NEUTRINO MIXING WITH NON-ZERO $\theta_{13}$ AND CP VIOLATION IN THE 3-3-1 MODEL BASED ON $S_4$ FLAVOR SYMMETRY

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The 3-3-1 model proposed in 2011 based on discrete symmetry $S_4$ responsible for the neutrino and quark masses is updated, in which the non-zero $\theta_{13}$ is focused. Neutrino masses and mixings are consistent with the most recent data on neutrino oscillations without perturbation. The new feature is adding a new $SU(3)_L$ anti-sextet lying in doublet under $S_4$ which can result the non-zero $\theta_{13}$ without perturbation, and consequently, the number of Higgs multiplets required is less than those of other models based on non-Abelian discrete symmetries and the 3-3-1 models. The exact tribimaximal form obtained with the breaking $S_4 \rightarrow Z_3$ in charged lepton sector and $S_4 \rightarrow K$ in neutrino sector. If both breakings $S_4 \rightarrow K$ and $K \rightarrow Z_2$ are taken place in neutrino sector, the realistic neutrino spectrum is obtained without perturbation. The upper bound on neutrino mass and the effective mass governing neutrinoless double beta decay at the tree level are presented. The model predicts the Dirac CP violation phase $\delta = 292.45^\circ$ in the normal spectrum (with $\theta_{23} \neq \frac{\pi}{4}$) and $\delta = 303.14^\circ$ in the inverted spectrum.

Keywords: Neutrino mass and mixing; Non-standard-model neutrinos, right-handed neutrinos; Flavor symmetries; Discrete symmetries; Models beyond the standard model.

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1. Introduction

Nowadays, particle physicists are attracted by two exciting subjects: Higgs and neutrino physics. The neutrino mass and mixing are the first evidence of beyond Standard Model physics. Many experiments show that neutrinos have tiny masses and their mixing is still mysterious\cite{footnote1}. The tri-bimaximal form for explaining the lepton mixing scheme was first proposed by Harrison-Perkins-Scott (HPS), which
apart from the phase redefinitions, is given by
defined by the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and they cannot be explained by the Standard Model. It is an interesting challenge to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons given in a completely natural way as first approximations. This has stimulated work on flavor symmetries and non-Abelian discrete symmetries are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and lepton. There are many recent models based on the non-Abelian discrete symmetries, such as $A_4$ (Refs. [11]–[28]), $A_5$ (Refs. [29]–[41]), $S_3$ (Refs. [42]–[83]), $S_4$ (Refs. [84]–[112]), $D_4$ (Refs. [113]–[124]), $D_5$ (Refs. [125]–[126]), $T'$ (Refs. [127]–[131]), $T''$ (Refs. [132]–[140]), and so forth. In our previous works [133], the discrete symmetries have been explored to the 3-3-1 models. In Ref. [133] we have studied the 3-3-1 model with neutral fermions based on $S_4$ group, in which most of the Higgs multiplets are in triplets under $S_4$ except $\chi$ lying in a singlet, and the exact tribimaximal form is obtained, wherein $\theta_{13} = 0$.

As we know, the recent considerations have implied $\theta_{13} \neq 0$ [141], but small as given in Eqs. (2) and (3). This problem has been improved in Ref. [134] by adding a new triplet $\rho$ put in $1'$ under $S_3$ and another antiseptet $s'$ put in $2$ under $S_3$, in which $s'$ is regarded as a small perturbation, or a new triplet $\rho$ put in $1''$ under $D_4$ regarded as a small perturbation [135]. Therefore the models contain up to eight Higgs multiplets, and the scalar potential of the model is quite complicated.

In this paper, we introduce another $SU(3)_L$ antiseptet lying in $2$ under $S_4$ which can result the non-zero $\theta_{13}$ without perturbation. The rest of this work is organized as follows. In Sec. [2] we review some main results from Ref. [133]. Sec. [3] is devoted for the neutrino mass and mixing. Sec. [4] presents the remark on the vacuum alignments and $\rho$ parameter. We summarize our results in the Sec. [5] (Appendix A) is devoted
to $S_4$ group with its Clebsch-Gordan coefficients. Appendix B presents the lepton numbers and lepton parities of model particles. Appendix C provides the breakings of $S_4$ group by triplets $3'$ and $3''$.

2. The model

The fermions in this model under $[SU(3)_L, U(1)_X, U(1)_E, S_4]$ symmetries, respectively, transform as $^{133}$

$$\
\psi_L \equiv \psi_{1,2,3L} = \left( \begin{array}{c} \nu_{1,2,3L} \\ l_{1,2,3L} \\ N_{1,2,3R}^c \end{array} \right) \sim [3, -1/3, 2/3, 3],
$$

$$\
l_{1R} \sim [1, -1, 1, 1], \quad l_R \equiv l_{2,3R} \sim [1, -1, 1, 1],
$$

$$\
Q_{3L} = \left( \begin{array}{c} u_{3L} \\ d_{3L} \\ U_L \end{array} \right) \sim [3, 1/3, -1/3, 3],
$$

$$\
Q_L \equiv Q_{1,2L} = \left( \begin{array}{c} d_{1,2L} \\ -u_{1,2L} \\ D_{1,2L} \end{array} \right) \sim [3^*, 0, 1/3, 3],
$$

$$\
u_R \equiv u_{1,2,3R} \sim [1, 2/3, 0, 3], \quad d_R \equiv d_{1,2,3R} \sim [1, -1/3, 0, 3],
$$

$$\
U_L \sim [1, 2/3, -1, 1], \quad D_R \equiv D_{1,2R} \sim [1, -1/3, 1, 1],
$$

where the numbered subscripts on field indicate respective families and define components of their $S_4$ multiplet representation. Note that the $2$ for quarks meets the requirement of anomaly cancelation where the last two left-quark families are in $3^*$ while the first one as well as the leptons are in $3$ under $SU(3)_L$. All the $L$ charges of the model multiplets are listed in the square brackets.

