansion being integrals of some invariants that depend on the symbols of both operators. The first two coefficients of the asymptotic expansion are computed explicitly.
1 Introduction

Elliptic operators on manifolds, in particular, first and second-order partial differential operators, play a crucial role in global analysis, spectral geometry and mathematical physics [14, 10, 9, 2, 4, 5]. The study of the spectrum of elliptic operators is of paramount importance since it describes various important objects in quantum field theory and differential geometry such as correlation functions, functional determinants, integrals of infinite-dimensional Hamiltonian systems etc. The spectrum of elliptic operators does, of course, depend on the geometry of the manifold. Therefore, one can ask the question: “To what extent does the spectrum of a single elliptic operator describe the geometry?”, or, as M. Kac put it “Can one hear the shape of a drum?” In general one cannot compute the spectrum exactly. One usually studies the spectrum indirectly by studying some spectral invariants such as the heat trace or zeta function [14]. These spectral invariants only depend on the eigenvalues of the operators and do not depend on the eigensections. It is well known now that the answer to this question is negative, that is, there are non-isometric manifolds that have the same spectrum. The classical heat trace of Laplace type operators has been studied for decades going back to H. Weyl [21] and S. Minakshisundaram and A. Plejel [16]. There is a vast literature on the subject (see [14, 10, 2, 4, 5] and references therein).

In [8] and [6] we studied more general spectral invariants that appear naturally in quantum statistical physics and geometry. In the present paper we introduce and study new relative spectral invariants of two elliptic operators. We hope that these new invariants could shed new light on the old questions of spectral geometry. We generalize the question as follows: “Does the spectral data of two elliptic operators determine the geometry?” These invariants depend both on the eigenvalues and the eigensections and contain much more information about geometry. Such relative spectral invariants appear naturally, in particular, in the study of particle creation in quantum field theory and quantum gravity [11, 12, 7]. They determine the number of created particles from the vacuum when the dynamical operator depends on time.

In Sec. 2 we motivate the study of the relative spectral invariants. We describe the so-called Bogolyubov invariant in quantum field theory and show how it can be expressed in terms of the relative spectral invariant. We consider a smooth n-dimensional compact manifold $M$ without boundary and a vector bundle $V$ over the manifold $M$. Let $L_\pm$ be two self-adjoint elliptic second-order partial differential operators acting on smooth sections of the vector bundle $V$ with a positive definite scalar leading symbols of Laplace type. Let $D_\pm$ be two self-adjoint elliptic first-
order partial differential operators acting on smooth sections of the vector bundle \( V \) of Dirac type such that the squares \( L_\pm = D_\pm^2 \) are Laplace type operators. The spectral information about the operators \( L_\pm \) and \( D_\pm \) are contained in the classical heat traces

\[
\Theta_\pm(t) = \text{Tr} \exp(-tL_\pm),
\]
\[
H_\pm(t) = \text{Tr} D_\pm \exp(-tD_\pm^2).
\]

We show that the Bogolyubov invariants can be expressed in terms of the traces

\[
\Psi(t, s) = \text{Tr} \{ \exp(-tL_+) - \exp(-tL_-) \} \{ \exp(-sL_+) - \exp(-sL_-) \},
\]
\[
\Phi(t, s) = \text{Tr} \{ D_+ \exp(-tD_+^2) - D_- \exp(-tD_-^2) \} \{ D_+ \exp(-sD_+^2) - D_- \exp(-sD_-^2) \},
\]

that we call relative spectral invariants; which can further be expressed further in terms of the classical heat traces and the combined heat traces

\[
X(t, s) = \text{Tr} \exp(-tL_+) \exp(-sL_-),
\]
\[
Y(t, s) = \text{Tr} D_+ \exp(-tD_+^2) D_- \exp(-sD_-^2),
\]

by

\[
\Psi(t, s) = \Theta_+(t + s) - \Theta_-(t + s) - X(t, s) - X(s, t),
\]
\[
\Phi(t, s) = -\partial_t \Theta_+(t + s) - \partial_t \Theta_-(t + s) - Y(t, s) - Y(s, t).
\]

In Sec. 3 we describe the relevant differential operators and their spectral traces and introduce the relevant notation. The operators \( L_\pm \) naturally define the metrics \( g_\pm^{ij} \); the connection one forms \( \mathcal{A}_j^\pm \) and the endomorphisms \( Q_\pm \) by

\[
L_\pm = -g_\pm^{-1/4}(\partial_i + \mathcal{A}_i^\pm)g_\pm^{1/2}g_\pm^{ij}(\partial_j + \mathcal{A}_j^\pm)g_\pm^{-1/4} + Q_\pm,
\]

where \( g_\pm^{ij} \) are the inverse metrics and \( g_\pm = \det g_\pm^{ij} \). Similarly, the operators \( D_\pm \) define the endomorphisms \( S_\pm \) by

\[
D_\pm = g_\pm^{1/4}i\gamma_\pm^i(\partial_j + \mathcal{A}_j^\pm)g_\pm^{-1/4} + S_\pm,
\]

where \( \gamma_\pm^i \) are the Dirac matrices satisfying (3.8). Here, the connection \( \mathcal{A}_j^\pm \) is supposed to satisfy the compatibility condition (3.9). Also, we suppose that the endomorphisms \( S_\pm \) anticommute with the Dirac matrices \( \gamma_\pm^i \), (3.10), so that the square of the Dirac type operator \( D_\pm^2 \) is a Laplace type operator with the potential

\[
Q_\pm = -\frac{1}{2} \gamma_\pm^{ij}\mathcal{R}_\pm^{ij} + S_\pm^2 + i\gamma_\pm^{ij}\nabla_\pm^i S_\pm,
\]
where $R^\pm_{ij}$ be the curvature of the connection $\mathcal{A}^\pm_i$ and $\gamma^i_j = \gamma^i_j \gamma^j_i$. We follow the standard convention [22] and denote the antisymmetrized products of Dirac matrices by $\gamma^i_{1\ldots k} = \gamma[i \ldots j].$

In Sec. 4 we present a detailed review of the Ruse-Synge function (which is equal to one half of the square of the geodesic distance between two points in a Riemannian manifold) with the particular emphasis on its dependence on the metric. We compute the diagonal values of the covariant derivatives (defined with respect to a metric $g$) of a Ruse-Synge function $\sigma^b(x, x')$ defined with respect to another metric $h$.

In Sec. 5 we study the asymptotics of the integrals of Laplace type and prove some important lemmas used in the proof of the main theorems. In Sec. 6 we study the asymptotics of the combined heat traces and prove the general theorems. The spectral information about the operators $L^\pm_\pm$ is contained in the classical heat traces (1.1). In particular, the asymptotic expansion as $t \to 0$

$$\Theta^\pm(t) \sim (4\pi)^{-n/2} \sum_{k=0}^\infty t^{k-n/2} A^\pm_k,$$

(1.12)

defines the sequence of spectral invariants

$$A^\pm_k = \frac{(-1)^k}{k!} \int_M dx \ g^{1/2}_\pm \text{tr}[a^\pm_k],$$

(1.13)

where $\text{tr}[a^\pm_k]$ are some scalar invariants which are polynomial in the jets of the symbols of the operators $L^\pm_\pm$, that is, in the covariant derivatives of the curvatures $R^\pm_{ijkl}$ of the metrics $g^\pm$, the curvatures $R^\pm_{ij}$ of the connections $\mathcal{A}^\pm_i$ and the potentials $Q^\pm_\pm$ (notice the different normalization factor in (1.12) compared to our earlier papers [1, 2, 4, 5]). It is well known that the first two classical heat kernel coefficients are [14, 2, 5]

$$A^\pm_0 = \int_M dx \ g^{1/2}_\pm \text{tr}I,$$

(1.14)

$$A^\pm_1 = \int_M dx \ g^{1/2}_\pm \text{tr}\left(\frac{1}{6} R^\pm I - Q^\pm_\pm\right),$$

(1.15)

where $\text{tr}$ is the fiber trace, $I$ is the identity endomorphism and $R^\pm_\pm$ is the scalar curvature of the metric $g^\pm$. Therefore, for the Dirac type operators the coefficient $A_1$ takes the form

$$A^\pm_1 = \int_M dx \ g^{1/2}_\pm \text{tr}\left(\frac{1}{6} R^\pm I + \frac{1}{2} \gamma^i_j \gamma^j_i R^\pm_{ij} - S^2_\pm\right).$$

(1.16)

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We study in this paper the asymptotics of the combined heat traces (1.5) and (1.6). We define the time-dependent metric $g_{ij} = g_{ij}(t, s)$ as the inverse of the matrix
\[
g_{ij}^{\text{ij}} = t g_{ij} + s g_{ij},
\]
with $t, s > 0$; throughout the paper we use the notation $g = \det g_{ij}$ for the determinant of the metric. Also, we define the time-dependent connection $\mathcal{A}_i = \mathcal{A}_i(t, s)$ by
\[
\mathcal{A}_i = g_{ij} \left( t g_{jk} \mathcal{A}_k + s g_{jk} \mathcal{A}_k \right).
\]
We omit the variables $t$ and $s$ where it does not cause any confusion. Note that the inverse metric $g^{ij}$ is a homogeneous function of $t$ and $s$ of degree 1, and, therefore, the metric $g_{ij}$ is a homogeneous function of $t$ and $s$ of degree $-1$, and the determinant $g = \det g_{ij}$ is a homogeneous function of $t$ and $s$ of degree $-n$; furthermore, the Christoffel symbols, $\Gamma_{ijk}$, the Riemann tensor $R_{ijkl}^g$ and the Ricci tensor $R_{ij}^g$ of the metric $g$ are homogeneous functions of $t$ and $s$ of degree 0. Similarly, the connection $\mathcal{A}_i$ and its curvature $R_{ij}^\mathcal{A}$ are homogeneous functions of $t$ and $s$ of degree 0.

**Theorem 1** There are asymptotic expansions as $\varepsilon \to 0$

\[
X(\varepsilon t, \varepsilon s) \sim (4\pi \varepsilon)^{-n/2} \sum_{k=0}^{\infty} \varepsilon^k B_k(t, s),
\]
\[
Y(\varepsilon t, \varepsilon s) \sim (4\pi \varepsilon)^{-n/2} \sum_{k=0}^{\infty} \varepsilon^{k-1} C_k(t, s),
\]

where

\[
B_k(t, s) = \int_M dx \, g^{1/2}(t, s) b_k(t, s),
\]
\[
C_k(t, s) = \int_M dx \, g^{1/2}(t, s) c_k(t, s).
\]

1. The coefficients $b_k(t, s)$ and $c_k(t, s)$ are scalar invariants built polynomially from the covariant derivatives (defined with respect to the metric $g_{ij}$ and the connection $\mathcal{A}_i$) of the metrics $g_{ij}^\pm$, the vectors $C_i^\pm$ and the potentials $Q_\pm$ and $S_\pm$. 
2. The coefficients \(b_k(t, s)\) and \(c_k(t, s)\) are symmetric under the exchange \((t, L_+) \leftrightarrow (s, L_-)\).

3. The coefficients \(b_k(t, s)\) are homogeneous functions of \(t\) and \(s\) of degree \(k\) and the coefficients \(c_k(t, s)\) are homogeneous functions of \(t\) and \(s\) of degree \((k - 1)\).

This gives the asymptotic expansion of the relative spectral invariants (1.3) and (1.4).

**Corollary 1** There are asymptotic expansions as \(\epsilon \to 0\)

\[
\Psi(\epsilon t, \epsilon s) \sim (4\pi \epsilon)^{-n/2} \sum_{m=0}^{\infty} \epsilon^m \Psi_k(t, s),
\]

(1.23)

\[
\Phi(\epsilon t, \epsilon s) \sim (4\pi \epsilon)^{-n/2} \sum_{k=0}^{\infty} \epsilon^{k-1} \Phi_k(t, s),
\]

(1.24)

where

\[
\Psi_k(t, s) = (t + s)^{k-n/2}(A_k^+ + A_k^-) - B_k(t, s) - B_k(s, t),
\]

(1.25)

\[
\Phi_k(t, s) = -(k - \frac{n}{2})(t + s)^{k-1-n/2}(A_k^+ + A_k^-) - C_k(t, s) - C_k(s, t).
\]

(1.26)

In Sec. 7 we consider some particular cases when the relative spectral invariants can be computed exactly in terms of the classical heat trace and compute explicitly the first two coefficients of the asymptotic expansions. To describe the main results we introduce a symmetric tensor \(G_{ij} = G_{ij}(t, s)\) (that we call the *dual metric*) by

\[
G_{ij} = sg_{ij}^+ + tg_{ij}^-,
\]

(1.27)

and its inverse \(G^{ij}\), which is related to the metric \(g^{ij}\), (1.17), by

\[
G_{ij} = g_{ik}^+ g_{lj}^+ + g_{ik}^- g_{lj}^-, \quad G^{ij} = g^{ip} g^{qj} g_{pq}^+ + g^{ip} g^{qj} g_{pq}^-.
\]

(1.28)

(1.29)

Notice that

\[
g_{ij}(1, 0) = G_{ij}(0, 1) = g_{ij}^+,
\]

(1.30)

\[
g_{ij}(0, 1) = G_{ij}(1, 0) = g_{ij}^-.
\]

(1.31)
Also, we introduce the non-compatibility tensors
\[ K^\pm_{ijk} = \nabla_i g^\pm_{jk}, \]  
(1.32)
\[ C^\pm_i = \mathcal{A}^\pm_i - \mathcal{A}_i, \]  
(1.33)
and the tensors
\[ W^\pm_{ijk} = \frac{1}{2} g^m_{\pm} \left( K^\pm_{jkm} + K^\pm_{kjm} - K^\pm_{mjk} \right), \]  
(1.34)
\[ W^\pm_{ij} = W^\pm_{i} = \nabla_j W^\pm, \]  
(1.35)
with
\[ W^\pm = \frac{1}{2} \log \left( \frac{g^\pm}{g} \right). \]  
(1.36)
Finally, we define
\[ W_i = \frac{1}{2} (W^+_i - W^-_i) = \frac{1}{2} \nabla^+_i (W^+_i + W^-_i), \]  
(1.37)
\[ W_{ij} = \frac{1}{2} \left( \nabla^+_j W_i + \nabla^-_j W_i \right) = \frac{1}{2} \nabla^+_i \nabla^+_j (W^+_i + W^-_i), \]  
(1.38)
\[ \Sigma_{ijk} = \frac{3}{2} s K^+_i (ijk) + \frac{3}{2} t K^\pm_{ijk}, \]  
(1.39)
\[ \Sigma_{ijkl} = s S^+_ijkl + t S^\pm_{ijkl}, \]  
(1.40)
where
\[ S^\pm_{ijkl} = 4 g^\pm_{m(i} \nabla^\pm_{jkl} W^m_{ij} + 4 g^\pm_{m(i} W^m_{ijkl} W^m_{jm} + 3 g^\pm W^m_{ij} W^m_{kl}). \]  
(1.41)
Here and everywhere below parenthesis denote symmetrization over all indices included. The indices excluded from the symmetrization are separated by vertical lines.

**Theorem 2** The first two coefficients of the asymptotic expansion of the combined heat trace \( X(t, s) \) are
\[ b_0(t, s) = \text{tr} I, \]  
(1.42)
\[ b_1(t, s) = \text{tr} \left\{ t \left( \frac{1}{6} R_+ I + Q_+ \right) + s \left( \frac{1}{6} R_- I - Q_- \right) + ts \left[ \frac{1}{6} G^{ij} \left( R^+_i + R^+_j - 2 R^+_i \right) I \right. \right. \]  
\[ + \frac{1}{6} G^{ij} \left( W_{ij} + W_j W_i \right) - G^{ij} G^{kl} \Sigma_{ijkl} W_j - \frac{1}{4} G^{ij} G^{kl} \Sigma_{ijkl} \]  
\[ + \frac{1}{12} \left( 2 G^{ij} G^{jm} + 3 G^{ij} G^{jm} \right) G^{lm} \Sigma_{ijkl} \Sigma_{lnm} \right\} \]  
+ G^{ij} (C^+_i - C^-_i)(C^+_j - C^-_j). \]  
(1.43)
Corollary 2  The first two coefficients of the asymptotic expansion of the relative spectral invariant $\Psi(t, s)$ are

$$\Psi_0(t, s) = \int_M dx \left\{ (t + s)^{-n/2} \left( g_{1/2}^+ + g_{1/2}^- \right) - g_{1/2}^+(t, s) - g_{1/2}^-(s, t) \right\} \text{tr}I,$$  \hspace{1cm} \left(1.44\right)

$$\Psi_1(t, s) = \int_M dx \left\{ (t + s)^{-n/2} \left[ g_{1/2}^+ \text{tr} \left( \frac{1}{6} R_+ I - Q_+ \right) + g_{1/2}^- \text{tr} \left( \frac{1}{6} R_- I - Q_- \right) \right] ight. \\
- g_{1/2}^+(t, s)b_1(t, s) - g_{1/2}^-(s, t)b_1(s, t) \right\}. \hspace{1cm} \left(1.45\right)$$

Further, we define the auxiliary tensors

$$N^{ijkl} = 2G^{ij}G^{kl} W_i - \frac{1}{3} \left( 2G^{ij}G^{jk} + 3G^{iq}G^{jk} \right) G^{pl} \Sigma_{ipq}, \hspace{1cm} \left(1.46\right)$$

$$M^{kl} = \left( G^{kl}G^{ij} + 2G^{ik}G^{jl} \right) (W_{ij} + W_i W_j)$$

$$- \left( 2G^{ij}G^{mk} G^{pl} + 2G^{im}G^{jk} G^{pl} + G^{kl} G^{im} G^{pj} \right) \Sigma_{ipm} W_j$$

$$- \frac{1}{4} \left( G^{pq} G^{kl} + 4G^{kp} G^{lq} \right) G^{pq} \Sigma_{ijpq}$$

$$+ \frac{1}{72} \left( 2G^{ij}G^{pr} G^{qs} G^{kl} + 3G^{ij}G^{pq} G^{rs} G^{kl} + 6G^{ik} G^{jl} G^{pq} G^{rs} \right)$$

$$+ 12G^{ij}G^{pq} G^{kl} G^{ls} + 12G^{ij}G^{pr} G^{ls} G^{kl} \right) \Sigma_{ipq} \Sigma_{jrs}. \hspace{1cm} \left(1.47\right)$$

and

$$V_{pijkl} = \text{Sym}(i, j, k, l) \left\{ \left( 4 g^+_m \nabla_k W^m_{+ ij} + 4 g^+_m W^m_{+ jk} W^m_{+ in} + 12 g^+_m W^m_{+ nk} W^m_{+ jp} \right) + 6 g^+_{mn} \right. \left( 4 g^-_m \nabla_k W^m_{- ij} + 4 g^-_m W^m_{- jk} W^m_{- in} \right)$$

$$- 6 g^+_m W^n_{+ k j} W^m_{+ n p} \left\} \right. \left\{ \left. 4 g^-_m \nabla_k W^m_{- ij} + 4 g^-_m W^m_{- jk} W^m_{- in} \right) + 12 g^-_m W^-_{- nk} W^-_{- jq} - 6 g^-_m \nabla_k W^-_{- kj} W^-_{- n m} \right) + 6 g^+_m W^m_{+ ij} g^-_{m n} W^-_{- kl} \right\}. \hspace{1cm} \left(1.48\right)$$

