Macroscopic Coulomb blockade in large Josephson junction arrays

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(Dated: June 26, 2008)

We investigate theoretically transport properties of one- and two-dimensional regular Josephson junction arrays (JJA\textsuperscript{s}) in an insulating state. We derive the low-temperature current-voltage characteristics (the I-V dependencies) for the current mediated by the Cooper pair transfer across the system. In the case where the screening length $\lambda_c$ associated with the capacitance of the islands to the ground is much larger than the island’s size $d$, we find that transport is governed by the macroscopic Coulomb blockade effect with the gap $\Delta_c$, well exceeding a single island charging energy $E_c$. In the limit of $\lambda_c \gg L$, where $L$ is the linear size of the array, the gap establishes the dependence on the array size, namely, $\Delta_c \approx E_c(L/d)$ in 1D and $\Delta_c \approx E_c \ln(L/d)$ in 2D arrays. We find two transport regimes: at moderate temperatures, $E_c < k_BT < \Delta_c$, the low bias transport is thermally activated with the resistance $R \propto \exp(T_0/T)$ where the activation energy $k_BT_0 = \Delta_c$. At ultra-low temperatures, $k_BT < E_c$, a JJA falls into a superinsulating state with $R \propto \exp[(\Delta_c/E_c) \exp(E_c/T)]$.

PACS numbers: 05.60.Gg, 74.81.Fa, 73.63.-b

I. INTRODUCTION

A theoretical and experimental study of large regular Josephson junction arrays (JJA\textsuperscript{s}), the systems comprised of small superconducting islands connected by Josephson junctions, has a long history\textsuperscript{1,2,3,4,5,6,7,8,9,10,11,12,13}, see\textsuperscript{14,15} for a review. The remarkable feature of these systems is that they experience a superconductor-to-insulator transition (SIT) as the Coulomb charging energy of a single island, $E_c$ (i.e. an energy cost to place a Cooper pair on such an island) compares to the Josephson coupling energy, $E_J$, measuring the strength of the phase coupling between the superconducting islands comprising a JJA\textsuperscript{16,17}. In arrays near the critical balance between $E_c$ and $E_J$, a SIT can be induced by the magnetic field\textsuperscript{18}. Studies on the superconductor-insulator transition in thin granular films, which are often modeled as JJA\textsuperscript{s}, revealed the similar behavior\textsuperscript{19,20,21,22,23,24,25,26,27,28,29,30,31,32}. Even more remarkably, critically disordered homogeneous superconducting films exhibited all the wealth of phenomena related to superconductor-insulator transition characteristics to granular superconducting systems\textsuperscript{33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50}. This brought about the idea that strong disorder induces an inhomogeneous spatial structure of isolated superconducting islands in thin homogeneously disordered films\textsuperscript{47,48,51,52,53,54}. Numerical simulations of the homogeneously disordered superconducting films confirmed that indeed in the high-disorder regime, the system breaks up into superconducting islands separated by an insulating state\textsuperscript{55,56,57}. Recent scanning tunneling microscope measurements of the local density of states in thin TiN films\textsuperscript{58} offered strong support to this hypothesis.

All the above together shows that the Josephson junction arrays offer a useful generic model that captures most essential features of the superconductor-insulator transition in a wide class of systems ranging from artificially manufactured Josephson junction arrays to superconducting granular systems and even the homogeneously disordered superconducting films (see also review\textsuperscript{59}) and allows for consideration of all of them on the common ground. Indeed, recent theoretical results describing insulating behavior of regular JJA\textsuperscript{s}, appeared to be in a striking quantitative accordance with the experimental findings in TiN and InO superconducting disordered films\textsuperscript{59,60}.

The common “working tool” for experimental study of the SIT is the measurements of transport characteristics of the systems in question. Altering various parameters of the system, such as tunnel resistance and transmittance in JJA\textsuperscript{s}, conditions of deposition, chemical composition, and thickness of the films, and applied magnetic field, one can drive the system directly from the superconducting to insulating state. The transition is observed as a set of the fan-shaped temperature dependences of the resistance $R(T)$, see Refs. \textsuperscript{1,3,4,6,9,12,13,19,21,24,27,28,29,30,31,32,33,34,35,36,38,39,40,41,42,43,44,45,46,47,48,49,50}, with the activation behavior

$$R(T) \propto e^{T_0/T}$$

on the insulating side of the transition, see Refs. \textsuperscript{5,6,8,12,13,19,35,36,43,47,48,49,50}, where $T_0$ is the activation temperature. Most pronounced features of this transition manifest themselves in the current-voltage characteristics (the I-V curves). On the superconducting side a system has very low resistance at low currents followed by jump in resistivity when current exceeds the critical value. On the insulating side the I-V characteristics show a mirror behavior: extremely high resistance at low voltages and abrupt jump in the conductivity at the threshold voltage $V_T$, see Refs. \textsuperscript{1,3,4,6,8,12,13,19,35,36,43,47,48,49,50,14,15,16,17,18,19,32,40,44,45,49,51}.

Yet the most startling observations that
come from the insulating side of the transition are: (i) the size dependent activation energy, \( \Delta_c \), (ii) hyperactivation temperature behavior of resistivity at ultra-low temperatures in JJAs, \( T \ll T_c \); (iii) size dependent threshold voltage, \( V_{th} \); (iv) peculiar interrelated magnetic field dependencies of the activation energy and threshold voltage in JJAs and superconducting films.

The above striking findings called for a theory capable of quantitative description of the accumulated wealth of the experimental results within a unified picture. The preceding publications offered such a description in a framework of the Cooper pair transport in large JJAs in the insulating region, where \( E_c \gg E_J, E_S = hI_c/2e \) \( (I_c \) is the critical current of a single Josephson junction). This paper is the extended version of earlier publications, presenting the details of the derivation of the current-voltage characteristics.

We briefly summarize our results: The insulating behavior of the large Josephson junction array is governed by the macroscopic Coulomb blockade effect with the Coulomb blockade activation energy

\[
\Delta_c = \begin{cases} 
E_c \left[ \Lambda/(2d) \right], & 1D \text{array} \\
E_c \ln(\Lambda/d), & 2D \text{array} 
\end{cases}
\]

where \( \Lambda = \min\{L, \lambda_c\}, L \) is the size of an array, \( d \) is the size of the elemental unit of JJA, and \( \lambda_c \) is the screening length related to the capacitance to the ground. Importantly, this activation energy can be much larger than the single junction charging energy \( E_c \). In the two-dimensional array the charge binding-unbinding Berezinskii-Kosterlitz-Thouless like transition takes place at \( T = T_{st} \approx E_c/k_B \) separating the insulating phase existing in the interval \( E_c < k_BT < \Delta_c \) and exhibiting the thermally activated resistivity \( (1) \) with \( k_BT_0 \equiv \Delta_c \), and the superinsulating state at \( T < E_c/k_B \), with \( R \propto \exp[(\Delta_c/E_c)\exp(E_c/k_BT)] \). In the one-dimensional arrays we expect a crossover between these two states.

The paper is organized as follows: in Section II we introduce the model and present the general equation for the dc I-V dependence, expressing it through the time-dependent correlation function \( K(t) \) of superconducting order parameter phases of the whole system. In Sections III and IV we discuss the properties of \( K(t) \) and the corresponding I-V characteristics above and below \( T_{st} \) respectively. Section V presents the discussion of the obtained results.

