Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables

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Received 17 February 2014
Accepted for publication 28 March 2014
Published 22 May 2014

Online at stacks.iop.org/JSTAT/2014/P05015
doi:10.1088/1742-5468/2014/05/P05015

Abstract. We solve the longstanding problem of defining a functional characterization of the spectrum of the transfer matrix associated with the most general spin-1/2 representations of the six-vertex reflection algebra for general inhomogeneous chains. The corresponding homogeneous limit reproduces the spectrum of the Hamiltonian of the spin-1/2 open XXZ and XXX quantum chains with the most general integrable boundaries. The spectrum is characterized by a second order finite difference functional equation of Baxter type with an inhomogeneous term which vanishes only for some special but yet interesting non-diagonal boundary conditions. This functional equation is shown to be equivalent to the known separation of variables (SOV) representation, hence proving that it defines a complete characterization of the transfer matrix spectrum. The polynomial form of the $Q$-function allows us to show that a finite system of generalized Bethe equations can also be used to describe the complete transfer matrix spectrum.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

ArXiv ePrint: 1401.4901
1. **Introduction**

The functional characterization of the complete transfer matrix spectrum associated with the most general spin-1/2 representations of the six-vertex reflection algebra on general...
inhomogeneous chains is a longstanding open problem. It has attracted much attention in the framework of quantum integrability, producing so far only partial results. This problem is extremely interesting from at least two points of view. On the one hand, the corresponding quantum integrable system, i.e. the open spin-1/2 XXZ quantum chain with arbitrary boundary magnetic fields, is an interesting physical quantum model. It appears, in particular, in the context of out-of-equilibrium physics ranging from the relaxation behavior of some classical stochastic processes, as the asymmetric simple exclusion processes [20, 21], to the transport properties of the quantum spin systems [56, 57]. Solution of these problems can lead to non-perturbative physical results and a complete and manageable functional characterization of their spectrum represents the first fundamental steps in this direction. On the other hand the analysis of the spectral problem of these integrable quantum models turned out to be quite involved by standard Bethe ansatz [7, 27] techniques. Therefore, these quantum models are natural laboratories where one can try alternative non-perturbative approaches. Indeed, the algebraic Bethe ansatz, introduced for open systems by Sklyanin [59] based on Cherednik’s reflection equation [14], in the case of open XXZ quantum spin chains can be applied directly only in the case of parallel z-oriented boundary magnetic fields. For these special boundary terms the spectrum is naturally described by a finite system of Bethe ansatz equations. Moreover, the dynamics of such systems can be studied by exact computation of correlation functions [34, 35], derived from a generalization of the method introduced in [36]–[38] for periodic spin chains.

Introducing a Baxter T–Q equation, Nepomechie [40, 41] first succeeded in describing the spectrum of the XXZ spin chain with non-diagonal boundary terms in the case of an anisotropy parameter associated with the roots of unity. Furthermore, this result was obtained only for the boundary terms satisfying a very particular constraint relating the magnetic fields on the two boundaries. This constraint was also used in [9] to introduce a generalized algebraic Bethe ansatz approach to this problem inspired by papers of Baxter [3, 4] and of Faddeev and Takhtajan [63] on the XYZ spin chain. This method has led to the first construction of the eigenstates of the XXZ spin chain with non z-oriented boundary magnetic fields and this construction has been obtained for a general anisotropy parameter, i.e., not restricted to the roots of unity cases. In [64] a different version of this technique based on the vertex-IRF transformation was proposed but in fact it required one additional constraint on the boundary parameters to work. We would like to mention that even if these constrained boundary conditions are satisfied and the generalized Bethe ansatz method gives a possibility to go beyond the spectrum, as was done for the diagonal boundary conditions, no representation of the scalar product of Bethe vectors and hence of the correlation functions was obtained.

This spectral problem in the most general setting has also been addressed by other approaches. An alternative functional method was presented in [22] leading to nested Bethe ansatz equations (analogous to those previously introduced in [39] by a generalized

3 Different methods leading to Bethe ansatz equations have been also proposed under the same boundary constraints using the framework of the Temperley–Lieb algebra in [23, 54] and making a combined use of coordinate Bethe ansatz and matrix ansatz in [15, 16].

4 Some partial results in this direction were achieved in [30] but only in the special case of double boundary constraints introduced in [64].
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T–Q formalism) for the eigenvalue characterization. The eigenstate construction has also been considered in these general settings in [1, 2] in the framework of the so-called $q$-Onsager algebra formalism. In this last case the spectrum is given by the classification of the roots of some characteristic polynomials. More recently, in [10] an ansatz $T$–$Q$ functional equation for the spin chains with non-diagonal boundaries has been proposed.

It is extremely important to remark that in general all methods based on the Bethe ansatz (or the generalized Bethe ansatz) are lacking proofs of the completeness of the spectrum and in most cases the only evidence of completeness is based on numerical checks for short length chains. This is true, in particular, for the XXZ chain with non-diagonal boundary matrices satisfying the boundary constraint. In this case the completeness of the description by the corresponding system of Bethe equations has been studied numerically [43, 44]. In the case of the XXZ chain with completely general non-diagonal boundary matrices some numerical analysis is also presented in [10]. Further numerical analysis has been developed in a much simpler case of the isotropic XXX spin chain where the most general boundary conditions can always be reduced using the $SU(2)$ symmetry to one diagonal boundary matrix and one non-diagonal boundary matrix. For the XXX chains the ansatz introduced in [12] was also applied and the completeness of the Bethe ansatz spectrum was checked numerically [33]. It is also important to mention a simplified ansatz proposed by Nepomechie based on a standard second order difference functional $T$–$Q$ equation with an additional inhomogeneous term. The completeness of the Bethe ansatz spectrum has been verified numerically for small XXX chains in [42] while in [45] the problem of the description of some thermodynamic properties has been addressed.

These interesting developments attracted our attention in connection with the quantum separation of variables (SOV) method pioneered by Sklyanin [58, 62]. The first analysis of the spin chain in the classical limit from this point of view was performed in [60, 61]. This alternative approach allows one to obtain (mainly by construction) the complete set of eigenvalues and eigenvectors of quantum integrable systems. In particular, it was recently developed [24]–[26], [28, 29], [46]–[53] for a large variety of quantum models not solvable by algebraic Bethe ansatz. Important results were also previously obtained for the non-compact cases, see in particular [8], [17]–[19]. Moreover it has been shown first in [24] that once the SOV spectrum characterization is achieved manageable and rather universal determinant formulas can be derived for the scalar products of separate states and for the matrix elements of local operators between transfer matrix eigenstates. In particular, this SOV method was first developed in [48] for the spin-1/2 representations of the six-vertex reflection algebra with quite general non-diagonal boundaries and then generalized to the most general boundaries in [28]. It gives the complete spectrum (eigenvalues and eigenstates) and already allows one to compute matrix elements of some local operators within this most general boundary framework. However, it is important to underline that this SOV characterization of the spectrum is somehow unusual in comparison with more traditional descriptions like those obtained from Bethe ansatz techniques. More precisely, the spectrum is described not in terms of the set of solutions to a standard system of Bethe

\[ \text{See also the papers [11]–[13] for the application of the same method to different models.} \]
ansatz equations but is given in terms of sets of solutions to a characteristic system of \(N\) quadratic equations in \(N\) unknowns, \(N\) being the number of sites of the chain. Although the SOV method permits one to characterize completely the spectrum without introducing any ansatz, and this is its clear advantage, we have to stress that the classification of the sets of solutions of the SOV system of quadratic equations represents a new problem in quantum integrability which requires a deeper and systematic analysis.

The aim of this paper is to show that the SOV analysis of the transfer matrix spectrum for generic inhomogeneous spin-1/2 chains with most general boundary matrices is strictly equivalent to a system of generalized Bethe ansatz equations. This ensures that this system of Bethe equations characterizes automatically the entire spectrum of the transfer matrix. More precisely, we prove that the SOV characterization is equivalent to a second order finite difference functional equation of Baxter type,

\[
\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta) + F(\lambda),
\]

which contains an inhomogeneous term \(F(\lambda)\) independent of the \(\tau\) and \(Q\)-functions and entirely fixed by the boundary parameters. One central requirement in our construction of this functional characterization is the polynomial form of the \(Q\)-function. Indeed, it is this requirement that allows us to show that a finite system of generalized Bethe equations can be used to describe the complete transfer matrix spectrum. Finally, we would like to point out that similar results on the reformulation of the SOV spectrum characterization in terms of functional \(T\)–\(Q\) equations with \(Q\)-function solutions in a well defined model dependent set of polynomials were previously derived \([26, 46, 47]\) for the cases of transfer matrices associated with cyclic representations of the Yang–Baxter algebra. The analysis presented here is also interesting as it introduces the main tools to generalize this type of reformulation to other classes of integrable quantum models.

This paper is organized as follows. In section 2 we introduce the main notations and we recall the main results of previous papers on SOV necessary for our purposes. Section 3 contains the main results of the paper with the reformulation of the SOV characterization of the transfer matrix spectrum in terms of the inhomogeneous Baxter functional equation and the associated finite system of generalized Bethe ansatz equations. In section 4 we define the boundary conditions for which the inhomogeneous term in the Baxter equation identically vanishes, in this way deriving the completeness of standard Bethe ansatz equations. Moreover, we classify the remaining boundary conditions compatible with homogeneous Baxter equations. Section 5 contains the description of a set of discrete symmetries of the transfer matrices. These symmetries are used to find equivalent functional equation characterizations of the spectrum generalizing the results described in sections 3 and 4. In section 6 we present the SOV characterization of the spectrum for the rational case and show the completeness of the description in terms of the corresponding inhomogeneous Baxter equation. Finally, in section 7, we present a comparison with the known numerical results for both the XXZ and XXX chains. The evidenced compatibility suggests that even in the homogeneous limit our spectrum description is still complete.
2. Separation of variables for spin-1/2 representations of the reflection algebra

2.1. Spin-1/2 representations of the reflection algebra and open XXZ quantum chain

The representation theory of the reflection algebra can be studied in terms of the solutions $\mathcal{U}(\lambda)$ (monodromy matrices) of the following reflection equation:

$$R_{12}(\lambda - \mu)\mathcal{U}_1(\lambda)R_{21}(\lambda + \mu - \eta)\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)R_{12}(\lambda + \mu - \eta)\mathcal{U}_1(\lambda)R_{21}(\lambda - \mu).$$

(2.1)

Here we consider the reflection equation associated with the six-vertex trigonometric $R$ matrix

$$R_{12}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix} \in \text{End}(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

(2.2)

where $\mathcal{H}_a \simeq \mathbb{C}^2$ is a two-dimensional linear space. The six-vertex trigonometric $R$-matrix is a solution of the Yang–Baxter equation, $H_{12}R_{123}(\lambda) = R_{123}(\lambda)H_{12}$. The most general scalar solution $(2 \times 2)$ of the reflection equation reads

$$K(\lambda; \zeta, \kappa, \tau) = \frac{1}{\sinh \zeta} \begin{pmatrix} \sinh(\lambda - \eta/2 + \zeta) & \kappa e^\tau \sinh(2\lambda - \eta) \\ \kappa e^{-\tau} \sinh(2\lambda - \eta) & \sinh(\zeta - \lambda + \eta/2) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \simeq \mathbb{C}^2),$$

(2.3)

where $\zeta, \kappa$ and $\tau$ are arbitrary complex parameters. Using it and following [59] we can construct two classes of solutions to the reflection equation (2.1) in the $2^N$-dimensional representation space,

$$\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n.$$  

(2.4)

