CONSTRUCTING COMBINATORIAL 4-MANIFOLDS

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ABSTRACT. Every closed oriented PL 4-manifold is a branched cover of the 4-sphere branched over a PL-surface with finitely many singularities by Piergallini [Topology 34(3):497-508, 1995]. This generalizes a long standing result by Hilden and Montesinos to dimension four. Izmestiev and Joswig [Adv. Geom. 3(2):191-225, 2003] gave a combinatorial equivalent of the Hilden and Montesinos result, constructing closed oriented combinatorial 3-manifolds as simplicial branched covers of combinatorial 3-spheres. The construction of Izmestiev and Joswig is generalized and applied to the result of Piergallini, obtaining closed oriented combinatorial 4-manifolds as simplicial branched covers of simplicial 4-spheres.

1. Introduction

The main objective of this paper is to give a complete yet concise account of how to obtain closed oriented combinatorial 4-manifolds as simplicial branched covers, that is, as partial unfolding of simplicial 4-spheres. The construction is at times technical involved and the topological background extensive. Thus we abstain from discussing related material and omit some of the proofs. Complete proofs and plenty of further material can be found in the Chapters 1 to 3 of [36]. Concerning the construction of closed oriented PL 4-manifolds as branched covers we refer to Piergallini [32] and Montesinos [25]. For the partial unfolding and the construction of closed oriented combinatorial 3-manifolds Izmestiev & Joswig [18] is mandatory reading. (Their construction has recently be simplified significantly by Hilden, Montesinos-Amilibia, Tejada & Toro [14].) For those able to read German additional analysis and examples can be found in [35]. The partial unfolding is implemented in the software package polymake [10].

Branched covers form a major tool for the study, construction and classification of d-manifolds. First results are by Alexander [1] in 1920, who observed that any closed oriented PL d-manifold M is a branched cover of the d-sphere. Unfortunately Alexander’s proof does not allow for any (reasonable) control over the number of sheets of the branched cover, nor over the topology of the branching set: The number of sheets depends on the size of some triangulation of M and the branching set is the co-dimension 2-skeleton of the d-simplex.

At least to our knowledge, there are no non-trivial upper bounds for the number of sheets of such a branched cover for $d > 4$. On the contrary, Bernstein & Edmonds [2] showed that at least $d$ sheets are necessary in general (for example the $d$-torus $(S^1)^d$ exhibits such a behavior), and that the branching set can not be required to be non-singular for $d \geq 8$.

However, in dimension $d \leq 4$, the situation is fairly well understood. The 2-dimensional case is straightforward: any closed oriented surface $F_g$ of genus $g$ is a 2-fold branched cover of the 2-sphere branched over $2g + 2$ isolated points.

By results of Hilden [13] and Montesinos [24] any closed oriented 3-manifold $M$ arises as 3-fold simple branched cover of the 3-sphere branched over a link $L$. Labeling each bridge $b$ of a diagram of $L$ with the corresponding monodromy action of a meridian around $b$, we can represent $M$ as a labeled (or colored) link diagram.

In dimension 4 the situation becomes increasingly difficult. First Piergallini [32] showed how to obtain any closed oriented PL 4-manifold as a 4-fold branched cover of the 4-sphere branched over a PL-surface with a finite number of cusp and node singularities. Prior to Piergallini’s work...
Montesinos [25] gave a description of oriented 4-manifolds composed of 0-, 1-, and 2-handles only as a branched cover of the 4-ball. Montesinos’ result is essential for Piergallini’s construction of closed oriented PL 4-manifolds as branched covers. These two constructions are the “blue print” for the main result of this paper and they are reviewed in Section 3.1.

Piergallini and later Iori & Piergallini improved the results on the construction of closed oriented PL 4-manifolds further. First Piergallini [32] eliminated the cusp singularities of the branching set. This yields a branched cover with a transversally immersed PL-surface as its branching set. Iori & Piergallini [16] then proved that the branching set may be realized locally flat if one allows for a fifth sheet for the branched cover, thus proving a long-standing conjecture by Montesinos [25]. The question whether any closed oriented PL 4-manifold can be obtained as 4-fold cover of the 4-sphere branched over a locally flat PL-surface is still open. Although these later developments certainly ask for a combinatorial equivalent, we will not investigate these here, nor make use of these observations.

Outline of the paper. After some basic definitions and notations the partial unfolding \( \hat{K} \) of a simplicial complex \( K \) is introduced. The partial unfolding defines a projection \( p : \hat{K} \to K \) which is a simplicial branched cover if \( K \) meets certain connectivity assumptions. We define combinatorial models of key features of a branched cover, namely the branching set and the monodromy homomorphism.

Section 2 introduces a notion of equivalence of simplicial complexes which agrees with their unfolding behavior. We proceed by establishing further (technical) tools for the construction of combinatorial 4-manifolds in Section 3.

Finally Section 3 states and proofs the main result Theorem 3.11. The key idea is to construct a simplicial 4-sphere \( S \), such that the projection \( p : \hat{S} \to S \) is equivalent to a given branched cover \( r : M \to S^4 \). In particular, the equivalence of the branched covers \( p \) and \( r \) implies homeomorphy of the covering spaces \( \hat{K} \) and \( M \). In Theorem 3.11 we prove that this is indeed possible for the branched covers arising in the construction of closed oriented PL 4-manifolds by Piergallini [32]: For any given closed oriented PL 4-manifold \( M \) there is a simplicial 4-sphere \( S \) such that the partial unfolding \( \hat{S} \) is PL-homeomorphic to \( M \). We proceed by giving a construction of the simplicial 4-sphere \( S \). Prior to proving the main result, the topological constructions by Montesinos [25] and Piergallini [32] are reviewed.

1.1. Basic definitions and notations. Given some topological manifold \( M \), we call a simplicial complex \( K \) homeomorphic to \( M \) a triangulation of \( M \), or a simplicial manifold. A simplicial complex \( K \) is a combinatorial d-sphere or combinatorial d-ball if it is piecewise linear homeomorphic to the boundary of the \((d+1)\)-simplex, respectively to the \(d\)-simplex. Equivalently, \( K \) is a combinatorial \( d \)-sphere or \( d \)-ball if there is a common refinement of \( K \) and the boundary of the \((d+1)\)-simplex, respectively the \(d\)-simplex. A simplicial complex \( K \) is a combinatorial manifold if the vertex link of each vertex of \( K \) is a combinatorial sphere or a combinatorial ball. Note that combinatorial spheres and balls are combinatorial manifolds.

A manifold \( M \) where all charts are piecewise linear is called a PL-manifold. Up to dimension 3 there is no difference between topological, PL-, and differential manifolds, that is, every topological manifold allows for a PL- or differential atlas (or structure). The existence of a triangulation of \( M \) as a combinatorial manifold is equivalent to the existence of a PL-atlas for \( M \). For an introduction to PL-topology see Björner [3, Part II], Hudson [15], and Rourke & Sanderson [33].

Similarly to the topological situation, there is no difference between the notion of a simplicial and a combinatorial manifold in dimension \( d \leq 3 \), that is, every simplicial manifold (or sphere, or ball) is a combinatorial manifold (or sphere, or ball). But in dimension 4 the situation becomes more complicated. Freedman & Quinn [9] construct a 4-manifold which does not have a triangulation as a combinatorial manifold. In fact, there are 4-manifolds which can not be triangulated at all [22, p. 9]. The following unanswered question illustrates the subtleties of the 4-dimensional case like no other: Is a combinatorial manifold homeomorphic to the 4-sphere necessarily a combinatorial 4-sphere? Surprisingly, the answer to this question is affirmative in all dimensions \( d \neq 4 \); see Moise [23] and Kirby & Siebenmann [20].
Neither barycentric subdivision nor anti-prismatic subdivision (of a face) change the PL-type of a simplicial manifold, that is, the subdivision of a simplicial complex $K$ is a combinatorial manifold if and only if $K$ is a combinatorial manifold. The cone of a combinatorial sphere is a combinatorial ball and the suspension of a combinatorial sphere is again a combinatorial sphere.

The simplicial complexes considered in the following (and throughout this exposition) are always pure, that is, all the inclusion maximal faces, called the facets, have the same dimension. We call a co-dimension 1-face of a pure simplicial complex a ridge, and the dual graph $Γ^*(K)$ of a pure simplicial complex $K$ has the facets as its node set, and two nodes are adjacent if the corresponding facets share a ridge. We denote the 1-skeleton of $K$ by $Γ(K)$, its graph.

Further it is often necessary to restrict ourselves to simplicial complexes with certain connectivity properties: A pure simplicial complex $K$ is strongly connected if its dual graph $Γ^*(K)$ is connected, and locally strongly connected if the star $st_K(f)$ of $f$ is strongly connected for each face $f ∈ K$. If $K$ is locally strongly connected, then connected and strongly connected coincide. Further we call $K$ locally strongly simply connected if for each face $f ∈ K$ with co-dimension $≥ 2$ the link $lk_K(f)$ of $f$ is simply connected, and finally, $K$ is nice if it is locally strongly connected and locally strongly simply connected. Observe that connected combinatorial manifolds are always nice.

1.2. The branched cover. The concept of a covering of a space $Y$ by another space $X$ is generalized by Fox [S] to the notion of the branched cover. Here a certain subset $Y_{\text{sing}} ⊂ Y$ may violate the conditions of a covering map. This allows for a wider application in the construction of topological spaces. It is essential for a satisfactory theory of (branched) coverings to make certain connectivity assumption for $X$ and $Y$. The spaces mostly considered are Hausdorff, path connected, and locally path connected; see Bredon [B, III.3.1]. Throughout we will restrict our attention to coverings of manifolds and we assume $Y$ to be connected, hence they meet the connectivity assumptions in [B].

Consider a continuous map $h : Z → Y$, and assume the restriction $h : Z → h(Z)$ to be a covering. If $h(Z)$ is dense in $Y$ (and meets certain additional connectivity conditions) then there is a surjective map $p : X → Y$ with $Z ⊂ X$ and $p|Z = h$. The map $p$ is called a completion of $h$, and any two completions $p : X → Y$ and $p' : X' → Y$ are equivalent in the sense that there exists a homeomorphism $ϕ : X → X'$ satisfying $p' ◦ ϕ = p$ and $ϕ|Z = \text{Id}$. The map $p : X → Y$ obtained this way is a branched cover, and we call the unique minimal subset $Y_{\text{sing}} ⊂ Y$ such that the restriction of $p$ to the preimage of $Y \setminus Y_{\text{sing}}$ is a cover, the branching set of $p$. The restriction of $p$ to $p^{-1}(Y \setminus Y_{\text{sing}})$ is called the associated cover of $p$. If $h : Z → Y$ is a cover, then $X = Z$, and $p = h$ is a branched cover with empty branching set.

**Example 1.1.** For $k ≥ 2$ consider the map $p_k : \mathbb{C} → \mathbb{C}$. The restriction $p_k|_{\mathbb{D}^2}$ is a $k$-fold branched cover $\mathbb{D}^2 → \mathbb{D}^2$ with the single branch point $\{0\}$.

We define the monodromy homomorphism
\[ m_p : π_1(Y \setminus Y_{\text{sing}}, y_0) → \text{Sym}(p^{-1}(y_0)) \]
of a branched cover for a point $y_0 ∈ Y \setminus Y_{\text{sing}}$ as the monodromy homomorphism of the associated cover: If $[α] ∈ π_1(Y \setminus Y_{\text{sing}}, y_0)$ is represented by a closed path $α$ based at $y_0$, then $m_p$ maps $[α]$ to the permutation $(x_i → α_i(1))$, where $\{x_1, x_2, \ldots, x_k\} = p^{-1}(y_0)$ is the preimage of $y_0$ and $α_i : [0,1] → X$ is the unique lifting of $α$ with $p ◦ α_i = α$ and $α_i(0) = x_i$; see Munkres [28, Lemma 79.1] and Seifert & Threlfall [31, § 58]. The monodromy group $ℳ_p$ is defined as the image of $m_p$.

Two branched covers $p : X → Y$ and $p' : X' → Y'$ are equivalent if there are homeomorphisms $ϕ : X → X'$ and $ψ : Y → Y'$ with $ψ(Y_{\text{sing}}) = Y_{\text{sing}}'$, such that $p' ◦ ϕ = ψ ◦ p$ holds. The well known Theorem 1.2 is due to the uniqueness of $Y_{\text{sing}}$, and hence the uniqueness of the associated cover; see Piergallini [31, p. 2].

