RELATIVELY GEOMETRIC ACTIONS OF COMPLEX HYPERBOLIC LATTICES ON CAT(0) CUBE COMPLEXES

COREY BREGMAN AND KEJIA ZHU

ABSTRACT. We prove that for \( n \geq 2 \), a non-uniform lattice in \( \text{PU}(n,1) \) does not admit a relatively geometric action on a CAT(0) cube complex, in the sense of [EG20]. As a consequence, we prove that if \( \Gamma \) is a non-uniform lattice in a non-compact semisimple Lie group \( G \) that admits a relatively geometric action on a CAT(0) cube complex, then \( G \) is isomorphic to \( \text{SO}(n,1) \).

1. Introduction

A finitely generated group is called cubulated if it acts properly cocompactly on a CAT(0) cube complex. Agol [Ago13], building on the work of Wise [Wis12] and many others, proved that cubulated hyperbolic groups enjoy many important properties, and used this to solve several open conjectures in 3-manifold topology. In an effort to transfer some of these techniques to the relatively hyperbolic setting, Einstein–Groves define the notion of a relatively geometric action of a group pair \( (\Gamma, \mathcal{P}) \) on a CAT(0) cube complex [EG20]. For such an action, elements of \( \mathcal{P} \) are allowed to act non-properly. Relatively geometric actions are a natural generalization of proper actions and share many of the same features as in the proper case, especially when \( \Gamma \) is hyperbolic relative to \( \mathcal{P} \) [EG22, EGN21].

Uniform lattices in \( \text{SO}(3,1) \) always act geometrically thus relatively geometrically on CAT(0) cube complexes [BW12]. More generally, Einstein–Groves proved that non-uniform lattices in \( \text{SO}(3,1) \) also admit relatively geometric actions, relative to their cusp subgroups [EG20]. In fact, they prove that if \( (G, \mathcal{P}) \) is hyperbolic relative to free abelian subgroups and the Bowditch boundary \( \partial(G, \mathcal{P}) \) is homeomorphic to \( S^2 \), then \( G \) is isomorphic to a non-uniform lattice in \( \text{SO}(3,1) \) if and only if \( (G, \mathcal{P}) \) admits a relatively geometric action on a CAT(0) cube complex. This result is a relative version of the work of Markovic [Mar13] and Haïssinsky [Haïr11–15] in the convex-cocompact setting, giving an equivalent formulation of the Cannon conjecture in terms of actions on hyperbolic CAT(0) cube complexes.

It is not known in general whether the above results extend to lattices in \( \text{SO}(n,1) \) for \( n \geq 3 \). In contrast, work of Delzant–Gromov implies that uniform lattices in \( \text{PU}(n,1) \) are not cubulated [DG05]. Recall that a group \( \Gamma \) is Kähler if \( \Gamma \cong \pi_1(X) \) for some compact Kähler manifold \( X \). If \( \Gamma \leq \text{PU}(n,1) \) is a torsion-free, uniform lattice, then \( \Gamma \) acts freely, properly discontinuously cocompactly on complex hyperbolic \( n \)-space \( \mathbb{H}_c^n \). The quotient \( M = \Gamma \backslash \mathbb{H}_c^n \) is a closed, negatively curved Kähler manifold, and in particular \( \Gamma \) is a hyperbolic, Kähler group. In this context, Delzant–Gromov showed that any Kähler group that is hyperbolic and cubulated is commensurable to a surface group of genus \( g \geq 2 \) [DG05]. Thus \( \Gamma \) is not cubulated for \( n \geq 2 \). Since every uniform lattice in

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PU(n, 1) is virtually torsion-free, it follows that uniform lattices in PU(n, 1) are not cubulated if \( n \geq 2 \).

On the other hand, uniform lattices in \( PU(1, 1) = SO(2, 1) \), are finite extensions of hyperbolic surface groups, hence are cubulated hyperbolic. Similarly, non-uniform lattices in PU(1, 1) are the orbifold fundamental groups of surfaces with finitely many cusps, hence virtually free. Such lattices admit both proper cocompact and relatively geometric actions on CAT(0) cube complexes. This leaves open the question of whether non-uniform lattices in PU(n, 1) admit relatively geometric actions on CAT(0) cube complexes for \( n \geq 2 \). Our main theorem answers this question in the negative.

**Theorem 1.1.** Let \( \Gamma \leq PU(n, 1) \) be a non-uniform lattice with \( n \geq 2 \). Then \((\Gamma, P)\) does not admit a relatively geometric action on a CAT(0) cube complex, where \( P \) is the collection of cusp subgroups of \( \Gamma \).

Our strategy for proving Theorem 1.1 will be to consider certain peripherally finite Dehn fillings of \((\Gamma, P)\), using work of Groves–Manning [GM18]. As a consequence of the assumption that \((\Gamma, P)\) admits a relatively geometric action, after replacing \( \Gamma \) with a finite index subgroup \( \Gamma_0 \) we find an infinite quotient \( \varphi: \Gamma_0 \to \Gamma_0 \) that is

1. Torsion-free hyperbolic,
2. Admits a proper cocompact action on a finite-dimensional CAT(0) cube complex \( X \).

We then show that the homomorphism \( \varphi: \Gamma_0 \to \Gamma_0 \) factors through a Kähler group \( \Delta \) which is not a surface group and derive a contradiction from the result of Delzant–Gromov mentioned above. The construction of the cubulated hyperbolic quotient \( \Gamma_0 \) can be found in Theorem 2.6 below.

In fact, the construction of the quotient \( \Gamma_0 \) allows us to classify which lattices in non-compact, semisimple Lie groups may admit proper or relatively geometric actions on CAT(0) cube complexes:

**Corollary 1.2.** Let \( \Gamma \) be a lattice in a non-compact semisimple Lie group \( G \) without compact factors. If either

1. \( \Gamma \) is uniform and cubulated hyperbolic, or
2. \( \Gamma \) is non-uniform, hyperbolic relative to its cusp subgroups \( P \), and \((\Gamma, P)\) admits a relatively geometric action on a CAT(0) cube complex,

then \( G \) is commensurable to \( SO(n, 1) \) for some \( n \geq 1 \).