Indeed, starting from

$$K_-(\lambda) = K(\lambda; \zeta_-, \kappa_-, \tau_-), \quad K_+(-\lambda) = K(\lambda - \eta; \zeta_+, \kappa_+, \tau_+),$$

(2.5)

where $\zeta_\pm, \kappa_\pm, \tau_\pm$ are the boundary parameters, the following boundary monodromy matrices can be introduced:

$$\mathcal{U}_-(\lambda) = M_0(\lambda)K_-(\lambda)M_0(\lambda) = \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H}),$$

(2.6)

$$\mathcal{U}_+^{0}(\lambda) = M_0^0(\lambda)K_+^{0}(\lambda)M_0^0(\lambda) = \begin{pmatrix} A_+^{0}(\lambda) & C_+^{0}(\lambda) \\ B_+^{0}(\lambda) & D_+^{0}(\lambda) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H}).$$

(2.7)

These matrices $\mathcal{U}_-(\lambda)$ and $\mathcal{U}_+^{0}(\lambda) = \mathcal{U}_+^{0}(-\lambda)$ define two classes of solutions of the reflection equation (2.1). Here, we have used the notations,

$$M_0(\lambda) = R_{0N}(\lambda - \xi_N - \eta/2) \ldots R_{01}(\lambda - \xi_1 - \eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

(2.8)

where $\xi_n$ are the boundary parameters.
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\[
\hat{M}(\lambda) = (-1)^N \sigma_0^x M_0^\eta(-\lambda) \sigma_0^y,
\]

(2.10)

where \(M_0(\lambda) \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H})\) is the bulk inhomogeneous monodromy matrix (the \(\xi_j\) are the arbitrary inhomogeneity parameters) satisfying the Yang–Baxter relation,

\[
R_{12}(\lambda - \mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}(\lambda - \mu).
\]

(2.11)

The main interest of these boundary monodromy matrices is the property shown by Sklyanin [59] that the following family of transfer matrices,

\[
\mathcal{T}(\lambda) = \text{tr}_0\{K_+(\lambda) M(\lambda) K_-(\lambda)\hat{M}(\lambda)\} = \text{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}
\]

(2.12)

defines a one-parameter family of commuting operators in \(\text{End}(\mathcal{H})\). The Hamiltonian of the open XXZ quantum spin 1/2 chain with the most general integrable boundary terms can be obtained in the homogeneous limit (\(\xi_m = 0\) for \(m = 1, \ldots, N\)) from the following derivative of the transfer matrix (2.12):

\[
H = \left. \frac{2(\sinh \eta)^{1-2N}}{\text{tr}\{K_+(\eta/2)\} \text{tr}\{K_-(\eta/2)\}} \frac{d}{d\lambda} \mathcal{T}(\lambda) \right|_{\lambda = \eta/2} + \text{constant},
\]

(2.13)

and its explicit form reads

\[
H = \sum_{i=1}^{N-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh \eta \sigma_i^z \sigma_{i+1}^z)
\]

\[
+ \frac{\sinh \eta}{\sinh \zeta_-} [\sigma_1^z \cosh \zeta_- + 2\kappa_- (\sigma_1^x \cosh \tau_- + i \sigma_1^y \sinh \tau_-)]
\]

\[
+ \frac{\sinh \eta}{\sinh \zeta_+} [\sigma_N^z \cosh \zeta_+ + 2\kappa_+ (\sigma_N^x \cosh \tau_+ + i \sigma_N^y \sinh \tau_+)].
\]

(2.14)

Here \(\sigma_i^a\) are local spin 1/2 operators (Pauli matrices), \(\Delta = \cosh \eta\) is the anisotropy parameter and the six complex boundary parameters \(\zeta_{\pm}, \kappa_{\pm}\) and \(\tau_{\pm}\) define the most general integrable magnetic interactions at the boundaries.

### 2.2. Some relevant properties

The following quadratic linear combination of the generators \(\mathcal{A}_-(\lambda), \mathcal{B}_-(\lambda), \mathcal{C}_-(\lambda)\) and \(\mathcal{D}_-(\lambda)\) of the reflection algebra,

\[
\frac{\det_q \mathcal{U}_-(\lambda)}{\sinh(2\lambda - 2\eta)} = \mathcal{A}_-(\epsilon \lambda + \eta/2)\mathcal{A}_-(\eta/2 - \epsilon \lambda) + \mathcal{B}_-(\epsilon \lambda + \eta/2)\mathcal{C}_-(\eta/2 - \epsilon \lambda)
\]

\[
= \mathcal{D}_-(\epsilon \lambda + \eta/2)\mathcal{D}_-(\eta/2 - \epsilon \lambda) + \mathcal{C}_-(\epsilon \lambda + \eta/2)\mathcal{B}_-(\eta/2 - \epsilon \lambda),
\]

(2.15)

(2.16)

where \(\epsilon = \pm 1\), is the quantum determinant. It was shown by Sklyanin that it is a central element of the reflection algebra

\[
[\det_q \mathcal{U}_-(\lambda), \mathcal{U}_-(\mu)] = 0.
\]

(2.17)

doi:10.1088/1742-5468/2014/05/P05015
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The quantum determinant plays a fundamental role in the characterization of the transfer matrix spectrum and it admits the following explicit expressions:

\[
\det_q U_-(\lambda) = \det_q K_-(\lambda) \det M_0(\lambda) \det M_0(-\lambda) = \sinh(2\lambda - 2\eta)A_-(\lambda + \eta/2)A_-(-\lambda + \eta/2),
\]

where

\[
\det_q M(\lambda) = a(\lambda + \eta/2)d(\lambda - \eta/2)
\]

is the bulk quantum determinant and

\[
\det_q K_\pm(\lambda) = \mp \sinh(2\lambda \pm 2\eta)g_\pm(\lambda + \eta/2)g_\pm(-\lambda + \eta/2).
\]

Here, we used the following notations:

\[
A_-(\lambda) = g_-(\lambda)a(\lambda)d(-\lambda), \quad d(\lambda) = a(\lambda - \eta), \quad a(\lambda) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n + \eta/2),
\]

and

\[
g_\pm(\lambda) = \frac{\sinh(\lambda + \alpha_\pm - \eta/2) \cosh(\lambda \mp \beta_\pm - \eta/2)}{\sinh \alpha_\pm \cosh \beta_\pm},
\]

where \(\alpha_\pm\) and \(\beta_\pm\) are defined in terms of the boundary parameters by

\[
\sinh \alpha_\pm \cosh \beta_\pm = \frac{\sinh \zeta_\pm}{2\kappa_\pm}, \quad \cosh \alpha_\pm \sinh \beta_\pm = \frac{\cosh \zeta_\pm}{2\kappa_\pm}.
\]

**Proposition 2.1** (Proposition 2.3 of [48]). The transfer matrix \(T(\lambda)\) is an even function of the spectral parameter \(\lambda\),

\[
T(-\lambda) = T(\lambda),
\]

and it is central for the following special values of the spectral parameter:

\[
\lim_{\lambda \to \pm \infty} e^{2\lambda(N+1)} \mathcal{T}(\lambda) = 2^{-(2N+1)\kappa_+\kappa_-} \cosh(\tau_+ - \tau_-) \sinh \zeta_+ \sinh \zeta_-, \quad \mathcal{T}(\pm \eta/2) = (-1)^N 2 \cosh \eta \det q M(0), \quad \mathcal{T}(\pm(\eta/2 - i\pi/2)) = -2 \cosh \eta \coth \zeta_- \coth \zeta_+ \det q M(i\pi/2).
\]

Moreover, the monodromy matrix \(U_\pm(\lambda)\) satisfies the following transformation properties under Hermitian conjugation.

- **Under the condition \(\eta \in i\mathbb{R}\) (massless regime), it holds,**

\[
U_\pm(\lambda)^\dagger = [U_\pm(-\lambda^*)]^\dagger, \quad \text{for } \{i\tau_\pm, i\kappa_\pm, i\zeta_\pm, \xi_1, \ldots, \xi_N\} \in \mathbb{R}^{N+3}.
\]
Under the condition $\eta \in \mathbb{R}$ (massive regime), it holds,

$$U_{\pm}(\lambda)^\dagger = [U_{\pm}(\lambda^*)]^{t_0},$$  \hspace{1cm} (2.30)

for $\{\tau_{\pm}, \kappa_{\pm}, \zeta_{\pm}, i\xi_1, \ldots, i\xi_N\} \in \mathbb{R}^{N+3}$.

So under the same conditions on the parameters of the representation it holds,

$$T(\lambda)^\dagger = T(\lambda^*),$$  \hspace{1cm} (2.31)

i.e. $T(\lambda)$ defines a one-parameter family of normal operators which are self-adjoint both for $\lambda$ real and purely imaginary.

2.3. SOV representations for the $T(\lambda)$-spectral problem

Let us recall here the characterization obtained in [28, 48] by the SOV method of the spectrum of the transfer matrix $T(\lambda)$. First we introduce the following notations:

$$X_{k,m}^{(i,r)}(\tau_{\pm}, \alpha_{\pm}, \beta_{\pm}) \equiv (-1)^i (1 - r) \eta + \tau_- - \tau_+ + (-1)^k (\alpha_- + \beta_-) - (-1)^m (\alpha_+ - \beta_+) + i\pi (k + m),$$  \hspace{1cm} (2.32)

and by using these linear combinations of the boundary parameters we introduce the set $N_{SOV} \subset \mathbb{C}^6$ of boundary parameters for which the separation of variables cannot be applied directly. More precisely

$$(\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in N_{SOV},$$

if $\exists(k, h, m, n) \in \{0, 1\}$ such that

$$X_{k,m}^{(0,N)}(\tau_{\pm}, \alpha_{\pm}, \beta_{\pm}) = 0 \quad \text{and} \quad X_{h,n}^{(1,N)}(\tau_{\pm}, \alpha_{\pm}, \beta_{\pm}) = 0.$$  \hspace{1cm} (2.33)

All the results in the following will be obtained for the generic values of the boundary parameters, not belonging to this set. The SOV method applicability can be further extended applying the discrete symmetries discussed in section 5.

Following [28] we define the functions as follows:

$$g_a(\lambda) = \frac{\cosh^2 2\lambda - \cosh^2 \eta}{\cosh^2 2\zeta_a^{(0)} - \cosh^2 \eta} \prod_{b \neq a}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(0)}} \quad \text{for } a \in \{1, \ldots, N\},$$  \hspace{1cm} (2.34)

$$A(\lambda) = (-1)^N \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} \frac{g_+(\lambda)g_-(\lambda)\alpha(\lambda)d(-\lambda)}{A(\eta/2)},$$  \hspace{1cm} (2.35)

and

$$f(\lambda) = \frac{\cosh 2\lambda + \cosh \eta}{2 \cosh \eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh \eta - \cosh 2\zeta_b^{(0)}} A(\eta/2) - (-1)^N \frac{\cosh 2\lambda - \cosh \eta}{2 \cosh \eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh \eta + \cosh 2\zeta_b^{(0)}} A(\eta/2 + i\pi/2).$$
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\[ + 2^{(1-N)} \frac{\kappa_+ \kappa_- \cosh(\tau_+ - \tau_-)}{\sinh \zeta_+ \sinh \zeta_-} \left( \cosh^2 2\lambda - \cosh^2 \eta \right) \]

\[ \times \prod_{b=1}^{N} \left( \cosh 2\lambda - \cosh 2\zeta_b^{(0)} \right), \quad (2.36) \]

where

\[ \zeta_n^{(h_n)} = \xi_n + (h_n - \frac{1}{2}) \eta \quad \forall n \in \{1, \ldots, N\}, \quad h_n \in \{0, 1\}. \quad (2.37) \]

We can now recall the main result on the characterization of the set \( \Sigma_T \) formed by all the eigenvalue functions of the transfer matrix \( T(\lambda) \).

