The Ising and BEG models critical behavior is analyzed in 2D and 3D by means of a renormalization group scheme on small clusters made of a few lattice cells. Different kinds of cells are proposed for both ordered and disordered model cases. In particular, cells preserving a possible antiferromagnetic ordering under decimation allow for the determination of the Néel critical point and its scaling indices. These also provide more reliable estimates of the Curie fixed point than those obtained using cells preserving only the ferromagnetic ordering. In all studied dimensions, the present procedure does not yield the strong disorder critical point corresponding to the transition to the spin-glass phase. This limitation is thoroughly analyzed and motivated.

PACS numbers:

I. INTRODUCTION

In this work we shall discuss the real space Renormalization Group (RG) study of critical behavior of spin systems interacting via different types of magnetic interaction. We will consider Ising models and the Blume-Emery-Griffiths (BEG) model, where spins can take either the value ±1, magnetic site, or 0, hole.

The real space RG analysis involves the study of small clusters, or “block”, of the Bravais lattice in finite dimension, such as those proposed for Ising spin models in the seventies, e.g., in Refs. 1, 2. While the real space RG is quite powerful for studying the Paramagnetic (PM) – Ferromagnetic (FM) transition, it often fails to detect more complex phases, such as the antiferromagnetic (AFM) phase in system with antiferromagnetic interactions or the spin-glass (SG) phase in disordered systems. Starting from the cluster approximation for ferromagnets used by Berker and Wortis 2 we will consider possible generalizations to more structured block RG transformations to capture the Néel point of antiferromagnetic systems, and analyze the robustness of both the FM Curie and AFM Néel critical points to a small amount of disorder. We shall also investigate the possibility of the onset of a SG critical point in the case of strong quenched disorder. The construction of the block RG transformation is ruled by two opposite requirements, minimal cluster structure to capture the properties of the phases and numerical feasibility.

We will consider the critical behavior in both 2D and 3D dimensions, and compare our results to the outcome of numerical simulations and, for small disorder, to the predictions of duality analysis of Nishimori 3.

The paper is organized as follows. Section II is devoted to 2D Ising models. Here we also recall the real space block RG transformation procedure, and its extension to the case of (quenched) random interactions. We also introduce the generalization of the block RG transformation used to tackle with the antiferromagnetic and disorder interactions. In Sec. III we extend the analysis to 3D Ising models, and in Sec. IV to the Blume-Emery-Griffiths model.

Finally, in Sec. V we summarize our findings and comment about the inability to locate an SG critical point for strong disorder, and how it might be overcome.

II. CLUSTER RENORMALIZATION GROUP FOR THE 2D ISING MODEL

The real space block RG transformation dates back to the 70’s, and consists of the following steps:

1. group spins on the real space Bravais lattice into blocks, called cells, with a given geometry;
2. replace each block by a new spin variable, cell-spin, whose value is dictated by the values of all the spins inside cell through a projection matrix;
3. decimate the spins by summing in the partition sum over all spins inside the cells for fixed value of the cell-spins;
4. rescale the lattice-space to its original value and compute the new, renormalized, values of interactions among the cell-spins leaving the partition function invariant.

When points 1 to 4 are iterated they yield the RG flow $\mathcal{K}^R = \mathcal{R}(\mathcal{K})$ in the interaction parameters space $\mathcal{K}$. Starting from the initial physical values the renormalized parameters flow towards a fixed point $\mathcal{K}^* = \mathcal{R}(\mathcal{K}^*)$ that characterizes the phase of the system. The stability matrix of the fixed point gives the critical exponents.
In this Section we apply this procedure to the 2D Ising model with quenched disordered bimodal ferromagnetic/antiferromagnetic interactions. The Hamiltonian, expressed in a form suitable for the RG study, is

\[-\beta H(s) = \sum_{ij} \left[ J_{ij} s_i s_j + h + s_i + s_j 2 + h_{ij} s_i - s_j 2 \right], \tag{1} \]

where \(ij\) denotes the ordered sum over near-neighbor sites on the 2D Bravais lattice. As usual in RG studies, we use reduced parameters where the temperature is absorbed into the interactions parameters.

The initial (physical) probability distribution of the couplings is

\[P(K_{ij}) = P(J_{ij}) P(h_{ij}) P(h^\dagger_{ij}) = [(1 - p) \delta(J_{ij} + J) + p \delta(J_{ij} - J)] \times \delta(h_{ij} - h) \delta(h^\dagger_{ij}), \tag{2} \]

where \(K = \{J, h, h^\dagger\} \).

\[ e^{-\beta H(\sigma)} = \sum_s \prod_c M(\sigma_c, s_{i \in c}) e^{-\beta H(s)}. \tag{3} \]

for the cell-spin.

The procedure must conserve the partition function. Therefore the final step is the replacement \(\sigma \rightarrow s\) and a rescaling that changes \(\mathcal{H}'\) back to the original form of the Hamiltonian in the new spin \(s\):

\[-\beta H_R(s) = \alpha \left( J_R s_a s_b + h_R \frac{s_a + s_b}{2} + h^\dagger_R \frac{s_a - s_b}{2} \right), \tag{4} \]

with the renormalized interactions:

\[ J_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--}}{x_{+-} x_{-+}} \right), \]

\[ h_R = \frac{1}{2\alpha} \log \left( \frac{x_{++}}{x_{-+}} \right), \tag{5} \]

\[ h^\dagger_R = \frac{1}{2\alpha} \log \left( \frac{x_{+-}}{x_{-+}} \right), \]

where

\[ x_{\sigma_a \sigma_b} = \sum_s M(\sigma_a, s_{i \in a}) M(\sigma_b, s_{i \in b}) e^{-\beta H(s)} \tag{6} \]

are the so-called edge Boltzmann factors. The coefficient \(\alpha\) is the number of near-neighbor sites on the lattice, 4.
for the 2D case. Note that if \( h = 0 \) then \( \mathcal{H}(s) = \mathcal{H}(-s) \) then \( x_+ = x_- \) and \( x_+ = x_- \) so that \( h_R = h_R^* = 0 \).

Equations (3) define the RG flow \( \mathbf{K}^R = \mathcal{R}(\mathbf{K}) \).

The critical exponents are obtained from the eigenvalues of the stability matrix \( \partial \mathbf{K}^R / \partial \mathbf{K} \) evaluated at the fixed point \( \mathbf{K}^* \), which can be written in terms of

\[
\frac{\partial x_{s_i a_i}}{\partial J} = \sum_{(ij)} M(\sigma_a, s_i, \epsilon_{a i}) M(\sigma_b, s_i, \epsilon_{b i}) s_i s_j e^{-\beta \mathcal{H}(s)},
\]

\[
\frac{\partial x_{s_i a_i}}{\partial h} = \sum_{(ij)} M(\sigma_a, s_i, \epsilon_{a i}) M(\sigma_b, s_i, \epsilon_{b i}) \frac{s_i + s_j}{2} e^{-\beta \mathcal{H}(s)},
\]

\[
\frac{\partial x_{s_i a_i}}{\partial h^*} = \sum_{(ij)} M(\sigma_a, s_i, \epsilon_{a i}) M(\sigma_b, s_i, \epsilon_{b i}) \frac{s_i - s_j}{2} e^{-\beta \mathcal{H}(s)}.
\]

The nontrivial fixed point(s) are for \( h = h^* = 0 \). In this case the stability matrix is diagonal and the relevant scaling exponent are \( \nu = \log_b(\partial J / \partial J^R) \) and \( \eta = \log_b(\partial h / \partial h^R) \), where \( b \) is the lattice scaling factor under decimation, equal to 2 for the SQ2 cluster of Fig. 4. The critical exponent are then

\[
\nu = \frac{1}{y_T}, \quad \eta = d + 2 - 2 y_h. \quad (7)
\]

The others follow from the scaling law.

