Gromov-Witten theory of target curves and the
taxtological ring

Felix Janda *

May 11, 2014

Abstract

In the Gromov-Witten theory of a target curve we consider descendent
integrals against the virtual fundamental class relative to the forgetful
morphism to the moduli space of curves. We show that cohomology classes
obtained in this way lie in the tautological ring.

0 Introduction

Let $X$ be an algebraic curve of genus $h$ over $\mathbb{C}$ and $1, \alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h, \omega$ be a basis of $H^*(X, \mathbb{C})$ such that $1$ is the identity of the cup product, $\omega$ is
the Poincaré dual of a point and the $\alpha_i \in H^{1,0}(X, \mathbb{C})$ and $\beta_i \in H^{0,1}(X, \mathbb{C})$
form a symplectic basis of $H^1(X, \mathbb{C})$, i.e. $\alpha_i \cup \beta_i = \omega$, $\beta_i \cup \alpha_i = -\omega$ for
all $i$, and all other cup products vanish.

There exists a fine moduli stack $M_{g,n}(X,d)$ parametrising degree $d$
stable maps $C \to X$ of $n$ pointed nodal, not necessarily connected
curves $C$ to the target curve $X$. It comes together with a projection map
$\pi : \overline{M}_{g,n}(X,d) \to M_{g,n}$ to the moduli of curves which forgets the stable
map and contracts unstable components. Furthermore each marking $i$ gives an evaluation map $ev_i : \overline{M}_{g,n}(X,d) \to X$. Similarly to the moduli
space of stable curves, there exists a universal curve over $\overline{M}_{g,n}(X,d)$,
which can be used to define a cotangent line class $\psi_i \in H^2(\overline{M}_{g,n}(X,d))$
for each marking $i$.

The space $\overline{M}_{g,n}(X,d)$ admits a virtual fundamental class

$$[\overline{M}_{g,n}(X,d)]^{\text{virt}} \in H_{2(2g-2+n)}(\overline{M}_{g,n}(X,d)).$$

Descendent invariants in the Gromov-Witten theory of $X$ are integrals of
monomials of $\psi$ classes and classes pulled back from $X$ along the eval-
uation maps against the virtual fundamental class. More explicitly the

*Supported by the Swiss National Science Foundation grant SNF 200021_143274

1Alternatively, one can define these cotangent line classes as the pull-backs of the corre-
sponding cotangent lines classes of $\overline{M}_{g,n}$ via $\pi$. 


disconnected $n$-pointed degree $d$ genus $g$ Gromov-Witten invariants of $X$
with descendent insertions $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{C})$ are
\[ \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,d}^X := \int_{[\overline{M}_{g,n}(X,d)]^{vir}} \prod_{i=1}^n \psi_{k_i}^! \ev_i^*(\gamma_i) \in \mathbb{Q} \]
for various $n$-tuples $k = (k_1, \ldots, k_n)$. The usual connected Gromov-Witten
invariants are related to the disconnected ones by combinatorial
formulae. In a series of articles ([2], [3], [4]) an effective way to calculate
these integrals was given.

Here we will more generally study the classes obtained by capping
with the virtual fundamental class and pushing down via $\pi$, instead of
integrating against it. We will use a nonstandard notation similar to the
bracket notation of Gromov-Witten invariants
\[ [\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)]_{r,d}^X := \pi_* \left( \prod_{i=1}^n \psi_{k_i}^! \ev_i^*(\gamma_i) \cap [\overline{M}_{g,n}(X,d)]^{vir} \right) \in H_{2r}(\overline{M}_{g,n}, \mathbb{Q}), \]
where $r = 2g - 2 + d(2 - 2h) + \sum_{i=1}^n (k_i + \text{codim}(\gamma_i))$ is the complex
dimension of the class we push-forward. Since the value of $r$ implicitly determines $g$ we have left out $g$ in the notation. If the value of $g$
obtained from $r$ would be a half-integer, we define the corresponding
class to be zero. Note that in the case of $r = 0$ using the canonical iso-
morphism $\mathbb{H}_0(\overline{M}_{g,n}, \mathbb{Q}) \cong \mathbb{Q}$ we re-obtain the usual descendent Gromov-Witten
invariants. We will call these enriched classes Gromov-Witten
push-forwards (GWpfs) in the sequel.

The tautological rings $RH^*(\overline{M}_{g,n})$ of $\overline{M}_{g,n}$ are defined (see [5]) as the
smallest system of subrings of $H^*(\overline{M}_{g,n})$ stable under push-forward and
pull-back by the maps
- $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ forgetting one of the markings,
- $\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ gluing two curves at a point,
- $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ gluing together two markings of a curve.

While this definition seems restrictive many geometric classes lie in the
tautological ring. In fact the aim of this article is to prove the following
theorem.

**Theorem 1.** The GWpfs
\[ [\tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)]_{r,d}^X \]
lie in the tautological ring $RH_{2r}(\overline{M}_{g,n}, \mathbb{Q})$ for any choices of insertions $\tau_{k_i}(\gamma_i)$.

This theorem was already known to be true in the case of $X = \mathbb{P}^1$
relative to a number of points [5] and a large part of the proof of Theorem [1]
is a reduction to this case.

