CALOGERO-MOSER SPACE AND KOSTKA POLYNOMIALS

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Abstract. We consider the canonical map from the Calogero-Moser space to symmetric powers of the affine line, sending conjugacy classes of pairs of $n \times n$-matrices to their eigenvalues. We show that the character of a natural $\mathbb{C}^*$-action on the scheme-theoretic zero fiber of this map is given by Kostka polynomials.

1. Introduction

1.1. The aim of this paper is to prove a refined version of Conjecture 17.14 of [EG]. To explain our result, recall the so-called Calogero-Moser space $\mathcal{C}_n$, a $2n$-dimensional complex algebraic manifold introduced by Kazhdan-Kostant-Sternberg [KKS], and studied further by G. Wilson [W]. It is defined as $\mathcal{C}_n := \mathcal{CM}_n//\text{PGL}_n$, the quotient by the natural (free) conjugation-action of the group $\text{PGL}_n$ on the set

$$\mathcal{CM}_n := \{(X,Y) \in \text{Mat}_n \times \text{Mat}_n \mid [X,Y] + \text{Id} = \text{rank 1 matrix}\}. \quad (1.1)$$

Let $A^{(n)}$ denote the set of unordered $n$-tuples of complex numbers. The assignment $(X,Y) \mapsto (\text{Spec}(X), \text{Spec}(Y))$, sending a pair of $n \times n$-matrices to the corresponding pair of $n$-tuples of their eigenvalues gives a map $p : \mathcal{C}_n \to A^{(n)} \times A^{(n)}$. The zero fiber $p^{-1}(0,0)$ of this map is formed by the conjugacy classes of nilpotent pairs $(X,Y) \in \mathcal{CM}_n$. This fiber is known to be a finite set labelled naturally by partitions of $n$. Given such a partition $\lambda$, let $p^{-1}(0,0)_{\lambda}$ be the corresponding point in the zero fiber.

The Conjecture 17.14 of [EG] states that, for any partition $\lambda$, the corresponding point in the (scheme-theoretic) zero fiber of $p$ comes with multiplicity equal to $(\dim V_{\lambda})^2$, where $V_{\lambda}$ is an irreducible representation of the symmetric group $\mathfrak{S}_n$ attached to the partition $\lambda$ in the standard way, see [M].

1.2. In the present paper we propose and prove the following $q$-analogue of the above conjecture. Observe that the complex torus $\mathbb{C}^*$ acts naturally on $\mathcal{CM}_n$ by $z : (X,Y) \mapsto (z^{-1} \cdot X, z \cdot Y)$, $\forall z \in \mathbb{C}^*$. This $\mathbb{C}^*$-action descends to the Calogero-Moser space $\mathcal{C}_n$ and preserves the zero fiber $p^{-1}(0,0)$. Now given $\lambda$, a partition of $n$, let $p^{-1}(0,0)_{\lambda}$ be the corresponding irreducible component of the zero fiber viewed as a non-reduced scheme (set theoretically concentrated at one point). The $\mathbb{C}^*$-action keeps these points (set theoretically) fixed hence, for each $\lambda$, induces a $\mathbb{C}^{*}$-action on the coordinate ring of the scheme $p^{-1}(0,0)_{\lambda}$, a finite dimensional vector space. The character
of this finite dimensional \( \mathbb{C}^* \)-module may be viewed as a Laurent polynomial \( ch_\lambda \in \mathbb{Z}[q,q^{-1}] \). Now, recall that for each partition \( \lambda \) one defines the Kostka polynomial \( K_\lambda(q) \in \mathbb{Z}[q] \) which is a certain \( q \)-analogue of \( \dim V_\lambda \), the dimension of the corresponding irreducible \( \mathfrak{S}_n \)-representation, see e.g. (\[M\] III.6).

Our result reads

**Theorem.** For any partition \( \lambda \) (of \( n \)), we have: \( ch_\lambda = K_\lambda(q) \cdot K_\lambda(q^{-1}) \).

1.3. This result has a natural generalization to other finite complex reflection groups \( W \) in a vector space \( \mathfrak{h} \). In more details, in [EG] the authors associate to a pair \((\mathfrak{h}, W)\) a Calogero-Moser space \( C_W \) together with a finite map \( p : C_W \to \mathfrak{h}/W \times \mathfrak{h}/W \). In the special case \( \mathfrak{h} = \mathbb{C}^n \) and \( W = \mathfrak{S}_n \), the space \( C_W \) reduces to the variety \( C_n \), and the map \( p : C_W \to \mathfrak{h}/W \times \mathfrak{h}/W \) reduces to the map \( p : C_n \to \mathbb{A}^{(n)} \times \mathbb{A}^{(n)} \) considered above.

More generally, in this paper we will consider the case where \( \mathfrak{h} = \mathbb{C}^n \) and \( W = \Gamma \sim \mathfrak{S}_n \) is a wreath product of \( \mathfrak{S}_n \) and \( \Gamma = \mathbb{Z}/NZ \), a cyclic group of some fixed order \( N \) (thus, \( W = \mathfrak{S}_n \times (\mathbb{Z}/NZ)^n \)), acting naturally in \( \mathfrak{h} \). It has been proved in [EG] that the corresponding Calogero-Moser space \( C_{\Gamma,n} := C_W \) is a smooth affine algebraic variety isomorphic to a certain Nakajima’s Quiver variety for a cyclic quiver.

The Conjecture 17.14 of [EG] says that the reduced fiber of \( p \) over \((0,0) \in \mathfrak{h}/W \times \mathfrak{h}/W \) can be identified with the set \( Irrep(W) \) of isomorphism classes of irreducible representations of \( W \), and the multiplicity of the point in this fiber corresponding to \( \rho \in Irrep(W) \) equals \((\dim \rho)^2\).

