SPECTRA FOR GELFAND PAIRS ASSOCIATED WITH THE FREE TWO STEP NILPOTENT LIE GROUP

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ABSTRACT. Let $F(n)$ be a connected and simply connected free 2-step nilpotent lie group and $K$ be a compact subgroup of $\text{Aut}(F(n))$. We say that $(K, F(n))$ is a Gelfand pair when the set of integrable $K$-invariant functions on $F(n)$ forms an abelian algebra under convolution. In this paper, we consider the case when $K = O(n)$. In [1], we know the only possible Gelfand pairs for $(K, F(n))$ is $(O(n), F(n))$, $(SO(n), F(n))$. So we just consider the case $(O(n), F(n))$, the other case can be obtained in the similar way. We study the natural topology on $\Delta(O(n), F(n))$ given by uniform convergence on compact subsets in $F(n)$. We show $\Delta(O(n), F(n))$ is a complete metric space. Our main result gives a necessary and sufficient result for the sequence of the "type 1" bounded $O(n)$-spherical functions uniform convergence to the "type 1" bounded $O(n)$-spherical function on compact sets in $F(n)$. What’s more, the "type 1" bounded $O(n)$-spherical functions are dense in $\Delta(O(n), F(n))$. Further, we define the Fourier transform according to the "type 2" bounded $O(n)$-spherical functions and give some basic properties of it.

1. Introduction

Given a locally compact group $G$ and compact subgroup $K \subseteq G$, the pair $(G, K)$ is called a Gelfand pair if $L^1(G//K)$, the space of integrable, $K$-bi-invariant functions on $G$, is commutative. Perhaps the best known examples are those defining symmetric spaces, that is, when $G$ is a connected semisimple lie group of finite center, and $K$ is a maximal compact subgroup. The analysis associated with such pairs plays an important role in the representation theory of semisimple lie groups and has been extensively developed in the last four decades. (cf. e.g. [2], [3]). In sharp contrast to this case, one might begin by assuming that $G$ is a solvable lie group. But then, if $G$ is simply connected for example, there maybe no non-trivial compact subgroups. One can, however, consider pairs of the form $(K \rtimes G, K)$, where $K$ is a compact subgroup of $\text{Aut}(G)$, the group of automorphism of $G$.

We study the connected, simply connected free two-step nilpotent lie groups $F(n)$ for two reasons. Firstly, for any nilpotent lie group $N$ and a compact group $K \in \text{Aut}(N)$, $(K \rtimes N, K)$ is a Gelfand pair if and only if $N$ is at most two step. Secondly, every two-step nilpotent lie group is a

1991 Mathematics Subject Classification. 22E46, 22E47.
Key words and phrases. Gelfand pairs, $O(n)$-bounded spherical functions.
quotient of a free two step nilpotent lie group. Therefore, the connected, simply connected free two-step nilpotent lie group $F(n)$ is defined in Section 2. We construct an isomorphism between the heisenberg group and some group with respect to $F(n)$. Then we get the "type 1" and "type 2" bounded $O(n)$-spherical functions. Note that for the 2-step nilpotent lie group $N$, and the Gelfand pair $(K,N)$, the bounded $K$-spherical functions are the same as the positive definite $K$-spherical functions.[4]. In fact, this is not true in general for the semisimple case.

Our focus here is on the topology of the bounded "type 1", "type 2" $O(n)$-spherical functions respectively, where the usual weak*-topology coincides with the compact-open topology on $\Delta(O(n),F(n))$. Our main result is stated as Theorem 4.1. It asserts that a "type 1" bounded $O(n)$-spherical sequence $(\psi N)_{\infty}^\gamma$ converges to a "type 1" bounded $O(n)$-spherical function $\psi$ if and only if $\hat{L}_{\gamma_i}(\omega_{\lambda N,\alpha N}) \to \hat{L}_{\gamma_j}(\omega_{\lambda,\alpha})$, where $i = 1, \cdot \cdot \cdot , n$; $\hat{T}(\omega_{\lambda N,\alpha N}) \to \hat{T}(\omega_{\lambda,\alpha}), \gamma N \to \gamma$. Here $L_{\gamma_1}, \cdot \cdot \cdot , L_{\gamma_n}, T$ is a generator of the algebra $D_k(H_n)$, which means the left-$H_n$-invariant and $K$-invariant differential operators on $H(n)$. And $\omega_{\lambda N,\alpha N}$, $\omega_{\lambda,\alpha}$ are the "type 1"bounded $K$ spherical functions on $H(n)$. $\gamma N, \gamma$ are the parameters that will be introduced later. We require a careful analysis of the behavior of such eigenvalues, and these results are described in Section 3.

We refine our description of the topology on the Gelfand space by proving two final results. Theorem 4.5 asserts that $\Delta(K,F(n))$ is complete. That is, if a sequence of bounded $K$-spherical functions converge to some function in the compact-open topology, then the limit is necessary a bounded $K$-spherical function. Later, we will assert the the "type 1" bounded $O(n)$-spherical functions are dense in $\Delta(O(n),F(n))$. Section 5 contains a description of the Godemental-Plancherel measure and the $O(n)$-spherical transform and then gives a definition for the Fourier transform induced by the "type 2" bounded $O(n)$-spherical functions. Also, we obtain some important properties about it.

2. Notation and Preliminaries

In this section, we will introduce some basic knowledge and significant results about the free two-step nilpotent lie group.

First Definition. Let $\mathcal{N}_p$ be the (unique up to isomorphism) free two-step nilpotent Lie algebra with $p$ generators. The definition using the universal property of the free nilpotent Lie algebra can be found in [7, Chapter V 5]. Roughly speaking, $\mathcal{N}_p$ is a (nilpotent)Lie algebra with $p$ generators $X_1, \cdot \cdot \cdot , X_p$, such that the vectors $X_1, \cdot \cdot \cdot , X_p$ and $X_{i,j} = [X_i,X_j], i < j$ form a basis; we call this basis the canonical basis of $\mathcal{N}_p$.

We denote by $\mathcal{V}$ and $\mathcal{Z}$, the vectors spaces generated by the families of vectors $X_1, \cdot \cdot \cdot , X_p$ and $X_{i,j} = [X_i,X_j], 1 \leq i < j \leq p$ respectively; these families become the canonical base of $\mathcal{V}$ and $\mathcal{Z}$. Thus $\mathcal{N}_p = \mathcal{V} \oplus \mathcal{Z}$, and $\mathcal{Z}$ is the center of $\mathcal{N}_p$. With the canonical basis, the vector space $\mathcal{Z}$ can be
identified with the vector space of antisymmetric $p \times p$-matrices $A_p$. Let $z = \dim Z = p(p - 1)/2$.

The connected simply connected nilpotent Lie group which corresponds to $N_p$ is called the free two-step nilpotent Lie group and is denoted $N_p$. We denote by $\exp: N_p \to N_p$ the exponential map.

In the following, we use the notations $X + A \in N, \exp(X + A) \in N$ when $X \in \mathcal{V}, A \in Z$. We write $p = 2p' or 2p' + 1$.

A Realization of $N_p$. We now present here a realization of $N_p$, which will be helpful to define more naturally the action of the orthogonal group and representations of $N_p$.

Let $(\mathcal{V}, <,>)$ be an Euclidean space with dimension $p$. Let $O(\mathcal{V})$ be the group of orthogonal transformations of $\mathcal{V}$, and $SO(\mathcal{V})$ is its special subgroup. Their common Lie algebra denoted by $Z$, is identified with the vector space of antisymmetric transformations of $\mathcal{V}$. Let $N = \mathcal{V} \oplus Z$ be the exterior direct sum of the vector spaces $\mathcal{V}$ and $Z$.

Let $[,] : V \times V \to Z$ be the bilinear application given by:

$$[X,Y](V) = \langle X, V \rangle Y - \langle Y, V \rangle X$$

where $X, Y, V \in \mathcal{V}$.

