BOUNDEDNESS PROPERTIES OF PSEUDO-DIFFERENTIAL OPERATORS AND CALDERÓN-ZYGMUND OPERATORS ON MODULATION SPACES

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Abstract. In this paper, we study the boundedness of pseudo-differential operators with symbols in $S^m_{\rho,\delta}$ on the modulation spaces $M^{p,q}$. We discuss the order $m$ for the boundedness $\text{Op}(S^m_{\rho,\delta}) \subset L(M^{p,q}(\mathbb{R}^n))$ to be true. We also prove the existence of a Calderón-Zygmund operator which is not bounded on the modulation space $M^{p,q}$ with $q \neq 2$. This unboundedness is still true even if we assume a generalized $T(1)$ condition. These results are induced by the unboundedness of pseudo-differential operators on $M^{p,q}$ whose symbols are of the class $S^0_{1,\delta}$ with $0 < \delta < 1$.

1. Introduction

The modulation spaces $M^{p,q}$ introduced by Feichtinger [5, 6] are fundamental function spaces of time-frequency analysis which is originated in signal analysis or quantum mechanics. See also [7] or Triebel [22]. Recently these spaces have been also recognized as a useful tool for the theory of pseudo-differential operators (see Gröchenig [10]). The objective of this paper is to discuss the boundedness of pseudo-differential operators and also the unboundedness of Calderón-Zygmund operators on modulation spaces.

The boundedness of pseudo-differential operators on the modulation spaces was studied by many authors, for example, Gröchenig and Heil [11], Tachizawa [19], and Toft [20]. They proved that pseudo-differential operators with symbols in some modulation space are $M^{p,q}$-bounded, and as a corollary we have the $M^{p,q}$-boundedness of pseudo-differential operators with symbols in Hörmander’s class $S^0_{0,0}$. A pioneering work of Sjöstrand [15] should be also mentioned here which proved the $L^2$-boundedness by introducing a symbol class based on the spirit of time-frequency analysis. We note here that $L^2 = M^{2,2}$.

In this paper, we consider the case of general symbol classes $S^m_{\rho,\delta}$. First we recall the result of Calderón and Vaillancourt [1], which shows that pseudo-differential operators with symbols in $S^0_{0,\delta}$ with $0 < \delta < 1$ (hence in $S^0_{0,\delta}$ with $0 \leq \delta \leq \rho < 1$, $\delta < 1$) are $L^2$-bounded by reducing them to the case of $S^0_{0,0}$. The proof was carried out by “dilation argument” based on the fact

$$\|f(\lambda x)\|_{L^2(\mathbb{R})} = \lambda^{-n/2}\|f(x)\|_{L^2(\mathbb{R})}, \quad \lambda > 0.$$
We can conclude that the operator norm of $\sigma(X, D)$ is equal to that of $\sigma(\lambda X, \lambda^{-1} D)$ from this equality. By following the same argument, we can also expect that pseudo-differential operators with symbols in $S^0_{\rho, \delta}$ with $0 < \delta < 1$ are $M^{p, q}$-bounded. However, the dilation property of the modulation spaces $M^{p, q}$ is rather different as was investigated in author’s previous paper [17], and due to the property, we can give the following negative answer to this expectation:

**Theorem 1.1.** Let $1 < q < \infty$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Then we have the boundedness $\text{Op}(S^m_{\rho, \delta}) \subset \mathcal{L}(M^{p, q}(\mathbb{R}^n))$ if and only if $m \leq -|1/q - 1/2|\delta n$.

The “only if” part of Theorem 1.1 is a restricted version of Theorem 3.6 which says that $\text{Op}(S^m_{\rho, \delta}) \subset \mathcal{L}(M^{p, q}(\mathbb{R}^n))$ only if $m \leq -|1/q - 1/2|\delta n$. This result was first proved in author’s paper [18]. The “if” part is a restricted version of Theorem 3.1 in Section 3 which treats the general $M^{p, q}$-boundedness with $m \leq m(p, q)$, where $m(p, q)$ is an index such that $m(2, q) = -|1/q - 1/2|\delta n$. We remark that the boundedness $\text{Op}(S^m_{\rho, \delta}) \subset \mathcal{L}(M^{p, q}(\mathbb{R}^n))$ with $m < -|1/q - 1/2|\delta n$ is a straightforward consequence of the dilation argument stated in the above if we directly use the dilation property (Proposition 2.1) of the modulation spaces. The main contribution of Theorem 1.1 is the boundedness result with the critical order for $m$.

Theorem 1.1 also indicates a difference between $L^p$ spaces and the modulation spaces $M^{p, q}$. Fefferman [4] proved that $\text{Op}(S^m_{\rho, \delta}) \subset \mathcal{L}(L^p(\mathbb{R}^n))$ if $m \leq -|1/p - 1/2|(1 - \rho)n$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Moreover, it is known that this order of $m$ is critical (10 Chapter 7, 5.12). We remark that, for the $L^p$-boundedness of pseudo-differential operators with symbols in $S^m_{\rho, \delta}$, the critical order of $m$ is determined by $\rho$. On the other hand, Theorem 1.1 says that, for the $M^{p, q}$-boundedness, the critical order of $m$ is determined by $\delta$ (at least, in the case $p = 2$).

By developing the investigation of pseudo-differential operators, we can also discuss the boundedness property of Calderón-Zygmund operators on the modulation spaces. A Calderón-Zygmund operator is an $L^2$-bounded linear mapping whose distributional kernel is a function $K(x, y)$ outside the diagonal $\{x = y\}$ satisfying

$$|K(x, y)| \leq C|x - y|^{-n},$$

$$|K(x, y) - K(x', y)| \leq C \frac{|x - x'|^\epsilon}{|x - y|^{n+\epsilon}} \quad \text{for } |x - y| > 2|x - x'|,$$

$$|K(x, y) - K(x, y')| \leq C \frac{|y - y'|^\epsilon}{|x - y|^{n+\epsilon}} \quad \text{for } |x - y| > 2|y - y'|,$$

where $0 < \epsilon \leq 1$. It is well known that Calderón-Zygmund operators are $L^p$-bounded for $1 < p < \infty$ (see, for example, [3] Theorem 5.10). In this paper, we consider the problem “Are Calderón-Zygmund operators $M^{p, q}$-bounded?” It is easy to show that the $L^p$-boundedness of the operators of convolution type induces the $M^{p, q}$-boundedness. Hence, Calderón-Zygmund operators of convolution type (Riesz transforms, for example) are always $M^{p, q}$-bounded. We will prove that it is not generally true if remove the convolution type assumption. The following theorem is a simplified version of Theorem 4.1 in Section 4.

**Theorem 1.2.** Let $1 < p, q < \infty$. If $q \neq 2$, then there exists a Calderón-Zygmund operator which is not bounded on $M^{p, q}(\mathbb{R}^n)$. 

Theorem 1.2 is a direct consequence of the unboundedness of pseudo-differential operators on the modulation space $\dot{M}^{p,q}$ whose symbols are of the class $S^0_{1,\delta}$ with $0 < \delta < 1$ (see Theorem 3.33). In fact, it is known that such pseudo-differential operators are Calderón-Zygmund operators (see Chapter 7, Sections 1 and 2).

