Noncommutative geometry as a functor

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Abstract

In this note the noncommutative geometry is interpreted as a functor, whose range is a family of the operator algebras. Some examples are given and a program is sketched.

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Introduction

A point made in this note is that some noncommutative spaces (i.e. C*-algebras, Banach or associative algebras) can be viewed as a generalized homology in the sense that there exist functors with the range in the noncommutative spaces. The domain of the functors can be any interesting category, e.g. the Hausdorff spaces, manifolds, Riemann surfaces, etc. We shall give examples of such functors.

The above functors have a long history, rather natural and familiar to specialists. A foundational example is given by the Gelfand-Naimark functor, which maps the category of the Hausdorff spaces to the category of commutative C*-algebras. It was conjectured by Novikov and proved by Kasparov [7] & Mishchenko [8], that in many cases the higher signatures of smooth n-dimensional manifolds are invariants of a certain class of the C*-algebras. The respective functor is known as an assembly map. In dynamics, Cuntz & Krieger [1] constructed a functor from the category of topological Markov chains to a category of the C*-algebras (Cuntz-Krieger algebras). There are many more examples to add to the list.

As long as a functor is constructed, one can calculate the noncommutative invariants attached to it. On the face of it, the C*-algebras are a way more complex than the abelian groups. However, many important families of the operator algebras have been lately classified in terms of the algebraic K-theory [1] and more developments will appear in the future. The invariants of the C*-algebras produce new (and old) invariants of the objects in the initial category.

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Thus, the problem of an interpretation of the noncommutative invariants in terms of the initial category arises.

In relation to the traditional parts of noncommutative geometry (e.g. the index theory, cyclic cohomology, quantum groups, etc), the functorial approach means a switch from a ‘romantic’ to ‘pragmatic’ relationship, in the sense that the noncommutative spaces become a toolkit in the study of the classical spaces. The problem has two parts: (i) to map a given category into a family of the noncommutative spaces and (ii) to prove that the mapping is a functor. Note that (ii) is the hardest part of the problem.

The note is organized as follows. In section 1 some examples of functors with the range in a category of the operator algebras are considered and their noncommutative invariants are analyzed. In section 2 draft of a program is sketched.

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1 Three examples

In this section, some examples of functors with the range in a family of the noncommutative spaces are given. In the two of three cases, the functors are non-injective. The list is by no means complete and the reader is encouraged to add examples of his own.

1.1 Gelfand and Naimark functor

A. This is a foundational example. Let \( X \) be a locally compact Hausdorff space. By \( C(X) \) one understands a commutative \( C^* \)-algebra of all functions \( f : X \to \mathbb{C} \), which vanish at infinity. The norm on \( C(X) \) is the supremum norm. Recall that every point \( x \in X \) can be thought of as a linear multiplicative functional \( \hat{x} : C(X) \to \mathbb{C} \). The Gelfand transform \( F : X \to C(X) \) is defined by the formula \( x \mapsto f \), where \( f \in C(X) \) is such that \( \hat{x}(f) = f(x) \).

B. Let \( h : X \to Y \) be a continuous map between the Hausdorff spaces \( X \) and \( Y \). It can be easily shown that the map \( h_* = F^{-1} \circ h \circ F \) is a homomorphism from the \( C^* \)-algebra \( C(Y) \) to \( C(X) \). In other words, \( F \) is a contravariant functor from the locally compact Hausdorff spaces to the commutative \( C^* \)-algebras:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{continuous map}} & Y \\
F & & F \\
C(X) & \xleftarrow{\text{homomorphism}} & C(Y)
\end{array}
\]
C. Note that $F$ is an injective functor. The functor $F$ does not produce new invariants of the Hausdorff spaces, because of the following isomorphism: $K^{alg}(C(X)) \cong K^{top}(X)$, where $K^{alg}$ and $K^{top}$ are the algebraic and the topological $K$-theory, respectively.

