Two-Qubits Entanglement Induced by a Faster Data Bus

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We propose a theoretical protocol to create the entanglement of two qubits via the Born-Oppenheimer (BO) approximation. In our scheme, each qubit is coupled to a faster data bus whose frequency is much larger than the energy spacing of the qubits and thus the BO approximation is valid. Then the adiabatic separation of qubits from the data bus can induce an effective potential to couple the two qubits, which can be utilized to create a quantum logic gate. We also discuss the quantum decoherence caused by the adiabatic entanglement between the two qubits and the external field.

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I. INTRODUCTION

In quantum information theory, it is important to create a "maximum entanglement" between two qubits. It has been shown by Barenco et. al. \cite{1} that, together with single-bit operations, the entanglement based two qubit gates except the classical SWAP gate (including, the CNOT \cite{2} and controlled phase gates) forms a universal set of logic gates of quantum computation. In other words, any unitary transformation of single or many-qubit system can be decomposed into the quantum networks of single-bit gates and non-trivial two-qubit gates. In the last few years, many schemes to create two-qubit interaction have been suggested and experimentally implemented \cite{3-8}. In this paper, we will propose a new scenario to induce a kind of two qubit interaction creating efficient quantum entanglement. In our proposal, each qubit interacts with a common quantum field as a data bus. If the frequency of the quantum field is much larger than the energy spacing of the qubit, the degree of freedom of the quantum field and the variables of the two qubit system can be separated adiabatically. In this situation, an effective two-qubit interaction can be induced by means of the Born-Oppenheimer (BO) approximation. To explain the main idea in our proposal in details, we will first briefly review the relevant quantum adiabatic theorem and the BO approximation as follows.

Adiabatic approximation \cite{9} is widely used in many fields of physics including classical physics and quantum physics. In the latter, one has a rich applications of quantum adiabatic approximation (QAA). We consider a time-dependent quantum system. Its Hamiltonian $H(t) = H[R_i(t)]$ is controlled by some adiabatically changing parameters $R_i(t)$. If the system is initially prepared in the $n$-th instantaneous eigen-state of the Hamiltonian $H(0)$ at $t = 0$, then during the evolution the quantum system will still persist in the $n$-th eigen-state of the instantaneous Hamiltonian $H(t)$ at another instance. The evolution of the system under the control of $R_i(t)$s can be evaluated by QAA, but the back action of the quantum system on the controlling parameters $R_i(t)$ is neglected. Of course, the QAA can only work well when $R_i(t)$ behave classically.

When the Hamiltonian of the quantum system depends on the variables of another slowly varying quantum system, (e.g., the motion of an electron near nuclear), one needs a more exact approximation, in which the interactions between the fast system and the slow one, especially for the "back action" of the fast one on the slow one, are considered. To this end, in 1930 \cite{10}, Born and Oppenheimer suggested a high order approximation (we now call it Born-Oppenheimer (BO) approximation) to tackle this problem. In the view of the BO approximation, to solve the dynamics of a composite system with fast and slowly changing parts, we can first solve the Schrödinger equation of the fast part for each fixed slow variable $q$. Then the obtained $q$-dependent eigen-value $V_n(q)$ provides the slow part with an effective potential when the motion of the slow part can not excite the transitions among those fast states, i.e., the adiabatic condition holds. In this sense, the time dependent wave function of the composite system can be written as

$$
\Psi(q, t) = \psi_n(q, x) \phi(q, t).
$$

Here, $x$ is the variable of the fast part, $\psi_n(q, x)$ the $q$-dependent eigen-state of the fast part corresponding to eigenvalue $V_n(q)$; and the slow part $\phi(q, t)$ evolves governed by the effective Hamiltonian $\hat{H}_f + V_n(q)$, i.e. $\phi(q, t) = e^{-i[\hat{H}_f + V_n(q)]t}\phi(q, 0)$, where $\hat{H}_s$ is the free Hamiltonian of the slow part.

In this paper we make a crucial observation for the creation of the nontrivial two qubits interaction of logical gate in quantum information as a result from the application of the BO approximation to a three body system. According to above discussion, it is very interesting that the slow part can be divided into two subsystems with variables $q_1$ and $q_2$ without interaction between them. In this situation, the Hamiltonian of the composite system can be generally written as

$$
H = H_1(q_1) + H_2(q_2) + W_1(q_1, x) + W_2(q_2, x) + H_f(x)
$$

(2)
where $H_1(q_1)$, $H_2(q_2)$ and $H_f(x)$ are the free Hamiltonians of the two slow subsystems (we can note them as $q_1$, $q_2$) and the fast subsystem. $W_1(q_1, x)$ ($W_2(q_2, x)$) are the interaction Hamiltonians between $q_1$, $q_2$ and $x$. As we have emphasized, Eq. (2) implies that there is no direct interaction between $q_1$ and $q_2$. Under adiabatic condition, the eigenvalue $V_n(q_1, q_2)$ of the Hamiltonian $W_1(q_1, x) + W_2(q_2, x) + H_f(x)$ of fast part induces an effective potential on the slow variables $q_1$ and $q_2$. In usual case, $V_n(q_1, q_2)$ can not be decomposed into the sum of two independent potentials of $q_1$ and $q_2$. Therefore, an effective interaction between $q_1$ and $q_2$ is induced via the BO approximation and then one can regard the fast variable $x$ as a data bus which can be removed. Indeed, in quantum information process, the inter-qubit interaction is very crucial to create the nontrivial interaction between two qubits $q_1$ and $q_2$ if we can regard the above mentioned two subsystems of $q_1$ and $q_2$ as two qubits.

We should point out that, in many previous scenarios of quantum logical gate, an external field was usually used as a data bus where the quantum information carried by a qubit is stored and then transferred to another qubit. Most of those proposals rely on the rotation wave approximation (RWA). The effective interaction between the two qubit system can be obtained through the adiabatic elimination of the external field [8].

However, to adiabatically eliminate the degree of freedom of the external field, the coupling strength $g$ between the qubit and the external field ought to be weak enough so that the ratio $\frac{g}{\Delta}$ is very small. Here, $\Delta$ is the detuning i.e. $\Delta = |\omega_a - \omega|$ where $\omega_a$ is the transition frequency of the qubit and $\omega$ is the frequency of the external field. On the other hand, since the RWA requires $\Delta \leq \omega_a$, the above large detuning condition implies that $g << \omega_a$. Then the effective coupling strength between the two qubit system $\frac{g}{\Delta}$ is much smaller than $\omega_a$. This weak coupling may reduce the “quality factor” $Q$ of the logic gate. Here $Q$ can be understood as the ratio between the atomic life time and the logic operation time $\tau_1 \sim \left(\frac{2\Delta}{g^2}\right)^{-1}$ since the Rabi transition frequency can be defined as an upper limit of the operation frequency.

In our proposal based on the BO approximaiton, the frequency of the external field can be significantly larger than the energy spacing of the qubits. Then $g$ may be larger than $\omega_a$, or, at least, have the value in the same order as $\omega_a$. The ratio between effective interaction strength $\frac{g}{\Delta}$ and $\omega_a$ can be much stronger than that in the previous scenarios mentioned above. In the next section, there is a brief introduction of our quantum logic gate model. The precise analysis of our model based on the perturbation theory is given in section III. In section IV, the adiabatic decoherence effect is analyzed detailedly in association with the idea of the adiabatic entanglements presented recently by Sun et. al. [11]. We also obtain the condition under which the decoherence effect can be neglected and the two-qubit interaction model works well. There are some conclusions and discussions about the physical realization of our protocol in the last section.