On the other hand, we know some boundedness results on the Besov spaces $B^{s,q}_p$ and Triebel-Lizorkin spaces $\dot{F}^{s,q}_p$. By the same reason described above, Calderón-Zygmund operators of convolution type are always $\dot{B}^{s,q}_p$-bounded. Furthermore, in the case of non-convolution type, the following generalized $T(1)$ condition is useful to discuss the $\dot{B}^{s,q}_p$-boundedness of $T$:

\begin{equation}
T(P) = 0 \quad \text{for all polynomials } P \text{ such that } \deg P \leq \ell.
\end{equation}

This condition, together with an extra condition on the smoothness of the kernel of $T$ and the weak boundedness property which will be defined in Section 4, induce the $\dot{B}^{s,q}_p$-boundedness of $T$, where $s$ is determined by the order of the polynomials $\ell$ and the smoothness order of the kernel (see Lemarié 13, Meyer and Coifman 14, p.114]). For the $\dot{F}^{s,q}_p$-boundedness of tempered distributions, respectively. We define the Fourier transform of $\dot{F}^{s,q}_p$-boundedness, we need more conditions on the transpose $T^*$ of $T$, that is, a smoothness condition of the kernel of $T^*$ and the condition

\begin{equation}
T^*(P) = 0 \quad \text{for all polynomials } P \text{ such that } \deg P \leq \ell^*
\end{equation}

(see Frazier, Torres and Weiss [8]). We remark that the boundedness on the homogeneous spaces $\dot{B}^{s,q}_p$, $\dot{F}^{s,q}_p$ induces that on the inhomogeneous spaces $(B^{s,q}_p, F^{s,q}_p)$ under suitable conditions.

On account of these results, we can expect the $M^{p,q}$-boundedness of Calderón-Zygmund operators $T$ which satisfy the smoothness conditions on the kernels, conditions (1.1)-(1.2), and the weak boundedness property. But even if we assume these reasonable conditions, we can never prove the $M^{p,q}$-boundedness of Calderón-Zygmund operators. See Theorem 4.1 for the detailed statement.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi.
$$

We introduce the modulation spaces based on Gröchenig [9]. Fix a function $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (called the window function). Then the short-time Fourier transform $V_{\gamma}f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $\gamma$ is defined by

$$
V_{\gamma}f(x, \xi) = (f, M_{\xi}T_{x}\gamma) \quad \text{for } x, \xi \in \mathbb{R}^n,
$$

where $M_{\xi}\gamma(t) = e^{i\xi \cdot t}\gamma(t)$, $T_{x}\gamma(t) = \gamma(t-x)$ and $(\cdot, \cdot)$ denotes the inner product on $L^2(\mathbb{R}^n)$. We note that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, $V_{\gamma}f$ is continuous on $\mathbb{R}^{2n}$ and $|V_{\gamma}f(x, \xi)| \leq C(1 + |x| + |\xi|)^N$ for some constants $C, N \geq 0$ (9 Theorem 11.2.3]). Let $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{M^{p,q}} = \|V_{\gamma}f\|_{L^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_{\gamma}f(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right\}^{1/q} < \infty.
$$
We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ (Proposition 11.3.1) and $M^{p,q}(\mathbb{R}^n)$ is a Banach space (Proposition 11.3.5). The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\gamma \in S(\mathbb{R}^n) \setminus \{0\}$, that is, different window functions yield equivalent norms (Proposition 11.3.2). We denote by $L(M^{p,q}(\mathbb{R}^n))$ the space of all bounded linear operators on $M^{p,q}(\mathbb{R}^n)$.

In order to state the dilation property of the modulation spaces, we introduce the indices. For $1 \leq p \leq \infty$, $p'$ is the conjugate exponent of $p$ (that is, $1/p + 1/p' = 1$). We define subsets of $(1/p, 1/q) \in [0, 1] \times [0, 1]$ in the following way:

$I_1 : \min(1/q, 1/2) \geq 1/p, \quad I_1^* : \max(1/q, 1/2) \leq 1/p$,

$I_2 : \min(1/p, 1/p') \geq 1/q, \quad I_2^* : \max(1/p, 1/p') \leq 1/q$,

$I_3 : \min(1/q, 1/2) \geq 1/p', \quad I_3^* : \max(1/q, 1/2) \leq 1/p'$.

We introduce the indices

$\mu_1(p, q) = \max\{0, 1/q - \min(1/p, 1/p')\} - 1/p, \quad \mu_2(p, q) = \min\{0, 1/q - \max(1/p, 1/p')\} - 1/p.$

Then

$\mu_1(p, q) = \begin{cases} 
-2/p + 1/q & \text{if } (1/p, 1/q) \in I_1, \\
-1/p & \text{if } (1/p, 1/q) \in I_2, \\
1/q - 1 & \text{if } (1/p, 1/q) \in I_3,
\end{cases}$

$\mu_2(p, q) = \begin{cases} 
-2/p + 1/q & \text{if } (1/p, 1/q) \in I_1^*, \\
-1/p & \text{if } (1/p, 1/q) \in I_2^*, \\
1/q - 1 & \text{if } (1/p, 1/q) \in I_3^*.
\end{cases}$

We define the dilation operator $\Lambda_a$ by $\Lambda_a f(x) = f(ax)$, where $a > 0$. The following proposition plays an important role in the proofs of Theorem 3.1 and Proposition 3.2.

**Proposition 2.1** ([17, Theorem 1.1]). Let $1 \leq p, q \leq \infty$. Then the following are true:

1. There exists a constant $C > 0$ such that

$$\|\Lambda_a f\|_{M^{p,q}} \leq C a^{\mu_1(p,q)} \|f\|_{M^{p,q}}$$

for all $f \in M^{p,q}(\mathbb{R}^n)$. 


(2) There exists a constant $C > 0$ such that
$$\|A_f\|_{M^{p,q}} \leq C a^{\mu_2(p,q)} \|f\|_{M^{p,q}} \text{ for all } f \in M^{p,q}(\mathbb{R}^n) \text{ and } 0 < a \leq 1.$$  

The optimality of the power of $a$ in Proposition 2.1 is also discussed in [17].

3. The boundedness of pseudo-differential operators

Let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. The symbol class $S^m_{\rho,\delta}$ consists of all $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that
$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}$$
for all $\alpha, \beta \in \mathbb{Z}_+^n = \{0,1,\ldots\}^n$. We denote by $|\cdot|_{S^m_{\rho,\delta},N}$, $N = 0,1,\ldots$, the semi-norms on $S^m_{\rho,\delta}$, that is,
$$|\sigma|_{S^m_{\rho,\delta},N} = \max_{|\alpha + \beta| \leq N} \sup_{x,\xi \in \mathbb{R}^n} (1 + |\xi|)^{-(m - \rho|\alpha| + \delta|\beta|)} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi)|.$$

For $\sigma \in S^m_{\rho,\delta}$, the pseudo-differential operator $\sigma(X,D)$ is defined by
$$\sigma(X,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \tilde{f}(\xi) \, d\xi$$
for $f \in \mathcal{S}(\mathbb{R}^n)$. We denote by $\text{Op}(S^m_{\rho,\delta})$ the class of all pseudo-differential operators with symbols in $S^m_{\rho,\delta}$. Given a symbol $\sigma \in S^m_{\rho,\delta}$ with $\delta < 1$, the symbol $\sigma^*$ defined by

$$\sigma^*(x,\xi) = \text{Os}_\sigma \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\zeta} \sigma(x+y,\xi+\zeta) \, dy \, d\zeta$$

$$= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\zeta} \chi(\epsilon y, \epsilon \zeta) \sigma(x+y,\xi+\zeta) \, dy \, d\zeta$$
satisfies $\sigma^* \in S^{m\epsilon}_{\rho,\delta}$ and

(3.2) $$\langle \sigma(X,D)f, g \rangle = \langle f, \sigma^*(X,D)g \rangle \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n),$$

where $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfies $\chi(0,0) = 1$ ([12] Chapter 2, Theorem 2.6]). Note that oscillatory integrals are independent of the choice of $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfying $\chi(0,0) = 1$ ([12] Chapter 1, Theorem 6.4]), and the derivatives of $\sigma^*(x,\xi)$ can be written as

$$\partial_\xi^\alpha \partial_\eta^\beta \sigma^*(x,\xi) = \text{Os}_\sigma \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\zeta} (\partial_\xi^\alpha \partial_\eta^\beta \sigma)(x+y,\xi+\zeta) \, dy \, d\zeta$$
in virtue of [12] Chapter 1, Theorem 6.6] (see also [12] p.70, (2.23)).

Our main result on the boundedness of pseudo-differential operators is the following:

**Theorem 3.1.** Let $1 < p, q < \infty$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. If $m \leq -(\mu_1(p,q) - \mu_2(p,q)) \delta n$, then $\text{Op}(S^m_{\rho,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$.

