Discriminant of the ordinary transversal singularity type

Maxim Kazarian, Dmitry Kerner and András Némethi

Abstract. Consider a space $X$ with a non-isolated singular locus $Z$. The transversal type of $X$ along $Z$ is generically constant but at some points of $Z$ it degenerates. We introduce the discriminant of the transversal type, the subscheme of $Z$ that reflects these degenerations whenever the generic transversal type is ‘ordinary’. The scheme structure is imposed by various compatibility properties and is often non-reduced.

In the global case, $Z$ being complete, we compute the class of the discriminant in the Picard group $\text{Pic}(Z)$. Further, we define the natural stratification of the singular locus and compute the classes of the simplest strata.

As immediate applications, we bound the jumps of multiplicity (when the singular locus is one-dimensional) and (re)derive the multi-degrees of the classical discriminant of complete intersections.

Contents

1. Introduction 1
2. Preliminaries 4
3. Transversal singularity type 5
4. The discriminant and its scheme structure 7
5. The equivalence classes of discriminant and strata 12
6. Applications 16
Appendix A. Discriminant of transversal type in more general settings 16
References 17

For the general introduction to singularities see [AGLV-book1], [Dimca-book], [Looijenga-book], [Seade-book]. The relevant facts from algebraic geometry (in particular intersection theory) can be found in [Fulton-book].

For a collection of facts about the discriminants of isolated complete intersections cf. [AGLV-book2, §I.2.2].

1. Introduction

1.1. Setup. Let $k$ be an algebraically closed field of zero characteristic, e.g. $k = \mathbb{C}$. Let $M$ be either a smooth variety (over $k$), or, for $k = \mathbb{C}$, a complex analytic manifold. Let $X \subset M$ be a reduced subspace. Let $Z$ be a connected component of the singular locus $\text{Sing}(X)$ of $X$ of positive dimension. We always consider $Z$ and $\text{Sing}(X)$ as (algebraic or analytic) sets, i.e. with their reduced structures. A possible strategy to introduce a discriminant of the transversal type along $Z$ would be the following. For each smooth point $o \in Z$ we consider the transversal singularity type, by intersecting the germ $(X,o)$ with a ‘transversal linear space’ $(L^\perp,o) \subset (M,o)$. The transversal type is generically constant along $Z$ but degenerates at some points. The set of such points forms a subspace, $\Delta^\perp \subset Z$, often with non–reduced structure. We call it the discriminant of the transversal type, it is the target of this note. We can address both the local questions, regarding the local scheme structure of $\Delta^\perp$, and the global ones (the equivalence class of $\Delta^\perp$ in $\text{Pic}(Z)$).

Nevertheless, at this generality, the definition of transversal singularity type and the discriminant is out of reach. By the lack of good equisingularity theory it is hard to identify and put in a family the different sections $(L^\perp \cap X,o)_{o \in Z}$. Therefore in this work we make the following assumptions:

- $X \subset M$ is a strictly locally complete intersection (this is a strengthening of locally complete intersection, see §2.2);
- $Z \subset M$ is a locally complete intersection (except §A);
- the transversal type of $X$ along $Z$ is generically ‘ordinary’. By this we mean: for sufficiently generic point $o \in Z$
Z, the projectivized tangent cone, \( \mathbb{P}T_{(L \cap X,o)} \), is a smooth complete intersection of expected dimension, see §2.2. Furthermore, we assume that if Z has several irreducible components then the multidegree of \( \mathbb{P}T_{(L \cap X,o)} \) at generic points of each component is the same.

1.2. Scheme structure on the discriminant. Before we provide the precise definition we analyze some guiding examples. The following example explains why it is natural to endow \( \Delta^\bot \) with a non-reduced space structure.

**Example 1.1.** Consider the singular surface \( X = \{ x^2z^q = y^q + x^3 \} \subset \mathbb{A}^3 \) for \( q \in \mathbb{Z}_{>0} \). Its (reduced) singular locus is the line \( x = y = 0 \). For \( q = 1 \) the singularity is the classical Whitney umbrella. For the generic point of \( \text{Sing}(X) \), i.e. for \( z \neq 0 \), the transversal singularity type is \( A_1 \), i.e. two smooth non-tangent branches. At the origin the transversal singularity type degenerates, hence this point (presumably) belongs to the discriminant. Consider the deformation: 
\[ Z = \{ x^2z^q - t = y^q + x^3 \} \subset \mathbb{A}^3 \text{ for } t \in (k,o). \]
For \( t \neq 0 \) the discriminant point \((0,0,0) \in \Delta^\bot \) with multiplicity \( q \) (or a multiple of \( q \)). One can say roughly that for \( q > 1 \) the transversal type degenerates (as \( z \to 0 \)) ‘faster’, or with higher multiplicity. (In example 4.1 we give another reason for non-reducedness of \( \Delta^\bot \).)

This and the next examples show the first basic property which should be satisfied by \( \Delta^\bot \): certain deformations of \( X \) should induce flat deformations of \( \Delta^\bot \); the precise formulation is given in §2.

**Example 1.2.** Consider the projective surface \( \{ (xy)^p = (zw)^p \} \subset \mathbb{P}^3 \), \( p \geq 2 \). The singular locus, \( Z = \text{Sing}(X) = \{ xy = zw = 0 \} \subset \mathbb{P}^3 \), is a singular curve. Should the singular points of \( Z \) necessarily belong to the discriminant? Consider the family of surfaces \( \{ (xy + \epsilon Q_1)^p = (zw + \epsilon Q_2)^p \} \subset \mathbb{P}^3 \), where \( Q_1, Q_2 \) are generic quadrics. This flat family induces the (flat) smoothing of the singular locus: \( Z_\epsilon = \{ xy + \epsilon Q_1 = zw + \epsilon Q_2 = 0 \} \). For any fixed \( \epsilon \neq 0 \) and for any \( o \in Z_\epsilon \), the transversal type of \( X_o \) along \( Z_o \) is an ordinary multiple point (a plane curve singularity with \( p \) smooth pairwise non-tangent branches). Hence \( (\Delta^\bot)_{x \neq 0} = \emptyset \). This gives the first sign that \( \Delta^\bot = Z \), although the ‘naive’ transversal type at the singular points of \( Z \) degenerates. Here, as we will see, \( \Delta^\bot = Z \) means that the family of transversal types of the punctured neighbourhood is ‘trivial’, and can be completed (in a certain universal space) to a non-degenerate fiber even over the points \( \text{Sing}(Z) \), cf. §4.2.

This last example also shows that the naive definition of transversal type, as in §1.1, is not suitable at the singular points of \( Z \). Rather, we need to use the algebraic version, defining the transversal type by the virtual normal cone, cf. §3.2.

In §4 we analyze the local scheme structure of the the transversal discriminant, \( \Delta^\bot \subset (Z,o) = \text{Sing}(X,o) \). Here \( \Delta^\bot \) is (algebraic/formal or analytic for \( k = \mathbb{C} \)) subspace whose defining ideal is principal, i.e. the discriminant is locally Cartier. The definition is based on the classical discriminant in the parameter space of projective complete intersections. Globally these germs glue to an effective Cartier divisor on \( Z \).

\( \Delta^\bot \) satisfies various natural properties as well, e.g. it deforms accordingly to the flat deformations of \( Z \) and \( X \). Moreover, if \( \phi : Bl_ZM \to M \) is the blowup of \( M \) along \( Z \) with exceptional divisor \( E \), and \( \tilde{X} \) is the strict transform of \( X \), then \( \Delta^\bot \) can be reinterpreted in terms of the ramification divisor of the map \( \tilde{X} \cap E \to Z \). More precisely, \( \Delta^\bot \) is the image of the critical locus of \( \pi \), with the fitting scheme structure, cf. §4.3.

The considerations in §3.2 and §4 are local and the properties hold in algebraic/analytic/formal settings.

In §4.4 we define the stratification of \( Z \) by the singularity type of \( \mathbb{P}T_{(L \cap X,o)} \).

**Remark 1.3.** A natural (and often considered) scheme structure on the singular locus is the one coming from the jacobian ideal, [Aluffi-1995], [Aluffi-2005]. It also reflects the degenerations of transversal type. However, we emphasize, that this jacobian scheme structure is incompatible with flat deformations and it differs from the scheme structure of our paper.

1.3. The class of the discriminant. In §5 we consider the global case, \( Z \) is a complete space and the germ \( (X,Z) \) is a strictly locally complete intersection in \( (M,Z) \). The scheme structure of \( \Delta^\bot \) is defined locally, nevertheless, the global object obtained in this way (and the associated global invariants) also carry deep geometric meanings. Let us introduce the relevant notation:

- \( [\Delta] \) denotes the class of \( \Delta^\bot \) in \( \text{Pic}(Z) \) (the divisor class group, identified with the isomorphism classes of line bundles);
- \( [X] \in \text{Pic}(Z) \) denotes the class of \( \mathcal{O}_M(X)|_Z \), whenever \( X \subset M \) is a hypersurface;
- \( N_{Z/M} \) denotes the normal bundle of \( Z \) in \( M \). Its determinant line bundle is \( \text{det}(N_{Z/M}) \).

**Theorem 1.4.** Let \( X \subset M \) be a hypersurface, and let \( Z \subset \text{Sing}(X) \) be a (complete) connected component of the singular locus. Suppose \( Z \) is a locally complete intersection of codim_MZ = k; the generic multiplicity of \( X \) along \( Z \) is \( p \), and the generic transversal type of \( X \) along \( Z \) is ordinary, see §3. Then
\[
[\Delta] = (p - 1)^{k-1} \left( k[X] - p \cdot [\text{det}(N_{Z/M})] \right) \in \text{Pic}(Z).
\]
Note that $\Delta^\perp$ is always an effective divisor, thus $[\Delta^\perp] = 0$ is equivalent to $\Delta^\perp = \emptyset$. We remark also, that the last formula can be rewritten as: $[\Delta^\perp] = (p-1)^{k-1}\left(k[X] + p(K_M|_Z - \omega_Z)\right) \in \text{Pic}(Z)$, where $K_M$ and $\omega_Z$ stay for the dualizing sheaves. Here the right hand side is meaningful for an arbitrary reduced Gorenstein scheme $Z$, this suggests possible generalizations.

Example 1.5. 1. Let $Z \subset \mathbb{P}^{n+1}$ be a smooth projective curve, a complete intersection of multidegree $(d_1, \ldots, d_n)$. Suppose $Z = \text{Sing}(X)$, where $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $d$. Then

$$\text{deg}[\Delta^\perp] = \left(\prod d_i\right)(p-1)^{n-1}(nd - p \sum d_i).$$

2. Consider the blowup $M$ of $\mathbb{P}^{n+1}$ at a point $p$, and let $E \cong \mathbb{P}^n$ be the exceptional divisor. Let $Z \subset E$ be a smooth complete intersection of dimension one of multidegree $(d_1, \ldots, d_{n-1})$. Suppose $X \subset M$ is an irreducible hypersurface, $Z = \text{Sing}(X)$ and the intersection $X \cap E$ is of degree $q$ in $E$. Then

$$\text{deg}[\Delta^\perp] = \left(\prod d_i\right)(p-1)^{n-1}(nq - p \sum d_i).$$

3. Let $X = \bigcup_{i=1}^{p} X_i \subset M$ be the union of smooth hypersurfaces intersecting pairwise transversally. Suppose that for any pair $i \neq j$ the intersection $X_i \cap X_j$ is the same subspace of $M$, denoted by $Z$. Hence $Z$ is smooth and the transversal type of $X$ along $Z$, at any point, is an ordinary multiple point with multiplicity $p$. Thus $\Delta^\perp = \emptyset$. Compare with Theorem 1.4. Indeed, we have $\text{codim}_M Z = 2$, $\det(N_{Z/M}) = [X_1] + [X_2]$, thus $[\Delta^\perp] = (p-1)(2 \sum_{i=1}^{p} [X_i] - p([X_1] + [X_2])) = 0$ (since for $p > 2$ one has $[X_i] = [X_j]$ in Pic($Z$)). As $\Delta^\perp$ is effective, of expected dimension, we get again: $\Delta^\perp = \emptyset$.

In §5.2 we compute the class $[\Delta^\perp]$ for $Z$ a projective algebraic locally complete intersection and the germ $(X, Z)$ a strictly complete intersection. In this case $[\Delta^\perp]$ sits in the Chow group $A^1(Z)$, and in the proof we rely on the Porteus-Thom formula. The setup is the following. Choose the defining hypersurfaces $X = \bigcap_{i=1}^{p} X_i \subset M$, whose strict transforms define $\tilde{X} \subset BZ(M)$, i.e. $\tilde{X} = \bigcap_{i=1}^{p} \tilde{X}_i$.

Theorem 1.6. Suppose the generic transversal type of $X$ along $Z$ is ordinary. Let the codimension of $Z$ in $M$ be $k$, suppose the generic multiplicity of $X_i$ along $Z$ is $p_i$. Then

$$A^i(Z)[\Delta^\perp] = \text{Coeff}_{f_k}f_{k-r}(1-t)^k\left(\prod_{i=1}^{p} \frac{t^{p_i}}{1-t^{p_i}}\right)\left(\sum_{j=1}^{r} \frac{[X_j]}{p_j - 1}\right) = \left(\prod p_i\right)\left(\sum_{j=1}^{r} \frac{[X_j]}{p_j}\right)\sum_{k=0}^{r} (p_j-1)^{k}\prod_{j=1}^{r} (p_j-1)^{k} S_{k-r-a-(c_1(N_{Z/M})S_{k-1}).}

$$

where $S_N := \sum_{k \geq 0, k_i = N} \prod_{i=1}^{r} (p_i - 1)^{k_i}$.

