Semiclassical back reaction around a cosmic dislocation

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(Dated: September 27, 2004)

The energy-momentum vacuum average of a conformally coupled massless scalar field vibrating around a cosmic dislocation — a cosmic string with a dislocation along its axis — is taken as source of the linearized semiclassical Einstein equations. The solution up to first order in the Planck constant is derived. Motion of a test particle is then discussed, showing that under certain circumstances a helical-like dragging effect, with no classical analogue around the cosmic dislocation, is induced by back reaction.

PACS numbers: 04.60.-m, 04.62.+v, 11.27.+d

I. INTRODUCTION

Various aspects of fields around cosmic strings have been considered in the literature [1, 2]. The usual motivation for such investigations is that these objects may play some role in the cosmological scenario. There is though another equally appealing motivation. Namely, gravitational fields of cosmic strings are locally flat solutions of the Einstein equations, resulting that one can carry out calculations without meeting serious obstacles, and still revealing non trivial effects. Moreover, the physical content of such effects may not only be relevant in the context of classical and semiclassical general relativity, but also in condensed matter physics, where a geometric interpretation can be used to describe physical properties of some linear defects in solids [3].

As is well known, there is no Newtonian potential around an ordinary cosmic string of mass density $\mu$ [1]. Accordingly, a particle left at rest near such object will remain at rest. The study of semiclassical back reaction on the gravitational field of an ordinary cosmic string has shown that vacuum fluctuations (of a conformally coupled massless scalar field) induce a Newtonian potential ($c = 1$, unless stated otherwise),

$$\Phi(\rho) = -\frac{G\hbar \mathcal{F}(\mu)}{\rho^2},$$  \hspace{1cm} (1)

at a distance $\rho$ from the cosmic string [1, 2, 6, 7]. Subjected to $\Phi(\rho)$, a particle initially at rest experiences a presumably small, but non vanishing “quantum mechanical” force when $\mu \neq 0$.

The gravitational field of an ordinary cosmic string corresponds to the geometry of a conical spacetime, where the deficit angle is proportional to $\mu$ [1]. It has been conjectured in Ref. [8] that the gravitational field of a certain type of chiral cosmic string [9] may correspond to the geometry of a cosmic dislocation — a conical spacetime with a helical structure. It seems pertinent to extend investigations on semiclassical back reaction to this background, and that is the issue addressed here.

In Sec. II, the geometry of a cosmic dislocation is presented. In the following section, the vacuum average of the energy-momentum tensor for a conformally coupled massless scalar field [10] is used as source of the Einstein equations. The solution up to first order in $\hbar$ is determined, and then used in Sec. IV to discuss the induced force acting on a test particle. Sec. V closes the work with a summary and additional remarks.

II. THE CLASSICAL BACKGROUND

A cosmic dislocation has as line element [8, 11]

$$ds^2 = dt^2 - dr^2 - \alpha^2 r^2 d\theta^2 - (dz + \kappa d\theta)^2,$$  \hspace{1cm} (2)
with the usual identification \((t, r, \theta, z) \sim (t, r, \theta + 2\pi, z)\). The geometry in Eq. (2) is obtained from that of a conical spacetime with deficit angle \(2\pi(1 - \alpha)\), replacing \(dz\) by \(dz + \kappa d\theta\). When the cone parameter \(\alpha\) equals unity and the dislocation parameter \(\kappa\) vanishes, Eq. (2) becomes the Minkowski line element written in circular cylindrical coordinates.

In fact, by defining new coordinates \(\varphi := \alpha \theta\) and \(Z := z + \kappa \theta\), Eq. (2) becomes

\[
ds^2 = dt^2 - dr^2 - r^2 d\varphi^2 - dZ^2,
\]

where the identification

\[
(t, r, \varphi, Z) \sim (t, r, \varphi + 2\pi \alpha, Z + 2\pi \kappa)
\]
must be observed. It is clear from Eq. (3) the locally flat nature of the background, and from Eq. (4) its helical structure. A particle at rest will remain at rest and (locally) geodesics are simply straight lines.

It should be remarked that the geometry in Eq. (2) fits in general relativity as well as in the Einstein-Cartan theory \([11]\). In the context of general relativity there is a curvature singularity along the symmetry axis. In the Einstein-Cartan theory, there is also a torsion singularity along the symmetry axis, when \(\kappa \neq 0\).