To generate masses for the charged leptons, we have introduced two $SU(3)_L$ scalar triplets $\phi$ and $\phi'$ lying in $3$ and $3'$ under $S_4$, respectively, with the VEVs $\langle \phi \rangle = (v, v, v)$ and $\langle \phi' \rangle = (v', v', v')$ written as those of $S_4$ components $^{133}$, i.e., $S_4$ is broken into $Z_3$ that consists of the element $\{1, T, T^2\}$. From the invariant Yukawa interactions for the charged leptons, we obtain $m_e = \sqrt{3}h_1v$, $m_\mu = \sqrt{3}(h_2v - h_3v')$, $m_\tau = \sqrt{3}(h_2v + h_3v')$, and the left and right-handed charged leptons mixing matrices are given $^{133}$

$$\
U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = 1.
$$

In similarity to the charged lepton sector, to generate the quark masses, we have additionally introduced three scalar Higgs triplets $\chi, \eta, \eta'$ lying in $1, 3$ and $3'$

---

*With the VEV alignment: $\langle \phi_1 \rangle = \langle \phi_2 \rangle = (\phi_3) \neq 0$, $S_4$ group is broken into $S_3$ which consisting of the elements $\{1, T, T^2, TSN^2, STS\}$; with the VEV alignment: $\langle \phi'_1 \rangle = \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$, $S_4$ is broken into $Z_3$ that consists of the elements $\{1, T, T^2\}$ as presented in Appendix C.
3' under $S_4$, respectively. Quark masses can be derived from the invariant Yukawa interactions for quarks with supposing that the VEVs of $\eta$, $\eta'$ and $\chi$ are $u$, $u'$ and $v_\chi$, respectively, where $u = \langle \eta^0 \rangle$, $u' = \langle \eta'^0 \rangle$, $v_\chi = \langle \chi^0 \rangle$ and the other VEVs $\langle \eta^0 \rangle$, $\langle \eta'^0 \rangle$, and $\langle \chi^0 \rangle$ vanish due to the lepton parity conservation. The exotic quarks get masses $m_U = f_3 v_\chi$ and $m_{D_{1,2}} = f v_\chi$. The masses of ordinary up-quarks and down-quarks are:

$$
m_u = -\sqrt{3}(h^u v + h^{u'} v'), \quad m_c = -\sqrt{3}(h^u v - h^{u'} v'), \quad m_t = \sqrt{3}h^u u,
nm_d = \sqrt{3}(h^d u + h^{d'} u'), \quad m_s = \sqrt{3}(h^d u - h^{d'} u'), \quad m_b = \sqrt{3}h^d v. \quad (6)
$$

The unitary matrices, which couple the left-handed up- and down-quarks to those in the mass bases, are $U^d_L = 1$ and $U^d_L = 1$, respectively. Therefore we get the quark mixing matrix $U_{\text{CKM}} = U^d_L U^d_L = 1$. For a detailed study on charged lepton and quark mass, the reader is referred to Ref. 133. In this work, we add a new $SU(3)_L$ anti-sextet lying in 2 under $S_4$ responsible for the non-zero $\theta_{13}$ without perturbation which is different from those in Refs. 133–135. The vacuum alignments and the gauge boson masses and mixings are similar to those in Refs. 135–141 so we will not discuss it further in this work.

3. Neutrino mass and mixing

In this type of the models, the neutrino masses arise from the couplings of $\bar{\psi}_L^c \psi_L$ to scalars, where $\bar{\psi}_L^c \psi_L$ transforms as $3^* \oplus 6$ under $SU(3)_L$ and $\mathbf{1} \oplus \mathbf{2} \oplus 3 \oplus 3'$ under $S_4$. For the known scalar triplets $(\phi, \phi', \chi, \eta, \eta')$, the available interactions are only $(\bar{\psi}_L^c \psi_L)^c \phi$ and $(\bar{\psi}_L^c \psi_L)^c \phi'$, but explicitly suppressed because of the $L$-symmetry. We will therefore propose new $SU(3)_L$ antisextets, lying in either $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, or $3'$ under $S_4$, which interact with $\bar{\psi}_L^c \psi_L$ to produce masses for the neutrinos. In Ref[133] we have introduced two $SU(3)_L$ antisextets $\sigma, s$ transform as follows

$$
\sigma = \begin{pmatrix}
\sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\
\sigma^+_{12} & \sigma^0_{22} & \sigma^+_{23} \\
\sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33}
\end{pmatrix}
\sim [6^*, 2/3, -4/3, 1],
$$

$$
s = \begin{pmatrix}
\sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\
\sigma^+_{12} & \sigma^0_{22} & \sigma^+_{23} \\
\sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33}
\end{pmatrix}
\sim [6^*, 2/3, -4/3, 2],
$$

with the VEV of $s$ is set as $(\langle s_1 \rangle, 0, 0)$ under $S_4$, where

$$
\langle s_1 \rangle = \begin{pmatrix}
\lambda_s & 0 & v_s \\
0 & 0 & 0 \\
v_s & 0 & \Lambda_s
\end{pmatrix},
$$

and the VEV of $\sigma$ is

$$
\langle \sigma \rangle = \begin{pmatrix}
\lambda_\sigma & 0 & v_\sigma \\
0 & 0 & 0 \\
v_\sigma & 0 & \Lambda_\sigma
\end{pmatrix}. \quad (9)
$$
With these $SU(3)_L$ anti-sextets, the exact tribimaximal form was obtained, in which $\theta_{13} = \Theta_{13}^{\text{max}}$. However, the recent experimental data have implied $\theta_{13} \neq 0$ as given in Eqs. (2) and (3). So that we need to modify the neutrino mass matrix to fit the recent data.

Notice that the VEV alignment as in (5), $S_4$ is broken into a group which is isomorphic to Klein four group$^{53}$ that consists of the elements $\mathcal{K} = \{1, S^2, TST^2, TST \}$. To obtain a realistic neutrino spectrum, in this work we additionally introduce another $SU(3)_L$ anti-sextet $(s')$ which lies in 2 under $S_4$ and responsible for the breaking $\mathcal{K} \rightarrow Z_2$. This happens in any case below: $\langle s' \rangle = (\langle s'_1 \rangle, 0)$, with

$$\langle s'_1 \rangle = \begin{pmatrix} \lambda' & 0 \\ 0 & v''_s \\ 0 & 0 & \Lambda'_s \end{pmatrix}. \tag{10}$$

The VEV alignment of $s'$ as in (10) will break $\mathcal{K}$ into $Z_2$ that consists of the elements $\{1, A^2\}$ (instead of $S_4$ is broken into another Klein four group$^{53}$ that consists of the elements $\{1, S^2, TST^2T^2, T^3S^2T \}$).