We also denote by $[,]$ the bilinear application extended to $N \times N \to N$ by:

$$[,]_{N \times Z} = [,]_{Z \times N} = 0$$

This application is a Lie bracket. It endows $\mathcal{V}$ with the structure of a two-step nilpotent Lie algebra.

As the elements $[X, Y], X, Y \in \mathcal{V}$ generate the vector space $Z$, we also define a scalar product $< >$ on $Z$ by:

$$<[X,Y], [X', Y']> = \langle X, X' \rangle Y' - \langle Y, X' \rangle X'$$

where $X, Y, X', Y' \in \mathcal{V}$.

It is easy to see $\mathcal{V}$ as a realization of $N_p$ when an orthonormal basis $X_1, \cdots, X_p$ of $(\mathcal{V}, <,>)$ is fixed.

We remark that $<[X, Y], [X', Y']> = <[X, Y]X', Y'>$, and so we have for an antisymmetric transformation $A \in Z$, and for $X, Y \in \mathcal{V}$:

$$<A, [X, Y]> = <A.X, Y>$$

This equality can also be proved directly using the canonical basis of $N_p$.

Actions of Orthogonal Groups. We denote by $O(\mathcal{V})$ the group of orthogonal linear maps of $(\mathcal{V}, <,>)$, and by $O_p$ the group of orthogonal $p \times p$-matrices.

On $N_p$ and $N_p$. The group $O(\mathcal{V})$ acts on the one hand by automorphism on $\mathcal{V}$, on the other hand by the adjoint representation $Ad_Z$ on $Z$. We obtain an action of $O(\mathcal{V})$ on $N = \mathcal{V} \oplus Z$. Let us prove that this action respects the Lie bracket of $N$. It suffices to show for $X, Y, Z \in \mathcal{V}$ and $k \in O(\mathcal{V})$:

$$[k.X, k.Y](V) = \langle k.X, V \rangle k.Y - < k.Y, V > k.X$$

$$k.(< X, k^i.V > Y - < Y, k^i.V > X$$

$$= k.[X,Y](k^{-1}.V) = Ad_Z k.[X, Y].$$
We then obtain that the group $O(V)$ and also its special subgroup $SO(V)$ acts by automorphism on the Lie algebra $\mathcal{N}$, and finally on the Lie group $\mathcal{O}$.

Suppose an orthonormal basis $X_1, \ldots, X_p$ of $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is fixed; then the vectors $X_{i,j} = [X_i, X_j], 1 \leq i < j \leq p$, form an orthonormal basis of $\mathcal{V}$ and we can identify:

- the vector space $\mathcal{Z}$ and $\mathcal{A}_p$,
- the group $O(V)$ with $O_p$,
- the adjoint representation $Ad_{\mathcal{Z}}$ with the conjugate action of $O_p$ and $\mathcal{A}_p$: $k.A = kAk^{-1}$, where $k \in O_p, A \in \mathcal{A}_p$.

Thus the group $O_p \sim O(V)$ acts on $\mathcal{V} \sim \mathbb{R}^p$ and $\mathcal{Z} \sim \mathcal{A}_p$, and consequently on $\mathcal{N}_p$. Those actions can be directly defined; and the equality $[k.X,k.Y] = k.[X,Y], k \in O_p, X,Y \in \mathcal{V}$, can then be computed.

On $\mathcal{A}_p$. Now we describe the orbits of the conjugate actions of $O_p$ and $SO_p$ on $\mathcal{A}_p$. An arbitrary antisymmetric matrix $A \in \mathcal{A}_p$ is $O_p$-conjugated to an antisymmetric matrix $D_2(\lambda)$ where $\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p$ and:

$$D_2(\lambda) = \begin{bmatrix}
\lambda_1 J & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 0 & \lambda_p J & 0 \\
0 & 0 & 0 & (0)
\end{bmatrix}$$

where $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

($(0)$ means that a zero appears only in the case $p = 2p^\prime + 1$) Furthermore, we can assume that $\lambda$ is in $\mathcal{L}$, where we denote by $\mathcal{L}$ the set of $\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p$ such that $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$.

Parameters. To each $\lambda \in \mathcal{L}$, we associate two numbers $p_0$ the number of $\lambda_i \neq 0$, $p_1$ the number of distinct $\lambda_i \neq 0$, and $\mu_1, \cdots, \mu_{p_1}$ such that:

$$\{\mu_1 > \mu_2 > \cdots > \mu_{p_1} > 0\} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p_0} > 0\}$$

We denote by $m_j$ the number of $\lambda_i$ such that $\lambda_i = \mu_j$, and we put $m_0 := m_0 := 0$ and for $j = 1, \cdots, p_1$ $m_j := m_1 + \cdots + m_j$.

For $j = 1, \cdots, p_1$, let $pr_j$ be the orthogonal projection of $\mathcal{V}$ onto the space generated by the vectors $X_{2i-1}, X_{2i}$, for $i = m_{j-1} + 1, \cdots, m_j$.

Let $\mathcal{M}$ be the set of $(r,\lambda)$ where $\lambda \in \mathcal{L}$, and $r \geq 0$, such that $r = 0$ if $2p_0 = p$.

Expression of the bounded spherical functions. The bounded spherical functions of $(\mathcal{N}_p, K)$ for $K = O_p$, are parameterized by

$$(r,\lambda) \in \mathcal{M} \quad \text{with the previous notations } p_0, p_1, \mu_i, pr_j \text{ associated to } \lambda,$$

$l \in \mathbb{N}^{p_1}$ if $\lambda \neq 0$, otherwise $\emptyset$.

Let $(r,\lambda,l)$ be such parameters. Then we have the following two types of bounded $O(n)$-spherical functions:

- Type 1: $\phi^{r,\lambda,l}(n) = \int_{K} e^{i \langle X^*_p, k.X \rangle} \omega_{\lambda,l}(\Psi(k,n))dk$.
Type 2: $\phi^v(n) = \int_K e^{iv\cdot X_p^k \cdot k \cdot X} dk$. Here $X_p^*$ is the unit $K_p$-fixed invariant vector. For a Gelfand pair $(H^{p_0}, K(m;p_1;p_0))$, we have $\omega_{\Lambda,\nu}$ is the "type 1" bounded $K(m;p_1;p_0)$-spherical functions for the Heisenberg group $H^{p_0}$. We will introduce it next. $\Psi_2$ bounded $K$ and $X$.

We introduce the following law of the Heisenberg group $H^{p_0}$:

\[ h = (z_1, \ldots, z_{p_0}, t) \quad h' = (z_1', \ldots, z_{p_0}', t') \in \mathbb{H}^{p_0} = \mathbb{C}^{p_0} \times \mathbb{R} \]

\[ h \cdot h' = (z_1 + z_1', \ldots, z_{p_0} + z_{p_0}', t + t' + \frac{1}{2} \sum_{i=1}^{p_0} \tilde{g}_i \tilde{z}_i) \]

The unitary $p_0 \times p_0$ matrix group $U_{p_0}$ acts by automorphisms on $\mathbb{H}^{p_0}$. Let us describe some subgroups of $U_{p_0}$. Let $p_0, p_1 \in \mathbb{N}$, and $m = (m_1, \ldots, m_{p_1}) \in \mathbb{N}^{p_1}$ be fixed such that $\sum_{j=1}^{p_1} m_j = p_0$. Let $K(m;p_1;p_0)$ be the subgroup of $U_{p_0}$ given by:

\[ K(m;p_1;p_0) = U_{m_1} \times \cdots \times U_{m_{p_1}}. \]

The expression of spherical functions of $(\mathbb{H}^{p_0}, K(m;p_1;p_0))$ can be found in the same way as in the case $m = (p_0)$, $p_1 = 1$ i.e. $K = U_{p_0}$.

