Constructing Landau framework for topological order: quantum chains and ladders

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Received 12 January 2017, revised 1 March 2017
Accepted for publication 1 March 2017
Published 3 April 2017

Abstract. We studied quantum phase transitions in the antiferromagnetic dimerized spin-$\frac{1}{2}$ $XY$ chain and two-leg ladders. From analysis of several spin models we present our main result: the framework to deal with topological orders and hidden symmetries within the Landau paradigm. After mapping of the spin Hamiltonians onto the tight-binding models with Dirac or Majorana fermions and, when necessary, the mean-field approximation, the analysis can be done analytically. By utilizing duality transformations the calculation of nonlocal string order parameters is mapped onto the local order problem in some dual representation and done without further approximations. Calculated phase diagrams, phase boundaries, order parameters and their symmetries for each of the phases provide a comprehensive quantitative Landau description of the quantum critical properties of the models considered. Complementarily, the phases with hidden orders can also be distinguished by the Pontryagin (winding) numbers which we have calculated as well. This unified framework can be straightforwardly applied for various spin chains and ladders, topological insulators and superconductors. Applications to other systems are under way.

Keywords: quantum criticality, spin chains, ladders and planes, integrable spin chains and vertex models, topological phases of matter
1. Introduction

According to the Landau theory of phase transitions, the phases on both sides of a critical point can be distinguished by different types of the long-range order, or its absence. The ordering is described by an appropriately chosen local order parameter. In the Landau framework phases are distinct due to their different symmetries, and a continuous phase transition is always related to spontaneous breaking of system’s Hamiltonian symmetry [1].

However, there is a mounting number of examples of systems where the states of matter and their change cannot be classified in terms of the apparent symmetry breaking and/or local order parameter, most notably a particular class of gapped systems known as quantum spin liquids [2, 3]. One of the best known example of such quantum liquids is realized in the fractional quantum Hall effect. Most likely, such a quantum (resonating valence bond) spin liquid is also seen in the pseudogap phase of the high-$T_c$ cuprates [2]. Low-dimensional and/or frustrated systems provide a lot of possibilities for realization of various exotic quantum liquid states, where a local long-range order is prevented even at zero temperature. In low-dimensional quantum magnets, fermions, and topological insulators/superconductors the existence of gapped phases and transitions between them are not necessarily accompanied by breaking of some symmetry associated with local observables. See [2–13] and more references there.

https://doi.org/10.1088/1742-5468/aa657c
To characterize these states a concept of *topological order* was introduced [2]. The exact definition of this concept is a really subtle issue [14]. It is often associated with the nontrivial Pontryagin, Chern or similar topological numbers (indices) [3, 13, 15]. Hatsugai and co-workers [16] proposed the quantized Berry phase as a quantum order parameter to probe nonlocal orders. This scheme was successfully applied, e.g. for several quantum spin liquid systems. It turns out that the change of the quantized Berry phase can diagnose also crossovers between the states which are not separated by gap closing, local symmetry or order changes [17–19].

It has been also suggested that topological entanglement, derived from the von Neumann entropy, provides another quantitative measure of topological order since quantum critical points separating different phases can be identified with extrema of the entanglement entropy [20].

To characterize phases with hidden nonlocal or topological orders the concept of string order parameter (SOP) introduced by den Nijs and Rommelse is particularly instrumental [21]. The appearance of such nonlocal order is accompanied by a hidden symmetry breaking [22]. SOPs are successfully applied to characterize quantum phases with hidden orders in various spin chains and ladders at zero temperature [5, 7–9, 12, 22–24].

The important property of SOP is that it can be used for both quantum and thermal phase transitions. For instance, the SOP can be used for analysis of the thermal transitions into a Kosterlitz–Thouless-type phase with an algebraic order [21]. Very importantly, the SOP is a proper order parameter in the sense of Landau: for the integrable models one can calculate exactly the critical indices $\beta$ and $\eta$ from the SOP and the string–string correlation function, resp., near the critical point [23] and check explicitly that such $\beta$, $\eta$ along with other critical indices satisfy the standard hyperscaling relations [24]. Since the SOP is a limit of a nonlocal correlation function, its analytical calculation is a difficult task even for exactly solvable models. For example, for the dimerized Heisenberg spin-$\frac{1}{2}$ chain the calculation is quite simple in the leading sine-Gordon approximation for the Hamiltonian [23], while taking into account the marginal terms renders the problem much more difficult [25], and more work is still needed. The other subtlety is that only for a spin chain or a two-leg ladder the definition of the SOPs is straightforward, while already for a three-leg ladder [8, 9, 24], to say nothing about a 2D lattice, there are many ways to define the SOP. Potential multitude of SOPs and their critical properties need further analysis.

There is an extensive recent literature on hidden orders in massive phases of Heisenberg ladders. In particular, the critical properties of the dimerized two- and three-leg ladders have been studied in numerical and analytical works [9, 24, 26–32]. Quantum critical lines or even gapless regions in the parameter space adjacent to various massive phases of spin ladders may be also induced by frustrations, magnetic fields, or four-spin interactions [5, 6].

The nonlocal SOP can be defined for bosonic systems as well [33, 34]. Moreover, this nonlocal order was even observed for bosons in optical lattice [35].

Another interesting class of models where nonlocal string-like order parameters turned out to be essential are the probabilistic cellular automata which have been experiencing a steadily growing interest during the last several decades. It was shown
recently [36] that the active phases of the (1 + 1)-dimensional automation models possess numerous percolative nonlocal order parameters emerging as cascades of geometric phase transitions.

There has been an opinion expressed in the literature that various low-dimensional fermionic or spin systems with hiddenly ordered phases and transitions between them, defy completely the Landau paradigm. It appears to us that such a claim is too radical, and one can formulate the theory of such transitions consistent with the Landau framework using the nonlocal SOPs [37]. The Landau picture needs however to be amended, since some facets of topological order quantified by topological indices, Berry phases, entanglement, are not reducible even to nonlocal SOPs. The major problem is that the current treatment of such systems in most cases is heavily numerical and the qualitative physical picture is obscured. It is really warranted to put more weight first on advancing relatively simple analytical approaches based on the effective (mean-field) models, similar, e.g. to the Kitaev model [11, 12], or the single-particle tight-binding models used for the analyses of the topological insulators and superconductors [13]. Results based on those models clearly indicate that interactions are not an indispensable element of the topological order framework.

In the present paper we carry out such program from analyses of the dimerized $XY$ chain and the two-leg Heisenberg ladders with staggered and columnar dimerizations. The first exactly-solvable model is equivalent to noninteracting Jordan–Wigner fermions. To treat the ladders, we use the Jordan–Wigner tranformation and the mean-field approximation to obtain an effective free-fermionic Hamiltonian for the spin ladder. In the earlier related work it has been shown that such approximation is quite adequate, even quantitatively [24, 31, 38]. By utilizing duality transformations we demonstrate that it is possible to map the problem of calculation of nonlocal SOPs in the chain and ladders onto the problem of a local (Landau) order parameter in the dual representation, where the dual models are given by exactly-solvable Hamiltonians and their order parameters can be calculated exactly. It is then also straightforward to relate appearance of the SOP with the spontaneous symmetry breaking in terms of the dual Hamiltonian. In addition we show that the gapped phases not only have distinct SOPs, but they can also be distinguished by the winding (Pontryagin) number.

These results allow us to stress close similarities between various gapped phases occurring in spin models and topological insulators/superconductors. The latter are of great current interest [3, 13]. The proposed approach provides a unifying theoretical framework to deal with nonlocal order parameters in such seemingly different systems within the Landau paradigm.

