WEIGHT DISTRIBUTION OF TWO CLASSES OF CYCLIC CODES WITH RESPECT TO TWO DISTINCT ORDER ELEMENTS

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Abstract. Cyclic codes are an interesting type of linear codes and have wide applications in communication and storage systems due to their efficient encoding and decoding algorithms. Cyclic codes have been studied for many years, but their weight distribution are known only for a few cases. In this paper, let $F_r$ be an extension of a finite field $F_q$ and $r = q^m$, we determine the weight distribution of the cyclic codes $C = \{c(a, b) : a, b \in F_r\}$,

$$c(a, b) = (\text{Tr}_{r/q}(ag_1^0 + bg_2^0), \ldots, \text{Tr}_{r/q}(ag_1^{n-1} + bg_2^{n-1})),$$

$g_1, g_2 \in F_r,$

in the following two cases: (1) $\text{ord}(g_1) = n, n | r - 1$ and $g_2 = 1$; (2) $\text{ord}(g_1) = n, g_2 = g_1^m, \text{ord}(g_2) = \frac{r-1}{n}$, and $2(r - 1) | (q + 1)$.

1. Introduction

Let $F_q$ be a finite field with $q$ elements, where $q = p^s$, $p$ is a prime, and $s$ is a positive integer. An $[n, k, d]$ linear code $C$ is a $k$-dimensional subspace of $F_q^n$ with minimum distance $d$. It is called cyclic if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C$. By identifying the vector $(c_0, c_1, \ldots, c_{n-1}) \in F_q^n$ with

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1} \in F_q[x]/(x^n - 1),$$

any code $C$ of length $n$ over $F_q$ corresponds to a subset of $F_q[x]/(x^n - 1)$. Then $C$ is a cyclic code if and only if the corresponding subset is an ideal of $F_q[x]/(x^n - 1)$. Note that every ideal of $F_q[x]/(x^n - 1)$ is principal. Hence there is a monic polynomial $g(x)$ of the least degree such that $C = \langle g(x) \rangle$ and $g(x) | (x^n - 1)$. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is called the parity-check polynomial of the cyclic code $C$. Suppose that $h(x)$ has $t$ irreducible factors over $F_q$, we call $C$ the dual of the cyclic code with $t$ zeros.

Let $A_i$ be the number of codewords with Hamming weight $i$ in the code $C$ of length $n$. The weight enumerator of $C$ is defined by

$$1 + A_1 x + A_2 x^2 + \cdots + A_n x^n.$$

The sequence $(1, A_1, A_2, \ldots, A_n)$ is called the weight distribution of the code $C$. In coding theory it is often desirable to know the weight distributions of the codes because they can be used to estimate the error correcting capability and the error
probability of error detection and correction with respect to some algorithms. This is quite useful in practice. Unfortunately, it is a very hard problem in general and remains open for most cyclic codes.

Let \( r = q^m \) for a positive integer \( m \) and \( \alpha \) a generator of \( \mathbb{F}_r^* \). Let \( h(x) = h_1(x)h_2(x) \cdots h_t(x) \), where \( h_i(x) (1 \leq i \leq t) \) are distinct monic irreducible polynomials over \( \mathbb{F}_q \). Let \( g_i^{-1} = \alpha^{-s_i} \) be a root of \( h_i(x) \) and \( n_i \) the order of \( g_i \) for \( 0 \leq s_i \leq r - 2 (1 \leq i \leq t) \). Denote \( \delta = \gcd(r - 1, s_1, s_2, \ldots, s_t) \) and \( n = \frac{r - 1}{\delta} \). The cyclic code \( C \) can be defined by

\[
C = \{ c(a_1, a_2, \ldots, a_t) : a_1, a_2, \ldots, a_t \in \mathbb{F}_r \},
\]

where

\[
c(a_1, a_2, \ldots, a_t) = (\Tr_{r/q}(\sum_{i=1}^{t} a_i g_i^0), \Tr_{r/q}(\sum_{i=1}^{t} a_i g_i^1), \ldots, \Tr_{r/q}(\sum_{i=1}^{t} a_i g_i^{n_i - 1}))
\]

and \( \Tr_{r/q} \) denotes the trace function from \( \mathbb{F}_r \) to \( \mathbb{F}_q \). It follows from Delsarte's Theorem \[7\] that the code \( C \) is an \([n, k]\) cyclic code over \( \mathbb{F}_q \) with the parity-check polynomial \( h(x) \), where \( k = \deg(h_1(x)) + \deg(h_2(x)) + \cdots + \deg(h_t(x)) \). The weight distributions of such cyclic codes have been studied for many years and are known in some cases. We describe the known results as follows.

1. For \( t = 1 \), \( C \) is called an irreducible cyclic code. The weight distributions of irreducible cyclic codes have been extensively studied and can be found in \([1, 2, 8, 9, 12, 23, 24]\).

2. For \( t = 2 \), i.e., \( h(x) = h_1(x)h_2(x) \). The duals of the cyclic codes with two zeros have been well investigated when \( \deg(h_1(x)) = \deg(h_2(x)) \). If \( g_1 \) and \( g_2 \) have the same order in \( \mathbb{F}_r^* \), we know \( \deg(h_1(x)) = \deg(h_2(x)) \). Then the weight distribution of such cyclic codes had been determined for some special cases \([11, 13, 15, 21, 23, 29, 30, 31, 32, 36]\). If \( \mathbb{F}_r^* = \langle g_1 \rangle = \langle g_2 \rangle \), then the weight distribution of the code \( C \) which is called the dual of primitive cyclic code with two zeros had been studied in \([3, 4, 5, 6, 14, 20, 22, 25, 27, 34]\).

3. For \( t = 3 \). The results on the weight distribution of cyclic codes \( C \) with three zeros can be found in \([19, 35, 37]\).

4. For arbitrary \( t \). Yang et al. \[33\] described a class of the duals of cyclic codes with \( t \) zeros and determined their weight distributions under special conditions. Li et al. \[17\] also studied such cyclic codes and developed a connection between the weight distribution and the spectra of Hermitian forms graphs.

