Scaling solutions in a continuous dimension

Alessandro Codello
SISSA, Via Bonomea 265-34136, Trieste, Italy
E-mail: codello@sissa.it

Received 16 August 2012, in final form 8 October 2012
Published 31 October 2012
Online at stacks.iop.org/JPhysA/45/465006

Abstract
We study scaling solutions of the RG flow equation for $\mathbb{Z}_2$-effective potentials in a continuous dimension $d \geq 2$. As the dimension is lowered from $d = 4$ we first observe the appearance of the Ising scaling solution and successively the appearance of multi-critical scaling solutions of arbitrary order. Approaching $d = 2$ these multi-critical scaling solutions converge to the unitary minimal models found in conformal field theory.

PACS numbers: 05.10.Cc, 11.10.Hi

(Some figures may appear in colour only in the online journal)

Introduction

One of the most valuable properties of the functional RG formalism based on the effective average action [1] is that its use is not restricted to any particular dimension, as instead are methods like the $\epsilon$-expansion, Monte Carlo simulations or conformal field theory (CFT) techniques. Often one can obtain RG flow equations where $d$ can be considered as an external tunable parameter and it is thus possible to enquire how properties of the RG ‘theory space’ evolve as the dimension is varied. In particular one can observe universality classes, i.e. fixed-points of the RG flow, emerge or disappear at various threshold or critical dimensions. The study of how $\mathbb{Z}_2$-universality classes depend on $d$ is particularly interesting since it offers the possibility of observing how the fixed-point structure of theory space interpolates between the upper critical dimension $d = 4$, where there is only the Gaussian fixed-point, and $d = 2$ where one expects, from the CFT analysis, an infinite discrete sequence of fixed-points describing multi-critical universality classes of arbitrary order. Indeed this is what we are able to observe.

Flow equation for the effective potential

The effective average action $\Gamma_k[\phi]$ is a generalization of the standard effective action that depends on the infrared scale $k$ [1]. The effective potential $V_k(\phi)$ is found by evaluating
the effective average action at a constant field configuration \( \varphi(x) = \varphi \) where \( \Gamma_{k}[\varphi] = \int d^{d}x \, V_{k}(\varphi) \). In this approximation theory space is projected into the infinite dimensional functional space of effective potentials and the RG flow is represented by an exact equation that describes the scale dependence of the effective potential [2]. In terms of dimensionless variables \( \varphi = Z_{k}^{-1/2}k^{d/2-1}\tilde{\varphi} \) and \( V(\varphi) = k^{d}\tilde{V}(\tilde{\varphi}) \), where \( Z_{k} \) is the running wave-function renormalization constant, the exact flow equation for the effective potential reads [3]

\[
\frac{\partial}{\partial t} \tilde{V}(\tilde{\varphi}) + d\tilde{V}(\tilde{\varphi}) - \frac{d-2 + \eta_{k}}{2} \tilde{V}_{\prime}(\tilde{\varphi}) = c_{d} \frac{1 - \eta_{k}}{1 + \tilde{V}_{\prime\prime}(\tilde{\varphi})},
\]

with \( c_{d}^{-1} = (4\pi)^{d/2}\Gamma(d/2 + 1) \). Note that equation (1) is a non-linear partial differential equation (PDE). The scale dependent anomalous dimension, introduced in (1), is defined in terms of the wave-function renormalization by \( \eta_{k} = -\partial \log Z_{k} \); it can be obtained from the dimensionless effective potential using the following relation [3]:

\[
\eta_{k} = c_{d} \frac{[\tilde{V}_{\prime\gamma}(\tilde{\varphi}_{0})]^{2}}{[1 + \tilde{V}_{\gamma}(\tilde{\varphi}_{0})]^{2}}. \tag{2}
\]

In (2) \( \tilde{\varphi}_{0} \) is the minimum of the dimensionless effective potential, i.e. \( \tilde{V}_{\gamma}(\tilde{\varphi}_{0}) = 0 \).

An interesting feature of equation (1) is that it is valid for any value of \( d \); in principle it contains information about universality classes in any dimension. In particular we expect to find non-trivial behaviors in \( d = 3 \), where equation (1) has been extensively studied [4], and in \( d = 2 \) where one expects a rich behavior, since the CFT analysis implies the existence of an infinite number of non-trivial RG fixed-points [5].

