Gauge fields with respect to $d = (3 + 1)$ in the Kaluza–Klein theories and in the spin-charge-family theory

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Abstract It is shown that in the spin-charge-family theory (Mankoč Borštnik in arXiv:1607.01618v2, 2016, Phys Rev D 91:065004. arxiv:1409.7791, 2015, J Mod Phys 6:2244. doi:10.4236/jmp.2015.615230. arXiv: 1409.4981, 2015, J Mod Phys 4:823. doi: 10.4236/jmp.2013.46113. arxiv:1312.1542, 2013, arxiv:1409.4981, 2014) as well as in all the Kaluza–Klein like theories (Blagojević in Gravitation and gauge symmetries, IoP Publishing, Bristol, 2002, An introduction to Kaluza–Klein theories, World Scientific, Singapore, 1983), vielbeins and spin connections manifest in $d = (3 + 1)$ space equivalent vector gauge fields, when space with $d \geq 5$ has a large enough symmetry. The authors demonstrate this equivalence in spaces with the symmetry of the metric tensor in the space out of $d = (3 + 1) - g^{\sigma \tau} = \eta^{\sigma \tau} f^2$ – for any scalar function $f$ of the coordinates $x^\sigma$, where $x^\sigma$ denotes the coordinates of space out of $d = (3 + 1)$. Also the connection between vielbeins and scalar gauge fields in $d = (3 + 1)$ (offering the explanation for the Higgs scalar) is discussed.

1 Introduction

The spin-charge-family theory [1–5] explains, starting with the simple action (Eq. (1)) in $d > (3 + 1)$, all the assumptions of the standard model, as well as other phenomena, like the matter–antimatter asymmetry, dark matter appearance and others. In this theory the spin-connection fields manifest in the low energy regime as the known vector gauge fields as well as the Higgs scalars (and the Yukawa couplings), while in the Kaluza–Klein theories [6,7] vielbeins (or rather metric tensors) are usually used to represent vector gauge fields.

We demonstrate in this paper that in $d$-dimensional spaces with the symmetry of the metric tensor in $(d-4)$-dimensional space $g_{\sigma \tau} = \eta_{\sigma \tau} f^{-2}$ [where $(x^\sigma, x^t)$ determine the coordinates of the almost compactified space [8–10]. $\eta_{\sigma \tau}$ is the diagonal matrix in this space and $f$ is any scalar function of these coordinates] both procedures – the ordinary Kaluza–Klein one with vielbeins and the one with spin connections (related to the vielbeins, Eq.(17)), used in the spin-charge-family theory ([1–5] and the references therein) – lead in $d = (3 + 1)$ to the same vector gauge fields. That either the vielbeins or the spin connections represent in symmetric enough $(d - 4)$ spaces in $d = (3 + 1)$ the same vector gauge fields has been known for a long time [6,7,9].

This paper is to clarify the equivalence of representing in theories with higher dimensional spaces vector gauge fields either with spin connections or with vielbeins, but is also to show that expressing the gauge fields with spin connections rather than with vielbeins makes the spin-charge-family theory transparent and correspondingly elegant, so that it is easier to recognize that the origin of the charges of the observed spinors, vector gauge fields, Higgs scalar and Yukawa couplings might really be in $(d - 4)$ space, and that this explanation might show a possible next step beyond the standard model.

Let us remind the reader that the vector gauge fields, which carry the space index $m = (0, 1, 2, 3)$, as well as the spinor fields, both observed in $d = (3 + 1)$, have in the Kaluza–Klein theories and in the spin-charge-family theory all the charges defined by the symmetry in $(d - 4)$-dimensional space, while the (observed) dynamics of these fields is defined in $(3 + 1)$ space.\footnote{It is demonstrated in the special case in Ref. [8] that the observed charges of spinors and of vector gauge fields (this is true also for the charges of the scalar fields) originate in the lowest value of $M^\mu$, that is, in $S^{\mu\nu}$.}

We present also the relation between the vielbeins and the spin-connection fields for the scalar gauge fields – for the same symmetry of $d$-dimensional spaces $(g_{\sigma \tau} = \eta_{\sigma \tau} f^{-2}$ in $(d - 4)$-dimensional space). While the vector gauge fields carry the space index $m = (0, 1, 2, 3)$, the scalar gauge fields carry the space index $(s \geq 5)$. Scalar gauge fields, carrying
the space index \( s = (7, 8) \), manifest in \( d = (3 + 1) \) as the Higgs scalar of the standard model, carrying the weak and the hyper charges \((\pm \frac{1}{2}, \mp \frac{1}{2})\), respectively [1–4]. Scalar gauge fields carry besides the properties defined by the space index (like there are the weak and hyper charges when the space index \( s = (7, 8) \)) also the charges defined by the superposition of \( S^{\alpha \beta} \) (superposition are determined by the symmetry of \( (d - 4) \) space).

There are spinor fields (and possibly also scalar gauge fields) which are responsible for curling \((d - 4)\) space, forcing the space to manifest the required symmetry (Eqs. (5)–(8)). Consequently vielbeins and spin connections of \((d - 4)\) space reflect this symmetry and correspondingly these spinors (or possibly as well scalar gauge fields) do not enter into the relation of Eq. (4).²

Let us start with the action of the spin-charge-family theory [1–5]. In this simple action in an even dimensional space \((d = 2n, d > 5)\) fermions interact with the vielbeins \( f^{\alpha \beta}_a \) and the two kinds of the spin-connection fields – \( \omega_{ab} \) and \( \tilde{\omega}_{ab} \) – the gauge fields of \( S^{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) \) and \( \tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \), respectively:

\[
A = \int d^d x \left( \frac{1}{2} \left( \bar{\Psi} \gamma^a \gamma^b \omega_{ab} \right) + h.c. \right) + \int d^d x \left( \frac{1}{2} \{ S^{ab} \omega_{ab} + \frac{1}{2} \frac{1}{2} S^{ab} \omega_{ab} \} \right),
\]

here \( \omega_{ab} = f^{\alpha \beta}_a p_{0a} + \frac{1}{2} \frac{1}{2} S^{ab} \omega_{ab} \),

\[
R = \frac{1}{2} \left( f^{\alpha \beta}_a f^{\beta \alpha}_b \right) \left( \omega_{ab, \beta} - \omega_{\alpha \beta, \alpha} \alpha \epsilon \beta \right) + h.c.,
\]

\[
\tilde{R} = \frac{1}{2} \left( f^{\alpha \beta}_a f^{\beta \alpha}_b \right) \left( \tilde{\omega}_{ab, \beta} - \tilde{\omega}_{\alpha \beta, \alpha} \alpha \epsilon \beta \right) + h.c. 4.
\]

