Polchinski’s exact renormalisation group for tensorial theories: Gaußian universality and power counting

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Received 16 February 2016, revised 26 July 2016
Accepted for publication 29 July 2016
Published 26 August 2016

Abstract
In this paper, we use the exact renormalisation in the context of tensor models and group field theories. As a byproduct, we rederive Gaussian universality for random tensors and provide a general power counting for Abelian tensorial field theories with a closure constraint, leading us to only five renormalisable theories.

Keywords: tensor models, group field theories, renormalization group, quantum gravity

1. Introduction

Constructing a quantum theory of gravity remains one of the most tantalising open problems in theoretical physics. There are currently multiple approaches: string theory, loop quantum gravity, non-commutative quantum field theory, asymptotic safety, dynamical triangulations, and Regge calculus, to name a few. Here we deal with tensor models and group field theories that have initially been introduced in the context of quantum gravity, merging ideas from loop quantum gravity, dynamical triangulations and Regge calculus (see for instance [1, 2] for a recent review).

Tensor models are higher dimensional generalisations of matrix models whose basic variable is a rank $D$ tensor $T_{i_1 \ldots i_D}$ instead of a matrix $M_{ij}$. Matrices are useful in two-dimensional quantum gravity because their perturbative expansions are sums over ribbon
graphs that reproduce triangulated surfaces. Perturbative expansions of higher rank tensors provide sums over $D$-dimensional triangulations which may be considered as a first step towards a path integral over geometries. In group field theories, the discrete indices are replaced by group elements (typically the rotation or Lorentz group) whose perturbative expansion reproduces a sum over spin foam amplitudes.

While tensor models and group field theories have been known since more than two decades, decisive breakthroughs only occurred in the recent years, thanks to the advent of colored models [3] and multi-orientable models [4] which contains a larger class of the former one. Among these important results, let us mention the Gaussian universality for random tensors [5] and the construction of renormalisable group field theories, first in the Abelian case [6], then in the non-Abelian case [7].

In this letter, we are interested in the applications of the exact renormalisation group equation, in the form derived by Polchinski [8], to tensor models and tensorial field theories. This is a continuation of our previous work [9], where we have formulated Polchinski’s exact renormalisation group equation in the tensorial context. In particular, we revisit two of the aforementioned results: Gaussian universality and the Abelian power counting for theories with a closure constraint. While these results are known, we want to stress that the technique we use here is radically different: we do not compute any Feynman graph but simply use the scaling properties of the exact renormalisation group equation.

Note that the exact renormalisation group equation (in the form of Wetterich’s equation) has also been studied [10, 11] and used to investigate the fixed point structure of tensor models [12] and tensorial field theories [13, 14]. A discrete formulation of the flow equation, based on multiscale analysis, has also been proposed in [15].

It is organised as follows. In section 2 we present tensor models and their effective actions. Then, we derive the exact renormalisation group equation in section 3 and prove Gaussian universality in section 4. Section 5 deals with tensorial field theories and their power counting. An appendix collects some explicit low order computations in the invariant tensor models, for $D = 3$ and $D = 4$.

## 2. Random tensor models and their effective actions

Random tensors are generalisations of random matrices whose basic variables are a rank $D$ complex tensor $F_{i_1,\ldots,i_D}$ and its complex conjugate $\overline{F}_{i_1,\ldots,i_D}$, with indices $i_k \in \{1,\ldots,N\}$, the integer $N$ being the size of the tensor. The tensors are not supposed to obey any invariance properties under permutations of its indices, as is the case in colored models.

In general, the main objects of interest are the expectation values of certain observables $O$,

$$
\langle O \rangle = \frac{1}{Z} \int d\Phi d\overline{\Phi} \, O(\Phi, \overline{\Phi}) \exp \{-\Phi \cdot C^{-1} \cdot \Phi + S_{\text{int}}(\Phi, \overline{\Phi})\},
$$

where $Z = \int d\Phi d\overline{\Phi} \exp \{-\Phi \cdot C^{-1} \cdot \Phi + S_{\text{int}}(\Phi, \overline{\Phi})\}$ is the partition function. $C$ is the covariance and $S_{\text{int}}$ the interacting part of the action. Because the tensor is complex, it is convenient to consider its complex conjugate $\overline{F}_{i_1,\ldots,i_D}$ as an independent variable. As a convention, we always denote by a bar the indices of $i$, and another index independent of $i$.

The precise form of the argument of the exponential is dictated by the fact that there are particularly simple tensor models which are the Gaussian ones. The latter are defined by retaining only quadratic terms in the argument of the exponential. Introducing a multi-index $I = (i_1,\ldots,i_D)$, the latter are defined by the quadratic term

$$
\Phi \cdot C^{-1} \cdot \Phi = \sum_{i,j} C_{ij} \Phi_i \Phi_j,
$$

and

$$
S_{\text{int}}(\Phi, \overline{\Phi}) = \sum_{i,j} S_{ij} \Phi_i \Phi_j + \sum_{i,j} T_{ij} \Phi_i \overline{\Phi}_j + \sum_i U_i \Phi_i
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In the Gaussian case, the expectation value of products of tensors is evaluated using Wick’s theorem

\[ N^{-1} \int d\bar{\Phi} d\Phi \exp \left\{ -\Phi \cdot C^{-1} \cdot \Phi \right\} \Phi_{i_1} \cdots \Phi_{i_p} \bar{\Phi}_{j_1} \cdots \bar{\Phi}_{j_p} = \sum_{\sigma \in \mathfrak{S}_n} C_{\sigma} \cdot \Phi_{\sigma(i_1)} \cdots \Phi_{\sigma(i_p)}, \tag{3} \]

and vanishes if there is not an equal number of $\Phi$ and $\bar{\Phi}$. $\sigma \in \mathfrak{S}_n$ is a permutation of $n$ elements, i.e. a bijection from $\{1, 2, \ldots, n\}$ onto itself. $N = \int d\Phi d\bar{\Phi} \exp \left\{ -\Phi \cdot C^{-1} \cdot \Phi \right\}$ is a normalisation factor and the positive hermitian matrices $C_{ij}$ and $C_{ij}^{-1}$ are inverse of the other. Note that the covariance $C$ is enough to define a Gaussian integration and the latter makes sense even if $C$ is not invertible, tensors in the kernel of $C_{ij}$ are simply set to 0. This can be easily done by adding a $\varepsilon > 0$ to $C$ to get $C_{ij} + \varepsilon \delta_{ij}$ instead of $C_{ij}$. In that case, $\varepsilon$ can be chosen in such a way that $C_{ij} + \varepsilon \delta_{ij}$ is invertible. Then, the limit $\varepsilon \to 0$ provides the Gaussian integration formula (3), whether $C$ is invertible or not.

We are interested in perturbations of a Gaussian measure using a suitable class of functions $S_{\text{int}}(\bar{\Phi}, \Phi)$ that can be constructed using bubbles. Recall that for a rank $D$ tensor, a bubble $B$ is $D$-colored bipartite graph. This means that there is an equal number of white and black vertices and that any edge connects a black vertex and a white vertex. Moreover, we assume that exactly $D$ edges are incident to each vertex and that every edge can be assigned a color in $\{1, \ldots, D\}$ in such a way that all the edges incident to a given vertex carry different colors. Note that we do not impose any connectivity requirements on the bubbles $B$. Refer to figure 1 for examples of such bubbles. Then, the action is expanded over bubbles as

\[ S_{\text{int}}(\bar{\Phi}, \Phi) = \sum_B \text{Tr}_{G} \lambda_B(\bar{\Phi}, \Phi) \]

where the sum runs over the pair of indices $\{i_1, \ldots, i_p\}$ ranging from 1 to $N$ for each edge of $B$. $I_B(\nu)$ (respectively $I_B(\tau)$) is the $D$-tuple of indices $\{i_1\}$ (respectively $\{\bar{i}_1\}$) pertaining to the lines of color 1, $\ldots$, $D$ attached to the white vertex $\nu$ (respectively black vertex $\tau$). The symmetry factor $\sigma_B$ is equal to the order of the symmetry group of $B$ (transformations of edges and vertices that preserve incidence relations and colorings) and is included for further convenience. Note that we generically consider bubbles differing solely by their color assignment as different. If not, we can always collect contributions from such bubbles into a single term. Finally, $\lambda_B(\{i_1, \ldots, i_p\})$ plays a role analogue to a coupling constant. The notation suggests that this generalises the expansion of a matrix interaction over traces of products of $MM'$. If we require the action to be invariant under the $U(N)^D$ transformation

\[ \bar{\Phi}_{h, \ldots, h_p} \rightarrow \sum_{j_1, \ldots, j_p} U^{1}_{j_1 h_1} \cdots U^{D}_{j_p h_p} \bar{\Phi}_{j_1, \ldots, j_p} \quad \Phi_{h, \ldots, h_p} \rightarrow \sum_{j_1, \ldots, j_p} U^{1}_{j_1 h_1} \cdots U^{D}_{j_p h_p} \Phi_{j_1, \ldots, j_p}, \tag{5} \]

then the couplings must be expressed in terms of the unique rank 2 invariant tensor, $\lambda_B(\{i_1\}) = \lambda_B \prod_{i} \delta_{i_1 i_2}$, with $\lambda_B$ a scalar. This is the conventional setting for random tensors which we refer to as the invariant random tensors. Our generalisation has been introduced to encompass the case of a covariance that depends on indices (with a view to write a renormalisation group equation) and to extend easily to tensorial field theories (see section 5). Let us sketch how non-Gaussian integral can be expanded as a perturbation of the Gaussian one over $(D + 1)$-colored bipartite graphs $G$ as...
This is obtained by expanding \( \exp S_{\text{int}}(\overline{\Phi}, \Phi) \) and using Wick’s theorem (3). Each occurrence of a covariance between two tensors \( \overline{\Phi} \) and \( \Phi \) leads to a new color 0 line which is only a half-edge if one of the tensors pertains to the \( n \)-point function observables \( \overline{\Phi}_{L_1} \ldots \overline{\Phi}_{L_{D-1}} \). The color 0 edges connect with covariances \( C \) the various bubbles involving colors \( \{1, \ldots, D\} \) so that we obtain a graph with \( D + 1 \) colors and, possibly, external color 0 edges (see figure 2 for an example).
Finally, we sum over all indices of the bubbles $\mathcal{B}$ and covariances following the graph $\mathcal{G}$, with the indices of the half-edges held fixed. It is nothing but the conventional Feynman graph expansion adapted to random tensors. This is the initial formulation of colored models \cite{3} while the formulation we use has been introduced in \cite{16}.

