Wave functions in disordered wires in a weak magnetic field

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Abstract

Using the supersymmetry technique combined with the transfer matrix approach we calculate different physical quantities characterizing localization in disordered wires. In particular, we analyze the density-density correlation function and its moments and study effects of an external magnetic field $H$ on tails of wave functions. At zero and very strong magnetic fields, we obtain explicit expressions for all moments and for the entire distribution of the wave functions. The crossover between the two limiting cases is more difficult and calculations are performed for weak magnetic fields only. We found that the far tail of the average density-density correlation function and of its moments is strongly influenced by a weak magnetic field and decays twice as slow as their main body. Extending Mott’s physical picture for the localized states we present also a qualitative description of the crossover in the magnetic field. From both the analytical calculations and the qualitative description we argue that the slower decay of the averaged quantities is a consequence of rare but large splashes of wave functions at large distances. The distribution of the logarithm of the wave function should also be affected by the magnetic field. The splashes and the two-scale decay of the averaged correlation functions can be relevant for the conductance of a system of disordered wires connected in parallel.

72.15.Rn, 73.20.Fz, 72.20.Ee
I. INTRODUCTION

Localization of all states in one-dimensional chains and quasi-one-dimensional wires with a finite thickness is a very well known phenomenon in the theory of disordered metals. Following the first works\textsuperscript{1,2} where the localization was predicted, a number of analytical methods has been developed to treat both the chains\textsuperscript{3–5} and wires\textsuperscript{6–8} quantitatively. A detailed discussion of the localization in the chains and wires can be found, e.g., in Refs.\textsuperscript{9–12}

The localization of electron waves in weakly disordered chains and wires occurs due to destructive interference and can be destroyed at finite temperatures by inelastic scattering. The phenomenon of the electron localization in thick wires is richer than in chains. The localization length $L_c$ in chains is of the order of the mean free path $l$. Electrons move at distances smaller than the localization length ballistically without being scattered by impurities and get localized at distances exceeding $L_c$. In contrast, the localization length $L_c$ in wires is larger than the mean free path $l$ by a factor proportional to the number of channels of the transversal quantization $N$. At distances larger than $l$ but smaller than $L_c$, the electrons diffuse and become localized only at distances larger than $L_c$.

One more important difference between chains and wires manifests itself if an external magnetic field $H$ is applied. In chains the magnetic field cannot influence the electron motion, whereas the motion in thick wires is quite sensitive to it. It turns out that the magnetic field does not destroy the localization of the wave functions but changes the localization length. Remarkably, the localization length $L_c$ in thick wires exactly doubles when applying a sufficiently strong magnetic field.\textsuperscript{6,7}

For analytic calculations for thick wires a supermatrix $\sigma$-model\textsuperscript{7,12} proved to be a powerful tool that allows one to reduce the calculation of kinetic quantities at arbitrary frequencies $\omega$ to solving of a system of differential equations. The localization length $L_c$ is obtained from an exponential decay of the density-density correlation function. The most interesting limit $\omega \rightarrow 0$ can be investigated explicitly and one obtains relatively simple formulae for different physical quantities proving the localization of all states.

Calculation of the transmittance of a finite sample with ideal leads is the basis of many numerical investigations of the localization properties. The equivalence of different methods of calculation of the localization length $L_c$ is traditionally guaranteed by the Borland conjecture\textsuperscript{13} stating a complete independence of the localization properties of the boundary conditions.

The limiting cases of a strong magnetic field and no magnetic field, the unitary and the orthogonal ensembles, respectively, are relatively well studied in quasi 1D, whereas the crossover between these two limits has not been well understood. It was assumed that the localization length changed smoothly\textsuperscript{14,15} between $L_c$ and $2L_c$ and one had to calculate a curve connecting $L_c$ and $2L_c$ when increasing the magnetic field from 0 to $\infty$.

At the same time, investigation of the crossover can have quite interesting applications now because a systematic experimental study of the localization has been performed recently. In the experiment\textsuperscript{16,17} a large number of submicrometer-wide wires prepared from Si $\delta$-doped GaAs were connected in parallel to insure statistical averaging. This enabled accurate measurements of the conductivity as a function of the temperature $T$ and the magnetic field $H$ applied perpendicular to the wires. The regimes of the weak and strong localization were observed and the experimental data in the regime of strong localization were used to extract
the dependence of the localization length $L_c$ on $T$ and $H$. The localization length $L_c$ was shown to double in a strong magnetic field as compared with the limit of zero magnetic field $H$, which is in complete agreement with the theory. As the measurements are done at an arbitrary magnetic field, study of the crossover between the orthogonal and unitary ensembles may become very important.

In a recent publication, we discovered that already an arbitrarily weak magnetic field drastically changes the tails of the averaged amplitudes of the wave functions. At sufficiently large distances depending on the magnetic field, the exponential decay with the localization length $L_c$ changes to a decay with the length $2L_c$. This result is of interest not only from the theoretical point of view but it might also be important for the transport measurements, since the localization length $L_c$ is relevant for the hopping conductivity at low temperature.

However, a numerical study of the averaged logarithm of the transmittance performed for sufficiently long wires did not manifest any two-scale behavior showed a smooth variation of the localization length between $L_c$ and $2L_c$. No sharp change in the far tail of the logarithm of the transmittance was observed in a weak magnetic field. As a possible explanation of the contradiction several reasons were suggested. As one of the possibilities, the difference between the logarithm of the averaged wave functions, the quantity we have considered, and the averaged logarithm of the wave functions was mentioned.

In this paper, we give details of the calculations presented in our short publication and discuss the question in what situations our effect of the changing of the tails of the wave functions by a weak magnetic field can be identified. To understand better the shape of the wave functions we calculate not only the density-density correlation function but also its moments. The changes in the tails of the wave functions are seen in all moments of the density-density correlation functions and all of them are proportional at large distances $x$ to $\exp(-|x|/4L_{cu})$, where $L_{cu}$ is the localization length for the unitary ensemble. The independence of the exponential decay of the asymptotics of the number of the moment signals that they are formed by strong rare splashes of the wave functions.

The situation with the averaged logarithm of the wave function is more delicate. The large splashes contribute also to the averaged logarithm of the wave functions at finite distances. As a result, some broad distribution of logarithms occurs. Unfortunately, we are not able to predict how this distribution function develops in the limit of infinitely long distances. However, our results provide us reasons enough to suggest that the logarithmically normal distribution implied within the Borland conjecture is not universally applicable for length of the sample $x \gg L_c$ for the description of the crossover region. Existing numerical studies might be interpreted as the head of some logarithmically normal distribution changes smoothly with $H$. Our results do not exclude a smooth behavior of the average logarithm of the wave function. We show, however, that another length scale, $x_H \sim L_c \ln(1/H)$ enters the theory of localization, up to which the distribution should differ from the standard logarithmically normal one. Thus, the whole distribution function of the logarithm is of interest.

For calculations we use the supersymmetry technique that makes possible consideration of disordered wires with any kind of boundary conditions. In particular, one can study the transmittance of the disordered wires with metallic leads. An external magnetic field is naturally accounted for within supersymmetry formalism allowing study of the crossover...
between the unitary and orthogonal ensembles.

The localization of the wave functions discussed in the present paper is extracted from the moments of the density-density correlation function

$$p^{(n)}_{\infty}(x - x') = \left\langle \sum_{\alpha} |\psi_{\alpha}(x)|^{2n} |\psi_{\alpha}(x')|^{2n} \delta(\varepsilon - \varepsilon_{\alpha}) \right\rangle$$

(1)

where $\psi_{\alpha}(x)$ and $\varepsilon_{\alpha}$ are the eigenfunction and eigenenergy of a state $\alpha$, $x$ and $x'$ are the coordinates along the wire, and the angle brackets stand for the averaging over the disorder. This quantity was previously studied for a closed finite multichannel wire, when the wave functions were taken at the ends of the wire. Generalizing the expressions for arbitrary $n$, the complete distribution was also obtained.

Naturally, the very presence of hard boundaries or ideal leads inevitably changes localized states near the ends of the disordered sample. These states should differ from those localized in the bulk. Such a difference has been found indeed for the density of states of a single chain.

The localization properties extracted from the transmittance of an open system and from the density-density correlator of a closed one manifest a generic universality: All moments of the both quantities decay with the same rate, $\exp(-|x - x'|/4L_c)$, and the logarithm of the both quantities is normally distributed. The fact that the decay rate is the same for all moments shows that an important contribution comes from rare strong splashes of the wave functions. Due to these splashes the moments decay slower with the length $4L_c$ and not with the localization length $L_c$. At the same time, the logarithm of physical quantities like the transmittance or the product of the amplitudes of the wave functions at different points is a self-averaging quantity and approaches the value $-|x - x'|/L_c$ for $|x - x'| \to \infty$. This quantity characterizes typical wave functions. The question what happens with the wave functions if a weak magnetic field is switched on has not been discussed.

In the subsequent chapters we present details of our analysis. We use an approach based on the Lebedev-Kontorovich transformation to find the exact distribution of any quantity of interest in quasi-1D wires for the orthogonal and unitary ensemble. This allows us to calculate the correlator, Eq. (1), and the Landauer-like conductivity. Exact expressions valid at arbitrary distances are obtained for all moments for the pure orthogonal and unitary ensemble. Using the formulae for the moments we derive the entire distribution function. At a weak magnetic field, we show that a peculiar two-scale behavior characterizes any moment of the wave functions, Eq. (1).

The paper is organized as follows: In Sec. II we introduce the quantities to be studied. A transfer-matrix formalism used for the calculations is described in Sec. III. Reduction to an effective Schrödinger equation is carried out in Sec. IV. Then, in Sec. V we calculate the moments and the distribution functions in the limiting cases of the orthogonal and unitary ensembles. In Sec. VI we present results for the wire in a weak magnetic field. Using a standard qualitative picture for localized states we discuss the physics involved in Sec. VII. Our arguments allow us to obtain without any calculation the characteristic length scales of the problem and reach a better understanding of the localization in the magnetic field. We discuss the results and make conclusions in Sec. VIII.
II. CORRELATION FUNCTIONS

Localization in disordered metals can be well characterized by different correlation and distribution functions. For an infinite sample one can consider the density-density correlation function and its moments. For a finite wire connected to metallic leads the transmittance and its moments are of interest.

These quantities can be efficiently studied using the supersymmetry method. Following this approach one should reduce calculation of the correlation functions of interest to computation of correlation functions of a supermatrix $\sigma$-model. In the standard formulation the $\sigma$-model contains $8 \times 8$ supermatrices $Q$. Fluctuations of the supermatrices $Q$ are strongly influenced by a magnetic field and one half of the “degrees of freedom” is suppressed if the magnetic field is sufficiently strong. This corresponds to reducing the size of the supermatrices $Q$ to $4 \times 4$. Keeping this size of the supermatrices is sufficient to calculate such physical quantities as conductivity or density-density correlation function.

