The doublet of Dirac fermions in the field of the non-Abelian monopole, isotopic chiral symmetry, and parity selection rules

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Abstract

The paper concerns a problem of Dirac fermion doublet in the external monopole potential obtained by embedding the Abelian SU(2) monopole solution in the non-Abelian scheme. In this particular case, the doublet-monopole Hamiltonian is invariant under some symmetry operations consisting of a (complex and one parametric) Abelian subgroup in the complex rotational group \( SO(3,\mathbb{C}) \): \([\hat{H}, \hat{F}(A)]_\pm = 0, \hat{F}(A) \in SO(3,\mathbb{C}) \). This symmetry results in a certain freedom in choosing a discrete operator \( \hat{N}_A \) (\( A \) is a complex number) entering the complete set of quantum variables. The same complex number \( A \) represents an additional parameter at the basis wave functions \( \Psi^{A}_{j m \delta \mu}(t, r, \theta, \phi) \). The generalized inversion-like operator \( \hat{N}_A \) implies its own (\( A \)-dependent) definition for scalar and pseudoscalar, and further affords certain generalized \( N_A \)-parity selection rules. All the different sets of basis functions \( \Psi^{A}_{j m \delta \mu}(x) \) determine globally the same Hilbert space. The functions \( \Psi^{A}_{j m \delta \mu}(x) \) decompose into linear combinations of \( \Psi^{A=0}_{j m \delta \mu}(x) \). However, the bases considered turn out to be nonorthogonal ones when \( A^* \neq A \); the latter correlates with the non-self-conjugacy property of the operator \( \hat{N}_A \) at \( A^* \neq A \). The meaning of possibility to violate the known quantum-mechanical regulation on self-conjugacy as regards the inversion-like operator \( \hat{N}_A \) is discussed. The question of possible physical understanding the complex expectation values for \( \hat{N}_A \) (at \( A^* \neq A \)) is examined. Also, the problem of possible physical status for the matrix \( \hat{F}(A) \) at \( A^* = A \) is considered in full detail: since the matrix belongs formally to the gauge group \( SU(2)^{\text{gauge}}_{\text{loc.}} \), but in the same time, being a symmetry operation for the Hamiltonian under consideration, this operator generates linear transformations on basis wave functions. It is emphasized that interpretation of the \( A \)-freedom as exclusively a gauge one is not justified since this will leads to a logical collision with the quantum superposition principle, and besides, there will arise the conclusion that two sorts of basis states \( \Psi^{A=0}_{j m, +1, \mu}(x) \) and \( \Psi^{A=0}_{j m, -1, \mu}(x) \) are to be physically identical. The latter could be interpreted only as a return to the Abelian scheme again.

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1. Introduction

While there not exists at present definitive succeeded experiments concerning monopoles, it is nevertheless true that there exists a veritable jungle of literature on the monopole theories. Moreover, properties of more general monopoles, associated with large gauge groups now thought to be relevant in physics. As evidenced even by a cursory examination of some popular surveys (see, for example, \[1,2\]), the whole monopole area covers and touches quite a variety of fundamental problems. The most outstanding of them are: the electric charge quantization \[3-10\], \(P\)-violation in purely electromagnetic processes \[11-16\], scattering on the Dirac string \[17-19\], spin from monopole and spin from isospin \[20-23\], bound states in fermion-monopole system and violation of the Hermiticity property \[24-38\], fermion-number breaking in the presence of a magnetic monopole and monopole catalysis of baryon decay \[39-41\]. The tremendous volume of publications on monopole topics (and there is no hint that its raise will stop) attests the interest which they enjoy among theoretical physicists, but the same token, clearly indicates the unsettled and problematical nature of those objects: the puzzle of monopole seems to be one of the still yet unsolved problems of particle physics.

In the same time, the study of monopoles has now reached a point where further progress depends on a clearer understanding of this object that had been available so far. Apparently what is needed is neither the search of decided experiments, which are unlikely to be successful, nor a new solution of some nonlinear systems of equations, but rather the analysis and careful criticism of already considered results. In reference to this, leaving aside a major part of various monopole problems, much more comprehensive in themselves, just some aspects of the \(SU(2)\)-model will be a subject of the present work. That of course seriously restricts the generality of consideration, but it should be emphasized at once that though much more involved monopole-like configurations are consistently (and somewhat routinely) invented and reported in the literature; in the same time we should recognize that certain purely Abelian or, the most contiguous with it, \(SU(2)\)-model’s aspects came to light when considering those generalized systems. In view of that, the particular \(SU(2)\)-model’s features, being considered here, might be of reasonable interest for a more large number of non-Abelian models.

Once the non-Abelian monopole had been brought by ’t Hooft and Polyakov \[42-44\] into scientific usage, its main properties had been noted and examined. The background of thinking of the whole (non-Abelian) monopole problem in that time can be easily traced: it was obviously tied up with the most outstanding points of its Abelian counterpart.

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1 A more short version of the paper may be seen in [101]

2 Very physicists have contributed to investigation of the monopole-based theories. The wide scope of the field and the prodigious number of investigators associated with various of its developments make it all but hopeless to list even the principal contributors. The present study does not pretend to be a survey in this matter, so I give but a few of the most important references which may be useful to the readers who wish some supplementary material or are interested in more technical developments beyond the scope of the present treatment.

3 Some more discussion on possible extension to any different gauge theories are given in the conclusive Sec. 11; and that possibility of generalization lends interest to the present study.
Dirac monopole; namely singularity properties, quantization conditions, and some other contiguous to them. This reflected the impress of old attitude towards the monopole, which had been imbibed by physicists from early Dirac’s investigation on this matter and consists in drawing special attention to the known singularity aspects. Evidently, the most significant and noticeable achievement of that new theory was the elimination of the singularity aspect from the theory. It should be noted that, from the very beginning, the main emphasis had been drawn to just a spatially radial non-singular structure of that new monopole-like system. Thus, the required absence of singularity had been achieved and therefore the clouds over this part of the subject had been dispersed.

Much less attention has been given to a number of other sides of that non-Abelian construct. For instance: To the ’t Hooft-Polyakov construction has been assigned a monopole-like status; what is a distinctive physical feature of it that provides the principal grounds for such an assignment? Or, which part of this system represents a trace of Abelian monopole and therefore bears this same old quality, and which one is referred to its own and purely non-Abelian nature? So, the question in issue is the role and status of the Abelian monopole in the non-Abelian theory. Although a number of general and rigorous relationships connecting these aspects of these two models have been established to date a little work has been done so far in linking them at the level of a specifically contrasting examination where all detailed calculations have been carried out. This article will endeavor to supply this work.

In general, there are several ways of approaching the monopole problems. As known, together with geometrically topological way of exploration into them, another approach to studying such configurations is possible, namely, that concerns any physical manifestations of monopoles when they are considered as external potentials. Moreover, from the physical standpoint, this latter method can thought of as a more visualizable one in comparison with less obvious and more direct topological language. So, the basic frame of our further investigation is the study of a particle multiplet in the external monopole potentials. Much work in studying quantum mechanical particles in the monopole potentials, in both the Abelian and non-Abelian cases, has been done in the literature; see, respectively, in [45-48] and [49-54]. For definiteness, we restrict ourselves to the simplest doublet case; taking special attention to any manifestations of just the Abelian monopole on the non-Abelian background. Many properties of this system turn out to be of interest in themselves, producing in their totality an example of the theory of what may be called ‘Abelian monopole in non-Abelian embodiment’. Generally speaking, this theory is the most important aspect of the present study.

Now, for convenience of the readers, some remarks about the approach and technique used in the work are to be given. The primary technical ‘novelty’ is that, in the paper, the tetrad (generally relativistic) method [55-63] of Tetrode-Weyl-Fock-Ivanenko (TWFI)

4Sometimes, it is considered to be solely a subsidiary construct, being appropriate to mimic the ’t Hooft-Polyakov potential: at least, it can exactly simulate the latter far away from the region \( r = 0 \) (at spatial infinity).

5Though evidently, ultimate answers have not been found by this work as well, it might hoped that a certain exploration into and clearing up this matter have been achieved.
for describing a spinor particle will be exploited. So, the matter equation for an isotopic doublet of spinor particles in the field of the non-Abelian monopole is taken in the form

\[
[i \gamma^\alpha(x) (\partial_\alpha + \Gamma_\alpha(x) - i e t^\alpha W^{(a)}) - (m + \kappa \Phi^{(a)} t^a)] \Psi(x) = 0. \tag{1.1}
\]

where \(\gamma^\alpha(x)\) are the generalized Dirac matrices, \(\Gamma_\alpha(x)\) stands for the bispinor connection; \(e\) and \(\kappa\) are certain constants. The choice of the formalism to deal with the monopole-doublet problem has turned out to be of great fruitfulness for examining this system.

Taking of just this method is not an accidental step. It is matter that, as known (but seemingly not very vastly), the use of a special spherical tetrad in the theory of a spin 1/2 particle had led Schrödinger and Pauli [64, 65] to a basis of remarkable features. In particular, the following explicit expression for (spin 1/2 particle’s) momentum operator components had been calculated

\[
J_1 = l_1 + \frac{i\sigma^{12} \cos \phi}{\sin \theta}, \quad J_2 = l_2 + \frac{i\sigma^{12} \sin \phi}{\sin \theta}, \quad J_3 = l_3 \tag{1.2}
\]

just that kind of structure for \(J_i\) typifies this frame in bispinor space. This Schrödinger’s basis had been used with great efficiency by Pauli in his investigation [65] on the problem of allowed spherically symmetric wave functions in quantum mechanics. For our purposes, just several simple rules extracted from the much more comprehensive Pauli’s analysis will be quite sufficient (those are almost mnemonic working regulations).

They can be explained on the base of \(S = 1/2\) particle case. To this end, using any representation of \(\gamma\) matrices where \(\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)\) (throughout the work, the Weyl’s spinor frame is used) and taking into account the explicit form for \(\vec{J}^2, J_3\) according to (1.2), it is readily verified that the most general bispinor functions with fixed quantum numbers \(j, m\) are to be

\[
\Phi_{jm}(t, r, \theta, \phi) = \begin{pmatrix}
  f_1(t, r) D^j_{-m,-1/2}(\phi, \theta, 0) \\
  f_2(t, r) D^j_{-m,+1/2}(\phi, \theta, 0) \\
  f_3(t, r) D^j_{-m,-1/2}(\phi, \theta, 0) \\
  f_4(t, r) D^j_{-m,+1/2}(\phi, \theta, 0)
\end{pmatrix} \tag{1.3}
\]

where \(D^j_{mm'}\) designates the Wigner’s \(D\)-functions (the notation and subsequently required formulas, according to [66], are adopted). One should take notice of the low right indices \(-1/2\) and \(+1/2\) of \(D\)-functions in (1.3), which correlate with the explicit diagonal structure of the matrix \(\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)\). The Pauli criterion allows only half integer values for \(j\).

So, one may remember some very primary facts of \(D\)-functions theory and then produce, almost automatically, proper wave functions. It seems rather likely, that there may exist a generalized analog of such a representation for \(J_i\)-operators, that might be successfully used whenever in a linear problem there exists a spherical symmetry, irrespective of the concrete embodiment of such a symmetry. In particular, the case of electron in the external Abelian monopole field, together with the problem of selecting the allowed wave functions as well as the Dirac charge quantization condition, completely come under
that Shrödinger-Pauli method. In particular, components of the generalized conserved momentum can be expressed as follows (for more detail, see [67])

\[ j_{g1} = l_1 + \frac{(i\sigma^{12} - eg)\cos\phi}{\sin\theta}, \quad j_{g2} = l_2 + \frac{(i\sigma^{12} - eg)\sin\phi}{\sin\theta}, \quad j_{g3} = l_3 \]  \hspace{1cm} (1.4)

where \( e \) and \( g \) are an electrical and magnetic charge, respectively. In accordance with the above regulations, the corresponding wave functions are to be built up as (also see (4.3a,b))

\[ \Phi^{eg}_{jm}(t,r,\theta,\phi) = \begin{pmatrix}
  f_1(t,r) D^j_{m,eg-1/2}(\phi,\theta,0) \\
  f_2(t,r) D^j_{m,eg+1/2}(\phi,\theta,0) \\
  f_3(t,r) D^j_{m,eg-1/2}(\phi,\theta,0) \\
  f_4(t,r) D^j_{m,eg+1/2}(\phi,\theta,0)
\end{pmatrix}. \] \hspace{1cm} (1.5)

The Pauli criterion produces two results: first, \( |eg| = 0,1/2,1,3/2,\ldots \) (what is called the Dirac charge quantization condition; second, the quantum number \( j \) in (1.5) may take the values \( |eg| - 1/2, |eg| + 1/2, |eg| + 3/2, \ldots \) that selects the proper spinor particle-monopole functions.

So, it seems rather a natural step: to try exploiting some generalized Schrödinger’s basis at analyzing the problem of \( SU(2) \) multiplet in the non-Abelian monopole field, if by no reason than curiosity or search of some points of unification.

There exists else one line justified the interest to just the aforementioned approach: the Shrödinger’s tetrad basis and Wigner’s \( D \)-functions are deeply connected with what is called the formalism of spin-weight harmonics [68-70] developed in the frame of the Newman-Penrose method of light (or isotropic) tetrad. Some relationships between spin-weight and spinor monopole harmonics have been examined in the literature [71-73], the present work follows the notation used in [67].

There is an additional reason for special attention just to the Scrödinger’s basis on the background of non-Abelian monopole matter. As will be seen subsequently, that basis can be associated with the unitary isotopic gauge in the non-Abelian monopole problem\(^6\); in Sec.2 the latter fact will be discussed in full detail.

Thus, the present work is, in its working mathematical language, somewhat reminiscent in contrast to prodigious modern investigations based on the abstruse geometrical theory underlying the monopole matter; it exploits rather conventional (if not ancient) mathematical and physical methods and intends to trace some unifying points among them (tetrad approach, Wigner’s functions, selection of allowable wave function, non-Abelian theories, and monopole area). After all these opening and general statements, some more concrete remarks referring to our further work and designated to delineate its content are to be given.

Sec.2 determines explicitly all the technical facts necessary to follow the subsequent content of the work in full detail; in that sense, it plays a subsidiary role. Here, in the first place, the question of intrinsic structure of the t’Hooft-Polyakov potentials \( (\Phi^a(x),W^a_\mu) \)

\(^6\)There is something rather enigmatic in such a relation between those, apparently not-touching each other, matters. In the same time, those always will be attractive points for theoreticians
is reexamined (to be more exact, the dyon ansatz of Julia-Zee [74] is taken initially). The main guideline consists in the following: It is well-known that the usual Abelian monopole potential generates a certain non-Abelian potential being a solution of the Yang-Mills \((Y - M)\) equations. First, such a specific non-Abelian solution was found out in [75]. A procedure itself of that embedding the Abelian 4-vector \(A_{\mu}(x)\) in the non-Abelian scheme: \(A_{\mu}(x) \rightarrow A_{\mu}^{(a)}(x) \equiv (0, 0, A_{3}^{(3)} = A_{\mu}(x))\) ensures automatically that \(A_{\mu}^{(a)}(x)\) will satisfy the free \(Y - M\) equations. Thus, it may be readily verified that the vector \(A_{\mu}(x) = (0, 0, 0, A_{\phi} = g \cos \theta)\) obeys the Maxwell general covariant equations in every curved space-time with the spherical symmetry:

\[
dS^2 = [e^{2\nu}(dt)^2 - e^{2\mu}(dr)^2 - r^2((d\theta)^2 + \sin^2 \theta(d\phi)^2)] ; A_{\phi} = g \cos \theta \rightarrow F_{\theta\phi} = -g \sin \theta;
\]

here we will face a single equation that coincides with the Abelian one. In turn, the non-Abelian tensor \(F_{\mu\nu}^{(a)}(x)\) defined by \(F_{\mu\nu}^{(a)}(x) = \nabla_{\mu} A_{\nu}^{(a)} - \nabla_{\nu} A_{\mu}^{(a)} + e \epsilon_{abc} A_{\mu}^{(b)} A_{\nu}^{(c)}\) and associated with the \(A_{\mu}^{(a)}\) above has a very simple isotopic structure: \(F_{\theta\phi}^{(3)} = -g \sin \theta\) and all other \(F_{\mu\nu}^{(a)}\) are equal to zero. So, this substitution \(F_{\mu\nu}^{(a)} = (0, 0, F_{\theta\phi}^{(3)} = -g \sin \theta)\) leads the \(Y - M\) equations to the single equation of the Abelian case. Therefore, strictly speaking, we cannot state that \(A_{\mu}^{(a)}(x)\) obeys a certain set of really nonlinear equations (it satisfies linear rather than nonlinear ones). Thus, this potential may be interpreted as a trivially non-Abelian solution of \(Y - M\) equations. Supposing that such a sub-potential is presented in the well-known monopole solutions:

\[
\Phi^{(a)}(x) = x^a \Phi(r), \quad W_0^{(a)}(x) = x^a F(r), \quad W_i^{(a)}(x) = \epsilon_{iab} x^b K(r) \quad (1.6)
\]

we can try to establish explicitly that constituent structure. The use of the spherical coordinates and special gauge transformation enables us to separate the trivial and nontrivial (in other terms, Abelian and genuinely non-Abelian) parts of the potentials (1.6) into different isotopic components (see the formula (2.7b) ). In the process of this rearrangement of the monopole’s constituents, heuristically useful concepts of three gauges: Cartesian (associated with the representation (1.6)), Dirac and Schwinger’s (both latter are unitary ones) in isotopic space are defined. In order to avoid possible confusion with all the different bases we will label the functions and operators by signs of used gauges.

The abbreviations \(S., D., C.\) will be associated, respectively, with the Schwinger, Dirac, and Cartesian gauges in the isotopic space, whereas the abbreviation sph. and Cart. are referred to the spherical and Cartesian tetrads, respectively.

Also, in Sec.2, we briefly review several, the most important for our work, facts concerning the generally relativistic Dirac’s equation. Besides, for convenience of the readers, some information about the aforementioned Pauli’s criterion is given.

Sec.3 begins analyzing the doublet-monopole problem. Starting from Schwinger unitary gauge ( \(\Phi_{(\alpha)\beta}^{S}, W_{(\alpha)\beta}^{S}\) ) and the spherical tetrad-based matter equation (1.1), the problem of obtaining and partly analyzing the relevant radial equations is studied here.
all over again. At the correlated choice of frames in both Lorentzian and isotopic space, an explicit form of the total momentum operator

\[ J^S_1 = l_1 + \frac{(i\sigma^{12} + t^3) \cos \phi}{\sin \theta}, \quad J^S_2 = l_2 + \frac{(i\sigma^{12} + t^3) \sin \phi}{\sin \theta}, \quad J^S_3 = l_3 \]  

characterizes this particular composite frame as of Schrödinger’s type; so, the above general technique is quite applicable. The \((\theta, \phi)\)-dependence in the relevant wave function is described by \(D\)-functions of three kinds: \(D^j_{m,m'}, m' = 0, -1, +1\) (also see in (3.3)):

\[ \Psi_{\epsilon jm}(x) = e^{-i\epsilon t} \left[ T_{+1/2} \otimes F(r, \theta, \phi) + T_{-1/2} \otimes G(r, \theta, \phi) \right], \]

\[ F = \begin{pmatrix} f_1(r)D_{-1} & f_2(r)D_0 \\ f_3(r)D_{-1} & f_4(r)D_0 \end{pmatrix}, \quad G = \begin{pmatrix} g_1(r)D_0 & g_2(r)D_{-1} \\ g_3(r)D_0 & g_4(r)D_{-1} \end{pmatrix}, \quad T_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

\(D_\sigma \equiv D^j_{-m,\sigma}(\phi, \theta, 0)\). The Pauli criterion (see sec.2) allows here all positive integer values for \(j: j = 0, 1, 2, 3, \ldots\)

The separation of variables in the equation is accomplished by the conventional \(D\)-function recursive relation techniques. Moreover, it turns out that only two relationships from the enormous \(D\)-function apparatus are really needed in doing this separation [66]:

\[ \frac{\partial}{\partial \beta} D^j_{mm'}(\alpha, \beta, \gamma) = + \frac{1}{2} \sqrt{(j + m')(j - m' + 1)} e^{-i\gamma} D^j_{m,m' - 1} - \frac{1}{2} \sqrt{(j - m')(j + m' + 1)} e^{i\gamma} D^j_{m,m' + 1} ; \]

\[ \frac{m - m' \cos \theta}{\sin \theta} D^j_{mm'}(\alpha, \beta, \gamma) = - \frac{1}{2} \sqrt{(j + m')(j - m' + 1)} e^{-i\gamma} D^j_{m,m' - 1} - \frac{1}{2} \sqrt{(j - m')(j + m' + 1)} e^{i\gamma} D^j_{m,m' + 1}. \]  

As known, an important case in theoretical investigation is the electron-monopole system at the minimal value of the quantum number \(j\); so, the case \(j = 0\) should be considered especially carefully, and we do this. In the chosen frame, it is the independence on \(\theta, \phi\)-variables that sets the wave functions of minimal \(j\) apart from all other particle multiplet states (certainly, functions \(f_1(r), f_2(r), g_2(r), g_4(r)\) in the substitution (1.8) must be equated to zero at once). Correspondingly, the relevant angular term in the wave equation will be effectively eliminated.

