Topological singular set of vector-valued maps, I: Applications to manifold-constrained Sobolev and BV spaces

Giacomo Canevari* and Giandomenico Orlandi†

August 10, 2018

Abstract

We introduce an operator $S$ on vector-valued maps $u$ which has the ability to capture the relevant topological information carried by $u$. In particular, this operator is defined on maps that take values in a closed submanifold $N$ of the Euclidean space $\mathbb{R}^m$, and coincides with the distributional Jacobian in case $N$ is a sphere. The range of $S$ are flat chains with coefficients in a suitable normed abelian group. In this paper, we use $S$ to characterise strong limits of smooth, $N$-valued maps with respect to Sobolev norms, extending a result by Pakzad and Rivière. We also discuss applications to the study of manifold-valued maps of bounded variation. In a companion paper, we will consider applications to the asymptotic behaviour of minimisers of Ginzburg-Landau type functionals.

1 Introduction

Let $N$ be a smooth, closed Riemannian manifold, isometrically embedded in a Euclidean space $\mathbb{R}^m$, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded, smooth domain of dimension $d \geq 2$. Functional spaces of maps $u: \Omega \rightarrow N$ (e.g., Sobolev or BV) have been extensively studied in the literature, in connection with manifold-constrained variational problems, in order to detect the topological information encoded by $u$.

In this paper, instead of dealing directly with $N$-valued maps, we consider vector-valued maps $u: \Omega \rightarrow \mathbb{R}^m$, which we think of as approximations of a map $v: \Omega \rightarrow N$. This point of view also arises quite naturally from variational problems, such as the penalised harmonic map problem, the Ginzburg-Landau model for superconductivity or other models from material science that share a common structure, e.g. the Landau-de Gennes model for nematic liquid crystals. Moreover, working with vector-valued, instead of manifold-valued, maps allows for more flexibility. On the other hand, if $u: \Omega \rightarrow \mathbb{R}^m$ does not take values uniformly close to $N$ but only close in, say, an integral sense (e.g. $\int_{\Omega} \text{dist}(u, N)$ is small) then it might not be obvious.

*BCAM — Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Spain.
E-mail address: gcanevari@bcamath.org

†Dipartimento di Informatica — Università di Verona, Strada le Grazie 15, 37134 Verona, Italy.
E-mail address: giandomenico.orlandi@univr.it
to extract the topological information carried by \( u \). For instance, in the Ginzburg-Landau theory, this task is accomplished by means of the distributional Jacobian. However, this tool is only available when the distinguished manifold \( \mathcal{M} \) has a special structure — typically, when \( \mathcal{M} \) is a sphere — and cannot be applied to some cases that are relevant to applications, for instance, when \( \mathcal{M} \) is a real projective plane \( \mathbb{R}P^2 \), as is the case in many models for liquid crystals.

The goal of this paper is to define an operator, \( S \), such that \( S(u) \) corresponds to the set of topological singularities of \( u \) and plays the rôle of a “generalised Jacobian”, which can be applied to more general target manifolds \( \mathcal{M} \). The properties of \( S \) are stated in our main result, Theorem 3.1 in Section 3.1 below. As the distributional Jacobian, this operator captures topological information and enjoys compactness properties, and in fact it reduces to the distributional Jacobian in the special case \( \mathcal{M} \simeq S^n \). The construction of \( S \) is carried out in the setting of flat chains with coefficients in a normed abelian group. This approach has been proposed by Pakzad and Rivière [46], in the context of manifold-valued maps, in order to characterise strong limits of smooth \( \mathcal{M} \)-valued maps in \( W^{1,p}(B^d, \mathcal{M}) \). Because we are interested in vector-valued maps, our construction is different from theirs, and relies on the “projection trick” devised by Hardt, Kinderlehrer and Lin [33]. Eventually, we generalise Pakzad and Rivière’s main result to a broader range of values for the exponent \( p \), see Theorem 1 in Section 1.3.

In this paper, we discuss some applications of the operator \( S \) to the study of manifold-valued functional spaces. In addition to the aforementioned generalisation of the result by Pakzad and Rivière (Theorem 1), we study manifold-valued spaces of functions of bounded variation. We show weak density of smooth maps in \( BV(\Omega, \mathcal{M}) \), see Theorem 2 in Section 1.3, thus generalising a result by Giaquinta and Mucci [30]. We also discuss the lifting problem in \( BV \) (see, for instance, [22]) for a larger class of manifolds \( \mathcal{M} \), see Theorem 3 in Section 1.3. Further applications to variational problems, including the Landau-de Gennes model for liquid crystals, will be investigated in forthcoming work [20]. As is the case for the distributional Jacobian in the Ginzburg-Landau theory, we expect that \( S \) might be used to identify the set where the energy concentrates and characterise the limiting energy densities.

The plan of the paper is the following. After recalling some background in Section 1.1, we sketch our construction in Section 1.2, and we present the statements of Theorems 1, 2, 3 in Section 1.3. In Section 2, we review some preliminary material about flat chains (Section 2.1), topology (Sections 2.2–2.3), and manifold-valued Sobolev spaces (Section 2.4). The main technical result of this paper, Theorem 3.1, which gives the existence of the operator \( S \), is stated in Section 3.1. The rest of Section 3 is devoted to the proof of Theorem 3.1 and of Theorem 1, which we recover as a corollary of Theorem 3.1. Finally, Section 4 contains the applications to manifold-valued BV spaces, with the proofs of Theorem 2 and 3.

1.1 Background and motivation

For the sake of motivation, consider the Ginzburg-Landau functional:

\[
(1) \quad u \in W^{1,2}(\Omega, \mathbb{R}^2) \mapsto E_{GL}^\varepsilon(u) := \int_\Omega \left\{ \frac{1}{2} \left| \nabla u \right|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right\},
\]

where \( \varepsilon > 0 \) is a small parameter. Functionals of this form arise as variational models for the study of type-II superconductivity. In this context, \( u(x) \) represents the magnetisation vector at
a point \( x \in \Omega \) and the energy favours configurations with \( |u(x)| = 1 \), which have a well-defined direction of magnetisation as opposed to the non-superconducting phase \( u = 0 \). Let \( S^1 \) denote the unit circle in the plane \( \mathbb{R}^2 \). As is well known, minimisers \( u_\varepsilon \) subject to a (\( \varepsilon \)-independent) boundary condition \( u_{\varepsilon}\big|_{\partial \Omega} = u_{bd} \in W^{1,2}(\partial \Omega, S^1) \) satisfy the sharp energy bound \( E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon| \) for some \( \varepsilon \)-independent constant \( C \) (see e.g. [14, Proposition 2.1]). In particular, \( u_\varepsilon \) takes values “close” to \( S^1 \) when \( \varepsilon \) is small, in the sense that \( \int_\Omega (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2 |\log \varepsilon| \). Despite the lack of uniform energy bounds, under suitable conditions on \( u_{bd} \), minimisers \( u_\varepsilon \) converge to a limit map \( u_0 : \Omega \to S^1 \), which is smooth except for a singular set of codimension two (see e.g. [9, 33, 10, 40, 2, 49, 13]). Moreover, the singular set of \( u_0 \) is itself a minimiser — in a suitable sense — of some “weighted area” functional [2]. The emergence of singularities in the limit map \( u_0 \) is related to topological obstructions, which may prevent the existence of a map in \( W^{1,2}(\Omega, S^1) \) that satisfies the boundary conditions.

There are other functionals, arising as variational models for material science, which share a common structure with (2), i.e. they can be written in the form

\[
(2) \quad u \in W^{1,k}(\Omega, \mathbb{R}^m) \mapsto E_\varepsilon(u) := \int_\Omega \left\{ \frac{1}{k} |\nabla u|^k + \frac{1}{\varepsilon^2} f(u) \right\}.
\]

Here \( f : \mathbb{R}^m \to \mathbb{R} \) is a non-negative, smooth potential that satisfies suitable coercivity and non-degeneracy conditions, and \( \mathcal{N} := f^{-1}(0) \) is assumed to be a non-empty, smoothly embedded, compact, connected submanifold of \( \mathbb{R}^m \) without boundary. The elements of \( \mathcal{N} \) correspond to the ground states for the material, i.e. the local configurations that are most energetically convenient. An important example is the Landau-de Gennes model for nematic liquid crystals (in the so-called one-constant approximation, see e.g. [23]). In this case, \( k = 2 \) and the distinguished manifold is a real projective plane \( \mathcal{N} = \mathbb{RP}^2 \), whose elements describe the locally preferred direction of alignment of the constituent molecules (which might be schematically described as un-oriented rods).

As in the Ginzburg-Landau case, topological obstructions may imply the lack of an extension operator \( W^{1-1/k,k}(\partial \Omega, \mathcal{N}) \to W^{1,k}(\Omega, \mathcal{N}) \). As a consequence, minimisers \( u_\varepsilon \) subject to a Dirichlet boundary condition \( u_\varepsilon = u_{bd} \in W^{1-1/k,k}(\partial \Omega, \mathcal{N}) \) may not satisfy uniform energy bounds with respect to \( \varepsilon \). Compactness results in the spirit of the Ginzburg-Landau theory have been shown for minimisers of the Landau-de Gennes functional [13, 18, 31, 19]. However, some points that are understood in the Ginzburg-Landau theory — for instance, a variational characterisation of the singular set of the limit or a description of the problem in terms of \( \Gamma \)-convergence, as in [40, 2, 3] — are still missing, even for the Landau-de Gennes functional.

A key tool in the analysis of the Ginzburg-Landau functional is the distributional Jacobian. In case \( d = m = 2 \), the distributional Jacobian \( Ju \) of a map \( u \in (L^\infty \cap W^{1,1})(\mathbb{R}^2, \mathbb{R}^2) \) is defined as the distributional curl of the field \( \tfrac{1}{2}(u^1 \partial_1 u^2 - u^2 \partial_1 u^1, u^1 \partial_2 u^2 - u^2 \partial_2 u^1) \). Equivalently, in the language of differential forms, \( Ju := \ast du^* \omega_{S^1} \), where \( \ast \) denotes the Hodge duality operator and \( \omega_{S^1}(y) := \tfrac{1}{2}(y^1 dy^2 - y^2 dy^1) \) is the 1-homogeneous extension of the renormalised volume form on \( S^1 \). The purpose of the distributional Jacobian is two-fold: on one hand, it captures topological information associated with \( u \), as is demonstrated by several formulas relating the Jacobian with the topological degree (see e.g. [15, Theorem 0.8]); on the other hand, it enjoys compactness properties — for instance, despite being a quadratic operator, it is stable under...
weak $W^{1,2}$-convergence. Unfortunately, an adequate notion of Jacobian may be missing for general manifolds $\mathcal{N}$. Consider the following simple example: let $S$ be a $(d-k)$-plane in $\mathbb{R}^d$, and let $u: \Omega \setminus S \to \mathcal{N}$ be a material configuration that is smooth everywhere, except at $S$. Then $S$ can be encircled by a $(k-1)$-dimensional sphere $\Sigma \subseteq \Omega \setminus S$, and the (based) homotopy class of $u|_{\Sigma} : \Sigma \to \mathcal{N}$ defines an element of $\pi_{k-1}(\mathcal{N})$ which, roughly speaking, characterises the behaviour of the material around the defect. (This is the basic idea of the topological classification of defects in ordered materials; see e.g. [44] for more details.) If $\pi_{k-1}(\mathcal{N})$ contains elements of finite order, these cannot be realised via integration of a differential form, so no notion of Jacobian that can be expressed as a differential form is able to capture such homotopy classes of defects. An example is provided by the Landau-de Gennes model for nematic liquid crystals, where $k = 2$, $\mathcal{N} \simeq \mathbb{R}P^2$ and $\pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2\mathbb{Z}$.

The aim of this paper is to construct an object that (i) brings topological information and (ii) enjoys compactness properties even when the distributional Jacobian is not defined, in particular when $\pi_{k-1}(\mathcal{N})$ contains elements of finite order. A notion of “set of topological singularities” for a manifold-valued Sobolev map was already introduced by Pakzad and Rivière [46], using the language of flat chains. Roughly speaking, a flat chain of dimension $n$ with coefficients in an abelian group $G$ is described by a collection of $n$-dimensional sets, carrying multiplicities that are elements of $G$ (see [27, 28]). The group of flat $n$-chains with coefficients in $G$ can be given a norm, called the flat norm, which satisfies useful compactness properties. Given integers numbers $2 \leq k \leq d$ and $u \in W^{1,k-1}(B^d, \mathcal{N})$, the topological singular set of $u$ ‘à la Pakzad-Rivière’ is a flat chain $S_{PR}(u)$ of dimension $(d-k)$ with coefficients in $\pi_{k-1}(\mathcal{N})$, and has the following property: $u$ can be $W^{1,k-1}$-strongly approximated by smooth maps $\Omega \to \mathcal{N}$ if and only if $S_{PR}(u) = 0$ [46, Theorem II]. The construction we carry out here is different (and relies on ideas from [33]), as we want to deal with vector-valued maps $u: \Omega \to \mathbb{R}^m$ instead of manifold-valued ones. However, following Pakzad and Rivière, we work in the formalism of flat chains. We discuss the link between Pakzad and Rivière’s construction and the one presented here in Section 5.4

### 1.2 Sketch of the construction

Throughout the paper, $d$, $m$, $k$ will be integer numbers with $\min\{d, m\} \geq k \geq 2$, $\Omega$ will be a smooth, bounded domain in $\mathbb{R}^d$, and $\mathcal{N}$ will denote a smooth submanifold of $\mathbb{R}^m$ without boundary. We make the following assumption on $\mathcal{N}$ and $k$:

(H) $\mathcal{N}$ is compact and $(k-2)$-connected, that is $\pi_0(\mathcal{N}) = \pi_1(\mathcal{N}) = \ldots = \pi_{k-2}(\mathcal{N}) = 0$. In case $k = 2$, we also assume that $\pi_1(\mathcal{N})$ is abelian.

The integer $k$ is thus related to the topology of $\mathcal{N}$, and represents the codimension of the (highest-dimensional) topological singularities for $\mathcal{N}$-valued maps. Under the assumption (H), the group $\pi_{k-1}(\mathcal{N})$ is abelian, and will be the coefficient group for our flat chains. As noted above, $\pi_{k-1}(\mathcal{N})$ classifies the topological defects of $\mathcal{N}$-valued maps. We will endow $\pi_{k-1}(\mathcal{N})$ with a norm, see Section 2.2.

The construction we carry out has been introduced by Hardt, Kinderlehrer and Lin [33] as a method to produce manifold-valued comparison maps with suitable properties; we sketch the
main idea. It is impossible to construct a smooth projection of \( \mathbb{R}^n \) onto a closed manifold \( \mathcal{N} \).

However, as noted by Hardt and Lin [34, Lemma 6.1], under the assumption (11) it is possible to construct a smooth projection \( \varrho: \mathbb{R}^m \setminus \mathcal{X} \to \mathcal{N} \), where \( \mathcal{X} \) is a union of \((m-k)\)-manifolds. Given a smooth map \( u: \mathbb{R}^d \to \mathbb{R}^m \), one could identify the set of topological singularities of \( u \) with \( u^{-1}(\mathcal{X}) \), which is exactly the set where the reprojectioon \( \varrho(u) \) fails to be well-defined, but \( u^{-1}(\mathcal{X}) \) may be very irregular even if \( u \) is smooth. However, Thom transversality theorem implies that, for a.e. \( y \in \mathbb{R}^m \), the set \((u - y)^{-1}(\mathcal{X})\) is indeed a union of \((d-k)\)-dimensional manifolds. This set can be equipped, in a natural way, with multiplicities in \( \pi_{k-1}(\mathcal{N}) \), so to define a flat chain \( S_y(u) \) of dimension \( d-k \). Thus, we define the set of topological singularities of \( u \) as a map \( y \in \mathbb{R}^m \to S_y(u) \) with values in the group of flat chains.

By integrating over \( y \in \mathbb{R}^m \) according to the strategy devised in [33], and applying the coarea formula, one obtains estimates on \( S_y(u) \) depending on the Sobolev norms of \( u \). Then, by density, one can define \( S_y(u) \) in case \( u \) is a Sobolev map, thus obtaining an operator

\[
S: (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m) \to L^1(\mathbb{R}^m; \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})))
\]

Here \( \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})) \) denotes the normed \( \pi_{k-1}(\mathcal{N})\)-module of \((d-k)\)-dimensional flat chains in \( \Omega \) with coefficients in \( \pi_{k-1}(\mathcal{N}) \) (see Section 2.1), and \( L^1(\mathbb{R}^m; \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N}))) =: Y \) is the set of Lebesgue-measurable maps \( S: \mathbb{R}^m \to \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})) \) such that

\[
\|S\|_Y := \int_{\mathbb{R}^m} \mathbb{F}_{\Omega}(S_y) \, dy < +\infty
\]

(\( \mathbb{F}_{\Omega} \) being the natural norm on \( \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})) \), see Section 2.1). In general, \( Y \) is not a vector space but it is a \( \pi_{k-1}(\mathcal{N})\)-module, and the left hand side of (3) defines a norm on \( Y \). The operator \( S \) is continuous in the following sense: if \((u_j)_{j \in \mathbb{N}}\) is a sequence of maps such that \( u_j \to u \) strongly in \( W^{1,k-1} \) and \( \sup_j \|u_j\|_{L^\infty} < +\infty \), then \( \|S(u_j) - S(u)\|_Y \to 0 \). The same remains true if the sequence \((u_j)_{j \in \mathbb{N}}\) is assumed to converge only weakly in \( W^{1,k} \) and to be uniformly bounded in \( L^\infty \); therefore, some of the compensation compactness properties that are typical of the Jacobian are retained by \( S \). Moreover, \( S \) carries topological information on the map \( u \). Indeed, the intersection (in a suitable sense: see Section 2.1) between \( S_y(u) \) and, say, a \( k\)-disk \( R \) completely determines the homotopy class of \( \varrho(u - y) \) on \( \partial R \). A precise statement of these properties, which requires some notation, is given in Theorem 5.1.

In the special case \( \mathcal{N} = S^{k-1} \) (the unit sphere in \( \mathbb{R}^k \)), \( \mathcal{X} = \{0\} \subseteq \mathbb{R}^k \) and \( \varrho: \mathbb{R}^k \setminus \{0\} \to S^{k-1} \) is the radial projection given by \( \varrho(y) = y/|y| \), we have \( \pi_{k-1}(S^{k-1}) \cong \mathbb{Z} \) and so elements of \( \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(S^{k-1})) \) have an alternative description as integer currents. Moreover, \( S_y(u) \) is related to the distributional Jacobian, as for any \( u \in (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^k) \) there holds

\[
J_u = \frac{1}{\omega_k} \int_{\mathbb{R}^m} S_y(u) \, dy,
\]

where \( \omega_k \) is the volume of the unit \( k\)-disk and the integral in the right-hand side is intended in the sense of distributions (see e.g. [11, Theorem 1.2]). However, if \( \pi_{k-1}(\mathcal{N}) \) is a finite group (or, more generally, if it only contains elements of finite order), then there is no meaningful way to define the integral of \( S_y(u) \) with respect to the Lebesgue measure \( dy \), as \( \pi_{k-1}(\mathcal{N}) \otimes \mathbb{R} = 0 \).
It is worth noticing that the proof of our main result, Theorem 3.1, does not strictly rely upon the manifold structure of $\mathcal{N}$. What is needed, is the existence and regularity of the exceptional set $\mathcal{E}$ and the retraction $\varrho$, in order to be able to apply Thom transversality theorem. This suggests a possible extension to more general targets $\mathcal{N} \subseteq \mathbb{R}^m$ such as, for instance, finite simplicial complexes.

1.3 Applications

We have chosen to work with vector-valued maps, instead of manifold-valued ones, as we were motivated by the applications to variational problems, such as \cite{2}. We expect that the results presented in this paper could be used as tools to obtain energy lower bounds for \cite{2} in the spirit of \cite{48, 39}, or even $\Gamma$-convergence results along the lines of \cite{2}. These questions will be addressed in a forthcoming work \cite{20}. Instead, we discuss here a few applications of this approach to classical questions in the theory of manifold-valued function spaces.

The first application concerns density of smooth maps. We define $W^{1,p}(B^d, \mathcal{N})$ as the set of maps $u \in W^{1,p}(B^d, \mathbb{R}^m)$ such that $u(x) \in \mathcal{N}$ for a.e. $x \in \Omega$, and endow it with the distance induced by $W^{1,p}(B^d, \mathbb{R}^m)$. Bethuel \cite{7} showed that smooth maps are dense in $W^{1,p}(B^d, \mathcal{N})$ if and only if $\pi_{[p]}(\mathcal{N}) = 0$ or $p \geq d$. Maps that belong to the strong-$W^{1,p}$ closure of $C^\infty(B^d, \mathbb{S}^{k-1})$ have been characterised in \cite{6}, in case $p = k - 1$, and in \cite{12}, in case $k - 1 < p < k$, using the distributional Jacobian. Pakzad and Rivièrè \cite{46, 38} generalised this result to other target manifolds, working in the setting of flat chains. As a corollary of our construction, we recover Pakzad and Rivièrè’s result.

**Theorem 1.** Let $d \geq 2$ be an integer, let $1 \leq p < d$, and let $\mathcal{N}$ be a compact, smooth, $([p]-1)$-connected manifold without boundary. In case $1 \leq p < 2$, we also suppose that $\pi_1(\mathcal{N})$ is abelian. Then, there exists a continuous map

$$S_{\text{PR}}: W^{1,p}(B^d, \mathcal{N}) \to \mathbb{F}_{d-[p]-1}(\overline{B}^d; \pi_{[p]}(\mathcal{N}))$$

such that $S_{\text{PR}}(u) = 0$ if and only if $u$ is a strong $W^{1,p}$-limit of smooth maps $\overline{B}^d \to \mathcal{N}$.

In contrast with Pakzad and Rivièrè, we do not need to impose the technical restriction $[p] \in \{1, d-1\}$. The arguments in \cite{46} rely on fine results in Geometric Measure Theory \cite{29} (which require $[p] \in \{1, d-1\}$); instead, the proof of Theorem II follows directly from our main construction, which is based essentially on the coarea formula, combined with the “removal of the singularities” results in \cite{46}. It is worth mentioning that the theorem may fail if the domain is not a disk (see the counterexamples in \cite{32} and the discussion in \cite{46}).

We next drive our attention to manifold-valued BV-maps. Recall that the space $BV(\Omega, \mathbb{R}^m)$, by definition, consists of those functions $u \in L^1(\Omega, \mathbb{R}^m)$ whose distributional derivative $Du$ is a finite Radon measure. The BV-norm is defined by $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, where $|\cdot|$ denotes the total variation measure. We say that $u \in SBV(\Omega, \mathbb{R}^m)$ if there exist Borel functions $\psi_0, \psi_1: \Omega \to \mathbb{R}^{m \times d}$ such that $\psi_j$ is $\mathcal{H}^{d-j}$-integrable, for $j \in \{0, 1\}$, and $Du = \psi_0\mathcal{H}^{d} + \psi_1\mathcal{H}^{d-1}$. We say that a sequence $u_j$ of BV-functions converges weakly to $u$ if and only if $u_j \rightharpoonup^* u$ strongly in $L^1$ and $Du_j \rightharpoonup^* Du$ weakly* as elements of the dual $C_0(\Omega, \mathbb{R}^m)'$. We define $BV(\Omega, \mathcal{N})$ (resp.,
SBV(Ω, ℳ)) as the set of maps \( u \in BV(Ω, \mathbb{R}^m) \) (resp., \( u \in SBV(Ω, \mathbb{R}^m) \)) such that \( u(x) \in ℳ \) for a.e. \( x \in Ω \).