In order to calculate higher moments of these quantities or distribution functions one has to increase the size of the supermatrices, which would make the theory very complicated. Fortunately, calculation of moments of certain quantities does not demand increasing the size of the supermatrices, which allows to compute them explicitly.

In all such cases the free energy functional $F[Q]$ describing fluctuations of the $8 \times 8$ supermatrices $Q$ can be written as

$$F[Q] = \frac{\pi \nu}{8} \text{Str} \int \left[ D \left( \nabla_r Q(r) - \frac{i e}{\hbar c} A[Q(r), \tau_3] \right)^2 + 2i \omega \Lambda Q(r) \right] dr.$$  \hspace{1cm} (2)

The $\sigma$-model, Eq. (2), appears after averaging over disorder and we are to carry out computations with the regular model. In Eq. (2) $D$ is the classical diffusion coefficient, $\nu$ is the density of states, and $\omega$ is the external frequency. The supermatrix $Q$ obeys the constraint $Q^2 = 1$, $A$ is the vector potential corresponding to an external magnetic field $H$, and the standard notations for the supertrace Str and matrices $\Lambda, \tau_3$ are used. Any quantity of interest in the present formulation is expressed as a functional integral over $Q(r)$ with the weight $\exp(-F[Q])$ and a combination of elements of the supermatrix $Q$ in the pre-exponential.

We study properties of wave functions calculating their moments written in Eq. (1). As the first step, we express the moments of the wave functions in terms of retarded $G^R$ and advanced $G^A$ Green functions that can be written as

$$G_{\varepsilon}^{R,A}(x, x') = \sum_\alpha \frac{\psi_\alpha(x) \psi^*_\alpha(x')}{\varepsilon - \varepsilon_\alpha},$$  \hspace{1cm} (3)

where $\psi_\alpha$ and $\varepsilon_\alpha$ are eigenfunctions and eigenenergies of the electron states in the disordered system. Using the spectral expansion, Eq. (3), assuming that all states are localized, such that the spectrum is discrete, and considering the most divergent terms at vanishing frequency $\omega \to 0$ one can come to the following relation

$$[G^R_\varepsilon(x, x')G^A_{\omega + \varepsilon}(x', x)]^n = \frac{2\pi i (2n - 2)! (-1)^{n-1}}{((-1)^n)!^2 \omega^{2n-1}} \sum_\alpha |\psi_\alpha(x)|^{2n} |\psi_\alpha(x')|^{2n} \delta(\varepsilon - \varepsilon_\alpha)$$  \hspace{1cm} (4)
Although Eq. (4) contains an arbitrary power of the Green functions we can express the LHS in terms of 8-component supervectors. This possibility is due to the fact that the Green functions are taken at two different points only. The next step is the averaging over disorder and the decoupling the effective “interaction” by integration over the supermatrices $Q$. A functional integral over $Q$ is simplified in the saddle-point approximation and we come after standard manipulations to the following expression

$$\left\langle \left[ G^R_{\omega}(x, x') G^A_{\omega+\nu}(x', x) \right]^n \right\rangle = (\pi \nu)^{2n} \left\langle \left[ Q^{12}_{33}(x) \right]^n \left[ Q^{21}_{33}(x') \right]^n \right\rangle_F \ .$$

(5)

In the LHS and RHS of Eq. (5), averaging over the impurity potential and averaging over the free energy, Eq. (2), is implied, respectively. The superscripts and subscripts of the supermatrices $Q$ stand for certain matrix elements.

In Section V, exact expressions for the moments of the density-density correlation function, Eq. (1), as well as its entire distribution function

$$P_\psi(t) = \left\langle \sum_\alpha \delta \left( t - |\psi_\alpha(x)|^2 |\psi_\alpha(x')|^2 \right) \delta (\varepsilon - \varepsilon_\alpha) \right\rangle \ .$$

(6)

will be obtained. Hereinafter we use dimensionless expression for the wave functions $\psi SL_c \rightarrow \psi$, where $S$ is the cross-section of the wire and $L_c$ is the localization length. To calculate the distribution function $P_\psi(t)$, Eq. (6), we introduce an auxiliary correlation function

$$P_{a,b}(t_1, t_2) = \left\langle \delta \left( t_1 - aQ^{12}_{33}(x_1) \right) \delta \left( t_2 - bQ^{21}_{33}(x_2) \right) \right\rangle_F \ ,$$

(7)

where $a$ and $b$ are some parameters. Once the correlator Eq. (7) is known, any other quantity of interest can be found by integrations of the type $\int_1^\infty P_{a,b(n)} \ da$ using a proper choice of $a, b(n)$. As an example, we will calculate the function $P_\psi$ and the distribution

$$P_Q = \left\langle \delta \left( t + (i\omega L_c\nu S)^2 Q^{12}_{33}(x) Q^{21}_{33}(x') \right) \right\rangle_F \ ,$$

(8)

characterizing the Landauer-type conductivity. Hereinafter, all the correlators are studied in the limit $\omega \rightarrow 0$.

To express the distribution Eq. (8) in terms of a functional integral over the supermatrices $Q$, we relate the numerical coefficients of all moments of the distribution functions $P_\psi$ and $P_Q$, using the identity

$$\frac{(2n-2)!(-1)^n}{((n-1)!)^2} = \left. \frac{d}{d\beta} \right|_{\beta=1} \int_0^1 dp \ (\beta p)^n (1 - p)^{n-1} \ .$$

(9)

Thus, we reduce the entire distribution $P_\psi$ to $P_Q$

$$P_\psi(t) = \left. \frac{d}{d\beta} \right|_{\beta=1} \int_0^1 \frac{dp}{\beta p(1 - p)} \left\langle \delta \left( t + (i\omega L_c S)^2 \beta^2 p(1 - p) Q^{12}_{33}(x) Q^{21}_{33}(x') \right) \right\rangle_F \ .$$

(10)

Eq. (10) can be proven taking $P_\psi(t)$ from Eq. (8) and expanding both sides in $\psi_\alpha$ and $Q$ respectively. Then, one should use Eqs. (4, 5) and this gives finally Eq. (10) with the
functional $F$ from Eq. (2). Eq. (10) allows us to calculate the distribution function of the amplitudes of the wave functions at two different points of an infinite sample.

Although we calculate here the distribution functions of the quantities written at two different points $x$ and $x'$, our procedure can be extended to many-point functions.

In quasi-1D case, one can calculate the functional integral, Eq. (7), exactly using the transfer-matrix technique. Within this method the functional integration is reduced to solving an effective “Schrödinger equation” in the space of variables parametrizing $Q$-matrices. Solving the “Schrödinger equation” and calculating certain “matrix elements” that are definite integrals over $Q$ one can obtain any physical quantity of interest. For instance, the moments corresponding to the distribution function $P_{a,b}$, Eq. (7), read

$$T_{mn} = \langle t^n t_m \rangle_{\mathcal{P}_Q} = a^n b^m \int \Psi(Q) \left( Q_{33}^{13} \right)^m \Gamma(x,x';Q,Q') \left( Q_{33}^{21} \right)^n \Psi(Q') \, dQdQ', \quad (11)$$

where the function $\Psi(Q)$ is the partition functions of the parts of the wire to the left from the point $x$ and to the right from the point $x'$, and the function $\Gamma$ represents the partition functions of the segment between $x$ and $x'$.

The meaning of the functions $\Psi$ and $\Gamma$ becomes clear after the reduction of the functional integrals to the effective Schrödinger equation. The function $\Psi(Q)$ appears to be the ground state of the effective Hamiltonian $\mathcal{H}_Q$ acting in the space of the elements of the $Q$-matrix whereas the function $\Gamma$ is the Green function in this space. They satisfy the following equations

$$\mathcal{H}_Q \Psi(Q) = 0 \quad (12)$$

and

$$\left( \frac{\partial}{\partial x} + \mathcal{H}_Q \right) \Gamma(x,x';Q,Q') \, dx = \delta(x-x')$$

The fact that the energy of the ground state is zero is a consequence of the supersymmetry of the initial $\sigma$-model, Eq. (3).

The explicit form of the Hamiltonian $\mathcal{H}_Q$ depends on the parametrization of the $Q$-matrix and its choice depends on the quantities calculated and the physics involved.

The supermatrices $Q$ consist of compact and non-compact blocks containing circular and hyperbolic functions, respectively. In the cases of zero and strong magnetic fields only the non-compact sector survives in the limit $\omega \to 0$, which greatly simplifies the whole analysis.

If a non-compact variable $\lambda_1$ is relevant only, the function $\Psi$ takes in the limit of small $\omega$ the well-known form

$$\Psi = 2z^{1/2} K_1(2z^{1/2}), \quad z = i\omega \lambda_1$$

valid both for chains and thick wires. Using the modified Bessel functions $K_{i\rho}$ one can write in this limit the solution for the function $\Gamma$ as

$$\Gamma(x,x';z,z') = \frac{8}{\pi^2} \int_0^\infty e^{-|x-x'|(1+\rho^2)/4L_c} z^{1/2} K_{i\rho}(2z^{1/2}) z'^{1/2} K_{i\rho}(2z'^{1/2}) \, \sinh(\pi \rho) \rho \, d\rho. \quad (14)$$

where $L_c$ is the localization length. Eq. (14) will be proven in Sec. V. Actually, it follows from the orthonormality of the modified Bessel functions $K_{i\rho}$ (see Eq. (42)) and the fact that
the functions $2z^{1/2}K_{iρ}(2z^{1/2})$ are eigenfunctions of the “heat” equation with the Laplacian in the space of the matrices $Q$.

The expressions for $Ψ$ and $Γ$, Eqs. (13, 14), are sufficient to calculate all the moments, Eq. (11), for the orthogonal and unitary ensembles. This way is more straightforward and more economical, as compared to the usual way of representing the moments as solutions of differential equations obtained within the transfer matrix technique. At the same time, the conventional approach is more general and is applicable also in the crossover regime.

It is worth mentioning that our analysis of the localization properties in infinite wires is somewhat easier than that for conductance and wave functions in finite wires. This is because only large values of the non-compact variables $λ_1$ are important for the calculations in our case. This is the simplification that allows us to calculate all moments as well as the entire distribution functions for the ‘pure’ (orthogonal and unitary) ensembles (see Sec. V).