It is well known that the irreducible \( \Gamma \sim \mathfrak{S}_n \)-modules are naturally parametrized by the set \( \mathfrak{P}_\Gamma(n) \) of \( \Gamma^\vee \)-partitions of \( n \), see e.g. (\[M\], Part I, Appendix B). Here \( \Gamma^\vee \) is the set of irreducible characters of \( \Gamma \), and a \( \Gamma^\vee \)-partition \( \Lambda \) is a collection \((\lambda_\chi, \chi \in \Gamma^\vee)\) of ordinary partitions such that \( \sum_\chi |\lambda_\chi| = n \). It is known that the points of reduced fiber of \( C_{\Gamma,n} \) over \((0,0) \) are also naturally numbered by \( \mathfrak{P}_\Gamma(n) \) (in case of trivial \( \Gamma \) it was proved in [W], and in the general case in [K2]). By abuse of notation we will denote the point in the fiber corresponding to \( \Lambda \in \mathfrak{P}_\Gamma(n) \) by \( \Lambda \) as well.

1.4. The cyclic Calogero-Moser space \( C_{\Gamma,n} \) has a natural \( \mathbb{C}^* \)-action, such that its fixed point set \( C_{\Gamma,n}^{\mathbb{C}^*} \) coincides with the reduced zero fiber. We consider the character of induced \( \mathbb{C}^* \)-action in the Artin coordinate ring \( O_\Lambda \) of the component \( p^{-1}(0,0)_\Lambda \) of the fiber concentrated at the point \( \Lambda \in C_{\Gamma,n}^{\mathbb{C}^*} \).

1.5. For an arbitrary cyclic group \( \Gamma = \mathbb{Z}/NZ \) and \( \Lambda \in \mathfrak{P}_\Gamma(n) \), we introduce a polynomial \( K_\Lambda(q) \in \mathbb{Z}[q] \) which is a \( q \)-analogue of \( \dim V_\Lambda \), the corresponding irreducible \( \mathfrak{S}_n \)-module, see [3.1], and prove the following

**Theorem.** The character of \( \mathbb{C}^* \)-module \( O_\Lambda \) equals \( K_\Lambda(q) \cdot K_\Lambda(q^{-1}) \).

\[1\]In the main body of the paper we use a minor modification of the standard \( K_\Lambda(q) \).
1.6. Our proof is a straightforward application of the remarkable work [W].

G. Wilson has studied the reduced fibers of the second projection $p_2 : \mathcal{C}_n \to A^{(n)}$ and identified them as certain products of Schubert cells in Grassmannians. His results reduce our problem to some classical computations in Grassmannians.

One ingredient in the proof of Theorem 1.4 is a relative Drinfeld compactification $\mathcal{T}_n$ of the Calogero-Moser space $\mathcal{C}_n$ (such that the projection $p_2$ extends to the proper projection $p_2 : \mathcal{T}_n \to A^{(n)}$, see 2.5) and its cyclic version, see 5.3. Though it enters our proof only at some technical point, we believe that $\mathcal{T}_n$ is a very interesting object in itself.

The space $\mathcal{T}_n$ was, in fact, implicitly introduced in [W] where Wilson studied the embedding of $\mathcal{C}_n$ into the adelic Grassmannian $Gr_{ad}$ (the cyclic version of this embedding is studied in [BGK]). Wilson constructed a set-theoretic partition $Gr_{ad} = \bigsqcup_{k \in \mathbb{N}} C_n$. However, it turns out that the union $\bigsqcup_{0 \leq k \leq n} C_k$ can not be equipped with the structure of an algebraic variety. The algebraic variety $\mathcal{T}_n$ has, on the other hand, a natural partition $\mathcal{T}_n = \bigsqcup_{0 \leq k \leq n} C_k \times A^{(n-k)}$, see 2.6, into smooth locally-closed strata (similar in spirit to the stratification used in [K]) and may be viewed as an algebraic ‘resolution’ of $\bigsqcup_{0 \leq k \leq n} C_k$, a nonalgebraic substack of $Gr_{ad}$. The name “Drinfeld’s compactification” is suggested by a close analogy with Drinfeld’s quasimap spaces, cf. [K].

In 2.7 we propose an alternative conjectural definition of $\mathcal{T}_n$ as a step towards its generalization for other Nakajima quiver varieties.

1.7. Acknowledgments. We are deeply obliged to A. Kuznetsov for very useful discussions and comments. M.F. is grateful to the University of Chicago for the wonderful working conditions, and to V. Vologodsky for patient explanations of the trivia of intersection theory. This research was conducted by M.F. for the Clay Mathematics Institute.

2. Wilson’s embedding into a relative Grassmannian

2.1. The Calogero-Moser space. Fix a positive integer $n$ and consider the space $\mathcal{CM}_n$ defined in (1.1). Then $\mathcal{CM}_n$ is smooth, and the action of $\text{PGL}_n$ by the simultaneous conjugation is free (see [W]). The quotient space $\mathcal{C}_n := \mathcal{CM}_n/\text{PGL}_n$ is a $2n$-dimensional smooth affine algebraic variety, the Calogero-Moser space. For $n = 0$ we define $\mathcal{C}_0$ to be a point.

Recall that $\mathbb{A}^{(n)} := \mathbb{A}^\times / S_n$. The assignment $Y \mapsto Spec(Y)$, sending a matrix $Y \in \text{Mat}_n$ to the $n$-tuple of its eigenvalues viewed as a finite subscheme of $\mathbb{A}^1$ given by zeros of the characteristic polynomial of $Y$, yields an isomorphism of algebraic varieties: $\text{Mat}_n/\text{PGL}_n \xrightarrow{\sim} \mathbb{A}^{(n)}$ (where $\text{Mat}_n/\text{PGL}_n$ denotes the categorical quotient). The second projection $\mathcal{CM}_n \to \text{Mat}_n$, $(X, Y) \mapsto Y$, descends to the projection $\pi_n : \mathcal{C}_n \to \mathbb{A}^{(n)}$. Wilson has determined all the reduced fibers of $\pi_n$. Namely, he constructed an embedding of any fiber into a certain product of (finite dimensional) Grassmann varieties, and identified the image with a union of products of
certain Schubert cells. Let us formulate his results more precisely. Till the end of this section fiber means reduced fiber, and we write \( \pi^{-1}(-) \) instead of \( \pi^{-1}(-)_{\text{reduced}} \).

2.2. **Theorem.** (Wilson, [W], 7.1) Suppose a divisor \( D = D_1 + D_2 \in \mathbb{A}^{(n)} \) is a sum of divisors \( D_1 \in \mathbb{A}^{(m)}, D_2 \in \mathbb{A}^{(k)} \) with disjoint supports. Then there is a canonical isomorphism \( \pi_n^{-1}(D) \simeq \pi_m^{-1}(D_1) \times \pi_k^{-1}(D_2) \).