The numerical implementation of this procedure gives for the ordered ferromagnetic 2D Ising model \((p = 1)\) the critical temperature \( T_c = J^{-1} = 1.896 \) for the PM/FM transition, and scaling exponents \( y_T = 0.727 \) e \( y_h = 1.942 \)

[see also Ref. 4]. The value \( y_h \) is less than the dimension of the space, implying that the transition is of the second order. The values of the critical exponent are shown in the first row of Table II.

By comparing with the exact Onsager solution 3, the critical temperature deviates of about 20% from the exact result \( T_c^{\text{Ons}} = 2 / (1 + \sqrt{2}) = 2.2692... \) and the values of the critical exponent all suffer major deviations. We postpone the discussion on how this estimates could be improved.

### B. Disordered 2D Ising Model

In presence of quenched disorder the RG flow cannot be restricted to a single interactions value \( \mathbf{K} \), and necessarily involves the whole coupling probability distribution \( P(\mathbf{K}) \). The RG equation then becomes

\[
P_R(\mathbf{K}^R) = \int d\mathbf{K} P(\mathbf{K}) \delta [\mathbf{K}^R - \mathcal{R}(\mathbf{K})]. \quad (8)
\]

The block RG transformation must then be repeated starting from interaction parameters configurations \( \mathbf{K} \) extracted with probability \( P(\mathbf{K}) \). The outcomes \( \mathbf{K}^R \) are then used to construct the renormalized probability distribution \( P_R(\mathbf{K}) \), which in turn is used as entry for the next iteration. And so on.

In a numerical study the number of possible interaction parameters configurations that one can consider is finite. The flow of the renormalized probability distribution \( P_R(\mathbf{K}) \) can then be followed by using a method initially suggested by Thouless, cfr. Ref. 3. One first sets up a starting pool of \( M \gg 1 \) different randomly chosen real numbers produced according to the initial probability of the couplings, Eq. (6) for the bimodal Ising Model. Then a coupling configuration \( \mathbf{K} \) is constructed by randomly picking numbers from the pool and assigning them to the couplings. A renormalized \( \mathbf{K}^R \) is thus, evaluated. The procedure is repeated \( M \) times obtaining a new pool that represents the renormalized probability distribution, from which one can compute the moments and estimate \( P_R(\mathbf{K}) \) from the frequency histogram.

In Fig. 5 we show the flow of the probability distribution \( P(J_{i j}) \) of a single pool in the disordered 2D Ising model \( \alpha \) with \( h = 0 \) and \( p = 0.9 \) generated by the block RG transformation on the SQ2 cluster. In the upper figure, \( T = J^{-1} = 1.4 \), the probability moves towards smaller values of \( J \) and shrinks. This signals a PM phase, with the PM fixed point probability distribution function of mean \( \mu_J \to 0 \) and variance \( \sigma_{J}^2 \to 0 \). In the lower figure, \( T = J^{-1} = 1.2 \), the probability narrows while shifting towards larger value of \( \mu_J \). This denotes a FM phase, with the FM fixed point probability specified by \( \mu_J \to \infty \) and \( \sigma_{J} \to 0 \).

To reduce the possible bias introduced by the choice of the initial pool, Nobre et al. 3 have proposed to repeat the block RG transformations using a set of \( N_s \) samples with different initial pools of size \( M \). When close to a critical point flows originating from different pools may flow towards different fixed point distributions. The size of the region where the phase is not uniquely identified gives the uncertainty on the critical value obtained with a the pools of size \( M \).

In our numerical study of the disordered 2D Ising model we have used \( N_s = 20 \) pools of size \( M = 10^6 \) each, and we have assumed a phase uniquely defined if

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( \nu \) | \( \eta \) |
|---|---|---|---|---|---|
| -0.7523 | 0.08038 | 2.592 | 33.24 | 1.376 | 0.1168 |
| -0.1233 | 0.1383 | 1.847 | 14.35 | 1.062 | 0.2606 |
| -1.426 | 0.05884 | 3.309 | 57.23 | 1.713 | 0.06870 |
| -0.6545 | 0.2141 | 2.226 | 11.40 | 1.327 | 0.3226 |
| -0.1524 | 0.1915 | 1.769 | 10.24 | 1.076 | 0.3559 |
| -0.4458 | 0.0779 | 1.490 | 4.118 | 1.222 | 0.7815 |

**TABLE I: Critical exponents of the ferromagnetic 2D Ising Model obtained with the different cluster discussed in this work compared with the known exact results.**
at least 80% of the RG flows flow towards the same fixed distribution. With this choice the uncertainty is generally less than 0.1% and the systematic error considerably decreased.

The \((p, T)\) phase diagram of the disordered 2D Ising model obtained using the SQ\(_2\) cluster is shown in Fig. 3 (black squares). As the probability \(p\) of the ferromagnetic bonds is lowered the critical temperature decreases until, for low enough \(p\), the FM phase disappears. In the figure it is also shown the Nishimori line \[8\]:

\[
\frac{1}{T} = \frac{1}{2} \log \frac{p}{1-p}
\]

that follows from duality. This line crosses the PM/FM critical line at the “multicritical” point \(p_{mc} = 0.8667\), \(T_{mc} = 1.070\), below which, \(p < p_{mc}\), duality imposes no FM ordering. Inspection of the Figure shows that not only the method fails to predict the correct critical temperature \(T^\text{Ons}_c\) of the pure ferromagnetic model, but also the requirements following from duality.

One can try to improve the numerical estimates tuning the parameter \(t\) in the projection matrix to fix some known points in the \((p, T)\) diagram. We consider two possible choices: fixing the critical temperature of the pure system to the exact value or the crossing point with the Nishimori line to the multicritical point. The requirement \(T_c = T^\text{Ons}_c\) leads to \(t = -0.06453\), while the requirement \(T_c(p_{mc}) = T_{mc}\) to \(t = 0.0304\). Note the “unphysical” negative value of \(t\), also used by Berker and Wortis \[2\], which implies that under the block transformation the contribution of some spin configuration of the cell to the partition sum can be negative. The transition lines obtained with these choices for \(t\) are shown in Fig. 3. In both cases, and besides the unphysical values of \(t\), the slope of the transition line increase as \(p\) decreases, but still no re-entrance or vertical line is recovered. In either cases the only critical point remains the FM fixed point at \(p = 1\) with scaling exponents \(\alpha = 0.9419\) and \(\gamma_h = 1.870\) for \(t = -0.06453\), and \(\gamma_T = 0.5837\) and \(\gamma_h = 1.965\) for \(t = 0.0304\). The numerical values of the critical exponents are shown in the second and third row of Table \[3\] respectively. Note that in all cases \(\alpha < 0\). According to the Harris criterion, \[3\] this indicates that the FM fixed point is stable against the introduction of a small amount of quenched disorder.

