Bounded states for breathers–soliton and breathers of sine–Gordon equation

Man Jia

1 Introduction

The sine–Gordon (sG) equation first arose in the theory of differential geometry and later was used to describe the charged particles in electromagnetic field with characteristics coordinates [1,2]. Also, sG equation arises widely in many physical systems, such as the kinks bounded by fermions [3], the one-dimensional domain wall motion in biaxial ferromagnets [4], the propagation of magnetic flux in Josephson junction [5,6], the problem of observer-based boundary control [7] and the quantum field theories [8]. Due to its wide applications, the sG equation has attracted great interests in both physics and mathematics. It has been proved the sG equation possesses Hamiltonian formalism, the bilinear form, Darboux transformations and Bäcklund transformations, Lax pairs, etc. The sG equation is found to possess not only the kink solutions and breather solutions, but also multiply periodic wave solutions by means of the new deformation relations [9].

Recently, the studies on soliton molecules have been a popular topic in nonlinear systems. SMs are considered as a special coherent states or bounded states for multiple soliton solitons. Many theories on SMs and BMs have been presented, and some experiments have been conducted to search for the special states, especially in nonlinear optics. Some experiments have successfully observed the special states. It is particular that the BSM is also reported theoretically in the mKdV-sG system describing few-optical-cycle solitons [10].
under some approximation. The kink bounded states by fermions have been found in $\phi^4$, $\phi^6$ and $sG$ models [3]. Actually, this static multiple soliton bound states are just the coherent state, or bounded state for the soliton solutions. Because the $sG$ equation possesses abundant kinks (solitons) and breathers structures, it is possible for us to search for more coherent and bounded state of the solitons. Furthermore, the findings on coherent and bounded states of the solitons may help to search for new physics theoretically.

Studies [11] show that $sG$ equation is a special case for the mKdV-$sG$ model describing few-optical-cycle solitons. As the mKdV-$sG$ model possesses the bounded states for solitons and breather, it is interesting to search for the similar bounded states for solitons and breathers.

In this manuscript, we first revisit the Wronskian solutions to the $sG$ equation and then present a novel expression for the multiple soliton solution to the $sG$ equation. The novel multiple soliton solution is different from other results because it has a nonzero background term. The background term has been proved to play an important role in some nonlinear phenomenon [12,13]. Then, to generate the bounded states for solitons and breather, we introduce some velocity resonant conditions. The $sG$ equation is found to possess the two bounded states, the bounded state for breathers–soliton, named as breather–soliton molecules (BSMs), and the bounded state for breathers, named as breather molecules (BMs). Though $sG$ equation can be considered as a special case of mKdV-$sG$ equation, the results are different from the mKdV-$sG$ equation that possesses bounded states for soliton molecules (BMs), BMs and BSMs. In addition, an approximate bounded state for solitons is also demonstrated if the parameters are nearly the same. Further studies on the interactions among BSMs, BMs, solitons and breathers show the nonelastic properties for the interactions.

2 Wronskian solutions to $sG$ equation

The Lagrangian density defined by

$$L = \frac{1}{2} \partial_\nu \phi \partial^\nu \psi - U(\phi),$$

(1)

where $\phi$ is a real scalar field and $U(\phi)$ is a real non-negative function of $\phi$ admits many classical models by selecting different potential forms. For the particular potential form $U(\phi) = \mu + \nu \phi^2 + \lambda \phi^4$, the Euler–Lagrange field equation that follows from Eq. (1) is the $\phi^4$ equation which has been proved to possess the kink solution [14], the periodic wave solution and the soliton solution [15]. For the form of $U(\phi) = \mu \phi^2 + \nu \phi^4 + \lambda \phi^6$, the Euler–Lagrange field equation from Eq. (1) will give the $\phi^6$ equation. In [16], the periodic–periodic interaction solutions and periodic–solitary wave interaction solutions of the $\phi^6$ equation have been constructed by using deformation relations among the solutions of the $\phi^6$ and $\phi^4$ equation. Furthermore, with an infinite number of degenerate vacua $\psi_0 = 2n\pi$, $n \in \mathbb{Z}$ for the potential $U(\psi) = 1 - \cos \phi$, the Euler–Lagrange field equation that follows from Eq. (1) is the famous nonlinear $sG$ equation [3,14,17,18]

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = \partial_\mu \partial^\mu \phi + \sin(\phi) = 0,$$

(2)

which is generally written in the form

$$u_{1T} = \sin u.$$  

(3)

The multiply periodic, quasi-periodic and nonperiodic waves solutions to the $sG$ equation have been obtained in [9].

