ON THE DENSENESS OF SOME SPARSE HOROCYCLES

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Abstract. Let $\Gamma$ be a non-uniform lattice in $\text{PSL}(2, \mathbb{R})$. In this note, we show that there exists a constant $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$, any one-parametrer unipotent subgroup $\{u(t)\}_{t \in \mathbb{R}}$ and any $p \in \text{PSL}(2, \mathbb{R})/\Gamma$ which is not $u(t)$-periodic, the orbit $\{u(n^{1+\gamma})p : n \in \mathbb{N}\}$ is dense in $\text{PSL}(2, \mathbb{R})/\Gamma$. We also prove that there exists $N \in \mathbb{N}$ such that for the set $\Omega$ of $N$-almost primes, the orbit $\{u(x)p : x \in \Omega\}$ is dense in $\text{PSL}(2, \mathbb{R})/\Gamma$. Moreover, for a one-parameter unipotent flow $u(t)$ on $\text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z})$, we construct points $x$ with the property that the orbit $\{u(n^2)x : n \in \mathbb{N}\}$ is dense in $\text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z})$.

1. Introduction

The equidistribution of unipotent orbits has been studied extensively over the past few decades, and many important and fundamental tools and machinaries to study this topic have been established, one of which is the celebrated Ratner’s Theorem [2, 8, 9, 14, 15]. Using Ratner’s Theorem, we can investigate the distribution of a subset in a homogeneous space which is invariant under an (almost)-unipotent group action. As for subsets which do not exhibit any invariance of group actions, their distributions in homogeneous spaces are still far from being well-known and it is mysterious how to apply Ratner’s Theorem in this situation. One of the special but interesting cases is the following: Let $\{u(t)\}_{t \in \mathbb{R}}$ be a unipotent flow on a homogeneous space $G/\Gamma$ and consider the parameter $t$ sampled in an irregular discrete subset $\Omega$ of $\mathbb{R}$. Then the distribution of $\{u(t)p : t \in \Omega\}$ in $G/\Gamma$ is an important problem, and it is listed as one of the open problems in homogeneous dynamics [4].

Sometimes the irregular subset $\Omega$ of $\mathbb{R}$ is chosen to be sparse relative to $\mathbb{R}$ in the sense that the density of $\Omega$ in $\mathbb{R}$ is zero. In this situation, the problem is called sparse equidistribution

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The sparse equidistribution problem has recently attracted much attention, and one can refer to [1, 6, 7] for discussions of this topic on general homogeneous spaces.

In this note, we investigate the topological version of the sparse equidistribution problem in the case where the homogeneous space $G/\Gamma = \text{PSL}(2, \mathbb{R})/\Gamma$. Specifically, let $\{u(t)\}$ be a one-parameter unipotent subgroup of $\text{PSL}(2, \mathbb{R})$, and suppose that the irregular discrete subset $\Omega$ is either $\{n^{1+\gamma} : n \in \mathbb{N}\}$ ($\gamma > 0$) or a subset of almost primes. We consider the subset $\{u(t)p : t \in \Omega\}$ ($\forall p \in G/\Gamma$), and would like to know if the closure of $\{u(t)p : t \in \Omega\}$ is the entire space $\text{PSL}(2, \mathbb{R})/\Gamma$ or a sub-homogeneous space in $\text{PSL}(2, \mathbb{R})/\Gamma$.

If $\Gamma$ is a uniform lattice in $\text{PSL}(2, \mathbb{R})$, quite a few results regarding this topic have been established. For instance, if $\Omega$ is the subset $\{n^{1+\gamma} : n \in \mathbb{N}\}$ for a sufficiently small constant $\gamma > 0$, then it is shown [18, 19] that $\{u(t)p : t \in \Omega\}$ is uniformly distributed in $\text{PSL}(2, \mathbb{R})/\Gamma$. If $\Omega$ is a subset of almost primes, then $\{u(t)p : t \in \Omega\}$ is uniformly distributed as well [11].

As for a non-uniform lattice $\Gamma$, much less is known. It is shown [20] that if $\Omega$ is the subset $\{n^{1+\gamma} : n \in \mathbb{N}\}$ for a sufficiently small constant $\gamma > 0$ and $p$ is a Diophantine point, then $\{u(t)p : t \in \Omega\}$ is uniformly distributed in $\text{PSL}(2, \mathbb{R})/\Gamma$. For $\Gamma = \text{PSL}(2, \mathbb{Z})$ and $\Omega$ the subset of primes, it is proved quantitatively [16] that any limiting distribution of $\{u(t)p : t \in \Omega\}$ is absolutely continuous with respect to the Haar measure on $\text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z})$. If $\Gamma = \text{PSL}(2, \mathbb{Z})$ and $\Omega$ is a subset of almost primes, then it is proved in [11, 16] that $\{u(t)p : t \in \Omega\}$ is dense in the ambient space. Note that the results in [11] are actually proved on the homogeneous space $\text{SL}(n, \mathbb{R})/\Gamma$ where $\Gamma = \text{SL}(n, \mathbb{Z})$ or a uniform lattice in $\text{SL}(n, \mathbb{R})$ ($n \geq 2$).

In this paper, we establish some results regarding the topological version of the sparse equidistribution problem for any non-uniform lattice in $\text{PSL}(2, \mathbb{R})$, which extend the work of [16] and [11]. We remark here that our method relies on a Diophantine analysis on the initial point $p$. If $p$ is Diophantine of type $\kappa$ for some $\kappa > 0$, then the orbit $\{u(t)p\}$ exhibits certain chaos and in this case we can use effective Dani-Smillie theorem (a special case of Ratner’s theorem) to show that the orbit $\{u(t)p : t \in \Omega\}$ is dense. If $p$ is not Diophantine of type $\kappa$, then the orbit $\{u(t)p\}$ is approximated by a sequence of periodic $u(t)$-orbits, and we can then apply related results on the circle in this case. Parts of our analysis may appear implicitly in [16] in a different language, and our method here is simpler and of dynamical flavor. This method is quite general, and can be used to study the problem for quite a large class of sample subsets $\Omega$.

Now we introduce the notation and state our main results in the paper. Let $G = \text{PSL}(2, \mathbb{R})$ and $\Gamma$ a non-uniform lattice in $G$. An element $g \in G$ is unipotent if 1 is the only eigenvalue
of \( g \). A one-parameter unipotent subgroup \( \{u(t)\}_{t \in \mathbb{R}} \) of \( G \) is a one-parameter subgroup of \( G \) where \( u(t) \) is unipotent for any \( t \in \mathbb{R} \). In this note, we prove the following

**Theorem 1.1.** Let \( \Gamma \) be a non-uniform lattice in \( \text{PSL}(2, \mathbb{R}) \). Then there exists a constant \( \gamma_0 > 0 \) such that for any \( 0 < \gamma < \gamma_0 \), any one-parameter unipotent subgroup \( \{u(t)\}_{t \in \mathbb{R}} \) and any \( p \in \text{PSL}(2, \mathbb{R})/\Gamma \) which is not \( \{u(t)\}_{t \in \mathbb{R}} \)-periodic, the orbit \( \{u(n^{1+\gamma})p : n \in \mathbb{N}\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\Gamma \).

**Remark 1.1.** If \( \Gamma \) is a uniform lattice in \( \text{PSL}(2, \mathbb{R}) \), then the denseness and the equidistribution of the orbit \( \{u(n^{1+\gamma})p\} \) (for a sufficiently small \( \gamma > 0 \) and for every \( p \in \text{PSL}(2, \mathbb{R})/\Gamma \)) is proved by [18, 19].