In order to clarify $\mu_1(p,q) - \mu_2(p,q)$, we divide $I_1,\ldots,I_5$ in the following way:

$$J_1 : (I_1 \cap I_2^c) \cup (I_2 \cap I_1^c), \quad J_2 : (I_1 \cap I_3^c) \cup (I_3 \cap I_1^c), \quad J_3 : (I_2 \cap I_3^c) \cup (I_3 \cap I_2^c).$$

See the following figure:
2

Let \( \psi \| (3.4) \)

sequence of the dilation property of the modulation spaces (Proposition 2.1).

Let Proposition 3.2.

Then we have

\[ \sigma < \delta \]

Since the decomposition (3.4), we have

\[ \delta < 1 \]

The following weak form of Theorem 3.1 (when \( \delta < 1 \)) is a straightforward consequence of the dilation property of the modulation spaces (Proposition 2.1).

**Proposition 3.2.** Let \( 1 < p, q < \infty, m \in \mathbb{R} \) and \( 0 \leq \delta \leq \rho \leq 1 \). If \( m < -(\mu_1(p, q) - \mu_2(p, q)) \delta n \), then \( \text{Op}(S^m_{p, \delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \).

**Proof.** Our proof is based on that of [16, Chapter 7, Theorem 2]. Let \( \sigma \in S^m_{p, \delta} \). By the decomposition (3.4), we have

\[ \sigma(x, \xi) = \sum_{j=0}^{\infty} \sigma(x, \xi) \psi_j(\xi) = \sum_{j=0}^{\infty} \sigma_j(x, \xi). \]

Since \( \sigma_j(X, D) = \Lambda_{2j^\delta} \sigma_j(2^{-j^\delta} X, 2^{j^\delta} D) \Lambda_{2^{-j^\delta}}, \) by Proposition 2.1 we see that

\[ \| \sigma(X, D) f \|_{M^{p,q}} \leq \sum_{j=0}^{\infty} \| \sigma_j(X, D) f \|_{M^{p,q}} \]

\[ = \sum_{j=0}^{\infty} \| \Lambda_{2j^\delta} \sigma_j(2^{-j^\delta} X, 2^{j^\delta} D) \Lambda_{2^{-j^\delta}} f \|_{M^{p,q}} \]

\[ \leq \sum_{j=0}^{\infty} 2^{j^\delta n}(\mu_1(p,q) - \mu_2(p,q)) \| \sigma_j(2^{-j^\delta} X, 2^{j^\delta} D) \|_{\mathcal{L}(M^{p,q})} \| f \|_{M^{p,q}} \]

for all \( f \in \mathcal{S}(\mathbb{R}^n) \). Since \( S^m_{p, \delta} \subset S^m_{\delta, \delta} \) and \( 1 + 2\delta |\xi| \sim 2j \) on \( \text{supp} \psi_j(2^{j^\delta} \cdot) \), if we set \( \tau_j(x, \xi) = 2^{-j^m} \sigma_j(2^{-j^m} x, 2^{j^m} \xi) \), then for any \( \alpha, \beta \in \mathbb{Z}_+^n \)

\[ |\partial_x^\alpha D_\xi^\beta \tau_j(x, \xi)| \leq C_{\alpha, \beta} |\sigma|_{S^m_{p,q,|\alpha+\beta|}} \]

\[ \mu_1(p, q) - \mu_2(p, q) \]
for all $j \in \mathbb{Z}_+$. Hence, by $\text{Op}(S^0_{0,0}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$ and $m + (\mu_1(p,q) - \mu_2(p,q))\delta n < 0$, we have

$$
\sum_{j=0}^{\infty} 2^{j\delta n(\mu_1(p,q) - \mu_2(p,q))} \|\sigma_j(2^{-j\delta} X, 2^{j\delta} D)\|_{\mathcal{L}(M^{p,q})} \|f\|_{M^{p,q}}
$$

$$
= \sum_{j=0}^{\infty} 2^{j(m + (\mu_1(p,q) - \mu_2(p,q))\delta n)} \|\tau_j(X, D)\|_{\mathcal{L}(M^{p,q})} \|f\|_{M^{p,q}}
$$

$$
\leq C \left( \sum_{j=0}^{\infty} 2^{j(m + (\mu_1(p,q) - \mu_2(p,q))\delta n)} \right) \|f\|_{M^{p,q}} = C \|f\|_{M^{p,q}}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. The proof is complete. \qed

In view of Proposition 3.2, the non-trivial part of Theorem 3.1 is the boundedness with the critical order $m = - (\mu_1(p,q) - \mu_2(p,q))\delta n$. The rest of this section is devoted to the proof of it. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$
(3.5) \quad \text{supp } \varphi \subset [-1,1]^n \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \varphi(\xi - k) \equiv 1.
$$

By the decompositions (3.4) and (3.5), we have

$$
1 \equiv \sum_{j=0}^{\infty} \psi_j(\xi) = \sum_{j=0}^{\infty} \psi_j(\xi) \left( \sum_{k \in \mathbb{Z}^n} \varphi(2^{-j\delta} y - k) \right) \left( \sum_{\ell \in \mathbb{Z}^n} \varphi(2^{-j\delta} \xi - \ell) \right).
$$

Hence,

$$
(3.6) \quad \sum_{k \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} \sum_{j=0}^{\infty} \varphi(2^{-j\delta} y - k) \varphi(2^{-j\delta} \xi - \ell) \psi_j(\xi) = 1
$$

for all $(y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. For $\sigma \in S^m_{\rho,\delta}$, we set

$$
\sigma^{k,\ell}_j(x, \xi) = \varphi(2^{-j\delta} D_x - k) \sigma(x, \xi) \varphi(2^{-j\delta} \xi - \ell) \psi_j(\xi),
$$

where

$$
\varphi(2^{-j\delta} D_x - k) \sigma(x, \xi) = \mathcal{F}_1^{-1}[\varphi(2^{-j\delta} \cdot - k) \mathcal{F}_1 \sigma(\cdot, \xi)](x)
$$

$$
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} \varphi(2^{-j\delta} y - k) \mathcal{F}_1 \sigma(y, \xi) dy,
$$

$\mathcal{F}_1$ and $\mathcal{F}_1^{-1}$ are the partial Fourier transform and inverse Fourier transform in the first variable, respectively.

**Lemma 3.3.** Let $1 \leq p \leq \infty$, $m \in \mathbb{R}$, $0 \leq \delta \leq 1$ and $\sigma \in S^m_{\rho,\delta}$. Then there exists a constant $C > 0$ such that

$$
\|\sigma^{k,\ell}_j(X, D)f\|_{L^p} \leq C 2^j m (1 + |k|)^{-n-1} \|f\|_{L^p}
$$

for all $j \in \mathbb{Z}_+$, $k, \ell \in \mathbb{Z}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, where $\sigma^{k,\ell}_j$ is defined by (3.7).

**Proof.** Using $\sigma^{k,\ell}_j(X, D) = \Lambda_{2^{-j}S} \sigma_j^{k,\ell}(2^{-j\delta} X, 2^{j\delta} D) \Lambda_{2^{-j}S}$, we have

$$
\|\sigma^{k,\ell}_j(X, D)f\|_{L^p} = 2^{-j\delta n/p \|\sigma_j^{k,\ell}(2^{-j\delta} X, 2^{j\delta} D) \Lambda_{2^{-j}S} f\|_{L^p}}
$$
for all \( f \in S(\mathbb{R}^n) \). Set
\[
K_j^{k,\ell}(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \sigma_j^{k,\ell}(2^{-j\delta} x, 2^{j\delta} \xi) \, d\xi.
\]
Since
\[
\sigma_j^{k,\ell}(2^{-j\delta} X, 2^{j\delta} D)\Lambda_{2^{-j\delta}} f(x) = \int_{\mathbb{R}^n} K_j^{k,\ell}(x, y) (\Lambda_{2^{-j\delta}} f)(y) \, dy,
\]
if
\[
|K_j^{k,\ell}(x, y)| \leq C 2^{jm} |k|^{-n-1} (1 + |x - y|)^{-n-1}
\]
for all \( j \in \mathbb{Z}_+ \) and \( k, \ell \in \mathbb{Z}^n \), then
\[
||\sigma_j^{k,\ell}(X, D)||_{L^p} \leq C 2^{-j\delta n/p} 2^{jm} |k|^{-n-1} \left[ (1 + |\cdot|)^{-n-1} \ast |\Lambda_{2^{-j\delta}} f| \right]_{L^p}
\]
\[
\leq C 2^{-j\delta n/p} 2^{jm} |k|^{-n-1} |\Lambda_{2^{-j\delta}} f|_{L^p}
\]
\[
= C 2^{jm} |k|^{-n-1} |f|_{L^p}
\]
for all \( j \in \mathbb{Z}_+, k, \ell \in \mathbb{Z}^n \) and \( f \in S(\mathbb{R}^n) \). This is the desired result. We prove (3.8).

