Canonical Equations of Singular Mechanical Systems in Terms of Quasi-coordinates

Zheng Mingliang

1Associate Professor, Department of Mechanical Engineering, Taihu University of Wuxi, Wuxi, Jiangsu, China.
Email: zhmlwxstu@163.com

Corresponding Author: Zheng Mingliang

https://doi.org/10.26782/jmcms.2019.08.00001

Abstract

The constrained mechanical systems by quasi-coordinates are more universal than by generalized coordinates. In this paper, the motion equations of non-conservative singular mechanical systems by quasi-coordinates in phase space are studied. The regularization forms of Boltzmann-Hamel equations for general holonomic and nonholonomic singular mechanical systems are derived. The results show that the canonical equations expressed by quasi-coordinates and quasi-velocities have a completely single structure, which do not depend on the constraints or not. The nonholonomic singular mechanical system is a natural extension of the general holonomic singular mechanical system.

Keywords: Quasi coordinates, Singular mechanical systems, Canonical Equations

I. Introduction

The systems described by singular Lagrange function are called singular systems, and the Hamilton form in phase space is also called constrained Hamilton systems [VIII]. Many important dynamical problems in mathematical physics and engineering technology are in line with the model of constrained Hamilton systems, such as electromagnetic field, transverse phenomenon of light, quantum electrodynamics and superstring theory. Dirac [I-II] and Li [IX], [X], [XI],[XII] firstly studied Noether symmetries and conserved quantities of Hamilton canonical equations and many physical applications for singular systems. Mei [XV] and Zhang [XXI] studied Lie symmetries and conserved quantities for singular systems in configuration space and phase space, Luo [XIII] studied Mei symmetries and conserved quantities of Hamilton canonical equations for singular systems, and the relationship between Mei symmetry and Noether symmetry and Lie symmetry were explained, Li [VII] also studied regular symmetry theory of constrained Hamilton systems under external constraints and compared it with non-singular systems. At present, the study of singular systems is mostly based on generalized coordinates, however, the quasi-coordinates is more universal in the study of analytical mechanics, and quasi-coordinates are used reasonably will bring great convenience to some
problems. Noether symmetries and Noether-type conserved quantities of holonomic systems by quasi-coordinates were discussed in references[XVI-XVII]. Lie symmetries and Hojman-type conserved quantities of holonomic systems by quasi-coordinates were given in reference [XVIII]. Mei symmetries and conserved quantities of holonomic systems by quasi-coordinates were studied in reference [XX]. Lie symmetries and conserved quantities of nonholonomic nonsingular mechanical systems by quasi-coordinates were studied in reference [V]. The latest literatures about quasi-coordinates include [VI], [XIV].

Research Significance

Through literature review, the research foundation of constrained mechanical systems by quasi-coordinates is carried out in configuration space, that is, the equations of motion for the system are Lagrange form, are not Hamilton form. It is well known that Hamilton mechanics have a set of perfect essential characteristics (such as integral theory and canonical transformation theory). Therefore, the study of singular systems by quasi-coordinates in phase space is very important to analytical mechanics. This paper is inspired by the fact that the canonical equations of holonomic and nonholonomic nonsingular systems by quasi-coordinates are similar form, and can be extended to the singular systems in phase space, which lays a foundation for further study of symmetries and conserved quantities by quasi-coordinates for singular systems.

II. Hamilton's canonical equation for holonomic singular systems by quasi-coordinates

Assume that the configuration of the system is specified by n generalized coordinates \( q_s (s = 1, \ldots, n) \), the Lagrange function is \( L(t, \mathbf{q}, \dot{\mathbf{q}}) = T - V \), the nonconservative generalized forces imposed on the system are \( \mathbf{Q}_s (t, \mathbf{q}, \dot{\mathbf{q}}) \), taking quasi velocities are:

\[
\omega_k = \omega_k (q_s, \dot{q}_s, t) = a_{ks}(q_s) \dot{q}_s (s, k = 1, 2, \ldots, n)
\]

Inverse solution form (1) \((\det[a_{ks}] \neq 0)\) are:

\[
\dot{q}_s = \dot{q}_s (q_k, \omega_k, t) = b_{sk}(q_k) \omega_k
\]

Here \( b_{sk} \) is the inverse elements of matrix \([a_{sk}]\). Introducing \( \pi_s \) is called quasi-coordinate:

\[
\omega_s = \dot{\pi}_s
\]

Here \( \dot{\pi}_s \) is a symbol and is not derivatives to time.