Theorem 3  The first two coefficients of the asymptotic expansion of the combined
heat trace \( Y(t, s) \) are

\[
c_0(t, s) = \frac{1}{2} g_{ij}(t, s) \text{tr} \left( \gamma^+_i \gamma^-_j \right), \tag{1.49}
\]

\[
c_1(t, s) = \text{tr} \left( \frac{1}{6} \frac{1}{2} g_{pq} R_+ - g_{gq} g_{gj} R^+_{ij} \right) \gamma^+_i \gamma^-_j + \frac{1}{6} s \left( \frac{1}{2} g_{pq} R_- - g_{p} g_{gj} R^+_{ij} \right) \gamma^+_i \gamma^-_j
\]

\[
+ \frac{1}{4} t g_{pq} \gamma^+_i \gamma^-_j R^+_{ij} + \frac{1}{4} s g_{pq} \gamma^+_i \gamma^-_j R^+_{ij} + S + S - \frac{1}{2} t g_{pq} \gamma^+_i \gamma^-_j S^+ - \frac{1}{2} s g_{pq} \gamma^+_i \gamma^-_j S^+
\]

\[
- \frac{1}{2} t g_{pq} \gamma^+_i \gamma^-_j \nabla^+_j S - \frac{1}{2} s g_{pq} \gamma^+_i \gamma^-_j \nabla^+_j S
\]

\[
+ ts \left[ \frac{1}{12} \left( g^{kl} G^{ij} + 2 G^{ik} G^{jl} \right) \left( R^+_{ij} + R^-_{ij} - 2 R^0_{ij} \right) g^{+}_{k} g^{+}_{l} \gamma^+_i \gamma^-_j \right.
\]

\[
+ \frac{1}{8} G^{ij} G^{kl} V_{pqijkl} \gamma^+_i \gamma^-_j + \frac{3}{4} N^{ijkl}(g^{+}_{mp} W^+_{m} (jk g^{+}_{l}) - g^{-}_{mq} W^{-}_{m} (jk g^{+}_{l})) \gamma^+_i \gamma^-_j
\]

\[
+ \frac{1}{2} M^{ijkl} g^{+}_{p} g^{-}_{q} \gamma^+_i \gamma^-_j - \frac{3}{4} G^{ijkl} g^{+}_{mp} g^{+}_{q} \gamma^+_i \gamma^-_j \nabla^{g,R} (C^+_i - C^-_i)
\]

\[
- \frac{3}{4} \left( G^{ijkl} \left[ g^{+}_{mp} W^+_{m} (jk g^{+}_{l}) + g^{-}_{mq} W^{-}_{m} (jk g^{+}_{l}) \right] + N^{ijkl} g^{+}_{pk} g^{+}_{q} \right) \left( \gamma^+_i \gamma^-_j \gamma^+_k \gamma^-_l \gamma^+_k \gamma^-_l \right)
\]

\[
- 2 C^+_i C^-_j \gamma^+_i \gamma^-_j - 2 C^+_i C^-_j \gamma^+_j \gamma^-_j] \right).
\]

**Corollary 3** The first two coefficients of the asymptotic expansion of the relative spectral invariant \( \Phi(t, s) \) are

\[
\Phi_0(t, s) = \int_M dx \left\{ \frac{n}{2} (t + s)^{-1-n/2} \left( g^{+}_{1} + g^{-}_{1} \right) \text{tr} \right.
\]

\[
- \frac{1}{2} \left[ g^{1/2}(t, s) g_{ij}(t, s) + g^{1/2}(s, t) g_{ij}(s, t) \right] \text{tr} \left( \gamma^+_i \gamma^-_j \right) \right\}, \tag{1.51}
\]

\[
\Phi_1(t, s) = \int_M dx \left\{ - g^{1/2}(t, s) c_1(t, s) - g^{1/2}(s, t) c_1(s, t) \right.
\]

\[
+ \left( \frac{n}{2} - 1 \right) (t + s)^{-n/2} \left[ g^{1/2} \left( \frac{1}{6} R_{+} - Q_{+} \right) + g^{1/2} \left( \frac{1}{6} R_{-} - Q_{-} \right) \right] \right\}, \tag{1.52}
\]

where \( Q_{\pm} \) are given by \( \tag{L.11} \).
2 Bogolyubov Invariant

We motivate the definition of the relative spectral invariants by quantum field theory. We will be very brief here, the detailed exposition will appear elsewhere \cite{[7]}. We describe now the standard method for calculation of particles creation via the Bogolyubov transformation \cite{[11],[12]}. Let \((\mathcal{M}, h)\) be a pseudo-Riemannian \((n + 1)\)-dimensional assume that \((\mathcal{M}, h)\) is globally hyperbolic so that there is a foliation of \(\mathcal{M}\) with space slices \(M_t\) at a time \(t\), moreover, we assume that there is a global time coordinate \(t\) varying from \(-\infty\) to \(+\infty\) and that at all times \(M_t\) is a compact \(n\)-dimensional Riemannian manifold without boundary. We will also assume that there are well defined limits \(M_\pm\) as \(t \to \pm\infty\). For simplicity, we will just assume that the manifold \(M\) has two cylindrical ends, \((-\infty, \beta) \times M\) and \((\beta, \infty) \times M\) for some positive parameter \(\beta\). So, the foliation slices \(M_t\) depend on \(t\) only on a compact interval \([-\beta, \beta]\). Let \(\mathcal{W}\) be a Hermitian vector bundle over \(\mathcal{M}\) and \(\mathcal{V}_t\) be the corresponding time slices (vector bundles over \(M_t\)).

In quantum field theory there are two types of particles, bosons and fermions. The bosonic fields are described by second order Laplace type partial differential operators whereas the fermionic fields are described by first order Dirac type partial differential operators. Let \(L_t\) be a one-parameter family of positive self-adjoint elliptic second-order partial differential operators of Laplace type acting on smooth sections of the vector bundle \(\mathcal{V}_t\). We assume that there are well defined limits \(L_\pm\) as \(t \to \pm\infty\). Let \(D_t\) be a one-parameter family of self-adjoint elliptic first-order partial differential operators of Dirac type acting on sections of the vector bundles \(\mathcal{V}_t\) such that its square \(L_t = D_t^2\) is a self-adjoint second-order positive elliptic partial differential operator of Laplace type. We assume that there are well defined limits \(D_\pm\) as \(t \to \pm\infty\). Then one defines so-called the in-vacuum and the out-vacuum and the corresponding in-particles and out-particles. Then the out-vacuum contains some in-particles (and vice versa). The total number of in-particles in the out-vacuum is determined by the so-called Bogolyubov invariant.

Let \(E_{b,f,0}\) be the functions defined by

\[
E_f(x) = \frac{1}{e^x + 1}, \quad (2.53)
\]
\[
E_b(x) = \frac{1}{e^x - 1}, \quad (2.54)
\]
\[
E_0(x) = \frac{1}{2 \sinh x}, \quad (2.55)
\]
and $\omega_\pm$ are pseudo-differential operators defined by

$$\omega_\pm = \sqrt{L_\pm}. \quad (2.56)$$

Then in some approximation (for details, see [7]) the Bogolyubov invariants for bosons and fermions are determined by the following traces

$$B_b(\beta) = \text{Tr} \left\{ E_f(\beta \omega_+) - E_f(\beta \omega_-) \right\} \left\{ E_b(\beta \omega_+) - E_b(\beta \omega_-) \right\}, \quad (2.57)$$

$$B_f(\beta) = 2\beta^2 \text{Tr} \left\{ D_+ E_0(\beta \omega_+) - D_- E_0(\beta \omega_-) \right\}^2. \quad (2.58)$$

The Bogolyubov invariants can be expressed in terms of the relative spectral invariants $\Psi(t, s)$ and $\Phi(t, s)$ defined in (1.3) and (1.4). Let $h_{b,f,0}$ be the functions defined by

$$h_f(t) = \frac{1}{2\pi} \int dp \ p \tan \left( \frac{p}{2} \right) \exp(-tp^2) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^\infty (-1)^{k+1} k \exp \left( -\frac{k^2}{4t} \right), \quad (2.59)$$

$$h_b(t) = \frac{1}{2\pi} \int dp \ p \cot \left( \frac{p}{2} \right) \exp(-tp^2) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^\infty k \exp \left( -\frac{k^2}{4t} \right), \quad (2.60)$$

$$h_0(t) = \frac{1}{2\pi} \int dp \ \frac{p}{\sin p} \exp(-tp^2) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=0}^\infty (2k + 1) \exp \left( -\frac{(2k + 1)^2}{4t} \right). \quad (2.61)$$

where the integrals are taken in the principal value sense. Then the Bogolyubov invariants take the form

$$B_b(\beta) = \int_0^\infty dt \int_0^\infty ds \ h_f(s) h_b(t) \Psi \left( \beta^2 s, \beta^2 t \right), \quad (2.62)$$

$$B_f(\beta) = \int_0^\infty dt \int_0^\infty ds \ h_0(s) h_0(t) 2\beta^2 \Phi \left( \beta^2 t, \beta^2 s \right). \quad (2.63)$$
It is the relative spectral invariants $\Psi(t, s)$ and $\Phi(t, s)$ that we study in the present paper. Obviously, the combined heat traces $X(t, s)$ and $Y(t, s)$ contains information about the spectra of both operators $L_\pm$ (and $D_\pm$) since, in particular,

\[ X(0, s) = \Theta_-(s), \quad X(t, 0) = \Theta_+(t), \quad Y(0, s) = H_-(s), \quad Y(t, 0) = H_+(t) \]  

(2.64) 

(2.65)

Also, although for any $t, s > 0$

\[ \Psi(0, s) = \Psi(t, 0) = \Phi(0, s) = \Phi(t, 0) = 0, \]  

(2.66)

the asymptotics as $t, s \to 0$ are non-trivial. It is these asymptotics that we study in the present paper.

We can also define the corresponding \textit{relative zeta functions}

\[ Z_\Psi(p, q) = \frac{1}{\Gamma(p) \Gamma(q)} \int_0^\infty dt \int_0^\infty ds \ t^{p-1} s^{q-1} \Psi(t, s), \]  

(2.67)

\[ Z_\Phi(p, q) = \frac{1}{\Gamma(p) \Gamma(q)} \int_0^\infty dt \int_0^\infty ds \ t^{p-1} s^{q-1} \Phi(t, s), \]  

(2.68)

and, similarly, $Z_X(p, q)$ and $Z_Y(p, q)$. Then

\[ Z_X(p) = \text{Tr} L^{-p}_+ L^{-q}_-, \]  

(2.69)

\[ Z_Y(p) = \text{Tr} D^{-2p+1}_+ D^{-2q+1}_- \]  

(2.70)

and

\[ Z_\Psi(p, q) = \text{Tr} \left( L^{-p}_+ - L^{-p}_- \right) \left( L^{-q}_- - L^{-q}_+ \right), \]  

(2.71)

\[ Z_\Phi(p, q) = \text{Tr} \left( D^{-2p+1}_+ - D^{-2p+1}_- \right) \left( D^{-2q+1}_- - D^{-2q+1}_+ \right). \]  

(2.72)

To avoid confusion the complex power of the operator $D_\pm$ (which is not positive) is defined as follows $D_\pm^{-2p+1} = D_\pm (D^2_\pm)^{-p}$.

For the Dirac case one can also introduce more general traces

\[ W_\pm(t, \alpha) = \text{Tr} \exp(-tD^2_\pm + i\alpha D_\pm), \]  

(2.73)

\[ V(t, s; \alpha, \beta) = \text{Tr} \exp(-tD^2_\pm + i\alpha D_\pm) \exp(-sD^2_- + i\beta D_-). \]  

(2.74)
Then, obviously,
\[
\Theta_{\pm}(t) = W_{\pm}(t, 0), \quad (2.75)
\]
\[
X(t, s) = V(t, s; 0, 0), \quad (2.76)
\]
\[
Y(t, s) = -\frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} V(t, s; \alpha, \beta) \bigg|_{\alpha=\beta=0}. \quad (2.77)
\]
Therefore, all traces can be obtained from the traces (2.73) and (2.74).
Notice that the trace \(W(t, \alpha)\) can be written in the form
\[
W_{\pm}(t, \alpha) = (4\pi t)^{-1/2} \int_{\mathbb{R}} d\alpha' \exp \left\{ -\frac{(\alpha - \alpha')^2}{4t} \right\} T_{\pm}(\alpha'), \quad (2.78)
\]
where
\[
T_{\pm}(\alpha) = \text{Tr} \exp (i\alpha D_{\pm}), \quad (2.79)
\]
which should be understood in the distributional sense. Similarly, the invariant \(V(t, s; \alpha, \beta)\) can be written in the form
\[
V(t, s; \alpha, \beta) = (4\pi^{-1} ts)^{-1/2} \int_{\mathbb{R}^2} d\alpha' d\beta' \exp \left\{ -\frac{(\alpha - \alpha')^2}{4t} - \frac{(\beta - \beta')^2}{4s} \right\} S_{\pm}(\alpha', \beta'), \quad (2.80)
\]
where
\[
S_{\pm}(\alpha, \beta) = \text{Tr} \exp (i\alpha D_{\pm}) \exp (i\beta D_{-}). \quad (2.81)
\]
In this paper we will be interested primarily in the asymptotic expansion of the combined heat traces as \(t, s \to 0\).

3 Generalized Heat Traces

3.1 Differential Operators

Let \(M\) be a compact \(n\)-dimensional Riemannian manifold without boundary. Throughout the whole paper we denote tensor indices by Latin letters and use Einstein summation convention. We use parenthesis for the symmetrization of indices and square brackets for the anti-symmetrization. The indices excluded from the symmetrization or anti-symmetrization are separated by vertical lines. Also, we denote the local coordinates by \(x^i\) and the partial derivatives by \(\partial_i\). Let \(V\) be a vector bundle of densities of weight \(1/2\) over \(M\), \(L^2(V)\) be the corresponding Hilbert
space; we use the notation $\text{tr}$ for the fiber trace and $\text{Tr}$ be the corresponding $L^2$ trace. We study positive self-adjoint elliptic second-order partial differential operators $L$ with a scalar positive definite leading symbol of Laplace type acting on smooth sections of the bundle $\mathcal{V}$. A Laplace type operator $L$ naturally defines a Riemannian metric $g$ and a connection $\nabla^g$ on the vector bundle with a connection one-form $\mathcal{A}_i$. Since we will be working with different operators we do not have a single metric, then, following [3], we prefer to work with the vector bundle of densities of weight $1/2$ and with the Lebesgue measure $dx$ instead of the Riemannian one. Then the heat kernel $U(t; x, x')$ of the heat semigroup $\exp(-tL)$ is also a density of weight $1/2$ at each point $x$ and $x'$, and the heat kernel diagonal $U(t; x, x)$ is a density of weight 1. Then a Laplace type operator has the form

$$L = g^{1/4} \left( -\Delta^g + Q \right) g^{-1/4},$$

where $\Delta^{g,\mathcal{A}} = g^{ij} \nabla^{g,\mathcal{A}}_i \nabla^{g,\mathcal{A}}_j$ is the Laplacian, $g = \det g_{ij}$, and $Q$ is some smooth endomorphism of the vector bundle $\mathcal{V}$; locally it has the form

$$L = -g^{-1/4} (\partial_i + \mathcal{A}_i) g^{1/2} g^{ij} (\partial_j + \mathcal{A}_j) g^{-1/4} + Q. \quad (3.2)$$

Let $L_{\pm}$ be two Laplace type operators defined by the metrics $g_{\pm}^{ij}$, the connections $\mathcal{A}_{\pm}^i$ and the potential terms $Q_{\pm}$. By using the metric $g_{ij}(t, s)$, (1.17), the connection $\mathcal{A}_i(t, s)$, (1.18), and the identity

$$t g^{ij}_+ C^+_j + s g^{ij}_- C^-_j = 0 \quad (3.3)$$

one can rewrite now the operators $L_{\pm}$ in the form

$$L_{\pm} = g^{1/4} \left( -\nabla^g_{\pm} \nabla^g_{\pm} - g_{\pm}^{ij} C^\pm_i C^\pm_j - \nabla^g_{\pm} g^\pm_{ij} W^\pm_i W^\pm_j + q_{\pm} \right) g^{-1/4}, \quad (3.4)$$

where

$$q_{\pm} = (t g^+_{ij} C^+ i C^+ j + s g^-_{ij} C^- i C^- j) + \frac{1}{2} \nabla^g_i (g^\pm_{ij} W^\pm_j) + \frac{1}{4} g^\pm_{ij} W^+_i W^-_j, \quad (3.5)$$

where $W^\pm_j$ is defined by (1.35).

Notice that the sum of Laplace type operators is a Laplace type operator, in particular, the operator

$$L(t, s) = t L_+ + s L_- \quad (3.6)$$

where

$$L(t, s) = g^{1/4} \left( -\Delta^g + Q \right) g^{-1/4}$$
is a Laplace type operator with the metric $g_{ij}(t, s)$, (1.17), the connection $\nabla_i(t, s)$, (1.18), and the potential form

$$
Q(t, s) = tQ_+ + sQ_- - t^{ij}C_i^jC_j^s - sg_{ij}^-C_i^j 
$$

$$
+ \frac{1}{2} s^i\nabla_j^g(g_{ij}^+W^i_j) + \frac{1}{4} t^{ij}W^+_iW^-_j 
+ \frac{1}{2} s^i\nabla_j^g(g_{ij}^-W^i_j) + \frac{1}{4} sg_{ij}^-W^+_iW^-_j.
$$

Now, assume that $\mathcal{V}$ is a Clifford bundle. Let $D_\pm$ be two self-adjoint first-order elliptic partial differential operators of Dirac type acting on sections of the bundle $\mathcal{V}$ such that its square $L_\pm = D_\pm^2$ is a self-adjoint second-order positive elliptic partial differential operator. Let $\gamma_\pm : T^*M \to \mathcal{V}$ be the Clifford map (traceless matrices) satisfying

$$
\gamma_\pm^i \gamma_\pm^j + \gamma_\pm^j \gamma_\pm^i = 2g_{ij}^\pm I,
$$

where $I$ is the identity endomorphism. The connection $\nabla_\pm$ is defined by requiring it to satisfy

$$
\nabla_\pm^i \gamma_\pm^j = \partial_i \gamma_\pm^j + \Gamma_\pm^i_{\pm j} \gamma_\pm^k + [\mathcal{A}_\pm^i, \gamma_\pm^j] = 0,
$$

where $\Gamma_\pm^i_{\pm j}$ are Christoffel symbols of the metric $g_{ij}^\pm$. Then the Dirac type operators have the form

$$
D_\pm = \frac{1}{4} i\gamma_\pm^i (\nabla_\pm^i + S_\pm) g_\pm^{-1/4}
$$

$$
= \frac{1}{4} i\gamma_\pm^i (\partial_i + \mathcal{A}_\pm^i) g_\pm^{-1/4} + S_\pm,
$$

(3.10)

where $S_\pm$ are some endomorphisms of the vector bundle $\mathcal{V}$. We suppose that $S_\pm$ anticommute with $\gamma_\pm^i$. By using the representation of the Dirac matrices in terms of the orthonormal frames, $\gamma_\pm^i(x) = e_\pm^a(x)\gamma^a$, this means that the matrices $S_\pm$ anti-commute also with the Dirac matrices $\gamma_\pm^i$, that is,

$$
[S_\pm, \gamma_\pm^j] = [S_\pm, \gamma_\pm^i] = 0.
$$

(3.11)

Then $D_\pm^2$ is a Laplace type operator of the form

$$
D_\pm^2 = g_\pm^{1/4} (-\Delta_\pm + Q_\pm) g_\pm^{-1/4},
$$

(3.12)

where

$$
Q_\pm = -\frac{1}{2} \gamma_\pm^{ij} \mathcal{R}_\pm^{ij} + S_\pm^2 + i\gamma_\pm^j \nabla_j S_\pm.
$$

(3.13)

$\gamma_\pm^{ij} = \gamma_\pm^{ij}\gamma_\pm^j$ and $\mathcal{R}_\pm^{ij}$ is the curvature of the connection $\mathcal{A}_\pm^i$. If the Clifford bundle is a twisted spinor bundle then the connection $\mathcal{A}_\pm^i$ has the form

$$
\mathcal{A}_\pm^i = \frac{1}{4} \omega_\pm^{ab} \gamma_\pm^a + E_\pm^i,
$$

(3.14)
where \( \omega_{abi}^\pm \) is the spin connection, and the curvature has the form
\[
\mathcal{R}_{ij}^\pm = \frac{1}{4} R_{abij}^\pm \gamma^{ab} + \mathcal{F}_{ij}^\pm,
\]
where \( \mathcal{F}_{ij}^\pm \) is the curvature of the connection \( \mathcal{E}_{ij}^\pm \) and \( R_{abij}^\pm \) is the Riemann tensor of the metric \( g_{ij}^\pm \).