**II. MODEL AND CVC CALCULATION**

The current-voltage characteristics of Josephson systems in an insulating state have been a subject of intense discussions over the decades. Single junctions were considered and the behavior of two-junction systems was studied. Only a few works were addressing the Cooper-pair current in large JJAs.

In this paper we will discuss Cooper pair transport (Josephson current) and the corresponding I-V characteristics in large JJAs, leaving the calculation of the quasiparticle contribution to the forthcoming publication. This implies, in particular, that we neglect interactions of the internal phases with the thermal bath, since the latter is equivalent to switching on quasiparticle current. Let us consider \( N \times M \) superconducting islands comprising a one- \((M = 1)\) or two-dimensional array closed by a small (as compared to the quantum resistance for Cooper pairs \( R_{C_0} = h/4e^2 \approx 6.45 \text{ k}\Omega \)) external resistance, \( R_{ext} \), see Fig. 1. Note, that in this kind of circuits with a small external load resistance both, a single Josephson junction and two Josephson junctions in series, are always in a superconducting state irrespectively to relation between \( E_c \) and \( E_J \). We are interested in a low-temperature transport, \( T \ll T_c \), where \( T_c \) is the critical temperature of a single superconducting island, and, therefore, we can neglect the fluctuations of the amplitude of the order parameter. We assign the fluctuating order parameter phase \( \chi_{ij}(t) \) to the \( \{i,j\} \)-th superconducting island (see Fig. 1). Josephson relation connects these phases with the fluctuating voltage drops between the adjacent islands. We denote the phases of the left- and right leads as \( \chi_L(t) \) and \( \chi_R(t) \), correspondingly. The finite voltage \( V \) applied to a JJA generates the alternating Josephson currents proportional to \( E_J \sin(\theta t + \{\chi_{ij}(t)\}) \). The dc component of the Josephson current results from the time averaging of the ac currents and is thus determined by the correlations of the time-dependent fluctuations of the Josephson phases \( \{\chi_{ij}\}(t) \) across the array.

We discriminate between the Josephson coupling ener-
gies of intrinsic junctions $\tilde{E}_j$ and the Josephson coupling energies $E_J$ between leads and first and $N$-th rows. Single

gling out the terms containing the leads phases $\chi_L$ and $\chi_R$ explicitly we present the array Hamiltonian as:

$$H = H_0 + H_{\text{bath}} + H_{\text{int}}(\chi_L - \chi_R) + \frac{\hbar^2}{4E_c} \sum_{j=1}^{M} [(\dot{\chi}_1(t) - \dot{\chi}_L)^2 + (\dot{\chi}_N(t) - \dot{\chi}_R)^2]$$

$$\quad - E_J \sum_{j=1}^{M} \left\{ \cos[\chi_L(t) - \chi_{1J}(t)] + \cos[\chi_R(t) - \chi_{Nj}(t)] \right\}. \quad (3)$$

Here

$$H_0 = \sum_{i,j,k,l} \frac{\hbar^2}{4E_c} (\dot{\chi}_{ij} - \dot{\chi}_{kl} - 2eV_{ij-kl}/\hbar)^2 - \tilde{E}_j \cos(\chi_{ij} - \chi_{kl}) \right] + \sum_{ij} \frac{\hbar^2}{4E_{c0}} \chi_{ij}^2, \quad (4)$$

the brackets $(ij, kl)$ denote summation over the pairs of adjacent junctions, and the last term in (3) represents the self-charge energies of superconducting islands. We introduced here the dc voltage drops on the junctions $V_{ij-kl}$. The charging energies $E_c$ and $E_{c0}$ are determined by the junction capacitance $C$ and capacitance to the ground $C_0$, as $E_c = 2e^2/C$ and $E_{c0} = 2e^2/C_0$, respectively. The $H_{\text{bath}}$ is the Hamiltonian characterizing the thermal bath, which can be modeled as a set of harmonic oscillators with coordinates $\xi_{j, k, l}$, i.e.

$$H_{\text{bath}} = \sum_i \frac{\hbar^2}{2M_i} [\dot{\xi}_i]^2 + M_i \omega_i^2 \xi_i^2, \quad (5)$$

where $\omega_i$ and $M_i$ are the frequency and mass of harmonic oscillators. The $H_{\text{int}}$ term in (3) describes a bilinear coupling of phases on the leads to the thermal heat bath as

$$H_{\text{int}} = \sum_i A_i \xi_i (\chi_L - \chi_R), \quad (6)$$

where $A_i$ are the coupling constants. In the circuits presented in Fig. 1, the coupling constants $A_i$ are determined by the external resistance $R_{CP}$, $R_{ext}$. Since we consider Josephson currents only, we will neglect the phase coupling to the thermal heat bath in the internal part of the array: dissipation on the internal islands implies the presence of the current of quasiparticles. This means that we can treat the evolution of the internal phases as a non-dissipative quantum dynamics.

All the above implies that the dynamics of the phases in the leads $\chi_L$ and $\chi_R$ interacting with the heat bath differs from that of the internal phases, i.e. the phases on superconducting islands $\chi_{ij}$. Namely, whereas the phases in the leads are to be treated as classical dynamical variables satisfying the Langevin stochastic equation, the phases in the internal superconducting islands are the quantum-mechanical variables with the dynamics described by the quantum-mechanical Hamiltonian $H_0$. Moreover, the phases in the leads and the intrinsic phases interact through the Josephson coupling terms [the last two terms in the array Hamiltonian (3)]. Now, shifting all the phases over $(\chi_L + \chi_R)/2$, i.e replacing $\chi_{ij} \rightarrow \chi_{ij} - (\chi_L + \chi_R)/2$, we obtain:

$$H = H_0 + H_{\text{bath}} + H_{\text{int}}(\chi_L - \chi_R) + \frac{\hbar^2}{8E_c} \sum_{j=1}^{M} (\dot{\chi}_R - \dot{\chi}_L - \dot{\chi}_{1J} + \dot{\chi}_{Nj})^2 + \frac{\hbar^2}{8E_{c0}} \sum_{j=1}^{M} \dot{\chi}_{1J}(t) + \dot{\chi}_{Nj}(t) \right) \right], \quad (7)$$

where a new independent variable $\phi_{ij} = (\chi_{1J} + \chi_{Nj})/2$ had been introduced. The dc Josephson current through the array is calculated as $I_s = \langle \partial H/\partial (\chi_L - \chi_R) \rangle$. Considering $\chi_R - \chi_L$ as a parameter, one can apply the Hellmann-
Feynman theorem and reduce this to

\[ I_\phi(V) = \left( \frac{\partial (H)}{\partial \phi} \right) \chi_{R - \chi} \chi_{\chi} \chi_{ij}. \]  

(8)

The term in the total Hamiltonian depending explicitly on the variables \( \phi_j \) is

\[ \langle \cos \phi_j \rangle \propto \langle n \rangle \langle \cos \phi_j \rangle \langle n \rangle = 0, \quad \text{where} \quad \langle n \rangle = \prod_{j=1}^M \frac{1}{\sqrt{2\pi}} \exp(i\phi_j n) \quad \text{are the wave functions corresponding to the first term in the Hamiltonian (9), i.e. the wave functions in a zero order of the perturbation theory.} \]

