Cramér type moderate deviations for trimmed $L$-statistics

Nadezhda Gribkova

*St. Petersburg State University, Mathematics and Mechanics Faculty, 199034, Universitetskaya nab. 7/9, St. Petersburg, Russia*

**Abstract.** We establish Cramér type moderate deviation (MD) results for heavy trimmed $L$-statistics; we obtain our results under a very mild smoothness condition on the inversion $F^{-1}$ ($F$ is the underlying distribution of i.i.d. observations) near two points, where trimming occurs, we assume also some smoothness of weights of the $L$-statistic. Our results complement previous work on Cramér type large deviations (LD) for trimmed $L$-statistics by Gribkova (2016) and Callaert et al. (1982).

**Keywords:** moderate deviations; large deviations; trimmed $L$-statistics

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1 Introduction and main results

The theory of large deviations is one of the main branches in the probability theory and its applications. There is an extensive literature on this subject for the various classes of statistics, especially for the classical case of sums of independent random variables (see, e.g., [Petrov (1975); Saulis and Statulevičius (1991)]) and for some types of sums of dependent variables, e.g., for $U$-statistics (see, e.g., [Borovskikh and Weber (2003); Lai et al. (2011)], and the references therein).

In contrast, there are only a few papers on this topic for $L$-statistics. In the case of non-trimmed $L$-statistics with coefficients generated by a smooth on $(0,1)$ weight function, the Cramér type large and moderate deviations were studied by [Vandemaeele and Veraverbeke (1982); Aleskevicius (1991)]. A highly sharp result on Cramér type large deviations for non-trimmed $L$-statistics with a smooth weight function was established by [Bentkus and Zitikis (1990)].

For the case of heavy truncated $L$-statistics, i.e., the case when the weight function is zero outside some interval $[\alpha, \beta] \subset (0,1)$, a result on Cramér type large deviations was first obtained by [Calloert et al. (1982)]; more recently, the latter result was extended and strengthened in Gribkova (2016), where a different approach than in [Calloert et al. (1982)] was proposed and implemented.
To conclude this introduction we want to mention a paper by Gao and Zhao (2011), where a general delta method in the theory of Chernoff’s type large and moderate deviations suggested and illustrated by many examples including M-estimators and L-statistics. Some interesting results on Chernoff’s type large deviations for (non-trimmed) $L$-statistics with smooth weight function were obtained also by Boistard (2007).

In this article we supplement our previous work on Cramér type large deviations for trimmed $L$-statistics (cf. Gribkova (2016)) by some results on moderate deviations. Our approach here is the same as in Gribkova (2016): we approximate the trimmed $L$-statistic by a non-trimmed $L$-statistic with coefficients generated by a smooth on $(0, 1)$ weight function, where the approximating (non-trimmed) $L$-statistic is based on order statistics corresponding to a sample of auxiliary i.i.d. Winsorized observations. We apply a result on moderate deviations due to Vandemaele and Veraverbeke (1982) to the approximating $L$-statistic and estimate suitably the remainder term of our approximation.

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) real-valued random variables (r.v.’s) with common distribution function $F$, and for each integer $n \geq 1$ let $X_{1: n} \leq \cdots \leq X_{n: n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$.

Consider the trimmed $L$-statistic given by

$$L_n = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i:n} X_{i:n},$$

(1.1)

where $c_{i:n} \in \mathbb{R}$, $k_n, m_n$ are two sequences of integers such that $0 \leq k_n < n - m_n \leq n$. Put $\alpha_n = k_n/n$, $\beta_n = m_n/n$. It will be assumed throughout this paper that

$$\alpha_n \to \alpha, \quad \beta_n \to \beta, \quad 0 < \alpha < 1 - \beta < 1,$$

(1.2)

as $n \to \infty$, i.e. we focus on the case of heavy trimmed $L$-statistic.

Define the left-continuous inverse of $F$: $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$, and let $F_n, F_n^{-1}$ denote the empirical distribution function and its inverse respectively.

We will consider also the trimmed $L$-statistics with coefficients generated by a weight function:

$$L_n^0 = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i:n}^0 X_{i:n} = \int_{\alpha_n}^{1-\beta_n} J(u) F_n^{-1}(u) \, du,$$

(1.3)

where $c_{i:n}^0 = n \int_{(i-1)/n}^{i/n} J(u) \, du$, and $J$ is a function defined in an open set $I$ such that $[\alpha, 1 - \beta] \subset I \subseteq (0, 1)$.

To state our results, we will need the following set of assumptions.

(i) $J$ is Lipschitz in $I$, i.e. there exists a constant $C \geq 0$ such that

$$|J(u) - J(v)| \leq C|u - v|, \quad \forall \ u, v \in I.$$
(ii) There exists a positive ε such that for each \( t \in \mathbb{R} \)

\[
F^{-1}(\alpha + t\sqrt{\log n/n}) - F^{-1}(\alpha) = O((\log n)^{-1+\varepsilon}), \\
F^{-1}(1 - \beta + t\sqrt{\log n/n}) - F^{-1}(1 - \beta) = O((\log n)^{-1+\varepsilon})
\]

as \( n \to \infty \).

(iii) \( \max(|\alpha_n - \alpha|, |\beta_n - \beta|) = O(\sqrt{\log n/n}) \) as \( n \to \infty \).