1.2 Anosov automorphisms of a two-dimensional torus

A. Let us consider a non-trivial application of the operator algebras to a problem in topology. Recall that an automorphism $\phi : T^2 \rightarrow T^2$ of the two-dimensional torus is called Anosov, if it is given by a matrix $A_\phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2, \mathbb{Z})$, such that $|a_{11} + a_{22}| > 2$. We wish to construct a functor (an assembly map) $\mu : \phi \mapsto A_\phi$, such that for every $h \in Aut(T^2)$ the following diagram commutes:

$$
\begin{array}{ccc}
\phi & \overset{\text{conjugacy}}{\longrightarrow} & \phi’ = h \circ \phi \circ h^{-1} \\
\mu & & \mu \\
A_\phi \otimes K & \overset{\text{isomorphism}}{\longrightarrow} & A_{\phi'} \otimes K,
\end{array}
$$

where $A_\phi$ is an $AF$-algebra and $K$ is the $C^*$-algebra of compact operators on a Hilbert space. In other words, if $\phi, \phi’$ are conjugate automorphisms, then the $AF$-algebras $A_\phi, A_{\phi’}$ are stably isomorphic.

B. The map $\mu : \phi \mapsto A_\phi$ is as follows. For simplicity, let $a_{11} + a_{22} > 2$. Note that without loss of generality, one can assume that $a_{ij} \geq 0$ for a proper basis in the homology group $H_1(T^2; \mathbb{Z})$. Consider an $AF$-algebra, $A_\phi$, given by the following periodic Bratteli diagram:

$$
A_\phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
$$

where $a_{ij}$ indicate the multiplicity of the respective edges of the graph. We encourage the reader to verify that $\mu : \phi \mapsto A_\phi$ is a correctly defined function on the set of Anosov automorphisms given by the hyperbolic matrices with the
non-negative entries. Note that \( \mu \) is not injective, since \( \phi \) and all its powers map to the same \( AF \)-algebra.

C. Let us show that if \( \phi, \phi' \in Aut \ (T^2) \) are the conjugate Anosov automorphisms, then \( \mathbb{A}_\phi, \mathbb{A}_{\phi'} \) are the stably isomorphic \( AF \)-algebras. Indeed, let \( \phi' = h \circ \phi \circ h^{-1} \) for an \( h \in Aut \ (X) \). Then \( A_{\phi'} = T A_{\phi} T^{-1} \) for a matrix \( T \in GL(2, \mathbb{Z}) \). Note that \((A_{\phi}')^n = (T A_{\phi} T^{-1})^n = T A_{\phi}^n T^{-1} \), where \( n \in \mathbb{N} \).

We shall use the following criterion \([2, \text{Theorem 2.3}]\): the \( AF \)-algebras \( \mathbb{A}_\phi, \mathbb{A}_{\phi'} \) are stably isomorphic if and only if their Bratteli diagrams contain a common block of an arbitrary length. Consider the following sequences of matrices: \( A_\phi A_\phi \ldots A_\phi \) and \( T A_\phi A_\phi \ldots A_\phi T^{-1} \), which mimic the Bratteli diagrams of \( \mathbb{A}_\phi \) and \( \mathbb{A}_{\phi'} \). Letting the number of blocks \( A_\phi \) tend to infinity, we conclude that \( \mathbb{A}_\phi \otimes K \cong \mathbb{A}_{\phi'} \otimes K \).

D. The conjugacy problem for the Anosov automorphisms can now be recast in terms of the \( AF \)-algebras: find invariants of the stable isomorphism classes of the stationary \( AF \)-algebras. One such invariant is due to Handelman \([5]\). Consider an eigenvalue problem for the hyperbolic matrix \( A_\phi \in GL(2, \mathbb{Z}) \): \( A_\phi v = \lambda v \), where \( \lambda > 0 \) is the Perron-Frobenius eigenvalue and \( v = (v^{(1)}, v^{(2)}) \) the corresponding eigenvector with the positive entries normalized so that \( v^{(i)} \in K = \mathbb{Q}(\lambda) \). Denote by \( m = \mathbb{Z}v^{(1)} + \mathbb{Z}v^{(2)} \) a \( \mathbb{Z} \)-module in the number field \( K \).

Recall that the coefficient ring, \( \Lambda \), of module \( m \) consists of the elements \( \alpha \in K \) such that \( \alpha m \subseteq m \). It is known that \( \Lambda \) is an order in \( K \) (i.e. a subring of \( K \) containing 1) and, with no restriction, one can assume that \( m \subseteq \Lambda \). It follows from the definition, that \( m \) coincides with an ideal, \( I \), whose equivalence class in \( \Lambda \) we shall denote by \( [I] \).

