GLOBAL EXISTENCE OF STRONG SOLUTIONS TO A GROUNDWATER FLOW PROBLEM

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Abstract. In this paper we study the initial boundary value problem for the system $\Delta v = u_{x_1}$, $u_t - \text{div} \left( \left( (a|q| + m)I + (b - a)\frac{q \otimes q}{|q|} \right) \nabla u \right) = -\nabla u \cdot q$, where $q = (-v_{x_2}, v_{x_1})^T$, $q \otimes q = qq^T$. This problem has been proposed as a model for a fluid flowing through a porous medium under the influence of gravity and hydrodynamic dispersion. For each $T > 0$ we obtain a weak solution $(v, u)$ in the function space $L^\infty(0, T; W^{1, \infty}(\Omega))$, where $\Omega$ is a bounded domain in $\mathbb{R}^2$. The key ingredient in our approach is the decomposition $A^2 = \text{tr}(A)A - \det A I$ for any $2 \times 2$ symmetric matrix $A$. By exploring this decomposition, we are able to derive an equation of parabolic type for the function $\left( \left( (a|q| + m)I + (b - a)\frac{q \otimes q}{|q|} \right) \nabla u \cdot \nabla u \right)_j$, $j \geq 1$. With the aid of this equation we obtain an uniform bound for $\nabla u$.

1. Introduction

Let $\Omega$ be a bounded domain in the $x = (x_1, x_2)$ plane with boundary $\partial \Omega$ and $T$ any positive number. We study the problem

(1.1) $\Delta v = u_{x_1}$ in $\Omega_T \equiv \Omega \times (0, T)$,

(1.2) $u_t - \text{div} \left( \left( (a|q| + m)I + (b - a)\frac{q \otimes q}{|q|} \right) \nabla u \right) = -\nabla u \cdot q$ in $\Omega_T$,

(1.3) $\left( (a|q| + m)I + (b - a)\frac{q \otimes q}{|q|} \right) \nabla u \cdot \nu = 0$ on $\Sigma_T \equiv \partial \Omega \times (0, T)$,

(1.4) $v = 0$ on $\Sigma_T$,

(1.5) $u(x, 0) = u_0(x)$ on $\Omega$,

where

(1.6) $q = \begin{pmatrix} -v_{x_2} \\ v_{x_1} \end{pmatrix}$,

$q \otimes q = qq^T$, $I$ is the $2 \times 2$ identity matrix, $a, b, m$ are positive numbers with $b > a$, and $\nu$ is the unit outward normal to the $\partial \Omega$.

This system arises in the description of the movement of a fluid of variable density $u$ through a porous medium under the influence of gravity and hydrodynamic dispersion [1]. The first equation (1.1) is derived from Darcy’s law, while the second equation (1.2) describes the mass balance. See [1, 6] for details. In a slightly different form, problem (1.1)-(1.5) was studied by Su [8] using classical partial differential equation (PDE) methods. In [1] the problem was formulated as abstract evolution equations in Banach spaces. Two cases were considered. First, if the coefficient matrix

(1.7) $D \equiv (a|q| + m)I + (b - a)\frac{q \otimes q}{|q|}$

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can be taken as an identity matrix, the resulting problem has a classical solution. If not only local existence of weak solutions in $W^{1,p}(\Omega)$ was obtained. The global existence was thereby proposed as an open problem. In particular, the fact that the coefficient matrix $D$ is not differentiable at the origin was mentioned as an impediment to the existence of classical solutions.

By the definition of $q$, we always have

$$\text{div} q = 0.$$  

Moreover,

$$\frac{|q \otimes q|}{|q|} \leq |q|.$$  

It is natural for us to define

$$|q \otimes q| = 0 \text{ whenever } q = 0.$$  

Thus the coefficient matrix $D$ is well-defined and satisfies

$$m|\xi|^2 \leq D\xi \cdot \xi = (a|q| + m)|\xi|^2 + \frac{b - a}{|q|} (q \cdot \xi)^2 \leq (b|q| + m)|\xi|^2 \text{ for each } \xi \in \mathbb{R}^2.$$  

The second equation in our system becomes singular on the set where $|q|$ is infinity. As observed in [1], each entry of $D$ is a Lipschitz function of $q$. In particular, we have

$$|D_t| \leq c|\nabla v_t|,$$

$$|D_{x_1}| + |D_{x_2}| \leq c|\nabla^2 v|,$$

where $\nabla^2 v$ denotes the Hessian of $v$. The letter $c$ here and in what follows represents a generic positive number whose value can be derived from the given data at least in theory. Our main result is:

**Theorem 1.1 (Main Theorem).** Let $a, b, m$ be given as before. Assume:

(H1) $\Omega$ is a bounded domain in $\mathbb{R}^2$ with $C^3$ boundary $\partial \Omega$;

(H2) $u_0 \in W^{2,\infty}(\Omega)$.

Then for each $T > 0$ there is a weak solution $(v, u)$ to (1.1)-(1.5) with $|\nabla u|, |\nabla v| \in L^\infty(\Omega_T)$.

Of course, the fact that $|\nabla u|, |\nabla v| \in L^\infty(\Omega_T)$ can yield more regularity results on the weak solution. We will not elaborate on this. Our approach is based upon the following observation: Let $A$ be an $2 \times 2$ symmetric matrix. Then we have

$$A^2 = \text{tr}(A)A - \det(A)I.$$  

The proof of this formula is very simple. Indeed, denote by $a_{ij}$ the $ij$ entry of $A$. We calculate

$$A^2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{12} + a_{12}a_{22} & a_{12}^2 + a_{22}^2 \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{11}a_{22} - \det(A) & a_{12}(a_{11} + a_{22}) \\ a_{12}(a_{11} + a_{22}) & a_{11}a_{22} - \det(A) + a_{22}^2 \end{pmatrix} = (a_{11} + a_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} - \det(A)I.$$  

For all practical purposes the last term in (1.14) behaves like a scalar function. The formula (1.14) says that a quadratic function of a matrix can be decomposed into a linear function of the matrix plus a roughly scalar function. That is, the first term in the decomposition has weaken the non-linearity, while the second term has reduced the dimensionality. In this work, we do not actually use
the formula directly. What we use is the idea behind the proof of this formula. That is, whenever we can represent certain terms in a high-powered matrix in terms of a determinant, something good follows. It is this that enables us to derive an equation for the function

\[(1.16) \quad \psi = (D\nabla u \cdot \nabla u)^j, \quad j \geq 1.\]

To be specific, we establish that for each \(j \geq 1\) the function \(\psi\) above satisfies

\[(1.17) \quad \frac{1}{\psi}\psi_t - \text{div} \left( \frac{1}{\psi} D\nabla \psi \right) = \frac{1}{\psi} H \cdot \nabla \psi + jh + j\text{div}K \quad \text{in } \{ |\nabla u| > 0 \}.\]

Here the coefficients \(H, h, K\) are bounded by \(u_t\), the entries of \(D\) and their partial derivatives \(D_t, D_{x_1}, D_{x_2}\). It turns out that each of those partial derivatives can also be bounded by \(\nabla u\). The key to our success is that we can choose \(j\) to be sufficiently large. This creates a situation where a term with large power is bounded by the same term with small power, from which estimates follow.

We believe that decomposition such as (1.14) is a very powerful tool. It may find applications elsewhere [9]. For example, one can explore the relationship between the cubic power of a three-by-three matrix and its determinant.

This work is organized as follows: Section 2 is largely devoted to the derivation of (1.17). In section 3, we assume that problem (1.1)-(1.5) has a classical solution and proceed to derive a priori estimates for the solution. The main theorem is established as a consequence of these estimates. In Section 4 we construct a sequence of smooth approximate solutions, thereby justifying the regularity assumption on solutions in Sections 2 and 3 and their subsequent calculations.

2. Derivation of Equation (1.17)

In this section we derive (1.17). It is essentially the parabolic version of the result in [9]. But before we do that, we recall some definitions and known results and formulae.

If \(A(x)\) is a matrix-valued function then

\[(2.1) \quad \text{div} A(x) = \text{the row vector whose } i\text{-th entry is the divergence of the } i\text{-th column of } A = (\text{div} A_1, \text{div} A_2).\]

When \(G(x)\) is a vector-valued function, then

\[(2.2) \quad \nabla G(x) = \text{the } 2 \times 2 \text{ matrix whose } ij\text{-entry is } (g_j(x))_{x_i} = (\nabla g_1, \nabla g_2).\]

Denote by \(\nabla^2 u\) the Hessian of \(u\). Then we have

\[(2.3) \quad \nabla |\nabla u|^2 = 2\nabla^2 u \nabla u.\]

The following identities will be frequently used

\[(2.4) \quad \nabla (F \cdot G) = \nabla F G + \nabla G F,\]

\[(2.5) \quad \text{div} (AF) = A : \nabla F + \text{div} AF,\]

\[(2.6) \quad \nabla (AF) = \nabla FA^T + (A_{x_1} F, A_{x_2} F)^T,\]

\[(2.7) \quad \text{div}(uA) = u\text{div} A + (\nabla u)^T A.\]

We also need the interpolation inequality

\[(2.8) \quad \|u\|_q \leq \varepsilon \|u\|_r + \varepsilon^{-\mu} \|u\|_\ell,\]

where \(1 \leq \ell \leq q \leq r\) with \(\mu = \left( \frac{1}{q} - \frac{1}{r} \right) / \left( \frac{1}{\ell} - \frac{1}{q} \right)\).

The next lemma deals with sequences of non-negative numbers which satisfy certain recursive inequalities.
Lemma 2.1. Let \( \{y_n\}, n = 0, 1, 2, \cdots \), be a sequence of positive numbers satisfying the recursive inequalities
\[
y_{n+1} \leq cb^n y_n^{1+\alpha}
\]
for some \( b > 1, c, \alpha \in (0, \infty) \).

If
\[
y_0 \leq e^{-\frac{b}{\alpha} - \frac{1}{\alpha^2}},
\]
then \( \lim_{n \to \infty} y_n = 0 \).