(iv) For some \( \tilde{\varepsilon} > 0 \)

\[
\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c_{i,n}^0| = O\left(\frac{1}{\log^{\tilde{\varepsilon}} n} \sqrt{\frac{n}{\log n}}\right).
\]

Define a sequence of centering constants

\[
\mu_n = \int_{\alpha_n}^{1-\beta_n} J(u)F^{-1}(u) \, du.
\]

Since \( \alpha_n \to \alpha \), \( \beta_n \to \beta \) as \( n \to \infty \), both variables \( L_n^0 \) and \( \mu_n \) are well defined for all sufficiently large \( n \).

It is well known (see, e.g., Mason and Shorack (1990); Stigler (1974); van der Vaart (1998)) that when the inverse \( F^{-1} \) is continuous at two points \( \alpha \) and \( 1 - \beta \), the smoothness condition (1.4) implies the weak convergence to the normal law: \( \sqrt{n}(L_n^0 - \mu_n) \Rightarrow N(0, \sigma^2) \), where

\[
\sigma^2 = \sigma^2(J, F) = \int_\alpha^{1-\beta} \int_\alpha^{1-\beta} J(u) J(v) (u \land v - uv) \, dF^{-1}(u) \, dF^{-1}(v),
\]

where \( u \land v = \min(u, v) \); we will also use the notation \( u \lor v \) for \( \max(u, v) \).

Here and in the sequel, we use the convention that \( \int_a^b = \int_{[a,b)} \) when integrating with respect to the left continuous integrator \( F \). All along the article, we assume that \( \sigma > 0 \).

Define the distribution functions of the normalized \( L_n \) and \( L_n^0 \) respectively

\[
F_{L_n}(x) = \mathbb{P}\{\sqrt{n}(L_n - \mu_n)/\sigma \leq x\}, \quad F_{L_n^0}(x) = \mathbb{P}\{\sqrt{n}(L_n^0 - \mu_n)/\sigma \leq x\}.
\]

Let \( \Phi \) denote the standard normal distribution function. Here is our first result on Cramér type moderate deviations for \( L_n \).

**Theorem 1.1** Suppose that \( F^{-1} \) satisfies condition (i) and that condition (iii) holds for the sequences \( \alpha_n \) and \( \beta_n \). In addition, assume that there exists a function \( J \) satisfying condition (i) such that (iv) holds for the weights \( c_{i,n} \). Then

\[
1 - F_{L_n}(x) = [1 - \Phi(x)](1 + o(1)), \\
F_{L_n}(-x) = \Phi(-x)(1 + o(1)),
\]

as \( n \to \infty \), uniformly in the range \(-A \leq x \leq c\sqrt{\log n} \), for each \( c > 0 \) and \( A > 0 \).
The proof of our results is relegated to Section \S 3.

Theorem 1.1 directly implies the following corollary.

\textbf{Corollary 1.1} Let \( c_{i,n} = c_{0,n} = n \int_{(i-1)/n}^{i/n} J(u) \, du \) \( (k_n + 1 \leq i \leq n - m_n) \), where \( J \) is a function satisfying (i). Assume that conditions (ii) and (iii) are satisfied.

Then relations (1.10) with \( L_n = L_0^n \) hold true, for each \( c > 0 \) and \( A > 0 \), uniformly in the range \(-A \leq x \leq c \sqrt{\log n} \).

Finally, we state a version of Theorem 1.1, where the scale factor \( \sigma/n^{1/2} \) is replaced by \( \sqrt{\text{Var}(L_n)} \). A very mild moment condition will be required now to ensure the existence of the variance of \( L_n \).

\textbf{Theorem 1.2} Suppose that the conditions of Theorem 1.1 hold true. In addition, assume that \( E|X_1|^{\gamma} < \infty \) for some \( \gamma > 0 \). Then

\[
\frac{\sqrt{\text{Var}(L_n)}}{\sigma/\sqrt{n}} = 1 + O\left((\log n)^{-1+2\nu}\right),
\]

where \( \nu = \varepsilon \land \varepsilon' \) (\( \varepsilon, \varepsilon' \) are as in (1.6) and (1.5) respectively).

Moreover, relations (1.10) remain valid for each \( c > 0 \) and \( A > 0 \), uniformly in the range \(-A \leq x \leq c \sqrt{\log n} \), if we replace \( \sigma/n^{1/2} \) in definition of \( F_{L_n}(x) \) (cf. (1.9)) by \( \sqrt{\text{Var}(L_n)} \).

2 Stochastic approximation for \( L_0^n \)

Let \( \xi_{\nu} = F^{-1}(\nu) \), \( 0 < \nu < 1 \), be the \( \nu \)-th quantile of \( F \) and \( W_i \) denote \( X_i \) Winsorized outside of \( (\xi_{\alpha}, \xi_{1-\beta}) \). In other words

\[
W_i = \begin{cases} 
\xi_\alpha, & X_i \leq \xi_\alpha, \\
X_i, & \xi_\alpha < X_i \leq \xi_{1-\beta}, \\
\xi_{1-\beta}, & \xi_{1-\beta} < X_i.
\end{cases}
\]  

Let \( W_{i:n} \) denote the order statistics, corresponding to \( W_1, \ldots, W_n \), the sample of \( n \) i.i.d. auxiliary random variables.