Another essential feature of the given frame is the appearance of a very simple expression for the term which mixes up together two distinct components of the isotopic doublet (see (3.1)). Moreover, it is evident at once that both these features will be retained, with no substantial variations, when generalizing this particular problem to more complex ones with other fixed Lorentzian or isotopic spin.
The system of radial equations found by separation of variables (4 and 8 equations in the cases of $j = 0$ and $j > 0$, respectively) are rather complicated. They are simplified by searching a suitable operator that could be diagonalized simultaneously with $\vec{J}^2, J_3$. The usual space reflection ($P$-inversion) operator for a bispinor doublet field has to be followed by a certain discrete transformation in the isotopic space, so that a required quantity could be constructed. The solution of this problem which has been established to date (see, for example, in [1, 11-16, 76-85]) is not general as much as possible. For this reason, the question of reflection symmetry in the doublet-monopole system is reexamined here all over again. As a result we find out\footnote{And this is a crucial moment in subsequent construction of the present work.} that there are two different possibilities depending on what type of external monopole potential is taken. So, in case of the non-trivial potential, the composite reflection operator with required properties is (apart from an arbitrary numerical factor)

$$\hat{N}^S = \hat{\pi} \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{\pi} = +\sigma_1$$

(1.10)

here, the quantities $\hat{\pi}$ and $\hat{P}_{\text{bisp.}}$ represent fixed matrices acting in the isotopic and bispinor space, respectively, and changing simultaneously with any variations of relevant bases (see (3.8a)). A totally different situation occurs in case of the simplest monopole potential. Now, a possible additional operator, suitable for separating the variables, depends on an arbitrary complex numerical parameter $A$ ($\Delta = e^{iA}, e^{iA} \neq 0, \infty$):

$$\hat{N}^S_A = \hat{\pi}_A \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{\pi}_A = e^{iA}\sigma_3\sigma_1.$$  

1.11a

The same quantity $A$ appears also in expressions for the corresponding wave functions $\Psi^A_{\epsilon jm\delta}(t, r, \theta, \phi)$ (the eigenvalues $N_A = \delta(-1)^{j+1}; \delta = \pm 1$):

$$\Psi^A_{\epsilon jm\delta}(x) = \left[ T_{+1/2} \otimes F(x) + \delta e^{iA} T_{-1/2} \otimes G(x) \right]$$

(1.11b)

the additional limitations (3.10a) are imposed on the radial functions in $F$ and $G$. Further, throughout all the sections 4-11, we look into the fermion doublet just in this simplest monopole field, and all results and discussion concern only this particular system unless the inverse is indicated.

Sec.4 , in the first place, finished the work on searching a remaining operator from a supposedly complete set: $\{ \hat{H}, \vec{J}^2, J_3, \hat{N}_A, \hat{K} =? \}$. That $\hat{K}$ is determined as a natural extension of the well-known (Abelian) Dirac operator to the non-Abelian case. Correspondingly, the set of radial equations is eventually reduced to a set of two ones; the latter is well known and coincides with that relating to the Abelian electron-monopole system; which has been studied by many authors. Then, on simple comparing the non-Abelian doublet functions with the Abelian ones, we arrive at an explicit factorization of the doublet functions by Abelian ones and isotopic basis vectors (see (4.4)). The relevant decompositions have been found for the composite states with all values of $J$, including the minimal one $J_{\text{min.}} = 0$ too. As known, the case of minimal $J$ in the Abelian theory
supplies some unexpected and rather singular features: in particular, it gives a candidate for a possible bound state in the electron-monopole system, it significantly touches the Hamiltonian self-conjugacy property and some others. Thus, as evidenced by the factorization, all those purely Abelian peculiarities, concerning the $J_{min}$-state, likewise turn out to be represented, in a practically unchanged form, in the non Abelian theory (it should be remembered that here the case of special non-Abelian monopole field is meant).

Else one fact associated with the above decomposition should be noted. It is matter that the $A$-ambiguity in determining the discrete operator $\hat{N}_A$ ranges from zero to infinity: $0 < |e^A| < \infty$. Therefore, the two distinct isotopic doublet components (see (1.11b)), proportional respectively to $T_{-1/2}$ and $T_{+1/2}$, cannot be eigenfunctions of the $\hat{N}_A$ whatever the values of $A$-parameter may be. In other words, the above $\hat{N}_A$ is to be considered as a specifically non-Abelian operator, and the parameter $e^{iA}$ itself may be regarded as a quantity measuring violation of Abelicity in the composite non-Abelian wave function. Of course, these two Abelian-like doublet states can formally be obtained from (1.11b) too: it suffices to put $e^{iA} = 0$ or $\infty$, but those singular cases are not covered by the above-mentioned complete set of operators. In that sense, the two bound values $A = 0$ and $\infty$ represent singular transition points between the Abelian and non-Abelian theories.

Sec. 5 concerns distinctions between Abelian and non-Abelian monopoles. We consider the question: what in the stated above (Sections 2-4) is dictated solely by presence of the external field and what is determined, in turn, only by isotopic multiplet’s structure. To this end, we compare the fermion doublet multiplet, being subjected to the monopole effect, with a free one. We draw attention to the fact that these two systems have their spherical symmetry operators $\vec{J}^2, J_3, \hat{N}$ identically coincided. Correspondingly, the relevant wave functions do not vary at all in their dependence on angular variables $\theta, \phi$; instead, a single difference appears in one parametric function entering the systems of radial equations. These non-Abelian wave functions’ property sharply contrasts with the Abelian one. Indeed, as well known, particle’s wave functions and all spherical symmetry operators undergo substantial changes (see (1.4) and (1.5) as the external monopole field is in effect. To clarify and spell out all the significance of such a ‘minor’ alteration in (1.5) as the simple displacement in a single index, we look at just one mathematical characteristic of the $D$-functions involved in the particle wave functions: namely, their boundary properties at the points $\theta = 0$ and $\theta = \pi$ (see Tables 1-3 in Sec.4). On comparing those characteristics for $D_{m,\pm 1/2}(\phi, \theta, 0)$ and $D_{m,\pm 1/2}(\phi, \theta, 0)$ we can conclude that these sets of $D$-functions provide us with the bases in different functional spaces $\{F^{eg=0}(\theta, \phi)\}$ and $\{F^{eg\neq 0}(\theta, \phi)\}$. Every of those functional spaces is characterized by its own behavior at limiting points, which is irreconcilable with that of any other space.

So, from the very beginning, in the Abelian monopole situation, we face a fact being crucial one by its further implications: the space of quantum-mechanical states of a particle in the monopole field is quite a contrasting one to a free particle’s space.

This circumstance implies a lot of hampering implications. In particular, we discuss some relations of them to quantum-mechanical superposition principle. Also, else one awkward question of that kind is: what is the meaning of the relevant scattering theory, if even at infinity itself, some manifestation of the magnetic charge presence does not vanish
By contrast, in the non-Abelian theory, there not exist any problems of such a kind. Even more, we may state that one of the substantial features characterizing the non-Abelian monopole is that such a potential, does not destroy the isotopic angular structure of the particle multiplet. From this point of view, this potential represents a certain analog of a spherically symmetrical Abelian potential \( A_\mu = (A_0(r), 0, 0, 0) \) rather than of the Abelian monopole one. In this connection, one additional remark might be useful: one should give attention to the fact that the designation monopole in the non-Abelian terminology, anticipates tacitly interpretation of \( W^{a}_\mu(x) \) as ones carrying, in a new situation, the essence of the well-known Abelian monopole, although really, as evidenced by the above arguments and some other, the real degree of their similarity may be probably less than one might expect. Also, the following point should be stressed. Though as was mentioned above, certain close relationships between the non-Abelian doublet wave functions and Abelian fermion-monopole functions occur (see the formulas (4.4a,b)), the non-Abelian situation, in reality, is intrinsically non-monopole-like (= non-singular one). The following aspect is meant: in the non-Abelian case, the totality of possible transformations (upon the relevant wave functions) which bear the gauge status are materially different from ones that there are in effect in the purely Abelian theory. As a consequence of this, the non-Abelian fermion doublet wave functions (1.8) can be readily transformed, by carrying out together the gauge transformations in Lorentzian and isotopic spaces \((S. \rightarrow C. \text{ and sph.} \rightarrow \text{Cart.})\), into the form (see the formulas (A.9) and (A.10)) where they will be single-valued functions of spatial points. In the Abelian monopole situation, the representation for particle-monopole functions can by no means be translated to any single-valued one.

So, in a sense, the whole multiplicity of Abelian monopole manifestations seems to be much more problematical than non-Abelian monopole’s. Furthermore, as it appears to be likely, examination of the non-Abelian case does not lead up to solving some purely Abelian problems. These two mathematical and physical theories should be only associated heuristically.

As else one confirmation to this general view, in Sec.5, we will compare two different situations relating to discrete symmetry problems tied up respectively with the Abelian and non-Abelian models (see also in [1,11-16,76-85]). We explain carefully how the Abelian \( P \)-symmetry problem (to be exact, its violation by a magnetic charge) is embedded in the non-Abelian model and the way Abelian \( P \)-violation results in the discrete composite symmetry in the non-Abelian theory. The main ideas are as follows: In virtue of the well-known Abelian monopole \( P \)-violation, the usual bispinor particle \( P \)-inversion operator \( \hat{P}_{bisp.} \otimes \hat{P} \) does not commute with the Hamiltonian \( \hat{H}^{eq} \). The way of how to obtain a certain formal covariance of the monopole-containing system with respect to \( P \)-symmetry there has been a subject of special interest in the literature. All the suggestions represent, in the essence, a single one: the magnetic charge characteristic is to be considered as a pseudo scalar quantity. For the subject under consideration, this assumption implies that one

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\(^{10}\)One should take into account that this, as it is, applies only to the Schwinger basis; the use of the Dirac gauge or any other, except Wu-Yang’s, implies quite definite modifications in representation of...
ought to accompany the ordinary $P$-transformation with a formal operator $\hat{\pi}$ changing the parameter $g$ into $-g$. Correspondingly, the composite Abelian discrete operator
\[ \hat{M} = \hat{\pi}_{\text{Abel}} \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P} \]
will commute with the relevant Hamiltonian indeed. Besides, this $\hat{M}$ can be diagonalized on the functions $\Psi^{eg}$: $\hat{M} \Psi^{eg}_{ejm\delta} = \delta \ (-1)^{j+1} \Psi^{eg}_{ejm\delta}$. However, as evidenced in Sec.5, this operator $\hat{M}$ does not result in a basic structural condition
\[ \Phi(t,-\vec{x}) = (4 \times 4 - \text{matrix}) \Phi(t,\vec{x}) \]  
which would guarantee indeed the existence of certain selection rules with respect to a discrete operator. In place of (1.12a), there exists just the following one
\[ \Phi^{+eg}_{ejm\delta}(t,-\vec{x}) = \delta(-1)^{j+1} \hat{P}_{\text{bisp.}}. \Phi^{-eg}_{ejm\delta}(t,\vec{x}) \]  
take notice of change in the sign at $eg$ parameter; this minor alteration is completely detrimental to the possibility of producing any selection rules: those do not exist whatever. Therefore, no discrete symmetry-based selection rules in presence of Abelian monopole are possible; those will be really achieved only if any relation with the general structure ((1.12a), apart from modification due to the type of particle, which would influence a matrix involved) exists.

However, a relation of required structure there occurs in the non-Abelian model:
\[ \Psi_{ejm\delta}(t,-\vec{x}) = \delta(-1)^{j+1}(\sigma^2 \otimes \hat{P}_{\text{bisp.}}) \Psi_{ejm\delta}(t,\vec{x}) \]  
Correspondingly, the relevant selection rules with respect to that composite $N$-parity can be established, and we did it.

Sec.6 concerns some technical details related to explicit expressions of the doublet wave functions and discrete operator $\hat{N}_A$ in the two other gauges: Dirac’s unitary and Cartesian ones. The following expression for the above non-Abelian $\hat{\pi}_A$-operation, now referring to the Cartesian gauge, has been calculated: $\hat{\pi}_{A}^C = (-i) \exp(iA\vec{\sigma}\vec{n}_{\theta,\phi})$, where $\vec{n}_{\theta,\phi}$ stands for the ordinary radial unit vector. If $A = 0$ then the form of the operator $\hat{N}_A^{C=0}$ (it does not involve any isotopic transformation) might be a source of some speculation about an extremely significant role of the Abelian $P$-symmetry in the non-Abelian model. Some subtle considerations related to this matter are discussed. In particular, it should be remembered that a genuinely Abelian fermion $P$-symmetry implies both a definite explicit

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11 It should be emphasized that some unexpected peculiarities with that procedure, in reality, occur as we turn to the states of minimal values of $j$: for more detail see in [67]

12 This observation can be conceptualized in more formal mathematical terms: this inversion-like operator $\hat{M}$ turns out to be a non-self-adjoint one; therefore it might not follow all the familiar patterns of behavior for self-adjoint quantities.

13 Among other thing, this material is designated to ease understanding in full for readers preferring the use of these gauges.
expression for $P$-operation and definite properties of the corresponding wave functions. In this connection, the relevant decomposition of fermion doublet wave functions in terms of Abelian fermion-like functions and unit isotopic vectors is given. From that it is evident that the usual Abelian fermion wave functions and non-Abelian doublet ones belong to substantially different types (see (6.9) at $A = 0$), so that the $P$-inversion operator plays only a subsidiary role in forming the composite functions.

Sec. 7 turns to the question of how the above complex parameter $A$ can manifests itself physically in matrix elements. On that line, as a natural illustration, the problem of parity selection rules is looked into again, but now depending on the $A$-background. The notions of a composite $N_A$-scalar and $N_A$-pseudoscalar related to some quantities with non-trivial isotopic structure are given; then the corresponding selection rules are found. For every fixed $A$, these rules imply their own special limitations on composite scalars and pseudoscalars, which are individualized by this $A$; correspondingly, selection rules arising in sequel for matrix elements (if a quantity belongs to the scalars or pseudoscalars) differ basically from each other.

Sec. 8 is interested in the question: Where does the above $A$-ambiguity come from? As shown, the origin of such a freedom lies in the existence of an additional (one parametric) operation $U(A)$ that leaves the doublet-monopole Hamiltonian invariant. Just this operation $U(A)$ changes $\hat{N}_{A=0}$ into $\hat{N}_A$. Different values for $A$ lead to the same whole functional space; each fixed $A$ governs only the basis states $\Psi^A_{\epsilon jm\delta}(x)$ of it, and the symmetry operation acts transitively on those states: $\Psi^A'_{\epsilon jm\delta}(x) = U(A'-A)\Psi^A_{\epsilon jm\delta}(x)$. An analogy between that isotopic symmetry and a more familiar example of Abelian chiral ($\gamma^5$) symmetry in massless Dirac field theory \cite{95,96} is drawn\footnote{So, the author suggests the term ‘isotopic chiral symmetry’. Besides, to be terminologically exact, one should split the notion of $\gamma^5$ complex chiral symmetry into two ones; properly $\gamma^5$-symmetry (when $A$ is real number) and conformal symmetry (when $A$ is purely imaginary number); but for simplicity, we will use the term ‘complex chiral symmetry’}. The role of the Abelian $\gamma^5$-matrix is taken by the isotopic $\sigma_3$-matrix: its form in the $S$-gauge is $U^S(A) = \exp(A/2) \exp(iA/2\sigma_3)$. Some additional technical details touching this operation are given; in particular, we find expressions for $U(A)$ in the Cartesian gauge. In Cartesian frame, this symmetry transformation takes the form

$$U_{C.}(A) = e^{+iA/2} \exp\left[-i \frac{A}{2} \vec{\sigma} \vec{n}_{\theta,\phi}\right]$$

(1.13)

where the second factor represents a 2-spinor local transformation from the 3-dimensional complex rotation group $SO(3,C)$. The explicit coordinate dependence appearing in Cartesian gauge results from the non-commutation $\sigma_3$ with a gauge transformation involved into transition from Shwinger’s to Cartesian isotopic basis. In the analogous Abelian situation, the form of the chiral transformation remains the same because $\gamma^5$ and the relevant gauge matrix (that belongs to the bispinor local representation of the group $SL(2,C)$) are commutative with each other.

In Sec. 9, we look into some qualitative peculiarities of the $A$-freedom placing special notice to the division of $A$-s values into the real and complex ones. All those values (complex as well as real) are equally permissible: they only govern bases of the same Hilbert
space of quantum states, which can be related to each other by the use of the ordinary superposition principle. However, a material distinction between real and complex $A$-s will appear, if one turns to the orthogonality properties of those basis states $\Psi_{j|m=0}^A(x)$. As will be seen, at $A^* \neq A$, the states $\Psi_{j|m=-1}^A(x)$ and $\Psi_{j|m,+1}^A(x)$ are not orthogonal to each other. Such specific (non-orthogonal) bases, though not being of very common use and having a number of peculiar features, are allowed to be exploited in conventional quantum theory.

Else one fact associated with the above (real-complex) division of $A$-s is that the discrete operator $\hat{N}_A$ represents a non-self-adjoint quantity as $A^* \neq A$. In this point, there are two possibilities to choose from: whether we restrict ourselves to the real $A$-values (correspondingly, no problems with self-conjugacy there arise) or we exploit the complex $A$-values as well as the real ones, and thereby, the non-orthogonal bases and non-self-adjoint character of the discrete operator, are allowed in the theory. We have chosen to accept and look into the second possibility.

Further, we consider narrowly such a specific nature of the $\hat{N}_A$ since, as the complex $A$-values are allowed, we will violate the well-known quantum-mechanical regulation about the self-adjointness of measurable physical quantities. The main guideline ideas in clearing up the problem faced us here is as follows. One should notice the fact that the single relation $(\hat{N}_A)^2 = I$ is abundantly sufficient one to produce real proper values: $+1$ and $-1$. Furthermore, as it was stressed above, just real values are not material here whatever; instead, the only required consequence of this symmetry is the mere distinction between two different quantum possibilities. In the light of this, the automatic incorporation of all discrete operators into class of self-adjoint ones does not seem inevitable. But accepting this, there is a problem to solve: what is the meaning of complex expectation values of such non-self-adjoint operators. We carefully explain how one may interpret all such complex values as being physically measurable ones.

Finally, we devote Sec.10 to clearing up else one, and rather important from the physical viewpoint, peculiarity of the doublet-monopole system. It is matter that if the parameter $A$ is real one, then the matrix translating $\Psi_{j|m=0}^A(x)$ into $\Psi_{j|m=0}^A(x)$ coincides (apart from a phase factor $e^{iA/2}$) with a matrix lying in the group $SU(2)$. However, the group $SU(2)$ has the status of gauge one for the system under consideration. So, the point of view might be brought to light: one could claim that the two functions $\Psi^A(x)$ and $\Psi^{A'}(x)$, referring respectively to the different values $A$ and $A'$, represent in reality only the transforms of each other in the sense of $SU(2)$ gauge theory. And further, as a direct consequence, one could insist on the impossibility in principle to observe indeed any physical distinctions between those wave functions. If the above transformation gets estimated so, then ultimately one will conclude that the above $N_A$-parity selection rules (explicitly depended on $A$ which is real in that case) are the mathematical fiction only, since the transformation $U(A)$ is not physically observable.

For answering this question, we carefully follow some interplay between the quantum-mechanical superposition principle and the concepts of gauge and non-gauge symmetries. As will be seen, the general outlook prescribing to interpret the transformation $U(A)$ as exclusively a gauge one contradicts with some basic regulations stemming from the super-
position principle. Even more, as shown, starting from the exclusively gauge understanding of that transformation, one can arrive at the requirement of physical identification of the two states $\Psi^A_{\epsilon jm\delta=-1}(x)$ and $\Psi^A_{\epsilon jm\delta=+1}(x)$. However, in a sense, this is equivalent to the effective returning into the Abelian scheme, that hardly can be desirable effect. Nevertheless, the matrix $U(A)$ belongs to $SU(2)_{gauge}^{loc.}$ (apart from $U(1)$ factor). In order not to reach a deadlock, in author’s opinion, there exists just one and very simple way out of this situation, which consists in the following: The complete symmetry group of the system under consideration is of the form $\hat{F}(A) \otimes SL(2.C)_{gauge}^{loc.} \otimes SU(2)_{gauge}^{loc.}$. This group, in particular, contains the gauge and non-gauge symmetry operations which both have the same mathematical form but different physical status.

Finally, in Sec.11, we briefly discuss extension of the present analysis to other situations, with different values of isotopic and Lorentzian spin, and gauge groups. It is argued that some facts discerned in the present work for $SU(2)$-model might bear upon similar aspects of other gauge group-based theories.

In Supplement A, we consider some additional relationships between explicit forms of the fermion-monopole functions in the bases of spherical and Cartesian tetrads.

2 Dirac and Schwinger gauges in isotopic space

This section deals with some representation of the non-Abelian monopole potential, which will be the most convenient one to formulate and analyze the problem of isotopic multiplet in this field. Let us begin describing in detail this matter. The well-known form of the monopole solution introduced by t’Hooft and Polyakov ([42-44]; see also Julia-Zee [74]) may be taken as a starting point. The field $W^{(a)}_\alpha$ represents a covariant vector with the usual transformation law $W^{(a)}_\beta = (\partial x^i/\partial x^\beta)W^{(a)}_i$ and our first step is the change of variables in 3-space. Thus, the given potentials $(\Phi^{(a)}(x), W^{(a)}_\alpha(x))$ convert into $(\Phi^{(a)}(x), W^{(a)}_t(x), W^{(a)}_r(x), W^{(a)}_\theta(x), W^{(a)}_\phi(x))$. Our second step is a special gauge transformation in the isotopic space. The required gauge matrix can be determined (only partly) by the condition $(O_{ab}\Phi^b(x)) = (0, 0, r\Phi(r))$. This equation has a set of solutions since the isotopic rotation by every angle about the third axis $(0, 0, 1)$ will not change the finishing vector $(0, 0, r\Phi(r))$. We fix such an ambiguity by deciding in favor of the simplest transformation matrix. It will be convenient to utilize the known group $SO(3.R)$ parameterization through the Gibbs 3-vector

\[
O = O(\vec{c}) = I + 2 \frac{\vec{c}^\times + (\vec{c}^\times)^2}{1 + c^2} , \quad (\vec{c}^\times)_{ac} = -\epsilon_{acb} c_b . \tag{2.1}
\]

According to [98], the simplest rotation above is $\vec{B} = O(\vec{c})\vec{A}$, $\vec{c} = [\vec{B}\vec{A}]/(\vec{A} + \vec{B})\vec{A}$, therefore,

if $\vec{A} = r\Phi(r) \vec{n}_{\theta,\phi}$, $\vec{B} = r\Phi(r)(0, 0, 1)$, then $\vec{c} = \frac{\sin \theta}{1 + \cos \theta} \left( \begin{array}{c} \sin \Phi, -\cos \Phi, 0 \end{array} \right) . \tag{2.2}$

\^15The author highly recommends the book [97] for many further details developing the Gibbs approach to groups $SO(3.R), SO(3.C), SO_b(3.1)$, etc.
Together with varying the scalar field $\Phi^\alpha(x)$, the vector triplet $W_{\beta}^{(a)}(x)$ is to be transformed from one isotopic gauge to another under the law [99]

$$W_{\alpha}^{(a)}(x) = O_{ab}(\vec{c}(x)) W_{\alpha}^{(b)}(x) + \frac{1}{e} f_{ab}(\vec{c}(x)) \frac{\partial c_b}{\partial x^\alpha}, \quad f(\vec{c}) = -2 \frac{1 + \vec{c}^2}{1 + \vec{c}^2}. \quad (2.3)$$

With the use of (2.3), we obtain the new representation

$$\Phi_{D.}^{(a)} = r \Phi(r) \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \quad W_{\theta}^{D.} = (r^2 K + 1/e) \left( \begin{array}{c} -\sin \phi \\ + \cos \phi \\ 0 \end{array} \right), \quad W_{\vec{r}}^{D.} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \quad W_{\phi}^{D.} = \left( \begin{array}{c} -(r^2 K + 1/e) \sin \theta \cos \phi \\ -(r^2 K + 1/e) \sin \theta \sin \phi \\ \frac{1}{e} \cos \theta - 1 \end{array} \right). \quad (2.4)$$

It should be noticed that the factor $(r^2 K(r) + 1/e)$ will vanish when $K = -1/er^2$. Thus, only the delicate fitting of the single proportional coefficient (it must be taken as $-1/e$) results in the actual formal simplification of the non-Abelian monopole potential.