**Theorem 2.** Let ℳ be a smooth, compact, connected manifold without boundary, with abelian \( π_1(ℳ) \). Then, \( C^∞(B^d, ℳ) \) is sequentially weakly dense in \( BV(B^d, ℳ) \).

A similar result has been obtained by Giaquinta and Mucci [30, Theorem 2.13], who worked in the framework of currents (more precisely, in the class of cartesian currents, see [29]). Giaquinta and Mucci need the additional assumption that \( π_1(ℳ) \) contains no element of finite order, in order to apply the formalism of currents. By working in the setting of flat chains, instead of currents, this assumption is not required any more, although we still need that \( π_1(ℳ) \) be abelian. In contrast with the scalar case, it may not be possible to construct approximating maps \( u_j \in C^∞(B^d, ℳ) \) in such a way that \( |Du_j|(B^d) \to |Du|(B^d) \) (see [30]).

The proofs of Theorems 1 and 2 follow a strategy that was adopted by Bethuel, Brezis and Coron in [8]: first we control the flat norm of the topological singular set, by means of the results in Section 3, then we “remove the singularities” using the results of [46]. The flat norm of the topological singular set coincides with what Bethuel, Brezis and Coron referred to as “minimal connection”.

Finally, we consider the lifting problem in BV. Let \( π: ℰ \to ℳ \) be the universal covering of ℳ. We choose a metric on ℰ and an isometric embedding \( ℰ \to \mathbb{R}^d \) in such a way that \( π \) is a local isometry. We say that \( v \in BV(Ω, ℰ) \) is a lifting for \( u \in BV(Ω, ℳ) \) if \( u = π \circ v \) a.e. on \( Ω \).

**Theorem 3.** Let \( Ω \subseteq \mathbb{R}^d \) be a smooth, bounded domain with \( d \geq 2 \), and let ℳ be a smooth, compact, connected manifold without boundary, with abelian \( π_1(ℳ) \). There exists a constant \( C \) such that any \( u \in BV(Ω, ℳ) \) admits a lifting \( v \in BV(Ω, ℰ) \) satisfying \( |Dv|(Ω) \leq C|Du|(Ω) \). Moreover, if \( u \in SBV(Ω, ℳ) \) then any lifting \( v \) of \( u \) belongs to \( SBV(Ω, ℰ) \).

The lifting problem in manifold-valued Sobolev spaces was studied by Bethuel and Chiron [11], who proved that any map \( v \in W^{1,p}(Ω, ℳ) \) with \( Ω \) simply connected and \( p \geq 2 \) has a lifting \( v \in W^{1,p}(Ω, ℰ) \). (The particular case \( ℳ \simeq \mathbb{R}^2 \), with applications to liquid crystals, was also studied by Ball and Zarnescu [3].) As conjectured by Bethuel and Chiron [11, Remark 1], Theorem 3 implies that any map \( u \in W^{1,p}(Ω, ℳ) \), with \( p \geq 1 \), has a lifting \( v \in BV(Ω, ℰ) \) (which may not belong to \( W^{1,p} \), see [11, Lemma 1]). The lifting problem in the space \( W^{s,p}(Ω, S^1) \) has been extensively studied by Bourgain, Brezis, and Mironescu, see e.g. [14, 15]. In the setting of BV-spaces, the lifting problem has been previously studied by Davila and Ignat [22], Ignat [37] in case \( ℳ = S^1 \), and recently by Ignat and Lamy [38], in case \( ℳ = \mathbb{R}P^n \). In contrast with Theorem 3, the results in [37, 38] are sharp, in the sense that they provide the optimal constant \( C \) such that \( |Dv|(Ω) \leq C|Du|(Ω) \); however, Theorem 3 is robust, in that it applies to more general manifolds. The proof of this theorem combines properties of the singular set \( S_μ(u) \) with a classical argument in topology, which gives the existence of the lifting for smooth functions \( u \), and which we revisit here in case the function \( u \) has jumps.

As remarked above, the techniques presented in this paper apply to quite general target manifolds, but not all. In particular, closed manifolds \( ℳ \) with non-abelian \( π_1(ℳ) \) are excluded, because the theory of flat chains with coefficients in a group \( G \) requires \( G \) to be abelian. However, in the topological obstruction theory, this kind of restriction can be removed by using suitable
technical tools (homology with local coefficients systems). This leaves a hope to extend, at least partially, some of the results in this paper to the case of non-abelian $\pi_1(\mathcal{M})$. Density (in the sense of biting convergence) of smooth maps in $W^{1,1}(\Omega, \mathcal{M})$ with non-abelian $\pi_1(\mathcal{M})$ has been proven by Pakzad [34].

2 Notation and preliminaries

2.1 Flat chains over an abelian coefficient group

Let $(G, |\cdot|)$ be a normed abelian group, that is, an abelian group (we will use additive notation for the operation on $G$) together with a non-negative function $|\cdot| : G \to [0, +\infty)$ that satisfies

(i) $|g| = 0$ if and only if $g = 0$

(ii) $|-g| = |g|$ for any $g \in G$

(iii) $|g + h| \leq |g| + |h|$ for any $g, h \in G$.

Throughout the following, we will assume that the norm $|\cdot|$ satisfies

$|g| \geq 1$ for any $G \setminus \{0\}$.

In order to fix some notation, and for the convenience of the reader, we recall some basic definitions and facts about flat chains with multiplicities in $G$. We follow the approach in [34, 28, 52], to which we refer the reader for further details.

For $n \in \mathbb{Z}$, $1 \leq n \leq d$, consider the free $G$-module generated by compact, convex, oriented polyhedra of dimension $n$. (In other words, we consider the set of all formal sums of polyhedra as above, with coefficients in $G$; there is a natural notion of sum which makes this set an abelian group.) We quotient this module by the equivalent relation $\sim$, requiring $-\sigma \sim \sigma'$ if $\sigma'$ and $\sigma$ only differ for the orientation, and $\sigma \sim \sigma_1 + \sigma_2$ if $\sigma$ is obtained by gluing $\sigma_1$, $\sigma_2$ along a common face (with the correct orientation). The quotient group is called the group of polyhedral $n$-chain with coefficients in $G$, and is denoted $P_n(\mathbb{R}^d; G)$. Every element $S \in P_n(\mathbb{R}^d; G)$ can be represented as a finite sum

$$S = \sum_{i=1}^{p} \alpha_i [\sigma_i],$$

where $\alpha_i \in G$, the $\sigma_i$’s are compact, convex, non-overlapping $n$-dimensional polyhedra, and $[\cdot]$ denotes the equivalence class modulo the relation $\sim$ defined above.

The mass of a polyhedral chain $S \in P_n(\mathbb{R}^d; G)$, presented in the form (2.2), is defined by $M(S) := \sum_{i}|\alpha_i| \mathcal{H}^n(\sigma_i)$. A linear operator $\partial : P_n(\mathbb{R}^d; G) \to P_{n-1}(\mathbb{R}^d; G)$, called the boundary operator, is defined in such a way that, for a single polyhedron $\sigma$, $\partial[\sigma]$ is the sum of the boundary faces of $\sigma$, with the orientation induced by $\sigma$ and multiplicity 1. The boundary operator satisfies $\partial \circ \partial = 0$. The flat norm of a polyhedral $n$-dimensional chain $S$ is defined by

$$F(S) := \inf \left\{ M(P) + M(Q) : P \in P_{n+1}(\mathbb{R}^d; G), Q \in P_n(\mathbb{R}^d; G), S = \partial P + Q \right\}.$$
It can be showed (see e.g. [28 Section 2]) that $\mathcal{F}$ indeed defines a norm on $\mathbb{P}_n(\mathbb{R}^d; \mathbb{G})$, in such a way that the group operation on $\mathbb{P}_n(\mathbb{R}^d; \mathbb{G})$ is Lipschitz continuous. The completion of $(\mathbb{P}_n(\mathbb{R}^d; \mathbb{G}), \mathcal{F})$, as a metric space, will be denoted $\mathbb{F}_n(\mathbb{R}^d; \mathbb{G})$. It can be given the structure of a $\mathbb{G}$-module, and it is called the group of flat $n$-chain with coefficients in $\mathbb{G}$. Moreover, the mass $\mathcal{M}$ extends to a $\mathcal{F}$-lower semi-continuous functional $\mathbb{F}_n(\mathbb{R}^d; \mathbb{G}) \to [0, +\infty]$, still denoted $\mathcal{M}$, and it remains true that

\begin{equation}
\mathcal{F}(S) := \inf \left\{ \mathcal{M}(P) + \mathcal{M}(Q) : P \in \mathbb{P}_{n+1}(\mathbb{R}^d; \mathbb{G}), Q \in \mathbb{P}_n(\mathbb{R}^d; \mathbb{G}), S = \partial P + Q \right\}
\end{equation}

for any $S \in \mathbb{F}_n(\mathbb{R}^d; \mathbb{G})$ [28 Theorem 3.1]. We let $\mathbb{M}_n(\mathbb{R}^d; \mathbb{G})$ be the set of flat $n$-chains $S$ with $\mathcal{M}(S) < +\infty$, and we let

$$\mathbb{N}_n(\mathbb{R}^d, \mathbb{G}) := \left\{ S \in \mathbb{M}_n(\mathbb{R}^d, \mathbb{G}) : \mathcal{M}(S) + \mathcal{M}(\partial S) < +\infty \right\}.$$ 

In fact, $\mathcal{M}$ is a norm on $\mathbb{M}_n(\mathbb{R}^d; \mathbb{G})$.

**Operations with flat chains.** Any Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^N$ induces group homeomorphisms $f_* : \mathbb{F}_n(\mathbb{R}^d; \mathbb{G}) \to \mathbb{F}_n(\mathbb{R}^N; \mathbb{G})$, for $0 \leq n \leq d$, called the push-forward via $f$. One first defines the push-forward of a a single polyhedron, $f_*[\sigma]$, by approximating $f$ with piecewise-affine maps (see [54 p. 297]). Then, $f_*$ extends to polyhedral chains by linearity, and to arbitrary chains by approximation with polyhedral chains. The push-forward commutes with the boundary, that is $\partial(f_* S) = f_*(\partial S)$. If $S$ is a flat $n$-chain and $\lambda$ is a Lipschitz constant for $f$, then

\begin{equation}
\mathcal{M}(f_* S) \leq \lambda^n \mathcal{M}(S), \quad \mathcal{F}(f_* S) \leq \max\{\lambda^n, \lambda^{n+1}\} \mathcal{F}(S)
\end{equation}

(see e.g. [28 Section 5]). A chain of the form $f_* S$, where $S$ is polyhedral and $f$ is Lipschitz (resp., smooth), will be called a Lipschitz (resp., smooth) chain. By a remarkable result by Fleming [28], later improved by White [52], if $\mathbb{G}$ satisfies [2.1] then Lipschitz chains are dense in $\mathbb{M}_n(\mathbb{R}^d; \mathbb{G})$ with respect to the $\mathcal{M}$-norm, and in particular $\text{spt} S$ is a rectifiable set for any $S \in \mathbb{M}_n(\mathbb{R}^d; \mathbb{G})$. (However, we will not need this result in our arguments.)

Given a chain $S$ of finite mass and a Borel set $A \subseteq \mathbb{R}^d$, one can define the restriction of $S$ to $A$, denoted $S|_A$, which roughly speaking represents the portion of $S$ contained in $A$. Again, this is obtained via approximation with polyhedral chains (see [28 Section 4]). Then, for fixed $A$, the functional $S \mapsto \mathcal{M}(S|_A)$ is $\mathcal{F}$-lower continuous, while for fixed $S$, $A \mapsto \mathcal{M}(S|_A)$ is a Radon measure.

A flat chain $S$ is said to be supported in a closed set $K \subseteq \mathbb{R}^d$ if, for any open neighbourhood $U$ of $K$, there exists a sequence of polyhedral chains $(P_j)_{j \in \mathbb{N}}$ that lie in $U$ (i.e., every cell of $P_j$ is contained in $U$) and $\mathcal{F}$-converges to $S$. If $S$, $R$ are supported in a closed set $K$, then $\partial S$, $S + R$ are also supported in $K$. The support of a chain $S$, noted $\text{spt} S$, is defined as the smallest $K$ such that $S$ is supported in $K$. If $S$ has finite mass, $\text{spt} S$ coincides with the support of the measure $A \mapsto \mathcal{M}(S|_A)$ (see [28 Sections 3 and 4]).

For $K$ closed set in $\mathbb{R}^d$, we denote by $\mathbb{F}_n(K; \mathbb{G})$ (resp., $\mathbb{M}_n(K; \mathbb{G})$, $\mathbb{N}_n(K; \mathbb{G})$) the set of chains $S \in \mathbb{F}_n(\mathbb{R}^d; \mathbb{G})$ (resp., $S \in \mathbb{M}_n(\mathbb{R}^d; \mathbb{G})$, $S \in \mathbb{N}_n(\mathbb{R}^d; \mathbb{G})$) that are supported in $K$. It follows from the definition of $\text{spt} S$, and from the lower semi-continuity of the mass, that the sets $\mathbb{F}_n(K; \mathbb{G})$, $\mathbb{M}_n(K; \mathbb{G})$, $\mathbb{N}_n(K; \mathbb{G})$ are closed under $\mathcal{F}$-convergence.

Finally, we recall the following property of 0-dimensional flat chains.
Lemma 2.1 ([52, Theorem 2.1]). There exists a unique group homomorphism \( \chi : \mathbb{F}_0(\mathbb{R}^d; G) \to G \) that satisfies the following properties:

(i) \( \chi(\sum_{j=1}^q g_j[x_j]) = \sum_{j=1}^q g_j \) for \( g_j \in G \) and \( x_j \in \mathbb{R}^d \), \( j \in \{1, \ldots, q\} \).

(ii) \( \chi(\partial R) = 0 \).

(iii) \( |\chi(S)| \leq F(S) \) for any \( S \in \mathbb{F}_0(\mathbb{R}^d; G) \).

The map \( \chi \) is sometimes called the augmentation homomorphism.

Remark 2.1. Lemma 2.1(iii) and our assumption (2.1) imply that \( \chi(S_0) = \chi(S_1) \) if the chains \( S_0, S_1 \in \mathbb{F}_0(\mathbb{R}^d; G) \) are such that \( \mathbb{F}(S_0 - S_1) < 1 \).

Relative flat chains on an open set. In view of our applications, we will need to consider flat chains defined in an open set \( U \subseteq \mathbb{R}^d \). A definition of the space \( \mathbb{F}_n(U; G) \) is given in several places in the literature (see, e.g., [25, 29, 40, ...]) but, to the best of the authors’ knowledge, it is usually required that the elements of \( \mathbb{F}_n(U; G) \) be compactly supported in \( U \), which is not convenient for our purposes. We discuss here an alternative definition and present some basic results for the sake of completeness, being aware that these facts might be well-known by the experts of the field.

Let \( U \subseteq \mathbb{R}^d \) be a non-empty open set, and let \( K \) be a closed set that contains \( U \). Recall that we have defined \( \mathbb{F}_n(K; G) \) as the set of chains in \( \mathbb{F}_n(\mathbb{R}^d; G) \) that are supported in \( K \). We now define

\[
\mathbb{F}_n(U; G) := \mathbb{F}_n(K; G)/\mathbb{F}_n(K \setminus U; G).
\]

\( \mathbb{F}_n(K \setminus U; G) \) is a \( G \)-submodule of \( \mathbb{F}_n(K; G) \) and is closed with respect to the \( \mathbb{F} \)-norm because \( K \setminus U \) is closed, therefore \( \mathbb{F}_n(U; G) \) is a complete normed \( G \)-module, with respect to the quotient norm:

\[
F_U(S) := \inf \left\{ F(R) : R \in \mathbb{F}_n(\mathbb{R}^d; G), \ spt(R) \subseteq K, \ spt(R - S) \subseteq K \setminus U \right\},
\]

for \( S \in \mathbb{F}_n(K; G) \) — by abuse of notation, we denote by the same symbol the chain \( S \) and its equivalence class in \( \mathbb{F}_n(U; G) \). The boundary operator \( \partial \) induces a well-defined, continuous operator \( \mathbb{F}_n(U; G) \to \mathbb{F}_{n-1}(U; G) \), still denoted \( \partial \). We now give an alternative characterisation of the norm \( F_U \).

Lemma 2.2. For any \( S \in \mathbb{F}_n(K; G) \), there holds

\[
F_U(S) = \inf \left\{ M(P \mathbf{1} U) + M(Q \mathbf{1} U) : P \in M_{n+1}(\mathbb{R}^d; G), \ Q \in M_n(\mathbb{R}^d; G), \ spt(S - \partial P - Q) \subseteq \mathbb{R}^d \setminus U \right\}.
\]

Proof. Denote by \( \hat{F}_U(S) \) the right-hand side. For any \( \varepsilon > 0 \), using the definition (2.5) of \( F_U \) and the characterisation (2.3) of the flat norm, we find \( P \in M_{n+1}(\mathbb{R}^d; G), \ Q \in M_n(\mathbb{R}^d; G) \) such that \( spt(\partial P + Q) \subseteq K \), \( spt(\partial P + Q - S) \subseteq K \setminus U \) and \( M(P) + M(Q) \leq F_U(S) + \varepsilon \). This shows the inequality \( \hat{F}_U(S) \leq F_U(S) \).
Before checking the opposite inequality, we remark that, for any chain \( T \) of finite mass and any open set \( W \subseteq \mathbb{R}^d \), there holds

\[
(2.6) \quad \text{spt}(T - T \llcorner W) \subseteq \mathbb{R}^d \setminus W, \quad \text{spt}(\partial T - \partial(T \llcorner W)) \subseteq \mathbb{R}^d \setminus W.
\]

(The first inclusion holds true because \((T - T \llcorner W) \llcorner W = T \llcorner W - T \llcorner W = 0\); the second one follows from the first, because the boundary of chain supported in \( \mathbb{R}^d \setminus W \) is also supported in \( \mathbb{R}^d \setminus W \).) Now, we fix \( S \in \mathbb{F}_n(K; G) \), \( P \in \mathbb{M}_{n+1}(\mathbb{R}^d; G) \) and \( Q \in \mathbb{M}_n(\mathbb{R}^d; G) \) such that \( \text{spt}(S - \partial P - Q) \subseteq \mathbb{R}^d \setminus U \). Let \( K_0 \) be the interior of \( K \), and let \( P' := P \llcorner K_0 \), \( Q' := Q \llcorner K_0 \). Then \( P' \), \( Q' \) are supported in \( K \), and so is \( S - \partial P' - Q' \). Moreover, there holds

\[
S - \partial P' - Q' = S - \partial P - Q + \partial P' - \partial P + Q' - Q
\]

and the three terms that are indicated by underbraces are all supported out of \( U \) (the first one is supported in \( \mathbb{R}^d \setminus U \) by assumption, the second and the third ones are supported in \( \mathbb{R}^d \setminus K_0 \subseteq \mathbb{R}^d \setminus U \) by \((2.6)\)). Therefore, \( \text{spt}(S - \partial P' - Q') \subseteq K \setminus U \). Finally, we have

\[
S = (P' \llcorner U) + Q' \llcorner U + \partial P' - \partial(P' \llcorner U) + Q' - Q \llcorner U + S - \partial P' - Q'
\]

The chain \( R \) is supported in \( \overline{U} \subseteq K \), and all the terms in the right-hand side but \( R \) are supported in \( K \subseteq U \), thanks to \((2.6)\). Therefore, by the definition \((2.5)\) of \( \mathbb{F}_U \) and \((2.3)\), we deduce that \( \mathbb{F}_U(S) \leq \mathbb{F}(R) \leq \mathbb{M}(P \llcorner U) + \mathbb{M}(Q \llcorner U) \) and hence, by arbitrariness of \( P, Q \), that \( \mathbb{F}_U(S) \leq \mathbb{F}(S) \).

The right-hand side of Lemma \(2.2\) do not depend on \( K \). Therefore, the space \( \mathbb{F}_n(U; G) \) is indeed independent of the choice of \( K \), in the following sense: for any closed sets \( K_1, K_2 \) with \( K_1 \supseteq K_2 \supseteq U \), there exists an isometric isomorphism

\[
\mathbb{F}_n(K_1; G)/\mathbb{F}_n(K_1 \setminus U; G) \to \mathbb{F}_n(K_2; G)/\mathbb{F}_n(K_2 \setminus U; G).
\]

The isomorphism is obtained by considering the map \( \mathbb{M}_n(K_1; G) \to \mathbb{F}_n(K_2; G)/\mathbb{F}_n(K_2 \setminus U; G) \) induced by the restriction operator \( S \mapsto S \llcorner K_2 \), extending it by density to a map \( \mathbb{F}_n(K_1; G) \to \mathbb{F}_n(K_2; G)/\mathbb{F}_n(K_2 \setminus U; G) \), with the help of Lemma \(2.2\) and passing to the quotient. The inverse is induced by the inclusion \( K_2 \hookrightarrow K_1 \). Because of the existence of an isomorphism, it makes sense to omit \( K \) in the notation. In the rest of this section, we assume that \( K = \overline{U} \), but other choices of \( K \) might be convenient.

In a similar fashion, from Lemma \(2.2\) we can derive the following compatibility property with respect to restrictions. For notational convenience, we set \( \mathbb{F}_{\mathbb{R}^d} := \mathbb{F} \).

**Lemma 2.3.** Let \( U_1, U_2 \) be non-empty, open sets in \( \mathbb{R}^d \) with \( U_1 \subseteq U_2 \). Then, there exists a continuous map

\[
\Psi : \mathbb{F}_n(U_2; G) \to \mathbb{F}_n(U_1; G)
\]

such that \( \Psi(S) = S \llcorner U_1 \) for any \( S \in \mathbb{M}_n(U_2; G) \).
We omit the proof of this lemma. An analogous compatibility property with respect to restrictions does not hold, in general, for the $F$-norm. (For instance, let $R_j \in M_2(\mathbb{R}^2; \mathbb{Z})$ be the chain carried by the rectangle $[-1/j, 1/j] \times [-1, 1]$ with standard orientation; then $\partial R_j F$-converges to zero but $\partial R_j (0, +\infty) \times \mathbb{R}$ does not.) Moreover, if $S$ has infinite mass, the restriction $S|U$ might not be well defined in $F_0(\mathbb{R}^d; G)$: for example, consider the $0$-chain with coefficients in $\mathbb{Z}/2\mathbb{Z}$ carried by the set $U_j = \{(2^j, j), (2^{j+1}, j)\} \subseteq \mathbb{R}^2$ and $U = (0, +\infty) \times \mathbb{R}$. In this case, $S|U$ is not well-defined in $F_0(\mathbb{R}^2; \mathbb{Z}/2\mathbb{Z})$, even though $\Psi(S)$ is well-defined. We recall the proof in the following lemma:

**Lemma 2.4.** Let $U \subseteq \mathbb{R}^d$ be a non-empty open set, and let $\rho_0$ be a positive number. For $\rho \in (0, \rho_0]$, set $\rho_0 = \{x \in U : \text{dist}(x, \partial U) > \rho\}$. Let $(S_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{F}_n(U; G)$ and let $S \in \mathbb{F}_n(U; G)$ be such that $\mathbb{F}_U(S_j - S) \to 0$ as $j \to +\infty$. Then, for a.e. $\rho \in (0, \rho_0]$ the restrictions $S_j|_U \rho$, $S|_U \rho$ are well-defined and $\mathbb{F}(S_j|_U \rho - S|_U \rho) \to 0$ as $j \to +\infty$.