**Theorem 1.2.** Let $p : X → Y$ be a branched cover of a connected manifold $Y$. Then $p$ is uniquely determined up to equivalence by the branching set $Y_{\text{sing}}$, and the monodromy homomorphism $m_p$. In particular, the covering space $X$ is determined up to homeomorphy.
Let $Y$ be a connected manifold and $Y_{\text{sing}}$ a co-dimension 2 submanifold, possibly with a finite number of singularities. We call a branched cover $p$ simple if the image $m_p(m)$ of any meridional loop $m$ around a non-singular point of the branching set is a transposition in $\mathbb{M}_p$. Note that the $k$-fold branched cover $p_k \mid_{\mathbb{D}^2} : \mathbb{D}^2 \to \mathbb{D}^2$ presented in Example 1.1 is not simple for $k \geq 3$.

1.3. The partial unfolding. The partial unfolding $K$ of a simplicial complex $K$ first appeared in a paper by Izmestiev & Joswig [19], with some of the basic notions already developed in Joswig [19]. The partial unfolding is closely related to the complete unfolding, also defined in [18], but we will not discuss the latter. The partial unfolding is a geometric object defined entirely by the combinatorial structure of $K$, and comes along with a canonical projection $p : K \to K$.

However, the partial unfolding $K$ may not be a simplicial complexes. In general $K$ is a pseudo-simplicial complexes: Let $\Sigma$ be a collection of pairwise disjoint geometric simplices, with simplicial attaching maps for some pairs $(\sigma, \tau) \in \Sigma \times \Sigma$, mapping a subcomplex of $\sigma$ isomorphically to a subcomplex of $\tau$. Identifying the subcomplexes accordingly yields the quotient space $\Sigma/\sim$, which is called a pseudo-simplicial complex if the quotient map $\Sigma \to \Sigma/\sim$ restricted to any $\sigma \in \Sigma$ is bijective. The last condition ensures that there are no self-identifications within each simplex $\sigma \in \Sigma$.

The group of projectivities. Let $\sigma$ and $\tau$ be neighboring facets of a finite, pure simplicial complex $K$, that is, $\sigma \cap \tau$ is a ridge. Then there is exactly one vertex in $\sigma$ which is not a vertex of $\tau$ and vice versa, hence a natural bijection $\langle \sigma, \tau \rangle$ between the vertex sets of $\sigma$ and $\tau$ is given by

$$
\langle \sigma, \tau \rangle : V(\sigma) \to V(\tau) : v \mapsto \begin{cases} v & \text{if } v \in \sigma \cap \tau \\ \tau \setminus \sigma & \text{if } v \in \sigma \setminus \tau. \end{cases}
$$

The bijection $\langle \sigma, \tau \rangle$ is called a perspectivity from $\sigma$ to $\tau$.

A facet path in $K$ is a sequence $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ of facets, such that the corresponding nodes in the dual graph $\Gamma^*(K)$ form a path, that is, $\sigma_i \cap \sigma_{i+1}$ is a ridge for all $0 \leq i < k$; see Figure 1. Now a projectivity $\langle \gamma \rangle$ along $\gamma$ is defined as the composition of perspectivities $\langle \sigma_i, \sigma_{i+1} \rangle$, thus $\langle \gamma \rangle$ maps $V(\sigma_0)$ to $V(\sigma_k)$ bijectively via

$$
\langle \gamma \rangle = \langle \sigma_{k-1}, \sigma_k \rangle \circ \cdots \circ \langle \sigma_1, \sigma_2 \rangle \circ \langle \sigma_0, \sigma_1 \rangle.
$$

We write $\gamma \gamma' = (\sigma_0, \sigma_1, \ldots, \sigma_k, \ldots, \sigma_{k+l})$ for the concatenation of two facet paths $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ and $\gamma' = (\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+l})$, denote by $\gamma^{-1} = (\sigma_k, \sigma_{k-1}, \ldots, \sigma_0)$ the inverse path of $\gamma$, and we call $\gamma$ a closed facet path based at $\sigma_0$ if $\sigma_0 = \sigma_k$. The set of closed facet paths based at $\sigma_0$ together with the concatenation form a group, and a closed facet path $\gamma$ based at $\sigma_0$ acts on the set $V(\sigma_0)$ via $\gamma \cdot v = \langle \gamma \rangle(v)$ for $v \in V(\sigma_0)$. Via this action we obtain the group of projectivities $\Pi(K, \sigma_0)$ given by all permutations $\langle \gamma \rangle$ of $V(\sigma_0)$. The group of projectivities is a subgroup of the symmetric group $\text{Sym}(V(\sigma_0))$ on the vertices of $\sigma_0$.

![Figure 1. A projectivity from $\sigma$ to $\tau$ along the facet path $\gamma$.](image)
The projectivities along null-homotopic closed facet paths based at \( \sigma_0 \) generate the subgroup \( \Pi_0(K, \sigma_0) < \Pi(K, \sigma_0) \), which is called the \textit{reduced group of projectivities}. Finally, if \( K \) is strongly connected then \( \Pi(K, \sigma_0) \) and \( \Pi(K, \sigma_0') \), respectively \( \Pi_0(K, \sigma_0) \) and \( \Pi_0(K, \sigma_0') \), are isomorphic for any two facets \( \sigma_0, \sigma_0' \in K \). In this case we usually omit the base facet in the notation of the (reduced) group of projectivities, and write \( \Pi(K) = \Pi(K, \sigma_0) \), respectively \( \Pi_0(K) = \Pi_0(K, \sigma_0) \).

The odd subcomplex. Let \( K \) be locally strongly connected; in particular, \( K \) is pure. The link of a co-dimension 2-face \( f \) is a graph which is connected since \( K \) is locally strongly connected, and \( f \) is called even if the link \( l_K(f) \) of \( f \) is bipartite, and odd otherwise. We define the odd subcomplex of \( K \) as all odd co-dimension 2-faces (together with their proper faces), and denote it by \( K_{\text{odd}} \) (or sometimes odd(\( K \))).

Assume that \( K \) is pure and admits a \((d+1)\)-coloring of its graph \( \Gamma(K) \), that is, we assign one color of a set of \( d+1 \) colors to each vertex of \( \Gamma(K) \) such that the two vertices of any edge carry different colors. Observe that the \((d+1)\)-coloring of \( K \) is minimal with respect to the number of colors, and is unique up to renaming the colors if \( K \) is strongly connected. Simplicial complexes that are \((d+1)\)-colorable are called foldable, since a \((d+1)\)-coloring defines a non-degenerated simplicial map of \( K \) to the \((d+1)\)-simplex. In other places in the literature foldable simplicial complexes are sometimes called balanced.

**Lemma 1.3.** The odd subcomplex of a foldable simplicial complex \( K \) is empty, and the group of projectivities \( \Pi(K, \sigma_0) \) is trivial. In particular we have \( \langle \alpha \rangle = \langle \beta \rangle \) for any two facet paths \( \alpha \) and \( \beta \) from \( \sigma \) to \( \tau \) for any two facets \( \sigma, \tau \in K \).

We leave the proof to the reader. As we will see in Theorem \ref{thm:1.4} the odd subcomplex is of interest in particular for its relation to \( \Pi_0(K, \sigma_0) \) of a nice simplicial complex \( K \). A projectivity around an odd co-dimension 2-face \( f \) is a projectivity along a facet path \( \gamma \l_\gamma^- \), where \( l \) is a closed facet path in \( \delta K(f) \) based at some facet \( \sigma \in \delta K(f) \), and \( \gamma \) is a facet path from \( \sigma_0 \) to \( \sigma \). The path \( \gamma \l_\gamma^- \) is null-homotopic since \( K \) is locally strongly simply connected.

**Theorem 1.4** (Izvestiev & Joswig \cite{Izvestiev2003} Theorem 3.2.2). The reduced group of projectivities \( \Pi_0(K, \sigma_0) \) of a nice simplicial complex \( K \) is generated by projectivities around the odd co-dimension 2-faces. In particular, \( \Pi_0(K, \sigma_0) \) is generated by transpositions.

Consider a geometric realization \([K]\) of \( K \). To a given facet path \( \gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k) \) in \( K \) we associate a \( (\text{piecewise linear}) \) path \( \overline{\gamma} \) in \([K]\) by connecting the barycenter of \( \sigma_i \) to the barycenters of \( \sigma_i \cap \sigma_{i-1} \) and \( \sigma_i \cap \sigma_{i+1} \) by a straight line for \( 1 \leq i < k \), and connecting the barycenters of \( \sigma_0 \) and \( \sigma_0 \cap \sigma_1 \), respectively \( \sigma_k \) and \( \sigma_k \cap \sigma_{k-1} \). The fundamental group \( \pi_1([K] \setminus [K_{\text{odd}}], y_0) \) of a nice simplicial complex \( K \) is generated by paths \( \overline{\gamma} \), where \( \gamma \) is a closed facet path based at \( \sigma_0 \), and \( y_0 \) is the barycenter of \( \sigma_0 \); see \cite{Izvestiev2003} Proposition A.2.1. Furthermore, due to Theorem \ref{thm:1.4} we have the group homomorphism

\begin{equation}
\hat{h}_K : \pi_1([K] \setminus [K_{\text{odd}}], y_0) \to \Pi_0(K, \sigma_0) : [\overline{\gamma}] \mapsto \langle \gamma \rangle,
\end{equation}

where \([\overline{\gamma}]\) is the homotopy class of the path \( \overline{\gamma} \) corresponding to a facet path \( \gamma \).

**The partial unfolding.** Let \( K \) be a pure simplicial \( d \)-complex and set \( \Sigma \) as the set of all pairs \( ([\sigma], v) \), where \( \sigma \in K \) is a facet and \( v \in \sigma \) is a vertex. Thus each pair \( ([\sigma], v) \in \Sigma \) is a copy of the geometric simplex \( [\sigma] \) labeled by one of its vertices. For neighboring facets \( \sigma \) and \( \tau \) of \( K \) we define the equivalence relation \( \sim \) by attaching \( ([\sigma], v) \in \Sigma \) and \( ([\tau], w) \in \Sigma \) along their common ridge \( [\sigma \cap \tau] \) if \( \langle \sigma, \tau \rangle(v) = w \) holds. Now the partial unfolding \( \hat{K} \) is defined as the quotient space \( \hat{K} = \Sigma / \sim \). The projection \( p : \hat{K} \to K \) is given by the factorization of the map \( \Sigma \to K : ([\sigma], v) \mapsto \sigma \); see Figure \ref{fig:2}. The partial unfolding of a strongly connected simplicial complex is not strongly connected in general. We denote by \( \hat{K}_{([\sigma], v)} \) the connected component containing the labeled facet \(([\sigma], v) \).

Clearly, \( \hat{K}_{([\sigma], v)} = \hat{K}_{([\tau], w)} \) holds if and only if there is a facet path \( \gamma \) from \( \sigma \) to \( \tau \) in \( K \) with \( \langle \gamma \rangle(v) = w \). It follows that the connected components of \( \hat{K} \) correspond to the orbits of the action of \( \Pi(K, \sigma_0) \) on \( V(\sigma_0) \). Note that each connected component of the partial unfolding is strongly
Figure 2. The starred triangle and its partial unfolding. The complex on the right is the non-trivial connected component of the partial unfolding, indicated by the labeling of the facets by the vertices $v_1$, $v_2$, and $v_3$. The second connected component is a copy of the starred triangle with all facets labeled $v_0$; see also Example 1.1 for $k = 2$.

connected and locally strongly connected \cite{35, Satz 3.2.2}. Therefore we do not distinguish between connected and strongly connected components of the partial unfolding.

The problem that the partial unfolding $\tilde{K}$ may not be a simplicial complex can be addressed in several ways. Izmestiev & Joswig \cite{18} suggest barycentric subdivision of $\tilde{K}$, or anti-prismatic subdivision of $K$. A more efficient solution (with respect to the size of the resulting triangulations) is given in \cite{35}.