**Theorem 2.1** (Theorem 5.3 and corollary 5.1 of [28]). Let \( (\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus \text{NSOV} \) and let the inhomogeneities \( \{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N \) be generic,

\[ \xi_a \neq \pm \xi_b + r \eta \text{ mod } 2\pi \quad \forall a \neq b \in \{1, \ldots, N\} \quad \text{and} \quad r \in \{-1, 0, 1\}, \quad (2.38) \]

then \( T(\lambda) \) has a simple spectrum and the set of its eigenvalues \( \Sigma_T \) is characterized by

\[ \Sigma_T = \left\{ \tau(\lambda) : \tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda) x_a, \forall \{x_1, \ldots, x_N\} \in \Sigma_T \right\}, \quad (2.39) \]

where \( \Sigma_T \) is the set of solutions to the following inhomogeneous system of \( N \) quadratic equations:

\[ x_n \sum_{a=1}^{N} g_a(\xi_n^{(1)}) x_a + x_n f(\xi_n^{(1)}) = q_n, \]

\[ q_n = \frac{\det_q K_+(\xi_n) \det_q U_-(\xi_n)}{\sinh(\eta + 2\xi_n) \sinh(\eta - 2\xi_n)}, \quad \forall n \in \{1, \ldots, N\}, \quad (2.40) \]

in \( N \) unknowns \( \{x_1, \ldots, x_N\} \).

**3. The inhomogeneous Baxter equation**

Here we show that the SOV characterization of the spectrum admits an equivalent formulation in terms of a second order functional difference equation of Baxter type,

\[ \tau(\lambda) Q(\lambda) = A(\lambda) Q(\lambda - \eta) + A(-\lambda) Q(\lambda + \eta) + F(\lambda), \quad (3.1) \]

which contains a non-zero inhomogeneous term \( F(\lambda) \) non-zero for generic integrable boundary conditions, and the \( Q \)-functions are trigonometric polynomials. In this paper we will call \( f(\lambda) \) a trigonometric polynomial of degree \( M \) if \( e^{M\lambda} f(\lambda) \) is a polynomial of \( e^{2\lambda} \) of degree \( M \). Most trigonometric polynomials we will consider in the following sections will be even functions of \( \lambda \) and will satisfy an additional condition \( f(\lambda + i\pi) = f(\lambda) \). It is easy to see in this situation that such functions can be written as polynomials of \( \cosh 2\lambda \).
3.1. The main functions in the functional equation

Let \( Q(\lambda) \) be an even trigonometric polynomial of degree \( 2N \). It can be written in the following form:

\[
Q(\lambda) = \sum_{a=1}^{N} \prod_{b \neq a}^{N} \left( \cosh 2\lambda - \cosh 2\zeta_a^{(0)} \right) + 2^N \prod_{a=1}^{N} \left( \cosh 2\lambda - \cosh 2\zeta_a^{(0)} \right) + 2^N \prod_{a=1}^{N} \left( \cosh 2\lambda - \cosh 2\lambda_a \right),
\]

where from now on the \( Q(\zeta_a^{(0)}) \) are arbitrary complex numbers or similarly the \( \lambda_a \) are arbitrary complex numbers. Then, introducing the function

\[
Z_Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta)
\]

we can prove the following lemma.

**Lemma 3.1.** Let \( Q(\lambda) \) be any function of the form (3.3) then the associated function \( Z_Q(\lambda) \) is an even trigonometric polynomial of degree \( 4N + 4 \) of the following form:

\[
Z_Q(\lambda) = \sum_{a=0}^{2(N+1)} z_a \cosh^a 2\lambda,
\]

with

\[
z_{2(N+1)} = \frac{2\kappa_+\kappa_- \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- - (N+1)\eta) \sinh \zeta_+ \sinh \zeta_-}{\sinh \zeta_+ \sinh \zeta_-}.
\]

**Proof.** The fact that the function \( Z_Q(\lambda) \) is even in \( \lambda \) is a trivial consequence of the fact that \( Q(\lambda) \) is even; in fact, it holds,

\[
Z_Q(-\lambda) = A(-\lambda)Q(-\lambda - \eta) + A(\lambda)Q(-\lambda + \eta)
\]

\[
= A(-\lambda)Q(\lambda + \eta) + A(\lambda)Q(\lambda - \eta) = Z_Q(\lambda).
\]

The fact that \( Z_Q(\lambda) \) is indeed a trigonometric polynomial follows from its definition once we observe that \( \lambda = 0 \) is not a singular point and the following identity holds:

\[
\lim_{\lambda \to 0} Z_Q(\lambda) = 2g_+(0)g_-(-\eta)Q(0) \cosh \eta.
\]

Now the functional form of \( Z_Q(\lambda) \) is a consequence of the following identities:

\[
Z_Q(\lambda + i\pi) = Z_Q(\lambda),
\]

\[
\lim_{\lambda \to \pm\infty} \frac{Z_Q(\lambda)}{\cosh^{4N+1} \lambda} = \frac{\kappa_+\kappa_- \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- - (N+1)\eta) \sinh \zeta_+ \sinh \zeta_-}{2^{2(N+1)} \sinh \zeta_+ \sinh \zeta_-},
\]

where the second identity follows from

\[
\lim_{\lambda \to \pm\infty} e^{\mp(2N+1)\lambda} A(\lambda) = 2^{-2(N+1)} \frac{\kappa_+\kappa_- \exp \pm(\alpha_+ + \alpha_- - \beta_+ + \beta_- + (N-1)\eta) \sinh \zeta_+ \sinh \zeta_-}{\sinh \zeta_+ \sinh \zeta_-},
\]

\[
\lim_{\lambda \to \pm\infty} e^{\mp2N\lambda} Q(\lambda) = 1.
\]

\[\square\]
3.2. On the need of an inhomogeneous term in the functional equation

Here, we would like to point out that it is simple to define the boundary conditions for which one can prove that the homogeneous version of the Baxter equation (3.1) does not admit trigonometric polynomial solutions for $\tau(\lambda) \in \Sigma_T$.

Lemma 3.2. Assume that the boundary parameters satisfy the following conditions:

$$\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad Y^{(i,r)}(\tau_\pm, \alpha_\pm, \beta_\pm) \neq 0 \quad \forall i \in \{0, 1\}, \quad r \in \mathbb{Z}$$  \hspace{1cm} (3.11)

where we have defined

$$Y^{(i,r)}(\tau_\pm, \alpha_\pm, \beta_\pm) \equiv \tau_- - \tau_+ + (-1)^i [(\mathbb{N} - 1 - r) \eta + (\alpha_- + \alpha_+ + \beta_- - \beta_+)],$$  \hspace{1cm} (3.12)

then for any $\tau(\lambda) \in \Sigma_T$ the homogeneous Baxter equation,

$$\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta),$$  \hspace{1cm} (3.13)

does not admit any (non identically zero) $Q(\lambda)$ of Laurent polynomial form in $e^{\lambda}$.

Proof. If we consider the following function,

$$Q(\lambda) = \sum_{a=-s}^{r} y_a e^{a\lambda}, \quad \text{with } r, s \in \mathbb{N},$$  \hspace{1cm} (3.14)

we can clearly always choose the coefficients $y_a$ such that the rhs of the homogeneous Baxter equation has no poles as required. However, it is enough to consider now the asymptotics

$$\lim_{\lambda \to +\infty} e^{(2\mathbb{N}+4+r)\lambda} \frac{[A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta)]}{2^{2(\mathbb{N}+1)} \sinh \zeta_+ \sinh \zeta_-} = \frac{y_r \kappa_+ \kappa_- \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- + (\mathbb{N} - 1 - r)\eta)}{2^{2(\mathbb{N}+1)} \sinh \zeta_+ \sinh \zeta_-}$$  \hspace{1cm} (3.15)

$$\lim_{\lambda \to +\infty} e^{-(2\mathbb{N}+4+r)\lambda} \tau(\lambda)Q(\lambda) = \frac{y_r \kappa_+ \kappa_- \cosh(\tau_+ - \tau_-)}{2^{2(\mathbb{N}+1)} \sinh \zeta_+ \sinh \zeta_-}$$  \hspace{1cm} (3.16)

and use the conditions (3.11) to observe that for any $r \in \mathbb{Z}$ the asymptotic of the homogeneous Baxter equation cannot be satisfied, which implies the validity of the lemma. \hfill \Box

3.3. The SOV spectrum in terms of the inhomogeneous Baxter equation

We introduce now the following function of the boundary parameters:

$$F_0 = \frac{2\kappa_+ \kappa_- (\cosh(\tau_+ - \tau_-) - \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- + (\mathbb{N} + 1)\eta))}{\sinh \zeta_+ \sinh \zeta_-},$$  \hspace{1cm} (3.17)
and then the function,

\[
F(\lambda) = 2^N F_0 (\cosh^2 2\lambda - \cosh^2 \eta) a(\lambda) a(-\lambda) d(\lambda) d(-\lambda)
\]

(3.18)

\[
= F_0 (\cosh^2 2\lambda - \cosh^2 \eta) \prod_{b=1}^{N} \prod_{i=0}^{1} (\cosh 2\lambda - \cosh 2\zeta^{(i)}_b).
\]

(3.19)

We introduce also the set of functions \( \Sigma_Q \) such that \( Q(\lambda) \in \Sigma_Q \) if it has a form (3.3) and

\[
\tau(\lambda) = \frac{Z_Q(\lambda) + F(\lambda)}{Q(\lambda)}
\]

is a trigonometric polynomial. We are now ready to prove the main theorem of this paper.

**Theorem 3.1.** Let the inhomogeneities \( \{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N \) be generic (2.38) and let the boundary parameters \( (\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus N_{SOV} \) satisfy the following conditions:

\[
\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad Y^{(i,2r)}(\tau_+, \alpha_+, \beta_+) \neq 0 \quad \forall i \in \{0,1\},
\]

(3.20)

then \( \tau(\lambda) \in \Sigma_T \) if and only if \( \exists! Q(\lambda) \in \Sigma_Q \) such that

\[
\tau(\lambda)Q(\lambda) = Z_Q(\lambda) + F(\lambda).
\]

(3.21)

**Proof.** First we prove that if \( \tau(\lambda) \in \Sigma_T \) then there is a trigonometric polynomial \( Q(\lambda) \in \Sigma_Q \) satisfying the inhomogeneous functional Baxter equation,

\[
\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta) + F(\lambda).
\]

(3.22)

To prove it we will show that there is the unique set of values \( Q(c^{(1)}_b) \) such that \( Q(\lambda) \) of the form (3.2) satisfies this equation.