There exists close connection between $W_{\phi}^{D.}(x)$ from (2.4) and the Dirac’s expression for the Abelian monopole potential (supposing that $\vec{n} = (0, 0, -1)$):

$$A_\alpha^{D.} = g \left( 0, \frac{[\vec{n} \, \vec{r}]}{(r + \vec{r} \vec{n}) \, r} \right), \quad \text{or} \quad A_\phi^{D.} = -g \left( \cos \theta - 1 \right). \quad (2.5)$$

So, $W_{\phi}^{triv.}(x)$ from (2.4) (produced by setting $K = -1/er^2$) can be thought of as the result of embedding the Abelian potential (2.5) in the non-Abelian gauge scheme: $W_{\alpha}^{(a)}(x) \equiv (0, 0, A_\alpha^{D.}(x))$. The quantity $W_{\alpha}^{(a)D.}(x)$ labelled with symbol $D.$ will be named after its Abelian counterpart; in other words, this potential will be treated as relating to the Dirac’s non-Abelian gauge in the isotopic space.

In Abelian case, the Dirac’s potential $A_\alpha^{D.}(x)$ can be converted into the Schwinger form $A_\alpha^{S.}(x)$

$$A_\alpha^{S.} = \left( 0, g \frac{[\vec{r} \, \vec{n}]}{(r^2 - (\vec{r} \vec{n})^2) \, r} \right), \quad \text{or} \quad A_\phi^{S.} = g \cos \theta \quad (2.6)$$

by means of the following transformation

$$A_\alpha^{S.} = A_\alpha^{D.} + \frac{hc}{ie} S \frac{\partial}{\partial x^\alpha} S^{-1}, \quad S(x) = \exp(-i \frac{e \theta}{hc}).$$

It is possible to draw an analogy between the Abelian and non-Abelian models. That is, we may introduce the Schwinger non-Abelian basis in the isotopic space:

$$\left( \Phi_{D.}^{(a)}, W_{\alpha}^{D.} \right) \rightarrow \left( \Phi_{S.}^{(a)}, W_{\alpha}^{S.} \right), \quad \vec{c}' = (0, 0, -\tan \phi/2); \quad (2.7a)$$

where

$$O(\vec{c}') = \left( \begin{array}{ccc} \cos \phi & \sin \phi & 0 \\
- \sin \phi & \cos \phi & 0 \\
0 & 0 & 1 \end{array} \right).$$
Now an explicit form of the monopole potential is given by

\[
\begin{align*}
W^{S,(a)}_{\theta} &= \begin{pmatrix} 0 & 0 \\ (r^2 K + 1/e) & 0 \end{pmatrix}, & W^{S,(a)}_{\phi} &= \begin{pmatrix} -(r^2 K + 1/e) & 0 \\ \frac{1}{e} \cos \theta & 0 \end{pmatrix}, \\
W^{S,(a)}_{x} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & W^{S,(a)}_{i} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \Phi^{S,(a)} &= \begin{pmatrix} 0 \\ r \Phi(r) \end{pmatrix}
\end{align*}
\]

(2.7b)

where the symbol \( S. \) stands for the Schwinger gauge.

Both \( D. \) - and \( S. \)-gauges (see (2.4) and (2.7b)) are unitary ones in the isotopic space due to the respective scalar fields \( \Phi^{D,(a)}(x) \) and \( \Phi^{S,(a)}(x) \) are \( x_3 \)-unidirectional, but one of them (Schwinger’s) seems simpler than another (Dirac’s).

For the following it will be convenient to determine the matrix \( O(\vec{c}''') \) relating the Cartesian gauge of isotopic space with Schwinger’s:

\[
O(\vec{c}''') = O(\vec{c}') O(\vec{c}) = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix},
\]

\[
\vec{c}''' = (+ \tan \theta/2 \tan \phi/2, - \tan \theta/2, - \tan \phi/2).
\]

This matrix \( O(\vec{c}''') \) is also well-known in other context as a matrix linking Cartesian and spherical tetrads in the space-time of special relativity (as well as in a curved space-time of spherical symmetry)

\[
x^\alpha = (x^0, x^1, x^2, x^3), \quad dS^2 = [(dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2], \quad e^{\alpha}_{(a)}(x) = \delta^{\alpha}_{a} \quad (2.9a)
\]

and

\[
x'^\alpha = (t, r, \theta, \phi), \quad dS^2 = [dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)],
\]

\[
e'^{\alpha}_{(0)} = (1, 0, 0, 0), \quad e'^{\alpha}_{(1)} = (0, 0, 1/r, 0),
\]

\[
e'^{\alpha}_{(2)} = (0, 0, 0, 1/r \sin \theta), \quad e'^{\alpha}_{(3)} = (0, 1, 0, 0). \quad (2.9b)
\]

Below we review briefly some relevant facts about the tetrad formalism. In the presence of an external gravitational field, the starting Dirac equation \( (i\gamma^a \partial/\partial x^a - m)\Psi(x) = 0 \) is generalized into [55-63]

\[
[ i \gamma^\alpha(x) \left( \partial_\alpha + \Gamma_\alpha(x) \right) - m ] \Psi(x) = 0 \quad (2.10)
\]

where \( \gamma^\alpha(x) = \gamma^a e^{\alpha}_{(a)}(x) \), and \( \epsilon^{\alpha}_{(a)}(x), \Gamma_\alpha(x) = \frac{1}{2} \sigma^{ab} e^{\beta}_{(a)} \nabla_\alpha (e^{\alpha}_{(b)}), \nabla_\alpha \) stand for a tetrad, the bispinor connection, and the covariant derivative symbol, respectively. In the spinor basis:

\[
\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \sigma^a = (I, +\sigma^k), \quad \bar{\sigma}^a = (I, -\sigma^k)
\]

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(\sigma^k are the two-row Pauli spin matrices; k = 1, 2, 3) we have two equations

\[ \begin{aligned}
&i\sigma^a(x) \left[ \partial_\alpha + \Sigma_\alpha(x) \right] \xi(x) = m \eta(x), \quad i\bar{\sigma}^a(x) \left[ \partial_\alpha + \bar{\Sigma}_\alpha(x) \right] \eta(x) = m \xi(x) \quad (2.11)
\end{aligned} \]

where the symbols \( \sigma^a(x) \), \( \bar{\sigma}^a(x) \), \( \Sigma_\alpha(x) \), \( \bar{\Sigma}_\alpha(x) \) denote respectively

\[ \sigma^a(x) = \sigma^a e^a_\alpha(x), \quad \bar{\sigma}^a(x) = \bar{\sigma}^a e^a_\alpha(x), \]

\[ \Sigma_\alpha(x) = \frac{1}{2} \Sigma^{ab} e^\beta_\alpha \nabla_\alpha (e^\beta_b \nabla_\beta), \quad \bar{\Sigma}_\alpha(x) = \frac{1}{2} \bar{\Sigma}^{ab} e^\beta_\alpha \nabla_\alpha (e^\beta_b \nabla_\beta), \]

\[ \Sigma^{ab} = \frac{1}{4}(\bar{\sigma}^a \sigma^b - \sigma^a \bar{\sigma}^b), \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a). \]

Setting \( m \) equal to zero, we obtain Weyl equation for neutrino \( \eta(x) \) and anti-neutrino \( \xi(x) \), or the Dirac equation for a massless particle (the latter will be used further in Sec.8).

The form of equations (2.10), (2.11) implies quite definite their symmetry properties. It is common, considering the Dirac equation in the same space-time, to use some different tetrads \( e^\beta_\alpha(x) \) and \( e^{\beta'}_\alpha(x) \), so that we have the equation (2.10) and an analogous one with a new tetrad mark. In other words, together with (2.10) there exists an equation on \( \Psi'(x) \), where quantities \( \gamma^\alpha(x) \) and \( \Gamma'_\alpha(x) \), in contrast with \( \gamma^\alpha(x) \) and \( \Gamma_\alpha(x) \), are based on a new tetrad \( e^{\beta'}_\alpha(x) \) related to \( e^\beta_\alpha(x) \) through a certain local Lorentz matrix

\[ e^{\beta'}_\alpha(x) = L^a_b(x) e^\beta_\alpha(x). \quad (2.12a) \]

It may be shown that these two Dirac equations on functions \( \Psi(x) \) and \( \Psi'(x) \) are related to each other by a definite bispinor transformation:

\[ \xi'(x) = B(k(x))\xi(x), \quad \eta'(x) = B^+(k(x))\eta(x). \quad (2.12b) \]

Here, \( B(k(x)) = \sigma^a k_a(x) \) is a local matrix from the \( SL(2,C) \) group; 4-vector \( k_a \) is the well-known parameter on this group [100]. The matrix \( L^a_b(x) \) from (2.12a) may be expressed as a function of arguments \( k_a(x) \) and \( k^*_a(x) \):

\[ L^a_b(k, k^*) = \delta^c_b \left[ -\delta^c_a k^a n^* + k^c k^a^* + k^c k^a + i\epsilon^{c a m n} k^m k^*_n \right] \quad (2.12c) \]

where \( \delta^c_b \) is a special Croneker symbol:

\[ \delta^c_b = 0 \quad \text{if} \quad c \neq b; \quad = +1 \quad \text{if} \quad c = b = 0; \quad = -1 \quad \text{if} \quad c = b = 1, 2, 3. \]

By the way, it is normal practice that some different tetrads are used in examining the Dirac equation on the same Riemannian space-time background. If there is a need to analyze some correlation between solutions in those distinct tetrads, then it is important to know what are the relevant gauge transformations over the spinor wave functions. In particular, the matrix relating spinor wave functions in Cartesian and spherical tetrads (see (2.9)) is as follows

\[ B = \pm \left( \begin{array}{cc} \cos \theta/2 e^{i\phi/2} & \sin \theta/2 e^{-i\phi/2} \\ -\sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{-i\phi/2} \end{array} \right) \equiv B(\vec{c}'') = \pm \frac{I - i\bar{\sigma}\vec{c}''}{\sqrt{1 - (\vec{c}'')^2}}. \quad (2.12d) \]
The vector matrix $L_a^a(\theta, \phi)$ referring to the spinor’s $B(\theta, \phi)$ is the same as $O(\vec{c}''')$ from (2.8). It is significant that the two gauge transformations, arising in quite different contexts, correspond so closely with each other.

This basis of spherical tetrad will play a substantial role in our subsequent work. This Schrödinger frame of spherical tetrad [64] was used with great efficiency by Pauli [65] when investigating the problem of allowed spherically symmetrical wave functions in quantum mechanics. Below, we briefly review some results of this investigation.

Let the $J_i^\lambda$ denote

\[ J_1 = (l_1 + \lambda \frac{\cos \phi}{\sin \theta}), \quad J_2 = (l_2 + \lambda \frac{\sin \phi}{\sin \theta}), \quad J_3 = l_3. \]

At an arbitrary $\lambda$, as readily verified, those $J_i$ satisfy the commutation rules of the Lie algebra $SU(2)$: $[J_a, J_b] = i \epsilon^{abc} J_c$. As known, all irreducible representations of such an abstract algebra are determined by a set of weights $j = 0, 1/2, 1, 3/2, \ldots$ ($\dim j = 2j + 1$). Given the explicit expressions of $J_a$ above, we will find functions $\Phi_{jm}^\lambda(\theta, \phi)$ on which the representation of weight $j$ is realized. In agreement with the generally known method, those solutions are to be established by the following relations

\[ J_+ \Phi_{jj}^\lambda = 0, \quad \Phi_{jm}^\lambda = \sqrt{(j + m)! (j - m)! (2j)!} J_{j-m}^\lambda \Phi_{jj}^\lambda, \quad (2.13) \]

\[ J_\pm = (J_1 \pm iJ_2) = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} + \frac{\lambda}{\sin \theta} \right]. \]

From the equations $J_+ \Phi_{jj}^\lambda = 0$ and $J_3 \Phi_{jj}^\lambda = \pm \Phi_{jj}^\lambda$, it follows that

\[ \Phi_{jj}^\lambda = N_{jj}^\lambda e^{ij\phi} \sin^j \theta \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{\lambda/2} \times \]

\[ \frac{1}{\sqrt{2\pi}} \frac{1}{2^j} \sqrt{\frac{(2j + 1)}{\Gamma(j + m + 1) \Gamma(j - m + 1)}}. \]

Further, employing (2.13) we produce the functions $\Phi_{jm}^\lambda$

\[ \Phi_{jm}^\lambda = N_{jm}^\lambda e^{im\phi} \frac{1}{\sin^m \theta} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{\lambda/2} \times \]

\[ \left( \frac{d}{d \cos \theta} \right)^{j-m} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] \quad (2.14) \]

where

\[ N_{jm}^\lambda = \frac{1}{\sqrt{2\pi 2^j}} \sqrt{\frac{(2j + 1) (j + m)!}{2(j - m)! \Gamma(j + \lambda + 1) \Gamma(j - \lambda + 1)}}. \]

The Pauli criterion tells us that the $(2j + 1)$ functions $\Phi_{jm}^\lambda(\theta, \phi)$, $m = -j, \ldots, +j$ so constructed, are guaranteed to be a basis for a finite-dimension representation, providing that the functions $\Phi_{j,-j}^\lambda(\theta, \phi)$, found by this procedure, obey the identity

\[ J_- \Phi_{j,-j}^\lambda = 0. \quad (2.15a) \]
After substituting the function $\Phi_{j,-j}^\lambda(\theta, \phi)$, the relation (2.15a) reads

$$J_\lambda \Phi_{j,-j}^\lambda = N_{j,-j}^\lambda e^{-i(j+1)\phi} (\sin \theta)^{j+1} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{\lambda/2} \times$$

$$\left( \frac{d}{d \cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0$$

(2.15b)

which in turn gives the following restriction on $j$ and $\lambda$

$$\left( \frac{d}{d \cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0$$

(2.15c)

But the relation (2.15c) can be satisfied only if the factor $P(\theta)$, subjected to the operation of taking derivative $(d/d \cos \theta)^{2j+1}$, is a polynomial of degree $2j$ in $\cos \theta$. So, we have (as a result of the Pauli criterion)

1. the $\lambda$ is allowed to take values $+1/2, -1/2, +1, -1, \ldots$

Besides, as the latter condition is satisfied, $P(\theta)$ takes different forms depending on the $(j, \lambda)$-correlation:

$$P(\theta) = (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} = P^{2j}(\cos \theta), \quad \text{if} \quad j = |\lambda|, |\lambda| + 1, \ldots$$

or

$$P(\theta) = \frac{P^{2j+1}(\cos \theta)}{\sin \theta}, \quad \text{if} \quad j = |\lambda| + 1/2, |\lambda| + 3/2, \ldots$$

so that the second necessary condition resulting from the Pauli criterion is

2. given $\lambda$ according to 1., the number $j$ is allowed to take values $j = |\lambda|, |\lambda| + 1, \ldots$

Hereafter, these two conditions: 1 and 2 will be termed, respectively, as the first and the second Pauli consequences\[^{16}\]. Also, it should be noted that the angular variable $\phi$ is not affected (charged) by the Pauli criterion; instead, a variable that works above is the $\theta$.

Significantly, in the contrast to this, the well-known procedure $[3-10]$ of deriving the electric charge quantization condition from investigating continuity properties of quantum mechanical wave functions, such a working variable is the $\phi$.

If the first and second Pauli consequences fail, then we face rather unpleasant mathematical and physical problems\[^{17}\]. As a simple illustration, we may indicate the familiar case

\[^{16}\]We draw attention to that the Pauli criterion $J_\lambda \Phi_{j,-j}(t, r, \theta, \phi) = 0$ affords the condition that is invariant relative to possible gauge transformations. The function $\Phi_{j,m}(t, r, \theta, \phi)$ may be subjected to any gauge transformation. But if all the components $J_i$ vary in a corresponding way too, then the Pauli condition provides the same result on $(j, \lambda)$-quantization. In contrast to this, the common requirement to be a single-valued function of spatial points, often applied to produce a criterion on selection of allowable wave functions in quantum mechanics, is not invariant under gauge transformations and can easily be destroyed by a suitable gauge one.

\[^{17}\]Reader is referred to the Pauli article $[65]$ for more detail about those peculiarities.
when $\lambda = 0$; if the second Pauli condition is violated, then we will have the integer and half-integer values of the orbital angular momentum number $l = 0, 1/2, 1, 3/2, \ldots$

As regards the Dirac electron with the components of the total angular momentum in the form (1.2), we have to employ the above Pauli criterion in the constituent form owing to $\lambda$ changed into $\Sigma_3$. Ultimately, we obtain the allowable set $J = 1/2, 3/2, \ldots$

A fact of primary practical importance to us is that the functions $\Phi^\lambda_{jm}(\theta, \phi)$ constructed above relate directly to the known Wigner $D$-functions [66]: $\Phi^\lambda_{jm}(\theta, \phi) = (-1)^{j-m} D^j_{m,\lambda}(\phi, \theta, 0)$.

3. Separation of variables and a composite inversion operator

We will utilize the general relativity covariant formalism when a fundamental Dirac equation is (1.1). In the spherical tetrad basis (2.9b) and the Schwinger unitary gauge of the monopole potentials (2.7b), the matter equation (1.1) takes the form

$$
\begin{align*}
\left[ \gamma^0 (i \partial_t + e r F(r) t^3) + i \gamma^3 (\partial_r + \frac{1}{r}) + \frac{1}{r} \Sigma^S_{\theta,\phi} + \\
\frac{er^2 K(r) + 1}{r} (\gamma^1 \otimes t^2 - \gamma^2 \otimes t^1) - (m + \kappa r \Phi(r) t^3) \right] \Psi^S = 0 , \\
\Sigma^S_{\theta,\phi} = \left[ i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + (i \sigma^{12} + t^3) \cos \theta}{\sin \theta} \right]
\end{align*}
$$

here $t^j = (1/2) \sigma^j$. The equation’s representation (3.1) itself is remarkable: the choice of working basis automatically produces a required rearrangement of its terms. It is useful to look at all the particular ones in (3.1) and further to trace their respective and rather distinctive contributions; as will be seen, each of them has its practical side in the subsequent formation of the doublet-monopole system’s properties. In particular, just one term in (3.1), proportional to $(e r^2 K(r) + 1)$, mixes up together the components of the multiplet and this term vanishes in case of the simplest monopole potential. The peculiarity of both $e r F(r) t^3$ and $\kappa r \Phi(r) t^3$ terms, at least as far as they are really touched in the present work, will be brought to light when we turn to the diagonalization of a composite (isotopic-Lorentzian) discrete operator. In the given basis, the components of total conserved momentum are determined by (1.7), and correspondingly, the starting doublet wave function $\Psi_{\epsilon jm}(x)$ is as in (1.8).