The relationship between quasi-coordinates and generalized coordinates are [IV]:

\[
\delta q_s = \frac{\partial \dot{q}_s}{\partial \omega_k} \delta \pi_k, \delta \pi_k = \frac{\partial \omega_k}{\partial \dot{q}_s} \delta q_s
\]

\[
\frac{\partial}{\partial \pi_s} = b_{ks} \frac{\partial}{\partial q_k}
\]
The Lagrange function $L$ and non-conservative forces $Q^r$ of the system expressed by quasi-velocities have a certain transformation relationship with the corresponding function by generalized coordinates:

$$L^* = L(t, q, \omega) = L(t, q, b_k \omega_k)$$

$$Q^s = Q^s(t, q, \omega) = Q_k b_k$$

(s, k = 1, 2, ..., n) (5)

Introducing generalized momentums are:

$$p_s = \frac{\partial L^*}{\partial \omega_s} \Rightarrow \omega_s = d_{sk} p_k + e_s$$

(6)

According to the inverse theorem of Legendre transformation, the Hamilton function $H(t, q, p)$ by quasi-coordinates is:

$$H(t, q, p) = (p_s \omega_s - L^*)_{\omega_s \to p_s}$$

(7)

The generalized Boltzmann-Hamel equations for constrained Hamilton systems by quasi-coordinates are [III], [XIX]:

$$\frac{d}{dt} \frac{\partial L^*}{\partial \pi_s} - \frac{\partial L^*}{\partial q_s} + \sum_{r=1}^{n} \sum_{\nu=1}^{n} \frac{\partial L^*}{\partial \omega_r} \gamma^r_{\nu s} \omega_s = Q^s_s (\sigma = 1, 2, ..., n)$$

(8)

Here $\gamma^r_{\nu s}$ is a symbol of Boltzmann.

$$\gamma^r_{\nu s} = \sum_{m=1}^{n} \sum_{\nu=1}^{n} \left( \frac{\partial a_{r m}}{\partial q_m} - \frac{\partial a_{s m}}{\partial q_s} b_{r \nu} b_{s \nu} \right)$$

(9)

Because the system is singular, that is $\det \left[ \frac{\partial^2 L^*}{\partial \omega_s \partial \omega_t} \right] = 0$, it is impossible to solve all generalized quasi-accelerations by (8), but a part of generalized quasi-accelerations can be solved and there are $r$ relations:

$$\beta_j (t, q, \omega) \bigg|_{\omega_s \to p_s} = 0 (j = 1, 2, ..., r)$$

(10)

The variational principle of non-conservative singular systems by quasi-coordinates in phase space should be as follows:

$$\delta \mathcal{A} = \delta \int_{t_1}^{t_2} (p_s \delta \omega_s - H) dt = \int_{t_1}^{t_2} p_s \delta \omega_s + \omega_s \delta p_s - \frac{\partial H}{\partial \pi_s} \delta \pi_s - \frac{\partial H}{\partial p_s} \delta p_s dt$$

$$= \int_{t_1}^{t_2} \left[ (p_s - \frac{\partial H}{\partial \pi_s}) \delta \pi_s + (\omega_s - \frac{\partial H}{\partial p_s}) \delta p_s \right] dt + \int_{t_1}^{t_2} \frac{d}{dt} (p_s \delta \pi_s) dt$$

$$= \int_{t_1}^{t_2} \left[ (p_s - \frac{\partial H}{\partial \pi_s}) \delta \pi_s + (\omega_s - \frac{\partial H}{\partial p_s}) \delta p_s \right] dt + \left[ p_s \delta \pi_s \right]_{t_1}^{t_2}$$

