Classifying spaces for homogeneous manifolds and their related Lie isometry deformations†

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Abstract

Among plenty of applications, low-dimensional homogeneous spaces appear in cosmological models as both, classical factor spaces of multidimensional geometry and minisuperspaces in canonical quantization.

Here a new tool to restrict their continuous deformations is presented: Classifying spaces for homogeneous manifolds and their related Lie isometry deformations.

The adjoint representation of \( n \)-dimensional real Lie algebras induces a natural topology \( \kappa^n \) on their classifying space \( K^n \). \( \kappa^n \) encodes the natural algebraic relationship between different Lie algebras in \( K^n \). For \( n \geq 2 \) this topology is not Hausdorffian. Even more it satisfies only the separation axiom \( T_0 \), but not \( T_1 \), i.e. there is a constant sequence in \( K^n \) which has a limit different from the members of the sequence. Such a limit is called a transition.

Recently it was found that transitions in \( K^n \) are the natural generalization and transitive completion of the well-known Inönü-Wigner contractions. For \( n \leq 4 \) the relational classifying spaces \( (K^n, \kappa^n) \) are constructed explicitly.

A Lie algebra \( A_n \) of low dimension \( n \leq 4 \) naturally acts as an isometry \( G_n = e^{A_n} \) of some homogeneous space; so (locally) a homogeneous Riemannian 3-space is either of Kantowski-Sachs type, with a transitive \( G_4 \), or it corresponds to one of the Bianchi types with a transitive \( G_3 \).

Calculating their characteristic scalar invariants via triad representations of the characteristic isometry, local homogeneous Riemannian 3-spaces are classified in their natural geometrical relations to each other. Their classifying space is a composition of pieces with different isometry types. Although it is Hausdorffian, different \( \kappa^3 \)-transitions to the same limit in \( K^3 \) may induce locally non-Euclidean regions (e.g. at Bianchi VII\(_0 \)).

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1 Introduction

Classical cosmological models are formulated in terms of homogeneous Riemannian 3-manifolds evolving through a 1 + 3-dimensional space-time. They can be naturally generalized to hypersurface homogeneous models of arbitrary dimension, including additional internal spaces, each of which is homogeneous and Riemannian. A multidimensional model is one for which all higher dimensional hypersurfaces decompose into a direct product of Riemannian factor spaces, with the same fixed dimensions on all hypersurfaces. (A generalization with dynamical dimensions has been considered recently in [1].) For a hypersurface homogeneous multidimensional model, the scales of the homogeneous factor spaces provide the only dynamical degrees of freedom. They coordinize the minisuperspace, which in this case is a Minkowski space with dimension equal to the number of Riemannian factor spaces of the underlying multidimensional hypersurface. In a more general model, the homogeneous factor spaces might no longer be independent, but their scales may interact locally. Then, the minisuperspace of this generalized multidimensional model is a homogeneous Lorentzian manifold.

Hence, a classification of local homogeneous manifolds of given dimension, is a clue to a systematic understanding of the possibilities for dynamical deformations of both, the factor space geometries of homogeneous generalized multidimensional cosmological models and the minisuperspace in canonical quantization. In the former application, the relevant manifolds are Riemannian, in the latter one Lorentzian. In general however, the problem to determine the proper classifying spaces is too hard to be solved. Therefore in this work, we will restrict to a complete description just for local Riemannian 3-manifolds. (In [2] the Lorentzian 3-manifolds were examined analogously, for specific orientations of their null hypersurfaces.)

Since the local isometry subgroups of a local homogeneous manifold are Lie groups, it is useful, before trying to classify the homogeneous manifolds, to find first all possible contractions and, more generally, all possible limit transitions between real (or complex) Lie algebras of fixed dimension, and to uncover the natural topological structure of the space of all such Lie algebras. The topology is given by the algebraic properties of the Lie algebras. The space \( W^n \) of all structure constants of real \( n \)-dimensional Lie algebras carries the subspace topology induced from the Euclidean \( \mathbb{R}^{n^3} \) (see [3]). The quotient topology \( \kappa^n \), obtained from this topology w.r.t. equivalence by \( \text{GL}(n) \) isomorphisms, renders the space \( K^n \) of all \( n \)-dimensional Lie algebras into a \( T_0 \) topological space, which is not \( T_1 \) for \( n \geq 2 \). This non-\( T_1 \) topology has first been described in [4]. In Sec. 3 below, we derive it with some new method, employing an index function on the algebra. This approach is somehow inspired by Morse theory. For \( n \geq 2 \), the space \( K^n \) contains some non-closed point \( A \), which has a special limit, to another point \( B \). The inverse of such a transition from \( A \) to \( B \) is a deformation of the algebra of \( B \) into the algebra \( A \). Note that, unlike in [5, 6], here transitions are defined to include also trivial constant limits. This has the advantage that also a trivial contraction (e.g. in the sense of Inönü-Wigner) is a transition. This definition is fully compatible with a partial order \( A \geq B \), which is taken to be the specialization order already used in [5, 6]. This choice of partial order is naturally related to a Morse like potential \( J \), decomposing \( K^n \) into subsets of different level.

Once the structure of the classifying space \( K^n \) is known, this information can be used as a first ingredient to construct the space of local Riemannian \( n \)-manifolds. This is demonstrated explicitly for \( n = 3 \) below. The paper is organized by the following sections.

Sec. 2 reviews the relevant local features of homogenous spaces. Sec. 3 derives and describes the classifying spaces \((K^n, \kappa^n)\) of \( n \)-dimensional Lie algebras. An index function \( J : K^n \to \mathbb{N}_0 \) is related naturally to the topology \( \kappa^n \).

Sec. 4 shows how the well known Inönü-Wigner [7] and Saletan [8] contractions correspond to special transitions. Unlike general transitions, the Inönü-Wigner contractions are well...
studied in the literature (e.g. up to real dimension 4 they have been classified in [9]).

In Sec. 5 we review the Bianchi Lie algebras, which constitute the elements of $K^3$. Using the index $J$ of Sec. 3, the topology $\kappa^3$ is described explicitly for both, the real and complex case.

Sec. 6 relates the topology $\kappa^n$ to the Zariski topology, and explains, via Lie algebra cohomology, why semisimple Lie algebras, and more generally all rigid ones, do not admit deformations in the category given by $K^n$.

Sec. 7 then classifies all local homogeneous Riemannian 3-manifolds using their isometries in $K^3$, and Sec. 8 discusses the results in the light of possible consequences for an evolution of the characteristic symmetries of fundamental theories like quantum gravity and cosmology.

\section{Homogenous manifolds}

Here we want to investigate the data which characterizes the local structure of a homogeneous Riemannian or pseudo Riemannian manifold $(M, g)$. If we consider for arbitrary dimension $n$ the different possible signatures modulo the reflection $g \rightarrow -g$, then $1 + \left[\frac{n}{2}\right]$ different signature classes are distinguished by the codimension $s = 0, \ldots, n - \left[\frac{n}{2}\right]$ of the characteristic null hypersurface in the tangent space. For Lorentzian signature $s = 1$ the latter is an $(n-1)$-dimensional open double cone at the basepoint, in the Riemannian case $s = 0$ it is just the basepoint itself.