An important case in theoretical investigation is the electron-monopole system at the minimal value of quantum number $j$. The allowed values for $j$ are $0, 1, 2, \ldots$; the case of $j = 0$ needs a careful separate consideration. If $j = 0$, then the used symbols $D^0_{0,\pm 1}$ (in (1.8)) are meaningless, and the wave function $\Psi_{\epsilon 0}(x)$ has to be constructed as

$$
\Psi_{\epsilon 0} = \frac{e^{-i \epsilon t}}{r} \left[ T_{+1/2} \otimes \begin{pmatrix} 0 \\ f_2(r) \\ 0 \\ f_4(r) \end{pmatrix} + T_{-1/2} \otimes \begin{pmatrix} g_1(r) \\ 0 \\ g_3(r) \\ 0 \end{pmatrix} \right].
$$

(3.3)
Using the required recursive relations for Wigner functions (see (1.9)) \( \nu = \sqrt{j(j+1)}, \omega = \sqrt{(j-1)(j+2)}, j \neq 0 \)

\[
\begin{align*}
\partial_\theta D_{-1} &= \frac{1}{2}(\omega D_{-2} - \nu D_0), \quad m - \cos \theta \overline{D}_{-1} = \frac{1}{2}(\omega D_{-2} + \nu D_0), \\
\partial_\theta D_0 &= \frac{1}{2}(\nu D_{-1} - \nu D_{+1}), \quad m \overline{D}_0 = \frac{1}{2}(\nu D_{-1} + \nu D_{+1}), \\
\partial_\theta D_{+1} &= \frac{1}{2}(\nu D_0 - \omega D_{+2}), \quad m + \cos \theta \overline{D}_{+1} = \frac{1}{2}(\nu D_0 + \omega D_{+2})
\end{align*}
\]

we find

\[
\Sigma_{\theta, \phi}^S \Psi_{jm}^S = \nu \begin{pmatrix} T_{+1/2} \otimes \begin{pmatrix} -if_4 D_{-1} \\ +if_3 D_0 \\ +if_2 D_{-1} \\ -if_1 D_0 \end{pmatrix} + T_{-1/2} \otimes \begin{pmatrix} -ig_4 D_0 \\ +ig_3 D_{+1} \\ +ig_2 D_0 \\ -ig_1 D_{+1} \end{pmatrix} \end{pmatrix}.
\]

Further, let us write down the expression for the term that mixes up the isotopic components

\[
\frac{er^2 K(r) + 1}{2r} \begin{pmatrix} \gamma^1 \otimes t^2 - \gamma^2 \otimes t^1 \end{pmatrix} \Psi_{jm} = \frac{er^2 K(r) + 1}{2r} \times \begin{pmatrix} T_{+1/2} \otimes \begin{pmatrix} 0 \\ +ig_3 D_0 \\ 0 \\ -ig_1 D_0 \end{pmatrix} + T_{-1/2} \otimes \begin{pmatrix} -if_4 D_0 \\ 0 \\ +if_2 D_0 \\ 0 \end{pmatrix} \end{pmatrix}.
\]

After a simple calculation one finds the system of radial equations (for shortness we set \( W \equiv (e r^2 K(r) + 1)/2, \overline{F} \equiv e r F(r)/2, \overline{\Phi} \equiv \kappa r \Phi(r)/2 \))

\[
\begin{align*}
(-i \frac{d}{dr} + \epsilon + \overline{F}) f_3 - i \frac{\nu}{r} f_4 - (m + \overline{\Phi}) f_1 &= 0 \\
(+i \frac{d}{dr} + \epsilon + \overline{F}) f_4 + i \frac{\nu}{r} f_3 + i \frac{W}{r} g_3 - (m + \overline{\Phi}) f_2 &= 0 \\
(+i \frac{d}{dr} + \epsilon + \overline{F}) f_1 + i \frac{\nu}{r} f_2 - (m + \overline{\Phi}) f_3 &= 0 \\
(-i \frac{d}{dr} + \epsilon + \overline{F}) f_2 - i \frac{\nu}{r} f_1 - i \frac{W}{r} g_1 - (m + \overline{\Phi}) f_4 &= 0 \\
(-i \frac{d}{dr} + \epsilon - \overline{F}) g_3 - i \frac{\nu}{r} g_4 - i \frac{W}{r} f_4 - (m - \overline{\Phi}) g_1 &= 0 \\
(+i \frac{d}{dr} + \epsilon - \overline{F}) g_4 + i \frac{\nu}{r} g_3 - (m - \overline{\Phi}) g_2 &= 0
\end{align*}
\]
would have the structure

\( (+i \frac{d}{dr} + \epsilon - \tilde{F})g_1 + i \nu r' g_2 + i \frac{W}{r} f_2 - (m - \tilde{\Phi})g_3 = 0 \)

\( (-i \frac{d}{dr} + \epsilon - \tilde{F})g_2 - i \nu r' g_1 - (m - \tilde{\Phi})g_4 = 0 \).

(3.6)

When \( j \) takes on value 0 (then \( \Sigma_{\theta,\phi} \Psi_{e0} \equiv 0 \)), the radial system is

\( (+i \frac{d}{dr} + \epsilon + \tilde{F})f_4 + i \frac{W}{r} g_3 - (m + \tilde{\Phi})f_2 = 0 \)

\( (-i \frac{d}{dr} + \epsilon + \tilde{F})f_2 - i \frac{W}{r} g_1 - (m + \tilde{\Phi})f_4 = 0 \)

\( (-i \frac{d}{dr} + \epsilon - \tilde{F})g_3 - i \frac{W}{r} f_4 - (m - \tilde{\Phi})g_1 = 0 \)

\( (+i \frac{d}{dr} + \epsilon - \tilde{F})g_1 + i \frac{W}{r} f_2 - (m - \tilde{\Phi})g_3 = 0 \).

(3.7)

Both these systems (3.6) and (3.7) are sufficiently complicated. To proceed further in a situation like that, it is normal practice to search a suitable operator which could be diagonalized additionally. It is known that the usual \( P \)-inversion operator for a bispinor field cannot be completely appropriate for this purpose and a required quantity has to be constructed as a combination of bispinor \( P \)-inversion operator and a certain discrete transformation in the isotopic space. Indeed, considering that the usual \( P \)-inversion operator for a bispinor field (in the basis of Cartesian tetrad, it is \( \hat{P}_{\text{bisp.}} \otimes \hat{P} = i \gamma^0 \otimes \hat{P} \), where \( \hat{P} \) causes the usual \( P \)-reflection of space coordinates) is determined in the given (spherical) basis as

\[ \hat{P}_{\text{bisp.}} \otimes \hat{P} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{P} = - (\gamma^5 \gamma^1) \otimes \hat{P} \]  

(3.8a)

and it acts upon the wave function \( \Psi_{jm}(x) \) as follows (the factor \( e^{i\alpha} / r \) is omitted)

\[ (\hat{P}_{\text{bisp.}} \otimes \hat{P}) \Psi_{jm}(x) = (-1)^{j+1} \left[ T_{+1/2} \otimes \begin{pmatrix} f_4 & D_0 \\ f_3 & D_{+1} \\ f_2 & D_0 \\ f_1 & D_{+1} \end{pmatrix} \right] + T_{-1/2} \otimes \begin{pmatrix} g_4 & D_{-1} \\ g_3 & D_0 \\ g_2 & D_{-1} \\ g_1 & D_0 \end{pmatrix} \].

(3.8b)

This relationship points the way towards the search for a required discrete operator: it would have the structure

\[ \hat{N}_{\text{sph.}}^S \equiv \hat{\pi}^S \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{\pi}^S = (a \sigma^1 + b \sigma^2), \quad \hat{\pi}^S \cdot T_{\pm 1/2} = (a \pm ib) T_{\mp 1/2} \].

(3.9)

The total multiplier at the quantity \( \hat{\pi}^S \) is not material one for separating the variables, below one sets \( (\hat{\pi}^S)^2 = (a^2 + b^2) = +1 \).
From the equation $\hat{N}_{\text{sph.}}^{S} \Psi_{jm} = N \Psi_{jm}$ one finds two proper values $N$ and corresponding limitation on the functions $f_{i}(r)$ and $g_{i}(r)$:

$$N = \delta (-1)^{j+1}, \quad \delta = \pm 1 : \quad g_{1} = \delta (a + ib) f_{4}, \quad g_{2} = \delta (a + ib) f_{3},$$

$$g_{3} = \delta (a + ib) f_{2}, \quad g_{4} = \delta (a + ib) f_{1}. \quad (3.10a)$$

Taking into account the relations (3.10a), one produces the equations ($\Delta \equiv (a + ib)$)

$$(-i \frac{d}{dr} + \epsilon + \tilde{F}) f_{3} - \frac{\nu}{r} f_{4} - (m + \tilde{\Phi}) f_{1} = 0$$

$$(+i \frac{d}{dr} + \epsilon + \tilde{F}) f_{4} + \frac{\nu}{r} f_{3} + i \frac{W}{r} - \delta f_{2} - (m + \tilde{\Phi}) f_{2} = 0$$

$$(-i \frac{d}{dr} + \epsilon + \tilde{F}) f_{1} + \frac{\nu}{r} f_{2} - (m + \tilde{\Phi}) f_{3} = 0$$

$$(-i \frac{d}{dr} + \epsilon + \tilde{F}) f_{2} - \frac{\nu}{r} f_{1} - i \frac{W}{r} \delta f_{4} - (m + \tilde{\Phi}) f_{4} = 0$$

$$(-i \frac{d}{dr} + \epsilon + \tilde{F}) f_{2} - \frac{\nu}{r} f_{1} - i \frac{W}{r} \Delta^{-1} \delta f_{4} - (m - \tilde{\Phi}) f_{4} = 0$$

$$(-i \frac{d}{dr} + \epsilon - \tilde{F}) f_{1} + \frac{\nu}{r} f_{2} - (m - \tilde{\Phi}) f_{3} = 0$$

$$(-i \frac{d}{dr} + \epsilon - \tilde{F}) f_{1} + \frac{\nu}{r} f_{2} - (m - \tilde{\Phi}) f_{3} = 0$$

$$(-i \frac{d}{dr} + \epsilon - \tilde{F}) f_{4} + \frac{\nu}{r} f_{3} + i \frac{W}{r} \Delta^{-1} \delta f_{2} - (m - \tilde{\Phi}) f_{2} = 0$$

$$(-i \frac{d}{dr} + \epsilon - \tilde{F}) f_{4} + \frac{\nu}{r} f_{3} - (m - \tilde{\Phi}) f_{1} = 0. \quad (3.10b)$$

It is evident at once that the system (3.10b) would be compatible with itself provided that $\tilde{F}(r) = 0$ and $\tilde{\Phi}(r) = 0$. In other words, the above-mentioned operator $\hat{N}_{\Delta}^{S}$ can be diagonalized on the functions $\Psi_{\epsilon jm}(x)$ if and only if $W_{\epsilon}^{(a)} = 0$ and $\kappa = 0$; below we suppose that these requirements will be satisfied. Moreover, given this limitation satisfied, it is necessary to draw distinction between two cases depending on expression for $W(r)$. If $W(r) = 0$, the difference between $\Delta$ and $\Delta^{-1}$ in the equations (3.10b) is not essential in simplifying these equations (because the relevant terms just vanish).

Thus, for the first case, the system (3.10b) converts into (the symbol $\Delta$ at $\hat{N}$ stands for the $a$- and $b$-dependence):

$$W(r) = 0, \quad \hat{N}_{\Delta}^{S} = (a \sigma^{1} + b \sigma^{2}) \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P} :$$

$$(-i \frac{d}{dr} + \epsilon) f_{3} - \frac{\nu}{r} f_{4} - m f_{1} = 0, \quad (+i \frac{d}{dr} + \epsilon) f_{4} + \frac{\nu}{r} f_{3} - m f_{2} = 0,$$

$$(-i \frac{d}{dr} + \epsilon) f_{1} + \frac{\nu}{r} f_{2} - m f_{3} = 0, \quad (-i \frac{d}{dr} + \epsilon) f_{2} - \frac{\nu}{r} f_{1} - m f_{4} = 0. \quad (3.11)$$

There exists sharply distinct situation at $W \neq 0$. Here, the equations are consistent with each other only if $\Delta = \Delta^{-1}$; therefore $\Delta = (a + ib) = \pm 1$. Combining this relation
with the normalizing condition \((a + ib)(a - ib) = 1\), one gets \(a = \pm 1\) and \(b = 0\) (for definiteness, let this parameter \(a\) be equal +1). The corresponding set of radial equations, obtained from (3.10b), is

\[
\hat{N}^S = (\sigma^1 \otimes \hat{P}_{\text{isp}} \otimes \hat{P}), \quad \hat{N} = \delta(-1)^{j+1}:
\]

\[
(-i \frac{d}{dr} + \epsilon) f_3 - \frac{\nu}{r} f_4 - mf_1 = 0, \quad (+i \frac{d}{dr} + \epsilon) f_4 + \frac{\nu}{r} f_3 + i \frac{W}{r} \delta f_2 - mf_2 = 0,
\]

\[
(+i \frac{d}{dr} + \epsilon) f_1 + \frac{\nu}{r} f_2 - mf_3 = 0, \quad (-i \frac{d}{dr} + \epsilon) f_2 - \frac{\nu}{r} f_1 - i \frac{W}{r} \delta f_4 - mf_4 = 0. \quad (3.12)
\]

In the same way, the case \(j = 0\) can be considered. Here, the proper values and limitation are

\[
N = -\delta, \quad \delta = \pm 1: \quad g_1(r) = \delta \Delta f_1(r), \quad g_3(r) = \delta \Delta f_2(r). \quad (3.13a)
\]

Further, the quantities \(\tilde{F}\) and \(\tilde{\Phi}\) are to be equated to zero; again there are two possibilities depending on \(W\):

\[
W(r) = 0: \quad (i \frac{d}{dr} + \epsilon) f_4 - mf_2 = 0, \quad (-i \frac{d}{dr} + \epsilon) f_2 - mf_4 = 0; \quad (3.13b)
\]

\[
W(r) \neq 0: \quad (i \frac{d}{dr} + \epsilon) f_4 - (m - i \frac{\delta}{r} W) f_2 = 0, \quad (-i \frac{d}{dr} + \epsilon) f_2 - (m + i \frac{\delta}{r} W) f_4 = 0. \quad (3.13c)
\]

The explicit forms of the wave functions \(\Psi_{e jm \delta}(x)\) and \(\Psi_{e 0 \delta}(x)\) are as follows:

The case \(W(r) \neq 0, \ j > 0\),

\[
\Psi_{e jm}(x) = \frac{e^{-i t / r}}{r} \left[ T_{+1/2} \otimes \begin{pmatrix} f_1 D_{-1} \\ f_2 D_0 \\ f_3 D_{-1} \\ f_4 D_0 \end{pmatrix} + \delta T_{-1/2} \otimes \begin{pmatrix} f_4 D_0 \\ f_3 D_{+1} \\ f_2 D_0 \\ f_1 D_{+1} \end{pmatrix} \right]; \quad (3.14a)
\]

The case \(W(r) \neq 0, \ j = 0\),

\[
\Psi_{e 0} = \frac{e^{-i t / r}}{r} \left[ T_{+1/2} \otimes \begin{pmatrix} f_2(r) \\ 0 \\ f_4(r) \end{pmatrix} + \delta T_{-1/2} \otimes \begin{pmatrix} f_4(r) \\ 0 \\ f_2(r) \end{pmatrix} \right]; \quad (3.14b)
\]

where \(\delta T_{-1/2}\) is to be changed for \(\delta \Delta T_{-1/2}\) when \(W = 0\).

These formulas point to the non-featureless and non-formal union of the one particle pattern with another (two distinct isotopic components), but a structural and specific one; at that, the second term in the composite doublet wave function is strictly determined up to the whole phase factor, by the first term, so that this system is not a plain sum of two components without any intrinsic structure.
4 Analyzis of the particular case of simplest monopole field

Now, some added aspects of the simplest monopole are examined more closely. The system of radial equations, specified for this potential, is basically simpler than in general case, so that the whole problem including the radial functions can be carried out to its complete conclusion.

Actually, the equation (3.11) admits of some further simplifications owing to diagonalizing the operator $\hat{K}_{(),\phi} = -i\gamma^0,\gamma^5\Sigma_{(),\phi}$. From the equation $\hat{K}_{(),\phi} \Psi_{jm} = \lambda \Psi_{jm}$, it follows that $\lambda = -\mu \sqrt{j(j+1)}$, $\mu = \pm 1$ and

$$f_4 = \mu f_1, \quad f_3 = \mu f_2, \quad g_4 = \mu g_1, \quad g_3 = \mu g_2.$$  (4.1)

Correspondingly, the system (3.11) yields

$$(-i\frac{d}{dr} + \epsilon)f_1 + i\frac{\nu}{r} f_2 - \mu m f_2 = 0, \quad (-i\frac{d}{dr} + \epsilon)f_2 - i\frac{\nu}{r} f_1 - \mu m f_1 = 0.$$  (4.2a)

The wave function with quantum numbers $(\epsilon, j, m, \delta, \mu)$ has the form

$$\Psi_{\epsilon jm\delta\mu}^{\Delta}(x) = \frac{e^{-it}}{r} \left[ T_{+1/2} \otimes \begin{pmatrix} f_1 D_{-1} \\ f_2 D_0 \\ \mu f_3 D_{-1} \\ \mu f_4 D_0 \end{pmatrix} + \Delta \mu \delta T_{-1/2} \otimes \begin{pmatrix} f_4 D_0 \\ f_3 D_{-1} \\ \mu f_2 D_0 \\ \mu f_1 D_{-1} \end{pmatrix} \right].$$  (4.2b)

We will not consider these systems of two radial equations; this would represent an easy problem concerning the well-known spherical Bessel functions. Instead, we relate these functions (4.2b) (also $\Psi_{\epsilon 0\delta}(x)$) with the wave functions satisfying the Dirac equation in the Abelian monopole potential. Those latter were investigated by many authors; below we will use the notation according to [67]). At $j > j_{\text{min}}$ these Abelian functions are described as in (1.5) with taking into account the additional relation

$$f_4 = \mu f_1, \quad f_3 = \mu f_2, \quad \mu = \pm 1.$$  

For the minimal values $j = j_{\text{min}} = |\epsilon g| - 1/2$, they are\footnote{Just these functions can be referred to the solutions of third type in terminology used by Kazama, Yang, and Goldhaber; see in [25].}

$$eg = +1/2, -1, +3/2, ... \quad \Phi^{(eg)}_{\epsilon 0}(t, r, \theta, \phi) = \begin{pmatrix} f_1(t, r) D_{-m,-1/2}^j(\phi, \theta, 0) \\ f_3(t, r) D_{-m,-1/2}^j(\phi, \theta, 0) \\ 0 \\ 0 \end{pmatrix}; \quad (4.3a)$$

$$eg = -1/2, -1, -3/2, ... \quad \Phi^{(eg)}_{\epsilon 0}(t, r, \theta, \phi) = \begin{pmatrix} f_2(t, r) D_{-m,+1/2}^j(\phi, \theta, 0) \\ f_4(t, r) D_{-m,+1/2}^j(\phi, \theta, 0) \\ 0 \\ 0 \end{pmatrix}. \quad (4.3b)$$
On comparing the formulas (3.14a,b) with (1.5) and ((4.3a,b), the following expansions can be easily found (respectively, for \( j > 0 \) and \( j = 0 \) cases):

\[
\Psi_{\Delta j\mu}^{\Delta\delta}(x) = \left[ T_{+1/2} \otimes \Phi_{\Delta j\mu}^{eg=-1/2}(x) + \mu \delta \Delta T_{-1/2} \otimes \Phi_{\Delta j\mu}^{eg=+1/2}(x) \right], \quad (4.4a)
\]

\[
\Psi_{e\delta}(x) = \left[ T_{+1/2} \otimes \Phi_{e\delta}^{eg=-1/2}(x) + \delta \Delta T_{-1/2} \otimes \Phi_{e\delta}^{eg=+1/2}(x) \right]. \quad (4.4b)
\]

In reference with the formulas (4.4a,b), one additional remark should be given. Though, as evidenced by (4.4a,b), definite close relationships between the non-Abelian doublet wave functions and Abelian fermion-monopole functions can be explicitly discerned, in reality, the non-Abelian situation is intrinsically non-monopole-like (non-singular one). Indeed, in the non-Abelian case, the totality of possible transformations (upon the relevant wave functions) which bear the gauge status are very different from ones that there are in the purely Abelian theory. In a consequence of this, the non-Abelian fermion doublet wave functions (1.8) can be readily transformed, by carrying out the gauge transformations in Lorentzian and isotopic spaces together (\( S. \rightarrow C. \) and \( sph. \rightarrow Cart. \)), into the form (see the formulas (A.9) and (A.10)) where they are single-valued functions of spatial points. In the Abelian monopole situation, the analogous particle-monopole functions can by no means be translated to any single-valued ones (see also in Supplement A).

5 On distinction between manifestation of the Abelian and non-Abelian monopoles. Some comments about parity selection rules

The problem that we have discussed so far concerned solely an isotopic doublet affected by the external monopole field. Let us now pass on to somewhat another subject, namely, we consider what in the everything having stated above was dictated by the presence of the non-Abelian external field and what was fixed only by the isotopic multiplet structure. To this end, it suffices to compare the doublet-monopole system with a free doublet.

A free wave equation is as follows

\[
\left[ \left( i\gamma^0 \partial_t + i\gamma^3 (\partial_r + \frac{1}{r}) + \frac{\gamma^1 \otimes t^2 - \gamma^2 \otimes t^1}{r} \right) + \frac{1}{r} \left( i\gamma^1 + \gamma^2 \partial_\phi + (i\sigma^{12} + t^3) \cos \theta \right) \right. \left. \frac{\sin \theta}{m} \right] \Psi_{sph.}^{S.} = 0. \quad (5.1)
\]

We draw attention to the term \((\gamma^1 \otimes t^2 - \gamma^2 \otimes t^1)/r\) mixing both isotopic components, which somewhat evicts out any contrast between these two physical systems, so that everything said above and concerned the case \( W \neq 0 \) is valid for this particular situation too\[^{19}\]. A single distinction is the explicit form of the factor at \((\gamma^1 \otimes t^2 - \gamma^2 \otimes t^1)/r\)-term.

The presence of such a mixing term in the equation referring to the free doublet might seem rather surprising fact. Nevertheless, as can be easily shown, its origin is due

\[^{19}\)Correspondingly, no \( A \)-freedom in choosing the composite inversion-like operator occurs in case of free fermion doublet.\]
to a gauge transformation. Actually, the corresponding free equation in the Cartesian isotopic gauge (compare it with (1.1))

$$[i \not{\gamma}(x)(\partial_{\alpha} + \Gamma_{\alpha}(x)) \otimes I - m] \Psi_{C}^{0} = 0 \quad (5.2a)$$

takes on the following explicit form in the basis of spherical tetrad

$$\left[i \gamma^{0} \partial_{t} + i \gamma^{3} \left(\partial_{r} + \frac{1}{r}\right) + \frac{1}{r} \left(i \gamma^{1} + \gamma^{2} \frac{i \partial_{\phi} + i \cos \theta}{\sin \theta} - m\right)\right] \Psi_{C}^{0} = 0 \quad (5.2b)$$

Applying the isotopic gauge transformation to $\Psi_{C}^{0}$ (see (2.12d)): $\Psi_{S}^{0}(x) = B(\theta, \phi) \Psi_{C}^{0}(x)$, one can bring the equation (5.2b) to the form

$$\left[i \gamma^{0}(x) \left(\partial_{\alpha} + \Gamma_{\alpha}(x)\right) \otimes I + i \gamma^{\alpha}(x) \otimes \left(\frac{\partial B^{-1}}{\partial x^{\alpha}}\right) - m\right] \Psi_{S}^{0} = 0 \quad (5.3a)$$

where

$$i \gamma^{\alpha}(x) \otimes \left(\frac{\partial B^{-1}}{\partial x^{\alpha}}\right) = \frac{1}{r} \left(\gamma^{1} \otimes r^{2} - \gamma^{2} \otimes t^{1}\right) + \gamma^{2} \otimes \frac{t^{3} \cos \theta}{r \sin \theta} \quad (5.3b)$$

The first term in (5.3b) will mix the isotopic components, the second represents a term being an essential addition to angular operator $\Sigma_{\theta, \phi}$, and both of them have arisen out of the above gauge transformation. Their correlated appearing may be regarded as a formal mathematical description of efficient linking the radial functions through the kinematical coupling of two isotopic components by diagonalization of the total angular momentum operators. In this context, the above-mentioned simplification of radial equations at $W = 0$ can be interpreted as follows: an efficient cinematical mixing (owing to the ordinary scheme of angular momentum addition) the different isotopic components is destroyed through simple placing that system into the external trivial monopole field, so that the angular coupling is conserved but the efficient linking through radial functions no longer obtains. In other words, these two factors cancel out each other.