Summarizing the results: the block RG transformation based on the SQ\(_2\) cluster finds no true multicritical point nor a “strong disorder” fixed point, and hence no change in the universality class of the critical behavior.
C. Antiferromagnetic order: need for “SSQ₂”

Another important issue of the block RG transformation discussed so far is the absence of an AFM phase. Below some critical value of \( p \) and down to \( p = 0 \), only the PM phase is found. This failure might also strongly bias the quest for a spin-glass phase in dimension higher than two.

By analyzing the block RG transformation used so far, we see that it assigns the same weight to symmetric configurations (e.g., ++−−) of the spins of the cell, regardless of their ordering. As a consequence, it is not able to identify an antiferromagnetic ordering, and a staggered magnetization cannot be properly defined.

We thus need a cluster construction that distinguishes the symmetry breaking ordering associated with the AFM phase. By referring to labeling of Fig. 1 we then assign the spins \( \{s_1, s_3, s_5, s_7\} \) to the cell \( a \) and the spins \( \{s_2, s_4, s_6, s_8\} \) to the cell \( b \), in a zig-zag order shaping a staggered topology (“SSQ₂”, in the following). The projection matrix of the cell remains unchanged. The phase diagram obtained through this block RG transformation is shown in Fig. 4. The improvement with respect to the SSQ₂ cluster is evident. The \( p = 0 \) antiferromagnetic critical point is now found, as well as a PM/AFM transition line for \( p > 0 \). Since in this model, for \( h = 0 \) the symmetry \( (p, J) \leftrightarrow (1 - p, -J) \) holds and the staggered cluster preserves AFM ordering, the PM/AFM line is symmetric to the PM/FM line with respect to \( p = 1/2 \). The behavior of the critical line below \( \rho_{\text{unc}} \), however, still violates the requirement imposed by the duality analysis.

The SSQ₂ cluster improves the estimate of the pure critical fixed point \( (p = 1) \). The critical temperature turns out \( T_c(p = 1) = 2.352 \) and deviates of about 3.5% from Onsager result. The scaling exponents are \( \nu_T = 0.7534 \) and \( \nu_h = 1.839 \), and the associated critical exponents are reported in Table 1. Though they display differences of 20% to 40% from the exact values, their estimates are sensitively better than those obtained with the classic SSQ₂ cluster.

As the AFM transition is concerned, the behavior is specular to that of the FM transition. The points along the AFM critical line are attracted by a unique second order AFM Néel fixed point at \( p = 0 \) at the same critical temperature \( T_c(p = 0) = 2.352 \) with scaling exponents \( \nu_T = 0.7534 \), as found for the FM fixed point, and \( \nu_h = 0.01565 \). The symmetry of the RG equations implies that the PM/AFM and PM/FM fixed points have the same \( \nu_T \). The values of \( \nu_h \) are, however, quite different, the AFM one being almost zero. The reason is that the magnetization is not the correct order parameter for the AFM transition, as it remains zero on both sides of the transition. If, rather, the staggered magnetization is considered, and hence a staggered field \( h_s \) is introduced in the Hamiltonian, then the relevant scaling exponent turns out to be \( \nu_{h_s} = \log_2 \partial h_s h_s^R = 1.797 \gg \nu_h \).

D. 4-square cells cluster (“SQ₄”)

The SSQ₂ cluster leads to an AFM fixed point, and improves both the analysis of the AFM and FM phases. However, it does not allow for possible frustrated configurations in the renormalized cells. In an attempt to circumvent this problem we extend the cluster from two to four cells.

As the number of cells increases so does the possible grouping of spins. We found that the best cell geometry, in terms of similarity with the exact results, is the one shown in Fig. 5 with the following decimation rule:

\[
\begin{align*}
\{s_0, s_4, s_8, s_{12}\} & \rightarrow s_a \\
\{s_1, s_5, s_9, s_{13}\} & \rightarrow s_b \\
\{s_2, s_6, s_{10}, s_{14}\} & \rightarrow s_c \\
\{s_3, s_7, s_{11}, s_{15}\} & \rightarrow s_d
\end{align*}
\]

The block RG transformation now generates, besides near-neighbor interactions, also next near-neighbor and “plaquette” interactions. To avoid truncation we have
then to start from the more general Hamiltonian:

\[
-\beta H(s) = \frac{1}{2} \sum_{i} \sum_{k=1}^{4} J_{i,i+\mu_{k}} s_{i+\mu_{k}} + \frac{1}{2} \sum_{i} \sum_{k=1}^{4} K_{i,i+\eta_{k}} s_{i+\eta_{k}} + \sum_{i} \sum_{k=1}^{4} D_{i} \xi_{k},
\]

where \( i = (i_x, i_y) \) denotes a site on the 2D lattice, \( \mu \) the relative position of the near-neighbor sites, \( \eta \) the relative position of the next near-neighbor sites and \( \xi \) the relative position of the plaquette sites:

\[
\mu_{1} = (0, 1), \quad \mu_{2} = (1, 0), \quad \mu_{3} = (0, -1), \quad \mu_{4} = (-1, 0)
\]

\[
\eta_{1} = (1, 1), \quad \eta_{2} = (1, -1), \quad \eta_{3} = (-1, -1), \quad \eta_{4} = (-1, 1)
\]

\[
\xi_{1} = (0, 0), \quad \xi_{2} = (0, 1), \quad \xi_{3} = (1, 1), \quad \xi_{4} = (1, 0).
\]

The initial distributions of the couplings is

\[
P(K_{ij}) = [(1-p)\delta(J_{ij} + J) + p\delta(J_{ij} - J)] \delta(K_{ij}) \delta(D_{i}).
\]

In the SQ\(_{4}\) cluster, including the periodic boundary conditions, the ratio between the different type of interactions is \( J : K : D = 2 : 2 : 1 \), cfr. Fig. 5. After the block RG transformation, however, we have 4 renormalized \( J \), but only 2 renormalized \( K \) and one \( D \); so it is not possible to maintain the correct ratio and correlation among the renormalized couplings. To overcome this problem we arrange the renormalized interactions into two configurations of 2 \( J \), 2 \( K \) and 1 \( D \) as shown in Fig. 6. The new cluster is built by extracting 16 of these configurations and assigning them as shown in the Figure.

