Stiff Matter Solution in Brans-Dicke Theory and The General Relativity Limit

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Abstract

Generally the Brans-Dicke theory reduces to General Relativity in the limit \( \omega \to \infty \) if the scalar field goes as \( \phi \propto 1/\omega \). However, it is also known that there are examples with \( \phi \propto 1/\sqrt{\omega} \) that does not tend to GR. We discuss another case: a homogeneous and isotropic universe filled with stiff matter. The power of time dependence of these solutions do not depend on \( \omega \), and there is no General Relativity limit even though we have \( \phi \propto 1/\omega \). A perturbative and a dynamical system analysis of this exotic case are carried out.

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1 Introduction

General Relativity (GR) is a very successful theory to describe gravitational interaction. It is a theoretically consistent and experimentally tested theory. So far, GR has been confirmed by every experiment in the solar system [1-4] and astrophysical phenomena such as the emission of gravitational waves by binary systems and it is in accordance with the bounds on the velocity of gravitational waves [5, 6]. However, there are at least three reason to seriously consider alternative theories of gravity. The necessity of the dark sector (dark energy and dark matter) on the standard cosmological model; the theoretical motivation to unify the gravitational interaction with the quantum interactions in a single theoretical framework; and the epistemological fact that alternative theories can be used to highlight the intrinsic properties of GR by showing how it could be otherwise.

The prototype of alternative theory of gravity is Brans-Dicke theory (BD). Historically, it is one of the most important alternative to the standard General Relativity theory, which was introduced by C. Brans and R. H. Dicke [7] as a possible implementation of Mach’s principle in a relativistic theory. Solar System time-delay experiments set a lower bound in the dimensionless parameter of \( \omega > 500 \) [1, 2], which means that BD has to be observationally indistinguishable from GR in the solar system dynamics. Notwithstanding, there is phenomenological applications for BD in cosmology and indeed it has received recently much attention of the scientific community [8-18].

In the present work, we display a particular solution of BD with matter content described by a stiff matter barotropic perfect fluid. This is a very interesting solution with exotic characteristics revealing some of the new features, for better or worst, that one can expect to find in BD-like alternative theories of gravity [19-22]. In particular, the time evolution of the system is independent of the value of the parameter \( \omega \). The evolution of the perturbations has only growing modes, which is also another distinct feature of this solution. In addition, the scale factor evolution behaves as \( a \propto t^{1/2} \), typical of radiation dominated epoch in GR, hence this configuration might be used to study the early universe. Recently, this period of the universe filled with stiff matter was also studied in the context of \( f(R) \) theories [23].

It is argued in the literature that BD approaches GR in the \( \omega \to \infty \) limit [24]. The crucial point behind this argument is that when the parameter \( \omega \gg 1 \), the field
equations seem to show that $\Box \varphi = O(\frac{1}{\omega})$ and hence

$$\varphi = \frac{1}{G_N} + O\left(\frac{1}{\omega}\right)$$  \hspace{1cm} (1)

$$G_{\mu \nu} = 8\pi G_N T_{\mu \nu} + O\left(\frac{1}{\omega}\right)$$  \hspace{1cm} (2)

where $G_N$ is Newton’s gravitational constant, $G_{\mu \nu}$ is the Einstein tensor and assume natural unit where $c = 1$. However, there are some examples [25–40] where exact solutions can not be continuously deformed into the corresponding GR solutions by taking the $\omega \to \infty$ limit. Their asymptotic behavior differs exactly because these solutions do not decay as Eq. (1) but instead as

$$\varphi = \frac{1}{G} + O\left(\frac{1}{\sqrt{\omega}}\right)$$  \hspace{1cm} (3)

Our particular solution, to be developed in section 4, has the novelty of having the appropriate asymptotic behavior given by Eq. (1) (see Eq. (31)) but no GR limit.

The paper is organized as follows. In section 2 we briefly describe the system and its equation of motion. In section 3 we review the general solution discovered by Gurevich et al. In section 4 and 5 we present our solution and develop the perturbation over this specific background, respectively. In section 6 we perform a dynamical system analysis of the system and finally in section 7 we end with some final remarks.

