A normal form without small divisors
(Draft)

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Abstract

Following the techniques of [4], we formulate a Normal Form Lemma suited to close-to-be-integrable Hamiltonian systems where not all the coordinates are action–angles. The Lemma turns to be useful in the theory of KAM tori of Sun-Earth-Asteroids systems (work in progress of the author; arXiv: 1702.03680).

1 Set Up

Consider the $2(n + m + 1)$-dimensional phase space

$$\mathcal{P} = \mathcal{R} \times \mathcal{I} \times \Xi \times T^n \times B^{2m}_\delta$$

where $\mathcal{R} \subset \mathbb{R}$, $\mathcal{I} \subset \mathbb{R}^n$, $0 \in \Xi \subset \mathbb{R}$ are open, connected and bounded, while $B^{2m}_\delta$ denotes the ball of radius $\delta$ in $\mathbb{R}^{2m}$ centered at $0 \in \mathbb{R}^{2m}$. Let $\mathcal{P}$ be equipped with set of canonical coordinates $(r, \mathcal{I}, x, \varphi, p, q) \in \mathcal{P}$ with respect to the standard two–form

$$\Omega = dr \wedge dx + d\mathcal{I} \wedge d\varphi + dp \wedge dq = dr \wedge dx + \sum_{i=1}^n d\mathcal{I}_i \wedge d\varphi_i + \sum_{j=1}^m dp_j \wedge dq_j$$

and consider, on $\mathcal{P}$, a Hamiltonian of the form

$$H(r, \mathcal{I}, p, x, \varphi, q) = H_0(r, \mathcal{I}, J(p, q)) + f(r, \mathcal{I}, p, x, \varphi, q)$$

(1)

where

$$J(p, q) = (p_1 q_1, \cdots, p_m q_m).$$

Note that we are not assuming that $f$ is periodic in $x$.  

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Setting $f$ to zero, the Hamiltonian $H = H_0$ has the motions

$$
\begin{align*}
I &= I_0, \\
\mathbf{r} &= \mathbf{r}_0, \\
p &= p_0 e^{-\omega\mathbf{t}},
\end{align*}
$$

where

$$
\omega_{1,0} := \frac{\partial_{1,0}}{h}(\mathbf{r}, \mathbf{I}, J)
$$

We consider the problem of the continuation of such motions to the full system (1).

The problem may be regarded as a generalization of problems that have been widely investigated in the framework of KAM and Nekhorossev theory.

In fact, if $H_0$ was taken to be independent of $r$, we would be in the setting of (partially hyperbolic) KAM theory, where the perturbing function will depend, in addition, on the “degenerate” couple $(r, x)$. Such case has been investigated in the literature, starting with V.I. Arnold and N.N. Nekhorossev [1, 3]. Refinements have been given by L. Chierchia and G. Pinzari in the case of properly–degenerate KAM theory [2], by J. Pöschel in the case of Nekhorossev theory [4]. Such papers are addressed to the study of Hamiltonian systems (named “properly–degenerate”) of the form

$$
H = H_0(I) + f(I, \varphi, p, q)
$$

where the unperturbed part $H_0$ has strictly less degrees of freedom than the whole system. For such systems standard techniques do not apply since, on one side, as for KAM theory, usual non-degeneracy assumptions are strongly prevented and, on the other side, as for Nekhorossev theory, one has to control the variation of the “degenerate” coordinates $(p, q)$. For the way how such difficulties have been overcome, we refer to the dedicated literature (recalled in [2] and references therein).

The generalization studied in this paper with respect to the previous mentioned cases is precisely related to the rôle of the coordinate $x$: we are not assuming that this is a periodic coordinate, henceforth, standard KAM theories do not apply. In this setting, one cannot reasonably expect, at least in general, that its linear motion of $x$ in (2) is preserved at any time.

As an example, let us look at the clock Hamiltonian

$$
H_\varepsilon = \frac{r^2}{2} + \varepsilon^2 x^2 + \frac{\varepsilon^2}{2}.
$$

For $\varepsilon = 0$, $H_\varepsilon$ reduces to the free Hamiltonian $H_0 = \frac{r^2}{2}$ whose motions are

$$
\begin{align*}
r_0(t) &= r_0, \\
x_0(t) &= x_0 + r_0(t - t_0).
\end{align*}
$$