In this paper, we shall determine the weight distributions of two classes of cyclic codes whose duals have two zeros. Let \( \alpha \) be a primitive element of \( \mathbb{F}_r \) and \( r - 1 = nN \) for two positive integers \( n > 1 \) and \( N > 1 \). We mainly consider the following two cases of the cyclic code \( C \).
(1) Assume that the order of \( g_1 \) is \( n \) and the order of \( g_2 \) is 1. Then we can set \( g = g_1 = \alpha^N, g_2 = 1 \), and 
\[ C_1 = \{ c(a, b) : a, b \in \mathbb{F}_r \}, \]
where 
\[ c(a, b) = (\text{Tr}_{r/q}(ag^0 + b), \text{Tr}_{r/q}(ag^1 + b), \ldots, \text{Tr}_{r/q}(ag^{n-1} + b)). \]
It is obvious that the order of \( g \) is not equal to 1 and thus the parity-check polynomial of \( C_1 \) is \((x - 1)h_{g-1}(x)\), where \( h_{g-1}(x) \) is the minimal polynomial of \( g^{-1} \) over \( \mathbb{F}_q \). The lower bound on the minimum weight of \( C_1 \) had been given by Ding [10]. We can also get a tight bound on the minimum weight of such cyclic code.

(2) Assume that the order of \( g_1 \) is \( n (n \text{ is even}) \) and the order of \( g_2 \) is \( \frac{n}{2} \). Then we can set \( g = g_1 = \alpha^N \) and \( g_2 = \mu g^2 \) for \( \mu \in \mathbb{F}_r^* \), where the order of \( \mu \) is a divisor of \( \frac{n}{2} \). Denote 
\[ C_2 = \{ c(a, b) : a, b \in \mathbb{F}_r \}, \]
where 
\[ c(a, b) = (\text{Tr}_{r/q}(ag^0 + b(\mu g^2)^0), \text{Tr}_{r/q}(ag^1 + b(\mu g^2)^1), \ldots, \text{Tr}_{r/q}(ag^{n-1} + b(\mu g^2)^{n-1})). \]
It is easily known that \( g \) and \( \mu g^2 \) are not conjugates of each other due to their distinct orders and then \( h_{g-1}(x) \) and \( h_{(\mu g^2)-1}(x) \) are distinct, where \( h_{g-1}(x) \) and \( h_{(\mu g^2)-1}(x) \) are the minimal polynomial of \( g^{-1} \) and \( (\mu g^2)^{-1} \) over \( \mathbb{F}_q \), respectively. We know that the parity-check polynomial of \( C_2 \) is \( h_{g-1}(x)h_{(\mu g^2)-1}(x) \). For convenience, we shall present a method to determine the weight distribution of the cyclic code \( C_2 \) when \( \mu = 1 \). The general case can be also dealt with by our method and the method in [11] or [21]. In addition, we present a tight bound on the minimum weight of cyclic code \( C_2 \).

This paper is organized as follows. In Section 2, we introduce some results about Gauss periods. In Section 3 and 4, we shall determine the weight distributions of the cyclic codes \( C_1 \) and \( C_2 \), respectively.

2. Gauss periods

Let \( \mathbb{F}_r \) be the finite field with \( r \) elements, where \( r \) is a power of prime \( p \). For any \( a \in \mathbb{F}_r \), we can define an additive character of the finite field \( \mathbb{F}_r \) as follows:
\[ \psi_a : \mathbb{F}_r \to \mathbb{C}^*, \psi_a(x) = \zeta_p^\text{Tr}_{r/p}(ax), \]
where \( \zeta_p = e^{2\pi i / p} \) is a \( p \)-th primitive root of unity and \( \text{Tr}_{r/p} \) denotes the trace from \( \mathbb{F}_r \) to \( \mathbb{F}_p \). If \( a = 1 \), then \( \psi_1 \) is called the canonical additive character of \( \mathbb{F}_r \). The
orthogonal property of additive characters which can be found in [18] is given by

\[ \sum_{x \in \mathbb{F}_r} \psi_1(ax) = \begin{cases} r, & \text{if } a = 0; \\ 0, & \text{if } a \in \mathbb{F}_r^*. \end{cases} \]

Let \( r - 1 = nN \) and \( \alpha \) a fixed primitive element of \( \mathbb{F}_r \), where \( r = q^m = p^en \). We define \( C_i^{(N,r)} = \alpha^i \langle \alpha^N \rangle \) for \( i = 0, 1, \ldots, N - 1 \), where \( \langle \alpha^N \rangle \) denotes the subgroup of \( \mathbb{F}_r^* \) generated by \( \alpha^N \). The Gauss periods of order \( N \) are given by

\[ \eta_i^{(N,r)} = \sum_{x \in C_i^{(N,r)}} \psi(x), \]

where \( \psi \) is the canonical additive character of \( \mathbb{F}_r \) and \( \eta_i^{(N,r)} = \eta_i^{(N,r)}(\mod N) \) if \( i \geq N \). In general, the explicit evaluation of Gauss periods is a very difficult problem. However, they can be computed in a few cases.

**Lemma 2.1.** [26] When \( N = 2 \), the Gauss periods are given by

\[ \eta_0^{(2,r)} = \begin{cases} -1 + (1 - 1)^{\frac{1}{2}} \sqrt[3]{2}, & \text{if } p \equiv 1 \pmod{4}, \\ -1 - (1 - 1)^{\frac{1}{2}} \sqrt[3]{2}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \]

and \( \eta_1^{(2,r)} = -1 - \eta_0^{(2,r)} \).

**Lemma 2.2.** [26] Let \( N = 3 \). If \( p \equiv 1 \pmod{3} \) and \( sm \equiv 0 \pmod{3} \), then

\[ \eta_0^{(3,r)} = \frac{-1 + c_1 r^\frac{1}{3}}{3} \]

and

\[ \{ \eta_1^{(3,r)}, \eta_2^{(3,r)} \} = \{ \frac{-1 - \frac{1}{2}(c_1 + 9d_1)r^\frac{1}{3}}{3}, \frac{-1 - \frac{1}{2}(c_1 - 9d_1)r^\frac{1}{3}}{3} \}, \]

where \( c_1 \) and \( d_1 \) are given by \( 4p^{sm} = c_1^2 + 27d_1^2 \), \( c_1 \equiv 1 \pmod{3} \), and \( \gcd(c_1, p) = 1 \).

