ASYMPTOTIC STUDY OF LERAY SOLUTION OF
3D-NSE WITH EXPONENTIAL DAMPING

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Abstract. We study the uniqueness, the continuity in $L^2$ and the large time decay for the Leray solutions of the $3D$ incompressible Navier-Stokes equations with the nonlinear exponential damping term $a(e^{b|u|^2} - 1)u$, $(a, b > 0)$ studied by the second author in [2].

1. Introduction

In this paper, we investigate the questions of the existence, uniqueness and asymptotic study of global weak solution to the following modified incompressible Navier-Stokes equations in $\mathbb{R}^3$

$$\begin{cases}
\partial_t u - \nu \Delta u + u.\nabla u + a(e^{b|u|^2} - 1)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u_0(x) & \text{in } \mathbb{R}^3,
\end{cases}$$

where $u = u(t, x) = (u_1, u_2, u_3)$, $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The function $\nu$ denotes the viscosity of fluid and $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$ is the initial given velocity. The damping of the system comes from the resistance to the motion of the fluid. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [7, 8, 14, 15] and references therein). The fact that div $u = 0$, allows to write the term $(u.\nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$ in the following form div $(u \otimes u) := (\text{div } (u_1 u), \text{div } (u_2 u), \text{div } (u_3 u))$. If the initial velocity $u^0$ is quite regular, the divergence free condition determines uniquely the pressure $p$.

Date: June 8, 2022.
2020 Mathematics Subject Classification. Primary 35-XX, 35Q30, 76D05, 76N10.

Key words and phrases. Navier-Stokes Equations, Friedrich method, global weak solution.
Without loss of generality and in order to simplify the proofs of our results, we consider the viscosity unitary ($\nu = 1$).

The global existence of weak solution of initial value problem of classical incompressible Navier-Stokes were proved by Leray and Hopf (see [13]-[16]) long before. Uniqueness remains an open problem if the dimension $d \geq 3$.

The polynomial damping $\alpha |u|^{\beta-1}u$ is studied in [10] by Cai and Jiu. They proved the global existence of weak solution in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3))$.

The exponential damping $a(e^{b|u|^2}-1)u$ is studied in [2] by J. Benameur. He proved the existence of global weak solution in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b,$

where $\mathcal{E}_b = \{f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} :$ measurable, $(e^{b|f|^2} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$.

The purpose of this paper is to prove the uniqueness and the continuity of the global solution given in [2]. Using Friedrich method, we construct approximate solutions and we make more delicate estimates to proceed the compactness arguments. In particular, we obtain some new a priory estimates:

$$\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2} ds + 2a \int_0^t (e^{b|u|^2} - 1)|u(s)|^2 \|L^1 ds \leq \|u^0\|^2_{L^2},$$

To prove the uniqueness we use the energy method and the approximate systems. The proof of the asymptotic study is based on a decomposition of the solution in high and low frequencies and the uniqueness of such solution in a well chosen time $t_0$.

In our case of exponential damping, we find more regularity of Leray solution in $\cap_p L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))$. In particular, we give a new energy estimate. Our main result is the following:

**Theorem 1.1.**

Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence free vector fields, then there is a unique global solution of the system $(S)$: $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b$. Moreover, for all $t \geq 0$

$$(1.1) \quad \|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2} ds + 2a \int_0^t (e^{b|u|^2} - 1)|u(s)|^2 \|L^1 ds \leq \|u^0\|^2_{L^2}.$$

Moreover, we have

$$(1.2) \quad \limsup_{t \to \infty} \|u(t)\|_{L^2} = 0.$$
Remark 1.2.

(1) The new results in this theorem is the uniqueness, the continuity of the global week solution in $L^2(\mathbb{R}^3)$ and its asymptotic behavior at infinity.

(2) The uniqueness of weak solution implies that

\[
\|u(t_2)\|_{L^2}^2 + 2 \int_{t_1}^{t_2} \|\nabla u(s)\|_{L^2}^2 ds + 2a \int_{t_1}^{t_2} \|e^{|u(s)|^2 - 1}\|_{L^1} ds \leq \|u(t_1)\|_{L^2}^2,
\]

which implies that $(t \to \|u(t)\|_{L^2})$ is decreasing.

2. Notations and Preliminary Results

The Friedrich operator $J_R$ is defined for $R > 0$ by: $J_R(D)f = \mathcal{F}^{-1}(\chi_{B_R}(\hat{f}))$, where $B_R$ is the ball of center 0 and radius $R$ and $f \in L^2(\mathbb{R}^3)$. The Leray operator $\mathbb{P}$ is the projector operator of $(L^2(\mathbb{R}^3))^3$ on the space of divergence-free vector fields $L^2_0(\mathbb{R}^3)$.