In calculation, combining both cases we have the Yukawa interactions responsible for neutrino mass:

$$-\mathcal{L}_\nu = \frac{1}{2} \left[ x (\bar{\psi}_L^c \psi_L) s + \frac{1}{2} y (\bar{\psi}_L^c \psi_L) \right] s + \frac{1}{2} z (\bar{\psi}_L^c \psi_L) s' + H.c.$$ 

$$= \frac{1}{2} x (\bar{\psi}_L^c \psi_L) s + \frac{1}{2} y (\bar{\psi}_L^c \psi_L) s + \frac{1}{2} z (\bar{\psi}_L^c \psi_L) s' + H.c.$$ 

The mass Lagrangian for the neutrinos is given by

$$-\mathcal{L}_\nu^{\text{mass}} = \frac{1}{2} \left[ x (\lambda_s \bar{\nu}_L \nu_L + v_s \bar{\nu}_L N_1^c \nu_L + v_s \bar{\nu}_L N_2^c \nu_L + \lambda'_s \bar{\nu}_L N_1^c \nu_L) + \frac{1}{2} y (\bar{\nu}_L^c \nu_L + \bar{\nu}_L^c N_2^c \nu_L) + \frac{1}{2} z (\bar{\nu}_L^c \nu_L + \bar{\nu}_L^c N_1^c \nu_L) + H.c. \right] \tag{11}$$
We can rewrite the mass Lagrangian for the neutrinos in the matrix form:

\[-\mathcal{L}_\nu^{\text{mass}} = \frac{1}{2} \chi_L \nu_L \chi_L + H.c.,\]

\[\chi_L \equiv \begin{pmatrix} \nu_L \\ N_R \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix},\]

where \(\nu = (\nu_1, \nu_2, \nu_3)^T\) and \(N = (N_1, N_2, N_3)^T\). The mass matrices are then obtained by

\[M_{L,D,R} = \begin{pmatrix} a_{L,D,R} + d_{L,D,R} & 0 & 0 \\ 0 & a_{L,D,R} + \omega d_{L,D,R} & b_{L,D,R} \\ b_{L,D,R} & a_{L,D,R} + \omega^2 d_{L,D,R} \end{pmatrix},\]

where

\[a_L = \lambda_\sigma x, \quad a_D = v_\sigma x, \quad a_R = \Lambda_\sigma x,\]

\[b_L = \lambda_y x, \quad b_D = v_y y, \quad b_R = \Lambda_y y,\]

\[d_L = \lambda_\sigma z, \quad d_D = v_\sigma z, \quad d_R = \Lambda_\sigma z.\]

The VEVs \(\Lambda_{\sigma,s}\) break the 3-3-1 gauge symmetry down to that of the standard model, and provide the masses for the neutral fermions \(N_R\) and the new gauge bosons: the neutral \(Z'\) and the charged \(Y^\pm\) and \(X^{0,0}\). The \(\lambda_{\sigma,s}\) and \(v_{\sigma,s}\) belong to the second stage of the symmetry breaking from the standard model down to the \(\text{SU}(3)_C \otimes U(1)_Q\) symmetry, and contribute the masses to the neutrinos. Hence, to keep a consistency we assume that \(\Lambda_{\sigma,s} \gg v_{\sigma,s},\lambda_{\sigma,s}\). The natural smallness of the lepton number violating VEVs \(\lambda_{\sigma,s}\) and \(v_{\sigma,s}\) was explained in Ref\(^{[133]}\). Three active-neutrinos therefore gain masses via a combination of type I and type II seesaw mechanisms derived from \(^{[133]}\) as

\[M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 \\ 0 & B_1 & D \end{pmatrix},\]

where

\[A = a_L + d_L - \frac{(a_D + d_D)^2}{a_R + d_R}, \quad D = \frac{a_2 - b_2}{a_R^2 + d_R^2 - a_R d_R - b_R^2},\]

\[B_1 = -\frac{a_1 + b_1 \omega^2 + c_1 \omega}{a_R^2 + d_R^2 - a_R d_R - b_R^2}, \quad B_2 = \frac{a_1 + b_1 \omega + c_1 \omega^2}{a_R^2 + d_R^2 - a_R d_R - b_R^2},\]

with

\[a_1 = a_D^2 a_R + 2 a_D (d_D d_R - b_D b_R) + a_R (b_D^2 - d_L d_R) - a_L (a_L^2 - b_R^2 + d_R^2) + a_L a_R d_R,\]

\[b_1 = a_D^2 d_R + a_R (d_D^2 - d_L d_R),\]

\[c_1 = 2 d_D (a_D a_R - b_D b_R) + (b_D^2 + d_D^2) d_R - d_L (a_L^2 - b_R^2 + d_R^2),\]

\[a_2 = a_D^2 b_R - d_D (a_D a_R + d_D d_R) + a_R (b_D^2 - b_L b_R + d_D^2) + b_L d_R^2,\]

\[b_2 = -a_R b_D d_R + a_D b_R d_R - a_D b_D d_R + a_R b_L d_R.\]
We can diagonalize the mass matrix (15) as follows:

\[ U_\nu^T M_{\text{eff}} U_\nu = \text{diag}(m_1, m_2, m_3), \]

where

\[ m_1 = \frac{1}{2} \left( B_1 + B_2 + \sqrt{4D^2 + (B_1 - B_2)^2} \right), \]

\[ m_2 = A, \]

\[ m_3 = \frac{1}{2} \left( B_1 + B_2 - \sqrt{4D^2 + (B_1 - B_2)^2} \right), \]

and the corresponding eigenstates put in the lepton mixing matrix:

\[ U_\nu = \begin{pmatrix}
0 & 1 & 0 \\
1 & \frac{K}{\sqrt{K^2 + 1}} & \frac{-i}{\sqrt{K^2 + 1}} \\
0 & \frac{1}{\sqrt{K^2 + 1}} & \frac{1 + K}{\sqrt{K^2 + 1}}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i
\end{pmatrix}, \]

(19)

where

\[ K = \frac{B_1 - B_2 - \sqrt{4D^2 + (B_1 - B_2)^2}}{2D}. \]

(20)

The lepton mixing matrix is defined as

\[ U_{\text{lep}} \equiv U_L^\dagger U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix}
\frac{1-K}{\sqrt{K^2 + 1}} & \frac{1+K}{\sqrt{K^2 + 1}} & \frac{\omega(1-K\omega)}{\sqrt{K^2 + 1}} \\
\frac{\omega(1+K\omega)}{\sqrt{K^2 + 1}} & \frac{\omega(1-K\omega)}{\sqrt{K^2 + 1}} & \frac{1}{\sqrt{K^2 + 1}} \\
\frac{\omega(1-K\omega)}{\sqrt{K^2 + 1}} & \frac{\omega(1+K\omega)}{\sqrt{K^2 + 1}} & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i
\end{pmatrix}. \]

(21)

It is easy to check that \( U_L \) in (15) is an unitary matrix. So, if \( U_\nu \) in (19) is unitary then \( U_{\text{lep}} \) in (21) is unitary. Here, we will only consider real values for \( K \) since the unitary condition of \( U_{\text{lep}} \). Furthermore, it is worth noting that in the case of the subgroup \( K \) is unbroken, i.e, without contribution of \( s' \) (or \( \lambda' = \lambda'_s = 0 \)), the lepton mixing matrix (21) being equal to \( U_{HPS} \) as given in (1).