1.4. The partial unfolding as a branched cover. As preliminaries to this section we state two theorems by Fox \cite{8} and Izmestiev & Joswig \cite{18}. Together they imply that under the “usual connectivity assumptions” the partial unfolding of a simplicial complexes is indeed a branched cover as suggested in the heading of this section. For simplicial complexes the analog of these topological connectivity requirements are nice simplicial complexes.

**Theorem 1.5** (Izmestiev & Joswig \cite{18, Theorem 3.3.2}). Let $K$ be a nice simplicial complex. Then the restriction of $p : \tilde{K} \to K$ to the preimage of the complement of the odd subcomplex is a covering.

**Theorem 1.6** (Fox \cite[p. 251]{8}; Izmestiev & Joswig \cite{18, Proposition 4.1.2}). Let $J$ and $K$ be nice simplicial complexes and let $f : J \to K$ be a simplicial map. Then the map $f$ is a simplicial branched cover if and only if

$$\text{codim } K_{\text{sing}} \geq 2.$$ 

Since the partial unfolding of a nice simplicial complex is nice Corollary 1.7 follows immediately.

**Corollary 1.7.** Let $K$ be a nice simplicial complex. The projection $p : \tilde{K} \to K$ is a branched cover with the odd subcomplex $K_{\text{odd}}$ as its branching set.

For the rest of this section let $K$ be a nice simplicial complex and let $y_0$ be the barycenter of $|\sigma_0|$. The projection $p : \tilde{K} \to K$ is a branched cover with $K_{\text{sing}} = K_{\text{odd}}$ by Corollary 1.7 and Izmestiev & Joswig \cite{18} proved that there is a bijection $\iota : p^{-1}(y_0) \to V(\sigma_0)$ that induces a group isomorphism $\iota_* : \text{Sym}(p^{-1}(y_0)) \to \text{Sym}(V(\sigma_0))$ such that the following Diagram (2) commutes.

$$\pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \xrightarrow{m_p} M_p \xrightarrow{\iota_*|m_p} \Pi(S, \sigma_0) \xrightarrow{\Pi} \Pi(S, \sigma_0)$$

In the case that the action of $\Pi(K, \sigma_0)$ on $V(\sigma_0)$ has only one non-trivial orbit we refer to the unique non-trivial connected component of $\tilde{K}$ corresponding to the non-trivial orbit as the
Consider a nice simplicial complex $K$, and a branched cover $r : X \to Z$. Assume that there is a homomorphism of pairs $\varphi : (Z, Z_{\text{sing}}) \to (|K|, |K_{\text{odd}}|)$, that is, $\varphi : Z \to |K|$ is a homomorphism with $\varphi(Z_{\text{sing}}) = |K_{\text{odd}}|$. Then Theorem 1.8 gives sufficient conditions for $p : \hat{K} \to K$ and $r : X \to Z$ to be equivalent branched covers. It is the key tool in the proof of the main Theorem 3.11 in Section 3.

Theorem 1.8. Let $K$ be a nice simplicial complex, and let $r : X \to Z$ be a simple branched cover. Further assume that there is a homomorphism of pairs $\varphi : (Z, Z_{\text{sing}}) \to (|K|, |K_{\text{odd}}|)$, and let $z_0 \in Z$ be a point such that $y_0 = \varphi(z_0)$ is the barycenter of $|\sigma_0|$ for some facet $\sigma_0 \in K$. The branched covers $p : \hat{K} \to K$ and $r : X \to Z$ are equivalent if there is a bijection $\iota : r^{-1}(z_0) \to V(\sigma_0)$ that induces a group isomorphism $\iota_* : M_r \to \Pi(K, \sigma_0)$ such that the diagram

$$
\begin{array}{ccc}
\pi_1(Z \setminus Z_{\text{sing}}, z_0) & \xrightarrow{\psi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \\
m_r & & m_p \\
\Pi(K, \sigma_0) & \xrightarrow{\iota_*} & \Pi(K, \sigma_0)
\end{array}
$$

commutes. In particular, we have $\hat{K} \cong X$.

Proof. Corollary 1.7 ensures that $p : \hat{K} \to K$ is indeed a branched cover, and commutativity of Diagram 2 and Diagram 3 proves commutativity of their composition:

$$
\begin{array}{ccc}
\pi_1(Z \setminus Z_{\text{sing}}, z_0) & \xrightarrow{\psi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \\
m_r & & m_p \\
\Pi(K, \sigma_0) & \xrightarrow{\iota_*^{-1} \circ \iota_*} & \Pi(K, \sigma_0)
\end{array}
$$

Theorem 1.2 completes the proof. \qed

2. COLOR EQUIVALENCE OF SIMPLICIAL COMPLEXES

Consider two nice simplicial complexes $K$ and $K'$. The partial unfoldings of two homeomorphic simplicial complexes need not to be homeomorphic in general. Here we present sufficient criteria for $p : \hat{K} \to K$ and $p' : \hat{K'} \to K'$ to be equivalent branched covers. Assume $K \cong K'$ and that the odd subcomplexes $K'_{\text{odd}}$ and $K'_{\text{odd}}$ are equivalent, that is, there is a homomorphism of pairs $\varphi : (|K|, |K_{\text{odd}}|) \to (|K'|, |K'_{\text{odd}}|)$. Let $\sigma_0 \in K$ be a facet, and $y_0$ the barycenter of $\sigma_0$, and assume that the image $y'_0 = \varphi(y_0)$ is the barycenter of $|\sigma_0'|$ for some facet $\sigma_0' \in K'$. Now $K$ and $K'$ are color equivalent if there is a bijection $\psi : V(\sigma_0) \to V(\sigma_0')$, such that

$$
\psi_* \circ h_K = h_{K'} \circ \varphi_*
$$

holds, where the maps $\varphi_* : \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \to \pi_1(|K'| \setminus |K'_{\text{odd}}|, y'_0)$ and $\psi_* : \text{Sym}(V(\sigma_0)) \to \text{Sym}(V(\sigma_0'))$ are the group isomorphisms induced by $\varphi$ and $\psi$, respectively. Observe that this is indeed an equivalence relation. The name “color equivalent” suggests that the pairs $(K, K_{\text{odd}})$ and $(K', K'_{\text{odd}})$ are equivalent, and that the “colorings” of $K_{\text{odd}}$ and $K'_{\text{odd}}$ by the $\Pi(K)$-action, respectively $\Pi(K')$-action, of projectivities around odd faces are equivalent. Proposition 2.1 justifies this name.

Proposition 2.1. Let $K$ and $K'$ be color equivalent simplicial complexes. Then the branched covers $p : \hat{K} \to K$ and $p' : \hat{K'} \to K'$ are equivalent.
Proof. With the notation of Equation (4) we have that
\[
\pi_1([K \setminus |K_{\text{odd}}|], y_0) \xrightarrow{\varphi_*} \pi_1([K' \setminus |K'_{\text{odd}}|], y'_0)
\]
commutes, since the Diagram (2) commutes and Equation (4) holds. Theorem 1.2 completes the proof.

The anti-prismatic subdivision. Let \(c_k\) be the simplicial complex obtained from the boundary complex of the \((k+1)\)-dimensional cross polytope by removing one facet. Alternatively, define \(c_k\) as the simplicial complex arising from the Schlegel diagram of the \((k+1)\)-dimensional cross polytope. To be more explicit, let \(\sigma = \{+v_i\}_{0 \leq i \leq k}\) be the vertices of the \(k\)-simplex. Then the facets of \(c_k\) are defined as all subsets \(\sigma' \neq \sigma\) of \(\{\pm v_i\}_{0 \leq i \leq k}\) such that either \(+v_i \in \sigma'\) or \(-v_i \in \sigma'\) holds. The complex \(c_k\) and the \(k\)-simplex are PL-homeomorphic with isomorphic boundaries, and \(c_k\) is \((k+1)\)-colorable by assigning the same color to \(+v_i\) and \(-v_i\), since \(\{+v_i, -v_i\}\) is not an edge. The anti-prismatic subdivision \(a_f(K)\) of a \(k\)-face \(f\) of a simplicial \(d\)-complex \(K\) is obtained from \(K\) by replacing \(\text{st}_K(f)\) by the join of \(c_k\) with \(\text{lk}_K(f)\), that is
\[
a_f(K) = (K \setminus \text{st}_K(f)) \cup (c_k * \text{lk}_K(f)).
\]

See Figure 3 for an example of an anti-prismatic subdivision of an edge and a triangle of a foldable simplicial complex.

![Figure 3](image)

**Figure 3.** Anti-prismatic subdivision of the edge \(e\) and the triangle \(\sigma\) of a foldable simplicial complex.

The anti-prismatic subdivision \(a(K)\) of a simplicial complex \(K\) is defined by recursively anti-primitively subdividing all faces of \(K\) from the facets down to the edges. Observe that \(a_f(K)\), and hence \(a(K)\), are PL-homeomorphic to \(K\), and that \(a_f(K)\) and \(a(K)\) inherit niceness from \(K\). For sake of brevity we omit the (lengthy but straight forward) proof of Proposition 2.2. The result is similar to Proposition A.1.3 and following in [18], and explicit proofs can be found in [36] Section 1.3.1.

**Proposition 2.2.** Let \(K\) be a nice simplicial complex and \(f \in K\) a face. The simplicial complexes \(a_f(K), a(K)\) and \(K\) are color equivalent.

2.1. **Prescribing the odd subcomplex.** Theorem 1.8 made it clear, that it is essential to control the odd subcomplex if one tries to determine the behavior of the partial unfolding as a branched cover, e.g. in the construction of combinatorial 4-manifolds in Section 3.

Let \(K\) be a strongly connected and foldable simplicial complex of dimension \(d\), and fix a \((d+1)\)-coloring using the colors \([d+1] = \{0, 1, \ldots, d\}\). Then the \(\{i_0, i_1, \ldots, i_k\}\)-skeleton is the subcomplex of \(K\) induced by the vertices colored \(\{i_0, i_1, \ldots, i_k\}\). Observe that the \(\{i_0, i_1, \ldots, i_k\}\)-skeleton is a pure simplicial complex of dimension \(k\).
Proposition 2.3. Let $K$ be a foldable combinatorial manifold of dimension $d$ and let $F$ be a co-dimension 1-manifold (possibly with more than one connected component) embedded in the $\{i_0, i_1, \ldots, i_{d-1}\}$-skeleton of $K$. Further assume that all facets (and their proper faces) of the boundary $\partial F$ of $F$ not contained entirely in $\partial K$, for short the closure $\text{cl}(\partial F \setminus \partial K)$, are embedded in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton. Then $\text{cl}(\partial F \setminus \partial K)$ can be realized as the odd subcomplex of some simplicial complex $K'$, that arises from $K$ by stellar subdivision of edges in the $\{i_{d-1}, i_d\}$-skeleton. The complex $K'$ is $(d+2)$-colorable by extending the coloring of $K$, and the odd subcomplex lies in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton.

Proof. Every $(d-1)$-simplex in $F$ has exactly one $i_{d-1}$-colored vertex since $F$ is foldable. Hence the vertex stars of all $i_{d-1}$-colored vertices cover $F$, that is,

$$ F = \bigcup_{v \text{ is } i_{d-1}\text{-colored}} \text{st}_F(v), \quad (5) $$

and the vertex stars intersect in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton. Further, a $(d-2)$-face $g \in F$ (a ridge in $F$) is contained in an odd number of vertex stars of $i_{d-1}$-colored vertices of $F$ if and only if $g \in \partial F$ since $F$ is an embedded combinatorial manifold. Observe that stellar subdivision of an edge $e$ changes the parity of $\text{lk}_K(g)$ of each co-dimension 2-face $g \in \text{lk}_K(e) \setminus \partial K$. The odd subcomplex resulting from a series of stellar subdivisions of edges is the symmetric difference of the edge links.