So let us recall the definition of relative Gromov-Witten invariants
of a curve $X$ relative to a collection of points $q_1, \ldots, q_m$. For these one
considers stable maps relative to $q_1, \ldots, q_m$ i.e. stable maps in the usual
sense such that the preimages of the marked points are finite sets disjoint from the markings and nodes of the domain. Then it is natural to consider the moduli space $\overline{M}_{g,n}(X, \eta_1, \ldots, \eta_m)$ of relative stable maps to $X$ with prescribed splitting $\eta_i$ at $q_i$; the $\eta_i$ are partitions of $d$. As in the absolute case there is a projection map $\pi : \overline{M}_{g,n}(X, \eta_1, \ldots, \eta_m) \to \overline{M}_{g,n+\ell(\eta)}$, where $\ell(\eta)$ is the sum of lengths of the partitions $\eta_1, \ldots, \eta_m$. One can define GWpfs of these as in the absolute case.

$$[\tau_{\eta_1}(\gamma_1) \cdots \tau_{\eta_m}(\gamma_n)]_{\eta_1, \ldots, \eta_m}^X := \pi_* \left( \prod_{i=1}^{n} \psi_i^{k_i} \ev_i^*(\gamma_i) \cap [\overline{M}_{g,n}(X, \eta_1, \ldots, \eta_m)]^\text{virt} \right) \in H_{2r}(\overline{M}_{g,n+\ell(\eta)}, \mathbb{Q}),$$

where $r$ implicitly determines $g$ as before. We also have left out the index $d$ since it is implicit in size of any partition $\eta_i$.

If there are only even insertions, we can use a degeneration formula (see [6]) to calculate the GWpf in terms of GWpfs of $\mathbb{P}^1$ relative to a point and by the results of [5] these are also tautological. This will be done in Section 1.

In the presence of odd insertions new things can happen. For example we might obtain odd classes in $H^*(\overline{M}_{g,n})$. Those can only be tautological if they vanish, since by definition tautological classes are algebraic. More generally one might obtain classes of non balanced Hodge type.

**Corollary.** All non balanced GWpfs vanish.

Actually we will first prove this corollary in Section 4 and use it as a lemma for the proof of Theorem 1.

The balanced case remains. In Section 5 we want to give an algorithm to calculate the GWpfs in the presence of odd cohomology in terms of GWpfs with only even insertions. It is a straight generalization of the algorithm given in [4].

If there are odd insertions, we cannot use a degeneration formula to reduce to the case of $\mathbb{P}^1$. Still it is possible to deform $X$ into a chain of elliptic curves to reduce to the genus 0 and genus 1 cases. This is done in Section 2. Therefore starting from Section 3 we will assume $X$ to be of genus one.

As in [4] we will use the following properties of Gromov-Witten theory to relate GWpfs with odd insertions to those with only even insertions.

- algebracity of the virtual fundamental classes
- invariance under monodromy transformations of $X$
- degeneration formulae
- vanishing relations from the group structure on an elliptic curve

We will study relations coming from the monodromy invariance of Gromov-Witten theory and the group structure of an elliptic curve in sections 3.1 and 3.2 respectively.

For the proof of the Corollary we will only need the results from sections 2, 3.1 and 4. It is even possible to adapt the proof so that the use
the reduction to genus 1 is not necessary. Its proof is the main new part of this article.

I would like to thank my supervisor Rahul Pandharipande for the introduction to the problem, his support and many helpful discussions.

1 Even classes

There is a nonsingular family $X_t$ of curves of genus $h > 0$ over $\mathbb{C}$ such that $X_t \cong X$ for $t \neq 0$ and $X_0$ is an irreducible curve of geometric genus $h - 1$ with a node. The degeneration formula relates the GWpfs of $X$ to the GWpfs of the normalization $\tilde{X}_0$ of $X_0$ relative to the two preimages of the marked point. Then the even classes of $X$ can be naturally lifted to $\tilde{X}_0$. This generalizes to the situation of $X$ relative to marked points $q_1, \ldots, q_m$.

Let

$$M = \prod_{h \in H} \tau_{\alpha_h}(1) \prod_{h' \in H'} \tau_{\alpha_{h'}}(\omega)$$

be a monomial in insertions of even classes and $\eta_1, \ldots, \eta_m$ be choices of splittings at the relative points. Since the target curve is irreducible the degeneration formula [6] in this case says that

$$[M|\eta_1, \ldots, \eta_m]_X^X = \sum_{|\mu| = d} z(\mu) [M|\eta_1, \ldots, \eta_m, \mu, \mu]_{\tilde{X}_0},$$

where the automorphism factor $z(\mu)$ is defined by

$$z(\mu) = |\text{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i.$$

By using this formula repeatedly we can reduce the genus $h$ until we arrive at the case of $X = \mathbb{P}^1$ relative to $q_1, \ldots, q_m$, which has been studied in [5]. This implies that Theorem 1 is true in the case that all $\gamma_i$ are even classes.

2 Reduction to genus 1

Recall that we have chosen a symplectic basis $\alpha_i, \beta_i \in H^1(X, \mathbb{C})$. There is a deformation $Y \to \mathbb{P}^1$ of $X$ into $\tilde{X} = E \cup X'$, a curve of genus one and a curve of genus $h - 1$ connected at a node $p$. Moreover the symplectic basis of $H^1(X, \mathbb{C})$ can be lifted to $Y$ such that over $\tilde{X}$ the classes $\alpha_1, \beta_1$ give a symplectic basis of $H^1(E, \mathbb{C})$ and the other $\alpha_i$ and $\beta_i$ give a symplectic basis of $H^1(X', \mathbb{C})$. Furthermore the deformation can be chosen such that $\omega$ deforms to the Poincaré dual class of a point on the genus 1 curve. Similarly in the relative theory the deformations of the relative points $q_1, \ldots, q_m$ can be assumed to lie on the genus 1 component.