The aim of this paragraph is to describe the stability group $K_\rho$ of $\rho \in T_{r_X^*} + D_2(\Lambda)$. Before this, let us recall that the orthogonal $2n \times 2n$ matrices which commutes with $D_2(1, \ldots, 1)$ have determinant one and form the group $Sp_n \cap O_{2n}$. This group is isomorphism to $U_n$; the isomorphism is denoted $\psi^{(n)}_1$, and satisfies:

\[ \forall k, X: \psi^{(n)}_1(k,X) = \psi^{(n)}_1(K)\psi^{(n)}_1(X), \]

where $\psi^{(n)}_c$ is the complexification:

\[ \psi^{(n)}_c(x_1, y_1; \ldots; x_n, y_n) = (x_1 + iy_1, \ldots, x_n + iy_n). \]

Now, we can describe $K_\rho$.

**Proposition 2.2.** Let $(r, \Lambda) \in \mathcal{M}$. Let $p_0$ be the number of $\lambda_i \neq 0$, where $\Lambda = (\lambda_1, \ldots, \lambda_{p_0})$, and $p_1$ the number of distinct $\lambda_i \neq 0$. We set $\Lambda = (\lambda_1, \ldots, \lambda_{p_0}) \in \mathbb{R}^{p_0}$. Let $\rho \in T_f$ where $f = rX_p^* + D_2(\Lambda)$. If $\Lambda = 0$, then $K_\rho$ is the subgroup of $K$ such that $k \cdot rX_p^* = rX_p^*$ for all $k \in K_\rho$.

If $\Lambda \neq 0$, then $K_\rho$ is the direct product $K_1 \times K_2$, where:

\[ K_1 = \{ k_1 = \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & I_d \end{bmatrix} \mid \tilde{k}_1 \in SO(2p_0), D_2(\tilde{\Lambda})\tilde{k}_1 = \tilde{k}_1 D_2(\tilde{\Lambda}) \} \]

\[ K_2 = \{ k_2 = \begin{bmatrix} I_d & 0 \\ 0 & \tilde{k}_1 \end{bmatrix} \mid \tilde{k}_2, rX_p^* = rX_p^* \}. \]

Furthermore, $K_1$ is isomorphism to the group $K(m;p_0;p_1)$.

**Proof.** We keep the notations of this proposition, and we set $A^* = D_2(\Lambda)$ and $X^* = rX_p^*$. It is easy to prove:

\[ K_\rho = \{ k \in K : kA^* = A^*k and kX^* = X^*k \}. \]

If $\Lambda = 0$, since $K_\rho$ is the stability group in $K$ of $X^* \in V^* \sim \mathbb{R}^p$. So the first part of Proposition 2.2 is proved.
Let us consider the second part. $\land \neq 0$ so we have

$$A^* = \begin{bmatrix} D_2(\tilde{\lambda}) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad D_2(\tilde{\lambda}) = \begin{bmatrix} \mu_1 J_{m_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mu_p J_{m_p} \end{bmatrix}$$

Let $k \in K_\rho$. From above computation, the matrices $k$ and $A^*$ commute and we have:

$$k = \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & \tilde{k}_2 \end{bmatrix} \quad \text{with} \quad \tilde{k}_1 \in O(2p_0) \quad \text{and} \quad \tilde{k}_2 \in O(p - 2p_0)$$

Furthermore, $\tilde{k}_2 X^* = X^*$, and the matrices $\tilde{k}_1$ and $D_2(\tilde{\lambda}^*)$ commute. So $\tilde{k}_1$ is the diagonal block matrix, with block $[\tilde{k}_1]^i_j \in O(m_j)$ for $i = 1, \ldots, p_1$.

Each block $[\tilde{k}_1]^i_j \in O(m_j)$ commutes with $J_{m_j}$. So on one hand, we have $\det[k_{1}] = 1$, $\det[k_{2}] = 1$, and one the other hand, $[\tilde{k}_1]^i_j \in O(m_j)$ corresponds to a unitary matrix $\psi_1^{(m_j)}([\tilde{k}_1]^i_j)$.

Now we set for $k_1 \in K_1$:

$$\begin{align*}
\Psi_1(k_1) &= (\psi_1^{(m_j)}([\tilde{k}_1]^i_j)_1, \ldots, \psi_1^{(m_j)}([\tilde{k}_1]^i_j)_p) \\
\Psi_1 : K_1 &\rightarrow K(m; p_0; p_1)
\end{align*}$$

$\square$

Quotient group $\overline{N} = N/\ker \rho$. In this paragraph, we describe the quotient groups $N/\ker \rho$ and $G/\ker \rho$, for some $\rho \in \overline{N}$. This will permit in the next paragraph to reduce the construction of the bounded spherical functions on $N_\rho$ to known questions on Euclidean and Heisenberg groups. For a representation $\rho \in \overline{N}$, we will denote by:

- $\ker \rho$ the kernel of $\rho$.
- $N/\ker \rho$ its quotient group and $\overline{N}$ its lie algebra.
- $(\mathcal{H}, \overline{\rho})$ the induced representation on $\overline{N}$.
- $\overline{n} \in \overline{N}$ and $\overline{y} \in \overline{N}$ the image of $n \in N$ and $y \in N$ respectively by the canonical projections $N \rightarrow \overline{N}$ and $\mathcal{N} \rightarrow \overline{N}$.

Now, with the help of the canonical basis, we choose $E_1 = \mathbb{R} X_1 \bigoplus \cdots \bigoplus \mathbb{R} X_{2p_0 - 1}$

as the maximal totally isotropic space for $\omega_{D_2(\lambda), v}$. The quotient lie algebra $\overline{N}$ has the natural basis: $\cdot$. You can refer to [6].

Here, we have denoted $|\land| = (\sum_{j=1}^{p} \lambda_j^2)^{1/2} = |D_2(\land)|$ (for the Euclidean norm on $\mathcal{Z}$).

Let $\overline{N}_1$ be the Lie sub-algebra of $\overline{N}$, with basis $\overline{X}_1, \ldots, \overline{X}_{2p_0}, \overline{B}$, and $\overline{N}_1$ be its corresponding connected simply connected nilpotent lie group. We define the mapping:

$$\Psi_2 : \mathbb{H}^{p_0} \rightarrow \overline{N}_1 \quad \text{for} \quad h = (x_1 + iy_1, \ldots, x_{p_0} + iy_{p_0}, t) \in \mathbb{H}^{p_0} \quad \text{by}$$

$$\Psi_2(h) = \exp(\sum_{j=1}^{p_0} \sqrt{\lambda_j^2}(x_j \overline{X}_{2j - 1} + y_j \overline{X}_{2j}) + t \overline{B})$$

We compute that each lie bracket of two vectors of this basis equals zeros, except:

$$[\overline{X}_{2i-1}, \overline{X}_{2i}] = \frac{\lambda_i}{|\land|} \overline{B}, \quad i = 1, \cdots, p_0.$$

From this, it is easy to see:

**Theorem 2.3.** $\Psi_2$ is a group isomorphism between $\overline{N}_1$ and $\mathbb{H}^{p_0}$.
Finally, we note that \( \overline{\psi} : N \to \mathcal{N}_1 \) is the canonical projection.

3. SOME ANALYSIS ON THE HEISENBERG GROUP \( H_n \)

The whole parts of this section can be referred from [7]. A result due to Howe and Umeda (cf. [8]) shows that \( \mathbb{C}[v_R]^K \) is freely generated as an algebra. So there are polynomials \( \gamma_1, \cdots, \gamma_d \in \mathbb{C}[v_R]^K \) so that \( \mathbb{C}[v_R]^K = \mathbb{C}[\gamma_1, \cdots, \gamma_d] \).

We call \( \gamma_1, \cdots, \gamma_d \) the fundamental invariants.

Invariant differential operators. The algebra \( \mathbb{D}(H_n) \) of left-invariant differential operators on \( H_n \) is generated by \( \{ Z_1, \cdots, Z_n, \overline{Z}_1, \cdots, \overline{Z}_n, T \} \). We denote the subalgebra of \( K \)-invariant differential operators by \( \mathbb{D}_K(H_n) := \{ D \in \mathbb{D}(H_n) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K, f \in C^\infty(H_n) \} \).

