COPRODUCT CANCELLATION ON ACT-S

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Abstract. The themes of cancellation, internal cancellation, substitution have led to a lot of interesting research in the theory of modules over commutative and noncommutative rings. In this paper, we introduce and study cancellation problem in the theory of acts over monoids. We show that if \( A \) is an \( S \)-act and \( A = \bigcup_{i \in I} A_i \) is the unique decomposition of \( A \) into indecomposable subacts \( A_i, i \in I \) such that the set \( P = \{ \text{Card}[i] \mid i \in I \} \) is finite, then \( A \) is cancellable if and only if the equivalence class \( [i] = \{ j \in I \mid A_i \cong A_j \} \) is finite, for every \( i \in I \). Likewise, we prove that every \( S \)-act is cancellable if and only if it is internally cancellable. Thus, the concepts cancellation and internal cancellation coincide here.

1. Introduction and Preliminaries

Jonsson and Tarski were considered cancellation problem initiatory in 1947 (see [5]). In the study of any algebraic system in which there is a notion of a direct sum, the theme of cancellation arises very naturally: if \( A \oplus B \cong A \oplus C \) in the given system, can we conclude that \( B \cong C \)? The answer is, perhaps not surprisingly, sometimes “yes” and sometimes “no”: it all depends on the algebraic system, and it depends heavily on the choice of \( A \) as well.

Importance of cancellation problem is obvious, since Serre’s famous conjecture on the freeness of f.g. projective modules over a polynomial ring \( R = K[x_1, \cdots, x_n] \) (for a field \( K \)) boiled down to a statement about the cancellability of \( R \) (see [2] and [9]).

Starting with a simple example, we all know that, by the Fundamental Theorem of Abelian Groups, the category of finitely generated abelian groups satisfies cancellation. If \( A \) is a finitely generated abelian group, then for any abelian groups \( B \) and \( C \), \( A \oplus B \cong A \oplus C \) still implies \( B \cong C \). There exists many

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torsionfree abelian groups of rank 1 that are not cancellable in the category of torsionfree abelian groups of finite rank, according to [4].

Now let $R$ be an associative ring with an identity. We say that $R$ has stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists some $y \in R$ such that $a + by$ is unit. If $\text{End}_R(A)$ has stable range one, then $A \oplus B \cong A \oplus C$ implies that $B \cong C$ for any right $R$-modules $B, C$ (see [8, Theorem 2]). Since every local ring has stable range one, therefore every strongly indecomposable module $A$ (that is, $\text{End}(A)$ is local) is cancellable and every simple module is cancellable by [1, Lemma 4.13.3].

On the other hand, in categories of modules over rings there are several variations on the notion of cancellation. For instance, for given module $A$, $A = K \oplus N = K' \oplus N'$ with $N \cong N'$, does it follow that $K \cong K'$? If the answer is always “yes”, $A$ is said to satisfy internal cancellation (or $A$ is internally cancellable). Another variations are the “Substitution” and “Dedekind-finite” properties (see [8] for the definitions). These properties are easily seen to be related as follows:

Substitution $\Rightarrow$ Cancellation $\Rightarrow$ Internal cancellation $\Rightarrow$ Dedekind-finite.

We encourage the reader to consult [1, 7, 8] about cancellation problem on the category of modules on arbitrary rings.

Let $S$ be a monoid with identity 1. Recall that a (right) $S$-act $A$ is a non-empty set equipped with a map $\lambda : A \times S \to A$ called its action, such that, denoting $\lambda(a,s)$ by $as$, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s,t \in S$. The category of all $S$-acts, with action-preserving ($S$-act) maps $(f : A \to B$ with $f(as) = f(a)s$, for $s \in S, a \in A)$ is denoted by $\text{Act}-S$. Clearly $S$ itself is an $S$-act with its operation as the action. Throughout this paper, all $S$-acts will be right $S$-act.