**Scaling solutions**

Scaling solutions are solutions of \( \partial_{t} \tilde{V}_{\gamma}(\tilde{\varphi}) = 0 \) and correspond to RG fixed-points in the functional space of effective potentials. Every scaling solution, together with its domain of attraction, defines a different universality class.

We start our analysis by considering the case where the anomalous dimension is set to zero \( \eta = 0 \). From (1) we see that a scaling solution satisfies the following ordinary differential equation (ODE):

\[
-d\tilde{V}(\tilde{\varphi}) + \frac{d - 2}{2} \tilde{V}_{\prime}(\tilde{\varphi}) + c_{d} \frac{1}{1 + \tilde{V}_{\prime\prime}(\tilde{\varphi})} = 0. \tag{3}
\]

The \( Z_{2} \)-symmetry of the effective potential requires that its first derivative vanishes at the origin \( \tilde{V}_{\prime}(0) = 0 \); (3) then implies \( \tilde{V}_{\prime}(0) = \frac{c_{d}}{1 + \tilde{V}_{\prime\prime}(0)} \). Since equation (3) is a second order non-linear ODE, we need to use numerical methods to solve it\(^1\). It is easy to set up the initial value problem as a function of the parameter \( \sigma = \tilde{V}_{\prime\gamma}(0) \) using the two initial conditions just given.

One immediately observes that for most values of the parameter \( \sigma \) the solution ends up in a singularity at a finite value of the dimensionless field. For every \( d \) and \( \sigma \) we can call this value \( \tilde{\varphi}_{0}^{d}(\sigma) \), in this way defining a function [6]. Requiring a scaling solution to be well defined for any \( \tilde{\varphi} \in \mathbb{R} \) restricts the admissible initial values of \( \sigma \) to a discrete set \( \{ \sigma_{\pm i}^{d} \} \) (labeled by \( i \)). One can now construct a numerical plot of the function \( \tilde{\varphi}_{0}^{d}(\sigma) \) to find the \( \sigma_{\pm i}^{d} \) as those values where the function \( \tilde{\varphi}_{0}^{d}(\sigma) \) has a ‘spike’, since a singularity in \( \tilde{\varphi}_{0}^{d}(\sigma) \) implies that the relative scaling solution, obtained by integrating the ODE (3), is a well defined function for every \( \tilde{\varphi} \in \mathbb{R} \). For

\(^{1}\) All the numerical analyses have been performed employing standard routines for solving ODE present in symbolic manipulation software packages.
any $d$, the function $\tilde{\phi}_d^d(\sigma)$ gives us a snapshot of theory space, where dimensionless effective potentials are parametrized by $\sigma$ and where RG fixed-points, i.e. scaling solutions, appear as spikes. By studying $\tilde{\phi}_d^d(\sigma)$ we will be able to follow the evolution of universality classes as we vary the dimension.

We can start by studying the function $\tilde{\phi}_4^d(\sigma)$ for $d = 4$. One finds only one spike at $\sigma^*_1 = 0$ corresponding to the Gaussian scaling solution $\tilde{V}_c(\tilde{\phi}) = \frac{1}{2} \tilde{\phi}^2$ (the singularity at $\sigma = -1$ is due to the structure of equation (3) and does not correspond to any scaling solution). The function $\tilde{\phi}_4^d(\sigma)$ is shown in figure 1. We find the same qualitative result for any $d \geq 4$ as expected by the fact that four is the upper critical dimension for the Ising (or Wilson–Fisher) universality class.

As we decrease the dimension from $d = 4$ we observe a new spike branching to the left of the Gaussian spike: this corresponds to the Ising scaling solution. As we continue to lower $d$ the spike moves to the left and for $d = 3$ the function $\tilde{\phi}_3^d(\sigma)$ looks as in figure 1. As one expects, the value of $\sigma^*_2$ at which we observe the Ising spike is negative, indicating that the relative scaling solution obtained by integrating the fixed-point equation (3) is concave at the origin.