II. EFFECTIVE INTERACTION BETWEEN TWO QUBITS

We generally consider a system consisting of two qubits with the same energy spacing $\omega_a$ and an external field which is described as a dynamic data bus. We can treat the external field as a harmonic oscillator with frequency $\omega$ (see Fig. 1).

The Hamiltonian of the composite system reads

$$H = \omega_a \left(\hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)}\right) + g \left(\hat{a}^\dagger + \hat{a}\right) \left(\hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)}\right) + \omega \hat{a}^\dagger \hat{a}$$

(3)

where $\hat{\sigma}_x^{(i)} = |e\rangle_i \langle e| - |g\rangle_i \langle g|$, $\hat{\sigma}_z^{(i)} = |e\rangle_i \langle g| + |g\rangle_i \langle e|$, $|e\rangle_i$ and $|g\rangle_i$ are the excited and ground states of the i-th qubit; $\hat{a}^\dagger$ and $\hat{a}$ the creation and annihilation operator of the external field, and $g$ the coupling strength between the qubits and this field. This model system can be realized in many practical physical situations such as the two trapped ion with the spatial oscillation mode, two atoms in a single mode cavity and the two Josephson charge qubits coupling to a large Josephson or nonamechanical resonator [12]. We assume $\omega$ is much larger than $\omega_a$. In this case, the RWA could not be applied since the condition for the validity of RWA is $|\omega + \omega_a| >> |\omega - \omega_a|$, and the usual adiabatic elimination technique does not work well for this case. We note that the above Hamiltonian depicts a typical spin-boson (continuous variable) system.

In the following parts of this paper, with the generalized BO approximation [11], and perturbation theory, we will prove that under the conditions $\omega >> \omega_a$ and $\omega >> g$, the evolution of the two qubit system is approximately governed by the effective Hamiltonian

$$\hat{h} = \omega_a \left(\hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)}\right) - 2\frac{g^2}{\omega} \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)}$$

(4)
and the decoherence effect caused by adiabatic entanglement [13] between the two qubits system and the data bus can be neglected.

Under the adiabatic condition that the external field does not excite the transitions among the internal states of qubits, the BO approximation determines a formally -factorized eigen-state of $H$ for the total composite system.

$$
\langle \lambda_1, \lambda_2 | \Psi \rangle = \phi_{n\alpha} \langle \lambda_1, \lambda_2 | n[\lambda_1, \lambda_2] \rangle \quad (5)
$$

where $| \lambda_1, \lambda_2 \rangle$ is the common eigen-state $| \lambda_1, \lambda_2 \rangle$ of $\{ \hat{\sigma}_x^{(1)}', \hat{\sigma}_x^{(2)}' \}$ with eigen-values $\{ \lambda_1, \lambda_2 = \pm 1 \}$ and $| n[\lambda_1, \lambda_2] \rangle$ the eigen-state of the effective Hamiltonian

$$
H_f[\lambda_1, \lambda_2] = \omega a \hat{a} + g (\hat{a}^+ + \hat{a}) (\lambda_1 + \lambda_2) \quad (6)
$$

of the external field for the given state $| \lambda_1, \lambda_2 \rangle$ of the two qubit system with corresponding eigen-value

$$
V_n(\lambda_1, \lambda_2) = n\omega - \frac{g^2}{\omega} (\lambda_1 + \lambda_2)^2 \quad (7)
$$

$$
= n\omega - 2\frac{g^2}{\omega} \lambda_1 \lambda_2.
$$

Here, we have neglected a constant term. For different $n$, $V_n(\lambda_1, \lambda_2)$ contribute different effective potentials for the slow part so that the wave function $\phi_{n\alpha}$ of the two qubit system is determined by

$$
\tilde{H}_{eff}(n) | \phi_{n\alpha} \rangle = E_{n\alpha} | \phi_{n\alpha} \rangle \quad (8)
$$

where the effective Hamiltonian is

$$
\tilde{H}_{eff}(n) \equiv \tilde{H}_{eff}(n; \hat{\sigma}_x^{(1)}, \hat{\sigma}_x^{(2)}) = \omega a \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} \right) + V_n \left( \hat{\sigma}_x^{(1)}', \hat{\sigma}_x^{(2)}' \right) = \hat{h} + n\omega
$$

where $\hat{h}$ defined in Eq. (4) is the $n$-independent part of the effective Hamiltonian.

So far, we have formally obtained a $X - X$-type of interbit coupling based on the BO approximation, which can be conveniently used to formulate an efficient scheme for quantum-computing. Recently, a similar interaction generating the entanglement of two Josephson junction charge qubits was proposed by Averin [14]. In ref. [14], two Josephson junction charge qubits are assumed to be coupled with another junction which works as a faster data bus. With the BO approximation, the energy of the lowest band of the latter junction can be considered as the effective interaction between the two Josephson junction charge qubits.

In the next section, we give a detailed derivation of the effective Hamiltonian (9) based on the perturbation theory. The decoherence of the quantum state of the two qubit system induced by the adiabatic entanglement with the external field is analyzed in section IV.

III. GENERALIZED BORN-OPPENHEIMER APPROXIMATION FOR DISCRETE SYSTEM

In the above section, we obtain the effective Hamiltonian (4) for two qubit system through a intuitional analysis from the BO approximation directly. However, the BO approximation is usually used to separate two continuous quantum systems (e.g. a melocauer and a electron) or a continuous system and a discrete one (e.g. the spatial motion and spin of a neutral particle [13]), rather than the two discrete subsystems. In this section, however, we can generalize the BO approximaiton to deal with the separation of two discrete quantum systems, i.e. the two qubit system and the external field. We now can give a rigorous proof to the intuitional argument in the above section based on the perturbation theory. In this way, we obtain the effective Hamiltonian (9) naturally.

A. Matrix Representation of Motion Equation

In section II, we define $| n(\lambda_1, \lambda_2) \rangle$ as the eigen-state of the Hamiltonian $H_f[\lambda_1, \lambda_2]$ of the external field and $| \lambda_1, \lambda_2 \rangle$ the eigen-states of $\{ \hat{\sigma}_x^{(1)}, \hat{\sigma}_x^{(2)} \}$. With the completeness relation [11]

$$
\sum_{\lambda_1, \lambda_2} \sum_n | n(\lambda_1, \lambda_2) \rangle \langle n(\lambda_1, \lambda_2) | \otimes | \lambda_1, \lambda_2 \rangle \langle \lambda_1, \lambda_2 | = 1,
$$

the quantum state of the composite system can be expanded as

$$
| \Psi \rangle = \sum_{\{ n, \lambda_1, \lambda_2 \}} C_{\lambda_1, \lambda_2} (n) | n | \lambda_1, \lambda_2 \rangle \quad (11)
$$

where

$$
| n | \lambda_1, \lambda_2 \rangle \equiv | n(\lambda_1, \lambda_2) \rangle | \lambda_1, \lambda_2 \rangle.
$$