**Lemma 2.3.** [26] Let \( N = 4 \). If \( p \equiv 1 \pmod{4} \) and \( sm \equiv 0 \pmod{4} \), then

\[ \eta_0^{(4,r)} = \frac{-1 - r^\frac{1}{2} - 2s_1 r^\frac{1}{4}}{4}, \eta_2^{(4,r)} = \frac{-1 - r^\frac{1}{2} + 2s_1 r^\frac{1}{4}}{4} \]

and

\[ \{ \eta_1^{(4,r)}, \eta_3^{(4,r)} \} = \{ \frac{-1 + r^\frac{1}{2} + 4t_1 r^\frac{1}{4}}{4}, \frac{-1 + r^\frac{1}{2} - 4t_1 r^\frac{1}{4}}{4} \}, \]

where \( s_1 \) and \( t_1 \) are given by \( p^{sm} = s_1^2 + 4t_1^2 \), \( s_1 \equiv 1 \pmod{4} \), and \( \gcd(s_1, p) = 1 \).

The Gauss periods in the semi-primitive case are known and are described in the following lemma.

**Lemma 2.4.** [26] Assume that there exists a least positive integer \( e \) such that \( p^e \equiv -1 \pmod{N} \). Let \( r = p^{2ef} \) for some positive integer \( f \).

1. If \( f, p, \) and \( \frac{p^{e} + 1}{N} \) are all odd, then

\[ \eta_i^{(N,r)} = \frac{(N - 1)\sqrt{r} - 1}{N}, \eta_i^{(N,r)} = -\frac{\sqrt{r} + 1}{N} \] for \( i \neq N/2 \).
(2) In all other cases,
\[ \eta_0^{(N,r)}(x) = \frac{(-1)^{N+1}(N-1)\sqrt{r} - 1}{N}, \eta_i^{(N,r)} = \frac{(-1)^i\sqrt{r} - 1}{N} \text{ for } i \neq 0. \]

The Gauss periods in the index 2 case can be described in the following lemma.

**Lemma 2.5.** [12, 26] Let \( N > 3 \) be a prime with \( N \equiv 3 \pmod{4} \), \( p \) a prime such that \( |Z_N^* : \langle p \rangle| = 2 \), and \( r = p^{N-1}/k \) for some positive integer \( k \). Let \( h \) be the class number of \( \mathbb{Q}(\sqrt{-N}) \) and \( a, b \) the integers satisfying
\[
\begin{aligned}
4p^b &= a^2 + Nb^2 \\
-2p^{\frac{N-1+2b}{2}}& \equiv a \pmod{N} \\
b &> 0, p \nmid b.
\end{aligned}
\]

Then the Gauss periods of order \( N \) are given by
\[
\begin{align*}
\eta_i^{(N,r)} &= \frac{1}{N}(P^{(k)}A^{(k)}(N-1) - 1), \\
\eta_0^{(N,r)} &= \frac{1}{N}(P^{(k)}A^{(k)} + P^{(k)}B^{(k)}N + 1), \text{ if } \left(\frac{a}{N}\right) = 1, \\
\eta_0^{(N,r)} &= \frac{1}{N}(P^{(k)}A^{(k)} - P^{(k)}B^{(k)}N + 1), \text{ if } \left(\frac{a}{N}\right) = -1,
\end{align*}
\]
where
\[
\begin{align*}
P^{(k)} &= (-1)^{k-1}p^{\frac{k}{2}(N-1-2b)} \\
A^{(k)} &= Re\left(\frac{a+b\sqrt{-N}}{2}\right)^k \\
B^{(k)} &= Im\left(\frac{a+b\sqrt{-N}}{2}\right)^k / \sqrt{N}.
\end{align*}
\]

In the following lemma, we introduce a bound on the values of Gauss periods which can be found in [12].

**Lemma 2.6.** [12] For all \( i \) with \( 0 \leq i \leq N-1 \), we have
\[ |\eta_i^{(N,r)} + \frac{1}{N}| \leq \frac{(N-1)\sqrt{r}}{N}. \]

3. The weight distribution of \( C_1 \)

Let \( \alpha \) be a primitive element of \( \mathbb{F}_r \), where \( r = q^m \). Let \( r-1 = nN \) for two positive integers \( n > 1 \) and \( N > 1 \). For \( g = \alpha^N \) we define a cyclic code over \( \mathbb{F}_q \) by
\[ C_1 = \{c(a,b) : a, b \in \mathbb{F}_r\}, \]
where
\[ c(a,b) = (\text{Tr}_{r/q}(ag^0 + b), \text{Tr}_{r/q}(ag^1 + b), \ldots, \text{Tr}_{r/q}(ag^{n-1} + b)). \]

Then \( C_1 \) is an \( [n,k+1] \) cyclic code with parity-check polynomial \( (x-1)h_{g^{-1}}(x) \), where \( k \) is the order of \( g \) modulo \( n \) and \( h_{g^{-1}}(x) \) is the minimal polynomial of \( g^{-1} \) over \( \mathbb{F}_q \).

The lower bound on the minimum weight of \( C_1 \) had been given by Ding [10]. For any \( a, b \in \mathbb{F}_r \), the Hamming weight of \( c(a,b) \) is equal to
\[ W_H(c(a,b)) = n - Z(r,a,b), \]
where
\[ Z(r,a,b) = |\{x \in C_0^{(N,r)} : \text{Tr}_{r/q}(ax + b) = 0\}|. \]
Let $\phi$ be the canonical additive character of $\mathbb{F}_q$. Then $\psi = \phi \circ \text{Tr}_{r/q}$ is the canonical additive character of $\mathbb{F}_r$. By the orthogonal property of additive characters we have

$$Z(r, a, b) = \sum_{x \in C^{(N,r)}_0} \frac{1}{q} \sum_{y \in \mathbb{F}_q} \phi(y \text{Tr}_{r/q}(ax + b))$$

$$= \sum_{x \in C^{(N,r)}_0} \frac{1}{q} \sum_{y \in \mathbb{F}_q} \phi(\text{Tr}_{r/q}(yax + yb))$$

$$= \frac{1}{q} \cdot \frac{r - 1}{N} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C^{(N,r)}_0} \psi(yax + yb)$$

$$= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \psi(yb) \sum_{x \in C^{(N,r)}_0} \psi(yax).$$

To compute the values of $Z(r, a, b)$, we introduce the following lemma which can be found in [16].