Mathematicians [10,19,20] have connected the $N$-soliton solutions of nonlinear evolution systems obtained from inverse scattering theory to Wronskian forms. They believe the Wronskian forms are one of the most effective methods to search for the interactions of various types of solutions.

It is known the $N$th-order Wronskian is defined as a $N \times N$ determinant

$$W(\phi_1, \phi_2, \cdots, \phi_N) = | \phi_2, \phi^{(1)}_1, \cdots, \phi^{(N)}_1 | \cdots | \phi_N, \phi^{(1)}_1, \cdots, \phi^{(N)}_1 |$$

(4)

where $\phi$ is named the entry vector of the Wronskian with $\phi = (\phi_1, \phi_2, \cdots, \phi_N)^T$ and $\phi^{(m)} = \frac{\partial^m \phi}{\partial x^m}$. For simplicity, the Wronskian Eq. (4) is generally rewritten in the compact form [20]

$$W(\phi) = | \phi, \phi^{(1)}, \cdots, \phi^{(N)} |$$

(5)

where $N - m$ denotes the set of consecutive columns $0, 1, \cdots, N - m$. It is well known that the transformation [21]

$$u = 2i \ln \frac{f^*}{f},$$

(6)
where \( f^* \) means the complex conjugate of \( f \) transforms the sG equation Eq. (3) into the bilinear form

\[
(D_x D_t) f \cdot f^* - \frac{1}{2} (f^2 - f^{*2}) = 0.
\]

Here, \( D_t \) and \( D_x \) are the famous bilinear operators first introduced by Hirota [22] with the definitions being

\[
D^m_x D^n_t f(t, x) \cdot g(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} f(t + s, x + y) \times g(t - s, x - y) |_{s=0, y=0}.
\]

Equation (7) is completely equivalent to the sG equation, so once \( f \) is known, the solution to sG equation is obtained by the transformation Eq. (6). According to [10], the solution \( f \) takes the following Wronskian form

\[
f = W(\phi) = |N - 1|,
\]

where the entry vector \( \phi \) is determined by

\[
\phi_x = B \phi^*, \quad \phi_0 = \frac{1}{4} \partial^{-1} \phi,
\]

with \( B \) being an \( N \)-th-order nonsingular complex constant matrix and \( |B| \in \mathbb{R} \) and \( \partial^{-1} \partial = \partial \partial^{-1} = 1 \).

Upon different choice of \( B \), some exact solutions of the sG equation Eq. (3) can be constructed. Here, we just show some special known results to the sG equation Eq. (3) according to the complex constant matrix \( B \).

(1). Soliton solution

The soliton solution of the sG equation can be obtained by selecting

\[
B = \text{Diag}(k_1, k_2, \ldots, k_N),
\]

where \( k_j \) is real and \( |k_i| \neq |k_j| (i \neq j) \), with the solution \( \phi = (\phi_1, \phi_2, \ldots, \phi_N)^T \) to Eq. (9) being expressed by

\[
\phi_j = a_j e^{i\eta_j} + a_j^* e^{-i\eta_j}, \quad \eta_j = k_j x - \frac{t}{k_j} + \eta_j^{(0)},
\]

where \( a_j^+, a_j^-, k_j \) and \( \eta_j^{(0)} \) are all arbitrary real constants. Particularly, if \( a_j^+ = (-1)^{j-1}, a_j^- = -1 \), the Wronskian solution \( f \) is simplified to the following form [10]

\[
f = \sum_{\nu=0,1} \exp \left[ \sum_{j=1}^{N} v_j \left( \eta_j + i \frac{\pi}{2} \right) + \sum_{1 \leq j < l} v_j v_l A_{jl} \right],
\]

where the sum over \( \nu \) = 0, 1 refers to each of \( \nu = 0, 1 \) for \( j = 1, 2, \ldots, N \) and

\[
e^{A_{jl}} = \left( \frac{k_l - k_j}{k_l + k_j} \right)^2.
\]

(2). Breather solution

When the \( N \)-th-order nonsingular complex constant matrix \( B \) is written as

\[
B = \text{Diag}(\theta_1, \theta_2, \ldots, \theta_N)_{2N}, \quad \theta_j = \left( \begin{array}{c} 0 \\ k_j^* \\ 0 \end{array} \right),
\]

with \( k_j \in \mathbb{C} \) and \( k_j^* \) being the complex conjugate of \( k_j \), the Wronskian entry vector \( \phi = (\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \phi_{N1}, \ldots, \phi_{N2})^T \) has the solution

\[
\phi_{jj} = a_j e^{i\eta_j} + b_j e^{-i\eta_j},
\]

\[
\phi_{2j} = a_j^* e^{i\eta_j} - b_j^* e^{-i\eta_j},
\]

\[
\eta_j = k_j x - \frac{t}{k_j} + \eta_j^{(0)},
\]

where \( a_j, b_j, \eta_j^{(0)} \) are all complex. The structure of \( B \) and \( \theta_j \) for breathers in Eq. (14) was early found in [19] and [24]. From Eqs. (8), (14)–(15), the breathers solution to sG equation is thus constructed.