### III. TRANSFER-MATRIX TECHNIQUE

At not very high frequencies, the zero transversal space harmonics of the supermatrix $Q$ in Eq. (2) gives the main contribution. In order to neglect the non-zero harmonics one should choose the London gauge for the vector potential $A$. Writing $A = (Hy, 0, 0)$, where $H$ is the external magnetic field perpendicular to the wire we represent the free energy, Eq. (2), for the quasi-1D sample in the form

$$F[Q] = \frac{L_{cu}}{16} \text{Str} \int \left( (\nabla x Q)^2 + \frac{X^2}{16L_{cu}^2}\langle Q, τ_3 \rangle^2 + \frac{2iωΛQ(x)}{D} \right) dx,$$

where $L_{cu} = 2πSνD$ is the localization length for the unitary ensemble. The crossover parameter $X$ depends on the geometry of the sample, $X = 2πφ/φ_0$, $φ_0 = hc/e$ is the flux quantum, and $φ = HL_{cu} \langle y^2 \rangle_{\text{sec}}^{1/2}$ is the magnetic flux through the area limited by the localization length. The brackets imply averaging across the wire over the coordinate $y$ which is chosen to be perpendicular both to the direction of the wire and the magnetic field. This averaging gives $\langle y^2 \rangle^{1/2} = d/√12$, where $d$ is the width of the wire, for “flat” wires made on the basis of a 2D gas. For wires with a circular cross-section, $\langle y^2 \rangle^{1/2} = d/4$, where $d$ is the diameter.

To determine the functions $Ψ$ and $Γ$ in Eq. (11), we discretize the wire by introducing sites on which the $Q$-supermatrix varies. The function $Ψ(Q)$ is the partition function of a semi-infinite sample and therefore it does not depend on the coordinate $x$. This allows us to write immediately the a recurrence equation. Relating the function $Ψ(Q)$ on one site and its value $Ψ(Q')$ on the neighboring one we obtain

$$Ψ(Q) = \int N(Q, Q')Z_0(Q')Ψ(Q')dQ'.$$

where the function $N(Q, Q')$ describing the coupling between the sites

$$N(Q, Q') = \exp \left( \frac{L_{cu}}{4Δf} \text{Str} QQ' \right),$$

originates from the kinetic term $(∇Q)^2$ in the free energy, Eq. (15). The function $Z_0(Q)$,
\[ Z_0(Q) = \exp\left( \frac{i\omega L_{cu} \Delta l}{4D} \text{Str} \Lambda Q + \frac{X^2 \Delta l}{16L_{cu}} \text{Str} [Q, \tau_3]^2 \right) \]  

(18)

describes the remaining terms in the free energy functional \( F \), Eq. (15), and should be integrated on each site. In order to reduce the integral Eq. (16) to the differential “Schrödinger equation” for the functions \( \Psi \) (and similarly for \( \Gamma \)), one should take the continuous limit \( \Delta l \to 0 \), where \( \Delta l \) is the length of the separation between the sites. In this continuous limit, \( \Delta l \) drops out from final expressions and therefore can be chosen to be arbitrarily short.

Since the kernel \( \Gamma \) enters equations for each moment only in a combination with \((Q_{21})^n \Psi(Q)\), it is advantageous to introduce a matrix function \( P^{(n)}_k \) such that

\[ P^{(n)}_k(Q) = \sum_x \exp[i(x - x')k] \int \Gamma(k; Q, Q')(Q'_{21})^n \Psi(Q') \, dQ', \]  

(19)

where \( \Gamma(k; Q, Q') \) is the Fourier transform of \( \Gamma(x - x'; Q, Q') \). The matrix function \( P^{(n)}_k(Q) \) on a site is related to its value \( P^{(n)}_k(Q') \) on the neighboring site as

\[ P^{(n)}_k(Q) = \exp(-ik\Delta l) \int N(Q, Q')Z_0(Q')P^{(n)}_k(Q')dQ' = (Q^{21})^n \Psi(Q). \]  

(20)

The moments of the distribution \( P_Q \), expressed in terms of the functions \( \Psi \) and \( P_k \), are given by

\[ T_{mn} = \int \Psi(Q)(Q_{33}^{21})^n(P^{(m)}_{k33} + P^{(m)}_{-k33})dQ, \]  

(21)

Solving Eqs. (19-20) and calculating the integral in Eq. (21) one can find, in principle, all the moments. However, further simplifications are necessary to obtain the results explicitly.

A spectral expansion is convenient for the analysis of the function \( P^{(n)}_k \). So, we expand the functions \( P^{(n)}_k \) in eigenfunctions \( \phi_E(Q) \), introduced as

\[ \int N(Q, Q')Z_0(Q')\phi_E(Q')dQ' = \mathcal{E}\phi_E(Q). \]  

(22)

The matrix functions \( \phi_E(Q) \) have the same structure as \( Q^{21} \). Their orthogonality and the normalization properties can be written in the form

\[ \int [\phi^+_E(Q)]_{33}[\phi_E(Q)]_{33}dQ = \delta(\mathcal{E} - \mathcal{E}'). \]  

(23)

Expanding \( P^{(m)}_k \) in the complete set of the functions \( \phi_E \), Fourier transforming it to the coordinate representation, and substituting the result into Eq. (21), we arrive at

\[ T_{nn} = -\sum_E \exp(-\mathcal{E}|x - x'|) \int (Q^{12}_{33})^m[\phi_E(Q)]_{33}\Psi(Q) \, dQ \int (Q^{21}_{33})^n[\phi_E(Q)]_{33}\Psi(Q) \, dQ. \]  

(24)

Eq. (24) contains the sum over all eigenstates that should be found from Eq. (22) and establishes a link between the eigenvalue problem in the space of parameters of the \( Q \)-matrix and the localization properties. The lowest non-zero eigenenergy corresponding to the first excited state determines the exponential decay of the correlation functions at large distances.
IV. EFFECTIVE HAMILTONIANS

Eqs. (11, 22, 24) form a closed system completely solving the problem. To perform explicit calculations a parametrization of the $Q$-matrices should be chosen. For the problem involved one can use either the “standard” or the “magnetic” parametrization. The former parametrization has been used, e.g., for calculation of the density-density correlation function in disordered wires for the orthogonal and unitary ensembles, while the latter one helped to obtain explicit formulae for the crossover between the ensembles in the zero-dimensional situation.

For the problem of the localization in wires the standard parametrization allows to calculate any quantity of interest for the orthogonal and unitary ensembles and it will be used in Sec. V for calculations of all moments as well as of the distribution function of the density-density correlator. At the same time, the standard parametrization is of little help in the crossover regime between the ensembles and we will use in this case the magnetic parametrization.

The basic equations and their solutions for the orthogonal and unitary ensembles can be found in Refs. 7, 12 and we do not write them here. Instead, let us present explicit differential equations that can be obtained within the magnetic parametrization from Eqs. (16, 22, 24).

The $Q$-matrix in this parametrization is represented as

$$Q = V_dQ_c\bar{V}_d,$$

where the supermatrix $Q_c, Q_c^2 = 1$, contains cooperon “degrees of freedom” and $V_d, \bar{V}_dV_d = 1$, diffuson ones. To make the reading easier, we present here the essential structure of the block $Q^{12}$ in the magnetic parametrization, keeping only those terms that contribute to the correlators considered

$$Q^{12} = u_d(1 - \hat{\lambda}_d^2)^{1/2}[\hat{\lambda}_c + 2(\eta_c\eta_c^* - \kappa_c\kappa_c^*)](\lambda_{1c} - \lambda_c)] \bar{v}_d,$$

where $\hat{\lambda}_{c,d} = \text{diag}(\cos \theta_{c,d}, \cosh \theta_{1c,1d}), v_d$ and $u_d$ are the standard $4 \times 4$ unitary matrices in the $V_d$ (diffuson) block, and $\eta_c, \eta_c^*, \kappa_c, \kappa_c^*$ are the Grassmann variables from the cooperon block.

An advantageous feature of the magnetic parametrization is that the term containing the magnetic field $H$ in the free energy has a simple form

$$\text{Str}[Q, \tau_3]^2 = 16(\lambda_{1c}^2 - \lambda_c^2).$$

In contrast, the term with the frequency $\omega$ is quite complex since it contains the Grassmann variables

$$\text{Str}(\Lambda Q) = 4[\lambda_{1d}\lambda_{1c} - \lambda_d\lambda_c + 2(\eta_c\eta_c^* - \kappa_c\kappa_c^*)](\lambda_{1c} - \lambda_c)(\lambda_{1d} - \lambda_d)].$$

The functions $\Psi$ and $\phi_E$ correspond to states having respectively the “angular momentum” zero and one with respect to the unitary rotations $V_d$. The form of the functions $\phi_E$
is determined by the RHS of Eq. (20) and, as a result, it should be searched for in the class of functions with the same structure as the matrix $Q^{12}$

$$
\phi_\varepsilon = v_d \begin{pmatrix}
(1 - \lambda_d^2)^{1/2} f & 0 \\
0 & i(\lambda_1^2 - 1)^{1/2} f_1
\end{pmatrix} \bar{u}_d,
$$

(29)

where $v_d$ and $u_d$ are unitary matrices containing Grassmann variables only. The different forms of the functions $\Psi$ and $\phi_\varepsilon$ lead to different forms of the effective Hamiltonians determining the functions $\Psi$ and $\phi_\varepsilon$. The functions $f$ and $f_1$ do not contain the diffusion Grassmann variables but may contain all other variables.

The Schrödinger equations corresponding to Eqs. (16) and (22) are obtained considering slow variations of $Q$ on neighboring sites. The matrix $Q'$ is expanded near $Q$ and integrated over small variations of the matrix elements. The expansion up to quadratic variations is sufficient and therefore, the quadratic form obtained is identical to the one describing the square of the elementary length in the space of the supermatrices $Q$. This elementary length determines the Berezinian corresponding to a chosen parametrization. Correspondingly, the “Laplacian” entering the effective Schrödinger equation contains the Berezinian of the transformation. The Berezinian of the magnetic parametrization reads

$$
J = J_c J_d J_{cd}, \quad J_{c,d} = \frac{1}{2\pi^2 (\lambda_{1c,d} - \lambda_{c,d})^2}, \quad J_{cd} = \frac{4\lambda_{1c}^2}{(\lambda_{1c} + \lambda_{1d})^2},
$$

(30)

The parts $J_d$ and $J_{cd}$ of the Berezinian Eq. (30) originate from the diffusion Grassmann variables $\eta_d, \kappa_d$, whereas $J_c$ comes from the cooperon ones $\eta_c, \kappa_c$. The diffusion Grassmann variables do not enter Eqs. (27) and (28). In some sense, the diffusion variables decouple from other variables and this simplifies calculations.