From now on, we always suppose \((K, H_n)\) is a Gelfand pair, and if this is true, \( \mathbb{D}_K(H_n) \) is an abelian algebra.

We define \( p(Z, \overline{Z}) \) as follows:

\[
p(Z, \overline{Z}) := \sum c_{a,b} Z^a \overline{Z}^b = \sum c_{a,b} Z_1^a_1 \cdot \cdots \cdot Z_n^a_n \overline{Z}_1^b_1 \cdot \cdots \cdot \overline{Z}_n^b_n
\]

where \( \psi \in \mathbb{D}_K(H_n) \). Note that the operator \( L_p \) is intrinsically defined, whereas \( p(Z, \overline{Z}) \) depends on the basis used to identify \( V \) with \( \mathbb{C}^n \). One has \( L_p = Sym(p(Z, \overline{Z})) \), where \( Sym \) is the linear map characterized by

\[
Sym(Z^a \overline{Z}^b) = \frac{1}{\|a\| + \|b\|} \sum_{\sigma \in S_{\|a\|+\|b\|}} \sigma(Z^a \overline{Z}^b).
\]

Here, as usual, \( \|a\| = a_1 + \cdots + a_n \) and \( \sigma(Z^a \overline{Z}^b) \) denotes the result of applying the permutation \( \sigma \) to the \( |a| + |b| \) terms in \( Z^a \overline{Z}^b \).

For a d-multi-index \( a \), let \( \gamma^a := \gamma_1^{a_1} \cdots \gamma_n^{a_n} \), \( \|a\| := a_1 |\delta_1| + \cdots + a_n |\delta_n| \) (the homogeneous degree of \( \gamma^a \)), and \( L_\gamma^a := L_1^{a_1} \cdots L_n^{a_n} \). Using the definition of the map \( \sim \), together with the fact that \( |Z_j, \overline{Z}_j| = -2iT \), one sees that

\[
L_\gamma^a = L_\gamma^a + \sum_{\|b\| < \|a\|} c_{a,b} L_\gamma^b T^{|b| - |a|}
\]

for some coefficients \( c_{a,b} \in \mathbb{C} \). Since \( \gamma_1, \cdots, \gamma_d \) generates \( \mathbb{C}[v_R]^K \), it follows easily that \( \{ L_{\gamma_1}, \cdots, L_{\gamma_d}, T \} \) generates the algebra \( \mathbb{D}_K(H_n) \).

Therefore, for any \( H_n \)-spherical function \( \psi \), \( \psi \) is an eigenfunction for every \( D \in \mathbb{D}_K(H_n) \) if and only if \( \psi \) is an eigenfunction for each of \( L_{\gamma_1}, \cdots, L_{\gamma_d}, T \).

We write \( \hat{D}(\psi) \) for the eigenvalue of \( D \in \mathbb{D}_K(H_n) \), that is \( D(\psi) = \hat{D}(\psi) \psi \).

Note that since \( \psi(0,0) = 1 \), one has \( \hat{D}(\psi) = D(\psi)(0,0) \).

**Theorem 3.1.** The bounded \( K \)-spherical functions on \( H_n \) are parametrized by the set \( \{ \psi | \psi \in \Delta(K, H_n) \} \) via

\[
\Delta(K, H_n) = \{ \phi_{\lambda, \alpha} \mid \lambda \in \mathbb{R}^\times, \alpha \in \wedge \} \cup \{ \eta_{\omega} \mid \omega \in V \}
\]

Note that, for \( \psi \in \Delta(K, H_n) \), one has \( \psi(z, t) = e^{i\lambda t} \psi(z, 0) \), where \( \lambda = -i\hat{T}(\psi) \in \mathbb{R} \).

**Lemma 3.2.** For \( p \in \mathbb{C}[v_R]^K \) and \( \psi \in \Delta(K, H_n) \), one has

\[
\hat{L}_p(\psi) = \partial_p(\psi)(0,0).
\]
where \( \partial_p = p(2 \frac{\partial}{\partial z}, 2 \frac{\partial}{\partial \bar{z}}) \). That is \( \partial_p \) is the operator obtained by replacing each occurrence of \( z_j \) in \( p \) by \( 2 \frac{\partial}{\partial z_j} \) and each \( \bar{z}_j \) by \( 2 \frac{\partial}{\partial \bar{z}_j} \).

**Lemma 3.3.** \( \hat{L}_{p_n}(\eta_\omega) = (-1)^{|\alpha|} P_\alpha(\omega) \)

**Lemma 3.4.** \( \hat{L}_{p_n}(\phi_{\lambda,\beta}) = \hat{L}_{p_n}(\phi_\beta) \) for \( \alpha, \beta \in \Lambda, \lambda \in \mathbb{R}^\times \).

**Lemma 3.5.** \( \hat{L}_{p_n} \) is a real number with sign \( (-1)^{|\alpha|} \) for all \( \alpha \in \Lambda \) and \( \psi \in \Delta(K,H_n) \).

**Lemma 3.6.** The eigenvalues for \( L_{\gamma_0} \) on the \( U(n) \)-spherical functions of type 1 are \( \hat{L}_{\gamma_0}(\phi_{\lambda,r}) = -|\lambda|(2r + n) \)

**Lemma 3.7.** \( \left| \hat{L}_{p_m}(\phi_r) \right| \leq \frac{(n + r + m - 1)}{m} \)

where \( p_m \) is the \( U(n) \)-invariant polynomial obtained from \( P_m(V) \)

**Theorem 3.8.** For \( \psi \in \Delta(K,H_n) \), one has

\[
\psi(z,0) = \sum_{\delta \in \Lambda} \frac{L_{p_\delta}(\psi)}{\dim(P_\delta)} p_\delta(z),
\]

where the series converges absolutely and uniformly on compact subsets in \( V \). Thus we have the following series expansions for the \( K \)-spherical functions of types 1 and 2 respectively:

\[
\phi_{\lambda,\alpha}(z,t) = e^{i\lambda t} \sum_{\delta \in \Lambda} \frac{|\lambda|^{|\delta|} L_{p_\delta}(\phi_\delta)}{\dim(P_\delta)} p_\delta(z)
\]

\[
\eta_\omega(z,t) = \sum_{\delta \in \Lambda} \frac{(-1)^{|\delta|} \hat{L}_{p_\delta}(\omega)}{\dim(P_\delta)} p_\delta(z).
\]

Here, convergence is absolute and uniform on compact subsets in \( H_n \).

**Proof.** The expansions for \( \phi_{\lambda,\alpha}(z,t) \) and \( \eta_\omega(z,t) \) follow immediately from that for \( \psi(z,0) \) together with Lemmas 3.4 and 3.3. It is a general fact that the spherical functions for a Gelfand pair \( (G,K) \) are real analytic (cf. Proposition 1.5.15 in [1]). For pairs of the form \( (K,H_n) \), one can see this directly from the functional forms of the two types of \( K \)-spherical functions. Write the Taylor series expansion of \( \psi(z,0) \) centered at \( z = 0 \) as \( \psi(z,0) = \sum_{m=0}^{\infty} h_m(z) \), where \( h_m(z) \) is a homogeneous polynomial of degree \( m \) on \( V_\mathbb{R} \) (i.e. in the variables \((z,\bar{z})\)). Since \( \psi \) is \( K \)-invariant, one sees by \( K \)-averaging this expression that each \( h_m \) is \( K \)-invariant. As \( \{p_\delta \mid \delta \in \Lambda \} \) is a basis for \( C[V_\mathbb{R}]^K \), we can rewrite the Taylor series as

\[
\psi(z,0) = \sum_{\delta \in \Lambda} c_\delta p_\delta(z)
\]

for some coefficients \( c_\delta \). Note that since Taylor series converge absolutely, it is not necessary to specify an ordering on the set \( \Lambda \) of indices for this sum. We use Lemma 3.2 and perform term-wise differentiation of this Taylor series to obtain \( \hat{L}_{p_n}(\psi) = \partial_{p_n}(\psi)(0,0) = \sum_{\delta \in \Lambda} c_\delta \partial_{p_n}(p_\delta)(0) \).