Recall that a natural number \( x \) is said to be an \( N \)-almost prime for some \( N \in \mathbb{N} \) if there are at most \( N \) prime factors (with multiplicities) in the decomposition of \( x \). We denote by \( \Omega(N) \) the subset of \( N \)-almost primes in \( \mathbb{N} \). Using the same techniques as in the proof of Theorem 1.1, we can prove the following

**Theorem 1.2.** Let \( \Gamma \) be a non-uniform lattice in \( \text{PSL}(2, \mathbb{R}) \). Then there exists \( N \in \mathbb{N} \) such that for any one-parameter unipotent subgroup \( u(t) \) and any \( p \in \text{PSL}(2, \mathbb{R})/\Gamma \) which is not \( \{u(t)\}_{t \in \mathbb{R}} \)-periodic, the orbit \( \{u(x)p : x \in \Omega(N)\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\Gamma \).

**Remark 1.2.** If \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) or a uniform lattice in \( \text{PSL}(2, \mathbb{R}) \), this result is established in [11, 16].

The following theorem may be of independent interest, and it gives initial points \( p \) for which \( \{u(n^{2})p : n \in \mathbb{N}\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z}) \).

**Theorem 1.3.** For a one-parameter unipotent subgroup \( u(t) \), there is an uncountable dense subset \( E \) in \( \text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z}) \) such that for any \( x \in E \), the orbit \( \{u(n^{2})x : n \in \mathbb{N}\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z}) \). Moreover, the Hausdorff dimension of \( E \) is greater than 2.

The strategy of the proof of Theorem 1.1 is the following: If \( p \) is Diophantine of type \( \kappa \) for some constant \( \kappa \), then the result is known and it is a corollary of [20]. So it suffices to deal with the case where \( p \) is not Diophantine of type \( \kappa \). In this case, we can find a sequence of \( \{u(t)\}_{t \in \mathbb{R}} \)-periodic points \( \{q_k\}_{k \in \mathbb{N}} \) converging to the point \( p \), and then the orbit \( \{u(n^{1+\gamma})p : n \in \mathbb{N}\} \) can be approximated by the orbit \( \{u(n^{1+\gamma})q_k : n \in \mathbb{N}\} \) as \( k \to \infty \). Note that the orbit \( \{u(n^{1+\gamma})q_k : n \in \mathbb{N}\} \) is a subset in the circle \( \{u(t)q_k : t \in \mathbb{R}\} \). As \( k \) goes to
infinity, the circle \( \{ u(t)q_k : t \in \mathbb{R} \} \) becomes dense in the homogeneous space PSL(2, \( \mathbb{R} \))/\( \Gamma \), while using Fourier Analysis we can also prove that the orbit \( \{ u(n^{1+\gamma})q_k : n \in \mathbb{N} \} \) becomes dense in the circle \( \{ u(t)q_k : t \in \mathbb{R} \} \). By standard approximation argument, this would then imply that the orbit \( \{ u(n^{1+\gamma})p : n \in \mathbb{N} \} \) is dense in PSL(2, \( \mathbb{R} \))/\( \Gamma \) for the point \( p \) which is not Diophantine of type \( \kappa \). The strategies of the proofs of Theorems 1.2 and 1.3 are similar.

The note is organized as follows. In section 2, we list the notation and preliminaries in this note. In section 3, we give a proof of Theorem 1.1 which is based on [20, 21]. Then we use the same idea to prove Theorem 1.2 in section 4. In the last section, we construct points \( p \) in PSL(2, \( \mathbb{R} \))/PSL(2, \( \mathbb{Z} \)) with the property that the orbit \( \{ u(n^2)p : n \in \mathbb{N} \} \) is dense.

2. Notation and Preliminaries

In this section, we introduce the notation and preliminaries in the paper. Let

\[
U := \left\{ u_0(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}
\]

be the upper triangular one-parameter unipotent subgroup in PSL(2, \( \mathbb{R} \)). Then any one-parameter unipotent subgroup of PSL(2, \( \mathbb{R} \)) is conjugate to \( \{ u_0(s) : s \in \mathbb{R} \} \) or \( \{ u_0(-s) : s \in \mathbb{R} \} \). In the following, we will fix the symbol \( u_0(s) \) for the one-parameter unipotent subgroup \( U \). We denote by \( U_- \) the lower triangular unipotent subgroup in \( G \), \( A \) the diagonal group

\[
A = \left\{ a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} : t \in \mathbb{R} \right\}
\]

and

\[
\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

the non-trivial Weyl element in \( G = \text{PSL}(2, \mathbb{R}) \). In the following, for any two quantities \( c_1, c_2 > 0 \) we write \( c_1 \ll c_2 \) if there exists a constant \( C > 0 \) such that \( c_1 \leq Cc_2 \). If \( c_1 \ll c_2 \) and \( c_2 \ll c_1 \), then we write \( c_1 \sim c_2 \). We will specify the implicit constants in the context if necessary.

We denote by \( C^\infty(G/\Gamma) \) the space of all smooth functions on \( G/\Gamma \), and \( C^\infty_c(G/\Gamma) \) the space of all compactly supported smooth functions on \( G/\Gamma \). For \( f \in C^\infty(G/\Gamma) \), let \( \| f \|_{p,k} \) be the Sobolev \( L^p \)-norm of \( f \) involving all the Lie derivatives of order \( \leq k \). For any smooth function \( f \) on \( \mathbb{R} \), we will use the same notation \( \| f \|_{p,k} \) for the \( L^p \)-norm of \( f \) involving derivatives of order up to \( k \). Note that \( \| f \|_{\infty,0} \) is the supremum norm of \( f \).
Since \( \Gamma \) is a non-uniform lattice in \( G \), there exist \( U_- \)-periodic points in \( G/\Gamma \). Without loss of generality, we may assume that \( e\Gamma \) is a \( U_- \)-periodic point. In the following, we recall several notions regarding Diophantine points in homogeneous spaces \([20, 21]\), which will be used frequently in the note. For any \( p \in G/\Gamma \), we denote by \( \text{Stab}(p) \) the stabilizer of \( p \) in \( G \). If \( p = g\Gamma \), then \( \text{Stab}(p) = g\Gamma g^{-1} \). We fix a norm \( \| \cdot \|_g \) on the Lie algebra \( g \) of \( G \), and denote by \( d_G(\cdot, \cdot) \) and \( d_{G/\Gamma}(\cdot, \cdot) \) the induced distances on \( G \) and \( G/\Gamma \) by \( \| \cdot \|_g \) respectively.

**Definition 2.1** ([20, 21]). For any \( p \in G/\Gamma \), we define the injectivity radius at \( p \) by

\[
\eta(p) = \inf_{v \in \text{Stab}(p) \setminus e} d_G(v, e).
\]

**Definition 2.2** ([20, 21]). A point \( p \in G/\Gamma \) is Diophantine of type \( \kappa \) (with respect to the diagonal group \( \{a_t\}_{t \in \mathbb{R}} \)) if there exists a constant \( C > 0 \) such that

\[
\eta(a_t p) \geq Ce^{-\kappa t} \text{ for all } t > 0.
\]

Note that \( 0 \leq \kappa < 1 \).

**Definition 2.3** ([20, 21]). A point \( p \in G/\Gamma \) is called rational if \( \text{Stab}(p) \cap U \neq \{e\} \). Note that in this case, \( p \) is a \( U \)-periodic point in \( G/\Gamma \).

**Definition 2.4** ([20, 21]). Define the denominator of a rational point \( p \in G/\Gamma \) by

\[
d(p) = \inf_{v \in \text{Stab}(p) \setminus \{U \cdot e\}} \| \log v \|_g.
\]

Here \( \log \) is the inverse of the exponential map from \( U \) to the Lie algebra of \( U \). Note that in this case, \( d(p) \) is the period of the \( U \)-periodic orbit \( U \cdot p \).