The action introduces two kinds of the Clifford algebra objects, \( \gamma^a \) and \( \tilde{\gamma}^a \),

\[
\{ \gamma^a, \gamma^b \}_+ = 2 \eta^{ab} = \{ \tilde{\gamma}^a, \tilde{\gamma}^b \}_+.
\]

² If there are additional spinors, which do strongly influence the relation among vielbeins and spin connections, the spin connections are not any longer uniquely determined by the vielbeins, as demonstrated in Eq. (4). Then the symmetry of \((d - 4)\) space might change further. It can happen, like in Refs. [8–10], that some of the spinors stay massless after the break and the others do not, or like at the electroweak break when the symmetry of \((d - 4)\) space breaks so that the weak and hyper charges break, keeping the electromagnetic charge unbroken [1–5], while some scalars gain constant values (called in the standard model the nonzero vacuum expectation values) independent of \((3 + 1)\) space coordinates.

In the spin-charge-family theory \( d \) is chosen to be \((13 + 1)\), which makes the simple starting action in \( d \) to manifest in \((3 + 1)\) in the low energy regime all the observed degrees of freedom, explaining all the assumptions of the standard model as well as other observed phenomena [1–5].

⁴ Whenever two indices are equal the summation over these two is meant.
are no spinor sources present carrying $\tilde{\delta}^{ab}$. Equation (4) manifests that the last terms with $\delta^a_\alpha$ and $\delta^a_\beta$ do not contribute when the vector gauge fields $\omega_{\mu}^{ab}$, $\omega_{\mu}^{ab}$, $(s, t) = (5, 6, \ldots, d)$ and $m = (0, 1, 2, 3)$ are under consideration.

We demonstrate in this paper, see Sect. 2, that in the spaces with the maximal number of the Killing vectors ([6], p. (331–340)) and with no spinor sources present (which would change the symmetry of $(d - 4)$ space), the vielbeins $f^a_m$ and the spin connections $\omega_{\mu}^{ab}$ are related in the Kaluza–Klein theories [6,7]. We find, with Eqs. (14) and (18), $f^a_m = -\frac{1}{2} E_{\mu}^{\alpha} \omega_{\mu}^{ab} (x^\nu)$. When the vector gauge fields are a superposition of the spin-connection fields $(A^{st}_{\mu} = \sum_{s,t} c^{st}_{\mu} \omega_{\mu}^{st} m)$, the relations among the vielbeins and spin connections are correspondingly: $f^a_m = \sum_A \tau^{\mu\nu} A^A_{\mu}$, as presented in Eqs. (20)–(23). Spinors, vector gauge fields and scalar gauge fields, manifested in $d = (3 + 1)$ as dynamical fields, can be treated as weak fields, which do not influence the symmetry of $(d - 4)$ space. When these fields start to be strong the symmetry of the curved space changes (Let us mention that, for example, scalar fields at the electroweak break do break the symmetry of $(d - 4)$ space).

Since the vielbeins $f^a_a$ and inverted vielbeins $e^a_a$ (Eq. (3)) appear in the metric tensor as a product $(g^{ab} = f^a_a f^b_b, \; g_{ab} = e^a_a e^b_b)$, also tensors of the vector gauge fields appear in $d = (3 + 1)$ in the curvature $R^{(d)}$ as is expected for the vector gauge fields, Eqs. (23), (24) and (25):

\[ R^{(d)} = R^{(d)} + \frac{1}{4} \varepsilon_{\alpha\beta\gamma} E_{\sigma}^{\alpha} E_{\tau}^{\beta} F_{\mu \nu}^{\gamma} F^{\tau} \delta^{\mu \nu}. \]

We demonstrate in Sect. 3 that also spin-connection fields $\omega_{\mu}^{st}$ (with the index $s'$ from $(d - 4)$ space, and accordingly scalar with respect to $(3 + 1)$ space) are uniquely expressible by vielbeins, see Eqs. (39) and (40), as long as the curved space has a large enough symmetry. Consequently also the superpositions of the scalar spin-connection fields are expressible by the vielbeins.

### 2 Proof that spin connections and vielbeins lead to the same vector gauge fields in $(3 + 1)$-dimensional space-time

We discuss relations between spin connections and vielbeins when space in $(d - 4)$ demonstrates the desired isometry in order to prove that both ways, either using vielbeins or spin connections, lead to equivalent vector gauge fields in $(3 + 1)$.

We point out that spin connections manifest (charges and properties of) vector gauge fields more transparently (and elegantly) than vielbeins.\(^6\)

\(^6\) In addition: At low energies there are superpositions of spins of spinors, which manifest charges of spinors in $(3 + 1)$, and there are superpositions of $S^{st}$ acting on superpositions of spin-connection fields

\[ \omega_{\mu}^{ab} (s, t) = (5, 6, \ldots, d) \]

\[ m = (0, 1, 2, 3) \]

Let $(d - 4)$ space manifest the rotational symmetry, determined by the infinitesimal coordinate transformations of the kind

\[ x^\mu = x^\mu, \]

\[ x^\alpha = x^\alpha + \varepsilon^{st} (x^\mu) E^{\alpha}_{st} (x^\nu = x^\nu - i \varepsilon^{st} (x^\mu) M_{st} x^\nu, \]