The rest of the paper is organized as follows: in section 2 we introduce and discuss the dimerized 1D $XY$ model. Using the duality transformation, the SOPs and local order parameters are calculated exactly. We also find the winding numbers for different phases of the model. In section 3 we present analogous results for the dimerized two-leg ladders. A mean-field approximation is used to construct the effective Hamiltonian for the Jordan–Wigner fermions which were mapped back onto exactly solvable (dual) spin models. The appendix contains details of the order parameter calculation for the quantum Ising chain with the three-spin interaction which seems was never done before. The results are summarized and discussed in the concluding section 4.
2. Dimerized $XY$ chain

2.1. Transformations of the Hamiltonian

To analyze in depth various aspects of the conventional Landau and the topological orders we start from the dimerized quantum $XY$ chain:

$$
H = \frac{J}{4} \sum_{i=1}^{N} \left[ (1 + \gamma)\sigma_i^x \sigma_{i+1}^x + (1 - \gamma)\sigma_i^y \sigma_{i+1}^y + \delta(-1)^i(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x \sigma_{i+1}^x) \right] + \frac{1}{2}(-1)^i h\sigma_i^z, \tag{1}
$$

where $\sigma$-s are the standard Pauli matrices, $J$ is the nearest-neighbor exchange coupling (we assume it to be antiferromagnetic), and $\gamma$ and $\delta$ are the dimensionless parameters of anisotropy and dimerization, respectively. This exactly-solvable model was first introduced and analyzed by Perk et al. [39] in the presence of uniform and alternating magnetic fields. (See also [40] for more recent work.) To make connections with ladder models discussed below we take the Hamiltonian (1) with an alternating transverse field. The standard Jordan–Wigner transformation (JWT) [41] maps spins onto free fermions

$$
H = \frac{1}{2} \sum_k \Psi_k^\dagger \mathcal{H}(k) \Psi_k, \tag{2}
$$

where the Fourier transforms of the Jordan–Wigner fermions residing on the even/odd sites of the chain are unified in the Nambu spinor

$$
\Psi_k = (d_k^1, d_k^0, d_{-k}, d_{-k}), \tag{3}
$$

with the wavenumbers restricted to the Brillouin zone (BZ) $k \in [-\pi/2, \pi/2]$ and we set the lattice spacing $a = 1$. The $4 \times 4$ Hamiltonian can be written as

$$
\mathcal{H}(k) = J \cos k \Gamma_3 + J\delta \sin k \Gamma_3 + J\gamma \sin k \Gamma_{13}, \tag{4}
$$

where the five Dirac matrices are:

$$
\Gamma_{1,...,5} = \{ \sigma_1 \otimes 1, \sigma_2 \otimes 1, \sigma_3 \otimes \sigma_1, \sigma_3 \otimes \sigma_2, \sigma_3 \otimes \sigma_3 \} \tag{5}
$$

and $\Gamma_{13} = [\Gamma_1, \Gamma_3] / 2i = -\sigma_2 \otimes \sigma_1$. This Hamiltonian has four eigenvalues $\pm \epsilon^\pm$ where

$$
\epsilon^\pm(k) = J\sqrt{\cos^2 k + \left(\frac{h}{J}\right)^2 + \delta^2 \sin^2 k \pm \gamma \sin k}. \tag{6}
$$

From (6) we infer a remarkable property of the model (1): each of the relevant perturbations $h, \delta, \gamma$ makes the uniform isotropic $XX$ chain gapfull with the gap

$$
\Delta = J\sqrt{(h/J)^2 + \delta^2 \pm \gamma}, \tag{7}
$$

while those terms cancel along the lines of quantum criticality

$$
\gamma = \pm \sqrt{(h/J)^2 + \delta^2}. \tag{8}
$$
and the model becomes gapless. The same cancelation effect happens in dimerized ladders considered in the next section, but contrary to ladders, the present model can be treated exactly at each step.

When we turn off the field and restrict ourselves to two relevant perturbations, pertinent to other systems considered in this work, the spectrum of the model (1) becomes:

\[ \epsilon \pm(k) = J \sqrt{\cos^2 k + (\delta \pm \gamma)^2 \sin^2 k} \]  

with the gap

\[ \Delta = J|\gamma \pm \delta|. \]  

Now we will analyze the nature of order and symmetry changes on the lines of the quantum phase transition \( \gamma = \pm \delta \) on the parameter plane \((\delta, \gamma)\). We utilize the duality transformation:

\[ \tau_n^x = -\tau_{n-1}^x \tau_n^z \]  

\[ \sigma_n^y = \prod_{l=n}^N \tau_{l}^z, \]  

where \( \tau \)-s obey the standard algebra of the Pauli operators, and they reside on the sites of the dual lattice, which can be placed between the sites of the original chain where the operators \( \sigma \) reside. In terms of the dual operators, the Hamiltonian (1) becomes a sum of two completely decoupled 1D quantum Ising model (QIM) Hamiltonians defined on the even and odd sites of the dual lattice:

\[ H = H_{\text{even}} + H_{\text{odd}} \]

\[ H_{\text{even}} = \frac{1}{4} \sum_{l=1}^{N/2}(J^{++} \tau_{2l-1}^x \tau_{2l}^x + J^{-+} \tau_{2l}^z) \]  

\[ H_{\text{odd}} = \frac{1}{4} \sum_{l=1}^{N/2}(J^{++} \tau_{2l-1}^x \tau_{2l}^x + J^{-+} \tau_{2l}^z), \]  

with the notations

\[ J^{\pm \pm} = J(1 \pm \gamma \pm \delta). \]  

2.2. Local and string order parameters

We define the string operator as

\[ O_\alpha^m = \exp \left[ i \pi \sum_{k=1}^{m-1} \sigma_k^\alpha \right], \]  

where \( \alpha = x, y, z \) and the summation is carried out along all sites of the string, left from the \( m \)th site. In case of the chain model this is unambiguous and means \( k < m \), while for ladders or other models the path of the string must be specified, as we discuss in the
following sections. The string order parameter (SOP) $O_n$ is determined from the limit $n - m \to \infty$ of the string–string correlation function

$$\langle O_m^n O_n^\alpha \rangle = (-1)^{m-1} \left\langle \exp \left[ \frac{i \pi}{2} \sum_{k=m}^{n-1} \sigma_k^\alpha \right] \right\rangle.$$  \hspace{1cm} (17)

Taking $m = 1$ and $n = 2l + 1$ in (17) (note that SOP for odd number of spins vanishes due to symmetry), the SOP is introduced as

$$O^2_\alpha = \lim_{l \to \infty} (-1)^l \left\langle \prod_{k=1}^{2l} \sigma_k^\alpha \right\rangle.$$ \hspace{1cm} (18)

In the original proposal by den Nijs and Rommelse [21] and in the subsequent work on the spin chains (see, e.g. [22, 23]) the SOP was identified (up to some minor variations) with the limit of the string–string correlation function, which is not convenient, since such SOP has a wrong dimension of square of the order parameter. The definition we use here [33] is more consistent with the standard theory of critical phenomena and is in line with the definition of the Landau order parameter via a correlation function of local operators. We will show that in the critical region $O_\alpha \propto |t|^\beta$, where $t$ is a distance from the critical point and the critical index of the order parameter $\beta$ correctly enters all the scaling relations.

Using the duality transformation (11) in equation (18) we get

$$O^2_x = \lim_{l \to \infty} (-1)^l \langle \tau^{x^x}_{x^\alpha} \rangle.$$ \hspace{1cm} (19)

So, the nonlocal SOP defined on the sites of the direct lattice becomes a local order parameter on the dual lattice. This implies that for the phase transitions with nonlocal orders the conventional Landau framework can be recovered via duality.