**Lemma 3.1.** Let $H$ and $K$ be two subgroups of a finite Abelian group $G$. Then we have

$$h_1K = h_2K \text{ if and only if } h_1(H \cap K) = h_2(H \cap K)$$

for $h_1, h_2 \in H$. Moreover,

$$HK/K \cong H/(H \cap K) \text{ and } [HK : K] = [H : (H \cap K)],$$

where $HK = \{hk : h \in H, k \in K\}$.

**Theorem 3.2.** Let $d = \gcd(N, \frac{r-1}{q-1})$. Note that $c(a, b_1) = c(a, b_2)$ if $\text{Tr}_{r/q}(b_1) = \text{Tr}_{r/q}(b_2)$. Then the weight distribution of $C_1$ is given by Table 1. In especial, if $N \mid \frac{r-1}{q-1}$, then the weight distribution of $C_1$ is given by Table 2.

**Table 1. Weight distribution of $C_1$.**

| Weight | Frequency ($0 \leq j \leq N - 1$) |
|--------|----------------------------------|
| $0$    | $1$                              |
| $\frac{(q-1)(r-1)}{qN} - \frac{1}{q} \sum_{l=0}^{N-1} \eta_d \left( \frac{(N^2)_l}{q^{l+j}} \right) \left( \frac{N^2}{q} \right)^k$ | $\frac{(q-1)(r-1)}{qN}$ |
| $\frac{(q-1)(r-1)}{qN}$ | $\frac{d(q-1)(r-1)}{N^2}$ |
Proof. Since $\alpha$ is a primitive element of $\mathbb{F}_r$, we have $\mathbb{F}_q^* = \langle \alpha^{\frac{r-1}{q-1}} \rangle$. Denote $H = \mathbb{F}_q^*$ and $K = C_0^{(N,r)} = \langle \alpha^N \rangle$. Then

$$HK = C_0^{(d,r)} = \langle \alpha^d \rangle, H \cap K = \langle \alpha^{\frac{r-1}{q-1}} \frac{N}{q} \rangle,$$

where $d = \gcd(N, \frac{r-1}{q-1})$. By Lemma 3.1 we have $[HK : K] = \frac{N^d}{d} | (q-1)$ and

$$\mathbb{F}_q^* = \bigcup_{i=0}^{\frac{N}{d}-1} C_i^{\left(\frac{N}{q} \cdot d\right)};$$

where

$$C_i^{\left(\frac{N}{q} \cdot d\right)} = \alpha^{\frac{r-1}{q-1}} \langle \alpha^{\frac{r-1}{q-1}} \frac{N}{q} \rangle = \alpha^{\frac{r-1}{q-1}} (H \cap K) \text{ for } 0 \leq i \leq \frac{N}{d} - 1.$$

Then by Lemma 3.1 again we have

$$HK = \mathbb{F}_q^* \cdot C_0^{(N,r)} = \bigcup_{i=0}^{\frac{N}{d}-1} \alpha^{\frac{r-1}{q-1} i} C_0^{(N,r)} = \bigcup_{i=0}^{\frac{N}{d}-1} C_i^{\left(\frac{N}{q} \cdot d\right)}.$$
Using the orthogonal property of additive characters again, we have
\[
\sum_{y \in \mathbb{F}_q^*} \psi(yb) = \sum_{y \in \mathbb{F}_q^*} \phi(\text{Tr}_{r/q}(yb))
\]
\[
= \sum_{y \in \mathbb{F}_q^*} \phi(y\text{Tr}_{r/q}(b))
\]
\[
= \begin{cases} 
q - 1, & \text{if } \text{Tr}_{r/q}(b) = 0, \\
-1, & \text{if } \text{Tr}_{r/q}(b) \neq 0.
\end{cases}
\]

It is known that \( \text{Tr}_{r/q} \) maps \( \mathbb{F}_r \) onto \( \mathbb{F}_q \) and
\[|\{b \in \mathbb{F}_r : \text{Tr}_{r/q}(b) = c\}| = \frac{r}{q}\]
for each \( c \in \mathbb{F}_q \). Note that \( c(a, b_1) = c(a, b_2) \) if \( \text{Tr}_{r/q}(b_1) = \text{Tr}_{r/q}(b_2) \). Then we can determine the values of \( Z(r, a, b) \) and their frequencies as follows.

(1) If \( a = 0, \text{Tr}_{r/q}(b) = 0 \). We have
\[
Z(r, a, b) = \frac{1}{q} \cdot \frac{r-1}{N} + \frac{1}{q} \cdot (q-1) \cdot \frac{r-1}{N} = \frac{r-1}{N}.
\]
This value occurs once.

(2) If \( a = 0, \text{Tr}_{r/q}(b) \neq 0 \). We have
\[
Z(r, a, b) = \frac{1}{q} \cdot \frac{r-1}{N} + \frac{1}{q} \cdot (-1) \cdot \frac{r-1}{N} = 0.
\]
This value occurs \( q-1 \) times.

(3) If \( a \in C_{l}^{(d,r)}, \text{Tr}_{r/q}(b) = 0 \). Note that \( HK = C_{0}^{(d,r)} = \bigcup_{i=0}^{\frac{d-1}{q}} C_{i}^{(N,r)} \). We have
\[
Z(r, a, b) = \frac{1}{q} \cdot \frac{r-1}{N} + \frac{1}{q} \sum_{i=0}^{\frac{d-1}{q}} \frac{(q-1)d}{N} \sum_{x \in C_{i}^{(N,r)}} \psi(ax)
\]
\[
= \frac{r-1}{qN} + \frac{(q-1)d}{qN} \sum_{x \in C_{i}^{(d,r)}} \psi(ax)
\]
\[
= \frac{r-1}{qN} + \frac{(q-1)d}{qN} \eta_{i}^{(d,r)}.
\]
This value occurs \( \frac{r-1}{N} \cdot \frac{q-1}{q} \) times.

(4) If \( a \in C_{j}^{(N,r)}, \text{Tr}_{r/q}(b) \in C_{k}^{(\frac{N}{q}-q)} \). We have
\[
Z(r, a, b) = \frac{1}{q} \cdot \frac{r-1}{N} + \frac{1}{q} \sum_{i=0}^{\frac{N}{q}-1} \eta_{i}^{(N,r)} \eta_{i+j}^{(\frac{N}{q}-q)}
\]
\[
= \frac{r-1}{N} \cdot \frac{q-1}{q} = \frac{d(q-1)(r-1)}{N^2} \] times.