2 The classical equations of motion

In Brans-Dicke theory the scalar field is understood as part of the geometrical degrees of freedom. This theory has a non-minimal coupling between gravity and the scalar field. The action reads

$$S = \int \mathrm{d}x^4 \sqrt{-g} \left[ \frac{1}{16\pi} \left( \frac{\phi R - \frac{\omega}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi}{\phi} \right) + \mathcal{L}_m \right],$$  \hspace{1cm} (4)

where $\mathcal{L}_m$ is the ordinary matter and $\omega$ is the scalar field coupling constant. Variation of the action Eq. (4) with respect to the metric and the scalar field give,
respectively, the following fields equations

\[ G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \omega \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} \left( \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right), \quad (5) \]

\[ \Box \phi = -\frac{\phi}{2\omega} R + \frac{1}{2\phi} \left( \nabla^\alpha \phi \nabla_\alpha \phi \right) = \frac{8\pi}{3 + 2\omega} T, \quad (6) \]

where we have used the trace of Eq. (5) in the last step of Eq. (6). The action Eq. (4) is diffeomorphic invariant and since all variables are dynamic fields we have conservation of energy-momentum, i.e.

\[ \nabla_\mu T^{\mu\nu} = 0. \quad (7) \]

We shall consider the matter content described by a perfect fluid such that the energy-momentum tensor is

\[ T_{\mu\nu} = (\rho + p) u^\mu u^\nu - pg^{\mu\nu}, \quad (8) \]

and a barotropic equation of state \( p = \alpha \rho \) with \( 0 \leq \alpha \leq 1 \). The equation of state parameter \( \alpha \) is bounded from above in order to avoid superluminal speed of sound. In the extreme case, \( \alpha = 1 \), the speed of sound equals the speed of light, which corresponds to stiff matter. This equation of state was first proposed by Zeldovich as an attempt to describe matter in extremely dense states such as in the very early universe.

We shall restrict our analysis to the Friedmann-Lemaître-Robertson-Walker (FLRW) universes where the metric has a preferred foliation given by homogeneous and isotropic spatial sections. In this particular spherical coordinate system, the line element has the form

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2(\theta) d\phi \right) \right], \quad (9) \]

where \( a(t) \) is the scale factor function and \( k = 0, \pm 1 \) defines the spatial section curvature.

A stiff matter fluid has equation of state \( p = \rho \). Thus, conservation of the energy-momentum tensor in a FLRW universe implies \( \rho = \rho_0 (a_0/a)^6 \) with \( \rho_0 \) and \( a_0 \) two constants of integration. The FLRW flat case has been analyzed in a quite general form in Ref. [41]. However, not all solutions were exploited there. In particular, we intend to describe a very peculiar solution associated with
this equation of state. After integrating the conservation of energy-momentum
equation, the two remaining independent equations become
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi\rho_0}{3\phi} \left( \frac{a_0}{a} \right)^6 + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\dot{\phi}} \right)^2 - \frac{\dot{a}}{a} \frac{\dot{\phi}}{\dot{\phi}},
\]
\[ (10) \]
\[
\dot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} = -\frac{16\pi\rho_0}{(3 + 2\omega)} \left( \frac{a_0}{a} \right)^6,
\]
\[ (11) \]
where a dot denotes differentiation with respect to the cosmic time \( t \).

3 Gurevich’s Families of Solutions

In this section we present the general solutions of scale factor and the scalar field
for the FLRW flat case in the Brans-Dicke theory obtained by Gurevich et al. [41]. We shall follow closely their presentation but adapting specifically for the
stiff matter case. Gurevich et al. obtained a class of flat space solutions for the
equation of state \( P = \alpha \rho \), where \( 0 \leq \alpha \leq 1 \). However, their analysis focused only
on \( 0 \leq \alpha < 1 \) since \( \alpha = 1 \) has distinct behavior as can be seen below.

There are two families of solutions depending on the value of \( \omega \). For \( \omega < -\frac{3}{2} \)
the general solutions for the scale factor and scalar field \( \phi \) are
\[
a = a_0 \left[ (\theta + \theta_-)^2 + \theta_+^2 \right]^{(1+(1-\alpha)\omega)/A} e^{\pm 2f(\theta)/A},
\]
\[ (12) \]
\[
\phi = \phi_0 \left[ (\theta + \theta_-)^2 + \theta_+^2 \right]^{(1-3\alpha)/A} e^{\pm 6(1-\alpha)f(\theta)/A},
\]
\[ (13) \]
where
\[
f(\theta) = \sqrt{\frac{2|\omega| - 3}{3}} \arctan \left( \frac{\theta + \theta_-}{\theta_+} \right)
\]
\[ (14) \]
\[
A = 2 \left( 2 - 3\alpha \right) + 3\omega (1 - \alpha)^2
\]
\[ (15) \]
and \( \theta_+ > 0 \) and \( \theta_- \) are constants of integration. The variable \( \theta \) is connected with
cosmic time through the relation \( dt = a^{3\alpha} d\theta \). For the stiff matter case, the solutions
(12)-(13) simplify to
\[
\begin{align*}
\dot{a} &= a_0 \left[ (\theta + \theta_-)^2 + \theta_+^2 \right]^{\frac{1}{2}} \exp \left\{ \pm \sqrt{\frac{2|\omega| - 3}{3}} \arctan \left( \frac{\theta + \theta_-}{\theta_+} \right) \right\},
\end{align*}
\]
\[ (16) \]
\[
\phi = \phi_0 \left[ (\theta + \theta_-)^2 + \theta_+^2 \right].
\]
\[ (17) \]
Note that the scalar field dynamics does not depend on $\omega$ anymore and the scale factor has a mild dependence on this parameter. The asymptotic behavior of the Eq.s (16)-(17) for $\theta \to \pm \infty$ are