This expansion can be given the following simplicial interpretation, useful in the context of discretised approaches to quantum gravity and random geometry. Each bubble $\mathcal{B}$ with $D$ colors represents a triangulation of dimension $D - 1$ (space) whose simplices of dimension $D - 1$ are the vertices of $\mathcal{B}$ attached along their $D - 2$ faces following the edges of $\mathcal{B}$. Similarly, a graph $\mathcal{G}$ with $D + 1$ colors represents a triangulation of dimension $D + 1$ (space–time). Finally, the contraction of two vertices $\nu$, $\nu'$ in $\mathcal{G}$ joined by a line of color 0 is defined by removing the color 0 line, $\nu$, and $\nu'$ and joining the remaining half edges of the same colors in 1, \ldots, $D$. Contracting all the pairs of vertices joined by a color 0 line and removing the color 0 half edge, we obtain a bubble $\mathcal{B}$ with $D$ colors. At the level of triangulations, this bubble corresponds to the boundary, $\mathcal{B} = \partial \mathcal{G}$.

### 3. Exact renormalisation group for tensors

In complete analogy with the case of quantum field theory, let us define the effective action for a tensor model as
\[ S_{t_0}(\Xi, \bar{\Xi}) = \log \int \frac{d\Xi d\bar{\Xi}}{N_{t_0}} \exp \{-\Xi \cdot C_{t_0}^{-1} \cdot \Xi + S_0(\Xi + \bar{\Xi}, \Xi + \bar{\Xi})\}, \quad (7) \]

with \( N_{t_0} = \int d\Xi d\bar{\Xi} \exp \{-\Xi \cdot C_{t_0}^{-1} \cdot \Xi\} \) a suitable normalisation factor. The parameter \( t \) allows us to follow the flow of effective actions, starting from a bare action \( S_0 \) at \( t_0 \). In conventional QFT, \( t = \log(\Lambda/\mu) \), where \( \Lambda \) is a varying cut-off and \( \mu \) is a given reference scale. We assume that the covariance can be written as \( C_{t_0} = \int_{t_0}^{\infty} dt' K_t \) so that the semi-group property holds, and

\[ S_{t_0}(\Xi, \bar{\Xi}) = \log \int \frac{d\Xi d\bar{\Xi}}{N_{t_0}} \exp \{-\Xi \cdot C_{t_0}^{-1} \cdot \Xi + S_{t_0}(\Xi + \bar{\Xi}, \Xi + \bar{\Xi})\}, \quad (8) \]

with \( C_{t_0} = C_{t_0} - C_{t_0} \). In particular, \( S_0 = S_0 \) can be identified with the bare action in quantum field theory and \( S_t \) is an effective action obtained by a partial integration over some modes of the tensors. The semi-group property means that integrating from \( t_0 \) to \( t_2 \) with covariance \( C_{t_0} \) and the action \( S_{t_0} \) is equivalent to integrating from \( t_1 \) to \( t_2 \) with covariance \( C_{t_0} \) and the effective action \( S_{t_0} \).

The effective action obeys the following differential equation, which is an infinitesimal version of the semi-group equation with \( t_1 = t \) and \( t_2 = t + \delta t \),

\[ \frac{\partial S_{\text{int}}}{\partial t} = \sum_{t_1 \leq t} K_{t,t} \left( \frac{\partial S_{\text{int}}}{\partial \Xi_t} \frac{\partial S_{\text{int}}}{\partial \bar{\Xi}_t} + \frac{\partial^2 S_{\text{int}}}{\partial \Xi_t \partial \bar{\Xi}_t} \right), \quad (9) \]

where we have dropped the explicit dependence of \( S_{\text{int}} \) on \( t \) and \( t_0 \) and of \( K \) on \( t \).

This is a straightforward adaptation of the Polchinski’s equation in quantum field theory (see [8]) where we have used the fact that \( \frac{\partial C_{t}}{\partial t} = K_t \). Let us also note that there is no factor \( \frac{1}{2} \) because we are working with a complex tensor. Figure 3 represents a graphical notation of such Polchinski’s equation.

At a formal perturbative level, this differential equation together with the boundary condition \( S_{t_0} = S_0 \) is equivalent to the integral representation (7). In order to use it in an efficient way, it is necessary to translate it into a system of differential equation for the bubble couplings \( \lambda_b(\{l_i, l_j\}) \).

To this aim, let us define for \( k \in \{0, \ldots, D\} \) a \( k \)-cut \( c \) in a bubble \( \mathcal{B} \) as a subset of edges \( \{e_1, \ldots, e_k\} \) of \( \mathcal{B} \) with different colors. The cut bubble \( \mathcal{B}_c \) is the bubble obtained from \( \mathcal{B} \) by cutting the \( k \) edges \( \{e_1, \ldots, e_k\} \) into half-edges, attaching to them a new black \( \tau \) and white vertex \( v \) and joining \( v \) and \( \tau \) by \( D - k \) edges carrying the colors not in \( \{e_1, \ldots, e_k\} \) (see figure 4).

This ensures that \( \mathcal{B}_c \) is a bubble with \( D \) colors. In particular, if \( c \) is a 0-cut, \( \mathcal{B}_c \) is just the disjoint union of \( \mathcal{B} \) with a dipole. A 1-cut on an edge \( e \) is just the insertion on \( e \) of a pair of vertices joined by \( D - 1 \) edges carrying the colors different from that of \( e \). Substituting the bubble expansion (4) into the flow equation (9), we arrive at the system of differential equations governing the bubble couplings

\[ \frac{\partial \lambda_b(\{l_i, l_j\})}{\partial t} = \sum_{k=0}^{D} \sum_{k \text{-cut } c} \sum_{\{j, j\}} K(\{j, j\}) \lambda_b(\{\{l_i, l_j\}, e_b, \{j, j\}\}) \]

\[ + \sum_{(c,c_c)} \sum_{\{j, j\}} K(\{j, j\}) \lambda_{b'}(\{\{l_i, l_j\}, e_{b'}, \{j, j\}\}) \]

\[ \times \lambda_{b''}(\{\{l_i, l_j\}, e_{b''}, \{j, j\}\}). \quad (10) \]
In this equation, the sum runs over cuts $c$ and $D$ pairs of tensor indices $\{j_i, j'_i\}_{1 \leq i \leq D}$. The tensor indices attached to the bubble $B$ are naturally attached to corresponding half-edges in $B_c$ and are completed by the new indices $\{j_i, j'_i\}_{1 \leq i \leq D}$. The second term involves a summation over $D$-cut that increases the number of connected components $\kappa$ and over ways of writing $B_c$ as a disjoint union of a bubble $B'$ containing $v$ and a bubble $B''$ containing $\bar{v}$. In the equation, the bubbles $B' - c$ and $B'' - c$ are the bubbles $B'$ and $B''$ with the edge $c$ removed.

These equations have been derived in a very general framework, not assuming unitary invariance. This may be useful to implement a renormalisation group idea in the context of tensor models, if they are assumed to distinguish between ‘high’ and ‘low’ modes.

