DYNAMICS OF HOMOGENEOUS SCALAR FIELDS WITH GENERAL
SELF-INTERACTION POTENTIALS: COSMOLOGICAL AND GRAVITATIONAL
COLLAPSE MODELS

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ABSTRACT. The general relativistic dynamics of a wide class of self-interacting, self-gravitating homogeneous scalar fields models is analyzed. The class is characterized by certain general conditions on the scalar field potential, which include both asymptotically polynomial and exponential behaviors. Within this class, we show that the generic evolution is always divergent in a finite time, and then make use of this result to construct cosmological models as well as radiating collapsing star models of the Vaidya type. It turns out that blackholes are generically formed in such models.

1. INTRODUCTION

Scalar fields as sources of self-gravitating models have attracted a great deal of attention in cosmology and in relativistic astrophysics. It is, indeed, worth mentioning the fundamental role that they play as models of the early universe and, on the other hand, the relevance of self-gravitating scalar field as test models for gravitational collapse (see e.g. [8] and references therein). There is, however, a crucial difference in the way scalar fields have been treated in these two scenarios.

First, consider the cosmological models. Here, the scalar field is always assumed as self-interacting; in other words, the cosmological expansion is "driven" by a non-vanishing scalar field potential, and the "free" case (i.e. the case of the quadratic field lagrangian) corresponds, as is well known, just to the case of a dynamical cosmological constant. Thus, in cosmology, the presence of a non vanishing field potential is a key point, and indeed many efforts have been made to study the possible dynamical behaviors of the models in dependence of the choice of the potential, and even to try to put in evidence possible large-scale observable effects (see e.g. [16, 17] and references therein). The situation is even more intriguing when issues from string theories come into play; indeed, here one is lead to test simple models which are asymptotically anti-De Sitter, and the potential for the scalar field might have, as a consequence, "non-standard" behaviors (we shall come back on this interesting issue later on). All in all, self-interacting fields with a non trivial potential are a key ingredient in cosmological models and therefore, of course, in the nature of the cosmological singularities.

Consider, now, the "astrophysical counterpart" of the cosmological solutions, that is, gravitational collapse. Here, as is well known, there still exist, unsolved, the issue of the cosmic censorship, that is whether the singularities which form at the end state of the collapse are always covered by an event horizon, or not [4, 5]. The scalar field model of course proposes itself a good test-bed also in this different scenario. However, in this context the scalar fields sources have usually been considered as free, i.e. minimally coupled to gravity via equivalence principle only. For this very special case the existence of naked singularities has been shown, but it has been also shown that such singularities are in a sense non generic with respect to the choice of the initial data [3, 4]. Only in recent years, a few results have been added to this scenario with the study of gravitational collapse of homogeneous self-interacting scalar fields. In these papers formation of naked singularities has been found [3, 4], but with the choice of very special potentials.

All in all, in both applications (cosmology, and gravitational collapse) it would be especially welcome a treatment of self-interacting scalar field dynamics able to treat the models in a unified manner, and to
predict the qualitative behavior in dependence of the choice of the potential, viewed as an element of a space of admissible "equations of state" for the matter source. It is the aim of the present paper to provide this unified framework, at least in the special case of homogeneous scalar fields. Indeed, we study here the dynamical behavior of such fields for a very general class of potentials, which satisfies simple physical requirements - the potentials are bounded from below and the weak energy condition is satisfied - as well as a few more technical assumptions to be discussed below.

2. Expanding and collapsing models

We consider homogeneous, spatially flat spacetimes

\begin{equation}
\begin{aligned}
g &= -dt \otimes dt + a^2(t) \left[ dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \right]
\end{aligned}
\end{equation}

where gravity is coupled to a scalar field \( \phi \), self-interacting with a scalar potential \( V(\phi) \). Assigning this function is equivalent to assign the physical properties of the matter source, and thus the choice of the potential can be seen as a sort of choice of the equation of state for the matter model. Therefore, as occurs in models with "ordinary" matter (e.g., perfect fluids), the choice of the "equation of state" must be restricted by physical considerations on stability, positivity of energy, and so on. We introduce the following conditions:

1. \( V(\phi) \) is a \( C^2 \) function bounded from below;
2. The critical points of \( V \) are isolated. They are either minimum points or nondegenerate maximum points.

In the present paper we are going to assume that \( V \) belongs to the set

\begin{equation}
\mathcal{V} = \{ V : \mathbb{R} \to \mathbb{R} : (1) \text{ and } (2) \text{ hold} \}.
\end{equation}

The Einstein field equations are given by the following

\begin{align}
\begin{aligned}
(G_0^0 &= 8 \pi T_0^0) : \quad - \frac{3 \dot{a}^2}{a^2} = -(\dot{\phi}^2 + 2V(\phi)), \\
(G_1^1 &= 8 \pi T_1^1) : \quad - \frac{\dot{a}^2 + 2a\ddot{a}}{a^2} = (\ddot{\phi}^2 - 2V(\phi)),
\end{aligned}
\end{align}

and have as a consequence the Bianchi identity

\begin{equation}
T^\mu_{0;\mu} = -2 \dot{\phi} \left( \ddot{\phi} + V'(\phi) + 3 \frac{\dot{a}}{a} \dot{\phi} \right) = 0.
\end{equation}

Denoting the energy density of the scalar field by

\begin{equation}
\epsilon = \dot{\phi}^2 + 2V(\phi),
\end{equation}

We will actually consider the following system:

\begin{align}
\begin{aligned}
\left( \frac{\dot{a}}{a} \right)^2 &= \frac{\epsilon}{3}, \\
\ddot{\phi} + V'(\phi) &= -3 \frac{\dot{a}}{a} \dot{\phi},
\end{aligned}
\end{align}

Of course, it is mandatory to extract the square root in the first equation. To do this, we introduce the sign function

\( \chi(t) := \text{sgn}(\dot{a}(t)) \).
Physically, this function obviously specifies if the solution describes an expanding (respectively, collapsing or re-collapsing) model at time \( t \). Thus our final system is composed by

\[
\begin{align*}
(2.7a) & \quad \frac{\dot{a}}{a} = \chi \sqrt{\frac{\epsilon}{3}} \\
(2.7b) & \quad \ddot{\phi} + V'(\phi) = -\chi \sqrt{3 \epsilon} \phi.
\end{align*}
\]

Let us observe that, using (2.5) and (2.7b), the following relation can be seen to hold:

\[
\dot{\epsilon} = -2 \chi \sqrt{3 \epsilon} \phi^2,
\]

that is \( \epsilon(t) \) is monotone in each interval in which \( \dot{a} \) is.

In what follows, we are going to focus on solutions of the equations (2.6a)–(2.6b) – or, better, (2.7a)–(2.7b). Actually, it can be proved \([6]\) that, if \( a \neq 0 \) for all \( t \geq 0 \) and \((a, \phi)\) are \( C^2 \) functions that solve (2.3a)–(2.3b) (with a non-constant \( \phi(t) \)) then they are solutions of (2.6a)–(2.6b) as well. To prove the converse, it suffices to derive (2.6a) w.r.t. time and use (2.6b) to deduce that if (in some interval \((a, b)\), \( \dot{a} \) is not identically zero – that is, \( \epsilon \neq 0 \) – then solutions of (2.6a)–(2.6b) are also solutions of (2.3a)–(2.3b).

Finally, the unique case in which \( \epsilon(t) = 0 \), \( \forall t > 0 \) and the Einstein Field Equations are satisfied is given by the trivial solution \( a(t) = a_0, \phi(t) = \phi_0 \), with \( V(\phi_0) = 0 \).

We shall divide our discussion of the properties of the solutions into the two following sections, the first dealing with the expanding case and the other with the collapsing case. Of course, the two cases are not disconnected, because in the expanding case, where \( \epsilon(t) \) is decreasing, there might be the possibility of reaching a vanishing \( \epsilon \) in a finite time \( t_0 \). If this happens, the model will be ruled from \( t_0 \) onward by the equations for the collapsing situation, where \( \epsilon(t) \) is increasing, and it will actually be proved to be divergent at some finite time for almost every choice of the initial data, thus yielding a singularity. Moreover this fact, and of course the particular form of system (2.7a)–(2.7b) allows us to perform the same analysis to study also backwards qualitative behavior of the solution. Indeed, if the model is expanding in a right neighborhood of \( t = 0 \), and we want to study what happens for \( t < 0 \), replacing \( t \) with \(-t\) amounts to consider a collapsing model for positive \( t \). Then we can conclude that the expanding model “collapses in the past”, i.e. it originates from a big–bang singularity.

We conclude this section proving a result that will be used throughout all the paper. It essentially states that the velocity of all (finite energy) solutions which extend infinitely in the future asymptotically vanishes:

**Lemma 2.1.** Let \( \phi(t) \) a solution of (2.7b) that can be extended for all \( t > 0 \), and such that \( \epsilon(t) \) and \( V'(\phi(t)) \) are bounded. Then

\[
\lim_{t \to +\infty} \dot{\phi}(t) = 0.
\]

If, in addition \( V''(\phi(t)) \) is also bounded, then

\[
\lim_{t \to +\infty} V'(\phi(t)) = 0.
\]

**Proof.** Using (2.8) we have

\[
\int_0^{+\infty} \dot{\phi}(t)^2 \, dt = \left| \int_0^{+\infty} \frac{\dot{\epsilon}(t)}{2 \sqrt{3 \epsilon} \dot{\epsilon}(t)} \, dt \right| < +\infty,
\]

so there exists a sequence \( t_k \to +\infty \) such that \( \dot{\phi}(t_k) \to 0 \). By contradiction, suppose that \( \exists \rho > 0 \), and a sequence \( s_k \to t_k \) – that can be taken such that \( s_k > t_k \) – with \( |\dot{\phi}(s_k)| > \rho \). Let \( \bar{k} \) such that \( |\dot{\phi}(t_k)| < \frac{\rho}{2}, \forall k \geq \bar{k} \), and let \( \tau_k, \sigma_k \) sequences such that \( t_k \leq \tau_k < \sigma_k \leq s_k \), and

\[
|\dot{\phi}(\tau_k)| = \frac{\rho}{2}, \quad |\dot{\phi}(\sigma_k)| = \rho, \quad \frac{\rho}{2} \leq |\dot{\phi}([\tau_k, \sigma_k])| \leq \rho.
\]
Note that $\epsilon$ bounded implies $\dot{\phi}$ bounded, while by assumption, $V'(\phi(t))$ is bounded. Therefore by (2.6b), there exists $M > 0$ such that

$$|\dot{\phi}(t)| \leq M \quad \forall t.$$ 

Therefore

$$\frac{\rho}{2} \leq |\dot{\phi}(\sigma_k) - \dot{\phi}(\tau_k)| \leq \int_{\tau_k}^{\sigma_k} |\dot{\phi}(t)| \, dt \leq M(\sigma_k - \tau_k),$$

that is $\sigma_k - \tau_k \geq \frac{\rho}{2M}$, and therefore

$$\int_{0}^{+\infty} \phi(t)^2 \, dt \geq \sum_{k} \int_{\tau_k}^{\sigma_k} \phi(t)^2 \, dt \geq \sum_{k} (\sigma_k - \tau_k) \cdot \left(\frac{\rho}{2}\right)^2 \geq \sum_{k} \frac{\rho^3}{8M},$$

that diverges. This is a contradiction, and then (2.9) must hold.