**Remark 1.3.** In general, it is still not known whether all lattices in \( SO(n, 1) \) admit a relatively geometric action, except when \( n = 2, 3 \).

**Proof.** Recall that a uniform lattice (resp. non-uniform lattice) \( \Gamma \) in a semisimple Lie group \( G \) is hyperbolic (resp. hyperbolic relative to its cusp subgroups \( P \)) if and only if \( G \) has rank 1 [BDM09]. Any rank 1 noncompact semisimple Lie group is commensurable with one of \( SO(n, 1) \), \( PU(n, 1) \), \( Sp(n, 1) \) for \( n \geq 2 \), or the isometry group of the octonionic hyperbolic plane \( H^3_\mathbb{O} \). The latter and \( Sp(n, 1) \) have Property (T), while \( SO(n, 1) \) and \( PU(n, 1) \) do not. Hence if \( G \) is commensurable with a lattice in \( Sp(n, 1) \) or \( Isom(H^3_\mathbb{O}) \), then \( \Gamma \) has (T).

By a result of Niblo–Reeves, any proper action of a group with Property (T) on a CAT(0) cube complex must have a global fixed point [NR97]. Thus if \( \Gamma \) is a uniform lattice, then \( \Gamma \) cannot be cubulated. On the other hand, if \( \Gamma \) is non-uniform and \((\Gamma, P)\) admits a relatively geometric action, let \( \Gamma_0 \leq \Gamma \) be the finite index subgroup whose existence is guaranteed by Theorem 2.6, with quotient homomorphism \( \varphi: \Gamma_0 \to \Gamma_0 \). Because Property (T) passes to finite index subgroups and quotients, this implies that \( \Gamma_0 \) also has Property (T) so the action of \( \Gamma_0 \) on \( X \) has a global fixed point,
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contradicting Theorem 2.6(1). Hence $\Gamma$ must be commensurable to a lattice in either $\text{PU}(n,1)$ or $\text{SO}(n,1)$. For $n \geq 2$, the uniform case of $\Gamma \leq \text{PU}(n,1)$ is eliminated by work Delzant–Gromov [DG05]. The corollary now follows from Theorem 1.1.

Outline: In Section 2, we review the definition of a relatively geometric action of a group pair on a CAT(0) cube complex, and prove the main technical result of this paper, namely Theorem 2.6. In Section 3, after reviewing the Borel–Serre and Toroidal compactifications of non-uniform quotients of complex hyperbolic space, we prove Theorem 1.1.

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2. Actions on CAT(0) Cube Complexes

In this section, we review the notion of a relatively geometric action of a group pair $(\Gamma, P)$ on a CAT(0) cube complex, defined by Einstein and Groves in [EG20]. We then introduce the Dehn fillings of group pairs and prove Theorem 2.6.

Definition 2.1. An action of $\Gamma$ on a CAT(0) cube complex $X$ is relatively geometric with respect to $P$ if

1. $X/\Gamma$ is compact;
2. Each element of $P$ acts elliptically on $X$;
3. Each cell stabilizer in $X$ is either finite or else conjugate to a finite index subgroup of an element of $P$.

Recall that if $(\Gamma, P)$ is a relatively hyperbolic group pair and $\Gamma_0 \leq \Gamma$ has finite index then $(\Gamma_0, P_0)$ is also a relatively hyperbolic group pair, where $P_0$ is the set of representatives of the conjugacy classes of

\[ \{ P^g \cap \Gamma_0 | g \in \Gamma, P \in P \} \]

Since $[\Gamma:\Gamma_0]$ is finite, $P_0$ is still a finite collection of subgroups. It follows that if $\Gamma$ admits a relatively geometric action on a CAT(0) cube complex $X$, then $(\Gamma_0, P_0)$ also admits a relatively geometric action on $X$ by restriction. Indeed, (2) and (3) in Definition 2.1 follow immediately and (1) follows from the fact that under the natural map $X/\Gamma_0 \to X/\Gamma$, each cell of $X/\Gamma$ has at most $[\Gamma:\Gamma_0] < \infty$ pre-images. Hence if $X/\Gamma$ is compact, so is $X/\Gamma_0$. We have just proven

Lemma 2.2. Let $\Gamma_0 \leq \Gamma$ be a finite index subgroup. If $(\Gamma, P)$ has a relatively geometric action on a CAT(0) cube complex $X$, then the restriction of this action to $(\Gamma_0, P_0)$ is also relatively geometric, where $P_0$ is defined as in Equation 1.

2.1. Dehn fillings.

Dehn fillings first appeared in the context 3-manifold topology and were subsequently generalized to the group-theoretic setting by Osin [Osi07] and Groves and Manning [GM08]. We now recall the notion of a Dehn filling of a group pair $(G, P)$:

Definition 2.3 (Dehn Filling). Given a group pair $(G, P)$, where $P = \{ P_1, ..., P_m \}$ and a choice of normal subgroups of peripheral groups $N = \{ N_i \leq P_i \}$, the Dehn filling of $(G, P)$ with respect to $N$
is the pair \((\overline{G}, \mathcal{P})\) where \(\overline{G} = G/K\) and \(K = \langle \bigcup N_i \rangle\) is the normal closure of the group generated by \(\{\cup_i N_i\}\) and \(\mathcal{P}\) is the set of images of \(\mathcal{P}\) under this quotient. The \(N_i\) are called the filling kernels. When we want to specify the filling kernels we will write \(G(N_1, \ldots, N_m)\) for the quotient \(\overline{G}\).

**Definition 2.4** (Periodically finite). If each normal subgroup \(N_i\) has finite index in \(P_i\), the filling is said to be **periodically finite** or PF.

**Definition 2.5** (Sufficiently long). We say that a property \(X\) holds for all sufficiently long Dehn fillings of \((G, \mathcal{P})\) if there is a finite subset \(B \subset G \setminus 1\) so that whenever \(N_i \cap B = \emptyset\) for all \(i\), the corresponding Dehn filling \(G(N_1, \ldots, N_m)\) has property \(X\).

