Difference equations for correlation functions of $A^{(1)}_{n-1}$-face model with boundary reflection

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Abstract

The $A^{(1)}_{n-1}$-face model with boundary reflection is considered on the basis of the boundary CTM bootstrap. We construct the fused boundary Boltzmann weights to determine the normalization factor. We derive difference equations of the quantum Knizhnik-Zamolodchikov type for correlation functions of the boundary model. The simplest difference equations are solved in the case of the free boundary condition.

1 Introduction

Many two dimensional integrable models without boundary have been solved by using the representation theory of affine quantum group [1, 2]. The integrability of bulk models is ensured by the factorized scattering condition or the Yang–Baxter equation, in addition to the unitarity and crossing symmetry condition [3, 4].

Cherednik [5, 6] showed that the integrability in the presence of reflecting boundary is ensured by the reflection equation (boundary Yang–Baxter equation) and the Yang–Baxter equation for bulk theory. A systematic treatment of determining the spectrum of integrable models with boundary reflection was initiated by Sklyanin [7] in the framework of the algebraic Bethe Ansatz. The boundary interaction is specified by the boundary S-matrix for massive quantum theories [8], by the reflection matrix for lattice vertex models [7], and by the boundary weights for lattice face models [9, 10].

In our previous paper [11] Belavin’s $Z_n$-symmetric elliptic vertex model with boundary reflection is considered on the basis of the boundary CTM(corner transfer matrix) bootstrap formulated in [12]. We derived a set of difference equations called the boundary quantum Knizhnik-Zamolodchikov equation for correlation functions in the boundary Belavin’s model. Furthermore, we obtained the boundary spontaneous polarization by solving the simplest difference equations. The resulting quantities are exactly equal to the square of that for bulk spontaneous polarization [13], up to a phase factor.

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In this paper we consider the $A_{n-1}^{(1)}$-face model \cite{14} with a boundary on the basis of boundary CTM bootstrap. The $\mathbb{Z}_n$-symmetric model and the $A_{n-1}^{(1)}$-face model are related by the vertex-face correspondence \cite{14} in the bulk theory. Thus we wish to find the similar structure in the boundary $A_{n-1}^{(1)}$-face model as the one observed in \cite{11}.

Integrable face models on a half infinite lattice have been studied in \cite{15,16,17,18}. In \cite{15} the transfer matrix of the boundary ABF model (the boundary $A_1^{(1)}$-face model) was diagonalized by constructing the boundary vacuum states, under the Ansatz \cite{19} that the boundary vacuum states should be obtained from the Fock vacuums by applying the exponential of the infinite sum of the quadratic bosonic oscillators associated with the bulk ABF models \cite{20}. In \cite{16} the solution to the reflection equation was given for the boundary $A_{n-1}^{(1)}$-face model. In \cite{17} free energy and critical exponents was obtained for $n = 2$, the boundary ABF model case. In order to discuss the higher $n$ cases we shall establish the boundary crossing symmetry in section 3. In \cite{18} correlation functions for the boundary $XYZ$ model were obtained in terms of those for the boundary ABF model \cite{15} by using the vertex-face transformation method \cite{21}.

The rest of this paper is organized as follows. In section 2 we review the boundary $A_{n-1}^{(1)}$-face model, thereby fixing our notation. In section 3 we introduce the fusions of $K$-matrix to determine the normalization factors of $K$-matrix. We also establish the boundary crossing symmetry. In section 4 we construct lattice realization of the vertex operators on the basis of the boundary CTM bootstrap approach. Furthermore, we derive difference equations for correlation functions of the boundary $A_{n-1}^{(1)}$-face model. In section 5 we give some concluding remarks. In Appendix A we give the explicit expressions of fused Boltzmann weights of the bulk $A_{n-1}^{(1)}$-face model \cite{22}. In Appendix B we give a simple sketch of the proof of the reflection equation.

2 Boundary $A_{n-1}^{(1)}$-face model

The present section aims to formulate the problem, thereby fixing the notation.

2.1 Theta functions

Throughout this paper we fix the integers $n$ and $r$ such that $r \geq n + 2$, and also fix the parameter $x$ such that $0 < x < 1$. We will use the abbreviations,

$$[z] = x^{\frac{z^2}{2}} \Theta_{z^2}(x^{2z}),$$ (2.1)

where the Jacobi theta function is given by

$$\Theta_q(\zeta) = (\zeta; q)_\infty (q \zeta^{-1}; q)_\infty (q, q)_\infty,$$ (2.2)

$$(\zeta; q_1, \ldots, q_m) = \prod_{i_1, \ldots, i_m \geq 0} (1 - \zeta q_1^{i_1} \cdots q_m^{i_m}).$$ (2.3)

For an additive parameter $z$, we often use the corresponding multiplicative parameter $\zeta = x^{2z}$.

For later conveniences we also introduce the following symbols

$$r_m(z) = \zeta^{-r \frac{n-m}{n}} \frac{g_m(\zeta)}{g_m(\zeta^{-1})}, \quad g_m(\zeta) = \frac{x^{2n+2r-m-1} \zeta \{x^{m+1} \zeta\} \{x^{2r+m-1} \zeta\}}{x^{2n-m+1} \zeta \{x^{2r+m-1} \zeta\}}.$$ (2.4)
where \( 1 \leq m \leq n - 1 \) and
\[
\{ \zeta \} = (\zeta; x^{2n}, x^{2r})_\infty.
\]
(2.5)
In particular the function \( r_1(z) \) will appear in the expression of the Boltzmann weights of the \( A^{(1)}_{n-1} \)-face model in the regime III.

### 2.2 The weight lattice of \( A^{(1)}_{n-1} \)

Let \( V = \mathbb{C}^n \) and \( \{ \varepsilon_\mu \}_{0 \leq \mu \leq n-1} \) be the standard orthogonal basis with the inner product \( \langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu} \). The weight lattice of \( A^{(1)}_{n-1} \) is defined as follows:
\[
P = \bigoplus_{\mu=0}^{n-1} \mathbb{Z} \bar{\varepsilon}_\mu,
\]
(2.6)
where
\[
\bar{\varepsilon}_\mu = \varepsilon_\mu - \varepsilon, \quad \varepsilon = \frac{1}{n} \sum_{\mu=0}^{n-1} \varepsilon_\mu.
\]

We denote the fundamental weights by \( \omega_\mu (1 \leq \mu \leq n - 1) \)
\[
\omega_\mu = \bar{\varepsilon}_0 + \bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_{\mu-1}.
\]
Since \( \omega_n = 0 \), you can define \( \omega_\mu \) for \( \mu \in \mathbb{Z} \) by setting \( \omega_{\mu+n} = \omega_\mu \). For \( a \in P \) we set
\[
a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu, \quad \bar{a}_\mu = \langle a + \rho, \varepsilon_\mu \rangle, \quad \rho = \sum_{\mu=1}^{n-1} \omega_\mu.
\]
(2.7)
We also set
\[
P^+_l = \left\{ \sum_{\mu=1}^{n-1} a^\mu \omega_\mu \Bigg| a^1, \cdots, a^{n-1} \in \mathbb{Z}_{\geq 0}, \sum_{\mu=1}^{n-1} a^\mu \leq l \right\}.
\]
We may denote \( a \in P^+_l \) by \( a = \sum_{\mu=1}^{n-1} a^\mu \omega_\mu + \left( l - \sum_{\mu=1}^{n-1} a^\mu \right) \omega_0 \).

### 2.3 The \( A^{(1)}_{n-1} \) face model

The \( A^{(1)}_{n-1} \) face model is the one whose local state \( a \) is restricted such that \( a \in P^+_{r-n} \). An ordered pair \( (a, b) \in P^2 \) is called admissible if \( b = a + \bar{\varepsilon}_\mu \), for a certain \( \mu (0 \leq \mu \leq n - 1) \). In what follows we denote \( b \leftarrow a \) when \( (a, b) \in P^2 \) is admissible. Furthermore, an ordered quartet \( (a, b, c, d) \in P^4 \) is called admissible if the four pairs \( (a, b), (a, d), (b, c) \) and \( (d, c) \) are admissible.