The following proposition plays a crucial role in our analysis.

**Proposition 2.1.** [21, Proposition 6.3] Let \( p \in U_-(e\Gamma) \subset G/\Gamma \). If \( p \) is not Diophantine of type \( \kappa \) (\( \kappa > 0 \)), then there exist a constant \( C > 0 \) depending only on \( G/\Gamma \) and \( \kappa \), and a sequence of distinct \( U \)-periodic points \( q_k \in U_-(e\Gamma) \) with \( U \)-period \( d(q_k) \to \infty \) such that

\[
d_{G/\Gamma}(p, q_k) \leq Cd(q_k)^{-\frac{1}{1-\kappa}}.
\]
3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We know that any one-parameter unipotent subgroup is conjugate to \( \{ u_0(s) : s \in \mathbb{R} \} \) or \( \{ u_0(-s) : s \in \mathbb{R} \} \). Let
\[
a_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and define \( \theta : G \to G \) by
\[
\theta(g) = a_0 g a_0^{-1}.
\]
Then \( \theta \) is an automorphism of \( G \) and it defines a homeomorphism between \( G/\Gamma \) and \( G/\theta(\Gamma) \). Therefore, for any \( p \in G/\Gamma \), the subset \( \{ u_0(n^{1+\gamma})p : n \in \mathbb{N} \} \) is dense in \( G/\Gamma \) if and only if \( \{ u_0(-n^{1+\gamma})\theta(p) : n \in \mathbb{N} \} \) is dense in \( G/\theta(\Gamma) \). In the following, without loss of generality, we may assume that the one-parameter unipotent subgroup is equal to \( \{ u_0(s) : s \in \mathbb{R} \} \). By Bruhat decomposition, we know that
\[
G = U A U_\omega \cup \omega A U_\omega \text{ and } G/\Gamma = U A U_\omega / \Gamma \cup \omega A U_\omega / \Gamma.
\]
Since any point in \( \omega A U_\omega / \Gamma \) is a \( U \)-periodic point, to prove Theorem 1.1, we only need to consider the initial point \( p \in U A U_\omega / \Gamma \). Now write \( p = uan\Gamma \) for \( u \in U, a \in A \) and \( n \in U_- \). Since \( a \) normalizes \( U \), we have
\[
u_0(t) \cdot uan\Gamma = (ua) \cdot u_0(ct) \cdot n\Gamma \quad (\forall t \in \mathbb{R})
\]
for some constant \( c > 0 \). Hence to prove Theorem 1.1, it suffices to prove the following

**Theorem 3.1.** Let \( \Gamma \) be a non-uniform lattice in \( \text{PSL}(2, \mathbb{R}) \). Then there exists a constant \( \gamma_0 > 0 \) such that for any \( p \in U_\omega (e\Gamma) \) which is not \( U \)-periodic, any \( 0 < \gamma < \gamma_0 \) and any \( c > 0 \), the orbit \( \{ u_0(cn^{1+\gamma})p : n \in \mathbb{N} \} \) is dense in \( \text{PSL}(2, \mathbb{R})/\Gamma \).

The rest of this section is devoted to the proof of Theorem 3.1. The following proposition is well-known.

**Proposition 3.1 (5).** Let \( f \) be a smooth function on \( [a,b] \) such that \( f''(x) \geq \delta > 0 \). Then
\[
\left| \sum_{n=a}^{b} e^{2\pi if(n)} \right| \ll \frac{f'(b) - f'(a) + 1}{\sqrt{\delta}}.
\]

As a corollary of Proposition 3.1, we deduce the following
Lemma 3.1. Let \( f(x) = cx^{1+\gamma} \) \((c, \gamma > 0)\). For each \( k \in \mathbb{R} \setminus \{0\} \), we have
\[
\left| \sum_{n=1}^{N} e^{2\pi ikf(n)} \right| \ll |k| \frac{1}{2} N^{(1+\gamma)/2} + |k|^{-\frac{1}{2}} N^{(1-\gamma)/2}.
\]

Proof. Let \( a = 1 \) and \( b = N \) in Proposition 3.1, and we take
\[
\delta = |ck|(1 + \gamma) N^{\gamma-1}.
\]
Then we have
\[
\left| \sum_{n=a}^{b} e^{2\pi ikf(n)} \right| \ll \frac{|ck|(1 + \gamma) N^{\gamma} - |ck|(1 + \gamma) + 1}{\sqrt{|ck|(1 + \gamma) N^{\gamma-1}}}
\ll |k| \frac{1}{2} N^{(1+\gamma)/2} + |k|^{-\frac{1}{2}} N^{(1-\gamma)/2}.
\]
This completes the proof of the lemma.

Lemma 3.2. Let \( f(x) \) be a smooth periodic function on \( \mathbb{R} \) with period \( l \). Then
\[
\left| \sum_{n=1}^{N} f(cn^{1+\gamma}) - N \cdot \frac{1}{l} \int_{0}^{l} f(x) dx \right| \ll l^{3} N^{(1+\gamma)/2} \|f\|_{2,\infty}.
\]

Proof. Write the Fourier series of \( f \) on \([0, l] \) as
\[
f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi ikx/l}
\]
where \( a_k = \frac{1}{l} \int_{0}^{l} f(x) e^{-2\pi ikx/l} dx \). Since \( f \in C^\infty(\mathbb{R}) \), using integration by parts, we have
\[
|a_k| \leq \frac{l^2}{4\pi^2 k^2} \|f\|_{2,\infty}.
\]
Therefore, by Lemma 3.1, we have
\[
\left| \sum_{n=1}^{N} f(cn^{1+\gamma}) - N \cdot \frac{1}{l} \int_{0}^{l} f(x) dx \right|
\leq \sum_{k \neq 0} |a_k| \left| \sum_{n=1}^{N} e^{2\pi ikcn^{1+\gamma}/l} \right|
\ll \sum_{k \neq 0} |a_k| N^{(1+\gamma)/2} |k|^{1/2} l^{1/2}
\ll l^{3} N^{(1+\gamma)/2} \|f\|_{2,\infty}.
\]
This completes the proof of the lemma.
Lemma 3.3. Let $c > 0$, $f \in C_c^\infty(\mathrm{PSL}(2, \mathbb{R})/\Gamma)$, $p = \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) \Gamma$ and $q = \left( \begin{array}{cc} 1 & 0 \\ y & 1 \end{array} \right) \Gamma$. Then
\[
\left| \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})p) - \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})q) \right| \ll N^{3+2\gamma}|x - y|\|f\|_{1,\infty}
\]
Here the implicit constant depends only on $c$ and $\gamma$.

Proof. Note that
\[
\left( \begin{array}{ccc} 1 & s \\ 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 \\ \delta & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -s \\ 1 & 1 - \delta s \end{array} \right) = \left( \begin{array}{ccc} 1 + s\delta & -s^2\delta \\ \delta & 1 - \delta s \end{array} \right)
\]
and
\[
u_0(s)q = \left( \begin{array}{ccc} 1 & s \\ 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 \\ y - x & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -s \\ 0 & 1 \end{array} \right) \nu_0(s)p.
\]
Hence
\[
d_G \left( \nu_0(s) \left( \begin{array}{ccc} 1 & 0 \\ y & 1 \end{array} \right) , \nu_0(s) \left( \begin{array}{ccc} 1 & 0 \\ x & 1 \end{array} \right) \right) \leq \max\{s^2, s, 1\}|x - y|.
\]
The lemma then follows from direct computations. \qed

The following proposition is crucial in the proof of Theorem 3.1.

