TOTAL SCISSORS CONGRUENCE

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Abstract. We introduce a Grothendieck group $E_n$ for bounded polytopes in $\mathbb{R}^n$. It differs from the usual Euclidean scissors congruence group in that lower-dimensional polytopes are not ignored. We also introduce analogous groups $L_n$ which bear the same relation to spherical scissors congruence. This provides a natural setting for generalized Dehn invariants.

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INTRODUCTION

The idea behind this paper is to define something like an $n$-dimensional scissors congruence group without ignoring everything of lower dimension. We make such a definition and then explore the consequences, finding some appealing structure in the new objects. It is reasonable to think that this method might eventually lead to new results about scissors congruence and homology of Lie groups. There may also be a relation with Riemannian geometry, since some of what we do has a manifold flavor. However, at this point it is more a matter of building a theory than of proving a theorem.

Let $E_n$ be the abelian group with a generator $[P]_n$ for every bounded polytope $P$ in $\mathbb{R}^n$, subject to relations equating any two congruent polytopes, killing the empty set, and making $[P \cup Q]_n$ equal to $[P]_n + [Q]_n - [P \cap Q]_n$.

These groups form a graded ring $E$ having the Euclidean scissors congruence ring $E'$ as a quotient.

There is an involution $I : E \to E$ such that when an element $\xi \in E_n$ is given by a polytope that is an $n$-dimensional PL manifold $M$ then $I(\xi)$ is the difference between $M$ and $\partial M$.

Investigation of $E$ leads to another graded ring $L$, a local analogue of $E$, which interacts with it. Elements of $L_n$ are given by germs of polytopes at a point rather than by compact polytopes. (There is an equivalent description using cones instead of germs.) The cokernel $L'_n$ of the obvious map $L_{n-1} \to L_n$ is the $(n-1)$st spherical scissors congruence group.

$L$ has features similar to those of $E$, including an analogous involution $I$. It also has another involution $D$ (commuting with $I$ up to a sign), defined by sending a cone $P$ to its dual: the cone consisting of all points $w$ such that the nearest point to $w$ in $P$ is the apex of the cone.

We consider $k$-valued global and local multiplicative invariants, meaning ring maps $E \to k$ and $L \to k$, where $k$ is any commutative ring. The most prominent global multiplicative invariants are the volume $V : E \to \mathbb{R}$ and the Euler characteristic
\( \chi : E \to \mathbb{Z} \). On the local side there is a volume invariant \( U : L \to \mathbb{R} \) and there are two \( \mathbb{Z} \)-valued invariants called 1 and \( e \), which take the germ of a based polytope \((P,v)\) to 1 and to \( \chi(P) - \chi(P-v) \) respectively.

It turns out that there is a composition law, defined along the lines of the Dehn invariant, for combining two multiplicative local invariants \( F : L \to k \) and \( G : L \to k \) to make a third one \( F \star G : L \to k \). It is defined only if \( F(s) = G(t) \), where \( s \) and \( t \) are the particular elements of \( L_1 \) corresponding to the germ of a line at a point and the germ of a point at a point respectively. For any ring \( k \) this leads to a groupoid whose objects are the elements of \( k \), and in which a morphism from \( a \) to \( b \) is a \( k \)-valued multiplicative invariant \( F \) such that \( F(s) = a \) and \( F(t) = b \). The inverse of the morphism \( F \) is \( F \circ D \).

The geometric meaning of \( E \), and of the interaction between \( E \) and \( L \), is not transparent when \( E \) is written as \( E'[x] \). That non-obvious isomorphism uses both the groupoid formalism and a deep result about scissors congruence [Sah], Lemma ??: the fact that for \( n > 0 \) the group \( E_n \) is canonically a real vector space.

Conceptually \( E \) has an advantage over \( E'[x] \) because it has more geometry in its definition. The later parts of the paper are mainly about elements of \( E \) and \( L \) that are given by manifolds.

We end with a list of questions and other unfinished business.
1. Polytopes

1.1. The group $E_n$ and additive invariants. A Euclidean space is a finite-dimensional real affine space $V$ equipped with a positive-definite inner product on the underlying vector space.

A bounded convex polytope in $V$ is the convex hull of a (possibly empty) finite set. A bounded polytope in $V$ is any finite union of bounded convex polytopes, or equivalently any subset of $V$ that can be triangulated by finitely many linear simplices, in other words a compact PL subspace of $V$. General closed PL subspaces of $V$ will be called polytopes in $V$.

Define an abelian group $E_n$ by giving a generator $[P]_n$ for every bounded polytope $P$ in $\mathbb{R}^n$, with three generating relations:

1. $[g(P)]_n = [P]_n$ if $g : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry,
2. $[\emptyset]_n = 0$,
3. $[P \cup Q]_n = [P]_n + [Q]_n - [P \cap Q]_n$

whenever all four terms are defined.

Equations (2) and (3) imply the more general inclusion-exclusion formula: Suppose that $P = P_1 \cup \cdots \cup P_k$, and write $P_S = \cap_{i \in S} P_i$. Then

$$ [P]_n = \sum_S (-1)^{|S|-1} [P_S]_n, $$

where the summation is over nonempty subsets of $\{1, \ldots, k\}$.

The more familiar scissors congruence group is obtained by replacing (2) by a stronger relation: $[P]_n$ is zero if the dimension of $P$ less than $n$. We denote this quotient of $E_n$ by $E'_n$ and write $[P]'_n$ for the element of $E'_n$ determined by $P$.

We can go further, defining $[X]_n$ when $X$ is any constructible set in $\mathbb{R}^n$, meaning any set obtainable from bounded convex polytopes by repeatedly taking finite union, pairwise intersection, and relative complement. There is a unique way to do this such that (3) is still valid. In particular if $P$ and $A \subset P$ are bounded polytopes we write $[P - A]_n = [P]_n - [A]_n$. For example, if $P$ is a bounded convex polytope then there is the element $[\text{int } P]_n = [P]_n - [\partial P]_n$ given by its interior. (Whenever we mention the interior of a polytope $P$ we will assume that $P$ is a PL manifold, and we mean the interior in that sense. If $P$ is nonempty and convex then this is the same as the topological interior of $P$ in the affine space that it spans.)

We will sometimes use (4) in the following form: Suppose that $P$ contains $P_1 \cup \cdots \cup P_k$. Then

$$ [P - \cup_i P_i]_n = \sum_S (-1)^{|S|} [P_S]_n. $$
The summation is now over all subsets of \( \{1, \ldots, k\} \), with \( P_S = \cap_{i \in S} P_i \) as before and \( P_\emptyset = P \).

We can also extend the definition of \( X \mapsto [X]_n \) from constructible sets in \( \mathbb{R}^n \) to constructible sets of dimension at most \( n \) in any Euclidean space. In this context the relation (1) should be understood as referring to isometries between affine subspaces of possibly different Euclidean spaces.

The function \( P \mapsto [P]_n \) is the universal example of an additive invariant of level \( n \): a rule \( F \) assigning an element \( F(P) \in G \) of some abelian group \( G \) to every bounded polytope \( P \) in \( \mathbb{R}^n \) in such a way that it is invariant:

\[
F(gP) = F(P)
\]

and additive:

\[
F(\emptyset) = 0 \\
F(P \cup Q) = F(P) + F(Q) - F(P \cap Q).
\]

We ordinarily use the same symbol for such an invariant and for the corresponding homomorphism \( E_n \to G \). Thus

\[
F(P) = [P]_n.
\]

Any additive invariant may be routinely extended to constructible sets.

The first examples of additive invariants of level \( n \) are the Euler characteristic and the volume:

\[
\chi : E_n \to \mathbb{Z} \\
V_n : E_n \to \mathbb{R}.
\]

It is clear that an additive invariant is determined by its value on convex bounded polytopes. In fact, we have the following convenient rule for creating invariants.

**Lemma 1.1.** A function \( F \) of bounded convex polytopes extends to a unique additive invariant if in addition to being invariant under isometry and vanishing at \( \emptyset \) it satisfies

\[
(6) \\
F(P) = F(P \cap H^+) + F(P \cap H^-) - F(P \cap H)
\]

whenever \( P \) is a bounded convex polytope, \( H \) is a hyperplane, and \( H^+ \) and \( H^- \) are the two closed half-spaces determined by \( H \).

**Proof.** We first show that \( F \) satisfies (6) when a bounded convex polytope \( P \) is the union of bounded convex polytopes \( P_1, \ldots, P_k \). We argue by induction on \( d \), and for fixed \( d \) by induction on the number \( m \) of hyperplanes \( H \) such that \( H \) contains a \((d - 1)\)-dimensional face \( F \) of some \( P_i \) such that \( F \) is not in \( \partial P \). If there is some such \( H \) then \( F(P \cap H^+) \) is the alternating sum of \( F(P_S \cap H^+) \) by induction on \( m \), \( F(P \cap H^-) \) is the alternating sum of \( F(P_S \cap H^-) \) for the same reason, and \( F(P \cap H) \) is the alternating sum of \( F(P_S \cap H) \) by induction on \( d \). It follows that \( F(P) \) is the
alternating sum of $F(P_5)$. If $m = 0$ then some $P_i$ is $P$ and the conclusion is obvious. The case $d = 0$ is trivial as well.

Now we extend $F$ to all bounded polytopes $P$ by writing $P$ as a union of convex polytopes $aP_i$ and using Equation (4) as a definition. This is independent of the choice of $P_1, \ldots, P_k$ because if an extraneous convex polytope $Q$ is added to the list the new term $F(Q)$ will equal the alternating sum of the other new terms $F(Q \cap P_S)$. To obtain (4) for the extended $F$ it is enough to verify the $k = 2$ case (3); the general case follows by induction. The proof of (3) is straightforward. □

It is clear that the group $E_n$ is generated by simplices in $\mathbb{R}^n$. Here are two stronger statements:

**Proposition 1.2.** $E_n$ is generated by $n$-simplices in $\mathbb{R}^n$.

*Proof.* It suffices to show that when $\sigma$ is an $m$-simplex, $0 \leq m < n$, then $[\sigma]_n$ is in the subgroup generated by $(m+1)$-simplices. Since $\sigma$ can always be expressed as the intersection of two $(m+1)$-simplices whose union is an $(m+1)$-simplex, this follows from (3). □

**Proposition 1.3.** If $P$ is a bounded polytope in any Euclidean space and $P$ has dimension at most $n$, then for any triangulation of $P$ (by linear simplices) we have

\begin{equation}
[P]_n = \sum_{\sigma} (1 - \chi L(\sigma, P)) [\sigma]_n.
\end{equation}

where the sum is over simplices (of arbitrary dimension) of the triangulation and $L(\sigma, P)$ is the link of the simplex $\sigma$ in the polytope $P$.

*Proof.* Begin with the case when $P$ is a simplex triangulated in the simplest possible way. For $\sigma \neq P$ the link $L(\sigma, P)$ is contractible and $1 - \chi L(\sigma, P) = 0$. For $\sigma = P$ the link is empty and $1 - \chi L(\sigma, P) = 1$.

In all other nonempty cases $P$ may be expressed as a union $Q \cup R$ of proper subcomplexes. The reader may verify that (7) holds for $Q \cup R$ if it holds for $Q$, $R$, and $Q \cap R$. Thus the proof can be completed by induction on the number of simplices in the triangulation. □

A triangulation of a polytope will always mean a triangulation by linear simplices.

**Remark 1.4.** We will occasionally have use for more general (linear) cell structures on a polytope. By a cell structure on $P$ we will mean a finite set of nonempty convex polytopes (the cells) such that the set $P$ is the disjoint union of their interiors, and such that the boundary of each cell is a union of cells. Proposition 1.3 can be extended to this setting, with the same proof.
Note that if the triangulated bounded Euclidean polytope \( M \) is a PL manifold of dimension \( m \leq n \) then by Proposition 1.3 we have
\[
|M|_n = \sum_{\sigma \subset \text{int } M} (-1)^{m-\text{dim } \sigma} [\sigma]_n,
\]
where the summation is over the interior simplices (that is, those which are not contained in \( \partial M \)). Indeed, the quantity \( 1 - \chi L(\sigma, M) \) vanishes for a boundary simplex and is \((-1)^{m-d}\) for an interior simplex of dimension \( d \).

Of course \( \chi(gP) = \chi(P) \) for all affine linear isomorphisms \( g \), not just isometries. Because any two \( n \)-simplices are affinely equivalent, Proposition 1.2 shows that \( \chi \) is the universal example of a level \( n \) invariant possessing this strong invariance.

There is a homomorphism from \( E_{n-1} \) to \( E_n \), the inclusion map, given by
\[
[P]_{n-1} \mapsto [P]_n.
\]
Its cokernel is \( E'_n \). It is not obviously injective, but in [2,2] we will see that it is.

Any level \( n \) additive invariant may be composed with inclusion to obtain a level \( n-1 \) invariant, its restriction. The former is then called an extension of the latter.

A level \( n \) invariant that restricts to zero at level \( n-1 \) is a scissors congruence invariant. The universal such invariant is \( P \to [P]_n \in E'_n \).

Write \( E_\infty \) for the direct limit of \( E_n \) with respect to the inclusion maps. This can also be defined more directly by giving a generator \([P]_{\infty}\) for every bounded Euclidean polytope, with no restriction on dimension, subject to appropriate versions of (1), (2), and (3). Homomorphisms \( E_\infty \to G \) correspond to absolute additive invariants, defined for bounded polytopes of arbitrary dimension. An example is the Euler characteristic.

1.2. The graded ring \( E \) and multiplicative invariants. Cartesian product gives a multiplication \( E_p \times E_q \to E_{p+q} \). The resulting graded ring \( E \) is associative, commutative, and unital. The ring \( E_0 \) is isomorphic to \( \mathbb{Z} \).

The element 1 is given by a point. Let \( p \) be the element of \( E_1 \) given by a point. Then \( p^n \in E_n \) is given by a point, and multiplication by \( p \) gives the inclusion \( E_{n-1} \to E_n \).

Let \( k \) be any commutative unital ring. A ring map \( F : E \to k \) corresponds precisely to a \( k \)-valued multiplicative invariant for Euclidean polytopes, meaning an infinite sequence \((F_0, F_1, \ldots)\) in which \( F_n \) is a \( k \)-valued level \( n \) additive invariant and the relations
\[
F_{p+q}(P \times Q) = F_p(P) \times F_q(Q)
\]
\[
F_0(\ast) = 1
\]
are satisfied.

Note that, for a polytope \( P \) and a multiplicative invariant \( F \), the quantity \( F_n(P) = F[P]_n \) is defined for all \( n \geq \text{dim } P \) but is not necessarily independent of \( n \). It is
independent of $n$ precisely if $F(p) = 1$. In that case $F$ is an absolute multiplicative invariant, and can be described as a ring homomorphism

$$E_\infty = E/(1-p)E \to k.$$ 

The Euler characteristic is a $\mathbb{Z}$-valued example.

At the other extreme are the multiplicative scissors congruence invariants, those multiplicative invariants $F$ such that $F(p) = 0$. An example is the volume invariant $V : E \to \mathbb{R}$ defined by $V[P]_n = V_n(P)$. Such invariants correspond to ring homomorphisms

$$E' \to k,$$

where $E' = E/pE$ is the Euclidean scissors congruence ring.

