Knot Invariants & Topological Quantum Field Theory

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Abstract
An elementary introduction to knot theory and its link to quantum field theory is presented with an intention to provide details of some basic calculations in the subject, which are not easily found in texts. Study of Chern-Simons theory with gauge group $G$, along with the Wilson lines carrying some representation is explained in generality, and a vital calculation of the Chern-Simons propagator is done. Explicit calculation for $U(1)$ Chern-Simons theory is presented, which leads to the topological invariants, and finally to knot invariants. Further, using this result along with the Gauss linking number formula, the expectation value of Wilson loops are calculated. Colored knot invariants are also discussed along with more advanced knot invariants which are obtained using Homology theory, i.e., categorification of Jones and HOMFLY polynomials. Various knot invariants for $SU(N)$ gauge group are also introduced, along with a brief introduction to A-polynomials and super A-polynomials. Recent developments in the field are explored, and we discuss a conjectured formula for colored superpolynomials, closed-form expression for HOMFLY polynomials, and conjectured expression for $6j$ symbol for $U_q(sl_N)$ for multiplicity free case. Also, a MATHEMATICA program based on the conjectured formula had been developed, which can compute the $6j$-symbols and the desired duality matrices which are needed to use the closed-form expression for HOMFLY polynomials.
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Introduction

Knot theory is the study of mathematical knots. The classification of knots/links had been one of the major problem of the Knot Theory. Various techniques where developed ranging from tricolorability of knots to Alexander Polynomial to Jones Polynomial to HOMFLY and beyond. But it turns out that the classification problems is not completely solved. There exists knots which are different isotopically but have same invariants which humanity had discovered yet. Also the Skein recursive Relation makes the computation of Knot Polynomials mathematically challenging.

Why as a physicist should we care about Knots? Edward Witten [1] while answering Michael Atiyah showed that the Chern-Simons Quantum Field Theory has observables which are related to knot invariants. He specifically showed how Jones Polynomials can be extracted from Topological Quantum Field Theory (TQFT) via Wilson Loop Observables, and thus established a strong link between TQFT and Knot Theory. Categorification theory further give us more knot invariants employing the Graded Homological Theory. So it turns out that Observables of TQFT which we physicist like to calculate and study due to its various application ranging from Condensed Matter to Conformal Field Theory to lower dimensional Quantum Gravity models and beyond, has strong connection to Knot and Knot Invariants.

We start with a brief overview of Mathematical Theory of knots, and then look at its connection to Chern-Simons theory. Then we study more refined knot invariants found through Categorifications. And finally, we review some recent developments on conjectured expression for Colored Superpolynomials for a certain class of knots known as Twist knots. We also discuss computation of HOMFLY polynomial using the closed form expression for the same. A mathematica program had been developed based on the conjectured expression for quantum $6j$ symbols for $U_q(\mathfrak{sl}_N)$, which further helps in computing the duality matrices. The program is linked as an ancillary files.

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Chapter 1

Basics of Knot Theory

A knot is a one-dimensional subset of $\mathbb{R}^3$ that is homeomorphic to $S^1$. We can specify a knot $K$ by specifying an embedding (smooth injective) $f : S^1 \to \mathbb{R}^3$ so that $K = f(S^1)$. Given two knots $K_1$ and $K_2$, how can we say that they are equivalent knots or different knots? Our word for equivalence of knots is ambient isotopy, the process of deforming knots without passing through itself. This refers to the fact that the homeomorphism that is witness to the equivalence of the knots acts on the ambient space the knots lives and not only on the knot itself.

For knots $K_1$, $K_2$ knots, we say that $K_1 \cong_\text{isotopic} K_2$ if there exists a (orientation-preserving) homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K_1) = K_2$. More precisely we require that there exists a 1-parameter family $\{f_t\} 0 \leq t \leq 1$ of smooth homeomorphisms
such that $f_0 = 1$ (identity map) and $f_1 = f$. In particular, we cannot have an isotopy that shrinks a knot to a point. This aligns pleasingly for the intuitive notion that knots are equal when they can be deformed to each other: $f$ here is just the global deformation.

A knot $K$ is said to be knotted or not an unknot (the trivial knot) if $K \not\approx S^1$.

**Planar diagram of a knot** When we visualize knots we make some projection $\mathbb{R}^3 \to \mathbb{R}^2$ (with defined coordinate system). The planar diagram is this projection with crossing information captured, i.e. overcrossing and undercrossing clearly marked as depicted in fig.1.1.

We can also orient the knot and associate the number $\sigma = \pm 1$ to each crossing depending on whether it is left handed($-$) or right handed($+$).

There is prescribed minimal set of moves known as Reidmeister moves which enables us to determine whether any two knots are isotopically equivalent or not.

![Figure 1.2: Reidmeister moves](image)

Figure 1.2: Reidmeister moves

![Reidemeister moves.](image)

The figure-eight knot is equivalent to its mirror image.

![Figure 1.3: Application of Reidmeister moves, some examples.](image)
The Reidemeister moves are:

R I  We can untwist the twist, and vice versa.

R II We can unpoke the poke, and vice versa.

R III We can slide a line behind an intersection across the intersection, and vice versa.

1.1 Polynomial Invariants

Associating polynomial invariants to every knot is one of the possible methods of distinguishing inequivalent knots (accordingly it follows that the polynomial invariants are invariant under Reidmeister moves). Note that, the construction of Polynomial invariants not necessarily imply that inequivalent knots will in general have different polynomial invariants. More generally we could also have links, a collection of non intersecting knots components say $K_1, K_2, ..., K_s$, i.e. $L = \bigcup_{i=1}^{s} K_i$

1.1.1 Alexander Polynomial

Alexander Polynomial is the earliest known Knot invariant polynomial due to Alexander\[2\], defined by the recursive relations as follows. Multivariate Alexander polynomial for a link $L_+$, is a polynomial $q$ obtained from the following recursive relation known as Skein Relation discovered by J.H. Conway involving three links/knots

\[
\Delta_{L_+}(q) - \Delta_{L_-}(q) = (q^{1/2} - q^{-1/2})\Delta_{L_0}(q)
\]  

The subscripts $+, -, 0$ refers to the three link projections identical everywhere except at any one crossing, where they have overcrossing (right handed crossing denoted by $+$), undercrossing (left handed crossing denoted by $-$) and no crossing (denoted by $0$). Let $U$ denote the unknot, we set $\Delta_{U}(q) = 1$ as normalization. Alexander polynomial was a major development in classification problem for knots, but it turned out that it is not good enough. Alexander polynomial cannot distinguish knots from there mirror images, and Alexander polynomial for disjoint union of knots is zero, i.e. The Alexander polynomial of a splittable link is always zero. So we need to develop more powerful invariants which could distinguish inequivalent knots which don’t get distinguished by just studying Alexander Polynomials.
1.1.2 Jones Polynomial

After 60 years of discovery of Alexander polynomial, Jones polynomial [3] was discovered which could distinguish handedness (i.e. chiral knots from their mirror images) in most of the cases. The recursion relation for Jones polynomial is obtained by modifying the Alexander Skein relation as follows

\[ q^{-1}V_{L_+}(q) - qV_{L_-}(q) = (q^{1/2} - q^{-1/2})V_{L_0}(q) \]  \hspace{1cm} (1.2)

Let \( K^* \) denote the mirror image of the knot \( K \), then it can be easily shown that \( V_{K^*}(q) = V_K(q^{-1}) \). Also it can be proved that the Jones polynomials for disjoint links is a product of the invariants of the disjoint components. The Jones polynomial for the knot sum \( L = K_1 \# K_2 \) satisfies \( V_L(q) = V_{K_1}(q)V_{K_2}(q) \). More generally if \( L = K_1 \# K_2 \# \cdots \# K_s \), then

\[ V_L(q) = \prod_{i=1}^{s} V_{K_i}(q) \]  \hspace{1cm} (1.3)

The above property will be shared in Quantum Field Theory because the expectation value of uncorrelated observables gets factorised.

1.1.3 HOMFLY Polynomial

A 2-variable generalization of the Jones polynomial is known as HOMFLY [4] after its co-discoverers: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter (Freyd et al. 1985). Independent work related to the HOMFLY polynomial was also carried out by Prztycki and Traczyk (1987). Prztycki and Traczyk (1987) also carried out work independently, this is why it is sometimes known as HOMFLY-PT polynomial. HOMFLY polynomial is defined by the following modified skein relationship

\[ w^{-\frac{1}{2}}q^{-\frac{1}{2}}P_{L_+}(w,q) - w^{\frac{1}{2}}q^{\frac{1}{2}}P_{L_-}(w,q) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})P_{L_0}(w,q) \]  \hspace{1cm} (1.4)

1. **Alexander Polynomial** HOMFLY polynomial reduces to Alexander polynomial for \( w = q^{-1} \), i.e. \( P_K(q^{-1}, q) = \Delta_K(q) \).

2. **Jones Polynomial** HOMFLY polynomial reduces to Jones polynomial for \( w = q \), i.e. \( P_K(q, q) = V_K(q) \).

Edward Witten [1] demonstrated that the Jones and HOMFLY polynomials are expectation values of Wilson loop observables carrying the defining representation in \( SU(2) \) and \( SU(N) \) Chern-Simons theory respectively.

It turns out that HOMFLY polynomial does not solves the classification problem. There are examples of various knots/links which are not related by Reidmeister moves, i.e. are isotopically inequivalent but still have same HOMFLY polynomial.

1.2 Braids and Braid Group

Braid are formed by connecting some \( n \) number of points in lower horizontal bar with \( n \) points in the upper horizontal bar with strings. Geometrically, an \( n \)-braid is a collection of \( n \) disjoint strings where the endpoints are fixed. When there is no crossing between
the strings, we call it the trivial braid or identity braid \((e)\); it turns out that this behaves as identity element in the group of braids. Let \(b_i\) denote the crossing of the \(i^{th}\) string over the \((i + 1)^{th}\) string; and we take \(b_i\) as a generator which acts on the trivial braid or some braid, and generate the further braiding. The inverse operation will be denoted by \(b_i^{-1}\) which will correspond to the \((i + 1)^{th}\) string crossing over the \(i^{th}\) string. Let \(B_n\) denote the set of all braids with \(n\) strings. It can be shown that the \(B_n\) forms a group. Any element of the braid group can be obtained by acting the generators \(b_i\) and \(b_i^{-1}\), for \(i \in \{1, 2, ..., n-1\}\) in different pattern on the identity braid. The sequence of the generator acting on the identity braid is called the Braid word.

![Figure 1.5: Braid Group. (a) Identity element \(e\). (b) The braiding generator \(b_i\).](image)

The defining relation of the Braid group are

\[
\begin{align*}
    b_ib_j &= b_jb_i & |i - j| > 1 \\
    b_ib_{i+1}b_i &= b_{i+1}b_ib_{i+1}
\end{align*}
\]

Consider the group operation to be just joining the Braid Words. Consider \(A, B\) be some braid word, then we define the operation \(A \ast B = AB\), just join the braid words. We can show that the inverse of the Braid word \(\sigma_{i_1}^{k_1}\sigma_{i_2}^{k_2}\cdots\sigma_{i_l}^{k_l}\) is \(\sigma_{i_1}^{-k_1}\sigma_{i_2}^{-k_2}\cdots\sigma_{i_l}^{-k_l}\). \(e\) is the identity element of the braid group \(B_n\). Also associativity holds for this operation. Hence it is established that the \(B_n\) is indeed a group. By joining the opposite ends of the braid we get knots/links.