However, when $\varepsilon \neq 0$, the motions of $H_\varepsilon$, given by

$$
\begin{align*}
r_\varepsilon(t) &= r_0 \cos \varepsilon(t - t_0) - \varepsilon x_0 \sin \varepsilon(t - t_0), \\
x_\varepsilon(t) &= \frac{r_0}{\varepsilon} \sin \varepsilon(t - t_0) + x_0 \cos \varepsilon(t - t_0)
\end{align*}
$$

are effectively close one to the one of $H_0$ for $|t - t_0|$ of the order $\varepsilon^{-1}$. For larger times, the two Hamiltonians generate a completely different dynamics, since the former has only unbounded motions, while the latter has bounded ones. The same conclusion could be reached, instead of solving the motion equations, looking at the phase portrait of $H_\varepsilon$, which consists of ellipses with semi–axes $\sqrt{2\varepsilon}$, $\varepsilon^{-1}\sqrt{2\varepsilon}$ which tend to the straight lines $r = \sqrt{2\varepsilon}$ as $\varepsilon \to 0$.

Similarly, one sees that still in the case of the Hamiltonian

$$
\tilde{H}_\varepsilon = \frac{r^2}{2} - \varepsilon^2 x^2
$$

which has unbounded motions for all $\varepsilon$, the dynamics of $\tilde{H}_0$ and $\tilde{H}_\varepsilon$ with $\varepsilon \neq 0$ are very far one from the other for $|t - t_0| \gg \varepsilon^{-1}$. 

\[2\]
For these reasons, we divide the problem of the study of the dynamics of the full Hamiltonian (1) in two steps. As a first step, which is actually the purpose of this note, we consider the intermediate problem of constructing a normal form for $H$ for very large (exponentially long) times, without any attempt to normalize the evolution of the couple $(r, x)$. Such normal form will be defined on a suitable sub-domain $\mathcal{P}_N \subset \mathcal{P}$, and will be of the kind

$$H_N = h_0(r, I, p, q) + h_1(r, I, pq, x) + f_N(r, I, p, x, \varphi, q)$$

where $f_N$ is a very (exponentially) small remainder which we shall quantify. Clearly, when dealing with concrete applications, such step should be followed by a second step where one verifies that the evolution generated by $H_N$ remains in the prescribed domain $\mathcal{P}_N$ for all such time. Such idea of “a posteriori check” goes back to N.N. Nekhoroshev [3], who indeed was able to establish the validity, to the $N$–body problem Hamiltonian written in Poincaré coordinates

$$H_P = h_0(\Lambda) + f_P(\Lambda, \lambda, p, q)$$

of a normal form of the kind

$$H_N = h_0(\Lambda) + h_1(\Lambda, p, q) + f_N(\Lambda, \lambda, p, q)$$

where $f_N$ is exponentially small, just controlling that the “degenerate” coordinates $(p, q)$ did not escape their domain for all that time. Before stating our result, let us fix the following

**Notations** We consider the complex neighborhood

$$\mathcal{P}_{r, p, \xi, \delta} = \mathcal{R}_r \times \mathcal{I}_p \times \mathcal{Z}_\xi \times \mathbb{T}_\delta^m \times B^m_\delta,$$

of $\mathcal{P}$ where, as usual, $A_\delta := \cup_{x_0 \in A} \{B_\delta(x_0)\}$, while $\mathbb{T}_\delta := \mathbb{T} + i[-s, s]$, with $\mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z})$ the standard torus.

We denote as $\mathcal{O}_{r, p, \xi, \delta}$ the set of complex holomorphic functions $\phi : \mathcal{P}_{r, p, \xi, \delta} \to \mathbb{C}$ for some $\tilde{r} > r$, $\tilde{p} > p$, $\tilde{\xi} > \xi$, $\tilde{s} > s$, $\tilde{\delta} > \delta$.

We equip $\mathcal{O}_{r, p, \xi, \delta}$ with the norm

$$\|\phi\|_{r, p, \xi, \delta} := \sum_{k, h, j} \|\phi_{khj}\|_{r, p, \xi} e^{i|k|} \delta^{h+j}$$

where $\phi_{khj}(r, I, x)$ are the coefficients of the Taylor–Fourier expansion

$$\phi = \sum_{k, h, j} \phi_{khj}(r, I, x) e^{ikx + h\tilde{p} q^j},$$

and $\|\phi_{khj}\|_{r, p, \xi} := \sup_{r_x \times I_p \times \mathcal{Z}_\xi} |\phi_{khj}|$. Observe that $\|g_{khj}\|_{r, p, \xi}$ is well defined because of the boundedness of $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{Z}$, while $\|\phi\|_{r, p, \xi, \delta}$ is well defined by the usual properties of holomorphic functions.