By Proposition 1.2 the ring $E$ is generated by elements $[\sigma]_n$ where $\sigma$ is an $n$-simplex and $n$ ranges through positive integers.

If $F$ is any multiplicative invariant, then $F_0$ must be the unique ring homomorphism from $E_0 \cong \mathbb{Z}$ to $k$. Denote by $\iota$ the most trivial example of a multiplicative invariant, namely the one satisfying $\iota_n = 0$ for $n > 0$.

If $F$ is a multiplicative $k$-valued invariant, then for any $a \in k$ we define another such invariant $aF$ by $aF_n(P) = a^nF_n(P)$. Note that $1F = F$ and $(ab)F = a(bF)$, and that $0F = \iota$. Call $aF$ the scaling of $F$ by $a$. (For example, volume measured in inches, square inches, cubic inches, and so on is the scaling by twelve of volume measured in feet, square feet, cubic feet, and so on.) Scaling by an indeterminate leads to a graded ring map

$$xF : E \to k[x].$$

The scaled Euler characteristic

$$x\chi : E \to \mathbb{Z}[x]$$

is the universal example of a strong multiplicative invariant.

1.3. $E_1$ and $E_2$. We now establish isomorphisms

$$E_1 \cong \mathbb{Z} \times \mathbb{R}$$

$$E_2 \cong \mathbb{Z} \times \mathbb{R} \times \mathbb{R}.$$ 

Along the way we name some more elements and some more invariants.

In $E_1$ the element given by a 1-simplex of length $\lambda$ will be denoted by $p + q(\lambda)$; in other words for $\lambda > 0$ we let $q(\lambda) \in E_1$ be the element given by a half-open line segment of length $\lambda$. Since $q(\lambda + \mu) = q(\lambda) + q(\mu)$, this determines an additive homomorphism $q : \mathbb{R} \to E_1$. Clearly

$$\chi(q(\lambda)) = 0$$

$$V(q(\lambda)) = \lambda.$$
We claim that the map \( \mathbb{Z} \times \mathbb{R} \to E_1 \) given by
\[
(k, \lambda) \mapsto kp + q(\lambda)
\]
is an isomorphism. It is surjective because \( E_1 \) is generated by 0-simplices and 1-simplices. Injectivity follows from the equation
\[
(\chi(kp + q(\lambda)), V(kp + q(\lambda))) = (k, \lambda).
\]
To apply the same method to \( E_2 \), first note that we have a group homomorphism \( \mathbb{Z} \times \mathbb{R} \times (\mathbb{R} \otimes \mathbb{Z}) \to E_2 \) given by
\[
(k, \lambda, \mu \otimes \nu) \mapsto kp^2 + pq(\lambda) + q(\mu)q(\nu).
\]
Its image contains \( pE_1 \), the subgroup generated by \( p \) and \( pq(\mathbb{R}) \).

It is well known that every two-dimensional convex polygon is scissors congruent to a rectangle with width one. (See [Dupont], pp. 3-4. The key steps are replacing a triangle by a parallelogram, replacing a parallelogram by another parallelogram having a side of arbitrary length, and replacing a parallelogram by a rectangle having one side in common with it.) That is, modulo \( pE_1 \) the group \( E_2 \) is generated by elements \( q(\mu)q(1) \). It follows that the map \( \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to E_2 \) given by
\[
(k, \lambda, \mu) \mapsto kp^2 + pq(\lambda) + q(\mu)q(1)
\]
is surjective. We claim that it is an isomorphism.

To prove this, we need three invariants. The maps \( \chi : E_2 \to \mathbb{Z} \) and \( V_2 : E_2 \to \mathbb{R} \) take the image of \( (k, \lambda, \mu) \) to \( k \) and \( \mu \) respectively. We complete the proof by making an invariant \( V(1) : E_2 \to \mathbb{R} \) that takes the image of \( (k, \lambda, \mu) \) to \( \lambda \).

Define \( V(1, P) \in \mathbb{R} \) for convex polytopes \( P \subset \mathbb{R}^2 \) as follows. If \( P \) is empty or a point then \( V(1, P) = 0 \). If \( P \) is a line segment of length \( \lambda \) then \( V(1, P) = \lambda \). If \( P \) is a convex polygon then \( V(1, P) \) is one half the perimeter of \( P \). The reader may verify that \( V(1, P \cup Q) = V(1, P) + V(1, Q) - V(1, P \cap Q) \) in all cases. The result is a level two invariant \( V(1) \) extending the level one invariant \( V_1 \). The reader may verify that \( V(1, q(\mu)q(\nu)) = 0 \).

Note that \( q(\mu)q(\nu) \) is equal to \( q(\mu \nu)q(1) \), since they look the same to each of the three invariants \( \chi, V(1) \), and \( V_2 \).

**Remark 1.5.** More generally the \( n \)-dimensional volume invariant \( V_n : E_n \to \mathbb{R} \) can be extended to a level \( n+1 \) invariant by associating to each \( (n+1) \)-dimensional convex polytope one half of the \( n \)-dimensional volume of its boundary. In §2.2 we will do better: for each \( n \) we will extend the level \( n \) invariant \( V_n \) to an absolute additive invariant \( V(n) : E_\infty \to \mathbb{R} \) using the following formula:
\[
(9) \quad V(n, P) = \sum_{|\sigma|=n} B_\sigma(P) \times V_n(\sigma),
\]
Here \( P \) is any bounded Euclidean polytope, the summation is over all \( n \)-simplices in a triangulation of \( P \), and \( B_\sigma(P) \) is a real number, the bending of \( P \) along \( \sigma \), which
will be defined in §2.2. We will see that the combined invariant
\[(10)\quad P \mapsto V(0, P) + V(1, P)x + \cdots + V(n, P)x^n + \ldots\]
is an absolute multiplicative invariant taking values in the polynomial ring $\mathbb{R}[x]$.

*Remark 1.6.* The invariant $V(0)$ will turn out to coincide with $\chi$ (in particular it takes values not just in $\mathbb{R}$ but in $\mathbb{Z}$). The real-valued invariants $V(0)$, $V(1)$, $V(2)$ suffice to detect all of $E_2$. The analogous statement for $E_3$ is false, essentially because 3-dimensional volume does not detect $E'_3$ (see [Sah] p. 2). To detect all of $E_3$ we can refine $V(1)$ to an invariant taking values in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$, essentially the Dehn invariant.

### 2. Polytope germs

We now introduce a theme that will play a big role in the rest of this work: germs of polytopes at a point. In §2.1 and §2.2 we will pursue this far enough to explain equation (9), and also to split the inclusion map $E_{n-1} \rightarrow E_n$. Beginning in §2.3 we will develop the local theory more systematically, introducing a graded ring $L$ analogous to $E$.

#### 2.1. Germs, cones, volume, and duality.

Fix a Euclidean space $V$ and a base-point $v \in V$. Two polytopes $P$ and $Q$ in $V$ have the same *germ at* $v$ if there is a neighborhood $N$ of $v$ such that $P \cap N = Q \cap N$. In this context it is convenient to allow unbounded polytopes (that is, arbitrary closed PL subspaces of $V$). This allows us to say that every polytope germ except that of the empty set is represented by a unique cone.

By a *cone at* $v$, or a *cone based at* $v$, or a *cone with apex* $v$, we mean a subset of $V$ that is the union of $v$ and some set of half-lines having $v$ as endpoint. Of course, the set of all cones at $v$ is in bijection with the set of all subsets of any sphere centered at $v$.

For any subset $S$ of $V$ there is a smallest *convex cone at* $v$ that contains $S$, namely the set of all points $x$ such that the vector $x - v$ is a finite nonnegative linear combination of $\{s - v \mid s \in S\}$. A convex cone generated in this way by a *finite* set $S$ is called a *convex conical polytope* at $v$. These sets may also be characterized as those which are intersections of finitely many closed half-spaces each of which has $v$ in its bounding hyperplane.

Among the convex conical polytopes are the *conical simplices*, those convex cones generated by sets $S$ such that the vectors $\{s - v \mid s \in S\}$ are linearly independent. Conical simplices are a special case of *conical cells*, those convex conical polytopes which do not contain any line through the base point. In general a conical convex polytope is the product of a conical cell and a Euclidean space orthogonal to it.

Every convex conical polytope is a PL manifold. It has nonempty boundary unless the polytope is a Euclidean space.
Any finite union of convex conical polytopes at \( v \) is called a \textit{conical polytope} at \( v \). The nonempty conical polytopes at \( v \) are the only cones that will seriously concern us. They correspond bijectively with germs at \( v \) of polytopes containing \( v \). Our main interest is in germs of polytopes; conical polytopes provide a convenient way of discussing them. A conical simplex corresponds to the germ of a simplex at a vertex. More generally a conical cell corresponds to the germ of a convex polytope at an extreme point.

At times we will speak of polytopes in a based Euclidean space; at other times we will speak of based polytopes in a Euclidean space; but there will never be more than one base point in play. By the base point of a cone we always mean the apex. Basepoints will usually go unnamed.

Note that we are calling the empty set a conical polytope at \( v \). On the whole this seems best (even though we are not calling it a cone and we are not calling it a convex conical polytope.) It corresponds to the empty germ: the germ of the empty set, or of any polytope that does not contain \( v \).

2.1.1. \textit{Normal cones.} The immediate reason for considering germs and cones is the following. To a simplex \( \sigma \) in a triangulation of a Euclidean polytope \( P \) in a Euclidean space \( V \) we associate a convex conical polytope \( \nu_P \sigma \) called its \textit{normal cone in} \( P \), as follows. Choose a point \( v \) in the interior of \( \sigma \). Let \( \sigma_v^+ \subset V \) be the affine space that is the orthogonal complement at \( v \) to the affine space spanned by \( \sigma \). The normal cone is the cone in \( \sigma_v^+ \) at \( v \) having the same germ as \( P \cap \sigma_v^+ \). Up to translation in \( V \), this is independent of the choice of point \( v \in \text{int} \ \sigma \). This makes sense more generally for (linear) cells in a cell structure.

2.1.2. \textit{Duals of convex cones.} For a closed convex cone \( P \) in \( V \) with apex \( v \), define its \textit{dual cone} \( DP \) to consist of those points \( w \in V \) such that, of all the points in \( P \), the one closest to \( w \) is \( v \). Equivalently it consists of those \( w \) such that for every \( u \in P \) the inner product \( (w - v)(u - v) \) is nonnegative. This is again a closed convex cone at \( v \). Because every closed convex cone at \( v \) is an intersection of closed half-planes having \( v \) in the boundary, the dual of \( DP \) is \( P \).

If \( P \) is a conical convex polytope then \( DP \) is again a conical convex polytope. The dual of a Euclidean space through \( v \) is its orthogonal complement. In particular \( DV = \{v\} \) and \( D\{v\} = V \). [rewrite]In an \( n \)-dimensional Euclidean space the dual of a conical \( n \)-simplex is again a conical \( n \)-simplex and the dual of a conical \( n \)-cell is again a conical \( n \)-cell. In general a conical convex polytope is the orthogonal product of a \( p \)-dimensional conical cell and a \( q \)-dimensional Euclidean space for some \( p \geq 0 \) and \( q \geq 0 \), and its dual is the orthogonal product of a \( p \)-dimensional conical cell and an \( (n - p - q) \)-dimensional Euclidean space.
If $P$ and $Q$ are two closed convex cones at $v$ then $P \cap Q$ is another. The cone $P \cup Q$ is not necessarily convex. If it is convex, then
\begin{equation}
D(P \cup Q) = DP \cap DQ \tag{11}
\end{equation}
and
\begin{equation}
D(P \cap Q) = DP \cup DQ. \tag{12}
\end{equation}
If $P$ and $Q$ are closed convex cones in $V$ and $W$ respectively, then in $V \times W$
\begin{equation}
DP \times DQ = D(P \times Q). \tag{13}
\end{equation}

2.1.3. Local volume. For a conical polytope $P$ in a based $n$-dimensional Euclidean space $V$, the local volume is the real number
\[
U_n(P) = \frac{Vol(P \cap S)}{Vol(S)},
\]
the quotient of the $(n-1)$-dimensional volume of the intersection of $P$ with a sphere centered at the basepoint by the $(n-1)$-dimensional volume of that sphere. In the case when $V$ is 0-dimensional a special convention is needed:
\[
U_0(V) = 1 \\
U_0(\emptyset) = 0.
\]
$U_n$ is invariant under (basepoint-preserving) isometry. It is additive in the sense that it satisfies
\[
U_n(P \cup Q) = U_n(P) + U_n(Q) - U_n(P \cap Q).
\]
It is also multiplicative: for convex conical polytopes $P$ and $Q$ in Euclidean spaces of dimensions $p$ and $q$ we have
\begin{equation}
U_{p+q}(P \times Q) = U_p(P) \times U_q(Q). \tag{14}
\end{equation}
We also speak of the local volume of a polytope at a point $v$, meaning the local volume of the unique conical polytope at $v$ that has the same germ.

In §2.3 we will develop the notions of local invariant and multiplicative local invariant more carefully and systematically.

2.1.4. Dual volume. For a convex conical polytope $P$ in an $n$-dimensional Euclidean space $V$, define its dual volume by $U^*(P) = U_n(DP)$. Using (11) and (12) and the additivity of $U_n$, we find that $U^*$ is additive. More precisely: it is initially defined only for convex conical polytopes in $V$, it is additive in that context, and it therefore extends uniquely to an additive invariant of all conical polytopes in $V$. In view of (13) and (14), $U^*$ is multiplicative as well.

The reason for not putting a subscript on $U^*$ is that the dimension of the ambient space is irrelevant: If $DP$ is the dual of $P$ in a $p$-dimensional Euclidean space $V$,
and if $W$ is a $q$-dimensional Euclidean space, then the dual of $P \times \{w\}$ in $V \times W$ is $DP \times W$, and

$$U_{p+q}(DP \times W) = U_p(DP) \times U_q(W) = U_p(DP).$$

In other words, $U^*$ is an absolute multiplicative invariant of conical polytopes.

Again we can use local rather than conical language, defining the dual volume of $P$ at $v$ to be that of the corresponding cone.

If $V$ is a Euclidean space of positive dimension then $U^*(V)$ is zero. Therefore

$$U^*(P \times V) = U^*(P) \times 0 = 0$$

for any conical polytope $P$. In particular whenever a convex conical polytope is not a conical cell then its dual volume is zero.

### 2.2. Some applications of dual volume.

The bending of a polytope $P$ along a simplex $\sigma$ in a triangulation of $P$ is the dual volume of the normal cone:

$$B_\sigma(P) = U^*(\nu_P\sigma).$$

This definition extends easily to cells in a cell structure.

**Example 2.1.** If $P$ is a triangulated convex polygon in the plane, then $B_\sigma(P)$ is 1 if $\sigma$ is a 2-simplex, $\frac{1}{2}$ if $\sigma$ is a boundary 1-simplex, 0 if $\sigma$ is an interior 1-simplex, $\frac{\pi-\theta}{2\pi}$ if $\sigma$ is a boundary 0-simplex and $\theta$ is the internal angle of $P$ at that vertex, and 0 if $\sigma$ is an interior 0-simplex.