To be explicit the RG procedure leads to the renormalized interactions

\[
J_{\pm}^{R} = \frac{1}{16} \left( \log \frac{x_{+++} x_{---} \pm \log x_{++} x_{---}}{x_{++} x_{---} \pm \log x_{++} x_{---}} \right)
\]

\[
J_{\pm}^{R} = \frac{1}{16} \left( \log \frac{x_{+++} x_{---} \pm \log x_{++} x_{---}}{x_{++} x_{---} \pm \log x_{++} x_{---}} \right)
\]

\[
K_{\pm}^{R} = \frac{1}{32} \left( \log \frac{x_{+++} x_{++} \pm \log x_{++} x_{++}}{x_{++} x_{++} \pm \log x_{++} x_{++}} \right)
\]

\[
P_{\pm}^{R} = \frac{1}{32} \left( \log \frac{x_{+++} x_{---} x_{++} x_{+++} \pm \log x_{++} x_{+++} x_{+++}}{x_{++} x_{+++} x_{+++} x_{+++} \pm \log x_{++} x_{+++} x_{+++}} \right)
\]

with the edge Boltzmann factor

\[
x_{\sigma_{a} \sigma_{b} \sigma_{c} \sigma_{d}} = \sum_{s} M_{a} M_{b} M_{c} M_{d} e^{-\beta H(s)},
\]

where \( M_{x} \equiv M(\sigma_{x}, s_{i=\pm}) \) are the cell projection matrices. The renormalized interactions are divided into the two configurations \{ \( J_{\pm}^{R}, J_{\pm}^{R}, K_{\pm}^{R}, D_{\pm}^{R} \) \} and \{ \( J_{\pm}^{R}, J_{\pm}^{R}, K_{\pm}^{R}, D_{\pm}^{R} \) \}.

The phase diagram obtained with \( N_{a} = 10 \) pools of size \( M = 10^{6} \) is shown in Fig. 7. All the points on the critical line are attracted by the pure fixed point at \( p = 1 \) and critical temperature of \( T_{c}(p = 1) = 2.391 \), about 5% off the exact 2D result.

To evaluate the critical exponents we have to include in the Hamiltonian an external magnetic field, and hence consider also the three spins interactions \( \sum_{k=1}^{4} s_{i+\mu_{k}} s_{i+\mu_{k}} \) generated by the RG. This gives a total of five parameters. At the pure fixed point only two are relevant with scaling exponent \( y_{F} = 0.9292 \) and \( y_{B} = 1.822 \). The values of the associated critical exponents are reported in the fifth line of Table 1.

The re-entrance of the critical line below the multicritical point \( T_{c}(p) < T_{ac} \) is still absent. However the line appears steeper than those obtained with the previous block RG transformations, approaching the expected behavior of the model. We stress as this effect is only present if the ratio and correlations among the renormalized interactions is maintained using the block RG transformation shown in Fig. 6. If this is removed, the transition line becomes straight, as found with the SQ\(_{2}\) cluster.

Despite this qualitative improvement, the intersection between the transition line and the Nishimori line occurs sensitively above the exact multicritical point, cfr. Table 1 and, as in the previous cases, it does not correspond to a real multicritical point.

The RG analysis indeed does not show critical fixed points besides the pure critical point \( p = 1 \). The so called strong disorder fixed point \( 10 \) is missing and the crossing is not associated with flows towards the FM and strong disorder fixed points.
TABLE II: Estimate of the FM critical point \((p = 1)\) temperature \((T_{\text{Ons}})\) and the intersection of coordinate point between the PM/FM transition line with the Nishimori line \((p_{\text{mc}}, T_{\text{mc}})\) for the disordered bimodal 2D Ising model obtained with the different block RG transformations discussed in this work, compared with the exact locations known for the 2D lattice.

| Cluster | \(T_{\text{Ons}}\) | \(p_{\text{mc}}\) | \(T_{\text{mc}}\) |
|---------|------------------|-----------------|------------------|
| SQ\(_2\) | 1.896            | 0.867           | 1.070            |
| tSQ\(_2\) Ons. | 2.269            | 0.834           | 1.242            |
| tSQ\(_2\) Nish. | 1.714            | 0.88997         | 0.9567           |
| SSQ\(_2\) | 2.352            | 0.827           | 1.277            |
| SQ\(_4\) | 2.391            | 0.835           | 1.231            |
| SSQ\(_4\) | 2.802            | 0.809           | 1.388            |
| 2D \([5, 11, 12]\) | 2.269...         | 0.88997         | 0.95673           |

The multi critical point is missing since we do not find any strong disorder fixed point. No SG phase can be tested because we are in dimension \(d < 2.5\). Therefore in the next Section we move to the 3D case.

E. 4-staggered cells cluster (“SSQ\(_4\)”)

As found for the SQ\(_2\) cluster, the SQ\(_4\) cluster does not show an AFM fixed point and the PM/AFM transition is missing. To recover it we then consider the generalization to a staggered grouping of spins for the four cells cluster (“SSQ\(_4\)”). By referring to the numbering of Fig. 3:

\[
\begin{align*}
\{s_0, s_1, s_2, s_3\} & \rightarrow s_a, \\
\{s_4, s_5, s_6, s_7\} & \rightarrow s_b, \\
\{s_8, s_9, s_{10}, s_{11}\} & \rightarrow s_c, \\
\{s_{12}, s_{13}, s_{14}, s_{15}\} & \rightarrow s_d.
\end{align*}
\]

The phase diagram obtained with this block RG transformation is shown in Fig. 4. Though we can now identify the PM/AFM transition, we observe a worsening of the estimates of the critical points: \(T_c = 2.802\) for both the Curie and the Néel points. The points along the PM/FM transition line flow towards the FM fixed point at \(p = 1\), while those on the PM/AFM transition line are attracted by the AFM fixed point at \(p = 0\). Therefore also in this case we do not find a strong disorder fixed point. We stress that again maintaining the ratio and correlations among the renormalized couplings is critical for the bending of the PM/FM transition line.

The two relevant scaling exponents of the stability matrices at the FM critical fixed point are \(\gamma_T = 0.8177\) and \(y_h = 1.609\); see Table I for the corresponding critical exponents.

For the AFM fixed point we have \(\gamma_T = 0.8177\), the same of the FM fixed point. As discussed previously, for the AFM transition the relevant order parameter is the staggered magnetization, and the scaling exponent of the staggered field is \(y_h = 1.569\).

To summarize, for the 2D Ising model with bimodal disorder, Eq. (2), we have evidence for both PM/FM and PM/AFM transition for large enough \(|p|\). Quantitatively, the best estimates for the Curie and Néel critical points are obtained in the SSQ\(_2\) cluster scheme (cfr. Table I).

III. CLUSTER RENORMALIZATION GROUP FOR THE 3D ISING MODEL

In this Section we extend the method based on the SQ\(_2\) cluster to the three dimensional case, by using the cluster of two cubic cells with periodic boundary conditions shown in Fig. 7 (referred as “CB\(_2\)” for the study of the 3D Ising model.

The associated projection matrix is

\[
M(1, s_{i\in c}) = s_{i\in c}
\]

\[
\begin{array}{cccccccc}
1 & + & + & + & + & + & + & + \\
1 - t_6 & + & + & + & + & + & + & + \\
1 - t_4 & + & + & + & + & + & + & + \\
1 - t_2 & + & + & + & + & + & + & + \\
1/2 & + & + & + & + & + & + & + \\
t_6 & + & + & + & + & + & + & + \\
t_4 & + & + & + & + & + & + & + \\
t_2 & + & + & + & + & + & + & + \\
0 & + & + & + & + & + & + & + \\
\end{array}
\]

and \(M(-1, -s_{i\in c}) = M(1, s_{i\in c})\), which, for \(t_i = 0\), reduces to the majority rule.