It has been proved by Handelman, that the triple \((\Lambda, [I], K)\) is an arithmetic invariant of the stable isomorphism class of \( \mathbb{A}_\phi \): the \( \mathbb{A}_\phi, \mathbb{A}_{\phi'} \) are stably isomorphic \( AF \)-algebras if and only if \( \Lambda = \Lambda' \), \( [I] = [I'] \) and \( K = K' \). It is interesting to compare the operator algebra invariants with those obtained in \([9]\).

E. Let \( M_\phi \) be a mapping torus of the Anosov automorphism \( \phi \), i.e. a three-dimensional manifold \( \{ T^2 \times [0,1] \ | \ (x,0) \mapsto (\phi(x),1) \ \forall x \in T^2 \} \). The \( M_\phi \) is known as a solvmanifold, since it is the quotient space of a solvable Lie group.

It is an easy exercise to show that the homotopy classes of \( M_\phi \) are bijective with the conjugacy classes of \( \phi \). Thus, the noncommutative invariant \((\Lambda, [I], K)\) is a homotopy invariant of \( M_\phi \).

1.3 Complex tori and Effros-Shen algebras

A. Let us consider an application of the operator algebras to a problem in conformal geometry. Let \( \tau \in \mathbb{H} := \{ z \in \mathbb{C} \ | \ \text{Im} \ (z) > 0 \} \) be a complex number. Recall that the quotient space \( E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) is called a complex torus. It is well-known that the complex tori \( E_\tau, E_{\tau'} \) are isomorphic, whenever \( \tau' \equiv \tau \mod SL(2, \mathbb{Z}) \), i.e. \( \tau' = \frac{a + b \tau}{c + d} \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \).

B. Let \( 0 < \theta < 1 \) be an irrational number given by the regular continued
fraction:
\[ \theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \]

By the Effros-Shen algebra \([3]\), \(A_\theta\), one understands an AF-algebra given by the Bratteli diagram:

![Bratteli Diagram](image)

Figure 2: The Effros-Shen algebra \(A_\theta\).

where \(a_i\) indicate the number of edges in the upper row of the diagram. Recall that \(A_\theta, A_{\theta'}\) are said to be stably isomorphic if \(A_\theta \otimes K \cong A_{\theta'} \otimes K\). It is known that \(A_\theta, A_{\theta'}\) are stably isomorphic if \(\theta' \equiv \theta \mod \text{SL}(2, \mathbb{Z})\). Comparing the categories of complex tori and Effros-Shen algebras, one cannot fail to observe that for the generic objects, the corresponding morphism are isomorphic as groups. Let us show that the observation has a ground – there exists a functor, \(F\), which makes the following diagram commute:

![Diagram](image)

C. To construct the map \(F : E_\tau \rightarrow A_\theta\), we shall use a Hubbard-Masur homeomorphism \(h : \mathbb{H} \rightarrow \Phi_{T^2}\), where \(\Phi_{T^2}\) is the space of measured foliations on the two-torus \([6]\). Each measured foliation \(F_\mu^\theta \in \Phi_{T^2}\) looks like a family of the parallel lines of a slope \(\theta\) endowed with an invariant transverse measure \(\mu\) (Fig.3). If \(\phi\) is a closed 1-form on \(T^2\), then the trajectories of \(\phi\) define a measured foliation \(F_\mu^\theta \in \Phi_{T^2}\) and vice versa. It is not hard to see that \(\mu = \int_{\gamma_1} \phi\) and \(\theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi\), where \(\{\gamma_1, \gamma_2\}\) is a basis in \(H_1(T^2; \mathbb{Z})\). Denote by \(\omega_N\) an invariant (Néron) differential of the complex torus \(\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})\). It is well known that \(\omega_1 = \int_{\gamma_1} \omega_N\) and \(\omega_2 = \int_{\gamma_2} \omega_N\). Let \(\pi\) be a projection acting by the formula \((\theta, \mu) \mapsto \theta\). The assembly map \(F\) is given by the composition \(F = \pi \circ h\), where \(h\) is the Hubbard-Masur homeomorphism. In other words, the assembly
map $E_{\tau} \mapsto \mathbb{A}_\theta$ can be written explicitly as:

$$E_{\tau} = E\left(\int_{\gamma_2} \omega_N)/(\int_{\gamma_1} \omega_N\right) \xrightarrow{h} \pi\left(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi\right) \xrightarrow{\pi} \mathbb{A}_\left(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi\right) = \mathbb{A}_\theta.$$

\[\text{Figure 3: The measured foliation } F^\mu \text{ on } T^2 = \mathbb{R}^2/\mathbb{Z}^2.\]