The proof of the next theorem will rely on the notion of a \(Q\)-filling of a collection of subgroups \(Q\) of \(G\). Recall from [GM18] that given a subgroup \(Q < G\), we call \(G(N_1, \ldots, N_m)\) a \(Q\)-filling if for all \(g \in G\), and \(P_i \in \mathcal{P}\), \(|Q \cap P_i^n| = \infty\) implies \(N_i^g \subseteq Q\). If \(Q = \{Q_1, \ldots, Q_t\}\) is a family of subgroups, then \(G(N_1, \ldots, N_m)\) is a \(Q\)-filling if it is a \(Q\)-filling for every \(Q \in Q\).

**Theorem 2.6.** Let \((\Gamma, \mathcal{P})\) be a relatively hyperbolic pair such that the elements of \(\mathcal{P}\) are residually finite. If \((\Gamma, \mathcal{P})\) admits a relatively geometric action on a CAT(0) cube complex \(X\) then there exists a finite index subgroup \(\Gamma_0 \leq \Gamma\) and a quotient \(\varphi : \Gamma_0 \to \overline{\Gamma_0}\) satisfying:

1. \(\overline{\Gamma_0}\) is infinite, torsion-free hyperbolic, and acts properly, discontinuously cocompactly on a finite-dimensional CAT(0) cube complex \(\overline{X}\).
2. If \(K = \ker \varphi\), then \(K\) is the normal closure of the subgroup generated by the peripheral subgroups in \(\mathcal{P}_0\), where \(\mathcal{P}_0 = \{P_{0,1}, \ldots, P_{0,r}\}\) is the collection of peripheral subgroups induced by \(\mathcal{P}\) as in Equation 1.
3. There is an isomorphism
   \[K \cong \bigast_{\mathcal{P}_0} P_{0,1} \ast \cdots \ast P_{0,r}\]
   where the outer free product is over \(\overline{\Gamma_0}\) orbits of elements of \(\mathcal{P}_0\).

**Proof.** Let \(\mathcal{P} = \{P_1, \ldots, P_m\}\) and suppose \((\Gamma, \mathcal{P})\) admits a relatively geometric action on a CAT(0) cube complex \(X\). Since the action is cocompact, there are only finitely many \(\Gamma\)-orbits of cubes in \(X\). By assumption, the stabilizer of each cube is either finite or conjugate into a finite index subgroup of some \(P_i \in \mathcal{P}\), and conversely each element of \(P_i \in \mathcal{P}\) stabilizes some cube. In particular, the stabilizer of each cube in \(X\) is full relatively quasi-convex and the action of \(\Gamma\) on \(X\) verifies the hypotheses of Corollary 6.6 of [GM18].

Let \(\sigma_1, \ldots, \sigma_k\) to be representatives of the \(\Gamma\)-orbits of cubes in \(X\), and let \(Q = \{Q_1, \ldots, Q_k\}\), where \(Q_i\) is the finite index subgroup of the stabilizer of \(\sigma_i\) that fixes \(\sigma_i\) pointwise. Observe that each element of \(Q\) is either finite or there exists \(g \in G\) such that \(Q^g_i\) is a finite index subgroup of \(P_j^g\). Thus, given any finite index subgroup \(N_j < P_j\), the subgroup \(N_j^g \cap Q_i < Q_i\) still has finite index in \(P_j^g\). Since the elements of \(Q\) are residually finite and there are only finite many \(Q_i\), we can therefore always find a sufficiently long peripherally finite Dehn filling \(N_j < P_j\) such that \(N_j^g < Q_i < P_j^g\) whenever \(Q_i < P_j\), to avoid any finite subset of \(G \setminus \{1\}\). Corollary 6.6 of [GM18] now asserts that for any sufficiently long \(Q\)-filling \(\Gamma \to \overline{\Gamma} = \Gamma(N_1, \ldots, N_m)\) with kernel \(K\), the quotient \(K\backslash X\) is a CAT(0) cube complex.

Thus there exists a peripherally finite Dehn filling
\[1 \to K \to \Gamma \to \overline{\Gamma} = \Gamma(N_1, \ldots, N_m) \to 1\]
such that the quotient space $\overline{X} = K\backslash X$ is still CAT(0). Since the filling was peripherally finite, this implies $\overline{\Gamma}$ is hyperbolic, since it is hyperbolic relative to the collection $\mathcal{P}$ of finite groups. Moreover, $\overline{\Gamma}$ acts properly discontinuously cocompactly on $\overline{X}$, since $\overline{\Gamma\backslash X} = \overline{K\backslash X}$ is cocompact, and cell stabilizers in $\overline{X}$ are finite (either the image of a finite subgroup of $\Gamma$, or one of the elements of $\mathcal{P}$).

As $\overline{\Gamma}$ is cubulated hyperbolic, it is virtually special by Agol’s theorem [Ago13]. In particular, $\overline{\Gamma}$ is residually finite and virtually torsion-free, thus there exists a torsion-free normal subgroup of finite index subgroup $\Gamma_0 \leq \overline{\Gamma}$. Then $\Gamma_0$ is also hyperbolic and acts freely, properly, cocompactly on $\overline{X}$. Let $\overline{\Gamma}_0 \leq \overline{\Gamma}$ be its preimage. By construction, $\Gamma_0$ is a finite index subgroup of $\Gamma$ that is hyperbolic relative to $\mathcal{P}_0$ defined as in Equation 1, and which contains $K$. This proves (1).

Let $\mathcal{P}_0 = \{P_{0,1}, \ldots, P_{0,r}\}$ be the peripheral subgroups of $\Gamma_0$. We claim that each $P_{0,i}$ is conjugate to one of the filling kernels $N_j$. Indeed, by Equation 1, each $P_{0,i}$ is of the form $\Gamma_0 \cap P^g_j$ for some $P_j \in \mathcal{P}$ and $g \in \Gamma$. Since $K \leq \Gamma_0$ this implies $N_j^g$ is a finite index subgroup of $\Gamma_0 \cap P^g_j = P_{0,i}$. On the other hand, by Theorem 1.1 of [Osi07], we may assume that $P_j/N_j \to \overline{\Gamma}$ is injective, hence $P^g_j/N_j^g \to \overline{\Gamma}$ is injective as well. But any nontrivial element of $P^g_j/N_j^g$ has finite order, and $\Gamma_0$ is torsion-free, so $(\Gamma_0 \cap P^g_j) \setminus N_j^g = \emptyset$. We conclude that each $P_{0,i}$ is a conjugate of some filling kernel, as desired.