The value of the Jarlskog invariant \( J_{CP} \), which gives a convention-independent measure of CP violation, is defined from (21) as

\[ J_{CP} = \text{Im}[U_{21}U_{31}^* U_{22}^* U_{32}] = \frac{1 - K^2}{6\sqrt{3}(1 + K^2)}. \]

(22)

Until now the values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos are unknown. The neutrino mass spectrum can be the normal hierarchy (\(|m_1| \approx |m_2| < |m_3|\)), the inverted hierarchy (\(|m_3| < |m_1| \approx |m_2|\)) or nearly degenerate (\(|m_1| \approx |m_2| \approx |m_3|\)). An upper bound on the absolute value of neutrino mass was found from the analysis of the cosmological data\(^{112}\)

\[ m_i \leq 0.6 \text{ eV}, \]

(23)

while the upper limit on the sum of neutrino masses given in\(^{113}\)

\[ \sum_{i=1}^{3} m_i < 0.23 \text{ eV} \]

(24)
In the case of 3-neutrino mixing, the two possible signs of $\Delta m_{23}^2$ corresponding to two types of neutrino mass spectrum can be provided as follows:

- **Normal hierarchy (NH):** $|m_1| \simeq |m_2| < |m_3|$, $\Delta m_{23}^2 = m_3^2 - m_2^2 > 0$.
- **Inverted hierarchy (IH):** $|m_3| < |m_1| \simeq |m_2|$, $\Delta m_{23}^2 = m_3^2 - m_2^2 < 0$.

As will be discussed below, the model under consideration can provide both normal and inverted mass hierarchy.

### 3.1. Normal case ($\Delta m_{23}^2 > 0$)

In the Normal Hierarchy, combining (22) with the data in Ref. 8, $J_{CP} = -0.032$, we get

$$K = -1.41297,$$

and the lepton mixing matrices are obtained as

$$U_{lep} = \begin{pmatrix} 0.805 & \frac{1}{\sqrt{3}} & 0.138 \\ -0.402 + 0.119i \frac{1}{\sqrt{3}} & 0.069 + 0.697i \\ -0.402 - 0.119i \frac{1}{\sqrt{3}} & 0.069 - 0.697i \end{pmatrix} \times P,$$

or

$$|U_{lep}| = \begin{pmatrix} 0.805 & 0.577 & 0.138 \\ 0.420 & 0.577 & 0.700 \\ 0.420 & 0.577 & 0.700 \end{pmatrix}.$$  

In the standard parametrization, the lepton mixing matrix can be parametrized as

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot \mathcal{P},$$

where $\mathcal{P} = \text{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}$ with $\theta_{12}, \theta_{23}$ and $\theta_{13}$ being the solar, atmospheric and reactor angles, respectively. $\delta = [0, 2\pi]$ is the Dirac CP violation phase while $\alpha$ and $\beta$ are two Majorana CP violation phases.

Using the parametrization in Eq. (28) we get

$$J_{CP} = \frac{1}{8} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \delta.$$  

With the help of (2), (25) and (29) we have $\sin \delta_{CP} = -0.9242$, i.e, $\delta_{CP} = -67.55^\circ$ or $\delta_{CP} = 292.45^\circ$.

From Eqs. (20) and (25) we get

$$B_1 = B_2 - 0.705241 D.$$  

In the normal case, i.e, $\Delta m_{23}^2 = m_3^2 - m_2^2 > 0$, taking the central values of neutrino mass squared difference from the data in Ref. 8 as shown in (2): $\Delta m_{21}^2 = 7.53 \times 10^{-5} \text{ eV}^2$.
and $\Delta m^2_{32} = 2.44 \times 10^{-3}$ eV$^2$, with $m_{1,2,3}$ given in Eq. (18), we get a solution (in [eV])

$$B_2 = -0.5\sqrt{4A^2 - 0.0005} - 0.707729D,$$

$$D = 0.471543 \left( \sqrt{A^2 + 2.44 \times 10^{-3}} - \sqrt{A^2 - 7.53 \times 10^{-5}} \right). \tag{31}$$

With $B_{1,2}$ and $D$ in Eqs. (31) and (30), $m_{1,2,3}$ depends only on one parameter $A$, so we will consider $m_{1,2,3}$ as functions of $A$. By using the upper bound on the absolute value of neutrino mass in (23) we can restrict the values of $A$: $|A| \leq 0.6$ eV. However, in this case, $A \in (0.0087, 0.05)$ eV or $A \in (-0.05, -0.0087)$ eV are good regions of $A$ that can reach the realistic neutrino mass hierarchy.

In Fig. 1, we have plotted the absolute value $|m_{1,2,3}|$ as functions of $A$ with $A \in (0.0087, 0.05)$ eV and $A \in (-0.05, -0.0087)$ eV, respectively. This figure shows that there exist allowed regions for values of $A$ where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained if $|A| \in [0.05 \text{eV}, +\infty)$. However, $|A|$ must be small enough because of the scale of $|m_{1,2,3}|$. The normal mass hierarchy will be obtained if $A$ takes the values around $(0.0087, 0.05)$ eV or $(-0.05, -0.0087)$ eV. The sum of neutrino masses in the normal case $\sum_{i=1}^{N} |m_i|$ with $A \in (0.0087, 0.05)$ eV is depicted in Fig. 2 which is consistent with the upper limit given in Eq. (24).

Fig. 1. $|m_{1,2,3}|$ as functions of $A$ in the case of $\Delta m^2_{32} > 0$ with a) $A \in (0.00867, 0.05)$ eV and b) $A \in (-0.05, -0.00867)$ eV.

\[\text{In fact, this system of equations has four solutions, however, these equations differ only by the sign of } m_{1,2,3} \text{ that it is not appear in the neutrino oscillation experiments. So, here we only consider in detail the solution in (31).}\]
From the expressions (20), (21), (30) and (31), it is easily to obtain the effective masses governing neutrinoless double beta decay,

\[ m_{\text{ee}}^N = \sum_{i=1}^{3} U_{ei}^2 |m_i|, \quad m_{\beta}^N = \left( \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \right)^{1/2} \]  

which is plotted in Fig. 3 with \( A \in (0.0087, 0.05) \) eV in the case of \( \Delta m_{32}^2 > 0 \). We also note that in the normal spectrum, \( |m_1| \approx |m_2| < |m_3| \), so \( m_1 \) given in (18) is the lightest neutrino mass, which is denoted as \( m_1 \equiv m_{\text{light}}^N \).