Proposition 3.2. Let $c > 0$, $f \in C_c^\infty(\mathrm{PSL}(2, \mathbb{R})/\Gamma)$ and $p \in U(e\Gamma) \subset \mathrm{PSL}(2, \mathbb{R})/\Gamma$. Suppose that $p$ is neither a $U$-periodic point nor Diophantine of type $\kappa$. Let $\{q_k\}_{k \in \mathbb{N}}$ be a sequence of $U$-periodic points which approach $p$ as in Proposition 2.1. Then there is a constant $C > 0$ such that
\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})p) - \frac{1}{N} \int_{0}^{d(q_k)} f(u_0(s)q_k) \, ds \right| \leq C(N^{2+2\gamma}d(p, q_k) + d(q_k)^3N^{(\gamma-1)/2})\|f\|_{2,\infty}
\]

Proof. By Lemma 3.3, we have
\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})p) - \frac{1}{N} \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})q) \right| \leq N^{2+2\gamma}d(p, q_k)\|f\|_{1,\infty}.
\]
By Lemma 3.2, we have
\[
\left| \frac{1}{N} \sum_{n=1}^{N} f(u_0(cn^{1+\gamma})q_k) - \frac{1}{d(q_k)} \int_{0}^{d(q_k)} f(u_0(s)q_k) \, ds \right| \leq d(q_k)^3N^{2-1}\|f\|_{2,\infty}.
\]
The proposition then follows by combining these inequalities.

Proof of Theorem 3.1. Let \( p \in \text{PSL}(2, \mathbb{R})/\Gamma \) be a point in \( U_-(e\Gamma) \) such that \( p \) is not a \( U \)-periodic point.

Case 1: \( p \) is Diophantine of type \( \kappa = 1 - 1/100 \). Then by [20, Theorem 1.1], Theorem 3.1 holds for an appropriate constant \( \gamma_0 > 0 \).

Case 2: Suppose that \( p \) is not Diophantine of type \( \kappa = 1 - 1/100 \). Then by Proposition 2.1, there exists a sequence of \( U \)-periodic points \( \{q_k\}_{k \in \mathbb{N}} \) such that \( q_k \to p \) and \( d(p, q_k) \leq d(q_k)^{-50} \).

Now according to Proposition 3.2, we have

\[
\left| \frac{1}{N} \sum_{n=1}^{N} f \left( u_0(cn^{1+\gamma})p \right) - \frac{1}{d(q_k)} \int_0^{d(q_k)} f \left( u_0(s)q_k \right) ds \right| \leq C \left( N^{2+2\gamma} d(p, q_k) + d(q_k)^3 N^{(\gamma-1)/2} \|f\|_{2,\infty} \right)
\]

Let \( \gamma_0 = 0.1 \) and \( N = N_k := d(q_k)^{10} \). Then for any \( 0 < \gamma < \gamma_0 \), we have

\[
N^{2+2\gamma} d(p, q_k) \leq d(q_k)^{-28}, \quad d(q_k)^3 N^{(\gamma-1)/2} \leq d(q_k)^{-1.5}.
\]

Therefore, as \( k \to \infty \), we have

\[
\left| \frac{1}{N_k} \sum_{n=1}^{N_k} f \left( u_0(cn^{1+\gamma})p \right) - \frac{1}{d(q_k)} \int_0^{d(q_k)} f \left( u_0(s)q_k \right) ds \right| \to 0.
\]

It is known that

\[
\frac{1}{d(q_k)} \int_0^{d(q_k)} f \left( u_0(s)q_k \right) ds \to \int_{G/\Gamma} f(g) d\mu_{G/\Gamma}(g)
\]

where \( \mu_{G/\Gamma} \) is the Haar measure on \( G/\Gamma \). Hence

\[
\frac{1}{N_k} \sum_{n=1}^{N_k} f \left( u_0(cn^{1+\gamma})p \right) \to \int_{G/\Gamma} f(g) d\mu_{G/\Gamma}(g)
\]

which implies that the orbit \( \{u_0(cn^{1+\gamma})p : n \in \mathbb{N}\} \) is dense in the space \( \text{PSL}(2, \mathbb{R})/\Gamma \).

We complete the proof of Theorem 3.1 by combining Case 1 and Case 2. \( \square \)

4. Linear sieve: the Jurkat-Richert theorem and the proof of Theorem 1.2

In this section, we prove Theorem 1.2. Similar to the discussion at the beginning of section 3, without loss of generality, we may assume that the one-parameter unipotent subgroup in Theorem 1.2 is equal to \( \{u_0(s) : s \in \mathbb{R}\} \). For any \( x \in \mathbb{R} \), we denote by \( [x] \) the largest integer \( \leq x \). We will need the Jurkat-Richert theorem about the linear sieve.
Theorem 4.1. [12, Theorem 9.7] Let $A = \{a(n)\}_{n \in \mathbb{N}}$ be a sequence of non-negative numbers such that
\[ |A| = \sum_{n=1}^{\infty} a(n) < \infty. \]

Let $\mathcal{P}$ be a set of prime numbers and for $z \geq 2$, let
\[ P(z) = \prod_{p \in \mathcal{P}, p < z} p. \]

Let
\[ S(A, P, z) = \sum_{n=1}^{\infty} a(n). \]

For every $n \geq 1$, let $g_n(d)$ be a multiplicative function such that
\[ 0 \leq g_n(p) < 1, \quad \text{for all } p \in \mathcal{P}. \]

Define $r(d)$ by
\[ |A_d| = \sum_{n=1,d|n}^{\infty} a(n) = \sum_{n=1}^{\infty} a(n)g_n(d) + r(d). \]

Let $Q$ be a finite subset of $\mathcal{P}$, and let $Q$ be the product of the primes in $Q$. Suppose that, for some $\epsilon$ satisfying $0 < \epsilon < 1/200$, the inequality
\[ \prod_{p \in \mathcal{P} \setminus Q, u \leq p < z} (1 - g_n(p))^{-1} < (1 + \epsilon) \frac{\log z}{\log u} \]
holds for all $n$ and $1 < u < z$. Then for any $D \geq z$ there is the upper bound
\[ S(A, \mathcal{P}, z) < (F_0(s) + \epsilon e^{14-s})X + R \]
and for any $D \geq z^2$ there is the lower bound
\[ S(A, \mathcal{P}, z) > (f_0(s) - \epsilon e^{14-s})X - R \]
for some functions $F_0(s)$ and $f_0(s)$ where $s = \frac{\log D}{\log z}$, $F_0(s) = 1 + O(e^{-s})$, $f_0(s) = 1 - O(e^{-s})$,
\[ X = \sum_{n=1}^{\infty} a(n) \prod_{p \in P(z)} (1 - g_n(p)) \]
and the remainder term is
\[ R = \sum_{d | P(z), d < DQ} |r(d)|. \]
If there is a multiplicative function $g(d)$ such that $g_n(d) = g(d)$ for all $n$, then

$$X = V(z) |A|$$

where

$$V(z) = \prod_{p | P(z)} (1 - g(p)).$$

**Remark 4.1.** The explicit expressions of the functions $F_0(s)$ and $f_0(s)$ can be found in [12, Theorem 9.4].

The following theorem verifies one assumption in Theorem 4.1.

**Theorem 4.2.** [12, Theorem 6.9] For any $\epsilon > 0$, there exists a number $u_1(\epsilon) > 0$ such that

$$\prod_{u \leq p < z} \left(1 - \frac{1}{p}\right)^{-1} < (1 + \epsilon/3) \frac{\log z}{\log u}$$

for any $u_1(\epsilon) \leq u < z$.

