Asynchronous Iterations in Optimization: New Sequence Results and Sharper Algorithmic Guarantees

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Abstract

We introduce novel convergence results for asynchronous iterations which appear in the analysis of parallel and distributed optimization algorithms. The results are simple to apply and give explicit estimates for how the degree of asynchrony impacts the convergence rates of the iterates. Our results shorten, streamline and strengthen existing convergence proofs for several asynchronous optimization methods, and allow us to establish convergence guarantees for popular algorithms that were thus far lacking a complete theoretical understanding. Specifically, we use our results to derive better iteration complexity bounds for proximal incremental aggregated gradient methods, to provide less conservative analyses of the speedup conditions for asynchronous block-coordinate implementations of Krasnosel’ski–Mann iterations, and to quantify the convergence rates for totally asynchronous iterations under various assumptions on communication delays and update rates.

1 Introduction

With the ubiquitous digitalization and automation of society, decision problems are rapidly expanding in size and scope. Increasingly often, we face problems where data, computations, and decisions need to be distributed on multiple nodes. These nodes may be individual cores in a CPU, different processors in a multi-CPU platform, or servers in a geographically dispersed cluster. Representative examples include control and coordination of infrastructure-scale systems, machine learning on data sets that are too large to conveniently store in a single computer, and real-time decision-making based on high-velocity data streams.

Insisting that such multi-node systems operate synchronously limits their scalability, since the performance is then dictated by the slowest node, and the system becomes fragile to node failures. Hence, there is a strong current interest in developing asynchronous algorithms for optimal decision-making (see, e.g., \cite{Bertsekas1997,Tsitsiklis1986,Chazan1969,Chazan1972} and references therein). Well-established models for parallel computations, such as bulk synchronous parallel \cite{Chazan1972} or MapReduce \cite{Dean2008}, are now being complemented by stale-synchronous parallel models \cite{Li2010} and fully asynchronous processing paradigms \cite{Li2010}. In many of these frameworks, the amount of asynchrony is a design parameter: in some systems, the delay is proportional to the number of parallel workers deployed \cite{Li2010}; and other systems use communication primitives which enforce a hard limit on the maximum information delay \cite{Li2010}. It is therefore useful to have theoretical results which characterize the level of asynchrony that can be tolerated by a given algorithm. To this end, this paper develops several theoretical tools for studying the convergence of asynchronous iterations.

The dynamics of asynchronous iterations are much richer than their synchronous counterparts and quantifying the impact of asynchrony on the convergence rate is mathematically challenging. Some of the first results on the convergence of asynchronous iterations were derived by Chazan and Miranker \cite{Chazan1969} for solving linear equations. This work was later extended to nonlinear iterations involving maximum norm contractions \cite{Chazan1972} and monotone mappings \cite{Chazan1972}. Powerful convergence results for broad classes of asynchronous iterations under different assumptions on communication delays and update rates were presented by Bertsekas \cite{Bertsekas1997}, Tsitsiklis et al. \cite{Tsitsiklis1993}, and in the celebrated book of Bertsekas and Tsitsiklis \cite{Bertsekas1997}. Although the framework for modeling
asynchronous iterations in [11] is both powerful and elegant, the most concrete results only guarantee asymptotic convergence and do not give explicit bounds on convergence times. Execution time guarantees are essential when iterative algorithms are used to find a decision under stringent real-time constraints. In this paper, we derive a number of convergence results for asynchronous iterations which explicitly quantify the impact of asynchrony on the convergence times of the iterates.

The convergence guarantees for influential asynchronous optimization algorithms such as Hogwild! [55], Delayed Sgd [1], AsySCD [44], ARock [57] and Asaga [38] have been established on a per-algorithm basis, and are often based on intricate induction proofs. Such proofs tend to be long and sources of conservatism are hard to isolate. A closer analysis of these proofs reveals that they rely on a few common principles. In this paper, we attempt to unify these ideas, derive general convergence results for the associated sequences, and use these results to systematically provide stronger guarantees for several popular asynchronous algorithms. In contrast to the recent analysis framework proposed in [46], which model the effect of asynchrony as noise, our results attempt to capture the inherent structure in the asynchronous iterations. In this way, we are able to derive convergence results for complex optimization algorithms in a systematic and transparent manner, without introducing unnecessary conservatism. We make the following specific contributions:

- We identify two important families of sequences that appear when common optimization algorithms are implemented in an asynchronous manner. For each family, we derive convergence results that allow to characterize how the degree of asynchrony affects the convergence rate guarantees. These results extend and streamline our earlier work [5 23 25], and recover the optimal convergence results for synchronous iterations in the absence of asynchrony.

- We use these sequence results to analyze several popular asynchronous optimization algorithms. First, we derive stronger convergence guarantees for the proximal incremental gradient method and provide a larger range of admissible step-sizes. Second, we give an improved analysis of the ARock framework for asynchronous block-coordinate updates of Krasnosel’ski–Mann iterations, proving faster convergence rates and better scaling properties with respect to the number of parallel computing elements. Finally, we present a uniform treatment of asynchronous iterations involving block-maximum norm contractions under partial and (several versions of) total asynchronism.

1.1 Notation and Preliminaries

Here, we introduce the notation and review the key definitions that will be used throughout the paper. We let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of real numbers, natural numbers, and the set of natural numbers including zero, respectively. For any $n \in \mathbb{N}$,

$$[n] := \{1, \ldots, n\}.$$  

For a real number $a$, we denote the largest integer less than or equal to $a$ by $\lfloor a \rfloor$ and define

$$(a)_+ := \max\{a, 0\}.$$  

We use $\| \cdot \|$ to represent the standard Euclidean norm on $\mathbb{R}^d$ and $\langle x, y \rangle$ to denote the Euclidean (dot) inner product of two vectors $x, y \in \mathbb{R}^d$. We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is $L$-smooth if it is differentiable and its gradient is Lipschitz continuous with constant $L$, i.e.,

$$\| \nabla f(y) - \nabla f(x) \| \leq \| y - x \|, \quad \forall x, y \in \mathbb{R}^d.$$  

We say a convex function $f : \mathbb{R}^d \to \mathbb{R}$ is $\mu$-strongly convex if

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \| y - x \|^2, \quad \forall x, y \in \mathbb{R}^d.$$  

The notation $g(t) = O(h(t))$ means that there exist positive constants $M$ and $t_0$ such that $g(t) \leq Mh(t)$ for all $t \geq t_0$, while $g(t) = o(h(t))$ means that $\lim_{t \to \infty} g(t)/h(t) = 0$. 

2
2 Novel Sequence Results for Asynchronous Iterations

Convergence proofs for optimization algorithms are usually based on induction and often presented without invoking any general theorems. For more complex algorithms, this leads to lengthy derivations where it is difficult to distinguish mathematical innovations. The need to systemize convergence proofs for optimization algorithms was recognized in Polyak’s insightful textbook [58]. Polyak argued that most results concerning convergence and rate of convergence of optimization algorithms can be derived using Lyapunov’s second method, with typical Lyapunov functions being the objective function value, the norm of its gradient, or the squared distance between the current iterate and the optimum, or the rate of convergence to the optimum and the optimal set. In addition, he derived and collected a number of useful sequence results which allowed to shorten, unify, and clarify many convergence proofs [58, Chapter 2]. To make these ideas more concrete, consider the simple gradient descent method

\[ x_{k+1} = x_k - \gamma \nabla f(x_k), \quad k \in \mathbb{N}_0. \]

Assume that \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \mu \)-strongly convex and \( L \)-smooth. If \( x^\star \) is the minimum point of \( f \) on \( \mathbb{R}^d \), the standard convergence proof (e.g., [52, Theorem 2.1.15]) establishes that the iterates satisfy

\[ \|x_{k+1} - x^\star\|^2 \leq \left(1 - \frac{2\gamma \mu L}{\mu + L}\right) \|x_k - x^\star\|^2 - \gamma \left(\frac{2}{\mu + L} - \gamma\right) \|
abla f(x_k)\|^2. \]

We can rewrite this as

\[ V_{k+1} \leq \left(1 - \frac{2\gamma \mu L}{\mu + L}\right) V_k - \gamma \left(\frac{2}{\mu + L} - \gamma\right) W_k, \]

where \( V_k = \|x_k - x^\star\|^2 \) and \( W_k = \|\nabla f(x_k)\|^2 \). For step-sizes \( \gamma \in (0, 2/(\mu + L)] \), the second term on the right-hand side of (1) can be dropped and, hence, the Lyapunov function \( V_k \) is guaranteed to decay by at least a factor \( q \in (0, 1) \):

\[ V_{k+1} \leq q V_k, \quad k \in \mathbb{N}_0. \]

This implies linear convergence of the iterates to the optimum, i.e.,

\[ V_k \leq q^k V_0, \quad k \in \mathbb{N}_0. \]

When \( f \) is convex, but not necessarily strongly convex, and \( L \)-smooth, the iterates satisfy

\[ 2\gamma (f(x_{k+1}) - f(x^\star)) + \|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - \gamma \left(\frac{1}{L} - \gamma\right) \|
abla f(x_k)\|^2. \]

Let \( V_k \) and \( W_k \) be as before, while \( X_k = 2\gamma (f(x_k) - f(x^\star)) \). Then, the inequality above can be rewritten as

\[ X_{k+1} + V_{k+1} \leq V_k - \gamma \left(\frac{1}{L} - \gamma\right) W_k. \]

If we let \( \gamma \in (0, 1/L] \), the second term on the right-hand side is non-positive and can be dropped. Summing both sides of (2) over \( k = 0, \ldots, K - 1 \) and using telescopic cancellation then gives

\[ \sum_{k=1}^K X_k + V_K \leq V_0, \quad K \in \mathbb{N}. \]

The well-known \( \mathcal{O}(1/k) \) convergence rate of \( f(x_k) \) to \( f(x^\star) \) follows from the fact that \( V_k \) is non-negative and that \( X_k \) is non-increasing [27, Theorem 10.21].

In this section, we will provide sequence results similar to (1) and (2) adapted for asynchronous iterations. Because of asynchrony, the right-hand side of the inequalities will involve delayed versions of \( V_k \) and \( W_k \) that perturb the convergence of the synchronous iteration. For example, in the analysis of the incremental aggregated gradient method [27], accelerated incremental aggregated gradient method with curvature information [67],
asynchronous quasi-Newton method \[21\], and asynchronous forward–backward method for solving monotone inclusion problems \[60\], one can establish iterate relationships on the form

\[ V_{k+1} \leq qV_k + p \max_{(k-\tau_k)+ \leq \ell \leq k} V_\ell - rW_k, \quad k \in \mathbb{N}_0. \]  

(3)

In this case, the perturbation caused by asynchrony at iteration \(k\) is modeled as a function on the order of \(V_\ell\) scaled by a factor \(p\), where \(\ell \in [k-\tau_k, k]\) and \(\tau_k\) is the age of the outdated information. We will show in this paper how iterates of the proximal incremental aggregated gradient method \[5\] and asynchronous Krasnosel’skiĭ–Mann method \[57\] for pseudo-contractive operators satisfy relationships on the form

\[ X_{k+1} + V_{k+1} \leq qV_k + p \sum_{\ell=(k-\tau_k)+}^k W_\ell - rW_k, \quad k \in \mathbb{N}_0. \]  

(4)

Here, the perturbation due to asynchrony does not introduce delayed \(V_k\)-terms, but manifests itself through the presence of delayed \(W_k\) terms instead.

In the remainder of this section, we will develop specific convergence results for iterations on the form \[3\] and \[4\]. Our results attempt to balance simplicity, applicability and power, and provide explicit bounds on how the amount of asynchrony affects the guaranteed convergence rates. As we will demonstrate later, the results allow for a simplified and uniform treatment of several asynchronous optimization algorithms.

Our first result, introduced in \[23\], establishes convergence properties of iterations on the form \[3\] when delays are bounded.

**Lemma 2.1.** Let \(\{V_k\}\) be a non-negative sequence satisfying

\[ V_{k+1} \leq qV_k + p \max_{(k-\tau_k)+ \leq \ell \leq k} V_\ell, \quad k \in \mathbb{N}_0, \]  

(5)

for some non-negative constants \(q\) and \(p\). Suppose there is a non-negative integer \(\tau\) such that

\[ 0 \leq \tau_k \leq \tau, \quad k \in \mathbb{N}_0. \]

If \(q + p < 1\), then

\[ V_k \leq \rho^k V_0, \quad k \in \mathbb{N}_0, \]

where \(\rho = (q + p)^{1/\tau}\).

**Proof.** See \[23\] Lemma 3].

Consider the delay-free counterpart of \[5\]:

\[ V_{k+1} \leq (q + p)V_k, \quad k \in \mathbb{N}_0. \]

Clearly, if \(q + p < 1\), the sequence \(\{V_k\}\) converges linearly at a rate of \(\rho = q + p\). Lemma 2.1 shows that the convergence rate of \(\{V_k\}\) is still linear in the presence of bounded delays. Lemma 2.1 also gives an explicit bound on the impact that an increasing delay has on the convergence rate. As can be expected, the guaranteed convergence rate deteriorates with increasing \(\tau\). More precisely, \(\rho\) is monotonically increasing in \(\tau\), and approaches one as \(\tau\) tends to infinity.

The next result extends Lemma 2.1 to a family of unbounded delays, which allows to deal with *totally asynchronous iterations* \[11\] Chapter 6], and shows that the sequence \(\{V_k\}\) can still be guaranteed to converge.

**Lemma 2.2.** Let \(\{V_k\}\) be a non-negative sequence such that

\[ V_{k+1} \leq qV_k + p \max_{(k-\tau_k)+ \leq \ell \leq k} V_\ell, \quad k \in \mathbb{N}_0, \]  

(6)
for some non-negative scalars $q$ and $p$. Suppose that the delay sequence $\{\tau_k\}$ satisfies
\[
\lim_{k \to +\infty} k - \tau_k = +\infty. \tag{7}
\]

If $q + p < 1$, then $\{V_k\}$ asymptotically converges to zero:
\[
\lim_{k \to +\infty} V_k = 0.
\]

**Proof.** See Appendix A.1. ■

Lemma 2.2 provides a test for asymptotic convergence of asynchronous iterations with delays satisfying condition (7). Roughly speaking, (7) implies that as $k$ increases, the delay $\tau_k$ grows slower than time itself. This constraint on delays is typically satisfied in asynchronous optimization algorithms. For example, assume that the update step in the gradient descent method is based on gradients computed at stale iterates rather than the current iterate, i.e.,
\[
x_{k+1} = x_k - \gamma \nabla f(x_{k-\tau_k}).
\]

If the delay $\tau_k$ satisfies (7), then given any $K_1 \in \mathbb{N}$, there exists $K_2 \in \mathbb{N}$ such that
\[
k - \tau_k \geq K_1, \quad \forall k \geq K_2.
\]

This means that gradients computed with respect to out-of-date information prior to $K_1$ will not be used to update the current iterate after iteration $K_2$. Thus, (7) guarantees that old information is eventually purged from the computations. Note that assumption (7) holds for bounded delays, irrespectively of whether they are constant or time-varying, and unbounded delays such as $\tau_k = \lfloor 0.2k \rfloor$ and $\tau_k = \lfloor \sqrt{k} \rfloor$.

Although Lemma 2.2 establishes convergence guarantees for the sequence $\{V_k\}$ also under unbounded delays, it no longer provides any finite-time guarantee or rate of convergence. The next result demonstrates that such guarantees can be obtained when we restrict how the possibly unbounded delay sequence is allowed to evolve.

**Lemma 2.3.** Let $\{V_k\}$ be a non-negative sequence satisfying
\[
V_{k+1} \leq qV_k + p \max_{(k-\tau_k) + \ell \leq k} V_\ell, \quad k \in \mathbb{N}_0, \tag{8}
\]

for some non-negative constants $q$ and $p$ such that $q + p < 1$. In addition, assume that there exists a function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that the following conditions hold:

(i) $\Lambda(0) = 1$.

(ii) $\Lambda$ is non-increasing.

(iii) $\lim_{k \to +\infty} \Lambda(k) = 0$ and
\[
(q + p)\Lambda(k - \tau_k) \leq \Lambda(k + 1), \quad k \in \mathbb{N}_0. \tag{9}
\]

Then $V_k \leq \Lambda(k)V_0$ for all $k \in \mathbb{N}_0$.

**Proof.** See Appendix A.2. ■

According to Lemma 2.3, any function $\Lambda$ satisfying conditions (i)–(iii) can be used to quantify how fast the sequence $\{V_k\}$ converges to zero. For example, if $\Lambda(t) = \rho^t$ with $\rho \in (0, 1)$, then $\{V_k\}$ converges at a linear rate; and if $\Lambda(t) = t^{-\eta}$ with $\eta > 0$, then $\{V_k\}$ is upper bounded by a polynomial function of time. Given $q$ and $p$, it is clear from (9) that the admissible choices for $\Lambda$ and, hence, the convergence bounds that we are able to guarantee depend on the delay sequence $\{\tau_k\}$. To clarify this statement, we will analyze a few special cases of unbounded delays in detail. First, assume that $\{\tau_k\}$ can grow unbounded at a linear rate, i.e.,
\[
\tau_k \leq \alpha k + \beta, \quad k \in \mathbb{N}_0, \tag{10}
\]

where $\alpha \in (0, 1)$ and $\beta \geq 0$. The associated convergence result reads as follows.
**Corollary 2.4.** Let \( \{V_k\} \) be a non-negative sequence such that

\[
V_{k+1} \leq qV_k + p \max_{(k-\tau_k) \leq \ell \leq k} V_\ell, \quad k \in \mathbb{N}_0,
\]

for some non-negative scalars \( q \) and \( p \). Suppose the delay sequence \( \{\tau_k\} \) satisfies (10). If \( q + p < 1 \), then

\[
V_k \leq \left( \frac{\alpha k}{1 - \alpha + \beta} + 1 \right)^{-\eta} V_0, \quad k \in \mathbb{N}_0,
\]

where \( \eta = \ln(q + p) / \ln(1 - \alpha) \).