In the similar spirit, in §5.3, we compute the classes of some low-codimension strata of the discriminant, namely: $[\Sigma_{A_2}], [\Sigma_{A_4, A_1}]$ in the Chow group $A^2(Z)$.

Finally, in §6.1, we consider the case when the singular locus is one-dimensional. In this case the discriminant is a collection of points, thus we can bound the total ”amount of degenerations” along the singular locus. We give explicit bound for the jumps of multiplicity.

In 6.2 we consider a very particular case: $M = Z \times M'$, with $\text{Sing}(X) = Z \times \{0\}$. In this case, the general formulas give the multi-degrees of the classical discriminant, recovering results of [Benoist2012].

1.4. History. The discriminant of transversal singularity type appears naturally in geometry and singularity theory and in some particular cases was considered already by Salmon, Cayley, Noether and Zeuthen, see [Pie1977]. One context where it appears is the image of the generic map from a smooth $n$–fold into $\mathbb{P}^{n+1}$. The image has non-isolated ordinary singularities (not to be confused with the ‘ordinary transversal type’ used in this paper). The natural question is to understand their degenerations, as one runs along the singular locus.

The class of $\Delta^\perp$ for projective surface, $X \subset \mathbb{P}^3$, with ordinary singularities goes back (probably) to the early history. For a computation see [Pie1977] (among various other invariants).

The case of one-dimensional singular locus, i.e. $Z$ is a curve, with the generic transversal type $A_1$, was thoroughly studied by Siersma, e.g. [Siersma2000]. The local degree of the discriminant, called also ‘the virtual number of $D_\infty$ points’ was studied in [Pelli1985], [Pelli1990] and [de Jong1990]. In particular, the authors show pathological behavior when $Z$ is not a locally complete intersection. In [de Jong-de Jong1990] the degree of $[\Delta^\perp]$ is computed for the case $X \subset M$ is a hypersurface, $Z = \text{Sing}(X)$ is of (pure) dimension one and the generic transversal type is $A_1$. For the review of various related result cf. [AGLV-book2, §1.4.6]. For the recent results and applications to real singularities cf. [van Straten2011].
We remark, that in Pelikan-de Jong’s approach the scheme structure on the discriminant is compatible with flat deformations and the discriminant is reduced for Whitney’s umbrella. These two conditions determine the scheme structure uniquely, therefore their and our scheme structures (for non-isolated singularities of surfaces) coincide.

2. Preliminaries

2.1. Local neighborhoods. In this paper $k$ is an algebraically closed field of zero characteristic, e.g. $\mathbb{C}$. When working locally, we consider the germ of an affine space, $(k^N, o)$, and its subspaces. This germ can be algebraic, analytic (for $k = \mathbb{C}$) or formal. Accordingly the local ring of regular functions, $\mathcal{O}_{(k^N, o)}$ is one of $k[x_1, \ldots, x_n]_m$, $k[x_1, \ldots, x_n]$, or the quotient by an ideal. In many cases the algebraic germs are ‘too large and rigid’, e.g. when speaking of irreducible components or rectifying locally a smooth variety. In such cases we take completion or henselization (i.e. work with formal of henselian germs).

2.2. Tangent cone and strictly complete intersections. Let $(X, o) \subset (k^N, o)$ be a (algebraic/analytic/formal) germ, let $I_{(X, o)} \subset \mathcal{O}_{(k^N, o)}$ be its defining ideal. Let $m \subset \mathcal{O}_{(k^N, o)}$ be the maximal ideal, we denote its restriction onto $(X, o)$ by the same letter: $m \subset \mathcal{O}_{(X, o)}$. The tangent cone $T_{(X, o)}$ is defined as the spectrum of the associated graded ring: $\oplus_{i \geq 0} m^i/m^{i+1}$. For example, in the hypersurface case, $(X, o) = \{f_\alpha(x) + f_{\beta+1}(x) + \cdots = 0\} \subset (k^N, o)$, the tangent cone is also a hypersurface: $T_{(X, o)} = \{f_\alpha(x) = 0\} \subset k^N$. If $(X, o)$ is a complete intersection then its tangent cone, in general, is not a complete intersection.

Example 2.1. For the complete intersection $(X, o) = \{x^2 + zy^3 - xy + z^3 = 0\} \subset (k^3, o)$ the tangent cone is $T_{(X, o)} = \{x^2 = xy = xz^3 = z^6 - zy^3 = 0\}$. Indeed, [Eisenbud-book, §15.10.3], it is enough to check the Groebner basis of the homogenized ideal, $\{w^2x^2 + zy^3, wxy + z^3\}$, with respect to any monomial ordering. For the ordering $x > y > z > w$ the Groebner basis is:

$$\{w^2x^2 + zy^3, wxy + z^3, wxyz, w^3z^3 - v^4, z^6 - z^5\}.$$  

By sending $w \to 1$ and taking the leading terms we get the projectivized tangent cone. Now, by direct check, this projectivization $PT_{(X, o)} \subset \mathbb{P}^2$ is a collection of smooth (!) points, whose defining ideal is not a complete intersection.

To avoid such pathologies we call $(X, o)$ a strictly complete intersection, s.c.i., if it is a complete intersection and its tangent cone is a complete intersection too. (This name seems to be coined by [Bennett-1977, pg.31]. The name "strong complete intersection" is used in commutative algebra to denote "geometric" complete intersections, i.e. rings of the form $S/(f_1, \ldots, f_r)$, where $S$ is a regular local ring and $\{f_i\}$ is a regular sequence, [Heitmann-Jorgensen]. The name "absolute complete intersection" would suggest that both the germ and all its proper transforms and exceptional loci in the resolution are locally complete intersections.)

In this case it is always possible to choose a basis for the defining ideal, $I_{(X, o)} = \langle f_1, \ldots, f_r \rangle$, such that $T_{(X, o)} \subset k^N$ is defined by the leading terms of $\{f_1\}$. Explicitly, expand $\{f_1\}$ in Taylor series $\{f_1 = f_{r_1}(x) + f_{r_2}(x) + \cdots\}$, then $T_{(X, o)} = \{f_{1}(p_i), \ldots, f_{r_p}(p_i)\}$. We call such a basis a good basis. A good basis is not unique, however the associated multiplicity sequence, $(p_1, \ldots, p_r)$, is unique up to a permutation.

Example 2.2. • A hypersurface singularity is obviously a strictly complete intersection.

• More generally, consider a complete intersection $(X, o) := \cap_i (X_i, o)$, suppose $(X_1, o)$ is smooth. Then $(X, o)$ is a s.c.i. iff $\cap_i (X_i, o)$ is a s.c.i.

• Let $(X, o)$ be a s.c.i. with a good basis $\{f_1, \ldots, f_r\}$. Consider a small deformation $\{f_1 + \epsilon g_1, \ldots, f_r + \epsilon g_r\}$ that preserves the multiplicity sequence. The generic member of this family is a s.c.i.

If the germ is taken not at a point, but along a subvariety then the definition is similar. We call the germ $(X, Z) \subset (M, Z)$ a s.c.i. (along $Z$) if there exists a basis $(X, Z) = \cap_i (X_i, Z)$ such that the strict transforms of $X, \{X_i\}$ under the blowup of $M$ along $Z$ satisfy: $\tilde{X} = \cap_i \tilde{X}_i \subset Bl_Z(M)$.

The following technical result is frequently used later, note that here $(Z, o)$ is just a complete intersection, not necessarily s.c.i.

Lemma 2.3. Let $(Z, o) \subset (M,o)$ be a reduced complete intersection. Suppose the generic multiplicity of $(X, o)$ along $(Z, o)$ is at least $p$. Then $I_{(X/M, o)} \subset P_{(Z/M, o)}^p$. Explicitly, if $I_{(Z/M, o)} = \langle g_1, \ldots, g_k \rangle$ and $f \in I_{(X/M, o)}$ then $f = \sum g_1^{m_1} \cdots g_k^{m_k} a_{m_1 \ldots m_k}$, for $a_{m_1 \ldots m_k} \in \mathcal{O}_{(M, o)}$.

Proof. Suppose $(Z, o) \subset (M,o)$ is a hypersurface, i.e. $k = 1$. Then $I_{(Z/M, o)} = \langle g \rangle$ and $f$ is divisible by $g^p$ generically on $(Z, o)$. Thus the function $\frac{f}{g^p}$ is regular outside a set of codimension two in $M$. So, $\frac{f}{g^p}$ is regular on $(M, o)$.

The case $k > 1$ is reduced to the hypersurface case, by considering the conditions in $\mathcal{O}_{(M, o)}/(g_1, \ldots, g_{k-1})$. The image of $g_k$ in $\mathcal{O}_{(M, o)}/(g_1, \ldots, g_{k-1})$ is reduced (square-free), so by the $k = 1$ case: $f \in (g_k^p)(\text{mod}(g_1, \ldots, g_{k-1}))$. By changing
the basis we get $f \in \langle (g_1, \ldots, g_k)^p \rangle$. ■

2.3. Ordinary multiple points. Recall that an isolated hypersurface singularity, \( \{ f_p + f_{p+1} + \cdots = 0 \} \subset (k^{n+1}, o) \), is called an ordinary multiple point if its projectivized tangent cone, \( \{ f_p = 0 \} \subset \mathbb{P}^n \), is smooth. In the case $k = \mathbb{C}$ this can be stated as: the hypersurface germ is topologically equisingular to \( \{ \sum x_i^p = 0 \} \subset (\mathbb{C}^{n+1}, o) \). More generally, we call a s.c.i. singularity an ordinary multiple point if its projectivized tangent cone is smooth. Example 2.1 shows that the s.c.i assumption is necessary.

2.4. The classical discriminant of projective complete intersections. While the discriminant of projective hypersurfaces in $\mathbb{P}^n$ has been classically studied, [Gelfand-Kapranov-Zelevinsky], we do not know a classical/canonical reference on the discriminant of (projective) globally complete intersections. Therefore we rederive some standard results.

For a given multi-degree, \( (p_1, \ldots, p_r) \), consider the multi-projective space \( \prod_i |O_{\mathbb{P}^n}(p_i)| \). A point of this space, \( (f_1, \ldots, f_r) \in \prod_i H^0(\mathbb{P}^n(p_i)) \), corresponds to a subscheme \( X = \{ f_1 = \cdots = f_r = 0 \} \subset \mathbb{P}^n \). This subscheme is a globally complete intersection, unless the polynomials \( \{ f_i \} \) are algebraically dependent. Thus there exists a Zariski open subset \( \mathcal{U} \subset \prod_i |O_{\mathbb{P}^n}(p_i)| \), whose points correspond to globally complete intersections. (Note that the complement, \( \prod_i |O_{\mathbb{P}^n}(p_i)| \setminus \mathcal{U} \), is of high codimension.) Therefore we consider \( \mathcal{U} \) as a parameter space for globally complete intersections, eventhough for \( r > 1 \) the map \( \{ (f_1, \ldots, f_r) \} \to X \subset \mathbb{P}^N \) is far from being injective.

**Definition 2.4.** The (classical) discriminant of complete intersections, \( \Delta \subset \prod_i |O_{\mathbb{P}^n}(p_i)| \), is a subset of points corresponding to singular schemes or schemes of non-expected dimension.

This definition is set-theoretic, it does not specify the scheme structure on \( \Delta \). The precise definition goes via the critical locus of a map, as follows. Consider the universal family of complete intersections, and the projection onto the parameter space:

\[
X = \{ (x, f_1, \ldots, f_r) \mid f_1(x) = 0, \ldots, f_r(x) = 0 \} \subset \mathbb{P}^N \times \prod_i |O_{\mathbb{P}^n}(p_i)|, \quad X \xrightarrow{\pi} \prod_i |O_{\mathbb{P}^n}(p_i)|
\]

Take the critical locus of the projection, \( \text{Crit}(\pi) \), the discriminant is the image, \( \Delta = \pi(\text{Crit}(\pi)) \subset \prod_i |O_{\mathbb{P}^n}(p_i)| \), with the fitting scheme structure. Namely, the defining ideal of \( \Delta \) is \( F_{N-r}(\pi, \mathcal{O}_{\text{Crit}(\pi)}) \), the fitting ideal of \( \mathcal{O}_{\prod_i |O_{\mathbb{P}^n}(p_i)|} \) module \( \pi_* \mathcal{O}_{\text{Crit}(\pi)} \), here \( (N-r) \) is the dimension of the fibres of \( \pi \).

**Proposition 2.5.** The discriminant is a (reduced) irreducible hypersurface.

**Proof.** Consider the multi-Veronese embedding \( \mathbb{P}^N \overset{\nu_{p_1, \ldots, p_r}}{\to} \prod_i |O_{\mathbb{P}^n}(p_i)| \). Each point \( \{ L_i \} \in \prod_i |O_{\mathbb{P}^n}(p_i)| \) corresponds to a collection of hyperplanes in \( \{ |O_{\mathbb{P}^n}(p_i)| \}_i \), whose intersection gives a codimension-\( r \) plane in \( \prod_i |O_{\mathbb{P}^n}(p_i)| \). Dually, this point corresponds to a subscheme in \( \mathbb{P}^N \) that is obtained as the intersection \( (\cap L_i) \cap \nu_{p_1, \ldots, p_r}(\mathbb{P}^N) \). This subscheme is singular or not of expected codimension if the intersection is non-transversal. Therefore, the discriminant points of \( \prod_i |O_{\mathbb{P}^n}(p_i)| \) are precisely the points of the multi-dual variety to the image of multi-Veronese embedding. As this image is irreducible and smooth, we get the irreducibility of the discriminant.