(5) where $M_{st} = S_{st} + L_{st}$, $L_{st} = x^\nu p^\nu - x^\nu p^\nu$, $S_{st}$ concern internal degrees of freedom of boson and fermion fields, $\{M_{st}, M_{st}^t\}_\tau = i (\eta^{tt} M_{st}^t + \eta^{st} M_{st}^t - \eta^{tt} M_{st}^t - \eta^{tt} M_{st}^t). \]

From Eq. (5) it follows

\[ - i M_{st} x^\sigma = E^\sigma_{st} = x^\nu f^\sigma t - x^\nu f^\sigma s, \]

\[ E^\sigma_{st} = (e_{st} f^\sigma t - e_{st} f^\sigma s) x^\nu, \]

\[ M_{st}^\sigma := i E^\sigma_{st}, \]

(6)

and correspondingly: $M_{st} = E^\sigma_{st} p_\sigma$. One derives, when taking into account Eq. (6) and the commutation relations among generators of the infinitesimal rotations, the equation for the Killing vectors $E^\sigma_{st}$

\[ E_{st} p_\sigma E_{s't'} p_{t'} - E_{s't'} p_{t'} E_{st} p_\sigma = -i (\eta_{st} E_{s't'} + \eta_{st} E_{s't'} - \eta_{st} E_{s't'} - \eta_{st} E_{s't'}) p_\tau, \]

(7)

and the Killing equation

\[ D_\sigma E_{st} + D_t E_{st} = 0, \]

\[ D_\sigma E_{st} = \partial_{\sigma} E_{st} - \Gamma_{\tau}^{\tau} E_{s't'}. \]

(8)

Let the corresponding background field $(g^{ab} = e^a_a e^b_b)$ be

\[ \delta^a_a = \left( \begin{array}{cc} \delta^m_m & 0 \\ \delta^m_m & \delta^m_m \end{array} \right), \]

\[ f^a_a = \left( \begin{array}{cc} f^m_m & 0 \\ f^m_m & f^m_m \end{array} \right), \]

(9)

so that the background field in $d = (3 + 1)$ is flat. From $e^a_a f^m_m = \delta^m_m = 0$ it follows

\[ e^\sigma_{st} = -\delta^m_m e^\sigma_{st} f^m_m. \]

(10)

This leads to

\[ g_{ab} = \left( \begin{array}{cc} \eta_{mn} + f^m_m n^e e^a e_{st} - f^m_m e^a e_{st} \\ -f^m_m n^e e^a e_{st} \end{array} \right), \]

(11)

Footnote 6 continued

which manifest as the charges of vector (and scalar) gauge fields (vectors manifest in addition to charges – originating in $(d - 4)$ – in $SO(3 + 1)$ the $SU(2) \times SU(2)$ spin structure, while scalars carry besides charges – originating in $(d - 4)$, of the same origin as there are charges of vector gauge fields – also the properties defined by the space index in $(d - 4)$). All these facts support the idea that the origin of vector (as well as scalar) gauge fields might indeed be in higher dimensional space.

While $L_{st}$ act on coordinates, $S_{st}$ act on spinor fields, on vector gauge fields (they are superposition of $\omega_{\mu}^{ab}$, $(s, t)$ belonging to $(d - 4)$ space, $m$ to $(3 + 1)$ space) and on scalar gauge fields (they are superposition of $\omega_{\mu}^{ab}$, $(s, t)$ belonging to $(d - 4)$ space), the charges of which originate in higher dimensional space and correspondingly $S_{st}$ act on their charges (which are the superposition of $S^{st}$). For example, $S_{ab}$ act on gauge fields $A^{ab}$ as follows: $S_{ab} A^{ab} = i (\eta^{ab} A^{ab} - \eta^{ab} A^{ab})$.\(^7\)
and

$$g^{\alpha\beta} = \left( \eta^{mn} f^{\sigma_m}_s f^{\sigma_m}_m f^{\sigma_m}_{tm} \right).$$

(12)

We have $\Gamma^{\tau}_{\sigma\tau} = \frac{1}{2} g^{\tau\tau'} (g_{\sigma\tau},_t + g_{\tau\sigma},_t - g_{\sigma\tau},_t).$

One can check properties of $f^{\sigma_m}_m \delta^m_{\mu}$ under general coordinate transformations: $x'^{\mu} = x^{\mu}(x^\nu), x'^{\alpha} = x^{\alpha}(x^\tau),$

$$f^{\sigma_m}_m \delta^m_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial x^{\alpha}}{\partial x'^{\tau}} f^{\tau}_{v}.\tag{13}$$

Let us introduce the vector gauge field $\Omega^s_{\mu}(x^\nu)$, which depends only on the coordinates in $d = (3 + 1)$, as follows

$$f^{\sigma_m}_m := - \frac{1}{2} E^{\sigma s}_{s t} \Omega^s_{\mu}(x^\nu),\tag{14}$$

with $E^{\sigma s}_{s t} = - i M^{\alpha\beta}_{s t} \sigma$ defined in Eq. (6). $f^{\sigma_m}_m$ depends on the $(3 + 1)$ coordinates through $\Omega^s_{\mu}$ and on $(d - 4)$ coordinates through $E^{\sigma s}_{s t}$. From Eqs. (13) and (14) the transformation properties of $\Omega^s_{\mu}$ under the coordinate transformations of Eq. (5) follow.