**Proof.** Conditions (i)–(iii) of Lemma 2.3 are satisfied by the function

\[
\Lambda(t) = \left( \frac{\alpha t}{1 - \alpha + \beta} + 1 \right)^{-\eta}.
\]

See Appendix A.3 for a detailed proof.

Corollary 2.4 shows that for unbounded delays satisfying (10), the convergence rate of the sequence \( \{V_k\} \) is \( O(k^{-\eta}) \). Note that \( \alpha \), the rate at which the unbounded delays grow large, affects \( \eta \). Specifically, \( \eta \) is monotonically decreasing with \( \alpha \) and approaches zero as \( \alpha \) tends to one. Hence, the guaranteed convergence rate slows down as the growth rate of the delays increases.

The following result gives a bound on the convergence rate of the sequence \( \{V_k\} \) for another class of unbounded delays whose order of growth is larger than that of delays satisfying (10).

**Corollary 2.5.** Suppose that the non-negative sequence \( \{V_k\} \) satisfies

\[
V_{k+1} \leq qV_k + p \max_{(k-\tau_k) \leq \ell \leq k} V_\ell, \quad k \in \mathbb{N}_0,
\]

for some non-negative constants \( q \) and \( p \). Suppose also there exist \( \alpha \in (0, 1) \) and \( \beta \geq 0 \) such that

\[
\tau_k \leq k - k^\alpha + \beta, \quad k \in \mathbb{N}_0.
\]

(11)

If \( q + p < 1 \), then

\[
V_k \leq \left( \frac{\ln(\delta)}{\ln(k + \delta)} \right)^{\eta} V_0, \quad k \in \mathbb{N}_0,
\]

where \( \delta = (1 + \beta) / (1 - \alpha) \) and \( \eta = \ln(q + p) / \ln(\alpha) \).

**Proof.** Conditions (i)–(iii) of Lemma 2.3 are satisfied by the function

\[
\Lambda(t) = \left( \frac{\ln(\delta)}{\ln(t + \delta)} \right)^{\eta}.
\]

See Appendix A.4 for a detailed proof.

In accordance with Lemma 2.2, the condition \( p + q < 1 \) is sufficient for asymptotic convergence also in Corollaries 2.4 and 2.5. However, both the value of \( p + q \) and the rate at which the delays may grow large affect the convergence guarantee. The quicker the delays are allowed to tend to infinity, the slower the convergence. Specifically, the convergence rate is \( O(k^{-\eta}) \) if the delays satisfy (10), and \( O((\ln(k))^{-\eta}) \) if the delays satisfy (11).
Remark 2.6. It is straightforward to verify that the function
\[ \Lambda(t) = (q + p) \frac{t}{1 + \tau} \]
satisfies conditions (i)–(iii) of Lemma 2.3 when \( \tau_k \leq \tau \) for all \( k \in \mathbb{N}_0 \). Therefore, Lemma 2.3 recovers Lemma 2.1 as a special case.

We will now shift our attention to the convergence result for iterations on the form (4). This result adds a lot of flexibility in how we can model and account for different perturbations that appear in asynchronous optimization algorithms, and will be central to the developments in Subsections 3.1 and 3.2.

Lemma 2.7. Let \( \{V_k\}, \{W_k\}, \) and \( \{X_k\} \) be non-negative sequences satisfying
\[ X_{k+1} + V_{k+1} \leq qV_k + p_k \sum_{\ell=(k-\tau_k)_+}^k W_\ell - r_k W_k, \quad k \in \mathbb{N}_0, \]
where \( q \in [0, 1] \) and \( p_k, r_k \geq 0 \) for all \( k \). Suppose there is a non-negative integer \( \tau \) such that
\[ 0 \leq \tau_k \leq \tau, \quad k \in \mathbb{N}_0. \]
For every \( K \in \mathbb{N} \), the following statements hold:

1. If \( q = 1 \) and
\[ \sum_{\ell=0}^{\tau} p_k + \ell \leq r_k \]
is satisfied for all \( k \in \mathbb{N}_0 \), then \( V_K \leq V_0 \) and
\[ \sum_{k=1}^K X_k \leq V_0. \]

2. If \( q \in (0, 1) \), \( p_k = p > 0 \), \( r_k = r > 0 \), and
\[ 2\tau + 1 \leq \min \left\{ \frac{1}{1 - q}, \frac{r}{p} \right\}, \]
then \( V_K \leq q^K V_0 \) and
\[ \sum_{k=1}^K q^{-k} X_k \leq V_0. \]

Proof. See Appendix A.5

Consider the non-delayed counterpart of (12) with \( p_k = p > 0 \) and \( r_k = r > 0 \):
\[ X_{k+1} + V_{k+1} \leq qV_k + (p - r)W_k, \quad k \in \mathbb{N}_0. \]
Assume that \( q \in (0, 1) \) and \( p \leq r \), or equivalently,
\[ 1 \leq \frac{1}{1 - q} \quad \text{and} \quad 1 \leq \frac{r}{p}. \]
In this case, the sequence \( \{V_k\} \) converges linearly to zero at a rate of \( q \). In general, the existence of delays may impair performance, induce oscillations and even instability. However, Lemma 2.7 shows that for the
Figure 1: The three classes of algorithms studied in Section 3 and the typical parallel architectures on which they are implemented. The PIAG algorithm on the parameter server (left) parallelizes data over multiple nodes able to evaluate the corresponding loss function gradients, while the master maintains and updates the decision vector. In the ARock framework (middle), multiple computing units access to shared memory and update (randomly selected) sub-vectors of the overall decision vector in parallel. Finally, the totally asynchronous framework (right) allows to model loosely coupled distributed architectures where computing nodes retrieve parts of the global decision vector from remote nodes, and evaluate components of an operator to update their local decisions.

delayed iteration (12), the convergence rate of \( \{V_k\} \) is also \( q \) if the maximum delay bound \( \tau \) satisfies

\[
2\tau + 1 \leq \frac{1}{1-q} \quad \text{and} \quad 2\tau + 1 \leq \frac{r}{p}
\]

This means that up to certain value of the delay, the iteration (12) and its delay-free counterpart have the same guaranteed convergence rate.

**Remark 2.8.** When \( p_k = p \) and \( r_k = r \) for all \( k \in \mathbb{N}_0 \), the convergence condition (13) in part 1 of Lemma 2.7 is simplified to

\[
p(\tau + 1) \leq q.
\]

### 3 Applications to Asynchronous Optimization Algorithms

Data-driven optimization problems can grow large both in the number of decision variables and in the number of data points that are used to define the objective and constraints. It may therefore make sense to parallelize the associated optimization algorithms over both data and decision variables, see Figure 1. One popular framework for parallelizing algorithms in the data dimension is the parameter server [41]. Here, a master node (the server) maintains the decision vector, while the data is divided between a number of worker nodes. When a worker node is queried by the server, it computes and returns the gradient of the part of the objective function defined by its own data. The master maintains an estimate of the gradient of the full objective function, and executes a (proximal) gradient update whenever it receives gradient information from one of the workers. As soon as the master completes an update, it queries idle worker nodes with the updated decision vector. If the asynchrony, measured in terms of the maximum number of iterations carried out by the master between two consecutive gradient updates from any worker, is bounded, then convergence can be guaranteed under mild assumptions on the objective function [5, 40].

The natural way to parallelize problems with high-dimensional decision vectors is to use block-coordinate updates. In these methods, the decision vector is divided into sub-vectors, and different processing elements update the sub-vectors in parallel. In the partially and totally asynchronous models of Bertsekas and Tsitsiklis [11], each processing element is responsible for storing and updating one sub-vector, and it does so using delayed information of the remaining decision variables retrieved from the other (remote) processors. Under weak assumptions on the communications delays and update rates of individual processors, convergence
can be proven for contraction mappings with respect to the block-maximum norm [11, Section 6.3]. However, only some special combinations of algorithms and optimization problems result in iterations that are contractive with respect to the block-maximum norm [11, Section 3.1]. Another type of block-coordinate updates are used in the ARock framework [57]. Here, the decision vector is stored in shared memory, and the parallel computing elements pick sub-vectors uniformly at random to update whenever they terminate their previous work. Under an assumption of bounded asynchrony, convergence of ARock can be established for a wide range of objective functions [57].

In the remaining parts of this paper, we demonstrate how the sequence results introduced in the previous section allows to strengthen the existing convergence guarantees for the algorithms discussed above. Specifically, we improve iteration complexity bounds for the proximal incremental aggregated gradient (PIAG) method, which is suitable for implementation in the parameter server framework, with respect to both the amount of asynchrony and the problem conditioning; we prove the linear rate of convergence for ARock under larger step-sizes and provide better scalability properties with respect to the number of parallel computing elements; and we describe a unified Lyapunov-based approach for analysis of totally and partially asynchronous iterations involving maximum norm contractions, that allows to derive convergence rate guarantees also outside the partially asynchronous regime.

### 3.1 Proximal Incremental Aggregated Gradient Method

We begin by considering composite optimization problems of the form

\[
\begin{align*}
\text{minimize } & \quad P(x) := F(x) + R(x). \\
\text{subject to } & \quad x \in \mathbb{R}^d.
\end{align*}
\]

(14)

Here, \(x\) is the decision variable, \(F\) is the average of many smooth component functions \(f_i\), i.e.,

\[
F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]

and \(R\) is a proper closed convex function that may be non-differentiable and extended real-valued. We use \(\mathcal{X}^*\) to denote the set of optimal solutions of (14) and \(P^*\) to denote the corresponding optimal value. We impose the following assumptions on Problem (14).

**Assumption 3.1.** The optimal set \(\mathcal{X}^*\) is non-empty.

**Assumption 3.2.** Each function \(f_i : \mathbb{R}^d \to \mathbb{R}, i \in [n]\), is convex and \(L_i\)-smooth.

Note that under Assumption 3.2 the average function \(F\) is \(L_F\)-smooth [68], where

\[
L_F \leq L := \frac{1}{n} \sum_{i=1}^{n} L_i.
\]

(15)

In the optimization problem (14), the role of the regularization term \(R\) is to favor solutions with certain structure. Common choices of \(R\) include: the \(\ell_1\) norm, \(R(x) = \lambda \|x\|_1\) with \(\lambda > 0\), used to promote sparsity in solutions; and the indicator function of a non-empty closed convex set \(\mathcal{X} \subseteq \mathbb{R}^d\),

\[
R(x) = \begin{cases} 0, & \text{if } x \in \mathcal{X}, \\ +\infty, & \text{otherwise}, \end{cases}
\]

used to force the admissible solutions to lie in \(\mathcal{X}\). A comprehensive catalog of regularization terms is given in [27].

Optimization problems on the form (14) are known as *regularized empirical risk minimization* problems and arise often in machine learning, signal processing, and statistical estimation (see, e.g., [32]). In such problems, we are given a collection of \(n\) training samples \(\{(a_1, b_1), \ldots, (a_n, b_n)\}\), where each \(a_i \in \mathbb{R}^d\) is a
feature vector, and each \( b_i \in \mathbb{R} \) is the desired response. A classical example is least-squares regression where the component functions are given by

\[
f_i(x) = \frac{1}{2}(a_i^T x - b_i)^2, \quad i \in [n],
\]

and popular choices of the regularization terms include \( R(x) = \lambda_1 \| x \|_2^2 \) (ridge regression), \( R(x) = \lambda_2 \| x \|_1 \) (Lasso), or \( R(x) = \lambda_1 \| x \|_2^2 + \lambda_2 \| x \|_1 \) (elastic net) for some non-negative parameters \( \lambda_1 \) and \( \lambda_2 \). Another example is logistic regression for binary classification problems, where each \( b_i \in \{-1, 1\} \) is the desired class label and the component functions are

\[
f_i(x) = \log \left(1 + \exp(-b_i a_i^T x)\right), \quad i \in [n].
\]

A standard method for solving Problem (14) is the \textit{proximal gradient} (PG) method, which consists of a gradient step followed by a proximal mapping. More precisely, the PG method is described by Algorithm 1 where \( \gamma \) is a positive step-size, and the prox-operator (proximal mapping) is defined as

\[
\text{prox}_\gamma R(x) = \arg\min_{u \in \mathbb{R}^d} \left\{ \frac{1}{2} \| u - x \|^2 + \gamma R(u) \right\}.
\]

Under Assumptions 3.1 and 3.2 the iterates generated by the PG method with step-size \( \gamma = \frac{1}{L} \) satisfy

\[
P(x_k) - P^* \leq \frac{L \| x_0 - x^* \|^2}{2k},
\]

for all \( k \in \mathbb{N} \) [7, Theorem 10.21]. This means that Algorithm 1 achieves an \( \mathcal{O}(1/k) \) rate of convergence in function values to the optimal value.

\begin{algorithm}
\caption{Proximal Gradient (Pg) Method}
\begin{algorithmic}[1]
\Input \( x_0 \in \mathbb{R}^d \), step-size \( \gamma > 0 \), number of iterations \( K \in \mathbb{N} \)
\State Initialize \( k \leftarrow 0 \)
\While {\( k < K \)}
\State Set \( g_k \leftarrow \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_k) \)
\State Set \( x_{k+1} \leftarrow \text{prox}_{\gamma R}(x_k - \gamma g_k) \)
\State Set \( k \leftarrow k + 1 \)
\EndWhile
\end{algorithmic}
\end{algorithm}

Each iteration of the PG method requires computing the gradients for all \( n \) component functions. When \( n \) is large, this per iteration cost is expensive, and hence often results in slow convergence. An effective alternative is the \textit{proximal incremental aggregated gradient} (PIAG) method that exploits the additive structure of (14) and operates on a single component function at a time, rather than on the entire cost function [63]. The PIAG method evaluates the gradient of only one component function per iteration, but keeps a memory of the most recent gradients of all component functions to approximate the full gradient \( \nabla F \). Specifically, at iteration \( k \), the method will have stored \( \nabla f_i(x_{[i]}) \) for all \( i \in [n] \), where \( x_{[i]} \) represents the latest iterate at which \( \nabla f_i \) was evaluated. An integer \( j \in [n] \) is then chosen and the full gradient \( \nabla F(x_k) \) is approximated by

\[
g_k = \frac{1}{n} \left( \nabla f_j(x_k) - \nabla f_j(x_{[j]}) + \sum_{i=1}^n \nabla f_i(x_{[i]}) \right).
\]

The aggregated gradient vector \( g_k \) is employed to update the current iterate \( x_k \) via

\[
x_{k+1} = \text{prox}_{\gamma R}(x_k - \gamma g_k).
\]

Thus, the PIAG method uses outdated gradients from previous iterations for the components \( f_i, i \neq j \), and does need not to compute gradients of these components at iteration \( k \).
Algorithm 2 Proximal Incremental Aggregated Gradient (PiAG) Method

Input: \( x_0 \in \mathbb{R}^d \), step-size \( \gamma > 0 \), number of iterations \( K \in \mathbb{N} \)

1: for \( i = 1 \) to \( n \) do
2: \hspace{1em} Compute \( \nabla f_i(x_0) \)
3: \hspace{1em} Store \( \nabla f_i(x[i]) \leftarrow \nabla f_i(x_0) \)
4: end for
5: Initialize \( k \leftarrow 0 \)
6: while \( k < K \) do
7: \hspace{1em} Choose \( j \) from \( \{1, \ldots, n\} \)
8: \hspace{1em} Compute \( \nabla f_j(x_k) \)
9: \hspace{1em} Set \( g_k \leftarrow \frac{1}{n} (\nabla f_j(x_k) - \nabla f_j(x[j]) + \sum_{i=1}^{n} \nabla f_i(x[i])) \)
10: \hspace{1em} Store \( \nabla f_j(x[j]) \leftarrow \nabla f_j(x_k) \)
11: \hspace{1em} Set \( x_{k+1} \leftarrow \text{prox}_{\gamma f}(x_k - \gamma g_k) \)
12: \hspace{1em} Set \( k \leftarrow k + 1 \)
13: end while

A formal description of the PiAG method is presented as Algorithm 2. Let us define \( s_{i,k} \) as the iteration number in which the gradient of the component function \( f_i \) is updated for the last time after the completion of the \( k \)th iteration. Then, we can rewrite the aggregated gradient vector \( g_k \) as

\[
g_k = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_{s_{i,k}}).\]

Note that \( s_{i,k} \in \{0, \ldots, k\} \) for each \( i \in [n] \) and \( k \in \mathbb{N}_0 \). In a traditional (serial) implementation of PiAG, the selection of \( j \) on line 7 follows a deterministic rule (for example, the cyclic rule) and the gradient of the component function \( f_j \) is computed at the current iterate, i.e., \( s_{j,k} = k \). In the parameter server implementation, the index \( j \) will not be chosen but rather assigned to the identity of the worker that returns its gradient to the master at iteration \( k \). Since workers can exchange information with the master independently of each other, worker \( j \) may evaluate \( \nabla f_j \) at an outdated iterate \( x_{s_{j,k}} \), where \( s_{j,k} < k \), and send the result to the server. We assume that each component function is sampled at least once in the past \( \tau \) iterations of the PiAG method. In other words, there is a fixed non-negative integer \( \tau \) such that the indices \( s_{i,k} \) satisfy

\[
(k - \tau)_+ \leq s_{i,k} \leq k, \quad i \in [n], \; k \in \mathbb{N}_0.
\]

The value \( \tau \) can be viewed as an upper bound on the delay encountered by the gradients of the component functions. For example, if the component functions are chosen one by one using a deterministic cyclic order on the index set \( \{1, \ldots, n\} \), then \( \tau = n - 1 \).