According to the classical results [43, 44] for the 1D QIM with the Hamiltonian

$$H = \sum_{i=1}^{N} \left[ \frac{J \tau_i^x \tau_{i+1}^x}{h} + h \tau_i^z \right],$$ \hspace{1cm} (20)

the model is disordered at $\lambda \equiv J/h \leq 1$, while the long-range order $m_x \neq 0$ appears at $\lambda > 1$. The order parameter is obtained from the correlation function:

$$\lim_{l \to \infty} \langle \tau_{i}^x \tau_{i+l}^x \rangle = (-1)^l (1 - \lambda^{-2}) \frac{1}{l} = (-1)^l m_x^2.$$ \hspace{1cm} (21)

From comparison of the Hamiltonians (13), (14) and (20) we see that if $\lambda_t > 1$, where

$$\lambda_{t,o} = \frac{1 + \gamma \mp \delta}{1 - \gamma \pm \delta},$$ \hspace{1cm} (22)

then $\tau^x$ is ordered on either even or odd dual sublattices. So we should distinguish between the even and odd SOP defined on the sites of the even or odd dual sublattices, respectively. As one can see, the SOP defined by equation (19) is even, and we will denote it by $O_{z,e}$ from now on. To define the odd SOP ($O_{z,o}$) we can take $m = 2$ and $n = 2l + 2$ in equation (17), then $O_{z,o}$ is given by equation (19) where the correlation
function on its rhs is now \( \langle \tau_1^z \tau_{2l-1}^z \rangle \). Using the above formulas the even and odd SOP can be calculated exactly:

\[
O_{x,e/o} = \begin{cases} 
2^{1/4} \left( \frac{t_+}{1 + t_+^2} \right)^{1/8} & t_+ \geq 0 \\
0 & t_+ < 0 
\end{cases}
\]  

(23)

where we denote

\[
t_\pm = \gamma \pm \delta
\]  

(24)

Let us now calculate the \( y \) component of the SOP. The duality transformations (11) and (12) with the interchange \( x \leftrightarrow y \) map the Hamiltonian (1) again onto a sum of two even and odd QIM Hamiltonians:

\[
H_{\text{even}} = \frac{1}{4} \sum_{l=1}^{N/2} (J^{-+} \tau_{2l-1}^y \tau_{2l}^y + J^{++} \tau_{2l}^z)
\]  

(25)

\[
H_{\text{odd}} = \frac{1}{4} \sum_{l=1}^{N/2} (J^{-+} \tau_{2l-1}^y \tau_{2l+1}^y + J^{++} \tau_{2l}^z)
\]  

(26)

Following the same steps as above we easily find:

\[
O_{y,e/o} = \begin{cases} 
0 & t_+ \geq 0 \\
O_{x,e/o}(t_-) & t_- < 0 
\end{cases}
\]  

(27)

We show the phase diagram of the dimerized XY model in figure 1. The two lines of quantum phase transitions \( \gamma = \pm \delta \) divide the parameter plane into four regions denoted
by A-D. For each region we indicate nonvanishing order parameters which characterize different phases located there.

The next important step in our analysis is to establish relation between the conventional long-range order with local order parameter(s) and the nonlocal (topological) SOPs. Using the duality transformation (11) and decoupling of the dual Hamiltonian into the even and odd terms, we find the magnetization \( m_x \) of the spins of the original Hamiltonian (1) as:

\[
m_x^2 = \lim_{l \to \infty} \langle \sigma_1^z \sigma_{2l}^z \rangle = \lim_{l \to \infty} \langle (\tau_0^x \tau_{2l}^x)(\tau_1^x \tau_{2l-1}^x) \rangle.
\]

The \( y \) component of the magnetization can be treated in a similar way, so we get

\[
m_y = O_{\alpha, e} O_{\alpha, o}, \quad \alpha = x, y.
\]

As we see from the model’s phase diagram in figure 1, the even and odd SOPs are mutually exclusive in the regions B and D. Those are the regions without conventional local (Landau) order. The phases in the regions B and D possess only a nonlocal (topological) order which is quantified by the SOP. As we conclude from the duality mapping, the dimerized XY model (1) possesses the hidden \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) symmetry which was also noticed earlier in various spin chains and ladders [22]. In the present analytically solvable model the origin of such symmetry can be easily traced to the \( \mathbb{Z}_2 \) symmetry of the even and odd Ising models (13) and (14). Nonzero even SOP \( O_{x, e} \) and odd SOP \( O_{x, o} \) in the sectors B and D are due to spontaneous breaking of the \( \mathbb{Z}_2 \) symmetry in the even (13) or odd (14) sectors of the dual Hamiltonian, respectively. These SOPs signal appearance of the spontaneous ‘dual’ magnetization \( \langle \tau_1^x \rangle \) or \( \langle \tau_0^y \rangle \). We have to emphasize an important issue: since mappings of the original model (1) onto the dual models (13), (14) or (25), (26) are results of the different dual transformations, the analyses in terms of the SOP \( O_x \) or \( O_y \) are complimentary, and the order parameters \( O_x \) and \( O_y \), even if they both are found to be nonzero in some parts of the phase diagram should not be understood as coexistent.

We conclude from equations (23) and (27) (see also figure 1) that contrary to common belief, the even and odd SOPs can coexist. In the region A of the phase diagram which corresponds to \( \gamma^2 > \delta^2, \gamma > 0 \) both even and odd SOPs \( O_{x,e/o} \) are nonvanishing, and then, as follows from equation (29) the local order (magnetization) \( m_x \neq 0 \) appears in the original model (1). Similarly, when \( \gamma^2 > \delta^2, \gamma < 0 \) (region C of the phase diagram), the magnetization \( m_y \neq 0 \). The appearance of the local order, or, equivalently for this model, the coexistence of the even and odd SOPs are the result of the spontaneous \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) symmetry breaking simultaneously in both even and odd sectors of the dual Hamiltonian. From Equations (23), (27) and (29) we find the numerical values of the local order parameters:

\[
m_x(\gamma) = m_y(-\gamma) = \frac{\gamma^2 - \delta^2}{(1 + \gamma)^2 - \delta^2}^{1/8}.
\]

In the limit \( \delta = 0 \) the above equation coincides with the result of Barouch and McCoy [44].

Another useful limit of the model (1) is \( \gamma = 0 \) when it becomes a dimerized XX chain. This model does not have a local long-range order, and its phase diagram can be read from the line \( \gamma = 0 \) in figure 1. The topological order in the phases \( \delta > 0 \) or \( \delta < 0 \) is
characterized by a pair of mutually exclusive SOPs $O_{\alpha,o}$ and $O_{\alpha,e}$ (we can choose $\alpha = x$ or $\alpha = y$) separated by a quantum critical point $\delta = 0$ where all SOPs vanish. From (23) and (27) we find

$$O_{x,o} = O_{y,e} = 2^{1/4} \left( \frac{\delta}{(1 + \delta)^2} \right)^{1/8}, \quad \delta > 0. \tag{31}$$

The magnitudes of the SOPs are symmetric with respect to $\delta$ and $O_{x,o}(\delta) = O_{y,o}(\delta) = O_{x,o}(\delta)$ for $\delta < 0$.

If we define a primary order parameter as the one which is nonzero exclusively for a given phase, then we can identify $m_x$ and $m_y$ as the primary order parameters for the magnetized phases residing, respectively, in the sectors A and C of the phase diagram. The topological order of the phases B and D is characterized by the pair of the primary (string) parameters $O_{x,e}$ and $O_{x,o}$ for the choice (11) and (12) of the dual transformation. (If we swap $x \leftrightarrow y$, we end up with the dual representation where the primary order parameters are $O_{y,e}$ and $O_{y,o}$.) From behavior of the gap (10) and the primary order parameter (see (23), (27) and (30), whichever applicable) near the lines $\gamma = \pm \delta$ of the quantum phase transition we find the critical indices of the order parameter $\beta = 1/8$ (see its definition below equation (18)) and of the correlation length $\nu = 1$. (The latter can be read from the gap equation (10) since $\xi^{-1} \propto \Delta \propto |t|^\nu$. So the critical behavior of the model on the phase boundaries belongs to the 2D Ising universality class as it must, since the Hamiltonian (1) is equivalent to free fermions.