This lemma can be found in ([2], p.12).

We write (1.2) in the form
\[
(2.9) \quad u_t - D : \nabla^2 u = \text{div} D \nabla u - \nabla u \cdot q \equiv w.
\]
Denote by \( d_{ij} \) the entry of \( D \) that lies in the \( i \)th row and the \( j \)th column. Then we have
\[
(2.10) \quad u_t - (d_{11} u_{x_1} + 2d_{12} u_{x_1 x_2} + d_{22} u_{x_2}) = w.
\]
We introduce the following quantities:
\[
\begin{align*}
(2.11) \quad a &= D \nabla u \cdot \nabla u = d_{11} u_{x_1}^2 + 2d_{12} u_{x_1} u_{x_2} + d_{22} u_{x_2}^2, \\
(2.12) \quad D_1 &= \begin{pmatrix} d_{11}(d_{11} u_{x_1} + d_{12} u_{x_2}) & d_{12}(d_{11} u_{x_1} - (d_{22} d_{11} - 2d_{12}^2) u_{x_2} \\ d_{11}(d_{12} u_{x_2} + d_{22} u_{x_2}) & d_{22}(d_{11} u_{x_1} + d_{12} u_{x_2}) \end{pmatrix}, \\
(2.13) \quad D_2 &= \begin{pmatrix} d_{11}(d_{12} u_{x_1} + d_{22} u_{x_2}) & d_{22}(d_{11} u_{x_1} + d_{12} u_{x_2}) \\ -(d_{22} d_{11} - 2d_{12}^2) u_{x_2} + d_{12} d_{22} u_{x_2} & d_{22}(d_{11} u_{x_1} + d_{12} u_{x_2}) \end{pmatrix}, \\
(2.14) \quad D_3 &= -D \nabla u (D \nabla u)^T, \\
(2.15) \quad G &= a^{-1} \begin{pmatrix} D_{x_1} \nabla u \cdot \nabla u \\ D_{x_2} \nabla u \cdot \nabla u \end{pmatrix}.
\end{align*}
\]

Theorem 2.2. For each \( j \geq 1 \) the function \( \psi = a^j \) satisfies the equation
\[
(2.16) \quad \frac{1}{\psi} \psi_t - \text{div} \left( \frac{1}{\psi} D \nabla \psi \right) = \frac{1}{\psi} H \cdot \nabla \psi + jh + j \text{div} K \quad \text{in } \{|\nabla u| > 0\},
\]
where
\[
\begin{align*}
(2.17) \quad H &= DG + 2a^{-1} u_t D \nabla u + \frac{1}{\det(D) a} \left( D_1^T \nabla \det(D), D_2^T \nabla \det(D) \right) \nabla u, \\
(2.18) \quad K &= -DG + 2a^{-1} u D \nabla u, \\
(2.19) \quad h &= \frac{1}{\det(D) a} \nabla \det(D) \cdot (D_1 G, D_2 G) \nabla u + \frac{2(u_t - w)}{\det(D) a^2} \nabla \det(D) \cdot D_3 \nabla u - 2a^{-1} u_t (u_t + \nabla u q) + a^{-1} D_1 \nabla u \cdot \nabla u - 2a^{-1} (u_t - w) D \nabla u \cdot G - D G \cdot G.
\end{align*}
\]

Proof. This theorem is a parabolic version of a result in [9]. As is done there, we first derive an equation for
\[
(2.20) \quad b = \ln a.
\]
We calculate
\[
\begin{align*}
(2.21) \quad b_t - \text{div} (D \nabla b) &= \frac{1}{a} a_t - \text{div} \left( \frac{1}{a} D \nabla a \right) \\
&= \frac{1}{a} a_t - \frac{1}{a} \text{div} (D \nabla a) + \frac{1}{a^2} D \nabla a \cdot \nabla a \\
&= \frac{1}{a} \left( a_t - \text{div} (D \nabla a) + \frac{1}{a} D \nabla a \cdot \nabla a \right).
\end{align*}
\]
Take the gradient of (2.9) and take the dot-product of the resulting equation with $D\nabla u$ to obtain
\begin{equation}
D\nabla u \cdot \nabla u_t - D\nabla u \cdot \nabla (D : \nabla^2 u) = D\nabla u \cdot \nabla w.
\end{equation}

Subsequently,
\begin{align}
a_t &= 2D\nabla u \cdot \nabla u_t + D_t \nabla u \cdot \nabla u \\
&= 2D\nabla u \cdot \nabla (D : \nabla^2 u) + 2D\nabla u \cdot \nabla w + D_t \nabla u \cdot \nabla u \\
&= 2D\nabla u \cdot \nabla (D : \nabla^2 u) + 2\text{div}(wD\nabla u) - 2w\text{div}(D\nabla u) + D_t \nabla u \cdot \nabla u.
\end{align}

We evaluate
\begin{align}
D\nabla u \cdot \nabla (\nabla^2 u : D) &= (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})x_1 (d_{11}u_{x_1} + d_{12}u_{x_2}) \\
&+ (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})x_2 (d_{12}u_{x_1} + d_{22}u_{x_2}) \\
&= \text{div} \left( (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}) (d_{11}u_{x_1} + d_{12}u_{x_2}) \right) \\
&+ (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}) (d_{12}u_{x_1} + d_{22}u_{x_2}) \\
&= \text{div}(D\nabla u) - (\nabla^2 u : D) \text{div}(D\nabla u) \\
&= \text{div} \left( (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}) (d_{11}u_{x_1} + d_{12}u_{x_2}) \right) \\
&+ (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}) (d_{12}u_{x_1} + d_{22}u_{x_2}) \\
&= (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})(d_{11}u_{x_1} + d_{12}u_{x_2}) \\
&+ (d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})(d_{12}u_{x_1} + d_{22}u_{x_2}) \\
&= -u_t - w \text{div}(D\nabla u).
\end{align}

Then calculate from (2.10) and (2.15) that
\begin{align}
D\nabla a &= D \left( \frac{2d_{11}u_{x_1} + 2d_{12}(u_{x_1}u_{x_1} + u_{x_1}u_{x_2}) + 2d_{22}u_{x_2}x_2}{2d_{11}u_{x_1x_1} + 2d_{12}(u_{x_1}u_{x_1} + u_{x_1}u_{x_2}) + 2d_{22}u_{x_2}x_2} \right) \\
&+ aDG \\
&= 2D \left( \frac{u_{x_1x_1}(d_{11}u_{x_1} + d_{12}u_{x_2}) + u_{x_2x_2}(d_{11}u_{x_1} + d_{12}u_{x_2})}{u_{x_1x_2}(d_{11}u_{x_1} + d_{12}u_{x_2}) + u_{x_2x_2}(d_{11}u_{x_1} + d_{12}u_{x_2})} \right) + aDG \\
&= 2 \left( \frac{(d_{11}u_{x_1x_1} + d_{12}u_{x_1x_2})(d_{11}u_{x_1} + d_{12}u_{x_2})}{(d_{12}u_{x_1x_1} + d_{22}u_{x_2x_2})(d_{11}u_{x_1} + d_{12}u_{x_2})} \right) \\
&+ 2 \left( \frac{(d_{11}u_{x_1x_2} + d_{12}u_{x_2x_2})(d_{12}u_{x_1} + d_{22}u_{x_2})}{(d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})(d_{12}u_{x_1} + d_{22}u_{x_2})} \right) + aDG.
\end{align}

As we mentioned in the introduction, we try to represent the difference between $2D\nabla u \cdot \nabla (D : \nabla^2 u)$ and $\text{div}(D\nabla a)$ in terms of determinants. To this end, we compute
\begin{align}
2D\nabla u \cdot \nabla (D : \nabla^2 u) - \text{div}(D\nabla a) &= 2\text{div} \left( (d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2})(d_{11}u_{x_1} + d_{12}u_{x_2}) - (d_{11}u_{x_1x_1} + d_{12}u_{x_1x_2})(d_{12}u_{x_1} + d_{22}u_{x_2}) \right) \\
&- 2(u_t - w)\text{div}(D\nabla u) - \text{div}(aDG) \\
&= 2\text{div} \left( \frac{\det(D)(u_{x_1x_2}u_{x_1} - u_{x_1}u_{x_1x_2})}{\det(D)(u_{x_1x_1}u_{x_2} - u_{x_1x_2}u_{x_1})} \right) - 2(\nabla^2 u : D)\text{div}(D\nabla u) - \text{div}(aDG) \\
&= 2\text{div} \left( \frac{u_{x_2x_2} - u_{x_1x_2}}{u_{x_1x_1} - u_{x_1}} \right) \nabla u + 4\det(D)\det(\nabla^2 u) \\
&= -2(u_t - w)\text{div}(D\nabla u) - \text{div}(aDG).
\end{align}