### 3.2 Heat Traces

Let \( \{\lambda_k^\pm\}_{k=1}^\infty \) be the eigenvalues (counted with multiplicities and ordered in nondecreasing order) and \( \{\varphi_k^\pm\}_{k=1}^\infty \) be the corresponding orthonormal sequence of eigensections of the operator \( L_\pm \). The heat kernel of the operator \( L_\pm \) has the following spectral representation
\[
U_\pm(t; x, x') = \sum_{k=1}^\infty \exp\left(-t\lambda_k^\pm\right) \varphi_k^\pm(x)\varphi_k^{\pm*}(x').
\]
Then the classical heat trace (1.1) has form
\[
\Theta_\pm(t) = \sum_{k=1}^\infty \exp\left(-t\lambda_k^\pm\right)
= \int_M dx \text{tr} U_\pm(t; x, x)
\]
and the combined heat trace and (1.3) is
\[
X(t, s) = \sum_{k,j=1}^\infty \exp\left(-t\lambda_k^\pm - s\lambda_j^-\right) |(\varphi_j^-, \varphi_k^+)|^2
= \int_{M \times M} dx\, dx' \text{ tr} \{U_+(t; x, x')U_-(s; x', x)\}. \tag{3.18}
\]

Let \( \{\mu_k^\pm\}_{k=1}^\infty \) be the eigenvalues of the operator \( D_\pm \) (counted with multiplicities and ordered in nondecreasing order of the absolute value) and \( \{\varphi_k^\pm\}_{k=1}^\infty \) be the corresponding orthonormal sequence of eigensections of the operator. The integral kernel of the heat semigroups \( \exp(-tD_\pm^2 + i\alpha D_\pm) \) and \( \exp(-tD_\pm^2) \) have the form
\[
V_\pm(t, \alpha; x, x') = \sum_{k=1}^\infty \exp\left[-t(\mu_k^\pm)^2 + i\alpha \mu_k\right] \varphi_k^\pm(x)\varphi_k^{\pm*}(x'),
\]
\[
U_\pm(t; x, x') = \sum_{k=1}^\infty \exp\left[-t(\mu_k^\pm)^2\right] \varphi_k^\pm(x)\varphi_k^{\pm*}(x'). \tag{3.20}
\]
Then the classical heat trace (1.2) has the form

\[ H_{\pm}(t) = \sum_{k=1}^{\infty} \mu_k^{\pm} \exp \left[ -t(\mu_k^{\pm})^2 \right] \]

\[ = \int_M dx \operatorname{tr} \{ D_\pm U_\pm(t; x, x) \}, \quad (3.21) \]

where the operators \( D_\pm \) act only on the first argument of the heat kernel, and the combined heat trace (1.6) is

\[ Y(t, s) = \sum_{k, j=1}^{\infty} \exp \left[ -t(\mu_k^{\pm})^2 - s(\mu_j^{-})^2 \right] \mu_k^{\pm} \mu_j^{-} \left| (\varphi_j^{-}, \varphi_k^{\pm}) \right|^2 \]

\[ = \int_{M \times M} dx \, dx' \operatorname{tr} \{ D_+ U_+(t; x, x') D_- U_-(s; x', x) \}, \quad (3.22) \]

where the differential operators act on the first spatial argument of the heat kernel.

The generalized traces (2.73) and (2.74) have the form

\[ W_{\pm}(t, \alpha) = \sum_{k=1}^{\infty} \exp \left[ -t(\mu_k^{\pm})^2 + i\alpha \mu_k^{\pm} \right], \]

\[ = \int_M dx \operatorname{tr} V_{\pm}(t, \alpha; x, x), \quad (3.23) \]

\[ V(t, s; \alpha, \beta) = \sum_{k, j=1}^{\infty} \exp \left[ -t(\mu_k^{\pm})^2 + i\alpha \mu_k^{\pm} - s(\mu_j^{-})^2 + i\beta \mu_j^{-} \left| (\varphi_j^{-}, \varphi_k^{\pm}) \right|^2 \right]. \]

\[ = \int_{M \times M} dx \, dx' \operatorname{tr} \{ V_+(t, \alpha; x, x') V_-(s, \beta; x', x) \}. \quad (3.24) \]

We would like to stress that whereas the classical invariants \( \Theta_{\pm}(t), H_{\pm}(t) \) and \( W_{\pm}(t) \) depend only on the eigenvalues of the operators the new invariants \( X(t, s), Y(t, s) \) and \( V(t, s; \alpha, \beta) \) depend on the eigenfunctions as well and, therefore, contain much more information about the relative spectrum of these operators.

## 4 Ruse-Synge Function

In this section we follow our books [2] [5]. We fix the notation for the rest of the paper. Let \( x' \) be a fixed point in a manifold \( M \). We denote indices of tensors in the
tangent space at the point \( x' \) by prime Latin letters. The derivatives with respect to coordinates \( x'^i \) will be denoted by prime indices as well. We will also use the notation for the partial derivatives of a scalar function \( f \) with respect to \( x \) and \( x' \) by just adding indices to the function after comma, e.g. \( f_{,ij} = \partial_i \partial_j f \). Obviously, the derivatives with respect to \( x \) and with respect to \( x' \) commute. Finally, everywhere below the square brackets denote the diagonal value of a two-point function \( f(x, x') \), that is, \([f] = f(x', x')\). It is also easy to see that the derivatives of the coincidence limits are equal to the sum of the coincidence limits of the derivative with respect to \( x \) and \( x' \)

\[ [f]_{,i} = [f_{,i}] + [f_{,j}] \]  \( (4.1) \)

Let \( g \) be a Riemannian metric and \( r_{\text{inj}}(M, g) \) be the injectivity radius of the manifold \( M \). Let \( B_r(x') \) be the geodesic ball of radius \( r \) less than the injectivity radius of the manifold, \( r < r_{\text{inj}}(M, g) \). Let \( U \subset B_r(x') \) be a sufficiently small neighborhood of the point \( x' \) in the ball \( B_r(x') \) so that it is covered by a single coordinate patch with coordinates \( x^i \).

Each point \( x \) in the neighborhood \( U \) can be connected with the point \( x' \) by a unique geodesic. The Ruse-Synge function \( \sigma(x, x') \) is a symmetric smooth function defined as one half of the square of the geodesic distance \( d(x, x') \) between the points \( x \) and \( x' \),

\[ \sigma(x, x') = \frac{1}{2} d^2(x, x'); \]  \( (4.2) \)

it was introduced by Ruse \([19]\) and used extensively by Synge \([20]\) and others \([12,11]\) in general relativity under the name world function. There are many ways to show that the Ruse-Synge function satisfies the (modified) Hamilton-Jacobi equation

\[ \sigma = \frac{1}{2} g^{ij}(x) \sigma_{,i} \sigma_{,j} = \frac{1}{2} g^{ij}(x') \sigma_{,i} \sigma_{,j}, \]  \( (4.3) \)

with the initial conditions

\[ [\sigma] = [\sigma_{,i}] = [\sigma_{,j}] = 0. \]  \( (4.4) \)

Furthermore, by differentiating eq. \( (4.3) \) and taking the coincidence limit it is easy to see that

\[ [\sigma_{,ij}] = [\sigma_{,i} \sigma_{,j}] = -[\sigma_{,i} \sigma_{,j}] = g_{ij}. \]  \( (4.5) \)

The Hamilton-Jacobi equation \( (4.3) \) with the above initial conditions \( (4.4) \) has a unique solution; it can be solved, for example, in form of a (noncovariant) Taylor
series

\[
\sigma(x, x') = \sum_{k=2}^{\infty} \frac{1}{k!} [\sigma_{ij\ldots k}] (x')^{y^1} \ldots y^k,
\]  

(4.6)

where \( y^i = x^i - x'^i \). The coincidence limits of partial derivatives of higher orders \([\sigma_{ij\ldots k}], k \geq 3\), are uniquely determined in terms of some polynomials in the partial derivatives of the metric \( g_{ijm_1 \ldots m_p} \) and the metrics \( g_{ij} \) and \( g^{ij} \), that is, some polynomials in the partial derivatives \([\sigma_{ij}], m_1 \ldots m_p \) and the matrix \([\sigma_{ij}]\) and its inverse. Therefore, there are non-trivial relations between the coincidence limits of partial derivatives. By using these equations one can obtain the coincidence limits of partial derivatives

\[
[\sigma_{ijk}] = 3g_{ml}\Gamma_{ij}^m = \frac{3}{2}g_{(ijk)},
\]  

(4.7)

\[
[\sigma_{ijkl}] = 4g_{ml}

\Gamma_{ij}^m + 4g_{ml}G^{mn}_{ij} \Gamma_{k}^m + 3g_{nm}G^{mn}_{ij} \Gamma_{k}^m,\]

(4.8)

\[
[\sigma_{ij}'] = -g_{ml}G^{m}_{ij} - g_{ml}(\Gamma_{ij}^m),
\]  

(4.9)

where \( \Gamma^{jk} \) are the Christoffel symbols for the metric \( g \). Here and everywhere below the parenthesis denote the symmetrization over all included indices and the vertical lines denote the indices excluded from the symmetrization.

By differentiating eq. (4.3) we also find

\[
\sigma_{ijk} = g^{ij} \sigma_{jk}' \sigma_{,ij}.
\]  

(4.10)

Let \( \gamma^{ij} \) be the inverse of the matrix of mixed derivatives \( \sigma_{,jk}' \) (it should not be confused with Dirac matrices). Then we obtain

\[
\gamma^{ij} \sigma_{,k} = \sigma^{ij} \sigma_{,j},
\]  

(4.11)

and, therefore, the Ruse-Synge function satisfies a non-trivial equation without any metric

\[
\sigma = \frac{1}{2} \gamma^{ij} \sigma_{,i} \sigma_{,j}.
\]  

(4.12)

This enables one to compute the Ruse-Synge function in terms of diagonal values of its own partial derivatives. The usual Taylor series (4.6) is not symmetric whereas the function \( \sigma(x, x') \) is. Thus, it is more appropriate to represent it in the manifestly symmetric Taylor series. Let us introduce new coordinates

\[
z^i = x^i + x'^i, \quad y^i = x^i - x'^i.
\]  

(4.13)
Then the Ruse-Syngne function is a function of $z$ and $y$

$$\sigma(x, x') = f(z, y).$$  \hfill (4.14)

Then the derivatives are related by

$$\partial_i^x = \frac{1}{2} (\partial_i^x + \partial_i^y), \quad \partial_i^y = \frac{1}{2} (\partial_i^x - \partial_i^y),$$

$$\partial_i^x = \partial_i^x + \partial_i^y, \quad \partial_i^y = \partial_i^x - \partial_i^y.$$  \hfill (4.15)

We can expand the Ruse-Syngne function in the Taylor series in the variables $y$ with coefficients depending on the variables $z$. Since it is symmetric it will only have even powers of $y$,

$$\sigma(x, x') = \sum_{k=1}^{\infty} \frac{1}{(2k)!} F_{i_1 \ldots i_{2k}} (z) y^{i_1} \ldots y^{i_{2k}},$$  \hfill (4.17)

Then the derivatives of the Ruse-Syngne function are

$$\sigma_i = A_i + B_i,$$

$$\sigma_{i'} = A_i - B_i,$$

$$\sigma_{ij} = -F_{ij} + C_{ij} + D_{ij},$$  \hfill (4.20)

where

$$A_i = \sum_{k=1}^{\infty} \frac{1}{(2k)!} F_{i_1 \ldots i_{2k}} y^{i_1} \ldots y^{i_{2k}},$$

$$B_i = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} F_{i_1 \ldots i_{2k+1}} y^{i_1} \ldots y^{i_{2k+1}},$$

$$C_{ij} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (F_{i_1 \ldots i_{2k},ij} - F_{i_1 \ldots i_{2k}ij}) y^{i_1} \ldots y^{i_{2k}},$$

$$D_{ij} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (F_{i_1 \ldots i_{2k+1},ij} - F_{i_1 \ldots i_{2k+1}ij}) y^{i_1} \ldots y^{i_{2k+1}}.$$  \hfill (4.23)

Now, by using these expansions one can compute the expansion of the matrix $\gamma^{ij}$ and then use the equation (4.12) to obtain recursive relation for the coefficients $F_{i_1 \ldots i_k}$. All of the higher-order coefficients $F_{i_1 \ldots i_k}$, with $k \geq 4$, will be determined by the derivatives of the first coefficient $F_{ij}$. 

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*rsi.tex; August 6, 2019; 0:48; p. 19*
The diagonal values of the covariant derivatives of the Ruse-Synge function are expressed in terms of the polynomials of the covariant derivatives of the curvature tensor, in particular,

\[
\nabla^g_i \nabla^g_j \nabla^g_k \sigma = \nabla^g_{k'} \nabla^g_j \nabla^g_i \sigma = \frac{2}{3} R^g_{i(klj)}.
\] (4.26)

That is, the diagonal values of all higher order covariant derivatives of the Ruse-Synge function \( \nabla^g_{i_1} \cdots \nabla^g_{i_k} \nabla^g_j \nabla^g_i \sigma \), with \( k + m \geq 4 \), are expressed in terms of the derivatives \( \sigma_{ij}, i_1 \cdots i_k \) of the diagonal values of the second derivatives \( \sigma_{ij} \).

One can also show that it also satisfies the following coincidence limits \([2]\): for any \( k \geq 2 \),

\[
\nabla^g_{i_1} \cdots \nabla^g_{i_k} \nabla^g_j \nabla^g_i \sigma = \nabla^g_{i_1} \cdots \nabla^g_{i_k} \nabla^g_j \sigma = 0.
\] (4.27)

An important ingredient is the Van Vleck-Morette determinant, defined by

\[
M(x, x') = \det (-\sigma_{ij}(x, x')).
\] (4.28)

it is a two-point density of weight 1 at each point (we denote it by \( M(x, x') \) instead of the usual \( D(x, x') \) to avoid confusion with the Dirac type operators \( D_x \)). Therefore, we find it convenient to define the function

\[
\zeta(x, x') = \frac{1}{2} \log \left( g^{-1/2}(x)M(x, x')g^{-1/2}(x') \right),
\] (4.29)

which is a scalar function at each point. The first coincidence limits of this function are \([2]\)

\[
\begin{align*}
[\zeta] &= [\zeta_{,i}] = 0, \\
[\nabla^g_i \nabla^g_j \zeta] &= \frac{1}{6} R^g_{ij}, \\
[\nabla^g_{(i} \nabla^g_j \nabla^g_{k)} \zeta] &= \frac{1}{4} \nabla^g_{(i} R^g_{jk)}, \\
[\nabla^g_{(i} \nabla^g_j \nabla^g_{k} \nabla^g_{l)} \zeta] &= \frac{3}{10} \nabla^g_{(i} R^g_{jk)} + \frac{1}{15} R^g_{m(i} n R^g_{j} m j)_{n}.
\end{align*}
\] (4.30)

One can also show that the tangent vector to the geodesic connecting the points \( x' \) and \( x \) at the point \( x' \) pointing to the point \( x \) is given by the derivative of the Ruse-Synge function \([2]\)

\[
\xi^j = -g^{jj'} \sigma_{,j'},
\] (4.34)
so that

\[ \sigma = \frac{1}{2} g_{ij} \xi^i \xi^j. \]  

(4.35)

The variables \( \xi^i \) are related to the so called Morse variables; they provide the normal coordinates in geometry. The Jacobian of the transformation \( x \mapsto \xi \) is expressed in terms of the Van Vleck-Morette determinant and for sufficiently close points \( x \) and \( x' \) is not equal to zero. The volume element and the derivatives in these coordinates have the form

\[ dx = M^{-1}(x, x') g(x') d\xi \]
\[ = g^{1/2}(x') g^{-1/2}(x) e^{-2\xi(x, x')} d\xi. \]  

(4.36)

\[ \frac{\partial}{\partial x^i} = -\sigma_{ij} g^{k'} \frac{\partial}{\partial \xi^j}. \]  

(4.37)

Then an arbitrary analytic scalar function \( f \) can be expanded in the covariant Taylor series, \[2\]

\[ f = \sum_{k=0}^{\infty} \frac{1}{k!} f_{i_1 \cdots i_k} \xi^{i_1} \cdots \xi^{i_k}, \]  

(4.38)

where \( f_{i_1 \cdots i_k} = [\nabla_{i_1} \cdots \nabla_{i_k} f](x') \).

One can show that the metric is determined by the Ruse-Synge function as follows. Let \( V \) be the matrix defined by

\[ V_{k'l'} = \sigma_{ij} \gamma^{i'}_{j'} \sigma_{k'l'}. \]  

(4.39)

Further, let \( Y \) be a matrix defined by

\[ Y_{k'l'} = \sigma_{k'l'} - V_{k'l'}, \]  

(4.40)

and \( X = (X^r'_{i'}) \) be the inverse of the matrix \( Y \). Then the matrix \( X \) is given by the series

\[ X = \sum_{n=0}^{\infty} (\beta V)^n \beta, \]  

(4.41)

where \( \beta^{i'j'} \) is the inverse of the matrix \( \sigma_{k'l'} \), that is,

\[ X^{i'j'} = \beta^{i'j'} + \beta^{i'k'} V_{k'p'} \beta^{p'j'} + \beta^{i'k'} V_{k'p'} \beta^{p'q'} V_{q'r'} \beta^{r'j'} + \cdots \]

By differentiating eq. (4.11) with respect to \( x'^i \) we obtain

\[ g^{ij} \sigma_{ij} = \gamma^{i'}_{j'} Y_{k'l'}. \]  

(4.42)

Finally, by multiplying by the matrix \( \gamma^{i'j'} \) we prove the following lemma.
Lemma 1 The metric is uniquely determined by the partial derivatives of the Ruse-Synge function by

\[ g_{ij} = \gamma^k \gamma^l Y_{k'l}, \]  
\[ g_{ij} = \sigma_{ik} \sigma_{jl} X_{k'l}. \]  

Even though the metric is determined by the off-diagonal derivatives of \( \sigma \) it does not depend on the point \( x' \). Also, of course for \( x = x' \) we get \( g_{ij}(x') = g_{ij}(x) \).

Notice that the matrix \( V \) is of first order in \( y^i = x^i - x'^i \); therefore, this power series is well defined near diagonal. Thus, we obtain for the metric

\[ g_{ij} = \sigma_{ik} \sigma_{jl} \beta_{k'l} + \sigma_{ik} \sigma_{jl} \beta_{k'm'} V_{m'q'} \beta_{q'l'} + \cdots \]  

Therefore, one can find the metric in terms of the Taylor series

\[ g_{ij}(x) = \sum_{k=0}^{\infty} \frac{1}{k!} g_{ij,k_{i_1}...k_{i_k}}(x') y^{i_1} \cdots y^{i_k}, \]  
\[ g_{ij}(x) = \sum_{k=0}^{\infty} \frac{1}{k!} g_{ij,k_{i_1}...k_{i_k}}(x') y^{i_1} \cdots y^{i_k}. \]  

The Taylor coefficients \( g_{ij,k_{i_1}...k_{i_k}}(x') \) and \( g_{ij,k_{i_1}...k_{i_k}}(x') \) are expressed in terms of polynomials in the coincidence limits \([\sigma_{k'l_i} ... i_p] \) and \([\sigma_{k'p_{i_1} ... i_p}] \) and the metric \( g_{ij}(x') \). This gives an expression for the metric entirely in terms of the partial derivatives of the Ruse-Synge function.

Finally, we study the dependence of the Ruse-Synge function on the metric. Let \( h_{ij} \) be another metric and \( \sigma^h(x,x') \) be the Ruse-Synge function for the metric \( h \). We will need to study the covariant Taylor expansion of this function in the power series in the variables \( \xi^i \) defined by (4.34) (with respect to the metric \( g \)), that is,

\[ \sigma^h(x,x') = \sum_{k=2}^{\infty} \frac{1}{k!} \left[ \nabla^g_{i_1} \cdots \nabla^g_{i_k} (\sigma^h)(x') \right] \xi^{i_1} \cdots \xi^{i_k}. \]  

To avoid confusion we use the notation \( \nabla^g \) and \( \nabla^h \) to denote the covariant derivatives with respect to the metrics \( g \) and \( h \). All indices will be raised and lowered by the metric \( g \).

The function \( \sigma^h \) satisfies the equation

\[ \sigma^h = \frac{1}{2} h^{ij} \sigma^h_{ij}. \]
with the initial conditions
\[ [\sigma^h] = [\sigma^g] = 0; \] (4.50)
therefore, the first two terms in the Taylor series of the function \( \sigma^h \) vanish.

