Schur-Weyl reciprocity between the quantum superalgebra and the Iwahori-Hecke algebra

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Abstract

In this paper, we establish Schur-Weyl reciprocity between the quantum general super Lie algebra $U_q^{\sigma}(\mathfrak{gl}(m,n))$ and the Iwahori-Hecke algebra $H_{Q(q),r}(q)$. We introduce the sign $q$-permutation representation of $H_{Q(q),r}(q)$ on the tensor space $V^\otimes r$ of $(m+n)$ dimensional $\mathbb{Z}_2$-graded $Q(q)$-vector space $V = V_0 \oplus V_1$. This action commutes with that of $U_q^{\sigma}(\mathfrak{gl}(m,n))$ derived from the vector representation on $V$. Those two subalgebras of $\text{End}_{Q(q)}(V^\otimes r)$ satisfy Schur-Weyl reciprocity. As special cases, we obtain the super case ($q \to 1$), and the quantum case ($n = 0$). Hence this result includes both the super case and the quantum case, and unifies those two important cases.

1 Introduction

In the representation theory, the classification and the construction of the irreducible representations are essential themes. In the first half of the twentieth century, I. Schur\cite{11} introduced a prominent method to obtain the finite dimensional irreducible representations of the general linear group $\text{GL}(n, \mathbb{C})$, or equivalently of its Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, which we call Schur-Weyl reciprocity at present. Schur applied this method to the permutation action of the symmetric group $\mathfrak{S}_r$ and the diagonal action of $\text{GL}(n, \mathbb{C})$ on the tensor powers $V^\otimes r$ of the $n$ dimensional complex vector space $V$.

After this work, Schur-Weyl reciprocity has been extended to various groups and algebras. Brauer\cite{1} obtained the centralizer algebra of the orthogonal Lie group $O(n)$. Sergeev\cite{12} and Berele-Regev\cite{4} extended the Schur’s result to the general super Lie algebra $\mathfrak{gl}(m,n)$. Jimbo\cite{6} extended it to the $q$-analogue case. He established Schur-Weyl reciprocity between the quantum enveloping algebra $U_q(\mathfrak{gl}_{n+1})$ and the Iwahori-Hecke algebra of type $A$. As in the book of Curtis-Reiner\cite{2}, the representation theory of Iwahori-Hecke algebras is an important part in representation theories of finite groups of Lie type. Hence we will focus on the representation theory of Iwahori-Hecke algebras.
In [9], we defined a \(q\)-deformation of the alternating group as a subalgebra of the Iwahori-Hecke algebra, and determined all the isomorphism classes of (ordinary) irreducible representations. After [9], we intended to compute character values of irreducible representations directly using combinatorial methods. But this strategy did not go well because the notion of conjugacy classes of the \(q\)-deformation of the alternating group is obscure, hence we could not apply the classical \((q = 1)\) case which is found in [5]. Thereby we will take a representational approach to obtain the character table.

In [10], Regev obtained double centralizer properties for alternating groups. Roughly speaking, his result claims when one restrict the representation of the symmetric group on tensor space to the alternating group, the corresponding centralizer algebra enlarges in “super case” while does not change in “normal case”. Those facts suggest that Schur-Weyl reciprocity for \(U_q(\mathfrak{gl}(m, n))\) is more suitable to describe the representation theory of the alternating group than that for \(U_q(\mathfrak{sl}(n))\).

In this paper, we establish Schur-Weyl reciprocity between the quantum superalgebra \(U_{\sigma q}(\mathfrak{gl}(m, n))\) and the Iwahori-Hecke algebra \(H_{\mathbb{Q}(q)}(q)\). We define the sign \(q\)-permutation representation of \(H_{\mathbb{Q}(q)}(q)\) on \(V^\otimes r\) using an operator \(T\) on \(V^\otimes 2\) defined by:

\[
v_k \otimes v_l T = \begin{cases} 
\frac{(-1)^{|v_k|}(q + q^{-1}) + q - q^{-1}}{2} v_k \otimes v_l & \text{if } k = l, \\
(-1)^{|v_k||v_l|} v_l \otimes v_k + (q - q^{-1})v_k \otimes v_l & \text{if } k < l, \\
(-1)^{|v_k||v_l|} v_l \otimes v_k & \text{if } k > l,
\end{cases}
\]

where \(V\) is an \((m + n)\) dimensional \(\mathbb{Z}_2\)-graded \(\mathbb{Q}(q)\)-vector space and \(|\cdot|\) is the degree map. This action reduces to the sign permutation action (see [4]) of the symmetric group when \(q \to 1\) and to the well-known action of \(H_{\mathbb{Q}(q)}(q)\) obtained from Drinfeld-Jimbo solutions to the Yang-Baxter equation when \(n = 0\).

The quantum superalgebra has been defined in several articles such as [3], [8], or [13]. \(U_q'(\mathfrak{gl}(m, n))\) is a Hopf algebra obtained from the “naive” quantum superalgebra \(U_q(\mathfrak{gl}(m, n))\), which is a Hopf superalgebra, by adding an involutive element \(\sigma\). We show that the vector representation of \(U_q'(\mathfrak{gl}(m, n))\) on \(V^\otimes r\), which is found in [9], and the sign \(q\)-permutation representation of \(H_{\mathbb{Q}(q)}(q)\) are commuting elements in \(V^\otimes r\). Furthermore, extending the base field to the algebraic closure and applying the double centralizer theorem, we obtain the tensor space decomposition of \((V \otimes \mathbb{Q}(q))^\otimes r\) as \(H_{\mathbb{Q}(q)}(q) \otimes (U_q'(\mathfrak{gl}(m, n)) \otimes \mathbb{Q}(q))^\otimes r\)-modules. Our result will be very useful to the representation theory of the \(q\)-deformation of the alternating group.

2 The sign \(q\)-permutation representation of the Iwahori-Hecke algebra of type \(A\)

In this section, we shall define the sign \(q\)-permutation representation of the Iwahori-Hecke algebra of type \(A\), which is a \(q\)-deformation of the sign permutation module introduced by several precedent works such as [4], [12].
Let \((W, S = \{s_1, \ldots, s_r\})\) be a Coxeter system of rank \(r\). Let \(R\) be a commutative domain with 1, and let \(q_i (i = 1, \ldots, r)\) be any invertible elements of \(R\) such that \(q_i = q_j\) if \(s_i\) is conjugate to \(s_j\) in \(W\).