If we look for the transformation properties of the superposition of the fields $\Omega_{stm}$, let say

$$A^{\mu}_{s t} = \sum_{s,t} c^{A s t} \Omega_{s t m},$$

which are the gauge fields of $\tau^{A \mu}$ (with the commutation relations $\{\tau^{A \mu}, \tau^{B \nu}\}_- = i \delta^{A}_{B} f^{A j k} \tau^{A k}$, where $\tau^{A \mu} = \sum_{s,t} c^{A s t} M^{s t}$ and $f^{A j k}$ are the structure constants of the corresponding gauge groups), under the coordinate transformations of Eq. (5), one finds $\delta_0 A^{\mu}_{s t} = c^{A s t}_{s t}, + i f^{A j k} c^{i}_{s t} \delta^{A k}_{\sigma}.$

Let us make a choice of $f^{\sigma s}_{m}$

$$f^{\sigma s}_{m} = f^{\delta_{\mu}}_{s},
\epsilon^{\sigma} = f^{-1} \delta^{\eta}_{\sigma},\tag{15}$$

for which $E^{\sigma s}_{s t}$ is equal to

$$E^{\sigma s}_{s t} = (\eta_{s t} \delta^{\alpha}_{\mu} - \eta_{s t} \delta^{\alpha}_{\nu}) x^{\tau},\tag{16}$$

solving the Killing Eq. (8) if $f$ is the scalar function of the coordinates. Let us put the expression for $f^{\sigma m}_{m}$, Eq. (14), into Eq. (4) to see the relation among $\omega_{stm}$ and $f^{\sigma m}_{m}$. One finds

$$\omega_{stm} = \frac{1}{2E} \left\{ f^{\sigma m}_{m} \left[ e_{\sigma} \partial_{\tau} (E f^{t \tau}_{s}) - e_{\sigma} \partial_{\nu} (E f^{t \nu}_{s}) \right] + e_{\sigma} \partial_{\nu} \left[ E (f^{\sigma m}_{m} f^{t \nu}_{s} - f^{t m}_{m} f^{\sigma s}_{s}) \right] - e_{\sigma} \partial_{\nu} \left[ E (f^{\sigma m}_{m} f^{t \nu}_{s} - f^{t m}_{m} f^{\sigma s}_{s}) \right] \right\}.\tag{17}$$

(Since we study only the relation between vielbeins and spin connections when there are no spinor sources present, either weak or strong, the term $\psi^\dagger \gamma^0 \gamma^s S_{sl} \psi$ is dropped. Studying problems with the weak spinor sources present would only slightly complicate the problem, while it would make the proof less transparent).

Using the inverse vielbeins $e^s_{\alpha} = f^{-1} \delta_{\eta}^{\alpha}$ and

$$det(e^s_{\alpha}) = E = f^{-(d-4)}$$

(Eq. (9)) and taking $\Omega_{stm} = \Omega_{stm}(x^\eta)$, as assumed above, it follows (after using Eq. (14) and recognizing that $f^{\sigma m}_{m} = - \frac{1}{2} (e_{s}^{\nu} \partial^{\nu} f^{s}_{m} f^{\alpha}_{\nu} - e_{s}^{\nu} f^{s}_{m} f^{\alpha}_{\nu} x^{\gamma} \Omega^{s t \gamma})$

$$\omega_{stm} = \frac{1}{2} (\eta_{s t} \delta^{\alpha}_{\tau} - \eta_{s t} \delta^{\alpha}_{\nu}) \partial_{\tau} f^{\sigma}_{m},$$

$$\omega_{stm} = \Omega_{stm}.\tag{18}$$

It is therefore proven for the vielbeins

$$f^{\sigma m}_{m} = - \frac{1}{2} E^{\sigma s}_{s t} \Omega_{s t m}(x^\nu),$$

see Eq. (14), where in $d \geq 5$ vielbeins solve the Killing equation (8), that the spin connections determine the gauge vector fields in $d = (3 + 1)$.

Statement: Let the space with $s \geq 5$ have the symmetry allowing the infinitesimal transformations of the kind

$$x^{\mu} = x^{\mu}, \quad x^{\alpha} = x^{\alpha} - i \sum_{A,i,s,t} e^{A i}(x^\mu) c^{A s t}_{A i s t} x^{s t} x^{\alpha},\tag{19}$$

then the vielbeins $f^{\sigma m}_{m}$ in Eq. (9) manifest in $d = (3 + 1)$ the vector gauge fields $A^{\mu}_{s t}$$

$$f^{\sigma m}_{m} = \sum_{A} A^{\mu}_{s t} A^{A s t}_{m},\tag{20}$$

where

$$\tau^{A i} = \sum_{s,t} c^{A i s t}_{s t} M^{s t},$$

$$\{\tau^{A i}, \tau^{B j}\}_- = i \sum_{s,t} f^{A j k} c^{i}_{s t} \delta^{A B},$$

$$\tau^{A i} = \tau^{A s t} p_{s t} = \tau^{A \tau} x^{\tau} p_{\tau},$$

$$\tau^{A i s} = \sum_{s,t} - i c^{A s t}_{s t} M^{s t}.\tag{20}$$

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The relation between $\omega^{st\prime}$ and vielbeins is determined by Eq. (17).

We have to express $A_{m}^{Ai} = \sum_{s,t} c_{s,t}^{Ai} \omega_{st\prime}$ using Eq. (17). Then it is not difficult to see that we end up with the relation

$$A_{m}^{Ai} = A_{m}^{Ai},$$

leading to the equation

$$f_{\sigma}^{m} = \sum_{A} \tau^{A\sigma} A_{m}^{A}.$$

The Lagrange function for these vector gauge fields follows from the curvature in $d$ dimensional space

$$R = R^{d} \beta_{\alpha\beta} g^{\beta\gamma},$$

after using Eqs. (11) and (12) in the relation for $\Gamma^{\alpha\beta\gamma} = \frac{1}{2} g^{\beta\delta} (g_{\gamma\delta,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta})$ and after taking into account this relation in the Riemann tensor $\Gamma_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha}[\gamma^\delta]$, where $\cdot$ denotes the derivative with respect to $x^{\delta}$ and the parentheses require antisymmetrization of the two indices.