Furthermore, the limit soliton solutions and limit solution of breathers can also be obtained by different \( N \)-th-order nonsingular complex constant matrix of \( B \). Based on the eigenfunctions of the matrix \( A = BB^* \), a full classification for the solutions to the mKdV equation has been done [19].

3 A Novel soliton solution to sG equation

In this section, we present a novel expression for soliton solution to sG equation by utilizing the expression Eq. (12). The \( N \)-soliton solutions to sG equation Eq. (3) on a nonzero background are given by

\[
u = N\pi + 2i \left[ \ln \sum_{j=1}^{N} K_j \cosh(\frac{1}{2} \sum_{j=1}^{N} v_j \eta_j^{(0)}) \right] / \sum_{j=1}^{N} K_j \cosh(\frac{1}{2} \sum_{j=1}^{N} v_j \eta_j^{(0)})
\]

\[
= N\pi + 4 \arctan \left[ \frac{\sum_{j=1}^{N} K_j \sinh(\frac{1}{2} \sum_{j=1}^{N} v_j \eta_j)}{\sum_{j=1}^{N} K_j \cosh(\frac{1}{2} \sum_{j=1}^{N} v_j \eta_j)} \right],
\]

\[
\eta_j^{(0)} = k_j x + \frac{t}{k_j} + \eta_{j0} \pm \frac{\pi}{2} \equiv \eta_j \pm \frac{\pi}{2},
\]

\( \odot \) Springer
where the summation of \( v \) should be done to all the permutations of \( \{v_1, v_2, \ldots \} \) for \( v = 1, -1 \), the summation of \( v_e \) means all nondual even permutations of \( \{v_1, v_2, \ldots v_N\} \) for \( v = 1, -1 \) and the summation of \( v_o \) means all nondual odd permutations of \( \{v_1, v_2, \ldots v_N\} \) for \( v = 1, -1 \), with
\[
K_v \equiv \prod_{i>j}(k_i - v_i v_j k_j), \quad (17)
\]
\( \eta_{j0} \) being an arbitrary real constant denoting the initial positions of every solitons.

One can directly check the results by substituting the solution Eq. (16) into the sG equation Eq. (3) by using the expression Eq. (17).

With Eq. (16), the one soliton solution to sG equation is written down
\[
u = \pi + 4 \arctan \left[ \frac{\tanh \left( \frac{k_1 x + t/k_1 + \eta_{10}}{2} \right)}{2} \right], \quad (18)
\]
which is a kink wave or an anti-kink wave (actually, a soliton by \( u_x \)) for different selection of \( k_1 \) with the propagation speed provided by
\[
u_s = \frac{1}{k_1^2}. \quad (19)
\]
The character of the kink wave can be made by examining the asymptotic behavior of \( u \) as \( t \to \infty \) [25]. Here, we do not show it in detail.

For \( N = 2 \), the soliton solution is expressed as
\[
u = 2\pi + 4 \arctan \left[ \frac{(k_2 - k_1) \sinh \frac{\eta_{11} + \eta_{21}}{2}}{(k_2 + k_1) \cosh \frac{\eta_{11} - \eta_{21}}{2}} \right], \quad (20)
\]
with \( \eta_j = k_j x + \frac{1}{k_j} + \eta_{j0}, j = 1, 2 \). This solution Eq. (20) describes the interactions among kink (or anti-kink) and kink (or anti-kink).