The Iwahori-Hecke algebra \(\mathcal{H}_R(W, S)\) is an \(R\)-algebra generated by \(\{T_{s_i} | s_i \in S\}\) with the relations:

(H1) \(T_{s_i}^2 = (q_i - q_i^{-1})T_{s_i} + 1\) if \(i = 1, 2, \ldots, r\),

(H2) \((T_{s_i}T_{s_j})^{k_{ij}} = (T_{s_j}T_{s_i})^{k_{ij}}\) if \(m_{ij} = 2k_{ij}\),

(H3) \((T_{s_i}T_{s_j})^{k_{ij}}T_{s_i} = (T_{s_j}T_{s_i})^{k_{ij}}T_{s_j}\) if \(m_{ij} = 2k_{ij} + 1\),

where \(m_{ij}\) is the order of \(s_is_j\) in \(W\). We define \(T_w = T_{s_{i_1}}T_{s_{i_2}} \cdots T_{s_{i_k}}\) where \(w = s_{i_1}s_{i_2} \cdots s_{i_k}\) is a reduced expression of \(w\). It is known that \(T_w\) is well defined because two elements \(T_w\) and \(T_{w'}\), where \(w\) and \(w'\) are reduced expressions of an element of \(W\), coincide and that \(\{T_w | w \in W\}\) form a basis of \(\mathcal{H}_R(W, S)\) as free \(R\)-modules. The relations (H1)-(H3) is equivalent to the following two relations:

(h1) \(T_{s_i}T_w = T_{s_iw}\) if \(l(w) < l(s_iw)\),

(h2) \(T_{s_i}T_w = (q_i - q_i^{-1})T_w + T_{s_iw}\) if \(l(w) > l(s_iw)\),

or equivalently,

(h'1) \(T_wT_{s_i} = T_{ws_i}\) if \(l(w) < l(ws_i)\),

(h'2) \(T_wT_{s_i} = (q_i - q_i^{-1})T_w + T_{ws_i}\) if \(l(w) > l(ws_i)\),

where \(l(w)\) means the length of \(w\). We write \(T_i = T_{s_i}\) for brevity.

If \((W, S)\) is of type \(A\) and of rank \(r-1\), then \(W\) is isomorphic to the symmetric group \(\mathfrak{S}_r\). Furthermore, all the elements of \(S\) are conjugate to each other, hence we may assume \(q_1 = \cdots = q_{r-1} = q\). The Iwahori-Hecke algebra \(\mathcal{H}_{R,q}(q) = \mathcal{H}_R(W, S)\) of type \(A\) has defining relations:

(A1) \(T_i^2 = (q - q^{-1})T_i + 1\) if \(i = 1, 2, \ldots, r-1\),

(A2) \(T_iT_{i+1}T_i = T_{i+1}T iT_{i+1}\) if \(i = 1, 2, \ldots, r-2\),

(A3) \(T_iT_j = T_jT_i\) if \(|i-j| > 1\).

Let \(V = \bigoplus_{k=1}^{m+n} Rv_k\) be a \(Z_2\)-graded \(R\)-module of rank \(m + n\). By \(Z_2\)-graded, we mean that \(V\) is a direct sum of two submodules \(V_0 = \bigoplus_{k=1}^{m} Rv_k\) and \(V_1 = \bigoplus_{k=m+1}^{m+n} Rv_k\), and that for each homogeneous element the degree map \(|\cdot|\)

\[
|v| = \begin{cases} 
0 & \text{if } v \in V_0, \\
1 & \text{if } v \in V_1,
\end{cases}
\]

is given. In order to define a representation of \(\mathcal{H}_{R,q}(q)\) on \(V^\otimes r\), we define a right operator \(T\) on \(V^\otimes V\) as follows.

\[
v_k \otimes v_l T = \begin{cases} 
-1)^{|v|}|v_k \otimes v_l \otimes v_l & \text{if } k = l, \\
\frac{1}{2}(-1)^{|v_k|}|(q + q^{-1})v_k \otimes v_l & \text{if } k < l, \\
(-1)^{|v_l|}|v_l \otimes v_k \otimes v_k & \text{if } k > l.
\end{cases}
\] (2.1)
Theorem 2.1. $\pi_r$ defines a representation of $\mathcal{H}_{R,r}(q)$ on $V^r$.

Proof. One can check that the above operators satisfy the defining relations (A1)–(A3) by a direct computation. For example, the relation (A1) is shown as follows.

case 1: $k = l$

$$v_k \otimes v_k T^2 = \frac{1}{4} \{ (-1)^{|v_k|} (q + q^{-1}) + q - q^{-1} \}^2 v_k \otimes v_k$$

$$= \frac{1}{2} \{ q^2 + q^{-2} + (-1)^{|v_k|} (q + q^{-1})(q - q^{-1}) \} v_k \otimes v_k$$

$$v_k \otimes v_k \{ (q - q^{-1})T + 1 \} = \frac{(q - q^{-1})}{2} \{ (-1)^{|v_k|} (q + q^{-1}) + q - q^{-1} \} v_k \otimes v_k + v_k \otimes v_k$$

$$= \frac{1}{2} \{ q^2 + q^{-2} + (-1)^{|v_k|} (q + q^{-1})(q - q^{-1}) \} v_k \otimes v_k$$

case 2: $k < l$

$$v_k \otimes v_l T^2 = \{ (-1)^{|v_k||v_l|} v_l \otimes v_k + (q - q^{-1})v_k \otimes v_l \} T$$

$$= v_k \otimes v_l + (q - q^{-1}) \{ (-1)^{|v_k||v_l|} v_l \otimes v_k + (q - q^{-1})v_k \otimes v_l \}$$

$$= (q - q^{-1}) \{ (-1)^{|v_k||v_l|} v_l \otimes v_k + (q^2 + q^{-2} - 1) v_k \otimes v_l \}$$

$$v_k \otimes v_l \{ (q - q^{-1})T + 1 \} = \{ (q - q^{-1}) \{ (-1)^{|v_k||v_l|} v_l \otimes v_k + (q - q^{-1})v_k \otimes v_l \} \} + v_k \otimes v_l$$

$$= (q - q^{-1}) \{ (-1)^{|v_k||v_l|} v_l \otimes v_k + (q^2 + q^{-2} - 1) v_k \otimes v_l \}$$

case 3: $k > l$

$$v_k \otimes v_l T^2 = (-1)^{|v_k||v_l|} \{ (-1)^{|v_k||v_l|} v_k \otimes v_l + (q - q^{-1})v_l \otimes v_k \}$$

$$= v_k \otimes v_l + (-1)^{|v_k||v_l|} (q - q^{-1}) v_l \otimes v_k$$

$$v_k \otimes v_l \{ (q - q^{-1})T + 1 \} = \{ (q - q^{-1}) (-1)^{|v_k||v_l|} v_l \otimes v_k + v_k \otimes v_l \}$$

(A2) can be shown in a similar manner to (A1), albeit slightly lengthy. The relation (A3) is obvious.

