Chapter 7

Canonical Systems of Differential Equations

Boundary value problems for regular and singular canonical systems of differential equations are investigated. After a brief introduction to Hilbert spaces of $\mathbb{C}^2$-valued vector functions which are square-integrable with respect to some $2 \times 2$ matrix-valued function in Section 7.1, the class of canonical systems to be studied here is introduced in Section 7.2. In Section 7.3 the notions of regular, quasiregular, and singular endpoints for canonical systems are explained. The number of square-integrable solutions at an endpoint of the interval is studied in Section 7.4. Together with a monotonicity principle from Chapter 5, this leads to a limit-circle/limit-point classification of singular endpoints in the same way as in Weyl’s alternative in Chapter 6. The important concept of definiteness of canonical systems is defined and studied in Section 7.5, and a cut-off technique for solutions is provided. Afterwards, in Section 7.6, a symmetric minimal relation in the appropriate $L^2$-Hilbert space and its adjoint, the maximal relation, are associated with real definite canonical systems. The defect numbers of the minimal relation are specified for regular endpoints and for endpoints in the limit-circle or limit-point case. Boundary triplets and Weyl functions for canonical systems in the limit-circle case are constructed in Section 7.7, while the limit-point case is treated in Section 7.8. The connection between subordinate solutions and properties of the Weyl function, as well as the description of absolutely continuous and singular spectrum are studied in Section 7.9. Finally, in Section 7.10 some special classes of canonical systems of differential equations are discussed, among them weighted Sturm–Liouville equations.
7.1 Classes of integrable functions

The purpose of this section is to introduce classes of vector functions which are locally square-integrable with respect to a measurable nonnegative matrix function and to collect some useful properties of such functions.

Let \( \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) be an interval, not necessarily bounded, with endpoints \( a < b \), not necessarily belonging to \( \mathcal{I} \). In the following an integral of a vector function or a matrix function over \( \mathcal{I} \) or over a subinterval is always understood in the componentwise sense. The linear space \( \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) of locally integrable \( \mathbb{C}^{2} \)-valued vector functions consists of all measurable \( \mathbb{C}^{2} \)-valued vector functions \( f \) defined almost everywhere on \( \mathcal{I} \) such that for each compact subinterval \( K \subset \mathcal{I} \)

\[
\int_{K} |f(s)| \, ds < \infty.
\]

Here \( |x| \) denotes the Euclidean norm of \( x \) in \( \mathbb{C}^{2} \). Note that for \( f \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) and each compact subinterval \( K \subset \mathcal{I} \) the norm inequality

\[
\left| \int_{K} f(s) \, ds \right| \leq \int_{K} |f(s)| \, ds \tag{7.1.1}
\]

holds. A \( \mathbb{C}^{2} \)-valued vector function \( f \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) is said to be integrable at the left endpoint \( a \) of the interval \( \mathcal{I} \) or integrable at the right endpoint \( b \) of the interval \( \mathcal{I} \) if for some, and hence for all \( c \in \mathbb{R} \) with \( a < c < b \)

\[
\int_{a}^{c} |f(s)| \, ds < \infty \quad \text{or} \quad \int_{c}^{b} |f(s)| \, ds < \infty, \tag{7.1.2}
\]

respectively. Similarly, a measurable \( 2 \times 2 \) matrix function \( \Phi \) is locally integrable on \( \mathcal{I} \) if for each compact subinterval \( K \subset \mathcal{I} \)

\[
\int_{K} |\Phi(s)| \, ds < \infty;
\]

here and in the following \( |A| \) stands for the operator norm of a \( 2 \times 2 \) matrix \( A \). In particular,

\[
\left| \int_{K} \Phi(s) \, ds \right| \leq \int_{K} |\Phi(s)| \, ds. \tag{7.1.3}
\]

The linear space consisting of all locally integrable \( 2 \times 2 \) matrix functions on \( \mathcal{I} \) will also be denoted by \( \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \); it will be clear from the context if the values of the functions in \( \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) are vectors in \( \mathbb{C}^{2} \) or \( 2 \times 2 \) matrices. A \( 2 \times 2 \) matrix function \( \Phi \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{I}) \) is said to be integrable at the left endpoint \( a \) of the interval \( \mathcal{I} \) or integrable at the right endpoint \( b \) of the interval \( \mathcal{I} \) if (7.1.2) holds for some, and hence for all \( c \in \mathbb{R} \) with \( a < c < b \) and \( f \) replaced by \( \Phi \).
7.1. Classes of integrable functions

Note that all norms on the linear space of $2 \times 2$ matrices are equivalent and that the operator norm $|A|$ of a $2 \times 2$ matrix $A$ can be estimated by the Hilbert–Schmidt matrix norm as follows:

$$|A| \leq \|A\|_2 \leq \sqrt{2}|A|,$$

where $\|A\|_2 := \sqrt{\sum_{i,j=1}^{2} |a_{ij}|^2}$. (7.1.4)

For the product of $2 \times 2$ matrices $A$ and $B$ one has

$$|AB| \leq |A||B| \quad \text{and} \quad \|AB\|_2 \leq \|A\|_2 \|B\|_2.$$ (7.1.5)

In the case where the $2 \times 2$ matrix $A$ is nonnegative the trace norm of $A$ can be estimated by the Hilbert–Schmidt matrix norm:

$$\|A\|_2 \leq \text{tr}A \leq \sqrt{2}\|A\|_2,$$

where $\text{tr}A = a_{11} + a_{22}$. (7.1.6)

The following definition introduces the semi-inner product space $L^2_\Delta(\mathbb{i})$ of $\mathbb{C}^2$-valued functions which are square-integrable with respect to $\Delta$; it is assumed that $\Delta$ is a $2 \times 2$ matrix function on $\mathbb{i}$ and that $\Delta(s) \geq 0$ for almost every $s \in \mathbb{i}$. In order to express the seminorm on $L^2_\Delta(\mathbb{i})$ the notation

$$f(s)^*\Delta(s)f(s) = (\Delta(s)f(s), f(s))$$

will be useful. Here $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbb{C}^2$ and $f$ is any $\mathbb{C}^2$-function defined on the interval $\mathbb{i}$.

**Definition 7.1.1.** Let $\mathbb{i} \subset \mathbb{R}$ be an interval and let $\Delta$ be a measurable $2 \times 2$ matrix function such that $\Delta(s) \geq 0$ for almost every $s \in \mathbb{i}$. Then $L^2_\Delta(\mathbb{i})$ denotes the linear space of all measurable functions $f$ on $\mathbb{i}$ with values in $\mathbb{C}^2$ which are square-integrable with respect to $\Delta$, that is,

$$\|f\|^2_\Delta = \int_{\mathbb{i}} f(s)^*\Delta(s)f(s) \, ds = \int_{\mathbb{i}} |\Delta(s)^{1/2}f(s)|^2 \, ds < \infty.$$ (7.1.7)

The semidefinite inner product $(\cdot, \cdot)_\Delta$ on $L^2_\Delta(\mathbb{i})$ corresponding to the seminorm $\|\cdot\|_\Delta$ in (7.1.7) is given by

$$(f, g)_\Delta = \int_{\mathbb{i}} g(s)^*\Delta(s)f(s) \, ds, \quad f, g \in L^2_\Delta(\mathbb{i}).$$ (7.1.8)

**Theorem 7.1.2.** Let $\mathbb{i} \subset \mathbb{R}$ be an interval and let $\Delta$ be a measurable $2 \times 2$ matrix function such that $\Delta(s) \geq 0$ for almost every $s \in \mathbb{i}$. Then the linear space $L^2_\Delta(\mathbb{i})$ equipped with the seminorm (7.1.8) is complete.
This shows that the transformation $U$ is integrable with respect to $\Delta$ for each compact subinterval $i$. Hence, one has for all measurable functions $f$ with values in $\mathbb{C}^2$ on $i$ that

$$\int f(s)^* \Delta(s) f(s) \, ds = \int (Uf)(s)^* \Xi(s)(Uf)(s) \, ds.$$  

Written out in components this gives

$$\int f(s)^* \Delta(s) f(s) \, ds = \int |(Uf)_1(s)|^2 e_1(s) \, ds + \int |(Uf)_2(s)|^2 e_2(s) \, ds$$

$$= \int |(Uf)_1(s)|^2 d\mu_1(s) + \int |(Uf)_2(s)|^2 d\mu_2(s),$$

(7.1.9)

where the measures $\mu_1$ and $\mu_2$ are absolutely continuous with respect to the Lebesgue measure $m$ and their Radon–Nikodým derivatives are given by $e_1$ and $e_2$, respectively. Therefore, it is now clear that $f \in L^2_\Delta(i) \iff (Uf)_1 \in L^2_{d\mu_1}(i)$ and $(Uf)_2 \in L^2_{d\mu_2}(i)$.

This shows that the transformation $U$ maps the space $L^2_\Delta(i)$ bijectively onto $L^2_{d\mu_1}(i) \times L^2_{d\mu_2}(i)$ and from (7.1.9) one sees that the seminorms in $L^2_\Delta(i)$ and $L^2_{d\mu_1}(i) \times L^2_{d\mu_2}(i)$ are preserved. Therefore, the completeness of $L^2_\Delta(i)$ is a consequence of the completeness of $L^2_{d\mu_1}(i)$ and $L^2_{d\mu_2}(i)$. \hfill \square

The space $L^2_\Delta(i)$ has the following approximation property.

**Lemma 7.1.3.** Each element of the seminormed space $L^2_\Delta(i)$ can be approximated by functions in $L^2_\Delta(i)$ which have compact support.

**Proof.** Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of nondecreasing compact intervals such that $i = \bigcup_{n=1}^{\infty} K_n$. For $f \in L^2_\Delta(i)$ put $f_n(s) = f(s)$ for $s \in K_n$ and $f_n(s) = 0$ elsewhere. Then $f_n \in L^2_\Delta(i)$, $f_n$ has support in $K_n$, and

$$\|f - f_n\|_\Delta^2 = \int (f(s) - f_n(s))^* \Delta(s) (f(s) - f_n(s)) \, ds \to 0$$

as $n \to \infty$, by dominated convergence. \hfill \square

The space $L^2_{\Delta, loc}(i)$ consists of all $\mathbb{C}^2$-valued functions which are square-integrable with respect to $\Delta$ for each compact subinterval $K \subset i$, i.e.,

$$\int_K f(s)^* \Delta(s) f(s) \, ds < \infty.$$
A function $f \in \mathcal{L}^2_{\Delta \text{loc}}(i)$ is said to be "square-integrable with respect to $\Delta$" at the left endpoint $a$ of the interval $i$ or "square-integrable with respect to $\Delta$" at the right endpoint $b$ of the interval $i$ if for some, and hence for all $c \in \mathbb{R}$ with $a < c < b$,

$$\int_a^c f(s)^* \Delta(s) f(s) \, ds < \infty \quad \text{or} \quad \int_c^b f(s)^* \Delta(s) f(s) \, ds < \infty,$$

respectively. A function $f \in \mathcal{L}^2_{\Delta \text{loc}}(i)$ belongs to $\mathcal{L}^2_{\Delta}(i)$ if and only if $f$ is square-integrable with respect to $\Delta$ at both endpoints $a$ and $b$ of $i$.

Clearly, if $\Delta$ is a nonnegative matrix function and $f$ is a vector function, then

$$|\Delta(s) f(s)| = |\Delta(s)^{\frac{1}{2}} \Delta(s)^{\frac{1}{2}} f(s)| \leq |\Delta(s)^{\frac{1}{2}}| |\Delta(s)^{\frac{1}{2}} f(s)|.$$

Now the statement in the next lemma is a consequence of the Cauchy–Schwarz inequality and the fact that

$$|\Delta(s)^{\frac{1}{2}}|^2 = |\Delta(s)^{\frac{1}{2}} \Delta(s)^{\frac{1}{2}}| = |\Delta(s)|.$$

**Lemma 7.1.4.** Let $\Delta$ be a locally integrable nonnegative $2 \times 2$ matrix function on $i$ and let $K \subset i$ be compact. If $f \in \mathcal{L}^2_{\Delta}(K)$, then $\Delta f \in \mathcal{L}^1(K)$ and

$$\int_K |\Delta(s) f(s)| \, ds \leq \left( \int_K |\Delta(s)| \, ds \right)^\frac{1}{2} \left( \int_K f(s)^* \Delta(s) f(s) \, ds \right)^\frac{1}{2}.$$

In particular, if $f \in \mathcal{L}^2_{\Delta}(i)$, then $\Delta f \in \mathcal{L}^1_{\text{loc}}(i)$ and for all compact $K \subset i$

$$\int_K |\Delta(s) f(s)| \, ds \leq \left( \int_K |\Delta(s)| \, ds \right)^\frac{1}{2} \|f\|_{\Delta}.$$

Let $\mathcal{R} = \{f \in \mathcal{L}^2_{\Delta}(i) : \|f\|_{\Delta} = 0\}$, so that $\mathcal{R}$ is a linear space, and consider the quotient space

$$L^2_{\Delta}(i) := \mathcal{L}^2_{\Delta}(i)/\mathcal{R}$$

equipped with the scalar product induced by (7.1.8), that is, $(f, g)_{\Delta} = (\tilde{f}, \tilde{g})_{\Delta}$, where $\tilde{f}, \tilde{g} \in \mathcal{L}^2_{\Delta}(i)$ are representatives in the equivalence classes $f, g \in L^2_{\Delta}(i)$. From Theorem 7.1.2 it is clear that $L^2_{\Delta}(i)$ is a Hilbert space. When no confusion can arise, the equivalence classes in $L^2_{\Delta}(i)$ will also be referred to as functions that are square-integrable with respect to $\Delta$. Note that the compactly supported functions in $L^2_{\Delta}(i)$ are dense in $L^2_{\Delta}(i)$ by Lemma 7.1.3.

Recall that a $\mathbb{C}^2$-valued vector function $f$ on an open interval $i$ is "absolutely continuous" if there exists a $\mathbb{C}^2$-valued vector function $h \in \mathcal{L}^1_{\text{loc}}(i)$ such that

$$f(t) - f(s) = \int_s^t h(u) \, du \quad (7.1.10)$$
for all $s,t \in i$. In this case, $f$ is differentiable and $f' = h$ almost everywhere. The space of absolutely continuous $C^2$-valued vector functions is denoted by $AC(i)$. When $a \in \mathbb{R}$, then $AC[a,b]$ stands for the subclass of $f \in AC(a,b)$ for which $h \in L^1_{\text{loc}}(a,b)$ in (7.1.10) additionally belongs to $L^1(a,a')$ for some, and hence for all $a < a' < b$, in which case

$$f(t) - f(a) = \int_a^t h(u) \, du$$

holds for all $t \in (a,b)$ and thus $f(a) = \lim_{t \to a} f(t)$. When $b \in \mathbb{R}$ there is a similar notation $AC(a,b]$ and for $f \in AC(a,b]$ one has $f(b) = \lim_{t \to b} f(t)$. The notation $AC[a,b]$ is analogous.

### 7.2 Canonical systems of differential equations

This section offers a brief review of so-called $2 \times 2$ canonical systems of differential equations. The existence and uniqueness result for linear systems of differential equations will be discussed and properties of the corresponding fundamental matrices will be derived.

Let $i = (a,b) \subset \mathbb{R}$ be an open, not necessarily bounded, interval and let $H$ and $\Delta$ be $2 \times 2$ matrix functions defined almost everywhere on $i$ such that

$$H, \Delta \in L^1_{\text{loc}}(i), \quad H(t) = H(t)^*, \quad \Delta(t) \geq 0$$

(7.2.1)

for almost every $t \in i$. Furthermore, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(7.2.2)

and note that $J^* = -J = J^{-1}$. A canonical system is a system of differential equations of the form

$$J f'(t) - H(t) f(t) = \lambda \Delta(t) f(t) + \Delta(t) g(t), \quad t \in i, \quad \lambda \in \mathbb{C},$$

(7.2.3)

where $g \in L^2_{\Delta_{\text{loc}}}(i)$ is a function that is locally square-integrable with respect to $\Delta$ with values in $\mathbb{C}^2$. The condition $g \in L^2_{\Delta_{\text{loc}}}(i)$ implies that $\Delta g$ is locally integrable; cf. Lemma 7.1.4. In the general case of (7.2.3) one speaks of an inhomogeneous system, while if the term involving $\Delta g$ is absent, that is,

$$J f'(t) - H(t) f(t) = \lambda \Delta(t) f(t), \quad t \in i, \quad \lambda \in \mathbb{C},$$

(7.2.4)

one speaks of the corresponding homogeneous system.

A function $f$ on $i$ with values in $\mathbb{C}^2$ is said to be a solution of the canonical system (7.2.3) if $f$ belongs to $AC(i)$ and the equation (7.2.3) holds for almost every
Observe that if $f$ is a solution of (7.2.3), then $f$ is also a solution of (7.2.3) when $g \in \mathcal{L}^2_{\Delta,\text{loc}}(\iota)$ is replaced by $\tilde{g} \in \mathcal{L}^2_{\Delta,\text{loc}}(\iota)$ with $\Delta(g - \tilde{g}) = 0$. Furthermore, if $f$ is a solution of (7.2.3) and $h$ is a solution of (7.2.4), then $f + h$ is a solution of (7.2.3). In fact, the collection of all solutions of the homogeneous system (7.2.4) forms a linear space. The following result on the existence and uniqueness of solutions of initial value problems for inhomogeneous canonical systems will be useful.

**Theorem 7.2.1.** Let $g \in \mathcal{L}^2_{\Delta,\text{loc}}(\iota)$ and $\lambda \in \mathbb{C}$. Fix some $c_0 \in \iota = (a, b) = (a, b)$ and $\gamma \in \mathbb{C}^2$. Then the initial value problem

$$Jf'(t) - H(t)f(t) = \lambda \Delta(t)f(t) + \Delta(t)g(t), \quad f(c_0) = \gamma,$$

(7.2.5)

admits a unique solution $f \in AC(\iota)$. Moreover, the mapping $\lambda \mapsto f(t, \lambda)$ is entire for every fixed $t \in \iota$.

In order to prove this theorem one replaces the initial value problem (7.2.5) by an equivalent integral equation; recall that the functions $H$ and $\Delta$ are locally integrable. The integral equation can be solved, for instance, by successive iterations, which also leads to the statement concerning the mapping $\lambda \mapsto f(t, \lambda)$ being entire, see, e.g., [754, Theorem 2.1].

In the next lemma a *Lagrange identity* for solutions of the inhomogeneous canonical system is obtained.

**Lemma 7.2.2.** Assume that $\lambda, \mu \in \mathbb{C}$ and that $g, k \in \mathcal{L}^2_{\Delta,\text{loc}}(\iota)$. Let $f, h$ be solutions of the inhomogeneous equations

$$Jf'(t) - H(t)f(t) = \lambda \Delta(t)f(t) + \Delta(t)g(t),$$
$$Jh'(t) - H(t)h(t) = \mu \Delta(t)h(t) + \Delta(t)k(t),$$

respectively. Then for every compact interval $[\alpha, \beta] \subset \iota$,

$$h(\beta)^*Jf(\beta) - h(\alpha)^*Jf(\alpha) = \int_{\alpha}^{\beta} \left( h(s)^*\Delta(s)g(s) - k(s)^*\Delta(s)f(s) \right) ds + (\lambda - \mu) \int_{\alpha}^{\beta} h(s)^*\Delta(s)f(s) ds.$$

**Proof.** The assumptions that $J$ is skew-adjoint and that $H(t)$ and $\Delta(t)$ are self-adjoint almost everywhere on $\iota$ lead to the identities

$$(h^*Jf)' = h^*(Jf') - (Jh^*)f$$
$$= h^*(\lambda \Delta f + \Delta g + Hf) - (\mu \Delta h + \Delta k + Hh)^*f$$
$$= h^*\Delta g - k^*\Delta f + (\lambda - \mu)h^*\Delta f,$$

which are valid almost everywhere on $\iota$. Integration over the interval $[\alpha, \beta]$ completes the argument. \(\square\)
Taking $\lambda = \mu = 0$ in Lemma 7.2.2, one obtains the following corollary. It provides the form of the Lagrange identity that will be studied in detail later in this chapter.

**Corollary 7.2.3.** Assume that $g, k \in L^2_{\Delta, \text{loc}}(\iota)$. Let $f, h$ be solutions of the inhomogeneous equations

\begin{align*}
Jf'(t) - H(t)f(t) &= \Delta(t)g(t), \\
Jh'(t) - H(t)h(t) &= \Delta(t)k(t),
\end{align*}

respectively. Then for every compact interval $[\alpha, \beta] \subset \iota$,

\[ h(\beta)^* Jf(\beta) - h(\alpha)^* Jf(\alpha) = \int_{\alpha}^{\beta} (h(s)^* \Delta(s)g(s) - k(s)^* \Delta(s)f(s)) \, ds. \]

There is also a corollary of Lemma 7.2.2 involving solutions of the corresponding homogeneous system. Let $Y_1(\cdot, \lambda)$ and $Y_2(\cdot, \lambda)$ be solutions of (7.2.4) and define the *solution matrix*

\[ Y(\cdot, \lambda) = \begin{pmatrix} Y_1(\cdot, \lambda) & Y_2(\cdot, \lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}, \]

which is a $2 \times 2$ matrix function for each $\lambda \in \mathbb{C}$. Then the matrix function $Y(\cdot, \lambda)$ solves the equation (7.2.4) in the sense that it actually solves the matrix version of (7.2.4),

\[ JY'(t, \lambda) - H(t)Y(t, \lambda) = \lambda \Delta(t)Y(t, \lambda), \quad t \in \iota. \]

**Corollary 7.2.4.** Let $Y(\cdot, \lambda)$ be a solution matrix of the homogeneous canonical system (7.2.4). Then for every compact interval $[\alpha, \beta] \subset \iota$ and all $\lambda, \mu \in \mathbb{C}$,

\[ Y(\beta, \mu)^* JY(\beta, \lambda) - Y(\alpha, \mu)^* JY(\alpha, \lambda) = (\lambda - \mu) \int_{\alpha}^{\beta} Y(s, \mu)^* \Delta(s)Y(s, \lambda) \, ds. \]

In particular, for all $[\alpha, \beta] \subset \iota$ and all $\lambda \in \mathbb{C},$

\[ Y(\beta, \lambda)^* JY(\beta, \lambda) = Y(\alpha, \lambda)^* JY(\alpha, \lambda). \]

It is a consequence of Corollary 7.2.4 that for every solution matrix $Y(\cdot, \lambda)$ the function

\[ t \mapsto Y(t, \lambda)^* JY(t, \lambda) \]

is constant on $\iota$. Hence, if for some $c_0 \in \iota$

\[ Y(c_0, \lambda)^* JY(c_0, \lambda) = J, \]

then $Y(t, \lambda)^* JY(t, \lambda) = J$ for all $t \in \iota$. This shows that

\[ Y(t, \lambda)^{-1} = -JY(t, \lambda)^* J \quad \text{and} \quad Y(t, \lambda)^* = -JY(t, \lambda)J, \quad t \in \iota, \]

(7.2.9)
and thus it also follows that
\[ Y(t, \lambda) J Y(t, \lambda)^* = J, \quad t \in \mathfrak{I}. \quad (7.2.10) \]

Let \( X \) be an invertible matrix which does not depend on \( \lambda \) and assume that \( X^* J X = J \). Let \( Y(\cdot, \lambda) \) be the solution matrix which is fixed by the initial condition
\[ Y(c_0, \lambda) = X \quad (7.2.11) \]
for some \( c_0 \in \mathfrak{I} \) and all \( \lambda \in \mathbb{C} \). Then (7.2.8) is valid and hence (7.2.9) and (7.2.10) are satisfied; the matrix \( Y(\cdot, \lambda) \) is a fundamental matrix, that is, its columns are linearly independent solutions of the homogeneous canonical system (7.2.4) on \( \mathfrak{I} \).

Frequently the fundamental matrix \( Y(\cdot, \lambda) \) will be fixed by the initial condition
\[ Y(c_0, \lambda) = I \quad (7.2.12) \]
for some \( c_0 \in \mathfrak{I} \).

According to Theorem 7.2.1, there is a unique solution of the initial value problem (7.2.5). It is possible to express this unique solution in terms of the fundamental matrix \( Y(\cdot, \lambda) \) determined by the initial condition (7.2.12) (and in a similar way with the initial condition (7.2.11)). In fact, for any \( \lambda \in \mathbb{C} \), any \( \gamma \in \mathbb{C}^2 \), and any \( g \in \mathcal{L}^2_{\Delta, \text{loc}}(\mathfrak{I}) \), the unique solution of the inhomogeneous initial value problem
\[ J f' - H f = \lambda \Delta f + \Delta g, \quad f(c_0) = \gamma, \quad (7.2.13) \]
is provided by the variation of constant formula:
\[ f(t) = Y(t, \lambda) \gamma + Y(t, \lambda) \int_{c_0}^t Y(s, \lambda)^{-1} J^{-1} \Delta(s) g(s) \, ds. \quad (7.2.14) \]
This can be seen by verifying that the second term on the right-hand side is a solution of the inhomogeneous equation that vanishes at \( c_0 \). Making use of (7.2.9), one recasts (7.2.14) as
\[ f(t) = Y(t, \lambda) \gamma - Y(t, \lambda) \int_{c_0}^t Y(s, \lambda)^* \Delta(s) g(s) \, ds. \quad (7.2.15) \]
In terms of the notation (7.2.7) for the fundamental matrix \( Y(\cdot, \lambda) \) fixed by (7.2.12) the unique solution (7.2.15) of (7.2.13) can be written as
\[ f(t) = Y_1(t, \lambda) \gamma_1 + Y_2(t, \lambda) \gamma_2 \]
\[ + Y_1(t, \lambda) \int_{c_0}^t Y_2(s, \lambda)^* \Delta(s) g(s) \, ds \]
\[ - Y_2(t, \lambda) \int_{c_0}^t Y_1(s, \lambda)^* \Delta(s) g(s) \, ds, \quad (7.2.16) \]
where \( \gamma = (\gamma_1, \gamma_2)^\top \). This form of the solution will be used later.
The general form of the inhomogeneous equation (7.2.3) can be simplified by a transformation of the system. This transformation will be employed in Corollary 7.4.8.

**Lemma 7.2.5.** Let \( \lambda_0 \in \mathbb{R} \), \( c_0 \in \mathcal{T} \), and let \( U(\cdot, \lambda_0) \) be a solution matrix which satisfies

\[
JU'(\cdot, \lambda_0) - HU(\cdot, \lambda_0) = \lambda_0 \Delta U(\cdot, \lambda_0), \quad U(c_0, \lambda_0)^* JU(c_0, \lambda_0) = J. \tag{7.2.17}
\]

Assume that \( g \in L^2_{\Delta, \text{loc}}(\mathcal{T}) \) and let \( f \) be a solution of the inhomogeneous equation (7.2.3). Define the functions \( \tilde{f}, \tilde{g}, \) and \( \tilde{\Delta} \) by

\[
\tilde{f} = U(\cdot, \lambda_0)^{-1} f, \quad \tilde{g} = U(\cdot, \lambda_0)^{-1} g, \quad \tilde{\Delta}(\cdot) = U(\cdot, \lambda_0)^* \Delta(\cdot) U(\cdot, \lambda_0). \tag{7.2.18}
\]

Then \( \tilde{\Delta} \) is a locally integrable nonnegative measurable matrix function,

\[
\tilde{f}^* \tilde{\Delta} \tilde{f} = f^* \Delta f \quad \text{and} \quad \tilde{g}^* \tilde{\Delta} \tilde{g} = g^* \Delta g, \tag{7.2.19}
\]

and, in particular, \( \tilde{g} \in L^2_{\tilde{\Delta}, \text{loc}}(\mathcal{T}) \). Moreover, the function \( \tilde{f} \) is a solution of the system of differential equations

\[
J\tilde{f}' = (\lambda - \lambda_0) \tilde{\Delta} \tilde{f} + \tilde{\Delta} \tilde{g}. \tag{7.2.20}
\]

Conversely, if

\[
\tilde{f}, \tilde{g} \in L^2_{\tilde{\Delta}, \text{loc}}(\mathcal{T}) \quad \text{and} \quad \tilde{\Delta}(\cdot) = U(\cdot, \lambda_0)^* \Delta(\cdot) U(\cdot, \lambda_0)
\]

satisfy the equation (7.2.20), then \( f = U(\cdot, \lambda_0) \tilde{f} \) and \( g = U(\cdot, \lambda_0) \tilde{g} \) satisfy the inhomogeneous equation (7.2.3).

**Proof.** First observe that it is a direct consequence of (7.2.17) that the function \( U(\cdot, \lambda_0) \) satisfies

\[
U(\cdot, \lambda_0)^* JU(\cdot, \lambda_0) = J;
\]

cf. (7.2.8). In particular, this shows that \( U(t, \lambda_0) \) is invertible for each \( t \in \mathcal{T} \).

Let \( \tilde{f}, \tilde{g}, \) and \( \tilde{\Delta} \) be defined by (7.2.18). Then it is clear that (7.2.19) holds. Since \( g \in L^2_{\Delta, \text{loc}}(\mathcal{T}) \) it also follows that \( \tilde{g} \in L^2_{\tilde{\Delta}, \text{loc}}(\mathcal{T}) \). Moreover,

\[
Jf' - Hf = \lambda \Delta f + \Delta g \tag{7.2.21}
\]

holds by assumption. Substituting \( f = U(\cdot, \lambda_0) \tilde{f} \) and \( g = U(\cdot, \lambda_0) \tilde{g} \) in (7.2.21), multiplying by \( U(\cdot, \lambda_0)^* \) from the left, and using (7.2.17) a straightforward calculation leads to (7.2.20). Similarly, one verifies by a direct calculation that the converse statement holds. \( \square \)
It follows from (7.2.19) that the functions $\tilde{f}$ or $\tilde{g}$ in (7.2.18) are square-integrable with respect to $\tilde{\Delta}$ if and only if $f$ or $g$ are square-integrable with respect to $\Delta$, respectively. The transformation in Lemma 7.2.5 implies that the boundary terms $h(x)^* Jf(x)$ in the Lagrange formula of the original equation in Lemma 7.2.2 can be written in terms of the boundary terms of the corresponding solutions of the transformed equation (7.2.20).

**Corollary 7.2.6.** Assume that $\lambda, \mu \in \mathbb{C}$, $g, k \in L^2_{\Delta, \text{loc}}(\mathbb{R})$, and that $f, h$ are solutions of the inhomogeneous equations

$$Jf'(t) - H(t)f(t) = \lambda \Delta(t) f(t) + \Delta(t) g(t),$$
$$Jh'(t) - H(t)h(t) = \mu \Delta(t) h(t) + \Delta(t) k(t).$$

Assume that $U(\cdot, \lambda_0)$ with $\lambda_0 \in \mathbb{R}$ is a solution matrix which satisfies (7.2.17) and define the functions $\tilde{f} = U(\cdot, \lambda_0)^{-1} f$ and $\tilde{h} = U(\cdot, \lambda_0)^{-1} h$ as in (7.2.18). Then for each $t \in \mathbb{R}$

$$h(t)^* Jf(t) = \tilde{h}(t)^* J\tilde{f}(t).$$

Recall that the functions $H$ and $\Delta$ were assumed to be $2 \times 2$ matrix functions with complex entries. When these functions are real we enjoy a certain symmetry property.

**Definition 7.2.7.** The canonical system $Jf' - Hf = \lambda \Delta f$ is said to be real if the entries of the $2 \times 2$ matrix functions $H$ and $\Delta$ in (7.2.1) are real functions.

To deal with real canonical systems the notion of conjugate matrices is useful. For a matrix $T$ the conjugate matrix $\overline{T}$ is the matrix whose entries are the complex conjugates of the entries of $T$. Let $T$ and $S$ be matrices, not necessarily of the same size, for which the matrix product $TS$ is defined. Then clearly

$$\overline{TS} = \overline{T} \overline{S}. \quad (7.2.22)$$

**Lemma 7.2.8.** Assume that the canonical system (7.2.4) is real. Let $Y(\cdot, \lambda)$ be a solution matrix of (7.2.4) such that for all $\lambda \in \mathbb{C}$

$$Y(c_0, \overline{\lambda}) = Y(c_0, \lambda) \quad (7.2.23)$$

for some point $c_0 \in \mathbb{R}$. Then

$$Y(\cdot, \overline{\lambda}) = Y(\cdot, \lambda) \quad (7.2.24)$$

for all $\lambda \in \mathbb{C}$. In particular, (7.2.24) holds when $Y(\cdot, \lambda)$ is a fundamental matrix fixed by (7.2.11) or (7.2.12).