We will refer to this result as the factorization property of the projection \( \pi_n \) (or rather of the collection of maps \( \pi_n \) over \( n \in \mathbb{N} \)).

2.3. In view of the above theorem, in order to describe an arbitrary fiber of \( \pi_n \), it suffices to describe the fiber over the principal diagonal, \( \pi_n^{-1}(ny), y \in \mathbb{A}^1 \). To this end, consider the polynomial algebra \( \mathbb{C}[z] \) and, for any \( y \in \mathbb{C} \) write \( m_y = (z - y) \cdot \mathbb{C}[z] \) for the corresponding maximal ideal. Let \( \text{Gr}(n, y) \) be Grassmannian of \( n \)-dimensional subspaces in the vector space \( \mathbb{C}[z]/m_y^{2n} \).

The vector space \( \mathbb{C}[z]/m_y^{2n} \) comes equipped with a distinguished complete flag

\[
0 \subset m_y^{2n-1} / m_y^{2n} \subset m_y^{2n-2} / m_y^{2n} \subset \ldots \subset m_y / m_y^{2n} \subset \mathbb{C}[z]/m_y^{2n}
\]

(quotients of ideals). This flag defines the Schubert stratification of \( \text{Gr}(n, y) \). Let \( \text{Sch}_n(y) \subset \text{Gr}(n, y) \) denote the locally closed subvariety formed by all the Schubert cells of dimension \( n \).

**Theorem.** ([W], 6.4) There is a canonical isomorphism \( \pi_n^{-1}(ny) \simeq \text{Sch}_n(y) \).

2.4. Wilson also describes the way the above fibers glue together. In order to formulate his result, we recall that \( \mathbb{A}^{(n)} \) may be viewed as the space of all codimension \( n \) ideals \( I \subset \mathbb{C}[z] \), and introduce the following definition.

**Definition.** The relative Grassmannian \( \mathcal{G}_n \) is the space of pairs \( (I, W) \) where \( I \subset \mathbb{C}[z] \) is a codimension \( n \) ideal, and \( W \subset \mathbb{C}[z]/I^2 \) is an \( n \)-dimensional linear subspace.

Clearly, \( \mathcal{G}_n \) is a quasiprojective variety equipped with a projection \( \pi_n : \mathcal{G}_n \twoheadrightarrow \mathbb{A}^{(n)}, \ I, W \mapsto I \). For any \( I \in \mathbb{A}^{(n)} \) we have: \( \pi_n^{-1}(I) \simeq \text{Gr}(n, 2n) \).

Wilson considers an open subset \( C_n^{\text{reg}} \subset C_n \) formed by the (conjugacy classes of) pairs \( (X, Y) \) such that \( Y \) is diagonalizable and has pairwise distinct eigenvalues. Each element in \( C_n^{\text{reg}} \) has a unique representative of the form \( Y = \text{diag}(y_1, \ldots, y_n), X = \|x_{ij}\| \), with \( x_{ij} = (y_i - y_j)^{-1} \), for \( i \neq j \), and \( x_{ii} = \alpha_i \). To the \( n \)-tuple \( (y_1, \ldots, y_n) \) we associate the \( n \)-tuple \( (1 - \alpha_i(z - y_i)) \subset \mathbb{C}[z]/m_y^{2i}, i = 1, \ldots, n \), in the corresponding 2-planes. Wilson defines an embedding \( \beta : C_n^{\text{reg}} \to \mathcal{G}_n \) by the formula \( \beta : (X, Y) \mapsto (I, W) \) where \( I = (z - y_1) \cdots (z - y_n) \), and \( W \) is set to be a direct sum of the lines \( W_i \), that is:

\[
I = m_{y_1} \cdots m_{y_n}, \quad W = \oplus_i W_i \subset \mathbb{C}[z]/I^2 \equiv \mathbb{C}[z]/m_{y_1}^2 \oplus \ldots \oplus \mathbb{C}[z]/m_{y_n}^2.
\]
Theorem. (Wilson, [W], 5.1) (i) The map \( \beta : C_n \to G_n \) commuting with the projections \( \pi_n \): 
(ii) Given \( D = \sum_{k=1}^l n_k y_k \in \mathbb{A}^{(n)} \) and \( C \in \pi_n^{-1}(D) \subset C_n \) write \( C = (W_1, \ldots, W_l) \), \( W_k \in \text{Sch}_{n_k}(y_k) \). Then, under the natural identification \( \mathbb{C}[z]/\prod_{k=1}^l m_{y_k}^{2n_k} \equiv \bigoplus_{k=1}^l \mathbb{C}[z]/m_{y_k}^{2n_k} \), we have 
\[
\beta : C \mapsto (\prod_{k=1}^l m_{y_k}^{n_k}, \bigoplus_{k=1}^l W_k).
\]

2.5. Drinfeld relative compactification. We define \( \overline{C}_n \subset G_n \) as the closure of \( \beta(C_n) \) or, equivalently, of \( \beta(C_n^{rel}) \). Specifically, consider the open stratum of the diagonal stratification \( \hat{\mathbb{A}}^{(n)} \subset \mathbb{A}^{(n)} \) formed by all the \( n \)-tuples of pairwise distinct points. Consider the locally closed subvariety \( \overline{C}_n^{reg} \subset \pi_n^{-1}(\hat{\mathbb{A}}^{(n)}) \subset G_n \) formed by all the pairs
\[
\left\{ (I, W) \mid I = m_{y_1} \cdots m_{y_n}, \ W \subset \mathbb{C}[z]/I^2 \equiv \mathbb{C}[z]/m_{y_1}^2 \oplus \cdots \oplus \mathbb{C}[z]/m_{y_n}^2, \text{ such that } W \cap (\mathbb{C}[z]/m_{y_i}^2) \neq 0, \forall i. \right\}
\]

Thus, \( W \) is a direct sum of lines \( W_i \subset \mathbb{C}[z]/m_{y_i}^2 \).

Definition. The Drinfeld compactification \( \overline{C}_n \subset G_n \) is defined as the closure of \( \overline{C}_n^{reg} \) in \( G_n \). The restriction of \( \pi_n : G_n \to \mathbb{A}^{(n)} \) to \( \overline{C}_n \) is also denoted by \( \pi_n \).