Per definition, a homogeneous manifold admits a transitive action of its isometry group. Let us restrict here to the case where it has even more a simply transitive subgroup of the isometry group. In this case we can solder the metric to an orthogonal frame spanned by the Lie algebra generators $e_i$ in the tangent space, i.e.

$$g_{\mu\nu} = e^a_{\mu} e^b_{\nu} g_{ab}$$

where $e^a = e^a_{\mu} dx^\mu = g^{ai} e_i$, $e_i = e^\mu_i \frac{\partial}{\partial x^\mu}$, $g^{ab} g_{ij} = \delta^a_i \delta^b_j$, with the constant metric

$$(g_{ab}) = \begin{pmatrix}
\epsilon_1 e^s & 0 & 0 \\
0 & \epsilon_2 e^{s+w-t} & 0 \\
0 & 0 & \epsilon_3 e^{s-t}
\end{pmatrix}. \quad (2.2)$$

Here $s$ fixes the overall scale, while $t$ and $w$ parametrize the anisotropies related respectively to the $e_1$ and $e_2$ direction (maintaining isotropy in the respective orthogonal planes).

The local data can be rendered in form of (i) the local scales of (2.2), (ii) the covariant derivatives

$$De^k = e^k_{ij} e^i e^j := e^k_{\alpha\beta} dx^\alpha dx^\beta$$

of the dual generators $e^k$ in the cotangent frame, (iii) the corresponding Lie algebra

$$[e_i, e_j] = C^k_{ij} e_k, \quad (2.4)$$

and (iv) the orientation of the $n-s$-dimensional null hyperspace in the tangent space. For the Lorentzian case $s = 1$, this orientation is described by the future oriented normal vector $n$ along the central axis of the double cone,

$$n = n^a e_a, \quad (2.5)$$

where its triad frame components $n^a$ have to be coordinate independent, since the manifold is assumed to be homogeneous.
Let us consider now $n = 3$. In this case, there are no further signature cases besides the Riemannian and Lorentzian ones. For the 3 special cases $n^a = \delta^a_i$, $i = 1, \ldots, 3$ an explicit description of the Lorentzian 3-spaces of nonflat Bianchi type has been given in [2]. However a complete classification of all homogeneous Lorentzian 3-spaces need to control systematically the effect of different orientations (2.3). Presently, this problem still remains to be solved.

Therefore let us consider in the following only the Riemannian case, where the datum (iv) is trivial. Sec. 8 will give the complete classification of local homogeneous 3-spaces with some isometry subgroup in $K^3$. Moreover, the Kantowski-Sachs (KS) spaces, here the only exception not admitting a simply-transitive subgroup of their isometry group, can be obtained as a specific limits of Bianchi IX spaces. The global geometrical correspondence of such a limit is given by a hyper-cigar like 3-ellipsoid of topology $S^3 \times \mathbb{R}$. So finally we will have a classification of all local homogeneous Riemannian 3-manifolds.

3 Classifying spaces $K^n$ of $n$-dimensional Lie algebras

A (real) (finite-dimensional) Lie algebra is a (real) vector space $V$ of dimension $n$, equipped with a skew symmetric bilinear product $[\cdot, \cdot]$, satisfying the Jacobi condition $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. The evaluation of the Lie bracket $[\cdot, \cdot]$ on a complete set of basis vectors $\{e_i\}_{i=1,\ldots,n}$ yields a description of the Lie algebra by a set of structure constants $\{C^k_{ij}\}_{i,j,k=1,\ldots,n}$ from Eq. (2.4). Equivalently the endomorphisms $C_i := \text{ad}(e_i)$, $i = 1, \ldots, n$, from the adjoint representation $\text{ad} : e_i \rightarrow [e_i, \cdot]$, carry the same information on the algebra. Note that this description is overcomplete: Due to its antisymmetry, the Lie algebra is already completely described by the $(n-1)$ matrices $C_{<i>}$, $i = 1, \ldots, n$, each with components $C^k_{ij}$, $j, k = 1, \ldots, n-1$. But, as we will see, also this description may still carry redundancies.

The bracket $[\cdot, \cdot]$ defines a Lie algebra, iff the structure constants satisfy the $n\{(\binom{n}{2}) + (\binom{n}{3})\}$ antisymmetry conditions

$$C^k_{[ij]} = 0,$$

(3.1)

and, corresponding to the Jacobi condition, the $n \cdot (\binom{n}{3})$ quadratic compatibility constraints

$$C^l_{[ij]} C^m_{lk} = 0$$

(3.2)

with nondegenerate antisymmetric indices $i, j, k$.

Here we only deal with finite-dimensional Lie algebras. Hence the adjoint representation in $\text{End}(V)$ gives a natural associiative matrix representation of the algebra, generated by the matrices $C_i$. Using this representation, the associativity of the matrix product $C_i \cdot C_j$ implies $[C_i, C_j] = C^k_{ij} C^l_{lk}$ that the Jacobi condition (3.2) is an identity following already from Eq. (3.1). However, if we do not use this extra knowledge from the adjoint representation, then, for $n > 2$, Eq. (3.2) yields algebraic relations independent of Eq. (3.1).

For $n \geq 2$ there exists an irreducible tensor decomposition $C = D + V$, i.e.

$$C^k_{ij} = D^k_{ij} + V^k_{ij},$$

(3.3)

where $D$ is the tracefree part, i.e. $\text{tr}(D_i) := D^k_{ik} = 0$, and $V$ is the vector part,

$$V^k_{ij} := \delta^k_i v_{kj},$$

(3.4)

given by $v_i := \frac{1}{2n} \text{tr}(C_i)$, $i = 1, \ldots, n$. The Lie algebra is tracefree (corresponding Lie groups are unimodular) iff $V \equiv 0$, and it is said to be of pure vector type iff $D \equiv 0$. For each $n$, there
exists exactly one non-Abelian pure vector type Lie algebra, denoted by $V^n$. For $n = 3$, the latter is the Bianchi type $V$, and the decomposition \((3.3)\) is given by

$$
C^k_{ij} = \varepsilon_{ijkl}n^{lk} + \varepsilon^{ijk}a_m, \quad D^k_{ij} = \varepsilon_{ijkl}n^{lk}, \quad v_i = 2a_i, \quad (3.5)
$$

where $n^{ij}$ is symmetric and $\varepsilon^{ijk}$ is the usual antisymmetric tensor (cf. also [10]). Hence for $n = 3$, the Jacobi condition \((3.2)\) can be written as

$$
n^{lm}a_m = 0. \quad (3.6)
$$

These 3 nontrivial relations are in general independent of Eq. \((3.1)\).

For arbitrary $n$, the space of all sets $\{C^k_{ij}\}$ satisfying the Lie algebra conditions \((3.1)\) and \((3.2)\) is a subvariety $W^n \subset \mathbb{R}^{n^3}$, with a dimension

$$
\dim W^n \leq n^3 - \frac{n^2(n + 1)}{2} = \frac{n^2(n - 1)}{2}, \quad (3.7)
$$

bounded by Eq. \((3.1)\). For $n \geq 3$ the inequality is strict, because \((3.2)\) is non-trivial in general. For $n = 3$, the bound \((3.7)\) reads $\dim W^n \leq 9$, and taking into account the 3 additional relations of Eq. \((3.6)\) actually yields $\dim W^n = 6$.