The situation with the cooperon variables is more difficult. Since the free energy, Eqs. (15, 28), contains explicitly the cooperon Grassmann variables $\eta_c, \kappa_c$, so do the functions $\Psi$ and $\phi_\varepsilon$. In order to write equations for these functions explicitly one should expand them in powers of the Grassmann variables. Proceeding in this way one obtains a system of differential equations for the functions entering the expansion. Another complication occurs due to a non-trivial angular dependence on $Q_c$. This matrix is surrounded by the diffusion modes and couples to them. As a result, one comes to extremely cumbersome equations that can hardly be solved analytically for arbitrary $X$. The solution can be found for $X \gg 1$ but this limit corresponds to the well studied unitary ensemble.

Fortunately, an important information about correlation functions can be extracted in the limit $X \ll 1$. To understand why this limit helps to study the problem we remind the reader that the possibility to obtain closed expressions for correlation functions in the orthogonal and unitary ensembles is due to the fact that the main contribution to integrals comes in the limit of small frequencies $\omega$ from large values of the non-compact variables (in the unitary ensemble $\lambda_{1d} \sim 1/\omega$). For large values of $\lambda_{1d}$ the partial differential equations for the functions $\Psi$ and $\phi_\varepsilon$ are sufficiently simple and can be solved. In the crossover regime when $X$ is finite, the equations are still very complicated even in the limit $\omega \to 0$. However, in the limit $X \ll 1$, some quantities of interest are determined by large $\lambda_{1c} \sim 1/X \gg 1$. This leads to an additional simplification of the equations and the possibility to make estimates. At the same time, not any correlation function can be calculated in this way, which limits the applicability of the trick.
In the continuous limit, Eq. (10) can be reduced to the form

$$\mathcal{H}_X \Psi = 0$$  \hspace{1cm} (31)

Due to the presence of the Grassmann variables in the free energy functional, Eqs. (15, 28), the solution $\Psi$ must contain them too, which is in contrast to the solutions for the orthogonal and unitary ensembles. The function $\Psi$ can be represented as

$$\Psi = \Psi_0 (\lambda_1, \lambda_c) + \Psi_1 (\lambda_c, \lambda_1) (\kappa_c \eta_c^* - \kappa_c \kappa_c^*) + \Psi_2 (\lambda_1, \lambda_c) \eta_c \eta_c^* \kappa_c \kappa_c^*$$  \hspace{1cm} (32)

Then, one obtains a system of partial differential equations for the functions $\Psi_0$, $\Psi_1$, $\Psi_2$

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_0 + 2 \frac{\lambda_1 \lambda_c - 1}{(\lambda_1 - \lambda_c)^2} \Psi_1 = 0$$

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_1 - \frac{\lambda_1 \lambda_c - 1}{(\lambda_1 - \lambda_c)^2} \Psi_2 + \tilde{\omega} x_2 \Psi_0 = 0$$

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_2 - 2 \tilde{\omega} x_2 \Psi_1 = 0$$  \hspace{1cm} (33)

where

$$\mathcal{H}_X^{(0)} = - \sum_{i=d,1d} \frac{1}{J_d} \frac{\partial}{\partial \lambda_i} J_d [1 - \lambda_i^2] \frac{\partial}{\partial \lambda_i} - \sum_{i=c,1c} \frac{1}{J_c J_{cd}} \frac{\partial}{\partial \lambda_i} J_c J_{cd} \lambda_i^2 - 1 \frac{\partial}{\partial \lambda_i} + X^2 (\lambda_1 C - \lambda_c^2) .$$  \hspace{1cm} (34)

$$\tilde{\omega} = 2\pi^2 (\nu S)^2 D \omega$$

and

$$x_1 = \lambda_1 \lambda_c - \lambda_c \lambda_c, \hspace{0.5cm} x_2 = 2 (\lambda_1 - \lambda_c) (\lambda_1 d - \lambda_d)$$  \hspace{1cm} (35)

In the limit $\lambda_1 d, \lambda_1 c \gg 1$, the difference between equations for the functions $\Psi$ and $\phi_c$ is negligible because it comes from the averages of the type $\langle \Delta \eta_i \Delta \eta_i^* \rangle \sim 1/\lambda_{1i}, \langle \Delta \kappa_i \Delta \kappa_i^* \rangle \sim 1/\lambda_{1i}, i = c, d$. In this limit, Eqs. (33, 34, 35) take a simpler form

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_0 = 0$$

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_1 + \tilde{\omega} x_2 \Psi_0 = 0$$

$$\left(\mathcal{H}_X^{(0)} + \tilde{\omega} x_1\right) \Psi_2 - 2 \tilde{\omega} x_2 \Psi_1 = 0$$  \hspace{1cm} (36)

with
\[ H_X^{(0)} = -\lambda_1^2 \frac{\partial^2}{\partial \lambda_1^2} - \lambda_1 \frac{\partial^2}{\partial \lambda_1^2} + 2\lambda_1 \frac{\partial}{\partial \lambda_1} + X^2 \lambda_1^2 \]  

\[ x_1 = x_2/2 = \lambda_1 \lambda_1 \]

The same equations can be written in the limit \( \lambda_1 \lambda_1 \gg 1 \) for the functions \( \phi_\mathcal{E} \).

It is clear that Eqs. \( \text{(32, 35)} \) cannot be solved analytically for an arbitrary \( X \) and the only hope is to solve Eqs. \( \text{(36, 37)} \). Of course, one is restricted then by calculation of those quantities for which large \( \lambda_1 \lambda_1 \) give the main contribution.

In the opposite limiting case \( X \to \infty \) but \( \omega \to 0 \) all the cooperon variables are frozen and one comes to the effective Hamiltonian

\[ H = -\lambda_1^2 \frac{\partial^2}{\partial \lambda_1^2} - \frac{2i\omega L_{cu}^2}{D} \lambda_1 \]  

(38)

After a proper change of the localization length the same Hamiltonian can be written in the orthogonal case using the standard parametrization. For completeness let us remind that the localization lengths for the orthogonal and unitary ensembles, \( L_{co} \) and \( L_{cu} \) respectively, are given by the following expressions

\[ L_{cu} = 2\pi \nu SD, \quad L_{co} = \pi \nu SD, \]  

(39)

In these cases the function \( \Psi \) does not contain Grassmann variables. The equations to be solved take the form

\[ H \Psi = 0, \quad H \phi_\mathcal{E} = L_{\mathcal{E}} \phi_\mathcal{E} \]  

(40)

where the localization length \( L_{\mathcal{E}} \) equals either \( L_{co} \) or \( L_{cu} \) depending on the ensemble.

In the next Section, solving Eqs. \( \text{(40)} \) we will calculate all moments and the entire distribution function of the density-density correlations. This will help us to come to certain conclusions about properties of the electron wave functions in the orthogonal and unitary ensembles.

**V. CALCULATION OF MOMENTS AND DISTRIBUTIONS FOR THE ORTHOGONAL AND UNITARY ENSEMBLES.**

Calculation of the moments and of the distribution functions is important for the understanding of the structure of the wave functions and properties of different averages. Higher moments and the distribution function for infinitely long wires have not been considered yet although the results are known for finite wires.\(^{12}\) So, calculation of the moments and of the distribution function can be interesting on its own but, what is more important, this information will also help us to make certain conclusions about properties of the wave functions in a weak magnetic field where the possibility of exact calculations is more limited. This will help us to extend the results of Ref.\(^{13}\) where only the averaged density-density correlation function was calculated.

Solutions \( \Psi \) and \( \phi_\mathcal{E} \) of Eqs. \( \text{(38, 40)} \) are well known (see e.g.\(^{14,15}\)) and can be written as
\[ \Psi = 2z^{1/2}K_1(2z^{1/2}), \quad \phi_C = 2z^{1/2}K_{i\rho}(2z^{1/2}), \quad \mathcal{E} = (1 + \rho^2)/4L_c, \]  

(41)

where \( z = 2i\omega L_c^2/D \). Comparing Eqs. (24) and (11), we can understand that the propagator has the form of Eq. (14). The numerical coefficient in Eq. (14) is found using the Lebedev-Kontorovich transformation applicable for an arbitrary function \( \varphi(y) \)

\[ \tilde{\varphi}(\rho) = \int_0^{\infty} \varphi(y)K_{i\rho}(y)\frac{dy}{y}, \quad \varphi(y) = \frac{2}{\pi^2y} \int_0^{\infty} \tilde{\varphi}(\rho)K_{i\rho}(y)\rho \sinh(\pi\rho)d\rho. \]  

(42)

Now we can calculate all the moments of the two-point correlator, Eq. (11). Substituting Eqs. (41) into Eq. (24), and calculating integrals over all elements of the supermatrices \( Q \) except \( \lambda_1 \), yields

\[ T_{mn} = \langle t_1^n t_2^m \rangle_Q = \frac{32}{\pi^2} \int_0^{\infty} d\rho \rho \sinh(\pi\rho)e^{-\frac{\rho^2+1}{4}}M_n(\rho)M_m(\rho). \]  

(43)

where

\[ M_n = \int dt_1 \left( \frac{t_1}{2} \right)^{2n-1} n^2K_1(t_1)K_{i\rho}(t_1). \]

and \( u = |x - x'|/L_c \).

The structure of Eq. (13) is not complicated and the integration over the variable \( t_1 \) can be performed exactly. As a result, we come to general expressions for all moments of the density-density correlations \( p_{\infty}^{(n)}(u) \) introduced in Eq. (41)

\[ p_{\infty}^{(n)}(u) = \frac{2n}{\pi (2n-2)!\Gamma((n-1)!)^2} \int_0^{\infty} d\rho \rho \sinh(\pi\rho)e^{-\frac{\rho^2+1}{4}} [P^{(n)}(\rho)]^2, \]  

(44)

\[ P^{(n)}(\rho) = \frac{n^2}{(2n-1)!} \Gamma \left( n + \frac{1}{2} + i\rho/2 \right) \Gamma \left( n + \frac{1}{2} - i\rho/2 \right) \Gamma \left( n - \frac{1}{2} + i\rho/2 \right) \Gamma \left( n - \frac{1}{2} - i\rho/2 \right). \]

For \( n = 1 \), Eq. (44) reduces to the expression obtained by Gogolin for a single chain. Eq. (44) is applicable at any \( x \) including the values smaller and of the order of \( L_c \) for both the unitary and orthogonal ensembles. Comparing the moments, Eqs. (44), with those obtained for finite wires, we see that their structure is similar, although they are not completely identical.