4. Gaussian limit at large $N$ limit revisited

As an application of the previous formalism, let us show how it can be used to give another proof of Gurau’s Gaussian universality result, see [5]. It states that at leading order in $N$ when $N \to \infty$, the expectation value of a bubble in a random tensor model for $D > 2$ is Gaussian

$$\langle \text{Tr}_g(\Psi, \bar{\Psi}) \rangle = \frac{1}{Z} \int d\Psi d\bar{\Psi} \text{Tr}_g(\Psi, \bar{\Psi}) \exp \{-N^{D-1}\Psi \cdot \bar{\Psi} + S_0(\Psi, \bar{\Psi})\}$$

$$= c_G N^{\frac{\nu_g}{2D^2}} G_T^2 + \cdots, \quad (11)$$

where … represents subdominant terms when $N \to \infty$ and $\nu_g$ the number of vertices of $B$. Here, we work with unitarily invariant random tensors so that the action only involves invariants

$$S_0(\Psi, \bar{\Psi}) = \sum_B N^{D-\kappa_B} \frac{\lambda_B^0}{\sigma_B^0} \text{Tr}_g(\Psi, \bar{\Psi}), \quad (12)$$

where $\lambda_B^0$ is a set of parameters necessary to define $S_0$, and $\kappa_B$ is the number of connected components of $B$ and the invariants are defined by

$$\text{Tr}_g(\Psi, \bar{\Psi}) = \sum_{(I, I')} \prod_e \delta_{i_e, i'_e} \prod_v \Psi_{v_i} \prod_{\tau} \bar{\Psi}_{\tau_j} \quad (13)$$

Because of unitary invariance, the couplings do not depend on edge indices $\{i_e, i'_e\}$. The large $N$ behavior singles out the so-called melonic bubbles which may be defined as follows: a bubble is melonic if for every white vertex $v$, there is a black vertex $\bar{v}$ such that removing $v$ and $\bar{v}$ increases the number of connected components by $D - 1$. For melonic bubbles, $c_g = 1$ and $\sigma_g = \kappa_B$ while for non-melonic bubbles we have $\sigma_g < \nu_g$. Finally, $G$ only depends on the melonic couplings in $S_0$ and can be obtained by solving the equation

$$1 - G = -\sum_B \frac{\nu_g \lambda_B^0}{2\sigma_B^0} G_T^2, \quad (14)$$

where we denote the effective covariance to be $G$ and the sum runs over all melonic bubbles in $S_0$. This result can be derived from the generating function defined by

$$Z(J, \bar{J}) = \frac{1}{Z} \int d\Psi d\bar{\Psi} \exp \{-N^{D-1}\Psi \cdot \bar{\Psi} + S_0(\Psi, \bar{\Psi}) + \bar{J} \cdot \bar{\Psi} + \Psi \cdot J\} \quad (15)$$

where $J$ is also a rank $D$ complex tensor. It is normalised in such a way that $Z(0, 0) = 1$.

Using the effective action and its flows, we give an alternative proof of these results. We show that a leading order in $N$ this generating function is Gaussian.
where the dots represent subdominant terms when $N \to \infty$ and $G$ only depends on the melonic couplings $\lambda_{\beta}^G$. The cornerstone of our proof is the system of differential equations for the bubble couplings, (21).

To begin with, let us first show that this generating function is related to the effective action by a simple change of variables. We define an effective action with a diagonal covariance $C = \frac{\tau}{N^{D-1}} \delta_{ij}$ and set $t_0 = 0$ for simplification. Thus, the effective action, normalised by the free energy $F = \log Z$ reads

$$
\int d\Phi d\Phi \exp \left\{-\frac{N^{D-1}}{t} \frac{\Phi}{N^{D-1}} \cdot \Phi + S_0(\Phi) + \Phi(\Phi + \Phi + \Phi) \right\}.
$$

(17)

In this section, the parameter $t$ controls the flow of effective actions and does not directly refer to a cut-off, since there is no such notion of high or low momenta in this example (see section 5) for a field theoretical example, where the flow is controlled by a true cut-off.

Then, let us shift the integration variables $\Phi \to \Phi - \Phi$ and $\Phi \to \Phi - \Phi$ so that we get

$$
\int d\Phi d\Phi \exp \left\{-\frac{N^{D-1}}{t} \frac{\Phi}{N^{D-1}} \cdot \Phi + S_0(\Phi, \Phi) + \frac{N^{D-1}}{t} \frac{\Phi}{N^{D-1}} \cdot \Phi + \frac{N^{D-1}}{t} \Phi \cdot \Phi \right\}.
$$

(18)

In this equation, the parameter $t$ is just a bookkeeping device, the term of order $t^n$ contains the contribution of all connected graphs $G$ with $n$ edges.

Introducing the sources $J = \frac{\Phi}{N^{D-1}}$ and $\Phi = \frac{\Phi}{N^{D-1}}$, the effective action at $t = 1$ is related to the generating function by

$$
Z(J, \Phi) = \exp \left\{ S_{t=1} \left( \frac{J}{N^{D-1}}, \frac{\Phi}{N^{D-1}} \right) - S_{t=1}(0, 0) + \frac{1}{N^{D-1}} J \cdot \Phi \right\}.
$$

(19)

In order to understand the large $N$ behavior of the effective action, it is convenient to expand it over trace invariants as in (12)

$$
S_{t}(\Phi, \Phi) = \sum_{g} N^{D-k_{g}} \frac{\lambda_{g}(t)}{\sigma_{g}} \text{Tr}_{g}(\Phi, \Phi).
$$

(20)

The couplings $\lambda_{g}(t)$ obey a system of differential equations that can be deduced from (9). Because of unitary invariance, it greatly simplifies and the sum over indices reduces to a factor of $N^{D-k}$ for a $k$-cut. Moreover, our normalisation in powers of $N$ of the covariance and of the couplings allows us to have only constant or negative powers of $N$. Thus, the evolution of a couplings reads

$$
\frac{\partial \lambda_{g}}{\partial t} = \sum_{i=0}^{D} \sum_{k \text{-cut } c} N^{k_{g} - k} \lambda_{R_{i}} + \sum_{D \text{-cut } c} \sum_{k(\beta) > k(\beta)} \lambda_{\beta'} \lambda_{\beta''},
$$

(21)

where $\sigma_{g,c}$ is defined as the number of connected components of $\beta$ containing edges of the cut, except for a $D$-cut that increases the number of connected components of $\beta$, in which
case \( \kappa_{B, c} = 0 \). We always have \( k \geq \kappa_{B, c} \) and \( k \geq \kappa_{C, c} \) if only all edges of the cut belong to different connected components of \( \mathcal{B} \). In particular, this holds for a 0-cut and a 1-cut.

There are two important consequences of this result. First, because the exponent of \( N \) is negative or 0, the couplings \( \lambda_B \) only contain constants or negative powers of \( N \), so that they remain bounded as \( N \to \infty \). Therefore, as \( N \to \infty \), only the quadratic term survives in

\[
\exp \left\{ \frac{G}{N^{D-1}} \mathcal{T} \cdot \mathcal{T} + \cdots \right\},
\]

where as usual the dots stand for subleading powers of \( N \). Comparing both sides shows that the effective covariance reads

\[
G = 1 + \lim_{N \to \infty} \lambda_{\sim} (t = 1).
\]

Second, consider the large \( N \) limit in (21)

\[
\frac{\partial \tilde{\lambda}_B}{\partial t} = \sum_{k=0}^{D} \sum_{\text{k-cut } c} \tilde{\lambda}_{B, c} + \sum_{\mathcal{E}, \mathcal{E}' \in \text{Cut } c} \sum_{\mathcal{E} \subset \mathcal{E}', \mathcal{E} \cup \mathcal{E}' \text{ is connected}} \tilde{\lambda}_{B, c} \tilde{\lambda}_{B', c'},
\]

where \( \tilde{\lambda}_B = \lim_{N \to \infty} \lambda_B \). We refer to appendices A.1 and A.2 for a few low order examples in \( D = 3 \) and \( D = 4 \).

Since we are interested in the dipole coupling \( \lambda_{\sim} \), we may start solving this system. Even if this involves an infinite number of equations, at a given order in the initial conditions \( \lambda_B (0) = \lambda^0_B \), there are only finitely many terms involved if the action \( S_0 \) we start with contains a finite number of bubble couplings.

It is instrumental to notice that starting with the dipole coupling \( \lambda_{\sim} \), only melonic couplings are involved at leading order in \( 1/N \). This is easily established by induction: if \( \mathcal{B} \) is melonic, so are \( \mathcal{B}', \mathcal{B}' \) and \( \mathcal{B}' \). Therefore, at leading order in \( 1/N \), \( \lambda_{\sim} (t) \) only depends on the initial couplings \( \lambda_B (0) \) for \( \mathcal{B} \) melonic. Then, the expectation value of an arbitrary bubble can be derived from the generating function using the identity

\[
\langle \text{Tr}_B(\mathcal{G}, \mathcal{G}) \rangle = \text{Tr}_B \left( \frac{\partial}{\partial \mathcal{J}} \right) \left( \frac{\partial}{\partial \mathcal{J}} \right) \langle Z(\mathcal{J}, \mathcal{J}) \rangle \bigg|_{\mathcal{J} = \mathcal{J} = 0}.
\]