The degeneration formula is a bit more complicated to write down in this case since there is a choice for the splitting of the domain curve into
two parts, one for each component of $\tilde{X}$, and a choice of splitting $\mu$ at $p$.
For each partition $g = g_1 + g_2 + \ell(\mu) - 1$ of $g$ there is a gluing map
\[ \iota : \overline{M}_{g_1,n_1+\ell(\eta)+\ell(\mu)} \times \overline{M}_{g_2,n_2+\ell(\mu)} \to \overline{M}_{g,n_1+n_2+\ell(\eta)}. \]

By choosing a suitable loop in the moduli space $\overline{M}_{1,1}$ starting at the point corresponding to $(X,p)$ around the point corresponding to the nodal...
elliptic curve we obtain a deformation of of $X$ to itself which leaves the even cohomology invariant while it acts on $H^1(X, \mathbb{C})$ via
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \phi \begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\]
In fact the monodromy group acts as the identity outside $H^1(X, \mathbb{C})$, where it acts as $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}^2$.

Because of the deformation invariance of Gromov-Witten theory applying these transformations to all the descendent insertions leaves the GWpf invariant. This gives a relation between GWpf.

We will use only these relations to establish the vanishing of unbalanced classes in Section 4.

For the proof of Theorem 2 we will consider certain linear combinations of these relations which have a nice form if one assumes that the vanishing of GWpf of unbalanced classes has already been shown. Let $I$ and $J$ be index sets of the same order and
\[
\begin{align*}
\mathbf{n} : I &\to \Psi_{\mathbb{Q}} \\
\mathbf{m} : J &\to \Psi_{\mathbb{Q}}
\end{align*}
\]
be refined descendant assignments. Here a refined descendant assignment is a formal $\mathbb{Q}$-linear combination of usual descendant assignments. Monomials of descendents with such assignments are just expanded multilinearly. Refined descendant assignments will only serve as a formal tool here. We consider the resulting GWpf to lie in the $\mathbb{Q}$-vector space
\[H^*(\overline{M}_{g,n+\ell(\eta)}).\]

Generalizing the definition of the map $\iota_*$ suitably we can apply the degeneration formula also to GWpf involving refined descendant assignments.

For a subset $\delta \subset I$ let $S(\delta)$ be the set of all subsets of $I \cup J$ of cardinality $|I|$ containing $\delta$. For any $D \subseteq I \cup J$ we may consider the class
\[
\tau_{\mathbf{n}, \mathbf{m}}(D) := \prod_{i \in I} \tau_{\mathbf{n}_i}(\gamma^D_i) \prod_{j \in J} \tau_{\mathbf{m}_j}(\gamma^D_j),
\]
where
\[
\gamma^D_k = \begin{cases} 
\alpha & \text{if } k \in D \\
\beta & \text{else.}
\end{cases}
\]
Finally we will consider a monomial
\[
N = \prod_{h \in H} \tau_{\mathbf{n}_h}(1) \prod_{h' \in H'} \tau_{\mathbf{m}_h'}(\omega)
\]
in the monodromy invariant insertions.

**Proposition 1.** The monodromy relation $R(N, \mathbf{n}, \mathbf{m}, \delta) = 0$ holds for any proper subset $\delta \subset I$. Here
\[
R(N, \mathbf{n}, \mathbf{m}, \delta) = \sum_{D \in S(\delta)} [N \tau_{\mathbf{n}, \mathbf{m}}(D)]^X_{\mathbf{n}, \mathbf{m}}.
\]
Proof. Consider the application of the monodromy transform $\phi$ to
\[ \left[ N \prod_{i \in I} \tau_{n_i}(\gamma_i^\delta) \prod_{j \in J} \tau_{m_j}(\beta_j) \right]_d^X. \]
This class vanishes since it is unbalanced because $\delta \subset I$ is a proper subset.
After applying $\phi$, all terms but those with exactly $|I|$ insertions of $\alpha$ vanish.
The sum of these remaining terms is exactly $R(N,n,m,\delta)$. \qed

3.2 from the elliptic action

Using the group structure of $X$ induced by identifying $X$ with its Jacobian via a point $0 \in X$ gives another set of relations.

Let the small diagonal of $X^r$ be the subset
\[ \{(x,\ldots,x) : x \in X\} \subset X^r \]
and $\Delta_r \in H^r(X^r,\mathbb{C})$ be its Poincaré dual. We will use the fact that $\Delta_r$ is invariant under the diagonal action of the elliptic curve $X$ on $X^r$, and the Künneth decomposition of $\Delta_r$ to obtain the relations.