**Example 2.2.** If $P$ is a triangulated closed PL surface in a Euclidean space, then $B_\sigma(P)$ is 1 if $\sigma$ is a 2-simplex, 0 if $\sigma$ is a 1-simplex, and $\frac{2\pi-\theta}{2\pi}$ if $\sigma$ is a 0-simplex and $\theta$ is the angle sum of the 2-simplices around that vertex.

Now that the right hand side of (9) has been defined, we can state and prove the following.

**Proposition 2.3.** Equation (9) defines a real-valued absolute additive invariant $V(n)$ whose restriction to level $n$ is the $n$-dimensional volume $V_n$.

**Proof.** The main point is to show that the right hand side is independent of the choice of triangulation and therefore a well-defined function of the bounded polytope $P$. Suppose that we have two triangulations, one a refinement (subdivision) of the other. Write $\tau$ for any simplex of the finer triangulation and $\sigma$ for any simplex of the coarser. We must show that

$$\Sigma_{|\tau|=n} B_\tau(P) \times V_n(\tau) = \Sigma_{|\sigma|=n} B_\sigma(P) \times V_n(\sigma),$$

where each sum is over all of the $n$-simplices of a triangulation. For each $\tau$ the interior of $\tau$ is contained in the interior of a unique $\sigma$; say that $\tau$ belongs to $\sigma$. If $\sigma$ has the same dimension as $\tau$, then $\nu_P\tau = \nu_P\sigma$ and $B_\tau(P) = B_\sigma(P)$. Otherwise $\sigma$ has higher dimension, in which case $B_\tau(P) = 0$ because $\nu_P\tau$ is the product of
with a positive-dimensional Euclidean space. For each $n$-dimensional $\sigma$ the term $B_{\sigma}(P) \times V_n(\sigma)$ on the right is equal to the sum of those terms on the left for which $\tau$ belongs to $\sigma$. The remaining terms on the left vanish.

To verify additivity, triangulate $P \cup Q$ in such a way that $P$ and $Q$ are subcomplexes, apply (9) to $P \cup Q$, $P$, $Q$, and $P \cap Q$, and compare terms. The key observation is that for a simplex $\sigma$ in $P \cap Q$

$$B_{\sigma}(P \cup Q) = B_{\sigma}(P) + B_{\sigma}(Q) - B_{\sigma}(P \cap Q).$$

This follows from

$$\nu_{P \cup Q} = \nu_P \cup \nu_Q,$$

and

$$\nu_{P \cap Q} = \nu_P \cap \nu_Q,$$

using the additivity of $U^*$. □

Remark 2.4. $V(0)$ is the Euler characteristic. That is, $\chi(P)$ is the sum of $B_v(P)$ over vertices $v$ of a triangulation. To prove this, it is enough to consider the case when $P$ is convex and nonempty. In this case it says that for an observer far away from $P$ in a Euclidean space the nearest point in $P$ is almost certainly a vertex.

The next result uses much the same argument. It also uses the fact ([Sah]) that for $n > 0$ the scissors congruence group $E'_n$ has a canonical real vector space structure.

**Theorem 2.5.** The inclusion map $E_{n-1} \to E_n$ is a split injection for all $n \geq 1$.

**Proof.** Recall that we are writing $E'_n$ for the cokernel $E_n/pE_{n-1}$. We will show that the direct limit system

$$0 \to E_0 \to E_1 \to E_2 \to \ldots$$

is isomorphic to the system

$$0 \to E'_0 \to E'_0 \oplus E'_1 \to E'_0 \oplus E'_1 \oplus E'_2 \to \ldots$$

in which the maps are the obvious split injections. For this it is enough to make a group map

$$\alpha_n : E'_n \to E'_n$$

for every $n$, in such a way that the composed map $E_n \to E'_n \to E'_n$ is the quotient map. In other words for each $n$ we need an absolute additive invariant $\alpha_n$ with values in $E'_n$ such that when $\sigma$ is an $n$-simplex then $\alpha_n(\sigma) = [\sigma]_n'$. Write

$$\alpha_n(P) = \Sigma_{|\sigma|=n} B_{\sigma}(P) [\sigma]_n'$$

for any triangulated bounded polytope $P$. Because of the vector space structure on $E'_n$, the right hand side is meaningful. This quantity is independent of triangulation, by an argument exactly as in the proof of Theorem 2.3. Invariance under isometry
is clear. Additivity follows from \( [15] \) as before. Clearly \( \alpha_n \) takes the desired value on an \( n \)-simplex. This completes the proof, except for one thing: the formula for \( \alpha_n \) does not apply when \( n = 0 \), because \( E_0' = E_0 = \mathbb{Z} \) is not a vector space.

We can fill this gap by writing

\[
\alpha_0(P) = \chi(P).
\]

(In fact Equation \( [18] \), suitably interpreted, is valid even in the \( n = 0 \) case, by Remark 2.4.) \( \square \)

We record the same conclusion in a more detailed form:

**Corollary 2.6.** There are group isomorphisms

\[
E_n \cong E'_0 \oplus E'_1 \oplus \cdots \oplus E'_n
\]

given by

\[
[P]_n \mapsto \alpha_0(P) + \alpha_1(P) + \cdots + \alpha_n(P).
\]

In the direct limit this becomes a ring isomorphism

\[
E_\infty \cong E'.
\]

**Proof.** For the ring statement we must verify that

\[
(19) \quad \alpha_n(P \times Q) = \sum_{i+j=n} \alpha_i(P) \times \alpha_j(Q).
\]

To do so, choose a cell structure for \( P \) with cells \( \sigma \) and a cell structure for \( Q \) with cells \( \tau \). This makes a cell structure for \( P \times Q \) with cells \( \sigma \times \tau \). The normal cones are given by

\[
\nu_{P \times Q}(\sigma \times \tau) = \nu_P(\sigma) \times \nu_Q(\tau),
\]

and their dual volumes by

\[
B_{\sigma \times \tau}(P \times Q) = B_{\sigma}(P) \times B_{\tau}(Q),
\]

so that

\[
B_{\sigma \times \tau}(P \times Q)[\sigma \times \tau]_{i+j} = B_{\sigma}(P)[\sigma]_i \times B_{\tau}(Q)[\tau]_j.
\]

When this is summed over all pairs \((\sigma, \tau)\) with \(|\sigma| = i\) and \(|\tau| = j\), the right hand side becomes \( \alpha_i(P) \times \alpha_j(Q) \). Summing then over all pairs \((i, j)\) such that \( i + j = n \) gives \(19\). Note that we have used that multiplication \( E'_p \times E'_q \rightarrow E'_{p+q} \) is \( \mathbb{R} \)-linear in each variable when this makes sense. \( \square \)

The scissors congruence class \( \alpha_n(P) \in E'_n \) might be thought of as the purely \( n\)-dimensional part of \( P \). Note that its \( n\)-dimensional volume is \( V(n, P) \).

Because of the ring isomorphism \( E_\infty \cong E' \), there is a canonical bijection between \( k\)-valued absolute multiplicative invariants and \( k\)-valued multiplicative scissors invariants. If \( F = (F_0, F_1, \ldots) \) is a scissors invariant, then the corresponding absolute invariant is given by

\[
P \mapsto \sum_{n \geq 0} F_n(\alpha_n(P)).
Example 2.7. If \( k = \mathbb{R}[x] \) and \( F = \ast V \) is the universal scaling of the volume, then the corresponding absolute invariant is given by (10).

Example 2.8. If \( k = \mathbb{Z} \) and \( F = \iota \) is the trivial invariant, then the corresponding absolute invariant is \( \chi \).

From a slightly different point of view the Theorem says

**Corollary 2.9.** \( E \) is isomorphic to the polynomial ring \( E'[x] \) via the map

\[
[P]_n \mapsto \alpha_n(P) + \alpha_{n-1}(P)x + \cdots + \alpha_0(P)x^n.
\]

Here we view \( E'[x] \) as a graded ring by giving \( E' \) its usual grading and letting \( x \) have degree one. Note that \( p \in E_1 \) corresponds to \( x \).

Later on we will make much more use of formulas of the same general type as (9) and (18). For now, we mention just two more examples.

First, there is a local analogue of Proposition 2.3 giving an absolute extension of \( U_n \).

Second, the absolute \( \mathbb{R} \)-valued invariant \( V(n) \) may be refined to an absolute \( \mathbb{R} \otimes \mathbb{Z} \mathbb{R} \)-valued invariant by writing

\[
\tilde{V}(n, P) = \sum_{|\sigma| = n} B_\sigma(P) \otimes \mathbb{Z} V_n(\sigma).
\]

For example, \( \tilde{V}(1) \) gives a map \( E_3' = E_3/pE_2 \to \mathbb{R} \otimes \mathbb{Q} \mathbb{R} \) that takes \( pE_2 \) into \( \mathbb{Q} \otimes \mathbb{R} \cong \mathbb{R} \) in such a way that the induced map

\[
E_3' = E_3/pE_2 \to (\mathbb{R}/\mathbb{Q}) \otimes \mathbb{Z} \mathbb{R}
\]

is essentially the Dehn invariant. By a theorem of Sydler ([Sah], p. 139), \( E_3' \) is detected by the Dehn invariant together with three-dimensional volume. It follows that all of \( E_3 \) is detected by the invariants \( \chi \), \( \tilde{V}(1) \), \( V(2) \), and \( V(3) \). The construction of \( \tilde{V}(n) \) will be generalized in §3.7.

### 2.3. The graded ring \( L \)

We now imitate the definition of \( E \) with germs instead of bounded polytopes.

Define the abelian group \( L_n \) by giving a generator \( \langle P \rangle_n \) for each conical polytope \( P \) in \( \mathbb{R}^n \) based at the origin, with generating relations analogous to (1), (2), and (3):

\[
\langle P \rangle_n = \langle g(P) \rangle_n
\]

if \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a linear isometry,

\[
\langle \emptyset \rangle_n = 0,
\]

and

\[
\langle P \cup Q \rangle_n = \langle P \rangle_n + \langle Q \rangle_n - \langle P \cap Q \rangle_n
\]

when all four terms are defined.
We routinely extend the notation \(\langle P \rangle_n\) to the case when \(P\) is any conical polytope of dimension at most \(n\) in any based Euclidean space, not necessarily \(\mathbb{R}^n\). We may also extend it to constructible sets in an appropriate sense. See the Appendix for details.

In order to have the option of using local rather than conical language, we agree that if \(P\) is a based polytope in \(V\) then \(\langle P \rangle_n\) is equal to \(\langle \hat{P} \rangle_n\), where \(\hat{P}\) is the conical polytope at \(v\) having the same germ as \(P\) at the base point. If \(P\) is a polytope not containing the base point then \(\langle P \rangle_n = 0\).

Since a cone \(P\) is determined by its intersection with the unit sphere, all of this can be phrased in terms of spherical geometry. If \(P\) is a nonempty conical polytope and \(A\) is its intersection with the unit sphere, then we call \(P\) the cone on \(A\) and write \(P = CA\).

A conical \(n\)-simplex \(\sigma\) has \(2^n\) faces. For convenience, let the base point be the origin and let \(\sigma\) be generated by linearly independent vectors \(v_1, \ldots, v_n\). For each subset \(S \subset \{1, \ldots, n\}\) let the face \(\sigma_S\) be the conical simplex spanned in the same way by the vectors \(\{v_i | i \in S\}\). For \(n \geq 1\) a conical \(n\)-simplex is the cone on a spherical \((n - 1)\)-simplex, and for \(0 < m \leq n\) the \(m\)-dimensional faces of the conical simplex are the cones on the \((m - 1)\)-dimensional faces of the spherical simplex.

A conical triangulation of the conical polytope \(P\) is a way of writing it as a union of conical simplices (with a common base point), such that the intersection of any two of these is a face of each. For example, if \(Q\) is a triangulated polytope and \(v\) is a vertex of \(Q\) then the (cones corresponding to the) simplices containing \(v\) give a conical triangulation of the (cone corresponding to the) germ of \(P\) at \(v\). If \(P = CA\) then a conical triangulation of \(P\) corresponds precisely to a triangulation of \(A\) by spherical simplices. A conical simplex has a normal cone \(\nu_P\sigma\) as in the global case. More generally we may speak of a conical cell structure of \(P\).

Clearly the group \(L_n\) is generated by elements \(\langle \sigma \rangle_n\) where \(\sigma\) is a conical \(m\)-simplex with \(0 \leq m \leq n\). Better:

**Proposition 2.10.** \(L_n\) is generated by elements \(\langle \sigma \rangle_n\) where \(\sigma\) is either a conical \(n\)-simplex or the unique conical \(0\)-simplex.

**Proof.** For \(0 < m < n\) a conical \(m\)-simplex in \(\mathbb{R}^n\) may be written as the intersection of two conical \((m + 1)\)-simplices whose union is a conical \((m + 1)\)-simplex (all with the same base point). (Note that this fails for \(m = 0\)). \(\square\)

**Proposition 2.11.** When a based polytope \(P\) is triangulated in such a way that the base point is a vertex, then we have

\[
\langle P \rangle_n = \sum_{\sigma} (1 - \chi L(\sigma, P)) \langle \sigma \rangle_n,
\]

where the sum is over all simplices of the triangulation that contain the base point.
(Equivalently: Equation (24) holds for any conical triangulation of a nonempty conical polytope $P$, where the summation is over the conical simplices.)

**Proof.** This is just like the proof of Proposition 1.3. There is a straightforward generalization to conical cell structures. □

There is a local analogue of (8): if the based polytope $P$ is a triangulated PL manifold of dimension $m \leq n$ with a vertex as base point then

$$
\langle M \rangle_n = \sum_{\text{int } \sigma \subseteq \text{int } M} (-1)^{m-|\sigma|} \langle \sigma \rangle_n
$$

where the summation is over the interior simplices containing the vertex.

There is an “inclusion map” $L_{n-1} \to L_n$ taking $\langle P \rangle_{n-1}$ to $\langle P \rangle_n$, but we do not assert that it is injective for general $n$. The cokernel $L'_n$ is the $(n-1)$st spherical scissors congruence group. The direct limit of $L_n$ is called $L_\infty$.

$L$ is a graded ring, with multiplication given by cartesian product. The ring $L_0$ is isomorphic to $\mathbb{Z}$, with 1 given by a point.

Let $t$ be the element of $L_1$ given by a point, so that “inclusion” is multiplication by $t$.

Let $d \in L_1$ be given by a conical 1-simplex – that is, a closed half-line with its endpoint as basepoint. It is easy to see that $t$ and $d$ constitute a $\mathbb{Z}$-basis for $L_1$.

The element of $L_1$ given by a line is called $s$ and satisfies $s + t = 2d$. We also write $d' = d - t$.

By Proposition 2.10 the ring $L$ is generated by $t$ together with all elements $\langle \sigma \rangle_n \in L_n$ where $\sigma$ is a conical $n$-simplex, $n > 0$.

There is an obvious notion of **level $n$ local invariant** with values in an abelian group $G$. These correspond to group maps $L_n \to G$. **Absolute** local invariants correspond to group maps $L_\infty \to G$.