Substituting Eq. (11) into the Schrödinger equation $H | \Psi \rangle = E | \Psi \rangle$, we obtain the effective equations of the coefficients $C_{\lambda_1, \lambda_2} (n)$:

$$
\sum_{\lambda_1', \lambda_2'} H_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n) C_{\lambda_1', \lambda_2'} (n) + \sum_{\lambda_1', \lambda_2'} F_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n) C_{\lambda_1', \lambda_2'} (n) \quad (12)
$$

$$
+ \sum_{\lambda_1, \lambda_2, m \neq n} O_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n, m) C_{\lambda_1', \lambda_2'} (m) = EC_{\lambda_1, \lambda_2} (n)
$$

where we have defined some complicated notations:

$$
H_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n) = \langle \lambda_1, \lambda_2 | \tilde{H}_{eff}(n) | \lambda_1', \lambda_2' \rangle \quad (13)
$$

$$
F_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n) = \langle \lambda_1, \lambda_2 | \omega a \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} \right) | \lambda_1', \lambda_2' \rangle 
$$

$$
\times \left[ 1 - \langle n(\lambda_1, \lambda_2) | n(\lambda_1', \lambda_2') \rangle \right]
$$

$$
O_{\lambda_1', \lambda_2'}^{\lambda_1, \lambda_2} (n, m) = \langle \lambda_1, \lambda_2 | a \hat{a} \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} \right) \right| \langle \lambda_1', \lambda_2' | a \hat{a} \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} \right) \rangle
$$

$$
| \lambda_1, \lambda_2 \rangle.
$$
\[
O^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n, m) = \langle \lambda_1, \lambda_2 | \omega_a \left( \hat{C}^{(1)} + \hat{C}^{(2)} \right) | \lambda_1', \lambda_2' \rangle \times \langle n(\lambda_1, \lambda_2) | m(\lambda_1', \lambda_2') \rangle.
\] (15)

Obviously, only when \( F^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) \) and \( O^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n, m) \) are negligible, can we obtain the effective Hamiltonian \( H_{eff} (n) \). With a straightforward calculation (see the Appendix A), \( \langle n(\lambda_1, \lambda_2) | m(\lambda_1', \lambda_2') \rangle \), \( F^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) \) and \( O^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n, m) \) can be expressed as power series of the parameter \( g \):

\[
\langle m(\lambda_1, \lambda_2) | n(\lambda_1', \lambda_2') \rangle = \exp \left[ -\frac{1}{2} \frac{g}{\omega} \sum^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} \right] \times \sum_{i=0}^{m} \frac{(-1)^{n-m} \left( \frac{g}{\omega} \sum^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} \right)^{2l+(n-m)}}{l! (l+n-m)!} \times \sqrt{n(n-1) \cdots (m-l+1) / m(m-1) \cdots (m-l+1)},
\] (16)

\[
F^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) = \omega_a \left[ (n+1) \frac{g^2}{\omega^2} \sum^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} + \cdots \right],
\] (17)

\[
O^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n, m) = \omega_a \left[ -\sqrt{n} \frac{g}{\omega} \sum^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} \delta_{n,m+1} + \cdots \right]
\] (18)

where we have assumed \( n - m \geq 0 \) and \( \sum^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} = \lambda_1 + \lambda_2 - \lambda_1' - \lambda_2' \). According to Eq. (17), \( F^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) \) can be neglected if \( n \) and \( g \) are small enough. Considering the condition \( \omega_a << \omega \) mentioned in the above section, in the following discussion, we assume \( n \ll 1 \), \( g << \omega \) and \( \omega_a << \omega \). For simplicity, we also assume \( g/\omega \) and \( \omega_a/\omega \) is a small parameter.

In order to solve Eq. (12) with the standard perturbation theory and thus give a proof for the generalized BO approximation for the discrete case, we rewrite Eq. (12) in matrix-value form as

\[
\hat{H} C = EC
\] (19)

where the total Hamiltonian \( \hat{H} \) and the eigen-vector \( C \) are defined as

\[
\hat{H} = \begin{bmatrix}
\hat{H}(0) + \hat{F}(0) & \hat{O}(0,1) \\
\hat{O}(1,0) & \hat{H}(1) + \hat{F}(1) 
\end{bmatrix},
\]  
\( C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \).

Here, the vector \( C \) contains two sub-vectors

\( C_i = \begin{pmatrix} C_{11} (i), C_{1-1} (i), C_{-11} (i), C_{-1-1} (i) \end{pmatrix}^T, i = 0, 1. \)

Each of them is a 4-dimension vector; \( \hat{H}(i), \hat{F}(i) \) and \( \hat{O}(i,j) (i, j = 0, 1) \) are 4 x 4 matrices whose elements are \( H^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (i), F^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (i) \) and \( O^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (i, j) \).

Making use of Eq. (17) and Eq. (18), we express the matrix \( \hat{H} \) as a power series of \( g/\omega \):

\[
\hat{H} \simeq \hat{H}(0) + \frac{g}{\omega} \hat{H}(1) + \left( \frac{g}{\omega} \right)^2 \left( \hat{V}(2) + \hat{O}(2) \right)
\] (20)

where

\[
\hat{H}(0) = \omega \begin{bmatrix} 0, 0 \\
0, 1 \end{bmatrix},
\] (21)

\[
\hat{H}(1) = \omega \begin{bmatrix}
\frac{\omega_a}{g} \left( \hat{C}^{(1)} + \hat{C}^{(2)} \right), & 0 \\
0, & \frac{\omega_a}{g} \left( \hat{C}^{(1)} + \hat{C}^{(2)} \right) 
\end{bmatrix},
\] (22)

\[
\hat{V}(2) = \omega \begin{bmatrix}
\hat{C}^{(1)} \hat{C}^{(2)}, & 0 \\
0, & \hat{C}^{(1)} \hat{C}^{(2)} 
\end{bmatrix},
\] (23)

and

\[
\hat{O}(2) = -\omega \begin{bmatrix}
0, & \frac{\omega_a}{g} \Sigma \\
\frac{\omega_a}{g} \Sigma, & 0 
\end{bmatrix}.
\] (24)

Here, \( I \) is the unit 4 x 4 matrix and \( \Sigma \) a 4 x 4 matrix whose elements are \( \Sigma^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} \). Only the terms of zeroth, first and second order in \( g/\omega \) are considered in Eq. (20). The terms of higher order are neglected. According to Eq. (17), \( \hat{F}(i) (i = 0, 1) \) does not contain the terms of zeroth, first and second order. Therefore, \( \hat{F}(i) \) are neglected in Eq. (20).

**B. Perturbation Solution of Motion Equation**

To apply the perturbation theory, we split \( \hat{H} \) into two parts:

\[
\hat{H} = \hat{H}_0 + \hat{W}
\] (25)

where

\[
\hat{H}_0 = \hat{H}(0) + \frac{g}{\omega} \hat{H}(1)
\] (26)

is treated as the unperturbed Hamiltonian and

\[
\hat{W} = \left( \frac{g}{\omega} \right)^2 \left( \hat{V}(2) + \hat{O}(2) \right)
\] (27)

as the perturbation term.