Set \( \tau_j(x, \xi) = 2^{-jm} \sigma(2^{-j\delta} x, 2^{j\delta} \xi) \psi_j(2^{j\delta} \xi) \). Since
\[
\sigma_j^{k,\ell}(x, \xi) = \left( \int_{\mathbb{R}^n} e^{i k \cdot (x-z)} \Phi(2^{j\delta} x - z) \sigma(2^{-j\delta} z, \xi) \psi_j(\xi) \, dz \right) \varphi(2^{-j\delta} \xi - \ell),
\]
we have
\[
K_j^{k,\ell}(x, y) = \frac{2^{jm}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \left\{ \left( \int_{\mathbb{R}^n} e^{i k \cdot (x-z)} \Phi(x - z) \tau_j(z, \xi) \, dz \right) \varphi(\xi - \ell) \right\} \, d\xi,
\]
where \( \Phi = \mathcal{F}^{-1} \varphi \). On the other hand, for any \( \alpha, \beta \in \mathbb{Z}^n_+ \),
\[
|\partial_\xi^\alpha \partial_\ell^\beta \tau_j(x, \xi)| \leq C_{\alpha,\beta} |\sigma|_{S_{\alpha,\beta}}
\]
for all \( j \in \mathbb{Z}_+ \). Let \( \alpha, \beta \in \mathbb{Z}^n_+ \) be such that \( |\alpha|, |\beta| \leq n + 1 \). Using
\[
(x - y)^\alpha k^\beta K_j^{k,\ell}(x, y)
\]
\[
= C_{\alpha,\beta} 2^{jm} \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta} C_{\alpha_1,\beta_1}^\alpha C_{\alpha_2,\beta_2}^\beta \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \left\{ \left( \int_{\mathbb{R}^n} e^{i k \cdot (x-z)} \partial_\xi^\alpha \Phi(x - z) \partial_\ell^\beta \tau_j(z, \xi) \, dz \right) \varphi(\xi - \ell) \right\} \, d\xi,
\]
we see that
\[
|(x - y)^\alpha k^\beta K_j^{k,\ell}(x, y)| \leq \left( C_{\alpha,\beta} |\sigma|_{S_{\alpha,\beta}} \sup_{\alpha_1 \leq \alpha} \| \partial_\xi^\alpha \varphi \|_{L^1} \sup_{\beta_1 \leq \beta} \| \partial_\ell^\beta \Phi \|_{L^1} \right) 2^{jm}.
\]
This implies (3.8). The proof is complete. \( \square \)

For \( 0 \leq \delta < 1 \), we take a sufficiently large integer \( j_0 \) such that
\[
2^{j_0(1-\delta) - 3} \geq \sqrt{n}.
\]
We recall that \( \varphi, \psi \in S(\mathbb{R}^n) \) satisfy \( \varphi \subset [-1, 1]^n \) and \( \psi \subset \{ \xi : 2^{-1} \leq |\xi| \leq 2 \} \) (see (3.4) and (3.5)).
Lemma 3.4. Let $0 \leq \delta < 1$ and $\ell \in \mathbb{Z}^n$. If there exists a positive integer $j(\ell) \geq j_0 + 1$ such that $\text{supp} \varphi(2^{-j(\ell)\delta} \cdot \ell) \cap \text{supp} \psi(2^{-j(\ell)\delta} \cdot \ell) \neq \emptyset$, then

$$\text{supp} \varphi(2^{-j\delta} \cdot \ell) \cap \text{supp} \psi(2^{-j\delta} \cdot \ell) = \emptyset \quad \text{for all } |j - j(\ell)| \geq j_0,$$

where $j_0$ is defined by (3.9).

Proof. Since $\text{supp} \varphi(2^{-j\delta} \cdot \ell) \subset \{ |\xi| - 2^j \delta \leq 2^j \delta \sqrt{n} \}$ and $\text{supp} \psi(2^{-j\delta} \cdot \ell) \subset \{ |\xi| \leq 2^{j+1} \}$, our assumption $\text{supp} \varphi(2^{-j(\ell)\delta} \cdot \ell) \cap \text{supp} \psi(2^{-j(\ell)\delta} \cdot \ell) \neq \emptyset$ gives

$$2^{j(\ell)-1} - 2^{j(\ell)\delta} \sqrt{n} \leq 2^{j(\ell)\delta} |\ell| \leq 2^{j(\ell)+1} + 2^{j(\ell)\delta} \sqrt{n}. \tag{3.10}$$

Let $|j - j(\ell)| \geq j_0$. We consider the case $j - j(\ell) \leq -j_0$. By (3.9), (3.10) and $j(\ell) \geq j_0 + 1$, we see that

$$2^{j\delta}|\ell| - 2^{j\delta} \sqrt{n} = 2^{j\delta-j(\ell)\delta+j(\ell)\delta}|\ell| - 2^{j\delta} \sqrt{n} \geq 2^{j\delta-j(\ell)\delta+1} \sqrt{n} \geq 2^{j\delta+1} \sqrt{n} \geq 2^{j\delta+1}.$$

Hence,

$$\text{supp} \varphi(2^{-j\delta} \cdot \ell) \subset \{ |\xi| \geq 2^{j\delta} |\ell| - 2^{j\delta} \sqrt{n} \} \subset \{ |\xi| > 2^{j+1} \}$$

for all $j - j(\ell) \leq -j_0$. This implies $\text{supp} \varphi(2^{-j\delta} \cdot \ell) \cap \text{supp} \psi(2^{-j\delta} \cdot \ell) = \emptyset$ for all $j - j(\ell) \leq -j_0$.

In the same way, we can prove

$$\text{supp} \varphi(2^{-j\delta} \cdot \ell) \subset \{ |\xi| \leq 2^{j\delta} |\ell| + 2^{j\delta} \sqrt{n} \} \subset \{ |\xi| < 2^{j-1} \}$$

for all $j - j(\ell) \geq j_0$. This implies $\text{supp} \varphi(2^{-j\delta} \cdot \ell) \cap \text{supp} \psi(2^{-j\delta} \cdot \ell) = \emptyset$ for all $j - j(\ell) \geq j_0$. \hfill $\square$

Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp} \eta$ is compact and $| \sum_{\nu \in \mathbb{Z}^n} \eta(\xi - \nu)| \geq C > 0$ for all $\xi \in \mathbb{R}^n$. It is well known that

$$\| f \|_{\mathcal{M}_{p,q}} \sim \left( \sum_{\nu \in \mathbb{Z}^n} \| \eta(D - \nu)f \|_{L_p}^q \right)^{1/q}, \tag{3.11}$$

where $\eta(D - \nu)f = \mathcal{F}^{-1}[\eta(\cdot - \nu)\hat{f}]$ (see, for example, [22]).