To prove (2.10), let us first observe that $\exists \delta_k \to +\infty$ such that $V'(\phi(t_k)) \to 0$ — otherwise, there would exist $\kappa > 0$ such that $|V'(\phi(t_k))| \geq \kappa$ definitely, which would imply, in view of (2.6b), that $|\dot{\phi}(t)| \geq \kappa/2$, that is absurd since $\lim_{t \to +\infty} \phi(t) = 0$.

Then, let us suppose by contradiction the existence of a constant $\rho > 0$, and a sequence $s_k \to +\infty$ such that $t_k < s_k$ and $|V'(\phi(s_k))| \geq \rho$. Therefore, one can choose $\sigma_k, \tau_k$ sequences such that $t_k \leq \tau_k < \sigma_k \leq s_k$, and

$$|V'(\phi(\tau_k))| = \frac{\rho}{2}, \quad |V'(\phi(\sigma_k))| = \rho, \quad \frac{\rho}{2} \leq |V'(\phi(t))| \leq \rho, \quad \forall t \in [\tau_k, \sigma_k].$$

Then, since by assumption $V''(\phi(t))$ is bounded, there exists a constant $L > 0$ such that

$$\frac{\rho}{2} = |V'(\phi(\sigma_k)) - V'(\phi(\tau_k))| \leq \int_{\tau_k}^{\sigma_k} |V''(\phi(t))| \, |\dot{\phi}(t)| \, dt \leq L(\sigma_k - \tau_k).$$

But for sufficiently large $k$ let us observe that (2.6b) implies $|\ddot{\phi}(t)| \geq \rho/4, \forall t \in [\tau_k, \sigma_k]$, and therefore $|\dot{\phi}(\sigma_k) - \dot{\phi}(\tau_k)| = |\int_{\tau_k}^{\sigma_k} \ddot{\phi}(t) \, dt| \geq \frac{\rho^2}{8L}$ that is a contradiction since $\dot{\phi} \to 0$. \hfill \Box

3. Qualitative Analysis of the Expanding Models

In this section we study the global behavior of the solutions of (2.7a), (2.7b) in the expanding ($\chi = 1$) case. Regarding the potential, throughout the section we shall assume that the function $V(\phi)$ belongs to the following subset $E \subseteq \mathfrak{V}$ (2.3):

$$(3.1) \quad E = \{ V \in \mathfrak{V} : \lim_{|\phi| \to \infty} V(\phi) = +\infty \}$$

Remark 3.1. The above growth condition on the potential is introduced here to avoid cases when $V(\phi)$ admits a flat plateau at infinity; on such situations, the argument of Lemma 3.2 below applies, but the function $\phi(t)$ may possibly also diverge as $t \to \infty$, with $\epsilon(t)$ approximating twice the plateau of $V(x)$. In this case, the critical point at infinity is non hyperbolic and arguments exploited, for instance, in Lemma 3.3 below do not apply anymore.

With an argument that uses of the Center Manifold Theorem, a result of local asymptotical instability for critical point at infinity is found in [11, 12], where the potential is supposed to be $V_\infty(1 - e^{-\sqrt{3/2}\phi})^2$. It can be seen that the Center Manifold Theorem applies to extend results of [11, 12] to a more general situation where, as $\phi \to +\infty$, $V(\phi)$ has a finite limit, $V'(\phi) \to 0^+$, and

$$\lim_{\phi \to +\infty} \frac{V''(\phi)}{V'(\phi)}$$

exists finite (and negative).

We begin our study proving the following:
Lemma 3.2. Let $\phi$ be a solution of (2.7B) with $\chi = 1$ such that $\phi(0) = \phi_0$, $\dot{\phi}(0) = v_0$ with $v_0^2 + 2V(\phi(0)) \equiv \epsilon_0 > 0$ and $(\phi(0), v_0) \neq (\phi_*, 0)$ for any $\phi_*$ critical point of $V$. Then, either there $T_* = T(\phi_0, v_0) > 0$ such that $\epsilon(T) = 0$, or $\phi(t)$ is defined in $\mathbb{R}^+$ and there exists $\phi_*$ critical point of $V$ such that:

$$\lim_{t \to +\infty} \phi(t) = \phi_*, \lim_{t \to +\infty} \dot{\phi}(t) = 0.$$ 

Proof. Due to (2.8), there is $T > 0$ such that $\sqrt{3} \int_0^T \dot{\phi}(s)^2 \, ds = \sqrt{\epsilon(0)}$ or

$$\int_0^t \dot{\phi}(s)^2 \, ds < \frac{\epsilon(0)}{3}, \quad \forall t > 0.$$ 

In the latter case, $\epsilon(t) \in [0, \epsilon_0], \forall t > 0$. Since $V$ is bounded from below we obtain that the solution is defined for all $t > 0$. Then, by (3.1) we have that $\phi(t)$ is bounded. Therefore, by Lemma 2.1

$$\lim_{t \to +\infty} \dot{\phi}(t) = 0, \lim_{t \to +\infty} V'(\phi(t)) = 0,$$

and by (2.2), we immediately see that $\phi$ must be convergent to a critical point of $V$. \hfill $\square$

We now proceed to study the two possible cases described in Lemma 3.2, starting from the behavior of the solutions for which $\epsilon$ vanishes in a finite time $T > 0$. Observe first that this situation may happen only if $V(\phi(T)) \leq 0$. However, we show in the Appendix B (see remark B.2) that if $\inf V < 0$ this situation is generic, in the sense that solutions of this kind exist, and that the set of initial data leading to solutions such that $V(\phi(T)) < 0$ is open. As said before, for $t > T$ the system will be ruled by the equations for the collapsing solution, discussed in Section 3.

Now, here remains to see what happens if $\phi(t)$ converges to a critical point for $V(\phi)$. In the following, we shall study first the behavior of the solution near a local minimum point of $V$.

Lemma 3.3. Let $\phi_*$ be a local minimum point of $V$ (not necessarily nondegenerate). Assume that there exists a bounded open interval $]a, \beta[ \ni \phi_*$, such that

(3.2) $V'(\phi) > 0, \forall \phi \in ]\phi_*, \beta[,$ and $V'(\phi) < 0, \forall \phi \in [a, \phi_*];$

(3.3) $V(a) = V(\beta) = E > 0;$

(3.4) $V(\phi_*) \geq 0.$

Let $\phi$ be a solution such that $0 < \epsilon(0) \leq E$ and $\phi_0 \in [a, \beta].$ Then $\phi$ is defined on all $\mathbb{R}^+$ and

$$\lim_{t \to +\infty} \phi(t) = \phi_*, \lim_{t \to +\infty} \dot{\phi}(t) = 0.$$ 

Proof. Let us consider initial data such that $\phi(0) = \phi_0$, $2V(\phi_0) = E$, $\dot{\phi}(0) = 0$. Since $\epsilon(t)$ is non-increasing we have

$$\epsilon(t) \leq E, \forall t \geq 0.$$ 

Moreover

$$\dot{\phi}(0) = -V'(\phi_0).$$

If $\phi_0 > \phi_*$, it is $V'(\phi_0) > 0$, then $\phi$ moves towards $\phi_*$. Analogously if $\phi_0 < \phi_*$, the solution $\phi$ moves towards $\phi_*$. Now consider an initial data $(\phi_0, \dot{\phi}(0))$ such that $\epsilon(0) = E$ and $\dot{\phi}(0) \neq 0$, so that $2V(\phi_0) = E - \phi(0)^2 < E$. Note that, by (3.3), it must be $2V(\phi(t)) \leq E$. Moreover, if there exists a $t$ such that $2V(\phi(t)) = E$ it must be $\dot{\phi}(t) = 0$ and arguing as above we see that $\phi$ moves towards $\phi_*$. Therefore $\alpha \leq \phi(t) \leq \beta$ for all $t$. Then, by Lemma 3.2 we deduce that $\lim_{t \to +\infty} \phi(t) = \phi_*$ which is the only critical point of $V$ in the interval $]a, \beta[$, and $\lim_{t \to +\infty} \dot{\phi}(t) = 0.$ \hfill $\square$
Remark 3.4. Whenever \( V(\phi_*) = 0 \) further information about the asymptotic behavior of the solution near the minimum point \( \phi_* \) can be obtained \([5]\). In that paper it is indeed proved the oscillation of \( \phi \) around the minimum point using asymptotic analysis. If \( \phi_* \) is a nondegenerate minimum point (which is the more relevant physical case) we can prove the oscillatory character also using an alternative argument as follows. Suppose, by contradiction, that \( \phi \) does not oscillate around \( \phi_* \), this means that there exists \( t_1 > 0 \) such that

\[
\phi(t) \neq \phi_*, \quad \forall t \geq t_1.
\]