As a first step, we perform averaging over \( \phi_j \). In the insulator regime, which we address here, \( E_c, E_{c0} \gg E_j \), and the averaging procedure is carried out by making use of the perturbation theory with respect to \( E_j/E_c \) [the last term in Eq. (9)]. Such a procedure is similar to the one used in the case of the Cooper pair two-junctions transistor. In the zero order perturbation theory

\[ \Psi_n = \langle n \rangle - 2E_j \cos \left( \frac{\chi_R - \chi_L - \chi_{ij}(t) + \chi_{Nj}(t)}{2} \right) \sum_{j=1}^M \sum_{m} \frac{\langle n \rangle \langle \cos \phi_j \rangle \langle m \rangle}{E_n^{(0)} - E_m^{(0)}} \cos \frac{\chi_R - \chi_L - \chi_{ij}(t) + \chi_{Nj}(t)}{2}. \]  

(10)

where

\[ E_n^{(0)} = \frac{E_c E_{c0} n^2}{2(E_c + E_{c0})} \]

are the energy levels obtained in a zero order of the perturbation theory. Therefore, in the first order one finds

\[ \langle \Psi_n \rangle \langle \cos \phi_j \rangle \langle \Psi_n \rangle_{\rho_T(\phi_j)} = -E_j(E_c + E_{c0}) \sum_{n=-\infty}^{\infty} e^{-\frac{E_n^{2}n^2}{2(2n^2 - 1/2)}} \times \cos^2 \left( \frac{\chi_R - \chi_L - \chi_{ij}(t) + \chi_{Nj}(t)}{2} \right). \]

(11)

Here, we used that for finite temperatures the quantum-mechanical distribution of \( \phi_j \) is determined by the equilibrium density matrix corresponding to a first term in the Hamiltonian (9), i.e.

\[ \rho_T(\phi_j) \propto \sum_n e^{-\frac{E_n^{2}n^2}{2(2n^2 - 1/2)}} |n\rangle \langle n|. \]

Further, to deal with less awkward formulas, we consider the case \( C \gg C_0 \) (the general situation with the arbitrary relation between \( C \) and \( C_0 \) is straightforwardly recovered as needed) and introduce the parameter

\[ \alpha = \sum_{n=-\infty}^{\infty} e^{-\frac{E_n^{2}n^2}{2(2n^2 - 1/2)}}. \]

The effective Hamiltonian \( H_{\text{eff}} \) depending on the variables \( \chi_R - \chi_L \) and \( \chi_{ij} \) assumes the form:

\[ H_{\text{eff}} = H_0 + H_{\text{bath}} + H_{\text{int}} \{ \chi_R - \chi_L \} + H^*, \]

(12)

where

\[ H^* = \sum_{j=1}^M \frac{2\alpha E_j}{E_c} E_j \cos \left( \frac{\chi_R - \chi_L - \chi_{ij}(t) + \chi_{Nj}(t)}{2} \right). \]

(13)
and, accordingly, the dc component of the non-dissipative Cooper pair Josephson current across the array is

$$I_s(V) = \left\langle \frac{\partial H^*}{\partial [\chi_R - \chi_L]} \right\rangle = I_c M \frac{2\alpha E_j}{E_c} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \left\langle \sin \left[ \chi_R - \chi_L - \chi_{ij}(t) + \chi_{Nj}(t) \right] \right\rangle,$$  \hspace{1cm} (14)

where, the brackets (...) stand for a quantum mechanical averaging over phases of internal junctions, \(\chi_{ij}(t)\).

The phase difference in the leads is fixed by the applied voltage bias, \(\chi_R - \chi_L = 2eVt/\hbar\), giving rise to a steady non-dissipative current. We will take into account the interaction with the thermal bath in the leads by adding the Langevin thermal force \(\dot{\psi}(t)\) generating phase fluctuations in the leads. This means that the effective constraint on the phase on the leads can be written as:

$$\chi_R - \chi_L = 2eVt/\hbar + \dot{\psi}(t),$$  \hspace{1cm} (15)

where the phase correlation function in the leads \(K_{\text{leads}}\) is determined by the classical Nyquist noise in an external resistance:\ref{4,64,65,66}:

$$K_{\text{leads}} \equiv \langle \exp[\psi(t)-\psi(0)] \rangle_T = \exp \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{R_{\text{ext}}}{\omega R_{\text{ext}} C^2 + 1} \left[ \coth \left( \frac{\hbar \omega}{2k_BT} \right) \left[ \cos(\omega t) - 1 \right] - i \sin(\omega t) \right] \right\}. \hspace{1cm} (16)$$

Thus the correlation function \(K_{\text{leads}}\) determines the interaction of the JJA with the thermal bath, and the energy relaxation takes place only in the leads, but not inside the array (\((...)_T\) stands for thermodynamic average). Shifting all internal phases \(\chi_{ij} - \chi_{kl}\) as \(\chi_{ij} - \chi_{kl} = \frac{2V_{ij-kl}t}{\hbar} + \tilde{\chi}_{ij} - \tilde{\chi}_{kl}\), we bring \(H_0\) to the form (we omit the tilde-sign):

$$H_0 = \sum_{(ij,kl)} \frac{\hbar^2}{4e^2 \epsilon} (\tilde{\chi}_{ij} - \tilde{\chi}_{kl})^2 - \tilde{E}_J \cos \left( \frac{2eV_{ij-kl}t}{\hbar} + \chi_{ij} - \chi_{kl} \right) + \sum_{ij} \frac{\hbar^2}{4e^2 \epsilon} \tilde{\chi}_{ij}^2,$$  \hspace{1cm} (17)

where \(V_{ij-kl}(t)\) are voltage drops between the adjacent islands. Plugging (15) into (13) we find that the term \(H^*\) in the Hamiltonian can be viewed as a time-dependent perturbation (the ac Josephson current) oscillating with the frequency \(\omega = 2e(V_{ij} + V_{Nj})/\hbar\):

$$H^* = \frac{2\alpha E_j}{E_c} \sum_{j=1}^M \cos \left[ \frac{2e(V_{ij} + V_{Nj})t}{\hbar} + \psi(t) - \chi_{ij}(t) + \chi_{Nj}(t) \right]. \hspace{1cm} (18)$$