GL($n$) basis transformations act on a given set of structure constants as GL($n$) tensor transformations:

$$
C^k_{ij} \rightarrow \tilde{C}^k_{ij} := (A^{-1})^k_h C^h_{fg} A^f_j A^g_i \quad \forall A \in \text{GL}(n). \quad (3.8)
$$

On $W^n$ this yields a natural equivalence relation $C \sim \tilde{C}$, defined by

$$
C^k_{ij} \sim \tilde{C}^k_{ij} : \Leftrightarrow \exists A \in \text{GL}(n) : \tilde{C}^k_{ij} = (A^{-1})^k_h C^h_{fg} A^f_j A^g_i, \quad (3.9)
$$

with associated projection $\pi$ to the quotient space,

$$
\pi : \begin{cases} 
W^n \rightarrow K^n := W^n/\text{GL}(n) \\
C \rightarrow [C]
\end{cases} \quad (3.10)
$$

$$
\dim W^n > \dim K^n \geq \dim W^n - n^2. \quad (3.11)
$$

The upper bound in Eq. \((3.11)\) is a strict one, because multiples of $\mathbb{1} \in \text{GL}(n)$ give rise to equivalent points of $W^n$. Note however that, while, for a given $C \in W^n$, certain transformations $A \in \text{GL}(n)$ transform $C \rightarrow \tilde{C} \neq C$, others keep $C = \tilde{C}$ invariant. The latter transformations constitute the automorphism group $\text{Aut}(C) \subset \text{GL}(n)$ of the adjoint representation associated with $C$. In general, the GL($n$) action on $W^n$ is not free, i.e. there exist points $C$ with $\dim \text{Aut}(C) > 0$. So, Eqs. \((3.7)\) and \((3.11)\) provide only very weak bounds on $\dim K^n$, which is still unknown for general $n$ (in the complex case, a more sophisticated upper bound estimate has been given in [11]). Note also that, e.g. for $n = 3$, the lower bound is trivial, because $\dim W^3 - 3^2 = -3 < 0$. Actually $\dim \text{Aut}(C) \geq 3$ for all $C \in W^3$. In general, let us define the automorphic dimension of $W^n$ as

$$
\dim_{\text{Aut}}(W^n) := \min_{C \in W^n} \{\dim \text{Aut}(C)\}. \quad (3.12)
$$

For any $A \in K^n$, consider a 1-parameter family of neighbourhoods $U_\varepsilon(A) \subset H(A)$ within the Hausdorff connected component $H(A)$ of $A$. Let us define the dimension of the infinitesimal Hausdorff connected neighbourhood of $A$ as

$$
\dim H(A) := \lim_{\varepsilon \to 0} \dim U_\varepsilon(A) \quad (3.13)
$$
Then,
\[ \dim W^n = \max_{C \in W^n} \{ \dim \pi^{-1}([C]) + \dim H([C]) \} \]
\[ \leq \max_{C \in W^n} \{ \dim \pi^{-1}([C]) \} + \max_{C \in W^n} \{ \dim H([C]) \} \]
\[ = n^2 - \min_{C \in W^n} \{ \dim \text{Aut}(C) \} + \dim K^n. \quad (3.14) \]

Using (3.12), the lower bound of Eq. (3.11) can be sharpened yielding
\[ \dim W^n > \dim K^n \geq \dim W^n - n^2 + \dim \text{Aut}(W^n). \quad (3.15) \]
Note that \( \dim \text{Aut}(C) = \dim \text{Aut}([C]) \) for any \( C \in W^n \). So \( \dim \text{Aut}(W^n) \) actually depends only on \( K^n \), and
\[ \dim \text{Aut}(W^n) = \max_{C \in W^n} \{ \dim \text{Aut}([C]) \} = \min_{A \in K^n} \{ \dim \text{Aut}(A) \} =: \dim \text{Aut}(K^n) \quad (3.16) \]
is the automorphic dimension of \( K^n \).

The space \( K^n \) of isomorphism classes of \( n \)-dimensional Lie algebras is naturally rendered a topological space \((K^n, \kappa^n)\), where the quotient topology \( \kappa^n \) is generated by the projection \( \pi \) from the subspace topology on \( W^n \subset \mathbb{R}^n \). In order to describe \((K^n, \kappa^n)\), let us first recall the axioms of separation (German: Trennung; cf. e.g. [12]):

\( T_0 \): For each pair of different points there is an open set containing only one of both.
\( T_1 \): Each pair of different points has a pair of open neighbourhoods with their intersection containing none of both points.
\( T_2 \) (Hausdorff): Each pair of different points has a pair of disjoint neighbourhoods.

It holds: \( T_2 \Rightarrow T_1 \Rightarrow T_0 \). Often it is more convenient to use the equivalent characterization of the separation axioms in terms of sequences and their limits:
\( T_0 \Leftrightarrow \) For each pair of points there is a sequence converging only to one of them.
\( T_1 \Leftrightarrow \) Each constant sequence has at most one limit.
\( T_2 \Leftrightarrow \) Each sequence, indexed by a directed partially ordered set, has at most one limit.
\( T_1 \) is equivalent to the requirement that each 1-point set is closed. Actually, for \( n \geq 2 \), the topology \( \kappa^n \) is not \( T_1 \), but only \( T_0 \). This means that there exists some point \( A \in K^n \), which is not closed, or in other words, there is a non-trivial transition from \( A \) to \( B \neq A \) in \( \text{cl}\{A\} \). Non-trivial \((A \neq B)\) transitions are special limits, which exist only due to the non-\( T_1 \) property of \( \kappa^n \). Here transitions from \( A \) to \( B \) are defined by
\[ A \geq B : \Leftrightarrow B \in \text{cl}\{A\}. \quad (3.17) \]

By this definition, transitions are transitive and yield a natural partial order. A transition \( A \geq B \) is non-trivial, if \( A > B \).

In the following we want to construct a minimal graph for the classifying space \((K^n, \kappa^n)\). Let us associate an arrow \( A \rightarrow B \) to a pair of algebras \( A, B \in K^n \), with \( A > B \), such that there exists no \( C \in K^n \) with \( A > C > B \). We call \( A \) the source and \( B \) the target of the arrow \( A \rightarrow B \). Now we define a discrete index function \( J : K^n \rightarrow \mathbb{N}_0 \) as following: We start with the unique minimal element \( \Gamma^n \), to which we assign the minimal index \( J(\Gamma^n) = 0 \). Then, for \( i \in \mathbb{N}_0 \), we assign the index \( J(S) = i + 1 \) to the source algebra \( S \) of any arrow pointing towards a target algebra \( T \) of index \( J(T) = i \), until, eventually for some index \( J = i_{\text{max}} \) there is no arrow to any target algebra \( T \) with \( J(T) = i_{\text{max}} \). Let us denote the subsets of all elements with index \( i \) as levels \( L(i) \subset K^n \).

For \( n \geq 2 \), \( K^n \) is directed towards its minimal element, the Abelian Lie algebra \( \Gamma^n \), constituting its only closed point. For \( n \geq 3 \) there are points in \( K^n \) which are neither open nor closed.
Open points correspond to locally rigid Lie algebras \( C \), i.e. those which cannot be deformed to some \( A \geq C \) with index \( J(A) > J(C) \). In this sense, the open points in \( K^n \) are its locally maximal elements.

