POINCARÉ TYPE INEQUALITIES FOR TWO DIFFERENT BILATERAL GRAND LEBESGUE SPACES

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Abstract.

In this paper we obtain the non-asymptotic inequalities of Poincaré type between function and its weak gradient belonging the so-called Bilateral Grand Lebesgue Spaces over general metric measurable space. We also prove the sharpness of these inequalities.

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1. Introduction

Let \((X, M, \mu, d)\) be metric measurable space with finite non-trivial measure \(\mu : 0 < \mu(X) < \infty\) and also with finite non-trivial distance function \(d = d(x, y) : 0 < \text{diam}(X) := \sup_{x,y \in X} d(x, y) < \infty\).

Define also for arbitrary numerical measurable function \(u : X \to R\) the following average

\[
u_X = \frac{1}{\mu(X)} \int_X u(x) \, d\mu(x),
\]

\[
||u||_p = \left[ \int_X |u(x)|^p \, d\mu(x) \right]^{1/p}, \quad p = \text{const} \in [1, \infty],
\]

\(g(x) = \nabla[u](x)\) will denote a so-called minimal weak upper gradient of the function \(u(\cdot)\), i.e. the (measurable) minimal function such that for any rectifiable curve \(\gamma : [0, 1] \to X\)

\[
|u(\gamma(1)) - u(\gamma(0))| \leq \int_{\gamma} g(s) \, ds.
\]

Note that if the function \(u(\cdot)\) satisfies the Lipschitz condition:

\[
|u(x) - u(y)| \leq L \cdot d(x, y), \quad 0 \leq L = \text{const} < \infty,
\]

then the function \(g(x) = \nabla[u](x)\) there exists and is bounded: \(g(x) \leq L\).
"The term Poincaré type inequality is used, somewhat loosely, to describe a class of inequalities that generalize the classical Poincaré inequality"

\[ \int_D |u(z)|^p \, dx \leq A_m(p, D) \int_D |\nabla u(z)|^p \, dz, \quad A_m(p, D) = \text{const} < \infty, \quad (1.0) \]

see [1], chapter 8, p.215, and the source work of Poincaré [36].

A particular case done by Wirtinger: "an inequality ascribed to Wirtinger", see [25], p. 66-68; see also [33], [38], [39].

In the inequality (1.0) \( D \) may be for instance open bounded non empty convex subset of the whole space \( \mathbb{R}^m \) and has a Lipschitz or at last Hölder boundary, or consists on the finite union of these domains, and \( |\nabla u(z)| \) is ordinary Euclidean \( \mathbb{R}^m \) norm of "natural" distributive gradient of the differentiable a.e. function \( u \).

The generalized Sobolev’s norm, more exactly, semi-norm \( ||f||_{W^1_p} \) of a "weak differentiable" function \( f : X \to \mathbb{R} \) may be defined by the formula

\[ ||f||_{W^1_p} \overset{\text{def}}{=} \left[ \int_X |\nabla f|^p \, d\mu(x) \right]^{1/p} = ||\nabla f||_p. \]

We will call "the Poincaré inequality" , or more precisely "the Poincaré \((L(p), L(q))\) inequality" more general inequalities of the forms

\[ \mu(X)^{-1/q}||u - u_X||_q \leq K_P(p, q) \text{diam}(X) \mu(X)^{-1/p} ||\nabla u||_p = 
\]

\[ K_P(p, q) \text{diam}(X) \mu(X)^{-1/p} ||u||_{W^1_p}, \quad (1.1) \]

where

\[ 1 \leq p < s = \text{const} > 1, \quad 1 \leq q < \frac{ps}{s-p} \quad (1.1a) \]

the Poincaré-Lebesgue-Riesz version; or in the case when \( p > s \) and after (possible) redefinition of the function \( u = u(x) \) on a set of measure zero

\[ |u(x) - u(y)| \leq K_L(s, p) d^{1-s/p}(x, y) \mu(X) ||\nabla u||_p = 
\]

\[ K_L(s, p) \mu(X) ||u||_{W^1_p} \quad (1.2) \]

the Poincaré-Lipshitz version; the case \( p = s \) in our setting of problem, indeed, in the terms of Orlicz’s spaces and norms, is considered in [19].