Solving the Eq. (20) up to the first order in \(H^*\) we obtain

$$\rho(t) = \rho_\beta - \frac{i}{\hbar} \int_0^t ds e^{-L_0(t-s)} [H^*, \rho_\beta], \hspace{1cm} (21)$$

where \(\rho_\beta\) is the equilibrium density matrix corresponding to the Hamiltonian \(H_0 + H_{\text{bath}} + H_{\text{int}},\) the Liouville operator \(L_0\) is determined as \(L_0 \chi = (1/\hbar)[H_0 + H_{\text{bath}} + H_{\text{int}}, \chi].\) The expression for the average dc Cooper pair current assumes the form:

$$I_s(V) = M L_c \frac{4E_j}{\hbar} \frac{\alpha^2 E_j^2}{E_c^2} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \int_0^t ds \sin \left[ \frac{2eV}{\hbar} (t-s) \right] \left\langle [\hat{F}_1(t-s), \hat{F}_1] + [\hat{F}_2(t-s), \hat{F}_2] \right\rangle_{\rho_\beta}, \hspace{1cm} (22)$$

where \([\hat{F}_{1,2}(t-s), \hat{F}_{1,2}]\) is the commutator of the operator \(\hat{F}_{1,2}\) and the corresponding Heisenberg operator \(\hat{F}_{1,2}(t-s)\). The functions \(F_{1,2}\) are determined as

$$F_1 = \cos \left[ \frac{\psi - \chi_{ij} + \chi_{Nj}}{2} \right], \hspace{1cm} F_2 = \sin \left[ \frac{\psi - \chi_{ij} + \chi_{Nj}}{2} \right]. \hspace{1cm} (23)$$
We define the time-dependent correlation function $K_{tot}(t)$ of a whole system (including leads) as

$$K_{tot}(t) = \left\langle \exp \left\{ i [\psi(t) - \psi(0) - \chi_{ij}(t) + \chi_{ij}(0) + \chi_{Nj}(t) - \chi_{Nj}(0)] \right\} \right\rangle_{H_0, \psi}. \quad (24)$$

Using the property of average values $\langle F(t)G(0) \rangle_{\rho_{st}} = \langle F(0)G(-t) \rangle_{\rho_{st}}$ and expressing all the commutators through $K_{tot}(t)$ we arrive at the general equation for the dc current

$$I_s(V) = 8n^2 M I_e E_j E_c^2 \int_{-\infty}^{\infty} dt \sin \left[ \frac{2e(V_1 + V_N)t}{\hbar} \right] 3m[K_{tot}(t)]. \quad (25)$$

Since $H_0$ does not contain the variable $\psi$ and $H_{int}$, in its turn, does not depend on the intrinsic variables $\chi_{ij}$, the correlation function $K_{tot}$ factorizes:

$$K_{tot}(t) = K_{leads}(t)K(t), \quad (26)$$

where the correlation function of the phase noise in leads is determined by Eq. (29) reflecting as we have already mentioned the interaction with the thermal bath (see (5) and (6)) and

$$K(t) = \langle \exp i [\chi_{ij}(t) - \chi_{ij}(0) - \chi_{Nj}(t) + \chi_{Nj}(0)] \rangle_{H_0}, \quad (27)$$

is the time-dependent correlation function of the intrinsic part of the system with $H_0$ defined in (17).

Note here that in a two-junction system (a single Cooper-pair transistor) where $\chi_{ij} = \chi_{Nj}$ and $K(t) \equiv 1$, the current-voltage characteristics $I(V)$ is determined by the external resistance $R_{ext}$. We calculate the $I$-$V$ curve for the current flowing through a single Cooper-pair transistor as an example of an application of the general formalism. If the external resistance $R_{ext} < R_{CP}$, the current displays a peak in the low-voltage region (this state is often referred to as the “pseudo-superconducting state”):

$$I(V) \approx M I_e E_j \left[ \frac{E_j}{E_c} \right]^2 \frac{R_{ext}}{R_{CP}} \frac{eV}{2eV^2 + (k_B T R_{ext}/R_{CP})^2}. \quad (28)$$

Proportionality of $I(V)$ to $E_j^2$ indicates the Cooper-pair cotunneling type of transport. Note, that the Cooper-pair current in the two-junction system does not display Coulomb blockade effect when the external resistance $R_{ext} < R_{CP}$. In the opposite case, $R_{ext} > R_{CP}$, the Eq. (29) with $K(t) = 1$ yields the Gaussian $I(V)$ dependence $I \propto \exp[-(eV - E_c)^2/(k_B T E_c)]$ (see Ref. [24]).

Now we discuss the general case of a large size JJA with the number of Josephson junctions larger than two. We consider the case of the small external resistance, $R_{ext} < R_{CP}$, which is the most frequent experimental situation. Expression (29) shows that the dc current depends explicitly upon the voltage drops $V_1$ and $V_N$ on the leftmost and rightmost junctions, while the voltage drops $V_{ij-kl}$ on the internal parts of the array come in through the correlation function $K(t)$. To evaluate the voltage distribution along the system, we notice that in the insulating domain, $E_j \ll E_c$, the correlation function $K(t)$ oscillates with the high frequency determined by the macroscopic collective Coulomb gap $\Delta_c \gg k_B T \gg E_c$ given by Eq. (2), and that in the time interval, from which the main contribution is coming from, $K_{leads} \approx 1$ (see two next Sections), giving rise to the exponentially low conductance $G(T) \propto G_0 e^{-\Delta_c/(k_B T)}$ (where $G_0$ is the non-activated factor in the conductance characterizing an individual junction), i.e.

$$I \approx G(T) (E_j/E_c)^4 (V_1 + V_N). \quad (29)$$

Then the dc current passing through an internal junction can be estimated as [in the first order approximation with respect to the time-dependent terms $\hat{E}_j \cos(2eV_{ij-kl}t/\hbar) + \chi_{ij} - \chi_{kl}$ in the Hamiltonian $H_0$ of (17)]

$$I^{(int)}_{dc} \approx G_0 V_{ij-kl} (\hat{E}_j/E_c)^2 \quad (30)$$

[cf Eqs. (21)-(25)]. Utilizing the dc current conservation law (the Kirchhoffs law), i.e. the fact that $I = I^{(int)}_{dc}$, we find:

$$V_1 + V_N \approx \frac{V}{1 + \left[ \frac{N E_j^2}{(\hat{E}_j E_c)^2} \right] G(T)/G_0}, \quad (31)$$

and, therefore, as long as

$$N \frac{E_j^2}{(\hat{E}_j E_c)^2} \frac{G(T)}{G_0} \ll 1, \quad (32)$$

the highly inhomogeneous voltage distribution takes place, i.e. almost all the applied dc bias $V$ drops on the first and the $N$-th rows of junctions, $V_1 + V_N \approx V$. In this regime, the dc current $I \approx G(T) (E_j/E_c)^4 V$ flowing through the array seems not to depend on the Josephson coupling of intrinsic Josephson junctions $\hat{E}_j$. However, as $\hat{E}_j$ becomes too small and/or, talking about the 2D case, the number $N$ of junctions is growing too large, the condition (32) breaks down. This gives rise
to the even distribution of the total voltage drop along the whole array. Yet due to extremely small values of $G(T)$ in the insulating regime, there exists a wide range of parameters where the highly inhomogeneous voltage distribution giving rise to the synchronized collective behavior is realized.

The above estimates suggest the following simple picture of the macroscopic Coulomb blockade governing the Cooper pair insulator dynamics: the applied voltage distributes mostly between the leftmost and rightmost rows of junctions, while the internal part of an array acts as a coherent superconducting island providing the macroscopic Coulomb barrier $\Delta_c \simeq (2e)^2/2C_{tot}$, where $C_{tot}$ is a total capacitance of the array.

III. TRANSPORT IN JJAS: THE INSULATING REGIME

In this Section we calculate the current defined by Eqs. (25) - (27), in the insulating state, $E_c \gg E_J$, at moderately low temperatures $\Delta_c > k_B T \gg E_c$.