Since the normal closure of $\{N_j^g\}$ equals the normal closure of $\{N_j\}$, we have that $K = \langle \cup_i P_{0,i} \rangle$. This proves (2). It now follows from Theorem 4.8 of [GMS19] that for a sufficiently long Dehn filling, we may assume $K$ is the free product of the conjugates of the $\{N_j\}$ indexed by the elements of $\overline{\Gamma}$, that is,

$$K \cong \ast_{g \in \overline{\Gamma}} (N_j \ast \ldots \ast N_m)$$

Under the conjugation action of $\Gamma_0$, we can rewrite this as

$$K \cong \ast_{h \in \Gamma_0} \ast_{g \in \Gamma_0} (N_j \ast \ldots \ast N_m)$$

Since each $P_{0,i}$ is a conjugate of some $N_j$, as $h$ ranges over elements of $\Gamma/\Gamma_0$, the conjugates $N_j$ in the inner free product above range over peripheral subgroups $P_{0,i}$. This implies that the $K$ has the free product decomposition as in (3), which completes the proof of the theorem. □

**Remark 2.7.** In [GM22, Definition 1.9], Groves and Manning introduce the notion of a weakly relatively geometric action on a CAT(0) cube complex. Using Theorem 4.5 of [GM22], we can repeat the argument above to get the same conclusion as Theorem 2.6 after replacing “relatively geometric” with “weakly relatively geometric.” Note, however, that in this case $\overline{X}$ will not be $K\backslash X$ where $K$ is the kernel of the filling homomorphism $\Gamma \to \overline{\Gamma}$. Indeed, the action of $\overline{\Gamma}$ on $K\backslash X$ will in general have cell stabilizers that are virtually free. Nevertheless, Theorem D of [GM18] implies that in this case $\overline{\Gamma}$ is still cubulated hyperbolic. It will follow from the results of Section 3 that Theorem 1.1 also holds if we replace “relatively geometric” with “weakly relatively geometric.”

### 3. Relatively geometric actions of lattices in $\text{PU}(n,1)$

Let $\Gamma$ be a non-uniform lattice in $\text{PU}(n,1)$. Then $\Gamma$ acts properly discontinuously on complex hyperbolic space $H^n_\mathbb{C}$ and the quotient, which we will henceforth denote by $M = \Gamma\backslash H^n_\mathbb{C}$, is a non-compact orbifold of finite volume with finitely many cusps. Each cusp corresponds to a conjugacy
class of subgroups stabilizing a parabolic fixed point in \( \partial_{\infty} \mathbb{H}^C_n \). It is well-known that \( \Gamma \) is hyperbolic relative to the collection of these cusp subgroups, which we denote by \( \mathcal{P} \). We wish to understand in what ways \( \Gamma \) can act on a CAT(0) cube complex. In this section, we first show in Proposition 3.3 that \( \Gamma \) does not admit a proper action, and then prove Theorem 1.1, namely that \((\Gamma, \mathcal{P})\) does not admits a relatively geometric action on a CAT(0) cube complex.

Throughout the course of the proof, we will freely pass to finite index subgroups by invoking Lemma 2.2. In order to streamline the exposition, we will not refer to Lemma 2.2 each time. First we reduce the Theorem 1.1 to the case where \( \Gamma \) is torsion-free.

**Lemma 3.1.** \( \Gamma \) has a torsion-free subgroup of finite index.

**Proof.** We have a short exact sequence
\[
1 \to \mathbb{Z}/(n + 1)\mathbb{Z} \to \text{SU}(n, 1) \to \text{PU}(n, 1) \to 1.
\]
Restricting to \( \Gamma \), we get a short exact sequence
\[
1 \to \mathbb{Z}/(n + 1)\mathbb{Z} \to \Lambda \to \Gamma \to 1,
\]
where \( \Lambda \) is the pre-image of \( \Gamma \) in \( \text{SU}(n, 1) \). Since \( \Gamma \) is finitely generated and \( \mathbb{Z}/(n + 1)\mathbb{Z} \) is finite, \( \Lambda \) is finitely generated. As \( \text{SU}(n, 1) \) is linear, Selberg’s lemma implies that \( \Lambda \) has a finite index torsion-free subgroup, say, \( \Lambda_0 \). Thus \( \Lambda_0 \cap \mathbb{Z}/(n + 1)\mathbb{Z} = 1 \) and hence it is mapped isomorphically to finite index subgroup \( \Gamma_0 \leq \Gamma \).

Following Lemma 3.1, for the remainder of this section we will assume that \( \Gamma \leq \text{PU}(n, 1) \) is torsion-free.