**Proof.** By definition, the solution matrix $Y(\cdot, \lambda)$ satisfies

$$JY'(\cdot, \lambda) - HY(\cdot, \lambda) = \lambda \Delta Y(\cdot, \lambda). \quad (7.2.25)$$
By assumption, the entries of $J$, $H$, and $\Delta$ are real; hence taking complex conjugates and using (7.2.22) one sees that

$$J\bar{Y}(\cdot, \lambda) - H\bar{Y}(\cdot, \lambda) = \bar{\Delta}\bar{Y}(\cdot, \lambda).$$

Therefore, the matrix function $\bar{Y}(\cdot, \bar{\lambda})$ satisfies the same equation (7.2.25) as $Y(\cdot, \lambda)$ and by (7.2.23) these matrix functions satisfy the same initial condition at $c_0$. Now the uniqueness in Theorem 7.2.1 leads to (7.2.24). \hfill \Box

The following observation is an easy consequence of Lemma 7.2.8.

**Corollary 7.2.9.** Let the system $Jf' - Hf = \lambda \Delta f$ be real and let a fundamental matrix $Y(\cdot, \lambda_0)$ be fixed by the initial condition $Y(c, \lambda_0) = I$ for some $a < c < b$. Then for every $u \in \mathbb{C}^2$

$$\int_i u^*Y(s, \lambda)^*\Delta(s)Y(s, \lambda)u \, ds = \int_i \bar{\pi}^*Y(s, \bar{\lambda})^*\Delta(s)Y(s, \bar{\lambda})\bar{\pi} \, ds.$$

In particular,

$$Y(\cdot, \lambda)u \in L^2_\Delta(i) \Leftrightarrow Y(\cdot, \bar{\lambda})\bar{\pi} \in L^2_\Delta(i).$$

**Proof.** Clearly, for any $u \in \mathbb{C}^2$ and all $s \in i$, one has

$$u^*Y(s, \lambda)^*\Delta(s)Y(s, \lambda)u \geq 0.$$

Therefore,

$$u^*Y(s, \lambda)^*\Delta(s)Y(s, \lambda)u = \bar{u}^*Y(s, \bar{\lambda})^*\Delta(s)Y(s, \bar{\lambda})\bar{u} = \bar{\pi}^*Y(s, \bar{\lambda})^*\Delta(s)Y(s, \bar{\lambda})\bar{\pi},$$

which gives the assertion. \hfill \Box

### 7.3 Regular and quasiregular endpoints

In this section the notions of regular and quasiregular for an endpoint of the interval $i$ are introduced; this makes it possible to extend Theorem 7.2.1, so that one may solve an initial value problem in an endpoint.

The following definition gives a classification for the endpoints of the canonical system (7.2.3).

**Definition 7.3.1.** An endpoint of the interval $i$ is said to be a quasiregular endpoint of the canonical system (7.2.3) if the locally integrable functions $H$ and $\Delta$ in (7.2.1) are integrable up to that endpoint. A finite quasiregular endpoint is called regular. An endpoint is said to be singular when it is not regular. The canonical system (7.2.3) is called regular if both endpoints are regular; otherwise it is called singular.
The main result in this section implies that if the term \( g \in L^2_{\Delta, \text{loc}}(\iota) \) in (7.2.3) is square-integrable with respect to \( \Delta \) at an endpoint which is regular or quasiregular, then every solution of the inhomogeneous equation has a continuous extension to that endpoint, so that it is square-integrable with respect to \( \Delta \) there.

**Proposition 7.3.2.** Assume that the endpoint \( a \) or \( b \) of \( \iota = (a, b) \) is regular or quasiregular and that \( g \in L^2_{\Delta, \text{loc}}(\iota) \) is square-integrable with respect to \( \Delta \) at \( a \) or \( b \), respectively. Then each solution \( f \) of (7.2.3) is square-integrable with respect to \( \Delta \) at \( a \) or at \( b \) and the limits

\[
f(a) := \lim_{t \to a} f(t) \quad \text{or} \quad f(b) := \lim_{t \to b} f(t) \quad (7.3.1)
\]

exist, respectively. Moreover, for each \( \gamma \in \mathbb{C}^2 \) there exists a unique solution \( f \) of (7.2.3) such that \( f(a) = \gamma \) or \( f(b) = \gamma \), respectively, and the corresponding function \( \lambda \mapsto f(t, \lambda) \) is entire for every \( t \in \iota \) and \( t = a \) or \( t = b \), respectively.

**Proof.** It suffices to consider the case of the endpoint \( b \). So let \( b \) be a regular or quasiregular endpoint, let \( \lambda \in \mathbb{C} \), and fix \( c \in (a, b) \). The proof is split in three separate steps.

**Step 1.** Any solution \( f \) of (7.2.3) with \( f(c) = \eta \) satisfies

\[
f(t) = \eta + \int_c^t J^{-1}(\lambda \Delta(s) + H(s)) f(s) \, ds + \int_c^t J^{-1} \Delta(s) g(s) \, ds \quad (7.3.2)
\]

with \( t \in \iota \). Recall that, since \( g \) is square-integrable with respect to \( \Delta \) at \( b \), it follows that \( \Delta g \) is integrable on \([c, b)\); cf. Lemma 7.1.4. By definition also \( \lambda \Delta + H \) is integrable on \([c, b)\). Hence, Gronwall’s lemma in Section 6.13 (see Lemma 6.13.2) shows that

\[
|f(t)| \leq \left( |\eta| + \int_c^t |\Delta(s) g(s)| \, ds \right) e^{\int_c^t |\lambda \Delta(s) + H(s)| \, ds}, \quad c \leq t < b. \quad (7.3.3)
\]

Thus, the solution \( f \) is bounded on \([c, b)\): \( |f(s)| \leq M, \ c < s < b \). In particular, this shows that

\[
\int_c^b f(s)^2 \Delta(s) f(s) \, ds \leq M^2 \int_c^b |\Delta(s)| \, ds < \infty,
\]

and hence \( f \) is square-integrable with respect to \( \Delta \) at \( b \). Moreover, it is clear from (7.3.2) that the limit \( f(b) = \lim_{t \to b} f(t) \) in (7.3.1) exists.

**Step 2.** In the special case where \( h \) is a solution of the homogeneous system (7.2.4) with \( h(c) = \eta \) it follows from (7.3.2) that

\[
h(b) = \eta + \int_c^b J^{-1}(\lambda \Delta(s) + H(s)) h(s) \, ds. \quad (7.3.4)
\]
The solution \( h(\cdot, \lambda) \) actually depends on \( \lambda \) and according to Theorem 7.2.1 for each \( c \leq t < b \) the function \( \lambda \mapsto h(t, \lambda) \) is entire. It will be shown that also \( \lambda \mapsto h(b, \lambda) \) is entire. In fact, from (7.3.4) it is clear that it suffices to prove that the mapping

\[
\lambda \mapsto \int_{c}^{b} J^{-1} (\lambda \Delta(s) + H(s)) h(s, \lambda) \, ds \tag{7.3.5}
\]

is entire. To see this note that (7.3.3) and the equality \( h(b) = \lim_{t \to b} h(t) \) imply that, for each compact set \( K \subset \mathbb{C} \),

\[
|h(t, \lambda)| \leq C_K e^{\int_{t}^{b} (|\Delta(s)| + |H(s)|) \, ds}
\]

for all \( c \leq t \leq b \) and for all \( \lambda \in K \). Hence, by dominated convergence, the mapping in (7.3.5) is continuous, and an application of Morera’s theorem implies that this mapping is holomorphic. Therefore, \( \lambda \mapsto h(b, \lambda) \) is entire.

**Step 3.** Let \( Z(\cdot, \lambda) \) be a fundamental matrix of the homogeneous equation (7.2.4) fixed by \( Z(c, \lambda) = I \). Then, according to Step 1 and Step 2, one has

\[
Z(t) = I + \int_{c}^{t} J^{-1} (\lambda \Delta(s) + H(s)) Z(s) \, ds
\]

and Gronwall’s lemma yields the estimate

\[
|Z(t, \lambda)| \leq e^{\int_{t}^{b} |\lambda \Delta(s) + H(s)| \, ds}, \quad c \leq t < b. \tag{7.3.6}
\]

Thus, \( Z(b, \lambda) = \lim_{t \to b} Z(t, \lambda) \) exists and it follows from Step 2 that the mapping \( \lambda \mapsto Z(b, \lambda) \) is entire. Moreover, from \( Z(t, \bar{\lambda})^* J Z(t, \lambda) = J \) for \( c \leq t < b \) one concludes by taking the limit \( t \to b \) that the matrix \( Z(b, \lambda) \) is invertible for all \( \lambda \in \mathbb{C} \). It is also clear that \( Z(b, \bar{\lambda})^* J Z(b, \lambda) = J \). Thus, the function \( U(\cdot, \lambda) \) defined by

\[
U(t, \lambda) = Z(t, \lambda) Z(b, \lambda)^{-1}
\]

is a fundamental matrix of the homogeneous equation which satisfies \( U(b, \lambda) = I \) and \( \lambda \mapsto U(t, \lambda) \) is entire for \( c \leq t \leq b \). Therefore, if \( \gamma \in \mathbb{C}^2 \) is fixed one sees that

\[
f(t) = U(t, \lambda) \gamma + U(t, \lambda) \int_{t}^{b} JU(s, \bar{\lambda})^* \Delta(s) g(s) \, ds
\]

is the unique solution of the inhomogeneous equation with \( f(b) = \gamma \); cf. (7.2.15). It remains to verify that \( \lambda \mapsto f(t, \lambda) \) is entire for \( c \leq t \leq b \). For this it suffices to check that

\[
\lambda \mapsto U(t, \lambda) \int_{t}^{b} JU(s, \bar{\lambda})^* \Delta(s) g(s) \, ds = Z(t, \lambda) \int_{t}^{b} JZ(s, \bar{\lambda})^* \Delta(s) g(s) \, ds
\]

is entire for \( c \leq t \leq b \), which can be seen with the help of (7.3.6) in the same way as in Step 2. □
Corollary 7.3.3. Assume that the endpoints $a$ and $b$ of the canonical system (7.2.3) are regular or quasiregular and that $g \in L^2_{\Delta}(\mathfrak{i})$. Then each solution $f$ of (7.2.3) belongs to $L^2_{\Delta}(\mathfrak{i})$ and both limits in (7.3.1) exist.

The next statement follows from Corollary 7.2.3 and Corollary 7.3.3.

Corollary 7.3.4. Assume that the endpoints $a$ and $b$ of the canonical system (7.2.3) are regular or quasiregular and that $g, k \in L^2_{\Delta}(\mathfrak{i})$. Let $f, h$ be solutions of the inhomogeneous equations (7.2.6). Then

$$h(b)^* Jf(b) - h(a)^* Jf(a) = \int_a^b \left( h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s) \right) ds.$$ 

Finally, the next statement is a consequence of Proposition 7.3.2 and identity (7.2.10).

Corollary 7.3.5. Assume that the endpoint $a$ or $b$ of the canonical system (7.2.3) is regular or quasiregular and let $Y(\cdot, \lambda)$ be a fundamental matrix of the canonical system (7.2.3). Then $Y(\cdot, \lambda) \phi$ is square-integrable with respect to $\Delta$ at $a$ or $b$ for every $\phi \in C^2_{\mathfrak{i}}$ and $Y(\cdot, \lambda)$ admits a unique continuous extension to $a$ or $b$ such that $Y(a, \lambda)$ or $Y(b, \lambda)$ is invertible, respectively. In particular, the point $c_0$ in (7.2.12) can be chosen to be $a$ or $b$, respectively.

### 7.4 Square-integrability of solutions of real canonical systems

Let $\mathfrak{i} = (a, b)$ be an open interval and consider on this interval the homogeneous system $Jf' - Hf = \lambda \Delta f$. Recall that a solution $f$, depending on $\lambda \in \mathbb{C}$, is called square-integrable with respect to $\Delta$ at $a$ or $b$ if for some $c \in \mathfrak{i}$

$$\int_a^c f(s)^* \Delta(s) f(s) ds < \infty \quad \text{or} \quad \int_c^b f(s)^* \Delta(s) f(s) ds < \infty,$$

respectively. In this section the existence of such solutions is studied for real canonical systems; cf. Definition 7.2.7. The first main result asserts that if there are two linearly independent solutions which are square-integrable with respect to $\Delta$ at an endpoint for some $\lambda \in \mathbb{C}$, then for any $\lambda \in \mathbb{C}$ all solutions are square-integrable with respect to $\Delta$ at that endpoint. The second main result states that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is at least one solution that is square-integrable with respect to $\Delta$ at an endpoint. A combination of these two results gives a general description of the existence of the solutions that are square-integrable with respect to $\Delta$ at an endpoint and leads to the limit-point and limit-circle classification.

In the rest of this section it will be assumed that the system (7.2.3) is real and the symmetry result in Corollary 7.2.9 will be used throughout.
Theorem 7.4.1. Assume that for $\lambda_0 \in \mathbb{C}$ the equation $Jf' - Hf = \lambda_0 \Delta f$ has two linearly independent solutions which are square-integrable with respect to $\Delta$ at $a$ or $b$. Then for any $\lambda \in \mathbb{C}$ each solution of $Jf' - Hf = \lambda \Delta f$ is square-integrable with respect to $\Delta$ at $a$ or $b$, respectively.

Proof. It is sufficient to show the result for one endpoint, say $b$. Assume without loss of generality that the endpoint $a$ of the canonical system (7.2.3) is regular. Fix a fundamental solution $Y(\cdot, \lambda_0)$ by the initial condition $Y(a, \lambda_0) = I$. The columns $Y_1(\cdot, \lambda_0)$ and $Y_2(\cdot, \lambda_0)$ of $Y(\cdot, \lambda)$ belong to $\mathcal{L}_\Delta^2(i)$ by assumption. As the system is assumed to be real, one has

$$
\int_a^b |\Delta(s)^{\frac{1}{2}} Y_i(s, \lambda)|^2 \, ds = \int_a^b |\Delta(s)^{\frac{1}{2}} Y_i(s, \lambda_0)|^2 \, ds, \quad i = 1, 2; \quad (7.4.1)
$$

cf. Corollary 7.2.9.

Let $\lambda \in \mathbb{C}$ and let $f(\cdot, \lambda)$ be any solution of $Jf' - Hf = \lambda \Delta f$. It will be shown that $f(\cdot, \lambda)$ is square-integrable with respect to $\Delta$ at $b$. Since the function $f(\cdot, \lambda)$ satisfies

$$
Jf'(\cdot, \lambda) - Hf(\cdot, \lambda) = \lambda_0 \Delta f(\cdot, \lambda) + (\lambda - \lambda_0) \Delta f(\cdot, \lambda),
$$

it follows from (7.2.16) (with $g = (\lambda - \lambda_0)f(\cdot, \lambda)$) that $f(\cdot, \lambda)$ can be written as

$$
f(t, \lambda) = Y_1(t, \lambda_0) \alpha_1 + Y_2(t, \lambda_0) \alpha_2 + (\lambda - \lambda_0) [Y_1(t, \lambda_0) y_2(t, \lambda) - Y_2(t, \lambda_0) y_1(t, \lambda)], \quad (7.4.2)
$$

where $f(a, \lambda) = (\alpha_1, \alpha_2)^T$ and $y_i(\cdot, \lambda)$ is defined by

$$
y_i(t, \lambda) = \int_a^t Y_i(s, \lambda_0)^* \Delta(s) f(s, \lambda) \, ds, \quad i = 1, 2,
$$

respectively. By applying the Cauchy–Schwarz inequality in the definition of $y_i(t, \lambda)$ and using (7.4.1) one obtains for $i = 1, 2$,

$$
|y_i(t, \lambda)| \leq \sqrt{\int_a^t |\Delta(s)^{\frac{1}{2}} Y_i(s, \lambda)|^2 \, ds} \sqrt{\int_a^t |\Delta(s)^{\frac{1}{2}} f(s, \lambda)|^2 \, ds}.
$$

Introduce the number $\alpha \geq 0$ and the nonnegative function $\varphi$ by

$$
\alpha = \max \{|\alpha_1|, |\alpha_2|\}, \quad \varphi(t) = \max \{|\Delta(t)^{\frac{1}{2}} Y_1(t, \lambda_0)|, |\Delta(t)^{\frac{1}{2}} Y_2(t, \lambda_0)|\},
$$

where $\Delta(t)^{\frac{1}{2}}$ is the principal square root of $\Delta(t)$.
so that $\varphi \in L^2(a, b)$. Multiply both sides in the identity in (7.4.2) from the left by $\Delta(t)^{\frac{1}{2}}$, then

$$|\Delta(t)^{\frac{1}{2}} f(t, \lambda)|$$

$$\leq 2\alpha \varphi(t) + 2|\lambda - \lambda_0| \varphi(t) \sqrt{\int_a^b \varphi(s)^2 \, ds} \sqrt{\int_a^t |\Delta(s)^{\frac{1}{2}} f(s, \lambda)|^2 \, ds}.$$  

Therefore, one obtains that

$$|\Delta(t)^{\frac{1}{2}} f(t, \lambda)|^2 \leq \varphi(t)^2 \left(A + B \int_a^t |\Delta(s)^{\frac{1}{2}} f(s, \lambda)|^2 \, ds\right),$$

where

$$A = 8\alpha^2, \quad B = 8|\lambda - \lambda_0|^2 \int_a^b \varphi(s)^2 \, ds.$$  

It follows from (7.4.3) by means of Lemma 6.1.4 with $u(t) = |\Delta(t)^{\frac{1}{2}} f(t, \lambda)|$, $\varphi$ as above, and $r = 1$, that the function $f(\cdot, \lambda)$ is square-integrable with respect to $\Delta$ at $b$. □

Next it will be shown that for each endpoint and any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is at least one solution of the homogeneous canonical system (7.2.4) which is square-integrable with respect to $\Delta$ at that endpoint. The proof of this fact is based on the monotonicity principle in Section 5.2; cf. Corollary 5.2.14. To apply this result, let $Y(\cdot, \lambda)$ be a fundamental matrix of the canonical system (7.2.3) fixed as in (7.2.12) and consider the $2 \times 2$ matrix function $D(\cdot, \lambda)$ on $\mathfrak{i}$ defined by

$$D(t, \lambda) = Y(t, \lambda)^*(-iJ)Y(t, \lambda), \quad t \in \mathfrak{i}, \quad \lambda \in \mathbb{C}.\quad (7.4.4)$$

Observe that the function $t \mapsto D(t, \lambda)$, $t \in \mathfrak{i}$, is absolutely continuous for every $\lambda \in \mathbb{C}$ and that the matrices $D(t, \lambda)$ are self-adjoint and invertible for all $t \in \mathfrak{i}$ and $\lambda \in \mathbb{C}$.

According to the following theorem, the matrix function in (7.4.4) admits self-adjoint limits at $a$ and $b$, which may be either self-adjoint matrices or self-adjoint relations with a one-dimensional domain and a one-dimensional multivalued part. Furthermore, the dimensions of the domains of the limit relations are directly connected with the number of linearly independent solutions of the homogeneous canonical system (7.2.4) that are square-integrable with respect to $\Delta$.

**Theorem 7.4.2.** For $\lambda \in \mathbb{C}^+$ or $\lambda \in \mathbb{C}^-$ the $2 \times 2$ matrix function $t \mapsto D(t, \lambda)$ is nondecreasing or nonincreasing on $\mathfrak{i}$, respectively. There exist self-adjoint relations $D(b, \lambda)$ and $D(a, \lambda)$ in $\mathbb{C}^2$ such that

$$D(t, \lambda) \to D(a, \lambda) \quad \text{and} \quad D(t, \lambda) \to D(b, \lambda)$$

in the (strong) resolvent sense when $t \to a$ and $t \to b$, respectively, and

$$1 \leq \dim \left(\operatorname{dom} D(a, \lambda)\right) \leq 2 \quad \text{and} \quad 1 \leq \dim \left(\operatorname{dom} D(b, \lambda)\right) \leq 2.$$
Furthermore, \( \phi \in \text{dom } D(a, \lambda) \) or \( \phi \in \text{dom } D(b, \lambda) \) if and only if \( Y(\cdot, \lambda)\phi \) is a solution of (7.2.4) that is square-integrable with respect to \( \Delta \) at \( a \) or \( b \), respectively.

**Proof.** It follows from Corollary 7.2.4 that

\[
D(\beta, \lambda) - D(\alpha, \lambda) = 2 \text{Im } \lambda \int_{\alpha}^{\beta} \Delta(s) Y(s, \lambda) ds, \quad \lambda \in \mathbb{C}, \quad (7.4.5)
\]

holds for any compact interval \([a, \beta] \subset \mathbb{R}\). Hence, the matrix function \( D(\cdot, \lambda) \) is nondecreasing for \( \lambda \in \mathbb{C}^+ \) and nonincreasing for \( \lambda \in \mathbb{C}^- \). It follows from Corollary 5.2.14 that there exist self-adjoint relations \( D(a, \lambda) \) and \( D(b, \lambda) \) such that

\[
\lim_{t \to a} (D(t, \lambda) - \mu)^{-1} = (D(a, \lambda) - \mu)^{-1}, \quad \mu \in \mathbb{C} \setminus \mathbb{R},
\]

and

\[
\lim_{t \to b} (D(t, \lambda) - \mu)^{-1} = (D(b, \lambda) - \mu)^{-1}, \quad \mu \in \mathbb{C} \setminus \mathbb{R}.
\]

Next it will be shown that the dimension of the domains of the self-adjoint relations \( D(a, \lambda) \) and \( D(b, \lambda) \) is at least one. For this it is sufficient to prove that there exists at least one (finite) eigenvalue.

Note first that, by (7.4.4) and (7.2.12),

\[
D(c_0, \lambda) = Y(c_0, \lambda)^* (-iJ) Y(c_0, \lambda) = -iJ
\]

and hence the eigenvalues of \( D(c_0, \lambda) \) are \( \nu_-(c_0) = -1 \) and \( \nu_+(c_0) = 1 \). As the function \( D(\cdot, \lambda) \) is continuous on \( \mathbb{R} \), the same holds true for its eigenvalues \( \nu_-(\cdot) \) and \( \nu_+(\cdot) \). Since the matrices \( D(t, \lambda) \) are self-adjoint and invertible for all \( t \in \mathbb{R} \) it follows that \( \nu_-(t) < 0 \) and \( \nu_+(t) > 0 \) for all \( t \in \mathbb{R} \). Recall that

\[
\nu_-(t) = \inf_{|x|=1} (D(t, \lambda)x, x) \quad \text{and} \quad \nu_+(t) = \sup_{|x|=1} (D(t, \lambda)x, x),
\]

and since \( D(t_1, \lambda) \leq D(t_2, \lambda) \), \( t_1 \leq t_2 \), it follows that

\[
\nu_-(t_1) \leq \nu_-(t_2) \quad \text{and} \quad \nu_+(t_1) \leq \nu_+(t_2), \quad t_1 \leq t_2.
\]

Therefore, it is clear that the limits of \( \nu_-(t) \) and \( \nu_+(t) \) exist and that

\[
\nu_-(b) = \lim_{t \to b} \nu_-(t) \leq 0 \quad \text{and} \quad 0 < \nu_+(b) = \lim_{t \to b} \nu_+(t) \leq \infty.
\]

In order to see the connection of these limits with the self-adjoint relation \( D(b, \lambda) \) observe that for \( \mu \in \mathbb{C} \setminus \mathbb{R} \)

\[
\frac{1}{\nu_-(t) - \mu} \quad \text{and} \quad \frac{1}{\nu_+(t) - \mu}
\]

are the eigenvalues of the matrix \( (D(t, \lambda) - \mu)^{-1} \). Therefore, again by continuity, one sees that

\[
\frac{1}{\nu_-(b) - \mu} \quad \text{and} \quad \frac{1}{\nu_+(b) - \mu}
\]
are the eigenvalues of the matrix \((D(b,\lambda) - \mu)^{-1}\). Hence, \(\nu_-(b)\) is a nonpositive eigenvalue of the self-adjoint relation \(D(b,\lambda)\), which implies \(\dim(\text{dom } D(b,\lambda)) \geq 1\). More precisely, if \(\nu_-(b) < \infty\), then \(\nu_+(b)\) is a positive eigenvalue of \(D(b,\lambda)\), in which case \(\dim(\text{dom } D(b,\lambda)) = 2\), while if \(\nu_+(b) = \infty\), then \(D(b,\lambda)\) has a one-dimensional multivalued part and \(\dim(\text{dom } D(b,\lambda)) = 1\). Similar observations may be made for the self-adjoint relation \(D(a,\lambda)\). In particular, it follows that \(\dim(\text{dom } D(a,\lambda)) \geq 1\).

Finally, it will be shown that \(\phi \in \text{dom } D(b,\lambda)\) if and only if the solution \(Y(\cdot,\lambda)\) of (7.2.4) is square-integrable with respect to \(\Delta\) at \(b\); the argument for the left endpoint \(a\) is the same. Suppose that \(\lambda \in \mathbb{C}^+\), so that \(D(\cdot,\lambda)\) is nondecreasing on \(I\). In this case it follows from Corollary 5.2.13 and Corollary 5.2.14 that

\[
\text{dom } D(b,\lambda) = \{ \phi \in C^2 : \lim_{t \to b} \phi^* D(t,\lambda) \phi < \infty \}
\]

and hence (7.4.5) implies that \(\phi \in \text{dom } D(b,\lambda)\) if and only if

\[
\int_a^b \phi^* Y(s,\lambda)^* \Delta(s) Y(s,\lambda) \phi \, ds < \infty,
\]

that is, the solution \(Y(\cdot,\lambda)\phi\) is square-integrable with respect to \(\Delta\) at \(b\). The case where \(\lambda \in \mathbb{C}^-\) is dealt with in a similar way.

A combination of Theorems 7.4.1 and 7.4.2 leads to the following observation.

**Corollary 7.4.3.** If for some \(\lambda_0 \in \mathbb{C} \setminus \mathbb{R}\) the equation \(Jf' - Hf = \lambda_0 \Delta f\) has, up to scalar multiples, only one nontrivial solution which is square-integrable with respect to \(\Delta\) at \(a\) or \(b\), then for any \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) the equation \(Jf' - Hf = \lambda \Delta f\) has, up to scalar multiples, precisely one nontrivial solution which is square-integrable with respect to \(\Delta\) at \(a\) or \(b\), respectively.

**Proof.** It is sufficient to consider the endpoint \(b\). Assume that for some \(\lambda_0 \in \mathbb{C} \setminus \mathbb{R}\) the equation \(Jf' - Hf = \lambda_0 \Delta f\) has, up to scalar multiples, only one nontrivial solution that is square-integrable with respect to \(\Delta\) at \(b\), and suppose that for some \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) with \(\lambda \neq \lambda_0\) the equation \(Jf' - Hf = \lambda \Delta f\) does not have, up to scalar multiples, only one nontrivial solution which is square-integrable with respect to \(\Delta\) at \(b\). Since

\[
1 \leq \dim(\text{dom } D(b,\lambda)) \leq 2
\]

by Theorem 7.4.2 there exist two linearly independent solutions of \(Jf' - Hf = \lambda \Delta f\) which are square-integrable with respect to \(\Delta\) at \(b\). But then Theorem 7.4.1 implies that there also exist two linearly independent solutions of \(Jf' - Hf = \lambda_0 \Delta f\) that are square-integrable with respect to \(\Delta\) at \(b\); a contradiction.

Theorem 7.4.1 and Corollary 7.4.3 yield the limit-point and limit-circle classification for real canonical systems in the next definition and corollary. The terminology is inspired by the terminology for Sturm–Liouville equations in Section 6.1.
Definition 7.4.4. For a real canonical system the endpoint $a$ or $b$ of the interval $I$ is said to be in the limit-circle case if for some, and hence for all $\lambda \in \mathbb{C}$ there exist two linearly independent solutions of $Jf' - Hf = \lambda \Delta f$ that are square-integrable with respect to $\Delta$ at $a$ or $b$, respectively. The endpoint $a$ or $b$ of the interval $I$ is said to be in the limit-point case if for some, and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists, up to scalar multiples, only one nontrivial solution of $Jf' - Hf = \lambda \Delta f$ that is square-integrable with respect to $\Delta$ at $a$ or $b$, respectively.

Note that, by Theorem 7.4.2, at any endpoint of the interval $I$ there is at least one nontrivial solution that is square-integrable with respect to $\Delta$ and there are at most two linearly independent solutions that are square-integrable with respect to $\Delta$. This leads to Weyl’s alternative for canonical systems.

Corollary 7.4.5. For a real canonical system each of the endpoints of the interval is either in the limit-circle case or in the limit-point case.

For completeness also the special case of regular and quasiregular endpoints is briefly discussed. The next corollary is an immediate consequence of Corollary 7.3.5.

Corollary 7.4.6. A regular or quasiregular endpoint of a real canonical system is in the limit-circle case.

A simple but useful characterization of the limit-point case is given in the following corollary. It is stated for the endpoint $b$, but clearly there is a similar statement for the endpoint $a$.

Corollary 7.4.7. Let the canonical system be real and assume that the endpoint $a$ is regular or quasiregular. Then the following statements hold:

(i) If the endpoint $b$ is in the limit-point case, then for all $\lambda \in \mathbb{R}$ the equation $Jf' - Hf = \lambda \Delta f$ has, up to scalar multiples, at most one nontrivial solution that is square-integrable with respect to $\Delta$ at $b$.

(ii) If there exists $\lambda_0 \in \mathbb{R}$ such that the equation $Jf' - Hf = \lambda_0 \Delta f$ has, up to scalar multiples, at most one nontrivial solution that is square-integrable with respect to $\Delta$ at $b$, then the endpoint $b$ is in the limit-point case.

Proof. (i) If there exists $\lambda \in \mathbb{R}$ for which the homogeneous equation has two linearly independent solutions that are square-integrable with respect to $\Delta$ at $b$, then by Theorem 7.4.1, for each $\lambda \in \mathbb{C}$ all nontrivial solutions are square-integrable with respect to $\Delta$ at $b$. Hence, $b$ is in the limit-circle case; a contradiction.

(ii) If $b$ is in the limit-circle case, then for all $\lambda \in \mathbb{C}$, and hence for $\lambda \in \mathbb{R}$, the homogeneous equation has two linearly independent solutions which are square-integrable with respect to $\Delta$ at $b$. This implies (ii). □

If, for instance, the endpoint $b$ is regular or quasiregular, then any solution of (7.2.3) with $g$ square-integrable with respect to $\Delta$ at $b$ has a limit at $b$ by
Proposition 7.3.2 and \( b \) is in the limit-circle case by Corollary 7.4.6. However, if \( b \) is in the limit-circle case, then the solutions of (7.2.3) are square-integrable with respect to \( \Delta \) at \( b \), but they do not necessarily have a limit at \( b \). It will be shown in this case that there exists a natural transformation which turns the system into one where \( b \) is quasiregular; cf. Lemma 7.2.5.

Corollary 7.4.8. Assume that \( a \) is regular and that \( b \) is in the limit-circle case. Let \( g \in \mathcal{L}^2_{\Delta}(a,b) \) and let \( f(\cdot, \lambda) \) be a solution of

\[
Jf' - Hf = \lambda \Delta f + \Delta g.
\]

Let \( U(\cdot, \lambda_0), \lambda_0 \in \mathbb{R} \), be a matrix function as in (7.2.17). Then the limit

\[
\tilde{f}(b) = \lim_{t \to b} U(t, \lambda_0)^{-1} f(t) \tag{7.4.6}
\]

exists in \( \mathbb{C}^2 \). Moreover, for each \( \gamma \in \mathbb{C}^2 \) there exists a unique solution \( f(\cdot, \lambda) \) of (7.2.3) such that \( \tilde{f}(b) = \gamma \) and the corresponding function

\[
\lambda \mapsto \lim_{t \to b} U(t, \lambda_0)^{-1} f(t, \lambda)
\]

is entire.