$$a(\theta) \propto \exp\left(\frac{\epsilon \pi}{2} \sqrt{\frac{2|\omega| - 3}{3}} \right) \frac{1}{\theta},$$  
(18)

$$\phi(\theta) \propto \theta^2,$$  
(19)

where $\epsilon = \pm 1$ for $\theta \to -\infty$ and $\epsilon = \pm 1$ for $\theta \to +\infty$. In this asymptotic behavior, the cosmic time goes as $t \propto \theta^{-2}$, hence, the scale factor Eq. (18) goes as $a \propto t^{1/2}$ in both asymptotic limits $t \to \pm \infty$. Similarly, the asymptotic behavior of the scalar field is $\phi \propto \theta^2 \propto t^{-1}$. The universe behaves as a GR radiation dominated phase in the remote past and in the far future with the scalar field decreasing inversely proportional to the cosmic time.

For the sake of completeness, we also display the general solutions for $\omega > -\frac{3}{2}$

$$a = a_0 (\theta - \theta_+)^{\omega/3A_+} (\theta - \theta_-)^{\omega/3A_-},$$  
(20)

$$\phi = \phi_0 (\theta - \theta_+)^{(1 \pm \sqrt{3+2\omega}/3)/A_+} (\theta - \theta_-)^{(1 \pm \sqrt{3+2\omega}/3)/A_-},$$  
(21)

$$A_{\pm} = 1 + \omega(1 - \alpha) \pm \sqrt{(3 + 2\omega)/3}$$  
(22)

where $\theta_+$ and $\theta_-$ are again constants of integration but now $\theta_+ > \theta_-$. The solutions for the stiff matter are

$$a = a_0 [\theta - \theta_+]^{\omega/[3(1 \pm \sqrt{3+2\omega}/3)]} (\theta - \theta_-)^{\omega/[3(1 \pm \sqrt{3+2\omega}/3)]},$$  
(23)

$$\phi = \phi_0 (\theta - \theta_+) (\theta - \theta_-).$$  
(24)

The asymptotic behavior of the Eq.s (23)-(24) show that for $|\theta| \gg \theta_+$ we have $a \propto \theta^{-1} \propto t^2$ and $\phi \propto \theta^2 \propto t^{-1}$. Thus, we have the same asymptotic behavior as the $\omega < -\frac{3}{2}$ given by Eq.s (18)-(19).

Another interesting asymptotic limit is given by finite time but allowing the BD parameter to increase boundlessly. Eq.s (12)-(13) show that the limit $\omega \to \infty$ depend crucially if $\alpha \neq 1$ or not. For $\alpha \neq 1$ and $|\omega| \to \infty$ we have $A = 3\omega(1 - \alpha)^2$, $f \propto 1/\sqrt{\omega} \arctan (1 + \theta/\theta_+) \to 0$, hence

$$\lim_{\omega \to \infty} a = a_0 [(\theta + \theta_-)^2 + \theta_+^2]^{1/3(1-\alpha)},$$  
(25)

$$\lim_{\omega \to \infty} \phi = \phi_0,$$  
(26)
Therefore, if $\alpha \neq 1$, independently of the sign of $\omega$, both families of solutions asymptotically approach GR. However, that is not the case for stiff matter. For $\alpha = 1$, Eq.s (16)-(17) and Eq.s (23)-(24) show that $\phi$ does not go to a constant and the scalar field does not go to its GR limit. Indeed, the $\alpha = 1$ case has to be studied separately, which is what we shall analyze in the next section.

### 4 Distinct Exact Power Law Solution

Let us study the dynamic for FLRW universe filled with a stiff matter perfect fluid. Instead of using Gurevich’s et al. family of solutions, we propose a power law ansatz such that

$$a = a_0 t^r, \quad \phi = \phi_0 t^s. \quad (27)$$

Equating the power in the Klein-Gordon Eq. (11), it is easy to check $6r + s = 2$. Furthermore, the coefficients of Eq.s (10)-(11) imply

$$3r^2 + 3rs - \frac{\omega}{2} s^2 = \frac{8\pi \rho_0}{\phi_0} - \frac{3k}{a_0^2}, \quad (28)$$

$$s(s - 1) + 3rs = -\frac{16\pi \rho_0}{(3 + 2\omega)\phi_0}. \quad (29)$$

Using the previous relation for $r$ and $s$ and combine these two equations, we find