Remark 2.2. Another definition of the Iwahori-Hecke algebra, which is frequently used

(A′1) $\tilde{T}_i^2 = (q' - 1)\tilde{T}_i + q'$ if $i = 1, 2, \ldots, r - 1$,

(A′2) $\tilde{T}_i\tilde{T}_{i+1}\tilde{T}_i = \tilde{T}_{i+1}\tilde{T}_i\tilde{T}_{i+1}$ if $i = 1, 2, \ldots, r - 2$,

(A′3) $\tilde{T}_i\tilde{T}_j = \tilde{T}_j\tilde{T}_i$ if $|i - j| > 1$, 

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can be obtained from previous definition (A1)–(A3) by letting \( q' = q^2 \) and \( T_i = qT_i \). Accordingly, we get another right operator \( \tilde{T} \) on \( V \otimes V \) as follows.

\[
v_k \otimes v_l \tilde{T} = \begin{cases} 
(1)^{|v_k|}(q' + 1) + q' - 1 v_k \otimes v_k & \text{if } k = l, \\
\frac{-1}{2} q^{|v_k|}q^{|v_l|} (q' - 1) v_k \otimes v_l & \text{if } k < l, \\
(1)^{|v_k|} q^{|v_l|} (q' - 1) v_l \otimes v_k & \text{if } k > l.
\end{cases}
\] (2.2)

We can also define the action of the alternative generators \( \tilde{T}_i \) of the Iwahori-Hecke algebra in a similar manner.

This representation \( \pi_r \) is reduced to the (normal) \( q \)-permutation representation of \( H_{R,r}(q) \) obtained from Drinfeld-Jimbo solutions to the Yang-Baxter equation when \( n = 0 \) and to the sign permutation action (see [1]) of the symmetric group when \( q \to 1 \).

### 3 The quantum superalgebra \( U_q^{\sigma}(\mathfrak{gl}(m,n)) \) and the vector representation

In Kac’s paper[7], classical superalgebras have been classified and studied in detail. Quantum superalgebras have been defined in several articles such as [3],[8] or [13]. Each definition of them seems to be essentially. \( U_q^{\sigma}(\mathfrak{gl}(m,n)) \) is a Hopf algebra obtained from the “naive” quantum superalgebra \( U_q(\mathfrak{gl}(m,n)) \), which is a Hopf superalgebra, by adding an involutive element \( \sigma \). According to [3], we adopt \( U_q^{\sigma}(\mathfrak{gl}(m,n)) \) to construct the vector representation on the tensor space \( V^\otimes r \).

Let \( \Pi = \{ \alpha_i \}_{i \in I} \) be a set of simple roots with the index set \( I = \{ 1, \ldots, r \} \). We assume that \( I \) is a disjoint union of two subsets \( I_{\text{even}} \) and \( I_{\text{odd}} \). We define a map \( p : I \to \{ 0, 1 \} \) to be such that

\[
p(i) = \begin{cases} 
0 & \text{if } i \in I_{\text{even}}, \\
1 & \text{if } i \in I_{\text{odd}}.
\end{cases}
\]

Let \( P \) be a free \( \mathbb{Z} \)-module which includes all \( \alpha_i \in P(i \in I) \). We assume that a \( \mathbb{Q} \)-valued symmetric bilinear form on \( P \), \( \langle \cdot, \cdot \rangle : P \times P \to \mathbb{Q} \) is defined and that the simple coroots \( h_i \in P^*(i \in I) \) are given as data. The natural pairing \( \langle \cdot, \cdot \rangle : P^* \times P \to \mathbb{Z} \) between \( P \) and \( P^* \) is assumed to satisfy

\[
\langle h_i, \alpha_j \rangle = \begin{cases} 
2 & \text{if } i = j \text{ and } i \in I_{\text{even}}, \\
0 & \text{if } i = j \text{ and } i \in I_{\text{odd}}, \\
\leq 0 & \text{if } i \neq j.
\end{cases}
\]

We denote by \( \Pi^\vee = \{ h_i | i \in I \} \) the set of all coroots. Furthermore, for each \( i \in I \) we assume that there exists a nonzero integer \( \ell_i \) such that \( \ell_i(h_i, \lambda) = (\alpha_i, \lambda) \) for every \( \lambda \in P \). Then we immediately have the Cartan matrix \( A = [\langle h_i, \alpha_j \rangle]_{ij} \) is symmetrizable because \( \ell_i(h_i, \alpha_j) = (\alpha_i, \alpha_j) = (\alpha_j, \alpha_i) = \ell_j(h_j, \alpha_i) \). We
mention that the symmetrized matrix is $A^\text{sym} = \text{diag}(\ell_1, \ldots, \ell_r) A = [(\alpha_i, \alpha_j)]_{ij}$. Let $\mathfrak{h} = P^* \otimes \mathbb{Q}$. Then $\Phi = (\mathfrak{h}, \Pi^+, \Pi)$ is said to be a fundamental root data associated to $A$. Let $\mathfrak{g} = \mathfrak{g}(\Phi)$ be the contragredient Lie superalgebra obtained from $\Phi$ and $p$. According to [9], we define the quantized enveloping algebra $U_q(\mathfrak{g})$ to be the unital associative algebra over $\mathbb{Q}(q)$ with generators $q^h (h \in P^*)$, $e_i, f_i (i \in I)$, which satisfy the following defining relations (compare [8] and [13]):

(Q1) $q^h = 1$ for $h = 0$,
(Q2) $q^{h_1} q^{h_2} = q^{h_1 + h_2}$ for $h_1, h_2 \in P^*$,
(Q3) $q^h e_i = q^{(h, \alpha_i)} e_i q^h$ for $h \in P^*$ and $i \in I$,
(Q4) $q^h f_i = q^{-(h, \alpha_i)} f_i q^h$ for $h \in P^*$ and $i \in I$,
(Q5) $[e_i, f_j] = \delta_{ij} q_i^{\ell_i} - q_j^{-\ell_j}$ for $i, j \in I$,

where $[e_i, f_j]$ means the supercommutator $[e_i, f_j] = e_i f_j - (-1)^{p(i)p(j)} f_j e_i$.

We assume further conditions (bitransitivity condition, see [7] p.19):

(Q6) If $a \in \sum_{i \in I} U_q(\mathfrak{n}_+) e_i U_q(\mathfrak{n}_+)$ satisfies $f_i a \in U_q(\mathfrak{n}_+) f_i$ for all $i \in I$, then $a = 0$,
(Q7) If $a \in \sum_{i \in I} U_q(\mathfrak{n}_-) f_i U_q(\mathfrak{n}_-) e_i$ for all $i \in I$, then $a = 0$,

where $U_q(\mathfrak{n}_+)$ (resp. $U_q(\mathfrak{n}_-)$) is the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i | i \in I\}$ (resp. $\{f_i | i \in I\}$). $U_q(\mathfrak{g})$ is a Hopf superalgebra whose comultiplication $\Delta$, counit $\varepsilon$, antipode $S$ are as follows.

\[
\begin{align*}
\Delta(q^h) &= q^h \otimes q^h \quad \text{for } h \in P^*, \\
\Delta(e_i) &= e_i \otimes q^{-\ell_i} + 1 \otimes e_i \quad \text{for } i \in I, \\
\Delta(f_i) &= f_i \otimes 1 + q^{\ell_i} e_i \otimes f_i \quad \text{for } i \in I, \\
\varepsilon(q^h) &= 1 \quad \text{for } h \in P^*, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0 \quad \text{for } i \in I, \\
S(q^h) &= q^{-h} \quad \text{for } h \in P^*, \\
S(e_i) &= -e_i q^{\ell_i}, \quad S(f_i) = -q^{-\ell_i} f_i \quad \text{for } i \in I.
\end{align*}
\]