If $f$ is in the Schwartz space $(S(\mathbb{R}^n))^3$,

\[
\mathcal{F}(\mathbb{P}f) = \hat{f}(\xi) - (\hat{f}(\xi), \frac{\xi}{|\xi|}) \frac{\xi}{|\xi|} = M(\xi)\hat{f}(\xi)
\]

and $(\mathbb{P}f)_k(x) = \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} \left( \delta_{kj} - \frac{\xi_k \xi_j}{|\xi|^2} \right) \hat{f}_j(\xi) e^{i\xi \cdot x} d\xi$, where $M(\xi)$ is the matrix $(\delta_{k,\ell} - \frac{\xi_k \xi_\ell}{|\xi|^2})_{1 \leq k, \ell \leq 3}$.

Define also the operator $A_R(D)$ on $L^2(\mathbb{R}^3)$ by:

\[
A_R(D)u = \mathbb{P}J_R(D)u = \mathcal{F}^{-1}(M(\xi)\chi_{B_R}(\xi)\hat{u}).
\]

To simplify the exposition of the main result, we first collect some preliminary results and we give some new technical lemmas.

Proposition 2.1. ([9])

Let $H$ be a Hilbert space.

(1) The unit ball is weakly compact, that is: if $(x_n)$ is a bounded sequence in $H$, then there is a subsequence $(x_{\varphi(n)})$ such that

\[
(x_{\varphi(n)}|y) \rightarrow (x|y), \ \forall y \in H.
\]

(2) If $x \in H$ and $(x_n)$ a bounded sequence in $H$ such that $\lim_{n \rightarrow +\infty} (x_n|y) = (x|y)$, for all $y \in H$, then $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$.

(3) If $x \in H$ and $(x_n)$ is a bounded sequence in $H$ such that

\[
\lim_{n \rightarrow +\infty} (x_n|y) = (x|y), \ \forall y \in H \text{ and } \limsup_{n \rightarrow \infty} \|x_n\| \leq \|x\|,
\]

then $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. 
We recall the following product law in the homogeneous Sobolev spaces:

**Lemma 2.2.** (111)
Let $s_1, s_2$ be two real numbers and $d \in \mathbb{N}$.

1. If $s_1 < \frac{d}{2}$ and $s_1 + s_2 > 0$, there exists a constant $C_1 = C_1(d, s_1, s_2)$ such that: if $f, g \in \hat{H}^{s_1}(\mathbb{R}^d) \cap \hat{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \hat{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and
   \[
   \|fg\|_{\hat{H}^{s_1+s_2-\frac{d}{2}}} \leq C_1(\|f\|_{\hat{H}^{s_1}}, \|g\|_{\hat{H}^{s_2}} + \|f\|_{\hat{H}^{s_2}} \|g\|_{\hat{H}^{s_1}}).
   \]

2. If $s_1, s_2 < \frac{d}{2}$ and $s_1 + s_2 > 0$ there exists a constant $C_2 = C_2(d, s_1, s_2)$ such that: if $f \in \hat{H}^{s_1}(\mathbb{R}^d)$ and $g \in \hat{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \hat{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and
   \[
   \|fg\|_{\hat{H}^{s_1+s_2-\frac{d}{2}}} \leq C_2\|f\|_{\hat{H}^{s_1}}, \|g\|_{\hat{H}^{s_2}}.
   \]

**Lemma 2.3.** (33)
Let $\beta > 0$ and $d \in \mathbb{N}$. Then, for all $x, y \in \mathbb{R}^d$, we have

\[
(2.1) \quad \langle |x|^\beta x - |y|^\beta y, x - y \rangle \geq \frac{1}{2}(\langle |x|^\beta + |y|^\beta \rangle |x - y|^2),
\]

and

\[
(2.2) \quad \langle (e^{b|x|^2} - 1)x - (e^{b|y|^2} - 1)y, x - y \rangle \geq \frac{1}{2}(\langle (e^{b|x|^2} - 1) + (e^{b|y|^2} - 1) \rangle |x - y|^2).
\]