Let \( \{v_1, \ldots, v_{\dim(P_\alpha)}\} \) be an orthonormal basis for \( P_\alpha \) and \( \{u_1, \ldots, u_{\dim(P_\alpha)}\} \) be an orthonormal basis for \( P_\delta \). Thus \( p_\alpha = \sum v_j(z) \overline{v}_j(\overline{\omega}) \), \( p_\delta = \sum u_j(z) \overline{u}_j(\overline{\omega}) \) and
Theorem 4.1. Taylor series, the convergence is absolute and uniform on compact sets.

Let $\psi$ be a compact set of $\phi$. Suppose that $T$ is a compact set if and only if $\psi$ is a bounded $O(n)$-spherical function of type 1. Therefore, $A = e^{\omega \hat{L}}$. Hence $\hat{L}(\omega_{n,m}) = \hat{L}(\omega,\hat{L})$ and $r_N \to r$.

4. The proof of the main theorem

Theorem 4.1. Let $(\psi_N)_{N=1}^{\infty}$ be a sequence of bounded $O(n)$-spherical functions of type 1, and $\psi$ is a bounded $O(n)$-spherical function of type 1. Then $\psi_N$ converges to $\psi$ in the topology of $\Delta(O(n), F(n))$ i.e. uniformly on compact sets) if and only if $\hat{L}_{\gamma_i}(\omega_{n,m}) = \gamma_i$ and $\hat{L}_{\gamma_i}(\omega_{n,m}) = \gamma_i$ for $i = 1, \ldots, d$.

Proof. Suppose that $(\phi_{n,m,n})_{N=1}^{\infty}$ converges uniformly to $\phi_{n,m,n}(n)$ on compact sets of $F(n)$. From Theorem 3.8, we obtain

\begin{equation}
(4.2)
\end{equation}

\begin{equation}
\phi_{n,m,n}(n_t) = \int_{K} e^{irN X^t_p k.X} \sum_{\delta \in \Lambda} \hat{L}_{\delta} \left( \frac{\omega_{n,m,n}}{\dim(P_b)} \right) P_d(\Psi^{-1}(q_1(k.exp(tX)))) dk
\end{equation}
\[
\phi_{\gamma,N,\lambda,N,\alpha,N}(\text{exp} A) = \int_K \omega_{\lambda,N,\alpha,N}(\Psi^{-1}_2(q_1(k,\text{exp}(tA))))dk =
\int_K e^{T(\omega_{\lambda,N,\alpha,N})\Psi^{-1}_2(q_1(k,\text{exp}(tA)))}dk.
\]
\[
\phi_{\gamma,\lambda,\alpha}(\text{exp} A) = \int_K \omega_{\lambda,\alpha}(\Psi^{-1}_2(q_1(k,\text{exp}(tA))))dk =
\int_K e^{T(\omega_{\lambda,\alpha})\Psi^{-1}_2(q_1(k,\text{exp}(tA)))}dk.
\]
Since \( \phi_{\gamma,N,\lambda,N,\alpha,N}(\text{exp} A) \) converges to \( \phi_{\gamma,\lambda,\alpha}(\text{exp} A) \) uniformly for all \(-\infty < t < \infty\).

Differentiate with respect to \( t \) respectively and let \( t = 0 \), we obtain
\[
\dot{\hat{T}}(\omega_{\lambda,N,\alpha,N}) \int_K \Psi^{-1}_2(q_1(k,\text{exp}(A)))dk \rightarrow \dot{\hat{T}}(\omega_{\lambda,\alpha}) \int_K \Psi^{-1}_2(q_1(k,\text{exp}(A)))dk.
\]
Therefore, \( \dot{T}(\omega_{\lambda,N,\alpha,N}) \rightarrow \dot{T}(\omega_{\lambda,\alpha}) \).

Finally, suppose \(|\lambda| = n\), we find \( X_1, \ldots, X_n \in \mathcal{N}_p^\ast \) such that \( < X_p^\ast, X_1 > , \ldots, < X_p^\ast, X_n > \) are different from each other as well as
\[
\left| \int_K P_{\delta_1}(\Psi^{-1}_2(q_1(k,\text{exp}(X_1))))dk \right| \cdots \left| \int_K P_{\delta_n}(\Psi^{-1}_2(q_1(k,\text{exp}(X_n))))dk \right| 
eq 0.
\]
Then
\[
\phi_{\gamma,N,\lambda,N,\alpha,N}(\text{exp} X_m) = \sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,N,\alpha,N})}{\dim(P_\delta)} \int_K e^{irN<\delta X_m^\ast,X_m>|t} P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(tX_m))))dk.
\]
Similarly, \( \phi_{\gamma,N,\alpha,N}(\text{exp} X_m) = \sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,\alpha})}{\dim(P_\delta)} \int_K e^{ir<\delta X_m^\ast,X_m>|t} P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(tX_m))))dk.\)

Here \( m = 1, \ldots, n \).

Since \( \phi_{\gamma,N,\lambda,N,\alpha,N}(\text{exp} X_m) \) converges uniformly to \( \phi_{\gamma,\lambda,\alpha}(\text{exp} X_m) \) and \( \gamma \rightarrow \gamma \). We obtain:
\[
\sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,N,\alpha,N})}{\dim(P_\delta)} \int_K e^{ir<\delta X_m^\ast,X_m>|t} P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(tX_m))))dk \text{ converges uniformly to } \sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,\alpha})}{\dim(P_\delta)} \int_K e^{ir<\delta X_m^\ast,X_m>|t} P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(tX_m))))dk.
\]

Here \( m = 1, \ldots, n \).

Differentiate with respect to \( t \) \( j \) times, where \( j = 1, \ldots, n - 1 \) and let \( t = 0 \), we get
\[
\sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,N,\alpha,N})}{\dim(P_\delta)} \int_K (P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(X_m)))))|^{j,\tau,j} < X_p^\ast, X_m >^{j,\tau,j} \text{ converges uniformly to } \sum_{\delta \in \Lambda} \frac{\hat{L}_{P_\delta}(\omega_{\lambda,\alpha})}{\dim(P_\delta)} \int_K (P_{\delta}(\Psi^{-1}_2(q_1(k,\text{exp}(X_m)))))|^{j,\tau,j} < X_p^\ast, X_m >^{j,\tau,j}.
\]
Therefore, if \( N \) large enough, for any \( \varepsilon > 0 \), we have
\[
\begin{bmatrix}
-\varepsilon \frac{\delta}{\alpha} \\
& \varepsilon \frac{\delta}{\alpha} \end{bmatrix} < 
\begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
-\varepsilon \frac{\delta}{\alpha} \\
& \varepsilon \frac{\delta}{\alpha} \end{bmatrix} < 
\begin{bmatrix}
\frac{\varepsilon}{M_1} \\
& \cdots \\
\frac{\varepsilon}{M_1} \\
\end{bmatrix},
\]
Therefore, if \( M_1 \) large enough, we have
\[
\left| \sum_{\delta \in \Lambda} \frac{\hat{L}_{P_{\delta}}(\omega_{\lambda,\alpha N}) - \hat{L}_{P_{\delta}}(\omega_{\lambda,\alpha})}{\dim(P_{\delta})} P_{\delta}^{-1}(q_1(k,exp(X))) \right| < \frac{\varepsilon}{M^2},
\]
where \(m = 1, \cdots n\).