**Alexander Theorem** Every Knot/Link can be represented by closure of some braid.

![Figure 1.6: Closure of the given braid in the diagram gives Figure 8 knot.](image)
For example, the closure of the braid represented by the braid word $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ gives Figure 8 knot as shown in fig.1.6. The braid word corresponding to a knot is not unique. Under Markov moves, the braids get transformed, but the knot/link obtained by the braid closure is the same. There are two Markov moves:

\[
[M I] \quad AB \rightarrow BA \\
[M II] \quad A \rightarrow Ab_n^{\pm 1}
\]

where $A, B \in B_n$, and $Ab_n^{\pm 1} \in B_{n+1}$.

To obtain topological invariants for knots/link, we first construct a braid group representation $\mathcal{D}(B_n)$, and then we find trace operation in this representation $\mathcal{D}(B_n)$ such that it is invariant under Markov moves. The obtained quantity is an invariant for the knot.

For more discussion on the subject I refer to my handwritten notes on Mathematical Theory of Knots [5].
Chapter 2

Chern-Simons Theory and Knot Invariants

Chern-Simons theory is a quantum gauge theory involving a rather subtle action principle. It leads to quantum field theory in which many natural questions can be explicitly answered because the usual difficulties of quantum field theory are exchanged for questions in topology, and the topological questions turn out to be fairly accessible. Chern-Simons theory is basically a topological theory and hence it does not have local observables. It is thus very computable but highly nontrivial example of a quantum field theory. It has wide variety of applications, from condensed matter physics, through string theory, and in pure mathematics. In 1989, Edward Witten showed [1] that Jones Polynomial could be extracted from Chern-Simons Theories. The paper answered a major question posed by Michael Atiyah: “What is the physical interpretation of the Jones polynomial?”.

2.1 Chern-Simons Theory

Consider a manifold $\mathcal{M}$ with gauge group $G$. Chern-Simons Action on the manifold is

$$S_{c.s.} = \frac{k}{4\pi} \int_{\mathcal{M}} Tr_{R[G]} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

(2.1)

where $k$ is some dimensionless coupling constant, $R[G]$ is some representation of the gauge group $G$, and $Tr_{R[G]}$ is trace over that representation. $A$ is connection 1-form

$$A = A_\mu dx^\mu$$

where $A_\mu = A^a_\mu T_a$  

(2.2)

and $T_a$ are the generators of the gauge group $G$ depending on the Representation $R[G]$, $A_\mu$ is $n \times n$ matrix. In the present context we are interested in Knot invariants and Knots. Knots are closed, non intersecting curve that is embedded in three dimensions. In higher dimension all knots are homeomorphic to Unkot. Its a theorem that, an $n$ dimensional object can be knotted in $n + 2$ dimensional space. If we embed it in lower dimension than $n + 2$, then knotting is not possible (since While Making knots we keep the object self intersection), and in higher dimension than $n + 2$, all knots formed from $n$ dimensional objects becomes unkots. Here, we are interested in one dimensional closed
non intersecting curve; so non trivial knotting is only possible in when this is visualized as embedded in 3 dimensional space. So, we work with knots embedded in 3 dimensional manifold, generically denoted by $\mathcal{M}_3$. Let $K \subset \mathcal{M}_3$ be a closed non intersecting loop (knot). Chern-Simons theory is a topological theory and it has gauge invariant Wilson Loop as observables associated to a knot $K$ carrying the representation $R[\mathcal{G}]$ is defined as follows

$$W_{R[\mathcal{G}]}(K) = Tr_{R[\mathcal{G}]} \mathcal{P} \exp \left( i \oint_K A \right)$$

(2.3)

where $\mathcal{P}$exp stands for path ordered exponential.

The Partition Function or Generating Function for Chern-Simons theory is (using Feynman path integral)

$$Z = \int_{\mathcal{A}[\mathcal{M}_3]/\mathcal{G}} [DA] \exp \left( \frac{ik}{4\pi} \int_{\mathcal{M}_3} Tr_{R[\mathcal{G}]} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right)$$

(2.4)

Here $\mathcal{A}[\mathcal{M}_3]$ is the space of one forms on the manifold $\mathcal{M}_3$. The path integral is taken over the quotient space $\mathcal{A}[\mathcal{M}_3]/\mathcal{G}$ because of the gauge group $\mathcal{G}$.

The vacuum expectation value of the Wilson Loop Operator $W_{R[\mathcal{G}]}(K)$ which is a topological invariant is

$$\langle W_{R[\mathcal{G}]}(K) \rangle = \frac{1}{Z} \int_{\mathcal{A}[\mathcal{M}_3]/\mathcal{G}} [DA] \exp (iS_{c.s.}) W_{R[\mathcal{G}]}(K)$$

(2.5)

Similarly if we consider link of Knots $L = \prod_{i=1}^{s} K_i$ made up of the components Knots $K_1, K_2, ..., K_s$ carrying the representation $R_1[\mathcal{G}], R_2[\mathcal{G}], ..., R_s[\mathcal{G}]$ respectively, then we can write the vacuum expectation value of the correlators

$$\langle W_{R[\mathcal{G}]}(L) \rangle \equiv \left( \prod_{i=1}^{s} W_{R_i[\mathcal{G}]}(K) \right) = \frac{1}{Z} \int_{\mathcal{A}[\mathcal{M}_3]/\mathcal{G}} [DA] \exp (iS_{c.s.}) \prod_{i=1}^{s} W_{R_i[\mathcal{G}]}(K)$$

(2.6)

### 2.2 $U(1)$ Chern-Simons Theory

For simplicity we work for now with abelian gauge gorup, $\mathcal{G} = U(1)$; it is relatively simple to solve in the context of Knot Theory and also its a quick verification for this simple $U(1)$ case that Chern-Simon Theory is indeed topological and has topological invariants. Let $R$ be the finite dimensional representation of $U(1)$. Since $U(1)$ is abelian group, it has one generator; hence the path ordering in the definition of Wilson Loop drops out.

$$W_R(K) = Tr_R \exp \left( i \oint_K A \right)$$

(2.7)

The action for $U(1)$ Chern-Simons becomes

$$S_{c.s.} = \frac{k}{8\pi} \int_{\mathcal{M}_3} d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

(2.8)
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Note that the $A \wedge A \wedge A$ term is not there in this action since it vanishes identically for $U(1)$ gauge group. Before moving ahead, let us calculate the Classical Equation of motions. The Euler Lagrange equation becomes

$$\partial_\nu \left( \frac{\partial L}{\partial (\partial_\nu A_\lambda)} \right) = \frac{\partial L}{\partial A_\lambda}$$

(2.9)

where the Chern-Simons lagrangian is $L = \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$. We solve the Euler Lagrange equation as follows

$$\partial_\nu (\epsilon^{\mu\nu\lambda} A_\mu) = \epsilon^{\lambda\sigma\mu} \delta_{\mu\lambda} \partial_\nu A_\sigma$$

$$= \epsilon^{\lambda\nu\mu} \partial_\nu A_\lambda$$

$$= \epsilon^{\lambda\mu\nu} \partial_\lambda A_\nu$$

$$= \epsilon^{\mu\lambda\nu} \partial_\mu A_\nu$$

$$\Rightarrow \epsilon^{\mu\lambda\nu} \partial_\nu A_\mu = \epsilon^{\mu\lambda\nu} \partial_\mu A_\nu$$

(2.10)

Define the curvature of the Potential (also known as field strength) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In terms of this, the Euler Lagrange equation becomes $\epsilon^{\mu\sigma\nu} F_{\nu\mu} = 0$. Since $F_{\mu\nu}$ is asymmetric so $F_{\mu\mu} = 0$. So from the equation of motion and along with the antisymmetry of the Field Strength we finally conclude that the classical equation of motion for $U(1)$ Chern-Simons is basically the Flat Gauge Filed condition, i.e.

$$F_{\mu\nu} = 0$$

(2.11)

2.2.1 Wilson Operator in irreducible representation

Since $U(1)$ is compact Lie group; its continuous, complex, finite dimensional representations are unitary and so direct sum of irreducible representations because of Peter Weyl Theorem. By Schur’s Lemma, such irreps\(^1\) of $U(1)$ are all 1-dimensional, i.e. they are given by $\chi(t) = t^n$ for some $n \in \mathbb{Z}$ (identifying $U(1)$ with the unit circle in $\mathbb{C}$). The representation of $U(1)$ are so given by $t \rightarrow (t^{n_1}, t^{n_2}, ..., t^{n_k})$ over some basis of $\mathbb{C}^k$ for $n_i \in \mathbb{Z}$.

Let $g \in U(1)$, then $g = e^{i\theta}$ (The group manifold $U(1)$ is action circle) for $\theta \in [0, 2\pi)$. Hence in the one dimensional irreducible representation $g \rightarrow g^n$ it becomes $g^n$, i.e. $\chi_n(g) = g^n = e^{i\theta n}$. This implies that the generator of this representation is $n$. The one form $A$ was $A = A_\mu T_\mu dx^\mu$, since we have only one generator $T = n$ for the irrep of $U(1)$ which is in concern here; the one form actually becomes

$$A[\chi_n] = A_\mu n dx^\mu$$

(2.12)

Note that here we have used a notation for one form which explicitly mentions the representation in which it is being written. Now we define the one form $A = A_\mu dx^\mu$, then the one form in the irrep $\chi_n$ can be written compactly as

$$A[\chi_n] = nA$$

(2.13)

\(^1\)Abbreviation for Irreducible Representation
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Since $\chi_n$ is one dimensional irrep, taking trace over this representation has trivial effect, hence the expression for Wilson Loop Operator becomes

$$W_{\chi_n}(K) = \exp \left( in \oint_K A \right) \quad (2.14)$$

### 2.2.2 Wilson Operator in higher dimensional reducible representation

Consider a Knot carrying a higher dimensional irreducible representation of $U(1)$ gauge group $\Gamma$. From Schur’s Lemma we know that $\Gamma$ can be written as direct sum of one dimensional irreps $\chi_n$.

$$\Gamma = \chi_{n_1} \oplus \chi_{n_2} \oplus \cdots \oplus \chi_{n_p} \quad (2.15)$$

Since $\Gamma$ is not 1 dimensional, then the trace operation will have non trivial effect. Due to decomposition of $\Gamma$ into irreps, trace over $\Gamma$ becomes

$$W_{\Gamma}(K) = Tr_{\Gamma} \exp \left( i \oint_K A[\Gamma] \right) = \prod_{i=1}^{p} Tr_{\chi_{n_i}} \exp \left( i \oint_K A[\chi_i] \right) = \prod_{i=1}^{p} W_{\chi_{n_i}}(K) \quad (2.16)$$

because the different space don’t talk because of blob diagonality when we split it in terms of irreps. Using eqn.2.14 we conclude

$$W_{\Gamma}(K) = \prod_{i=1}^{p} \exp \left( in_i \oint_K A \right) \quad (2.17)$$

Since $A$ appearing in eqn.2.17 is one dimensional object, we can take up the product appearing inside the exponential to make it summation there, i.e.

$$W_{\Gamma}(K) = \exp \left( i \sum_{i=1}^{p} n_i \oint_K A \right) \quad (2.18)$$

Now, if we define $n_{\Gamma} = \sum_{i=1}^{p} n_i$, then the Wilson loop Operator for this reducible representation can be written as

$$W_{\Gamma}(K) = \exp \left( in_{\Gamma} \oint_K A \right) \quad (2.19)$$

So we conclude that, as far as Wilson Loop Operators are concerned, for any higher dimensional reducible representation $\Gamma$, there exists $n_{\Gamma} = \sum_{i=1}^{p} n_i$ such that we can associate the one dimensional irreducible representation $\chi_{n_{\Gamma}}$ with the Knot, and the Topological invariant Wilson loop operator is unchanged.