For \( (a, b, c, d) \in P^4 \) let
\[
W(a, b, c, d| z_1 - z_2) = W \left( \begin{array}{c|c} c & d \\ \hline b & a \end{array} \right) z_1 - z_2
\]
be the Boltzmann weight of the \( A^{(1)}_{n-1} \) model for the state configuration \( (a, b, c, d) \) round a face. Here the four states \( a, b, c \) and \( d \) are ordered clockwise from the SE corner, and the oriented broken lines in the
above figure carry spectral parameters. In this model \( W(a, b, c, d | z) = 0 \) unless the quartet \((a, b, c, d)\) is admissible. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter \( z \) as follows:

\[
W \left( \begin{array}{cc} a + 2\tilde{\epsilon}_\mu & a + \tilde{\epsilon}_\mu \\ a + \tilde{\epsilon}_\mu & a \end{array} \right) | z \rangle = r_1(z),
\]

\[
W \left( \begin{array}{cc} a + \tilde{\epsilon}_\mu + \tilde{\epsilon}_\nu & a + \tilde{\epsilon}_\nu \\ a + \tilde{\epsilon}_\nu & a \end{array} \right) | z \rangle = r_1(z) \frac{[z + a_{\mu\nu} - 1]}{[z + 1][a_{\mu\nu}]} \quad (\mu \neq \nu),
\]

\[
W \left( \begin{array}{cc} a + \tilde{\epsilon}_\mu + \tilde{\epsilon}_\nu & a + \tilde{\epsilon}_\nu \\ a + \tilde{\epsilon}_\nu & a \end{array} \right) | z \rangle = r_1(z) \frac{[z + a_{\mu\nu}][1]}{[z + 1][a_{\mu\nu}]} \quad (\mu \neq \nu),
\]

where \(-\frac{\pi}{2} < z < 0\) in the regime \(\text{III}\).

The Boltzmann weights (2.8) solve the face-type Yang-Baxter equation [14]:

\[
\sum_g W \left( \begin{array}{ccc} d & e & z_1 \\ c & g & z_2 \end{array} \right) W \left( \begin{array}{cc} c & g \\ b & a \end{array} \right) W \left( \begin{array}{cc} e & f \\ g & a \end{array} \right) z_{12} = \sum_g W \left( \begin{array}{ccc} g & f & z_1 \\ b & a & z_2 \end{array} \right) W \left( \begin{array}{ccc} d & e & z_1 \\ g & f & z_2 \end{array} \right) W \left( \begin{array}{cc} d & g \\ c & b \end{array} \right) z_{12}.
\]

Some numerical calculations concerning the hard hexagon model in [4] suggest that the corner transfer matrix (CTM) is well defined in the thermodynamic limit if the normalization factor \( r_1(z) \) is chosen such that the partition function per site is equal to unity. In order to fix \( r_1(z) \) the following two inversion relations are useful [14]:

\[
\sum_g W \left( \begin{array}{cc} c & g \\ b & a \end{array} \right) | -z \rangle W \left( \begin{array}{cc} c & d \\ g & a \end{array} \right) | z \rangle = \delta_{bd},
\]

\[
\sum_g G_g W \left( \begin{array}{cc} g & b \\ d & c \end{array} \right) | -n - z \rangle W \left( \begin{array}{cc} g & d \\ b & a \end{array} \right) | z \rangle = \delta_{ac} \frac{G_b G_d}{G_a},
\]

where

\[
G_a = \prod_{0 \leq \mu < \nu \leq n-1} [a_{\mu\nu}].
\]

From the inversion trick based on these relations we get the expression of \( r_1(z) \) in the regime \(\text{III}\).

The Boltzmann weights (2.8) also have \(\sigma\)-invariance [14]:

\[
W \left( \begin{array}{cc} \sigma(c) & \sigma(d) \\ \sigma(b) & \sigma(a) \end{array} \right) | z \rangle = W \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) | z \rangle
\]

where \(\sigma\) is the diagram automorphism of \(A_{n-1}^{(1)}\) defined by \(\sigma(\omega_\mu) = \omega_{\mu+1}\).
2.4 Solution to the reflection equation

Let us consider the interaction at the boundary, which is specified by the boundary Boltzmann weight or the $K$-matrix:

$$K(a, b, c|z) = K\left(\begin{array}{c} b \\ a \\ c \end{array} \bigg| z \right) = 
\begin{array}{c} b \\ a \\ c \end{array}$$

Here $(b, a)$ and $(c, a)$ are admissible. The integrability condition of the boundary face model is the face-type reflection equation [9, 10]:

$$\sum_{d, e} K\left(\begin{array}{c} f \\ d \\ g \end{array} \bigg| z_2 \right) W\left(\begin{array}{c} c \\ d \\ e \end{array} \bigg| z_1 + z_2 \right) K\left(\begin{array}{c} e \\ b \\ a \end{array} \bigg| z_1 \right) W\left(\begin{array}{c} d \\ b \\ e \end{array} \bigg| z_1 - z_2 \right)
= \sum_{d, e} W\left(\begin{array}{c} c \\ d \\ g \end{array} \bigg| z_1 - z_2 \right) K\left(\begin{array}{c} e \\ b \\ a \end{array} \bigg| z_1 \right) W\left(\begin{array}{c} d \\ b \\ e \end{array} \bigg| z_1 + z_2 \right) K\left(\begin{array}{c} e \\ b \\ a \end{array} \bigg| z_2 \right).$$

(2.13)

For the bulk Boltzmann weights of the $A_{n-1}^{(1)}$-face model [14] the diagonal solution to eq. (2.13) is given as follows [14]:

$$K\left(\begin{array}{c} a + \xi \mu \\ a \end{array} \bigg| z \right) = f_\mu(z) \frac{\delta_{\nu \mu + \eta + z}}{\delta_{\nu \mu + \eta - z}} \delta_{\alpha \alpha'},$$

(2.14)

where $\eta = \eta(a)$ may depend on $a$ but a constant with respect to $z$. In the present paper we take an $a$-independent constant $\eta$ for simplicity. The paper [15] employed the diagonal $K$-matrix for the boundary ABF model $(n = 2$ case) with $\eta(a) = k/2 + c$ for $a = (k-1)\omega_1 + (r-1-k)\omega_0$. We also notice that the opposite admissible conditions such that $(a, b)$ and $(a, c)$ are admissible were used in [16]. Thus we give a simple sketch of the proof of the claim that (2.14) solves the reflection equation (2.13) in Appendix B.

Let $a \in P_{r-n}^+$ be the local state. Then in regime III any ground state configuration is labeled by some $b \in P_{r-n-1}^+$, where the cyclic sequence of $b, b + \omega_1, \cdots, b + \omega_{n-1}$ [14]. In the ‘low temperature’ limit $x \to 0$, one of such ground states is realized. In what follows we fix one of them (say, labeled by $b$) and define all the correlation functions in terms of the ‘low-temperature’ series expansion, the formal power series with respect to $x$. Then the fixed ground state configuration gives the lowest order. Furthermore, any finite order contribution results from the configurations which differ from that of the fixed ground state by altering a finite number of local states. Thus the infinite number of states at far enough sites should coincide with the fixed ground state configuration labeled by $b$. Such one-to-one correspondence with the ground states configuration allows us to specify the boundary conditions by the same index $b \in P_{r-n-1}^+$.

In the presence of the boundary weight (2.14), the $\sigma$-invariance (2.12) is broken when we fix $a \in P_{r-n}^+$ at the right-most corner. In order to determine which $K(a + \xi, a, a|z)$ takes the largest among $K(a + \xi, a, a|z)$’s, let us consider the boundary Boltzmann weights in the ‘low temperature’ limit $x \to 0$. 

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Here we assume that the constant \( \eta \) belongs to one of the following \( n \) disjoint intervals

\[
-\bar{a}_{n-1} < \eta < \frac{r - \bar{a}_0 - \bar{a}_{n-1}}{2}; \quad -\bar{a}_{i-1} + \bar{a}_i < \eta < -\bar{a}_i, \quad (1 \leq i \leq n-1).
\]

Note that

\[
\bar{a}_{n-1} < \cdots < \bar{a}_1 < \bar{a}_0 < r + \bar{a}_{n-1},
\]

for \( a \in P^{-}_{r-n} \). We further restrict the spectral parameter \( z \) to satisfy

\[
-\frac{n}{2} < -\bar{a}_{n-1} - \eta < z < 0, \quad \text{if} \quad -\bar{a}_{n-1} < \eta < \frac{r - \bar{a}_0 - \bar{a}_{n-1}}{2}, \tag{2.16}
\]

\[
-\frac{n}{2} < \bar{a}_i + \eta < z < 0, \quad \text{if} \quad -\bar{a}_{i-1} + \bar{a}_i < \eta < -\bar{a}_i, \quad (1 \leq i \leq n-1). \tag{2.17}
\]

When \(-\bar{a}_{n-1} < \eta < \frac{r - \bar{a}_0 - \bar{a}_{n-1}}{2}\), by using (2.15) and (2.16) we have

\[
\bar{a}_\mu + \eta - z > \bar{a}_\mu + \eta + z \geq \bar{a}_{n-1} + \eta + z > 0, \quad (0 \leq \mu \leq n-1),
\]

\[
r - \bar{a}_\mu - \eta - z > r - \bar{a}_\mu - \eta + z \geq r - \bar{a}_0 - \eta + z > 0, \quad (0 \leq \mu \leq n-1).
\]

Thus the boundary Boltzmann weights behave like

\[
f_a(z)^{-1}K\left((a + \xi_\mu \begin{vmatrix} a \\ a' \end{vmatrix} z, \begin{vmatrix} a \\ a' \end{vmatrix} \right) \sim \zeta^{\frac{2(\bar{a}_\mu + n)}{r + 1}}, \quad (x \to 0). \tag{2.18}
\]

We therefore find from (2.18) that \( K(a + \xi_0, a, a|z) \) takes the largest for \(-\bar{a}_{n-1} < \eta < \frac{r - \bar{a}_0 - \bar{a}_{n-1}}{2}\).