Isolated open points correspond to rigid Lie algebras \( C \), i.e. those which cannot be deformed to any \( A \in K^n \) with \( A \not\leq C \) and index \( J(A) \geq J(C) \). In this sense, the isolated open points are the locally isolated maximal elements. In Sec. 7 below, the isolated open points are considered also from a dual perspective.

\( K_1 \) contains only the Abelian algebra \( I^1 \). \( K_2 \) contains 2 algebras, the Abelian \( I^2 \) and the isolated open point \( V^2 \), with a non-trivial transition \( V^2 \rightarrow I^n \). At the end of Sec. 6 the minimal graph for \( K_3 \) is given explicitly. In \([5,6]\) the topological structure of \( K_4 \) (which will not be displayed here) has been constructed likewise.

4 The special transitions of Inönü-Wigner and Saletan

Special kinds of transitions on a certain 2-point set \( \{ A, B \} \) of Lie algebra isomorphism classes are the contractions of Inönü-Wigner [8] and their generalization by Saletan [9].

Consider a 1-parameter set of matrices \( A_t \in \text{GL}(n) \) with \( 0 < t \leq 1 \), having a well defined matrix limit

\[
A_0 := \lim_{t \to 0} A_t
\]

which is singular, i.e. \( \det A_0 = 0 \).

For given structure constants \( C_{ij}^k \) of a Lie algebra \( A \) let us define for \( 0 < t \leq 1 \) further structure constants

\[
C_{ij}^k(t) := (A_t^{-1})_{ik} C_{fg}^h (A_t)_i^f (A_t)_j^g,
\]

which, according to (3.9), all describe the same Lie algebra \( A \).

If there is a well defined limit \( C_{ij}^k(0) := \lim_{t \to 0} C_{ij}^k(t) \), which satisfies conditions (3.1) and (3.2), yielding well defined structure constants of a Lie algebra \( B \), then the associated transition \( A \leq B \) is called a (Saletan) contraction.

Moreover a contraction is called Inönü-Wigner contraction if there is a basis \( \{ e_i \} \) in which

\[
A(t) = \begin{pmatrix}
E_m & 0 \\
0 & t \cdot E_{n-m}
\end{pmatrix}
\]

\( \forall t \in [0,1] \),

where \( E_k \) denotes the \( k \)-dimensional unit matrix (cf. [8] and [13]). Given the decomposition (1.3), it was shown in [8] that, the limit \( C_{ij}^k(0) \) exists if \( e_i, i = 1, \ldots, m \) span a subalgebra \( W \) of \( A \), which then characterizes the contraction.

Saletan [8] gives also a technical criterion for the existence of the limit \( C_{ij}^k(0) \) defining his general contractions. In contrast to the case of Inönü-Wigner contractions, a general Saletan contraction might be nontrivially iterated.

Not every transition \( A \leq B \) corresponds to a contraction. While neither Inönü-Wigner nor Saletan contractions are transitive, general transitions are transitive. In this sense, the topological space \( K^n \) provides the natural transitive completion of the well known contractions of Inönü-Wigner and Saletan.

5 The classifying space \( K^3 \) of Bianchi Lie algebras

The elements of \( K^3 \) are well known to correspond to the famous Bianchi Lie algebras, classified independently by Lie [14] and Bianchi [15]. For all types of Bianchi Lie algebras I up to IX an explicit description can be given in terms of the nonvanishing matrices \( C_{<ij>}^k, i = 1, \ldots, 3 \), of some adjoint representation. This representation can be normalized modulo an overall scale
of the basis $e_1, e_2, e_3$, and moreover $C_3$ can be chosen in some normal form (use the Jordan normal form).

In the semisimple representation category, there are only the simple Lie algebras VIII $\equiv so(1,2) = su(1,1)$ and IX $\equiv so(3) = su(2)$, given by

$$
\begin{align*}
C_{<3>}(\text{VIII}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & C_{<1>}(\text{VIII}) &= C_{<2>}(\text{VIII}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
C_{<3>}(\text{IX}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & C_{<1>}(\text{IX}) &= C_{<2>}(\text{IX}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
$$

(5.1)

All other algebras are in the solvable representation category. They all have an Abelian ideal span{$e_1, e_2$}. Hence, with vanishing $C_{<1>} = C_{<2>} = 0$, they are described by $C_{<3>}$ only.

The "inhomogeneous" algebras VI$_0$ $\equiv e(1,1) = \text{iso}(1,1)$ (local isometry of a Minkowski plane) and VII$_0$ $\equiv e(2) = \text{iso}(2)$ (local isometry of an Euclidean plane) are determined by

$$
\begin{align*}
C_{<3>}(\text{VI}_0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & C_{<3>}(\text{VII}_0) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{align*}
$$

(5.2)

It holds $C_{<3>}(\text{VIII}) = C_{<3>}(\text{VII}_0) = C_{<3>}(\text{IX})$. Furthermore, $C_{<1>}(\text{VIII}) = C_{<3>}(\text{VI}_0)$ and $C_{<1>}(\text{IX}) = C_{<3>}(\text{VII}_0)$. So we find both transitions VIII $\leq$ VII$_0$ and IX $\leq$ VII$_0$, but only VIII $\leq$ VII$_0$. These inhomogeneous algebras are endpoints $h = 0$ of two 1-parameter sets of algebras, VI$_h$ and VII$_h$, for $h > 0$ given respectively as

$$
\begin{align*}
C_{<3>}(\text{VI}_h) &= \begin{pmatrix} h & 1 \\ 1 & h \end{pmatrix}, & C_{<3>}(\text{VII}_h) &= \begin{pmatrix} h & 1 \\ -1 & h \end{pmatrix}.
\end{align*}
$$

(5.3)

Note also that the 1-parameter set of algebras VI$_h$, $0 \leq h < \infty$ contains an exceptional decomposable point III := VI$_1 = V^2 \oplus \mathbb{R}$, where $V^2$ is the unique non-Abelian algebra of $K^2$. $C_{<3>}(\text{III})$ has exactly 1 zero eigenvalue, while for all other algebras VI$_h$ and VII$_h$, $0 \leq h < \infty$ the matrix $C_{<3>}$ has two different non-zero eigenvalues, which become equal only in the limit $h \to \infty$. If the geometric multiplicity of this limit is 1, then the latter corresponds to the Bianchi Lie algebra IV, representable with

$$
C_{<3>}(\text{IV}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

(5.4)

By geometric specialization of the algebra IV the geometric multiplicity of its eigenvalue is increases to 2, yielding the pure vector type algebra V, given by

$$
C_{<3>}(\text{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

(5.5)

By algebraic specialization of the algebra IV the eigenvalue becomes zero, yielding the Heisenberg algebra II, given by

$$
C_{<3>}(\text{II}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

(5.6)

This algebra can also be reached by via a direct transition from any of the algebras VI$_h$ and VII$_h$, $0 \leq h < \infty$. Finally both, geometric specialization of II and algebraic specialization of V, yield the unique Abelian Lie algebra I, given by

$$
C_{<i>}(\text{I}) = 0, \quad i = 1, \ldots, 3.
$$

(5.7)
Fig. 1 shows on each horizontal level the algebras of equal index, which are the sources for the level below, and possible targets for the level above.