The non-compatibility of the metric \( h \) and \( g \) is measured by the non-metricity tensor
\[ K_{ijk} = \nabla_i h_{jk} \] (4.51)
and the disformation tensor
\[ W^i_{jk} = \Gamma^i_{hjk} - \Gamma^i_{gjk}, \] (4.52)
where \( \Gamma^i_{hjk} \) are the Levi-Civita connections of the metrics \( h \) and \( g \). These two tensors are related by
\[ W^i_{jk} = \frac{1}{2} h^{im} \left( K_{jkm} + K_{kmj} - K_{mjk} \right), \] (4.53)
\[ K_{ijk} = h_{km} W^m_{ij} + h_{jm} W^m_{ik}. \] (4.54)

The covariant derivatives with respect to the metrics \( g \) and \( h \) are related by
\[ \nabla^h_i T^k_{jl} = \nabla^g_i T^k_{jl} + W^l_{im} T^m_{ij} - W^m_{ij} T^l_{m}. \] (4.55)

We introduce the following scalar \( W = \frac{1}{2} \log \left( \frac{h}{g} \right) \) with \( h = \det h_{ij} \), \( g = \det g_{ij} \), and the vector \( W_j = \partial_j W \), then
\[ W^i_{ij} = \frac{1}{2} h^{kl} K_{jkl} = W_j. \] (4.56)

The Riemann tensors are related by
\[ R^h_{i jkl} = R^g_{i jkl} + \nabla^g_k W^i_{lj} - \nabla^g_l W^i_{kj} + W^i_{km} W^m_{lj} - W^m_{ij} W^l_{mk}. \] (4.57)

One has to be careful with this equation when lowering or raising indices. For example, the Ricci tensors are obtained by just contracting the indices
\[ R^h_{jl} = R^g_{jl} + \nabla^g_i W^i_{lj} - \nabla^g_j W^i_{li} + W^i_{m} W^m_{lj} - W^l_{mi} W^m_{ij}. \] (4.58)
but for the Riemann tensor \( R^h_{ijkl} \) with all indices lowered we have to use the metric \( h_{ij} \) and, therefore, it will not be directly related to the Riemann tensor \( R^g_{ijkl} \), which is obtained by using the metric \( g_{ij} \), that is,
\[ R^h_{ijkl} = h_{ni} g^{im} R^g_{m jkl} + h_{ni} \nabla^g_k W^i_{lj} - h_{ni} \nabla^g_l W^i_{kj} + h_{ni} W^i_{km} W^m_{lj} - h_{ni} W^l_{mi} W^m_{ij}. \] (4.59)
We will need to compute the following tensors determined by the diagonal values of the symmetrized covariant derivatives with respect to the metric $g$ of the Ruse-Synge function $\sigma^h$ of the metric $h$ and the vectors $\sigma^j$ and $\sigma^h$,

\begin{align}
S_{i_1 \ldots i_k} &= [\nabla^g_{(i_1} \cdots \nabla^g_{i_k)} \sigma^h], \\
T_{j_1 \ldots j_k} &= [\nabla^g_{(j_1} \cdots \nabla^g_{j_k)} \nabla^g \sigma^h], \\
V_{j_1 \ldots j_k} &= [\nabla^g_{(j_1} \cdots \nabla^g_{j_k)} \nabla^g \sigma^h].
\end{align}

(4.60) (4.61) (4.62)

We know that

\begin{align}
[\nabla^h_i \sigma^j] &= 0, \\
[\nabla^h_j \nabla^h_i \sigma^j] &= h_{ij}, \\
[\nabla^h_k \nabla^h_j \nabla^h_i \sigma^j] &= [\nabla^h_k \nabla^h_j \nabla^h_i \sigma^h] = 0, \\
[\nabla^h_j \nabla^h_k \nabla^h_i \sigma^j] &= -[\nabla^h_j \nabla^h_k \nabla^h_i \sigma^h] = -\frac{2}{3} R^h_{i(jk)}. \tag{4.66}
\end{align}

By using these equations and the relation (4.55) between the covariant derivatives we obtain

\begin{align}
[\nabla^g_i \sigma^h] &= S_i = T_i = V_i = 0, \\
[\nabla^g_j \nabla^g_i \sigma^h] &= -[\nabla^g_j \nabla^g_i \sigma^h] = S_{ij} = T_{ij} = -V_{ij} = h_{ij}, \\
[\nabla^g_{k j} \nabla^g_{i j} \sigma^h] &= S_{ijk} = T_{ijk} = 3 h_{mi} W^m_{jk} = \frac{3}{2} K_{i(jk)}. \tag{4.69}
\end{align}

Also, by using the relation (4.1) and (4.69) (or (4.65) and (4.55)) we obtain

\begin{align}
[\nabla^g_i \nabla^g_j \nabla^g_l \sigma^h] &= V_{ijk} = -h_{mi} W^m_{kj}. \tag{4.70}
\end{align}

Similarly, we compute

\begin{align}
[\nabla^g_{(i} \nabla^g_{k j)} \nabla^g_{l j)} \nabla^g_{p j} \sigma^h] &= T_{ijkl} \\
&= 3 h_{mi} \nabla^g_{(i} W^m_{jl)} + h_{mi} \nabla^g_{(j} W^m_{kl)} + 3 h_{mi} W^m_{kl} W^m_{lj} \\
&+ h_{mi} W^m_{(jk} W^m_{lj)} + 3 h_{mn} W^m_{(jk} W^m_{lj)}, \tag{4.71}
\end{align}

\begin{align}
[\nabla^g_{(i} \nabla^g_{k j)} \nabla^g_{l j)} \nabla^g_{p j} \sigma^h] &= V_{ijkl} \\
&= -h_{mi} \nabla^g_{(i} W^m_{kl)} - h_{mi} W^m_{(jk} W^m_{lj)}, \tag{4.72}
\end{align}

\begin{align}
[\nabla^g_{(i} \nabla^g_{k j)} \nabla^g_{p j} \nabla^g_{l j)} \sigma^h] &= S_{ijkl} \tag{4.73}
\end{align}

\begin{align}
&= 4 h_{mi} \nabla^g_{(i} W^m_{kl)} + 4 h_{mi} W^m_{(jk} W^m_{lj)} + 3 h_{mn} W^m_{(ij} W^m_{kl)}.
\end{align}
This can also be written in terms of the tensors $K_{ij}$

$$
S_{ijkl} = 2\nabla^g_{(i} K_{jkl)} - h_{ij} K_{km} K_{k^{lm}} + h_{ik} K_{j^{lm} K_{m}^{jl}} - \frac{1}{4} h_{ij} K_{m(j} K_{k^{lm} K_{l}^{ik})}.
$$

(4.74)

Let $\nabla^g$ be a connection on a vector bundle $V$ over a manifold $M$. It defines the operator of parallel transport $P_{g;A}(x, x')$ of sections of the vector bundle $V$ along geodesics of the metric $g$ from the point $x'$ to the point $x$. It satisfies the equation of parallel transport \cite{2}

$$
g^{ij} \sigma_{ij}^g \nabla^g_{i} P_{g;A} = 0
$$

(4.75)

with the initial condition

$$
[P_{g;A}] = I,
$$

(4.76)

where $I$ is the identity endomorphism. By using these equations we obtain the coincidence limits of partial derivatives

$$
[P_{g;A}] = -A_i
$$

(4.77)

$$
[P_{g;A}]_{ij} = -A_{i(j} + A_{i[j} A_{j]}.
$$

(4.78)

The coincidence limits of covariant derivatives are

$$
[\nabla^g_{i} P_{g;A}] = 0,
$$

(4.79)

$$
[\nabla^g_{i} \nabla^g_{j} P_{g;A}] = \frac{1}{2} R_{ij}^g,
$$

(4.80)

where $R_{ij}^g$ is the curvature of the connection $\nabla^g$. Moreover, one can show that the diagonal values of the symmetrized covariant derivatives vanish,

$$
[\nabla^g_{(i} \cdots \nabla^g_{k) P_{g;A}}] = 0.
$$

(4.81)

Now, suppose that there is another metric $h$ and another connection $\nabla^h$ and $P_{h;B}$ be the corresponding operator of parallel transport. We need to compute the diagonal values of the derivatives $[\nabla^g_{(i} \cdots \nabla^g_{k) P_{h;B}}]$. The difference of the connection one-forms defines the tensor

$$
C_i = B_i - A_i,
$$

(4.82)

so that

$$
\nabla^g_{i} B_{j} = \nabla^h_{i} B_{j} - C_i P_{h;B}.
$$

(4.83)

By using the eqs. (4.79), (4.80), (4.83) we obtain

$$
[\nabla^g_{i} P_{h;B}] = -C_i,
$$

(4.84)

$$
[\nabla^g_{i} \nabla^g_{j} P_{h;B}] = -\nabla^g_{(i} C_{j)} + C_i C_j.
$$

(4.85)
5 Asymptotics of Integrals

We use the Laplace method to compute the asymptotics as \( \varepsilon \to 0 \) of Laplace type integrals

\[
F(\varepsilon) = (4\pi\varepsilon)^{-n/2} \int_{U} dx \exp\left(-\frac{1}{2\varepsilon}\Sigma(x, x')\right) \varphi(x),
\]

with some positive smooth function \( \Sigma \) and a smooth function \( \varphi \) over a sufficiently small neighborhood \( U \) of a point \( x' \) in a manifold \( M \).

5.1 Gaussian Integrals on Riemannian Manifolds

First of all, we recall the standard Gaussian integrals. Let \( G_{ij} \) be a real symmetric positive matrix, \( G^{ij} \) be its inverse, \( G = \det G_{ij} \) and \( \langle y, Gy \rangle = G_{ij}y^iy^j \). We define the Gaussian average of a smooth function \( f \) on \( \mathbb{R}^n \) by

\[
\langle f \rangle_G = (4\pi)^{-n/2}G^{1/2} \int_{\mathbb{R}^n} dy \exp\left(-\frac{1}{4}\langle y, Gy \rangle\right)f(y). \tag{5.2}
\]

Then the Gaussian average of the odd monomials vanish and the average of the monomials for any \( k \geq 0 \) are (see, e.g. [5, 18])

\[
\langle y^{i_1}\cdots y^{i_{2k}} \rangle_G = \frac{(2k)!}{k!} G^{i_1i_2}\cdots G^{i_{2k-1}i_{2k}}. \tag{5.3}
\]

By integrating by parts it is easy to obtain a useful relation for the averages of derivatives

\[
\langle \partial_{i_1}\cdots \partial_{i_k} f \rangle_G = \langle H_{i_1...i_k} f \rangle_G, \tag{5.4}
\]

where \( H_{i_1...i_k} \) are Hermite polynomials defined by [13]

\[
H_{i_1...i_k}(y) = (-1)^k \exp\left(\frac{1}{4}\langle y, Gy \rangle\right) \partial_{i_1}\cdots \partial_{i_k} \exp\left(-\frac{1}{4}\langle y, Gy \rangle\right) = (-1)^k D_{i_1}\cdots D_{i_k} \cdot 1, \tag{5.5}
\]

and

\[
D_i = \partial_i - \frac{1}{2} G_{ij}y^j. \tag{5.6}
\]

As a result, the Gaussian average of a Hermite polynomial of degree \( k \) with any polynomial \( f \) of degree less than \( k \) vanishes

\[
\langle H_{i_1...i_k} f \rangle_G = 0, \tag{5.7}
\]
and the average of the product of Hermite polynomials of the same degree is
\[ \langle \mathcal{H}_{i_1...i_k} \mathcal{H}_{j_1...j_k} \rangle_G = \frac{k!}{2^k} G_{i_1(i_1} \cdots G_{j_k)j_k}. \] (5.8)

By the same trick one could get the relations
\[ \langle y^i f \rangle_G = 2G^{ij} \langle \partial_j f \rangle_G, \] (5.9)
\[ \langle y^i y^j f \rangle_G = 2G^{ij} \langle f \rangle_G + 4G^{ik}G^{jm} \langle \partial_k \partial_m f \rangle_G, \] (5.10)

etc.

**Lemma 2** Let \( U \) be an open set in \( \mathbb{R}^n \) containing the origin and \( \varphi \) be a smooth real function on \( U \). Let \( \varepsilon > 0 \) be a positive real parameter, and
\[ F(\varepsilon) = (4\pi\varepsilon)^{-n/2} \int_U dy \exp \left( -\frac{1}{4\varepsilon} \langle y, Gy \rangle \right) \varphi(y). \] (5.11)

Then there is the asymptotic expansion of the integral \( F(\varepsilon) \) as \( \varepsilon \to 0^+ \), independent of \( U \),
\[ F(\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k c_k, \] (5.12)
where
\[ c_k = \frac{1}{k!} G^{i_1i_2} \cdots G^{i_{2k-1}i_{2k}} G^{-1/2} \varphi_{i_1...i_{2k}}, \] (5.13)
and \( \varphi_{i_1...i_k} = \varphi_{i_1...i_k}(0) \).

**Remark.** This can also be written as
\[ c_k = \frac{1}{k!} (\Delta_G G^{-1/2} \varphi)(0), \] (5.14)
where
\[ \Delta_G = G^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}. \] (5.15)

**Proof.** This lemma can be proved by using the Taylor expansion. The open set \( U \) must contain an open ball \( B_\delta(0) \) of some radius \( \delta > 0 \) centered at the origin. After rescaling of the variables \( y^i \mapsto \sqrt{\varepsilon} y^i \) the domain of the integration becomes \( U_\varepsilon \) containing the ball \( B_{\delta/\sqrt{\varepsilon}}(0) \) of radius \( \delta/\sqrt{\varepsilon} \) and as \( \varepsilon \to 0 \) it becomes the whole space \( \mathbb{R}^n \). The calculation of Gaussian average gives then the result.
We will need the following Lemma to compute the coefficients of the asymptotic expansion. We use the notation introduced at the beginning of this section. We pick a metric $g$ and let $R^g_{i,j,k,l}$ be the Riemann tensor, $R^g_{i,j}$ be the Ricci tensor, $R_g$ be the scalar curvature, $\nabla^g_i$ be the covariant derivative (also denoted by the semicolon ;) and $\Delta_g$ be the scalar Laplacian of the metric $g$. Further, let $\sigma$ be the Ruse-Synge function of this metric and $\zeta$ be the modified Van Vleck-Morette determinant. We generalize the Gaussian integrals in the Euclidean space to Riemannian manifolds by replacing the quadratic form in the exponential by the Ruse-Synge function.

**Lemma 3** Let $U$ be a sufficiently small neighborhood of a fixed point $x'$ in a manifold $M$, $\varphi(x,x')$ be a smooth scalar density of weight 1, and

$$F(\varepsilon) = (4\pi\varepsilon)^{-n/2} \int_U dx \exp \left( -\frac{\sigma(x,x')}{2\varepsilon} \right) \varphi(x).$$

(5.16)

Then as $\varepsilon \to 0^+$ there is the asymptotic expansion independent of $U$

$$F(\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k c_k,$$

(5.17)

where

$$c_k = \frac{1}{k!} g^{i_1 i_2} \cdots g^{i_{2k-1} i_{2k}} \hat{\varphi}_{i_1 \cdots i_{2k}},$$

(5.18)

and

$$\hat{\varphi}_{i_1 \cdots i_k} = \left[ \left( \nabla^g_{i_1} - 2\zeta_{i_1} \right) \cdots \left( \nabla^g_{i_k} - 2\zeta_{i_k} \right) \left( g^{-1/2} \varphi \right) \right](x').$$

(5.19)

The coefficients $c_k$ are polynomial in the derivatives of the curvature of the metric $g$ and linear in the derivatives of the function $\varphi$. In particular,

$$c_0 = g^{-1/2}[\varphi],$$

(5.20)

$$c_1 = [\Delta_g g^{-1/2} \varphi] - \frac{1}{3} R_g g^{-1/2}[\varphi],$$

(5.21)

$$c_2 = \frac{1}{2} g^{i j} g^{k l} \left[ \nabla^g_{(i} \nabla^g_{j} \nabla^g_{k} \nabla^g_{l)} (g^{-1/2} \varphi) \right] - \frac{2}{3} R_g^{ij} \left[ \nabla^g_{(i} \nabla^g_{j)} (g^{-1/2} \varphi) \right]$$

$$- \frac{1}{3} R_g [\Delta_g (g^{-1/2} \varphi)] - \frac{2}{3} R_g^{ij} [\nabla^g_{i} (g^{-1/2} \varphi)]$$

$$+ \left( \frac{1}{18} R_g^2 - \frac{1}{5} (\Delta_g R_g) + \frac{4}{45} R_g^{ij} R_g^{ij} - \frac{1}{30} R_g^{ijkl} R_g^{ijkl} \right) g^{-1/2}[\varphi].$$

(5.22)
Proof. Let \( \hat{\xi}^g = -g^{ij} \sigma_{j}^{g} \) and \( |\hat{\xi}|^2 = g_{ij} \hat{\xi}^i \hat{\xi}^j \). Then by changing the variables \( x \mapsto \xi \) and using eq. (4.36) we obtain

\[
F(\varepsilon) = (4\pi \varepsilon)^{-n/2} \int_{\mathcal{U}} d\xi \ g^{1/2}(x') \exp \left( -\frac{1}{4\varepsilon} |\xi|^2 \right) \hat{\phi}(\xi). \tag{5.23}
\]

where \( \mathcal{U} \) is the corresponding domain in the variables \( \xi \) and

\[
\hat{\phi}(\xi) = \exp \{-2\zeta(x, x') g^{-1/2}(x) \varphi(x)\}. \tag{5.24}
\]

We rescale the variables \( \xi \mapsto \sqrt{\varepsilon} \xi \). Then as \( \varepsilon \to 0 \) we can extend the integration domain to the whole space \( \mathbb{R}^n \); this does not affect the asymptotic expansion. Therefore, the asymptotic expansion is determined by the Gaussian average

\[
F(\varepsilon) \sim \langle \hat{\phi}(\sqrt{\varepsilon} \xi) \rangle_g. \tag{5.25}
\]

Next, we expand the function \( \hat{\phi} \) in the covariant Taylor series

\[
\hat{\phi}(\sqrt{\varepsilon} \xi) = \sum_{k=0}^{\infty} \frac{\xi^{k/2}}{k!} \hat{\phi}_{i_1 \cdots i_k} \xi^{i_1} \cdots \xi^{i_k}, \tag{5.26}
\]

where

\[
\hat{\phi}_{i_1 \cdots i_k} = \left[ \nabla_{i_1} \cdots \nabla_{i_k} e^{-2\zeta g^{-1/2} \varphi} \right](x')
\]

\[
= \left[ \left( \nabla_{i_1} g^{-1/2} - 2\zeta_{i_1} \right) \cdots \left( \nabla_{i_k} g^{-1/2} - 2\zeta_{i_k} \right) \right] (g^{-1/2} \varphi)(x'). \tag{5.27}
\]

and compute the Gaussian average over \( \xi \) to get the result.

Notice that the diagonal values of the derivatives of the function \( \zeta \), and, therefore, the coefficients \( \hat{\phi}_{i_1 \cdots i_k} \) and \( c_k \), are polynomial in the derivatives of the curvature of the metric \( g \). By using (4.30) we obtain, in particular,

\[
\hat{\phi} = g^{-1/2}[\varphi], \tag{5.28}
\]

\[
\hat{\phi}_i = [\nabla_i (g^{-1/2} \varphi)], \tag{5.29}
\]

\[
\hat{\phi}_{ij} = \left[ \nabla^g_{\tilde{i}} \nabla^g_{\tilde{j}} (g^{-1/2} \varphi) \right] - 2[\zeta_{ij} g^{-1/2} \varphi], \tag{5.30}
\]

\[
\hat{\phi}_{ijk} = \left[ \nabla^g_{\tilde{i}} \nabla^g_{\tilde{j}} \nabla^g_{\tilde{k}} (g^{-1/2} \varphi) \right] - 6[\zeta_{ij} \nabla^g_{\tilde{k}} (g^{-1/2} \varphi)] + 2[\zeta_{ijk} g^{-1/2} \varphi], \tag{5.31}
\]

\[
\hat{\phi}_{ijkl} = \left[ \nabla^g_{\tilde{i}} \nabla^g_{\tilde{j}} \nabla^g_{\tilde{k}} \nabla^g_{\tilde{l}} (g^{-1/2} \varphi) \right] - 12[\zeta_{ij} \nabla^g_{\tilde{k}} \nabla^g_{\tilde{l}} (g^{-1/2} \varphi)] - 8[\zeta_{ijkl} g^{-1/2} \varphi]
\]

\[+ \left( 12[\zeta_{ij} \zeta_{kl} \zeta_{ij} \zeta_{kl}] - 2[\zeta_{ijkl} \zeta_{ijkl}] \right) g^{-1/2}[\varphi]. \tag{5.32}
\]

Finally, by using the diagonal values of the derivatives of the function \( \zeta \), (4.30), and (4.31), (4.33), we obtain the coefficients \( c_0, c_1 \) and \( c_2 \). Of course, in the case of the flat metric we recover the earlier result (5.13).
5.2 Morse Lemma

We say that a smooth real valued symmetric function \( \Sigma(x, x') \) on \( M \times M \) has a non-degenerate critical point on the diagonal if:

1. the first derivatives vanish on the diagonal, \( [\Sigma_j] = 0 \), and
2. the Hessian is positive definite on the diagonal, \( G_{ij} = [\Sigma_{ij}] > 0 \).