### 2.2.3 Wilson Correlators

From sec2.2.2 we conclude that the generic Wilson correlator of interest is the one in which the Knot carries some irreducible representation $\chi_n$ of $U(1)$ gauge group. The generic correlators is of the form

$$\prod_{\alpha=1}^{s} W_{\chi_{n_{\alpha}}}(K_{\alpha}) = \exp \left( i \sum_{\alpha=1}^{s} n_{\alpha} \oint_{K_{\alpha}} A \right) \quad (2.20)$$
So the vacuum expectation value of the Link \( L \) becomes

\[
\langle W_{R[U(1)]}(L) \rangle = \left\langle \prod_{\alpha=1}^{s} W_{\chi_{n_{\alpha}}}(K_{\alpha}) \right\rangle = \frac{1}{Z} \int_{A[\mathcal{M}_{3}]_{U(1)}} [DA] \exp (iS_{c.s.}) \exp \left( i \sum_{\alpha=1}^{s} n_{\alpha} \oint_{K_{\alpha}} A \right)
\]

where \( S_{c.s} \) is \( U(1) \) Chern-Simons action given by eqn.2.8.

### 2.3 Feynman Propagator for \( U(1) \) Chern-Simons Theory

To find the Wilson Correlators we will need Propagators. So here we develop the theory of propagator and calculate it for our \( U(1) \) Chern Simon.

#### 2.3.1 An aside on Propagators

Let \( \phi \) be a Bosonic Field with action \( S[\phi] \). We write the Partition Function using Feynman Path Integral as follows

\[
Z = \int [D\phi] \exp \left( \frac{i}{\hbar} S[\phi] \right)
\]

Let \( \mathcal{O} \) be some operator, then its vacuum expectation value is given by

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int [D\phi] \exp \left( \frac{i}{\hbar} S[\phi] \right) \mathcal{O}
\]

When Lagrangian can be written in Quadratic form

\[
\mathcal{L} = -\frac{1}{2} \phi \mathcal{T} \phi
\]

where \( \mathcal{T} \) is a differential operator acting on \( \phi \). Now if we consider the vacuum expectation value of \( \phi(x_1)\phi(x_2) \), i.e.

\[
\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{Z} \int [D\phi] \exp \left( -\frac{i}{\hbar} \int d^{d+1}x \frac{1}{2} \phi \mathcal{T} \phi \right) \phi(x_1)\phi(x_2)
\]

Note that Path Integral takes care of ordering of operators. Recall the result of Gaussian Integrals, then we see that the field integral appearing is just \( N \) dimensional Gaussian integral with \( N \to \infty \), i.e. working in continuum limit; we conclude that

\[
\langle \phi(x_1)\phi(x_2) \rangle = (\mathcal{T}^{-1})_{x_1,x_2}
\]

where \( (\mathcal{T}^{-1})_{x_1,x_2} \) is the infinite dimensional matrix element inverse of the operator \( \frac{i}{\hbar} \mathcal{T} \), i.e., it satisfies

\[
\frac{i}{\hbar} \mathcal{T}( (\mathcal{T}^{-1})_{x_1,x_2} ) = \delta^{d+1}(x_1 - x_2)
\]
Note that we get dirac delta on RHS of the above equation, since it is just the continuum limit of Kronekar Delta. \((\mathcal{T}^{-1})_{x_1 x_2}\) is then basically greens function for the \(\mathcal{T}\) operator. Define \(G(x_1 - x_2) \equiv (\mathcal{T}^{-1})_{x_1 x_2}\), then it satisfies the equation
\[
\mathcal{T} G(x_1 - x_2) = -i\hbar \delta^{d+1}(x_1 - x_2)
\] (2.28)
\[
\langle \phi(x_1)\phi(x_2) \rangle = G(x_1 - x_2)
\] is propagator of the theory, and can be obtained by solving the above differential equation.

### 2.3.2 \(U(1)\) Chern-Simons Propagator

To find the propagator \(G_{\mu\nu}(x - y) = \langle A_\mu(x)A_\nu(y)\rangle\), let us first find the Differential equation it satisfies following the lines of previous section. We note that, \(U(1)\) Chern-Simons Lagrangian can be written as
\[
\mathcal{L} = -\frac{1}{2} A_\mu \left( -\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \right) A_\lambda
\] (2.29)
So the differential operator \(\mathcal{T}\) is
\[
\mathcal{T}^{\mu\lambda} = -\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \partial_\nu
\] (2.30)
The propagator satisfies the matrix equation as follows
\[
[\mathcal{T}][G(x - y)] = 1_{3\times3}\delta^3(x - y)
\] (2.31)
Writing the above equation in component form gives
\[
\mathcal{T}^{\mu\nu}G_{\nu\lambda}(x - y) = \delta^\lambda_\mu \delta^3(x - y)
\]
\[
e^{\mu\nu\sigma} \partial_\sigma G_{\nu\lambda}(x - y) = \frac{4i\pi\hbar}{k} \delta^\lambda_\mu \delta^3(x - y)
\] (2.32)
No we stress on the property of fully antisymmetric tensor \(\epsilon^{\mu\nu\lambda}\), and note that Laplacian in 3 dimension can be written as follows
\[
\partial^2 = \frac{1}{2} \left( \epsilon_{\mu\nu\lambda} \partial^\nu \left( \epsilon^{\mu\sigma\lambda} \partial_\sigma \right) \right)
\] (2.33)
So we see that the Laplacian \(\partial^2\) behaves like square of the operator \(\mathcal{T}\), i.e, \(\partial^2 \sim \mathcal{T}^2\). So the inverse of laplacian \((\partial^2)^{-1}_{x_1 x_2}\) behaves as square of \((\mathcal{T}^{-1})_{x_1 x_2}\). This implies that the greens function for laplacian \(G_{\alpha\beta}\) behaves as \((\mathcal{T}^{-2})_{x_1 x_2}\), i.e. \(G_{\alpha\beta} \sim (\mathcal{T}^{-2})\). Here we are interested in greens function for \(U(1)\) Chern-Simons, which is \((\mathcal{T}^{-1})\); and we note that \(\mathcal{T}(\mathcal{T}^{-2}) = (\mathcal{T}^{-1})\). So applying \(\mathcal{T}\) on the greens function for Laplacian will give greens function for \(U(1)\) Chern-Simons upto some factor. Recall that Greens function for Laplacian in 3 dimension is proportional to \(\frac{1}{|x - y|}\), i.e.
\[
G_{\alpha\beta} \propto \frac{1}{|x - y|}
\] (2.34)
From the discussion above we conclude that
\[
G_{\mathcal{T}}(x - y) = \xi \mathcal{T} \left( \frac{1}{|x - y|} \right)
\]
\[
G_{\mu\nu}(x - y) = \xi \epsilon_{\mu\sigma\nu} \partial^\sigma \left( \frac{1}{|x - y|} \right)
\]
\[
= -\xi \epsilon_{\mu\sigma\nu} \frac{(x - y)^\sigma}{|x - y|^3}
\] (2.35)
where $\xi$ some constant to be determined. We plug the expression for $G_{\mu\nu}(x - y)$ in eqn.2.32, and then find $\xi$ which makes the equations consistent. Solving we get $\xi = i\hbar k$. So we get

$$G_{\mu\nu}(x - y) = \frac{i\hbar}{k} \epsilon_{\mu\sigma} \frac{(x - y)^\sigma}{|x - y|^3}$$

(2.36)

So the propagator for $U(1)$ Chern-Simons Theory comes out to be

$$\langle A_\mu(x)A_\nu(y) \rangle = \frac{i\hbar}{k} \epsilon_{\mu\sigma} \frac{(x - y)^\sigma}{|x - y|^3}$$

(2.37)

This basic result is quoted in literature whenever needed but the proof is usually not available. I had tried here to present a complete simple proof of this result which is of immense importance.

### 2.4 Wilson Observables for $U(1)$ Chern-Simons

Now we come back to the problem of calculating expectation value of vacuum correlators. The calculation may seem intensive, but it has an important concept of Combinatorics which gives us closed form expression when all the perturbative Feynman Graph are summed over. Consider a knot $K$ carrying the irreducible representation $\chi_n$ of $U(1)$, then the Wilson loop operator is given by eqn.2.14. Expanding the exponential gives

$$W_{\chi_n}(K) = \sum_{m=0}^\infty \frac{(i\hbar)^m}{m!} \oint_K \oint_K \cdots \oint_K A_{\mu_1}(x_1)A_{\mu_2}(x_2) \cdots A_{\mu_m}(x_m) dx_1^{\mu_1} dx_2^{\mu_2} \cdots dx_m^{\mu_m}$$

(2.38)

Taking expectation value on both sides gives

$$\langle W_{\chi_n}(K) \rangle = \sum_{m=0}^\infty \frac{(i\hbar)^m}{m!} \oint_K \oint_K \cdots \oint_K \langle A_{\mu_1}(x_1)A_{\mu_2}(x_2) \cdots A_{\mu_m}(x_m) \rangle dx_1^{\mu_1} dx_2^{\mu_2} \cdots dx_m^{\mu_m}$$

(2.39)

Now we need to evaluate $\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2) \cdots A_{\mu_m}(x_m) \rangle$. We will resort to Wicks Theorem from Quantum Field Theory. In terms of Feynman diagrams, the propagator is represented as straight line connecting two vertices labelled as follows

$$\langle A_\mu(x)A_\nu(y) \rangle = \frac{x}{\mu} \frac{y}{\nu}$$

(2.40)

Expectation value of odd integrals is zero. Intuitively we can think of that, since we have only one kind of propagator which connects two points, and the theory has no interaction term, so if we had odd number of fields and we do the expansion using Wicks Theorem then one point will be left unpaired and will not be joined to any other point through the propagator. Hence we conclude

$$\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2) \cdots A_{\mu_{2m+1}}(x_{2m+1}) \rangle = 0$$

(2.41)

So only the expectation value of even number of fields is non zero. Let us consider a simple non trivial case of 4 point correlator for field $\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)A_{\mu_4}(x_4) \rangle$ which will
give us the idea of combinatorics being involved. Using Wicks Theorem we can write it as

$$\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)A_{\mu_4}(x_4) \rangle = \begin{array}{ccc}
1 & 4 & 1 \\
2 & 3 & 2 \\
3 & 3 & 3
\end{array} + \begin{array}{ccc}
1 & 4 & 1 \\
2 & 3 & 2 \\
3 & 2 & 3
\end{array} + \begin{array}{ccc}
1 & 4 & 1 \\
2 & 3 & 2 \\
3 & 3 & 2
\end{array}$$  \hspace{2cm} (2.42)

The quantity of interest is the integral of this which simplifies as follows

$$\oint_K \oint_K \oint_K \oint_K \langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)A_{\mu_4}(x_4) \rangle \, dx_1^{\mu_1} \, dx_2^{\mu_2} \, dx_3^{\mu_3} \, dx_4^{\mu_4} =$$

$$= \oint_K \oint_K \oint_K \oint_K \left( \begin{array}{ccc}
1 & 4 & 1 \\
2 & 3 & 2 \\
3 & 3 & 2
\end{array} \right) \, dx_1^{\mu_1} \, dx_2^{\mu_2} \, dx_3^{\mu_3} \, dx_4^{\mu_4}$$

$$= 3 \oint_K \oint_K \oint_K \oint_K \langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)A_{\mu_4}(x_4) \rangle \, dx_1^{\mu_1} \, dx_2^{\mu_2} \, dx_3^{\mu_3} \, dx_4^{\mu_4}$$