Proof. Let \( g \in \mathcal{L}^2_{\Delta}(a,b) \) and let \( f \) be a solution of (7.2.3). Since \( b \) is in the limit-circle case, there exists for \( \lambda_0 \in \mathbb{R} \) and \( \epsilon_0 \in [a,b) \) a matrix function \( U(\cdot, \lambda_0) \) satisfying (7.2.17) that is square-integrable with respect to \( \Delta \) at \( b \). Thus, the function \( \tilde{\Delta} \) defined in (7.2.18) is integrable at \( b \), which means that the endpoint \( b \) for the system in (7.2.20) is quasiregular. Since \( g \) is square-integrable with respect to \( \Delta \) at \( b \), the function \( \tilde{g} \) in Lemma 7.2.5 is square-integrable with respect to \( \tilde{\Delta} \) at \( b \). Therefore, the assertion is clear from Proposition 7.3.2 as \( \tilde{f} \) is a solution of (7.2.20). \( \square \)

Let the endpoint \( a \) be regular or quasiregular. Let \( g, k \in \mathcal{L}^2_{\Delta}(a) \) and let \( f, h \) be solutions of the inhomogeneous equations (7.2.6) such that \( f, h \in \mathcal{L}^2_{\Delta}(a) \). Then for \( a \leq t < b \) one has

\[
h(t)^* Jf(t) - h(a)^* Jf(a) = \int_a^t (h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s)) \, ds \tag{7.4.7}
\]

by the Lagrange identity in Corollary 7.2.3. It follows from (7.4.7) that the limit

\[
\lim_{t \to b} h(t)^* Jf(t)
\]

exists. Of course, when \( b \) is regular or quasiregular, then the individual limits \( \lim_{t \to b} f(t) \) and \( \lim_{t \to b} h(t) \) exist by Proposition 7.3.2, see also Corollary 7.3.4. In general the existence of the individual limits \( \lim_{t \to b} f(t) \) and \( \lim_{t \to b} h(t) \) is not guaranteed. However, in the case where \( b \) is in the limit-circle case but not quasiregular the next corollary suggests to employ the limits in (7.4.6).
Corollary 7.4.9. Assume that the endpoint \( a \) is regular and that \( b \) is in the limit-circle case. Let \( g, k \in \mathcal{L}_2^2(\iota) \) and let \( f, h \) be solutions of the inhomogeneous equations (7.2.6) such that \( f, h \in \mathcal{L}_2^2(\iota) \). Then
\[
\lim_{t \to b} h(t)^* J f(t) = \tilde{h}(b)^* J \tilde{f}(b),
\]
where \( \tilde{f}(b) \) and \( \tilde{h}(b) \) are as in (7.4.6). Moreover,
\[
\tilde{h}(b)^* J \tilde{f}(b) - h(a)^* J f(a) = \int_a^b \left( h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s) \right) ds.
\]
Proof. It follows by taking limits in (7.4.7) that
\[
\lim_{t \to b} h(t)^* J f(t) - h(a)^* J f(a) = \int_a^b \left( h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s) \right) ds.
\]
Now apply Corollary 7.2.6 and Corollary 7.4.8. Take the limit \( t \to b \) and (7.4.8) and (7.4.9) follow. \( \square \)

7.5 Definite canonical systems

The general class of canonical differential equations as in (7.2.3) will now be narrowed down by imposing a definiteness condition; see Definition 7.5.5. This condition will be assumed in the rest of this chapter. In this section various equivalent formulations of the definiteness condition will be presented. Moreover, it will be shown that the solution of a definite canonical system (7.2.3) can be cut off near an endpoint of the interval \( \iota \), in the sense that the solution is modified in such a way that it becomes trivial in a neighborhood of that endpoint.

It will be convenient to begin the discussion of definiteness of the canonical system (7.2.3) with the notion of definiteness when the system is restricted to an arbitrary subinterval \( j \subset \iota \).

Definition 7.5.1. Let \( j \subset \iota \) be a nonempty interval. The canonical system (7.2.3) is said to be definite on \( j \) if for each solution \( f \) of \( J f' - H f = 0 \) on \( j \) one has
\[
\Delta(t) f(t) = 0, \ t \in j \ \Rightarrow \ f(t) = 0, \ t \in j.
\]

Observe that if a solution \( f \) of the canonical system (7.2.3) vanishes on a nonempty subinterval \( j \subset \iota \), then \( f(t) = 0 \) for \( t \in \iota \); cf. Theorem 7.2.1. Hence, it is clear that if the canonical system (7.2.3) is definite on \( j \), then it is also definite on every interval \( \bar{j} \) with the property that \( j \subset \bar{j} \subset \iota \). Also observe that with the subinterval \( j \subset \iota \) and a continuous function \( f \) one has
\[
\Delta(t) f(t) = 0, \ t \in j \ \Leftrightarrow \ \int_j f(s)^* \Delta(s) f(s) ds = 0.
\]
Clearly, if $\Delta(t)$ has full rank for almost all $t \in j$, then the canonical system is automatically definite on $j$.

**Lemma 7.5.2.** Let $j \subset \bar{i}$ be a nonempty interval. The canonical system (7.2.3) is definite on the interval $j \subset \bar{i}$ if and only if for all $\lambda \in \mathbb{C}$ and for each solution $f$ of $Jf' - Hf = \lambda \Delta f$ on $j$ one has

$$\Delta(t)f(t) = 0, \quad t \in j \quad \Rightarrow \quad f(t) = 0, \quad t \in j.$$

**Proof.** Assume that the canonical system is definite on $j$. Choose $\lambda \in \mathbb{C}$ and let $f$ be a solution of $Jf' - Hf = \lambda \Delta f$ on $j$ with $\Delta(t)f(t) = 0$ for almost all $t \in j$. Thus, $f$ is a solution of $Jf' - Hf = 0$ with $\Delta(t)f(t) = 0$ for almost all $t \in j$. By assumption this implies that $f(t) = 0$ for $t \in j$. The converse statement is trivial. □

The following result is an alternative useful version of Lemma 7.5.2 in terms of a fundamental matrix $Y(\cdot, \lambda)$.

**Corollary 7.5.3.** Let $Y(\cdot, \lambda)$, $\lambda \in \mathbb{C}$, be a fundamental matrix for (7.2.3) and let $I \subset \bar{i}$ be a compact interval. Then the system (7.2.3) is definite on $I$ if and only if the $2 \times 2$ matrix

$$\int_I Y(s, \lambda)^* \Delta(s)Y(s, \lambda) \, ds \quad (7.5.2)$$

is invertible for some, and hence for all $\lambda \in \mathbb{C}$.

**Proof.** Assume that (7.2.3) is definite on $I$. If the (nonnegative) matrix in (7.5.2) is not invertible, then there exists a nontrivial $\gamma \in \mathbb{C}^2$ for which

$$\gamma^* \left( \int_I Y(s, \lambda)^* \Delta(s)Y(s, \lambda) \, ds \right) \gamma = 0, \quad (7.5.3)$$

or alternatively $\Delta(t)Y(t, \lambda)\gamma = 0$ for $t \in I$; cf. (7.5.1). Since $Y(\cdot, \lambda)\gamma$ is a solution of $Jf' - Hf = \lambda \Delta f$, it follows from the definiteness that $Y(t, \lambda)\gamma = 0$ for $t \in I$, which implies $\gamma = 0$. This contradiction shows that the matrix in (7.5.2) is invertible.

Conversely, assume that the (nonnegative) matrix in (7.5.2) is invertible. In order to show that (7.2.3) is definite, let

$$Jf'(t) - Hf(t) = \lambda \Delta(t)f(t), \quad \Delta(t)f(t) = 0, \quad t \in I.$$

Since $Y(\cdot, \lambda)$ is a fundamental matrix of $Jf' - Hf = \lambda \Delta f$, every solution of this equation can be written in the form $f = Y(\cdot, \lambda)\gamma$ with a unique $\gamma \in \mathbb{C}^2$. The condition $\Delta(t)f(t) = 0$, $t \in I$, implies that (7.5.3) holds. Therefore, $\gamma = 0$ and thus the system (7.2.3) is definite. □

The next proposition shows that there is no difference between global definiteness and local definiteness.
Proposition 7.5.4. The canonical system (7.2.3) is definite on \( \mathbb{I} \) if and only if there exists a compact interval \( I \subset \mathbb{I} \) such that the canonical system (7.2.3) is definite on the interval \( I \).

Proof. If the canonical system (7.2.3) is definite on the interval \( I \), then it is clearly definite on the larger interval \( \mathbb{I} \).

To see the converse statement, let the canonical system (7.2.3) be definite on the interval \( \mathbb{I} \); in other words, assume that for each solution \( f \) of \( Jf' - Hp = 0 \) on \( \mathbb{I} \) one has

\[
\Delta(t)f(t) = 0, \quad t \in \mathbb{I} \quad \Rightarrow \quad f(t) = 0, \quad t \in \mathbb{I}.
\]

Introduce for each compact subinterval \( K \) of \( \mathbb{I} \) the subset \( d(K) \) of \( C^2 \) by

\[
d(K) = \{ \phi \in C^2 : |\phi| = 1, \int_K \phi^*Y(s,0)^*\Delta(s)Y(s,0)\phi ds = 0 \}.
\]

Clearly, \( d(K) \) is compact and \( K \subset \bar{K} \) implies \( d(\bar{K}) \subset d(K) \). Now choose an increasing sequence of compact intervals \( (K_n) \) such that their union equals the interval \( \mathbb{I} \). Then

\[
\bigcap_{n \in \mathbb{N}} d(K_n) = \emptyset. \tag{7.5.4}
\]

Indeed, assume that there exists an element \( \phi \in C^2 \) with \( |\phi| = 1 \), such that

\[
\int_{K_n} \phi^*Y(s,0)^*\Delta(s)Y(s,0)\phi ds = 0
\]

for every \( n \in \mathbb{N} \). Then, by monotone convergence,

\[
\int_{\mathbb{I}} \phi^*Y(s,0)^*\Delta(s)Y(s,0)\phi ds = 0.
\]

As the canonical system (7.2.3) is definite, this implies by (7.5.1) that \( Y(\cdot,0)\phi = 0 \), which leads to \( \phi = 0 \); a contradiction. Therefore, the identity (7.5.4) is valid. Since each of the sets \( d(K_n) \) in (7.5.4) is compact, it follows that there exists a compact interval \( K_m \) such that \( d(K_m) = \emptyset \). Hence, \( I = K_m \) satisfies the requirements. To see this, let \( Jf' - Hp = 0 \) on \( K_m \) and assume that \( \Delta(t)f(t) = 0, \quad t \in K_m \), or, equivalently, \( \int_{K_m} f(s)^*\Delta(s)f(s) = 0; \) cf. (7.5.1). Since \( d(K_m) = \emptyset \) one concludes that \( f = 0 \). \( \square \)

In the rest of the text one often speaks of definite systems in the following sense.

Definition 7.5.5. The canonical system (7.2.3) is said to be definite if it is definite on \( \mathbb{I} \).

The next result is about smoothly cutting off the solution of a definite canonical system (7.2.3) near an endpoint of the interval \( \mathbb{I} \), i.e., modifying the solution so that it becomes trivial in a neighborhood of that endpoint. The following proposition and corollary will be used in Section 7.6.
Proposition 7.5.6. Let the canonical system (7.2.3) be definite and choose a compact interval \([ \alpha, \beta ] \subset \mathbb{i}\) such that the system is definite on \([ \alpha, \beta ]\). Let \(g \in L^2_{\Delta, \text{loc}} (\mathbb{i})\) and let \(f \in AC (\mathbb{i})\) be a solution of the inhomogeneous equation (7.2.3) for some \(\lambda \in \mathbb{C}\). Then there exist functions \(f_a \in AC (\mathbb{i})\) and \(g_a \in L^2_{\Delta, \text{loc}} (\mathbb{i})\) satisfying
\[
J f_a'(t) - H(t) f_a(t) = \lambda \Delta(t) f_a(t) + \Delta(t) g_a(t)
\]
such that
\[
f_a(t) = \begin{cases} f(t), & t \in (a, \alpha], \\ 0, & t \in [\beta, b), \end{cases} \quad \text{and} \quad g_a(t) = \begin{cases} g(t), & t \in (a, \alpha], \\ 0, & t \in [\beta, b). \end{cases}
\]
Similarly, there exist functions \(f_b \in AC (\mathbb{i})\) and \(g_b \in L^2_{\Delta, \text{loc}} (\mathbb{i})\) satisfying
\[
J f_b'(t) - H(t) f_b(t) = \lambda \Delta(t) f_b(t) + \Delta(t) g_b(t)
\]
such that
\[
f_b(t) = \begin{cases} 0, & t \in (a, \alpha], \\ f(t), & t \in [\beta, b), \end{cases} \quad \text{and} \quad g_b(t) = \begin{cases} 0, & t \in (a, \alpha], \\ g(t), & t \in [\beta, b). \end{cases}
\]

Proof. Let the functions \(f\) and \(g\) be as indicated. The result will be proved for the functions \(f_b\) and \(g_b\); the proof for the functions \(f_a\) and \(g_a\) is similar.

Let \([ \alpha, \beta ] \subseteq \mathbb{i}\) be a compact interval on which the canonical system (7.2.3) is definite; cf. Proposition 7.5.4. Let \(k \in L^2_{\Delta}(\alpha, \beta)\) and fix a fundamental system \(Y(\cdot, \lambda)\) by the initial condition \(Y(\alpha, \lambda) = I\). According to (7.2.15), the function defined by
\[
h(t) = -Y(t, \lambda) \int_{\alpha}^{t} JY(s, \lambda)^* \Delta(s) k(s) \, ds 
\]
satisfies the inhomogeneous equation
\[
J h'(t) - H(t) h(t) = \lambda \Delta(t) h(t) + \Delta(t) k(t), \quad \alpha < t < \beta,
\]
and in the endpoints it has the values
\[
h(\alpha) = 0 \quad \text{and} \quad h(\beta) = -Y(\beta, \lambda) \int_{\alpha}^{\beta} JY(s, \lambda)^* \Delta(s) k(s) \, ds.
\]
It will be shown that there exists a function \(k \in L^2_{\Delta}(\alpha, \beta)\) such that \(h(\beta) = f(\beta)\).

In order to verify this, observe that \(Y(\beta, \lambda)\) is invertible and that the integral operator
\[
\ell \mapsto \int_{\alpha}^{\beta} JY(s, \lambda)^* \Delta(s) \ell(s) \, ds
\]
taking $L^2_\Delta(\alpha, \beta)$ into $\mathbb{C}^2$ is surjective. To see this, assume that $\gamma \in \mathbb{C}^2$ is orthogonal to the range of this integral operator, that is,

$$0 = \gamma^* \int_{\alpha}^{\beta} JY(s, \lambda) \Delta(s) \ell(s) \, ds = \int_{\alpha}^{\beta} (Y(s, \lambda)J^*\gamma)^* \Delta(s) \ell(s) \, ds$$

for all $\ell \in L^2_\Delta(\alpha, \beta)$. With $\ell(s) = Y(s, \lambda)J^*\gamma$ it then follows from (7.5.1) and Lemma 7.5.2 that $\ell(s) = 0$ for $s \in (\alpha, \beta)$, which implies that $\gamma = 0$. Thus, the integral operator is surjective.

Now choose $k \in L^2_\Delta(\alpha, \beta)$ as above, so that $h$ defined by (7.5.5) satisfies $h(\alpha) = 0$ and $h(\beta) = f(\beta)$. Hence, the functions $f_b$ and $g_b$ defined by

$$f_b(t) = \begin{cases} 0, & t \in (a, \alpha], \\ h(t), & t \in (\alpha, \beta), \\ f(t), & t \in [\beta, b), \end{cases} \quad \text{and} \quad g_b(t) = \begin{cases} 0, & t \in (a, \alpha], \\ k(t), & t \in (\alpha, \beta), \\ g(t), & t \in [\beta, b), \end{cases}$$

satisfy the appropriate inhomogeneous canonical equations on $(a, \alpha)$, $(\alpha, \beta)$, and $(\beta, b)$. Since $f_\alpha(\alpha) = h(\alpha)$ and $f_\beta(\beta) = h(\beta)$ it follows that $f_b \in AC(i)$. \qed

In particular, if $f$ is a solution of the homogeneous system (7.2.4), then $f$ can be localized as indicated above. The following restatement of this fact in terms of matrix functions (groupings of column vector functions) is useful. Note that the modification of the solutions of the homogeneous equation involves a solution of the inhomogeneous equation.

**Corollary 7.5.7.** Let the canonical system (7.2.3) be definite and choose a compact interval $[\alpha, \beta] \subset i$ such that the system is definite on $[\alpha, \beta]$. Let $Y(\cdot, \lambda)$ be a fundamental matrix of (7.2.4). Then there exist a $2 \times 2$ matrix function $Y_a(\cdot, \lambda) \in AC(i)$ and a $2 \times 2$ matrix function $Z_a(\cdot, \lambda)$ whose columns belong to $L^2_\Delta(i)$, satisfying

$$JY'_a(t, \lambda) - H(t)Y_a(t, \lambda) = \lambda \Delta(t)Y_a(t, \lambda) + \Delta(t)Z_a(t, \lambda)$$

such that

$$Y_a(t, \lambda) = \begin{cases} Y(t, \lambda), & t \in (a, \alpha], \\ 0, & t \in [\beta, b), \end{cases} \quad \text{and} \quad Z_a(t, \lambda) = \begin{cases} 0, & t \in (a, \alpha], \\ 0, & t \in [\beta, b). \end{cases}$$

Similarly, there exist a $2 \times 2$ matrix function $Y_b(\cdot, \lambda) \in AC(i)$ and a $2 \times 2$ matrix function $Z_b(\cdot, \lambda)$ whose columns belong to $L^2_\Delta(i)$, satisfying

$$JY'_b(t, \lambda) - H(t)Y_b(t, \lambda) = \lambda \Delta(t)Y_b(t, \lambda) + \Delta(t)Z_b(t, \lambda)$$

such that

$$Y_b(t, \lambda) = \begin{cases} 0, & t \in (a, \alpha], \\ Y(t, \lambda), & t \in [\beta, b), \end{cases} \quad \text{and} \quad Z_b(t, \lambda) = \begin{cases} 0, & t \in (a, \alpha], \\ 0, & t \in [\beta, b). \end{cases}$$
With \( \phi \in C^2 \) observe that the function \( Y_a(\cdot, \lambda)\phi \) belongs to \( L^2_\Delta(i) \) if and only if \( Y(\cdot, \lambda)\phi \) is square-integrable with respect to \( \Delta \) at \( a \), and, likewise, that the function \( Y_b(\cdot, \lambda)\phi \) belongs to \( L^2_\Delta(i) \) if and only if \( Y(\cdot, \lambda)\phi \) is square-integrable with respect to \( \Delta \) at \( b \).

It is useful to have a special notation for the elements that modify the pairs \( \{Y(\cdot, \lambda), \lambda Y(\cdot, \lambda)\} \) in Corollary 7.5.7. Define the matrix functions \( Y_a(\cdot, \lambda) \) and \( Y_b(\cdot, \lambda) \) by

\[
Y_a(\cdot, \lambda) := \{Y_a(\cdot, \lambda), \lambda Y_a(\cdot, \lambda) + Z_a(\cdot, \lambda)\}, \\
Y_b(\cdot, \lambda) := \{Y_b(\cdot, \lambda), \lambda Y_b(\cdot, \lambda) + Z_b(\cdot, \lambda)\},
\]

that is, for \( \phi \in C^2 \) one has

\[
Y_a(\cdot, \lambda)\phi = \{Y_a(\cdot, \lambda)\phi, \lambda Y_a(\cdot, \lambda)\phi + Z_a(\cdot, \lambda)\phi\}, \\
Y_b(\cdot, \lambda)\phi = \{Y_b(\cdot, \lambda)\phi, \lambda Y_b(\cdot, \lambda)\phi + Z_b(\cdot, \lambda)\phi\}.
\]

Note that \( Y_a(\cdot, \lambda) \) and \( Y_b(\cdot, \lambda) \) satisfy

\[
Y_a(t, \lambda) = \begin{cases} 
\{Y(t, \lambda), \lambda Y(t, \lambda)\}, & a < t \leq \alpha, \\
\{0, 0\}, & \beta \leq t < b,
\end{cases}
\]

\[
Y_b(t, \lambda) = \begin{cases} 
\{0, 0\}, & a < t \leq \alpha, \\
\{Y(t, \lambda), \lambda Y(t, \lambda)\}, & \beta \leq t < b.
\end{cases}
\]

It is clear from the construction that the columns of \( Y_a(\cdot, \lambda) \) or \( Y_b(\cdot, \lambda) \) are square-integrable on \((a, b)\) with respect to \( \Delta \) if and only if the corresponding columns of \( Y(\cdot, \lambda) \) have this property at \( a \) or \( b \), respectively.

### 7.6 Maximal and minimal relations for canonical systems

In this and later sections it will be assumed that the canonical system \((7.2.3)\) is real as in Definition 7.2.7 and definite as in Definition 7.5.5: such systems will be called real definite canonical systems. In this context the central Hilbert space will be \( L^2_\Delta(i) \), in which the maximal and minimal relations associated with the real definite canonical system \((7.2.3)\) will be defined. In principle, both these relations may be multivalued. The results from Section 7.4 and Section 7.5 make it possible to consider the limit-circle case and the limit-point case from the point of view of the maximal and minimal relations.

The real definite canonical system \((7.2.3)\) induces the maximal relation \( T_{\text{max}} \) in \( L^2_\Delta(i) \) defined by

\[
T_{\text{max}} = \{\{f, g\} \in L^2_\Delta(i) \times L^2_\Delta(i) : Jf' - Hf = \Delta g\}.
\]
Since the elements of \( L_\Delta^2(\iota) \) are equivalence classes, the definition of \( T_{\text{max}} \) needs the following explanation: an element \( \{f, g\} \in L_\Delta^2(\iota) \times L_\Delta^2(\iota) \) belongs to \( T_{\text{max}} \) if and only if the equivalence class \( f \) contains an absolutely continuous representative \( \tilde{f} \) such that the inhomogeneous equation \( Jf'(t) - H(t)f(t) = \Delta(t)\tilde{g}(t) \) is satisfied for almost every \( t \in \iota \). Here \( \tilde{g} \) is any representative of \( g \in L_\Delta^2(\iota) \); observe that the function \( \Delta(t)\tilde{g}(t) \) is independent of the representative. The above argument also shows that the relation \( T_{\text{max}} \) is linear.

Since the canonical system (7.2.3) is assumed to be definite, the absolutely continuous representative is unique.

**Lemma 7.6.1.** If \( \{f, g\} \in T_{\text{max}} \), then the equivalence class \( f \) has a unique absolutely continuous representative.

**Proof.** Let \( \{f, g\} \in T_{\text{max}} \) and let \( \tilde{f}_1 \) and \( \tilde{f}_2 \) be absolutely continuous representatives of \( f \). Then \( J(\tilde{f}_1 - \tilde{f}_2)' - H(\tilde{f}_1 - \tilde{f}_2) = 0 \) holds and

\[
\Delta(t)(\tilde{f}_1 - \tilde{f}_2)(t) = 0, \quad t \in \iota.
\]

Therefore, by Definition 7.5.5, it follows that \( \tilde{f}_1(t) = \tilde{f}_2(t) \) for all \( t \in \iota \). \( \square \)

It will be shown that \( T_{\text{max}} \) is the adjoint of a symmetric relation whose defect numbers are equal and at most \((2, 2)\). Let \( T_0 \) be the preminimal relation, i.e., the restriction of the maximal relation \( T_{\text{max}} \) to the elements where the first component has compact support in \( \iota \):

\[
T_0 = \{ \{f, g\} \in T_{\text{max}} : f \text{ has compact support} \}.
\]

More precisely, an element \( \{f, g\} \in L_\Delta^2(\iota) \times L_\Delta^2(\iota) \) belongs to \( T_0 \) if and only if the equivalence class \( f \) contains an absolutely continuous representative \( \tilde{f} \) with compact support such that the inhomogeneous equation \( J\tilde{f}'(t) - H(t)\tilde{f}(t) = \Delta(t)\tilde{g}(t) \) is satisfied for almost every \( t \in \iota \). Here \( \tilde{g} \) is any representative of \( g \in L_\Delta^2(\iota) \). The minimal relation \( T_{\text{min}} \) is defined as \( T_{\text{min}} = \overline{T_0} \).

**Theorem 7.6.2.** The closure \( T_{\text{min}} = \overline{T_0} \) of \( T_0 \) is a closed symmetric relation in \( L_\Delta^2(\iota) \) and it satisfies

\[
T_{\text{min}} \subset (T_{\text{min}})^* = T_{\text{max}},
\]

and, consequently, \( T_{\text{min}} = (T_{\text{max}})^* \).

**Proof.** Step 1. It will be shown that

\[
T_{\text{max}} \subset (T_0)^*.
\]

(7.6.1)

For this purpose, let \( \{f, g\} \in T_{\text{max}}, \{h, k\} \in T_0 \), and choose an interval \([\alpha, \beta] \subset \iota\) containing the support \( h \) (and hence the support of \( \Delta k \)). Then

\[
(g, h)_\Delta - (f, k)_\Delta = \int_{\alpha}^{\beta} h(s)\Delta(s)g(s) \, ds - \int_{\alpha}^{\beta} k(s)\Delta(s)f(s) \, ds
\]

\[
= h(\beta)^* Jf(\beta) - h(\alpha)^* Jf(\alpha)
\]

\[
= 0
\]
by Corollary 7.2.3. Here $(\cdot, \cdot)_{\Delta}$ denotes the scalar product in $L^2_{\Delta}(i)$, $f$ and $h$ are the uniquely defined absolutely continuous representatives, while $g$ and $k$ are arbitrary representatives. Observe that the integral does not depend on the particular choice of $g$ and $k$. This shows $\{f, g\} \in (T_0)^*$ and hence (7.6.1) follows.

Step 2. It will be shown that

$$(T_0)^* \subset T_{\text{max}}. \quad (7.6.2)$$

For this, let $\{f, g\} \in (T_0)^*$. By Theorem 7.2.1, there exists a nontrivial absolutely continuous function $u$ on $i$ such that $Ju' - Hu = \Delta g$. The aim is to show that for any representative $f$ the difference $f - u$ is absolutely continuous modulo an element whose $L^2_{\Delta}(a,b)$-norm is zero. Recall that the system is assumed to be definite, and hence there exists a compact interval $[\alpha_0, \beta_0]$ on which it is definite; cf. Proposition 7.5.4. Choose an interval $[\alpha, \beta] \subset T$ which contains $[\alpha_0, \beta_0]$; then the system is also definite on $[\alpha_1, \beta_1]$.

It is convenient to introduce the subspace

$$M_1 := \left\{ k \in L^2_{\Delta}(\alpha_1, \beta_1) : Jh' - Hh = \Delta k \text{ for some } h \in AC[\alpha_1, \beta_1] \right\}.$$

Let $k \in M_1$ and let $h \in AC[\alpha_1, \beta_1]$ be a solution of $Jh' - Hh = \Delta k$ for which $h(\alpha_1) = h(\beta_1) = 0$. It follows from (7.2.15) that

$$h(t) = Y(t,0)J^{-1} \int_{\alpha_1}^{t} Y(s,0)^* \Delta(s)k(s) \, ds, \quad t \in [\alpha_1, \beta_1], \quad (7.6.3)$$

where the fundamental matrix $Y(\cdot, \lambda)$ is fixed by $Y(\alpha_1, \lambda) = I$. Note that the condition $h(\beta_1) = 0$ implies

$$\int_{\alpha_1}^{\beta_1} Y(s,0)^* \Delta(s)k(s) \, ds = 0. \quad (7.6.4)$$

Conversely, if $k \in L^2_{\Delta}(\alpha_1, \beta_1)$ satisfies (7.6.4), then $k \in M_1$ since $h \in AC[\alpha_1, \beta_1]$ in (7.6.3) satisfies $Jh' - Hh = \Delta k$ and $h(\alpha_1) = h(\beta_1) = 0$. In other words, one has

$$M_1 = \left\{ k \in L^2_{\Delta}(\alpha_1, \beta_1) : \int_{\alpha_1}^{\beta_1} Y(s,0)^* \Delta(s)k(s) \, ds = 0 \right\}.$$

Now let $k \in M_1$ and let $h$ be defined by (7.6.3). Then the pair of functions $\{h, k\}$ can be trivially extended to all of $i$ and the extended pair, which will also be denoted by $\{h, k\}$, belongs to $T_0$. As $\{f, g\} \in (T_0)^*$, one has $(h, g)_{\Delta} = (k, f)_{\Delta}$, and since the supports of $h$ and $k$ are inside $[\alpha_1, \beta_1]$ it follows that

$$\int_{\alpha_1}^{\beta_1} g(s)^* \Delta(s)h(s) \, ds = \int_{\alpha_1}^{\beta_1} f(s)^* \Delta(s)k(s) \, ds. \quad (7.6.5)$$
Consider the pair \( \{u, g\} \) on \([\alpha_1, \beta_1]\). Note that on this interval \( u \) is absolutely continuous and \( g \) is square-integrable with respect to \( \Delta \). It follows from the Lagrange identity in Corollary 7.2.3 applied to the pairs \( \{h, k\} \) and \( \{u, g\} \), and \( h(\alpha_1) = h(\beta_1) = 0 \) that

\[
\int_{\alpha_1}^{\beta_1} g(s)^* \Delta(s) h(s) \, ds = \int_{\alpha_1}^{\beta_1} u(s)^* \Delta(s) k(s) \, ds. \tag{7.6.6}
\]

Combining (7.6.5) and (7.6.6), one obtains that

\[
\int_{\alpha_1}^{\beta_1} (f(s) - u(s))^* \Delta(s) k(s) \, ds = 0
\]

for all \( k \in \mathcal{M}_1 \). In other words, the restriction of \( f - u \) to \([\alpha_1, \beta_1]\) is orthogonal to \( \mathcal{M}_1 \) in the semidefinite Hilbert space \( L^2_\Delta(\alpha_1, \beta_1) \). Furthermore, by (7.6.4) one has that \( Y(\cdot, 0)^* \) is orthogonal to \( \mathcal{M}_1 \) in \( L^2_\Delta(\alpha_1, \beta_1) \) for all \( \gamma \in \mathbb{C}^2 \), and since the same is true for \( f - u \) it follows that \( f - u - Y(\cdot, 0)^* \gamma \) is orthogonal to \( \mathcal{M}_1 \) in \( L^2_\Delta(\alpha_1, \beta_1) \) for all \( \gamma \in \mathbb{C}^2 \).