$$r = \frac{1}{2} \left( 1 \pm \frac{k}{a_0} \right), \quad s = -\left( 1 \pm \frac{3k}{a_0} \right). \quad (30)$$

for spatial curvature $k = 0, -1$. For positive spatial curvature ($k = 1$) there is no power law solution like Eq. (27). Note also that the equations require the constraint

$$\phi_0 = -\frac{32\pi}{(3 + 2\omega)\rho_0}. \quad (31)$$

Therefore, in order to have an attractive gravity with positive energy density, i.e. $\rho_0$ and $\phi_0 > 0$, we must have $\omega < -\frac{3}{2}$. This power law solution has a distinct behavior compared with Gurevich’s solution. During the whole evolution the scale factor mimics a GR radiation dominated expansion, namely $a \propto t^{1/2}$, while the scalar field is always decreasing inversely proportional to the cosmic time $\phi \propto t^{-1}$.
There is no GR limit in the sense that the evolution does not depend on the BD parameter $\omega$. Furthermore, Eq. (31) seems to show that the limit $\omega \to \infty$ is not even well defined since the gravitation strength is inversely proportional to the scalar field, i.e $G \sim \phi^{-1}$. Notwithstanding, this power law solution is completely consistent for finite values of $\omega$.

In the next section we shall study the cosmological perturbations over this particular solution. From now on, we shall consider only the flat spatial section case $k = 0$.

5 Cosmological Perturbation

Consider the background solution found above for a flat FLRW universe filled with stiff matter in BD theory, i.e.

$$a = a_0 t^{1/2}, \quad \phi = \phi_0 t^{-1}. \quad (32)$$

In Ref. [42] the general perturbed equations for a fluid with equation of state of the type $p = \alpha \rho$, with $\alpha$ constant, has been established for the Brans-Dicke cosmology. The full perturbed dynamical system is given by the perturbed version of Eqs. (5)-(6). However, the evolution of the matter density perturbation can be analyzed using only the perturbed version of the time-time Einstein’s equations, the Klein-Gordon, and the conservation of energy-momentum tensor.

The metric perturbation is defined as $h_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(0)}$, where $g_{\mu\nu}^{(0)}$ is given by the FLRW solution with Eq. (32) and $k = 0$. Following Ref. [42], we adopt the synchronous gauge where $h_{0i} = 0$. It is straightforward to calculate the perturbed Ricci tensor, which has time-time component given by

$$\delta R_{00} = \frac{1}{a^2} \left[ h_{kk} - 2 \frac{\dot{a}}{a} h_{kk} + 2 \left( \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right) h_{kk} \right].$$

The time-time component of the perturbed energy-momentum tensor and its trace read

$$\delta T^{00} = \delta \rho , \quad \delta T = \delta \rho - 3\delta \rho. \quad (33)$$

Similarly, the perturbation of the scalar field is defined as $\delta \phi(x) = \phi(x) - \phi^{(0)}(t)$. The d’Alembertian of the scalar field is

$$\delta \Box \phi = \delta \dddot{\phi} + a \dot{\delta} h_{kk} \dot{\phi} - \frac{1}{2a^2} \dddot{\phi} h_{kk} \dot{\phi} + 3 \frac{\dot{a}}{a} \delta \phi - \frac{1}{a^2} \nabla^2 \delta \phi . \quad (34)$$
It is convenient to define new variables. In particular, we define the usual expression for the density contrast, a similar version for the perturbation of the scalar field, the divergence of the perturbation of the perfect fluid’s velocity field \((\delta u')\), and a normalized version of the metric perturbation. They are defined, respectively, as

\[
\delta = \frac{\delta \rho}{\rho}, \quad \lambda = \frac{\delta \phi}{\phi}, \quad U = \delta u', \quad h = \frac{h_{kk}}{a^2}.
\] (35)

Using (35) and decomposing them in Fourier modes \(n\), the time-time BD and the Klein-Gordon equations read respectively

\[
\ddot{h} + \frac{1}{2}H\dot{h} = \frac{8\pi \rho}{\phi} \left( \frac{2 + \omega + 3\alpha(1 + \omega)}{3 + 2\omega} \right) (\delta - \lambda) + \dot{\lambda} + 2(1 + \omega)\frac{\dot{\phi}}{\phi} \lambda,
\] (36)

\[
\ddot{\lambda} + \left( 3H + 2\frac{\dot{\phi}}{\phi} \right)\dot{\lambda} + \left[ \frac{n^2}{a^2} + \frac{\dddot{\phi}}{\phi} + 3H\frac{\ddot{\phi}}{\phi} \right] \lambda = \frac{8\pi(1 - 3\alpha)\rho}{(3 + 2\omega)\phi} \delta + \frac{\dot{\phi}}{\phi} \frac{\dot{h}}{2}.
\] (37)