The analogue of Lemma 1.1 holds:

**Lemma 2.12.** A function $F$ of conical convex polytopes extends to a unique additive invariant if in addition to being invariant under isometry it satisfies

$$
F(P) = F(P \cap H^+) + F(P \cap H^-) - F(P \cap H)
$$

whenever $P$ is a conical convex polytope, $H$ is a hyperplane containing the base point, and $H^+$ and $H^-$ are the two closed half-spaces determined by $H$.

We may speak of multiplicative local invariants $F = (F_0, F_1, \ldots)$ with values in a ring $k$. These correspond to ring maps $L \to k$. As in the global case, $k$-valued multiplicative local invariants can be scaled by elements of $k$.

Here are some key examples.
The yes/no invariant is the absolute local multiplicative invariant
\[ 1 : L_\infty \to \mathbb{Z} \]
defined by \( 1(P) = 1 \) if \( P \) is any convex conical polytope. More generally \( 1(P) \) is 1 for any nonempty conical polytope. For a conical constructible set it takes the value 1 or 0 according to whether the base point is present. For germs of polytopes, it takes the value 1 at every nonempty germ and 0 at the empty germ.

The local Euler invariant is another absolute local multiplicative invariant
\[ e : L_\infty \to \mathbb{Z} \]
For a polytope \( P \) with base point \( v \) we define \( e(P) \) as the relative Euler characteristic of the pair \( (P, P - v) \). If \( P \) is bounded then this is \( \chi(P) - \chi(P - v) \). If \( P \) is a nonempty conical polytope, say \( CA \) for a spherical polytope \( A \), then \( e(P) = 1 - \chi(A) \). Note that when \( P \) is an \( m \)-manifold then \( e(P) \) is \( (-1)^m \) if the base point is in the interior and zero if the base point in the boundary. In particular if \( P \) is a convex conical polytope then \( e(P) \) is zero except in the special case when \( P \) is a Euclidean space.

The local volume \( U_n \) defined in §2.1 is a level \( n \) additive local invariant. It is a scissors congruence invariant, that is, it factors through \( L'_n \). Together these give a multiplicative local invariant \( U = (U_0, U_1, U_2, \ldots) \).

We record the values 1, \( e \), and \( U \) on named elements:
\[
1(t) = 1(s) = 1(d) = 1, 1(d') = 0 \\
e(t) = 1, e(s) = -1, e(d) = 0, e(d') = -1 \\
U(t) = 0, U(s) = 1, U(d) = U(d') = \frac{1}{2}
\]
In the subgroup of \( L_2 \) generated by \( t^2, td, \) and \( d^2 \) are also \( ts, ds, \) and \( s^2 \). These six elements correspond respectively to the following conical polytopes in the plane: the origin, a closed half-line, a closed quadrant, a line, a closed half-plane, and the entire plane.

The remaining convex conical polytopes in the plane are those closed sectors (conical 2-simplices) which are not right-angled. A sector with angle \( \theta \) gives an element which will be denoted by \( td + a(\theta) \). That is, for \( 0 < \theta < \pi \) we define \( a(\theta) \in L_2 \) to be the difference between this conical 2-simplex and a conical 1-simplex. Since \( a(\theta + \phi) = a(\theta) + a(\phi) \), this gives an additive homomorphism \( a : \mathbb{R} \to L_2 \). Note that
\[
dd' = d^2 - dt = a\left(\frac{\pi}{2}\right) \\
d(s - t) = 2dd' = a(\pi) \\
s^2 - t^2 = (s + t)(s - t) = 2d(s - t) = a(2\pi)
\]
and
\[ 1(a(\theta)) = 0 \]
\[ e(a(\theta)) = 0 \]
\[ U(a(\theta)) = \frac{\theta}{2\pi}. \]

There is an isomorphism
\[ \mathbb{Z} \times \mathbb{Z} \times \mathbb{R} \cong L_2 \]
given by
\[ (k, \ell, \theta) \mapsto kt^2 + \ell td' + a(\theta). \]

To see that it is surjective, note that its image contains \( tL_1 \) (because \( tL_1 \) is generated by \( t^2 \) and \( td' \)) and that the quotient \( L'_2 = L_2/tL_1 \) is generated by based 2-simplices, thus by the image of \( a \). To see that it is injective, use the invariants \( 1, e \) and \( U_2 \).

The inclusion map \( L_1 \to L_\infty \) is split injective, since the absolute invariant \((1, e)\) takes a \( \mathbb{Z} \)-basis for \( L_1 \) to a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^2 \). Thus for \( n \geq 1 \) the group \( L_n \) is the direct sum
\[ (t^{n-1}L_1) \oplus (ker(1) \cap ker(e)) \]
of a subgroup isomorphic to \( L_1 \) and a subgroup isomorphic to \( coker(L_1 \to L_n) \). The first summand has a \( \mathbb{Z} \)-basis consisting of the point element \( t^n \) and the half-line element \( t^{n-1}d \). The second summand is 2-divisible; this follows from the fact (\[\text{[Dupont], page ?}\]) that for \( n \geq 2 \) the spherical scissors congruence group \( L'_n = coker(L_{n-1} \to L_n) \) is 2-divisible.

In particular the structure of \( L/2L = L \otimes (\mathbb{Z}/2\mathbb{Z}) \) is easily described:

**Proposition 2.13.** For \( n > 0 \) a \( \mathbb{Z}/2\mathbb{Z} \)-basis for \( L_n/2L_n \) is given by the point and the half-line. As a ring \( L/2L \) is generated by \( t \) and \( d \) subject to the relations \( td = d^2 \) and \( 2 = 0 \).

1 and \( e \) are strong invariants: they are invariant under all (basepoint-preserving) affine isomorphisms, not just isometries. For each \( n > 0 \) the map \((1, e) : L_n \to \mathbb{Z} \times \mathbb{Z}\) is the universal local strong invariant of level \( n \), by Proposition 2.10.

Scaling the invariants 1 and \( e \) by an indeterminate yields multiplicative local invariants
\[ x1 : L \to \mathbb{Z}[x] \]
\[ xe : L \to \mathbb{Z}[x]. \]

Combining these, we have a graded ring map \( L \to \mathbb{Z}[x] \times \mathbb{Z}[x] \). Call its image \( L^{str} \).

This has an additive \( \mathbb{Z} \)-basis consisting of \( 1 = (1, 1) \) and of \((x^n, 0)\) and \((0, x^n)\) for \( n > 0 \). The elements \((x, 0)\) and \((0, x)\) are the images of \( d \) and \( t - d = -d' \). Thus as a ring \( L^{str} \) is generated by two elements \( d \) and \( t \) subject only to the relation \( td = d^2 \).

It is clear that this surjective map \( L \to L^{str} \) is the universal example of a multiplicative strong local invariant. As a result, we can classify all such invariants:
Lemma 2.14. A ring map \( F : L \rightarrow k \) corresponding to a strong invariant is determined by the two elements \( F(d) \) and \( F(t) \), and the only constraint on these is
\[
F(d)(F(d) - F(t)) = 0.
\]
If \( F(d) - F(t) \) is zero, then \( F \) is the scaling of the yes/no invariant by \( a = F(t) = F(d) \). If \( F(d) = 0 \) then \( F \) is the scaling of the local Euler invariant by \( a = F(t) \).
In particular if \( k \) is a domain then the only strong multiplicative invariants \( L \rightarrow k \) are the scalings of 1 and the scalings of \( e \).

Note that the canonical map \( L/2L \rightarrow L^{str}/2L^{str} \) is an isomorphism. More generally, we have:

Theorem 2.15. If \( k \) is any ring whose additive group has no nontrivial 2-divisible subgroup, then every \( k \)-valued multiplicative invariant is strongly invariant. That is,
\[
k \otimes L \rightarrow k \otimes L^{str}
\]
is an isomorphism.

Proof. \( k \otimes \ker(L \rightarrow L^{str}) = 0 \) because \( \ker(L \rightarrow L^{str}) = \ker(1) \cap \ker(e) \) is 2-divisible. \( \square \)

2.4. Duality in \( L \). By (11) and (12), and using Lemma 2.12, the duality operation on convex conical polytopes yields a group map \( D : L_n \rightarrow L_n \); the rule is
\[
D\langle P \rangle_n = \langle DP \rangle_n
\]
whenever \( P \) is a convex conical polytope in an \( n \)-dimensional Euclidean space. Because of (13), these maps constitute a graded ring map \( D : L \rightarrow L \). It is an involution: the composed map \( D \circ D \) is the identity.

In \( L_1 \) we have
\[
Dt = s, Ds = t, Dd = d, Dd' = -d'.
\]

In \( L_2 \) we have
\[
Da(\theta) = -a(\theta).
\]

It is important to remember that geometrically the dual of a convex conical polytope depends on which vector space it is regarded as being in: if \( DP \) is the dual of \( P \) in \( V \), then in \( V \times W \) the dual of \( P \times 0 \) is \( DP \times W \). Algebraically this is reflected in the equation \( D(t\xi) = sD(\xi) \in L_{n+1} \), valid for \( \xi \in L_n \).

It is also important to remember that, whereas the behavior of \( D \) on convex conical polytopes is given in a direct geometric way, its behavior on more general sets uses additivity. We will not ordinarily refer to one conical polytope as the dual of another unless they are convex.

The yes/no invariant is unaltered by duality:
\[
1 \circ D = 1.
\]
We claim that the local Euler invariant is altered as follows:

\[ e \circ D = e \circ \epsilon, \]

where \( \epsilon : L \to L \) is the grading involution given by \( \epsilon(\xi) = (-1)^n \xi \) if \( \xi \in L_n \). (We will use the symbol \( \epsilon \) for the grading involution of any graded ring.) That is, \( \epsilon(D\xi) = (-1)^n \epsilon(\xi) \) for every \( \xi \in L_n \). To see this, it is enough to verify it for all \( \xi \) in a generating set for the ring \( L \). Both \( e(\xi) \) and \( e(D\xi) \) vanish when \( \xi \in L_n \) is given by a conical simplex of positive dimension, and for \( \xi = t \in L_1 \) both of them are equal to \(-1\).

Since \( e \circ \epsilon \) is the scaling of \( e \) by \(-1\), we can also write

\[ (27) \quad e \circ D = -1 \cdot e. \]

An even less computational way of proving (27) is to use Lemma 2.14.

The dual volume

\[ U^* = U \circ D \]

is an absolute \( \mathbb{R} \)-valued multiplicative invariant. It satisfies

\[ U^*(t) = 1, U^*(s) = 0, U^*(d) = \frac{1}{2}, U^*(d') = -\frac{1}{2} \]

\[ U^*(a(\theta)) = -\frac{\theta}{2\pi}. \]

We finish this section by giving a relation between interiors of conical simplices and the duality involution: up to sign, the interior of the dual is algebraically the dual of the interior.

If \( P \) is a convex conical polytope based at \( v \) then \( \partial P \) is a conical polytope based at \( v \). Consider the case when \( P = \sigma \) is a conical \( n \)-simplex. For convenience let the origin be the base point, so that \( \sigma \) consists of the nonnegative linear combinations of some independent vectors \( v_1, \ldots, v_n \). Then \( \text{int}(\sigma) \) consists of the linear combinations with strictly positive coefficients. Because \( \partial \sigma \) is the union of the \((n-1)\)-dimensional faces, and because \( \sigma_{S \cap T} = \sigma_S \cap \sigma_T \), the inclusion-exclusion formula (4) gives

\[ (28) \quad \langle \text{int} \sigma \rangle_n = \langle \sigma - \partial \sigma \rangle_n = \Sigma_{S \subseteq \{1, \ldots, n\}} (-1)^{n-|S|} \langle \sigma_S \rangle_n. \]

**Theorem 2.16.** If \( \sigma \) is a conical \( n \)-simplex in an \( n \)-dimensional Euclidean space, then

\[ D\langle \text{int} \sigma \rangle_n = (-1)^n \langle \text{int} D\sigma \rangle_n. \]

**Proof.** From (28) we obtain

\[ (29) \quad D\langle \text{int} \sigma \rangle_n = (-1)^n \Sigma_S (-1)^{|S|} \langle D\sigma_S \rangle_n. \]
Let $V$ be the ambient Euclidean space. The set $D\sigma_S$ consists of all those vectors $v \in V$ such that $v.v_i \leq 0$ for all $i \in S$. Let $\Delta$ be the union of $D\sigma_S$ over all nonempty $S$. Using inclusion-exclusion again, we have

\[
\langle V - \Delta \rangle_n = \sum_S (-1)^{|S|} \langle D\sigma_S \rangle_n,
\]

whence

\[
D\langle \text{int } \sigma \rangle_n = (-1)^n \langle V - \Delta \rangle_n.
\]

But $V - \Delta$, the set of all $v$ such that $v.e_i > 0$ for all $i$, is related to $\text{int}(D\sigma)$ by the isometry $v \mapsto -v$. □

**Corollary 2.17.** For a conical $n$-simplex $U^*(\text{int } \sigma) = (-1)^n U^*(\sigma)$.

**Proof.** Since $D\sigma$ is again a conical $n$-simplex,

\[
U^*(\text{int } \sigma) = U_n D\langle \text{int } \sigma \rangle_n = (-1)^n U_n \langle \text{int } D\sigma \rangle_n = (-1)^n U_n \langle D\sigma \rangle_n = (-1)^n U^*(\sigma).
\]

Thus when $n$ is even $U^*(\partial \sigma) = U^*(\sigma) - U^*(\text{int } \sigma) = 0$: the dual volume of the boundary of an even-dimensional conical simplex is zero. In Theorem 4.4 this will be generalized to all odd-dimensional conical manifolds without boundary.

### 3. The groupoid of local multiplicative invariants

We now turn to a general way of making new multiplicative invariants from old. This places the constructions and arguments of §2.2 in a wider setting. Throughout §3 the word “invariant” will ordinarily mean “multiplicative invariant” (either local or global). When a mere additive invariant is meant, we will say so.

If $F$ and $G$ are local invariants with values in some ring $k$, and if $F(s) = G(t)$, we will define another such invariant $F \star G$ with $(F \star G)(s) = G(s)$ and $(F \star G)(t) = F(t)$. We will see that with this composition law the $k$-valued local invariants become the morphisms of a category $\mathcal{G}(k)$ whose objects are the elements of $k$. The source and target of $F$ are $F(s)$ and $F(t)$, the ring elements given by evaluating the level one invariant $F_1$ on the germ of a line and the germ of a point. The identity morphism of the object $a \in k$ is $a^1$, the scaling of the yes/no invariant by $a$. The category is a groupoid, and the inverse of a morphism $F$ is made by composing (as ring maps) the local invariant $F : L \to k$ and the duality involution $D : L \to L$. The scaling action of the multiplicative monoid of $k$ on morphisms respects the groupoid structure. All of this is natural in $k$.

**Remark 3.1.** In algebraic geometry language, we have an affine groupoid scheme $\mathcal{G}$ whose object scheme is the affine line $\text{Spec } \mathbb{Z}[x]$ and whose morphism scheme is $\text{Spec } L$. Scaling is an action of the multiplicative monoid scheme. This action arises from the fact that the representing rings $L$ and $\mathbb{Z}[x]$ are nonnegatively graded (and the fact that the groupoid structure maps are given by graded ring maps.)
The relative Euler invariant \(e\) is a \(\mathbb{Z}\)-valued morphism from \(-1\) to \(1\). The local volume \(U\) is an \(\mathbb{R}\)-valued morphism from \(1\) to \(0\). The dual volume \(U^*\) is its inverse, an \(\mathbb{R}\)-valued morphism from \(0\) to \(1\).