It is significant that in both cases, the wave functions obeying the free equation and the equation with external monopole potentials, respectively, do not vary at all in their $\theta, \phi$-dependence. A single manifestation of the external monopole field is the change in the single parametric function $W(r)$: the quantity $W^{0}(r) = 1$ is to be replaced with another $W(r) = (1 + e r^{2} K(r))$. The above correlates with the fact that the operators of spherical symmetry $\vec{J}^{2}, J_{3}, N$ of these two different physical systems exactly coincide.

Totally different from this is the situation in the Abelian problem when the spherical symmetry operators and wave functions are both basically transformed (see (1.3) and (1.4)) in presence of the Abelian monopole. The free basic wave functions (setting $eg = 0$ in (1.3)) $\Phi_{0}^{JMA}(t, r, \theta, \phi)$ and the monopole ones $\Phi_{eJM\mu}^{eg}(t, r, \theta, \phi)$ vary noticeably in their boundary properties at $\theta = 0, \pi$. Let us consider this question in some more detail.

To clarify all the significance of the mere displacement in a single index at the wave functions in (1.5), we are going to look at just one mathematical characteristic of those $D$-functions involved in the particle wave functions: namely, their boundary properties
at the points \( \theta = 0 \) and \( \theta = \pi \). So, the following Tables can be produced (only some of them are written out):

Table 1a \( D_{m,+1/2}^j \):

| \( j \) | \( m \) | \( \theta = 0 \) | \( \theta = \pi \) |
|-------|-------|---------------|---------------|
| \( 1/2 \) | \(-1/2\) | 0 | \( e^{+i\phi/2} \) |
|       | \(+1/2\) | \( e^{-i\phi/2} \) | 0 |
| \( 3/2 \) | \(-1/2\) | 0 | \( e^{+i\phi/2} \) |
|       | \(+1/2\) | \( e^{-i\phi/2} \) | 0 |
|       | \(-3/2\) | 0 | 0 |
|       | \(+3/2\) | 0 | 0 |

| \( 5/2,\ldots \) |

Table 1b \( D_{m,-1/2}^j \):

| \( j \) | \( m \) | \( \theta = 0 \) | \( \theta = \pi \) |
|-------|-------|---------------|---------------|
| \( 1/2 \) | \(-1/2\) | \( e^{+i\phi/2} \) | 0 |
|       | \(+1/2\) | 0 | \( e^{-i\phi/2} \) |
| \( 3/2 \) | \(-1/2\) | \( e^{+i\phi/2} \) | 0 |
|       | \(+1/2\) | 0 | \( e^{-i\phi/2} \) |
|       | \(-3/2\) | 0 | 0 |
|       | \(+3/2\) | 0 | 0 |

| \( 5/2,\ldots \) |

Table 2a \( D_{m,+1}^j \):

| \( j \) | \( m \) | \( \theta = 0 \) | \( \theta = \pi \) |
|-------|-------|---------------|---------------|
| \( 1 \) | \( 0 \) | 0 | 0 |
|       | \(-1\) | 0 | \( e^{+i\phi} \) |
|       | \(+1\) | \( e^{-i\phi} \) | 0 |
| \( 2 \) | \( 0 \) | 0 | 0 |
|       | \(-1\) | 0 | \( e^{+i\phi} \) |
|       | \(+1\) | \( e^{-i\phi} \) | 0 |
|       | \(-2\) | 0 | 0 |
|       | \(+2\) | 0 | 0 |

| \( 3,\ldots \) |
Table 2b $D^j_{m,-1}$:

| j = 1 | m = 0 | 0 | 0 |
| m = -1 | $e^{+i\phi}$ | 0 |
| m = +1 | 0 | $e^{-i\phi}$ |

| j = 2 | m = 0 | 0 | 0 |
| m = -1 | $e^{+i\phi}$ | 0 |
| m = +1 | 0 | $e^{-i\phi}$ |
| m = -2 | 0 | 0 |
| m = +2 | 0 | 0 |

j = 3, ...

Table 3a $D^j_{m,3/2}$:

| j = 3/2 | m = -1/2 | 0 | 0 |
| m = +1/2 | 0 | 0 |
| m = -3/2 | 0 | $e^{+i3\phi/2}$ |
| m = +3/2 | $e^{-i3\phi/2}$ | 0 |

| j = 5/2 | m = -1/2 | 0 | 0 |
| m = +1/2 | 0 | 0 |
| m = -3/2 | 0 | $e^{+i3\phi/2}$ |
| m = +3/2 | $e^{-i3\phi/2}$ | 0 |
| m = -5/2 | 0 | 0 |
| m = +5/2 | 0 | 0 |

j = 7/2, ...
Table 3b  \( D^j_{m,-3/2} : \)

\[
\begin{matrix}
\theta = 0 & \theta = \pi \\
\hline
j = 3/2 & \\
m = -1/2 & 0 & 0 \\
m = +1/2 & 0 & 0 \\
m = -3/2 & e^{+i3\phi/2} & 0 \\
m = +3/2 & 0 & e^{-i3\phi/2} \\
\end{matrix}
\]

\[
\begin{matrix}
\hline
j = 5/2 & \\
m = -1/2 & 0 & 0 \\
m = +1/2 & 0 & 0 \\
m = -3/2 & e^{+i3\phi/2} & 0 \\
m = +3/2 & 0 & e^{-i3\phi/2} \\
m = -5/2 & 0 & 0 \\
m = +5/2 & 0 & 0 \\
\end{matrix}
\]

\[
\begin{matrix}
\hline
j = 7/2, ... \\
\end{matrix}
\]

On comparing such characteristics for \( D^j_{m,\pm1/2}(\phi, \theta, 0) \) and \( D^j_{-m,\pm1/2}(\phi, \theta, 0) \), we can immediately conclude that these sets of \( D \)-functions provide us with the bases in different functional spaces \( \{ F^{eg=0}(\theta, \phi) \} \) and \( \{ F^{eg\neq0}(\theta, \phi) \} \) (all various values of the parameter \( eg \) lead to different functional spaces as well). Each of those spaces is characterized by its own behavior at limiting points, which is irreconcilable with that of any other space.

These peculiarities are rather crucial on their implications. For example, that difference leads to some obvious problems referring to the basic superposition principle of quantum mechanics. Indeed, it is understandable that any possible series decompositions of particle wave functions \( \Phi^{eg\neq0} \) by \( \Phi^{eg=0} \) and inversely:

\[
\Phi^{eg}_{\epsilon jm\mu}(t, r, \theta, \phi) = \sum C^{\epsilon JM\delta}_{\epsilon jm\mu} \Phi^0_{\epsilon JM\phi}(t, r, \theta, \phi),
\]

\[
\Phi^0_{\epsilon JM\delta}(t, r, \theta, \phi) = \sum C^{\epsilon jm\mu}_{\epsilon JM\delta} \Phi^{eg}_{\epsilon jm\mu}(t, r, \theta, \phi)
\]
cannot be correct at the whole \( x_3 \)-axis. The latter leads to a very interesting, if not serious, question as to the physical status of the monopole potential. It is matter that conventional (one particle-based) quantum mechanics presupposes tacitly that any quantum object remains intrinsically the same as a certain identified entity when this object is placed into an arbitrary external field. For example, an isolated free electron and an electron in the Coulomb field, in both cases, are represented by the same single entity, \textit{electron}, just situated in the two different conditions. Mathematically, this tacit assumption is expressed as the possibility to exploit together the fundamental superposition principle and presupposedly identity of those Hilbert spaces: \( \{ \Phi^{\text{free}} \} \equiv \{ \Phi^{\text{ext.field}} \} \). So, we always can obtain extensions of any wave functions of the type \( \Phi^{\text{ext.field}} \) in terms of functions \( \Phi^{\text{free}} \), or inversely\textsuperscript{20}.

\textsuperscript{20}For instance, there exists the momentum representation \( \Phi(p) \) for the Coulomb wave functions, which may be considered just as an illustration to the above.
It is easily understandable that limitations imposed by this condition on external fields are quite restrictive, and we could show that many commonly used potentials, referring to real sources, satisfy them. Whereas, evidently, the presupposed existence of a magnetic charge is inconsistent with this basic proposition of the theory. Indeed, at the whole $x_3$ axis, some of particle-monopole functions cannot, in principle, be represented as any linear combinations of free particle functions: those later $\{\Phi^{g=0}(x_3,0,0)\}$ do not contain at all any functions required to describe some representatives from $\{\Phi^{g\neq0}(x_3,0,0)\}$ (see Tables 1-3 and Supplement A).

Furthermore, one might regard the above criterion as, in a sense, a superselection principle yielding the definite separation of any mathematically possible potentials into the two classes: real and not real ones (respectively, not changing and changing the relevant Hilbert spaces). In such terms, the Abelian monopole potential should be thought of as unphysical and forbidden; whereas the non-Abelian potential may be regarded as quite allowable.

In this connection, some more remarks might be added. As a matter of fact, the status of monopoles in physics is, in general, rather peculiar. Indeed, a modern age in understanding electricity and magnetism (EM) was ushered in by Dirac [3] who had argued on the line of quantum mechanical arguments that electric charge $e$ must be quantized as an isolated magnetic charge $g$ exists, thereby he had gave an appreciably significant piece of EM essentials, now known as the Dirac’s (electric charge) quantization condition. Certainly, the electric charge had been quantized. However — to say exactly, from the very beginning, this Dirac’s condition might be interpreted a bit differently — to say the least: one could say that a new introduced quantity (magnetic charge) must be quantized as the electric charge exists.

In any case, the problem is just one: any experimental observation of a magnetic charge, unfortunately or may be luckily, has not been registered to date. Therefore, at the present time we have to count solely on the theoretical investigation of those constructs. Furthermore, for the reason alone of such a consistently invisible character of a magnetic charge, at least as it concerns experiments, there appear grounds for special seeking some rationalization of such a strange, if not enigmatic, and persistent disinclination of the monopoles to be seen experimentally. Even more, it is the time to have searched some possible formally conceptualized grounds for forbidding, in principle, this charge from existence in nature. Therefore, some provisional steps in this direction might be made yet today.

In reference to this, else one peculiarity of the above property of the particle-monopole functions deserves to be especially noticed: the irreconcilable character of monopole-affected and free functions, respectively, might be interpreted in physical terms as the fundamental impossibility to eliminate the monopole influence on the particle wave functions over all space up to infinitely distance points. This property implies a lot of hampering implications, in turn giving rise to some awkward questions. For instance, the question of that kind is: what is the meaning of the relevant scattering theory, if even at infinity itself, some manifestation of the magnetic charge presence does not vanish (just because of the given $\theta, \phi$-dependence). That point finds its natural corollary in giving rise the well-
known difficulties in the relevant scattering theory [17-19,86-94]: the particle-monopole functions, being regarded at the asymptotical infinity (far away from the region of \( r = 0 \)), exhibit such kind of behavior that does not fit it to be appropriate for a free quantum-mechanical particle. So, we cannot get rid of the Abelian monopole effects up to infinitely distant points, and such a property is removed far from what is familiar when a situation is less singular (for instance, of external electric charge presence).

Now, as a way of further contrasting the Abelian and non-Abelian models, we are going to pass on another subject and consider the question of discrete symmetry (restricting ourselves to \( P \)-transformation-involving operations) in both these theories (see also in [1,11-16,76-85]).

In the Abelian case (when \( eg \neq 0 \)), the monopole wave functions cannot be proper functions of the usual space reflection operator for a particle (for definiteness, the bispinor field case is meant). There exists only the following relationship (\( j > j_{\text{min.}} \))

\[
(\hat{P}_{\text{bisp.}} \otimes \hat{P}) \psi_{eg}^{j}(x) = e^{-iet} \mu (-1)^{j+1} \begin{pmatrix} f_{1}(r) D_{m,-eg+1/2}^{j} \\ f_{2}(r) D_{m,eg-1/2}^{j} \\ \mu f_{2}(r) D_{m,eg+1/2}^{j} \\ \mu f_{1}(r) D_{m,-eg-1/2}^{j} \end{pmatrix}
\]

(5.4a)

take notice of the sign ‘minus’ at the parameter \( eg \). By contrast, the required relation for free wave functions occurs

\[
(\hat{P}_{\text{bisp.}} \otimes \hat{P}) \Phi_{eg}^{0}(x) = \delta (-1)^{j+1} \Phi_{eg}^{0}(x).
\]

(5.4b)

It should be emphasized that a certain, diagonalized on the functions \( \psi_{eg}^{j} \), discrete operator can be obtained through multiplying the usual \( P \)-inversion bispinor operator by the formal one \( \hat{\pi}_{\text{Abel.}} \) that affects the \( eg \)-parameter in the wave functions as follows: \( \hat{\pi}_{\text{Abel.}} \psi_{eg}^{j}(x) = \psi_{eg}^{j}(x) \). Thus, we have

\[
\hat{M} = \hat{\pi}_{\text{Abel.}} \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}, \quad \hat{M} \psi_{eg}^{j}(x) = \mu (-1)^{j+1} \psi_{eg}^{j}(x)
\]

(5.4c)

but, as may be seen, the latter fact does not allow us to obtain any \( M \)-parity selection rules. Actually, a matrix element for some physical observable \( \mathcal{G}(x) \) is to be

\[
\int \psi_{eg}^{j}(x) \mathcal{G}(x) \psi_{eg}^{j'}(x) dV \equiv \int r^{2} dr \int f(\vec{x}) d\Omega.
\]

First we examine the case \( eg = 0 \), in order to compare it with the situation at \( eg \neq 0 \). Let us relate \( f(\vec{x}) \) with \( f(\vec{x}) \). Considering the relation (and the same with \( J' M' \delta' \))

\[
\Phi_{eg}^{0}(x) \equiv (-1)^{j+1} \hat{P}_{\text{bisp.}} \Phi_{eg}^{0}(x)
\]

(5.5a)

we get

\[
f(\vec{x}) = \delta \delta' (-1)^{j+j'+1} \Phi_{eg}^{0}(\vec{x}) \left[ \hat{P}_{\text{bisp.}} \mathcal{G}(\vec{x}) \hat{P}_{\text{bisp.}} \right] \Phi_{eg}^{0}(\vec{x}).
\]

(5.5b)
Thus, if the quantity $\hat{G}^0(x)$ obeys the equation

$$\hat{P}^+_{\text{bisp.}} \hat{G}^0(-x) \hat{P}_{\text{bisp.}} = \omega^0 \hat{G}^0(x)$$

(5.5c)

here $\omega^0$ defined to be $+1$ or $-1$ relates to the scalar and pseudoscalar, respectively, then the relationship (5.5b) comes to

$$f(-x) = \omega \delta \delta' (-1)^{j+j'+1} f(x)$$

(5.5d)

that generates the well-known $P$-parity selection rules.

In contrast to everything just said, the situation at $eg \neq 0$ is completely different because any equality in the form (5.5a) or (5.5b) does not appear there; instead, the relations (5.4a) and (1.12b) only occur. So, there not exist any $M$-parity selection rules in the presence of the Abelian monopole. In accordance with this, for instance, the expectation value for the usual operator of space coordinates $x$ need not equal zero and it follows this (see, for example, in [79-85]).

Now, let us return to the non-Abelian problem when there exists the relationship of required form:

$$\Psi_{ejmd}(x) = (\sigma^2 \otimes \hat{P}_{\text{bisp.}}) \delta (-1)^{j+1} \Psi_{ejmd}(x)$$

(5.6a)

owing to the $N$-reflection symmetry; so that

$$f(-x) = \delta \delta' (-1)^{j+j'} \Psi_{ejmd}(x) \left[ (\sigma^2 \otimes \hat{P}_{\text{bisp.}})^+ \hat{G}(-x) (\sigma^2 \otimes \hat{P}_{\text{bisp.}}) \right] \Psi_{ej'm'd}(x).$$

(5.6b)

If a certain quantity $\hat{G}(x)$ depending on isotopic coordinates

$$\hat{G}(x) = \left( \frac{\hat{g}_{11}(x)}{\hat{g}_{21}(x)} \frac{\hat{g}_{12}(x)}{\hat{g}_{22}(x)} \right) \otimes \hat{G}^0(x)$$

obeys the following structural condition (what is the definition of the composite scalar and pseudoscalar)

$$(\sigma^2 \otimes \hat{P}_{\text{bisp.}})^+ \hat{G}(-x) (\sigma^2 \otimes \hat{P}_{\text{bisp.}}) = \Omega \hat{G}(x)$$

(5.6c)

where $\Omega$ defined to be $+1$ or $-1$, the relationship (5.6b) converts into

$$f(-x) = \Omega \delta \delta' (-1)^{j+j'} f(x)$$

(5.6d)

that results in the evident $N$-parity selection rules. For instance, applying these rules to $\hat{G}(x) \equiv x$, we found out

$$< \Psi_{ejmd}(x) | x | \Psi_{ejmd}(x) > \sim [ 1 - \delta^2 (-1)^{2j} ] \equiv 0.$$  

6. Some additional remarks on $\hat{N}_A$ operator in Cartesian gauge

Now we proceed further with studying the reflection symmetry for the fermion-monopole system and consider the question of explicit form of the discrete operator $\hat{N}_A$ in several other gauges. All our calculations so far (in sections 3-5) have been tied with the Schwinger
isotopic frame, now let us turn to the unitary Dirac and the Cartesian (both isotopic) gauges. Simple calculations result in

\[
\hat{\pi}_{D,\Delta}^C = \begin{pmatrix}
0 & -i (a - i b) e^{-i\phi} \\
+i (a + i b) e^{+i\phi} & 0
\end{pmatrix},
\]

(6.1a)

and the corresponding wave functions

\[
\Phi_{\epsilon jm\delta}^{C,\Delta} = \frac{e^{-i\epsilon t}}{r} \left[ e^{+i\phi/2} T_{+1/2} \otimes F + \Delta \delta e^{-i\phi/2} T_{-1/2} \otimes G \right],
\]

(6.2a)

\[
\Psi_{\epsilon jm\delta}^{C,\Delta}(x) = \frac{e^{-i\epsilon t}}{r} \left[ \left( \cos \theta/2 \ e^{+i\phi/2} \right) \otimes F + \Delta \delta \left( \sin \theta/2 \ e^{-i\phi/2} \right) \otimes G \right].
\]

(6.2b)

It is convenient for our further work to rewrite the expression (6.1b) for the matrix \( \hat{\pi}_{\Delta}^C \) in the form

\[
\hat{\pi}_{\Delta}^C = \left[ -i \frac{\Delta + \Delta^{-1}}{2} + i \frac{\Delta - \Delta^{-1}}{2} \left( \sigma \tilde{n}_{\theta,\phi} \right) \right].
\]

(6.3)

Setting \( \Delta = 1 \), it follows from (6.3) that \( \hat{\pi}_{\Delta}^C = -i I \). Therefore, the above \( N \)-reflection operator in the Cartesian gauge takes on the form (at \( \Delta = 1 \))

\[
\hat{N}^C = (-i I) \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}
\]

(6.4)

thus, the \( \hat{N}^C \) does not involve any transformation on the isotopic coordinates. In other words, the ordinary \( P \)-reflection operator for a bispinor field can be diagonalized upon the composite (doublet) wave functions. But this fact is not of primary or conceptualizable importance; mainly because it is not gauge invariant. Therefore, relying on this relationship, we cannot come to the conclusion that non-Abelian problem of monopole discrete symmetry amounts to the Abelian (monopole free) problem of discrete symmetry.

Furthermore, that non-Abelian theory’s discrete symmetry features have no relationship to the Abelian monopole case. The clue to understanding this is that the Abelian fermion-monopole wave functions \( F(x) \) and \( G(x) \) (see in (6.2b)) are represented in the non-Abelian functions \( \Psi_{\epsilon jm\delta}^{C,\Delta}(x) \) only as constructing elements

\[
\Psi_{\epsilon jm\delta}^{C,\Delta}(x) = \frac{e^{-i\epsilon t}}{r} \left[ T_{+1/2} \otimes \left( \cos \theta/2 \ e^{-i\phi/2} F(x) + \Delta \delta \sin \theta/2 \ e^{-i\phi/2} G(x) \right) \right. +
\]

\[
T_{-1/2} \otimes \left( \sin \theta/2 \ e^{+i\phi/2} F(x) + \Delta \delta \cos \theta/2 \ e^{+i\phi/2} G(x) \right) \bigg]
\]

(6.5)

but the multiplying functions at \( (T_{+1/2} \otimes) \) and \( (T_{-1/2} \otimes) \), in themselves, cannot be obtained by any \( U(1) \)-gauge transformation from the real Abelian particle-monopole functions \( F(x) \) and \( G(x) \), and we should set a higher value on this than on the form of \( \hat{N}^C \) in (6.4). In other words, at the price of the gauge transformation used above (\( S \rightarrow C_\cdot \)),
we only have carried the non-null action upon isotopic coordinates (generated by \(N^S\)-inversion) into a null action upon these coordinates and a concomitant vanishing of all individual Abelian-like qualities (belonging solely to the \(F\) and \(G(x)\)).