The breather solution is obtained from the two-soliton solution Eq. (20) directly by selecting \( k_1 = k_2^* = \lambda_1 + i\mu_1 \) and \( \eta_{10} = \eta_{20} = x_1 + i t_1 \),
\[
u = 2\pi + 4 \arctan \left[ \frac{\lambda_1 \sin \left( \mu_1 x - \frac{\mu_1 t}{k_1^2 + \mu_1^2} + t_1 \right)}{\mu_1 \cosh \left( \lambda_1 x + \frac{\lambda_1 t}{k_1^2 + \mu_1^2} + x_1 \right)} \right]. \quad (21)
\]
The breather solution Eq. (21) shows an oscillating periodically localized wave propagating along a straight line with the speed
\[
u_B = \frac{1}{\lambda_1^2 + \mu_1^2}. \quad (22)
\]

The behavior of the breather has also been studied in [25] by using the dressing method. A sketch of breather is exhibited in Fig. 1 with the parameters selected as \( \lambda_1 = \mu_1 = 1, x_0 = t_0 = 0 \).

It can be concluded that there are abundant different solution structures according to the choice of the parameters \( k_i \). For example, for \( N = 3 \) in Eq. (16), if \( k_i \in \mathbb{R}, i = 1, 2, 3 \), the solution denotes three solitons interactions by \( u_x \). If any two \( k_i \) is complex, the solution \( u_x \) stands for the interactions between a soliton and breathers. The density plot for the interaction between a soliton and a breather is given in Fig. 2 where the parameters are \( k_1 = k_2^* = 4/5 + 4i/5, k_3 = 3/5, \eta_{10} = \eta_{20} = \eta_{30} = 0 \) in Eq. (16).

It should be pointed out that \( u_x \) is shown because \( u_x \) stands for different physical meaning. For example, \( u_x \) is the electric field intensity when the sG equation is used to describe the ultrashort optical pulse propagation in resonant medium [1] and the short-wave approximation regime [26].

4 Coherent structures for soliton–breather molecules and breather molecules

The existence of the abundant structures of the solutions to the sG equation Eq. (3) makes it possible to search for some special coherent structures or bounded states of solitons and breathers by using the velocity resonant conditions [11, 27, 28]. The velocity resonance condition means the solitons propagate with the same speed, and thus, they will form some special structures that can be used to describe significant physical phenomena, such as the few-cycle solitons in optics, the nonlinear nonintegrable optical model with dispersions and nonlinear effects including self-steeping, Raman scattering and nonlinear dispersion. The coherent structures are named as soliton molecules in nonlinear optical physics [29]. By now, the BMs, the SMs and the BSMs are found in most nonlinear systems, even in the case of nonintegrable systems [30].

For the sG equation, we find it possesses the bounded state for breathers and soliton, i.e., the BSMs, and the bounded state for breathers, i.e., the BMs by utilizing the velocity resonant conditions.
4.1 Bounded states for breathers and soliton

If \( m \) breathers and one-soliton share the same the velocity, say

\[
\frac{1}{k_{2m+1}} = \frac{1}{\lambda_1 + \mu_1} = \cdots = \frac{1}{\lambda_m + \mu_m},
\]

(23)

they form a special coherent structure of BSM by suitable parameters selections. In other words, the BSM structure contains one soliton and \( m \) breathers. This BSM structure is generally found in \((2m + 1)\)-soliton solutions where the parameters are selected as \( k_i = k_{i+1} = \lambda_i + i\mu_i, i = 1, 2, \ldots, m \) and \( k_{2m+1} \) is real, and all the parameters are determined by the velocity resonant conditions Eq. (23).

Here, we show two particular bounded states for BSMs. The first case is \( m = 1 \) for the three-soliton solution. With the choice of the parameters

\[
k_1 = k_2^* = 1 + \frac{4}{5}i, \quad \eta_{10} = \eta_{20} = 0, \quad k_3 = \sqrt{\frac{4T}{5}}, \quad \eta_{30} = 10,
\]

(24)

being constrained by the velocity resonant condition Eq. (23), a BSM consisted of one breather and one soliton is thus observed. The corresponding density plot for \( u_x \) is illustrated in Fig. 3.

The second case is for \( m = 2 \) in the five-soliton solution to sG equation. It is not difficult to find that the soliton solution for \( N = 5 \) leads to a BSM contained with one soliton and two breathers by using Eq. (23). Such a BSM is exhibited in Fig. 4 by the choice of parameters being

\[
k_1 = k_2^* = \frac{3}{5} + \frac{1}{2}i, \quad k_3 = k_4^* = \frac{61}{100} + \frac{2379}{100}, \quad k_5 = \sqrt{\frac{61}{10}}.
\]

(25)
In general, the bounded states for BSMs depend on not only the parameters selections $k_i$, but also the number of solitons, i.e., $N$.