Note that \( W_{H}(c(a, b)) = n - Z(r, a, b) \). Then Table 1 can be obtained and Table 2 follows from Table 1. This completes the proof. □
We have determined the weight distribution of the cyclic code $C_1$ when Gauss periods of order $N$ are known. Then we give the following examples.

**Example 3.3.** The case $N = 2$.

1. Let $q = 3$ and $r = 27$. Then $d = \gcd(N, \frac{r-1}{q-1}) = 1$. By Lemma 2.1 we have
   \[ \eta_0^{(2,27)} = \frac{-1 - 3\sqrt{-3}}{2}, \quad \eta_1^{(2,27)} = \frac{-1 + 3\sqrt{-3}}{2} \]
   and
   \[ \eta_0^{(2,3)} = \frac{-1 + \sqrt{-3}}{2}, \quad \eta_1^{(2,3)} = \frac{-1 - \sqrt{-3}}{2}. \]

   Then by Table 1 we know that the code $C_1$ is a $[13, 4, 7]$ cyclic code over $F_3$ with the weight enumerator
   \[ 1 + 26x^7 + 26x^9 + 26x^{10} + 2x^{13}. \]

2. Let $q = 5$ and $r = 25$. Then $N \mid \frac{r-1}{q-1}$. By Lemma 2.1 we have
   \[ \eta_0^{(2,25)} = -3, \quad \eta_1^{(2,25)} = 2. \]

   Then by Table 2 we know that the code $C_1$ is a $[12, 3, 8]$ cyclic code over $F_5$ with the weight enumerator
   \[ 1 + 12x^8 + 48x^9 + 48x^{10} + 16x^{12}. \]

**Example 3.4.** For the case $N = 3$, we have $N \mid \frac{r-1}{q-1}$ under the conditions of Lemma 2.2. Let $q = 7$ and $r = 7^3$. By Lemma 2.2 we have
   \[ \eta_0^{(3,7^3)} = 2, \quad \eta_1^{(3,7^3)} = -12, \quad \eta_2^{(3,7^3)} = 9. \]

   Then by Table 2 we know that the code $C_1$ is a $[114, 4, 90]$ cyclic code over $F_7$ with the weight enumerator
   \[ 1 + 114x^{90} + 798x^{96} + 684x^{98} + 684x^{99} + 114x^{108} + 6x^{114}. \]

**Example 3.5.** For the case $N = 4$, we have $N \mid \frac{r-1}{q-1}$ under the conditions of Lemma 2.3. Let $q = 5$ and $r = 5^4$. By Lemma 2.3 we have
   \[ \eta_0^{(4,5^4)} = 1, \quad \eta_2^{(4,5^4)} = -14, \quad \text{and} \quad \{\eta_1^{(4,5^4)}, \eta_3^{(4,5^4)}\} = \{1, 11\}. \]

   Then by Table 2 we know that the code $C_1$ is a $[156, 5, 116]$ cyclic code over $F_5$ with the weight enumerator
   \[ 1 + 156x^{116} + 624x^{112} + 312x^{124} + 1248x^{125} + 624x^{127} + 156x^{136} + 4x^{156}. \]

**Example 3.6.** For the semi-primitive case. Let $q = 5$ and $N = 3$. Then there exists the least positive integer $e = 1$ such that $5 \equiv -1 \pmod{3}$. Let $r = 5^2$. By Lemma 2.4 we have
   \[ \eta_0^{(3,5^2)} = 3, \quad \eta_1^{(3,5^2)} = \eta_2^{(3,5^2)} = -2. \]
Then by Table 2 we know that the code $C_1$ is an $[8, 3, 4]$ cyclic code over $\mathbb{F}_5$ with the weight enumerator
\[1 + 8x^4 + 64x^6 + 32x^7 + 20x^8.\]

**Example 3.7.** For the index 2 case. Let $q = 2$, $N = 7$, and $r = 2^6$. By Lemma 2.5 we have
\[\eta_0(7, 2^6) = 5, \eta_1(7, 2^6) = \eta_2(7, 2^6) = \eta_4(7, 2^6) = -3,\]
and
\[\eta_3(7, 2^6) = \eta_5(7, 2^6) = \eta_6(7, 2^6) = 1.\]
Then by Table 2 we know that the code $C_1$ is a $[9, 7, 2]$ cyclic code over $\mathbb{F}_2$ with the weight enumerator
\[1 + 9x^2 + 27x^3 + 27x^4 + 27x^5 + 27x^6 + 9x^7 + x^9.\]

We know that the explicit values of the Gauss periods are very hard to determine. Then we have the following tight bound on the minimum weight of $C_1$ which is denoted by $W_H(C_1)$.

**Theorem 3.8.** If $N | \frac{r - 1}{q - 1}$, $q \geq 3$, and $N < \sqrt{r}$, then we have
\[W_H(C_1) \geq \frac{(q - 1)(r - (N - 1)(q - 1))}{q^N}.\]

**Proof.** It is immediate from Lemma 2.6 and Theorem 3.2. \qed

### 4. The weight distribution of $C_2$

Let $\alpha$ be a primitive element of $\mathbb{F}_r$ and $r - 1 = nN$ for two positive integers $n > 1$ and $N > 1$. For $g = \alpha^N$ we define a cyclic code over $\mathbb{F}_q$ by
\[C_2 = \{c(a, b) : a, b \in \mathbb{F}_r\},\]
where
\[c(a, b) = (\text{Tr}_{r/q}(ag^0 + b(g^2)^0), \text{Tr}_{r/q}(ag^1 + b(g^2)^1), \ldots, \text{Tr}_{r/q}(ag^{n-1} + b(g^2)^{n-1})).\]

It is known that the parity-check polynomial of $C_2$ is $h_{g^{-1}}(x)h_{g^{-2}}(x)$, where $h_{g^{-1}}(x)$ and $h_{g^{-2}}(x)$ are the minimal polynomial of $g^{-1}$ and $g^{-2}$ over $\mathbb{F}_q$, respectively.