Similarly as in Gribkova (2016), we will approximate \( L_n \) by a linear combination of the order statistics \( W_{i:n} \) with the coefficients generated by the following weight function

\[
J_w(u) = \begin{cases} 
J(\alpha), & u \leq \alpha, \\
J(u), & \alpha < u \leq 1 - \beta, \\
J(1-\beta), & 1 - \beta < u
\end{cases}
\]  

defined in \([0, 1]\). It is obvious that if \( J \) is Lipschitz in \( I \), i.e. satisfies condition (1.4) with some positive constant \( C \), then the function \( J_w \) is Lipschitz in \([0, 1]\) with some constant \( C_w \leq C \).

Consider the auxiliary non-truncated \( L \)-statistic given by

\[
\tilde{L}_n = n^{-1} \sum_{i=1}^n \tilde{c}_{i,n} W_{i:n} = \int_0^1 J_w(u) G_n^{-1}(u) \, du,
\]  

where
where \( \tilde{c}_{i,n} = n \int_{(i-1)/n}^{i/n} J_w(u) \, du \). Define the centering constants

\[
\mu_{\tilde{L}_n} = \int_0^1 J_w(u) G^{-1}(u) \, du.
\]

Since \( W_i \) has the finite moments of any order and because \( J_w \) is Lipschitz, the distribution of the normalized \( \tilde{L}_n \) tends to the standard normal law (see, e.g., Stigler (1974))

\[
\sqrt{n}(\tilde{L}_n - \mu_{\tilde{L}_n}) / \sigma(J_w, G) \Rightarrow N(0,1),
\]

where the asymptotic variance is given by

\[
\sigma^2(J_w, G) = \int_0^1 \int_0^1 J_w(u) J_w(v) (u \wedge v - uv) \, dG^{-1}(u) \, dG^{-1}(v).
\]

Observe that for \( u \in (\alpha, 1 - \beta) \) we have \( J_w(u) = J(u) \), \( G^{-1}(u) = F^{-1}(u) \), and that \( dG^{-1}(u) \equiv 0 \) for \( u \notin (\alpha, 1 - \beta) \). This yields the equality of the asymptotic variances

\[
\sigma^2(J_w, G) = \sigma^2(J, F) = \sigma^2
\]

of the truncated \( L \)-statistic \( L^0_n \) and the non-truncated \( L \)-statistic \( \tilde{L}_n \) based on the Winsorized random variables.

Define the binomial random variable \( N_\nu = \# \{ i : X_i \leq \xi_\nu \} \), where \( 0 < \nu < 1 \). Put \( A_n = N_\alpha / n \), \( B_n = (n - N_{1-\beta}) / n \).

The following lemma provides us a useful representation which is crucial in our proofs. This lemma is proved in Gribkova (2016, Lemma 2.1), therefore here we present only its statement.

**Lemma 2.1** (Gribkova (2016))

\[
L^0_n - \mu_n = \tilde{L}_n - \mu_{\tilde{L}_n} + R_n,
\]

where \( R_n = R_n^{(1)} + R_n^{(2)} \),

\[
R_n^{(1)} = \int_\alpha^{A_n} J_w(u)[F_n^{-1}(u) - \xi_\alpha] \, du - \int_{1-\beta}^{1-B_n} J_w(u)[F_n^{-1}(u) - \xi_{1-\beta}] \, du
\]

and

\[
R_n^{(2)} = \int_\alpha^{1-B_n} J(u)[F_n^{-1}(u) - F^{-1}(u)] \, du - \int_{1-\beta}^{1-B_n} J(u)[F_n^{-1}(u) - F^{-1}(u)] \, du.
\]

**Remark 2.1** It should be noted that the method based on the \( L \)-statistic approximation was first applied in Gribkova (2016); it can be viewed as a development of the approach proposed in Gribkova and Helmers (2006, 2007, 2014), where the second order asymptotic properties (Berry–Esseen bounds and one term Edgeworth type expansions) for (intermediate) trimmed means and their Studentized and
bootstrapped versions were established. In the articles mentioned we constructed
$U$-statistic type approximations for (intermediate) trimmed means, where we used
sums of auxiliary i.i.d. Winsorized observations as the linear terms; in order to get
the second (quadratic) $U$-statistic terms, we applied some special Bahadur–Kiefer
representations of von Mises statistic type for (intermediate) sample quantiles
(cf. Gribkova and Helmers (2012)).

3 Proofs

Proof of Theorem 1.1

Obviously it suffices to prove the first of relations (1.10). Set

$$V_n = L_n - L^0_n = n^{-1} \sum_{i=k_n+1}^{n-m_n} (c_{i,n} - c^0_{i,n}) X_{i:n}. \quad (3.1)$$

Lemma 2.1 and relation (3.1) yield

$$L_n - \mu_n = \tilde{L}_n - \mu_{\tilde{L}_n} + R_n + V_n. \quad (3.2)$$

An application of the classical Slutsky argument to (3.2) gives that, for $\delta > 0$, $1 - F_{L_n}(x)$ is bounded above and below by

$$\text{P}\{\sqrt{n}(\tilde{L}_n - \mu_{\tilde{L}_n})/\sigma > x - 2\delta\} + \text{P}\{\sqrt{n}|R_n|/\sigma > \delta\} + \text{P}\{\sqrt{n}|V_n|/\sigma > \delta\} \quad (3.3)$$

and

$$\text{P}\{\sqrt{n}(\tilde{L}_n - \mu_{\tilde{L}_n})/\sigma > x + 2\delta\} - \text{P}\{\sqrt{n}|R_n|/\sigma > \delta\} - \text{P}\{\sqrt{n}|V_n|/\sigma > \delta\} \quad (3.4)$$