Clearly, \( \pi_n : \overline{C}_n \to \mathbb{A}^{(n)} \) is a projective morphism.

2.6. Twist by a divisor. The rest of this section will not be used elsewhere in the paper but it helps to understand better the structure of \( \overline{C}_n \).

For \( 0 \leq k \leq n \) we will define a map \( \text{twist}^{(n-k)}_k : \overline{C}_k \times \mathbb{A}^{(n-k)} \to \overline{C}_n \) (twist by a divisor). To this end, given an ideal \( I \subset \mathbb{C}[z] \) of codimension \( n-k \), and \( (J, W) \in \overline{C}_k \), take the preimage of \( W \) under the natural projection \( \mathbb{C}[z]/IJ^2 \to \mathbb{C}[z]/J^2 \), and let \( W' \subset I/I^2J^2 \subset \mathbb{C}[z]/I^2J^2 \) correspond to this preimage under the natural identification \( I/I^2J^2 \cong \mathbb{C}[z]/IJ^2 \). We set \( \text{twist}^{(n-k)}_k ((J, W), I) := (IJ, W') \).

From now on we will identify \( C_n \) with its image \( \beta(C_n) \subset \overline{C}_n \subset G_n \). Given \( y \in \mathbb{A} \), write \( \text{Sch}_m(y) \subset \text{Gr}(m, \mathbb{C}[z]/m_{2m}^2) \) for the union of Schubert cells of dimension \( \leq m \). Wilson’s theorem [2,4] immediately implies the following

Theorem. (i) Let \( D = \sum_{k=1}^l n_k y_k \in \mathbb{A}^{(n)} \). Then \( \pi_n^{-1}(D) \subset \overline{C}_n \) equals \( \prod_{k=1}^l \text{Sch}_{n_k}(y_k) \). Specifically, under the natural identification \( \mathbb{C}[z]/\prod_{k=1}^l m_{y_k}^{2n_k} \equiv \bigoplus_{k=1}^l \mathbb{C}[z]/m_{y_k}^{2n_k} \), a point \( (W_1, \ldots, W_l) \), \( W_k \in \text{Sch}_{n_k}(y_k) \), corresponds to \( \bigoplus_{k=1}^l W_k \).

(ii) \( \overline{C}_n \times C_n = \text{twist}^{(n-1)}_{n-1}(\overline{C}_{n-1} \times \mathbb{A}^1) \), where the RHS is a closed subvariety.

(iii) \( \overline{C}_n \) is a disjoint union of the locally closed subvarieties:
\[
\overline{C}_n = \bigsqcup_{k=1}^n \text{twist}^{(n-k)}_k(C_k \times \mathbb{A}^{(n-k)}).
\]

Part (i) implies, in particular, that the map \( \pi_n : \overline{C}_n \to \mathbb{A}^{(n)} \) enjoys the factorization property.
2.7. Remark. One would like to find a construction of $\mathcal{C}_n$ in the ordinary Calogero-Moser setup of $[2,3]$ avoiding the use of adelic Grassmannian. Here is a conjectural definition. Recall that $\mathcal{CM}_n \subset \text{Mat}_n \times \text{Mat}_n$ is a smooth closed subvariety. Now $\text{Mat}_n$ can be viewed as an open subset of $\text{Gr}(n, 2n)$ via identifying a matrix $X$ with the graph $W_X \subset \mathbb{C}^n \oplus \mathbb{C}^n$ of the corresponding linear map $\mathbb{C}^n \rightarrow \mathbb{C}^n$. Let $\mathcal{CM}'_n$ be the closure of $\mathcal{CM}_n$ in $\text{Gr}(n, 2n) \times \text{Mat}_n$.

The group $\text{GL}_n$ acts on $\mathcal{CM}'_n$ naturally: $g(W, Y) = (gWgY^{-1})$. Let $\mathcal{C}'_n$ be the GIT quotient of $\mathcal{CM}'_n$ with respect to $\text{GL}_n$. 

**Question.** Is there an isomorphism $\mathcal{C}'_n \simeq \mathcal{C}_n$ which is the identity on the common open subset $\mathcal{C}'_n \cap \mathcal{C}_n \subset \mathcal{C}_n$?

3. $\mathbb{C}^*$-action on Schubert cells

3.1. Write $\text{Gr}(n)$ instead of $\text{Gr}(n, 0)$ for the Grassmannian of $n$-dimensional subspaces of $\mathbb{C}[z]/(z^{2n})$. We have the standard complete flag in $\mathbb{C}[z]/(z^{2n})$ (see $[2,3]$):

$$0 \subset m_y^{2n-1}/m_y^{2n} \subset m_y^{2n-2}/m_y^{2n} \subset \cdots \subset m_y/m_y^{2n} \subset \mathbb{C}[z]/m_y^{2n}$$

Recall that $\text{Sch}_n \subset \text{Gr}(n)$ is a disjoint union of the $n$-dimensional cells, which are known to be exactly the cells $\text{Sch}_\lambda$ numbered by the set $\mathfrak{P}(n)$ of partitions of $n$. In more detail, given a partition $\lambda = (l_1, \ldots, l_n)$, $0 \leq l_1 \leq \cdots \leq l_n$, $l_1 + \cdots + l_n = n$, we have

$$\text{Sch}_\lambda = \{ W \in \text{Gr}(n) \mid \dim(W \cap (m_y^{2n-l_i}/m_y^{2n})) = i, \forall i = 1, \ldots, n\}.$$