The last inequality (1.2) may be reformulated in the terms of the module of continuity of the function \( u \):

\[ \omega(u, \tau) := \sup_{x,y: d(x,y) \leq \tau} |u(x) - u(y)|, \quad \tau \geq 0. \]

Namely,

\[ \omega(u, \tau) \leq K_L(s, p) \tau^{1-s/p} \mu(X) ||u||_{W^1_p}. \]
We will name following the authors of articles \[13\], \[19\] etc. all the spaces \((X, M, \mu, d)\) which satisfied the inequalities (1.1) or (1.2) for each functions \(\{u\}\) having the weak gradient correspondingly as a Poincaré-Lebesgue spaces or Poincaré-Lipshitz spaces.

As for the constants \(s, K_P(s, p), K_L(s, p)\). The value \(s\) may be defined from the following condition (if there exists)

\[
\inf_{x_1,x_2 \in X} \left\{ \frac{\mu(B(x_1, r_1))}{\mu(B(x_2, r_2))} \right\} \geq C \cdot \left[ \frac{r_1}{r_2} \right]^s, \quad C = \text{const} > 0,
\]

where as ordinary \(B(x, r)\) denotes a closed ball relative the distance \(d(\cdot, \cdot)\) with the center \(x\) and radii \(r, r > 0:\)

\[B(x, r) = \{y, \ y \in X, \ d(x, y) \leq r\}.\]

This condition is equivalent to the so-called double condition, see \[13\], \[19\] and is closely related with the notion of Ahlfors \(Q\) – regularity

\[C_1 r^Q \leq \mu(B(x, r)) \leq C_2 r^Q, \quad C_1, C_2, Q = \text{const} > 0,\]

see \[19\], \[26\].

Further, we will understand as a capacity of the values \(K_P(s, p), K_L(s, p)\) its minimal values, namely

\[
K_P(p, q) \overset{\text{def}}{=} \sup_{0 < ||\nabla u||_p < \infty} \left\{ \frac{\mu(X)^{-1/q} ||u - u_X||_q}{\text{diam}(X) \mu(X)^{-1/p} ||\nabla u||_p} \right\}, \quad (1.3a)
\]

\[
K_L(s, p) \overset{\text{def}}{=} \sup_{0 < ||\nabla u||_p < \infty} \left\{ \frac{|u(x) - u(y)|}{d^{1-s/p}(x, y) \mu(X) ||\nabla u||_p} \right\}. \quad (1.3b)
\]

We will denote for simplicity

\[s = \text{order } X = \text{order}(X, M, \mu, d).\]

There are many publications about grounding of these inequalities under some conditions and about its applications, see, for instance, in articles \[6\], \[7\], \[11\], \[13\], \[14\], \[19\], \[20\], \[23\], \[30\], \[34\], \[37\], \[39\] and in the classical monographs \[3\], \[12\]; see also reference therein.

Our aim is a generalization of the estimation (1.1) and (1.2) on the so-called Bilateral Grand Lebesgue Spaces \(BGL = BGL(\psi) = G(\psi)\), i.e. when \(u(\cdot) \in G(\psi)\) and to show the precision of obtained estimations.

We recall briefly the definition and needed properties of these spaces. More details see in the works \[9\], \[10\], \[15\], \[16\], \[28\], \[29\], \[21\], \[17\], \[18\] etc. More about rearrangement invariant spaces see in the monographs \[1\], \[22\].

For \(b = \text{const}, \ 1 < b \leq \infty\), let \(\psi = \psi(p), p \in [1, b)\), be a continuous positive function such that there exists a limits (finite or not) \(\psi(1 + 0)\) and \(\psi(b - 0)\), with conditions \(\inf_{p \in (1, b)} \psi(p) > 0\) and \(\min\{\psi(1 + 0), \psi(b - 0)\} > 0\). We will denote the set of all these functions as \(\Psi(b)\) and \(b = \text{supp } \psi\).
The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : R^d \rightarrow R$ endowed with the norm

$$||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (a, b)} \left[ \frac{|f|^p}{\psi(p)} \right],$$

if it is finite.

In the article [29] there are many examples of these spaces.

The $G(\psi)$ spaces over some measurable space $(X, M, \mu)$ with condition $\mu(X) = 1$ (probabilistic case) appeared in [21].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [2], [5], [17], [18].

It was proved also that in this case each $G(\psi)$ space coincides only under some additional conditions: convexity of the functions $p \rightarrow p \cdot \ln \psi(p)$, $b = \infty$ etc. [29] with the so-called exponential Orlicz space, up to norm equivalence.

In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

**Remark 1.1** If we introduce the discontinuous function

$$\psi_r(p) = 1, \ p = r; \psi_r(p) = \infty, \ p \neq r, \ p, r \in (a, b) \quad (1.5)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides formally with the $L_r$ norm:

$$||f||_{G(\psi_r)} = |f|_r. \quad (1.5a)$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces $L_r$.

