CHARACTERIZATIONS OF PRODUCT HARDY SPACES ON STRATIFIED GROUPS BY SINGULAR INTEGRALS AND MAXIMAL FUNCTIONS

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Abstract. A large part of the theory of Hardy spaces on products of Euclidean spaces has been extended to the setting of products of stratified Lie groups. This includes characterisation of $H^1$ by square functions and by atomic decompositions, proof of the duality of $H^1$ with $\text{BMO}$, and description of many interpolation spaces. Until now, however, two aspects of the classical theory have been conspicuously absent: the characterisation of $H^1$ by singular integrals (of Christ–Geller type) or by (vertical or nontangential) maximal functions. In this paper we fill in these gaps by developing new techniques on products of stratified groups, using the ideas in [4] on the Heisenberg group with flag structure.

1. Introduction and statement of main results

Hardy spaces first appeared in the study of boundary behaviour of holomorphic functions on the disc and upper half plane. The modern theory of Hardy spaces began in 1960, when E. M. Stein and G. Weiss [37] considered functions defined on $\mathbb{R}^n \times \mathbb{R}^+$, and it took off in the early 1970s, with the remarkable work of C. Fefferman and Stein [12] and then R. R. Coifman and Weiss [7]. Much of this theory has been extended to more general spaces of homogeneous type, in the sense of Coifman and Weiss [6, 7]. In the late 1970s, G. B. Folland and Stein [17] characterised the Hardy space $H^1(G)$ on a stratified group $G$ in terms of atomic decompositions, square functions, area functions, and maximal functions. The area integrals and maximal functions involve taking integrals or suprema over cones in $G \times \mathbb{R}^+$. Soon after, M. Christ and D. Geller [11] showed that there are singular integral operators $R_0, \ldots, R_n$ on a stratified Lie group such that $f \in H^1(G)$ if and only if all $R_j f \in L^1(G)$. Here $R_0$ is the identity operator and the other $R_j$ are Riesz transformations, that is, convolutions with derivatives of a potential.

Harmonic analysis on product spaces $\mathbb{R}^m \times \mathbb{R}^n$ was born in the late 1970s and studied extensively in the 1980s, in particular by S.-Y. A. Chang, R. Fefferman, R. F. Gundy, J.-L. Journé, J. Pipher, and Stein (see [3, 14, 21, 29, 31, 34]), motivated by problems on the boundary behavior of holomorphic functions in several complex variables, which require consideration of approach regions that behave differently in different variables. Harmonic analysis on product spaces is influenced by classical harmonic analysis, but is different in that the different factors in the product may be dilated independently. The terms one-parameter and multiparameter are often used to highlight the different structures of the dilations considered. As in classical harmonic analysis, an important part of the theory is the development of Hardy and BMO spaces, their duality and the connections to atomic decompositions. A key ingredient is Journé’s covering lemma, which provides a tool to replace general open sets by rectangles with controlled geometry.
Since the 1980s, the development of multiparameter harmonic analysis proceeded apace; recent contributions in the area include [16, 28, 30, 33, 32]. Much of the product space theory on \( \mathbb{R}^m \times \mathbb{R}^n \) has been extended to more general product spaces, including the duality of \( \text{H}^1 \) with BMO, characterisation of \( \text{H}^1 \) by square functions and atomic decompositions, and description of various interpolation spaces. In [31, 22, 23, 24], the theory of Hardy spaces \( H^p \), for \( p \) less than and close to 1, has been developed on products \( X_1 \times X_2 \) of spaces of homogeneous type. Hence on products \( G_1 \times G_2 \) of stratified Lie groups, there is already a well-defined Hardy space \( \text{H}^1(G_1 \times G_2) \) that may be characterised by atomic decompositions and by square or area functions.

Two aspects of the classical theory that have been conspicuous by their absence until now are a singular integral characterisation of Christ–Geller type and a maximal function characterisation. The main difficulty is that the geometrical structure is harder to handle than in the one-parameter case. For example, one may obtain the atomic decomposition from the nontangential maximal function in the one-parameter case by using the classical Calderón–Zygmund and Whitney decompositions involving cubes, but these decompositions are absent in the multiparameter case.

This paper fills these gaps for products of stratified Lie groups, with Theorems 1.1 and 1.3 below. For simplicity, and because new methods would otherwise be needed, we consider products of only two groups. Our new techniques come from [4], where similar results are proved on the Heisenberg group with its flag structure.

Unexplained definitions may be found below.

**Theorem 1.1.** The double Riesz transformations \( \mathcal{R}^{[1]}_{j_1} \otimes \mathcal{R}^{[2]}_{j_2} \) characterise the Hardy space \( \text{H}^1(G_1 \times G_2) \). That is, \( f \in \text{H}^1(G_1 \times G_2) \) if and only if each \( \mathcal{R}^{[1]}_{j_1} \otimes \mathcal{R}^{[2]}_{j_2} f \) is in \( L^1(G_1 \times G_2) \), and moreover

\[
\|f\|_{\text{H}^1(G_1 \times G_2)} \approx \sum_{j_1=0}^{d_1} \sum_{j_2=0}^{d_2} \left\| \mathcal{R}^{[1]}_{j_1} \otimes \mathcal{R}^{[2]}_{j_2} f \right\|_{L^1(G_1 \times G_2)}.
\]

Using Theorem 1.1 and the \( \text{H}^1\)-BMO duality (see for example [22]), we obtain a decomposition of functions in the product space \( \text{BMO}(G_1 \times G_2) \).

**Corollary 1.2.** For a function \( u \) on \( G_1 \times G_2 \), the following are equivalent:

(a) \( u \in \text{BMO}(G_1 \times G_2) \);

(b) there exist \( g_{j_1,j_2} \in L^\infty(G_1 \times G_2) \) such that \( u = \sum_{j_1=0}^{d_1} \sum_{j_2=0}^{d_2} \mathcal{R}^{[1]}_{j_1} \otimes \mathcal{R}^{[2]}_{j_2}(g_{j_1,j_2}) \).

Write \( \Gamma(g_1, g_2) \) for the product \( \Gamma_1(g_1) \times \Gamma_2(g_2) \) of the cones treated by Folland and Stein [21], and for suitable functions \( \psi^{[i]}_t \) on \( G_t \), define the nontangential maximal function:

\[
\mathcal{N}_\psi(f)(g_1, g_2) := \sup \left\{ \left| f \ast (\psi^{[1]}_{t_1} \otimes \psi^{[2]}_{t_2})(h_1, h_2) \right| : (h_1, h_2) \in \Gamma(g_1, g_2), t_1, t_2 \in \mathbb{R}_+ \right\},
\]

where \( \psi^{[i]}_t \) is a normalised dilate of \( \psi^{[i]} \).

**Theorem 1.3.** The nontangential maximal operator \( \mathcal{N}_\psi \) characterises the Hardy space \( \text{H}^1(G_1 \times G_2) \). That is, \( f \in \text{H}^1(G_1 \times G_2) \) if and only if \( \mathcal{N}_\psi f \) is in \( L^1(G_1 \times G_2) \); moreover

\[
\|f\|_{\text{H}^1(G_1 \times G_2)} \approx \left\| \mathcal{N}_\psi(f) \right\|_{L^1(G_1 \times G_2)}.
\]
This paper is organised as follows. In Section 2, we remind the reader of some background on stratified Lie groups and analysis thereupon, and introduce some notation to simplify the formulae in the case of products of such groups. In Section 3, we review some of the main results on Hardy spaces on products of stratified groups; many of these are valid in the more general context of products of spaces of homogeneous type. Then we prove our main theorems on the characterisations of $H^1(G_1 \times G_2)$, by Riesz transforms in Sections 4 and by maximal functions in Section 5. The results proved are actually somewhat more general than stated in Theorems 1.1 and 1.3, but precise statements require more notation than we have established at this point.

"Constants" are always positive real numbers; we write $A \lesssim B$ when there is a constant $C$ such that $A \leq CB$, and $A \asymp B$ when $A \lesssim B$ and $B \lesssim A$. We denote the identity of a group by $o$, and the indicator function of a set $E$ by $\chi_E$.

2. Preliminaries

2.1. Stratified nilpotent Lie groups. Let $G$ be a (real and finite dimensional) stratified nilpotent Lie group of step $k$ with Lie algebra $\mathfrak{g}$. This means that we may write $\mathfrak{g}$ as a vector space direct sum $\mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_k$, where $[\mathfrak{v}_1, \mathfrak{v}_j] = \mathfrak{v}_{j+1}$ when $1 \leq j \leq k$; here $\mathfrak{v}_{k+1} = \{0\}$. Let $Q$ denote the homogeneous dimension $\sum_{j=1}^k \dim \mathfrak{v}_j$ of $G$.

There is a one-parameter family of automorphic dilations $\delta_t$ on $\mathfrak{g}$, given by $$\delta_t(X_1 + X_2 + \cdots + X_k) = tX_1 + t^2X_2 + \cdots + t^kX_k;$$ here each $X_j \in \mathfrak{v}_j$ and $t > 0$. The exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism, and we identify $\mathfrak{g}$ and $G$. The dilations extend to automorphic dilations of $G$, also denoted by $\delta_t$, by conjugation with $\exp$. The natural bi-invariant Haar measure on $G$ is the Lebesgue measure on $\mathfrak{g}$, lifted to $G$ using $\exp$.

By [25], the group $G$ may be equipped with a smooth subadditive homogeneous norm $\rho$, a continuous function from $G$ to $[0, \infty)$ that is smooth on $G \setminus \{o\}$ and satisfies

(a) $\rho(g^{-1}) = \rho(g)$;
(b) $\rho(xy) \leq \rho(x) + \rho(y)$
(c) $\rho(\delta_t(g)) = t\rho(g)$ for all $g \in G$ and $t > 0$;
(d) $\rho(g) = 0$ if and only if $g = o$.

Abusing notation, we define $\rho(g, g') = \rho(g^{-1}g')$ for all $g, g' \in G$; this defines a metric on $G$. We write $B(g, r)$ for the open metric ball with centre $g$ and radius $r$ with respect to $\rho$: $$B(g, r) = gB(o, r) = g\{h \in G : \rho(h) < 1\}.$$ The metric space $(G, \rho)$ is geometrically doubling; that is, there exists $N \in \mathbb{N}$ such that every metric ball $B(x, 2r)$ may be covered by at most $N$ balls of radius $r$.