There is also a very similar rule for making a global invariant \(F \ast G\) when \(F\) and \(G\) are local and global invariants respectively and \(F(s)\) coincides with \(G(p)\). For example, \(U^* \ast x^* V\) is the \(\mathbb{R}[x]\)-valued invariant appearing in (10).

3.1. Definition of the groupoid \(G(k)\). We define the composition law and verify that it makes a groupoid.

Suppose that \(F\) and \(G\) are two multiplicative \(k\)-valued invariants and that \(F(s) = G(t)\).

Define

\[
(F \ast G)_n(P) = \sum_{\sigma} F_{n-|\sigma|}(\nu_P \sigma) \times G_{|\sigma|}(\text{int } \sigma).
\]

Here \(P\) is any nonempty conical polytope of dimension at most \(n\) and \(\sigma\) runs through the simplices of a conical triangulation of \(P\).

3.1.1. Composition is well-defined. Let us verify that

- the right-hand side of (30) is independent of the choice of conical triangulation of \(P\),
- the function \((F \ast G)_n\) is an additive local invariant,
- the sequence \(((F \ast G)_0, (F \ast G)_1, (F \ast G)_2, \ldots)\) is multiplicative,
- \((F \ast G)(s) = G(s)\)
- \((F \ast G)(t) = F(t)\)

Suppose that two triangulations are given, one a refinement of the other. Denote the simplices of the coarser triangulation by \(\sigma\) and those of the finer by \(\tau\). For each \(\tau\) there is a unique \(\sigma\) such that \(\text{int } \tau \subseteq \text{int } \sigma\); say that \(\tau\) belongs to \(\sigma\). Each term in the sum

\[
\sum_{\tau} F_{n-|\tau|}(\nu_P \tau) \times G_{|\tau|}(\text{int } \tau)
\]

can be rewritten in terms of the simplex \(\sigma\) that \(\tau\) belongs to. Indeed, the polytope \(\nu_P \tau\) is congruent to \(\nu_P \sigma \times \mathbb{R}^{[\sigma]-[\tau]}\), so that

\[
F_{n-|\tau|}(\nu_P \tau) \times G_{|\tau|}(\text{int } \tau) = F_{n-|\sigma|}(\nu_P \sigma) \times F_1(\mathbb{R})^{[\sigma]-[\tau]} \times G_{|\sigma|}(\text{int } \tau);
\]

and because

\[
F_1(\mathbb{R}) = F(s) = G(t) = G_1(\{0\})
\]

this becomes

\[
F_{n-|\sigma|}(\nu_P \sigma) \times G_1(\{0\})^{[\sigma]-[\tau]} \times G_{|\sigma|}(\text{int } \tau) = F_{n-|\sigma|}(\nu_P \sigma) \times G_{|\sigma|}(\text{int } \tau).
\]

The sum of these terms over all those \(\tau\) which belong to a given \(\sigma\) is

\[
F_{n-|\sigma|}(\nu_P \sigma) \times G_{|\sigma|}(\text{int } \sigma),
\]
because \( \text{int } \sigma \) is the union of the disjoint sets \( \text{int } \tau \). Therefore the sum over all \( \tau \) is equal to the sum of \( F_{n-|\sigma|}(\nu_P\sigma) \times G_{|\sigma|}(\text{int } \sigma) \) over all \( \sigma \), as asserted.

In fact, by the same argument one can see that Equation (30) is valid more generally for a conical cell structure instead of a conical triangulation, with summation over the cells.

We must verify that \((F \star G)_n\) is an additive local invariant. It is certainly invariant under isometry. To prove that

\[
(F \star G)_n(P \cup Q) = (F \star G)_n(P) + (F \star G)_n(Q) - (F \star G)_n(P \cap Q),
\]

choose a conical triangulation of \( P \cup Q \) that makes \( P \) and \( Q \) subcomplexes and then use (16) and (17). Details are left to the reader.

It is easy to see that \((F \star G)_0(*) = 1\). To see that

\[
(F \star G)_{p+q}(P \times Q) = (F \star G)_p(P) \times (F \star G)_q(Q),
\]

choose conical cell structures of \( P \) and \( Q \). As \( \sigma \) ranges over the cells of \( P \) and \( \tau \) ranges over those of \( Q \), the products \( \sigma \times \tau \) are the cells of a conical cell structure for \( P \times Q \). Moreover,

\[
\nu_{P \times Q}(\sigma \times \tau) = \nu_P(\sigma) \times \nu_Q(\tau)
\]

and

\[
\text{int } (\sigma \times \tau) = \text{int } \sigma \times \text{int } \tau.
\]

The left side of (31) is equal to the sum of

\[
F_{p+q-|\sigma \times \tau|}(\nu_{P \times Q}(\sigma \times \tau)) \times G_{|\sigma \times \tau|}(\text{int } (\sigma \times \tau))
\]

over all pairs \((\sigma, \tau)\) with \(|\sigma| + |\tau| = n\). This is the same as the sum of

\[
F_{p-|\sigma|}(\nu_P(\sigma)) \times G_{|\sigma|}(\text{int } \sigma) \times F_{q-|\tau|}(\nu_Q(\tau)) \times G_{|\tau|}(\text{int } \tau),
\]

which is equal to the right hand side of (31).

To verify that

\[
(F \star G)(s) = G(s),
\]

note that the left hand side is \((F \star G)_1(\mathbb{R})\) and use (30). The only conical triangulation of \( \mathbb{R} \) has one 0-simplex and two 1-simplices. The 0-simplex \(*\) contributes the term

\[
F_1(\nu_{\mathbb{R}*}) \times G_0(\text{int } *) = F_1(\mathbb{R}) \times G_0(*) = F(s) \times G(1) = F(s) = G(t),
\]

while each 1-simplex \( \sigma \) contributes the term

\[
F_0(\nu_{\mathbb{R}\sigma}) \times G_1(\text{int } \sigma) = F_0(*) \times G_1(\sigma - *) = F(1) \times G(d') = G(d').
\]

The total is \( G(t) + 2G(d') = G(s) \).

To verify that

\[
(F \star G)(t) = F(t),
\]
note that the left hand side is \((F \ast G)_1(*)\). The only conical triangulation of \(*\) has just one (0-dimensional) simplex. In this case the one term in (30) is
\[
F_1(\nu \ast t) \times G_0(\text{int } \ast) = F_1(*) \times G_0(*) = F(t) \times G(1) = F(t).
\]

3.1.2. **Composition is associative.** First note that when \(\tau\) is a simplex in a conical triangulation of \(P\) then \(\nu_p \tau\) has a conical triangulation whose simplices are the \(\nu_\sigma \tau\) for all simplices \(\sigma\) containing \(\tau\).

Now suppose that \(F, G,\) and \(H\) are multiplicative \(k\)-valued invariants with \(F(s) = G(t)\) and \(G(s) = H(t)\). Both \(((F \ast G) \ast H)_n(P)\) and \((F \ast (G \ast H))_n(P)\) are equal to the sum, over all pairs \(\tau \subset \sigma\) in the conical triangulation, of
\[
F_{n-|\sigma|}(\nu_p \sigma) \times G_{|\sigma| - |\tau|}(\text{int } \nu_\sigma \tau) \times H_{|\tau|}(\text{int } \tau).
\]

3.1.3. **Objects have identity morphisms.** For any \(a \in k\) the scaling \(^a1\) of the yes/no invariant takes both \(s\) and \(t\) to \(a\). The claim is that it is a two-sided identity for \(a\).

We verify that \(^a1\) is a left identity. For any local invariant \(G\) with \(G(t) = a\), evaluate \((^a1 \ast G)_n\) on a triangulated conical polytope \(P\) of dimension at most \(n\). For every simplex \(\sigma\) of the triangulation, the term \((^a1)_n(\nu_p \sigma) \times G_{|\sigma|}(\text{int } \sigma)\) is
\[
G(t)^{|\sigma|} \times G_{|\sigma|}(\text{int } \sigma) = G_{n-|\sigma|}(\ast) \times G_{|\sigma|}(\text{int } \sigma) = G_n(\text{int } \sigma).
\]

The sum of this over all \(\sigma\) is \(G_n(P)\) because \(P\) is the union of the disjoint constructible sets \(\text{int } \sigma\).

We verify that \(^a1\) is a right identity. For any local invariant \(F\) with \(F(s) = a\), evaluate \(F \ast ^a1\) on \(P\). For each \(\sigma\) except the 0-simplex \(\ast\), the term
\[
F_{n-|\sigma|}(\nu_p \sigma) \times (^a1)_{|\sigma|}(\text{int } \sigma) = F_{n-|\sigma|}(\nu_p \sigma) \times a^{|\sigma|} \times 1(\text{int } \sigma)
\]

is zero because
\[
1(\text{int } \sigma) = 1(\sigma) - 1(\partial \sigma) = 1 - 1 = 0.
\]

The remaining term is \(F_n(\nu_p \ast) \times F_0(\text{int } \ast) = F_n(P) \times 1 = F_n(P)\) because \(\nu_p \ast\) is \(P\) and \(\text{int } \ast = \ast\).

3.1.4. **Morphisms have inverses.** Note that for a multiplicative invariant \(F\) the source and target of \(F \circ D\) are respectively \(F(Ds) = F(t)\) and \(F(Dt) = F(s)\), the target and source of \(F\). We will show that \(F \circ D\) is a left inverse for \(F\). It follows as a special case that \((F \circ D) \circ D\) is a left inverse for \(F \circ D\), so that \(F \circ D\) is in fact a two-sided inverse for \(F\).

We evaluate \((F \circ D) \ast F\) on any conical simplex \(\sigma \subset \mathbb{R}^n:\)
\[
((F \circ D) \ast F)_n(\sigma) = \Sigma_\tau F_{n-|\tau|}(D \nu_\sigma \tau) \times F_{|\tau|}(\text{int } \tau) = \Sigma_\tau F_n(D \nu_\sigma \tau \times \text{int } \tau).
\]

For every vector \(v \in \mathbb{R}^n\) there is a unique nearest point to \(v\) in \(\sigma\). Express \(\mathbb{R}^n\) as the union of disjoint constructible sets \(X_\tau\), one for each face of \(\sigma\): a vector \(v\) belongs to
if and only if the point in $\sigma$ nearest to $v$ belongs to the interior of $\tau$. For every $\tau$ the set $X_{\tau}$ is congruent to the product $Dv_\tau \times \text{int } \tau$. It follows that

$$(F \circ D) * F_n(\sigma) = F_n(\mathbb{R}^n) = F(s)^n = (F(s))_n(\sigma),$$

so that $(F \circ D) * F = F(s)1$.

3.1.5. Scaling. Recall that the multiplicative monoid of $k$ acts on the morphisms of $G(k)$ by scaling. If we make it act on the objects as well by saying that the scaling of $a$ by $c$ is $ca$, then this becomes an action on the groupoid. That is, scaling is compatible with source, target, identity, and composition. Explicitly,

$$(cF)(s) = cF(s)$$

$$(cF)(t) = cF(t)$$

$$ca1 = c(a1)$$

$$(cF \star G) = cF \star cG.$$

3.2. The $\star$-product of a local and a global invariant. Again suppose that $F$ is a multiplicative local invariant, but now let $G$ be a multiplicative global invariant. If $F(s) = G(p)$ then equation (30) defines a multiplicative invariant $F \star G$ for bounded Euclidean polytopes. Now the summation is over simplices in a triangulation (or cells in a cell structure) of a bounded polytope $P$. As in the local case, this is independent of triangulation (precisely because $F(s) = G(p)$), and invariant, additive, and multiplicative. We have

$$(F \star G)(p) = F(t)$$

$$(F \star G) \star H = F \star (G \star H)$$

when $F$ and $G$ are local and $H$ is global (assuming that $F(s) = G(t)$ and $G(s) = H(p)$, so that the two sides are defined). We also have

$$G(p)1 \star G = G.$$

The action is compatible with scaling:

$$cF \star cG = c(F \star G).$$

We omit the proofs, which are exactly as in the local case.

This may be summarized by saying that we have a functor from $G(k)$ to the category of sets, taking the object $a \in k$ to the set of all global $k$-valued invariants $G$ such that $G(p) = a$, and taking the morphism $F$ to the map $G \mapsto F \star G$.

We will sometimes say that the global invariant $G$ is located at the ring element $G(p)$. If $F$ is a morphism from $a$ to $b$ and $G$ is located at $a$, then $F \star G$ is located at $b$. We may speak of using $F$ to transport $G$ from $a$ to $b$.

The $\mathbb{Z}$-valued invariant $\chi$ is located at $1$, like any absolute global invariant. The $\mathbb{R}$-valued invariant $V$ is located at $0$, like any global scissors congruence invariant.
Also located at 0 is the trivial invariant \(\iota\) introduced in \(\text{[12]}\).

The following is a restatement of Remark 2.4:

(32) \[ U^* \ast \iota = \chi. \]

An equivalent statement is

\[ U \ast \chi = \iota. \]

3.3. Extended volume and extended local volume. \(V\) is a multiplicative real-valued Euclidean invariant located at 0. The invariant \(xV\), its scaling by an indeterminate \(x\), is also located at 0. Transporting by the local invariant \(U^*\), which is a morphism from 0 to 1, we obtain a global invariant \(U^* \ast xV\) located at 1, in other words an absolute global invariant. It takes values in the polynomial ring \(\mathbb{R}[x]\).

Explicitly, \((U^* \ast xV)(P)\) is

\[ \sum_\sigma U^*(\nu_\sigma) \times xV_{|\sigma|(\text{int } \sigma)} = \sum_n (\Sigma_{|\sigma|=n} B_\sigma(P) \times V_n(\sigma)) \times x^n. \]

Thus its coefficients are the extended volume invariants given by \(\text{[9]}\); we have

\[ (U^* \ast xV)(P) = V(0, P) + V(1, P)x + \cdots + V(n, P)x^n + \ldots. \]

If \(P\) is \(d\)-dimensional, then the degree of the polynomial is \(d\) and the leading coefficient \(V(d, P)\) is the volume \(V_d(P)\). If \(M\) is a \(d\)-dimensional manifold with boundary then \(V(d - 1, M) = \frac{1}{2}V_{d-1}(\partial M)\). The constant term \(V(0, P)\) is \(\chi(P)\).

All of the above can be repeated in the local case, using \(U\) instead of \(V\). The result is a (multiplicative) absolute local invariant

\[ (U^* \ast xU)(P) = U(0, P) + U(1, P)x + \cdots + U(n, P)x^n + \ldots, \]

with

\[ U(n, P) = \sum_{|\sigma|=n} B_\sigma(P) \times U_n(\sigma). \]

Note that

\[ U(0) = U^* \ast 0U = U^*, \]

because \(0U = 01\) is the identity morphism of the object 0 in the groupoid. If the conical polytope \(P\) is \(d\)-dimensional, then the degree of the polynomial is \(d\) and the leading coefficient \(U(d, P)\) is \(U_d(P)\). Thus \(U(n)\) is an absolute local additive invariant extending the level \(n\) additive local invariant \(U_n\).

