ON RICCI SOLITONS WHOSE POTENTIAL IS CONVEX

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Abstract. In this paper we consider the Ricci curvature of a Ricci soliton. In particular, we have showed that a complete gradient Ricci soliton with non-negative Ricci curvature possessing a non-constant convex potential function having finite weighted Dirichlet integral satisfying an integral condition is Ricci flat and also it isometrically splits a line. We have also proved that a gradient Ricci soliton with non-constant concave potential function and bounded Ricci curvature is non-shrinking and hence the scalar curvature has at most one critical point.

1. Introduction and preliminaries

In 1982, Hamilton introduced the concept of Ricci flow. The Ricci flow is defined by an evolution equation for metrics on the Riemannian manifold $(M, g₀)$:

$$\frac{\partial}{\partial t}g(t) = -2\text{Ric}, \quad g(0) = g₀.$$ 

A complete Riemannian manifold $(M, g)$ of dimension $n ≥ 2$ with Riemannian metric $g$ is called a Ricci soliton if there exists a vector field $X$ satisfying

$$\text{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where $\lambda$ is a constant and $\mathcal{L}$ denotes the Lie derivative. The vector field $X$ is called potential vector field. The Ricci solitons are self-similar solutions to the Ricci flow. Ricci solitons are natural generalization of Einstein metrics, which have been significantly studied in differential geometry and geometric analysis. A Ricci soliton is an Einstein metric if the vector field $X$ is zero or Killing. Throughout the paper by $M$ we mean an $n$-dimensional, $n ≥ 2$, complete Riemannian manifold endowed with Riemannian metric $g$. Let $C^\infty(M)$ be the ring of smooth functions on $M$. If $X$ is the gradient of some function $u ∈ C^\infty(M)$, such a manifold is called a

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gradient Ricci soliton, and then (1) reduces to the form

\[ \nabla^2 u + \text{Ric} = \lambda g, \]

where \( \nabla^2 u \) is the Hessian of \( u \) and the function \( u \) is called potential function. The Ricci soliton \((M, g, X, \lambda)\) is called shrinking, steady and expanding according as \( \lambda > 0 \), \( \lambda = 0 \) and \( \lambda < 0 \), respectively. Each type of Ricci solitons determines some unique topology of the manifold. For example, if the scalar curvature of a complete gradient shrinking Ricci soliton is bounded, then the manifold has finite topological type [5]. Munteanu and Wang proved that an \( n \)-dimensional gradient shrinking Ricci soliton with non-negative sectional curvature and positive Ricci curvature must be compact [11], for more results see [10, 12]. Perelman [13] proved that a compact Ricci soliton is always gradient Ricci soliton. For the detailed treatment on Ricci solitons and their interaction to Ricci flow, we refer to [2, 4]. A smooth function \( \varphi : M \to \mathbb{R} \) is said to be convex [15, 16] if for any \( p \in M \) and for any vector \( v \in T_p M \)

\[ \langle \text{grad} \varphi, v \rangle_p \leq \varphi(\exp_p v) - \varphi(p). \]

If \( \varphi \) is convex, then \(-\varphi\) is called concave.

The paper is arranged as follows: In the first section, we have proved that a complete non-compact gradient Ricci soliton with non-negative Ricci curvature possessing a non-constant convex function with finite weighted Dirichlet integral satisfying an integral condition is Ricci flat and also it isometrically splits a line. We have also deduced a corollary relating to the Ricci soliton and harmonic function. In the last section, we have proved that if in a complete gradient Ricci soliton, the potential function is a non-constant concave function with bounded Ricci curvature then the scalar curvature possesses at most one critical point, see Theorem 3.3.

2. Ricci soliton and Ricci flat manifold

**Lemma 2.1.** Let \((M, g)\) be a complete Riemannian manifold with non-negative Ricci curvature. If \( u \in C^\infty(M) \) is a non-constant convex function with finite weighted Dirichlet integral, i.e.,

\[ \int_{M - B(p, r)} d(x, p)^{-2} |\nabla u|^2 < \infty, \]

then
and also satisfies the relation
\begin{equation}
\int_{M-B(p,r)} d(x,p)^{-2}u < \infty,
\end{equation}
where $B(p,r)$ is an open ball with center $p$ and radius $r$, then the hessian of $u$ vanishes in $M$.