Proposition 3.5. Let $1 \leq p \leq \infty$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $\sigma \in \mathcal{S}_{p,\delta}^m$. If $m \leq - (\mu_1(p, \infty) - \mu_2(p, \infty)) \delta n$, then there exists a constant $C > 0$ such that

$$\| \sigma(X,D)f \|_{\mathcal{M}_{p,\infty}} \leq C \| f \|_{\mathcal{M}_{p,\infty}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. In view of (3.11), we estimate $\sup_{\nu \in \mathbb{Z}^n} \| \varphi(D - \nu)(\sigma(X,D)f) \|_{L_p}$, where $\varphi$ is as (3.3) and $f \in \mathcal{S}(\mathbb{R}^n)$. By the decomposition (3.6), we have

$$\sigma(x, \xi) = \sum_{k \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} \sum_{j=0}^{\infty} \sigma_{k,\ell}^j(x, \xi)$$

$$= \sum_{k \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} \sum_{j=0}^{j_0} \sigma_{k,\ell}^j(x, \xi) + \sum_{k \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} \sum_{j=j_0+1}^{\infty} \sigma_{k,\ell}^j(x, \xi),$$
where \( \sigma_{j}^{k,l}(x, \xi) \) is defined by (3.7), and \( j_{0} \) is defined by (3.9). We only consider the second sum since the estimate for the first sum can be carried out in a similar way. By Lemma 3.3, we see that
\[
\sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{j=j_{0}+1}^{\infty} \sigma_{j}^{k,l}(x, \xi) = \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \sigma_{j}^{k,l}(x, \xi).
\]

Since \( \varphi(2^{-j_{0}} \cdot -k) \subset 2^{j_{0}}k + [-2^{j_{0}}, 2^{j_{0}}]^{n} \) and
\[
\mathcal{F}[\sigma_{j}^{k,l}(X, D)f](y) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} \mathcal{F} \varphi(2^{-j_{0}}D_{x} - k) \sigma(x, \xi) \varphi(2^{-j_{0}}\xi - \ell) \psi(2^{-j_{0}}\xi) \hat{f}(\xi) \, d\xi,
\]
we see that \( \mathcal{F}[\sigma_{j}^{k,l}(X, D)f] \subset 2^{j_{0}}(k + \ell) + [-2^{j_{0}+1}, 2^{j_{0}+1}]^{n} \). This gives
\[
\varphi(D - \nu) \left( \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \sigma_{j}^{k,l}(X, D)f \right) = \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \varphi(D - \nu \sigma_{j}^{k,l}(X, D)f).
\]

Let \( \eta \in \mathcal{S}(\mathbb{R}^{n}) \) be such that \( \text{supp} \eta \) is compact, \( \eta = 1 \) on \( \text{supp} \varphi \) and \( |\sum_{\nu \in \mathbb{Z}^{n}} \eta(\xi - \nu)| \geq C > 0 \) for all \( \xi \in \mathbb{R}^{n} \). Note that \( \sigma_{j}^{k,l}(x, \xi) = \sigma_{j}^{k,l}(x, \xi) \eta(2^{-j_{0}}\xi - \ell) \). By Lemma 3.3 we have
\[
\left\| \varphi(D - \nu) \left( \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \sigma_{j}^{k,l}(X, D)f \right) \right\|_{L^{p}} \leq \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \left\| \varphi(D - \nu) \sigma_{j}^{k,l}(X, D)f \right\|_{L^{p}}
\]
\[
\leq C \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} \left\| \sigma_{j}^{k,l}(X, D)f \eta(2^{-j_{0}}D - \ell)f \right\|_{L^{p}}
\]
\[
\leq C \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} 2^{jm} (1 + |k|)^{-n-1} \left\| \eta(2^{-j_{0}}D - \ell)f \right\|_{L^{p}}.
\]
Since \( \eta(2^{-j_{0}}D - \ell)f = \Lambda_{2^{-j_{0}}} \eta(D - \ell)f \), \( \Lambda_{2^{-j_{0}}}f \), \( \mu_{1}(p, \infty) - \mu_{2}(p, \infty) \delta n \leq 0 \), by (3.11) and Proposition 2.1 we see that
\[
\sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} 2^{jm} (1 + |k|)^{-n-1} \left\| \eta(2^{-j_{0}}D - \ell)f \right\|_{L^{p}}
\]
\[
= \sum_{k \in \mathbb{Z}^{n}} \sum_{\ell \in \mathbb{Z}^{n}} \sum_{i=1, \ldots, n} \sum_{j \geq j_{0}+1} \sum_{\|j-j_{0}\| < j_{0}} 2^{j(m-\delta n/p)} (1 + |k|)^{-n-1} \left\| \eta(D - \ell)(\Lambda_{2^{-j_{0}}}f) \right\|_{L^{p}}.
This implies

$$\text{On the other hand, it is well known that } \text{Op}(S(3.13)) \subset \bigcap_{r>0} L^{r,\infty}(\mathbb{R}^n).$$

Hence,

$$(3.12) \quad \text{Op}(S_{\rho,\delta}^{m-n/r}) \subset \mathcal{L}(M^{r,\infty}(\mathbb{R}^n)).$$

On the other hand, it is well known that $\text{Op}(S_{\rho,\delta}^0) \subset \mathcal{L}(L^2(\mathbb{R}^n))$ (see Chapter 7, Theorem 2). Hence,

$$(3.13) \quad \text{Op}(S_{\rho,\delta}^0) \subset \mathcal{L}(M^{2,2}(\mathbb{R}^n)).$$
By interpolation, (3.12) and (3.13) give $\text{Op}(S^{\rho,\sigma+(-\delta_0,\rho)(1-\theta)}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$.

Since $(1-\theta)/r = 1/p - 1/q$ and $M^{p,q}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^n)$, we obtain

$$\text{Op}(S^{(-1/p-1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_1 \text{ with } 1/p \geq 1/2.$$  

In the same way, we can prove

$$\text{Op}(S^{(-1/p-1/q+1)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_3 \text{ with } 1/p \leq 1/2.$$  

Note that $\mu_1(p, q) - \mu_2(p, q) = -1/p - 1/q + 1$ if $(1/p, 1/q) \in J_3$ with $1/p \leq 1/2$.

We next consider the case $(1/p, 1/q) \in J_2$. Then $\mu_1(p, q) - \mu_2(p, q) = -1/p + 1/q$. Let $\sigma \in S^{(-1/p+1/q)\delta_0}_{p,\delta}$. Then $\sigma^*(X, D)$ satisfies (3.12), where $\sigma^* \in S^{(-1/p+1/q)\delta_0}_{p,\delta}$ is defined by (3.11). Let $p', q'$ be the conjugate exponents of $p, q$, respectively. Since $-(-1/p+1/q)\delta_0 = -(1/p'-1/q')\delta_0$ and $(1/p', 1/q') \in J_1$ with $1/p' \geq 1/2$, by (3.14), we see that $\sigma^*(X, D)$ is bounded on $M^{p',q'}(\mathbb{R}^n)$. Then, by duality and (3.15), we obtain that $\sigma(X, D)$ is bounded on $M^{p,q}(\mathbb{R}^n)$, that is,

$$\text{Op}(S^{(-1/p+1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_1 \text{ with } 1/p \leq 1/2.$$  

In the same way, using duality and (3.15), we can prove

$$\text{Op}(S^{-(1/p+1/q-1)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_3 \text{ with } 1/p \geq 1/2.$$  

Note that $\mu_1(p, q) - \mu_2(p, q) = 1/p + 1/q - 1$ if $(1/p, 1/q) \in J_3$ with $1/p \geq 1/2$.

Finally, we consider the case $(1/p, 1/q) \in J_2$. Let $1/p \leq 1/2$. Then $\mu_1(p, q) - \mu_2(p, q) = -2/p + 1$. Since $(1/p, 1/p') \in J_1$ with $1/p \leq 1/2$ and $(1/p, 1/p') \in J_3$ with $1/p \geq 1/2$, (3.16) and (3.15) imply $\text{Op}(S^{(-1/p+1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p',q'}(\mathbb{R}^n))$ and $\text{Op}(S^{(-2/p+1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,p}(\mathbb{R}^n))$. Then, by interpolation, we have

$$\text{Op}(S^{(-1/p+1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)),$$

where $1/q = \rho/p' + (1-\theta)/p$, that is,

$$\text{Op}(S^{(-2/p+1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_2 \text{ with } 1/p \leq 1/2.$$  

In the same way (or duality with (3.18)), using interpolation, (3.14) and (3.17), we can prove

$$\text{Op}(S^{(2/p-1/q)\delta_0}_{p,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)) \text{ if } (1/p, 1/q) \in J_2 \text{ with } 1/p \geq 1/2.$$  

Note that $\mu_1(p, q) - \mu_2(p, q) = 2/p - 1$ if $(1/p, 1/q) \in J_2$ with $1/p \geq 1/2$. The proof is complete.