It remains to check the co-dimension, this is done by the standard lifting to a bigger space. Consider the incidence variety,

\[
\tilde{\Delta} = \{ (o, \{ f_i \}) \mid f_1|_o = \cdots = f_r|_o = 0, \quad \text{rank}(\text{Jac}(f_1, \ldots, f_r)|_o) < r \} \subset \mathbb{P}^N \times \prod_i |O_{\mathbb{P}^n}(p_i)|.
\]

Here \( \text{Jac}(f_1, \ldots, f_r) \) is the Jacobian matrix of the forms, the rank condition means that all the maximal minors vanish. The projection \( \tilde{\Delta} \to \Delta \) is the normalization, in particular the two varieties have the same dimension. The codimension of \( \tilde{\Delta} \) is \( r + (N + 1 - r) \). Thus \( \text{dim}(\tilde{\Delta}) = \text{dim}(\prod_i |O_{\mathbb{P}^n}(p_i)|) - 1 \). Thus the codimension of \( \Delta \subset \prod_i |O_{\mathbb{P}^n}(p_i)| \) is one. ■

We mention several recent works on the classical discriminant of projective complete intersections: [Esterov2011], [Benoist2012], [C.C.D.R.S.2011]. In particular, in many cases the multi-degrees were computed.

3. Transversal singularity type

At smooth points of $Z$ the transversal type can be defined geometrically, by taking a transversal slice. At singular points of $Z$ the definition of transversal type is algebraic. The situation is similar to replacing the local triviality of fibre bundles over a smooth base by the local freeness of sheaves over an arbitrary base.
3.1. **At smooth points of** $Z$. Let $o \in Z$ be a smooth point of $Z$, and let $(L^\perp, o) \subset (M, o)$ be the germ of a smooth variety of dim $L^\perp = k$, transversal to $(Z, o)$, so that $(L^\perp \cap X, o) = \{o\}$. For example, in some convenient local coordinates of $(M, o)$, (going to henselization or completion of the local ring), $L^\perp$ is the germ of a linear subspace. The intersection $(L^\perp \cap X, o)$ can have non-isolated singularities. Even if the singularity of $(L^\perp \cap X, o)$ is isolated, its embedded (topological for $k = \mathbb{C}$) singularity type depends on the choice of $L^\perp$.

**Example 3.1.** Consider Whitney umbrella, $\{x^2z = y^2\} \subset (\mathbb{k}^3, o)$. For the generic transversal $L^\perp = \{z = \alpha x + \beta y + \cdots\}$, $\alpha \neq 0$ the intersection $(L^\perp \cap X, o)$ is a plane curve singularity of type $A_2$, i.e. a cusp. For $\{z = \delta x\}$ the intersection is of type $A_{p+1}$, for $\{z = 0\}$ the intersection is a non-isolated singularity, a double line.

Nevertheless, for $k = \mathbb{C}$, the singularity $(L^\perp \cap X, o)$ is isolated and its topological type does not depend on the choice of $L^\perp$ for $L^\perp$ generic enough (corresponding to a dense open subset of a jet space). We call this the ‘transversal topological singularity type’ of $(X, o)$. For an arbitrary field $k$ we proceed as follows.

**Definition 3.2.** Let $(Z, o)$ be smooth and $(X, o) \subset (M, o)$ be a s.c.i. The transversal type of $(X, o)$ along $(Z, o)$ is called ordinary at $o$ if the multiplicity of $X$ along $Z$ is locally constant near $o$ and for some choice of $(L^\perp, o)$ the (strictly complete) intersection $(L^\perp \cap X, o)$ is an isolated ordinary multiple point, in the sense of §2.3.

Explicitly: the condition is that the projectivized tangent cone $\mathbb{P}T_{(L^\perp \cap X, o)} \subset \mathbb{P}L^\perp$ is a smooth complete intersection with equations inherited from the equations of $(L^\perp \cap X, o)$. By example 2.2, being of ordinary type is an open property, if one considers deformations preserving the multiplicity sequence.

**Lemma 3.3.** Suppose for some choice of $L^\perp$ the intersection $(L^\perp \cap X, o)$ is an isolated ordinary multiple point. Then such an intersection is an ordinary multiple point for any other choice of $L^\perp$, transversal to $Z$. In particular, the property ‘$(X, o)$ has an ordinary type along $(Z, o)$’ is independent of the choice of $L^\perp$ (and the multidegree of $\mathbb{P}T_{(X \cap L^\perp, o)}$ is well defined).

Moreover, for any $o' \in Z$ sufficiently close to $o$ the type of $(X, o')$ is also ordinary (and the multidegree of $\mathbb{P}T_{(X \cap L^\perp, o)}$ is locally constant).

**Proof.** Choose the local coordinates $(x_1, \ldots, x_N)$ in $(M, o)$ such that $(Z, o) = \{x_1 = \cdots = x_k = 0\}$ and $(L^\perp, o) = \{x_{k+1} = \cdots = x_N = 0\}$ and $(L^\perp \cap X, o)$ is an isolated ordinary multiple point. By lemma 2.3 we can choose the generators $\langle f_1, \ldots, f_r \rangle = I_{(X, o)}$ satisfying

$$f_i = \sum_{m_1 + \cdots + m_k = p_i} x_1^{m_1} \cdots x_k^{m_k} a^{(i)}_{m_1, \ldots, m_k} (x_{k+1}, \ldots, x_N) + \text{(terms of higher order in } x_1, \ldots, x_k\text{)}$$

and the projective complete intersection, $\mathbb{P}T_{(L^\perp \cap X, o)} = \bigcap_{i=1}^r \left\{ \sum_{m_1 + \cdots + m_k = p_i} x_1^{m_1} \cdots x_k^{m_k} a^{(i)}_{m_1, \ldots, m_k} (0) = 0 \right\} \subset \mathbb{P}^{k-1}$, is smooth of expected dimension. Any other transversal section (after a possible linear reparametrization of the last $N - k$ coordinates) can be presented as

$$L_i^\perp = \{x_{k+1} + l_{k+1}(x_1, \ldots, x_k) + g_{k+1}(x_1, \ldots, x_N) = \cdots = x_N + l_N(x_1, \ldots, x_k) + g_N(x_1, \ldots, x_N) = 0\},$$

where all $\{g_i\}$ are of order at least two. Therefore $\mathbb{P}T_{(L_i^\perp \cap X, o)} = \mathbb{P}T_{(L^\perp \cap X, o)}$, hence the first statement.

Since all the $\{p_i\}$ are locally constant, the projectivized tangent cones $\mathbb{P}T_{(L_i^\perp \cap X, o')}$ form a flat family as $o'$ moves in $(Z, o)$. As the central fibre of this family is smooth is smooth all the nearby fibres are smooth.

**Remark 3.4.** Both assumptions in definition 3.2 (being s.c.i. and of locally constant multiplicity) are important, as the following examples show.

* Consider $(X, o) = \{x^3 + zy^2 + y^5 = 0\} \subset (\mathbb{k}^3, o)$, here $(Z, o)$ is the $z$-axis. For the generic section (those $\{z = \alpha x + \beta y + \cdots\}$ for which the cubic polynomial $(x^3 + \alpha xy^2 + \beta y^3)$ has distinct roots) the intersection $(X \cap L^\perp, o)$ is an isolated ordinary multiple point. But e.g. for the section $z = 0$ the intersection $(L^\perp \cap X, o)$ has a triple line as the tangent cone, i.e. non-isolated singularity. Moreover, for all the other points of $Z$ the intersection $(L^\perp \cap X, o')$ is never an ordinary multiple point.

* Consider the complete intersection $X = \{ty^2 + x^2 + zy^3 = 0 = xy + z^3\} \subset \mathbb{k}^4$. Here the singular locus is the $t$-axis, $\text{Sing}(X) = \{x = y = z = 0\}$. For $t \neq 0$ the singularity is a s.c.i. Consider the following two sections at the origin: $t = 0$ and $t = y$; both hyperplanes are transverse to $\text{Sing}(X)$. We have $T_{X \cap \{t = 0\}, o} = \{x^2 = yx = xz^3 = z^5 + 3y^5 = 0\}$ so $\mathbb{P}T_{(X \cap \{t = 0\}, o)} \subset \mathbb{P}^2$ is smooth (a reduced point). On the other hand: $T_{X \cap \{t = y\}, o} = \{x^2 = xy = xz^3 - y^4 = y^5 = 0\}$ so $\mathbb{P}T_{(X \cap \{t = y\}, o)} \subset \mathbb{P}^2$ is a non-reduced point.
3.2. The definition of transversal type at singular points of $Z$. Consider now the general case, $(Z, o) \subset (M, o)$ is a c.i., not necessarily smooth or s.c.i. Take some basis, $I_{Z/M} = \langle g_1, \ldots, g_k \rangle$. Let $(X, o)$ be a s.c.i. and consider a good basis $I_X = \langle f_1, \ldots, f_r \rangle$, in the sense of §2.2. Suppose the generic multiplicity of $f_i$ along $(Z, o)$ is $p_i$. Then, by lemma 2.3, $f_i = \sum_{m_j = p_i} y_1^{m_1} \cdots y_k^{m_k} a_{m_1 \cdots m_k}(x)$. Consider the ‘virtual normal cone’ of $(X, o)$ along $(Z, o)$:

$$\mathcal{N}_{vir} := \cap_{i=1}^r \{ \sum_{m_j = p_i} y_1^{m_1} \cdots y_k^{m_k} a_{m_1 \cdots m_k}(i)| o = 0 \} \subset k^{y_1 \cdots y_k}.$$

Definition-Proposition 3.5. 1. The germ $(X, o)$ is said to have an ordinary transversal type along $(Z, o)$, at $o$, if $\mathbb{P}(\mathcal{N}_{vir}) \subset \mathbb{P}^{k-1}$ is a smooth complete intersection of dimension $\dim(X) - \dim(Z) - 1$.

2. This definition does not depend on the choices of basis of $I_{Z/M}$ or the choice of good basis of $I_{X/M}$.

3. If the transversal type is ordinary at $o$ then it is ordinary at the nearby points.

4. If $(Z, o)$ is smooth then the current definition coincides with definition 3.2.

Proof. 2. The proof is the same as that of Lemma 3.3.

3. Note that the expansion $f_i = \sum_{m_j = p_i} y_1^{m_1} \cdots y_k^{m_k} a_{m_1 \cdots m_k}(i)$ holds in some neighborhood of $o \in M$. Thus the collection $\{a_{m_1 \cdots m_k}(i)\}$, evaluated at some $o'$ is obtained by a deformation of the collection at $o$. And a small deformation of a smooth projective complete intersection (of expected dimension) is again of that type.

4. If $(Z, o)$ is smooth then the generators $\{g_1, \ldots, g_k\}$ can be chosen as local coordinates. Then we get the definition 3.2.

Example 3.6. Consider a reduced complete intersection $(Z, o) = \cap_i \{g_i = 0 \} \subset (k^N, o)$. Consider the hypersurface $(X, o) = \{ \sum g_i^p (1 + b_i(x)) = 0 \} \subset (k^p, o)$, where $b_i(x) \in m, p \geq 2$. Then $Sing(X, o) = (Z, o)$, as a set, and the transversal type of $(X, o)$ along $(Z, o)$ is ordinary. Compare to example 1.2.

Finally, we remark that the property of being of ordinary type is open. More precisely, take sufficiently small representatives of the germs, $Z \subset X \subset M$, i.e. work in a henselian of formal neighborhoods of $Z$. Consider an equimultiple deformation of $X$, i.e. the family $\{(X_t, Z_t)\}$ satisfying:

* both $\{X_t\}$ and $\{Z_t\}$ are flat families

* $Sing(X_t) = Z_t$, the good basis of $I_{X/M}$ deforms, $\{f(t, x)\}$, and mult$_{Z_t}(f(t, x)) = p_i$.

Then, if $(X, o)$ is of ordinary type, the singularity type of $X_t$ along $Z_t$ is ordinary for $t$ in a sufficiently small neighborhood of the origin.

4. The discriminant and its scheme structure

Let $X \subset M$ be as in §1.1, and let $Z$ be a positive dimensional connected component of $Sing(X)$. We assume that $X$ and $Z$ are locally complete intersections, hence both are pure-dimensional. Let $\text{codim}_M(Z) = k, \text{codim}_X(Z) = r$ and $\dim(X) = n$, so that $n + r = N = \dim(M)$. We assume that $(X, o)$ has ordinary transversal type along $(Z, o)$ at its generic points. The other points are candidates for $\Delta^\perp$.

4.1. Some motivational examples. Example 1.1 shows that it is natural to assign some multiplicities to the points of $\Delta^\perp$, i.e. this discriminant is often a non-reduced scheme. Already that example suggests how to choose the scheme structure, the guiding principle is the flatness of $\Delta^\perp$ under deformation:

Let $\{X^{(t)}\}_t$ be a flat family with the common singular locus, $Sing(X^{(t)}) = \text{const}$. Suppose for a connected component $Z \subset Sing(X^{(t)})$ the generic multiplicity mult$_Z(X^{(t)})$ is constant, and the generic transversal singularity type of $X^{(t)}$ along $Z$ is ordinary for any $t$. Then the discriminants $\Delta^\perp_t$ must form a flat family of embedded subschemes of $Z$.