respectively. Fix arbitrary $c > 0$ and $A > 0$. Set $\delta = \delta_n = (\log(n + 1))^{-1/2 - \varepsilon_1}$, where $0 < \varepsilon_1 < \varepsilon \land \bar{\varepsilon}$, and $\varepsilon$, $\bar{\varepsilon}$ are as in conditions (ii) and (iv) respectively (cf. (1.5)-(1.6)). From (3.3) and (3.4) it immediately follows that to prove our theorem it suffices to show that

$$\text{P}\{\sqrt{n}(\tilde{L}_n - \mu_{\tilde{L}_n})/\sigma > x \pm 2\delta_n\} = [1 - \Phi(x)](1 + o(1)), \quad (3.5)$$

$$\text{P}\{\sqrt{n}|R_n|/\sigma > \delta_n\} = [1 - \Phi(x)]o(1), \quad (3.6)$$

$$\text{P}\{\sqrt{n}|V_n|/\sigma > \delta_n\} = [1 - \Phi(x)]o(1), \quad (3.7)$$

uniformly in the range $-A \leq x \leq c\sqrt{\log n}$.

Let us prove (3.5). Observe that $\tilde{L}_n$ represents a non-truncated $L$-statistic
based on the sample $W_1, \ldots, W_n$ of i.i.d. bounded random variables; and
since its weight function $J_w$ is Lipschitz in $[0, 1]$, we can apply a result by
Vandemaele and Veraverbeke (1982). Set $\Delta = 2 \sup_{n \geq 1} \delta_n = 2/(\log 2)^{1/2 + \varepsilon_1}$. Since $E|W_i|^p < M < \infty$ (for each $p > 0$ with some $M > 0$), by Theorem 1 (i) of
Vandemaele and Veraverbeke (1982)

$$\text{P}\{\sqrt{n}(\tilde{L}_n - \mu_{\tilde{L}_n})/\sigma > x \pm 2\delta_n\} = [1 - \Phi(x \pm 2\delta_n)](1 + o(1)), \quad (3.8)$$
uniformly with respect to $x$ such that $-(A + \Delta) \leq x \leq c_1 \sqrt{\log n}$, where we may take $c_1 > c$. Hence (3.8) holds uniformly in the range $-A \leq x \leq c\sqrt{\log n}$ for all sufficiently large $n$. Further, we apply Lemma A1 of Vandemaele and Veraverbeke (1982) (in which the required asymptotic property of $\Phi$ is given in a very convenient form). Since $\delta_n \sqrt{\log n} = o(1)$, due to that lemma we obtain that $1 - \Phi(x \pm \delta_n) = |1 - \Phi(x)|(1 + o(1))$ uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. Summarizing, we find that (3.5) is valid, uniformly in the range required.

Let us prove (3.6). First, we argue similarly to the corresponding place in (Gribkova, 2016, Theorem 1.1). Let $I_1^{(j)}$ and $I_2^{(j)}$ denote the first and the second terms of $R_n^{(j)}$ (cf. (2.8)–(2.9)) respectively, $j = 1, 2$. Then $R_n = I_1^{(1)} - I_2^{(1)} + I_1^{(2)} - I_2^{(2)}$ and

$$P\{\sqrt{n}|R_n|/\sigma > \delta_n\} \leq \sum_{k=1}^{2} P\{\sqrt{n}|I_k^{(1)}|/\sigma > \delta_n/4\} + \sum_{k=1}^{2} P\{\sqrt{n}|I_k^{(2)}|/\sigma > \delta_n/4\}. $$

Notice that for $x \in [-A, c\sqrt{\log n}]$

$$\frac{1}{1 - \Phi(x)} \leq \frac{1}{1 - \Phi(c\sqrt{\log n})} \sim c\sqrt{\log n}n^{2/2}. $$

(3.10)

Hence, it suffices to show that for each positive $C$ (in particular for $C = \sigma/4$),

$$P\{\sqrt{n}|I_k^{(j)}| > C\delta_n\} = o\left((\log n)^{-1/2}n^{-2/2}\right), \ k, j = 1, 2, $$

(3.11)

as $n \to \infty$. We will prove (3.11) for $I_1^{(1)}$ and $I_1^{(2)}$ (the treatment of $I_2^{(1)}$ and $I_2^{(2)}$ is similar and therefore omitted).

Consider $I_1^{(1)}$. First, note that if $\alpha < A_n$, then $\max_{u \in (\alpha, A_n)}|F_n^{-1}(u) - \xi_\alpha| = \xi_\alpha - X_{[\alpha]n} + 1 > \xi_\alpha - X_{[\alpha]n}$, as $F_n^{-1}$ is monotonic. Here and in what follows $[x]$ represents the greatest integer function. Similarly we find that if $A_n \leq \alpha$, then $\max_{u \in (A_n, \alpha)}|F_n^{-1}(u) - \xi_\alpha| = X_{[\alpha]n} - \xi_\alpha$. Furthermore, by the Lipschitz condition for $J$, there exists a positive $K$ such that $\max_{u \in [0,1]} J_w(u) \leq \sup_{u \in I} J(u) \leq K$. This yields

$$|I_1^{(1)}| = \left|\int_{\alpha}^{A_n} J_w(u)[F_n^{-1}(u) - \xi_\alpha] \, du\right| \leq K|A_n - \alpha||X_{[\alpha]n} - \xi_\alpha|. $$