Recall that the category $\text{Act}-S$ has coproducts of any non-empty families of $S$-acts. More Precisely, if $I$ is a non-empty set and $X_i \in \text{Act} - S, i \in I$ then by [6, Proposition 2.1.8] the coproduct of $\{X_i : i \in I\}$ is their disjoint union $\bigcup_{i \in I} X_i$. Likewise, we recall that an $S$-act $A$ decomposable if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$. In this case $A = B \cup C$ is called a decomposition of $A$. Otherwise $A$ is called indecomposable. By [6, theorem 1.5.10], every $S$-act $A$ has a unique decomposition into indecomposable subacts. We will use this unique decomposition frequently. For more information about $S$-acts we encourage the reader to see [6].
In this paper, we investigate the cancellation problem in the category of $S$-acts.

2. CANCELLATION ON $\textbf{Act}-S$

In this paper, we give some results for cancellation problem in $\textbf{Act}-S$. We start with a definition:

**Definition 2.1.** An $S$-act $A$ satisfies cancellation, if for any $B, C \in \textbf{Act} - S$ that $A \dot{\cup} B \cong A \dot{\cup} C$ implies $B \cong C$. If $A$ satisfies cancellation we call $A$ is cancellable.

There exist examples that cancellation in $\textbf{Act}-S$ always does not satisfy.

**Example 2.2.** Let $S$ be a monoid.

(i) Given two non-isomorphic $S$-acts $B$ and $C$, let

\[
A := C \dot{\cup} B \dot{\cup} C \dot{\cup} \cdots
\]

then

\[
A \dot{\cup} B \cong A \dot{\cup} C,
\]

and we can not cancel $A$.

(ii) Take an indecomposable $S$-act $A$ and an arbitrary infinite set $I$. Then $B = \dot{\cup}_{i \in I} A_i$ in which $A_i = A$ for any $i \in I$ is not cancellable, because

\[
B \dot{\cup} A \cong B \dot{\cup} (A \dot{\cup} A)
\]

but

\[
A \dot{\cup} A \not\cong A.
\]

**Theorem 2.3.** Let $A$ and $B$ are $S$-acts. Then $S$-act $A \dot{\cup} B$ is cancellable if and only if $A$ and $B$ themselves are.

**Proof.** Let $A \dot{\cup} B$ is cancellable and $A \dot{\cup} C \cong A \dot{\cup} D$ in which $C, D$ are arbitrary $S$-acts. We have $B \dot{\cup} (A \dot{\cup} C) \cong B \dot{\cup} (A \dot{\cup} D)$ then $(A \dot{\cup} B) \dot{\cup} C \cong (A \dot{\cup} B) \dot{\cup} D$. Since $A \dot{\cup} B$ is cancellable therefore $C \cong D$. Hence $A$ is cancellable. In a similar vein, $B$ is cancellable.

Conversely, let $A, B$ are cancellable and $(A \dot{\cup} B) \dot{\cup} C \cong (A \dot{\cup} B) \dot{\cup} D$ in which $C, D$ are arbitrary $S$-acts. We have $A \dot{\cup} (B \dot{\cup} C) \cong A \dot{\cup} (B \dot{\cup} D)$ then $B \dot{\cup} C \cong B \dot{\cup} D$, because $A$ is cancellable. As $B \dot{\cup} C \cong B \dot{\cup} D$ and $B$ is cancellable we have $C \cong D$. Therefore $A \dot{\cup} B$ is cancellable. \qed
Proposition 2.4. Every indecomposable $S$-act is cancellable.

Proof. Let $A$ be an indecomposable $S$-act and $B$ and $C$ are two arbitrary $S$-acts in which

\[ A \cup B \cong A \cup C. \]  

(2.5)

We will show that $B \cong C$. We may assume without loss of generality that $A \cap B = A \cap C = \emptyset$. Let $f : A \cup B \to A \cup C$ be an $S$-isomorphism. By [6, Theorem 1.5.10] we can write $B = \bigcup_{i \in I} B_i$ and $C = \bigcup_{j \in J} C_j$ where all $B_i$’s and $C_j$’s are indecomposable and $B_i \cap B_{i'} = \emptyset, C_j \cap C_{j'} = \emptyset$ for any $i, i' \in I$ and $j, j' \in J$. Since $f$ is an isomorphism we get

\[ f(A \cup \bigcup_{i \in I} B_i) = f(A) \cup \bigcup_{i \in I} f(B_i) = A \cup \bigcup_{j \in J} C_j. \]