This is not a Hopf algebra. In order to give a Hopf algebra structure to $U_q(\mathfrak{g})$, we define an involutive operator $\sigma$ on $U_q(\mathfrak{g})$ by $\sigma(q^h) = q^h$ for all $h \in P^*$ and $\sigma(e_i) = (-1)^{p(i)} e_i$, $\sigma(f_i) = (-1)^{p(i)} f_i$ for all $i \in I$. Let $U_q^\sigma(\mathfrak{g}) = U_q(\mathfrak{g}) \oplus U_q(\mathfrak{g}) \sigma$. Then $U_q^\sigma(\mathfrak{g})$ is the algebra with the additional multiplication law given by $\sigma^2 = 1$ and $\sigma^{-1} x \sigma = \sigma(x)$ for any $x \in U_q(\mathfrak{g})$. $U_q(\mathfrak{g})$ is a Hopf algebra whose comultiplication $\Delta_\sigma$, counit
Let $\varepsilon_\sigma$, antipode $S_\sigma$ are as follows.
\[
\begin{align*}
\triangle_\sigma(\sigma) &= \sigma \otimes \sigma, \\
\triangle_\sigma(q^h) &= q^h \otimes q^h \quad \text{for } h \in P^*, \\
\triangle_\sigma(e_i) &= e_i \otimes q^{-\ell, h_i} + \sigma^{p(i)} \otimes e_i \quad \text{for } i \in I, \\
\triangle_\sigma(f_i) &= f_i \otimes 1 + \sigma^{p(i)} q^{\ell, h_i} \otimes f_i \quad \text{for } i \in I, \\
\varepsilon_\sigma(\sigma) &= \varepsilon_\sigma(q^h) = 1 \quad \text{for } h \in P^*, \\
\varepsilon_\sigma(e_i) &= \varepsilon_\sigma(f_i) = 0 \quad \text{for } i \in I, \\
S_\sigma(\sigma) &= \sigma, \\
S_\sigma(q^h) &= q^{\mp h} \quad \text{for } h \in P^*, \\
S_\sigma(e_i) &= -\sigma^{p(i)} e_i q^{\ell, h_i}, \\
S_\sigma(f_i) &= -\sigma^{p(i)} q^{-\ell, h_i} f_i \quad \text{for } i \in I.
\end{align*}
\]

The quantized enveloping algebra $U_q(\mathfrak{gl}(m,n))$ is obtained from the fundamental root data as follows.

- $I = I_{\text{even}} \cup I_{\text{odd}}$ is defined by $I_{\text{even}} = \{1, 2, \ldots, m-1, m+1, \ldots, m+n-1\}$ and $I_{\text{odd}} = \{m\}$.
- $P = \bigoplus_{b \in B} \mathbb{Z} \xi_b$, where $B = B_+ \cup B_-$ with $B_+ = \{1, \ldots, m\}$ and $B_- = \{m+1, \ldots, m+n\}$.
- $(\cdot, \cdot) : P \times P \rightarrow \mathbb{Q}$ is the symmetric bilinear form on $P$ defined by
  \[
  (\epsilon_a, \epsilon_{a'}) = \begin{cases} 
  1 & \text{if } a = a' \in B_+, \\
  -1 & \text{if } a = a' \in B_-, \\
  0 & \text{otherwise},
  \end{cases}
  \]
- $\Pi = \{\alpha_i | i \in I\}$ is defined by $\alpha_i = \epsilon_i - \epsilon_{i+1}$.
- $\Pi' = \{h_i | i \in I\}$ is uniquely determined by the formula $\ell_i(h_i, \lambda) = (\alpha_i, \lambda)$ for any $\lambda \in P$, where
  \[
  \ell_i = \begin{cases} 
  1 & \text{if } i = 1, \ldots, m, \\
  -1 & \text{if } i = m+1, \ldots, m+n-1.
  \end{cases}
  \]

Let $V$ be as in section 2, and suppose $R = \mathbb{Q}(q)$. The vector representation $(\rho, V)$ of $U_q^\sigma(\mathfrak{gl}(m,n))$ on $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ (recall that $V_0 = \bigoplus_{i=1}^{m} Rv_i, V_1 = \bigoplus_{i=m+1}^{m+n} Rv_i$) is defined by
\[
\begin{align*}
\rho(\sigma)v_j &= (-1)^{|v_j|}v_j \quad \text{for } j = 1, \ldots, m + n, \\
\rho(q^h)v_j &= q^{\ell_j(h)}v_j \quad \text{for } h \in P^*, j = 1, \ldots, m + n, \\
\rho(e_j)v_{j+1} &= v_j \quad \text{for } j = 1, \ldots, m + n - 1, \\
\rho(f_j)v_{j+1} &= v_{j+1} \quad \text{for } j = 1, \ldots, m + n - 1, \\
&\quad \text{otherwise } 0.
\end{align*}
\]

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This representation can be extended to the representation on the tensor space $V^\otimes r$. Let $\rho_r$ be the map from $U_q^\sigma(\mathfrak{gl}(m,n))$ to $\text{End}_{Q(q)}(V^\otimes r)$ defined by

\[
\begin{align*}
\rho_r(\sigma) &= \rho(\sigma)^\otimes r, \\
\rho_r(q^h) &= \rho(q^h)^\otimes r \quad \text{for } h \in P^*, \\
\rho_r(e_i) &= \sum_{k=1}^N \rho(\sigma^{p_{\sigma}}(i)) \otimes \rho(e_i) \otimes \rho(q^{-e_i,h_i})^{\otimes r-k} \quad \text{for } i \in I, \\
\rho_r(f_i) &= \sum_{k=1}^r \rho(\sigma^{p_{\sigma}}(i) q^{f_i,h_i}) \otimes \rho(f_i) \otimes \text{Id}^{\otimes r-k} \quad \text{for } i \in I.
\end{align*}
\]

(3.2)

**Proposition 3.1 (3 Proposition3.1).** $\rho_r$ gives a completely reducible representation of $U_q^\sigma(\mathfrak{gl}(m,n))$ on $V^\otimes r$ for $r \geq 1$.

Making use of the comultiplication of Hopf algebra, we obtain $\rho_r$. Let $\Delta^{(1)} = \Delta_{\sigma}$ at first and set $\Delta^{(k)} = (\Delta_{\sigma} \otimes \text{Id}^{\otimes k-1}) \Delta^{(k-1)}$ inductively. Then from the definition of $\Delta_{\sigma}$, we have $\rho_r(x) = \rho^{\otimes r} \circ \Delta^{(r-1)}(x)$ for $x \in U_q^\sigma(\mathfrak{gl}(m,n))$ immediately.