**Lemma 4** Let \( U \) be a sufficiently small open set in a manifold \( M \), \( \Sigma : U \times U \to \mathbb{R} \) be a smooth real valued symmetric non-negative function that has a non-degenerate critical point on the diagonal and vanishes on the diagonal, that is, \( [\Sigma] = [\Sigma_{i}j] = 0 \) and \( G_{ij} = [\Sigma_{ij}] > 0 \). Then there exists a local diffeomorphism \( \eta^i = \eta^i(x, x') \) such that the function \( \Sigma \) has the form

\[
\Sigma(x, x') = \frac{1}{2} G_{ab}(x') \eta^a(x, x') \eta^b(x, x').
\] (5.33)

**Proof.** We pick some metric \( g_{ij} \) and define the corresponding Ruse-Synge function \( \sigma(x, x') \) and the variables \( \xi^i = -g^{ij} \sigma_{j} \) introduced in (4.34). We expand the function \( \Sigma \) in the covariant Taylor series

\[
\Sigma(x, x') = \sum_{k=2}^{\infty} \frac{1}{k!} \Sigma_{i_1...i_k} \xi^{i_1} \cdots \xi^{i_k},
\] (5.34)

where \( \Sigma_{i_1...i_k} = [\Sigma_{ij...ik}] (x') \). This can be written in the form

\[
\Sigma(x, x') = \frac{1}{2} A_{\nu \rho}(x, x') \xi^{\nu} \xi^{\rho},
\] (5.35)

where

\[
A_{\nu \rho}(x, x') = \sum_{k=0}^{\infty} \frac{2}{(k+2)!} \Sigma_{\nu \rho i_1...i_k} \xi^{i_1} \cdots \xi^{i_k}.
\] (5.36)

The matrix \( A \) is real and symmetric, so it can be written (nonuniquely) in the form \( A = B^T H B \); that is,

\[
A_{\nu \rho} = G_{ab} B^{\nu \rho}_{a} B^{b \rho}.
\] (5.37)

The matrix \( B \) is defined up to an orthogonal matrix, that is, up to a transformation \( B \mapsto UB \) with the matrix \( U \) satisfying \( U^T G U = G \). Then the function \( \Sigma \) takes the Morse form (5.33) with \( \eta^i = B^{a \nu} \xi^{a} \). The Morse diffeomorphism is obviously also defined up to an orthogonal transformation \( \eta \mapsto U \eta \).
The Morse diffeomorphism \( \eta^a = \eta^a(x, x') \) can be computed explicitly in terms of the Taylor series
\[
\eta^a = \sum_{k=1}^{\infty} \frac{1}{k!} \eta^a_{i_1 \ldots i_k} \xi^{i_1} \cdots \xi^{i_k};
\]
(5.38)
the coefficients \( \eta^a_{i_1 \ldots i_k} \) will be expressed in terms of the derivatives \( \Sigma^a_{i_1 \ldots i_k} \) of the function \( \Sigma \) on the diagonal at the point \( x' \). They can be obtained by substituting this Taylor series in (5.33) and comparing it with (5.34). The solution is not unique. The first coefficient can be chosen to be a frame of vectors \( \eta^a_{i_1} \) at the point \( x' \) determined by
\[
G_{ab} \eta^a_{i_1} \eta^b_{i_1} = \Sigma^a_{i_1};
\]
(5.39)
Of course, the matrix \( \eta^a_{i_1} \) is defined up to an orthogonal transformation.

### 5.3 Asymptotics of Laplace Type Integrals

We will need the following lemma. We fix some metric \( g_{ij} \); all covariant derivatives and the curvature are defined with respect to this metric.

**Lemma 5** Let \( x' \) be a point in a manifold \( M \) and \( U \) be a sufficiently small neighborhood of this point. Let \( \Sigma : U \times U \to \mathbb{R} \) be a smooth real valued symmetric non-negative function that has a non-degenerate critical point on the diagonal and vanishes on the diagonal. Let \( \varphi(x, x') \) be a smooth scalar density of weight 1 and
\[
F(\epsilon) = (4\pi\epsilon)^{-n/2} \int_U dx \exp\left(-\frac{1}{2\epsilon} \Sigma(x, x')\right) \varphi(x, x').
\]
(5.40)
Then there is the asymptotic expansion as \( \epsilon \to 0^+ \)
\[
F(\epsilon) \sim \sum_{k=0}^{\infty} \epsilon^k F_k.
\]
(5.41)
The coefficients \( F_k \) do not depend on the domain \( U \); they depend only on the derivatives of the functions \( \varphi \) and \( \Sigma \) at the point \( x' \).

Let \( \Sigma^g_{i_1 \ldots i_k} = [\nabla_{i_1} \cdots \nabla_{i_k} \Sigma](x') \) be the symmetrized covariant derivatives of the function \( \Sigma \) on the diagonal at the point \( x' \), in particular, let \( G_{ij} = [\Sigma_{ij}](x') \) be the Hessian on the diagonal, \( G^{ij} \) be the inverse of this matrix and \( G = \det G_{ij} \) be its determinant. Let \( \phi^g_{i_1 \ldots i_k} = [\nabla_{i_1} \cdots \nabla_{i_k} G^{-1/2} \phi](x') \) be the symmetrized covariant derivatives of the function \( \phi \) on the diagonal at the point \( x' \). Then:
1. the coefficients $F_k$ have the form $F_k = G^{-1/2} \tilde{F}_k$, where

2. $\tilde{F}_k$ are linear in the derivatives, $\varphi_{i_1, \ldots, i_k}$, of the function $\varphi$ at the point $x'$ and

3. polynomial in the inverse Hessian, $G^i_j$, and the derivatives, $\Sigma_{i_1, \ldots, i_k}$, $k > 2$, of the function $\Sigma$ on the diagonal at $x'$ of order higher than 2.

4. The first two coefficients are

\begin{align*}
F_0 &= G^{-1/2}[\varphi], \\
F_1 &= G^{-1/2}g^{1/2} \left\{ G^{ij} [\nabla_i^g \nabla_j^g (g^{-1/2} \varphi)] - G^{ij} G^{pq} \Sigma_{ijpq} [\nabla_i^g (g^{-1/2} \varphi)] \right\} + \left[ \frac{1}{3} G^{ij} R_{ij}^g + \frac{1}{12} \left( 2 G^{ij} G^{jm} + 3 G^{ij} G^{ln} \right) G^k_l \Sigma_{ijkl} \Sigma_{mn} \right. \\
&\left. - \frac{1}{4} G^{ij} G^{kl} \Sigma_{ijkl} \right] g^{-1/2}[\varphi].
\end{align*}

5. In the case when the function $\varphi$ and its first derivative vanish on the diagonal, $[\varphi] = \nabla_i^g [\varphi] = 0$, the third coefficient is

\begin{align*}
F_2 &= g^{1/2}G^{-1/2} \left\{ \frac{1}{2} G^{ij} G^{kl} [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g g^{-1/2} \varphi] - \frac{1}{3} \left( 2 G^{ij} G^{qk} + 3 G^{iq} G^{jk} \right) G^{pq} \Sigma_{ijpq} [\nabla_i^g \nabla_k^g \nabla_l^g g^{-1/2} \varphi] \right. \\
&\left. + \frac{1}{12} \left( G^{ij} G^{kl} + 2 G^{ik} G^{jl} \right) R_{ij}^g - \frac{1}{4} \left( G^{pq} G^{kl} + 4 G^{kp} G^{qj} \right) G^{pq} \Sigma_{ijpq} \right. \\
&\left. + \frac{1}{72} \left( 2 G^{ij} G^{pq} G^{qr} G^{st} + 3 G^{ij} G^{pq} G^{rs} G^{kl} + 6 G^{ik} G^{jl} G^{pq} G^{rs} \right. \\
&\left. + 12 G^{ij} G^{pq} G^{kr} G^{ls} + 12 G^{ij} G^{pq} G^{kl} G^{rs} \right) \Sigma_{ijpq} \Sigma_{jrs} \left[ \nabla_i^g \nabla_j^g g^{-1/2} \varphi \right] \right\}.
\end{align*}

Remark. Notice that the metric $g$ is arbitrary, in particular, it could be taken to be equal to the Hessian $g_{ij} = G_{ij}$.

Proof. First, it is easy to show that the asymptotic expansion does not depend on the size of the domain $U$; so, it can be assumed to be sufficiently small. Then
for a sufficiently small $U$ there is a Morse diffeomorphism $\eta^a = \eta^a(x, x')$ so that 
\[ \Sigma = \frac{1}{2} G_{ab} \eta^a \eta^b. \]
Then the integral takes the form
\[ F(\varepsilon) = (4\pi \varepsilon)^{-n/2} \int_{\tilde{U}} d\eta \exp \left( -\frac{1}{4\varepsilon} \langle \eta, G\eta \rangle \right) f(\eta), \tag{5.46} \]
where $\tilde{U}$ is the corresponding domain for the variables $\eta$ and
\[ f(\eta) = \left( \det \left( \frac{\partial \eta^a}{\partial x^i} \right) \right)^{-1} \varphi(x(\eta)). \tag{5.47} \]
Now, by applying Lemma 2 we get the asymptotic expansion
\[ F(\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k F_k, \tag{5.48} \]
where $F_k = \frac{1}{\tilde{U}} G^{-1/2} (\Delta^0_U f)(0)$.

The coefficients $F_k$ can be computed explicitly now by using the Taylor series. We decompose the function $\Sigma$ via
\[ \Sigma = \frac{1}{2} G_{\xi^j} \xi^j \xi^j + \hat{\Sigma}, \tag{5.49} \]
where $\xi^j$ are the variables introduced in (4.34) and
\[ \hat{\Sigma}(\xi) = \sum_{k=3}^{\infty} \frac{1}{k!} \Sigma_{\xi^j_1...\xi^j_k} \xi^j_1 \cdots \xi^j_k, \tag{5.50} \]
where $\Sigma_{\xi^j_1...\xi^j_k} = [\Sigma_{\delta j_1...\delta j_k}](x')$. Then by changing the variables $x \mapsto \xi$ and using (4.36) the integral takes the form
\[ F(\varepsilon) = (4\pi \varepsilon)^{-n/2} \int_{\tilde{U}} d\xi \ g^{1/2}(x') \exp \left( -\frac{1}{4\varepsilon} \langle \xi, G\xi \rangle \right) \psi(\xi, \varepsilon), \tag{5.51} \]
where $\tilde{U}$ is the modified domain and
\[ \psi(\xi, \varepsilon) = \exp \left( -\frac{1}{2\varepsilon} \hat{\Sigma}(\xi) \right) \hat{\varphi}(\xi), \tag{5.52} \]
with $\hat{\varphi}(\xi)$ defined by (5.24).
Next, by rescaling the variables \( \xi' \mapsto \sqrt{\varepsilon} \xi' \), we extend the integration to the whole space \( \mathbb{R}^n \) so that the asymptotics of the integral is given by the Gaussian average

\[
F(\varepsilon) \sim g^{1/2} G^{-1/2} \langle \psi(\sqrt{\varepsilon} \xi, \varepsilon) \rangle_G. \tag{5.53}
\]

Now, we expand the function \( \psi(\sqrt{\varepsilon} \xi, \varepsilon) \) in powers of \( \varepsilon \)

\[
\psi(\sqrt{\varepsilon} \xi, \varepsilon) = \sum_{k=0}^\infty \varepsilon^{k/2} \psi_{k/2}(\xi), \tag{5.54}
\]

where \( \psi_{k/2}(\xi) \) are polynomials in \( \xi \). It is easy to see that the half-integer order coefficients \( \psi_{k+1/2}(\xi) \) are odd polynomials and the integer order coefficients \( \psi_k(\xi) \) are even polynomials. Therefore, the Gaussian average of the half-integer order coefficients vanish, \( \langle \psi_{k+1/2}(\xi) \rangle_G = 0 \). Thus, finally we obtain the asymptotic expansion (5.41) with only integer powers of \( \varepsilon \) with

\[
F_k = g^{1/2} G^{-1/2} \langle \psi_k(\xi) \rangle_G. \tag{5.55}
\]

Then by computing the Gaussian average (5.3) we get the explicit form of the coefficients of the asymptotic expansion.

By using the covariant Taylor expansions of the function \( \hat{\varphi} \), (5.26), and of the function \( \hat{\Sigma} \), (5.50), we obtain

\[
\psi_0(\xi) = \hat{\varphi}, \tag{5.56}
\]

\[
\psi_1(\xi) = \frac{1}{2} \hat{\varphi}_{ij} \xi^i \xi^j - \frac{1}{48} \left( 4 \Sigma_{ijkl} \hat{\varphi}_{kl} + \Sigma_{ijkl} \hat{\varphi}_{kl} \right) \xi^i \xi^j \xi^k \xi^l
\]

\[
+ \frac{1}{288} \Sigma_{ijkl} \Sigma_{mnp} \xi^i \xi^j \xi^k \xi^l \xi^m \xi^n \hat{\varphi}, \tag{5.57}
\]

where \( \hat{\varphi}, \hat{\varphi}, \) and \( \hat{\varphi}_{ij} \) are given by (5.28)-(5.30).

By computing the Gaussian average this gives

\[
F_0 = g^{1/2} G^{-1/2} \hat{\varphi}, \tag{5.58}
\]

\[
F_1 = g^{1/2} G^{-1/2} \left\{ G^{ij} \hat{\varphi}_{ij} - G^{ij} G^{kl} \Sigma_{(ijk)} \hat{\varphi} - \frac{1}{4} G^{ij} G^{kl} \Sigma_{ijkl} \hat{\varphi}
\]

\[
+ \frac{5}{12} G^{ij} G^{kl} G^{mn} \Sigma_{(ijkl)lmn} \hat{\varphi} \right\}. \tag{5.59}
\]

By using Lemma 2.1 of [17] we get

\[
G^{ij} G^{kl} G^{mn} \Sigma_{(ijk)lmn} = \frac{1}{5} G^{ij} G^{kl} G^{mn} \left( 2 \Sigma_{ikm} \Sigma_{jln} + 3 \Sigma_{ijn} \Sigma_{klm} \right). \tag{5.60}
\]
Therefore,

\[ F_1 = g^{1/2}G^{-1/2} \left\{ G^{ij} \hat{\phi}_{ij} - G^{ij}G^{kl} \Sigma_{ijkl} \hat{\phi}_{kl} - \frac{1}{4} G^{ij}G^{kl} \Sigma_{ijkl} \hat{\phi} + \frac{1}{12} \left( 2G^{ij}G^{lm} + 3G^{ij}G^{im} \right) G^{kn} \Sigma_{ijkl} \hat{\phi} \right\}. \]  

(5.61)

Finally, by using eqs. (5.28)-(5.30) we get the result (5.44).

In the case when the function \( \varphi \) and its first derivative vanish on the diagonal we also get

\[ \psi_2(\xi) = \frac{1}{24} \hat{\phi}_{ijkl} \xi^i \xi^j \xi^k \xi^l - \frac{1}{72} \sum_{ijkl} \hat{\phi}_{ijkl} \xi^i \xi^j \xi^k \xi^l + \frac{1}{576} \sum_{ijkl} \hat{\phi}_{ijkl} \xi^i \xi^j \xi^k \xi^l \xi^p \xi^q \xi^r \xi^s. \]  

(5.62)

By computing the Gaussian average over \( \xi \) we get

\[ F_2 = g^{1/2}G^{-1/2} \left\{ \frac{1}{2} G^{ij}G^{kl} \hat{\phi}_{ijkl} - \frac{5}{3} G^{ij}G^{kl} G^{mn} \Sigma_{ijkl} \hat{\phi}_{mn} \right\}. \]  

(5.63)

(5.64)

We use Lemma 2.1 of [17] to get

\[ G^{ij}G^{kl}G^{mn} \Sigma_{ijkl} \hat{\phi}_{mn} = \frac{1}{5} G^{ij}G^{kl}G^{mn} \left( \Sigma_{ijkl} \hat{\phi}_{mn} + 4 \Sigma_{ikmn} \hat{\phi}_{jl} \right). \]  

(5.65)

\[ G^{ij}G^{kl}G^{mn} G^{pq} \Sigma_{ijkl} \hat{\phi}_{pq} = \frac{1}{35} G^{ij}G^{kl}G^{mn} G^{pq} \left( 2 \Sigma_{ikmn} \hat{\phi}_{jq} + 3 \Sigma_{ijkl} \hat{\phi}_{mn} \right) + \frac{1}{6} \Sigma_{ijkl} \hat{\phi}_{pq} + 12 \Sigma_{ijkl} \hat{\phi}_{pq} + 12 \Sigma_{ikmn} \hat{\phi}_{jq} + 12 \Sigma_{ikmn} \hat{\phi}_{jq} \right) \]  

(5.66)
Now, by using these equations together with (5.60) we obtain
\[ F_2 = g^{1/2}G^{-1/2} \left( \frac{1}{2} G^{ij} G^{kl} \hat{\phi}_{ijkl} - \frac{1}{3} \left( 2G^{ij} G^{jk} + 3G^{iq} G^{jk} \right) G^{pq} \Sigma_{ipq} \hat{\phi}_{ijkl} \right) \]
\[ + \left[ \frac{1}{4} \left( G^{pq} G^{kl} + 4G^{kp} G^{pq} \right) G^{pq} \Sigma_{ijpq} \right. \]
\[ + \frac{1}{72} \left( 2G^{ij} G^{pq} G^{kl} + 3G^{ij} G^{pq} G^{rs} G^{kl} + 6G^{ik} G^{jl} G^{pq} G^{rs} \right. \]
\[ + 12G^{ij} G^{pq} G^{kl} \right] \Sigma_{ipq} \sum_{jr s} \hat{\phi}_{kl} \].

Finally, by using eqs. (5.28)-(5.32) and (4.30)-(4.33) we obtain (5.45).

Of course, in the particular case when $\Sigma$ is the Ruse-Synge function, $\Sigma = \sigma^g$ of the metric $g$, all its symmetrized covariant derivatives of order higher than two vanish on the diagonal and the second derivative is equal to the metric, which gives the earlier result (5.21)-(5.22).

### 6 Asymptotics of Heat Traces

#### 6.1 Classical Heat Trace

First of all, it is easy to see that the asymptotics of the classical heat trace as $t \to \infty$ are determined by the bottom eigenvalue
\[ \Theta_+(t) \sim d_1^+ \exp (-t\lambda_1^+), \]
where $d_1^+ = \text{tr} P_1^+$ is the multiplicity of the first eigenvalue and $P_1^+$ is the projection to the first eigenspace. We will be primarily interested in the asymptotics as $t \to 0$.

For Laplace type operators $L_\pm$ there is an asymptotic expansion of the heat kernel $U_\pm(t; x, x')$ in the neighborhood of the diagonal as $t \to 0$ (see e.g. [1, 2, 4, 5])
\[ U_\pm(t; x, x') \sim (4\pi)^{-n/2} \exp \left( -\frac{\sigma_\pm}{2t} \right) \sum_{k=0}^{\infty} t^{k-n/2} \tilde{a}_k^\pm(x, x'), \]
where
\[ \tilde{a}_k^\pm(x, x') = \frac{(-1)^k}{k!} M_\pm^{1/2}(x, x') \mathcal{P}_\pm(x, x') a_k^\pm(x, x') \]
\[ = \frac{(-1)^k}{k!} g_\pm^{1/4}(x) g_\pm^{1/4}(x') e^{\varepsilon_\pm(x, x')} \mathcal{P}_\pm(x, x') a_k^\pm(x, x'), \]
\[ \varepsilon_\pm = \begin{cases} \lambda_\pm & \text{for } \lambda_\pm \geq 0, \\
-1 & \text{for } \lambda_\pm < 0. \end{cases} \]
σ_± = σ_±(x, x') is the Ruse-Synge function of the metric g_±, M_± = M_±(x, x') is the Van Vleck-Morette determinant, P_± = P_±(x, x') is the operator of parallel transport of sections along the geodesic in the connection ∇± and the metric g_± from the point x' to the point x and a_±^k = a_±^k(x, x') are the usual heat kernel coefficients. In particular, [2]

\[ a_0^± = I, \]  

and

\[ [a_1^±] = Q_± - \frac{1}{6} R_± I, \]  

where R_± is the scalar curvature of the metric g_± and for the Dirac operator Q_± is given by (3.13).