The following result shows that in the analysis of the PiAG method, we can establish iterate relationships on the form \( \text{(3.6)} \).

Lemma 3.3. Suppose Assumptions 3.1 and 3.2 hold. Suppose also that \( \{\alpha_k\} \) is a sequence defined by

\[
\alpha_k = k + \alpha_0, \quad k \in \mathbb{N}_0,
\]

where \( \alpha_0 \) is a non-negative scalar. Let

\[
V_k = 2\gamma \alpha_k (P(x_k) - P^*) + \|x_k - x^*\|^2
\]

and \( W_k = \|x_{k+1} - x_k\|^2 \) for \( k \in \mathbb{N}_0 \). Then, the iterates \( \{x_k\} \) generated by Algorithm 2 satisfy

\[
V_{k+1} \leq V_k + \gamma L (\alpha_k + \tau + 1) \sum_{\ell=(k-\tau)_+}^{k} W_\ell - (2\alpha_k + 1 - \gamma L \tau \alpha_k) W_k, \quad k \in \mathbb{N}_0.
\]

Proof. See Appendix B.1
Using this iterate relationship, the sequence result in Lemma 2.7 yields the following convergence guarantee for the PiAG method.

**Theorem 3.4.** Let Assumptions 3.1 and 3.2 hold. Suppose that \( \gamma \in (0, \gamma_{\text{max}}] \) with \( \gamma_{\text{max}} = \frac{1}{L(2\tau + 1)} \). Then, for every \( k \in \mathbb{N} \) and any \( x^* \in \mathcal{X}^* \), the iterates \( \{x_k\} \) generated by Algorithm 2 satisfy

\[
P(x_k) - P^* \leq \frac{1}{2\gamma} \|x_0 - x^*\|^2 + \frac{\mu}{\gamma} \left( P(x_0) - P^* \right)
\]

**Proof.** See Appendix B.2 ■

According to Theorem 3.4, the PiAG iterations converge at a rate of \( O(1/k) \) when the constant step-size \( \gamma \) is appropriately tuned. The convergence rate depends on the choice of \( \gamma \). For example, if we pick \( \gamma = \frac{1}{L(2\tau + 1)} \), then the corresponding \( \{x_k\} \) converges in terms of function values with the rate

\[
P(x_k) - P^* \leq \frac{L}{2\gamma} \|x_0 - x^*\|^2 + 2\tau \left( L \|x_0 - x^*\|^2 + P(x_0) - P^* \right) \frac{1}{k + \tau}, \quad k \in \mathbb{N}
\]

(18)

In the case that \( \tau = 0 \), the preceding guaranteed bound reduces to (16), which is achieved by the Pg method. From (18), we can see that if

\[
k \geq K_{\epsilon} = \frac{L}{2\gamma} \|x_0 - x^*\|^2 + 2\tau \left( L \|x_0 - x^*\|^2 + P(x_0) - P^* \right) - \tau, \quad k \in \mathbb{N}
\]

then PiAG with the step-size choice (17) achieves an accuracy of \( P(x_k) - P^* \leq \epsilon \). This shows that the PiAG method has an iteration complexity of \( O(L(\tau + 1)/\epsilon) \) for convex problems. Therefore, as \( \tau \) increases, the complexity bound deteriorates linearly with \( \tau \). Note that the linear dependence on the maximum delay bound \( \tau \) is unavoidable and cannot further be improved [3].

Next, we restrict our attention to composite optimization problems under the following quadratic functional growth condition.

**Assumption 3.5.** There exists a constant \( \mu > 0 \) such that

\[
P(x) - P^* \geq \frac{\mu}{2} \|x - \Pi_{X^*}(x)\|^2, \quad \forall x \in \text{dom } R,
\]

(19)

where \( \text{dom } R \) is defined as \( \text{dom } R := \{x \in \mathbb{R}^d \mid R(x) < +\infty\} \), is the effective domain of the function \( R \), and \( \Pi_{X^*}(\cdot) \) denotes the Euclidean-norm projection onto the optimal set \( \mathcal{X}^* \), i.e.,

\[
\Pi_{X^*}(x) = \arg \min_{u \in \mathcal{X}^*} \|u - x\|.
\]

Assumption 3.5 implies that the objective function grows faster than the squared distance between any feasible point and the optimal set. While every strongly convex function satisfies the quadratic functional growth condition (19), the converse is not true in general [50]. For example, if \( A \in \mathbb{R}^{m \times d} \) is rank deficient and \( b \in \mathbb{R}^m \), the function \( F(x) = \|Ax - b\|^2 \) is not strongly convex, but it satisfies Assumption 3.5. Other examples of objective functions which satisfy the quadratic functional growth condition can be found in [50]. Let us define the condition number of the optimization problem (14) as \( Q = L/\mu \). The role of \( Q \) in determining the linear convergence rate of (proximal) gradient methods is well-known [52].

We next show that under Assumption 3.5, the iterates of the PiAG method satisfy relationships on the form (4).
Lemma 3.6. Suppose that Assumptions 3.1, 3.2 and 3.5 hold. Let
\[ V_k = \frac{2}{L} (P(x_k) - P^*) + \|x_k - \Pi_{X^*}(x_k)\|^2, \]
and \( W_k = \|x_{k+1} - x_k\|^2 \) for \( k \in \mathbb{N}_0 \). Then, the iterates \( \{x_k\} \) generated by Algorithm 2 satisfy
\[ V_{k+1} \leq \left( \frac{1}{1 + \gamma \mu \theta} \right) V_k + \frac{1 + \gamma L (\tau + 1)}{1 + \gamma \mu \theta} \sum_{\ell=(k-\tau)_+}^k W_\ell - \frac{2}{1 + \gamma \mu \theta} W_k, \]
where \( \theta = \frac{Q + 1}{\tau + 1} \).

Proof. See Appendix B.3

We use Lemma 3.6 together with Lemma 2.7 to derive the convergence rate of the PiAG method for optimization problems whose objective functions satisfy the quadratic functional growth condition.

Theorem 3.7. Let Assumptions 3.1, 3.2 and 3.5 hold. Suppose that the step-size \( \gamma \) is set to
\[ \gamma = \frac{h}{L(2\tau + 1)}, \quad h \in (0, 1]. \]
Then, for every \( k \in \mathbb{N} \), the iterates \( \{x_k\} \) generated by Algorithm 2 satisfy
\[ \|x_k - \Pi_{X^*}(x_k)\|^2 \leq \left( 1 - \frac{1}{1 + (Q + 1)(2\tau + 1)/h} \right)^k \left( \frac{2}{L} (P(x_0) - P^*) + \|x_0 - \Pi_{X^*}(x_0)\|^2 \right), \]
\[ P(x_k) - P^* \leq \left( 1 - \frac{1}{1 + (Q + 1)(2\tau + 1)/h} \right)^k \left( P(x_0) - P^* \right) + \frac{L}{2} \|x_0 - \Pi_{X^*}(x_0)\|^2. \]

Proof. See Appendix B.4

Theorem 3.7 demonstrates that under Assumption 3.5, the PiAG method is linearly convergent by taking a constant step-size inversely proportional to the maximum delay bound \( \tau \). The best convergence rate is obtained for the step-size
\[ \gamma = \frac{1}{L(2\tau + 1)}. \]
With this choice of \( \gamma \), the iterates converge linearly in terms of function values with the rate
\[ P(x_k) - P^* \leq \left( 1 - \frac{1}{1 + (Q + 1)(2\tau + 1)} \right)^k \epsilon_0, \]
where \( \epsilon_0 = P(x_0) - P^* + \frac{L}{2} \|x_0 - \Pi_{X^*}(x_0)\|^2 \). Taking logarithm of both sides yields
\[ \log(P(x_k) - P^*) \leq k \log \left( 1 - \frac{1}{1 + (Q + 1)(2\tau + 1)} \right) + \log (\epsilon_0). \]
Since \( \log(1 + x) \leq x \) for any \( x > -1 \), it follows that
\[ \log(P(x_k) - P^*) \leq - \frac{k}{1 + (Q + 1)(2\tau + 1)} + \log (\epsilon_0). \]
Therefore, if the number of iterations satisfy
\[ k \geq K_\epsilon = (1 + (Q + 1)(2\tau + 1)) \log \left( \frac{\epsilon_0}{\epsilon} \right), \]
then $P(x_k) - P^* \leq \epsilon$. We conclude that Piag achieves an iteration complexity of $O(Q(\tau+1)\log(1/\epsilon))$ for optimization problems satisfying the quadratic functional growth condition. Note that when $\tau = 0$, this bound becomes $O(Q\log(1/\epsilon))$, which is the iteration complexity for the PG method [27, Theorem 10.30].

As discussed before, if the component functions are selected in a fixed cyclic order, then $\tau = n - 1$. It follows from Theorem 3.4 and Theorem 3.7 that the iteration complexity of the Piag method with cyclic sampling is $O(nL/\epsilon)$ for convex problems and $O(nQ\log(1/\epsilon))$ for problems whose objective functions satisfy the quadratic functional growth condition. Each iteration of the Piag method requires only one gradient evaluation compared to $n$ gradient computations in the PG method. Therefore, in terms of the total number of component gradients evaluated to find an $\epsilon$-optimal solution, the iteration complexity of the cyclic Piag and PG are the same.

**Comparison to Related Work.** In the seminal work [13], Blatt et al. proposed the incremental aggregated gradient (IAG) method for solving unconstrained smooth optimization problems (the case where $R(x) \equiv 0$). They showed that in the special case where each component function is quadratic, the IAG method with a constant step-size achieves a linear rate of convergence. However, neither an explicit convergence rate nor an explicit upper bound on the step-size that can ensure linear convergence was given. Tseng and Yun [63] proved global convergence and local linear convergence for the Piag method in a more general setting where the gradients of the component functions are Lipschitz continuous and each component function satisfies a local error bound assumption. While the results are more general than those in [13], as the authors did not limit the objective function to be strongly convex and quadratic, explicit rate estimates and a characterization of the step-size needed for linear convergence were still missing in [63]. Gürbüzbalaban et al. [27] provided the first explicit linear rate result for the IAG method and proved an iteration complexity of $O(Q^2\tau^2\log(1/\epsilon))$ for strongly convex functions. The quadratic dependence of the iteration complexity on the condition number $Q$ was improved by Aytekin et al. [5], who showed that the iteration complexity of the Piag method can be $O(Q\tau^2\log(1/\epsilon))$ for strongly convex composite objectives. Later, Zhang et al. [72] proved an iteration complexity of $O(Q^2\tau^2\log(1/\epsilon))$ for the Piag method under the quadratic functional growth condition. Vanli et al. [66] improved the quadratic dependence on $\tau$ to a linear one by showing that Piag returns an $\epsilon$-optimal solution after at most $50Q(\tau+1)\log(1/\epsilon)$ iterations. In comparison with [66], Theorem 5.7 allows the Piag method to use larger step-sizes that leads to an iteration complexity approximately 12 times lower. Note that contrary to [5, 27, 72], our step-size selection rule in the strongly convex case is independent of the problem parameter $\mu_*$, which is typically unknown and hard to estimate. This also makes the Piag algorithm adaptive to the level of strong convexity naturally present, i.e., the algorithm will automatically converge faster for problems with a higher degree of local strong-convexity around the optimal solution [59].

Recently, Sun et al. [62] analyzed convergence of the Piag method under Assumptions 3.1 and 3.2 and proved $O(C/k)$ convergence rate, where $C$ is a positive constant. While the constant $C$ is implicit in [62], Huang et al. [34, Table 1] showed that the analysis in [62] guarantees an $O(\tau^2L^2/\epsilon)$ iteration complexity for the Piag method. In comparison with this result, Theorem 3.4 gives a better dependence on the Lipschitz constant ($L$ vs. $L^2$) and on the maximum delay bound ($\tau$ vs. $\tau^3$) in the iteration complexity.

Le Roux et al. [69] proposed a randomized variant of the IAG method, called stochastic average gradient (SAG), where the component functions are sampled uniformly at random. The iteration complexity of the SAG method, in expectation, is $O(\max\{n, Q\}\log(1/\epsilon))$ for strongly convex problems and $O((n+L)/\epsilon)$ for convex problems. The maximum allowable step-size for the SAG method is larger than that of the Piag method, which can lead to improved empirical performance (see Figure 1 in [59]). Note, however, that in some applications, the component functions must be processed in a particular deterministic order and, hence, random sampling is not possible. For example, in source localization or distributed parameter estimation over wireless networks, sensors may only communicate with their neighbors subject to certain constraints in terms of geography and distance, which can restrict the updates to follow a specific deterministic order [13]. Inspired by the SAG method, there has been extensive work on stochastic gradient aggregation algorithms including, to name a few, SVRG [35], SAGA [20], MISO [45], SARAH [54], Katyusha [2], and RPDG [37].

**Remark 3.8.** One important drawback of the Piag and SAG methods is the need to store $n$ gradient vectors, which amounts to an $O(nd)$ storage. This can be prohibitive when both $d$ and $n$ are large. Note, however, that the memory requirements can be reduced to $O(n)$ for optimization problems where each component function $f_i$ only depends on a linear function of $x$, as in least-squares and logistic regression [59].
3.2 Asynchronous Coordinate Update Methods

Many popular optimization algorithms, such as gradient descent, projected gradient descent, proximal-point, forward-backward splitting, Douglas-Rachford splitting, and the alternating direction method of multipliers (ADMM), can be viewed as special instances of the Krasnosel’ski–Mann (KM) method for operators \([57]\). The only way they differ is in their choice of operator. The KM method has the form

\[
x_{k+1} = (1 - \gamma)x_k + \gamma T(x_k), \quad k \in \mathbb{N}_0,
\]

where \(\gamma \in (0, 1)\) is the step-size, and \(T : \mathbb{R}^d \to \mathbb{R}^d\) is a nonexpansive operator, i.e.,

\[
\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.
\]

Any vector \(x^* \in \mathbb{R}^d\) satisfying \(T(x^*) = x^*\) is called a fixed point of \(T\) and the KM method can be viewed as an algorithm for finding such a fixed point. Let \(S = I_d - T\), where \(I_d\) is the identity operator on \(\mathbb{R}^d\). Then, \((20)\) can be rewritten as

\[
x_{k+1} = x_k - \gamma S(x_k), \quad k \in \mathbb{N}_0.
\]

This shows that the KM method can be interpreted as a taking a step of length \(\gamma\) in the opposite direction of \(S\) evaluated at the current iterate. For example, the gradient descent method for minimization of a \(L\)-smooth convex function \(f : \mathbb{R}^d \to \mathbb{R}\) can be formulated as the KM method with

\[
S = \frac{2}{L} \nabla f, \quad \text{or equivalently,} \quad T = I_d - \frac{2}{L} \nabla f.
\]

We represent \(x\) as \(x = ([x]_1, \ldots, [x]_m)\), where \([x]_i \in \mathbb{R}^{d_i}\) with \(d_1, \ldots, d_m\) being positive integer numbers satisfying \(d = d_1 + \ldots + d_m\). We denote by \(S_i : \mathbb{R}^d \to \mathbb{R}^{d_i}\) the \(i\)th component of \(S\), so \(S(x) = (S_1(x), \ldots, S_m(x))\).

Thus, we can rewrite \((21)\) as

\[
[x_{k+1}]_i = [x_k]_i - \gamma S_i(x_k), \quad i \in [m].
\]

ARock is an algorithmic framework for parallelizing the KM method in an asynchronous fashion \([57]\). In ARock, multiple agents (machines, processors, or cores) have access to a shared memory for storing the vector \(x\), and are able to read and update \(x\) simultaneously without using locks. Conceptually, ARock lets each agent repeat the following steps:

- Read \(x\) from the shared memory without software locks and save it in a local cache as \(\hat{x}\);
- Choose an index \(i \in [m]\) uniformly at random and use \(\hat{x}\) to compute \(S_i(\hat{x})\);
- Update component \(i\) of the shared \(x\) via

\[
[x]_i \leftarrow [x]_i - \gamma S_i(\hat{x}).
\]

Since the agents are being run independently without synchronization, while one agent is busy reading \(x\) and evaluating \(S_i(x)\), other agents may repeatedly update the value stored in the shared memory. Therefore, the value \(\hat{x}\) read from the shared memory may differ from the value of \(x\) to which the update is made later. In other words, each agent can update the shared memory using possibly out-of-date information.

**Algorithm 3** ARock

**Input:** \(x_0 \in \mathbb{R}^d\), step-size \(\gamma > 0\), number of iterations \(K \in \mathbb{N}\)

1. Initialize global counter \(k \leftarrow 0\)
2. while \(k < K\) every agent asynchronously and continuously do
3. Read \(\hat{x}_k\) from shared memory
4. Sample \(i_k\) from \([1, \ldots, m]\) with equal probability \(\frac{1}{m}\)
5. Set \([x_{k+1}]_{i_k} \leftarrow [x_k]_{i_k} - \gamma S_{i_k}(\hat{x}_k)\)
6. Set \(k \leftarrow k + 1\)
7. end while
Algorithm 3 describes ARock. We assume that the write operation on line 5 is atomic, in the sense that the updated result will successfully appear in the shared memory by the end of the execution. In practice, this assumption can be enforced through compare-and-swap operations [55]. Updating a scalar is a single atomic instruction on most modern hardware. Thus, the atomic write assumption in Algorithm 3 naturally holds when each block is a single scalar, i.e., \( m = d \) and \( d_i = 1 \) for all \( i \in [m] \).