When \(-\frac{\bar{a}_{i-1} + \bar{a}_i}{2} < \eta < -\bar{a}_i \quad (1 \leq i \leq n-1)\), by using (2.15) and (2.17) we have

\[
\bar{a}_\mu + \eta - z > \bar{a}_\mu + \eta + z \geq \bar{a}_{i-1} + \eta + z > 0, \quad (0 \leq \mu \leq i - 1),
\]

\[
\bar{a}_\mu + \eta + z < \bar{a}_\mu + \eta - z \leq \bar{a}_i + \eta - z < 0, \quad (i \leq \mu \leq n - 1),
\]

\[
r - \bar{a}_\mu - \eta - z > r - \bar{a}_\mu - \eta + z \geq r - \bar{a}_0 - \eta + z > 0, \quad (0 \leq \mu \leq n - 1).
\]

Thus the boundary Boltzmann weights behave like

\[
f_a(z)^{-1}K\left((a + \xi_\mu \begin{vmatrix} a \\ a' \end{vmatrix} z, \begin{vmatrix} a \\ a' \end{vmatrix} \right) \sim \begin{cases} 
\zeta^{\frac{2(\bar{a}_\mu + n)}{r + 1}}, & (0 \leq \mu \leq i - 1), \\
\zeta^{\frac{2(\bar{a}_\mu + n)}{r + 1}}, & (i \leq \mu \leq n - 1), 
\end{cases} \quad (x \to 0). \tag{2.19}
\]

We therefore find from (2.19) that \( K(a + \xi_i, a, a|z) \) takes the largest for \(-\frac{\bar{a}_{i-1} + \bar{a}_i}{2} < \eta < -\bar{a}_i \quad (1 \leq i \leq n - 1)\).

As is seen above, which \( K(a + \xi_i, a, a|z) \) takes the largest depends on the value of \( \eta \). Suppose that we fix \( \eta \) such that \( K(a + \xi_i, a, a|z) \) takes the largest value among \( K(a + \xi_\mu, a, a|z) \)'s. Then the boundary condition is be labeled by

\[
b = a - \omega_i \quad (0 \leq i \leq n - 1). \tag{2.20}
\]

For fixed \( i \) in (2.20) we should rather rewrite (2.14) to the expression normalized by \( K(a + \xi_i, a, a|z) \):

\[
K(i)\left((a + \xi_\mu \begin{vmatrix} a \\ a' \end{vmatrix} z, \begin{vmatrix} a \\ a' \end{vmatrix} \right) = f_a^{(i)}(z)\left[\frac{\bar{a}_i + \eta - z}{\bar{a}_i + \eta + z}\right]\delta_{a,a'}.
\tag{2.21}
\]
3 Fusion of $K$-matrices

In order to determine the normalization factor $f_a^{(i)}(z)$ let us introduce the fusion of $K$-matrices. The fusion hierarchy of the boundary AFB model (boundary $A_1^{(1)}$-face model) was constructed in [10], and the fusion procedure of the boundary vertex models was considered in [24, 25].

3.1 Bulk and boundary face operators

In this subsection we reformulate the bulk Boltzmann weight $W$ and the boundary Boltzmann weight $K$ as elements of the bulk and boundary face operators. Let

$$\Omega_z^{(b,a)} = \begin{cases} C v^{(b,a)}(a, b) & \text{if } (a, b) \in P^2 \text{ is admissible}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Omega_z = \bigoplus_{a,b} \Omega_z^{(b,a)}.$$

Then the $W$-operator is defined as [23]

$$W\Omega_z^{(d,a)} \otimes \Omega_{-z}^{(c,b)} = \sum_b v^{(c,b)}(a, b) W\left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \Omega_z^{(b,a)} \otimes \Omega_{-z}^{(c,b)}.$$

Furthermore, if we introduce the $K$-operator by

$$K(z) : \Omega_z \rightarrow \Omega_{-z},$$

$$K(z)v^{(b,a)} = \sum_{a'} v^{(b,a')} K\left( \begin{array}{cc} b & a \\ a' & z \end{array} \right),$$

the reflection equation (2.13) can be regarded as the equality of linear operators:

$$K_2(z_2)W_{21}^{\Omega_{z_2}^{(d,a)} \otimes \Omega_{-z_2}^{(c,b)}} K_1(z_1)W_{12}^{\Omega_{z_1}^{(d,a)} \otimes \Omega_{-z_1}^{(c,b)}} = W_{21}^{\Omega_{z_1}^{(d,a)} \otimes \Omega_{-z_1}^{(c,b)}} K_1(z_1)W_{12}^{\Omega_{z_2}^{(d,a)} \otimes \Omega_{-z_2}^{(c,b)}} K_2(z_2),$$

where the subscripts of $W$’s and $K$’s denote the spaces on which they nontrivially act. The both sides of (3.3) map $\Omega_{z_1} \otimes \Omega_{z_2}$ to $\Omega_{-z_1} \otimes \Omega_{-z_2}$.

3.2 Fusion procedure of $W$-operator

For fixed $m (2 \leq m \leq n)$ let $z_j = z + \frac{m+1}{2} - j$ $(1 \leq j \leq m)$, and for $0 \leq \mu_1 < \cdots < \mu_m \leq n - 1$ we denote

$$a_0 = a, \quad a_j = a + \varepsilon_{\mu_1} + \cdots + \varepsilon_{\mu_j} (1 \leq j \leq m),$$

and

$$a_\sigma^j = a + \varepsilon_{\mu_{\sigma(1)}} + \cdots + \varepsilon_{\mu_{\sigma(j)}} (1 \leq j \leq m),$$

for $\sigma \in S_m$. Note that $a_m^m = a_m$. Let

$$\wedge^m(\Omega_z^{(a_m,a_0)}) = \bigoplus_{\sigma \in S_m} (\text{sgn } \sigma) \Omega_z^{(a_\sigma^m,a_0)} \otimes \Omega_z^{(a_\sigma^{m-1},a_1)} \otimes \cdots \otimes \Omega_z^{(a_m,a_0)}.$$
and let \( v^{(a_m,a)} \) stand for the one dimensional basis of \( \wedge^m(\Omega_z^{a_m,a}) \). Then \( W_{\Omega_2,\Omega_1}^{\wedge^2(\Omega_z)} \) is the fusion operator associated with \( \wedge^2(\Omega_z) \), because \( \text{Im} (W_{\Omega_2,\Omega_1}^{\wedge^2(\Omega_z),\Omega_z} | \Omega_z \otimes \Omega_z) = \wedge^2(\Omega_z) \). For general \( m \), the fusion operator \( \pi_{\pm}^{(m)} \) associated with \( \wedge^m(\Omega_z) \)'s are given as follows [23]:

\[
\begin{align*}
\pi_{\pm}^{(m)} &= W_{\Omega_z,\Omega_{m-1}}^{\wedge^m(\Omega_z),\Omega_z} \cdots W_{\Omega_z,\Omega_{2}}^{\wedge^3(\Omega_z),\Omega_z} W_{\Omega_z,\Omega_{1}}^{\wedge^2(\Omega_z),\Omega_z}, \\
\pi_{\mp}^{(m)} &= W_{\Omega_{m-1},\Omega_z}^{\wedge^2(\Omega_z)} W_{\Omega_{m-2},\Omega_z}^{\wedge^3(\Omega_z)} \cdots W_{\Omega_{1},\Omega_z}^{\wedge^m(\Omega_z),\Omega_z} \cdots W_{\Omega_{m-1},\Omega_z}^{\wedge^2(\Omega_z),\Omega_z}.
\end{align*}
\]

The \( m \)-fold fused \( W \)-operator as an intertwiner on \( \Omega_z \otimes \wedge^m(\Omega_z) \) should be defined as

\[
W_{\Omega_z,\wedge^m(\Omega_z)} v^{(d,a)} \otimes v^{(d,m)} = \sum_{a_m} v^{(d_m,a_m)} \otimes v^{(a_m,a)} W^{(1,m)}(d_m a_m a | z_1 - z_2),
\]

where \( W^{(1,m)} \)'s are the horizontal \( m \)-fold fused Boltzmann weights, and \( d_m - d = \varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_m} \) with \( 0 \leq \lambda_1 < \cdots < \lambda_m \leq n - 1 \). Another \( m \)-fold fused \( W \)-operator as an intertwiner on \( \wedge^m(\Omega_z) \otimes \Omega_z \) should be defined as

\[
W^{\wedge^m(\Omega_z)} v^{(a_m,a)} \otimes v^{(b_m,b)} = \sum_{b} v^{(b_m,b)} \otimes v^{(b,a)} W^{(m,1)}(b_m a_m b a | z_1 - z_2),
\]

where \( W^{(m,1)} \)'s are the vertical \( m \)-fold fused Boltzmann weights, and \( b_m - b = \varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_m} \) with \( 0 \leq \lambda_1 < \cdots < \lambda_m \leq n - 1 \). See Ref. [22] as for the definitions of horizontal and vertical fused Boltzmann weights. The results are also summarized in Appendix A of the present paper.