At large distances, \( |x - x'| \gg L_c \), essential contribution to the integral over \( \rho \) in Eq. (44) comes from small \( \rho \ll 1 \) and the integral can be easily calculated. We see that the moments of all quantities decay similarly as in infinite wires, Eqs. (14), (13), finite closed wires, as well as in open chains

\[ p_{\infty}^{(n)}(x) = C_n \left( \frac{1}{|x|} \right)^{3/2} \exp \left( -\frac{|x|}{4L_c} \right). \]  

(45)

where \( C_n \) is a constant depending on \( n \). The exponential decay of all moments seen from Eq. (13) manifests the localization of the wave functions. A very important feature of the moments \( p_{\infty}^{(n)}(x) \), Eq. (13), is that the dependence on the coordinate is the same for all \( n \).
This does not allow to interpret \( p^{(n)}_\infty (x) \), Eq. (43), as functions characterizing the typical shape of the wave functions.

The exact representation of the moments of the wave functions, Eq. (43), allows to obtain the entire distribution function of the density-density correlations. Of the crucial importance is the fact that Eq. (43) is valid for all \( n \) including the limit \( n \to \infty \). It is clear, if the \( n \)th moment of a quantity can be represented in a form of an integral of a function multiplied by the \( n \)th power of the variable of integration, then this function is the distribution of this quantity. In the case under consideration, Eq. (43) contains, besides the necessary power of the variable \( t \), the factor \( n^2 \), originating from the expansion of the element \( (Q^{12}_{33})^n \) in the Grassmann variables. This does not make the calculation of the distribution function more difficult because the presence of this factor leads merely to additional derivatives, and we obtain for the distribution function \( \mathcal{P}(t_1, t_2) \), Eq. (7)

\[
\mathcal{P}(t_1, t_2) = \frac{2}{\pi^2} \int_0^\infty d\rho \rho \sinh(\pi \rho) e^{-\frac{\pi}{2}(\rho^2+1)} \Pi_{\rho}(t_1) \Pi_{\rho}(t_2), \tag{46}
\]

where

\[
\Pi_{\rho}(t) = \left( [K_1(2t^{1/2})K_{i\rho}(2t^{1/2})]' t \right)' . \tag{47}
\]

and \( ' \) stands for the derivative over \( t \). The two-point distribution function \( \mathcal{P}(t_1, t_2) \) can be used to obtain any other correlator of interest. For instance, the distribution of the Landauer-type conductivity \( \mathcal{P}_Q(t) \), Eq. (8) can be represented as the convolution

\[
\mathcal{P}_Q = \int d\tau \langle \delta(t - \tau \nu L_c) S_{12}^{12}(x_1) \delta(\tau - i \nu L_c S_{21}^{21}(x_2)) \rangle_F . \tag{48}
\]

Then, we obtain for the function \( \mathcal{P}_Q(t) \)

\[
\mathcal{P}_Q(t) = \frac{2\nu}{\pi^2} \int_0^\infty d\rho \rho \sinh(\pi \rho) e^{-\frac{\pi}{2}(\rho^2+1)} \int_0^\infty d\tau \Pi_{\rho}(\tau) \Pi_{\rho}(t/\tau) , \tag{49}
\]

Eq. (49) is also exact and applicable at any distances. It is interesting to find this function near its maximum and this can be done easily in the limit \( |x - x'| \gg L_c \). At very large separation between \( x \) and \( x' \), typical values of \( t \) are exponentially small in distributions \( \mathcal{P}_Q \) and \( \mathcal{P}_\psi \), Eqs. (8, 10). A dominant contribution to the integral, Eq. (19), comes from the region \( \tau \sim t \ll 1 \). Eq. (17) allows us to write the asymptotics of the function \( \Pi_{\rho}(\tau) \) at \( \tau \ll 1 \) and \( \tau \gg 1 \)

\[
\Pi_{\rho}(\tau) \simeq \Re \left[ \Gamma(i\rho)/4\tau^{3/2+i\rho/2} \right] , \quad \tau \ll 1
\]

\[
\Pi_{\rho}(\tau) \simeq \pi \exp \left( -4\tau^{1/2} \right) /\tau^{1/2} , \quad \tau \gg 1 \tag{50}
\]

which ensures convergence of the integral over \( \tau \) at \( \tau \to 0 \) in Eq. (49). Performing integration in the resulting expression

\[
\mathcal{P}_Q(-\ln t) = \frac{\pi}{32} \int_{-\infty}^\infty d\rho \rho \sinh(\pi \rho) \frac{\Gamma(i\rho)\Gamma(2+i\rho)}{t^{1/2}} \exp \left( -\frac{u}{4} \frac{\ln^2 t}{4u} - \frac{u}{4}(\rho + \frac{i\ln t}{u})^2 \right) ,
\]
in the saddle point approximation, we obtain

\[ \mathcal{P}_Q(-\ln t) \simeq \frac{1}{2\pi^{1/2}u^{1/2}} \exp \left( -\frac{1}{4u}(u + \ln t)^2 \right). \]  

(51)

Considering the distribution \( \mathcal{P}_\psi \), Eq. (10), under the same assumptions it is not difficult to check that in the limit \(|x - x'| \gg L_c\) it reduces to Eq. (51).

The saddle-point approximation used for the derivation of Eq. (51) is applicable at large distances \(|x - x'| \gg L_c\) when \(u\) is large. Eq. (51) shows that in this limit the distribution functions are log-normal. The result about the log-normal distribution of physical quantities like conductance agrees with the corresponding result obtained for chains and is universal. It shows that the localization length of the wave functions or in other words, the logarithm of the conductance is a self-averaging quantity. Considering very long samples one has to obtain the localization length \(L_c\) and fluctuations of this quantity can be neglected.

At the same time, the log-normal distribution, Eq. (51), is applicable for not very large values of \(t\) only. The far tail of the distribution function \( \mathcal{P}_\psi \) decays more slowly. The moments of the density-density correlations \( p_{\Psi}^{(n)} \), Eq. (13), cannot be obtained from Eq. (51) because the main contribution comes from the far tails of the distribution function. All these peculiarities are discussed in details in Refs. 29, 30.

What is the physical meaning of the fact that all the moments \( p_{\Psi}^{(n)} \), Eq. (13) decay in the same way with a larger length \(4L_c\)? This can be understood if we imagine that there are big splashes of the wave functions even far from the localization center. Although, they can give a considerable contribution to the shape of the wave function, their probability decreases exponentially with \(|x|\) (is proportional to \(\exp (-|x|/4L_c)\)). Then, such splashes would give a similar contribution to all the moments. As concerns typical values of the squares of wave functions, they decay faster with the localization length \(L_c\) and give a smaller contribution to the moments.

The situation is opposite if one calculates the average logarithm of the wave functions. For the logarithm, the splashes do not give a specially large contribution, being at the same time very rare. So, their contribution to the average logarithm can be neglected. In other words, the average logarithm of the correlations of wave functions characterizes typical wave functions, whereas the moments describe the rare splashes of the wave functions. This understanding will be important for interpretation of the results in a weak magnetic fields obtained in the next Section.

VI. EFFECT OF MAGNETIC FIELD ON THE TAILS OF WAVE FUNCTIONS

This section is central in our paper. We try now describe electron wave functions in disordered wires in a weak magnetic field, such that \(X \ll 1\). General equations describing this situation are very difficult even when they are written for the “ground state” function \(\Psi\), Eqs. (31-35). At the same time, any attempt to consider the weak magnetic field as a perturbation fails in the most interesting limit \(\omega \to 0\) (one obtains terms like \(X^2/\tilde{\omega}\)). This reflects a clear physics: travelling a sufficiently long time the electron ‘feels’ any weak magnetic field.
The failure of the perturbation theory signals immediately that the effect of the weak magnetic field is more complicated than a slight changing of the eigenenergies of the effective Schrödinger equation and hence the localization length. The first excited state for the orthogonal ensemble has the eigenenergy $E_0 = 1/2$ and this energy definitely changes if the magnetic field is applied. But, as we will try to show below, an additional eigenstate with the eigenenergy $E_u = 1/4$, the energy of the unitary ensemble, appears (strictly speaking, a continuum of states with the energies $E_\rho = (1 + \rho^2)/4$, where $\rho$ is a continuous variable, should be added). Although at $X \ll 1$ this state contributes to correlation functions with a small weight, the value $E_u = 1/4$ does not depend on $X$.

As concerns the eigenenergy of the ground state, it is exactly zero due to the supersymmetry of the model. The eigenfunction $\Psi$ of the ground state is a complicated function $Q$ depending on the ratio $X^2/\tilde{\omega}$. We did not manage to find it for an arbitrary value of this ratio and can write in the limits $X^2/\tilde{\omega} \to 0$ (orthogonal ensemble) and $X^2/\tilde{\omega} \to \infty$ (the limit we are interested in now) only. Although the solution for the orthogonal ensemble is very well known in the standard parametrization, it has not been written before in the magnetic parametrization. To find it in the limit $\lambda_1d, \lambda_1c \gg 1$ we use Eqs. (36, 37) taken at $X = 0$. The solution for the functions $\Psi_i$ ($i = 0, 1, 2$) can be sought in the form

$$\Psi_i (\lambda_{1d}, \lambda_{1c}) = \lambda_{1d} R_i (z) , \ z = \lambda_{1d}\lambda_{1c} \tag{52}$$

Then, we obtain the following equation for $R_i$

$$(\mathcal{H}_o + \tilde{\omega}z) R_0 (z) = 0 \tag{53}$$

$$(\mathcal{H}_o + \tilde{\omega}z) R_1 (z) + 2\tilde{\omega}z R_0 (z) = 0 \tag{54}$$

$$(\mathcal{H}_o + \tilde{\omega}z) R_2 - 4\tilde{\omega}z R_1 (z) = 0 \tag{55}$$

where

$$\mathcal{H}_o = -2z^2 \frac{d^2}{dz^2} \tag{56}$$

Eq. (53) is exactly the final equation obtained for the orthogonal ensemble in the standard parametrization. Writing the eigenfunction of the ground state $\Psi$ in the form

$$\Psi = \lambda_{1d} R \tag{57}$$

we find for $R$

$$R = R_0 (z (1 + 2 (n\eta^* - \kappa\kappa^*))) \tag{58}$$

In the opposite limit $X^2/\tilde{\omega} \to \infty$, the solution of Eqs. (36, 37) is completely different. Putting $\tilde{\omega} = 0$ in this equations we find a solution

$$\Psi_0 (\lambda_{1c}) = (1 + X\lambda_{1c}) \exp (-X\lambda_{1c}) \tag{59}$$

$$\Psi_1 = \Psi_2 = 0$$
Formally, one could write non-zero solutions for $\Psi_1$ and $\Psi_2$ proportional to $\Psi_0$. However, we understand that in the magnetic parametrization the ground state function $\Psi$ should not contain at $\omega = 0$ Grassmann variables because the effective Hamiltonian does not contain them.