Because \(U^* \ast 1U = U^* \ast U = 11 = 1\) is the yes/no invariant, we have

(33) \[ 1 = U(0, P) + U(1, P) + U(2, P) + \cdots + U(d, P) \]

for every \(d\)-dimensional conical polytope \(P\).

Because \(U^* \ast xU\) has the same source as \(xU\), we have

\[ (U^* \ast xU)(s^d) = (xU)(s^d) = x^d. \]

This says that the value of \(U(n)\) at the germ of a \(d\)-dimensional Euclidean space is 1 if \(d\) is \(n\) and 0 otherwise.
3.4. Calculating dual volume. Solving (33) for \( U(0, P) = U^*(P) \), we obtain
\[
U^*(P) = 1 - \sum_{n \geq 1} \sum_{|\sigma| = n} U^*(\nu_P \sigma) U_n(\sigma).
\]
In every term of the right hand side, the invariant \( U^* \) is applied to a simplex of lower dimension than \( P \). Therefore this equation can serve as a recursive formula for \( U^* \).

Applying this in low-dimensional cases, we find:

If \( P \) is a conical 0-simplex, the cone on the empty set, then \( U^*(P) = 1 \) and the polynomial is 1.

If \( P \) is a conical 1-simplex, the cone on a point, then \( U^*(P) = \frac{1}{2} \) and the polynomial is \( \frac{1}{2} + \frac{1}{2}x \).

If \( P \) is a conical 2-simplex, the cone on a circular arc of length \( 2\pi L \), then \( U^*(P) = \frac{1}{2} - L \) and the polynomial is \( \left( \frac{1}{2} - L \right) + \frac{1}{2}x + Lx^2 \).

If \( P \) is a conical 3-simplex, the cone on a spherical triangle \( \sigma \) with perimeter \( 2\pi L \) and area \( 4\pi A \), then \( U^*(P) = \frac{1 - L}{2} \) and the polynomial is \( \frac{1 - L}{2} + \left( \frac{1}{2} - A \right)x + \frac{L}{2}x^2 + Ax^3 \).

3.5. The morphisms \( T^{x,y} \). Here is a slightly different way of organizing the additive local invariants \( U(n) \). Define
\[
T^{x,y} = xU^* \ast yU.
\]
This is a local invariant with values in \( \mathbb{R}[x, y] \), and a morphism from \( y \) to \( x \). In §3.3 we looked at the special case when \( x \) was 1 and the invariant was absolute. The equation
\[
*U \ast xU = T^{1,x} = U(0) + U(1)x + U(2)x^2 + \ldots
\]
from §3.3 becomes
\[
T^{x,y}_d = U(0)x^d + U(1)x^{d-1}y + U(2)x^{d-2}y^2 + \ldots + U(d)y^d.
\]

More or less directly from (34) we have
\[
T^{0,y} = yU
\]
\[
T^{x,0} = xU^*
\]
\[
T^{x,x} = x1
\]
\[
T^{x,y} \ast T^{y,z} = T^{x,z}
\]
\[
c(T^{x,y}) = T^{c,c}.
\]
The inverse $T^{x,y} \circ D$ of $T^{x,y}$ is $T^{y,x}$, which means that if $DP$ is the dual of the convex conical polytope $P$ in $\mathbb{R}^d$ then

$$T^{x,y}(DP)_d = T^{y,x}(P)_d.$$ Equating coefficients of $x^n y^{d-n}$, this says

$$U(n, DP) = U(d-n, P).$$

Note that

$$T^{x,y} \alpha(\theta) = \theta^2 \pi (y^2 - x^2).$$

3.6. The isomorphism $E \cong E'[x]$ revisited. The isomorphism of Corollary 2.6 viewed as a multiplicative global invariant with values in $E'[x]$, may be expressed as a $\ast$-product.

To be precise, we must introduce a slightly larger ring than $E'[x]$ for this purpose. Recall again that for $n > 0$ the group $E'_n$ has a real vector space structure. For $p > 0$ and $q > 0$ the multiplication $E'_p \times E'_q \to E'_{p+q}$ is $\mathbb{R}$-bilinear. If we replace the ring $E'_0 \cong \mathbb{Z}$ by a copy of $\mathbb{R}$ but leave the other groups $E'_n$ unchanged, we obtain a graded $\mathbb{R}$-algebra containing the graded scissors congruence ring $E'$. Call it $E'_R$.

Now the $\mathbb{R}[x]$-valued invariant $x U^*$ can be viewed as $E'_n[x]$-valued. Let $\varphi$ be the quotient map $E \to E'$ viewed as a global $E'_R[x]$-valued invariant. Then $x U^* \ast \varphi$ is defined, and is given by

$$(x U^* \ast \varphi)[P]_n = \sum_{k \geq 0} \alpha_k(P)x^k.$$ 

3.7. Generalized Dehn invariants. The only ring maps $L \to \mathbb{R}$ that we can produce by these methods are those obtained from $T^{x,y}$ by specializing $x$ and $y$ to real numbers. These form a groupoid, a subcategory of $\mathcal{G}(\mathbb{R})$, of a very trivial kind, having exactly one morphism between any two objects.

We can make something more interesting by introducing the ring $\mathbb{R} \otimes \mathbb{R}$. The global $(\mathbb{R} \otimes \mathbb{R})$-valued invariants $\tilde{V}(n)$ introduced at the end of §2.2 can be treated as a single multiplicative $(\mathbb{R} \otimes \mathbb{R})[x]$-valued invariant

$$(U^* \otimes 1) \ast (1 \otimes x \tilde{V}) = \tilde{V}(0) + \tilde{V}(1)x + \tilde{V}(2)x^2 + \ldots.$$ 

Here, for example, $1 \otimes x \tilde{V}$ denotes the $(\mathbb{R} \otimes \mathbb{R})[x]$-valued invariant obtained from the $\mathbb{R}[x]$-valued invariant $x \tilde{V}$ by applying the ring map $\mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ given by $\xi \mapsto 1 \otimes \xi$. Let us make this procedure systematic, starting with the local version.

Write

$$T^{x,y,z} = (T^{x,y} \otimes 1) \ast (1 \otimes T^{y,z}).$$

This is the $\ast$-product of two multiplicative local invariants with values in $(\mathbb{R} \otimes \mathbb{R})[x, y, z]$, one a morphism from $z$ to $y$ and one a morphism from $y$ to $x$.
Likewise define

\[ T^{x,y,z,w} = (T^{x,y} \otimes 1 \otimes 1) \star (1 \otimes T^{y,z} \otimes 1) \star (1 \otimes 1 \otimes T^{y,z}) . \]

This takes values in \((R \otimes R \otimes R)[x, y, z, w]\). In general we obtain an \((R \otimes R)[x_0, \ldots, x_r]\)-valued local invariant \(T^{x_0, \ldots, x_r}\) with source \(x_r\) and target \(x_0\). When \(r\) is zero the convention that \(T^x\) is \(z^1\), with values in \(Z[x]\), and with source and target \(x\).

In a sense every local invariant that can be made from \(U\) by a combination of scaling, groupoid composition, groupoid inversion, and precomposing an invariant with a ring map arises from these invariants \(T^{x_0, \ldots, x_r}\). A natural question is whether such invariants are enough to detect all of \(L_n\).

There are relations between them. For example, \(T^{x,y,y,z}\) is obtained from \(T^{x,y,z}\) by the ring map \(\xi \otimes \eta \mapsto \xi \otimes 1 \otimes \eta\). For another kind of example, \(T^{x,y,z,w}\) becomes \(T^{x,z,w}\) upon precomposition with the ring map \(\xi \otimes \eta \otimes \zeta \mapsto \xi \eta \otimes \zeta\).

Of course, we can also make global invariants of the same kind:

\[ (T^{x_0, \ldots, x_r, 0} \otimes 1) \star (1 \otimes \cdots \otimes y V) . \]

We plan to return to the study of these generalized Dehn invariants in a future paper.

4. The local Euler invariant \(e\) and the interior operator \(I\)

We now investigate the role of the local Euler invariant in the groupoid formalism. The first thing to notice is that up to scaling it is its own inverse: (27) implies that \(e^{-1}\) is the \(-1\) scaling of \(e\) and more generally

\[ (a e)^{-1} = -ae . \]

4.1. The operator \(I\) in the global case. Any global multiplicative invariant \(F : E \to k\) can be transported by a scaling of \(e\) to make another such invariant \(-F(p)e \star F\). In the universal example, \(F\) is the identity map \(I_E : E \to E\). Let \(I : E \to E\) be the ring map defined by

\[ I = -p e \star I_E . \]

Explicitly this means that for a triangulated polytope \(P\) of dimension \(\leq n\)

\[ I[P]_n = \Sigma_{\sigma} (-p)^{n-|\sigma|} e(\nu p \sigma)[int \sigma]_\sigma = \Sigma_{\sigma} (-1)^{n-|\sigma|} e(\nu p \sigma)[int \sigma]_\sigma , \]

where the summation is over the simplices of the triangulation. We will find some simpler formulas for \(I\), namely (38) and (40).

For any ring map \(F : E \to k\) we have

\[ F \circ I = -F(p)e \star F . \]

(Equation (36) is the universal example of this.)
In particular,
\[ I \circ I = p_{e} \ast I = p_{e} \ast -p_{e} \ast I = I = I \ast E = E, \]
so that \( I \) is an involution of the graded ring \( E \).

Note that \( e(\nu_{P}(\sigma)) \) is the same number \( 1 - \chi L(\sigma, P) \) that occurs in (7). In the case when a polytope \( M \) is an \( m \)-dimensional PL manifold, \( e(\nu_{M}(\sigma)) = 0 \) when \( \sigma \subset \partial M \) and \( e(\nu_{M}(\sigma)) = (-1)^{m-|\sigma|} \) otherwise. Thus for an \( m \)-manifold polytope we have

(38) \[ I[M]_{n} = (-1)^{n-m}[\text{int } M]_{n}. \]

In particular (38) holds when \( \sigma \) is a simplex, so

(39) \[ I[\text{int } \sigma]_{n} = (-1)^{n-|\sigma|}[\sigma]_{n}. \]

Equation (7) can now be understood as saying \( E = p_{e} \ast I \).

Given a general triangulated polytope \( P \) (bounded, of dimension \( \leq n \)), we may triangulate it and sum both sides of (39) over all simplices of the triangulation to obtain the general formula

(40) \[ I[P]_{n} = \Sigma_{\sigma} (-1)^{n-|\sigma|}[\sigma]_{n}. \]

Remark 4.1. There is also the operator \( eI \) given by composing \( I \) with the grading involution. This satisfies

(41) \[ eI[M]_{n} = (-1)^{m}[\text{int } M]_{n} \]

for \( m \)-manifolds and

(42) \[ eI[P]_{n} = \Sigma_{\sigma} (-1)^{|\sigma|}[\sigma]_{n}. \]

for triangulated polytopes. In contrast to (38) and (40) which have signs according to “codimension”, these have signs according to dimension.

The behavior of \( I \) on named elements of \( E \) and \( L \) is given by

(43) \[ I(p) = -p \]

(44) \[ I(q(\lambda)) = q(\lambda) \]

There are the following relations with named global invariants:

(45) \[ \chi \circ eI = \chi \]

(46) \[ V \circ I = V \]

Equation (45) holds because for an \( m \)-dimensional convex polytope \( P \)
\[ \chi(eI[P]_{n}) = (-1)^{m}\chi[\text{int } P]_{n} = (-1)^{m}(\chi(P) - \chi(\partial P)) = 1 = \chi(P). \]

Equation (46) applied to an \( m \)-manifold asserts that
\[ (-1)^{n-m}V_{n}(\text{int } M) = V_{n}(M). \]
This is clearly true if $m = n$, and if $m < n$ both sides are zero.

4.2. **The operator $I$ in the local case.** All of the above can be repeated using $L$ instead of $E$. We obtain a graded ring involution $I : L \to L$ defined by

$$I = -t \ast I_L$$

and satisfying

$$F \circ I = -F(t) \ast F$$

for every local multiplicative invariant $F$.

For an $m$-manifold germ it is given by

$$I \langle M \rangle_n = (-1)^{n-m} \langle \text{int } M \rangle_n.$$  

For a triangulated germ it is given by

$$I \langle P \rangle_n = \Sigma_{\sigma} (-1)^{n-|\sigma|} \langle \sigma \rangle_n.$$  

with summation over the simplices containing the base vertex.

Note that $\langle \text{int } M \rangle_n = \langle M \rangle_n - \langle \partial M \rangle_n$ if the base point is in $\partial M$, while $\langle \text{int } M \rangle_n = \langle M \rangle_n$ if the base point is in the interior.

Again, it follows that $\epsilon I$ satisfies similar equations, with sign according to dimension instead of codimension:

$$\epsilon I \langle M \rangle_n = (-1)^m \langle \text{int } M \rangle_n$$

for $m$-manifold germs and

$$\epsilon I \langle P \rangle_n = \Sigma_{\sigma} (-1)^{|\sigma|} \langle \sigma \rangle_n.$$  

for triangulated germs. We have

$$I(t) = -t$$

$$I(d) = d'$$

$$I(s) = s$$

$$I(d') = d$$

$$I(a(\theta)) = a(\theta)$$

and

$$1 \circ \epsilon I = e$$

$$e \circ \epsilon I = 1$$

$$U \circ I = U$$
Equation (57) holds because for a based $m$-manifold $M$ we have
\[ 1(\epsilon I(M)_n) = (-1)^m 1(\text{int } M) = (-1)^m (1(M) - 1(\partial M)) = 0 = e(M) \]
if the base point is on the boundary and
\[ 1(\epsilon I(M)_n) = (-1)^m 1(\text{int } M) = (-1)^m 1(M) = (-1)^m = e(M). \]
if the base point is in the interior. Equation (58) follows because $\epsilon I \circ \epsilon I = \mathbb{I}_L$.
The proof of (59) is just like that of (46).

In the local case we may also consider the relation of $I$ with the duality operator. From Theorem 2.16 we have
\[ D \circ I = \epsilon I \circ I. \]
When expressed in terms of the groupoid, this becomes

**Theorem 4.2.** For any multiplicative local invariant $F$ we have
\[ F \star F(e) = F(I) e \star^{-1} F \]

**Proof.** The proof is easier to read if scaling factors of $e$ are left unspecified. The desired statement is that $F \star \star e$ is the $-1$ scaling of $e \star F$.
Since $D \circ I \circ D = \epsilon I$, the invariant $F \circ D \circ I \circ D$ is the $-1$ scaling of $F \circ I$. We know that $F \circ D$ is the inverse of $F$ in the groupoid. Using this principle twice, we obtain
\[ F \circ D \circ I \circ D = (F^{-1} \circ I)^{-1}. \]
By (47)
\[ (F^{-1} \circ I)^{-1} = (e \star F^{-1})^{-1} = F \star (e^{-1}), \]
and by (38) this is $F \star \star e$. Thus $F \star \star e$ is indeed the $-1$ scaling of $F \circ I = e \star F$. □

We may also express this conclusion by the equation
\[ (61) \quad F \star \star e = F \circ \epsilon I. \]

**4.3. Duality, interior, and volume.** If $F$ is a morphism from $a$ to $0$, then Theorem 4.2 gives $F \star a e = 0 e \star^{-1} F$, equivalently $F \star a e = -1 F$, equivalently
\[ a e = F^{-1} \star^{-1} F. \]
In particular taking $F$ to be $^x U$ we have
\[ (62) \quad ^x e = ^x U \star \star ^{-x} U = T^{x,-x}. \]
Explicitly the content of the equation $T^{1,-1} = e$ is that for any conical polytope $P$
\[ U(0, P) - U(1, P) + U(2, P) - \cdots = e(P). \]
Combining this with Equation (33) we obtain:

\[ U(0, P) + U(2, P) + U(4, P) + \cdots = \frac{1 + e(P)}{2} \]

and

\[ U(1, P) + U(3, P) + U(5, P) + \cdots = \frac{1 - e(P)}{2}. \]

**Remark 4.3.** It is noteworthy that \( T_x, -x \) takes values in \( \mathbb{Z}[x] \), not merely in \( \mathbb{R}[x] \).