Example 4.1. Example 1.1 shows that $\Delta^\perp$ can be non-reduced if the degeneration occurs ‘faster than normally’. Another reason for being non-reduced is when the degeneration is not ‘minimal’. Consider the surface $X = \{ x^p z = y^p + x^{p+1} \} \subset k^3$. Its singular locus is the line $Z = \{ x = 0 = y \}$. Consider the projection $k^3 \rightarrow Z, (x, y, z) \rightarrow z$, and the fibres $\pi^{-1}(z)$. Then we have a family of plane curve singularities, $\pi^{-1}(t) \cap X \subset \pi^{-1}(t) = (k^2, o)$, for $t \in Z$. This family is equimultiple, thus the projectivized tangent cones of these curve singularities form the flat family: $\{\sigma_x z = \sigma_y\} \subset P^1_{\sigma_x, \sigma_y} \times k^1_z$. For each $z \neq 0$ there are $p$ distinct roots, while for $z = 0$ all these roots coincide. Under a generic deformation this multiple root at $z = 0$ splits into several double roots near $z = 0$. In our case one can take $\{\sigma_x z = \sigma_y - \epsilon \sigma_x^{p-1} \sigma_y\} \subset P^1_{\sigma_x, \sigma_y} \times k^1_z$. By direct check, for each fixed $\epsilon \neq 0$ the number of the double roots near $z = 0$ is $(p - 1)$. So, by the principle above, the multiplicity of $\Delta^\perp$ for the initial surface should be $(p - 1)$. This multiplicity can be obtained also in another way, as the degree of ramification. Blow-up $k^3$ along the line $Z = \{ x = 0 = y \}$, let
$E \subset Bl_Z \mathbb{k}^3$ be the exceptional divisor, consider the strict transform $\bar{X} \subset BL_Z \mathbb{k}^3$ and the projection $\bar{X} \cap E \to Z$. Explicitly: $Z \times \mathbb{P}^1 = E \supset \bar{X} \cap E = \{x^r z = \sigma_y^p\} \to \{z\} \subset Z$. This is a $p : 1$ covering, totally ramified over $z = 0$ and the ramification degree is $(p - 1)$.

This example suggests the second guiding property which has to be satisfied by $\Delta^\perp$:

Additionally to the previous assumptions, suppose $Z$ is smooth and consider the blowup $Bl_Z M \to M$. Then the scheme structure of $\Delta^\perp$ should reflect the ramification properties of the map $\bar{X} \cap E \to Z$.

**Example 4.2.** Consider the hypersurface $X = \{z(x^p + y^r) = x^q + y^r\} \subset \mathbb{k}^3$, with $p < q, r$. Again, $Sing(X) = \{x = y = z = 0\}$ as a set. The deformation $\{xz^p + (z - \epsilon)y^r = x^q + y^r\}$ splits the discriminant into two: at $z = 0$ and at $z = \epsilon$. The previous example suggests that both points should have multiplicity $(p - 1)$, regardless of $q, r$. Hence the multiplicity in the current case should be $2(p - 1)$.

This suggests the third principle:

The scheme structure of $\Delta^\perp$ is determined solely by the ‘first infinitesimal neighborhood’ of $Sing(X)$ in $X$, more precisely, by the exceptional divisor of blowup: $(E, \bar{X} \cap E) \subset (Bl_Z M, \bar{X})$ or by the co-normal sheaves $I_{Z/M}/I_{Z/M}^2$ and $I_{X/M}/I_{X/M}^2$.

### 4.2. The definition of the subscheme

$\Delta^\perp \subset Z$. We will define $\Delta^\perp$ as a Cartier divisor in $Z$: hence at any point $p \in Z$ we need to define a local equation. Let the reduced germ $(Z, o) \subset (M, o)$ be defined by the ideal $I_{Z/M,o} = \langle g_1, \ldots, g_k \rangle$, for $g_i \in O_{(M,o)}$. Let $(X, o) \subset (M, o)$ be a strictly complete intersection with singular set $(Z, o)$.

**4.2.1.** We fix good generators $\{f_i\}_{i=1}^r$ of $I_{X/(M,o)}$, as in §2.2. Let the generic multiplicity of $f_i$ along $(Z, o)$ be $p_i$. If $(Z, o)$ is not locally irreducible, we assume that the corresponding multidegree of $(X, o)$ at generic point of each component of $(Z, o)$ is the same. Then, by Lemma 2.3, $f_i \in (I_{Z/(M,o)})^{p_i}$, that is,

$$f_i = \sum_{m_1 + \cdots + m_k = p_i} g_1^{m_1} \cdots g_k^{m_k} a_{m_1, \ldots, m_k}^{(i)};$$

where $\{a_{m_1, \ldots, m_k}^{(i)} \mid \sum_j m_j = p_i, 1 \leq i \leq r\}$ is a collection of functions in $O_{(M,o)}$. Their values at different points $o' \in (Z, o)$ define a family of intersections \{P(o')\}_{o' \in (Z, o)} with multidegree $(p_1, \ldots, p_r)$ parameterized by $(Z, o)$:

$$\{P(o')\}_{o' \in (Z, o)} = \bigcap_{i=1}^r \{ \sum_{m_1 + \cdots + m_k = p_i} g_1^{m_1} \cdots g_k^{m_k} a_{m_1, \ldots, m_k}^{(i)} | o' \in (Z, o) \} \subset (Z, o) \times \mathbb{P}^{k-1}_{y_1, \ldots, y_k}.$$ 

By our assumptions, for generic $o' \in Z \setminus \{o\}$ the corresponding complete intersection is smooth of codimension $r$.

Let $U(o)$ be a small open neighborhood of $o$ in $Z$ where all the involved functions $g_j, f_i$ and $a_{m_1, \ldots, m_k}^{(i)}$ are defined.

Let $\Delta$ be the classical discriminant in the parameter space of projective complete intersections in $\mathbb{P}^{k-1}$ of codimension $r$ and multidegree $(p_1, \ldots, p_r)$, see §2.4. It is a hypersurface, defined by one equation $\Delta = \{\mathcal{D} = 0\}$. We define the local Cartier divisor $\Delta^\perp$ in $U(o)$ by

$$\Delta^\perp(U(o)) := \{\mathcal{D}^\perp(o') := \mathcal{D}(\{a_{m_1, \ldots, m_k}^{(i)} \mid \sum m_i = p_i\}) = 0\} \subset U(o).$$

By the definition of $\mathcal{D}$: $o' \in \Delta^\perp$ if and only if the transversal type of $(X, o')$ along $(Z, o')$ is not ordinary, in the sense of definition 3.5. Note that $\Delta^\perp$ is defined also at the points where the singularities of $P(O)$ are non-isolated or this space is not of expected codimension.

**Proposition 4.3.** The ideal generated by $\mathcal{D}^\perp \subset O_{U(o)}$ is independent, of the choices of the generators $\langle g_1, \ldots, g_k \rangle$ of $I_{Z/(M,o)}$, of the good generators $f_1, \ldots, f_r$ of $I_{X/(M,o)}$, and of the coefficients $\{a_{m_1, \ldots, m_k}^{(i)}\}$ in (5).

In particular, for a covering $\{U(o)\}_o$ of $Z$, the collection of local divisors $\Delta^\perp(U(o))$ defines a Cartier divisor $\Delta^\perp \subset Z$.

**Proof.** Any two choices of $\{g_i\}$ and $\{a_{m_1, \ldots, m_k}^{(i)} \mid (Z, o)\}_{\sum m_i = p_i}$ are related by invertible (linear) transformations: $g_i \rightarrow \sum G_{ij} g_j$ and $\{a_{m_1, \ldots, m_k}^{(i)} \mid (Z, o)\}_{\sum m_i = p_i} \rightarrow A\{a_{m_1, \ldots, m_k}^{(i)} \mid (Z, o)\}_{\sum m_i = p_i}$, here $A$ is a huge tensor, both $G$ and $A$ depend on $o \in Z$. Both transformations $G$ and $A$ result in a linear transformation in $\mathbb{P}^{k-1}$, which does not change the (non-)smoothness of the complete intersection $\mathbb{P}(I/I^2)|_{pt} \subset \mathbb{P}^{k-1}$. Hence the independence of the choice of basis. ■
4.2.2. We restate the construction of $\Delta^\perp$ in a more invariant language. Let $(Z, o) \subset (M, o)$ be a locally complete intersection. Choose some representatives of the germs and blowup the ambient space: \((Bl_Z M, E) \to (M, Z)\). As the normal sheaf $N^\perp_Z/M$ is locally free at $o \in Z$, we can trivialize the fibration $E \to Z$ near $o \in Z$ by $\phi : E|_{(Z, o)} \simto (Z, o) \times \mathbb{P}^{k-1}$. (Here $k$ is the codimension of $Z$ in $M$ and the trivialization implies that some homogeneous coordinates in $\mathbb{P}^{k-1}$ are fixed.) Then the strict transform $\tilde{X} \subset Bl_Z M$ produces the flat family of embedded projective varieties: $\tilde{X} \cap E|_{(Z, o)} \simeq \{Y_t\}_t \subset \mathbb{P}^{k-1} \times (Z, o)$, cf. equation (6). As $X$ is a strictly complete intersection at each point, each fibre $Y_t \subset \mathbb{P}^{k-1}$ is a complete intersection with the chosen basis. Thus we get a map $(Z, o) \xrightarrow{j_\phi} \prod_i |O_{p_k-1}(p_i)|$. Note that the map $j_\phi$ depends on the trivialization $\phi$. But the comparison to the previous subsection gives:

**Corollary 4.4.** The discriminant of the transversal type is the pullback: $(\Delta^\perp, o) = j^\ast_\phi(\Delta, j_\phi(o))$. In particular the germ $(\Delta^\perp, o) \subset (Z, o)$ does not depend on the choice of trivialization $\phi$.

**Remark 4.5.** One is naturally tempted to define $\Delta^\perp$ in the case when $Z$ has arbitrary (not l.c.i.) singularities as follows.

The last definition provides the rational map $Z \xrightarrow{j^\ast_\phi} \prod_i |O_{p_k-1}(p_i)|$ defined on the smooth points of $Z$. Resolve the indeterminacies of $j_\phi$, as on the diagram. Then the map $j_\phi$ can be lifted to the map $\tilde{j}_\phi$, defined everywhere on $\tilde{Z}$. Hence the natural definition would be: $\Delta^\perp := \nu_\ast(\tilde{j}_\phi^* (\Delta))$, corresponding to the topological closure of $\tilde{j}_\phi(Smooth(Z))$ in $\tilde{Z} \xrightarrow{j_\phi} \prod_i |O_{p_k-1}(p_i)|$. But the part of $\Delta^\perp$ at singular points of $Z$ depends on the choice of the trivialization $\phi$! Consider example 1.2, there $Z$ contains the line $\{x = 0 = z \} \subset \mathbb{P}^3$. Consider an open subset: $U = \{x = 0 = z, y \neq 0, w = 1\}$. Choose the natural local trivialization: $E|_U \xrightarrow{j_\phi} \mathbb{P}^1_{y_\sigma, \sigma_z} \times k_y^1$. Then $\tilde{X} \cap E|_U \xrightarrow{j_\phi} \{\sigma_\tau^2 y = \sigma_i^2\} \subset \mathbb{P}^1 \times U$. And the closure of the image intersects the classical discriminant. On the other hand, for the local trivialization $E|_U \xrightarrow{j_\phi} \mathbb{P}^1_{y_\sigma, \sigma_z} \times k_y^1$ the image is just one point, not in the discriminant.

4.2.3. The definition of $\Delta^\perp$ as the pullback of the classical discriminant is rather theoretical, as in most cases it is extremely difficult to write down the classical discriminant explicitly. (Recall that even in the hypersurface case, $r = 1$, $\mathfrak{D}$ is a polynomial of degree $k(p-1)^k-1$ in $\binom{p+k}{k}$ variables.) Yet, some consequences are obtained immediately.

Let $\{X(t)\}_t \subset M \times (k^1, o)$ be a flat family of subvarieties (strictly complete intersections at each point). Suppose a connected component of their singular loci deforms accordingly, i.e. there exists a flat family of locally complete intersections $\{Z_t\}_t$ such that for any $t$: $Z_t \subset Sing(X(t))$ is a connected component. Suppose the generic multiplicity sequence $(p_1, \ldots, p_r)$ of $X(t)$ along $Z_t$ is preserved and for each $t$ the generic transversal type is ordinary.

**Corollary 4.6.** Under these assumptions the following holds:

1. The family $\{\Delta^\perp_t\}_{t \in (k^1, o)}$ is flat.
2. If in the above situation the singular locus, $\{Z_t\}_{t \in (k^1, o)}$, is preserved, then the equivalence class of $[\Delta^\perp]$ in the Neron-Severi group $NS(Z)$ is preserved.

**Proof.** 1. As $Z_t$ form a flat family we can choose a local basis $I(z_i, o_t) = \langle f_1(t), \ldots, f_k(t) \rangle$, where $f_i \in O_{(M, o)} \otimes O_{(k^1, o)}$. Similarly, $(X(t), o_t) = \cap_i (X_i(t), o_t)$ giving rise to the families of functions $\{a_{m_1, \ldots, m_k}(z, o)\}_{\sum m_j = p_j(t)}$. Therefore the polynomial $\mathfrak{D}(\{a_{m_1, \ldots, m_k}(z, o)\}_{\sum m_j = p_j(t)})$ is a power series in $t$. Finally, $\mathfrak{D}(\{a_{m_1, \ldots, m_k}(z, o)\}_{\sum m_j = p_j(t) = 0}) \neq 0$, as the generic transversal type of $(X_{t=0}, o)$ is ordinary. Thus the flatness of the family of subschemes $\Delta^\perp(t) \subset Z_t$.