For a flat four dimensional space ($R^{(4)} = 0$) it follows for the curvature ([6], Eq. (10.41))

$$R^{(d)} = R^{(d-4)} - \frac{1}{4} \sum_{s,t} \omega_{st}^{A} E_{s}^{A} E_{t}^{A} F_{mn}^{sst} F_{n}^{i'm'm'},$$

$$F_{mn}^{st} = \partial_{m} A_{n}^{st} - \partial_{n} A_{m}^{st} - f_{st}^{i'm'} A_{m}^{i't} A_{n}^{i'm'},$$

$$f_{\sigma}^{m} = -\frac{1}{2} E_{st}^{A} \omega_{\sigma sm} f_{\mu}^{m},$$

$$E_{st}^{A} = -i M_{st}^{A} x^{A} = (e_{s} f^{A} - e_{t} f^{A}) x^{t},$$

where $R^{(d-4)}$ determines the curvature in $(d-4)$ dimensional space and $f_{st}^{i'm'}$ can be obtained from the commutation relations \{$M^{st}, M^{s't'}\} = i(\eta^{st} M^{s't'} + \eta^{s't'} M^{st} - \eta^{st} M^{s't'} - \eta^{s't'} M^{st})$. Vielbein $f_{\sigma}^{m}$ simplifies, when $f_{\sigma}^{A} = f_{\sigma}^{A}$ and $d = (3 + 1)$ is a flat space, to $f_{\sigma}^{m} = \omega^{\sigma} x^{m}$.

When $(d -4)$ space manifests the symmetry of Eq. (20) ($f_{\sigma}^{m} = \sum_{A} \tau^{A\sigma} A_{m}^{A}$) and $d = (3 + 1)$ is a flat space, the curvature $R^{(d)}$ becomes equal to [6] (Eq. (10.41))

$$R^{(d)} = R^{(d-4)} - \frac{1}{4} \sum_{A, i, a} \tau^{A\sigma} \omega^{\sigma}_{st} F_{mn}^{Ai} F_{n}^{i'm'm'},$$

$$F_{mn}^{Ai} = \partial_{m} A_{n}^{Ai} - \partial_{n} A_{m}^{Ai} - i f_{Aij} \omega_{st}^{A} A_{j}^{Ai} A_{k}^{Ak},$$

with $E_{st}^{A}$ defined in Eq. (21).

The integration of the action $\int E d^{d}x d^{(d-4)}x R^{(d)}$ over an even dimensional $(d - 4)$ space leads to the well known effective action for the vector gauge fields in $d = (3 + 1)$ space: $\int E' d^{d}x (\frac{1}{4} \sum_{A, i, m, n} F_{Ai}^{m} F_{Ai}^{n})$, where $E'$ is determined by the gravitational field in $(3 + 1)$ space ($E' = 1$, if $(3 + 1)$ space is flat). All the vector gauge fields (manifesting in $d = (3 + 1)$, $x^{m}$ are coordinates in a flat $(3 + 1)$ space) are superposition of the spin-connection fields: $A_{m}^{Ai} = \sum_{s, t} c_{s, t}^{Ai} \omega_{st\prime}^{m}$, the charges of which ($\tau^{Ai}$ = $\sum_{s, t} c_{s, t}^{Ai} S^{m}$) are determined by the symmetry of $(d - 4)$ space.

This completes the proof of the above statement that the vielbeins $f_{\sigma}^{A} m$, $\sigma = (5, 6, \ldots, d)$, $m = (0, 1, 2, 3)$, are expressible with the spin-connection fields $\omega_{st\prime}$: $f_{\sigma}^{m} = \sum_{A, i, s, t} \tau^{A\sigma} c_{A}^{st} \omega_{st\prime}$.

Since vector gauge fields are direct ($c_{A}^{st}$ are complex numbers) superposition of spin-connection fields, the spin-connection fields offer an elegant and transparent description of the vector gauge fields. This is what the spin-charge-family theory is using.

In Sect. 2.1 we demonstrate the connection among the spin-connection fields $\omega_{st\prime}$ and the vielbeins $f_{\sigma}^{m}$ when $(d -4)$ space manifests the SU(2) $\times$ SU(2) symmetry. Generalization to any symmetry in $(d - 4)$ space goes in a similar way, leading to the corresponding expressions for the vector gauge fields in $d = (3 + 1)$.

2.1 Vector gauge fields $SU(2) \times SU(2)$ as the superposition of the spin connections

Let us demonstrate the statement that all the vector gauge fields are a superposition of the spin-connection fields in the case that the space of the symmetry $SO(3, 1)$ breaks into $SO(3, 1) \times SU(2) \times SU(2)$.

One finds the coefficients $c_{A}^{st}$ for the two $SU(2)$ generators, $\tau^{1} = \sum_{s, t} c_{A}^{st} M^{st}$ and $\tau^{2} = \sum_{s, t} c_{A}^{st} M^{st}$ by requiring the commutation relations $\{\tau^{Ai}, \tau^{Bj}\} = \delta^{A^{R}} f_{A}^{Ak} \tau^{Ak}$. 

$$\tau^{1} = \frac{1}{2} \left(M^{58} - M^{67}, M^{57} + M^{68}, M^{56} - M^{78}\right)$$

$$\tau^{2} = \frac{1}{2} \left(M^{58} + M^{67}, M^{57} - M^{68}, M^{56} + M^{78}\right).$$

Reference [11], Sect. 5.3, deriving the Lagrange function for the gauge fields by using the Clifford algebra space, allows both, the curvature $R$ and its quadratic form $R^{2}$, Eq. (240).
while one finds coefficients $e^{1t}_{st}$ and $e^{2t}_{st}$ for the corresponding gauge fields,

$$A^1_\sigma = \frac{1}{2} (\omega_{58a} - \omega_{67a}, \omega_{57a} + \omega_{68a}, \omega_{56a} - \omega_{78a})$$

$$A^2_\sigma = \frac{1}{2} (\omega_{58a} + \omega_{67a}, \omega_{57a} - \omega_{68a}, \omega_{56a} + \omega_{78a}), \quad (27)$$

from the relation

$$\sum_A \tau^A A^A_m = \sum_{s,t} M^{st} \omega_{stm}. \quad (28)$$

Taking into account Eq. (6) one finds

$$\tau^1_\sigma = \tau^{1\sigma} p_\sigma = \tau^{1\sigma} x^\tau p_\tau,$$

$$\tau^2_\sigma = \tau^{2\sigma} p_\sigma = \tau^{2\sigma} x^\tau p_\tau,$$

$$\tau^{1\sigma}_\kappa = \frac{1}{2} (e^5_\tau f^{\sigma \kappa} - e^8_\tau f^{\sigma \kappa} - e^6_\tau f^{\sigma 7} + e^7_\tau f^{\sigma 6},$$