We define one iteration of ARock as a modification on any block of \( x \) in the shared memory. A global counter \( k \) is introduced to track the total number of iterations so that \( x_k \) is the value of \( x \) in the shared memory after \( k \) iterations. We use \( i_k \) to denote the component that is updated at iteration \( k \), and \( \hat{x}_k \) for value of \( x \) that is used in the calculation of \( S_{i_k}() \). Since every iteration changes one block of the shared memory and all writes are required to be atomic,

\[
x_k = \hat{x}_k + \sum_{j \in J_k} (x_{j+1} - x_j),
\]

where \( J_k \subseteq \{0,\ldots,k-1\} \) is a subset of index numbers of previous iterations. Roughly speaking, the sum represents all updates of the shared vector that have occurred from the time that agent \( i_k \) begins reading \( \hat{x}_k \) from memory until it finishes evaluating \( S_{i_k}(\hat{x}_k) \) and writes the result to memory; see [57] for a more detailed discussion. Note that individual blocks of the shared vector may be updated by multiple agents while it is in the process of being read by another agent. Therefore, the components of \( \hat{x}_k \) may have different ages and the vector \( \hat{x}_k \) may never actually exist in the shared memory during the execution of Algorithm 3. This phenomenon is known as inconsistent read [49].

To analyze the convergence of ARock, we need to make a few assumptions similar to [57].

**Assumption 3.9.** For Algorithm 3, the following properties hold:

1. **(Pseudo-contractivity)** The operator \( T = I_d - S \) has a fixed point and is pseudo-contractive with respect to the Euclidean norm with contraction modulus \( c \). That is, there exists \( c \in (0,1) \) such that
   \[ \|T(x) - x^*\| \leq c\|x - x^*\|, \quad \forall x \in \mathbb{R}^d. \]

2. **(Bounded delay)** There is a non-negative integer \( \tau \) such that
   \[ (k - \tau)_+ \leq \min\{j \mid j \in J_k\}, \quad \forall k \in \mathbb{N}_0. \]

3. **(Independence)** All random variables \( i_k \) for \( k = 0,1,\ldots,K \) are independent of each other.

Under Assumption 3.9.1, the serial KM iteration [21] with \( \gamma \in (0,1] \) converges to the fixed point \( x^* \) at a linear rate [11, Chapter 3]. Assumption 3.9.2 guarantees that during any update cycle of an agent, the vector \( x \) in the shared memory is updated at most \( \tau \) times by other agents. Therefore, no component of \( \hat{x}_k \) is older than \( \tau \) for all \( k \in \mathbb{N}_0 \). The value of \( \tau \) is an indicator of the degree of asynchrony in ARock. In practice, \( \tau \) will depend on the number of agents involved in the computation. If all agents are working at the same rate, we would expect \( \tau \) to be a multiple of the number of agents [49]. Similar to [57] and most results on asynchronous stochastic optimization for shared memory architecture (e.g., [43, 44, 55]), we assume that the age of the components of \( \hat{x}_k \) is independent of the block \( i_k \) being updated at iteration \( k \). However, this may not hold in practice if, for example, some blocks are more expensive to update than others [39]. This independence assumption can be relaxed using several techniques such as before read labeling [40], after read labeling [39], single coordinate consistent ordering [18], and probabilistic models of asynchrony [14, 61].

The following result shows that in the analysis of ARock, we can establish iterate relationships on the form [4].

**Lemma 3.10.** Suppose that Assumption 3.9 holds. Let \( V_k = \mathbb{E}\left[\|x_k - x^*\|^2\right] \) and \( W_k = \mathbb{E}\left[\|S(\hat{x}_k)\|^2\right] \) for \( k \in \mathbb{N}_0 \), where the expectation is over all choices of index \( i_k \) up to step \( k \). Then, the iterates generated by
Algorithm 3 satisfies
\[ V_{k+1} \leq \left( 1 - \frac{\gamma(1-c^2)}{m(1+\gamma\left(\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}\right))} \right) V_k + \frac{2\gamma^2}{m\tau} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \sum_{\ell=(k-\tau)}^{k-1} W_\ell \\
- \frac{\gamma}{m} \left( 1 - \gamma \left( 1 + \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right) W_k. \]

Proof. See Appendix C.2 \[\Box\]

Using this iterate relationship, the sequence result in Lemma 2.7 allows us to derive new convergence guarantees for ARock.

**Theorem 3.11.** Suppose that Assumption 3.9 holds and that the step-size \( \gamma \) is set to 
\[ \gamma = \frac{h}{1 + 5 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right)} \]
with \( h \in (0,1] \). Then, the iterates generated by Algorithm 3 satisfy
\[ \mathbb{E}\left[ \| x_k - x^* \|^2 \right] \leq \left( 1 - \frac{h(1-c^2)}{m(1 + 6 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right))} \right)^k \| x_0 - x^* \|^2 \]
for all \( k \in \mathbb{N}_0 \). Moreover, the algorithm reaches an accuracy of \( \mathbb{E}\left[ \| x_k - x^* \|^2 \right] \leq \epsilon \) after
\[ k \geq K_\epsilon = \frac{m \left( 1 + 6 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right)}{h(1-c^2)} \log \left( \frac{\| x_0 - x^* \|^2}{\epsilon} \right) \]
iterations.

Proof. See Appendix C.3 \[\Box\]

Theorem 3.11 shows that for pseudo-contractive operators, ARock converges in expectation at a linear rate. To quantify how \( \tau \) can affect the convergence of ARock, let us define
\[ \Gamma = \frac{\tau}{m} + \sqrt{\frac{\tau}{m}}. \]
Clearly, \( \Gamma \) is monotonically increasing in \( \tau \) and is equal to zero for \( \tau = 0 \). The maximum allowable step-size and the linear rate of ARock depend on \( \Gamma \). As \( \Gamma \) tends to infinity, the maximum allowable step-size decreases and approaches zero while the convergence factor increases and approaches one. Therefore, ARock should take a smaller step-size for a larger \( \Gamma \), which will lead to a slower convergence rate. Note that if \( \Gamma \approx 0 \), or equivalently, \( \tau \ll m \), then ARock with step-size \( \gamma \approx 1 \) has the linear rate 
\[ \rho = 1 - \frac{1-c^2}{m}, \]
which is precisely the rate for the serial stochastic KM method. As discussed before, \( \tau \) is related to the number of agents used in the algorithm. Therefore, the number of agents can be of the order of \( o(m) \) without appreciably degrading the convergence rate of ARock.

In order to satisfy \( \| x_k - x^* \|^2 \leq \epsilon \), the serial KM method (21) with step-size \( \gamma_{KM} = h \in (0,1] \) needs
\[ k \geq \frac{1}{h(1-c^2)} \log \left( \frac{\| x_0 - x^* \|^2}{\epsilon} \right) \]
iterations. Each iteration of (21) requires computing all \( m \) blocks of \( S \) and updating the whole vector \( x \). Thus, the overall complexity of the serial KM method, in terms of the total number of blocks updated to find
an $\epsilon$-optimal solution, is
\[
K_{KM} = \frac{m}{h(1-c^2)} \log \left( \frac{\|x_0 - x^*\|^2}{\epsilon} \right).
\]

On the other hand, according to Theorem 3.11 ARock with step-size $\gamma_{ARock} = h/(1 + 5\Gamma)$ performs
\[
K_{ARock} = \frac{m(1+6\Gamma)}{h(1-c^2)} \log \left( \frac{\|x_0 - x^*\|^2}{\epsilon} \right)
\]
component updates to return an $\epsilon$-optimal solution in average. In the case that $\Gamma \approx 0$, we have $\gamma_{ARock} \approx \gamma_{KM}$ and $K_{ARock} \approx K_{KM}$. Hence, as long as $\tau$ is bounded by $o(m)$, ARock can take the same step-size as the serial KM method and achieve the same iteration bound. Furthermore, since ARock runs on $p$ agents in parallel, updates can occur roughly $p$ times more frequently. This means that ARock can enjoy the near-linear speedup in the number of agents.

**Remark 3.12.** Peng et al. [57] proposed ARock for finding fixed points of nonexpansive operators in an asynchronous parallel fashion. They proved that for pseudo-contractive operators, ARock with step-size
\[
\gamma = \frac{1}{1 + \mathcal{O} \left( \frac{x^2}{\sqrt{m}} \right)}
\]
converges linearly to a fixed point and achieves the linear speedup for $\tau \leq \mathcal{O} \left( m^{1/4} \right)$. Hannah and Yin [30] improved this result by showing that
\[
\gamma = \frac{1}{1 + \mathcal{O} \left( \frac{x}{\sqrt{m}} \right)}
\]
guarantees the linear convergence of ARock and $\tau \leq \mathcal{O} \left( m^{1/2} \right)$ ensures the linear speedup. In comparison with [30], not only can Theorem 3.11 provide a larger value for the maximum allowable step-size, but it also improves the requirement for the linear speedup property from $\tau \leq \mathcal{O} \left( m^{1/2} \right)$ to $\tau \leq o(m)$.

**Remark 3.13.** There is a gap between ideal and realizable speedups when exploiting a large number of agents due to associated overheads such as communication costs and (memory access) resource conflicts. This commonly observed phenomenon is referred to as speedup saturation [70]. Nevertheless, a larger and less delay-sensitive step-size is likely to result in better practical performance [39].

**A Special Case: Asynchronous Coordinate Descent Method.** We now present a special case of ARock, namely the asynchronous coordinate descent algorithm for minimizing a class of composite objective functions. Specifically, we consider the optimization problem
\[
\min_{x \in \mathbb{R}^d} P(x) := F(x) + R(x), \tag{25}
\]
where $F : \mathbb{R}^d \to \mathbb{R}$ is $\mu$-strongly convex and $L$-smooth, and $R : \mathbb{R}^d \to \mathbb{R}$ is separable in all coordinates, i.e.,
\[
R(x) = \sum_{i=1}^{m} r_i([x]_i).
\]
Here, each $r_i : \mathbb{R}^{d_i} \to \mathbb{R} \cup \{+\infty\}$, $i \in [m]$, is a closed, convex, and extended real-valued function. The best known examples of separable regularizers include $\ell_1$ norm, $\ell_2$ norm square, and the indicator function of box constraints [22]. The minimizer of Problem (25) is the unique fixed point of $T_{\text{prox}}$ defined as
\[
T_{\text{prox}}(x) = \text{prox}_{\frac{2}{\mu+L} R} \left( x - \frac{2}{\mu + L} \nabla F(x) \right).
\]
The operator $T_{\text{prox}}$ is contractive with contraction modulus
\[ c = \frac{Q - 1}{Q + 1}, \]
where $Q = L/\mu$ [9]. To solve (25), we apply ARock to $S = I_d - T_{\text{prox}}$. Then, the update rule of Algorithm 3 at the $k$th iteration becomes
\[
[x_{k+1}]_i \leftarrow [x_k]_i - \gamma \left( \tilde{x}_k - \text{prox}_{\frac{2}{\mu + L r_i}} \left( \frac{2}{\mu + L \nabla_i F(\tilde{x}_k)} \right) \right),
\]
where $\nabla_i F(x)$ denotes the partial gradient of $F$ with respect to $[x]_i$. According to Theorem 3.11, the iterates generated by ARock with step-size $\gamma = 1/(1 + 5\Gamma)$ satisfy
\[
E[\|x_k - x^*\|^2] \leq \left( 1 - \frac{4Q}{m(1 + 6\Gamma)(Q + 1)^2} \right)^k \|x_0 - x^*\|^2, \quad k \in \mathbb{N}_0.
\]
If $\Gamma \approx 0$, or equivalently, $\tau \ll m$, then ARock using step-size $\gamma \approx 1$ converges linearly at a rate of
\[ \rho_{\text{ARock}} = 1 - \frac{4Q}{m(Q + 1)^2}. \]
Note that when $m = 1$, the preceding linear rate reduces to
\[ \rho_{\text{GD}} = \left( \frac{Q - 1}{Q + 1} \right)^2, \]
which is the best convergence rate for the gradient descent method applied to strongly convex optimization.

**Remark 3.14.** Liu and Wright [43] proposed an asynchronous coordinate descent algorithm for solving composite optimization problems of the form (25). They proved that the linear speedup is achievable if $\tau \leq O(m^{1/4})$. For the special case of Problem (25) where $R(x) \equiv 0$, Liu et al. [44] showed that the asynchronous coordinate descent method can enjoy the linear speedup if $\tau \leq O(m^{1/2})$. In comparison with [43] and [44], our requirement for the linear speedup property is $\tau \leq o(m)$ and, hence, allows a larger value for $\tau$. Recently, Cheung et al. [18] analyzed convergence of the asynchronous coordinate descent method for composite objective functions without assuming independence between $i_k$ and $\tilde{x}_k$. Their analysis guarantees the linear speedup for $\tau \leq O(m^{1/2})$.

**Remark 3.15.** In Theorem 3.11, the linear convergence of ARock is given in terms of the expected quadratic distance from the iterates to the fixed point. Note however that the literature on coordinate descent algorithms (e.g., [43, 44, 51]) usually establishes convergence results using coordinate-wise Lipschitz constants of the function $F$. This allows to provide larger step-sizes which can lead to potentially better convergence bounds, especially in terms of the function values $P(x_k) - P^*$.

### 3.3 A Lyapunov Approach to Analysis of Totally Asynchronous Iterations

Finally, we study iterations involving maximum norm pseudo-contractions under the general asynchronous model introduced by Bertsekas and Tsitsiklis [11], which allows for heterogeneous and time-varying communication delays and update rates. Such iterations arise, for example, in algorithms for the solution of certain classes of linear equations, optimization problems and variational inequalities [11, 28, 49], optimum multiuser detection algorithms [69], distributed algorithms for averaging [47], power control algorithms for wireless networks [26], and reinforcement algorithms for solving discounted Markov decision processes [71].

We will demonstrate how the convergence results in [11] for maximum norm contractions can be derived and extended using Lemmas 2.1, 2.3. This allows to unify and expand the existing results for partially and totally asynchronous iterations.
We consider iterative algorithms on the form

\[ x_{k+1} = T(x_k), \quad k \in \mathbb{N}_0, \]

where \( T : \mathbb{R}^d \to \mathbb{R}^d \) is a continuous mapping. This iteration aims to find a fixed point of \( T \), that is, a vector \( x^* \in \mathbb{R}^d \) satisfying \( x^* = T(x^*) \). Similar to Subsection 3.2 we decompose the space \( \mathbb{R}^d \) as a Cartesian product of \( m \) subspaces:

\[ \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}, \quad d = \sum_{i=1}^m d_i. \]

Accordingly, we can partition any vector \( x \in \mathbb{R}^d \) as \( x = ([x_1], \ldots , [x_m]) \) with \([x_i] \in \mathbb{R}^{d_i}, i \in [m] \). We denote by \( T_i : \mathbb{R}^d \to \mathbb{R}^{d_i} \) the \( i \)th component of \( T \), so \( T(x) = (T_1(x), \ldots , T_m(x)) \). Then, we rewrite (27) as

\[ [x_{k+1}]_i = T_i([x_k]_1, \ldots , [x_k]_m), \quad i \in [m]. \] (28)

This iteration can be viewed as a network of \( m \) agents, each responsible for updating one of the \( m \) blocks of \( x \) so as to find a global fixed point of the operator \( T \).

Let us fix some norms \( \| \cdot \|_i \) for the spaces \( \mathbb{R}^{d_i} \). The block-maximum norm \( \| \cdot \|_{b, \infty} \) on \( \mathbb{R}^d \) is defined as

\[ \|x\|_{b, \infty} = \max_{i \in [m]} w_i \| [x]_i \|_i, \]

where \( w_i \) are positive constants. Note that when \( d_i = 1 \) for each \( i \in [m] \), the block-maximum norm reduces to the maximum norm

\[ \|x\|_{\infty} = \max_{i \in [m]} w_i \| [x]_i \|. \]

The following definition introduces pseudo-contractive mappings with respect to the block-maximum norm.

**Definition 3.16.** A mapping \( T : \mathbb{R}^d \to \mathbb{R}^d \) is called a pseudo-contraction with respect to the block-maximum norm if it has a fixed-point \( x^* \) and the property

\[ \| T(x) - x^* \|_{b, \infty} \leq c \| x - x^* \|_{b, \infty}, \quad \forall x \in \mathbb{R}^d, \]

where \( c \), called the contraction modulus of \( T \), is a constant belonging to \((0, 1)\).

Note that we follow [11] and include the existence of a fixed point in the definition of pseudo-contractions. Examples of iterative algorithms that involve pseudo-contractions with respect to the block-maximum norm can be found in [11] Chapter 3. An important property of pseudo-contractions is that they have a unique fixed point, to which the iterates produced by (28) converge at a linear rate [11 Proposition 3.1.2].

More precisely, using the Lyapunov function \( V_k = \| x_k - x^* \|_{b, \infty} \), one can show that the sequence \( \{x_k\} \) generated by (28) with initial vector \( x_0 \in \mathbb{R}^d \) satisfies \( V_{k+1} \leq c V_k \) for all \( k \in \mathbb{N}_0 \). This implies that

\[ \| x_k - x^* \|_{b, \infty} \leq c^k \| x_0 - x^* \|_{b, \infty}, \quad k \in \mathbb{N}_0. \]

The algorithm described by (28) is synchronous in the sense that all agents update their states at the same time and have instantaneous access to the states of all other agents. Synchronous execution is possible if there are no communication failures in the network and if all agents operate in sync with a global clock. In practice, these requirements are hard to satisfy: local clocks in different agents tend to drift, global synchronization mechanisms are complex to implement and carry substantial execution overhead, and the communication latency between agents can be significant and unpredictable. Insisting on synchronous operation in an inherently asynchronous environment forces agents to spend a significant time idle, waiting for the slowest agents (perhaps due to lower processing power or higher workload per iteration) to complete its work.