Fig. 1: The topological space $K^3$ (right and left images have to be identified for the algebras IV and V; the locally maximal algebras IV, VI$^h$ and VII$^h$, $0 \leq h < \infty$, form a 1-parameter set of sources of arrows).
Fig. 2 gives the analogous picture for the space $K^3_{\mathbb{C}}$ of 3-dimensional complex Lie algebras.

Fig. 2: The topological space $K^3_{\mathbb{C}}$ (the locally maximal algebras IV, $e_h$, $0 \leq h < \infty$, form a 1-parameter set of sources of arrows).
6 Zariski dual topology and Lie algebra cohomology

Now note that for any topology there exists a dual topology by exchanging open and closed sets. Applied to the topology \( \kappa^n \), open points of \( K^n \) become closed and closed points become open for the dual topology. Furthermore source and target of arrows interchange in the dual topology, i.e. their arrows change their direction. In the dual topology the rigid Lie algebras correspond to isolated closed points. Actually all semisimple Lie algebras are such isolated closed points.

Recall now, that on any algebraic variety there is a unique topology, called the Zariski topology, such that its closed subsets correspond to algebraic subvarieties. In this sense the topology \( \kappa^n \) turns naturally out to be the dual of the Zariski topology on \( K^n \). So, what is the meaning of this Zariski dual topology of \( K^n \) and, more specifically of its closed points and isolated closed points? To answer that question, note first that the reversed arrows of the dual topology correspond to some "inverse limit" of the transitions along them. More generally the dual of any transition might be called spontaneous deformation in \( K^n \). This has to be distinguished from a parametrical deformation in \( K^n \), which is given by a continuous change of parameters within a Hausdorff connected component of \( K^n \). In the Zariski dual topology of \( K^n \), the closed points cannot be source of spontaneous deformations in \( K^n \), and the isolated closed points admit neither spontaneous nor parametrical deformations.

Actually deformations of Lie algebras can also be considered from a slightly different point of view, using Lie algebra cohomology, introduced in [16] and [17]. Let us consider a cochain complex

\[
\delta_0, \delta_1 : 0 \to C^0 \to C^1 \to C^2 \to \ldots
\]

Its cochain spaces

\[
C^k(A, \mathcal{R}) := \left\{ f : A \otimes \cdots \otimes A \to \mathcal{R} \mid f \text{ linear, antisymmetric} \right\}
\]

(6.2)

can generally be defined for any \( A \text{-module } \mathcal{R} \) over some Lie algebra \( A \). The coboundary operators are given by

\[
\delta_k f(x_1, \ldots, x_n) := \sum_{i=1}^k (-1)^{k+i} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_n)
+ \sum_{i,j=1}^k (-1)^{i+j} f(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n, [x_i, x_j]).
\]

(6.3)

\( f \) is an \( k \)-cocycle iff \( \delta_k f = 0 \). \( Z^k(A, \mathcal{R}) := \ker \delta_k \). \( f \) is an \( k \)-coboundary iff \( f = \delta_{k-1} g \). \( B^k(A, \mathcal{R}) := \text{im} \delta_{k-1} \). The coboundary \( (6.3) \) satisfies \( \delta^2 = 0 \), hence \( B^n(A, \mathcal{R}) \subset Z^n(A, \mathcal{R}) \) and the \( k \)-th cohomology is defined as \( H^k(A, \mathcal{R}) := Z^k(A, \mathcal{R})/B^k(A, \mathcal{R}) \).

Let us restrict now for simplicity to complexes \( (6.1) \) with \( \mathcal{R} = A \), where the left multiplication by \( x \) is just given by the adjoint action \( \text{ad}_x := [x, \cdot] \), and write \( C^k \equiv C^k(A, A) \) and \( H^k \equiv H^k(A, A) \), keeping in mind that these quantities all depend on the algebra \( A \in K^n \).

The deformation \( [\cdot, \cdot]_\varepsilon \) of the product \( [\cdot, \cdot] \) of some algebra \( A \), can be written as a formal power series

\[
[x, y]_\varepsilon = [x, y] + \varepsilon F_1(x, y) + \varepsilon^2 F_2(x, y) + \ldots
\]

(6.4)

If the product \( [\cdot, \cdot] \) is defined in some category, e.g. the category Lie products \( W^n \) of Lie algebras in \( K^n \), the formally deformed product \( [\cdot, \cdot]_\varepsilon \) is in general not well defined on the same category, but only on some extended category. In order to be still a product in the same category, here in \( W^n \), the deformed product \( [\cdot, \cdot]_\varepsilon \) has to satisfy an infinite number of
deformation equations, namely for all \( k \in \mathbb{N}_0 \) the coefficients of the formal power series have to satisfy
\[
\sum_{x,y,z \text{ cyclic}} \sum_{i+j=k} F_i(F_j(x,y),z) + F_j(F_i(x,y),z) = 0,
\]
(6.5)
with \( F_0 \equiv [\cdot,\cdot] \). For Lie products in \( W^n \), the equation for \( k = 0 \) corresponds to the Jacobi condition, and the \textit{infinitesimal deformation equation}, i.e. the equation for \( k = 1 \), can be expressed with \( \delta \equiv \delta_2 \) from (6.3) as
\[
\delta F_1 = 0.
\]
(6.6)
So we see that \( Z^2 \) is the set of \textit{infinitesimal deformations} of elements of \( W^n \). Some of these deformations yield the again the original algebra \( A \in K^n \), i.e. they are deformations along the \( \text{GL}(n) \)-orbit through \( F_0 \in W^n \). These \textit{trivial infinitesimal deformations} are elements of \( B^2 \). Hence \( H^2 \) contains just the \textit{non-trivial infinitesimal deformations}. If some algebra \( A \in K^n \) satisfies \( H^2 \equiv H^2(A,A) = 0 \), then this algebra cannot be source of \textit{infinitesimal deformations}. The latter may, corresponding to the definitions above, be divided into \textit{infinitesimal spontaneous deformations} and \textit{infinitesimal parametrical deformations}, where the former are the duals of transitions and the latter generate parametrical deformations within the Hausdorff connected component of \( A \). So, if \( A \in K^n \) is an isolated open point w.r.t. the original topology of \( K^n \) or, equivalently, an isolated closed point w.r.t. the Zariski dual topology of \( K^n \), then there are no non-trivial infinitesimal deformations of the product \( F_0 \in W^n \); rather all infinitesimal deformations are within \( \pi^{-1}(A) \subset W^n \). For these algebras \( H^2 = 0 \). In particular, the latter is known to be true for all semisimple Lie algebras.

7 Classifying local homogeneous Riemannian 3-manifolds

Now we will construct a classifying space for local homogeneous Riemannian 3-manifolds.

The KS spaces appear as a limit of Bianchi IX spaces, in which the Bianchi IX isometry is still maintained, but no longer transitive. Hence it is sufficient to consider local 3-manifolds of Bianchi Lie isometry.