$$e^5_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 5} + e^6_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 6},$$

$$e^5_\tau f^{\sigma 6} - e^6_\tau f^{\sigma 5} - e^7_\tau f^{\sigma 8} + e^8_\tau f^{\sigma 7}), \quad (29)$$

$$\tau^{2\sigma}_\kappa = \frac{1}{2} (e^5_\tau f^{\sigma \kappa} - e^8_\tau f^{\sigma \kappa} + e^6_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 6},$$

$$e^5_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 5} - e^6_\tau f^{\sigma 8} + e^8_\tau f^{\sigma 6},$$

$$e^5_\tau f^{\sigma 6} - e^6_\tau f^{\sigma 5} + e^7_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 7}).$$

The expressions for $f^{\sigma}_m$ are correspondingly

$$f^{\sigma}_m = (\tau^{1\sigma} A^1_m + \tau^{2\sigma} A^2_m) x^\tau. \quad (30)$$

Expressing the two $SU(2)$ gauge fields, $A^1_m$ and $A^2_m$, with $\omega_{stm}$ as required in Eq. (27), then using for each $\omega_{stm}$ the expression presented in Eq. (17), in which $f^{\sigma}_m$ is replaced by the relation in Eq. (30), then taking for $f^{\sigma}_s = f^{\delta^\sigma}_s$, where $f$ is a scalar function of the coordinates $x^\sigma$, $\sigma = (5, 6, \ldots, 8)$ (in this case $e^{\mu}_\sigma = -\delta^{\mu\sigma} f^{\sigma}_m$, Eq. (10)), it follows after a longer but straightforward calculation that

$$A^1_m = A^1,$$

$$A^2_m = A^2. \quad (31)$$

One obtains this result for any component of $A^1_m$ and $A^2_m$, $i = 1, 2, 3$, separately.

It is not difficult to see that the gauge fields, which are a superposition of $\omega_{stm}$, $(s, t) = (5, 6, \ldots, d)$, demonstrate in $d = (3 + 1)$ the isometry of the space of $SO(d - 4)$, Eq. (9), with

$$e^\sigma = f^{-1} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$  

The space breaks into $SO(3 + 1) \times SO(d - 4)$ and $f$ is any scalar field of the coordinates:

$$f = f \left( \frac{\sum_\kappa (x^\kappa)^2}{\rho_0^2} \right), \quad (33)$$

while $\rho_0$ is the radius of the $(d - 4)$ sphere and

$$f^{\sigma}_m = \sum_\kappa A^\kappa_m x^{\kappa \sigma} x^\tau, \quad (34)$$

where the $A^\kappa_m$ are the superpositions of $\omega^{st}_m$,

$$A^\kappa_m = \sum_\kappa e^{\kappa t}_{st} \omega^{st}_m,$$

demonstrating the symmetry of space with $s \geq 5$. This illustrates the proof of the statement in Sect. 2.

3 Relations between vielbeins and spin connections for scalars

The spin-charge-family theory offers the explanation for the origin of the Higgs scalar and the Yukawa couplings: The scalar gauge fields – the gauge fields of the charges described by the two kinds of Clifford algebra objects [1,2], the $\gamma^a$ and the $\bar{\gamma}^a$, Eq. (1)) – take care of the masses of spinors after the electroweak break.

We discuss here only the relation between vielbeins and spin connections for scalars the charges of which have the same origin as the charges of the vector gauge fields and only as long as $(d - 4)$ space manifests the isometry presented in Eqs. (5)–(9) with the choice of $f^{\sigma}_s = f^{\delta^\sigma}_s$, Eq. (15), (which solves the Killing Eq. (8), if $f$ is the scalar function of the coordinates $x^\sigma$). We do not include fermion sources which would change the symmetry of $(d - 4)$ space, while $f^{\sigma}_s$ are (in the low energy regime) weak fields. This section is only to point out the differences between the relation of spin connection – vielbeins for vector and scalar gauge fields.

Let us add that while the spin of the vector gauge fields in $(3 + 1)$ determines with respect to the space index $m = (0, 1, 2, 3)$ the $SU(2) \times SU(2)$ structure of their spin, the space index $s$ of the superposition of the scalar spin-connection fields $- \sum_{t, t} e^{Ai t}_{st} \omega^{st}_s$ – manifests for $s = (7, 8)$ the weak and hyper charges of the Higgs scalar: $(\pm \frac{1}{2}, \pm \frac{1}{2})$, respectively. A superposition of the spin-connection fields with the space indices $> 8$ takes care of
transitions from matter to antimatter and back, contributing to the matter–antimatter asymmetry of our universe.

To find the relation between vielbeins and spin connections we need to express the curvature $R^a_{\tau\sigma}\tau^\tau$ for $(d-4)$ space, where the Riemann tensor and $\Gamma^\sigma_{\tau\sigma'}$ for this space are

$$R^a_{\tau\sigma}\tau^\tau = \Gamma^\sigma_{\tau[\tau',\sigma']} + \Gamma^\sigma_{\tau'[\sigma',\tau]} - \Gamma^\sigma_{\tau[\sigma',\tau']} - \Gamma^\sigma_{\tau'[\tau',\sigma']},$$

$$\Gamma^\sigma_{\tau\sigma'} = \frac{1}{2} g^{\sigma\tau'} (g_{\sigma\tau'} + g_{\tau\sigma'} - g_{\tau\tau'}) \Gamma^\tau_{\tau\tau'}' (35),$$

in terms of vielbeins $g^{\sigma\tau} = f^{\sigma}\, f^{\tau}$, which is in our case $g^{\sigma\tau} = f^2\, \eta^{\sigma\tau}$, while $\eta^{\sigma\tau} = f^{-2}\, \eta_{\sigma\tau}$ ($\delta$ again denotes the derivative with respect to $x^\delta$ and $t$ the anti symmetrization with respect to particular two indices) and compare this expression with the corresponding one when $R$ is expressed with spin connections (and with the vielbeins).