(11)

$$= \int_{t_1}^{t_2} \left[ (p_s - \frac{\partial H}{\partial \pi_s}) \delta \pi_s + (\omega_s - \frac{\partial H}{\partial p_s}) \delta p_s \right] dt + \int_{t_1}^{t_2} \left( p_s \gamma^r_{\nu s} \frac{\partial H}{\partial p_s} + Q^s_s \delta \pi_s \right) dt$$

(s, r, v = 1, 2, ..., n)

For singular systems, internal constraints (10) is suitable for $q_i, p_i$.
Using (4), have
\[ \frac{\partial \beta}{\partial q_i} \frac{\partial}{\partial q_i} + \frac{\partial \beta}{\partial p_i} \frac{\partial}{\partial p_i} = 0 (j = 1, 2, \ldots, r) \]  \hspace{1cm} (12)

Because of the independence of \( \delta \pi_s, \dot{\delta} \pi_s \), the canonical equations of holonomic singular systems by quasi-coordinates in phase space can be obtained as follows:

\[ \left\{ \begin{array}{l}
\dot{\pi}_i = \frac{\partial H}{\partial \pi_i} + \lambda_j \frac{\partial \beta_j}{\partial p_i} \\
\dot{p}_i = -\frac{\partial H}{\partial \pi_i} - \lambda_j \frac{\partial \beta_j}{\partial \pi_i} - \gamma_i^s \frac{\partial H}{\partial p_i} + Q_i^s \end{array} \right. \hspace{1cm} (l, r, \nu = 1, 2, \ldots, n; j = 1, 2, \ldots, r) \]  \hspace{1cm} (14)

The form of (14) is completely similar to the holonomic non-conservative non-singular systems, but there are internal constrained equations (10) and additional equations (12). It shows that the singularity of the system by quasi-coordinates does not affect the single structure of the canonical equations, but strengthens the restriction conditions of motion.

III. Hamilton's canonical equations for nonholonomic singular systems by quasi-coordinates

Set the motion of the system is subject to \( g \) ideal nonholonomic constraints:

\[ f_i(t, q, \dot{q}) = 0, (i = 1, \ldots, g) \]  \hspace{1cm} (15)

Reasonable selecting the coefficient matrix function \( [a_{ks}(q)] \) of quasi-velocity, so (15) and generalized momentum can be transformed into:

\[ \omega_{\varepsilon \nu} = 0, \delta \pi_{\varepsilon \nu} = 0, p_{\varepsilon \nu} = 0 (\varepsilon = n - g) \]  \hspace{1cm} (16)

Continuing to repeat the derivation in Section II. The variational principle of nonconservative singular systems by quasi-coordinates in phase space under nonholonomic constraints (16) becomes:

\[ \delta l = \delta \int_{h_1}^{i_1} (p_i \omega_i - H) dt = \int_{h_1}^{i_1} p_i (\omega_i - H) dt + \int_{h_1}^{i_1} \frac{d}{dt} (p_i \delta \pi_i) dt \]

\[ = \int_{h_1}^{i_1} [(- \dot{p}_i - \frac{\partial H}{\partial \pi_i}) \delta \pi_i + (\omega_i - \frac{\partial H}{\partial p_i}) \delta p_i] dt \]  \hspace{1cm} (17)

\[ = \int_{h_1}^{i_1} [(- \dot{p}_i - \frac{\partial H}{\partial \pi_i}) \delta \pi_i + (\omega_i - \frac{\partial H}{\partial p_i}) \delta p_i] dt \]  \hspace{1cm} (17)

\[ = \int_{h_1}^{i_1} [(- \dot{p}_i - \frac{\partial H}{\partial \pi_i}) \delta \pi_i + (\gamma_i^s \frac{\partial H}{\partial p_i}) \delta p_i] dt \]  \hspace{1cm} (17)