Let $K$ and $H$ be two ordered index sets and $P$ a set partition of $K$ into subsets of size at least 2. For any part $p$ of $P$ we have a product evaluation map
\[ \phi_p : \overline{M}_{g,K\cup H}(X,d) \to X|p|. \]

Let $l : K \to \Psi$ be an assignment of descendents. Finally let $M$ be a monomial in insertions of the identity
\[ M = \prod_{h \in H} \tau_{n_h}(1). \]

**Proposition 2.** The elliptic vanishing relation $V(M,P,l) = 0$ holds. Here
\[ V(M,P,l) := \pi_* \left( \prod_{h \in H} \psi_h^{n_h} \prod_{k \in K} \psi_k^{l_k} \prod_{p \in P} \phi_p^* (\Delta_{|p|}) \cap [\overline{M}_{g,K\cup H}(X,d)]^{virt} \right). \]

Notice that no insertions of $\omega$ appear and that we do not work in the relative theory. There is a natural generalization to a more general assignment $1 : K \to \Psi_Q$.

**Proof.** The elliptic curve $X$ acts on the moduli space $\overline{M}_{g,H\cup K}(X,d)$ by the action induced from the group operation $X \times X \to X$. The action can be used to fix the image in $X$ of one marked point $p$. This gives an $X$ equivariant splitting
\[ \overline{M}_{g,H\cup K}(X,d) \cong \text{ev}_p^{-1}(0) \times X. \]

In particular there exists an algebraic quotient
\[ \overline{M}_{g,H\cup K}(X,d)/X \cong \text{ev}_p^{-1}(0) \]
of $\overline{M}_{g,H\cup K}(X,d)$. 7
Notice that the integrand is pulled back via the projection map from an analogous class on the quotient space. Furthermore by its construction the virtual fundamental class also pulled back from the quotient. The push-pull formula applied to the projection map tells us that the GWpf must vanish.

To apply these relations we need to reformulate them as relations between GWpf's of $X$. In order to rewrite the $\phi_p$ pullbacks as products of usual pullbacks via the evaluation maps we Künneth decompose the classes $\Delta_r$. For $\Delta_2$ and $\Delta_3$ we have for example

\[
\Delta_2 = 1 \otimes \omega + \omega \otimes 1 - \alpha \otimes \beta + \beta \otimes \alpha \\
\Delta_3 = 1 \otimes \omega \otimes \omega + \omega \otimes 1 \otimes \omega + \omega \otimes \omega \otimes 1 - \omega \otimes \alpha \otimes \beta + \omega \otimes \beta \otimes \alpha \\
- \alpha \otimes \omega \otimes \beta + \beta \otimes \omega \otimes \beta - \alpha \otimes \beta \otimes \omega + \beta \otimes \alpha \otimes \omega.
\]

In general $\Delta_r$ is a sum $\Delta_r = \Delta_{r, \text{even}} + \Delta_{r, \text{odd}}$ where $\Delta_{r, \text{even}}$ is the sum of the $r$ classes of the form

\[
\omega \otimes \cdots \otimes \omega \otimes 1 \otimes \omega \cdots \otimes \omega
\]

and $\Delta_{r, \text{odd}}$ is the sum of the $\binom{r}{2}$ linear combinations of classes

\[
-\omega \otimes \cdots \otimes \omega \otimes \alpha \otimes \omega \cdots \otimes \omega \otimes \beta \otimes \omega \cdots \otimes \omega \\
+\omega \otimes \cdots \otimes \omega \otimes \beta \otimes \omega \cdots \otimes \omega \otimes \alpha \otimes \omega \cdots \otimes \omega.
\]

We will mostly be interested in the odd summand since the even summand will usually already be known by an induction hypothesis.

4 Unbalanced classes

Let us fix a monomial $M$ in insertions of even classes of $X$. We want to show that the classes

\[
[M \cdot N|]\eta_r^X
\]

vanish for any unbalanced monomial $N$ in insertions of $\alpha$ and $\beta$. Let us call $N$ of type $(a, b)$ if has $a$ and $b$ insertions of $\alpha$ and $\beta$ respectively. We will use only the invariance under the monodromy transformation $\phi$ and linear algebra in order to show that all such classes for $N$ of type $(a, b)$ with $a > b$ vanish. The claim then follows by symmetry.

It will even be enough to show it only for monomials $N$ of type $(a, b)$ with $a = b + 1$ since any $N$ of type $(a, b)$ with $a > b$ is product of an $N'$ of type $(b+1, b)$ with a monomial in $\alpha$ insertions only. Since $\alpha$ is invariant under the monodromy transformation the argument for $N'$ translates directly to the argument for $N$.

The argument for $N$ of type $(b+1, b)$ is by induction over $b$. So let us assume the claim is shown for all $N'$ of type $(b' + 1, b')$ with $0 \leq b' < b$. Then we have seen that this implies the claim for all $N'$ of type $(a', b')$ with $a' > b'$ and $b' \leq b$ so in particular for all $(a', b')$ with $a' > b'$ and $a' + b' = (b+1) + b$ but $(a', b') = (b+1, b)$.