Furthermore, we denote the dual space of \( \Omega_z \) by \( \Omega_z^* \cong \wedge^{n-1}(\Omega_z) \). Let

\[
\Omega_z^{*,(b,a)} = \begin{cases} 
\bullet v^{*(b,a)} & \text{if } (b,a) \in P^2 \text{ is admissible}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\Omega_z^* = \bigoplus_{a,b} \Omega_z^{*,(b,a)}.
\]

The dual \( W \)-operators are defined as

\[
\begin{align*}
W_{\Omega_z,\Omega_z^*} v^{(d,a)} \otimes v^{*(c,d)} &= \sum_{b} v^{*(c,b)} \otimes v^{*(b,a)} W_{\Omega_z,\Omega_z^*}(c d b a | z_1 - z_2), \\
W_{\Omega_z^*,\Omega_z} v^{*(d,a)} \otimes v^{(c,d)} &= \sum_{b} v^{*(c,b)} \otimes v^{(b,a)} W_{\Omega_z^*,\Omega_z}(c d b a | z_1 - z_2), \\
W_{\Omega_z^*,\Omega_z^*} v^{*(d,a)} \otimes v^{*(c,d)} &= \sum_{b} v^{*(c,b)} \otimes v^{*(b,a)} W_{\Omega_z^*,\Omega_z^*}(c d b a | z_1 - z_2).
\end{align*}
\]

The dual Boltzmann weights are graphically represented as follows:
Here, \( b \rightarrow \ldots a \) implies that \((b, a) \in P^2\) is admissible.

### 3.3 Fusion procedure of \( K \)-operator

Now we wish to construct the \( m \)-fold fusion of \( K \)-operator mapping \( \wedge^m(\Omega_z) \) to \( \wedge^m(\Omega_{-z}) \):

\[
K^{(m)}(z)\nu(a_m, a) = \sum_{a'} q(a_m, a') K^{(m)} \left( \begin{array}{c|c} a_m & a' \\ \hline z_1 & z \\ \end{array} \right).
\]

(3.7)

Here we use the same notation \( z_j, a_j, a'_j \) (\( 1 \leq j \leq m \)) as in the previous subsection.

Let us define the \( m \)-fold fusion of \( K \) matrices in an inductive manner as follows. For \( m = 2 \) let

\[
K^{(2)}_+ \left( \begin{array}{c|c} a_2 & a \\ \hline a & z \\ \end{array} \right) = \sum_{\sigma \in S_2} \text{sgn} \sigma K_1 \left( \begin{array}{c|c} a_1 & a \\ \hline z_1 & z_1 \\ \end{array} \right) W_{12} \left( \begin{array}{c|c} a_2 & a_1 \\ \hline a_{1'} & z_1 + z_2 \\ \end{array} \right) K_2 \left( \begin{array}{c|c} a_{1'} & a \\ \hline z_2 & z_2 \\ \end{array} \right)
\]

(3.8)
and for $m > 2$ let

$$K_+^{(m)} \left( \begin{array}{c|c}
  a_m & a \\
  a & a \\
\end{array} \right) z \right) = \sum_{\sigma \in S_m} \sgn \sigma K_+^{(m-1)} \left( \begin{array}{c|c}
  a_{m-1} & a \\
  a & a \\
\end{array} \right) \left( z + \frac{1}{2} \right)$$

$$\times W^{(m-1,1)} \left( \begin{array}{c|c}
  a & a_{m-1} \\
  a & a \\
\end{array} \right) \left( z + \frac{1}{2} + z_m \right) K_+ \left( \begin{array}{c|c}
  a & a \\
  a & a \\
\end{array} \right) \left( z_m \right)$$

$$= \sum_{\sigma \in S_m} \sgn \sigma K \left( \begin{array}{c|c}
  a_1 & a \\
  a & a \\
\end{array} \right) W^{(1,m-1)} \left( \begin{array}{c|c}
  a & a_1 \\
  a & a \\
\end{array} \right) \left( z - \frac{1}{2} + z_1 \right)$$

$$\times K_+^{(m-1)} \left( \begin{array}{c|c}
  a_{m-1} & a \\
  a & a \\
\end{array} \right) \left( z - \frac{1}{2} \right)$$

(3.9)

Here $W^{(1,m-1)}$ and $W^{(m-1,1)}$ are horizontal and vertical fused Boltzmann weights, respectively.

Successive application of the Yang-Baxter equation (2.9) and the reflection equation (2.13) indicate the reflection equations involving $K_+^{(m)}(z)$:

$$K_2(z_2) W_{21}^{\Omega_{z_2}, \wedge^m(\Omega_{-z_1}) \Omega_{z_1}} K_+^{(m)}(z_1) W_{12}^{\wedge^m(\Omega_{z_1}), \Omega_{z_2}}$$

$$= W_{21}^{\Omega_{-z_2}, \wedge^m(\Omega_{-z_1}) \Omega_{z_1}} K_+^{(m)}(z_1) W_{12}^{\wedge^m(\Omega_{z_1}), \Omega_{z_2}} K_+^{(m)}(z_2)$$

$$K_+^{(m)}(z_2) W_{21}^{\wedge^m(\Omega_{z_2}), \wedge^m(\Omega_{-z_1}) \Omega_{z_1}} K_+^{(m)}(z_1) W_{12}^{\wedge^m(\Omega_{z_1}), \wedge^m(\Omega_{-z_2})}$$

(3.10)

The both sides of the first one of eqs. (3.10) map $\wedge^m(\Omega_{z_1}) \otimes \Omega_{z_2}$ to $\wedge^m(\Omega_{-z_1}) \otimes \Omega_{-z_2}$, while those of the second one map $\wedge^m(\Omega_{z_1}) \otimes \wedge^m(\Omega_{z_2})$ to $\wedge^m(\Omega_{-z_1}) \otimes \wedge^m(\Omega_{-z_2})$.

Another fused $K$-matrix is defined in an inductive manner as follows. For $m = 2$ let

$$K_+^{(2)} \left( \begin{array}{c|c}
  a_2 & a \\
  a & a \\
\end{array} \right) z \right) = \sum_{\sigma \in S_2} \sgn \sigma K_2 \left( \begin{array}{c|c}
  a_1 & a \\
  a & a \\
\end{array} \right) W_{21} \left( \begin{array}{c|c}
  a_2 & a_1 \\
  a_1 & a \\
\end{array} \right) \left( z_2 \right) W_{21}^{\wedge^m(\Omega_{z_1}), \wedge^m(\Omega_{-z_2})} K_+^{(m)} \left( \begin{array}{c|c}
  a_1 & a \\
  a & a \\
\end{array} \right) \left( z_1 \right),$$

(3.11)
and for $m > 2$ let

$$K_{-}^{(m)} \left( \begin{array}{c} a_m \ a \\ a \ a \end{array} \right| z ) = \sum_{\sigma \in \mathcal{E}_m} \text{sgn} \sigma \ K_{-}^{(m-1)} \left( \begin{array}{c} a_{m-1} \ a \\ a \ a \end{array} \right| z - \frac{1}{2} \right)$$

$$\times \ W^{(m-1,1)} \left( \begin{array}{c} a \ a_{m-1} \\ a_1 \ a \end{array} \right| z - \frac{1}{2} + z_1 \right) K \left( \begin{array}{c} \sigma a_1 \ a \\ a_{m-1} \ a \end{array} \right| z_1 \right)$$

$$= \sum_{\sigma \in \mathcal{E}_m} \text{sgn} \ K \left( \begin{array}{c} a_1 \ a \\ a \ a \end{array} \right| z_m \right) W^{(1,m-1)} \left( \begin{array}{c} a \ a_1 \\ \sigma a_{m-1} \ a \end{array} \right| z + \frac{1}{2} + z_m \right)$$

$$\times \ K_{+}^{(m-1)} \left( \begin{array}{c} \sigma a_{m-1} \ a \\ a \ a \end{array} \right| z + \frac{1}{2} \right) .$$

(3.12)

Two kinds of $m$-fold fused $K$-matrices are related as follows:

$$K_{+}^{(m)}(z) \pi_{+}^{(m)} = \pi_{-}^{(m)} K_{-}^{(m)}(z) ,$$

(3.13)

Note that (3.13) with $m = 2$ directly follows from the reflection equation (2.13), and also that (3.13) with $m > 2$ follows from the Yang-Baxter equation (2.9) and the reflection equation (2.13). The relation (3.13) implies that $\text{Im} (K_{+}^{(m)}(z)|\wedge^m(\Omega_z)) = \wedge^m(\Omega_{-z})$, and hence $K_{+}^{(m)}(z)$ can be regarded as a bona-fide $K$-matrix mapping $\wedge^m(\Omega_z)$ to $\wedge^m(\Omega_{-z})$ satisfying (3.10).