To get explicit values of the model parameters, we set \( A = 10^{-2} \) eV, which is
safely small. The other physical neutrino masses are explicitly given as

\[ |m_1| \approx 4.97 \times 10^{-3} \text{ eV}, \quad |m_2| = 10^{-2} \text{ eV}, \quad |m_3| \approx 5.04 \times 10^{-2} \text{ eV}. \]  

(33)

It follows that

\[ |m_{\text{ee}}^N| \approx 7.50 \times 10^{-3} \text{ eV}, \quad |m_{\beta}^N| = 9.87 \times 10^{-3} \text{ eV}, \]  

(34)

\[ B_1 = -3.523 \times 10^{-2} \text{ eV}, \quad B_2 = -2.013 \times 10^{-2} \text{ eV}, \quad D = 2.142 \times 10^{-2} \text{ eV}. \]  

(35)

This solution means a normal mass spectrum as mentioned above. Furthermore, by assuming that

\[ \lambda_s = \lambda_s^\prime = \lambda_\sigma = 1 \text{ eV}, \quad v_s = v_s^\prime = v_\sigma, \]  

(36)

we obtain a solution

\[ x \approx (2.0 + 0.2i) \times 10^{-3}, \quad y \approx -(6.1 + 0.61i) \times 10^{-3}, \]  

\[ z = -(4.85 + 0.48) \times 10^{-3}, \quad a \approx 0.222 + 0.017i. \]  

(37)

3.2. Inverted case (\( \Delta m_{32}^2 < 0 \))

For inverted hierarchy, the data in Ref. 8 implies \( J_{CP} = -0.029 \). Hence, we get

\[ K = -1.36483, \]  

(38)

and the lepton mixing matrices are obtained as

\[
U_{\text{lep}} = \begin{pmatrix}
0.807 & -0.125 \\
-0.403 + 0.108i & 0.602 + 0.699i \\
-0.403 - 0.108i & 0.602 - 0.699i
\end{pmatrix} \times P, 
\]  

(39)

or

\[
|U_{\text{lep}}| = \begin{pmatrix}
0.807 & 0.577 & 0.125 \\
0.418 & 0.577 & 0.701 \\
0.418 & 0.577 & 0.701
\end{pmatrix}.
\]  

(40)

Combining 33, 20, and 38 yields \( \sin \delta_{CP} = -0.8371 \), i.e., \( \delta_{CP} = -56.84^\circ \) or \( \delta_{CP} = 303.14^\circ \).

From Eqs. (20) and (38) we get

\[ B_1 = B_2 - 0.632138D. \]  

(41)

In the inverted case, \( \Delta m_{32}^2 = m_3^2 - m_2^2 < 0 \), taking the central values of neutrino mass squared difference from the data in Ref. 8 as shown in 33: \( \Delta m_{21}^2 = 7.53 \times 10^{-5} \text{ eV}^2 \)

\(^c\)The values of the parameters \( \lambda_s, \lambda_s^\prime, \lambda_\sigma, v_s, v_s^\prime, v_\sigma, \Lambda_s, \Lambda_s^\prime, \Lambda_\sigma \) have not been confirmed by experiment, however, their hierarchies were given in Ref. 134. The parameters in Eqs. 36 and 37 is a set of the model parameters that can fit the experimental data on neutrino given in 21.
and $\Delta m^2_{32} = -2.52 \times 10^{-3}$ eV$^2$, with $m_{1,2,3}$ given in Eq. (18), we get a solution in [eV]$^\dagger$

$$B_2 = -0.5 \sqrt{4A^2 - 0.0003 - 0.732692D},$$
$$D = -0.476753 \left( \sqrt{A^2 + 2.52 \times 10^{-3}} + \sqrt{A^2 - 7.53 \times 10^{-5}} \right). \quad (42)$$

With $B_{1,2}$ and $D$ in Eqs. (41) and (42), $m_{1,2,3}$ depends only on one parameter $A$, so we will consider $m_{1,2,3}$ as functions of $A$. In this case, $A \in (0.05, 0.1)$ eV or $A \in (-0.1, -0.05)$ eV are good regions of $A$ that can reach the realistic neutrino mass hierarchy.

In Fig. 4, we have plotted the absolute value $|m_{1,2,3}|$ as functions of $A$ with $A \in (0.05, 0.1)$ eV and $A \in (-0.1, -0.05)$ eV, respectively. We see that there exist allowed regions for values of $A$ where either inverted or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained if $|A| \in [0.1 \text{ eV}, +\infty)$. However, $|A|$ must be small enough because of the scale of $|m_{1,2,3}|$. The inverted mass hierarchy will be obtained if $|A|$ takes the values around $(0.05, 0.1)$ eV. The sum of neutrino masses in the normal case $\sum^N_i = \sum^3_i |m_i|$ with $A \in (0.0087, 0.05)$ eV is depicted in Fig. 2 which is consistent with the upper limit given in Eq. (24). The effective masses governing neutrinoless double beta decay defined in (32) is plotted in Fig. 6 with $A \in (0.05, 0.1)$ eV in the case of $\Delta m^2_{32} < 0$. We also note that in the inverted spectrum, $|m_3| \approx |m_2| \simeq |m_1|$, so $m_3$ given in (18) is the lightest neutrino mass, which is denoted as $m_3 \equiv m^I_{\text{light}}$.

\dagger Similarly to the normal case, there are four solutions in the inverted hierarchy. Here we only consider in detail the solution in (42).
To get explicit values of the model parameters, we set $A = 5 \times 10^{-2}$ eV, which is safely small. The other physical neutrino masses are explicitly given as

$$|m_1| \simeq 4.924 \times 10^{-2} \text{ eV}, \quad |m_2| = 5 \times 10^{-2} \text{ eV}, \quad |m_3| \simeq 4.472 \times 10^{-3} \text{ eV}. \quad (43)$$

It follows that

$$|m_{\text{ee}}'| \simeq 4.88 \times 10^{-2} \text{ eV}, \quad |m_3'| = 4.91 \times 10^{-2} \text{ eV},$$

$$B_1 = (1.72 - 0.29i) \times 10^{-2} \text{ eV}, \quad B_2 = (3.204 - 0.156i) \times 10^{-2} \text{ eV},$$

$$D = (2.348 + 0.213i) \times 10^{-2} \text{ eV}. \quad (45)$$
This solution means an inverted mass spectrum. Furthermore, by assuming that
\[ \lambda_s = \lambda'_s = \lambda = a, \quad v_s = v'_s = -v'_\sigma, \quad \Lambda'_s = \Lambda_s = -\Lambda, \]
\[ \Lambda_s = v_s^2, \quad \Lambda'_s = v'^2, \quad \Lambda = -v^2, \]
we obtain a solution
\[ x \simeq (3.192 - 0.452i) \times 10^{-2}, \quad y \simeq (-2.563 + 0.294i) \times 10^{-2}, \]
\[ z = -(1.910 + 0.825i) \times 10^{-2}, \quad a \simeq 0.105 - 0.186i. \]