Since $K$ is a combinatorial manifold the vertex star $\text{st}_K(v)$ of an $i_{d-1}$-colored vertex $v \in F$ is a $d$-ball, which is the join of $v$ with an $(i_0, i_1, \ldots, i_{d-2}, i_d)$-colored $(d-1)$-ball if $v \in \partial K$, and which is the join with an $(i_0, i_1, \ldots, i_{d-2}, i_d)$-colored $(d-1)$-sphere otherwise. The vertex star $\text{st}_F(v)$ divides $\text{st}_K(v)$ into two connected components, and we will call these two connected components of $|\text{st}_K(v)| \setminus |\text{st}_F(v)|$ the two sides of $\text{st}_F(v)$, mimicking the topological concept of a two-sided manifold (embedded in an orientable space); see Figure 4 for a 3-dimensional example. The link $\text{lk}_K(\{v, w\})$ of an $\{i_{d-1}, i_d\}$-colored edge $\{v, w\} \in \text{st}_K(v)$ is a $(d-2)$-sphere in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton of $\partial \text{st}_K(v)$. Moreover, the vertex stars of all $\{i_{d-1}, i_d\}$-colored edges $\{v, w\} \in \text{st}_K(v)$ cover $\text{lk}_K(v)$. Thus if we stellar subdivide all $\{i_{d-1}, i_d\}$-edges in one side of $\text{st}_F(v)$ we obtain $\text{lk}_F(v)$ as the odd subcomplex.

Finally we construct the desired odd subcomplex $\text{cl}(\partial F \setminus \partial K)$ as the symmetric difference of vertex links of all $i_{d-1}$-colored vertices in $F$.

The resulting complex $K'$ is $(d+2)$-colorable by assigning a new color to the vertices introduced by stellar subdivision of edges. If an edge $e$ is subdivided twice, use the original colors of $e$ to color the two new vertices. \qed
Remark 2.4. Observe that a projectivity based at \( \sigma_0 \) around an odd co-dimension 2-face constructed via Proposition 2.3 exchanges the two vertices of \( \sigma_0 \) colored \( i_{d-1} \) and \( i_d \).

We conclude this section with a characterization of some co-dimension 2-manifolds which by Proposition 2.3 can be realized as an odd subcomplex in the \( \{i_0, i_1, \ldots, i_{d-2}\} \)-skeleton.

Lemma 2.5. Let \( |K| \) be some geometric realization of a foldable combinatorial d-manifold \( K \). An orientable PL \((d-1)\)-manifold \( F \) may be embedded in the \( \{i_0, i_1, \ldots, i_{d-2}, d-1\} \)-skeleton of \((a refinement of) \ K \) with boundary \( \partial F \) embedded in the \( \{i_0, i_1, \ldots, i_{d-2}\} \)-skeleton if there is an embedding \( F \rightarrow |K| \). Note that the last color in the coloring of the embedding of \( F \) is explicitly required to be \( d-1 \).

Proof. Simplicial approximation of the embedding \( F \rightarrow |K| \) yields an embedding of \( F \) in the co-dimension 1-skeleton of some refinement \( K' \) of \( K \). Let \( b(K') \) be the barycentric subdivision of \( K' \) with each vertex colored by the dimension of its originating face. The embedding \( F \rightarrow K' \) yields an embedding \( i: F \rightarrow b(K') \) of \( F \) in the \( \{0, 1, \ldots, d-1\} \)-skeleton of \( b(K') \), with \( \partial F \) embedded in the \( \{0, 1, \ldots, d-2\} \)-skeleton. Further we have that the vertex stars of all \((d-1)\)-colored vertices cover \( i(F) \); see Equation (5).

It remains to show, how to “push” \( F \) into the desired skeleton. The \( \{0, 1, \ldots, d-2, d-1\} \)-skeleton of \( b(K') \) differs from the \( \{i_0, i_1, \ldots, i_{d-2}, d-1\} \)-skeleton by one color \( c = \{0, 1, \ldots, d-2\} \setminus \{i_0, i_1, \ldots, i_{d-2}\} \), that is, replacing \( c \) by \( d \) in \( \{0, 1, \ldots, d-2, d-1\} \) yields \( \{i_0, i_1, \ldots, i_{d-2}, d-1\} \). For each \((d-1)\)-colored vertex \( v \in i(F) \) choose one of the two sides of \( st_{i(F)}(v) \) consistent with the orientation of \( F \). This may be done since \( F \) is orientable. Let \( v \in i(F) \) be \((d-1)\)-colored, let \( D_v \) be the chosen side of \( st_{i(F)}(v) \), and let \( V_c \) denote the set of all \( c \)-colored vertices in \( lk_{i(F)}(v) \); see Figure 4. Now we obtain the desired embedding \( i': F \rightarrow b(K') \) by replacing \( st_{i(F)}(v) \) with

\[
\bigcup_{w \in V_c} v \ast ((lk_{b(K')}(\{v, w\}) \cap D_v) \cong \mathbb{D}^{d-1}.
\]

Here it is important that the triangulation of \( b(K') \) may have to be refined further by antiprismatic subdivision. The map \( i': F \rightarrow b(K') \) is an embedding of \( F \) since we replace \((d-1)\)-balls by \((d-1)\)-balls, and two \((d-1)\)-balls in \( i'(F) \) intersect as in \( i(F) \) due to the consistent choice of the sides of \( st_{i(F)}(v) \). \( \square \)

2.2. Extending triangulations. We present a technique how to extend a partial triangulation of some topological space to the entire space (e.g. a triangulation of \( S^{d-1} \) to \( \mathbb{D}^d \)), while considering certain restraints on the colorability of the triangulations. This technique is crucial in the constructions in Section 3.

A first assault on how to extend triangulation and coloring is by Goodman & Onishi [11], who proved that a 4-colorable triangulation of \( S^2 \) may be extended to a 4-colorable triangulation of \( \mathbb{D}^3 \). Their result was improved independently by Izmestiev [17] and [36, 35] to arbitrary dimensions. In [36, Theorem 2.3] further restrictions (e.g. regularity) are taken into account.

Lemma 2.6. Let \( S \) be a \( k \)-colored combinatorial \((d-1)\)-sphere. Then there exists a combinatorial \( d \)-ball \( B \) with boundary \( \partial B \) equal to \( S \) such that the \( k \)-coloring of \( S \) may be extended to a max\{\( k, d+1 \)\}-coloring of \( B \).

We sketch the construction in [36]. Set \( B_0 = a \ast S \) as the cone over \( S \) with apex \( a \). We have \( k \geq d \) and in the case \( k = d \) set \( B = B_0 \) and color \( a \) with a new color. Otherwise fix an ordering \( c_0 < c_1 < \cdots < c_d \) of the first \( d+1 \) colors in the coloring of \( S \) and color \( a \) by \( c_0 \). For \( 1 \leq i \leq d \) we obtain \( B_i \) from \( B_{i-1} \) by stellarly subdividing all edges with both vertices colored \( c_{i-1} \). The new vertices are colored \( c_i \). Theorem 2.3 in [36] ensures that \( B = B_d \) is properly colored; see Figure 5.

Let \( X = X^d \) be a finite CW-complex of dimension \( d \) with \( l \)-cells \( \{e^l_\alpha\}_\alpha \), closed cells \( \{C^l_\alpha\}_\alpha = \{cl(e^l_\alpha)\}_\alpha \), attaching maps \( \varphi_\alpha: \partial C^l_\alpha \rightarrow X^{l-1} \), and \( l \)-skeleton

\[
X^l = X^{l-1} \cup (\bigcup_\alpha e^l_\alpha).
\]
A finite CW-complex is regular if the attaching maps \( \varphi_\alpha : \partial C_\alpha^l \to X^{l-1} \) (restricted to their image) are homeomorphisms for all \( 1 \leq l \leq d \); see Hatcher [12, p. 5]. We call a simplicial complex \( K \cong X \) a triangulation of \( X \), if \( K \) refines the cell structure of \( X \), that is, the \((d-1)\)-skeleton of \( K \) is a triangulation of the CW-complex \( X^{d-1} \).

A subset \( Y \subset \{ e_\alpha \} \) is a subcomplex if for each closed cell \( C_\alpha^l \in Y \) all cells in the image of \( \varphi_\alpha : \partial C_\alpha^l \to X^{l-1} \) are also in \( Y \). Hence \( Y \) is also a CW-complex, and \( Y \) is regular if \( X \) is regular. For example, any \( l \)-skeleton \( X^l \) is a subcomplex of \( X \). We call a triangulation of a subcomplex \( Y \subset X \) a partial triangulation of \( X \).

**Proposition 2.7.** Let \( X \) be a finite regular CW-complex of dimension at most 4, and let \( Y \subset X \) be a subcomplex. Then any triangulation and \( k \)-coloring of \( Y^l \) can be extended to a triangulation and \( \max\{k, l+1\} \)-coloring of \( X^l \).

**Proof.** We prove by induction on \( 0 \leq i \leq l \) that there exists a triangulation of the \( i \)-skeleton \( X^i \) which can be colored with \( \max\{k, i+1\} \) colors such that the triangulation and coloring of \( X^i \) extends the triangulation and coloring of \( Y^i \). This clearly holds for \( i = 0 \), and for \( i = l \) we obtain Proposition 2.7.

Let \( i \geq 1 \) and let \( e_\alpha^i \) be an \( i \)-cell of \( X^i \) not contained in \( Y^i \). By induction \( X^{i-1} \) is triangulated and colored using \( \max\{k, i+1\} \) colors, and the triangulation of \( X^{i-1} \) extends triangulation and coloring of \( Y^{i-1} \). Since \( X \) is regular, the image of the attaching map \( \varphi_\alpha : \partial C_\alpha^i \to X^{i-1} \) is a \((i-1)\)-sphere with a triangulation induced by the triangulation of \( X^{i-1} \). Since \( i \leq d \) is at most 4, every simplicial \((i-1)\)-sphere is a combinatorial \((i-1)\)-sphere. Now Lemma 2.6 extends triangulation and coloring of \( \varphi_\alpha(\partial C_\alpha^i) \) to the entire \( i \)-ball \( C_\alpha^i \). Since the \( i \)-balls \( \{C_\alpha^i\} \) intersect pairwise only in \( X^{i-1} \), extending the triangulation of \( \partial C_\alpha^i \) to its interior for each \( i \)-cell \( e_\alpha^i \) yields the desired triangulation of \( X^i \).

**Remark 2.8.** A similar result to Propositions 2.7 for a partial triangulation \( R \) of a relative handlebody decomposition \(|R| = N^{-1} \subset N^0 \subset \cdots \subset N^4 = N \) of the pair \((|R|, N)\) can be found in [39, Proposition 2.12].

**Remark 2.9.** Proposition 2.7 is not applicable in higher dimensions. For example, let \( H \) be a triangulation of the Poincaré homology sphere; see Björner & Lutz [3, 4, 35]. The double suspension \( \text{susp}^2(H) \) is homeomorphic to \( S^3 \) [21, 7], yet not a combinatorial sphere: There are two vertices with \( \text{susp}(H) \not\approx S^4 \) as vertex links. Consider the cell decomposition of the 6-ball given by the triangulation \( \text{susp}^2(H) \) of \( S^3 \) plus an additional 6-cell. Now, when attaching the final 6-cell, one can not apply the inductive argument for the two vertices with \( \text{susp}(H) \) as vertex links.

3. **Constructing Combinatorial 4-Manifolds**

The main result, the construction of a simplicial 4-sphere \( S \) such that the partial unfolding \( \hat{S} \) is PL-homeomorphic to a given closed oriented PL 4-manifold \( M \), is developed in Section 3.2. Prior to giving a combinatorial construction of \( M \), we will review the topological situation.

Let \( W^3 \) be a 3-manifold. Following Montesinos [27], we call two given branched covers \( p_1, p_2 : W^3 \to S^3 \) branched over links \( L_1 \) and \( L_2 \), respectively, cobordant if there exists a branched cover...
$p : W^3 \times [0,1] \to S^3 \times [0,1]$ which is equal to $p_1$ if restricted to $W^3 \times \{0\}$, and equal to $p_1$ if restricted to $W^3 \times \{1\}$, and is branched over an immersed PL $2$-manifold with a boundary equal to the disjoint union $L_1 \cup L_2$. The branched cover $p$ is called a cobordism.

A (surprisingly) useful technique is to attach a trivial sheet. Given a $k$-fold branched cover $p : X \to D^4$, (with sheets numbered $0,1,\ldots,k-1$) we want to add another sheet to the covering without changing the topology of the covering space $X$. To this end add a $2$-ball $D$ to the branching set of $p$ such that $\partial D$ is contained in $\partial D^4$, and let a meridional loop around $D$ correspond to the transposition $(1,k)$. The covering space $X'$ of the branched cover obtained this way is the union of $X$ and a $4$-ball attached to $\partial X$ along a $3$-ball, thus $X' \cong X$ holds.