In order to prove the “only if” part of Theorem 1.1, we introduce a special symbol. Let $\varphi, \eta \in \mathcal{S}(\mathbb{R}^n)$ be real-valued functions satisfying

$$\varphi: \text{ supp } \varphi \subset \{ \xi : |\xi| \leq 1/8 \}, \quad \int_{\mathbb{R}^n} \varphi(\xi) d\xi = 1,$$

$$\eta: \text{ supp } \eta \subset \{ \xi : 2^{-1/2} \leq |\xi| \leq 2^{1/2} \}, \quad \eta = 1 \text{ on } \{ \xi : 2^{-1/4} \leq |\xi| \leq 2^{1/4} \}.$$  

Moreover, we assume that $\varphi$ is radial. This assumption implies that $\Phi$ is also real-valued, where $\Phi = F^{-1}\varphi$. Then we define

$$\sigma(x, \xi) = \sum_{j=0}^{\infty} 2^{jm} \left( \sum_{0 < |k| \leq 2^{j/2}} e^{-ik(2^{j/2}\xi - 2^{j/2}x - k)} \Phi(2^{j/2}x - k) \right) \eta(2^{-j}\xi),$$  

(3.19)
where \(0 < \delta < 1\) and \(j_0 \in \mathbb{Z}_+\) is chosen to satisfy
\[
1 + 2^{j_0(\delta - 1) + 1} \leq 2^{1/4}, \quad 1 - 2^{j_0(\delta - 1) + 1} \geq 2^{-1/4}, \quad 2^{-j_0 \delta / 2} \sqrt{n} \leq 2^{-3}.
\]
The symbol \(\sigma^*\) is constructed from \(\sigma\) using the oscillatory integral (3.1). The following is the “only if” part of Theorem 3.6.

**Theorem 3.6** ([18 Theorem 1.1]). Let \(1 < p, q < \infty, 0 \leq \delta < 1\) and \(m > -1/q - 1/2|\delta n|\). Then the symbols \(\sigma\) and \(\sigma^*\) defined by (3.11) belong to the class \(S^{m}_{1,\delta}\). Moreover, if \(q \geq 2\) (\(q \leq 2\) resp.), then the corresponding operator \(\sigma(X, D)(\sigma^*(X, D)\) resp.) is not bounded on \(M^{p,q}(\mathbb{R}^n)\).

Precisely speaking, [18] treated only the case \(0 < \delta < 1\), but the case \(\delta = 0\) can be also included by a simple argument. If \(m > 0\) and \(\sigma(\xi) = (1 + |\xi|^2)^{m/2}\), then \(\sigma(D)\) is not bounded on \(L^2(\mathbb{R}^n)\). Indeed,
\[
\|\sigma(D)(M_j f)\|_{L^2} = (2\pi)^{-n/2}\|\hat{\sigma}(D)(M_j f)\|_{L^2} = (2\pi)^{-n/2}\|\hat{f}(\cdot - j)\|_{L^2} \geq C|j|^m
\]
for all \(j \in \mathbb{Z}^n\), where \(f \in \mathcal{S}(\mathbb{R}^n)\) such that \(\text{supp} \hat{f} \subset \{|\xi| \leq 1/2\}\). On the other hand, \(\|M_j f\|_{L^2} = \|f\|_{L^2}\) for all \(j \in \mathbb{Z}^n\). Hence, \(\sigma(D)\) is not bounded on \(L^2(\mathbb{R}^n)\). Moreover, \(\sigma(D)\) is not bounded on \(M^{p,q}(\mathbb{R}^n)\) for any \(1 < p, q < \infty\). Indeed, since \(\sigma(D)^* = \sigma(D)\), if \(\sigma(D)\) is bounded on \(M^{p,q}(\mathbb{R}^n)\), then \(\sigma(D)\) is also bounded on \(M^{p',q'}(\mathbb{R}^n)\). Then, by interpolation, \(\sigma(D)\) is bounded on \(M^{2,2}(\mathbb{R}^n)\). However, since \(M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)\), this is a contradiction. Hence, \(\sigma(D)\) is not bounded on \(M^{p,q}(\mathbb{R}^n)\) for any \(1 < p, q < \infty\). Note that \(\sigma(\xi) = (1 + |\xi|^2)^{m/2}\) belongs to \(S^{m}_{1,0}\).

**4. The unboundedness of Calderón-Zygmund operators**

We recall the theory of Calderón-Zygmund operators based on [8] (see also [21]). Let \(\mathcal{D}(\mathbb{R}^n)\) be the space of all infinitely differentiable functions with compact support. For \(\ell \in \mathbb{Z}_+\), we denote by \(\mathcal{D}_\ell(\mathbb{R}^n)\) the space of all \(g \in \mathcal{D}(\mathbb{R}^n)\) such that \(\int_{\mathbb{R}^n} x^\beta g(x) \, dx = 0\) for all \(|\beta| \leq \ell\). Let \(T : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)\) be a continuous linear operator, where \(\mathcal{D}'(\mathbb{R}^n)\) is the dual space of \(\mathcal{D}(\mathbb{R}^n)\). We denote by \(K\) the distributional kernel of \(T\), that is,
\[
\langle Tf, g \rangle = \langle K, g \otimes f \rangle \quad \text{for all } f, g \in \mathcal{D}(\mathbb{R}^n).
\]
We say that \(T\) is a generalized Calderón-Zygmund operator of smoothness \(\ell + \epsilon\) (we write \(T \in \text{CZO}(\ell + \epsilon)\)), where \(\ell \in \mathbb{Z}_+\) and \(0 < \epsilon < 1\), if the restriction of \(K\) to \(\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}\) is a function with continuous partial derivatives in the variable \(x\) up to order \(\ell\) which satisfy
\[
|\partial_\alpha^0 K(x, y)| \leq C|x - y|^{-n - |\alpha|} \quad \text{for } |\alpha| \leq \ell,
\]
\[
|\partial_\alpha^0 K(x, y) - \partial_\alpha^0 K(x', y)| \leq C \frac{|x - x'|^\ell}{|x - y|^{n + \ell + \epsilon}}
\]
for \(|\alpha| = \ell\) and \(|x - y| > 2|x - x'|\). Let \(\phi \in \mathcal{D}(\mathbb{R}^n)\) be such that \(\phi = 1\) on \(\{x : |x| \leq 1\}\). If \(T \in \text{CZO}(\ell + \epsilon)\) and \(f \in C^\infty(\mathbb{R}^n)\) satisfies \(f(x) = O(|x|^\ell)\) as \(|x| \to \infty\), then the limit \(\lim_{n \to \infty} \langle T(\phi(x/\sqrt{n}) f), g \rangle\) exists for all \(g \in \mathcal{D}_1(\mathbb{R}^n)\) ([8] Lemma 1.19, [21] Lemma 2.2.12). In particular, \(T(x^\beta)\) can be defined as an element of \(\mathcal{D}'(\mathbb{R}^n)\), where \(|\beta| \leq \ell\) and \(\mathcal{D}'(\mathbb{R}^n)\) is the dual space of \(\mathcal{D}_1(\mathbb{R}^n)\). Note that this limit \(\lim_{n \to \infty} \langle T(\phi(x/\sqrt{n}) f), g \rangle\) is independent of the choice of \(\phi\) ([8, p.49]). It is known that, if \(T \in \text{CZO}(\ell + \epsilon)\) is a translation invariant operator, then \(T(x^\beta) = 0\) as an element of \(\mathcal{D}'_k(\mathbb{R}^n)\) for
all $|β| = k ≤ ℓ$ (21). A continuous linear operator $T$ is said to satisfy the weak boundedness property, if for each bounded subset $B$ of $D(ℝ^n)$ there exists a constant $C = C(ℳ) > 0$ such that

$$|⟨Tf_+,g_+⟩| ≤ CR^n$$

for all $f, g ∈ B$, $x ∈ ℝ^n$ and $R > 0$, where $f_+,g_+ = f((y - x)/R)$. The transpose of $T$, $T^∗ : D(ℝ^n) → D' (ℝ^n)$ is defined by

$$⟨T^∗ f, g⟩ = ⟨T g, f⟩$$

for $f, g ∈ D(ℝ^n)$.