Next it will be shown that for some \( \gamma_1 \in \mathbb{C}^2 \) the function \( f - u - Y(\cdot, 0)^* \gamma_1 \) belongs to \( \mathcal{M}_1 \). In fact, first of all it is clear from (7.2.15) that for any \( \gamma \in \mathbb{C}^2 \)

\[
h(t) = Y(t, 0) J^{-1} \int_{\alpha_1}^{t} Y(s, 0)^* \Delta(s)(f(s) - u(s) - Y(s, 0)^* \gamma) \, ds
\]

satisfies \( J h' - H h = \Delta(f - u - Y(\cdot, 0)^* \gamma) \) and \( h(\alpha_1) = 0 \). To satisfy the boundary condition \( h(\beta_1) = 0 \), choose \( \gamma = \gamma_1 \in \mathbb{C}^2 \) such that

\[
\int_{\alpha_1}^{\beta_1} Y(s, 0)^* \Delta(s)(f(s) - u(s)) \, ds = \int_{\alpha_1}^{\beta_1} Y(s, 0)^* \Delta(s) Y(s, 0)^* \gamma_1 \, ds;
\]

this is possible since the system is definite on \([\alpha_1, \beta_1]\) and hence the matrix

\[
\int_{\alpha_1}^{\beta_1} Y(s, 0)^* \Delta(s) Y(s, 0) \, ds
\]

is invertible; cf. Corollary 7.5.3. Therefore, \( f - u - Y(\cdot, 0)^* \gamma_1 \in \mathcal{M}_1 \). Since the element \( f - u - Y(\cdot, 0)^* \gamma_1 \) is orthogonal to \( \mathcal{M}_1 \) in \( L^2_\Delta(\alpha_1, \beta_1) \), this yields

\[
\int_{\alpha_1}^{\beta_1} (f(s) - u(s) - Y(s, 0)^* \gamma_1)^* \Delta(s)(f(s) - u(s) - Y(s, 0)^* \gamma_1) \, ds = 0,
\]

and hence there exists a function \( \omega_1 \) on \([\alpha_1, \beta_1]\) such that

\[
f(s) = u(s) + Y(s, 0)^* \gamma_1 + \omega_1(s), \quad \Delta(s) \omega_1(s) = 0, \quad s \in [\alpha_1, \beta_1].
\]
Likewise, on any interval $[\alpha_2, \beta_2]$ extending $[\alpha_1, \beta_1]$ the same argument shows that there exist $\gamma_2 \in \mathbb{C}^2$ and a function $\omega_2$ such that
\[
f(s) = u(s) + \bar{Y}(s,0)\gamma_2 + \omega_2(s), \quad \Delta(s)\omega_2(s) = 0, \quad s \in [\alpha_2, \beta_2];
\]
and recall from Theorem 1.4.11 that the relation $\gamma = (\gamma,\gamma)$ is closed and symmetric, while $\bar{Y}(\cdot,\cdot) \subseteq \gamma$.

Step 3. It follows from (7.6.1) and (7.6.2) that $\max = (\max)^*$ and, in particular, this implies that $\max$ is closed. Hence, the fact that $\max \subseteq \max$ and the definition $\min = \max$ imply that
\[
\min = \max \subseteq \max = (\max)^* = (\min)^*.
\]
Thus, $\min$ is a (closed) symmetric relation and $\min = (\max)^*$.

At this stage note that $\min$ is a closed symmetric relation which need not be densely defined in $L^2_\Delta(i)$. Consider the orthogonal decomposition
\[
L^2_\Delta(i) = (\text{mul} \min)^\perp \oplus \text{mul} \min = \overline{\text{dom} \max} \oplus \text{mul} \min
\]
and recall from Theorem 1.4.11 that $\min$ admits the corresponding orthogonal sum decomposition
\[
\min = (\min)^\perp \oplus \{0\} \times \text{mul} \min.
\]
\[(7.6.7)\]
The operator part $(\min)^\perp$ is not necessarily densely defined in $\overline{\text{dom} \max}$ and $\{0\} \times \text{mul} \min$ is the purely multivalued self-adjoint relation in $\text{mul} \min$.

Since by Theorem 7.6.2 the relation $\min$ is closed and symmetric, while $(\min)^* = \max$, it follows from the von Neumann decomposition, as given in Theorem 1.7.11, that the relation $\max$ has the componentwise sum decomposition
\[
\max = \min \oplus \overline{\mathfrak{R}_\lambda(\max)} \oplus \overline{\mathfrak{R}_\mu(\max)}, \quad \lambda \in \mathbb{C}^+, \quad \mu \in \mathbb{C}^-,
\]
\[(7.6.8)\]
where the sums are direct. Now assume that \( f \in \mathcal{H}_\zeta(T_{\text{max}}) \) with \( \zeta \in \mathbb{C} \setminus \mathbb{R} \). Then \( \{ f, \zeta f \} \in T_{\text{max}} \), which is equivalent to \( f \in L^2_{\Delta}(i) \) having an absolutely continuous representative \( \tilde{f} \) such that \( J\tilde{f}' - H\tilde{f} = \zeta \Delta f \). Since the system consists of \( 2 \times 2 \) matrix functions, there are precisely two linearly independent solutions of this homogeneous equation and at most two linearly independent solutions that are square-integrable with respect to \( \Delta \). Furthermore, since the canonical system is assumed to be real, the number of solutions at \( \zeta \in \mathbb{C}^+ \) that are square-integrable with respect to \( \Delta \) coincides with the number of solutions at \( \bar{\zeta} \in \mathbb{C}^- \) that are square-integrable with respect to \( \Delta \) by Corollary 7.2.9. Taking into account Corollary 7.4.5 and Corollary 7.4.6 one then obtains the following statement. The case where both endpoints of the interval \( i \) are in the limit-point case will be dealt with in Corollary 7.6.9.

**Corollary 7.6.3.** Let \( T_{\text{min}} \) be the minimal symmetric relation associated with the real definite canonical system (7.2.3) in \( L^2_{\Delta}(i) \). Then the following statements hold:

(i) If both endpoints of \( i \) are regular or quasiregular, then the defect numbers of \( T_{\text{min}} \) are \( (2, 2) \).

(ii) If one endpoint of \( i \) is regular or quasiregular and one endpoint is in the limit-point case, then the defect numbers of \( T_{\text{min}} \) are \( (1, 1) \).

Recall that elements \( \{ f, g \} \in T_{\text{max}} \) satisfy the equation \( Jf' - Hf = \Delta g \) and that the entries also satisfy the integrability condition \( f, g \in L^2_{\Delta}(i) \). These two ingredients make it possible to extend the usual Lagrange identity in Corollary 7.2.3 on a compact subinterval to all of \( i \). This new Lagrange identity for the elements in \( T_{\text{max}} \) will play an important role in the rest of this chapter.

**Lemma 7.6.4.** Let \( T_{\text{max}} \) be the maximal relation associated with the real definite canonical system (7.2.3) in \( L^2_{\Delta}(i) \). Then for all \( \{ f, g \}, \{ h, k \} \in T_{\text{max}} \) one has

\[
(g, h)_\Delta - (f, k)_\Delta = \lim_{t \to b} h(t)^* Jf(t) - \lim_{t \to a} h(t)^* Jf(t),
\]

(7.6.9)

where \( f(t) \) and \( h(t) \) denote the values of the unique absolutely continuous representatives of \( f \) and \( h \), respectively.

**Proof.** First observe that for all elements \( \{ f, g \}, \{ h, k \} \in T_{\text{max}} \) and every compact subinterval \([\alpha, \beta] \subset i \) one has

\[
\int_{\alpha}^{\beta} (h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s)) \, ds = h(\beta)^* Jf(\beta) - h(\alpha)^* Jf(\alpha)
\]

by the Lagrange identity in Corollary 7.2.3; here \( f(t) \) and \( h(t) \) denote the values of the unique absolutely continuous representatives of \( f \) and \( h \), and \( g(t) \) and \( k(t) \) are the values of some representatives of \( g \) and \( k \). Observe that the integral on the left-hand side does not depend on the choice of the representatives of \( g \) and
The limit of the left-hand side exists as $\beta \rightarrow b$ and $\alpha \rightarrow a$, respectively, since $f, g, h, k \in L^2_\Delta(t)$. As a consequence, one sees that each of the limits
\[
\lim_{t \rightarrow a} h(t)^*Jf(t) \quad \text{and} \quad \lim_{t \rightarrow b} h(t)^*Jf(t)
\]
exists and hence the Lagrange identity takes the limit form (7.6.9).

**Remark 7.6.5.** Observe that in (7.6.9) one uses the values $f(t)$ and $h(t)$ of the unique absolutely continuous representatives of $f$ and $h$ in $\text{dom } T_{\text{max}}$, respectively. For instance, if $\{0, g\} \in T_{\text{max}}$ and $\{h, k\} \in T_{\text{max}}$, then there exists an absolutely continuous function $\tilde{f}$ such that $\Delta(t)\tilde{f}(t) = 0$ and $H(t)\tilde{f}(t) = \Delta(t)\tilde{g}(t)$ is satisfied for almost every $t \in t$, where $\tilde{g}$ is any representative of $g \in L^2_\Delta(t)$. In this situation the identity (7.6.9) has the form
\[
(g, h)_{\Delta} - (0, k)_{\Delta} = \lim_{t \rightarrow b} h(t)^*J\tilde{f}(t) - \lim_{t \rightarrow a} h(t)^*J\tilde{f}(t).
\]

The elements in the minimal symmetric relation $T_{\text{min}} = T_0$ can be easily characterized in terms of these limits.

**Corollary 7.6.6.** Let $T_{\text{min}}$ and $T_{\text{max}}$ be the minimal and maximal relations associated with the real definite canonical system (7.2.3) in $L^2_\Delta(t)$ and let $\{f, g\} \in T_{\text{max}}$. Then $\{f, g\} \in T_{\text{min}}$ if and only if
\[
\lim_{t \rightarrow a} h(t)^*Jf(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow b} h(t)^*Jf(t) = 0 \quad (7.6.10)
\]
for all $\{h, k\} \in T_{\text{max}}$, where $f(t)$ and $h(t)$ denote the values of the unique absolutely continuous representatives of $f$ and $h$, respectively.

**Proof.** Observe first that since $T_{\text{min}} = (T_{\text{max}})^*$ one has $\{f, g\} \in T_{\text{min}}$ if and only if $(g, h)_{\Delta} = (f, k)_{\Delta}$ for all $\{h, k\} \in T_{\text{max}}$. Hence, it follows from the Lagrange identity (7.6.9) that $\{f, g\} \in T_{\text{min}}$ if and only if
\[
\lim_{t \rightarrow b} h(t)^*Jf(t) = \lim_{t \rightarrow a} h(t)^*Jf(t) \quad (7.6.11)
\]
for all $\{h, k\} \in T_{\text{max}}$. To see that for $\{f, g\} \in T_{\text{min}}$ each of the limits in (7.6.11) is zero, consider $\{h, k\} \in T_{\text{max}}$ and use Proposition 7.5.6 (with $\lambda = 0$) to obtain an element $\{h_a, k_a\} \in T_{\text{max}}$ that coincides with $\{h, k\}$ in a neighborhood of $a$ and with $\{0, 0\}$ in a neighborhood of $b$. Then (7.6.11) implies
\[
\lim_{t \rightarrow a} h(t)^*Jf(t) = \lim_{t \rightarrow a} h_a(t)^*Jf(t) = \lim_{t \rightarrow b} h_a(t)^*Jf(t) = 0,
\]
and hence (7.6.10) follows together with (7.6.11). Conversely, if (7.6.10) holds for some $\{f, g\} \in T_{\text{max}}$ and all $\{h, k\} \in T_{\text{max}}$, then the identity (7.6.11) holds for all $\{h, k\} \in T_{\text{max}}$ and hence $\{f, g\} \in T_{\text{min}}$. \qed
The main difficulty when dealing with the boundary value problems associated with the system (7.2.3) is to break the limits in (7.6.9) and in (7.6.10) into limits of the separate factors. The case where the endpoints are regular, quasiregular, or in the limit-circle case will be pursued in Section 7.7. If one of the endpoints is in the limit-point case, the situation is somewhat simpler, since one of the limits in (7.6.9) automatically vanishes, as will be shown now. A further discussion of the remaining limit will be pursued in Section 7.8.

Lemma 7.6.7. Let $T_{\text{max}}$ be the maximal relation associated with the real definite canonical system (7.2.3) in $L^2_{\Delta}(i)$, let $\lambda \in \mathbb{C}$, and let $Y_a(\cdot, \lambda)$ and $Y_b(\cdot, \lambda)$ be as in (7.5.6). Then the following statements hold:

(i) Let $a$ be a regular or quasiregular endpoint and let $b$ be in the limit-point case. Then for $\lambda \in \mathbb{C}$

$T_{\text{max}} = T_{\text{min}} \hat{+} \{ Y_a(\cdot, \lambda) \phi : \phi \in \mathbb{C}^2 \},$  

where the sum is direct.

(ii) Let $b$ be a regular or quasiregular endpoint and let $a$ be in the limit-point case. Then for $\lambda \in \mathbb{C}$

$T_{\text{max}} = T_{\text{min}} \hat{+} \{ Y_b(\cdot, \lambda) \phi : \phi \in \mathbb{C}^2 \},$

where the sum is direct.

Proof. It suffices to consider the case (i) since the case (ii) can be proved in a similar way. Let $Y(\cdot, \lambda)$ be a fundamental matrix of (7.2.4). Note that if $a$ is regular or quasiregular, then it follows from Corollary 7.3.3 and (7.5.7) that $Y_a(\cdot, \lambda) \phi \in L^2_{\Delta}(i) \times L^2_{\Delta}(i)$, and Corollary 7.5.7 implies that $Y_a(\cdot, \lambda) \phi \in T_{\text{max}}$ for all $\phi \in \mathbb{C}^2$.

As $T_{\text{min}} \subset T_{\text{max}}$, it is clear that the right-hand side of (7.6.12) is contained in $T_{\text{max}}$. By assumption and Corollary 7.6.3 (ii) $T_{\text{max}}$ is a two-dimensional extension of $T_{\text{min}}$ and hence it suffices to show that the elements $Y_a(\cdot, \lambda) \phi$, $\phi \in \mathbb{C}^2$, span a two-dimensional subspace of $T_{\text{max}}$ which has a trivial intersection with $T_{\text{min}}$. In other words, it remains to check that $Y_a(\cdot, \lambda) \phi \in T_{\text{min}}$ if and only if $\phi = 0$. Suppose that $Y_a(\cdot, \lambda) \phi \in T_{\text{min}}$ for some $\phi \in \mathbb{C}^2$. For all $\psi \in \mathbb{C}^2$ one has $Y_a(\cdot, \lambda) \psi \in T_{\text{max}}$ and therefore, by Corollary 7.6.6,

$$0 = \lim_{t \to a} \psi^* Y_a(t, \lambda)^* J Y_a(t, \lambda) \phi.$$  

Since $Y_a(\cdot, \lambda) = Y(\cdot, \lambda)$ and $Y_a(\cdot, \lambda) = Y(\cdot, \lambda)$ in a neighborhood of $a$, it follows that

$$0 = \lim_{t \to a} \psi^* Y(t, \lambda)^* J Y(t, \lambda) \phi$$

for all $\psi \in \mathbb{C}^2$. Fix the fundamental matrix $Y(\cdot, \lambda)$ by $Y(a, \lambda) = I$. This leads to $\psi^* J \phi = 0$ for all $\psi \in \mathbb{C}^2$, which implies $\phi = 0$. Hence, the right-hand side of (7.6.12) is a two-dimensional extension of $T_{\text{min}}$ which is contained in $T_{\text{max}}$, and therefore coincides with $T_{\text{max}}$. \hfill \Box
In the next lemma the case of a singular endpoint in the limit-point case is discussed.

**Lemma 7.6.8.** Let $T_{\text{max}}$ be the maximal relation associated with the real definite canonical system (7.2.3) in $L^2_{\Delta}(\dot{i})$. Then the endpoint $a$ or $b$ of the interval $\dot{i}$ is in the limit-point case if and only if for all $\{f, g\}, \{h, k\} \in T_{\text{max}}$ one has

$$\lim_{t \to a} h(t)^* Jf(t) = 0 \quad \text{or} \quad \lim_{t \to b} h(t)^* Jf(t) = 0,$$

respectively. Here $f(t)$ and $h(t)$ denote the values of the unique absolutely continuous representatives of $f$ and $h$, respectively.

**Proof.** It suffices to consider the case that the endpoint $a$ is regular. The proof of the case where $b$ is regular is similar. As usual the fundamental matrix is fixed by $Y(a, \lambda) = I$.

Assume that $b$ is in the limit-point case. In this case one has the decomposition (7.6.12) in Lemma 7.6.7. Let $\{f, g\}, \{h, k\} \in T_{\text{max}}$ be decomposed in the form

$$\{f, g\} = \{f_0, g_0\} + Y_a(\cdot, \lambda) \phi \quad \text{and} \quad \{h, k\} = \{h_0, k_0\} + Y_a(\cdot, \lambda) \psi,$$

where $\{f_0, g_0\}, \{h_0, k_0\} \in T_{\text{min}}$ and $\phi, \psi \in \mathbb{C}^2$. Then it follows from (7.5.7) that

$$\lim_{t \to b} h(t)^* Jf(t) = \lim_{t \to b} h_0(t)^* Jf_0(t) = 0,$$

where Corollary 7.6.6 was used in the last step.

Conversely, assume that for all $\{f, g\}, \{h, k\} \in T_{\text{max}}$

$$\lim_{t \to b} h^*(t) Jf(t) = 0.$$

Then $b$ is in the limit-point case. To see this, assume that $b$ is not in the limit-point case, so that $b$ is in the limit-circle case by Corollary 7.4.5. It then follows that for $\lambda_0 \in \mathbb{R}$ the columns of the matrix function $Y_b(\cdot, \lambda_0)$ are square-integrable with respect to $\Delta$ at $b$. Consider $\{f, g\} = \{h, k\} = Y_b(\cdot, \lambda_0) \phi \in T_{\text{max}}$ for some $\phi \in \mathbb{C}^2$ such that $\phi^* J\phi \neq 0$. Using (7.5.7) and $Y(t, \lambda_0)^* Y(t, \lambda_0)$ (see (7.2.8)), one computes

$$\lim_{t \to b} h^*(t) Jf(t) = \lim_{t \to b} \phi^* Y(t, \lambda_0)^* Y(t, \lambda_0) \phi = \phi^* J\phi \neq 0,$$

which contradicts (7.6.13). \qed

**Corollary 7.6.9.** Let $T_{\text{min}}$ be the minimal symmetric relation associated with the real definite canonical system (7.2.3) in $L^2_{\Delta}(\dot{i})$ and assume that both endpoints of $\dot{i}$ are in the limit-point case. Then the defect numbers of $T_{\text{min}}$ are $(0, 0)$ and $T_{\text{min}} = T_{\text{max}}$ is self-adjoint in $L^2_{\Delta}(\dot{i})$. 

Proof. Assume that the system is definite on the interval \([\alpha, \beta] \subset I\). Then the system is also definite on the intervals \((a, \beta')\) and \((\alpha', b)\), with \(\beta' \in (\beta, b)\) and \(\alpha' \in (a, \alpha)\), respectively. Denote the maximal relation in \(L^2_\Delta(\alpha', b)\) associated with the canonical system by \(T_{\max}(\alpha', b)\). It follows that the endpoint \(\alpha'\) is regular and that the endpoint \(b\) is in the limit-point case for the canonical system on \((\alpha', b)\).

To see the last assertion, assume that there are two linearly independent solutions on \((\alpha', b)\) which are square-integrable with respect to \(\Delta\) at \(b\). Since these solutions admit unique extensions to solutions on \((a, b)\), one obtains a contradiction. In particular, for all \(\{f_b, g_b\}, \{h_b, k_b\} \in T_{\max}(\alpha', b)\) one concludes from Lemma 7.6.8 that \(\lim_{t \to b} h_b(t)^* J f_b(t) = 0\).

Now consider \(\{f, g\}, \{h, k\} \in T_{\max}\) and let \(f_b, g_b, h_b, k_b\) be the restrictions of \(f, g, h, k\) to the interval \((\alpha', b)\). Then one has \(\{f_b, g_b\}, \{h_b, k_b\} \in T_{\max}(\alpha', b)\), and hence

\[
\lim_{t \to b} h(t)^* J f(t) = \lim_{t \to b} h_b(t)^* J f_b(t) = 0.
\]

A similar argument applies to the canonical system on \((a, \beta')\) and shows that

\[
\lim_{t \to a} h(t)^* J f(t) = 0
\]

for all \(\{f, g\}, \{h, k\} \in T_{\max}\). Therefore, Lemma 7.6.4 implies

\[
(g, h)_\Delta - (f, k)_\Delta = \lim_{t \to b} h(t)^* J f(t) - \lim_{t \to a} h(t)^* J f(t) = 0
\]

for all \(\{f, g\}, \{h, k\} \in T_{\max}\), and hence \(T_{\max} \subset T_{\max}^*\). From Theorem 7.6.2 one now concludes \(T_{\min} = T_{\max}\) and thus it follows that the defect numbers of \(T_{\min}\) are \((0, 0)\).

**7.7 Boundary triplets for the limit-circle case**

Assume that the system (7.2.3) is real and definite, and assume that the endpoints of the system are both in the limit-circle case. A boundary triplet will be presented for \(T_{\max} = (T_{\min})^*\) and the self-adjoint extensions of \(T_{\min}\) will be described in terms of the boundary triplet. For a straightforward presentation the case where the endpoints are regular or quasiregular is discussed first. At the end of the section it will be explained what modifications are necessary for endpoints which are in the limit-circle case and which are not regular or quasiregular.

The symmetric relation \(T_{\min} = T_0\) will now be described when \(a\) and \(b\) are regular or quasiregular.

**Lemma 7.7.1.** Assume that \(a\) and \(b\) are regular or quasiregular endpoints for the canonical system (7.2.3). Then the minimal relation \(T_{\min}\) is given by

\[
T_{\min} = \{\{f, g\} \in T_{\max} : f(a) = f(b) = 0\},
\]
where \( f(a) \) and \( f(b) \) denote the boundary values of the unique absolutely continuous representatives of \( f \).

**Proof.** According to Corollary 7.6.6, the element \( \{ f, g \} \in T_{\text{max}} \) belongs to \( T_{\text{min}} \) if and only if

\[
\lim_{t \to a} h(t)^* J f(t) = 0 \quad \text{and} \quad \lim_{t \to b} h(t)^* J f(t) = 0
\]

for all \( \{ h, k \} \in T_{\text{max}} \). Since the endpoints are regular or quasiregular, these conditions are the same as

\[
h(a)^* J f(a) = 0 \quad \text{and} \quad h(b)^* J f(b) = 0
\]

for all \( \{ h, k \} \in T_{\text{max}} \). Now observe that for any \( \gamma \in \mathbb{C}^2 \) and \( k \in L_\Delta^2 (\iota) \) there exists an element \( h \in L_\Delta^2 (\iota) \) such that \( \{ h, k \} \in T_{\text{max}} \) and \( h(a) = \gamma \) or \( h(b) = \gamma \). Hence, it follows that \( f(a) = 0 \) and \( f(b) = 0 \).

When the endpoints of the interval \( \iota = (a, b) \) are regular or quasiregular for the canonical system, then the solutions of \( J f' - H f = \lambda \Delta f, \lambda \in \mathbb{C} \), automatically belong to \( L_\Delta^2 (\iota) \) and thus \( \dim \ker (T_{\text{max}} - \lambda) = 2 \), so that the defect numbers of \( T_{\text{min}} \) are \((2, 2)\); cf. Corollary 7.6.3. In the next theorem a boundary triplet for \( (T_{\text{min}})^* = T_{\text{max}} \) is provided and the corresponding \( \gamma \)-field and Weyl function are obtained in terms of an arbitrary fundamental matrix \( Y(\cdot, \lambda) \) fixed by \( Y(c, \lambda) = I \) for some \( c \in [a, b] \).

**Theorem 7.7.2.** Assume that \( a \) and \( b \) are regular or quasiregular endpoints for the canonical system (7.2.3) and let the fundamental matrix \( Y(\cdot, \lambda) \) of (7.2.4) be fixed by \( Y(c, \lambda) = I \) for some \( c \in [a, b] \). Then \( \{ \mathbb{C}^2, \Gamma_0, \Gamma_1 \} \), with

\[
\Gamma_0 \{ f, g \} = \frac{1}{\sqrt{2}} (f(a) + f(b)) \quad \text{and} \quad \Gamma_1 \{ f, g \} = -\frac{J}{\sqrt{2}} (f(a) - f(b)),
\]

where \( \{ f, g \} \in T_{\text{max}} \), is a boundary triplet for \( (T_{\text{min}})^* = T_{\text{max}} \); here \( f(a) \) and \( f(b) \) denote the boundary values of the unique absolutely continuous representative of \( f \). The corresponding \( \gamma \)-field and Weyl function are given by

\[
\gamma(\lambda) = \sqrt{2} Y(\cdot, \lambda) (Y(a, \lambda) + Y(b, \lambda))^{-1}, \quad \lambda \in \rho(A_0),
\]

and

\[
M(\lambda) = -J (Y(a, \lambda) - Y(b, \lambda)) (Y(a, \lambda) + Y(b, \lambda))^{-1}, \quad \lambda \in \rho(A_0).
\]

**Proof.** Let \( \{ f, g \}, \{ h, k \} \in T_{\text{max}} \). Since the endpoints \( a \) and \( b \) are regular or quasiregular, one has the Lagrange identity

\[
(g, h)_\Delta - (f, k)_\Delta = \int_a^b (h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s)) \, ds
\]

\[
= h(b)^* J f(b) - h(a)^* J f(a);
\]
cf. Corollary 7.3.4. On the other hand, a straightforward calculation shows that
\[(\Gamma_1\{f, g\}, \Gamma_0\{h, k\}) - (\Gamma_0\{f, g\}, \Gamma_1\{h, k\})\]
\[= -\frac{1}{2}(h(a) + h(b))^* J(f(a) - f(b)) + \frac{1}{2}(h(a) - h(b))^* J^*(f(a) + f(b))\]
and hence the boundary mappings \(\Gamma_0\) and \(\Gamma_1\) satisfy the abstract Green identity (2.1.1). Furthermore, the mapping \((\Gamma_0, \Gamma_1)^\top : T_{\text{max}} \to \mathbb{C}^4\) is surjective. To see this, observe first that
\[
\begin{pmatrix}
(\Gamma_0\{f, g\}) \\
(\Gamma_1\{f, g\})
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
I & I \\
-J & J
\end{pmatrix} \begin{pmatrix}
f(a) \\
f(b)
\end{pmatrix}, \quad \{f, g\} \in T_{\text{max}},
\]
and that the \(4 \times 4\) matrix on the right-hand side is invertible. Hence, it suffices to check that for any \(\gamma_a, \gamma_b \in \mathbb{C}^2\) there exists \(\{f, g\} \in T_{\text{max}}\) such that
\[
\begin{pmatrix}
f(a) \\
f(b)
\end{pmatrix} = \begin{pmatrix}
\gamma_a \\
\gamma_b
\end{pmatrix}.
\]
Choose a solution of the equation \(Jh' - Hh = 0\) such that \(h(a) = \gamma_a\) and modify \(h\) as in Proposition 7.5.6, so that it becomes a solution \(h_a\) of an inhomogeneous equation \(Jh'_a - Hh_a = \Delta k_a\) which coincides with \(h\) in a neighborhood of \(a\) and vanishes in a neighborhood of the endpoint \(b\). Then one has \(\{h_a, k_a\} \in T_{\text{max}}\) and \(h_a(a) = \gamma_a\) and \(h_a(b) = 0\). The same argument shows that there exists an element \(\{h_b, k_b\} \in T_{\text{max}}\) such that \(h_b(b) = \gamma_b\) and \(h_b(a) = 0\). Thus, for \(f = h_a + h_b\) and \(g = k_a + k_b\) one has \(\{f, g\} \in T_{\text{max}}\) and (7.7.1) holds. It follows that the mapping \((\Gamma_0, \Gamma_1)^\top : T_{\text{max}} \to \mathbb{C}^4\) is surjective, as claimed. Therefore, \(\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}\) is a boundary triplet for \((T_{\text{min}})^* = T_{\text{max}}\).

To obtain the expressions for the associated \(\gamma\)-field and Weyl function, let \(\lambda \in \rho(A_0)\), where \(A_0 = \ker \Gamma_0\), and note that
\[\mathfrak{H}_\lambda(T_{\text{max}}) = \{Y(\cdot, \lambda)\phi : \phi \in \mathbb{C}^2\}, \quad \lambda \in \mathbb{C}.
\]
Hence, for \(\tilde{f}_\lambda = \{Y(\cdot, \lambda)\phi, \lambda Y(\cdot, \lambda)\phi\}, \phi \in \mathbb{C}^2\), and \(\lambda \in \rho(A_0)\) one has
\[\Gamma_0\tilde{f}_\lambda = \frac{1}{\sqrt{2}} (Y(a, \lambda) + Y(b, \lambda))\phi \quad \text{and} \quad \Gamma_1\tilde{f}_\lambda = -\frac{J}{\sqrt{2}} (Y(a, \lambda) - Y(b, \lambda))\phi,
\]
which leads to
\[\gamma(\lambda) = \left\{ \left\{ \frac{1}{\sqrt{2}} (Y(a, \lambda) + Y(b, \lambda))\phi, Y(\cdot, \lambda)\phi \right\} : \phi \in \mathbb{C}^2 \right\}
\]
and
\[M(\lambda) = \left\{ \left\{ \frac{1}{\sqrt{2}} (Y(a, \lambda) + Y(b, \lambda))\phi, -\frac{J}{\sqrt{2}} (Y(a, \lambda) - Y(b, \lambda))\phi \right\} : \phi \in \mathbb{C}^2 \right\}.
\]
cf. Definition 2.3.1 and Definition 2.3.4. Now observe that for \( \lambda \in \rho(A_0) \) the matrix \( Y(a, \lambda) + Y(b, \lambda) \) is invertible, as otherwise \( (Y(a, \lambda) + Y(b, \lambda)) \psi = 0 \) for some nontrivial \( \psi \in \mathbb{C}^2 \) would imply that \( \lambda \) is an eigenvalue of the self-adjoint relation \( A_0 = \ker \Gamma_0 \) with corresponding eigenfunction \( Y(\cdot, \lambda) \psi \); a contradiction. Therefore, the formulas for the \( \gamma \)-field and the Weyl function follow from the above identities for \( \gamma(\lambda) \) and \( M(\lambda) \).