Here we have used the fact that
\begin{equation}
\text{div} \left( \frac{u_{x_2x_2}u_{x_1} - u_{x_1}u_{x_1x_2}}{u_{x_1x_1}u_{x_2} - u_{x_1x_2}u_{x_1}} \right) = 2\det(\nabla^2 u).
\end{equation}
On the other hand, we have
\[ \nabla a = \nabla^2 u D \nabla u + \nabla (D \nabla u) \nabla u \]
(2.28)
\[ = 2 \nabla^2 u D \nabla u + a G. \]
Consequently,
\[ D \nabla a \cdot \nabla a = (\nabla a)^T D \nabla a \]
\[ = (2(D \nabla u)^T \nabla^2 u + a G^T) D \nabla^2 u \nabla u + a G \]
(2.29)
\[ = 4(D \nabla u)^T \nabla^2 u D \nabla^2 u D \nabla u + 4a D \nabla^2 u \nabla u \cdot G + a^2 DG \cdot G. \]
Set
\[ B = \nabla^2 u D \nabla^2 u. \]
We also represent the four entries of \( B \) in terms of determinants as follows
\[ b_{11} = d_{11}u_{x_1x_1}^2 + 2d_{12}u_{x_1x_1}u_{x_1x_2} + d_{22}u_{x_2x_2}^2 \]
\[ = d_{11}u_{x_1x_1}^2 + 2d_{12}u_{x_1x_1}u_{x_1x_2} + d_{22}(u_{x_1x_1}u_{x_2x_2} - \det(\nabla^2 u)) \]
\[ = u_{x_1x_1}(d_{11}u_{x_1x_1} + 2d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}) - d_{22}\det(\nabla^2 u) \]
\[ = u_{x_1x_1}(u_t - w) - d_{22}\det(\nabla^2 u), \]
\[ b_{21} = d_{11}u_{x_1x_1}u_{x_1x_2} + d_{12}u_{x_1x_1}u_{x_2x_2} + d_{12}u_{x_1x_2}^2 + d_{22}u_{x_2x_2}u_{x_1x_2} \]
\[ = d_{11}u_{x_1x_1}u_{x_1x_2} + d_{12}(u_{x_1x_2}^2 + \det(\nabla^2 u)) + d_{12}u_{x_1x_2} + d_{22}u_{x_2x_2}u_{x_1x_2} \]
\[ = u_{x_1x_2}(u_t - w) + d_{12}\det(\nabla^2 u), \]
\[ b_{12} = b_{21}, \]
\[ b_{22} = d_{11}u_{x_2x_2}^2 + 2d_{12}u_{x_1x_2}u_{x_2x_2} + d_{22}u_{x_2x_2}^2 \]
\[ = d_{11}(u_{x_1x_1}u_{x_2x_2} - \det(\nabla^2 u)) + 2d_{12}u_{x_1x_2}u_{x_2x_2} + d_{22}u_{x_2x_2}^2 \]
\[ = u_{x_2x_2}(u_t - w) - d_{11}\det(\nabla^2 u). \]
That is,
\[ B = (u_t - w)\nabla^2 u - \det(\nabla^2 u)\det(D) D^{-1}. \]
Plug this into (2.29) to obtain
\[ D \nabla a \cdot \nabla a = 4(D \nabla u)^T ((u_t - w)\nabla^2 u - \det(\nabla^2 u)\det(D) D^{-1}) D \nabla u \]
\[ + 4a D \nabla^2 u D \nabla u \cdot G + a^2 DG \cdot G \]
(2.32)
\[ = -4a\det(\nabla^2 u)\det(D) + 4D \nabla^2 u D \nabla u \cdot (aG + (u_t - w)\nabla u) + a^2 DG \cdot G. \]
Equipped with the preceding results, we can evaluate
\[ a_t - \text{div} (D \nabla a) \]
\[ = 2\nabla \det(D) \cdot \left( \begin{array}{c} u_{x_2x_2} \\ -u_{x_1x_2} \\ -u_{x_1x_1} \end{array} \right) \nabla u \]
\[ -2u_t \text{div} (D \nabla u) - \text{div} (aDG - 2wD \nabla u) + D_t \nabla u \cdot \nabla u \]
\[ + 2a^{-1}D(\nabla a - a \nabla G) \cdot (aG + (u_t - w) \nabla u) + aDG \cdot G \]
\[ = 2\nabla \det(D) \cdot \left( \begin{array}{c} u_{x_2x_2} \\ -u_{x_1x_2} \\ -u_{x_1x_1} \end{array} \right) \nabla u \]
\[ -2u_t \text{div} (D \nabla u) - \text{div} (aDG - 2wD \nabla u) + D_t \nabla u \cdot \nabla u \]
\[ + (2DG + 2a^{-1}(u_t - w)D \nabla u) \cdot \nabla a - 2(u_t - w)D \nabla u \cdot G - aDG \cdot G \]
(2.33)
Subsequently,

\[
b_t - \text{div} (D \nabla b) = \frac{1}{a} \left( a_t - \text{div} (D \nabla a) + \frac{1}{a} D \nabla a \cdot \nabla a \right)
\]

\[
= 2a^{-1} \nabla \det(D) \cdot \begin{pmatrix} u_{x_2 x_2} & -u_{x_1 x_2} \\ -u_{x_1 x_2} & u_{x_1 x_1} \end{pmatrix} \nabla u
\]

\[
-2a^{-1} u_t \text{div} (D \nabla u) - a^{-1} \text{div} (aDG - 2wD \nabla u) + a^{-1} D_t \nabla u \cdot \nabla u
\]

\[
+ (2a^{-1} DG + 2a^{-2} (u_t - w) D \nabla u) \nabla a - 2a^{-1} (u_t - w) D \nabla u \cdot G - DG \cdot G
\]

\[
= 2a^{-1} \nabla \det(D) \cdot \begin{pmatrix} u_{x_2 x_2} & -u_{x_1 x_2} \\ -u_{x_1 x_2} & u_{x_1 x_1} \end{pmatrix} \nabla u
\]

\[
-2a^{-1} u_t (u_t + \nabla u w) - \text{div} (DG - 2a^{-1} wD \nabla u) + a^{-1} D_t \nabla u \cdot \nabla u
\]

\[
+ (a^{-1} DG + 2a^{-2} u_t D \nabla u) \cdot \nabla a - 2a^{-1} (u_t - w) D \nabla u \cdot G - DG \cdot G.
\]

(2.34)

We still need to eliminate the second partial derivatives of \( u \) on the right hand side of the preceding equation. To this end, we deduce from (2.28) and (2.10) that

\[
(2.35) \quad (d_{11} u_{x_1} + d_{12} u_{x_2}) u_{x_1 x_1} + (d_{12} u_{x_1} + d_{22} u_{x_2}) u_{x_1 x_2} = \frac{1}{2} (a_{x_1} - a g_1),
\]

\[
(2.36) \quad (d_{11} u_{x_1} + d_{12} u_{x_2}) u_{x_1 x_2} + (d_{12} u_{x_1} + d_{22} u_{x_2}) u_{x_2 x_2} = \frac{1}{2} (a_{x_2} - a g_2),
\]

\[
(2.37) \quad d_{11} u_{x_1 x_1} + 2d_{12} u_{x_1 x_2} + d_{22} u_{x_2 x_2} = u_t - w.
\]

Denote by \( E \) the coefficient matrix of the above system. Then

\[
\det E = \det \begin{pmatrix} d_{11} u_{x_1} + d_{12} u_{x_2} & d_{12} u_{x_1} + d_{22} u_{x_2} & 0 \\ 0 & d_{11} u_{x_1} + d_{12} u_{x_2} & d_{12} u_{x_1} + d_{22} u_{x_2} \\ d_{11} + 2d_{12} & d_{22} & d_{22} \end{pmatrix}
\]

\[
= (d_{11} u_{x_1} + d_{12} u_{x_2}) [d_{22} (d_{11} u_{x_1} + d_{12} u_{x_2}) - 2d_{12} (d_{12} u_{x_1} + d_{22} u_{x_2})]
\]

\[
+ d_{11} (d_{12} u_{x_1} + d_{22} u_{x_2})^2
\]

\[
= (d_{11} u_{x_1} + d_{12} u_{x_2}) \left[ (d_{22} d_{11} - 2d_{12}^2) u_{x_1} - d_{22} d_{12} u_{x_2} \right]
\]

\[
+ d_{11} (d_{12} u_{x_1} + d_{22} u_{x_2})^2
\]

\[
= d_{11} (d_{22} d_{11} - d_{22}^2) u_{x_1}^2 + 2d_{11} d_{22} d_{12} - d_{22}^2) u_{x_1} u_{x_2} + (d_{11} d_{22}^2 - d_{22}^2) u_{x_2}^2
\]

\[
= \det(D) \left( \frac{d_{11} u_{x_1}^2}{a_1} + 2d_{12} u_{x_1} u_{x_2} + d_{22} u_{x_2}^2 \right)
\]

(2.38)

By Cramer’s rule, we have

\[
u_{x_1 x_1} = \frac{1}{2 \det(D) a} \left[ (d_{22} d_{11} - 2d_{12}^2) u_{x_1} - d_{12} d_{22} u_{x_2} (a_{x_1} - a g_1) - d_{22} (d_{11} u_{x_1} + d_{22} u_{x_2}) (a_{x_2} - a g_2) \right]
\]

\[
+ \frac{1}{\det(D) a} (u_t - w) (d_{12} u_{x_1} + d_{22} u_{x_2})^2,
\]

\[
u_{x_1 x_2} = \frac{1}{2 \det(D) a} \left[ d_{11} (d_{12} u_{x_1} + d_{22} u_{x_2}) (a_{x_1} - a g_1) + d_{22} (d_{11} u_{x_1} + d_{12} u_{x_2}) (a_{x_2} - a g_2) \right]
\]

\[
- \frac{1}{\det(D) a} (u_t - w) (d_{11} u_{x_1} + d_{12} u_{x_2}) (d_{12} u_{x_1} + d_{22} u_{x_2}),
\]

\[
u_{x_2 x_2} = -\frac{1}{2 \det(D) a} \left[ d_{11} (d_{12} u_{x_1} + d_{22} u_{x_2}) (a_{x_1} - a g_1) + (d_{12} d_{11} u_{x_1} - (d_{11} d_{22} - 2d_{12}^2) u_{x_2}) (a_{x_2} - a g_2) \right]
\]