4. Remark on the vacuum alignments and $\rho$ parameter

In the model under consideration, to generate masses for all fermions, we need eight Higgs scalars $\phi, \phi', \chi, \eta, \eta', \sigma, s, s'$. It is important to note that $\chi$ and $s'$ do not break $S_4$ since they are put in $S_4$ under $S_4$ while $s', \phi, \eta, \phi'$ can break $S_4$ into its subgroups since they are put in non-trivial representations $2, 3, 3'$ of $S_4$. The breaking of $S_4$ group depends on the vacuum alignment of the flavons.

For doublets $2 \ (s')$ we have two followings alignments. The first alignment, $0 \neq \langle s'_1 \rangle \neq \langle s'_2 \rangle = 0$ or $0 \neq \langle s'_1 \rangle \neq \langle s'_2 \rangle = 0$ or $0 \neq \langle s'_1 \rangle \neq \langle s'_2 \rangle = 0$ then $S_4$ is broken into a group which is isomorphic to Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ that consists of the elements $\{1, T, S, T S, S^2, T S^2, T^2 S, T^2 S^2\}$. The second alignment, $\langle s'_1 \rangle = \langle s'_2 \rangle = 0$ then $S_4$ is broken into $D_4$ consists of the elements $\{1, T S, T S^2, T S T, S, S^3, T S^2 T, S^2, T^2 S T\}$. For triplets $3$ and $3'$ the breakings of $S_4$ are given in Appendix C.

To obtain a realistic neutrino spectrum, in this work, we argue that the breaking $S_4 \rightarrow Z_3$ is taken place in charged lepton sector while both breakings $S_4 \rightarrow K$ and $K \rightarrow Z_2$ must be taken place in neutrino sector.

Note that $\Lambda_4, \Lambda_s, \Lambda_{s'}$ are needed to the same order and not to be so large that can naturally be taken at TeV scale as the VEV $v_\chi$ of $\chi$. This is because $v_\sigma, v_s$ and $v_{\sigma'}$ carry lepton number, simultaneously breaking the lepton parity which is naturally constrained to be much smaller than the electroweak scale.\[122,123,150,151\] This is also behind a theoretical fact that $v_\chi, \Lambda_4, \Lambda_s, \Lambda_{s'}$ are scales for the gauge symmetry breaking in the first stage from $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y$ in the original form of $3-3-1$ models.\[120,124\] They provide masses for the new gauge bosons $Z'$, $X$ and $Y$. Also, the exotic quarks gain masses from $v_\chi$ while the neutral fermions masses arise from $\Lambda_4, \Lambda_s, \Lambda_{s'}$. The second stage of the gauge symmetry breaking from $SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q$ is achieved by the electroweak scale VEVs such as $u, v$ responsible for ordinary quark masses. In combination with those of type II seesaw as determined, in this type of the model, the following limit is often taken into account\[122,123,150,151\]:
\[ (eV)^2 \sim \lambda^2, \lambda_4^2, \lambda_{s'}^2 \ll v^2, v_s^2, v_{\sigma'}^2 \ll u^2, u'^2, v^2, v'^2 \]
\[ \ll v^2 \sim \Lambda_4^2 \sim \Lambda_s^2 \sim \Lambda_{s'}^2 \sim (\text{TeV})^2. \]  

\[ \Box \]

\*The values of the parameters $\lambda, \lambda', \lambda_4, v_s, v_{\sigma'}, v_\sigma, \Lambda_4, \Lambda_s, \Lambda_{s'}$ have not been confirmed by experiment, however, their hierarchies were given in Ref.\[122\] The parameters in Eqs.\[140\] and \[141\] is a set of the model parameters that can fit the experimental data on neutrino given in \[42\].
On the other hand, our model can modify the precision electroweak parameter such as $\rho$ parameter at the tree-level. To see this let us approximate the masses of $W$ and $Z$ bosons:

$$M_W^2 \simeq 2g^2(3u^2 - v_3^2), \quad M_Z^2 \simeq \frac{g^2 u^2}{c_W^2} \left(6 - \frac{v_3^2}{12}\right), \quad (49)$$

$$M_Y^2 \simeq \frac{g^2}{2} \left(6\Lambda_x^2 + 4\Lambda_y^2 + 2\Lambda_z^2 + v_3^2\right). \quad (50)$$

The $\rho$ parameter is defined as

$$\rho = \frac{M_W^2}{c_W^2 M_Z^2} \simeq 1 - \frac{v_3^2}{3u^2}. \quad (51)$$

It is easily to see that the $\rho$ parameter in (51) is absolutely close to the unity since $v_3^2 \ll u^2$ and this is in agreement with the data in Ref.8.

The mixings between the charged gauge bosons $W - Y$ and the neutral ones $Z' - W_4$ are in the same order since they are proportional to $\frac{v_3}{\Lambda_x}$, and in the limit $v_3 \ll \Lambda_x$, these mixing angles tend to zero. In addition, from (48) and (49), (50), it follows that $M_W^2$ is much smaller than $M_Y^2$.

5. Conclusions

In this paper, we have modified the previous 3-3-1 model combined with discrete $S_4$ symmetry to adapt the most recent neutrino mixing with non-zero $\theta_{13}$. We have shown that the realistic neutrino masses and mixings can be obtained if the two directions of the breakings $S_4 \rightarrow K$ and $K \rightarrow Z_2$ simultaneously take place in neutrino sector and are equivalent in size, i.e, the contributions due to $s$, $\sigma$ and $s'$ are comparable. The new feature is adding a new $SU(3)_L$ anti-sextet lying in 2 under $S_4$ which can result the non-zero $\theta_{13}$ without perturbation, and consequently, the number of Higgs multiplets required is less than those of other models based on non-Abelian discrete symmetries and the 3-3-1 models. The exact tribimaximal form obtained with the breaking $S_4 \rightarrow Z_3$ in charged lepton sector while $S_4 \rightarrow K$ in neutrino sector. If both the breakings $S_4 \rightarrow K$ and $K \rightarrow Z_2$ are taken place in neutrino sector, the realistic neutrino spectrum is obtained without perturbation. The upper bound on neutrino mass as well as the effective mass governing neutrinoless double beta decay at the level are presented. The model predicts the Dirac CP violation phase $\delta = 292.45^\circ$ in the normal spectrum (with $\theta_{23} \neq \frac{\pi}{4}$) and $\delta = 303.14^\circ$ in the inverted spectrum. We have found some regions of model parameters that can fit the experimental data in 2014 on neutrino masses and mixing without perturbation.