### 3.1. The structure of cusps

We now briefly review the geometric structure of cusps in \( \mathbb{H}^C_n \). For more details see Gol99. Recall that up to scaling each horosphere in \( \mathbb{H}^C_n \) is isometric to \( \mathcal{H}_{2n-1}(\mathbb{R}) \), the \((2n-1)\)-dimensional Heisenberg group, equipped with a left-invariant metric. The Heisenberg group is a central extension
\[
1 \to \mathbb{R} \to \mathcal{H}_{2n-1}(\mathbb{R}) \to \mathbb{R}^{2n-2} \to 1
\]
with extension 2-cocycle equal to the standard symplectic form
\[
\omega = 2 \sum_{i=1}^{n-1} dx_i \wedge dy_i,
\]
where \((x_1, y_1, \ldots, x_{n-1}, y_{n-1})\) are coordinates on \( \mathbb{R}^{2n-2} \). The Lie algebra \( \mathfrak{h}_{2n-1} \) is 2-step nilpotent with basis \( \{X_1, Y_1, \ldots, X_n, Y_n, Z\} \) where
\[
[X_i, Y_j] = Z,
\]
and all other brackets vanish. Thus \( Z \) generates the center of \( \mathfrak{h}_{2n-1} \) representing the kernel \( \mathbb{R} \) in Equation 2, while the remaining coordinates project to the generators of \( \mathbb{R}^{2n-2} \). Choosing the identity matrix \( I_{2n-1} \) as the inner product on \( \mathfrak{h}_{2n-1} \), we see that the isometry group of \( \mathcal{H}_{2n-1}(\mathbb{R}) \) is isomorphic the \( \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n - 1) \), where the \( \mathcal{H}_{2n-1}(\mathbb{R}) \) factor is the action of \( \mathcal{H}_{2n-1}(\mathbb{R}) \) on itself by left translation, and the unitary group \( U(n - 1) \) is the stabilizer of the identity. Indeed, any isometry which fixes \( 1 \in \mathcal{H}_{2n-1}(\mathbb{R}) \) must also be a Lie algebra isomorphism; it therefore preserves the center \( (Z) \) and induces an isometry of \( \mathbb{R}^{2n-2} \cong \langle X_1, Y_1, \ldots, X_{n-1}, Y_{n-1} \rangle \) preserving \( \omega \). We conclude that such an isometry lies in \( U(n - 1) = O_{2n-2}(\mathbb{R}) \cap \text{Sp}_{2n-2}(\mathbb{R}) \).

**Definition 3.2.** Let \( \pi: \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n - 1) \to U(n - 1) \) be the projection. For any \( g \in \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n - 1) \), we call \( \pi(g) \) the rotational part of \( g \).
Since the center of $\mathcal{H}_{2n-1}(\mathbb{R})$ is invariant under any isometry we have a short exact sequence
\begin{equation}
1 \to \mathbb{R} = Z(\mathcal{H}_{2n-1}(\mathbb{R})) \to \mathcal{H}_{2n-1}(\mathbb{R}) \times U(n-1) \to \mathbb{R}^{2n-2} \times U(n-1) \to 1
\end{equation}

Since $\Gamma$ is torsion-free, each cusp subgroup $P \leq \Gamma$ is isomorphic to a discrete, torsion-free, cocompact subgroup of Isom($\mathcal{H}_{2n-1}(\mathbb{R})$). In particular, $P_0 = P \cap \mathcal{H}_{2n-1}(\mathbb{R})$ is a discrete cocompact subgroup and $P \cap Z(\mathcal{H}_{2n-1}(\mathbb{R})) \cong \mathbb{Z}$. By Equation 3, $P$ fits into a short exact sequence
\begin{equation}
1 \to \mathbb{Z} = P \cap Z(\mathcal{H}_{2n-1}(\mathbb{R})) \to P \to \Lambda \to 1
\end{equation}

where $\Lambda$ is a discrete cocompact subgroup of $\mathbb{R}^{2n-2} \times U(n-1)$. It follows that $\Lambda$ has a finite index subgroup $\Lambda_0$ isomorphic to $\mathbb{Z}^{2n-2}$, which is the image of $P_0$.

On the level of quotient spaces, the sequence in Equation 4 has the following translation. The quotient space $\mathcal{O} = \Lambda \backslash \mathbb{R}^{2n-2}$ is a Euclidean orbifold finitely covered by the $(2n-2)$-dimensional torus $T = \Lambda_0 \backslash \mathbb{C}^{n-1}$, and $\Sigma = \mathcal{P} \backslash \mathcal{H}_{2n-1}(\mathbb{R})$ is the total space of an $S^1$-bundle over $\mathcal{O}$, i.e., there is a fiber sequence
\begin{equation}
S^1 \to \Sigma \to \mathcal{O}
\end{equation}

Since $\mathcal{O}$ need not be smooth, this is not generally a locally trivial fibration. However, as $P$ is torsion-free, $\Sigma$ is smooth. Passing to the torus cover, we obtain an actual fiber bundle
\[ S^1 \to \Sigma \to T \]

The finite group $F = P/P_0$ acts on $\Sigma$ preserving the fibration, hence defines a finite group of isometries of $T$. Thus the stabilizer of a point in $T$ acts freely on the $S^1$ fiber. Since the action of $F$ on $\Sigma$ is free, it follows that point stabilizers in $T$ must be cyclic of finite order, and act by rotations on the fiber. Since $F \leq U(n-1)$, any abelian subgroup is diagonalizable. Thus, locally each point in $N$ has a neighborhood of the form $(S^1 \times \mathbb{D}^{n-1})/(\mathbb{Z}/m\mathbb{Z})$ where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk, and $\mathbb{Z}/m\mathbb{Z}$ acts on $S^1$ by rotation by $2\pi/m$ and on the polydisk $\mathbb{D}^{n-1}$ by a diagonal unitary matrix $\Delta = \text{diag}\left(e^{\frac{2\pi i k_1}{m}}, \ldots, e^{\frac{2\pi i k_{n-1}}{m}}\right)$, where at least one $k_i$ is coprime to $m$. See Figure 1 for a schematic.

Since $F$ acts by rotation on each fiber, $\Sigma$ is the boundary of a disk bundle over $\mathcal{O}$, which we will denote by $E_\mathcal{O}$.

Recall that the center of $\mathcal{H}_{2n-1}(\mathbb{R})$ is quadratically distorted. It follows that the center of $P$ is quadratically distorted as well. One consequence of this fact is that the action of $\Gamma$ on a CAT(0) cube complex can never be proper:

**Proposition 3.3.** Let $\Gamma \leq \text{PU}(n,1)$ be a torsion-free non-uniform lattice, and suppose $\Gamma$ acts on a CAT(0) cube complex $X$. Then the action of each cusp subgroup of $\Gamma$ is not proper.

**Proof.** By a result of Haglund (see also [Woo17]), if a group $G$ acts on a CAT(0) cube complex, then every element is semi-simple, i.e., either acts elliptically or preserves a bi-infinite combinatorial axis. In particular, every infinite order element which acts loxodromically must be undistorted [Hag21].