For a given vector–valued function $\tilde{\phi} = (\phi_1, \cdots, \phi_k) \in \mathcal{O}^k_{r, p, \xi, \delta}$, we let

$$\|\tilde{\phi}\|_{r, p, \xi, \delta} := \sum_{i=1}^k \|\phi_i\|_{r, p, \xi, \delta} .$$

If $\tilde{\phi} \in \mathcal{O}_{r, p, \xi, \delta}$, we define its “off–average” and “average” as

$$\tilde{\phi} := \sum_{(k, h, j) \neq (0, 0)} g_{khj}(r, I, x)e^{ikx + h\tilde{p} q^j}, \quad \overline{\phi} := \tilde{\phi} - \tilde{\phi} .$$

Then we define the “zero–average” and the the “normal” classes as

$$Z_{r, p, \xi, \delta} := \{ \phi \in \mathcal{O}_{r, p, \xi, \delta} : \phi = \overline{\phi} \} = \{ \phi \in \mathcal{O}_{r, p, \xi, \delta} : \overline{\phi} = 0 \} \quad \text{(3)}$$

$$N_{r, p, \xi, \delta} := \{ \phi \in \mathcal{O}_{r, p, \xi, \delta} : \phi = \overline{\phi} \} = \{ \phi \in \mathcal{O}_{r, p, \xi, \delta} : \overline{\phi} = 0 \} .$$

respectively. Obviously, one has the decomposition

$$\mathcal{O}_{r, p, \xi, \delta} = Z_{r, p, \xi, \delta} \oplus N_{r, p, \xi, \delta} .$$
The beginning is just as in the standard case. We follow the well–settled framework acknowledged to Jürgen Pöschel \cite{4}, which is also largely studied in the aforementioned papers, where \( f \) is a perturbation of the unperturbed part which is naturally small. We shall show a situation where indeed this is the case in a forthcoming paper.

We assume that, at a certain step, we have a system of the form \( H = H_0(r, I, J(p, q)) + g(r, I, J(p, q), x) + f(r, I, x, \varphi, p, q) \) (7)

where \( f \in \mathcal{O}_{r,\rho,\xi,\alpha,\delta} \), while \( H_0 \) is independent of \( x \) (the first step corresponds to take \( g \equiv 0 \)).

After splitting \( f \) on its Taylor–Fourier basis

\[ f = \sum_{k, h, j} f_{k, h, j}(r, I, x) e^{i k \varphi} p^h q^j. \]

one looks for a time–1 map

\[ \Phi = e^{\mathcal{E} \varphi} \]

generated by a small Hamiltonian \( \phi \) which will be taken in the class \( \mathcal{Z}_{r,\rho,\xi,\alpha,\delta} \) in (3). One lets

\[ \phi = \sum_{(k, h, j)} \phi_{k, h, j}(r, I, x) e^{i k \varphi} p^h q^j. \]
The operation 
\[\phi \rightarrow \{\phi, H_0\}\]
acts diagonally on the monomials in the expansion (8), carrying
\[\phi_{khj} \rightarrow - (\omega_i \partial_{x_i} \phi_{khj} + \lambda_{khj} \phi_{khj})\]
with \(\lambda_{khj} := (h - j) \cdot \omega_3 + i k \cdot \omega_1\). (9)
Therefore, one defines
\[\{\phi, H_0\} = - D_\omega \phi.\]
The formal application of \(\Phi = e^{\epsilon \phi}\) yields:
\[e^{\epsilon \phi} H = e^{\epsilon \phi} (H_0 + g + f) = H_0 + g - D_\omega \phi + f + \Phi_2 (H_0) + \Phi_1 (g) + \Phi_1 (f)\] (10)
where the \(\Phi_h\)’s are the queues of \(e^{\epsilon \phi}\), defined in Section 2.
Next, one requires that the residual term \(- D_\omega \phi + f\) lies in the class \(N_{r, \rho, \xi, s, \delta}\) in (4). This amounts to solve the “homological” equation
\[\tilde{(-D_\omega \phi + f)} = 0\] (11)
for \(\phi\).
Since we have chosen \(\phi \in Z_{r, \rho, \xi, s, \delta}\), by (9), we have that also \(D_\omega \phi \in Z_{r, \rho, \xi, s, \delta}\). So, Equation (11) becomes
\[\tilde{-D_\omega \phi + \tilde{f}} = 0.\] (12)
In terms of the Taylor–Fourier modes, the equation becomes
\[\omega_i \partial_{x_i} \phi_{khj} + \lambda_{khj} \phi_{khj} = f_{khj} \forall (k, h, j) : (k, h - j) \neq (0, 0).\] (13)
In the standard situation, one typically proceeds to solve such equation via Fourier series:
\[f_{khj}(r, I, x) = \sum_{\ell} f_{khj\ell}(r, I) e^{i \ell x}, \quad \phi_{khj}(r, I, x) = \sum_{\ell} \phi_{khj\ell}(r, I) e^{i \ell x}\] (14)
so as to find \(\phi_{khj\ell} = \frac{f_{khj\ell}}{\mu_{khj\ell}}\) with the usual denominators \(\mu_{khj\ell} := \lambda_{khj} + i \ell \omega_i\) which one requires not to vanish via, e.g., a “diophantine inequality” to be held for all \((k, h, j, \ell)\) with \((k, h - j) \neq (0, 0)\). Observe that, in the classical case, there is not much freedom in the choice of \(\phi\). In fact, such solution is determined up to solutions of the homogenous equation
\[D_\omega \phi_0 = 0\] (15)
which, in view of the Diophantine condition, has the only trivial solution \(\phi_0 \equiv 0\).