The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}[z]$ by $(c, z^i) \mapsto c^{-i}z^i$. This action induces a natural action on $\text{Sch}_\lambda \subset \text{Gr}(n)$ contracting this Schubert cell to the unique fixed point $W_\lambda := \langle z^{2n-l_1-1}, z^{2n-l_2-2}, \ldots, z^{2n-l_n-n} \rangle$. The tangent space $T_{W_\lambda} \text{Sch}_\lambda$ at the point $W_\lambda$ is naturally isomorphic to

$$\text{Hom}(\mathbb{C} \langle z^{2n-l_1-1}, z^{2n-1}, \ldots, z^{2n-l_1} \rangle) \times$$

$$\text{Hom}(\mathbb{C} \langle z^{2n-l_2-2}, z^{2n-1}, \ldots, z^{2n-l_2-1}, \ldots, z^{2n-l_1-1} \rangle) \times \cdots \times$$

$$\text{Hom}(\mathbb{C} \langle z^{2n-l_n-n}, z^{2n-1}, \ldots, z^{2n-l_n-1}, \ldots, z^{2n-l_n-1}, \ldots, z^{2n-l_n-n+1} \rangle)$$

where $\langle \rangle$ means omission of an element. From this we read off easily the character of $\mathbb{C}^*$ on $T_{W_\lambda} \text{Sch}_\lambda$. Specifically, write $h_\lambda(u)$ for the hook length of a box $u$ in the Young diagram attached naturally to a partition $\lambda$. Below, we use the notation $q^i$ for the character $\mathbb{C}^* \rightarrow \mathbb{C}^*$, $c \mapsto c^i$, and write $ch V$ for the character of a finite dimensional $\mathbb{C}^*$-module $V$.

3.2. Lemma. We have: $ch(T_{W_\lambda} \text{Sch}_\lambda) = \sum_{u \in \lambda} q^{-h_\lambda(u)}$. \hfill $\square$

4. Nilpotent extensions of Schubert cells

4.1. Recall the map $\pi_n: \mathcal{C}_n \rightarrow A^{(n)}$, $(X, Y) \mapsto \text{Spec}(Y)$. Denote this map by $p_2$, and similarly, consider the other projection $p_1: \mathcal{C}_n \rightarrow A^{(n)}$, $(X, Y) \mapsto \text{Spec}(X)$. Note that there is an involution $\omega$ on $\mathcal{C}_n$ such that $\omega: (X, Y) \mapsto$...
(Y^t, X^t), and we have: \( p_1 = p_2 \circ \omega \). Let \( p = (p_1, p_2) \) stand for the simultaneous projection \( (p_1, p_2) : C_n \to \mathbb{A}^{(n)} \times \mathbb{A}^{(n)} \). To distinguish between the two copies of \( \mathbb{A}^{(n)} \) we will use the notation \( p : C_n \to \mathbb{A}_1^{(n)} \times \mathbb{A}_2^{(n)} \). According to [EG], the map \( p \) is a finite morphism.

The scheme theoretic fiber \( p_2^{-1}(0) \) is a disjoint union of schemes \( p_2^{-1}(0) \lambda \) such that the underlying reduced scheme is \( \text{Sch}_\lambda \), to be denoted \( \text{Sch}_\lambda^2 \) from now on. Similarly, the scheme theoretic fiber \( p_1^{-1}(0) \) is a disjoint union of schemes \( p_1^{-1}(0) \lambda \) such that the underlying reduced scheme is denoted by \( \text{Sch}_\lambda^1 \).

Our goal is to compute the scheme theoretic fiber \( p^{-1}(0, 0) \). It is well known that the corresponding reduced scheme is a disjoint union of points: the \( \mathbb{C}^* \)-fixed points of \( \text{Sch}_1^1 \) (or equivalently, \( \text{Sch}_2^2 \)). Abusing the language we will denote the \( \mathbb{C}^* \)-fixed point of \( \text{Sch}_\lambda^1 \) by \( \lambda \); thus \( \text{Sch}_1^1 \cap \text{Sch}_\lambda^2 = \lambda \). We will denote the connected component of \( p^{-1}(0, 0) \) concentrated at \( \lambda \) by \( p^{-1}(0, 0) \lambda \).

Note that \( p^{-1}(0, 0) \lambda \) is the fiber over \( 0 \in \mathbb{A}_1^{(n)} \) with respect to the projection \( p_1 : p_2^{-1}(0) \lambda \to \mathbb{A}_1^{(n)} \). Our first step will be to compute the fiber over \( 0 \in \mathbb{A}_1^{(n)} \) with respect to the projection \( p_1 : \text{Sch}_\lambda^1 \to \mathbb{A}_1^{(n)} \).

4.2. Recall that the Kostka polynomial associated to a Young diagram \( \lambda \) is a polynomial in the variable ‘\( q \)’ given by the formula: \( q^m(\lambda) (1 - q) \ldots (1 - q^n) \prod_{u \in \lambda} (1 - q^{h_\lambda(u)})^{-1} \), where \( m(\lambda) \) is a certain positive integer, see (M, page 143, Example 2). This is a \( q \)-analogue of the dimension \( \dim V_\lambda \) of the irreducible representation \( V_\lambda \) of the symmetric group \( S_n \). We will consider a version of Kostka polynomial with the lowest term equal to 1, that is, we put

\[
K_\lambda(q) := (1 - q) \ldots (1 - q^n) \prod_{u \in \lambda} (1 - q^{h_\lambda(u)})^{-1}.
\]

**Proposition.** We have:

\[ ch \mathcal{O}(\text{Sch}_\lambda^2 \cap p_1^{-1}(0) \lambda) = K_\lambda(q) \quad \text{and} \quad ch \mathcal{O}(\text{Sch}_\lambda^1 \cap p_2^{-1}(0) \lambda) = K_\lambda(q^{-1}). \]

**Proof.** The two formulas are analogous, so we only prove the first one. We compute the geometric fiber of the sheaf \( (p_1)_* \mathcal{O}(\text{Sch}_\lambda^2) \) at the point \( 0 \in \mathbb{A}_1^{(n)} \). This is a locally free coherent sheaf, that is a (trivial) vector bundle, so to compute the character of its geometric fiber at 0 it suffices to know the character \( ch \mathcal{O}(\text{Sch}_\lambda^2) \) of its space of global sections, and the character of \( \mathcal{O}(\mathbb{A}_2^{(n)}) \). Now we pass to the formal completions at 0 and \( \lambda \). Thus, we are reduced to finding the characters of tangent spaces \( T_0 \mathbb{A}_1^{(n)} \) and \( T_\lambda \text{Sch}_\lambda^2 \). The former character equals \( 1 + q^{-1} + \ldots + q^{-n} \), while the latter character was computed in the Lemma 3.3. We conclude that \( ch \mathcal{O}(\mathbb{A}_1^{(n)}, 0) = (1 - q)^{-1} \ldots (1 - q^n)^{-1} \), and \( ch \mathcal{O}(\text{Sch}_\lambda^2, \lambda) = \prod_{u \in \lambda} (1 - q^{h_\lambda(u)})^{-1} \). Thus, we get

\[
ch \mathcal{O}(\text{Sch}_\lambda^2 \cap p_1^{-1}(0) \lambda) = ch \mathcal{O}(\text{Sch}_\lambda^2, \lambda)/ch \mathcal{O}(\mathbb{A}_1^{(n)}, 0) = K_\lambda(q).
\]

\( \Box \)
4.3. We are going to compute \( ch \mathcal{O}(p^{-1}(0,0)_\lambda) \) along similar lines. To this end it suffices to compute the character of the completion \( ch \tilde{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} \).