$$R = \frac{1}{2} \left[ f^{\sigma\tau} \omega^d_{\alpha\beta} - \omega_{\alpha\alpha} \omega^d_{\beta\delta} \right] + h.c. \quad (36)$$

One finds that $\Gamma^\sigma_{\tau\sigma'}$ is for $f^{\sigma} = f \delta^{\sigma}_t$ equal to $\Gamma^\sigma_{\tau\sigma'} = f^{-1} (\delta^{\sigma}_t f_{\tau t} + \delta^{\sigma}_t f_{\sigma t} - \eta_{\sigma\tau} f^\tau)$, while one finds for $\omega^\tau_{\sigma' t} = \omega^\tau_{\sigma' t} + f\, \delta^{\sigma'}_t - f \delta^{\sigma'}_t + f \, \delta^{\sigma'}_t$ and for $\omega^{\tau\sigma} = \omega^{\tau\sigma} + f \delta^{\sigma\tau}$, that

$$R = R^a_{\tau\sigma\tau'} g^{\tau\tau}$$

$$= (d-4) \{ 2 - (d-4) \} \cdot f_{\tau t} \frac{\omega_{\tau\sigma} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t}}{f_{\tau t}} \quad (37)$$

We take into account Eq. (4) and evaluate Eq. (36), obtaining

$$f^{\sigma\tau} \omega_{\tau\sigma t} = 2 (d-4-1) \{ f_{\tau t} \frac{\omega_{\tau\sigma} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t}}{f_{\tau t}} \},$$

$$= (-1 + d - 4) (2 - d - 4) f_{\tau t} \frac{\omega_{\tau\sigma} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t}}{f_{\tau t}},$$

which leads to

$$\frac{1}{2} \left[ f^{\sigma\tau} \omega_{\tau\sigma t} \left( \omega_{\tau\sigma t} - \omega_{\tau\tau} \omega_{\sigma t} \right) \right] + h.c.$$

$$= (d-4) \{ 2 - (d-4-2) \} \cdot f_{\tau t} \frac{\omega_{\tau\sigma} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t} - \omega_{\tau\tau} \omega_{\sigma t}}{f_{\tau t}} \quad (38)$$

We conclude: If $f^{\sigma\tau} = \delta^{\sigma}_t \, f$, where $f = f(x^\tau, x_t)$, then both expressions for the curvature of $(d-4)$ space – the one with the metric tensor (Eq. 35) and the one with the spin connection (Eq. 36) – lead, as expected, to the same expression

$$R = R^a_{\tau\sigma}\tau^\tau$$

$$= \frac{1}{2} \left[ f^{\sigma\tau} \omega_{\tau\sigma t} \left( \omega_{\tau\sigma t} + \omega_{\tau\tau} \omega_{\sigma t} \right) \right] + h.c., \quad (39)$$

where

$$\omega_{\tau\sigma} = \omega_{\tau\sigma} + f \omega_{\tau\sigma} = -f^{-1} (f^{-1} \delta^{\sigma}_t - f \delta^{\sigma}_t) \quad (40)$$

The result is valid also for the case that vielbeins and spin connections depend on the coordinates of $(3+1)$ space: $f = f(\rho, x^m), \omega_{\tau\sigma} = \omega_{\tau\sigma} (x^\sigma, x^m), m = (0, 1, 2, 3), (s, t, t') = (5, 6, \ldots, d)$.

That spin connections and vielbeins lead to the same Lagrange density in $(d-4)$ space, although as expected, contributes to better understanding how in the low energy regime, after the electroweak break, scalar fields expressed with spin connections $\omega_{\tau\sigma'}, t' = (7, 8)$, offer the explanation for the Higgs scalar and the Yukawa couplings [1,3].

4 Conclusions

In the Kaluza–Klein theories the vector gauge fields – the gauge fields of the charges originating in higher dimensional spaces – are represented through the vielbeins $f^a_{\tau m}$ (Eq. (9)) or rather with the corresponding metric tensors (Eqs. (11) and (12)). In the spin-charge-family theory the vector gauge fields are expressed as superposition of the spin-connection fields $A^\mu_m = \sum_{t', t} e^{\tau\tau} \omega_{\tau\tau} m$. This presentation offers an elegant and transparent understanding of the appearance of the vector gauge fields $A^\mu_m$, the charges of which originate in this theory (and in the Kaluza–Klein theories) in higher dimensional spaces, while dynamics is determined in $(3+1)$. Also the scalar (gauge) fields of the spin-charge-family theory originate in higher dimensional spaces, offering the explanation for the origin of the Higgs scalar and Yukawa couplings of the standard model – when the scalar gauge fields of both charges, $S^\tau$ and $S^\tau$ (Eq. (2)), are taken into account [1]. Their dynamics is (like in the case of the vector gauge fields) determined in $(3+1)$. We discuss in this paper only gauge fields of $S^\tau$ for either vector or scalar fields.

We presented the proof that the vielbeins $f^\sigma_{\tau m}$ (Einstein index $\sigma \geq 5, m = 0, 1, 2, 3$) lead in $d = (3 + 1)$ to the vector gauge fields, which are the superposition of the spin-connection fields $\omega_{\tau\sigma m} = \sum_{t, t} A^A_{\tau m} A_{\tau t}$, with $A_{\tau m} = \sum_{s, t} e^{\tau\rho} \omega_{\tau\tau} m$, when the metric in $(d-4)$, $g_{\tau t}$, is invariant under the coordinate transformations $x^\sigma = x^\sigma + \sum_{A, i, s, t} e^{\tau\rho} (x^m) e_{\tau\sigma} m E^{\sigma\tau} (x^t)$ and $\sum_{s, t} e_{\tau\sigma} m E^{\sigma\tau} = e^{\tau\sigma} m$, while $A^{\tau\sigma}$ solves the Killing equation (8): $D_\sigma \tau^{\tau} + D_{\tau}^{\tau} A^{\tau} A_{\tau t} = 0$ (where $D_\sigma \tau^{\tau} = \partial_\sigma \tau^{\tau} - \Gamma_{\tau t} \tau_{\tau t} + \tau_{\tau t}$).