**Proof.** Since $u \in C^\infty(M)$ is a non-constant convex function on $M$, it follows that $M$ is non-compact. Now, we consider the cut-off function, introduced in [3], $\varphi_r \in C^2_0(B(p,2r))$ for $r > 0$ such that
\begin{align*}
0 \leq \varphi_r &\leq 1 \quad \text{in } B(p,2r) \\
\varphi_r = 1 &\quad \text{in } B(p,r) \\
|\nabla \varphi_r|^2 &\leq \frac{C}{r^2} \quad \text{in } B(p,2r) \\
\Delta \varphi_r &\leq \frac{C}{r^2} \quad \text{in } B(p,2r).
\end{align*}
Then for $r \to \infty$, we have $\Delta \varphi_r^2 \to 0$ as $\frac{\Delta \varphi_r^2}{r^2}$. Since $u$ is a smooth convex function, $u$ is also subharmonic [6], i.e., $\Delta u \geq 0$. Now using integration by parts, we have
\begin{equation}
\int_M u \Delta \varphi_r^2 = \int_M \Delta u \varphi_r^2.
\end{equation}
Since $\varphi_r \equiv 1$ in $B(p,r)$, using (5), we get
\begin{equation}
\int_{B(p,r)} \Delta u = 0.
\end{equation}
Again, using the integration by parts and also by our assumption, we obtain
\begin{equation}
0 \leq \int_{B(p,2r)} \varphi_r^2 \Delta u = \int_{B(p,2r)-B(p,r)} u \Delta \varphi_r^2 \leq \int_{B(p,2r)-B(p,r)} \frac{u C}{r^2} \to 0,
\end{equation}
as $r \to \infty$. Hence we have
\begin{equation}
\int_M \Delta u = 0.
\end{equation}
But $\Delta u \geq 0$. Therefore, $\Delta u = 0$ in $M$, i.e., $u$ is a harmonic function. The Bochner formula [1] for the Riemannian manifold is written as
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + g(\nabla u, \nabla \Delta u) + Ric(\nabla u, \nabla u).
\end{equation}
Since $u$ is harmonic, so $\Delta u = 0$. Therefore, the above equation reduces to
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + Ric(\nabla u, \nabla u).
\end{equation}
Combining $\varphi_r^2$ with (6) and then integrating we obtain
\[ \int_M \left\{ |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) \right\} \varphi_r^2 = \int_M \frac{1}{2} \Delta |\nabla u|^2 \varphi_r^2. \]

Using integration by parts we get
\[ \int_M \frac{1}{2} \Delta |\nabla u|^2 \varphi_r^2 = \int_M \frac{1}{2} |\nabla u|^2 \Delta \varphi_r^2. \]

Then the above equation and the property of $\varphi_r$ together imply
\[ \int_{B(p,2r)-B(p,r)} \frac{1}{2} \Delta |\nabla u|^2 \varphi_r^2 \leq \int_{B(p,2r)-B(p,r)} \frac{C}{2r^2} |\nabla u|^2 \to 0 \]
as $r \to \infty$. And also in $B(p,r)$ we have
\[ \int_{B(p,r)} \left\{ |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) \right\} = \int_{B(p,r)} \frac{1}{2} |\nabla u|^2 \Delta \varphi_r^2 = 0, \]
since $\varphi_r^2 \equiv 1$ in $B(p,r)$. Therefore
\[ \lim_{r \to \infty} \left( \int_{B(p,2r)-B(p,r)} + \int_{B(p,r)} \right) \left\{ |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) \right\} = 0, \]
which implies that $\nabla^2 u = 0$. \(\square\)

Lemma 2.2. [14, Lemma 2.3] Let $u$ be a smooth function in a complete Riemannian manifold $(M,g)$. Then the following conditions are equivalent:
(i) $u$ is an affine function,
(ii) Hessian of $u$ vanishes everywhere in $M$,
(iii) $\nabla u$ is a Killing vector field with $|\nabla u|$ is constant.

Theorem 2.3. [9, Theorem 1] If a complete Riemannian manifold $(M,g)$ admits a non-constant smooth affine function, then $M$ is isometric to $N \times \mathbb{R}$ for a totally geodesic submanifold $N$ of $M$.

Theorem 2.4. Let $M$ be a complete Riemannian manifold with non-negative Ricci curvature. If $M$ admits a non-constant convex function satisfying (3) and (4), then $M$ is isometric to the Riemannian product $N \times \mathbb{R}$, where $N$ is a totally geodesic submanifold of $M$. 
Proof. In view of Lemma 2.1, it follows that hessian of $u$ vanishes. Again, Lemma 2.2 implies that $u$ is an affine function. Therefore, using Theorem 2.3 we conclude that $M$ is isometric to the Riemannian product $N \times \mathbb{R}$, where $N$ is a totally geodesic submanifold of $M$. □

**Theorem 2.5.** Let $(M, g, u)$ be a complete gradient Ricci soliton with non-negative Ricci curvature. If $u$ is a non-constant convex function on $M$ satisfying (3) and (4), then $M$ is Ricci flat. Moreover, $\nabla u$ is a Killing vector field with $|\nabla u|$ is constant.