The initial probability distribution of the interactions is given in Eq. (2), and we used \(N_c = 10\) pools of size \(M = 10^6\). The phase diagram for CB\(_2\) cluster is shown in Fig. 8. Once again, only the pure fixed point at \(p = 1\) controlling the PM/FM transition is found.

For the choice \(t_i = 0\) the critical temperature is \(T_c = 4.0177\), which compared with the estimation from numerical simulations \(T_c = 4.5115\) [13], has a difference of about 12%. The scaling exponents of the fixed point are \(\gamma_T = 1.253\) and \(y_h = 2.684\), the value of the critical exponents are reported in Table III. The PM/FM transition line crosses the Nishimori line at the point \(p_{\text{mc}} = 0.76793\) and \(T_{\text{mc}} = 1.6721\), compatible with the multicritical point obtained for the 3D Ising model on a the cubic lattice [15]: \(p_{\text{mc}} = 0.7673(4)\), \(T_{\text{mc}} = 1.676(3)\).
to have a staggered topology, and hence preserving a possible antiferromagnetic ordering in the decimation process. The resulting phase diagram is shown in Fig. VII line “SCB″. Now, besides the FM fixed point at p = 1, a symmetric AFM fixed point at p = 0 appears. The critical temperature is \( T_c = 4.5537 \), closer to the FM critical temperature found from from numerical simulations (cfr. Table II). The scaling exponents for the FM fixed point are \( y_T = 1.039 \) and \( y_h = 2.487 \), see Table III for comparison of the corresponding critical exponents. The exponent \( \alpha \) is negative, even more than the previous cases, signaling according to the Harris criterion \([9]\) the absence of other fixed points for \( 1/2 \leq p < 1 \).

For the AFM critical fixed point we get \( y_T = 1.039 \) and \( y_h = 0.6075 \), while that of the staggered magnetic field is \( y_h = 1.487 \).

The re-entrance of the transition line below the Nishimori line is missing, confirming also for the 3D case the limitations of these block RG transformations based on the cluster renormalization scheme.

In conclusion, the phase diagram obtained for the 2D and 3D Ising models are qualitatively similar, with the notable absence of any SG phase in the 3D case.

The extension to larger cells, similar to the one discussed in Sec. III for the 2D case, becomes readily unfeasible for 3D lattices. For example with 8 cubic cells one should sum over the configurations of \( 4^3 \) spins, more than \( 10^{14} \) times the configurations of the two cells cluster.

However, based on the results of the 2D case, we do not expect that such an extension would solve the problem of the SG phase. To catch the SG phase one has to look for different block RG transformation strategies.

### 2-staggered cubic cells cluster (“SCB″)

As done for the 2D case, to locate the AFM fixed point we modify the two cells cluster of Fig. VII to have a staggered topology, and hence preserving a possible antiferromagnetic ordering in the decimation process. The resulting phase diagram is shown in Fig. VII line “SCB″. Now, besides the FM fixed point at p = 1, a symmetric AFM fixed point at p = 0 appears. The critical temperature is \( T_c = 4.5537 \), closer to the FM critical temperature found from numerical simulations (cfr. Table II). The scaling exponents for the FM fixed point are \( y_T = 1.039 \) and \( y_h = 2.487 \), see Table III for comparison of the corresponding critical exponents. The exponent \( \alpha \) is negative, even more than the previous cases, signaling according to the Harris criterion \([9]\) the absence of other fixed points for \( 1/2 \leq p < 1 \).

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The re-entrance of the transition line below the Nishimori line is missing, confirming also for the 3D case the limitations of these block RG transformations based on the cluster renormalization scheme.

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However, based on the results of the 2D case, we do not expect that such an extension would solve the problem of the SG phase. To catch the SG phase one has to look for different block RG transformation strategies.

Despite this agreement the transition line, however, does not show any re-entrance.

When the parameters \( t_i \) are fixed by the condition \( T_c(p = 1) = 4.5115 \) \([3]\), leading to \( t_2 = 0.011 \), \( t_3 = -0.010 \) and \( t_6 = -0.050 \), the transition line shows a sharp increase of the slope for \( p < p_{mc} \), yet no re-entrance nor vertical part are observed, see tCB2 line in Fig. VIII. There is still only the critical point at \( p = 1 \) and the crossing with the Nishimori line does not correspond to a real multicritical point. The scaling exponent are \( y_T = 1.303 \) and \( y_h = 2.425 \), and one observes a slight improvement of the values of the critical exponents, see Table III.

The condition \( T_c(p_{mc}) = T_{mc} \) is compatible with \( t_i = 0 \) and does not give new results.

The 3D Ising model is known to present a multicritical point where PM, FM and SG phase meet \([8]\). Contrarily to numerical simulation predictions, where \( \alpha = 2 - d \nu > 0 \) \([14]\), in both cases we find a negative \( \alpha \), indicating that in the RG analysis the FM p = 1 fixed point is stable against the introduction of quenched disorder. Indeed, as noted above, the RG based on the CB2 cluster fails to locate any fixed point different from the PM/FM p = 1 critical fixed point.

**TABLE III: FM critical exponents of the 3D Ising model obtained with the block RG transformation using the two cells clusters CB2 and tCB2 (see text).** The tCB2 method the values of \( t_i \) are fixed by the requirement \( T_c(p = 1) = 4.5115 \).

|       | α     | β     | γ     | δ     | ν     | η     |
|-------|-------|-------|-------|-------|-------|-------|
| CB2   | -0.3952 | 0.2521 | 1.891 | 8.499 | 0.7984 | -0.3684 |
| tCB2  | -0.3015 | 0.4413 | 1.419 | 4.215 | 0.7672 | 0.1505  |
| SCB2  | -0.8887 | 0.4944 | 1.900 | 4.843 | 0.9629 | 0.02963 |
| sim. [14] | 0.1101 | 0.3265 | 1.2373 | 4.789 | 0.6301 | 0.03645 |

**FIG. 8:** Phase diagram in the \((p, T)\) plane of the \(\pm J\) 3D Ising model obtained with the block RG transformation using the two cells clusters CB2 and tCB2 (see text). The dashed line is the Nishimori line. The \( t \neq 0 \) curve is obtained by fixing the values \( t_i \) by the requirement \( T_c(p = 1) = 4.5115 \).

**FIG. 9:** Phase diagram in the \((p, T)\) plane of the \(\pm J\) 3D Ising model obtained using the 2-cubic cell cluster with \( t_i = 0 \), line CB2, and the 2-staggered cell cluster, line SCB2. The dashed line is the Nishimori line.
TABLE IV: Estimate of the FM critical fixed point and of the intersection between the PM/FM transition line with the Nishimori line for the disordered bimodal 3D Ising model obtained using the two cell clusters discussed in the text. In the last line we compare with the values for the 3D Bravais lattice.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
 & \(T_c(p = 1)\) & \(p_{mc}\) & \(T_{mc}\) \\
\hline
CB\(_2\) & 4.0177 & 0.7679 & 1.672 \\
tCB\(_2\) & 4.5115 & 0.7562 & 1.767 \\
SCB\(_2\) & 4.5537 & 0.7445 & 1.870 \\
3D \cite{15} & 4.5115 & 0.7673(4) & 1.676(3) \\
\hline
\end{tabular}
\end{center}

that account for SG local order. In particular, the spins could not be right variables to be decimated because local magnetization is not a meaningful parameter for the SG phase.