To fix our ideas suppose \( \phi(t) > \phi_* \forall t \geq t_1 \). Then, if \( t \geq t_1 \) and \( \dot{\phi}(t) = 0 \) it is \( \ddot{\phi}(t) = -V'(\phi(t)) < 0 \) (since \( V' > 0 \) in a right neighborhood of \( \phi_* \)). This implies that there exists \( t_2 > 0 \) such that

\[
\dot{\phi}(t) < 0, \quad \forall t \geq t_2.
\]

Moreover there are not sequences \( t_k \to +\infty \) where \( \ddot{\phi}(t_k) = 0 \). Otherwise we should have \( \dddot{\phi}(t_k) = -(V''(\phi(t_k)) + \sqrt{3}k^2)\dot{\phi}(t_k) = -(V''(\phi(t_k)) - 12k^2\phi(t_k))^2\dot{\phi}(t_k) \), so \( \dot{\phi}(t_k) < 0 \) for any \( k \) sufficiently large, getting a contradiction. Therefore there exists \( t_3 > 0 \) such that

\[
\ddot{\phi}(t) > 0, \quad \forall t \geq t_3.
\]

Then, by (2.7b),

\[
-V'(\phi) - \sqrt{3}k\dot{\phi} > 0, \quad \forall t \geq t_3,
\]

that implies (recalling that \( \dot{\phi} < 0 \))

\[
3(\dot{\phi})^4 + 6V(\phi)^2 - (V')^2 \geq 0, \quad \forall t \geq t_3
\]

and

\[
(\dot{\phi})^2 \geq \sqrt{V^2 + (V')^2} - V, \quad \forall t \geq t_3.
\]

Now \( V(\phi) = a(\phi - \phi_*)^2 + o(\phi - \phi_*)^2 \) with \( a > 0 \), so there exists \( b > 0 \) and \( t_4 > 0 \) such that

\[
|\dot{\phi}| \geq b\sqrt{\phi - \phi_*}, \quad \forall t \geq t_4,
\]

(recall that \( \phi(t) > \phi_* \)). Finally \( \forall t \geq t_4 \)

\[
\dot{\phi} \leq -b\sqrt{\phi - \phi_*},
\]

and integrating we obtain,

\[
\sqrt{\phi(t) - \phi_*} - \sqrt{\phi(t_4) - \phi_*} \leq -\frac{b}{2}t, \quad \forall t \geq t_4,
\]

in contradiction with \( \phi(t) \geq \phi_* \) for any \( t \geq t_4 \).

Now we shall study the behavior near a local nondegenerate maximum point of \( V \).

**Lemma 3.5.** Let \( \phi_* \) be a local nondegenerate maximum point of \( V \) such that \( V(\phi_*) \geq 0 \). Then there are only two solutions \( \phi_1 \) and \( \phi_2 \) of (2.7b) such that \( \lim_{t \to +\infty} \phi_i(t) = \phi_* \), \( \lim_{t \to +\infty} \dot{\phi}_i(t) = 0 \), \( i = 1, 2 \). Moreover there exists \( t_* > 0 \) such that \( \phi_1(t) > \phi_* \) for all \( t \geq t_* \) and \( \phi_2(t) < \phi_* \) for all \( t \geq t_* \).

**Proof.** Consider the first order system associated to (2.7b)

\[
(3.6) \quad \begin{cases}
\dot{x} = -V'(y) - \sqrt{E(x,y)}x \\
\dot{y} = x,
\end{cases}
\]

where \( E(x, y) = x^2 + 2V(y) \).

Let \((0, y_*)\) be an equilibrium point of (3.6), namely \( V'(y_*) = 0 \). Suppose

\[
E(0, y_*) = 2V(y_*) > 0.
\]
The linearized system of (3.6) at \((0, y_*)\) is
\[
\begin{align*}
\dot{\alpha} &= -V''(y_*) \beta - \sqrt{2V(y_*)} \\
\dot{\beta} &= \alpha.
\end{align*}
\]

Since, by our assumption \(V''(y_*) < 0\), the matrix
\[
\begin{pmatrix}
-\sqrt{2V(y_*)} & -V''(y_*) \\
1 & 0
\end{pmatrix}
\]
has one positive eigenvalue and one negative eigenvalue. Therefore the proof follows immediately by classical theory on stability of nonlinear dynamical system in a neighborhood of an equilibrium point (cf. e.g. [13]).

Note that the function \(\sqrt{E(x, y)x}\) is not of class \(C^1\) in a neighborhood of \((0, y_*)\) whenever \(V(y_*) = 0\).

However the quantity \(\sqrt{E(x, y)x}\) is an infinitesimal of order greater than 1 in \((0, y_*)\), therefore the classical theory can be adapted to this case, studying the linear system
\[
\begin{align*}
\dot{\alpha} &= -V''(y_*) \beta \\
\dot{\beta} &= \alpha,
\end{align*}
\]
obtaining the same conclusion of the case \(V(y_*) > 0\).

Collecting the results shown in this Section (and in Appendix B) we obtain the following

**Proposition 3.6.** Suppose that \(V\) belongs to the set \(E\) (3.1). Let be \(\phi\) a solution of (2.7b) where \(\chi = 1\), with initial data \(\phi(0) = \phi_0, \dot{\phi}(0) = v_0, \) and \(v_0^2 + 2V(\phi_0) > 0\). Then, one of the two mutually exclusive situations occur:

either
\[
\exists T = T(\phi_0, v_0) > 0 \text{ such that } \epsilon(T) = (\dot{\phi}(T))^2 + 2V(\phi(T)) = 0,
\]
and the set of initial data such that (3.8) is satisfied with \(V(\phi(T)) < 0\) is open and not empty, while there are only a finite number of solutions satisfying (3.8) with \(V(\phi(T)) = 0\),

or
\[
\exists \phi_* \text{ critical point of } V \text{ with } V(\phi_*) \geq 0 \text{ such that } \lim_{t \to +\infty} \phi(t) = \phi_*, \text{ and } \lim_{t \to +\infty} \dot{\phi}(t) = 0,
\]
and the measure of the initial data set such that \(\phi(t) \to \phi_* \) maximum point is zero, while the set of initial data such that \(\phi(t) \to \phi_* \) minimum point is open and not empty.

Moreover, situation (3.8) occurs only if \(\inf V < 0\).

4. Qualitative Analysis of the Collapsing Models

The aim of the present section is to study the qualitative behavior of the solution of (2.7a)–(2.7b) in the collapsing case \(\chi = -1\), for which we will require that \(V(\phi)\) satisfies some further conditions. We first assume that \(\exists V^*\) positive constant such that:

(4.1) the set \( B := \{ \phi \in \mathbb{R} : V(\phi) \leq V^* \} \) is bounded, and

(4.2) \( \phi \geq \sup B \implies V'(\phi) > 0, \quad \phi \leq \inf B \implies V'(\phi) < 0. \)

Moreover, defining the function
\[
u(\phi) := \frac{V'(\phi)}{2\sqrt{3V(\phi)}},
\]
and introducing the growth conditions
\begin{align}
(4.4) \quad & \limsup_{\phi \to \pm \infty} |u(\phi)| < 1, \\
(4.5) \quad & \exists \lim_{\phi \to \pm \infty} u'(\phi) (= 0).
\end{align}
we assume that $V$ belongs to the subset $\mathcal{C} \subseteq \mathcal{V}$ defined as
\begin{equation}
\mathcal{C} = \{ V \in \mathcal{V} : V \text{ satisfies } (4.1) - (4.2) \text{ and } (4.4) - (4.5) \}.\end{equation}
We stress that this class contains the potentials which are usually considered in the physical applications. For instance, it contains the standard quartic potential ($V(\phi) = -\frac{1}{2}m^2 \phi^2 + \lambda^2 \phi^4$) and, more generally, all the potentials bounded from below whose asymptotic behavior is polynomial (i.e. $\phi^{2n}$ for $|\phi| \to \infty$, $n \geq 1$). Moreover, it obviously suffices for $V \in \mathcal{C}$ to diverge at infinity to be also in the class $\mathcal{E}$ studied in previous Section 3.

Let $t = 0$ be the initial time and let $\phi_0 = \phi(0) \in B$. We consider only data which satisfy the weak energy condition and thus, we assume
\begin{equation}
(4.7) \quad \epsilon_0 = \epsilon(0) = \dot{\phi}^2(0) + 2V(\phi(0)) \geq 0.
\end{equation}
We start considering initial data such that $\epsilon_0$ is "sufficiently large", more precisely
\begin{equation}
(4.8) \quad \epsilon_0 \geq 2V^*,
\end{equation}
where $V^*$ has been defined just above, and proceed to show that the solution $\phi(t)$ diverges in a finite time for almost every choice of the initial data (Proposition 4.11). Then, we will extend the result by removing the restriction (4.8).

Before starting the proof, let us notice the following:

**Remark 4.1.** If $\dot{\phi}(0) = 0$ then $\dot{\phi}(t) \neq 0$ in a right neighborhood of $t = 0$. Indeed, $2V(\phi_0) = \epsilon_0 \geq 2V^*$, then $\phi_0 \notin \overset{\circ}{B}$, the set of internal points of $B$. This implies by (4.2) that $V'(\phi_0) \neq 0$, so that (2.7b) shows $\dot{\phi}(0) \neq 0$.

**Remark 4.2.** Recall from (2.8) that $\epsilon(t)$ is monotonically non decreasing. Actually, we can also observe that, for all $t > 0$, $\epsilon(t)$ is strictly greater than $\epsilon_0$: it is straightforward if $\dot{\phi}(0) > 0$, but is also true, in view of Remark 4.1, if $\dot{\phi}(0) = 0$.

**Remark 4.3.** Given $t_0 > 0$, there exists a constant $\delta > 0$ such that, for all $t_1, t_2$ such that $t_0 < t_1 < t_2$, and $\phi([t_1, t_2]) \subseteq \overset{\circ}{B}$, $\dot{\phi}(t)^2 \geq \delta$, $\forall t \in [t_1, t_2]$. Indeed, let $\delta = \epsilon(t_0) - \epsilon_0 > 0$ (cf. Remark 4.2). Then, if $t \in [t_1, t_2]$, $\dot{\phi}(t)^2 = \epsilon(t) - 2V(\phi(t)) \geq \epsilon(t_1) - 2V(\phi(t)) \geq \delta + \epsilon_0 - 2V^* \geq \delta$, where we have used (4.1) and (4.8).