It is pointed out that we can also choose \( \hat{H}(0) \) as the unperturbed Hamiltonian, \( \frac{g}{\omega} \hat{H}(1) \) the first order perturbation term and \( \left( \frac{g}{\omega} \right)^2 \left( \hat{V}(2) + \hat{O}(2) \right) \) the second order one. It is easy to prove that, if \( \frac{g}{\omega} \hat{H}(1) \) is regarded as the first order perturbation Hamiltonian, the eigenenergy and eigenstate to the first order given by the perturbation theory is just the same as the ones we obtain by diagonalizing the matrix \( \hat{H}(0) + \frac{g}{\omega} \hat{H}(1) \) exactly. Therefore, our final result is not relevant to the choice of unperturbed Hamiltonian and perturbation terms.
The eigen-energy and eigen-state of unperturbed Hamiltonian $\hat{H}_0$ in Eq. (26) are easy to obtain. The eigen-energy can be expressed as:

$$E_{n}^{(0)} = \varepsilon_i + n\omega$$  \hspace{1cm} (28)

where $n = 0, 1$ and $i = 0, +1, -1$. Here, we have defined

$$\varepsilon_0 = 0, \quad \varepsilon_{\pm 1} = \pm 2\omega_a.$$  \hspace{1cm} (29)

The energy level structure of the composite system is illustrated in Fig. 2. Since $\omega_a$ is much smaller than $\omega$, it follows from Eq. (28) and Eq. (29) that $E_{n_1}^{(0)} - E_{n_0}^{(0)} \sim 0$. Noting $|E_{n_1}^{(0)} - E_{n_0}^{(0)}| = |\varepsilon_i - \varepsilon_j| \sim \omega_a$, we have $|E_{n_1}^{(0)} - E_{n_0}^{(0)}| \ll |E_{1i}^{(0)} - E_{0j}^{(0)}|$. Therefore, the energy spectrum of Hamiltonian $\hat{H}^{(0)}$ possesses a quasi-band structure: it consists of two quasi-bands corresponding to $n = 0$ and $1$.

\[ \begin{array}{c}
\varepsilon_i - \varepsilon_j \sim \omega_a \\
E_{n_1}^{(0)} - E_{n_0}^{(0)} \sim 0 \\
n = 1 \\
n = 0
\end{array} \]

Fig. 2. The energy levels of $\hat{H}^{(0)}$. Since $\omega > \omega_a$, we have $E_{n_1}^{(0)} - E_{n_0}^{(0)} > \varepsilon_i - \varepsilon_j$. Therefore, there are two subbands with respect to $n = 0$ and $n = 1$.

The eigen-states corresponding to $E_{n\pm}$

$$\left\langle E_{n\pm}^{(0)} \right| = (C_{\pm}, 0)^T, \quad \left\langle E_{n\pm}^{(0)} \right| = (0, C_{\pm})^T$$

(30)

can be written in terms of the unit vectors

$$C_+ = (1, 0, 0, 0)^T, \quad C_- = (0, 0, 0, 1)^T.$$  

It is obvious from Eq. (21) and Eq. (22) that the eigen-energies $E_{n0}^{(0)}$ are two-fold degenerate, which corresponds to the two degenerate eigen-vectors

$$\left| E_{n0; k} \right| = (C_k, 0)^T, \quad \left| E_{n0; k} \right| = (0, C_k)^T.$$  \hspace{1cm} (31)

where $k = 1, 2$ and

$$C_1 = (0, 1, 0, 0)^T, \quad C_2 = (0, 0, 1, 0)^T.$$  

By making use of the perturbation theory, the eigen-energies and eigen-states to the first order can be written as

$$E_{ni} = E_{n}^{(0)} + E_{n}^{(1)}$$

and

$$\left\| E_{n\pm 1} \right\| = \left\| E_{n}^{(0)} \right\| + \left\| E_{n}^{(1)} \right\|.$$  

According to the perturbation theory for degenerate levels, the zero-th order eigen-states $\left| E_{n0; k} \right|$ are linear combinations of $\left| E_{n0; 1} \right|$ and $\left| E_{n0; 2} \right|$. For any $k$ and $k'$, $\left\langle E_{n0; 1} \right| \hat{O}^{(2)} \left| E_{n0; 2} \right| = 0$, $\left| E_{n0; k} \right|$ can be obtained by diagonalizing the $2 \times 2$ matrices

$$\left[ \begin{array}{ccc}
\left\langle E_{n0; 1} \right| \hat{V}^{(2)} \left| E_{n0; 1} \right| , & \left\langle E_{n0; 1} \right| \hat{V}^{(2)} \left| E_{n0; 2} \right| , & \left\langle E_{n0; 2} \right| \hat{V}^{(2)} \left| E_{n0; 2} \right|
\end{array} \right].$$  \hspace{1cm} (32)

The first order corrections to the eigen-energies $E_{ni}$ and eigen-states can be calculated directly. Since we have $\left\langle E_{n\pm 1} \right| \hat{O}^{(2)} \left| E_{n\pm 1} \right| = 0$ for every given $n$, it is obviously that the contributions from $\hat{O}^{(2)}$ to the correction of the eigen-energies are zero. On another hand, the first order eigen-vector $\left| E_{n}^{(1)} \right|$ can be expressed as:

$$\left\| E_{n}^{(1)} \right\| = \left( \frac{g}{\omega} \right)^2 \left\langle E_{n-1}^{(0)} \right| \hat{V}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n-1}^{(1)} \right\|$$

$$+ \left( \frac{g}{\omega} \right)^2 \sum_{k=1,2} \frac{\left\langle E_{n0; k}^{(0)} \right| \hat{V}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n0; k}^{(1)} \right\|}{\varepsilon_i - \varepsilon_j}$$

$$+ \left( \frac{g}{\omega} \right)^2 \sum_{j=\pm 1} (-1)^{j+i} \frac{\left\langle E_{n-1; j}^{(0)} \right| \hat{O}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n-1; j}^{(1)} \right\|}{\varepsilon_i - \varepsilon_j}$$

$$+ \left( \frac{g}{\omega} \right)^2 \sum_{k=1,2} (-1)^{n+1} \frac{\left\langle E_{n-1; 0; k}^{(0)} \right| \hat{O}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n-1; 0; k}^{(1)} \right\|}{\varepsilon_i - \varepsilon_j}.$$  \hspace{1cm} (33)

The another eigenstate $\left| E_{n0; k}^{(1)} \right|$ have the similar expression and thus for simplicity, we need not give it's explicit expression here.