In addition, the conservation of energy-momentum tensor decompose in two equations, namely

\[
2\delta - (1 + \alpha) \left( h - 2U \right) = 0,
\] (38)

\[
(1 + \alpha) \left( \dot{U} + (2 - 3\alpha)HU \right) = \alpha \frac{n^2}{a^2} \delta.
\] (39)

where again \(n\) represents the Fourier mode. In the long wavelength limit, \(n \to 0\), Eq. (39) shows that the perturbation of the four velocity decouples. For a stiff matter fluid, \(\alpha = 1\), it becomes a growing mode with

\[
U \propto a.
\] (40)

This growing mode has nothing to do with BD’s extra scalar degree of freedom. Eq.s (38)-(39) come from perturbing the conservation of the energy-momentum Eq. (7), which is identical in GR. For equation of state lower than \(\alpha < 2/3\), such as for radiation, we have a decaying mode and can ignore it by setting \(U = 0\) that in turn implies \(\delta = h\). In contrast, in our case \(\alpha = 1\) the growing mode together with Eq. (38) implies

\[
\delta = h - \frac{4}{3}U_0 t^{3/2},
\] (41)
with $U_0$ a constant of integration. For this reason, we will retain this inhomogeneous term. The resulting dynamical system in the long wavelength limit reads

$$\ddot{\delta} + 2H\dot{\delta} = \frac{16\pi\rho}{\phi} \left( \frac{5 + 4\omega}{3 + 2\omega} \right) (\delta - \lambda) + 2\dot{\lambda} + 4(1 + \omega) \frac{\dot{\phi}}{\phi} \dot{\lambda}, \quad (42)$$

$$\ddot{\lambda} + \left( 3H + 2\frac{\dot{\phi}}{\phi} \right) \dot{\lambda} + \left( \frac{\ddot{\phi}}{\phi} + 3H \frac{\dot{\phi}}{\phi} \right) \lambda = -\frac{16\pi\rho}{(3 + 2\omega)\phi} (\delta - \lambda) + \frac{\dot{\phi}}{\phi} \ddot{\phi} \quad . \quad (43)$$

Using the background expressions (Eq. (32)), the long wavelength limit ($\eta \to 0$), and Eq. (41), the dynamical system simplify to

$$\frac{\dot{h}}{2} + \frac{\dot{h}}{2t} + \frac{(5 + 4\omega)}{4t^2} h = \dot{\lambda} - \frac{2(1 + \omega)\dot{\lambda}}{t} + \frac{(5 + 4\omega)}{4t^2} \dot{\lambda} + \frac{(5 + 4\omega)U_0}{3t^{1/2}}, \quad (44)$$

$$\frac{\dot{h}}{2} + \frac{h}{2t^2} = \dot{\lambda} - \frac{\lambda}{2t} + \frac{\lambda}{2t^2} + \frac{2U_0}{3t^{1/2}}. \quad (45)$$

These equations admit a solution under the form,

$$h = h_0 \eta^r + f_0 \eta^\frac{3}{2}, \quad \lambda = \lambda_0 \eta^r + g_0 \eta^\frac{3}{2}, \quad (46)$$

with $\lambda_0$, $h_0$, $f_0$ and $g_0$ are constants. Equating the power in the time parameter and the coefficients of the polynomials, we obtain a set of equations connecting these different constants of integration, namely

$$\left\{ r^2 + \frac{(5 + 4\omega)}{2} \right\} h_0 = 2 \left\{ r^2 - (3 + 2\omega)r + \frac{(5 + 4\omega)}{4} \right\} \lambda_0, \quad (47)$$

$$(1 - r)h_0 = 2 \left\{ r^2 - \frac{3}{2}r + \frac{1}{2} \right\} \lambda_0, \quad (48)$$

$$f_0 = -\frac{2}{3}(3 + 4\omega)g_0 + \frac{8}{27}(5 + 4\omega)U_0, \quad (49)$$

$$f_0 = -2g_0 + \frac{8}{3}U_0. \quad (50)$$

The constants $h_0$ and $\lambda_0$ give the homogeneous modes, while $f_0$ and $g_0$ give the inhomogeneous modes associated with the growing mode $U$. The homogeneous mode admit four power solution given by $r = 0, -1, \frac{1}{2}$ and 1. The solutions corresponding to $r = 0, -1$ are connected with the residual gauge freedom typical of the synchronous gauge. A remarkable novelty is that the physical solutions correspond only to growing modes. In fact, these modes appear also in the long wavelength limit of the radiative cosmological model both in the GR and BD theories [42, 43]. However, there are two interesting aspects connected with these
modes: there is no dependence on $\omega$, and when $r = 1$, $h_0 = \lambda_0 = 0$, while for $r = \frac{1}{2}$, $h = 0$ and $\lambda_0$ is arbitrary. Thus, the most important perturbative modes are the inhomogeneous modes, which are represented by $h = f_0 t^{3/2}$, $\lambda = g_0 t^{3/2}$, with

$$f_0 = \frac{2 (7 + 2\omega)}{9} \frac{U_0}{\omega},$$  
$$g_0 = -\frac{4 (7 + 6\omega)}{9} \frac{U_0}{\omega}. 
$$