Note the appearance of the factor 3, it is basically the combinatorial factor corresponding to the number of ways of connecting 4 points in pairs. If we had correlators of $2m$ number of fields then we would have the double integral appearing above raised to $m$ instead of 2 because we would get $m$ lines there connecting $2m$ points in pairs, and instead of the combinatorial factor 3, we would have some other combinatorial factor say $C_{2m}$ appearing. Hence we get

$$\oint_K \oint_K \ldots \oint_K \langle A_{\mu_1}(x_1)A_{\mu_2}(x_2) \ldots A_{\mu_{2m}}(x_{2m}) \rangle \, dx_1^{\mu_1} \, dx_2^{\mu_2} \ldots \, dx_{2m}^{\mu_{2m}} = C_{2m} \left( \oint_K \oint_K \langle A_{\mu}(x)A_{\nu}(y) \rangle \, dx^\mu \, dy^\nu \right)^m$$  \hspace{2cm} (2.44)

The combinatorial factor for connecting $2m$ points in pairs is

$$C_{2m} = \frac{(2m)!}{(2^m m!)}$$  \hspace{2cm} (2.45)

Plugging these in eqn.2.39 gives the following

$$\langle W_\chi_n(K) \rangle = \sum_{m=0}^{\infty} \frac{(in)^{2m}}{(2m)!} C_{2m} \left( \oint_K \oint_K \langle A_{\mu}(x)A_{\nu}(y) \rangle \, dx^\mu \, dy^\nu \right)^m$$  \hspace{2cm} (2.46)

Now we plug the expression for $C_{2m}$ the combinatorial factor, and this will lead to the closed form expression for the correlator in terms of exponential as follows

$$\langle W_\chi_n(K) \rangle = \sum_{m=0}^{\infty} \frac{(-n^2/2)^m}{m!} \left( \oint_K \oint_K \langle A_{\mu}(x)A_{\nu}(y) \rangle \, dx^\mu \, dy^\nu \right)^m$$

$$\Rightarrow \langle W_\chi_n(K) \rangle = \exp \left( \frac{-n^2}{2} \oint_K \oint_K \langle A_{\mu}(x)A_{\nu}(y) \rangle \, dx^\mu \, dy^\nu \right)$$  \hspace{2cm} (2.47)

Now we plug the expression for propagator we calculated from eqn.2.37 in above equation which then gives

$$\langle W_\chi_n(K) \rangle = \exp \left( \frac{-i \hbar n^2}{2k} \oint_K \oint_K \epsilon_{\mu \nu \sigma} \frac{(x - y)^\sigma}{|x - y|^3} \, dx^\mu \, dy^\nu \right)$$  \hspace{2cm} (2.48)
Now we change the order of integration appropriately, so that sign in the exponential changes (by redifing dummy variables, and then swapping one pair of indices in $\epsilon_{\mu\nu\sigma}$; we finally get

$$\langle W_{\chi_n}(K) \rangle = \exp \left( \frac{i \hbar n^2}{2k} \oint_K dx^\mu \oint_K dy^\nu \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3} \right)$$  \hspace{1cm} (2.49)

Now we go back in time to 1833 during the era of Carl Friedrich Gauss, and recall the formula for Linking Number given by Gauss in terms of integral along the knots \[6\]. Given $K_\alpha$ and $K_\beta$ be two knots in $\mathbb{R}^3$, then the linking number of these knots denoted by $\Phi(K_\alpha, K_\beta)$ is given by the following Gauss Linking number formula

$$\Phi(K_\alpha, K_\beta) = \frac{1}{4\pi} \oint_{K_\alpha} dx^\mu \oint_{K_\beta} dy^\nu \epsilon_{\mu\nu\sigma} \frac{(x-y)^\sigma}{|x-y|^3}$$  \hspace{1cm} (2.50)

We can easily check that the above expression for linking number formula is symmetric in $K_\alpha$ and $K_\beta$ which is matching with our intuition, i.e. $\Phi(K_\alpha, K_\beta) = \Phi(K_\beta, K_\alpha)$. When both the knots are taken to be same, say $K$ which is appearing in eqn.2.49, we get self linking number denoted by $\Phi(K, K)$. So making use of eqn. 2.50 gives

$$\langle W_{\chi_n}(K) \rangle = \exp \left( \frac{2\pi i \hbar}{k} n^2 \Phi(K, K) \right)$$  \hspace{1cm} (2.51)

### 2.5 Framing and Self Linking Number

The appearance of Gauss linking number very well illustrates the fact that Chern-Simons theory does leads to Topological Invariants as we were hoping. The self linking number appearing here is not that straight forward to incorporate, the integral will be ill defined near $x = y$, then how to we interpret $\Phi(K, K)$. In abelian gauge theory on general three manifold $\mathcal{M}_3$, on topological grounds, the self-linking number can be a non-zero fraction, defined modulo 1; so we cant just drop this term since its not zero. Topologically, we

[Figure 2.1: Framing of Trefoil. The dashed knot is the displaced trefoil knot along the framing vector field; and solid knot is the original Trefoil with which we started.]
slightly along the vector field and get a new knot say $K'$; and now the quantity $\Phi(K, K')$ is well defined. We can think of framing as thickening of the knot as shown in fig 2.1 for trefoil. Clear the self linking number depends on the choice of vector field which was used to displace the knot $K$ to obtain $K'$, but only on the Topological class of the vector field taken for framing. So, by framing we actually mean only the topological class from now on. Although a choice of framing gives a definition for self linking number of a knot $K$, but by choosing a some other choice of framing we could get any desired answer we want for the self linking number as illustrated in fig 2.2. A t-fold twist in the framing will change the value of self linking number by t. We can define some canonical choice of framing like, in $S^3$ we can choose a framing such that the self linking number of every knot is zero. This will make the linking integral zero fro abelian case, but in general not for non abelian gauge group. So, we give up on finding a natural choice for framing and we simply pick a framing and work with it. The physical results should come out to be independent of framing, sice we are not having any canonical choice. So we set a general rule for how the expectation value of Wilson Loop Observables change under change of framing. Note that there is no choice of zero in the set of possible framings (because we dont have any canonical choice of framing); but the self linking number calculated in two different frames differ by some integer $t \in \mathbb{Z}$ which is the relative twist of the framings going around the knot. In the abelian $U(1)$ theory we know how the expectation value of Wilson operator will change if we change the framing of the knot $K$ by t twist from eqn 2.51. The self linking number is changed when the framing is t-twisted, so the expectation value of Wilson observables changes by phase as follows

$$\langle W_{\chi_n}(K) \rangle \rightarrow \exp \left( 2\pi it \hbar \frac{n^2}{k} \right) \cdot \langle W_{\chi_n}(K) \rangle$$

(2.52)

The non abelian analogue for gauge group $G$, for a knot $K$ carrying the representation $R[G]$ is [1]

$$\langle W_{R[G]}(K) \rangle \rightarrow \exp (2\pi it \hbar \Delta) \cdot \langle W_{R[G]}(K) \rangle$$

(2.53)

where $\Delta$ is conformal weight of certain primary field living in 1 + 1 dimensional current algebra. This technical result for tranformation is key ingredient for consistency of Chern-Simons theory, i.e., although we could pick any framing for knots to find self-linking number; but there is no loss or creation of information since we now have a
definite transformation rule for physical observables under Framing. A similar situation arises while defining velocity in Newtonian physics, there is no absolute velocity and we must always pick a frame; changing the frame will change the velocity according to Galilean Transformation.

For physical interpretation of framing of Knots we note that Wilson line can be considered as the spacetime trajectory of a charged particle. Quantum mechanically, although any path is possible; if the particle travelled on a particular path \( K \), there is a probability amplitude for it to arrive at its destination through this path which depends on the path \( K \) itself. In Feynman Path integral we sum over all these path, so we have to also deal with the case when some paths are knotted. the quantum mechanical amplitude that the particle did travel along a given path \( K \) is given by the Wilson Operator \( W(K) \).

In \( 2 + 1 \) dimensions, it’s possible for a particle to possess fractional statistics, meaning that the quantum wave function changes by a phase \( e^{2\pi i \delta} \) under a \( 2\pi \) rotation [7]. For computing quantum amplitude with propagation of a particle of fractional statistics, it’s not enough to specify the orbit of the particle; it’s also necessary to also count the amount of \( 2\pi \) rotations that the particle undergoes within the course of its motion. The transformation rule eqn.2.52 and eqn.2.53 suggests that the particles represented by Wilson line operators within the Chern-Simons theory have fractional statistics with \( \delta = n^2/2k \) within the abelian \( U(1) \) theory and \( \delta = \Delta \) within the non-abelian theory. We consider different trajectories of the quantum particle in \( 2 + 1 \) spacetime as shown in fig.2.3 joining two spacetime points where some of the trajectories are knotted.

According to Feynman the particle could go along any trajectory, and we sum over all the trajectories, so we also get knotted trajectories as depicted in fig.2.3. Note that the
knot drawn in fig. 2.3 is more like a knot what we usually make in real life: it starts and ends at different points as depicted in fig., its tied down at ends and knotted in between. Its not a freely floating knot in empty space what we usually think in mathematics.

2.6 Canonical Quantisation

The strategy for solving the Yang-Mills theory with Chern-Simons action on an arbitrary three manifold \( M_3 \) is to develop a machinery for chopping \( M_3 \) in pieces and then solving the problem on the pieces, and gluing things back together. Consider a three manifold \( M_3 \) with Wilson lines \( K \), as shown Fig. 2.5. We cut \( M_3 \) along a Riemann surface \( \Sigma \). Near the cut, \( M_3 \) looks like \( \Sigma \times \mathbb{R}^1 \), and now we try to understand the theory on \( \Sigma \times \mathbb{R}^1 \) as a step towards understanding the theory on \( M_3 \).

The special case of a three manifold of the form \( \Sigma \times \mathbb{R}^1 \) is tractable by means of canonical quantization. Canonical quantization on \( \Sigma \times \mathbb{R}^1 \) will produce a Hilbert \( \mathcal{H}_\Sigma \), "the physical Hilbert space of the Chern-Simons theory quantized on \( \Sigma \)." These will turn out to be finite dimensional spaces.

![Figure 2.5](image.png)

Figure 2.5: (a) Cutting a three manifold \( M_3 \) on Riemann Surface \( \Sigma \), with a Wilson Line \( K \) piercing through the surface \( \Sigma \), and so \( \Sigma \) comes with some generic marked points. (b) Locally near \( \Sigma \), \( M_3 \) looks like \( \Sigma \times \mathbb{R}^1 \).

In general we can have a situation where Wilson lines \( K \) on \( M_3 \) are "cut" by \( \Sigma \), as in the figure. In the general set up, \( \Sigma \) can be presented with finitely many marked points \( P_1, P_2, ..., P_k \) with a representation \( R_i[G] \) of gauge group \( G \) attached to it, since each Wilson line has an associated representation. To this data, an oriented topological surface with marked points, and for each marked point a representation of \( G \); we wish to associate a vector space. If one reverses the orientation of \( \sigma \) and replaces the representations \( R_i[G] \) associated with the marked points with their complex conjugates \( \overline{R}_i[G] \), the vector space \( \mathcal{H}_\Sigma \) must be replaced with its dual denoted by \( \overline{\mathcal{H}}_\Sigma \).