Substituting Eq. (29-32), into Eq. (33), we can calculate the contribution from $\hat{O}^{(2)}$ and $\hat{V}^{(2)}$ to the first order correction of eigen-state. It is obvious that

$$\left( \frac{g}{\omega} \right)^2 \left\langle E_{n-1}^{(0)} \right| \hat{V}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n-1}^{(1)} \right\|$$

$$\left( \frac{g}{\omega} \right)^2 \left\langle E_{n0; k}^{(0)} \right| \hat{V}^{(2)} \left| E_{n}^{(0)} \right| \left\| E_{n0; k}^{(1)} \right\|.$$

5
\[ \sim \left( \frac{g}{\omega} \right)^2 \left| \begin{array}{cc} \langle E^{(0)}_{n0,k} | \bar{V}^{(2)} | E^{(0)}_{ni} \rangle \\ \varepsilon_i \end{array} \right| \]

\[ \sim \left( \frac{g}{\omega} \right)^2 \frac{\omega}{\omega_a} \sim \frac{g}{\omega} \] (34)

and

\[ \left( \frac{g}{\omega} \right)^2 \left| \begin{array}{cc} \langle E^{(0)}_{|n-1|,k} | \bar{O}^{(2)} | E^{(0)}_{ni} \rangle \\ (-1)^{n+1} \omega + \varepsilon_i - \varepsilon_k \end{array} \right| \]

\[ \sim \left( \frac{g}{\omega} \right)^2 \left| \begin{array}{cc} \langle E^{(0)}_{|n-1|,0,k} | \bar{O}^{(2)} | E^{(0)}_{ni} \rangle \\ (-1)^{n+1} \omega + \varepsilon_i \end{array} \right| \]

\[ \sim \left( \frac{g}{\omega} \right)^2 \] (35)

where we have used \( \frac{\omega}{\omega_a} \sim 1 \). Therefore, it can be seen from Eq. (33) that the contribution from \( \bar{O}^{(2)} \) to \( \| E^{(1)}_{ni} \| \) and \( \| E^{(1)}_{n0,k} \| \) are much smaller than the ones from \( \bar{V}^{(2)} \).

In summary, we have shown that one can reasonably neglect the first order corrections from \( \bar{O}^{(2)} \) to both the eigen-values and the eigen-states.

C. The Generalized BO Approximation From Perturbation Theory

The above result can be obtained with intuitive analysis. It can be seen from Eq. (24) and Eq. (23) that the matrix \( \bar{V}^{(2)} \) leads to the transitions in the same energy band and \( \bar{O}^{(2)} \) leads to the interband transitions (see Fig. 3).

![Fig. 3. The transitions induced by \( \bar{W} \). We have shown that, \( \left( \frac{g}{\omega} \right)^2 \bar{O}^{(2)} \) only induces the transitions between subbands and \( \left( \frac{g}{\omega} \right)^2 \bar{V}^{(2)} \) only induces the transitions in the same sub-band.](image)

As we have mentioned, the energy gap between the two bands has the same order of \( \omega \) and is much larger than the energy difference between the levels in the same band.

The latter have the same order of \( \omega_z \). Then the probabilities of the interband transitions are much smaller than the ones of the transitions in the same energy band.

With the above discussions, the Hamiltonian matrix \( \bar{H} \) can be written as

\[ \bar{H} \approx \bar{H}^{(0)} + \frac{g}{\omega} \bar{H}^{(1)} + \left( \frac{g}{\omega} \right)^2 \bar{V}^{(2)} \] (36)

Which is just the effective Hamiltonian for the BO approximation. In fact, substituting Hamiltonian Eqs. (23-21) into Eq. (36), we can write the eigen equation (19) as

\[ \left[ \begin{array}{cc} \omega_a \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) + \frac{\omega}{\omega_a} \omega_z \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) + \frac{\omega^2}{\omega_a} \sigma_x^{(2)} \end{array} \right] C = EC. \]

This equation equals to

\[ \sum_{\lambda_1, \lambda_2} H^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) C_{\lambda_1', \lambda_2'} (n) = EC_{\lambda_1, \lambda_2} (n) \] (37)

which is approximated form of Eq. (12). It can be seen from Eq. (13) that \( H^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) \) are the matrix elements of the effective Hamiltonian \( \bar{H}_{eff} (n) \) in Eq. (9). It is pointed out that, Eq. (37) is equivalent with Eq. (5) and Eq. (8) in the above section.

To solve the time-dependent Schrödinger equation which describe the time evolution of the composite system, we can calculate the approximate eigen-energies \( E \) of the Hamiltonian of the composite system with Eq. (9):

\[ E_{ni} = \varepsilon_i + n \omega \] (38)

where \( \varepsilon_i \) is the \( i \)-th eigen-energy of the Hamiltonian \( \hat{H} \) defined in Eq. (4). Eq. (13) is the time-independent Schrödinger equation under the BO approximation. The time-dependent Schrödinger equation can be written as:

\[ i \frac{d}{dt} C_{\lambda_1, \lambda_2} (n) = \sum_{\lambda_1', \lambda_2'} H^{\lambda_1, \lambda_2}_{\lambda_1', \lambda_2'} (n) C_{\lambda_1', \lambda_2'} (n). \] (39)

Making use of the above BO approximation for discrete system, we can calculate the evolution of the composite system with Eq. (39). If the system is initially prepared in the state

\[ |\Psi (0)\rangle = \sum_{n, (\lambda_1, \lambda_2)} C_{\lambda_1, \lambda_2} (n, 0) |n, [\lambda_1, \lambda_2]\rangle, \]

then with Eq. (38), Eq. (9) and Eq. (4), the quantum state at any instance can be expressed as:

\[ |\Psi (t)\rangle = \sum_{n, (\lambda_1, \lambda_2)} C_{\lambda_1, \lambda_2} (n, t) e^{-i \omega t} |n, [\lambda_1, \lambda_2]\rangle \]

\[ = \sum_{n, (\lambda_1, \lambda_2)} C_{\lambda_1, \lambda_2} (n, t) e^{-i \omega t} |\lambda_1, \lambda_2\rangle \otimes |n, (\lambda_1, \lambda_2)\rangle \] (40)
where
\[
C_{\lambda_1, \lambda_2} (n, t) = \sum_{\lambda'_1, \lambda'_2} \langle \lambda_1, \lambda_2 | e^{-iht} | \lambda'_1, \lambda'_2 \rangle C_{\lambda'_1, \lambda'_2} (n, 0).
\]

Apparently, the evolution of the two qubit system is governed by the Hamiltonian \( \hat{h} \) in Eq. (4) which contains the two qubit interaction \(-2 \beta \sum_{x} \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)} \). However, it can be seen from Eq. (40) that because of the dependence of \(|n(\lambda_1, \lambda_2)\rangle\) on \{\(\lambda_1, \lambda_2\)\}, the instantaneous state \(|\psi (t)\rangle\) of the composite system is usually an entangle state of the two qubit system and the external field. Thus, if one trace over the variable of the data bus, then the state of the two qubit system is not a pure sate but a mixed one. This adiabatic decoherence effect will be analyzed in next section.

\[\text{IV. DECOHERENCE EFFECT FROM THE ADIABATIC ENTANGLEMENT}\]

In the above section, we have obtained an approximate Hamiltonian \( \hat{H}_e(q) \equiv \hat{h} + n \omega t \) for the composite system with two qubits interacting with an external field. As we have mentioned, there is quantum entanglement between the states of the external field and those of the two qubit system because of the dependence of \(|n(\lambda_1, \lambda_2)\rangle\) on \{\(\lambda_1, \lambda_2\)\}. This is just the adiabatic quantum entanglement which has been studied in detail for the development of BO approximation [11]. It is easy to see that the adiabatic entanglement may cause the adiabatic decoherence of the quantum state of the two qubit system. Only when this decoherence effect can be neglected, can our effective Hamiltonian \( \hat{h} \) be used to create a two bit gate. In this section, the adiabatic decoherence effect in our problem will be analyzed in detail. It will be shown that, in some practical cases, this kind of decoherence effect is negligible and then the faster quantum entanglement can be created via the BO approximation.