The situation is different if \(f\) is not periodic in \(x\), or \(\phi\) is not needed so. In such a case, it is possible to find a solution of (13), corresponding to a non–trivial solution of (15), where small divisors do not appear.

This is
\[\phi_{khj}(r, I, x) = \frac{1}{\omega_i} \int_0^x f_{khj}(r, I, \tau) e^{\frac{\lambda_{khj}}{\omega_i} (r - \tau)} d\tau \quad \forall (k, h, j) : (k, h - j) \neq (0, 0)\] (16)
and \(\phi_{0,khj}(r, I, x) \equiv 0\). Note that in the particular case that \(f\) is periodic in \(x\), and hence it affords an expansion like (14), the solution (16) may be written as
\[\phi_{khj} = e^{-\frac{\lambda_{khj}}{\omega_i} x} \sum_{\ell} f_{khj\ell}(r, I) e^{\frac{\mu_{khj\ell}}{\mu_{khj\ell}}} - 1 = e^{-\frac{\lambda_{khj}}{\omega_i} x} \tilde{\phi}_{khj}(r, I, x, p, q)\]
As expected, such a solution provides, via (8), a function \(\phi\) that, in general, is not periodic in \(x\) for all \((r, I, \varphi, p, q)\) in their domain. Indeed, under the genericity assumption that the \(\tilde{\phi}_{khj}\)’s have no other common zero than \(x = 0\), since such \(\tilde{\phi}_{khj}\)’s are periodic in \(x\), we have that the \(\phi_{khj}\)’s are so only for \((r, I, p, q)\) such that \(\frac{\lambda_{khj}}{\omega_i} \in i \mathbb{Z}\). Henceforth, \(\phi\) is \(x\)-periodic only on the subset \((r, I, p, q, \varphi) \in \mathcal{R}_{cs} \times \mathbb{T}^n\), where \(\mathcal{R}_{cs}\) is the zero–measure subset of \(\mathcal{R} \times \mathcal{I} \times \mathcal{B}_{2^{20}}\) where \(\frac{\lambda_{khj}}{\omega_i} \in i \mathbb{Z}\) for all \((k, j, h)\) such that \((k, h - j) \neq (0, 0)\).

We conclude with a comment on the necessity of the two first inequalities in (5): the formula (16) involves some loss of analyticity for \(\phi\) whose strength we will evaluate to be of the order of the maximum of \(X\|f\|_{r, \rho}, X\|f\|_{r, \rho}\).
2 Proofs

Definition 2.1 (Time–one flows and their queues) Let \( L_\phi(\cdot) := \{\phi, \cdot\} \), where \( \{f, g\} := \sum_{i=1}^{\kappa} (\partial_{y_i} f \partial_{y_i} g - \partial_{y_i} g \partial_{y_i} f) \), where \( \Omega = \sum_{i=1}^{\kappa} dp_i \wedge dq_i \) is the standard two–form, denotes Poisson parentheses.

For a given \( \phi \in \mathcal{O}_{r,\rho,\xi,s,\delta} \), we denote as \( \Phi_h \), \( \Phi \) the formal series

\[
\Phi_h := \sum_{j \geq 0} \frac{L_\phi^j}{j!} \quad \Phi := \Phi_0.
\]

(17)

It is customary to let, also \( \Phi := e^{L_\phi} \).