**Proof.** We first claim that, for any $T \in \mathbb{F}_n(U; G)$ and a.e. $\rho \in (0, \rho_0]$, the restriction $T|_U \rho$ is well-defined and there holds

$$\int_0^{\rho_0} F(T|_U \rho) \, d\rho \leq (1 + \rho_0) F(T).$$

(As mentioned above, this fact is well-known and we include a proof here only for the sake of completeness.) Suppose first that $T$ has finite mass. Let $P \in M_{n+1}(\mathbb{R}^d; G)$, $Q \in M_n(\mathbb{R}^d; G)$ be such that $T = \partial P + Q$. Having assumed that $T$ has finite mass, it follows that $\partial P$ has finite mass. We also remark that, for any $\rho \leq \rho_0$, $\rho$ is a sublevel set for the signed distance function from $\partial U$ (i.e., the function $f$ defined by $f(x) := \text{dist}(x, \partial U)$ if $x \in U$, $f(x) := \text{dist}(x, \partial U)$ if $x \notin U$), which is $1$-Lipschitz continuous. Then, we can apply [28, Theorem 5.7], and deduce that $B_\rho := \partial (P|_U \rho) - (\partial P) \rho \rho$ is well-defined, and there holds

$$\int_0^{+\infty} M(B_\rho) \, d\rho \leq M(P|_U),$$

Moreover, there holds

$$T|_U \rho = \partial (P|_U \rho) + Q|_U \rho - B_\rho,$$

which yields

$$F(T|_U \rho) \leq M(P) + M(Q) + M(B_\rho).$$

By integrating this inequality with respect to $\rho \in (0, \rho_0]$, using (2.8), and taking the infimum with respect to all possible choices of $P$ and $Q$, we deduce that (2.7) holds, in case $T$ has finite mass. If $T$ has infinite mass, we recover the same result using that finite-mass chains are dense in $F_0(U; G)$.

Now, let $(S_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{F}_n(U; G)$ that $F_U$-converges to $S$. By possibly modifying the $S_j$'s out of $U$, we can assume that there exists a sequence $(R_j)_{j \in \mathbb{N}}$ in $\mathbb{F}_n(U; G)$ such that $\mathbb{F}(S_j - R_j) \to 0$ as $j \to +\infty$ and $R_j - S$ is supported out of $U$; for each $j$. By [28], $\mathbb{F}(S_j - R_j)|_U \rho \to 0$ as $j \to +\infty$, for a.e. $\rho$. But $R_j|_U \rho = S|_U \rho$, so the lemma is proved. □
Remark 2.2. Assume that \( H \subseteq U \) is a Borel set such that \( \text{dist}(H, \partial U) > 0 \) and let \( S \in M_n(U; G) \) be a finite-mass chain. By taking \( \rho_0 := \text{dist}(H, \partial U) \), noting that \( T \mathbb{L} U_\rho = (T \mathbb{L} H) + (T \mathbb{L} U_\rho \setminus H) \) for any \( \rho \in (0, \rho_0] \), and selecting a suitable \( \rho \), from (2.7) we deduce that

\[
F(T \mathbb{L} H) \leq \left( 1 + \text{dist}^{-1}(H, \partial U) \right) F(T) + M(T \mathbb{L} (U \setminus H)).
\]

By taking the infimum over all finite-mass \( T \)'s in a given equivalence class of \( F_n(U; G) \), we also deduce

\[
(2.9) \quad F(T \mathbb{L} H) \leq \left( 1 + \text{dist}^{-1}(H, \partial U) \right) F(U) + M(T \mathbb{L} (U \setminus H)).
\]

Lemma 2.5. Let \( (S_j)_{j \in \mathbb{N}} \) be a sequence in \( M_n(U; G) \), and let \( S \in F_n(U; G) \) be such that \( F(U)(S_j - S) \to 0 \) as \( j \to +\infty \). Then, \( S \mathbb{L} U \) is well-defined and there holds \( M(S \mathbb{L} U) \leq \liminf_{j \to +\infty} M(S_j \mathbb{L} U_j) \).

Proof. Let \( U_\rho \) be as in Lemma 2.4. By Lemma 2.4 and the \( F \)-lower semi-continuity of the mass, we deduce that \( M(S \mathbb{L} U_\rho) \leq \liminf_{j \to +\infty} M(S_j \mathbb{L} U_j) \) for a.e. \( \rho > 0 \). The lemma follows by letting \( \rho \to 0 \).

Finally, we establish a compactness result with respect to the norm \( F_U \). We first remark that, as a consequence of our assumption (2.1), the following property holds:

\[
(2.10) \quad \text{for any } \Lambda > 0, \text{ the set } \{ g \in G : |g| \leq \Lambda \} \text{ is compact.}
\]

Lemma 2.6. Assume that the coefficient group \( G \) satisfies (2.10). Let \( U \subseteq \mathbb{R}^d \) be a non-empty, bounded, open set, and let \( (S_j)_{j \in \mathbb{N}} \) be a sequence in \( M_n(U; G) \) such that

\[
(2.11) \quad \sup_{j \in \mathbb{N}} (M(S_j \mathbb{L} U) + M(\partial S_j \mathbb{L} U)) < +\infty.
\]

Then, there exists a subsequence (still denoted \( S_j \)) and a chain \( S \in M_n(U; G) \) such that \( F_U(S_j - S) \to 0 \) as \( j \to +\infty \).

Proof. As in Lemma 2.4 let \( U_\rho := \{ x \in U : \text{dist}(x, U) > \rho \} \), for \( \rho \in (0, \rho_0] \) and \( \rho_0 > 0 \) fixed. Consider the sequence of measures in \( C_0(U)' \) defined by \( A \mapsto M(S_j \mathbb{L} A) \), for \( A \subseteq U \) Borel set. This sequence is bounded due to (2.11), and therefore it converges weakly* (up to a subsequence) to a limit measure \( \mu \in C_0(U)' \). The boundedness of \( \mu \) implies that \( \mu(\partial U_\rho) = 0 \) for a.e. \( \rho \).

Setting \( B_{\rho,j} := \partial(S_j \mathbb{L} U_\rho) - (\partial S_j) \mathbb{L} U_\rho \), by [28, Theorem 5.7] and Fatou lemma we deduce that

\[
\int_0^\infty \liminf_{j \to +\infty} M(B_{\rho,j}) \, d\rho \leq \liminf_{j \to +\infty} M(S_j \mathbb{L} U) < +\infty.
\]

Therefore, for a.e. \( \rho \in (0, \rho_0] \) there exists a subsequence (still denoted \( S_j \)) such that

\[
\sup_{j \in \mathbb{N}} (M(S_j \mathbb{L} U_\rho) + M(\partial(S_j \mathbb{L} U_\rho))) \leq \sup_{j \in \mathbb{N}} (M(S_j \mathbb{L} U_\rho) + M((\partial S_j) \mathbb{L} U_\rho) + M(B_{\rho,j})) < +\infty.
\]
Due to the assumption (2.10), and the boundedness of $U$, for a.e. $\rho \in (0, \rho_0]$ we can apply the compactness result [28, Corollary 7.5]. With the help of a diagonal argument, we find a sequence $\rho_k \searrow 0$ and a subsequence of $j$ such that, for any $k \in \mathbb{N}$, the following properties hold:

\begin{align}
(2.12) & \quad \mu(\partial U_{\rho_k}) = 0 \\
(2.13) & \quad (S_j \mathcal{L} U_{\rho_k})_{j \in \mathbb{N}} \text{ F-} \text{converges to a limit, say } R_k \in \mathcal{M}_n(U_{\rho_{k+1}}; \mathbf{G}).
\end{align}

The uniqueness of the limit implies that $R_k \mathcal{L} U_{\rho_h} = R_h$, for any $h \leq k$. The sequence $(R_k)_{k \in \mathbb{N}}$ is $\mathcal{M}$-convergent, because (2.13) and the $\mathcal{F}$-lower semi-continuity of the mass imply

$$
\sum_{k \in \mathbb{N}} \mathcal{M}(R_{k+1} - R_k) \leq \liminf_{j \to +\infty} \sum_{k \in \mathbb{N}} \mathcal{M}(S_j \mathcal{L} (U_{\rho_{k+1}} \setminus U_{\rho_k})) = \liminf_{j \to +\infty} \mathcal{M}(S_j \mathcal{L} U) \stackrel{(2.11)}{=} +\infty.
$$

Let $S \in \mathcal{M}_n(U; \mathbf{G})$ be the $\mathcal{M}$-limit of the $R_k$’s; it remains to check that $\mathcal{F}_U(S_j - S) \to 0$. For a fixed $\varepsilon > 0$, let $k_\varepsilon \in \mathbb{N}$ be such that $\mu(U \setminus U_{\rho_{k_\varepsilon}}) \leq \varepsilon/2$. Due to (2.12), we have $\mathcal{M}(S_j \mathcal{L} (U \setminus U_{\rho_{k_\varepsilon}})) \to \mu(U \setminus U_{\rho_{k_\varepsilon}})$ and hence

$$
\limsup_{j \to +\infty} \mathcal{F}_U(S_j - S) \leq \limsup_{j \to +\infty} \left\{ \mathcal{F}_U((S_j - S) \mathcal{L} U_{\rho_{k_\varepsilon}}) + \mathcal{M}((S_j - S) \mathcal{L} (U \setminus U_{\rho_{k_\varepsilon}})) \right\}
\leq 2\mu(U \setminus U_{\rho_{k_\varepsilon}}) = \varepsilon,
$$

where we have used again the $\mathcal{F}$-lower semi-continuity of the mass and the fact that $\mathcal{F}_U \leq \mathcal{F}$. Since $\varepsilon > 0$ is arbitrary, the lemma follows. 

\textbf{Intersection index for flat chains.} For $y \in \mathbb{R}^d$, we denote by $\tau_y: x \in \mathbb{R}^d \mapsto x + y$ the translation map associated with $y$. Given chains $S \in \mathcal{F}_n(\mathbb{R}^d; \mathbf{G})$ and $R \in \mathcal{N}_m(\mathbb{R}^d; \mathbb{Z})$, with $n + m \geq d$, for a.e. $y \in \mathbb{R}^d$ we would like to define the intersection $S \cap \tau_y \cdot R$ as an element of $\mathcal{F}_m(\mathbb{R}^d; \mathbf{G})$. This construction has been described in [53, Section 5] but, for the convenience of the reader, we briefly recall it here.

Suppose first that $S$, $R$ are single polyhedra. By Thom transversality theorem, for a.e. $y$ the polyhedra $S$ and $\tau_y \cdot R$ intersect transversely, so the set $\sigma := [S] \cap [\tau_y \cdot R]$ is a finite union of polyhedra of dimension $n + m - d$. We orient $[S] \cap [\tau_y \cdot R]$ according to the convention of [52, Section 3], i.e., the orientation is chosen in such a way that the following holds: if $(u_1, \ldots, u_{n+m-d})$ is an oriented basis for the $(n + m - d)$-plane spanned by $[S] \cap [\tau_y \cdot R]$, $(u_1, \ldots, u_{n+m-d}, v_1, \ldots, v_{d-m})$ is an oriented basis for the $n$-plane spanned by $[S]$, and $(u_1, \ldots, u_{n+m-d}, w_1, \ldots, w_{d-m})$ is an oriented basis for the $m$-plane spanned by $[\tau_y \cdot R]$, then $(u_1, \ldots, u_{n+m-d}, v_1, \ldots, v_{d-m}, w_1, \ldots, w_{d-n})$ is a positively oriented basis for $\mathbb{R}^d$. Having chosen the orientation, we can regard the intersection $S \cap \tau_y \cdot R$ as a polyhedral $(n + m - d)$ chain, in the obvious way. This definition now extend by linearity to the case $S$, $R$ are polyhedral. Now, it can be showed [53, Theorem 5.3] that

\begin{align}
(2.14) & \quad \int_{\mathbb{R}^d} \mathcal{F}(S \cap \tau_y \cdot R) \, dy \leq \mathcal{F}(S)(\mathcal{M}(R) + \mathcal{M}(\partial R)).
\end{align}

As a consequence, we can extend $\cap$ by continuity so that, for a.e. $y \in \mathbb{R}^m$, any $S \in \mathcal{F}_n(\mathbb{R}^d; \mathbf{G})$ and $R \in \mathcal{N}_m(\mathbb{R}^d; \mathbb{Z})$, $S \cap \tau_y \cdot R$ is a well-defined element of $\mathcal{F}_{n+m-d}(\mathbb{R}^d; \mathbf{G})$. Moreover, for a
sequence \((S_j)_{j \in \mathbb{N}}\) that converges to \(S\) in the flat norm and a.e. \(y \in \mathbb{R}^d\), the chain \(S_j \cap \tau_{y_*, R}\) flat-converges to \(S \cap \tau_{y_*, R}\).

For the convenience of the reader, we sketch the proof of (2.14).

Proof of (2.14). Suppose that \(S, R\) are single polyhedra. Then, by applying the coarea formula, we deduce that

\[
\int_{\mathbb{R}^d} M(S \cap \tau_{y_*, R}) \, dy \leq M(S)M(R).
\]

This inequality can be extended by linearity to the case \(S, R\) are polyhedral chains. Now, it can be checked that, when \(A, B\) are polyhedral chains that intersect transversely, and with the orientation convention described above, there holds

\[
(2.15) \quad \partial(A \cap B) = (-1)^{d-m} \partial A \cap B + A \cap \partial B.
\]

Therefore, writing \(S = P + \partial Q\), we have

\[
S \cap \tau_{y_*, R} = P \cap \tau_{y_*, R} + (-1)^{d-m} Q \cap \tau_{y_*, \partial R} + (-1)^{d-m+1} \partial(Q \cap \tau_{y_*, R})
\]

By taking the flat norm and integrating with respect to \(y \in \mathbb{R}^d\), we see that the left-hand side of (2.14) is bounded by \(M(P)M(R) + M(Q)M(\partial R) + M(Q)M(R)\), and hence (2.14) follows.

In the rest of the paper, we will be interested in the case \(S, R\) are of complementary dimensions, that is, \(\dim(S) + \dim(R) = d\). In this case, \(S \cap \tau_{y_*, R}\) is a 0-chain, and we can consider the quantity \(\chi(S \cap \tau_{y_*, R} \in G)\), where \(\chi\) is the augmentation homomorphism given by Lemma 2.1. \(\square\)

Lemma 2.7. Suppose that (2.14) is satisfied. Let \(S \in \mathbb{F}_n(\mathbb{R}^d; G)\), \(R \in \mathbb{N}_{d-n}(\mathbb{R}^d; \mathbb{Z})\) be chains such that

\[
(2.16) \quad \text{spt}(\partial S) \cap \text{spt}(R) = \text{spt}(S) \cap \text{spt}(\partial R) = \emptyset.
\]

Then, there exists \(\delta = \delta(S, R) > 0\) such that, for a.e. \(y_1, y_2 \in B^d_\delta\), there holds

\[
\chi(S \cap \tau_{y_1, R}) = \chi(S \cap \tau_{y_2, R}).
\]

Proof. Suppose first that \(S, R\) are polyhedra of complementary dimensions that satisfy (2.16). Take \(y_1, y_2\) such that \(S\) intersects transversely \(\tau_{y_1, R}\) and \(\tau_{y_2, R}\) and \(y_2 - y_1\) does not belong to the linear subspace spanned by \(R\). If \(|y_1|, |y_2|\) are small enough, then the 0-chain \(S \cap (\tau_{y_2, R} - \tau_{y_1, R})\) is (either 0 or) the boundary of a segment whose length tends to zero as \(|y_2 - y_1| \to 0\), for fixed \(S, R\). (The assumption (2.16) is essential here.) Thus, \(\mathbb{F}(S \cap \tau_{y_2, R} - S \cap \tau_{y_1, R}) \to 0\) as \(|y_1|\) and \(|y_2|\) simultaneously tend to 0, and hence (2.16), together with Lemma 2.1, implies that \(\chi(S \cap \tau_{y_1, R}) = \chi(S \cap \tau_{y_2, R})\) for \(|y_1|, |y_2|\) small enough. By linearity and a density argument, using the stability of \(\cap\) and \(\chi\) with respect to the flat convergence (Equation (2.14) and Remark 2.1 respectively), the lemma follows. \(\square\)

By Lemma 2.7, the function \(y \in \mathbb{R}^d \mapsto \chi(S \cap \tau_{y_*, R}) \in G\) is equal a.e. to a constant, in a neighbourhood of 0. We call such constant the intersection product of \(S\) and \(R\), and we denote it by \(\llangle S, R \rrangle\). Note that \(\llangle S, R \rrangle\) is not well-defined if the condition (2.16) does not hold.
Let $U \subseteq \mathbb{R}^d$ be a non-empty, open set. Let $S$, $R$ satisfy (2.16) with $\text{spt}(R) \subseteq U$, and let $S' \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G})$ be such that $\text{spt}(S - S') \subseteq \mathbb{R}^d \setminus U$. By approximating $S$, $S'$, $R$ with polyhedral chains $S_j$, $S'_j$, $R_j$ such that $\text{spt}(S_j \setminus S'_j) \subseteq U \setminus \text{spt}(S_j - S'_j)$, it can be checked that $I(S, R) = I(S', R)$. Therefore, the intersection index $I(S, R)$ is well-defined when $S \in \mathbb{F}_n(U; \mathbf{G})$, provided that $R$ satisfies $\text{spt}(R) \subseteq U$ in addition to (2.16).

**Lemma 2.8.** The intersection product satisfies the following properties.

(i) $I(S, R) = 0$ if $\text{spt}(S) \cap \text{spt}(R) = \emptyset$.

(ii) $I$ is bilinear: $I(S_1 + S_2, R) = I(S_1, R) + I(S_2, R)$ and $I(S, R + R_2) = I(S, R_1) + I(S, R_2)$, as soon as all the terms are well-defined.

(iii) $I$ is stable with respect to $\mathbb{F}_U$-convergence: if $U \subseteq \mathbb{R}^d$ is a non-empty open set, $(S_j)_{j \in \mathbb{N}}$ is a sequence in $\mathbb{F}_n(U; \mathbf{G})$ that $\mathbb{F}_U$-converges to $S$, and if $R \in \mathbb{N}_{d-n}(\mathbb{R}^d; \mathbb{Z})$ satisfies

$$\text{spt}(R) \subseteq U, \quad \text{spt}(\partial S_j) \cap \text{spt}(R) = \text{spt}(\partial R) = \emptyset \quad \text{for any } j \in \mathbb{N},$$

then $I(S, R) = I(S_j, R)$ for any $j$ large enough.

(iv) $I$ is stable with respect to homology: for any $S \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G})$ and $R \in \mathbb{N}_{d-n+1}(\mathbb{R}^d; \mathbb{Z})$ such that $\text{spt}(\partial S) \cap \text{spt}(\partial R) = \emptyset$, there holds $I(S, \partial R) = (-1)^n I(\partial S, R)$. In particular, if $\text{spt}(\partial S) \cap \text{spt}(\partial R) = \emptyset$ then $I(S, \partial R) = 0$.

**Proof.** Properties (i), (ii) and (iii) follow in a straightforward way from (2.14) and Lemma 2.1. For (iv) we remark that, due to (2.15), there holds

$$(-1)^n \partial S \cap \tau_{g,*} R + S \cap \tau_{g,*}(\partial R) = \partial (S \cap \tau_{g,*} R).$$

By taking $\chi$ on both sides, and applying Lemma 2.1(ii) and (ii), we obtain (iv). \qed

### 2.2 The group $\pi_{k-1}(\mathcal{N})$

If the condition (H) is satisfied, in particular if $\pi_j(\mathcal{N}) = 0$ for any integer $0 \leq j \leq k - 2$ (and $\pi_1(\mathcal{N})$ is abelian in case $k = 2$), then the action of $\pi_1(\mathcal{N})$ over $\pi_{k-1}(\mathcal{N})$ is trivial. Therefore, we can and we shall identify the free homotopy classes of continuous maps $\mathbb{S}^{k-1} \to \mathcal{N}$ with the elements of the homotopy group $\pi_{k-1}(\mathcal{N})$. Moreover, we have an isomorphism

$$\pi_{k-1}(\mathcal{N}) \simeq H_{k-1}(\mathcal{N}),$$

due to Hurewicz theorem (see e.g. [33, Theorem 4.37 p. 371]). The group $H_{k-1}(\mathcal{N})$ is finitely generated because $\mathcal{N}$ can be given the structure of a finite CW complex, hence $\pi_{k-1}(\mathcal{N})$ is finitely generated. The choice of a finite generating set $\{\gamma_j\}_{j=1}^q$ for $\pi_{k-1}(\mathcal{N})$ induces a group norm on $\pi_{k-1}(\mathcal{N})$, in the following way: for $a \in \pi_{k-1}(\mathcal{N})$, $|a|$ is the smallest length of a sum of $\gamma_j$’s representing $a$, that is

$$|a| := \inf \left\{ \sum_{j=1}^q |d_j| : (d_j)_{j=1}^q \in \mathbb{Z}^q, \ a = \sum_{j=1}^q d_j \gamma_j \right\}.$$  

(2.17)

It can be easily checked that the right-hand side does define a group norm. This norm is integer-valued, so $\pi_{k-1}(\mathcal{N})$ is a discrete topological space, and the condition (2.17) is satisfied. Throughout the paper, we will consider flat chains with multiplicities in $\mathbf{G} = \pi_{k-1}(\mathcal{N})$.  

16
2.3 Smooth complexes and the retraction over $\mathcal{N}$

A compact set $\mathcal{X} \subseteq \mathbb{R}^m$ will be called a $n$-dimensional smooth (resp., Lipschitz) complex if and only if there exists a diffeomorphism (resp., a bilipschitz map), defined on a neighbourhood of $\mathcal{X}$, that takes $\mathcal{X}$ onto a finite $n$-dimensional simplicial complex $\mathcal{X}^s \subseteq \mathbb{R}^m$. For $j \in \mathbb{N}$ with $j \leq n$, we define the $j$-skeleton $\mathcal{X}^j$ of $\mathcal{X}$ as the union of all the cells of dimension $\leq j$.

We recall an important topological fact, upon which our construction is based.

**Lemma 2.9** (Lemma 6.1). Suppose that $\mathcal{X}$ is satisfied. Then, there exist a compact $(m-k)$-dimensional smooth complex $\mathcal{X} \subseteq \mathbb{R}^m$ and a locally smooth retraction $g: \mathbb{R}^m \setminus \mathcal{X} \to \mathcal{N}$ such that

$$|\nabla g(y)| \leq \frac{C}{\text{dist}(y, \mathcal{X})}$$

for any $y \in \mathbb{R}^m \setminus \mathcal{X}$ and some constant $C = C(\mathcal{N}, m, \mathcal{X}) > 0$, and $\nabla g$ has full rank on a neighbourhood of $\mathcal{N}$.

**Proof.** This is exactly the statement of Lemma 6.1, except that in Lemma 6.1, the set $\mathcal{X}$ is required to be a Lipschitz complex and $g$ is required to be a Lipschitz map. However, the same argument can be used to produce a smooth pair $(\mathcal{X}, g)$ with the same properties (one starts with a smooth triangulation of $\mathbb{S}^m = \mathbb{R}^m \cup \{\infty\}$, in place of a Lipschitz triangulation; the smoothness of $g$ can be achieved by a standard regularisation argument).