Let us fix the descendent assignment corresponding to $2b+1$ insertions of odd classes. Then for $N$ of type $(b+1, b)$ there remain in general $\binom{2b+1}{b}$
choices corresponding to the distribution of copies of \( \alpha \) and \( \beta \) among the markings. There are also \((2b+1)_b\) relations between GWpfs coming from the monodromy transformation applied to the \( N' \) of type \((b,b+1)\):

\[
[M \cdot N'][\eta]^X = [M \cdot \phi(N')[\eta]^X
\]

After subtracting the left hand side from the right hand side, by the form of the monodromy transformation, in these relations only classes corresponding to \( N' \) of type \((a',b')\) with \( a' > b' \) and \( a' + b' = 2b+1 \) appear. By the vanishing we have already shown they are even relations between the classes only corresponding to \( N' \) of type \((b+1,b)\). Let us denote the class corresponding to a subset \( S \) of \( \{1, \ldots, 2b+1\} \) of size \( b \) by \( C(S) \) and the relation corresponding to a subset \( T \) of \( \{1, \ldots, 2b+1\} \) of size \( b+1 \) by \( R(T) \). Then we have

\[
0 = R(T) = \sum_{S \subseteq T} C(S).
\]

To conclude the theorem it will be enough to write each \( C(S) \) in terms of relations \( R(T) \). But we have

\[
C(S) = \sum_{i=0}^{b} (-1)^{i+b} c_i^{-1} \sum_{|T \cap S| = i} R(T)
\]

for appropriate positive rational numbers

\[
c_i = (b+1) \binom{b}{i}.
\]

### 5 Balanced classes

We want to finish the proof of Theorem 2 in the remaining case of balanced classes here, therefore giving a proof of Theorem 1. We follow the discussion of [4, Section 5.5] and try to keep the notation as similar as possible. Compared to [4] there is one additional induction on the codimension.

The following lemma will be used to determine relative GWpfs from a set of related absolute GWpfs. Before stating the lemma we need to introduce a special refined descendent assignment.

Let \( P(d) \) be the set of partitions of \( d \) and \( \mathbb{Q}^{P(d)} \) the \( \mathbb{Q} \)-vector space of functions from \( P(d) \) to \( \mathbb{Q} \). Let

\[
\tilde{\tau}(\omega) = \sum_{q=0}^{\infty} c_q \tau_q(\omega)
\]

be a refined descendent of \( \omega \). The Gromov-Witten theory of \( \mathbb{P}^1 \) relative to a point gives for each \( v \geq 0 \) a function

\[
\gamma_v : P(d) \to \mathbb{Q}, \quad \eta \mapsto (\tau(\omega)^v|\eta)^{P1}.
\]
**Fact.** There exists a $\mathbb{Q}$-linear combination $\tilde{\tau}(\omega)$ depending on $d$ such that the set of functions
\[
\{\gamma_0, \gamma_1, \ldots \}
\]
spans $\mathbb{Q}P^2(d)$.

**Proof.** This is Lemma 5.6 in [4]. Its proof uses the Gromov-Witten Hurwitz correspondence [2].

We will fix such a refined descendent assignment $\tilde{\tau}(\omega)$. Let us define
\[
\tilde{\psi} = \sum_{q=0}^{\infty} c_q \psi^q
\]
so that formally $\tilde{\tau}(\omega) = \tau(\tilde{\psi}(\omega))$.

**Lemma 1.** Let $M$, $L$, $A$, $B$ be monomials in insertions of $1$, $\omega$, $\alpha$ and $\beta$ respectively
\[
M = \prod_{h \in H} \tau_{\alpha_h}(1), \quad L = \prod_{h' \in H'} \tau_{\alpha_{h'}}(\omega), \quad A = \prod_{i \in I} \tau_{\alpha_i}(\alpha), \quad B = \prod_{j \in J} \tau_{\alpha_j}(\beta)
\]
and $\eta \in P(d)$ be a splitting. Then the GWpf
\[
[MAB|\eta]_X^X, \quad [MLAB]|_{r,d}^X
\]
are tautological if the classes
\[
[M^v \tilde{\tau}(\omega)^v AB]|_{r,d}^X, \quad [M^v AB]|_{r,d}^X
\]
are tautological for arbitrary $v \geq 0$, $r' \leq r$, $\mu \in P(d)$ and divisors $M'$ of $M$ with the possible exception of the case $r' = r$, $M' = M$.

**Proof.** We first study the case $M = 1$, $r = 0$. There is a degeneration of $X$ into $X \cup_{pt} \mathbb{P}^1$ we have already studied in Section [2] The corresponding degeneration formula spells here
\[
[M^v \tilde{\tau}(\omega)^v AB]|_{r,d}^X = \sum_{|\eta| = d} x(\eta) \mu, \quad [AB]|_{0,d}^X, \quad [\tilde{\tau}(\omega)^v]|_{0,d}^X
\]
By the Fact letting $v$ vary this determines $[AB]|_{0,d}^X$ for all $\eta$. The degeneration formula
\[
[LAB]|_{0,d}^X = \sum_{|\eta| = d} x(\eta) \mu, \quad [LAB]|_{0,d}^X, \quad [\tilde{\tau}(\omega)^v]|_{0,d}^X
\]
then determines the second kind of GWpf if $M = 1$, $r = 0$.

In general there are additional sums in the degeneration formula: one for the distribution of the factors of $M$ and one for the splitting of the domain curve. However by the hypothesis of the lemma and the fact that we already have shown the tautologicalness of GWpf of $\mathbb{P}^1$, only the summand corresponding to the distribution of all of $M$ to $X$ and all of $r$ to $X$ may be non-tautological. But then we can mirror the above argument in the simple case. 

\[10\]
5.1 Simple case

To illustrate the principle of the proof we start with the GWpfs with only 2 odd insertions (one of each \( \alpha \) and \( \beta \)). So for descendent assignments \( n \), \( m \), a monomial of identity insertions

\[
M = \prod_{h \in H} \tau_{\alpha}(1)
\]

and the choice of splitting \( \mu \) for the relative point we wish to determine

\[
[M \tau_{\alpha}(\alpha) \tau_{\beta}(\beta)]_{\mu}^{X}
\]

in terms of GWpfs with only even insertions. By induction on \( r \) and \( M \) we will assume that this statement has already been proven for all \( r' \leq r \) and \( M'|M \) except the case \( r' = r \), \( M' = M \).