Lemma 2.1 ([4]) There exist an integer number \( \tilde{e}_{n,m} \) such that, for any \( \phi \in \mathcal{O}_{r,\rho,\xi,s,\delta} \) and any \( r' < r, s' < s, \rho' < \rho, \xi' < \xi, \delta' < \delta \) such that

\[
\sum_{m} \frac{\|\phi\|_{r,\rho,\xi,s,\delta}}{d} < 1 \quad d := \min \{\rho' \sigma', r' \xi', \delta'^2\}
\]

then the series in (17) converge uniformly so as to define the family \( \{\Phi_h\}_{h=0,1,\ldots} \) of operators

\[
\Phi_h : \mathcal{O}_{r,\rho,\xi,s,\delta} \to \mathcal{O}_{r-r',\rho-\rho',\xi-\xi',s-s',\delta-\delta'}.
\]

Moreover, the following bound holds (showing, in particular, uniform convergence):

\[
\|L_\phi^j[g]\|_{r-r',\rho-\rho',\xi-\xi',s-s',\delta-\delta'} \leq j! \left( \frac{\tilde{e}_{n,m} \|\phi\|_{r,\rho,\xi,s,\delta}}{d} \right)^j \|g\|_{r,\rho,\xi,s,\delta}.
\]

(18)

for all \( g \in \mathcal{O}_{r,\rho,\xi,s,\delta} \).

Remark 2.1 ([4]) The bound (18) immediately implies

\[
\|\Phi_h g\|_{r-r',\rho-\rho',\xi-\xi',s-s',\delta-\delta'} \leq \frac{\left( \frac{\|\phi\|_{r,\rho,\xi,s,\delta}}{d} \right)^h}{1 - \frac{\|\phi\|_{r,\rho,\xi,s,\delta}}{d}} \|g\|_{r,\rho,\xi,s,\delta} \quad \forall g \in \mathcal{O}_{r,\rho,\xi,s,\delta}.
\]

(19)

Lemma 2.2 (Iterative Lemma) There exists a number \( \tilde{e}_{n,m} > 1 \) such that the following holds. For any choice of positive numbers \( r', \rho', s', \xi', \delta' \) satisfying

\[
2r' < r, \quad 2\rho' < \rho, \quad 2\xi' < \xi
\]

(20)

\[
2s' < s, \quad 2\delta' < \delta, \quad X_\xi \left[ \frac{\omega_j}{\omega_i} \right]_{r,\rho} < s - 2s', \quad X_\xi \left[ \frac{\omega_j}{\omega_i} \right]_{r,\rho} < \log \frac{\delta}{2\delta'}
\]

(21)

and provided that the following inequality holds true

\[
\tilde{e}_{n,m} \frac{X_\xi}{d} \left[ \frac{1}{\omega_i} \right]_{r,\rho} \|f\|_{r,\rho,\xi,s,\delta} < 1 \quad d := \min \{\rho' \sigma', r' \xi', \delta'^2\}
\]

(22)

one can find an operator

\[
\Phi : \mathcal{O}_{r,\rho,\xi,s,\delta} \to \mathcal{O}_{r,\rho,\xi,s,\delta}.
\]

with

\[
r_+ := r - 2r', \quad \rho_+ := \rho - 2\rho', \quad \xi_+ := \xi - 2\xi', \quad s_+ := s - 2s' - X_\xi \left[ \frac{\omega_j}{\omega_i} \right]_{r,\rho}, \quad \delta_+ := \delta e^{-X_\xi} \left[ \frac{\omega_j}{\omega_i} \right]_{r,\rho} - 2\delta'
\]

which carries the Hamiltonian \( H \) in (7) to

\[
H_+ := \Phi[H] = H_0 + g + f + f_+
\]

where

\[
\|f_+\|_{r,\rho,\xi,s,\delta} \leq \tilde{e}_{n,m} \frac{X_\xi}{d} \left[ \frac{1}{\omega_i} \right]_{r,\rho} \|f\|_{r,\rho,\xi,s,\delta} + \|\{\phi, g\}\|_{r-r',\rho-\rho',\xi-\xi',s-s',\delta-\delta'}
\]

(23)
with 
\[ r_1 := r, \quad \rho_1 := \rho, \quad \xi_1 := \xi, \quad s_1 := s - \mathcal{X}\|\frac{\omega_1}{\omega_r}\|_{r,\rho}, \quad \delta_1 := \delta e^{-X\|\frac{\omega_1}{\omega_r}\|_{r,\rho}} \]
for a suitable \( \phi \in \mathcal{O}_{r_1, \rho_1, \xi_1, s_1, \delta_1} \) verifying
\[ \|\phi\|_{r_1, \rho_1, \xi_1, s_1, \delta_1} \leq \frac{X}{d}\|\frac{1}{\omega_r}\|_{r,\rho} \|f\|_{r,\rho, \xi, \delta}. \tag{24} \]

**Proof** Let \( \overline{c}_{n,m} \) be as in Lemma 2.1. We shall choose \( \overline{c}_{n,m} \) suitably large with respect to \( \overline{c}_{n,m} \).