**Canonical Formalism**

[8] Working on \( \Sigma \times \mathbb{R}^1 \), we choose the gauge \( A_0 = 0 \) with \( A_0 \) being the component of the connection in the \( \mathbb{R}^1 \) direction. In this gauge we immediately see that the Lagrangian becomes quadratic. It reduces to

\[
\mathcal{L} = \frac{k}{8\pi} \int dt \int_\Sigma d^2x \epsilon^{ij} \text{Tr} \left[ A_i \frac{d}{dt} A_j \right]
\] (2.54)
As we can easily see that the two components of the field are canonically conjugate to each other up to constant factor, so the Poisson Brackets becomes
\[ \{ A^a_i(x), A^b_j(y) \} = \frac{4\pi}{k} \varepsilon_{ij} \delta^{ab} \delta^2(x - y) \] (2.55)

The equation of motion comes out to be Gauss Law Constraint
\[ \varepsilon_{ij} F^a_{ij} = 0 \] (2.56)

We can now quantize the theory. In quantum field theory, one very often quantize first and then imposes the constraints (for example, imposing Gupta Bleuler condition). The situation that we are considering here is a situation in which it is far more illuminating to first impose the constraints and then quantize. Let \( \mathcal{A}_0 \) denote the phase space of connections before imposing the constraint, which is an infinite dimensional phase space; imposing the constraint will reduce to the finite dimensional phase space denoted by \( \mathcal{A} \). \( \mathcal{A} \) inherits a symplectic structure, that is from the structure of Poisson brackets on \( \mathcal{A}_0 \) as in eqn.2.55. \( \mathcal{A} \) is a compact space (with some singularities), and in particular its volume with the natural symplectic volume element is finite. Since in quantum mechanics there is one quantum state per unit volume in classical phase space, the finiteness of the volume of \( \mathcal{A} \) means that the quantum Hilbert spaces will be finite dimensional.

### 2.7 Calculation of Knot Invariants

Consider a three manifold \( \mathcal{M} \) made up of connected sum of manifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) joined along a two sphere \( S^2 \). There may be knots in \( \mathcal{M}_1 \) or \( \mathcal{M}_2 \), but if so they do not pass through the joining two sphere. For every three manifold \( \mathcal{M} \), let \( \mathcal{Z}(\mathcal{M}) \) denote the partition function or Feynman path integral. The physical Hilbert space associated with the \( S^2 \) is one dimensional (because it is Riemann Surface with no marked points) and hence the Feynman path integral on \( \mathcal{M}_1 \) is determined by a vector \( |\xi_1\rangle \in \mathcal{H} \), since \( \mathcal{M}_2 \) is oppositely oriented to \( S^2 \), the path integral associates a vector \( |\xi_2\rangle \in \bar{\mathcal{H}} \), i.e. in the dual of \( \mathcal{H} \). According to the general ideas of quantum field theory, the partition function of the connected sum \( \mathcal{M} = \mathcal{M}_1 \mathring{\#} \mathcal{M}_2 \) is given by
\[ \mathcal{Z}(\mathcal{M}) = \langle \xi_1 | \xi_2 \rangle \] (2.57)

We cannot evaluate eqn.2.57, since we do not know \( |\xi_2\rangle \) and \( |\xi_2\rangle \). Instead, let us consider some variations on this theme. The two sphere \( S^2 \) that separates the two parts of fig.2.6 could be embedded in \( S^3 \) in such a way as to separate \( S^3 \) into two three balls \( B_L \) and \( B_R \). Associating the vectors states \( |v_1\rangle \) and \( |v_2\rangle \) to the three balls \( B_L \) and \( B_R \), with oppositely oriented \( S^2 \) boundary gives the partition function for \( S^3 \) without any knots
\[ \mathcal{Z}(S^3) = \langle v_1 | v_2 \rangle \] (2.58)

Since the Hilbert space is one dimensional we note that
\[ |v_1\rangle \propto |\xi_1\rangle \]
\[ |v_2\rangle \propto |\xi_2\rangle \]
\[ \Rightarrow \langle \xi_1 | \xi_2 \rangle \langle v_1 | v_2 \rangle = \langle \xi_1 | v_2 \rangle \langle v_1 | \xi_2 \rangle \] (2.59)
Figure 2.6: (a) $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ joined along the two sphere $S^2$. (b) A three sphere $S^3$ cut along its equator. (c) The cut pieces of (a) and (b) can be rearranged to give disconnected sum of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Now we note that $\langle \xi_1 | v_2 \rangle$ is just $Z(\mathcal{M}_1)$, and $\langle v_1 | \xi_2 \rangle$ is $Z(\mathcal{M}_2)$, then we get

$$Z(\mathcal{M}_1)Z(S^3) = Z(\mathcal{M}_1)Z(\mathcal{M}_2)$$

$$\frac{Z(\mathcal{M})}{Z(S^3)} = \frac{Z(\mathcal{M}_1)}{Z(S^3)} \cdot \frac{Z(\mathcal{M}_2)}{Z(S^3)} \quad (2.60)$$

Similarly if we consider link of Knots $L = \prod_{i=1}^{s} C_i$ made up $s$ unlinked Knots $C_1, C_2, \ldots, C_s$ carrying the representation $R_1[G], R_2[G], \ldots, R_s[G]$ of the gauge group $G$ respectively. Let $Z(S^3; [C_1, R_1], [C_2, R_2], \ldots, [C_s, R_s])$ denote the partition function of $S^3$ with the collection of Wilson Loops, then by cutting the $S^3$ in such a way that the curves $C_i$ do not intersect the $S^2$ boundary. So the normalized expectation value of the link $L$ will be

$$\langle W_{R[G]}(L) \rangle = \frac{Z(S^3; [C_1, R_1], [C_2, R_2], \ldots, [C_s, R_s])}{Z(S^3)} = \prod_{k=1}^{s} \frac{Z(S^3; [C_k, R_k])}{Z(S^3)} \quad (2.61)$$
which can be rewritten as
\[
\left( \prod_{i=1}^{s} W_{R_i[G]}(C_i) \right) = \prod_{i=1}^{s} \langle W_{R_i[G]}(C_i) \rangle \tag{2.62}
\]
Note that the property as in eqn.2.62 is respected by Jones Polynomial as depicted in eqn.1.3

### 2.8 TQFT and Knot Invariants

In this section we develop the link between TQFT Chern-Simons theory and Knot Theory more seriously. First of all, let us note down the generalized expression for eqn.2.51 for the vacuum expectation value of the link \( L \) described as in sec.2.2.3. The generalization is straightforward, and goes on similar lines as the proof of eqn.2.51.

\[
\langle W_{R[U(1)]}(L) \rangle = \prod_{\alpha=1}^{s} W_{\chi_{\alpha}}(K_{\alpha}) = \exp \left( \frac{2\pi i \hbar}{k} \sum_{\alpha,\beta} n_{\alpha} n_{\beta} \Phi(K_{\alpha}, K_{\beta}) \right) \tag{2.63}
\]

The above formula again justifies that the Wilson Line Observables of \( U(1) \) Chern-Simons theory gives indeed topological invariants. Notice that the above result also contains self linking numbers \( \Phi(K_{\alpha}, K_{\alpha}) \), and for this we just choose any framing, and the transformation property as in eqn.2.52 takes care of the ambiguity in the choice of Framing.

When the link of Knots \( L = \prod_{i=1}^{s} K_i \) made up \( s \) Knots \( K_1, K_2, ..., K_s \) carry the representation \( R_1, R_2, ..., R_s \) of the gauge group \( G = SU(N) \) respectively; then it can be shown \cite{1, 8} that the expectation value \( \langle W_{R[SU(N)]}(L) \rangle \) satisfies the following recursion relation
\[
q^{\frac{N}{2}} \langle W_{R[SU(N)]}(L_+) \rangle - q^{-\frac{N}{2}} \langle W_{R[SU(N)]}(L_-) \rangle = (q^{1/2} - q^{-1/2}) \langle W_{R[SU(N)]}(L_0) \rangle \tag{2.64}
\]
where
\[
q = \exp \left( \frac{2\pi i}{k + N} \right) \tag{2.65}
\]
where \( k \) is Chern-Simon level. For \( N = 2 \) the above eqn.2.64 reduces to the Skein recursion relation for the Jones Polynomial
\[
q \langle W_{R[SU(2)]}(L_+) \rangle - q^{-1} \langle W_{R[SU(2)]}(L_-) \rangle = (q^{1/2} - q^{-1/2}) \langle W_{R[SU(2)]}(L_0) \rangle \tag{2.66}
\]
If we define \( w = q^{N-1} \), then the eqn.2.64 can be written in a form which manifest itself as the Skein recursion relation for HOMFLY polynomial
\[
w^{\frac{1}{2}} q^{\frac{1}{2}} \langle W_{R[SU(N)]}(L_+) \rangle - w^{-\frac{1}{2}} q^{-\frac{1}{2}} \langle W_{R[SU(N)]}(L_-) \rangle = (q^{1/2} - q^{-1/2}) \langle W_{R[SU(N)]}(L_0) \rangle \tag{2.67}
\]
For \( N = 0 \) limit, the recursion relation eqn.2.64 reduces to the recursion relation for the Alexander Polynomial.
\[
\lim_{N \to 0} \left( \langle W_{R[SU(0)]}(L_+) \rangle - \langle W_{R[SU(0)]}(L_-) \rangle \right) = (q^{1/2} - q^{-1/2}) \langle W_{R[SU(0)]}(L_0) \rangle \tag{2.68}
\]
These results establishes a strong connection between Chern-Simons theory and Knot Theory. We have presented that the observables of the Chern-Simons theory indeed yield Knot Invariant polynomials.
Chapter 3

Knot Homology and Knot Invariants

[9] Let’s consider the category of finite-dimensional vector spaces and linear maps. We naturally associated a number to each object in this category, the dimension of that vector space. Replacing some collection of vector spaces with a collection of numbers in this way can be thought of as a **decategorification**: by remembering only the dimension of each space, we keep some information, but lose all knowledge about say morphisms between spaces. In this sense, decategorification is like forgetting about geometry. **Categorification** can be thought of as the opposite procedure. Given some piece of information (an invariant of a topological space, for instance), one asks whether it arises in some natural way as a “decategorification” of some Categorical structure: a piece of data extracted out of a more geometrical or categorical invariant, which may carry more information and thus be a finer and more powerful tool, which turns out to be the categorification of that invariant.