We remind the reader that a stratified Lie group is a space of homogenous type in the sense of Coifman and Weiss [6, 7], and analysis on stratified Lie groups uses much from the theory of such spaces. In particular, we frequently deal with molecules, that is, functions $\psi$ that satisfy standard decay and smoothness conditions, meaning that there is a parameter $\varepsilon \in (0, 1]$, which we fix once and for all, and a constant $C$ such that

$$|\psi(g)| \leq C \frac{1}{(1 + \rho(g))^{Q+\varepsilon}}$$

(1) $$|\psi(g) - \psi(g')| \leq C \frac{\rho(g^{-1}g')^{\varepsilon}}{(1 + \rho(g) + \rho(g'))^{Q+2\varepsilon}}$$
for all $g, g' \in G$. We often impose an additional cancellation condition, namely

$$\int_G \psi(g) \, dg = 0.$$  

We write $\|\psi\|_{M(G)}$ for the least constant $C$ such that the conditions (1) hold, $M(G)$ for the Banach space of all such functions $\psi$, and $M_0(G)$ for the subspace of $M(G)$ of all $\psi$ that also satisfy condition (2).

The normalised dilate $f_t$ of a function $f$ on $G$ by $t > 0$ is given by $f_t := t^{-Q} f \circ \delta_{1/t}$, and the convolution $f * f'$ of measurable functions $f$ and $f'$ on $G$ is defined by

$$f * f'(g) = \int_G f(h) f'(h^{-1} g) \, dh = \int_G f(gh^{-1}) f'(h) \, dh.$$  

Take left-invariant vector fields $\mathcal{X}_1, \ldots, \mathcal{X}_n$ on $G$ that form a basis of $\mathfrak{g}_1$, and define the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (\mathcal{X}_j)^2$. Observe that each $\mathcal{X}_j$ is homogeneous of degree 1 and $\mathcal{L}$ is homogeneous of degree 2, in the sense that

$$\mathcal{X}_j (f \circ \delta_t) = t (\mathcal{X}_j f) \circ \delta_t$$  

and

$$\mathcal{L} (f \circ \delta_t) = t^2 (\mathcal{L} f) \circ \delta_t$$  

for all $t > 0$ and all $f \in C^2(G)$.

Associated to the sub-Laplacian, there are various Riesz potential operators $\mathcal{L}^{-\alpha}$, where $\alpha > 0$; these are convolution operators with homogeneous kernels—see Folland [13]. The Riesz transformation $\mathcal{R}_j := \mathcal{X}_j \mathcal{L}^{-1/2}$ is a singular integral operator, and is bounded on $L^p(G)$ when $1 < p < \infty$ as well as from the Folland–Stein Hardy space $H^1(G)$ to $L^1(G)$. We define $\mathcal{R}_0$ to be the identity operator $\mathcal{I}$.

The Hardy–Littlewood maximal operator $\mathcal{M}$ on $G$ is defined using the metric balls:

$$\mathcal{M} f(g) := \sup \left\{ \frac{1}{|B(g', r)|} \int_{B(g', r)} |f(g'')| \, dg'' : g \in B(g', r) \right\}.$$  

For future use, we note that the layer cake formula implies that, if $\mu$ is a radial decreasing function on $G$ (that is, $\mu(g)$ depends only on $\rho(g)$ and decreases as $\rho(g)$ increases), then

$$|f| \ast \mu_\varepsilon(g) \leq \|\mu\|_{L^1(G)} \mathcal{M} f(g) \quad \forall g \in G.$$  

2.2. Functional calculus for the sub-Laplacian. The sub-Laplacian $\mathcal{L}$ has a spectral resolution:

$$\mathcal{L}(f) = \int_0^\infty \lambda \, d\mathcal{E}_\mathcal{L}(\lambda) f \quad \forall f \in L^2(G),$$  

where $\mathcal{E}_\mathcal{L}(\lambda)$ is a projection-valued measure supported on $[0, \infty)$, the spectrum of $\mathcal{L}$. For a bounded Borel function $\eta : [0, \infty) \to \mathbb{C}$, we define the operator $F(\mathcal{L})$ spectrally:

$$\eta(\mathcal{L}) f = \int_0^\infty \eta(\lambda) \, d\mathcal{E}_\mathcal{L}(\lambda) f \quad \forall f \in L^2(G).$$  

This operator is a convolution with a Schwartz distribution on $G$.

Take a smooth function $\eta : \mathbb{R}_+ \to \mathbb{R}$, supported in $[1/2, 2]$, such that $\sum_{n \in \mathbb{Z}} \eta(2^{-n} s) = 1$ for all $s \in \mathbb{R}_+$. The convolution kernels $k_{\eta}(\mathcal{L}_i)$ of the operators $\eta(\mathcal{L}_i)$ on $G$ are Schwartz functions, by [20]. Moreover, we may write $\eta(t \mathcal{L}_i) = t \mathcal{L}_i \psi(t \mathcal{L}_i)$, where $\psi(t) := t^{-1} \eta(t)$ for all $t \in \mathbb{R}_+$ and $\text{supp } \psi \subset [1/2, 2]$, and deduce that

$$k_{\eta(t \mathcal{L}_i)} = t \mathcal{L}_i k_{\psi(t \mathcal{L}_i)}.$$
Integration by parts now implies that
\[
\int_G k_{0(t\mathcal{L}_1)}(g) \, dg = \int_G t\mathcal{L}_1 k_{\psi(t\mathcal{L}_1)}(g) \, dg = 0.
\]

2.3. The heat and Poisson kernels. Let \( p_t \) and \( P_t \), where \( t > 0 \), be the heat and Poisson kernels associated to the sub-Laplacian operator \( \mathcal{L} \), that is, the convolution kernels of the operators \( e^{t\mathcal{L}} \) and \( e^{t\sqrt{\mathcal{L}}} \) on \( G \). We write \( Q_t \) for \( t\partial_t P_t \), and to simplify notation later, we often write \( P \) instead of \( P_1 \). We warn the reader that \( P_t \) and \( Q_t \) are the normalised dilates of \( P_1 \) and \( Q_1 \) by the factor \( t \), but \( p_t \) is the normalised dilate of \( p_1 \) by a factor of \( t^{1/2} \). Let \( \nabla \) denote the subgradient on \( G \) and \( \nabla \) denote the gradient \((\nabla, \partial_t)\) on \( G \times \mathbb{R}_+ \).

**Lemma 2.1.** The kernels \( p_t \) and \( P_t \) are \( \mathbb{R}_+ \)-valued. Further, \( p_t \) and \( P_t \) have integral 1, while \( Q_t \) has integral 0 for all \( t \in \mathbb{R}_+ \). Finally, there exists a constant \( c \) such that
\[
\begin{align*}
p_t(g) &\leq t^{-Q/2} \exp \left( -\rho^2(g)/ct \right) \\
|\nabla p_t(g)| &\leq t^{-(Q+1)/2} \exp \left( -\rho^2(g)/ct \right) \\
P_t(g) &\approx \frac{t}{(t^2 + \rho(g)^2)^{(Q+1)/2}} \\
|\nabla P_t(g)| &\leq \frac{t}{(t^2 + \rho(g)^2)^{(Q+2)/2}}
\end{align*}
\]
for all \( g \in G \) and \( t \in \mathbb{R}_+ \).

**Proof.** For the heat kernel estimates, see [39, Theorem IV.4.2]. Note that there is a version of the first estimate with the opposite inequality and a different constant \( c \).

The estimates for \( P_t \) and \( Q_t \) follow from the subordination formula
\[
e^{-t\sqrt{\mathcal{L}}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{te^{-t^2/4v}}{\sqrt{v}} e^{-\psi dv} \, dv.
\]
For the case of the Heisenberg group, much of this is worked out in detail in [4]. \( \square \)

This lemma implies that the heat kernel \( p_1 \) and the Poisson kernel \( P_1 \) (and their derivatives) both satisfy the standard decay and smoothness conditions [1]; the derivatives also satisfy the cancellation condition [2].

Lemma 2.1 also implies the following standard corollary, whose proof we omit.

**Corollary 2.2.** Suppose that \( f \in L^p(G) \), where \( 1 \leq p \leq \infty \). Then \( \|f \ast P_t\|_{L^p(G)} \) and \( \|f \ast t\nabla P_t\|_{L^p(G)} \) are uniformly bounded as \( t \) runs over \( \mathbb{R}_+ \). Further,
\[
\lim_{t \to 0} f \ast P_t = f;
\]
the convergence is both pointwise almost everywhere, and in the \( L^p(G) \) norm if \( 1 \leq p < \infty \) and in the weak-star topology if \( p = \infty \). Finally,
\[
\lim_{t \to 0} f \ast t\nabla P_t = 0;
\]
the convergence is both pointwise almost everywhere, and in the strong operator topology if \( f \in L^1(G) \), in the \( L^p(G) \) norm if \( 1 < p < \infty \) and in the weak-star topology if \( p = \infty \).
2.4. Systems of pseudodyadic cubes. We use the Hytönen–Kairema [27] families of “dyadic cubes” in geometrically doubling metric spaces. We state a version of [27] Theorem 2.2 that is simpler, in that we work on well-behaved metric spaces rather than general pseudometric spaces. The Hytönen–Kairema construction builds on seminal work of Christ [10] and of Sawyer and Wheeden [35].

Theorem 2.3 ([27]). Let \((G, \rho)\) be a metric stratified group and \(c_0, C_0\) and \(\kappa\) constants such that \(0 < c_0 \leq C_0 < \infty\) and \(12C_0\kappa \leq c_0\). Then for all \(k \in \mathbb{Z}\), there exist families \(\mathcal{D}^k(G)\) of pseudodyadic cubes \(Q\) with centres \(z(Q)\), such that:

(a) \(G\) is the disjoint union of all \(Q \in \mathcal{D}^k(G)\), for each \(k \in \mathbb{Z}\);
(b) \(B(z(Q), c_0\kappa^k/3) \subseteq Q \subseteq B(z(Q), 2C_0\kappa^k)\) for all \(Q \in \mathcal{D}^k(G)\);
(c) if \(Q \in \mathcal{D}^k(G)\) and \(Q' \in \mathcal{D}^{k'}(G)\) where \(k \leq k'\), then either \(Q \cap Q' = \emptyset\) or \(Q \subseteq Q'\); in the second case, \(B(z(Q), 2C_0\kappa^k) \subseteq B(z(Q'), 2C_0\kappa^{k'})\).

The family of pseudodyadic cubes \(Q\) in \(\mathcal{D}^k(G)\), where \(k \in \mathbb{Z}\), of Theorem 2.3 will be called a Hytönen–Kairema set of cubes on \(G\). We write \(\mathcal{D}(G)\) for the union of all \(\mathcal{D}^k(G)\). Given a cube \(Q \in \mathcal{D}^k(G)\), we denote the quantity \(\kappa^k\) by \(\ell(Q)\), by analogy with the side-length of a Euclidean cube.

2.5. Products of stratified groups. We equip products of stratified groups \(G_1\) and \(G_2\) with a product structure: the basic geometric objects are rectangles, which are products of balls, and pseudodyadic rectangles, which are products of pseudodyadic cubes. We write \(\mathcal{P}^i(G)\) for the collection of all pseudodyadic rectangles that are products of cubes in \(\mathcal{D}^i(G_1)\) and in \(\mathcal{D}^j(G_2)\); \(\mathcal{P}(G)\) for the collection of all pseudodyadic rectangles, and \(\mathcal{R}(G)\) for the collection of all rectangles. We let \(\ell : \mathcal{P}(G) \to T\) be the function such that \(\ell_i(Q_1 \times Q_2) = \ell(Q_1)\), the “side-length” of \(Q_i\).