For any $a, b \in \mathbb{F}_r$, the Hamming weight of $c(a, b)$ is equal to
\[W_H(c(a, b)) = n - Z(r, a, b),\]
where
\[Z(r, a, b) = |\{x \in C_0^{(N,r)} : \text{Tr}_{r/q}(ax + bx^2) = 0\}|.\]
Let $\phi$ be the canonical additive character of $\mathbb{F}_q$. Then $\psi = \phi \circ \text{Tr}_{r/q}$ is the canonical additive character of $\mathbb{F}_r$. By the orthogonal property of additive characters we have

$$Z(r, a, b) = \frac{1}{q} \sum_{x \in C_0^{(N,r)}} \sum_{y \in \mathbb{F}_q} \phi(y \text{Tr}_{r/q}(ax + bx^2))$$

$$= \frac{1}{q} \sum_{x \in C_0^{(N,r)}} \sum_{y \in \mathbb{F}_q} \phi(\text{Tr}_{r/q}(yax + ybx^2))$$

$$= \frac{1}{q} \cdot \frac{r-1}{N}$$

$$+ \frac{1}{q} \sum_{x \in C_0^{(N,r)}} \sum_{y \in \mathbb{F}_q} \psi(yax + ybx^2).$$

**Theorem 4.1.** Let $r = q^2$ and $2N \mid (q + 1)$. Then the weight distribution of the cyclic code $C_2$ is given by Table 3 if $\frac{q+1}{2N}$ is even and is given by Table 4 if $\frac{q+1}{2N}$ is odd.

**Table 3. Weight distribution of $C_2$ when $\frac{q+1}{2N}$ is even**

| Weight | Frequency |
|--------|-----------|
| $q-1$  | $\frac{r-1}{N}$ (0 $\leq i \leq N-1$) |
| $q$    | $\frac{r-1}{N}$ (0 $\leq j \leq 2N-1$) |
| $q^2$  | $\frac{(N-1)(r-1)^2}{2N^2}$ (0 $\leq j \leq 2N-1$) |
| $q^3$  | $\frac{r-1}{N}(r-1)$ |
| $q^4$  | $\frac{r-1}{N}(r-1)^2 - q + 1$ |

| Frequency |
|-----------|
| $\frac{r-1}{N}(r-1)^2 - q + 1$ |

**Table 4. Weight distribution of $C_2$ when $\frac{q+1}{2N}$ is odd**

| Weight | Frequency |
|--------|-----------|
| $q-1$  | $\frac{r-1}{N}$ (0 $\leq i \leq N-1$) |
| $q$    | $\frac{r-1}{N}$ (0 $\leq j \leq 2N-1$) |
| $q^2$  | $\frac{(N-1)(r-1)^2}{2N^2}$ (0 $\leq j \leq 2N-1$) |
| $q^3$  | $\frac{r-1}{N}(r-1)$ |
| $q^4$  | $\frac{r-1}{N}(r-1)^2 - q + 1$ |

$\frac{(r-1)^2}{2N^2}$ (0 $\leq j \leq 2N-1$, $j \neq N$)

**Proof.** Let $\alpha$ be a primitive element of $\mathbb{F}_r$ and $\beta = \alpha^{\frac{q+1}{r-1}} = \alpha^{q+1}$. Then $\mathbb{F}_q^* = \langle \beta \rangle$ and $\mathbb{F}_q^* = C_0^{(N,r)} \cup \beta C_0^{(N,r)}$, where $C_0^{(N,r)} = \langle \beta^2 \rangle$. Since $q \equiv -1 \pmod{2N}$, we have $\mathbb{F}_q^* \subset C_0^{(2N,r)} \subset C_0^{(N,r)}$ and $yC_0^{(N,r)} = C_0^{(N,r)}$ for each $y \in \mathbb{F}_q^*$.

If $a = 0$ and $b = 0$, then we have

$$Z(r, a, b) = \frac{r-1}{N}.$$

This value occurs once.
If \( a \in C_i^{(N,r)} \) for some \( i(0 \leq i \leq N - 1) \) and \( b = 0 \), then we have

\[
Z(r, a, b) = \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yax)
= \frac{r - 1}{qN} + \frac{q - 1}{q} \sum_{x \in C_0^{(N,r)}} \psi(ax) = \frac{r - 1}{qN} + \frac{q - 1}{q} \eta_i^{(N,r)}.
\]

This value occurs \( \frac{r - 1}{N} \) times.

If \( a = 0 \) and \( b \neq 0 \), then we can let \( b \in C_j^{(2N,r)} \) for some \( j, 0 \leq j \leq 2N - 1 \), by \( 2N \mid (r - 1) \). We have

\[
Z(r, a, b) = \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(ybx^2) = \frac{r - 1}{qN} + \frac{q - 1}{q} \sum_{x \in C_0^{(N,r)}} \psi(bx^2)
= \frac{r - 1}{qN} + \frac{2(q - 1)}{q} \sum_{x \in C_0^{(2N,r)}} \psi(bx) = \frac{r - 1}{qN} + \frac{2(q - 1)}{q} \eta_j^{(2N,r)}.
\]

This value occurs \( \frac{r - 1}{2N} \) times.

Now we suppose that \( a \neq 0 \) and \( b \neq 0 \). Then we have

\[
Z(r, a, b) = \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yax + ybx^2)
= \frac{r - 1}{qN} + \frac{1}{q} \left( \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yax + ybx^2) \right) + \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yax + ybx^2)
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(y^2ax + y^2bx^2) + \frac{1}{2} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(\beta yax + \beta ybx^2)
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yaxy + b(xy)^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(\beta yaxy + \beta b(xy)^2)
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yaz + b^2z^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(\beta yaz + \beta b^2z^2)
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{x \in C_0^{(N,r)}} \psi(yaz + b^2z^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(\beta yaz) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(\beta b^2z^2)
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(baz + \beta baz^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(y Tr_{r/q}(az))
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(baz) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(\beta baz^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(y Tr_{r/q}(az))
= \frac{r - 1}{qN} + \frac{1}{q} \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(baz) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(\beta baz^2) + \sum_{y \in \mathbb{F}_q^*} \sum_{z \in C_0^{(N,r)}} \psi(y Tr_{r/q}(az))
\]

Note that \( \beta = \alpha^{q+1} \in \mathbb{F}_q^* \). Suppose that \( 0 \neq a \in C_0^{(N,r)} \) and \( Tr_{r/q}(az) = 0 \). Then we have \( az + (az)^q = 0 \) and

\[
z = a^{-1} \alpha^{2q+1} v \text{ for all } v \in \mathbb{F}_q^*.
\]
This means that there exist exactly \( q - 1 \) solutions \( z \in C_0^{(N,r)} \) such that \( \text{Tr}_{r/q}(az) = 0 \) if \( a \in C_0^{(N,r)} \) and there exists no solution \( z \in C_0^{(N,r)} \) such that \( \text{Tr}_{r/q}(az) = 0 \) if \( a \not\in C_0^{(N,r)} \).