For the present case of Riemannian 3-spaces \( g_{ab} \) has a definite sign. Let us consider the 3-geometry \textit{modulo} a transformation of the its global sign,
\[
g_{ab} \rightarrow -g_{ab}.
\]
(7.1)
Then we can normalize the global sign with \( \det(g_{ab}) > 0 \) to \( \epsilon_1 = \epsilon_3 = \epsilon_3 = 1 \) in Eq. (2.2).

The choice of real parameters \( s,t,w \) of Eq. (2.2) simplifies the following calculations in a specific triad basis for the Bianchi Lie algebras of Sec. 5. This basis is chosen in consistency with the representations of [10] and [18]. It can be represented by matrices \( (e^a_\alpha) \), with anholonomic \( a = 1,2,3 \) of the generators of the algebra, and holonomic coordinate columns \( \alpha = 1,2,3 \). The coordinates will also be denoted as \( x^1 =: x, x^2 =: y, x^3 =: z \). For the different Bianchi type these matrices take the following form:

Bianchi I:

\[
(e^a_\alpha) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix},
\]
(7.2)

Bianchi II:

\[
(e^a_\alpha) = \begin{bmatrix} 1 & -z \\ 0 & 1 \\ & & 1 \end{bmatrix},
\]
(7.3)
Bianchi IV:
\[
(e^a_α) = \begin{bmatrix} 1 & e^x & 0 \\ xe^x & e^x & 0 \end{bmatrix},
\] (7.4)

Bianchi V:
\[
(e^a_α) = \begin{bmatrix} 1 & e^x \\ e^x \\ e^x \end{bmatrix},
\] (7.5)

Bianchi VI, \( h = A^2 \):
\[
(e^a_α) = \begin{bmatrix} 1 & e^{Ax} \cosh x & -e^{Ax} \sinh x \\ -e^{Ax} \sinh x & e^{Ax} \cosh x & 0 \end{bmatrix},
\] (7.6)

Bianchi VII, \( h = A^2 \):
\[
(e^a_α) = \begin{bmatrix} 1 & e^{Ax} \cos x & -e^{Ax} \sin x \\ e^{Ax} \sin x & e^{Ax} \cos x & 0 \end{bmatrix},
\] (7.7)

Bianchi VIII:
\[
(e^a_α) = \begin{bmatrix} \cosh y \cos z & -\sin z & 0 \\ \cosh y \sin z & \cos z & 0 \\ \sinh y & 0 & 1 \end{bmatrix},
\] (7.8)

Bianchi IX:
\[
(e^a_α) = \begin{bmatrix} \cos y \cos z & -\sin z & 0 \\ \cos y \sin z & \cos z & 0 \\ -\sin y & 0 & 1 \end{bmatrix}.
\] (7.9)

For each of the lines \{VI\_h\} and \{VII\_h\} the parameter range is given by \( \sqrt{h} = A \in [0, \infty[ \). Let us also remind that, \( \text{III} := \text{VII}_1 \).

The structure constants can be reobtained from \( ds^2 \) and the triad by
\[
C_{ijk} = ds^2([e_i, e_j], e_k), \quad C_{ij}^k = C_{ijr}g^{rk}.
\] (7.10)

The metrical connection coefficients are determined as
\[
\Gamma^k_{ij} = \frac{1}{2}g^{kr}(C_{ijr} + C_{jri} + C_{irj}).
\] (7.11)

W.r.t. the triad basis, the curvature operator is defined as
\[
\Re_{ij} := \nabla_{[e_i,e_j]} - (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}),
\] (7.12)

the Riemann tensor components are
\[
R_{hijk} := < e_h, \Re_{ij} e_k >,
\] (7.13)

and the Ricci tensor is
\[
R_{ij} := R_{ikj} = \Gamma^f_{ij} \Gamma^e_{fe} - \Gamma^f_{ie} \Gamma^e_{jf} + \Gamma^e_{ij} C^f_{ej}.
\] (7.14)
From (7.14) we may form the following scalar invariants of the geometry: The Ricci curvature scalar
\[ R := R^i{}_i, \tag{7.15} \]
the sum of the squared eigenvalues
\[ N := R^i{}_j R^j_i, \tag{7.16} \]
the trace-free scalar
\[ S := S^i{}_j S^j{}_i = R^i{}_j R^j{}_k R^k{}_i - RN + \frac{2}{9} R^3, \tag{7.17} \]
where \( S^i{}_j := R^i{}_j - \frac{1}{3} \delta^i{}_j R \), and, related to the York tensor,
\[ Y := R^i_{k;j} g^{il} g^{jm} g^{kn} R_{lm;n}, \tag{7.18} \]
and, related to the York tensor,
\[ Y := R^i_{k;j} g^{il} g^{jm} g^{kn} R_{lm;n}, \tag{7.18} \]
with
\[ R^i_{j;k} := \epsilon^\alpha_i \epsilon^\beta_j \epsilon^\gamma_k R_{\alpha\beta\gamma} = \epsilon^\alpha_i \epsilon^\beta_j \epsilon^\gamma_k (\delta^m_i \delta^m_j + \delta^m_i \delta^m_j) R_{mn}. \]
The 4 scalar invariants above characterize a local homogeneous Riemannian 3-space.

It is \( N = 0 \), iff the Riemannian 3-space is the unique flat one. This has a transitive isometry of Bianchi type I, and also admits the left-invariant (but not transitive) action of the Bianchi group \( \text{VII}_0 \) on its 2-dimensional hyperplanes (cf. [19, 20]).

In the following we take the flat Riemannian 3-space as a center of projection for the non-flat Bianchi or KS geometries. These satisfy \( N \neq 0 \). The invariant \( N \) then parametrizes (like \( e^{-2s} \)) the homogeneous conformal scale on the 3-manifold under consideration. A homogeneous conformal, i.e. homothetic, rescaling of the metric,
\[ g_{ij} \rightarrow \sqrt{N} g_{ij}, \tag{7.19} \]
yields the following normalized invariants, which depend only on the homogeneously conformal class of the geometry:
\[ \hat{N} := 1, \quad \hat{R} := R/\sqrt{N}, \quad \hat{S} := S/N^{3/2}, \quad \hat{Y} := Y/N^{3/2}. \tag{7.20} \]

For a non-flat Riemannian space, the invariant \( \hat{Y} \) vanishes, iff the 3-geometry is conformally flat. Note that a general conformal transformation is not necessarily homogeneous. Hence there may exist homogeneous spaces, which are in the same conformal class, but in different homogeneously conformal classes.

Note that, under (7.1), \( N \) is invariant, while \( \hat{R}, \hat{S} \) and \( \hat{Y} \) just all reverse their sign. Furthermore, a rescaling (7.19) does not change the Bianchi or KS type of isometry.

So we can now concentrate on the classifying space of non-flat local homogeneous Riemannian 3-geometries modulo the global sign (7.1) and modulo homogeneous conformal transformations (7.19) for each fixed Bianchi type. This moduli space can be parametrized by the invariants \( \hat{R}, \hat{S} \) and \( \hat{Y} \), given for each fixed Bianchi type as a function of the anisotropy parameters \( t \) and \( w \).

A minimal cube, in which the classifying moduli space can be imbedded, is spanned by \( \hat{R}/\sqrt{3}, \sqrt{6} \hat{S} \in [-1, 1] \) and \( 2 \tanh \hat{Y} \in [0, 2] \).