We carry forward the notation from Section 2.1, modified by adding a subscript \(i\) or superscript \(i\) to clarify that we are dealing with \(G_i\). To shorten the formulae, we often use bold face type to indicate a product object: thus we write \(G, g, r\) and \(t\) in place of \(G_1 \times G_2, (g_1, g_2), (r_1, r_2)\) and \((t_1, t_2)\). For example, \(B_i(g_i, r_i)\) denotes the open ball on \(G_i\) with centre \(g_i\) and radius \(r_i\), with respect to the homogeneous norm \(\rho_i\), and a typical rectangle \(R(g, r)\) is then a product \(B_1(g_1, r_1) \times B_2(g_2, r_2)\). We also write \(t \, dt\) in place of \(t_1 \, dt_1 \wedge dt_2\), and \(T\) for the product parameter space \(\mathbb{R}_+ \times \mathbb{R}_+\).

The element of Haar measure on \(G\) is denoted \(dg\), but may be written as \(dg_1 \, dg_2\) for calculations. The convolution \(f * f'\) of functions \(f\) and \(f'\) on \(G\) is defined by

\[(f * f')(g) := \int_G f(h) f'(h^{-1}g) \, dh.\]

We define the strong maximal operator \(\mathcal{M}_S\) by

\[\mathcal{M}_S(f)(g) := \sup \left\{ \frac{1}{|R|} \int_R |f(h)| \, dh : R \ni g, R \in \mathcal{R}(G) \right\}.\]

It is a straightforward exercise to show that \(\mathcal{M}_S\) is dominated by the iterated Hardy–Littlewood maximal operators in the factors:

\[\mathcal{M}_S f \leq \mathcal{M}_1 \mathcal{M}_2(f) \quad \text{and} \quad \mathcal{M}_S f \leq \mathcal{M}_2 \mathcal{M}_1(f) \quad \forall f \in L^1_{\text{loc}}(G).\]

When \(1 < p \leq \infty\), the operators \(\mathcal{M}_1\) and \(\mathcal{M}_2\) in the factors are \(L^p\)-bounded, so the iterated maximal operators and the strong maximal operator are also \(L^p\)-bounded.

Given functions \(\psi^{[1]}\) on \(G_1\) and \(\psi^{[2]}\) on \(G_2\), we often deal with the product of their normalised dilates on \(G_1 \times G_2\), and we abbreviate this to \(\psi_t\):

\[\psi_t := \psi^{[1]}_t \otimes \psi^{[2]}_t.\]
If \( \psi^{[1]} \in \mathcal{M}(G_1) \) and \( \psi^{[2]} \in \mathcal{M}(G_2) \), then
\[
|f \ast \psi_t(g)| \lesssim \mathcal{M}_S(f)(g) \quad \forall g \in G \quad \forall f \in L^1(G),
\]
much as argued to prove (3), but with “biradial” in place of “radial”.

Given an open subset \( U \) of \( G \) with finite measure \(|U|\), we define the enlargement \( \widetilde{U} \) of \( U \) using the strong maximal operator \( \mathcal{M}_S \):
\[
\widetilde{U} := \{ g \in G : \mathcal{M}_S \chi_U(g) > \frac{1}{4} \}.
\]
We write \( \mathcal{M}(U) \) for the family of maximal pseudodyadic rectangles contained in \( U \).

We let \( P_t := P^{[1]}_{t_1} \otimes P^{[2]}_{t_2} \), when \( t_1 = 0 \) or \( t_2 = 0 \), we interpret this as a distribution supported in \( G_2 \) or in \( G_1 \) in the obvious way. We write \( Q^{[i]}_{t_i} \) for the convolution kernel of the operator \( t_i \partial_i e^{-t_i \sqrt{L_i}} \); then \( Q^{[i]}_{t_i} = t_i \partial_i P^{[i]}_{t_i} \). By arguing as in Corollary 2.2, it is easy to see that for any measurable subset \( V \) of \( G \),
\[
\lim_{t_i \to 0} \chi_V * (Q^{[1]}_{t_1} \otimes P^{[2]}_{t_2})(g) = 0
\]
for almost all \( g \) in \( G \) and in the weak-star topology of \( L^\infty(G) \).

The double Riesz transforms \( R^{[1]}_{j_1} \otimes R^{[2]}_{j_2} f \), where \( 0 \leq j_i \leq d_i \), of a suitable function \( f \) on \( G \) are defined in the obvious way: when \( j_1 \) and \( j_2 \) are nonzero,
\[
R^{[1]}_{j_1} \otimes R^{[2]}_{j_2} f := X^{[1]}_{j_1} \mathcal{L}_{1}^{-1/2} X^{[2]}_{j_2} \mathcal{L}_{2}^{-1/2} f,
\]
and if \( j_i = 0 \) we replace \( X^{[i]}_{j_i} \mathcal{L}_{i}^{-1/2} \) by the identity operator \( \mathcal{I}_i \).

3. The known product Hardy spaces

3.1. The atomic Hardy space. Fix a constant \( C \) and Hytönen–Kairema sets of pseudodyadic cubes in \( G_1 \) and \( G_2 \). A pseudodyadic rectangle \( R \) is a product \( Q_1 \times Q_2 \) of pseudodyadic cubes in the factors \( G_1 \) and \( G_2 \).

An integrable function \( a_R \) is said to be a particle associated to the pseudodyadic rectangle \( R \) if the following support and product cancellation conditions hold:
\[
supp a_R \subseteq CR \quad \text{and} \quad \int_{G_1} a_R(g_1, \cdot) \, dg_1 = 0 \quad \text{and} \quad \int_{G_2} a_R(\cdot, g_2) \, dg_2 = 0
\]
(almost everywhere).

A function \( a \) on \( G \) is said to be a product atom associated to an open subset \( U \) of \( G \) of finite measure if \( a \) satisfies the following support and size conditions:
\[
supp a \subset \widetilde{U} \quad \|a\|_{L^2(G)} \leq \|\widetilde{U}\|^{-1/2},
\]
and we may decompose \( a \) as a sum \( \sum_{R \in \mathcal{M}(U)} a_R \) of particles \( a_R \) associated to the pseudodyadic rectangles \( R \in \mathcal{M}(U) \) in such a way that
\[
\left( \sum_{R \in \mathcal{M}(U)} \|a_R\|_{L^2(G)}^2 \right)^{1/2} \leq |U|^{-1/2}.
\]
We work on the domain $G$. Recall that $\psi$ for all $\lambda_n \in \mathbb{R}^+$ for all $n$, and $\sum_{n \in \mathbb{N}} \lambda_n < \infty$. We define the norm $\|f\|_{H^1_{\text{atom}}(G)}$ to be the infimum of the sums $\sum_{n \in \mathbb{N}} \lambda_n$ over all such representations of $f$.

It is often more convenient to impose a stronger requirement on particles, namely, that $a_R = L_1^{N_1} L_{1/2}^{N_2} b_R$ for some $L^2(G)$ function $b_R$ in the domain of $L_1^{N_1} L_{1/2}^{N_2}$ and for large integers $N_1$ and $N_2$; this means that $a_R$ has many vanishing moments, which may make calculations easier. We may show that this stronger requirement on particles gives the same atomic Hardy space, using telescopic series arguments to make moments vanish.

### 3.2. Square function and area function Hardy spaces

For $g \in G$ and $\beta \in [0, \infty)$, we write $\Gamma^\beta(g)$ for the product cone $\Gamma_1^\beta(g_1) \times \Gamma_2^\beta(g_2)$, where

$$\Gamma_i^\beta(g_i) := \{(h_i, t_i) \in G_i \times \mathbb{R}_+ : \rho_i(g_i, h_i) \leq \beta t_i\}.$$ 

We work on the domain $G_1 \times G_2 \times \mathbb{R}_+ \times \mathbb{R}_+$.

Take functions $\psi^{(i)}$ on $G_i$ that satisfy the standard decay, smoothness and cancellation conditions [1] and [2]. Recall that $\psi_t$ denotes the product function $\psi_{t_1}^{(1)} \otimes \psi_{t_2}^{(2)}$.

### Definition 3.2.

For $\psi^{(i)}$ as above and $\beta > 0$, we define $S_{\psi, \beta}(f)(g)$ to be

$$\left(\int_{\Gamma^\beta(g)} |(f \ast \psi_t(h))^2| \frac{dh \, dg}{t}\right)^{1/2}$$

for all $g \in G$ and $f \in L^1(G)$. We also define

$$S_{\psi,0}(f)(g) := \left(\int_T |f \ast \psi_t(g)|^2 \frac{dg}{t}\right)^{1/2}$$

for all $g \in G$ and $f \in L^1(G)$. The Hardy space $H^1_{sq,\psi,\beta}(G)$ is defined to be the space

$$\{f \in L^1(G) : \|S_{\psi, \beta}(f)\|_{L^1(G)} < \infty\},$$

equipped with the norm

$$\|f\|_{H^1_{sq,\psi,\beta}(G)} := \|S_{\psi, \beta}(f)\|_{L^1(G)}.$$ 

Note that $S_{\psi, \beta}(f)$ tends to $S_{\psi,0}(f)$ as $\beta \to 0$, at least pointwise. There are also discrete versions of this definition, where the integrals over $\mathbb{R}_+$ are replaced by sums over powers of 2 (or some other base). We usually call $S_{\psi, \beta}(f)$ an area function when $\beta > 0$ and a square function when $\beta = 0$, but it is more efficient to treat these together.

As mentioned earlier, much is known about Hardy spaces defined as above, and we summarise some of the main results. From [22], the space $H^1_{sq,\psi,0}(G)$ is independent of the choice of the functions $\psi^{(i)}$, provided that they satisfy the decay, smoothness and cancellation conditions [1] and [2]; discrete square functions and area operators $S_{\psi,1}$ also characterise the same space, which we write simply as $H^1(G)$. The key technique to prove these equivalences is a Plancherel–Pólya inequality. From [23] and [5], we see also that $H^1(G)$ may be characterised using wavelet and atomic decompositions; more precisely, $H^1_{\text{atom}}(G) = H^1(G)$. Further, the double Riesz transformations $R_{j_1}^{[1]} \otimes R_{j_1}^{[1]}$ (see Definition 5) and similar singular integral operators are all bounded from $H^1(G)$.
to $L^1(G)$. Finally, from [22], the dual of $H^1(G)$ is the space $\text{BMO}$ defined in terms of (suitable product) Carleson measures on $G$.

In Section 3.3 below, we show that the space $H^1_{\text{sq},v,\theta}(G)$ is also independent of $\theta$.