To begin with, we find the correlation function $K(t)$ for a simple 1D case. In the first approximation we put $\tilde{E}_j = 0$, and, to simplify the notations, assume $E_{c,0} = \infty$ (the generalization to the finite $E_{c,0}$ case is almost straightforward). Then the Hamiltonian $H_0 = \sum_i H_i$, where $H_i$ are the Hamiltonians of individual Josephson junctions. Using the quantum-mechanical definition of $K(t)$ we obtain:\[ K(t) = \left[ A \sum_{n,m} e^{i(E_m - E_n)/\hbar} \langle n|e^{i\varphi}|m\rangle \right]^N, \]

where $E_n$'s are the energy levels of a single junction. These energy levels are determined by the charge energy $E_c$ as $E_n = (E_c n^2/2, (n|e^{i\varphi}|m\rangle$ are the matrix elements of the operator $e^{i\varphi}$ between the $n$-th and $m$-th states, and $A = \left[ \sum_n \exp(-E_n/k_B T) \right]^{-1}$ is the normalization coefficient (which cancel out from the final expression for $K(t)$). Therefore, the quantum mechanical dynamics of a single junction can be mapped onto a well studied behavior of quantum rotator which has the matrix elements for the operator $e^{i\varphi}$ between the states $n$ and $n + 1$ only. We find

$$K(t) = e^{iN E_c t/2} \left[ A \sum_n e^{iE_n t - E_n^2/2\hbar} \right]^N.$$\[ K(t) = e^{iN E_c t/2} \left[ A \int dx \exp\{iE_c tx - E_c x^2/(2k_B T)\} \right]^N, \]

and arrive at

$$K(t) = e^{iN E_c t/2 - NE_c k_B T^2/2}.$$\[ (33) \]

In a general 1D case with the finite value of $E_{c,0}$ the time dependence of the correlation function $K(t)$ preserves its form (33) but the quantity $NE_c$ has to be replaced by a more general expression determined by the full capacitance matrix of the array. A crucial conditions allowing to obtain this result are the bilinear form of the Hamiltonian $H_0$ in the momentum representation, and the factorization of the wave functions: $|n\rangle_{H_0} = \prod_{k=1}^N (1/\sqrt{2\pi}) \exp(i\phi_k n_j)$.

In the two-dimensional situation the calculations are more involved and the correlation function of the array, $K(t)$, is derived as an analytical continuation of the quantity $K(\tau)$, where $\tau$ is the imaginary time,

$$K(\tau) = \int D[\chi_{ij}] e^{i\chi_{ij}(\tau) - \chi_{ij}(0)} \exp \left( -\frac{\hbar}{4} \int_0^\hbar (k_B T) \sum_{(ij,kl)} \frac{[\chi_{ij}(\tau) - \chi_{kl}(\tau)]^2}{E_c} \right)$$

$$-\tilde{E}_j \cos[\chi_{ij}(\tau) - \chi_{kl}(\tau)] - \sum_{ij} \frac{[\chi_{ij}(\tau)]^2}{E_{c,0}} \right). \]

Note that the correlation function $K(t)$, determining the ac synchronization between the external leftmost, 1-st, and rightmost, $N$-th, Josephson junctions contacting with the left and right leads respectively, is not zero even in the zero-approximation, $\tilde{E}_j = 0$, with respect to the intrinsic Josephson coupling $\tilde{E}_j$. The phases $\chi_{ij}$ are written as $\chi_{ij} + 2\pi M_{ij}(k_B T)/\hbar$, where $\chi_{ij}$ is a periodic function on the interval $0 < \tau < 2\pi$ and $M_{ij}$ are the winding numbers.

In the high temperature regime $k_B T \gg E_c$ we can neglect all nonzero winding numbers. Indeed, the nonzero winding numbers contribution to the $K(\tau)$ can be estimated as:

$$K^{(M)}(\tau) \simeq \exp\{(M_{ij} - M_{kl})^2 / E_c \} \frac{2\pi k_B T}{\hbar} \frac{\pi^2 k_B T}{E_c} \left[ M_{ij} - M_{kl} \right]^2.$$\[ Therefore, on the time scale $\tau \ll \hbar / E_c$ the contribution of nonzero windings numbers is small. Since the characteristic time $\tau \simeq \hbar / (k_B T)$ in the integral over time in the Eq. (25), we neglect all nonzero winding numbers in the limit $E_c/k_B T \ll 1$. The nonzero winding numbers become important at low temperatures, $k_B T \ll E_c$.
Next, we expand the periodic phases $\chi_{ij}(\tau)$ over the Matsubara frequencies (see also Appendix):

$$
\chi_{ij}(\tau) = \sum_{\omega_n = 2\pi k_BTn/h} e^{i\omega_n \tau} \chi_{ij}(\omega_n),
$$

(35)

and change the variables in the integrals over $\chi_{ij}(\omega_n)$:

$$
\chi_{ij}(\omega_n) = x_{ij}(2E_c k_BT/\hbar\omega_n^2)[\exp(-i\omega_n \tau) - 1].
$$

(36)

Substituting (35) and (36) into (34) yields the correlation function $K(t)$ in the following form

$$
K(\tau) = \int D[x_{ij}] \exp \left\{ \sum_{\omega_n} \left[ \frac{16E_c k_BT \sin^2(\omega_n \tau/2)}{\hbar^2 \omega_n^2} \right. \right.
\left. \times \left[ i(x_{ij} - x_{Nj}) - \sum_{(ij,kl)} \frac{1}{2} (x_{ij} - x_{kl})^2 - \sum_{ij} \frac{E_c x_{ij}^2}{2E_c0} \right] \right\}.
$$

(37)

The function $K(\tau)$ is the periodic function of $\tau$ with the period $= \hbar/(k_BT)$. Therefore, it is enough to calculate the sum over $\omega_n$ in (37) for $0 < \tau < \hbar/(k_BT)$. In this range of $\tau$

$$
\sum_{\omega_n} \left[ \frac{16E_c k_BT \sin^2(\omega_n \tau/2)}{\hbar^2 \omega_n^2} \right] = \frac{4E_c}{\hbar^2} \tau^2 - \frac{4E_c k_BT}{\hbar^2} \tau^2,
$$

and one finally arrives at

$$
K(\tau) = \exp \left( \frac{4\Delta_c k_BT \tau^2}{\hbar^2} - \frac{4\Delta_c \tau}{\hbar} \right),
$$

(38)

where $\Delta_c$ is the macroscopic Coulomb gap for the Cooper pair propagation defined through the functional integral on the lattice:

$$
\exp(-\Delta_c/k_BT) = \int D[x_{ij}] \exp \left[ \frac{E_c}{k_BT} \left( i(x_{ij} - x_{Nj}) - \sum_{(ij,kl)} \frac{1}{2} (x_{ij} - x_{kl})^2 - \sum_{ij} \frac{E_c x_{ij}^2}{2E_c0} \right) \right] .
$$

(39)

The analytic continuation of the periodic function $K(\tau)$ to the real time $t$, and the corresponding calculation of $\Im mK(t)$ is to be carried out according to the general recipes of the statistical physics. First, we find the quantities $K(\omega_n)$ determined by the Matsubara frequencies ($\omega_n = 2\pi n(k_BT)/\hbar$):

$$
K(\omega_n) = \int_0^{\hbar/(k_BT)} K(\tau) e^{i\omega_n \tau} d\tau.
$$

(40)