An important feature of these inhomogeneous solutions is that the perturbation grows very quickly with the scale factor, $\delta \propto a^3$. It is also interesting to contrast with the same situation in GR where this inhomogeneous modes identically cancel for the pure stiff matter case. As a final remark, we should stress that these homogeneous and inhomogeneous modes have a well defined $\omega \to \infty$ limit. However, these solutions depend on the background solution, which is inconsistent in this limit.

6 Dynamical system analysis

The power law solution for stiff matter fluid in BD displayed in section 4 has distinct features compared with Gurevich’s families of solution 5. In order to compare these solutions, we perform a dynamical system analysis. Instead of using the conservation equation (7), in this section we shall use BD and Klein-Gordon equations of motion. For a flat FLRW, the dynamical system reads

$$\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{\phi}}{\phi}\right) = \frac{8\pi \rho}{3 \phi} + \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2,$$

$$\left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right) \left(\frac{\dot{\phi}}{\phi}\right) = \frac{8\pi (3 + \omega) \rho - 3\omega p}{2\omega + 3} - \frac{\omega}{3} \left(\frac{\dot{\phi}}{\phi}\right)^2,$$

$$\frac{\ddot{\phi}}{\phi} + 3 \left(\frac{\dot{a}}{a}\right) \left(\frac{\dot{\phi}}{\phi}\right) = \frac{8\pi (\rho - 3p)}{(3 + 2\omega) \phi}. 
$$

It is convenient to define the Hubble factor $H = \dot{a}/a$ and its analogous for the scalar field, namely $F = \dot{\phi}/\phi$. Restricting ourselves to the stiff matter ($p = \rho$), we can combine the hamiltonian constraint Eq. (53) with Eq. (54) and (55) obtaining
the following dynamical autonomous system:

\[
\dot{H} = -\frac{6}{(3 + 2\omega)} \left( (1 + \omega)H^2 + \frac{\omega}{3} HF + \frac{\omega}{12} F^2 \right), \quad (56)
\]

\[
\dot{F} = -\frac{6}{(3 + 2\omega)} \left( H^2 + \frac{(5 + 2\omega)}{2} HF + \frac{(3 + \omega)}{6} F^2 \right), \quad (57)
\]

In fact, note that Eq.s (56)-(57) can simultaneously describe the stiff matter and the vacuum \((p = \rho = 0)\). It is easy to check that there are two fixed points for this dynamical system

\[
H = F = 0 \quad \text{corresponding to the Minkowski case,} \quad (58)
\]

\[
H = -\frac{F}{3} \quad \text{with} \quad \omega = -\frac{4}{3}. \quad (59)
\]

We can also find the invariant rays defined by the condition \(F = qH\) with \(q\) constant, which correspond to power law solutions of the system. Using Eq. (27), this condition translates into \(q = s/r\), where \(r\) and \(s\) are the powers in time of the scale factor and of the scalar field, respectively. Imposing this condition and combining the resulting expressions we find the following third order polynomial for \(q\)

\[
(q + 2)\left(\frac{\omega}{2} q^2 - 3q - 3\right) = 0, \quad (60)
\]

with solutions given by

\[
q = -2, \quad q_{\pm} = \frac{3}{\omega} \left( 1 \pm \sqrt{1 + \frac{2}{3} \omega} \right). \quad (61)
\]

The first root corresponds to the power law solution found previously, for which gravity is attractive only if \(\omega < -3/2\). Indeed, using \(F = qH\), the constraint Eq. (53) reads

\[
\left( -\frac{\omega}{2} q^2 + 3q + 3 \right) H^2 = \frac{8\pi\rho}{\phi}. \quad (62)
\]

One can immediately see that if \(q = -2\) then the energy density is positive only for \(2\omega + 3 < 0\) as already argued in Eq. (31). The other two roots correspond to the vacuum solution. Again, Eq. (62) shows that for \(q = q_{\pm}\) the left hand-side of the above equation vanishes implying that \(\rho = 0\). Note also that the invariant rays
\( q = q_\pm \) disappear when \( \omega < -\frac{3}{2} \) (the roots become imaginary). Varying \( \omega \) into negative values makes the two \( q_\pm \) rays collapse into \( q = -2 \) when \( \omega = -3/2 \). For \( \omega < -3/2 \) only the \( q = -2 \) invariant ray remains (see Fig. 1).