Note that the kernel of $T^∗$ is given by $K^∗(x, y) = K(y, x)$. David and Journé [2] proved that, when $T, T^∗ ∈ CZO(ε)$, $T$ is bounded on $L^2(ℝ^n)$ if and only if $T(1), T^*(1) ∈ BMO(ℝ^n)$ and $T$ satisfies the weak boundedness property.

Our main result on the unboundedness of Calderón-Zygmund operators is the following:

**Theorem 4.1.** Let $1 < p, q < ∞$. If $q ≠ 2$, then there exists an operator $T : S(ℝ^n) → S'(ℝ^n)$ such that $T, T^∗ ∈ CZO(ℓ + ε)$ for all $ℓ ∈ ℤ_+$, where $0 < ε < 1$, $T(P) = T^∗(P) = 0$ for all polynomials $P$, $T$ satisfies the weak boundedness property, but $T$ is not bounded on $M^p(ℝ^n)$.

By David-Journé’s $T(1)$ theorem mentioned above, as a corollary of Theorem 4.1, we have Theorem 1.2. In the rest of this section, we prove Theorem 4.1.

**Lemma 4.2.** Let $σ$ be defined by (3.19) with $0 < δ < 1$ and $j_0$ satisfying (3.20) and $2j_0(δ−1)+2 < 2^{−1/2}$. Then $σ, σ^* ∈ S^m_{1,δ}$ and $(∂^ασ)(x, 0) = (∂^ασ^*)(x, 0) = 0$ for all $α ∈ ℤ^n_+$, where $σ^*$ is defined by (3.1).

Proof. By Theorem 3.6, we have $σ, σ^* ∈ S^m_{1,δ}$. Since $supp |s| = \{(x, ξ) : |ξ| ≥ 2^{j_0−1/2}\}$, we obtain $supp (σ^*)(x, 0) = 0$ for all $α ∈ ℤ^n_+$.

Let $χ_1, χ_2 ∈ S(ℝ^n)$ be such that $χ_1(0) = χ_2(0) = 1$ and $supp χ_1, supp χ_2 ⊂ \{|ξ| ≤ 1\}$. Since $supp η(2^{−j}) ⊂ \{2^{j−1/2} ≤ |ξ| ≤ 2^{j+1/2}\}$ and $supp χ_2(ε) ⊂ \{|ξ| ≤ 1/ε\}$, for each $0 < ε < 1$, there exists $N_ε$ such that $supp (η(2^{−j}) ∩ supp χ_2(ε) = ∅$ for all $j ≥ N_ε$. Hence, by (3.3), we see that

$$⟨∂^ασ^∗⟩(x, 0) = O_{α, β}(2π)^n \int_{ℝ^n} \int_{ℝ^n} e^{-iy·ζ} (∂^ασ)(x + y, ζ) dy dζ$$

$$= \lim_{ε→0} \frac{1}{2π)^n} \int_{ℝ^n} \int_{ℝ^n} e^{-iy·ζ} χ_1(εy) χ_2(εζ) \left\{\sum_{j=j_0}^{∞} 2^{j(m−|α|)} \int_{ℝ^n} \int_{ℝ^n} e^{-iy·ζ} \left(Φ(2^{jδ/2}(x + y)−k) (∂^αη)(2^{−j}ζ)\right)dy dζ \right\}$$

$$= \lim_{ε→0} \sum_{j=j_0}^{∞} 2^{j(m−|α|)} \sum_{0<|k|≤2^{jδ/2}} \frac{1}{2π)^n} \int_{ℝ^n} \int_{ℝ^n} e^{-iy·ζ} χ_1(εy) χ_2(εζ)$$

$$× e^{ik·(2^{jδ/2}(x + y)−k)} (Φ(2^{jδ/2}(x + y)−k) (∂^αη)(2^{−j}ζ) dy dζ$$

$$= \lim_{ε→0} \sum_{j=j_0}^{∞} 2^{j(m−|α|)} \sum_{0<|k|≤2^{jδ/2}} \frac{1}{2π)^n} \int_{ℝ^n} χ_2(εζ) (∂^αη)(2^{−j}ζ)$$

$$× \left(\int_{ℝ^n} e^{-iy·ζ} χ_1(εy) e^{ik·(2^{jδ/2}(x + y)−k)} (Φ(2^{jδ/2}(x + y)−k) dy dζ \right) dζ$$
Proof. By Plancherel’s theorem, we have

\[ \eta_{\epsilon,k}(x, \xi) = \int_{\mathbb{R}^n} e^{-i\epsilon \xi} \chi_1(\xi) e^{-ik \cdot (2^{j/2}(x+y) - k)} \Phi(2^{j/2}(x+y) - k) dy \]

\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\epsilon \xi} \chi_1(\xi + \eta) e^{-ik \cdot (2^{j/2}(x+y) - k)} \Phi(2^{j/2}(x+y) - k) d\eta. \]

Let \( j \geq j_0 \), \( 0 < |k| \leq 2^{j/2} \), \( 0 < \epsilon < 1 \) and \( x \in \mathbb{R}^n \). Since \( \chi_1(\cdot) \subset \{ |\xi| \leq 1 \} \) and \( \text{supp} \varphi(2^{-j/2} \cdot + k) \subset \{ |\xi| \leq 2^{j+1} \} \), we see that \( \text{supp} \eta_{\epsilon,k}(x, \cdot) \subset \{ |\xi| \leq 2^{j+1} \} \). Therefore, our assumption \( 2^{j_0(\delta-1)+2} < 2^{-1/2} \) gives \( \text{supp} \eta_{\epsilon,k}(x, \cdot) \subset \{ |\xi| < 2^{j_0/2} \} \). This implies \( \text{supp} (\partial^\alpha \eta)(2^{-j_0} \cdot) \cap \text{supp} \eta_{\epsilon,k}(x, \cdot) = \emptyset \), that is, \( (\partial^\alpha \sigma^*)(x,0) = 0 \). The proof is complete. \( \square \)

**Lemma 4.3** (Chapter 7, Section 1, Proposition 1, [21, Lemma 5.1.6]). Let \( \sigma \in S_{0,1}^0 \). Then, \( K \), the distributional kernel of \( \sigma(X, D) \) as an operator from \( S'(\mathbb{R}^n) \) to \( S'(\mathbb{R}^n) \), is a \( C^\infty \)-function on \( \{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y \} \) satisfying

\[ |\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_{\alpha,\beta} |x - y|^{-n-|\alpha|-|\beta|} \]

for all \( \alpha, \beta \in \mathbb{Z}_+^n \).

Lemma 4.3 says that, if \( \sigma \in S_{0,1}^0 \), then \( \sigma(X, D), \sigma(X, D)^* \in \text{CZO}(\ell + \epsilon) \) for all \( \ell \in \mathbb{Z}_+ \), where \( 0 < \epsilon < 1 \) ([21 p. 151, p.154]).

**Lemma 4.4** ([21 p.152]). Let \( \sigma \in S_{1,1}^0 \). Then

\[ \sigma(X, D)(x^\beta) = (-i)^{|\beta|} \partial^\beta_x (e^{i\epsilon \xi} \sigma(x, \xi)) \]

for all \( \beta \in \mathbb{Z}_+^n \).