We will prove that \( ch \tilde{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} = K_\lambda(q^{-1}) \prod_{u \in \lambda}(1 - q^{h_\lambda(u)})^{-1}. \) Hence, arguing exactly as in the proof of 4.2 we will be able to conclude that \( ch \mathcal{O}(p^{-1}(0,0)_\lambda) = K_\lambda(q)K_\lambda(q^{-1}), \) as required in the Theorem 1.4. Thus, to prove the Theorem it suffices to prove the following

**Proposition.** \( ch \tilde{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} = K_\lambda(q^{-1}) \prod_{u \in \lambda}(1 - q^{h_\lambda(u)})^{-1}. \)

4.4. We start the proof of the Proposition with the following

**Lemma.** \( \text{Sch}_\lambda^1 \) and \( \text{Sch}_\lambda^2 \) are transversal at \( \lambda. \)

**Proof.** The varieties \( \text{Sch}_\lambda^1 \) and \( \text{Sch}_\lambda^2 \) are smooth of complementary dimensions. Moreover, the character of \( T_\lambda \text{Sch}_\lambda^2 \) is a polynomial in \( q^{-1} \) without constant term, while the character of \( T_\lambda \text{Sch}_\lambda^1 \) is a polynomial in \( q \) without constant term. Hence, these two tangent spaces must have zero intersection, and we are done. \( \square \)

Thus the formal completion of \( \mathcal{C}_n \) at \( \lambda \) is isomorphic to a product of formal completions of \( \text{Sch}_\lambda^1 \) and \( \text{Sch}_\lambda^2 \) at \( \lambda. \) We will denote by \( pr_1 \) and \( pr_2 \) the projections arising this way. The fiber over \( \lambda \) of the restriction of \( pr_2 \) to the formal completion of \( p_2^{-1}(0)_\lambda \) equals: \( pr_2^{-1}(\lambda) = \text{Sch}_\lambda^1 \cap p_2^{-1}(0)_\lambda. \)

We already know formulas for \( ch \mathcal{O}(\text{Sch}_\lambda^1 \cap p_2^{-1}(0)_\lambda) \) and \( ch \tilde{\mathcal{O}}_{\text{Sch}_\lambda^1,\lambda} \), so to complete the proof it suffices to show that \( pr_2_* \tilde{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} \) is a (trivial) vector bundle on the completion of \( \text{Sch}_\lambda^2 \) at \( \lambda. \) To this end it suffices to show that the dimension of the generic fiber of \( (pr_2)_* \tilde{\mathcal{O}}_{p_2^{-1}(0)_\lambda,\lambda} \) equals \( \dim \mathcal{O}(\text{Sch}_\lambda^1 \cap p_2^{-1}(0)_\lambda) = d_\lambda = \dim V_\lambda. \) But the dimension of the generic fiber equals \( m_\lambda \), the multiplicity of the scheme \( p_2^{-1}(0) \) at the generic point of its reduced subscheme \( \text{Sch}_\lambda^2. \)

To compute this multiplicity \( m_\lambda \) we may as well work in the Drinfeld compactification \( \overline{\mathcal{C}}_n \) embedded into the relative Grassmannian \( \mathcal{G}_n \) over \( \mathbb{A}^{(n)}_2 \).

A general fiber \( p_2^{-1}(y) \subset \text{Gr}(n,y) \) is reduced at the generic point, so \( m_\lambda \) is the coefficient of the cycle class \( [p_2^{-1}(y)] \) with respect to the Schubert basis \( \{ [\text{Sch}_\lambda], \lambda \in \mathcal{P}(n) \} \) of the degree \( 2n \) homology group of \( \text{Gr}_n. \)

Now recall that a general \( n \)-tuple \( \underline{y} = (y_1, \ldots, y_n) \in \mathbb{A}^{(n)}_2 \) of pairwise distinct points gives rise to a direct sum decomposition \( C[z]/m_{y_1}^2 \cdot \cdots \cdot m_{y_n}^2 = \bigoplus_i C[z]/m_{y_i}^2, \) and \( p_2^{-1}(y) \subset \text{Gr}(n,y) \) is the product of corresponding projective lines: \( p_2^{-1}(y) = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \subset \text{Gr}_n. \) It is the classical result of Schubert calculus that for the corresponding homology classes one has an expansion: \( [\mathbb{P}^1 \times \cdots \times \mathbb{P}^1] = \sum \lambda m_\lambda \cdot [\text{Sch}_\lambda], \) where the coefficients \( m_\lambda \) can be read off from the formula: \( p^T_1 = \sum \lambda m_\lambda \cdot s_\lambda, \) an expansion of the \( n \)-th power of the first symmetric function \( p^T_1 \) with respect to the basis of Schur functions \( s_\lambda. \) The coefficients in the latter expansion are well-known to be equal to \( d_\lambda = K_\lambda(1), \) see e.g. \( ([M], \text{page 114}). \)
This completes the proof of Proposition 4.3 and the proof of Theorem 4.4. □

5. Cyclic Calogero-Moser space

5.1. Consider the action of $\Gamma = \mathbb{Z}/N\mathbb{Z} \subset \mathbb{C}^*$ on the Calogero-Moser space $C_{nN}$. The fixed-point subvariety $C_{nN}^\Gamma$ consists of various connected components. There is a single component characterized by the property that the representation of $\Gamma$ in the fiber of tautological bundle at any point in this component is a multiple of the regular representation, see [K2]. We will call this connected component $C_{\Gamma,n}$. According to loc. cit., $C_{\Gamma,n}$ is a special case of Nakajima’s Quiver variety (corresponding to $N$-cyclic quiver with $n$-dimensional spaces at all ”finite” vertices, 1-dimensional space at an ”extended” vertex, and a nonzero value of the diagonal moment map).