Notice that $\varrho_y: z \in \mathcal{N} \mapsto g(y - z)$, for $|y|$ small enough, defines a smooth family of maps $\mathcal{N} \to \mathcal{N}$ such that $\varrho_0 = \text{Id}_\mathcal{N}$. Therefore, the implicit function theorem implies that $\varrho_y$ has a smooth inverse $\varrho_y^{-1}: \mathcal{N} \to \mathcal{N}$ for $|y|$ sufficiently small.

2.4 Manifold-valued Sobolev maps

Given a bounded, smooth open set $U \subseteq \mathbb{R}^d$ and a number $1 \leq p < +\infty$, we let $H^{1,p}(U, \mathcal{N})$ denote the strong $W^{1,p}$-closure of $C^\infty_c(\overline{U}, \mathcal{N})$. We denote by $H^{1,p}_\text{loc}(U, \mathcal{N})$ the set of maps $u \in W^{1,p}(U, \mathcal{N})$ such that, for any point $x \in U$, there exists a ball $B_r(x) \subset U$ such that $u|_{B_r(x)} \in H^{1,p}(B_r(x), \mathcal{N})$. Clearly, we have the chain of inclusions

$$H^{1,p}(U, \mathcal{N}) \subseteq H^{1,p}_\text{loc}(U, \mathcal{N}) \subseteq W^{1,p}(U, \mathcal{N})$$

and a well-known result by Bethuel [7 Theorem 1] implies that the equality $H^{1,p}_\text{loc} = W^{1,p}$ holds if and only if $p \geq d$ or $\pi|_{\mathcal{Y}}(\mathcal{N}) = 0$. The equality $H^{1,p} = W^{1,p}$ has been characterised by Hang and Lin [32 Theorem 1.3] in terms of topological properties of $U$ and $\mathcal{N}$. In particular, we have

**Lemma 2.10.** Suppose that $U \subseteq \mathbb{R}^d$ is a smooth, bounded domain that has the same homotopy type of a smooth $(k-1)$-complex, and let $p \geq k - 1$. Then, $H^{1,p}(U, \mathcal{N}) = H^{1,p}_\text{loc}(U, \mathcal{N})$.

**Proof.** If $p \geq d$ then, arguing as in Proposition p. 267, one sees that $H^{1,p}(U, \mathcal{N}) = H^{1,p}_\text{loc}(U, \mathcal{N}) = W^{1,p}(U, \mathcal{N})$, so we can assume that $k - 1 \leq p < d$. Let $u \in H^{1,p}_\text{loc}(U, \mathcal{N})$ and $\varepsilon > 0$ be given. By reflection across $\partial\Omega$, we can extend $u$ to a map in $W^{1,p}(U', \mathcal{N})$, where $U' \supset U$ is a slightly larger domain that retracts onto $\overline{U}$ (see e.g. Lemma 8.1 and Remark 8.2)); thus, we can apply the methods of [32], even though $U$ has a boundary.
Thanks to [32, Theorem 6.1] (or [7, Theorem 2]), we find a smooth cell decomposition $M$ of $U'$, a dual $d - |p| - 1$-skeleton $L^{d-[p]-1}$ and a map $\tilde{u} \in W^{1,p}(U', \mathcal{N})$ that is continuous on $U' \setminus L^{d-[p]-1}$ and satisfies $\|u - \tilde{u}\|_{W^{1,p}} \leq \varepsilon$. It suffices to show that $\tilde{u}|_{M[p]}$ can be extended to a continuous map $U' \to \mathcal{N}$, as [32, Theorem 6.2] yields then $\tilde{u} \in H^{1,p}(U, \mathcal{N})$ and, by arbitrariness of $\varepsilon$, $u \in H^{1,p}(U, \mathcal{N})$.

For each cell $Q \in M[|p|+1]$, denote by $Q' \in L^{d-[p]-1}$ the dual cell and let $x$ such that $Q \cap Q' = \{x\}$. Since $u \in H^{1,p}(U, \mathcal{N})$, there exist $0 < \rho < \text{dist}(x, \partial Q)$ a sequence of smooth maps $u_j : B_\rho(x) \to \mathcal{N}$ that converges to $u$ in $W^{1,p}$. In case $p \notin \mathbb{Z}$, Fubini theorem and Sobolev embeddings imply that, for a.e. $r \in (0, \rho)$, $u_j \to u$ uniformly on $Q \cap \partial B_r(x)$, therefore the homotopy class of $u_j|_{Q \cap \partial B_r(x)}$ is trivial. In case $p \in \mathbb{Z}$, one can approximate $u_j$ with the continuous functions

$$u_j^\delta(y) := \int_{Q \cap \partial B_r(x) \cap B_\delta(y)} u_j(z) \, dz$$

for $y \in Q \cap B_r(x)$. By the Poincaré inequality, we deduce that $\sup_{j, \delta} \text{dist}(u_j^\delta(y), \mathcal{N}) \to 0$ as $\delta \to 0$ and $u_j^\delta \to u^\delta$ uniformly on $Q \cap B_r(x)$ as $j \to +\infty$, so the same conclusion follows. (Details of the argument can be found in [17] and [32, Lemma 4.4].) By similar arguments we also obtain that, if $\varepsilon$ is small enough, then the homotopy class of $\tilde{u}|_{Q \cap \partial B_r(x)}$ is trivial, hence the homotopy class of $\tilde{u}|_{Q}$ is trivial and $\tilde{u}|_{M[|p|]}$ has a continuous extension $M[|p|+1] \to \mathcal{N}$. Finally, by applying [32, Lemma 2.2] and reminding that $U$ is homotopy equivalent to a $(k-1)$-complex and that $p \geq k-1$, we conclude that $\tilde{u}|_{M[|p|]}$ has a continuous extension $U \to \mathcal{N}$.

**Push-forward of a chain by a Sobolev map and homology classes.** Let $S \in M_{k-1}(U; \mathbb{Z})$ be an integral chain with $\partial S = 0$, and let $u \in H^{1,k-1}(U, \mathcal{N})$. We aim at defining the homology class of the pushforward chain $u_*(S)$. To this end, we pick a sequence $(u_n)_{n \in \mathbb{N}}$ in $(C^{\infty} \cap W^{1,k-1})(U, \mathcal{N})$ that converges to $u$ in $W^{1,k-1}$, and a sequence $(S_j)_{j \in \mathbb{N}}$ of polyhedral chains supported in an open set $U' \subset U$, with $\partial S_j = 0$ for any $j \in \mathbb{N}$, that converges to $S$ in the flat-norm. (Such a sequence $S_j$ exists as a consequence of the deformation theorem; see e.g. [28, Theorem 5.6 and remark at p. 175].) We claim that, for any $n$, $m$, $i$, $j$ large enough,

$$[u_n, S_i(S_j)]_{H_{k-1}(\mathcal{N})} = [u_m, S_i(S_j)]_{H_{k-1}(\mathcal{N})}.
\tag{2.18}$$

This homology class does not depend on the choice of the sequences $(u_n)$ and $(S_j)$, for any two such pairs of sequences $(u_n, S_j)$ and $(u'_n, S'_j)$ can be restructured into a single converging one. We denote this homology class by $[u_*(S)]$. By the Hurewicz isomorphism [35, Theorem 4.37 p. 371], $[u_*(S)]$ defines a unique homotopy class in $\pi_{k-1}(\mathcal{N})$, which we denote by the same symbol. By an approximation argument, once Claim (2.18) is proved we can deduce

**Lemma 2.11.** If $(u_j)_{j \in \mathbb{N}}$ is a sequence in $H^{1,k-1}(U, N)$ that converges $W^{1,k-1}$-strongly to $u$, and $(S_j)_{j \in \mathbb{N}}$ is a sequence of cycles in $N_{k-1}(U; \mathbb{Z})$ that converges to $S$ in the flat-norm, then

$$[u_*(S)] = [u_j, S_j]$$

for any $j$ large enough.
Lemma 4.5] and the fact that \((u_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(W^{1,k-1}(U)\) that, for any \(n, m\) large enough and any \((k-1)\)-polyhedral complex \(K \subseteq U\), \(u_n|_K\) and \(u_m|_K\) belong to the same homotopy class of continuous maps \(K \to \mathcal{N}\). Since homotopic maps induce the same pushforward in homology, it follows that \([u_n, s](S_i)] = [u_m, s](S_i)]\) for any \(n, m\) large enough and any \(i\) and, hence, Claim (2.18) is proved.

\[\square\]

3 The construction of the sets of topological singularities

3.1 Statement of the main results

Let \(\Omega \subseteq \mathbb{R}^d\) be a smooth and bounded domain, \(d \geq k\). We consider the set \(X(\Omega) := (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m)\) with the direct limit topology induced by the increasing family of subspaces

\[X_\Lambda(\Omega) := \left\{ u \in W^{1,k-1}(\Omega, \mathbb{R}^m) : \|u\|_{L^\infty(\Omega)} \leq \Lambda \right\} \text{ for } \Lambda > 0\]

(each \(X_\Lambda(\Omega)\) is given the strong \(W^{1,k-1}\)-topology). This defines a metrisable topology on \(X(\Omega)\), and a sequence \((u_j)_{j \in \mathbb{N}}\) converges to \(u \in X(\Omega)\) if and only if \(u_j \to u\) strongly in \(W^{1,k-1}\) and \(\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty} < +\infty\). We also consider the set \(Y(\Omega) := L^1(\mathbb{R}^m, F_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})))\), whose elements are Lebesgue-measurable maps \(S : \mathbb{R}^m \to F_{d-k}(\Omega; \pi_{k-1}(\mathcal{N}))\) such that

\[\|S\|_Y := \int_{\mathbb{R}^m} F_\Omega(S_y) \, dy < +\infty.\]

The set \(Y(\Omega)\) is a complete normed \(\pi_{k-1}(\mathcal{N})\)-modulus, with respect to the norm \(\| \cdot \|_Y\). When no ambiguity arises, we will write \(X\), \(Y\) instead of \(X(\Omega)\), \(Y(\Omega)\). Recall that the assumption \((\mathcal{H})\) is in force, see Section 1.2.

Theorem 3.1. Suppose that \((\mathcal{H})\) is satisfied. Then, there exists a unique continuous map \(S : X \to Y\) that satisfies the following property:

\((P_1)\) For any \(u \in X\), a.e. \(y \in \mathbb{R}^m\), and any \(R \in \mathbb{N}_k(\overline{\Omega}; \mathbb{Z})\) such that \(\text{spt } R \subseteq \Omega\), \(\text{spt } (\partial R) \cap \text{spt } S_y(u)) = \emptyset\), there holds

\[\|S_y(u), R\| = \|\varrho(u - y), (\partial R)\|\]

Moreover, for any \(\Lambda > 0\) there exists \(C_\Lambda > 0\) such that, for any \(u \in X\) with \(\|u\|_{L^\infty(\Omega)} \leq \Lambda\) and a.e. \(y \in \mathbb{R}^m\), the following properties are satisfied.

\((P_2)\) \(\varrho(u - y) \in W^{1,k-1}(\Omega, \mathcal{N}) \cap H^{1,k-1}_{\text{loc}}(\Omega \setminus \text{spt } S_y(u), \mathcal{N})\).
(P₃) $S_y(u)$ is a relative boundary in $\Omega$: there exists $R_y \in M_{d-k+1}(\Omega; \pi_{k-1}({\mathcal N}))$ such that $\text{spt}(S_y(u) - \partial R_y) \subseteq \mathbb{R}^d \setminus \Omega$ and

$$\int_{\mathbb{R}^m} M(R_y) \, dy \leq C_\Lambda \| \nabla u \|_{L^{k-1}(\Omega)}^{k-1}.$$ 

(P₄) If, in addition, $u \in W^{1,k}(\Omega; \mathbb{R}^m)$ then, for a.e. $y \in \mathbb{R}^m$, the chain $S_y(u)$ has finite mass and there holds

$$\int_{\mathbb{R}^m} M(S_y(u)) \, dy \leq C_\Lambda \| \nabla u \|_{L^k(\Omega)}^k.$$ 

(P₅) If $u_0, u_1 \in X$ satisfy $\| u_0 \|_{L^\infty(\Omega)} \leq \Lambda$, $\| u_1 \|_{L^\infty(\Omega)} \leq \Lambda$, then

$$\int_{\mathbb{R}^m} F_\Omega(S_y(u_1) - S_y(u_0)) \, dy \leq C_\Lambda \int_{\Omega} |u_1 - u_0| \left( |\nabla u_1|^{k-1} + |\nabla u_0|^{k-1} \right).$$ 

Property (P₁) implies that $S_y(u)$ does capture topological information on $u$, and motivates the name “set of topological singularities”. Notice that both sides of (P₁) are well-defined, thanks to (P₂) and Lemmas 2.10, 2.11, (P₃) and (P₄) provide an integral control on the $F_\Omega$-norm and the mass norm of $S_y(u)$, respectively. Property (P₃) will be crucially exploited in the applications we present in Section 4, while (P₄) is important in applications to variational problems, along the lines of [2]. Finally, (P₂) is a continuity estimate. Note that, if $u_j \rightharpoonup u$ in $X$, then by applying Lebesgue dominated convergence theorem to the right-hand side of (P₅) (after, possibly, taking a subsequence so that the $|\nabla u_j|$’s are dominated) we obtain $\| S(u_j) - S(u) \|_Y \to 0$, so $S$ is indeed continuous as a map $X \to Y$. However, (P₆) also implies the stability with respect to weak convergence, as demonstrated by the following result.

**Corollary 3.2.** Suppose that (H) is satisfied. Then, for any $\Lambda > 0$ there exists $C_\Lambda > 0$ such that, for any $u_0, u_1 \in X$ satisfying $\| u_0 \|_{L^\infty(\Omega)} \leq \Lambda$, $\| u_1 \|_{L^\infty(\Omega)} \leq \Lambda$, there holds

$$\int_{\mathbb{R}^m} F_\Omega(S_y(u_1) - S_y(u_0)) \, dy \leq C_\Lambda \| u_0 - u_1 \|_{L^k(\Omega)} \left( \| \nabla u_0 \|_{L^{k-1}(\Omega)}^{k-1} + \| \nabla u_1 \|_{L^{k-1}(\Omega)}^{k-1} \right).$$

In particular, if a sequence $(u_j)_{j \in \mathbb{N}}$ converges $W^{1,k}$-weakly to $u$ and $\sup_j \| u_j \|_{L^\infty(\Omega)} < +\infty$, then $\| S(u_j) - S(u) \|_Y \to 0$.

Corollary 3.2 follows immediately from (H) by applying the Hölder inequality at the right-hand side. This result is similar to [10] Theorem 1), which implies the continuity of the Jacobian determinant with respect to the weak topology of $W^{1,k}(\Omega, \mathbb{R}^k)$, when $\Omega \subseteq \mathbb{R}^k$.

The uniqueness of the operator $S$, together with Lemma 2.3, implies the following property.

**Corollary 3.3.** Let $\Omega_1, \Omega_2$ be bounded, smooth domains in $\mathbb{R}^d$ with $\Omega_1 \subset \subset \Omega_2$, and let $S^{\Omega_1}, S^{\Omega_2}$ be the corresponding operators, given by Theorem 3.1. Let $\Psi: F_{d-k}(\Omega_2; \pi_{k-1}({\mathcal N})) \to F_{d-k}(\Omega_1; \pi_{k-1}({\mathcal N}))$ be the restriction map, given by Lemma 2.3. Then, for any $u \in X(\Omega_2)$ and a.e. $y \in \mathbb{R}^d$, there holds

$$S^{\Omega_1}_{y}(u|_{\Omega_1}) = \Psi(S^{\Omega_2}_{y}(u)).$$
Corollary 3.3 implies that the operator $S$ is local: if two maps $u_1, u_2 \in X(\Omega)$ coincide a.e. on a (not necessarily smooth) open subset $\omega \subseteq \Omega$, then $\text{spt}(S_y(u_2) - S_y(u_1)) \subseteq \overline{\Omega} \setminus \omega$ for a.e. $y \in \mathbb{R}^m$. If we had constructed $S$ as an operator with values in $\mathbb{P}_n(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$, then Corollary 3.3 would not hold, because the restriction $S_y^{bd}((u) \mathbb{L} \mathbb{T})$ need not be well-defined (see the discussion in Section 2.1).

We can, in the suitable sense, define "the trace of $S$" on the boundary of $\Omega$. More precisely, suppose that $d \geq k + 1$, consider the space $X^{bd} := (L^\infty \cap W^{1-1/k,k} (\partial \Omega, \mathbb{R}^m))$ and define a direct limit topology on it, in such a way that a sequence $(g_j)_{j \in \mathbb{N}}$ converges to $g$ in $X^{bd}$ if and only if $g_j \rightharpoonup g$ weakly in $W^{1-1/k,k}$ and $\sup_{j \in \mathbb{N}} \|g_j\|_{L^\infty(\partial \Omega)} < +\infty$. Let $Y^{bd} := L^1(\mathbb{R}^m, F_{d-k-1}(\partial \Omega; \pi_{k-1}(\mathcal{N})))$ be endowed with the norm

$$\|S\|_{Y^{bd}} := \int_{\mathbb{R}^m} \mathbb{F}(S_y) \, dy.$$ 

**Proposition 3.4.** There exists a sequentially continuous operator $S^{bd} : X^{bd} \to Y^{bd}$ with the following property: for any $g \in X^{bd}$, any open set $\Omega' \supset \Omega$, any $u \in (L^\infty \cap W^{1-1/k,k}) (\Omega', \mathbb{R}^m)$ such that $u = g$ on $\partial \Omega$ (in the sense of traces), and a.e. $y \in \mathbb{R}^m$, there holds $S_y^{bd}(g) = \partial(S_y(u) \mathbb{L} \Omega) = \partial(S_y(u) \mathbb{L} \mathbb{T})$.

Note that, for a.e. $y$, the restrictions $S_y(u) \mathbb{L} \Omega$, $S_y(u) \mathbb{L} \mathbb{T}$ are well-defined because $S_y(u)$ has finite mass, due to $(\overline{P}_4)$. The space $X^{bd}$ does not coincide with the image of $X$ under the trace operator, that is $\text{tr} (X) = (L^\infty \cap W^{1-1/(k-1),k-1}) (\partial \Omega, \mathbb{R}^m) \supsetneq X^{bd}$. In general, it is not possible to extend $S^{bd}$ to an operator $\text{tr} (X) \to Y^{bd}$ that is continuous with respect to the strong topology on $\text{tr} (X)$. In case $\mathcal{N} = \mathbb{S}^1$, $k = m = 2$, $\Omega$ is the unit ball in $\mathbb{R}^2$, if such an extension existed then $S_y^{bd}(g)$ would be defined for merely measurable maps $g: \mathbb{S}^2 \to \mathbb{S}^1$, and continuous with respect to strong $L^1$-convergence. But $C^\infty(\mathbb{S}^2, \mathbb{S}^1)$ is dense in $L^1(\mathbb{S}^2, \mathbb{S}^1)$ and $S_y^{bd}(g) = 0$ for any $g \in C^\infty(\mathbb{S}^2, \mathbb{S}^1)$ and a.e. $y \in \mathbb{R}^m$, so $S^{bd} = 0$. This is a contradiction, in view of $(\overline{H})$, as there are maps in $W^{1,1}(\mathbb{S}^2, \mathbb{S}^1) \subsetneq W^{1/2,2}(\mathbb{S}^2, \mathbb{S}^1)$ whose distributional Jacobian is non-zero.

Recall that two chains are said to be homologous (or cobordant) if they differ by a boundary. In case $u \in (L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$, the homology class of $S_y(u)$ is determined by the boundary conditions only. More precisely, we have the following

**Proposition 3.5.** For any $g \in X^{bd}$, any open set $\Omega' \supset \Omega$, any two maps $u_1$, $u_2 \in (L^\infty \cap W^{1,k})(\Omega', \mathbb{R}^m)$ with $u_1 = u_2 = g$ on $\partial \Omega$ (in the sense of traces) and a.e. $y_1, y_2 \in \mathbb{R}^m$ there exists a chain $R \in \mathbb{M}_{d-k+1}(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that

$$S_y(u_2) \mathbb{L} \overline{\Omega} - S_y(u_1) \mathbb{L} \overline{\Omega} = \partial R.$$ 

Let $\delta_0 := \text{dist}(\mathcal{N}, \mathcal{X})$. If, in addition, $g$ takes values in $\mathcal{N}$ then for a.e. $y_1, y_2 \in \mathbb{R}^m$ with $|y_1| < \delta_0$, $|y_2| < \delta_0$ there exists a chain $R \in \mathbb{M}_{d-k+1}(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that

$$S_y(u_2) \mathbb{L} \overline{\Omega} - S_y(u_1) \mathbb{L} \overline{\Omega} = \partial R.$$ 

As above, the previous result need not be true for $u_1 \in X$, $u_2 \in X$, because the restrictions $S_y(u_1) \mathbb{L} \overline{\Omega}$, $S_y(u_2) \mathbb{L} \overline{\Omega}$ may not be well-defined. However, when $u_1, u_2$ are merely in $X$ and have the same trace at the boundary it is possible to show that, a.e. $y$, there exists a chain $R$
of finite mass such that \( \text{spt}(\mathbf{S}_y(u_2) - \mathbf{S}_y(u_1) - \partial R) \subseteq \mathbb{R}^d \setminus \Omega \) (this follows by Proposition 3.10 below).

Finally, let us mention an additional property of \( \mathbf{S}_y(u) \), in case \( u \) is an \( \mathcal{N} \)-valued map.

**Proposition 3.6.** As above, let \( \delta_0 := \text{dist}(\mathcal{N}, \mathcal{X}) \). For any \( u \in W^{1,k-1}(\Omega, \mathcal{N}) \) and a.e. \( y_1, y_2 \in \mathbb{R}^m \) with \( |y_1| < \delta_0, |y_2| < \delta_0 \) there holds \( \mathbf{S}_{y_1}(u) = \mathbf{S}_{y_2}(u) \). Likewise, if \( g \in W^{1-1/k,k}(\partial \Omega, \mathcal{N}) \) then \( \mathbf{S}^{bd}_y(g) = \mathbf{S}^{bd}_y(g) \) for a.e. \( y_1, y_2 \) with \( |y_1| < \delta_0, |y_2| < \delta_0 \). Finally, if \( u \in W^{1,k}(\Omega, \mathcal{N}) \) then \( \mathbf{S}_y(u) = 0 \) for a.e. \( y \in \mathbb{R}^m \) with \( |y| < \delta_0 \).

In case \( u \) is \( \mathcal{N} \)-valued and \( |y| < \delta_0 \), the chain \( \mathbf{S}_y(u) \) actually agrees with the topological singular set as defined by Pakzad and Rivière in [66] (see Section 3.3).

### 3.2 The case of smooth maps

We first carry out the construction of \( \mathbf{S}_y(u) \) for a smooth map \( u \). In order to control the behaviour of \( u \) at the boundary, we assume that \( \Omega \) is compactly contained in a domain \( \Omega' \subseteq \mathbb{R}^d \), and we assume that \( u \) is smoothly defined on \( \Omega' \), with \( ||u||_{L^\infty(\Omega')} \leq \Lambda \). Throughout this section, we also tacitly assume that the condition \((\text{H})\) is satisfied.

