Theoretical analysis of fracture in non-linear elastic functionally graded beam of linearly changing thickness with three longitudinal cracks

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Abstract. The present paper is focussed on fracture analysis of a functionally graded non-linear elastic cantilever beam structure with three longitudinal cracks. The cross-section of the beam is a rectangle. The beam height increases linearly towards the clamped end. The material is functionally graded along the thickness of the beam. The $J$-integral approach is applied in the fracture analysis. The solutions to the $J$-integral for the three cracks are verified by deriving of the strain energy release rate. For this purpose, the complementary strain energy cumulated in the beam is differentiated with respect to the areas of the three cracks. A parametric investigation of the longitudinal fracture behaviour of the beam is carried-out by applying the solutions to the $J$-integral. The effects of the beam geometry, the crack length and the material gradient along the beam thickness on the longitudinal fracture behaviour are evaluated and discussed.

1. Introduction
The functionally graded materials whose properties vary continuously in the solid are employed in a wide range of engineering applications [1-4]. The great interest which has been shown recently towards the functionally graded materials is due to their intensive use in more and more sophisticated areas in aeronautics and automotive engineering [5]. The mechanical reliability of functionally graded structural members and components depends largely on their fracture behaviour [6]. It should be mentioned that functionally graded materials can be built-up layer by layer [5] which is a premise for appearance of longitudinal cracks between layers. Therefore, analyzing of longitudinal fracture is of paramount importance for guarantying the integrity and reliability of functionally graded structural members and components.

The present paper deals with fracture analysis of a functionally graded cantilever beam configuration with three longitudinal parallel cracks in contrast to previous papers which consider fracture behaviour of functionally graded non-linear elastic beams with one longitudinal crack [7-9]. The present study is motivated by the fact that the layered structure of the beams and material gradient in the thickness direction create a high risk of appearance of more parallel longitudinal cracks. The three longitudinal cracks are analyzed by applying the $J$-integral approach in the present paper. The fracture is analyzed also in terms of the strain energy release rate for verification.
2. Analysis of functionally graded non-linear elastic beam with three longitudinal cracks

The present paper studies fracture behaviour of the functionally graded non-linear elastic beam with three parallel longitudinal cracks shown in figure 1. The beam is clamped in its right-hand end. The loading consists of a bending moment \( M \), applied at the free end of the beam. The length of the beam is \( l \). The beam cross-section is a rectangle of width \( b \) and thickness \( h \).

![Figure 1. Geometry and loading of a functionally graded cantilever beam configuration of linearly changing thickness with three longitudinal cracks.](image)

The thickness varies linearly from \( h_0 \) at the free end of the beam to \( H \) at the clamping (figure 1)

\[
h = h_0 + \frac{H - h_0}{l} x_1. \tag{1}
\]

Three parallel longitudinal cracks are located in the beam as shown in figure 1. The lengths of cracks 1, 2 and 3 are \( a_1 \), \( a_2 \) and \( a_3 \), respectively. The beam is functionally graded in the thickness direction. The longitudinal fracture behaviour of the beam is analyzed by applying the \( J \)-integral approach. First, crack 1 is studied. The \( J \)-integral is solved along the integration contour \( B \), shown by a dashed line in figure 1. It should be mentioned that contours are chosen to facilitate the integration (for example, the stress along the upper and lower horizontal segments of the contours are zero). The solution of the \( J \)-integral is written as

\[
J = J_{B_1} + J_{B_2}, \tag{2}
\]

where \( J_{B_1} \) and \( J_{B_2} \) are the \( J \)-integral values in segments \( B_1 \) and \( B_2 \), of the integration contour (segments \( B_1 \) and \( B_2 \) coincide with the cross-sections of the lower arms of cracks 1 and 2, respectively). The \( J \)-integral in segment \( B_1 \) is written as

\[
J_{B_1} = \int_{B_1} \left[ u_{01} \cos \alpha - \left( p_x \frac{\partial u}{\partial x} + p_y \frac{\partial v}{\partial x} \right) \right] ds, \tag{3}
\]

where \( u_{01} \) is the strain energy density. The quantities involved in (3) are obtained as

\[
p_x = -\sigma; \quad p_y = 0; \quad ds = dz_3; \quad \cos \alpha = -1; \quad \frac{\partial u}{\partial x} = \epsilon, \tag{4}
\]
where \( \sigma \) is the stress, \( z_3 \) is the vertical centroidal axis of the cross-section of the lower arm of crack 1, \( \varepsilon \) is the strain. The stress is related to strain \( \varepsilon \), by the following non-linear constitutive law [10]:

\[
\sigma = \frac{E \varepsilon}{\sqrt{1 + \varepsilon^2}}
\]  
(5)

In (5), \( E \) is a material property that is distributed exponentially along the beam thickness

\[
E = E_0 e^{m z_2} \frac{h}{h},
\]  
(6)

where \( E_0 \) is the value of \( E \) at the upper surface of the beam, \( m \) is a parameter that controls the material gradient in the thickness direction, \( z_2 \) is the vertical centroidal axis of the beam. It should be noted that the stress \( \sigma \) in (4) is obtained by using (5).