IV. CLUSTER RENORMALIZATION GROUP FOR THE BLUME-EMERY-GRiffithS MODEL

In this Section we apply the RG analysis to the BEG model, a spin-1 model introduced for the study of the superfluid transition in He\(^3\)-He\(^4\) mixtures \cite{16}. The BEG model was originally studied in the mean-field approximation in Refs. \cite{14,15}. Finite dimensional analysis has been carried out by different means, e.g., series extrapolation techniques \cite{19}, RG analysis \cite{2}, Monte Carlo simulations \cite{20}, effective-field theory \cite{21} or two-particle cluster approximation \cite{22}. Extensions to quenched disordered, both perturbing the ordered fixed point and in the regime of strong disorder, have been studied throughout the years by means of mean-field approximation \cite{23,24}, real space RG analysis on Migdal-Kadanoff hierarchical lattices \cite{20,21} and Monte Carlo numerical simulations \cite{26,31}.

The model is known to have, besides a second order phase transition, a first order phase transition associated with the phase separation between the PM and FM phases in the ordered case, and between the PM and SG phases in the quenched disordered case. This rich phase diagram allows for a structured analysis of the RG approximations. In particular we shall go through a detailed study of the ordered 2D BEG model, to compare with the results of Ref. \cite{2}, and we shall show the main properties of the quenched disordered 3D BEG model, which is an important test model for RG methods of quenched disordered systems.

A. Ordered 2D BEG model

Following Berker and Wortis \cite{2} in the ordered case we write the BEG Hamiltonian as

\[-\beta H(\{s\}) = J \sum_{\langle ij \rangle} s_i s_j + K \sum_{\langle ij \rangle} s_i^2 s_j^2 - \Delta \sum_i s_i^2 + h \sum_i s_i + L \sum_{\langle ij \rangle} (s_i s_j^2 + s_i^2 s_j), \tag{12}\]

where \(s_i = 0, \pm 1\). In the limit \(\Delta \to -\infty\) the spin reduces to \(s_i \to \pm 1\) and one recovers the Ising model discussed in the previous Section. The BEG model has two order parameters: the magnetization \(M \equiv \langle s_i \rangle\) and the quadrupole order parameter \(Q \equiv \langle s_i^2 \rangle\) giving the density of magnetic or occupied sites.

We shall consider the block RG transformation based on the same clusters used for the 2D Ising model, with the cell projection matrix:

\[
M(1, s_{1\ell}, c) M(-1, s_{1\ell}, c) M(0, s_{1\ell}, c) s_{1\ell}, c
\]

which for \(t = 0\) reduces to the majority rule. With the \(\text{SQ}_2\) cluster we obtain results coinciding with those of Ref. \cite{2}.

The block RG transformation leads to the renormalized Hamiltonian for the new spin variables

\[-\beta H_R(s_a, s_b) = a \left[ J_R s_a s_b + K_R s_a^2 s_b^2 + L_R (s_a^2 s_b + s_a s_b^2) + - \Delta_R (s_a^2 + s_b^2) + h_R (s_a + s_b) \right] \tag{13}\]
with

\[
J_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--}}{x_{+-}} \right),
\]

\[
K_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--} x_{+-}^2 x_{00}^4}{x_{+-}^4 x_{00}^4} \right),
\]

\[
\Delta_R = \frac{1}{2} \log \left( \frac{x_{00}^2}{x_{+0} x_{-0}} \right),
\]

\[
L_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--}^2}{x_{+-} x_{-+}} \right),
\]

\[
h_R = \frac{1}{2} \log \left( \frac{x_{+0}}{x_{-0}} \right),
\]

\[
J = 0.5275, 0.4407, 0.4259, 0.4407
\]

\[
\Delta = -\infty, -\infty, -\infty, -\infty
\]

\[
J = 0.5822, 0.5319, 0.4994, 0.5026
\]

\[
J = 1.756, 1.476, 1.495, 1.508
\]

TABLE VI: Location of the fixed points C*, G* and P* for the ordered 2D BEG model obtained with all the 2 cells cluster discussed in the text compared to the exact results for the 2D lattice.

The locations of all the fixed points in the RG flow for the 2D BEG model obtained with the SSQ2 cluster are reported in Table V. The evaluation of the stability matrix can be problematic if the RG flux flows towards a fixed point where one of the parameters is infinite, e.g., \(\Delta \to -\infty\). In cases like this it is more convenient to use a variable remaining finite at the fixed point, e.g. \(A = e^{\Delta}\).

The locations of all the fixed points in the RG flow generated by the block RG based on the SSQ2 cluster are reported in Table V.

The fixed points C*, G* and P* are of particular interest for testing the RG procedure because they are known exactly. Moreover, the FM Ising fixed point C* and the Griffiths fixed point G* are related to each other. The first occurs for \(\Delta \to -\infty\), while the second at \(J = 0\), and

\[
\Delta_R = \frac{1}{2} \log \left( \frac{x_{00}}{x_{+0} x_{-0}} \right),
\]

\[
L_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--}^2}{x_{+-} x_{-+}} \right),
\]

\[
h_R = \frac{1}{2} \log \left( \frac{x_{+0}}{x_{-0}} \right),
\]

where \(x_{+0, -0}\) are the edge factors [6] and \(\alpha = 4\) for the 2D lattice. The RG fixed point stability matrix is reported in the Appendix. The evaluation of the stability matrix can be problematic if the RG flux flows towards a fixed point where one of the parameters is infinite, e.g., \(\Delta \to -\infty\). In cases like this it is more convenient to use a variable remaining finite at the fixed point, e.g. \(A = e^{\Delta}\).

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\[
\Delta_R = \frac{1}{2} \log \left( \frac{x_{00}}{x_{+0} x_{-0}} \right),
\]

\[
L_R = \frac{1}{4\alpha} \log \left( \frac{x_{++} x_{--}^2}{x_{+-} x_{-+}} \right),
\]

\[
h_R = \frac{1}{2} \log \left( \frac{x_{+0}}{x_{-0}} \right),
\]
point $P^*$ from the Potts-axis can then be used as an indicator of the error made with the cluster approximation used to build the block RG transformation. The distance of $P^*$ from the Potts-axis, over its distance from the origin, turn out to be $6 \times 10^{-4}$ for the SQ$_2$ cluster, $10^{-2}$ for the tuned tSQ$_2$ cluster and $4 \times 10^{-4}$ for the SSQ$_2$ cluster. Note, specifically, that a strong violation is obtained with the tSQ$_2$ cluster with the “unphysical” negative $t$.

Finally, in Table VII we show the five scaling exponents for the fixed points $G^*$, $C^*$, $L^*$ and $P^*$. We stress as the critical exponents obtained with the SSQ$_2$ cluster approximation are more precise than those obtained with the original square cells cluster SQ$_2$.