### 3.1. 4-Manifolds as branched covers

Every closed oriented PL $4$-manifold admits a handle representation

$$M = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4;$$

see [36, Example 2.2]. With $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$, and $M_B = H^0 \cup \gamma H^1$ by duality, we obtain $M$ as the union $M_A \cup h M_B$, where $h$ is the attaching map. That is, we paste $M_A$ and $M_B$ together along their common boundary $\gamma \# (S^1 \times S^2)$, the connected sum of $\gamma$ copies of $S^1 \times S^2$. In fact, Montesinos [26] proved that $H^0 \cup \lambda H^1 \cup \mu H^2$ alone topologically determines $M$.

The construction of the branched cover $r : M \to S^4$ proceeds in two steps. In the first step (see Montesinos [25]) the $4$-manifolds $M_A$ and $M_B$ are constructed as simple $3$-fold branched covers $r_A$ and $r_B$ of the $4$-ball $D^4$. Since $M_B = H^0 \cup \gamma H^1$ is of the form $H^0 \cup \lambda H^1 \cup \mu H^2$ it suffices to show how to construct $r_A : M_A \to D^4$.

Although $\partial M_A = \partial M_B$ holds, the branching sets of $r_A$ and $r_B$ restricted to the common boundary $\gamma \# (S^1 \times S^2)$ of $M_A$ and $M_B$ may not be equivalent, and $M_A \cup h M_B$ is not the covering space of a branched cover $r : M \to S^4$ in general. Hence in the second step we construct a cobordism between $r_A |_{\partial M_A}$ and $r_B |_{\partial M_B}$, that is, a branched cover $r_H : H \to S^3 \times [0,1]$ with covering space $H \cong (\gamma \# (S^1 \times S^2)) \times [0,1]$ which satisfies

$$(6) \quad r_H (\gamma \# (S^1 \times S^2)) \times \{0\} = r_A |_{\partial M_A} \quad \text{and} \quad r_H (\gamma \# (S^1 \times S^2)) \times \{1\} = r_B |_{\partial M_B}.$$

The cobordism $r_H$ is branched over a PL $2$-manifold with a finite number of singularities and boundary equal to the disjoint union of the branching sets of $r_A |_{\partial M_A}$ and $r_B |_{\partial M_B}$. The boundary of the covering space $H$ is homeomorphic to two disjoint copies of $\gamma \# (S^1 \times S^2)$, and we have $M \cong M_A \cup h H \cup h M_B$. Note that $r_H$ is a $4$-fold cover in general, thus we must add a fourth, trivial sheet to $r_A$, respectively $r_B$.

The existence of such a cobordism, and hence the representation of $M$ as a branched cover of $S^4$, was first observed by Piergallini [32]. The following diagram illustrates this approach.

\[
\begin{array}{cccccc}
M & \cong & M_A & \cup \text{Id} & H & \cup \text{Id} & M_B \\
r & \downarrow & r_A & \cup \text{Id} & r_H & \cup \text{Id} & r_B \\
S^4 & \cong & D^4 & \cup \text{Id} & S^3 \times [0,1] & \cup \text{Id} & D^4
\end{array}
\]

**Construction of $M_A$.** In the following we will sketch a construction of $r_A : M_A \to D^4$ as a $3$-fold branched cover branched over a ribbon manifold. This construction is due to Montesinos, and we omit the proofs; refer to [25] for further details.

First, consider the $4$-manifold $W = H^0 \cup \lambda H^1$ which consists of a single $4$-ball and $1$-handles only. It arises as the standard branched cover $r_W : W \to D^4$ branched along $\lambda + 2$ unlinked and unknotted copies of $D^2$. Let $u : \mathbb{R}^4 \to \mathbb{R}^4$ be the reflection on the hyperplane given by $x_1 = 0$, that is, $u$ maps $(x_1, x_2, x_3, x_4)$ to $(-x_1, x_2, x_3, x_4)$. The covering space $W$ is obtained from $[-1,1]^3 \times [-1,2]$ by the following identifications on its boundary. Consider the subset $A$ of $[-1,1]^3$ consisting of $2\lambda$ disjoint rectangles given by

\[
A = \bigcup_{i=1}^{\lambda} \left\{ (x_1, x_2, x_3) \in [-1,1]^3 \left| x_3 = 1 \text{ and } x_1 \in \left[ \frac{2i-1}{2\lambda+1}, \frac{2i}{2\lambda+1} \right] \right. \right\}.
\]
The single unknotted 2-ball \( Q \) and they are given by the equivalence relation given by the identifications described above. Figure 6 illustrates the equivalence relation given by these identifications, we have \( ([0, 1]) \times ([0, 1]) / \mathcal{R}' \) is the equivalence relation given by the identifications described above. Figure 6 illustrates the 3-dimensional case.

Now we are ready to define the covering map \( r_W \). For simplicity we identify \( W \) and \( \mathbb{D}^4 \) with \( ([−1, 1]^3 \times [−1, 2]) / \mathcal{R} \), respectively \( ([−1, 1]^3 \times [0, 1]) / \mathcal{R}' \), and let \( [x] \) denote the equivalence class of \( x \) in the quotient spaces \( W \) and \( \mathbb{D}^4 \), respectively.

\[
r_W : W \to \mathbb{D}^4 : [(x_1, x_2, x_3, x_4)] \mapsto \begin{cases} \{(-x_1, x_2, x_3, -x_4)\} & \text{if } -1 \leq x_4 \leq 0 \\ \{(-x_1, x_2, x_3, x_4)\} & \text{if } 0 < x_4 \leq 1 \\ \{(-x_1, x_2, x_3, 2 - x_4)\} & \text{if } 1 < x_4 \leq 2 \end{cases}
\]

The covering map \( r_W \) is well defined since it is compatible with \( \mathcal{R} \) and \( \mathcal{R}' \), and \( r_W \) is a branched cover. Note that the third sheet \( ([−1, 1]^3 \times [1, 2]) / \mathcal{R} \) is homeomorphic to \( \mathbb{D}^4 \) and does not contribute to the construction of the 1-handles; it is trivial so far. Yet it will be needed in the process of attaching the 2-handles.

We will distinguish the connected components of the branching set of \( r_W \) as follows. The branching set consists of \( \lambda + 1 \) pairwise disjoint unknotted 2-balls \( \{P_i\}_{0 \leq i \leq \lambda} \), and a single unknotted 2-ball \( Q \) disjoint to any of the \( P_i \). We denote the \( \lambda + 1 \) disjoint unknotted 2-balls by \( \mathcal{P} \), and they are given by

\[
\mathcal{P} = P_0 \cup P_1 \cup \cdots \cup P_\lambda = (\{0\} \times [−1, 1]^2 \times \{0\}) \cup (\mathcal{A} / \mathcal{R}' \times \{0\}).
\]

The single unknotted 2-ball \( Q \) is given by

\[
Q = \{0\} \times [−1, 1]^2 \times \{1\}.
\]
The 2-balls $\mathcal{P} \cup Q$ intersect the boundary of $\mathbb{D}^4$ in a system of $\lambda + 2$ unknotted and unlinked 1-spheres.

The preimage of a meridial loop $m \subset \mathbb{D}^4 \setminus (\mathcal{P} \cup Q)$ passing around any $P \in \mathcal{P}$ lies in the first and second sheet of $r_W$, that is, $r_W^{-1}(m)$ is contained in $([-1, 1]^3 \times [-1, 1]) / \mathcal{R}$. On the other hand, if a meridial loop $m' \subset \mathbb{D}^4 \setminus (\mathcal{P} \cup Q)$ passes around $Q$ we have $r_W^{-1}(m') \subset ([-1, 1]^3 \times [0, 2]) / \mathcal{R}$, and the preimage of $m'$ lies in the second and third sheet of $r_W$. Therefore the monodromy group $\mathfrak{M}_{r_W}$ of $r_W$ is isomorphic to the symmetric group $\Sigma_3$ on three elements. In the following we label the sheets 0, 1, and 2, and we assume $m$ and $m'$ to correspond to the (generating) transpositions $(0, 1) \in \Sigma_3$ and $(0, 2) \in \Sigma_3$, respectively.

**Attaching 2-handles.** Note the (non-trivial) fact that the 2-handles $\{H^2_i\}_{1 \leq i \leq \mu}$ may be attached independently to $W = H^0 \cup \lambda H^1$. Hence we may assume $M_A = W \cup h H^2$ to be obtained from $W$ by adding a single 2-handle $H^2 \cong \mathbb{D}^4$. The 2-handle $H^2$ is attached to $W$ along a solid 3-torus $S^1 \times \mathbb{D}^2$. To be more precise, a solid 3-torus $S^1 \times \mathbb{D}^2 \subset \partial H^2$ is embedded into $\partial W$ via the attaching map $h : S^1 \times \mathbb{D}^2 \to \partial W$. The attaching map $h$ is determined by the image of the meridian $S^1 \times \{0\} \subset S^1 \times \mathbb{D}^2$, a knot $L$ in $\partial W$. Using isotopy the knot $L$ may be placed in $\partial W$ such that its image $A = r_W(L) \subset \partial \mathbb{D}^4$ under $r_W$ is an arc which intersects the branching set $\mathcal{P} \cup Q$ of $r_W$ as follows: The arc $A$ intersects the $\lambda + 1$ connected components $\mathcal{P}$ in its end points only and does not intersect $Q$ at all. Conversely, the preimage $r_W^{-1}(A)$ of $A$ is the knot $L$ and a disjoint arc $A'$. The restriction $r_W|_{A'}$ is a 2-fold branched cover of $A$, and $r_W|_{A'}$ is a homeomorphism corresponding to the third, trivial sheet of $r_W$.

In order to represent $M_A$ as a 3-fold branched cover $r_A : M_A \to \mathbb{D}^4$, we attach another 4-ball $D$ to $\mathbb{D}^4$ along the 3-dimensional neighborhood $r_W \circ h(S^1 \times \mathbb{D}^2)$ of $A \subset \partial \mathbb{D}^4$. This neighborhood of $A$ is homeomorphic to $\mathbb{D}^3$ if the domain $S^1 \times \mathbb{D}^2 \subset \partial H^2$ of $h$ is chosen sufficiently small, and the resulting base space remains homeomorphic to $\mathbb{D}^4$. The preimage of $D$ is a collection of three copies of $D$, two of which form the 2-handle $H^2$ attached to $W$ via $h$. The third copy $D' \cong \mathbb{D}^4$ is attached to $W$ along a 3-dimensional neighborhood of $A'$, that is, we attach $D'$ to $W$ along a 3-ball, and attaching $D'$ does not alter the homeomorphic type of $M_A$.

The resulting branching set of $r_A$ is the union of the branching set of $r_W$, the 2-balls $\mathcal{P} \cup Q$, and a 2-ball $\overline{A} \supset A$ attached to $\mathcal{P}$ along two arcs $a$ and $a'$ in the boundary of $\mathcal{P}$, a ribbon manifold; see Figure 7. The two arcs $a$ and $a'$ are neighborhoods in $\partial \mathcal{P}$ of the two endpoints $A \cap \mathcal{P}$ of $A$. Note that $r_A$ is a “proper” 3-fold branched cover (the third sheet is non-trivial), since although $\overline{A}$ does not intersect $Q$, it might “weave around” $Q$ (and in fact also around any $P \in \mathcal{P}$).
Figure 8. The moves $C^\pm$ and $N^\pm$.

Fix a set of meridional loops as generators of $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$, that is, choose one meridional loop around each of the 2-balls in $\mathcal{P}$, and one meridional loop around $Q$. Let $P, P' \in \mathcal{P}$ with $a \subset P$ and $a' \subset P'$, and let $\beta, \beta' \in \pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ be the generators corresponding to the meridional loops around $P$ and $P'$, respectively. Then adding the ribbon $\overline{A}$ to the branching set introduces a new relation to the fundamental group, that is, the group $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ differs from $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ by the relation

$$\beta \alpha = \beta',$$

where the element $\alpha \in \pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ corresponds to the way $\overline{A}$ weaves around $\mathcal{P} \cup Q$. We summarize the construction above by the following Theorem 3.1.