[we remind that $K(\tau)$ is the periodic function of $\tau$ on the interval $(0, \hbar/(k_BT))]$. The next step is the analytic continuation $i\omega_n \to \omega + i\delta$, giving rise to the retarded correlation function $K^R$. Performing the inverse Fourier transformation, we get:

$$
K^R(t) = 2\Im mK(t) = \int_{-\infty}^{\infty} K^R \exp(-i\omega) \frac{d\omega}{2\pi}.
$$

To carry out the analytic continuation we transform the integral in (40) from the real axis to the contour in the complex plain, i.e. two lines $iz$ and $\hbar/(k_BT) + iz$, where $z$ runs first from $\infty$ to 0 and then from 0 to $\infty$ (see Fig. 2) and find

$$
K(\omega_n) = i \int_{0}^{\infty} dz \left[ K(iz) - K(i(\omega + \hbar/(k_BT))) e^{-\omega \cdot z} \right].
$$

(41)

After that, we change $i\omega_n$ to $\omega + i\delta$, and performing the inverse Fourier transform, we obtain

$$
\Im mK(t) = -i \left[ K(it) - K(it + \hbar/(k_BT)) \right],
$$

(42)

and in the limit $E_c \ll k_BT \ll \Delta_c$ we finally arrive at

$$
\Im mK(t) = \exp \left( -\frac{4\Delta_c k_BT t^2}{\hbar^2} \right) \sin \left( \frac{4\Delta_c t}{\hbar} \right).
$$

(43)

Calculating the integral over time in Eq. (25) we find the $I$-$V$ dependence in the following form

$$
I \sim (V_1 + V_N) \exp \left[ -\frac{(2\Delta_c - e(V_1 + V_N))^2}{4E_c k_BT} \right].
$$

(44)

Substituting the expression for $V_1 + V_N$ (31) in (41) we obtain the $I$($V$) characteristic of a large JJA in a form:

$$
I \propto \tilde{V} \exp \left[ -\frac{(2\Delta_c - e\tilde{V})^2}{4E_c k_BT\Delta_c} \right],
$$

(45)

with

$$
\tilde{V} = V \left[ 1 + N \frac{E_f^4}{(E_f E_c)^2} \frac{G(T)}{G_0} \right]^{-1}.
$$
Note that if we set $\tilde{E}_J = 0$ in Eq. (45), then the current through the system is zero, $I = 0$, as it should. On the other hand, if the condition \[32\] is satisfied, $V_1 + V_n \approx V$, and the $I$-$V$ curve assumes the simple final form

$$I \approx V \exp \left[-\frac{(2\Delta_c - eV)^2}{4k_B T \Delta_c} \right].$$

(46)

The Gaussian formula (46) was obtained earlier for a single JJ incorporated in a circuit with the high resistance; the corresponding peak in the $I$-$V$ curve was considered a manifestation of the “Coulomb blockade of Cooper-pair tunneling”\[65\]. Experimentally such a peak has been observed in Refs. \[67,68\]. In our case of large JJs one does not expect the similar Gaussian peak in the $I$-$V$ curve. The reason is that on approach of the bias $eV$ to $\Delta_c$, the conductance $G(T)$ grows appreciably and the condition (32) breaks down. The voltage distribution becomes homogeneous and formula (40) does not hold any more.

At low voltages, $eV \ll \Delta_c$, Eq. (46) yields the thermally activated behavior of the resistance:

$$R_{array} \propto \exp \left( \frac{\Delta_c}{k_B T} \right) ,$$

(47)

and therefore we identify the activation temperature $T_0$ of the experiment with the macroscopic Coulomb blockade barrier $\Delta_c/k_B$. In order to carry out calculations in Eq. (45) and determine $\Delta_c$, we consider the standard “spin-wave types” fluctuations:

$$x_{ij} = \int d\vec{p} \exp \left( i\vec{p} \cdot \vec{R}_{ij} \right) x(\vec{p}) .$$

Taking all the Gaussian integrals over $x(\vec{p})$ in Eq. (39) we obtain the expression for $\Delta_c$ in the following form (the vector $\vec{L}$ is directed along the current in JJA):

$$\Delta_c = 2E_c d^{n-2} \int \frac{d^np}{(2\pi)^n} \frac{\sin^2 \frac{p \vec{L}}{2}}{p^2 + (1/\lambda_c)^2} ,$$

(48)

where $\lambda_c = d \sqrt{E_{c0}/E_c}$ is the correlation length in the charged coupled tunnel junction arrays, and $n = 1, 2$ for the 1D and 2D JJs, correspondingly. For a large one-dimensional array, $L \gg \lambda_c$,

$$\Delta_c = E_c \lambda_c / (2d) .$$

(49)

In the opposite limit of shorter one-dimensional arrays, i.e. $L \ll \lambda_c$, the Coulomb gap increases with the array size $L$ linearly:

$$\Delta_c = E_c L / (2d) .$$

(50)

In two-dimensional junction arrays the Coulomb gap acquires the logarithmic form:

$$\Delta_c = E_c \ln \frac{\min \{ \lambda_c, L \}}{d} .$$

(51)

This concludes the description of thermally activated behavior in the temperature interval $E_c < k_B T < \Delta_c$.

IV. TRANSPORT IN JJAS: THE SUPERINSULATING REGIME

Now we turn to low temperatures, $k_B T \ll E_c$, where all the windings numbers $M_{ij}$ have to be taken into account. In a one-dimensional array the calculation of the correlation function $K(t)$ is straightforward. Consider, to be specific, the case where the screening length is larger than the size of a system $L$, then:

$$K(t) = e^{-\frac{n E_c}{2} \left( \sum_n e^{i E_c n t - \frac{n^2 E_c}{2 k_B T}} \right)^N} ,$$

$$A = \left[ \sum_n e^{-\frac{n^2 E_c}{2 k_B T}} \right]^{-1} ,$$

(52)

and the values $n = 0, \pm 1$ give the main contribution. As a result we arrive at

$$K(t) = \exp \left\{ i \Delta_c t + 2 N e^{-E_c/(2 k_B T)} \cos(E_c t - 1) \right\} .$$

(53)

Substituting (53) into Eq. (25) we find the expression for the current as

$$I(V) \propto \exp \left[ - \frac{(eV - \Delta_c)^2}{8 E_c^2 N k_B T} \right] .$$

(54)

This result holds in the temperature range $E_c / \ln(N) < k_B T \ll E_c$.