**Construction of \( \mathbf{S}_y(u) \).** Recall from Lemma 2.3 that, as a consequence of \((\text{H})\), there exists a smooth retraction \( q: \mathbb{R}^m \setminus \mathcal{X} \to \mathcal{N} \), where \( \mathcal{X} \) is a smooth \((m-k)\)-complex. Let \( K \) be a \((m-k)\)-dimensional cell of \( \mathcal{X} \), and let \( \alpha \) be a smooth \((m-k)\)-vector field that defines an orientation on \( K \). For any \( x \in K \setminus \partial K \) and \( r > 0 \), we consider the \( k \)-dimensional disk \( D^k_r(x) := B^m_r(x) \cap (T_x K)^\perp \), with the orientation induced by \( \sigma \). We suppose that \( r \) is so small that \( D^k_r(x) \cap \mathcal{X} = D^k_r(x) \cap K = \{x\} \). Then, we denote by \( \gamma(K, \sigma) := [q, \partial D^k_r(x)] \in \pi_{k-1}(\mathcal{N}) \) the homotopy class of \( q \) restricted to \( \partial D^k_r(x) \cong S^{k-1} \). (One easily checks that \( \gamma(K, \sigma) \) does not depends on the choice of \( x \) and \( r \), but only on \( K \) and \( \sigma \).)

By applying Thom parametric transversality theorem (see e.g. [66] Theorem 2.7 p. 79]) to the map \( (x, y) \in \Omega' \times \mathbb{R}^m \mapsto u(x) - y \), which is smooth and has surjective differential at every point, we deduce that, for a.e. \( y \in \mathbb{R}^m \), the map \( u - y \) is transverse to all the cells of \( \mathcal{X} \). Therefore, for any \( j \)-cell \( K \) of \( \mathcal{X} \) with \( m - d \leq j \leq m - k \), the set \( (u - y)^{-1}(\mathcal{X}) \) is a smooth \((d - m + j)\)-submanifold of \( \Omega' \) with \( \text{d}(u - y)^{-1}(K) \subseteq (u - y)^{-1}(\mathcal{X}^{j-1}) \), while \( (u - y)^{-1}(K) = \emptyset \) if \( j < m - d \). We subdivide each \( (u - y)^{-1}(K) \) into \((d - m + j)\)-cells, in such a way to make \( (u - y)^{-1}(\mathcal{X}) \) a smooth, finite complex. Using Thom transversality theorem again, we see that, for a.e. \( y \in \mathbb{R}^m \), the intersection of any cell of \( (u - y)^{-1}(\mathcal{X}) \) with \( \partial \Omega \) is a smooth manifold. Therefore, up to further subdivision, we can assume that each cell of \( (u - y)^{-1}(\mathcal{X}) \) is either \( \subset \Omega \) or \( \subset \Omega' \setminus \Omega \).

Let \( H \) be a \((d-k)\)-cell of \( (u - y)^{-1}(\mathcal{X}) \), oriented by a smooth \((d-k)\)-unit vector field \( \tau \). By construction, the image \( (u - y)(H) \) is contained in a \((m-k)\)-cell \( K \) of \( \mathcal{X} \). Let \( \sigma \) be a smooth \((m-k)\)-vector field associated with the orientation of \( K \). We define

\[
\epsilon(H, \tau, K, \sigma) := \begin{cases} 
1 & \text{if } (u - y)_*(\tau)^\perp \wedge \sigma \text{ is a positive } m\text{-vector field in } \mathbb{R}^m \\
-1 & \text{otherwise.}
\end{cases}
\]
Here $(\star \tau)^#$ denotes the $k$-vector field naturally associated with the Hodge dual of $\tau$, which induces an orientation on the normal bundle to $H$, and $(u-y)_*(\star \tau)^#$ denotes its push-forward through the differential of $u-y$. Then, we define

$$
S_y(u) := \sum_{(H, \tau)} \epsilon(H, \tau, K, \sigma) \gamma(K, \sigma)[H, \tau],
$$

where the sum is taken over all oriented $(d-k)$-cells $(H, \tau)$ of $(u-y)^{-1}(U)$. Then, $S_y(u)$ is a smooth $(m-k)$-chain with coefficients in $\pi_{k-1}(M)$. Each term of the sum in (3.3) is invariant under the changes of orientation $\tau \mapsto -\tau$ and $\sigma \mapsto -\sigma$. From now on, we will omit the $\tau$'s and $\sigma$'s in the notation.

**Lemma 3.7.** Let $\Sigma$ be a smoothly embedded $k$-disk that intersects transversely a $(d-k)$-cell $H$ of $(u-y)^{-1}(U)$ at a point $x \in H \setminus \partial H$, and suppose that $\Sigma \cap (u-y)^{-1}(U) = \Sigma \cap H = \{x\}$. Let $K$ be the $(m-k)$-cell of $U$ that contains $(u-y)(H)$. Then, we have

$$
\epsilon(H, K) \gamma(K) = [g(u-y)_*(\partial \Sigma)].
$$

**Proof.** Assume, for simplicity of notation only, that $y = 0$ and $u(x) = 0$. Let $D_r^k := B_r^d(x) \cap T_x \Sigma$, for $0 < r < \text{dist}(x, \partial \Sigma)$. The sphere $\partial \Sigma$ is homotopic to $\partial D_r^k$ (one contracts $\partial \Sigma$ towards $x$, then project it on the tangent space). Moreover, if $r$ is small enough, $u|_{\partial D_r^k}$ is homotopic to $\partial u_x|_{\partial D_r^k}$ because $\|u - du_x\|_{L^\infty(\partial D_r^k)} \to 0$ as $r \to 0$. Therefore, we have

$$
[(\partial \circ u)_*(\partial \Sigma)] = [(\partial \circ u)_*(\partial D_r^k(x))] = [(\partial \circ du_x)_*(\partial D_r^k(x))].
$$

Now, the transversality assumption yields $\partial u_x(T_x \Sigma) + T_0 K = \mathbb{R}^m$ and hence, by a dimension argument, $\partial u_x$ restricts to an isomorphism of $T_x \Sigma$ onto its image. Thus, we have

$$
[(\partial \circ du_x)_*(\partial D_r^k)] = \text{sign det}(d u_x|_{T_x \Sigma}) \left[\partial_*(d u_x|_{\partial D_r^k})\right] = \epsilon(H, K) \gamma(K).
$$

Combining this identity with (3.3), the lemma follows. \hfill \square

**Lemma 3.8.** $S_y(u)$ is a relative cycle, that is, $\partial(S_y(u)) \cdot \Omega = 0$.

**Proof.** By construction, $\partial(S_y(u))$ is supported on the $(d-k-1)$-skeleton of $(u-y)^{-1}(U)$. Let $H$ be a $(d-k-1)$-cell of $(u-y)^{-1}(U)$ that is contained in $\overline{U}$, and let $H_1, \ldots, H_q$ be the $(d-k)$-cells of $(u-y)^{-1}(U)$ that are adjacent to $H$. By composing with a diffeomorphism, we can assume WLOG that $H, H_1, \ldots, H_q$ are affine polyhedra. Moreover, since $S_y(u)$ is independent on the choice of the orientations on the cells, we can choose the orientation $\tau_j$ of $H_j$ in such a way that $\tau_j |_{H_j}$ agrees with the orientation of $H$. Take $x \in H \setminus \partial H$ and a radius $r > 0$ so small that, setting $D_{r}^{k+1}(x) := B_r^d(x) \cap H^\perp$, we have $D_{r}^{k+1}(x) \cap \partial H_j = D_{r}^{k+1}(x) \cap H = \{x\}$, for any $j \in \{1, \ldots, q\}$. Then $\partial D_{r}^{k+1}(x)$ intersects each $H_j$ at a single point, which we call $x_j$, and $g(u-y)$ restricts to a continuous map $\partial D_{r}^{k+1}(x) \setminus \{x_j\}_{j=1}^q \to \mathcal{N}$. Take a smooth $k$-disk $\Sigma \subseteq \partial D_{r}^{k+1}(x)$ such that $x_j \in \Sigma \setminus \partial \Sigma$ for any $j$. We endow $\Sigma$ with the orientation induced by $\partial D_{r}^{k+1}(x)$. Finally, for each $j$ we take a small $k$-disk $\Sigma_j \subseteq \Sigma$, in such a way that $x_j \in \Sigma_j \setminus \partial \Sigma_j$, $23$
and the $\Sigma_j$’s are pairwise disjoint. By Lemma 3.7, the multiplicity of $S_y(u)$ at $H_j$ is equal to $[\varrho(u - y)_*(\partial \Sigma_j)]$. Therefore, with our choice of the orientation, we have

$$\text{multiplicity of } \partial S_y(u) \text{ at } H = \sum_{j=1}^{k} \text{(multiplicity of } S_y(u) \text{ at } H_j)$$
$$= \sum_{j=1}^{k} [\varrho(u - y)_*(\partial \Sigma_j)] = [\varrho(u - y)_*(\partial \Sigma)].$$

On the other hand, $\varrho(u - y)|\partial \Sigma$ is null-homotopic, because $\varrho(u - y)$ is continuous on the set $\partial D_{r+1}(x) \setminus \Sigma$, which is diffeomorphic to a $k$-disk. Thus, we have $\partial(S_y(u)) \cup H = 0$. □

In the rest of this section, we check that $S_y(u)$ satisfies $[P_1], [P_5]$ in case $u$ is smooth. The extension to the Sobolev case is left to Section 3.3.

**$S_y(u)$ satisfies $[P_1]$.** By applying the deformation theorem [28, Theorem 7.3] on a grid of sufficiently small size, we can write

$$(3.5) \quad R = \sum_{\alpha=1}^{q} \lambda_\alpha [K_\alpha] + A + \partial B,$$

where $\lambda_\alpha \in \mathbb{Z}$, the $K_\alpha$’s are affine $k$-polyhedra, $A$ is a $k$-chain with finite mass and $B$ is a $(k+1)$-cell with finite mass, such that

$$(3.6) \quad \text{spt}(S_y(u)) \cap \text{spt}(P) = \emptyset, \quad \text{spt}(\partial S_y(u)) \cap \text{spt}(P) = \emptyset,$$

$$(3.7) \quad \text{spt}(S_y(u)) \cap \text{spt}(A) = \emptyset, \quad \text{spt}(\partial S_y(u)) \cap \text{spt}(B) = \emptyset.$$

By the transversality theorem, we can also assume WLOG that $\text{spt}(S_y(u)) \cap \partial K_\alpha = \emptyset$ and $K_\alpha$ is transverse to $\text{spt}(S_y(u))$ for any $\alpha$. Thanks to Lemma 2.8 and (3.7), we have $\llbracket S_y(u), A \rrbracket = \llbracket S_y(u), \partial B \rrbracket = 0$. Then, by bilinearity of the intersection product, we obtain

$$\llbracket S_y(u), R \rrbracket = \sum_{\alpha=1}^{q} \lambda_\alpha \llbracket S_y(u), [K_\alpha] \rrbracket.$$

Due to (3.7), $\varrho(u - y)$ is well-defined and continuous in a neighbourhood of $A$. Therefore, taking the homology classes in (3.5), we deduce

$$[\varrho(u - y)_*(\partial R)] = \sum_{\alpha=1}^{q} \lambda_\alpha [\varrho(u - y)_*[\partial K_\alpha]].$$

Thus, it suffices to show that $\llbracket S_y(u), [K_\alpha] \rrbracket = [\varrho(u - y)_*[\partial K_\alpha]]$. Because we assumed that $K_\alpha$ is transverse to $\text{spt}(S_y(u))$, their intersection is a finite set. Using again additivity on both sides, we reduce to the case $\#(\text{spt}(S_y(u)) \cap K_\alpha) = 1$, and then $[P_1]$ follows by Lemma 3.7. □
\( S_y(u) \) satisfies (P2). We have \( \text{spt}(S_y(u)) = (u - y)^{-1}(\mathcal{X}) \), so \( \varrho(u - y) \) is well defined and smooth away from \( \text{spt}(S_y(u)) \). We need to check that \( \varrho(u - y) \in W^{1,k-1}(\Omega, \mathbb{R}^m) \). Recall that, by assumption, \( \|u\|_{L^\infty(\Omega)} \leq \Lambda \). Thus, if

\[
(3.8) \quad |y| > M := \Lambda + \sup_{z \in \mathcal{X}} |z|,
\]

then \( (u - y)^{-1}(\mathcal{X}) = \emptyset \) and \( \varrho(u - y) \) is smooth on \( \overline{\Omega} \). Thus, we only need to consider the case \( y \in B_{M}^\circ \). We can now apply the arguments in [33, Lemma 2.3] or [32, Lemma 6.2], which we recall here for the convenience of the reader. In fact, we will prove a slightly stronger statement than (P2) because it will be useful later on.

**Lemma 3.9.** For any \( v \in X := (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m) \), let \( \Phi(v) \colon y \in \mathbb{R}^m \mapsto \varrho(v - y) \). Then, \( \Phi \) is a well-defined and continuous operator

\[
\Phi \colon X \to L^1_{\text{loc}}(\mathbb{R}^m, W^{1,k-1}(\Omega, \mathcal{N}))
\]

Moreover, for any positive \( M, \Lambda \) and any \( v \in X \) such that \( \|v\|_{L^\infty(\Omega)} \leq \Lambda \), there holds

\[
(3.9) \quad \int_{B_M^m} \|\nabla(\varrho(v - y))\|_{k-1}^{k-1}(\Omega) \, dy \leq C_\Lambda \|\nabla v\|_{k-1}^{k-1}(\Omega),
\]

where \( C_\Lambda \) is a positive constant that only depends on \( M, \Lambda, k, \mathcal{N} \) and \( \varrho \).

**Proof.** We first remark the following useful fact, which is the essence of the proof of [33, Lemma 2.3]: for any positive numbers \( M, \Lambda \), any \( v \in L^\infty(\Omega, \mathbb{R}^m) \) with \( \|v\|_{L^\infty(\Omega)} \leq \Lambda \), any measurable \( w \colon \Omega \to [0, +\infty) \) and any Borel function \( f \colon \mathbb{R}^m \to [0, +\infty) \), there holds

\[
(3.10) \quad \int_{B_M^m} \left( \int_{\Omega} w(x)f(v(x) - y) \, dx \right) \, dy \leq \int_{\Omega} w(x) \, dx \int_{B_{M+\Lambda}^m} f(z) \, dz.
\]

This follows by applying Fubini theorem, then making the change of variable \( z := v(x) - y \) in the integral with respect to \( y \). Another useful fact we will use in the proof is that

\[
(3.11) \quad \nabla \varrho \in L^{k-1}_{\text{loc}}(\mathbb{R}^m, \mathbb{R}^{m \times m}).
\]

Indeed, \( \|\nabla \varrho\| \leq C \text{dist}(\cdot, \mathcal{X})^{-1} \) by Lemma 2.9 and \( \text{dist}(\cdot, \mathcal{X})^{-k+1} \) is locally integrable on \( \mathbb{R}^m \) because \( \mathcal{X} \) is, up to a bounded change of metric in \( \mathbb{R}^m \), a finite union of simplices of codimension \( k \).

Let us now check that \( \Phi \) is well-defined. For any \( v \in X \), the set

\[
N := \{(x, y) \in \Omega \times \mathbb{R}^m : v(x) - y \in \mathcal{X}\}
\]

is measurable and \( \mathcal{H}^{d+m}(N) = 0 \), because each slice \( N \cap \{x \times \mathbb{R}^m\} = v(x) - \mathcal{X} \) has dimension \( m - k \). By Fubini theorem, it follows that \( \mathcal{H}^{d+m}(N \cap (\Omega \times \{y\})) = 0 \) for a.e. \( y \in \mathbb{R}^m \), so \( \varrho(u - y) \) is well-defined for a.e. \( y \in \mathbb{R}^m \), and belongs to \( L^1(\Omega, \mathcal{N}) \). By the chain rule, for a.e. \( (x, y) \in \Omega \times \mathbb{R}^m \) we have

\[
|\nabla(\varrho(v(x) - y))| = |(\nabla \varrho)(v(x) - y)||\nabla v(x)|
\]

25
and thus (3.9) follows by applying (3.10) with \( f = |\nabla g|^{k-1} \), \( w = |\nabla v|^{k-1} \) and using (3.11).

It only remains to check the continuity of \( \Phi \). Let \((v_j)_{j \in \mathbb{N}}\) be a sequence such that \( v_j \to v \) in \( X \) as \( j \to +\infty \), and let \( \Lambda > 0 \) be such that \( |v_j|_{L^\infty(\Omega)} \leq \Lambda \) for any \( j \in \mathbb{N} \). Up to extraction of a subsequence, we assume that \( v_j \to v \) a.e. Let \( M > 0 \) be fixed. By Fubini theorem and a change of variable as in (3.10), we obtain

\[
\int_{B_{M}^m} \|\phi(v_j - y) - \phi(v - y)\|_{L^{k-1}(\Omega)}^{k-1} dy \leq \int_{\Omega} \int_{B_{M+\Lambda}^m} \left( |\phi(z + \tilde{v}_j(x)) - \phi(z)|^{k-1} dy \right) dx,
\]

where \( \tilde{v}_j := v_j - v \). Since \( \phi(z + \tilde{v}_j(x)) \to \phi(z) \) for any \( z \in \mathbb{R}^m \setminus \mathcal{J} \) and a.e. \( x \in \Omega \), Lebesgue’s dominated convergence theorem implies that the right hand side converges to zero as \( j \to +\infty \).

Now, for fixed \( \epsilon > 0 \), let \( \varphi \in C^\infty(\mathbb{R}^m, \mathbb{R}^{m \times m}) \) be such that \( \|\nabla \varphi\|_{L^{k-1}(B_{M+\Lambda}^m)} \leq \epsilon \). The chain rule implies

\[
\int_{B_{M}^m} \|\nabla (\phi(v_j - y) - \phi(v - y))\|_{L^{k-1}(\Omega)}^{k-1} dy \leq 4^{k-2}(I_1 + I_2 + I_3 + I_4),
\]

where

\[
I_1 := \int_{B_{M}^m} \left( \int_{\Omega} |(\nabla g)(v_j - y)|^{k-1} |\nabla v_j - \nabla v|^{k-1} d\mathcal{H}^d \right) dy,
\]

\[
I_2 := \int_{B_{M}^m} \left( \int_{\Omega} |\nabla v|^{k-1} |(\nabla \phi)(v_j - y) - \varphi(v_j - y)|^{k-1} d\mathcal{H}^d \right) dy,
\]

\[
I_3 := \int_{B_{M}^m} \left( \int_{\Omega} |\nabla v|^{k-1} |\varphi(v_j - y) - \varphi(v - y)|^{k-1} d\mathcal{H}^d \right) dy,
\]

\[
I_4 := \int_{B_{M}^m} \left( \int_{\Omega} |\nabla v|^{k-1} |\varphi(v - y) - \nabla \phi(v - y)|^{k-1} d\mathcal{H}^d \right) dy.
\]

We apply (3.10) to each of this integrals. For the first one, we obtain

\[
I_1 \leq \|\nabla v_j - \nabla v\|_{L^{k-1}(\Omega)}^{k-1} \int_{B_{M+\Lambda}^m} |\nabla \phi(z)|^{k-1} dz,
\]

and the integral with respect to \( z \) in the right hand side is finite, due to (3.11). As for \( I_2 \), we have

\[
I_2 \leq \|\nabla v\|_{L^{k-1}(\Omega)}^{k-1} \|\nabla \varphi\|_{L^{k-1}(B_{M+\Lambda}^m)}^{k-1} \leq \epsilon \|\nabla v\|_{L^{k-1}(\Omega)}^{k-1},
\]

and the same holds for \( I_3 \). Finally, for \( I_4 \) we get

\[
I_3 \leq \int_{\Omega} |\nabla v(x)|^{k-1} \left( \int_{B_{M+\Lambda}^m} |\varphi(z + \tilde{v}_j(x)) - \varphi(z)|^{k-1} dz \right) dx
\]

where \( \tilde{v}_j := v_j - v \), and again we can apply Lebesgue’s dominated convergence theorem to show that the right hand side tends to zero as \( j \to +\infty \). Putting all together, we deduce

\[
\limsup_{j \to +\infty} \int_{B_{M}^m} \|\phi(v_j - y) - \phi(v - y)\|_{L^{k-1}(\Omega)}^{k-1} dy \leq 4^{k-3/2} \epsilon \|\nabla v\|_{L^{k-1}(\Omega)}^{k-1}
\]

for arbitrary \( \epsilon, M \), and hence the lemma follows. \( \square \)

26
\( S_y(u) \) satisfies (P4). Pick a positive constant \( C \) such that \( |\gamma(K)| \leq C \) for any \((m - k)\)-cell \( K \) of \(\mathcal{K} \). By the definition (3.3) of \( S_y(u) \), we have

\[
\mathcal{M}(S_y(u) \mathbf{L} \Omega) \leq C \sum_K \mathcal{H}^{d-k} ((u - y)^{-1}(K) \cap \Omega),
\]

the sum being taken over all \((m - k)\)-cells of \(\mathcal{K} \), and \( S_y(u) = 0 \) if \( |y| > M \) where \( M \) is defined in (3.5). Since \(\mathcal{K} \) only contains a finite number of cells, each of which is bilipschitz equivalent to an affine \((m - k)\)-polyedron, it suffices to show

\[
\int_{[-M, M]^m} \mathcal{H}^{d-k} ((u - y)^{-1}(V) \cap \Omega) \, dy \leq C \|\nabla u\|_{L^k(\Omega)}^k,
\]

where \( V \) is an affine \((m - k)\)-subspace of \(\mathbb{R}^m \). By composing with an isometry, we can assume WLOG that \( V = \{y \in \mathbb{R}^m : y_1 = \ldots = y_k = 0\} \). We denote the variable \( y = (z, z') \in V^\perp \times V \) and let \( p_\perp : \mathbb{R}^m \to V^\perp \) be the orthogonal projection onto \( V^\perp \). Then, (3.12) can be rewritten as

\[
\int_{[-M, M]^k \times [-M, M]^{m-k}} \mathcal{H}^{d-k} ((p_\perp \circ u)^{-1}(z) \cap \Omega) \, dz \, dz' \leq C \|\nabla u\|_{L^k(\Omega)}^k
\]

and this inequality follows from the coarea formula, applied to the smooth function \( p_\perp \circ u : \Omega \to V^\perp \simeq \mathbb{R}^k \). This concludes the proof of (P4).

\( S_y(u) \) satisfies (P3) and (P5). Properties (P3) and (P5) follow at once from the result below.

**Proposition 3.10.** Let \( \pi : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \) denote the canonical projection. Let \( u_0, u_1 \) be two smooth maps \( \Pi' \to \mathbb{R}^m \) and let \( u : [0, 1] \times \Pi' \to \mathbb{R}^m \) be given by \( u(t, x) := (1-t)u_0(x) + tu_1(x) \). For a.e. \( y \in \mathbb{R}^m \) there holds

\[
\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0) = \partial (\pi_* \mathbf{S}_y(u)) \quad \text{in} \ \Pi' \quad \text{if, in addition,} \quad u_0 = u_1 \text{ on } \Omega' \setminus \Omega.
\]

If, in addition, \( u_0 = u_1 \) on \( \Omega' \setminus \Omega \), then \( \pi_* \mathbf{S}_y(u) \) is supported in \( \Omega' \) for a.e. \( y \in \mathbb{R}^m \). Finally, for any positive \( \Lambda \) there exists a constant \( C_\Lambda \) such that, if \( u_0, u_1 \) satisfy \( \|u_0\|_{L^\infty(\Omega')} \leq \Lambda \), \( \|u_1\|_{L^\infty(\Omega')} \leq \Lambda \), then

\[
\int_{\mathbb{R}^k} \mathcal{M}(\pi_* \mathbf{S}_y(u) \mathbf{L} \Omega) \, dy \leq C \int_{\Omega} |u_1 - u_0| \left( |\nabla u_0|^{k-1} + |\nabla u_1|^{k-1} \right).
\]

Once the proposition is proved, in order to show (P3) we apply Proposition 3.10 with \( u_0 \) identically equal to 0, and notice that \( \mathbf{S}_y(0) = 0 \) for any \( y \in \mathbb{R}^m \setminus \mathcal{K} \). As for Property (P5), (3.13) and Lemma 2.2 imply \( \mathcal{F}_\Omega(\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0)) \leq \mathcal{M}(\pi_* \mathbf{S}_y(u) \mathbf{L} \Omega) \), so (P5) follows from (3.14).