It is straightforward to verify that if \( \tau(\lambda) \in \Sigma_T \) and \( Q(\lambda) \) has the form (3.3) then the left- and right-hand sides of the above equation are both even trigonometric polynomials of \( \lambda \) and both can be written (using the asymptotic behavior) in the form

\[
\frac{2\kappa_+ \kappa_- \cosh(\tau_+ - \tau_-) \prod_{b=1}^{2N+2} (\cosh 2\lambda - \cosh 2\eta^{\text{lhs}}/\text{rhs}_{b})}{\sinh \zeta_+ \sinh \zeta_-}.
\]

(3.23)

Then to prove that we can introduce a \( Q(\lambda) \) of the form (3.3) which satisfies the inhomogeneous Baxter equation (3.1) with \( \tau(\lambda) \in \Sigma_T \), we have only to prove that (3.1) is satisfied in \( 4N + 4 \) different values of \( \lambda \). As the \( \text{rhs} \) and \( \text{lhs} \) of (3.1) are even functions we need to check this identity only for \( 2N + 2 \) non-zero points \( \mu_j \) such that \( \mu_j \neq \pm \mu_k \). It is a simple exercise to verify that equation (3.1) is satisfied automatically for any \( Q(\lambda) \) of the form (3.3) in the following two points, \( \eta/2 \) and \( \eta/2 + i\pi/2 \):

\[
\tau(\eta/2)Q(\eta/2) = A(\eta/2)Q(\eta/2 - \eta) = A(\eta/2)Q(\eta/2),
\]

(3.24)
and

\[
\tau(\eta/2 + i\pi/2)Q(\eta/2 + i\pi/2) = A(\eta/2 + i\pi/2)Q(i\pi/2 - \eta/2)
= A(\eta/2 + i\pi/2)Q(\eta/2 + i\pi/2).
\] (3.25)

Indeed, these equations reduce to

\[
\tau(\eta/2) = A(\eta/2), \quad \tau(\eta/2 + i\pi/2) = A(\eta/2 + i\pi/2)
\] (3.26)

and so they are satisfied by definition for any \(\tau(\lambda) \in \Sigma_T\). Then we check the explicit form of equation (3.1) in the \(2N\) points \(\zeta^{(0)}_b\) and \(\zeta^{(1)}_b\),

\[
\tau(\zeta^{(0)}_b)Q(\zeta^{(0)}_b) = A(-\zeta^{(0)}_b)Q(\zeta^{(0)}_b + \eta) = A(-\zeta^{(0)}_b)Q(\zeta^{(1)}_b),
\] (3.27)

and

\[
\tau(\zeta^{(1)}_b)Q(\zeta^{(1)}_b) = A(\zeta^{(1)}_b)Q(\zeta^{(1)}_b - \eta) = A(\zeta^{(1)}_b)Q(\zeta^{(0)}_b).
\] (3.28)

They are equivalent to the following system of equations:

\[
\frac{A(\zeta^{(1)}_b)}{A(-\zeta^{(0)}_b)} = \frac{\tau(\zeta^{(0)}_b)}{\tau(-\zeta^{(0)}_b)} \quad \forall b \in \{1, \ldots, N\}
\] (3.29)

\[
\frac{Q(\zeta^{(0)}_b)\tau(\zeta^{(0)}_b)}{A(-\zeta^{(0)}_b)} = \sum_{a=1}^{N} \prod_{c=1}^{N} \frac{\cosh 2\zeta^{(1)}_b - \cosh 2\zeta^{(0)}_c}{\cosh 2\zeta^{(0)}_a - \cosh 2\zeta^{(1)}_c} Q(\zeta^{(0)}_a) + 2N \prod_{a=1}^{N} (\cosh 2\zeta^{(1)}_b - \cosh 2\zeta^{(0)}_a).
\] (3.30)

Now, using the following quantum determinant identity,

\[
\frac{\det_q K_+(\lambda - \eta/2) \det_q U_+(\lambda - \eta/2)}{\sinh(2\lambda + \eta) \sinh(2\lambda - \eta)} = A(\lambda)A(-\lambda + \eta),
\] (3.31)

it is easy to see that the system of equations (3.29) is certainly satisfied as \(\tau(\lambda) \in \Sigma_T\), once we recall the SOV characterization (2.39) of \(\Sigma_T\). Indeed there is a set \(\{x_1, \ldots, x_n\}\) satisfying the equations (2.40) and \(\tau(\zeta^{(0)}_b) = x_b\).

So we are left with (3.30) a linear system of \(N\) inhomogeneous equations with \(N\) unknowns \(Q(\zeta^{(0)}_a)\). Here, we prove that the matrix of this linear system

\[
c_{ab} \equiv \prod_{c=1}^{N} \frac{\cosh 2\zeta^{(1)}_b - \cosh 2\zeta^{(0)}_c}{\cosh 2\zeta^{(0)}_a - \cosh 2\zeta^{(1)}_c} - \delta_{ab} \frac{\tau(\zeta^{(0)}_b)}{A(-\zeta^{(0)}_b)} \quad \forall a, b \in \{1, \ldots, N\}
\] (3.32)

has non-zero determinant for the given \(\tau(\lambda) \in \Sigma_T\). Indeed, let us suppose that for some \(\tau(\lambda) \in \Sigma_T\),

\[
\det \left[ c_{ab} \right] = 0.
\] (3.33)
Then there is at least one nontrivial solution \( \{Q(\zeta_1^{(0)}), \ldots, Q(\zeta_N^{(0)})\} \neq \{0, \ldots, 0\} \) to the homogeneous system of equations,

\[
\frac{Q(\zeta_b^{(0)})\tau(\zeta_b^{(0)})}{A(-\zeta_b^{(0)})} = \prod_{a=1}^{N} \prod_{b \neq a}^{N} \frac{\cosh 2\zeta_b^{(1)} - \cosh 2\zeta_a^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_c^{(0)}} Q(\zeta_a^{(0)}),
\]

and hence we can define

\[
Q_M(\lambda) = \sum_{a=1}^{N} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(0)}} Q(\zeta_a^{(0)}) = \lambda^M \prod_{b=1}^{N} (\cosh 2\lambda - \cosh 2\lambda_b^{(M)}).
\]

It is an even trigonometric polynomial of degree \( 2M \) such that \( 0 \leq M \leq N - 1 \) fixed by the solution \( \{Q(\zeta_1^{(0)}), \ldots, Q(\zeta_N^{(0)})\} \). Now using the \( Q_M(\lambda) \) and \( \tau(\lambda) \in \Sigma_T \) we can define two functions,

\[
W_1(\lambda) = Q_M(\lambda)\tau(\lambda) \quad \text{and} \quad W_2(\lambda) = A(\lambda)Q_M(\lambda - \eta) + A(-\lambda)Q_M(\lambda + \eta),
\]

which are both even trigonometric polynomials of degree \( 2M + 2N + 4 \). Then it is straightforward to observe that the systems of equations (3.29) and (3.34) plus the conditions (3.24) and (3.25), which are also satisfied with the function \( Q_M(\lambda) \), imply that \( W_1(\lambda) \) and \( W_2(\lambda) \) coincide in \( 4N + 4 \) different values of \( \lambda \) \( \pm \eta/2, \pm (\eta/2 + i\pi/2), \pm \xi_0^{(0)} \) and \( \pm \xi_1^{(0)} \). This means that \( W_1(\lambda) \equiv W_2(\lambda) \), as these are two polynomials of maximal degree \( 4N + 2 \). So, we have shown that from the assumption \( \exists \tau(\lambda) \in \Sigma_T \) such that (3.33) holds it follows that \( \tau(\lambda) \) and \( Q_M(\lambda) \) have to satisfy the following homogeneous Baxter equation:

\[
\tau(\lambda)Q_M(\lambda) = A(\lambda)Q_M(\lambda - \eta) + A(-\lambda)Q_M(\lambda + \eta).
\]

Now we can apply the lemma 3.2 which implies that \( Q_M(\lambda) = 0 \) for any \( \lambda \), which contradicts the hypothesis of the existence of a nontrivial solution to the homogeneous system (3.34). Hence, we have proven that \( \det_{N}[c_{ab}] \neq 0 \). Therefore there is a unique solution \( \{Q(\zeta_1^{(0)}), \ldots, Q(\zeta_N^{(0)})\} \) of the inhomogeneous system (3.30) which defines one and only one \( Q(\lambda) \) of the form (3.2) satisfying the functional inhomogeneous Baxter equation (3.1).

We prove now that if \( Q(\lambda) \in \Sigma_Q \) then \( \tau(\lambda) = (ZQ(\lambda) + F(\lambda))/Q(\lambda) \in \Sigma_T \). By definition of the functions \( ZQ(\lambda), F(\lambda) \) and \( Q(\lambda) \) the function \( \tau(\lambda) \) has the desired form,

\[
\tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda)\tau(\zeta_a^{(0)}).
\]

To prove now that \( \tau(\lambda) \in \Sigma_T \) we have to write the inhomogeneous Baxter equation (3.1) in the \( 2N \) points \( \zeta_b^{(0)} \) and \( \zeta_b^{(1)} \). Indeed, we have already proved that this reproduces the systems (3.29) and (3.30) and it is simple to observe that the system of equations (3.29) just coincides with the inhomogeneous system of \( N \) quadratic equations,

\[
x_n \sum_{a=1}^{N} g_a(\zeta_n^{(1)})x_a + x_nf(\zeta_n^{(1)}) = q_n, \quad \forall n \in \{1, \ldots, N\},
\]
once we define \( x_a = \tau(\zeta_a^{(0)}) \) for any \( a \in \{1, \ldots, N\} \) and we write \( \tau(\zeta_n^{(1)}) \) in terms of the \( x_a \). Thus we show that

\[
\tau(\lambda) = \left( Z_Q(\lambda) + F(\lambda) \right) / Q(\lambda) \in \Sigma_T,
\]

completing the proof of the theorem.

We would like to mention that our characterization of the spectrum is compatible with the \( T\!-\!Q \) functional equations proposed by ansatz in [10]. More precisely, the ansatz proposed in [10] generally requires the use of two polynomials \( Q \), solutions of a coupled system of \( T\!-\!Q \) functional equations. However, this system reduces to a single inhomogeneous \( T\!-\!Q \) functional equation of the type (3.22) once one of the polynomial \( Q \)-functions is reduced to a constant. The possibility that this last equation alone may give the complete spectrum characterization was mentioned in the Conclusion of [10]. The ansatz in [10] has been introduced to satisfy the set of identities (3.29) between products of transfer matrix eigenvalues and the quantum determinant at specific points fixed by the inhomogeneities of the models. The approach based on the use of this type of identities to characterize the spectrum of integrable quantum models was for the first time introduced in [52], where these identities were derived at the operator level for the transfer matrices associated with both the dynamical six-vertex and eight-vertex Yang–Baxter algebras. In the case of the open chains these identities were first derived in [48] and subsequently generalized to the most general boundaries in [10] and in [28] by two different approaches. In the SOV framework developed in [28, 48] these identities were proven to characterize the full transfer matrix spectrum once solved in a model dependent class of functions. Indeed they lead to the spectrum characterization presented in theorem 2.1 that we have used here to prove the theorem 3.1.

### 3.4. Completeness of the Bethe ansatz equations

In section 3.3 we have shown that to solve the transfer matrix spectral problem associated with the most general representations of the trigonometric six-vertex reflection algebra we have just to classify the set of functions \( Q(\lambda) \) of the form (3.3) for which \( (Z_Q(\lambda) + F(\lambda))/Q(\lambda) \) is a trigonometric polynomial; i.e. the set of functions \( \Sigma_Q \) completely fixes the set \( \Sigma_T \). We can show now that the previous characterization of the transfer matrix spectrum allows us to prove that \( \Sigma_{InBAE} \subset \mathbb{C}^N \), the set of all the solutions of inhomogeneous Bethe equations

\[
\{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{InBAE},
\]

if

\[
A(\lambda_a)Q(\lambda_a - \eta) + A(-\lambda_a)Q(\lambda_a + \eta) = -F(\lambda_a), \quad \forall a \in \{1, \ldots, N\},
\]

(3.40)
defines the complete set of transfer matrix eigenvalues. In particular, the following corollary applies.

doi:10.1088/1742-5468/2014/05/P05015
Corollary 3.1. Let the inhomogeneities \( \{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N \) be generic (2.38) and let the boundary parameters \((\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus \text{SOV}\) satisfy (3.20) then \(\tau(\lambda) \in \Sigma_T\) if and only if \(\exists! \{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{InBAE}}\) such that
\[
\tau(\lambda) = \frac{Z_Q(\lambda) + F(\lambda)}{Q(\lambda)} \quad \text{with } Q(\lambda) = 2^N \prod_{a=1}^N (\cosh 2\lambda - \cosh 2\lambda_a). \tag{3.41}
\]
Moreover, under the condition of normality defined in proposition 2.1, the set \(\Sigma_{\text{InBAE}}\) of all the solutions to the inhomogeneous system of Bethe equations (3.40) contains \(2^N\) elements.