\[
+ \frac{1}{\det(D) a} (u_t - w) (d_{11} u_{x_1} + d_{12} u_{x_2})^2.
\]
Using (2.12)-(2.14) yields
\[
\begin{pmatrix}
-u_{x_2x_2} & u_{x_1x_2} \\
u_{x_1x_2} & -u_{x_1x_1}
\end{pmatrix}
= \frac{1}{2 \det(D)a} (D_1(\nabla a - a G), D_2(\nabla a - a G)) + \frac{(u_t - w)}{\det(D)a} D_3.
\]
Observe that
\[
\nabla \det(D) \cdot (D_1 \nabla a, D_2 \nabla a) \nabla u = (\nabla u)^T \begin{pmatrix}
(\nabla a)^T D_1^T \nabla \det(D) \\
(\nabla a)^T D_2^T \nabla \det(D)
\end{pmatrix}
= (\nabla u)^T \begin{pmatrix}
(\nabla \det(D))^T D_1 \nabla a \\
(\nabla \det(D))^T D_2 \nabla a
\end{pmatrix}
\]
(2.40)
\[
= (D_1^T \nabla \det(D), D_2^T \nabla \det(D)) \nabla u \cdot \nabla a.
\]
With this in mind, we derive from (2.34) that
\[
b_t - \text{div}(D \nabla b) \\
= 2a^{-1} \nabla \det(D) \cdot \begin{pmatrix}
u_{x_2x_2} & -u_{x_1x_2} \\
u_{x_1x_2} & -u_{x_1x_1}
\end{pmatrix} \nabla u
-2a^{-1} u_t (u_t + \nabla u q) - \text{div} (DG - 2a^{-1} w D \nabla u) + a^{-1} D_t \nabla u \cdot \nabla u
+(a^{-1} DG + 2a^{-2} u_t D \nabla u) \cdot \nabla a - 2a^{-1} (u_t - w) D \nabla u \cdot G - DG \cdot G
\]
\[
= 2a^{-1} \nabla \det(D) \cdot \left( \frac{1}{2 \det(D)a} (D_1(\nabla a - a G), D_2(\nabla a - a G)) + \frac{(u_t - w)}{\det(D)a} D_3 \right) \nabla u
-2a^{-1} u_t (u_t + \nabla u q) - \text{div} (DG - 2a^{-1} w D \nabla u) + a^{-1} D_t \nabla u \cdot \nabla u
+(a^{-1} DG + 2a^{-2} u_t D \nabla u) \cdot \nabla a - 2a^{-1} (u_t - w) D \nabla u \cdot G - DG \cdot G
\]
\[
= \left( DG + 2a^{-1} u_t D \nabla u + \frac{1}{\det(D)a} \left( D_1^T \nabla \det(D), D_2^T \nabla \det(D) \right) \nabla u \right) \cdot \nabla b
-\frac{1}{\det(D)a} \nabla \det(D) \cdot (D_1 G, D_2 G) \nabla u + \frac{2(u_t - w)}{\det(D)a^2} \nabla \det(D) \cdot D_3 \nabla u
-\text{div} (DG - 2a^{-1} w D \nabla u) - 2a^{-1} u_t (u_t + \nabla u q)
\]
\[\]
+a^{-1} D_t \nabla u \cdot \nabla u - 2a^{-1} (u_t - w) D \nabla u \cdot G - DG \cdot G.
\]
Using (2.17)-(2.19), we have
\[
b_t - \text{div}(D \nabla b) = H \cdot \nabla b + h + \text{div} K.
\]
To see (2.16), we compute
\[
b_t = a^{-1} a_t = \frac{1}{j} a^{-1} \psi^{j-1} \psi_t = \frac{1}{j} \psi^{j-1} \psi_t,
\]
(2.43)
\[
\nabla b = a^{-1} \nabla a = \frac{1}{j} a^{-1} \psi^{j-1} \nabla \psi = \frac{1}{j} \psi^{j-1} \nabla \psi.
\]
(2.44)
Substituting these into (2.42) gives the desired result. The proof is complete. □
3. A priori estimates

In this section we derive a priori estimates for solutions to (1.1)-(1.5). The main theorem will be established as a consequence of these estimates. We begin with the energy estimate.

Lemma 3.1. Assume \( u_0 \in L^2(\Omega) \). Then we have

\[
\frac{1}{2} \sup_{0 \leq t \leq T} \int_\Omega u^2 \, dx + \int_\Omega \nabla u \cdot \nabla u \, dx \, dt \leq \int_\Omega u_0^2(x) \, dx.
\]

Proof. Use \( u \) as a test function in (1.2) and keep in mind (1.11) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dx + \int_\Omega \left( (a|q| + m)|\nabla u|^2 + \frac{b-a}{|q|}(q \cdot \nabla u)^2 \right) dx = -\frac{1}{2} \int_\Omega q \cdot \nabla u^2 \, dx.
\]

By the definition of \( q \), we have

\[
\int_\Omega q \cdot \nabla u^2 \, dx = \int_\Omega (-v_{x_2}(u^2)_{x_1} + v_{x_1}(u^2)_{x_2}) \, dx = \int_\Omega \text{div} \left( \frac{v(u^2)_{x_2}}{-v(u^2)_{x_1}} \right) \, dx = 0.
\]

The last step is due to the boundary condition for \( v \). Plug this into (3.2) and integrate to obtain the desired result. \( \square \)

Lemma 3.2. The function \( u \) satisfies the weak maximum principle, i.e.,

\[
\sup_{\Omega_T} |u| \leq \|u_0\|_{\infty, \Omega}.
\]

This is a consequence of (1.8). Indeed, let \( K = \|u_0\|_{\infty, \Omega} \). Then we can write (1.2) in the form

\[
u_t - \text{div} (D \nabla u - (u - K)q) = 0 \quad \text{in} \quad \Omega_T,
\]

where \( D \) is given as in (1.7). Use \((u - K)^+\) as a test function in this equation and then apply (1.11) to get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega [(u - K)^+]^2 \, dx + \int_\Omega \left( (a|q| + m)|\nabla (u - K)^+|^2 + (b-a)(q \cdot \nabla (u - K)^+)^2 \right) dx = -\frac{1}{2} \int_\Omega q \cdot \nabla [(u - K)^+]^2 \, dx.
\]

As before, the last integral is zero. Thus we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega [(u - K)^+]^2 \, dx + m \int_\Omega |\nabla (u - K)^+|^2 \, dx \leq 0.
\]

Integrate to obtain the desired result.

According to a result in ([7], p.82), for each \( r > 1 \) there is a positive number \( c \) such that

\[
\sup_{0 \leq t \leq T} \|q\|_{r, \Omega} = \sup_{0 \leq t \leq T} \|\nabla q\|_{r, \Omega} \leq c \sup_{0 \leq t \leq T} \|u\|_{r, \Omega} \leq c.
\]

In addition, we can conclude from the classical Calderón-Zygmund estimate that

\[
\int_0^T \|v\|_{W^{2,2}(\Omega)}^2 dt \leq c \int_0^T \int_\Omega (u_{x_1})^2 dx dt \leq c.
\]

Now we set

\[
\phi = v_t, \quad \varphi = u_t.
\]

By (1.12) and (1.6), we have

\[
|D_t| \leq c|\nabla \phi|, \quad |q_t| = |\nabla \phi|.
\]
Differentiate (1.1) with respect to $t$ to get
\begin{equation}
\Delta \phi = \varphi_{x_1} \text{ in } \Omega_T.
\end{equation}
Moreover,
\begin{equation}
\phi \mid_{\partial \Omega} = 0.
\end{equation}
A result of [7] asserts that for each $r > 1$ there is a positive number $c$ such that
\begin{equation}
\|\nabla \phi\|_{r, \Omega} \leq c\|\varphi\|_{r, \Omega}.
\end{equation}
For Theorem 2.2 to be useful to us, we must be able to bound $u_t$ by $\nabla u$. The following two lemmas address this issue.

**Lemma 3.3.** We have
\begin{equation}
\int_{\Omega_T} u_t^2 \, dx \, dt + \sup_{0 \leq t \leq T} \int_{\Omega} D\nabla u \cdot \nabla u \, dx \leq c \int_{\Omega_T} |\nabla u|^4 \, dx \, dt + c.
\end{equation}

**Proof.** Use $u_t$ as a test function in (1.2) to obtain
\begin{equation}
\int_{\Omega} (u_t)^2 \, dx + \int_{\Omega} D\nabla u \cdot \nabla u_t = -\int_{\Omega} q \cdot \nabla uu_t \, dx.
\end{equation}

Note that $D$ is symmetric. Hence
\begin{equation}
\int_{\Omega} D_t \nabla u \cdot \nabla u dx \leq \left( \int_{\Omega} |D_t|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla u|^4 dx \right)^\frac{1}{2}
\end{equation}
\begin{equation}
\leq c \left( \int_{\Omega} |u_t|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla u|^4 dx \right)^\frac{1}{2}.
\end{equation}

Similarly,
\begin{equation}
-\int_{\Omega} q \cdot \nabla uu_t \, dx \leq \left( \int_{\Omega} |u_t|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla u|^4 dx \right)^\frac{1}{2} \left( \int_{\Omega} |q|^4 dx \right)^\frac{1}{4}
\leq c \left( \int_{\Omega} |u_t|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla u|^4 dx \right)^\frac{1}{4}.
\end{equation}

Here the last step is due to (3.8). Use the above two inequalities in (3.16) and integrate to derive
\begin{equation}
\int_{\Omega_T} (u_t)^2 dx dt + \sup_{0 \leq t \leq T} \int_{\Omega} D\nabla u \cdot \nabla u dx \leq c \int_{\Omega_T} |\nabla u|^4 dx dt + c \int_{\Omega} D(x,0) \nabla u_0 \cdot \nabla u_0 dx + c.
\end{equation}

Our assumptions on $u_0$ implies that
\begin{equation}
\int_{\Omega} D(x,0) \nabla u_0 \cdot \nabla u_0 dx \leq c.
\end{equation}

To see this, remember that $v_0 \equiv v(x,0)$ is the solution of
\begin{equation}
\Delta v_0 = (u_0)_{x_1} \text{ in } \Omega,
\end{equation}
\begin{equation}
v_0 = 0 \text{ on } \partial \Omega
\end{equation}
and
\begin{equation}
q(x,0) = \left( -v_0(x_2) \atop (v_0(x_2))_{x_1} \right).
\end{equation}
In fact, it is enough for us to assume that \( u_0 \in W^{1,p}(\Omega) \) for some \( p > 2 \) because this already implies \( |q(x, 0)| = |\nabla v_0| \in L^\infty(\Omega) \). Therefore,

\[
|D(x, 0)| = |(a|q(x, 0)| + m)I + \frac{b-a}{|q(x, 0)|}q(x, 0) \otimes q(x, 0)| \\
\leq b|q(x, 0)| + m \leq c. 
\]

The proof is complete. \( \square \)

**Lemma 3.4.** We have

\[
\|\varphi\|_{\infty, \Omega_T} \leq c\|\nabla u\|_{\infty, \Omega_T}^4 + c. 
\]

**Proof.** First we observe that

\[
\|\varphi(x, 0)\|_{\infty, \Omega} \leq c. 
\]

To see this, we let \( t = 0 \) in (1.2) to obtain

\[
\varphi(x, 0) = \text{div}(D(x, 0)\nabla u(x, 0)) - \nabla u(x, 0)q(x, 0) \\
= D(x, 0) : \nabla^2 u_0 + \text{div}D(x, 0)\nabla u_0 - \nabla u_0q(x, 0). 
\]

We can easily derive (3.26) from (H2), (3.24), (1.13), and (3.22).