□

Before formulating the next proposition some terminology is recalled. Let \( T \) be an integral operator of the form

\[
Tf(t) = \int_a^b K(t, s)f(s)\, ds,
\]

where \( f \) is a \( \mathbb{C}^2 \)-valued function and \( K \) is a \( \mathbb{C}^{2 \times 2} \)-valued measurable matrix kernel. If \( K \) is square-integrable with respect to the Lebesgue measure on \( i \times i \), that is,

\[
\int_a^b \int_a^b \|K(t, s)\|_2^2 \, ds \, dt < \infty,
\]

where \( \| \cdot \|_2 \) is the Hilbert–Schmidt matrix norm in (7.1.4), then \( T \) is a bounded linear operator from \( L^2(i) \) into itself, which belongs to the Hilbert–Schmidt class. Recall that a bounded linear operator from \( L^2(i) \) into itself belongs to the Hilbert–Schmidt class if for some, and hence for all orthonormal bases \( (\varphi_i) \) in \( L^2(i) \) one has

\[
\sum_{i,j} |(T\varphi_i, \varphi_j)|^2 < \infty.
\]

Proposition 7.7.3. Assume that \( a \) and \( b \) are regular or quasiregular endpoints for the canonical system (7.2.3) and let \( \{C^2, \Gamma_0, \Gamma_1\} \) be the boundary triplet for \( T_{\text{max}} \) in Theorem 7.7.2 with corresponding Weyl function \( M \). Let the fundamental matrix \( Y(\cdot, \lambda) \) be fixed by \( Y(a, \lambda) = I \). Then the self-adjoint relation \( A_0 = \ker \Gamma_0 \) is given by

\[
A_0 = \ker \Gamma_0 = \{ \{f, g\} \in T_{\text{max}} : f(a) + f(b) = 0 \},
\]

where \( f(a) \) and \( f(b) \) denote the boundary values of the unique absolutely continuous representative of \( f \). The resolvent of \( A_0 \) is an integral operator

\[
((A_0 - \lambda)^{-1}g)(t) = \int_a^b G_0(t, s, \lambda)\Delta(s)g(s)\, ds, \quad \lambda \in \rho(A_0),
\]

which belongs to the Hilbert–Schmidt class. The Green function \( G_0(t, s, \lambda) \) is given by

\[
G_0(t, s, \lambda) = G_{0,e}(t, s, \lambda) + G_{0,i}(t, s, \lambda),
\]

where the entire part \( G_{0,e} \) is given by

\[
G_{0,e}(t, s, \lambda) = Y(t, \lambda) \left[ \frac{1}{2} J \text{sgn} (s - t) \right] Y(s, \lambda)^*,
\]

where

\[
\begin{align*}
Y(t, \lambda), & \quad s < t, \\
\frac{1}{2} J Y(s, \lambda)^*, & \quad s > t,
\end{align*}
\]
and
\[ G_{0,i}(t, s, \lambda) = Y(t, \lambda) \left[ -\frac{1}{2}JM(\lambda)J \right] Y(s, \bar{\lambda})^*. \] (7.7.5)

**Proof. Step 1.** The resolvent of \( A_0 \) has the form (7.7.2) with \( G_0 \) as in (7.7.3). To see this, let \( \lambda \in \rho(A_0) \) and \( g \in L_2^\Delta(\bar{\iota}) \), and define the function
\[ f(t) = \int_a^b G_0(t, s, \lambda) \Delta(s) g(s) \, ds. \]

From the structure of the Green function in (7.7.3), (7.7.4), and (7.7.5), it follows that
\[ f(t) = \frac{1}{2} Y(t, \lambda)J \left( -\int_a^t Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds + \int_t^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds \right) \]
\[ + Y(t, \lambda) E_0(\lambda) \int_a^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds, \]
with \( E_0(\lambda) = -\frac{1}{2}JM(\lambda)J \). Hence, \( f \) is well defined and absolutely continuous. A straightforward computation together with (7.2.10) shows that
\[ Jf'(t) = \frac{1}{2} JY'(t, \lambda)J \left( -\int_a^t Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds + \int_t^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds \right) \]
\[ + \Delta(t) g(t) + JY'(t, \lambda) E_0(\lambda) \int_a^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds. \]

This implies that
\[ Jf' - Hf = \lambda \Delta f + \Delta g = \Delta(g + \lambda f), \]
and thus one has \( \{f, g + \lambda f\} \in T_{\max} \). Furthermore, it is clear from the definition of \( f \) and \( Y(a, \lambda) = I \) that
\[ f(a) = \left[ \frac{1}{2} J + E_0(\lambda) \right] \int_a^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds \]
and
\[ f(b) = Y(b, \lambda) \left[ -\frac{1}{2} J + E_0(\lambda) \right] \int_a^b Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds. \]

Since \( E_0(\lambda) = -\frac{1}{2}JM(\lambda)J = -\frac{1}{2}(I - Y(b, \lambda))(I + Y(b, \lambda))^{-1}J \), observe that
\[ \frac{1}{2} J + E_0(\lambda) = Y(b, \lambda)(I + Y(b, \lambda))^{-1}J, \]
\[ -\frac{1}{2} J + E_0(\lambda) = -(I + Y(b, \lambda))^{-1}J. \]

Thus,
\[ \left[ \frac{1}{2} J + E_0(\lambda) \right] + Y(b, \lambda) \left[ -\frac{1}{2} J + E_0(\lambda) \right] = 0, \]
and hence $\Gamma_0\{f, g + \lambda f\} = \frac{1}{\sqrt{2}} (f(a) + f(b)) = 0$, which implies $\{f, g + \lambda f\} \in A_0$. Therefore, $f = (A_0 - \lambda)^{-1} g$ and the resolvent of $A_0$ is given by (7.7.2).

Step 2. The kernel $G_\Delta(\cdot, \cdot, \lambda)$ defined by

$$G_\Delta(t, s, \lambda) = \Delta(t)^{\frac{1}{2}} G_0(t, s, \lambda) \Delta(s)^{\frac{1}{2}},$$

(7.7.6)

where the kernel $G_0(\cdot, \cdot, \lambda)$ is given in (7.7.3), satisfies

$$\int_a^b \int_a^b \|G_\Delta(t, s, \lambda)\|_2^2 ds \, dt < \infty;$$

(7.7.7)

here $\|\cdot\|_2$ is the Hilbert–Schmidt matrix norm. In fact, from (7.7.3), (7.7.4), (7.7.5), and (7.1.5) it follows that

$$\int_a^b \int_a^b \|\Delta(t)^{\frac{1}{2}} Y(t, \lambda)\|_2^2 \|Y(s, \lambda)^* \Delta(s)^{\frac{1}{2}}\|_2^2 ds \, dt < \infty.$$

To show that the right-hand side is finite, note that with $Y(\cdot, \lambda) = (Y_1(\cdot, \lambda) Y_2(\cdot, \lambda))$ one has

$$\int_a^b \|\Delta(t)^{\frac{1}{2}} Y(t, \lambda)\|_2^2 \, dt = \int_a^b \|\Delta(t)^{\frac{1}{2}} Y_1(t, \lambda)\|_2^2 \, dt + \int_a^b \|\Delta(t)^{\frac{1}{2}} Y_2(t, \lambda)\|_2^2 \, dt < \infty,$$

as the columns $Y_1(\cdot, \lambda)$ and $Y_2(\cdot, \lambda)$ of $Y(\cdot, \lambda)$ are square-integrable with respect to $\Delta$. Due to the identity $\|A\|_2 = \|A^*\|_2$, it follows that

$$\int_a^b \|Y(s, \lambda)^* \Delta(s)^{\frac{1}{2}}\|_2^2 ds = \int_a^b \|\Delta(s)^{\frac{1}{2}} Y(s, \lambda)\|_2^2 ds < \infty.$$

Therefore, the kernel $G_\Delta$ is square-integrable with respect to the Lebesgue measure on $[a, b] \times [a, b]$ and hence (7.7.7) holds. Consequently, the integral operator $T_\Delta$, defined by

$$T_\Delta f(t) = \int_a^b G_\Delta(t, s, \lambda) f(s) \, ds, \quad f \in L^2(\Omega),$$

(7.7.8)

belongs to the Hilbert–Schmidt class in $L^2(\Omega)$.

Step 3. The operator $(A_0 - \lambda)^{-1}$ belongs to the Hilbert–Schmidt class or, equivalently,

$$\sum_{i,j} \|((A_0 - \lambda)^{-1} u_i, u_j)\|_2^2 < \infty, \quad \lambda \in \rho(A_0),$$

(7.7.9)
for some, and hence for any orthonormal basis \((u_i)\) in \(L^2(i)\). To see (7.7.9), observe that \((\Delta^{1/2} u_i)\) is an orthonormal system in \(L^2(i)\) and that (7.7.2) and (7.7.6) give

\[
\begin{align*}
(A_0 - \lambda)^{-1} u_i, u_j \Delta &= \int_a^b \left( \int_a^b G_0(t, s, \lambda) \Delta(s) u_i(s) ds \right) dt \\
&= \int_a^b \int_a^b (\Delta(t)^{1/2} u_j(t))^* G_\Delta(t, s, \lambda) (\Delta(s)^{1/2} u_i(s)) ds dt \\
&= (T_\Delta \Delta^{1/2} u_i, \Delta^{1/2} u_j)_{L^2(i)},
\end{align*}
\]

where \(T_\Delta\) is the Hilbert–Schmidt operator (7.7.8) in \(L^2(i)\) whose kernel is given by (7.7.6). Hence, by Step 2 it follows that (7.7.9) holds, which implies that \((A_0 - \lambda)^{-1}\) is a Hilbert–Schmidt operator.

Since the resolvent of the self-adjoint relation \(A_0\) is a Hilbert–Schmidt operator, the spectrum of \(A_0\) is discrete. As the minimal relation \(T_{\min}\) has no eigenvalues, the next statement follows immediately from Proposition 3.4.8.

**Theorem 7.7.4.** Let \(\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}\) be the boundary triplet for \(T_{\max}\) as in Theorem 7.7.2. Then the operator part \((T_{\min})_{\text{op}}\) is simple in \(L^2_\Delta(i) \ominus \text{mul} T_{\min}\).

Theorem 7.7.4 together with the considerations in Section 3.5 and Section 3.6 ensure that the Weyl function \(M\) in Theorem 7.7.2 contains the complete spectral data of \(A_0\). In the present situation the eigenvalues of \(A_0\) coincide with the poles of the Weyl function and the multiplicities of the eigenvalues of \(A_0\) coincide with the multiplicities of the poles of \(M\).

Let \(\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}\) be the boundary triplet in Theorem 7.7.2 with corresponding \(\gamma\)-field \(\gamma\) and Weyl function \(M\). The self-adjoint (maximal dissipative, maximal accumulative) extensions \(A_\Theta \subset T_{\max}\) of \(T_{\min}\) are in a one-to-one correspondence to the self-adjoint (maximal dissipative, maximal accumulative) relations \(\Theta\) in \(\mathbb{C}^2\) via

\[
A_\Theta = \left\{ \{f, g\} \in T_{\max} : \Gamma_0 \{f, g\}, \Gamma_1 \{f, g\} \in \Theta \right\} = \left\{ \{f, g\} \in T_{\max} : \{f(a) + f(b), -Jf(a) + Jf(b)\} \in \Theta \right\},
\]

where \(f(a)\) and \(f(b)\) denote the boundary values of the unique absolutely continuous representative of \(f\). Recall from Theorem 2.6.1 that for \(\lambda \in \rho(A_\Theta) \cap \rho(A_0)\) the Kreǐn formula for the corresponding resolvents reads

\[
(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda) (\Theta - M(\lambda))^{-1} \gamma(\lambda)^*.
\]

Assume in the following that \(\Theta\) is a self-adjoint relation in \(\mathbb{C}^2\). Since the spectrum of \(A_0\) is discrete and the difference of the resolvents of \(A_0\) and \(A_\Theta\) is an operator of rank \(\leq 2\), it is clear that the spectrum of the self-adjoint relation
\( A_\Theta \) is also discrete. Note that \( \lambda \in \rho(A_0) \) is an eigenvalue of \( A_\Theta \) if and only if \( \ker(\Theta - M(\lambda)) \) is nontrivial, and that
\[
\ker(A_\Theta - \lambda) = \gamma(\lambda) \ker(\Theta - M(\lambda)).
\]
For the self-adjoint relation \( \Theta \) one may use a parametric representation with the help of \( 2 \times 2 \) matrices \( A \) and \( B \) as in Section 1.10 and give a complete description of the (discrete) spectrum of \( A_\Theta \) via poles of a transform of the Weyl function \( M \); cf. Section 3.8 and Section 6.3.

In the following paragraph and corollary it is assumed for simplicity that the relation \( \Theta \) in (7.7.10) is a self-adjoint \( 2 \times 2 \) matrix. In this case the self-adjoint relation \( A_\Theta \) in (7.7.10) is given by
\[
A_\Theta = \left\{ \{f, g\} \in T_{\max} : \Theta(f(a) + f(b)) = -Jf(a) + Jf(b) \right\}
\]
and according to Section 3.8 the spectral properties of \( A_\Theta \) can also be described with the help of the function
\[
\lambda \mapsto (\Theta - M(\lambda))^{-1};
\]
that is, the poles of the matrix function (7.7.13) coincide with the (discrete) spectrum of \( A_\Theta \) and the dimension of the eigenspace \( \ker(A_\Theta - \lambda) \) coincides with the dimension of the residue of the function in (7.7.13) at \( \lambda \). Now fix a fundamental matrix \( Y(\cdot, \lambda) \) by \( Y(a, \lambda) = I \) as in Proposition 7.7.3. By Proposition 7.7.3, the resolvent \( (A_0 - \lambda)^{-1} \) in the Kre\'in formula (7.7.11) is an integral operator. Since \( I + JM(\lambda) = 2(I + Y(b, \lambda))^{-1} \), the \( \gamma \)-field and Weyl function in Theorem 7.7.2 are connected in the present situation via
\[
\gamma(\cdot, \lambda) = \frac{1}{\sqrt{2}} Y(\cdot, \lambda)(I + JM(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
One verifies that
\[
\gamma(\overline{\lambda})^* g = \frac{1}{\sqrt{2}} (I - M(\lambda)J) \int_a^b Y(s, \overline{\lambda})^* \Delta(s) g(s) \, ds, \quad g \in L^2_\Delta(\iota),
\]
and this implies that the second term on the right-hand side of (7.7.11) applied to \( g \in L^2_\Delta(\iota) \) can be written as
\[
\frac{1}{2} Y(\cdot, \lambda)(I + JM(\lambda))(\Theta - M(\lambda))^{-1}(I - M(\lambda)J) \int_a^b Y(s, \overline{\lambda})^* \Delta(s) g(s) \, ds.
\]
Combining this expression with the entire part \( G_{0,e} \) in (7.7.4) and the part \( G_{0,i} \) in (7.7.5) for \( (A_0 - \lambda)^{-1} \), one sees that the resolvent of the self-adjoint extension \( A_\Theta \) is an integral operator in \( L^2_\Delta(\iota) \) of the form
\[
((A_\Theta - \lambda)^{-1} g)(t) = \int_a^b G_\Theta(t, s, \lambda) \Delta(s) g(s) \, ds, \quad \lambda \in \rho(A_\Theta) \cap \rho(A_0),
\]
(7.7.14)
where \( g \in L^2_\Delta(i) \). The Green function \( G_\Theta(t, s, \lambda) \) in (7.7.14) is given by

\[
G_\Theta(t, s, \lambda) = Y(t, \lambda) \left[ \frac{1}{2} J \text{sgn} (s - t) + E_\Theta(\lambda) \right] Y(s, \bar{\lambda})^*,
\]

(7.7.15)

where

\[
E_\Theta(\lambda) = - \frac{1}{2} J \left[ M(\lambda) + (M(\lambda) - J)(\Theta - M(\lambda))^{-1}(M(\lambda) + J) \right] J.
\]

(7.7.16)

In the next corollary the Green function in (7.7.14) is further decomposed in the case that the self-adjoint relation \( \Theta \) in \( C^2 \) is a self-adjoint matrix.

**Corollary 7.7.5.** Let \( a \) and \( b \) be regular or quasiregular endpoints for the canonical system (7.2.3) and let \( \{C^2, \Gamma_0, \Gamma_1\} \) be the boundary triplet for \( T_{max} \) in Theorem 7.7.2 with corresponding Weyl function \( M \). Assume that the fundamental matrix \( Y(\cdot, \lambda) \) is fixed by \( Y(a, \lambda) = I \), let \( \Theta \) be a self-adjoint matrix in \( C^2 \), and let \( A_\Theta \) be the self-adjoint extension in (7.7.12). Then the Green function \( G_\Theta(t, s, \lambda) \) in (7.7.14) has the decomposition

\[
G_\Theta(t, s, \lambda) = G_{\Theta, e}(t, s, \lambda) + G_{\Theta, i}(t, s, \lambda),
\]

where the entire part \( G_{\Theta, e} \) is given by

\[
G_{\Theta, e}(t, s, \lambda) = Y(t, \lambda) \left[ \frac{1}{2} J \text{sgn} (s - t) + \frac{1}{2} J \Theta J \right] Y(s, \bar{\lambda})^*,
\]

and

\[
G_{\Theta, i}(t, s, \lambda) = Y_\Theta(t, \lambda) \left[ \frac{1}{2} (\Theta - M(\lambda))^{-1} \right] Y_\Theta(s, \bar{\lambda})^*,
\]

where \( Y_\Theta(t, \lambda) = Y(t, \lambda)(I + J \Theta) \).

**Proof.** Since \( \Theta \) is a self-adjoint \( 2 \times 2 \)-matrix, one sees that

\[
(M(\lambda) - J)(\Theta - M(\lambda))^{-1}(M(\lambda) + J)
\]

\[
= -M(\lambda) - \Theta + (\Theta - J)(\Theta - M(\lambda))^{-1}(\Theta + J).
\]

Therefore, \( E_\Theta(\lambda) \) in (7.7.16) has the form

\[
E_\Theta(\lambda) = - \frac{1}{2} J \left[ -\Theta + (\Theta - J)(\Theta - M(\lambda))^{-1}(\Theta + J) \right] J
\]

\[
= \frac{1}{2} J \Theta J + (I + J \Theta) \left[ \frac{1}{2} (\Theta - M(\lambda))^{-1} \right] (I - \Theta J).
\]

The assertion now follows from this identity combined with (7.7.15). \( \square \)

At the end of this section the assumption is that the endpoints \( a \) and \( b \) are in the general limit-circle case, so that the assumption that \( a \) and \( b \) are regular or quasiregular is abandoned. The transformation in Lemma 7.2.5 will be useful as for any \( \lambda_0 \in \mathbb{R} \) the solution matrix \( U(\cdot, \lambda_0) \) is now square-integrable with respect
to $\Delta$. This implies that the transformed equation (7.2.20) is in the quasiregular case at $a$ and $b$. Then most of the above results remain true once the (limit) values $f(a)$ and $f(b)$ are replaced by the limits in (7.4.6). The next proposition is the counterpart of Theorem 7.7.2.

**Proposition 7.7.6.** Assume that $a$ and $b$ are in the limit-circle case and let $Y(\cdot, \lambda)$ be a fundamental matrix. Let $\lambda_0$ in $\mathbb{R}$, let $U(\cdot, \lambda_0)$ be a solution matrix as in Lemma 7.2.5, and consider the limits $\tilde{f}(a) = \lim_{t \to a} U(t, \lambda_0)^{-1} f(t)$ and $\tilde{f}(b) = \lim_{t \to b} U(t, \lambda_0)^{-1} f(t)$ for $\{f, g\} \in T_{\max}$; cf. Corollary 7.4.8. Then $\{C_2, \Gamma_0, \Gamma_1\}$, with

$$
\Gamma_0 \{f, g\} = \frac{1}{\sqrt{2}} (\tilde{f}(a) + \tilde{f}(b))
$$

and $\Gamma_1 \{f, g\} = -\frac{J}{\sqrt{2}} (\tilde{f}(a) - \tilde{f}(b))$, where $\{f, g\} \in T_{\max}$, is a boundary triplet for $(T_{\min})^* = T_{\max}$. The corresponding $\gamma$-field and Weyl function are given by

$$
\gamma(\lambda) = \sqrt{2} Y(\cdot, \lambda) (\tilde{Y}(a, \lambda) + \tilde{Y}(b, \lambda))^{-1}, \quad \lambda \in \rho(A_0),
$$

and

$$
M(\lambda) = -J (\tilde{Y}(a, \lambda) - \tilde{Y}(b, \lambda)) (\tilde{Y}(a, \lambda) + \tilde{Y}(b, \lambda))^{-1}, \quad \lambda \in \rho(A_0),
$$

where $\tilde{Y}(\cdot, \lambda) \phi = U(\cdot, \lambda_0)^{-1} Y(\cdot, \lambda) \phi$ for $\phi \in \mathbb{C}^2$ and $\lambda \in \rho(A_0)$.

**Proof.** Recall that due to Corollary 7.4.9 the Lagrange formula takes the form

$$
\tilde{h}(b)^* J \tilde{f}(b) - \tilde{h}(a)^* J \tilde{f}(a) = \int_a^b (h(s)^* \Delta(s) g(s) - k(s)^* \Delta(s) f(s)) \, ds
$$

for $\{f, g\}, \{h, k\} \in T_{\max}$. Now the same computation as in the proof of Theorem 7.7.2 shows that the abstract Green identity (2.1.1) is satisfied. The surjectivity of the map $(\Gamma_0, \Gamma_1)^\top : T_{\max} \to \mathbb{C}^4$, and the form of the $\gamma$-field and Weyl function follow in the same way as in the proof of Theorem 7.7.2.\qed

### 7.8 Boundary triplets for the limit-point case

Assume that the system (7.2.3) is real and definite, and assume that the endpoint $a$ is in the limit-circle case and the endpoint $b$ is in the limit-point case. A boundary triplet will be presented for $T_{\max} = (T_{\min})^*$ and will be used to describe the self-adjoint extensions of $T_{\min}$. To make the presentation straightforward, the case where the endpoint $a$ is regular or quasiregular is dealt with first. At the end of the section it will be explained what modifications are necessary if the endpoint $a$ is in the limit-circle case.

The symmetric relation $T_{\min} = T_{0}$ will now be described when $a$ is a regular or quasiregular endpoint and $b$ is in the limit-point case.
Lemma 7.8.1. Assume that the endpoint $a$ is regular or quasiregular and the endpoint $b$ is in the limit-point case. Then the minimal relation $T_{\min}$ is given by

$$T_{\min} = \{\{f, g\} \in T_{\max} : f(a) = 0\},$$

where $f(a)$ denotes the boundary value of the unique absolutely continuous representative of $f$.

Proof. According to Corollary 7.6.6 and Lemma 7.6.8, an element $\{f, g\} \in T_{\max}$ belongs to $T_{\min}$ if and only if

$$\lim_{t \to a} h(t)^* Jf(t) = 0$$

for all $\{h, k\} \in T_{\max}$. Since the endpoint $a$ is regular or quasiregular this condition is the same as

$$h(a)^* Jf(a) = 0$$

for all $\{h, k\} \in T_{\max}$. Now observe that for any $\gamma \in \mathbb{C}^2$ there exists $\{h, k\} \in T_{\max}$ such that $h(a) = \gamma$. In fact, choose a solution $Ju' - Hu = 0$ such that $u(a) = \gamma$ and use Proposition 7.5.6 to modify $u$ to a function $h \in L^2(\Delta)$ which coincides with $u$ in a neighborhood of $a$, vanishes in a neighborhood of $b$, and satisfies $Jh' - Hh = \Delta k$ with some $k \in L^2(\Delta)$, that is, $\{h, k\} \in T_{\max}$. Since $\gamma^* Jf(a) = 0$ for all $\gamma \in \mathbb{C}^2$, it follows that $f(a) = 0$. \hfill \Box

Let the endpoint $a$ be regular or quasiregular and let $b$ be in the limit-point case. Then there exists for some, and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, up to scalar multiples, one nontrivial solution of $Jf' - Hf = \lambda \Delta f$, which is square-integrable with respect to $\Delta$ at $b$ and thus $\dim \ker (T_{\max} - \lambda) = 1$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This implies that the defect numbers are $(1, 1)$; cf. Corollary 7.6.3. In the next theorem a boundary triplet is provided in this case. To avoid confusion, recall that $Y_1(\cdot, \lambda)$ and $Y_2(\cdot, \lambda)$ are the columns of a fundamental matrix $Y(\cdot, \lambda)$, whereas $f_1$ and $f_2$ stand for the components of the $2 \times 1$ vector function $f$.

Theorem 7.8.2. Assume that the endpoint $a$ is regular or quasiregular and that the endpoint $b$ is in the limit-point case. Let $Y(\cdot, \lambda)$ be a fundamental matrix fixed by $Y(a, \lambda) = I$. Then $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0\{f, g\} = f_1(a) \quad \text{and} \quad \Gamma_1\{f, g\} = f_2(a), \quad \{f, g\} \in T_{\max},$$

is a boundary triplet for $(T_{\min})^* = T_{\max}$; here $f_1(a)$ and $f_2(a)$ denote the boundary values of the components of the unique absolutely continuous representative of $f$. Moreover, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\chi(\cdot, \lambda)$ is a nontrivial element in $\mathfrak{N}_\lambda(T_{\max})$, then one has $\chi_1(a, \lambda) \neq 0$. For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the corresponding $\gamma$-field and Weyl function are given by

$$\gamma(\cdot, \lambda) = Y_1(\cdot, \lambda) + M(\lambda)Y_2(\cdot, \lambda) \quad \text{and} \quad M(\lambda) = \frac{\chi_2(a, \lambda)}{\chi_1(a, \lambda)}.$$
Proof. Since the endpoint $a$ is assumed to be regular or quasiregular, the elements \{f, g\}, \{h, k\} $\in T_{\text{max}}$ have boundary values $f(a), h(a) \in \mathbb{C}^2$. Due to Lemma 7.6.8, the Lagrange identity in Corollary 7.3.4 takes the form

\[
(f, h)_\Delta - (f, k)_\Delta = \int_a^b (h(s)^* \Delta(s)g(s) - k(s)^* \Delta(s)f(s)) \, ds
\]

\[
= -h(a)^* Jf(a)
\]

\[
= f_2(a)h_2(a) - f_1(a)h_2(a)
\]

\[
= (\Gamma_1\{f, g\}, \Gamma_0\{h, k\}) - (\Gamma_0\{f, g\}, \Gamma_1\{h, k\}).
\]

Hence, the boundary mappings $\Gamma_0$ and $\Gamma_1$ satisfy the abstract Green identity (2.1.1). In the proof of Lemma 7.8.1 it was shown that for $\gamma \in \mathbb{C}^2$ there exists \{h, k\} $\in T_{\text{max}}$ such that $h(a) = \gamma$, and so the mapping $(\Gamma_0, \Gamma_1)^\top : T_{\text{max}} \to \mathbb{C}^2$ is surjective. It follows that \{C, $\Gamma_0, \Gamma_1$\} is a boundary triplet for $(T_{\text{min}})^* = T_{\text{max}}$.

Due to the assumption that the endpoint $b$ is in the limit-point case, each eigenspace $\mathfrak{N}_\lambda(T_{\text{max}})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, has dimension 1. Hence, $\hat{f}_\lambda \in \mathfrak{N}_\lambda(T_{\text{max}})$ has the form $\hat{f}_\lambda = \{\chi(\cdot, \lambda)c, \lambda\chi(\cdot, \lambda)c\}$ for some $c \in \mathbb{C}$, where $\chi(\cdot, \lambda)$ is a nontrivial element in $\mathfrak{N}_\lambda(T_{\text{max}})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It follows from Definition 2.3.1 and Definition 2.3.4 that

\[
\gamma(\lambda) = \{\{\chi_1(a, \lambda)c, \chi(\cdot, \lambda)c\} : c \in \mathbb{C}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

and

\[
M(\lambda) = \{\{\chi_1(a, \lambda)c, \chi_2(a, \lambda)c\} : c \in \mathbb{C}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Observe that $\chi_1(a, \lambda) \neq 0$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, as otherwise $\lambda \in \mathbb{C} \setminus \mathbb{R}$ would be an eigenvalue of the self-adjoint relation $A_0 = \ker \Gamma_0$ and $\chi(\cdot, \lambda)$ would be a corresponding eigenfunction. Thus, one concludes that

\[
\gamma(\lambda) = \frac{\chi(\cdot, \lambda)}{\chi_1(a, \lambda)} \quad \text{and} \quad M(\lambda) = \frac{\chi_2(a, \lambda)}{\chi_1(a, \lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Note that $\chi(\cdot, \lambda) = \alpha_1Y_1(\cdot, \lambda) + \alpha_2Y_2(\cdot, \lambda)$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$ and that the assumption $Y(a, \lambda) = I$ yields

\[
\begin{pmatrix}
\chi_1(a, \lambda) \\
\chi_2(a, \lambda)
\end{pmatrix}
= \chi(a, \lambda) = \begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}.
\]

This implies

\[
\gamma(\lambda) = \frac{\alpha_1Y_1(\cdot, \lambda) + \alpha_2Y_2(\cdot, \lambda)}{\chi_1(a, \lambda)} = Y_1(\cdot, \lambda) + M(\lambda)Y_2(\cdot, \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

establishing the formulas for the $\gamma$-field and Weyl function. \hfill \Box

Note that the $\gamma$-field and Weyl function corresponding to the boundary triplet \{C, $\Gamma_0, \Gamma_1$\} in Theorem 7.8.2 are defined and analytic on the resolvent set of the
self-adjoint relation $A_0 = \ker \Gamma_0$. It follows in the same way as in Section 6.4 (see the discussion after the proof of Proposition 6.4.1) that the expressions for $\gamma$ and $M$ in Theorem 7.8.2 extend to points in $\rho(A_0) \cap \mathbb{R}$.

In the next proposition the resolvent of the self-adjoint relation $A_0$ is expressed as an integral operator.

**Proposition 7.8.3.** Assume that the endpoint $a$ is regular or quasiregular and that the endpoint $b$ is in the limit-point case. Let $Y(\cdot, \lambda)$ be a fundamental matrix fixed by $Y(a, \lambda) = I$. Let $\{C, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\text{max}}$ in Theorem 7.8.2 with corresponding Weyl function $M$. Then the self-adjoint relation $A_0 = \ker \Gamma_0$ is given by

$$A_0 = \ker \Gamma_0 = \{ \{f, g\} \in T_{\text{max}} : f_1(a) = 0 \},$$

where $f_1(a)$ denotes the boundary value of the first component of the unique absolutely continuous representative of $f$. The resolvent of $A_0$ is an integral operator

$$(A_0 - \lambda)^{-1}g(t) = \int_a^b G_0(t, s, \lambda)\Delta(s)g(s) \, ds, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (7.8.1)$$

where $g \in L^2_\Delta(i)$. The Green function $G_0(t, s, \lambda)$ is given by

$$G_0(t, s, \lambda) = G_{0,e}(t, s, \lambda) + G_{0,i}(t, s, \lambda), \quad (7.8.2)$$

where the entire part $G_{0,e}$ is given by

$$G_{0,e}(t, s, \lambda) = \begin{cases} Y_1(t, \lambda)Y_2(s, \bar{\lambda})^*, & s < t, \\ Y_2(t, \lambda)Y_1(s, \bar{\lambda})^*, & s > t, \end{cases} \quad (7.8.3)$$

and

$$G_{0,i}(t, s, \lambda) = Y_2(t, \lambda)M(\lambda)Y_2(s, \bar{\lambda})^*. \quad (7.8.4)$$

**Proof.** To prove the identity (7.8.1), consider $g \in L^2_\Delta(i)$ and define the function $f$ by the right-hand side of (7.8.1) with $G_0$ as in (7.8.2). In view of (7.8.3) and (7.8.4), this means that

$$f(t) = (Y_1(t, \lambda) + Y_2(t, \lambda)M(\lambda)) \int_a^t Y_2(s, \bar{\lambda})^*\Delta(s)g(s) \, ds + Y_2(t, \lambda) \int_t^b (Y_1(s, \bar{\lambda})^* + M(\bar{\lambda})^*Y_2(s, \bar{\lambda})^*)\Delta(s)g(s) \, ds. \quad (7.8.5)$$

Observe that, indeed, the integral near $b$ exists, since one has

$$\gamma(\cdot, \bar{\lambda}) = Y_1(\cdot, \bar{\lambda}) + M(\bar{\lambda})Y_2(\cdot, \bar{\lambda}) \in L^2_\Delta(i)$$
and $g \in L^2_{\Delta}(i)$. It follows that the function $f$ in (7.8.5) is well defined and absolutely continuous. Rewrite (7.8.5) in the form

$$
f(t) = Y(t, \lambda) \left(\begin{array}{cc} 1 & 0 \\ M(\lambda) & 1 \end{array}\right) \int_a^t \left(\begin{array}{c} 0 \\ 1 \end{array}\right) Y(s, \lambda) \Delta(s) g(s) \, ds + Y(t, \lambda) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \int_t^b \left(\begin{array}{c} 1 \\ M(\lambda)^* \end{array}\right) Y(s, \lambda) \Delta(s) g(s) \, ds.
$$

(7.8.6)

Then a straightforward calculation using the identity

$$
\left(\begin{array}{cc} 1 \\ M(\lambda) \end{array}\right) \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right) - \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) = -J
$$

(7.8.7)

and (7.2.10) shows that $f$ satisfies the inhomogeneous equation

$$
Jf' - Hf = \lambda \Delta f + \Delta g.
$$

(7.8.8)

Moreover, one sees from (7.8.5) that $f$ satisfies

$$
f(a) = \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \int_a^b \left(\begin{array}{cc} Y_1(s, \lambda) + Y_2(s, \lambda) M(\lambda)^* \end{array}\right) \Delta(s) g(s) \, ds = \left(\begin{array}{c} 0 \\ (g, \gamma(\lambda))_{\Delta} \end{array}\right).
$$

Now denote the function on the left-hand side of (7.8.1) by $h = (A_0 - \lambda)^{-1} g$. Then it is clear that

$$
\{ h, \lambda h + g \} = \{ (A_0 - \lambda)^{-1} g, g + \lambda (A_0 - \lambda)^{-1} g \} \in A_0 \subset T_{\max},
$$

(7.8.9)

so that $h$ also satisfies (7.8.8). Moreover, by (7.8.9) and Proposition 2.3.2 one obtains

$$
\begin{align*}
    h_1(a) &= \Gamma_0 \{ h, \lambda h + g \} = 0, \\
    h_2(a) &= \Gamma_1 \{ h, \lambda h + g \} = \gamma(\lambda)^* g = (g, \gamma(\lambda))_{\Delta}.
\end{align*}
$$

Thus, for a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the functions $h$ and $f$ satisfy the same inhomogeneous equation (7.8.8) and they have the same initial value $f(a) = h(a)$. Since the solution is unique, $h = f$. One concludes that $(A_0 - \lambda)^{-1} g$ is given by the right-hand side of (7.8.1). \hfill \square

Note that there exist canonical systems whose Weyl functions are of the form $M(\lambda) = \alpha + \beta \lambda$ where $\alpha \in \mathbb{R}$ and $\beta \geq 0$; cf. Example 7.10.4 for a special case. Hence, the functions $M$ and $G_{0,i}$ in (7.8.4) may be entire.