We demonstrated for the case when SO(7,1) breaks into SO(3,1) × SU(2) × SU(2) that $\sum_{A, i} A_{\tau m} A_{\tau t} = \sum_{t, t} S^{\tau\tau} \omega_{\tau\sigma m}$ and that the effective action in flat $(3+1)$ space for the vector gauge fields is $\int d^4x \left( -\frac{1}{4} F^A_{\tau m} - F^{A_{\tau m}} \right)$, where $F^A_{\tau m} = \partial_\tau A_{\tau m} - \partial_m A_{\tau m} - i f_{\tau, m}^{A_{\tau m}} A_{\tau m} A_{\tau m}$, and $f_{\tau, m}^{A_{\tau m}}$ are the structure constants of the corresponding gauge groups.
The generalization of the break of \( SO(13,1) \) into \( SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1) \), used in the spin-charge-family theory, goes equivalently. In a general case one has
\[
\sum_{A,i} \tau^{A_i} A^m_m = \sum_{s,i} S^m_i \omega_{s,m}, \quad * \text{ means that the sum-}
\]
\[
\text{mation concerns only those } (s,i), \text{ which appear in } \tau^{A_i} = \sum_{s,i} c^{A_i} S^m_i \text{. These vector gauge fields } A^m_m \text{, expressible with the connection fields, } A^m_m = \sum_{s,i} c^{A_i} \omega_{s,m}, \text{ offer an elegant explanation for the appearance of the vector gauge fields in the observed } (3+1) \text{ space. The proof is true for any } f \text{ which is a scalar function of the coordinates } x^\sigma, \sigma \geq 5.
\]

We demonstrated also the relation between the spin-connection fields and vielbeins for the scalar fields. While for the vector gauge fields the effective low energy action is in \( d = (3+1) \) equal to
\[
\int E^d \frac{d^d x}{12E} \int \frac{d^d x}{12E} \frac{1}{2} \tau^{A_i} A^m_m \text{, where } \tau^{A_i} = \sum_{s,i} c^{A_i} S^m_i \text{ and } A^m_m \text{ are structure constants of the corresponding gauge groups } \text{ – it follows for the scalar fields that, with Eqs. (35) and (36),}
\]
\[
R = \left[ \Gamma^s \tau_{[s,r]} + \Gamma^s \tau_{[s]r} \right] \frac{g_{r t}}{2} \left( \varphi_{s t, r} + \omega_{s t, r} \omega_{s t} \right) + h.c.
\]
\[
(\text{The corresponding action is proportional to } \int E^d d^d x \ R). \text{ Similar relation follows also for the superposition of the spin-connection fields.}
\]

If \( \omega_{s t, r} \) depend on \( x^m \) (\( x^m \) are coordinates in \( (3+1) \) space), the scalar fields are the dynamical fields in \( (3+1) \), explaining, for example, after the starting symmetry, the appearance of the Higgs scalars and the Yukawa couplings [1–5].

All these relations are valid as long as spinors and vector gauge fields are weak fields in comparison with the fields which force \( (d-4) \) space to be curled. When all these fields, with the scalar gauge fields included, start to be comparable with the fields (spinors or scalars), which determine the symmetry of \( (d-4) \) space, the symmetry of the whole space changes.

Appendix A: Derivation of the equality \( A^1_m = A^1_m \)

We demonstrate for the case \( A^1_m = (\omega_{s8m} - \omega_{67m}) \), Eq. (27), that this \( A^1_m \) is equal to \( A^1_m \), appearing in Eq. (30)

\[
f^m_m = \sum_{A,i} A^m_m \epsilon^{A_i} \epsilon^{x^*}. \quad (A.1)
\]

When using Eq. (17) for \( A^1_m = \omega_{s8m} - \omega_{67m} \) we end up with the expression

\[
A^1_m = \frac{1}{2E} \left[ f^{m}_m \left[ e^s_{m} \partial_t (E f^{t^5}) - e^s_{m} \partial_t (E f^{t^8}) \right] - f^{m}_m \left[ e^s_{m} \partial_t (E f^{t^6}) - e^s_{m} \partial_t (E f^{t^7}) \right] + e^s_{m} \partial_t \left( E f^{m_{f^t}} \right) + e^s_{m} \partial_t \left( E f^{m_{f^t}} \right) - e^s_{m} \partial_t \left( E f^{m_{f^t}} \right) - e^s_{m} \partial_t \left( E f^{m_{f^t}} \right) + e^s_{m} \partial_t \left( E f^{m_{f^t}} \right) \right]. \quad (A.2)
\]

Inserting for \( f^{m}_m \) the expression from Eq. (A.1) we obtain, when taking into account Eq. (29),

\[
A^1_m = \partial_8 (f_{5m}) - \partial_8 (f_{5m}) - \partial_8 (f_{5m}) + \partial_8 (f_{5m}). \quad (A.3)
\]

Inserting Eq. (A.1), in which we take into account Eq. (29) as well as that \( e^s_{m} = f^{-1} \delta^s_{m} \) and \( f^{m}_{5m} = f^{s}_{5m} \), into Eq. (A.3), we end up with

\[
A^1_m = \sum_{A,i} A^{A_i}_m \delta^{A_i}_1 \delta^1. \quad (A.4)
\]

Similarly one obtains for the gauge fields of both subgroups \( SU(2) \times SU(2) \) of the group \( SO(4) \)

\[
A^{A_i}_m = \sum_{B,j} A^{B_j}_m \delta^B_j \delta^1. \quad (A.5)
\]

Similar derivations go for any \( SO(n) \).

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