Let \( K_v \) be an index set with \( v + 2 \) elements. We first look at the elliptic vanishing relation \( V(M, \{ K_v \}, I) \) where \( I \) assigns \( \bar{\psi} \) to every element of \( K_v \). There are \( 2 \binom{v + 2}{2} \) summands which contain odd classes and in fact since the descendent assignment is identical for each element of \( K_v \) each of them is equal to

\[
- [M \bar{\tau}(\omega)^v \bar{\tau}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X},
\]

which we thus have determined in terms of even GWpfs.

Lemma \( \mathbb{I} \) and the induction hypothesis gives us the determination of the classes

\[
[M \bar{\tau}(\alpha) \bar{\tau}(\beta)]_{\eta}^{X}, [M L \bar{\tau}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X}
\]

for any monomial \( L \) in descendents of \( \omega \).

Next we look at the elliptic vanishing relation \( V(M, \{ K_v \}, I) \) where this time the descendent assignment \( I \) takes the value \( \bar{\psi} \) at all but the first element of \( K_v \) where it takes the value \( n \). The even terms are still of no relevance but now there are four kinds of odd summands. They are

\[
-(v + 1) [M \bar{\tau}(\omega)^v \tau_{\alpha}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X} \\
+(v + 1) [M \bar{\tau}(\omega)^v \tau_{\beta}(\beta) \bar{\tau}(\alpha)]_{r,d}^{X} \\
- \left( \frac{v + 1}{2} \right) [M \bar{\tau}(\omega)^{v-1} \tau_{\alpha}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X} \\
+ \left( \frac{v + 1}{2} \right) [M \bar{\tau}(\omega)^{v-1} \tau_{\beta}(\beta) \bar{\tau}(\alpha)]_{r,d}^{X}.
\]

We are only interested in the first pair of summands since the second two are determined by \( \mathbb{I} \). By applying the relation \( R(M \bar{\tau}(\omega)^v, \{ \psi^n \}, \{ \bar{\psi} \}, \emptyset) \) we see that the first two summands are equal. Therefore we now know

\[
[M \bar{\tau}(\omega)^v \tau_{\alpha}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X}
\]

and by Lemma \( \mathbb{I} \) also

\[
[M \tau_{\alpha}(\alpha) \bar{\tau}(\beta)]_{\eta}^{X}, [M L \tau_{\alpha}(\alpha) \bar{\tau}(\beta)]_{r,d}^{X}.
\]
Repeating this argumentation we successively determine

\[
\left[ M\tilde{\tau}(\alpha)\tau_m(\beta)\eta \right]_r^X, \left[ ML\tilde{\tau}(\alpha)\tau_m(\beta)\eta \right]_{r,d}^X, \quad (3)
\]

\[
\left[ M\tau_m(\alpha)\tau_m(\beta)\eta \right]_r^X.
\]

For (3) we need the elliptic vanishing relation \( V(M, \{ K_v \}, 1) \), where \( I \) takes the value \( \tilde{\psi} \) on all but the first and the last elements of \( K_v \) where it is \( \psi^m \). As before two terms in this relation are not yet determined and these are proportional to each other by the monodromy relation \( R(M\tilde{\tau}(\omega)^n, \{ \tilde{\psi} \}, \{ \psi^m \}, \emptyset) \).

For (4) we use the relation \( V(M, \{ K_v \}, 1) \) with \( I \) having the value \( \tilde{\psi} \) on all but the first and the last element of \( K_v \) where it takes the values \( n \) and \( m \) respectively. To see that there is only a pair of not yet determined terms we in particular need to use (2) and (4). We finish with the use of the relation \( R(M\tilde{\tau}(\omega)^n, \{ \psi^n \}, \{ \psi^m \}, \emptyset) \).

### 5.2 General case

Let \( I \) and \( J \) be two ordered index sets of the same size and

\[
\mathbf{n} : I \to \Psi_Q, \quad \mathbf{m} : J \to \Psi_Q
\]

be general descendent assignments. In order to prove Theorem 2 we need to calculate for a monomial \( M \) in insertions of the identity the GWpf:

\[
\left[ M \prod_{i \in I} \tau_{n_i}(\alpha) \prod_{j \in J} \tau_{m_j}(\beta)\eta \right]_{r,d}^X
\]

in terms of lower GWpf. This follows from the following lemma.