**Theorem 7.8.4.** Assume that the endpoint $a$ is regular or quasiregular and that the endpoint $b$ is in the limit-point case. Then the operator part $(T_{\min})_{\text{op}}$ is simple in the Hilbert space $L^2_{\Delta}(i) \ominus \text{mul} T_{\min}$. 
Proof. Step 1. Let \( g \in L^2_\Delta(i) \) and define \( f = (A_0 - \lambda)^{-1}g \). Then it is clear that \( \{f, g + \lambda f\} \in A_0 \) and one has

\[
f(t) = -Y(t, \lambda)J \int_a^t Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds + Y(t, \lambda) \begin{pmatrix} 0 \\ (g, \gamma(\bar{\lambda}))_\Delta \end{pmatrix},
\]

(7.8.10)

where \( Y(\cdot, \lambda) \) is the fundamental matrix fixed by \( Y(a, \lambda) = I \). In fact, it follows from Proposition 7.8.3 and its proof that \( f \) is given by (7.8.1) or, equivalently, by (7.8.6). Now on the right-hand side of (7.8.6) subtract and add the term

\[
Y(t, \lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_a^t \begin{pmatrix} 1 \\ M(\bar{\lambda})^* \end{pmatrix} Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds,
\]

and use (7.8.7) and \( \gamma(\cdot, \bar{\lambda}) = Y_1(\cdot, \bar{\lambda}) + M(\bar{\lambda}) Y_2(\cdot, \bar{\lambda}) \). This yields (7.8.10).

Step 2. The multivalued part mul \( T_{\min} \) is given by

\[
(\text{span} \{ \gamma(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \})^\perp = \text{mul} \, T_{\min},
\]

(7.8.11)

which, in view of Corollary 3.4.6, is equivalent to \( (T_{\min})_{\text{op}} \) being simple in the Hilbert space \( L^2_\Delta(i) \ominus \text{mul} \, T_{\min} \).

The identity (7.8.11) will be verified by exhibiting the corresponding inclusions. For the inclusion \( (\supset) \) in (7.8.11), let \( g \in \text{mul} \, T_{\min} \). Since \( \{0, g\} \in T_{\min} \) and \( \{\gamma(\lambda), \lambda \gamma(\lambda)\} \in T_{\max} \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) one sees that

\[
(g, \gamma(\lambda))_\Delta = (g, \gamma(\lambda))_\Delta - (0, \lambda \gamma(\lambda))_\Delta = 0.
\]

Hence, \( g \in \text{mul} \, T_{\min} \) is orthogonal to all \( \gamma(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \).

For the inclusion \( (\subset) \) in (7.8.11), assume that \( g \in L^2_\Delta(i) \) is orthogonal to all \( \gamma(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then it follows from (7.8.10) that

\[
f(t) = \left((A_0 - \lambda)^{-1}g\right)(t) = -Y(t, \lambda)J \int_a^t Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds.
\]

(7.8.12)

Clearly, \( f(a) = 0 \), so that, in fact,

\[
\{f, g + \lambda f\} \in T_{\min}.
\]

(7.8.13)

Let \( h \in L^2_\Delta(i) \) have compact support, say in \([a', b'] \subset i\). Then it follows from (7.8.12) that

\[
((A_0 - \lambda)^{-1}g, h)_\Delta = - \int_a^b \left( \int_a^t h(t)^* \Delta(t) Y(t, \lambda) J Y(s, \bar{\lambda})^* \Delta(s) g(s) \, ds \right) dt
\]

and due to the structure of the double integral the integration takes place only on the square \([a', b'] \times [a', b']\).
Now consider a bounded interval $\delta \subset \mathbb{R}$ such that the endpoints of $\delta$ are not eigenvalues of $A_0$. Then the spectral projection of $A_0$ corresponding to the interval $\delta$ is given by Stone’s formula (1.5.7) (see also Example A.1.4),

$$(E(\delta)g, h)_\Delta = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_\delta \left( ((A_0 - (\mu + i\epsilon))^{-1} - (A_0 - (\mu - i\epsilon))^{-1})g, h \right)_\Delta d\mu.$$  

Making use of the above integral for $((A_0 - \lambda)^{-1}g, h)_\Delta$ one has that

$$(E(\delta)g, h)_\Delta = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_\delta \int_a^b \int_a^t h(t)^* \Delta(t) F_\epsilon(t, s, \mu) \Delta(s)g(s) ds dt d\mu$$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_\delta \int_a^b \int_a^t h(t)^* \Delta(t) \left( \int_\delta F_\epsilon(t, s, \mu) d\mu \right) \Delta(s)g(s) ds dt,$$

where

$$F_\epsilon(t, s, \mu) = Y(t, \mu - i\epsilon) J Y(s, \mu - i\epsilon)^* - Y(t, \mu + i\epsilon) J Y(s, \mu + i\epsilon)^*.$$  

To justify the application of Fubini’s theorem above note that each of the functions

$$s \mapsto \Delta(s)g(s) \quad \text{and} \quad t \mapsto \Delta(t)h(t)$$

is integrable on $[a', b']$, due to $g, h \in L^2_\Delta(a, b)$ and Lemma 7.1.4, and that the function

$$(s, t, \lambda) \mapsto Y(t, \lambda) J Y(s, \lambda)^* - Y(t, \lambda) J Y(s, \lambda)^*, \quad s, t \in [a', b'], \lambda \in K,$$

where $K \subset \mathbb{C}$ is some compact set, is continuous and hence bounded on the set $[a', b'] \times [a', b'] \times K$. Since the mapping $\lambda \mapsto Y(t, \lambda)$ is entire, it follows that

$$\lim_{\epsilon \downarrow 0} \int_\delta F_\epsilon(t, s, \mu) d\mu = 0$$

and dominated convergence implies that $(E(\delta)g, h)_\Delta = 0$ for any $h \in L^2_\Delta(a)$ with compact support. Therefore, $E(\delta)g = 0$ for any bounded interval $\delta$ with endpoints not in $\sigma_p(A_0)$. With $\delta \to \mathbb{R}$ one concludes $E(\mathbb{R})g = 0$ and this implies $g \in \text{mul } A_0$. Since $\{f, g + \lambda f\} \in A_0$ and $\{0, g\} \in A_0$, it follows that $\{f, \lambda f\} \in A_0$ and hence $f = 0$, as $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is not an eigenvalue of $A_0$. Since $\{f, g + \lambda f\} \in T_{\min}$ by (7.8.13), one concludes $\{0, g\} \in T_{\min}$, that is, $g \in \text{mul } T_{\min}$. This shows the inclusion $(\subset)$ in (7.8.11). \[\square\]

Let $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ be the boundary triplet in Theorem 7.8.2. Then the self-adjoint extensions of $T_{\min}$ are in a one-to-one correspondence to the numbers $\tau \in \mathbb{R} \cup \{\infty\}$ via

$$A_\tau = \{\{f, g\} \in T_{\max} : \Gamma_1 \{f, g\} = \tau \Gamma_0 \{f, g\}\}. \quad (7.8.14)$$
Note that $A_0$ corresponds to $\tau = \infty$. For a given $\tau \in \mathbb{R} \cup \{\infty\}$ one can transform the boundary triplet $\{C, \Gamma_0, \Gamma_1\}$ as follows:

$$
\begin{pmatrix} \Gamma_0^\tau \\ \Gamma_1^\tau \end{pmatrix} = \frac{1}{\sqrt{\tau^2 + 1}} \begin{pmatrix} \tau & -1 \\ 1 & \tau \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}
$$

(7.8.15)

(see (2.5.19)), so that $A_\tau = \ker \Gamma_0^\tau$. Then, by (2.5.20), the $\gamma$-field and Weyl function corresponding to the new boundary triplet are given by

$$
M_\tau(\lambda) = \frac{1 + \tau M(\lambda)}{\tau - M(\lambda)} \quad \text{and} \quad \gamma_\tau(\lambda) = \frac{\sqrt{\tau^2 + 1}}{\tau - M(\lambda)} \gamma(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

(7.8.16)

The Weyl function $M_\tau$ and the $\gamma$-field $\gamma_\tau$ are connected by

$$
\frac{M_\tau(\lambda) - M_\tau(\mu)}{\lambda - \mu} = \gamma_\tau(\mu)^* \gamma_\tau(\lambda), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.
$$

(7.8.17)

Let $Y(\cdot, \lambda)$ be a fundamental matrix fixed by $Y(a, \lambda) = I$. In a similar fashion one can transform $Y(\cdot, \lambda)$ to a fundamental matrix $V(\cdot, \lambda)$ given by

$$
\begin{pmatrix} V_1(\cdot, \lambda) \\ V_2(\cdot, \lambda) \end{pmatrix} = \begin{pmatrix} Y_1(\cdot, \lambda) \\ Y_2(\cdot, \lambda) \end{pmatrix} \frac{1}{\sqrt{\tau^2 + 1}} \begin{pmatrix} \tau & 1 \\ -1 & \tau \end{pmatrix}.
$$

(7.8.18)

The next proposition is the counterpart of Proposition 7.8.3 for the self-adjoint extensions $A_\tau$.

**Proposition 7.8.5.** Assume that the endpoint $a$ is regular or quasiregular and that the endpoint $b$ is in the limit-point case. Let $Y(\cdot, \lambda)$ be a fundamental matrix fixed by $Y(a, \lambda) = I$. Let $\{C, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\text{max}}$ in Theorem 7.8.2. For $\tau \in \mathbb{R}$ let $A_\tau$ be a self-adjoint extension of $T_{\text{min}}$ given by (7.8.14) and let $M_\tau$ be as in (7.8.16). Then the resolvent of $A_\tau$ is an integral operator

$$
((A_\tau - \lambda)^{-1} g)(t) = \int_a^b G_\tau(t, s, \lambda) \Delta(s) g(s) \, ds, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
$$

(7.8.19)

where $g \in L^2_\Delta(\mathfrak{i})$. The Green’s function $G_\tau(t, s, \lambda)$ is given by

$$
G_\tau(t, s, \lambda) = G_{\tau,e}(t, s, \lambda) + G_{\tau,i}(t, s, \lambda),
$$

(7.8.20)
where the entire part $G_{\tau,e}$ is given by

$$G_{\tau,e}(t, s, \lambda) = \begin{cases} V_1(t, \lambda)V_2(s, \bar{\lambda})^*, & s < t, \\ V_2(t, \lambda)V_1(s, \bar{\lambda})^*, & s > t, \end{cases} \quad (7.8.21)$$

and

$$G_{\tau,i}(t, s, \lambda) = V_2(t, \lambda)M_\tau(\lambda)V_2(s, \bar{\lambda})^*. \quad (7.8.22)$$

**Proof.** The proof of Proposition 7.8.5 is similar to the proof of Proposition 7.8.3. In order to show the identity (7.8.19), consider $g \in L^2_{\Delta}(i)$ and define the function $f$ by the right-hand side of (7.8.19). Then one has

$$f(t) = (V_1(t, \lambda) + V_2(t, \lambda)M_\tau(\lambda)) \int_a^t V_2(s, \bar{\lambda})^*\Delta(s)g(s)\, ds$$

$$+ V_2(t, \lambda) \int_t^b (V_1(s, \bar{\lambda})^* + M_\tau(\bar{\lambda})V_2(s, \bar{\lambda})^*)\Delta(s)g(s)\, ds \quad (7.8.23)$$

and the same arguments as in the proof of Proposition 7.8.3 show that $f$ is well defined, absolutely continuous, and satisfies the inhomogenous equation

$$Jf' - Hf = \lambda \Delta f + \Delta g. \quad (7.8.24)$$

Moreover, one sees from (7.8.23) that $f$ satisfies

$$f(a) = \frac{1}{\sqrt{\tau^2 + 1}} \left( \frac{1}{\tau} \int_a^b (V_1(s, \bar{\lambda}) + V_2(s, \bar{\lambda})M_\tau(\bar{\lambda}))^*\Delta(s)g(s)\, ds \right)$$

$$= \frac{1}{\sqrt{\tau^2 + 1}} \left( \frac{1}{\tau} \right) (g, \gamma_\tau(\overline{\lambda}))_\Delta,$$

and hence

$$\frac{\tau f_1(a) - f_2(a)}{\sqrt{\tau^2 + 1}} = 0 \quad \text{and} \quad \frac{f_1(a) + \tau f_2(a)}{\sqrt{\tau^2 + 1}} = (g, \gamma_\tau(\overline{\lambda}))_\Delta.$$

Now denote the function on the left-hand side of (7.8.19) by $h = (A_\tau - \lambda)^{-1}g$. Then $h$ also satisfies (7.8.24) and from (7.8.15) and Proposition 2.3.2 one obtains

$$\frac{\tau h_1(a) - h_2(a)}{\sqrt{\tau^2 + 1}} = \Gamma_\tau^* \{h, \lambda h + g\} = 0,$$

$$\frac{h_1(a) + \tau h_2(a)}{\sqrt{\tau^2 + 1}} = \Gamma_\tau^* \{h, \lambda h + g\} = \gamma_\tau(\overline{\lambda})^*g = (g, \gamma_\tau(\overline{\lambda}))_\Delta.$$

Thus, for a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the functions $h$ and $f$ satisfy the same initial value problem and hence it follows that $h = f$. Therefore, $(A_\tau - \lambda)^{-1}g$ is given by the right-hand side of (7.8.19).
Let \( \{ C, \Gamma_0, \Gamma_1 \} \) be the boundary triplet for \( T_{\text{max}} \) in Theorem 7.8.2 and consider the self-adjoint extension \( A_\tau = \ker \Gamma_\tau \); cf. (7.8.14). Assume that the Weyl function \( M_\tau \) corresponding to \( \{ C, \Gamma_\tau_0, \Gamma_\tau_1 \} \) has the integral representation

\[
M_\tau(\lambda) = \alpha_\tau + \beta_\tau \lambda + \int_\mathbb{R} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\sigma_\tau(t),
\]

with \( \alpha_\tau \in \mathbb{R} \), \( \beta_\tau \geq 0 \), and \( \sigma_\tau \) a nondecreasing function with

\[
\int_\mathbb{R} \frac{1}{t^2 + 1} d\sigma_\tau(t) < \infty.
\]

Recall from Theorem 3.5.10 and Lemma A.2.6 that \( \text{mul} A_\tau \ominus \text{mul} T_{\text{min}} \) is nontrivial if and only if \( \beta_\tau > 0 \).

**Lemma 7.8.6.** Let \( \tau \in \mathbb{R} \cup \{ \infty \} \) and let \( E_\tau(\cdot) \) be the spectral measure of the self-adjoint relation \( A_\tau \). For \( h \in L^2(\Delta) \) with compact support define the Fourier transform \( \hat{f} \) by

\[
\hat{f}(\mu) = \int_a^b V_2(s, \mu)^* \Delta(s) f(s) ds, \quad \mu \in \mathbb{R},
\]

where \( V_2(\cdot, \mu) \) is the formal solution in (7.8.17). Let \( \sigma_\tau \) be the function in the integral representation (7.8.25) of the Weyl function \( M_\tau \). Then for every bounded open interval \( \delta \subset \mathbb{R} \) such that its endpoints are not eigenvalues of \( A_\tau \) one has

\[
(E_\tau(\delta)f, f)_\Delta = \int_\delta \hat{f}(\mu) \overline{\hat{f}(\mu)} d\sigma_\tau(\mu).
\]

**Proof.** Recall that \( (A_\tau - \lambda)^{-1} \) is given by (7.8.19), where the Green function \( G_\tau(t, s, \lambda) \) in (7.8.20) is given by (7.8.21) and (7.8.22). Assume that the function \( h \in L^2(\Delta) \) has compact support in \([a', b'] \subset \mathbb{R} \). Then

\[
((A_\tau - \lambda)^{-1} f, f)_\Delta = \int_a^b f(t)^* \Delta(t) \left( \int_a^b G_\tau(t, s, \lambda) \Delta(s) f(s) ds \right) dt
\]

for each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), where, in fact, the integration takes place only on the square \([a', b'] \times [a', b'] \).

Let \( \delta \subset \mathbb{R} \) be a bounded interval such that the endpoints of \( \delta \) are not eigenvalues of \( A_\tau \). Then the spectral projection of \( A_\tau \) corresponding to the interval \( \delta \) is given by Stone’s formula (1.5.7) (see also Example A.1.4)

\[
(E(\delta)f, f)_\Delta = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_\delta \left( ((A_\tau - (\mu + i\varepsilon))^{-1} - (A_\tau - (\mu - i\varepsilon))^{-1}) f, f \right)_\Delta d\mu
\]

\[
= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_\delta \left( \int_a^b f(t)^* \Delta(t) \left( G_\tau(t, s, \mu + i\varepsilon) - G_\tau(t, s, \mu - i\varepsilon) \right) \Delta(s) f(s) ds dt \right) d\mu.
\]
Decompose the Green function in (7.8.20) as in (7.8.21) and (7.8.22). Since the function \( \lambda \mapsto V(t, \lambda) \) is entire, one verifies in the same way as in the proof of Theorem 7.8.4 that

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\delta} \left( \int_{a}^{b} \int_{a}^{b} f(t)^* \Delta(t) (G_{\tau,e}(t, s, \mu + i\varepsilon) - G_{\tau,e}(t, s, \mu - i\varepsilon)) \Delta(s) f(s) ds \, dt \right) d\mu = 0.
\]

Therefore, it remains to consider the corresponding integral with \( G_{\tau,i} \), which takes the form

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\delta} \left( \int_{a}^{b} \int_{a}^{b} f(t)^* \Delta(t) \left[ (g_{t,s}M_{\tau})(\mu + i\varepsilon) - (g_{t,s}M_{\tau})(\mu - i\varepsilon) \right] \Delta(s) f(s) ds \, dt \right) d\mu,
\]

(7.8.27)

where \( g_{t,s} \) stands for the \( 2 \times 2 \) matrix function

\[ g_{t,s}(\eta) = V_2(t, \eta) V_2(s, \eta)^*. \]

For \( t, s \in [a', b'] \) this function is entire in \( \eta \). For \( \varepsilon_0 > 0 \) and \( A < B \) such that \( \delta \subset (A, B) \) consider the rectangle \( R = [A, B] \times [-i\varepsilon_0, i\varepsilon_0] \). Then the function \( \{ t, s, \eta \} \mapsto g_{t,s}(\eta) \) is bounded on \( [a', b'] \times [a', b'] \times R \), and since \( \Delta h \in L^1(a', b') \), it follows that for each fixed \( \varepsilon \) such that \( 0 < \varepsilon \leq \varepsilon_0 \)

\[
\frac{1}{2\pi i} \int_{\delta} \left( \int_{a}^{b} \int_{a}^{b} f(t)^* \Delta(t) \left[ (g_{t,s}M_{\tau})(\mu + i\varepsilon) - (g_{t,s}M_{\tau})(\mu - i\varepsilon) \right] \Delta(s) f(s) ds \, dt \right) d\mu = \frac{1}{2\pi i} \int_{a}^{b} \int_{a}^{b} f(t)^* \Delta(t) \left( \int_{\delta} \left[ (g_{t,s}M_{\tau})(\mu + i\varepsilon) - (g_{t,s}M_{\tau})(\mu - i\varepsilon) \right] d\mu \right) \Delta(s) f(s) ds \, dt.
\]

By the Stieltjes inversion formula in Lemma A.2.7 and Remark A.2.10, one sees

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\delta} \left[ (g_{t,s}M_{\tau})(\mu + i\varepsilon) - (g_{t,s}M_{\tau})(\mu - i\varepsilon) \right] d\mu = \int_{\delta} g_{t,s}(\mu) d\sigma_{\tau}(\mu)
\]

for all \( t, s \in [a', b'] \). To justify taking the limit \( \varepsilon \downarrow 0 \) inside the integral (7.8.27) one needs dominated convergence. For this purpose recall from Lemma A.2.7 and
Remark A.2.10 that there exists $m \geq 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ one has

$$\left| \int_{\delta} [(g_{t,s}M_\tau)(\mu + i\varepsilon) - (g_{t,s}M_\tau)(\mu - i\varepsilon)] \, d\mu \right| \leq m \sup\{ |g_{t,s}(\eta)|, |g'_{t,s}(\eta)| : t, s \in [a', b'], \lambda \in \mathbb{R} \},$$

(7.8.28)

where $R = [A, B] \times [-i\varepsilon_0, i\varepsilon_0]$. Since the functions

$$\{ t, s, \eta \} \mapsto g_{t,s}(\eta) \quad \text{and} \quad \{ t, s, \eta \} \mapsto g'_{t,s}(\eta)$$

are bounded on $[a', b'] \times [a', b'] \times \mathbb{R}$, it follows that the integral in (7.8.28), regarded as a function in $\{ t, s \}$ on $[a', b'] \times [a', b']$, is bounded by some constant for all $0 < \varepsilon \leq \varepsilon_0$. Furthermore, $\Delta f \in L^1(a', b')$ implies that there is an integrable majorant for (7.8.27), and so dominated convergence and Fubini’s theorem show that

$$(E(\delta)f, f)_{\Delta} = \int_{\delta} \int_{\delta} f(t)^* \Delta(t) \int_{\delta} g_{t,s}(\mu) \, d\sigma_\tau(\mu) \Delta(s)f(s) \, ds \, dt$$

$$= \int_{\delta} \left( \int_{\delta} (V_2(t, \mu)^* \Delta(t)f(t))^* \, dt \right) \left( \int_{\delta} V_2(s, \mu)^* \Delta(s)f(s) \, ds \right) \, d\sigma_\tau(\mu)$$

for every open interval $\delta$ such that the endpoints are not eigenvalues of $A_\tau$. This gives the formula in (7.8.26).

The next theorem is a consequence of Lemma 7.8.6 and Theorem B.2.3.

**Theorem 7.8.7.** Let $\tau \in \mathbb{R} \cup \{ \infty \}$, let $V_2(\cdot, \mu)$ be the formal solution in (7.8.17), and let $\sigma_\tau$ be the function in the integral representation of the Weyl function $M_\tau$. Then the Fourier transform

$$f \mapsto \hat{f}, \quad \hat{f}(\mu) = \int_a^b V_2(s, \mu)^* \Delta(s)f(s) \, ds, \quad \mu \in \mathbb{R},$$

extends by continuity from compactly supported functions $f \in L^2(\Delta)$ to a surjective partial isometry $\mathcal{F}$ from $L^2(\Delta)$ to $L^2_{\partial\sigma_\tau}(\mathbb{R})$ with $\ker \mathcal{F} = \text{mul}A_\tau$. The restriction $\mathcal{F}_{op} : L^2(\Delta) \cap \text{mul}A_\tau \to L^2_{\partial\sigma_\tau}(\mathbb{R})$ is a unitary mapping, such that the self-adjoint operator $(A_\tau)_{op}$ in $L^2(\Delta) \cap \text{mul}A_\tau$ is unitarily equivalent to multiplication by the independent variable in $L^2_{\partial\sigma_\tau}(\mathbb{R})$.

**Proof.** It follows from Lemma 7.8.6 that the condition (B.2.2) is satisfied. Furthermore, for every $\mu \in \mathbb{R}$ there exists a compactly supported function $f \in L^2(\Delta)$ such that

$$\hat{f}(\mu) = \int_a^b V_2(s, \mu)^* \Delta(s)f(s) \, ds \neq 0.$$

To see this, assume that for some $\mu \in \mathbb{R}$

$$\int_a^b V_2(s, \mu)^* \Delta(s)f(s) \, ds = 0$$
for all compactly supported $f \in L^2_\Delta (\mathfrak{i})$. This implies that $V_2(s, \mu)^* \Delta(s) = 0$ for a.e. $s \in (a, b)$. By definiteness one has $V_2(s, \mu) = 0$ for a.e. $s \in (a, b)$, which is a contradiction. Therefore, condition (B.2.9) is satisfied and the result follows from Theorem B.2.3. □

In the next lemma the Fourier transform $\mathcal{F}_\gamma^\tau$ of the $\gamma$-field in (7.8.16) corresponding to the boundary triplet $\{\mathcal{C}, \Gamma_0^\tau, \Gamma_1^\tau\}$ is computed; this allows to identify the model in Theorem 7.8.7 with the model for scalar Nevanlinna functions in Section 4.3.

**Lemma 7.8.8.** Let $\tau \in \mathbb{R} \cup \{\infty\}$ and let $\gamma^\tau$ be the $\gamma$-field in (7.8.16). Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has almost everywhere in the sense of $d\sigma^\tau$:

$$\left[\mathcal{F}_\gamma^\tau(\lambda)\right](\mu) = \frac{1}{\mu - \lambda}, \quad \mu \in \mathbb{R},$$

where $\mathcal{F}$ is the Fourier transform from $L^2_\Delta (\mathfrak{i})$ onto $L^2_{d\sigma^\tau}(\mathbb{R})$ in Theorem 7.8.7.

**Proof.** Recall first that for $g \in L^2_\Delta (\mathfrak{i})$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the function $(A_\tau - \lambda)^{-1}g$ is given by the identity (7.8.19), which holds for all $t \in \mathfrak{i}$. In fact, the Green function $G_\tau$ in (7.8.19) is a $2 \times 2$ matrix function and the following notation will be useful:

$$G_\tau(t, s, \lambda) = \begin{pmatrix} G_{11}(t, s, \lambda) & G_{12}(t, s, \lambda) \\ G_{21}(t, s, \lambda) & G_{22}(t, s, \lambda) \end{pmatrix},$$

where each of these components is a $1 \times 2$ matrix function. The fundamental matrix $V(\cdot, \lambda)$ in (7.8.17) is written in the form

$$(V_1(\cdot, \lambda) \quad V_2(\cdot, \lambda)) = \begin{pmatrix} V_{11}(\cdot, \lambda) & V_{12}(\cdot, \lambda) \\ V_{21}(\cdot, \lambda) & V_{22}(\cdot, \lambda) \end{pmatrix}.$$ 

Now observe that

$$G_{\tau,1}(t, s, \lambda) = \begin{cases} (V_{11}(t, \lambda) + M_\tau(\lambda)V_{12}(t, \lambda))V_2(s, \overline{\lambda})^*, & s < t, \\ V_{12}(t, \lambda)\gamma_\tau(s, \overline{\lambda})^*, & s > t, \end{cases} \quad (7.8.29)$$

and

$$G_{\tau,2}(t, s, \lambda) = \begin{cases} (V_{21}(t, \lambda) + M_\tau(\lambda)V_{22}(t, \lambda))V_2(s, \overline{\lambda})^*, & s < t, \\ V_{22}(t, \lambda)\gamma_\tau(s, \overline{\lambda})^*, & s > t, \end{cases} \quad (7.8.30)$$

which follows easily from (7.8.21) and (7.8.22); cf. (7.8.23). Note also that

$$V_{ij}(\cdot, \overline{\lambda}) = \overline{V_{ij}(\cdot, \lambda)}, \quad i, j = 1, 2,$$

since the system is real (see Lemma 7.2.8), and that (7.2.10) implies the useful identity

$$V_{11}(t, \lambda)V_{22}(t, \lambda) - V_{21}(t, \lambda)V_{12}(t, \lambda) = 1, \quad t \in \mathfrak{i}. \quad (7.8.31)$$
For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) one has

\[
((A_r - \lambda)^{-1} g)(t) = \left( \int_a^b G_{r,1}(t, s, \lambda) \Delta(s) g(s) \, ds \right) \left( \left( g, G_{r,1}(t, \cdot, \lambda)^* \right)_\Delta \right) = \left( g, G_{r,2}(t, \cdot, \lambda)^* \right)_\Delta.
\]

Since the Fourier transform \( \mathcal{F} : L^2_\Delta(i) \to L^2_{\Delta^*}(\mathbb{R}) \) in Theorem 7.8.7 is a partial isometry with \( \ker \mathcal{F} = \text{mul} A_r \), it follows from (1.1.10) that

\[
((A_r - \lambda)^{-1} g)(t) = \left( \int_{\mathbb{R}} \mathcal{F}g(\mu) \left[ \mathcal{F}G_{r,2}(t, \cdot, \lambda)^* \right](\mu) \, d\sigma(\mu) \right)
\]

is valid for all \( g \in \text{mul} A_r \). It is clear that (7.8.32) is also true for all \( g \in \text{mul} A_r \), since in this case \( (A_r - \lambda)^{-1} g = 0 \) and \( \mathcal{F}g = 0 \). Therefore, (7.8.32) is valid for all \( g \in L^2_\Delta(i) \). Moreover, if \( g \in L^2_\Delta(i) \) one has for almost all \( t \in i \)

\[
((A_r - \lambda)^{-1} g)(t) = \left( \int_{\mathbb{R}} \frac{V_2(t, \mu)}{\mu - \lambda} \mathcal{F}g(\mu) \, d\sigma(\mu) \right)
\]

see (B.2.5). Furthermore, if \( \mathcal{F}g \) has compact support, then the right-hand side of (7.8.33) is absolutely continuous, and hence in this case the equality holds for all \( t \in i \). Therefore, when \( \mathcal{F}g \) has compact support the right-hand sides of (7.8.32) and (7.8.33) are equal for all \( t \in i \), and hence one has

\[
\frac{V_{12}(t, \mu)}{\mu - \lambda} = \left[ \mathcal{F}G_{r,1}(t, \cdot, \lambda)^* \right](\mu) \quad \text{and} \quad \frac{V_{22}(t, \mu)}{\mu - \lambda} = \left[ \mathcal{F}G_{r,2}(t, \cdot, \lambda)^* \right](\mu)
\]

for all \( t \in i \). These identities hold for all \( \mu \in \mathbb{R} \setminus \Omega(t) \), where the set \( \Omega(t) \subset \mathbb{R} \) has \( d\sigma_{\tau} \)-measure 0. Hence, (by replacing \( \lambda \) with \( \overline{\lambda} \) and taking conjugates) one has for all \( t \in i \) and all \( \mu \in \mathbb{R} \setminus \Omega(t) \)

\[
\frac{V_{12}(t, \mu)}{\mu - \lambda} = \left[ \mathcal{F}G_{r,1}(t, \cdot, \overline{\lambda})^* \right](\mu) \quad \text{and} \quad \frac{V_{22}(t, \mu)}{\mu - \lambda} = \left[ \mathcal{F}G_{r,2}(t, \cdot, \overline{\lambda})^* \right](\mu).
\]

By means of (7.8.29), (7.8.30), (7.8.31), and (7.8.34) it is straightforward to verify that for all \( t \in i \) and all \( \mu \in \mathbb{R} \setminus \Omega(t) \),

\[
\frac{1}{\mu - \lambda} \left( V_{11}(t, \lambda) V_{22}(t, \mu) - V_{21}(t, \lambda) V_{12}(t, \mu) \right)
\]

\[
= V_{11}(t, \lambda) \left[ \mathcal{F}G_{r,2}(t, \cdot, \overline{\lambda})^* \right](\mu) - V_{21}(t, \lambda) \left[ \mathcal{F}G_{r,1}(t, \cdot, \overline{\lambda})^* \right](\mu) \quad \text{(7.8.35)}
\]

\[
= \mathcal{F} \left[ V_{11}(t, \lambda) G_{r,2}(t, \cdot, \overline{\lambda})^* - V_{21}(t, \lambda) G_{r,1}(t, \cdot, \overline{\lambda})^* \right](\mu)
\]

\[
= \mathcal{F} \left[ W(t, \cdot, \lambda) \right](\mu),
\]

where \( W(t, \cdot, \lambda) \) is given by (7.8.35).
where the $2 \times 1$ matrix function $W(\cdot, \cdot, \lambda)$ is given by

$$W(t, s, \lambda) = \begin{cases} M_\tau(\lambda)V_2(s, \lambda), & s < t, \\ \gamma_\tau(s, \lambda), & s > t. \end{cases} \quad (7.8.36)$$