Therefore, there is the asymptotic expansion (1.12) of the classical heat trace (3.17). This is the classical heat trace asymptotics of Laplace type operators. By using the off-diagonal expansion of the heat kernel (6.2) for the Laplace type operator (and using the diagonal values of the derivatives of the functions σ_±, M_±, P_±) one can also obtain the asymptotic expansion of the classical spectral invariant H(t), (1.2), for the Dirac type operator,

\[ H_±(t) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^k H_k^±, \]  

where

\[ H_k^± = \frac{(-1)^k}{k!} \int_M dx g_±^{1/2} \text{tr} [D_± a_k^±]. \]  

Here we used, in particular, a useful relation

\[ [D_± a_k] = \frac{(-1)^k}{k!} g_±^{1/2} [D_± a_k] \]  

Further, by using

\[ [D_± P_±] = S_± \]  

and the results of [1, 2, 5] we get

\[ H_0^± = \int_M dx g_±^{1/2} \text{tr} S_±, \]  

\[ H_1^± = \int_M dx g_±^{1/2} \text{tr} \left\{ i\gamma_± \left( \frac{1}{2} \nabla_± j Q_± - \frac{1}{6} \nabla_± R_±^k j \right) + S_± \left( \frac{1}{6} R_± - Q_± \right) \right\}. \]  

For the manifolds without boundary we can safely neglect the total derivative terms in (6.11).
6.2 Combined Heat Trace for Laplace Type Operators

To compute the asymptotics of the relative spectral invariants $\Psi(t, s)$ and $\Phi(t, s)$ we rescale the variables $t \mapsto \varepsilon t$ and $s \mapsto \varepsilon s$ and study the asymptotics as $\varepsilon \to 0$ or $\varepsilon \to \infty$. It is easy to see that the asymptotics of the combined heat traces $X(t, s)$, (1.5), and $Y(t, s)$, (3.22), as $\varepsilon \to \infty$ are determined by the bottom eigenvalues $\lambda_1^\pm$ and $\mu_1^\pm$,

\[ X(\varepsilon t, \varepsilon s) \sim \exp\left[-\varepsilon (t\lambda_1^+ + s\lambda_1^-)\right] \text{Tr} P_{P_1}^+ \text{Tr} P_{P_1}^- \tag{6.12} \]

\[ Y(\varepsilon t, \varepsilon s) \sim \exp\left[-\varepsilon (t\mu_1^+)^2 + s(\mu_1^-)^2\right] \mu_1^+ \mu_1^- \text{Tr} P_{P_1}^+ \text{Tr} P_{P_1}^- \tag{6.13} \]

We will be interested mainly in the asymptotics as $\varepsilon \to 0$. In this subsection we prove the Theorem I for the combined heat trace $X(t, s)$.

**Proof of Theorem I Part I.** The combined trace $X(t, s)$ is given by the integral (3.18) over $M \times M$ of the form

\[ X(t, s) = \int_{M \times M} dx \, dx' \, f_1(t, s; x, x'), \tag{6.14} \]

where

\[ f_1(t, s; x, x') = \frac{1}{2} \text{tr}\{U_+(t, x, x')U_-(s, x', x) + U_-(s, x, x')U_+(t, x', x)\}. \tag{6.15} \]

Notice that we made it manifestly symmetric under the exchange $(t, L_+) \leftrightarrow (s, L_-)$ by symmetrizing the integrand in $x$ and $x'$ (alternatively, we could do the symmetrization $(t, L_+) \leftrightarrow (s, L_-)$ at the end of the calculations).

Let $r_{inj}^\pm$ be the injectivity radii of the metrics $g_\pm$ and

\[ \rho = \min\{r_{inj}^+ , r_{inj}^- \}. \tag{6.16} \]

We fix a point $x'$ in the manifold $M$ (of course, we could instead fix the point $x$; we have to do it both ways to achieve the required symmetry $(t, L_+) \leftrightarrow (s, L_-)$ of the heat trace).

Let $B_r^\pm(x')$ be the geodesic balls in the metric $g^\pm_{ij}$ centered at $x'$ of radius $r < \rho$ smaller than the injectivity radii $r_{inj}^\pm$. Let $B_+(x') \subset B_r^+(x') \cap B_r^-(x')$ be an open set contained in both of these balls. We decompose the combined traces as follows

\[ X(t, s) = X_{\text{diag}}(t, s) + X_{\text{off-diag}}(t, s), \tag{6.17} \]
where

\[ X_{\text{diag}}(t, s) = \int_{M} dx' \int_{B_{r}(x')} dx f_{1}(t, s; x, x'), \tag{6.18} \]

\[ X_{\text{off-diag}}(t, s) = \int_{M} dx' \int_{M-B_{r}(x')} dx f_{1}(t, s; x, x'). \tag{6.19} \]

To estimate these integrals we will need the following lemma.

**Lemma 6** The off-diagonal part \( X_{\text{off-diag}}(\varepsilon t, \varepsilon s) \) of the combined heat trace is exponentially small as \( \varepsilon \to 0 \) and does not contribute to its asymptotic expansion, that is, as \( \varepsilon \to 0 \)

\[ X(\varepsilon t, \varepsilon s) \sim X_{\text{diag}}(\varepsilon t, \varepsilon s). \tag{6.20} \]

**Proof.** This can be proved by using the standard elliptic estimates of the heat kernel. For any \( x \in M - B_{r}(x') \) and \( 0 < t < 1 \) there is an estimate \[ |U_{\pm}(t; x, x'|) \leq C_{1} r^{-n/2} \exp \left( -\frac{r^{2}}{4t} \right), \tag{6.21} \]

where \( C_{1} = C_{1}(r) \) is some constant. Therefore,

\[ |f_{1}(t, s; x, x')| \leq C_{2} \varepsilon^{-n} t^{-n/2} s^{-n/2} \exp \left( -\frac{r^{2}}{4\varepsilon} \left( \frac{1}{t} + \frac{1}{s} \right) \right) \tag{6.22} \]

with some constant \( C_{2} \). This means that as \( \varepsilon \to 0 \)

\[ X_{\text{off-diag}}(\varepsilon t, \varepsilon s) \sim 0. \tag{6.23} \]

The statement follows.

Now, by using this Lemma 6 and the asymptotic expansion of the heat kernel (6.2) we obtain the asymptotic expansion of the combined heat trace (3.18) as \( \varepsilon \to 0 \),

\[ X(\varepsilon t, \varepsilon s) \sim (4\pi\varepsilon)^{-n/2} \sum_{m=0}^{\infty} \varepsilon^{m} X_{m}(\varepsilon, t, s), \tag{6.24} \]

where

\[ X_{m}(\varepsilon, t, s) = (4\pi\varepsilon s)^{-n/2} \int_{M} dx' \int_{B_{r}(x')} dx \exp \left\{ -\frac{1}{2\varepsilon s} \Sigma(t, s; x, x') \right\} \Lambda_{m}(t, s; x, x'), \tag{6.25} \]
and
\[
\Sigma(t, s; x, x') = s\sigma_+(x, x') + t\sigma_-(x, x'),
\]
(6.26)

\[
\Lambda_m(t, s; x, x') = \sum_{j=0}^{m} t^{m-j}s^j \frac{1}{2} \text{tr} \left\{ \tilde{a}_{m-j}^+(x, x') \tilde{a}_j^-(x', x) + \tilde{a}_{m-j}^-(x, x') \tilde{a}_j^+(x', x) \right\}.
\]
(6.27)

Next, we compute the asymptotic expansion of the function \( X_m(\varepsilon, t, s) \).

**Lemma 7** There is the asymptotic expansion as \( \varepsilon \to 0 \)
\[
X_m(\varepsilon, t, s) \sim \sum_{k=0}^{\infty} \varepsilon^k X_{m,k}(t, s),
\]
(6.28)

where
\[
X_{m,k}(t, s) = \int_M d\mathbf{x} g^{1/2}(t, s) b_{m,k}(t, s).
\]
(6.29)

The coefficients \( b_{m,k}(t, s) \) are scalar invariants constructed polynomially from the diagonal values of the derivatives of the function \( \Lambda_m \) and the derivatives of the function \( \Sigma(t, s) \) of order higher than 2 as well as the metrics \( g_{ij}, g_{ij}^+ \) and \( g_{ij}(t, s) \), (1.17). The coefficients \( b_{m,k}(t, s) \) are homogeneous functions of \( t \) and \( s \) of degree \( m+k \) and the coefficients \( X_{m,k}(t, s) \) are homogeneous functions of \( t \) and \( s \) of degree \( m+k-n/2 \). In particular,
\[
b_{m,0}(t, s) = \sum_{j=0}^{m} \frac{(-1)^m}{(m-j)!j!} t^{m-j}s^j \text{tr} \left\{ [a_{m-j}^+] [a_j^-] \right\}.
\]
(6.30)

**Proof.** The function \( \Sigma(t, s) = \Sigma(t, s; x, x') \), (6.26), is smooth and positive (here and below we omit the space variables \( x \) and \( x' \)). It has the absolute minimum on the diagonal, at \( x = x' \), equal to zero,
\[
[\Sigma(t, s)] = 0
\]
(6.31)

and has a non-degenerate critical point on the diagonal, that is, the first derivatives vanish on the diagonal
\[
[\Sigma_j(t, s)] = 0,
\]
(6.32)
and the Hessian $[\Sigma_{ij}(t, s)] = G_{ij}(t, s)$, which is exactly equal to the matrix $G_{ij}(t, s)$, defined by \((1.27)\), is positive on the diagonal. Also, by using \((1.28)\) we can see that the determinant of the Hessian has the form
\[
G = \det G_{ij} = \frac{g^+ g^-}{g}.
\] (6.33)

where $g = \det g_{ij}$ and $g_{\pm} = \det g^\pm_{ij}$.

Now, by using Lemma \[5\] we compute the asymptotic expansion of the integral \((6.25)\) which gives \((6.28)\) and proves the first part of the lemma. The coefficient $b_{m,0}(t, s)$ is given by \((5.42)\), so
\[
b_{m,0} = g^{-1/2}G^{-1/2}[\Lambda_m] = g^+_m g^-_m [\Lambda_m].
\] (6.34)

Further, we compute the diagonal value of the functions $\Lambda_m(t, s)$, \((6.27)\)
\[
[\Lambda_m(t, s)] = g^+_m g^-_m \sum_{j=0}^{m} \frac{(-1)^m}{(m-j)! j!} t^{m-j}s^{j} \text{tr}([a^+_{m-j}] [a^-_j]),
\] (6.35)

to get \((6.30)\). This proves Lemma \[7\].

Thus, by using \((6.24)\) and \((6.28)\) we obtain the asymptotic expansion \((1.19)\) of the combined heat trace $X(t, s)$ with the coefficients
\[
b_k(t, s) = \sum_{j=0}^{k} b_{k-j,0}(t, s).
\] (6.36)

This proves Theorem \[\text{II}\] for the trace $X(t, s)$. Finally, by using the relation \((1.7)\) and the asymptotic expansions \((1.12)\) and \((1.19)\) we obtain the asymptotic expansion \((1.23)\) of the relative spectral invariant $\Psi(t, s)$ with the coefficients \((1.25)\). This proves Corollary \[\text{II}\] for the function $\Psi(t, s)$.

### 6.3 Combined Heat Trace for Dirac Type Operators

The case for the Dirac type operators is similar to the Laplace type operators; so we will omit some details. In this subsection we prove the Theorem \[\text{II}\] for the combined heat trace $Y(t, s)$.

**Proof of the Theorem \[\text{III}\] Part II.** The combined heat trace $Y(t, s)$ is given by the integral \((3.22)\) over $M \times M$ of the form
\[
Y(t, s) = \int_{M \times M} dx \, dx' f_2(t, s; x, x'),
\] (6.37)
where
\[ f_2(t, s; x, x') = \frac{1}{2} \text{tr} \left\{ D_+ U_+(t, x, x') D_- U_-(s, x, x') + D_- U_-(s, x, x') D_+ U_+(t, x', x) \right\}, \]
(6.38)

where the operators \( D_\pm \) act on the first space argument of the heat kernels. The method for computing this integral is essentially the same as for the Laplace type operators. In this case we use the estimate for the derivative of the heat kernel: for any \( x \in M - B_r(x') \) and \( 0 < t < 1 \)
\[ |D_\pm U_\pm(t; x, x')| \leq C_3 t^{1-n/2} \exp \left( -\frac{r^2}{4t} \right), \]
(6.39)
where \( C_3 \) is some constant. By using this estimate it is easy to see that the off-diagonal part of the integral is exponentially small and does not contribute to the asymptotic expansion of the trace \( Y(\varepsilon t, \varepsilon s) \) as \( \varepsilon \to 0 \).

Therefore, we obtain
\[ Y(\varepsilon t, \varepsilon s) \sim (4\pi \varepsilon)^{-n/2} \sum_{m=0}^{\infty} \varepsilon^m \tilde{Y}_m(\varepsilon, t, s), \]
(6.40)
where
\[ \tilde{Y}_m(\varepsilon, t, s) = (4\pi \varepsilon)^{-n/2} \int_M dx' \int_{B_r(x')} dx \exp \left\{ -\frac{1}{2\varepsilon t s} \Sigma(t, s; x, x') \right\} \tilde{N}_m(\varepsilon, t, s, x, x'), \]
(6.41)

with
\[ \tilde{N}_m(\varepsilon, t, s, x, x') = \frac{1}{2} \sum_{j=0}^{m} t^{m-j} s^j \text{tr} \left\{ \left( D_+ - \frac{1}{2\varepsilon t} \nu_+(x, x') \right) \tilde{a}^+_m(x, x') \left( D_- - \frac{1}{2\varepsilon s} \nu_-(x', x) \right) \tilde{a}^-_j(x', x) \right\} + x \leftrightarrow x', \]
(6.42)
and
\[ \nu_\pm(x, x') = i \gamma_\pm(x) \sigma_\pm(x, x'). \]
(6.43)

To avoid confusion we stress here once again that the operators \( D_\pm \) act on the first space argument of the coefficients \( \tilde{a}^\pm_k \). The functions \( \tilde{N}_m \) depend on \( \varepsilon \) in the following way
\[ \tilde{N}_m = \frac{1}{\varepsilon^2} N_m^{(0)} + \frac{1}{\varepsilon} N_m^{(1)} + N_m^{(2)}, \]
(6.44)
where

\[ N_{m}^{(0)} = \frac{1}{8} (ts)^{-1} \sum_{j=0}^{m} t^{m-j} s^{j} \text{tr} \left\{ \nu_{+}(x, x') \tilde{a}_{m-j}^{+}(x, x') \nu_{-}(x', x) \tilde{a}_{j}(x', x) \right\} + x \leftrightarrow x', \quad (6.45) \]

\[ N_{m}^{(1)} = -\frac{1}{4} (ts)^{-1} \sum_{j=0}^{m} t^{m-j} s^{j} \text{tr} \left\{ s \nu_{+}(x, x') \tilde{a}_{m-j}^{+}(x, x') (D_{-} \tilde{a}_{j}(x', x)) + t (D_{+} \tilde{a}_{m-j}^{+}(x, x')) \nu_{-}(x', x) \tilde{a}_{j}^{-}(x', x) \right\} + x \leftrightarrow x', \quad (6.46) \]

\[ N_{m}^{(2)} = \frac{1}{2} \sum_{j=0}^{m} t^{m-j} s^{j} \text{tr} \left\{ (D_{+} \tilde{a}_{m-j}^{+}(x, x')) (D_{-} \tilde{a}_{j}(x', x)) \right\} + x \leftrightarrow x'. \quad (6.47) \]

Therefore,

\[ \tilde{Y}_{m} = \frac{1}{\varepsilon} Y_{m}^{(0)} + \frac{1}{\varepsilon} Y_{m}^{(1)} + Y_{m}^{(2)}, \quad (6.48) \]

with the obvious notation

\[ Y_{m}^{(i)}(\varepsilon, t, s) = (4\pi\varepsilon ts)^{-n/2} \int_{M} dx' \int_{B_{r}(x')} dx \exp \left\{ -\frac{1}{2\varepsilon ts} \Sigma(t, s; x, x') \right\} N_{m}^{(i)}(t, s, x, x'). \quad (6.49) \]

Therefore, by using (6.48) we have

\[ Y(\varepsilon t, \varepsilon s) \sim (4\pi\varepsilon)^{-n/2} \sum_{m=0}^{\infty} \varepsilon^{m-2} Y_{m}(\varepsilon, t, s), \quad (6.50) \]

where

\[ Y_{0} = Y_{0}^{(0)}, \quad (6.51) \]

\[ Y_{1} = Y_{1}^{(0)} + Y_{1}^{(1)}, \quad (6.52) \]

and for \( m \geq 2 \)

\[ Y_{m} = Y_{m}^{(0)} + Y_{m-1}^{(1)} + Y_{m-2}^{(2)}. \quad (6.53) \]

By using Lemma [5] again to compute the integral (6.49) we prove the following lemma.
Lemma 8  There is the asymptotic expansion as $\varepsilon \to 0$

$$Y_m^{(i)}(\varepsilon, t, s) \sim \sum_{k=0}^{\infty} \varepsilon^k Y_m^{(i)}(t, s), \quad (6.54)$$

where

$$Y_m^{(i)}(t, s) = \int_M dx \ g_{1/2}^{ij}(t, s) c_{m,i}^{(i)}(t, s). \quad (6.55)$$

The coefficients $c_{m,i}^{(i)}(t, s)$ are scalars constructed polynomially from the diagonal values of the derivatives of the functions $N_{m}^{(i)}$ and the derivatives of the function $\Sigma(t, s)$ of order higher than 2 as well as the metric $g_{ij}(t, s)$ and $g_{\pm}^{ij}$. The coefficients $c_{m,i}^{(i)}(t, s)$ are homogeneous functions of $t$ and $s$ of degree $(m + k + i - 2)$ and the coefficients $Y_{m}^{(i)}(t, s)$ are homogeneous functions of $t$ and $s$ of degree $(m + k + i - 2 - n/2)$. The first coefficients are

$$c_{m,0}^{(0)} = c_{m,0}^{(1)} = 0, \quad (6.56)$$

$$c_{m,0}^{(2)}(t, s) = \sum_{j=0}^{m} \frac{(-1)^m}{(m-j)!j!} t^{m-j} s^j \text{tr} \left\{ \left[ D_+ a_{m-j}^+ \right] \left[ D_- a_j^- \right] \right\}. \quad (6.57)$$

Proof. The proof of this lemma is essentially the same as that of the Lemma [7]. By using Lemma [5] we compute the asymptotic expansion of the integral (6.49) which gives (6.54) and proves the first part of the lemma.

The coefficient $c_{m,0}^{(i)}(t, s)$ is given by (5.42), so

$$c_{m,0}^{(i)} = g_{1/2}^{-1/2} G_{1/2}^{-1/2} [N_{m}^{(i)}] = g_{\pm}^{1/2} g_{\pm}^{-1/2} [N_{m}^{(i)}]. \quad (6.58)$$

Thus, we need to compute the diagonal values of the functions $N_{m}^{(i)}$. First of all, since the diagonal values of the function $\sigma_\pm$ and its first derivatives vanish it is easy to see that the diagonal values of the functions $\nu_\pm$, (6.43), vanish,

$$[\nu_\pm] = 0, \quad (6.59)$$

and, therefore,

$$[N_{m}^{(0)}] = [N_{m}^{(1)}] = 0. \quad (6.60)$$

This means that

$$c_{m,0}^{(0)} = c_{m,0}^{(1)} = 0. \quad (6.61)$$
The functions \( N^{(2)}_m \) are expressed in terms of the coefficients \( \tilde{a}_k^\pm \), which are related to the standard heat kernel coefficients \( a_k^\pm \) by (6.3). Therefore, by using the diagonal values of the functions \( \sigma_\pm, M_\pm, \mathcal{P}_\pm \), and their derivatives we obtain

\[
[N^{(2)}_m(t, s)] = 8^{1/2} g_+^{-1/2} \sum_{j=0}^{m} \frac{(-1)^m}{(m-j)!j!} t^{m-j} s^j \text{tr}\left\{[D_+a^+_{m-j}][D_-a^-_j]\right\}.
\] (6.62)

Now, by using (6.33) and (6.58) we get (6.57). This proves Lemma 8.