\*\*We have used the notation $s_W = \sin \theta_W$, $c_W = \cos \theta_W$, $t_W = \tan \theta_W$, and the continuation of the gauge coupling constant $g$ of the $SU(3)_L$ at the spontaneous symmetry breaking point was used.
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Appendix A. $S_4$ group and Clebsch-Gordan coefficients

$S_4$ is the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into 5 conjugacy classes, with $1, 1', 2, 3,$ and $3'$ as its 5 irreducible representations. Any element of $S_4$ can be formed by multiplication of the generators $S$ and $T$ obeying the relations $S^4 = T^3 = 1$, $ST^2S = T$. Without loss of generality, we could choose $S = (1234)$, $T = (123)$ where the cycle $(1234)$ denotes the permutation $(1, 2, 3, 4) \rightarrow (2, 3, 4, 1)$, and $(123)$ means $(1, 2, 3, 4) \rightarrow (2, 3, 1, 4)$. The conjugacy classes generated from $S$ and $T$ are

- $C_1 : 1$
- $C_2 : (12)(34) = TS^2T^2$, $(13)(24) = S^2$, $(14)(23) = T^2S^2T$
- $C_3 : (123) = T$, $(132) = T^2$, $(124) = T^2S^2$, $(142) = S^2T$, $(134) = S^2TS^2$, $(143) = STS$, $(234) = S^3T^2$, $(243) = TS^2$
- $C_4 : (1234) = S$, $(1243) = T^2ST$, $(1324) = ST$, $(1342) = TST$, $(1423) = TST$, $(1432) = S^3$
- $C_5 : (12) = STS^2$, $(13) = TSTS^2$, $(14) = ST^2$, $(23) = S^2TS$, $(24) = TST$, $(34) = T^2S$

The character table of $S_4$ is given as follows

| Class | $n$ | $h$ | $\chi_1$ | $\chi_1'$ | $\chi_2$ | $\chi_3$ | $\chi_3'$ |
|-------|-----|-----|----------|-----------|----------|----------|-----------|
| $C_1$ | 1   | 1   | 1        | 1         | 2        | 3        | 3         |
| $C_2$ | 3   | 2   | 1        | 1         | 2        | -1       | -1        |
| $C_3$ | 8   | 3   | 1        | 1         | -1       | 0        | 0         |
| $C_4$ | 6   | 4   | 1        | -1        | 0        | -1       | 1         |
| $C_5$ | 6   | 2   | 1        | -1        | 0        | 1        | -1        |

where $n$ is the order of class and $h$ is the order of elements within each class. Let us note that $C_{1,2,3}$ are even permutations, while $C_{4,5}$ are odd permutations. The two three-dimensional representations differ only in the signs of their $C_4$ and $C_5$ matrices. Similarly, the two one-dimensional representations behave the same.

We will work in the basis where $3, 3'$ are real representations whereas $2$ is com-
plex. One possible choice of generators is given as follows

\[
\begin{align*}
1 & : S = 1, \quad T = 1 \\
1' & : S = -1, \quad T = 1 \\
2 & : S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \\
3 & : S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
3' & : S = - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\end{align*}
\]

where \( \omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2 \) is the cube root of unity. Using them we calculate the Clebsch-Gordan coefficients for all the tensor products as given below.

First, let us put \( \mathbf{3}(1, 2, 3) \) which means some \( \mathbf{3} \) multiplets such as \( x = (x_1, x_2, x_3) \sim \mathbf{3} \) or \( y = (y_1, y_2, y_3) \sim \mathbf{3} \) or so on, and similarly for the other representations. Moreover, the numbered multiplets such as \( (..., i,...) \) mean \( (..., x_iy_j,...) \) where \( x_i \) and \( y_j \) are the multiplet components of different representations \( x \) and \( y \), respectively. In the following the components of representations in l.h.s will be omitted and should be understood, but they always exist in order in the components of decompositions in r.h.s:

\[
\begin{align*}
1 \otimes 1 & = 1(11), \quad 1' \otimes 1' = 1(11), \quad 1 \otimes 1' = 1'(11), \\
1 \otimes 2 & = 2(11, 12), \quad 1' \otimes 2 = 2(11, -12), \\
1 \otimes 3 & = 3(11, 12, 13), \quad 1' \otimes 3 = 3'(11, 12, 13), \\
1 \otimes 3' & = 3'(11, 12, 13), \quad 1' \otimes 3' = 3(11, 12, 13), \\
2 \otimes 2 & = 2(12 + 21) \oplus 1'(12 - 21) \oplus 2(22, 11), \\
2 \otimes 3 & = 3((1 + 2)1, (1 + \omega)2, \omega^2(1 + \omega^2)2) \\
& \oplus 3'(1 - 2)1, \omega(1 - \omega)2, \omega^2(1 - \omega^2)2) \\
2 \otimes 3' & = 3'(1 + 2)1, (1 + \omega)2, \omega^2(1 + \omega^2)2) \\
& \oplus 3((1 - 2)1, \omega(1 - \omega)2, \omega^2(1 - \omega^2)2), \\
2 \otimes 2 & = 2(11 + 22 + 23) \oplus 2(11 + \omega^222 + \omega33, 11 + \omega22 + \omega^233) \\
& \oplus 3(23 + 32, 31 + 13, 12 + 21) \oplus 3'(23 - 23, 32 - 31, 13, 12 - 21), \\
2' \otimes 2' & = 2(11 + 22 + 33) \oplus 2(11 + \omega22 + \omega33, 11 + \omega22 + \omega^233) \\
& \oplus 3(23 + 32, 31 + 13, 12 + 21) \oplus 3'(23 - 23, 32 - 31, 13, 12 - 21), \\
2 \otimes 2' & = 2(11 + 22 + 33) \oplus 2(11 + \omega^222 + \omega33, -11 - \omega22 - \omega^233) \\
& \oplus 3'(23 + 32, 31 + 13, 12 + 21) \oplus 3(23 - 32, 31 - 13, 12 + 21),
\end{align*}
\]

where the subscripts \( s \) and \( a \) respectively refer to their symmetric and antisymmetric product combinations as explicitly pointed out. We also notice that many group
multiplication rules above have similar forms as those of \( S_3 \) and \( A_4 \) groups.