**Lemma 2.** For \( s, t \geq 0 \) the GWpf

\[
\left[ M \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s < j \leq t} \tilde{\tau}(\alpha) \prod_{J \leq j \leq t} \tau_{m_j}(\beta) \prod_{t < j} \tilde{\tau}(\beta)\eta \right]_r^X,
\]

\[
\left[ ML \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s < j \leq t} \tilde{\tau}(\alpha) \prod_{J \leq j \leq t} \tau_{m_j}(\beta) \prod_{t < j} \tilde{\tau}(\beta)\eta \right]_{r,d}^X,
\]

for an arbitrary monomial \( L \) in insertions of the identity are determined in terms of the GWpf

\[
\left[ M' \prod_{i \leq s'} \tau_{n_i}(\alpha) \prod_{s' < j \leq t'} \tilde{\tau}(\alpha) \prod_{J \leq j \leq t'} \tau_{m_j}(\beta) \prod_{t' < j} \tilde{\tau}(\beta)\eta \right]_{r',s'}^X
\]

\[
\left[ M'L' \prod_{i \leq s'} \tau_{n_i}(\alpha) \prod_{s' < j \leq t'} \tilde{\tau}(\alpha) \prod_{J \leq j \leq t'} \tau_{m_j}(\beta) \prod_{t' < j} \tilde{\tau}(\beta)\eta \right]_{r',d}^X.
\]

where \( L' \) is an arbitrary monomial in insertions of the identity and \((r', s', t', M') < (r, s, t, M)\), and GWpf with strictly less odd insertions. Here we have used the partial order defined by \((r', s', t', M') \leq (r, s, t, M)\) if and only if \( r' \leq r, s' \leq s, t' \leq t \) and \( M' \mid \mathcal{M} \).

12
Proof. We need additional notation. For \( v \geq 0 \) let \( W \) be an index set of cardinality \( v \). Define \( K_v \) by

\[
K_v = I \sqcup W \sqcup J
\]

with order implicit in the notation. Let \( 1_{f[s|t]} : K_v \rightarrow \Psi_q \) be the descendant assignment with

\[
1_{f[s|t]}(k) = \begin{cases} 
  n_k, & \text{if } k \text{ is one of the first } s \text{ elements of } I, \\
  m_k, & \text{if } k \text{ is one of the first } t \text{ elements of } J, \\
  \phi_i, & \text{else.}
\end{cases}
\]

We call the \( s \) first elements of \( I \subset K_v \) and the \( t \) first elements of \( J \subset K_v \) special elements of \( K_v \) with respect to \((s,t)\).

Let \( \sigma : I \rightarrow J \) be a bijection, which we can, using the orders on \( I \) and \( J \), also interpret as a permutation of \( I \). Let \( P \sigma \) be the set partition of \( K_v \) with first part \( \{1,\sigma(1)\} \sqcup W \) and pairs \( \{i,\sigma(i)\} \) as the other parts.

Consider the relations \( V(M,P \sigma,1_{f[s|t]}) \) for varying \( \sigma \). By the induction hypothesis we only need to care about the terms from the Künneth decomposition with exactly \(|I| + |J|\) odd insertions. After expanding the product there are \( 2 \cdot \binom{s+2}{2} \cdot 2^{|I|-1} \) terms of this kind. If we consider the odd part of the Künneth decomposition corresponding to the part \( \{1,\sigma(1)\} \sqcup W \) of \( P \) in more detail, we see that depending on the \( s, t \) and \( \sigma(1) \) still different kinds of terms might occur. We only care about the terms such that the least possible amount of point classes \( \omega \) is distributed to the special elements of \( K_v \) with respect to \((s,t)\) since all possible other terms are of the form \( [\text{5}] \) for

\[
(s’,t’) \in \{(s-1,t),(s,t-1),(s-1,t-1)\}.
\]

The remaining terms occur still with a combinatorial multiplicity \( C_\sigma \) depending on the number of special elements in \( \{1,\sigma(1)\} \). These multiplicities are

\[
C_\sigma = \begin{cases} 
  1, & \text{if } \{1,\sigma(1)\} \text{ contains 2 special elements} \\
  v+1, & \text{if } \{1,\sigma(1)\} \text{ contains 1 special elements} \\
  \binom{s+2}{2}, & \text{if } \{1,\sigma(1)\} \text{ contains 0 special elements.}
\end{cases}
\]

The last case can only occur if \( s = 0 \).

Let \( V \) be the relation obtained by summing these relations over all permutations \( \sigma \) and weighting with \( C_\sigma^{-1} \) and a sign

\[
\sum_\sigma (-1)^{\binom{2}{2}} \text{sign}(\sigma) C_\sigma^{-1} V(M,P \sigma,1_{f[s|t]})
\]

and removing terms determined by the induction hypothesis and of the form \( [\text{5}] \) for \((s’,t’)\) as before. Using the notation from Section \( \text{[XX]} \), we can write

\[
V = \sum_{\delta \leq I} \sum_{D \in S^*(\delta)} (-1)^{|J|-|D|} |\delta|! (|I| - |\delta|)! 
\]

\[
\left[ M^\tau(\omega)^s \prod_{i \leq s} \tau_n(\gamma_i) \prod_{s < i \leq t} \tau_s(\gamma_i) \prod_{j \geq s \leq t} \tau_m(\gamma_i) \prod_{t < j \leq \tau} \tau_m(\gamma_j) \right]^X_{r,d},
\]
where \( S^*(\delta) \) denotes the set of all subsets of \( I \sqcup J \) such that \( D \cap I = \delta \).