(3.12)

Let $U_1, \ldots, U_n$ be a sample of independent $(0,1)$-uniform distributed random variables, $U_{i:n}$ – the corresponding order statistics. Set $M_\alpha = \sharp\{i : U_i \leq \alpha\}$. Since the joint distribution of $X_{i:n}$ and $N_\alpha$ coincides with the joint distribution of $F^{-1}(U_{i:n})$ and $M_\alpha$, $i = 1, \ldots, n$, we have

$$P\{\sqrt{n}|I_1^{(1)}| > C\delta_n\} \leq P\{n^{-1/2}|M_\alpha - \alpha n||F^{-1}(U_{[\alpha]n}) - F^{-1}(\alpha)| > C\delta_n\} \leq P_1 + P_2, $$

(3.13)
where
\[
P_1 = \mathbb{P}\{ |M_\alpha - \alpha n| > c_1 \sqrt{n \log n} \},
\]
\[
P_2 = \mathbb{P}\{ |F^{-1}(U_{[n\alpha]}; n) - F^{-1}(\alpha)| > C_1(\log(n + 1))^{-(1+\varepsilon_1)} \},
\]
where we choose \(c_1 > c\). Here and in the sequel, \(C\) stands for a positive constant not depending on \(n\), which may change its value from line to line.

For \(P_1\) by Bernstein’s inequality we obtain
\[
P_1 \leq 2 \exp(-h_n),
\]
with \(h_n = \frac{c^2_1 \log n}{2(1+O(\sqrt{\log n}))} \sim \frac{c^2_1 \log n}{2}\). Hence \(P_1 = o((\log n)^{-1/2} n^{-c^2/2})\). Next we estimate \(P_2\) on the r.h.s. in (3.13). To shorten notation, let \(k_\alpha = \lceil n\alpha \rceil\), \(p_\alpha = \mathbb{E}U_{k_\alpha;n} = k_\alpha/(n+1)\), and note that \(0 < \alpha - p_\alpha \leq (\alpha + 1)/(n+1) = O(1/n)\). Define \(V_n(p_\alpha) = \sqrt{n}(U_{k_\alpha;n} - p_\alpha)\) and let \(E\) denote the event \(\{ |V_n(p_\alpha)| \leq c_1 \sqrt{p_\alpha \log n} \}\), where as before \(c_1\) is an arbitrary number such that \(c_1 > c\). Put \(M_n = |F^{-1}(p_\alpha + c_1 \sqrt{p_\alpha \log n/n}) - F^{-1}(p_\alpha)| \vee |F^{-1}(p_\alpha - c_1 \sqrt{p_\alpha \log n/n}) - F^{-1}(p_\alpha)|\). Then we have
\[
P_2 \leq \mathbb{P}\{ M_n > C_1(\log(n + 1))^{-(1+\varepsilon_1)} \} + \mathbb{P}\{ \bar{E} \}.
\]
By condition (ii), and because of \(\varepsilon_1 > \varepsilon\), the first probability on the r.h.s. in (3.15) is zero for all sufficiently large \(n\). In order to estimate the second probability on the r.h.s. in (3.15), we can apply Inequality 1 given in (Shorack and Wellner, 1986, page 453). Then we obtain
\[
\mathbb{P}\{ \bar{E} \} \leq \exp\left[ -c^2_1 \frac{\log n}{2} \right] + \exp\left[ -c^2_1 \frac{\log n}{2} \bar{\psi}(t_n) \right],
\]
where \(\bar{\psi}\) is the function defined in (Shorack and Wellner, 1986, page 453, formula (2)), \(t_n = c_1 \sqrt{\frac{\log n}{n}}\). Since \(t_n \to 0\) as \(n \to \infty\), hence \(t_n > -1\) for all sufficiently large \(n\), and by Proposition 1 in (Shorack and Wellner, 1986, page 455, relation (12)), we find that \(\bar{\psi}(t_n) \leq \frac{1}{1+2t_n/3}\). This and relation (3.16) together imply that
\[
\mathbb{P}\{ \bar{E} \} \leq 2 \exp\left[ -c^2_1 \frac{\log n}{2} \right] = 2n^{-c^2/2},
\]
for each \(c_1\) such that \(c < c_2 < c_1\) and for all sufficiently large \(n\). Summarizing, we get that \(P_2 = o((\log n)^{-1/2} n^{-c^2/2})\), and the desired bound (3.11) for \(I_1^{(1)}\) follows.

Next we prove (3.11) for \(I_1^{(2)}\). Define a sequence of intervals \(\Gamma_n = [\alpha \wedge \alpha_n, \alpha \vee \alpha_n + 1/n]\), then we obtain
\[
|I_1^{(2)}| = \left| \int_{\alpha_n}^{\alpha} J(u)[F_1^{-1}(u) - F^{-1}(u)] \, du \right| \leq K |\alpha_n - \alpha| D_n,
\]
where \(D_n = \max_{i/n \in \Gamma_n} |X_{i:n} - F^{-1}(i/n)| \vee |X_{i:n} - F^{-1}((i-1)/n)|\). By condition (iii), the estimate (3.18) implies that
\[
\mathbb{P}\{ \sqrt{n} |I_1^{(2)}| > C\delta_n \} \leq \mathbb{P}\{ D_{n,u} > C(\log(n + 1))^{-(1+\varepsilon_1)} \},
\]
for
Define $\mathbb{V}_n(p_i) = \sqrt{n}(U_{i:n} - p_i)$, where $p_i = \mathbb{E}U_{i:n} = i/(n+1)$, and let $\mathcal{E}_i$ denote the event $\{\sqrt{n}(p_i) \leq c_1\sqrt{p_i\log n}\}$, where now we take $c_1$ such that $c_1^2 > c^2 + 1/2$. Then by condition (ii), we find that

$$\Pr\{\sqrt{n}|I_1^{(2)}| > C\delta_n\} \leq \Pr\{\bigcup_{i: i/n \in \Gamma_n} \mathcal{E}_i\} \leq \sum_{i: i/n \in \Gamma_n} \Pr(\mathcal{E}_i). \tag{3.20}$$

