TOPOLOGY OF MAXIMALLY WRITHE REAL ALGEBRAIC KNOTS

G. B. MIKHALKIN AND S. YU. OREVKOV

Abstract. Oleg Viro introduced an invariant of rigid isotopy for real algebraic knots in $\mathbb{R}P^3$ which can be viewed as a first order Vassiliev invariant. In this paper we look at real algebraic knots of degree $d$ with the maximal possible value of this invariant. We show that for a given $d$ all such knots are topologically isotopic and explicitly identify their knot type.

Introduction

A real algebraic curve in $\mathbb{P}^3$ is a (complex) one-dimensional subvariety $L$ in $\mathbb{P}^3 = \mathbb{C}P^3$ invariant under the involution of complex conjugation $\text{conj} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$, $(x_0 : x_1 : x_2 : x_3) \mapsto (\bar{x}_0 : \bar{x}_1 : \bar{x}_2 : \bar{x}_3)$. The $\text{conj}$-invariance is equivalent to the fact that $L$ can be defined by a system of homogeneous polynomial equations with real coefficients. The degree of $L$ is defined as its homological degree, i.e. the number $d$ such that $[L] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^3) \cong \mathbb{Z}$. A curve of degree $d$ intersects a generic complex plane in $d$ points.

We denote the set of real points of $L$ by $\mathbb{R}L$. We say that a real curve $L$ is smooth if it is a smooth complex submanifold of $\mathbb{P}^3$. In this case, $\mathbb{R}L$ is a smooth real submanifold of $\mathbb{R}P^3$ and if it is non-empty, we call it a real algebraic link or, more specifically, a real algebraic knot in the case when $\mathbb{R}L$ is connected.

Two real algebraic links are called rigidly isotopic if they belong to the same connected component of the space of smooth real curves of the same degree. A rigid isotopy classification of real algebraic rational curves in $\mathbb{P}^3$ is obtained in [1] up to degree 5 and in [2] up to degree 6. Also we gave in [2] a rigid isotopy classification for genus one knots and links up to degree 6 (here we speak of the genus of the complex curve $L$ rather than the minimal genus of a Seifert surface of $\mathbb{R}L$).

In all the above-mentioned cases, a rigid isotopy class is completely determined by the usual (topological) isotopy class, the complex orientation (for genus one links), and the invariant of rigid isotopy $w$ introduced by Viro [4] (called in [4] encomplexed writhe). This invariant is defined as the sum of signs of crossings of a generic projection but the crossings with non-real branches are also counted with appropriate signs; see details in [4] (the definition of $w$ is also reproduced in [2]).

Let $T(p, q) = \{(z, w) \mid z^p = w^q\} \cap S^3$, $p \geq q \geq 0$, be the $(p, q)$-torus link in the 3-sphere $S^3 \subset \mathbb{C}^2$. If $p \equiv q \mod 2$, we define the projective torus link $T_{\text{proj}}(p, q) = T(p, q)/(−1) \subset S^3/(−1) = \mathbb{R}P^3$.

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Let \( N_d = (d - 1)(d - 2)/2 \). By the genus formula, this is the maximal possible value of \( w \) for irreducible curves of degree \( d \) which can be attained on rational curves only. So, if a real algebraic curve \( K \) in \( \mathbb{P}^3 \) is smooth, irreducible, and \( |w(K)| = N_d \) where \( d = \deg K \) (and hence the genus of \( K \) is zero), then we call it \emph{maximally writhed} or \( MW \)-curve. The main result is the following.

**Theorem 1.** Let \( K \) be an \( MW \)-curve of degree \( d \geq 3 \), and \( w(K) = N_d \). Then \( \mathbb{R}K \) is isotopic to \( T_{proj}(d, d - 2) \).

**Corollary 1.** A plane projection of an \( MW \)-curve from any generic real point has \( N_d \) or \( N_d - 1 \) real double points with real local branches.

**Proof.** Follows from Murasugi’s result [3; Proposition 7.5] which states that any projection of a torus link \( T(p, q) \), \( 1 \leq q \leq p \), has at least \( p(q - 1) \) crossings. \( \Box \)

In Proposition 2 (see the end of the paper) we give a precision and a self-contained (i.e., not using [3]) proof to Corollary 1.