**Proposition 2.4.** (22)
Let $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$ and $f^0 \in L^2_2(\mathbb{R}^3)$.
For $n \in \mathbb{N}$, let $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a measurable function in $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ such that

\[
A_n(D)F_n = F_n, \quad F_n(0, x) = A_n(D)f^0(x)
\]

and

\[
(E1) \quad \partial_t F_n + \sum_{k=1}^3 \nu_k |D_k|^{2r_k} F_n + A_n(D) \text{div} (F_n \otimes F_n) + A_n(D) h(|F_n|) F_n = 0.
\]

\[
(E2) \quad \|F_n(t, \cdot)\|_{L^2}^2 + 2 \sum_{k=1}^3 \nu_k \int_0^t \| |D_k|^{r_k} F_n(s, \cdot) \|_{L^2}^2 ds + 2a \int_0^t \|h(|F_n(s, \cdot)|) F_n(s, \cdot)\|_{L^1} ds \leq \|f^0\|_{L^2}^2.
\]
where \( h(\lambda) = a(e^{b\lambda^2} - 1) \), with \( a, b > 0 \). Then: for every \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon, a, b, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0 \) such that: for all \( t_1, t_2 \in \mathbb{R}^+ \), we have

\[
(2.3) \quad \left( |t_2 - t_1| < \delta \implies \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon \right), \quad \forall n \in \mathbb{N},
\]

with \( s_0 \geq \max(3, 2r_1, 2r_2, 2r_3) \).

**Lemma 2.5.** Let \( a, b > 0 \), then there is a unique real \( \lambda_0 = \lambda_0(a, b) \geq 0 \) such that: For all \( \lambda \geq 0 \)

\[
a(e^{b\lambda} - 1) \leq \lambda \implies \lambda \in [0, \lambda_0].
\]

**Precisely**

- If \( ab \geq 1 \), we have \( \lambda_0 = 0 \),
- If \( ab < 1 \), we have \( \lambda_0 > \frac{1}{b} \log(1/ab) \).

### 3. Proof of the Main Theorem 1.1

The proof of the theorem is given in four steps:

**3.1. Existence of Week Solution.**

In this step, we build approximate solutions of the system \((S)\) inspired by the method used in [2, 11], hence we construct a global solution. For this, consider the approximate system with parameter \( n \in \mathbb{N} \):

\[
(S_n) \left\{ \begin{array}{l}
\partial_t u - \Delta J_n u + J_n(J_n u, \nabla J_n u) + aJ_n[(e^{b|J_n u|^2} - 1)J_n u] = -\nabla p_n \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
p_n = (-\Delta)^{-1}\left( \text{div } J_n(J_n u, \nabla J_n u) + a\text{div } J_n[(e^{b|J_n u|^2} - 1)J_n u] \right) \\
div u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) = J_n u_0(x) \quad \text{in } \mathbb{R}^3.
\end{array} \right.
\]

\( J_n \) is the Friedrich operator defined in the second section.

- By Cauchy-Lipschitz Theorem, we obtain a unique solution \( u_n \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^3)) \) of \((S_{2,n})\). Moreover, \( J_n u_n = u_n \) such that

\[
(3.1) \quad \|u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u_n\|_{L^2}^2 + 2a \int_0^t \|(e^{b|u_n|^2} - 1)|u_n|^2\|_{L^1} \leq \|u_0\|_{L^2}^2.
\]

- The sequence \((u_n)_n\) is bounded in \( L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \) and on \( L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)). \) Using Proposition 2.4 and the interpolation method, we deduce that the sequence \((u_n)_n\) is equicontinuous on \( H^{-1}(\mathbb{R}^3) \).
• Let \((T_q)_q\) be a strictly increasing sequence such that \(\lim_{q \to +\infty} T_q = \infty\). Consider a sequence of functions \((\theta_q)_q\) in \(C_0^\infty(\mathbb{R}^3)\) such that:
\[
\begin{cases}
\theta_q(x) = 1, & \text{for } |x| \leq q + \frac{5}{4} \\
\theta_q(x) = 0, & \text{for } |x| \geq q + 2 \\
0 \leq \theta_q \leq 1
\end{cases}
\]

Using (3.1), the equicontinuity of the sequence \((u_n)_n\) on \(H^{-1}(\mathbb{R}^3)\) and classical argument by combining Ascoli’s theorem and the Cantor diagonal process, there exists a subsequence \((u_{\varphi(n)}(n))_n\) and \(u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))\) such that:
\[
(3.2) \quad \lim_{n \to \infty} \|\theta_q(u_{\varphi(n)} - u)\|_{L^\infty([0, T_q], H^{-4})} = 0.
\]

In particular, the sequence \((u_{\varphi(n)}(t))_n\) converges weakly in \(L^2(\mathbb{R}^3)\) to \(u(t)\) for all \(t \geq 0\).

• Combining the above inequalities, we obtain:
\[
(3.3) \quad \|u(t)\|_{L^2} + 2 \int_0^t \|\nabla u(s)\|_{L^2} ds + 2a \int_0^t \|e^{b|u(s)|^2} - 1\|_{L^1} |u(s)|^2 ds \leq \|u^0\|_{L^2}.
\]

for all \(t \geq 0\).