4 Schur-Weyl reciprocity for $U_q^\sigma(\mathfrak{gl}(m,n))$ and $H_{Q(q),r}(q)$

In the preceding sections, we obtained the left action of $U_q^\sigma(\mathfrak{gl}(m,n))$ and the right one of $H_{Q(q),r}(q)$ on the tensor space. We notice that the element $q$ of $Q(q)$ is an indeterminate. In this section we derive the commutativity between these two actions. Furthermore, we establish Schur-Weyl reciprocity for these two algebras. We consider the relation between the vector representation $\rho_2$ and the operator $T$, both act on the tensor space $V^\otimes 2$, at first.

**Proposition 4.1.** $T$ commutes with the action of $U_q^\sigma(\mathfrak{gl}(m,n))$ on $V^\otimes 2$ which is given by the representation $\rho_2$.

*Proof.* The action $\rho_2(g) = (\rho \otimes \rho) \circ \Delta_{\sigma}(g)$ of $U_q^\sigma(\mathfrak{gl}(m,n))$ on $V^\otimes 2$ is defined in several cases depending upon the generator of $U_q^\sigma(\mathfrak{gl}(m,n))$ and the basis vector $v_i \otimes v_j$ of $V^\otimes 2$.

Case 1: $g = \sigma$

One can immediately check the commutativity between $\rho_2$ and $T$.

Case 2: $g = q^h (h \in P^*)$

It is clear in this case because of the equation,

\[
\rho_2(q^h) v_i \otimes v_j = q^{\epsilon_i(h)} q^{\epsilon_j(h)} v_i \otimes v_j = q^{(\epsilon_i + \epsilon_j)(h)} v_i \otimes v_j.
\]

Case 3: $g = e_k$ ($k = 1, 2, \ldots, m + n$)
We obtain the case-by-case definition of $\rho_2$ from (3.2) as follows.

$$\rho_2(e_k)v_i \otimes v_j = \begin{cases} 0 & \text{if } i, j \neq k + 1, \\ v_{i-1} \otimes v_j & \text{if } i = k + 1, j \neq k, k + 1, \\ q^{(-1)^{\varepsilon_j}}v_{i-1} \otimes v_j & \text{if } i = k + 1, j = k, \\ q^{(-1)^{\varepsilon_j}}v_{i-1} \otimes v_j & \text{if } i \neq k, j = k + 1. \\ +(-1)^{v_i|\delta_{km}v_j}v_{i-1} \otimes v_j & \text{if } i = k + 1, j = k + 1, \\ (-1)^{v_i|\delta_{km}v_i}v_{i+1} \otimes v_j & \text{if } i \neq k + 1, j = k + 1. \end{cases} \quad (4.1)$$

The operator $T$ has already defined in (2.1). One can check the commutativity between $T$ and $\rho_2$ by a direct computation in each case.

Case3–1 : $i, j \neq k + 1$

In this case, we immediately see $(\rho_2(e_k)v_i \otimes v_j)T = \rho_2(e_k)(v_i \otimes v_j)T = 0$.

Case3–2 : $i = k + 1, j > k + 1$

$$\{\rho_2(e_k)v_i \otimes v_j\}T = v_{i-1} \otimes v_j T = (-1)^{v_{i-1}|v_j}v_j \otimes v_{i-1} + (q - q^{-1})v_{i-1} \otimes v_j$$

$$\rho_2(e_k)(v_i \otimes v_j)T = \rho_2(e_k)\{(-1)^{v_i|v_j}v_j \otimes v_i + (q - q^{-1})v_i \otimes v_j\} = (-1)^{v_i|v_j + \delta_{km}}v_i \otimes v_j T.$$ If $k < m$, then $|v_{i-1}| = |v_i| = \delta_{km} = 0$. If $k = m$, then $|v_{i-1}| = 0, |v_i| = 1, \delta_{km} = 1$. If $k > m$, then $|v_{i-1}| = |v_i| = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i \otimes v_j\}T = \rho_2(e_k)(v_i \otimes v_j)T$ holds.

Case3–3 : $i = k + 1, j < k$

$$\{\rho_2(e_k)v_i \otimes v_j\}T = v_{i-1} \otimes v_j T = (-1)^{v_{i-1}|v_j}v_j \otimes v_{i-1}$$

$$\rho_2(e_k)(v_i \otimes v_j)T = \rho_2(e_k)\{(-1)^{v_i|v_j}v_j \otimes v_i\} = (-1)^{v_i|v_j + \delta_{km}}v_j \otimes v_{i-1}.$$ As in the case3–2, $\{\rho_2(e_k)v_i \otimes v_j\}T = \rho_2(e_k)(v_i \otimes v_j)T$ holds.

Case3–4 : $i = k + 1, j = k$

$$\{\rho_2(e_k)v_i \otimes v_j\}T = q^{(-1)^{\varepsilon_j}}v_{i-1} \otimes v_j T = q^{(-1)^{\varepsilon_j}}2^{-1}\{(-1)^{v_j}|(q + q^{-1}) + q - q^{-1}\}v_j \otimes v_j$$

$$\rho_2(e_k)(v_i \otimes v_j)T = \rho_2(e_k)\{(-1)^{v_i|v_j}v_j \otimes v_i\} = (-1)^{v_j + \delta_{km}}v_j \otimes v_{i-1}.$$ If $k < m$, then $v_j = \delta_{km} = 0$. If $k = m$, then $v_j = 0, \delta_{km} = 1$. If $k > m$, then $v_j = v_{j+1} = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i \otimes v_j\}T = \rho_2(e_k)(v_i \otimes v_j)T$ holds.
If $k < m$ then $|v_{i-1}| = |v_i| = \delta_{km} = 0$. If $k = m$, then $|v_{i-1}| = 0, |v_i| = \delta_{km} = 1$. If $k > m$, then $|v_{i-1}| = |v_i| = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i\otimes v_j\}T = \rho_2(e_k)(v_i\otimes v_jT)$ holds.

Case 3-6: $i < k$, $j = k + 1$

$$\{\rho_2(e_k)v_i\otimes v_j\}T = (-1)^{|v_i|}|\delta_{km}v_i\otimes v_{j-1}T$$
$$= (-1)^{|v_i|+|\delta_{km}|}|v_{j-1}\otimes v_i + (-1)^{|v_i|}|\delta_{km}(q-q^{-1})v_i\otimes v_{j-1}$$

$$\rho_2(e_k)(v_i\otimes v_j) = \rho_2(e_k)\{(-1)^{|v_i|}v_{j}\otimes v_i + (q-q^{-1})v_i\otimes v_j\}$$
$$= (-1)^{|v_i|}v_{j-1}\otimes v_i + (q-q^{-1})(-1)^{|v_i|}\delta_{km}v_i\otimes v_{j-1}$$

If $k < m$, then $|v_i| = |v_j| = \delta_{km} = 0$. If $k = m$, then $|v_i| = |v_j| = 0, |v_j| = \delta_{km} = 1$. If $k > m$, then $|v_i| = |v_j| = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i\otimes v_j\}T = \rho_2(e_k)(v_i\otimes v_jT)T$ holds.