We have the natural projection $p = (p_1, p_2) : C_{\Gamma,n} \to (A_1^{(nN)} \times A_2^{(nN)})^\Gamma$. Note that

$$(A_1^{(nN)} \times A_2^{(nN)})^\Gamma = (A_1^{(nN)})^\Gamma \times (A_2^{(nN)})^\Gamma$$

and

$$(A_1^{(nN)})^\Gamma = (A_1^{1} \times \Gamma^{(n)}).$$

We let $A_{\Gamma,1,2}^{(n)}$ denote the set on the right of this formula, and view $p$ as a projection $p = (p_1, p_2) : C_{\Gamma,n} \to A_{\Gamma,1}^{(n)} \times A_{\Gamma,2}^{(n)}$. The natural $\mathbb{C}^*$-action on $C_{nN}$ when restricted to $C_{\Gamma,n}$ factors through $\mathbb{C}^* \xrightarrow{\sim} \mathbb{C}^*$, and we will consider the resulting $\mathbb{C}^*$-action on $C_{\Gamma,n}$ (which is generically free).

5.2. Wilson’s embedding $\beta : C_{nN} \hookrightarrow G_{nN}$ is $\Gamma$-equivariant, and its image lands into a connected component $G_{\Gamma,n} \subset G_{nN}^\Gamma$ characterized by the property that the representation of $\Gamma$ in the fiber of tautological bundle at any point of this component is a multiple of the regular representation (to see the inclusion: $C_{\Gamma,n} \subset G_{\Gamma,n}$ it suffices to check it at any $\mathbb{C}^*$-fixed point, e.g. $\lambda = (nN)$). We will denote by $\beta : C_{\Gamma,n} \hookrightarrow G_{\Gamma,n}$ this cyclic version of Wilson’s embedding, and we will use it to describe the reduced fibers of $p_2$.

First of all, the action of $\Gamma$ on $\mathbb{C}[z]/(z^{2nN})$ splits it into a direct sum $\mathbb{C}[z]/(z^{2nN}) = \bigoplus_{\chi \in \Gamma^\vee} \mathbb{C}[z]/(z^{2nN})_{\chi}$ of $N$ $2n$-dimensional eigenspaces according to the characters of $\Gamma$. Note that we can canonically identify $\Gamma^\vee$ with $\mathbb{Z}/N\mathbb{Z}$, and then $\mathbb{C}[z]/(z^{2nN})_{\chi}$ is spanned by $\{z^k, k \equiv \chi \pmod{N}\}$. The fiber of $G_{\Gamma,n}$ over $nN \cdot 0 \in (A_2^{(nN)})^\Gamma$ is formed by all the $nN$-dimensional subspaces $W \subset \mathbb{C}[z]/(z^{2nN})$ such that $W = \bigoplus_{\chi \in \Gamma^\vee} W_{\chi}$, $W_{\chi} \subset \mathbb{C}[z]/(z^{2nN})_{\chi}$, and $\dim W_{\chi} = n$. Thus, this fiber equals $\prod_{\chi \in \Gamma^\vee} \text{Gr}(n, \mathbb{C}[z]/(z^{2nN})_{\chi})$. Each space $\mathbb{C}[z]/(z^{2nN})_{\chi}$ carries a natural complete flag (given by the intersections with powers of the maximal ideal). Thus, each variety $\text{Gr}(n, \mathbb{C}[z]/(z^{2nN})_{\chi})$ has a natural stratification into Schubert cells numbered by partitions.

Set $\Psi_{\Gamma}(n) := \{\chi \in \Gamma^\vee, \sum_{\chi} |\lambda_{\chi}| = n\}$, and given $\Lambda \in \Psi_{\Gamma}(n)$ put $\text{Sch}_{\Lambda} := \bigcap_{\chi \in \Gamma^\vee} \text{Sch}_{\lambda_{\chi}} \subset \prod_{\chi \in \Gamma^\vee} \text{Gr}(n, \mathbb{C}[z]/(z^{2nN})_{\chi})$. Now Wilson’s theorem 2.3 together with [K2] yield the following.
Proposition. The reduced fiber of \( C_{\Gamma,n} \) over \( 0 \in A_{\Gamma,2}^{(n)} \) is canonically isomorphic to \( \prod_{\Lambda \in \Phi(n)} \text{Sch}^2_{\Lambda} \). \( \square \)

Corollary. (i) Each component \( \text{Sch}^2_{\Lambda} \) contains a unique \( \mathbb{C}^* \)-fixed point \( \Lambda \in C_{\Gamma,n} \).

(ii) The reduced fiber of \( p : C_{\Gamma,n} \to A_{\Gamma,1}^{(n)} \times A_{\Gamma,2}^{(n)} \) over \((0,0)\) coincides with \( C^*_{\Gamma,n} = \Phi(n) \).

We will denote by \( p^{-1}(0,0) \) the connected component of the scheme theoretic fiber concentrated at the point \( \Lambda \), and we will denote by \( p_{1,2}^{-1}(0) \) the connected component of the scheme theoretic fiber concentrated at \( \text{Sch}_{\Lambda}^{1,2} \).

5.3. We define the Drinfeld compactification \( \overline{C}_{\Gamma,n} \supset C_{\Gamma,n} \) as the closure of \( C_{\Gamma,n} \) inside \( \mathcal{G}_{\Gamma,n} \).

We will need a description of a general fiber of \( p_2 : \overline{C}_{\Gamma,n} \to A_{\Gamma,2}^{(n)} \). If we choose a primitive \( N \)-th root of unity \( \zeta \) then a general point \( y \in A_{\Gamma,2}^{(n)} \) can be represented by a collection

\[ y = (y_1, \zeta y_1, \ldots, \zeta^{N-1} y_1, y_2, \ldots, \zeta^{N-1} y_2, \ldots, y_n, \ldots, \zeta^{N-1} y_n) \]

of distinct points of \( A_{\Gamma,2}^{(n)} \). The \( 2nN \)-dimensional vector space \( V = \mathbb{C}[z]/m_y^2 \ldots m_{\zeta^{N-1} y_n}^2 \) is acted upon by \( \Gamma \), and splits up into a direct sum of \( N \) \( 2n \)-dimensional eigenspaces \( V_\chi \) according to the characters of \( \Gamma \). We also have a direct sum decomposition \( V = U_1 \oplus \ldots \oplus U_n \) where \( U_i = \mathbb{C}[z]/m_{y_i}^2 \ldots m_{\zeta^{N-1} y_i}^2 \). Note that for any \( i \) and \( \chi \) the intersection \( U_i \cap V_\chi \) is \( 2 \)-dimensional. We will denote this intersection by \( UV_{i,\chi} \).