2. Note that two fibres of a flat deformation are algebraically equivalent. Therefore they have the same class in the Neron-Severi group.

**Remark 4.7.** 1. Though the discriminant is an effective Cartier divisor, it is not ample in general. It is not even numerically effective. For example, let $X \subset M$ be a hypersurface in a smooth complete four-fold, suppose $Z = Sing(X)$ is a smooth projective surface and $\emptyset \neq \Delta^\perp \subset Z$ a reduced curve. Suppose the generic transversal type of $X$ along $Z$ is ordinary, while at the generic point of $\Delta^\perp$ the local model is equisingular to:

$$(8) \qquad (X, o) = \{z(x^p + x^{p-1}y) + y^p f_{p-2}(x, y) + g_{> p}(x, y, z, w) = 0\} \subset (k^1, o).$$

Here $f_{p-2}$ is a generic enough homogeneous polynomials of degree $p - 2$, while $g_{> p}(x, y, z, w) \in \langle x, y \rangle^{p+1}$. Then, in the local coordinates, the discriminant is a line: $(\Delta^\perp, o) = \{x = y = z = 0\} \subset (Z, o) = \{x = y = 0\}$. If we blow up the generic point of $\Delta^\perp$ then we get a hypersurface $\tilde{X}$ whose singular locus is again a smooth surface, while the
discriminant is the strict transform \( \overline{\Delta}^\perp \). By blowing up several generic points of \( \Delta \) we get to the similar situation, but with negative self-intersection of \( \Delta^\perp \) in the surface \( \tilde{Z} \).

2. It would be interesting to obtain various other properties of \( \Delta^\perp \) from the properties of the classical discriminant \( \Delta \). For example, for \( k = \mathbb{C} \), when is the local complement \( (Z \setminus \Delta^\perp, pt) \) a \( K(\pi, 1) \) space (provided \( Z \) is good enough, e.g. smooth) etc.

4.3 A (computational) definition of \( \Delta^\perp \) via fitting ideals. Let \((X, o) \subset (M, o)\) be a s.c.i. of codimension \( r \). Suppose that the set \( \text{Sing}(X, o) = (Z, o) \) is a complete intersection in \((M, o)\), and the generic multiplicity sequence of \((X, o)\) is the same along all the irreducible components of \((Z, o)\), thus \((p_1, \ldots, p_r)\) is well defined.

Suppose further that the generic transversal type is ordinary along all the components of \((Z, o)\). Blowup \((M, o)\) along \((Z, o)\) and consider the corresponding diagram. As \((Z, o) \subset (M, o)\) is a complete intersection, the exceptional divisor can be locally trivialized: \( E \approx \mathbb{P}^{k-1} \times (Z, o) \).

As \((X, o) \subset (M, o)\) is a s.c.i., its strict transform gives a complete intersection: \( \tilde{X} \cap E = \bigcap_{i=1}^{r} \tilde{X}_i \cap E \). Thus over \((Z, o)\) we have: \( \tilde{X} \cap E = \{ f_1 = \cdots = f_r = 0 \} \subset \mathbb{P}^{k-1} \times (Z, o) \), with \( f_i \in H^0(\mathcal{O}_{\mathbb{P}^{k-1}}(p_i)) \otimes \mathcal{O}_{(Z, o)} \).

**Theorem 4.8.** 1. The critical locus of the projection is:

\[
\text{Crit}(\pi) := \{ f_1 = \cdots = f_r = 0, \ \text{rank} \left( \begin{array}{c} df_1 \\ \vdots \\ df_r \end{array} \right) < r \} \subset \mathbb{P}^{k-1} \times (Z, o),
\]

where \( \{ df_i \} \) are \( k \times 1 \) columns of partial derivatives of \( \{ f_i \} \), taken with respect to the homogeneous coordinates in \( \mathbb{P}^{k-1} \).

2. Suppose \( \text{mult}(X, o') \) is locally constant on \((Z, o)\), so the multiplicity sequence is locally constant too. Suppose the fiber \( \pi^{-1}(o) \subset \mathbb{P}^{k-1} \) has at most isolated singularities, so the map \( \pi \) is finite at \( o \in (Z, o) \). Then the discriminant is the image of \( \text{Crit}(\pi) \), with the fitting scheme structure, i.e. the defining ideal of \( \Delta^\perp \subset (Z, o) \) is: \( F_0(\pi_* \mathcal{O}_{\text{Crit}(\pi)}) \).

Here \( \pi_* \mathcal{O}_{\text{Crit}(\pi)} \) is the pushforward of the \( \mathcal{O}_{(Z, o)} \)-module \( \mathcal{O}_{\text{Crit}(\pi)} \) and \( F_0(...) \) is the minimal fitting ideal of a module, i.e. the ideal of maximal minors of a presentation matrix of the module.

**Proof.** 1. Recall the general definition of the critical locus of a map, [Teissier1976, pg.587]. Let \( X \xrightarrow{\pi} S \) be a flat map of (algebraic/analytic) spaces, with the fibres of pure dimension \( d \). Then \( \text{Crit}(\pi) \) is defined by the (coherent) sheaf of ideals \( F_d(\Omega^k_{X/S}) \). Here \( \Omega^k_{X/S} \) is the sheaf of relative differentials, while \( F_d(...) \) is the \( d \)th fitting ideal of an \( \mathcal{O}_S \)-module.

In our case we can work locally. Identify \( E |_{(Z, o)} \approx \mathbb{P}^{k-1} \times (Z, o) \), choose a point \((pt, o) \in \mathbb{P}^{k-1} \times (Z, o)\). Then we can consider the map of germs: \( (\mathbb{P}^{k-1}, pt) \times (Z, o) \supset (\tilde{X} \cap E) \xrightarrow{\pi} (Z, o) \). The module of relative differentials can be written as follows:

\[
\Omega^1 (\tilde{X} \cap E, pt \times o | (Z, o)) \approx \Omega^1 (\tilde{X} \cap E, pt \times o) / \pi^* \Omega^1 (Z, o) \approx \Omega^1 (\mathbb{P}^{k-1}, pt) \times \Omega^1 (Z, o) / \pi^* \Omega^1 (Z, o) \\
\approx \Omega^1 (\mathbb{P}^{k-1}, pt) \otimes \Omega^1 (Z, o) / \pi^* \Omega^1 (Z, o) = \Omega^1 (\tilde{X} \cap E, pt \times o | (Z, o))
\]

Here the derivatives of \( df_i \) are taken with respect to the local coordinates of \( (\mathbb{P}^{k-1}, pt) \). Thus we get the free resolution:

\[
\mathcal{O}_{(\tilde{X} \cap E, pt \times o | (Z, o))} (df_1, \ldots, df_r) / \mathcal{O}_{(\mathbb{P}^{k-1}, pt) \times \mathcal{O}_{(Z, o)}} (df_1, \ldots, df_r) \xrightarrow{r \times r \text{ matrix}} 0
\]

Here each \( df_i \) is taken as a \((k-1) \times 1 \) column, so the map is by \((k-1) \times r \) matrix. Recall that \( r < k \) and the dimension of fibres of \( \pi \) is \((k-1) \times r \). Thus the critical locus of the map is defined (inside \( \tilde{X} \cap E \)) by \( F_{k-1-r} (\Omega^1 (\tilde{X} \cap E, pt \times o | (Z, o)) \).

So, locally \( \text{Crit}(\pi) \) is defined by \( I_{\text{Crit}(\pi), pt \times o} = \{ f_1, \ldots, f_r, I_r (df_1, \ldots, df_r) \} \). Use Euler’s formula for homogeneous polynomial, \( \sum x_i \partial_x f_j = p_j f_j \), to get: \( I_{\text{Crit}(\pi), \pi^{-1} (o)} = \{ f_1, \ldots, f_r, I_r (df_1, \ldots, df_r) \} \), where now \((df_1, \ldots, df_r)\) is the \( k \times r \) matrix of partial derivatives in homogeneous coordinates.

2. By the assumption, the map \( \text{Crit}(\pi) \xrightarrow{\pi} (Z, o) \) is finite. Thus the discriminant of projection \( \tilde{X} \cap E \xrightarrow{\pi} (Z, o) \) is defined by fitting structure, [Teissier1976, pg.588], \( F_0(\pi_* \mathcal{O}_{\text{Crit}(\pi)}) \).

On the other hand, the discriminant of transversal type, \( \Delta^\perp \), was defined above as the pullback of the classical discriminant, \( \Delta \). But \( \Delta \) itself is defined as the discriminant of projection, see §2.4. Thus, the two definitions of \( \Delta^\perp \) coincide.

4.3.1. Using this approach we get the property: the definition of discriminant is compatible with the base change. Explicitly, denote \( X' = \tilde{X} \cap E \), so we consider the projection \( X' \xrightarrow{\pi} Z \) and its critical locus \( \text{Crit}(\pi) \). Given a map
(of algebraic/analytic/formal/.. spaces or germs), $S \xrightarrow{\phi} Z$, we consider the fibre product $X' \times Z$. Then one has the commutative diagram:

$$
\begin{array}{ccc}
\psi^*(\text{Crit}(\pi)) = \text{Crit}(\pi_S) & \subset X' \times Z & \xrightarrow{\phi} X' \\
\downarrow & \pi_S \downarrow & \downarrow \pi \\
\phi^*(\Delta_e) = \Delta_{\pi_S} & \subset S & \xrightarrow{\phi} Z \ni \Delta_x
\end{array}
$$

(11)

4.3.2. To compute the fitting ideal $F_0(\pi_*\mathcal{O}_{\text{Crit}(\pi)})$ can be unpleasant, in general. However, we can compute the transversal multiplicity of the discriminant, or the multiplicity of its intersections with germs, as follows.

**Proposition 4.9.** In the assumptions of the theorem above, consider a subgerm $(Z', o) \subseteq (Z, o)$, such that $(Z', o)$ is a one dimensional locally complete intersection and $(Z', o) \cap \Delta^\bot \subset (Z', o)$ is a Cartier divisor (in particular, a zero dimensional subscheme). Then

$$\text{deg}\left((Z', o) \cap \Delta^\bot\right) = \text{deg}\left(\text{Crit}(\pi) \cap \pi^{-1}(Z', o)\right).$$

**Proof.** The discriminant of the projection $(\pi^{-1}(Z', o) \cap \bar{X} \cap E) \xrightarrow{\pi|_{\bar{X}'}} (Z', o)$ is obtained by the base change, i.e. $\Delta_{Z'|Z}^\bot = \Delta_{Z'}^\bot|_{\bar{X}}$. Therefore we can restrict to $(Z', o)$. So, we assume that $(Z, o)$ is a one-dimensional locally complete intersection and $\Delta^\bot \subset (Z, o)$ is a Cartier divisor (in particular it is a zero dimensional subscheme). And we should prove: $\text{deg}(\Delta^\bot) = \text{deg}(\text{Crit}(\pi))$.

- We start from the case: $(Z, o)$ is smooth. Note that

$$\text{deg}(\Delta^\bot) = \dim_k \mathcal{O}_{\Delta^\bot} = \dim_k \mathcal{O}_{(Z, o)/\pi|_{\text{Crit}(\pi)}}$$

Thus, the statement to prove is: given a finite module $M$ over a one-dimensional regular local ring, $\mathcal{O}_{(Z, o)}$, the colength of the fitting ideal satisfies: $\text{colength}_{\mathcal{O}_{(Z, o)}}(F_0M) = \dim_k M$. This is a standard statement of commutative algebra.

Take the minimal free resolution: $\mathcal{O}_{(Z, o)}^{\oplus p} \xrightarrow{A} \mathcal{O}_{(Z, o)}^{\oplus q} \to M \to 0$. As $M$ is finite, it is supported at one point only, so $p \geq q$. Further, as the ring is local and regular, $A$ is equivalent, by $A \to UAV$, to a matrix with only one non-zero diagonal (and zeros outside this diagonal). Let $z$ be a generator of $\mathcal{O}_{(Z, o)}$, then $F_0(M) = F_0(A) = (z \Sigma d_i)$, here $d_i$ are the exponents of the diagonal. Thus $M \approx \oplus_i \mathcal{O}_{(Z, o)}/(z^{d_i})$ and colength$(F_0M) = \sum d_i = \dim_k M$. Proving that $\text{deg}(\Delta^\bot) = \text{deg}(\text{Crit}(\pi))$.

- Suppose $(Z, o)$ is a complete intersection (of dimension one), then it can be smoothed. Let $\{Z_i\}_{i \in \{k\}}$ be a smoothing, then we have the (flat) family of projections, $(\bar{X}_i \cap E) \xrightarrow{\pi} Z_i$. Explicitly, if $\bar{X} \cap E = \{\{f_i(x, z) = 0\}_i\}$ then $\bar{X} \cap E = \{\{f_i(x, z) = 0\}_i\}$.

This induces the flat family $\{\Delta_i^\bot\}$. Thus, for $t \in (k^1, o)$ small enough, we can fix some (small enough, Zariski open) neighborhood of $o \in M$ such that $\text{deg}(\Delta_{t=0}^\bot) = \text{deg}(\Delta_t^\bot)$, Here the r.h.s. is the total degree of $\Delta_t^\bot$ in the neighborhood. Note that $\Delta_t^\bot_{t\neq 0}$ is a subscheme of a smooth curve $Z_t$. Thus the statement holds for $\Delta_t^\bot_{t=0}$ and then, by flatness, for $\Delta_t^\bot_{t=0}$. \hfill \blacksquare

**Example 4.10.** 1. If $(Z, o)$ is already one dimensional, then from the proposition we get the multiplicity of the discriminant. One can re-compute the multiplicities from examples 1.1, 1.2, 4.1, 4.2.