The strain energy density is found by integrating of (5). The result is

\[
u_01 = E \left( \sqrt{1 + \varepsilon^2} - 1 \right).
\]  
(9)

The \( J \)-integral in segment \( B_2 \) is expressed as

\[
J_{B_2} = \int_{\partial B_2} \left[ u_{02} \cos \alpha_{B_2} - \left( p_{xB_2} \frac{\partial u}{\partial x_{B_2}} + p_{yB_2} \frac{\partial v}{\partial x_{B_2}} \right) \right] ds_{B_2},
\]  
(10)

where the strain energy density \( u_{02} \) is obtained by replacing of \( \varepsilon \) with \( \varepsilon_{B_2} \) in (9). The strain \( \varepsilon_{B_2} \) is found by replacing of \( \kappa_2, z_3 \) and \( z_{3n} \) with \( \kappa_2, z_4 \) and \( z_{4n} \) in (7). Here \( \kappa_2 \) is the curvature, \( z_4 \) is the vertical centroidal axis, \( z_{4n} \) is the coordinate of the neutral axis of the cross-section of the lower arm of crack 2. Equations (8) are used to determine \( \kappa_2 \) and \( z_{3n} \) by the MATLAB computer program.

The \( J \)-integral is used also to investigate crack 2. The integral is solved along the integration contour \( C \), shown by dashed line in figure 1.

Finally, the \( J \)-integral is applied to analyze crack 3. The integration is carried-out along the integration contour \( D \), shown by a dashed line in figure 1.
In order to verify the solutions to the $J$-integral, the tree cracks are studied also in terms of the strain energy release rate $G$. First, crack 1 is analyzed. The strain energy release rate is derived by applying the following dependence [11]:

$$G = \frac{dU^*}{bd\alpha_1},$$

(12)

where $U^*$ is the complementary strain energy, $d\alpha_1$ is an elementary increase of the length of crack 1. The complementary strain energy is obtained by integrating the complementary strain energy density in the volume of the beam. The complementary strain energy density $u_{01}^*$, in the lower arm of crack 1 is written as [9]

$$u_{01}^* = \sigma\varepsilon - u_{01}.$$

(13)

By combining of (5), (9) and (13), one obtains

$$u_{01}^* = E \left(1 - \frac{1}{\sqrt{1 + \varepsilon^2}}\right).$$

(14)

The complementary strain energy densities in the other parts of the beam are found by replacing of $\varepsilon$ with the corresponding strain in (14).

Cracks 2 and 3 are also studied in terms of the strain energy release rate. For this purpose, $d\alpha_2$ is replaced, respectively, with $d\alpha_2$ and $d\alpha_3$ in (12). It should be noted that the strain energy release rates found by (12) are exact matches of $J$-integral values which is a verification of the fracture analysis of the functionally graded non-linear elastic cantilever beam with three parallel longitudinal cracks developed in the present paper.

3. Numerical results

The numerical results reported here are obtained by applying the solutions to the $J$-integral derived in section 2 of the present paper. It is assumed that $b = 0.010$ m, $h_0 = 0.005$ m and $l = 0.300$ m.
Figure 3. The $J$-integral value in non-dimensional form plotted against 
\((l_1 + l_2)/l\) ratio (curve 1 – at $M = 4$ Nm, curve 2 – at $M = 5$ Nm and curve 
3 – at $M = 6$ Nm).

The $J$-integral value is presented in non-dimensional form by using the formula $J_N = J/(E_0 b)$.

The values of the $J$-integral in non-dimensional form are plotted against $H/h_0$ ratio in figure 2. 
One can observe in figure 2 that the $J$-integral value decreases with increasing of $H/h_0$ ratio. It can be 
obscerved also in figure 2 that the $J$-integral has highest, intermediate and lowest value for cracks 1, 2 and 3, respectively.

The location of crack 2 is characterized by $(l_1 + l_2)/l$ ratio. The $J$-integral value in non-
dimensional form is plotted against $(l_1 + l_2)/l$ ratio in figure 3 at three values of the bending moment $M$, by using the $J$-integral solution for crack 2. The curves in figure 3 indicate that the $J$-integral value decreases with increasing of $(l_1 + l_2)/l$ ratio.

The $J$-integral value in non-dimensional form is plotted against parameter $m$, in figure 4 at three 
$a_3/l$ ratios by applying the $J$-integral solution for crack 3. It can be observed in figure 4 that the $J$-
integral value decreases with increasing of $a_3/l$ ratio and $m$.

Figure 4. The $J$-integral value in non-dimensional form plotted against $m$ 
(curve 1 – at $a_3/l = 0.1$, curve 2 – at $a_3/l = 0.2$ and curve 3 – at $a_3/l = 0.3$).
4. Conclusions
Longitudinal fracture of a functionally graded non-linear elastic cantilever beam is analyzed. The basic novelty of the present study is in the fact that three parallel longitudinal cracks are analyzed in contrast to previous studies which deal with one crack only [7-9]. The appearance of more parallel cracks is due to the fact that functionally graded structural members and components can be built-up layer by layer. The thickness of the beam increases linearly from the free end towards the clamping. The beam is functionally graded in the thickness direction. The fracture is studied by applying the \( J \)-integral approach. The fracture is analyzed also in terms of the strain energy release rate for verification of the solutions to the \( J \)-integral. Numerical results are obtained by using the solutions to the \( J \)-integral. It is found that the \( J \)-integral value decreases with increasing of \( H/h_0 \) ratio. It is found also that the \( J \)-integral value for cracks 1, 2 and 3 is higher, intermediate and lower, respectively. The analysis reveals that the \( J \)-integral value for crack 2 decreases with increasing of \( (l_1 + l_2)/l \) ratio. The \( J \)-integral value for crack 3 decreases with increasing of \( \alpha_3 l/l \) ratio and \( m \). Form viewpoint of practical engineering, the results obtained in the present paper can be used in design of functionally graded non-linear elastic beam structures of continuously varying thickness in cases when the longitudinal fracture behavior has to be addressed. For example, the present paper clarifies the question for the influences of varying thickness, locations of and lengths of the longitudinal cracks on the fracture behavior.

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