**Remark 1.2.** Let $F = \{f_\alpha(x)\}, \ x \in X, \ \alpha \in A$ be certain family of numerical functions $f_\alpha(\cdot) : x \rightarrow R, \ A$ is arbitrary set, such that

$$\exists b > 1, \ \forall p < b \Rightarrow \psi_F(p) \overset{def}{=} \sup_{\alpha \in A} ||f_\alpha(\cdot)||_p < \infty. \quad (1.6)$$

The function $p \rightarrow \psi_F(p)$ is named ordinary as natural function for the family $F$. Evidently,

$$\forall \alpha \in A \Rightarrow f_\alpha(\cdot) \in G\psi_F$$

and moreover

$$\sup_{\alpha \in A} ||f_\alpha(\cdot)||_{G\psi_F} = 1. \quad (1.7)$$

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [9], [15], theory of probability in Banach spaces [24], [21], [28], in the modern non-parametrical statistics, for example, in the so-called regression problem [28].

We will denote as ordinary the indicator function

$$I(A) = I(x \in A) = 1, \ x \in A, \ I(x \in A) = 0, \ x \notin A;$$
here $A$ is a measurable set.

Recall, see, e.g. [4] that the fundamental function $\phi(\delta, S)$, $\delta > 0$ of arbitrary rearrangement invariant space $S$ over $(X, M, \mu)$ with norm $\| \cdot \|_S$ is

$$\phi(\delta, S) \overset{\text{def}}{=} \| I(A) \|_S, \mu(A) = \delta.$$ 

We have in the case of BGLS spaces

$$\phi(\delta, G\psi) = \sup_{1 \leq p < b} \left[ \delta^{1/p} \frac{\delta^{1/b}}{\psi(p)} \right].$$

(1.8)

This notions play a very important role in the functional analysis, operator theory, theory of interpolation and extrapolation, theory of Fourier series etc., see again [4]. Many properties of the fundamental function for BGLS spaces with considering of several examples see in the articles [29], [27].

**Example 1.1.** Let $\mu(X) = 1$ and let

$$\psi^{b,\beta}(p) = (b - p)^{-\beta}, \ 1 \leq p < b, \ b = \text{const} > 1, \ \beta = \text{const} > 0,$$

then as $\delta \to 0^+$

$$\phi(G\psi^{b,\beta}, \delta) \sim (\beta b^2/e)^{\beta} \cdot \delta^{1/b} \cdot |\ln \delta|^{-\beta}.$$ 

(1.9a)

**Example 1.2.** Let again $\mu(X) = 1$ and let now

$$\psi^{q}(p) = p^\beta, \ 1 \leq p < \infty, \ \beta = \text{const} > 0,$$

then as $\delta \to 0^+$

$$\phi(G\psi^{q}, \delta) \sim \beta^\beta |\ln \delta|^{-\beta}.$$ 

(1.10a)

2. **Main result: BGLS estimations for Poincaré-Lebesgue-Riesz version.**

**The case of probability measure.**

We suppose in this section without loss of generality that the measure $\mu$ is probabilistic: $\mu(X) = 1$ and that the source tetrad $(X, M, \mu, d)$ is Poincaré-Lebesgue space.

Assume also that the function $|\nabla u(x)|$, $x \in X$ belongs to certain BGLS $G\psi$ with supp $\psi = s = \text{order} \ X > 1$; the case when order $\psi = b \neq s$ may be reduced to considered here by transfiguration $s' := \min(b, s)$.

The function $\psi(\cdot)$ may be constructively introduced as a natural function for one function $|\nabla u| :$

$$\psi^{(0)}(p) := \| u \|_{W^1_p},$$

if there exists and is finite for at least one value $p$ greatest than one.

Define the following function from the set $\Psi$

$$\nu(q) := \inf_{p \in (qs/(q+s), s)} \{ K_P(p, q) \cdot \psi(p) \}, \ 1 \leq q < \infty.$$ 

(2.1)

**Proposition 2.1.**
\[ \|u - u_X\|_{G\nu} \leq \text{diam}(X) \cdot \|\nabla u\|_{G\psi}, \]  
(2.2)

where the "constant" \(\text{diam}(X)\) is the best possible.