So, in the limit $\omega \to 0$ the form of the ground state function $\Psi$ changes discontinuously from Eqs. (57) to Eq. (59) as a magnetic field is applied. Physically, it may correspond to lifting degeneracies of some states of the initial electron problem by the magnetic field.

The ground state solution, Eq. (59), is written for $\lambda_1 d, \lambda_1 c \gg 1$. We have checked that this state survives and does not considerably change its form solving Eqs. (33) in the limit $\omega = 0$. The limit of the vanishing frequency considerably simplifies the equations because the “diffuson variable” $\lambda_1 d$ decouples from the “cooperon variables” $\lambda_1 c$ and $\lambda_c$. Then, one can search for a solution $\Psi$ depending on the cooperon variables only. We used the standard over-relaxation method with the Chebyshev acceleration to found numerically that up to $X$ as large as 0.3, Eq. (59), describes the solution rather well. At higher values of $X$, no dramatic changes occurred to this state either.

It is clear that in the limit $X \to \infty$ essential values of the cooperon variables $\lambda_1 c$ and $\lambda_c$ are close to unity for an arbitrary $\omega$ and one obtains the function $\Psi$ for the unitary ensemble. The solution found at $\omega = 0$ can be used for non-zero frequencies provided the variable $\lambda_1 d$ is not very large. From Eqs. (36,37) we can estimate the region of the applicability of the solution as

$$\lambda_1 d \lambda_1 c \tilde{\omega} \ll 1$$

(60)

In order to find the density-density correlation function $p_{\infty}(x - x')$ and its moments $p^{(n)}_{\infty}(x - x')$ at large distances we have to find not only the ground state but also at least the first excited state $\phi_1$. After that one should calculate the proper “matrix elements” in Eq. (24) or use Eq. (21). Although we argue that the eigenvalue $E_u = 1/4$ of the lowest excited state is exact, we are not able, as previously, to write the proper wave function in all regions of the variables exactly. The solution for $\phi_1$ can be sought in the form, Eq. (29), corresponding to the symmetry of the block $Q^{12}$ of the supermatrix $Q$, which is the usual way of constructing the proper excited eigenstates. Explicit formulae can be written at $X \ll 1$ provided the variables $\lambda_1 d$, $\lambda_1 c$ are large, $\lambda_1 d, \lambda_1 d \gg 1$, but limited from above by the inequality (60). In this limit one may keep the function $f_1$ only and we come to the following equation

$$\left( \mathcal{H}_X^{(0)} - 2\lambda_1 d \frac{\partial}{\partial \lambda_1 d} \right) f_{1\rho} = \mathcal{E} (\rho) f_{1\rho}$$

(61)

$\mathcal{H}_X^{(0)}$ is given by Eq. (37) and $\rho$ is a continuous real variable. The variables $\lambda_1 d$ and $\lambda_1 c$ in Eq. (61) separate from each other and we write the solution $f_1$ (see Eq. (29) in the form

$$f_{1\rho} = \lambda_1^{-1/2} \chi_\rho \left( \ln \lambda_1 d \right) \Psi_0 (\lambda_1 c)$$

(62)

with the function $\Psi_0$, given by Eq. (59). The function $\chi (v)$ should be written as

$$\chi_\rho (v) = \left( \frac{2}{\pi} \right)^{1/2} \sin \left[ - (\rho/2) v + \delta (\rho) \right]$$

(63)
Eq. (63) corresponds to a picture of “free motion” suggested in Ref. 2 for a description of the region of not very large $\lambda_1$. In the case considered here, Eq. (62), is applicable in the region determined by the inequality (60). In calculation of asymptotics of the density-density correlator at large distances small values of $\rho \ll 1$ give the main contribution and, in this limit, the phase $\delta$ is also small, $\delta (\rho) \sim \rho$ (see Ref. 4).

From Eq. (62) we obtain immediately

$$E (\rho) = \frac{1}{4} (1 + \rho^2)$$

(64)

The form of the solution $f_1$, Eq. (62), differs from the ground state by a dependence on the diffusion variables $\lambda_1 d$. This is the same difference as the one for the unitary ensemble. It is not difficult to check that the variable $\lambda_1 d$ separates from the cooperon variables even if $\lambda_1 c$ is not large, $\lambda_1 c \sim 1$. Then, the solution (62) may still be used in the region specified Eq. (60) provided the function $\Psi_0 (\lambda_1 c)$ is replaced by the solution $\Psi$ (depending only on the cooperon variables) of Eqs. (32, 33) taken at $\omega = 0$.

Eq. (62) with the function $\Psi$ can be used also for an arbitrary $X$. In the limit $X \gg 1$, the function $\Psi$ decays fast as the variables $\lambda_1 c$ and $\lambda_c$ deviate from 1. This means that the cooperon variables are ‘frozen’ and we come to the unitary ensemble. Then, the restriction (60) is not important and all formulae can be written for arbitrary $\lambda_1 d$. Of course, in order to prove the existence of the states with the eigenvalues $E (\rho)$, (62), rigorously one should investigate the exact equations for arbitrary values of the cooperon and diffusion variables and prove that the solutions ‘behave well’ everywhere. It is a difficult task that can apparently be performed only numerically.

Assuming nevertheless that this state exists and is given for $\lambda_1 d$, $\lambda_1 c \gg 1$ by Eq. (62) (the inequality (60) should also be fulfilled) we can estimate the proper matrix elements, Eq. (24), determining the average density-density correlation function $p_\infty (x)$ and its moments $p_{n\infty} (x)$, Eq. (1). However, trying to calculate the integrals in Eq. (24) one encounters a difficulty. The integrals over the matrix elements of $Q$ can be reduced to integrals over the variables $\lambda_1 d$, $\lambda_1 c$, $\lambda_d$, $\lambda_c$ and the Grassmann variables. Proceeding in this way one should use the Berezinian $J$ of the transformation to these variables, Eq. (80), which is singular at $\lambda_1 c = \lambda_c = 1$. The singularity in the Berezinian leads usually to additional contributions. To avoid explicit calculations of these contributions it is more convenient to calculate not the physical quantities of interest themselves but their derivative over $X$. This leads to additional factors $\lambda_1 c^2 - \lambda_c^2$ in integrands, thus compensating the singularities of the Berezinians.

Calculation of the physical quantities using the spectral expansion, Eq. (24), is still very difficult. The alternative way of calculations is solving Eq. (20), and calculating the integral in Eq. (21). A simplified differential form of Eq. (20) can be written for the function $P^{(n)}_k$ determining the correlator $p_{n\infty}^{(n)}$ as

$$H_{X}^{(0)} P^{(n)}_k + i \kappa P^{(n)}_k = \lambda_{1d}^{n} \lambda_{1c}^{n} \Psi_0 (\lambda_{1c})$$

(65)

where $\kappa = k L_{cu}$. In Eq. (63), we do not write Grassmann variables explicitly implying that $P^{(n)}$ is the non-compact central part similar to $(1 - \lambda_{1c}^2)^{1/2} f_1$ in Eq. (24).

The derivative over the field $(T_{mn})_X$ of the correlator $T_{11}$, Eq. (21) can be reduced to the form

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\[
\frac{dT_{nn}(k)}{dX} \sim \frac{d}{dX} \int \lambda_{id}^{n-2} \lambda_{ic}^{n-4} \tilde{F}_{k3d}^{(n)} d\lambda_{id} d\lambda_{ic} \quad (66)
\]

Writing Eq. (66), the Berezinian \( J \), Eq. (31), was used in the limit \( \lambda_{id}, \lambda_{ic} \gg 1 \) and \( \tilde{F}_{k}^{(n)} = P_{k}^{(n)} + P_{\kappa k}^{(n)} \).

Using the fact that the Hamiltonian \( H_{X}^{(0)} \) contains the variables \( \lambda_{id} \) and \( \lambda_{ic} \) separately let us expand the function \( P_{k}^{(n)} (\lambda_{id}, \lambda_{ic}) \) in the eigenfunctions of the diffuson part of the Hamiltonian. Writing

\[
P_{k}^{(n)} (\lambda_{id}, \lambda_{ic}) = \sum_{\rho} \lambda_{id}^{1/2} \chi_{\rho} (\ln \lambda_{id}) P_{k\rho}^{(n)} (\lambda_{ic}) \quad (67)
\]

we obtain the following equation for \( P_{k\rho}^{(n)} (\lambda_{ic}) \)

\[
(\mathcal{E} (\rho) + i\kappa) P_{k\rho}^{(n)} (\lambda_{ic}) + \tilde{H}_{X}^{(0)} P_{k\rho}^{(n)} (\lambda_{ic}) = A (\lambda_{ic}) \quad (68)
\]

where

\[
A (\lambda_{ic}) = \int \lambda_{id}^{n-3/2} \chi_{\rho} (\ln \lambda_{id}) \lambda_{ic}^{n} \Psi \lambda_{ic} d\lambda_{id}
\]

The operator \( \tilde{H}_{X}^{(0)} \) is the part of the Hamiltonian \( H_{X}^{(0)} \) in Eq. (37) acting on the variables \( \lambda_{ic} \).

A similar expansion was done in Ref. \(^2\) to consider the case of a strong disorder. The main contribution in the integral over \( \lambda_{id} \) comes from large \( \lambda_{id} \) and the integral formally diverges. When considering the unitary ensemble one had to cutoff the integral by \( \lambda_{ic} \sim 1/\tilde{\omega} \). At weak magnetic fields considered now, the cutoff can be determined from Eqs. (15, 28). We see that the term with the frequency can be neglected if \( \lambda_{id} \lambda_{ic} \tilde{\omega} \ll 1 \), which leads to the cutoff \( \lambda_{id} \sim (\lambda_{ic} \tilde{\omega})^{-1} \). So, we estimate the function \( A (\lambda_{ic}) \) at \( \rho \ll 1 \) as

\[
A (\lambda_{ic}) \sim (-i\tilde{\omega})^{1/2-n} \rho \lambda_{ic}^{1/2} \Psi \lambda_{ic} (\lambda_{ic}) \quad (69)
\]

In a similar way we can reduce Eq. (66) to the form

\[
\frac{dT_{nn}(k)}{dX} \sim (-i\tilde{\omega})^{1/2-n} \frac{d}{dX} \int P_{k\rho}^{(n)} (\lambda_{ic}) \Psi \lambda_{ic} \frac{d\lambda_{ic}}{\lambda_{ic}^{7/2}} \quad (70)
\]