For values of the dimension lower than three the function $\tilde{\phi}_3^d(\sigma)$ becomes very interesting. Starting at $d = 3$ a new spike branches from the Gaussian one, this time to the right: this corresponds to the tri-critical Ising universality class, for which this is the upper critical dimension. When we reach $d = 2.6$ this spike is clearly visible, as is shown in figure 1. In $d = 2.4$ another spike has already emerged, this time to the left of the Gaussian spike: this corresponds to the tetra-critical Ising universality class. In the insets of figure 1 we show the
The function \( \tilde{\phi}_d^*(\sigma) \) for values of the dimension approaching two; (from bottom) \( d = 2.1, 2.01, 2.003, 2.001 \). One observes the increasing number of spikes corresponding to multi-critical scaling solution of increasing degree. An arbitrary number of spikes can be observed as \( d \to 2 \), the only constraint being the numerical resolution at which the function \( \tilde{\phi}_d^*(\sigma) \) is investigated.

As we lower further the dimension new spikes emerge alternatively on the left or on the right of the Gaussian spike. These correspond to multi-critical potentials of increasing order. In particular, one observes that the \( i \)th multi-critical scaling solution appears at the critical dimension \( d_{c,i} \); these can be understood as those dimensions where new operators become relevant as \( d \) is lowered. The dimension of the operator \( \phi^{2i} \) is \( 2i(2 - d) \), so the relative coupling \( \lambda_{2i} \) scales as \( d - 2i (\frac{d}{2} - 1) \). Setting this to zero gives \( d_{c,i} = \frac{2}{i+1} \); for integer values of \( i = 1, 2, 3, 4, ... \) we find the following sequence of critical dimensions \( d_{c,i} = \infty, 4, 3, \frac{11}{5}, \frac{1}{2}, \frac{19}{10}, \frac{1}{3} \).... Infinity is the upper critical dimension for the Gaussian universality class, four is the upper critical dimension for the Ising universality class, while three is the upper critical dimension for the tri-critical Ising universality class and so on. Note that the critical dimensions \( d_{c,i} \) accumulate at \( d = 2 \); this is a first hint that in two dimensions there are infinitely many different universality classes. By plotting the function \( \tilde{\phi}_d^*(\sigma) \) with the appropriate resolution one can check that the new spikes emerge precisely at the dimensions \( d_{c,i} \). It will be interesting to analytically confirm these results, as was done for the Polchinski analogue of (3) in [7].

As shown in figure 2, when we continue to decrease \( d \) towards two, the number of spikes of \( \tilde{\phi}_d^*(\sigma) \) grows as expected. In \( d = 2 \) we ‘lose track’ of the multi-critical scaling solutions; this is related to the fact that the first derivative term on the lhs of (3) vanishes, making the equation exactly integrable.

**Sine-Gordon model**

To integrate the fixed-point equation (3) when \( d = 2 \), we recast it in the following form:

\[
\ddot{V}_e^*(\tilde{\phi}) = -\frac{d}{d\tilde{V}_e} \left[ \dot{V}_e - \frac{1}{8\pi} \log \dot{V}_e \right].
\]
Equation (4) can be interpreted as Newton’s equation where \( \tilde{V}(\tilde{\varphi}) \leftrightarrow x(t) \) \cite{10}; its solution is thus implicitly given by

\[
\tilde{\varphi} = \int \frac{d\tilde{V}_s}{\sqrt{2\left[ \frac{c_d}{d}(1 + \sigma) - \tilde{V}_s \right] + \frac{1}{4\pi} \log \left[ \frac{(1 + \sigma)\tilde{V}_s}{c_d} \right]}}
\]

(5)

Since the ‘potential’ \( \tilde{V}_s - \frac{1}{4\pi} \log \tilde{V}_s \) in (4) is convex and bounded from below, the solutions (5) are periodic functions of \( \tilde{\varphi} \); the period depends on \( \sigma \) and the solution with zero period is the Gaussian scaling solution. We thus see that in \( d = 2 \) equation (3) has a continuum of periodic solutions. These are related to the critical sine-Gordon model \cite{8}. One can expand the dimensionless effective potential in Fourier series \( \tilde{V}_s(\tilde{\varphi}) = \sum_{n=1}^{\infty} v_{n,s} \cos(n\beta_s \tilde{\varphi}) \) and check that indeed one finds the fixed-point value \( \beta_s = \sqrt{8\pi} \) corresponding to the Coleman point. In the sine-Gordon model the wave-function renormalization is an essential coupling related to the vortex-fugacity; one can then use (1) and (2) to obtain beta functions exhibiting the Kosterlitz–Thouless–Berezinski phase transition \cite{9}.