**Conjecture 1.** In Theorem 1, \( K \) is rigidly isotopic to \( T_{proj}(d, d - 2) \).

**Conjecture 2.** If an algebraic knot \( \mathbb{R}K \) of degree \( d \) in \( \mathbb{R}\mathbb{P}^3 \) is isotopic to \( T_{proj}(d, d - 2) \), then \( w(K) = N_d \).

In a forthcoming paper we are going to give a proof of Conjecture 2 as well as a generalization of Theorem 1 for links of arbitrary genus.

The following differential geometric property of maximally writhed algebraic knots was communicated to us by Oleg Viro.

**Proposition 1.** Let \( K \) be as in Theorem 1. Then the torsion of \( \mathbb{R}K \) is everywhere positive.

1. \( MW \)-curves have everywhere positive torsion (proof of Proposition 1)

Recall that the sign of the (differential geometric) torsion of a curve \( t \mapsto r(t) \in \mathbb{R}^3, t \in \mathbb{R} \), coincides with the sign of \( \det(r', r'', r''') \) and it does not depend on the parametrization if \( r' \neq 0 \). The sign of the torsion of a curve in \( \mathbb{R}\mathbb{P}^3 \) does not depend on a choice of positively oriented affine chart.

**Lemma 1.** Let \( K \) be a real algebraic knot in \( \mathbb{P}^3 \) of genus 0 which is not contained in any plane. If the torsion of \( \mathbb{R}K \) vanishes at a point \( p \), then there exists an arbitrarily small deformation of \( K \) (in the class of real algebraic knots) which has points with negative torsion.

**Proof.** We can always choose affine coordinates \((x_1, x_2, x_3)\) centered at \( p \) such that a parametrization \( t \mapsto (x_1(t), x_2(t), x_3(t)) \) of \( K \) at \( p \) satisfies the condition \( \text{ord}_t x_1 < \text{ord}_t x_2 < \text{ord}_t x_3 \). If \( \text{ord}_t x_k > k \) for \( k = 1, 2, \) or \( 3 \), then \( x_k \) can be perturbed so that \( \text{ord}_t x_k = k \) and the \( k \)-th derivative \( x_k^{(k)}(0) \) has any sign we want. Indeed, let \((y_0 : y_1 : y_2 : y_3)\) be homogeneous coordinates such that \( x_i = y_i/y_0, \ i = 1, 2, 3 \). Then the parametrization can be chosen so that \( x_i(t) = y_i(t)/y_0(t), \ i = 1, 2, 3, \) where \( y_0(t), \ldots, y_3(t) \) are real polynomials of degree \( d = \deg K \), and \( y_0(0) > 0 \). Then the desired perturbation of \( x_k(t) \) is just \((c_k t^k + y_k(t))/y_0(t) \) where \( 0 < |c_k| \ll 1 \) and \( c_k \) has the prescribed sign. \( \Box \)

**Proof of Proposition 1.** By Lemma 1, it is enough to show that \( \mathbb{R}K \) does not have points with negative torsion. Suppose it does. Then, in an appropriate affine
chart, \( \mathbb{R}K \) admits a parametrization of the form \( t \mapsto (t, t^2 + O(t^3), -t^3 + O(t^4)) \). This means that in a sufficiently small neighbourhood of the origin, the curve is approximated by a negatively twisted rational cubic curve. Hence there is a projection with a negative crossing (see [4; Section 1.4]). Since \( w(K) \) is the sum of the signs of all real crossings and the number of them is at most \( N_d \), a single negative crossing makes impossible to attain the equality \( w(K) = N_d \). \( \square \)

2. **Uniqueness of MW-curves up to isotopy (proof of Theorem 1)**

Let \( K \) be as in Theorem 1. So, \( K \) is a smooth rational curve in \( \mathbb{P}^3 \) of degree \( d \geq 3 \) and \( w(K) = N_d \).

Given a point \( p \in \mathbb{P}^3 \), let \( \pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2 \) be the projection from \( p \) and let \( \hat{\pi}_p : K \to \mathbb{P}^2 \) be the restriction of \( \pi_p \) to \( K \). If \( p \in K \), then we extend \( \hat{\pi}_p \) to \( p \) by continuity, thus \( \pi_p^{-1}(\hat{\pi}_p(p)) = T_p \), where \( T_p \) is the tangent line to \( K \) at \( p \).