In the context of §3.7 this implies that \( T_x, -x \otimes 1 \) is the same as \( 1 \otimes T_x, -x \). This introduces more relations among the invariants \( T_{x, 0}, \ldots, x_r \). For example, since

\[(T^{x,y} \otimes 1) \ast (T^{y,-y} \otimes 1) \ast (1 \otimes T^{y,-z}) = (T^{x,y} \otimes 1) \ast (1 \otimes T^{y,-y}) \ast (1 \otimes T^{y,-z}),\]

we have

\[ T^{x, -y,z} = T^{x, y, z}. \]

From (60) and (59) we obtain

\[ U \circ D = U \circ I \circ D = U \circ D \circ \epsilon I, \]

whence

\[ U^* = U^* \circ \epsilon I. \]

That is, if \( M \) is a based Euclidean \( m \)-manifold then

\[ U^*(M) = (-1)^m U^*(\text{int } M). \]

If the base point is in the interior of \( M \) then \( \text{int } M \) has the same germ as \( M \) and the right hand side is \( (-1)^m U^*(M) \). We conclude that the dual volume of an odd-dimensional Euclidean manifold with interior base point is zero. Applying this to a normal cone, we obtain the following generalization:

**Theorem 4.4.** The bending \( B_\sigma(M) = U^*(\nu P \sigma) \) of a Euclidean manifold at an interior simplex of odd codimension is zero.

This has global consequences:

**Corollary 4.5.** If \( M \) is a closed \( m \)-manifold and \( m - k \) is odd, then \( V(k, M) = 0 \) and \( \alpha_k(M) = 0 \), where \( \alpha_k \) is as in the proof of Theorem 5.2.

### 4.4. Some spherical statements.

Let us rewrite some statements in spherical terms. Equations (63) and (64) applied to the cone \( CA \) on a spherical polytope \( A \) give

\[ U(0, CA) + U(2, CA) + U(4, CA) + \cdots = 1 - \frac{\chi(A)}{2} \]

and

\[ U(1, CA) + U(3, CA) + U(5, CA) + \cdots = \frac{\chi(A)}{2}. \]
Choose a spherical triangulation of $A$ and thus a conical triangulation of $CA$. The positive-dimensional conical simplices $\sigma$ of $CA$ are the cones on the spherical simplices $\tau$ of $A$. Thus by (49) \[
(−1)^n I⟨CA⟩_n = t^n - \Sigma_\tau (−1)^{|\tau|}⟨C\tau⟩_n.\]
Likewise by (7) \[
⟨CA⟩_n = (1 - \chi(A))t^n + \Sigma_\tau (1 - \chi(L(\tau, A)))⟨C\tau⟩_n.\]
If $A$ is a closed manifold of dimension $m$, then this last equation becomes \[
⟨CA⟩_n = (1 - \chi(A))t^n + \Sigma_\tau (−1)^m |\tau|⟨C\tau⟩_n.\]
This proves:

**Lemma 4.6.** If the spherical polytope $M$ is a closed combinatorial manifold of dimension $m < n$ then \[
⟨CM⟩_n + (−1)^{n−m}I⟨CM⟩_n = (1 + (−1)^m - \chi(M))t^n = (\chi(S^m) - \chi(M))t^n.\]
Note also that by (61) and (62) \[
T^{x: y} \circ I = T^{x: y} \star e = T^{x: y} \star T^{y: −y} = T^{x: −y}.\]
Equating coefficients of $x^k y^{n−k}$ in the equation $T_n^{x: y} \circ I = T_n^{x: −y}$ gives \[
(68) \quad U(k) \circ I = (−1)^k U(k),\]
which generalizes (59) and (65). This last identity has consequences when the spherical polytope is a closed manifold:

**Theorem 4.7.** If the spherical polytope $M$ is a closed odd-dimensional manifold, then $U(k, CM) = 0$ for all odd $k$. If it is a closed even-dimensional manifold, then $U(k, CM) = 0$ for all even positive $k$.

**Proof.** Evaluate $U(k)$ on all terms in (4.6) and use (68) to obtain \[
U(k, CM) + (−1)^{k+m}U(k, CM) = 0\]
for all $k > 0$. □

**Corollary 4.8.** If $P$ is the cone on a spherical polytope $M$ that is a closed even-dimensional manifold, then $U^∗(P) = 1 - \chi(M)/2$.

**Proof.** $U^∗(P)$ is $U(0, P)$, and all of the other terms in (66) are zero. □

Note that in the case when $M$ is topologically a sphere this was established in §4.3.
5. MANIFOLD ELEMENTS IN $E_n$ AND $L_n$

5.1. The subring $E^+$. Let $E_n^{ev}$ and $E_n^{od}$ be the subgroups of $E_n$ generated by closed manifolds of even and odd dimension respectively. We will see that $E$ is their direct sum and that they coincide with $\ker(I - \epsilon I)$ and $\ker(I + \epsilon I)$.

Up to factors of two, this can be proved using little more than the definition of $I$:

**Lemma 5.1.**

\[ \text{im}(I + \epsilon I) \subset E_n^{ev} \subset \ker(I - \epsilon I). \]

\[ \text{im}(I - \epsilon I) \subset E_n^{od} \subset \ker(I + \epsilon I). \]

**Proof.** If $\xi \in E_n$ is given by a closed $m$-manifold, then by (42) $\xi = (-1)^m \epsilon I(\xi)$, so that $\xi$ is in the kernel of $I - \epsilon I$ or $I + \epsilon I$ according to parity. If $\xi$ is given by an $m$-manifold $M$ with boundary then $\xi - (-1)^m \epsilon I(\xi)$ is given by the boundary $\partial M$ and $\xi + (-1)^m \epsilon I(\xi)$ is given by the double $\mathcal{D}M$. Therefore the image of $I + \epsilon I : E_n \to E_n$ is generated by certain closed even-dimensional manifold elements (the boundaries of odd-dimensional simplices and the doubles of even-dimensional simplices). Likewise, the image of $I - \epsilon I$ is generated by certain closed odd-dimensional manifold elements. \(\square\)

**Theorem 5.2.** $E_n$ is the direct sum of $\ker(I - \epsilon I)$ and $\ker(I + \epsilon I)$. The quotients $\ker(I - \epsilon I)/\text{im}(V + \epsilon I)$ and $\ker(I + \epsilon I)/\text{im}(I - \epsilon I)$ have order two and one respectively. Moreover, $\ker(I - \epsilon I) = E_n^{ev}$, $\ker(I + \epsilon I) = E_n^{od}$, and $\text{im}(I + \epsilon I) \subset E_n^{ev}$ is generated by the closed even-dimensional manifolds of even Euler characteristic.

**Proof.** The Euler characteristic splits $E_n$ as $\mathbb{Z} p^n \oplus \ker(\chi)$, and the involution $\epsilon I$ preserves this splitting. In the first summand, an infinite cyclic group, $\epsilon I$ acts trivially, so that for this summand $\ker(I - \epsilon I)/\text{im}(I + \epsilon I)$ is $\mathbb{Z}/2\mathbb{Z}$ and $\ker(I + \epsilon I)/\text{im}(I - \epsilon I)$ is trivial. By Theorem 2.5 the second summand is isomorphic to $E'_1 \oplus \cdots \oplus E'_n$ and is therefore uniquely 2-divisible (in fact, it is a real vector space), so that for this summand $\ker(I - \epsilon I)/\text{im}(I + \epsilon I)$ and $\ker(I + \epsilon I)/\text{im}(I - \epsilon I)$ are trivial. The statements about manifolds now follow using Lemma 5.1. \(\square\)

Let us look at this splitting in the light of Theorem 2.5. By Corollary 1.5 the isomorphism $E_n \to E'_0 \oplus \cdots \oplus E'_n$ of Corollary 2.6 takes $\ker(I - \epsilon I)$ and $\ker(I + \epsilon I)$ respectively into $E'_0 \oplus E'_2 \oplus \cdots$ and $E'_1 \oplus E'_3 \oplus \cdots$. Therefore under that isomorphism the involution $\epsilon I : E_n \to E_n$ corresponds to the grading involution.
It is worth restating the result in terms of \( I \) rather than \( \epsilon I \). Let \( E^+_n \subset E_n \) be \( \ker(I-I) \) and let \( E^-_n \) be \( \ker(I+I) \). By Theorem 5.2 \( E^+_n \) is generated by closed manifolds of even “codimension”, that is, of dimension \( n - 2j \leq n \), and \( E^-_n \) is generated by closed manifolds of odd codimension, that is, of dimension \( n - 1 - 2j \leq n \). Note that \( E^+ \) is a subring of \( E \). We conclude:

**Theorem 5.3.** As a module for the subring \( E^+ \), \( E \) has a basis consisting of 1 and \( p \).

**Proof.** Multiplication by \( p \) gives a group map \( E^+_{n-1} \to E^-_n \). It is injective by Theorem 2.5. It is surjective because \( n - 1 - 2j < n \). □

5.2. The boundary operator. In the local setting the analogue of “closed manifold” is “interior manifold germ”. There is an obvious analogue of Lemma 5.1: Let \( \mathcal{L}_{ev}^n \) and \( \mathcal{L}_{od}^n \) be the subgroups of \( L_n \) generated by interior manifold elements of even and odd dimension respectively.

**Lemma 5.4.**

\[
\text{im}(I + \epsilon I) \subset L_{ev}^n \subset \ker(I - \epsilon I).
\]

\[
\text{im}(I - \epsilon I) \subset L_{od}^n \subset \ker(I + \epsilon I).
\]

**Proof.** The argument is exactly as in the global case. □

Beyond this, the local picture diverges from the global. One difference is that, whereas \( E_n \) is generated by closed manifold elements, \( L_n \) is not generated by interior manifold elements:

**Proposition 5.5.** For every \( n > 0 \) the subgroup \( L_{ev}^n + L_{od}^n \) has index two in \( L_n \).

**Proof.** The subgroup contains \( 2\xi \) for every \( \xi \in L_n \), because by Lemma 5.4 it contains both \( \xi - \epsilon I(\xi) \) and \( \xi + \epsilon I(\xi) \). Therefore by Proposition 2.13 the quotient \( L_n/(M_{ev}^n + M_{od}^n) \) is killed by 2 and is generated by the point element \( t^n \) and the half-line element \( t^{n-1}d \). The former belongs to \( L_{ev}^n \). The latter does not belong to \( L_{ev}^n + L_{od}^n \) because the absolute additive local invariant \( 1 - \epsilon \) takes even values (0 or 2) on interior manifold elements and odd values (1) on boundary manifold elements. □

We will define a subring \( L^+ \subset L \) analogous to \( E^+ \) and prove a statement similar to Theorem 5.3, but by a different method. We first introduce an operator \( \delta : L_n \to L_{n-1} \).

**Remark 5.6.** There is an analogous operator \( \delta : E_n \to E_{n-1} \), but we will hardly mention it. It is easier to define than its local analogue, because the inclusion \( E_{n-1} \to E_n \) is known to be an injection. It is also less useful than its local analogue, because analogues of Equations (87) and (88) below are not available.
5.2.1. **Statements.**

**Theorem 5.7.** There is a group map \( \delta : L_n \to L_{n-1} \) such that for a based manifold, Euclidean and of dimension \( m \leq n \),

\[
\delta \langle M \rangle_n = \langle \partial M \rangle_{n-1} \text{ if } n - m \text{ is even}
\]

\[
\delta \langle M \rangle_n = \langle \mathcal{D}M \rangle_{n-1} \text{ if } n - m \text{ is odd}.
\]

One can also define \( \delta \) by the following formula, valid for any based triangulated polytope with a vertex as base point:

\[
\delta \langle P \rangle_n = \sum_{|\sigma| < n} (e(\nu_P \sigma) - (-1)^{n-|\sigma|}) \langle \sigma \rangle_{n-1},
\]

Summation is over all simplices of dimension less than \( n \) containing the base vertex.

**Remark 5.8.** Note that in the case when the base point is in the interior of \( n \) we have \( \langle \partial M \rangle_{n-1} = 0 \). By abuse of notation we are writing \( \langle \mathcal{D}M \rangle_{n-1} \) for \( 2\langle M \rangle_{n-1} \) in the same case. Thus \( \langle \mathcal{D}M \rangle_{n-1} \) means \( \langle M \rangle_{n-1} + \langle \text{int } M \rangle_{n-1} \) in all cases. We cannot say that \( \langle \partial M \rangle_{n-1} \) means \( \langle M \rangle_{n-1} - \langle \text{int } M \rangle_{n-1} \) in all cases, because \( \langle M \rangle_{n-1} \) is not defined if \( m = n \).

Before proving that \( \delta \) exists, we state some of its properties.

**Theorem 5.9.** The operator \( \delta \) satisfies

\[
\delta(\xi \eta) = \delta(\xi)\eta + I(\xi)\delta(\eta).
\]

\[
\delta \circ \delta = 0.
\]

It takes the following values on named elements:

\[
\delta(t) = 2
\]

\[
\delta(d) = 1
\]

\[
\delta(s) = 0
\]

\[
\delta(d') = -1
\]

\[
\delta(a(\theta)) = 0
\]

It also satisfies the following identities:

\[
t \delta(\xi) = \xi - I(\xi)
\]

\[
\delta(t \xi) = \xi + I(\xi)
\]

\[
I \circ \delta = \delta
\]

\[
\delta \circ I = -\delta.
\]
5.2.2. Proofs. To define the map $\delta$ we use Lemma 2.12. For an $m$-dimensional convex conical polytope $P$, define $\delta\langle P \rangle_n$ by (69)–(70). This is compatible with cutting by a hyperplane using the following result, which we leave as an exercise in inclusion-exclusion for the reader.

**Proposition 5.10.** Suppose that a Euclidean $m$-manifold $A \cup B$ is the union of Euclidean $m$-manifolds $A$ and $B$ such that $A \cap B$ is an $(m-1)$-manifold, and that a base point in $A \cap B$ is chosen. Then

$$\langle \partial A \rangle_m + \langle \partial B \rangle_m - \langle \partial (A \cup B) \rangle_m = \langle \mathcal{D}(A \cap B) \rangle_m$$

(83)

$$\langle \mathcal{D}A \rangle_m + \langle \mathcal{D}B \rangle_m - \langle \mathcal{D}(A \cup B) \rangle_m = \langle \partial (A \cap B) \rangle_m.$$  

(84)

We still have to extend (69)–(70) from convex polytopes to manifolds. In order to do so, we first prove the combinatorial formula (71).