• \(u\) is a solution of the system \((S)\).

### 3.2. Continuity of the Solution in \(L^2\).

In this section, we give a simple proof of the continuity of the solution \(u\) of the system \((S)\) and we prove also that \(u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3))\). The construction of the solution is based on the Friedrich approximation method. We point out that we can use this method to show the same results as in [15].

• By inequality (3.3), we get
\[
\limsup_{t \to 0} \|u(t)\|_{L^2} \leq \|u^0\|_{L^2}.
\]

Then, proposition 2.1-(3) implies that
\[
\limsup_{t \to 0} \|u(t) - u^0\|_{L^2} = 0,
\]

which ensures the continuity of \(u\) at 0.

• Consider the functions
\[
v_{n, \varepsilon}(t, \cdot) = u_{\varphi(n)}(t + \varepsilon, \cdot), \quad p_{n, \varepsilon}(t, \cdot) = p_{\varphi(n)}(t + \varepsilon, \cdot),
\]

for \(n \in \mathbb{N}\) and \(\varepsilon > 0\). We have:
By using Lemma 2.5, we get:

$$\| w_n,\|_{L^2}^2 + \| \nabla w_n,\|_{L^2}^2 + a J(\phi_n) \left( (e^{b|\phi_n|})^2 - 1 \right) u_{\phi(n)} - \left( e^{b|v_{n,\epsilon}|^2} - 1 \right) v_{n,\epsilon} = -\nabla p_{\phi(n)}$$

Taking the scalar product in $L^2(\mathbb{R}^3)$ with $w_{n,\epsilon}$ and using the properties div $w_{n,\epsilon} = 0$ and $\langle w_{n,\epsilon}, \nabla w_{n,\epsilon}, w_{n,\epsilon} \rangle = 0$, we get

$$\frac{1}{2} \frac{d}{dt} \| w_{n,\epsilon} \|_{L^2}^2 + \| \nabla w_{n,\epsilon} \|_{L^2}^2 + a J(\phi_n) \left( (e^{b|\phi_n|})^2 - 1 \right) u_{\phi(n)} - \left( e^{b|v_{n,\epsilon}|^2} - 1 \right) v_{n,\epsilon} \| w_{n,\epsilon} \|_{L^2}$$

Using inequality (2.2), we get

$$\langle J(\phi(n)) \left( (e^{b|\phi_n|})^2 - 1 \right) u_{\phi(n)} - \left( e^{b|v_{n,\epsilon}|^2} - 1 \right) v_{n,\epsilon} \rangle$$

$$= \langle (e^{b|\phi_n|})^2 - 1 \rangle u_{\phi(n)} - \left( e^{b|v_{n,\epsilon}|^2} - 1 \right) v_{n,\epsilon} \rangle_{L^2}$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} \left( (e^{b|\phi_n|})^2 - 1 \right) |w_{n,\epsilon}|^2$$

$$| \langle J(\phi(n)) \langle w_{n,\epsilon}, \nabla u_{\phi(n)} \rangle \rangle | \leq \int_{\mathbb{R}^3} |w_{n,\epsilon}| |u_{\phi(n)}| |\nabla w_{n,\epsilon}|$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^3} |w_{n,\epsilon}|^2 |u_{\phi(n)}|^2 + \frac{1}{2} \| \nabla w_{n,\epsilon} \|_{L^2}^2.$$

Combining the identity (2.2) and the inequality (3.4), we get

$$\frac{d}{dt} \| w_{n,\epsilon} \|_{L^2}^2 + \| \nabla w_{n,\epsilon} \|_{L^2}^2 + a \int_{\mathbb{R}^3} \left( (e^{b|\phi_n|})^2 - 1 \right) |w_{n,\epsilon}|^2 \leq \int_{\mathbb{R}^3} |w_{n,\epsilon}|^2 |u_{\phi(n)}|^2.$$

By using Lemma 2.5 we get:

If $ab \geq 1$

$$\frac{d}{dt} \| w_{n,\epsilon} \|_{L^2}^2 + \| \nabla w_{n,\epsilon} \|_{L^2}^2 \leq 0,$$
If \( ab < 1 \), we have
\[
\frac{d}{dt} \| w_{n, \varepsilon} \|_{L^2}^2 + \| \nabla w_{n, \varepsilon} \|_{L^2}^2 + a \int_{\mathbb{R}^3} (e^{b|\varphi_n(t)|^2} - 1)|w_{n, \varepsilon}|^2 \leq \lambda_0 \int_{\mathcal{A}_t} |w_{n, \varepsilon}|^2 |\varphi_n(t)|^2,
\]
where \( A_{n,t} = \{ x \in \mathbb{R}^3 / a(e^{b|\varphi_n(t)|^2} - 1) < |\varphi_n(t)|^2 \} \). Then, also by Lemma 2.5 we obtain
\[
x \in A_{n,t} \implies |\varphi_n(t, x)|^2 \leq \lambda_0,
\]
which implies
\[
\frac{d}{dt} \| w_{n, \varepsilon} \|_{L^2}^2 + \| \nabla w_{n, \varepsilon} \|_{L^2}^2 \leq \lambda_0 \int_{\mathcal{A}_t} |w_{n, \varepsilon}|^2 \leq \lambda_0 \| w_{n, \varepsilon} \|_{L^2}^2.
\]
In all cases, we get
\[
\frac{d}{dt} \| w_{n, \varepsilon} \|_{L^2}^2 \leq \lambda_0 \| w_{n, \varepsilon} \|_{L^2}^2.
\]
By Gronwall Lemma, we get
\[
\| w_{n, \varepsilon}(t) \|_{L^2}^2 \leq \| w_{n, \varepsilon}(0) \|_{L^2}^2 e^{\lambda_0 t}.
\]
Then
\[
\| u_{\varphi_n}(t + \varepsilon) - u_{\varphi_n}(t) \|_{L^2}^2 \leq \| u_{\varphi_n}(\varepsilon) - u_{\varphi_n}(0) \|_{L^2}^2 e^{\lambda_0 t}.
\]
For \( t_0 > 0 \) and \( \varepsilon \in (0, t_0) \), we have
\[
\| u_{\varphi_n}(t_0 + \varepsilon) - u_{\varphi_n}(t_0) \|_{L^2}^2 \leq \| u_{\varphi_n}(\varepsilon) - u_{\varphi_n}(0) \|_{L^2}^2 \exp(2\lambda_0 t_0),
\]
\[
\| u_{\varphi_n}(t_0 - \varepsilon) - u_{\varphi_n}(t_0) \|_{L^2}^2 \leq \| u_{\varphi_n}(\varepsilon) - u_{\varphi_n}(0) \|_{L^2}^2 \exp(2\lambda_0 t_0).
\]
So
\[
\| u_{\varphi_n}(\varepsilon) - u_{\varphi_n}(0) \|_{L^2}^2 = \| J_{\varphi_n} u_{\varphi_n}(\varepsilon) - J_{\varphi_n} u_{\varphi_n}(0) \|_{L^2}^2
\]
\[
\leq \| X_{\varphi_n}(\hat{u}_{\varphi_n}(\varepsilon) - \hat{u}_0) \|_{\varphi_n}^2
\]
\[
\leq \| u_{\varphi_n}(\varepsilon) - u_0 \|_{L^2}^2
\]
\[
\leq 2\| u_0 \|_{L^2}^2 - 2\text{Re} \langle u_{\varphi_n}(\varepsilon), u_0 \rangle.
\]
But
\[
\lim_{n \to +\infty} \langle u_{\varphi_n}(\varepsilon), u_0 \rangle = \langle u(\varepsilon), u_0 \rangle.
\]
Hence
\[
\lim_{n \to \infty} \| u_{\varphi_n}(\varepsilon) - u_{\varphi_n}(0) \|_{L^2}^2 \leq 2\| u_0 \|_{L^2}^2 - 2\text{Re} \langle u(\varepsilon); u_0 \rangle_{L^2}.
\]
Moreover, for all \( q, N \in \mathbb{N} \)
\[
\| J_N \left( \theta_q(u_{\varphi_n}(t_0 \pm \varepsilon) - u_{\varphi_n}(t_0)) \right) \|_{L^2}^2 \leq \| \theta_q(u_{\varphi_n}(t_0 \pm \varepsilon) - u_{\varphi_n}(t_0)) \|_{L^2}^2
\]
\[
\leq \| u_{\varphi_n}(t_0 \pm \varepsilon) - u_{\varphi_n}(t_0) \|_{L^2}^2.
\]
Using (3.2) we get, for $q$ big enough,
\[
\|J_N\left(\theta_q(u(t_0 \pm \epsilon) - u(t_0))\right)\|_{L^2}^2 \leq \lim_{n \to \infty} \|u_{\varphi(n)}(t_0 \pm \epsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.
\]
Then
\[
\|J_N\left(\theta_q(u(t_0 \pm \epsilon) - u(t_0))\right)\|_{L^2}^2 \leq 2\left(\|u^0\|_{L^2}^2 - \text{Re}\langle u(\epsilon); u^0\rangle_{L^2}\right) \exp(2\lambda_0 t_0).
\]
By applying the monotone convergence theorem in the order $N \to \infty$ and $q \to \infty$, we get
\[
\|u(t_0 \pm \epsilon, \cdot) - u(t_0, \cdot)\|_{L^2}^2 \leq 2\left(\|u^0\|_{L^2}^2 - \text{Re}\langle u(\epsilon); u^0\rangle_{L^2}\right) \exp(2\lambda_0 t_0).
\]
Using the continuity at 0 and make $\epsilon \to 0$, we get the continuity at $t_0$.