Let $\nabla_i$ and $L_i$ denote the subgradient and the sub-Laplacian on $G_i$; recall that $\nabla_i$ denotes the gradient $(\nabla_i, \partial_i)$ on $G_i \times \mathbb{R}_+$. The (vector-valued) convolution kernels of the operators $t_i L_i e^{-t_i L_i}$ and $t_i \nabla_i e^{-t_i \nabla_i}$ satisfy the decay, smoothness and cancellation conditions (11) and (2). Hence $H^1(G)$ may also be characterised via the Littlewood–Paley area functions and square functions defined using the heat and Poisson kernels.

3.3. Independence of cone angle. Recall that $R(g, t) := B_1(g_1, t_1) \times B_2(g_2, t_2)$. Fix a parameter $\theta$ in $(0, 1)$.

If $V$ is a closed subset of $G$, then we say that $g \in G$ has global $\theta$-density with respect to $V$ if

$$\frac{|V \cap R(g, t)|}{|R(g, t)|} \geq \theta$$

for all $t \in T$. Let $V^*$ be the set containing all points of global $\theta$-density of $V$, then $V^*$ is closed and $V^* \subseteq V$. Equivalently,

$$(V^*)^c = \{ g \in G : \mathcal{M}_S(\chi_{V^c})(g) > 1 - \theta \}.$$ 

It follows from the $L \log L \to L^{1,\infty}$ estimate for the strong maximal function (see, for example, [3]) that $|(V^*)^c| \leq c_\theta |V^c|$, where

$$c_\theta = \frac{C}{1 - \theta} \left( 1 + \log^+ \left( \frac{1}{1 - \theta} \right) \right).$$

For a closed subset $V$ of $G$, write

$$W^\beta(V) := \bigcup_{g \in V} \Gamma^\beta(g).$$

**Lemma 3.3.** Suppose that $V$ is a closed set in $G$ such that $|V^c| < \infty$. Then there exist constants $c_0 \leq 1/4$ and $C$ such that if $\beta > 1$ and $\theta = 1 - c_0 \beta^{-Q_1-Q_2}$, then

$$\int \int_{W^\beta(V^*)} F(g, t) |R(o, t)| \, dg \, dt \lesssim \int \int_{\Gamma(g)} F(h, t) \, dh \, dt$$

for all measurable nonnegative-real-valued functions $F$ on $G \times T$.

**Proof.** First, if $(h, t) \in W^\beta(V^*)$, then there exists $\tilde{g} \in V^* \cap R(h, \beta t)$. We see easily that

$$|R(\tilde{g}, \beta t) \cap R(h, t)^c| \leq (1 - 2c_0 \beta^{-Q_1-Q_2}) |R(\tilde{g}, \beta t)|,$$

for some constant $c_0 \leq 1/4$. Hence

$$|V \cap R(h, t)| \geq |V \cap R(\tilde{g}, \beta t)| - |R(\tilde{g}, \beta t) \cap R(h, t)^c|$$

$$\geq (\theta - 1 + 2c_0 \beta^{-Q_1-Q_2}) |R(\tilde{g}, \beta t)|$$

$$= c_0 \beta^{-Q_1-Q_2} |R(\tilde{g}, \beta t)| \geq C |R(g, t)|.$$
Now, by Fubini’s Theorem,
\[
\int_V \int_{\Gamma(g)} F(h, t) \, dh \, dt \, dg
= \int_{\mathcal{T} \times G} \int_V \chi_{R(o, t)}(h^{-1} g) F(h, t) \, dh \, dg \, dt
\]
\[
\geq \int_{W^\beta(V^*)} \int_G \chi_{R(o, t)}(h^{-1} g) F(h, t) \, dg \, dh \, dt
= \int_{W^\beta(V^*)} F(h, t) |R(o, t)| \, dh \, dt,
\]
as required. \(\square\)

**Proposition 3.4.** With the notation of Definition 3.2,
\[
H^1_{sq, \psi, \beta}(G) = H^1_{sq, \psi, 1}(G),
\]
and these spaces have equivalent norms for all \(\beta > 0\).

**Proof.** It suffices to suppose that \(\beta > 1\) and show that
\[
\|S_{\psi, \beta}(f)\|_{L^1(G)} \lesssim \beta^{Q_1 + Q_2} (1 + \log_2^\frac{1}{\beta}) \|S_{\psi, 1}(f)\|_{L^1(G)}.
\]
For all \(\lambda > 0\), set
\[
V = \{g \in G : S_{\psi, \beta}(f)(g) \leq \lambda\},
\]
and \(\theta = 1 - \beta^{-Q_1 - Q_2}/4\). Then, from Lemma 3.3 and Fubini’s theorem,
\[
\int_V S_{\psi, \beta}(f)(g)^2 \, dg = \int_{V^*} \int_{\Gamma(g)} \frac{|f \ast \psi_t(h)|^2}{|R(o, \beta t)|} \, dh \, dg \, dt
\]
\[
\lesssim \beta^{Q_1 + Q_2} \int_{W^\beta(V^*)} \frac{|f \ast \psi_t(h)|^2}{|R(o, \beta t)|} \, dh \, dg \, dt
\]
\[
\lesssim \beta^{Q_1 + Q_2} \int_V \int_{\Gamma(g)} \frac{|f \ast \psi_t(h)|^2}{|R(o, \beta t)|} \, dh \, dg \, dt
\]
\[
\approx \beta^{Q_1 + Q_2} \int_V S_{\psi, 1}(f)^2 \, dg.
\]
Therefore
\[
|\{g \in G : S_{\psi, \beta}(f)(g) > \lambda\}|
\]
\[
\leq |(V^*)^c| + \frac{C}{\lambda^2} \int_{V^*} S_{\psi, \beta}(f)(g)^2 \, dg
\]
\[
\leq C \beta^{Q_1 + Q_2} (1 + \log_2^\frac{1}{\beta}) \left(|(V^*)^c| + \frac{1}{\lambda^2} \int_V S_{\psi, 1}(f)(g)^2 \, dg\right).
\]
Integrating with respect to \(\lambda\) yields
\[
\|S_{\psi, \beta}(f)\|_{L^1(G)} \lesssim \beta^{Q_1 + Q_2} (1 + \log_2^\frac{1}{\beta}) \|S_{\psi, 1}(f)\|_{L^1(G)},
\]
which completes the proof of Proposition 3.4. \(\square\)
3.4. **Summary.** The known results cited in Section 3.2 and our additional material here may be summarised in the following proposition.

**Proposition 3.5.** The atomic Hardy space $H^1_{atomic}(G)$ and the square function and area function Hardy spaces $H^1_{sq,\psi,\beta}$ for different $\psi$ and $\beta$ coincide and have equivalent norms.

4. **The singular integral characterisation**

We consider a stratified Lie group $G$. Recall that $R_0$ is the identity operator $I$, and when $1 \leq j \leq d_i$, the $j$th Riesz operator $R_j$ on $G$ is defined by

$$R_j := \mathcal{X}_j(L)^{-1/2};$$

its convolution kernel, $k_j$ say, is smooth away from the identity of $G$, and homogeneous of degree $-Q$. According to Christ and Geller [11], $f \in H^1(G)$ if and only if all $R_j f \in L^1(G)$, and there is a corresponding norm equivalence. We say that the singular integral operators $R_j$, where $0 \leq j \leq d_j$, characterise $H^1(G)$.

**Definition 4.1.** Suppose that the singular integral operators $K^{[i]}_j$, where $0 \leq j \leq n_i$, characterise $H^1(G_i)$, in the sense above. The space $H^1_{SIO}(G)$ is defined to be the set of all $f \in L^1(G)$ such that

$$\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left\| K^{[1]}_{j_1} \otimes K^{[2]}_{j_2} f \right\|_{L^1(G)} < \infty,$$

with norm

$$\|f\|_{H^1_{SIO}(G)} := \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left\| K^{[1]}_{j_1} \otimes K^{[2]}_{j_2} f \right\|_{L^1(G)}.$$

In this section, we generalise Theorem [11] which states that the spaces $H^1_{Riesz}(G)$ and $H^1(G)$ coincide and have equivalent norms.

**Theorem 4.2.** Suppose that the singular integral operators $K^{[i]}_j$, where $0 \leq j \leq n_i$, characterise $H^1(G_i)$. Then the double singular integral operators $K^{[1]}_{j_1} \otimes K^{[2]}_{j_2}$ characterise the Hardy space $H^1(G)$. That is, $f \in H^1(G)$ if and only if each $K^{[1]}_{j_1} \otimes K^{[2]}_{j_2} f$ is in $L^1(G)$ and moreover

$$\|f\|_{H^1(G)} \approx \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left\| K^{[1]}_{j_1} \otimes K^{[2]}_{j_2} f \right\|_{L^1(G)}.$$

It is known (see [23] and [5]) that singular integral operators associated to homogeneous kernels are bounded from $H^1(G)$ to $L^1(G)$, so it suffices to show that if all the double singular integral transforms of a function $f$ are in $L^1(G)$ then $f \in H^1(G)$. Our proof of Proposition 4.3 below extends [4], which introduced a new method, using randomisation, to characterise flag Hardy space on Heisenberg groups by products of singular integrals.

4.1. **The square function and singular integral transforms.**

**Proposition 4.3.** Suppose that $f \in L^2(G)$, and $K^{[1]}_{j_1} \otimes K^{[2]}_{j_2} f \in L^1(G)$ when $j_i = 0, \ldots, n_i$. Then $f \in H^1(G)$, and

$$\|f\|_{H^1(G)} \leq C \|f\|_{H^1_{SIO}(G)}.$$


Proof. We use a randomisation argument coupled with the analogous one-parameter result of Christ and Geller [11]. Fix a smooth function \(\eta\) on \(\mathbb{R}_+\), supported in \([1/2, 2]\), such that \(\sum_{m \in \mathbb{Z}} \eta(2^{-m}t) = 1\) for all \(t \in \mathbb{R}_+\). By Section 2.2 the convolution kernels \(k_{\eta(L_i)}\) of the operators \(\eta(L_i)\) on \(G_i\) are Schwartz functions of mean 0.