**Remark 4.4.** Let $t > 0$ such that $\phi(t) \notin \overset{\circ}{B}$ and $\dot{\phi} = 0$. Then, (2.7b) and (4.2) imply that $t$ is a local extremum for $\phi$. In particular, it is a local maximum if $\phi(t) > \sup B$, and is a local minimum if $\phi(t) < \inf B$.

Let $I \subseteq [0, +\infty)$ be the maximal right neighborhood of $t = 0$ where the solution $\phi(t)$ is defined, and call $t_s = \sup I \in \mathbb{R}^+ \cup \{+\infty\}$. We can now state the following lemma.

**Lemma 4.5.** If $\phi(t)$ is bounded, also $\dot{\phi}(t)$ is bounded, and $t_s = +\infty$. 

Proof. Let $K$ a bounded set such that $\phi(t) \in K$, $\forall t \in I$, and argue by contradiction, assuming $\dot{\phi}$ not bounded, and supposing that $\sup_{t} \phi(t) = +\infty$ – the same argument can be used if $\dot{\phi}$ is unbounded only below. Let $\tilde{t}$ such that
\[
\dot{\phi}(\tilde{t}) > \sup_{K} \frac{V'(\phi)}{\sqrt{3\epsilon}}.
\]
therefore (2.7b) implies that $\ddot{\phi}(\tilde{t}) > 0$, and then $\dot{\phi}$ is increasing in $\tilde{t}$. But this means that $\dot{\phi}$ is increasing in $\tilde{t}$, therefore (2.7b) shows that $\dot{\phi} \geq 0$, $\forall t \geq \tilde{t}$, and then $\dot{\phi}$ is eventually increasing, namely $\lim_{t \to t_+} \phi(t) = +\infty$. This shows that $t_+ < +\infty$, otherwise it would be $\phi(t) - \phi_0 = \int_{0}^{1} \dot{\phi}(\tau) d\tau$, which would diverge as $t \to +\infty$. Then $t_+ \in \mathbb{R}$.

With the position
\[
\lambda(t) = \frac{1}{\phi(t)},
\]
equation (2.7b) implies
\[
\lambda(t) = \frac{V'(\phi(t))}{\phi(t)^2} - \sqrt{3} \sqrt{\frac{1 + \frac{2V(\phi(t))}{\phi(t)^2}}{t \to t_+}} - \sqrt{3},
\]
and $\lambda(t)$, near $t = t_+$, behaves like $\sqrt{3}(t_+ - t)$. Since $\dot{\phi}(t)$ positively diverges, $\lim_{t \to t_+} \phi(t) = \phi^*$ exists, and is finite since $\dot{\phi}(t)$ is bounded by hypothesis. But the quantity
\[
\dot{\phi}(t) - \phi^* = \int_{t_+}^{t} \dot{\phi}(\tau) d\tau = \int_{t}^{t_+} \frac{1}{\lambda(\tau)} d\tau
\]
diverges, which is absurd. This shows that $\dot{\phi}(t)$ must be bounded too.

To show that $t_+ = +\infty$, we proceed again by contradiction. Let $t_k \to t_+$ be a sequence such that the sequence $(\phi(t_k), \dot{\phi}(t_k))$ converges to a finite limit $(\phi_+, \dot{\phi}_+)$. Solving the Cauchy problem (2.7b) with initial data $(\phi_+, \dot{\phi}_+)$ shows that the solution $\phi(t)$ is $C^1$, the solution could be prolonged on a right neighborhood of $t_+$.

Proposition 4.6. The function $\dot{\phi}(t)$ is unbounded.

Proof. Let by contradiction $\dot{\phi}(t)$ be bounded. Then, by Lemma 4.5, $|\dot{\phi}(t)| \leq M$ for a suitable constant $M$, and $t_+ = \sup I = +\infty$. In particular $V(\phi(t)), V'(\phi(t)), \epsilon(t), \phi(t)$ are bounded also. Then Lemma 2.4 says that $\lim_{t \to +\infty} \dot{\phi}(t) = 0$, that implies $\lim_{t \to +\infty} 2V(\phi(t)) = \lim_{t \to +\infty} \epsilon(t) > \epsilon_0 \geq 2\epsilon$, and so $\phi(t) \not\in B$ for any $t$ sufficiently large. Then, for large $t$, $\phi(t)$ moves in a region where $V$ is invertible.

Since $V(\phi(t))$ converges this happen also for $\phi(t)$ which converges to a point $\phi_* \not\in B$. Then by (2.7b), $\dot{\phi}(t) \to -V'(\phi_*) \neq 0$ (since $\dot{\phi}(t) \to 0$), that is absurd. Then $\lim \sup_{t \to t_+} |\phi(t)| = +\infty$.

We have shown that $\phi(t)$ is not bounded. Now we want to show that, actually, $\phi(t)$ monotonically diverges in the approach to $t_+$.

Remark 4.7. We observe that the quantity
\[
\rho(t) := \frac{2V(\phi(t))}{\phi(t)^2},
\]
satisfies the equation
\[
\dot{\rho} = 2\sqrt{3} \dot{\phi} \rho \sqrt{(1 + \rho)} \left( u(\phi) \sqrt{1 + \rho} - \text{sgn}(\dot{\phi}) \right).
\]

Proposition 4.8. Let $V \in \mathcal{C}(4.6)$, and let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ the solution of (2.7b). Then $\dot{\phi}(t)$ is eventually not zero, and
\[
\lim_{t \to \sup I} |\phi(t)| = +\infty.
\]
Proof. By contradiction, let \( s_k \) be a sequence of local minima, and \( t_k \) a sequence of local maxima for \( \phi(t) \), both convergent to \( t_s \), and such that \( s_k < t_k < s_{k+1} < t_{k+1} \), for each \( k \) (recall that \(|\phi(t)|\) is unbounded). Then Remark 4.2 implies
\[
V_s \leq \frac{\epsilon_0}{2} \leq \frac{1}{2} \epsilon(s_k) \leq \frac{1}{2} \epsilon(t_k) \leq \frac{1}{2} \epsilon(s_{k+1}) \leq \frac{1}{2} \epsilon(t_{k+1}),
\]
that is
\[
V_s \leq V(s_k) \leq V(t_k) \leq V(s_{k+1}) \leq V(t_{k+1}).
\]
Recalling Remark 4.2, \( s_k \) is \( \inf B \) and \( t_k \geq \sup B \), that is \( \phi(t) \) crosses \( B \) infinitely many times. We claim that \( \lim_{t \to t_s} \epsilon(t) = +\infty \). Otherwise, \( \epsilon(t) \) would be bounded, and also \( \dot{\phi}(t) \) would be. In particular, taken \( \sigma_k, \tau_k \) sequences converging to \( t_s \) such that \( \sigma_k < \tau_k \), and
\[
\phi(\sigma_k) = \inf B, \qquad \phi(\tau_k) = \sup B, \qquad \phi([\sigma_k \tau_k]) \subseteq B,
\]
and called \( N = \sup B - \inf B \), it must be
\[
N = \phi(\tau_k) - \phi(\sigma_k) = \int_{\sigma_k}^{\tau_k} \dot{\phi}(t) dt \leq M(\tau_k - \sigma_k),
\]
and so \( \tau_k - \sigma_k \geq \frac{N}{M} \). Then, by (2.8), and Remark 4.3,
\[
\sqrt{\epsilon(\tau_k)} - \sqrt{\epsilon(\sigma_k)} = \int_{\sigma_k}^{\tau_k} \frac{\dot{\epsilon}}{2\sqrt{\epsilon}} dt = \sqrt{3} \int_{\sigma_k}^{\tau_k} \dot{\phi}^2 dt \geq \delta \sqrt{3}(\tau_k - \sigma_k) \geq \delta \sqrt{3} \frac{N}{M},
\]
and then \( \epsilon(t) \) must diverge.

Now, recalling 4.3, let \( \theta > 0 \) be a sufficiently small constant, and let \( r_k \to t_s \) a sequence such that \( \phi(r_k) = \tilde{\phi} \), with \( u(\tilde{\phi}) \leq 1 - \theta \) as \( \phi \geq \tilde{\phi} \). Then \( \phi(r_k)^2 = \epsilon(r_k) - 2V(\tilde{\phi}) \to +\infty \). Let us suppose that \( \dot{\phi}(r_k) \) is unbounded above – analogously one can argue if it is only unbounded below. Up to subsequences, we can suppose that \( \dot{\phi}(r_k) \) positively diverges. Moreover, recalling (4.3), \( \rho(r_k) \to 0 \) and then, if \( k \) is sufficiently large,
\[
|\rho(r_k)| < \left( \frac{1}{1 - \theta} \right)^2 - 1,
\]
so that, since \( \phi(r_k) = \tilde{\phi} \) and \( u(\tilde{\phi}) \leq 1 - \theta \), we have
\[
u(\phi(r_k)) \sqrt{1 + \rho(r_k)} - 1 < 0.
\]
Then, \( \dot{\rho}(r_k) < 0 \), that is \( \rho(t) \) is decreasing at \( r_k \). But we can observe that \( \phi(t) \) is increasing in \( r_k \), ensuring \( u(\phi(t)) < 1 - \theta \), and so, in a right neighborhood of \( r_k \),
\[
u(\phi(t)) \sqrt{1 + \rho(t)} - 1 < (1 - \theta)\sqrt{1 + \rho(t)} - 1
\]
that is decreasing. We conclude that the function \( \rho(t) \) decreases for \( t > r_k \), until \( t \) equals a local maximum \( t_k \), where \( \dot{\phi} \) vanishes. But since \( 2V(\phi(t)) \leq \kappa \phi(t)^2 \) for some \( \kappa > 0 \), this fact would imply \( \epsilon(\phi(t_k)) = 0 \), a contradiction. Then \( \dot{\phi} \) is eventually non zero. This implies that there exists \( \lim_{t \to +\infty} |\phi(t)| \)
so, by Proposition 4.6, it is \( +\infty \).