If \( p \in \mathbb{R}P^3 \), then we set \( C_p = \hat{\pi}_p(K) \). Note that

\[
\deg C_p = \begin{cases} 
  d, & p \notin K \\
  d - 1, & p \in K.
\end{cases}
\]

Recall that an algebraic curve \( C \) (maybe, singular) of degree \( m \) in \( \mathbb{R}P^2 \) is called \textit{hyperbolic} with respect to a point \( q \in \mathbb{R}P^2 \) (which may or may not belong to \( C \)), if any real line through \( q \) intersects \( C \) at \( m \) real points counting the multiplicities.

We denote:

\[
\text{hyp}(C) = \{q \mid C \text{ is hyperbolic with respect to } q\}.
\]  

It is easy to check that \( \text{hyp}(C) \) is either empty or a convex closed set. It is possible that \( \text{hyp}(C) \) contains only one point. In this case, the point should be singular. For example, if \( C \) is a cuspidal cubic, then \( \text{hyp}(C) \) consists of the cusp only.

Similarly, we say that \( K \) is hyperbolic with respect to a real line \( L \) if, for any real plane \( P \) passing through \( L \), each intersection point of \( K \) and \( P \setminus L \) is real. The following two properties of hyperbolic curves are immediate from the definition:

**Lemma 2.** Let \( C \) be a real plane curve, \( q \in \text{hyp}(C) \), and \( q_1 \in \mathbb{R}C \setminus \{q\} \). Then each local branch of \( C \) at \( q_1 \) is smooth, real, and transverse to the line \( (qq_1) \). The projection from \( q \) defines a covering \( \mathbb{R}C \to \mathbb{R}P^1 \) where \( \mathbb{R}C \) is the normalization of \( C \). \( \square \)

**Lemma 3.** Let \( L \) be a real line in \( \mathbb{P}^3 \) and \( p \in \mathbb{R}L \). Then \( K \) is hyperbolic with respect to \( L \) if and only if \( \pi_p(L) \in \text{hyp}(C_p) \). \( \square \)

**Lemma 4.** If \( p \in \mathbb{R}K \), then \( \hat{\pi}_p(p) \) is a smooth point of \( \mathbb{R}C_p \) and \( T_p \cap K = \{p\} \).

**Proof.** By Proposition 1 the torsion of \( \mathbb{R}K \) does not vanish at \( p \), hence the image of the germ \( (K, p) \) under the projection \( \hat{\pi}_p \) is a smooth local branch of \( C_p \) at \( q = \hat{\pi}_p(p) \). Suppose that \( C_p \) has another local branch at \( q \) which is the projection of a germ \( (K, p_1) \). Let \( p_0 \) be a real point on the line \( (pp_1), p_0 \notin \{p, p_1\} \). Then \( C_{p_0} \) has (at least) two branches at \( \pi_{p_0}(p) = \pi_{p_0}(p_1) \) one of whom is cuspidal. In this case by perturbing \( K \) we may obtain either a non-real crossings or a pair of real crossings of opposite signs which contradicts the maximality of \( w(K) \) (cp. the end of the proof of Proposition 1). \( \square \)
Lemma 5. If \( p \in \mathbb{R}K \), then \( \text{hyp}(C_p) \) is the closure of a component of \( \mathbb{R}P^2 \setminus \mathbb{R}C_p \) and \( \hat{\pi}_p(p) \) is a smooth point of its boundary.