2. (Extending example 4.1.) Consider the hypersurface singularity $X = \{x_1x_2^n = f_0(x_2, \ldots, x_{n-1}) + g_{>p}(x_1, \ldots, x_n)\} \subset (k^n, o)$, where $f_0(x_2, \ldots, x_{n-1})$ is a homogeneous form of degree $p$, while all the monomials of $g_{>p}(x_1, \ldots, x_n)$ are of degree bigger than $p$. Suppose $f_0$ is generic enough, so that $\{f_0(x_2, \ldots, x_{n-1}) = 0\} \subset \mathbb{P}^{n-3}$ is smooth. Suppose $g_{>p}$ contains a monomial $x_1^N$ for some $N$. Then $\text{Sing}(X) = \{x_1 = \cdots = x_{n-1} = 0\} \subset k^n$ and the generic transversal type (for $x_1 \neq 0$) is ordinary. The discriminant $\Delta^\bot \subset \text{Sing}(X)$ is supported at the point $\{x_n = 0\} \subset \text{Sing}(X)$ and its multiplicity equals the length of the scheme $\text{Crit}(\pi) = \{x_1^p = 0 = \partial f_0(\sigma_2, \ldots, \sigma_{n-1})\} \subset k_1^{x_1} \times \mathbb{P}^{n-2}_{\sigma_2, \ldots, \sigma_{n-1}}$. As the form $f_0$ is generic, this scheme coincides with the scheme $\{x_1^p = 0 = \partial f_0(\sigma_2, \ldots, \sigma_{n-1})\}$, whose degree is $q(p-1)^{n-2}$.

3. Consider the hypersurface singularity $(X, o) = \{x^p + x^{p+r}y^{r+1}z_1y^{p+2} + g_{>p}(x, y, z_1, z_2) = 0\} \subset (k_2^1, y, o) \times (k_2^1, z_1, z_2, o)$. Here $g_{>p}(x, y, z_1, z_2) \in (x, y)^{p+1}$. (The reduced) singular locus is the plane $(Z, o) = \{x = 0 = y\}$. If one blows-up along $(Z, o)$ then $\bar{X} \cap E = \{\sigma_x^p + \sigma_y^{p+r}\sigma_xz_1 + \sigma_y^{p+2}z_2 = 0\} \subset \mathbb{P}^1_{\sigma_x, \sigma_y} \times (k_2^1, z_1, z_2, o)$. The critical set is: $\text{Crit}(\pi) = (p \sigma_x^{p-1} + (p-r) \sigma_y^{p+r-1}\sigma_yz_1 + p \sigma_y^{p-2}z_2)$. Therefore $\text{Crit}(\pi)$ is located in the $\sigma_y = 1$ chart of $\mathbb{P}_1$. The discriminant $\Delta^\bot \subset (Z, o)$ is the image of $\text{Crit}(\pi)$ with the fitting structure.
We compute the multiplicity of the intersection $\Delta^\perp \cap \{z_2 = \alpha z_2^k\}$. Note that for $k = 1$ and generic $\alpha$ the germ $\{z_2 = \alpha z_2^k\} \subset (Z,o)$ is smooth and its tangent line is not in the tangent cone of $\Delta^\perp$. In this case the intersection multiplicity is just the multiplicity of $\Delta^\perp$ at $o$. By direct check:
\[(13)\]
$$
deg\left(\Delta^\perp \cap \{z_2 = \alpha z_2^k\}\right) = \deg\left(z_1 = 0 = \sigma^p_{z}^{-1}\right) + \deg\left(\sigma^p_{z}^{-r} + \alpha z^{k-1} = 0 = \sigma^p_{z}^{-r-1}\right) + \deg\left(\sigma^p_{z}^{-r} + \alpha z^{k-1} = 0 = \sigma^p_{z}^{-r} + z_1\right)
$$
If $\alpha = 0$ then in the second brackets we get infinity, thus the line $z_1 = 0$ belongs to $\Delta^\perp$. If $\alpha \neq 0$ then
\[(14)\]
$$
deg\left(\Delta^\perp \cap \{z_2 = \alpha z_2^k\}\right) = (p - 1) + (p - r - 1)(k - 1) + \min\left(p - r, r(k - 1)\right).
$$

**Corollary 4.11.** Suppose $\text{mult}(X,o)$ is locally constant along $(Z,o)$, hence the multiplicity sequence is locally constant too. Suppose the singularity of $(X \cap E)|o := \pi^{-1}(o)$ is isolated. Then the multiplicity of the discriminant is bounded:
\[
\text{mult}(\Delta^\perp,o) \geq \tilde{\tau}(\tilde{X} \cap \tilde{E}|o) \text{mult}(Z,o).
\]
Here $\tilde{\tau}(\tilde{X} \cap \tilde{E}|o)$ is the modified Tjurina number of the global complete intersection $(\tilde{X} \cap \tilde{E})|o \subset \mathbb{P}^{k-1}$. If the later is defined by $\{f_1 = \cdots = f_r = 0\} \subset \mathbb{P}^{k-1}$, then $\tilde{\tau}(\tilde{X} \cap \tilde{E}|o)$ is the length of the scheme
\[(15)\]
$$
\text{Crit}(\pi) \cap \pi^{-1}(o) = \{f_1|o = \cdots = f_r|o = 0, \text{ rank}\left(df_1, \ldots, df_r\right)|o < r\} \subset \mathbb{P}^{k-1}.
$$
For example, if $r = 1$ then $(\tilde{X} \cap \tilde{E})|o \subset \mathbb{P}^{k-1}$ is a hypersurface and $\tilde{\tau}(\tilde{X} \cap \tilde{E}|o)$ is the ordinary total Tjurina number of this hypersurface. For $r > 1$ this modified Tjurina number does not coincide with the usual Tjurina number. The difference occurs already for the simple singularities $I_{2k+1}, I_{2k+4}$, [AGLV-book2, I.2.1].

**Proof.** As the multiplicity is computed by the intersection with the generic smooth germ of complementary dimension, we can assume $\text{dim}(Z,o) = 1$. Further, as $\Delta^\perp$ is a Cartier divisor, under a flat deformation the discriminant splits into at least $\text{mult}(Z,o)$ points on the smooth part of $Z$. So, it is enough to check the case: $(Z,o)$ is smooth. Then $\text{mult}(\Delta^\perp,o)$ is just the length of the scheme $\text{Crit}(\pi) \cap \pi^{-1}(o)$, i.e. $\tilde{\tau}(\tilde{X} \cap \tilde{E}|o)$.

**Remark 4.12.** Note the difference with the classical results on the discriminant in the miniversal deformation, $\Delta \subset (\mathbb{C}^N,o)$: the multiplicity of that discriminant is the Milnor number of the fibre, [Teissier1976, §2], [Looijenga-book, Chapter 4]. In that case one considers the miniversal deformation and the critical locus of a function on $(\mathbb{C}^N,o)$, while in our case we consider the critical locus of the projection from $\tilde{X} \cap \tilde{E}$.

### 4.4. The stratification of singular locus.
In this section $(X,o) \subset (M,o)$ is a hypersurface singularity. Recall that the singularity type of $(L^\perp \cap X,o)$ depends on the choice of the section of the $L^\perp$, cf. example 3.1. But the singularities of $\mathbb{P}^{\text{GL}(P^{k-1})}$, or of $\mathbb{P}^{N^\text{vir}}$ if $Z$ is singular, are well defined.

**Lemma 4.13.** If $(Z,o) \subset (M,o)$ is a (locally) complete intersection then the projectivization of the ‘virtual normal cone’, $\mathbb{P}^{N^\text{vir}} \subset \mathbb{P}^{k-1}$, in the sense of §3.2, is well defined, its local equation is defined up to a $\mathbb{P}^{\text{GL}(P^{k-1})}$ transformation. In particular the singular points of $\mathbb{P}^{N^\text{vir}}$ are well defined.

Indeed, if $f = \sum x_1^{m_1} \cdots x_k^{m_k} a_{m_1,\ldots,m_k}(x)$ then the hypersurface $\{\sum \sigma_1^{m_1} \cdots \sigma_k^{m_k} a_{m_1,\ldots,m_k}(o) = 0\} \subset \mathbb{P}^{k-1}$ is defined up to $\mathbb{P}^{\text{GL}(P^{k-1})}$ transformation, cf. definition-proposition 3.5.

Thus, any stratification of $|O_{P^{k-1}}(p)|$ (e.g. the stratification by the singularity type, for a certain equivalence relation) induces a stratification of $\text{Sing}(X)$. We consider the standard stratification of $|O_{P^{k-1}}(p)|$: the strata of $\mu = \text{const}$ deformations. This defines the strata on $\text{Sing}(X)$:
\[(16)\]
$$
\Delta^\perp = \sum A_1 \cup \sum A_2 \cup \sum A_1, A_1 \cup \sum A_3, \sum A_1, A_2, \sum A_1, A_1, A_1, \cdots \sum D_4.
$$

### 5. The equivalence classes of discriminant and strata

#### 5.1. The class $[\Delta^\perp]$, the classical approach.

##### 5.1.1. The classical discriminant.
Consider a homogeneous polynomial of degree $p$ in $k$ variables
\[(17)\]
$$
f = \sum_{\sum m_i = p} a_{m_1,\ldots,m_k} x_1^{m_1} \cdots x_k^{m_k}.
$$
The corresponding projective hypersurface, $\{f = 0\} \subset \mathbb{P}^{k-1}$, is singular iff the coefficients $\{a_{*\star}\}$ satisfy the polynomial equation: $\mathcal{D}(\{a_{*\star}\}) = 0$. Here $\mathcal{D}$ is the classically studied discriminant, it is a homogeneous polynomial of degree $k(p - 1)^{k-1}$.

We need its ‘weighted degree’. Suppose the variables $\{x_i\}$ are given weights, say $w_i = w(x_i)$, while the weight of $f$ is determined to be $p$. This gives weights to the coefficients: $w(a_{m_1,\ldots,m_k}) = p - \sum w_i m_i$. The discriminant is
Let $Z \subseteq \mathbb{P}^{k-1}$. The fibers of the projection $p : E \to \mathbb{P}^{k-1}$ are projective varieties of dimension $(k-r-1)$. By theorem 4.8, the discriminant, as a scheme, is completely determined by the pair $(E, E \cap \tilde{X}) \subseteq (Bl_Z M, \tilde{X})$. We work in the setup of the diagram. Let $I = I_{Z/M}$ be the ideal sheaf of $Z$ in $M$, let $I/I^2$ be the co-normal sheaf, (which is locally free), and $E = P(\mathcal{N}_{Z/M})$ be the exceptional divisor.

5.2. The class $[\Delta^\perp]$, using Porteous-Thom formula. Here we prove Theorem 1.6. By theorem 4.8, the discriminant, as a scheme, is completely determined by the pair $(E, E \cap \tilde{X}) \subseteq (Bl_Z M, \tilde{X})$. We work in the setup of the diagram. Let $I = I_{Z/M}$ be the ideal sheaf of $Z$ in $M$, let $I/I^2$ be the co-normal sheaf, (which is locally free), and $E = P(\mathcal{N}_{Z/M})$ be the exceptional divisor.

The fibers of the projection $E \cap \tilde{X} \to Z$ are projective varieties of dimension $(k-r-1)$. By theorem 4.8, the class of $\Delta^\perp$
is obtained by the pushforward \( \pi_* \) of the class of critical locus \( \text{Crit}(\pi) \). The later class is obtained by Porteus-Thom formula, [Fulton-book, §14.4]:

\[
(20) \quad [\text{Crit}(\pi)] = (-1)^{k-r} c_{k-r}(T_{E \cap X} - \pi^* T_Z) \in A^{k-r}(E \cap \hat{X}).
\]

Here the relative Chern class is determined by the expansion, \( c_i(T_{E \cap X} - \pi^* T_Z) = \frac{c_i(T_{E \cap X})}{c_i(\pi^* T_Z)} \), of the total Chern polynomials, \( c_i(V) = \sum_{i \geq 0} t^i c_i(V) \).

Use the adjunction sequence \( 0 \rightarrow T_{E \cap X} \rightarrow T_E|_{E \cap X} \rightarrow N_{E \cap X} \rightarrow 0 \), where the later bundle is the normal bundle of \( E \cap X \) in \( E \). Recall that \( (X, E \cap X) \) is a complete intersection, thus \( c_i(N_{E \cap X}) = \prod(1 + t[\hat{X}_i]) \), where \( [\hat{X}_i] = \pi^*[X_i] + p_i c_i(O(E(1))) \), for the tautological bundle \( O_E(1) \).