The above identity and a limit process will give the desired result. In fact, first observe that according to the definition of $W(\cdot, \cdot, \lambda)$ in (7.8.36) one has

$$\|\gamma_\tau(\cdot, \lambda) - W(t, \cdot, \lambda)\|_\Delta = \int_a^t |\Delta(s)^{1/2}V_1(s, \lambda)|^2 ds \to 0 \quad as \quad t \to a,$$

and hence the continuity of $F : L^2_{\Delta}(\cdot) \to L^2_{d\sigma_\tau}(\R)$ implies that

$$\|\gamma_\tau(\cdot, \lambda) - W(t, \cdot, \lambda)\|_{L^2_{d\sigma_\tau}(\R)} \to 0 \quad as \quad t \to a.$$

Now approximate $a$ by a sequence $t_n \in \tau$. Then there exists a subsequence, again denoted by $t_n$, such that pointwise

$$[F\gamma_\tau(\cdot, \lambda)](\mu) = \lim_{n \to \infty} [FW(t_n, \cdot, \lambda)](\mu), \quad \mu \in \R \setminus \Omega,$$

where $\Omega$ is a set of measure 0 in the sense of $d\sigma_\tau$. Observe that (7.8.35) gives

$$[FW(t_n, \cdot, \lambda)](\mu) = \frac{1}{\mu - \lambda} \left( V_{11}(t_n, \lambda)V_{22}(t_n, \mu) - V_{21}(t_n, \lambda)V_{12}(t_n, \mu) \right)$$

for all $\mu \in \R \setminus \Omega(t_n)$. The limit on the right-hand side as $n \to \infty$ gives

$$\frac{1}{\mu - \lambda} \left( V_{11}(a, \lambda)V_{22}(a, \mu) - V_{21}(a, \lambda)V_{12}(a, \mu) \right) = \frac{1}{\mu - \lambda},$$

which follows from the form of the fundamental matrix $(V_1(\cdot, \cdot) V_2(\cdot, \cdot))$ in (7.8.17). Hence,

$$[F\gamma_\tau(\cdot, \lambda)](\mu) = \frac{1}{\mu - \lambda}, \quad \mu \in \R \setminus \left( \Omega \cup \bigcup_{n=1}^{\infty} \Omega(t_n) \right),$$

which completes the proof. \(\square\)

Lemma 7.8.8 will be used to explain the model in Theorem 7.8.7 with the model for scalar Nevanlinna functions discussed in Section 4.3. Without loss of generality it is assumed that $T_{\min}$ is simple; cf. Section 3.4. The Weyl function $M_\tau$ of the boundary triplet $\{C, \Gamma_0', \Gamma_1'\}$ for $T_{\max}$ has the integral representation (7.8.25). If $\beta = 0$, then the discussion in Chapter 6 following Lemma 6.4.8 applies in this case as well. Hence, assume $\beta > 0$ in (7.8.25). Then by Theorem 4.3.4 there is a closed simple symmetric operator $S$ in $L^2_{d\sigma_\tau}(\R) \oplus \C$ such that the Nevanlinna function $M_\tau$ in (7.8.25) is the Weyl function corresponding to the boundary triplet $\{C, \Gamma_0', \Gamma_1'\}$ for $S^*$ in Theorem 4.3.4. The $\gamma$-field corresponding to $\{C, \Gamma_0', \Gamma_1'\}$ is
denoted by $\gamma'$ and it is given by (4.3.16). Furthermore, the restriction $A'_0$ corresponding to the boundary mapping $\Gamma'_0$ is a self-adjoint relation in $L^2_{d\sigma_r}(\mathbb{R}) \oplus \mathbb{C}$ whose operator part $(A'_0)_{\text{op}}$ is the maximal multiplication operator by the independent variable in $L^2_{d\sigma_r}(\mathbb{R})$. By comparing with (4.3.16) one sees that, according to Lemma 7.8.8, the Fourier transform $\mathcal{F}_{\text{op}}$ from $L^2_A(i) \ominus \text{mul} A_r$ onto $L^2_{d\sigma_r}(\mathbb{R})$ as a unitary mapping satisfies

$$\mathcal{F}_{\text{op}} P\gamma(\lambda) = P'\gamma'(\lambda),$$

where $P$ and $P'$ stand for the orthogonal projections from $L^2_A(i)$ onto $(\text{mul} A_r)$ and from $L^2_{d\sigma_r}(\mathbb{R}) \oplus \mathbb{C}$ onto $L^2_{d\sigma_r}(\mathbb{R}) = (\text{mul} A'_0)^\perp$. Recall that $(I - P)\gamma(\lambda)$ and $(I - P')\gamma'(\lambda)$ are independent of $\lambda$ and belong to $\text{mul} A_r$ and $\text{mul} A'_0$, respectively; cf. Corollary 2.5.16. Hence, the mapping $\mathcal{F}_m$ from $\text{mul} A_r$ to $\text{mul} A'_0$ defined by

$$\mathcal{F}_m (I - P)\gamma(\lambda) = \beta \frac{1}{2} (I - P')\gamma'(\lambda)$$

is a one-to-one correspondence. In fact, $\mathcal{F}_m$ is an isometry due to Proposition 3.5.7. Define the mapping $U$ from the space $L^2_A(i)$ to the model space $L^2_{d\sigma_r}(\mathbb{R}) \oplus \mathbb{C}$ by

$$U = \begin{pmatrix} \mathcal{F}_{\text{op}} & 0 \\ 0 & \mathcal{F}_m \end{pmatrix} : \begin{pmatrix} (\text{mul} A_r)^\perp \\ \text{mul} A_r \end{pmatrix} \to \begin{pmatrix} (\text{mul} A'_0)^\perp \\ \text{mul} A'_0 \end{pmatrix}.$$  

Then it is clear that $U$ is unitary and that

$$U\gamma(\lambda) = \gamma'(\lambda).$$

Hence, by Theorem 4.2.6, it follows that the boundary triplet $\{C, \Gamma'_0, \Gamma'_1\}$ for $S^*$ and the boundary triplet $\{C, \Gamma_0, \Gamma'_1\}$ for $S^*$ are unitarily equivalent under the mapping $U$, in particular, one has

$$(A'_0)_{\text{op}} = \mathcal{F}_{\text{op}} (A_r)_{\text{op}} \mathcal{F}_{\text{op}}^{-1} \quad \text{and} \quad A'_0 = U A_r U^{-1}.$$  

At the end of this section the case where the endpoint $a$ is in the limit-circle case and the endpoint $b$ is in the limit-point case is briefly discussed. In a similar way as in the end of Section 7.7 one makes use of the transformation in Lemma 7.2.5. The next proposition is the counterpart of Theorem 7.8.2; it is proved in the same way.

**Proposition 7.8.9.** Assume that $a$ is in the limit-circle case and that $b$ is in the limit-point case. Let $\lambda_0$ in $\mathbb{R}$, let $U(\cdot, \lambda_0)$ be a solution matrix as in Lemma 7.2.5, and consider the limit

$$\tilde{f}(a) = \lim_{t \to a} U(t, \lambda_0)^{-1} f(t).$$
for \( \{f,g\} \in T_{\max} \); cf. Corollary 7.4.8. Then \( \{\mathbb{C}, \Gamma_0, \Gamma_1\} \), where

\[
\Gamma_0 \{f,g\} = \tilde{f}_1(a) \quad \text{and} \quad \Gamma_1 \{f,g\} = \tilde{f}_2(a), \quad \{f,g\} \in T_{\max},
\]

is a boundary triplet for \((T_{\min})^* = T_{\max}\). Let \( Y(\cdot, \lambda) \) be a fundamental matrix fixed in such a way that \( \tilde{Y}(\cdot, \lambda) = U(\cdot, \lambda_0)^{-1}Y(\cdot, \lambda) \) satisfies \( \tilde{Y}(a, \lambda) = I \). Then for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the \( \gamma \)-field \( \gamma \) and Weyl function \( M \) corresponding to \( \{\mathbb{C}, \Gamma_0, \Gamma_1\} \) are given by

\[
\gamma(\lambda) = Y_1(\cdot, \lambda) + M(\lambda)Y_2(\cdot, \lambda) \quad \text{and} \quad M(\lambda) = \frac{\tilde{\chi}_2(a, \lambda)}{\tilde{\chi}_1(a, \lambda)},
\]

where \( \tilde{\chi}(\cdot, \lambda) = U(\cdot, \lambda_0)^{-1}\chi(\cdot, \lambda) \) and \( \chi(\cdot, \lambda) \) is a nontrivial element in \( \mathcal{M}_\lambda(T_{\max}) \).

### 7.9 Weyl functions and subordinate solutions

Consider the real definite canonical system \((7.2.3)\) on the interval \( \iota = (a, b) \) and assume that the endpoint \( a \) is regular and that the endpoint \( b \) is in the limit-point case. Let \( \{\mathbb{C}, \Gamma_0, \Gamma_1\} \) be the boundary triplet for \( T_{\max} \) in Theorem 7.7.2 with \( \gamma \)-field \( \gamma \) and Weyl function \( M \). The spectrum of the self-adjoint extension

\[
A_0 = \ker \Gamma_0 = \{ \{f,g\} \in T_{\max} : f_1(a) = 0 \}
\]

will be studied by means of subordinate solutions of the equation \( Jy' - Hy = \lambda \Delta y \).

The discussion in this section is parallel to the discussion in Section 6.7.

It is useful to take into account all self-adjoint extensions of \( T_{\min} \). As in Section 7.8, there is a one-to-one correspondence to the numbers \( \tau \in \mathbb{R} \cup \{\infty\} \) as restrictions of \( T_{\max} \) via

\[
A_\tau = \{ \{f,g\} \in T_{\max} : \Gamma_1 \{f,g\} = \tau \Gamma_0 \{f,g\} \}, \quad (7.9.1)
\]

with the understanding that \( A_0 \) corresponds to \( \tau = \infty \). As before, let the boundary triplet \( \{\mathbb{C}, \Gamma^\tau_0, \Gamma^\tau_1\} \) be defined by the transformation \((7.8.15)\) with the Weyl function and \( \gamma \)-field given by

\[
M_\tau(\lambda) = \frac{1 + \tau M(\lambda)}{\tau - M(\lambda)} \quad \text{and} \quad \gamma_\tau(\lambda) = \frac{\sqrt{\tau^2 + 1}}{\tau - M(\lambda)^*} \gamma(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (7.9.2)
\]

Recall that the Weyl function and the \( \gamma \)-field are connected via

\[
\frac{M_\tau(\lambda) - M_\tau(\mu)^*}{\lambda - \mu} = \gamma_\tau(\mu)^* \gamma_\tau(\lambda), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \quad (7.9.3)
\]

The transformation \((7.8.15)\) also induces a transformation of the fundamental matrix \( Y(\cdot, \lambda) \) with \( Y(a, \lambda) = I \) as in \((7.8.17)\):

\[
(V_1(\cdot, \lambda) \quad V_2(\cdot, \lambda)) = (Y_1(\cdot, \lambda) \quad Y_2(\cdot, \lambda)) \frac{1}{\sqrt{\tau^2 + 1}} \begin{pmatrix} \tau & 1 \\ -1 & \tau \end{pmatrix}. \quad (7.9.4)
\]
Recall that
\[ \gamma_\tau(\lambda) = V_1(\cdot, \lambda) + M_\tau(\lambda)V_2(\cdot, \lambda) \]
is square-integrable with respect to \( \Delta \), while the second column of \( V(\cdot, \lambda) \) satisfies the (formal) boundary condition which determines \( A_\tau \); cf. (7.8.18).

In the next estimates it is more convenient to work in the semi-Hilbert space \( \mathcal{L}^2_\Delta(a) \) rather than in the Hilbert space \( \mathcal{L}^2_\Delta(i) \). In fact, for each \( x > a \) the notation \( \mathcal{L}^2_\Delta(a, x) \) stands for the semi-Hilbert space with the semi-inner product
\[ (f, g)_x = \int_a^x g(t)^* \Delta(t) f(t) \, dt, \quad f, g \in \mathcal{L}^2_\Delta(a, x), \]
and the seminorm corresponding to \((\cdot, \cdot)_x\) will be denoted by \( \| \cdot \|_x \); here the index \( \Delta \) is omitted. Hence, for fixed \( f, g \in \mathcal{L}^2_\Delta(a, c) \), \( a < c < b \), the function \( x \mapsto (f, g)_x \) is absolutely continuous and
\[ \frac{d}{dx} (f, g)_x = g(x)^* \Delta(x) f(x) \quad (7.9.5) \]
holds almost everywhere on \((a, c)\).

**Definition 7.9.1.** Let \( \lambda \in \mathbb{C} \). Then a solution \( h(\cdot, \lambda) \) of \( Jh' - Hh = \lambda \Delta h \) is said to be **subordinate** at \( b \) if
\[ \lim_{x \to b} \| h(\cdot, \lambda) \|_x = 0 \]
for every nontrivial solution \( k(\cdot, \lambda) \) of \( Jk' - Hk = \lambda \Delta k \) which is not a scalar multiple of \( h(\cdot, \lambda) \).

The spectrum of the self-adjoint extension \( A_0 \) will be studied in terms of solutions of the canonical system \( Jy' - Hy = \xi \Delta y \), \( \xi \in \mathbb{R} \), which do not necessarily belong to \( \mathcal{L}^2_\Delta(a, b) \). Observe that if a solution \( h(\cdot, \xi) \) of \( Jy' - Hy = \xi \Delta y \) belongs to \( \mathcal{L}^2_\Delta(a, b) \), then it is subordinate at \( b \) since \( b \) is in the limit-point case, and hence any other nontrivial solution which is not a multiple does not belong to \( \mathcal{L}^2_\Delta(a, b) \).

By means of the fundamental system \((V_1(\cdot, \lambda), V_2(\cdot, \lambda))\) in (7.8.17) define for any \( \lambda \in \mathbb{C} \) and \( h \in \mathcal{L}^2_\Delta(a, x) \), \( a < x < b \),
\[ (\mathcal{H}(\lambda)h)(t) = V_1(t, \lambda) \int_a^t V_2(s, \overline{\lambda})^* \Delta(s) h(s) \, ds \]
\[ - V_2(t, \lambda) \int_a^t V_1(s, \overline{\lambda})^* \Delta(s) h(s) \, ds, \quad t \in (a, x). \]
Thus, \( \mathcal{H}(\lambda) \) is a well-defined integral operator and it is clear that the function \( \mathcal{H}(\lambda)h \) is absolutely continuous. Using the identity (7.2.10) for \( V(\cdot, \lambda) \) (which holds because \( V(a, \overline{\lambda})^* JV(a, \lambda) = J \)) one sees in the same way as in (7.2.14)–(7.2.16) that \( f = \mathcal{H}(\lambda)h \) satisfies
\[ Jf' - Hf = \lambda \Delta f + \Delta h, \quad f(a) = 0. \]
In particular, $\mathcal{H}(\lambda)$ maps $L^2_\Delta(a, x)$ into itself. It follows directly that for $\lambda, \mu \in \mathbb{C}$ one has
\[ V_i(\cdot, \lambda) - V_i(\cdot, \mu) = (\lambda - \mu)\mathcal{H}(\lambda)V_i(\cdot, \mu), \quad i = 1, 2, \tag{7.9.6} \]
since the functions on the left-hand side and the right-hand side both satisfy the same equation $Jy' - Hy = \lambda \Delta y + (\lambda - \mu)\Delta V_i(\cdot, \mu)$ and both functions vanish at the endpoint $a$.

**Lemma 7.9.2.** Let $a < x < b$ and let $h \in L^2_\Delta(a, x)$. Then the operator $\mathcal{H}(\lambda)$ satisfies
\[ \|\mathcal{H}(\lambda)h\|^2_2 \leq 2\|V_1(\cdot, \lambda)\|^2_2 \|V_2(\cdot, \lambda)\|^2_2 \|h\|^2_2 \]
for each $x > a$.

**Proof.** The definition of $\mathcal{H}(\lambda)$ may be written as
\[ (\mathcal{H}(\lambda)h)(t) = V_1(t, \lambda)g_2(t, \lambda) - V_2(t, \lambda)g_1(t, \lambda), \tag{7.9.7} \]
with the functions $g_i(\cdot, \lambda)$, $i = 1, 2$, defined by
\[ g_i(t, \lambda) = \int_a^t V_i(s, \lambda)^* \Delta(s)h(s) \, ds. \]
The Cauchy–Schwarz inequality and Corollary 7.2.9 show that
\[ |g_i(t, \lambda)|^2 \leq \|V_i(\cdot, \lambda)\|^2_2 \|h\|^2_2 = \|V_i(\cdot, \lambda)\|^2_2 \|h\|^2_2, \quad i = 1, 2. \tag{7.9.8} \]
Multiplying $(\mathcal{H}(\lambda)h)(t)$ in (7.9.7) on the left by the matrix $\Delta(t)^{\frac{1}{2}}$, using the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, and using (7.9.8) one obtains
\begin{align*}
&|\Delta(t)^{\frac{1}{2}}(\mathcal{H}(\lambda)h)(t)|^2 \\
&\leq 2(|\Delta(t)^{\frac{1}{2}}V_1(t, \lambda)|^2 |g_2(t, \lambda)|^2 + |\Delta(t)^{\frac{1}{2}}V_2(t, \lambda)|^2 |g_1(t, \lambda)|^2) \\
&\leq 2(|\Delta(t)^{\frac{1}{2}}V_1(t, \lambda)|^2 \|V_2(\cdot, \lambda)\|^2_2 \|h\|^2_2 + |\Delta(t)^{\frac{1}{2}}V_2(t, \lambda)|^2 \|V_1(\cdot, \lambda)\|^2_2 \|h\|^2_2). \\
\end{align*}
Integration of this inequality over $(a, x)$ and (7.9.5) lead to
\begin{align*}
\|\mathcal{H}(\lambda)h\|^2_x &\leq 2 \int_a^x \left(|\Delta(t)^{\frac{1}{2}}V_1(t, \lambda)|^2 \|V_2(\cdot, \lambda)\|^2_2 \|h\|^2_t \\
&\quad + |\Delta(t)^{\frac{1}{2}}V_2(t, \lambda)|^2 \|V_1(\cdot, \lambda)\|^2_2 \|h\|^2_t \right) \, dt \\
&= 2 \int_a^x \left( \frac{d}{dt} \|V_1(\cdot, \lambda)\|^2_t \|V_2(\cdot, \lambda)\|^2_t \right) \|h\|^2_t \, dt \\
&\leq 2\|h\|^2_x \int_a^x \left( \frac{d}{dt} \|V_1(\cdot, \lambda)\|^2_t \|V_2(\cdot, \lambda)\|^2_t \right) \, dt,
\end{align*}
which implies the desired result. \qed
Since the system is assumed to be definite on \( \alpha = (a, b) \), there is a compact subinterval \([\alpha, \beta]\) such that the system is definite on \([\alpha, \beta]\) and hence on any interval \((a, x)\) with \(x > \beta\). This implies that for \(x > \beta\) both functions
\[
x \mapsto \|V_1(\cdot, \lambda)\|_x \quad \text{and} \quad x \mapsto \|V_2(\cdot, \lambda)\|_x
\]
have positive values; cf. (7.5.1) and Lemma 7.5.2.

**Lemma 7.9.3.** Let \( \xi \in \mathbb{R} \) be a fixed number. The function \( x \mapsto \varepsilon(\tau, x, \xi) \) given by
\[
\sqrt{2} \varepsilon(\tau, x, \xi) \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x = 1, \quad x > \beta,
\]
is well defined, continuous, nonincreasing, and satisfies
\[
\lim_{x \to b} \varepsilon(\tau, x, \xi) = 0.
\]

**Proof.** It is clear that \( \varepsilon(\tau, x, \xi) > 0 \) is well defined due to the assumption that \(x > \beta\). Note that \( x \mapsto \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \) is continuous and nondecreasing. The assumption that \(b\) is in the limit-point case implies that not both \(V_1(\cdot, \xi)\) and \(V_2(\cdot, \xi)\) belong to \(L^2(\alpha)\). Thus, the limit result follows. □

The function \( x \mapsto \varepsilon(\tau, x, \xi) \) appears in the estimate in the following theorem.

**Theorem 7.9.4.** Let \( M_\tau \) be the Weyl function in (7.9.1) corresponding to the boundary triplet \( \{C, \Gamma_0^\tau, \Gamma_1^\tau\} \). Assume that \( \xi \in \mathbb{R} \) and let \( \varepsilon(\tau, x, \xi) \) be as in Lemma 7.9.3. Then for \(a < \beta < x < b\)
\[
\frac{1}{d_0} \leq \frac{\|V_2(\cdot, \xi)\|_x}{\|V_1(\cdot, \xi)\|_x} |M_\tau(\xi + i\varepsilon(\tau, x, \xi))| \leq d_0,
\]
where \( d_0 = 1 + 2(\sqrt{2} + \sqrt{2} + \sqrt{2}) \).

**Proof.** Assume that \( \xi \in \mathbb{R} \) and let \( \varepsilon > 0 \). Define the function \( \psi(\cdot, \xi, \varepsilon) \) by
\[
\psi(\cdot, \xi, \varepsilon) = V_1(\cdot, \xi) + M_\tau(\xi + i\varepsilon)V_2(\cdot, \xi).
\]
(7.9.9)

For any \(a < x < b\) this leads to
\[
\frac{\|V_2(\cdot, \xi)\|_x}{\|V_1(\cdot, \xi)\|_x} |M_\tau(\xi + i\varepsilon)| - \|V_1(\cdot, \xi)\|_x \leq \|\psi(\cdot, \xi, \varepsilon)\|_x
\]
or, equivalently, when \( \beta < x < a \)
\[
\frac{\|V_2(\cdot, \xi)\|_x}{\|V_1(\cdot, \xi)\|_x} |M_\tau(\xi + i\varepsilon)| - 1 \leq \frac{\|\psi(\cdot, \xi, \varepsilon)\|_x}{\|V_1(\cdot, \xi)\|_x}.
\]
(7.9.10)

The term on the right-hand side of (7.9.10) will now be estimated in a suitable way. First note that it follows from (7.9.6) that for \( \lambda \in \mathbb{C} \) and \( \mu \in \mathbb{C} \setminus \mathbb{R} \) one obtains
\[
V_1(\cdot, \lambda) + M_\tau(\mu)V_2(\cdot, \lambda) - \gamma_\tau(\mu) = (\lambda - \mu)\mathcal{H}(\lambda)\gamma_\tau(\cdot, \mu).
\]
(7.9.11)
Applying the identity in (7.9.11) with \( \lambda = \xi \) and \( \mu = \xi + i\varepsilon \) one sees that
\[
\psi(\cdot, \xi, \varepsilon) = \gamma_\tau(\cdot, \xi + i\varepsilon) - i\varepsilon H(\xi)\gamma_\tau(\cdot, \xi + i\varepsilon),
\]
which expresses the function \( \psi(\cdot, \xi, \varepsilon) \) in (7.9.9) in terms of the \( \gamma \)-field \( \gamma_\tau \). Hence, it follows from Lemma 7.9.2 that
\[
\|\psi(\cdot, \xi, \varepsilon)\|_x \leq (1 + \sqrt{2}\varepsilon \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x)\|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_x.
\]

Therefore, the right-hand side of (7.9.10) is estimated by
\[
\frac{(1 + \sqrt{2}\varepsilon \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x) \|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_x}{\|V_1(\cdot, \xi)\|_x} = \frac{1 + \sqrt{2}\varepsilon \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \frac{1}{2} \|V_1(\cdot, \xi)\|_x \frac{1}{2} \|V_2(\cdot, \xi)\|_x \frac{1}{2}}{(\|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \frac{1}{2})} \|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_x.
\]

Now observe that \( \|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_x \leq \|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_b \) and it follows from (7.9.3) that
\[
\|\gamma_\tau(\cdot, \xi + i\varepsilon)\|_b \leq \sqrt{\frac{\text{Im} M_\tau(x, \xi)}{\varepsilon}} \leq \sqrt{\frac{\|M_\tau(x, \xi + i\varepsilon)\|}{\varepsilon}}.
\]

Thus, for any \( \varepsilon > 0 \) and \( \beta < x < b \) one obtains the inequality
\[
\frac{\|V_2(\cdot, \xi)\|_x \|M_\tau(x, \xi + i\varepsilon)\|_x}{\|V_1(\cdot, \xi)\|_x} |M_\tau(x, \xi + i\varepsilon) - 1| \leq \frac{1 + \sqrt{2}\varepsilon \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \frac{1}{2} \|V_2(\cdot, \xi)\|_x \frac{1}{2}}{(\varepsilon \|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \frac{1}{2})} \left(\frac{\|V_2(\cdot, \xi)\|_x \|M_\tau(x, \xi + i\varepsilon)\|_x}{\|V_1(\cdot, \xi)\|_x \|V_2(\cdot, \xi)\|_x \frac{1}{2}} \right)^{\frac{1}{2}}.
\]

Now for \( \xi \in \mathbb{R} \) and \( \beta < x < b \) choose \( \varepsilon = \varepsilon_\tau(x, \xi) \) in this estimate. This choice minimizes the first factor on the right-hand side to \( 2^{5/4} \). Hence, the nonnegative quantity
\[
Q = \frac{\|V_2(\cdot, \xi)\|_x \|M_\tau(x, \xi + i\varepsilon_\tau(x, \xi))\|_x}{\|V_1(\cdot, \xi)\|_x \|M_\tau(x, \xi + i\varepsilon_\tau(x, \xi))\|_x}
\]
satisfies the inequality
\[
|Q - 1| \leq 2^{5/4} Q^{\frac{1}{2}}
\]
or, equivalently, \( Q^2 - 2Q + 1 \leq 4\sqrt{2}Q \). Therefore, \( 1/d_0 \leq Q \leq d_0 \), which completes the proof.

The following result is now a direct consequence of Theorem 7.9.4.

**Theorem 7.9.5.** Let \( M \) be the Weyl function corresponding to the boundary triplet \( \{C, \Gamma_0, \Gamma_1\} \) and let \( \xi \in \mathbb{R} \). Then the following statements hold:
If $\tau \in \mathbb{R}$, then the solution $Y_1(\cdot, \xi) + \tau Y_2(\cdot, \xi)$ of the boundary value problem

$$Jf' - Hf = \xi \Delta f, \quad f_2(a) = \tau f_1(a),$$

which is unique up to scalar multiples, is subordinate if and only if

$$\lim_{\varepsilon \downarrow 0} M(\xi + i\varepsilon) = \tau.$$

If $\tau = \infty$, then the solution $Y_2(\cdot, \xi)$ of the boundary value problem

$$Jf' - Hf = \xi \Delta f, \quad f_1(a) = 0,$$

which is unique up to scalar multiples, is subordinate if and only if

$$\lim_{\varepsilon \downarrow 0} M(\xi + i\varepsilon) = \infty.$$

Proof. Since $x \mapsto \varepsilon \tau(x, \xi)$ is continuous, nonincreasing, and has limit 0 as $x \to b$, one obtains the identity

$$\lim_{\varepsilon \downarrow 0} M \tau(\xi + i\varepsilon) = \lim_{x \to b} M \tau(\xi + i\varepsilon \tau(x, \xi)).$$

(i) Assume that $\tau \in \mathbb{R}$ and note that

$$V_2(\cdot, \xi) = \frac{1}{\sqrt{\tau^2 + 1}}(Y_1(\cdot, \xi) + \tau Y_2(\cdot, \xi))$$

by (7.9.4). It will be shown that $|M \tau(\xi + i\varepsilon)| \to \infty$ for $\varepsilon \downarrow 0$ if and only if the solution $V_2(\cdot, \xi)$ is subordinate. To see this, assume first that $|M \tau(\xi + i\varepsilon)| \to \infty$. Then, by Theorem 7.9.4, it follows that

$$\lim_{x \to b} \frac{\|V_2(\cdot, \xi)\|_x}{\|V_1(\cdot, \xi)\|_x} = 0.$$  \hfill (7.9.12)

Hence, for any $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$, one obtains from (7.9.12) that

$$\lim_{x \to b} \frac{\|V_2(\cdot, \xi)\|_x}{\|c_1 V_1(\cdot, \xi) + c_2 V_2(\cdot, \xi)\|_x} = 0,$$  \hfill (7.9.13)

and therefore the solution $V_2(\cdot, \xi)$ is subordinate. Conversely, assume that $V_2(\cdot, \xi)$ is subordinate, so that (7.9.13) holds for all $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Then clearly (7.9.12) holds, and therefore it follows from Theorem 7.9.4 that $|M \tau(\xi + i\varepsilon)| \to \infty$.

It is a consequence of (7.9.2) that for $\varepsilon \downarrow 0$ one has

$$|M \tau(\xi + i\varepsilon)| \to \infty \quad \Leftrightarrow \quad M(\xi + i\varepsilon) \to \tau.$$  

This equivalence leads to the assertion for $\tau \in \mathbb{R}$.

(ii) The case $\tau = \infty$ can be treated in the same way as (i).  \hfill $\square$
The self-adjoint extension $A_0 = \ker \Gamma_0$ of $T_{\min}$ is given by
\[ A_0 = \{ \{f, g\} \in T_{\max} : f_1(a) = 0 \}; \] (7.9.14)
cf. (7.9.1). The boundary condition
\[ f_1(a) = 0 \] (7.9.15)
plays a central role in the following definition, which is based on Theorem 7.9.5; cf. Definition 6.7.6.

**Definition 7.9.6.** With the canonical system $J f' - H f = \xi \Delta f, \xi \in \mathbb{R}$, the following subsets of $\mathbb{R}$ are associated:

(i) $M$ is the complement of the set of all $\xi \in \mathbb{R}$ for which a subordinate solution exists that does not satisfy (7.9.15);
(ii) $M_{ac}$ is the set of all $\xi \in \mathbb{R}$ for which no subordinate solution exists;
(iii) $M_s$ is the set of all $\xi \in \mathbb{R}$ for which a subordinate solution exists that satisfies (7.9.15);
(iv) $M_{sc}$ is the set of all $\xi \in \mathbb{R}$ for which a subordinate solution exists that satisfies (7.9.15) and does not belong to $L^2_{\Delta}(\iota)$;
(v) $M_p$ is the set of all $\xi \in \mathbb{R}$ for which a subordinate solution exists that satisfies (7.9.15) and belongs to $L^2_{\Delta}(\iota)$.

It is a direct consequence of Definition 7.9.6 that
\[ \mathbb{R} = M^c \sqcup M_{ac} \sqcup M_s, \quad M = M_{ac} \sqcup M_s, \quad \text{and} \quad M_s = M_{sc} \sqcup M_p, \]
where $\sqcup$ stands for disjoint union.

Let the Weyl function $M$ of the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ have the integral representation
\[ M(\lambda) = \alpha + \beta \lambda + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\sigma(t), \] (7.9.16)
where $\alpha \in \mathbb{R}, \beta \geq 0$, and the measure $\sigma$ satisfies
\[ \int_{\mathbb{R}} \frac{1}{t^2 + 1} d\sigma(t) < \infty; \]
cf. Theorem A.2.5. The following proposition is based on Corollary 3.1.8, where minimal supports for the various parts of the measure $\sigma$ in the integral representation of $M$ are described in terms of the boundary behavior of the Nevanlinna function $M$. The proof of Proposition 7.9.7 is the same as the proof of Proposition 6.7.7 and will not be repeated.
Proposition 7.9.7. Let $M$ be the Weyl function associated with the boundary triplet $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ and let $\sigma$ be the corresponding measure in (7.9.16). Then the sets

$$M, M_{ac}, M_s, M_{sc}, M_p,$$

are minimal supports for the measures

$$\sigma, \sigma_{ac}, \sigma_s, \sigma_{sc}, \sigma_p,$$

respectively.

The minimal supports in Proposition 7.9.7 are intimately connected with the spectrum of $A_0$. For the absolutely continuous spectrum one obtains in the same way as in Theorem 6.7.8 the following result, where the notion of the absolutely continuous closure of a Borel set from Definition 3.2.4 is used. Similar statements (with an inclusion) can be formulated for the singular parts of the spectrum; cf. Section 3.6.