In the two-dimensional case, the contribution from the nonzero windings numbers is analogous to the vortex contribution which appears in the classical two-dimensional planar Heisenberg model (or classical Josephson junction arrays below the BKT). We will follow the procedure developed for calculation of the vortex contribution to various correlation functions in Ref. \[81\]. Namely, it was shown that the coordinate dependent correlation function

$$g_{(vortex)}^p(\mathbf{r} - \mathbf{r}') = \exp \left\{ i p \left[ \chi(\mathbf{r}) - \chi(\mathbf{r}') \right] \right\} ,$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the two points on the 2D lattice, can be expressed as

$$g_{(vortex)}^p(\mathbf{r} - \mathbf{r}') \simeq \exp \left( -\frac{\pi}{4} p^2 \xi \ln \frac{|\mathbf{r} - \mathbf{r}'|}{d} \right) ,$$

where $\xi = \sum_{n=0}^\infty r_0^2 (m(0)n(r_0))$ is the space correlation function of vortex (charge)-antivortex (anticharge) pairs, diverging near the binding-unbinding transition temperature. Using this result and taking into account the corresponding mapping $p = \tau E_c / h$, we find

$$K(t) = \exp \left( -\frac{\Delta_c E_c \xi t^2}{h^2} - i \frac{2 \Delta_c t}{h} \right) .$$

(55)
Plugging (55) into (25) and calculating integral over time, we obtain, at low voltages, the following expression for the resistance of the array:

$$R_{\text{array}} \propto \exp\left(\frac{\Delta c}{E_c \xi}\right).$$

At low temperatures, $k_B T \ll E_c$, the concentration of the charge-anticharge pairs is small and accordingly, $\xi = \text{const} \cdot \exp[-E_c/(k_B T)]$ (see Ref. [51]). This gives double-exponential behavior of the resistance

$$R \propto \exp\left[\frac{\Delta c}{E_c} \exp\left(\frac{E_c}{k_B T}\right)\right]$$

in the superinsulating regime. We would like to emphasize here that the double-exponential temperature dependence favors enormously the fulfilling the condition (32) for the inhomogeneous distribution of the voltage drop, which ensures the validity of our approach.

V. A QUALITATIVE PICTURE

To gain physical insight in the transport phenomena near SIT of the large one- and two-dimensional Josephson junction arrays and films, let us discuss the distribution of the electric field in the experimental systems in question. Consider first one- and two-dimensional JJAs. The arrays are comprised of overlapping superconducting platelets (islands) separated by thin oxide layers (see Fig. 3). The related junction capacitance, $C$, well exceeds the capacitance of each constitutive island to the ground, $C_0$. Thus, the total capacitance of the JJA is determined by the capacitance of a junction $C$. If now we place a charge in the array, the induced electric field will remain within the array plane. In other words, one- and two-dimensional arrays can be viewed as systems with the anomalously large dielectric constant $\varepsilon \approx C/C_0$. Accordingly, in 1D arrays charges interact linearly over distances $l \ll \lambda_c$, and in 2D arrays the charges interact logarithmically over scales $l \ll \lambda_c$ (Ref. [4]).

Turning to disordered superconducting films and granular superconductors near the SIT, we recall that in the two phase system in the vicinity of the percolation transition between the conducting and insulating phases, the dielectric constant diverges. The origin of this divergence can be understood from the simple picture of the percolation transition, see Fig. 4. The white spots represent superconducting clusters in the insulating sea (dark blue). The capacitance between the two adjacent clusters is proportional to the length of the insulating layer separating them. Upon approaching the transition from the insulating side of the SIT (see Fig. 3), the length of this layer diverges infinitely. It results in the divergent growth of the effective capacitance of the system, implying the divergence of the dielectric constant. Since the SIT in disordered films and granular superconductors is supposed to be of the percolative nature, one expects that these systems possess anomalously large $\varepsilon$ near the transition. Recently the enhanced dielectric constant was indeed observed near the SIT in ultrathin amorphous beryllium films. We therefore can conclude that superconducting films near the SIT exhibit two dimensional behavior with respect to Coulomb interaction. Note that from the viewpoint of the Coulomb interaction between charges, the 2D JJAs and disordered films near the SIT are alike: what matters is the logarithmically divergent internal structure between the systems in question are irrelevant (at least in the absence of the magnetic field).

The two-dimensional character of the Coulomb inter-
action has important implications for the systems with the size less than the electrostatic screening length $\lambda_c$. Unbinding of logarithmically interacting topological excitations gives rise to the celebrated universal Berezinskii-Kosterlitz-Thouless (BKT) phase transition first introduced in the context of vortices in XY-magnets and extended then to vortex-antivortex pairs in superfluid and superconducting films and Josephson junction arrays.53,65,66,87

On the superconducting side of the superconductor-insulator transition logarithmic interaction between vortices gives rise to BKT transition separating the superconducting low-temperature phase, where vortices and antivortices are bound in pairs, from the high-temperature phase with free vortices. In the high-temperature domain the free motion of vortices breaks down the phase coherence and a superconductor falls into the resistive state with the resistance much less than that in the normal state. At $T = T_{BKT} \approx E_s/k_B$ the phase coherence restores and the 2D array or film becomes superconducting. On the insulating side the film experiences charge binding-unbinding transition at the temperature $T = T_{si} \approx E_c/k_B$ [see Fig. 2(a)] dual to the BKT in the superconducting state. In the high temperature phase, $T > T_{si}$, the charges of either sign form a gas of free 2e charge Cooper pairs. At low temperatures, $T < T_{si}$, the charges of the opposite signs are bound in dipoles. This charge binding-unbinding BKT is a realization of the earlier theoretical observation that in a two-dimensional electrolyte with the logarithmic interaction between ions, the transition at which the ions of the opposite sign become bound in pairs, occurs upon lowering down the temperature of the system.23

Having established a background, we are now in a position to give a qualitative picture of the transport in a large JJA. The electric field induced by the charge placed on a superconducting island (or distributed over several islands) remains trapped within the JJA. Thus a JJA is a one- or two-dimensional system with respect to the electric field distribution. Let us consider the situation where the screening length, $\lambda_c$, appearing due to capacitance to the ground exceeds the sample size $L$. In this case the energy necessary to place an additional Cooper pair into the system is $\Delta_c = E_c(L/d)$ in one- and $\Delta_c = E_c \ln(L/d)$ in two-dimensional case. This Coulomb energy can be presented as $\Delta_c = (2e)^2/2C_{tot}$, where $C_{tot}$ is the total capacitance of the array. In an one-dimensional regular array $C_{tot} = C/N$ (the total capacitance for the system comprised of $N$ capacitors in series). Correspondingly, in a two-dimensional system $C_{tot} \approx C/\ln N$ for the array containing $N \times N$ junctions. The thermally activated transport is thus governed by the activation barrier $\Delta_c$ and the corresponding resistance is

$$R \propto \exp(\Delta_c/T),$$

reproducing the experimentally observed dependence (1) in the Cooper pair insulating state. The thermally activated resistance exists only in a moderate temperatures region $\Delta_c > k_BT > E_c$, above the transition temperature $T_{si}$, where the free charges can propagate across the array. To understand the dynamics below the $T_{si}$, let us notice first that the thermally activated behavior means that the whole array acts in a synchronized manner as a one single superconducting island with the characteristic capacitance $C_{tot}$. In the insulating state $E_c \gg E_f$ the charges at every junctions are fixed and thus by the quantum mechanical uncertainty principle the corresponding phases fluctuate loosely and so do the related local electric fields. However, as the current starts passing the array, all the internal phases synchronize in order to minimize the Joule losses. Thus the phase evolves coherently over the array implying that the whole system behaves as a single superconducting island.