### 2.3. Topological winding numbers

In recent years it has been proven that many quantum phase transitions with hidden orders are accompanied by a change of topological numbers [3, 13, 15–19]. In this section we calculate the winding number (or the Pontryagin index) in the dimerized XY spin chain to characterize its different gapped phases. These topological numbers were calculated recently in similar 1D systems [18, 45, 46]. Following the formalism of [47] we first rewrite (4) as the Bogoliubov–de Gennes Hamiltonian of the topological superconductor (class DIII):

$$\mathcal{H}(k) = \begin{pmatrix}
\hat{h}(k) & \hat{\Delta}(k) \\
\hat{\Delta}^\dagger(k) & -\hat{h}(k)
\end{pmatrix}, \tag{32}
$$

where

$$\hat{h}(k) \equiv J \cos k\sigma_1 + J\delta \sin k\sigma_2 + h\sigma_3, \tag{33}$$

$$\hat{\Delta}(k) \equiv -iJ\gamma \sin k\sigma_1. \tag{34}$$

By a unitary transformation the above Hamiltonian can be brought to the block off-diagonal form

$$\hat{\mathcal{H}}(k) = \begin{pmatrix}
0 & \hat{D}(k) \\
\hat{D}^\dagger(k) & 0
\end{pmatrix}, \tag{35}$$

where

$$\hat{D}(k) \equiv \begin{pmatrix}
\hat{h}(k) & \hat{\Delta}(k) \\
\hat{\Delta}^\dagger(k) & -\hat{h}(k)
\end{pmatrix}. \tag{36}$$

The magnitudes of the $D$ parameters are symmetric with respect to $\gamma$ and $\delta$ and $D_{\gamma,\delta} = D_{\gamma,\delta}$ for $\gamma < \delta$. If we define a primary order parameter as the one which is nonzero exclusively for a given phase, then we can identify $m_x$ and $m_y$ as the primary order parameters for the magnetized phases residing, respectively, in the sectors A and C of the phase diagram. The topological order of the phases B and D is characterized by the pair of the primary (string) parameters $O_{x,e}$ and $O_{x,o}$ for the choice (11) and (12) of the dual transformation. (If we swap $x \leftrightarrow y$, we end up with the dual representation where the primary order parameters are $O_{y,e}$ and $O_{y,o}$.) From behavior of the gap (10) and the primary order parameter (see (23), (27) and (30), whichever applicable) near the lines $\gamma = \pm \delta$ of the quantum phase transition we find the critical indices of the order parameter $\beta = 1/8$ (see its definition below equation (18)) and of the correlation length $\nu = 1$. (The latter can be read from the gap equation (10) since $\xi^{-1} \propto \Delta \propto |t|^\nu$. So the critical behavior of the model on the phase boundaries belongs to the 2D Ising universality class as it must, since the Hamiltonian (1) is equivalent to free fermions.

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where

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\hat{\Delta}^\dagger(k) & -\hat{h}(k)
\end{pmatrix}. \tag{36}$$
with the operator
\[ \hat{D}(k) = \hat{h}(k) + \hat{\Delta}(k), \]
which has two eigenvalues \( \pm \lambda(k) \) with
\[ \lambda(k) = J \sqrt{(h/J)^2 + 1 + (\delta^2 - \gamma^2 - 1) \sin^2 k - i \gamma \sin 2k}. \]

Note a useful relation between the eigenvalues of \( \hat{D}(k) \), \( \hat{D}^\dagger(k) \) and of the Hamiltonian (6):
\[ \epsilon^+(k) \epsilon^-(k) = \lambda(k) \lambda'(k) \]

In one spatial dimension the winding number defined as [47]
\[ N_w = \frac{1}{4\pi i} \int_{\text{BZ}} dk \text{Tr} \{ \partial_k \ln \hat{D} - \partial_k \ln \hat{D}^\dagger \} \]
can be readily calculated analytically for this model. We set \( h = 0 \) and calculated the winding number in all four regions of the phase diagram in figure 1. To make a connection between the definition (39) and a more intuitive definition of the winding number let us introduce a two-component unit vector \( \mathbf{n}(k) = (n_1(k), n_2(k)) \) constructed from the spectrum of the model. The eigenvalues (9) allow us to define two unit vectors with the components
\[ n_1(\epsilon^\pm(k)) = J \cos k \epsilon^\pm(k), \]
\[ n_2(\epsilon^\pm(k)) = J(\delta \pm \gamma) \sin k \epsilon^\pm(k), \]
which can also be related to the Bogoliubov angle \( e^{i\theta_k} = n_1(k) + in_2(k) \). (For the definition of \( \theta_k \) see, e.g. [48].) The winding number defined as [3, 15]
\[ N_\pm = \frac{1}{2\pi} \sum_{i,j=1}^2 \int_{\text{BZ}} dk \delta \gamma n_i(\epsilon^+(k)) \partial_k n_j(\epsilon^-(k)) \]
counts the number of loops wrapped by the unit vector around the origin while the wavevector spans over the Brillouin zone. One can show that the definition (39) yields
\[ N'_w = N_+ - N_- = \frac{1}{2} \{ \text{sign}(\delta - \gamma) - \text{sign}(\delta + \gamma) \}. \]

The topological number \( N'_w \) accounts for the ‘relative’ winding of two normalized eigenvalues of the Hamiltonian. We find the winding number \( N_w \) which adds up the loops made by each of the eigenvectors more convenient for various applications:
\[ N_w = N_+ + N_- = \frac{1}{2} \{ \text{sign}(\delta - \gamma) + \text{sign}(\delta + \gamma) \}. \]

As one check from equations (43) or (44), both topological numbers change by \( \pm 1 \) on the phase boundaries \( \gamma = \pm \delta \), signalling the phase transition. We choose \( N_w \) as a complimentary topological order parameter characterizing a given phase, since its behavior is in accord with the standard wisdom: as one can see in figure 1, \( N_w = 0 \) in the topologically trivial regions where a conventional local order exists.
3. Dimerized two-leg ladder

3.1. Model and effective mean-field Hamiltonian

Now we present the results for a two-legged ladder with intrinsic dimerization. In line with the earlier work [24, 31, 38] we consider the two possible dimerization patterns of the ladder: staggered and columnar, shown in figure 2. The Hamiltonian of the dimerized spin-$\frac{1}{2}$ ladder with two legs is given by:

$$H = \sum_{n=1}^{N} \sum_{\alpha=1}^{2} J_{0}(n) S_{\alpha}(n) \cdot S_{\alpha}(n+1) + J_{\perp} \sum_{n=1}^{N} S_{\alpha}(n) \cdot S_{\alpha+1}(n),$$

where the dimerization occurs along the chains ($\alpha = 1, 2$) only, with the rung coupling $J_{\perp}$ taken as constant. All the spin exchange couplings are antiferromagnetic. The dimerization patterns are then defined as:

$$J_{\alpha}(n) = J[1 - (-1)^{n + \alpha} \delta] \text{ (staggered)}$$

$$J_{\alpha}(n) = J[1 - (-1)^{n} \delta] \text{ (columnar)}$$

with periodic boundary conditions along the chains and open boundary conditions in the rung directions.

Critical properties of dimerized ladders are known from analytical and numerical studies: [9, 24, 26–32] the staggered ladder can undergo a continuous quantum phase transition and is gapless on the line of quantum criticality, while the columnar ladder is always gapped and does not undergo any transition. We infer also from our earlier work [24, 31] that the mean-field approximation for this model is quite adequate, even quantitatively. So we apply this approximation to map the spin ladder model onto a problem of effective quadratic fermionic Hamiltonian. The latter will be utilized to study analytically the critical properties of two dimerizations, calculation of SOPs and topological numbers, similarly to what was done above for the exactly solvable chain.