If \( a \in C_i^{(N,r)} \) for all \( i = 1, 2, \ldots, N-1 \) and \( b \in C_j^{(2N,r)} \) for some \( j \) \((0 \leq j \leq 2N-1)\), then we have \( \text{Tr}_{r/q}(az) \neq 0 \) and

\[
Z(r, a, b) = \frac{r - 1}{qN} + \frac{1}{2q} \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \cdot (-1)
\]

\[
= \frac{r - 1}{qN} - \frac{2}{q} \sum_{z \in C_0^{(2N,r)}} \psi(bz) = \frac{r - 1}{qN} - \frac{2}{q} \eta_j^{(2N,r)}.
\]

This value occurs \( \frac{r-1}{N} \cdot (N - 1) \cdot \frac{r-1}{2N} = \frac{(N-1)(r-1)^2}{2N^2} \) times.

In the following, we consider the case \( a \in C_0^{(N,r)} \). We have

\[
\Delta = \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \sum_{y \in F_q^*} \psi(y \text{Tr}_{r/q}(az))
\]

\[
= \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \cdot (q - 1)
\]

\[
+ \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \cdot (-1)
\]

\[
= \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \cdot q
\]

\[
+ \left( \sum_{z \in C_0^{(N,r)}} \psi(bz^2) + \sum_{z \in C_0^{(N,r)}} \psi(\beta bz^2) \right) \cdot (q - 1)
\]

\[
= q \sum_{v \in F_q^* \backslash \{0\}} \psi(b(a^{-1} \alpha^{q+1} v)^2) + q \sum_{v \in F_q^* \backslash \{0\}} \psi(\beta b(a^{-1} \alpha^{q+1} v)^2) - 2 \sum_{z \in C_0^{(N,r)}} \psi(bz^2)
\]

\[
= q \sum_{v \in F_q^* \backslash \{0\}} \psi(ba^{-2} \alpha v^2) + q \sum_{v \in F_q^* \backslash \{0\}} \psi(\beta ba^{-2} \alpha v^2) - 4 \sum_{z \in C_0^{(2N,r)}} \psi(bz)
\]

\[
= 2q \sum_{v \in C_0^{(2N,r)}} \psi(ba^{-2} v) + 2q \sum_{v \in C_0^{(2N,r)}} \psi(ba^{-2} \beta v) - 4 \sum_{z \in C_0^{(2N,r)}} \psi(bz)
\]

\[
= 2q \sum_{v \in F_q^* \backslash \{0\}} \psi(ba^{-2} v) - 4 \sum_{z \in C_0^{(2N,r)}} \psi(bz)
\]

\[
= 2q \sum_{v \in F_q^* \backslash \{0\}} \psi(\text{Tr}_{r/q}(ba^{-2})) - 4 \sum_{z \in C_0^{(2N,r)}} \psi(bz).
\]
Suppose that $\text{Tr}_{r/q}(ba^{-2}) = 0$. Then we have $ba^{-2} + b^q a^{-2q} = 0$ and
\[ b = a^2 \alpha^{\frac{q+1}{2}} \] for all $v \in \mathbb{F}_q^*.$
This means that there exist exactly $q-1$ solutions $b \in C(2N,r)$ such that
\[ \text{Tr}_{r/q}(ba^{-2}) = 0 \] for each $a \in C(0,N,r)$, where $C(2N,r) = C(2N,r) \equiv \frac{q+1}{2} \pmod{2N}$.

1. If $\frac{q+1}{2N}$ is even, then $\frac{q+1}{2} \equiv 0 \pmod{2N}$.
   If $a \in C(0,N,r)$ and $b \in C(0,2N,r)$ satisfy $\text{Tr}_{r/q}(ba^{-2}) = 0$, then we have
   \[ Z(r,a,b) = \frac{r-1}{qN} + q - 1 - \frac{2}{q} \eta_0(N,r). \]
   This value occurs $\frac{(q-1)(r-1)}{N}$ times.
   If $a \in C(0,N,r)$, $b \in C(0,2N,r)$, and $\text{Tr}_{r/q}(ba^{-2}) \neq 0$, then we have
   \[ Z(r,a,b) = \frac{r-1}{qN} - 1 - \frac{2}{q} \eta_0(N,r). \]
   This value occurs $\frac{r-1}{N} \cdot \frac{(r-1)}{2N}$ times.

2. If $\frac{q+1}{2N}$ is odd, then $\frac{q+1}{2} \equiv N \pmod{2N}$.
   If $a \in C(0,N,r)$ and $b \in C(2N,r)$ satisfy $\text{Tr}_{r/q}(ba^{-2}) = 0$, then we have
   \[ Z(r,a,b) = \frac{r-1}{qN} + q - 1 - \frac{2}{q} \eta_0(N,r). \]
   This value occurs $\frac{(q-1)(r-1)}{N}$ times.
   If $a \in C(0,N,r)$, $b \in C(2N,r)$, and $\text{Tr}_{r/q}(ba^{-2}) \neq 0$, then we have
   \[ Z(r,a,b) = \frac{r-1}{qN} - 1 - \frac{2}{q} \eta_0(N,r). \]
   This value occurs $\frac{r-1}{N} \cdot \frac{(r-1)}{2N}$ times.

Note that $W_H(c(a,b)) = n - Z(r,a,b)$. Then we can obtain Table 3 and Table 4. This completes the proof.

We have determined the weight distribution of the cyclic code $C_2$ when Gauss periods of order $2N$ are known. Then we have the following theorem.
Theorem 4.2. Assume that there exists the least positive integer $e$ such that $p^e \equiv -1 \pmod{2N}$, we know that $p$ is odd. Let $q = p^f$ for some positive integer $f$ and $r = q^2 = p^{2ef}$.

(1) If $f$ and $\frac{p^e+1}{2N}$ are both odd, then the weight distribution of $C_2$ can be given by Table 5.

(2) In all other cases, the weight distribution of $C_2$ can be given by Table 6.