We have shown so far that \( \phi(t) \) diverges, as \( t \) approaches \( t_s = \sup I \). In the following, we will show that \( t_s \in \mathbb{R} \) for almost every solution, in the sense that there exists a set of initial data, dense in the set of the admissible ones, such that the solution is defined until a certain finite comoving time \( t_s \) (depending on the data). Henceforth we will suppose (just to fix our ideas) \( \phi(t) \) positively diverging and \( \dot{\phi}(t) > 0 \) \( \forall t \geq t_s \). Then, in the interval \([t_s, +\infty) \), \( \rho \) can be seen as a function of \( \phi \), that satisfies by (4.10) the ODE
\[
\frac{d\rho}{d\phi} = 2\rho \sqrt{3(1 + \rho)} \left( u(\phi) \sqrt{1 + \rho} - 1 \right).
\]
Remark 4.9. It is not restrictive to study (4.11) for large and positive $\phi$. Indeed, consider the case $\phi \to -\infty$, and take $\tilde{\rho}(\psi) = \rho(-\psi), \tilde{V}(\psi) = V(-\psi)$. Then, (4.10) gives

$$
\frac{d\tilde{\rho}}{d\psi}(\psi) = -\frac{d\rho}{d\psi}(-\psi) = -\rho(-\psi)2\sqrt{3(1 + \rho(-\psi))} \left( \frac{V'(-\psi)}{2\sqrt{3}V(-\psi)} \sqrt{1 + \rho(-\psi) + 1} \right) = \\
\tilde{\rho}(\psi)2\sqrt{3(1 + \tilde{\rho}(\psi))} \left( \frac{\tilde{V}'(\psi)}{2\sqrt{3}\tilde{V}(\psi)} \sqrt{1 + \tilde{\rho}(\psi) - 1} \right),
$$

where $\psi$ is large and positive. Once we observe that $\tilde{V}$ satisfies, for large and positive $\psi$, the same assumptions as $V$, we have that (4.11) also controls the behavior of $\rho$ as $\phi \to -\infty$.

We now state the following crucial result.

Lemma 4.10. Except at most for a measure zero set of initial data, the function $\rho(t)$ goes to zero for $t \to \sup I$.

Proof. The proof will be carried on by studying qualitatively the solutions of the ODE (4.11). By virtue of Proposition 4.8, and Remark 4.9, we will be interested in those solutions which can be indefinitely prolonged on the right.

With the variable change $y = \sqrt{1 + \tilde{\rho}}$, (4.11) becomes

$$
(4.12) \quad \frac{dy}{d\phi} = \sqrt{3}(y^2 - 1) (u(\phi)y - 1),
$$

where we recall that $u(\phi)$ is given by (4.3).

Let us consider solutions of (4.12) defined in $[\phi_0, +\infty)$. If $\liminf_{\phi \to +\infty} |y(\phi)u(\phi) - 1| > 0$ then, necessarily, $u(\phi)y(\phi) - 1 < 0$ (and $y(\phi)$ decreases, and goes to 1), otherwise the solution would not be defined in a neighborhood of $+\infty$. In short, if $y(\phi)$ is a solution defined in a neighborhood of $+\infty$, the following behaviors are allowed:

1. either $y(\phi)$ is eventually weakly decreasing, and tends to 1 (so that $\rho$ tends to 0),
2. or $\liminf_{\phi \to +\infty} |y(\phi)u(\phi) - 1| = 0$.

With the variable and functions changes

$s = e^{-\sqrt{3}\phi}, \quad z = uy - 1,$

equation (4.12) takes the form

$$
(4.13) \quad (su) \ddot{z}(s) = -(z + 1)^2 z + (u^2 + su)z + s \dot{u}.
$$

But it can be easily seen, using (4.5), and the identity

$s \dot{u}(s) = -\frac{1}{\sqrt{s}} u'(\phi),$

that (4.13) satisfies the assumption of Theorem A.1 in Appendix A and so there exists a unique solution for the Cauchy problem given by (4.13) with the initial condition $z(0) = 0$, that is furthermore the only possible solution of the ODE (4.13) with $\liminf_{s \to 0} |z(s)| = 0$, and this fact results in a unique solution satisfying case (2) above, whereas all other situations lead to case (1). □

Now, we show that the singularity forms in a finite amount of time, for almost every choice of initial data.
Proposition 4.11. If $\epsilon_0 \geq 2V_*$, except at most for a measure zero set of initial data, there exists $t_s < +\infty$ such that $t_s = \sup I$, and the following facts holds:

\begin{align}
(4.14) \quad & \lim_{t \to t_s^-} \epsilon(t) = +\infty, \\
(4.15) \quad & \lim_{t \to t_s^-} \phi(t) = +\infty, \\
(4.16) \quad & \lim_{t \to t_s^-} a(t) = 0.
\end{align}

Proof. From (2.5) and (2.7a)-(2.7b), it easily follows that

\begin{equation}
(4.17) \quad \dot{\epsilon}(t) = 2\sqrt{3}\kappa^{3/2} \frac{1}{1 + \rho(t)}.
\end{equation}

But Lemma above ensures that $\rho$ (defined in (4.9)) goes to zero for almost every choice of initial data. For this choice, $\dot{\epsilon}(t) > \epsilon^{3/2}$ in a left neighborhood of $\sup I$. Then $t_s < +\infty$ and (4.14) easily follows, using comparison theorems in ODE.

We already know that $\phi(t) \to +\infty$. To prove (4.15) we observe that

\[ \frac{\epsilon(t)}{\phi(t)^2} = \rho(t) + 1 \to 1, \]

and then (4.14) implies (4.15).

To prove (4.16), we first prove that

\begin{equation}
(4.18) \quad \lim_{t \to t_s^-} \int_0^t \sqrt{\tau} \, d\tau = +\infty.
\end{equation}

Indeed, recalling that $\rho(t)$ is eventually bounded, $\exists \overline{\rho} > 0$ such that $\epsilon(t) = \dot{\phi}^2(t)(1 + \rho(t)) \geq \kappa^2 \dot{\phi}(t)^2$, for some suitable constant $\kappa > 0$, so $\sqrt{\epsilon(t)} > \kappa \phi(t)$, $\forall t > \bar{t}$, and then

\[ \int_\overline{\phi}^t \sqrt{\tau} \, d\tau \geq \kappa \int_\overline{\phi}^t \phi(\tau) \, d\tau = \kappa(\phi(t) - \phi(\overline{t})). \]

The righthand side above diverges because of Proposition 4.8, and (4.18) is proved. Moreover, using (2.7a),

\[ \log \frac{a(t)}{a(0)} = \int_0^t \frac{\dot{a}}{a} \, d\tau = -\frac{1}{\sqrt{3}} \int_0^t \sqrt{\tau} \, d\tau, \]

from which (4.16) follows, since by (4.18) the righthand side above negatively diverges as $t \to t_s$. \qed

4.1. The case $\epsilon_0 < 2V^*$. Now we are going to consider the case where (4.8) does not hold. This means that there exists at least one critical point $\phi_*$ for $V(\phi)$ such that

\begin{equation}
(4.19) \quad 2V^* > V(\phi_*) \geq \epsilon_0.
\end{equation}

First of all, we take care of the case $\epsilon_0 = 0$. Since the potential can be negative, one should consider the case in which the energy vanishes "dynamically" (that is, $\phi$ solves the equation $\dot{\phi}(t)^2 = -2V(\phi(t))$ $\forall t$)

However, these functions are not solutions of Einstein field equations (2.3a)-(2.3b), since $\epsilon(t) \equiv 0$ implies $a(t) = a_0$ and then $\phi(t) = 0$, $V(\phi(t)) = V(\phi_0) = 0$. Thus, one is left with a set of constant solutions with zero energy of the form $(a_0, \phi_0)$, with $a_0$ positive constant and $V(\phi_0) = 0$. This shows that at the "boundary of the weak energy condition" local uniqueness of the field equations may be violated if $V'(\phi_0) \neq 0$. However the set of the initial data for the expanding equations intersecting such solutions in a finite time has zero measure (note that the points $\phi_0$ such that $V(\phi_0) = 0$ and $V'(\phi_0) \neq 0$ are isolated).

Further, we prove (see Lemma 4.3 in Appendix B) a result of local existence and uniqueness for the solutions of equation (2.7b) satisfying $\epsilon(0) = 0$, but for which $\epsilon(t) > 0$ for $t > 0$.\"
Let us now study the evolution of $\phi(t)$ assuming $\epsilon(t) < 2V^*$, $\forall t \in I$ (otherwise the previously obtained results would apply). In this case $\phi(t) \in B$ for any $t \geq 0$ and, since $\dot{\phi}$ is bounded, $\sup I = +\infty$. Then Lemma 2.1 applies, to find $\lim_{t \to +\infty} \dot{\phi}(t) = 0$, $\lim_{t \to +\infty} V'(\phi(t)) = 0$. Moreover by assumption 2.2 there exists $\phi_\star$ critical point of $V$ such that $\lim_{t \to +\infty} \phi(t) = \phi_\star$, (with critical value $V(\phi_\star) = \lim_{t \to +\infty} \epsilon(t) \in ]0,2V_*]$. Whenever $\phi_\star$ is a (nondegenerate) maximum point we can study the linearization of the first order system equivalent to (2.7) in a neighborhood of the equilibrium point $(0, \phi_\star)$, as done in Lemma 3.3, obtaining a result totally analogous to Lemma 3.5. Moreover, by the results of Lemma 3.3 (which can be seen as the "time-reversed" version) we see if $\phi_\star$ is a minimum point and $\phi$ starts with initial data close to $(0, \phi_\star)$, then $\phi$ moves far away from $\phi_\star$.