We map the spin ladder Hamiltonian (45) onto fermions via a JWT. There are different ways to introduce this transformation when we depart from a single-chain problem (see, e.g. [12, 49, 50]). We chose the snake-like path for the JWT used in [50]. Labelling the sites of the ladder along the path as shown in figure 3, the JWT is defined as

$$\sigma_{n}^{+} = c_{n}^{\dagger} \exp \left( i \pi \sum_{l=1}^{n-1} c_{l}^{\dagger} c_{l} \right)$$

Figure 2. Dimerized two-leg ladder. Bold/thin/dashed lines represent the stronger/weaker chain coupling $J(1 \pm \delta)$ and rung coupling $J_{\perp}$, respectively. Dimerization patterns: (a)—staggered; (b)—columnar.
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\begin{align}
\sigma_n^z &= 2c_n^\dagger c_n - 1, \text{ where } \sigma_n^{\pm} = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y).
\end{align}

We treat the interaction terms via a standard mean-field decoupling:

\begin{align}
\left(\hat{n}_k - \frac{1}{2}\right)\left(\hat{n}_m - \frac{1}{2}\right) &\approx \frac{1}{4} - \langle \langle c_m^\dagger c_k \rangle \rangle + \langle c_k c_m + h.c. \rangle + |\langle c_k^\dagger c_k \rangle|^2. \tag{49}
\end{align}

Then the spin ladder (45) maps onto the following fermionic Hamiltonian:

\begin{align}
H_{\text{MF}} &= NC + \frac{1}{2} \sum_{m,\alpha} J_\alpha(m)\left[e^{i\Phi_\alpha(m)} + 2t_{\alpha\parallel}(m)\right]c_\alpha^\dagger(m)c_\alpha(m + 1)
&+ \frac{1}{2} J_\perp \sum_{m}[1 + 2t_{\perp}(m)]c_1^\dagger(m)c_2(m) + h.c. \tag{50}
\end{align}

Although we use the JWT path as shown in figure 3, we keep the labelling of fermions residing on the ladder in agreement with the notations of the original spin Hamiltonian (45). The phase entering equation (50) is defined as

\begin{align}
\hat{\Phi}_\alpha(m) &= \begin{cases} 0 & \alpha = 1, \ m = 2l \\
\pi(\hat{n}_2(m) + \hat{n}_2(m + 1)) & \alpha = 1, \ m = 2l - 1 \\
0 & \alpha = 2, \ m = 2l - 1 \\
\pi(\hat{n}_1(m) + \hat{n}_1(m + 1)) & \alpha = 2, \ m = 2l \end{cases} \tag{51}
\end{align}

and the effective hopping terms are

\begin{align}
t_{\alpha\parallel}(m) &= \langle c_\alpha(m)c_\alpha^\dagger(m + 1) \rangle \tag{52}
\end{align}

\begin{align}
t_{\perp}(m) &= \langle c_1(m)c_2^\dagger(m + 1) \rangle. \tag{53}
\end{align}

Note that the coupling $J_{\perp}$ spoils the commutation of $\hat{\Phi}_\alpha(m)$ and $H_{\text{MF}}$. We however will approximately treat the phase as a good quantum number, i.e. $\hat{\Phi}_\alpha(m) \approx \Phi_\alpha(m)$. The quantitative validity of this approximation was confirmed in earlier related work [31, 38, 49, 53]. Then according to the Lieb theorem [51] the ground state of this quadratic Hamiltonian is in the $\pi$-flux phase [52], which amounts to

\begin{align}
e^{i\Phi_\alpha(m)} + 2t_{\alpha\parallel}(m) &= (-1)^{m+\alpha-1}(1 + 2t_{\alpha\parallel}). \tag{54}
\end{align}

Assuming further $t_{\alpha\parallel} \approx t_{\parallel}$ and $t_{\perp} \approx t_{\perp}$ we get

\begin{align}
H_{\text{MF}} &= NC + \frac{1}{2} \sum_{n,\alpha} (-1)^{\alpha+1}\langle J_{\alpha\parallel}(n)c_\alpha^\dagger(n)c_\alpha(n + 1) + \frac{1}{2} J_{\perp} \sum_n c_1(n)c_2(n) + h.c. \tag{55}
\end{align}

Figure 3. Path of the Jordan–Wigner transformation used for the two-leg ladder.

\begin{align}
\begin{array}{cccc}
1 & 4 & 5 & 8 \\
2 & 3 & 6 & 7 \\
& & 10 &
\end{array}
\end{align}

https://doi.org/10.1088/1742-5468/aa657c
where $J_{nR}(n)$ is given by equations (46) and (47) with $J \mapsto J_R$, and the renormalized couplings are

$$J_R = J(1 + 2t_R)$$

$$J_{\perp R} = J(1 + 2t_{\perp})$$

The constant term

$$C = \frac{1}{4}(J_\perp + 2J) + 2Jt_{\parallel}^2 + J_\perp t_{\perp}^2.$$  \hspace{1cm} (58)

The renormalization parameters $t_{\parallel}$ and $t_{\perp}$ must be determined self-consistently from minimization of the mean-field Hamiltonian (55). In fact this was done before in a slightly more sophisticated mean-field treatment [31] with $t_{\parallel} = t_{2\parallel}$ (For earlier work on this mean-field approach, see [49].) The present mean-field Hamiltonian gives essentially the same results for the ground state energies, eigenvalues, gaps, renormalization parameters, and the phase diagram for the both staggered and columnar ladders [53] with some minor numerical differences. Keeping in mind that the mean-field parameters lie within the range $0 \leq (t_{\parallel}, t_{\perp}) \leq \frac{1}{2}$ for all possible values of the Hamiltonian’s parameters [53], it suffices for the goals of the present study to express results directly in terms of $J_R$ and $J_{\perp R}$. So in the following these renormalized couplings are treated as free model parameters of the effective Hamiltonian (55).

### 3.2. Spectra and winding numbers

#### 3.2.1. Staggered ladder

To further simplify the effective Hamiltonian (55) one can perform a canonical transformation mapping $(-1)^n c_\alpha^\dagger(n)c_\alpha(n + 1) \mapsto c_\alpha^\dagger(n)c_\alpha(n + 1)$. Introducing instead of $c_\alpha$ two distinct fermions residing on the even/odd sites $a_{\alpha, e/o}$ and Fourier transforming, we rewrite the Hamiltonian as

$$H_{\text{MF}} = NC + \sum_k \Psi^\dagger_k \mathcal{H}(k) \Psi_k,$$  \hspace{1cm} (59)

where the spinor

$$\Psi^\dagger_k = (d_{1, e}(k), d_{1, o}(k), d_{2, e}(k), d_{2, o}(k)),$$  \hspace{1cm} (60)

and the wavenumbers restricted to the Brillouin zone $k \in [-\pi/2, \pi/2]$. The $4 \times 4$ Hamiltonian of the staggered phase can be written as

$$\mathcal{H}_{\text{st}}(k) = \frac{1}{2} J_{\perp R} \Gamma_1 + J_R \cos k \Gamma_3 + J_R^\delta \sin k \Gamma_{35},$$  \hspace{1cm} (61)

where the Dirac matrices are defined in (5) and $\Gamma_{35} = -1 \otimes \sigma_2$. This Hamiltonian has four eigenvalues $\pm \epsilon^\pm$ where

$$\epsilon^\pm(k) = J_R \sqrt{\cos^2 k + \left(\delta \sin k \pm \frac{J_{\perp R}}{2J_R}\right)^2}.$$  \hspace{1cm} (62)
The spectrum has the gap
\[ \Delta = J_R \left| \delta \pm \frac{J_{\perp R}}{2J_R} \right| . \] (63)

Similar to the dimerized XY chain, two relevant perturbations cancel on the lines of quantum critical transition
\[ \frac{J_{\perp R}}{2J_R} = \pm \delta , \] (64)

where the staggered two-leg ladder becomes gapless. (A more refined mean-field treatment of this model’s phase diagram and comparison to numerical results can be found in [31].) From comparison of the eigenvalues of the staggered two-leg ladder becomes gapless. (A more refined mean-field treatment of this model’s phase diagram and comparison to numerical results can be found in [31].) From comparison of the eigenvalues of the effective Hamiltonian (62) and those of the XY chain in the alternating transverse field (6), we can infer the equivalence between the spectra of those models, and even match corresponding parameters:

| Staggered ladder | \( J_R \leftrightarrow J \) | \( \frac{1}{2} J_{\perp R} \leftrightarrow h \) | \( \delta \leftrightarrow \gamma \). |

Recall also that the fermionized Hamiltonian of the XY chain is equivalent to a topological superconductor (32). One can also establish equivalence and match parameters of the effective ladder Hamiltonian to those of the two-leg Kitaev ladder [12] or to the XY chain in a uniform transverse field and its dual model (for details of the latter, see next section). This is hardly surprising since all these models map onto quadratic fermionic Hamiltonians.