We assume the two qubit system and the external field are initially prepared in a factorizable state:
\[|\Psi (0)\rangle = |\psi \rangle \otimes |\phi \rangle\]
where \(|\psi \rangle\) is the initial state of the two qubits and \(|\phi \rangle\) the state of the external field. \(|\Psi (0)\rangle\) can be expanded with respect to the basis \(|n(\lambda_1, \lambda_2)\rangle\) as
\[|\Psi (0)\rangle = \sum_{n, (\lambda_1, \lambda_2)} C_{\lambda_1, \lambda_2} (n, 0) \otimes |n(\lambda_1, \lambda_2)\rangle\]
or equivalently in the \(|\lambda_1, \lambda_2\rangle\) picture
\[\langle \lambda_1, \lambda_2 | \Psi (0)\rangle = \sum_{n} C_{\lambda_1, \lambda_2} (n, 0) \otimes |n(\lambda_1, \lambda_2)\rangle\]

where
\[C_{\lambda_1, \lambda_2} (n, t) = \langle \lambda_1, \lambda_2 | \psi \rangle \cdot \langle n(\lambda_1, \lambda_2) | \phi \rangle.\]

is the projection of initial state onto the completeness basis \(|n(\lambda_1, \lambda_2)\rangle\). With the BO approximation generalized in the above section, the evolution of the composite system is depicted by Eq. (40).

Substituting Eq. (42) into Eq. (40), we have the wave function in the \(|\lambda_1, \lambda_2\rangle\) picture at time \(t\).
\[\langle \lambda_1, \lambda_2 | \Psi (t)\rangle = \sum_{\lambda'_1, \lambda'_2} \langle \lambda_1, \lambda_2 | e^{-iht} | \lambda'_1, \lambda'_2 \rangle \langle \lambda'_1, \lambda'_2 | \psi \rangle \otimes |\phi (t)\rangle\]

where
\[|\phi (t)\rangle = |\phi (t, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2)\rangle = \sum_n e^{-i\omega t} |n(\lambda'_1, \lambda'_2) \rangle |n(\lambda_1, \lambda_2)\rangle\]
depends on both coordinate sets \(|\lambda_1, \lambda_2\rangle\) and \(|\lambda'_1, \lambda'_2\rangle\). It should be noted that the factor \(|n(\lambda'_1, \lambda'_2) \rangle \langle n(\lambda_1, \lambda_2)\rangle\) has different pairs of indices \(|\lambda_1, \lambda_2\rangle\) and \(|\lambda'_1, \lambda'_2\rangle\). If \(|\phi (t)\rangle = |\phi (t, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2)\rangle\) were independent of \(\lambda_1, \lambda_2\) and \(\lambda'_1, \lambda'_2\), then we would have a factorized evolution described by \(|\Psi (t)\rangle = e^{-iht} |\psi \rangle \otimes |\phi (t)\rangle\). In this case, the two qubit system were just in the pure state controlled by the effective Hamiltonian \(\hat{h}\) and the decoherence phenomenon does not occur.

In the usual case, \(|\phi (t)\rangle = |\phi (t, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2)\rangle\) does depend on \(\lambda_1, \lambda_2, \lambda'_1, \lambda'_2\). Then the decoherence effect must result from the dependence \(|\phi (t)\rangle\) to different sets of \(|\lambda_1, \lambda_2, \lambda'_1, \lambda'_2\rangle\). In fact, we can calculate the elements of the reduced density matrix of the two qubit system
\[\rho_{\lambda_1, \lambda_2, \lambda'_1, \lambda'_2} (t) = \sum_{\mu_1, \mu_2, \mu'_1, \mu'_2} K_{\lambda_1, \lambda_2; \mu_1, \mu_2} [|\psi \rangle, t] K_{\lambda'_1, \lambda'_2; \mu'_1, \mu'_2}^* [|\psi \rangle, t] \times \langle \phi (t, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2) \rangle \langle \phi (t, \lambda_1, \lambda_2, \mu_1, \mu_2) \rangle\]

where the propagation functionals
\[K_{\lambda_1, \lambda_2; \mu_1, \mu_2} [|\psi \rangle, t] = \langle \lambda_1, \lambda_2 | e^{-iht} | \mu_1, \mu_2 \rangle \langle \mu_1, \mu_2 | \psi \rangle.\]

depend on the initial state \(|\psi \rangle\) of the external field. On the other hand, without decoherence, the reduced density matrix of the qubits is \(\rho' = e^{-iht} |\psi \rangle \langle \psi | e^{iht}\) with matrix elements
\[\rho_{\lambda_1, \lambda_2, \lambda'_1, \lambda'_2} (t) = \sum_{\mu_1, \mu_2, \mu'_1, \mu'_2} K_{\lambda_1, \lambda_2; \mu_1, \mu_2} [|\psi \rangle, t] K_{\lambda'_1, \lambda'_2; \mu'_1, \mu'_2}^* [|\psi \rangle, t]\]

depend on the initial state \(|\psi \rangle\) of the external field.
Comparing Eq. (46) with Eq. (44), we measure the extent of decoherence by the so called decoherence factor [15].

\[ F \equiv \langle \phi (t; \lambda_1', \lambda_2', \mu_1', \mu_2') | \phi (t, \lambda_1, \lambda_2; \mu_1, \mu_2) \rangle \quad (47) \]

As pointed in ref. [15], if \( F = 1 \), there is no decoherence. It should be pointed out that, not only the norm of \( F \) but also its phase are crucially important to the description of quantum decoherence process. For example, if \( F = \exp [i \Theta] \) and \( \Theta \) is a random phase with respect to the parameters \( \lambda_1, \lambda_2, \lambda_1', \lambda_2'; \mu_1', \mu_2', \mu_1, \mu_2, \) even if \( |F| = 1 \), the pure parameters \( F \) depends on the parameters and its phase are crucially important to the description of quantum decoherence process. For example, if \( F = \exp [i \Theta] \) and \( \Theta \) is a random phase with respect to the parameters \( \lambda_1, \lambda_2, \lambda_1', \lambda_2'; \mu_1', \mu_2', \mu_1, \mu_2, \) even if \( |F| = 1 \), \( \rho_{\lambda_1, \lambda_2, \lambda_1', \lambda_2'} (t) \) still have significant difference from the pure \( \rho_{\lambda_1, \lambda_2, \lambda_1', \lambda_2'} (t) \).