4. The homogeneous Baxter equation

4.1. Boundary conditions annihilating the inhomogeneity of the Baxter equation

The description presented in the previous sections can be applied to completely general integrable boundary terms including as a particular case the boundary conditions for which the inhomogeneous term in the functional Baxter equation vanishes. As these are still quite general boundary conditions it is interesting to point out how the previous general results explicitly look in these cases.

Theorem 4.1. Let \((\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus \text{SOV}\) satisfy the condition
\[
\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad \exists i \in \{0, 1\}: Y^{(i,2N)}(\tau_+, \alpha_+, \beta_+) = 0 \tag{4.1}
\]
and let the inhomogeneities \(\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N\) be generic (2.38), then \(\tau(\lambda) \in \Sigma_T\) if and only if \(\exists! Q(\lambda) \in \Sigma_Q\) such that
\[
\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta). \tag{4.2}
\]
Or equivalently, \(\tau(\lambda) \in \Sigma_T\) if and only if \(\exists! \{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{BAE}}\) such that
\[
\tau(\lambda) = \frac{A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta)}{Q(\lambda)} \quad \text{with } Q(\lambda) = 2^N \prod_{a=1}^N (\cosh 2\lambda - \cosh 2\lambda_a) \tag{4.3}
\]
where
\[
\Sigma_{\text{BAE}} = \{\{\lambda_1, \ldots, \lambda_N\} \in \mathbb{C}^N : A(\lambda_a)Q_\lambda(\lambda_a - \eta) + A(-\lambda_a)Q_\lambda(\lambda_a + \eta) = 0, \forall a \in \{1, \ldots, N\}\}. \tag{4.4}
\]
Moreover, under the condition of normality defined in proposition 2.1, the set \(\Sigma_{\text{BAE}}\) of the solutions to the homogeneous system of Bethe ansatz type equations (4.4) contains \(2^N\) elements.

Proof. This theorem is just a rewriting of the results presented in theorem 3.1 and corollary 3.1 for the case of a vanishing inhomogeneous term. Indeed if the conditions (4.4) are satisfied then automatically the conditions of the main theorem (3.20) are satisfied too, which implies that the map from the \(\tau(\lambda) \in \Sigma_T\) to the \(\{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{BAE}}\) is indeed an isomorphism. \(\square\)

doi:10.1088/1742-5468/2014/05/P05015
4.2. More general boundary conditions compatible with homogeneous Baxter equations

We address here the problem of describing the boundary conditions,

$$\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad \exists i \in \{0, 1\}, \quad M \in \{0, \ldots, N-1\} : Y^{(i,2M)}(\tau_\pm, \alpha_\pm, \beta_\pm) = 0,$$

(4.5)

for which the conditions (3.20) are not satisfied and then theorem 3.1 cannot be directly applied. In these $2N$ hyperplanes in the space of the boundary parameters we have just to modify this theorem to take into account that the Baxter equation associated with the choice of coefficient $A(\lambda)$ is indeed compatible with the homogeneous Baxter equation for a special choice of the polynomial $Q(\lambda)$. First we define the following functions:

$$Q_M(\lambda) = 2M \prod_{b=1}^{M}(\cosh 2\lambda - \cosh 2\lambda_b^{(M)}).$$

(4.6)

We introduce also the set of polynomials $\Sigma_Q^M$ such that $Q_M(\lambda) \in \Sigma_Q^M$ if $Q_M(\lambda)$ has a form (4.6) and

$$\tau(\lambda) = \frac{A(\lambda)Q_M(\lambda - \eta) + A(-\lambda)Q_M(\lambda + \eta)}{Q_M(\lambda)}$$

is a trigonometric polynomial. Then we can define the corresponding set $\Sigma_T^M$

$$\Sigma_T^M = \left\{ \tau(\lambda) : \tau(\lambda) \equiv \frac{A(\lambda)Q_M(\lambda - \eta) + A(-\lambda)Q_M(\lambda + \eta)}{Q_M(\lambda)} \text{ if } Q_M(\lambda) \in \Sigma_Q^M \right\}.$$

(4.7)

It is simple to prove the validity of the following.

Lemma 4.1. Let the boundary conditions (4.5) be satisfied, then $\Sigma_T^M \subset \Sigma_T$ and moreover for any $\tau(\lambda) \in \Sigma_T^M$ there exists one and only one $Q_M(\lambda) \in \Sigma_Q^M$ such that

$$\tau(\lambda)Q_M(\lambda) = A(\lambda)Q_M(\lambda - \eta) + A(-\lambda)Q_M(\lambda + \eta),$$

(4.8)

and for any $\tau(\lambda) \in \Sigma_T \setminus \Sigma_T^M$ there exists one and only one $Q(\lambda) \in \Sigma_Q$ such that

$$\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta) + F(\lambda).$$

(4.9)

Proof. The proof follows the one given for the main theorem 3.1; we have just to observe that thanks to the boundary conditions (4.5) the set $\Sigma_T^M$ is formed by transfer matrix eigenvalues as the Baxter equation implies that for any $\tau(\lambda) \in \Sigma_T^M$ the systems of equations (3.24), (3.25) and (3.29) are satisfied and moreover that the asymptotics of the $\tau(\lambda) \in \Sigma_T^M$ is exactly that of the transfer matrix eigenvalues.

Finally, it is interesting to remark that under the boundary conditions (3.20) the complete characterization of the spectrum of the transfer matrix is given in terms of the even polynomials $Q(\lambda)$ all of fixed degree $2N$ and form (3.3) which are solutions of the inhomogeneous/homogeneous Baxter equation. However, in the cases when the boundary parameters satisfy the constraints (4.5) for a given $M \in \{0, \ldots, N-1\}$ a part of the transfer matrix spectrum can be defined by polynomials of smaller degree; i.e. the $Q_M(\lambda) \in \Sigma_Q^M$ for the fixed $M \in \{0, \ldots, N-1\}$.  

doi:10.1088/1742-5468/2014/05/P05015
5. Discrete symmetries and equivalent Baxter equations

It is important to point out that we have some large amount of freedom in the choice of the functional reformulation of the SOV characterization of the transfer matrix spectrum. We have reduced it looking for trigonometric polynomial solutions $Q(\lambda)$ of the second order difference equations with coefficients $A(\lambda)$ which are rational trigonometric functions. It makes the finite difference terms $A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta)$ in the functional equation a trigonometric polynomial. Indeed, this assumption reduces the possibility of using the following gauge transformations of the coefficients allowed instead by the SOV characterization:

\[ A_\alpha(\lambda) = \alpha(\lambda)A(\lambda), \quad D_\alpha(\lambda) = \frac{A(-\lambda)}{\alpha(\lambda + \eta)}. \quad (5.1) \]

In the following we discuss simple transformations that do not modify the functional form of the coefficients allowing equivalent reformulations of the SOV spectrum by Baxter equations.

5.1. Discrete symmetries of the transfer matrix spectrum

It is not difficult to see that the spectrum (eigenvalues) of the transfer matrix presents the following invariance.

**Lemma 5.1.** We denote explicitly the dependence from the boundary parameters in the set of boundary parameters $\Sigma_T(\tau^+, \alpha^+ , \beta^+ , \tau^- , \alpha^- , \beta^-)$ of the eigenvalue functions of the transfer matrix $T(\lambda)$, then this set is invariant under the following $Z_2^3 \otimes 2^2$ transformations of the boundary parameters:

\[ \Sigma_T(\tau^+, \alpha^+ , \beta^+ , \tau^- , \alpha^- , \beta^-) \equiv \Sigma_T(\epsilon^\tau \epsilon^\alpha \epsilon^\beta, \epsilon^\tau \epsilon^\alpha \epsilon^\beta, \epsilon^\tau \epsilon^\alpha \epsilon^\beta) \]

\[ \forall (\epsilon^\tau, \epsilon^\alpha, \epsilon^\beta) \in \{-1, 1\} \times \{-1, 1\} \times \{-1, 1\}. \quad (5.2) \]

**Proof.** To prove this statement it is enough to look at the SOV characterization which defines completely the transfer matrix spectrum, i.e. the set $\Sigma_T$, and to prove that it is invariant under the above considered $Z_2^3 \otimes 2^2$ transformations of the boundary parameters. We have first to remark that the central values (2.26)–(2.28) of the transfer matrix $T(\lambda)$ are invariant under these discrete transformations and then the function $f(\lambda)$, defined in (2.36), is invariant too and the same is true for the form (2.39) of the interpolation polynomial describing the elements of $\Sigma_T$. Then the invariance of the SOV characterization (2.40) follows from the invariance of the quantum determinant

\[ \det K_+(\lambda) \det U_-(\lambda) = \sinh(2\eta - 2\lambda) \sinh(2\lambda + 2\eta)g_+(\lambda + \eta/2)g_+(-\lambda + \eta/2) \]

\[ \times g_-(\lambda + \eta/2)g_+(-\lambda + \eta/2)a(\lambda + \eta/2)d(\lambda - \eta/2) \]

\[ \times a(-\lambda + \eta/2)d(-\lambda - \eta/2) \quad (5.3) \]

under these discrete transformations. \(\square\)

doi:10.1088/1742-5468/2014/05/P05015
Theorem 5.1. Let the inhomogeneities of the boundary parameters do indeed change the transfer matrix $T(\lambda)$ and the Hamiltonian and so this invariance is equivalent to the statement that these different transfer matrices are all isospectral. In particular, it is simple to find the similarity matrices implementing the following $Z_2$ transformations of the boundary parameters:

$T(\lambda|\tau_+, -\zeta_+, \kappa_+ \ldots) = \Gamma_y T(\lambda|\tau_+, \zeta_+, \kappa_+ \ldots) \Gamma_y$, \quad $\Gamma_y \equiv \bigotimes_{n=1}^{N} \sigma^y_n$. \hspace{1cm} (5.4)

$T(\lambda|\tau_+, \zeta_+, -\kappa_+ \ldots) = \Gamma_z T(\lambda|\tau_+, \zeta_+, \kappa_+ \ldots) \Gamma_z$, \quad $\Gamma_z \equiv \bigotimes_{n=1}^{N} \sigma^z_n$. \hspace{1cm} (5.5)

5.2. Equivalent Baxter equations and the SOV spectrum

The invariance of the spectrum $\Sigma_T$ under these $Z_2$ transformations of the boundary parameters can be used to define equivalent Baxter equation reformulation of $\Sigma_T$. More precisely, let us introduce the following functions $A_{(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda)$ and $F_{(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda)$ obtained respectively by implementing the $Z_2$ transformations of the boundary parameters:

$$(\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \rightarrow (\epsilon_r \tau_+, \epsilon_\alpha \alpha_+, \epsilon_\beta \beta_+, \epsilon_r \tau_-, \epsilon_\alpha \alpha_-, \epsilon_\beta \beta_-), \hspace{1cm} (5.6)$$

then the following characterizations hold for any fixed $(\epsilon_r, \epsilon_\alpha, \epsilon_\beta) \in \{-1, 1\} \times \{-1, 1\}$.