The proof of Proposition 3.10 is in some sense a refinement of (P4). It will be convenient to work in the setting of differential forms and currents. We follow here the notation of [1, Section 7.4]. Given a smooth map \( v : [0, 1] \times \Omega' \to \mathbb{R}^k \), we define the Jacobian \( Jv \) as the pull-back of the standard volume form on \( \mathbb{R}^k \) through \( v \), i.e. \( Jv := v^{*}(dy^1 \wedge \ldots \wedge dy^k) \). If we denote by \( (x^1, \ldots, x^d) \) the coordinates on \( \Omega' \) and by \( x^0 = t \) the coordinate in \([0, 1] \), then we can write

\[
Jv = \sum_{\alpha \in I(k, d)} \det(\partial_{x^\alpha} v) \, dx^\alpha,
\]

27
where \( I(k, d) \) is the set of multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) such that \( 0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq d \), and
\[
\partial_\alpha v := (\partial_{\alpha_1} v, \ldots, \partial_{\alpha_k} v), \quad dx^\alpha := dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}.
\]

For any regular value \( y \in \mathbb{R}^k \) of \( v \) and any \( x \in v^{-1}(y) \), the Hodge dual \( *Jv(x) \) is a simple \((d - k + 1)\)-vector which spans \( T_xv^{-1}(y) \). By (3.15), we have \( |*Jv|^2 = |Jv|^2 = \det((\nabla v)(\nabla v)^T) \).

We define \( N_y(v) \) as the rectifiable current (in the ambient space \([0, 1] \times \Omega' \)) supported by \( v^{-1}(y) \), with orientation given by \( *Jv/|*Jv| \) and constant multiplicity 1. \( N_y(v) \) can be identified with an element of \( M_{d-k+1}([0, 1] \times \Omega'; \mathbb{Z}) \). The mass of \( N_y(v) \), whether it be regarded as a current or as a flat chain with coefficients in \( \mathbb{Z} \), coincides with the Hausdorff measure of the set \( N_y(v) \), because \( N_y(v) \) is rectifiable.

**Lemma 3.11.** Let \( v_0, v_1 \) be two smooth maps \( \Omega' \to \mathbb{R}^k \), and let \( v: [0, 1] \times \Omega' \to \mathbb{R}^k \) be defined by \( v(t, x) := (1 - t)v_0(x) + tv_1(x) \). Let \( K \subseteq [0, 1] \times \Omega' \) be a Borel set. Then, there holds
\[
\int K \mathbb{M}(\pi_*(N_y(v) \mathcal{L} K)) \, dy \leq C \int_{\pi(K)} |v_1 - v_0| \left( |\nabla v_0|^{k-1} + |\nabla v_1|^{k-1} \right)
\]

**Proof.** By definition of the mass of a current, we can write
\[
\mathbb{M}(\pi_*(N_y(v) \mathcal{L} K)) = \sup_\omega \langle N_y(v) \mathcal{L} K, \pi^* \omega \rangle,
\]
where the sup is taken over all smooth \((d - k + 1)\)-forms \( \omega \) supported in \( \Omega' \), such that the comass norm \( ||\omega(x)|| \leq 1 \) for any \( x \in \Omega' \). (This condition means \( \langle \omega(x), \xi \rangle \leq 1 \) for any unit, simple \((d - k + 1)\)-vector \( \xi \) and any \( x \in \Omega' \)). Any such form \( \omega \) can be written as
\[
\omega = \sum_{\beta \in I(d-k+1, d); \beta_1 > 0} \omega_\beta \, dx^\beta
\]
for some functions \( \omega_\beta \in C_c^\infty(\Omega') \) which satisfy
\[
\sum_\beta \omega_\beta^2(x) = ||\omega(x)||^2 \leq C ||\omega(x)||^2 \leq C \quad \text{for any } x \in \Omega'
\]
where \( C = C(d, k) \) is a positive constant. Using the properties of \( * \) and (3.15), we compute
\[
\langle N_y(v) \mathcal{L} K, \pi^* \omega \rangle = \int_{v^{-1}(y) \cap K} \langle \pi^* \omega, \frac{\star Jv}{|\star Jv|} \rangle \, d\mathcal{H}^{d-k+1} \]
\[
= (-1)^{k(d-k+1)} \int_{v^{-1}(y) \cap K} \frac{\star (\pi^* \omega \wedge Jv)}{|\star Jv|} \, d\mathcal{H}^{d-k+1} \]
\[
\leq \sum_{\alpha \in I(k, d); \alpha_1 = 0} \int_{v^{-1}(y) \cap K} |\omega_{\alpha}| \frac{|\det(\partial_\alpha v)|}{|\star Jv|} \, d\mathcal{H}^{d-k+1},
\]
where \( \bar{\alpha} \) denotes the unique element of \( I(d - k + 1, d) \) that complements \( \alpha \). Recalling the definition of \( v \), for any \( \alpha \in I(k, d) \) such that \( \alpha_1 = 0 \) we obtain
\[
|\det(\partial_\alpha v)| \leq |\partial_\alpha v| |\nabla v|^k \leq |v_0 - v_1| \left( |\nabla v_0| + |\nabla v_1| \right)^{k-1}.
\]
Then, using (3.16) and (3.17) as well, we have
\[
\mathcal{M}(\pi_*(N_y(v) \llcorner K)) \leq C \int_{\nu^{-1}(y) \cap K} |v_0 - v_1| \frac{(|\nabla v_0| + |\nabla v_1|)^{k-1}}{|* Jv|} \, d\mathcal{H}^{d-k+1}.
\]
By integrating this inequality with respect to \( y \in \mathbb{R}^k \), and applying the coarea formula, we conclude that
\[
\int_{\mathbb{R}^k} \mathcal{M}(\pi_*(N_y(v) \llcorner K)) \, dy \leq C \int_K |v_0 - v_1| (|\nabla v_0| + |\nabla v_1|)^{k-1} \, d\mathcal{H}^{d+1}
\]
whence the lemma follows.

\[\square\]

**Proof of Proposition 3.10.** We first prove (3.13). Pick \( y \in \mathbb{R}^k \) such that \( u_0 - y, u_1 - y \) and \( u - y \), together with their restrictions to \( \partial \Omega \), are transverse to all the cells of \( \mathcal{K} \). Then, up to subdivision, we can assume that all the cells of \((u - y)^{-1}(\mathcal{K})\) are contained either in \( \{0, 1\} \times \Omega \) or in \((0, 1) \times \Omega^\prime\). Then, (3.13) follows by the same argument of Lemma 3.8. In case \( u_0 = u_1 \) out of \( \Omega \), we have \( u(t, x) = u_0(x) \) for any \((t, x) \in [0, 1] \times (\Omega^\prime \setminus \overline{\Omega})\), so
\[
S_y(u) \llcorner ([0, 1] \times (\Omega^\prime \setminus \overline{\Omega})) = [[0, 1]] \times S_y(u_0) \llcorner (\Omega^\prime \setminus \overline{\Omega})
\]
and
\[
\pi_* S_y(u) \llcorner (\Omega^\prime \setminus \overline{\Omega}) = \pi_*(S_y(u) \llcorner ([0, 1] \times (\Omega^\prime \setminus \overline{\Omega}))) = \pi_*([0, 1]) \times S_y(u_0) \llcorner (\Omega^\prime \setminus \overline{\Omega}) = 0.
\]
Thus, \( \pi_* S_y(u) \) is supported in \( \overline{\Omega} \).

We now prove (3.14). Fix \( \Lambda > 0 \) such that \( \|u_0\|_{L^\infty(\Omega^\prime)} \leq \Lambda \) and \( \|u_1\|_{L^\infty(\Omega^\prime)} \leq \Lambda \). Then, we have \( \|u\|_{L^\infty([0, 1] \times \Omega^\prime)} \leq \Lambda \) and so \( S_y(u_0) = S_y(u_1) = 0 \) whenever
\[
|y| > M := \Lambda + \sup_{z \in \mathcal{K}} |z|.
\]
If we choose a constant \( C \) such that \( |\gamma(K)| \leq C \) for any \((m - k)\)-cell \( K \) of \( \mathcal{K} \), then the definition (3.3) of \( S_y(u) \) implies
\[
(3.18) \quad \mathcal{M}(\pi_* S_y(u) \llcorner \overline{\Omega}) \leq C \sum_K \mathcal{M}(\pi_* [(u - y)^{-1}(K)] \llcorner \overline{\Omega}).
\]
Fix a \((m - k)\)-cell \( K \) of \( \mathcal{K} \). By composing with a diffeomorphism, we can assume WLOG that \( K \) is an affine polyhedron contained in the \((m - k)\)-plane \( V := \{y \in \mathbb{R}^m: y_1 = \ldots = y_k = 0\} \). We denote the variable in \( \mathbb{R}^m \) by \( y = (z, z') \in V^\perp \times V \), and we let \( p, p_\perp \) be the orthogonal projections onto \( V, V^\perp \) respectively. For a suitable choice of the orientation of \( K \), we have
\[
[(u - y)^{-1}(K)] = N_{p_\perp(y)}(p_\perp \circ u) \llcorner (p \circ u - p(y))^{-1}(K).
\]
Thus, for any \( z' \in V \), by applying Lemma 3.11 to \( v := p_\perp \circ u \) and
\[
K_{z'} := (p \circ u - z')^{-1}(K) \cap ([0, 1] \times \overline{\Omega}),
\]

we obtain
\[
\int_{V^+} M \left( \pi_* \left( (u - (z, z'))^{-1}(K) \right) L \right) \, dz = \int_{V^+} M (\pi_*(N_z(v) L, K_z)) \, dz \leq C \int_{\Omega} |u_0 - u_1| \left( |\nabla u_0|^{k-1} + |\nabla u_1|^{k-1} \right).
\]
By integrating with respect to \( z' \in V \cap B^m_M \), summing over \( K \), and using (3.18), we obtain
\[
\int_{V^+ \times (V \cap B^m_M)} M(\pi_* S_g(u) L) \, dy \leq C \int_{\Omega} |u_0 - u_1| \left( |\nabla u_0|^{k-1} + |\nabla u_1|^{k-1} \right)
\]
for some constant \( C \) depending on \( M \) (hence on \( \Lambda \)) and on \( \mathcal{Y} \). Now, reminding that \( S_g(u) = 0 \) if \( |y| > M \), the proposition follows. \( \square \)

### 3.3 The case of Sobolev maps

In the previous section, we have defined \( S_g(u) \) in case \( u \) is smooth; we now have to extend the definition to the case \( u \) belongs to a suitable Sobolev space, and of course this is accomplished by a density argument. We will then provide the proof of the main theorem, Theorem 3.1, and of Proposition 3.1.

Since \( \Omega \) is assumed to be bounded and smooth, there exist a larger domain \( \Omega' \supset \supset \Omega \) and a linear, continuous operator \( E: X := (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m) \to (L^\infty \cap W^{1,k-1})(\Omega', \mathbb{R}^m) \) that satisfies \( E u |_{\Omega} = u \) and
\[
(3.19) \quad \int_{\Omega'} |E u_1 - E u_0| \left( |\nabla (E u_0)|^{k-1} + |\nabla (E u_1)|^{k-1} \right) \leq C \int_{\Omega} |u_1 - u_0| \left( |\nabla u_0|^{k-1} + |\nabla u_1|^{k-1} \right)
\]
for any \( u \in X \) and some constant \( C \) that only depends on \( \Omega \). Such an operator can be constructed, e.g., by standard reflection about the boundary \( \partial \Omega \).

Let \( \Psi: F_{d-k}(\Omega'; \pi_{k-1}(\mathcal{N})) \to F_{d-k}(\Omega; \pi_{k-1}(\mathcal{N})) \) be the restriction map given by Lemma 2.9. For any \( u \in E^{-1} C^\infty(\Omega', \mathbb{R}^m) \) and any \( y \in \mathbb{R}^m \), with a slight abuse of notation, we let \( S_g(u) := \Psi(S_g(E u)) = S_g(E u) L \). By Proposition 3.10 and (3.19), this defines a uniformly continuous operator \( S: E^{-1} C^\infty(\Omega', \mathbb{R}^m) \to Y \), if \( E^{-1} C^\infty(\Omega', \mathbb{R}^m) \) is given the topology of a subspace of \( X \). Since \( E^{-1} C^\infty(\Omega', \mathbb{R}^m) \) is dense in \( X \), we can extend \( S \) to a continuous operator \( X \to Y \), still denoted \( S \), that satisfies \( (P_5) \). Now, before completing the proof of Theorem 3.1 we state a useful lemma.

**Lemma 3.12.** Let \( \delta_0 := \text{dist}(\mathcal{N}, \mathcal{X}) \). For any smooth map \( u: \Omega' \to \mathbb{R}^m \) and a.e. \( y, y' \in \mathbb{R}^m \) with \( |y'| < \delta_0 \), there holds \( S_g(u) = S_g(y) (\varrho(u - y)) \).

For the sake of convenience of exposition, we leave the proof of Lemma 3.12 to Section 3.4. By the continuity of \( S \), and because \( \varrho(u_j - y) \to \varrho(u - y) \) in \( W^{1,k-1} \) for a.e. \( y \in \mathbb{R}^m \) if \( u_j \to u \) in \( X \) (Lemma 3.12), from Lemma 3.12 we derive

**Lemma 3.13.** As above, let \( \delta_0 := \text{dist}(\mathcal{N}, \mathcal{X}) \). For any \( u \in X \) and a.e. \( y, y' \in \mathbb{R}^m \) with \( |y'| < \delta_0 \), there holds \( S_g(u) = S_g(y) (\varrho(u - y)) \).
Proof of Theorem 3.1. We already know, by Proposition 3.10 and 3.19, that $S$ satisfies $\{P_2\}$. We need to check that it also satisfies $\{P_1\}$, $\{P_3\}$ and $\{P_4\}$. Properties $\{P_3\}$ and $\{P_4\}$ follow by a density argument, since we have already established that they hold for smooth maps, using the $\mathbb{F}_\Omega$-lower semi-continuity of the mass, Lemma 2.5. Property $\{P_2\}$ can be proved by a “removal of the singularity" technique, exactly as in [14, Theorem II].

We now check $\{P_1\}$. For fixed $u \in X$ and $y \in \mathbb{R}^m$, take a chain $R \in \mathbb{N}_k(\mathbb{R}^d; \mathbb{Z})$ such that $\text{spt}\, R \subseteq \Omega$ and $\text{spt}\,(\partial R) \cap \text{spt}(S_y(u)) = \emptyset$. Let $U$ be an open neighbourhood of $\text{spt}(\partial R)$ such that $U \cap \text{spt}(S_y(u)) = \emptyset$. Taking a smaller $U$ if necessary, we can assume that $U$ retracts by deformation over $\text{spt}(\partial R)$. Moreover, we can assume without loss of generality that $\partial R$ is polyhedral. For, due to the Deformation Theorem [28, Theorem 7.3], there is a $k$-chain of finite mass $\tilde{R}$, supported in $U$, such that $\partial \tilde{R} = \partial R$ is polyhedral. By Lemma 2.8, and because $\text{spt}\, \tilde{R} \subseteq U \subseteq \mathbb{R}^d \setminus \text{spt}(S_y(u))$, we have $\|S_y(u), \tilde{R}\| = 0$, so we may redefine $R := R - \tilde{R}$.

Under these conditions, we can apply $\{P_2\}$ and Lemma 2.11 to deduce that $\varrho(u - y) \in H^{1,k-1}(U, \mathcal{N})$. As a consequence, we can find an open set $U'$, with $\text{spt}(\partial R) \subseteq U' \subseteq U$, and a sequence $w_j \in C^\infty(\Omega, \mathbb{R}^m)$ such that $w_j(x) \in \mathcal{N}$ for any $x \in U'$ and any $j$, and $w_j \to \varrho(u - y)$ in $X$. Thus, for a.e. $y' \in \mathbb{R}^m$ with $|y'| < \text{dist}(\mathcal{N}, \mathcal{X})$ we have $\text{spt} (S_{y'} (w_j)) \cap U' = \emptyset$ and we can apply $\{P_1\}$ to $w_j$, because we have already proved $\{P_1\}$ for smooth maps. This gives

$$\|S_{y'}(w_j), R\| = \|\varrho(w_j - y')\|_\ast(\partial R).$$

Since $w_j \to \varrho(u - y)$ in $X$, and using Lemma 2.11 we see that

$$\|\varrho(w_j - y')\|_\ast(\partial R) = \|\varrho(u - y - y')\|_\ast(\partial R) = \|\varrho(u - y)\|_\ast(\partial R).$$

for $j$ large enough and a.e. $y$, $y'$ with $|y'| < \text{dist}(\mathcal{N}, \mathcal{X})$. The latter identity holds because the map $z \in \mathcal{N} \mapsto \varrho(z - y)$ is homotopic to the identity on $\mathcal{N}$ (a homotopy is given by $(z, t) \in \mathcal{N} \times [0, 1] \mapsto \varrho(z - ty)$). As for the left-hand side of (3.20), we use again that $w_j \to \varrho(u - y)$ in $X$, the continuity of $S$, Lemmas 2.8 and 3.13 to obtain that

$$\|S_{y'}(w_j), R\| = \|S_{y'}(\varrho(u - y)), R\| = \|S_y(u), R\|$$

for a.e., $y$, $y'$, provided that $j$ large enough and $|y'|$ is sufficiently small. Then, $\{P_1\}$ follows from (3.20), (3.21) and (3.22).

Finally, we prove the uniqueness part of the theorem. Let $S' : X \to Y$ be a continuous operator that satisfies $\{P_1\}$ and let $u \in C^\infty(\Omega', \mathbb{R}^m)$. Let $y \in \mathbb{R}^m$ be such that $u - y$ intersects transversely each cell of $\mathcal{X}$, and let $B \subseteq \Omega \setminus (u - y)^{-1}(\mathcal{X})$ be a ball. Since $\varrho(u - y)$ is well-defined and smooth on $B$, by $\{P_1\}$ we have

$$\|S_{y'}'(u), R\| = \|\varrho(u - y)\|_\ast(\partial R) = 0$$

for any $k$-disk $R$ supported in $B$ such that $\text{spt}(\partial R) \cap \text{spt}(S_{y'}'(u)) = \emptyset$. By approximating $S_{y'}'(u)$ with polyhedral chains, using the deformation theorem as stated in [51, Theorem 1.1] together with [51, Proposition 2.2], we deduce that $S_{y'}'(u) \mathbf{L} B = 0$, hence $\text{spt}(S_{y'}'(u)) \subseteq (u - y)^{-1}(\mathcal{X})$. However, using again $\{P_1\}$, we see that the multiplicity of $S_y(u)$ and $S_{y'}'(u)$ must agree on $(u - y)^{-1}(K)$, for any $(m - k)$-open cell $K$ of $\mathcal{X}$. Thus, $S_y(u) - S_{y'}'(u)$ must be supported on the $(d - k - 1)$-skeleton of $(u - y)^{-1}(\mathcal{X})$, and thus $S_y(u) = S_{y'}'(u)$ because no non-trivial $(d - k)$-chain can be supported on a $(d - k - 1)$-dimensional set [51, Theorem 3.1]. We have shown that $S'$ agrees with $S$ on smooth maps, and by continuity of $S'$, we must have $S' = S$. \qed
We now turn to the study of $S_{y}^{bd}$. Suppose that $d \geq k + 1$, and let $\Omega' \supset \Omega$ be an open set. For $g \in X^{bd} := (L^\infty \cap W^{1-k,k})(\partial \Omega, \mathbb{R}^m)$, take a map $u \in (L^\infty \cap W^{1,k})(\Omega', \mathbb{R}^m)$ that satisfies $u|_{\partial \Omega} = g$ in the sense of traces. Since $S_{y}(u) \in F_{d-k}(\Omega'; \pi_{k-1}(\mathcal{N}))$ (Corollary 3.3), we have that $\Omega'$ is well-defined, for a.e. $y$. Let $S_{y}^{bd}(g) := \partial(S_{y}(u)|_{\Omega})$.

**Proof of Proposition 3.4.** By construction, $S_{y}^{bd}(g)$ is supported in $\overline{\Omega}$. On the other hand, by noting that $S_{y}(u)$ has no boundary inside $\Omega'$ due to $[P_3]$, we see that

\begin{equation}
S_{y}^{bd}(g) = -\partial(S_{y}(u) - S_{y}(u)|_{\Omega}) = -\partial(S_{y}(u)|_{\Omega \setminus \partial \Omega})
\end{equation}

is supported in $\mathbb{R}^d \setminus \Omega$. Thus, $S_{y}^{bd}(g) \in F_{d-k-1}(\partial \Omega; \pi_{k-1}(\mathcal{N}))$ for a.e. $y$. In fact, the map $y \mapsto S_{y}^{bd}(g)$ belongs to $Y^{bd} := L^1(\mathbb{R}^m, F_{d-k-1}(\partial \Omega; \pi_{k-1}(\mathcal{N})))$, because $F(S_{y}^{bd}(g)) \leq M(S_{y}(u))$ by $[20]$ and the integral of $M(S_{y}(u))$ with respect to $y$ is finite, due to $[P_4]$. We now claim that

\begin{equation}
S_{y}(u)|_{\Omega \setminus \partial \Omega} = 0 \quad \text{for a.e. } y \in \mathbb{R}^m.
\end{equation}

Indeed, for $\rho \in (0, \text{dist}(\Omega, \partial \Omega')$, let $\Gamma_{\rho} := \{x \in \mathbb{R}^d : \text{dist}(x, \partial \Omega) < \rho\}$. Thanks to $[P_4]$ and the locality of $S$ (Corollary 3.3), we have

\begin{equation}
\int_{\mathbb{R}^d} M(S_{y}(u)|_{\Omega}) dy \leq \int_{\mathbb{R}^d} M(S_{y}(u)|_{\Gamma_{\rho}}) dy \leq C \|\nabla u\|_{L^{k}(\Gamma_{\rho})}^k
\end{equation}

and the right-hand side tends to zero as $\rho \to 0$, so (3.24) follows. As a consequence of (3.24), we have $S_{y}(u)|_{\Omega} = S_{y}(u)|_{\Omega \setminus \partial \Omega}$ for a.e. $y$.