**Proof.** We can suppose without loss of generality \(\|\nabla u\|_{G\psi} = 1\), then it follows by the direct definition of the norm in BGLS

\[ \|\nabla u\|_{p} \leq \psi(p), \; 1 \leq p < s. \]  
(2.3)

The inequality (1.1) may be rewritten in our case as follows:

\[ \|u - u_X\|_{q} \leq K_P(p, q) \cdot \text{diam}(X) \cdot \|\nabla u\|_{p}, \]  
therefore

\[ \|u - u_X\|_{q} \leq K_P(p, q) \cdot \text{diam}(X) \cdot \psi(p), \; 1 \leq p < s. \]  
(2.4)

Since the value \(p\) is arbitrary in the set \(1 \leq p < s\), we can take the minimum of the right-hand side of the inequality (2.4):

\[ \|u - u_X\|_{q} \leq \text{diam}(X) \cdot \inf_{1 \leq p < s} [K_P(p, q) \cdot \psi(p)] = \text{diam}(X) \cdot \nu(q), \]  
which is equivalent to the required estimate

\[ \|u - u_X\|_{G\nu} \leq \text{diam}(X) = \text{diam}(X) \cdot \|\nabla u\|_{G\psi}. \]

The exactness of the constant \(\text{diam}(X)\) in the inequality (2.2) follows immediately from theorem 2.1 in the article [32].

3. **Main result: BGLS estimations for Poincaré-Lebesgue-Riesz version.**

The general case of arbitrary measure.

The case when the value \(\mu(X)\) is variable, is more complicated. As a rule, in the role of a sets \(X\) acts balls \(B(x, r)\), see [13], [19].

**Definition 3.1.** We will say that the function \(K_P(p, q)\), \(1 \leq p < s, 1 \leq q < \infty\) allows factorable estimation, symbolically: \(K_P(\cdot, \cdot) \in AFE\), iff there exist two functions \(R = R(p) \in \Psi(s)\) and \(V = V(q) \in G\Psi(\infty)\) such that

\[ K_P(p, q) \leq R(p) \cdot V(q). \]  
(3.1)

**Theorem 3.1.** Suppose that the source tetrad \((X, M, \mu, d)\) is again Poincaré-Lebesgue space such that \(K_P(\cdot, \cdot) \in AFE\). Let \(\zeta = \zeta(q)\) be arbitrary function from the set \(\Psi(\infty)\).

Assume also as before in the second section that the function \(|\nabla u(x)|\), \(x \in X\) belongs to certain BGLS \(G\psi\) with supp \(\psi = s = \text{order} X > 1\); the case when order \(\psi = b \neq s\) may be reduced to considered here by transfiguration \(s' := \min(b, s)\).

Our statement:

\[ \frac{\|u - u_X\|_{G(V \cdot \zeta)}}{\phi(G\zeta, \mu(X))} \leq \text{diam}(X) \cdot \frac{\|\nabla u\|_{G\psi}}{\phi(R \cdot \psi, \mu(X))} \]  
(3.2)
and the "constant" diam(X) in (3.2) is as before the best possible.

Proof. Denote and suppose for brevity $u^{(0)} = u - u_X$, $\mu = \mu(X)$, diam(X) = 1, $||\nabla u||G\psi = 1$. The last equality imply in particular

$$||\nabla u||_p \leq \psi(p), \quad 1 \leq p < s.$$  

(3.3)

The inequality (1.1) may be reduced taking into account (3.3) as follows

$$\mu^{-1/q} ||u^{(0)}||_q \leq R(p) V(q) \mu(X)^{-1/p} \psi(p),$$

and after dividing by $\zeta(q)$ and by $\mu^{-1/q}$

$$\frac{||u^{(0)}||_q}{V(q) \zeta(q)} \leq R(p) \psi(p) \mu^{-1/p} \frac{\mu^{1/q}}{\zeta(q)}.$$  

(3.4)

We take the supremum from both the sides of (3.4) over $q$ using the direct definition of the fundamental function and norm for BGLS:

$$||u^{(0)}||G(V \cdot \zeta) \leq \frac{R(p) \psi(p)}{\mu^{1/p}} \cdot \phi(G \zeta, \mu).$$  

(3.5)

Since the left-hand side of relation (3.5) does not dependent on the variable $p$, we can take the infimum over $p$. As long as

$$\inf_p \left[ \frac{R(p) \psi(p)}{\mu^{1/p}} \right] = \left[ \sup_p \frac{\mu^{1/p}}{R(p) \psi(p)} \right]^{-1} = [\phi(G(R \cdot \psi), \mu)]^{-1},$$

we deduce from (3.5)

$$\frac{||u^{(0)}||G(V \cdot \zeta)}{\phi(G \zeta, \mu)} \leq \frac{1}{\phi(G(R \cdot \psi), \mu)} = \text{diam} X \cdot \frac{||\nabla u||G\psi}{\phi(G(R \cdot \psi), \mu)},$$

Q.E.D.