4.2 Bounded states for breathers

The velocity resonance condition may also arise in breathers. The $2m$-soliton solution to the sG equation becomes to $m$-breather interaction solution by the selections of

$$k_i = k^*_{i+1} = \lambda_i + i\mu_i \quad \text{with} \quad \xi_i = \xi^*_{i+1} = x_i + i t_i, \quad i = 1, 2, \cdots, m.$$ 

If $m$ breathers travel its own path with the same speed, they will form the bounded states called BMs. The generation of bounded states is also related to the velocity resonance condition, i.e.,

$$v_{B1} = v_{B2} = \cdots = v_{Bm},$$

or concretely,

$$\frac{1}{\lambda_1^2 + \mu_1^2} = \frac{1}{\lambda_2^2 + \mu_2^2} = \cdots = \frac{1}{\lambda_m^2 + \mu_m^2}.$$ 

(26)

Figure 5 exhibits a special BM consisted of two breathers for the density plot with the parameters selected as

$$k_1 = k_2^* = \frac{3}{5} + i\frac{1}{2}, \quad k_3 = k_4^* = \frac{61}{100} + \frac{i\sqrt{2379}}{100},$$

$$\eta_{10} = \eta_{20}^* = 5 + 5i, \quad \eta_{30} = \eta_{40}^* = -5 - 5i,$$

(27)

in the four-soliton solution Eq. (16). From the profile, we see the two breathers propagate their own straight line separately with same speed as expected.

Furthermore, by introducing another breather with the parameters

$$k_5 = k_6^* = \frac{11}{20} + \frac{\sqrt{123}}{20}i, \quad \eta_{50} = \eta_{60}^* = -1 - i,$$

(28)

for $N = 6$ in the solution Eq. (16), a BM containing three breathers is shown in Fig. 6.

In conclusion, the velocity resonant condition is an effective way to construct the bounded states for solitons and breathers. The parameters selections affect both the soliton shapes (soliton or breather) and the bounded states.

Unfortunately, the velocity resonant cannot take place in solitons because the speed of the soliton is determined by Eq. (19). So the sG equation does not possess the bounded states for SMs which is significantly different from the mKdV-sG equation and the mKdV equation [31]. The bounded state for kinks [3] in $(1 + 1)$-dimensional scalar field theories may approximately occur in some special areas for the parameters $k_i$. 

Fig. 4 Density plot for a BSM constituted of two breathers and one soliton with the parameters being selected as Eq. (25).

Fig. 5 Density plot of the BM for $N = 4$ with the parameters selected as Eq. (27).

Fig. 6 A BM consisted of three breathers for $N = 6$ with the parameters being Eq. (27)-(28).
being nearly the same. As an example, if the parameters for \( k_1 = 1, k_2 = 0.9999 \) and \( \eta_{10} = \eta_{20} = 0 \), the two solitons are approximately bounded together. Figure 7 shows the two solitons in approximately bounded state for \( t = 0 \). Because the velocities are determined by Eq. (19), we know the two solitons will meet eventually after sometime. Clearly, the approximately bounded state for solitons occurs in some special time and areas.

5 Interactions among BSMs, BMs, solitons and breathers

5.1 Interactions among BSMs, solitons and breathers

It is interesting the multiple soliton solutions exhibit the interactions among BSMs, BMs, solitons and breathers with suitable parameters selections. The interactions between BSMs, BMs, solitons and breathers in some particular senses may not be elastic which is different to the usual knowledge.

For example, with the choice of \( k_1 = k_2^* = \lambda_1 + i\mu_1 \) and \( \eta_{10} = \eta_{20}^* = x_{10} + it_{10} \) for \( N = 4 \), and \( k_3 \) being determined by the velocity resonant Eq. (23) in Eq. (16), an interaction between BSM and a soliton is therefore constructed. The profile of the special interaction is exhibited in Fig. 8.

From the profile, we see the distance between breather and soliton of the BSM becomes larger after the interaction with the other soliton. In other words, the size of the BSM has been changed after the interaction. From the point view of the BSM size changes, the interactions among BSM and soliton are considered as nonelastic. The same unelastic phenomena have also been reported in [11] for a model describing the few-cycle-optical pulses beyond slowly varying envelope approximation.