Therefore, under the assumptions made, for almost every choice of the initial data, the function $\phi(t)$ must be such that its evolution cannot be contained in the compact set $B$ defined in (4.1), namely, $\epsilon(t_1) \geq 2V_*$ for some $t_1 > 0$, which allows us to apply the theory we already know to show that the singularity forms for almost every choice of the initial data. The results are summarized in the following theorem:

**Theorem 4.12.** Suppose that $V(\phi)$ belongs to the set $\mathcal{C}$ (4.6). Then, except at most for a measure zero set of initial data satisfying weak energy condition, there exists $t_s \in \mathbb{R}$ such that the scalar field solution becomes singular at $t = t_s$, that is $\lim_{t \to t_s^-} \epsilon(t) = +\infty$, and $\lim_{t \to t_s^-} \phi(t) = +\infty$.

### 4.2. Examples

**Example 4.13.** The above results hold for all potentials with polynomial leading term at infinity (i.e. $\lambda^2 \phi^{2m}$). For instance, for a quartic potential $V(\phi) = -\frac{1}{2}m^2 \phi^2 + \lambda^2 \phi^4$, with $\lambda, m \neq 0$, the function $u(\phi)$ goes as $\frac{2}{\sqrt{3\phi}}$ for $\phi \to +\infty$, and all conditions listed above hold. The behavior for a particular example from this class of potential is given in Figure 1.

**Example 4.14.** In the case of exponential potentials, the results hold for asymptotic behaviors with leading term at infinity of the form $V_0 e^{-\sqrt{\lambda} |\phi|}$ with $\lambda < 1$. For instance for $V(\phi) = V_0 e^{2\sqrt{\lambda} \sqrt{\phi^2 + \gamma^2}}$, where $V_0, \lambda, \gamma > 0$, the quantity $u(\phi)$ goes like $\lambda$, and so (4.4) is verified if $\lambda < 1$. See a particular situation from this class represented in Figure 2.

**Figure 1.** Behavior of the function $\phi(t), \epsilon(t)$, and $\alpha(t)$ with potential given by $V(\phi) = 1 - \phi^2 + \phi^4$. The initial conditions are $\phi_0 = -0.6$, $\phi_0 = 0$, $\alpha_0 = 1$.

The time of collapse $t_s$ approximately equals $t_s = 2.1$. 

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Figure 2. Same as Figure 1, but now the potential is given by $V(\phi) = e^{2\sqrt{\frac{3}{2}(\phi^2 + 1)}}/100$. The initial conditions are $\phi_0 = 0.1$, $\dot{\phi}_0 = 0$, $a_0 = 1$. The time of collapse $t_s$ approximately equals $t_s = 15.3$.

Figure 3. Same as Figure 1, where now $V(\phi) = 2(1 - e^{-\sqrt{\phi^2 + 1}})^2 - \frac{4}{6}e^{-(x+4)^2} - \frac{1}{2}e^{-(x-4)^2} + 1$. The initial conditions are $\phi_0 = 0$, $\dot{\phi}_0 = \sqrt{-2V(0)}$, $a_0 = 1$. The time of collapse $t_s$ approximately equals $t_s = 1.8$.

Example 4.15. Decaying exponential potentials can be considered as well. For instance for $V(\phi) = (1 - e^{-\alpha\sqrt{\phi^2 + \gamma^2}})^2$ the function $u(\phi)$ behaves like $e^{-\alpha|\phi|}$, so it goes to zero at $\pm\infty$. Figure 3 represent a situation where some corrections terms have been added in order to obtain a potential with more than one critical points. Even choosing initial data such that $\epsilon_0 = 0$, the solution diverges in a finite amount of time, as showed in general in Subsection 4.1.
5. Gravitational collapse models

In what follows, we construct models of collapsing objects composed by homogeneous scalar fields. To achieve this goal we must match a collapsing solution (considered as the interior solution in matter) with an exterior spacetime. The natural choice for the exterior is the so-called generalized Vaidya solution

\begin{equation}
    ds^2_{\text{ext}} = - \left(1 - \frac{2M(U,Y)}{Y}\right) \, dU^2 - 2 \, dY \, dU + Y^2 \, d\Omega^2,
\end{equation}

where $M$ is an arbitrary (positive) function (we refer to [18] for a detailed physical discussion of this spacetime, which is essentially the spacetime generated by a radiating fluid). The matching is performed along a hypersurface $\Sigma$ which, in terms of a spherical system of coordinates for the interior, has the simple form $r = r_b = \text{const}$. The Israel junction conditions at the matching hypersurface read as follows (see [5, Proposition 4.1 and Remark 4.2]):

\begin{equation}
    M(U(t), Y(t)) = \frac{1}{2} r^2 a(t) \dot{a}(t)^2,
\end{equation}

\begin{equation}
    \frac{\partial M}{\partial Y}(U(t), Y(t)) = \frac{1}{2} r^3 b(\dot{a}(t)^2 + 2a(t)\ddot{a}(t)),
\end{equation}

where the functions $(Y(t), U(t))$ satisfy

\begin{equation}
    Y(t) = r_b a(t), \quad \frac{dU}{dt}(t) = \frac{1}{1 + \dot{a}(t)r_b}.
\end{equation}

The two equations (5.2)–(5.3) are equivalent to require that Misner–Sharp mass $M$ is continuous and $\frac{\partial M}{\partial r} = 0$ on the junction hypersurface $\Sigma$.

The endstate of the collapse of these “homogeneous scalar field stars” is analyzed in the following theorem.

**Theorem 5.1.** Except at most for a measure zero set of initial data, the scalar field model matched with the generalized Vaidya solution (5.1) collapses to a black hole.

**Proof.** We will make use of the result in [5, Theorem 5.2], that states that if, in the approach to the singularity, the quantity $\dot{a}^2 = a^2 \dot{\rho}^2$ is bounded, then the apparent horizon cannot form, and the singularity is naked. It is indeed easy to check that the equation of the apparent horizon for the metric (5.1) is given by $r^2 \ddot{a}(t)^2 = 1$. If $\dot{a}$ is bounded, one can choose the junction surface $r = r_b$ sufficiently small such that $\dot{a}^2(t) < \frac{1}{4}$, $\forall (t, r) \in [0, t_s] \times [0, r_b]$, and so $(1 - \frac{2M}{r^2})$ is bounded away from zero near the singularity. As a consequence, one can find in the exterior portion of the spacetime (5.1), null radial geodesics which meet the singularity in the past, and therefore the singularity is naked. Otherwise, if $\dot{a}^2$ is unbounded, the trapped region forms and the collapse ends into a black hole. Now, we have proved in Lemma 4.10 the existence of $\rho_\infty = \lim_{\phi \rightarrow \infty} \rho(\phi)$ (which is also equal to $\lim_{t \rightarrow t_s^-} \rho(t)$), and that, actually, $\rho_\infty$ vanishes except for a zero–measured set of initial data. On the other end, using (2.3b), we get

\begin{equation}
    \ddot{\rho} = -c_s^2 \left(\frac{2 - \rho}{1 + \rho}\right),
\end{equation}

and therefore $\ddot{\rho} \leq -\frac{c_s^2}{a}$ eventually holds in a left neighborhood of $t_s$, say $[t_0, t_s]$ where $\dot{\rho}$ is decreasing. It follows that $\forall t \in [t_0, t_s]$, $\ddot{\rho}(t) \leq \ddot{\rho}(t_0)$, which is negative by hypotheses. Then

$$
    \dot{\rho}(t) - \dot{\rho}(t_0) = \int_{t_0}^{t} \dot{\rho}(\tau) \, d\tau \leq -\int_{t_0}^{t} \frac{\dot{\rho}(\tau)^2}{a(\tau)} \, d\tau \leq -\dot{\rho}(t_0) \int_{t_0}^{t} \frac{\dot{\rho}(\tau)}{a(\tau)} \, d\tau,
$$

that diverges to $-\infty$ by (4.16). Then $\dot{\rho}(t)$ is unbounded. \hfill \Box
5.1. Examples of non-generic situations. In the crucial Lemma 4.10 the function $\rho$ has been shown to vanish, in the late time behavior, for almost every choice of the initial data. Now we want to reconsider the non generic situation where $\lim_{t \to t^-} y(\phi(t))u(\phi(t)) = 1$. Actually, if the potential $V(\phi)$ satisfies the condition
\[ \lim \inf_{\phi \to \pm \infty} |u(\phi)| > 0, \]
the argument of Proposition 4.11 may be used also in this (non generic case), to prove the formation of the singularity in a finite amount of time. Indeed, in this case $y$ is bounded above, and then $\rho$ is, so that one can use (4.17) to find a constant $\kappa > 0$ such that $\dot{\epsilon} > \kappa\epsilon^{3/2}$ in a left neighborhood of $\sup I$.

Moreover, recalling that $\rho = y^2 - 1$, and equation (5.5), if
\[ \lim \inf_{\phi \to \pm \infty} |u(\phi)| > \frac{\sqrt{3}}{3}, \]
the argument in the proof of Theorem 5.1 also applies here, and the collapse ends into a black hole. In particular, observe that in this case $2 - \rho$ is bounded away from zero, and positive. On the other side, if
\[ \lim \sup_{\phi \to \pm \infty} |u(\phi)| < \frac{\sqrt{3}}{3}, \]
then $\dot{a}$ is eventually positive, and this implies that $\dot{a}$ is bounded, so that the formation of the apparent horizon is forbidden. As a consequence, the collapse, in this non–generic situation, ends into a naked singularity.

Example 5.2. The potential in exponential form of Example 4.14 is such that $u(\phi)$ behaves like a positive constant $\lambda$, and then it always gives rise to a black hole. The non–generic – data producing the solution such that $\lim_{\phi \to \infty} \rho(\phi) = \frac{1}{\sqrt{\lambda}} - 1$ are discussed in [5, 7]: a choice of $\lambda < \frac{\sqrt{3}}{3}$ makes the limit $\rho_\infty$ greater than 2, and the apparent horizon cannot form, resulting in a naked singularity.