To calculate the topological winding number \( N_w \) in different phases of the staggered ladder we use a unitary transformation to bring the Hamiltonian (61) to the block diagonal form
\[ \hat{H}_t^{st}(k) = \begin{pmatrix} \hat{h}_+(k) & 0 \\ 0 & \hat{h}_-(k) \end{pmatrix} \] (65)

familiar from the context of topological insulators and superconductors [13], where \( \hat{h}_-(k) = \hat{h}_+(\kappa - k) \). Explicitly the operators \( \hat{h}_\pm(k) \) are
\[ \hat{h}_\pm(k) = i R J_R \cos k \sigma_1 - \left( J_R \delta \sin k \pm \frac{1}{2} J_{\perp R} \right) \sigma_2 , \] (66)

and their eigenvalues are given by equation (62). Using the eigenvalues (62) to yield the unit vector (40) and (41), we get from definition (42)
\[ N_{\pm} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk}{\cos^2 k + (\delta \sin k \pm \kappa)^2} \] (67)
where we denote

\[ \kappa \equiv \frac{J_{\perp R}}{2J_R}. \]  

(68)

One can check that \( N_+ = N_- \). The result for the winding number \( N_w \) can be cast in a simple form:

\[ N_w = N_- + N_+ = \frac{1}{2} [\text{sign}(\delta - \kappa) + \text{sign}(\delta + \kappa)]. \]  

(69)

We have shown the winding numbers for each phase on the phase diagram of the staggered ladder in figure 4. (Recall that we analyze the ladder with antiferromagnetic couplings.) The complimentary order parameter \( N_w \) changes by \( \Delta N_w = \pm 1 \) on the phase boundaries \( J_{\perp R}/2J_R = \pm \delta \). The sectors (A) and (C) correspond to the same topologically nontrivial phase.

3.2.2. Columnar ladder. The \( 4 \times 4 \) Hamiltonian of the columnar phase is

\[ \mathcal{H}^{col}(k) = \frac{1}{2} J_{\perp R} \Gamma_1 + J_R \cos k \Gamma_3 + J_R \delta \sin k \Gamma_4, \]  

(70)

and it has two two-fold degenerate eigenvalues \( \pm \epsilon \) where

\[ \epsilon(k) = J_R \sqrt{\cos^2 k + \delta^2 \sin^2 k + \left( \frac{J_{\perp R}}{2J_R} \right)^2}. \]  

(71)

Contrary to the staggered case, the columnar ladder is always gapped with the gap

\[ \Delta = J_R \sqrt{\delta^2 + \left( \frac{J_{\perp R}}{2J_R} \right)^2}, \]  

(72)

and no phase transition occurs on the whole plane \( (\delta, J_{\perp R}/2J_R) \). As in the case of staggered dimerization, from (6) and (71) we can establish equivalence between the spectra.
of the effective Hamiltonian of the columnar ladder and the dimerized XY model in the alternating transverse field:

\[
\begin{align*}
\text{Columnar ladder} & \quad \leftrightarrow \quad \text{XX chain (1), } \gamma = 0 \\
J_R & \quad \leftrightarrow \quad J \\
\frac{1}{2}J_{\perp R} & \quad \leftrightarrow \quad h \\
\delta & \quad \leftrightarrow \quad \delta.
\end{align*}
\]

The columnar phase is trivial not only in the sense that it is always gapped and noncritical, it is also trivial topologically, since no integer winding number can be assigned to this phase. Indeed, if we identify the opposite ends of the Brillouin zone, then the wavenumber \( k \) is defined in the space equivalent to the one-dimensional sphere \( S^1 \). This is also the space spanned by the two-component unit vector \( (40) \) and \( (41) \) of the staggered phase. The nontrivial winding number for that phase corresponds to the homotopy group \( \pi_1(S^1) = \mathbb{Z} \). As one can see from (71) the analogous unit vector is three-component in the columnar phase, and it spans the space \( S^2 \). The corresponding homotopy group is trivial \( \pi_1(S^2) = 0 \) and no integer winding number exists, since any closed loop on the sphere \( S^2 \) can be shrunk [54].

### 3.3. String order parameters

In order to analytically calculate the SOPs we rewrite the effective Hamiltonian (55) in terms of the Majorana operators

\[
a_n + ib_n \equiv 2c_n^\dagger.
\]

For the staggered configuration it reads (we drop the constant term \( NC \)):

\[
H^s = \frac{i}{4} J_R (1 - \delta) \sum_{l=1}^N (a_{2l-1}b_{2l+2} + a_{2l+2}b_{2l-1})
- \frac{i}{4} J_R (1 + \delta) \sum_{l=1}^N (a_{2l}b_{2l+1} + a_{2l+1}b_{2l})
- \frac{i}{4} J_{\perp R} \sum_{l=1}^N (a_{2l}b_{2l} + a_{2l+1}b_{2l+1}),
\]

where we labeled the sites of the ladder along the JWT path shown in figure 3. Using the JWT (48) in the Majorana representation

\[
\begin{pmatrix}
\sigma_n^x \\
\sigma_n^y \\
\sigma_n^z
\end{pmatrix} = 
\begin{pmatrix}
a_n \\
b_n \\
\prod_{l=1}^{n-1} [i a_l b_l]
\end{pmatrix}
\]

we can establish the following relations for the inverse JWT from Majorana fermions to the dual spin operators \( \tau \) (11) and (12):

\[
\sigma_n^x \sigma_{n+1}^x = ib_n a_{n+1} = \tau_{n-1}^x \tau_n^x
\]

\[
\sigma_n^y \sigma_{n+1}^y = -ia_n b_{n+1} = \tau_n^z.
\]

https://doi.org/10.1088/1742-5468/aa657c
In terms of the dual spin operators, the Hamiltonian (74) maps onto a sum of two decoupled exactly-solvable Hamiltonians defined on the even and odd sites of the snake-like string shown in figure 3:

\[
H^s_l = H^s_{el} + H^s_{o1}
\]

\[
H^s_{el} = \frac{1}{4} \sum_{l=1}^{N} (J_{\perp R} \tau^x_{2l-2} \tau^x_{2l} + J_R (1 - \delta) \tau^x_{2l-2} \tau^x_{2l+1} \tau^x_{2l+2} + J_R (1 + \delta) \tau^x_{2l})
\]

\[
H^s_{o1} = \frac{1}{4} \sum_{l=1}^{N} (J_R (1 + \delta) \tau^x_{2l-1} \tau^x_{2l+1} - J_R (1 - \delta) \tau^y_{2l-1} \tau^y_{2l+1} + J_{\perp R} \tau^x_{2l-1}).
\]

(78)

(79)

The QIM with the three-spin interactions (78) in the even sector of the Hamiltonian is discussed in the appendix. The odd sector (79) is the XY model in the transverse field with the anisotropy parameter \(\gamma = 1/\delta\).