**Lemma 5.11.** Under the hypothesis of (25), the sum

$$\sum_{|\sigma|<n}(e(\nu_M\sigma) - (-1)^{n-|\sigma|})\langle \sigma \rangle_{n-1}$$

is equal to $\langle \partial M \rangle_{n-1}$ if $n - m$ is even and to $\langle \mathcal{D}M \rangle_{n-1}$ if $n - m$ is odd. Here the summation is over all simplices of dimension less than $n$ containing the base vertex.

**Proof.** Consider the coefficient

$$e(\nu_M\sigma) - (-1)^{n-|\sigma|}$$

of a simplex $\sigma$. If $\sigma$ is in $\partial P$ then the coefficient is $(-1)^{n-1-|\sigma|}$. Thus the sum over all boundary simplices is

$$(-1)^{n-m}\sum_{\sigma \in \partial M} (-1)^{m-1-|\sigma|}\langle \sigma \rangle_{n-1} = (-1)^{n-m}\langle \partial M \rangle_{n-1},$$

by (25) applied to the interior based $(m-1)$-manifold $\partial M$. If $\sigma$ is not in $\partial P$ then the coefficient is 0 in the even case and $2(-1)^{m-|\sigma|}$ in the odd case. Thus the sum over all simplices is $\langle \partial M \rangle_{n-1}$ in the even case, and (by 25 again) in the odd case it is $2\langle M \rangle_{n-1} - \langle \partial M \rangle_{n-1} = \langle \mathcal{D}M \rangle_{n-1}$. □

In view of Lemma 5.11, Equation (69) or (70) for a given $M$ is equivalent to (71) for any given triangulation of the same $M$. For convex conical polytopes we have (69)–(70) by definition of $\delta$, and therefore we have (71). From (71) for simplices with the simplest possible triangulation we get (71) in general by the usual induction, and from this we get (69)–(70) in general.

This completes the proof of Theorem 5.7. We now prove Theorem 5.9.

For (72), it is enough to consider the case when $\xi \in E_p$ and $\eta \in E_q$ are additive generators. Suppose that they are given by manifolds $P$ and $Q$ of dimensions $p$ and $q$. Then

$$\delta(\xi\eta) = \delta[P \times Q]_{p+q} = [\partial(P \times Q)]_{p+q-1} = [\partial P]_{p-1}[Q]_q + [\text{int } P]_p[\partial Q]_{q-1} = \delta(\xi)\eta + I(\xi)\delta(\eta),$$

where $I$ is the identity map.
since \( \partial(P \times Q) \) is the union of the disjoint sets \( \partial P \times Q \) and \( \text{int} P \times \partial Q \).

Equations (74)-(78) are easy calculations using (69) and (70).

We verify Equation (79) as applied to an \( m \)-manifold element \( \langle M \rangle_n \). If \( n - m \) is even then
\[
t \delta \langle M \rangle_n = t \langle \partial M \rangle_{n-1} = \langle \partial M \rangle_n = \langle M \rangle_n - \langle \text{int} M \rangle_n = \langle M \rangle_n - I \langle M \rangle_n.
\]
If \( n - m \) is odd then
\[
t \delta \langle M \rangle_n = t \langle \partial M \rangle_{n-1} = \langle \partial M \rangle_n = \langle M \rangle_n + \langle \text{int} M \rangle_n = \langle M \rangle_n + I \langle M \rangle_n.
\]

We verify Equation (80) as applied to \( \langle M \rangle_n \). If \( n - m \) is even then
\[
\delta (t \langle M \rangle_n) = \delta \langle M \rangle_{n+1} = \langle D \rangle_n + \langle \text{int} M \rangle_n = \langle M \rangle_n + I \langle M \rangle_n
\]
If \( n - m \) is odd then
\[
\delta (t \langle M \rangle_n) = \delta \langle M \rangle_{n+1} = \langle \partial M \rangle_n = \langle M \rangle_n - \langle \text{int} M \rangle_n = \langle M \rangle_n + I \langle M \rangle_n.
\]

Remark 5.12. An analogous operator on \( E \) may be defined by the analogue of (79):
\[
 p \delta (\xi) = \xi - I(\xi).
\]

This is a legitimate definition because multiplication by \( p \) is injective and \( \xi - I(\xi) \) is always in its image. All of the properties listed above for the local \( \delta \) except that \( t \) should become \( p \), (75) through (78) should be omitted, and one can add \( \delta(q(\lambda)) = 0 \).

5.3. The subring \( L^+ \). Define
\[
 L_n^+ = \ker(\delta : L_n \to L_{n-1}).
\]

Because of the Leibniz rule (72), \( L^+ \) is a graded subring of \( L \).

Theorem 5.13. \( L^+ = \delta(L) \). As a module for \( L^+ \), \( L \) has a basis consisting of 1 and \( d \). For every \( n \) the group \( L_n^+ \) is generated by all interior manifold germs of dimension \( n - 2j \leq n \). If \( n > 0 \) then \( L_n^+ \) coincides with the image of \( 1 + I \).

Proof. \( \delta(L) \subset L^+ \) by (73). We now use the pair of equations
\[
(87) \quad \delta(d' \xi) = -\xi + d \delta(\xi)
\]
\[
(88) \quad \delta(d \xi) = \xi + d' \delta(\xi).
\]

These follow from the Leibniz rule, (77), (55), (75), and (53). The first shows that \( L^+ \subset \delta(L) \): if \( \delta(\xi) = 0 \) then \( \xi = \delta(d' \xi) \). It also shows that \( L = L^+ + dL^+ \): any element \( \xi \) can be written as \( -\delta(d' \xi) + d \delta(\xi) \). To see that 1 and \( d \) are linearly independent over \( L^+ \), use the second equation: if both \( \xi \) and \( d \xi \) are in the kernel of \( \delta \) then \( \xi = 0 \). That
every interior manifold germ of even codimension is in \( \text{ker}(\delta) \) follows from (69). That \( \text{im}(\delta) \) is generated by some such germs (boundaries and doubles) follows from (69)-(70). For the final statement, note that Proposition 2.13 implies that any element of \( L_{n+1} \) has the form \( 2\xi + t\eta \), and observe that

\[
\delta(2\xi + t\eta) = (\mathbb{1} + I)(\delta\xi + \eta)
\]

by (81) and (74).

\(\Box\)

**Remark 5.14.** Ultimately this proof works because \( \delta(d) = 1 \). This reflects the fact that locally a point is a boundary.

**Remark 5.15.** The unique monic degree two equation satisfied by \( d \) over \( L^+ \) is

\[
d^2 - sd + a\left(\frac{\pi}{2}\right) = 0.
\]

The other root of the same polynomial is \( d' \).

**Remark 5.16.** Both in \( E_n \) and in \( L_n \) we have the following relations between subgroups:

(89) \( \text{im}(\mathbb{1} + I) \subset \text{im}(\delta) \subset \text{ker}(\delta) \subset \text{ker}(\mathbb{1} - I) \),

where the quotient \( \text{ker}(\mathbb{1} - I)/\text{im}(\mathbb{1} + I) \) is obviously killed by 2. Let us summarize the more detailed information that we have in the two cases.

For \( E_n \) the quotient \( \text{ker}(\mathbb{1} - I)/\text{im}(\mathbb{1} + I) \) is trivial if \( n \) is odd and of order two if \( n \) is even. The inclusion that is not strict in the even case is the second one. The nontrivial element of \( \text{ker}(\delta)/\text{im}(\delta) \) is detected by mod 2 Euler characteristic and represented by the point element \( t^n \). The group \( E_n^+ = \text{ker}(\delta) = \text{ker}(\mathbb{1} - I) \) is \( E_n^e \) if \( n \) is even and \( E_n^o \) if \( n \) is odd.

For \( L_n \) the second inclusion is an equality, as well as the first if \( n > 0 \). The group \( L_n^+ = \text{im}(\delta) = \text{ker}(\delta) \) is \( L_n^e \) if \( n \) is even and \( L_n^o \) if \( n \) is odd. It is not clear whether this is the same as \( \text{ker}(\mathbb{1} - I) \). It is the same if the “inclusion” \( L_{n-1} \to L_n \) is injective.

5.4. **The subring** \( D(L^+) \). Because \( D \) is a ring involution, \( D(L^+) \) is also a graded subring of \( L \) and is the kernel of the operator

\[
\bar{\delta} = D \circ \delta \circ D : L_n \to L_{n-1}.
\]

Because \( Dt = s \) and \( D \circ I \circ D = \epsilon I \), \( \bar{\delta} \) satisfies

\[
\begin{align*}
    s\bar{\delta}(\xi) &= \xi - \epsilon I(\xi) \\
    \bar{\delta}(s\xi) &= \xi + \epsilon I(\xi).
\end{align*}
\]

**Theorem 5.17.** \( D(L^+) = \bar{\delta}(L) \). As a module for \( D(L^+) \), \( L \) has a basis consisting of 1 and \( d \). For every \( n \) the group \( D(L_n^+) \) is generated by all interior manifold germs of even dimension \( 2j \leq n \). If \( n > 0 \) then \( D(L_n^+) \) coincides with the image of \( \mathbb{1} + \epsilon I \).
Proof. Except for the statement about manifolds, all of this follows directly from Theorem 5.13 (Use the fact that $Dd = d$.)

Note that

\[
D(L^+_{2k}) = \text{im}(\mathbb{I} + \epsilon I) = \text{im}(\mathbb{I} + I) \\
D(L^+_{2k+1}) = \text{im}(\mathbb{I} + \epsilon I) = \text{im}(\mathbb{I} - I).
\]

\[
L^+_{2k} = L^{ev}_{2k} \\
L^+_{2k+1} = L^{od}_{2k+1}.
\]

It follows that

\[
tL^+_{2k} = L^{ev}_{2k+1} \\
tL^+_{2k+1} = L^{od}_{2k+2},
\]

simply because any even (resp. odd) dimension less than or equal to $2k + 1$ (resp. $2k + 2$) is less than or equal to $2k$ (resp. $2k + 1$). We have to show that

\[
D(L^+_{2k}) = L^{ev}_{2k} \\
D(L^+_{2k+1}) = L^{ev}_{2k+1}.
\]

For the first of these,

\[
D(L^+_{2k}) = \text{im}(\mathbb{I} + I) = L^+_{2k} = L^{ev}_{2k}.
\]

For the second,

\[
D(L^+_{2k+1}) = \text{im}(\mathbb{I} + \epsilon I) \subset L^{ev}_{2k+1}
\]

by Lemma 5.4, and

\[
L^{ev}_{2k+1} = tL^+_{2k} = (Ds)D(L^+_{2k}) = D(sL^+_{2k}) \subset D(L^+_{2k+1})
\]

since $s \in L^+_1$. \qed

Corollary 5.18. $L^+_{2k+1} = sL^+_2$

Let us write

\[
\Theta_{2k} = L^+_{2k} = M^{ev}_{2k} \\
\Theta_{2k+1} = 0.
\]

Then $\Theta$ is a graded subring contained in $L^+ \cap D(L^+)$, and potentially equal to it. Corollary 5.18 says that $L^+$ is generated as a $\Theta$-module by 1 and $s$. By Theorem 5.13 it follows that $L$ is generated as a $\Theta$-module by 1, $s$, $d$, and $ds$. 
5.5. **Possible 2-torsion in $L$.** Except for some small values of $n$, we do not know whether $L_n$ has any 2-torsion (or indeed any torsion at all). This is unfortunate, because the following types of elements of $L_n$ are all killed by 2:

- Any element of $\text{ker}(\mathbb{I} - I) \cap \text{ker}(\mathbb{I} + I)$.
- Any element of $L_{2k+1}^+ \cap D(L_{2k+1})$. In fact, this is contained in $\text{ker}(\mathbb{I} - I) \cap \text{ker}(\mathbb{I} + I)$.
- Any $\xi \in L_n^+$ such that $t\xi = 0$. Indeed, for such an element $0 = \delta(t\xi) = \xi + I(\xi) = 2\xi$.
- Any $\xi \in L_{2k}^+$ such that $s\xi = 0$. Indeed, for such an element $D\xi \in L_{2k}^+$ and $tD\xi = 0$.
- $\delta(\xi)$ if $I(\xi) = \xi$. Indeed, in that case $2\delta(\xi) = \delta(\xi + I(\xi)) = 0$.

Therefore, if it happens that $L$ has no 2-torsion, the following statements are true:

- $\Theta = L^+ \cap D(L^+)$
- As a $\Theta$-module, $L^+$ has a basis consisting of 1 and $s$.
- As a $\Theta$-module, $L$ has a basis consisting of 1, $s$, $d$, and $ds$.
- $\text{ker}(\delta) = \text{ker}(\mathbb{I} - I)$ in $L$.
- In $L_{2k+1}$ the subgroup generated by all interior manifold elements is the direct sum of
  \[
  L_{2k+1}^{ev} = t\Theta_{2k} = L_{2k+1}^+ = \text{ker}(\delta) = \text{ker}(\mathbb{I} - I)
  \]
  and
  \[
  L_{2k+1}^{od} = s\Theta_{2k} = D(L_{2k+1}^+) = \text{ker}(\bar{\delta}) = \text{ker}(\mathbb{I} + I).
  \]
- In $L_{2k}$ the subgroup generated by all interior manifold elements is the direct sum of
  \[
  L_{2k}^{ev} = \Theta_{2k} = L_{2k}^+ = \text{ker}(\delta) = \text{ker}(\bar{\delta}) = \text{ker}(\mathbb{I} - I)
  \]
  and
  \[
  L_{2k}^{od} = st\Theta_{2k-2} = \text{ker}(\mathbb{I} + I).
  \]

We repeat that these statements rely on the assumption that $L$ has no 2-torsion.

For what it is worth, these statements become true if $L$ is replaced by $L/J$, where $J$ is the ideal of 2-torsion elements.

6. **Questions and other unfinished business**

Here are some fundamental questions, closely related to fundamental questions about spherical scissors congruence: Is the “inclusion” map $L_{n-1} \to L_n$ injective? Is $L_n$ torsion free? Is it even free of 2-torsion? Do the generalized Dehn invariants of \[3.7\] detect all of $L_n$?

It appears that the subring $\Theta$ is the most important part of $L$. In a future paper we plan to show that ring maps $F : \Theta \to k$ are the objects of a groupoid $\mathcal{G}^+(k)$.
whose objects are again the elements of $k$; the source and target of $F$ are $F(s^2)$ and $F(t^2)$. In some sense this groupoid scheme is obtained by dividing $G$ by the “central involution” given by $e$ and its scalings. In order to get a geometric understanding of what such a map $F$ is an invariant of, it is worth giving an explicit presentation of $\Theta_{2m}$. We may take the generators to be manifold germs without boundary, of even dimension and codimension. It is interesting to work out a set of generating relations with a manifold flavor.

There may also be a useful relation with differential geometry. One may imagine interpolating between the usual Riemannian manifolds and our “piecewise Euclidean” manifolds by considering piecewise Riemannian manifolds as a common generalization. For example, a smooth simplex in a piecewise Riemannian manifold has a smoothly varying normal cone. Again, we hope to return to this theme.

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