However, as noted above, each peripheral subgroup of $\Gamma$ contains a central infinite cyclic subgroup that is quadratically distorted. Hence these central elements must act elliptically, and since they have infinite order, the action cannot be proper.

Thus, for any action of $\Gamma$ on a CAT(0) cube complex $X$, the center of any cusp subgroup must act elliptically. For that reason, it is natural to consider relatively geometric actions of the relatively hyperbolic pair $(\Gamma, \mathcal{P})$. 


3.2. The Borel–Serre compactification of $M$. Since $\Gamma$ is torsion-free, the quotient $M$ is a smooth, noncompact manifold of finite volume with finitely many cusps. Borel–Serre constructed a compactification $M_0$ of $M$ in [BS73]. Although their construction works in general for lattices in noncompact symmetric spaces, in this paper we will only deal with the rank 1 case when $\Gamma$ is torsion-free; in this situation $M_0$ is actually a smooth manifold with boundary.

Let $\{C_1, \ldots, C_r\}$ be the cusps of $M$. By the Margulis lemma, each cusp is diffeomorphic to $\Sigma_i \times [0, \infty)$, where $\Sigma_i$ is the quotient of $H_{2n-1}(\mathbb{R})$ by the peripheral subgroup $P_i$ (see for example, [BGS85, §10.3]). Lifting to the universal cover, the pre-image of $\sqcup_i C_i$ is a $\Gamma$-equivariant union of pairwise disjoint horoballs centered on the parabolic fixed points in $\partial_\infty H^n_{\mathbb{C}}$. The $\Gamma$-stabilizer of each parabolic fixed point is a conjugate of some $P_i$ that acts freely, properly discontinuously cocompactly on the corresponding horosphere with quotient $\Sigma_i$. The quotient of the complement of the interiors of these horoballs is therefore a compact manifold $M_0$ with boundary $\partial M_0 = \sqcup_i \Sigma_i$.

The pair $(M_0, \partial M_0)$ is called the Borel–Serre compactification of $M$. The boundary components $\{\Sigma_i\}$ are in one-to-one correspondence with the peripheral subgroups $\{P_i\}$. Thus, $M_0$ is obtained from $M$ by removing a neighborhood of the cusps. Thus $M$ and $M_0$ are homotopy equivalent and in particular, $\pi_1(M_0) = \pi_1(M) = \Gamma$.

3.3. The toroidal compactification of $M$. Another natural compactification of $M$ fills in the cusps with the Euclidean orbifolds described in Section 3.1. Let $O_i$ be the Euclidean orbifold quotient of $\Sigma_i$, with corresponding disk bundle $E_i$. Thus, we can identify $E_i \setminus O_i$ with the cusp $C_i$, then compactify $M$ by adding $\sqcup_i O_i$ at infinity. The result is a Kähler orbifold $T(M)$ with boundary divisor $D = \sqcup_i O_i$. The pair $(T(M), D)$ is called the toroidal compactification of $M$. See [HS96, AMRT10] for more details.
When the parabolic elements in \( \Gamma \) have trivial rotational part, then each \( O_i \) is a \((2n-2)\)-dimensional torus, \( T(M) \) is a smooth Kähler manifold and \( D \) is a smooth divisor in \( T(M) \). Moreover, Hummel–Schroeder show that \( T(M) \) admits a nonpositively curved Riemannian metric [HIS96]. In particular, \( T(M) \) is aspherical; if \( \Delta = \pi_1(T(M)) \) then \( T(M) \) is a \( K(\Delta, 1) \). The following lemma ensures that we can always find a finite cover of \( M \) whose toroidal compactification is smooth.

**Lemma 3.4.** Let \( \Gamma \leq \text{PU}(n, 1) \) be torsion-free and let \( M = \mathbb{H}^2_n/\Gamma \) be the quotient. There exists a finite cover \( M' \rightarrow M \) such that the toroidal compactification of \( M' \) is smooth.

**Proof.** By the main theorem of [Hum98] (p. 2453), there exists a finite subset \( F \subset \Gamma \) of parabolic isometries such that if \( N \not\subseteq \Gamma \) is a normal subgroup satisfying \( F \cap N = \varnothing \), then any parabolic isometry in \( N \) has no rotational part. Since \( \Gamma \) is residually finite and \( F \) is finite, we can find a finite index normal subgroup \( \Gamma' \triangleleft \Gamma \) such that \( \Gamma' \cap F = \varnothing \). Therefore the finite cover \( M' := \Gamma' \backslash \mathbb{H}^2_n \) of \( M \) admits a toroidal compactification which is smooth. \( \square \)

For the rest of this section, we assume that \( T(M) \) is smooth. Since \( M_0 \setminus \partial M_0 \cong M \cong T(M) \setminus D \), there is a natural map of pairs \( f: (M_0, \partial M_0) \rightarrow (T(M), D) \) that is a deformation retraction on the interior of \( M_0 \) and sends \( \partial M = \sqcup \Sigma_i \rightarrow D = \sqcup_i O_i \) via the fibering in Equation 5.

**Proposition 3.5.** The map \( f: (M_0, \partial M_0) \rightarrow (T(M), D) \) induces an isomorphism

\[
f_*: \mathcal{H}_{2n}(M_0, \partial M_0) \rightarrow \mathcal{H}_{2n}(T(M), D)
\]

**Proof.** Let \( E_i \) be the total space of the \( \mathbb{D} \)-bundle over \( O_i \) and set \( E = \sqcup_i E_i \). Since \( \partial E_i \cong \Sigma_i \), removing the interior of \( E \) yields a manifold diffeomorphic to \( M_0 \). By excision, we thus have \( H_*(M_0, \partial M_0) \cong H_*(T(M), E) \). On the other hand, since \( O_i \) is smooth, each \( E_i \) deformation retracts to \( O_i \) and therefore \( H_*(M_0, \partial M_0) \cong H_*(T(M), D) \). Restricted to \( M_0 \cong T(M) \setminus \text{Int}(E) \), this deformation retraction is exactly the map \( f \) defined above. \( \square \)

### 3.4. Filling the cusps

By Lemmas 3.1 and 3.4, we may assume that both \( M \) and its toroidal compactification \( T(M) \) are smooth. Observe that the same holds for any finite cover of \( M \). Let \( \tau = f_*: \Gamma \rightarrow \Delta \) be the homomorphism on \( \pi_1 \) induced by \( f \). The image \( \mathcal{D} \) of \( \mathcal{P} \) is a collection of free abelian subgroups of \( \Delta \) such that the pair \( (\Delta, \mathcal{D}) \) is relatively hyperbolic.