**Theorem 3.1** (Montesinos [25, Theorem 6]). Each 4-manifold $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$ is a simple 3-fold branched cover of the 4-ball, the branching set being a ribbon manifold.

**Construction of $H$.** The construction of the cobordism $r_H : H \to S^3 \times [0, 1]$ is rather straightforward once we have established its existence, which is provided by the Theorems 3.2 and 3.3. Note that the branched cover $r_H : H \to S^3 \times [0, 1]$ is already defined on the boundary of $H$ by the restrictions given in Equation (6): The boundary of $H$ is the disjoint union of two copies of the 3-manifold $\gamma \pitchfork (S^3 \times S^2)$, and the branching sets of the restrictions $r_A |_{\partial M_A}$ and $r_B |_{\partial M_B}$ are two links $L_A$ and $L_B$, respectively.

In general, any closed oriented 3-manifold $W^3$ arises as a simple 3-fold branched cover of $S^3$ branched over a link $L$, and the monodromy group $\mathfrak{M}$ of the branched cover is isomorphic to a subgroup of $\Sigma_3$ (generated by transpositions); see [13, 24]. After adding a fourth trivial sheet and thus a new generating transposition to $\mathfrak{M}$, $\mathfrak{M}$ becomes isomorphic to a subgroup of $\Sigma_4$. Consider a generic projection of $L$ to the plane with marked over and under crossings. Such a projection is called a diagram of $L$, and we call a strand which is not crossed by other strands of the diagram a bridge. Fix a set of meridional loops around the bridges of the diagram as generators of $\pi_1(S^3 \setminus L)$, and we identify the meridional loops around the bridges with transpositions in $\mathfrak{M}$ via the monodromy homomorphism $m : \pi_1(S^3 \setminus L) \to \mathfrak{M}$. Hence we can think of $L$ as a colored diagram: A bridge $b$ of the diagram is colored $(i, j)$ if the meridional loop around $b$ corresponds to the transposition $(i, j) \in \Sigma_4$. Further we define the moves $C^\pm$ and $N^\pm$ on a colored link as in Figure 8.

**Theorem 3.2** (Montesinos [27, p. 345]). Let $p_1, p_2 : W^3 \to S^3$ be 4-fold branched covers (coming from 3-fold covers by the addition of a trivial sheet) such that it is possible to pass from the branching set $L_1$ of $p_1$ to the branching set $L_2$ of $p_2$ by a sequence of moves $C^\pm$ and $N^\pm$. Then $p_1$...
and $p_2$ are cobordant and the branching set of the cobordism is an embedded PL 2-manifold with a cusp singularity (a cone over the trefoil) for each $C^\pm$-move and a node singularity (a cone over the Hopf link) for each $N^\pm$-move.

To understand the main idea of the proof it suffices to look at two branched covers $p_1, p_2 : W^3 \to S^3$ such that their branching sets $L_1$ and $L_2$ differ by exactly one $C^\pm$- or $N^\pm$-move $m$. Let $U \subset S^3$ be a closed neighborhood of the move $m$, that is, $L_1 \setminus U \subset S^3 \setminus U$ and $L_2 \setminus U \subset S^3 \setminus U$ are equivalent, and replacing $L_1 \cap U$ by $L_2 \cap U$ realizes the move $m$. The branching set in $(S^3 \setminus U) \times [0,1]$ is $(L_1 \setminus U) \times [0,1] \cong (L_2 \setminus U) \times [0,1]$. If $m$ is a $C^\pm$-move then the intersection of the branching set $(L_1 \setminus U) \times [0,1]$ with the boundary of $U \times [0,1]$ is the trefoil, otherwise the intersection is the Hopf link. In order to complete the base space of our cobordism, we replace $U \times [0,1]$ by a 4-ball with a cone over the trefoil or the Hopf link, respectively, as a branching set.

Theorem 3.2 together with the following Theorem 3.3 establish the existence of the cobordism $r_H$, and completes the construction of the branched cover $r : M \to S^4$. As observed by Montesinos [27], Theorem 3.4 then follows immediately.

**Theorem 3.3** (Piergallini [32, Theorem A]). Any two branching sets of 4-fold branched covers $p_1, p_2 : W^3 \to S^3$ obtained from 3-fold branched covers by adding a fourth, trivial sheet, which represent the same 3-manifold $W^3$, are related by a finite sequence of moves $C^\pm$ and $N^\pm$.

The proof extends over two papers by Piergallini. In [30] the number of different moves needed to relate any two such branching sets via a finite sequence of moves is brought down to four. Then in [32] each of these four moves is realized by a finite sequence of $C^\pm$- and $N^\pm$-moves, and the usage of a fourth, trivial sheet, thus establishing Theorem 3.3.

**Theorem 3.4.** Every closed oriented PL 4-manifold is a simple 4-fold branched cover of the 4-sphere branched over a PL-surface with a finite number of cusp and node singularities.

### 3.2. 4-Manifolds as Partial Unfoldings

Let $M$ be a closed oriented PL 4-manifold, and let $r : M \to S^4$ be the 4-fold branched cover with branching set $F$ described in Section 3.1. Hence $F$ is an immersed PL-surface with a finite number of cusp and node singularities by Theorem 3.3. In Theorem 3.11 we construct a triangulation $S$ of $S^4$ such that the branched cover given by the projection $p : \tilde{S} \to S$ is equivalent to $r : M \to S^4$. In particular, $\tilde{S}$ is PL-homeomorphic to $M$. Recall that we refer to the (unique non-trivial) connected component of the partial unfolding $\tilde{S}$ PL-homeomorphic to $M$ as the partial unfolding.

We outline the construction of $S$. The branched cover $r$ is characterized by $F$ and the monodromy isomorphism $m_r : \pi_1(S^4 \setminus F, y_0) \to \text{Sym}(r^{-1}(y_0))$, where $y_0$ is a point in $S^4 \setminus F$; see Section 1.2 and Theorem 1.2. Therefore we construct $S$ such that there is a homeomorphism of pairs $\varphi : (S^4, F) \to (|S|, |S_{\text{odd}}|)$ and $\varphi$ induces a group isomorphism $\varphi_* : \pi_1(S^4 \setminus F, y_0) \to \pi_1(|S| \setminus |S_{\text{odd}}|, \varphi(y_0))$. Further, assume that $\varphi(y_0)$ is the barycenter of some facet $\sigma_0 \subset S$. We construct $S$ such that the following Diagram (8) commutes for some bijection $\iota : r^{-1}(y_0) \to V(\sigma_0)$ and the induced group isomorphism $\iota_* : M_r \to \Pi(S, \sigma_0)$.

\[
\begin{array}{ccc}
\pi_1(S^4 \setminus F, y_0) & \xrightarrow{\varphi_*} & \pi_1(|S| \setminus |S_{\text{odd}}|, \varphi(y_0)) \\
\downarrow m_r & & \downarrow \iota_* \\
M_r & \xrightarrow{\iota_*} & \Pi(S, \sigma_0)
\end{array}
\]

This establishes Theorem 3.11 since the partial unfolding of a nice simplicial complex is a branched cover by Corollary 1.7 and since $\varphi(F) = |S_{\text{odd}}|$ and commutativity of Diagram (8) ensures that $p : \tilde{S} \to S$ and $r : M \to S^4$ are indeed equivalent by Theorem 1.3. The PL-properties follow once we proved $S$ to be a combinatorial manifold.

The construction of $S$ follows closely the construction of the branched cover $r : M \cong M_A \cup H \cup M_B \to \mathbb{D}^4 \cup (S^3 \times [0,1]) \cup \mathbb{D}^4$ reviewed in Section 3.1. First the combinatorial 4-balls $D_A$ and $D_B$ are constructed such that $\tilde{D}_A \cong M_A$ and $\tilde{D}_B \cong M_B$, respectively. The resulting complex $T_1$ is the disjoint union of $D_A$ and $D_B$. For each $C^\pm$ and $N^\pm$-move $m$ needed to relate
the odd subcomplexes of ∂DA and ∂DB such that the partial unfolding of DA ∪ DH = DA ∪ (∪m Dm) is PL-homeomorphic to MA ∪ H. We refer to the simplicial complex constructed as T2 = DA ∪ DH ∪ DB, and we have T1 ⊂ T2. In the last step we triangulate the remaining space S4 \ T2 | ≃ S3 × [0, 1], attaching DB to DA ∪ DH. This yields T3 = S. In each step T1, T2, and T3 of the construction of S we have to ensure

(I) that φ(F) ∪ |T3| = |odd(T3)| and

(II) that Diagram \(\Sigma\) restricted to T3 commutes.

Note that each of the complexes Ti has to be nice for hT3 to be well defined. Finally we may assume T3 to be a sufficiently fine triangulation. A fine triangulation can be obtained by anti-prismatic subdivision of faces at any stage of the construction by Proposition 2.2.

Construction of T1 = DA ∪ DB. We begin with constructing a triangulation DW of the base space of the branched cover rW : W = H0 ∪ λH1 → D4, that is, DW ≃ W. Then we modify odd(DW) by adding the branching set which produces the µ 2-handles in order to construct a triangulation DA of the base space of rA : MA = H0 ∪ λH1 ∪ µH2 → D4, that is DA ≃ MA. To this end let C be a sufficiently fine triangulated foldable combinatorial 4-ball obtained via the iterated barycentric subdivision of a 4-simplex. Since C arises as a barycentric subdivision there is a natural 5-coloring of the vertices of C by coloring each vertex v ∈ C by the dimension of the original face subdivided by v. Therefore ∂C lies in the \{0, 1, 2, 3\}-skeleton, and vertices colored 4 appear only in the interior of C. The triangulation DW of D4 (and later the triangulation DA) is obtained from C by a series of stellar subdivisions of edges. To cut down on notation we keep referring to our complex by C throughout all stages of the construction, and C is 6-colorable assigning a new color to all new vertices while preserving the original coloring otherwise; see Proposition 2.3.

In order to specify the isomorphism i0 : M0 → H(S, σ0) in Equation \(\Sigma\) fix a facet σ0 ∈ C and let i map the element xi ∈ p−1(yi) contained in the i-th sheet of r to the vertex of σ0 colored j ∈ {0, . . . , 4} via the permutation

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
3 & 1 & 2 & 4 & 0
\end{pmatrix}
\]

We will keep σ0 fixed throughout the construction of S. Although the choice for i may seem arbitrary, it turns out to be useful when applying Lemma 2.5 in the construction of DW.

Recall that subdividing an edge e in the \{i, j\}-skeleton yields lk(e) as the odd subcomplex in the complementary skeleton, that is, in the \{(0, . . . , 4) \ \{i, j\}\}-skeleton; see Proposition 2.3. A projectivity around a triangle in lk(e) exchanges the two vertices of σ0 colored i and j. Via i−1 such a projectivity corresponds to exchanging the elements of r−1(yi) contained in the sheets of r labeled i−1(i) and i−1(j).

We first realize the 2-balls in P as the odd subcomplex in the \{0, 2, 4\}-skeleton, since they correspond via i−1 to the transposition (0, 1) in M0. To this end we embed for each P ∈ P a 3-ball FP in the \{0, 2, 3, 4\}-skeleton with ∂FP in the \{0, 2, 4\}-skeleton, and P ≃ cl(∂FP \ ∂C). Such an embedding of FP exists by Lemma 2.5 since we assume C to be sufficiently finely triangulated, and we choose the \{FP\}P∈P pairwise disjoint. Now we obtain P as the odd subcomplex by stellar subdivision of \{1, 3\}-edges following Proposition 2.3.

The odd subcomplex representing Q is built in a similar fashion in the \{0, 1, 4\}-skeleton, since Q corresponds via i−1 to the transposition (0, 2) in M0. The 3-ball FQ with Q ≃ cl(∂FQ \ ∂C) is embedded in the \{0, 1, 3, 4\}-skeleton with ∂FQ in the \{0, 1, 4\}-skeleton. Proposition 2.3 is applicable since P and FQ are disjoint. Now Q is realized as the odd subcomplex in the \{0, 1, 4\}-skeleton by subdividing \{2, 3\}-edges. This completes the construction of DW. The odd subcomplex intersects ∂C in a system of λ + 1 unknotted and unlinked S1 in the \{0, 2\}-skeleton representing ∂P, and a single unknotted and unlinked S1 in the \{0, 1\}-skeleton representing ∂Q.