It is known that $G$ acts on the upper half plane $\mathcal{H}^2$ by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z := \frac{az + b}{cz + d}$$

and we denote by $\pi$ the projection from $\Gamma \backslash G$ to $\Gamma \backslash \mathcal{H}^2$ by sending $\Gamma g$ to $\Gamma g \cdot i$. We define the geodesic flow on $\Gamma \backslash G$ by

$$g_t(\Gamma g) = \Gamma g \left( \begin{array}{cc} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{array} \right) = \Gamma g \cdot a_{-t}$$

Fix a point $p_0 \in \Gamma \backslash \mathcal{H}^2$. Define

$$\text{dist}(p) = d_{\Gamma \backslash \mathcal{H}^2}(p_0, \pi(p))$$

where $d_{\Gamma \backslash \mathcal{H}^2}(\cdot, \cdot)$ is the hyperbolic distance on $\Gamma \backslash \mathcal{H}^2$. We will need the following effective equidistribution of discrete horocycle orbits. Note that in the statement, the homogeneous space is denoted by $\Gamma \backslash G$.

**Theorem 4.3.** [20, Theorem 1.2] Let $T > K \geq 1$ and $f \in C^\infty(\Gamma \backslash G)$ such that

$$\int_{\Gamma \backslash G} f \, d\mu = 0 \text{ and } \|f\|_{\infty, A} < \infty.$$
Suppose that $q \in \Gamma \backslash G$ satisfies

$$r = r(q, T) = T \cdot e^{-\operatorname{dist}(g_{\log} T(q))} \geq 1.$$ 

Then we have

$$\left| \frac{1}{T/K} \sum_{j \in \mathbb{Z}, 0 \leq K_j < T} f(qu_0(K_j)) \right| \ll \frac{K^{3/2} \ln^3(r + 2)}{r^{\beta/2}} \|f\|_{\infty, 4}$$

for some $\beta > 0$ depending only on the constant in the mixing property of the unipotent flow $u(t)$ and the spectral gap of the Laplacian on $\Gamma \backslash \mathcal{H}^2$.

**Remark 4.2.** Theorem 1.2 in [20] assumes that $T > K > 2$. We remark here that the condition $T > K \geq 1$ works equally well in the proof of [20, Theorem 1.2].

The following lemma gives an estimate for the function $r = r(q, T)$ in Theorem 4.3.

**Lemma 4.1.** [20, Lemmas 3.3, 3.4] Let $p \in \Gamma \backslash G$. We have

$$r(p, T) = T \cdot e^{-\operatorname{dist}(g_{\log} T(p))} \sim T \cdot \eta(g_{\log} Tp).$$

In particular, if $p$ is Diophantine of type $\gamma$, then $r(p, T) \gg T^{1-\gamma}$.

**Proof of Theorem 1.2.** Let $p \in \text{PSL}(2, \mathbb{R})/\Gamma$ such that $p$ is not a $U$-periodic point, and take $\mathcal{P}$ to be the set of all prime numbers in Theorem 4.1. We consider two cases.

Case 1: Let $p \in \Gamma \backslash G$ and suppose that $p$ is Diophantine of type $\kappa = 1 - 1/10000$. We prove that there exists a natural number $L \in \mathbb{N}$ such that the orbit $\{p \cdot u_0(x) : x \in \Omega(L)\}$ is dense in $\Gamma \backslash G$.

We remark here that in order to make the symbols consistent with the literature, in this case we write the homogeneous space as $\Gamma \backslash G = \text{PSL}(2, \mathbb{Z}) \backslash \text{PSL}(2, \mathbb{R})$.

We take a non-negative compactly supported smooth function $f \neq 0$ on $\Gamma \backslash G$ and fix $N \in \mathbb{N}$. Define

$$a(n) = \begin{cases} 
  f(p \cdot u_0(n)) & n \leq N \\
  0 & n > N 
\end{cases}$$

Then by Theorem 4.3, we have

$$|A_d| = \sum_{1 \leq n \leq N, d | n} a(n) = \sum_{1 \leq n \leq N, d | n} f(p \cdot u_0(n)) = \sum_{1 \leq dj < N} f(p \cdot u_0(jd))$$

$$\ll (N/d) \int_{\Gamma \backslash G} f \, d\mu + (N/d) \frac{d^{3/2} \ln^3(r + 2)}{r^{\beta/2}} \|f\|_{\infty, 4}$$
\[ \ll \sum_{1 \leq n \leq N} \frac{1}{d} a(n) + \left( \frac{N}{d} \right) \frac{d^{\frac{1}{2}} \ln^{3/2}(r + 2)}{r^{\beta/2}} \| f \|_{\infty, 4} + \frac{N \ln^{\frac{3}{2}}(r + 2)}{d} \frac{d^{\frac{1}{2}} \ln^{3/2}(r + 2)}{r^{\beta/2}} \| f \|_{\infty, 4} \]

According to Theorem 4.1, we take \( g(d) = 1/d \) in the linear sieve and
\[ |r(d)| \ll \left( \frac{N}{d} \right) \frac{2d^{\frac{1}{2}} \ln^{3/2}(r + 2)}{r^{\beta/2}} \| f \|_{\infty, 4}. \]

Note that by Lemma 4.1, we have \( r = r(p, N) \gg N^{1-\kappa} \).

Now let \( z = N^\alpha \) for some small constant \( \alpha > 0 \) which will be determined later. Let \( s > 100 \) be a sufficiently large number so that \( f_0(s) > 0.1 \), where \( f_0(s) \) is defined as in Theorem 4.1. Then by Theorem 4.1, if \( \alpha \) is sufficiently small, then
\[ S(A, P, z) > 0.01V(z)|A| - R \]

where using Theorem 4.2, Theorem 4.3 and Lemma 4.1 we have
\[ |A|/N \sim \int_{\Gamma \backslash G} f d\mu, \quad 1/\log N \ll V(z) \text{ and } |R| \ll N^{1-\delta} \]
for some small constant \( \delta > 0 \). Hence \( S(A, P, z) > 0 \) if \( N \) is sufficiently large. Note that in the formula of \( S(A, P, z) \), any \( n \) coprime to \( P(z) \) has at most \( \lfloor 1/\alpha \rfloor + 1 \) prime factors. If we take \( L = \lfloor 1/\alpha \rfloor + 1 \), then this implies that for the set \( \Omega(L) \) of \( L \)-almost primes, the orbit \( \{ pu_0(x) : x \in \Omega(L) \} \) is dense in \( G/\Gamma \). This completes the proof of Case 1.

Case 2: Let \( p \in G/\Gamma \) and suppose that \( p \) is not Diophantine of type \( \kappa = 1 - 1/10000 \). Then we prove that there exists \( L \in \mathbb{N} \) such that the orbit \( \{ u_0(x)p : x \in \Omega \} \) is dense in \( G/\Gamma \) for \( \Omega \) the subset of \( L \)-almost primes.

By Bruhat decomposition, we know that
\[ G = UAU \cup \omega AU \quad \text{and} \quad G/\Gamma = UAU/\Gamma \cup \omega AU/\Gamma. \]

Since any point in \( \omega AU/\Gamma \) is a \( U \)-periodic point, we have \( p \in UAU \). Now write \( p = uan\Gamma \) for \( u \in U, a \in A \) and \( n \in U_- \). Since \( a \) normalizes \( U \), we have
\[ u(t) \cdot uan\Gamma = (ua) \cdot u(ct) \cdot n\Gamma \]
for some constant \( c > 0 \). Hence to prove Theorem 1.2 in this case, it suffices to prove the following
Proposition 4.1. There exists $L \in \mathbb{N}$ such that for any $p \in U_-(e\Gamma)$ which is neither a $U$-periodic point nor Diophantine of type $\kappa = 1 - 1/10000$, and for any $c > 0$, the orbit $\{u_0(cx)p : x \in \Omega(L)\}$ is dense in $G/\Gamma$.