Let \( \phi_{khj} \) as in (16). Let us fix
\[ 0 < \tau \leq r, \quad 0 < \rho \leq \rho, \quad 0 < \xi \leq \xi, \quad 0 < \sigma < s, \quad 0 < \delta < \delta \tag{25} \]
and assume that
\[ \mathcal{X}\|\frac{\omega_1}{\omega_r}\|_{r,\rho} \leq s - \tau, \quad \mathcal{X}\|\frac{\omega_1}{\omega_r}\|_{r,\rho} \leq \log \frac{\delta}{\delta}. \tag{26} \]
Then we have
\[ \|\phi_{khj}\|_{\tau,\tau,\tau} \leq \|\frac{1}{\omega_r}\|_{\tau,\tau} \|f_{khj}\|_{\tau,\tau,\tau} \int_0^\tau |e^{-\lambda\frac{\tau}{\omega_r}} + \|X\|\|f_{khj}\|_{\tau,\tau,\tau} e^{X\|\frac{\omega_1}{\omega_r}\|_{r,\rho}} \]. \]
Since
\[ \|\lambda\|_{\tau,\tau,\tau} \leq (h + j)\|\frac{\omega_1}{\omega_r}\|_{\tau,\tau} + |k|\|\frac{\omega_1}{\omega_r}\|_{\tau,\tau} \]
we have, definitely,
\[ \|\phi_{khj}\|_{\tau,\tau,\tau} \leq \mathcal{X}\|\frac{1}{\omega_r}\|_{\tau,\tau} \|f_{khj}\|_{\tau,\tau,\tau} e^{(h+j)\|\frac{\omega_1}{\omega_r}\|_{r,\rho} + |k|\|\frac{\omega_1}{\omega_r}\|_{r,\rho}} \]
which yields (after multiplying by \( e^{(h+j)\|\frac{\omega_1}{\omega_r}\|_{r,\rho} + |k|\|\frac{\omega_1}{\omega_r}\|_{r,\rho}} \) and summing over \( k,j,h \) with \( (k,h-k) \neq (0,0) \)) to
\[ \|\phi\|_{\tau,\tau,\tau} \leq \mathcal{X}\|\frac{1}{\omega_r}\|_{\tau,\tau} \|f\|_{\tau,\tau,\tau,\tau,\tau,\tau} e^{X\|\frac{\omega_1}{\omega_r}\|_{r,\rho}} \]. \]
Note that the right hand side is well defined because of (26). In the case of the choice
\[ \tau = r := r_1, \quad \rho = \rho := \rho_1, \quad \xi = \xi := \xi_1, \quad \sigma = s - \mathcal{X}\|\frac{\omega_1}{\omega_r}\|_{r,\rho} := s_1 \quad \delta = \delta e^{-X\|\frac{\omega_1}{\omega_r}\|_{r,\rho} := \delta_1 \]
(which, in view of the two latter inequalities in (21), satisfies (25)–(26)) the inequality becomes (24). An application of Lemma 2.1, with \( r, \rho, \xi, s, \delta \) replaced by \( r_1 - r', \rho_1 - \rho', \xi_1 - \xi', \]
\( \sigma_1 - \sigma', \delta_1 - \delta' \), concludes with a suitable choice of \( \overline{c}_{n,m} > \overline{c}_{n,m} \) and (by (28))
\[ f_\delta := \Phi_2(H_0) + \Phi_1(g) + \Phi_1(f). \]
Observe that the bound (23) follows from Equations (19), (18) and the identities
\[ \Phi_2[H_0] = \sum_{j=2}^{\infty} \frac{L_j^2(H_0)}{j!} = \sum_{j=1}^{\infty} \frac{L_j^1(H_0)}{(j+1)!} = \sum_{j=1}^{\infty} \frac{L_j^2(f)}{(j+1)!} \]
\[ \Phi_1[g] = \sum_{j=1}^{\infty} \frac{L_j^2(g)}{j!} = \sum_{j=0}^{\infty} \frac{L_j^1(g)}{(j+1)!} = \sum_{j=0}^{\infty} \frac{L_j^2(g)}{(j+1)!} \]
with \( g_1 := L_0(g) = \{\phi, g\} \)