Now, we have to solve Eq. (68) and calculate the integral, Eq. (70). In order to solve Eq. (68) we introduce a Green function \( g \) of this equation

\[
(\mathcal{E} (\rho) + i\kappa) g_{\rho} (\lambda_{ic}, \lambda_{ic}') + \tilde{H}_{X}^{(0)} g_{\rho} (\lambda_{ic}, \lambda_{ic}') = \lambda_{ic}^{4} \delta (\lambda_{ic} - \lambda_{ic}') \quad (71)
\]

and write the solution as

\[
P_{k\rho}^{(n)} = \int g_{\rho} (\lambda_{ic}, \lambda_{ic}') A (\lambda_{ic}') \frac{d\lambda_{ic}'}{(\lambda_{ic}')^{4}} \quad (72)
\]

Then, Eq. (70) is reduced to the form
The last step to be done is to find the function $g$ and calculate the integral in Eq. (73). As usual, the combination $\mathcal{E}(\rho) + ik$ enters the final formulae and one should perform Fourier transform to get the coordinate dependence. The point $\kappa = i\mathcal{E}(\rho)$ is a branching point and the integral over $\kappa$ can be shifted such that the integration is performed along the edges of the cut $(i\mathcal{E}(\rho), i\infty)$. The point $i\mathcal{E}(0) = i/4$ gives the value of the localization length whereas the integration over $\rho$ leads to an additional power law prefactor $|x|^{-3/2}$. Being interested mainly in determining the exponential decay we can immediately understand that all correlators $T_{nn}$ decay as $\exp\left(-|x|/4L_{\text{cu}}\right)$. To calculate the prefactor we may put $\kappa = i\mathcal{E}(\rho)$ in Eq. (74) and find the proper solution $g_0(\lambda_{1c}, \lambda'_{1c})$, which satisfies the equation
\[
\tilde{H}_X^{(0)} g_0 (\lambda_{1c}, \lambda'_{1c}) = \lambda_{1c}^4 \delta (\lambda_{1c} - \lambda'_{1c})
\] (74)
and has to be substituted into Eq. (73).

The solution of Eq. (74) can be found easily because one of the solutions $\Psi_0(\lambda_{1c})$, Eq. (59), of the corresponding homogeneous equation is known. We can write immediately the second solution $\Phi_0(\lambda_{1c})$ in the form
\[
\Phi_0(\lambda_{1c}) = \Psi_0(\lambda_{1c}) \int_0^{\lambda_{1c}} \frac{\lambda^2}{\Psi_0^2(\lambda)} d\lambda
\] (75)
The function $\Phi_0(\lambda_{1c})$ grows exponentially at $\lambda_{1c} \to \infty$ and is proportional to $\lambda_{1c}^3$ for small $\lambda_{1c}$. Using the functions $\Psi_0(\lambda_{1c})$ and $\Phi_0(\lambda_{1c})$ we write the Green function $g_0(\lambda_{1c}, \lambda'_{1c})$ as
\[
g_0(\lambda_{1c}, \lambda'_{1c}) = \begin{cases} 
\Phi_0(\lambda_{1c}) \Psi_0(\lambda'_{1c}), & \lambda_{1c} \leq \lambda'_{1c} \\
\Psi_0(\lambda_{1c}) \Phi_0(\lambda'_{1c}), & \lambda_{1c} \geq \lambda'_{1c}
\end{cases}
\] (76)
Substituting the function $g_0$, Eqs. (76, 59, 74), into Eq. (73) we see that in the region $1 \ll \lambda_{1c}, \lambda'_{1c} \ll 1/X$ the integral is logarithmic and is safely cut from below and above by the limits of this region. It is important that we remain in the region of large $\lambda_{1c}$ where our approximations are valid. As a result of the calculation we obtain for the moment $p_{\infty}^{(n)}(x)$ of the density-density correlation function
\[
\frac{dp_{\infty}^{(n)}(x)}{dX} \sim X \ln(1/X) |x|^{-3/2} \exp\left(-|x|/4L_{\text{cu}}\right)
\] (77)
It is interesting to note that the same prefactor determines correlations at coinciding points. Writing $p_{\infty}^{(n)}(0)$ as
\[
\frac{dp_{\infty}^{(n)}(0)}{dX} \sim (-i\tilde{\omega})^{2n-1} \int \frac{(\lambda_{1d}\lambda_{1c})^{2n} d\lambda_{1d} d\lambda_{1c}}{\lambda_{1d}^2 \lambda_{1c}^2}
\] and integrating as previously over the region $0 < \lambda_{1d} \leq (\lambda_{1c}\tilde{\omega})^{-1}$, $1 \leq \lambda_{1c} \leq 1/X$ we come to the result
\[
\frac{dp_{\infty}^{(n)}(0)}{dX} \sim X \ln(1/X)
\] (78)
Eq. (77) shows that the derivative over the magnetic field of all moments of the density-density correlation function decays with the localization length $L_{cu}$ of the unitary ensemble. The moments themselves can be obtained by integrating Eq. (77) over the magnetic field. The result can be written as

$$p_{\infty}^{(n)}(x) \propto \left(\frac{1}{x}\right)^{3/2} \exp \left(-\frac{|x|}{4L_{co}}\right) + C_n X^2 \ln \left(\frac{1}{X}\right) \exp \left(-\frac{|x|}{4L_{cu}}\right).$$

(79)

where $C_n$ is a coefficient depending on $n$. The first term in Eq. (79) is added assuming that at zero magnetic field one should have the standard result for the orthogonal ensemble. (In our previous work we wrote $\ln^2 (1/X)$ instead of $\ln (1/X)$ which was a result of a not sufficiently accurate estimate). The second term in Eq. (79) becomes larger than the first one at distances exceeding

$$x_H \sim L_c \ln (1/X)$$

(80)

and decays in the same way for all the moments. As we have discussed in the previous Section, this should mean that this term is determined by rare splashes of the wave functions. These splashes can be due to a hybridization of states with eigenenergies very close to each other and it is not surprising that they are sensitive to the magnetic field.

A more delicate question concerns the distribution of the logarithm of the wave functions, governing the behavior of “typical” wave functions. This quantity is necessary for comparison with a recent numerical study, where the logarithm of the transmittance was shown to behave smoothly between the orthogonal and unitary ensembles.

Our new state with the eigenenergy $\mathcal{E} = (1 + \rho^2)/4$ contributes to the distribution of the logarithms also. Unfortunately, the procedure developed for calculation of the moments cannot be extended to a calculation of the distribution function in the most interesting relation of the variables $t \approx \exp (-|x|/L_{cu})$ which would give the distribution of the localization lengths. The problem is that one needs to know the analytical properties of the Green function $g_{k\rho}$, Eq. (71), for arbitrary complex $\rho$. One can understand this on the example of the pure ensembles. In Eq. (71), computation of the distribution function required shifting the contour of the integration over $\rho$ to the saddle point $\rho = -i \ln t/u \approx i$. One could perform this procedure there because the integrand was known.

In the crossover regime, one can rely only on the picture of the “free motion” applicable at small $\rho$, which make calculation of the distribution function not feasible. We can say only that the distribution function has an additional weight at $t \approx \exp (-|x|/L_{cu})$ so that the entire function is broader that for the pure ensembles. At the same time, the main body of the function is not known, which does not allow us to say much about “typical” wave functions. So, the results that we really have do not seem to contradict to the results of the numerical study of the average logarithms, where the second length was not seen. They suggest also that the information the Borland conjecture refers to, the behavior of the averaged logarithm of the wave function, may be not sufficient to characterize the crossover. Instead, the entire distribution function is of interest since it can differ from the standard logarithmically normal even for $x \gg L_c$.

As concerns a numerical observation of the second length $L_{cu}$ in the moments of transmittances or density-density correlations, it may be very difficult because one needs to make
computation for a very large number of configurations to be able to make a reliable statistics of the rare events. In addition, a smearing due to the presence of leads can make the observation even more difficult.

VII. DISCUSSION OF THE RESULTS

We see from Eq. (79) that for weak magnetic fields the first term dominates at small distances. However, whatever weak the magnetic field is, the second term always prevails at sufficiently large distances. This is due to the fact that, although the small crossover parameter \( X \) enters the pre-exponential factor, the exponential function containing the localization length \( L_{cu} = 2L_{co} \) decays much slower. Comparing these two terms, we obtained the characteristic distance \( x_H \), Eq. (80) where the both terms are of the same order of magnitude.

To explain Eq. (79) qualitatively one can follow argument suggested by Mott for describing the AC conductivity \(^{31,32}\) with minor modifications necessary in our case. Actually, the arguments presented below apply for localized states in any dimensionality and allow us to obtain the scale \( x_H \), Eq. (80), without any calculation. Naturally, the localization length \( L_c \) in Eq. (80) depends on the dimensionality of the space. However, to be precise, we will speak in terms of the localization in wires.

In quasi-one-dimensional samples all states are localized (in other words, \( \int_{-\infty}^{+\infty} p^{(1)}_x(x)dx = 1 \)). At the same time, for frequencies exceeding the energy \( E_c = D_0/L_c^2 \) the system behaves like a metal. Therefore, eigenenergies of states localized in a region of the wire of size \( L_c \) are separated by the mean level spacing of the order of \( E_c \). This picture allows us to estimate immediately the magnetic field \( H_c \) at which the crossover between the orthogonal and unitary ensembles occurs. One should simply consider the region of the size \( L_c \) as a closed grain and recall that the crossover between the orthogonal and unitary ensembles in a metallic grain occurs \(^{7,12}\) at flux \( \phi \) through the grain given by the relation

\[
E_T (\phi/\phi_0)^2 \sim \Delta_G ,
\]

where \( \phi_0 \) is the flux quantum, \( \Delta_G = (\nu V)^{-1} \) is the mean level spacing in the grain (\( V \) is the volume), having the same order of magnitude as the Thouless energy \( E_T = D_0/L_c^2 \). For a quasi-one-dimensional grain of length \( L = L_c \) one obtains the characteristic value \( X \sim 1 \), which means that the magnetic flux through the segment of the length \( L_c \) is equal to the flux quantum \( \phi_0 \). At such fields the localization length of the main body of the wave functions changes considerably and this was clearly seen in the numerical work \(^{21}\).

At the same time, we know \(^{21,12}\) that the low frequency asymptotics of the level-level correlation function in an isolated grain changes from \( \omega \) to \( \omega^2 \) as soon as an arbitrarily weak magnetic field is applied. This corresponds to a finite probability of having two different levels at any small distance \( \omega \). These levels hybridize by the magnetic field and this leads to the low frequency asymptotics \( \omega^2 \) typical for the unitary ensemble.

The two-scale localization considered in the previous sections is of a similar origin, namely, one can always find two localized states that are arbitrarily close to each other in energy, although they may be separated by a large distance in space. Using this fact Mott \(^{21,12}\) calculated the AC conductivity at finite frequencies \( \omega \). We do not study the conductivity itself but rather different kinetic quantities like e.g. the density-density correlation.
function. In our case, the role of an external perturbation is taken over by the magnetic field.