In the text we usually use the following notations, for example, \((xy')_3 = [xy']_3 = (x_2y_3' - x_3y_2', x_3y_1' - x_1y_3', x_1y_2' - x_2y_1')\) which is the Clebsch-Gordan coefficients of \( \mathbf{3}_1 \) in the decomposition of \( \mathbf{2} \otimes \mathbf{3}' \), where as mentioned \( x = (x_1,x_2,x_3) \sim \mathbf{2} \) and \( y' = (y_1',y_2',y_3') \sim \mathbf{3}' \).

The rules to conjugate the representations \( \mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{2}' \) and \( \mathbf{3}' \) are given by

\[
\begin{align*}
\mathbf{2}^*(1^*,2^*) & = \mathbf{2}(2^*,1^*), \\
\mathbf{1}^*(1^*) & = \mathbf{1}(1^*), \\
\mathbf{1}'' (1^*) & = \mathbf{1}'(1^*), \\
\mathbf{2}^*(1^*,2^*,3^*) & = \mathbf{3}(1^*,2^*,3^*), \\
\mathbf{2}'' (1^*,2^*,3^*) & = \mathbf{3}'(1^*,2^*,3^*),
\end{align*}
\]

where, for example, \( \mathbf{2}^*(1^*,2^*) \) denotes some \( \mathbf{2}^* \) multiplet of the form \( (x_1^*,x_2^*) \sim \mathbf{2}^* \).

**Appendix B. The numbers**

In the following we will explicitly point out the lepton number \( (L) \) and lepton parity \( (P_L) \) of the model particles (notice that the family indices are suppressed):

| Particles | \( L \)  | \( P_L \) |
|-----------|---------|---------|
| \( N_R, u, d, \phi_1^+, \phi_1^+, \phi_2^+, \phi_2^+, \eta_{12}^0, \eta_{12}^0, \eta_{12}^0, \eta_{12}^0, \chi_{3}, \sigma_{33}^0, s_{33}^0 \) | 0 | 1 |
| \( \nu, l, t, U^*, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+, \phi_3^+ \) | -1 | -1 |
| \( \sigma_{11}^0, \sigma_{12}^0, \sigma_{22}^0, s_{11}^0, s_{12}^0, s_{22}^0 \) | -2 | 1 |

**Appendix C. The breakings of \( S_4 \) by triplets \( \mathbf{2} \) and \( \mathbf{3}' \)**

For triplets \( \mathbf{3} \) we have the followings alignments:

1. The first alignment: \( \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \) then \( S_4 \) is broken into \( \{1\} \equiv \{1\} \), i.e. \( S_4 \) is completely broken.
2. The second alignment: \( 0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0 \) or \( 0 \neq \langle \phi_1 \rangle \neq \langle \phi_3 \rangle = \langle \phi_2 \rangle \neq 0 \) then \( S_4 \) is broken into \( Z_2 \) which consisting of the elements \{1, TSTS\} or \{1, TSS\} or \{1, S^2TS\}, respectively.
3. The third alignment: \( \langle \phi_1 \rangle = \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0 \) then \( S_4 \) is broken into \( S_3 \) which consisting of the elements \{1, T, T^2, TSTS, STS, S^2TS\}.
4. The fourth alignment: \( 0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle \neq \langle \phi_3 \rangle \neq 0 \) or \( 0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0 \) then \( S_4 \) is broken into \( Z_2 \) which consisting of the elements \{1, TSTS\} or \{1, TSS\} or \{1, S^2TS\}, respectively.
5. The fifth alignment: \( 0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle \neq \langle \phi_3 \rangle \neq 0 \) or \( 0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \) then \( S_4 \) is completely broken.
6. The sixth alignment: \( 0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle = 0 \) or \( 0 \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle = \langle \phi_1 \rangle = 0 \) or \( 0 \neq \langle \phi_3 \rangle \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0 \) then \( S_4 \) is broken into Klein four group \( K \) which
consisting of the elements \{1,S^2,STST^2,TST\} or \{1,T^2S^2,ST^2S^2\} or \{1,T^2S^2T,ST^2S^2\}, respectively.

For triplets 2' we have the followings alignments:

1. The first alignment: \(\langle \phi_1' \rangle \neq \langle \phi_2' \rangle \neq \langle \phi_3' \rangle\) then \(S_4\) is broken into \(\{1\} \equiv \{\text{identity}\}\), i.e. \(S_4\) is completely broken.
2. The second alignment: \(0 \neq \langle \phi_1' \rangle \neq \langle \phi_2' \rangle \neq \langle \phi_3' \rangle\) = 0 or \(0 \neq \langle \phi_1' \rangle \neq \langle \phi_2' \rangle \neq \langle \phi_3' \rangle\) = 0 then \(S_4\) is broken into \(\{1\} \equiv \{\text{identity}\}\), i.e. \(S_4\) is completely broken.
3. The third alignment: \(\langle \phi_1' \rangle = \langle \phi_2' \rangle = \langle \phi_3' \rangle\) \neq 0 then \(S_4\) is broken into \(Z_3\) which consists of the elements \(\{1,T,T^2\}\).
4. The fourth alignment: \(0 = \langle \phi_1' \rangle \neq \langle \phi_2' \rangle \neq \langle \phi_3' \rangle\) = 0 or \(0 = \langle \phi_1' \rangle \neq \langle \phi_2' \rangle \neq \langle \phi_3' \rangle\) = 0 then \(S_4\) is broken into \(Z_2\) which consisting of the elements \(\{1,T^2S\}\) or \(\{1,TST\}\) or \(\{1,ST^2\}\), respectively.
5. The fifth alignment: \(0 = \langle \phi_2' \rangle \neq \langle \phi_1' \rangle \neq \langle \phi_3' \rangle\) = 0 or \(0 = \langle \phi_2' \rangle \neq \langle \phi_1' \rangle \neq \langle \phi_3' \rangle\) = 0 then \(S_1\) is completely broken.
6. The sixth alignment: \(0 \neq \langle \phi_2' \rangle \neq \langle \phi_1' \rangle \neq \langle \phi_3' \rangle\) = 0 or \(0 \neq \langle \phi_2' \rangle \neq \langle \phi_1' \rangle \neq \langle \phi_3' \rangle\) = 0 then \(S_4\) is broken into a four-element subgroup generated by a four-cycle, which consisting of the elements \(\{1,S,S^2,S^3\}\) or \(\{1,TST^2,ST,ST^2S^2\}\) or \(\{1,TS^2ST,T^2S^2T\}\), respectively.

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