In our earlier work [24] we took the definition of the SOP for ladders which was proposed by Kim and coworkers [8] and used also in other works [9]. In this paper the ladder SOP is calculated via the string-string correlation function as defined in section 2. One can check that our SOP is proportional to the SOP of Kim and other workers, but it is given by a simpler analytical formula. The even and odd SOPs defined below have a clear connection to the even/odd sectors (78) and (79) of the Hamiltonian.

To calculate the first SOP we define the string operator (16) along the path shown in figure 3. In so doing we obtain the even SOP:

\[
O^2_{x,e} = \lim_{N \to \infty} (-1)^N \left\{ \prod_{k=1}^{2N} \sigma^x_k \right\} = \langle \tau^x_0 \tau^x_{2N} \rangle.
\]

(80)

Note that the string chosen allows to incorporate in principle all spins of the ladder in the calculation of \(O_{x,e}\). Let us now choose another string for the operator (16) shown in figure 5. In such case we obtain the odd SOP:

\[
O^2_{x,o} = \lim_{N \to \infty} (-1)^{N-1} \left\{ \prod_{k=2}^{2N-1} \sigma^x_k \right\} = \langle \tau^x_1 \tau^x_{2N-1} \rangle.
\]

(81)

We calculated the order parameter of the three-spin Ising chain (A.1) in the appendix. Using the results presented there, we obtain:
\[ \mathcal{O}_{x,e} = \begin{cases} \left( \frac{\kappa^2 - \delta^2}{1 + \kappa^2 - \delta^2} \right)^{1/8} & |\kappa\delta| \geq 1 \\ 0 & |\kappa\delta| < 1 \end{cases} \] (82)

where \( \kappa \) is given by (68). The odd SOP can be calculated from the results of Barouch and McCoy [44]

\[ \mathcal{O}_{x,o} = \begin{cases} 0 & |\kappa\delta| > 1 \\ \sqrt{\frac{2}{1 + \delta}} \left[ \delta^2 - \kappa^2 \right]^{1/8} & |\kappa\delta| \leq 1. \end{cases} \] (83)

In agreement with earlier results [8, 9, 24] we find that the even and odd SOPs are mutually exclusive. They detect hidden \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) symmetry breaking [22] and ordering in the even or odd sectors of the dual Hamiltonian. Note that contrary to its even counterpart, the string for the odd SOP misses two sites at the opposite ends of the ladder. The order probed by \( \mathcal{O}_{x,o} \) is topologically nontrivial, since it corresponds to nonzero winding numbers. According to conventional wisdom [13], there are zero Majorana modes at the ladder’s ends in this phase. We indicate the order parameters on the phase diagram of the staggered ladder in figure 4. From the behavior of the gap (63) and SOPs (82) and (83) near the lines of phase transitions \( J_{L,R} / 2J_R = \pm \delta \) we find the critical indices \( \nu = 1 \) and \( \beta = 1/8 \) of the 2D Ising universality class, but now their values are the artifact of the effective free-fermionic approximation.

Analysis of the columnar phase is done similarly. The sites of the columnar ladder are labelled along the JWT path in figure 3. The Majorana representation of the columnar Hamiltonian (55) can be mapped via transformations (76) and (77) onto dual spins. Again the dual Hamiltonian splits into even and odd sectors \( H_{\text{col}} = H_{\text{e}}^{\text{col}} + H_{\text{o}}^{\text{col}} \), where the odd term is the Hamiltonian of the dimerized XX (\( \gamma = 0 \)) chain in the alternating field (1), while the even term corresponds to its dual model which is the Ising chain in the alternating transverse field with the three-spin interactions. Since these models are always gapped (see Equation (72)) and do not undergo any phase transitions, we will not discuss the properties of the columnar phase in more detail.

Note a very remarkable property of the effective Hamiltonian (55): for both dimerization patterns it decouples into the reciprocally dual even and odd sectors \( H_{\text{str/col}} = H_{\text{o}}^{\text{str/col}} + H_{\text{e}}^{\text{str/col}} \), i.e. these Hamiltonians map onto each other \( H_{\text{o}}^{\text{str/col}} = H_{\text{e}}^{\text{str/col}} \) under the duality transformation (11) and (12).

4. Conclusion

In this work we studied quantum phase transitions in the antiferromagnetic dimerized spin-\( \frac{1}{2} \) XY chain and two-leg ladders. The common feature of these models is a subtle interplay of relevant perturbations each of which results in a gap creation in the spectrum. There are however lines of quantum criticality where those perturbations cancel.
and the models become gapless. We analyzed the properties of the different phases of these models.

The dimerized $XY$ chain is equivalent to the noninteracting fermions and is mapped by a duality transformation onto two 1D QIMs residing separately on even and odd sites of the dual lattice. The SOPs for the original chain are mapped onto the local (Landau) order parameters of these dual QIMs and thus calculated exactly. As follows from the duality mapping, the dimerized $XY$ model possesses the hidden $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ symmetry, and nonzero SOPs are due to spontaneous breaking of the $\mathbb{Z}_2$ symmetry in the even or odd sectors of the dual Hamiltonian.

The phase diagram of the model contains topologically trivial phases where the even and odd SOPs coexist along with the conventional long-range order (magnetization). There are also topologically nontrivial phases without conventional order where only the even or odd nonlocal SOPs exist. In addition the winding numbers are calculated. Their values are consistent with the topologically trivial or nontrivial nature of each phase.

These analyses and results are quite similar to those for the cluster Ising chain models which attracted attention in recent literature [55–59] in the context of quantum computation. These exactly solvable models with multi-spin interactions are equivalent to free fermions, and both the conventional and nonlocal SOPs they manifest are readily calculated, similarly to what we have done for the dimerized $XY$ chain.

To extend the framework shown to work so well for the exactly solvable models, the fermionization of the dimerized two-leg ladders was followed up by a mean-field approximation. The resulting Hamiltonian was then treated as a new effective free-fermionic model. The predictions of this effective Hamiltonian for the critical properties and phase diagram of the ladders with the staggered or columnar dimerizations are in agreement to what is known from earlier work. From the Majorana representation of the effective Hamiltonian we constructed its transformation to new dual spins. The staggered effective Hamiltonian maps onto the sum of the decoupled even and odd dual exactly-solvable Hamiltonians. The even term is the quantum Ising chain with three-spin interaction, while the odd term is the $XY$ chain in a uniform transverse field. Interestingly, the even and odd sectors map onto each other under the duality transformations. The SOPs of the original spin ladders are calculated as the local order parameters in the dual models. We defined two mutually exclusive (even and odd) SOPs which detect breaking of the hidden $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ ladder symmetry. One of those SOPs probes topologically trivial phase, while another corresponds to the topologically nontrivial phase. This statement is also corroborated by the calculation of the winding numbers for those phases. Similar mappings and dualities are found for the columnar dimerized ladder, but this case is not very interesting, since the columnar ladder is always gapped and does not undergo phase transitions.

Our main result is the framework to treat nonlocal orders and hidden symmetries which unifies the key elements of the Landau paradigm with the new concept of topological order. This unified framework can be straightforwardly applied for such important physical systems as spin chain and ladders, topological insulators and superconductors. As we have shown throughout our analysis, the fermionic Hamiltonians (exact or effective) we used for the chain or ladder models are equivalent to tight-binding Hamiltonians of topological materials. To probe the topological order in the latter one
needs the Majorana string operators which can be straightforwardly mapped onto the local parameters via the duality transformations we applied in this work.