Let \(r_m : [0, 1] \to \mathbb{R}\) be a collection of independent Rademacher random variables (see [20]). Fix \(i\), take \(\eta\) as above, and define
\[T_s(f) = \sum_{m \in \mathbb{Z}} r_m(s)\eta(2^{-m}L_i)f\]
for all \(f \in H^1(G_i)\) and all \(s \in [0, 1]\). Straightforward calculation shows that
\[\left| \xi^k \partial_x^k \left( \sum_{m \in \mathbb{Z}} r_m(s)\eta(2^{-m}\xi) \right) \right| \leq C_k \quad \forall \xi \in \mathbb{R}_+ \quad \forall k \in \mathbb{N},\]
and from the multiplier theorem (see for example, [17, Theorem 6.25]), the operator \(T_s\) is bounded from \(H^1(G_i)\) to \(L^1(G_i)\) with norm uniformly bounded for \(s \in [0, 1]\). Together with the Christ–Geller characterisation [11, Theorem A], this implies that
\[\left\| \sum_{m \in \mathbb{Z}} r_m(s)\eta(2^{-m}L_i)f \right\|_{L^1(G_i)} \lesssim \left( \sum_{j=0}^{n_i} \left\| K_{j_i}^{[j]}f \right\|_{L^1(G_i)} \right)^{n_i},\]
for all \(f \in L^1(G_i)\) such that \(K_{j_i}^{[j]}f \in L^1(G_i)\). Iteration of the argument shows that
\[\left\| \sum_{m \in \mathbb{Z}} r_m(s_1)\eta(2^{-m}L_1) \left( \sum_{n \in \mathbb{Z}} r_n(s_2)\eta(2^{-n}L_2)f \right) \right\|_{L^1(G)} \]
\[\lesssim \sum_{j_1=0}^{n_1} \left\| K_{j_1}^{[j_1]} \sum_{n \in \mathbb{Z}} r_n(s_2)\eta(2^{-n}L_2)f \right\|_{L^1(G)} \]
\[= \sum_{j_1=0}^{n_1} \left\| \sum_{n \in \mathbb{Z}} r_n(s_2)\eta(2^{-n}L_2)K_{j_1}^{[j_1]}f \right\|_{L^1(G)} \]
\[\lesssim \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left\| K_{j_1}^{[j_1]} \otimes K_{j_2}^{[j_2]}f \right\|_{L^1(G)} ,\]
because operators involving convolutions (even with distributions) on \(G_1\) and operators involving convolutions (even with distributions) on \(G_2\) commute. By Khinchin’s inequality (see, for example, [20, Appendix C.5]), this implies that
\[\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\eta(2^{-m}L_1)\eta(2^{-n}L_2)f|^2 \right)^{1/2} \right\|_{L^1(G)} \]
\[\lesssim \int_{[0, 1] \times [0, 1]} \left\| \sum_{m \in \mathbb{Z}} r_m(s_1)\eta(2^{-m}L_1) \sum_{n \in \mathbb{Z}} r_n(s_2)\eta(2^{-n}L_2)f \right\|_{L^1(G)} \, ds_1 \, ds_2 \]
\[\lesssim \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \left\| K_{j_1}^{[j_1]} \otimes K_{j_2}^{[j_2]}f \right\|_{L^1(G)} .\]
This ends the proof of Proposition 4.3. \(\square\)

Remark 4.4. It is straightforward to extend this result to products of more than two factors. It is just a matter of repeating the randomisation argument more times.
5. Maximal function characterisation

In this section, we prove Theorem 1.3. As we have already noted, in the one-parameter setting, a common strategy for showing that maximal functions characterise the Hardy space is to use atoms; this strategy does not work in the multi-parameter case. Merryfield managed to extend the one-parameter result to the product space on Heisenberg groups by maximal functions. Here we extend this method to product groups.

5.1. The maximal function Hardy spaces. Recall that $\Gamma^\beta(g)$ denotes the cone with vertex $g$ and angle $\beta$:

$$\Gamma^\beta(g) := \{(h, t) \in G \times T : \rho_i(g_i, h_i) \leq \beta t_i \text{ when } i = 1, 2\}.$$

**Definition 5.1.** Take functions $\zeta^{[i]} \in M(G_i)$, and define the maximal operator $M_{\zeta, \beta}$ by

$$M_{\zeta, \beta}(f)(g) := \sup_{h \in \Gamma^\beta(g)} |f \ast \zeta^{[i]}(h)| \quad \forall g \in G \quad \forall f \in L^1(G).$$

The Hardy space $H^1_{\text{max}, \zeta, \beta}(G)$ is defined to be the space

$$\{f \in L^1(G) : \|M_{\zeta, \beta} f\|_{L^1(G)} < \infty\}$$

equipped with the norm

$$\|f\|_{H^1_{\text{max}, \zeta, \beta}(G)} := \|M_{\zeta, \beta} f\|_{L^1(G)},$$

for the Hardy space, we require that the integrals of the $\zeta^{[i]}$ are nonzero.

It is obvious that $\|M_{\zeta, \gamma} f\|_{L^1(G)} \leq \|M_{\zeta, \beta} f\|_{L^1(G)}$ when $\gamma \leq \beta$.

We treat the cases when $\beta > 0$ and when $\beta = 0$ together. In the important special cases when the $\zeta^{[i]}$ coincide with the Poisson or heat kernels, we have additional tools, such as Harnack or Moser inequalities. The possibly less well known Plancherel–Pólya inequality provides similar results for more general $\zeta$.

To characterise $H^1(G)$ by maximal functions, we are going to show two results.

**Proposition 5.2.** If $\beta$ is large enough, then the spaces $H^1_{\text{sq}, P, 1}(G)$ and $H^1_{\text{max}, P, \beta}(G)$ coincide and have equivalent norms.

**Proposition 5.3.** For different choices of $\varphi$ and $\zeta$ and different choices of $\beta$ and $\gamma$, the spaces $H^1_{\text{max}, \varphi, \beta}(G)$ and $H^1_{\text{max}, \zeta, \gamma}(G)$ coincide and have equivalent norms.

Combining the above two results with Proposition 3.4 proves Theorem 1.3.

5.2. Part 1 of the proof of Proposition 5.2. Evidently $H^1_{\text{sq}, P, 1}(G) \subseteq H^1_{\text{max}, P, \beta}(G)$, and

$$\|M_{P, \beta} f\|_{L^1(G)} \lesssim \|S_{P, 1} f\|_{L^1(G)}.$$

Indeed, when $\|S_{P, \beta} f\|_{L^1(G)} < \infty$, then by [5], we may write $f = \sum_j \lambda_j a_j$, where each $a_j$ is an atom, and $\sum_j |\lambda_j| \lesssim \|S_{P, \beta} f\|_{L^1(G)}$. Thus, to prove (8), it suffices to verify that

$$\|M_{P, \beta}(a)\|_{L^1(G)} \lesssim 1.$$
for each product atom \( a \) as in Section 3.1. The Poisson kernels \( P_1^1 \) and \( P_1^2 \) satisfy the standard decay and smoothness conditions \([1]\), and the atom \( a \) satisfies the standard product cancellation condition \([2]\). Then the desired estimate of \( \| M_{P,\beta}(a) \|_{L^1(G)} \) follows from standard product arguments and Journé’s covering lemma.

We provide a brief outline of the proof here for completeness and for the reader’s convenience. It suffices to show that, for each product atom \( a \),

\[
\| M_{P,\beta}(a) \|_{L^1(G)} \lesssim 1.
\]

From Section 3.1, we may write \( a \) on \( G \) as a sum \( \sum_{R \in \mathcal{A}(U)} a_R \), where \( U \) is an open subset of \( G \) of finite measure, and the particles \( a_R \) satisfy support and size conditions.

There are two steps to the proof: first, we find some small positive \( \epsilon \), such that for any pseudodyadic rectangle \( S \) containing the pseudodyadic rectangle \( R \),

\[
\int_{G \setminus S} |M_{P,\beta}(a_R)(g)| \, dg \lesssim \left( \frac{\ell_1(R)}{\ell_1(S)} + \frac{\ell_2(R)}{\ell_2(S)} \right)^\epsilon |R|^{1/2} \| a_R \|_{L^2(G)},
\]

for all particles \( a_R \) associated to \( R \). The estimation of this expression may be achieved by writing \( S = \prod_{i=1}^d (Q_i)_x \) and breaking up \( G \setminus S \) into the three regions \((Q_1)_x \times (Q_2)_x\), \((Q_1)_x \times Q_2\) and \( Q_1 \times (Q_2)_x\). In the first region we use the support and cancellation conditions on \( a_R \) to estimate \( a_R \ast P_1(g) \), and show that

\[
\int_{(Q_1)_x \times (Q_2)_x} |M_{P,\beta}(a_R)(g)| \, dg \lesssim \left( \frac{\ell_1(R)}{\ell_1(S)} \right)^\epsilon |R|^{1/2} \| a_R \|_{L^2(G)};
\]

in the second region we use the support and cancellation conditions to control the \( g_1 \) variable and Hölder’s inequality and a Littlewood–Paley argument to control the \( g_2 \) variable, and show that

\[
\int_{(Q_1)_x \times Q_2} |M_{P,\beta}(a_R)(g)| \, dg \lesssim \left( \frac{\ell_1(R)}{\ell_1(S)} \right)^\epsilon |R|^{1/2} \| a_R \|_{L^2(G)};
\]

in the third region, we argue similarly, but with the roles of the variables reversed.

Once \([3]\) is proved, we apply Journé’s lemma to obtain an estimate for atoms rather than particles. Let \( a \) be an atom associated to the open set \( U \), and write \( a = \sum_{R \in \mathcal{A}(U)} a_R \). Define

\[
V = \{ g \in G : M_S \chi_U(g) > 1/4 \} \quad \text{and} \quad W = \{ g \in G : M_S \chi_V(g) > 1/4 \}.
\]

Then \( |W| \lesssim |V| \lesssim |U| \). Further,

\[
\int_G |M_{P,\beta}(a)(g)| \, dg = \int_W |M_{P,\beta}(a)(g)| \, dg + \int_{G \setminus W} |M_{P,\beta}(a)(g)| \, dg \leq |W|^{1/2} \left( \int_W |M_{P,\beta}(a)(g)| \, dg \right)^{1/2} + \sum_{R \in \mathcal{A}(U)} \int_{G \setminus W} |M_{P,\beta}(a_R)(g)| \, dg.
\]

The first term is estimated using the \( L^2 \)-boundedness of \( M_{P,\beta} \):

\[
|W|^{1/2} \left( \int_W |M_{P,\beta}(a)(g)| \, dg \right)^{1/2} \lesssim |W|^{1/2} \| a \|_{L^2(G)} \lesssim |U|^{1/2} |U|^{-1/2} = 1.
\]
The second term is estimates using Journé’s lemma, namely, for each $R \in m(U)$, we can find $S \in m(W)$ (depending on $R$) such that $R \subseteq S$ and 
\[
\sum_{r \in m(R)} \left( \frac{\ell_1(R)}{\ell_1(S)} + \frac{\ell_2(R)}{\ell_2(S)} \right)^\varepsilon |R| \lesssim |U|.
\]
Then from [9], Hölder’s inequality, Journé’s lemma, and the definitions, 
\[
\sum_{R \in m(U)} \int_{G \setminus S} |\mathcal{M}_{P,\beta}(a_R)(g)| \, dg
\]
\[
\leq \sum_{R \in m(U)} \int_{G \setminus S} |\mathcal{M}_{P,\beta}(a_R)(g)| \, dg
\]
\[
\lesssim \sum_{R \in m(U)} \left( \frac{\ell_1(R)}{\ell_1(S)} + \frac{\ell_2(R)}{\ell_2(S)} \right)^\varepsilon |R|^{1/2} \|a_R\|_{L^2(G)}
\]
\[
\lesssim \left( \sum_{R \in m(U)} \left( \frac{\ell_1(R)}{\ell_1(S)} + \frac{\ell_2(R)}{\ell_2(S)} \right)^{2\varepsilon} |R| \right)^{1/2} \left( \sum_{R \in m(U)} \|a_R\|_{L^2(G)}^2 \right)^{1/2}
\]
\[
\lesssim |U|^{1/2}|U|^{-1/2} = 1.
\]
Note that proving this result of products of more than two factors seems to be nontrivial; Journé’s lemma requires us to have more than one “improving factor” $\ell_f(R)/\ell_f(S)$.