Similarly as before, using Inequality 1 from Shorack and Wellner (1986), for each $i : i/n \in \Gamma_n$, we obtain that

$$\Pr(\mathcal{E}_i) \leq 2 \exp\left[-c_2^2 \log n \right] = 2n^{-c_2^2/2},$$

with some $c_2$ such that $c_1^2 > c_2^2 > c^2 + 1/2$, and since by condition (iii) $\{i : i/n \in \Gamma_n\} = O(\sqrt{n\log n})$, it follows from (3.20) that $\Pr\{\sqrt{n}|I_1^{(2)}| > C\delta_n\} = o\left((\log n)^{-1/2}n^{-c^2/2}\right)$. This completes the proof of (3.11), which implies that (3.6) holds true uniformly in the range $-A \leq x \leq c\sqrt{\log n}$.

Let us finally prove that (3.7) is valid uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. By condition (iv), there exists $b > 0$ such that

$$\sqrt{n}|V_n| \leq b(\log n)^{-1/2}\left(|X_{(k_{n+1})}\vert \vee |X_{(n-m_n)}|\right),$$

for all sufficiently large $n$. Thus,

$$\Pr\left(\sqrt{n}|V_n|/\sigma > \delta_n\right) \leq \Pr\left(|X_{(k_{n+1})}| \vee |X_{(n-m_n)}| > C(\log(n+1))^{\varepsilon-\varepsilon_1}\right) \leq \Pr_3 + \Pr_4,$$

where $\Pr_3 = \Pr\left(|X_{(k_{n+1})}| > C(\log(n+1))^{\varepsilon-\varepsilon_1}\right)$, $\Pr_4 = \Pr\left(|X_{(n-m_n)}| > C(\log(n+1))^{\varepsilon-\varepsilon_1}\right)$, and $\varepsilon - \varepsilon_1 > 0$ by the choice of $\varepsilon_1$. Let us estimate $\Pr_3$ (the treatment for $\Pr_4$ is same and therefore omitted). We have

$$\Pr_3 = \Pr\left(|F^{-1}(U_{(k_{n+1})})| > C(\log(n+1))^{\varepsilon-\varepsilon_1}\right) \leq \Pr\left(|F^{-1}(U_{(k_{n+1})}) - F^{-1}(\alpha)| + |F^{-1}(\alpha)| > C(\log(n+1))^{\varepsilon-\varepsilon_1}\right) = \Pr\left(|F^{-1}(U_{(k_{n+1})}) - F^{-1}(p_{\alpha_n})| > C(\log(n+1))^{\varepsilon-\varepsilon_1}(1 + o(1))\right), \tag{3.21}$$

where $p_{\alpha_n} = \mathbb{E}U_{(k_{n+1})}$. Arguing similarly as when estimating $\Pr_2$ (cf. (3.16)-(3.17)), we find that the r.h.s. of (3.21) is $o\left((\log n)^{-1/2}n^{-c^2/2}\right)$. This completes the proof of (3.7) and the theorem. □

**Proof of Theorem 1.2** Let us first prove relation (1.11). By Lemma 2.1 and relation (3.2), we have

$$\text{Var}(L_n) = \text{Var}(\bar{L}_n) + \text{Var}(R_n + V_n) + 2\text{cov}(\bar{L}_n, R_n + V_n).$$

Since $W_i$ are bounded, the conditions in (Vandamele and Veraverbeke, 1982, Theorem 2 (ii), page 431) are satisfied, and hence

$$\sigma^{-1}n^{1/2}\sqrt{\text{Var}(\bar{L}_n)} = 1 + O(n^{-1/2})$$
Further, we have
\[
n|\text{cov}(\tilde{L}_n, R_n + V_n)| \leq n[\text{Var}(\tilde{L}_n) \text{Var}(R_n + V_n)]^{1/2} = \sigma[n \text{Var}(R_n + V_n)]^{1/2}(1 + O(n^{-1/2})).
\]

The latter three relations imply that in order to prove (1.11), it suffices to show that
\[
n \text{Var}(R_n + V_n) = O((\log n)^{-1+2\nu}),
\]
where \( \nu = \varepsilon \wedge \bar{\varepsilon} \), and \( \varepsilon, \bar{\varepsilon} \) are the constants from conditions (ii) and (iv) respectively. We have
\[
n \text{Var}(R_n + V_n) \leq n \mathbb{E}(R_n + V_n)^2 \leq 5n \left[ \sum_{k,j=1}^{2} \mathbb{E}(I_k^{(j)})^2 + \mathbb{E}V_n^2 \right],
\]
where \( I_k^{(j)} \) are as in (3.9)-(3.11). We will show that
\[
n \mathbb{E}(I_1^{(j)})^2 = O((\log n)^{-2(1+\varepsilon)}), \quad n \mathbb{E}(I_2^{(j)})^2 = O((\log n)^{-1+2\varepsilon}), \quad j = 1, 2,
\]
and that
\[
n \mathbb{E}V_n^2 = O((\log n)^{-1+2\varepsilon}).
\]
Relations (3.23)-(3.25) imply the desired bound (3.22).