**Anomalous dimension**

In order to follow the evolution of scaling solutions down to two dimensions, we turn to consider the case of the non-vanishing anomalous dimension \( \eta \neq 0 \). For every scaling solution we will determine the allowed values of \( \eta \) by solving, self-consistently, the fixed-point equation

\[
-d\tilde{V}_s(\tilde{\varphi}) + \frac{d}{2} d + \frac{\eta}{2} \tilde{\varphi} \tilde{V}_s'(\tilde{\varphi}) + c_d \frac{1 - \eta}{1 + \tilde{V}_s'(\tilde{\varphi})} = 0,
\]

(6)

together with (2). Since now the second term on the lhs of (6) is non-zero even when \( d = 2 \), due to a non-vanishing anomalous dimension, we expect to find a discrete set of scaling solutions for every \( d \geq 2 \). Following the \( \eta = 0 \) case, we define the functions \( \tilde{\varphi}_{d,\eta}^d(\sigma) \) and identify the discrete values \( \{ \sigma_{s,n}^d \} \) for which the solutions of (6) are well defined for every \( \tilde{\varphi} \in \mathbb{R} \). The functions \( \tilde{\varphi}_{d,\eta}^d(\sigma) \), for \( \eta \neq 0 \), turn out to be qualitatively similar to their counter-parts \( \tilde{\varphi}_d^d(\sigma) \) studied previously.

We proceed as follows: we fix \( d \) and we start with an ansatz for \( \eta \); we compute \( \tilde{\varphi}_{d,\eta}^d(\sigma) \) from which we find the values \( \{ \sigma_{s,n}^d \} \); we use them to solve numerically the ODE (6) to obtain the relative scaling solutions; we estimate the anomalous dimension by employing (2); we use this value as the ansatz for the next iteration until we converge to a self-consistent solution of (6). Using this procedure we were able to find scaling solutions of (6) together with the relative anomalous dimensions. In particular we started at \( d = 4 \) where \( \eta = 0 \) and we followed the scaling solutions down to \( d = 2 \) by making steps in \( d \) of size 0.1. At every critical \( d_{\eta,\sigma} \), we were able to follow the emerging multi-critical scaling solution. This allowed us to calculate the anomalous dimensions \( \eta \) of the multi-critical fixed-points as a function of \( d \). In figure 3 we show the anomalous dimensions of the first five multi-critical scaling solutions in the range \( 2 \leq d \leq 3 \). In particular, in \( d = 2 \) our fixed-points can be directly related to the multi-critical unitary minimal models that are found in CFT.

In \( d = 2 \) we compared our estimates for the anomalous dimensions with those found by Morris \cite{10}, using the derivative expansion to order \( \beta^2 \), and with those obtained from the exact relation \( \eta_{\text{ex}} = \frac{3}{(1+1)^2} \), obtained by CFT techniques \cite{5}. We make the comparison for the first ten multi-critical scaling solutions in figure 4. Due to our elementary estimate (2), our values systematically over-estimate both the \( \beta^2 \) and exact values; still it is astonishing that we correctly reproduce the overall qualitative picture. We stress that we were able to
observe an arbitrary number of scaling solutions and not only the first ten as in [10]. Our RG analysis predicts an infinite sequence of multi-critical scaling solutions of increasing degree, with anomalous dimensions decreasing monotonically with \( i \). It is important to remark that nowhere did we assume conformal invariance, but just translation, rotation, \( \mathbb{Z}_2 \) and RG scale invariance! All our conclusions where drawn just from the analysis of the simple ODE (6).

Our analysis can be seen as an RG validation of the correspondence between Landau–Ginsburg actions and minimal models [11]; correspondence that cannot be seen at the level of beta functions alone, but requires at least a functional truncation of the RG theory space, since only in this case is it possible to distinguish spurious fixed-points from real ones and thus to disentangle the rich fixed-point structure observed near \( d = 2 \). This is probably one reason why the \( \epsilon \)-expansion fails to correctly detect minimal models in two dimensions [12].