Proof. Let \( q = \hat{\pi}_p(p) \). It is a smooth point of \( C_p \) by Lemma 4. For a real singular point \( u \) of \( C_p \), let \( \sigma(u) \) be its contribution to \( w(K) \), i.e. the sum of the signs of crossings of a nodal perturbation of \( u \). Then, similarly to [2; Proposition 21], we have

\[
  w(K) = i(q') + i(q'') + \sum_{u \in \text{Sing} (\mathbb{R}C_p)} \sigma(u),
\]

where \( q', q'' \not\in C_p \) are points close to \( q \) on different sides of \( \mathbb{R}C_p \), and \( i(x) \) for \( x \not\in C_p \) is one half of the image of \( \mathbb{R}C_p \) under the isomorphism \( H_1(\mathbb{R}P^2 \setminus \{x\}) \cong \mathbb{Z} \) (see [2; §6] for the choice of the orientations). It is clear that \( i(q') + i(q'') \leq \deg C_p - 1 = d - 2 \) and \( \sum \sigma(u) \leq \text{Card} \text{Sing} (\mathbb{R}C_p) \leq (d - 2)(d - 3)/2 \). The sum of these two upper bounds is \( N_d \), thus \( w(K) = N_d \) implies the equality sign in both estimates. It remains to note that \( i(q') + i(q'') = d - 2 \) implies \( q \in \text{hyp}(C_p) \). \( \square \)

Recall that the tangent line to \( K \) at \( p \in K \) is denoted by \( T_p \). Let us set

\[
  T = \bigcup_{p \in \mathbb{R}K} \mathbb{R}T_p
\]

Lemma 6. Suppose that \( K \) is hyperbolic with respect to a real line \( L \) and let \( p \in (\mathbb{R}K) \setminus L \). Then \( L \cap T_p = \emptyset \).

Proof. Combine Lemma 2 and Lemma 3. \( \square \)

Lemma 7. Let \( p_1 \) and \( p_2 \) be two distinct points on \( \mathbb{R}K \). Then \( T_{p_1} \cap T_{p_2} = \emptyset \).

Proof. Let \( L = T_{p_1} \). Then \( K \) is hyperbolic with respect to \( L \) by Lemma 5 combined with Lemma 3, and we have \( p_2 \not\in L \) by Lemma 4. Hence the result follows from Lemma 6. \( \square \)

Thus \( T \) is a disjoint union of a continuous family of real projective lines (topologically, circles) parametrized by \( \mathbb{R}K \). We are going to show that the pair \( (T, \mathbb{R}K) \) is isotopic in \( \mathbb{R}P^3 \) to a hyperboloid with a projective torus link \( T_{\text{proj}}(d, d - 2) \) sitting in it. Note that \( T \) is not smooth. It has a cuspidal edge along \( \mathbb{R}K \).

Lemma 8. There exist two real lines \( L_1 \) and \( L_2 \) such that \( K \) is hyperbolic with respect to each of them, \( L_1 \cap K = \emptyset \), and \( L_2 \) crosses \( K \) without tangency at a pair of complex conjugate points.

![Figure 1. Two perturbations of a cusp in the proof of Lemma 8](image)
Proof. Let us choose a point \( p \in \mathbb{R}K \) and let \( p_0 \in \mathbb{R}T_p \setminus \{ p \} \). Then \( \pi_p(T_p) = \hat{\pi}_p(p) \in \text{hyp}(C_p) \) by Lemma 5 whence \( K \) is hyperbolic with respect to \( T_p \) by Lemma 3. Let \( q_0 = \pi_{p_0}(p) \). Then, again by Lemma 3, we have \( q_0 \in \text{hyp}(C_{p_0}) \). The curve \( C_{p_0} \) has a cusp at \( q_0 \) because the torsion at \( p \) is nonzero. Let \( p_1 \) and \( p_2 \) be points close to \( p_0 \) and chosen on different sides of the osculating plane of \( \mathbb{R}K \) at \( p \). Then \( C_{p_1} \) and \( C_{p_2} \) are obtained from \( C_{p_0} \) by a perturbation of the cusp as shown in Figure 1 where \( q_2 \) is a solitary node of \( C_{p_2} \) (a point where two complex conjugate branches cross). Then we set \( L_j = \pi_{p_j}^{-1}(q_j), \ j = 1, 2 \), where the points \( q_1 \) and \( q_2 \) are chosen as in Figure 1. The fact that \( q_0 \in \text{hyp}(C_{p_0}) \) implies \( q_j \in \text{hyp}(C_{p_j}), \ j = 1, 2 \), whence the hyperbolicity of \( K \) with respect to \( L_j \) by Lemma 3. \( \square \)

Proof of Theorem 1. Let \( L_1 \) and \( L_2 \) be as in Lemma 8.