Table 5. The case (1) of Theorem 4.2.

| Weight                  | Frequency                  |
|-------------------------|----------------------------|
| $0$                     | $1$                        |
| $\frac{r-1}{N} - q + 1$ | $\frac{r-1}{N}$            |
| $\frac{r-1}{N}$        | $\frac{r-1}{N}$            |
| $\frac{r-1}{N} - 2q + 2$ | $\frac{r-1}{N}$            |
| $\frac{(q+1)(q-2)}{N} + 2$ | $\frac{r-1}{N}$            |
| $\frac{(q+1)(q-2)}{N}$ | $\frac{r-1}{N}$            |
| $\frac{(q+1)(q-2)}{N} - q + 3$ | $\frac{r-1}{N}$            |
| $\frac{(q+1)(q-2)}{N} + 3$ | $\frac{r-1}{N}$            |
| $\frac{(q+1)(q-2)}{N}$ | $\frac{r-1}{N}$            |

Table 6. The case (2) of Theorem 4.2.

| Weight                  | Frequency                  |
|-------------------------|----------------------------|
| $0$                     | $1$                        |
| $\frac{(q-1)(q+(-1)^f(N-1))}{N}$ | $\frac{r-1}{N}$            |
| $\frac{(q-1)(q-(-1)^f)}{N}$ | $\frac{r-1}{N}$            |
| $\frac{(q-1)(q+(-1)^f)(2N-1)}{N}$ | $\frac{r-1}{N}$            |
| $\frac{q^2-q-1+(-1)^f}{N} - 2 \cdot (-1)^f$ | $\frac{r-1}{N}$            |
| $\frac{q^2-q-1+(-1)^f}{N} - 2 \cdot (-1)^f - q + 1$ | $\frac{r-1}{N}$            |
| $\frac{q^2-q-1+(-1)^f}{N} - 2 \cdot (-1)^f + 1$ | $\frac{r-1}{N}$            |
| $\frac{q^2-q-1+(-1)^f}{N} + 1$ | $\frac{r-1}{N}$            |

Proof. (1) If $f, p$, and $\frac{p^e+1}{2N}$ are all odd, then by Lemma 2.4 we have

$$\eta_{N}^{(2N,r)} = \frac{(2N-1)q - 1}{2N}, \eta_{j}^{(2N,r)} = -\frac{q + 1}{2N}$$

for $0 \leq j \leq 2N - 1, j \neq N$.

Thus

$$\eta_{0}^{(N,r)} = \eta_{0}^{(2N,r)} + \eta_{N}^{(2N,r)} = \frac{(N-1)q - 1}{N},$$

$$\eta_{i}^{(N,r)} = \eta_{i}^{(2N,r)} + \eta_{i+N}^{(2N,r)} = -\frac{q + 1}{N}$$

for $1 \leq i \leq N - 1$.

Note that

$$\frac{q + 1}{2N} = p^f + 1 = \frac{p^f + 1}{2N} \cdot (p^e(1) + (-1)^{f-2}p^e(2) + \cdots + (-1)p^e + 1)$$
is odd since \( f, p, \) and \( \frac{p^e + 1}{2N} \) are all odd. Then Table 5 can be obtained by Table 4 of Theorem 4.1.

(2) In all other cases, by Lemma 2.4 we have

\[
\eta_0^{(2N,r)} = \frac{(-1)^{f+1}(2N - 1)q - 1}{2N}, \quad \eta_j^{(2N,r)} = \frac{(-1)^f q - 1}{2N} \text{ for } 1 \leq j \leq 2N - 1.
\]

Thus

\[
\eta_0^{(N,r)} = \eta_0^{(2N,r)} + \eta_N^{(2N,r)} = \frac{(-1)^f(N - 1)q - 1}{N},
\]

\[
\eta_i^{(N,r)} = \eta_i^{(2N,r)} + \eta_{i+N}^{(2N,r)} = \frac{(-1)^f q - 1}{N} \text{ for } 1 \leq i \leq N - 1.
\]

Note that there is at least one of \( f \) and \( \frac{p^e + 1}{2N} \) is even and \( p \) is odd. Then

\[
\eta_0^{(N,r)} = \eta_0^{(2N,r)} + \eta_N^{(2N,r)} = \frac{(-1)^f(N - 1)q - 1}{N}
\]

\[
\eta_i^{(N,r)} = \eta_i^{(2N,r)} + \eta_{i+N}^{(2N,r)} = \frac{(-1)^f q - 1}{N} \text{ for } 1 \leq i \leq N - 1.
\]

We also have a tight bound on the minimum weight of \( C_2 \) which is denoted \( W_H(C_2) \).

**Example 4.3.** Let \( q = p = 3, r = 9, \) and \( N = 2 \). By Table 5 we known that \( C_2 \) is a \([4, 3, 2]\) cyclic code over \( \mathbb{F}_3 \) and thus is optimal with respect to Singleton bound. The weight enumerator of such cyclic code is

\[
1 + 12x^2 + 8x^3 + 6x^4.
\]

**Example 4.4.** Let \( q = p = 7, r = 49, \) and \( N = 2 \). By Table 6 we known that \( C_2 \) is a \([24, 4, 12]\) cyclic code over \( \mathbb{F}_7 \) and the weight enumerator is

\[
1 + 12x^{12} + 144x^{16} + 24x^{18} + 864x^{20} + 864x^{21} + 288x^{22} + 144x^{23} + 60x^{24}.
\]

**Example 4.5.** Let \( q = p = 5, r = 25, \) and \( N = 3 \). By Table 5 we known that \( C_2 \) is an \([8, 3, 4]\) cyclic code over \( \mathbb{F}_5 \) and the weight enumerator is

\[
1 + 8x^4 + 64x^6 + 32x^7 + 20x^8.
\]

We also have a tight bound on the minimum weight of \( C_2 \) which is denote \( W_H(C_2) \).

**Theorem 4.6.** Let \( r = q^2, 2N | (q + 1), \) and \( 5 \leq N < \frac{\sqrt{r}}{2} \). Then we have

\[
W_H(C_2) \geq \frac{(q - 1)(r - (2N - 1)\sqrt{r})}{qN}.
\]

**Proof.** By Lemma 2.6 and Theorem 4.1 we only need to compare some values of the weights. It is not difficult and then we omit the proof here.
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