In view of the results of this paragraph, and of those of [10], one can expect to reproduce CFT results at the quantitative level by employing functional RG techniques.
Obviously the payback is that the functional RG analysis can be done independently and continuously in the dimension.

Critical exponents

Once we have found a scaling solution, together with its anomalous dimension, we can obtain the other critical exponents by studying linear perturbations around it. An easy way to do this [4] is by expanding the potential in a Taylor series (9) and to transform the PDE (4) in a system of $N_c$ coupled ODE for the beta functions $\tilde{\beta}_{2n} = \partial_2 \tilde{\lambda}_{2n}$ of the dimensionless couplings $\tilde{\lambda}_{2n} = \kappa^{d-2n}(\xi - 1 + \frac{d}{2})\lambda_{2n}$ (note that we consider here $\eta$ as an input from the preceding analysis). RG fixed-points now correspond to solutions of the following system of algebraic equations:

$$\tilde{\beta}_{2n} = 0, \quad n = 1, \ldots, N_c.$$  \hspace{1cm} (7)

As we vary the number $N_c$ of couplings in our truncation we usually find more and more solutions to the system (7). Most of these are spurious fixed-points: only with the previous analysis of the scaling solutions it is possible to understand which of these do correspond to admissible solutions of the fixed-point equation (6) and which instead correspond to singular solutions. Since $\tilde{\lambda}_2 = \sigma$, the values $\{\sigma_{i,n}^*\}$ that we previously obtained for given $d$ and $\eta$ now serve as a ‘map’ to correctly spot the desired solutions of the algebraic system (7). We clearly understand that the spurious solutions of (7) correspond to solutions of (6) which end up in a singularity [6].

It is useful to proceed as follows. Since the beta functions $\tilde{\beta}_{2n}$ are linear in the couplings $\tilde{\lambda}_{2n}$, we can iteratively solve (7) to find the couplings $\tilde{\lambda}_{2n}(\sigma)$ in terms of $\tilde{\lambda}_2 = \sigma$. Inserting these in $\tilde{\beta}_{2n+2} = 0$ makes it a polynomial equation in $\tilde{\lambda}_2 = \sigma$. This can be solved numerically; from the solutions we find, we pick the one that best approximates the value $\sigma_{i,n}$ relative to the scaling solution we are Taylor-expanding. Then we obtain the fixed-point values for all the dimensionless couplings $\tilde{\lambda}_{2n}(\sigma_{i,n})$. To determine the critical exponents, we just need to calculate the stability matrix and evaluate it at $\tilde{\lambda}_{2n}(\sigma_{i,n})$: $M_{nm} = \frac{\partial^2 \tilde{\beta}_{2n}}{\partial \tilde{\lambda}_{2m} \partial \tilde{\lambda}_{2n}}|_{\tilde{\lambda}_{2n}(\sigma_{i,n})}$.

We did this analysis for the Ising universality class. In this case the eigenvalues of $M_{nm}$ are such that $\Lambda_1(d) < 0 < \Lambda_2(d) < \Lambda_3(d) < \cdots$ for any $2 \leq d \leq 4$. By varying the number of couplings $N_c$ in our truncation we were able to obtain convergent estimates for the eigenvalues\footnote{We repeated the analysis by performing a Taylor expansion around the minimum of the dimensionless effective potential.}. We obtained estimates for the correlation length critical exponent $\nu_2(d) = -1/\Lambda_1(d)$ as a function of the dimension. We show the results of our analysis in figure 5, together with $\eta_2(d)$ and the other critical exponents that we obtained by employing standard scaling relations [13]. In two and three dimensions we found the results listed in table 1. Our estimates for the exponents $\nu_2(d)$ and $\gamma_2(d)$ turned out particularly good, both in $d = 2$ and $d = 3$. $\delta_2(d)$ and $\beta_2(d)$ are well estimated in $d = 3$ but not so well in $d = 2$, while the value of $\alpha_2(d)$ is poor in both cases. The quantitative estimate for $\eta_2(d)$, as we observed earlier, is more crude due to the elementary ansatz (2).