Finally we have to add the µ ribbons to the odd subcomplex in order to construct DA. To this end let yo be the barycenter of σ0, and fix a set of meridional loops as generators of π1(C \ (P ∪ Q), yo), that is, choose one meridional loop around each of the 2-balls in P, and one
meridional loop around $Q$. Further assume that the generators do not intersect the collection of 3-balls $\{ F_P \}_{P \in \mathcal{P}} \cup F_Q$. Then a projectivity along the image under $h_C$ of a generator around a 2-ball $P \in \mathcal{P}$ exchanges the vertices colored 1 and 3 of $\sigma_0$, and a projectivity along the image under $h_C$ of the generator around the 2-ball $Q$ exchanges the vertices colored 2 and 3.

Now let $A \in \partial C$ be the arc corresponding to a ribbon $\overline{A}$ and let $a \subset P$ and $a' \subset P'$ be the intersection of $\overline{A}$ with $\mathcal{P}$ as described in Section 3.1. Further let $\beta$ and $\beta'$ be the elements of $\pi_1(\mathcal{C} \setminus (\mathcal{P} \cup Q), y_0)$ corresponding to the meridional loops around $P$ and $P'$. In order to apply Proposition 2.3 choose a regular 4-dimensional neighborhood $U_A$ of $A$ in $C$. (Provided that $C$ is sufficiently fine triangulated one may choose $U_A = \bigcup_{v \in \mathcal{A}} \text{st}_C(v)$.) The neighborhood $U_A$ is 5-colorable since odd($U_A$) = $\emptyset$, and we may choose the coloring such that it coincides with the coloring of $C$ in neighborhoods of $a$ and $a'$, respectively. The later assumption holds since $\beta \alpha = \beta'$ holds by Equation (7), where $\alpha \in \pi_1(\mathcal{C} \setminus (\mathcal{P} \cup Q), y_0)$ corresponds to the way $A$ weaves around $\mathcal{P} \cup Q$. Observe that the 5-coloring $U_A$ does not coincide with the coloring of $C$ in general. It changes corresponding to the way $A$ weaves around the 2-balls $\mathcal{P} \cup Q$.

Now choose a 3-ball $F_{\overline{A}}$ according to Proposition 2.3 in the $\{0, 2, 3, 4\}$-skeleton of $U_A$ with $\partial F_{\overline{A}}$ in the $\{0, 2, 4\}$-skeleton, and $\overline{A} \cong \overline{\partial F_{\overline{A}} \setminus \partial C}$. If we color the vertices of $U_A$ by the coloring of $C$, then in general $\partial F_{\overline{A}}$ is partly embedded in the $\{0, 2, 4\}$-, $\{0, 1, 4\}$-, and $\{0, 3, 4\}$-skeleton, reflecting the fact that different parts of the ribbon correspond to different transposition $(0, 1)$, $(0, 2)$, and $(1, 2)$. The intersection of $\overline{A}$ with $\mathcal{P}$ however is always contained in the $\{0, 2\}$-skeleton.

The ribbon $\overline{A}$ is added to the odd subcomplex by stellarly subdividing edges in the $\{1, 3\}$-skeleton of $U_A$ by Proposition 2.3. Adding all ribbons $\bigcup_{i=1}^{n} \overline{A_i}$ to the odd subcomplex completes the construction of $D_A$.

The simplicial 4-balls $D_A$ and $D_B$ are indeed combinatorial 4-balls (and hence nice) since they are constructed by subdivision of faces from the 4-simplex, and they meet conditions (I) and (II) by construction. We have $T_1 = D_A \cup D_B$, and we summarize the construction of $D_A$ by the following proposition.

**Proposition 3.5.** For each PL 4-manifold $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$ there is a combinatorial 4-ball $D_A$ such that one of the connected components of the partial unfolding $\overline{D_A}$ is PL-homeomorphic to $M_A$. The projection $\overline{D_A} \to D_A$ is a simple 3-fold branched cover with a ribbon manifold as a branching set.

**Construction of $T_2 = D_A \cup D_H \cup D_B$.** For the construction of the cobordism $r_H : \mathbb{S}^3 \times [0, 1]$ we need $r_A$ and $r_B$ to be 4-fold branched covers obtained from a 3-fold branched cover by adding a trivial sheet. The fourth sheet is obtained by adding a 2-ball in the $\{0, 1, 2\}$-skeleton to the odd subcomplex via stellarly subdividing edges in the $\{3, 4\}$-skeleton by Proposition 2.3 and Lemma 2.3. A projectivity along a closed facet path based at $\sigma_0$ around a triangle of the newly added odd subcomplex exchanges the vertices of $\sigma_0$ colored 3 and 4, and corresponds via $r^{-1}$ to the transposition $(0, 3)$ in $\mathfrak{S}_n$.

We first construct $D_A \cup D_H$ such that its partial unfolding yields $M_A \cup H$. In particular the odd subcomplex of the boundary of $D_A \cup D_H$ is equivalent to the odd subcomplex of $D_B$. To this end a combinatorial 4-ball $D_m$ is attached to $\partial D_A$ successively for each of the $C^\pm$- and $N^\pm$-moves required to relate the odd subcomplex of $\partial \overline{D_A}$ and $\partial \overline{D_B}$. The 4-ball $D_m$ realizes $m$ in the sense that the odd subcomplexes of $\partial \overline{D_A}$ and $\partial (\overline{D_A} \cup D_m)$ differ by the move $m$. This produces the triangulation $T_2$. We then “identify” the boundaries of $D_B$ and $D_A \cup (\partial D_A \times [0, 1])$, thus completing the triangulation $S = T_3$. Keep in mind that we have to ensure conditions (I) and (II) to be valid throughout the construction.

The combinatorial 4-ball $D_m = \text{cone}(S_m)$ is constructed as the cone over a combinatorial 3-sphere $S_m$ with a trefoil knot or Hopf link as (colored!) odd subcomplex, respectively. The resulting odd subcomplex odd($D_m$) is a cusp or a node singularity depending on whether $m$ is a $C^\pm$- or $N^\pm$-move, since odd($D_m$) = odd($\text{cone}(S_m)$) = cone(odd($S_m$)) holds.

In general, the sphere $S_m$ may be obtained following the construction by Izvestiev & Joswig [18]. Alternatively, an explicit triangulation of $S_m$ for a $C^\pm$-move $m$ is available as an electronic model (polymake [10] file) by [33]. For a $N^\pm$-move $m$ a triangulation of $S_m$ may be
Let \( \sigma \) denote a face of \( K \) and combinatorial 3-manifolds (possibly with boundary). There is a triangulation \( T \) defining the color equivalence of \( N \) via extending the triangulations \( T \) respectively of \( M \). Choose 3-dimensional neighborhoods \( U \subset \partial D_A \) and \( U' \subset \partial D_m = S_m \), such that replacing \( U \) by \( S_m \setminus U' \) realizes the move \( m \). Now the move \( m \) is realized by identifying \( \{U\} \) and \( \{U'\} \). Since the triangulations \( U \) and \( U' \) are non-equal in general, we triangulate the space \( \{U\} \times [0,1] \), such that \( U \) triangulates \( \{U\} \times \{0\} \) and \( U' \) triangulates \( \{U\} \times \{1\} \), and such that the odd subcomplex is equivalent to the prism over \( \{U_{odd}\} \).

Attaching \( D_m \) to \( D_A \) by identifying \( \{U\} \) to \( \{U'\} \) is similar to the last remaining step in the construction of \( S \), where \( M_B \) is attached to \( M_A \cup H \) via identifying \( \partial(M_A \cup H) \) and \( \partial M_B \). We explain how to realize the identification of \( \{U\} \) and \( \{U'\} \), respectively of \( \partial(M_A \cup H) \) and \( \partial M_B \), via extending the triangulations \( U \) and \( U' \), respectively \( \partial M_B \) and \( \partial(M_A \cup H) \) in a more general setting, thus completing the construction of \( S \).

**Attaching along color equivalent subcomplexes.** Consider two combinatorial 4-manifolds \( K \) and \( K' \) and combinatorial 3-manifolds (possibly with boundary) \( U \subset \partial K \) and \( U' \subset \partial K' \) with \( U \cong U' \). Assume that there exist color equivalent regular 4-dimensional neighborhoods \( N \) and \( N' \) of \( U \), respectively \( U' \), such that \( |N_{odd}| \) (and hence also \( |N'_{odd}| \)) is equivalent to \( |U \cup N_{odd}| \times [0,1] \), and \( N_{odd} \) is a locally flat combinatorial 2-manifold. Note that \( \cup N_{odd} \) does not hold in general. Further let \( \varphi : \{N\} \rightarrow \{N'\}, \sigma_0 \in N, \sigma'_0 \in N' \), and \( \psi : V(\sigma_0) \rightarrow V(\sigma'_0) \) as in Equation (4), defining the color equivalence of \( N \) and \( N' \).

**Proposition 3.7.** There is a triangulation \( T \) of \( \{U\} \times [0,1] \) with \( |T_{odd}| \) equivalent to \( |U \cup N_{odd}| \times [0,1] \), such that \( T \) equals \( U \) on \( \{U\} \times \{0\} \) and \( U' \) on \( \{U\} \times \{1\} \), and such that \( odd(K \cup T \cup K') = K_{odd} \cup T_{odd} \cup K'_{odd} \), thus in effect attaching \( K' \) to \( K \) via identification of \( U \) and \( U' \). Here \( K \cup T \cup K' \) denotes the union of \( K \), \( K' \), and \( T \), attaching \( T \) to \( K \) and \( K' \) along \( U \), respectively \( U' \). The simplicial complex \( K \cup T \cup K' \) is a combinatorial 4-manifold.

In order to make the proof digestible it is split into the three Lemmas 3.8, 3.9, and 3.10. We denote a face \( f \in N \) which intersects \( U \) in all except one vertex by \( f = \{g, x_g\} \), where \( g \) is a face of \( U \) and \( x_g \) the one remaining vertex. Faces of \( N' \) intersecting \( U' \) in all except one vertex are denoted similarly. Throughout, \( \tau \in U \) will be a facet of \( U \), that is, a ridge of \( N \).

After possible refinements of \( N \) and \( N' \) via anti-prismatic subdivision there is a simplicial approximation \( \varphi' : N \rightarrow N' \) of \( \varphi \) which does not degenerate \( \sigma_0 \). Note that any simplicial approximation of \( \varphi \) maps \( N_{odd} \) to \( N'_{odd} \), and \( U \) to \( U' \); see [29] Lemma 14.4, Theorem 16.1. Let \( \sigma \in N \) be a facet, \( \gamma \) a facets path in \( n \) from \( \sigma_0 \) to \( \sigma \), and let \( \gamma' \) be the facet path in \( N' \).
defined by the non-degenerated images of facets in $\gamma$. Let $\kappa_\sigma$ be the last facet of $\gamma'$, hence $\sigma' = \varphi'(\sigma) \subset \kappa_\sigma$ in general, and $\sigma' = \kappa_\sigma$ if $\varphi'$ does not degenerate $\sigma$. We define the bijective map $\psi_\sigma : V(\sigma) \to V(\kappa_\sigma)$ by

$$\psi_\sigma = \langle \gamma' \rangle \circ \psi \circ \langle \gamma \rangle^{-1}.$$ 

Since $N$ and $N'$ are color equivalent, $\psi_\sigma$ is independent of the choice of $\gamma$ and hence well defined. Further note that $\psi_\sigma^{-1} |_{\sigma'}$ is injective, and that $\psi_\sigma(\sigma \cap N_{\text{odd}}) \subset N'_\text{odd}$ since $N$ and $N'$ are color equivalent.

Consider the following regular cell decomposition of $|U| \times [0,1]$. First the $i$-faces of $U$ and $U'$ form closed $i$-cells in the natural way. In particular, the vertices of $U$ and $U'$ are the 0-cells. Now we add a closed $(i+1)$-cell $C_{f_j}^{i+1}$ for each $i$-face $f \in U$. The $(i+1)$-cell $C_{f_j}^{i+1}$ is attached to the union of all $i$-cells along the cell decomposition of $S^i$ given by the cells $f$ (and its proper faces), $\varphi'(f)$ (and its proper faces), and all cells $C_{g}^{j+1}$ with $g \subset f$ is a $j$-face. The top dimensional cells are the 4-cells $\{C_{\tau}^{4}\}_{\tau \in U}$ corresponding to facets of $U$. Any two cells $C_{f_j}^{i+1}$ and $C_{g}^{j+1}$ intersect properly, that is, in the common cell corresponding to $f \cap g$, and the union of all cells equals $|U| \times [0,1]$.