An example of categorification can be seen in the reinterpretation of the Euler characteristic as the alternating sum of ranks of homology groups. Let \( M \) be a topological space, and \( H_n(M) \) denote the homology group. Then the Euler characteristic can be written as

\[
\chi(M) = \sum_{n \geq 0} (-1)^n \text{rank}(H_n(M))
\]  

(3.1)

Note that, \( b_n = \text{rank}(H_n(M)) \) are actually the Betti numbers. Thus, the homology of a manifold \( M \) can be seen as a categorification of its Euler characteristic: a more sophisticated and richly structured bearer of information, from which the Euler characteristic can be evaluated naturally. This shows that categorification can be of practical interest: by trying to categorify invariants, we can hope to construct stronger invariants, which can distinguished different knots which were not distinguishable by simple invariants like Alexander polynomial, Jones Polynomial, etc. Edward Witten [1] demonstrated that three dimensionaChern-Simon theory with a compact gauge group \( G \) renders us with a natural framework for the study of Knots and Links. Actually, the vacuum expectation value of the Wilson Loop observables along the knot \( K \) gives us the polynomial invariant \( V_R^G(K; q) \) of the knot \( K \). Here \( R \) denotes the representation of the group \( G \) and the polynomial variable \( q \) is \( \exp[2\pi i/(k + C_v)] \) where \( k \in \mathbb{Z} \) is the Chern-Simons level and \( C_v \) is the quadratic Casimir.
3.1 Colored Jones Polynomial

Before we step into the categorification, we introduce polynomial knot invariants. Indeed, we not only introduce just one polynomial invariant, but a whole sequence of knot polynomials $J_n(K; q) \in \mathbb{Z}[q, q^{-1}]$ called the Jones Polynomials. For each non-negative integer $n$, the $n$-colored Jones polynomial of a knot $K$ is the quantum group invariant associated to the decoration $\mathfrak{g} = \mathfrak{sl}(2)$ with its $n$-dimensional representation $R_n$. The ordinary Jones polynomial is $J_2(K, q)$. In Chern-Simons theory with gauge group $G = SU(2)$, we can think of $J_n(K; q)$ as the expectation value of a Wilson loop operator on $K$, colored by the $n$-dimensional representation of $SU(2)$.

The colored Jones polynomial obeys the following relations, known as \textbf{Cabling Formulas}, which follow directly from the rules of Chern-Simons TQFT\footnote{abbreviation for Topological Quantum Field Theory}:

\[
J_{\oplus_i R_i}(K; q) = \sum_i J_{R_i}(K; q) \tag{3.2}
\]

\[
J_{R^n}(K; q) = J_{R \otimes n}(K; q) \tag{3.3}
\]

Here $K^n$ is the $n$-cabling of the knot $K$, obtained by taking the path of $K$ and tracing it with a “cable” of $n$ strands. These equations allow us to compute the $n$-colored Jones polynomial, if we know how to compute the ordinary Jones polynomial and have some knowledge of representation theory. For instance, any knot $K$ has $J_1(K; q) = 1$ and $J_2(K; q) = J(K; q)$, the ordinary Jones polynomial. Using cabling formula we note the following examples:

\[
2 \otimes 2 = 1 \oplus 3 = J_3(K; q) = J(K^2; q) - 1
\]

\[
2 \otimes 2 \otimes 2 = 2 \oplus 2 \oplus 4 = J_4(K; q) = J(K^3; q) - 2J(K; q)
\]

We can switch to representations of lower dimension for considering more complicated links; however, the compatibility of the ordinary Jones polynomial gives us the stage for computing colored Jones polynomial.

For a class of $(n - 1)^{\text{th}}$ rank symmetric representation $R = S^{n-1}$ of $SU(2)$, the field theoretic invariant denoted by $V_n^{SU(2)}$ turns out to be proportional to the colored Jones Polynomial. With appropriate normalization, we have

\[
J_n(K; q) = \frac{V^{SU(2)}_n(K; q)}{V_n^{SU(2)}(\text{unknot}, q)} \tag{3.4}
\]

In the fundamental representation $n = 2$, it reduces to the original Jones Polynomial, i.e. $J_{n=2}(K, q) = J(K, q)$.

3.2 Colored HOMFLY Polynomial

Similar to the fact that HOMFLY-PT polynomial is a generalization of Jones polynomial, we have the generalization of the colored Jones Polynomials known as the colored HOMFLY-PT polynomial denoted by $P_n(K; w, q)$. The relation to the colored Jones polynomial is as follows:

\[
J_n(K; q) = P_n(K; w = q, q) \tag{3.5}
\]
The relation to the $SU(N)$ Chern-Simons invariant with the colored HOMFLY-PT polynomial is
\[ P_n(K; w = q^N, q) = \frac{V_n^{SU(N)}(K; q)}{V_n^{SU(N)}(\text{unknot}, q)} \] (3.6)

Even though the colored generalization of Jones and HOMFLY polynomial give us with an infinite sequence of two variable polynomial knot invariants, they are still not powerful enough to distinguish simple pairs of knots and links called mutants.

![Figure 3.1: Pair of Mutant knots: Kinoshita-Terasaka and Conway knots are related by Mutation.](image)

**Mutation** involves drawing a disc on a knot diagram such that two incoming and two outgoing strands pass its boundary, and then rotating the portion of the knot inside the disc by 180 degrees. The Kinoshita-Terasaka and Conway knots shown in fig. 3.1 are a famous pair of knots that are mutants of one another, but are actually distinct. It turns out that the homological knot invariants distinguishes them.

**Theorem** The colored Jones polynomial, the colored HOMFLY-PT polynomial, and the Alexander polynomial cannot distinguish knots which are related to each other by Mutation.

The above theorem motivates the need for finding more powerful knot invariants.

### 3.3 Classical A-Polynomial

Given a Knot $K$, let $N(K) \subset S^3$ be an open tubular neighbourhood of $K$. Tubular neighbourhood is actually an embedding
\[ f : K \times B^2 \to S^3 \] (3.7)
such that $f(x, 0) = x$, $x \in K$ and $B^2$ is the open unit disk. We define the knot complement to be $\mathcal{M}(K) = S^3 \setminus N(K)$. To every such manifold (not necessarily a knot complement) one can associate a planar algebraic curve
\[ \mathcal{C} = \{(x, y) \in \mathbb{C}^2 : A(x, y) = 0\} \] (3.8)

It turns out that the classical invariant of the manifold $\mathcal{M}(K)$ is its fundamental group $\pi_1(\mathcal{M}(K))$ known as the Knot group for the knot $K$. It contains a lot more information about $\mathcal{M}(K)$ and can distinguish knots much better than any of the polynomial invariants. This is a point which gives us the hint that analysing the Homology may also give us some invariants. The curve $\mathcal{C}$ also contains lot of information. $A(x, y)$ is known as the classical A-Polynomial. For more discussion and example look at [9].
3.4 Categorifications

Mikhail Khovanov [11] constructed a categorification of the Jones polynomial in 2000. Similar to the Jones polynomial, it is associated to the data $\mathfrak{g} = \mathfrak{sl}(2)$ and its fundamental representation $R = V_2$. Associates a chain complex to a diagram of a link $K$. The homology of this chain complex can be shown to be invariant under the Reidemeister moves, and therefore to be an invariant of $K$. Khovanov homology $\mathcal{H}_{i,j}(K)$ is doubly graded, and the Jones polynomial is its graded Euler characteristic actually. Evaluation of colored HOMFLY polynomials for knots gives Laurent series expansion [12]

$$P_{R}(a, q) = \sum_{i,j} c_{ij} q^i a^j$$

where $c_{ij}$ are integer coefficients so there may be some underlying topological interpretation for the integer coefficients, because usually the Topological Invariants are integers, like Genus, Betti numbers, etc.

Given $\mathcal{M}$ a topological space with finitely generated homology, the Poincaré polynomial of $\mathcal{M}$ is defined as the polynomial in $q$ which is the generating function of its Betti numbers, i.e. the polynomial where the coefficient of $q^n$ is $b_n(\mathcal{M}) = \text{rank}(H_n(\mathcal{M}))$. More generally, we can have a higher categorification, i.e. $l$ graded homology $H_{i_1,i_2,...,i_l}(\mathcal{M})$, the Poincaré polynomial is a polynomial in variables $q_1, q_2, ..., q_l$ which is the generating function of the Betti numbers such that the the coefficient of $q_1^{i_1} q_2^{i_2} ... q_l^{i_l}$ is $b_{i_1,i_2,...,i_l}(\mathcal{M}) = \text{rank}(H_{i_1,i_2,...,i_l}(\mathcal{M}))$.

$$\mathcal{P}(\mathcal{M}; q_1, q_2, ..., q_l) = \sum_{i_1,i_2,...,i_l} q_1^{i_1} q_2^{i_2} ... q_l^{i_l} \text{rank}(H_{i_1,i_2,...,i_l}(\mathcal{M}))$$

More generally, we can define the $(q_1, q_2, ..., q_{k-1}, \hat{q}_k, q_{k+1}, ..., q_l)$-graded Euler Characteristic (here $\hat{q}_k$ hat on $q_k$ denotes the omission of the variable $q_k$ in the list.) as follows:

$$\chi_k(\mathcal{M}; (q_1,q_2,\ldots,q_{k-1},\hat{q}_k,q_{k+1},\ldots,q_l)) = \sum_{i_1,i_2,\ldots,i_l} q_1^{i_1} q_2^{i_2} ... q_{k-1}^{i_{k-1}} (-1)^{i_k} q_{k+1}^{i_{k+1}} ... q_l^{i_l} \text{rank}(H_{i_1,i_2,\ldots,i_l}(\mathcal{M}))$$

3.4.1 Colored Jones polynomial as $q$-graded Euler characteristic

Khovanov introduced a categorification of the colored Jones polynomial as the doubly-graded homology theory known as the Colored $\mathfrak{sl}_2$ Knot Homology $\mathcal{H}^{\mathfrak{sl}_2}_{i,j} R(K)$, the corresponding polynomial invariant is known as Poincaré polynomial. The Poincaré polynomial of the colored $\mathfrak{sl}_2$ homology $\mathcal{H}^{\mathfrak{sl}_2}_{i,j} R(K)$ is given by

$$\mathcal{P}^{\mathfrak{sl}_2}_{R}(K; q, t) = \sum_{i,j} t^i q^j \dim(\mathcal{H}^{\mathfrak{sl}_2}_{i,j} R(K))$$

here the subscript $i$ is for the quantum polynomial grading, and $j$ is for the Homological grading. The $q$-graded Euler Characteristic of the $\mathfrak{sl}_2$ knot homology remarkably gives us the colored Jones polynomial.

$$J_R(K; q) = \mathcal{P}^{\mathfrak{sl}_2}_{R}(K; q, t = -1) = \sum_{i,j} (-1)^j q^j \dim(\mathcal{H}^{\mathfrak{sl}_2}_{i,j} R(K))$$
3.4.2 Colored HOMFLY polynomial as bi-graded Homology theory

The categorification of the higher rank gauge group $\mathcal{G} = SU(N)$ gives the Colored $\mathfrak{sl}_N$ Homology $\mathcal{H}_{i,j}^{\mathfrak{sl}_N,R}(K)$. The Poincaré polynomial of the colored $\mathfrak{sl}_N$ Knot homology is

$$P_{R}^{\mathfrak{sl}_N}(K; q,t) = \sum_{i,j} t^i q^j \dim(\mathcal{H}_{i,j}^{\mathfrak{sl}_N,R}(K))$$

These Poincaré polynomials are related to the Colored HOMFLY polynomial as follows:

$$P_{R}(K; a=q^N, q) = P_{R}^{\mathfrak{sl}_N}(K; q,t = -1)$$

3.4.3 Colored HOMFLY polynomial as $(a,q)$-graded Euler characteristic

We can have more advanced categorification. HOMFLY polynomials $P_{R}(K; a, q)$ can further be categorized as polynomials in two variables $(a, q)$, which actually lead to the triply-graded Homology theory known as the Colored HOMFLY homology $\mathcal{H}_{i,j,k}^{R}(K)$. The Poincaré polynomials for $\mathcal{H}_{i,j,k}^{R}(K)$ homology is called the Colored Superpolynomial

$$P_{R}(K; a, q, t) = \sum_{i,j,k} a^i q^j t^k \dim(\mathcal{H}_{i,j,k}^{R}(K))$$