It remains to prove the opposite inclusion: $H^1_{\text{max},P,\beta}(G) \subseteq H^1_{\text{sq},P,1}(G)$, and 
\[
\|S_{P,1}f\|_{L^1(G)} \lesssim \|\mathcal{M}_{P,\beta}f\|_{L^1(G)}.
\]
We first treat a stratified group, and then a product of stratified groups.

5.3. Part 2 of the proof of Proposition 5.2. In this part of the proof, we prove [10] for a stratified group $G$ with no product structure. This simplifies the notation. Later the group $G$ will be one of the factors of the product group $G$ that we wish to consider.

We are going to use integration by parts, and need to know about the behaviour of certain harmonic functions on $G \times \mathbb{R}_+$ at the boundaries of this region. Suppose that $f \in L^p(G)$, where $1 \leq p < \infty$, and consider the Poisson integral $f \ast P_t(g)$ and $f \ast Q_t(g)$, where $g \in G$ and $t \in \mathbb{R}_+$, whose behaviour as $t \to 0$ is discussed in Corollary 2.2.

From Lemma 2.1, if $f \in L^\infty(G)$, then $f \ast P_t$ and $f \ast Q_t$ are bounded in $L^\infty(G)$ as $t \to \infty$. If $f \in H^1_{\text{max},P,\gamma}(G)$, then $\|f_{1/s} \ast P_1\|_{H^1_{\text{max},P,\gamma}(G)}$ is bounded for all $s > 0$. Since 
\[
f_{1/s} \ast P_1 \to \left( \int_G f(g) \, dg \right) P_1 \quad \text{as } s \to \infty
\]
in $L^1(G)$ and $\sup_{t \geq 1} \|P_t(\cdot)\| \notin L^1(G)$ so $P_1 \notin H^1_{\text{max},P,\gamma}(G)$, we see that $f$ has mean 0. Thus 
\[
\|f \ast P_t\|_{L^1(G)} + \|f \ast Q_t\|_{L^1(G)} = \|f_{1/t} \ast P_1\|_{L^1(G)} + \|f_{1/t} \ast Q_1\|_{L^1(G)} \to 0
\]
and 
\[
\|f \ast P_t\|_{L^\infty(G)} + \|f \ast Q_t\|_{L^\infty(G)} = t^{-Q} \|f_{1/t} \ast P_1\|_{L^\infty(G)} + t^{-Q} \|f_{1/t} \ast Q_1\|_{L^\infty(G)} \to 0
\]
as $t \to \infty$. This convergence is also pointwise almost everywhere.

**Proposition 5.4.** Suppose that $G$ is a stratified Lie group and $\gamma > 0$. If $\beta$ is large enough, then

$$H^1_{\max, P, \beta}(G) \subseteq H^1_{\text{seq}, P, \gamma}(G),$$

and there is a corresponding norm inequality:

$$\|f\|_{H^1_{\max, P, \beta}(G)} \lesssim \|f\|_{H^1_{\max, P, \beta}(G)} \quad \forall f \in H^1_{\max, P, \beta}(G).$$

*Proof.* Take $f \in L^1(G)$ such that $M_{P, \beta}(f) \in L^1(G)$. We assume that $f$ is real-valued, for otherwise we may treat the real and imaginary parts separately. We may also suppose that $f$ is smooth, by a simple mollification argument.

Fix $\alpha > 0$, and define

$$L_\beta(\alpha) := \{g \in G : M_{P, \beta}(f)(g) \leq \alpha\}$$

$$A_\beta(\alpha) := \{g \in G : M_S(1 - \chi_{L_\beta(\alpha)})(g) < \frac{1}{4}\},$$

where $M_S$ is the strong maximal operator, which is $L^2$ bounded. Then

$$(11) \quad A_\beta(\alpha) \subseteq L_\beta(\alpha) \quad \text{and} \quad |(L_\beta(\alpha))^e| \leq |A_\beta(\alpha)| \lesssim |(L_\beta(\alpha))^e|.$$ 

Define also

$$W_\beta := \bigcup_{g \in A_\beta(\alpha)} \Gamma^\beta(g) \quad \text{and} \quad \widetilde{W}_\beta := \bigcup_{h \in L_\beta(\alpha)(f)} \Gamma^\beta(h).$$

We claim that there exists $C_0 \in (0, 1)$ such that

$$\chi_{L_\gamma(\alpha)} * P_t(g) \geq C_0 \quad \forall (g, t) \in W_\gamma.$$

Indeed, by definition, for such $(g, t)$,

$$(1 - \chi_{L_\gamma(\alpha)}) * \chi_{B(\alpha, \gamma t)} < \frac{1}{4} |B(\alpha, \gamma t)|,$$

that is,

$$\chi_{L_\gamma(\alpha)} * \chi_{B(\alpha, \gamma t)} \geq \frac{3}{4} |B(\alpha, \gamma t)|,$$

and the claim follows from Lemma 2.1. We also claim that if $\beta$ is large enough, then there is a constant $C_1 \in (0, C_0)$, such that if $(g, t) \notin \widetilde{W}_\beta$, then

$$\chi_{L_\beta(\alpha)} * P_t(g) \leq C_1.$$

Indeed, if $(g, t) \notin \widetilde{W}_\beta$ then $\rho(h^{-1} g) \geq \beta t$ for all $h \in L_\beta(\alpha)$. Hence,

$$\chi_{L_\beta(\alpha)} * P_t(g) = \int_G \chi_{L_\beta(\alpha)}(h) P_t(h^{-1} g) \, dh \leq \int_{B(g, \beta t)^c} P_t(h^{-1} g) \, dh$$

$$= \int_{B(g, \beta t)^c} P_1(h^{-1} g) \, dh \to 0$$

as $\beta \to \infty$, proving our claim.

Take a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(s) = 1$ when $s \geq C_0$ and $\eta(s) = 0$ when $s \leq C_1$. Define $H_t := \chi_{L_\beta(\alpha)} * P_t$. Then

$$t \partial_t H_t(g) = \chi_{L_\beta(\alpha)} * Q_t(g),$$

which is uniformly bounded for all $g \in G$ and $t \in \mathbb{R}_+$ and

$$t \partial_t H_t \to 0 \quad \text{as} \quad t \to 0.$$
pointwise almost everywhere, by Corollary 2.2.

It will suffice to show that

\[ \int_{A_{\gamma}(\alpha)} S_{P,\gamma}(f)(g)^2 \, dg \lesssim \int_{L_{\beta}(\alpha)} \mathcal{M}_{P,\beta}(f)(g)^2 \, dg + \alpha^2 |L_{\gamma}(\alpha)^c|. \]

Indeed, coupled with (11), this implies that

\[ \begin{align*}
|\{ g \in G : S_{P,\gamma}(f)(g) > \alpha \}| \\
& \leq |\{ g \in A_{\gamma}(\alpha)^c : S_{P,\gamma}(f)(g) > \alpha \}| + \{ g \in A_{\gamma}(\alpha) : S_{P,\gamma}(f)(g) > \alpha \} \\
& \leq |A_{\gamma}(\alpha)^c| + \frac{1}{\alpha^2} \int_{A_{\gamma}(\alpha)} S_{P,\gamma}(f)(g)^2 \, dg \\
& \lesssim |L_{\gamma}(\alpha)^c| + \frac{1}{\alpha^2} \int_{L_{\beta}(\alpha)} \mathcal{M}_{P,\beta}(f)(g)^2 \, dg.
\end{align*} \]

A standard integration with respect to \( \alpha \) then implies that

\[ \| S_{P,\gamma}(f) \|_{L^1(G)} \lesssim \| \mathcal{M}_{P,\beta}(f) \|_{L^1(G)}, \]

that is, the required estimate (10) holds.

We observe that

\[ \begin{align*}
\int_{A_{\beta}(\alpha)} S_{P,\gamma}(f)(g)^2 \, dg &= \int_{A_{\beta}(\alpha)} \int \int_{\Gamma_{\gamma}(g)} \left| \nabla (f * P_t)(h) \right|^2 \frac{t}{|B(o, t)|} \, dt \, dh \\
& \lesssim \int \int_{W_{\gamma}} \left| \nabla (f * P_t)(g) \right|^2 t \, dt \, dg \\
& \leq \int \int_{G \times \mathbb{R}_+} \left| \nabla (f * P_t)(g) \right|^2 |\eta(H_t(g))|^2 t \, dt \, dg.
\end{align*} \]

From (12), it will therefore suffice to show that

\[ I_0 := \int \int_{G \times \mathbb{R}_+} \left| \nabla (f * P_t)(g) \right|^2 |\eta(H_t(g))|^2 t \, dt \, dg \]

\[ \lesssim \int_{L_{\beta}(\alpha)} \mathcal{M}_{P,\beta}(f)(g)^2 \, dg + \alpha^2 |L_{\gamma}(\alpha)^c|. \]

We note that \( u : (g, t) \mapsto \mathcal{F} \mathcal{P}(F) \) is harmonic on \( G \times \mathbb{R}_+ \) for all \( F \in L^1(G) + L^\infty(G) \), in the sense that

\[ \mathcal{L} u(g, t) = 0, \]

where \( \mathcal{L} := \mathcal{L} - \partial_t^2 \). Consequently,

\[ \left| \nabla u(g, t) \right|^2 = -\frac{1}{2} \mathcal{L} \left( u^2(g, t) \right), \quad \forall (g, t) \in G \times \mathbb{R}_+. \]

Further, by our remark on harmonicity, \( \mathcal{L} \eta = 0 \), and so

\[ \begin{align*}
\mathcal{L} \eta(H_t(g)) &= \nabla \cdot (\eta'(H_t(g)) \nabla H_t(g)) \\
&= \eta''(H_t(g)) \left| \nabla H_t(g) \right|^2.
\end{align*} \]
It follows that
\[
|\nabla (f \ast P_t)(g)\eta (H_t(g))|^2 = \frac{1}{2} \mathcal{L} \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) - 4f \ast P_t(g)\eta (H_t(g)) \nabla (f \ast P_t)(g) \cdot \nabla \eta (H_t(g))
\]
\[
- |f \ast P_t(g)|^2 |\nabla \eta (H_t(g))|^2
- |f \ast P_t(g)|^2 \eta (H_t(g)) \eta''(H_t(g)) |\nabla H_t(g)|^2.
\]
(14)