Case 3-7: $i = k$, $j = k + 1$

$$\{\rho_2(e_k)v_i\otimes v_j\}T = (-1)^{|v_i|}|\delta_{km}v_i\otimes v_{j-1}T$$
$$= (-1)^{|v_i|}|\delta_{km}(q+q^{-1})+q-q^{-1}v_i\otimes v_i$$

$$\rho_2(e_k)(v_i\otimes v_j) = \rho_2(e_k)\{(-1)^{|v_i|}v_{j}\otimes v_i + (q-q^{-1})v_i\otimes v_j\}$$
$$= (-1)^{|v_i|}v_{j-1}\otimes v_i + (q-q^{-1})(-1)^{|v_i|}\delta_{km}v_i\otimes v_{j-1}$$

If $k < m$, then $|v_i| = |v_j| = \delta_{km} = 0$. If $k = m$, then $|v_i| = 0, |v_j| = \delta_{km} = 1$. If $k > m$, then $|v_i| = |v_j| = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i\otimes v_j\}T = \rho_2(e_k)(v_i\otimes v_jT)$ holds.

Case 3-8: $i > k + 1$, $j = k + 1$

$$\{\rho_2(e_k)v_i\otimes v_j\}T = (-1)^{|v_i|}|\delta_{km}v_i\otimes v_{j-1}T$$
$$= (-1)^{|v_{j-1}|+|\delta_{km}|}|v_{j-1}\otimes v_i$$

$$\rho_2(e_k)(v_i\otimes v_j) = \rho_2(e_k)\{(-1)^{|v_i|}v_{j}\otimes v_i + (q-q^{-1})v_i\otimes v_j\}$$
$$= (-1)^{|v_i|}v_{j-1}\otimes v_i$$

If $k < m$, then $|v_{j-1}| = |v_{j}| = \delta_{km} = 0$. If $k = m$, then $|v_{j-1}| = 0, |v_{j}| = \delta_{km} = 1$. If $k > m$, then $|v_{j-1}| = |v_{j}| = 1, \delta_{km} = 0$. In each case, $\{\rho_2(e_k)v_i\otimes v_j\}T = \rho_2(e_k)(v_i\otimes v_jT)$ holds.

Case 3-1 to Case 3-8 exhaust the possible cases in (4.1), thus we have checked the commutativity for case 3.
Case 4: $g = f_k$ ($k = 1, 2, \ldots, m + n$)

In a similar manner to case 3, we obtain

$$
\rho_2(f_k) v_i \otimes v_j = \begin{cases} 
0 & \text{if } i, j \neq k, \\
(-1)^{|\nu_i| \delta_{k = v_i} \otimes v_{j+1}} & \text{if } i \neq k, k + 1, j = k, \\
v_{i+1} \otimes v_j & \text{if } i = k, j = k, \\
(-1)^{|\nu_i| \delta_{k = q(-1)^{|\nu_i|} v_i} \otimes v_{j+1}} & \text{if } i = k + 1, j = k, \\
v_{i+1} \otimes v_j & \text{if } i = k, j \neq k, 
\end{cases}
$$

and one can check for this case, so we omit the detail.

Finally, these exhaust the entirely possible cases, thus we have completed the proof. \qed

**Proposition 4.2.** For every $g \in H_{\mathcal{Q}(q), r}(q)$ and $x \in U_q^{r}(\mathfrak{gl}(m, n))$, we have $\pi_r(g) \rho_r(x) = \rho_r(x) \pi_r(g)$.

**Proof.** When $r = 2$, we have already shown in proposition 4.1. It is enough to prove for $g \in \{T_1, \ldots, T_{r-1}\}$. We deduce from cocommutativity of $\Delta_\sigma$ that

$$
\Delta^{(r)} = (\Delta_\sigma \otimes \text{id}^{\otimes r-1}) \Delta^{(r-1)} \\
= (\text{id} \otimes \Delta_\sigma \otimes \text{id}^{\otimes r-2}) \Delta^{(r-1)} \\
= (\text{id}^{\otimes 2} \otimes \Delta_\sigma \otimes \text{id}^{\otimes r-3}) \Delta^{(r-1)} \\
= \ldots \\
= (\text{id}^{\otimes r-1} \otimes \Delta_\sigma) \Delta^{(r-1)}
$$

where id is the identity operator on $U_q^{r}(\mathfrak{gl}(m, n))$. For any $r$ with $r > 2$, we may write $\Delta^{(r-2)}(x) = x_1 \otimes \cdots \otimes x_{r-1}$ for some $x_1, \ldots, x_{r-1} \in U_q^{r}(\mathfrak{gl}(m, n))$. Then applying the case $r = 2$, we have the following.

$$
\pi_r(T_i) \rho_r(x) = \left\{ \text{Id}^{\otimes i-1} \otimes \pi_2(T_i) \otimes \text{Id}^{\otimes r-i-1} \right\} \left\{ \rho^{\otimes r} \circ \Delta^{(r-1)}(x) \right\} \\
= \left\{ \text{Id}^{\otimes i-1} \otimes \pi_2(T_i) \otimes \text{Id}^{\otimes r-i-1} \right\} \left\{ \rho^{\otimes r} \circ (\text{id}^{\otimes i-1} \otimes \Delta_\sigma \otimes \text{id}^{\otimes r-i-1}) \Delta^{(r-2)}(x) \right\} \\
= \left\{ \otimes_{k=1}^{i-1} \rho(x_k) \right\} \otimes \pi_2(T_i) \Delta_\sigma(x_i) \otimes \left\{ \otimes_{i+1}^{r-1} \rho(x_i) \right\} \\
= \left\{ \otimes_{k=1}^{i-1} \rho(x_k) \right\} \otimes \Delta_\sigma(x_i) \pi_2(T_i) \otimes \left\{ \otimes_{i+1}^{r-1} \rho(x_i) \right\} \\
= \left\{ \rho^{\otimes r} \circ (\text{id}^{\otimes i-1} \otimes \Delta_\sigma \otimes \text{id}^{\otimes r-i-1}) \Delta^{(r-2)}(x) \right\} \left\{ \text{Id}^{\otimes i-1} \otimes \pi_2(T_i) \otimes \text{Id}^{\otimes r-i-1} \right\} \\
= \rho_r(x) \pi_r(T_i)
$$

\qed

We may define $\mathcal{G}_q$ to be the subalgebra of $\text{End}_{R_0}((R_0^{m+n})^{\otimes r}) \cong \text{Mat}((m + n)^r, R_0)$ generated by the set $\{\rho_r(\sigma), \rho_r(q^h), \rho_r(e_i), \rho_r(f_i) | h \in P^*, i \in I\}$ because all the matrix elements of those generators are in
$R_0$ from (3.1) and (3.2). For the same reason we may also define $S_q$ the one generated by $\{\pi_r(T_j) | j = 1, \ldots, r - 1\}$. Let us define two subalgebras of Mat $((m + n)^r, R_0)$ as follows.