**Lemma 4.5.** Let \( \sigma \) and \( \sigma^* \) be defined by \( \text{3.20} \) with \( m \leq 0, 0 < \delta < 1 \) and \( j_0 \) satisfying \( \text{3.20} \) and \( 2^{j_0(\delta-1)+2} < 2^{-1/2} \), and \( \text{3.1} \), respectively. Then the following are true:

1. For all \( \ell \in \mathbb{Z}_+ \), \( \sigma(X, D), \sigma(X, D)^* \in \text{CZO}(\ell + \epsilon) \), where \( 0 < \epsilon < 1 \), and for all \( \beta \in \mathbb{Z}_+^n \), \( \sigma(X, D)(x^\beta) = \sigma(X, D)^*(x^\beta) = 0 \).

2. For all \( \ell \in \mathbb{Z}_+ \), \( \sigma(X, D), \sigma^*(X, D)^* \in \text{CZO}(\ell + \epsilon) \), where \( 0 < \epsilon < 1 \), and for all \( \beta \in \mathbb{Z}_+^n \), \( \sigma^*(X, D)(x^\beta) = \sigma^*(X, D)^*(x^\beta) = 0 \).

Proof. Let \( m \leq 0 \). By Lemma 4.2 we have \( \sigma, \sigma^* \in S_{1,3}^m \) and \( (\partial^\beta_x \sigma)(x,0) = (\partial^\beta_x \sigma^*)(x,0) \) for all \( \alpha \in \mathbb{Z}_+^n \). Then, by \( S_{1,3}^m \subset S_{1,3}^0 \) and the remark below Lemma 4.3 we get \( \sigma(X, D), \sigma(X, D)^* \in \text{CZO}(\ell + \epsilon) \) for all \( \ell \in \mathbb{Z}_+ \), where \( 0 < \epsilon < 1 \). On the other hand, by Lemma 4.4, \( (\partial^\beta_x \sigma)(x,0) = 0 \) for all \( \alpha \in \mathbb{Z}_+^n \), gives \( \sigma(X, D)(x^\beta) = 0 \) for all \( \beta \in \mathbb{Z}_+^n \). By \( \text{5.2} \), we have

\[ \langle \sigma(X, D)^* f, g \rangle = \langle \sigma(X, D)g, f \rangle = \langle \sigma(X, D)g, \overline{f} \rangle \]

\[ = \langle g, \sigma^*(X, D)\overline{f} \rangle = \langle g, \sigma^*(X, D)f \rangle \]

for all \( f, g \in S(\mathbb{R}^n) \). This implies \( \sigma(X, D)^* = \sigma^*(X, D) \). Hence, by Lemma 4.3 and \( (\partial^\alpha \sigma^*)(x,0) = 0 \) for all \( \alpha \in \mathbb{Z}_+^n \), we see that

\[ \sigma(X, D)^*(x^\beta) = (-i)^{|\beta|} \partial^\beta_x (e^{-i\epsilon \xi} \sigma(x, -\xi)) \]

\[ = 0 \]
for all $\beta \in \mathbb{Z}^n_+$. Therefore, we obtain Lemma 4.5 (1).

In the same way, we can prove Lemma 4.5 (2). Note that $\sigma^*(X, D)^* = \mathcal{F}(X, -D)$. $\square$

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1** Let $1 < p, q < \infty$ and $q \neq 2$. We only consider the case $q > 2$ since the case $q < 2$ can be carried out in a similar way. By Theorem 3.6, we see that $\sigma$ belongs to $S^m_{1,\delta}$, but $\sigma(X, D)$ is not bounded on $M^p,q(R^n)$, where $\sigma$ is defined by (3.19) with $m > (1/q - 1/2)\delta n$, $0 < \delta < 1$ and a sufficiently large integer $j_0$. Since $1/q - 1/2 < 0$, we may assume $m < 0$. Then, by Lemma 4.5 (1), we see that $\sigma(X, D), \sigma(X, D)^* \in CZO(\ell + \epsilon)$ for all $\ell \in \mathbb{Z}_+$, where $0 < \epsilon < 1$, and $\sigma(X, D)(x^\beta) = \sigma(X, D)^*(x^\beta) = 0$ for all $\beta \in \mathbb{Z}^n_+$. On the other hand, it is well known that, if $\delta < 1$, then pseudo-differential operators with symbols in $S^0_{1,\delta}$ are bounded on $L^2(R^n)$ (\cite[Chapter 2, Theorem 4.1]{12}, \cite[Chapter 7, Theorem 2]{16}). It is easy to prove that the $L^2$-boundedness of $\sigma(X, D)$ gives the weak boundedness property of $\sigma(X, D)$. Hence, $\sigma(X, D), \sigma(X, D)^* \in CZO(\ell + \epsilon)$ for all $\ell \in \mathbb{Z}_+$, $\sigma(X, D)(P) = \sigma(X, D)^*(P) = 0$ for all polynomials $P$, and $\sigma(X, D)$ satisfies the weak boundedness property, but $\sigma(X, D)$ is not bounded on $M^p,q(R^n)$.

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**REFERENCES**

[1] A.P. Calderón and R. Vaillancourt, A class of bounded pseudo-differential operators, Proc. Nat. Acad. Sci. U.S.A. 69 (1972), 1185-1187.
[2] G. David and J.L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. 120 (1984), 371-397.
[3] J. Duoandikoetxea, Fourier Analysis, Graduate Studies in Mathematics 29, Amer. Math. Soc., Providence, RI, 2001.
[4] C. Fefferman, $L^p$ bounds for pseudo-differential operators, Israel J. Math. 14 (1973), 413-417.
[5] H.G. Feichtinger, Banach spaces of distributions of Wiener’s type and interpolation, in: P. Butzer, B.Sz. Nagy and E. Görlich (Eds.), Proc. Conf. Oberwolfach, Functional Analysis and Approximation, August 1980, Int. Ser. Num. Math., Vol. 69, Birkhäuser-Verlag, Basel, Boston, Stuttgart, 1981, 153-165.
[6] H.G. Feichtinger, Modulation spaces on locally compact abelian groups, Technical Report, University of Vienna, Vienna, 1983.
[7] H.G. Feichtinger, Modulation spaces: Looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006), 109-140.
[8] M. Frazier, R. Torres and G. Weiss, The boundedness of Calderón-Zygmund operators on the spaces $\mathbb{F}_{p,q}^m$, Rev. Mat. Iberoam. 4 (1988), 41-72.
[9] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[10] K. Gröchenig, Time-Frequency analysis of Sjöstrand’s class, to appear in Rev. Mat. Iberoam.
[11] K. Gröchenig and C. Heil, Modulation spaces and pseudodifferential operators, Integral Equations Operator Theory 34 (1999), 439-457.
[12] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, 1981.
[13] P.G. Lemarié, Continuité sur les espaces de Besov des opérateurs définis par des intégrales singulières, Ann. Inst. Fourier (Grenoble) 35 (1985), 175-187.
[14] Y. Meyer and R. Coifman, Wavelets: Calderón-Zygmund and Multilinear Operators, Cambridge University Press, Cambridge, 1997.
[15] J. Sjöstrand, An algebra of pseudodifferential operators, Math. Res. L. 1 (1994), 185-192.
[16] E.M. Stein, Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.
[17] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, submitted. arXiv:math.FA/0606176.
[18] M. Sugimoto and N. Tomita, A counterexample for boundedness of pseudo-differential operators on modulation spaces, submitted. arXiv:math.FA/0701732.
[19] K. Tachizawa, The boundedness of pseudodifferential operators on modulation spaces, Math. Nachr. 168 (1994), 263-277.
[20] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus-I, J. Funct. Anal. 207 (2004), 399-429.
[21] R. Torres, Boundedness Results for Operators with Singular Kernels on Distribution Spaces, Mem. Amer. Math. Soc. No. 442, 1991.
[22] H. Triebel, Modulation spaces on the Euclidean n-spaces, Z. Anal. Anwendungen 2 (1983), 443-457.

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