3.3. **Uniqueness of the Solution.**

Let $u, v$ be two solutions of $(S)$ in the space
\[
C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b.
\]
The function $w = u - v$ satisfies the following:
\[
\partial_t w - \Delta w + a\left((e^{b|u|^2})u - (e^{b|v|^2})v\right) = -\nabla(p - \tilde{p}) + w.\nabla w - w.\nabla u - u.\nabla w.
\]
Taking the scalar product in $L^2$ with $w$, we get
\[
\frac{1}{2} \frac{d}{dt}\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} + a\langle\left((e^{b|u|^2})u - (e^{b|v|^2})v\right); w\rangle_{L^2} = -\langle w.\nabla u; w\rangle_{L^2}.
\]
The idea is to lower the term $\langle\left((e^{b|u|^2})u - (e^{b|v|^2})v\right); w\rangle_{L^2}$ with the help of the Lemma 2.3 and then divide the term find into two equal pieces, one to absorb the nonlinear term and the other is used in the last inequality.

By using inequality (2.2), we get
\[
\langle\left((e^{b|u|^2})u - (e^{b|v|^2})v\right); w\rangle_{L^2} \geq \frac{1}{2} \int_{\mathbb{R}^3} \left((e^{b|u|^4}) - (e^{b|v|^4})\right) |w|^2 \geq \frac{1}{2} \int_{\mathbb{R}^3} (e^{b|u|^4} - 1) |w|^2.
\]
Moreover, we have
\[
|\langle w.\nabla u; w\rangle_{L^2}| = |\langle \text{div} (w \otimes u); w\rangle_{L^2}| = |\langle w \otimes u; \nabla w\rangle_{L^2}|
\leq \int_{\mathbb{R}^3} |w|.|u|.|\nabla w| \leq \frac{1}{2} \int_{\mathbb{R}^3} |w|^2 |u|^2 + \frac{1}{2} \|\nabla w\|^2_{L^2}.
\]
Combining the above inequalities, we get
\[ \frac{d}{dt} \|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} + a \int_{\mathbb{R}^3} (e^{b|u|^2} - 1)|w|^2 \leq \int_{\mathbb{R}^3} |w|^2 |u|^2. \]
By using Lemma 2.5 and the set
\[ A_t = \{ x \in \mathbb{R}^3 / a(e^{b|u|^2} - 1) < |u(t, x)|^2 \}, \]
we get
\[ \frac{d}{dt} \|w\|^2_{L^2} \leq \lambda_0 \|w\|^2_{L^2}. \]
and, Gronwall Lemma gives
\[ \|w\|^2_{L^2} \leq \|w^0\|^2_{L^2} e^{\lambda_0 t}. \]
As \( w^0 = 0 \), then \( w = 0 \) and \( u = v \). Which implies the uniqueness.