The line \( L_1 \) is disjoint from \( T \) by Lemma 6. Let \( P \) be a real plane through \( L_1 \). Again by Lemma 6, \( P \) crosses each line \( T_p, p \in \mathbb{R}K \), at a single point. Let us denote this point by \( \xi_P(p) \). Then \( \xi_P : \mathbb{R}K \to \mathbb{R}P \) is a continuous mapping. It is injective by Lemma 7 and its image (which is \( T \cap \mathbb{R}P \)) is disjoint from \( L_1 \). Hence \( T \cap \mathbb{R}P \) is a Jordan curve in the affine real plane \( \mathbb{R}P \setminus L_1 \). Let \( D_P \) be the disk bounded by this Jordan curve and let \( U_1 = \bigcup_P D_P \) where \( P \) runs through all the real planes through \( L_1 \). Then \( U_1 \) is fibered by disks over a circle which parametrizes the pencil of planes through \( L_1 \). Since \( \mathbb{R}P^3 \) is orientable, this fibration is trivial, thus \( U_1 \) is a solid torus and \( \partial U_1 = T \). Each \( P \) transversally crosses \( K \) at \( d \) real points, thus \( \mathbb{R}K \) sits in \( T \) and it realizes the homology class \( d\alpha \) where \( \alpha \) is a generator of \( H_1(U_1) \).

The same arguments applied to the line \( L_2 \) show that \( T \) bounds a solid torus \( U_2 \) such that \( \mathbb{R}K \) realizes the homology class \( (d-2)\beta \) where \( \beta \) is a generator of \( H_1(U_2) \). We conclude that the lift of \( K \) on \( S^3 \) is \( T(d, d-2) \) and the result follows. \( \square \)

We see that \( T \) cuts \( \mathbb{R}P^3 \) into two solid tori \( U_1 \) and \( U_2 \) such that \( \mathbb{R}L_1 \subset U_2 \) and \( \mathbb{R}L_2 \subset U_1 \).

Proposition 2. (Compare with Corollary 1). Let \( p \) be a generic point of \( \mathbb{R}P^3 \). Then \( C_p \) has only real double points. If \( p \in U_1 \), then all the double points have real local branches and the interior of \( \text{hyp}(C_p) \) is non-empty. If \( p \in U_2 \), then one double point \( q \) is solitary (i.e. has complex conjugate local branches), all the other double points have real local branches, and \( \text{hyp}(C_p) = \{ q \} \).

Proof. Let us consider a generic path \( p(t) \) which relate the given point with a point on \( T \). It defines a continuous deformation of the knot diagram which is a sequence of Reidemeister moves \( (R1) - (R3) \). However, \( (R2) \) is impossible because it involves a negative crossing and \( (R1) \) may occur only when \( p(t) \) passes through \( T \). Thus the number and the nature of double points does not change during the deformation. The projection from a point of \( T \) is cuspidal and it is hyperbolic with respect to the cusp, so the result follows from Lemma 2.

Non-emptiness of the interior of \( \text{hyp}(C_p) \) in case \( p \in U_1 \), follows from the fact that \( \text{hyp}(C_p) \) can disappear only by a move \( (R3) \). This is however impossible because all crossings are positive and the boundary orientation on \( \partial(\text{hyp}(C_p)) \) agrees with an orientation of \( \mathbb{R}C_p \) due to Lemma 2 (see Figure 2). \( \square \)

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References

1. J. Björklund, Real algebraic knots of low degree, J. Knot Theory Ramifications 20:9 (2011), 1285–1309.
Figure 2. Impossibility of a move (R3) which eliminates $\text{hyp}(C_p)$

2. G. Mikhailik and S. Orevkov, *Real algebraic knots and links of low degree*, J. Knot Theory Ramifications 25 (2016), 1642010, 34 pages.

3. K. Murasugi, *On the braid index of alternating links*, Trans. Amer. Math. Soc. 326:1 (1991), 237–260.

4. O. Viro, *Encomplexing the writhe*, in: Topology, Ergodic Theory, Real Algebraic Geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, 2001, pp. 241–256; arxiv:math.0005162.