4. Main result: BGLS estimations for Poincaré-Lipschitz version.

Recall that we take the number $s$, $s > 1$ to be constant.

We consider in this section the case when $p \in (s, b)$, $s < b = \text{const} \leq \infty$.

Theorem 4.1. Suppose the fourth $(X, M, \mu, d)$ is Poincaré-Lipschitz space and that the function $|\nabla u(x)|, x \in X$ belongs to certain BGLS $G\psi$ with supp $\psi = b$. Then the function $u = u(x)$ satisfies after (possible) redefinition on a set of measure zero the inequality

$$|u(x) - u(y)| \leq \mu(X) \cdot \frac{d(x, y)}{\phi(G(K_L \cdot \psi), d^s(x, y))} \cdot ||\nabla u||G\psi,$$

(4.1)

or equally

$$\omega(u, \tau) \leq \mu(X) \cdot \frac{\tau}{\phi(G(K_L \cdot \psi), \tau^s)} \cdot ||\nabla u||G\psi,$$

(4.1a)

and this time the "constant" $\mu(X)$ is best possible.
Proof. Suppose for brevity \( \mu(X) = 1, \ ||\nabla u||G\psi = 1. \) The last equality imply in particular

\[
||\nabla u||_p \leq \psi(p), \ s < p < b. \quad (4.2)
\]

The function \( u(\cdot) \) satisfies the inequality (1.2) after (possible) redefinition of the function \( u = u(x) \) on a set of measure zero

\[
|u(x) - u(y)| \leq K_L(s,p) d^{1-s/p}(x,y) \mu(X) ||\nabla u||_p \leq K_L(s,p)\psi(p) \cdot d^{1-s/p}(x,y). \quad (4.3)
\]

The excluding set in (4.3) may be dependent on the value \( p \), but it sufficient to consider this inequality only for the rational values \( p \) from the interval \((s,b)\).

The last inequality may be transformed as follows

\[
|u(x) - u(y)| \leq K_L(s,p) \psi(p) \cdot d^{1-s/p}(x,y) \cdot \mu(X) \cdot \frac{d(x,y)}{\phi(G(K_L \cdot \psi), d^s(x,y))}.
\]

Since the left-hand side of (4.4) does not dependent on the variable \( p \), we can take the infimum from both all the sides of (4.4):

\[
|u(x) - u(y)| \leq \frac{1}{\phi(G(K_L \cdot \psi), d^s(x,y))} \cdot \mu(X) \cdot \frac{d(x,y)}{\phi(G(K_L \cdot \psi), d^s(x,y))} = \mu(X) \cdot \mu(X) \cdot \frac{d(x,y)}{\phi(G(K_L \cdot \psi), d^s(x,y))} \cdot ||\nabla u||G\psi.
\]

The exactness of the constant \( \mu(X) \) may be proved as before, by mention of the article [32].

This completes the proof of theorem 4.1.

Let us consider two examples.

Example 4.1. Suppose in addition to the conditions of theorem 4.1 that \( \mu(X) = 1 \) and

\[
K_L(s,p) \psi(p) = \psi(b,\beta)(p) = (b - p)^{-\beta}, \ 1 \leq p < b, \ b = \text{const} > 1, \ \beta = \text{const} > 0.
\]

We deduce taking into account the example 1.1 that for almost everywhere values \((x,y)\) and such that \(d(x,y) \leq 1/e\)

\[
|u(x) - u(y)| \leq C_1(b,\beta,s) d^{1-1/b}(x,y) \mid \ln d(x,y) \mid \beta \cdot ||\nabla u||G\psi.
\]

Example 4.2. Suppose in addition to the conditions of theorem 4.1 that \( \mu(X) = 1 \) and
\[ K_L(s,p) \cdot \psi(p) = \psi(\beta)(p) = p^\beta, \ 1 \leq p < \infty, \ \beta = \text{const} > 0. \]

We deduce taking into account the example 1.2 that for almost everywhere values \((x,y)\) and such that \(d(x,y) \leq 1/e\)

\[ |u(x) - u(y)| \leq C_2(\beta,s) \ d(x,y) \ |\ln d(x,y)|^\beta ||\nabla u||G\psi. \]

5. Concluding remarks

A. It may be interest by our opinion to investigate the weights generalization of obtained inequalities, alike as done for the classical Sobolev’s case, see for instance \([6],[23],[35],[37]\).

B. The physical applications of these inequalities, for example, in the uncertainty principle, is described in the article of C.Fefferman \([8]\).

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