The proof of Lemma 1.1 goes through iterate applications of Lemma 2.2. At this respect, we premise the following
Remark 2.2 Replacing conditions in (21) with the stronger ones

\[ 3s' < s , \quad 3\delta' < \delta , \quad \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} < s', \quad \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} < \frac{\delta'}{\delta} \]  

(27)

(and keeping (20), (22) unvaried) one can take, for \( s_+, \delta_+, s_1, \delta_1 \) the simpler expressions

\[ s_{+\text{new}} = s - 3s' , \quad \delta_{+\text{new}} = \delta - 3\delta' , \quad s_{1\text{new}} := s - s' , \quad \delta_{1\text{new}} = \delta - \delta' \]  

(while keeping \( r_+, \rho_+, \xi_+, r_1, \rho_1, \xi_1 \) unvaried). Indeed, since \( 1 - e^{-x} \leq x \) for all \( x \),

\[ \delta_1 = \delta_+ - \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} = \delta - \delta(1 - e^{-\lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta}}) \geq \delta - \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} \geq \delta - \delta' = \delta_{1\text{new}} . \]

This also implies \( \xi_+ = \delta_1 - \delta' \geq \delta - 2\delta' = \xi_{+\text{new}} \). That \( s_+ \geq s_{+\text{new}} , s_1 \geq s_{1\text{new}} \) is even more immediate.

Now we can proceed with the

Proof of Lemma 1.1 Let \( \varepsilon_{n,m} \) be as in Lemma 2.2. We shall choose \( \varepsilon_{n,m} \) suitably large with respect to \( \varepsilon_{n,m} \).

We apply Lemma 2.2 with

\[ 2r' = \frac{r}{3} , \quad 2\rho' = \frac{\rho}{3} , \quad 2\xi' = \frac{\xi}{3} , \quad 3s' = \frac{s}{3} , \quad 3\delta' = \frac{\delta}{3} , \quad g \equiv 0 . \]

We make use of the stronger formulation described in Remark 2.2. Conditions in (20) and the three former conditions in (27) are trivially true. The two latter inequalities in (27) reduce to

\[ \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} < \frac{s}{9} , \quad \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} < \frac{1}{9} \]

and they are certainly satisfied by assumption (5), for \( N > 1 \). Since

\[ d = \min \{ \rho s', r' \xi', \delta'^2 \} = \min \{ \rho s/36 , r \xi/54 , \delta^2 /81 \} \geq \frac{1}{81} \min \{ \rho s , r \xi , \delta^2 \} = \frac{d}{81} \]

we have that condition (22) is certainly implied by the last inequality in (5), once one chooses \( c_{n,m} > 81 \varepsilon_{n,m} \). By Lemma 2.2, it is then possible to conjugate \( H \) to

\[ H_1 = H_0 + f_1 \]

with \( f_1 \in \mathcal{O}_{(r^{(1)}, \rho^{(1)}, \xi^{(1)}, \delta^{(1)})}^{(2)} \), where \( (r^{(1)}, \rho^{(1)}, \xi^{(1)}, \delta^{(1)}) := 2/3 (r, \rho, \xi, \delta) \) and

\[ \| f_1 \|_{(r^{(1)}, \rho^{(1)}, \xi^{(1)}, \delta^{(1)})} \leq 81 \varepsilon_{n,m} \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} \| f \|_{(r, \rho, \xi, \delta)} \| f \|_{(r, \rho, \xi, \delta)} \leq \frac{\| f \|_{(r, \rho, \xi, \delta)}}{2} . \]  

(28)

since \( c_{n,m} \geq 162 \varepsilon_{n,m} \) and \( N \geq 1 \). Now we aim to apply Lemma 2.2 \( N \) times, each time with parameters

\[ r'_j = \frac{r}{6N} , \quad \rho'_j = \frac{\rho}{6N} , \quad \xi'_j = \frac{\xi}{6N} , \quad s'_j = \frac{s}{9N} , \quad \delta'_j = \frac{\delta}{9N} . \]