Following Mott’s arguments, let us try to find states whose energies slightly differ from the energy of a chosen state. As we have mentioned above, the typical separation in energy of states localized within a region of the length $L_c$ is of order $E_c$. In neighboring regions of the lengths $L_c$ one cannot find such states either: Even if two states were occasionally closely located in two initially isolated systems, they would hybridize and split after bringing these two segments of the wire in contact. The value of the splitting would be $E_c$ again. However, if two states are separated by a large distance $x \gg L_c$, the splitting energy $\Delta_x$ decays exponentially

$$\Delta_x \sim E_c \exp(-x/L_c) \quad (82)$$

due to exponentially decaying overlap between the localized states. (Mott considered a tight-binding model; hence, the energy characterizing the splitting in his consideration was $\Delta_x \sim I_0 \exp(-x/L_c)$, where $I_0$ is of the order of band width.)

Thus, one can find states $a$ and $b$ with an exponentially small energy difference considering states localized sufficiently far from each other. Now, turning on an external perturbation (magnetic field in our case) we want to mix these states, which, like in isolated granules, would lead to a behavior typical for the unitary ensemble (with the doubled localization length $L_{cu} = 2L_{co}$). However, in order to mix the states we need a sufficiently large matrix element $A_{ab}$ of the vector potential between the states. If the states did not hybridize at all this matrix element would be exponentially small and this is similar the situation one encounters when calculating the AC conductivity at low frequencies. Mott suggested to take into account the states when they start hybridizing and we use this idea for our estimates. In this case the matrix element $A_{ab}$ is not exponentially small because the hybridized wave function is concentrated equally in the both localization centers.

Using the modelling of the localization center in terms of an isolated granule of the length $L_c$ and applying the magnetic field $H$ we obtain easily a characteristic energy window $E_H$ of the order of

$$E_H \sim X^2E_c \quad . \quad (83)$$

This estimate follows from Eq. (15) by using the fact that in wires the energies $E_c$, $E_T$ and $\Delta_G$ are of the same order. In an isolated grain states within this window are mixed and their correlations obey the statistics of the unitary ensemble.

So, we come to the following picture:

If two localized states $a$ and $b$ are separated by a distance $x$ such that the corresponding overlap integral $\Delta_x$ exceeds $E_H$, the states are not considerably affected by the magnetic field and its influence can be considered perturbatively.

If two localized states $a$ and $b$ are separated by a very large distance $x$ such that the overlap $\Delta_x$ is much smaller than $E_H$, then the matrix $A_{ab}$ is exponentially small and the effect of the magnetic field on the states can be neglected.

The magnetic field is important if the distance $x$ between the localized states $a$ and $b$ is such that $\Delta_x \sim E_H$. For such states the matrix element $A_{ab}$ is not small but, at the same time, the levels are close to each other. The magnetic field $H$ mixes the states $a$ and $b$ and
one can expect for the hybridized states the statistics of the unitary ensemble. Comparing the energies $\Delta x$ and $E_H$

$$\exp(-x/L_c) \sim X^2,$$  \hspace{1cm} (84)

we obtain the characteristic distance $x_H$, Eq. (80).

Assuming that the mixing of the levels by the magnetic field leads to the correlations given by the unitary ensemble we come to the conclusion that the exponential decay of the density-density correlation function and other correlation functions characterizing wave functions should be determined at distances $x \gtrsim x_H$ by the localization length $L_{cu}$ of the unitary ensemble.

These simple arguments explain the two-scale behavior, Eq. (79), and give the correct second scale $x_H$, although the doubling of the localization length can be obtained by the exact calculation only. At the same time, the probability to obtain states at a desired distance with eigenenergies very close to each other is small. This small probability may be compensated by a large contribution to moments of the density and therefore can be essential when calculating these quantities. As concerns calculation of the logarithm the large splashes are not crucial and their contribution is small due to the small probability, which is in agreement with the results of the numerical investigation of the transmittance \cite{21} (actually, the authors of this work suggested as one of possible explanation of the difference between the averaged logarithm and the moments existence of “anomalously” localized states).

An additional suppression of the splashes can be a consequence of a smearing due to inelastic scattering and presence of leads. To observe the splashes one has to be at distances from the ends of the wire exceeding $x_H$. At distances smaller that $x_H$ the weak magnetic field cannot have any considerable effect of the wave functions because the validity of the perturbation theory in $H$ is determined by the value of the smearing and not by the energy separation of the relevant states. But the transmittance probes the wave functions at the end of the sample. In this region, the magnetic field can become important only if the relevant magnetic energy $E_H$ becomes of the order of the smearing energy and the latter is of the order of $E_c$ near the ends.

Recently, localization in disordered wires in a magnetic field was studied analyzing the autocorrelation function of spectral determinants (ASD) \cite{40} and no indication of the two-scale localization has been found. In our opinion, this cannot be a great surprise because the ASD is a quantity that can give only a rough picture of what is going in reality. By definition, the ASD probes only correlations of the energy levels and not the spatial structure of the wave functions. Any localized states contribute to the ASD independent of the distance between the centers of localization. This makes hardly possible to say anything about the overlap of the localized states, which is so important for the existence of the second scale. We think that the fact that the second scale was not identified in Ref. \cite{40} is not in contradiction with our finding but rather demonstrates that the ASD is not a proper quantity to describe the effect involved.

Let us now discuss how one can try to check experimentally our predictions. Considering the transmittance of single wire one should speak about the logarithm of the transmittance that is a self-averaging quantity. In this case, the rare splashes we discussed previously are not important and one should see only one localization length which is at weak magnetic fields close to the length $L_{co}$ of the orthogonal ensemble. However, one can measure the
conductance of a system of a large number of wires connected in parallel. This is exactly the set up of the experiment of Ref. 16, 17. The large number of wires in this case leads to averaging of the transmittance itself rather than of its logarithm. In such a situation one might see, in principle, the second scale \( L_{cu} \). Physically this means that although the splashes are rare, a wire, where such a splash occurs, gives the main contribution to the conductivity.

Unfortunately, in the experiment 16, 17, the localization length was not measured directly but was extracted from the dependence of the hopping conductivity on temperature.

It is known that the conductance at very low temperatures is possible due to activations through Mott resonances, which leads to the picture of the variable range hopping. The localization length \( L_c \) enters directly the activation energy in this conductivity

\[
\sigma = \sigma_0 \exp \left[ (-T_0/T)^{1/(1+d)} \right], \quad T_0 \sim (\nu L_c^d)^{-1}.
\]

(85)

where \( d \) is the dimensionality of the system. However, the localization length entering this formula is the length for a typical wave function and the splashes are not very important.

As a matter of fact, Eq. (85) does not apply to 1D samples. The conductivity of 1D chains and wires should obey a simple Arrhenius-type law instead

\[
\sigma = \sigma_0 \exp (-T_0/T), \quad T_0 \sim (\nu S L_c)^{-1}.
\]

(86)

Remarkably, even accounting for electron-electron interactions preserves this kind of dependence. It is this law that was discovered in recent experiments in the regime of strong localization and activation energy \( T_0 \) was used to extract the localization length \( L_c \). Which localization length enters the conductivity in the 1D case is not as clear as for higher dimensions and there could be a chance that the second scale can be seen here. Performing such experimental analysis at different temperatures one could try to observe the doubling of the localization length considered, because decreasing the temperature should result in a crossover from the activation behavior with the activation energy \( T_0 \) in Eq. (86), to the activation energy \( T_0/2 \). Indeed, at very low temperatures, \( T < (\nu S x_H)^{-1} \), the electron hopping due to the overlap of far tails plays the dominant role, so that the larger localization length \( L_{cu} \) should be used in Eq. (86). For temperatures \( (\nu S x_H)^{-1} < T < (\nu S L_c)^{-1} \), hopping between the main body of localized states is essential and the localization length \( L_{co} \) should be substituted for \( L_c \) in Eq. (86).

Needless to say that the second scale may be observed only if the wires are sufficiently long such that the main contribution to the resistivity comes from the variable-range hopping inside the sample and the wire can be considered effectively as infinite. In this limit the resistivity should be proportional to the length of the wire. Measuring the conductance of a finite wire at ultralow temperatures where the resistivity grows exponentially with the sample length cannot help extracting the two-scale behavior as it has been discussed previously.

**VIII. CONCLUSIONS**

Studying localization properties of infinitely long disordered wires within supersymmetry technique combined with the transfer-matrix technique, we managed to calculate correlation
functions of interest. This procedure is based on working in the coordinate representation, Eq. (14), and generating any quantity of interest from the two-point correlation function, Eq. (6). Using this approach we obtained all moments and distribution functions of the density-density correlator and Landauer-like conductivity for the orthogonal and unitary ensembles, Eqs. (46,49,44). These expressions are exact and they are valid for arbitrary distances.

Studying the crossover between the unitary and orthogonal ensembles for weak magnetic fields we have found moments of the density-density correlations at large distances $x \gg L_c$. We have demonstrated that the far tail of these correlation functions characterizing the wave functions decays at arbitrary weak magnetic field with the localization length which is double as large as that of the main part. As all the moments decay with the same length, we argue that this behavior is due to rare splashes of the wave functions that are very sensitive to the magnetic field. In contrast to this, the localization length characterizing a typical wave function might exhibit a smooth crossover.

We emphasize, however, that in the crossover regime, not only study of the typical localization length, but also of the whole distribution of the logarithm is of interest. It should differ from the standard log-normal one even for $x \gg L_c$ so long as $x \lesssim x_H$, Eq. (80), being asymmetric and broad. At present, we can not predict how the entire distribution function of the averaged logarithm of the wave function will develop in the limit of infinitely long distances.

Slightly revising the Mott arguments for the AC conductance,\textsuperscript{31,32} we have shown how one can extract the characteristic scale of the problem $x_H$, without performing any calculations. These arguments are quite general, which implies that our results may be valid for a disordered system of any dimension.

From the exact treatment as well as from the qualitative arguments of the previous section we understand that the effect we have found is very sensitive to different kinds of the level smearing. Therefore, it may be observed at low temperatures only.

As concerns a possibility of an experimental observation of the effect it might be seen at low temperatures in the regime of the hopping conductivity in a system of wires connected in parallel. The wires should be sufficiently long, such that the system would obey Ohm’s law. Then, if the tails of the wave functions determine the hopping conductivity, one should expect the Arrhenius law but with different activation energies in the limit of very low temperatures.

We hope that it can become possible to check our finding both numerically and experimentally, although this can be a difficult task due to a small probability of the wave functions with the high sensitivity to the magnetic field.

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