Recall that a number $x$ is Diophantine of type $\mu$ ($\mu \geq 2$) if for any $p/q \in \mathbb{Q}$ with $(p, q) = 1$ and $q > 0$

$$|x - p/q| \geq 1/q^\mu$$

We will need the following lemma to prove Proposition 4.1.

Lemma 4.2. Let $c$ and $l \in \mathbb{R}$ such that $c/l$ is a Diophantine number of type $\mu$ ($\mu \geq 2$), and $f$ a smooth function on the circle of radius $l/(2\pi)$. Then

$$\left| \frac{1}{K} \sum_{1 \leq j \leq K} f(cdj) - \frac{1}{l} \int_0^l f(x)dx \right| \ll \frac{d^{[\mu]+8}}{K} \|f\|_{\infty,[\mu]+8}$$

Proof. We write the Fourier series of $f$ as

$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi ikx/l}$$

where $a_k = \frac{1}{l} \int_0^l f(x)e^{-2\pi ikx/l}dx$. We have

$$\left| \frac{1}{K} \sum_{j=1}^K f(cdj) - \frac{1}{l} \int_0^l f(x)dx \right|$$

$$= \sum_{j=1}^K \sum_{k \neq 0} \frac{1}{K} a_k e^{2\pi ikcd/l}$$

$$= \sum_{k \neq 0} a_k \frac{1}{K} \sum_{j=1}^K e^{2\pi ikjcd/l}$$

$$\leq \sum_{k \neq 0} |a_k| \frac{2}{K} \left( \frac{1}{e^{2\pi kcd/l} - 1} \right)$$

$$\leq \sum_{k \neq 0} |a_k| \frac{2}{K} (kd)^\mu.$$
Therefore, we have
\[
\left| \frac{1}{K} \sum_{j=1}^{K} f(cdj) - \frac{1}{l} \int_{0}^{l} f(x)dx \right| \ll \frac{d\mu([\mu]+8)}{K} \|f\|_{\infty,[\mu]+8}.
\]
This completes the proof of the lemma.

Now we continue the proof of Theorem 1.2, and prove Proposition 4.1. By Proposition 2.1, we can find a sequence of distinct \(U\)-periodic points \(q_{k} \in U_{-}(e\Gamma)\) with period \(d(q_{k}) \to \infty\) such that
\[
d(p, q_{k}) \leq Cd(q_{k})^{-\frac{1}{d}}
\]
for some constant \(C > 0\). Then for any \(t > 0\), \(a_{t}q_{k}\) is still a \(U\)-periodic point with period \(e^{t}d(q_{k})\). Since the subset of Diophantine numbers of type \(\mu \ (\mu > 2)\) has full Lebesgue measure in \(\mathbb{R}\), by properly choosing a sufficiently small \(t > 0\), we obtain a \(U\)-periodic point \(q_{k}' = a_{t}q_{k}\) satisfying
\begin{enumerate}
  \item \(c/d(q_{k}')\) is a Diophantine number of type 2.1.
  \item \(d(q_{k}') \sim d(q_{k})\) and \(d_{G/\Gamma}(p, q_{k}') \leq 2Cd(q_{k})^{-\frac{1}{d}}\).
\end{enumerate}
Now we take a non-negative compactly supported smooth function \(f \neq 0\) on \(\Gamma \setminus G\), and define
\[
\begin{cases}
  a(n) := f(u_{0}(cn) \cdot q'_{k}) & n \leq N \\
  a(n) := 0 & n > N
\end{cases}
\]
Then by Lemma 4.2, we have
\[
|A_{d}| = \sum_{1 \leq n \leq N, d|n} a(n) = \sum_{1 \leq n \leq N, d|n} f(u_{0}(cn) \cdot q'_{k})
\]
\[
= \sum_{1 \leq dj \leq N} f(u_{0}(cdn) \cdot q'_{k})
\]
\[
\ll \frac{N}{d} \frac{1}{d(q_{k}')} \int_{0}^{d(q_{k}')} f(u_{0}(x) \cdot q'_{k})dx + d^{2.1}d(q_{k}')^{10}\|f\|_{\infty,10}
\]
\[
\ll \frac{1}{d} \sum_{1 \leq n \leq N} a(n) + 2d^{2.1}d(q_{k}')^{10}\|f\|_{\infty,10}.
\]
According to Theorem 4.1, we take \(g(d) = 1/d\) and
\[
|r(d)| \leq 2d^{2.1}d(q_{k}')^{10}\|f\|_{\infty,10}.
\]
Let $z = N^\alpha$ for some small constant $\alpha > 0$ which will be determined later. Let $s > 100$ be a sufficiently large number so that $f_0(s) > 0.1$ where $f_0(s)$ is the function defined in Theorem 4.1. Then by Theorem 4.1, we have

$$S(A, \mathcal{P}, z) > 0.01V(z)|A| - R$$

where by Lemma 4.2 we have

$$\frac{|A|}{N} \sim \frac{1}{d(q'_k)} \int_0^{d(q'_k)} f(u(x) \cdot q'_k) dx$$

and if $\alpha$ is sufficiently small, then

$$R \ll N^\delta d(q'_k)^{10} \|f\|_{\infty,10}$$

for some small constant $0 < \delta < 0.1$.

Now write $p = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \Gamma$ and $q'_k = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \Gamma$. For any $s \in \mathbb{R}$ and $s \geq 1$, we calculate $u_0(s)q'_k$ by

$$u_0(s)q'_k = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \Gamma$$

$$= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-2}s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y - x & 1 \end{pmatrix} \Gamma.$$

Hence by the argument of Lemma 3.3, the distance between $u_0(s)p$ and $u_0(s)q'_k$ is bounded by

$$|a - 1| + |a^{-1}s^2(y - x)| \leq 2s^2|y - x| = 2s^2d_{G/\Gamma}(p, q'_k).$$

Therefore, we have

$$|f(u_0(s)p) - f(u_0(s)q'_k)| \ll \|f\|_{\infty,1} s^2d_{G/\Gamma}(p, q'_k) \ll \|f\|_{\infty,1} s^2d(q'_k)^{-\frac{1}{1-\kappa}}$$

and

$$\left| \sum_{1 \leq n \leq N, (P(z), n) = 1} f(u_0(n)p) - S(A, \mathcal{P}, z) \right| \leq 2^2 \sum_{1 \leq n \leq N, (P(z), n) = 1} n^2d(q'_k)^{-\frac{1}{1-\kappa}} \|f\|_{\infty,1}$$

$$\leq 2N^3d(q'_k)^{-\frac{1}{1-\kappa}} \|f\|_{\infty,1}.$$
This implies that
\[
\sum_{1 \leq n \leq N_\kappa(P(z), n) = 1} f(u_0(n)p) \geq S(A, P, z) - 2N^3 d(q_k')^{-\frac{1}{10}} \|f\|_{\infty, 1}
\]
(1)
\[
\geq 0.01V(z)|A| - N^\delta d(q_k')^{10}\|f\|_{\infty, 10} - 2N^3 d(q_k')^{-\frac{1}{10}} \|f\|_{\infty, 1}.
\]