By using this lemma, we obtain the asymptotic expansion

\[
Y_m(\varepsilon, t, s) \sim \sum_{k=0}^\infty \varepsilon^k Y_{m,k}(t, s),
\] (6.63)

where

\[
Y_{m,k}(t, s) = \int_M dx \, g_+^{1/2}(t, s)c_{m,k}(t, s)
\] (6.64)

with

\[
c_{0,k} = c_{0,k}^{(0)},
\] (6.65)

\[
c_{1,k} = c_{1,k}^{(0)} + c_{1,k}^{(1)}
\] (6.66)

and for \( m \geq 2 \)

\[
c_{m,k} = c_{m,k}^{(0)} + c_{m-1,k}^{(1)} + c_{m-2,k}^{(2)}.
\] (6.67)

Now, by using (6.50) and the equations above we obtain

\[
Y(\varepsilon t, \varepsilon s) \sim (4\pi\varepsilon)^{-n/2} \sum_{k=-1}^\infty \varepsilon^{k-1} C_k(t, s),
\] (6.68)

with the coefficients

\[
C_k = \sum_{j=0}^{k+1} Y_{j,k+1-j}.
\] (6.69)

Finally, we notice that the first coefficient \( C_{-1} \) vanishes since

\[
C_{-1} = Y_{0,0} = Y_{0,0}^{(0)} = 0.
\] (6.70)

Thus, we obtain the asymptotic expansion (1.20) of the combined heat trace \( Y(t, s) \) with the coefficients

\[
c_k(t, s) = \sum_{j=0}^{k+1} c_{j,k+1-j}(t, s).
\] (6.71)
This proves Theorem II for the trace \( Y(t, s) \). Finally, by using the relation (1.8) and the asymptotic expansions (1.12) and (1.20) we obtain the asymptotic expansion (1.24) of the relative spectral invariant \( \Phi(t, s) \) with the coefficients (1.26). This proves Corollary II for the function \( \Phi(t, s) \).

### 6.4 Specific Cases

First of all, we notice that since for equal operators \( L_- = L_+ \) the combined trace \( X(t, s) \) can be expressed in terms of the classical heat trace

\[
X(t, s) = \Theta(t + s), \tag{6.72}
\]

then, by comparing (1.19) and (1.12) we see that in this case

\[
B_k(t, s) = (t + s)^{k-n/2} A_k. \tag{6.73}
\]

Similarly, since for equal operators \( D_- = D_+ \) the combined trace \( Y(t, s) \) can be expressed in terms of the classical heat trace

\[
Y(t, s) = -\frac{\partial}{\partial t} \Theta(t + s), \tag{6.74}
\]

then, by comparing (1.20) and (1.12) we see that in this case

\[
C_k(t, s) = -\left(k - \frac{n}{2}\right)(t + s)^{k-1-n/2} A_k. \tag{6.75}
\]

This gives non-trivial relations between the heat kernel coefficients and their derivatives and provides a useful check of the results. It is easy to see then that for equal operators \( L_- = L_+ \) and \( D_- = D_+ \) the relative spectral invariants \( \Psi(t, s) \) and \( \Phi(t, s) \) vanish.

If the Laplace type operators differ by just a constant,

\[
L_+ = L_- + M^2, \tag{6.76}
\]

then the metrics and the connections are the same and

\[
\Theta_+(t) = e^{-iM^2} \Theta_-(t), \tag{6.77}
\]

\[
X(t, s) = e^{-iM^2} \Theta_-(t + s), \tag{6.78}
\]

and, therefore,

\[
\Psi(t, s) = \left(e^{-iM^2} - 1\right)\left(e^{-sM^2} - 1\right) \Theta_-(t + s). \tag{6.79}
\]
In this case

\[ B_0(t, s) = (t + s)^{-n/2}A_0^-, \quad (6.80) \]
\[ B_1(t, s) = (t + s)^{-1-n/2}A_1^- - t(t + s)^{-n/2}M^2A_0^- . \quad (6.81) \]

For the Dirac case suppose that there is an endomorphism \( M \) such that it anticommutes with the operator \( D_- \),

\[ D_- M = -M D_-, \quad (6.82) \]
and \( M^2 \) is a scalar. Then it is easy to see that

\[ \text{Tr} M D_- \exp(-sD_-^2) = 0. \quad (6.83) \]

Now, suppose that

\[ D_+ = D_- + M, \quad (6.84) \]
so that (recall that \( L_+ = D_+^2 \))

\[ L_+ = L_- + M^2; \quad (6.85) \]

Then it is easy to show that

\[ H_+(t) = H_-(t) + \text{Tr} M \exp(-tD_-^2), \quad (6.86) \]
\[ Y(t, s) = -e^{-tM^2} \partial_\Theta_-(t + s), \quad (6.87) \]

and, hence,

\[ \Phi(t, s) = -\left(e^{-tM^2} - 1\right)\left(e^{-sM^2} - 1\right) \partial_\Theta_-(t + s) + M^2 e^{-(t+s)M^2} \Theta_-(t + s). \quad (6.88) \]

Therefore,

\[ C_0(t, s) = \frac{n}{2}(t + s)^{-1-n/2}A_0^-, \quad (6.89) \]
\[ C_1(t, s) = \left(\frac{n}{2} - 1\right)(t + s)^{-n/2}A_1^- - \frac{n}{2} t(t + s)^{-1-n/2}M^2A_0^- . \quad (6.90) \]

A more general case is the case of commuting operators; then the combined heat traces still simplify significantly, they can be expressed in terms of the classical one

\[ X(t, s) = \text{Tr} \exp(-tL_+ - sL_-), \quad (6.91) \]
\[ Y(t, s) = \text{Tr} D_- D_+ \exp(-tD_+^2 - sD_-^2). \quad (6.92) \]

Therefore, the asymptotics of the combined traces can be obtained from the classical ones. Notice that the leading symbol of the operators \( L = tL_+ + sL_- \) and \( L = tD_+^2 + sD_-^2 \) is determined exactly by the metric \( g^{ij}(t, s) \). Therefore, in this case the combined traces are given by the classical trace for the operator \( L = tL_+ + sL_- \).
7 Explicit Results

7.1 Laplace Type Operators (Proof of Theorem 2)

Coming back to the general case, it is easy to see that the first coefficients are the same as in the commuting case. The coefficient $B_0$ is obtained by using (6.36), (6.30) and (6.4)

\[ b_0 = b_{0,0} = \text{tr} I. \] (7.1)

This proves eq. (1.42).

The coefficient $b_1$ has the form

\[ b_1 = b_{1,0} + b_{0,1}. \] (7.2)

Here the first coefficient is easy to compute. By using the well known results (6.4), (6.5), for the coefficients $[a_0^\pm]$ we obtain

\[
b_{1,0} = \text{tr} \left\{ -t [a_1^+ a_0^-] - s [a_0^+ a_1^-] \right\}
= \text{tr} \left\{ t \left( \frac{1}{6} R_+ I - Q_+ \right) + s \left( \frac{1}{6} R_- I - Q_- \right) \right\}.
\] (7.3)

The coefficient $b_{0,1}$, (6.33), is determined by the second term of the asymptotics of the quantity $X_0(\varepsilon t, \varepsilon s)$, (6.25). By using (6.27), (6.3), (6.33) and (4.29) we have

\[
\Lambda_0 = g^{1/2}(x) G^{1/2}(x') e^{\omega(x,x')} \varphi_1(x,x'),
\] (7.4)

where

\[
\omega(x,x') = \frac{1}{4} \log \left( \frac{g_+(x) g_-(x')}{g(x) g(x')} \cdot \frac{g_+(x) g(x)}{g(x) g(x')} \right) + \zeta_+(x,x') + \zeta_-(x,x')
\] (7.5)

and

\[
\varphi_1(x,x') = \frac{1}{2} \text{tr} \left\{ \mathcal{P}_+(x,x') \mathcal{P}_-(x',x) + \mathcal{P}_-(x,x') \mathcal{P}_+(x',x) \right\}. \] (7.6)

By using the fact that $\mathcal{P}(x',x) = \mathcal{P}^{-1}(x,x')$ we find it convenient to rewrite the function $\varphi_1$ in the form

\[
\varphi_1 = \frac{1}{2} \text{tr} \left( \Pi + \Pi^{-1} \right),
\] (7.7)

where

\[
\Pi(x,x') = \mathcal{P}_-(x',x) \mathcal{P}_+(x,x') = \mathcal{P}^{-1}_- \mathcal{P}_+.
\] (7.8)
Here the function $\zeta_\pm(x, x')$ is defined by (4.29). Now, by using Lemma 5 and eq. (5.44) we obtain

$$b_{0,1} = tsG^{ij}[\nabla^g_i \nabla^g_j (e^{\omega_1} \varphi_1)] - tsG^{ij}G^{kl}\Sigma_{ijkl}[\nabla^g_i (e^{\omega_1} \varphi_1)]$$

$$+ts\left(-\frac{1}{3}G^{ij}R^g_{ij} - \frac{1}{4}G^{ij}G^{kl}\Sigma_{ijkl} + \frac{1}{6}G^{il}G^{jm}G^{kn}\Sigma_{ijkl}\Sigma_{lmn}\right)$$

$$+\frac{1}{4}G^{ij}G^{lm}G^{kn}\Sigma_{ijkl}\Sigma_{lmn}][\varphi_1],$$

(7.9)

where $\Sigma_{ijkl} = [\nabla^g_{i_1} \cdots \nabla^g_{i_k}]$ are the coincidence limits of symmetrized covariant derivatives of $\Sigma$ determined by the metric $g_{ij}$ (1.17).

First of all, we notice that $[\omega] = 0$. We will denote the diagonal values of the derivatives of the function $\omega$ by just adding indices, that is, $\omega_i = [\nabla^g_i \omega]$ and $\omega_{ij} = [\nabla^g_i \nabla^g_j \omega]$. By using (4.30) and (4.31) and the fact that $[\zeta^\pm_{i,j}] = 0$ we compute the diagonal values of the first two derivatives

$$\omega_i = W_i,$$

(7.10)

$$\omega_{ij} = \frac{1}{6}R^+_i + \frac{1}{6}R^-_i + W_{ij},$$

(7.11)

where $W_i$ and $W_{ij}$ are defined by (1.37) and (1.38).

Next, it is easy to see that

$$[\varphi_1] = \text{tr}I,$$

(7.12)

Next, since $[\nabla^g_i P_\pm] = 0$ and $[P_\pm] = I$ we have

$$[\nabla^g_i P_\pm(x, x')] = -[\nabla^g_i P_\pm(x, x')] = -A^i_\pm,$$

(7.13)

and

$$[\nabla^g_i \Pi] = -[\nabla^g_i \Pi^{-1}] = C_i^- - C_i^+;$$

(7.14)

therefore,

$$[\nabla^g_i \varphi_1] = 0,$$

(7.15)

and

$$[\nabla^g_i (e^{\omega} \varphi_1)] = \omega_i \text{tr}I.$$  

(7.16)

Further, we compute

$$[\nabla^g_i \nabla^g_j (e^{\omega} \varphi_1)] = [\nabla^g_i \nabla^g_j \varphi_1] + (\omega_{ij} + \omega_i \omega_j) \text{tr}I.$$  

(7.17)
By using (4.84) and (4.85) we compute
\[\nabla_i \nabla_j \phi_1 = \text{tr} \left( (C_{ij}^+ - C_{ij})(C_{ji}^+ - C_{ji}) \right).\] (7.18)

Finally, we obtain
\[\nabla_i \nabla_j (e^{\omega} \phi_1) = \left( \omega_{ij} + \omega_i \omega_j \right) \text{tr} I + \text{tr} \left( (C_{ij}^+ - C_{ij})(C_{ji}^+ - C_{ji}) \right).\] (7.19)

By collecting the above results we obtain
\[b_{0,1} = t \text{str} \left( \frac{1}{6} G^{ij} \left( R_{ij}^+ + R_{ij}^- - 2R_{ij}^x + W_{ij} + W_j I \right) - G^{ij} G^{kl} \Sigma_{ijkl} W_j - \frac{1}{4} G^{ij} G^{kl} \Sigma_{ijkl} \right.
\[+ \frac{1}{6} G^{ij} G^{lm} G^{kn} \Sigma_{ijkl} \Sigma_{lmn} + \frac{1}{4} G^{ij} G^{lm} G^{kn} \Sigma_{ijkl} \Sigma_{lmn} \left] \right. + G^{ij}(C_{ij}^+ - C_{ij})(C_{ji}^+ - C_{ji}) \right).\] (7.20)

Next, we compute the derivatives of the function \(\Sigma\) defined by (6.26). By using the eqs. (4.69) and (4.73) we obtain eqs. (1.39) and (1.40). By using the results (7.3) and (7.20) we obtain (1.43), which proves Theorem 2; the Corollary follows. It is easy to see that for equal operators \(L_+ = L_-\) the coefficient \(B_1\) is equal to \((t + s)^{1-n/2}A_1\), as it should.

7.2 Dirac Type Operators (Proof of Theorem 3)

7.2.1 Coefficient \(c_0\)

The coefficient \(c_0\) is given by (6.71),
\[c_0 = c_{0,0}^{(0)} + c_{0,1}^{(0)} + c_{1,0}^{(1)},\] (7.21)
and, since \(c_{0,0}^{(0)} = c_{0,1}^{(1)} = 0\) it is equal to \(c_0 = c_{0,1}^{(0)}\), which is determined by the second coefficient of the asymptotics of the function \(Y_0\), (6.54), which is equal to \(Y_0 = Y_0^{(0)}\) given by (6.49).

First, we have
\[K_0^{(0)} = \frac{1}{4} (ts)^{-1} G^{1/2} G^{1/2} e^{\omega(x,x')} \varphi_2(x,x'),\] (7.22)
where \(\omega\) is defined by (7.5) and
\[\varphi_2(x,x') = \frac{1}{2} \text{tr} \left( \mu_+(x,x') \mu_-(x',x) + \mu_-(x,x') \mu_+(x',x) \right),\] (7.23)
with

\[ \mu_\pm(x, x') = v_\pm(x, x')P_\pm(x, x') = \sigma_\pm^j(x, x')i\gamma_j'(x)P_\pm(x, x'). \quad (7.24) \]

We use Lemma 5, namely, eq. (5.44) to compute it. We notice that the diagonal values of the function \( \varphi \) and its first derivative vanish,

\[ [\varphi_2] = [\nabla_i^g \varphi_2] = 0. \quad (7.25) \]

Therefore, by using (5.44), (7.25) and (6.33) we get

\[ c_{0,1} = \frac{1}{4}G^{ij}[\nabla_i^g \nabla_j^g (e^\omega \varphi_2)]. \quad (7.26) \]

We use now the connection \( \nabla^{g,A} \) defined with respect to the metric \( g_{ij}(t, s) \) given by (1.17) and the connection \( \mathcal{A}_i(t, s) \) given by (1.18). Now, by using (6.43) and the diagonal values of the second derivatives of the Ruse-Synge function \( \sigma_\pm \), we compute

\[ [\nabla_j^g v_\pm(x, x')] = [\nabla_j^g \mu_\pm(x, x')] = -[\nabla_j^g \nu_\pm(x', x)] = -[\nabla_j^g \mu_\pm(x', x)] = ig_{ji}^\gamma \gamma_j'. \quad (7.27) \]

By using (6.45), the derivatives of the functions \( M_\pm \) and \( \mathcal{P}_\pm \), (4.30), (4.79), and (7.27), we obtain

\[ [\nabla_i^g \nabla_j^g (e^\omega \varphi_2)] = [\nabla_i^g \nabla_j^g \varphi_2] = 2g_{k(i}^g g_{j)mn} \text{tr} \left( \gamma_{+}^k \gamma_{-}^m \right), \quad (7.28) \]

and, therefore, by using (1.29) we obtain

\[ c_0(t, s) = \frac{1}{2}g_{ij}(t, s) \text{tr} \left( \gamma_+^i \gamma_-^j \right), \quad (7.29) \]

which gives (1.49).

### 7.2.2 Coefficient \( c_1 \)

The coefficient \( c_1 \) is given by (6.71)

\[ c_1 = c_{2,0}^{(0)} + c_{1,0}^{(1)} + c_{0,0}^{(2)} + c_{0,1}^{(1)} + c_{0,2}^{(0)} \quad (7.30) \]

and, since \( c_{1,0}^{(1)} = c_{2,0}^{(0)} = 0 \), is equal to

\[ c_1 = c_{0,0}^{(2)} + c_{1,1}^{(1)} + c_{0,1}^{(1)} + c_{0,2}^{(0)}. \quad (7.31) \]
The coefficient $c_{0,0}^{(2)}$ is given by (6.57)

$$c_{0,0}^{(2)} = \text{tr} \left[ [D_+ a_0^+] [D_- a_0^-] \right],$$

(7.32)

and, therefore, by using (6.9) we obtain

$$c_{0,0}^{(2)} = \text{tr} \left( S_+ S_- \right).$$

(7.33)

The coefficient $c_{1,1}^{(0)}$ is determined by the second coefficient of the asymptotic expansion of the integral $Y_1^{(0)}$, (6.49), of the function $N_1^{(0)}$, (6.45), which we can rewrite in the form

$$N_1^{(0)} = \frac{1}{4} (ts)^{-1} g^{1/2}(x) G^{1/2}(x') e^{\omega(x,x')} \varphi_3(x, x'),$$

(7.34)

where

$$\varphi_3(x, x') = -\frac{1}{2} \text{tr} \left\{ t \nu_+(x, x') a_+^t(x, x') \mu_-(x', x) + s \mu_+(x, x') \nu_-(x', x) a_+^t(x', x) + s \nu_-(x, x') a_-^t(x, x') \mu_+(x', x) + t \mu_-(x, x') \nu_+(x', x) a_-^t(x', x) \right\}. \quad (7.35)$$

We use Lemma 5 and eq. (5.44) to compute it. First of all, we notice that

$$[\varphi_3] = [\nabla_i^g \varphi_3] = 0.$$  

(7.36)

Therefore, we get

$$c_{1,1}^{(0)} = \frac{1}{4} G^{ij} [\nabla_i^g \nabla_j^g (e^{\omega} \varphi_3)].$$  

(7.37)

Next, by using (7.27) we compute the diagonal values of the second derivatives

$$[\nabla_i^g \nabla_j^g (e^{\omega} \varphi_3)] = [\nabla_i^g \nabla_j^g \varphi_3] = 2 g_{mi} g_{nj} \text{tr} \left\{ t \gamma_+^k c_+^m \left( \frac{1}{6} R_+ I - Q_+ \right) + s \gamma_+^m c_+^k \left( \frac{1}{6} R_- I - Q_- \right) \right\}. \quad (7.38)$$

This gives

$$c_{1,1}^{(0)} = \frac{1}{2} g_{ij} \text{tr} \left\{ t \gamma_+^j c_+^j \left( \frac{1}{6} R_+ I - Q_+ \right) + s \gamma_+^j c_+^j \left( \frac{1}{6} R_- I - Q_- \right) \right\}. \quad (7.39)$$

Recall that $Q_+$ for Dirac type operators is given by (3.13).
The coefficient $c^{(1)}_{0,1}$ is determined by the second coefficient of the asymptotic expansion of the integral $Y^{(1)}_0$, (6.49), of the function $N^{(1)}_0$, (6.46), which can be written in the form

$$N^{(1)}_0 = \frac{1}{4} (ts)^{-1} g^{1/2}(x) G^{1/2}(x') e^{i\omega(x,x')} \varphi_4(x, x'), \quad (7.40)$$

where

$$\varphi_4(x, x') = \text{tr} \left\{ -t \left( \theta_+(x, x') \mu_- (x', x) + \mu_- (x, x') \theta_+ (x', x) \right) \\
- s \left( \theta_-(x, x') \mu_+ (x', x) + \mu_+ (x, x') \theta_-(x', x) \right) \right\}, \quad (7.41)$$

with

$$\theta_\pm (x, x') = e^{-\xi \pm} D_\pm \left( e^{\xi \pm} \mathcal{P}_\pm \right) = \left\{ i\gamma^k \pm (x)(\nabla^\pm_k + \xi^\pm_k) + S_\pm (x) \right\} \mathcal{P}_\pm (x, x'), \quad (7.42)$$

We use Lemma 5 and eq. (5.44) to compute it. First of all, we notice that

$$[\varphi_4] = 0. \quad (7.43)$$

Next, by using (7.27) and the obvious limit

$$[\theta_\pm] = S_\pm \quad (7.44)$$

we compute

$$[\nabla^g_j \varphi_4] = 0. \quad (7.45)$$

Therefore,

$$c^{(1)}_{0,1} = \frac{1}{4} G^{ij} \left[ \nabla^g_i \nabla^g_j (e^{i\omega} \varphi_4) \right]. \quad (7.46)$$