One more point needs to be emphasized. In the present study we did not need to build up the full-size formalism based on the Ginzburg-Landau effective potential written in terms of the dual spins ($\tau$), since the quadratic Majorana Hamiltonian (exact, as in the case of the $XY$ chain, or an effective mean-field approximation, as in the case of ladders) maps onto exactly solvable dual 1D spin models. So, the key notions of the Landau framework: order parameters, spontaneous symmetry breaking, etc, are easily identified and calculated from those models without any additional approximations. That is why we obtain the critical indices of the 2D Ising universality class (e.g. $\beta = 1/8$), but not the mean-field ones ($\beta = 1/2$) which a naïve Landau theory would predict.

There are other topologically-ordered lattice systems, including those in higher dimensions which can be mapped via various duality transformations onto the systems with the Landau local orders [14, 56, 60]. We are currently working on applying the present approach to the spin-$\frac{1}{2}$ Heisenberg $n$-leg ladders and tubes. The key step is to obtain the effective mean-field quadratic fermionic Hamiltonian consistent with the $\pi$-flux theorem [51]. The transformations to the dual spins maps the Majorana fermion models onto exactly solvable Ising chains with multi-spin interactions. The local order parameters of these dual models (i.e. the SOPs of the original spin ladders or tubes) can be calculated via Toeplitz determinants [41, 67]. Another interesting developments of the proposed formalism would be its extension for the exotic thermal phases in classical spin models, as, e.g. the algebraically-ordered topological floating phase occurring in some frustrated 2D Ising models [61, 62].

Acknowledgments

We thank V Lahtinen, F Mila, Z Nussinov, and J H H Perk for correspondence and bringing important papers to our attention. Financial support from NSERC (Canada) and the Laurentian University Research Fund (LURF) is gratefully acknowledged. TP thanks the Ontario Graduate and the Queen Elizabeth II Fellowships for support.

Appendix. Transverse Ising chain with three-spin interactions

The Hamiltonian of the Ising chain in transverse field with three-spin interactions is defined as:

$$H = \sum_{i=1}^{N} (J_{x} \sigma_{x}^{i} \sigma_{x}^{i+1} + J_{z} \sigma_{z}^{i} \sigma_{z}^{i-1} \sigma_{z}^{i+1} + h \sigma_{z}^{i}).$$

(A.1)

The duality transformations (11) and (12) map the above Hamiltonian onto the anisotropic $XY$ chain in transverse field [63]. Some basic properties of the model (A.1) were analyzed in [64] and later in [46, 65]. The Jordan–Wigner transformation of (A.1) yields the free-fermionic model (up to an unimportant constant term):

$$H = \sum_{q} (2A(q)c_{q}^{\dagger}c_{q} - iB(q)(c_{q}^{\dagger}c_{-q} + c_{q}c_{-q})).$$

(A.2)
where

\[ A(q) \equiv h + J \cos q - J_3 \cos 2q \quad \text{(A.3)} \]

\[ B(q) \equiv J \sin q - J_3 \sin 2q, \quad \text{(A.4)} \]

and \( c_q^+, c_q \) are the Fourier transforms of the lattice fermion operators \( c_i^+, c_i \). The Bogoliubov transformation

\[ \eta_q = \cos \Theta_q c_q - i \sin \Theta_q c_{-q}^\dagger \quad \text{(A.5)} \]

with the Bogoliubov angle \( \Theta_q \) defined as \( \tan 2\Theta_q = B(q)/A(q) \), diagonalizes the Hamiltonian. The spectrum of the quasiparticles \( \eta_q, \eta_q^\dagger \) is

\[ \varepsilon(q) = \sqrt{A(q)^2 + B(q)^2} = \sqrt{h^2 + J^2 + J_3^2 + 2J(h - J_3) \cos q - 2hJ_3 \cos 2q}. \quad \text{(A.6)} \]

The model is gapless along the lines of quantum criticality \( h = J_3 \pm J \), and for the transverse fields within the range

\[ J_3 - J < h < J_3 + J \quad \text{(A.7)} \]

the model manifests the local long-range order [64] (magnetization) \( m_x = 0 \). Our goal is to calculate the order parameter \( m_x \). We will use the classical approaches [41, 44, 66]. More modern treatments and references can be found, e.g. in [48, 67]. The fermionic correlation function

\[ G_{l-m} \equiv \langle h_l a_m \rangle \quad \text{(A.8)} \]

is defined in terms of the site Majorana operators (73). It can be calculated as the Fourier transform

\[ G_n = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{in\varphi} G(\varphi) \quad \text{(A.9)} \]

of the generating function

\[ G(\varphi) = \frac{A(\varphi) + iB(\varphi)}{A(\varphi) - iB(\varphi)} = \left[ \frac{(e^{i\varphi} - \alpha_+)(e^{i\varphi} - \alpha_-)}{(e^{-i\varphi} - \alpha_+)(e^{-i\varphi} - \alpha_-)} \right]^{1/2} \quad \text{(A.10)} \]

where

\[ \alpha_\pm \equiv \frac{J \pm \sqrt{J^2 + 4hJ_3}}{2J_3}. \quad \text{(A.11)} \]

The spin-spin correlation function \( \langle \sigma_1^x \sigma_{N+1}^x \rangle \) is given by the \( N \times N \) Toeplitz determinant: [41]

\[ \langle \sigma_1^x \sigma_{N+1}^x \rangle = \begin{vmatrix} G_{-1} & \ldots & G_{-N} \\ \vdots & \ddots & \vdots \\ G_{N-2} & \ldots & G_{-1} \end{vmatrix} \equiv D_N. \quad \text{(A.12)} \]
Note that the singular points of \( G(\varphi) \) satisfy \( \alpha_+ \alpha_- = -h/J_3 \), and within the magnetically ordered range (A.7) the first root is bound \(-1 < \alpha_- < 0\), thus for \( J_3/h < 1 \) the inequality \( 0 < \alpha_-^{-1} < 1 \) for the second root holds. The generating function can be written as \( G(\varphi) = e^{i\varphi} \tilde{G}(\varphi) \) with

\[
\tilde{G}(\varphi) = \left[ \frac{(e^{-i\varphi} - \alpha_+^{-1})(e^{i\varphi} - \alpha_-)}{(e^{i\varphi} - \alpha_+^{-1})(e^{-i\varphi} - \alpha_-)} \right]^{1/2}.
\]

Then

\[
G_n = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i(n+1)\varphi} \tilde{G}(\varphi) = \tilde{G}_{n+1},
\]

and the determinant (A.12) can be written in terms of \( \tilde{G}_n \):

\[
D_N = \begin{vmatrix} \tilde{G}_0 & \ldots & \tilde{G}_{N-1} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{N-1} & \ldots & \tilde{G}_0 \end{vmatrix} \equiv \tilde{D}_N.
\]

Since zeros/singularities of the generating function \( \tilde{G}(\varphi) \) \(|\alpha_+^{-1}, |\alpha_-|\) < 1 lie inside the unit circle \( z = e^{i\varphi} \) on the complex plane, the conditions for Szegö’s theorem are satisfied [67], and it can be applied for calculation of the limit \( N \to \infty \) of the Toeplitz determinant \( \tilde{D}_N \). The latter result is well known and reads: [67]

\[
\lim_{N \to \infty} \tilde{D}_N = \left[ \frac{(1 - \alpha_-^2)(1 - \alpha_+^2)}{(1 - \alpha_-/\alpha_+^2)} \right]^{1/4}.
\]

Combining all formulae together, we obtain the sought result:

\[
m_z^2 = \lim_{N \to \infty} \langle \sigma_z^x \sigma_{N+1}^x \rangle = \left[ \frac{J_3^2 - (J_3 - h)^2}{J_3^2 + 4h J_3} \right]^{1/4}.
\]

The above formula for magnetization is analytic in the whole locally ordered range (A.7) across the line \( h = J_3 \). The validity of the formula for the order parameter (A.17) was checked by direct numerical calculation of the determinant (A.12) using Mathematica.

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https://doi.org/10.1088/1742-5468/aa657c
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