The properties of this unelastic interaction can be made evident by examining the asymptotic behavior of \( u(x, t) \) as \( t \to \pm \infty \) by using dressing method [25], the inverse scattering method [32], etc. According to [25], the positional shift due to collisions in the soliton (kink) and breather for the breather is

\[
\Delta_B = \frac{1}{\lambda_1} \ln \left( \frac{(k_4 + \lambda_1)^2 + \mu_1^2}{(k_4 - \lambda_1)^2 + \mu_1^2} \right) + \frac{1}{\lambda_1} \ln \left( \frac{(k_4 + \lambda_1)^2 + \mu_1^2}{(k_4 - \lambda_1)^2 + \mu_1^2} \right),
\]

while the position shift for the kink (soliton) in BSM caused by the interactions of solitons is

\[
\Delta_s = \frac{2}{k_3} \ln \left( \frac{k_4 - k_3}{k_4 + k_3} \right),
\]

and thus, the whole position shift for the BSM is given by

\[
\Delta_1 = \Delta_s + \Delta_B = 2 \frac{k_3}{k_4 + k_3} \ln \left( \frac{k_4 - k_3}{k_4 + k_3} \right) + \frac{1}{\lambda_1} \ln \left( \frac{(k_4 + \lambda_1)^2 + \mu_1^2}{(k_4 - \lambda_1)^2 + \mu_1^2} \right).
\]

By using the velocity resonant condition \( k_2^2 = \lambda_2^2 + \mu_2^2 \), the position shift generally holds \( \Delta_1 \neq 0 \). This will naturally lead to the size of BSM changes after the collision.
presented by action and breather–soliton interaction, with the result action is exactly determined by breather–breather interaction between BSM and soliton, the interaction between BSM and breather. Similar to the interaction of BSM and soliton, the interaction between BSM and breather. The position shift caused by BSM and breather interaction is determined by breather–breather interaction based on the velocity resonant condition Eq. (23),

\[ k_1 = k_2^* = \lambda_1 + i\mu_1 = 1 + i/2, \]
\[ k_3 = k_4^* = \lambda_2 + i\mu_2 = \sqrt{3}/5 + i/3, \]
\[ k_5 = 5, \]
\[ \eta_{10} = \eta_{20} = \eta_{30} = \eta_{40} = 0, \]
\[ \eta_{50} = 5. \] (30)

The position shift caused by BSM and breather interaction is exactly determined by breather–breather interaction and breather–soliton interaction, with the result presented by

\[ \Delta_2 = \frac{2}{\lambda_1} \ln \left[ \frac{(\lambda_1 - \lambda_2) + (\mu_1 - \mu_2 i)}{(\lambda_1 + \lambda_2) + (\mu_1 + \mu_2 i)} \right] \left[ \frac{(\lambda_1 + \lambda_2) - (\mu_1 - \mu_2 i)}{(\lambda_1 + \lambda_2) - (\mu_1 + \mu_2 i)} \right] \]
\[ + \frac{1}{\lambda_1} \ln \left( \frac{k_5 + \lambda_1}{k_5 - \lambda_1} \right)^2 + \frac{\mu_1^2}{\lambda_1^2}. \] (31)

Thus, the size of the BSM is also changed after the interaction of BSM and breather. Similar to the interaction between BSM and soliton, the interaction between BSM and breather is still inelastic by the meaning the size of the BSM changes after the collision.

Moreover, there exist BSM interactions if the velocities of the BSMs are different. By selecting \( k_i = k_{i+1}^* = \lambda_i + i\mu_i \) with \( \xi_{i0} = \xi_{i+10} = \xi_{i0} + i\eta_{i0}, i = 1, 2 \) in the six-soliton solution, where \( \{\lambda_1, \mu_1, k_5\} \) and \( \{\lambda_2, \mu_2, k_6\} \) are given by the velocity resonant condition, respectively, the interactions between BSMs are constructed and demonstrated in Eq. (32).

\[ k_1 = k_2^* = 1 + \frac{4\sqrt{3}}{15}i, \]
\[ k_3 = k_4^* = \frac{6}{5} + \frac{\sqrt{71}}{10}i, \quad k_5 = \frac{3}{5}, \]
\[ k_6 = -\frac{9}{10}, \quad \eta_{10} = \eta_{20}^* = 1 + i, \]
\[ \eta_{30} = \eta_{40}^* = -1 + i, \]
\[ \eta_{50} = 5, \quad \eta_{60} = -10. \] (32)

5.2 Interactions among BMs, solitons and breathers

There exist other kinds of interactions, such as the interaction between BMs and solitons and the interactions between BMs and breathers. One of the examples is for the five solitons. If the parameters are selected as

\[ k_1 = k_2^* = \frac{3}{5} + \frac{1}{2}i, \]
\[ k_3 = k_4^* = \frac{61}{100} + \frac{\sqrt{2379}}{100}i, \quad k_5 = \frac{1}{3}, \]
\[ \eta_{10} = \eta_{20}^* = 5 + 5i, \]
\[ \eta_{30} = \eta_{40}^* = -5 - 5i, \quad \eta_{50} = 0, \] (33)
the solution represents the interaction between a BM and a soliton which is demonstrated in Fig. 11. By examining the the distance between breathers that consisted of the BM, we believe the interaction between