We assume the initial state of the external field is in a coherent state, \( |\phi \rangle = |\alpha \rangle \) (noting the condition \( n \leq 1 \) in the above section, we assume \( |\alpha| < 0.5 \) here). By a straightforward calculation, we obtain

\[ |\phi (t, \lambda_1, \lambda_2, \lambda_1', \lambda_2') \rangle = e^{i \Phi (t, \lambda_1, \lambda_2, \lambda_1', \lambda_2')} \times \left[ |\alpha + \frac{g}{\omega} (\lambda_1' + \lambda_2') \rangle e^{-i \omega t} - \frac{g}{\omega} (\lambda_1 + \lambda_2) \right] \quad (48) \]

where the time-varying phase

\[ \Phi = \frac{g}{\omega} (\lambda_1' + \lambda_2'), \quad \Im \alpha^* \]

\[ - \left[ \frac{g}{\omega} (\lambda_1 + \lambda_2) \right] \Im (\alpha^* e^{i \omega t}) \]

\[ - \left[ \left( \frac{g}{\omega} \right)^2 (\lambda_1' + \lambda_2') (\lambda_1 + \lambda_2) \right] \sin \omega t \quad (49) \]

depends on the parameters \( \lambda_1, \lambda_2, \lambda_1' \), and \( \lambda_2' \) and \( \left[ |\alpha + \frac{g}{\omega} (\lambda_1' + \lambda_2') \rangle e^{-i \omega t} - \frac{g}{\omega} (\lambda_1 + \lambda_2) \right] \) is also a coherent state.

Considering the inner product of coherent state satisfies

\[ \langle \alpha | \alpha + p \rangle = \exp \left[ - |p|^2 \right] \exp [i \Re (p \alpha^*)] \quad (50) \]

we can express the decoherence factor

\[ F = \exp [i \Omega] \cdot \exp \left[ - |\Upsilon|^2 \right] \quad (51) \]

in terms of the phase factor \( \Omega \) and its norm \( \exp \left[ - |\Upsilon|^2 \right] \).

Here, the explicit expression of \( \Omega \) and \( \Upsilon \) can be obtained:

\[ \Omega = 2 \left[ \frac{g}{\omega} (\mu_1 + \mu_2 - \mu_1' - \mu_2') \right] \Im \alpha^* \]

\[ + 2 \left[ \frac{g}{\omega} (\lambda_1' + \lambda_2' - \lambda_1 - \lambda_2) \right] \Im (\alpha^* e^{i \omega t}) \]

\[ + \left( \frac{g}{\omega} \right)^2 (\mu_1 + \mu_2 - \mu_1' - \mu_2') \]

\[ \times (\lambda_1' + \lambda_2' + \lambda_1 + \lambda_2) \sin \omega t \quad (52) \]

and

\[ \Upsilon = \left[ \frac{g}{\omega} (\mu_1 + \mu_2 - \mu_1' - \mu_2') \right] e^{-i \omega t} \]

\[ - \frac{g}{\omega} (\lambda_1 + \lambda_2 - \lambda_1' - \lambda_2') \quad (53) \]

However, in the case with

\[ |\Omega| << 1, \quad |\Upsilon|^2 << 1 \quad (54) \]

we have \( \langle \phi (t; \lambda_1', \lambda_2'; \mu_1', \mu_2') | \phi (t, \lambda_1, \lambda_2; \mu_1, \mu_2) \rangle \simeq 1 \) and the decoherence effect can be neglected.

It is observed from the explicit expression for the decoherence factor that \( \Omega \) and \( \Upsilon \) are all periodic functions of \( t \) with frequency \( \omega \). The amplitudes of \( \Omega \) and \( \Upsilon \) do not vary with time (see Fig. 4). Therefore, in case the amplitudes of \( \Omega \) and \( \Upsilon \) are much smaller than 1, the inequalities in Eq. (54) can be satisfied at any instance.

![FIG. 4. The real part of the decoherence factor ReF as a function of t. The unit of t is $\omega^{-1}$. We assume $\alpha = 0$ and $\mu_1 + \mu_2 - \mu_1' - \mu_2' = \lambda_1 + \lambda_2 - \lambda_1' - \lambda_2' = 1$. The ratio $\frac{g}{\omega}$ is denoted as k. It can be seen from the figure that the decoherence factor is a periodic function of time.](image)

Because every \( \lambda_1 \) or \( \lambda_2 \) may be $\pm 1$, the sufficient condition of \( |\Upsilon|^2 << 1 \) is just \( \left| \frac{g}{\omega} \right|^2 << 1 \) and if \( \frac{g}{\omega} \) is small enough so that this condition can be satisfied, then the last term of the expression of \( \Omega \) can be neglected and the sufficient condition of \( |\Omega| << 1 \) becomes

\[ 10 \frac{g}{\omega} |\alpha| << 1 \quad (55) \]

This is a restriction for the strength of the external field. Therefore, we know that if the ratio \( \frac{g}{\omega} \) is small enough (for example, \( \frac{g}{\omega} \sim 0.01 \)) and the external field is weak enough (e.g. \( |\alpha| << 1 \)), the atomic state decoherence caused by the external field can be omitted and the two qubit system can evolve under the control of the effective Hamiltonian $\hat{h}$.

**V. CONCLUSION WITH REMARKS**

In this paper, we have proposed a new protocol for the creation of quantum entanglement between two qubits
interacting with a high frequency data bus. Unlike the previous schemes, our protocol does not require the rotation wave approximation for the interaction Hamiltonian. We can first assume the external field is prepared in the coherence state $|\alpha\rangle$. If the frequency of the external field is large enough and the strength of it is weak enough so that the following conditions

$$\omega_n \sim g << \omega, \quad 64 \frac{g^2}{\omega^2} << 1, \quad |\alpha| < 0.5, \quad \frac{16g}{\omega} |\alpha| << 1$$

(56)
can be satisfied, an effective coupling between the two qubit system can be obtained under the BO approximation and the decoherence caused by the cavity field can be neglected.

The explicit expression of the Hamiltonian of the composite system is given in Eq. (3). If the conditions by Eq. (56) are satisfied, the evolution of the two qubit system is governed by the effective Hamiltonian

$$\hat{h} = \omega_n \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} \right) - 2\frac{g^2}{\omega} \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)}. \quad (57)$$

In our previous calculation, we have assumed $\omega_n \sim g << \omega$. This is the sufficient condition to obtain the effective Hamiltonian (57). It is easy to prove that if $\omega_n$ is smaller than $g$ and $g << \omega$, the effective Hamiltonian (57) can be also obtained with the deduction in the above sections. To realize a logic gate with high efficiency, we suppose our proposal work under the condition $\omega_n << g$ and $g << \omega$ so that the interaction strength $2\frac{g^2}{\omega}$ is comparable with the strength $\omega_n$ of the free Hamiltonian.

As we have shown, the decoherence effect is determined by the factor $\Omega$ and $|\gamma|^2$ which are periodic functions of time. Then if the conditions in Eq. (56) are satisfied, the decoherence effect can be neglected no matter how long the operation takes.

The scheme proposed in this paper can be generalized to the system that $N$ qubits coupled to a data bus. With the same discussions as above, it is apparently that in the large detuning and weak coupling case where $\omega_n << \omega$, $g << \omega$, and the number of the qubits is not very large so that the condition $2N\omega_n << \omega$ is satisfied, an effective coupling between the qubits

$$V \sim \frac{g^2}{\omega} \left( \hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)} + ... + \hat{\sigma}_x^{(N)} \right) \sim 2\frac{g^2}{\omega} \hat{J}_x$$

can be obtained under the BO approximation. It is pointed out that with the “$J_x^2$ interaction” between $N$ qubits, the $N$-qubit GHZ state can be created easily (16). Then a physical qubit of the form $\alpha |0\rangle + \beta |1\rangle$ can be encoded into a logical qubit of the form $\alpha |000...\rangle + \beta |111...\rangle \ (17)$. It is also proved that with the interaction of this type, Shor code for error correction can be easily realized.