Theorem 7.9.8. Let $A_0$ be the self-adjoint relation in (7.9.14) and let $M_{ac}$ be as in Definition 7.9.6. Then

$$\sigma_{ac}(A_0) = \text{clos}_{ac}(M_{ac}).$$

7.10 Special classes of canonical systems

In this section two particular types of canonical systems are studied. First it is shown how a class of Sturm–Liouville problems, which are slightly more general than the equations treated in Chapter 6, fit in the framework of canonical systems. In this context the results from the previous sections can be carried over to Sturm–Liouville equations. The second class of canonical systems which is discussed here consists of systems of the form (7.2.3) with $H = 0$. In this situation a simple limit-point/limit-circle criterion is provided.

Weighted Sturm–Liouville equations

Let $\tau \subset \mathbb{R}$ be an open interval. Let $1/p, q, r, s \in L^1_{\text{loc}}(\tau)$ be real functions, assume $r(t) \geq 0$ for almost all $t \in \tau$, and define the $2 \times 2$ matrix functions $H$ and $\Delta$ by

$$H(t) = \begin{pmatrix} -q(t) & -s(t) \\ -s(t) & 1/p(t) \end{pmatrix} \quad \text{and} \quad \Delta(t) = \begin{pmatrix} r(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad (7.10.1)$$

respectively. Let the $2 \times 2$ matrix $J$ be as in (7.2.2). If the vector functions $f$ and $g$ satisfy the canonical system $Jf' - Hf = \Delta g$, then their first components $f = f_1$ and $g = g_1$ satisfy the weighted Sturm–Liouville equation

$$- (f^{[1]})' + sf^{[1]} + qf = rg, \quad \text{where} \quad f^{[1]} = p(f' + sf), \quad (7.10.2)$$
and, since \( f \in AC(i) \), it follows that \( f, f^{[1]} \in AC(i) \). Conversely, if \( f, f^{[1]} \in AC(i) \) and \( f, g \) satisfy (7.10.2), then the vector functions

\[
f = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} g \\ 0 \end{pmatrix},
\]

satisfy the canonical system (7.2.3) with the coefficients given by (7.10.1). Since the functions \( 1/p, q, r, s \) are assumed to be real, the system \( Jf' - Hf = \Delta g \) is real. Moreover, the canonical system corresponding to (7.10.1) is definite, when any solution \( f \) of \( Jf' - Hf = 0 \) which satisfies \( \Delta f = 0 \) vanishes or, equivalently,

\[-f'_2 + qf_1 + sf_2 = 0, \quad f'_1 + sf_1 - (1/p)f_2 = 0, \quad rf_1 = 0 \Rightarrow f = 0.
\]

In accordance with the definition of definiteness for canonical equations, the Sturm–Liouville equation (7.10.2) is said to be definite if

\[-(f^{[1]})' + sf^{[1]} + qf = 0, \quad rf = 0 \Rightarrow f = 0.
\]

In particular, the Sturm–Liouville equation (7.10.2) is definite if the weight function \( r \) is positive on an open interval.

The matrix function \( \Delta \) in (7.10.1) induces the spaces \( L^2_{\Delta}(i) \) and \( L^2_{\Delta}(i) \). With the weight \( r \) it is natural to introduce the space \( L^2_{r}(i) \) of all complex measurable functions \( \varphi \) for which

\[
\int_i |\varphi(s)|^2 r(s) \, ds = \int_i \varphi(s)^* r(s) \varphi(s) \, ds < \infty.
\]

The corresponding semi-inner product is denoted by \( (\cdot, \cdot)_r \) and the corresponding Hilbert space of equivalence classes of elements from \( L^2_{r}(i) \) is denoted by \( L^2_{r}(i) \). It is clear that the mapping \( R \) defined by

\[
f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2_{\Delta}(i) \mapsto f_1 \in L^2_{r}(i)
\]

is an isometry with respect to the semi-inner products, thanks to the identity

\[
(f, f)_\Delta = \int_i \begin{pmatrix} r(s) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds = (f_1, f_1)_r.
\]

Furthermore, this mapping is onto, since each function in \( L^2_{r}(i) \) can be regarded as the first component of an element in \( L^2_{\Delta}(i) \) with the understanding that the second component can be any measurable function. Therefore, the mapping \( R \) induces a unitary operator, again denoted by \( R \), from \( L^2_{\Delta}(i) \) onto \( L^2_{r}(i) \).

Assume now that the system or, equivalently, the Sturm–Liouville equation is definite. In the Hilbert space \( L^2_{\Delta}(i) \) there are the preminimal relation \( T_0 \), minimal relation \( T_{\text{min}} \), and maximal relation \( T_{\text{max}} \) associated with the canonical system \( Jf' - Hf = \Delta g \):

\[
T_0 \subset \overline{T_0} = T_{\text{min}} \subset T_{\text{max}} = (T_{\text{min}})^*.
\]
Likewise, one can define corresponding relations in the Hilbert space $L^2(r)$. The maximal relation $\mathcal{T}_{\text{max}}$ is defined as follows:

$$\mathcal{T}_{\text{max}} = \left\{ \{f, g\} \in L^2_r(i) \times L^2_r(i) : -(pf^{[1]})' + sf^{[1]} + qf = rg \right\},$$

in the sense that there exist representatives $f$ and $g \in L^2_r(i)$ of $f$ and $g$, respectively, such that $f \in AC(i)$, $p^{[1]}f \in AC(i)$, and (7.10.2) holds. It is clear that the definiteness of the canonical system or, equivalently, of the equation (7.10.2) implies that each element $f \in \text{dom} \mathcal{T}_{\text{max}}$ has a unique representative $\hat{f}$ such that $f \in AC(i)$, $p^{[1]} \hat{f} \in AC(i)$; cf. Lemma 7.6.1. The preminimal relation $\mathcal{T}_0$ and the minimal relation $\mathcal{T}_{\text{min}}$ are defined by

$$\mathcal{T}_0 = \left\{ \{f, g\} \in \mathcal{T}_{\text{max}} : f \text{ has compact support} \right\} \quad \text{and} \quad \mathcal{T}_{\text{min}} = \overline{\mathcal{T}_0}.$$

It is not difficult to see that the mapping $\hat{R}$ defined by

$$\hat{R}\{f, g\} = \{Rf, Rg\}, \quad \{f, g\} \in L^2_r(i) \times L^2_r(i),$$

takes $T_{\text{max}}$ one-to-one onto $\mathcal{T}_{\text{max}}$, including absolutely continuous representatives, and that with $\{f, g\} = \hat{R}\{f, g\}$ and $\{h, \xi\} = \hat{R}\{h, k\}$:

$$(g, h)_\Delta - (f, k)_\Delta = (g, h)_r - (f, \xi)_r, \quad \{f, g\}, \{h, k\} \in \mathcal{T}_{\text{max}}. \quad (7.10.3)$$

Similarly, $\hat{R}$ takes $T_0$ one-to-one onto $\mathcal{T}_0$ and hence $\hat{R}$ takes $T_{\text{min}}$ one-to-one onto $\mathcal{T}_{\text{min}}$.

In the Hilbert space $L^2_r(i)$ the relations $\mathcal{T}_0$, $\mathcal{T}_{\text{min}}$, and $\mathcal{T}_{\text{max}}$ associated with the Sturm–Liouville equation (7.10.2) satisfy

$$\mathcal{T}_0 \subset \overline{\mathcal{T}_0} = \mathcal{T}_{\text{min}} \subset \mathcal{T}_{\text{max}} = (\mathcal{T}_{\text{min}})^*.$$

Furthermore, $R$ maps $\text{ker}(T_{\text{max}} - \lambda)$ one-to-one onto $\text{ker}(\mathcal{T}_{\text{max}} - \lambda)$. Since the functions $p$, $q$, $s$, and $r$ are real, it follows that the defect numbers of $T_{\text{min}}$ and $\mathcal{T}_{\text{min}}$ are equal. Let $\{\mathcal{S}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $T_{\text{max}}$ and let $\mathcal{T}_{\text{max}}$ be the corresponding maximal relation for the Sturm–Liouville operator. Then the mappings $\Gamma_0'$ and $\Gamma_1'$ from $\mathcal{T}_{\text{max}}$ to $\mathcal{S}$ given by

$$\Gamma_0'\{f, g\} = \Gamma_0\{f, g\} \quad \text{and} \quad \Gamma_1'\{f, g\} = \Gamma_1\{f, g\}, \quad \{f, g\} = \hat{R}\{f, g\}, \quad (7.10.4)$$

form a boundary triplet for $\mathcal{T}_{\text{max}}$; cf. (7.10.3). The boundary triplet $\{\mathcal{S}, \Gamma_0, \Gamma_1\}$ and the one in (7.10.4) have the same Weyl function.

Via the above identification, the discussion and the results for canonical systems with regular, quasiregular, and singular endpoints in Section 7.7 and Section 7.8 remain valid for weighted Sturm–Liouville equations of the form (7.10.2). Note that in the special case where $s(t) = 0$ and $r(t) > 0$ for almost all $t \in \mathbb{I}$ the Sturm–Liouville expression (7.10.2) coincides with the Sturm–Liouville expression studied in Chapter 6.
Special canonical systems

This subsection is devoted to the special class of canonical differential equations which have the form

\[ Jf' = \lambda \Delta f + \Delta g \]  

(7.10.5)
on an open interval \( \mathfrak{i} = (a,b) \), i.e., the class of canonical systems of the form (7.2.3) with \( H = 0 \). It will be assumed that the system is real and definite on \( \mathfrak{i} \). Here definiteness means that the identity \( \Delta(t)e = 0 \) for some \( e \in \mathbb{C}^2 \) and all \( t \in \mathfrak{i} \) implies \( e = 0 \). Note that Lemma 7.2.5 shows that any real definite canonical system of the form (7.2.3) can be transformed into the form (7.10.5) with a possible real shift of the eigenvalue parameter. For this class of equations the limit-point and limit-circle classification at an endpoint can be characterized in terms of the integrability of the function \( \Delta \).

**Theorem 7.10.1.** Let the canonical system (7.10.5) be real and definite, and let the endpoint \( a \) be regular. Then there is the alternative:

(i) \( \Delta \) is integrable and the endpoint \( b \) is in the limit-circle case;

(ii) \( \Delta \) is not integrable and the endpoint \( b \) is in the limit-point case.

**Proof.** By assumption, the canonical system (7.10.5) is real and definite, and the endpoint \( a \) is regular. If \( \Delta \) is integrable on \( \mathfrak{i} \), then \( b \) is quasiregular, which implies that \( b \) is in the limit-circle case; see Corollary 7.4.6. Therefore, it suffices to show that if \( \Delta \) not integrable, then \( b \) is in the limit-point case. Hence, assume that \( \Delta \) is not integrable at \( b \), so that

\[ \infty = \int_a^b |\Delta(s)| \, ds \leq \int_a^b \text{tr} \Delta(s) \, ds, \]  

(7.10.6)

where the estimate \( |\Delta(s)| \leq \text{tr} \Delta(s) \) follows from (7.1.6). In order to show that the endpoint \( b \) is in the limit-point case one must verify that

\[ \lim_{t \to b} h(t)^* Jf(t) = 0 \]

for all \( \{f, g\}, \{h, k\} \in T_{\max} \); cf. Lemma 7.6.8. Since the limit on the left-hand side exists due to Lemma 7.6.4, it suffices to verify the weaker statement

\[ \liminf_{t \to b} h(t)^* Jf(t) = 0 \]  

(7.10.7)

for all \( \{f, g\}, \{h, k\} \in T_{\max} \). According to Corollary 7.6.6 and the von Neumann formula in Theorem 1.7.11, it then suffices to prove (7.10.7) for elements of the form \( f = u_f + v_f \) and \( h = u_h + v_h \), where

\[ \{u_f, \lambda u_f\}, \{u_h, \lambda u_h\} \in T_{\max} \quad \text{and} \quad \{v_f, \mu v_f\}, \{v_h, \mu v_h\} \in T_{\max} \]
with $\lambda$ and $\mu$ in different half-planes. Finally, by polarization, it is clearly sufficient to show that

$$\liminf_{t \to b} f(t)^* J f(t) = 0$$  \hspace{1cm} (7.10.8)

for $f = u + v$, where $\{u, \lambda u\}, \{v, \mu v\} \in T_{\max}$. The proof of (7.10.8) is carried out in five steps.

**Step 1.** Let $u$ be a solution of the homogeneous equation $Jy' = \lambda \Delta y$ which satisfies $u \in L^2_\Delta(\Omega)$. Then

$$u(t) = u(a) + \int_a^t J^{-1} \lambda \Delta(u)(s) u(s) \, ds. \hspace{1cm} (7.10.9)$$

Hence, it follows from (7.10.9) as in Lemma 7.1.4 that

$$|u(t)| \leq |u(a)| + |\lambda| \int_a^t |\Delta(u)(s)| \, ds \leq |u(a)| + |\lambda| \left( \int_a^t |\Delta(s)| \, ds \right)^{\frac{1}{2}} \left( \int_a^t |\Delta(s)^{\frac{1}{2}} u(s)|^2 \, ds \right)^{\frac{1}{2}} \leq |u(a)| + |\lambda| \left( \int_a^t |\Delta(s)| \, ds \right)^{\frac{1}{2}} \|u\|_\Delta. \hspace{1cm} (7.10.10)$$

Due to the estimate $|\Delta(s)| \leq \text{tr} \Delta(s)$ one obtains

$$|u(t)| \leq |u(a)| + |\lambda| \left( \int_a^t \text{tr} \Delta(s) \, ds \right)^{\frac{1}{2}} \|u\|_\Delta. \hspace{1cm} (7.10.11)$$

It is clear that for a solution $v$ of the homogeneous equation $Jy' = \mu \Delta y$ which satisfies $v \in L^2_\Delta(\Omega)$ one obtains the similar inequality

$$|v(t)| \leq |v(a)| + |\mu| \left( \int_a^t \text{tr} \Delta(s) \, ds \right)^{\frac{1}{2}} \|v\|_\Delta. \hspace{1cm} (7.10.11)$$

**Step 2.** Let $u$ and $v$ be as in Step 1 with $\lambda$ and $\mu$ in different half-planes. Due to the assumption $\int_a^b \text{tr} \Delta(s) \, ds = \infty$ in (7.10.6), one can choose $t_0 > a$ so large that $\int_a^t \text{tr} \Delta(s) \, ds \geq 1$ for $t \geq t_0$. Then it follows from the estimates (7.10.10) and (7.10.11) that

$$|u(t)| \leq \left( \int_a^t \text{tr} \Delta(s) \, ds \right)^{\frac{1}{2}} (|u(a)| + |\lambda| \|u\|_\Delta), \hspace{1cm} t \geq t_0,$$

$$|v(t)| \leq \left( \int_a^t \text{tr} \Delta(s) \, ds \right)^{\frac{1}{2}} (|v(a)| + |\mu| \|v\|_\Delta), \hspace{1cm} t \geq t_0.$$
Consequently, for \( f = u + v \),

\[
|f(t)| \leq C_f \left( \int_a^t \text{tr} \, \Delta(s) \, ds \right)^{\frac{1}{2}}, \quad t \geq t_0,
\]

where \( C_f = |u(a)| + |\lambda| \|u\|_{\Delta} + |v(a)| + |\mu| \|v\|_{\Delta} \).

**Step 3.** Define the \( 2 \times 2 \) matrix function \( \Delta_0 \) by

\[
\Delta_0(t) = \begin{cases} \left( \text{tr} \, \Delta(t) \right)^{-1} \Delta(t), & \text{tr} \, \Delta(t) \neq 0, \\ \frac{1}{2} I, & \text{tr} \, \Delta(t) = 0, \end{cases}
\]

for almost every \( t \in \mathcal{I} \). Since \( \text{tr} \, \Delta(t) = 0 \) implies \( \Delta(t) = 0 \) (see (7.1.6)), one has \( \Delta(t) = (\text{tr} \, \Delta(t)) \Delta_0(t) \). Then \( \Delta_0 \) is nonnegative and

\[
\Delta_0 = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix},
\]

where the functions \( \alpha \) and \( \delta \) are nonnegative with \( \alpha + \delta = 1 \), and the function \( \beta \) is real. Define the matrix function \( \Delta_1 \) by

\[
\Delta_1 = \left( \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} \right)^* \left( \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} \right) = \begin{pmatrix} \alpha & \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} \\ \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} & \delta \end{pmatrix};
\]

then \( \Delta_1 \) is nonnegative. Moreover, since \( \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} \geq \left( \text{sgn} \, \beta \right) \beta \) it follows that the matrix

\[
2\Delta_0 - \Delta_1 = \begin{pmatrix} \alpha & 2\beta - \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} \\ 2\beta - \left( \text{sgn} \, \beta \right) \alpha^{\frac{1}{2}} \delta^{\frac{1}{2}} & \delta \end{pmatrix}
\]

is nonnegative. Therefore, \( \Delta_1(t) \leq 2\Delta_0(t) \) and one has the estimate

\[
(\text{tr} \, \Delta(t)) \Delta_1(t) \leq 2(\text{tr} \, \Delta(t)) \Delta_0(t) = 2\Delta(t)
\]

for almost every \( t \in \mathcal{I} \).

**Step 4.** It will be shown that

\[
\liminf_{t \to b} \left[ f(t)^* \Delta_1(t) f(t) \int_a^t \text{tr} \, \Delta(s) \, ds \right] = 0
\]

for \( f \in \mathcal{L}^2_{\Delta}(\mathcal{I}) \) and, in particular, for \( f = u + v \) as in Step 2. In fact, assume that (7.10.15) does not hold. Then there exist \( a < a' < b \) and \( \varepsilon > 0 \) such that for all \( t \geq a' \)

\[
\varepsilon \leq f(t)^* \Delta_1(t) f(t) \int_a^t \text{tr} \, \Delta(s) \, ds
\]
or, equivalently,

\[ \varepsilon \frac{\text{tr} \Delta(t)}{\int_a^t \text{tr} \Delta(s) \, ds} \leq f(t)^* \Delta_1(t) f(t) \text{tr} \Delta(t). \]  

(7.10.16)

Integration of the right-hand side of (7.10.16) together with (7.10.14) lead to

\[ \int_a^b f(t)^* \Delta_1(t) f(t) \text{tr} \Delta(t) \, dt \leq 2 \int_a^b f(t)^* \Delta(t) f(t) \, dt < \infty, \]

while integration of the left-hand side of (7.10.16) gives

\[ \varepsilon \int_a^b \frac{\text{tr} \Delta(t)}{\int_a^t \text{tr} \Delta(s) \, ds} \, dt = \varepsilon \int_a^b \frac{d}{dt} \left( \log \int_a^t \text{tr} \Delta(s) \, ds \right) \, dt = \infty, \]

due to (7.10.6). This contradiction shows that (7.10.15) is valid.

Step 5. It will be shown that for \( f = u + v \) as in Step 2 the limit in (7.10.15) implies the limit in (7.10.8). It is helpful to introduce the notation

\[ \varphi = \left( (\text{sgn} \beta) \alpha \frac{1}{2} \delta \frac{1}{2} \right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \]

so that \( |\varphi|^2 = f^* \Delta_1 f; \) cf. (7.10.13). Then the limit result (7.10.15) can be written as

\[ \liminf_{t \to b} \left| \varphi(t) \right|^2 \int_a^t \text{tr} \Delta(s) \, ds = 0. \]  

(7.10.17)

Observe that the term \( f^* J f \) in (7.10.8) is given by

\[ f^* J f = 2 i \text{Im} (\overline{f_2} f_1). \]

(7.10.18)

To estimate the term \( |\text{Im} (\overline{f_2} f_1)| \) note that, by the definition of the function \( \varphi \),

\[ \overline{f_1} \varphi = (\text{sgn} \beta) \alpha \frac{1}{2} \overline{f_1} f_1 + \delta \frac{1}{2} \overline{f_1} f_2 \quad \text{and} \quad \overline{f_2} \varphi = (\text{sgn} \beta) \alpha \frac{1}{2} \overline{f_2} f_1 + \delta \frac{1}{2} \overline{f_2} f_2. \]

This yields the identities

\[ \text{Im} (\overline{f_1} \varphi) = \delta \frac{1}{2} \text{Im} (\overline{f_1} f_2) \quad \text{and} \quad \text{Im} (\overline{f_2} \varphi) = (\text{sgn} \beta) \alpha \frac{1}{2} \text{Im} (\overline{f_2} f_1). \]

Therefore, it is clear that

\[ \delta \frac{1}{2} |\text{Im} (\overline{f_1} f_2)| = |\text{Im} (\overline{f_1} \varphi)| \leq |f_1| |\varphi| \]  

(7.10.19)

and

\[ \alpha \frac{1}{2} |\text{Im} (\overline{f_2} f_1)| = |\text{Im} (\overline{f_2} \varphi)| \leq |f_2| |\varphi|. \]  

(7.10.20)

Since \( \alpha + \delta = 1 \), one has \( \alpha < \frac{1}{2} \) if and only if \( \delta \geq \frac{1}{2} \). Note that \( x \geq \frac{1}{2} \) if and only if \( 1/\sqrt{x} \leq \sqrt{2} \), so it follows from (7.10.18) and (7.10.19)–(7.10.20) that

\[ |f^* J f| \leq \begin{cases} 2 \sqrt{2} |\varphi||f_2|, & \alpha \leq \frac{1}{2}, \\ 2 \sqrt{2} |\varphi||f_1|, & \delta \geq \frac{1}{2}. \end{cases} \]
Therefore, if \( t \geq t_0 \), then (7.10.12) implies that
\[
|f(t)^* Jf(t)| \leq 2\sqrt{2} C_f |\varphi(t)| \left( \int_a^t \text{tr} \Delta(s) \, ds \right)^{\frac{1}{2}}, \quad t \geq t_0.
\]
Combined with (7.10.17) this shows that (7.10.8) is satisfied. \( \square \)

The next corollary follows from Theorem 7.10.1 and (7.1.6).

**Corollary 7.10.2.** Let the canonical system (7.10.5) be real and definite, and let the endpoint \( a \) be regular. Assume that \( \Delta \) is trace-normed in the sense that \( \text{tr} \Delta = 1 \).

Then the following alternative holds:

(i) if \( b \in \mathbb{R} \), then the limit-circle case prevails;

(ii) if \( b = \infty \), then the limit-point case prevails.

Next two simple examples for trace-normed canonical systems are discussed.

**Example 7.10.3.** Let \( \iota = (-1, 1) \) and define the matrix function \( \Delta \) by
\[
\Delta(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in (-1, 0), \quad \Delta(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in (0, 1).
\]

A measurable function \( f = (f_1, f_2)^\top \) belongs to \( L^2_\Delta(\iota) \) if and only if
\[
\int_{-1}^0 |f_1(t)|^2 \, dt < \infty \quad \text{and} \quad \int_{0}^1 |f_2(t)|^2 \, dt < \infty,
\]
and for \( f, g \in L^2_\Delta(\iota) \) the semi-inner product is given by
\[
(f, g)_\Delta = \int_{-1}^0 \overline{g}_1(t) f_1(t) \, dt + \int_{0}^1 \overline{g}_2(t) f_2(t) \, dt.
\]

Hence, an element \( f \in L^2_\Delta(\iota) \) has \( \Delta \)-norm 0 if and only if
\[
\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f_2(t) \end{pmatrix} \quad \text{for a.e. } t \in (-1, 0)
\]
and
\[
\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ 0 \end{pmatrix} \quad \text{for a.e. } t \in (0, 1),
\]
where \( f_2 \) on \((-1, 0)\) and \( f_1 \) on \((0, 1)\) are completely arbitrary complex measurable functions.

It is straightforward to see that the regular canonical system \( Jf' = \Delta g \) is definite. Hence, in the Hilbert space \( L^2_\Delta(\iota) \), the maximal relation
\[
T_{\text{max}} = \{ \{ f, g \} \in L^2_\Delta(\iota) \times L^2_\Delta(\iota) : Jf' = \Delta g \}.
\]
is well defined and for each \( \{f, g\} \in T_{\text{max}} \) the equivalence class \( f \) contains a unique absolutely continuous representative such that \( Jf' = \Delta g \); cf. Lemma 7.6.1. In fact, an absolutely continuous function \( f = (f_1, f_2)^\top \) satisfies \( Jf' = \Delta g \) with \( g \in L^2_\Delta(\iota) \) if and only if

\[
\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 + \int_t^0 g_1(s) \, ds \end{pmatrix} \quad \text{for a.e. } t \in (-1, 0)
\]

and

\[
\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} \gamma_1 + \int_0^t g_2(s) \, ds \\ \gamma_2 \end{pmatrix} \quad \text{for a.e. } t \in (0, 1)
\]

for some constants \( \gamma_1, \gamma_2 \in \mathbb{C} \). From the equality

\[
\int_{-1}^1 \left( f(t) - \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right)^* \Delta(t) \left( f(t) - \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \, dt = 0
\]

it follows that \( f = (f_1, f_2)^\top \) and \( (\gamma_1, \gamma_2)^\top \) are in the same equivalence class in \( L^2_\Delta(\iota) \). Therefore,

\[
\dim(\text{dom } T_{\text{max}}) = 2,
\]

and the functions \( \varphi = (1, 0)^\top \) and \( \psi = (0, 1)^\top \) form an orthonormal system in \( \text{dom } T_{\text{max}} \). Furthermore, it follows from the representation of \( T_{\text{min}} \) in Lemma 7.7.1 that \( \{f, g\} \in T_{\text{min}} \) if and only if

\[
\gamma_1 = 0, \quad \gamma_2 = 0, \quad \int_{-1}^0 g_1(t) \, dt = 0, \quad \int_0^1 g_2(t) \, dt = 0.
\]

Hence,

\[
\text{dom } T_{\text{min}} = \{0\} \quad \text{and} \quad \text{mul } T_{\text{min}} = (\text{dom } T_{\text{max}})^{\perp},
\]

and

\[
\text{mul } T_{\text{max}} = (\text{dom } T_{\text{min}})^{\perp} = L^2_\Delta(\iota).
\]

The boundary mappings in Theorem 7.7.2 are given by

\[
\Gamma_0\{f, g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2\gamma_1 + \int_{-1}^1 g_2(t) \, dt \\ 2\gamma_2 + \int_{-1}^0 g_1(t) \, dt \end{pmatrix} \quad \text{and} \quad \Gamma_1\{f, g\} = \frac{1}{\sqrt{2}} \begin{pmatrix} \int_{-1}^0 g_1(t) \, dt \\ \int_0^1 g_2(t) \, dt \end{pmatrix}.
\]

In order to compute the \( \gamma \)-field and Weyl function corresponding to the boundary triplet \( \{C^2, \Gamma_0, \Gamma_1\} \) fix a fundamental system by \( Y(-1, \lambda) = I \). Then

\[
Y(t, \lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda t - \lambda & 1 \end{pmatrix} \quad \text{for a.e. } t \in (-1, 0),
\]

\[
Y(t, \lambda) = \begin{pmatrix} -\lambda^2 t + 1 & \lambda t \\ -\lambda & 1 \end{pmatrix} \quad \text{for a.e. } t \in (0, 1).
\]
Hence, it follows from Theorem 7.7.2 that the γ-field γ and the Weyl function \( M \) are given by
\[
\gamma(\cdot, \lambda) = Y(\cdot, \lambda) \frac{\sqrt{2}}{4 - \lambda^2} \begin{pmatrix} 2 & -\lambda \\ \lambda & 2 - \lambda^2 \end{pmatrix} \quad \text{and} \quad M(\lambda) = \frac{1}{4 - \lambda^2} \begin{pmatrix} 2\lambda & -\lambda^2 \\ -\lambda^2 & 2\lambda \end{pmatrix}.
\]
In particular, the poles of \( M \) are \( \{-2, 2\} \) and hence the spectrum of \( A_0 \) consists of the eigenvalues 2 and \(-2\) which both have multiplicity 1.

The next example is a variant of Example 7.10.3 in the limit-point case.

**Example 7.10.4.** Let \( i = (-1, \infty) \) and define the matrix function \( \Delta \) by
\[
\Delta(t) = \begin{cases} 
1 & 0 \\
0 & 0 
\end{cases}, \quad t \in (-1, 0), \\
\begin{cases} 
0 & 0 \\
0 & 1 
\end{cases}, \quad t \in (0, \infty).
\]
As in Example 7.10.3, a measurable function \( f = (f_1, f_2)^\top \) belongs to \( L^2_\Delta(i) \) if and only if
\[
\int_{-1}^{0} |f_1(t)|^2 \, dt < \infty \quad \text{and} \quad \int_{0}^{\infty} |f_2(t)|^2 \, dt < \infty.
\]
The semi-inner product and the elements with \( \Delta \)-norm 0 are as in Example 7.10.3, except that the interval \((0, 1)\) has to be replaced by \((0, \infty)\). Furthermore, the canonical system \( Jf' = \Delta g \) is definite and in the limit-point case; cf. Corollary 7.10.2. Hence, the maximal relation
\[
T_{\max} = \{ \{f, g\} \in L^2_\Delta(i) \times L^2_\Delta(i) : Jf' = \Delta g \}
\]
is well defined in \( L^2_\Delta(i) \). In a similar way as in Example 7.10.3 it follows that an absolutely continuous function \( f = (f_1, f_2)^\top \in L^2_\Delta(i) \) satisfies \( Jf' = \Delta g \) with \( g \in L^2_\Delta(i) \) if and only if
\[
\begin{pmatrix} f_1(t) \\
f_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\
1 \end{pmatrix} \int_{t}^{0} g_1(s) \, ds \quad \text{for a.e. } t \in (-1, 0)
\]
and
\[
\begin{pmatrix} f_1(t) \\
f_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\
0 \end{pmatrix} \int_{0}^{t} g_2(s) \, ds \quad \text{for a.e. } t \in (0, \infty)
\]
hold for some constant \( \gamma_1 \in \mathbb{C} \). The functions \( f = (f_1, f_2)^\top \) and \((\gamma_1, 0)^\top\) are in the same equivalence class in \( L^2_\Delta(i) \) and therefore
\[
\dim (\text{dom } T_{\max}) = 1,
\]
and \( \text{dom } T_{\max} \) is spanned by the function \( \varphi = (1, 0)^\top \). It follows from Lemma 7.8.1 that \( \{f, g\} \in T_{\min} \) if and only if
\[
\gamma_1 = 0 \quad \text{and} \quad \int_{-1}^{0} g_1(t) \, dt = 0.
\]
Hence, $\text{dom } T_{\text{min}} = \{0\}$, and $\text{mul } T_{\text{min}}$ and $\text{mul } T_{\text{max}}$ are related as in Example 7.10.3.

The boundary mappings in Theorem 7.8.2 are given by

$$\Gamma_0 \{f, g\} = \gamma_1 \quad \text{and} \quad \Gamma_1 \{f, g\} = \int_{-1}^{0} g_1(t) \, dt.$$  

To compute the $\gamma$-field and Weyl function corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ use the fundamental system

$$Y(t, \lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda t - \lambda & 1 \end{pmatrix} \quad \text{for a.e. } t \in (-1, 0),$$

$$Y(t, \lambda) = \begin{pmatrix} -\lambda^2 t + 1 & \lambda t \\ -\lambda & 1 \end{pmatrix} \quad \text{for a.e. } t \in (0, \infty).$$

Clearly, not both columns of $Y(\cdot, \lambda)$ belong to $L^2_{\Delta} (i)$, but the function $\chi(\cdot, \lambda)$ given by

$$\begin{pmatrix} 1 \\ -\lambda t \end{pmatrix} \quad \text{for a.e. } t \in (-1, 0), \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for a.e. } t \in (0, \infty),$$

belongs to $L^2_{\Delta} (i)$ and satisfies $J \chi'(\cdot, \lambda) = \lambda \Delta \chi(\cdot, \lambda)$. Hence, by Theorem 7.8.2, the $\gamma$-field $\gamma$ and the Weyl function $M$ are given by

$$\gamma(\cdot, \lambda) = Y(\cdot, \lambda) \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \quad \text{and} \quad M(\lambda) = \lambda.$$