In an asynchronous implementation of the iteration (28), each agent can update its state at its own pace, using possibly outdated information from the other agents. This leads to iterations on the form

\[ [x_{k+1}]_i = \begin{cases} T_i([x_{s_{i,k}}]_1, \ldots , [x_{s_{i,m,k}}]_m), & k \in K_i, \\ [x_k]_i, & k \notin K_i. \end{cases} \] (29)
Here, \( \mathcal{K}_i \) is the set of times when agent \( i \) executes an update, and \( s_{ij,k} \) is the time at which the most recent version of \( [x]_j \) available to agent \( i \) at time \( k \) was computed. The sets \( \mathcal{K}_i \) need not be known to all the agents. Thus, there is no requirement for a shared global clock. Since the agents use only values computed in the previous iterations, \( s_{ij,k} \leq k \) for all \( k \in \mathbb{N}_0 \). We can view the value

\[
\tau_{ij,k} := k - s_{ij,k}
\]
as the communication delay from agent \( j \) to agent \( i \) at time \( k \). It is clear that the synchronous iteration is a special case of \( \tau_{ij,k} = 0 \) for all \( i, j \in [m] \) and all \( k \in \mathbb{N}_0 \).

Based on the assumptions on communication delays and update rates, Bertsekas and Tsitsiklis introduced a classification of asynchronous algorithms as either **totally asynchronous** or **partially asynchronous**. The totally asynchronous model is characterized by the following assumption.

**Assumption 3.17 (Total Asynchronism).** For the asynchronous iteration \( \tau_{ij} \), the following properties hold:

1. The sets \( \mathcal{K}_i \) are infinite subsets of \( \mathbb{N}_0 \) for each \( i \in [m] \).
2. \( \lim_{k \to \infty} s_{ij,k} = +\infty \) for all \( i, j \in [m] \).

Assumption 3.17.1 guarantees that no agent ceases to execute its update while Assumption 3.17.2 guarantees that outdated information about the agent updates is eventually purged from the computation (compare the discussion after Lemma 2.2). Under total asynchronism, the communication delays \( \tau_{ij,k} \) can become unbounded as \( k \) increases. This is the main difference with partially asynchronous iterations, where delays are bounded; in particular, the following assumption holds.

**Assumption 3.18 (Partial Asynchronism).** For the asynchronous iteration \( \tau_{ij} \), there exist non-negative integers \( B \) and \( D \) such that the following conditions hold:

1. At least one of the elements of the set \( \{k, k + 1, \ldots, k + B\} \) belongs to \( \mathcal{K}_i \) for each \( i \in [m] \) and all \( k \in \mathbb{N}_0 \).
2. \( k - D \leq s_{ij,k} \leq k \) for all \( i, j \in [m] \) and all \( k \in \mathcal{K}_i \).
3. \( s_{ii,k} = 0 \) for all \( i \in [m] \) and \( k \in \mathcal{K}_i \).

Assumptions 3.18.1 and 3.18.2 ensure that the time interval between updates executed by each agent and the communication delays are bounded by \( B \) and \( D \), respectively. This means that no agent waits an arbitrarily long time to compute or to receive a message from another agent. Assumption 3.18.3 states that agent \( i \) always uses the latest version of its own component \( [x]_i \). Note that when \( B = D = 0 \), the asynchronous iteration under partial asynchronism reduces to the synchronous iteration.

We will now present a uniform analysis of the iteration involving block-maximum norm pseudo-contractive under both partial and total asynchronism, and study its convergence rate under different assumptions on the communication delays and update rates. To this end, we introduce \( \tau_k \) to represent the maximum age of the outdated information being used to update blocks at global time \( k \in \mathbb{N}_0 \). Specifically, we define \( \tau_k \) as

\[
\tau_k := k - \min_{i \in [m]} \min_{j \in [m]} s_{ij, t_i(k)},
\]

where \( t_i(k) \) is the most recent update time of agent \( i \) at \( k \in \mathbb{N}_0 \), i.e.,

\[
t_i(k) = \max\{\kappa \mid \kappa \leq k \land \kappa \in \mathcal{K}_i\}.
\]

In this way, if \( k \in \mathcal{K}_i \) then \( t_i(k) = k \), and if \( k \in (\kappa^-, \kappa^+) \) for two consecutive elements \( \kappa^- \) and \( \kappa^+ \) of \( \mathcal{K}_i \), then \( t_i(k) = \kappa^- \). For simplicity, we assume that \( 0 \in \mathcal{K}_i \) for each \( i \), so that \( t_i(k) \) is well defined for all \( k \in \mathbb{N}_0 \).

The next result shows that for the asynchronous iteration, if we use \( V_k = \|x_k - x^*\|_\infty^{\mathbb{R}} \) as a candidate Lyapunov function (similarly to the convergence analysis for the synchronous case), we can establish iterate relationships on the form \( \|x_k - x^*\|_\infty^{\mathbb{R}} \leq \cdots \cdots \).
Lemma 3.19. Suppose that $T$ is pseudo-contractive with respect to the block-maximum norm with contraction modulus $c$. Let $V_k = \|x_k - x^*\|_{b,\infty}$. Then, the iterates $\{x_k\}$ generated by the asynchronous iteration (29) satisfy

$$V_{k+1} \leq c \max_{(k-k^*)^+ \leq t \leq k} V_t, \quad k \in \mathbb{N}_0,$$

where $r_k$ is defined in (30).

Proof. See Appendix D.1.

We apply Lemma 2.2 to show that for pseudo-contractions with respect to the block-maximum norm, the asynchronous iteration (29) converges asymptotically to the fixed point under total asynchronism.

Theorem 3.20. Let Assumption 3.17 hold. Suppose that $T$ is pseudo-contractive with respect to the block-maximum norm. Then, the sequence $\{x_k\}$ generated by the asynchronous iteration (29) converges asymptotically to the unique fixed point of $T$.

Proof. See Appendix D.2.

While convergent synchronous algorithms may diverge in the face of asynchronism, Theorem 3.20 shows that pseudo-contractive mappings in the block-maximum norm can tolerate arbitrarily large communication delays and update intervals satisfying Assumption 3.17. In addition, as the next result shows, these iterations admit an explicit convergence rate bound when they are executed in a partially asynchronous fashion.

Theorem 3.21. Let Assumption 3.18 hold. Suppose that $T$ is pseudo-contractive with respect to the block-maximum norm with contraction modulus $c$. Then, the sequence $\{x_k\}$ generated by the asynchronous iteration (29) satisfies

$$\|x_k - x^*\|_{b,\infty} \leq c^k \|x_0 - x^*\|_{b,\infty}, \quad k \in \mathbb{N}_0.$$

Proof. See Appendix D.3.

According to Theorem 3.21, the asynchronous iteration (29) involving block-maximum norm pseudo-contractions remains linearly convergent for bounded communication delays and update rates. Note that $c^{k+B+D+1} - \log(c)$ is monotonically increasing with $B$ and $D$, and approaches one as either $B$ or $D$ tends to infinity. Hence, the guaranteed convergence rate of (29) slows down as either the delays increase in magnitude or agents execute less frequently. The convergence rate is directly related to the number of iterations required for the algorithm to converge. It follows from Theorem 3.21 that the asynchronous iteration (29) needs

$$k \geq \frac{B + D + 1}{-\log(c)} \log \left( \frac{\|x_0 - x^*\|_{b,\infty}}{\epsilon} \right)$$

iterations to satisfy $\|x_k - x^*\|_{b,\infty} \leq \epsilon$. We can see that the time to reach a fixed target accuracy deteriorates linearly with both $B$ and $D$.

We now use Lemma 2.3 to develop a result that provides guaranteed convergence rates for the asynchronous algorithm (29) under a rather broad family of communication delays and update rates, in between the partially and totally asynchronous models.

Theorem 3.22. Suppose that $T$ is pseudo-contractive with respect to the block-maximum norm with contraction modulus $c$. Suppose also that there exists a function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that the following conditions hold:

(i) $\Lambda(0) = 1$.

(ii) $\Lambda$ is non-increasing.
\[ \lim_{k \to +\infty} \Lambda(k) = 0 \quad \text{and} \quad c \Lambda(k - \tau_k) \leq \Lambda(k + 1), \quad k \in \mathbb{N}_0, \]

where \( \tau_k \) is defined in (30).

Then, the iterates \( \{x_k\} \) generated by the asynchronous iteration (29) satisfy
\[ \|x_k - x^*\|_{\infty}^w \leq \Lambda(k)\|x_0 - x^*\|_{\infty}^w, \quad k \in \mathbb{N}. \]

**Proof.** See Appendix D.4. ■

According to Theorem 3.22, any function \( \Lambda \) satisfying conditions (i)–(iii) can be used to estimate the convergence rate of the asynchronous iteration (29). Condition (iii) implies that the admissible choices for \( \Lambda \) depend on \( \tau_k \). This means that the rate at which the nodes execute their updates as well as the way communication delays tend large both affect the guaranteed convergence rate of (29). For example, one can verify that under partial asynchronism, the function
\[ \Lambda(t) = c \frac{t^1}{1 - \alpha + \beta} + 1 \]
satisfies conditions (i)–(iii). In the following example, we use Theorem 3.22 and Corollary 2.4 to establish convergence rates for a particular class of totally asynchronous iterations, where the communication delays can grow unbounded at a linear rate.

**Example.** For the asynchronous iteration (29), suppose that \( K_i = \mathbb{N}_0 \) for each \( i \in [m] \). Suppose also that there exist scalars \( \alpha \in (0, 1) \) and \( \beta \geq 0 \) such that
\[ (1 - \alpha)k - \beta \leq s_{ij,k}, \quad i, j \in [m], \quad k \in K_i. \quad (31) \]

Note that (31) implies that the communication delays \( \tau_{ij,k} \) belong to the interval \([0, \alpha k + \beta]\) and may therefore grow unbounded. In this example, as \( t_i(k) = k \), we have
\[ (1 - \alpha)k - \beta \leq \min_{i \in [m]} \min_{j \in [m]} s_{ij,t_i(k)}. \]

Thus, according to Lemma 3.19, the iterates generated by the asynchronous iteration (29) satisfy
\[ V_{k+1} \leq qV_k + p \max_{(k - \tau_k) \leq t \leq k} V_t, \quad k \in \mathbb{N}_0, \]
with \( V_k = \|x_k - x^*\|_{\infty}^w, \; q = 0, \; p = c \), and \( \tau_k \leq \alpha k + \beta \). Since \( q + p = c < 1 \), it follows from Corollary 2.4 that the function
\[ \Lambda(t) = \left( \frac{\alpha t}{1 - \alpha + \beta} + 1 \right)^{-\eta} \]
with \( \eta = \ln(c)/\ln(1 - \alpha) \) satisfies conditions (i)–(iii) of Theorem 3.22. Therefore,
\[ \|x_k - x^*\|_{\infty}^w \leq \left( \frac{\alpha k}{1 - \alpha + \beta} + 1 \right)^{-\eta} \|x_0 - x^*\|_{\infty}^w, \quad k \in \mathbb{N}_0. \]

We can see that under unbounded communication delays satisfying (31), the convergence rate of the asynchronous iteration (29) is of order \( O(1/k^\eta) \).

Although pseudo-contractions in the block-maximum norm converge when executed in a totally asynchronous manner, we note that in many applications, it is rare that the associated fixed-point iterations are maximum norm contractions. For instance, the gradient descent iterations for unconstrained optimization problems are maximum norm contractions if the Hessian matrix of the cost function is diagonally dominant [11, Section 3.1.3]. The diagonal dominance assumption is quite strong and violated even by some strongly convex quadratic objective functions [11, Example 6.3.1].
Remark 3.23. In their textbook [11], Bertsekas and Tsitsiklis proved that contractions in the block-
maximum norm converge when executed in a totally asynchronous manner. However, they did not quantify
how bounds on the communication delays and update rates of agents affect the convergence rate of the
iterates. There are very few results in the literature on convergence rates of asynchronous iterations involving
block-maximum norm contractions. Notable exceptions are the works of [12, 71], where the convergence
rate of iterates was estimated under partial asynchronism. Theorems 3.20, 3.21 and 3.22 demonstrate that
not only do we recover the asymptotic convergence results in [11] for maximum norm pseudo-contractions,
but we also provide explicit bounds on the convergence rate of asynchronous iterations for various classes of
bounded and unbounded communication delays and update rates.

4 Conclusions

We have introduced a number of novel sequence results for asynchronous iterations that appear in the analysis
of parallel and asynchronous algorithms. In contrast to previous analysis frameworks, which have used
conservative bounds for the effects of asynchrony, our results attempt to capture the inherent structure in the
asynchronous iterations. The results balance simplicity, applicability and power, and provide explicit bounds
on how the amount of asynchrony affects the guaranteed convergence rates. To demonstrate the potential of
the sequence results, we illustrated how they can be used to improve our theoretical understanding of several
important classes of asynchronous optimization algorithms. First, we derived better iteration complexity
bounds for the proximal incremental aggregated gradient method, reducing the dependence of the convergence
times on the maximum delay and problem condition number. Second, we gave an improved analysis of the
ARock framework for asynchronous block-coordinate updates of Krasnosel’ski–Mann iterations, proving a
larger range of admissible step-sizes, faster convergence rates and better scaling properties with respect to
the number of parallel computing elements. Finally, we gave a uniform treatment of asynchronous iterations
involving block-norm contractions under partial and (several versions of) total asynchronism.

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Appendix

A   Proofs for Section 2

In this section, we prove the results presented in Section 2.

A.1 Proof of Lemma 2.2

The proof idea is inspired by the convergence theorem for totally asynchronous iterations [11, Proposition 6.2.1]. We first use perfect induction to prove that

\[ V_k \leq V_0, \quad k \in \mathbb{N}_0. \]  

(32)

Since \( V_0 \) satisfies (32), the induction hypothesis is true for \( k = 0 \). Now, assume that (32) holds for all \( k \) up to some \( K \in \mathbb{N}_0 \). This means that

\[ V_K \leq V_0 \quad \text{and} \quad \max_{(K - \tau_k)_{+} \leq t \leq K} V_t \leq V_0. \]

Thus, from (6), we have

\[ V_{K+1} \leq (q + p)V_0 \leq V_0, \]

where the second inequality follows from the fact that \( q + p \in (0, 1) \). Therefore, \( V_{K+1} \leq V_0 \) and, hence, the induction proof is complete.

Next, we prove that \( V_k \) converges to zero as \( k \to +\infty \). To this end, we use induction to show that for each \( m \in \mathbb{N}_0 \), there exists \( K_m \in \mathbb{N}_0 \) such that

\[ V_k \leq (q + p)^m V_0, \quad \forall k \geq K_m. \]

(33)

From (32), \( V_k \leq (q + p)^0 V_0 \) for all \( k \geq 0 \). Thus, the induction hypothesis holds for \( m = 0 \) (with \( K_0 = 0 \)). Assuming (33) is true for a given \( m \), we will show that there exists \( K_{m+1} \in \mathbb{N}_0 \) such that \( V_k \leq (q + p)^{m+1}V_0 \) for all \( k \geq K_{m+1} \). From (7), one can find a sufficiently large \( K_m \geq K_m \) such that \( k - \tau_k \geq K_m \) for \( k \geq K_m \). Since (33) holds by induction, we have

\[ \max_{(k - \tau_k)_{+} \leq t \leq k} V_t \leq (q + p)^m V_0, \quad \forall k \geq K_m. \]

It follows from (6) that

\[ V_{k+1} \leq (q + p)(q + p)^m V_0 = (q + p)^{m+1}V_0, \quad \forall k \geq K_m, \]

which implies that

\[ V_k \leq (q + p)^{m+1}V_0, \quad \forall k \geq K_m + 1. \]

Set \( K_{m+1} = K_m + 1 \). The induction proof is complete.

In summary, we conclude that for each \( m \), there exists \( K_m \) such that \( V_k \leq (q + p)^m V_0 \) for all \( k \geq K_m \). Since \( q + p < 1 \), \( (q + p)^m \) approaches zero as \( m \to +\infty \). Hence, the sequence \( V_k \) asymptotically converges to zero.
A.2 Proof of Lemma 2.3

We will show by induction that

\[ V_k \leq \Lambda(k) V_0, \quad k \in \mathbb{N}_0. \]  \hspace{1cm} (34)

Since \( \Lambda(0) = 1 \), the induction hypothesis is true for \( k = 0 \). Assume for induction that (34) holds for all \( k \) up to some \( K \). From (8), we have

\[
V_{K+1} \leq q \Lambda(K) V_0 + p \max_{(K-\tau_K)+1 \leq \ell \leq K} \Lambda(\ell) V_0 \\
\leq q \Lambda(K) V_0 + p \max_{K-\tau_K \leq \ell \leq K} \Lambda(\ell) V_0.
\]

Since \( \Lambda \) is non-increasing on \( \mathbb{R} \), we obtain

\[ V_{K+1} \leq (q + p) \Lambda(K - \tau_K) V_0. \]

It follows from (9) that

\[ V_{K+1} \leq \Lambda(K + 1) V_0. \]

Therefore, (34) holds for \( k = K + 1 \). The induction proof is complete.