![Fig. 9](image-url) Density profile for BSM and breather interaction with the parameters determined by Eq. (30)

![Fig. 10](image-url) The unelastic interaction between BSMs for \( N = 6 \) by selecting the parameters as Eq. (32)
Bounded states for breathers–soliton and breathers of sine–Gordon equation

The multiple soliton solution to the sG equation makes it possible for us to search for more meaningful coherent structures.

6 Summary and discussion

In summary, from the Wronskian forms of the sG equation, we present a novel expression of the $N$-soliton solution to the sG equation with a nonzero background. Compared with Hirota’s exponential polynomial form, the advantage of the Wronskian form is that it can freely provide interaction of different kinds of solutions. The construction of the multiple soliton solution is simple and straightforward. Furthermore, the nonzero background term motivates us to search for more meaningful solutions. Recent studies [12,13] show that some special soliton solutions related to the nonzero background of some high-order nonlinear systems have been found. Because almost all the integrable systems possess similar transformations and multiple soliton solutions, further studies may be done to make clear descriptions for the background.

The obtained multiple soliton solution is used to construct abundant coherent structures of solitons and breathers by introducing the velocity resonant condition. Thus, the mechanism of generating bounded states is related to the velocities of the solitons and breathers. Two special bounded states for sG equation are successfully constructed. In addition, the approximately bounded states for solitons (kinks or anti-kinks) are also studied in detail. Actually, introducing the velocity resonant condition is an effective way to construct SMs, BSMs and BMs. Because most of the integrable systems possess the same solution expressions, it may help to find more bounded states for solitons, kinks and breathers.

Furthermore, the interactions among BSMs, BMs, solitons and breathers are found to be nonelastic in some particular senses that the sizes of the BMs and BSMs change. This is different to the usual interactions among solitons and breathers. In addition, the size of BSM changes after collision is explicitly given based on some known results.

It is shown in [11] that the mKdV-sG model describing few-optical-cycle solitons possesses abundant SMs, BMs and BSMs structures. Though sG equation can be considered as a special case of the mKdV-sG equation, the sG equation does not possess SMs structures.
The results presented in this manuscript theoretically illustrate the bounded states for solitons and breathers. Actually, some experimental work related to BMs is observed in mode-locked fiber lasers by means of real-time ultrafast measurements [29,33]. It is possible and interesting to observe the bounded states of solitons and breathers in laboratories described by sG equation.

Because of the wide applications of sG equation, we believe it is worth further studying to find more possible applications in a wide type of physical systems.

Acknowledgements The author is grateful to Prof. S. Y. Lou for helpful and enlightening discussion. The author also acknowledges the support of NNSFC (No. 11675084) and K. C. Wong Magna Fund in Ningbo University.

Data Availability Statements Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