(2.6)

Here, by [6, Lemma 1.5.36] the subacts $f(A)$ and $f(B_i)$ are indecomposable for any $i \in I$. Furthermore, again by [6, Theorem 1.5.10] this decomposition is unique. Thus,

\[ f(A) = A \text{ or } f(A) = C_{j'} \text{ for some } j' \in J. \]

(2.7)

If $f(A) = A$ then for every $i \in I$ there exists a unique element $j \in J$ such that $f(B_i) = C_j$. Therefore

\[ B \cong f(B) = \bigcup_{i \in I} f(B_i) = \bigcup_{j \in J} C_j = C, \]

(2.8)

because $f$ is an isomorphism. If $f(A) = C_{j'}$ for some $j' \in J$ then $A = f(B_{i'})$ for some $i' \in I$. Therefore for every $i \neq i'$ there exists a unique element $j \neq j'$ such that $f(B_i) = C_j$ and this implies that

\[ \bigcup_{i \in I \setminus \{i'\}} f(B_i) = \bigcup_{j \in J \setminus \{j'\}} C_j. \]

(2.9)
Next from (2.9) we have

\[ B \cong f(B) = f(B_v) \cup \left( \bigcup_{i \in I \setminus \{v\}} f(B_i) \right) = A \cup \left( \bigcup_{j \in J \setminus \{j'\}} C_j \right) \cong f(A) \cup \left( \bigcup_{j \in J \setminus \{j'\}} C_j \right) = C_j \cup \left( \bigcup_{j \in J \setminus \{j'\}} C_j \right) = C, \]

i.e, \( B \cong C \), as required. \( \square \)

**Definition 2.5.** Let \( A = \bigcup_{i \in I} A_i \) be the unique decomposition of \( A \) into indecomposable subacts \( A_i, i \in I \). We call \( A \) finitely decomposable if \( 1 \leq |I| < \infty \). Otherwise \( A \) is called infinitely decomposable.

**Proposition 2.6.** Let \( S \) be a monoid. Then

1. Every cyclic \( S \)-act is cancellable.
2. Every simple \( S \)-act is cancellable.
3. Every monoid \( S \) is cancellable.
4. Every finitely decomposable \( S \)-act is cancellable.
5. Every finitely generated \( S \)-act is cancellable.

**Proof.** Since every cyclic \( S \)-act is indecomposable by [6, Proposition 1.5.8], (1)-(3) are clear by Proposition 2.4. Statements (4) and (5) are followed by Theorem 2.3 and Proposition 2.4. \( \square \)

**Corollary 2.7.** Let \( A \) be a free \( S \)-act with basis \( X \). Then \( A \) is cancellable if and only if the basis \( X \) is finite.

**Proof.** Since \( A \) is free \( S \)-act by [6, Theorem 1.5.13], \( A \cong \bigcup_{i \in I} S_i \) where \( S_i \cong S \) for any \( i \in I \) and \( |I| = |X| \). On the other hand, In Example 2.2 we have seen that if \( |X| = \infty \) then \( A \) is not cancellable. Therefore by Proposition 2.6 we get the result. \( \square \)

We have shown that every finitely decomposable \( S \)-act is cancellable. The converse is not true in general as the following theorem shows.
Theorem 2.8. Let $A$ be an infinitely decomposable $S$-act such that $A = \bigcup_{i \in I} A_i$ is the unique decomposition of $A$ into indecomposable subacts and $A_i \not\cong A_j$ for any pair of distinct elements $i, j \in I$. Then $A$ is cancellable.