Differentiate (1.2) with respect to \( t \) to get

\[
\varphi_t - \text{div}(D\nabla \varphi + D_t \nabla u) = -q \cdot \nabla \varphi - q_t \cdot \nabla u \quad \text{in } \Omega_T. 
\]

Furthermore,

\[
\partial_t (D\nabla u) \cdot \nu = 0 \quad \text{on } \Sigma_T. 
\]

Pick

\[
k \geq 2\|\varphi(x, 0)\|_{\infty, \Omega} 
\]

as below. Set

\[
k_n = k - \frac{k}{2^n+1}, \quad n = 0, 1, 2, \ldots . 
\]

Assume that

\[
\sup_{\Omega_T} \varphi = \|\varphi\|_{\infty, \Omega_T}. 
\]

Otherwise, consider \(-\varphi\). In view of (3.29), we can use \((\varphi - k_n)\) as a test function in (3.28) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega [(\varphi - k_n)^+]^2 dx + \int_\Omega D\nabla (\varphi - k_n)^+ \cdot \nabla (\varphi - k_n)^+ dx \\
= -\frac{1}{2} \int_\Omega q \cdot \nabla [(\varphi - k_n)^+]^2 dx - \int_\Omega D_t \nabla u \cdot \nabla (\varphi - k_n)^+ dx - \int_\Omega \nabla u \cdot q_t (\varphi - k_n)^+ dx. 
\]

Set

\[
S_n(t) = \{ x \in \Omega : \varphi(x, t) \geq k_n \}, \\
A_{\Omega, k}(t) = \frac{1}{|\Omega|} \int_\Omega (\varphi(x, t) - k_n)^+ dx. 
\]
By virtue of Poincaré’s inequality, for each \( r > 2 \) we have that

\[
- \int_{\Omega} \nabla u \cdot \mathbf{q}_t (\varphi - k_n)^+ \, dx \leq \| \nabla u \cdot \mathbf{q}_t \|_{S_n(t)} \left( \int_{\Omega} \left| \nabla (\varphi - k_n)^+ \right|^{\frac{2r}{r+2}} \, dx \right)^{\frac{r+2}{2r}} \\
- A_{\Omega, k}(t) \int_{S_n(t)} \nabla u \cdot \mathbf{q}_t \, dx \\
\leq c \| \nabla u \cdot \mathbf{q}_t \|_{S_n(t)} \left( \int_{\Omega} \left| \nabla (\varphi - k_n)^+ \right|^2 \, dx \right)^{\frac{1}{2}} \\
+ c \left( \int_{\Omega} \left[ (\varphi - k_n)^+ \right]^2 \, dx \right)^{\frac{1}{2}} \int_{S_n(t)} |\nabla u \cdot \mathbf{q}_t| \, dx.
\]

(3.36)

This together with (3.33) implies

\[
\sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx + \int_{\Omega_T} |\nabla (\varphi - k_n)^+|^2 \, dx \, dt \leq c \int_{0}^{T} \int_{S_n(t)} |D_t \nabla u - u \mathbf{q}_t|^2 \, dx \, dt.
\]

(3.37)

Here we have used the fact that \( \frac{t}{T} < 2 \). Let

\[
y_n = \int_{0}^{T} |S_n(t)| \, dt = \{|(x, t) \in \Omega_T : \varphi(x, t) \geq k_n\}.
\]

Let \( s \in (1, 2) \) be given. We estimate from Poincaré’s inequality that

\[
\int_{\Omega_T} [(\varphi - k_n)^+]^{2s} \, dx \, dt \\
= \int_{0}^{T} \left( \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \left( \int_{\Omega} \left[ (\varphi - k_n)^+ \right]^{\frac{2s}{r}} \, dx \right)^{\frac{2-r}{2}} \, dt \\
\leq 2^{s-1} \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \int_{0}^{T} \left( \int_{\Omega} \left| \nabla (\varphi - k_n)^+ \right|^2 \, dx \right)^{\frac{s}{2}} \, dt \\
+ 2^{s-1} \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \int_{0}^{T} A_{\Omega, k}(t) |S_n(t)|^{\frac{2-s}{2}} \, dt \\
\leq \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \int_{0}^{T} \int_{\Omega} |\nabla (\varphi - k_n)^+|^s \, dx \, dt \\
+ c \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \left( \int_{0}^{T} A_{\Omega, k}(t) \, dt \right)^{\frac{s}{2}} \frac{2-s}{2n} y_n^{1-s} \\
\leq c \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \left( \int_{\Omega_T} |\nabla (\varphi - k_n)^+|^2 \, dx \, dt \right)^{\frac{s}{2}} y_n^{1-s} \\
+ c \left( \sup_{0 \leq t \leq T} \int_{\Omega} [(\varphi - k_n)^+]^2 \, dx \right)^{\frac{s}{2}} \frac{2-s}{2n} y_n^{1-s} \\
\leq c \left( \int_{0}^{T} \int_{S_n(t)} |D_t \nabla u - u \mathbf{q}_t|^2 \, dx \, dt \right)^{\frac{s}{2}} y_n^{1-s} \\
(3.39) \leq c \|D_t \nabla u - u \mathbf{q}_t\|_{2q, \Omega_T}^{\frac{s(q-2)}{2q}} y_n^{\frac{1}{2} + \frac{s(q-2)}{2q}},
\]
where \( q > 2 \). On the other hand, we have

\[
\int_{\Omega_T} \left[ (\phi - k_n)^+ \right]^{2s} \, dx \, dt \geq (k_{n+1} - k_n)^{2s} y_{n+1} = \frac{k^{2s}}{4^s(n+1)} y_{n+1}.
\]

Combining this with (3.39) yields

\[
y_{n+1} \leq c \frac{4^s}{k^{2s}} \left\| D_t \nabla u - u q_t \right\|_{2q, \Omega_T}^{2s} y_n^{1+\frac{(q-2)}{2q}}.
\]

By Lemma 2.1, we have

\[
\lim_{n \to \infty} y_n = 0
\]

provided that

\[
y_0 \leq c \left( \frac{k^{2s}}{\| D_t \nabla u - u q_t \|_{2q, \Omega_T}^{2s}} \right)^{\frac{2q}{q(q-2)}}.
\]

This together with (3.30) implies

\[
\phi \leq k = c \| D_t \nabla u - u q_t \|_{2q, \Omega_T} + c \| \phi(x,0) \|_{\infty, \Omega}.
\]

Subsequently,

\[
\| \phi \|_{\infty, \Omega_T} \leq c \| \nabla u \|_{\infty, \Omega_T} \| D_t \|_{2q, \Omega_T} + c \| q_t \|_{2q, \Omega_T} + c
\]

\[
\leq c(\| \nabla u \|_{\infty, \Omega_T} + 1) \| \nabla \phi \|_{2q, \Omega_T} + c
\]

\[
\leq c(\| \nabla u \|_{\infty, \Omega_T} + 1) \| \phi \|_{2q, \Omega_T} + c.
\]

The last step is due to (3.14). In view of (2.8), we have

\[
\| \phi \|_{2q, \Omega_T} \leq c \| \phi \|_{\infty, \Omega_T} + \frac{1}{\varepsilon^{q-1}} \| \phi \|_{2, \Omega_T}.
\]

By choosing \( \varepsilon \) suitably, we arrive at

\[
\| \phi \|_{\infty, \Omega_T} \leq c \| \phi \|_{2, \Omega_T} (\| \nabla u \|_{\infty, \Omega_T} + 1) \frac{q}{q-1} + c
\]

\[
\leq c \| \phi \|_{2, \Omega_T} (\| \nabla u \|_{\infty, \Omega_T} + 1)^2 + c.
\]

The last step is due to the fact that \( \frac{q}{q-1} < 2 \) since \( q > 2 \). Use Lemma 3.3 to yield the desired result. \( \square \)

We are ready to prove the main theorem

**Proof of the Main Theorem.** By Theorem 2.2, the function \( \psi = a^j \) satisfies

\[
\frac{1}{\psi} \psi_t - \text{div} \left( \frac{1}{\psi} D \nabla \psi \right) = \frac{1}{\psi} \mathbf{H} \cdot \nabla \psi + j h + j \text{div} \mathbf{K} \text{ in } \{ \| \nabla u \| > 0 \}
\]

Now fix a point \( z_0 = (x_0, t_0) \in \Omega_T \). Then pick a number \( R \) from \((0, \min \{ \text{dist}(x_0, \partial \Omega), \sqrt{t_0} \})\).