The \( q = -2 \) invariant ray does not depend on \( \omega \) which means that is insensitive to the \( \omega \to \infty \) limit. On the other hand, the \( q = q_\pm \) decays as

\[
\lim_{\omega \to \infty} q_\pm = \pm \sqrt{\frac{6}{\omega}}.
\]  

Thus, since \( F = qH \), for arbitrary finite values of \( H \) we have \( \dot{\phi} \to 0 \) in the \( \omega \to \infty \) limit. Naively, one could expect that a vacuum solution with \( \dot{\phi} \to 0 \) should approach the Minkowski spacetime. However, the term \( \omega \dot{\phi}^2 \) does not go to zero in this limit producing a power law expansion with \( a \propto t^{1/3} \). Indeed, it has been shown in Ref. [18] that in this regime the \( \omega \dot{\phi}^2 \) term behaves as an effective stiff matter like term, which is responsible for the \( a \propto t^{1/3} \) evolution of the scale factor.

### 6.1 Singular Points at Infinity

The stability of the fixed point at the origin of the phase space can be inferred directly from the phase space diagrams. However, the stability of the invariant rays of the dynamical system must be analyzed at infinity. For this purpose we use the Poincaré central projection method \[44-46\], using the coordinate transformation

\[
H = \frac{h}{z}, \quad F = \frac{f}{z}, \quad \text{with} \quad h^2 + f^2 + z^2 = 1.
\]  

Eq.s (56)–(57) can be recast as \( P(H, F)\, dF - Q(H, F)\, dH = 0 \), which combined with equation Eq. (64) gives us

\[
-zQdf + zPdh + (hQ - fP)\, dz = 0.
\]  

where the functions \( P(h, f, z) \) and \( Q(h, f, z) \) are given by

\[
P(h, f, z) = -\frac{6}{(3 + 2\omega)} \left( (1 + \omega)h^2 + \frac{\omega}{3} hf + \frac{\omega}{12} f^2 \right),
\]  

\[
Q(h, f, z) = -\frac{6}{(3 + 2\omega)} \left( h^2 + \frac{(5 + 2\omega)}{2} hf + \frac{(3 + \omega)}{6} f^2 \right).
\]
Collecting all terms we can explicitly write Eq. (65) in terms of the projective coordinates as

\[ z \left[ 6h^2 + 3(2\omega + 3)hf + (3 + \omega)f^2 \right] df - z \left[ 6(1 + \omega)h^2 + 2\omega hf + \omega f^2 \right] dh - \left[ 6h^3 - \frac{\omega f^3}{2} + (3 - \omega)hf^2 + 9h^2 f \right] dz = 0. \]  

(68)

The singular points at infinity have projective coordinates in the plane \((h, f, z = 0)\). Given Eqs. (64) and (68), they are solutions of the system:

\[ h^2 + f^2 = 1, \]

\[ 6h^3 - \frac{\omega f^3}{2} + (3 - \omega)hf^2 + 9h^2 f = 0. \]  

(69)

In order to find the invariant rays, we substitute \(f/h = q\) in the system Eqs. (69). As expected, there are three invariant rays

\[ q = -2 : \quad h = \pm \frac{1}{\sqrt{5}}, \quad f = qh, \]  

(70)

\[ q_+ = \frac{3}{\omega} \left( 1 + \sqrt{1 + \frac{2}{3} \omega} \right) : \quad h = \frac{1}{\sqrt{1 + q_+^2}}, \quad f = q_+ h, \]  

(71)

\[ q_- = \frac{3}{\omega} \left( 1 - \sqrt{1 + \frac{2}{3} \omega} \right) : \quad h = \frac{1}{\sqrt{1 + q_-^2}}, \quad f = q_- h. \]  

(72)

The analytical expressions of the solutions of the scale factor and the scalar field that correspond to these rays are given by

For \( q = -2 \):

\[ a(t) \propto t^{1/2}, \quad \phi(t) \propto t^{-1}, \]  

(73)

For \( q_+ = \frac{3}{\omega} \left( 1 + \sqrt{1 + \frac{2}{3} \omega} \right) \):

\[ a(t) \propto t^{\omega (3 + q_+)/(3(4 + 3\omega))}, \quad \phi(t) \propto t^{(4 - \omega q_+)/\omega (4 + 3\omega)}, \]  

(74)

For \( q_- = \frac{3}{\omega} \left( 1 - \sqrt{1 + \frac{2}{3} \omega} \right) \):

\[ a(t) \propto t^{\omega (3 + q_-)/(3(4 + 3\omega))}, \quad \phi(t) \propto t^{(4 - \omega q_-)/(\omega (4 + 3\omega))}. \]  

(75)
The phase portrait for six different values of $\omega$: $-5$, $-4/3$, $-1$, $1$, $50$ and $500$ are plotted in Fig. 1. As mentioned before, the two spread invariant rays for $\omega > -3/2$ are related to the $q_\pm$ vacuum solution ($\rho = 0$), while the invariant ray in the middle is for $q = -2$, which corresponds to our solution Eq. (27). Increasing the value of $\omega$ makes the $q_\pm$ invariant rays to move away from the $q = -2$ invariant ray. As can be seen by Eq. (62), the region between the two invariant rays $q_\pm$ corresponds to negative values of the energy density, hence should be excluded on physical basis. Additionally, for large values of $\omega$ the $q_\pm$ invariant rays tend to lay along the $F = 0$ line, which represent the $\omega \to \infty$ limit.