### III. BACK REACTION

The non trivial global geometry encoded in Eq. (4) leads to vacuum polarization. Although vacuum averages of the fields themselves vanish, that is not necessarily the case for their renormalized energy-momentum tensors \(\langle T^\mu_\nu \rangle\). In particular, for a conformally coupled massless scalar field, the behavior of \(\langle T^\mu_\nu \rangle\) as \(\kappa/r \ll 1\) [in terms of the coordinates in Eq. (3)] is given by \([10]\)

\[
\langle T^\mu_\nu \rangle = \frac{\hbar}{r^4} \begin{pmatrix}
-A & 0 & 0 & 0 \\
0 & -A & 0 & 0 \\
0 & 0 & 3A & \kappa B/r^2 \\
0 & 0 & \kappa B & -A
\end{pmatrix},
\]

where \(A(\alpha) := (\alpha^{-4} - 1)/1440\pi^2\) and

\[
B(\alpha) := \frac{1}{32\pi^3 \alpha^2} \int_0^\infty d\tau \frac{\alpha \sin(\pi/\alpha)}{\cosh(\pi/\alpha) - \cos(\tau) + \tau \sinh(\tau)} \frac{\cos(\pi/\alpha) \cosh(\tau) - 1}{[\cosh(\tau) - \cos(\pi/\alpha)]^2 \cosh^4(\alpha \tau/2)}.
\]

Noting that \(A(1) = 0\) and

\[
B(1) = \frac{1}{60\pi^2},
\]

when \(\alpha = 1\) and \(\kappa = 0\) it follows that \(\langle T^\mu_\nu \rangle = 0\), which is consistent with the fact that the vacuum averages in Ref. \([10]\) are renormalized with respect to the Minkowski vacuum. [It should be pointed out that Eq. (5) holds for \(\kappa \neq 0\) only when \(\alpha \neq 1\), since for \(\alpha = 1\) the diagonal components vanish and subleading contributions depending on \(\kappa\) should be taken into account.]

The semiclassical approach to back reaction in general relativity is implemented by feeding the Einstein equations with the vacuum average in Eq. (5). Since \(\langle T^\mu_\nu \rangle\) is traceless, these equations are

\[
R^\mu_\nu = -8\pi G \langle T^\mu_\nu \rangle.
\]

The most general stationary axially symmetric line element, symmetric also with respect to translations along the axis, is

\[
ds^2 = g_{00}(r) dt^2 + 2g_{02}(r) dt \ d\varphi + g_{11}(r) dr^2 + g_{22}(r) d\varphi^2 + 2g_{23}(r) d\varphi \ dZ + g_{33}(r) dZ^2,
\]

whose form is clearly invariant under redefinition of the radial coordinate \(r \to \rho, r = f(\rho)\). This gauge freedom can be used to choose \(g_{22}(r) = -r^2\). The next step is to allow quantum perturbations of Eq. (4) consistent with Eq. (9). Then, discarding non physical solutions, the standard linearization procedure \([1, 2, 3, 4, 12]\) applied to Eqs. (9) leads to

\[
ds^2 = \left(1 - \frac{4\pi A G \hbar}{r^2}\right) (dt^2 - dZ^2) - \left(1 + \frac{16\pi A G \hbar}{r^2}\right) dr^2 - r^2 d\varphi^2 - \frac{4\pi \kappa B G \hbar}{r^2} \ d\varphi \ dZ,
\]
up to first order in $\hbar$. When $\kappa \neq 0$ and/or $\alpha \neq 1$, an inspection shows that the perturbed background is locally flat only asymptotically (i.e., as $r \to \infty$), and that the dislocation contributes ($\kappa \neq 0$) with non-vanishing off-diagonal components in the semiclassical metric tensor.

When $\kappa = 0$, Eq. (10) should reproduce the results in the literature regarding semiclassical back reaction on the gravitational field of an ordinary cosmic string. In order to implement this check, one uses the gauge freedom in choosing the radial coordinate, defining
\[
\rho := r + \frac{\lambda \pi AG \hbar}{r},
\]
where $\lambda$ is an arbitrary dimensionless parameter. Thus, Eq. (10) becomes
\[
ds^2 = \left(1 - \frac{4\pi AG \hbar}{\rho^2}\right)(dt^2 - d\varphi^2) - \left(1 + \left(\lambda + 8\frac{2\pi AG \hbar}{\rho^2}\right)\right) d\rho^2 - \rho^2 \left(1 - \frac{2\pi AG \hbar}{\rho^2}\right)\ d\varphi^2
\]
\[-\frac{4\pi \kappa BG \hbar}{\rho^2} d\varphi dZ.
\]
For $\kappa = 0$, the results in Refs. [4] and [7] are obtained from Eq. (12) by setting $\lambda = -10$ and $\lambda = -4$, respectively.