The relation (6.5) is interesting from another standpoint: it is convenient to produce some factorizations of the doublet-fermion functions by the Abelian fermion functions and the isotopic vectors \(T_{\pm 1/2}\). Indeed, taking into account the known recursive relations [66]

\[
\cos \frac{\beta}{2} e^{i(\alpha + \gamma)/2} D_{m+1/2,m'+1/2}^j(\alpha, \beta, \gamma) = \sqrt{(j + m + 1/2)(j + m' + 1/2)} \frac{D_{m,m'}^{j-1/2}(\alpha, \beta, \gamma) + D_{m,m'}^{j+1/2}(\alpha, \beta, \gamma)}{2j+1} \]

\[
\sin \frac{\beta}{2} e^{i(\alpha - \gamma)/2} D_{m+1/2,m'-1/2}^j(\alpha, \beta, \gamma) = -\sqrt{(j + m + 1/2)(j - m' + 1/2)} \frac{D_{m,m'}^{j-1/2}(\alpha, \beta, \gamma) + D_{m,m'}^{j+1/2}(\alpha, \beta, \gamma)}{2j+1} \]

the representation (6.5) can be transformed into the form:

\[
\Psi_{\epsilon jm\delta}^C(x) = \frac{e^{-iec}}{r} \times
\]

\[
\begin{bmatrix} T_{+1/2} \otimes \frac{\sqrt{j + m}}{2j+1} \left( \begin{array}{c} (\sqrt{j+1} f_1 + \delta \Delta \sqrt{j} f_4) \\ + \delta \Delta (\sqrt{j+1} f_2 + \delta \Delta^{-1} \sqrt{j+1} f_3) \\ - \delta \Delta (\sqrt{j+1} f_1 - \delta \Delta^{-1} \sqrt{j+1} f_4) \end{array} \right) \end{bmatrix} (D_{-m+1/2}^{j-1/2}) +
\]

\[
\begin{bmatrix} T_{+1/2} \otimes \frac{\sqrt{j - m}}{2j+1} \left( \begin{array}{c} (\sqrt{j} f_1 - \delta \Delta \sqrt{j+1} f_4) \\ (\sqrt{j+1} f_2 - \delta \Delta \sqrt{j} f_3) \\ - \delta \Delta (\sqrt{j+1} f_1 - \delta \Delta^{-1} \sqrt{j+1} f_4) \end{array} \right) \end{bmatrix} (D_{-m+1/2}^{j+1/2}) +
\]

\[
\begin{bmatrix} T_{-1/2} \otimes \frac{\sqrt{j - m}}{2j+1} \left( \begin{array}{c} (-\sqrt{j+1} f_1 + \delta \Delta \sqrt{j} f_4) \\ -\delta \Delta (-\sqrt{j} f_2 + \delta \Delta \sqrt{j+1} f_3) \\ - \delta \Delta (-\sqrt{j+1} f_1 + \delta \Delta^{-1} \sqrt{j+1} f_4) \end{array} \right) \end{bmatrix} (D_{-m-1/2}^{j-1/2}) +
\]

\[
\begin{bmatrix} T_{-1/2} \otimes \frac{\sqrt{j + m}}{2j+1} \left( \begin{array}{c} (\sqrt{j} f_1 + \delta \Delta \sqrt{j+1} f_4) \\ (\sqrt{j+1} f_2 + \delta \Delta \sqrt{j} f_3) \\ + \delta \Delta (\sqrt{j+1} f_1 + \delta \Delta^{-1} \sqrt{j+1} f_4) \end{array} \right) \end{bmatrix} (D_{-m-1/2}^{j+1/2}) \right] (6.6)
\]

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Correspondingly, using the notation according to the representation (6.6) can be rewritten as follows with the use of the following symbolic designation (here there are four different possibilities)

\[
D^{j \pm 1/2}_{-m \pm 1/2} = \begin{pmatrix}
D^{j+1/2}_{m+1/2,-1/2} \\
D^{j+1/2}_{m+1/2,+1/2} \\
D^{j+1/2}_{m-1/2,-1/2} \\
D^{j+1/2}_{m-1/2,+1/2}
\end{pmatrix}.
\]

Now, remembering the equality \((a^2 + b^2) = 1\), let us introduce a new variable \(A\) defined by \(\cos A = a\) and \(\sin A = b\), so the above operator \(\hat{\pi}_A^C\) is expressed as

\[
\hat{\pi}_A^C = (-i) \exp\{ iA \, \vec{\sigma} \, \vec{n}_{\theta,\phi} \} \tag{6.7}
\]

here the \(A\) is a complex parameter due to the quantities \(a\) and \(b\) are complex ones. Correspondingly, using the notation according to

\[
\begin{align*}
\sqrt{j + 1} f_1 + \delta e^{iA} \sqrt{j} f_4 &= K_A^A, \\
\sqrt{j} f_2 + \delta e^{iA} \sqrt{j + 1} f_3 &= L_A^A, \\
\sqrt{j} f_1 - \delta e^{iA} \sqrt{j + 1} f_4 &= M_A^{-A}, \\
\sqrt{j + 1} f_2 - \delta e^{iA} \sqrt{j} f_3 &= N_A^{-A}
\end{align*}
\]

the representation (6.6) can be rewritten as follows

\[
\begin{align*}
\Psi^{C,A}_{\epsilon j m \delta} (x) &= \frac{e^{-ist}}{r} \times \\
&\left[ T_{+1/2} \otimes \frac{\sqrt{j + m}}{2j + 1} \begin{pmatrix}
K_A^{+A} \\
L_A^{+A} \\
\delta e^{iA} L_A^{-A} \\
\delta e^{iA} K_A^{-A}
\end{pmatrix} \begin{pmatrix}
D_{+m+1/2}^{-1/2}
\end{pmatrix} \right] + \\
&\left[ T_{+1/2} \otimes \frac{\sqrt{j - m + 1}}{2j + 1} \begin{pmatrix}
M_A^{+A} \\
N_A^{+A} \\
-\delta e^{iA} N_A^{-A} \\
-\delta e^{iA} M_A^{-A}
\end{pmatrix} \begin{pmatrix}
D_{+m+1/2}^{+1/2}
\end{pmatrix} \right] + \\
&\left[ T_{-1/2} \otimes \frac{\sqrt{j - m}}{2j + 1} \begin{pmatrix}
-K_A^{+A} \\
-L_A^{+A} \\
\delta e^{iA} L_A^{-A} \\
\delta e^{iA} K_A^{-A}
\end{pmatrix} \begin{pmatrix}
D_{-m-1/2}^{-1/2}
\end{pmatrix} \right] + \\
&\left[ T_{-1/2} \otimes \frac{\sqrt{j + m + 1}}{2j + 1} \begin{pmatrix}
M_A^{+A} \\
N_A^{+A} \\
\delta e^{iA} N_A^{-A} \\
\delta e^{iA} M_A^{-A}
\end{pmatrix} \begin{pmatrix}
D_{-m-1/2}^{+1/2}
\end{pmatrix} \right]. \tag{6.8}
\end{align*}
\]

When \(A = 0\), then all the formulas in (6.8) will be significantly simplified, so that the familiar sub-structure of electronic wave functions with fixed P-parity can easily be seen...
This is what one might expect because, if \( A = 0 \), then the operator \( N_{A=0}^{C} \) does not involve any isotopic transformation. The latter might be a source of some speculation about an extremely significant role of the Abelian \( P \)-symmetry in the non-Abelian model. However, one should remember that a genuinely Abelian fermion \( P \)-symmetry implies both a definite explicit expression for \( P \)-operation and definite properties of the corresponding wave functions. The above decomposition (6.8) of fermion doublet wave functions in terms of Abelian fermion functions and unit isotopic vectors shows that the usual Abelian fermion particle wave functions and non-Abelian doublet ones belong to substantially different classes. (see (6.9) at \( A = 0 \)); so that the Abelian like \( P \)-operation plays only a subsidiary role in forming the whole composite wave functions; besides, as was mentioned above, this role will be completely negated in other isotopic gauges.

By the way, an analogous principle of checking the property of functional space (in particular, as to whether or not the relevant functions are single-valued ones, apart from any possible gauge transformations) might serve as a guideline argument to prevent some serious discussion (if not speculation) on fermion interpretation for bosons as well as boson-like interpretation for fermions [20-23] when, in the Abelian model, the monopole is in effect and the case of half-integer \( eg \)-values is realized. Of course, such possibilities seem striking and attractive for every physicist, however they are correct and quite satisfactory ones only at first glance. Evidently, such monopole-based fermions or bosons being produced from the usual bosons and fermions respectively, turn out to be tied with functional spaces which are absolutely different from those used in reference with the usual (fermion or boson) particles. In addition, the Lorentz group-based transformation characteristics of such new ‘fermions and boson’, in reality, will completely negate their new nature and will be dictated by their old classification assignment.

Also (in the author’s opinion), the vastly discussed producing ‘spin from isospin’ [20-23] belongs to the same class of striking but hardly realized possibilities. Many arguments against might be formulated; the simplest one is as follows: the correct understanding of the meaning of Lorentzian spin presupposes quite definite properties of this characteristic under Lorentz group transformations; however, such a ‘spin’, being produced from isospin, does not obey these regulations. Else one theoretical criterion might be given: placing such an object into the background of any Rimannian (curved) space-time. Evidently, the gravitational field will ignore any boson-fermion inverting based on the above mechanism.

7. Free parameter at discrete symmetry and \( N_{A} \)-parity selection rules

Now we proceed with analyzing the totality of the discrete operators \( \hat{N}_{A} \), which all are suitable for separation of variables. What is the meaning of the parameter \( A \)? In other words, how can this \( A \) manifest itself and why does such an unexpected ambiguity exist?

The \( A \) fixes up one of the complete set of operators \{ \( i \, \partial_{t}, \, \vec{J}^{2}, \, J_{3}, \, \hat{N}_{A}, \, \hat{K} \) \}, and correspondingly this \( A \) also labels all basic wave functions. It is obvious, that this parameter \( A \) can manifest itself in matrix elements of physical quantities. To see this, it suffices to look at the general structure of the relevant expectation value of those
observables\(^{21}\) (the \(\epsilon, J, M\) are omitted):

\[
\tilde{G} = \langle \Psi^A_{J\mu} | \hat{G} | \Psi^A_{J\mu} \rangle = \langle T_{+1/2} \otimes \Phi^{(+)}_{\mu}(x) | \hat{G} | T_{+1/2} \otimes \Phi^{(+)}_{\mu}(x) \rangle +
\]

\[
| e^{iA} | < T_{-1/2} \otimes \Phi^{(-)}_{\mu}(x) | \hat{G} | T_{-1/2} \otimes \Phi^{(-)}_{\mu}(x) \rangle +
\]

\[
2 \delta \mu \text{ Re } [ e^{iA} < T_{+1/2} \otimes \Phi^{(+)}_{\mu}(x) | \hat{G} | T_{-1/2} \otimes \Phi^{(-)}_{\mu}(x) \rangle ] . \tag{7.1}
\]

If such a \(\hat{G}\) has the diagonal isotopic structure

\[
\hat{G}(x) = \begin{pmatrix}
\hat{g}_{11}(x) & 0 \\
0 & \hat{g}_{22}(x)
\end{pmatrix}
\]

then the third term in (7.1) vanishes and the matrix element only depends on \(| e^{iA} |\).

As a simple example let us consider a new form of the above-mentioned selection rules depending on the \(A\)-parameter. Now, the matrix element examined is

\[
\int \tilde{\Psi}_{eJ\bar{M}\delta\mu}(x) \hat{G}(x) \Psi^A_{eJ\prime\bar{M}'\delta'\mu'}(x) dV \equiv \int r^2 dr \int f^A(x) d\Omega
\]

then

\[
f^A(-\vec{x}) = \delta \delta' (-1)^{J+J'} \tilde{\Psi}_{eJ\bar{M}\delta\mu}(x) \times
\]

\[
\left[ (a^* \sigma^1 + b^* \sigma^2) \otimes \hat{P}_{\text{bisp.}} \hat{G}(-\vec{x}) (a\sigma^1 + b\sigma^2) \otimes \hat{P}_{\text{bisp.}} \right] \Psi^A_{eJ\prime\bar{M}'\delta'\mu'}(\vec{x}) . \tag{7.3a}
\]

If this \(\hat{G}\) obeys the condition

\[
\left[ (a^* \sigma^1 + b^* \sigma^2) \otimes \hat{P}_{\text{bisp.}} \hat{G}(-\vec{x}) [(a\sigma^2 + b\sigma^1) \otimes \hat{P}_{\text{bisp.}} \right] = \Omega^A \hat{G}(\vec{x}) \tag{7.3b}
\]

which is equivalent to

\[
\begin{pmatrix}
e^{i(A-A^*)} \hat{g}_{22}(-\vec{x}) & e^{-i(A+A^*)} \hat{g}_{21}(-\vec{x}) \\
e^{i(A+A^*)} \hat{g}_{12}(-\vec{x}) & e^{-i(A-A^*)} \hat{g}_{11}(-\vec{x})
\end{pmatrix} \otimes
\]

\[
\left[ \hat{P}_{\text{bisp.}} \hat{G}^0(-\vec{x}) \hat{P}_{\text{bisp.}} \right] = \Omega^A \begin{pmatrix}
\hat{g}_{11}(\vec{x}) & \hat{g}_{12}(\vec{x}) \\
\hat{g}_{21}(\vec{x}) & \hat{g}_{22}(\vec{x})
\end{pmatrix} \otimes \hat{G}(\vec{x}) \tag{7.3c}
\]

where \(\Omega^A = +1\) or \(-1\), then the relationship (7.3a) comes to

\[
f^A(-\vec{x}) = \Omega^A \delta \delta' (-1)^{J+J'} f^A(\vec{x}) . \tag{7.3d}
\]

Taking into account (7.3d), we bring the matrix element’s integral above to the form

\[
\int \tilde{\Psi}_{eJ\bar{M}\delta\mu}(x) \hat{G}(x) \Phi^A_{eJ'\bar{M}'\delta'\mu'}(x) dV = \left[ 1 + \Omega^A \delta \delta' (-1)^{J+J'} \right] \int_{V_{1/2}} f^A(\vec{x}) dV \tag{7.4a}
\]

\(^{21}\)To be exact, any variations in this \(A\) will lead to alteration in normalization conditions for the relevant wave function \(\Psi^A_{eJ\bar{M}\delta\mu}\) (see in Sec.9), so that this circumstance should be taken into account; but for simplicity, we pass over those alterations.
where the integration in the right-hand side is done on the half-space. This expansion provides the following selection rules:

\[ ME \equiv 0 \quad \leftrightarrow \quad \left[ 1 + \Omega^A \delta \delta' (-1)^{J+J'} \right] = 0 . \quad (7.4b) \]

It is to be especially emphasized that the quantity \( \Omega^A \), defined to be +1 or −1, is not the same as the analogous that \( \omega \) in (5.6d). These \( \omega \) and \( \Omega^A \) involve their own particular limitations on composite scalar or pseudoscalar because they imply respective (and rather specific) configurations of their isotopic parts, obtained by delicate fitting all the quantities \( \hat{g}_{ij} \). Therefore, each of those \( A \) will generate its own distinctive selection rules.

8. Existence of the parameter \( A \) and isotopic chiral symmetry

Where does this \( A \)-ambiguity come from and what is the meaning of this parameter \( A \)? To proceed further with this problem, one is to realize that the all different values for \( A \) lead to the same whole functional space; each fixed value for \( A \) governs only the basis states \( \Psi_{eJ\delta\mu}(x) \) associated with \( A \), but with no change in the whole space. Connection between any two sets of functions \( \{ \Psi(x) \}^A \) and \( \{ \Psi(x) \}^{A'} \) is characterized by

\[
\Psi_{eJ\delta\mu}^{A'} = U_S(A' - A) \Psi_{eJ\delta\mu}^A(x) , \quad U_S(A' - A) = e^{-iA} \begin{pmatrix} e^{iA} & 0 \\ 0 & e^{iA'} \end{pmatrix} \otimes I . \quad (8.1a)
\]

Besides, it is readily verified that the operator \( \hat{N}_A^S \) (depending on \( A \)) can be obtained from the operator \( \hat{N}_S^S \) as follows

\[
\hat{N}_A^S = U_S(A) \hat{N}_S \hat{U}^{-1}(A) . \quad (8.1b)
\]

The matrix \( U_S(A' - A) \) is so simple only in the Schwinger basis; after translating that into Cartesian one we will have

\[
\Psi_{eJ\delta\mu}^{A'} = U_C(A' - A) \Psi_{eJ\delta\mu}^A(x) , \quad (8.1c)
\]

\[
S_C = \frac{1}{\Delta} \begin{pmatrix} (\Delta \cos^2 \theta/2 + \Delta' \sin^2 \theta/2) & \frac{1}{2} (\Delta - \Delta') \sin \theta e^{-i\phi} \\
\frac{1}{2} (\Delta - \Delta') \sin \theta e^{+i\phi} & (\Delta' \cos^2 \theta/2 + \Delta \sin^2 \theta/2) \end{pmatrix} \otimes I
\]

and \( S_C(A' - A) \) satisfies the equation

\[
\hat{N}_A^C = U_C(A) \hat{N}_C \hat{U}^{-1}_{Cart}(A) . \quad (8.1d)
\]

In connection with everything said above on parity selection rules and just ‘unexpectedly’ established relationship (8.1b) (or (8.1d)), we need to think this over again before finding a conclusive answer. Let us begin from some generalities. As well known, when analyzing any Lie group problems (or their algebra’s) there indeed exists a concept of equivalent representations: \( U M_k U^{-1} = M'_k \rightarrow M_k \sim M'_k \). In this context, the two
sets of operators \( \{ J_i^S, \hat{N}_A^S \} \) and \( \{ J_i^S, \hat{N}_A^S \} \) provide basically just the same representation of the \( O(3, R) \)-algebra

\[
\{ J_i^S, \hat{N}_A^S \} = U_S(A) \{ J_i^S, \hat{N}_A^S \} U_S^{-1}(A).
\] (8.2a)

The totally different situation in the context of the use of those two operator sets as physical observables concerning the system with the fixed Hamiltonian

\[
\{ \hat{J}_2^2, J_3^S, \hat{N}_A^S \} \hat{H} \quad \text{and} \quad \{ \hat{J}_2^2, J_3^S, \hat{N}_A^S \} \hat{H}.
\] (8.2b)

Actually, in this case the two operator sets represent different observables at the same physical system: both of them are followed by the same Hamiltonian \( \hat{H} \) and also lead to the same functional space, changing only its basis vectors \( \{ \Psi_{\epsilon JM\delta\mu}(x) \}^A \). Moreover, in the quantum mechanics it seems always possible to relate two arbitrary complete sets of operators by some unitary transformation:

\[
\{ \hat{X}_\mu, \mu = 1, \ldots \} \hat{H} \rightarrow \{ \hat{Y}_\mu, \mu = 1, \ldots \} \hat{H}, \{ \Phi_{x_1 \ldots x_s} \} \rightarrow \{ \Phi_{y_1 \ldots y_s} \}.
\]

But arbitrary transformations \( U \) cannot generate, through converting \( U \{ \hat{X}_\mu \} U^{-1} = \hat{Y}_\mu \), a new complete set of variables; instead, only some Hamiltonian symmetry’s operations are suitable for this: \( U \hat{H} U^{-1} = \hat{H} \).

In this connection, we may recall a more familiar situation for Dirac massless field [95,96]. The wave equation for this system was earlier mentioned (see Sec. 2) and that has the form

\[
i \sigma^a(x) (\partial_a + \Sigma_a) \xi(x) = 0, \quad i \sigma^a(x) (\partial_a + \Sigma_a) \eta(x) = 0.
\] (8.3a)

If the function \( \Phi(x) = (\xi(x), \eta(x)) \) is subjected to the transformation

\[
\begin{pmatrix}
\xi'(x) \\
\eta'(x)
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & z I
\end{pmatrix} \begin{pmatrix}
\xi(x) \\
\eta(x)
\end{pmatrix}
\] (8.3b)

where \( z \) is an arbitrary complex number, then the new function \( \Phi'(x) = (\xi'(x), \eta'(x)) \) satisfies again the equation in the form (8.3a). This manifests the Dirac massless field’s symmetry with respect to the transformation

\[
\hat{H}' = U \hat{H} U^{-1} = \hat{H}, \quad \Phi'(x) = U \Phi(x).
\] (8.3c)

The existence of the symmetry raises the question as to whether this symmetry affects determination of complete set of diagonalized operators and constructing spherical wave solutions. These solutions, conformed to diagonalizing the usual bispinor \( P \)-inversion operator, in addition to \( j_2^2 \) and \( j_3 \), are as in (5.4a) at \( eg = 0 \). In the same time, other spherical solutions, together with corresponding diagonalized discrete operator, can be produced:

\[
\Phi_{\epsilon j m\delta}^\pm = \frac{e^{-i\epsilon t}}{r} \left( \begin{array}{c}
f_1 D^j_{-m,-1/2} \\
f_2 D^j_{-m,+1/2} \\
z \delta f_2 D^j_{-m,-1/2} \\
z \delta f_1 D^j_{-m,+1/2}
\end{array} \right),
\] (8.4a)
$$U \left( \hat{P}_{\text{sph.}} \otimes \hat{P} \right) U^{-1} = \left[ \frac{1}{2} \left( z + \frac{1}{z} \right) (-\gamma^5 \gamma^1) + \frac{1}{2} \left( z - \frac{1}{z} \right) (-\gamma^1) \right] \otimes \hat{P}. \quad (8.4b)$$

Introducing another complex variable $A$ instead of the parameter $z$: $z = (\cos A + i \sin A) = e^{iA}$; so that the operator from (8.4b) is rewritten in the form

$$(\cos A + i \sin A \gamma^5) (-\gamma^5 \gamma^1) \otimes \hat{P} \equiv e^{+iA\gamma^5} \hat{P}_{\text{bisp.}} \otimes \hat{P} \quad (8.4c)$$

(8.3b) may be expressed as follows

$$\Phi'(x) = e^{+iA/2} e^{i\gamma^5 A/2} \Phi(x) \quad (8.4d)$$

Evidently, that translation of the basis of spherical tetrad into Cartesian tetrads will preserve the general structure of (8.4c): $e^{+iA\gamma^5} \hat{P}_{\text{Cart.}} \otimes \hat{P}$, since the gauge matrix $S(k(x), k^*(x))$ and matrix $\gamma^5$ are commutative with each other. In contrast to this, translation of the isotopic Schwinger frame into the Cartesian that does change the form $\hat{N}_A$: the initial one is

$$\hat{N}_A^S = (e^{-iA \sigma^3 n^S}) \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P} \quad (8.5a)$$

and the finishing form is

$$\hat{N}_A^C = (-i) \exp \left[ -i A \sigma^3 \tilde{n}_{\theta, \phi} \right] \otimes \hat{P}_{\text{bisp.}} \otimes \hat{P}. \quad (8.5b)$$

The appearance of this dependence on variables $\theta, \phi$ comes from noncommutation of the gauge transformation $B(\theta, \phi)$ and matrix $\sigma^3$ (the latter plays the role of $\gamma^5$ in case of Abelian chiral symmetry (8.4d)).