3.4. Asymptotic Study of the Global Solution.
In this subsection we prove the asymptotic behavior (1.2). Here, we use a modified version of Benameur-Selmi method ([5]). The idea is to write the nonlinear term of exponential type as follows
\[ a(e^{b|u|^2} - 1)u = a(e^{b|u|^2} - 1 - b|u|^2)u + a|u|^2u. \]
The first term, one treats it of the matter of [5], and the second and by specifying the difficulty in small frequencies, we descend into the Sobolev norm of negative index \( H^{-\sigma}(\mathbb{R}^3), \sigma > 0 \).
For this, let \( \varepsilon > 0 \) be positif real number. For \( \delta > 0 \), put the following functions
\[ v_\delta = \mathcal{F}^{-1}(1_{B(0,\delta)}(\xi)\widehat{u}(\xi)), \quad w_\delta = u - v_\delta. \]
We have
\[ v_\delta = \sum_{k=1}^{4} f_{\delta,k}(t), \]
where
\[
\begin{align*}
    f_{\delta,1} &= e^{t\Delta}v_\delta^0, \quad v_\delta^0 = \mathcal{F}^{-1}(1_{B(0,\delta)}(\xi)\widehat{u}^0(\xi)) \\
    f_{\delta,2} &= -\int_0^t e^{(t-z)\Delta}1_{B(0,\delta)}(D)\mathbb{P}\text{div} (u \otimes u) \\
    f_{\delta,3} &= -a\int_0^t e^{(t-z)\Delta}1_{B(0,\delta)}(D)\mathbb{P}(e^{b|u|^2} - 1 - b|u|^2)u \\
    f_{\delta,4} &= -\int_0^t e^{(t-z)\Delta}1_{B(0,\delta)}(D)\mathbb{P}(|u|^2u).
\end{align*}
\]
• We have
\[ \|f_{\delta,1}(t)\|_{L^2} \leq \|v_\delta^0\|_{L^2}. \]
As \( \lim_{\delta \to 0} \| \epsilon_0 \|_{L^2} = 0 \), then there is \( \delta_1 > 0 \) such that
\[
(3.5) \quad \sup_{t \geq 0} \| f_{\delta,1}(t) \|_{L^2} < \epsilon/8, \quad \forall 0 < \delta < \delta_1.
\]
- We have
\[
\| f_{\delta,2}(t) \|_{H^{-1/4}} \leq \int_0^t \| e^{(t-z)\Delta} 1_{B(0,\delta)}(D)\mathbb{P}(e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-1/4}}
\]
\[
\leq \int_0^t \| 1_{B(0,\delta)}(D)(e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-1/4}}
\]
\[
\leq \int_0^t \| \mathbb{P} \|_{H^{-1/4}}
\]
\[
\leq \delta \int_0^t \| e^{(t-z)\Delta} 1_{B(0,\delta)}(D)\mathbb{P}(e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-1/4}}
\]
\[
\leq \delta \int_0^t \| (e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-1/4}}
\]
\[
\leq C\delta \int_0^t \| u \|_{L^2}^2, \quad (s_1 + s_2 = 2, s_1 = s_2 = 1)
\]
\[
\leq C \| u \|_{L^2}^2 \delta^{1/4}.
\]
But
\[
\| f_{\delta,2}(t) \|_{L^2} \leq c_0 (1 + \delta^2)^{1/4} \| f_{\delta,2}(t) \|_{H^{-1/4}}
\]
Then
\[
\| f_{\delta,2}(t) \|_{L^2} \leq c_0 \| u \|_{L^2}^2 (1 + \delta^2)^{1/4} \delta^{1/4}.
\]
there is \( \delta_2 > 0 \) such that
\[
(3.6) \quad \sup_{t \geq 0} \| f_{\delta,2}(t) \|_{L^2} < \epsilon/8, \quad \forall 0 < \delta < \delta_2.
\]
- We have
\[
\| f_{\delta,3}(t) \|_{H^{-2}} \leq \int_0^t \| e^{(t-z)\Delta} 1_{B(0,\delta)}(D)\mathbb{P}(e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-2}}
\]
\[
\leq \int_0^t \| 1_{B(0,\delta)}(D)(e^{b|u|^2} - 1 - b|u|^2)u \|_{H^{-2}}
\]
\[
\leq R_\delta \int_0^t \| (e^{b|u|^2} - 1 - b|u|^2)u \|_{L^1},
\]
where
\[
R_\delta = \| 1_{B(0,\delta)}(D) \|_{H^{-2}} = \left( \int_{B(0,\delta)} \frac{1}{(1 + |\xi|^2)^2} \right)^{1/2} \leq \left( \int_{B(0,\delta)} \right)^{1/2} = c_0 \delta^{3/2}.
\]
By using the elementary inequality with $M_b > 0$
\[(e^{b\varepsilon} - 1 - b\varepsilon^2)z \leq M_b(e^{b\varepsilon} - 1)z^2, \quad \forall z \geq 0\]