A.3 Proof of Corollary 2.4

Since \( q + p \in (0, 1) \) and \( \alpha \in (0, 1) \), there exists a positive constant \( \eta \) such that

\[ q + p = (1 - \alpha)^\eta. \]  \hspace{1cm} (35)

To derive the convergence rate, we define the function \( \Lambda \) as

\[
\Lambda(t) = \begin{cases} 
(\frac{(1-\alpha)(1+\beta)}{1-\alpha+\beta})^{-\eta}, & t \leq -\beta, \\
(\frac{\alpha t}{1-\alpha+\beta} + 1)^{-\eta}, & t \geq -\beta.
\end{cases}
\]

It is clear that \( \Lambda(0) = 1 \) and \( \Lambda \) is non-increasing on \( \mathbb{R} \). For delays satisfying (10), we have

\[ k - \tau_k \geq (1 - \alpha)k - \beta, \quad k \in \mathbb{N}_0, \]

which implies that

\[ (q + p)\Lambda(k - \tau_k) \leq (q + p)\Lambda((1 - \alpha)k - \beta). \]

Since \((1 - \alpha)k - \beta \geq -\beta \) for \( k \in \mathbb{N}_0 \), we obtain

\[
(q + p)\Lambda(k - \tau_k) \leq (q + p)\left(\frac{\alpha((1 - \alpha)k - \beta)}{1 - \alpha + \beta} + 1\right)^{-\eta} \\
= (q + p)(1 - \alpha)^{-\eta} \left(\frac{\alpha(k + 1)}{1 - \alpha + \beta} + 1\right)^{-\eta}.
\]

It follows from (35) that

\[ (q + p)\Lambda(k - \tau_k) \leq \Lambda(k + 1). \]

Therefore, conditions (i)–(iii) of Lemma 2.3 holds. Hence, \( V_k \leq \Lambda(k) V_0 \) for all \( k \in \mathbb{N}_0 \).
A.4 Proof of Corollary 2.5

Let \( \delta = (1 + \beta)/(1 - \alpha) \). One can verify that

\[
1 + \alpha \delta = -\beta + \delta.
\]

(S6)

Since \( \alpha \in (0, 1) \), \((x + y)^{\alpha} \leq x^{\alpha} + y^{\alpha}\) for any non-negative real numbers \( x \) and \( y \) [31, Theorem 2.12.2]. Thus, for any \( k \in \mathbb{N}_0 \), we have

\[
(k + 1 + \delta)^{\alpha} \leq k^{\alpha} + (1 + \delta)^{\alpha}
\]

\[
\leq k^{\alpha} + 1 + \alpha \delta
\]

where the second inequality follows from Bernoulli’s inequality, i.e., \((1 + x)^{\rho} \leq 1 + \rho x\) for any \( \rho \in [0, 1] \) and any real number \( x \geq -1 \). This implies that

\[
\alpha \ln(k + 1 + \delta) \leq \ln(k^{\alpha} - \beta + \delta), \quad k \in \mathbb{N}_0.
\]

(S7)

Since \( q + p \in (0, 1) \) and \( \alpha \in (0, 1) \), there exists a positive constant \( \eta \) such that

\[
q + p = \alpha^\eta.
\]

(S8)

To derive the convergence rate, we define the function \( \Lambda \) as

\[
\Lambda(t) = \begin{cases} 
(\ln(\delta)/\ln(1 + \alpha \delta))^{\eta}, & t \leq -\beta, \\
(\ln(\delta)/\ln(k + 1 + \delta))^{\eta}, & t \geq -\beta.
\end{cases}
\]

Note that \( \Lambda(0) = 1 \) and \( \Lambda \) is non-increasing on \( \mathbb{R} \). Furthermore, for delays satisfying (11), we have

\[
k - \tau_{tk} \geq k^{\alpha} - \beta, \quad k \in \mathbb{N}_0,
\]

which implies that

\[
(q + p)\Lambda(k - \tau_{tk}) \leq (q + p)\Lambda(k^{\alpha} - \beta).
\]

Since \( k^{\alpha} - \beta \geq -\beta \) for \( k \in \mathbb{N}_0 \), we get

\[
(q + p)\Lambda(k - \tau_{tk}) \leq (q + p)\left(\frac{\ln(\delta)}{\ln(k^{\alpha} - \beta + \delta)}\right)^{\eta}
\]

\[
\leq (q + p)\left(\frac{\ln(\delta)}{\alpha \ln(k + 1 + \delta)}\right)^{\eta}
\]

\[
= \Lambda(k + 1).
\]

Therefore, conditions (i)–(iii) of Lemma 2.3 holds. Hence, \( V_k \leq \Lambda(k)V_0 \) for all \( k \in \mathbb{N}_0 \).

A.5 Proof of Lemma 2.7

Let us define \( W_k = 0 \) for \( k \in \{\ldots, -2, -1\} \). Dividing both sides of (12) by \( q^{k+1} \), summing from \( k = 0 \) to \( K - 1 \), and then using telescoping cancellation, we have

\[
\sum_{k=0}^{K-1} \frac{X_{k+1}}{q^{k+1}} + V_K \leq V_0 + \sum_{k=0}^{K-1} \sum_{\ell=(k+\tau_{tk})_{+}}^{k} p_k W_{\ell} q^{k+1} - \sum_{k=0}^{K-1} r_k W_k q^{k+1}
\]

\[
= V_0 + \sum_{k=0}^{K-1} \sum_{\ell=k-\tau_{tk}}^{k} p_k W_{\ell} q^{k+1} - \sum_{k=0}^{K-1} r_k W_k q^{k+1}.
\]
Since $\tau_k \leq \tau$ for all $k \in \mathbb{N}_0$, we obtain
\[
\sum_{k=0}^{K-1} \frac{X_{k+1}}{q^{k+1}} + \frac{V_K}{q^K} \leq V_0 + \sum_{k=0}^{K-1} \sum_{\ell=k}^\tau \frac{p_k W_{k-\ell}}{q^{k+1}} - \sum_{k=0}^{K-1} \frac{r_k W_k}{q^{k+1}}
\]
\[
= V_0 + \sum_{k=0}^{K-1} \sum_{\ell=k-\tau}^{\tau} \frac{p_k W_{k-\ell}}{q^{k+1}} - \sum_{k=0}^{K-1} \frac{r_k W_k}{q^{k+1}}
\]
\[
= V_0 + \sum_{\ell=0}^{\tau} \sum_{k=0}^{K-1} \frac{p_{k+\ell} W_k}{q^{k+\ell+1}} - \sum_{k=0}^{K-1} \frac{r_k W_k}{q^{k+1}}
\]
\[
\leq V_0 + \sum_{k=0}^{K-1} \left( r_k - \sum_{\ell=0}^{\tau} \frac{p_{k+\ell}}{q^\ell} \right) \frac{W_k}{q^{k+1}}.
\]
where the second inequality comes from our assumption that $W_k \geq 0$ for $k \geq 0$ and $W_k = 0$ for $k < 0$. Since $q \in (0, 1]$, we have $q^\tau \leq q^\ell$ for $\ell = 0, \ldots, \tau$. This implies that
\[
\sum_{k=0}^{K-1} \frac{X_{k+1}}{q^{k+1}} + \frac{V_K}{q^K} \leq V_0 - \sum_{k=0}^{K-1} \left( r_k - q^{-\tau} \sum_{\ell=0}^{\tau} p_{k+\ell} \right) \frac{W_k}{q^{k+1}}.
\]
We are now ready to prove Lemma 2.7.

1. In this case, $q = 1$. Thus, (39) simplifies to
\[
\sum_{k=0}^{K-1} X_{k+1} + V_K \leq V_0 - \sum_{k=0}^{K-1} \left( r_k - \sum_{\ell=0}^{\tau} p_{k+\ell} \right) W_k.
\]
The assumption that
\[
\sum_{\ell=0}^{\tau} p_{k+\ell} \leq r_k
\]
holds for every $k \in \mathbb{N}_0$ implies that
\[
\sum_{k=0}^{K-1} X_{k+1} + V_K \leq V_0.
\]
Since $\{X_k\}$ and $\{V_k\}$ are non-negative sequences, it follows that $\sum_{k=1}^K X_k \leq V_0$ and $V_K \leq V_0$ for $K \in \mathbb{N}$.

2. In this case, $q \in (0, 1)$, $p_k = p$, and $r_k = r$. Thus, (39) simplifies to
\[
\sum_{k=0}^{K-1} \frac{X_{k+1}}{q^{k+1}} + \frac{V_K}{q^K} \leq V_0 - \sum_{k=0}^{K-1} \left( r - q^{-\tau} p(\tau + 1) \right) \frac{W_k}{q^{k+1}}.
\]
By Bernoulli’s inequality, i.e., $(1 + x)^n \geq 1 + nx$ for any $n \in \mathbb{N}_0$ and any real number $x > -1$, we have
\[
q^\tau = (1 - (1 - q))^\tau \\
\geq 1 - (1 - q)\tau.
\]
We write the proximal mapping more explicitly as 

\[ \Pi_{\gamma}(g) \]

which establishes important recursions for the iterates generated by the \textit{PIAG} method.

This section provides the proofs for the results presented in Subsection 3.1. We first state two key lemmas which establish important recursions for the iterates generated by the \textit{PIAG} method.

\textbf{Lemma B.1.} Suppose Assumptions 3.1 and 3.2 hold. Let \( \{x_k\} \) be the sequence generated by Algorithm 2. Then, for any \( x^* \in \mathcal{X}^* \) and every \( k \in \mathbb{N}_0 \), we have

\[ P(x_{k+1}) - P^* + \frac{1}{2\gamma}\|x_{k+1} - x^*\|^2 \leq \frac{1}{2\gamma}\|x_k - x^*\|^2 + \frac{L(\tau + 1)}{2} \sum_{\ell=(k-\tau)_+}^{k} \|x_{\ell+1} - x_{\ell}\|^2 - \frac{1}{2\gamma}\|x_{k+1} - x_k\|^2. \]

\[ (42) \]

\textit{Proof.} At iteration \( k \in \mathbb{N}_0 \), the update rule in Algorithm 2 is of the form

\[ g_k = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_{s_i,k}), \]

\[ x_{k+1} = \text{prox}_{\gamma R}(x_k - \gamma g_k). \]

We write the proximal mapping more explicitly as

\[ x_{k+1} = \arg\min_{u \in \mathcal{R}^d} \left\{ \frac{1}{2\gamma}\|u - (x_k - \gamma g_k)\|^2 + \gamma R(u) \right\}. \]

\[ (43) \]

From the first-order optimality condition for the point \( u \) in the minimization problem \( (43) \), there is a sub-gradient \( \xi_k \in \partial R(x_{k+1}) \) such that

\[ \langle x_{k+1} - x_k + \gamma(g_k + \xi_{k+1}), x^* - x_{k+1} \rangle \geq 0. \]

\[ (44) \]

Since \( R \) is convex, we have

\[ R(x_{k+1}) \leq R(x^*) + \langle \xi_{k+1}, x_{k+1} - x^* \rangle \]

\[ \leq R(x^*) + \frac{1}{\gamma} \langle x_k - x_{k+1}, x_{k+1} - x^* \rangle - g_k, x_{k+1} - x^* \rangle \]

\[ = R(x^*) + \frac{1}{\gamma} \langle x_k - x_{k+1}, x_{k+1} - x^* \rangle - \langle g_k, x_{k+1} - x^* \rangle \]

\[ = R(x^*) + \frac{1}{2\gamma}(\|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2 - \|x_{k+1} - x^*\|^2) - \langle g_k, x_{k+1} - x^* \rangle, \]

\[ (45) \]
where the last equality follows from the fact
\[ 2(a - b, b - c) = \|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2, \quad a, b, c \in \mathbb{R}^d. \]

By Assumption 3.2 each component function \( f_i \) is \( L_i \)-smooth. Thus, from [32] Lemma 1.2.3, we have
\[
 f_i(x_{k+1}) \leq f_i(x_{s_i,k}) + \langle \nabla f_i(x_{s_i,k}), x_{k+1} - x_{s_i,k} \rangle + \frac{L_i}{2} \| x_{k+1} - x_{s_i,k} \|^2
 \]
\[
 \leq f_i(x^*) + \langle \nabla f_i(x_{s_i,k}), x_{k+1} - x^* \rangle + \frac{L_i}{2} \| x_{k+1} - x_{s_i,k} \|^2,
\]
where the second inequality follows from the convexity of \( f_i \). Dividing both sides of the above inequality by \( n \) and then summing from \( i = 1 \) to \( n \), we obtain
\[
 F(x_{k+1}) \leq F(x^*) + \langle g_k, x_{k+1} - x^* \rangle + \sum_{i=1}^{n} \frac{L_i}{2n} \| x_{k+1} - x_{s_i,k} \|^2. \tag{46}
\]

By adding inequalities (45) and (46), rearranging the terms, and recalling that \( P(x) = F(x) + R(x) \), we have
\[
 P(x_{k+1}) - P^* + \frac{1}{2\gamma} \| x_{k+1} - x^* \|^2 \leq \frac{1}{2\gamma} \| x_k - x^* \|^2 + \sum_{i=1}^{n} \frac{L_i}{2n} \| x_{k+1} - x_{s_i,k} \|^2 - \frac{1}{2\gamma} \| x_{k+1} - x_k \|^2. \tag{47}
\]

Next, we find an upper bound on the term \( \mathcal{H} \). We expand \( \mathcal{H} \) as follows:
\[
 \mathcal{H} = \sum_{i=1}^{n} \frac{L_i}{2n} \left( \sum_{\ell=1}^{k} x_{\ell+1} - x_\ell \right)^2 = \sum_{i=1}^{n} \frac{L_i}{2n} \left( \sum_{\ell=s_i,k}^{k} \frac{x_{\ell+1} - x_\ell}{k-s_i,k+1} \right)^2.
\]

The squared Euclidean norm \( \| \cdot \|^2 \) is convex. Thus,
\[
 \mathcal{H} \leq \sum_{i=1}^{n} \frac{L_i(k-s_i,k+1)}{2n} \sum_{\ell=s_i,k}^{k} \| x_{\ell+1} - x_\ell \|^2.
\]

Since \( (k - \tau)_+ \leq s_{i,k} \leq k \) for all \( i \in [n] \) and \( k \in \mathbb{N}_0 \), we have
\[
 \mathcal{H} \leq \sum_{i=1}^{n} \frac{L_i(\tau+1)}{2n} \sum_{\ell=(k-\tau)_+}^{k} \| x_{\ell+1} - x_\ell \|^2 \leq \frac{L(\tau+1)}{2} \sum_{\ell=(k-\tau)_+}^{k} \| x_{\ell+1} - x_\ell \|^2. \tag{48}
\]

Substituting (48) into the bound (47) concludes the proof. \( \blacksquare \)

**Lemma B.2.** Suppose Assumptions 3.1 and 3.2 hold. Let \( \{\alpha_k\} \) be a sequence of non-negative numbers. Then, for every \( k \in \mathbb{N}_0 \), the sequence \( \{x_k\} \) generated by Algorithm 2 satisfies
\[
 \alpha_k(P(x_{k+1}) - P^*) \leq \alpha_k(P(x_k) - P^*) + \frac{L\alpha_k}{2} \sum_{\ell=(k-\tau)_+}^{k} \| x_{\ell+1} - x_\ell \|^2 - \alpha_k \left( \frac{1}{\gamma} - \frac{L\tau}{2} \right) \| x_{k+1} - x_k \|^2. \tag{49}
\]

**Proof.** According to Assumption 3.2 each component function \( f_i, i \in [n] \), is \( L_i \)-smooth. It follows from [66] second inequality of (3.5) in Lemma 3.4 that
\[
 P(x_{k+1}) - P^* \leq P(x_k) - P^* + \frac{L}{2} \sum_{\ell=(k-\tau)_+}^{k} \| x_{\ell+1} - x_\ell \|^2 - \left( \frac{1}{\gamma} - \frac{L(\tau+1)}{2} \right) \| x_{k+1} - x_k \|^2
 \]
\[
 = P(x_k) - P^* + \frac{L}{2} \sum_{\ell=(k-\tau)_+}^{k} \| x_{\ell+1} - x_\ell \|^2 - \left( \frac{1}{\gamma} - \frac{L\tau}{2} \right) \| x_{k+1} - x_k \|^2.
\]

34
We will prove that if $\alpha$ according to Lemma 3.3, the iterates generated by Algorithm 2 satisfy

$$\tau$$

The second term on the right-hand side is non-negative for any $\tau \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Thus,

$$\frac{1}{2\tau + 1} \leq \frac{2\alpha_k + 1}{\tau \alpha_k + \sum_{\ell=0}^\tau (\alpha_{\ell+k} + \tau + 1)}, \quad k \in \mathbb{N}_0.$$

B.1 Proof of Lemma 3.3

Since $\alpha = k + \alpha_0$ with $\alpha_0 \geq 0$, the sequence $\{\alpha_k\}$ is non-negative and satisfies $\alpha_{k+1} = \alpha_k + 1$ for every $k \in \mathbb{N}_0$. Adding inequalities (42) and (49), and rearranging the terms, we have

$$\alpha_{k+1} (P(x_{k+1}) - P^*) + \frac{1}{2\gamma} \|x_{k+1} - x^*\|^2 \leq \alpha_k (P(x_k) - P^*) + \frac{1}{2\gamma} \|x_k - x^*\|^2$$

$$+ \frac{L(\alpha_k + \tau + 1)}{2} \sum_{\ell=(k-\tau)_+}^k \|x_{\ell+1} - x_\ell\|^2$$

$$- \frac{1}{2} \left( \frac{2\alpha_k + 1}{\gamma} - \alpha_k \gamma L \right) \|x_{k+1} - x_k\|^2.$$ (50)

Multiplying both sides by $2\gamma$ and then letting $V_k = 2\gamma \alpha_k (P(x_k) - P^*) + \|x_k - x^*\|^2$ and $W_k = \|x_{k+1} - x_k\|^2$, we can rewrite (50) as

$$V_{k+1} \leq V_k + \gamma L (\alpha_k + \tau + 1) \sum_{\ell=(k-\tau)_+}^k W_\ell - (2\alpha_k + 1 - \gamma \alpha_k \gamma L) W_k, \quad k \in \mathbb{N}_0.$$

The proof is complete.