The Chern class of the relative tangent bundle \( T_{E/Z} = T_E - \pi^* T_Z \) can be computed as follows A section of \( T_{E/Z} \) at each point over \( o \in Z \) correspond to a two dimension vector subspace in \( N_{E/M}|_o \). Such elements are produced as maps from the tautological line bundle on \( E \) to \( \pi^* N_{Z/M} \), so \( \text{Hom}(O_E(-1), \pi^* N_{Z/M}) \rightarrow T_{E/Z} \rightarrow 0 \). The kernel of this surjection consists of maps that send the line to itself, i.e.

\[
(21) \quad 0 \rightarrow \text{Hom}(O_E(-1), O_E(-1)) \rightarrow \text{Hom}(O_E(-1), \pi^* N_{Z/M}) \rightarrow T_{E/Z} \rightarrow 0.
\]

This gives the exact sequence, see also [Fulton-book, Example 3.2.11]:

\[
(22) \quad 0 \rightarrow O_E \rightarrow \pi^*(N_{Z/M}) \otimes O_E(1) \rightarrow T_E \rightarrow \pi^* T_Z \rightarrow 0.
\]

Thus \( \frac{c_i(T_E)}{c_i(\pi^* T_Z)} = c_i(\pi^*(N_{Z/M}) \otimes O_E(1)) \left( 1 + t c_1(O_E(1)) \right)^k t^r c_(\pi^*(N_{Z/M})) \), where \( \tau = \frac{k}{1 + t c_1(O_E(1))} \). Altogether:

\[
(23) \quad [\text{Crit}(\pi)] = (-1)^{k-r} \text{Coeff}_{t^k} \left( \sum_{j=0}^k \left( 1 + t c_1(O_E(1)) \right)^{k-j} t^i c_{j}(\pi^*(N_{Z/M})) \prod_{i=1}^{k} \left( 1 + t[X_i] \right) \right) \in A^{k-r}(E \cap \hat{X}).
\]

The class \( \lfloor \Delta \rceil \in A^1(Z) \) is obtained by pushforward \( \pi_*[\text{Crit}(\pi)] \). To compute this we first obtain the class \( j_*[\text{Crit}(\pi)] \in A^k(E) \), i.e. multiply the expression above by \( [E \cap \hat{X}] = \prod[X_i] \). Then apply the Gysin homomorphism, i.e. the projection \( A^k(E) \rightarrow A^1(Z) \).

Here is the computation. As the resulting class is in \( A^1(Z) \) we expand the expression in \( [X_i] \) and \( c_i \) and keep only the terms of order \( \leq 1 \). In particular we discard \( c_i(1) \) by \(-E\).

\[
\begin{align*}
\quad A^{k-r}(E) & \ni [\text{Crit}(\pi)] = (-1)^{k-r} \prod_{i=1}^r \left( [X_i] - p_i[E] \right) \text{Coeff}_{t^k} \left( \sum_{j=0}^k \left( 1 - \frac{t[X_i]}{p_i} \right)^{k-j} t^{i} c_{j}(\pi^*(N_{Z/M})) \prod_{i=1}^k \left( 1 - \frac{t[X_i]}{p_i} \right) \right) \\
& \sim (-1)^{k-r} \prod_{i=1}^r \left( [E]^r - [E]^{r-1} \prod_{i=1}^r \frac{[X_i]}{p_i} \right) \text{Coeff}_{t^k} \left( \left( 1 - \frac{t[X_i]}{p_i} \right)^{k-1} c_1(\pi^*(N_{Z/M})) \prod_{i=1}^k \left( 1 - \frac{t[X_i]}{p_i} \right) \right) =: B.
\end{align*}
\]

Applying the Gysin homomorphism amounts to the substitution (after expanding in powers of \( E \)): \( E^j \rightarrow (-1)^j s_{j-k+1} \), where \( s_{j-k+1} \) is the Segre class of \( N_{Z/M} \). Again, as the resulting class is in \( A^1(Z) \) we need only the terms linear in \( [X_i] \) or \( s_i(N_{Z/M}) \), but no higher terms. Hence we only need to extract the coefficients of \( E^k \) and \( E^{k-1} \):

\[
(24) \quad (-1)^k s_1(N_{Z/M}) \text{Coeff}_{E^k}((B) + (-1)^{k-1}\text{Coeff}_{E^{k-1}}(B) = (-1)^{k-1}\text{Coeff}_{E^k} \left( c_1(N_{Z/M})B + [E]B \right) =
\]

\[
(25) \quad \sum_{i=1}^r \text{Coeff}_{E^k} \left( c_1(N_{Z/M})([E]^{1-t} \prod_{i=1}^{r} \frac{[X_i]}{p_i}) - \frac{[E]}{1-t} \prod_{i=1}^{r} \frac{[X_i]}{p_i} \right) \]

\[
\begin{align*}
& = \text{Coeff}_{E^k} \left( (1-t)^k \sum_{i=1}^r \frac{[X_i]}{p_i} - (1-t)^{k-1} c_1(N_{Z/M}) \prod_{i=1}^{r} \frac{p_i}{1-p_i} \right) \\
& = \text{Coeff}_{E^k} \left( (1-t)^k \sum_{i=1}^r \frac{[X_i]}{p_i} - \frac{c_1(N_{Z/M})}{1-t} \prod_{i=1}^{r} \frac{p_i}{1-p_i} \right) \]

To get the first formula of the theorem we should prove:

\[
(26) \quad \text{Coeff}_{E^k} \left( (1-t)^{k-1} \prod_{i=1}^k (1-p_i t) \right) = \text{Coeff}_{E^k} \left( (1-t)^k \prod_{i=1}^k \frac{1}{1-p_i t} \right).
\]
Present the difference as the residue (remember that the residue of a derivative is always zero):

\[
Res_0 \frac{1}{t^{k-r+\epsilon}} \left( \frac{1}{\prod_i (1-p_i t)} \left( \frac{k}{1-t} - \sum_i \frac{1}{1-p_j t} \right) \right) = -Res_0 \frac{1}{t^2} f(\frac{1}{t}) = \\
= -Res_\infty t^{k-r-1} (t-1)^k \frac{\sum_j t^{-r-k}}{\prod_i (t-p_i t)^{-t} \left( \frac{k}{t-1} - \sum_j \frac{1}{t-p_j} \right)} = \\
= Res_\infty \left( \frac{d}{dt} \left( \frac{(1-t)^k}{\prod_i (1-p_i t)} \right) \right) = - \sum_i Res_{t=p_i} (B(t)')^0 = 0.
\]

Finally we write down the needed coefficient. The coefficient of \(c_1:\)

\[
Coef f_{v}\left( \frac{(1-t)^k}{\prod_i (1-p_i t)} \right) = (-1)^{k-r} Coef f_{v}\left( \frac{(1+x)^{k-1}}{\prod_i (1+p_i x)} \right) = \\
= (-1)^{k-r} Coef f_{v}\left( \sum_{\{k_i \geq 0\}} x^{\sum k_i} (1+x)^{k-r-1} \sum_{i} k_i \prod_i (1-p_i)^{k_i} \right)
\]

Here the second equality is proved in Proposition 2.1 of [Kerner-Nemethi-2012]. In the obtained expression, if \(\sum k_i \leq k-r-1\) then there is no term of order \((k-r).\) If \(\sum k_i \geq k-r\) then we have \(\frac{x^{\sum k_i}}{\prod_i (1+x)^{k_i-r+1}} = x^{\sum k_i} (1+\cdots).\) So, if \(\sum k_i > k-r\) then all the terms here are of order bigger than \((k-r).\) While for \(\sum k_i = k-r\) there is precisely one term, with coefficient 1. Therefore

\[
(-1)^{k-r} Coef f_{v}\left( \frac{1}{\prod_i 1-p_i t} \right) = \sum_{\{k_i \geq 0\}} x^{\sum k_i} (1+x)^{k-r-1} \sum_{i} k_i \prod_i (1-p_i)^{k_i}.
\]

By similar arguments we get: \(Coef f_{v}\left( \prod_i \frac{p_i}{1-p_i t (1-p_i t)^{2}} \right) = \frac{X_i}{p_i} \prod_i \frac{k_i}{p_i} = (p_j-1)^a S_{k-r-a}\) Hence the statement.

5.3. Classes of some further strata. We work in the setup of §5.2.

**Theorem 5.1.** Let \(Z \subset M\) be a reduced complete algebraic subspace which is a locally complete intersection. Let \((X,Z) \subset (M,Z)\) be a formal hypersurface germ (along \(Z\)) with \(Z \subset \text{Sing}(X)\) a connected component of the singular locus. Suppose the generic multiplicity of \(X\) along \(Z\) is \(p\) and the generic transversal type is ordinary. Set \(c_1 = c_i(N_{Z/M}).\) Then the classes of the codimension-two strata in \(A^2(Z)\) are given by:

\[
[\Sigma_{A_2}] = \frac{(k-1)(p-2)(p-1)-k^2}{2} (c_1)^2 p(2p-1) + [c_2] 2p(\frac{2k}{k-1} - p) - 2[c_1] X[p(k+1)] + [X]^2 k(k+1)
\]

\[
[\Sigma_{A_1,A_1}] = \frac{(p-1)^2k^2}{2} \left[ Xk - c_1 p \right]^2 + \frac{(p-1)^2k^2}{2} \left( \frac{[X]^2 k(6k^2 - 2 - p(k+2)(3k-2))}{4} + [p][c_1] \left( [p(k+1)(3k-2) - 2(3k^2 - 2)] + 2p[c_1] \left( [p^2(3k-2) - 4p(3k-1) + 12k] + 2p[c_1]^2 (p(9k-8) - 6(k-1) - p^2(3k-2)) \right) \right).
\]

**Proof.** By the general theory, the classes of the strata are obtained by specializing the relevant Thom polynomials (e.g. cf. Table 1 of [Kazarian2000] and Table 2 of [Kazarian2003]). In particular, in the low co-dimension cases we have the following classes in \(A^*(\text{Crit}(\pi)):\)

\[
[\Sigma_{A_1}] = 1, \quad [\Sigma_{A_2}] = a_1, \quad [\Sigma_{A_1,A_1}] = \frac{c_i(N_{A_2/M})-u-3a_1}{4}, \quad [\Sigma_{A_3}] = ua_1 + 3a_2, \quad [\Sigma_{D_4}] = a_1 a_2 - ua_2 - 2a_3 \ldots
\]

Here \(u = [\pi^* X] - p[E],\) while the classes \(\{a_i\}\) are defined by the formula:

\[
\sum_i a_i = c(T_{E/Z}^* \otimes \mathcal{O}_E(\bar{X}) - T_{E/Z}) = \frac{(1 + u + E)^k - (1 + u + E)^k - 1 + \cdots + c_k}{(1 - E)^k + (1 - E)^k - c_1 + \cdots + c_k} \quad \text{for } c_i = c_i(N_{Z/M}).
\]

As in §5.2 we pushforward these classes to \(A^*(E),\) i.e. multiply these expressions by \(j_{i,i_*}(1) = [\text{Crit}(\pi)] \in A^1(E \cap \bar{X})\) and by \([E \cap X] = [\pi^* X] - p[E].\) Note that the class \([\text{Crit}(\pi)]\) in the hypersurface case is

\[
c_k \left( \pi^* N_{Z/M}^* \otimes \mathcal{O}_E(\bar{X})(-1) \right) = (u + E)^k - (u + E)^k - 1 + \cdots + c_k(N_{Z/M}) + \cdots \pm c_k(N_{Z/M}).
\]
Then one applies the Gysin homomorphism to get the classes in $A^* (Z)$.

6. Applications

6.1. A bound on jumps of multiplicity. Let $X \subset M$ be a hypersurface of dimension $n$, suppose $Z = \text{Sing} (X)$ is a smooth compact connected curve, the generic multiplicity of $X$ along $Z$ is $p$ and the generic transversal type is ordinary. Suppose that at some points of $Z$ the multiplicity jumps: $\{\text{mult} (X, o_i) = p + p_i\}_i$. Using the degree of the discriminant we bound the number of such points.

As $\Delta^\perp \subset Z$ is zero dimensional, it is enough to compute the local degrees of $\Delta^\perp$ at the points $\{o_i\}_i$. Choose the local coordinates so that $(Z, o_i) = \{ x_1 = \cdots = x_n = 0 \} \subset (M, o_i)$ and $x_{n+1}$ is the local coordinate along $(Z, o_i)$. (If needed, we consider the formal germ $(Z, o_i)$.) As $\text{mult} (X, o_i) = p + p_i$ this hypersurface singularity can be deformed to a singularity of the type

$$\{x_{n+1}^p f_p (x_1, \ldots, x_n) + \sum_{j < k} x_j^{n+1} g_j = 0\} \subset (M, o_i),$$

where $f_p$ is a generic homogeneous form of degree $p$, and the order of $g_j$ at $o_i$ is bigger than $(p + p_i - j)$. So the local degree of $\Delta^\perp (X, o_i)$ is at least the local degree of the discriminant for this particular singularity.

Deform the later singularity to $\{ (x_{n+1}^p - \epsilon) f_p (x_1, \ldots, x_n) + \sum_{j < p_i} x_j^{n+1} g_j = 0\}$, this splits the initial point into $k_i$ discriminantal points. Further, each of these points deforms into $n$ distinct points: $\sum_{j=1}^n (x_{n+1}^p - \epsilon_j) x_j^{n+1} + \cdots$, i.e. all $\{\epsilon_j\}$ are distinct. For each such point the multiplicity of $X$ is locally constant, thus we can compute the local degree of the discriminant by using theorem 4.8. Namely, it is the degree of the scheme $\{\partial_1 f = \cdots = \partial_n f = 0\} = \{x_1^{p-1} = \cdots = x_n^{p-1} = 0\}$. Altogether, we get: $\text{deg} (\Delta^\perp) \geq \sum_i p_i n (p-1)^{n-1}$. Therefore, using theorem 1.4 (with $k = n$) we get:

$$\sum_i p_i \leq \text{deg} (|X|) - \frac{p}{n} \text{deg} (\text{det} (N_{Z/M}))$$

6.2. The multi-degrees of the classical discriminant. Fix a multi-degree $(p_1, \ldots, p_r)$, consider the parameter space of complete intersections of this multi-degree, $\prod_i |O_{p_{k-1}} (p_i)|$, cf. §5.1.1.