Proof. Assume that $A \cup B \cong A \cup C$ where $B$ and $C$ are $S$-act. We must show that $B \cong C$. We may assume without loss of generality that $A \cap B = A \cap C = \emptyset$. Let $f : A \cup B \to A \cup C$ be an isomorphism and $B = \bigcup_{k \in K} B_k, C = \bigcup_{j \in J} C_j$ are unique decompositions of $B, C$ into their indecomposable subacts, respectively. We have $f(A \cup B) = A \cup C$ then

\[ \bigcup_{i \in I} f(A_i) \cup \bigcup_{k \in K} f(B_k) = \bigcup_{i \in I} A_i \cup \bigcup_{j \in J} C_j. \]  

Note that by [6, Lemma 1.5.36], in (2.10) all the components on the two sides are indecomposable acts. Since $A_i \not\cong A_i'$ for any distinct elements $i, i' \in I$, by applying [6, Theorem 1.5.10], we get for every $i \in I$, $f(A_i) = A_i$ or $f(A_i) = C_j$ for some $j \in J$. Next Put

\[ I_1 = \{i \in I \mid f(A_i) = A_i\}, \]
\[ I_2 = \{i \in I \mid f(A_i) = C_j \text{ for some } j \in J\}, \]
\[ J_1 = \{j \in J \mid C_j = f(A_i) \text{ for some } i \in I\}, \]
\[ J_2 = \{j \in J \mid C_j = f(B_k) \text{ for some } k \in K\}, \]
and
\[ K_1 = \{k \in K \mid f(B_k) = C_j \text{ for some } j \in J\}, \]
\[ K_2 = \{k \in K \mid f(B_k) = A_i \text{ for some } i \in I\}. \]

Then it is clear that

\[ I_1 \cap I_2 = \emptyset, \quad J_1 \cap J_2 = \emptyset \quad \text{and} \quad K_1 \cap K_2 = \emptyset \]
and

\[ |I_2| = |J_1|, \quad |K_1| = |J_2| \quad \text{and} \quad |K_2| = |I_2|. \]

We have $\bigcup_{j \in J_1} C_j = \bigcup_{i \in I_2} f(A_i)$, because for any $i \in I_2$ there exists a unique element $j \in J_1$ in such a way that $f(A_i) = C_j$, and vice versa. Similarly

\[ \bigcup_{j \in J_2} C_j = \bigcup_{k \in K_1} f(B_k) \text{ and } \bigcup_{i \in I_2} A_i = \bigcup_{k \in K_2} f(B_k). \]
Now, since $f$ is an isomorphism, by (2.14) and (2.15) we obtain

$$C = \left( \bigcup_{j \in J_1} C_j \right) \cup \left( \bigcup_{j \in J_2} C_j \right)$$

$$= \left( \bigcup_{i \in I_2} f(A_i) \right) \cup \left( \bigcup_{k \in K_1} f(B_k) \right)$$

$$\cong \left( \bigcup_{i \in I_2} A_i \right) \cup \left( \bigcup_{k \in K_1} f(B_k) \right)$$

$$= \left( \bigcup_{k \in K_2} f(B_k) \right) \cup \left( \bigcup_{k \in K_1} f(B_k) \right)$$

$$= f(B).$$

Therefore, $C \cong f(B) \cong B$. □

Let $A$ be an $S$-act and $A = \bigcup_{i \in I} A_i$ be the unique decomposition of $A$ into indecomposable subacts. Define for $i, j \in I$, $i \sim j$ if and only if $A_i \cong A_j$. Then $\sim$ is an equivalence relation on $I$. The equivalence class $i \in I$ is given by $[i] = \{ j \in I \mid A_i \cong A_j \}$.

With this introduction we have

**Theorem 2.9.** Let $A$ be an $S$-act and let $A = \bigcup_{i \in I} A_i$ be the unique decomposition of $A$ into indecomposable subacts $A_i$, $i \in I$ such that the set $P = \{ \text{Card}[i] \mid i \in I \}$ is finite. Then $A$ is cancellable if and only if the equivalence class $[i]$ is finite for every $i \in I$.

**Proof.** Let $P = \{ \text{Card}[i] \mid i \in I \}$ be finite.

**Necessity.** If for some $i \in I$, $[i]$ is an infinite set then $\bigcup_{j \in [i]} A_j$ is not cancellable (see Example 2.2). Therefore by Theorem 2.3, $A$ is not cancellable which is a contradiction.