### 3.4 Boundary crossing symmetry

By taking account of the recursion relation (3.9), the explicit expression of $K_{+}^{(m)}(z)$ can be obtained as follows:

$$K_{+}^{(m)} \left( \begin{array}{c} a + \sum_{j=1}^{m} \xi_{\mu_j} \ a \\ a \ a \end{array} \right| z)$$

$$= (-1)^m C_2 \prod_{j<k} r_1(z_j + z_k) \prod_{j=1}^{m-1} \left[ \frac{2z - j}{2z + j} \right] \prod_{j=1}^{m} f_a^{(i)}(z_j) \frac{[\tilde{a}_i + \eta - z_j]}{[\tilde{a}_i + \eta + z_j]} \frac{[\tilde{a}_{\mu_j} + \eta + z_1]}{[\tilde{a}_{\mu_j} + \eta - z_n]} .$$

(3.14)

Note that the sign factor $(-1)^m C_2$ results from the permutation $(1, \cdots, m) \rightarrow (m, \cdots, 1)$. See the figure below eq. (3.9).

Since $\wedge^n(\Omega_z^{(a^o)}) \cong \mathbb{C}$, the $n$-fold fused $K$-matrix should be a scalar. By putting the scalar to be equal to unity we obtain the normalization factor $f_a^{(i)}(z)$ of $K$-matrix in (2.21) as follows:

$$f_a^{(i)}(z) = \zeta^{-\frac{2\pi a_i z_a - \pi a_i z_a}{2\pi}} \frac{g(\zeta) p_a^{(i)}(\zeta) p_a^{(i)}(x^2 \zeta^{-1})}{g(\zeta^{-1}) p_a^{(i)}(\zeta^{-1}) p_a^{(i)}(x^2 \zeta)} .$$

(3.15)
where \( \zeta = x^{2z} \), and

\[
g(\zeta) = \frac{(x^{2n+2}\zeta^2; x^{4n}, x^{2r})_\infty(x^{2(r+n-1)}\zeta^2; x^{4n}, x^{2r})_\infty}{(x^{2r}\zeta^2; x^{4n}, x^{2r})_\infty(x^{4n}\zeta^2; x^{4n}, x^{2r})_\infty},
\]

\[
p^{(i)}_a(\zeta) = \prod_{j=0}^{n-1} \frac{(x^{2(\bar{a}_i+\eta+j)\zeta}; x^{2n}, x^{2r})_\infty}{(x^{2(r-j-1-\bar{a}_i,-\eta)\zeta}; x^{2n}, x^{2r})_\infty} \frac{(x^{2(\bar{a}_j+\eta+n-1)\zeta}; x^{2n}, x^{2r})_\infty}{(x^{2(r-\bar{a}_j-n)\zeta}; x^{2n}, x^{2r})_\infty}.
\]

Let us define the dual \( K \)-operator and its matrix element as follows:

\[
K^*(z)v^{(b,a)} = \sum_{a'} v^{(b,a')} K^* \left( \begin{array}{c} a \\ a' \end{array} \right) | z \rangle.
\]  

(3.16)

When \( m = n - 1 \) we identify the dual \( K \)-matrix with the fused \( K \)-matrix as follows:

\[
K^* \left( \begin{array}{c} a - \bar{\varepsilon}_\mu \\ a' \end{array} \right) | z \rangle = K^{(n-1)}_+ \left( \begin{array}{c} a + \sum_{\nu=1}^{n} \bar{\varepsilon}_\nu \\ a' \end{array} \right) | z \rangle.
\]  

(3.17)

For later convenience we further introduce the \( \bar{K} \)-operator

\[
\bar{K}(z)v^{(b,a)} = \sum_{b'} v^{(b',a)} \bar{K}^* \left( \begin{array}{c} b \\ b' \end{array} \right) | z - \frac{n}{2} \rangle.
\]  

(3.18)

The boundary crossing symmetry can be obtained by substituting (3.17) into (3.4) with \( m = n \)

\[
K^* \left( \begin{array}{c} b \\ a' \end{array} \right) | z \rangle = \delta_{aa'} \sum_d \frac{G_d}{G_a} W \left( \begin{array}{c} d \\ a \end{array} \right) | 2z \rangle K \left( \begin{array}{c} d \\ a \end{array} \right) | -\frac{n}{2} - z \rangle.
\]  

(3.19)

The boundary crossing relations were found in [7, 25] for vertex-type models, and in [10, 16] for face-type models. As for \( A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)} \) and \( A_n^{(2)} \)-face model cases, see [10]. This is the first to derive the boundary crossing relation for the boundary \( A_{n-1}^{(1)} \)-face model with \( n > 2 \). The vertex-type version of (3.19) is given by (4.12) in [11].

## 4 Correlation functions and difference equations

### 4.1 Vertex operators and commutation relations

For \( b = a - \omega_i \) (\( 1 \leq i \leq n \)), let \( \mathcal{H}_{l,k} \) be the space of admissible paths \((\cdots, a_2, a_1, a_0)\) such that

\[
a_0 = a, \quad a_j - a_{j-1} \in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_{n-1}\}, \quad \text{for} \quad j = 1, 2, 3, \cdots, \quad a_j = b + \omega_{j-1}, \quad \text{for} \quad j \gg 1,
\]  

(4.1)

where \( k = a + \rho, l = b + \rho \).

Following [2, 23] we can identify the type I vertex operators with the half transfer matrices. Here we need four types of vertex operators:
In what follows we often suppress the letter $b$ specifying the boundary condition.

It follows from the Yang-Baxter equation \((2.9)\) and the boundary condition that these vertex operators satisfy the following commutation relations:

\[
\phi^{(c,d)}(z_1)\phi^{(d,a)}(z_2) = \sum_d W \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \phi^{(c,d)}(z_1)\phi^{(d,a)}(z_2),
\]

\[
\phi^{+(c,d)}(z_2)\phi^{(b,a)}(z_1) = \sum_d W^{\Omega_1^{*},\Omega_2} \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \phi^{(c,d)}(z_1)\phi^{+(d,a)}(z_2),
\]

\[
\phi^{(c,b)}(z_2)\phi^{+(b,a)}(z_1) = \sum_d W^{\Omega_1^{*},\Omega_2} \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \phi^{(c,d)}(z_1)\phi^{+(d,a)}(z_2),
\]

\[
\phi^{+(c,b)}(z_2)\phi^{+(b,a)}(z_1) = \sum_d W^{\Omega_1^{*},\Omega_2} \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) \phi^{+(c,d)}(z_1)\phi^{+(d,a)}(z_2).
\]

Here $W^{\Omega_1,\Omega_2}$, $W^{\Omega_1^{*},\Omega_2}$, $W^{\Omega_1,\Omega_2^{*}}$ and $W^{\Omega_1^{*},\Omega_2^{*}}$ denote the dual $W$-operators. See Appendix A concerning their matrix elements.

Furthermore, the unitarity relations with respect to the $W$-operators imply the inversion relation of the vertex operators:

\[
\sum_{\mu=0}^{n-1} \phi^{(a,a+\varepsilon_{\mu})}(-z)\phi^{(a+\varepsilon_{\mu},a)}(z) = 1.
\]

Thanks to the crossing symmetry with respect to $W$-operators (see Appendix A) we also have the duality
identities:
\[ \phi^{(a-\bar{\varepsilon}_{\mu},a)}(z) = G^{-1}_a \phi^{(a-\varepsilon_{\bar{\mu}},a)}(-z - \frac{\bar{n}}{T}), \quad \phi^{(a+\bar{\varepsilon}_{\mu},a)}(\bar{z}) = G_a \phi^{(a+\varepsilon_{\bar{\mu}},a)}(-z - \frac{n}{T}). \] (4.4)

Using the vertex operators introduced in the previous section, the transfer matrix for the semi-infinite lattice is expressed as follows:
\[ T^{(i)}_B(z) = \sum_{\mu=0}^{n-1} \phi^{(a,\bar{\varepsilon}_{\mu},\bar{\varepsilon}_{\mu})}(z) K^{(i)}(a + \bar{\varepsilon}_{\mu} a \mid z) \phi^{(a+\bar{\varepsilon}_{\mu},a)}(z). \] (4.5)