Let \( N := N_k = d(q_k')^{1/(100(1-\kappa))) = d(q_k')^{100} \), and we have
\[
N^\delta_k d(q_k')^{10} \ll N_k^{1-\epsilon} \quad \text{and} \quad 2N^3 d(q_k')^{-\frac{1}{10}} \ll N_k^{1-\epsilon}
\]
for some \( \epsilon > 0 \). Hence the main term in inequality (1) is 0.01V(z)|A|, where by Lemma 4.2 and by Theorem 4.2
\[
|A|/N_k \sim \frac{1}{d(q_k')} \int_0^{d(q_k')} f(u_0(x)q_k')dx \quad \text{and} \quad 1/\log N_k \ll V(z)
\]
for sufficiently large \( k \in \mathbb{N} \). Since
\[
\frac{1}{d(q_k')} \int_0^{d(q_k')} f(u_0(x)q_k')dx \to \int_{G/\Gamma} f(g)d\mu_{G/\Gamma}(g) > 0
\]
we have
\[
\sum_{1 \leq n \leq N_\kappa(P(z), n) = 1} f(u_0(n)p) > 0
\]
for sufficiently large \( k \). Note that any \( n \) coprime to \( P(z) \) has at most \( \lfloor 1/\alpha \rfloor + 1 \) prime factors. If we take \( L = \lfloor 1/\alpha \rfloor + 1 \) and \( \Omega(L) \) the set of \( L \)-almost primes, then it implies that \( \{u_0(x)p : x \in \Omega(L)\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\Gamma \). This completes the proof of Proposition 4.1.

Note that if \( L_1 \geq L_2 \), then \( \Omega(L_1) \supset \Omega(L_2) \). Now combining Case 1 and Case 2, we conclude that there exists \( L \in \mathbb{N} \) such that for any \( p \in G/\Gamma \) which is not \( u(t) \)-periodic, the subset \( \{u(x)p : x \in \Omega(L)\} \) is dense in \( G/\Gamma \). This completes the proof of Theorem 1.2. \( \square \)

5. Denseness of \( \{u(n^2)p\}_{n \in \mathbb{N}} \)

In the rest of the note, we prove Theorem 1.3. We will use the same techniques in sections 3 and 4 to construct points \( p \in \text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z}) \) such that \( \{u(n^2)p : n \in \mathbb{N}\} \) is dense in \( \text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z}) \). Without loss of generality, we may assume that the one-parameter unipotent subgroup \( \{u(s) : s \in \mathbb{R}\} \) is equal to \( \{u_0(s) : s \in \mathbb{R}\} \).

Recall that a number \( \alpha \) is badly approximable if the digits in the continued fraction of \( \alpha \) are bounded. We will need the following lemma.
Lemma 5.1 (van der Corput trick). Let $\alpha$ be a badly approximable number in $\mathbb{R}$. Let $k \in \mathbb{Z}$ ($k \neq 0$) and $l \in \mathbb{N}$. Then for any $\epsilon > 0$, there exists a constant $C > 0$ depending only on $\alpha$ and $\epsilon$ such that
\[
\left| \sum_{n=M+1}^{N} e^{2\pi k \alpha n^2 / l} \right| \leq C (N - M)^{1/2 + \epsilon} |k|^{1/2 + \epsilon} l^{1/2}.
\]

Proof. Denote by $\langle x \rangle$ the distance between $x$ and $\mathbb{Z}$. We have
\[
\left| \sum_{n=M+1}^{N} e^{2\pi k / lio n^2} \right|^2 = \left| \sum_{q=M+1}^{N} \sum_{p=M+1}^{N-q} e^{2\pi k / lio (q^2 - p^2)} \right|
\]
\[
= \left| \sum_{q=M+1}^{N-q} \sum_{p=M+1}^{N-q} e^{2\pi k / lio (q^2 - (q+p)^2)} \right|
\]
\[
= \left| \sum_{q=M+1}^{N-q} \sum_{p=M+1}^{N-q} e^{-2\pi k / lio (p^2 + 2qp)} \right|
\]
\[
= \sum_{p=M+1}^{N-1-M} e^{-2\pi k / lio p^2} \min(N, N-p) \sum_{q=\max(M+1, M+1-p)}^{\min(N, N-p)} e^{-2\pi k / lio 2qp}
\]
\[
\leq \sum_{p=1+M-N}^{N-M-1} e^{-2\pi k / lio 2qp} \min(N, N-p) \sum_{q=\max(1, 1-p)}^{\min(N, N-p)} e^{-2\pi k / lio 2qp}
\]
\[
= N - M + \sum_{M+1-N \leq p \leq N-M-1, p \neq 0} e^{-2\pi k / lio 2qp} \min(N, N-p) \sum_{q=\max(1, 1-p)}^{\min(N, N-p)} e^{-2\pi k / lio 2qp}
\]
\[
\leq N - M + \sum_{1+M-N \leq p \leq N-M-1, p \neq 0} \frac{2}{1 - e^{-4\pi k / lio p}}
\]
\[
= N - M + 2 \sum_{p=1}^{N-M-1} \frac{1}{|\sin(2\pi k / lio p)|}
\]
\[
\leq N - M + \frac{1}{2} \sum_{p=1}^{N-M-1} \frac{1}{(k\alpha / l)(1 - 2(k\alpha / l))}
\]

By [13, Theorem 4.2.11], the Dirichlet series
\[
\lambda_\alpha(s) = \sum_{n \in \mathbb{N}} \frac{1}{(n\alpha)n^s}
\]
Lemma 5.2. This completes the proof of the lemma.

Proof. Write the Fourier series of the function \( f \) as

\[
f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x / l}
\]

where \( a_k = \frac{1}{l} \int_0^l f(x) e^{-2\pi i k x / l} dx \). Using integration by parts, we have

\[
|a_k| \leq \frac{l^2}{4\pi^2 k^2} \|f\|_{\infty,2}
\]

By Lemma 5.1, we have

\[
\left| \sum_{n=M+1}^N f(\alpha n^2) - \left( N - M \right) \frac{1}{l} \int_0^l f(x) dx \right|
\]

\[
= \left| \sum_{k \neq 0} a_k \left( \sum_{n=M+1}^N e^{2\pi i \alpha n^2 / l} \right) \right|
\]

\[
\leq \sum_{k \neq 0} |a_k| \left| \sum_{n=M+1}^N e^{2\pi i \alpha n^2 / l} \right|
\]

converges if \( s > 1 \). Hence, for any \( s > 1 \) we have

\[
\left| \sum_{n=M+1}^N e^{2\pi i \alpha n^2 / l} \right|^2 \leq N - M + \frac{1}{2} \sum_{p=1}^{N-M-1} \frac{1}{|k\alpha p / l|} + \frac{1}{2} \sum_{p=1}^{N-M-1} \frac{1}{|k\alpha p / l| - 1/2}
\]

\[
\leq N - M + \frac{1}{2} \sum_{p=1}^{N-M-1} \frac{l}{|k\alpha p|} + \frac{1}{2} \sum_{p=1}^{N-M-1} \frac{2l}{|2k\alpha p|}
\]

\[
\leq N - M + \sum_{p=1}^{N-M-1} \frac{l}{|k\alpha p|} (kp)^s + \sum_{p=1}^{N-M-1} \frac{l}{|2k\alpha p|} (2kp)^s
\]

\[
\leq N - M + (\lambda_\alpha(s) + \lambda_\alpha(s))(2(N - M)k)^s l.
\]

This completes the proof of the lemma. \( \square \)

As a corollary of Lemma 5.1, we deduce the following

Lemma 5.2. Let \( f \) be a smooth periodic function on \( \mathbb{R} \) with period \( l \), and \( \alpha \in \mathbb{R} \) a badly approximable number. Then for any \( \epsilon > 0 \), there exists a constant \( C > 0 \) depending only on \( \alpha \) and \( \epsilon \) such that

\[
\left| \sum_{n=M+1}^N f(\alpha n^2) - (N - M) \frac{1}{l} \int_0^l f(x) dx \right| \leq l^3 C (N - M)^{\frac{1}{2} + \epsilon} \|f\|_{\infty,2}.
\]

Proof. Write the Fourier series of the function \( f(x) \) as

\[
f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x / l}
\]
\[ \leq \sum_{k \neq 0} |a_k| C(N - M)^{1/2 + \epsilon} |k|^{1/2 + \epsilon} l^{1/2} \]
\[ \leq C(N - M)^{1/2 + \epsilon} l^{3} \|f\|_{\infty,2}. \]

This completes the proof of the lemma. \(\square\)

We calculate the period of a \(U\)-periodic orbit in the following lemma.