Because at leading order \( Z(\mathcal{J}, \mathcal{J}) \) is Gaussian, this simply amounts to sum over all possible Wick contraction on the invariant \( \text{Tr}_B(\mathcal{G}, \mathcal{G}) \) with a covariance \( G \). It leads to

\[
\langle \text{Tr}_B(\mathcal{G}, \mathcal{G}) \rangle = c_B N^{\kappa_B} \left( \frac{G}{N^{D+1}} \right)^{\frac{D}{2}},
\]

where \( c_B \) is the number of Wick contraction giving a leading order contribution of \( N^{\kappa_B} \). At fixed number of vertices \( v_B \), \( c_B \) is maximal for a melonic bubble. This is established by contracting pairs of conjugate vertices in a melonic bubble. This unique Wick contraction leads to \( \kappa_B = (D - 1) \frac{D}{2} + \kappa_B \), which provides the announced scaling law for a melonic bubble given in (11).
Finally, let us compute the effective covariance $G$. It is known that this can be achieved using the Schwinger–Dyson equations [17]. Alternatively, one may observe that, using the flow equation (24) that $\frac{\partial \lambda_\theta}{\partial t} = \lambda_{\theta \theta}$. The empty bubble (with $\kappa_\theta = 0$) is related to the free energy

$$\mathcal{N}_0 \lambda_\theta = S_0(0, 0),$$

$$= \log \int d\Psi d\overline{\Psi} \exp \left\{ - \frac{\mathcal{N}_0^{-1}}{t} \Psi \cdot \Psi + S_0(\Psi, \overline{\Psi}) \right\}$$

$$- \log \int d\Psi d\overline{\Psi} \exp \left\{ - \frac{\mathcal{N}_0^{-1}}{t} \Psi \cdot \Psi \right\},$$

where the last term arises from the normalisation. By rescaling the $\Psi$ and $\overline{\Psi}$, one immediately shows that $S_0(0, 0)$ is invariant under the rescaling of $t \rightarrow s^{-1} t$ and of the couplings in $S_0$. $\lambda_0^0 \rightarrow s^{v_0} \lambda_0^0$. This leads to the differential equation

$$-2t \frac{\partial S_0(0, 0)}{\partial t} + \sum_B v_B \lambda_B^0 \frac{\partial}{\partial \lambda_B^0} S_0(0, 0) = 0$$

which is equivalent to

$$-2t \mathcal{N}_0 \lambda_{\theta \theta}(t) + \sum_B v_B \lambda_B^0 \frac{\partial}{\partial \lambda_B^0} \mathcal{N}_0^{D-\kappa_B} \langle \text{Tr}_B(\Psi, \overline{\Psi}) \rangle = 0.$$  

Then, using (26), we obtain at $t = 1$ and for $N \rightarrow \infty$

$$G - 1 - \sum_B v_B \frac{\lambda_B^0}{2 \sigma_B} G_{B^g} = 0,$$

where we have discarded non-melonic couplings in the large $N$ limit since they are non-leading. This equation can be used to determine $G$ as a formal power series in the bubble couplings $\lambda_B^0$.

The arguments we have presented hold for $D > 2$ only. In the matrix model case, the situation is more involved. Indeed, even if (24) still holds and the generating function $Z(J, \overline{J})$ is still Gaussian, the expectation value of a bubble is no longer given by Wick contrition only. Indeed, in this case the differential operator in (25) can generate higher powers of $N$ in such a way that non-quadratic terms admit a finite limit.

5. Group field theory

Group field theories are specific quantum field theories defined on $D$ copies of a group, usually SU(2), U(1)$^d$, $\mathbb{R}^d$ or SL$_2(\mathbb{C})$. In quantum gravity applications, $D$ denotes the space-time dimension and $d$ is the dimension of the group. Group field theories are constructed from tensor models by replacing tensors by tensorial fields

$$\Phi_{b_1 \ldots b_d} \rightarrow \Phi(g_{11}, \ldots, g_{d1})$$

$$\overline{\Phi}_{b_1 \ldots b_d} \rightarrow \overline{\Phi}(g_{12}, \ldots, g_{d2}),$$

(32)

Note that $\overline{g}$ is not the complex conjugate of $g$ but an independent group element that is an argument of the complex conjugate field $\overline{\Phi}$.

The effective action is expanded over bubbles as in (4), except that indices are replaced by group elements and sums by integration over the group using the Haar measure.
where capital letters $G_B(v)$ and $\overline{G}_B(\tau)$ indicate the $D$-tuplets of group elements (ordered by their colors) incident to $v$ and $\tau$ in $\mathcal{B}$. We also assume that the couplings and the covariance only depend on the products $g_e \overline{g}_e^{-1}$. In some cases it is also assumed that the action and the propagator are invariant under the gauge transformations $g_e \rightarrow h_e g_e$ and $\overline{g}_e \rightarrow \overline{h}_e \overline{g}_e$, where $v$ (respectively $\tau$) is the white vertex (respectively black vertex) attached to $e$. This translates into the following conditions obeyed by the covariance and by the couplings 

$$C((g_e \overline{g}_e^{-1})) = C((h_e g_e \overline{g}_e^{-1} \overline{h}_e^{-1})), \quad \lambda_B((g_e \overline{g}_e^{-1})) = \lambda_B((h_v g_e \overline{g}_e^{-1} \overline{h}_e^{-1})),$$

where $v_B(e)$ (respectively $\overline{v}_B(e)$) is the white (respectively black) vertex of $B$ attached to $e$. Finally, it is convenient to write the covariance as the integral of a product of $D$ averaged heat kernels $H_B$ on the group. We also use a more conventional notation for quantum fields, writing $t = \log(\Lambda/\mu)$ and interpreting $\Lambda$ as a momentum space cut-off. The covariance reads

$$C_{\Lambda, \Lambda}((g_e \overline{g}_e^{-1})) = \int \frac{d\alpha}{\Lambda^D} K_{\alpha}((g_e \overline{g}_e^{-1})),
\text{with} \quad K_{\alpha}((g_e \overline{g}_e^{-1})) = \int d\theta \overline{d}\tau \prod_{v \in B} H_B(h_v g_e \overline{g}_e^{-1} \overline{h}_e^{-1}).$$

In the last equation, $H_B(g)$ refers to the heat kernel on the group at time $\alpha$. It is the solution of the heat equation $\frac{\partial H_B}{\partial \alpha} = \Delta H_B$ such that $\lim_{\alpha \rightarrow 0} H_B(g) = \delta(g)$, with $\delta$ the Dirac distribution concentrated at the identity. Then, the exact renormalisation group equation for the effective action can be written as

$$\Lambda \frac{\partial S_{\text{int}}}{\partial \Lambda} = -\frac{2}{\Lambda^D} \int \prod_{1 \leq e \leq D} \text{d}g_e \text{d}\overline{g}_e K_{\frac{1}{\Lambda}}((g_e \overline{g}_e^{-1})) \left( \frac{\partial S_{\text{int}}}{\partial \overline{\mathcal{I}}(G)} \frac{\partial S_{\text{int}}}{\partial \mathcal{I}(G)} + \frac{\partial^2 S_{\text{int}}}{\partial \mathcal{I}(G) \partial \overline{\mathcal{I}}(G)} \right).$$

In the SU(2) case which is involved in many spin-foam models, it is possible to expand all the bubble couplings on spin networks, see [9], rather than just bubbles.

In this article, we are mostly interested in deriving scaling properties of the bubble couplings from the exact renormalisation group equation. For simplicity, we restrict our attention to Abelian models. Elements of $U(1)$ are written as $g = e^{2\pi i \mathcal{F}}$ with $\mathcal{F} \in \{0, L\}$, with $L$ being a length. The propagator is given by

$$C_{\Lambda, \Lambda}((\theta_e - \overline{\theta}_e)) = \int \frac{1}{\Lambda^D} \text{d}\mathcal{F} \sum_{|\mathcal{F}| \in \mathbb{Z}^D} \exp \left\{ \alpha \sum_i \mathcal{F}_i^2 + i \sum_i \mathcal{F}_i (\theta_i - \overline{\theta}_i) \right\} \delta_{\sum_i \mathcal{F}_i, 0},$$

where we have enforced the condition $\sum_i \mathcal{F}_i = 0$ so that the theory is invariant under the gauge transformation of the field, $\mathcal{I}(g_1, \ldots, g_L) \rightarrow \mathcal{I}(h_1 g_1, \ldots, h_L g_L)$ and $\overline{\mathcal{I}}(g_1, \ldots, g_L) \rightarrow \overline{\mathcal{I}}(h_1 g_1, \ldots, h_L g_L)$.

At first sight, the bubble couplings depend on $D$ group elements for each bubble vertex, so that there is a total of $v_D D$ group elements. It is also possible to consider that there are two group elements $\theta_e$ and $\overline{\theta}_e$ per edge. Then, the bubble couplings only depend on the differences, $\lambda_B((\theta_e - \overline{\theta}_e))$. By Fourier transform, there is a single momentum per edge (because only the differences $\theta_e - \overline{\theta}_e$), which we conventionally orient from the black to white vertices.
Therefore the bubble couplings are denoted by $\lambda_B(\{p_i\})$. Furthermore, gauge invariance is equivalent to the constraint $\sum_{\text{incident}} p_e = 0$ that has to be implemented at every vertex $v$.