The classical discriminant $\Delta \subset \prod_i |O_{p_{k-1}} (p_i)|$, is a hypersurface, denote its multi-degree by $d = (d_1, \ldots, d_r)$. To compute $d$ we consider a particular case of the general problem. Suppose $M = Z \times M'$, for $Z = \mathbb{P}^1$ and we have a complete intersection $X \subset M$, such that $\text{Sing} (X) = Z \times \{o\}$ for some point $o \in M'$. This gives a one-dimensional family of complete intersections in $M'$. Then $\mathbb{P} N_{Z/X}$ induces the map $Z \xrightarrow{j} \prod_i |O_{p_{k-1}} (p_i)|$ such that $\mathbb{P} N_{Z/X}$ is the pullback of the universal family. For this map: $|X| = j^* c_1 (O_{\mathbb{P}^{r-1}} (p_i)) \subset A^1 (Z)$, hence the class $[\Delta^\perp] \in A^1 (Z)$ provide the multi-degree:

$$d_j = \frac{1}{p_j} \text{Coeff} f_{k-\cdots} \left[ \frac{(1 - t)^k}{1 - p t} \prod_{i=1}^r \frac{p_i}{1 - p_i t} \right]$$

By direct check, this coincides e.g. with the degrees of [Benois2012, Theorem 1.3].

Appendix A. Discriminant of transversal type in more general settings

Here we consider the case of a non-isolated hypersurface singularity $(X, o) \subset (M, o)$, whose singular locus, $(Z, o)$, is not necessarily a locally complete intersection, and the generic multiplicities of $(X, o)$ on the components of $(Z, o)$ can vary.

Consider the decomposition $(Z, o) = (\cup_i Z_i, o)$ so that the generic multiplicity of $(X, o)$ along $(Z_i, o)$ is (well defined) $p_i$. (Each $(Z_i, o)$ can be further reduced.) The generic transversal type of $X$ on any of $Z_i$ is assumed to be ordinary. Throughout this section we assume that the symbolic powers and the ordinary powers of the ideals coincide:

$$I_i^{(k)} (Z_i, o) = I_i^{(k)} (Z_i, o), \quad I_i^{(k)} (Z, o) = I_i^{(k)} (Z, o), \quad I_i (Z, o) = \cap_i I_i (Z_i, o) = \prod_i I_i (Z_i, o) \cap I_i (X, o) \subset \cap_i I_i^{(p_i)} (Z_i, o).$$

This is a significant restriction, it is not always satisfied when $(Z_i, o)$ is not l.c.i. For example, it is not satisfied for the surface singularity $\{xyz = 0\} \subset (K^3, o)$. This coincidence of powers of ideals is a weakening of the l.c.i. condition but it is restrictive enough to ensure the needed generalization.

According to the decomposition of $(Z, o)$ we define the collection of discriminants, $\{\Delta_i^\perp\}$, each of them lives naturally on $(Z_i, o)$. Fix some $i$, consider the defining ideal $I_i (Z_i, o) = (s_1, \ldots, s_k) \subset \mathcal{O} (M, o)$. Let $(X, o) = \{f = 0\}$, by the assumption we can expand: $f = \sum_{\sum_i a_j = p_i} a_{m_1, \ldots, m_k} s_1^{m_1} \cdots s_k^{m_k}$. Here the coefficients $\{a_{m_1, \ldots, m_k}\}$ are regular functions on $(M, o)$. 

16 Maxim Kazarian, Dmitry Kerner and András Némethi
Definition A.1. The collection of coefficients \( \{a_{m_j}\} \) is called non-degenerate at \( o \) if for any small deformation \( \{X_t\}_{t \in (k^1, o)} \) that preserves all the generic multiplicities \( \{p_j\} \), there exists a neighborhood \( o \in U_i \subset Z_t \) and a neighborhood \( 0 \in B \subset k^1 \) such that for any \( t \in B \) and any point \( pt \in U_t \cap \text{Smooth}(Z) \) the transversal type of \( X_t \) at \( pt \) is ordinary, i.e. \( \Delta_{pt}^{\perp}(X_t) = \emptyset \).

(Note that in the definition \( pt \) does not belong to the other components, \( \{Z_j\}_{j \neq i} \).)

The non-degeneracy is a generic property, as the following lemma shows. Consider the evaluation map of the coefficients, \( ev_o : \{a_{m_j}\} \to \{a_{m_j}(o)\} \). To an array of regular functions it assigns an array of numbers. As the coefficients vary, call the image of this map \( Val \), it is an affine space of a big dimension.

Lemma A.2. There exists a Zariski-dense open subset of \( Val \) whose preimage under \( ev_o \) gives non-degenerate collections of coefficients.

Proof. Suppose the collection \( \{a_{m_j}\} \) is degenerate. Then there exists a family of points in \( pt_t \in (Z_t \cap \text{Smooth}(Z), o) \) with \( \Delta_{pt_t}^{\perp}(X_t) \neq \emptyset \). So, the corresponding family of the coefficients of transversal type of \( (X_t, pt_t) \) satisfies the discriminant equation, \( \Delta \{b_j\} = 0 \). Note that the coefficients \( \{b_j\} \) are regular functions of \( \{a_{m_j}\} \). Thus the equation above gives an (algebraic/analytic/formal) equation on \( \{a_{m_j}\} \). Finally, this equation is non-trivial as the equation \( \Delta \{b_j\} = 0 \) is non-trivial (as the generic type of \( X \) along \( Z_t \) is ordinary). Therefore the evaluations \( \{a_{*}(0)\} \) lie in a hypersurface inside \( Val \). ■

Example A.3. Consider the union of a line and a plane, \( (Z, o) = (Z_1 \cup Z_2, o) \subset \{x = y = w = 0\} \cup \{z = w = 0\} \subset (k_t^{2yzw}, o) \). Let \( (X, o) = \{f = 0\} \subset (k_t^4, o) \) be a hypersurface singularity satisfying

\[
\sum_{i+j \leq p_1} a_{ij} z^i w^{p_2-i} + \sum_{i+j \leq p_2} b_{ij} z^i y^{p_1-p_2+i} z^j w^{p_2-j}.
\]

Suppose \( p_1 \leq p_2 \) then \( f = \sum_{i+j \leq p_1} a_{ij} z^i w^{p_2-i} \). The transversal type of \( X \) at any point \( o \in Z_2 \) is not ordinary if the homogeneous polynomial \( \sum_z z^i w^{p_2-i} \) has multiple roots. \( \\) Now, the discriminant is defined by deforming to a non-degenerate collection of coefficients. Let \( \{X_t\}_{t \neq 0} \subset (k_t^{2yzw}, o) \) be a family whose fibres for \( t \neq 0 \) have non-degenerate collection of coefficients at 0. Define the discriminant for \( t \neq 0 \) fibres by the closure \( \Delta_{Z_t}(X_t) = \Delta_{Z_t}(X_t) \subset \text{Smooth}(Z_t) \). As was proved above, the family \( \{\Delta_{Z_t}(X_t)\}_{t \neq 0} \) is flat. Thus, we take the flat limit at the origin, i.e. get the flat family \( \Delta^{\perp}(X) = \{\Delta_{Z_t}(X_t)\}_{t \in (k^1, o)} \).

Definition A.4. \( \Delta^{\perp}_{Z_t}(X) := \Delta^{\perp}(X)|_{t = 0} \).

Proposition A.5. 1. This definition does not depend on the choice of deformation, \( \{X_t\}_t \), neither on the choice of basis \( I_{Z_t}^{(p)}(o) \).
2. \( \Delta_{Z_t}(X) \subset Z_t \) is an effective Cartier divisor.
3. Suppose \( (Z_t, o) \subset (M, o) \) is a locally complete intersection, then the current definition coincides with the definition of \( \Delta_{\text{def}} \).

Proof. 1. By the assumption \( I_{Z_t}^{(p)}(o) = I_{Z_t}^{(p)}(o) \), therefore the deformation space is linear. More precisely, any two families \( \{X_t\}, X_t \) with the fixed generic multiplicity sequence \( \{p_i\} \) can be combined into a two-dimension family \( \{X_{t, \tau}\} \) each member of it again has generic multiplicity sequence \( \{p_i\} \). Thus, from the flat families \( \{\Delta_{Z_t}^{\perp}\}, \{\Delta_{X_t}^{\perp}\} \) one constructs the two dimensional flat family \( \{\Delta_{X_{t, \tau}}^{\perp}\} \). Thus the limit scheme \( \lim_{\{t, \tau\} \to (0, 0)} \Delta_{X_{t, \tau}}^{\perp} \) is unique.

Further, the change of basis of \( I_{Z_t}^{(p)}(o) \) results in a linear transformation on the coefficients \( \{a_{*, *}\} \).

2. By construction \( \Delta_{Z_t}^{\perp}(X) \) is a flat limit of Cartier divisors, thus it is a Cartier divisor too.

3. As mentioned in §4: \( \Delta_{\text{def}}^{\perp} \) is completely determined by its compatibility with flat deformations. Thus it is enough to compare the current definition and the definition of §4 for the minimal possible degeneration. And this is immediate. ■

References
[Aluffi-1995] P. Aluﬃ, Singular schemes of hypersurfaces. Duke Math. J. 80 (1995), no. 2, 325–351
[Aluﬃ-2005] P. Aluﬃ Characteristic classes of singular varieties. Topics in cohomological studies of algebraic varieties, 1-32, Trends Math., Birkhäuser, Basel, 2005.
[AGL-book1] V.I. Arnol’d, V.V. Goryunov, O.V. Lyashko, V.A. Vasil’ev, *Singularity theory. I.* Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993]. Springer-Verlag, Berlin, 1998.

[AGL-book2] V.I. Arnol’d, V.A. Vasil’ev, V.V. Goryunov, O.V. Lyashko, *Singularities. II. Classification and applications.* (Russian) With the collaboration of B. Z. Shapiro. Itogi Nauki i Tekhniki, Current problems in mathematics. Fundamental directions, Vol. 39, 5256, Akad. Nauk SSSR, 1989.

[Benett-1977] B. Bennett, *Normally flat deformations.* Trans. Amer. Math. Soc. 225 (1977), 1–57

[Benoist2012] O. Benoist, *Degrés d’homogénéité de l’ensemble des intersections complètes singulières.* Ann. Inst. Fourier (Grenoble) 62 (2012), no. 3, 1189–1214.

[B.C.D.R.S.2011] E. Cattani, M.A. Cueto, A. Dickenstein, S. Di Rocco, B. Sturmfels, *Mixed Discriminants,* arXiv:1112.1012

[Bennett-1977] B. Bennett, *Normally flat deformations.* Trans. Amer. Math. Soc. 225 (1977), 1–57

[Dimca-book] A. Dimca, *Singularities and topology of hypersurfaces.* Universitext. Springer-Verlag, New York, 1992.

[Eisenbud-book] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[Esterov2011] A. Esterov, *Discriminant of system of equations,* arXiv:1110.4060

[Fulton-book] W. Fulton, *Intersection theory.* Second edition. A Series of Modern Surveys in Mathematics, 2, Springer-Verlag, Berlin, 1998.

[Gelfand-Kapranov-Zelevinsky] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, *Discriminants, resultants and multidimensional determinants.* Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. x+523 pp.

[Heitmann-Jorgensen] R.C. Heitmann, D.A. Jorgensen, *Are complete intersections complete intersections?* J. Algebra 371 (2012), 276-299.

[Heitmann-Jorgensen] R.C. Heitmann, D.A. Jorgensen, *Are complete intersections complete intersections?* J. Algebra 371 (2012), 276-299.

[Kazarian2000] M. Kazarian, *Classifying spaces of singularities and Thom polynomials.* New developments in singularity theory (Cambridge, 2000), 117–134, NATO Sci. Ser. II Math. Phys. Chem., 21, Kluwer Acad. Publ., Dordrecht, 2001

[Kazarian2003] M. Kazarian, *Thom polynomials.* Singularity theory and its applications, 85–135, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006

[Kerner-Nemethi-2012] D. Kerner, A. Nemethi, *The ‘corrected’ Durfee’s inequality for homogeneous complete intersections,* to appear in Mathematische Zeitschrift.

[Looijenga-book] E. Looijenga, *Isolated Singular Points on Complete Intersections* London Math. Soc. LNS 77, CUP, 1984

[Siersma2000] D. Siersma, *The vanishing topology of non isolated singularities.* New developments in singularity theory (Cambridge, 2000), 447–472, NATO Sci. Ser. II Math. Phys. Chem., 21, Kluwer Acad. Publ., Dordrecht, 2001

[Seade-book] J. Seade, *On the Topology of Isolated Singularities in Analytic Spaces.* Progress in Mathematics 241, Birkhäuser 2006.

[Teissier1976] B. Teissier, *The hunting of invariants in the geometry of discriminants.* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 565–678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

**STEKLOV MATHEMATICAL INSTITUTE, RAS AND THE PONCELET LABORATORY, INDEPENDENT UNIVERSITY OF MOSCOW**

**E-mail address:** kazarian@mccme.ru

**Department of Mathematics, Ben Gurion University of the Negev, Israel**

**E-mail address:** dmitry.kerner@gmail.com

**RÉNYI INSTITUTE OF MATHEMATICS, BUDAPEST,**

**E-mail address:** nemethi@renyi.hu