we get
\[
\int_0^t \|(e^{b|u|^2} - 1 - b|u|^2)u\|_{L^1} \leq M_b \int_0^t \|(e^{b|u|^2} - 1)|u|^2\|_{L^1} \leq (2a)^{-1}M_b\|u^0\|_2^2.
\]
Combining the above inequalities we get
\[
\|f_{\delta,3}(t)\|_{H^{-2}} \leq c_0(2a)^{-1}M_b\|u^0\|_2^2 \delta^{3/2}.
\]
But
\[
\|f_{\delta,3}(t)\|_{L^2} \leq c_0(1 + \delta^2)^2\|f_{\delta,2}(t)\|_{H^{-2}}
\]
then
\[
\|f_{\delta,3}(t)\|_{L^2} \leq c_0(2a)^{-1}M_b\|u^0\|_2^2 (1 + \delta^2)^2 \delta^{3/2}.
\]
Then there is $\delta_3 > 0$ such that
\[
(3.7) \quad a \sup_{t \geq 0} \|f_{\delta,3}(t)\|_{L^2} < \varepsilon/8, \quad \forall 0 < \delta < \delta_3.
\]
- By using Lemma 2.2 with the well choice of $s_1$ and $s_2$, we get
\[
\|f_{\delta,4}(t)\|_{H^{-1/2}} \leq \int_0^t \left\| e^{(t-z)\Delta} 1_{B(0,\delta)}(D) P|u|^2 u \right\|_{H^{-1/2}}
\]
\[
\leq \int_0^t \left\| 1_{B(0,\delta)}(D)|u|^2 u \right\|_{H^{-1/2}}
\]
\[
\leq \int_0^t \left\| 1_{B(0,\delta)}(D)|u|^2 u \right\|_{\dot{H}^{-1/2}}
\]
\[
\leq \int_0^t \left\| 1_{B(0,\delta)}(D)|D|^{1/2}|D|^{-1/2}|u|^2 u \right\|_{\dot{H}^{-1/2}}
\]
\[
\leq \delta^{1/2} \int_0^t \left\| 1_{B(0,\delta)}(D)|D|^{-1/2}|u|^2 u \right\|_{\dot{H}^{-1/2}}
\]
\[
\leq \delta^{1/2} \int_0^t \left\| |D|^{-2} |u|^2 u \right\|_{\dot{H}^{-1/2}}
\]
\[
\leq \delta^{1/2} \int_0^t \left\| |u|^2 u \right\|_{H^{-1}}
\]
\[
\leq \delta^{1/2} C \int_0^t \left\| |u|_{H^{1/2}} u \right\|_{L^2}, \quad (s_1 + s_2 = 1/2, \quad s_1 = 0, \quad s_2 = 1/2)
\]
\[
\leq \delta^{1/2} C' \int_0^t \left\| |u|_{H^{1/2}} u \right\|_{H^1}, \quad (s_1 + s_2 = 3/2, \quad s_1 = 1/2, \quad s_2 = 1),
\]
\[
\leq \delta^{1/2} C' \int_0^t \left\| |u|_{L^2} u \right\|_{H^1}, \quad (by \ interpolation),
\]
\[
\leq \delta^{1/2} C'' \|u^0\|_{L^2} \int_0^t \|\nabla u\|_{L^2}^2
\]
\[
\leq \delta^{1/2} C'' \|u^0\|_{L^2}^3.
\]
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But
\[ \|f_{\delta,4}(t)\|_{L^2} \leq c_0(1 + \delta^2)^{1/2}\|f_{\delta,2}(t)\|_{H^{-1/2}} \]
then
\[ \|f_{\delta,4}(t)\|_{L^2} \leq c_0C''\|u^0\|_{L^2}^3(1 + \delta^2)^{1/2}\delta^{1/2}. \]
Then there is \( \delta_4 > 0 \) such that
\[ (3.8) \sup_{t \geq 0} \|f_{\delta,4}(t)\|_{L^2} < \varepsilon/8, \forall \delta < \delta_4. \]
Combining the above equations (3.5)-(3.6)-(3.7)-(3.8) we get
\[ (3.9) \sup_{t \geq 0} \|v_{\delta_0}(t)\|_{L^2} < \varepsilon/2, \quad \delta_0 = \frac{1}{2} \min_{1 \leq i \leq 4} \delta_i. \]

In other hand we have
\[
\int_0^\infty \|w_{\delta_0}(t)\|_{L^2}^2 dt \leq \delta_0^{-2} \int_0^\infty \|\nabla w_{\delta_0}(t)\|_{L^2}^2 dt \\
\leq \delta_0^{-2} \int_0^\infty \|\nabla w_{\delta_0}(t)\|_{L^2}^2 dt \\
\leq \delta_0^{-2} \|u^0\|_{L^2}^2.
\]
As \( (t \to \|w_{\delta_0}(t)\|_{L^2}^2) \) is continuous, then there is a time \( t_0 \geq 0 \) such that
\[ (3.10) \|w_{\delta_0}(t_0)\|_{L^2}^2 < \varepsilon/2. \]
Combining inequalities (3.9)-(3.10) we get
\[ \|u(t_0)\|_{L^2} \leq \|v_{\delta_0}(t_0)\|_{L^2} + \|w_{\delta_0}(t_0)\|_{L^2} < \varepsilon. \]
As \( (t \to \|w_{\delta_0}(t)\|_{L^2}^2) \) is decreasing, then
\[ \|u(t)\|_{L^2} < \varepsilon, \forall t \geq t_0. \]
which complete the proof.

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