It can be shown that the $(a,q)$-graded Euler characteristic of the triply-graded Homology theory $\mathcal{H}_{i,j,k}^{R}(K)$ is equivalent to the Colored HOMFLY polynomial

$$P_{R}(K; a, q) = P_{R}(K; a, q, t = -1) = \sum_{i,j,k} (-1)^k a^i q^j \dim(\mathcal{H}_{i,j,k}^{R}(K))$$

3.4.4 Colored Superpolynomial

In the papers [13, 14, 15], it was established that the colored Superpolynomial is not same as the Poincaré polynomial of the colored $\mathfrak{sl}_N$ homology, i.e. $P_{R}(K; a = q^N, q, t) \neq P_{R}^{\mathfrak{sl}_N}(K; q,t)$ in general. [12] They are actually related by the differentials $(\mathcal{H}_{a, \alpha, \beta, \gamma}, d_N) \cong \mathcal{H}_{a, \alpha, \beta, \gamma}^{R}$ for $N \in \mathbb{Z}$. Here we had dropped off the triply-graded Homology groups $d_N : \mathcal{H}_{i,j,k} \to \mathcal{H}_{i+\alpha,j+\beta,k+\gamma}$ paired via the differential (the boundary operator) $d_N$ of degree $(\alpha, \beta, \gamma)$ corresponding to $(a,q,t)$ grading, being exact. The Poincaré polynomial can be written in the form

$$P_{R}(K; a, q, t) = R_{R}^{\mathfrak{sl}_N}(K; a, q, t) + (1 + a^\alpha q^\beta t^\gamma)Q_{R}^{\mathfrak{sl}_N}(K; a, q, t)$$

here $R_{R}^{\mathfrak{sl}_N}(K; a, q, t)$ and $Q_{R}^{\mathfrak{sl}_N}(K; a, q, t)$ are polynomials with non-negative coefficients, where $Q_{R}^{\mathfrak{sl}_N}(K; a, q, t)$ correspond to the Homological part which is annihilated by the differential $d_N$. The bi-graded homological knot invariant is then $P_{R}^{\mathfrak{sl}_N}(K; q,t) = R_{R}^{\mathfrak{sl}_N}(K; a = q^N, q, t)$. 

Note that the structure of the eqn.3.13 is same as the eqn.3.1; and this justifies the name $q$-graded Euler characteristic, and in fact the interpretation of the colored Jones as the $q$-graded Euler characteristic of the $\mathfrak{sl}_2$ knot homology. So the coefficient of Jones polynomial can be interpreted as dimension of vector spaces of Homological theory.
Chapter 4

Recent Developments

There had been immense development in the field after the pioneering work of Edward Witten [1]. Homological theory gave us lot more invariants; but it turns out that we don’t have a general formula for those Knot Invariants polynomials. Here I review recent developments in the field, and some conjectures for calculating Knot invariants are reviewed. We discuss a conjectured formula for colored Superpolynomials, closed form expression for HOMFLY polynomials and conjectured expression for $6j$ symbol for $U_q(sl_N)$ for multiplicity free case. Also a Mathematica program had been developed based on the Conjectured expression for $6j$-symbols.

4.1 Nawata-Ramadevi-Zodinmawia Conjecture for Twist Knots

In the paper [12] the formula for Colored Superpolynomial for a certain class of knots, known as twist Knots $K_p$, where $p$ denotes the number of full twist.

![Twist Knot](image)

Figure 4.1: Twist Knot $K_p$ with $p$ full twist.

Here $p$ counts the number of right-handed-full-twists for $p > 0$, number of left-handed-full-twists for $p < 0$. It turns out that there is no general technique developed for computing Colored Superpolynomial $P_n(K_p; a, q, t)$ for twist knots because $K_p≠1$ are hyperbolic knots. The Rolfsen’s table [16] gives the relation between known friendlier knots and the
Twist knots. Some examples can be found in fig.4.2. The planar projection of the twist knots $K_p$ has $2|p| + 2$ crossings where $2|p|$ contribution comes from the $p$ full twists, and the 2 comes from the negative clasp, i.e. the the crossing displayed in the upper half of the fig.4.1.

4.1.1 Multi-sum Formula for Colored Jones Polynomial

In an attempt to get unified Witten-Reshetikhin-Turaev (WRT) invariants for integral homology three-spheres, Habiro introduced the cyclotomic expansions of the colored Jones polynomials of the trefoil and the figure-eight knot by using quantum group $U_q(sl_2)$. Motivated by this, Masbaum found the cyclotomic expansion of the colored Jones polynomial of the twist knot $K_p$ by using Skein theoretic techniques.

Masbaum expansion for colored Jones [17] is displayed as follows

$$J_n(K_p; q) = \sum_{k=0}^{\infty} C_{K_p}(k) \frac{\{n-k\} \{n-k+1\} \cdots \{n+k\}}{n} \quad (4.1)$$

where

$$C_{K_p}(k) = q^{k(k+3)/2} \sum_{l=0}^{k} (-1)^l q^{l(l+1)p} \frac{\{k\}!}{\{k+l+1\}!\{k-l\}!} \quad (4.2)$$

where we have used the following notations for q-numbers and q-factorial

$$\{n\} = q^n - q^{-n}, \quad q^n = q, \quad \{n\}! = \{n\} \{n-1\} \cdots \{1\} \quad (4.3)$$

Using the q-Pochhammer symbol $(z; q)_k = \prod_{j=0}^{k-1}(1 - zq^j)$, the Masbaum formula as in eqn.4.1 can be rearranged to the following

$$J_n(K_p; q) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} q^k (q^{1-n}; q)_k (q^{1+n}; q)_k (-1)^l q^{l(l+1)p+l(l-1)/2} (1 - q^{2l+1}) \frac{(q; q)_k}{(q)_k} \frac{(q; q)_{k+l+1}(q; q)_k}{(q)_k} \quad (4.4)$$

Using lengthy algebra, it can be shown that the two expression for Masbaum formula eqn.4.1 and eqn.4.4 are indeed equivalent. For the proof of the equivalence of this, I refer to [18] which contains the explicit hand written calculation done by me; and some results related to q-Pochhammer symbols which are vital to the proof: which can be in a way
considered as an appendix to the present text. The expression for Masbaum formula in terms of double sum as in eqn.4.4 can be written in terms of Multi Sum formula as follows

\[ J_n(K_{p>0}; q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} \infty q^{s_p(q^{1-n}; q) s_p(q^{1+n}; q) s_p \prod_{i=1}^{p-1} \sum_{i=1}^{s_i(s_i+1)} q^{s_i(s_i+1)} \left( \frac{s_i+1}{s_i} \right) q \]  

(4.5)

\[ J_n(K_{p<0}; q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} \infty (-1)^{s_p} q^{-s_p(q^{1-n}; q) s_p(q^{1+n}; q) s_p \prod_{i=1}^{p-1} \sum_{i=1}^{s_i(s_i+1)} q^{s_i(s_i+1)} \left( \frac{s_i+1}{s_i} \right) q \]  

(4.6)

where we have used the following notation for q-binomial

\[ \binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \]  

(4.7)

A brief guide for the equivalence of double sum expression for Masbaum formula as in eqn.4.4 and the multisum formula as in eqn.4.5,4.6 is presented in the appendix of [12] using Bailey’s Lemma and Bailey’s sequences. For the complete derivation of this equivalence I again refer to my handwritten proof [18]. For the simple knots \( K_1 \) (trefoil) and \( K_{-1} \), the summation over \( l \) as in eqn.4.4 can be explicitly carried out and then the colored Jones polynomials for them can be written in terms of single sum expression as

\[ J_n(K_1, q) = \sum_{k=0}^{\infty} q^k(q^{1-n}; q)_k(q^{1+n}; q)_k \]  

(4.8)

\[ J_n(K_{-1}, q) = \sum_{k=0}^{\infty} (-1)^k q^{-k+1} q^{k+1-n} q^{1-n} q^{1+n} q_k \]  

(4.9)

If we closely observes the above expressions, we will see that the multi-sum formula as in eqn.4.5,4.6 is built upon the expression for the colored Jones polynomial for trefoil \( K_1 \) and Figure eight \( K_{-1} \) as in eqn.4.8,4.9 respectively with additional \( p-1 \) twisting factors.

The relation between twisting factor for right-handed full twist and left-handed full twist is

\[ q^{s_i(s_i+1)} \left( \frac{s_i+1}{s_i} \right) q \leftarrow (q \leftrightarrow q^{-1}) \rightarrow q^{-s_i(s_i+1)} \left( \frac{s_i+1}{s_i} \right) q \]  

(4.10)

### 4.1.2 Conjecture for Colored Superpolynomial for Twist Knots

From the analysis over the colored Jones Polynomials in previous section, it is very natural to expect that the Colored Superpolynomial for the Twist Knot may have very similar structure and they can be build upon the expression for Colored Superpolynomial for Trefoil \( K_1 \) and Figure eight knot \( K_{-1} \), by appropriately inserting some twisting factor. The colored Superpolynomial for \( K_1 \) and \( K_{-1} \) are already known in more concise form in [15]. But here we presented an expanded version which will be easy to generalize. Before introducing the result for \( K_1 \) and \( K_{-1} \), let us defined few functions, which will make the formula look somewhat neat.

\[ F_{n,k}(a, q, t) \equiv (-t)^{1-n} q^k \left( \frac{-atq^{-1}; q)_k}{(q; q)_k} \right) \]  

(4.11)
The relation between twisting factor for right-handed full twist and left-handed full twist

\[ G_{n,k}(a, q, t) \equiv (-at^2)^{-k}q^{-k(k-3)/2}(-at^{-1}q^{-1}; q)_k(q^{1-n}; q)_k(-at^2q^{n-1}; q)_k \]  \quad (4.12)

Note that the defined function \( F_{n,k} \) and \( G_{n,k} \) are related as follows:

\[ F_{n,k}(a, q, t) = (-t)^{1-n}(-at^2k^{-k(k-1)/2})G_{n,k}(a, q, t) \]  \quad (4.13)

In terms of these functions, the known result of the Colored Superpolynomial for \( K_1 \) and \( K_{-1} \) can be written as

\[ P_n(K_1; a, q, t) = \sum_{k=0}^{\infty} F_{n,k}(a, q, t) \]  \quad (4.14)
\[ P_n(K_{-1}; a, q, t) = \sum_{k=0}^{\infty} G_{n,k}(a, q, t) \]  \quad (4.15)

We further note that the known results of \( K_2 \) and \( K_{-2} \) can be written as

\[ P_2(K_2; a, q, t) = F_{2,0}(a, q, t) + (1 + at^2)F_{2,1}(a, q, t) \]  \quad (4.16)
\[ P_3(K_2; a, q, t) = F_{3,0}(a, q, t) + (1 + at^2)F_{3,1}(a, q, t) + (1 + at^2(1 + q) + a^2t^4q^2)F_{3,2}(a, q, t) \]
\[ P_2(K_{-2}; a, q, t) = G_{2,0}(a, q, t) + (1 + a^{-1}t^2)G_{2,1}(a, q, t) \]
\[ P_3(K_{-2}; a, q, t) = G_{3,0}(a, q, t) + (1 + a^{-1}t^2)G_{3,1}(a, q, t) + (1 + a^{-1}t^2(1 + q^{-1}) + a^{-2}t^{-4}q^{-2})G_{3,2}(a, q, t) \]  \quad (4.17)