The exact renormalisation group equation written in terms of bubble couplings is

$$
\Lambda \frac{\delta \lambda_B(\{p_i\})}{\delta \Lambda} = -\frac{2}{N^2} \sum_{k=0}^{D} \sum_{k \text{-cut } \{p_i\} \notin \mathcal{B}_k} \sum_{\mathcal{B} \ni \{p_i\}} \delta \sum_{\mathcal{E} \ni \{p_i\}} + \sum_{\mathcal{E} \ni \{p_i\}} \frac{\delta}{\delta \sum_{\mathcal{E} \ni \{p_i\}}} \lambda_B(\{\{p_i\}\} \in \mathcal{B}_k, \{p_i\} \notin \mathcal{B}_k) \times \epsilon \frac{\delta}{\delta \sum_{\mathcal{E} \ni \{p_i\}}} \lambda_B(\{\{p_i\}\} \in \mathcal{B}_k, \{p_i\} \notin \mathcal{B}_k) - \frac{2}{N^2} \sum_{\mathcal{B} \ni \{p_i\}} \sum_{\mathcal{B} \in \mathcal{B}^\prime} \sum_{\mathcal{B}^\prime \notin \mathcal{B}^\prime} \delta \sum_{\mathcal{E} \ni \{p_i\}} + \sum_{\mathcal{E} \ni \{p_i\}} \frac{\delta}{\delta \sum_{\mathcal{E} \ni \{p_i\}}} \lambda_B^\prime(\{\{p_i\}\} \in \mathcal{B}^\prime, \{p_i\} \in \mathcal{B}^\prime) \lambda_B^\prime(\{\{p_i\}\} \in \mathcal{B}^\prime, \{p_i\} \in \mathcal{B}^\prime). \quad (38)
$$

As in (10), we perform $k$-cuts with $k \in \{0, \ldots, D\}$. The first term generates the analogue of the loops in QFT. It involves a summation over the momenta associated with the colors not in the cut. The second one corresponds to the QFT trees and involves $D$-cuts that separate the bubbles into $\mathcal{B}$ and $\mathcal{B}^\prime$. Note that this is nothing but the tensorial exact renormalisation group equation (10), with the constraint that the 2 edge indices coincide, $i_e = \tilde{i}_e = p_e$ and take their values in $\mathcal{D}_L$.

In the last part of this section, we aim at addressing the issue of the renormalisability of the model, from the point of view of the exact renormalisation group equation (38). This implies the need of the definition of a proper scaling dimension of the bubble coupling, which we first present from a heuristic point of view. To ease the discussion, we assume $L$ to be large enough ($L \ll \Lambda$) so that momenta can be treated as continuous variables, then we can trade sums for integrals

$$
\sum \rightarrow \int d \mathbf{p} \quad \text{and} \quad \delta \sum_{\mathcal{E} \ni \{p_i\}} \rightarrow \frac{1}{L^d} \delta \left( \sum \mathbf{p} \right). \quad (39)
$$

Otherwise, a more precise but cumbersome analysis can be performed using the Poisson resummation. Finally, we also allow derivatives of the bubble couplings with respect to momenta and rotational invariance in momentum space. This last hypothesis is fulfilled if we assume that the initial couplings are invariant under rotation since the propagator is.

In the remainder of this section, we introduce the canonical dimension for bubble couplings. We motivate our choice by a heuristic analysis. This is later justified by the fact that dimensionless couplings obey an equation with only negative powers of the cut-off, see equation (45).

The normalisation of the effective action at $\varphi = \varphi = 0$ involves a subtraction of the divergent quantity $\log \det G_{\Lambda, \Lambda}$, which behaves for large $\Lambda$ as $(L \Lambda)^{d(D-1)}$ up to an additive constant, as is easily seen using a free field theory. Therefore, it is natural to assume that the action behaves as $S(0, 0) \sim (L \Lambda)^{d(D-1)}$ for large $\Lambda$, up to a multiplicative constant. As far as the scaling of $\Lambda$ is concerned, we attribute to the action the scaling dimension $[S] = d(D - 1)$, even if it is dimensionless (i.e. invariant if we rescale both $\Lambda \rightarrow s \Lambda$ and $L \rightarrow L/s$).

Moreover, our procedure involves integrations over shells of momenta $p \sim \Lambda$, so that $[p] = 1$. Then, the dimension of the field can be obtained by examining the quadratic terms

$$
[S] = d(D - 1) + 2 + 2[\varphi]. \quad (40)
$$
where we recall that there are $dD - d = d(D - 1)$ momenta ($D$ momenta in $\mathbb{Z}^d/L$ and one constraint). Out of the three terms on the rhs, the first term is the dimension of the integration over momenta and the second term is the dimension of the Laplacian. Because we also have $[S] = d(D - 1)$, the scaling dimension in $\Lambda$ of the field is $[\Phi] = -1$. This does not mean that $[\Phi]$ has the same units as $1/\Lambda$, but rather that we should scale $\Phi$ as $\Lambda^{-1}$ if we want to have a well defined limit for large $\Lambda$ at fixed $L$.

The dimension of a bubble coupling obeys a similar equation

$$[S] = \delta_B + d(e_B - \nu_B + \kappa_B) + \nu_B[\Phi],$$

(41)

where $\delta_B = [\lambda_B]$ is the dimension of the bubble coupling $\lambda_B$. We denote by $e_B$ the number of edges, by $\nu_B$ the number of vertices and by $\kappa_B$ is the number of connected components of $B$. $(e_B - \nu_B + \kappa_B)$ represents the number of independent integrations over momenta in $\mathbb{R}^d$ since there is one such momentum for each edge, one constraint at each vertex and a remaining constraint for each connected component. Taking into account the relation $2e_B = D\nu_B$, it leads to the following scaling dimension for a bubble coupling

$$\delta_B = d(D - 1) - d\nu_B - (d(D - 2) - 2)\nu_B/2.

(42)

Since a bubble coupling depends on several momenta, it is helpful to perform a Taylor expansion to order $n$. This leads to a sum of homogenous polynomials of total degree less than $n$.

$$\lambda_B([p_i]) = \sum_n \lambda_{B,n} \times \text{homogeneous polynomial of degree } n \text{ in momenta}. \quad (43)$$

Accordingly, the dimension of coefficients of the Taylor expansion are

$$\lambda_B([p_i]) = \sum_n \lambda_{B,n} \times \text{homogeneous polynomial of degree } n \text{ in momenta}. \quad (43)$$

Note that the couplings depend explicitly on the scale $\Lambda$ and on momenta. In term of these new couplings, the exact renormalisation group equation (38) takes the form

$$\Lambda \frac{\partial u_B([q_i])}{\partial \Lambda} = -\delta_B u_B([q_i]) + \sum q_e \frac{\partial u_B([q_i])}{\partial q_e}$$

$$-2 \sum_{k=0}^D \sum_{L \text{-cut } e \nu_B} \frac{L^{d(D-k-1)}}{N^{d-\nu_B}} \prod_{i \in e} dq_i \left( \sum_{q_i} + \sum_{q_e} \right) e^{-\left( \sum_{i \in e} q_i^2 + \sum_{e \in \bar{e}} q_e^2 \right)} u_{\bar{B}}\left( \{q_i\}_{i \in \bar{B}}, \{q_j\}_{j \in \bar{e}} \right)$$

$$-2 \sum_{k=0}^D \sum_{L \text{-cut } e \nu_B} \frac{L^{d(D-k-1)}}{N^{d-\nu_B}} \prod_{i \in e} dq_i \left( \sum_{q_i} + \sum_{q_e} \right) e^{-\left( \sum_{i \in e} q_i^2 + \sum_{e \in \bar{e}} q_e^2 \right)} u_{\bar{B}}\left( \{q_i\}_{i \in \bar{B}}, \{q_j\}_{j \in \bar{e}} \right) u_B\left( \{q_i\}_{i \in B}, \{q_j\}_{j \in B} \right), \quad (45)$$

with $\kappa_B$, the number of connected components of $B$ containing edges of the cut, except if we cut $D$ colors and disconnect the graph, in which case $\kappa_B = 0$. In order to derive (45) we insert $\lambda_B([p_i], \Lambda) = N^\nu u_B([q_i], \Lambda)$ in (38) with $q_e = \frac{p}{\Lambda}$.
The first two terms on the rhs in (45) follows from
\[
\Lambda \frac{\partial \lambda_G (\{ p_i \}, \Lambda)}{\partial \Lambda} = \partial_G \Lambda^\nu_G u_G (\{ q_c \}, \Lambda) - \Lambda^\nu_c \sum_c q_c \frac{\partial}{\partial q_c} u_G (\{ q_c \}, \Lambda) \\
+ \Lambda^\nu_e \frac{\partial u_G (\{ q_c \}, \Lambda)}{\partial \Lambda}.
\] (46)

To understand the origin of the powers of \( \Lambda \) in the third term in (45), observe that a \( k \)-cut on \( \kappa_{G,c} \) different connected components gathers these connected components into a single one. Thus, \( \kappa_{B,c} = \kappa_{G,c} + 1 \). Since \( B_c \) has two more vertices, we have \( \partial_G = \partial_B + \partial_{G,c} = (D - 1)d + 2 \). In addition, there are \( D - k \) integrations over \( p_t \) for \( t \not\in c \) and one constraint, so that, after the change of variables \( p_t \rightarrow q_c \) we get an extra power \( \Lambda^{(D - k) - d} \). Taking all terms into account, the exponent of \( \Lambda \) is \( \partial_B + D - k - d - 2 - \partial_G = d(\kappa_{G,c} - k) \), which is the announced result.

The last term involves no power of \( \Lambda \) since \( \kappa_{B,c} = \kappa_{G,c} + \kappa_{B,c} - 1 \), so that \( \partial_{B,c} = \partial_{G,c} + \partial_{B,c} - 2 \).