Using the substitution

\[
e_k = \sum_{|\delta|=k} \sum_{D \in S^*(\delta)} [M \tau(\omega)^n \prod_{i \leq s} \tau_n_i(\gamma_i^D) \prod_{s < j \leq t} \tau(\gamma_i^D) \prod_{J \ni j} \tau_m_j(\gamma_j^D) \prod_{t < j} \tau(\gamma_j^D)]_{r,d}
\]

we can write \( V \) more simply as

\[
V = \sum_{k=0}^{|I|} (-1)^{|I|-k} k!(|I| - k)! e_k.
\]

We wish to eliminate \( e_0, \ldots, e_{|I|-1} \) from \( V \) to obtain a formula for

\[
e_{|I|} = \left[ M \tau(\omega)^n \prod_{i \leq s} \tau_n_i(\alpha) \prod_{s < j \leq t} \tau(\alpha) \prod_{J \ni j} \tau_m_j(\beta) \prod_{t < j} \tau(\beta) \right]_{r,d}.
\]

Let \( R(\ell) \) be the sum

\[
R(\ell) = \sum_{|\delta|=\ell} R(M \tau(\omega)^n, n', m', \delta).
\]

Here \( n' \) and \( m' \) are the restrictions of \( 1_{[s], [t]} \) to \( I \) and \( J \) respectively.

Since unbalanced GWpsfs vanish we have the expansion

\[
R(\ell) = \sum_{|\delta| \geq \ell} \sum_{D \in S^*(\delta)} \binom{|\delta|}{\ell} [M \tau(\omega)^n \prod_{i \leq s} \tau_n_i(\gamma_i^D) \prod_{s < j \leq t} \tau(\gamma_i^D) \prod_{J \ni j} \tau_m_j(\gamma_j^D) \prod_{t < j} \tau(\gamma_j^D)]_{r,d}
\]

\[
= \sum_{k \geq \ell} \binom{k}{\ell} e_k.
\]

The following lemma in linear algebra gives us the formula for the desired \( e_{|I|} \).

**Lemma 3.** Let \( e_0, \ldots, e_n \) be a basis of the vector space \( \mathbb{Q}^{n+1} \). Then the vectors

\[
V := \sum_{k=0}^n (-1)^{n-k} k!(n-k)! e_k
\]

and

\[
R(\ell) := \sum_{k \geq \ell} \binom{k}{\ell} e_k
\]

for \( 0 \leq \ell < n \) form a basis of \( \mathbb{Q}^{n+1} \).
Proof. Note that by formally extending the definition of $R(\ell)$ to $R(n)$ we obtain an $(n+1) \times (n+1)$ lower uni-triangular matrix $R$ with coefficients

$$R_{ab} = \begin{pmatrix} a \\ b \end{pmatrix}.$$  

$R$ is therefore invertible and the coefficients of its inverse $R^{-1}$ are

$$(R^{-1})_{ab} = (-1)^{a+b} \begin{pmatrix} a \\ b \end{pmatrix}.$$  

In particular the $R(0), \ldots, R(n-1)$ are linearly independent. In order to show that $V$ is not a linear combination of these vectors we expand $V$ in terms of the basis corresponding to $R$

$$V = \sum_{\ell=0}^{n} c_{\ell} R(\ell)$$  

and check that the coefficient $c_n$ is nonzero:

$$c_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (-1)^{n-k} k!(n-k)! = (n+1)!$$

We next apply Lemma 1 to determine

$$\left[ M \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s \prec i \in I} \tilde{\tau}(\alpha) \prod_{J \ni j \leq t} \tau_{m_j}(\beta) \prod_{t < j} \tilde{\tau}(\beta) \right]_{r,d}^{X}$$

using the induction hypothesis for the $r$ induction.

By a degeneration argument as in the simple case we finally obtain a formula for

$$\left[ ML \prod_{i \leq s} \tau_{n_i}(\alpha) \prod_{s \prec i \in I} \tilde{\tau}(\alpha) \prod_{J \ni j \leq t} \tau_{m_j}(\beta) \prod_{t < j} \tilde{\tau}(\beta) \right]_{r,d}^{X}.$$

References

[1] K. Behrend, “Gromov-Witten invariants in algebraic geometry.”
Invent. Math. 127 no. 3, (1997) 601–617 [arXiv:alg-geom/9601011]
http://dx.doi.org/10.1007/s002220050132

[2] A. Okounkov and R. Pandharipande, “Gromov-Witten theory,
Hurwitz theory, and completed cycles,”
Ann. of Math. (2) 163 no. 2, (2006) 517–560 [arXiv:math/0204305]
http://dx.doi.org/10.4007/annals.2006.163.517
[3] A. Okounkov and R. Pandharipande, “The equivariant Gromov-Witten theory of $\mathbb{P}^1$,” *Ann. of Math. (2)* 163 no. 2, (2006) 561–605, arXiv:math/0207233
http://dx.doi.org/10.4007/annals.2006.163.561

[4] A. Okounkov and R. Pandharipande, “Virasoro constraints for target curves,” *Invent. Math.* 163 no. 1, (2006) 47–108, arXiv:math/0308097
http://dx.doi.org/10.1007/s00222-006-0455-y

[5] C. Faber and R. Pandharipande, “Relative maps and tautological classes,” *J. Eur. Math. Soc. (JEMS)* 7 no. 1, (2005) 13–49, arXiv:math/0304485
http://dx.doi.org/10.4171/JEMS/20

[6] J. Li, “A degeneration formula of GW-invariants.” *J. Differential Geom.* 60 no. 2, (2002) 199–293, arXiv:math/0110113
http://projecteuclid.org/getRecord?id=euclid.jdg/1090351102

Departement Mathematik
ETH Zürich
felix.janda@math.ethz.ch