The transformation $U(A)$, after translating it to the Cartesian basis (see (8.1c)), can be brought to the form

$$U^C(A) = \left[ \frac{1 + e^{iA}}{2} + \frac{1 - e^{iA}}{2} \sigma \tilde{n}_{\theta, \phi} \right]. \quad (8.6)$$

Separating out the factor $e^{iA/2}$ in the right-hand side of this formula, we can rewrite the $U^C$ in the form

$$U^C = e^{iA/2} \exp \left[ -i \frac{A}{2} \sigma \tilde{n}_{\theta, \phi} \right]$$

where the second factor lies in the (local) spinor representation of the 3-dimensional complex rotational group $SO(3.C)$. This matrix provides a very special transformation upon the isotopic fermion doublet and can be thought of as an analogue of the Abelian chiral symmetry transformation; it may be also termed as the transformation of isotopic (complex) chiral symmetry. This symmetry leads to the $A$-ambiguity (8.5) and permits to choose an arbitrary reflection operator from the totality $\{\hat{N}_A\}$. 

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9. Complex values of the $A$ and interplay between the quantum mechanical superposition principle and self-conjugacy requirement

In this section, let us look closely at some qualitative peculiarities of the above considered $A$-freedom placing special notice to the division of $A$-s into the real and complex values. It is convenient to work at this matter in the Schwinger unitary basis. Recall that the $A$-freedom tell us that simultaneously with $\hat{H}, \hat{j}_2^2, \hat{j}_3$, else one discrete operator $\hat{N}_A$, that depends generally on a complex number $A$, can be diagonalized on the wave functions. Correspondingly, the basis functions associated with the complete set ($\hat{H}, \hat{j}_2^2, \hat{j}_3, \hat{N}_A$) besides being certain determined functions of the relevant quantum numbers ($\epsilon, j, m, \delta$), are subject to the $A$-dependence.

In other words, all different values of this $A$ lead to different quantum-mechanical bases of the system. There exists a set of possibilities, but one can relate every two of them by means of a respective linear transformation. For example, the states $\Psi_{\epsilon jm\delta}(x)$ decompose into the following linear combinations of the initial states $\Psi_{\epsilon jm\delta=0}(x)$ (further, this $A=0$ index will be omitted):

$$\Psi_{\epsilon jm\delta}(x) = \left[ \frac{1 + \delta e^{iA}}{2} \Psi_{\epsilon jm,+1} + \frac{1 - \delta e^{iA}}{2} \Psi_{\epsilon jm,-1} \right]. \quad (9.1)$$

One should give heed to that, no matter what an $A$ is (either real or complex one), the new states (9.1), being linear combinations of the initial states, are permissible as well as old ones. This added aspect of the allowance of the complex values for $A$ conforms to the quantum-mechanical superposition principle: the latter presupposes that arbitrary complex coefficients $c_i$ in a linear combination of some basis states $\Sigma c_i \Psi_i$ are acceptable.

However, an essential and subtle distinction between real and complex $A$-s comes straightforward to light as we turn to the matter of normalization and orthogonality for $\Psi_{\epsilon jm\delta}(x)$. An elementary calculation gives

$$< \Psi_{\epsilon jm,\delta}^A | \Psi_{\epsilon jm,\delta}^A > = \frac{1 + e^{i(A-A^*)}}{2} \Psi_{\epsilon jm,+1} ; \quad < \Psi_{\epsilon jm,\delta}^A | \Psi_{\epsilon jm,-\delta}^A > = \frac{1 - e^{i(A-A^*)}}{2} \quad (9.2)$$

i.e. if $A \neq A^*$ then the normalizing condition for $\Psi_{\epsilon jm\delta}(x)$ does not coincide with that for $\Psi_{\epsilon jm\delta}(x)$, and what is more, the states $\Psi_{\epsilon jm,-1}(x)$ and $\Psi_{\epsilon jm,+1}(x)$ are not mutually orthogonal. The latter means that we face here the non-orthogonal basis in Hilbert space and the pure imaginary part of the $A$ plays a crucial role in the description of its non-orthogonality property.

The oblique character of the basis $\Psi_{\epsilon jm\delta}(x)$ (if $A \neq A^*$ ) exhibits its very essential qualitative distinction from perpendicular one for $\Psi_{\epsilon jm\delta}(x)$. However, those specific bases in quantum mechanics, though not being of very common use and having a number of peculiar features, are allowed to be exploited in conventional quantum theory. Even more, in a sense, the existence itself of the non-orthogonal bases in the Hilbert space represents
a direct consequence of the quantum-mechanical superposition principle.  

Up to this point, the complex \( A \)-s seem to be good as well as the real ones. Now, it is the moment to point to some clouds handing over this part of the subject. Indeed, as readily verified, the operator \( \hat{N}_A \) does not represent a self-conjugated (self-adjoint) one: 

\[
< \hat{N}_A \Phi(x) \mid \Psi(x) > = < \Phi(x) \mid e^{i(A-A^*)\sigma_3} \hat{N}_A \Psi(x) > .
\]

It is understandable that this (non-self-conjugacy) property correlates with the above-mentioned nonorthogonality conditions: as well known, a self-conjugated operator entails both real its eigenvalues and the orthogonality of its eigenfunctions. As already noted, the eigenvalues of \( \hat{N}_A \) are real ones and this conforms to the general statement that all inversion-like operators possess the property of the kind: if \( \hat{G}^2 = I \) then \( \lambda \) is a real number, as \( \hat{G} \Phi = \lambda \Phi \).

So, we have got into a point to choose: whether one has to reject all complex values for \( A \) and thereby narrow (if not violate) the one quantum mechanical principle of major generality (of superposition) or whether it is remain to accept all complex \( A \)-s as well as real ones and thereby, in turn, stretch another quantum-mechanical regulation about the self-adjoint character of physical quantities.

We have chosen to accept and look into the second possibility. In the author’s opinion, one should accord the primacy of the general superposition principle over the self-adjointness requirement. In support of this point of view, there exist clear-cut physical grounds.

Indeed, recall the quantum-mechanical status of all inversion-like quantities: they serve always to distinguish two quantum-mechanical states. Moreover, to those quantum variables there not correspond any classical variables; the latter correlates with that any classical apparatus measuring those discrete variables does not exist whatsoever. In contrast to this, one should recollect why the self-adjointness requirement had been imposed on physical quantum operators. The reason is that such operators imply all their eigenvalues to be real. Besides, that limitation on physical quantum variables had been put, in the first place, for quantum variables having their classical counterparts (with the continuum of classical values measured). And after this, in the second place, the discrete quantities such as \( P \)-inversion and like it were tacitly incorporated into a set of self-adjoint mathematical operations, as a natural extrapolation. But one should notice (and the author inclines to place a special emphasis on this) the fact that the single relation \( \hat{N}_A^2 = I \) is completely sufficient that the eigenvalues of \( \hat{N}_A \) to be real. In the light of this, the above-mentioned automatic incorporation of those discrete operators into a set of self-adjoint ones does not seem inevitable. But admitting this, there is a problem to solve: what is the meaning of complex expectation values of such non self-adjoint discrete

\[\text{Footnote 22: For this reason, a prohibition against complex } A \text{-s could be partly a prohibition against the conventional superposition principle too (narrowing it); since all complex values for } A \text{, having forbidden, imply specific limitations on two coefficients in (9.1); but those are not presupposed by the superposition principle itself.}\]

\[\text{Footnote 23: The author is grateful to Dr. E.A.Tolkachev for pointing out that it is so.}\]
operators; since, evidently, the conventional formula $< \Psi | \hat{N}_A | \Psi >$ provides us with complex values. Indeed, let $\Psi(x)$ be $\Psi(x) = [m \Psi_{+1}(x) + n \Psi_{-1}(x)]$, then

$$< \Psi | \hat{N}_A | \Psi > = < m \Psi_{+1}(x) + n \Psi_{-1}(x) | m \Psi_{+1}(x) - n \Psi_{-1}(x) > =$$

$$\left[ (m^*m - n*n) \frac{1 + e^{i(A-A^*)}}{2} + (n^*m - n*m^*) \frac{1 - e^{i(A-A^*)}}{2} \right]. \tag{9.3}$$

Must one be skeptical about those complex $\hat{N}_A$, or treat them as physically acceptable quantities? Let us examine this problem in more detail. It is reasonable to begin with an elementary consideration of the measuring procedure of the $\hat{N} = \hat{N}_{A=0}$. Let a wave function $\Psi(x)$ decompose into the combination

$$\Psi(x) = \left[ e^{i\alpha} \cos^2 \Gamma \Psi_{+1}(x) + e^{i\beta} \sin^2 \Gamma \Psi_{-1}(x) \right] \tag{9.4a}$$

where $\alpha$ and $\beta \in [0, 2\pi]$, and $\Gamma \in [0, \pi/2]$. For the $\hat{N}$ expectation value, one gets

$$\bar{N} = < \Psi | \hat{N} | \Psi > = (-1)^{j+1} (\cos^2 \Gamma - \sin^2 \Gamma) = (-1)^{j+1} \cos 2\Gamma. \tag{9.4b}$$

From (9.4b), one can conclude that $\bar{N}$, after having measured, provides us only with the information about the parameter $\Gamma$ at (9.4a), but does not furnish any information on the phase factors $e^{i\alpha}$ and $e^{i\beta}$ (or their relative factor $e^{i(\alpha-\beta)}$). This interpretation of measured $\bar{N}$ as receptacle of the quite definite information about superposition coefficients in the decomposition (9.4a), represents one and only physical meaning of the $\bar{N}$.

Now, returning to the case of $\hat{N}_A$ operation, one should put an analogous question concerning the $\hat{N}_A$. The material question is: what kind of information about $\Psi(x)$ can be extracted from the measured $\bar{N}_A$. It is convenient to rewrite the above function $\Psi(x)$ as a linear combination of functions $\Psi_{ejm,+1}^A$ and $\Psi_{ejm,-1}^A$. Thus inverting the relations (9.1), we get

$$\Psi_{ejm,+1} = \left[ \frac{1 + e^{-iA}}{2} \Psi_{ejm,+1}^A + \frac{1 - e^{-iA}}{2} \Psi_{ejm,-1}^A \right];$$

$$\Psi_{ejm,-1} = \left[ \frac{1 - e^{-iA}}{2} \Psi_{ejm,+1}^A + \frac{1 + e^{-iA}}{2} \Psi_{ejm,-1}^A \right]$$

and then $\Psi(x)$ takes the form (the fixed quantum numbers $\epsilon, j, m$ are omitted)

$$\Psi(x) = \left[ \left( e^{i\alpha} \cos \Gamma \frac{1 + e^{-iA}}{2} + e^{i\beta} \sin \Gamma \frac{1 - e^{-iA}}{2} \right) \Psi_{+1}^A(x) + \right. \tag{9.5a}$$

$$\left. \left( e^{i\alpha} \cos \Gamma \frac{1 - e^{-iA}}{2} + e^{i\beta} \sin \Gamma \frac{1 + e^{-iA}}{2} \right) \Psi_{+1}^A(x) \right] = \left[ m \Psi_{+1}^A(x) + n \Psi_{-1}^A(x) \right].$$

Although the quantity $A$ enters the expansion (9.5a), but really $\Psi(x)$ only contains three arbitrary parameters: those are $\Gamma, e^{i\alpha},$ and $e^{i\beta}$. After simple calculation one gets

$$\bar{N}_A = < \Psi | \hat{N}_A | \Psi > = (-1)^{j+1} (\rho \cosh g + i\sigma \sinh g), \tag{9.5b}$$

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\[\rho = \cos 2\Gamma \cos f + \sin 2\Gamma \sin f \sin(\alpha - \beta) , \quad \sigma = - \cos 2\Gamma \sin f + \sin 2\Gamma \cos f \sin(\alpha - \beta)\]

where \(f\) and \(g\) are real parameters defined by \(A = f + ig\). Examining this expression, one may single out four particular cases for separate consideration. Those are:

1. \(g = 0, f = 0\), \(\bar{N}_A = (-1)^{j+1} \cos 2\Gamma\) \(\tag{9.6a}\)
here, the \(\bar{N}\) only fixes \(\Gamma\), but \(e^{i(\alpha-\beta)}\) remains indefinite.

2. \(g = 0, f \neq 0\), \(\bar{N}_A = (-1)^{j+1} (\cos 2\Gamma \cos f + \sin 2\Gamma \sin f \sin(\alpha - \beta))\) \(\tag{9.6b}\)
here, the measured \(\bar{N}_A\) does not fix \(\Gamma\) and \((\alpha - \beta)\), but only imposes a certain limitation on both these parameters.

3. \(g \neq 0, f = 0\), \(\bar{N}_A = (-1)^{j+1} (\cos 2\Gamma \cosh g + i \sin 2\Gamma \sin(\alpha - \beta) \sinh g)\) \(\tag{9.6c}\)
here, the \(\bar{N}_A\) determines both \(\Gamma\) and \((\alpha - \beta)\); and thereby this complex \(\bar{N}_A\) is the physical quantity being quite interpreted one. Finally, for the fourth case \((g \neq 0, f \neq 0)\), it follows

4. \(\cos 2\Gamma = (\rho \cos f - \sigma \sin f), \quad \sin 2\Gamma \sin(\alpha - \beta) = (\rho \cos f + \sigma \sin f)\) \(\tag{9.6d}\)
i.e. the complex \(\bar{N}_A\) also gives some information about \(\Gamma\) and \((\alpha - \beta)\) and therefore has character of a physically interpreted quantity.

10. Why \(A\)-freedom is not a gauge one? On logical collision between concepts of gauge and non-gauge symmetries

There exists else one cloud over the subject under consideration\[\text{\footnote{The author is grateful to E. A. Tolkachev, L. M. Tomil’chik, and Ya. M. Shnir for the fruitful discussion on this matter}}\]. Indeed, if the parameter \(A\) is a real number, then the matrix \(S(A)\) translating \(\Psi_{ejmb}(x)\) into \(\Psi_{ejmb}^A(x)\) coincides (apart from a phase factor \(e^{iA/2}\)) with a matrix lying in the group \(SU(2)\): \(\hat{F}(A) \equiv e^{-iA/2}S(A)\) \(\in SU(2)_{\text{loc.}}, \quad \Psi_{ejmb}^A(x) \equiv \hat{F}(A)\Psi_{ejmb}(x) = e^{-iA/2}\Psi_{ejmb}^A(x). \tag{10.1}\)

However, the group \(SU(2)_{\text{loc.}}\) has the status of gauge one for this system. So, else one point of view could be brought to light: one could claim that two functions \(\Psi_{ejmb}(x)\) and \(\Psi_{ejmb}^A(x)\) (at \(A^* = A\)) are related by means of a gauge transformation: and therefore the \(\Psi_{ejmb}^A(x)\) exhibits in other ways the same physical state \(\Psi_{ejmb}(x)\). And further, as a direct consequence, one could insist on the impossibility in principle to observe indeed any physical distinctions between the wave functions \(\Psi_{ejmb}(x)\) and \(\Psi_{ejmb}^A(x)\). If the \(\hat{F}(A)\) transformation gets estimated so, then ultimately one concludes that the above \(N_{A}\)-parity selection rules (explicitly depended on \(A\) which is the real for this case) are only a mathematical fiction since the transformation \(\hat{F}(A)\) is not physically observable.

In this point we run across a problem of material physical significance, in which one could perceive the tense interplay of the quantum-mechanical superposition principle and
the subtle distinction between the concepts of gauge and non-gauge symmetries. In examining
of this phenomenon, one should accord the primacy of careful coordination of foregoing
quantum-mechanical generalities over all other considerations.

So, a question of principle is either the \( \hat{F}(A) \) transformation provides us with a gauge
one or not? The same question can be reformulated as follows: is the fact \( \hat{F}(A) \in SU(2)_{\text{loc.}} \)
sufficient to interpret \( \hat{F}(A) \) exclusively as the transformation with gauge status?

For the moment let us suppose that the \( \hat{F}(A) \) is exclusively a gauge transformation and
no other else. Then all functions \( \Psi^A_{ejm\delta}(x) \equiv \hat{F}(A) \Psi_{ejm\delta}(x) \) represent the same physically
identified state which had been described already by the initial function \( \Psi_{ejm\delta}(x) \). In
other words, the function \( \Psi_{ejm\delta}(x) \) and the following
\[
\Psi^A_{ejm\delta} = \left[ \frac{e^{-iA/2} + \delta e^{iA/2}}{2} \Psi_{ejm,+1} + \frac{e^{-iA/2} - \delta e^{iA/2}}{2} \Psi_{ejm,-1} \right] \tag{10.2}
\]
are both only different representatives of the same physical state. However, such an out-
look is not acceptable on several physical grounds. For clearing up this matter it is
sufficient to have recourse again to the quantum-mechanical superposition principle and
its concomitant requirements. Indeed, the possibility not to accompany the transition of
form \( \Psi_{ejm\delta}(x) \to \Psi^A_{ejm\delta}(x) \) by the similarity transformation on all physical operators ( it
is meant \( \hat{G} \to \hat{G}' = \hat{F}(A)\hat{G}\hat{F}^{-1}(A) \)) is generally supposed to be an essential constituent
part in understanding the conventional superposition principle. Evidently, the essence of
the superposition principle in quantum mechanic consists in just this assertion but not
in a simple fixation and reminding of the linearity property of the matter equation. In
contrast to this, the gauge-like interpretation of the transformation \( \hat{F}(A) \) makes us ac-
company the change \( \Psi_{ejm\delta}(x) \to \Psi^A_{ejm\delta}(x) \) by a similarity transformation on \( \hat{G} \). Thus,
a general outlook prescribing to interpret the transformation \( \hat{F}(A) \) as exclusively a gauge
one, contradicts with regulations stemming from the superposition principle.

One could suggest that any contradiction does not arise here if all genuine observ-
able operators are invariant under the similarity transformation above and all other (not
obeyed it) operators are unphysical fictions. In this connection, let us look more closely
at the character of limitations imposed on \( \hat{G} \) by this condition \( \hat{G} = \hat{F}(A)\hat{G}\hat{F}^{-1}(A) \); an el-
ementary analysis shows that the \( \hat{G} \) is to be of diagonal isotopic structure, i.e. \( \hat{g}_{12}(x) \) and
\( \hat{g}_{21}(x) \) must be equated to zero. All other possibilities for \( \hat{G} \) are associated with the as-
sertion that a certain physical distinction between \( \Psi_{ejm\delta}(x) \) and \( \Psi^A_{ejm\delta}(x) \) is observable.
But there are no grounds for the use of physical operators with this diagonal structure
only, and all the more, for imposing the limitation of the form \( A^* = A \).

Besides, the simultaneous acceptance of operators (with a status of physical ones)
of diagonal isotopic structure only and the added limitation in the form \( A^* = A \), are
indissolubly tied up with a very definite conceiving of the particle doublet itself. Indeed,
this can be physically interpreted as an exclusively additive character of the particle
doublet. In other words, it can be considered as follows: one must, in the first place,
measure separately the quantities \( \hat{g}_{11}(x) \) and \( \hat{g}_{22}(x) \) and after, in the second place, one
can sum up both results. In author’s opinion, having supposed such an attitude for
the particle doublet essence, which in turn presupposes a quite definite measurement procedures as allowed, it is a mystic and fruitless outlook that further we can hope to have found any real correlations of quantum-mechanical nature between components $T_{+1/2} \otimes \Phi^+(x)$ and $T_{-1/2} \otimes \Phi^-(x)$. If such a point of view is recognized as a truly physical one, then they automatically give the understanding of particle doublet as an entity to some mystic powers which are not controlled by the quantum-mechanical mathematical formalism. Instead, in author’s opinion, a truly quantum-mechanical nature of particle doublet conception envisages that some physical operators of non-diagonal form must exist really.

By the way, if one insists on a diagonal form only as possible form of physical operators, one should consider a further dimension to the problem under consideration: what is the meaning of the $A$-freedom. Indeed, let us turn back to (10.1) and (10.2) again and set $e^{iA} = 1$ ($A = \pi$), then these relations, in particular, give

$$\hat{F}(A = \pi) \Psi_{\epsilon jm,-1}(x) \equiv \Psi_{\epsilon jm,+1}(x).$$

The latter shows that if one decides in favor of the gauge character only of the $A$-freedom, then one faces a very strange case. Since two consistently distinguishable thus for and linearly independent of each other solutions $\Psi_{\epsilon jm,-1}(x)$ and $\Psi_{\epsilon jm,+1}(x)$ turn out to be only different representatives of a single invariant state. But then the natural and legitimate question arises: what is the meaning of such a physical situation. Recalling that, generally speaking, the quantum doublet states $\Psi^A_{\epsilon jm\delta\mu}(x)$ (above the number $\mu$ was often omitted) bear five quantum numbers in place of four ones ($\epsilon, j, m, \mu$) in the Abelian case, and that distinction (4 from 5) between Abelian and non-Abelian situations seems to be quite understandable and natural, as a result of addition by hands a new degree of freedom at going over to the non-Abelian case. In the light of this, it is easy to realize that the physical identification of the functions $\Psi_{\epsilon jm,-1,\mu}(x)$ and $\Psi_{\epsilon jm,+1,\mu}(x)$, being effectively generated (through the transformation $\hat{F}(A)$) just from the gauge understanding of the $A$-freedom, represents a return to the Abelian scheme again. But what is the meaning of such a strange reversion? Thus, seemingly, the interpretation of the $A$-freedom as exclusively a gauge one is not justified since this leads to a logical collision with the quantum superposition principle and also entails the return to the Abelian scheme.

However, the matrix $F(A) \in SU(2)_{\text{loc. gauge}}$. In author’s opinion, there exists just one and very simple way out of this situation which consists in the following: The complete symmetry group of system under consideration is (apart from a rotational symmetry related with $\vec{j}_2^2, j_3$ that has non-gauge character) of the form $\hat{F}(A) \otimes SL(2,C)_{\text{loc. gauge}} \otimes SU(2)_{\text{loc. gauge}}$. This group, in particular, contains the gauge and non-gauge symmetry operations which both have the same mathematical form but different physical status. Only such a way of understanding allows us not to reach a deadlock.