$$\tilde{S}_q = \{ X \in \text{Mat} \left( (m + n)^r, R_0 \right) | XY = YX \text{ for all } Y \in S_q \}$$

$$\tilde{G}_q = \{ X \in \text{Mat} \left( (m + n)^r, R_0 \right) | XY = YX \text{ for all } Y \in G_q \}$$

Let $R_0 = \mathbb{Q}[q, q^{-1}]$ be the Laurent polynomial ring. Let us define the specialization to a nonzero complex number $t$ to be a ring homomorphism $\varphi_t : R_0 \to \mathbb{C}$ with the condition $\varphi_t(q) = t$. $\mathbb{C}$ becomes $(\mathbb{C}, R_0)$-bimodule, with $R_0$ acting from the right via $\varphi_t$. Applying the specialization $\varphi_t$, we obtain the specialized algebras $G_t = \mathbb{C} \otimes_{R_0} G_q$ and $S_t = \mathbb{C} \otimes_{R_0} S_q$ which are subalgebras of Mat $((m + n)^r, \mathbb{C})$. We also have $\tilde{G}_t = \mathbb{C} \otimes_{R_0} \tilde{G}_q$ and $\tilde{S}_t = \mathbb{C} \otimes_{R_0} \tilde{S}_q$ immediately.

**Proposition 4.3.** $G_q = \tilde{S}_q$ and $S_q = \tilde{G}_q$ hold.

**Proof.** Since $R_0$ is a principal ideal domain, the submodules $G_q$ and $S_q$ of the free $R_0$-module Mat $((m + n)^r, R_0)$ are also free. Let $X_i(q) \in \text{Mat} \left( (m + n)^r, R_0 \right) (i = 1, \ldots, N)$ be a basis of $G_q$ and $x_i^{k,l}(q) \in R_0$ the $(k, l)$-entry of $X_i(q)$. Then we immediately have that the specialized elements $X_i(t) (i = 1, \ldots, N)$ generate $G_t$ and $\text{dim}_\mathbb{C} G_t \leq \text{rank}_{R_0} G_q$. Because $X_i(q)$ are linearly independent, $\sum_{i=1}^N \alpha_i(q)x_i^{k,l}(q) = 0$ for $\alpha_1(q), \ldots, \alpha_N(q) \in R_0$ and for all $k, l$ imply $\alpha_1(q) = \cdots = \alpha_N(q) = 0$. We consider the system of linear equations as follows.

$$\begin{bmatrix}
x_1^{1,1}(q) & x_2^{1,1}(q) & \cdots & x_N^{1,1}(q) \\
x_1^{1,2}(q) & x_2^{1,2}(q) & \cdots & x_N^{1,2}(q) \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{(m+n)^r, (m+n)^r-1}(q) & x_2^{(m+n)^r, (m+n)^r-1}(q) & \cdots & x_N^{(m+n)^r, (m+n)^r-1}(q) \\
x_1^{(m+n)^r, (m+n)^r}(q) & x_2^{(m+n)^r, (m+n)^r}(q) & \cdots & x_N^{(m+n)^r, (m+n)^r}(q)
\end{bmatrix} \begin{bmatrix}
\alpha_1(q) \\
\alpha_2(q) \\
\vdots \\
\alpha_N(q)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}$$

This has only the trivial solution. But applying the specialization $\varphi_t$ to the above system, we possibly obtain a non-trivial solution. If there exists a non-trivial solution, then $t$ must be a zero of some Laurent polynomial whose coefficients are in $\mathbb{Q}$. Therefore if $t$ is a transcendental number, the above system has only the trivial solution and hence $\text{dim}_\mathbb{C} G_t = \text{rank}_{R_0} G_q$. In the same manner, we also have that if $t$ is a transcendental number, then $\text{dim}_\mathbb{C} \tilde{S}_t = \text{rank}_{R_0} \tilde{S}_q$.

We shall show that $\text{rank}_{R_0} \tilde{S}_q = \text{dim}_\mathbb{C} \tilde{S}_q$. One can readily see that $\text{rank}_{R_0} \tilde{S}_q \geq \text{dim}_\mathbb{C} \tilde{S}_q$ where $\tilde{S}_q$ is the specialized algebra of $\tilde{S}_q$. Assume that $X(q) = (x^{k,l}(q)) \in \text{Mat} \left( (m + n)^r, R_0 \right) \in \tilde{S}_q$. Then $X(q)$ commutes with $\pi_r(T_i)$ for all $i$, hence the matrix elements $x^{k,l}(q) (k, l = 1, \ldots, (m + n)^r)$ satisfy linear equations of coefficients in $R_0$. In the same manner as $G_q$, one can find that the commutativity condition turns out to be the condition of solubilities of certain Laurent polynomials of coefficients in $\mathbb{Q}$. Thus if $t$ is a transcendental number, only the trivial equation exists, so $\text{rank}_{R_0} \tilde{S}_q = \text{dim}_\mathbb{C} \tilde{S}_q$ holds. In a similar manner, we also have that if $t$ is a transcendental number, then $\text{rank}_{R_0} \tilde{G}_q = \text{dim}_\mathbb{C} \tilde{G}_q$. Now we fix a transcendental number $t$. In the preceding work such like [4, 12], it has already been shown that $\tilde{S}_1 = G_1$. From this fact and the inequality

$$\text{dim}_\mathbb{C} G_1 \leq \text{dim}_\mathbb{C} G_2 \leq \text{dim}_\mathbb{C} \tilde{S}_1 \leq \text{dim}_\mathbb{C} \tilde{S}_q,$$
we deduce that \( \dim_{\mathbb{C}} \mathcal{G}_t = \dim_{\mathbb{C}} \mathcal{S}_t \). Because \( \mathcal{G}_t \subseteq \mathcal{S}_t \), we obtain \( \mathcal{G}_t = \mathcal{S}_t \). It is known that the specialized algebra \( \mathcal{H}_{\mathbb{C},r}(t) = \mathbb{C} \otimes_{R_0} \mathcal{H}_{R_0,r}(q) \) is (split) semisimple. Therefore applying the double centralizer theorem, we also obtain \( \mathcal{S}_t = \mathcal{S}_t \). Using the properties,

\[
\begin{align*}
\text{rank}_{R_0} \mathcal{G}_t & = \dim_{\mathbb{C}} \mathcal{G}_t, \\
\text{rank}_{R_0} \mathcal{S}_t & = \dim_{\mathbb{C}} \mathcal{S}_t, \\
\text{rank}_{R_0} \mathcal{G}_t & = \dim_{\mathbb{C}} \mathcal{G}_t, \\
\text{rank}_{R_0} \mathcal{S}_t & = \dim_{\mathbb{C}} \mathcal{S}_t,
\end{align*}
\]

which are already shown in the previous discussion, we readily see that \( \mathcal{G}_q = \mathcal{S}_q \) and \( \mathcal{S}_q = \mathcal{S}_q \).