Our strategy now is to find a hyperbolic Dehn filling of \( \Gamma \) which is cubulated and which factors through \( \Delta \). Let \( \Gamma_0 \leq \Gamma \) be the finite index subgroup whose existence is guaranteed by Theorem 2.6. Replacing \( \Gamma \) with \( \Gamma_0 \), we may assume that there exists a quotient homomorphism \( \varphi: \Gamma \rightarrow \overline{\Gamma} \) such that

- \( \overline{\Gamma} \) is infinite, torsion-free hyperbolic and acts properly discontinuously cocompactly on a finite-dimensional CAT(0) cube complex \( \mathcal{X} \).
- Let \( \mathcal{P} = \{P_1, \ldots, P_r\} \) the peripheral subgroups. Then \( K = \ker \varphi \) is the normal closure of the subgroup of \( \Gamma \) generated by elements of \( \mathcal{P} \) and moreover \( K \) can be written as a free product

\[
K \cong \ast_{g \in \overline{\Gamma}} (P_1 \ast \cdots \ast P_r)
\]
In particular, since $K$ is a normal subgroup containing each peripheral subgroup, and $\mathcal{T}(M)$ is obtained by killing a subgroup of each peripheral subgroup, we have our desired factorization:

$$
(\Gamma, \mathcal{P}) \xrightarrow{\varphi} (\Gamma, 1) \xrightarrow{\kappa} (\Delta, \mathcal{D})
$$

Since $\Delta$ is Kähler and $\overline{\Gamma}$ is cubulated hyperbolic, by the corollary on page 52 of [DG05], the homomorphism $\kappa: \Delta \to \overline{\Gamma}$ must admit a further factorization

$$
(\Delta, \mathcal{D}) \xrightarrow{\kappa} (\overline{\Gamma}, 1)
$$

where $\overline{\Delta} \leq \text{SO}(2,1)$ is a cocompact lattice, and the image $\overline{\mathcal{D}}$ of $\mathcal{D}$ is a collection of quasi-convex subgroups of $\overline{\Delta}$. Thus, in order to prove Theorem 1.1, it suffices to prove that $\kappa: \Delta \to \overline{\Gamma}$ does not admit such a factorization. Recall that $\mathcal{T}(M)$ is aspherical hence $H_*(\mathcal{T}(M); \mathbb{Z}) \cong H_*(\Delta; \mathbb{Z})$, and since $\mathcal{T}(M)$ is a smooth, orientable manifold of real dimension $2n$, we know that $\mathcal{H}_{2n}(\Delta; \mathbb{Z}) \cong \mathbb{Z}$. Since $2n \geq 4$, in order to show the factorization in Diagram 8 is not possible, we will prove that $\varphi$ induces an isomorphism $\kappa_*: \mathcal{H}_{2n}(\Delta; \mathbb{Z}) \to \mathcal{H}_{2n}(\overline{\Gamma}; \mathbb{Z})$. For this, we first need to construct an explicit $K(\overline{\Gamma}, 1)$ model.

3.5. **Collapsing the cusps.** Write $\partial M_0 = \Sigma_1 \cup \cdots \cup \Sigma_r$, where $\pi_1(\Sigma_i) = P_i$. Each $\Sigma_i$ is aspherical, since it is a quotient of the real Heisenberg group $\mathcal{H}_{2n-1}(\mathbb{R})$ by a torsion-free, cocompact subgroup. By construction, $\overline{\Gamma}$ is obtained from $\Gamma$ by killing the peripheral subgroups $P_i$. Now define

$$
\mathcal{C}(M) := M_0 \sslash \partial M_0
$$

where the notation “$\sslash$” indicates that we collapse each component of $\partial M_0$ separately to a point. Let $s_i \in \mathcal{C}(M)$ be the image of $\Sigma_i$ and set $S = \{s_1, \ldots, s_r\}$. Observe that $\mathcal{C}(M)$ is homeomorphic to the space obtained from $M_0$ by gluing on the cone $\Sigma_i$ for each component $\Sigma_i \subset \partial M_0$. A schematic picture for each of the three compactifications of $M$ is shown in Figure 2.

**Lemma 3.6.** $\mathcal{C}(M)$ is aspherical and $\pi_1(\mathcal{C}(M)) = \overline{\Gamma}$.

**Proof.** The fact that $\pi_1(\mathcal{C}(M)) = \overline{\Gamma}$ follows easily from van Kampen’s theorem since a neighborhood of each $s_i$ is homeomorphic to the cone on $\Sigma_i$. Hence we must show that $\mathcal{C}(M)$ is aspherical. Consider the short exact sequence

$$
1 \to K \to \Gamma \to \overline{\Gamma} \to 1.
$$

Let $p_K: M_K \to M$ be the covering space of $M_0$ corresponding to $K$. Since $M_0$ is a deformation retract of $M$ and the universal cover of $M$ is complex hyperbolic space $\mathbb{H}^n$, we know that $M_K$ is aspherical and has fundamental group $K$. Observe that the universal cover $Y$ of $\mathcal{C}(M)$ is obtained from $M_K$ by collapsing each boundary component of $M_K$ separately to a point. Since $K$ contains each cusp subgroup, each boundary component of $M_K$ projects diffeomorphically under $p_K$ onto some boundary component of $M_0$. By Equation 6, $K$ is free product

$$
K \cong \ast_{g \in \overline{\Gamma}} (P_1 \ast \cdots \ast P_r)
$$
As noted above, each $\Sigma_i$ is a $K(P_i,1)$, hence we obtain

\[ H_* (\Sigma) = H_* (K; \mathbb{Z}) = \bigoplus_{g \in \Gamma} (H_* (P_1; \mathbb{Z}) \oplus \cdots \oplus H_* (P_r; \mathbb{Z})) = \bigoplus_{\Sigma \subseteq \partial M_K} H_* (\Sigma). \]