B.2 Proof of Theorem 3.4

According to Lemma 3.3, the iterates generated by Algorithm 2 satisfy

$$V_{k+1} \leq V_k + \gamma L (\alpha_k + \tau + 1) \sum_{\ell=(k-\tau)_+}^k W_\ell - r_k W_k, \quad k \in \mathbb{N}_0,$$

where $p_k = \gamma L (\alpha_k + \tau + 1)$, $r_k = 2\alpha_k + 1 - \gamma L \alpha_k$, and $\{\alpha_k\}$ is a non-negative sequence defined as $\alpha_k = k + \alpha_0$. To apply Lemma 2.7, we need to enforce that the convergence condition

$$\sum_{\ell=0}^\tau \gamma L (\alpha_{\ell+k} + \tau + 1) \leq 2\alpha_k + 1 - \gamma L \alpha_k$$

is satisfied for every $k \in \mathbb{N}_0$. This inequality is equivalent to

$$\gamma L \leq \frac{2\alpha_k + 1}{\tau \alpha_k + \sum_{\ell=0}^\tau (\alpha_{\ell+k} + \tau + 1)}.$$ (51)

We will prove that if $\alpha_0 = \tau$ and $\gamma L (2\tau + 1) \leq 1$, then (51) holds for all $k \in \mathbb{N}_0$. Replacing $\alpha_k = k + \tau$ in (51), we have

$$\frac{2\alpha_k + 1}{\tau \alpha_k + \sum_{\ell=0}^\tau (\alpha_{\ell+k} + \tau + 1)} = \frac{2k + 2\tau + 1}{(2\tau + 1)k + \frac{2}{2\tau + 1} + \frac{1}{2\tau + 1}} = \frac{1}{2\tau + 1} + \frac{(2\tau + 1)k + \frac{1}{2}\tau (\tau + 1)}{(2\tau + 1) \left( (2\tau + 1)k + \frac{2}{2\tau + 1} + \frac{1}{2\tau + 1} \right)}.$$

The second term on the right-hand side is non-negative for any $\tau \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. Thus,

$$\frac{1}{2\tau + 1} \leq \frac{2\alpha_k + 1}{\tau \alpha_k + \sum_{\ell=0}^\tau (\alpha_{\ell+k} + \tau + 1)}, \quad k \in \mathbb{N}_0.$$
This shows that if
\[ \gamma L \leq \frac{1}{2\tau + 1}, \]
then \((51)\) holds for \(k \in \mathbb{N}_0\), and hence the convergence condition \((13)\) in Lemma \((2.7)\) is satisfied. It follows from part 1 of Lemma \((2.7)\) that \(V_k \leq V_0\) for \(k \in \mathbb{N}\). Recalling that \(V_k = 2\gamma \alpha_k (P(x_k) - P^*) + \|x_k - x^*\|^2\), we obtain
\[ 2\gamma (k + \tau) (P(x_k) - P^*) + \|x_k - x^*\|^2 \leq 2\gamma \tau (P(x_0) - P^*) + \|x_0 - x^*\|^2, \quad k \in \mathbb{N}. \]
By dropping the second term on the left-hand side and dividing both sides by \(2\gamma (k + \tau)\), we finish the proof.

### B.3 Proof of Lemma \((3.6)\)

According to Lemma \((B.1)\), the relation \((42)\) holds for every \(x^* \in \mathcal{X}^*\). Taking \(x^* = \Pi_{\mathcal{X}^*}(x_k)\) in \((42)\), we get
\[ P(x_{k+1}) - P^* + \frac{1}{2\gamma} \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_k)\|^2 \leq \frac{1}{2\gamma} \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2 \\
+ \frac{L(\tau + 1)}{2} \sum_{\ell=(k-\tau)_+}^k \|x_{\ell+1} - x_\ell\|^2 - \frac{1}{2\gamma} \|x_{k+1} - x_k\|^2. \] 
(52)

By the projection property, we have \(\|x_{k+1} - \Pi_{\mathcal{X}^*}(x_{k+1})\| \leq \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_k)\|\). Combining this inequality with \((52)\) yields
\[ P(x_{k+1}) - P^* + \frac{1}{2\gamma} \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_k)\|^2 \leq \frac{1}{2\gamma} \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2 \\
+ \frac{L(\tau + 1)}{2} \sum_{\ell=(k-\tau)_+}^k \|x_{\ell+1} - x_\ell\|^2 - \frac{1}{2\gamma} \|x_{k+1} - x_k\|^2. \] 
(53)

Adding inequalities \((19)\) and \((53)\), and setting \(\alpha_k = \alpha\) for some \(\alpha > 0\), we have
\[ (1 + \alpha) (P(x_{k+1}) - P^*) + \frac{1}{2\gamma} \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_k)\|^2 \leq \alpha (P(x_k) - P^*) + \frac{1}{2\gamma} \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2 \\
+ \frac{L(\alpha + \tau + 1)}{2} \sum_{\ell=(k-\tau)_+}^k \|x_{\ell+1} - x_\ell\|^2 - \frac{1}{2\gamma} \left(\frac{2\alpha + 1}{\gamma} - \alpha L\tau\right) \|x_{k+1} - x_k\|^2. \] 
(54)

Let \(\theta = \frac{\alpha}{\alpha + \frac{1}{\gamma}}\). Note that \(\theta \in (0, 1)\). It follows from Assumption \((3.5)\) that
\[ (1 + \alpha) (P(x_{k+1}) - P^*) = \theta (P(x_{k+1}) - P^*) + (1 - \theta + \alpha) (P(x_{k+1}) - P^*) \\
\geq \frac{\mu\theta}{2} \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_{k+1})\|^2 + (1 - \theta + \alpha) (P(x_{k+1}) - P^*). \]

Combining the above inequality with \((51)\) and then multiplying both sides by \(2\gamma\), we obtain
\[ 2\gamma (1 - \theta + \alpha) (P(x_{k+1}) - P^*) + (1 + \gamma \mu \theta) \|x_{k+1} - \Pi_{\mathcal{X}^*}(x_{k+1})\|^2 \leq 2\alpha^2 (P(x_k) - P^*) + \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2 \\
+ \gamma L (\alpha + \tau + 1) \sum_{\ell=(k-\tau)_+}^k \|x_{\ell+1} - x_\ell\|^2 - \left(2\alpha + 1 - \gamma \alpha L\tau\right) \|x_{k+1} - x_k\|^2. \] 
(55)

By letting \(\alpha = \frac{1}{\gamma\tau}\), \(V_k = \frac{2}{\gamma^2} (P(x_k) - P^*) + \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2\) and \(W_k = \|x_{k+1} - x_k\|^2\), the inequality \((55)\) can be rewritten as
\[ (1 + \gamma \mu \theta) V_{k+1} \leq V_k + (1 + \gamma L(\tau + 1)) \sum_{\ell=(k-\tau)_+}^k W_\ell - \left(\frac{2}{\gamma^2} + 1 - \tau\right) W_k. \]

Dividing both sides by \(1 + \gamma \mu \theta\) completes the proof.
B.4 Proof of Theorem 3.7

According to Lemma 3.6, the iterates generated by Algorithm 2 satisfy

$$V_{k+1} \leq qV_k + p \sum_{\ell=(k-\tau)_+}^k W_\ell - rW_k$$

for every $k \in \mathbb{N}_0$, where

$$q = \frac{1}{1 + \gamma \mu \theta}, \quad p = \frac{1 + \gamma L(\tau + 1)}{1 + \gamma \mu \theta} \quad \text{and} \quad r = \frac{2\tau + 1 - \tau}{1 + \gamma \mu \theta}.$$

To apply Lemma 2.7, we need to ensure that

$$2\tau + 1 \leq \min \left\{ \frac{1}{\gamma \mu \theta} + 1, \frac{2\tau + 1 - \tau}{1 + \gamma L(\tau + 1)} \right\}.$$

This convergence condition is equivalent to

$$\begin{align*}
2\gamma \mu \theta \tau &\leq 1, \\
\gamma^2 L^2 (2\tau + 1)(\tau + 1) + 3\gamma L \tau &\leq 2.
\end{align*} \tag{56}$$

Define $h = \gamma L(2\tau + 1)$. The inequalities (56) can be rewritten in terms of $h$, $Q$ (recall that $\theta = \frac{Q}{Q + 1}$), and $\tau$ as

$$\begin{align*}
2h\tau &\leq (Q + 1)(2\tau + 1), \\
h^2(\tau + 1) + 3h\tau &\leq 2(2\tau + 1).
\end{align*} \tag{57}$$

For any fixed $\tau$, the left-hand side of inequalities (57) is non-decreasing in $h \geq 0$ and smaller than the right-hand side for $h = 1$. Thus, the inequalities (57) hold for any $h \in [0, 1]$. This shows that if $\gamma$ is set to

$$\gamma = \frac{h}{L(2\tau + 1)}, \quad h \in (0, 1],$$

then the convergence condition (56) is satisfied. Therefore, by part 2 of Lemma 2.7, $V_k \leq q^k V_0$ for $k \in \mathbb{N}$. Since $V_k = \frac{2}{L} (P(x_k) - P^*) + \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2$, it follows that

$$\frac{2}{L} (P(x_k) - P^*) + \|x_k - \Pi_{\mathcal{X}^*}(x_k)\|^2 \leq \left(1 - \frac{1}{1 + \gamma \mu \theta}\right)^k \left(\frac{2}{L} (P(x_0) - P^*) + \|x_0 - \Pi_{\mathcal{X}^*}(x_0)\|^2\right) \left(1 - \frac{1}{1 + (Q + 1)(2\tau + 1)/h}\right)^k \left(\frac{2}{L} (P(x_0) - P^*) + \|x_0 - \Pi_{\mathcal{X}^*}(x_0)\|^2\right).$$

This concludes the proof.

C Proofs for Subsection 3.2

In this section, we provide the proofs for the results presented in Subsection 3.2. We start with some preliminaries, then proceed to the proofs of Lemma 3.10 and Theorem 3.11.

C.1 Preliminaries

Let $I \in \mathbb{R}^{d \times d}$ be the identity matrix. We define the matrices $U_i \in \mathbb{R}^{d \times d}$, $i \in [m]$, for which $I = [U_1, \ldots, U_m]$. Then, any vector $x = ([x]_1, \ldots, [x]_m) \in \mathbb{R}^d$ can be represented as

$$x = \sum_{i=1}^m U_i [x]_i, \quad [x]_i \in \mathbb{R}^d, \quad i \in [m].$$
Since \( [x_{k+1}]_j = [x_k]_j - \gamma S_j(\vec{x}_k) \) for \( j = i_k \) and \( [x_{k+1}]_j = [x_k]_j \) for \( j \neq i_k \), the update formula of Algorithm 3 can be written as
\[
x_{k+1} = x_k - \gamma U_{i_k} S_{i_k}(\vec{x}_k), \quad k \in \mathbb{N}_0.
\]

Note that \( x_k \) depends on the observed realization of the random variable \( \xi_{k-1} := \{i_0, \ldots, i_{k-1}\} \) but not on \( i_j \) for any \( j \geq k \). For convenience, we define \( \xi_{-1} = \emptyset \). We use \( \mathbb{E} \) to denote the expectation over all random variables, and \( \mathbb{E}_k \) to denote the conditional expectation in term of \( i_k \) given \( \xi_{k-1} \).

For any random vector \( x \in \mathbb{R}^d \), the variance can be decomposed as
\[
\mathbb{E} \left[ \|x - \mathbb{E}[x]\|^2 \right] = \mathbb{E} \left[ \|x\|^2 \right] - \|\mathbb{E}[x]\|^2.
\]

For any vectors \( a, b \in \mathbb{R}^d \) and any constant \( \eta > 0 \), the inequalities
\[
\langle a, b \rangle \leq \frac{\eta \|a\|^2}{2} + \frac{\|b\|^2}{2\eta},
\]
\[
\|a + b\|^2 \leq (1 + \eta) \|a\|^2 + \left( 1 + \frac{1}{\eta} \right) \|b\|^2,
\]
and
\[
-\|a\|^2 \leq -\frac{\|b\|^2}{1 + \eta} + \frac{\|a - b\|^2}{\eta},
\]
hold by the Cauchy-Schwarz inequality.

We need the following two lemmas in the convergence analysis of ARock. The first one provides an upper bound on the expectation of \( \|x_k - \vec{x}_k\|^2 \).

**Lemma C.1.** Let \( \{x(k)\} \) be the sequence generated by Algorithm 3 Then, it holds that
\[
\mathbb{E}[\|x_k - \vec{x}_k\|^2] \leq \frac{\gamma^2 (\sqrt{m} + \sqrt{7})^2}{m^2} \sum_{\ell = (k-\tau)_{+}}^{k-1} \mathbb{E}[\|S(\vec{x}_\ell)\|^2], \quad k \in \mathbb{N}_0.
\]

**Proof.** Let \( \eta \) be a positive constant. From (58), we have
\[
\mathbb{E}[\|x_k - \vec{x}_k\|^2] = \mathbb{E} \left[ \left\| \sum_{j \in J_k} (x_{j+1} - x_j) \right\|^2 \right] = \gamma^2 \mathbb{E} \left[ \left\| \sum_{j \in J_k} U_{i_j} S_{i_j}(\vec{x}_j) \right\|^2 \right] = \gamma^2 \mathbb{E} \left[ \left\| \sum_{j \in J_k} (U_{i_j} S_{i_j}(\vec{x}_j) - \frac{1}{m} S(\vec{x}_j)) + \frac{1}{m} \sum_{j \in J_k} S(\vec{x}_j) \right\|^2 \right] \leq \gamma^2 (1 + \eta) \mathbb{E} \left[ \left\| \sum_{j \in J_k} (U_{i_j} S_{i_j}(\vec{x}_j) - \frac{1}{m} S(\vec{x}_j)) \right\|^2 \right] + \frac{\gamma^2}{m^2} \left( 1 + \frac{1}{\eta} \right) \mathbb{E} \left[ \left\| \sum_{j \in J_k} S(\vec{x}_j) \right\|^2 \right].
\]

We will find upper bounds on the quantities \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). We expand \( \mathcal{H}_1 \) as follows
\[
\mathcal{H}_1 = \sum_{j \in J_k} \mathbb{E} \left[ \left\| U_{i_j} S_{i_j}(\vec{x}_j) - \frac{1}{m} S(\vec{x}_j) \right\|^2 \right] + \sum_{j, j' \in J_k, j < j'} 2 \mathbb{E} \left[ \langle U_{i_j} S_{i_j}(\vec{x}_j) - \frac{1}{m} S(\vec{x}_j), U_{i_{j'}} S_{i_{j'}}(\vec{x}_{j'}) - \frac{1}{m} S(\vec{x}_{j'}) \rangle \right] = \sum_{j \in J_k} \mathbb{E} \left[ \left\| U_{i_j} S_{i_j}(\vec{x}_j) - \frac{1}{m} S(\vec{x}_j) \right\|^2 \right],
\]

38
where the second equality uses the fact that
\[ \sum_{j,j' \in J_k} \mathbb{E} \left[ \langle U_{ij} S_{ij}(\tilde{x}_j) - \frac{1}{m} S(\tilde{x}_j), U_{ij'} S_{ij}(\tilde{x}_{j'}) - \frac{1}{m} S(\tilde{x}_{j'}) \rangle \right] = \sum_{j,j' \in J_k} \mathbb{E} \left[ \mathbb{E}_j \left[ \langle U_{ij} S_{ij}(\tilde{x}_j) - \frac{1}{m} S(\tilde{x}_j), U_{ij'} S_{ij}(\tilde{x}_{j'}) - \frac{1}{m} S(\tilde{x}_{j'}) \rangle \right] \right]
\]
\[ = \sum_{j,j' \in J_k} \mathbb{E} \left[ \mathbb{E}_j \left[ \langle U_{ij} S_{ij}(\tilde{x}_j) - \frac{1}{m} S(\tilde{x}_j), U_{ij'} S_{ij}(\tilde{x}_{j'}) - \frac{1}{m} S(\tilde{x}_{j'}) \rangle \right] \right] = 0. \]

We then bound \( \mathcal{H}_1 \) by
\[ \mathcal{H}_1 = \sum_{j \in J_k} \mathbb{E}_j \left[ \| U_{ij} S_{ij}(\tilde{x}_j) - \frac{1}{m} S(\tilde{x}_j) \|^2 \right] \]
\[ \leq \frac{1}{m} \sum_{j \in J_k} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right] - \frac{1}{m^2} \sum_{j \in J_k} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right] \]
\[ \leq \frac{1}{m} \sum_{j \in J_k} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right]. \]  

(64)

Next, we turn to \( \mathcal{H}_2 \). We rewrite \( \mathcal{H}_2 \) as
\[ \mathcal{H}_2 = |J_k|^2 \mathbb{E} \left[ \left\| \sum_{j \in J_k} \frac{S(\tilde{x}_j)}{|J_k|} \right\|^2 \right]. \]

By convexity of the squared Euclidean norm \( \| \cdot \|^2 \), we get
\[ \mathcal{H}_2 \leq |J_k| \sum_{j \in J_k} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right]. \]  

(65)

Substituting (64) and (65) into (63) yields
\[ \mathbb{E} \left[ \| x_k - \tilde{x}_k \|^2 \right] \leq \frac{\gamma^2}{m^2} \left( (1 + \eta) m + \left( 1 + \frac{1}{\eta} \right) |J_k| \right) \sum_{j \in J_k} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right]. \]

From Assumption 3.9, \( (k - \tau)_+ \leq j \leq k - 1 \) for any \( j \in J_k \) and \( |J_k| \leq \tau \) for all \( k \in \mathbb{N}_0 \). Thus,
\[ \mathbb{E} \left[ \| x_k - \tilde{x}_k \|^2 \right] \leq \frac{\gamma^2}{m^2} \left( (1 + \eta) m + \left( 1 + \frac{1}{\eta} \right) \tau \right) \sum_{j=(k-\tau)_+}^{k-1} \mathbb{E} \left[ \| S(\tilde{x}_j) \|^2 \right]. \]

It is easy to verify that the optimal choice of \( \eta \), which minimizes the right-hand-side of the above inequality, is \( \eta = \sqrt{\tau/m} \). Using this choice of \( \eta \) and the change of variable \( \ell = j \) completes the proof. \( \blacksquare \)

As a consequence of this lemma, we derive a bound on the expectation of \( \langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle \).