Therefore,
\[
\left[ -\frac{\varepsilon}{M^2} \right] < \left[ \begin{array}{ccc}
\int_K P_{\delta_1}(\Psi_2^{-1}(q_1(k,exp(X)))) & \cdots & \int_K P_{\delta_n}(\Psi_2^{-1}(q_1(k,exp(X)))) \\
\vdots & \ddots & \vdots \\
\int_K P_{\delta_1}(\Psi_2^{-1}(q_1(k,exp(X)))) & \cdots & \int_K P_{\delta_n}(\Psi_2^{-1}(q_1(k,exp(X)))) \\
\end{array} \right] \times
\]
\[
\left[ \begin{array}{c}
\hat{L}_{P_{\delta_1}}(\omega_{\lambda,\alpha N}) - \hat{L}_{P_{\delta_1}}(\omega_{\lambda,\alpha}) \\
\vdots \\
\hat{L}_{P_{\delta_n}}(\omega_{\lambda,\alpha N}) - \hat{L}_{P_{\delta_n}}(\omega_{\lambda,\alpha}) \\
\end{array} \right] < \left[ \frac{\varepsilon}{M^2} \right]
\]

Therefore, if \(M_2\) large enough, we have \(\left| \hat{L}_{P_{\delta m}}(\omega_{\lambda,\alpha N}) - \hat{L}_{P_{\delta m}}(\omega_{\lambda,\alpha}) \right| < \varepsilon\) where \(m = 1, \cdots n\).

Thus, \(\hat{L}_{P_{\delta m}}(\omega_{\lambda,\alpha N}) \rightarrow \hat{L}_{P_{\delta m}}(\omega_{\lambda,\alpha})\), for \(m = 1, \cdots n\).

Since \(\{\gamma_1, \cdots, \gamma_d\} \subset \{P_{\delta} \mid \delta \in \Lambda\}\), this shows in particular that \(\hat{L}_{\gamma_j}(\omega_{\lambda,\alpha N}) \rightarrow \hat{L}_{\gamma_j}(\omega_{\lambda,\alpha})\), for \(j = 1, \cdots d\).

Conversely, suppose \(\hat{L}_{\gamma_j}(\omega_{\lambda,\alpha N}) \rightarrow \hat{L}_{\gamma_j}(\omega_{\lambda,\alpha})\), for \(j = 1, \cdots d\); \(\hat{T}(\omega_{\lambda,\alpha N}) \rightarrow \hat{T}(\omega_{\lambda,\alpha})\) and \(r_N \rightarrow r\). It follows that \(\hat{L}_P(\omega_{\lambda,\alpha N}) \rightarrow \hat{L}_P(\omega_{\lambda,\alpha})\) for every \(P \in \mathbb{C}[V_R]^K\).

Indeed, each \(P \in \mathbb{C}[V_R]^K\) is a linear combination of monomials \(\gamma^a\) in the fundamental invariants,

\[
(4.3) \quad \lim_{N \to \infty} (L^a_{\gamma})^N(\omega_{\lambda,\alpha N}) = \lim_{N \to \infty} (L^a_{\gamma}(\omega_{\lambda,\alpha N}) + \sum_{\|b\| < \|a\|} c_{a,b} \hat{L}^b_{\gamma}(\omega_{\lambda,\alpha N}) \hat{T}(\omega_{\lambda,\alpha N})^{\|a\|-\|b\|})
= \lim_{N \to \infty} (L^a_{\gamma}(\omega_{\lambda,\alpha}) + \sum_{\|b\| < \|a\|} c_{a,b} \hat{L}^b_{\gamma}(\omega_{\lambda,\alpha}) \hat{T}(\omega_{\lambda,\alpha})^{\|a\|-\|b\|})
= (L^a_{\gamma})^N(\omega_{\lambda,\alpha}).
\]

Suppose that \(\psi_N = \phi_{\gamma N,\alpha N}\) and \(\psi_N \rightarrow \psi\), where \(\psi = \phi_{\gamma,\alpha}\) is a bounded \(O(n)\)-spherical function.

Since \(\hat{T}(\omega_{\lambda,\alpha N}) = i\lambda N\) converges to \(\hat{T}(\omega_{\lambda,\alpha}) = i\lambda\), we have \(\lambda N \rightarrow \lambda\).

Since \(\hat{L}_{\gamma_0}(\omega_{\lambda,\alpha N}) = |\lambda N| \hat{L}_{\gamma_0}(\omega_{\lambda,\alpha N}) = |\lambda N| (2|\alpha N| + n)\) converges to \(\hat{L}_{\gamma_0}(\omega_{\lambda,\alpha}) = |\lambda| (2|\alpha| + n)\). Also, \(r_N := |\alpha N|\) must converges to \(|\alpha|\). Thus, both \(\lambda N\) and \((\gamma N)\) are bounded sequences and we choose constants \(C_1, C_2\) with \(|\lambda N| \leq C_1, 0 \leq \gamma N \leq C_2\) for all \(N\). Therefore, \(|\lambda| \leq C_1, 0 \leq \gamma \leq C_2\).

Choose constant \(C_3\) and \(C_4\) with
Theorem 4.4. Let $S$ be a sequence of bounded $O(n)$-spherical functions of type 2, and $\psi$ is a bounded $O(n)$-spherical function of type 2. Then $\psi_N$ converges to $\psi$ in the topology of $\Delta(O(n), F(n))$ (i.e., uniformly on compact sets) if and only if $r_N \to r$, where $\psi_N(n) = \int_K e^{ir_N X^*_p,k.X} dk, \psi(n) = \int_K e^{irX^*_p,k.X} dk, n = \exp(X + A)$. 

Proof. If $\psi^v_N(n)$ converges uniformly to $\psi^v(n)$. Take $X \in N_p$, such that $\int_K < X^*_p,k.X > dk \neq 0$.

Then $\psi_N(\exp(0X)) = \int_K e^{ir_N X^*_p,k.X} dk \to \psi(\exp(X)) = \int_K e^{irX^*_p,k.X} dk$ uniformly, where $-\infty < t < \infty$.

Differentiate with respect to $t$ and let $t = 0$, we obtain $ir_N \int_K < X^*_p,k.X > dk \to ir \int_K < X^*_p,k.X > dk$.

Therefore, $r_N \to r$.

Conversely, if $r_N \to r$, its obvious that $\psi_N(n) = \int_K e^{ir_N X^*_p,k.X} dk \to \psi(n) = \int_K e^{irX^*_p,k.X} dk$. 

Theorem 4.5. $\Delta(O(n), F(n))$ is a complete metric space. That is, if $(\psi_N)_{N=1}^\infty$ is a sequence of bounded-$O(n)$-spherical functions that converges uniformly to $\psi$ on compact subset in $F(n)$, then $\psi$ is a bounded $O(n)$-spherical function.
Proof. It is clear that $\psi$ is continuous, $O(n)$-invariant and bounded and $\psi(e) = 1$. Moreover, if $f, g \in L^1_K(F(n))$ have compact support then
\[
\int_{F(n)} \psi(n)(f \ast g)(n) dn = \lim_{N \to \infty} \int_{F(n)} \psi_N(n)(f \ast g)(n) dn
\]
\[
= \lim_{N \to \infty} \int_{F(n)} \psi_N(n)f(n) dn \int_{F(n)} \psi_N(n)g(n) dn
\]
\[
= \int_{F(n)} \psi(n)f(n) dn \int_{F(n)} \psi(n)g(n) dn
\]
(4.6)

Thus, $f \to \int_{F(n)} \psi(n)f(n) dn$ defines a continuous non-zero algebra homomorphism $L^1_K(F(n)) \to \mathbb{C}$. It follows that $\psi \in \triangle(O(n), F(n))$. \hfill \Box

5. THE SECOND MAIN THEOREM AND SOME OTHER RESULTS

The $O(N)$-Spherical transform for $f \in L^1_K(F(n))$ is the function
\[
\hat{f} : \triangle(O(n), F(n)) \to \mathbb{C}, \hat{f}(\psi) = \int_{F(n)} f(n) \psi(n^{-1}) dn.
\]
Here $dn$ denote the haar measure for the group $F(n)$.

One has
\[
(f \ast g)^\wedge(\psi) = \hat{f}(\psi)\hat{g}(\psi)
\]
(5.1)
and
\[
(f^*)^\wedge(\psi) = \overline{f(\psi)}
\]
(5.2)
for $f, g \in L^1_K(F(n)), \psi \in \triangle(O(n), F(n)), f^*(n) = f(n^{-1})$.