Using, alternatively, the free $t$ trick, the critical exponents are more similar to the known exact ones respect to the staggered cells cluster case. Especially, the exactly known exponents for $C^*$, $G^*$ and $L^*$ are considerably better approached with the tSQ$_2$ cluster. This is not surprising since $t = -0.06453$ fixes the exact location for $C^*$ (and $G^*$), and we, then, expect that also the estimates of their scaling exponents improve. The known exponents of $P^*$ show, instead, only a slight improvement.

B. 3D BEG with quenched disorder

In this Section we extend the analysis to the quenched disordered BEG model in three dimensions. The quenched disordered 3D BEG model represents a relevant test for the cluster RG applied to disordered systems. Monte Carlo numerical simulations show a critical transition line between the PM phase and a SG phase, which, similar to what found in the mean-field study, consists of a second order transition terminating in a tricritical point from which a first order inverse transition starts. Furthermore, a re-entrance of the first order transition line is present for positive, finite values of the chemical potential of the holes, yielding the so-called inverse freezing phenomenon. The real space RG study of Ozcelik and Berker based on Migdal-Kadanoff cells does not reveal any first order phase transitions, nor any re-entrance. When the real space RG is extended to more structured hierarchical lattices the re-entrance can be recovered, but no tricritical point and first order transition are found.

The Hamiltonian of the disordered BEG model suitable for the RG study is

$$-\beta H = \sum_{(ij)} J_{ij} s_i s_j + \sum_{(ij)} K_{ij} s_i^2 s_j^2 +$$

$$- \sum_{(ij)} \Delta_{ij} \left( s_i^2 + s_j^2 \right) - \sum_{(ij)} \Delta_{ij}^\dagger \left( s_i^2 - s_j^2 \right)$$

(17)

where the couplings are quenched random variables with the probability distribution

$$P(K_{ij}) = \left[ (1-p) \delta(J_{ij} + J) + p \delta(J_{ij} - J) \right]$$

$$\times \delta (K_{ij} - K) \delta (\Delta_{ij} - \Delta) \delta \left( \Delta_{ij}^\dagger \right).$$

(18)

If an external field $h$ is added, besides the single site term, one has to include also the odd interaction term $s_i s_j^2$.

The model has been studied using the CB$_2$ cluster shown in Fig. 7 and its staggered version SCB$_2$ using in both cases $N = 10$ pools of size $M = 10^6$.

Similar to what seen in the previous Section, only the PM and the FM phases are found, while the SG phase remains undetected in the whole phase diagram. Two typical flows of the probability distribution towards the PM and FM fixed points are shown in Figs. 11 and 12. In the PM phase the average value of $J_{ij}$ goes to zero, while in the FM it moves towards $+\infty$. In both cases the distributions become narrower and narrower under the block RG transformation.

The PM/FM critical surface in the space $(T, \Delta/J, p)$ for the $K = 0$ case obtained with the SCB$_2$ cluster is shown in Fig. 12. All the points on the critical surface flow under RG towards one of the two ordered fixed points at $p = 1$ with mean value $\mu_2 \to \pm \infty$ and variance $\sigma_2^2 \to 0$. The analysis of the critical properties is then

|       | SQ$_2$ [2] | tSQ$_2$ | SSQ$_2$ | 2D |
|-------|-----------|---------|---------|----|
| $y_2$ | 0.7267    | 0.9419  | 0.7534  | 1  |
| $y_4$ | -1.0492   | -1.644  | -0.2714 |    |
| $C^*$ | $-\infty$ | $-\infty$| $-\infty$|    |
| $y_1$ | 1.942     | 1.870   | 1.839   | 1.875 |
| $y_3$ | 0.3792    | -0.3556 | 0.3408  |    |
| $y_2$ | 0.7267    | 0.9419  | 0.7534  | 1  |
| $y_4$ | 2.000     | 2.000   | 2.000   |    |
| L$^*$ | $-\infty$ | -0.5095 | $-\infty$|    |
| $y_1$ | 1.942     | 1.870   | 1.839   | 1.875 |
| $y_3$ | 0.2355    | -0.3208 | 0.3428  |    |
| $y_2$ | 1.942     | 1.870   | 1.854   | 1.865 |
| $y_4$ | 0.8327    | 1.106   | 0.8958  | 1.2 |
| $P^*$ | 0.4645    | 0.5248  | 0.4383  |    |
| $y_1$ | 1.936     | 1.869   | 1.837   |    |
| $y_3$ | 0.3846    | 0.5304  | 0.3021  |    |

TABLE VII: Scaling exponents of the fixed points $C^*$, $G^*$, $L^*$ and $P^*$ obtained by means of different cell clusters. The parity of the scaling exponent index refers to the parity of the interaction. The exponent $y_{2C} = y_{2G} = y_{2L} = 1$ corresponds to the thermal eigenvalue of the Onsager transition ($y_{1T}$), while the exponent $y_{4C} = y_{4G} = y_{4L} = 1.875$ corresponds to the magnetic eigenvalue one ($y_{1M}$) [2]. The exact critical exponents for the $P^*$ fixed point correspond instead to the transition in the three-state Potts model [23].
FIG. 10: Flow of the renormalized probability distribution $P(\mathbf{K})$ for the disordered 3D BEG model in the paramagnetic phase: $J = 4$, $K = 0$, $\Delta = 0.4$ and $p = 0.6$ on the SCB$_2$ cluster. The parameters $K$ and $\Delta^\dagger$ are integrated.

FIG. 11: Flow of the renormalized probability distribution $P(\mathbf{K})$ for the disordered 3D BEG model in the paramagnetic phase: $J = 4$, $K = 0$, $\Delta = 0.4$ and $p = 0.7$ on the SCB$_2$ cluster. The parameters $K$ and $\Delta^\dagger$ are integrated.

FIG. 12: PM/FM critical surface in the $(T, \Delta/J, p)$ parameter space for the 3D BEG model with $K = 0$ obtained with the SCB$_2$ cluster.

reduced to the study of an ordered model. In particular the fixed point at $\mu_\Delta = -\infty$ corresponds at the critical fixed point of the 3D Ising model discussed in Sec. [III]

V. DISCUSSION AND CONCLUSIONS

In this paper we have presented an extension of the real space cluster RG method with two cells proposed by Berker and Wortis [2] by considering a staggered topology for the clusters. This not only makes the antiferromagnetic phase detectable, but leads to a sensible improvement of the numerical estimates of the critical exponents and location of the critical points for both the Ising and BEG models.

The two staggered cells cluster gives generally better results also with respect to the tuned version of the square cells cluster approach where one, or more, free parameters in the cell projection matrix are fixed by the knowledge of some points in the phase diagrams. The later tuning method is not only less predictive, requiring as input some known points, but it may lead to an “unphysical” projection matrix [2]. We have seen, indeed, that in certain cases, for example when fixing the critical temperature of the 2D Ising model to the exact value, the resulting projection matrix assigns a negative contribution to some spin configurations to the partition sum. A choice not providing any physical insight. The staggered cells cluster, instead, is physically motivated: the request of an antiferromagnetic ordering. It is remarkable that this request quantitatively improves the results also for the pure ferromagnetic models.