IV. DRAGGING EFFECT

A physical consequence of the perturbed geometry in Eq. (12) is revealed when studying the geodesic motion. In order to get rid of “inertial forces”, one sets $\lambda = 2$ in Eq. (12), resulting
\[
ds^2 = \left(1 - \frac{4\pi AG \hbar}{\rho^2}\right)(dt^2 - d\varphi^2 - d\rho^2) - \left(1 + \frac{20\pi AG \hbar}{\rho^2}\right) d\rho^2 - \frac{4\pi \kappa BG \hbar}{\rho^2} d\varphi dZ.
\]
Using the coordinate time $t$ (instead of the proper time) as an affine parameter, the geodesic equations corresponding to this gauge, up to first order in $\hbar$, yield (inserting dimensionful $c$)
\[
a_\rho = \frac{4\pi G \hbar}{c \rho^3} \left[-A \left(1 - \frac{7v_\rho^2}{c^2} + 5\frac{v_\varphi^2}{c^2} - \frac{v_Z^2}{c^2}\right) - \frac{\kappa B v_\varphi v_Z}{\rho c^2}\right],
\]
\[a_\varphi = \frac{4\pi G \hbar k B}{c \rho^2} \frac{v_\rho v_Z}{c^2},
\]
\[a_Z = \frac{8\pi G \hbar k B}{c \rho^2} \frac{v_\rho v_\varphi}{c^2},
\]
where $(v_\rho, v_\varphi, v_Z)$ and $(a_\rho, a_\varphi, a_Z)$ are the usual circular cylindrical components of the particle velocity and acceleration in Euclidean space (i.e., $v_\rho = \dot{\rho}$, $v_\varphi = \rho \dot{\varphi}$, $v_Z = \dot{Z}$, and $a_\rho = \ddot{\rho} - \rho \dot{\varphi}^2$, $a_\varphi = \rho \ddot{\varphi} + 2\dot{\rho} \dot{\varphi}$, $a_Z = \ddot{Z}$, with the overdot denoting differentiation with respect to the coordinate time $t$). In order to interpret Eqs. (14) to (16) in the context of a possibly realistic scenario, one should recall that the physics of formation of ordinary cosmic strings suggests $\alpha < 1$ and very close to unity [1], in which case Eq. (4) is a good approximation.

As the quantum corrections in Eq. (13) are “small”, the corresponding frame is nearly inertial. The right-hand sides of Eqs. (14) to (16), after multiplied by the mass of a particle, can be interpreted as the components of a gravitational force acting on the particle. By setting $\kappa = 0$, the last term in Eq. (13) as well as $a_\varphi$ and $a_Z$ vanish. Additionally, if the velocity is much smaller than $c$, $a_\rho = -4\pi G \hbar A/c \rho^3$, which matches the Newtonian potential in Eq. (4).

The contributions due to the dislocation ($\kappa \neq 0$) in the equations of motion (14) to (16) are non-vanishing, only if the corresponding transverse velocities are both non-vanishing. If the particle is initially moving on a plane defined by $v_\rho = 0$ with $v_\varphi v_Z > 0$, then $a_\varphi = a_Z = 0$, whereas the last term in Eq. (14) corresponds to an attractive or repulsive force, for $\kappa > 0$ or $\kappa < 0$, respectively. If the initial motion takes place on a plane defined by $v_\varphi = 0$ with $v_\rho v_Z > 0$, $a_Z$ and the last term in $a_\rho$ vanish, and the force corresponding to $a_\varphi \neq 0$ sweeps the particle around the symmetry axis, counterclockwise or clockwise, for $\kappa > 0$ or $\kappa < 0$, respectively. Finally, if the particle is initially moving on a plane defined by $v_Z = 0$ with $v_\rho v_\varphi > 0$, $a_\varphi$ and the last term in $a_\rho$ vanish, whereas the force corresponding to $a_Z \neq 0$
pushes the particle up or down, for \( \kappa > 0 \) or \( \kappa < 0 \), respectively [this is the only effect which has a classical analogue, as the equations of motion are recast in terms of the coordinates \((t, r, \theta, z)\)].

In considering the “dragging” effect described above, a pertinent issue that might be raised concerns the dependence of it on the choice of a particular radial coordinate [such as the one corresponding to \( \lambda = 2 \) in Eq. (11)]. This effect amounts to state that, for \( \dot{\varphi} = 0 \) and non vanishing transverse velocities, \( \ddot{\varphi} \neq 0 \) when \( \kappa \neq 0 \). By observing Eq. (15), one sees that redefinition of the radial coordinate \( r = f(\rho) \) does not change this fact. In other words, the dragging effect is gauge invariant.

V. FINAL REMARKS

In summary, this work extended investigations on semiclassical general relativity in ordinary conical spacetime to a cosmic dislocation, whose geometry has been conjectured to be associated with the gravitational field of a certain chiral cosmic string. By perturbing the locally flat metric tensor, the main result is that back reaction due to a conformally coupled massless scalar field leads to non vanishing off-diagonal components in the semiclassical metric tensor, resulting in a helical-like dragging effect on the motion of test particles. As the dragging effect has no classical counterpart around the cosmic dislocation, it might (in principle) play a role in astrophysical or cosmological scenarios.

Before closing, it should be mentioned that semiclassical dragging effects around cosmic strings were first suggested in Ref [13], in the context of vacuum polarization around a spinning cosmic string (a cosmic string with a time dislocation \([8, 11, 14]\)). The associated background, however, is not globally hyperbolic leading to pathological vacuum fluctuations \([15]\).

Acknowledgments

This work was partially supported by the Brazilian research agencies CNPq and FAPEMIG.

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