As a next step, one can realize that the Cooper pair propagation across the array can be viewed as a propagation of a charge soliton which is not necessarily confined to a one island. Following49, we introduce the local charge density, $n_s(r)$, which is normalized to give the total soliton energy as $\Delta_c = \int d^2r n_s^2(r)$. The probability for such a local density to appear at point $r$ is proportional to $\exp[-n_s^2(r)/(2\langle \delta n^2 \rangle)]$, where $\langle \delta n^2 \rangle$ is the mean square fluctuation of the local charge density. The Cooper pair current is proportional then to the number of solitons generated per unit time and traversing the array. The latter is proportional to a product of all the above local probabilities at all the points of the system: $8\pi \propto \prod_r [\exp(-n_s^2(r)/(2\langle \delta n^2 \rangle))] = \exp(-1/[2\langle \delta n^2 \rangle]) \propto \exp(-\Delta_c/(2E_c \langle \delta n^2 \rangle))$. At temperatures above the charge binding-unbinding transition, $T_{si} \approx E_c/k_B$, the solitons are unbound and, according to the equipartition theorem, $\langle \delta n^2 \rangle = k_BT/E_c$, giving rise to thermally activated resistance $R \propto \exp(\Delta_c/(k_BT))$. At low temperatures, $T < T_{si}$, the charge solitons and antisolitons are bound, and therefore $\langle \delta n^2 \rangle$ is the probability of breaking these pairs, i.e. $\exp(-E_c/(k_BT))$. This yields a double-exponential resistivity in the superinsulating phase:

$$R \propto \exp \left[ \frac{\Delta_c}{E_c} \exp \left( \frac{E_c}{k_BT} \right) \right].$$

The transition from the thermally activated insulating to superinsulating behavior can be viewed as a manifestation of the fact that $\langle \delta n^2 \rangle$ represents the mean filling density $\overline{n}$ for the energy state $E = E_c$ by the Cooper pairs. The filling density $\overline{n}$, in its turn, is given by the Bose statistics: $\delta n^2 \equiv \overline{n} = [\exp(E_c/(k_BT)) - 1]^{-1}$.

The outlined picture of the Cooper pair transport implies that the internal part of an array acts coherently as a single superconducting island, while the most of the applied voltage drops at the leftmost and rightmost junctions. In other words the system can be viewed as a two-junction system with the capacitance between the central island and leads equal to the total capacitance of the array. The criterion for this scenario to hold at temperatures $T > E_c/k_B$ can be presented as [see Eq. (52)]
above\): \[
N \left( \frac{E_c}{E_c} \right)^2 \exp \left( -\frac{\Delta_c}{k_B T} \right) \ll 1. \tag{60}
\]

One sees that in a one-dimensional system, where \( \Delta_c \simeq N E_c \), the larger the system, the better the criterion \((60)\) is satisfied in compliance with the experimental observation of the insulating behavior of the 1D Josephson array becomes more pronounced with the increase of the system length. In the 2D case the situation is more complicated, but this criterion is satisfied pretty well at low enough temperatures \( k_B T \gtrsim E_c \). In the superinsulating phase, \( k_B T < E_c \), the corresponding criterion following from Eq. \((52)\) is met very well.

We expect that the model of the large Josephson junction array applies fairly well to thin films in the critical region of the superconductor-to-insulator transition, since as we have already noticed, the internal structure is irrelevant with respect to Coulomb properties of the system.

In conclusion, we have shown that the Cooper pair transport in the insulating state of one- and two-dimensional Josephson junction arrays is governed by the macroscopic Coulomb blockade. The macroscopic Coulomb blockade energy \( \Delta_c \simeq E_c (L/d) \) in 1D and \( \Delta_c \simeq E_c \ln(L/d) \) in 2D systems. We have shown that the charge binding-unbinding BKT-like transition separates the insulating high temperature state with the thermally activated conductivity from the low temperature superinsulating state. We have determined the conditions under which the macroscopic Coulomb blockade is realized and the conditions under which the Coulomb blockade activation energy exhibits the system size dependence.

The questions that remain open include:

1. The contribution of the quasiparticle current into the transport properties of large Josephson junction arrays.

2. The role of quantum fluctuations in the transport properties of large Josephson junction arrays.

These and related topics will be a subject of forthcoming publication.

**Acknowledgements**

We are deeply indebted to Yu. Galperin for most useful discussions. This work was supported by the U.S. Department of Energy Office of Science through contract No. DE-AC02-06CH11357, SFB 491 and Alexander von Humboldt Foundation (Germany), and RFBR Grant No. 06-02-16704. The authors will be most grateful for sending copies (in the pdf format) of all the relevant works that were overlooked in this paper, so that we could properly refer to them in our forthcoming publications.\(^{89}\)

**APPENDIX**

As an illustration, we calculate the correlation function for a single JJ case with the Hamiltonian

\[
H_0 = \frac{\hbar^2}{4 E_c} \dot{\phi}^2,
\]

where \( \phi \) is the Josephson phase. In this case,

\[
K(\tau) = \int D\phi e^{i[\phi(\tau) - \phi(0)]} \exp \left[ -\frac{\hbar}{4 E_c} \int_0^{h/(k_B T)} d\tau \dot{\phi}^2 \right],
\]

The calculation is done via expanding \( \dot{\phi}(\tau) \) into a Fourier series:

\[
\dot{\phi}(\tau) = \sum_{\omega_n = \pm 2\pi k_B T n / \hbar} e^{i \omega_n \tau} \dot{\phi}_n.
\]

Replacing the functional integration over \( \dot{\phi}(\tau) \) by integration over Fourier coefficients \( \dot{\phi}_n \) yields

\[
K(\tau) = \prod_n \int d\phi_n \exp \left\{ -\sum_n \left[ \frac{\hbar^2 \omega_n^2 \dot{\phi}_n^2}{4 E_c k_B T} + \phi_n (e^{i \omega_n \tau} - 1) \right] \right\}
\]

\[
= \exp \left[ -\sum_n \frac{4 E_c k_B T}{\hbar^2 \omega_n^2} \sin^2(\omega_n \tau / 2) \right].
\]

In the interval \( 0 < \tau < h/(k_B T) \) the sum over \( n \) yields:

\[
\sum_n \frac{4 E_c k_B T \sin^2(\omega_n \tau / 2)}{\hbar^2 \omega_n^2} = \frac{E_c}{\hbar} \tau - \frac{E_c k_B T \tau^2}{\hbar^2},
\]

and the correlation function assumes the form

\[
K(\tau) = \exp \left( -\frac{E_c \tau}{\hbar} + \frac{E_c k_B T \tau^2}{\hbar^2} \right).
\]

Finally, the changing \( \tau \) to it gives

\[
K(t) = \exp \left( -i \frac{E_c t}{\hbar} - \frac{E_c k_B T t^2}{\hbar^2} \right).
\]

This result coincides with the one obtained by direct calculations using the quantum-mechanical definition of \( K(t) \) (Ref. 77) and 88.

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Notice here, that the method of the analytic continuation used in order to obtain the Eq. 43 leads to the identical results as the direct change $\tau$ to $it$ in the Eq. 25.

Moreover, the same method of the analytic continuation, determined by Eqs. 10 and 11, allows to obtain the Eq. 24 in the general case as the nonzero winding numbers are taken into account.

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