From (4.3) and (4.4) we also have another expression
\[ T^{(i)}_B(z) = \sum_{\mu=0}^{n-1} G_{a+\varepsilon_{\mu}} \phi^{(a,a+\varepsilon_{\mu})}(z) K^{(i)}(a + \varepsilon_{\mu} a \mid z) \phi^{(a+\varepsilon_{\mu},a)}(z). \] (4.6)

4.2 Derivation of difference equations

In sections 2 and 3 we fix the normalization of $W$ and $K$ such that the maximal eigenvalues of the boundary transfer matrix $T^{(i)}_B(z)$ is equal to unity in the thermodynamic limit. Thus the boundary vacuum state $|k - \omega_i, k\rangle_B$ in $H_{k-\omega_i,k}$ and its dual $B(k - \omega_i, k)$ in $H^*_{k-\omega_i,k}$ should satisfy
\[ K^{(i)}(a + \varepsilon_{\mu} a \mid z) \phi^{(a+\varepsilon_{\mu},a)}(z)|k - \omega_i, k\rangle_B = \phi^{(a+\varepsilon_{\mu},a)}(-z)|k - \omega_i, k\rangle_B, \] (4.7)

Define the $(N + 1)$-point correlation function as
\[ F^{(i)}(z_1, z_2, \ldots, z_N|a_0,a_1,a_2,\ldots,a_{N-1},a_N) = B(k - \omega_i, k)|\phi^{(a,a_1)}(z_1)\phi^{(a_1,a_2)}(z_2)\ldots\phi^{(a_{N-1},a_N)}(z_N)|k - \omega_i, k\rangle_B, \] (4.8)

where we assume that $N \equiv 0 \mod n$ for simplicity. It follows from (4.2), (4.7) and (4.4) that correlation functions should satisfy
1. R-matrix symmetry:
\[ F^{(i)}(\cdots, z_{j+1}, z_j, \cdots, a_j, a_{j+1}, \cdots) = \sum_{a_j} W \left( \begin{array}{c} a_{j+1} \ a_j \\ a_j \ a_{j-1} \end{array} \right) z_1 - z_2 F^{(i)}(\cdots, z_j, z_{j+1}, \cdots, a_{j-1}, a_j, a_{j+1}, \cdots). \] (4.9)
2. Reflection properties:

\[ F^{(i)}(\cdots, -z_N)_{\cdots,a_{N-1},a_N} = K^{(i)} \left( \begin{array}{c} a_{N-1} \\ a_N \end{array} \right) F^{(i)}(\cdots, z_N)_{\cdots,a_{N-1},a_N}, \quad (4.10) \]

\[ F^{(i)}(-z_1 - \frac{n}{2}, \cdots)_{a_0,a_1,\cdots} = \hat{K}^{(i)} \left( \begin{array}{c} a_1 \\ a_0 \end{array} \right) F^{(i)}(z_1, \cdots)_{a_0,a_1,\cdots}. \quad (4.11) \]

These relations can be shown by the same discussion as in \[12, 11\].

Let

\[ F^{(i)}(z_1, z_2, \cdots, z_N) = \sum_{a_0,a_1,\cdots,a_N} v^{(a_0,a_1)} \otimes v^{(a_1,a_2)} \otimes \cdots \otimes v^{(a_N-1,a_N)} F^{(i)}(z_1, z_2, \cdots, z_N)_{a_0,a_1,\cdots,a_{N-1},a_N}. \]

Then we conclude from (4.9, 4.11) that the \( \Omega_{z_1} \otimes \cdots \otimes \Omega_{z_N} \)-valued correlation function \( F^{(i)}(z_1, \cdots, z_N) \) should satisfy the following difference equations

\[ T_j F^{(i)}_N(z, z'|z_1, \cdots, z_N) = W^{\Omega_{z_j-\mu} \cdot \Omega_{z_j-\nu} \cdots \Omega_{z_{j+1}-\mu} \cdot \Omega_{z_{j+1}-\nu}} K_j(-z_j) \times W^{\Omega_{z_j-\mu} \cdot \Omega_{z_j-\nu} \cdots \Omega_{z_{j+1}-\mu} \cdot \Omega_{z_{j+1}-\nu}} K_j(z_j) W^{\Omega_{z_j} \cdot \Omega_{z_{j+1}} \cdots \Omega_{z_N}} F^{(i)}_N(z, z'|z_1, \cdots, z_N), \quad (4.12) \]

where

\[ T_j f(z, z'|z_1, \cdots, z_j, \cdots, z_N) = f(z, z'|z_1, \cdots, z_j - n, \cdots, z_N). \]

4.3 Simple difference equations

In this subsection we consider the correlation functions of the form

\[ P_i^{(a+\bar{\xi}, a)}(z_1, z_2) = B(k - \omega_1, k)_{a,a + \bar{\xi}}(z_1)_{a + \bar{\xi}}(z_2)_{a + \bar{\xi}} B(k - \omega_1, k)_{a,a + \bar{\xi}} \quad (4.13) \]

You can show by the similar way as in (4.12) that this correlation function satisfy the following difference equations:

\[
T_i P_i^{(a+\bar{\xi}, a)}(z_1, z_2) = \sum_{\mu \nu} G_{\alpha+\bar{\xi}} G_{\alpha-\bar{\xi}} \frac{K^{(i)}}{G^2} \left( \begin{array}{c} a + \bar{\xi}_\lambda \\ a \end{array} \right)_{z_1 - n} \times W \left( \begin{array}{c} a + \bar{\xi}_\lambda \\ a \end{array} \right)_{z_1 - z_2} K^{* (i)} \left( \begin{array}{c} a - \bar{\xi}_\mu \\ a \end{array} \right)_{z_1 - \frac{n}{2}} \times W \left( \begin{array}{c} a + \bar{\xi}_\nu \\ a \end{array} \right)_{z_2 - z_1} P_i^{(a+\bar{\xi}, a)}(z_1, z_2),
\]
\[ T_2 P_i^{(a+\vec{\epsilon},a)}(z_1, z_2) = \sum_{\mu \nu} \frac{G_{a+\vec{\epsilon}_\lambda} G_{a-\vec{\epsilon}_\mu}}{G_{\bar{a}}} W \left( \begin{array}{cc} a+\vec{\epsilon}_\lambda & a \\ a & a-\vec{\epsilon}_\mu \end{array} \right) z_1 - z_2 \]
\[ \times K^{(i)} \left( a-\vec{\epsilon}_\mu \right) z_2 - \frac{a}{2} W \left( \begin{array}{cc} a+\vec{\epsilon}_\nu & a \\ a & a-\vec{\epsilon}_\mu \end{array} \right) - z_1 - z_2 \right) \]
\[ \times K^{(i)} \left( a+\vec{\epsilon}_\nu \right) z_1 - n P_i^{(a+\vec{\epsilon},a)}(z_1, z_2). \] (4.15)

Set
\[ P_i^{(a)}(z_1, z_2) = \sum_{\lambda=0}^{n-1} P_i^{(a+\vec{\epsilon}_\lambda,a)}(z_1, z_2), \]
and we restrict ourselves to the limiting case that \( K \)-matrix is a certain scalar as done in [12, 13]. Then from (4.14, 4.15) the difference equations for \( P_i^{(a)}(z_1, z_2) \) are given as
\[ \frac{T_1 P_i^{(a)}(z_1, z_2)}{P_i^{(a)}(z_1, z_2)} = \left( x^{-2n} \zeta_1^{n-1} \right) \sum_{\lambda=0}^{n-1} r_1(2z_1 - n) r_1(-z_+) \sum_{\lambda=0}^{n-1} \frac{a_{\lambda+\vec{\epsilon}_\lambda}}{a_{\lambda}} \] \[ = r_1(z) \frac{[z+n]}{[z+1]}, \] (4.16)
\[ \frac{T_2 P_i^{(a)}(z_1, z_2)}{P_i^{(a)}(z_1, z_2)} = \left( x^{-2n} \zeta_2^{n-1} \right) \sum_{\mu=0}^{n-1} r_1(2z_2 - n) r_1(-z_+) \sum_{\mu=0}^{n-1} \frac{a_{\lambda-\vec{\epsilon}_\mu}}{a_{\lambda}} \] \[ = r_1(z) \frac{[z+n]}{[z+1]} \] (4.17)

The both formulae (4.17) are equivalent to
\[ \prod_{\lambda=0}^{n-1} \frac{a_{\lambda+\vec{\epsilon}_\lambda} + 1}{a_{\lambda}} \prod_{\mu=0}^{n-1} \frac{z + a_{\lambda+\vec{\epsilon}_\lambda}}{z+1} \prod_{\lambda \neq \mu} \frac{a_{\lambda+\vec{\epsilon}_\lambda}}{a_{\lambda}} = \frac{[z+n]}{[z+1]} \] (4.18)