We check that $S_{y}^{bd}(g)$ is independent of the choice of $u$. Let $u_1, u_2$ be two maps in $(L^\infty \cap W^{1,k})(\Omega', \mathbb{R}^m)$ such that $u_1 = u_2 = g$ on $\partial \Omega$ in the sense of traces. Define the map $u_*$ by

\begin{equation}
u_* := \begin{cases}
u_1 & \text{on } \Omega \\
u_2 & \text{on } \mathbb{R}^d \setminus \Omega,
\end{cases}
\end{equation}

and note that $u_* \in (L^\infty \cap W^{1,k})(\Omega', \mathbb{R}^m)$. By locality of the operator $S$ (Corollary 3.3), we have $S_{y}(u_*)|_{\Omega} = S_{y}(u_1)|_{\Omega}$ and $S_{y}(u_*)|_{\Omega \setminus \partial \Omega} = S_{y}(u_2)|_{\Omega \setminus \partial \Omega}$ and hence

\begin{equation}
\partial(S_{y}(u_1)|_{\Omega}) = \partial(S_{y}(u_2)|_{\Omega \setminus \partial \Omega}) = -\partial(S_{y}(u_*)|_{\Omega \setminus \partial \Omega}) = \partial(S_{y}(u_2)|_{\Omega}).
\end{equation}

It only remains to prove the sequential continuity of $S_{y}^{bd}$. Let $(g_j)_{j \in \mathbb{N}}$ be a sequence that converges to $g$ weakly in $W^{1-1/k,k}(\partial \Omega, \mathbb{R}^m)$, and suppose that $\Lambda := \sup_j \|g_j\|_{L^\infty(\partial \Omega)} < +\infty$. By Rellich-Kondrakov theorem, we know that $g_j \to g$ strongly in $L^k(\Omega, \mathbb{R}^m)$. We can find an open set $\Omega' \supset \Omega$ and functions $u_j, u \in W^{1,k}(\Omega', \mathbb{R}^m)$ such that $u_j|_{\partial \Omega} = g_j$, $u|_{\partial \Omega} = g$ in the sense of traces, and

\begin{align}
\|u_j - u\|_{L^k(\Omega')} &\leq C \|g_j - g\|_{L^k(\partial \Omega)} , \\
\|\nabla u_j\|_{L^k(\Omega')} &\leq C \|g_j\|_{W^{1-1/k,k}(\partial \Omega)} , \\
\|\nabla u\|_{L^k(\Omega')} &\leq C \|g\|_{W^{1-1/k,k}(\partial \Omega)} .
\end{align}
By a truncation argument, we can also assume that \( \sup_j \|u_j\|_{L^\infty(\Omega)} \leq \Lambda, \|u\|_{L^\infty(\Omega)} \leq \Lambda \). For any \( \rho \in (0, \text{dist}(\Omega, \partial\Omega)), \) let \( \Omega_\rho := \Omega \cup \Gamma_\rho = \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) < \rho\} \). By applying (2.9) with \( U = \Omega_\rho, H = \Omega, \) and using that \( F_{\Omega_\rho} \leq F_{\Omega'} \) (as a consequence of Lemma 2.2), we obtain
\[
F(S_y^b(d_j) - S_y^b(g)) \leq F((S_y(u_j) - S_y(u)) \mathbf{1}_\Omega) \\
\leq (1 + \rho^{-1}) F_{\Omega'}((S_y(u_j) - S_y(u)) + \mathbf{M}(S_y(u_j) - S_y(u)) \chi_{\Gamma_\rho}).
\]
We integrate with respect to \( y \) and apply \([P_4]\) Corollary 3.2 to deduce
\[
\int_{\mathbb{R}^m} F(S_y^b(d_j) - S_y^b(g)) \, dy \leq C(1 + \rho^{-1}) \|u_j - u\|_{L^k(\Omega')} \left( \|\nabla u_j\|_{L^k(\Omega')}^k + \|\nabla u\|_{L^k(\Gamma')}^k \right) \\
+ \|\nabla u_j\|_{L^k(\Gamma')} + \|\nabla u\|_{L^k(\Gamma')} \\
\leq C(1 + \rho^{-1}) \|g_j - g\|_{L^k} \left( \|g_j\|_{W^{1/k,k}_1}^k + \|g\|_{W^{1/k,k}_1}^k \right) \\
+ \|\nabla u_j\|_{L^k(\Gamma')} + \|\nabla u\|_{L^k(\Gamma')}.
\]
By letting \( j \to +\infty \) first, and then \( \rho \to 0 \), we deduce that \( S^b \) is sequentially continuous.

**Proof of Proposition 3.3.** We first prove (3.1). Let \( u \in (L^\infty \cap W^{1,k}_1) (\Omega', \mathbb{R}^m) \) be such that \( u = g \) on \( \partial\Omega \), in the sense of traces. For \( j \in \{1, 2\} \), define
\[
\tilde{u}_j := \begin{cases} 
  u_j & \text{on } \overline{\Omega}, \\
  u & \text{on } \Omega' \setminus \overline{\Omega}.
\end{cases}
\]
Let \( \rho_\varepsilon \) be a standard mollifier supported in \( B^d_\varepsilon \), and let \( v_{j,\varepsilon} := \tilde{u}_j * \rho_\varepsilon \). By taking a smaller \( \Omega' \), we have that \( v_{j,\varepsilon} \) is well-defined and smooth on \( \Omega' \), for any \( \varepsilon \) small enough. Setting \( \Omega_{\varepsilon} := \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) < \varepsilon\} \), we have \( v_{1,\varepsilon} = v_{2,\varepsilon} \) on \( \Omega' \setminus \overline{\Omega_{\varepsilon}} \). Therefore, by Proposition 3.10 for a.e. \( y \in \mathbb{R}^m \) and any \( \varepsilon \) there exists a smooth chain \( R_\varepsilon \in \mathbb{M}_{d-k+1}(\overline{\Omega_{\varepsilon}}; \pi_{k-1}(\mathcal{N})) \) such that
\[
S_y(v_{2,\varepsilon}) \mathbf{1}_{\overline{\Omega_{\varepsilon}}} - S_y(v_{1,\varepsilon}) \mathbf{1}_{\overline{\Omega_{\varepsilon}}} = \partial R_\varepsilon
\]
and \( \sup_{\varepsilon} \mathbf{M}(R_\varepsilon) < +\infty \). Up to extraction of a subsequence, we have \( F(R_\varepsilon - R) \to 0 \) as \( \varepsilon \to 0 \), for some \( R \in \mathbb{M}_{d-k+1}(\overline{\Omega}; \pi_{k-1}(\mathcal{N})) \). Therefore, (3.1) follows if we show that \( S_y(v_{j,\varepsilon}) \mathbf{1}_{\overline{\Omega_{\varepsilon}}} \) \( F \)-converges to \( S_y(u_j) \mathbf{1}_{\Omega'} \mathbf{1}_{\overline{\Omega_{\varepsilon}}} \), for \( j \in \{1, 2\} \) and a.e. \( y \). To this end, let us fix \( \varepsilon_0 > 0 \) and take \( 0 < \varepsilon < \varepsilon_0 \). We apply (2.9) and Lemma 2.2 to obtain
\[
\mathbf{F}(S_y(v_{j,\varepsilon}) \mathbf{1}_{\overline{\Omega_{\varepsilon}}} - S_y(\tilde{u}_2) \mathbf{1}_{\overline{\Omega_{\varepsilon}}}) \leq (1 + \varepsilon^{-1}) \mathbf{F}_{\Omega'}(S_y(v_{j,\varepsilon}) - S_y(v_2)) + \mathbf{M}(S_y(v_{j,\varepsilon})) \mathbf{1}_{\overline{\Omega_{0 \varepsilon}}} \mathbf{1}_{\overline{\Omega_{\varepsilon}}})
\]
for \( j \in \{1, 2\} \). For a.e. \( y \), the first term in the right-hand side converges to zero as \( \varepsilon \to 0 \), because \( v_{j,\varepsilon} \to v_j \) in \( (L^\infty \cap W^{1,k}_1)(\Omega', \mathbb{R}^m) \) and because of \([P_5]\) As for the right-hand side, we have
\[
\sup_{0 < \varepsilon < \varepsilon_0} \int_{\mathbb{R}^d} \mathbf{M}(S_y(v_{j,\varepsilon})) \mathbf{1}_{\overline{\Omega_{0 \varepsilon}}} \mathbf{1}_{\overline{\Omega_{\varepsilon}}}) \, dy \leq C \sup_{0 < \varepsilon < \varepsilon_0} \|\nabla v_{j,\varepsilon}\|_{L^k(\Omega_{0 \varepsilon})} \leq C \|\nabla v_j\|_{L^k(\Gamma_{2 \varepsilon})}
\]
where \( \Gamma_{2\varepsilon_0} := \{ x \in \mathbb{R}^d : \text{dist}(x, \partial \Omega) < 2\varepsilon_0 \} \). (We have applied here Young’s inequality for the convolution.) Since the right-hand side converges to zero as \( \varepsilon_0 \to 0 \), we conclude the proof of (3.1).

We turn now to the proof of (3.2). In view of (3.1), we can assume w.l.o.g. that \( u_1 = u_2 \). Let \( 0 < \theta < 1 \) be fixed. Let \( y_1 \in \mathbb{R}^m \) with \( |y_1| \leq \theta \delta_0 = \theta \text{dist}(\mathcal{N}, \mathcal{X}) \). Let \( \xi \in C_0^\infty(\mathbb{R}^m) \) be a cut-off function such that \( 0 \leq \xi \leq 1 \), \( \xi = 0 \) in a neighbourhood of \( \mathcal{N} \) and \( \xi = 1 \) in a neighbourhood of \( \mathcal{X} + y_1 \), and let \( \phi: \mathbb{R}^m \to \mathbb{R}^m \) be given by \( \phi(z) := z - y_0 \xi(z) \) for a fixed \( y_0 \in \mathbb{R}^m \). There exists \( \delta > 0 \) such that, for \( |y_0| \leq \delta \), the map \( \phi \) is a diffeomorphism. In fact, since the \( C^1 \)-norm of \( \xi \) can be bounded in terms of \( \theta, \delta_0 \), we can choose \( \delta = \delta(\theta, \delta_0) \) uniformly with respect to \( y_1 \in B_{\delta_0}^m \). Then, for \( |y_0| \leq \delta \) and a.e. \( y \) in a neighbourhood of \( y_1 \), there holds

\[
S_y(\phi(u_1)) = S_{y+y_0}(u_1).
\]

This equality is readily checked in case \( u_1 \) is smooth, and remains true in general by the continuity of \( S \). Since \( \phi(u_1) \) has trace \( g \) on \( \partial \Omega \) (because \( g \) is \( \mathcal{N} \)-valued), we can apply (3.1) and deduce that (3.2) holds, provided that \( |y_1| \leq \theta \delta_0 \) and \( |y_1 - y_2| \leq \delta \). Since \( B_{\delta_0}^m \) can be covered by finitely many balls of diameter \( \delta \), (3.2) remains true when \( |y_1| \leq \theta \delta_0 \), \( |y_2| \leq \theta \delta_0 \), and the proposition follows by letting \( \theta \nearrow 1 \).

### 3.4 The topological singular set of \( \mathcal{N} \)-valued maps

In this section, we study the special case of \( \mathcal{N} \)-valued Sobolev maps. We show that Pakzad and Rivière’s construction \( \mathbf{S}^{\mathbf{PR}} \) of a topological singular set

\[
\mathbf{S}^{\mathbf{PR}}: W^{1,k-1}(\Omega, \mathcal{N}) \to \mathbb{F}_{d-k}(\overline{\Omega}, \pi_{k-1}(\mathcal{N}))
\]

is essentially equivalent to \( \mathbf{S} \), that is, one can reconstruct the operator \( \mathbf{S} \) given \( \mathbf{S}^{\mathbf{PR}} \), and conversely. As a consequence, we prove Theorem [11] which extends the results in [46].

We first recall the definition of \( \mathbf{S}^{\mathbf{PR}} \). Let \( R_\infty^p(\Omega, \mathcal{N}) \) (resp. \( R_0^p(\Omega, \mathcal{N}) \)) be the class of maps \( u \in W^{1,p}(\Omega, \mathcal{N}) \) that are smooth (resp., continuous) on \( \overline{\Omega} \) away from the skeleton of a polyhedral \( (d-|p|-1) \)-complex. The set \( R_\infty^p(\Omega, \mathcal{N}) \) is dense in \( W^{1,p}(\Omega, \mathcal{N}) \) [7, Theorem 2] (and so is, fortiori, \( R_0^p(\Omega, \mathcal{N}) \)). Let \( u \in R_0^p(\Omega, \mathcal{N}) \) and let \( Z \) be a polyhedral \( (d-|p|-1) \)-complex such that \( u \in C^0(\Omega \backslash Z) \). For each \( (d-|p|-1) \)-cell \( H \) of \( Z \), we take a \( (|p|+1) \)-disk \( B_H \) that intersects transversely \( H \) at a unique point \( x_H \), and do not intersect any other cell of \( Z \). We orient \( H \) and \( B_H \) in such a way that \( T_{x_H}H \oplus T_{x_H}B_H \) induces the standard orientation on \( \mathbb{R}^d \).

We set

\[
(3.27) \quad \mathbf{S}^{\mathbf{PR}}(u) := \sum_H [u_*(\partial B_H)][H],
\]

the sum being taken over all \( (d-|p|-1) \)-cells \( H \) of \( Z \). Pakzad and Rivière [40, Theorem II] showed that, in case \( \Omega = B^d \) and \( p \in [1, 2) \cup [d-1, d) \), the map \( \mathbf{S}^{\mathbf{PR}} \) can be extended continuously to \( W^{1,k-1}(B^d, \mathcal{N}) \).

**Lemma 3.14.** Let \( \delta_0 := \text{dist}(\mathcal{N}, \mathcal{X}) \). For any \( u \in R_{k-1}^0(\Omega, \mathcal{N}) \) and a.e. \( y \in \mathbb{R}^m \) with \( |y| < \delta_0 \), there holds \( S_y(u) = \mathbf{S}^{\mathbf{PR}}(u) \).
Proof. Choose a number $0 < \delta < \delta_0$, and pick a function $u \in R_{k-1}^0(\Omega, \mathcal{N})$. By reflection (see e.g. [2] Lemma 8.1), we can extend $u$ to a new map defined on a slightly larger domain $\Omega' \supset \Omega$ that retracts onto $\Omega$, in such a way that $u \in W^{1,k-1}(\Omega', \mathcal{N})$. Let $\rho$ be a standard mollifier supported in $B^\varepsilon_z$, and let $u_\varepsilon := u \ast \rho \varepsilon$. For any $0 < \varepsilon < \operatorname{dist}(\Omega, \partial \Omega')$, $u_\varepsilon$ is a well-defined map in $C^\infty(\Omega', \mathbb{R}^m)$. Let $Z$ be a polyhedral $(d-k)$-complex such that $u \in C^0(\Omega \setminus Z)$, and for any $\eta > 0$, let $V_\eta$ be the closed $\eta$-neighborhood of $Z$. Since $u$ is $\mathcal{N}$-valued and uniformly continuous on $\Omega \setminus V_\eta$, for $\varepsilon$ small enough and any $x \in \Omega \setminus V_\eta$ we have $\operatorname{dist}(u_\varepsilon(x), \mathcal{N}) < \operatorname{dist}(\mathcal{N}, \mathcal{X}) - \delta$. Thus, $S_y(u_\varepsilon)(\Omega \setminus V_\eta) = 0$ for any $y$ such that $|y| \leq \delta$. Taking the limit as $\varepsilon \to 0$ with the help of $[P_3]$ and using that the flat-convergence preserves the support, we conclude that

$$\operatorname{spt}(S_y(u)) \subseteq \bigcap_{\eta > 0} V_\eta = Z \quad \text{for any } y \text{ with } |y| \leq \delta.$$

Moreover, $S_y(u)$ is a cycle relative to $\Omega$, being the flat limit of the relative cycles $S_y(u_\varepsilon)$. Therefore, the constancy theorem [25, Theorem 7.1] implies that, for any open $(d-k)$-cell $H$ of $Z$, there exists $\alpha(H) \in \pi_{k-1}(\mathcal{N})$ such that $S_y(u) \mathcal{L} H = \alpha(H) \llbracket H \rrbracket$. In fact, we also have

$$S_y(u) = \sum_H \alpha(H) \llbracket H \rrbracket,$$

because no non-trivial $(d-k)$-chain can be supported on the $(d-k-1)$-skeleton of $Z$ [21, Theorem 3.1]. Finally, let $B_H$ be a closed $k$-disk that intersects transversely $H$ at a single point, and does not intersect any other cell of $Z$. Arguing as above, we see that $\operatorname{spt}(S_y(u_\varepsilon)) \cap \partial B_H = \emptyset$ for any $y$ such that $|y| \leq \delta$ and for $\varepsilon$ small enough. Therefore, using the stability of $\llbracket$ with respect to flat convergence (Lemma 2.8) and $[P_1]$ we conclude that

$$\alpha(H) = \llbracket(S_y(u), [B_H]) = \llbracket(S_y(u_\varepsilon), [B_H]) = [\varrho(u - y)_* \partial B_H].$$

Now, when $|y| \leq \delta < \operatorname{dist}(\mathcal{N}, \mathcal{X})$, the map $z \in \mathcal{N} \mapsto \varrho(z - y)$ is homotopic to the identity on $\mathcal{N}$; a homotopy is given by $(t, z) \in [0, 1] \times \mathcal{N} \mapsto \varrho(z - ty)$. Therefore, we have $[\varrho(u - y)_* \partial B_H] = [u_* \partial B_H]$ and hence $S_y(u) = S^{\mathcal{P}}(u)$ for a.e.-$y$ with $|y| \leq \delta$. By letting $\delta \nearrow \delta_0$, the lemma follows. \[\Box\]

Remark 3.1. Note that, in the proof of Lemma 3.14, we only need to apply Property $[P_1]$ to smooth maps, so Lemma 3.14 only relies on the results in Section 3.2 and the continuity of $S$.

Proof of Lemma 3.12. If $u : \Omega' \to \mathbb{R}^m$ is smooth then, for a.e. $y \in \mathbb{R}^m$, there holds $\varrho(u - y) \in R^{1,k-1}(\Omega, \mathcal{N})$. Thus, Lemma 3.14 (see also Remark 3.1) and the very definition of $S_y(u)$ imply

$$S_y'((\varrho(u - y)) = S^{\mathcal{P}}((\varrho(u - y)) = S_y(u)$$

for a.e. $y'$ with $|y'| < \delta_0$. \[\Box\]

Proof of Proposition 3.6. In case $u \in W^{1,1-k}(\Omega, \mathcal{N})$, the statement follows immediately from Lemma 3.14 combined with a density argument. For $g \in X^{1d}$, the statement follows by taking the boundary of both sides of (3.2), and using Proposition 3.3. Finally, an arbitrary map $u \in W^{1,k}(\Omega, \mathcal{N})$ can be approximated (in the $W^{1,k}$-norm) by maps $\tilde{u} : \Omega \to \mathcal{N}$ that are smooth.
away from the skeleton of a smooth complex of dimension \(d - k - 1\). By Lemma [3.14] for a.e. \(y\) with \(|y| < \delta_0\) we have \(S_y(\tilde{u}) = S^{PR}(\tilde{u})\), and the latter must be zero because no non-trivial, smooth \((d - k)\)-chain can be supported on a \((d - k - 1)\)-dimensional set. The proposition follows by a density argument. \(\square\)

We conclude this section by giving the proof of Theorem II.

**Proof of Theorem II**. For any \(u \in R^{1,p} (B^d, \mathcal{N})\), \(S^{PR}(u)\) is defined by (3.27), as in [46]. For any two maps \(u_0, u_1 \in R^{1,p} (B^d, \mathcal{N})\), Lemma [3.14] and (P5) (with the choice \(k = \lfloor p \rfloor + 1\)) imply that

\[
\mathcal{F}\left(S^{PR}(u_1) - S^{PR}(u_0)\right) \leq C \int_{B^d_0} |u_1 - u_0| \left(|\nabla u_0|^p + |\nabla u_1|^p + 1\right)
\]

for some constant \(C = C(\mathcal{N}, \mathcal{N}^\epsilon, p)\). Then, by applying Lebesgue dominated theorem to the right-hand side of this inequality, we deduce that \(S^{PR}\) maps Cauchy sequences in \(R^{1,p} (B^d, \mathcal{N})\) into Cauchy sequences in \(\mathcal{F}_{d-p} - 1(B^d; \pi_p(\mathcal{N}))\). Thus, \(S^{PR}\) admits a continuous extension to \(W^{1,p} (B^d, \mathcal{N})\). Now, the theorem follows by the same arguments of [46] Theorem II. \(\square\)

4 Applications to \(\mathcal{N}\)-valued BV spaces

4.1 Density of smooth, \(\mathcal{N}\)-valued maps in BV

In this section, we consider the space \(BV(\Omega, \mathbb{R}^m)\), consisting of functions \(u \in L^1(\Omega, \mathbb{R}^m)\) whose distributional derivative \(Du\) is a finite Radon measure, endowed with the norm \(||u||_{BV(\Omega)} := ||u||_{L^1(\Omega)} + |Du|(\Omega)\). We also consider the semi-norm \(||u||_{BV(\Omega)} := |Du|(\Omega)\). The distributional derivative of a BV-function has the following representation:

\[
Du = \nabla u J^d + DC^u + DJ^u,
\]

where \(\nabla u\) is the approximate gradient of \(u\), \(DC^u\) and \(DJ^u\) are, respectively, the Cantor and the jump part. The latter is supported on a \((d - 1)\)-rectifiable set \(J_u\), called the jump set, and we have

\[
DJ^u = (u^+ - u^-) \otimes \nu_u J^d \mathcal{H}^{d-1} L J_u
\]

where \(\nu_u\) is the approximate unit normal to \(J_u\) and \(u^+, u^-\) are the approximate traces of \(u\) from either side of \(J_u\). We define \(SBV(\Omega, \mathbb{R}^m)\) as the set of all functions \(u \in BV(\Omega, \mathbb{R}^m)\) such that \(DC^u = 0\). We refer the reader, e.g., to [2] for more details and notation. We define \(BV(\Omega, \mathcal{N})\) (resp., \(SBV(\Omega, \mathcal{N})\)) as the set of maps \(u \in BV(\Omega, \mathbb{R}^m)\) (resp., \(u \in SBV(\Omega, \mathbb{R}^m)\)) such that \(u(x) \in \mathcal{N}\) for a.e. \(x \in \Omega\). We say that a sequence \(u_j\) of BV-functions converges weakly to \(u\) if and only if \(u_j \rightharpoonup u\) strongly in \(L^1\) and \(D\) weakly* as elements of the dual \(C_0(\Omega, \mathbb{R}^m)'\).

**Proof of Theorem [2]**. Let \(u_j \in C^\infty (B^d, \mathbb{R}^m)\) be a sequence of smooth maps that converges to \(u\) weakly in BV and a.e., with \(||\nabla u_j||_{L^1(B^d)} \leq C ||Du||_{L^1(B^d)}\) (see e.g. [1] Theorem 3.5]). Since \(\mathcal{N}\) is compact, by a truncation argument we can make sure that \(||u||_{L^\infty(B^d)} \leq \Lambda\), for some constant \(\Lambda\) that only depends on the embedding of \(\mathcal{N}\) in \(\mathbb{R}^m\). Since \(\mathcal{N}\) is connected and \(\pi_1(\mathcal{N})\) is abelian,
we can apply Theorem[5.1] to $u_j$ with $k = 2$. In particular, by \{P_3\} for any $j \in \mathbb{N}$ and a.e. $y \in B^m_{\delta_0}$ (where $\delta_0 := \text{dist}(\mathcal{N}, \mathcal{X})$) there exists a $(d - 1)$-chain $R^j_y$ such that $(\partial R^j_y - S_y(u_j)) \mathbf{1} B^d = 0$ and

$$
\int_{B^m_{\delta_0}} \mathbb{M}(R^j_y) \, dy \leq C \|\nabla u_j\|_{L^1(B^d)} \leq C |\text{Du}|(B^d).
$$

Moreover, by Lemma[5.3] we have

$$
\int_{B^m_{\delta_0}} \left( \int_{B^d} |\nabla (\varrho(u_j - y))(x)| \, dx \right) \, dy \leq C \|\nabla u_j\|_{L^1(B^d)} \leq C |\text{Du}|(B^d).
$$

By an average argument we deduce that, for each $j \in \mathbb{N}$ and $\delta \in (0, \delta_0)$, there exists $y(j) \in B^m_{\delta}$ such that

$$
\|\nabla (\varrho(u_j - y(j)))\|_{L^1(B^d)} \leq C |\text{Du}|(\Omega), \quad \mathbb{M}(R^j_{y(j)}) \leq C |\text{Du}|(B^d).
$$

for some constant $C$ that depends on $\delta$. By choosing $\delta$ small enough, we can make sure that the map $\varrho_y: z \in \mathcal{N} \mapsto \varrho(z - y)$ has a smooth inverse $\varrho^{-1}_y: \mathcal{N} \rightarrow \mathcal{N}$ for any $y \in B^m_{\delta}$.