Below, Fig. 3 describes those points of the moduli space which are of Bianchi types VI/VII or lower level, Fig. 4 likewise points of Bianchi types VIII/IX.
Fig. 3: Riemannian Bianchi geometries II, IV, V, VI$_h(w = 0)$, VII$_h(\sqrt{\kappa} = 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 1)$, VIII$_h(\sqrt{\kappa} = 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 1)$; w.r.t. the common origin, the axes of the 3 planar diagrams, are: $\hat{R}/\sqrt{3}$ to the right, $\sqrt{6}S$ up, and $2 \tanh \hat{Y}$ both, left and down.
Fig. 4: Riemannian Bianchi geometries II, V, VI₀, VI₁, VII₀, VIII(t, w) (t = −5, −1, 0, 1, 5), IX(t, w) (t = 0, ½, 1, 2, 5); w.r.t. the common origin, the axes of the 3 planar diagrams, are: \( \tilde{R}/\sqrt{3} \) to the right, \( \sqrt{6} S \) up, and 2 tanh \( \tilde{Y} \) both, left and down.
For a homogeneous space with 2 equal Ricci eigenvalues the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a double line $L_2$, which has a range defined by $|\hat{R}| \leq \sqrt{3}$ and satisfies the algebraic equation
\begin{equation}
162\hat{S}^2 = (3 - \hat{R}^2)^3.
\end{equation}
(7.21)
All other algebraically possible points of the $\hat{R}$-$\hat{S}$-plane lie inside the region surrounded by the line $L_2$. At the branch points $\hat{R} = \pm\sqrt{3}$ of $L_2$ all Ricci eigenvalues are equal. These homogeneous spaces possess a 6-dimensional isometry group. Homogeneous spaces possessing a 4-dimensional isometry group are represented by points on $L_2$.

If one Ricci eigenvalue equals $R$, i.e. if there exists a pair $(a, -a)$ of Ricci eigenvalues, the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a line $L_{+\pm}$, defined by the range $|\hat{R}| \leq 1$ and the algebraic equation
\begin{equation}
\hat{S} = \frac{11}{9}\hat{R}^3 - \hat{R}.
\end{equation}
(7.22)
In the case that one eigenvalue of the Ricci tensor is zero, the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a line $L_0$, defined by the range $|\hat{R}| \leq \sqrt{2}$ and the algebraic equation
\begin{equation}
\hat{S} = \frac{\hat{R}}{2}(1 - \frac{5}{9}\hat{R}^2).
\end{equation}
(7.23)

For Eqs. (7.21),(7.22),(7.23) see also [21].

At the branch points of the curve $L_2$ the Ricci tensor has a triple eigenvalue, which is negative for geometries of Bianchi type V, and positive for type IX geometries with parameters $(t, w) = (0, 0)$. These constant curvature geometries are all conformally flat with $\hat{Y} = 0$. Besides the flat Bianchi I geometry, the remaining conformally flat spaces with $\hat{Y}$ are the KS space $(\hat{R}, \hat{S}, \hat{Y}) = (\sqrt{2}, -\frac{\sqrt{2}}{18}, 0)$ and, point reflected, the Bianchi type IIIc, corresponding to the initial point of a Bianchi III line segment ending at the Bianchi II point in Fig. 3.

The point $(-1, 0, 0)$ of Fig. 3 admits both types, Bianchi V and VIIh with $h > 0$. Nevertheless, this point corresponds only to one homogeneous space, namely the space of constant negative curvature. This is possible, because this space has a 6-dimensional Lie group, which contains the Bianchi V and VIIh subgroups. Note that in the flat limit $V \rightarrow I$, the additional Bianchi groups VIIh change with $h \rightarrow 0$.

Similarly, the Bianchi III points of Fig. 3 lie on the curve $L_2$ of the $\hat{R}$-$\hat{S}$ diagram. However, these points are also of Bianchi type VIII. In fact, each of them correspond to one homogeneous geometry only. However, the latter admits a 4-dimensional isometry group, which has two 3-dimensional subgroups, namely Bianchi III and VIII, both containing the same 2-dimensional non-Abelian subgroup.

Altogether, the location of Riemannian Bianchi (and KS) spaces is consistent with the topology $\kappa^3$ of the space of Bianchi Lie algebras.

Our classifying moduli space of local homogeneous Riemannian 3-spaces is a $T_2$ (Hausdorff) space. But it is not a topological manifold: The line of VIIh moduli is a common boundary of 3 different 2-faces, namely that of the IX moduli, that of the VIII moduli, and with $h \rightarrow 0$ that of all moduli of type VIIh with $h > 0$. Like the moduli space, also the full classifying space is not locally Euclidean; rather both are stratifiable varieties.
8 Discussion: Evolution of fundamental symmetries

In this section we show up the possible application of our results to fundamental symmetries of physics. All applications typically involve changes of the symmetry of the system under consideration; they differ in the kind of system that evolves the considered Lie symmetries. Here only two examples of fundamental systems shall be mentioned: The evolution of cosmological models and the connection dynamics of quantum gravity.

The traditionally considered 1 + 3-dimensional inhomogeneous cosmological models with homogenous Riemannian 3-hypersurfaces (see [21, 22, 23]), and a more general class of multidimensional geometries, admit deformations between Riemannian Bianchi geometries. Such a deformation may also induce a change of the spatial anisotropy of the universe, which essentially affects physical quantities like its tunnelling rate [24]. Even for the more general multidimensional case, where \( M = \mathbb{R} \times M_1 \times \ldots \times M_n \) with homogeneous \( M_1 \), our results provide an important piece of information, namely the complete control over the possible continuous deformations of the homogeneous external 3-space \( M_1 \).

Furthermore, the superspace of homogeneous Riemannian 3-geometries plays a key role for an understanding of the canonical quantization of a homogeneous universe. The homogeneous conformal modes are just the homothetic scales, which span a 3-dimensional minisuperspace underlying the conformally equivariant quantization scheme [25, 26], yielding the Wheeler-deWitt equation for a given point of the moduli space of local homogeneous 3-geometries. Global properties of the homogeneous 3-geometry have not considered here. However it should be clear that a given global geometry according to one of the Thurston types exists only for a specific local geometries specified by characteristic points in our moduli space (cf. [27]).

The short distance regime of quantum gravity might be described in terms of connection dynamics, recently also related to spin networks [28]. While the standard theory is worked out for the compact structure group \( SU(2) \), the topology of \( K^3 \) suggests that this structure group could change by a transition to the noncompact group \( E(2) \) and similarly further, until the 3-dimensional Abelian group is reached. It remains an interesting question, what happens to connections, and moreover to the holonomy groups, under such a deformation. In [28] a q-deformation of \( SU(2) \) was suggested, in order to regularize infrared divergences. However, infrared divergences are obviously related to the macroscopic limit. Since in this limit the discrete structure of space-time is expected to become replaced by a continuous structure, we have no reason to expect that original spin network, or some related braid structures, might be pertained in the macroscopic theory. So a transition within the category of Lie algebras is more likely to provide the solution of the infrared problem, even more, since \( K^3 \)-transitions appear naturally in the cosmological evolution (e.g. related to the isotropization of a homogeneous universe).