Let us prove the first relation in (3.24). We will consider in detail only the case \( k = 1 \) (the treatment in the case \( k = 2 \) is same and therefore omitted). Let as before \( k_\alpha = [\alpha n] \) and \( k_n = \alpha_n n \). By (3.12) and the Schwarz inequality, we have
\[
\mathbb{E}(I_1^{(1)})^2 \leq K^2[\mathbb{E}(A_n - \alpha)^4\mathbb{E}(X_{k_\alpha:n} - \xi_\alpha)^4]^{1/2} = K^2n^{-2}[\mathbb{E}(N_\alpha - \alpha n)^4\mathbb{E}(X_{k_\alpha:n} - \xi_\alpha)^4]^{1/2}.
\]

By well-known formula for 4-th moments of a binomial random variable, we have
\[
\mathbb{E}(N_\alpha - \alpha n)^4 = 3\alpha^2(1 - \alpha^2)n^2(1 + o(1)).
\]
Thus, there exists a positive constant \( C \) independent of \( n \) such that
\[
n \mathbb{E}(I_1^{(1)})^2 \leq C[\mathbb{E}(X_{k_\alpha:n} - \xi_\alpha)^4]^{1/2}
\]
for all sufficiently large \( n \). Fix arbitrary \( c > 0, A > 0 \). Let \( p_\alpha, \forall n(p_\alpha) \) and the event \( \mathcal{E} \) be as when estimating \( P_2 \) in the proof of Theorem 1.1 and \( c_1 \) is an arbitrary constant such that \( c_1 > c \). Then we can write
\[
\mathbb{E}(X_{k_\alpha:n} - \xi_\alpha)^4 = \mathbb{E}[(F^{-1}(U_{k_\alpha:n}) - F^{-1}(\alpha))^4\mathbb{1}_{\mathcal{E}}] + \mathbb{E}[(F^{-1}(U_{k_\alpha:n}) - F^{-1}(\alpha))^4\mathbb{1}_{\bar{\mathcal{E}}}] - \mathbb{E}[(F^{-1}(U_{k_\alpha:n}) - F^{-1}(\alpha))^4].
\]

By a well known property of the order statistics (see, e.g., (Gribkova, 1995, Theorem 1), and due to our moment assumption, \( \mathbb{E}|F^{-1}(U_{k_\alpha:n})|^k \) is bounded from above for each \( k > 0 \). Then by condition (ii), the latter quantity is of the order \( O((\log n)^{-4(1+\varepsilon)} + P(\bar{\mathcal{E}})) = O((\log n)^{-4(1+\varepsilon)}) \) (cf. (3.17)). This bound and (3.20) together imply that
\[
n \mathbb{E}(I_1^{(1)})^2 = O((\log n)^{-2(1+\varepsilon)}).
\]
Consider $I_1^{(2)}$. By condition (iii), there exists $L > 0$ such that $(\alpha_n - \alpha)^2 \leq L \log n/n$, for all sufficiently large $n$. Then in view of (3.18) we obtain

$$nE(I_1^{(2)})^2 \leq nK^2(\alpha_n - \alpha)^2ED_n^2 \leq LK^2 \log nED_n^2. \tag{3.27}$$

Hence, to get the second bound in (3.24), it suffices to show that

$$ED_n^2 = O((\log n)^{-2(1+\varepsilon)}), \tag{3.28}$$

and since $D_{n,u} = D_{n,u}$, it suffices to prove (3.28) for $D_{n,u}$. Let $p_i, V_n(p_i)$ and the event $E_i$ be as in the proof of Theorem 1.1 when estimating of $I_1^{(2)}$ (cf. (3.20)), where we now take $c_1$ such that $c_1^2 > c^2 + 1/2$. Let $1_{E_i}$ denote the indicator of the event $E_i$. Then we have

$$ED_{n,u}^2 = E(D_{n,u}^2 1_{\bigcup_i i/n \in \Gamma_n \cap E_i}) + E(D_{n,u}^2 1_{\bigcup_i i/n \in \Gamma_n \cap E_i^c}). \tag{3.29}$$

By condition (ii) the first term on the r.h.s. in (3.29) is $O((\log n)^{-2(1+\varepsilon)})$, and since $ED_{n,u}^2$ is bounded from above for each $k > 0$, there exists a positive constant $M$, not depending on $n$, such that for all sufficiently large $n$

$$E(D_{n,u}^2 1_{\bigcup_i i/n \in \Gamma_n \cap E_i}) \leq MP\{\bigcup_{i: i/n \in \Gamma_n} E_i\} \leq M \sum_{i: i/n \in \Gamma_n} P\{E_i\}. \tag{3.30}$$

Similarly to the proof of Theorem 1.1 (cf. (3.20)), we find that the magnitude on the r.h.s. in (3.30) is of the order $o((\log n)^{-1/2}n^{-c^2/2})$. Summarizing, we obtain the validity of the second relation in (3.24).