![Figure 1: Compacted phase portraits using the variables $h$, $f$ that are respectively related to $H$ and $F$ by Eq. (64). Each portrait uses a different value of BD parameter $\omega$. In particular, we used the values $\omega = -5, -4/3, -1, 1, 50, 500$ respectively from the top left to the bottom right. The unity circle corresponds to the projected singular points at infinity. The dashed straight lines depict the invariant rays Eqs. (70-72) and the empty region has been excluded since it corresponds to the unphysical situation of negative values of the energy density.](image)

7 Discussion

In this paper we analyzed the cosmological solution for a perfect fluid in Brans-Dicke theory and showed that stiff matter ($p = \rho$) is a very particular solution.
There are power law solutions with the scale factor and the scalar field, respectively, proportional to \( a \propto t^{1/2} \) and \( \phi \propto t^{-1} \). Even though the matter content behaves as stiff matter, the cosmological evolution mimics a radiation dominated dynamics in General Relativity. Furthermore, the scalar field gives the effective gravitational strength, hence, gravity becomes stronger with the expansion of the universe.

Eq. (31) shows that the scalar field is inversely proportional to the BD parameter \( \omega \). This condition is commonly understood as a sufficient condition for a well defined GR limit. However, we have shown that this is not the case for the power law solution (27). Furthermore, this is not a special case of the Gurevich’s families of solution since there is no choice of parameter that reduce Eq.s (16)-(17) or Eq.s (23)-(24) into (27). The Gurevich’s solution tend to (27) only in the remote past or the far future times.

The scalar cosmological perturbation also has interesting features. The velocity field for the stiff matter fluid has a growing mode \( U \) that is proportional to scale factor. This extra contribution produces new polynomials solutions for the density contrast \( \delta = \delta \rho / \rho \), the fractional scalar field perturbation \( \lambda = \delta \phi / \phi \) and the tensor perturbation \( h = h_{kk} / a^2 \). The homogeneous mode has four power solution in cosmic time \( t' \) with \( r = -1,0,1/2,1 \). The first two are connected with the residual freedom of the synchronous gauge and the other two are the physical solutions corresponding to two growing modes. There is no decaying mode. The inhomogeneous mode related to the growing mode \( U \) goes as \( t^{3/2} \propto a^3 \), hence it is a steep growth if compared with the standard cosmological model.

The dynamical systems analysis developed in section 6 shows the existence of three invariant rays associated with the system Eq.s (56)-(57). The first correspond to the power law solution of section 4 with constant of proportionally \( q = -2 \). The other two are vacuum solutions with constant of proportionally \( q_{\pm} \) given by Eq. (61). For \( \omega < -3/2 \) there is only one invariant ray associated with \( q = -2 \) while for \( \omega > -3/2 \) there are two vacuum invariant rays \( q_{\pm} \). The region in the phase space diagram between these two rays has negative energy density, which is unphysical. Increasing the value of the BD parameter \( \omega \), these two rays rotate away from each other expanding the unphysical region. The limiting case is for \( \omega \to \infty \) when both \( q_{\pm} \) tends to zero. This limit has a vacuum solution with constant scalar field \( F = 0 \) but the scale factor increases as \( a \propto t^{1/3} \), which is characteristic of a FLRW universe with stiff matter in GR. Even though this is the vacuum case and the scalar field is constant, we do not approach Minkowski spacetime.

It is well known in the literature that there are examples where the BD parameter scales as \( \phi \sim 1/\sqrt{\omega} \) but the system does not approach a GR regime in the
limit $\omega \to \infty$. Nevertheless, it is commonly expected to recover GR in this limit if $\phi \sim 1/\omega$ and the matter energy-momentum tensor has a nonzero trace. We have explicit showed an exact BD solution with $\phi \sim 1/\omega$ and $T^{\mu\nu} \neq 0$ that does not approach GR in the limit $\omega \to \infty$.

**Acknowledgments**

The authors would like to thank and acknowledge financial support from the National Scientific and Technological Research Council (CNPq, Brazil) the State Scientific and Innovation Funding Agency of Espírito Santo (FAPES, Brazil) and the Brazilian Federal Agency for Support and Evaluation of Graduate Education (CAPES, Brazil).

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