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APPENDIX A: THE CALCULATION OF THE OVERLAP FACTOR $\langle M (\lambda_1, \lambda_2) | N (\lambda_1, \lambda_2) \rangle$

In this appendix, we calculate the explicit expression of $\langle m (\lambda_1, \lambda_2) | n (\lambda_1', \lambda_2') \rangle$. It can be seen from Eq. (6) that the Hamiltonian $H_f (\lambda_1, \lambda_2)$ is just the Hamiltonian of a displaced harmonic oscillator and can be expressed as

$$H_f (\lambda_1, \lambda_2) \simeq \omega \hat{A}_1^\dagger \hat{A}_1 + \omega \hat{A}_2^\dagger \hat{A}_2 - 2\frac{g^2}{\omega} \lambda_1 \lambda_2 \quad (A1)$$

where the constant term independent on $\lambda_1, \lambda_2$ is omitted. Here,

$$\hat{A}_1, \hat{A}_2 = \hat{a} + \frac{g}{\omega} (\lambda_1 + \lambda_2)$$

is a displaced boson operator that satisfies $[\hat{A}_1, \hat{A}_2^\dagger, \hat{A}_1^\dagger] = 1$. With Eq. (A1), the eigen-state of $H_f (\lambda_1, \lambda_2)$ can be written as:

$$|n(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{n!}} \hat{A}_1^n |0 (\lambda_1, \lambda_2)\rangle = D \left( -\frac{g}{\omega} (\lambda_1 + \lambda_2) \right) |n\rangle \quad (A2)$$

in terms of the displacement operator $D \left( -\frac{g}{\omega} (\lambda_1 + \lambda_2) \right)$ and the natural Fock state $|n\rangle = \frac{1}{\sqrt{n!}} \hat{a}^n |0\rangle$. Here, $|0\rangle$ is the realistic vacuum state satisfies $\hat{a} |0\rangle = 0$ and $|0 (\lambda_1, \lambda_2)\rangle$ the displaced vacuum state satisfies $\hat{A}_1, \hat{A}_2 |0 (\lambda_1, \lambda_2)\rangle = 0$. It is easy to prove that $|0 (\lambda_1, \lambda_2)\rangle$ is actually a coherence state which can be defined as

$$|0 (\lambda_1, \lambda_2)\rangle = D \left( -\frac{g}{\omega} (\lambda_1 + \lambda_2) \right) |0\rangle.$$

With Eq. (A2), the explicit value of $\langle m (\lambda_1, \lambda_2) | n (\lambda_1', \lambda_2') \rangle$ can be obtained easily:

$$\langle m (\lambda_1, \lambda_2) | n (\lambda_1', \lambda_2') \rangle = \langle m | D \left[ \frac{g}{\omega} \sum_{l=1}^{\lambda_1} \hat{a}^l \right] D \left[ -\frac{g}{\omega} \sum_{l=1}^{\lambda_1'} \hat{a}^l \right] | n \rangle$$

$$= e^{-\frac{g^2}{\omega^2} \sum_{l=1}^{\lambda_1} \hat{a}^l \hat{a}^{l+\lambda_2} + \frac{g^2}{\omega^2} \sum_{l=1}^{\lambda_1'} \hat{a}^l \hat{a}^{l+\lambda_2'}} \langle m | e^{-\sum_{l=1}^{\lambda_1} \hat{a}^l \hat{a}^{l+\lambda_2}} e^{-\sum_{l=1}^{\lambda_1'} \hat{a}^l \hat{a}^{l+\lambda_2'}} | n \rangle$$

$$= e^{-\frac{g^2}{\omega^2} \sum_{l=1}^{\lambda_1} \hat{a}^l \hat{a}^{l+\lambda_2}} \sum_{m=0}^{n} (-1)^{n-m} \frac{\left( \frac{g}{\omega} \sum_{l=1}^{\lambda_1} \hat{a}^l \right) !}{m! (l + n - m)!} \langle m \rangle \langle m (\lambda_1, \lambda_2) | n \rangle$$

$$\sqrt{n (n - 1) ... (m - l + 1)} \sqrt{m (m - 1) ... (m - l + 1)}.$$

This is just Eq. (16) in section III. Substituting Eq. (16) into Eq. (14) and Eq. (15), we can get the expression of $F_{\lambda_1, \lambda_2}^\dagger (n)$ and $O_{\lambda_1, \lambda_2}^\dagger (n)$ i.e. Eq. (17) and Eq. (18).
[1] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
[2] A. Barenco, D. Deutsch, and A. Ekert, Phys. Rev. Lett. 74, 4083 (1995).
[3] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. 75, 4710 (1995); A. rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 83, 5166 (1999); S. Osnaghi, P. Bertet, A. Auffeves, P. Maioli, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 87, 037902 (2001).
[4] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. 75, 4714 (1995).
[5] N. A. Gershenfeld and I. L. Chuand, Science 275, 350 (1997); D. G. Cory, A. F. Fahmy, and T. F. Havel, Proc. Natl. Acad. Sci. U.S.A. 94, 1634 (1997); J. A. Jones, M. Mosca, and R. H. Hansen, Nature (London) 393, 344 (1998).
[6] Yu. A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D. V. Averin, and J. S. Tsai, Nature 421, 823 (2003). A. J. Berkley,* H. Xu, R. C. Ramos, M. A. Gubrud, F. W. Strauch, P. R. Johnson, J. R. Anderson, A. J. Dragt, C. J. Lobb, F. C. Wellstood, Science, 300, 1548.
[7] P. Domokos, J. M. Raimond, M. Brune, and S. Haroche, Phys. Rev. A 52, 3554 (1995); T. Pellizzari, S. Gardiner, J. I. Cirac, P. Zoller, Phys. Rev. Lett. 75, 3788 (1995); L. You, X. X. Yi, and X. H. Su, Phys. Rev. A 67, 032308 (2003); X. X. Yi, X. H. Su and L. You, Phys. Rev. Lett 90, 097902 (2003).
[8] S. B. Zheng and G. C. Guo, Phys. Rev. Lett. 85, 2392 (2000).
[9] A. Messiah Quantum Mechanics vol 2 (Amsterdam: North-Holland).
[10] M. Born, R. Oppenheimer, Ann, Physik 84, 457 (1930).
[11] C. P. Sun, X. F. Liu, D. L. Zhou and S. X. Yu, Phys. Rev. A 63, 062111 (2000); C. P. Sun, D. L. Zhou, S. Y. Yu and X. F. Liu, Eur. Phys. D, 13, 145 (2001).
[12] Xue Ming Henry Huang, Christian A. Zorman, Mehran Mehregany, and M. L. Roukes, Nature (London) 421, 496 (2003).
[13] C.P.Sun, M.L.Ge, Phys.Rev.D, 41, 1349, 1990.
[14] D. V. Averin and C. Bruder, Phys. Rev. Lett. 91, 057003(2003).
[15] C. P. Sun, H. Zhan, X. F. Liu, Phys. Rev. 58, 1810(1998).
[16] K. Molmer and A. Sorensen, Phys. Rev. Lett. 82, 1835(1999).
[17] B. Zeng, D. L. Zhou, C. P. Sun and L. You, private communication.