Finally it is suggestive that, the index technique introduced in Sec. 3 might provide some kind of potential \( J \) on \( K^n \) determining the evolution of the symmetries under consideration. For \( n = 3 \), the \( SU(2) \) symmetry would be metastable and decay after some time to a state of \( K^3 \) which has a lower potential level, and so on, until the minimal state of Abelian symmetry is reached. The metastability of a higher level in the potential could have an explanation in the higher dimension of the corresponding subspaces of local homogeneous moduli of just that symmetry. Note that for symmetries in \( K^3 \), the dimensionality of the subspace of corresponding moduli increases with the level of the potential.
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Appendix

Here we list up the scalar invariants $\hat{R}_I$, $\hat{S}_I$ and $\hat{Y}_I$ for all non-flat Riemannian Bianchi geometries:

\[
\begin{align*}
\hat{R}_{II} &= -\frac{\sqrt{3}}{3} \\
\hat{S}_{II} &= \frac{16 \sqrt{3}}{81} \\
\hat{Y}_{II} &= \frac{8 \sqrt{3}}{9} \\
\hat{R}_{IV} &= -\frac{12 e^w + 1}{\sqrt{48 e^{2w} + 16 e^w + 3}} \\
\hat{S}_{IV} &= \frac{16 + 72 e^w}{9 (48 e^{2w} + 16 e^w + 3)^{3/2}} \\
\hat{Y}_{IV} &= \frac{8 + 8 e^w + 32 e^{2w}}{(48 e^{2w} + 16 e^w + 3)^{3/2}} \\
\hat{R}_V &= -\sqrt{3} \\
\hat{S}_V &= 0 \\
\hat{Y}_V &= 0
\end{align*}
\]

(A.1)

In the next formulas an auxiliary invariant $D$ simplifies the notation.

\[
\begin{align*}
D_{VI_h} &:= 3 + 4 (4 h + 1) e^w + 2 \left(1 + 16 h + 24 h^2 \right) e^{2w} \\
&+ 4 (4 h + 1) e^{3w} + 3 e^{4w} \\
\hat{R}_{VI_h} &= -(D_{VI_h})^{-\frac{1}{2}} \left(1 + 2 \left(1 + 6 h \right) e^w + e^{2w} \right) \\
\hat{S}_{VI_h} &= \frac{8}{9} (D_{VI_h})^{-\frac{3}{2}} (e^w + 1)^4 \left(2 + (9 h - 5) e^{w} + 2 e^{2w} \right) \\
\hat{Y}_{VI_h} &= 8 (D_{VI_h})^{-\frac{3}{2}} (e^w + 1)^2 \left(e^{4w} + (h - 1) e^{3w} \right) \\
&+ 2 \left(2 h^2 - 5 h + 2 \right) e^{2w} + (h - 1) e^w + 1
\end{align*}
\]

(A.2)

\[
\begin{align*}
D_{VII_h} &:= 3 + 4 (4 h - 1) e^w + 2 \left(1 - 16 h + 24 h^2 \right) e^{2w} \\
&+ 4 (4 h - 1) e^{3w} + 3 e^{4w} \\
\hat{R}_{VII_h} &= -(D_{VII_h})^{-\frac{1}{2}} \left(1 + 2 \left(6 h - 1 \right) e^w + e^{2w} \right) \\
\hat{S}_{VII_h} &= \frac{8}{9} (D_{VII_h})^{-\frac{3}{2}} (e^w - 1)^4 \left(2 + (9 h + 5) e^{w} + 2 e^{2w} \right) \\
\hat{Y}_{VII_h} &= 8 (D_{VII_h})^{-\frac{3}{2}} (e^w - 1)^2 \left(e^{4w} + (h + 1) e^{3w} \right) \\
&+ 2 \left(2 h^2 + 5 h + 2 \right) e^{2w} + (h + 1) e^w + 1
\end{align*}
\]

(A.3)
\[ D_{\text{VIII}} := 2 e^{-2 w t} + 3 e^{-2 w} + 4 e^{-2 w t} - 4 e^t + 4 e^w + 3 e^{2 w} \\
+ 4 e^{-w} + 3 e^{-2 w} - 4 e^{-w} + 4 e^{-2 w t} + 3 e^{-2 w t} + 2 + 4 e^{-w t} \\
- 4 e^t - 4 e^{-w} + 3 e^{2 t} \]

\[ \dot{R}_{\text{VIII}} = -\left( D_{\text{VIII}} \right)^{-\frac{1}{2}} (2 e^{-w} + e^{-w} + e^{-w} + 2 e^t) \]

\[ \dot{S}_{\text{VIII}} = -\frac{8}{9} (D_{\text{VIII}})^{-\frac{1}{2}} \left( 6 e^{w+2 t} + 3 e^{t+2 w} + 3 e^{5 t-2 w} + 14 e^{-3 w+3 t} \\
+ 6 e^{-3 w+2 t} + 6 e^{-3 w} + 3 e^{-3 w+4 t} - 3 e^{-3 w+3 t} + 6 e^{-w+4 t} - 14 e^t \\
- 3 e^{-3 w+5 t} - 2 e^3 - 2 e^{-3 w} - 15 e^{-w+t} - 3 e^{-2 w} \\
+ 15 e^{-2 w+3 t} + 18 e^w + 15 e^{-w+3 t} - 15 e^t - 3 e^3 \\
+ 15 e^t + 18 e^{-2 w+4 t} - 18 e^{-2 w+2 t} + 6 e^w - 42 e^{-w} + \left( 6 e^w - 15 e^{-2 w+2 t} - 2 e^{-3 w+6 t} + 14 \right) \right) \] (A.6)

\[ D_{\text{IX}} := 2 + 3 e^2 w + 3 e^2 w - 4 e^{-w} - 3 e^{-2 w+4 t} + 2 e^t \\
+ 3 e^{-2 w+2 t} + 4 e^{-w} - 4 e^{-2 w+3 t} - 4 e^t \\
+ 4 e^{-w+t} - 4 e^{-w+3 t} - 4 e^w + t \]

\[ \dot{R}_{\text{IX}} = (D_{\text{IX}})^{-\frac{1}{2}} \left( -e^w - e^{-w+2 t} - e^{-w} + 2 e^{-w+t} + 2 e^t \right) \]

\[ \dot{S}_{\text{IX}} = -\frac{8}{9} (D_{\text{IX}})^{-\frac{1}{2}} \left( 6 e^{w+2 t} + 3 e^{t+2 w} + 3 e^{5 t-2 w} + 14 e^{-3 w+3 t} \\
+ 6 e^{-3 w+2 t} + 6 e^{-3 w} + 3 e^{-3 w+4 t} - 3 e^{-3 w+3 t} + 6 e^{-w+4 t} - 14 e^t \\
+ 3 e^{-3 w+5 t} - 2 e^3 - 2 e^{-3 w} - 15 e^{-w+t} - 3 e^{-2 w} \\
+ 15 e^{-2 w+3 t} + 18 e^w + 15 e^{-w+3 t} - 15 e^t - 3 e^3 \\
+ 15 e^t - 18 e^{-2 w+4 t} - 18 e^{-2 w+2 t} + 6 e^w - 42 e^{-w} + \left( 6 e^w + 15 e^{-2 w+2 t} - 2 e^{-3 w+6 t} - 14 \right) \right) \] (A.7)
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