Further, by using (7.44) and (7.27) and omitting all terms that vanish on the diagonal we obtain

$$[\nabla^g_i \nabla^g_j (e^{i\omega} \varphi_4)] = [\nabla^g_i \nabla^g_j \varphi_4] = -2 \text{tr} \left\{ \frac{1}{2} t S_+ \left[ \nabla^g_i \nabla^g_j \mu_- (x', x) + \nabla^g_i \nabla^g_j \mu_- (x, x') \right] \\
+ \frac{1}{2} s S_- \left[ \nabla^g_i \nabla^g_j \mu_+ (x', x) + \nabla^g_i \nabla^g_j \mu_+ (x, x') \right] \\
+ t i \gamma^k \pm \left[ \nabla^g_j \theta_+ (x', x) - \nabla^g_j \theta_+ (x, x') \right] \\
+ s i \gamma^k \pm \left[ \nabla^g_j \theta_- (x', x) - \nabla^g_j \theta_- (x, x') \right] \right\}. \quad (7.47)$$
Next, by using (7.13) and (7.27) we compute

\[ [\nabla_{(i}^g \nabla_{j)}^g \mu_{\pm}(x, x')] = [\nabla_{(i}^g \nabla_{j)}^g \nu_{\pm}(x, x')] - 2i\gamma^k_{\pm}g^k_{kl}(\mathcal{A}^\pm_{ij}), \quad (7.48) \]

\[ [\nabla_{(i}^g \nabla_{j)}^g \mu_{\pm}(x', x)] = [\nabla_{(i}^g \nabla_{j)}^g \nu_{\pm}(x', x)] - 2i\gamma^k_{\pm}g^k_{kl}(\mathcal{A}^\pm_{ij}). \quad (7.49) \]

Now, by using (3.9) and (4.55) we have

\[ \nabla_j^g \gamma_{\pm} = -W_{\pm}^{k} \eta_{\pm}^m - [\mathcal{A}^\pm_i, \gamma^k_{\pm}] \quad (7.50) \]

By using this equation and (4.69) we compute

\[ [\nabla_{(i}^g \nabla_{j)}^g \nu_{\pm}(x, x')] = -2g_{kl}^\pm i[\mathcal{A}_{ij}^\pm, \gamma^k_{\pm}] + i\gamma^k_{\pm}g_{mk}^\pm W_{\pm}^{m}ij. \quad (7.51) \]

Further, by using (4.70) we have

\[ [\nabla_{(i}^g \nabla_{j)}^g \nu_{\pm}(x', x)] = -i\gamma^k_{\pm}g_{mk}^\pm W_{\pm}^{m}ij. \quad (7.52) \]

Therefore,

\[ [\nabla_{(i}^g \nabla_{j)}^g \mu_{\pm}(x, x')] = i\gamma^k_{\pm}g_{mk}^\pm W_{\pm}^{m}ij - 2\mathcal{A}_{ij}^\pm g_{mk}^\pm i\gamma^k_{\pm}, \quad (7.53) \]

\[ [\nabla_{(i}^g \nabla_{j)}^g \mu_{\pm}(x', x)] = -i\gamma^k_{\pm}g_{mk}^\pm W_{\pm}^{m}ij - 2i\gamma^k_{\pm}g_{mk}^\pm \mathcal{A}_{ij}^\pm. \quad (7.54) \]

Next, by using (4.31) and (4.80) we compute the diagonal values

\[ [\nabla_j^g \theta_+(x, x')] = i\gamma^k_{\pm}S^+_{jk} - i\gamma^k_{\pm} \mathcal{A}_{jk}^+. \quad (7.55) \]

\[ [\nabla_j^g \theta_-(x, x')] = -i\gamma^k_{\pm} \mathcal{A}_{jk}^- \quad (7.56) \]

where

\[ \Omega_{jk}^\pm = \frac{1}{2} R_{jk}^\pm + \frac{1}{6} R_{jk}^\pm I. \quad (7.57) \]

To avoid confusion, we note that the derivatives \( \nabla^g \) here do not include the connection \( \mathcal{A} \); therefore,

\[ [\nabla_j^g \theta_+(x, x')] = i\gamma^k_{\pm} \Omega_{jk}^+ + \nabla_j^g S^+ - \mathcal{A}_{jk}^+ S^+, \quad (7.58) \]

\[ [\nabla_j^g \theta_-(x', x)] = -i\gamma^k_{\pm} \Omega_{jk}^- + S^+ \mathcal{A}_{jk}^- \quad (7.59) \]

By using the above results we compute

\[ [\nabla_j^g \nabla_j^g \varphi_4] = -2\text{tr}\left\{ -ti\gamma_{-}g_{mi}^\pm \nabla_j^g S^+ + tS^+ \left[ (C^+_{ij} - C^-_{ij}) g^+_{jk} + i\gamma^k_{\pm}g^+_{kl}(C^+_{jk} - C^-_{jk}) \right] 
- st\gamma^m_{-}g_{mi}^\pm \nabla_j^g S^- - sS^- \left[ (C^+_{ij} - C^-_{ij}) g^+_{jk} + i\gamma^k_{\pm}g^+_{kl}(C^+_{jk} - C^-_{jk}) \right] 
+ 2t\gamma^m_{-}g^+_{mi}^\pm \Omega^+_{jk} - 2s\gamma^m_{-}g^+_{mi}^\pm \Omega^-_{jk} \right\}. \quad (7.60) \]
Now, by using (3.11) and the cyclicity of the trace we obtain a simpler form

\[ [\nabla_i^g \nabla_j^g \varphi_2] = -2\text{tr} \left\{ -t \gamma_m^m g_m^m (\nabla_j^g)^+ S_+ - \text{si} \gamma_m^m g_m^m (\nabla_j^g)^- S_- \right\} + 2t \gamma_m^m \nabla_m^m (\nabla_j^g)^+ \Omega_j^+ + 2 \gamma_m^m \nabla_m^m (\nabla_j^g)^- \Omega_-. \] (7.61)

Thus, we obtain the coefficient \( c_{0,1}^{(1)} \) from (7.46)

\[ c_{0,1}^{(1)} = \frac{1}{2} G^{ij} \text{tr} \left\{ -t \gamma_m^m g_m^m (\nabla_j^g)^+ S_+ - \text{si} \gamma_m^m g_m^m (\nabla_j^g)^- S_- \right\} + 2t \gamma_m^m \nabla_m^m (\nabla_j^g)^+ \Omega_j^+ + 2 \gamma_m^m \nabla_m^m (\nabla_j^g)^- \Omega_- \} \] (7.62)

The coefficient \( c_{0,2}^{(0)} \) is determined by the third coefficient of the asymptotic expansion of the integral \( \gamma_0^{(0)} \), (6.49), of the function \( N_0^{(0)} \), (7.22). We use Lemma 5 to compute it. Since the function \( \varphi_2 \) and its first derivative vanish on the diagonal, (7.25), it is given by the eq. (5.45). We use the equations

\[ [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] = [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] + 4 \omega_{ij} [\nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] + 6 \left( \omega_{ij} + \omega_{ij} \right) [\nabla_k^g \nabla_l^g (e^\omega \varphi_2)], \] (7.63)

\[ [\nabla_i^g \nabla_j^g \nabla_k^g (e^\omega \varphi_2)] = [\nabla_i^g \nabla_j^g \nabla_k^g (e^\omega \varphi_2)] + 3 \omega_{ij} [\nabla_j^g \nabla_k^g (e^\omega \varphi_2)], \] (7.64)

and (7.11) to obtain

\[ c_{0,2}^{(0)} = \frac{1}{4} t s \left\{ \frac{1}{2} G^{ij} G^{kl} [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] + N^{ijkl} [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] \right\} \] (7.66)

\[ + \left\{ \frac{1}{6} \left( G^{kl} G^{ij} + 2 G^{ik} G^{jl} \right) \left( R_{ij}^+ + R_{ij}^- - 2 R_{ij}^0 \right) + M^{ijkl} \right\} \] [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)],

where \( N^{ijkl} \) and \( M^{ijkl} \) are defined by (1.46) and (1.47).

The second derivative of \( \varphi_2 \) was computed in (7.28). So, we compute the third derivative; we have

\[ [\nabla_i^g \nabla_j^g \nabla_k^g \nabla_l^g (e^\omega \varphi_2)] = \frac{3}{2} \left[ [\nabla_i^g \nabla_j^g \mu_+(x, x') \nabla_k^g \mu_-(x', x) + \nabla_i^g \mu_+(x, x') \nabla_j^g \nabla_k^g \mu_-(x', x) + \nabla_i^g \nabla_k^g \mu_+(x', x) + \nabla_i^g \nabla_j^g \mu_+(x', x') \nabla_k^g \mu_-(x', x)] \right]. \] (7.67)
By using (7.27), (7.53) and (7.54) we get

\[ [\nabla^g_j \nabla^g_i \nabla^g_k \varphi_2] = 3 \text{tr} \left\{ W^m_{+\pm \pm} g^+_{klq} g^-_{mp} + W^-_{+\pm \pm} g^+_{klp} g^-_{mq} \right\} \gamma^g_p \gamma^g_q \]

\[ + g^-_{q(i)} (C^+_{j} - C^-_{j}) g^+_{klp} \gamma^g_p \gamma^g_q - g^-_{p(i)} (C^+_{j} - C^-_{j}) g^-_{klq} \gamma^g_p \gamma^g_q \right\}. \]  

(7.68)

Finally, we compute the forth derivative of the function \( \varphi_2 \); we rewrite it in the form

\[ \varphi_2 = -\frac{1}{2} A_{pq} (x, x') B^{pq} (x, x') \]  

\[ (+ \leftrightarrow -), \]  

(7.69)

where

\[ A_{pq}' = \sigma^+ \sigma^- \]  

(7.70)

and

\[ B^{pq} = \text{tr} \left\{ P^-_{-\pm} (x, x') \gamma^g_p (x) P_+ (x, x') \gamma^g_q (x') \right\}, \]  

(7.71)

and the symbol \((+ \leftrightarrow -)\) indicates that one should add the same term with + and – switched. The diagonal value of the forth symmetrized derivative is then

\[ [\nabla^g_j \nabla^g_i \nabla^g_k \nabla^g_l \varphi_2] = -\frac{1}{2} [\nabla^g_j \nabla^g_i \nabla^g_k A_{pq}] \text{tr} (\gamma^g_p \gamma^g_q) - 2 [\nabla^g_j \nabla^g_k A_{pq}'] [\nabla^g_l B^{pq}'] \]

\[ - 3 [\nabla^g_i A_{pq}'] [\nabla^g_k B^{pq}'] + (+ \leftrightarrow -). \]  

(7.72)

First of all, it is easy to get

\[ [\nabla^g_i \nabla^g_j A_{pq}'] = 2 g^+_{pq(i) g^-_{jk)}, \]  

(7.73)

\[ [\nabla^g_j \nabla^g_k A_{pq}'] = 3 V^-_{q(i) g^+_{jk}} + 3 T^+_{p(jk) g^-_{lq}}, \]  

(7.74)

where the tensors \( V_{ijk} \) and \( T_{ijk} \) are given by (4.70), (4.69)

\[ T_{ijk}^+ = 3 a_{+mi} W_{\pm jk}, \]  

(7.75)

\[ V_{ijk}^+ = -a_{+mi} W_{\pm jk}, \]  

(7.76)

Similarly, we obtain

\[ [\nabla^g_i \nabla^g_j \nabla^g_k A_{pq}'] = 4 V^-_{q(ij) g^+_{lp}} + 6 V^-_{q(ij) T^+_{klp}} - 4 T^+_{p(ijk) g^-_{lpq}}, \]  

(7.77)

where the tensors \( T^\pm_{ijk} \) and \( V^\pm_{ijk} \) are given by (4.71), (4.72)

\[ T^\pm_{ijk} = 3 a_{+mi} \nabla^g_{+ij} W_{\pm lj} + a_{+mi} \nabla^g_{-ij} W_{\pm l|k} + 3 a_{+mi} W_{\pm l|k} W_{\pm lj} \]

\[ + g^+_{mi} \nabla^g_{\pm lj} W_{\pm l|k} + 3 g^+_{mi} W_{\pm l|k} W_{\pm l|j}, \]  

(7.78)

\[ V^\pm_{ijk} = -a^+_{mi} \nabla^g_{\pm ij} W_{\pm lj} - a^-_{mi} W_{\pm l|j}, \]  

(7.79)
Next, by using
\[ [\nabla^+_i B^{pq}] = -\text{tr} \left[ (C^+_i - C^-_i) \gamma^p_+ \gamma^q_- \right] \] (7.80)
we compute
\[ [\nabla^g_i B^{pq}] = \text{tr} \left[ -(C^+_i - C^-_i) \gamma^p_+ \gamma^q_- - W^+_p m \gamma^m_+ \gamma^q_- \right] . \] (7.81)
Further, we have
\[
[\nabla^g_(k \nabla^g_i) B^{pq}] = \text{tr} \left\{ -\nabla^g_k W^+_p \gamma^{pq}_{lm} + 3 W^+_p m(k \nabla^g_i W^+_n) \gamma^{pq}_{lm} - W^+_n \nabla^g_i W^+_p \gamma^{pq}_{nm} \right\} \gamma^m_+ \gamma^q_- \\
+ \text{tr} \left\{ -\nabla^g_k \nabla^g_i W^+_p \gamma^{pq}_{lm} + 3 W^+_p m(k \nabla^g_i W^+_n) \gamma^{pq}_{lm} - W^+_n \nabla^g_i W^+_p \gamma^{pq}_{nm} \right\} \gamma^m_+ \gamma^q_- \\
+ [\nabla^g_(k \nabla^g_i) B^{pq}] . \] (7.82)
Next, by using the equations \([\nabla^g_i \mathcal{P}_+] = [\nabla^g_i \nabla^g_j \mathcal{P}_+] = 0, \nabla^g_i \gamma^j_- = 0, \) and
\[
[\nabla^g_i \mathcal{P}_+] = C^+_i - C^-_i , \\
[\nabla^g_i \nabla^g_j \mathcal{P}_+] = \nabla^g_i (C^+_j - C^-_j) + (C^+_i - C^-_i)(C^+_j - C^-_j) , \] (7.83)
(7.84)
we obtain
\[
[\nabla^g_i \nabla^g_j B^{pq}] = \text{tr} \left\{ -\nabla^g_i (C^+_j - C^-_j) + (C^+_i - C^-_i)(C^+_j - C^-_j) \right\} \gamma^p_+ \gamma^q_- \\
= \text{tr} \left\{ -\nabla^g_i (C^+_j - C^-_j) + W^+_m (C^+_s - C^-_s) \\
- 2C^+_i (C^+_j - C^-_j) + C^+_i (C^+_j + C^-_i) \right\} \gamma^p_+ \gamma^q_- , \] (7.85)
and, therefore,
\[
[\nabla^g_(k \nabla^g_j) B^{pq}] = \text{tr} \left\{ -\nabla^g_k (W^+_p \gamma^{pq}_{lm}) + 3 W^+_p m(k \nabla^g_i W^+_n) \gamma^{pq}_{lm} - W^+_n \nabla^g_i W^+_p \gamma^{pq}_{nm} \right\} \gamma^m_+ \gamma^q_- \\
+ 2 W^+_p m(k \nabla^g_i \nabla^g_i W^+_n \gamma^{pq}_{lm}) - \nabla^g_k \nabla^g_i (C^+_j - C^-_j) \gamma^p_+ \gamma^q_- \\
+ \left( C^+_i (C^+_j - C^-_j) + C^+_i (C^+_i - C^-_j) \right) \gamma^p_+ \gamma^q_- . \] (7.86)
Finally, by collecting all these results we obtain

\[
[\nabla^g_{0j} \nabla^g_{k} \nabla^g_{lj} \nabla^g_{jk} \varphi_2] = \text{Sym}(i, j, k, l) \text{tr} \left\{ V_{pji,kl} \gamma^p_i \gamma^q_j - 6 g^p_{ip} g^q_{kj} [\gamma^p_i, \gamma^q_j] \nabla^g_{k} (C^+_i - C^-_i) \right. \\
-6 \left( g^p_{mp} W^m_{+ij} g^q_{kq} + g^p_{kp} g^q_{mq} W^m_{-ij} \right) [\gamma^p_i, \gamma^q_j] (C^+_i - C^-_i) \\
+6 g^p_{pj} g^q_{jq} (C^+_k C^+_i + C^-_k C^-_i) (\gamma^p_i \gamma^q_j + \gamma^q_i \gamma^p_j) \\
-12 g^p_{pj} g^q_{jq} C^+_k C^-_i \gamma^p_i \gamma^q_j - 12 g^p_{pq} g^q_{jq} C^+_k C^-_i \gamma^q_j \gamma^p_i \right\}. 
\]

(7.87)

where \( V_{pji,kl} \) is defined by (1.48).

This enables us to compute the coefficient \( c^{(0)}_{0,2} \) (7.66),

\[
c^{(0)}_{0,2} = \frac{1}{4} t_5 \text{tr} \left\{ \frac{1}{3} \left( G^{kl} G^{ij} + 2 G^{jk} G^{il} \right) (R^+_i + R^-_i - 2 R^g_{ij}) g^p_{lp} g^q_{lj} \nabla^p \gamma^q_j \right. \\
+ \left[ \frac{1}{2} G^{(ij)(kl)} V_{pji,kl} + 3 N^{kl} \left( g^p_{mp} W^m_{+ij} g^q_{lq} + g^p_{mq} W^m_{-ij} g^q_{lp} \right) \right] \nabla^p \gamma^q_j \\
+ 2 M^{kl} g^p_{lp} g^q_{lj} \nabla^p \gamma^q_j + 3 G^{(ij)} G^{(kl)} g^p_{lp} g^q_{lj} \left( \nabla^p \gamma^q_j - \gamma^q_j \gamma^p_i + \gamma^q_j \gamma^p_i \right) \nabla^p \gamma^q_j \left( C^+_i - C^-_i \right) \\
+ \left[ 3 G^{(ij)} G^{(kl)} \left( g^p_{mp} W^m_{+ij} g^q_{lq} + g^p_{mq} W^m_{-ij} \right) + 3 N^{kl} g^p_{lp} g^q_{lj} \right] \nabla^p \gamma^q_j \left( \nabla^p \gamma^q_j - \gamma^q_j \gamma^p_i + \gamma^q_j \gamma^p_i \right) g^p_{lp} g^q_{lj} \\
\left. \right\} \left( C^+_i - C^-_i \right) \\
+ 3 G^{(ij)} G^{(kl)} (C^+_k C^+_i + C^-_k C^-_i) (\gamma^p_i \gamma^q_j + \gamma^q_i \gamma^p_j) g^p_{lp} g^q_{lj} \\
-6 G^{(ij)} G^{(kl)} C^+_k C^-_i \nabla^p \gamma^q_j - 6 G^{(ij)} G^{(kl)} C^+_k C^-_i \gamma^q_j \nabla^p \gamma^q_j - 6 G^{(ij)} G^{(kl)} C^+_k C^-_i \gamma^q_j \nabla^p \gamma^q_j \left( C^+_i - C^-_i \right) \right\}. 
\]

(7.88)

Thus (after some tedious but straightforward manipulations; by using (3.13) and many well known algebraic properties of the Dirac matrices (22)) we obtain the coefficient \( c^{(1)}_1 \) (1.50). This proves Theorem [3] the Corollary [3] follows.

For equal operators \( D_+ = D_- \) the coefficient \( C_1 \) takes the form

\[
C_1 = (t + s)^{-n/2} \int_M dx g^{1/2} \text{tr} \left\{ \frac{n}{2} - 1 \left( \frac{1}{6} R^{ij} + \frac{1}{2} \gamma^{ij} \nabla^2_i - S \right) - \frac{(n - 1)}{2} i \gamma^i \nabla_j S \right\}. 
\]

(7.89)

Notice that since \( \nabla_j S \) anticommutes with \( \gamma^j \), the last term here vanishes and \( C_1 \) is indeed equal to \( (n/2 - 1)(t + s)^{-n/2} A_1 \), with \( A_1 \) given by (1.16).
8 Conclusion

The primary goal of this paper was to introduce and to study some new spectral invariants of a pair of elliptic partial differential operators on manifolds, that we call the relative spectral invariants and the combined heat traces. Of special interest are the asymptotics of these invariants. We established a general asymptotic expansion of these invariants and computed the first two coefficients of the asymptotic expansions.

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