Note that \( \Lambda \) only appears with a non-positive exponent identical to that of \( N \) in the tensor model (21). It is also useful to notice that the operator \( \sum_c q_c \frac{\partial}{\partial q_c} \) is diagonalised by homogeneous polynomials in \( \{ q_c \} \) with eigenvalues being the total degree of the polynomials since it is the generator of momentum rescaling.

6. Renormalisable Abelian group field theories with a closure constraint

In analogy with ordinary quantum field theory (see [8]), this equation can serve as a basis for identifying the renormalisable interactions in group field theories. Indeed, the group field theory flow equation (45) differs from the quantum field theory flow equation only by some non-positive powers of the cut-off. Indeed, the exact renormalisation group equation for dimensionless variables (45) can be written as
\[
\Lambda \frac{\partial u_G}{\partial \Lambda} = \sum_{n \geq 0} \frac{\beta_G^n}{n!} (u_G),
\] (47)
where the functionals \( \beta_G^n \) are independent of \( \Lambda \). In quantum field theory, one usually has only \( \beta_G^0 \). In our case, there are, for a given \( B \), finitely many negative powers of \( \Lambda \). These are not specific to tensorial field theories but can be traced back to the second scale \( L \). However, it is not possible to take the infinite volume limit \( \Lambda \rightarrow 0 \) in a naive way because of the closure constraint.

Therefore, it is reasonable to expect these terms in the flow equation to have no effect for large \( \Lambda \) and do not alter the power counting. This follows from a standard argument in quantum field theory. Suppose that we momentarily disregard the nonlinear terms in the flow equation (45), the latter reduces to \( \Lambda \frac{\partial u_G}{\partial \Lambda} = - \partial_G u_G \). With a boundary condition at \( \Lambda_0 \), it leads to the solution \( u_G (\Lambda) = (\Lambda/\Lambda_0)^{\partial_G} u_G (\Lambda_0) \) if the boundary conditions are imposed at the high energy scale \( \Lambda_0 \). Therefore, variables with positive dimensions diverge as \( \Lambda_0 \) is taken to infinity. Then, renormalisation amounts to impose boundary condition at a low energy scale \( \Lambda_r \) for those variables. Finally, it may be shown (see [8]) that this picture remains valid in perturbation theory when the nonlinear corrections are added. Indeed, one can inductively prove that \( u_G \) obeys some inequalities whose leading term is the previous scaling law corrected by some polynomial in \( \log (\Lambda/\Lambda_0) \).
Admitting that this picture remains valid in our context, only the interactions with a positive scaling dimension in $\Lambda$ require renormalisation. Consequently, the only renormalisable (for simplicity we do not distinguish between the super-renormalisable ($\delta > 0$) and the just renormalisable ($\delta = 0$) cases) interactions are the homogeneous monomials of degree $n$ in the momenta of a bubble $B$ whose dimension obeys

$$\hat{\delta}_{g,n} = (D - 1)d - \kappa g d - [(D - 2)d - 2]\frac{v_g}{2} - n \geq 0. \quad (48)$$

In particular, among all the renormalisable interactions, we always have the mass term $\delta_{\infty, 0} = 2$ and for the kinetic term $\delta_{\infty, 2} = 0$.

Let us end by listing all renormalisable interactions in Abelian models with a closure constraint. The condition $\hat{\delta}_{g} \geq 0$ implies $v_g \leq 2 + \frac{d(D - 2)}{4}$. Non-trivial interactions involve bubbles with at most four vertices, so that we are left with only two possibilities: $d(D - 2) = 4$ and $d(D - 2) = 3$, which leads to the following 5 renormalisable theories. This list, with connected interaction only, was first obtained in the study of a non-Abelian model [7] using the multiscale analysis of Feynman graphs. This means that for $d = 3$ we obtain the same result for an Abelian model based on $U(1)^3$ and a non-Abelian one based on $SU(2)$. We emphasise that this is due to the fact that the most divergent graphs are dual to simply connected triangulations, for which the power counting only depends on the dimension of the group, see [18]. Super-renormalisable Abelian models (with $D = 4$ and $d = 1$ so that $d(D - 2) = 2$, thus leading to arbitrary polynomial interactions) have also been studied in [19].

- $D = 3$ and $d = 4$ so that $\delta = 8 - 4\kappa - \nu$. The renormalisable interactions is quartic and melonic

$$\includegraphics[width=1cm]{fig1.png} \ (\delta = 0).$$

The fixed point structure of a non-Abelian version of this model has been studied in [13].

- $D = 4$ and $d = 2$ so that $\delta = 6 - 2\kappa - \nu$. The renormalisable interactions are quartic

$$\includegraphics[width=1cm]{fig2.png} \ (\delta = 0), \quad \includegraphics[width=1cm]{fig3.png} \ (\delta = 0).$$

Note that the second interaction is not melonic (called necklace in [20]).

- $D = 6$ and $d = 1$ so that $\delta = 5 - \kappa - \nu$. The renormalisable interactions are quartic

$$\includegraphics[width=1cm]{fig4.png} \ (\delta = 0), \quad \includegraphics[width=1cm]{fig5.png} \ (\delta = 0), \quad \includegraphics[width=1cm]{fig6.png} \ (\delta = 0).$$

The last two interactions are not melonic. This model was shown to be renormalisable in [21] and its fixed point structure was further investigated in [14]. Its solution at leading order can be found in [22].

- $D = 3$ and $d = 3$ so that $\delta = 6 - 3\kappa - \nu/2$. The renormalisable interactions are quartic and sextic

$$\includegraphics[width=1cm]{fig7.png} \ (\delta = 1), \quad \includegraphics[width=1cm]{fig8.png} \ (\delta = 0), \quad \includegraphics[width=1cm]{fig9.png} \ (\delta = 0), \quad \includegraphics[width=1cm]{fig10.png} \ (\delta = 0).$$
The first term is in fact superrenormalisable ($\delta > 0$) and the last one is not melonic. Its non-Abelian counterpart is the tensorial $\text{SU}(2)$ group field theory in $D = 3$ which corresponds to three-dimensional Euclidian quantum gravity without the cosmological constant, first shown to be renormalisable in [7].

- $D = 5$ and $d = 1$ so that $\delta = 4 - \kappa - \nu/2$. The renormalisable interactions are quartic and sextic. We first have the melonic ones

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (n1) at (0,0) [circle,draw] {};
\node (n2) at (1,0) [circle,draw] {};
\end{tikzpicture}
\end{array} & (\delta = 1),
\begin{array}{c}
\begin{tikzpicture}
\node (n1) at (0,0) [circle,draw] {};
\node (n2) at (1,0) [circle,draw] {};
\node (n3) at (2,0) [circle,draw] {};
\end{tikzpicture}
\end{array} & (\delta = 0),
\begin{array}{c}
\begin{tikzpicture}
\node (n1) at (0,0) [circle,draw] {};
\node (n2) at (1,0) [circle,draw] {};
\node (n3) at (2,0) [circle,draw] {};
\node (n4) at (3,0) [circle,draw] {};
\end{tikzpicture}
\end{array} & (\delta = 0),
\begin{array}{c}
\begin{tikzpicture}
\node (n1) at (0,0) [circle,draw] {};
\node (n2) at (1,0) [circle,draw] {};
\end{tikzpicture}
\end{array} & (\delta = 0).
\end{align*}
\]

The last interaction is not connected ($\kappa = 2$). This model, with the non-connected interaction, was shown to be renormalisable in [21]. Besides, there are also non-melonic renormalisable interactions.

Note that all these bubble couplings with $\nu \geq 4$ have dimension 0 or 1, so that there is no possibility of adding a derivative coupling. Indeed, the latter corresponds to a term with $n > 0$ in (48), which would lead to a negative dimension, the case $n = 1$ being forbidden by rotational symmetry. Of course, we could always add quadratic terms (mass and kinetic terms) to the list of interaction. Then, the kinetic term can play the role of a derivative coupling.

It is also worthwhile to notice that some of the renormalisable interactions are not necessarily melonic. As for unitarily invariant random tensors, non-melonic interaction appear with a negative power of $\Lambda$ in the flow of renormalisable melonic interactions. Therefore, they do not require renormalisation if the bare theory only contains melonic interactions. However, if the bare theory contains non-melonic renormalisable interactions, the latter requires renormalisation. A rigorous analysis of the role of the non-melonic couplings, based on bounds on the aforementioned bounds on the couplings, will be presented in a separate publication.

7. Conclusion and outlook

In this paper, we have introduced an exact renormalisation group equation for tensor models and tensorial field theories, in the form proposed by Polchinski in the context of quantum field theories. We have expanded the Wilsonian effective action on bubble couplings $\lambda_{\delta}$ and formulated a system of differential equations for these couplings, both for invariant tensors (21) and Abelian group field theories with a closure constraint (38). These differential equations have provided us with two applications.

- We have rederived, using the flow equation, Gurau’s universality theorem for invariant random tensors. This study could be pursued by further investigating the phase structure of random tensor models.
- In the case of Abelian group field theories, we have introduced a suitable notion of the dimension for bubble couplings, which led us to a list of only five renormalisable theories. Of course, this is only a first step and non-Abelian models with a simplicity constraint should be considered.