Example 5.3. Let $V$ the potential given by the polynomial in Example 5.13. We already know that almost every choice of initial data forms a singularity, which happens to be a black hole. Note that, in the general case of a polynomial potential with leading term $\lambda^2\phi^{2n}$, with $n > 2$, it can be shown that every initial data gives rise to a singularity, although the function $u(\phi)$ diverges and therefore the above arguments cannot be applied. Indeed, if $y(\phi)$ goes like $u(\phi)^{-1}$ – that in this case behaves like $\frac{1}{\sqrt{\phi}}$ – then using (4.9) it is easily seen that $\phi \approx \phi^{n-1}$, since
\[ \frac{3\phi^2}{n^2} \approx y^2 = 1 + \rho \approx 1 + \frac{2\lambda^2\phi^{2n}}{\phi^2}, \]
and then if $n > 2$ the solution must diverge in a finite amount of time. The non generic data yielding this particular situation, moreover, forbid apparent horizon formation, and so the resulting singularity is naked.

6. Discussion and Conclusions

We have discussed here the qualitative behavior of the solutions of the Einstein field equations with homogeneous scalar fields sources in dependence of the choice of the self-interacting potential. In the case of cosmological models, our results widely extend the recent relevant results obtained by Rendall ([14, 15]) and by Miritzis ([13]). Using asymptotic analysis, Rendall has been able to show that scalar fields exhibit an oscillating behavior if the potential is quadratic (that is, if the field is free but not massless) or if $V$ is of the form $a\phi^2 + O(\phi^3)$, while Miritzis has classified the limiting behaviour of solutions for a large class of non-negative scalar field potentials. In both these cases however, global existence is guaranteed. Our approach here extends the “spectrum” of available potential, and in doing so includes the cases in which there is no global in time evolution. This allows us to treat in a unified manner also the case of gravitational collapse, in which a singularity is always formed in the future. Matching these solutions with a Vaidya “radiating star” exterior we obtained models of gravitational collapse which can be
viewed as the scalar-field generalization of the Oppenheimer-Snyder collapse model, in which a dust homogeneous universe is matched with a Schwarzschild solution (the Schwarzschild solution can actually be seen as a special case of Vaidya).

The Oppenheimer-Snyder model, as is well known, describes the formation of a covered singularity, i.e., a blackhole (it is actually the first model of blackhole formation ever discovered). The same occurs here: indeed we show that homogeneous scalar field collapse generically forms a blackhole. The examples of naked singularities which were found in recent papers [5, 7] turn out to correspond to very special cases which, mathematically, are not generic. Therefore, our results here show that naked singularities are not generic in homogeneous, self-interacting scalar field collapse, at least for the considered (but wide) class of physically relevant potentials. Non-genericity is already well known for non self-interacting (i.e. $V(\phi) = 0$) spherically symmetric scalar fields. Whether this result can actually be shown to hold also in the much more difficult case of both inhomogeneous and self-interacting scalar fields remains an open problem.

**APPENDIX A. AN EXISTENCE/UNIQUENESS THEOREM FOR A KIND OF SINGULAR ODE**

In this section we prove a result of existence, uniqueness, and instability of solution for a particular kind of ordinary differential equation of first order to which the standard theory does not apply straightforwardly. To the best of our knowledge this result, needed in the proof of Lemma 4.10, although relatively simple, does not appear in the literature.

**Theorem A.1.** Let us consider the Cauchy problem

$$h(s)\dot{z}(s) = f(s, z(s)) + g(s), \quad z(0) = 0.$$  

and $\beta$ is a positive constant such that the following conditions hold:

1. $h \in C^0([0, \beta], \mathbb{R})$ such that $h(0) = 0$, $h(s) > 0$ in $[0, \beta]$, and $h(s)^{-1}$ is not integrable in $[0, \beta]$;
2. $f \in C^0([0, \beta], \mathbb{R})$ such that $g(0) = 0$;
3. $f \in C^0([0, \beta] \times \mathbb{R}, \mathbb{R})$ such that $f(s, 0) \equiv 0$, $\frac{\partial f}{\partial z}(0, 0) < 0$, and
4. $\exists \rho > 0$ such that $\frac{\partial f}{\partial z}(s, z)$ is uniformly Lipschitz continuous with respect to $z$ in $[0, \beta] \times [-\rho, \rho]$, that is $\exists L > 0$ such that, if $|z_1|, |z_2| \leq \rho$, $s \in [0, \beta]$, then $|\frac{\partial f}{\partial z}(s, z_1) - \frac{\partial f}{\partial z}(s, z_2)| \leq L|z_1 - z_2|$.  

Then, there exists $\alpha < \beta$ such that the above Cauchy problem admits a unique solution $z(s)$ in $[0, \alpha]$. Moreover, this solution is the only function solving the differential equation $h(s)\dot{z}(s) = f(s, z(s)) + g(s)$ with the further property $\liminf_{s \to 0^+} |z(s)| = 0$.

**Proof.** Let us set $\alpha < \beta$ free to be determined later, and let $\mathcal{X}_\alpha$ be the space

$$\mathcal{X}_\alpha = \{ z \in C^0([0, \alpha]) \cap C^1([0, \alpha]) : z(0) = 0, \lim_{s \to 0^+} h(s)\dot{z}(s) = 0 \}.$$  

It can be proved that $\mathcal{X}_\alpha$ is a Banach space, endowed with the norm $||z||_\alpha = ||z||_\infty + ||h\dot{z}||_\infty$.

Let also be

$$\mathcal{Y}_\alpha = \{ \lambda \in C^0([0, \alpha]) : \lambda(0) = 0 \}$$

a (Banach) space endowed with the $L^\infty$-norm and let us consider the functional

$$\mathcal{F} : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha, \quad \mathcal{F}(z)(s) = h(z(s)) - f(s, z(s)).$$

It is easily verified that $\mathcal{F}$ is a $C^1$ functional, with tangent map at a generic element $z \in \mathcal{X}_\alpha$ given by

$$(d\mathcal{F}(z)[\xi])(s) = h(s)\dot{\xi}(s) - \frac{\partial f}{\partial z}(s, z(s))\xi(s),$$

where $\xi \in \mathcal{X}_\alpha$. Observing that $g \in \mathcal{Y}_\alpha$, we want to find $\alpha$ such that the equation

$$\mathcal{F}(z) = g$$

has a unique solution $z \in \mathcal{X}_\alpha$. To this aim, we will exploit an Inverse Function scheme, and we will prove that $\mathcal{F}$ is a local homeomorphism from a neighborhood of $z_0 \equiv 0$ in $\mathcal{X}_\alpha$ onto a neighborhood of...
$F(z_0) \equiv 0$ in $Y_\alpha$, that includes $g$. This will be done using neighborhoods with radius independent of $\alpha$ and this will be crucial to obtain the uniqueness result.

In the following we review the classic scheme (see for instance [1]) for reader’s convenience. Let $R(z) = F(z) - F(0) - dF(0)[z]$; then (A.1) is equivalent to find $z$ such that $R(z) + dF(0)[z] = g$, and therefore, if $dF(0)$ is invertible, to prove the existence of a unique fixed point of the application $T$ on $\mathcal{X}_\alpha$.

(A.2) \[ T(z) = (dF(0))^{-1}[g - R(0, z)]. \]

We first show that $T$ is a contraction map from the ball $B(0, \delta) \subseteq \mathcal{X}_\alpha$ in itself, provided that $\delta$ and $\|g\|_\infty$ are sufficiently small (independently by $\alpha$). The following facts must be proven to this aim:

1. there exists a constant $M$, independent on $\alpha$, such that, $\forall z_1, z_2 \in \mathcal{X}_\alpha$ with $\|z_1\|_\alpha, \|z_2\|_\alpha \leq 1$, it is

(A.3) \[ \|dF(z_1) - dF(z_2)\| \leq M\|z_1 - z_2\|_\alpha \]

(the norm on the left hand side refers to the space of linear applications from $\mathcal{X}_\alpha$ to $Y_\alpha$).

2. there exists a constant $C$, independent on $\alpha$, such that

(A.4) \[ \|dF(0)^{-1}\| \leq C \]

(here the norm refers to the space of linear applications from $Y_\alpha$ to $\mathcal{X}_\alpha$).

If the above facts hold, given $z_1, z_2 \in \mathcal{X}_\alpha$, then

\[ dF(0)[T(z_1) - T(z_2)] = R(z_2) - R(z_1) = F(z_2) - F(z_1) - dF(0)[z_2 - z_1] = \int_0^1 (dF(tz_2 + (1 - t)z_1) - dF(0)) [z_2 - z_1] dt, \]

hence, if in addition $z_1, z_2 \in B(0, \delta)$,

\[ \|T(z_1) - T(z_2)\|_\alpha \leq \|dF(0)^{-1}\| \left( \int_0^1 \| (dF(tz_2 + (1 - t)z_1) - dF(0)) \| dt \right) \|z_1 - z_2\|_\infty \leq 2MC\delta\|z_1 - z_2\|_\alpha. \]

and therefore:

- $T$ is a contraction, taking $\delta$ such that $K := 2MC\delta < 1$;
- since $\|T(z)\|_\alpha \leq \|T(z) - T(0)\|_\alpha + \|T(0)\|_\alpha \leq K\|z\|_\alpha + \|(dF(0)^{-1}[g])\|_\alpha \leq K\|z\|_\alpha + C\|g\|_\infty$, then $T$ maps $B(0, \delta)$ in itself, provided that $\|g\|_\infty \leq C^{-1}(1 - K)\delta$.

Observe that the first of these two facts fixes the value of $\delta$, whilst the last inequality holds choosing $\alpha$ – free so far – small enough. This is one of the reasons why the constants $M$ and $C$ must be independent on $\alpha$. Then $T$ admits a unique fixed point on $B(0, \delta)$, which is the solution to our problem (A.1) on the interval $[0, \alpha]$. In other words, the function $F$ is a local homeomorphism from $B(0, C^{-1}(1 - K)\delta) \subseteq Y_\alpha$.