The fiber of \( \mathcal{G}_{\Gamma,n} \) over \( y \) equals \( \prod_{\chi \in \Gamma^\vee} \text{Gr}(n, V_\chi) \). The fiber of \( \overline{\mathcal{C}}_n \) over \( y \) is isomorphic to \( \prod_{1 \leq i \leq n} \mathbb{P}^1 \); let us explain how it is embedded into \( \prod_{\chi \in \Gamma^\vee} \text{Gr}(n, V_\chi) \). We have a direct sum decomposition \( U_i = \bigoplus_{\chi \in \mathbb{Z}/N} \mathbb{C}[z]/m_{\zeta^i y_i}^2 \), and the action of \( \Gamma \) on \( U_i \) permutes these summands. Hence \( \mathbb{C}[z]/m_{y_i}^2 \) projects isomorphically onto any \( UV_{i,\chi} \). Given a line \( \ell_i \in \mathbb{P}^1(\mathbb{C}[z]/m_{y_i}^2) \) we denote by \( \ell_{i,\chi} \subset UV_{i,\chi} \) its image under the above isomorphism projection. Finally, for a collection \( \{\ell_i\} \in \prod_{1 \leq i \leq n} \mathbb{P}^1(\mathbb{C}[z]/m_{y_i}^2) \) the corresponding point of \( \prod_{\chi \in \Gamma^\vee} \text{Gr}(n, V_\chi) \) is the collection of subspaces \( (\bigoplus_i \ell_{i,\chi} \subset V_\chi) \).

5.4. Our aim is to compute the character of \( \mathbb{C}^* \)-action on the Artin ring \( \mathcal{O}(p^{-1}(0,0)_{\Lambda}) \), that is, to prove Theorem 4.2. The proof is entirely similar to that of 4.4. Let us spell out the intermediate steps. First, we define:

\[ K_\Lambda(q) := (1-q) \ldots (1-q^n) \prod_{\chi \in \Gamma^\vee} (1-q^{h_\chi(w)})^{-1} \]

Analogously to Proposition 4.2, we obtain

Proposition. We have:

\[ ch \mathcal{O}(\text{Sch}^2_{\Lambda} \cap p_{1}^{-1}(0)_{\Lambda}) = K_\Lambda(q) \quad \text{and} \quad ch \mathcal{O}(\text{Sch}^1_{\Lambda} \cap p_{2}^{-1}(0)_{\Lambda}) = K_\Lambda(q^{-1}). \]
Further, an analogue of Proposition 4.3 reads

5.5. Proposition. \( ch \hat{\partial}_{p_2^{-1}(0)_A} = K_{\Lambda}(q^{-1}) \prod_{\chi \in \Gamma^\vee} (1 - q^{h_{\Lambda}(u)})^{-1}. \)

To prove this last Proposition we argue, as in 4.4, that it suffices to check if the generic multiplicity \( m_{\Lambda} \) of \( p_2^{-1}(0)_A \) equals \( d_{\Lambda} := K_{\Lambda}(1) \). To this end we turn to the cyclic version of Drinfeld compactification \( \mathcal{C}_{\Gamma,n} \), see 5.3. A general fiber \( p_2^{-1}(y) \) being reduced at the generic point, \( m_{\Lambda} \) are the coefficients of the cycle class \( [p_2^{-1}(y)] \) with respect to the Schubert basis \( \{ [S_{\chi\Lambda}], \Lambda \in \mathcal{P}_\Gamma(n) \} \) of the degree 2 homology group of \( \prod_{\chi \in \Gamma^\vee} \text{Gr}_n \). Our description of the general fiber \( p_2^{-1}(y) \) in 5.3 boils down to the following.

Take the diagonal embedding \( \mathbb{P}^1 = \Delta_{\mathbb{P}^1} \hookrightarrow \prod_{\chi \in \Gamma^\vee} \mathbb{P}^1_{\chi} \). For each \( \chi \in \Gamma^\vee \) we have an embedding \( (\mathbb{P}^1_{\chi})^n \hookrightarrow \text{Gr}_n \) as in 4.4. Now form the composition

\[
(\mathbb{P}^1)^n = (\Delta_{\mathbb{P}^1})^n \hookrightarrow \prod_{\chi \in \Gamma^\vee} (\mathbb{P}^1_{\chi})^n \hookrightarrow \prod_{\chi \in \Gamma^\vee} \text{Gr}(n, V_{\chi}).
\]

The homology class of \( [\Delta_{\mathbb{P}^1}] \) in the 2-homology of \( \prod_{\chi \in \Gamma^\vee} \mathbb{P}^1_{\chi} \) equals \( \sum_{\chi} [\mathbb{P}^1_{\chi}] \), the sum of degree 2 the generators of the homology groups of the factors. As in 4.4, we conclude that \( [(\Delta_{\mathbb{P}^1})^n] = \sum_{\Lambda} m_{\Lambda} \cdot [S_{\chi\Lambda}] \) where the coefficients \( m_{\Lambda} \) equal the coefficients of \( (\sum_{\chi} p_{i,\chi})^n \) with respect to the basis of Schur functions \( S_{\Lambda} \) (here \( p_{i,\chi} \) is the first power sum symmetric function in the variables \( x_{i,\chi}, \ 1 \leq i < \infty \), and \( S_{\Lambda} = \prod_{\chi} s_{\Lambda}(x_{i,\chi}) \), see [M], part I, Appendix B.) The latter coefficients are in turn equal to:

\[
n! / \prod_{\chi \in \Gamma^\vee} h_{\Lambda}(u) = d_{\Lambda} = K_{\Lambda}(1), \quad \text{see loc. cit. (9.6) on page 178.}
\]

This completes the proof of Proposition 5.3, hence the proof of Theorem 1.5. 

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