To conclude, let us emphasise that the originality of our method lies in the fact that we have not evaluated a single $(D + 1)$-dimensional Feynman graphs. Our equations only involve $D$-dimensional graphs, which collect the contribution of all space-time triangulations.
with a fixed boundary. Therefore, it is technically simpler than evaluating the Feynman graphs and may be applied to a large class of models, including models with a simplicity constraint. We note here that we were able to obtain our results without evaluating any Feynman amplitude because we deal with a purely combinatorial quantum field theory model where the propagator is trivial and therefore only counting (graphs, faces, etc) issues are taken into consideration.

Furthermore, let us note that although our study is exact (no truncations of the equations have been used), it remains perturbative. Non-perturbative studies, based on truncations to a couple of bubble couplings, have been recently presented in the literature [12–14]. These works used Wetterich’s formulation of the exact renormalisation group equation, more suitable than Polchinski’s for truncations. For tensor models, Wetterich’s equation can also be formulated using cuts and lead to a system of differential equations for the couplings.

Acknowledgments

T K thanks ANR JCJC CombPhysMat2Tens grant for partial support and R Gurau for interesting and helpful discussions. T K and R T thank ESI Vienna for hospitality during the program ‘Combinatorics, Geometry and Physics’.

Appendix. Exact renormalisation group equation for low order bubble couplings

In this appendix, we give explicit evolution equations for low order bubble couplings in invariant tensor models. We order terms in decreasing powers of $N$.

A.1. Couplings in rank $D = 3$ tensor models

We consider here the evolution equations for the couplings for a few lowest order tensor invariants in rank 3 tensor models given in terms of the couplings of other higher order tensor invariants

Figure A1. 4 possible 1-link-cut operations which are counted in the second term in rhs of (50).

Figure A2. 2 possible 1-link-cut operations which are counted in the sixth term in rhs of (50).
Note for example, the combinatorial factor 4 in the second term in the rhs of (50) comes from 4 possible 1-link-cut operations one can do on the four-point function graph shown above and the resulting graphs for rank \( D = 3 \). Note that the resulting graphs can be disconnected as in the third row and can be non-planar as in the last row.

**Table A1.** The figure shows link-cut operations denoted by / on the four-point function graph shown above and the resulting graphs for rank \( D = 3 \). Note that the resulting graphs can be disconnected as in the third row and can be non-planar as in the last row.

\[
\frac{\partial}{\partial t} \lambda \quad = \quad \left[ \lambda \quad \overset{\text{1 cut}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_0 + \left[ \frac{\lambda^3}{3} \quad \overset{\text{1 cut}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_1 + \left[ \frac{\lambda^2}{2} \quad \overset{\text{2 cuts}}{\underset{\text{1 cut}}{\longrightarrow}} \right]_3 + \frac{1}{N} \left[ \lambda \quad \overset{\text{2 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_2 + \frac{1}{N^3} \left[ \lambda \quad \overset{\text{3 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_3,
\]

(49)

\[
\frac{\partial}{\partial t} \lambda \quad = \quad \left[ \lambda \quad \overset{\text{1 cut}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_0 + \frac{1}{N} \left[ \lambda \quad \overset{\text{2 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_2 + \frac{1}{N^2} \left[ \lambda \quad \overset{\text{3 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_3 + \frac{1}{N^3} \left[ \lambda \quad \overset{\text{4 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \right]_3.
\]

(50)

Note for example, the combinatorial factor 4 in the second term in the rhs of (50) comes from 4 possible 1-link-cut operations one can do on the four point function graph \( \overset{\text{1 cut}}{\underset{\text{0 cut}}{\longrightarrow}} \) in the lhs of (50). Similarly, the combinatorial factor 2 in the sixth term on the rhs of (50) counts the two possible 2-link-cut operations one can perform on the graph \( \overset{\text{2 cuts}}{\underset{\text{0 cut}}{\longrightarrow}} \). See figures A1 and A2 for how these particular combinatorial factors arise and refer to table A1 for examples of
consequences of link-cut operation on $\lambda \xrightarrow{\text{cut}}$.

$$
\frac{\partial}{\partial t} \lambda \xrightarrow{\text{cut}} = \left[ \lambda \xrightarrow{\text{cut}} \right]_{0 \text{ cuts}} + \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{1 \text{ cuts}} + \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{2 \text{ cuts}} + \left[ 4 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N} \left\{ \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{2 \text{ cuts}} + \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}} \right\}
$$

$$
+ \frac{1}{N^3} \left[ 2 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}},
$$

(51)

$$
\frac{\partial}{\partial t} \lambda \xrightarrow{\text{cut}} = \left[ \lambda \xrightarrow{\text{cut}} \right]_{0 \text{ cuts}} + \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{1 \text{ cuts}} + \left[ 6 \lambda \xrightarrow{\text{cut}} \right]_{2 \text{ cuts}} + \left[ 3 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N} \left\{ 3 \lambda \xrightarrow{\text{cut}} + 6 \lambda \xrightarrow{\text{cut}} + 12 \lambda \xrightarrow{\text{cut}} + 6 \lambda \xrightarrow{\text{cut}} \right\}_{2 \text{ cuts}}
$$

$$
+ \frac{1}{N^2} \left[ 12 \lambda \xrightarrow{\text{cut}} + 6 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}} + \frac{1}{N^3} \left[ 6 \lambda \xrightarrow{\text{cut}} + 3 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

(52)

$$
\frac{\partial}{\partial t} \lambda \xrightarrow{\text{cut}} = \left[ \lambda \xrightarrow{\text{cut}} \right]_{0 \text{ cuts}} + \left[ 4 \lambda \xrightarrow{\text{cut}} \right]_{1 \text{ cuts}} + \left[ 2 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N} \left\{ 6 \lambda \xrightarrow{\text{cut}} + 6 \lambda \xrightarrow{\text{cut}} + 3 \lambda \xrightarrow{\text{cut}} + 10 \lambda \xrightarrow{\text{cut}} + 2 \lambda \xrightarrow{\text{cut}} \right\}_{2 \text{ cuts}}
$$

$$
+ \frac{1}{N^2} \left[ 6 \lambda \xrightarrow{\text{cut}} + \lambda \xrightarrow{\text{cut}} + 8 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N^3} \left[ 6 \lambda \xrightarrow{\text{cut}} + 2 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}},
$$

(53)

$$
\frac{\partial}{\partial t} \lambda \xrightarrow{\text{cut}} = \left[ \lambda \xrightarrow{\text{cut}} \right]_{0 \text{ cuts}} + \left[ 9 \lambda \xrightarrow{\text{cut}} \right]_{1 \text{ cuts}} + \left[ 18 \lambda \xrightarrow{\text{cut}} \right]_{2 \text{ cuts}}
$$

$$
+ \left[ 6 \lambda \xrightarrow{\text{cut}} + 6 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N} \left\{ 9 \lambda \xrightarrow{\text{cut}} + 2 \lambda \xrightarrow{\text{cut}} \right\}_{2 \text{ cuts}} + \left[ 18 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}}
$$

$$
+ \frac{1}{N^3} \left[ 3 \lambda \xrightarrow{\text{cut}} \right]_{3 \text{ cuts}},
$$

(54)
A.2. Couplings in rank $D = 4$ tensors models

We consider here the evolution equations for the couplings for a few lowest order tensor invariants in rank 4 tensor models given in terms of the couplings of other higher order tensor invariants

\[
\frac{\partial}{\partial t} \lambda_{\square} = [\lambda_{\square} \rightarrow 3_0 \text{ cut}] + [8 \lambda_{\square} \rightarrow 4_1 \text{ cut}] + [4 \lambda_{\square} \rightarrow 2_2 \text{ cut}] + [2 \lambda_{\square} \rightarrow \lambda_3 \text{ cut}] + [2 \lambda_{\square} \rightarrow \lambda_3 \text{ cut}] + \frac{1}{N} \left\{ 3 \lambda_{\square} \rightarrow 1_1 \text{ cut} + 8 \lambda_{\square} \rightarrow 1_1 \text{ cut} + 12 \lambda_{\square} \rightarrow 1_1 \text{ cut} \right\} + \frac{1}{N^2} [4 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^2} [6 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [4 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [6 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [8 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [10 \lambda_{\square} \rightarrow 1_1 \text{ cut}] + \cdots \right]
\]

\[
\frac{\partial}{\partial t} \lambda_{\bullet} = [\lambda_{\bullet} \rightarrow 3_0 \text{ cut}] + [2 \lambda_{\bullet} \rightarrow 4_1 \text{ cut}] + [4 \lambda_{\bullet} \rightarrow 1_2 \text{ cut}] + [4 \lambda_{\bullet} \rightarrow 1_2 \text{ cut}] + \frac{1}{N} \left\{ 6 \lambda_{\bullet} \rightarrow 1_1 \text{ cut} + 18 \lambda_{\bullet} \rightarrow 1_1 \text{ cut} + 6 \lambda_{\bullet} \rightarrow 1_1 \text{ cut} \right\} + \frac{1}{N^2} [12 \lambda_{\bullet} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^2} [12 \lambda_{\bullet} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [4 \lambda_{\bullet} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [4 \lambda_{\bullet} \rightarrow 1_1 \text{ cut}] + \cdots \right]
\]

\[
\frac{\partial}{\partial t} \lambda_{\ast} = [\lambda_{\ast} \rightarrow 3_0 \text{ cut}] + [8 \lambda_{\ast} \rightarrow 4_1 \text{ cut}] + [4 \lambda_{\ast} \rightarrow 2_2 \text{ cut}] + \frac{1}{N} \left\{ 4 \lambda_{\ast} \rightarrow 1_1 \text{ cut} + 16 \lambda_{\ast} \rightarrow 1_1 \text{ cut} + 4 \lambda_{\ast} \rightarrow 1_1 \text{ cut} \right\} + \frac{1}{N^2} [16 \lambda_{\ast} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^2} [16 \lambda_{\ast} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [8 \lambda_{\ast} \rightarrow 1_1 \text{ cut}] + \frac{1}{N^3} [10 \lambda_{\ast} \rightarrow 1_1 \text{ cut}] + \cdots \right]
\]
A.3. Solutions of the Polchinski equations in large $N$ limit

For a concrete illustration, we consider a specific model given earlier in the section 4 with a diagonal covariance $C = \frac{i}{N^{\alpha}} \epsilon_{\gamma,T}$, where $\epsilon$ is a parameter to control the flow. When $\epsilon = 0$, $C^{-1} = 0$ so that there is no integration at all and the effective action equals the bare action.