D. Let us show that the map $F$ is a covariant functor. Indeed, an isomorphism $E_{\tau} \rightarrow E_{\tau'}$ is induced by an automorphism $\varphi \in Aut(T^2)$ of the two-torus. Let $A_{\varphi} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right) \in GL(2;\mathbb{Z})$ be a matrix realizing such an automorphism. From the formulas for $F$, one gets

$$\tau' = \frac{\int_{c\gamma_1 + d\gamma_2} \omega_N)/(\int_{a\gamma_1 + b\gamma_2} \omega_N) = \frac{c+d\tau}{a+b\tau},$$

and

$$\theta' = \frac{\int_{c\gamma_1 + d\gamma_2} \phi)/(\int_{a\gamma_1 + b\gamma_2} \phi) = \frac{c+d\theta}{a+b\theta}.$$ 

Thus, $F$ sends isomorphic complex tori to the stably isomorphic Effros-Shen algebras. Moreover, the formulas imply that $F$ is a covariant functor. Note, that since $F$ contains a projective map $\pi$, $F$ is not an injective functor.

E. Finally, let us consider a noncommutative invariant coming from the functor $F$. The $E_{CM}$ is said to have a complex multiplication, if the endomorphism ring of the lattice $\mathbb{Z} + \mathbb{Z}\tau$ exceeds $\mathbb{Z}$. It is an easy exercise to show (in view of the explicit formulas for $F$) that $F(E_{CM}) = \mathbb{A}_\theta$, where $\theta$ is a quadratic irrationality. In this case the continued fraction of $\theta$ is eventually periodic and we let $r$ be the length of the minimal period of $\theta$. Clearly, the integer $r$ is an invariant of the stable isomorphism class of the $AF$-algebra $\mathbb{A}_\theta$. To interpret the noncommutative invariant $r$ in terms of $E_{CM}$, recall that $E_{CM}$ is isomorphic to a projective elliptic curve defined over a subfield $K = \mathbb{Q}(j(E_{CM}))$ of $\mathbb{C}$, where $j(E_{CM})$ is the $j$-invariant. It is known that the $K$-rational points of $E_{CM}$ make an abelian group, whose infinite part has rank $R \geq 0$. We conclude by the following

\textbf{Conjecture 1} For every elliptic curve with a complex multiplication $R = r - 1$.

2 Sketch of a program

One can outline a program by indicating: (i) an object of study, (ii) a typical problem and (iii) a set of exercises. A functorial noncommutative geometry (FNCG) can be described as follows.

\textbf{Object of study.} The FNCG studies non-trivial functors from a category of the classical objects, $\mathfrak{S}$, to a category of the noncommutative spaces (operator,
Banach or associative algebras), $\mathcal{A}$. The functor can be non-injective. The category $\mathcal{A}$ is (possibly) endowed with a good set of invariants.

Typical problem. The main problem of FNCG is construction of new invariants of the objects in $\mathcal{G}$ from the known noncommutative invariants of $\mathcal{A}$. A reconstruction of the classical invariants from the noncommutative invariants is regarded as a partial solution of the main problem.

Exercises. Let $\mathcal{A}$ be a category of:

(i) the $UHF$ algebras;
(ii) the Cuntz-Krieger algebras $\mathcal{O}_A$ with $\det(A) = \pm 1$.

Find a category $\mathcal{G}$ corresponding to $\mathcal{A}$ and solve the typical problem. (Hint: for the Cuntz-Krieger algebras of type (ii), the category $\mathcal{G}$ consists of the homotopy classes of the torus bundles $M_A$ over $S^1$ with the monodromy given by the matrix $A$.)

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