Let’s compute equation (5.2) for example,
\[
(f^*)^\wedge(\psi) = \int_{F(n)} f^*(n)\psi(n^{-1}) dn
\]
\[
= \int_{F(n)} \overline{f(n^{-1})}\psi(n^{-1}) dn = \int_{F(n)} \overline{f(n)}\psi(n) dn
\]
\[
= \int_{F(n)} \overline{f(n)} \times \psi(n^{-1}) dn = \int_{F(n)} \overline{f(n)}\psi(n^{-1}) dn
\]
(5.3)
\[
= \overline{\hat{f}(\psi)}
\]

The compact open topology is the smallest topology makes all of the maps $\{\hat{f} \mid f \in L^1_K(F(n))\}$ continuous. Since $L^1_K(F(n))$ is a Banach, $*$-algebra with respect to the involution $f \to f^*$, it follows that $\hat{f}$ belongs to the space $C_0(\triangle(O(n), F(n)))$ of continuous functions on $\triangle(O(n), F(n))$ that vanish at infinity. Moreover, we have $\|\hat{f}\|_\infty \leq \|f\|_1$ for $f \in \triangle(O(n), F(n))$.

This follows immediately from the fact that for $\psi \in \triangle(O(n), F(n))$ one has $|\psi(n)| \leq \psi(e) = 1$, since $\psi$ is positive definite.

Godement’s Plancherel Theory for Gelfand pairs (cf.[9] or Section 1.6 in [1]) ensures that there exists a unique positive Borel measure $d\mu$ on the
space \( \triangle(O(n), F(n)) \) for which

\[
(5.4) \quad \int_{F(n)} |f(n)|^2 \, dn = \int_{\triangle(O(n), F(n))} |\hat{f}(\psi)| \, d\mu(\psi)
\]

for all continuous functions \( f \in L^1_K(F(n)) \cap L^2_K(F(n)) \). If \( f \in L^1_K(F(n)) \cap L^2_K(F(n)) \) is continuous and \( \hat{f} \) is integrable with respect to \( d\mu \), then one has the Inversion Formula.

\[
(5.5) \quad f(n) = \int_{\triangle(O(n), F(n))} \hat{f}(\psi)\psi(n^{-1}) \, d\mu(\psi)
\]

In particular, this formula holds when \( f \) is continuous, positive definite and \( K \)-invariant. Moreover, the spherical transform \( f \rightarrow \hat{f} \) extends uniquely to an isomorphism between \( L^2_K(F(n)) \) and \( L^2(\triangle(O(n), F(n)), d\mu) \).

Let \( \mathcal{L} \) be the set of \( \wedge = (\lambda_1, \cdots, \lambda_{p'}') \in \mathbb{R}^{p'} \) such that \( \lambda_1 > \cdots > \lambda_{p'}' > 0 \). We define the following measure on \( \mathcal{L} \):

\[
d\eta'(\wedge) = \begin{cases} 
-c \prod_{i=1}^{p'} \lambda_i \prod_{j<k} (\lambda_j^2 - \lambda_k^2) \, d\lambda_1 \cdots d\lambda_{p'}' & \text{if } p = 2p' \\
-c \prod_{i=1}^{p'} \lambda_i^3 \prod_{j<k} (\lambda_j^2 - \lambda_k^2) \, d\lambda_1 \cdots d\lambda_{p'}', & \text{if } p = 2p' + 1
\end{cases}
\]

where the constant \( c \) is some constant.

Over \( \mathbb{R}^+ \), we define the measure \( \tau \) given as the Lebesgue measure if \( p = 2p' + 1 \), and the Dirac measure in 0, if \( p = 2p' \).

\[
c(p) = \begin{cases} 
(2\pi)^{-\frac{p(p-1)}{2}+p'} & \text{if } p = 2p' \\
2(2\pi)^{-\frac{p(p-1)}{2}+p'-1} & \text{if } p = 2p' + 1
\end{cases}
\]

**Theorem 5.6.** \( m^* \) is the radial Plancherel measure for \( (N_p, O_p) \), i.e. for a \( K \)-invariant function \( \psi \in L^2(N) \), we have

\[
(5.7) \quad \|\psi\|^2_{L^2(N)} = \int \left| \langle \psi, \phi^{r,\wedge,l} \rangle \right|^2 \, dm^*(r, \wedge, l).
\]

Note that \( m^* \) is given as the tensor product of \( \eta' \) on \( \mathcal{L} \), and the counting measure \( \sum \) on \( \mathbb{N}^{p'} \), and the measure \( \tau \) on \( \mathbb{R}^+ \), up to the normalizing constant \( c(p) \). According to the definition of \( d\eta' \), the second type of the bounded \( O(n) \)-spherical functions has no compact on the above formula.

**Theorem 5.8.** The bounded \( O(n) \)-spherical functions of "type 1" are dense in the space \( \triangle(O(n), F(n)) \).

**Proof.** Take a point \( r \in R^+ \), and suppose that \( \phi^{r,0} \) is not in the closure of \( \{\phi^{r,|\lambda|,\alpha} \mid r \in R^+, |\lambda| \in R^+, \alpha \in \wedge \} \). \( \triangle(O(n), F(n)) \) is metrizable, hence it is completely regular. So we can find a continuous function \( J : \triangle(O(n), F(n)) \rightarrow \mathbb{R} \) with \( J(\phi^{r,0}) = 1 \), \( J(\phi^{r,|\lambda|,\alpha}) = 0 \) for all \( r \in R^+, |\lambda| \in R^+, \alpha \in \wedge \). We can assume that \( J \) has compact support.

The equation (5.2) ensures that \( L^1_K(F(n)) \) is a symmetric banach *-algebra.

It follows that \( \{\hat{f} \mid f \in L^1_K(F(n)) \} \) is dense in \( (C_0(\triangle(O(n), F(n))), \|\cdot\|_\infty) \).
(See for example, §14 in chapter 3 of [12].) Thus we can find a sequence
\((j_N)\) in \(L^1_K(F(n))\) with \(\hat{j}_N \to J\) uniformly on \(\Delta(O(n), F(n))\). We can as-
sume that each \(j_N\) is continuous and compactly supported. Moreover, since \(J\) is real-valued, we can assume that \(j_N^* = j_N\).

Similar to the proof of Proposition 3 in [10] shows that one can find an
approximate identity \((a_s)_{s>0}\) in \(L^1_K(F(n))\) with \(\hat{a}_s\) compactly supported in
\(\Delta(O(n), F(n))\) for all \(s > 0\). For \(s\) sufficiently small, one has \(\hat{a}_s(\phi^{r,0}) > \frac{3}{4}\).
moreover, for each \(s\) one sees that \((a_s * j_N)\) converges uniformly to
\(a_sJ\) as \(N \to \infty\). Thus we can choose \(s_0\) sufficiently small and \(N_0\) sufficiently
large that \(g = a_{s_0} * j_{N_0}\) satisfies \(\hat{a}_s(\phi^{r,0}) > \frac{1}{2}\) and \(|\hat{a}_s(\phi^{r,|\lambda|,\alpha})| < \frac{1}{4}\) for all \(r \in R^+, |\lambda| \in R^+, \alpha \in \Lambda\).

Note that \(g\) is continuous, integrable, square-integrable and \(g^* = g\).

Dixmier’s functional Calculus (cf. [11]) ensures that “sufficiently smooth
functions operate on \(L^1_K(F(n))\).” Thus if \(\zeta : \mathbb{R} \to \mathbb{R}\) is sufficiently smooth
with \(\zeta\) and its derivatives integrable and \(\zeta(0) = 0\), then there is a function
\(f := \zeta \{g\} \in L^1_K(F(n)) \cap L^2_K(F(n)) \cap C(F(n))\) with the property that \(\hat{f} = \zeta \hat{g}\).
We choose a \(\zeta\) with \(\zeta(t) = 1\) for \(t > \frac{1}{2}\) and \(\zeta(t) = 0\) for \(t < \frac{1}{4}\). Then \(F = \tilde{f} =
\zeta \{g\}\) satisfies \(F(\phi^{r,0}) = 1\) and \(F(\phi^{r,|\lambda|,\alpha}) = 0\) for all \(r \in R^+, |\lambda| \in R^+, \alpha \in \Lambda\).