**Sufficiency.** Assume that the equivalence class $[i]$ is finite for every $i \in I$. Let $m_1, \cdots, m_n$ are distinct elements of $P$, where $m_1$ and $m_n$ are the smallest and the greatest elements of $P$ respectively. We define for $1 \leq k \leq n$:

$$I_{m_k} = \{ i \in I \mid \text{card}[i] = m_k \}$$

In fact we realize that $I_{m_k}$ is the union of classes that each has $m_k$ elements. Note that it is possible that $I_{m_k}$ to be infinite. Now we define subsets $I_{m_k}^1, I_{m_k}^2, \cdots, I_{m_k}^n$ of $I_{m_k}$ recursively as follows:

Put $I_{m_k}^1$ to be the subset $I_{m_k}$ which consists of elements that we choose from
each classes one element. In a similar vein, suppose that we have defined subsets \( I_{m_k}^1, \ldots, I_{m_k}^{l-1} \). Then define \( I_{m_k}^l \) to be the set of elements of \( I_{m_k} \) that are not in the earlier subsets \( I_{m_k}^1, I_{m_k}^2, \ldots, I_{m_k}^{l-1} \). Summarizing for \( 1 \leq l \leq m_k \) we have,

\[
I_{m_k}^l = \{ i \in I_{m_k} \mid i \notin I_{m_k}^l \text{ for any } l_1 < l \} \text{ and } [i] \neq [i'], \quad \forall i \neq i' \in I_{m_k}^l.
\]

Set

\[
C_{m_k}^l = \bigcup_{i \in I_{m_k}^l} A_i.
\]

Then

\[
A = \bigcup_{k=1}^{n} \bigcup_{l=1}^{m_k} C_{m_k}^l
\]

for \( 1 \leq k \leq n \) and \( 1 \leq l \leq m_k \). By Theorem 2.8 \( C_{m_k}^l \) is cancellable for every \( 1 \leq k \leq n \) and \( 1 \leq l \leq m_k \), because for every distinct pair of elements \( i, j \in I_{m_k}^l \) we have \( A_i \nsubset A_j \). Therefore \( A \) is cancellable by Theorem 2.3. \( \square \)

**Theorem 2.10.** Let \( A = \bigcup_{i \in I} A_i \) be the unique decomposition of \( A \) into indecomposable subacts \( A_i, i \in I \). Furthermore, assume that the set of equivalence classes of \( I, I/\sim \), is finite. Then \( A \) is cancellable if and only if \( A \) is finitely decomposable.

**Proof.** Suppose that \( A \) is cancellable \( S \)-act. If in contrary \( A \) is infinitely decomposable, then \( I \) is infinite. Since \( |I/\sim| < \infty \) there exists an infinite subset \( J \subseteq I \) such that \( A_i \cong A_j \) for any \( i, j \in J \). Since \( \cup_{j \in J} A_j \) is not cancellable (see Example 2.2) therefore by Theorem 2.3 \( A \) is not cancellable, a contradiction. Therefore \( A \) is finitely decomposable. Then converse is true by Proposition 2.6. \( \square \)

**Remark 2.11.** In corollary 2.7 we have seen that a free \( S \)-act is cancellable if and only if it is finitely decomposable. This is easy by Theorem 2.10 because for each free \( S \)-act \( A \), we have \( A \cong \bigcup_{i \in I} S_i \) which \( S_i = S \) for any \( i \in I \), by [6, Theorem 1.5.13], therefore \( |I/\sim| = 1 \).

Let \( E(S) \) be the set of all idempotents of \( S \). By [6, Theorem 3.17.8], an \( S \)-act \( P \) is projective if and only if \( P = \bigcup_{i \in I} P_i \) where \( P_i \cong e_i S \) for idempotents \( e_i \in S, i \in I \). We define an equivalence relation on \( E(S) \), \( e \sim f \) if and only if
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eS \cong fS\text{ which }e, f \in E(S).\text{ Let } E(S)/\sim = \{[e]; e \in E(S)\} \text{ where } [e] = \{f \in E(S); fS \cong eS\}.

\textbf{Corollary 2.12.} Let \(S\) be a monoid in which \(|E(S)/\sim| < \infty\) and let \(P\) be a projective \(S\)-act. Then \(P\) is cancellable if and only if \(P\) is finitely decomposable.