Note that the LHS of (4.18) is not singular at \( \bar{a}_i = \bar{a}_j (0 \leq i < j \leq n-1) \), and also that the LHS vanishes at \( z = -n \). Hence the LHS is equal to the RHS up to a constant \( C \). In order to show that \( C = 1 \), it is sufficient to set \( z = 0 \) and
\[ a_{\mu \nu} = \begin{cases} n + \nu - \mu & (\nu = 0, 1, \cdots, \mu - 1), \\ \nu - \mu & (\nu = \mu + 1, \cdots, n - 1). \end{cases} \]

Thus we find from (4.16) and (4.17) that the correlation function has the form
\[ P_i^{(a)}(z_1, z_2) = C_i^{(a)} A(z_1) A(z_2) B(z_+) B(z_-), \] (4.19)

where \( C_i^{(a)} \) is a constant, and
\[ A(z) = a(\zeta^2) a(\zeta^{-2}), \quad a(\zeta) = \frac{(x^{4n+2r-2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{2n+2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty}{(x^{2n+2r} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{4n} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{2n+2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty}, \]
\[ B(z) = b(\zeta) b(\zeta^{-1}), \quad b(\zeta) = \frac{(x^{2r} \zeta; x^{2n}, x^{2r})_\infty (x^{4n} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{2n+2r} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty}{(x^{2n+2r-2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{4n+2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty (x^{2n+2} \zeta; x^{2n}, x^{4n}, x^{2r})_\infty}. \]
5 Concluding remarks

There are two ways in order to proceed further. The one is solving the difference equations (4.12) to obtain the corresponding local state probabilities, while the other is constructing the boundary vacuum states in terms of bosonized vertex operators to do the same thing. Let us discuss the latter way here.

Consider the bosons $B^j_m$ $(1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\})$ with the commutation relations \[26, 27\]

\[
[B^j_m, B^k_{m'}] = \begin{cases} 
  m \frac{[(n-1)m]_x}{[nm]_x} \frac{[(r-1)m]_x}{[rm]_x} \delta_{m+m',0}, & (j = k) \\
  -m x^{\text{sgn}(j-k)nm} \frac{[m]_x}{[nm]_x} \frac{[(r-1)m]_x}{[rm]_x} \delta_{m+m',0}, & (j \neq k),
\end{cases}
\]

where the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. Define $B^a_n$ by

\[
\sum_{j=1}^{n} x^{-2jm} B^j_m = 0.
\]

Using these oscillators $B^j_m$’s the bosonization of the type I vertex operators for the $A^{(1)}_{n-1}$-face model was given in \[22\] on the Fock space $\mathcal{F}_{l,k}$. Furthermore, we make the Ansatz \[14, 15\] such that the boundary vacuum states and their dual have the form

\[
|k - \omega_i, k\rangle_B = \exp(F^{(i)}_a)|k - \omega_i, k\rangle, \quad B\langle k - \omega_i, k| = B\langle k - \omega_i, k| \exp(G^{(i)}_a),
\]

where $|l, k\rangle$ is the highest weight of $\mathcal{F}_{l,k}$, and

\[
F^{(i)}_a = -\frac{1}{2} \sum_{m > 0} \sum_{s,t=1}^{n-1} \alpha_{m,s}^i B^s_m B^t_{-m} + \sum_{m > 0} \sum_{t=1}^{n-1} \beta^{t,(i)}_{m,a} B^t_{-m},
\]

\[
G^{(i)}_a = -\frac{1}{2} \sum_{m > 0} \sum_{s,t=1}^{n-1} \gamma_{m,s}^t B^s_m B^t_{-m} + \sum_{m > 0} \sum_{t=1}^{n-1} \delta^{t,(i)}_{m,a} B^t_{-m}.
\]

Since $\mathcal{H}_{l,k}$ and $\mathcal{F}_{l,k}$ have different characters, the bosonized expressions of correlation functions cannot be identified with the one defined in \[13, 8\]. In order to obtain the correct bosonized formulae for correlation functions, we have to construct the BRST cohomology of appropriate complex which realizes the space of physical states $\mathcal{H}_{l,k}$ as subquotients of $\mathcal{F}_{l,k}$’s. We wish to address this problem in a separate paper.

Acknowledgement

The author would like to thank K. Hasegawa and M. Okado for useful discussion.

A Appendix A. Fused $A^{(1)}_{n-1}$ Boltzmann weight

Let us introduce the fusion of Boltzmann weights $W$ \[22\]
Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be a subset of $N = \{0, 1, \ldots, n\}$ such that $\lambda_1 < \cdots < \lambda_m$. For $\kappa, \mu \in N$ set $\mu = \kappa$ if $\kappa \in \Lambda$, otherwise set $\mu \in \Lambda \cup \{\kappa\}$. For given $\kappa, \mu, \Lambda$ let $0 \leq \nu_1 < \cdots < \nu_m \leq n - 1$ be such that $\bar{\xi}_\mu + \bar{\xi}_{\nu_1} + \cdots + \bar{\xi}_{\nu_m} = \bar{\xi}_\kappa + \bar{\xi}_{\lambda_1} + \cdots + \bar{\xi}_{\lambda_m}$.

The fusion of $W$ in the horizontal direction is constructed as follows. Let $a, b, c, d = d_0, d_1, \ldots, d_{m-1}, d_m = c \in P$ satisfy

$$c = b + \bar{\xi}_\mu, \quad d_j - d_{j-1} = \bar{\xi}_{\lambda_j} \quad (1 \leq j \leq m), \quad d = a + \bar{\xi}_\kappa.$$ 

Note that $b = a + \bar{\xi}_{\nu_1} + \cdots + \bar{\xi}_{\nu_m}$ from the definition of $\nu_j$'s. Let $\sigma \in S_m$ be a permutation of $(1, \cdots, m)$, and set

$$a_0^\sigma = a, \quad a_j^\sigma = b_j^\sigma - 1 + \bar{\xi}_{\nu_j} \quad (1 \leq j \leq m), \quad a_m^\sigma = b.$$

Then $m$-fold anti-symmetric fusion of $W$ in the horizontal direction is given as

$$W^{(1,m)} \left( \begin{array}{ccc} c & d \\ b & a \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W \left( \begin{array}{ccc} d_j & d_{j-1} \\ a_j^\sigma & a_{j-1}^\sigma \end{array} \right) \left( z + \frac{m+1}{2} - j \right).$$ (A.1)

Note that $W^{(1,m)}$ is anti-symmetric with respect to $(\lambda_1, \cdots, \lambda_m)$.

Next consider the fusion in the vertical direction. We use the same $\kappa, \mu, \lambda_j$'s and $\nu_j$'s as before. Now we set

$$b = a + \bar{\xi}_\mu, \quad a_j - a_{j-1} = \bar{\xi}_{\lambda_j} \quad (1 \leq j \leq m), \quad c = d + \bar{\xi}_\kappa,$$

where $a_0 = a, a_m = d$. We have $c = b + \bar{\xi}_{\nu_1} + \cdots + \bar{\xi}_{\nu_m}$. For $\sigma \in S_m$ set

$$b_0^\sigma = b, \quad b_j^\sigma = b_j^\sigma - 1 + \bar{\xi}_{\nu_j} \quad (1 \leq j \leq m), \quad b_m^\sigma = c.$$

Then $m$-fold anti-symmetric fusion of $W$ in the vertical direction is given as

$$W^{(m,1)} \left( \begin{array}{ccc} c & d \\ b & a \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W \left( \begin{array}{ccc} b_j^\sigma & a_j \\ b_{j-1}^\sigma & a_{j-1} \end{array} \right) \left( z - \frac{m+1}{2} + j \right).$$ (A.2)

Note that $W^{(m,1)}$ is anti-symmetric with respect to $(\lambda_1, \cdots, \lambda_m)$.