Now theorem 5.6 shows

\[
\|\psi\|_{L^2(\mathcal{N})}^2 = \int_{\mathcal{N}} |\psi, \phi^{r,\lambda,\alpha}|^2 dm^*(r, \lambda, l)
\]

\[
= \int \int |\int_{F(n)} \psi(x) \phi^{r,\lambda,\alpha}(x) |^2 dx dm^*(r, \lambda, l) = \int \int \int_{F(n)} \psi(x) \phi^{r,\lambda,\alpha}(x) |^2 dx dm^*(r, \lambda, l)
\]

\[
= \int \int |\tilde{\psi}(\phi^{r,\lambda,\alpha})|^2 dm^*(r, \lambda, l)
\]

From this and equation (5.4), we obtain \(F = 0\) a.e. on \(\Delta(O(n), F(n))\).
In particular, \(F\) is integrable on \(\Delta(O(n), F(n))\) and we apply formula (5.5)
to conclude \(f \equiv 0\) on \(F(n)\). This implies that \(F = \tilde{f}\) is identically zero on
\(\Delta(O(n), F(n))\), which contradicts \(F(\phi^{r,0}) = 1\). \qed

We define the Fourier transform \(\mathcal{F}_\mathcal{N}(f) : \mathcal{N} \to \mathbb{C}\) of \(f \in L^1(\mathcal{N})\) by

\[
\mathcal{F}_\mathcal{N}(f)(w) = \int_{\mathcal{N}} f(z) e^{-ir<z,w>} dz
\]

where \(dz\) denotes Euclidean measure on \(\mathcal{N}_\mathbb{R}\) and \(\mathcal{N}\) is the lie algebra of \(N_p\).
With this normalization, one has \(\|\mathcal{F}_\mathcal{N}(f)\|_2 = (2\pi)^n \|f\|_2\). This is because
There are several properties about the Fourier transform, they are as follows:

(a) \( F_N(c_1 f_1 + c_2 f_2) = c_1 F_N(f_1) + c_2 F_N(f_2) \),
for \( c_1, c_2 \in \mathbb{C} \) and \( f_1, f_2 \in L^1(N) \).
(b) \( |F_N(f)(w)| \leq \int_N |f(z)| \, dz = \|f\|_1 \) for all \( f \in L^1(N) \).
(c) \( F_N(f * g)(w) = F_N(f)(w) F_N(g)(w) \) for all \( f, g \in L^1(N) \).
(d) \( F_N(f) = \overline{F_N(f)} \) for all \( f \in L^1(N) \).
(e) Let \( (L_z f)(z) = f(z - z') \), then \( (L_z f)(w) = e^{-ir<z,w>} F_N(f)(w) \).
Conversely, \( F_N(e^{-ir<z,w>} f)(w) = (L_z F_N(f))(w) \) for all \( f \in L^1(N) \).

Proof. (a)

\[
F_N(c_1 f_1 + c_2 f_2) = \int_N (c_1 f_1 + c_2 f_2)(z) e^{-ir<z,w>} \, dz
\]

\[
= c_1 \int_N f_1(z) e^{-ir<z,w>} \, dz + c_2 \int_N f_2(z) e^{-ir<z,w>} \, dz
= c_1 F_N(f_1) + c_2 F_N(f_2)
\]

(b) \( |F_N(f)(w)| = \int_N |f(z)| e^{-ir<z,w>} \, dz \)

\[
\leq \int_N |f(z)| \, dz = \|f\|_1.
\]

(c)

\[
F_N(f * g)(w)
= \int_N \int_N f(z - x) g(x) e^{-ir<x,w>} \, dz \, dx
= \int_N \int_N f(y) g(x) e^{-ir<y,z+w>} \, dx \, dy
= \int_N \int_N f(y) g(x) e^{-ir<y,z,w>} e^{-ir<x,w>} \, dx \, dy
= \int_N f(y) e^{-ir<y,w>} \, dy \int_N g(x) e^{-ir<x,w>} \, dx
= F_N(f)(w) F_N(g)(w)
\]

(5.11)
\[ F_N(\tilde{f})(\omega) = \int_N \tilde{f}(z) e^{-ir\langle z, w \rangle} dz \]
\[ = \int_N \tilde{f}(-z) \times e^{-ir\langle -z, w \rangle} dz \]
\[ = \int_N f(z) e^{-ir\langle z, w \rangle} dz \]
\[ = F_N(f)(\omega) \quad \text{(5.14)} \]

\[ (L_z f)(w) \]
\[ = \int_N (L_z f)(z) e^{-ir\langle z, w \rangle} dz \]
\[ = \int_N f(z - z') e^{-ir\langle z, w \rangle} dz \]
\[ = \int_N f(z) e^{-ir\langle z + z', w \rangle} dz \]
\[ = e^{-ir\langle z', w \rangle} F_N(f)(w) \quad \text{(5.15)} \]

Conversely,
\[ F_N(e^{ir\langle z', z \rangle} f(z))(w) \]
\[ = \int_N e^{ir\langle z', z \rangle} f(z) e^{-ir\langle z, w \rangle} dz \]
\[ = \int_N f(z) e^{-ir\langle z, w - z' \rangle} dz \]
\[ = F_N(f)(w - z') \]
\[ = L_{z'} F_N(f)(w) \quad \text{(5.16)} \]

Finally, since we know the two different types bounded \( O(n) \)-spherical functions, we compute the spherical transform of them respectively. Remember the two different types bounded \( O(n) \)-spherical functions are as follows: For \( n = exp(X + A) \in N \).

Type 1: \( \phi^{r,\wedge, l}(n) = \int_K e^{ir\langle X^*_p, k.X \rangle} \omega^{\wedge, l}(\Psi^{-1}_2(\gamma(k.n))) dk \).

Type 2: \( \phi^{\nu}(n) = \int_K e^{ir\langle X^*_p, k.X \rangle} dk \).
For any \( f \in L^1_K(F(n)) \) and type 1 bounded \( O(n) \)-spherical functions, we have
\[
\hat{f}(\phi^{r,\lambda,l}) = \int_N f(n) \int_K e^{-ir\langle X_p^k, X \rangle} \omega_{\lambda,l}(\Psi_2^{-1}(\overline{\Omega}(k.n^{-1}))) dkd\mathfrak{n}
\]
\[= \int_N f(k.n) \int_K e^{-ir\langle X_p^k, X \rangle} \omega_{\lambda,l}(\Psi_2^{-1}(\overline{\Omega}(n^{-1}))) dkd\mathfrak{n}
\]
\[= \int_K \int_N f(n) \int_K e^{-ir\langle X_p^k, X \rangle} \omega_{\lambda,l}(\Psi_2^{-1}(\overline{\Omega}(n^{-1}))) dndk
\]
\[= \int_N f(n) e^{-ir\langle X_p^k, X \rangle} \omega_{\lambda,l}(\Psi_2^{-1}(\overline{\Omega}(n^{-1}))) dn
\]
(5.17)

For any \( f \in L^1_K(F(n)) \) and type 2 bounded \( O(n) \)-spherical functions, we have
\[
\hat{f}(\phi^{\upsilon}) = \int_N f(n) \int_K e^{-ir\langle X_p^k, X \rangle} dkd\mathfrak{n}
\]
\[= \int_N f(k.n) \int_K e^{-ir\langle X_p^k, X \rangle} dkd\mathfrak{n}
\]
\[= \int_K \int_N f(n) \int_K e^{-ir\langle X_p^k, X \rangle} dndk
\]
\[= \int_N f(n) e^{-ir\langle X_p^k, X \rangle} dn
\]
(5.18)

\[\square\]

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