**Theorem 5.1.** Let the inhomogeneities $\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N$ be generic (2.38) and let the boundary parameters $(\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus \text{NSOV}$ satisfy the following conditions:

$$\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad Y^{(i, 2r)}(\epsilon_r \tau_+, \epsilon_\alpha \alpha_+, \epsilon_\beta \beta_+) \neq 0 \quad \forall i \in \{0, 1\}, \quad r \in \{0, \ldots, N-1\}, \hspace{1cm} (5.7)$$

then $\tau(\lambda) \in \Sigma_T$ if and only if $\exists! Q(\lambda) \in \Sigma_Q$ such that

$$\tau(\lambda) Q(\lambda) = Z_{Q,(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda) + F_{(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda), \hspace{1cm} (5.8)$$

where

$$Z_{Q,(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda) = A_{(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(\lambda) Q(\lambda - \eta) + A_{(\epsilon_r, \epsilon_\alpha, \epsilon_\beta)}(-\lambda) Q(\lambda + \eta). \hspace{1cm} (5.9)$$

**Proof.** The proof follows step by step the one given for the main theorem 3.1. $\square$

5.3. General validity of the inhomogeneous Baxter equations

The previous reformulations of the spectrum in terms of different inhomogeneous Baxter equations and the observation that the conditions under which the theorem does not apply are related to the choice of the $(\epsilon_r, \epsilon_\alpha, \epsilon_\beta) \in \{-1, 1\} \times \{-1, 1\} \times \{-1, 1\}$ allow us to prove that unless the boundary parameters are lying on a finite lattice of step $\eta$ we can always use an inhomogeneous Baxter equation to completely characterize the spectrum of the...
transfer matrix. More precisely, let us introduce the following hyperplanes in the space of the boundary parameters:

$$
M \equiv \left\{ (\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 : \exists (r_{+-}, r_{-+}, r_{--}) \in \{0, \ldots, N-1\} \right. \\
\left. \quad \text{such that:} \begin{cases} 
  r_{+-} + r_{-+} - r_{--} \in \{0, \ldots, N-1\} \\
  \alpha_+ + \alpha_- = (r_{--} - r_{++})\eta \\
  \beta_- - \beta_+ = (r_{++} - r_{--})\eta \\
  \tau_- - \tau_+ = (N - 1 + r_{--} - 3r_{++})\eta 
\end{cases} \right\} \quad (5.10)
$$

then the following theorem holds.

**Theorem 5.2.** Let the inhomogeneities \( \{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N \) satisfy the conditions (2.38) and let \((\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus (M \cup N_{SOV})\) then we can always find a \((\epsilon_\tau, \epsilon_\alpha, \epsilon_\beta) \in \{-1,1\} \times \{-1,1\} \times \{-1,1\}\) such that \(\tau(\lambda) \in \Sigma_T\) if and only if \(\exists Q(\lambda) \in \Sigma_Q\) such that

$$
\tau(\lambda)Q(\lambda) = Z_{Q,(\epsilon_\tau,\epsilon_\alpha,\epsilon_\beta)}(\lambda) + F_{(\epsilon_\tau,\epsilon_\alpha,\epsilon_\beta)}(\lambda). \quad (5.11)
$$

**Proof.** Theorem 5.1 does not apply if \(\exists i \in \{0,1\}\) and \(\exists r \in \{0,\ldots,N-1\}\) such that the following system of conditions on the boundary parameters is satisfied:

$$
Y^{(i,2r)}(\epsilon_\tau \tau_\pm, \epsilon_\alpha \alpha_\pm, \epsilon_\beta \beta_\pm) = 0 \quad \forall (\epsilon_\tau, \epsilon_\alpha, \epsilon_\beta) \in \{-1,1\} \otimes^3, \quad (5.12)
$$

then by simple computations it is possible to observe that the set \(M\) defined in (5.10) indeed coincides with the following set:

$$
\left\{ (\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 : \exists i \in \{0,1\}, r \in \{0,\ldots,N-1\} \right. \\
\left. \quad \text{such that (5.12) is satisfied} \right\}, \quad (5.13)
$$

from which the theorem clearly follows.

**5.4. The homogeneous Baxter equation**

The discrete symmetries of the transfer matrix also allow us to define the general conditions on the boundary parameters for which the spectrum can be characterized by a homogeneous Baxter equation. In particular the following corollary holds.

**Corollary 5.1.** Let \((\tau_+, \alpha_+, \beta_+, \tau_-, \alpha_-, \beta_-) \in \mathbb{C}^6 \setminus N_{SOV}\) satisfy the condition

$$
\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad \exists i \in \{0,1\}, \exists (\epsilon_\tau, \epsilon_\alpha, \epsilon_\beta) \in \{-1,1\} \times \{-1,1\} \times \{-1,1\} : Y^{(i,2N)}(\epsilon_\tau \tau_\pm, \epsilon_\alpha \alpha_\pm, \epsilon_\beta \beta_\pm) = 0 \quad (5.14)
$$

and let the inhomogeneities \(\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N\) be generic (2.38), then \(\tau(\lambda) \in \Sigma_T\) if and only if \(\exists Q(\lambda) \in \Sigma_Q\) such that

$$
\tau(\lambda)Q(\lambda) = A_{(\epsilon_\tau,\epsilon_\alpha,\epsilon_\beta)}(\lambda)(\lambda)Q(\lambda - \eta) + A_{(\epsilon_\tau,\epsilon_\alpha,\epsilon_\beta)}(-\lambda)Q(\lambda + \eta). \quad (5.15)
$$

\(\text{doi:10.1088/1742-5468/2014/05/P05015}\)
Or equivalently we can define the set of all the solutions of the Bethe equations

\[ \Sigma_{\text{BAE}} = \{ \{ \lambda_1, \ldots, \lambda_N \} \in \mathbb{C}^N : A(\lambda_a)Q_\lambda(\lambda_a - \eta) + A(-\lambda_a)Q_\lambda(\lambda_a + \eta) = 0, \forall a \in \{1, \ldots, N\} \}. \] (5.16)

Then \( \tau(\lambda) \in \Sigma_T \) if and only if \( \exists \{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{BAE}} \) such that

\[ \tau(\lambda) = \frac{A(\epsilon, \epsilon, \epsilon, \epsilon_b)(\lambda)Q(\lambda - \eta) + A(\epsilon, \epsilon, \epsilon, \epsilon_b)(-\lambda)Q(\lambda + \eta)}{Q(\lambda)}, \] (5.17)

with

\[ Q(\lambda) = 2^N \prod_{a=1}^{N} (\cosh 2\lambda - \cosh 2\lambda_a). \]

Moreover, under the condition of normality defined in proposition 2.1, the set \( \Sigma_{\text{BAE}} \) of the solutions to the homogeneous system of Bethe ansatz type equations (4.4) contains \( 2^N \) elements.

6. XXX chain by SOV and Baxter equation

The construction of the SOV characterization can be naturally applied in the case of the rational six-vertex \( R \)-matrix, which in the homogeneous limit reproduces the XXX open quantum spin-1/2 chain with general integrable boundary conditions. Let us define

\[ R_{12}(\lambda) = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix} \in \text{End}(\mathcal{H}_1 \otimes \mathcal{H}_2). \] (6.1)

Due to the \( SU(2) \) invariance of the bulk monodromy matrix the boundary matrices defining the most general integrable boundary conditions can always be recast in the following form:

\[ K_-(\lambda; p) = \begin{pmatrix} \lambda - \eta/2 + p & 0 \\ 0 & p - \lambda + \eta/2 \end{pmatrix}, \]
\[ K_+(\lambda; q, \xi) = \begin{pmatrix} \lambda + \eta/2 + q & \xi(\lambda + \eta/2) \\ \xi(\lambda + \eta/2) & q - (\lambda + \eta/2) \end{pmatrix}, \] (6.2)

leaving only three arbitrary complex parameters, here denoted by \( \xi, p \) and \( q \). Then the one-parameter family of commuting transfer matrices,

\[ T(\lambda) = tr_0 \{ K_+(\lambda) M(\lambda) K_-(\lambda) \hat{M}(\lambda) \} \in \text{End}(\mathcal{H}), \] (6.3)

6 Here we use notations similar to those introduced in papers [33] and [42] where some inhomogeneous Baxter equation ansatz appears with the aim of making simpler for the reader a comparison when the limit of the homogeneous chain is implemented.
in the homogeneous limit leads to the following Hamiltonian:
\[ H = \sum_{n=1}^{N} \left( \sigma^x_n \sigma^x_{n+1} + \sigma^y_n \sigma^y_{n+1} + \sigma^z_n \sigma^z_{n+1} \right) + \frac{\sigma^z_0 + \xi \sigma^z}{q}. \] (6.4)

It is simple to show that the following identities hold:
\[ \det_q K_+ (\lambda) \det_q U_- (\lambda) = 4(\lambda^2 - \eta^2)(\lambda^2 - p^2)((1 + \xi^2)\lambda^2 - q^2) \times \prod_{b=1}^{N} (\lambda^2 - (\xi_n + \eta)^2)(\lambda^2 - (\xi_n - \eta)^2). \] (6.5)

We define
\[ A(\lambda) = (-1)^N \frac{2\lambda + \eta}{2\lambda} (\lambda - \eta/2 + p) \left( \sqrt{(1 + \xi^2)(\lambda - \eta/2)} + q \right) \prod_{b=1}^{N} \left( \lambda - \zeta_b^{(0)} \right) (\lambda + \zeta_b^{(1)}), \] (6.6)
then it is easy to derive the following quantum determinant identity:
\[ \frac{\det_q K_+ (\lambda) \det_q U_- (\lambda)}{(4\lambda^2 - \eta^2)} = A(\lambda + \eta/2)A(-\lambda + \eta/2). \] (6.7)

From the form of the boundary matrices it is clear that for the rational six-vertex case one can directly derive the SOV representations using the method developed in [48] without any need to introduce Baxter’s gauge transformations. Some results in this case also appeared in [31, 32] based on a functional version of the separation of variables of Sklyanin, a method which allows one to define the eigenvalues and wavefunctions but which does not allow one to construct in the original Hilbert space of the quantum chain the transfer matrix eigenstates.

The separation of variables description in this rational six-vertex case reads as follows.

**Theorem 6.1.** Let the inhomogeneities \( \{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N \) be generic,
\[ \xi_a \neq \pm \xi_b + r\eta \quad \forall a \neq b \in \{1, \ldots, N\} \quad \text{and} \quad r \in \{-1, 0, 1\}, \] (6.8)
then \( T(\lambda) \) has a simple spectrum and \( \Sigma_T \) is characterized by
\[ \Sigma_T = \left\{ \tau(\lambda) : \tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda)x_a, \forall \{x_1, \ldots, x_N\} \in \Sigma_T \right\}, \] (6.9)
where
\[ g_a(\lambda) = \frac{4\lambda^2 - \eta^2}{4\zeta_a^{(0)^2} - \eta^2} \prod_{b=1}^{N} \left( \frac{\lambda^2 - \zeta_b^{(0)^2}}{\zeta_a^{(0)^2} - \zeta_b^{(0)^2}} \right) \quad \text{for} \quad a \in \{1, \ldots, N\}, \] (6.10)
and
\[ f(\lambda) = \prod_{b=1}^{N} \left( \frac{\lambda^2 - \zeta_b^{(0)^2}}{\zeta_a^{(0)^2} - \zeta_b^{(0)^2}} \right) A(\eta/2) + 2 \left( 4\lambda^2 - \eta^2 \right) \prod_{b=1}^{N} \left( \frac{\lambda^2 - \zeta_b^{(0)^2}}{\zeta_a^{(0)^2} - \zeta_b^{(0)^2}} \right), \] (6.11)
and $\Sigma_T$ is the set of solutions to the following inhomogeneous system of $N$ quadratic equations:

$$x_n \sum_{a=1}^{N} g_a(\zeta_n^{(1)}) x_a + x_n f(\zeta_n^{(1)}) = q_n,$$

$$q_n = \frac{\det_q K_+(\xi_n) \det q U_-(\xi_n)}{\eta - 4\xi_n^2}, \quad \forall n \in \{1, \ldots, N\},$$

in $N$ unknowns $\{x_1, \ldots, x_N\}$.