We denote \( \pi_r(\mathcal{H}_{\mathbb{Q}(q),r}(q)) \) by \( \mathcal{A}_q \) and \( \rho_r(\mathcal{U}_q^\sigma(^m_n(\mathfrak{gl}(m,n)))) \) by \( \mathcal{B}_q \). Then we have the following.

**Theorem 4.4.** \( \text{End}_{\mathcal{B}_q} V^\otimes r = \mathcal{A}_q \) and \( \text{End}_{\mathcal{A}_q} V^\otimes r = \mathcal{B}_q \) hold.

**Proof.** Obviously \( \mathcal{A}_q \cong \mathcal{S}_q \otimes_{R_0} \mathbb{Q}(q) \) and \( \mathcal{B}_q \cong \mathcal{G}_q \otimes_{R_0} \mathbb{Q}(q) \) as \( \mathbb{Q}(q) \)-algebras. From proposition 4.3 we obtain that \( \text{End}_{\mathcal{B}_q} V^\otimes r = \mathcal{A}_q \) and \( \text{End}_{\mathcal{A}_q} V^\otimes r = \mathcal{B}_q \), and we have completed the proof.

\[\square\]

## 5 Decomposition of the tensor space

Let \( \mathbb{Q}(q) \) be the algebraic closure of \( \mathbb{Q}(q) \). We define \( \mathcal{U}_q^\sigma(^m_n(\mathfrak{gl}(m,n))) = \mathcal{U}_q^\sigma(^m_n(\mathfrak{gl}(m,n))) \otimes \mathbb{Q}(q) \), \( \mathcal{A}_q = \mathcal{A}_q \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q) \), \( \mathcal{B}_q = \mathcal{B}_q \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q) \). Then, \( \pi_r(\mathcal{H}_{\mathbb{Q}(q),r}(q)) = \mathcal{A}_q \) and \( \rho_r(\mathcal{U}_q^\sigma(^m_n(\mathfrak{gl}(m,n)))) = \mathcal{B}_q \) as \( \mathbb{Q}(q) \)-algebras of operators on \( V^\otimes r = (V \otimes \mathbb{Q}(q) \mathcal{Q}(q))^\otimes r \). Theorem 4.4 still holds when we exchange the base field from \( \mathbb{Q}(q) \) to \( \mathbb{Q}(q) \), namely, \( \text{End}_{\mathcal{B}_q} V^\otimes r = \mathcal{A}_q \) and \( \text{End}_{\mathcal{A}_q} V^\otimes r = \mathcal{B}_q \).

We denote by \( \text{Par}(r) \) the set of all partitions of \( r \). By the double centralizer theorem, there is a subset \( \Gamma \) of \( \text{Par}(r) \) such that \( \mathcal{A}_q \cong \oplus_{\lambda \in \Gamma} \mathcal{A}_{q,\lambda} \) where \( \mathcal{A}_{q,\lambda} \) is the Wedderburn component corresponding to the irreducible representation of \( \mathcal{H}_{\mathbb{Q}(q),r}(q) \) indexed by \( \lambda \). Moreover, we also obtain the decomposition of \( \mathcal{H}_{\mathbb{Q}(q),r}(q) \otimes \mathcal{U}_q^\sigma(\mathfrak{gl}(m,n)) \)-modules,

\[
V^\otimes r = \bigoplus_{\lambda \in \Gamma} \mathcal{H}_\lambda \otimes V_\lambda,
\]

where \( \mathcal{H}_\lambda \) is the irreducible representation of \( \mathcal{H}_{\mathbb{Q}(q),r}(q) \) indexed by \( \lambda \), and \( V_\lambda \) is the one of \( \mathcal{U}_q^\sigma(\mathfrak{gl}(m,n)) \) such that \( V_\lambda \not\cong V_\mu \) if \( \lambda \neq \mu \). Our subject in this chapter is to determine \( \Gamma \).

Let \( H(m,n;r) = \{ \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(r) | \lambda_j \leq n \text{ if } j > m \} \). Diagrams of elements of \( H(m,n;r) \) are exactly those contained by the \( (m,n) \)-hooks. We shall show that \( \Gamma = H(m,n;r) \).

**Theorem 5.1.** \( \mathcal{A}_q = \bigoplus_{\lambda \in H(m,n;r)} \mathcal{A}_{q,\lambda} \). Hence \( V^\otimes r = \bigoplus_{\lambda \in H(m,n;r)} \mathcal{H}_\lambda \otimes V_\lambda \) holds.

**Proof.** When \( q = 1 \), then Berele and Regev have already shown that
Theorem 5.2 (3.20 The Hook Theorem). Let $F$ be an algebraic closed field of characteristic 0 and $\rho$ the sign permutation representation on $V^{\otimes n}$ where $V$ is a $(k, l)$-dimensional vector space over $F$. Then

$$\rho(F[\mathfrak{S}_n]) = \bigoplus_{\lambda \in H(k,l;n)} A_{\lambda} \cong \bigoplus_{\lambda \in H(k,l;n)} I_{\lambda}$$

where each $A_{\lambda}$ is the Wedderburn component corresponding to $\lambda$, and $I_{\lambda}$ is a simple subalgebra of $F[\mathfrak{S}_n]$ such that $\rho(I_{\lambda}) = A_{\lambda}$.

Thus $\Gamma = H(m,n;r)$ holds for $q = 1$. Let $t$ be a transcendental number. We have already shown in the proof of proposition 4.3 that $\dim_C G_t = \dim_C G_1$ and $\dim_C S_t = \dim_C S_1$. Let $S_t = \bigoplus_{\lambda \in \text{Par}(r)} S_{t,\lambda}$ be the Wedderburn decomposition. Then, by Theorem 5.2, we have $S_{1,\lambda} = 0$ if and only if $\lambda \not\in H(m,n;r)$. Because $\dim_C S_{t,\lambda} \geq \dim_C S_{1,\lambda}$ for every $\lambda \in \text{Par}(r)$ and $\dim_C S_t = \dim_C S_1$, we have $\dim_C S_{t,\lambda} = \dim_C S_{1,\lambda}$ for every $\lambda \in \text{Par}(r)$. Thus we obtain that $S_t = \bigoplus_{\lambda \in H(m,n;r)} S_{t,\lambda}$. Since $\bar{A}_q = \left(S_q \otimes \mathbb{Q}(q)\right) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)$ and $\dim_{\mathbb{Q}(q)} \bar{A}_q = \dim_C S_t$, we immediately get $\dim_{\mathbb{Q}(q)} \bar{A}_q = \dim_C S_t$. Because $t$ is transcendental, $\dim_{\mathbb{Q}(q)} \bar{A}_{q,\lambda} = \dim_C S_{t,\lambda}$ for every $\lambda \in \text{Par}(r)$. Thus we conclude that $\bar{A}_q = \bigoplus_{\lambda \in H(m,n;r)} \bar{A}_{q,\lambda}$. The second statement is the direct consequence of the double centralizer theorem.

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