Define a sequence of cylinders \( Q_{R_n}(z_0) \) in \( \Omega_T \) as follows:

\[
Q_{R_n}(z_0) = B_{R_n}(x_0) \times (t_0 - R_n^2, t_0),
\]

where

\[
R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \quad n = 0, 1, 2, \ldots
\]
and $B_{R_n}(x_0)$ is the open ball centered at $x_0$ with radius $R_n$. Choose a sequence of smooth functions $\theta_n$ so that

\begin{align}
\theta_n(x,t) &= 1 \quad \text{in } Q_{R_n}(z_0), \\
\theta_n(x,t) &= 0 \quad \text{outside } Q_{R_{n-1}}(z_0),
\end{align}

\begin{align}
|\partial_t \theta_n(x,t)| &\leq \frac{c4^n}{R^2} \quad \text{on } Q_{R_{n-1}}(z_0), \\
|\nabla \theta_n(x,t)| &\leq \frac{c2^n}{R} \quad \text{on } Q_{R_{n-1}}(z_0), \quad \text{and}
\end{align}

\begin{align}
0 &\leq \theta_n(x,t) \leq 1 \quad \text{on } Q_{R_{n-1}}(z_0).
\end{align}

Select

\begin{align}
K > 2
\end{align}

as below. Set

\begin{align}
K_n = K - \frac{K}{2^{n+1}}, \quad n = 0, 1, 2, \cdots.
\end{align}

We use $\theta_{n+1}^2(\psi - K_{n+1})^+$ as a test function in (3.48) to obtain

\begin{align}
\frac{d}{dt} \int_0^{\ln \psi} (e^s - K_{n+1})^+ ds \theta_{n+1}^2 dx + \int_\Omega \frac{1}{\psi} D\nabla \psi \cdot \nabla (\psi - K_{n+1})^+ \theta_{n+1}^2 dx
&= 2 \int_0^{\ln \psi} (e^s - K_{n+1})^+ ds \theta_{n+1} \partial_t \theta_{n+1} dx - 2 \int_\Omega \frac{1}{\psi} D\nabla \psi \cdot \nabla \theta_{n+1}(\psi - K_{n+1})^+ \theta_{n+1} dx \\
&\quad + \int_\Omega \frac{1}{\psi} H\nabla \psi \theta_{n+1}^2 (\psi - K_{n+1})^+ dx + j \int_\Omega h \theta_{n+1}^2 (\psi - K_{n+1})^+ dx \\
&\quad - j \int_\Omega K \cdot \nabla (\psi - K_{n+1})^+ \theta_{n+1}^2 dx - 2j \int_\Omega K \cdot \nabla \theta_{n+1}(\psi - K_{n+1})^+ \theta_{n+1} dx.
\end{align}

We easily evaluate

\begin{align}
\int_0^{\ln \psi} (e^s - K_{n+1})^+ ds = (\psi - K_{n+1})^+ - K_{n+1} (\ln \psi - \ln K_{n+1})^+.
\end{align}

We claim that

\begin{align}
(\psi - K_{n+1})^+ - K_{n+1} (\ln \psi - \ln K_{n+1})^+ \geq \left[ \left( \sqrt{\psi} - \sqrt{K_{n+1}} \right)^+ \right]^2.
\end{align}

To see this, we consider the function

\begin{align}
g(s) = \frac{2}{\sqrt{K_{n+1}}} \left( \sqrt{s} - \sqrt{K_{n+1}} \right) - \ln s + \ln K_{n+1} \quad \text{on } [K_{n+1}, \infty).
\end{align}

A simple calculation shows that

\begin{align}
g(K_{n+1}) &= 0, \\
g'(s) &= \frac{1}{\sqrt{s}} \left( \frac{1}{\sqrt{K_{n+1}}} - \frac{1}{\sqrt{s}} \right) > 0 \quad \text{for } s > K_{n+1}.
\end{align}

This immediately implies that

\begin{align}
\frac{2}{\sqrt{K_{n+1}}} \left( \sqrt{\psi} - \sqrt{K_{n+1}} \right)^+ \geq (\ln \psi - \ln K_{n+1})^+.
\end{align}

It is not difficult to see that this inequality is equivalent to (3.60).

Note that

\begin{align}
\nabla \psi = \nabla (\psi - K_{n+1})^+ \quad \text{on } S_{n+1}(t),
\end{align}
where

\[ S_{n+1}(t) = \{ x \in B_n(x_0) : \psi(x,t) \geq K_{n+1} \}. \]

By (1.11), we have

\[
\begin{align*}
D \nabla \psi \cdot \nabla (\psi - K_{n+1})^+ &= (a|q| + m)|\nabla (\psi - K_{n+1})^+|^2 + \frac{b-a}{|q|}(q \cdot \nabla (\psi - K_{n+1})^+)^2, \\
D \nabla \psi (\psi - K_{n+1})^+ &= (a|q| + m)\nabla (\psi - K_{n+1})^+(\psi - K_{n+1})^+ \\
&+ \frac{b-a}{|q|}(q \cdot \nabla (\psi - K_{n+1})^+)(\psi - K_{n+1})^+ q.
\end{align*}
\]

Integrate (3.58) with respect to \( t \) and then incorporate the preceding results in the resulting equation to deduce

\[
\begin{align*}
\int_{\Omega} \left[ \sqrt{\psi} - \sqrt{K_{n+1}}^+ \right]^2 \theta_{n+1}^2 dx + \int_{\Omega_r} \frac{1}{\psi}(a|q| + m)|\nabla (\psi - K_{n+1})^+|^2 \theta_{n+1}^2 dx dt \\
\leq \frac{c^4}{R^2} \int_{Q_{R_n}(z_0)} (\psi - K_{n+1})^+ dx dt + \frac{c^4}{R^2} \int_{Q_{R_n}(z_0)} \frac{(a|q| + m)}{\psi} [(\psi - K_{n+1})^+]^2 dx dt \\
+ \int_{\Omega_r} \frac{c}{\psi} |H|^2 \theta_{n+1}^2 \left[(\psi - K_{n+1})^+\right]^2 dx dt + c \int_{\Omega_r} |H| \theta_{n+1}^2 (\psi - K_{n+1})^+ dx dt \\
+ c \int_{t_0-R_n^2}^{t_0} \int_{S_{n+1}(t)} \psi |K|^2 \theta_{n+1}^2 dx dt + \frac{c^2}{R} \int_{\Omega_r} |K|(\psi - K_{n+1})^+ \theta_{n+1} dx dt,
\end{align*}
\]

where \( \Omega_r = \Omega \times (0, \tau) \) for \( \tau \in (0, t_0] \). The last term in (3.69) can be estimated as follows:

\[
\frac{2^n}{R} \int_{\Omega_r} |K|(\psi - K_{n+1})^+ \theta_{n+1} dx dt \leq \frac{c^4}{R^2} \int_{t_0-R_n^2}^{t_0} \int_{S_{n+1}(t)} \frac{1}{\psi} [(\psi - K_{n+1})^+]^2 dx dt \\
+ c \int_{t_0-R_n^2}^{t_0} \int_{S_{n+1}(t)} \psi |K|^2 \theta_{n+1}^2 dx dt.
\]

Observe that

\[
\begin{align*}
\frac{1}{\psi} |\nabla (\psi - K_{n+1})^+|^2 &= 4|\nabla (\sqrt{\psi} - \sqrt{K_{n+1}}^+)|^2, \\
\frac{1}{\psi} \left[(\psi - K_{n+1})^+\right]^2 &= \frac{1}{\psi} \left[ \left( \sqrt{\psi} - \sqrt{K_{n+1}}^+ \right)^2 \left( \sqrt{\psi} + \sqrt{K_{n+1}} \right)^2 \right] \\
&= \left[ \left( \sqrt{\psi} - \sqrt{K_{n+1}}^+ \right)^2 \left( 1 + \frac{\sqrt{K_{n+1}}}{\sqrt{\psi}} \right) \right] \\
&\leq 4 \left( \sqrt{\psi} - \sqrt{K_{n+1}}^+ \right)^2.
\end{align*}
\]

Notice that

\[
\frac{\sqrt{K_{n+1}} - \sqrt{K_n}}{\sqrt{K_{n+1}}} = \frac{\sqrt{1 - \frac{1}{2^n + 2}} - \sqrt{1 - \frac{1}{2^{n+1}}}}{\sqrt{1 - \frac{1}{2^{n+1}}}}
\]

\[
= \frac{1}{2^n + 2} \left( \sqrt{1 - \frac{1}{2^n + 2}} + \sqrt{1 - \frac{1}{2^{n+1}}} \right) \sqrt{1 - \frac{1}{2^{n+2}}}
\]