Proof. From Lemma 2.1, we get $\nabla^2 u = 0$ and $Ric(\nabla u, \nabla u) = 0$, since Ricci curvature is non-negative. Now from (2), we get

$$Ric(\omega, \omega) = \lambda g(\omega, \omega), \text{ for } \omega \in TM,$$

which implies that

$$Ric(\nabla u, \nabla u) = \lambda g(\nabla u, \nabla u) = 0.$$

Therefore, we conclude that $\lambda = 0$. And hence, (2) implies that Ricci curvature of $M$ vanishes in $M$, i.e., $M$ is a Ricci flat manifold. Also, Lemma 2.1 and Lemma 2.2 together imply that $\nabla u$ is Killing vector field with $|\nabla u|$ is constant. □

**Corollary 2.5.1.** Let $(M, g, u)$ be a complete non-compact gradient Ricci soliton satisfying (2) with non-negative Ricci curvature. If $u \in C^\infty(M)$ is a harmonic function with finite weighted Dirichlet integral, i.e.,

$$\int_{M-B(p,r)} d(x,p)^{-2}|\nabla u|^2 < \infty,$$

then $M$ is a Ricci flat manifold.

Proof. The proof is same as that of Theorem 2.5 except the part where we have proved the harmonicity of the function $u$ and hence we omit. □

3. **Ricci soliton and critical points**

**Theorem 3.1.** Let $(M, g)$ be a complete gradient Ricci soliton satisfying (2). If $u \in C^\infty(M)$ is a non-constant concave function and $(M, g)$ has bounded Ricci curvature, i.e., $|Ric| \leq K$ for some constant $K > 0$, then the Ricci soliton is non-shrinking.
Proof. Since \( u \) is a non-trivial concave function in \( M \), the function \( -u \) is non-constant convex and it implies that the manifold \( M \) is non-compact. Let us consider a length minimizing normal geodesic \( \gamma : [0, t_0] \to M \) for some arbitrary large \( t_0 > 0 \). Take \( p = \gamma(0) \) and \( X(t) = \gamma'(t) \) for \( t > 0 \). Then \( X \) is the unit tangent vector along \( \gamma \). Now integrating (2) along \( \gamma \), we get

\[
\int_0^{t_0} \text{Ric}(X, X) = \int_0^{t_0} \lambda g(X, X) - \int_0^{t_0} \nabla^2 u(X, X)
\]

(8)

Again, by the second variation of arc length, we have

\[
\int_0^{t_0} \varphi^2 \text{Ric}(X, X) \leq (n - 1) \int_0^{t_0} |\varphi'(t)|^2 dt,
\]

(9)

for every non-negative function \( \varphi \) defined on \([0, t_0]\) with \( \varphi(0) = \varphi(t_0) = 0 \). We now choose the function \( \varphi \) as the following:

\[
\varphi(t) = \begin{cases} 
  t & t \in [0, 1] \\
  1 & t \in [1, t_0 - 1] \\
  t_0 - t & t \in [t_0 - 1, t_0]. 
\end{cases}
\]

Then

\[
\int_0^{t_0} \text{Ric}(X, X) dt = \int_0^{t_0} \varphi^2 \text{Ric}(X, X) dt + \int_0^{t_0} (1 - \varphi^2) \text{Ric}(X, X) dt \\
\leq (n - 1) \int_0^{t_0} |\varphi'(t)|^2 dt + \int_0^{t_0} (1 - \varphi^2) \text{Ric}(X, X) dt \\
\leq 2(n - 1) + \sup_{B(p, 1)} |\text{Ric}| + \sup_{B(\gamma(t_0), 1)} |\text{Ric}|.
\]

(10)

Combining the equations (2) and (10), we get

\[
\lambda t_0 - \int_0^{t_0} \nabla^2 u(X, X) \leq 2(n - 1) + \sup_{B(p, 1)} |\text{Ric}| + \sup_{B(\gamma(t_0), 1)} |\text{Ric}|
\]

(11)

\[
= 2(n - 1) + 2K.
\]

Therefore, taking limit \( t_0 \to \infty \) on both sides of (11), we can write

\[
\lim_{t_0 \to \infty} \lambda t_0 - \lim_{t_0 \to \infty} \int_0^{t_0} \nabla^2 u(X, X) \leq 2(n - 1) + 2K.
\]
Now \( \lim_{t_0 \to \infty} \int_0^{t_0} \nabla^2 u(X, X) \leq 0 \), since \( u \) is a concave function. If \( \lambda > 0 \), then \( \lim_{t_0 \to \infty} \lambda t_0 = +\infty \), which contradicts the inequality (12). Thus \( \lambda \leq 0 \), i.e., the Ricci soliton is non-shrinking. \( \square \)

**Lemma 3.2.** [7] Let \((M, g)\) be a steady gradient Ricci soliton with positive Ricci curvature. Then there is at most one critical point of \( R \).

**Theorem 3.3.** Let \((M, g)\) be a complete non-compact gradient Ricci soliton satisfying

\[ \nabla^2 u + \text{Ric} = \lambda g, \]

with \( \lambda \geq 0 \). If \( u \in C^\infty(M) \) is a non-constant concave function and Ricci curvature of \( M \) satisfies \( 0 < \text{Ric} \leq K \) for some constant \( K > 0 \), then there is at most one critical point of the scalar curvature \( R \).

**Proof.** Since \( \lambda \geq 0 \), by using Theorem [3.1] we can prove that \( \lambda = 0 \). Therefore, \( M \) is a steady Ricci soliton. Now using Lemma [3.2] the result easily follows. \( \square \)

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