We estimate the second, third, and fourth terms on the right hand side of (14) as follows. First, by the arithmetic–geometric mean inequality and the chain rule,
\[
|4f \ast P_t(g)\eta (H_t(g)) \nabla (f \ast P_t)(g) \cdot \nabla \eta (H_t(g))| = \frac{1}{2} \nabla P_t \ast f(g) |\nabla \eta (H_t(g))|^2 + 8 |f \ast P_t(g)|^2 |\nabla \eta (H_t(g))|^2
\]
\[
\leq \frac{1}{2} |\nabla P_t \ast f(g)|^2 |\nabla \eta (H_t(g))|^2 + 8 |f \ast P_t(g)|^2 |\nabla \eta (H_t(g))|^2
\]
\[
\leq \frac{1}{2} \left| f \ast \nabla P_t(g) \right|^2 |\eta (H_t(g))|^2 + 8 \|\eta'\|_{L^\infty(\mathbb{R})} |f \ast P_t(g)|^2 |\nabla H_t(g)|^2
\]
and we can move the first term on the right hand side of this inequality to the left hand side of (14). Next,
\[
|f \ast P_t(g)|^2 |\nabla \eta (H_t(g))|^2 \leq \|\eta'\|_{L^\infty(\mathbb{R})} |f \ast P_t(g)|^2 |\nabla H_t(g)|^2
\]
and similarly,
\[
|f \ast P_t(g)|^2 |\eta (H_t(g))|^2 |\eta''(H_t(g))|^2 |\nabla H_t(g)|^2
\]
\[
\leq \|\eta\|_{L^\infty(\mathbb{R})} \|\eta''\|_{L^\infty(\mathbb{R})} |f \ast P_t(g)|^2 |\nabla H_t(g)|^2.
\]

We conclude that
\[
|\nabla (f \ast P_t)(g)\eta (H_t(g))|^2
\]
\[
\leq - \mathcal{L} \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) + C(\eta) |f \ast P_t(g)|^2 |\nabla H_t(g)|^2
\]
\[
=: f_1(g, t) + f_2(g, t),
\]
say. The proof of (13) is now straightforward.

Evidently, $I_0 \leq I_1 + I_2$, where
\[
I_j = \left| \int \int_{G \times R^+} f_j(g, t) t \, dg \, dt \right|
\]

To treat the term $I_1$, we recall that $\mathcal{L} = \Delta - \partial^2_t$. Integration by parts and the decay of the Poisson integral $f \ast P_t$ at infinity imply that
\[
\int_G \mathcal{L} \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) t \, dg = 0
\]
for all $t > 0$, and also that
\[
\int_{R^+} \partial^2_t \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) t \, dt
\]
\[
= \left[ t \partial_t \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) \bigg|_{t=0}^{t=\infty} - \int_{R^+} \partial_t \left( |f \ast P_t(g)\eta (H_t(g))|^2 \right) dt \right]
\]
\[
= 2 \left[ (f \ast P_t(g)\eta (H_t(g))) t \partial_t (f \ast P_t(g)\eta (H_t(g))) \bigg|_{t=0}^{t=\infty} - \left[ |f \ast P_t(g)\eta (H_t(g))|^2 \right]_{t=0}^{t=\infty} \right]
\]
\[
= |f(g) \chi_{L^2(\alpha)}(g)|^2.
\]
Again we may and do assume that $f$ where $M$ many of the terms here when $t = 0$ vanish by our remarks before the enunciation of this proposition. Therefore

\[ I_1 = \int_{L_{\beta}(a)} |f(g)|^2 \, dg, \]

which is the first term on the right hand side of (13).

Next, since $|f \ast P_t(g)| \leq \mathcal{M}_{P,\beta}(g) \leq \alpha$ when $(g, t) \in W_\beta$, and $\nabla P_t$ has mean 0,

\[ I_2 = \int_{W_\beta} |f \ast P_t(g)|^2 \, \left| \nabla H_t(g) \right|^2 t \, dt \, dg \]

\[ \leq \alpha^2 \int_{W_\beta} \left| \nabla H_t(g) \right|^2 t \, dt \, dg \]

\[ \leq \alpha^2 \int_{G \times \mathbb{R}_+} \left| \chi_{L_{\beta}(a)} \ast t \nabla P_t(g) \right|^2 \frac{dt}{t} \, dg \]

\[ = \alpha^2 \int_{G \times \mathbb{R}_+} \left| (1 - \chi_{L_{\beta}(a)}) \ast t \nabla P_t(g) \right|^2 \frac{dt}{t} \, dg \]

\[ \approx \alpha^2 \left| L_{\beta}(a) \right|^2, \]

by Littlewood–Paley theory. This is the second term on the right hand side of (13), and the proposition is now proved.

\[ \square \]

Remark 5.5. We summarise the first step of this proof as the application of harmonicity to estimate the desired square function as a sum of two terms in (15). The “main term” $I_1$ gives us the function $f$ that we started with, while the “error term” $I_2$ gives us an expression that we can handle by using Littlewood–Paley arguments.

5.4. Part 3 of the proof of Proposition 5.2. It remains to take a product group $G$, prove the inclusion and inequality

\[ H^1_{\max,P,\beta}(G) \subseteq H^1_{sq,P,1}(G) \]

\[ \|S_{P,1}f\|_{L^1(G)} \lesssim \|\mathcal{M}_{P,\beta}f\|_{L^1(G)} \quad \forall f \in H^1_{\max,P,\beta}(G). \]

Again we may and do assume that $f$ is real-valued and smooth.

The initial definitions are the same as in the one-parameter case. Take $f \in L^1(G)$ such that $\mathcal{M}_{P,\beta}(f) \in L^1(G)$ and $\alpha > 0$. Define

\[ L_{\beta}(\alpha) := \{ g \in G : \mathcal{M}_{P,\beta}(f)(g) \leq \alpha \}, \]

\[ A_{\beta}(\alpha) := \left\{ g \in G : \mathcal{M}_{S}(1 - \chi_{L_{\beta}(\alpha)})(g) < \frac{1}{4} \right\}, \]

where $\mathcal{M}_{S}$ denotes the strong maximal operator. By the same argument as in the one-parameter case,

\[ A_{\beta}(\alpha) \subseteq L_{\beta}(\alpha) \quad \text{and} \quad |A_{\beta}(\alpha)^c| \lesssim |L_{\beta}(\alpha)^c|. \]

Define also

\[ W_{\beta} := \bigcup_{g \in A_{\beta}(\alpha)} \Gamma_\beta(g) \quad \text{and} \quad \overline{W_{\beta}} := \bigcup_{h \in L_{\beta}(\alpha)(f)} \Gamma_\beta(h). \]

As in the one-parameter case, there exists $C_0 \in (0,1)$ and $C_1 \in (0, C_0)$, such that

\[ \chi_{L_{\beta}(\alpha)} \ast P_t(g) \geq C_0 \quad \forall (g, t) \in W_\gamma \]

\[ \chi_{L_{\beta}(\alpha)} \ast P_t(g) \leq C_1 \quad \forall (g, t) \in (\overline{W_{\beta}})^c, \]
provided that $\beta$ is large enough.

We let $H_t := \chi_{L^2(\alpha)} * P_t$; here $t_1, t_2 \geq 0$. Take a smooth real-valued function $\eta$ on $\mathbb{R}$ such that $\eta(s) = 1$ when $s \geq C_0$ and $\eta(s) = 0$ when $s \leq C_1$. By definition,

$$t_1 \partial_{t_1} H_t(g) = \chi_{L^2(\alpha)} * (Q_{t_1} \otimes P_{t_2})(g),$$

and this is uniformly bounded for all $g \in G$ and $t_1, t_2 \geq 0$; further, by (14),

$$t_1 \partial_{t_1} H_t(g) \to 0 \quad \text{as } t_1 \to 0$$

for almost all $g \in G$.

Again, it will suffice to show that

$$\int \int_{G \times T} \left| \nabla_1 \nabla_2 (f * P_t)(g) \right|^2 |\eta(H_t(g))|^2 t \, dt \, dg$$

$$\lesssim \int_{L^2(\alpha)} \mathcal{M}_{P,\beta}(f)(g)^2 \, dg + \alpha^2 |L_1(\alpha)|^2.$$

We do this by extending the computation for a single homogeneous group.

First, we fix the variables $g_2$ and $t_2$. By the one-parameter case,

$$\left\| S_{P,\gamma}(f) ; g_2 \right\|_{L^1(G_1)} \lesssim \left\| \mathcal{M}_{P,\beta}(f)(\cdot , g_2) \right\|_{L^1(G_1)},$$

whence

$$\left\| S_{P,\gamma}(f) \right\|_{L^1(G)} \lesssim \left\| \mathcal{M}_{P,\beta}(f) \right\|_{L^1(G)} \leq \left\| \mathcal{M}_{P,\beta}(f) \right\|_{L^1(G)}$$

by integration over $G_2$ and the pointwise inequality $\mathcal{M}_{P,\beta}(f) \leq \mathcal{M}_{P,\beta}(f)$. A similar result holds for the Littlewood–Paley operator acting in the second variable only.

The function $(g, t) \mapsto f * P_t(g)$ is harmonic in the $g_1$ and $t_1$ variables, and in the $g_2$ and $t_2$ variables. This leads to a more complicated analogue of (15), with four terms, $I_{11}, I_{12}, I_{12},$ and $I_{22}$, where the subscript $i_1 i_2$ indicates a term like $I_{i_1}$ in the first factor, and a term like $I_{i_2}$ in the second factor.

There is one “main term” $I_{11}$ with a double sub-Laplacian, namely,

$$\mathcal{E}_1 \mathcal{E}_2 \left( |f * P_t(g) \eta(H_t(g))|^2 \right).$$

When integrated, by iterating the argument used to treat $I_1$ in Section 5.3, $I_{11}$ gives

$$\int_{L^1(\alpha)} |f(g)|^2 \, dg.$$

The “mixed terms” $I_{12}$ and $I_{12}$, with a sub-Laplacian in one variable and a square function in the other, may be treated by (16) and its analogue with $G_1$ and $G_2$ interchanged. Finally, the “double error term” $I_{22}$ may be treated using Littlewood–Paley theory, much as we treated $I_2$ before.