\[
\geq \frac{1}{2^{n+2}}.
\]
With this in mind, we estimate
\[
\left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \geq \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \chi_{S_{n+1}}(t) \\
= \frac{1}{2} \left( (\sqrt{\psi} - \sqrt{K_n})^+ (\sqrt{\psi} + \sqrt{\psi}) \left( 1 - \frac{\sqrt{K_n}}{\sqrt{\psi}} \right) \chi_{S_{n+1}}(t) \right) \\
\geq \frac{1}{2} \left( (\sqrt{\psi} - \sqrt{K_n})^+ (\sqrt{\psi} + \sqrt{K_{n+1}}) \left( 1 - \frac{\sqrt{K_n}}{\sqrt{K_{n+1}}} \right) \chi_{S_{n+1}}(t) \right) \\
\geq \frac{1}{2^{n+1}} (\psi - K_{n+1})^+. \\
\tag{3.74}
\]
Here \( \chi_{S_{n+1}}(t) \) is the indicator function of the set \( S_{n+1}(t) \). Similarly,
\[
\left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \geq \psi \left[ (1 - \frac{\sqrt{K_n}}{\sqrt{\psi}}) \right]^2 \chi_{S_{n+1}}(t) \geq \frac{1}{2^{2(n+3)}} \psi \chi_{S_{n+1}}(t). \\
\tag{3.75}
\]
Plugging the preceding results into (3.69), we obtain
\[
J_n + \int_{\Omega_T} \left| \nabla (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right|^2 \theta_{n+1}^2 dxdt \\
\leq \frac{c^8 n}{R^2} \int_{Q_n(z_0)} (1 + |q|) \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 dxdt \\
+ c \int_{\Omega_T} \left| H \right|^2 \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 \theta_{n+1}^2 dxdt + c_{2n} \int_{\Omega_T} |h| \theta_{n+1}^2 \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 dxdt \\
\tag{3.76}
+ c_{2n} \int_{\Omega_T} \left| K \right|^2 \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 \theta_{n+1}^2 dxdt,
\]
where
\[
J_n = \sup_{t_0 - R_2^2 \leq t \leq t_0} \int_{\Omega} \left| (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right|^2 \theta_{n+1}^2 dxd.
\tag{3.77}
\]
We pick a number \( r \) from the interval \((1, \infty)\). Define
\[
y_n = \left( \int_{Q_{R_n}(z_0)} \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^{2r} dxdt \right)^{\frac{1}{2r}}. \\
\tag{3.78}
\]
Set
\[
Q_{n+1} = \{ (x, t) \in Q_{R_n}(z_0), \psi \geq K_{n+1} \}. \\
\tag{3.79}
\]
We conclude from (3.76) that
\[
J_n + \int_{\Omega_T} \left| \nabla (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right|^2 \theta_{n+1}^2 dxdt \\
\leq \frac{c^8 n}{R^2} \| (1 + |q|) \|_{r^{-1}, Q_1} y_n + c \| H \|_{r^{-1}, Q_1} y_n + c_{2n} \| h \|_{r^{-1}, Q_1} y_n \\
+ c_{2n} \| K \|_{r^{-1}, Q_1} y_n \\
\tag{3.80}
\leq \frac{c^8 n}{R^2} \Gamma y_n,
\]
where
\[
\Gamma = \| (1 + |q|) \|_{r^{-1}, Q_1} + R^2 \left( \| H \|_{r^{-1}, Q_1} + \| h \|_{r^{-1}, Q_1} + \| K \|_{r^{-1}, Q_1} \right). \\
\tag{3.81}
\]
Now we further require \( r \) to be less than 2. With the aid of Poincaré’s inequality, we derive
\[
\int_{t_0 - R_n^2}^{t_0} \int_{\Omega} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right)^{2r} dx dt
\]
\[
\leq \int_{t_0 - R_n^2}^{t_0} \int_{\Omega} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right)^{2} dx \left( \int_{\Omega} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right)^{2r} dx \right)^{\frac{2}{2r}} dt
\]
\[
\leq c J_n^r \int_{Q_{R_n(z_0)}} \left| \nabla \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right) \right|^r dx dt
\]
\[
\leq c J_n^r \left( \int_{Q_{R_n(z_0)}} \left| \nabla \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right) \right|^2 \right)^{\frac{r}{2}} \left| Q_{n+1} \right|^{1 - \frac{r}{2}}
\]
\[
\leq c J_n^r \left( \int_{Q_{R_n(z_0)}} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} dx \right)^2 \right)^{\frac{r}{2}} \left| Q_{n+1} \right|^{2 - \frac{r}{2}}
\]
\[
+ \frac{c^4 n J_n}{R^2} \int_{Q_{R_n(z_0)}} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \right)^2 \theta_{n+1} dx dt
\]
\[
\leq \left( \frac{c^8 n}{R^2} \Gamma y_n \right)^r \left| Q_{n+1} \right|^{2 - \frac{r}{2}} + \left( \frac{c^4 n}{R^2} \right)^r \Gamma^2 y_n \left| Q_{n+1} \right|^{\frac{1}{2}}.
\]
Consequently,
\[
y_{n+1} \leq \left( \int_{t_0 - R_n^2}^{t_0} \int_{\Omega} \left( \sqrt{\psi - \sqrt{K_{n+1}}}^+ \theta_{n+1} \right)^{2r} dx dt \right)^{\frac{1}{2}}
\]
\[
\leq \frac{c^8 n}{R^2} \Gamma y_n \left| Q_{n+1} \right|^{2 - \frac{r}{2}} + \frac{c^4 n}{R^2} \Gamma^2 y_n \left| Q_{n+1} \right|^{\frac{1}{2}}
\]
\[
\leq \frac{c^8 n}{R^2} \left( \Gamma + R^{2(r-1)} \right) y_n \left| Q_{n+1} \right|^{2 - \frac{r}{2}}.
\]
\[
(3.82)
\]
We easily see that
\[
y_n \geq \left( \int_{Q_{n+1}} \left( \sqrt{K_{n+1}} - \sqrt{K_n} \right)^{2r} dx dt \right)^{\frac{1}{r}} \geq \frac{K}{2^{2(n+3)}} \left| Q_{n+1} \right|^{\frac{1}{r}}.
\]
\[
(3.83)
\]
Substituting this into (3.82) yields
\[
y_{n+1} \leq \frac{c^8 n}{R^2 K^{2 - \frac{r}{2}}} \left( \Gamma + R^{2(r-1)} \right) y_n \left| Q_{n+1} \right|^{1 + \frac{2r}{2r - r}}.
\]
\[
(3.84)
\]
In view of Lemma 2.1 and (3.56), it is enough for us to take
\[
K = \frac{c}{R^{2-r}} y_0 \left( \Gamma + R^{2(r-1)} \right)^{\frac{2}{2r}} + 2
\]
to obtain
\[
(3.85)
\]
\[
\sup_{Q_{R_n(z_0)}} \psi \leq K = \frac{c}{R^{2-r}} y_0 \left( \Gamma + R^{2(r-1)} \right)^{\frac{2}{2r}} + 2.
\]
\[
(3.86)
\]
Now we proceed to estimate \( \Gamma \). The boundedness of the first term in \( \Gamma \) is a direct consequence of (3.8). As for the remaining terms, we first observe from (1.7) that
\[
(3.87)
\]
\[
|D| \leq a |q| + m.
\]
Subsequently,
\begin{align}
|D_1| & \leq (c_1|\mathbf{q}|^2 + c_2)|\nabla u|, \quad i = 1, 2, \\
|D_3| & \leq (c_1|\mathbf{q}|^2 + c_2)|\nabla u|^2.
\end{align}

It follows from (1.11) and (1.13) that
\begin{equation}
|\mathbf{G}| \leq c|\nabla^2 v|.
\end{equation}

Note that
\begin{equation}
\det(D) = (a|\mathbf{q}| + m)(b|\mathbf{q}| + m).
\end{equation}

Thus we have
\begin{equation}
|\nabla \det(D)| \leq c(|\mathbf{q}| + 1)|\nabla^2 v|.
\end{equation}

We are ready to estimate
\begin{equation}
|\mathbf{K}| = \left|-DG + 2a^{-1}(\text{div}D\nabla u - \nabla u \cdot \mathbf{q})D\nabla u\right|
\leq c(|\mathbf{q}| + 1)|\nabla^2 v| + c(|\mathbf{q}|^2 + |\mathbf{q}|).
\end{equation}

Note that
\begin{equation}
\psi \geq 1 \quad \text{if and only if} \quad a \geq 1.
\end{equation}

By (1.11), we have
\begin{equation}
|\nabla u|^2 \geq \frac{1}{b|\mathbf{q}| + m} \quad \text{in} \quad Q_1,
\end{equation}

from whence follows
\begin{equation}
|a^{-1}u_t D\nabla u| \leq \frac{c|u_t|(b|\mathbf{q}| + m)}{|\nabla u|} \leq c|u_t|(b|\mathbf{q}| + m)\frac{3}{2} \quad \text{in} \quad Q_1.
\end{equation}

We are in a position to estimate
\begin{equation}
|\mathbf{H}| = \left|DG + 2a^{-1}u_tD\nabla u + \frac{1}{\det(D)a} \left(D_1^T \nabla \det(D), D_2^T \nabla \det(D)\right) \nabla u\right|
\leq c(|\mathbf{q}| + 1)|\nabla^2 v| + c|u_t|(b|\mathbf{q}| + m)\frac{3}{2} \quad \text{in} \quad Q_1.
\end{equation}

As for \( h \), we first note that all our previous calculations are still valid if we drop the two non-positive terms in \( h \). Keeping this in mind, we estimate
\begin{equation}
|h| \leq \left|\frac{1}{\det(D)a} \nabla \det(D) \cdot (D_1 \mathbf{G}, D_2 \mathbf{G}) \nabla u + \frac{2(u_t - \text{div}D\nabla u + \nabla u \cdot \mathbf{q})}{\det(D)a^2} \nabla \det(D) \cdot D_3\nabla u\right|
\end{equation}
\begin{align*}
&+ \left|-2a^{-1}u_t \nabla u \mathbf{q} + a^{-1}D_1 \nabla u \cdot \nabla u - 2a^{-1}(u_t - \text{div}D\nabla u + \nabla u \cdot \mathbf{q})D\nabla u \cdot \mathbf{G}\right|
\end{align*}
\begin{equation}
\leq c(|\mathbf{q}| + 1)|\nabla^2 v|^2 + c|u_t|(b|\mathbf{q}| + m)\frac{3}{2}|\nabla^2 v| + c|\mathbf{q}|(|\mathbf{q}| + 1)|\nabla^2 v|^2 + c|u_t|(b|\mathbf{q}| + m)\frac{3}{2}
\end{equation}
\begin{equation}
+ c|\nabla u_t| + c|\mathbf{q}|(|\mathbf{q}| + 1)|\nabla^2 v|.
\end{equation}

With the aid of (3.8), we estimate for \( s > \frac{2r}{r-1} \) that
\begin{equation}
\left(\int_{Q_1} |\mathbf{H}|^{\frac{2r}{r-1}} dx dt\right)^{\frac{r-1}{r}} \leq c \left(\int_{Q_1} \left[\left(|\mathbf{q}| + 1\right)|\nabla^2 v|^2\right]^{\frac{2r}{r-1}} dx dt\right)^{\frac{r-1}{r}}
\end{equation}
\begin{equation}
+ c \left(\int_{Q_1} \left[|u_t|(b|\mathbf{q}| + m)\frac{3}{2}\right]^{\frac{2r}{r-1}} dx dt\right)^{\frac{r-1}{r}}
\end{equation}
\begin{equation}
\leq c\|\nabla^2 v\|_{s, Q_1}^2 + c\|u_t\|_{s, Q_1}^2.
\end{equation}
Similarly, for the same $s$ we have

$$\|K\|_{s,Q_1} \leq c\|\nabla^2 v\|_{s,Q_1}^2 + c,$$

$$\|h\|_{s,Q_1} \leq c\|\nabla^2 v\|_{s,Q_1}^2 + c\|u_t\|_{s,Q_1}^2 + c\|\nabla v_t\|_{r^{-1},Q_1} + c.$$  