We set $w_j := (\varrho^{-1}_{y(j)} \circ \varrho)(u_j - y(j))$. Then, $w_j \in R^{1,1}(B^d, \mathcal{N})$ and the $L^1$-norm of $\nabla w_j$ is bounded by the total variation of $\text{Du}$. By applying the “removal of the singularity” technique in \cite{16} Proposition 5.1] we find a map $v_j \in C^\infty(B^d, \mathcal{N})$ such that

$$
\|v_j - w_j\|_{L^1(B^d)} \leq j^{-1},
$$

$$
\|\nabla v_j\|_{L^1(B^d)} \leq \|\nabla w_j\|_{L^1(B^d)} + C\mathbb{M}(R^j_{y(j)}) + Cj^{-1} \leq C |\text{Du}|(B^d).
$$

Now, $(w_j)_{j \in \mathbb{N}}$ is bounded in the BV-norm and hence, modulo extraction of a subsequence, $w_j$ converges weakly in BV and a.e. to some limit $w \in BV(B^d, \mathbb{R}^m)$. On the other hand, it can be easily checked that, up to subsequences, $v_j$ converges to $u$ a.e. out of the $\mathcal{H}^d$-negligible set $\cup_j \text{spt} R^j_{y(j)}$, and hence by \{3.1\} we have $w = u$. \hfill \square

### 4.2 Lifting results in BV

In this section, we consider the lifting problem in BV. Let $\pi: \mathcal{E} \rightarrow \mathcal{N}$ be the universal covering of $\mathcal{N}$. We endow $\mathcal{E}$ with the pull back metric $\pi^*(h_{\mathcal{E}})$, $h_{\mathcal{E}}$ being the metric of $\mathcal{N}$, so that $\pi$ is a local isometry. We also identify $\mathcal{E}$ with an isometrically embedded submanifold of some Euclidean space $\mathbb{R}^\ell$, and we define $BV(\Omega, \mathcal{E})$ as the set of functions $u \in BV(\Omega, \mathbb{R}^m)$ such that $u(x) \in \mathcal{E}$ for a.e. $x \in \Omega$. We say that $v \in BV(\Omega, \mathcal{E})$ is a lifting for $u \in BV(\Omega, \mathcal{N})$ if $u = \pi \circ v$ a.e. on $\Omega$. When the domain is a ball, the existence of a lifting in BV could be deduced by a density argument, based on Theorem[2], but we give below a different proof which works on more general domains.

**Proof of Theorem[3]** We choose a norm $|\cdot|$ on $\pi_1(\mathcal{N})$ that, in addition to \{2.17\}, satisfies

$$
\inf \left\{ \int_{\Sigma^1} |\gamma'(s)| \, ds : \gamma \in g \cap W^{1,1}(\Sigma^1, \mathcal{N}) \right\} \leq C |g|
$$

for any $g \in \pi_1(\mathcal{N})$ and some $g$-independent constant $C$. Such a norm exists. Indeed, the left-hand side itself of \{4.3\} defines a norm on $\pi_1(\mathcal{N})$ that satisfies \{2.17\} up to a multiplicative
factor, as any loop whose length is less than the injectivity radius of \( \mathcal{N} \) is contained in a contractible geodesic ball.

We also need to fix some notation. Given a smooth chain \( R \in M_{d-1}(\mathbb{R}^d; \pi_1(\mathcal{N})) \), we can always assign an orientation to each \((d-1)\)-cell of \( R \). We can then write \( R = \sum_i g_i [H_i] \), where the \( H_i \) are oriented, smooth \((d-1)\)-polyhedra with pairwise disjoint interiors and \( g_i \in \pi_1(\mathcal{N}) \). We denote the local multiplicity of \( R \) at a point \( x \in H_i \setminus \partial H_i \) by \( \mathfrak{g}[R](x) := g_i \). Note that \( \mathfrak{g}[R] \) depends on the choice of the orientation on \( H_i \), and \( \mathfrak{g}[R](x) \) changes into \(-\mathfrak{g}[R](x)\) when the orientation of \( H \) is flipped.

**Step 1** (Construction of an approximating sequence). Let \( \Omega' \) be an open cube (i.e., \( \Omega' = (-L, L)^d \) for some \( L > 0 \)) that contains \( \Omega \). Let \( u_\Omega := \mathcal{H}^d(\Omega)^{-1} \int_{\Omega} u \in \mathbb{R}^m \) be the average of \( u \) over \( \Omega \). Thanks to [4, Proposition 3.21] and the BV-Poincaré inequality [4, Theorem 3.44], we can extend \( u - u_\Omega \) to a map \( u_* \in (L^\infty \cap BV)(\Omega', \mathbb{R}^m) \) that satisfies \( |u_*|_{BV(\Omega')} \leq C |u|_{BV(\Omega)} \) for some constant \( C = C(\Omega) \). Thus, redefining \( u := u_* + u_\Omega \), we have constructed an extension of the given map \( u \) that belong to \((L^\infty \cap BV)(\Omega', \mathbb{R}^m)\) and satisfies \( |u|_{BV(\Omega')} \leq C |u|_{BV(\Omega)} \).

Take a sequence of smooth functions \( u_j \in C^\infty(\Omega', \mathbb{R}^m) \) that converges to \( u \) BV-weakly and a.e., and is uniformly bounded in \( L^\infty \). By applying Theorem 3.1 to \( u_j \), with the choice \( k = 2 \), and using an average argument as in the proof of Theorem 3 for any \( j \in \mathbb{N} \) we find \( y(j) \in \mathbb{R}^m \) and a smooth \((d-1)\)-chain \( R_j := R_{y(j)} \in M_{d-1}(\Omega'; \pi_1(\mathcal{N})) \) such that, setting \( S_j := S_{y(j)}(u_j) \) and \( w_j := (\partial_{y(j)}^{-1}) \circ \gamma)(u_j - y(j)) \), the following properties hold:

\begin{align}
(4.4) & \quad w_j \in W^{1,1}(\Omega', \mathcal{N}) \cap C^\infty(\Omega' \setminus spt S_j, \mathcal{N}) \\
(4.5) & \quad w_j \to u \quad \text{a.e. on } \Omega \\
(4.6) & \quad \|\nabla w_j\|_{L^1(\Omega')} + M(R_j) \leq C |u|_{BV(\Omega)} \\
(4.7) & \quad (\partial R_j - S_j) \cup_{\Omega'} = 0.
\end{align}

**Step 2** (Construction of a lifting for \( w_j \)). For each \( j \in \mathbb{N} \), we will construct a lifting \( v_j \) of \( w_j \) such that \( v_j \in C^\infty(\Omega' \setminus spt R_j, \mathcal{E}) \) and

\[ v_j^+(x) = (-1)^d g[R_j](x) \cdot v_j^-(x) \quad \text{for } x \in spt R_j \setminus spt R_j^{d-2}. \]

To this end, we adapt a well-known topological construction (see e.g. [35, Proposition 1.33]). We choose base points \( x_0 \in \Omega' \setminus spt R_j \) and \( \gamma \in \pi^{-1}(x_0) \). For any \( x \in \Omega' \setminus spt R_j \), we take a smooth path \( \gamma : [0, 1] \to \Omega' \setminus spt S_j \) from \( x_0 \) to \( x \). We suppose that \( \gamma \) crosses transversely each cell of \( R_j \), which is generically the case, by Thom’s transversality theorem. In particular, there exists finitely many \( t_i \in (0, 1) \) such that \( \gamma(t_i) \in spt R_j \); moreover, each \( \gamma(t_i) \) lie in the interior of a \((d-1)\)-cell. We define \( g_i \in \pi_1(\mathcal{N}) \) by

\[ g_i := \begin{cases} (-1)^d g[R_j](\gamma(t_i)) & \text{if } \gamma'(t_i) \text{ agrees with the orientation of } R_j \\ (-1)^{d-1} g[R_j](\gamma(t_i)) & \text{otherwise.} \end{cases} \]

We define a path \( \alpha : [0, 1] \to \mathcal{E} \) in the following way: on the interval \([0, t_1]\), \( \alpha \) is the lifting of \( w_j \circ \gamma_{x_0}^{[0,t_1]} \) starting from the point \( e_j \); on \([t_1, t_2]\), \( \alpha \) is the lifting of \( w_j \circ \gamma_{x_1}^{[t_1,t_2]} \) starting from \( g_1 \cdot \alpha(t_1) \), and so on. Note that \( \alpha \) is uniquely defined by \( \gamma \). Then, we set \( v_j(x) := \alpha(1) \in \mathcal{E} \).
We need to check that \( v_j \) is well-defined. Let \( \gamma, \eta \) be two smooth paths from \( x_0 \) to \( x \), and let \( \alpha, \beta \) be the corresponding paths in \( \mathcal{E} \) obtained via the previous construction. Let \( g_1, \ldots, g_p \), resp. \( h_1, \ldots, h_q \), be the elements of \( \pi_1(\mathcal{N}) \) associated with \( \gamma \), resp. \( \eta \), via (4.9). We denote by \( \gamma \ast \eta \) the loop obtained by first travelling along \( \gamma \) then along \( \eta \), the opposite way from \( x \) to \( x_0 \). Since \( \Omega' \) is a cube, hence a simply connected set, \( \gamma \ast \eta \) can be seen as the boundary of a smooth chain \( T \in \mathcal{M}_2(\Omega' ; \mathbb{Z}) \). By definition of the \( g_i \)'s and \( h_k \)'s and by Lemma 2.8, we have

\[
(4.10) \quad - \sum_{i=1}^{p} g_i + \sum_{k=1}^{q} h_k = (-1)^{d-1} \mathbb{1}(R_j, \partial T) = \mathbb{1}(\partial R_j, T) \quad \text{where} \quad \mathbb{1}(S, T) \left( \frac{P_1}{P_1} \right) [w_{j,s}(\partial T)].
\]

Let \( \sigma : [0, 1] \to \mathcal{E} \), resp. \( \tau : [0, 1] \to \mathcal{E} \), be liftings for \( w_j \circ \gamma \), resp. \( w_j \circ \eta \), with \( \sigma(0) = \tau(0) = e_j \). Then, by construction of \( \alpha, \beta \), we have

\[
\alpha(1) = \sum_{i=1}^{p} g_i \cdot \sigma(1), \quad \beta(1) = \sum_{k=1}^{q} h_k \cdot \tau(1)
\]

and \( \sigma(1) = [w_{j,s}(\partial T)] \cdot \tau(1) \). From these identities and (4.10), it follows that \( \alpha(1) = \beta(1) \), so \( v_j(x) \) is well-defined. Now, arguing exactly as in [35, Proposition 1.33], one sees that \( v_j \) is smooth on \( \Omega' \setminus \spt R_j \), and (4.8) is satisfied by construction.

**Step 3** (Passage to the limit). Since \( \pi \) is a local isometry and \( w_j = \pi \circ v_j \), we have that \( |\nabla v_j| = |\nabla w_j| \) on \( \Omega' \setminus \text{spt} R_j \); moreover, for any \( y \in \mathcal{E} \) and \( g \in \pi_1(\mathcal{N}) \) there holds

\[
|y - g \cdot y| \leq \text{dist}_{\mathcal{E}}(y, g \cdot y) = \inf_{\gamma \in g} \int_{S^1} |\gamma'(s)| \mathrm{d}s \leq C |g|,
\]

where \( \text{dist}_{\mathcal{E}} \) denotes the geodesic distance in \( \mathcal{E} \). Together with (4.8), this yields \( |D^1 v_j|(\Omega') \leq M(R_j) \) and hence, by (4.10),

\[
(4.11) \quad |v_j|_{BV(\Omega')} \leq C |u|_{BV(\Omega)}.
\]

Now, thanks to the BV-Poincaré-type inequality [21, Lemma 6, Eq. (16)], for each \( j \) we find \( \xi_j \in \mathcal{E} \) such that

\[
(4.12) \quad \int_{\Omega'} \text{dist}_{\mathcal{E}}(v_j(x), \xi_j) \mathrm{d}x \leq C |v_j|_{BV(\Omega')}.
\]

Since the group \( \pi_1(\mathcal{N}) \) acts isometrically on \( \mathcal{E} \), and since \( \mathcal{E} \) admits a cover of the form \( \{g \cdot U \}_{g \in \pi_1(\mathcal{N})} \) where \( U \subseteq \mathcal{E} \) is bounded, by multiplying each \( v_j \) by a suitable element of \( \pi_1(\mathcal{N}) \) we can assume w.l.o.g. that the \( \xi_j \)'s are uniformly bounded. Then, (4.11) and (4.12) imply that \( (v_j)_{j \in \mathbb{N}} \) is bounded in \( BV \). We extract a subsequence that converges \( BV \)-weakly and a.e. to a limit \( v \in BV(\Omega, \mathcal{E}) \); by (4.5) and (4.11), \( v \) is a lifting of \( u \) with the desired properties.

**Step 4** (The case \( u \in SBV \)). Let \( \iota \) be the canonical embedding \( \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{2\ell} \). We first construct a smooth immersion \( \tilde{\pi} : \mathbb{R}^\ell \to \mathbb{R}^{m+2\ell} \) that restricts to \( \iota \circ \pi \) on \( \mathcal{E} \subseteq \mathbb{R}^\ell \). We consider a tubular neighbourhood \( U \) of \( \mathcal{E} \) together with the nearest-point projection \( \tau : U \to \mathcal{E} \), which is well-defined and smooth. We take smooth cut-off functions \( \xi_0, \xi_1 \) such that \( \xi_0 = 0 \) and \( \xi_1 = 1 \)
in a neighbourhood of $\mathcal{E}$, $\text{spt}(\xi_1) \subseteq U$ and $\text{spt}(1 - \xi_0)$ is contained in the interior of $\xi_1^{-1}(1)$ (so that, for any $x \in \mathbb{R}^\ell$, either $\xi_0$ or $\xi_1$ is equal to 1 in a neighbourhood of $x$). We set

$$\tilde{\pi}(x) := (\xi_1(x)\pi(\tau(x)), \xi_1(x)(x - \tau(x)), \xi_0(x)x) \quad \text{for } x \in \mathbb{R}^\ell.$$  

Using the fact that $\pi: \mathcal{E} \to \mathcal{N}$ is a local isometry, and in particular an immersion, it can be checked that $\tilde{\pi}$ has injective differential at any point; moreover, $\tilde{\pi}|_{\mathcal{E}} = \iota \circ \pi$. Take now a map $u \in \text{SBV}(\Omega, \mathcal{N})$ and a lifting $v \in \text{BV}(\Omega, \mathcal{E})$. Then $\tilde{\pi} \circ v = \iota \circ u \in \text{SBV}(\Omega, \mathbb{R}^{m+2\ell})$ and hence the chain rule for BV-functions [4, Theorem 3.96] implies $\nabla \tilde{\pi}(\bar{v})D_v = D(\iota \circ u) = 0$, where $\bar{v}$ is the precise representative of $v$ (see, e.g., [4, Corollary 3.80]). Since $\nabla \tilde{\pi}(y)$ is injective for any $y \in \mathbb{R}^{m+2\ell}$, we conclude that $D_v = 0$, that is, $v \in \text{SBV}(\Omega, \mathcal{E})$.

\section*{Acknowledgements}

G. C.’s research was supported by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 291053; by the Basque Government through the BERC 2014-2017 program; and by the Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa accreditation SEV-2013-0323. G. O. was partially supported by GNAMPA-IndAM.

\section*{References}

[1] G. Alberti, S. Baldo, and G. Orlandi. Functions with prescribed singularities. *J. Eur. Math. Soc. (JEMS)*, 5(3):275–311, 2003.

[2] G. Alberti, S. Baldo, and G. Orlandi. Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.*, 54(5):1411–1472, 2005.

[3] R. Alicandro and M. Ponsiglione. Ginzburg-Landau functionals and renormalized energy: a revised $\Gamma$-convergence approach. *J. Funct. Anal.*, 266(8):4890–4907, 2014.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] J. M. Ball and A. Zarnescu. Orientability and energy minimization in liquid crystal models. *Arch. Rational Mech. Anal.*, 202(2):493–535, 2011.

[6] F. Bethuel. A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 7(4):269 – 286, 1990.

[7] F. Bethuel. The approximation problem for Sobolev maps between two manifolds. *Acta Math.*, 167(3-4):153–206, 1991.

[8] F. Bethuel, H. Brezis, and J.-M. Coron. *Relaxed Energies for Harmonic Maps*, pages 37–52. Birkhäuser Boston, Boston, MA, 1990.
[9] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau Vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[10] F. Bethuel, H. Brezis, and G. Orlandi. Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions. *J. Funct. Anal.*, 186(2):432–520, 2001.

[11] F. Bethuel and D. Chiron. Some questions related to the lifting problem in Sobolev spaces. *Contemporary Mathematics*, 446:125–152, 2007.

[12] F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein. *A Cohomological Criterion for Density of Smooth Maps in Sobolev Spaces Between Two Manifolds*, pages 15–23. Springer Netherlands, Dordrecht, 1991.

[13] F. Bethuel, G. Orlandi, and D. Smets. Convergence of the parabolic ginzburg–landau equation to motion by mean curvature. *Comptes Rendus Mathematique*, 336(9):719 – 723, 2003.

[14] J. Bourgain, H. Brezis, and P. Mironescu. Lifting in Sobolev spaces. *Journal d’Analyse Mathématique*, 80(1):37–86, 2000.

[15] J. Bourgain, H. Brezis, and P. Mironescu. Lifting, degree, and distributional jacobian revisited. *Communications on Pure and Applied Mathematics*, 58(4):529–551, 2005.

[16] H. Brezis and H.-M. Nguyen. The Jacobian determinant revisited. *Invent. Math.*, 185(1):17–54, 2011.

[17] H. Brezis and L. Nirenberg. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.*), 1(2):197–263, 1995.

[18] G. Canevari. Biaxiality in the asymptotic analysis of a 2D Landau-de Gennes model for liquid crystals. *ESAIM : Control, Optimisation and Calculus of Variations*, 21(1):101–137, 2015.

[19] G. Canevari. Line defects in the small elastic constant limit of a three-dimensional Landau-de Gennes model. *Archive for Rational Mechanics and Analysis*, 223(2):591–676, Feb 2017.

[20] G. Canevari and G. Orlandi. Topological singular set of vector-valued maps, II: $\Gamma$-convergence for Ginzburg-Landau type functionals. In preparation.

[21] D. Chiron. On the definitions of Sobolev and BV spaces into singular spaces and the trace problem. *Commun. Contemp. Math.*, 9(4):473–513, 2007.

[22] J. Dávila and R. Ignat. Lifting of BV functions with values in $S^1$. *Comptes Rendus Mathematique*, 337(3):159–164, 2003.

[23] P. G. De Gennes and J. Prost. *The Physics of Liquid Crystals*. International series of monographs on physics. Clarendon Press, 1993.
[24] T. De Pauw and R. Hardt. Rectifiable and flat $G$ chains in a metric space. *Amer. J. Math.*, 134(1):1–69, 2012.

[25] T. De Pauw and R. Hardt. Some basic theorems on flat $G$ chains. *J. Math. Anal. Appl.*, 418(2):1047–1061, 2014.

[26] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[27] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[28] W. H. Fleming. Flat chains over a finite coefficient group. *Trans. Amer. Math. Soc.*, 121:160–186, 1966.

[29] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations. I*, volume 37 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998. Cartesian currents.

[30] M. Giaquinta and D. Mucci. The BV-energy of maps into a manifold: relaxation and density results. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 5(4):483–548, 2006.

[31] D. Golovaty and J. A. Montero. On minimizers of a Landau-de Gennes energy functional on planar domains. *Arch. Rational Mech. Anal.*, 213(2):447–490, 2014.

[32] F. Hang and F.-H. Lin. Topology of Sobolev mappings. II. *Acta Math.*, 191(1):55–107, 2003.

[33] R. Hardt, D. Kinderlehrer, and F.-H. Lin. Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys.*, 105(4):547–570, 1986.

[34] R. Hardt and F.-H. Lin. Mappings minimizing the $L^p$ norm of the gradient. *Comm. Pure Appl. Math.*, 40(5):555–588, 1987.

[35] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[36] M. W. Hirsch. *Differential topology*. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.

[37] R. Ignat. The space $BV(S^2, S^1)$: minimal connection and optimal lifting. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 22(3):283–302, 2005.

[38] R. Ignat and X. Lamy. Lifting of $\mathbb{RP}^{d-1}$-valued maps in $BV$ and applications to uniaxial $Q$-tensors. With an appendix on an intrinsic BV-energy for manifold-valued maps. Preprint arXiv: 1706.01281, 2017.
[39] R. L. Jerrard. Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.*, 30(4):721–746, 1999.

[40] R. L. Jerrard and H. M. Soner. The Jacobian and the Ginzburg-Landau energy. *Cal. Var. Partial Differential Equations*, 14(2):151–191, 2002.

[41] R. L. Jerrard and H. M. Soner. Rectifiability of the distributional Jacobian for a class of functions. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(8):683–688, 1999.

[42] F.-H. Lin and T. Riviè re. Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents. *J. Eur. Math. Soc. (JEMS)*, 1(3):237–311, 1999.

[43] A. Majumdar and A. Zarnescu. Landau-De Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. *Arch. Rational Mech. Anal.*, 196(1):227–280, 2010.

[44] N. D. Mermin. The topological theory of defects in ordered media. *Rev. Modern Phys.*, 51(3):591–648, 1979.

[45] M. R. Pakzad. Weak density of smooth maps in $W^{1,1}(M, N)$ for non-abelian $\pi_1(N)$. *Ann. Global Anal. Geom.*, 23(1):1–12, Mar 2003.

[46] M. R. Pakzad and T. Riviè re. Weak density of smooth maps for the Dirichlet energy between manifolds. *Geom. Funct. Anal.*, 13(1):223–257, 2003.

[47] T. Riviè re. Dense subsets of $H^{1/2}(S^2, S^1)$. *Ann. Global Anal. Geom.*, 18(5):517–528, 2000.

[48] É. Sandier. Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.*, 152(2):379–403, 1998. see Erratum, ibidem 171, 1 (2000), 233.

[49] É. Sandier and S. Serfaty. *Vortices in the magnetic Ginzburg-Landau model*. Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston, Inc., Boston, MA, 2007.

[50] R. Schoen and K. Uhlenbeck. Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom.*, 18(2):253–268, 1983.

[51] B. White. The deformation theorem for flat chains. *Acta Math.*, 183(2):255–271, 1999.

[52] B. White. Rectifiability of flat chains. *Ann. of Math. (2)*, 150(1):165–184, 1999.

[53] B. White. Topics in Geometric Measure Theory, 2012. Lecture notes taken by O. Chodosh for a course given at Stanford University, available at the URL: https://web.math.princeton.edu/~ochodosh/GMTnotes.pdf.

[54] H. Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.