\textit{Proof.} Since every projective \(S\)-act is of the form \(P \cong \bigcup_{e \in E(S)} eS\), the result is clear by Theorem 2.10. \(\square\)

Here, we introduce the concept of internal cancellation in \(\text{Act}-S\). As we have mentioned in the abstract, we shall show that this coincides with cancellation.

\textbf{Definition 2.13.} An \(S\)-act \(A\) satisfies internal cancellation if, for any subacts \(C, D, E,\) and \(F\) of \(A\), \(A = C \cup D = E \cup F\) and \(C \cong E\) implies that \(D \cong F\). If \(A\) satisfies internal cancellation we call \(A\) is internally cancellable.

There exist examples that internal cancellation in the category \(\text{Act}-S\) always does not satisfy. Let us to provide an example.

\textbf{Example 2.14.} Let \(S\) be a monoid. As \(S\) with its operation is an \(S\)-act then all \(S\)-acts

\[ C = S \times \{1\}, D = \bigcup_{i \in \mathbb{N}} (S \times \{i + 1\}), E = S \times \{1\} \cup S \times \{2\}, F = \bigcup_{i \in \mathbb{N}} (S \times \{i + 2\}) \]

with actions induced by the action of \(S\) are \(S\)-acts. Furthermore, we have \(C \cup D = E \cup F\) and \(D \cong F\), but \(C \not\cong E\). It is means that, the \(S\)-act \(A = \bigcup_{i \in \mathbb{N}} (S \times \{i\})\) is not internally cancellable.

\textbf{Theorem 2.15.} Let \(A\) be an \(S\)-act. Then \(A\) is cancellable if and only if \(A\) is internally cancellable.

\textit{Proof. Necessity.} Assume \(A = C \cup D = E \cup F\) in which \(C, D, E, F\) are subacts of \(A\) and \(C \cong E\). Then \(C \cup D \cong C \cup F\). By Theorem 2.3 \(C\) is cancellable and then \(D \cong F\). Therefore \(A\) is internally cancellable.

\textit{Sufficiency.} Suppose \(A\) is an internally cancellable \(S\)-act and \(A \cup B \cong A \cup C\) in which \(B, C \in \text{Act}-S\). We may assume without loss of generality that \(A \cap B = A \cap C = \emptyset\). Let \(f : A \cup B \rightarrow A \cup C\) be an \(S\)-isomorphism. Since \(f(A) \cup f(B) = A \cup C\), intersect this equation once with \(f(A)\) and once more with \(A\) we get

\[ f(A) = (A \cap f(A)) \cup (f(A) \cap C) \]
and

\[(2.22) \quad A = (f(A) \cap A) \cup (f(B) \cap A).\]

Combine together equations (2.21) and (2.22) gives us

\[(2.23) \quad A = f^{-1}(A \cap f(A)) \cup f^{-1}(f(A) \cap C) = (f(A) \cap A) \cup (f(B) \cap A),\]

Since

\[(2.24) \quad f^{-1}(f(A) \cap A) \cong f(A) \cap A\]

and \(A\) is internally cancellable we deduce that

\[(2.25) \quad f^{-1}(f(A) \cap C) \cong f(B) \cap A.\]

i.e.,

\[(2.26) \quad A \cap f^{-1}(C) \cong f(B) \cap A.\]

Since \(f^{-1}\) is an isomorphism we get

\[(2.27) \quad f^{-1}(f(B) \cap C) \cong f(B) \cap C\]

i.e.,

\[(2.28) \quad B \cap f^{-1}(C) \cong f(B) \cap C.\]

In a similar way, as we did in (2.21) we have

\[(2.29) \quad f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)\]

and

\[(2.30) \quad f(B) = (f(B) \cap A) \cup (f(B) \cap C).\]

As \(f\) and \(f^{-1}\) are isomorphism we have \(B \cong f(B), C \cong f^{-1}(C)\). Now by (2.26), (2.28), (2.29) and (2.30) we deduce \(f^{-1}(C) \cong f(B)\) and so \(B \cong C\). Therefore \(A\) is cancellable. \(\square\)

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