11. Discussion and some generalities

In conclusion, some additional general notices are to be given. The specific analysis implemented in the above study may play a part in considering analogous situations for more
complicated gauge groups [1], serving as some guidelines. Since the case of freedom in choosing an explicit form of certain discrete operators, which has its roots in the fact that a certain subgroup $G'$ of a complete gauge group $G$ commutes with a Hamiltonian can appear. Then those symmetry operations $G'$ will generate some linear transformations in a set of basis functions, which in their mathematical form will coincide with a matrix (independent of space coordinates) lying formally in the gauge group $G$. It appears that analogous studying such Abelian monopole manifestations on the background of other (big) gauge groups $G$ is feasible and would require no large departures from the present scheme. Those latter, seemingly would be completely determined by the inner structure of the relevant Lie algebras, in particular, their respective Cartan’s sub-algebras. The number of elements in those sub-algebras would coincide with the number of (one-parametric) generalized chiral symmetry transformations.

Else one remark may be given. Existence of the above isotopic chiral symmetry is not relevant to whether a particle multiplet carries the isotopic spin $T = 1/2$ and the Lorentzian spin $S = 1/2$. The general structure of the matter equation (see (3.1)) will remain the same if one extends the problem to any other values of $T$ and $S$ (to retain the formal similarity, one ought to exploit the first order wave equation formalism)[25]. Moreover, all the problem can be easily extended to an arbitrary curved space-time of spherical symmetry.

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\footnote{For instance, the author has carried out all required calculations for $T = 1$ case. Technically, this is some more laborious task but considered conclusions are in part similar. The main difference arisen is that now there exist two independent one-parametric symmetries of the triplet-monopole Hamiltonian (instead of the one discerned by the present work for $T = 1/2$ case), these symmetry operations vary substantially in their mathematical forms and physical manifestations. An account of this has appeared to be rather unwieldy, so that it has not been included in the present paper.}
Supplement A. Connection between electron-monopole functions in spherical and Cartesian bases

Let us consider relationships between fermion-monopole functions in spherical and Cartesian bases. First, we look at connection between $D$-functions used above and the so-called spinor monopole harmonics. To this end, one ought to perform subsequently two translations: from the spherical tetrad and 2-spinor (by Weyl) frame in bispinor space into, respectively, the Cartesian tetrad and the so-called Pauli’s (bispinor) frame. In the first place, it is convenient to accomplish those translations for a free electronic function; so as, in the second place, to follow this pattern further in the monopole case.

So, subjecting that free electronic function to the local bispinor gauge transformation (associated with the change $sph. \rightarrow Cart.$)

$$\Phi_{Cart.} = \begin{pmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix} \Psi_{sph.}, \quad U^{-1} = \begin{pmatrix} \cos \theta/2 e^{-i\phi/2} & -\sin \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{i\phi/2} \end{pmatrix}$$

and further, taking the bispinor frame from the Weyl 2-spinor form into the Pauli’s

$$\Phi_{Cart.}^{Pauli} = \begin{pmatrix} \varphi \\ \xi \end{pmatrix}, \quad \Phi_{Cart.}^{Weyl} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \varphi = \frac{\xi + \eta}{\sqrt{2}}, \quad \chi = \frac{\xi - \eta}{\sqrt{2}}$$

we get

$$\varphi = \left[ \frac{f_1 + f_3}{\sqrt{2}} \begin{pmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \end{pmatrix} D_{-1/2} + \frac{f_2 + f_4}{\sqrt{2}} \begin{pmatrix} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{i\phi/2} \end{pmatrix} D_{+1/2} \right]; \quad (A.1a)$$

$$\chi = \left[ \frac{f_1 - f_3}{\sqrt{2}} \begin{pmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \end{pmatrix} D_{-1/2} + \frac{f_2 - f_4}{\sqrt{2}} \begin{pmatrix} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{i\phi/2} \end{pmatrix} D_{+1/2} \right]; \quad (A.1b)$$

Further, for the above solutions with fixed proper values of $P$-operator, we produce

$$P = (-1)^{j+1} : \Phi_{Cart.}^{Pauli} = \frac{e^{-i\eta t}}{r \sqrt{2}} \begin{pmatrix} (f_1 + f_2)(\chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2}) \\ (f_1 - f_2)(\chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2}) \end{pmatrix} \quad (A.2a)$$

$$P = (-1)^{j} : \Phi_{Cart.}^{Pauli} = \frac{e^{-i\eta t}}{r \sqrt{2}} \begin{pmatrix} (f_1 - f_2)(\chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2}) \\ (f_1 + f_2)(\chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2}) \end{pmatrix} \quad (A.2b)$$

where $\chi_{+1/2}$ and $\chi_{-1/2}$ designate the columns of matrix $U^{-1}(\theta, \phi)$ (in the literature they are termed as helicity spinors)

$$\chi_{+1/2} = \begin{pmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{i\phi/2} \end{pmatrix}. \quad (A.2c)$$

Now, using the known extensions for spherical spinors $\Omega_{jm}^{\pm 1/2}(\theta, \phi)$ in terms of $\chi_{\pm 1/2}$ and $D$-functions $[66]$:

$$\Omega_{jm}^{(+)} = (-1)^{m+1/2} \sqrt{(2j+1)/8\pi} \left( + \chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2} \right),$$

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we eventually arrive at the common representation of the spinor spherical solutions:

\[
\Omega_{jm}^{(-)} = (-1)^{m+1/2} \sqrt{(2j+1)/8\pi} \left( -\chi_{+1/2}^1 D_{-1/2} + \chi_{-1/2}^1 D_{+1/2} \right)
\]

The Abelian monopole situation can be considered in the same way. As a result, we produce the following representation of the monopole-electron functions in terms of ‘new’ angular harmonics \((k \equiv eg)\)

\[
P = (-1)^{j+1} : \quad \Phi_{\text{Pauli}}^\text{Cart.} = \frac{e^{-i\omega t}}{r} \left( \begin{array}{c} +f(r) \Omega_{jm}^{(+)}(\theta, \phi) \\ -i \frac{g(r)}{f(r)} \Omega_{jm}^{(-)}(\theta, \phi) \end{array} \right); \quad (A.3a)
\]

\[
P = (-1)^j : \quad \Phi_{\text{Pauli}}^\text{Cart.} = \frac{e^{-i\omega t}}{r} \left( \begin{array}{c} -i g(r) \Omega_{jm}^{(-)}(\theta, \phi) \\ f(r) \Omega_{jm}^{(+)}(\theta, \phi) \end{array} \right); \quad (A.3b)
\]

Here, the two column functions \(\xi_{jm}^{(1)}(\theta, \phi)\) and \(\xi_{jm}^{(2)}(\theta, \phi)\) denote the special combinations of \(\chi_{\pm 1/2}(\theta, \phi)\) and \(D_{-m, eg/\hbar \pm 1/2}(\phi, \theta, 0)\):

\[
\xi_{jm}^{(1)} = (+\chi_{-1/2} D_{k+1/2} + \chi_{+1/2} D_{k-1/2}) , \quad \xi_{jm}^{(2)} = (+\chi_{-1/2} D_{k+1/2} - \chi_{+1/2} D_{k-1/2}); \quad (A.5)
\]

compare them with analogous extensions for \(\Omega_{jm}^{\pm 1/2}(\theta, \phi)\). These 2-component and \((\theta, \phi)\)-dependent functions \(\xi_{jm}^{(1)}(\theta, \phi)\) and \(\xi_{jm}^{(2)}(\theta, \phi)\) just provide what is called spinor monopole harmonics. It should be useful to write down the detailed explicit form of these generalized harmonics. Given the known expressions for \(\chi\) - and \(D\)-functions, the formulae yield the following

\[
\xi_{jm}^{(1,2)}(\theta, \phi) = \left[ e^{im\phi} \left( \frac{-\sin \theta/2 e^{-i\phi/2}}{\cos \theta/2 e^{i\phi/2}} \right) d_{-m,k+1/2}(\cos \theta) \pm \right. \left. e^{im\phi} \left( \frac{\cos \theta/2 e^{-i\phi/2}}{\sin \theta/2 e^{i\phi/2}} \right) d_{-m,k-1/2}(\cos \theta) \right]; \quad (A.6)
\]

where the signs + (plus) and − (minus) refer to \(\xi^{(1)}\) and \(\xi^{(2)}\), respectively. One can equally work whether in terms of monopole harmonics \(\xi^{(1,2)}(\theta, \phi)\) or directly in terms of \(D\)-functions, but the latter alternative has an advantage over the former because of the straightforward access to the ‘unlimited’ \(D\)-function apparatus, instead of proving and producing just disguised old results.

Above, at translating the electron-monopole functions into the Cartesian tetrad and Pauli’s spin frame, we had overlooked the case of minimal \(j\). Turning to it, on straightforward calculation we find (for \(k < 0\) and \(k > 0\), respectively)

\[
k > 0 : \quad \Phi_{\text{min.}}^{(eg)\text{Cart.}} = \frac{e^{-i\omega t}}{\sqrt{2}r} \left( \frac{(f_1 + f_3)}{(f_1 - f_3)} \right) \chi_{+1/2}(\theta, \phi) D_{-m,k-1/2}^{k-1/2}(\theta, \phi, 0); \quad (A.7a)
\]
\[ k < 0 : \quad \Phi_{j_{\min}}^{(eg)Cart.} = \frac{e^{-i e t}}{\sqrt{2r}} \begin{pmatrix} f_2 + f_4 \\ f_2 - f_4 \end{pmatrix} \begin{pmatrix} f_2 + f_4 \\ f_2 - f_4 \end{pmatrix} \chi_{-1/2}(\theta, \phi) D_{-m,k+1/2}^{(m-1/2)}(\theta, \phi, 0). \quad (A.7b) \]

In addition, now it is a convenient point to clarify the way how the used gauge transformation \( U^{-1}(\theta, \phi) \), not being a single-valued matrix-function of spatial points, affects the continuity (or discontinuity) properties of the wave functions under consideration.

Returning to the \( \varphi(x) \) from (A.1a) at the points \( \theta = 0, \pi \) (the functions \( \chi(x) \) from (A.1b) are completely analogous ones), one gets

\[
\varphi(\theta = 0) \sim \left[ \frac{f_1 + f_3}{\sqrt{2}} \begin{pmatrix} e^{-i \phi/2} \\ 0 \end{pmatrix} D_{-m,0}^j(\phi, \theta = 0, 0) + \right.
\]

\[
\left. \frac{f_2 + f_4}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i \phi/2} \end{pmatrix} D_{-m,0}^j(\phi, \theta = 0, 0) \right]; \quad (A.8a)
\]

\[
\varphi(\theta = \pi) \sim \left[ \frac{f_1 + f_3}{\sqrt{2}} \begin{pmatrix} e^{-i \phi/2} \\ 0 \end{pmatrix} D_{-m,0}^j(\phi, \theta = \pi, 0) + \right.
\]

\[
\left. \frac{f_2 + f_4}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i \phi/2} \end{pmatrix} D_{-m,0}^j(\phi, \theta = \pi, 0) \right]. \quad (A.8b)
\]

Further, allowing for the relevant relations from Tables 1a, b, one produces: \( \varphi(\theta = 0) \) and \( \varphi(\theta = \pi) \) are single-valued functions.

In turn, for the monopole case, in place of (A.7a,b) one gets (for definiteness, let \( eg = +1/2 \))

\[
\varphi^{(eg=+1/2)}(\theta = 0) \sim \left[ \frac{f_1 + f_3}{\sqrt{2}} \begin{pmatrix} e^{-i \phi/2} \\ 0 \end{pmatrix} D_{-m,0}^j(\phi, \theta = 0, 0) + \right.
\]

\[
\left. \frac{f_2 + f_4}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i \phi/2} \end{pmatrix} D_{-m,0}^j(\phi, \theta = 0, 0) \right]; \quad (A.9a)
\]

\[
\varphi^{(eg=+1/2)}(\theta = \pi) \sim \left[ \frac{f_1 + f_3}{\sqrt{2}} \begin{pmatrix} e^{-i \phi/2} \\ 0 \end{pmatrix} D_{-m,0}^j(\phi, \theta = \pi, 0) + \right.
\]

\[
\left. \frac{f_2 + f_4}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i \phi/2} \end{pmatrix} D_{-m,0}^j(\phi, \theta = \pi, 0) \right]. \quad (A.9b)
\]

From that, allowing for the relations from Tables 2a, b, one finds that the totality of all \( \varphi^{(eg=+1/2)}(x) \) consists of both regular and non-regular (non-single-valued) functions at the \( x_3 \) axis; these latter behave like \( e^{-i \phi/2} \) and \( e^{i \phi/2} \) at the half-axes \( \theta = 0 \) and \( \theta = \pi \), respectively.

The case of minimal \( j \) follows the same behavior: for example, if \( eg = \pm 1/2 \) then one gets

\[
eg \quad = 1/2 : \quad \Phi_{j_{\min}}^{(eg=+1/2)Cart.} = \frac{e^{-i e t}}{\sqrt{2r}} \begin{pmatrix} f_1 + f_3 \\ f_1 - f_3 \end{pmatrix} \chi_{+1/2}(\theta, \phi) ; \quad (A.10a)
\]

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\[
\begin{align*}
\frac{eg}{r} &= -1/2 : \quad \Phi_{\text{eg}=-1/2}^{\text{Cart.}} = \frac{e^{-i\epsilon t}}{\sqrt{2r}} \begin{pmatrix}
(f_2 + f_4) \\ (f_2 - f_4)
\end{pmatrix} \chi_{-1/2}(\theta, \phi).
\end{align*}
\]  

(A.10b)

Now, let us turn to the non-Abelian doublet-fermion case. The task is to translate the composite functions \(\Psi_{\text{ejm}^\delta}^{\text{sph.}(A)}\) (see (6.9)) into the Cartesian tetrad basis:

\[
\Psi_{\text{ejm}^\delta}^{\text{sph.}} \rightarrow \Psi_{\text{ejm}^\delta}^{\text{Cart.}} = \begin{pmatrix}
\sum_{\text{ejm}^\delta}^{(+)}(x) \\
\sum_{\text{ejm}^\delta}^{(-)}(x)
\end{pmatrix}
\]

where the 2-component structure in Lorentzian space is explicitly detailed. For those composite two-column functions \(\sum_{\text{ejm}^\delta}^{(\pm)}(x)\) one gets

\[
\sum_{\text{ejm}^\delta}^{(\pm)}(x) = \frac{e^{-i\epsilon t}}{r} \times
\]

(A.11)

\[
\left\{ 
\begin{array}{c}
T_{+1/2} \otimes \frac{\sqrt{j + m + 1}}{2j + 1} \left[ \left( K^A_\delta \pm \delta e^{iA} L^{-A}_\delta \right) \chi_{1/2} D^{j-1/2}_{-m+1/2,-1/2} + \\
( L^A_\delta \pm \delta e^{iA} K^{-A}_\delta ) \chi_{-1/2} D^{j-1/2}_{-m+1/2,+1/2} \right] + \\
T_{+1/2} \otimes \frac{\sqrt{j - m + 1}}{2j + 1} \left[ \left( M^A_\delta \mp \delta e^{iA} N^{-A}_\delta \right) \chi_{1/2} D^{j+1/2}_{-m+1/2,-1/2} + \\
( N^A_\delta \mp \delta e^{iA} M^{-A}_\delta ) \chi_{-1/2} D^{j+1/2}_{-m+1/2,+1/2} \right] + \\
T_{-1/2} \otimes \frac{\sqrt{j - m}}{2j + 1} \left[ \left( -K^A_\delta \pm \delta e^{iA} L^{-A}_\delta \right) \chi_{1/2} D^{j-1/2}_{-m-1/2,-1/2} + \\
(- L^A_\delta \pm \delta e^{iA} K^{-A}_\delta ) \chi_{-1/2} D^{j-1/2}_{-m-1/2,+1/2} \right] + \\
T_{-1/2} \otimes \frac{\sqrt{j + m + 1}}{2j + 1} \left[ \left( M^A_\delta \pm \delta e^{iA} N^{-A}_\delta \right) \chi_{1/2} D^{j+1/2}_{-m-1/2,-1/2} + \\
( N^A_\delta \pm \delta e^{iA} M^{-A}_\delta ) \chi_{-1/2} D^{j+1/2}_{-m-1/2,+1/2} \right] \}
\]

With the use of four formulas [16]

\[
\chi_{\pm 1/2} \quad D^{j-1/2}_{-m+1/2,\mp 1/2} = \left( \Omega_{-1/2, m-1/2}^{(+)} \mp \Omega_{-1/2, m-1/2}^{(-)} \right) \frac{\sqrt{4\pi}}{2\sqrt{J}(-1)^m};
\]

\[
\chi_{\pm 1/2} \quad D^{j+1/2}_{-m+1/2,\mp 1/2} = \left( \Omega_{1/2, m+1/2}^{(+)} \mp \Omega_{1/2, m+1/2}^{(-)} \right) \frac{\sqrt{4\pi}}{2\sqrt{J}+1(-1)^m};
\]

\[
\chi_{\pm 1/2} \quad D^{j-1/2}_{-m-1/2,\mp 1/2} = \left( \Omega_{1/2, m+1/2}^{(+)} \mp \Omega_{1/2, m+1/2}^{(-)} \right) \frac{\sqrt{4\pi}}{2\sqrt{J}(-1)^{m+1}};
\]

\[
\chi_{\pm 1/2} \quad D^{j+1/2}_{-m-1/2,\mp 1/2} = \left( \Omega_{1/2, m+1/2}^{(+)} \mp \Omega_{1/2, m+1/2}^{(-)} \right) \frac{\sqrt{4\pi}}{2\sqrt{J}+1(-1)^{m+1}};
\]

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the above expression (A.11) for \( \Sigma_{\epsilon jm\delta}(x) \) can be rewritten in the form

\[
\Sigma_{\epsilon jm\delta}(x) = \frac{e^{-i\epsilon t}}{r} \times \ (A.12)
\]

\[
T_{+1/2} \otimes B \left\{ \left[ (K^A_\delta + \delta e^{iA} K^{-A}_\delta) + (L^A_\delta + \delta e^{iA} L^{-A}_\delta) \right] \Omega^{(+)}_{j-1/2,m-1/2} \right. \\
+ \left. \left[ (-K^A_\delta + \delta e^{iA} K^{-A}_\delta) + (-L^A_\delta + \delta e^{iA} L^{-A}_\delta) \right] \Omega^{(-)}_{j-1/2,m-1/2} \right\} + \\
T_{+1/2} \otimes C \left\{ \left[ (M^A_\delta + \delta e^{iA} M^{-A}_\delta) + (N^A_\delta + \delta e^{iA} N^{-A}_\delta) \right] \Omega^{(+)}_{j+1/2,m-1/2} \right. \\
+ \left. \left[ (-M^A_\delta + \delta e^{iA} M^{-A}_\delta) + (-N^A_\delta + \delta e^{iA} N^{-A}_\delta) \right] \Omega^{(-)}_{j+1/2,m-1/2} \right\} + \\
T_{-1/2} \otimes D \left\{ \left[ (-K^A_\delta + \delta e^{iA} K^{-A}_\delta) + (-L^A_\delta + \delta e^{iA} L^{-A}_\delta) \right] \Omega^{(+)}_{j-1/2,m+1/2} \right. \\
+ \left. \left[ (K^A_\delta + \delta e^{iA} K^{-A}_\delta) + (-L^A_\delta + \delta e^{iA} L^{-A}_\delta) \right] \Omega^{(-)}_{j-1/2,m+1/2} \right\} + \\
T_{-1/2} \otimes E \left\{ \left[ (M^A_\delta + \delta e^{iA} M^{-A}_\delta) + (N^A_\delta + \delta e^{iA} N^{-A}_\delta) \right] \Omega^{(+)}_{j+1/2,m+1/2} \right. \\
+ \left. \left[ (-M^A_\delta + \delta e^{iA} M^{-A}_\delta) + (-N^A_\delta + \delta e^{iA} N^{-A}_\delta) \right] \Omega^{(-)}_{j+1/2,m+1/2} \right\}
\]

where the symbols \( B, C, D, E \) denote respectively

\[
B = \frac{\sqrt{j+m}}{2j+1} \frac{1}{\sqrt{2}} \frac{\sqrt{4\pi}}{2\sqrt{j-1}^{m}}, \quad C = \frac{\sqrt{j-m+1}}{2j+1} \frac{1}{\sqrt{2}} \frac{\sqrt{4\pi}}{2\sqrt{j+1}^{m+1}}.
\]

\[
D = \frac{\sqrt{j-m}}{2j+1} \frac{1}{\sqrt{2}} \frac{\sqrt{4\pi}}{2\sqrt{j-1}^{m+1}}, \quad E = \frac{\sqrt{j-m+1}}{2j+1} \frac{1}{\sqrt{2}} \frac{\sqrt{4\pi}}{2\sqrt{j+1}^{m+1}}.
\]

The representation (A.12) will be significantly simplified if \( A = 0 \); so one can find

\[
A = 0 : \quad \Sigma_{\epsilon jm\delta}(x) = \frac{e^{-i\epsilon t}}{r} \times \ (A.13)
\]

\[
\left[ T_{+1/2} \otimes B \left( \pm \delta K_\delta + L_\delta \right) \Omega^{(\pm\delta)}_{j-1/2,m-1/2} \right. \right. \\
+ \left. \left. T_{+1/2} \otimes C \left( \mp \delta M_\delta + N_\delta \right) \Omega^{(\mp\delta)}_{j+1/2,m-1/2} \right. \right. \\
+ \left. \left. T_{-1/2} \otimes D \left( \pm \delta K_\delta - L_\delta \right) \Omega^{(\pm\delta)}_{j-1/2,m+1/2} \right. \right. \\
+ \left. \left. T_{-1/2} \otimes E \left( \pm \delta M_\delta + N_\delta \right) \Omega^{(\pm\delta)}_{j+1/2,m+1/2} \right].
\]

In particular, the formula (A.13) apparently exhibits the Abelian fermion-like substructure that stems from the Abelian-like \( P \)-inversion operation (6.4).
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