References

1. Lamb, G.L.: Analytical descriptions of ultrashort optical pulse propagation in a resonant medium. Rev. Mod. Phys. 43, 99–124 (1971)
2. Coleman, S.: Quantum sine-Gordon equation as the massive Thirring model. Phys. Rev. D 11, 2088–2097 (1975)
3. Perapechka, I., Shnir, Y.: Kinks bounded by fermions. Phys. Rev. D 101, 021701 (2020)
4. Rama-Eiroa, R., Otxoa, R.M., Roy, P.E., Guslienko, K.Y.: Steady one-dimensional domain wall motion in biaxial ferromagnets: Mapping of the Landau-Lifshitz equation to the sine-Gordon equation. Phys. Rev. B 101, 094416 (2020)
5. Krasnov, V.M.: Josephson junctions in a local inhomogeneous magnetic field. Phys. Rev. B 101, 144507 (2020)
6. Benaballah, A., Caputo, J.G., Scott, A.C.: Exponentially tapered Josephson flux-flow oscillator. Phys. Rev. B 54, 16139–16146 (1996)
7. Dolgopolik, M., Fradkov, A.L., Andrievsky, B.: Observer-based boundary control of the sine-Gordon model energy. Automatica 113, 108682 (2020)
8. Hegedűs, Á.: Finite volume expectation values in the sine-Gordon model. J. High Energy Phys. 2020, 122 (2020)
9. Hu, H.C., Lou, S.Y., Chow, K.: New interaction solutions of multiply periodic, quasi-periodic and non-periodic waves for the $(n+1)$-dimensional double sine-Gordon equations. Chaos, Solitons Fractals 31, 1213–1222 (2007)
10. Sun, Y.Y., Wu, H.: New breather solutions of the model describing few-optical-cycle solitons beyond the slowly varying envelope approximation. Phys. Scr. 88, 065001 (2013)
11. Jia, M., Lin, J., Lou, S.Y.: Soliton and breather molecules in few-cycle-pulse optical model. Nonlinear Dyn. 100, 3745–3757 (2020)
12. Wang, W., Yao, R.X., Lou, S.Y.: Abundant traveling wave structures of $(1+1)$-dimensional Sawada-Kotera equation: few cycle solitons and soliton molecules. Chin. Phys. Lett. 37, 100501 (2020)
13. Chen, Z.T., Jia, M.: Novel travelling wave structures: few-cycle-pulse solitons and soliton molecules. Commun. Theor. Phys. 73, 025003 (2021)
14. Manton, N., Sutcliffe, P.: Topological Solitons. , Cambridge (2004)
15. Jia, M., Lou, S.Y.: New Types of Exact Solutions for $(N+1)$-Dimensional $\phi^4$ Model. Commun. Theor. Phys. 46, 91–96 (2006)
16. Jia, M., Lou, S.Y.: New deformation relations and exact solutions of the high-dimensional $\Phi^6$ field model. Phys. Lett. A 353, 407–415 (2006)
17. Brihaye, Y., Delsate, T.: Remarks on bell-shaped lumps: Stability and fermionic modes. Phys. Rev. D 78, 025014 (2008)
18. Chu, Y.Z., Vachaspati, T.: Fermions on one or fewer kinks. Phys. Rev. D 77, 025006 (2008)
19. Zhang, D.J., Zhao, S.L., Sun, Y.Y., Zhou, J.: Solutions to the modified Korteweg-de Vries equation. Rev. Math. Phys. 26, 1430006 (2014)
20. Freeman, N., Nimmo, J.: Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: The wronskian technique. Phys. Lett. A 95, 1–3 (1983)
21. Ashlowitz, M.J., Segur, H.: Solitons and the inverse scattering transform. Society for Industrial and Applied Mathematics, Philadelphia (1981)
22. Hirota, R.: A new form of Bäcklund transformations and its relation to the inverse scattering problem. Prog. Theor. Phys. 52, 1498–1512 (1974)
23. Chen, D.Y., Zhang, D.J., Deng, S.F.: The novel multi-soliton solutions of the mKdV-sine Gordon Equations. J. Phys. Soc. Jpn. 71, 658–659 (2002)
24. Zhou, J., Zhang, D.J., Zhao, S.L.: Breathers and limit solutions of the nonlinear lumped self-dual network equation. Phys. Lett. A 373, 3248–3258 (2009)
25. Mikhailov, A.V., Papamikos, G., Wang, J.P.: Dressing method for the vector sine-Gordon equation and its soliton interactions. Phys. D. 325, 53–62 (2016)
26. Leblond, H., Mihalache, D.: Models of few optical cycle solitons beyond the slowly varying envelope approximation. Phys. Rep. 523, 61–126 (2013)
27. Lou, S.Y.: Soliton molecules and asymmetric solitons in three fifth order systems via velocity resonance. J. Phys. Commun. 4, 041002 (2020)
28. Zhang, Z., Yang, X., Li, B.: Soliton molecules and novel smooth positions for the complex modified KdV equation. Appl. Math. Lett. 103, 106168 (2020)
29. Peng, J., Boscolo, S., Zhao, Z., Zeng, H.: Breathing dissipative solitons in mode-locked fiber lasers. Sci. Adv. 5,(2019)
30. Xu, D.H., Lou, S.Y.: Dark soliton molecules in nonlinear optics (In Chinese). Acta Phys. Sin. 69, 014208 (2020)
31. Jia, M., Chen, Z.T.: Coherent structures for breather-soliton molecules and breather molecules of the modified KdV equation. Phys. Scr. 95, 105210 (2020)
32. Faddeev, L.D., Takhtajan, L.A.: Hamiltonian methods in the theory of solitons. Springer, Berlin (1987)
33. Xu, G., Gelash, A., Chabchoub, A., Zakharov, V., Kibler, B.: Breather wave molecules. Phys. Rev. Lett. 122(8), 084101 (2019)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.