**Lemma C.2.** The iterates \( \{x(k)\} \) generated by Algorithm 3 satisfy
\[ \mathbb{E} \left[ \langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle \right] \leq \frac{\gamma}{2} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \left( 1 + \frac{k-1}{\tau} \right) \sum_{\ell=(k-\tau)_+}^{k-1} \mathbb{E} \left[ \| S(\tilde{x}_\ell) \|^2 \right] + \mathbb{E} \left[ \| S(\tilde{x}_k) \|^2 \right]. \]
Proof. Let \( \eta \) be a positive constant. From (60), we have
\[
\mathbb{E}[\langle \tilde{x}_k - x^*, S(\tilde{x}_k) \rangle] \leq \frac{\eta}{2} \mathbb{E}[\|x_k - \tilde{x}_k\|^2] + \frac{1}{2\eta} \mathbb{E}[\|S(\tilde{x}_k)\|^2].
\]
It follows from Lemma C.1 that
\[
\mathbb{E}[\langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle] \leq \frac{\eta^2}{2m^2} \mathbb{E}[\|S(\tilde{x}_k)\|^2] + \frac{1}{2\eta} \mathbb{E}[\|S(\tilde{x}_k)\|^2].
\]
Substituting
\[
\eta = \frac{m}{\gamma \sqrt{m + \sqrt{\tau}}}
\]
into the above inequality proves the statement of the lemma. \(\blacksquare\)

C.2 Proof of Lemma 3.10

We reuse the proof technique from [30]. Our point of departure with [30] is to use Lemmas C.1 and C.2 to bound \( \mathbb{E}_k[\|x_k - \tilde{x}_k\|^2] \) and \( \mathbb{E}_k[\langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle] \). Let \( x^* \) be a fixed point of the operator \( T \). By subtracting \( x^* \) from both sides of (58) and then taking norm squares, we have
\[
\|x_{k+1} - x^*\|^2 = \|x_k - x^*\|^2 - 2\gamma \langle x_k - x^*, U_{i_k} S_{i_k}(\tilde{x}_k) \rangle + \gamma^2 \|U_{i_k} S_{i_k}(\tilde{x}_k)\|^2. \tag{66}
\]
By taking conditional expectation on both sides of (66) with respect to only the random variable \( i_k \), we obtain
\[
\mathbb{E}_k[\|x_{k+1} - x^*\|^2] = \|x_k - x^*\|^2 - 2\gamma \langle \tilde{x}_k - x^*, S(\tilde{x}_k) \rangle + \gamma^2 \|S(\tilde{x}_k)\|^2
\]
\[
= \|x_k - x^*\|^2 - 2\gamma \langle \tilde{x}_k - x^*, S(\tilde{x}_k) \rangle + 2\gamma \langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle + \gamma^2 \|S(\tilde{x}_k)\|^2. \tag{67}
\]
We will use \( \mathcal{H} \) to generate a \( \|x_k - x^*\|^2 \) term to help prove linear convergence. Since \( T \) is pseudo-contractive with contraction modulus \( c \) and \( S(x^*) = 0 \), it follows from [30] Lemma 8] that
\[
-\mathcal{H} \leq -\frac{1 - c^2}{2} \|\tilde{x}_k - x^*\|^2 - \frac{1}{2} \|S(\tilde{x}_k)\|^2. \tag{68}
\]
We then use (62) to convert \( \|\tilde{x}_k - x^*\|^2 \) to \( \|x_k - x^*\|^2 \) as follows
\[
-\|\tilde{x}_k - x^*\|^2 \leq -\left( \frac{1}{1 + \eta} \right) \|x_k - x^*\|^2 + \frac{1}{\eta} \|x_k - \tilde{x}_k\|^2,
\]
where \( \eta \) is a positive constant. Combining this inequality and (68), we have
\[
-\mathcal{H} \leq -\frac{1 - c^2}{2(1 + \eta)} \|x_k - x^*\|^2 + \frac{1 - c^2}{2\eta} \|x_k - \tilde{x}_k\|^2 - \frac{1}{2} \|S(\tilde{x}_k)\|^2
\]
\[
\leq -\frac{1 - c^2}{2(1 + \eta)} \|x_k - x^*\|^2 + \frac{1}{2\eta} \|x_k - \tilde{x}_k\|^2 - \frac{1}{2} \|S(\tilde{x}_k)\|^2.
\]
Substituting the above inequality into (67) yields
\[
\mathbb{E}_k[\|x_{k+1} - x^*\|^2] \leq \left( 1 - \frac{\gamma(1 - c^2)}{m(1 + \eta)} \right) \|x_k - x^*\|^2 + \frac{2\gamma}{m} \langle \tilde{x}_k - x_k, S(\tilde{x}_k) \rangle
\]
\[
+ \frac{\gamma}{mn} \|x_k - \tilde{x}_k\|^2 - \frac{\gamma}{m} (1 - \gamma) \|S(\tilde{x}_k)\|^2.
\]
After taking the full expectation on both sides, it follows from Lemmas C.1 and C.2 that

\[ E[\|x_{k+1} - x^*\|^2] \leq \left( 1 - \frac{\gamma (1 - c^2)}{m(1 + \eta)} \right) E[\|x_k - x^*\|^2] \]

\[ + \frac{\gamma^2}{m\tau} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \left( 1 + \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \frac{\gamma}{\eta} \right) \sum_{\ell=(k-\tau)_+}^{k-1} E[\|S(\hat{x}_\ell)\|^2] \]

\[ - \frac{\gamma}{m} \left( 1 - \gamma \left( 1 + \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right) E[\|S(\hat{x}_k)\|^2]. \]

(69)

Let \( V_k = E[\|x_k - x^*\|^2] \), \( W_k = E[\|S(\hat{x}_k)\|^2] \), and

\[ \eta = \gamma \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right). \]

Then, the inequality (69) can be rewritten as

\[ V_{k+1} \leq \left( 1 - \frac{\gamma (1 - c^2)}{m(1 + \gamma \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right))} \right) V_k + \frac{2\gamma^2}{m\tau} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \sum_{\ell=(k-\tau)_+}^{k-1} W_\ell \]

\[ - \frac{\gamma}{m} \left( 1 - \gamma \left( 1 + \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right) W_k. \]

This completes the proof.

C.3 Proof of Theorem 3.11

According to Lemma 3.10, the iterates generated by Algorithm 3 satisfy

\[ V_{k+1} \leq qV_k + p \sum_{\ell=(k-\tau)_+}^{k} W_\ell - rW_k \]

for every \( k \in \mathbb{N}_0 \), where

\[ q = 1 - \frac{\gamma (1 - c^2)}{m(1 + \gamma \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right))}, \quad p = \frac{2\gamma^2}{m\tau} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right), \]

and

\[ r = \frac{\gamma}{m} \left( 1 - \gamma \left( 1 + \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right) + \frac{2\gamma^2}{m\tau} \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right). \]

To apply Lemma 2.7 we need to enforce that

\[ 2\tau + 1 \leq \min \left\{ \frac{1}{1-q}, \frac{r}{p} \right\}. \]

This convergence condition is equivalent to

\[ \begin{cases} \gamma (1 - c^2)(2\tau + 1) \leq m \left( 1 + \gamma \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right), \\ \gamma (1 + 5 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right)) \leq 1. \end{cases} \]

(70)

Using the change of variable

\[ h = \gamma \left( 1 + 5 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) \right), \]
the inequalities (70) can be rewritten as
\[
\begin{align*}
\left\{ \frac{h(1-c^2)(2\tau+1)}{h} \leq m \left( (5+h) \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) + 1 \right) , \\
\leq 1. \right. 
\end{align*}
\] (71)

Since \( m \geq 1 \) and \( c \in (0, 1) \), we have
\[
(1-c^2)(2\tau+1) \leq (5+h)\tau + m = m \left( (5+h) \left( \frac{\tau}{m} + 1 \right) \right) \leq m \left( (5+h) \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) + 1 \right).
\]
Thus, the inequalities (71) hold for any \( h \in [0, 1] \). This shows that if the step-size \( \gamma \) is set to
\[
\gamma = \frac{h}{1+5 \left( \frac{\tau}{m} + \sqrt{\frac{\tau}{m}} \right) }, \quad h \in (0, 1],
\]
then the convergence condition (70) is satisfied. Therefore, by part 2 of Lemma 2.7, \( V_k \leq q^k V_0 \) for all \( k \in \mathbb{N}_0 \).

Since \( V_k = \mathbb{E}[\|x_k - x^*\|^2] \), it follows that
\[
\mathbb{E}[\|x_k - x^*\|^2] \leq \left( 1 - \frac{h(1-c^2)}{m (1 + 6 (\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}))} \right)^k \|x_0 - x^*\|^2 \leq \left( 1 - \frac{h(1-c^2)}{m (1 + 6 (\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}))} \right)^k \|x_0 - x^*\|^2,
\]
where the second inequality is due to that \( h \leq 1 \). This matches the upper bound (23) on the convergence rate.

Finally, we derive the iteration complexity bound (24). Taking logarithm of both sides of (23) yields
\[
\log (\mathbb{E}[\|x_k - x^*\|^2]) \leq k \log \left( 1 - \frac{h(1-c^2)}{m (1 + 6 (\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}))} \right) + \log (\|x_0 - x^*\|^2).
\]
Since \( \log(1+x) \leq x \) for any \( x > -1 \), it follows that
\[
\log (\mathbb{E}[\|x_k - x^*\|^2]) \leq - \left( \frac{h(1-c^2)}{m (1 + 6 (\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}))} \right) k + \log (\|x_0 - x^*\|^2).
\]
Therefore, for any \( k \) satisfying
\[
- \left( \frac{h(1-c^2)}{m (1 + 6 (\frac{\tau}{m} + \sqrt{\frac{\tau}{m}}))} \right) k + \log (\|x_0 - x^*\|^2) \leq \log(\epsilon),
\] (72)
we have \( \log (\mathbb{E}[\|x_k - x^*\|^2]) \leq \log(\epsilon) \), implying that \( \mathbb{E}[\|x_k - x^*\|^2] \leq \epsilon \). Rearranging terms in (72) completes the proof.

D Proofs for Subsection 3.3

This section provides the proofs for the results presented in Subsection 3.3.
D.1 Proof of Lemma 3.19

For each \( i \in [m] \), let \( \kappa_i \) and \( \kappa'_i \) be two arbitrary consecutive elements of \( \mathcal{K}_i \). From (29), we have

\[
[x_{k+1}]_i = T_i \left( [x_{s_1,\kappa_i}]_1, \ldots, [x_{s_m,\kappa_i}]_m \right), \quad k \in [\kappa_i, \kappa'_i].
\]

As \( \tau_i(k) = \kappa_i \) for \( k \in [\kappa_i, \kappa'_i] \), we obtain

\[
[x_{k+1}]_i = T_i \left( [x_{s_1,\tau_i(k)}]_1, \ldots, [x_{s_m,\tau_i(k)}]_m \right), \quad k \in [\kappa_i, \kappa'_i].
\]

Since \( \kappa_i \) and \( \kappa'_i \) are two arbitrary consecutive elements of \( \mathcal{K}_i \) and \( 0 \in \mathcal{K}_i \) for each \( i \), we can rewrite the asynchronous iteration (29) as

\[
[x_{k+1}]_i = T_i \left( [x_{s_1,\tau_i(k)}]_1, \ldots, [x_{s_m,\tau_i(k)}]_m \right), \quad k \in \mathbb{N}_0.
\] (73)

Let \( V_k = \|x_k - x^*\|_{b,\infty}^w \). From the definition of \( \| \cdot \|_{b,\infty}^w \), we have

\[
V_{k+1} = \max_{i \in [m]} w_i \| [x_{k+1}]_i - [x^*]_i \|_i
\]

\[
\overset{(73)}{\leq} \max_{i \in [m]} w_i \| T_i \left( [x_{s_1,\tau_i(k)}]_1, \ldots, [x_{s_m,\tau_i(k)}]_m \right) - [x^*] \|_i
\]

\[
\overset{}{\leq} \max_{i \in [m]} \| T \left( [x_{s_1,\tau_i(k)}]_1, \ldots, [x_{s_m,\tau_i(k)}]_m \right) - x^* \|_{b,\infty}^w,
\]

where the inequality follows from the fact that \( w_i \| [x]_i \|_i \leq \| x \|_{b,\infty}^w \) for any \( x \in \mathbb{R}^d \) and \( i \in [m] \). Since \( T \) is pseudo-contractive with respective to the block-maximum norm with contraction modulus \( c \), we obtain

\[
V_{k+1} \leq c \max_{i \in [m]} \left( \| [x_{s_1,\tau_i(k)}]_1, \ldots, [x_{s_m,\tau_i(k)}]_m \| - x^* \|_{b,\infty}^w \right)
\]

\[
= c \max_{i \in [m]} \max_{j \in [m]} w_j \left( \| x_{s_j,\tau_i(k)} \|_j - [x^*]_j \right),
\]

where the equality follows from the definition of \( \| \cdot \|_{b,\infty}^w \). Since \( w_j \| [x]_j \|_j \leq \| x \|_{b,\infty}^w \) for any \( x \in \mathbb{R}^d \) and \( j \in [m] \), we have

\[
V_{k+1} \leq c \max_{i \in [m]} \max_{j \in [m]} \left( \| x_{s_j,\tau_i(k)} \|_j - x^* \|_{b,\infty}^w \right)
\]

\[
= c \max_{i \in [m]} \max_{j \in [m]} V_{s_j,\tau_i(k)}.
\] (74)

From (30), \( k - \tau_k \leq s_{ij,\tau_i(k)} \) for all \( i, j \in [m] \) and \( k \in \mathbb{N}_0 \). Thus, \( s_{ij,\tau_i(k)} \in \{ k - \tau_k, \ldots, k \} \), and hence

\[
V_{s_{ij,\tau_i(k)}} \leq \max_{k - \tau_k \leq \ell \leq k} V_{\ell}.
\]

This together with (74) implies that

\[
V_{k+1} \leq c \max_{k - \tau_k \leq \ell \leq k} \max_{\ell \in [m]} V_{\ell},
\]

which is the desired result.

D.2 Proof of Theorem 3.20

According to Lemma 3.19, the iterates generated by (29) satisfy

\[
V_{k+1} \leq q V_k + p \max_{(k - \tau_k)_+ \leq \ell \leq k} V_{\ell}, \quad k \in \mathbb{N}_0,
\]

43
with \( q = 0, p = c \), and
\[
\tau_k = k - \min_{i \in [m]} \min_{j \in [m]} s_{ij,t_i(k)}.
\]

Since \( t_i(k) \in \mathcal{K}_i \) for all \( i \in [m] \) and \( k \in \mathbb{N}_0 \), it follows from Assumption 3.17.1 that \( \lim_{k \to \infty} t_i(k) = \infty \). Thus, by Assumption 3.17.2, \( s_{ij,t_i(k)} \to \infty \) as \( k \to \infty \) for all \( i, j \in [m] \). This implies that
\[
\lim_{k \to \infty} k - \tau_k = \infty.
\]

Therefore, since \( c < 1 \), using Lemma 2.2 with \( q = 0 \) and \( p = c \) completes the proof.

**D.3 Proof of Theorem 3.21**

From Assumption 3.18.2, \( k - D \leq s_{ij,k} \forall \{i,j\} \subseteq [m] \) and \( k \in \mathbb{N}_0 \). Since \( t_i(k) \in \mathcal{K}_i \) for \( k \in \mathbb{N}_0 \), we have
\[
t_i(k) - D \leq s_{ij,t_i(k)}.
\]

By Assumption 3.18.1, \( k - B \leq t_i(k) \) for each \( i \) and all \( k \in \mathbb{N}_0 \). Thus, \( k - B - D \leq s_{ij,t_i(k)} \), implying that
\[
\tau_k = k - \min_{i \in [m]} \min_{j \in [m]} s_{ij,t_i(k)} \leq B + D.
\]

It follows from Lemma 3.19 that the iterates generated by (29) satisfy
\[
V_{k+1} \leq c \max_{k - \tau_k \geq f \leq k} V_f, \quad k \in \mathbb{N}_0,
\]
with \( V_k = \|x_k - x^*\|_{b,\infty} \) and \( \tau_k \leq B + D \). Since \( c < 1 \), using Lemma 2.1 with \( q = 0, p = c \) and \( \tau = B + D \) leads to
\[
V_k \leq c^{\frac{k}{B+D}} V_0, \quad k \in \mathbb{N}_0.
\]

The proof is complete.

**D.4 Proof of Theorem 3.22**

According to Lemma 3.19, the iterates generated by (29) satisfy (8) with \( V_k = \|x_k - x^*\|_{b,\infty} \), \( q = 0 \), \( p = c \), and \( \tau_k \) defined in (30). Since \( q + p = c < 1 \), it follows from Lemma 2.5 that \( V_k \leq \Lambda(k)V_0 \). This completes the proof.