\[ = \int_{h_1}^{i_1} (\dot{p}_i - \frac{\partial H}{\partial p_i}) \delta \pi_i + (\gamma_i^s \frac{\partial H}{\partial p_i}) \delta p_i] dt \]  \hspace{1cm} (17)

\[ (\sigma = 1, 2, \ldots, \varepsilon; s, r, \nu = 1, 2, \ldots, n) \]
The (12) can be transformed into:

$$\frac{\partial \beta_j}{\partial \pi_s} \delta \pi_s + \frac{\partial \beta_j}{\partial p_\sigma} \delta p_\sigma = 0 (j = 1, 2, \ldots, r)$$

(18)

So we also get the canonical equations of nonholonomic singular systems by quasi-coordinates in phase space as follows:

$$\dot{\pi}_\sigma = \frac{\partial H}{\partial p_\sigma} + \lambda_j \frac{\partial \beta_j}{\partial \pi_s}$$

$$\dot{\gamma}_\sigma = \frac{\partial H}{\partial \pi_j} - \lambda_j \frac{\partial \beta_j}{\partial \pi_s} - \gamma'_\sigma p_\sigma \frac{\partial H}{\partial p_\sigma} + Q_\sigma$$

(19)

We find that the (19) is formal similarity to the canonical equations (14) of holonomic singular system by quasi-coordinates, but due to the external constraints, the number of differential equations of the system is reduced and is replaced by the algebraic equations (16).

IV. Example illustration

Let the Lagrange function and the non-potential generalized forces of a two-degree-of-freedom holonomic mechanical system are:

$$L = \frac{1}{2} \dot{q}_1^2 + q_1 \dot{q}_2$$

$$Q_1 = -\frac{q_1}{q_1}, Q_2 = q_1 \dot{q}_2$$

(20)

Solving the Hamilton canonical equations by quasi-coordinates.

The rank of the Hess matrix of Lagrange function $1 < 2$, which is a singular system. The quasi-velocities are taken $\omega_1 = \dot{q}_1, \omega_2 = \dot{q}_1 - q_1 \dot{q}_2$, the inverse solution can be obtained as $\dot{q}_1 = \omega_1, \dot{q}_2 = (\omega_1 - \omega_2) \frac{1}{q_1}$. The generalized momentums are:

$$p_1 = \omega_1 + 1, p_2 = -1$$

(21)

The Hamilton function is:

$$H = p_1 \omega_1 + p_2 \omega_2 - \frac{1}{2} \omega_1^2 - \omega_1 + \omega_2$$

(22)

After calculating the symbol of Boltzmann $\gamma'_\sigma = 0$, then the internal constraint equation of the system is:

$$\omega_2 = 0$$

(23)

The Laplace multiplier is $\lambda = -q_1^2$.

Then the canonical equations of the system are:

$$\dot{\pi}_1 = p_1 - 1$$

$$\dot{\pi}_2 = \omega_2 = 0$$

$$\dot{p}_1 = -\frac{q_1}{\omega_1} - q_1 \omega_1$$

$$\dot{p}_2 = 0$$

(24)
As can be seen from the example, the differential equations of motion of the system can be described simply by using the quasi-coordinates.

V. Results and Discussions

In summary, the canonical equations of non-conservative singular mechanical systems by quasi-coordinates in phase space are studied in this paper. By reasonably constructing the quasi-velocity and combining the modified variational principle, the regular form of Boltzmann-Hamel equations for holonomic and nonholonomic singular systems is obtained, which greatly simplifies the influence of constrained multipliers. The equations retain the single structural form of canonical equations for nonsingular mechanical systems. The quasi-coordinates can effectively deal with the inherent constraints caused by singularity, and can be further extended to the symmetries and conserved quantities of singular systems.

VI. Acknowledgements

The author wish to gratefully acknowledge the support of Natural Science Foundation of Jiangsu higher education institutions (18KJB460027), China.