To this end, we let

\[ r^{(j+1)} := r^{(1)} - \frac{r}{3N} , \quad \rho^{(j+1)} := \rho^{(1)} - \frac{\rho}{3N} , \quad \xi^{(j+1)} := \xi^{(1)} - \frac{\xi}{3N} \]

\[ s^{(j+1)} := s^{(1)} - \frac{s}{3N} , \quad \delta^{(j+1)} := \delta^{(1)} - \frac{\delta}{3N} \]

\[ r_1^{(j)} := r^{(j)} , \quad \rho_1^{(j)} := \rho^{(j)} , \quad \xi_1^{(j)} := \xi^{(j)} , \quad s_1^{(j)} := s^{(j)} - \frac{s}{9N} \]

\[ \delta_1^{(j)} := \delta^{(j)} - \frac{\delta}{9N} , \quad \lambda \| \frac{\omega}{\omega_t} \|_{\rho, \delta} := \sup \{ |x| : x \in \Xi_{\xi_j} \} \]

with \( 1 \leq j \leq N \).
We assume that for a certain $1 \leq i \leq N$ and all $1 \leq j \leq i$, we have $H_j \in \mathcal{O}_{r(1), \rho(1), \xi(1), \rho(1), \delta(1)}$ of the form

$$H_j = H_0 + g_{j-1} + f_j, \quad g_{j-1} \in \mathcal{N}_{r(1), \rho(1), \xi(1), \rho(1), \delta(1)}, \quad g_{j-1} - g_{j-2} = \mathcal{T}_{j-1}$$

(29)

$$\|f_j\|_{r(1), \rho(1), \xi(1), \rho(1), \delta(1)} \leq \frac{\|f_i\|_{r(1), \rho(1), \xi(1), \rho(1), \delta(1)}}{2^{j-1}}$$

(30)

with $g_{-1} \equiv 0, g_0 = f_0 = \mathcal{T}$. If $i = N$, we have nothing more to do. If $i < N$, we want to prove that Lemma 2.2 can be applied so as to conjugate $H_i$ to a suitable $H_{i+1}$ such that (29)–(30) are true with $j = i + 1$. To this end, we have to check

$$\mathcal{X}_i \| \frac{\omega_i}{\omega_j} \| r, \rho_i < s'_i, \quad \mathcal{X}_i \| \frac{\omega_j}{\omega_i} \| r, \rho_i < \frac{\delta'_j}{\delta_i}$$

(31)

$$\tilde{c}_{n,m} \frac{\mathcal{X}_i}{d_i} \| \frac{1}{\omega_i} \| r, \rho_i \| f_i \| r, \rho_i, \xi, \alpha_i, \delta_i < 1$$

(32)

where $d_i := \min\{\rho_i s'_i, r_i \xi'_i, \delta'^2_i\}$. Conditions (31) are certainly verified, since in fact they are implied by the definitions above (using also $\delta_i \leq \frac{\delta}{3}, \mathcal{X}_i \leq \mathcal{X}$) and the two former inequalities in (5). To check the validity of (32), we firstly observe that

$$d_i = \min\{\rho'_i s'_i, r'_i \xi'_i, \delta'^2_i\} \geq \frac{d}{81N^2}$$

Using then $c_{n,m} > 162\tilde{c}_{n,m}, \mathcal{X}_i < \mathcal{X}$, Equation (28), the inequality in (30) with $j = i$ and the last inequality in (5), we easily conclude

$$\|f_i\|_{r, \rho, \xi_i, \alpha_i, \delta_i} \leq \|f_i\|_{r(1), \rho(1), \xi(1), \rho(1), \delta(1)} \leq 81 \tilde{c}_{n,m} \frac{\mathcal{X}_i}{d_i} \| \frac{1}{\omega_i} \| r, \rho \| f \|_{r, \rho, \xi, \alpha, \delta}$$

$$\leq \frac{1}{\tilde{c}_{n,m}} \frac{d_i}{81N^2} \frac{1}{\mathcal{X}_i} \| \frac{1}{\omega_i} \| r, \rho_i \| r, 1 \rangle^{-1} \leq \frac{1}{\tilde{c}_{n,m}} \frac{d_i}{81N^2} \frac{1}{\mathcal{X}_i} \| \frac{1}{\omega_i} \| r, \rho_i \| r, 1 \rangle^{-1}$$

(33)

which is just (32). Then the Iterative Lemma is applicable to $H_i$, and Equations (29) with $j = i + 1$ follow from it. The proof that also (30) holds (for a possibly larger value of $c_{n,m}$) when $j = i + 1$ proceeds along the same lines as in [4, proof of the Normal Form Lemma, p. 194–95] and therefore is omitted. The same for the proof of the first inequality in (6), for $g_N := H_1$. ■
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