**Lemma 5.3.** The point \( \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} \) PSL(2, Z) is a \(U\)-periodic point with period \(q^2\).

**Proof.** Suppose that
\[
\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} PSL(2, \mathbb{Z}) = \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} PSL(2, \mathbb{Z}).
\]

Then
\[
\begin{pmatrix} 1 & 0 \\ -\frac{p}{q} & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} PSL(2, \mathbb{Z}) = PSL(2, \mathbb{Z})
\]
\[
\begin{pmatrix} 1 + sp/q & s \\ -p^2 s/q^2 & 1 - ps/q \end{pmatrix} PSL(2, \mathbb{Z}) = PSL(2, \mathbb{Z}).
\]

This implies that \(q^2|s\) and hence the period of the \(U\)-orbit of \( \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} PSL(2, \mathbb{Z}) \) is \(q^2\). \(\square\)

**Lemma 5.4.** Let \( f \in C^\infty_c(PSL(2, \mathbb{R})/PSL(2, \mathbb{Z})) \) and \( \alpha \) a badly approximable number. Let \( x \in \mathbb{R} \) and \( \frac{p}{q} \in \mathbb{Q} \). Then
\[
\left| \sum_{n=1}^{N} f \left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} Z^2 \right) - \sum_{n=1}^{N} f \left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} Z^2 \right) \right| \leq N^5 |x - p/q| \|f\|_{\infty,1}
\]

The implicit constant depends only on \(\alpha\).

**Proof.** By the argument of Lemma 3.3, we known that the distance between
\[
u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \text{ and } \nu_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ \frac{p}{q} & 1 \end{pmatrix}
\]
is bounded by
\[ \max\{\alpha^2 n^4, \alpha n^2, 1\}|x - p/q|. \]
The lemma then follows from simple calculations. \(\square\)

The following lemma is crucial in our proof of Theorem 1.3.

**Lemma 5.5.** Let \( f \in C^\infty_c(\PSL(2,\mathbb{R})/\PSL(2,\mathbb{Z})) \) and \( \alpha \) a badly approximable number. Then for any \( \varepsilon > 0 \), there exists a constant \( C > 0 \) depending only on \( \varepsilon \) and \( \alpha \) such that
\[
\left| \frac{1}{N} \sum_{n=1}^{N} f \left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) Z^2 \right| \leq C(N^4|x - p/q| + q^6 N^{-\frac{1}{2} + \varepsilon})\|f\|_{\infty,2}.
\]

**Proof.** By Lemmas 5.2, 5.3 and 5.4, we have
\[
\left| \frac{1}{N} \sum_{n=1}^{N} f \left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) Z^2 \right| - \frac{1}{q^2} \int_0^{q^2} f \left( u_0(s) \begin{pmatrix} 1 & 0 \\ p/q & 1 \end{pmatrix} \right) Z^2 \, ds \leq C(N^4|x - p/q| + q^6 N^{-\frac{1}{2} + \varepsilon})\|f\|_{\infty,2}.
\]

This completes the proof of the lemma. \(\square\)

**Proof of Theorem 1.3.** Let \( \alpha \) be a badly approximable number. Let \( x \in \mathbb{R} \) such that \( x \) is not a Diophantine number of type 100, i.e. there exist infinitely many rational numbers \( p_k/q_k \) with \( (p_k, q_k) = 1 \) and \( q_k \to \infty \) such that
\[
|x - \frac{p_k}{q_k}| \leq \frac{1}{q_k^{100}}.
\]

We take \( f \in C^\infty_c(\PSL(2,\mathbb{R})/\PSL(2,\mathbb{Z})). \) Let \( \varepsilon = 0.01, N = N_k := q_k^{24} \) and \( p/q = p_k/q_k \) in Lemma 5.5, and we have
\[
\left| \frac{1}{N_k} \sum_{n=1}^{N_k} f \left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) Z^2 \right| - \frac{1}{q_k^4} \int_0^{q_k^4} f \left( u_0(s) \begin{pmatrix} 1 & 0 \\ p_k/q_k & 1 \end{pmatrix} \right) Z^2 \, ds \leq C(N_k^4|x - p_k/q_k| + q_k^6 N_k^{-\frac{1}{2} + \varepsilon})\|f\|_{\infty,2} \leq 2C/q_k^4.
for some constant $C > 0$ depending only on $\alpha$. Hence
\[
\frac{1}{N_k} \sum_{n=1}^{N_k} f\left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbb{Z}^2 \right) - \frac{1}{q_k^2} \int_0^{q_k^2} f\left( u_0(s) \begin{pmatrix} 1 & 0 \\ \frac{p_k}{q_k} & 1 \end{pmatrix} \mathbb{Z}^2 \right) ds \to 0.
\]
It is known that
\[
\frac{1}{q_k^2} \int_0^{q_k^2} f\left( u_0(s) \begin{pmatrix} 1 & 0 \\ \frac{p_k}{q_k} & 1 \end{pmatrix} \mathbb{Z}^2 \right) ds \to \int_{G/\Gamma} f \, d\mu_{G/\Gamma}.
\]
Therefore, we have
\[
\frac{1}{N_k} \sum_{n=1}^{N_k} f\left( u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbb{Z}^2 \right) \to \int_{G/\Gamma} f \, d\mu_{G/\Gamma}
\]
and the orbit
\[
\left\{ u_0(\alpha n^2) \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbb{Z}^2 : n \in \mathbb{N} \right\}
\]
is dense in $\text{PSL}(2,\mathbb{R})/\text{PSL}(2,\mathbb{Z})$. This implies that for any $s \in \mathbb{R}$, any badly approximable number $\alpha$ and any $x \in \mathbb{R}$ which is not Diophantine of type 100, the orbit
\[
\left\{ u_0(n^2) \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbb{Z}^2 : n \in \mathbb{N} \right\}
\]
is dense in $\text{PSL}(2,\mathbb{R})/\text{PSL}(2,\mathbb{Z})$. Let $S_1$ be the set of badly approximable numbers and $S_2$ the subset of numbers which are not Diophantine of type 100. We define
\[
E = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbb{Z}^2 : s \in \mathbb{R}, \alpha \in S_1, x \in S_2 \right\}.
\]
By Bruhat decomposition, $E$ is dense in $\text{PSL}(2,\mathbb{R})/\text{PSL}(2,\mathbb{Z})$. Note that $E$ is uncountable. It is known that the Hausdorff dimensions of $S_1$ and $S_2$ are 1 and 1/50 respectively, and the (1/50)-Hausdorff measure of $S_2$ is positive. By Marstrand slicing theorem ([3, Theorem 5.8] and [10]), the Hausdorff dimension of $E \geq 2 + 1/50 > 2$. This completes the proof of Theorem 1.3. □

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