To see that the solution is globally unique on $\mathcal{X}_\alpha$, let us argue as follows. Suppose $\tilde{z} \in \mathcal{X}_\alpha \setminus B(0, \delta)$ ($\tilde{z} \neq z$) solves the problem, and let $\alpha_1 \leq \tilde{\alpha}$ sufficiently small such that $\|\tilde{z}\|_{\alpha_1} \leq \delta$. Then, observing $\|g\|_{L^\infty([0, \alpha_1])} \leq \|g\|_{L^\infty([0, \alpha])} \leq C^{-1}(1 - K)\delta$, and recalling that estimates (A.3)–(A.4) do not depend on $\alpha$, one can argue as before to find that $\tilde{z}|_{[0, \alpha_1]}$ is the unique element of $B(0, \delta) \subseteq \mathcal{X}_{\alpha_1}$ mapped into $g|_{[0, \alpha_1]} \in B(0, C^{-1}(1 - K)\delta) \subseteq Y_{\alpha_1}$. But, of course, $\|z\|_{\alpha_1} \leq \|z\|_\alpha < \delta$, so $z$ and $\tilde{z}$ coincide on $[0, \alpha_1]$, and therefore on all $[0, \alpha]$.

Therefore, to complete the proof of the existence and uniqueness for the given Cauchy problem, just (A.3)–(A.4) are to be proven. The first equation is a consequence of local uniform Lipschitz continuity.
of $f_z$. The second one needs some more care: taken $\lambda \in \mathcal{Y}_\alpha$, we must consider the Cauchy problem
\begin{equation}
(A.5) \quad h(s)\dot{\xi}(s) = \ell(s)\xi(s) + \lambda(s), \quad \xi(0) = 0,
\end{equation}
where $\ell(s) := \frac{\partial f_z(s, 0)}{\partial s}$, that without loss of generality we can suppose negative, $\forall s \in [0, \beta]$. First, it is easily seen that $(A.5)$ admits the unique solution $\xi \in \mathcal{X}_\alpha$.

\[\xi(s) = e^{-\int_0^s \ell(t)h(t)^{-1} \, dt} \int_0^s \frac{\lambda(t)}{h(t)} e^{\int_\tau^s \ell(t)h(t)^{-1} \, dt} \, d\tau.\]

Then $\|d\mathcal{F}(0)^{-1}\lambda\|_\alpha = \|\xi\|_\alpha \leq (1 + \ell_1)\|\xi\|_\infty + \|\lambda\|_\infty$, where $\ell_1 = \|\ell\|_{L^\infty([0,\beta])}$. Moreover, called $\ell_0 = \sup_{[0,\beta]} \ell(s) < 0$, it is easily seen that $\|\xi\|_\infty \leq -\frac{1}{\ell_0} \|\lambda\|_\infty$, so it suffices to choose $C = 1 - \frac{\ell_1 + 1}{\ell_0}$, and $(A.4)$ is proven.

To prove last claim of the Theorem, let us suppose that $w(s)$ is a function defined in $[0, \alpha]$ such that $h(s)\dot{w}(s) = f(s, w(s)) + g(s)$, and that $s_h$ is an infinitesimal and monotonically decreasing sequence such that $w(s_h) \to 0$ as $h \to \infty$. We want to prove that $w = z$, and therefore, it will suffice to show that $\lim_{s_h \to 0} w(s) = 0$.

First of all, observe that from the hypotheses, the equation $f(s, z) + g(s) = 0$ defines a continuous function $\zeta(s) : [0, \delta] \to \mathbb{R}$, such that $\zeta(0) = 0$. In particular, since $\frac{\partial f}{\partial s}(0, 0) < 0$, $\exists \rho > 0$ such that, in the rectangle $[0, \delta] \times [-\rho, \rho]$, it must be $f(s, z) + g(s) < 0$ (resp. $> 0$) if $z > \zeta(s)$ (resp. $z < \zeta(s)$).

Let us now argue by contradiction, supposing the existence of an infinitesimal sequence $s_h$ that can be chosen with the property $s_h < s_h$, such that $|w(s_h)| > \theta$ for some given constant $\theta$. Now, up to taking a smaller constant $\delta$, then $|\zeta(s)| < \frac{\rho}{2}$, $\forall s \in [0, \delta]$. Then, for $k$ sufficiently large, $|w(s_h)| \geq \theta > \frac{\rho}{2} \geq \sup_{[0, \delta]} |\zeta(s)|$. Therefore, $\forall s < s_h$, $|w(s)| > |w(s_h)|$, which is a contradiction since $w(s_h) \to 0$. Then $\lim_{s_h \to 0} w(s) = 0$, and the proof is complete.

\section*{Appendix B. Local existence/uniqueness of solutions with initial zero–energy}

\begin{lemma}
Let $\phi_0$, $v_0$ such that $v_0^2 + 2V(\phi_0) = 0$. Then, there exists $t_0 > 0$ such that the Cauchy problem
\begin{equation}
(B.1) \quad \begin{cases}
\ddot{\phi}(t) = -V'(\phi(t)) + \sqrt{3(\dot{\phi}(t)^2 + 2V(\phi(t)))}\dot{\phi}(t), \\
\phi(0) = \phi_0, \\
\dot{\phi}(0) = v_0,
\end{cases}
\end{equation}
has a unique solution $\phi(t)$ defined in $[0, t_0]$ with the property
\begin{equation}
(B.2) \quad \epsilon(t) = \frac{3}{2} \left(\int_0^t \dot{\phi}(s)^2 \, ds\right)^2, \quad \forall t \in [0, t_0].
\end{equation}

Moreover if $(\phi_{0,m}, v_{0,m}) \to (\phi_0, v_0)$, $(v_{0,m})^2 + 2V(\phi_{0,m}) = 0$ and $\phi_m$ is the solution of $(B.1)$ with initial data $(\phi_{0,m}, v_{0,m})$ satisfying condition $[B.2]$, it is $\phi_m \to \phi$ with respect to the $C^2$-norm in the interval $[0, t_0]$.

\end{lemma}

\begin{proof}
Let us consider the "penalized" problem
\begin{equation}
(B.3) \quad \begin{cases}
\ddot{\phi}(t) = -V'(\phi(t)) + \sqrt{3(\dot{\phi}(t)^2 + 2V(\phi(t))) + \frac{1}{m}}\dot{\phi}(t), \\
\phi(0) = \phi_0, \\
\dot{\phi}(0) = v_0,
\end{cases}
\end{equation}
that has a unique local solution $\phi_m$. If $\phi_m$ is not defined $\forall t \geq 0$, let $I_m$ be the set

\[I_m = \{ t \in \mathbb{R} : |\phi_m(s)| \leq |\phi_0| + 1, \quad |\dot{\phi}_m(s)| \leq |v_0| + 1, \quad \forall s \geq t \}.\]
Of course, \( I_n \neq \emptyset \) and, called \( t_n = \sup I_n \), if \( t_n \) is finite, then \( |\dot{\phi}_n(t_n)| = |v_0| + 1 \), or \( |\phi_n(t_n)| = |\phi_0| + 1 \). Now assume \( |\phi_n(t_n)| = |\phi_0| + 1 \). Then
\[
1 = |\phi_n(t_n) - \phi_0| \leq \int_0^{t_n} |\dot{\phi}_n(s)| \, ds \leq (|v_0| + 1)t_n.
\]

Analogously if \( |\dot{\phi}_n(t_n)| = |v_0| + 1 \) we have
\[
1 = |\phi_n(t_n) - \phi_0| \leq \int_0^{t_n} |\dot{\phi}_n(s)| \, ds.
\]

Since \( |\dot{\phi}_n(t)| \leq |\phi_0| + 1 \) and \( |\phi_n(s)| \leq |\phi_0| + 1 \) for all \( t \in [0, t_n] \), and \( \phi_n \) solves \( (B.3) \), we see that there exists \( C \) independent of \( n \) such that \( |\phi(t)| \leq C \) for all \( t \in [0, t_n] \). Therefore in this second case we obtain \( 1 \leq Ct_n \).

Then \( t_* := \inf_t t_n > 0 \) (we set \( t_* = +\infty \) \( \forall n \)). Moreover \( |\dot{\phi}_n| \) is uniformly bounded in \( [0, t_*] \), then up to subsequences, there exists a \( C^1 \) function \( \phi(t) \), solution of \( (B.1) \), such that \( \phi_n \to \phi \) and \( \dot{\phi}_n \to \dot{\phi} \) uniformly on \( [0, t_*] \).

Now, consider \( \epsilon_n(t) := \phi_n(t)^2 + 2V(\phi_n(t)) + \frac{1}{\sqrt{n}} \). We have
\[
\dot{\epsilon}_n(t) = 2\sqrt{3} \sqrt{\epsilon_n(t)} \dot{\phi}_n(t)^2.
\]

Then \( \epsilon \) is not decreasing, while \( \epsilon(0) = \frac{1}{n} \). Then is uniformly bounded away from zero and therefore by \( (B.4) \), dividing by \( \sqrt{n} \) and integrating gives \( \sqrt{\epsilon_n(t)} = \frac{1}{\sqrt{n}} + \sqrt{3} \int_0^t \dot{\phi}_n(s)^2 \, ds \). Therefore passing to the limit in \( n \) we obtain \( \epsilon(t) = \dot{\phi}(t)^2 + 2V(\phi(t)) = 3 \left( \int_0^t \dot{\phi}(s)^2 \, ds \right)^2 \) for all \( t \in [0, t_*] \) obtaining the proof of the existence of a solution.

The uniqueness of such a solution can be obtained by a contradiction argument. Assuming \( \phi \) and \( \psi \) solutions, and called \( \theta = \phi - \psi \), one can obtain, using \( (B.1) \), the estimate
\[
|\dot{\theta}(t)| \leq K_1 \int_0^t |\dot{\theta}(s)| \, ds + K_2 \int_0^t |\theta(s)| \, ds,
\]
for suitable constants \( K_1, K_2 \). Setting \( \rho(t) = |\theta(t)| + |\dot{\theta}(t)| \), and observing that \( \rho(0) = 0 \), it is not hard to get the estimate \( \rho(t) \leq (K_1 + K_2 + 1) \int_0^t \rho(s) \, ds \), and then \( \rho \equiv 0 \) from Gronwall’s inequality.

Finally using Gronwall’s Lemma as above we obtain also the continuity with respect to the initial data.

\[\square\]

**Remark B.2.** Reversing time direction in the above discussed problem \( (B.1) \) yields a results of genericity for expanding solutions such that the energy \( \epsilon(t) \) vanishes at some finite time \( T \).

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