We now turn to the proof of (3.25). We have

$$nEV_n^2 \leq n^{-1}\left(\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c_{0,i,n}|\right)^2 E(X_{k_n+1:n}^2 \vee X_{n-m_n,n}^2).$$

Due to our moment assumption we have $E(X_{k_n+1:n}^2 \vee X_{n-m_n,n}^2) = O(1)$, and by condition (iv) we get $(\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c_{0,i,n}|)^2 = O(n(\log n)^{(1+c^2)}).$ These bounds and the latter displayed estimate yield (3.25). Thus, relation (1.11) is proved.

In order to complete the proof of our theorem, it remains to argue the possibility of the replacement $\sigma/n^{1/2}$ by $\sqrt{Var(L_n)}$ in (1.10) without affecting the result. We prove it for the first relation in (1.10), for the second one it will then follow from the first one if we replace $c_{i,n}$ by $-c_{i,n}$.

Fix arbitrary $c > 0$ and $A > 0$, set $\lambda_n = \sigma^{-1}n^{1/2}\sqrt{Var(L_n)}$ and write

$$\frac{P((L_n - \mu_n)/\sqrt{Var(L_n)}) > x}{1 - \Phi(x)} = \frac{1 - F_{L_n}(\lambda_n x)}{1 - \Phi(\lambda_n x)} \frac{1 - \Phi(\lambda_n x)}{1 - \Phi(\lambda_n x)}. \tag{3.31}$$

Set $B = A\sup_{n\in\mathbb{N}} \lambda_n$. Since $\lambda_n \to 1$, the number $B$ exists. Hence by Theorem 1.1 the first ratio on the r.h.s. in (3.31) tends to 1 as $n \to \infty$, uniformly in $x$ such that $-B \leq \lambda_n x \leq c\sqrt{\log n}$, hence in particular uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. Furthermore, we see that $|\lambda_n - 1|^{1/2} = O((\log n)^{-1/2+\nu})$. Hence, by Lemma A1 from Van der Maele and Veraverbeek (1982), the second ratio on the r.h.s. in (3.31) also tends to 1, uniformly in the range $-A \leq x \leq c\sqrt{\log n}$. The theorem is proved. $\Box$
References

Aleskevičiene, A. (1991). Large and moderate deviations for L-statistics. *Lithuanian Math. J.*, 33:145–156.

Bentkus, V. and Zitikis, R. (1990). Probabilities of large deviations for L-statistics. *Lithuanian Math. J.*, 30:215–222.

Boistard, H. (2007). Large deviations for L-statistics. *Statist. & Decis.*, 25:89–125.

Borovskikh, Y. and Weber, N. (2003). Large deviations of U-statistics. I-II. *Lithuanian Math. J.*, 43: 11–33, 241–261.

Callaert, H., Vandemaele, M., and Veraverbeke, N. (1982). A Cramér type large deviations theorem for trimmed linear combinations of order statistics. *Comm. Statist. Th. Meth.*, 11: 2689–2698.

Gao, F. and Zhao, X. (2011). Delta method in large deviations and moderate deviations for estimators. *Ann. Statist.*, 39: 1211–1240.

Gribkova, N. (1995). Bounds for absolute moments of order statistics. In (Skorokhod, A.V., Borovskikh, Yu.V. eds.) *Exploring Stochastic Laws: Festschrift in Honor of the 70th Birthday of Acad. V.S. Korolyuk*, pages 129–134. VSP, Utrecht. Available at arXiv:1607.08066v2[math.PR].

Gribkova, N. V. (2016). Cramér type large deviations for trimmed L-statistics. *Probab. Math. Statist.* (to appear). Available at arXiv:1507.02403[math.PR].

Gribkova, N. V. and Helmers, R. (2006). The empirical Edgeworth expansion for a Studentized trimmed mean. *Math. Methods Statist.*, 15(1): 61–87.

Gribkova, N. V. and Helmers, R. (2007). On the Edgeworth expansion and the M out of N bootstrap accuracy for a Studentized trimmed mean. *Math. Methods Statist.*, 16(2): 142–176.

Gribkova, N. V. and Helmers, R. (2012). On a Bahadur–Kiefer representation of von Mises statistic type for intermediate sample quantiles. *Probab. Math. Statist.*, 32(2): 255–279.

Gribkova, N. V. and Helmers, R. (2014). Second order approximations for slightly trimmed means. *Theory Probab. Appl.*, 58(3): 383–412.

Lai, T., Shao, Q., and Wang, Q. (2011). Cramér type moderate deviations for Studentized U-statistics. *ESAIM: Probability and Statistics*, 15: 168–179.

Mason, D. and Shorack, G. (1990). Necessary and sufficient conditions for asymptotic normality of trimmed L-statistics. *J. Statist. Plan. Inference*, 25: 111–139.

Petrov, V. V. (1975). *Sums of independent random variables*. Springer-Verlag, New York.
Saulis, L. and Statulevičius, V. (1991). *Limit theorems for large deviations*. Kluwer Academic Publishers, Dordrecht.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical processes with application in statistics*. Wiley, New York.

Stigler, S. M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.*, 2: 676–693.

van der Vaart, A. (1998). *Asymptotic statistics*. Cambridge Univ. Press, Cambridge.

Vandemaele, M. and Veraverbeke, N. (1982). Cramér type large deviations for linear combinations of order statistics. *Ann. Probab.*, 10: 423–434.