We are now ready to present the following equivalent characterization of the transfer matrix spectrum.

**Theorem 6.2.** Let the inhomogeneities $\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N$ be generic (6.8), then for $\xi \neq 0$ the set of transfer matrix eigenvalue functions $\Sigma_T$ is characterized by

$$\tau(\lambda) \in \Sigma_T \text{ if and only if } \exists Q(\lambda) = \prod_{b=1}^{N} (\lambda^2 - \lambda_b^2) \text{ such that } \tau(\lambda)Q(\lambda) = ZQ(\lambda) + F(\lambda),$$

with

$$F(\lambda) = 2 \left(1 - \sqrt{(1 + \xi^2)}\right) (4\lambda^2 - \eta^2) \prod_{b=1}^{N} \prod_{i=0}^{1} (\lambda^2 - \zeta^{(0)b}_i).$$

**Proof.** The proof presented in theorem 3.1 applies with small modifications also to the rational case. \qed

The previous characterization of the transfer matrix spectrum allows us to prove that the set $\Sigma_{\text{InBAE}} \subset \mathbb{C}^N$ of all the solutions of the Bethe equations

$$\{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{InBAE}}$$

if

$$A(\lambda_a)Q(\lambda_a - \eta) + A(-\lambda_a)Q(\lambda_a + \eta) = -F(\lambda_a), \quad \forall a \in \{1, \ldots, N\},$$

defines the complete set of transfer matrix eigenvalues. In particular, the following corollary can be proved.

**Corollary 6.1.** Let the inhomogeneities $\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N$ satisfy the following conditions (2.38), then $T(\lambda)$ has a simple spectrum and for $\xi \neq 0$ then $\tau(\lambda) \in \Sigma_T$ if and only if $\exists! \{\lambda_1, \ldots, \lambda_N\} \in \Sigma_{\text{InBAE}}$ such that

$$\tau(\lambda) = \frac{ZQ(\lambda) + F(\lambda)}{Q(\lambda)} \text{ with } Q(\lambda) = \prod_{b=1}^{N} (\lambda^2 - \lambda_b^2).$$

Finally, we would like to mention that the inhomogeneous Baxter equation (4.9) has been used for the XXX spin chain with general boundary conditions in [5] as the starting point for the construction of the transfer matrix eigenstates of algebraic Bethe ansatz type\(^7\). While the construction presented in [5] is so far restricted to quantum chains containing only few quantum sites, it is interesting and potentially useful to introduce the algebraic Bethe ansatz approach for other integrable quantum models characterized by inhomogeneous Baxter equations.

\(^7\) See also [6] and [55] for first steps towards the construction of these Bethe states.
7. Homogeneous chains and existing numerical analysis

It is important to stress that the spectrum construction together with the corresponding statements of completeness presented in this paper work strictly for the most general spin 1/2 representations of the six-vertex reflection algebra, and only for generic inhomogeneous chains. However, it is worth mentioning that the transfer matrix as well as the coefficients and the inhomogeneous term in our functional equation characterization of the SOV spectrum are analytical functions of the inhomogeneities \( \{\xi_j\} \) so we can take without any problem the homogeneous limit \( (\xi_a \to 0 \ \forall a \in \{1, \ldots, N\}) \) in the functional equations. The main problem to be addressed then is the completeness of the description by this functional equation. Some initial understanding of this central question can be derived looking at the numerical analysis \([43, 44]\) of the completeness of Bethe ansatz equations when the boundary constraints are satisfied and for the open XXX chain with general boundary terms \([42]\).

7.1. Comparison with numerical results for the XXZ chain

The numerical checks of the completeness of Bethe Ansatz equations for the open XXZ quantum spin 1/2 chains were first done in \([43]\) for the chains with non-diagonal boundaries satisfying boundary constraints,

\[
\kappa_+ \neq 0, \quad \kappa_- \neq 0, \quad \exists i \in \{0, 1\}, M \in \mathbb{N} : Y^{(\tau_\pm, \alpha_\pm, \beta_\pm)} = 0. \tag{7.1}
\]

Indeed, under these conditions some generalizations of algebraic Bethe Ansatz can be used and so the corresponding Bethe equations can be defined.

In particular, the Nepomechie–Ravanini numerical results reported in \([43, 44]\) suggest the completeness of the Bethe ansatz equations \((4.4)\) in the homogeneous limit for the roots of the \(Q\) function

\[
Q(\lambda) = 2^M \prod_{a=1}^{M} (\cosh 2\lambda - \cosh 2\lambda_a), \tag{7.2}
\]

with the degree \(M\) obtained from the boundary constraint.

- For \(M = N\) they define the complete transfer matrix spectrum.
- For \(M < N\) the complete spectrum of the transfer matrix contains two parts described by different Baxter equations. The first one has trigonometric polynomial solutions of degree \(2M\), and the second one has trigonometric polynomial solutions of degree \(2N - 2 - 2M\).
- For \(M > N\) they define the complete spectrum of the transfer matrix spectrum plus \(\tau(\lambda)\) functions do not belong to the spectrum of the transfer matrix.

These results seem to be compatible with our characterization for the inhomogeneous chains. Indeed, the case \(M = N\) coincides with the case in which our Baxter functional equation becomes homogeneous. Theorem 4.1 states that in this case for generic

doi:10.1088/1742-5468/2014/05/P05015
inhomogeneities the Bethe ansatz is complete so we can expect (from the numerical analysis) that completeness will survive in the homogeneous limit.

In the case $M < N$, our description of the spectrum by lemma 4.1 separates the spectrum into two parts. The first part of the spectrum is described by trigonometric polynomial solutions of degree $2M$ to the homogeneous Baxter equation (4.8) and the second part is instead described by trigonometric polynomial solutions of degree $2N$ of the inhomogeneous Baxter equation (4.9). However, by implementing the following discrete symmetry transformations $\alpha_\pm \to -\alpha_\pm$, $\beta_\pm \to -\beta_\pm$, $\tau_\pm \to -\tau_\pm$ and applying the same lemma 4.1 w.r.t. the Baxter equations with coefficients $A_{(-,-,-)}(\lambda)$ we get an equivalent description of the spectrum separated into two parts. One part of the spectrum is described in terms of the solutions of the transformed homogeneous Baxter equation which should be trigonometric polynomials of degree $2M'$, with $M' = N - 1 - M$, and the second part by the inhomogeneous Baxter equation. The comparison with the numerical results then suggests that, at least in the limit of homogeneous chains, the part of the spectrum generated by the trigonometric polynomial solutions of degree $2N$ of the inhomogeneous Baxter equation (4.9) coincides with the part generated by the trigonometric polynomial solutions of degree $2M'$ of the transformed homogeneous Baxter equation.

Finally, in the case $M > N$ we have a complete characterization of the spectrum given by an inhomogeneous Baxter functional equation, however there is nothing to prevent consideration of solutions to the homogeneous Baxter equation once we take the appropriate $Q$-function with $M > N$ Bethe roots. The numerical results however seem to suggest that considering the homogeneous Baxter equations is not the proper thing to do in the homogeneous limit.

The previous analysis seems to support the idea that in the limit of the homogeneous chain our complete characterization still describes the complete spectrum of the homogeneous transfer matrix.

### 7.2. Comparison with numerical results for the XXX chain

In the case of the open spin $1/2$ XXX chain an ansatz based on two $Q$-functions and an inhomogeneous Baxter functional equation has been first introduced in [12]; the completeness of the spectrum obtained by that ansatz has been later verified numerically for small chains [33]. Using these results Nepomechie has introduced a simpler ansatz and developed some further numerical analysis in [42] confirming once again that the ansatz defines the complete spectrum for small chains. Here, we would like to point out that our complete description of the transfer matrix spectrum in terms of an inhomogeneous Baxter functional equation obtained for the inhomogeneous chains has the following well defined homogeneous limit:

$$\tau(\lambda)Q(\lambda) = A(\lambda)Q(\lambda - \eta) + A(-\lambda)Q(\lambda + \eta) + F(\lambda) \quad (7.3)$$

where

$$F(\lambda) = 8 \left( 1 - \sqrt{(1 + \xi^2)} \right) (\lambda^2 - (\eta/2)^2)^{2N+1}, \quad (7.4)$$

$$A(\lambda) = (-1)^N \frac{2\lambda + \eta}{2\lambda} (\lambda - \eta/2 + p) \left( \sqrt{(1 + \xi^2)(\lambda - \eta/2)} + q \right) (\lambda^2 - (\eta/2)^2)^N. \quad (7.5)$$

doi:10.1088/1742-5468/2014/05/P05015
Taking into account the shift in our definition of the monodromy matrix which ensures that the transfer matrix is an even function of the spectral parameter, the limit of our inhomogeneous Baxter functional equation coincides with the ansatz proposed by Nepomechie in [42]. Then the numerical evidence of completeness derived by Nepomechie in [42] suggests that the exact and complete characterization that we get for the inhomogeneous chain is still valid and complete in the homogeneous limit.

8. Conclusion and outlook

In this paper we have shown that the transfer matrix spectrum associated with the most general spin-1/2 representations of the six-vertex reflection algebras (rational and trigonometric), on general inhomogeneous chains is completely characterized in terms of a second order difference functional equation of Baxter $T$–$Q$ type with an inhomogeneous term depending only on the inhomogeneities of the chain and the boundary parameters. This functional $T$–$Q$ equation has been shown to be equivalent to the SOV complete characterization of the spectrum when the $Q$-functions belong to a well defined set of polynomials. The polynomial form of the $Q$-function is a central feature of our approach which allows us to introduce an equivalent finite system of generalized Bethe ansatz equations. Moreover, we have explicitly proven that our functional characterization holds for all the values of the boundary parameters for which SOV works, clearly identifying the only three-dimensional hyperplanes in the six-dimensional space of the boundary parameters where our description cannot be applied. We have also clearly identified the five-dimensional hyperplanes in the space of the boundary parameters where the spectrum (or a part of the spectrum) can be characterized in terms of a homogeneous $T$–$Q$ equation and the polynomial character of the $Q$-functions is then equivalent to a standard system of Bethe equations. Completeness of this description is a built in feature due to the equivalence to the SOV characterization.

The equivalence between our functional $T$–$Q$ equation and the SOV characterization holds for generic values of the $\xi_a$ in the $N$-dimensional space of the inhomogeneity parameters however there exist hyperplanes for which the conditions (2.38) are not satisfied and so a direct application of the SOV approach is not possible (at least for the separate variables described in [28]) and the limit of homogeneous chains ($\xi_a \to 0 \forall a \in \{1, \ldots, N\}$) clearly belongs to these hyperplanes. From the analyticity of the transfer matrix eigenvalues, of the coefficients of the functional $T$–$Q$ equation and of the inhomogeneous term in it with respect to the inhomogeneity parameters, it is possible to argue that these functional equations still describe transfer matrix eigenvalues on the hyperplanes where the SOV method cannot be applied, in particular, in the homogeneous limit. However, in all these cases the statements about the simplicity of the transfer matrix spectrum and the completeness of the description by our functional $T$–$Q$ equation are not anymore granted and they require independent proof. These fundamental issues will be addressed in a future publication. Here we want just to recall that the comparison with the few existing numerical results on the subject seems to suggests that the statement of completeness should be satisfied even in the homogeneous limit of special interest as it allows us to reproduce the spectrum of the Hamiltonian of the spin-1/2 open XXZ quantum chains under the most general integrable boundary conditions.
Finally, it is important to note that the form of the Baxter functional equation for
the most general spin-1/2 representations of the six-vertex reflection algebras, and in
particular the necessity of an inhomogeneous term, is mainly imposed by the requirement
that the set of solutions is restricted to polynomials. Then the problem to get homogeneous
Baxter equations relaxing this last requirement remains an interesting open problem.