We describe how to triangulate $C_{\tau}^{4}$ for each facet $\tau \in U$. Note that apart from $\tau$ and $\tau' = \varphi'(\tau)$ there might be already a triangulation induced on some cells of $\partial C_{\tau}^{4}$ via the triangulation of neighboring cells of $C_{\tau}^{4}$. Fix a 5-coloring on the vertices of $\{\tau, x_\tau\} \in N$, and color each vertex of $\tau'$ with the color of its preimage under $\psi_{\{\tau, x_\tau\}}$.

**Lemma 3.8.** The 5-coloring of $\tau$ and $\tau'$ can be extended to a 5-coloring of the cells of $\partial C_{\tau}^{4}$ already triangulated.

**Proof.** Let us call any strongly connected subcomplex of $N$ with trivial group of projectivities which contains a facet $\sigma \in N$ a *trivial domain* of $\sigma$, and consider the trivial domain of $\{\tau, x_\tau\}$

$$O = \bigcup_{v \in \{\tau, x_\tau\}\setminus N_{\text{odd}}} \text{st}_N(v),$$

defined by the union of the stars of all vertices of $\{\tau, x_\tau\}$ not contained in $N_{\text{odd}}$. This is indeed a trivial domain if $N$ is triangulated sufficiently fine (there are no identifications in $\partial O$), since no star of an odd triangle is contained in $O$, and since any facet path in $O$ is contractible. For each cell $C_{f_j}^{i+1}$ of $\partial C_{\tau}^{4}$ already triangulated there is a facet $\rho \in U$ in with $f = \tau \cap \rho$, and in the case $f \notin N_{\text{odd}}$ we have $\{\rho, x_\rho\} \in O$. Hence the 5-coloring of $\{\tau, x_\tau\}$ extends uniquely to the triangulation of $C_{f_j}^{i+1}$. Furthermore, if there are two facets $\rho$ and $\overline{\rho}$ with $f = \tau \cap \rho = \tau \cap \overline{\rho}$, both facets $\rho$ and $\overline{\rho}$ produce the same coloring of the triangulation of $C_{f_j}^{i+1}$ since $\text{st}_N(f) \subset O$, and since $O$ is a trivial domain.

In the case where $f \in N_{\text{odd}}$ consider the subcomplex $\overline{O} = \bigcup_{v \in \{\tau, x_\tau\}} \text{st}_N(v)$, a regular neighborhood of $\{\tau, x_\tau\}$. Assuming a sufficiently fine triangulation of $N$ and that $N_{\text{odd}}$ is locally flat, we have $\overline{O} \cong D^4$, $\overline{C}_{\text{odd}} \cong D^2$ with $\overline{C}_{\text{odd}} \cap \partial \overline{O} \cong S^1$, and $\Pi(\overline{O}) \cong \Sigma_2$. Therefore 3 colors of the 5-coloring of $\{\tau, x_\tau\}$ corresponding to the three trivial orbits of $\Pi(\overline{O})$, and let us call these three colors the *stable colors*. Propagating the 5-coloring of $\{\tau, x_\tau\}$ along any facets path in $\overline{O}$ from $\{\tau, x_\tau\}$ to any facet $\{\rho, x_\rho\} \in \overline{O}$ with $f = \tau \cap \rho$ yields the same coloring for the triangulation of $C_{f_j}^{i+1}$ using only the three stable colors, since the vertices of $\{f, x_f\}$ correspond to trivial orbits of $\Pi(\overline{O})$. \hfill $\square$

Now the partial triangulation and 5-coloring of $\partial C_{\tau}^{4}$ is extended to a triangulation and 5-coloring of the entire cell $C_{\tau}^{4}$ using Proposition 2.7. The triangulation and 5-coloring of $C_{\tau}^{4}$ is extended in two steps. First, let $f = \tau \cap N_{\text{odd}}$, and triangulate $C_{f}^{i+1}$ applying Proposition 2.7 using only the three stable colors, unless, of course, $C_{f}^{i+1}$ is already triangulated. Then using Proposition 2.7 once more, the triangulation and 5-coloring is extended to the entire cell $C_{\tau}^{4}$.

**Lemma 3.9.** The odd subcomplex of $K \cup T \cup K'$ is $K_{\text{odd}} \cup K'_{\text{odd}}$, and the union of all triangles in $\bigcup_{f \in U \cap K_{\text{odd}}} C_{f}^{i+1}$. 
Proof. We first prove that a triangle in the interior of a cell $C_{f}^{i+1}$ is even if $f \not\in K_{\text{odd}}$. To this end let $\tau \in U$ be a facet with $f \in \tau$ and let $O$ be the trivial domain of $\{\tau, x_{\tau}\}$ as described above. If $v \not\in O\partial U$ then $\text{lk}(v, D_{f}^{i})$ is even.

By construction of $T$ there is a 5-coloring of the triangulation of $\bigcup_{f \in O} C_{\tau}^{4} \supset \text{st}(T)$, thus $t$ is even.

Any triangle $t$ in $U$, respectively $U'$, is even in $K \cup T \cup K'$, since for any facet $\tau \in U$ the 5-coloring of the cell $C_{\tau}^{4}$ extends the 5-coloring of $\{\tau, x_{\tau}\}$ and $\{\tau', x_{\tau'}\}$ by construction of $T$, hence $\text{st}(K \cup T \cup K(t))$ is 5-colorable and $t$ is even.

It remains to determine the parity of the triangles in the union $\bigcup_{f \in U \cap N_{\text{odd}}} C_{f}^{i+1}$, which form a PL 2-manifold (with boundary) equivalent to $U \cap N_{\text{odd}} \times [0,1]$, and we denote the triangles in question suggestively by $T_{0}$. Let $e$ be an interior co-dimension 3-face of a combinatorial manifold, hence we have $\text{lk}(e) \cong S^{2}$. It is immediate by double counting facet-ridge incidences in any simplicial pseudo manifold without boundary, that the number of facets is even, thus $\text{lk}(e)$, and consequently $\text{st}(e)$ has an even number of facets. We double count the number of incidences of co-dimension 2-faces $\{e, x\} \in \text{st}(e)$ incident to $e$, and facets of $\text{st}(e)$

$$\sum_{\{e, x\} \in \text{st}(e)} \# \{\sigma \in \text{st}(e) \mid \{e, x\} \subset \sigma\} = \sum_{\sigma \in \text{st}(e)} 3.$$

The left hand side equals the number of odd co-dimension 2-faces incident to $e$ modulo 2, and the right hand side is even since there is an even number of facets $\sigma \in \text{st}(e)$.

Returning to our triangulation $K \cup T \cup K'$, we have that any edge $e \in \partial T_{0}$ is contained in none or two odd triangles ($e$ is a ridge of the 2-manifold $T_{0}$). Therefore if there is one odd triangle in a (strongly) connected component of $T_{0}$, then all triangles in the connected component of $T_{0}$ must be odd faces of $T$, and each connected component of $T_{0}$ intersects $K_{\text{odd}}$ in at least one edge. Thus all triangles in $\bigcup_{f \in U \cap K_{\text{odd}}} C_{f}^{i+1}$ are odd, and we proved $T_{0} = T_{\text{odd}}$.\qed

Lemma 3.10. The simplicial complex $K \cup T \cup K'$ is a combinatorial 4-manifold. In particular, $K \cup T \cup K'$ is a nice simplicial complex.

Proof. It suffices to prove that the vertex link of each vertex in $T$ is a simplicial 3-sphere or simplicial 3-ball (in $K \cup T \cup K'$), and hence a combinatorial sphere, respectively ball. Let $f \in U$ be an $i$-face, $C_{f}^{i+1}$ the corresponding closed cell of the regular cell decomposition of $|U| \times [0,1]$, and let $v \in T$ be a vertex contained in the triangulation of $C_{f}^{i+1}$. Further let $g \in U$ be an $j$-face containing $f$ (thus $i \leq j$). In the case $v \in U \subset T$ (or $v \in U' \subset T$) we have

$$D(v, g) = \left| \text{lk}_{T}(v) \right| \cap C_{g}^{i+1} \cong \text{cone}(\partial \text{st}_{U}(f) \cap g),$$

and otherwise

$$D(v, g) = \left| \text{lk}_{T}(v) \right| \cap C_{g}^{i+1} \cong \text{susp}(\partial \text{st}_{U}(f) \cap g).$$

Observe that if $i = 3$, that is, $f$ is a ridge of $N$ (and a facet of $U$), then $\text{lk}_{T}(v)$ is a 3-ball (if $v \in \partial T$) or 3-sphere completely contained in $C_{f}^{4}$. Otherwise $D(v, g)$ is a 3-ball in the case $v \in U \cup U'$ as well as in the case $v \not\in U \cup U'$ for $j \neq 0$. In the remaining case $v \not\in U \cup U'$ and $j = 0$ we have $D(v, g) \cong S^{0}$. (Recall that $\text{cone}(\emptyset) \cong D^{0}$ and $\text{susp}() \cong S^{0}$ holds by definition.)

For $i < 3$ let $\tau, \tau' \in \text{st}_{U}(f)$ be facets intersecting in $g = \tau \cap \tau' \supset f$. Then the two 3-balls $D(v, \tau)$ and $D(v, \tau')$ intersect in $D(v, g)$. Assume that $f \not\in \partial U$ holds. Since $\text{st}_{U}(f)$ is a combinatorial 3-ball (and $\partial \text{st}_{U}(f)$ a combinatorial 2-sphere) we have

$$\text{lk}_{T}(v) \cong \bigcup_{\tau \in \text{st}_{U}(f)} D(v, \tau) \cong \text{cone}(\partial \text{st}_{U}(f))$$

if $v \in U \cup U'$, and

$$\text{lk}_{T}(v) \cong \bigcup_{\tau \in \text{st}_{U}(f)} D(v, \tau) \cong \text{susp}(\partial \text{st}_{U}(f))$$

otherwise. The case $f \in \partial U$ is treated similarly, except we consider the 2-ball $\text{cl}(\partial \text{st}_{U}(f) \setminus \partial U)$ instead of the entire boundary of $\text{st}_{U}(f)$. Thus $T$ is a combinatorial 4-manifold.

It remains to prove that $\text{lk}_{K \cup T \cup K'}(v)$ is a 3-sphere or 3-ball for a vertex $v \in U \subset T$ (or $v \in U' \subset T$). This follows since $\text{lk}_{K \cup T \cup K'}(v)$ is the union of the two combinatorial 3-balls $\text{lk}_{T}(v)$ and $\text{lk}_{K}(v)$, respectively $\text{lk}_{K'}(v)$.\qed
Attaching $\bigcup_m D_m$ to $D_A$ producing $D_A \cup D_H$, and then attaching $D_B$ to $D_A \cup D_H$ as described above completes the construction of $S$, a combinatorial manifold homeomorphic to $S^4$ (note the difference to a combinatorial 4-sphere). The partial unfolding $\hat{S}$ is a combinatorial manifold by Remark 3.6.

It remains to verify conditions (I) and (II). As for condition (I), $T_{\text{odd}}$ is homotopy equivalent to $U_{\text{odd}}$. Further, any path around an odd triangle in the triangulation of some cell $C^{i+1}$, where $f$ is an edge in $U \cap N_{\text{odd}}$, is homotopy equivalent to a path around the (unique) triangle $\{f, x_f\} \cup N_{\text{odd}}$. This settles condition (II). We summarize the construction in the following Theorem 3.11 which states the main result of this paper.

**Theorem 3.11.** For every closed oriented PL 4-manifold $M$ there is a combinatorial manifold $S \cong S^4$ such that one of the connected components of the partial unfolding $\hat{S}$ of $S$ is a combinatorial 4-manifold PL-homeomorphic to $M$. The projection $\hat{S} \to S$ is a simple 4-fold branched cover branched over a PL-surface with a finite number of cusp and node singularities.

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