We further introduce the fusion of $W$ in both horizontal and vertical directions. Let $0 \leq \kappa_1 < \cdots < \kappa_m \leq n - 1$, $0 \leq \mu_1 < \cdots < \mu_m \leq n - 1$, $0 \leq \lambda_1 < \cdots < \lambda_m' \leq n - 1$ and $0 \leq \nu_1 < \cdots < \nu_{m'} \leq n - 1$ satisfy

$$\sum_{j=1}^{m} \bar{\xi}_{\kappa_j} + \sum_{j=1}^{m'} \bar{\xi}_{\lambda_j} = \sum_{j=1}^{m} \bar{\xi}_{\mu_j} + \sum_{j=1}^{m'} \bar{\xi}_{\nu_j}.$$ 

Let $a, b, c, d \in P$ satisfy

$$d = a + \sum_{j=1}^{m} \bar{\xi}_{\kappa_j}, \quad c = d + \sum_{j=1}^{m'} \bar{\xi}_{\lambda_j}, \quad b = a + \sum_{j=1}^{m'} \bar{\xi}_{\nu_j}, \quad c = b + \sum_{j=1}^{m} \bar{\xi}_{\mu_j}.$$ 

Then the $m \times m'$-fold fusion of $W$ is defined as the antisymmetrized product of the $m'$-fold fusion of $W$ in the horizontal direction:

$$W^{(m,m')} \left( \begin{array}{ccc} c & d \\ b & a \end{array} \right) = \sum_{\sigma \in S_m} \text{sgn} \sigma \prod_{j=1}^{m} W^{(1,m')}_{I} \left( \begin{array}{ccc} b_j^\sigma & a_j \\ b_{j-1}^\sigma & a_{j-1} \end{array} \right) \left( z - \frac{m+1}{2} + j \right).$$ (A.3)
where
\[ b_0^\sigma = b, \ b_j^\sigma = b_{j-1}^\sigma + \bar{\varepsilon}_{\nu_j(j)} \ (1 \leq j \leq m), \ b_m^\sigma = c. \]

The \( W^{(m,m')} \) can be also defined as the antisymmetrized product of the \( m \)-fold fusion of \( W \) in the vertical direction:
\[
W^{(m,m')}(c \ d \ b \ a \ z) = \sum_{\sigma \in \mathfrak{S}_{m'}} \text{sgn}\sigma \prod_{j=1}^{m'} W^{(m,1)}(d_j \ a_{j-1} \ a \ a \ z + m'+1 - j), \tag{A.4}
\]
where
\[
a_0^\sigma = a, \ a_j^\sigma = a_{j-1}^\sigma + \bar{\varepsilon}_{\nu_j(j)} \ (1 \leq j \leq m'), \ a_{m'}^\sigma = b.
\]

The explicit expression of fused Boltzmann weights in both the horizontal and vertical directions are given as follows [22]:

\[
W^{(1,m)}(a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a \ a \ a \ z) = (-1)^{m-1}r_m(z) \prod_{j=1}^{m} \frac{[a_{\lambda_j \lambda_j} - 1]}{[a_{\lambda_j \lambda_j}]}, \tag{A.5a}
\]

\[
W^{(1,m)}(a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a \ a \ z) = (-1)^{m-1}r_m(z) \prod_{j=2}^{m} \frac{[a_{\lambda_j \lambda_j}]}{[a_{\lambda_j \lambda_j} + 1]}, \tag{A.5b}
\]

\[
W^{(1,m)}(a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a \ a \ z) = (-1)^{m-1}r_m(z) \prod_{j=2}^{m} \frac{[a_{\lambda_j \lambda_j}]}{[a_{\lambda_j \lambda_j} + 1]}, \tag{A.5c}
\]

\[
W^{(m,1)}(a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a \ a \ a \ z) = (-1)^{m-1}r_m(z) \prod_{j=1}^{m} \frac{[a_{\lambda_j \lambda_j} - 1]}{[a_{\lambda_j \lambda_j} + 1]}, \tag{A.5d}
\]

\[
W^{(m,1)}(a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a + \bar{\varepsilon}_L \ a \ a \ z) = (-1)^{m-1}r_m(z) \prod_{j=1}^{m} \frac{[a_{\lambda_j \lambda_j} - 1]}{[a_{\lambda_j \lambda_j} + 1]}, \tag{A.5e}
\]

Here we denote \( \bar{\varepsilon}_{\lambda_1} + \cdots + \bar{\varepsilon}_{\lambda_m} \) by \( \bar{\varepsilon}_L \) for simplicity.

When \( m = n - 1 \) we identify \( W^{\Omega_2,\Omega_2^*} \) (resp. \( W^{\Omega_2^*,\Omega} \)) with \( W^{(1,n-1)} \) (resp. \( W^{(n-1,1)} \)) as follows:
\[
W^{\Omega_2,\Omega_2^*}(c \ d \ b \ a \ z_1 - z_2) = (-1)^{n-1+\lambda+\nu} W^{(1,n-1)}(c \ d \ b \ a \ z_1 - z_2), \tag{A.6}
\]

where \( b - a = -\bar{\varepsilon}_\nu, \ c - d = -\bar{\varepsilon}_\lambda \), and \((b,c), (a,d)\) are admissible; and
\[
W^{\Omega_2^*,\Omega_2}(c \ d \ b \ a \ z_1 - z_2) = (-1)^{n-1+\mu+\kappa} W^{(n-1,1)}(c \ d \ b \ a \ z_1 - z_2), \tag{A.7}
\]
where \( d - a = -\bar{\xi}_\kappa, \ c - b = -\bar{\xi}_\mu, \) and \((a, b), (d, c)\) are admissible.

The crossing symmetries are as follows:

\[
W^{\Omega_1, \Omega_2} \begin{pmatrix} c & d \\ b & a \end{pmatrix} = \frac{G_b}{G_c} W \begin{pmatrix} d & a \\ c & b \end{pmatrix} \left[ z_2 - z_1 - \frac{n}{2} \right], \tag{A.8}
\]

and

\[
W^{\Omega_1, \Omega_2} \begin{pmatrix} c & d \\ b & a \end{pmatrix} = \frac{G_b}{G_a} W \begin{pmatrix} b & c \\ a & d \end{pmatrix} \left[ z_2 - z_1 - \frac{n}{2} \right]. \tag{A.9}
\]

### B Appendix B. Proof of the reflection equation

The aim of this Appendix is to give a simple sketch of the proof of the claim that \( (2.14) \) solves the reflection equation \( (2.13) \). Since \( K\)-matrix is diagonal in the sense that \( K(a, b, c | z) = 0 \) unless \( b = c \), it is sufficient to consider the case \( a = c = g \) in \( (2.13) \). In this case the reflection equation reduces to

\[
\sum_d K \begin{pmatrix} d & a \\ a & z_1 \end{pmatrix} K \begin{pmatrix} f & a \\ a & z_2 \end{pmatrix} W \begin{pmatrix} c & f \\ d & a \end{pmatrix} \left[ z_1 + z_2 \right] W \begin{pmatrix} c & d \\ b & a \end{pmatrix} \left[ z_1 - z_2 \right] = \sum_d K \begin{pmatrix} d & a \\ a & z_1 \end{pmatrix} K \begin{pmatrix} b & a \\ a & z_2 \end{pmatrix} W \begin{pmatrix} c & f \\ d & a \end{pmatrix} \left[ z_1 - z_2 \right] W \begin{pmatrix} c & d \\ b & a \end{pmatrix} \left[ z_1 + z_2 \right]. \tag{B.1}
\]

Note that \( (B.1) \) holds as \( 0 = 0 \) unless the quartet \((a, b, c, f)\) is admissible. Assume that \((a, b, c, f)\) is admissible. Then there are the following three cases:

(i) \( b = f = a + \bar{\xi}_\mu, \ c = a + 2\bar{\xi}_\mu; \)

(ii) \( b = f = a + \bar{\xi}_\mu, \ c = a + \bar{\xi}_\mu + \bar{\xi}_\nu (\mu \neq \nu); \)

(iii) \( b = a + \bar{\xi}_\mu, \ f = a + \bar{\xi}_\nu, \ c = a + \bar{\xi}_\mu + \bar{\xi}_\nu (\mu \neq \nu). \)

For the case (i) the equation \( (B.1) \) is trivial because the only non-zero terms of both sides result from \( d = a + \bar{\xi}_\mu. \) It is also easy to prove the case (ii). Up to now we did not use the explicit form of \( K\)-matrix \( (2.14) \) except for its diagonal property.

Let us prove the case (iii). By substituting \((2.8)\) and \((2.14)\) into \((B.1)\), the reflection equation \( (2.13) \) is equivalent to

\[
\frac{[\bar{a}_\nu + \eta + z_2]}{[\bar{a}_\mu + \eta - z_2]} \left[ [\bar{a}_\mu + \eta + z_1] [z_1 + z_2] [a_{\mu\nu} - z_1 + z_2] + \frac{[\bar{a}_\nu + \eta + z_1]}{[\bar{a}_\nu + \eta - z_1]} [z_1 - z_2] [a_{\mu\nu} + z_1 + z_2] \right]

= \frac{[\bar{a}_\mu + \eta + z_2]}{[\bar{a}_\mu + \eta - z_2]} \left[ [\bar{a}_\mu + \eta + z_1] [z_1 + z_2] [a_{\mu\nu} + z_1 + z_2] + \frac{[\bar{a}_\nu + \eta + z_1]}{[\bar{a}_\nu + \eta - z_1]} [z_1 - z_2] [a_{\mu\nu} - z_1 + z_2] \right].
\]

Let \( f \) stand for the difference of both sides as the function of \( z_1. \) It is easy to show that the poles at \( z_1 = a_\mu + \eta \) and \( z_2 = a_\nu + \eta \) are spurious. Thus the function \( f \) is an entire function of \( z_1 \) with the quasi double periodicities. Suppose that \( f \) is not identically zero. Then the transformation properties of \( f \) contradict the positions of zeros at \( z_1 = \pm z_2. \) We therefore obtain \( f = 0 \) and conclude that \( (2.14) \) solves the reflection equation \( (2.13). \)
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