References

I. Dirac P. A. M., “Quantization of Singular systems”, Can J Math, vol.2, pp:122-129, 1950.
II. Dirac P. A. M., “Lecture on Quantum Mechanics”, Book, Yeshi-va University Press, pp:8-16, 1964.
III. Dong L. X., Liang J. H., “Lie Symmetries and Conserved Quantities of Nonholonomic Singular Mechanical Systems in Terms of Quasi-coordinates”, Jiangxi Science, vol.33, no.1, pp:61-65, 2015.
IV. Fasso F., Sansonetto N., “Conservation of 'Moving' Energy in Nonholonomic Systems with Affine Constraints and Integrability of Spheres on Rotating Surfaces”, Journal of Non-Linear Science, vol.26, no.2, pp:519-544, 2016.
V. Fu J. L., Liu R. W., “Lie symmetries and conserved quantities of nonholonomic mechanical systems in quasi coordinates”, Journal of Mathematical Physics, vol.20, no.1, pp:63-69, 2000.
VI. Jahromi A. F., Bhat R. B., Xie W. F., “Integrated ride and handling vehicle model using Lagrangian quasi-coordinates”, International Journal of Automotive Technology, vol.16, no.2, pp:239-251, 2015.
VII. Li A. M., “Canonical symmetry of nonholonomic constraint systems”, Journal of Wuhan Technical College of Communication, vol.15, no.3, pp:1-3, 2013.
VIII. Li Z. P., “Constrained Hamiltonian systems and their symmetry properties”, Book, Beijing University of Technology Press, pp:3-8, 1999.
IX. Li Z. P., Jiang J. H., “Symmetries in Constrained Canonical System”, Book, Science Press, pp:9-19, 2002.

X. Li Z. P., “Generalized Noether theorem of regular form of nonholonomic singular systems and its inverse theorem”, HuangHuai Journal, vol.3, no.1, pp:8-16, 1992.

XI. Li Z. P., “Noether theorem of regular form and its application”, Science Bulletin, vol.36, no.12, pp:954-958, 1991.

XII. Li Z. P., “Classical and quantum symmetry properties of Constrained Systems”, Book, Beijing University of Technology Press, pp:32-46, 1993.

XIII. Luo S. K., “Mei symmetries, Noether symmetries and Lie symmetries of Hamilton canonical equations of singular systems”, Acta phys Sinica, vol.53, no.1, pp: 5-11, 2004.

XIV. Mahmoudkhani S., “Dynamics of a mass-spring-beam with 0:1:1 internal resonance using the analytical and continuation method”, International Journal of Non-Linear Mechanics, vol.97, pp:48-67, 2017.

XV. Mei F. X., Zhu H. P., “Lie symmetries and conserved quantities of singular Lagrange systems”, Journal of Beijing Institute of Technology, vol.9, no.1, pp:11-14, 2000.

XVI. Mei F. X., “The application of Li Group and Lie algebra to the constrained mechanical system”, Book, Science Press, pp:126-137, 1999.

XVII. Mei F. X., “Analytical mechanics (2)”, Beijing Institute of Technology Press, Book, pp:145-162, 2013.

XVIII. Qiao Y. F., Zhao S. H., “Lie symmetric theorem and inverse theorem of generalized mechanical systems in quasi coordinates”, Acta phys Sinica, vol.50, no.1, pp:1-7, 2001.

XIX. Wang X. J., Zhao X. L., Fu J. L., “Noether symmetry of nonholonomic systems in quasi coordinates in phase space”, Journal of Henan Institute of Education, vol.13, no.2, pp:21-23, 2004.

XX. Xu Z. X., Mei F. X., “Unified symmetry of general holonomic systems under quasi coordinates”, Acta phys Sinica, vol.54, no.12, pp:5521-5524, 2005.

XXI. Zhang Y., Xue Y., “Lie symmetry constraint Hamilton systems with the second type of constraints”, Acta phys Sinica, vol.50, no.5, pp:816-819, 2001.