The above expression seems to suggest that the factors appearing alongside \( F_{n,k} \) and \( G_{n,k} \), play the role of twisting factors similar to the Colored Jones polynomial case. The analysis of previous section helps us to guess the following multi-sum expression for Superpolynomials for \( K_2 \) and \( K_{-2} \) twist knot.

\[ P_n(K_2; a, q, t) = \frac{\sum_{s_2 \geq s_1 \geq 0}^{n-1} F_{n,s_2}(a, q, t)(at^2)^{s_1}q^{s_1(s_1-1)}(s_2\text{q}^{s_2})}{s_1} q \]  \quad (4.18)
\[ P_n(K_{-2}; a, q, t) = \frac{\sum_{s_2 \geq s_1 \geq 0}^{n-1} G_{n,s_2}(a, q, t)(at^2)^{-s_1}q^{-s_1(s_1-1)}(s_2\text{q}^{s_2})}{s_1} q \]

These patterns very well motivated Nawata-Ramadevi-Zodinmawia to conjecture the general expression for Colored Superpolynomial for the twist knots

\[ P_n(K_{p>0}; a, q, t) = \sum_{s_{p}\geq \cdots \geq s_1 \geq 0}^{\infty} F_{n,s_{p}}(a, q, t) \prod_{i=1}^{p-1} (at^2)^{s_i}q^{s_i(s_i-1)}(s_{i+1}\text{q}^{s_2}) \]  \quad (4.19)
\[ P_n(K_{p<0}; a, q, t) = \sum_{s_{|p|}\geq \cdots \geq s_1 \geq 0}^{\infty} G_{n,s_{|p|}}(a, q, t) \prod_{i=1}^{|p|-1} (at^2)^{-s_i}q^{-s_i(s_{i+1}-1)}(s_{i+1}\text{q}^{s_2}) \]

The relation between twisting factor for right-handed full twist and left-handed full twist is

\[ (at^2)^{s_i}q^{s_i(s_i-1)}(s_{i+1}\text{q}^{s_2}) \rightarrow (a, t, q \leftrightarrow a^{-1}, t^{-1}, q^{-1}) \rightarrow (at^2)^{-s_i}q^{-s_i(s_{i+1}-1)}(s_{i+1}\text{q}^{s_2}) \]  \quad (4.20)
The above formula is a guesswork, and we don’t give the proof here; but just an intuition with the observed patterns. In the original paper [12], the authors have presented several checks of the formula. We can write the above conjectured formula in terms of double sum formula which is convenient for the computation of Super-A-Polynomials.

\[
\mathcal{P}_n(K_{p>0}; a, q, t) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} F_{n,k}(a, q, t)(-1)^l(at^2)^p q^{(p+1)/2}(l-1) \left( \frac{1 - at^2 q^{2l-1}}{(at^2 q^{l-1}; q)_{k+1}} \right)^l (l)_{q}^{k}
\]

\[
\mathcal{P}_n(K_{p<0}; a, q, t) = (t)^{n-1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} F_{n,k}(a, q, t)(-1)^l(at^2)^p q^{(p+1)/2}(l-1) \left( \frac{1 - at^2 q^{2l-1}}{(at^2 q^{l-1}; q)_{k+1}} \right)^l (l)_{q}^{k}
\]

Note that, in these expression; the summation does not actually runs to infinity. After a stage in summation, man terms like the q-binomial, etc. will becomes ill defined, and we take the convention that we make those ill defined terms to be zero; and thus our virtual infinite series finally gets terminated.

4.2 Ramadevi-Govindarajan-Kaul closed form expression for HOMFLY polynomial

We had seen in sec.2.8, that the expectation value of Wilson Line satisfy the Skein Recursion relation for HOMFLY for $SU(N)$. In the paper [19], closed form expression for HOMFLY polynomial from $SU(N)$ Chern-Simons theory was developed. Here I don’t present the proof and result of the theorems which was developed in [19]; but present an application of those theorem’s. Here I will assume the notations which are being used in [19], and will then present calculation for Knot Invariants for some knots. Note that these calculation had been done in the paper [19]; but what here I plan to do is give explicit guideline on how those results are arrived by using the general formula. The examples which will be solved will will be sufficient to teach how to use the formula in general case, i.e. all possible major variants have been taken care.

4.2.1 Unknot $O$

The invariant for unknot is the $q$-dimension of the representation living on it. Let $R_n$ be the representation living on the unknot $O$, then

\[
V[O, R_n] = \dim_q R_n
\]

4.2.2 Trefoil ($3_1$), $5_1$ and $7_1$ knot

The braid which closes to form $3_1$ is presented in fig.4.3. Now we refer to fig.4 of the paper [19], which then suggest that the braid for trefoil is actually $\mathcal{L}_3(R, R)$. Using theorem 5, eqn. 5.1 & fig.4 of [19] we get

\[
V[3_1, R_n] = V[\mathcal{L}_3(R_n, R_n)]
\]

Similarly, for $5_1$ and $7_1$ knot; we get $V[5_1, R_n] = V[\mathcal{L}_5(R_n, R_n)]$ and $V[7_1, R_n] = V[\mathcal{L}_7(R_n, R_n)]$ respectively.
4.2.3 Figure 8 knot $(4_1)$

The calculation for figure 8 knot is presented in the fig.4.4 Using eqn.4.5,4.6,5.2 & fig.6
from [19], we get

$$V[4_1, R_n] = \left\langle \chi_q^{-1} \left( \hat{Q}_2^V \left( \hat{R}_n \ R_n \ \hat{R}_n \right) \right) \left| \chi_q \left( \hat{Q}_2^H \left( \hat{R}_n \ R_n \ \hat{R}_n \right) \right) \right\rangle$$  \hspace{1cm} (4.24)$$

which then simplifies to

$$V[4_1, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \dim_q \hat{\rho}_l \cdot a_{\hat{\rho}, \hat{\rho}} \cdot w^{j-l} q^{j^2-l^2}}$$  \hspace{1cm} (4.25)$$

### 4.2.4 6_1 Knot

Using the fig.4.5 and the similar analysis for 4_1 knot we get

Figure 4.5: 6_1 knot and HOMFLY polynomial calculation

$$V[4_1, R_n] = \left\langle \chi_q^{-1} \left( \hat{Q}_2^V \left( \hat{R}_n \ R_n \ \hat{R}_n \right) \right) \left| \chi_q \left( \hat{Q}_4^H \left( \hat{R}_n \ R_n \ \hat{R}_n \right) \right) \right\rangle$$  \hspace{1cm} (4.26)$$

which then simplifies to

$$V[4_1, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \dim_q \hat{\rho}_l \cdot a_{\hat{\rho}, \hat{\rho}} \cdot w^{j-l} q^{j^2-l^2}}$$  \hspace{1cm} (4.27)$$

### 4.2.5 5_2 Knot

Figure 4.6: Calculation of HOMFLY polynomial for 5_2 Knot
The calculation for $5_2$ knot is presented in the fig.4.7. Using eqn.5.1,5.2, fig.5 & Theorem 7 from [19], we get

$$V[5_2, R_n] = \left\langle \psi_q \left( \hat{Q}_3^H \left( \bar{R}_n \bar{R}_n \right) \right) \right| \psi_q \left( \hat{Q}_2^V \left( \bar{R}_n \bar{R}_n \right) \right) \right\rangle$$

which then simplifies to

$$V[5_2, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \dim_q \hat{\rho}_l} \cdot a_{\hat{\rho}_j \hat{\rho}_l} w^{n+3j/2} q^{n(n+1)-(l(l+1)+3j^2/2)}$$

(4.29)

4.2.6 $7_2$ Knot

![Figure 4.7: Calculation of HOMFLY polynomial for $7_2$ Knot](image)

The calculation for $7_2$ knot is presented in the fig.4.7. Using eqn.5.1,5.2, fig.5 & Theorem 7 from [19], we get

$$V[7_2, R_n] = \left\langle \psi_q^{-1} \left( \hat{Q}_5^H \left( \bar{R}_n \bar{R}_n \right) \right) \right| \psi_q^{-1} \left( \hat{Q}_2^V \left( \bar{R}_n \bar{R}_n \right) \right) \right\rangle$$

which then simplifies to

$$V[7_2, R_n] = \sum_{l,j=0}^{n} \sqrt{\dim_q \hat{\rho}_j \dim_q \hat{\rho}_l} \cdot a_{\hat{\rho}_j \hat{\rho}_l} w^{-n-5j/2} q^{-[b(n+1)-(l(l+1))-5j^2/2}$$

(4.31)

Note the appearance of the duality matrices $a_{\hat{\rho}_j \hat{\rho}_l}$ in the above result. For the above examples the result are known and is presented in [19]. However, the general result for these duality matrices is not known. Hence, we are not able to calculate knot invariants using the formula developed in [19] because of the incomplete knowledge of the duality matrices.

4.3 Nawata-Ramadevi-Zodinmawia Conjecture for Multiplicity free quantum $6j$-symbols for $U_q(\mathfrak{sl}_N)$

The need for computing duality matrices is very well motivated from the previous section. It turns out that these duality matrices can be written in terms of quantum $6j$-symbols.
The complete solution for quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$ is not known. In the paper \cite{20} Nawata, Ramadevi, and Zodinmawia conjectured an expression for Multiplicity free quantum 6j-symbols for $U_q(\mathfrak{sl}_N)$. The paper \cite{20} discusses the conjectured expression in a very concise and understandable language. I will not write about it here, since it will then become just a rewriting.

4.3.1 Mathematica Program developed by Shoaib Akhtar

A Mathematica program \cite{21} is build upon the Conjectured expression. The program is made user-friendly; the user can put the input data and will get the desired result in the matrix form. The program computes the duality matrices using the conjectured formula and gives the output in user-friendly matrix form, and thus solving the problem which was mentioned in sec.4.2; assuming the conjecture to be true. The program is expected to be self explanatory, and is readable; and all the functions which were needed in the computation has also been explicitly defined.

The link to the program is  
https://www.wolframcloud.com/obj/shoaib0092/Published/6j-Symbol.nb
Conclusion

We started with a brief review on Knot Theory. Then we established relationship between Knot Theory and Chern-Simons theory. We saw how the Wilson Loop observables in Chern-Simons theory with $SU(N)$ gague group give us HOMFLY polynomial invariant for knots/links, and $SU(2)$ theory gives Jones Polynomials, and how the limiting $SU(0)$ theory renders us with the Alexander Polynomial. We also encountered Colored Knot Invariants and Classical A polynomial. We noted that these invariants where not sufficient to solve our classification problem in Knot Theory; for example colored Jones and HOMFLY polynomials were not able to distinguish Mutants. Then we studied the refinement of these invariants through categorification using graded Homological Theory, giving us Colored Superpolynomials. We noted that the explicit result of Superpolynomials for general knot is not known. We then reviewed Nawata-Ramadevi-Zodinmawia Conjecture which give us a promissing expression for Colored Superpolynomial for a special class of knots called Twist Knots. Closed form expression for HOMFLY polynomials, and then conjectured expression for $6j$ symbol for $U_q(sl_N)$ for multiplicity free case were discussed. Also a Mathematica Program had been developed which computed the $6j$-symbols and the desired Duality Matrices which were need to use the closed form expression for HOMFLY polynomials.

Future Work direction

At present, there is lot more to explore in this subject. Tremendous development in the field has taken place since the foundational work of Edward Witten [1]. But still, the theory is far from completeness. Interesting links to String Theory involving BPS state had been developed; which gives us BPS $q$-series. There is lot more to be discovered, and these notes is an attempt to introduce the reader with this growing field.

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