Discussion and outlook

In this paper we studied how universality classes of scalar theories with $Z_2$-symmetry evolve with the dimension $d$. Using functional RG techniques based on the effective average action, we were able to follow, continuously with $d$, the evolution of RG fixed-points (represented by
Figure 5. Critical exponents for the Ising universality class in the range $2 \leq d \leq 4$. From top-left $\delta_2(d)/5$ (empty circles), $\gamma_2(d)$ (downward triangles), $\nu_2(d)$ (filled circles), $\eta_2(d)$ (squares), $\beta_2(d)$ (upward triangles) and $\alpha_2(d)$ (rhombuses).

Table 1. Calculated critical exponents for the Ising universality class in two and three dimensions compared to exact results [5] and ‘world best’ estimates [13].

|       | $d = 2$ |       | $d = 3$ |
|-------|---------|-------|---------|
|       | This work | Exact | This work | World best |
| $\eta$ | 0.436 | 0.25 | 0.11 | 0.036 |
| $\nu$ | 1.05 | 1 | 0.65 | 0.63 |
| $\alpha$ | −0.11 | 0 | 0.06 | 0.11 |
| $\beta$ | 0.23 | 0.125 | 0.36 | 0.33 |
| $\gamma$ | 1.65 | 1.75 | 1.22 | 1.24 |
| $\delta$ | 10.17 | 15 | 4.60 | 4.79 |

scaling solutions) through the functional theory space of effective potentials. Even if all our analysis was based on the study of a simple ODE, we were able to observe a very rich behavior.

Above four dimensions we found only the Gaussian universality class; at $d = 3$ the Ising or Wilson–Fisher universality class appeared; while in two dimensions we found a countable infinity of multi-critical fixed-points corresponding to the unitary minimal models of CFT. More importantly, we presented the continuum evolution with $d$ of these universality classes; we observed a succession of critical dimensions $d_{c,i}$ (that accumulate at $d = 2$) where new universality classes branched from the Gaussian one. The picture that we presented captured all possible critical behavior associated to $\mathbb{Z}_2$-symmetry in any dimension $d \geq 2$. We were also able to give quantitative estimates for critical exponents as a function of $d$.

It will be very interesting to find real physical systems, of effective fractal dimension $d$, where to observe the phase transitions related to these universality classes, and to compare our critical exponents with experiments. Otherwise these systems could be studied on the lattice [14]. Extensions of the approach presented in this paper can be very fruitful to deal with the general problem of classifying universality classes, and to study how these depend continuously on $d$. 
Acknowledgments

We would like to thank R Percacci, O Zanusso, G D’Odorico and A Trombettoni for useful and stimulating discussions.

References

[1] Berges J, Tetradis N and Wetterich C 2002 Phys. Rep. 363 223 (arXiv:hep-ph/0005122)
[2] Wetterich C 1993 Phys. Lett. B 301 90
[3] Ballhausen H, Berges J and Wetterich C 2004 Phys. Lett. B 582 144 (arXiv:hep-th/0310213)
[4] Litim D F 2001 Phys. Rev. D 64 105007 (arXiv:hep-th/0103195)
[5] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Nucl. Phys. B 241 333
[6] Morris T R 1994 Phys. Lett. B 334 355 (arXiv:hep-th/9405190)
[7] Felder G 1987 Commun. Math. Phys. 111 101
[8] Nandori I, Nagy S, Sailer K and Trombettoni A 2009 Phys. Rev. D 80 025008 (arXiv:0903.5524 [hep-th])
[9] Nagy S, Nandori I, Polonyi J and Sailer K 2009 Phys. Rev. Lett. 102 241603 (arXiv:0904.3689 [hep-th])
[10] Morris T R 1995 Phys. Lett. B 345 139 (arXiv:hep-th/9410141)
[11] Zamolodchikov A B 1986 Sov. J. Nucl. Phys. 44 529
     Zamolodchikov A B 1986 Yad. Fiz. 44 821
[12] Howe P S and West P C 1989 Phys. Lett. B 223 371
[13] Pelissetto A and Vicari E 2002 Phys. Rep. 368 549 (arXiv:cond-mat/00012164)
[14] Vezzani A 2003 J. Phys. A: Math. Gen. 36 1593 (arXiv:cond-mat/0212497)