We, further, reported that the extension of the two cells cluster method, even in the staggered version, is not consistent with exact and established results for the corresponding regular lattice, in particular the ferromagnetic phase is detected also beyond the intersection with Nishimori line.

In the two dimensional case we have also extended the analysis by considering four cells in the cluster. Although the approximation is not systematic, with a square cell arrangements a clear improvement is achieved in the pure model. When a staggered topology is considered, instead, it does not work comparably well as in the two cells case. We observe that a four cells cluster is the minimal requirement to preserve plaquette frustration in presence of bond disorder under the RG process, which is necessary to identify a spin glass critical point (at $T = 0$ in 2D). The requirement is necessary but non sufficient. Indeed, the phase diagram of four cells cluster, besides a minor improvement in the slope of the critical line is achieved, shows the same issues of the two cells case.

In the three dimensional case, a similar scenario is ob-
tained: in the pure case the staggered version shows a clear improvement, while the quenched disordered extension is always ineffective and the expected spin glass phase remains undetected for both the Ising and BEG models.

This failure follows previous attempts of generalizing real space RG methods conceived for ordered systems to disordered systems. The generalization to disordered systems has led in the past to ambiguous results. On the one hand the cumulant expansion has provided evidence for a spin glass phase in dimension 2, lower than the lower critical dimension 2.5. On the other hand, however, the attempt to extend the block RG transformation on spin clusters did not yield any spin glass fixed point, even in dimension 3.

The lack of a spin glass phase is related to the incorrect location of the boundary of the ferromagnetic phase in the disordered region. We have seen, indeed, that in the disordered Ising model the ferromagnetic phase enters also in the region forbidden by duality analysis. This occurs with all clusters used. The problem is partly mitigated when the tuned cluster is used, cfr. Figs. 3 and 8, but its uncontrolled nature does not allow for any further physical insight. A milder, but more recognizable attenuation, is obtained with the four cells cluster if the correlation between the interactions is properly taken in account along the RG flow, cfr. Fig. 4.

The connection between the problem in the ferromagnetic critical line and the detection of the spin glass phase is highlighted by looking at the single RG flow: the probability distributions always shrink under the block RG transformations, cfr. Figs. 2 and 11. The paramagnetic and ferromagnetic phases only differ for the trend of the average $\mu_J$ of the couplings $J_{ij}$, while the variance $\sigma_J^2$ goes quickly to zero in all the detected phases. This does not happen, for example, in the real space RG on hierarchical lattice, where the variance $\sigma_J^2$ of the couplings increases in the FM phase, even though $\sigma_J/\mu_J \to 0$, and the spin glass phase is detected as the region of the phase diagram where $\sigma_J/\mu_J \to \infty$.

It is quite clear that in order to build a valuable generalization of the RG cluster method to strong disorder, the first and crucial step is to obtain the correct evolution of the FM phase as disorder is added to the ordered system. The improvement achieved with the staggered cells clusters shows that to detect the antiferromagnetic phase it is essential that the ground state is invariant under the block RG transformation, and clearly this is not the case with a possible spin glass phase.

The analysis of the limits of the cluster RG present in this work makes eventually clear that, while the method works in an effective and controlled way for pure or weakly disordered, ferromagnetic or antiferromagnetic, systems, the generalization to the case of strong disorder calls for a different decimation procedure for the block RG transformation.

The decimation via the majority rule, or its tuned improvement, yields a local magnetization of the coarse grained cell. This is meaningful as far as magnetization is the relevant order parameter of the transition. In the spin-glass transition, though, magnetization is zero and the relevant order parameter is the “replica” overlap. To put forward a decimation procedure based on the overlap coarse graining one has, thus, to resort to replicated clusters. Such a generalization is currently under investigation.

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Appendix: RG Stability Matrix for the BEG model

The critical exponents are obtained from the eigenvalue of the stability matrix $\partial K^R/\partial K$ evaluated at the fixed point $K^*$. For the BEG model $K = \{J, K, \Delta, L, h\}$ and the elements of the stability matrix are

$$\frac{\partial J_R}{\partial K} = \frac{1}{4\alpha} \left( \frac{x'_{++}}{x_{++}} + \frac{x'_{-+}}{x_{-+}} - \frac{2x'_{+}}{x_{+}} \right)$$

$$\frac{\partial K_R}{\partial K} = \frac{1}{4\alpha} \left( \frac{x'_{++}}{x_{++}} + \frac{x'_{-+}}{x_{-+}} + \frac{2x'_{+}}{x_{+}} + \frac{4x'_{+0}}{x_{00}} - \frac{4x'_{+0}}{x_{+0}} \right)$$

$$\frac{\partial \Delta_R}{\partial K} = \frac{1}{2} \left( \frac{x'_{+0}}{x_{+0}} + \frac{x'_{-0}}{x_{-0}} - \frac{2x'_{0}}{x_{0}} \right)$$

$$\frac{\partial L_R}{\partial K} = \frac{1}{4\alpha} \left( \frac{x'_{++}}{x_{++}} + \frac{2x'_{-0}}{x_{-0}} - \frac{x'_{-+}}{x_{-+}} - \frac{2x'_{+0}}{x_{+0}} \right)$$

$$\frac{\partial h_R}{\partial K} = \frac{1}{4} \left( \frac{x'_{+0}}{x_{+0}} - \frac{x'_{-0}}{x_{-0}} \right)$$

where $x'_{\sigma_a \sigma_b} = \partial x_{\sigma_a \sigma_b} / \partial K$ and $\alpha = 2d$, with $d$ the space dimension. The derivative of the Boltzmann factors can
be expressed as

\[
\frac{\partial x_{\sigma_a \sigma_b}}{\partial J} = \sum_a M_a M_b \left[ \alpha \sum_{(ij)} s_i s_j \right] e^{-\beta H(s)}
\]

\[
\frac{\partial x_{\sigma_a \sigma_b}}{\partial K} = \sum_a M_a M_b \left[ \alpha \sum_{(ij)} s_i^2 s_j^2 \right] e^{-\beta H(s)}
\]

\[
\frac{\partial x_{\sigma_a \sigma_b}}{\partial D} = \sum_a M_a M_b \left[ -\sum_i s_i^2 \right] e^{-\beta H(s)}
\]

\[
\frac{\partial x_{\sigma_a \sigma_b}}{\partial L} = \sum_a M_a M_b \left[ \alpha \sum_{(ij)} (s_i^2 s_j + s_i s_j^2) \right] e^{-\beta H(s)}
\]

\[
\frac{\partial x_{\sigma_a \sigma_b}}{\partial h} = \sum_a M_a M_b \left[ \sum_i s_i \right] e^{-\beta H(s)}
\]

where \( M_x \equiv M(\sigma_x, s_{i \in x}) \) are the cell projection matrices.

When the fixed point is at \( L = h = 0 \) the even and odd couplings decouples and the stability matrix is block-diagonal, with a 3 \times 3 block for even couplings and a 2 \times 2 block for odd ones.

The scaling exponents controlling the stability of the fixed point are \( y_i = \log_\lambda \lambda_i \), where \( \lambda_i \) are the eigenvalues of the stability matrix evaluated at the fixed point, and \( b \) the scaling factor of the RG scheme, \( b = 2 \) in this work.
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