Since $Y$ is connected and simply connected, by the Hurewicz theorem it suffices to show that $H_p (Y) = 0$ is trivial for all $p \geq 2$. Let $\overline{V}$ be a neighborhood of $S$ which is evenly covered by the covering map $p: Y \to C(M)$. Take an open cover of $Y$ by two sets $U, V$ where $U = p^{-1} (C(M) \setminus S)$ and $V = p^{-1} (\overline{V})$. In particular, each component of $V$ is homeomorphic to the cone on some $\Sigma_i$. We choose $U$ and $V$ so that $U \cap V$ is a disjoint union of connected components, each of which is diffeomorphic to $\Sigma \times (0,1)$ for some $\Sigma \subseteq \partial M_K$. In particular, $H_p (U \cap V)$ is isomorphic to a direct sum $\bigoplus_{\Sigma \subseteq \partial M_K} H_p (\Sigma)$. Now by Mayer–Vietoris applied to $U \cup V$, we have

\[ \cdots \to H_p (U \cap V) \to H_p (U) \oplus H_p (V) \to H_p (Y) \to H_{p-1} (U \cap V) \to H_{p-1} (U) \oplus H_{p-1} (V) \to \cdots \]

Each connected component of $V$ is a cone, and hence contractible. So for $p > 0$, $H_p (V) = 0$. On the other hand, $U$ is homeomorphic to $M_K$, so $H_p (U) \cong H_p (M_K)$ for all $p$. By our observation above, $H_p (U \cap V) \cong \bigoplus_{\Sigma \subseteq \partial M_K} H_p (\Sigma)$, hence we obtain

\[ \cdots \to \bigoplus_{\Sigma \subseteq \partial M_K} H_p (\Sigma) \to H_p (M_K) \to H_p (Y) \to \bigoplus_{\Sigma \subseteq \partial M_K} H_{p-1} (\Sigma) \to H_{p-1} (M_K) \to \cdots \]
for all $p \geq 2$. Now by Equation 9, this means $H_p(Y)$ is sandwiched between two isomorphisms for all $p \geq 2$. Therefore $H_p(Y) = 0$ for all $p \geq 2$, as desired. □

It follows from Lemma 3.6 that $H_*(\Gamma; \mathbb{Z}) \cong H_*(C(M))$. Now consider the natural quotient map $q: M_0 \to C(M)$ which collapses each boundary component of $M_0$ separately to a point. Then $q$ induces a map of pairs $(M_0, \partial M_0) \to (C(M), S)$. Since the dimension of $C(M)$ is $2n \geq 4$, from the long exact sequence of the pair $(C(M), S)$ we have an isomorphism induced by inclusion

$$\tag{10} H_{2n}(C(M)) \cong H_{2n}(C(M), S)$$

**Lemma 3.7.** The induced map $q_*: H_{2n}(M_0, \partial M_0) \to H_{2n}(C(M), S)$ is an isomorphism. In particular, there is a natural isomorphism $H_{2n}(M_0, \partial M_0) \to H_{2n}(C(M), S)$ and $H_{2n}(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$.

**Proof.** Since $(M_0, \partial M_0)$ is a good pair (see Theorem 2.13 of [Hat02]), the map $M_0 \to M_0/\partial M_0$ induces isomorphism on homology $H_*(M_0, \partial M_0) \cong H_*(M_0/\partial M_0, \{\ast\})$, where $\{\ast\}$ is the image of $\partial M_0$. Similarly, $(C(M), S)$ is a good pair, so the quotient map $C(M) \to C(M)/S$ also yields an isomorphism in homology $H_*(C(M), S) \cong H_*(C(M)/S, \{\ast\})$, where now $\{\ast\}$ represents the common image of $\{c_1, \ldots, c_r\}$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
H_*(M_0, \partial M_0) & \xrightarrow{q_*} & H_*(C(M), S) \\
\downarrow & & \downarrow \\
H_*(M_0/\partial M_0, \ast) & \xrightarrow{h_*} & H_*(C(M)/S, \ast)
\end{array}
$$

where the horizontal map $h_*$ is induced from the obvious homeomorphism

$h: (M_0/\partial M_0, \ast) \to (C(M)/S, \ast).$

Since the vertical maps are isomorphisms, this implies $q_*$ is as well. By Poincaré-Lefschetz duality, $H_{2n}(M_0, \partial M_0) \cong \mathbb{Z}$. The final statement now follows from the isomorphism in Equation 10 and the fact that $C(M)$ is a $K(\Gamma, 1)$ by Lemma 3.6. □

3.6. **Proof of the main theorem.** We now have all the ingredients necessary to prove Theorem 1.1.

**Proof of Theorem 1.1.** Suppose $(\Gamma, P)$ admits a relatively geometric action on a CAT(0) cube complex $X$. We then construct $\Gamma, \Delta$ and $\kappa$ as in Diagram 7. Since $\Delta$ is Kähler, by the corollary on page 52 of [DG05], we obtain a factorization of $\kappa$ as in Diagram 8, where $\Delta \leq \text{SO}(2, 1)$ is a cocompact lattice. Since the virtual cohomological dimension of $\Delta$ is 2, and $\dim(T(M)) = 2n \geq 4$, this implies that $\kappa_*: H_{2n}(\Delta) \to H_{2n}(\Gamma)$ must be the zero map. On the other hand, by Lemma 3.7, $\varphi$ induces an isomorphism $\varphi_*: H_{2n}(\Gamma, P) \to H_{2n}(\Gamma, \{1\}) \cong H_{2n}(\Gamma) \cong \mathbb{Z}$, which factors through the isomorphism $\tau_*: H_{2n}(\Gamma, P) \to H_{2n}(\Delta, D)$ by Proposition 3.5. Therefore $\varphi_*: H_{2n}(\Delta, D) \to H_{2n}(\Gamma, \{1\}) = \mathbb{Z}$ must be an isomorphism, which is a contradiction. □

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