The partition function is the effective action (given by (17)) evaluated at 0 field. In addition to the set of differential equations for couplings given in (49)–(55), we have

$$
\frac{\partial}{\partial \tau} \lambda_{\sigma} = \lambda_{\sigma},
$$

where $\lambda_{\sigma}$ is the coupling for an empty graph denoted by $\sigma$, on which one can only perform 0-cut, therefore obtaining the dipole graph on the rhs. Using these differential equations, one can readily solve this set of differential equation iteratively with an ansatz for the partition function as it can be written as the sum of all possible vacuum diagrams

$$
\lambda_{\sigma}(t) = \lambda_{\sigma}(0) + \sum_m a_{\sigma_m} t^m,
$$

where $t$ is the momentum associated with the propagator (in other words, 0 color line), and $a_m$ is the coupling associated with the $2m$-point function. Here, for simplicity, we will not distinguish the colors.

We remark here that the leading contribution in the large $N$ limit in the Polchinski equation selects only melonic graphs with the normalization given in (17). However, once we start including the terms of $O\left(\frac{1}{N}\right)$, non-planar terms also appear in the evolution equation.

We rewrite the ansatze for each couplings

$$
\lambda_{\sigma}(t) = \lambda_{\sigma}(0) + \sum_m a_{\sigma_m} t^m + O(t^{m+1}),
$$

$$
\lambda_{\sigma}(t) = \lambda_{\sigma}(0) + \sum_{m-1} a_{\sigma_m} t^{m-1} + O(t^m),
$$

$$
\lambda_{\sigma}(t) = \lambda_{\sigma}(0) + \sum_{m-2} a_{\sigma_m} t^{m-2} + O(t^{m-1}),
$$

$$
\lambda_{\sigma}(t) = \lambda_{\sigma}(0) + \sum_{m-2} a_{\sigma_m} t^{m-2} + O(t^{m-1}),
$$

etc, where constants $a_m$’s are to be determined in terms of the couplings at $t = 0$. The above ansatze are consistent with the orders in $t$ for each couplings observing the set of differential equations given.
A.3.1. The leading order in large $N$ limit. We will first only solve the set of differential equations (49)–(55) and (60) in the leading order in large $N$ limit. Starting with $m = 1$, we obtain:

$$
\begin{align*}
\lambda_{\sigma}(t) &= \lambda_{\sigma}(0) + \lambda_{\sigma}(0) t + O(t^2), \\
\lambda_{\varnothing}(t) &= \lambda_{\varnothing}(0) + O(t).
\end{align*}
$$

We proceed to $m = 2$ using the above result to obtain the higher order corrections in $t$;

$$
\begin{align*}
\lambda_{\sigma}(t) &= \lambda_{\sigma}(0) + \lambda_{\sigma}(0) t + \frac{1}{2} \left( \lambda_{\varnothing}(0) + 3 \lambda_{\varnothing}(0) + \lambda_{\varnothing}(0) \right) t^2 + O(t^3) \\
\lambda_{\varnothing}(t) &= \lambda_{\varnothing}(0) + \left( \lambda_{\varnothing}(0) + 3 \lambda_{\varnothing}(0) + \lambda_{\varnothing}(0) \right) t + O(t^2) \\
\lambda_{\lambda}(t) &= \lambda_{\varnothing}(0) + O(t) \\
\lambda_{\chi}(t) &= \lambda_{\chi}(0) + O(t).
\end{align*}
$$

Including up to $m = 3$ in the coupling for vacuum, we obtain:

$$
\begin{align*}
\lambda_{\sigma}(t) &= \lambda_{\sigma}(0) + \lambda_{\sigma}(0) t + \frac{1}{2} \left( \lambda_{\varnothing}(0) + 3 \lambda_{\varnothing}(0) + \lambda_{\varnothing}(0) \right) t^2 \\
&\quad+ \frac{1}{3} \left( \frac{1}{2} \lambda_{\varnothing}(0) + \frac{9}{2} \lambda_{\varnothing}(0) \right) t^3 + O(t^4) \\
\lambda_{\varnothing}(t) &= \lambda_{\varnothing}(0) + \left( \lambda_{\varnothing}(0) + 3 \lambda_{\varnothing}(0) + \lambda_{\varnothing}(0) \right) t^2 + O(t^3) \\
&\quad+ \left( \frac{1}{2} \lambda_{\varnothing}(0) + \frac{9}{2} \lambda_{\varnothing}(0) \right) t^3 + O(t^4) \\
\lambda_{\lambda}(t) &= \lambda_{\varnothing}(0) \\
&\quad+ \left( \lambda_{\varnothing}(0) + 6 \lambda_{\varnothing}(0) + 6 \lambda_{\varnothing}(0) \right) t + O(t^2) \\
\lambda_{\chi}(t) &= \lambda_{\chi}(0) + O(t^2) \\
&\quad+ \left( \lambda_{\varnothing}(0) + 4 \lambda_{\varnothing}(0) + 2 \lambda_{\varnothing}(0) + 4 \lambda_{\varnothing}(0) \right) t + O(t^2),
\end{align*}
$$

$$
\begin{align*}
\lambda_{\sigma}(t) &= \lambda_{\varnothing}(0) + O(t) \\
\lambda_{\varnothing}(t) &= \lambda_{\varnothing}(0) + O(t) \\
\lambda_{\lambda}(t) &= \lambda_{\varnothing}(0) + O(t) \\
\lambda_{\chi}(t) &= \lambda_{\varnothing}(0) + O(t).
\end{align*}
$$

(68)
A.3.2. To $\mathcal{O}(1/N)$. Performing a similar calculation as for the leading order in $N$, up to $m = 3$ in the coupling for vacuum, we obtain:

$$
\lambda_{\varphi}(t) = \lambda_{\varphi}(0) + \lambda_{\varphi}(0) t \\
+ \frac{1}{2} \left[ \lambda_{\varphi}(0) + \left( 3 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) + \lambda_{\varphi}^2(0) \right] t^2 \\
+ \frac{1}{3} \left[ \frac{1}{2} \lambda_{\varphi}(0) + \left( \frac{9}{2} + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \\
+ \left( 9 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \right] t^3 + \mathcal{O}(t^4),
$$

$$
\lambda_{\varphi}(t) = \lambda_{\varphi}(0) + \left[ \lambda_{\varphi}(0) + \left( 3 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) - \lambda_{\varphi}^2(0) \right] t \\
+ \left[ \frac{1}{2} \lambda_{\varphi}(0) + \left( \frac{9}{2} + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \\
+ \left( 9 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \right] t^2 + \mathcal{O}(t^3),
$$

$$
\lambda_{\varphi}(t) = \lambda_{\varphi}(0) + \left[ \lambda_{\varphi}(0) + \left( 6 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \\
+ 6 \lambda_{\varphi}(0) + 4 \lambda_{\varphi} \lambda_{\varphi}(0) + \frac{1}{N^2} 6 \lambda_{\varphi} \lambda_{\varphi} \lambda_{\varphi}(0) \right] t + \mathcal{O}(t^2),
$$

$$
\lambda_{\varphi}(t) = \lambda_{\varphi}(0) + \left[ \lambda_{\varphi}(0) + \left( 4 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) \\
+ \left( 2 + \frac{1}{N^2} \right) \lambda_{\varphi}(0) + 4 \lambda_{\varphi} \lambda_{\varphi}(0) + \frac{1}{N^2} 2 \lambda_{\varphi} \lambda_{\varphi} \lambda_{\varphi}(0) \right] t + \mathcal{O}(t^2),
$$
$$\lambda \quad \lambda \quad \lambda(t) = \lambda \quad \lambda \quad \lambda(0) + \mathcal{O}(t),$$

$$\lambda \quad \lambda \quad \lambda(t) = \lambda \quad \lambda \quad \lambda(0) + \mathcal{O}(t),$$

$$\lambda \quad \lambda \quad \lambda(t) = \lambda \quad \lambda \quad \lambda(0) + \mathcal{O}(t),$$

$$\lambda \quad \lambda \quad \lambda \quad \lambda(t) = \lambda \quad \lambda \quad \lambda \quad \lambda(0) + \mathcal{O}(t).$$