Acknowledgments

The authors would like to thank E Sklyanin and V Terras for discussions. JMM and GN
are supported by CNRS. NK and JMM are supported by ANR grant ‘DIADEMS’. NK
would like to thank LPTHE, University Paris VI and Laboratoire de Physique, ENS Lyon
for hospitality.

References

[1] Baseilhac P, The q-deformed analogue of the Onsager algebra: beyond the Bethe ansatz approach, 2006
Nucl. Phys. B 754 309
[2] Baseilhac P and Koizumi K, A deformed analogue of Onsager’s symmetry in the XXZ open spin
chain, 2005 J. Stat. Mech. P10005
[3] Baxter R, Partition function of the eight-vertex lattice model, 1972 Ann. Phys., NY 70 193
[4] Baxter R J, One-dimensional anisotropic Heisenberg chain, 1972 Ann. Phys., NY 70 323
[5] Belliard S and Crampé N, Heisenberg XXX model with general boundaries: eigenvectors from algebraic
Bethe ansatz, 2013 SIGMA 9 072
[6] Belliard S, Crampé N and Ragoucy E, Algebraic Bethe ansatz for open XXX model with triangular
boundary matrices, 2013 Lett. Math. Phys. 103 493
[7] Bethe H, Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen Atomketten, 1931 Z. Phys. 71 205
[8] Bytsko A G and Teschner J, Quantization of models with non-compact quantum group symmetry: modular
XXZ magnet and lattice sinh–Gordon model, 2006 J. Phys. A: Math. Gen. 39 12927
[9] Cao J, Lin H-Q, Shi K-J and Wang Y, Exact solution of XXZ spin chain with unparallel boundary
fields, 2003 Nucl. Phys. B 663 487
[10] Cao J, Yang W, Shi K and Wang Y, Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains
with arbitrary boundary fields, 2013 Nucl. Phys. B 887 152
[11] Cao J, Yang W, Shi K and Wang Y, Off-diagonal Bethe ansatz and exact solution a topological spin
ring, 2013 Phys. Rev. Lett. 111 137201
[12] Cao J, Yang W, Shi K and Wang Y, Off-diagonal bethe ansatz solution of the XXX spin-chain with
arbitrary boundary conditions, 2013 Nucl. Phys. B 875 152
[13] Cao J, Yang W, Shi K and Wang Y, Spin-1/2 XYZ model revisit: general solutions via off-diagonal Bethe
ansatz, 2013 arXiv:1307.0280
[14] Cherednik I V, Factorizing particles on a half-line and root systems, 1984 Theor. Math. Phys. 61 977
[15] Crampé N, Ragoucy E and Simon D, Eigenvectors of open XXZ and ASEP models for a class of
non-diagonal boundary conditions, 2010 J. Stat. Mech. P11038
[16] Crampé N and Ragoucy E, Generalized coordinate Bethe ansatz for non-diagonal boundaries, 2012 Nucl.
Phys. B 858 502
[17] Derkachov S E, Korchemsky G P and Manashov A N, Noncompact Heisenberg spin magnets from
high-energy QCD. 1. Baxter Q operator and separation of variables, 2001 Nucl. Phys. B 617 375
[18] Derkachov S E, Korchemsky G P and Manashov A N, Separation of variables for the quantum SL(2,R) spin
chain, 2003 J. High Energy Phys. JHEP07(2003)047
[19] Derkachov S E, Korchemsky G P and Manashov A N, Baxter Q operator and separation of variables for the
open SL(2,R) spin chain, 2003 J. High Energy Phys. JHEP10(2003)053
[20] de Gier J and Essler F H L, Bethe ansatz solution of the asymmetric exclusion process with open
boundaries, 2005 Phys. Rev. Lett. 95 240601
[21] de Gier J and Essler F H L, Exact spectral gaps of the asymmetric exclusion process with open
boundaries, 2006 J. Stat. Mech. P12011
doi:10.1088/1742-5468/2014/05/P05015
Open spin chains with generic integrable boundaries

[22] Galleas W, Functional relations from the Yang–Baxter algebra: eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions, 2008 Nucl. Phys. B 790 524

[23] de Gier J and Pyatov P, Bethe ansatz for the Temperley–Lieb loop model with open boundaries, 2004 J. Stat. Mech. P03002

[24] Grosjean N, Maillet J M and Niccoli G, On the form factors of local operators in the lattice sine–Gordon model, 2012 J. Stat. Mech. P10006

[25] Grosjean N, Maillet J-M and Niccoli G, On the form factors of local operators in the Bazhanov–Stroganov and chiral Potts models, 2014 Ann. Henri Poincaré at press (arXiv:1309.4701)

[26] Grosjean N and Niccoli G, The $\tau_2$-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin’s SOV method, 2012 J. Stat. Mech. P11005

[27] Faddeev L D, Sklyanin E K and Takhtajan L A, Quantum inverse problem method I, 1979 Theor. Math. Phys. 40 688

[28] Faldella S, Kitanine N and Niccoli G, Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms, 2014 J. Stat. Mech. P01011

[29] Faldella S and Niccoli G, SOV approach for integrable quantum models associated to the most general representations on spin-1/2 chains of the eight-vertex reflection algebra, 2014 J. Phys. A: Math. Theor. 47 115202

[30] Filali G and Kitanine N, Spin chains with non-diagonal boundaries and trigonometric SOS model with reflecting end, 2011 SIGMA 7 012

[31] Frahm H, Seel A and Wirth T, Separation of variables in the open XXX chain, 2008 Nucl. Phys. B 802 351

[32] Frahm H, Grelik J H, Seel A and Wirth T, Functional Bethe ansatz methods for the open XXX chain, 2011 J. Phys. A: Math. Theor. 44 015001

[33] Jiang Y, Cui S, Cao J, Yang W-L and Wang Y, Completeness and Bethe root distribution of the spin-1/2 Heisenberg chain with arbitrary boundary fields, 2013 arXiv:1309.6456v1

[34] Kitanine N, Kozlowski K, Maillet J, Niccoli G, Slavnov N and Terras V, On correlation functions of the open XXZ chain I, 2007 J. Stat. Mech. P10009

[35] Kitanine N, Kozlowski K, Maillet J, Niccoli G, Slavnov N and Terras V, On correlation functions of the open XXZ chain II, 2008 J. Stat. Mech. P07010

[36] Kitanine N, Maillet J M and Terras V, Form factors of the XXZ Heisenberg spin-1/2 finite chain, 1999 Nucl. Phys. B 554 647

[37] Kitanine N, Maillet J M and Terras V, Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field, 2000 Nucl. Phys. B 567 554

[38] Maillet J M and Terras V, On the quantum inverse scattering problem, 2000 Nucl. Phys. B 575 627

[39] Murgan R and Nepomechie R I, Bethe ansatz derived from the functional relations of the open XXZ chain for new special cases, 2005 J. Stat. Mech. P05007

[40] Nepomechie R I, Solving the open XXZ spin chain with nondiagonal boundary terms at roots of unity, 2002 Nucl. Phys. B 622 615

[41] Nepomechie R I, Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms, 2004 J. Phys. A: Math. Gen. 37 433 (special issue on recent advances in the theory of quantum integrable systems)

[42] Nepomechie R I, Inhomogeneous T–Q equation for the open XXX chain with general boundary terms: completeness and arbitrary spin, 2013 J. Phys. A: Math. Theor. 46 442002

[43] Nepomechie R I and Ravanini F, Completeness of the Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms, 2003 J. Phys. A: Math. Gen. 36 11391

[44] Nepomechie R I and Ravanini F, Addendum to ‘Completeness of the Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms’, 2004 J. Phys. A: Math. Gen. 37 1945

[45] Nepomechie R I and Wang C, Boundary energy of the open XXX chain with a non-diagonal boundary term, 2014 J. Phys. A: Math. Theor. 47 032001

[46] Niccoli G, Reconstruction of Baxter Q-operator from Sklyanin SOV for cyclic representations of integrable quantum models, 2010 Nucl. Phys. B 835 263

[47] Niccoli G, Completeness of Bethe ansatz by sklyanin SOV for cyclic representations of integrable quantum models, 2011 J. High Energy Phys. JHEP03(2011)123

[48] Niccoli G, Non-diagonal open spin 1/2 XXZ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators, 2012 J. Stat. Mech. P10025

[49] Niccoli G, On the developments of Sklyanin’s quantum separation of variables for integrable quantum field theories, 2013 ICMP13 Proc. by World Scientific [arXiv:1301.4924]

[50] Niccoli G, Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and form factors, 2013 Nucl. Phys. B 870 397

doi:10.1088/1742-5468/2014/05/P05015
Open spin chains with generic integrable boundaries

[51] Niccoli G, Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables, 2013 J. Math. Phys. 54 053516
[52] Niccoli G, Antiperiodic dynamical six-vertex model I: complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model, 2013 J. Phys. A: Math. Theor. 46 075003
[53] Niccoli G and Teschner J, The sine–Gordon model revisited: I, 2010 J. Stat. Mech. P09014
[54] Nichols A, Rittenberg V and de Gier J, One-boundary Temperley–Lieb algebras in the XXZ and loop models, 2005 J. Stat. Mech. P03003
[55] Pimenta R A and Lima-Santos A, Algebraic Bethe ansatz for the six vertex model with upper triangular $K$-matrices, 2013 J. Phys. A: Math. Theor. 46 455002
[56] Prosen T, Open XXZ spin chain: nonequilibrium steady state and a strict bound on ballistic transport, 2011 Phys. Rev. Lett. 106 217206
[57] Sirker J, Pereira R G and Affleck I, Diffusion and ballistic transport in one-dimensional quantum systems, 2009 Phys. Rev. Lett. 103 216602
[58] Sklyanin E K, The quantum Toda chain, 1985 Nonlinear Equations in Classical and Quantum Field Theory (Meudon/Paris, 1983/1984) (Lecture Notes in Phys. vol 226) (Berlin: Springer) pp 196–233
[59] Sklyanin E, Boundary conditions for integrable quantum systems, 1988 J. Phys. A: Math. Gen. 21 2375
[60] Sklyanin E K, Poisson structure of a periodic classical XYZ chain, 1989 J. Sov. Math. 46 1664
[61] Sklyanin E K, Poisson structure of classical XXZ chain, 1989 J. Sov. Math. 46 2104
[62] Sklyanin E K, Quantum inverse scattering method. Selected topics, 1992 Quantum Group and Quantum Integrable Systems (Nankai Lectures in Mathematical Physics) ed M-L Ge (Singapore: World Scientific) pp 63–97 (arXiv:hep-th/9211111)
[63] Takhtajan L A and Faddeev L D, The quantum method of the inverse problem and the Heisenberg XYZ model, 1979 Russ. Math. Surv. 34 11
[64] Yang W-L and Zhang Y-Z, On the second reference state and complete eigenstates of the open XXZ chain, 2007 J. High Energy Phys. JHEP04(2007)044

doi:10.1088/1742-5468/2014/05/P05015