Recall that

$$y_0 = \left( \int_{Q_R(z_0)} \left[ \left( \sqrt{\psi} - \sqrt{K/2} \right)^{2r} \right] dxdt \right)^{\frac{1}{2}} \leq \|a\|_{j,r,Q_R(z_0)}.$$  

Collecting the preceding estimates in (3.86) and taking the $j$th root of the resulting inequality, we arrive at

$$\sup_{Q_R(z_0)} a \leq \frac{c}{R^{(2-r)j}} \left( \|\nabla^2 v\|_{s,Q_1}^2 + \|u_t\|_{s,Q_1}^2 + \|\nabla v_t\|_{r^{-1},Q_1} + 1 \right)^{\frac{2}{(2-r)j}} + c.$$  

By an argument in ([4], p. 303), we can extend the above estimate to the whole $\Omega_T$. That is, we have

$$\sup_{\Omega_T} a \leq c\|a\|_{j,r,T} \left( \|\nabla^2 v\|_{s,T}^2 + \|u_t\|_{s,T}^2 + \|\nabla v_t\|_{r^{-1},T} + 1 \right)^{\frac{2}{(2-r)j}} + c.$$  

We deduce from the classical Calderón-Zygmund inequality, (3.14), and Lemma 3.4 that

$$\sup_{\Omega_T} a \leq c\|a\|_{j,r,T} \left( \|\nabla u\|_{s,T}^8 + \|u_t\|_{s,T}^2 + \|u_t\|_{r^{-1},T} + 1 \right)^{\frac{2}{(2-r)j}} + c.$$  

(3.105)

Remember from Lemma 3.1 that

$$\int_{\Omega_T} a \pi dxdt = \int_{\Omega_T} D\nabla u \cdot \nabla u dxdt \leq c.$$  

On account of (2.8), we have

$$\|a\|_{j,r,T} \leq \varepsilon\|a\|_{\infty,T} + \frac{1}{\varepsilon^{j/r-1}} \|a\|_{1,T}$$

(3.107)

$$\leq \varepsilon\|a\|_{\infty,T} + \frac{c}{\varepsilon^{j/r-1}}, \quad \varepsilon > 0.$$  

Substitute this into (3.104) to obtain

$$\sup_{\Omega_T} a \leq c \left( \|\nabla u\|_{\infty,T}^8 + 1 \right)^{\frac{2}{(2-r)j}} \left( 1 + \frac{1}{j/r-1} \right) + c.$$  

(3.108)

Pick $j$ so large that

$$\frac{8}{(2-r)j} \left( 1 + \frac{1}{j/r-1} \right) < 1.$$  

Then we have

$$\|\nabla u\|_{\infty,T} \leq c.$$  

(3.110)

This completes the proof. □
In this section we design an approximation scheme for (1.1)-(1.5). The key is to find a way to smooth the term $|q|$, while maintaining the basic structure of the original system as much as possible so that all the calculations in the preceding two sections are valid. To do this, let $\zeta$ be a mollifier on $\mathbb{R}^3$. That is, $\zeta$ is a compactly supported $C^\infty$ function with the properties

\begin{align}
(4.1) \quad &\int_{\mathbb{R}^3} \zeta \, dz = \int_{\mathbb{R}^3} \zeta \, dx \, dt = 1, \\
(4.2) \quad &\lim_{\varepsilon \to 0^+} \zeta_\varepsilon(z) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \zeta \left( \frac{z}{\varepsilon} \right) = \delta(z) \text{ in the sense of distributions,}
\end{align}

where $\delta(z)$ is the Dirac delta function. Then set

\begin{align}
(4.3) \quad v_\varepsilon &= \zeta_\varepsilon \ast v, \\
(4.4) \quad q_\varepsilon &= \left( -(v_\varepsilon)_{x_2} \quad (v_\varepsilon)_{x_1} \right).
\end{align}

Outside $\Omega_T$ the function $v$ is understood to be the extension of $v$ in the sense of the Sobolev extension theorem ([3], p.135). Let

\begin{equation}
(4.5) \quad D_\varepsilon = \left( a(|q_\varepsilon|^2 + \varepsilon)^{\frac{1}{2}} + m \right) I + \frac{b-a}{(|q_\varepsilon|^2 + \varepsilon)^{\frac{1}{2}}} q_\varepsilon \otimes q_\varepsilon.
\end{equation}

Obviously, $D_\varepsilon$ is infinitely differentiable for each $\varepsilon > 0$. Moreover,

\begin{equation}
(4.6) \quad m|\xi|^2 \leq D_\varepsilon \xi \cdot \xi = \left( a(|q_\varepsilon|^2 + \varepsilon)^{\frac{1}{2}} + m \right) |\xi|^2 + \frac{b-a}{(|q_\varepsilon|^2 + \varepsilon)^{\frac{1}{2}}} (\xi \cdot q_\varepsilon)^2 \leq \left( b(|q_\varepsilon|^2 + \varepsilon)^{\frac{1}{2}} + m \right) |\xi|^2
\end{equation}

for each $\xi \in \mathbb{R}^2$. We form our approximate problems as follows:

\begin{align}
(4.7) \quad &\Delta v = u_{x_1} \text{ in } \Omega_T, \\
(4.8) \quad &u_t - \text{div} \left( D_\varepsilon \nabla u \right) = -\nabla u \cdot q \text{ in } \Omega_T, \\
(4.9) \quad &v = D_\varepsilon \nabla u \cdot \nu = 0 \text{ on } \Sigma_T, \\
(4.10) \quad &u(x,0) = u_0(x) \text{ on } \Omega.
\end{align}

As before, here $q = (-v_{x_2}, v_{x_1})$.

The existence of a solution to (4.7)-(4.10) can be obtained via the Leray-Schauder fixed point theorem ([4], p.280). To see this, we define an operator $T$ from $L^\infty(\Omega_T)$ into itself as follows: We say $T(u) = u$ if $u$ is the solution of the problem

\begin{align}
(4.11) \quad &u_t - \text{div} \left( D_\varepsilon \nabla u \right) = -\nabla u \cdot q \text{ in } \Omega_T, \\
(4.12) \quad &D_\varepsilon \nabla u \cdot \nu = 0 \text{ on } \Sigma_T, \\
(4.13) \quad &u(x,0) = u_0(x) \text{ on } \Omega.
\end{align}

The functions $q, q_\varepsilon$ in the problem are given by first solving

\begin{align}
(4.14) \quad &\Delta v = w_{x_1} \text{ in } \Omega_T, \\
(4.15) \quad &v = 0 \text{ on } \Sigma_T
\end{align}

and then let

\begin{equation}
(4.16) \quad q = \begin{pmatrix} -v_{x_2} \\ v_{x_1} \end{pmatrix}, \quad q_\varepsilon = \begin{pmatrix} -(v_\varepsilon)_{x_2} \\ (v_\varepsilon)_{x_1} \end{pmatrix}.
\end{equation}

As before, $v_\varepsilon = \zeta_\varepsilon \ast v$. To see that $T$ is well-defined, we conclude from a result in ([7], p.82) that for each $r > 1$ there is a positive number $c$ such that

\begin{equation}
(4.17) \quad \|\nabla v\|_{r,\Omega} \leq c \|w\|_{r,\Omega} \leq c.
\end{equation}
In particular, the number $c$ in the above inequality is independent of $t$. Thus $q$ is a function in $L^\infty(0, T; (L^r(\Omega))^2)$ for each $r > 1$. For each $\varepsilon > 0$ equation (4.11) is linear and uniformly parabolic. Classical results assert that there is a unique solution $u$ to (4.11)-(4.13) in the space $C[0, T; L^2(\Omega)] \cap L^2(0, T; W^{1,2}(\Omega))$. Furthermore, $u$ is Hölder continuous on $\Omega_T$. Thus we can conclude that $T$ is continuous and maps bounded sets into precompact ones. It remains to be seen that we have

$$\|u\|_{\infty, \Omega_T} \leq c$$

for all $u \in L^\infty(\Omega_T)$ and all $\sigma \in (0, 1)$ satisfying $u = \sigma T(u)$. This equation is equivalent to the following

$$\begin{align*}
\Delta v &= u_{x_1} \text{ in } \Omega_T, \\
u_t - \text{div}(D_\varepsilon \nabla u) &= -\nabla u \cdot q \text{ in } \Omega_T, \\
v &= D_\varepsilon \nabla u \cdot \nu = 0 \text{ on } \Sigma_T, \\
T(x, 0) &= \sigma u_0(x) \text{ on } \Omega.
\end{align*}$$

We can easily infer (4.18) from Lemma 3.2.

We can employ a bootstrap argument to gain high regularity on the solution $(v, u)$. We begin with

$$u \in C^{\alpha, \alpha/2}(\Omega_T) \cap L^2(0, T; W^{1,2}(\Omega)) \text{ for some } \alpha \in (0, 1).$$

This together with the Calderón-Zygmund inequality and (4.17) implies

$$v \in L^\infty(0, T; (W^{1,r}(\Omega))^2) \cap L^2(0, T; W^{2,2}(\Omega)) \text{ for each } r > 1.$$  

Remember that entries of $D_\varepsilon$ are infinitely differentiable. Classical results in ([5], Chap. IV) become applicable. Upon using them appropriately, we can conclude that $|\nabla u| \in L^r(\Omega_T)$ for each $r > 1$. This, in turn, implies that $u_t, \Delta u \in L^r(\Omega_T)$ for each $r > 1$, from which it follows that $\nabla v \in L^r(0, T; (W^{2,2}(\Omega))^2)$ for each $r > 1$. By differentiating (4.8) with respect to $x_i, i = 1, 2$, we arrive at $(u_{x_i})_t, \Delta u_{x_i} \in L^r(\Omega_T)$ for each $r > 1$. This combined with the fact that $L^p$ norms of $q_\varepsilon$ (resp. its partial derivatives) are bounded by their corresponding norms of $q$ (resp. its partial derivatives) is sufficient for all the calculations in the preceding sections to carry through here.

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