5.5. A reproducing formula. We shall use the discrete Calderón reproducing formula from [22, Theorem 2.9]. We first give a definition of the space $\mathcal{M}(G, r, g)$ of molecules of scale $r$ near a point $g$ on a group $G$ of homogeneous dimension $Q$, and then define the analogous space on a product group. In the more general setting of spaces of homogenous type, this space was introduced in [21]. Recall that $\varepsilon \in (0, 1]$ is a fixed parameter.
Definition 5.6. Fix $r > 0$ and $g \in G$. We say that a function $f$ on $G$ is in $M(G, r, g)$ if there is a constant $C$ such that
\begin{equation}
|f(h)| \leq C \frac{r^\varepsilon}{(r + \rho(g^{-1}h))^{Q + \varepsilon}}
\end{equation}
(17)
\begin{equation}
|f(h) - f(h')| \leq C \frac{\rho(h^{-1}h')^{\varepsilon}}{(r + \rho(g^{-1}h) + \rho(g^{-1}h'))^{Q + \varepsilon}}
\end{equation}
for all $h, h' \in G$. If moreover $f$ satisfies the cancellation condition
\[ \int_G f(g) \, dg = 0, \]
then we write $f \in M_0(G, r, g)$. The norm $\|f\|_{M(G, r, g)}$ is defined to be the least constant $C$ such that the inequalities (17) both hold.

Clearly, the space $M(G)$ of (11) is equal to $M(G, 1, o)$, and $f \in M(G)$ if and only if $f \in M(G, r, g)$ for all $r > 0$ and all $g \in G$. Changing $r$ or $g$ changes the norms.

We now define the molecular space $M(G, r, g)$ on the product group $G$ as follows.

Definition 5.7. Fix $r \in T$ and $g \in G$. We say that $\psi : G \to \mathbb{C}$ is in $M(G, r, g)$ if $\psi(\cdot, h_2) \in M(G_1, r_1, g_1)$ for all $h_2 \in G_2$ and $\psi(h_1, \cdot) \in M(G_2, r_2, g_2)$ for all $h_1 \in G_1$, and
\begin{equation}
\|\psi(\cdot, h_2)\|_{M(G_1, r_1, g_1)} \leq C \frac{r_2^{\varepsilon}}{(r_2 + \rho_2(g_2^{-1}h_2))^{Q_2 + \varepsilon}}
\end{equation}
\begin{equation}
\|\psi(h_1, \cdot)\|_{M(G_2, r_2, g_2)} \leq C \frac{r_1^{\varepsilon}}{(r_1 + \rho_1(g_1^{-1}h_1))^{Q_1 + \varepsilon}}
\end{equation}
(18)
\begin{equation}
\|\psi(\cdot, h_2) - \psi(\cdot, h'_2)\|_{M(G_1, r_1, g_1)} \leq C \frac{\rho_2(h_2^{-1}h'_2)^{\varepsilon}}{(r_2 + \rho_2(g_2^{-1}h_2) + \rho_2(g_2^{-1}h'_2))^{Q_2 + \varepsilon}}
\end{equation}
\begin{equation}
\|\psi(h_1, \cdot) - \psi(h'_1, \cdot)\|_{M(G_2, r_2, g_2)} \leq C \frac{\rho_1(h_1^{-1}h'_1)^{\varepsilon}}{(r_1 + \rho_1(g_1^{-1}h_1) + \rho_1(g_1^{-1}h'_1))^{Q_1 + \varepsilon}}
\end{equation}
for all $h, h' \in G$. If moreover $\psi$ satisfies the cancellation conditions
\[ \int_{G_1} \psi(g_1, \cdot) \, dg_1 = 0 \quad \text{and} \quad \int_{G_2} \psi(\cdot, g_2) \, dg_2 = 0, \]
then we write $\psi \in M_0(G, r, g)$. The norm $\|\psi\|_{M(G, r, g)}$ is defined to be the least constant $C$ such that the inequalities (18) above all hold.

Evidently, if $\psi_1 \in M_0(G_1, r_1, g_1)$ and $\psi_2 \in M_0(G_2, r_2, g_2)$, then $\psi_1 \otimes \psi_2 \in M_0(G, r, g)$. It is easy to check that
\[ \|f\|_{M(G, r, g)} = \|r_1^{Q_1} r_2^{Q_2} \psi(h^{-1} \sigma(\cdot))\|_{M(G_1 \times G_2)}, \]

Hence the $L^1(G)$ norms of elements of a bounded subset of $M(G, r, g)$ are bounded.

We are now ready to state the version of the Calderón reproducing formula that we are going to use. Let $\sigma : \mathcal{P}(G) \to G$ be an arbitrary function such that $\sigma(R) \in \hat{R}$ for all $R \in \mathcal{P}(G)$ and let $\ell : \mathcal{P}(G) \to T$ be the function such that $\ell_i(Q_1 \times Q_2) = \ell(Q_i)$, the "side-length" of $Q_i$. For $\varphi^{[1]} \in M(G_1)$ and $\varphi^{[2]} \in M(G_2)$, we take $\varphi$ to be $\varphi^{[1]} \otimes \varphi^{[2]}$, and define $\varphi_R$ to be the function $h \mapsto [\varphi]_{\ell(R)} (h^{-1} \sigma(R))$. Observe that
\begin{equation}
\int_G f(h) \varphi_R(h) \, dh = \int_G f(h) [\varphi]_{\ell(R)} (h^{-1} \sigma(R)) \, dh = f * [\varphi]_{\ell(R)} (\sigma(R)).
\end{equation}
The point of the following theorem is that the collection  \{ |R|^{1/2} \varphi_R : R \in \mathcal{P}(G) \} is a well-behaved frame in \( L^2(G) \), with a well-behaved dual frame  \{ |R|^{1/2} \tilde{\varphi}_R : R \in \mathcal{P}(G) \}.

By well-behaved, we mean that \( \tilde{\varphi}_R \) exist functions \( \psi \) as discussed above. Then, after possible replacing (22, Theorem 2.9) Theorem 5.8

\[ R \]

for all \( \psi \in M_0(G) \), which may also depend on \( \sigma \), such that

\[ \psi = \sum_{R \in \mathcal{P}(G)} |R| \langle \psi, \varphi_R \rangle \tilde{\varphi}_R \]

for every \( \psi \in M_0(G) \). Further,

\[ \| \varphi_R \|_{M(G,\ell(R),\sigma(R))} + \| \tilde{\varphi}_R \|_{M_0(G,\ell(R),\sigma(R))} \]

is uniformly bounded, irrespective of \( R \) and the choice of \( \sigma(R) \).

As \( L^1(G) \) is a subspace of the dual space of \( M_0(G) \), by 19 and a duality argument,

\[ f * \psi(g) = \sum_{R \in \mathcal{P}(G)} \left( |R| f * [\varphi]_{\ell(R)}(\sigma(R)) \right) \tilde{\varphi}_R * \psi(g) \quad \forall g \in G \]

for all \( \psi \in M_0(G) \) and all \( f \in L^1(G) \).

5.6. **Proof of Proposition 5.3**. We are going to prove Proposition 5.3. Again, we need to extend what we know about homogeneous groups to product groups. We shall prove a stronger result concerning the grand maximal function, which we now introduce.

We define

\[ F(G) := \left\{ \zeta^{[1]} \otimes \zeta^{[2]} : \| \zeta^{[1]} \|_{M(G_1)} \leq 1, \| \zeta^{[2]} \|_{M(G_2)} \leq 1 \right\}, \]

and the grand maximal operator \( \mathcal{G} \):

\[ \mathcal{G}(f)(g) := \sup \{ |f * \zeta_t(g)| : \zeta \in F(G), t \in T \} \quad \forall g \in G \]

for all \( f \in L^1(G) \). We write \( R_h \) for the operator of right translation by \( h \in G \), that is, \( R_h \zeta(g) = \zeta(gh) \) for all \( g \in G \) and \( \zeta \in F(G) \). Since

\[ \mathcal{M}_{\zeta,\beta}(f)(g) = \sup \{ |f * \zeta_t(g')| : g' \in R(g,\beta t), t \in T \} \]

\[ = \sup \{ |f * (R_h \zeta)_t(g)| : h \in R(o,\beta,\beta), t \in T \}, \]

and \( R_h \zeta \) is a uniformly bounded (\( \beta \)-dependent) multiple of a function in \( F(G) \) when \( h \in R(o,\beta,\beta) \) we deduce that

\[ \mathcal{M}_{\zeta,\beta} f(g) \lesssim_\beta \mathcal{G}(f)(g) \quad \forall g \in G. \]

Take functions \( \varphi^{[i]} \) on \( G_i \) such that \( \| \varphi^{[i]} \|_{M(G_i)} \leq 1 \) and \( \int_{G_i} \varphi^{[i]} \, \text{d}g_i \neq 0 \). From the discussion above, it will suffice to prove that there exists \( \theta \in (0,1) \) such that

\[ \mathcal{M}_{\zeta,0} f(g) \lesssim \left( \mathcal{M}_S(\| \mathcal{M}_{\varphi,0}(f)(g) \|^\theta) \right)^{1/\theta} \quad \forall g \in G \]

for all \( \beta \geq 0 \), all \( \zeta \in F(G) \), and all \( f \in L^1(G) \), for then the \( L^{1/\theta}(G) \) boundedness of \( \mathcal{M}_S \) shows that

\[ \| \mathcal{G}(f) \|_{L^1(G)} \lesssim \int_G \left( \mathcal{M}_S(\| \mathcal{M}_{\varphi,0}(f) \|^\theta) \right)^{1/\theta} \, \text{d}g \lesssim \| \mathcal{M}_{\varphi,0}(f) \|_{L^1(G)}, \]
which implies the required result. We may assume that $f$ is continuous, by mollification.

To prove (21), we make and confirm three claims, which together imply the result.

**Claim 1:** for $\theta$ less than but close to 1,

\begin{equation}
|f \ast \zeta_t| \lesssim \|\psi\|_{M_0(G)} \left( M_S\left( |M_{\varphi,0}(f)|^\theta \right) \right)^{1/\theta}
\end{equation}

for all $f \in L^1(G)$, all $t \in T$ and all $\zeta$ of the form $\psi_1 \otimes \psi_2$, where $\psi_1 \in M_0(G_1)$ and $\psi_2 \in M_0(G_2)$. To prove this claim, we use Theorem 5.8 which tells us that

\begin{equation}
f \ast \psi_t(g) = \sum_{R \in \mathcal{R}(G)} |R| \langle f, \varphi_R \rangle \tilde{\varphi}_R \ast \psi_t(g).
\end{equation}

On the one hand, once $f$ is given, we may choose $\sigma$ such that

\[ |\langle f, \varphi_R \rangle| = |(f \ast [\varphi]_{t(R)})(\sigma(R))| = \min\{ |(f \ast [\varphi]_{t(R)})(h)| : h \in \bar{R} \}, \]

and on the other, by the almost orthogonality estimate of [24 (4.4)], for all choices of $\sigma$ and all choices of $h$ in $R$,

\[ |\tilde{\varphi}_R \ast \psi_t(g)| \lesssim \mu_{j,t}(h^{-1}g) \]

\[ \sim \frac{(\ell_1(R) \land t_1)^\varepsilon (\ell_1(R)t_1)^{-\varepsilon}}{(\ell_1(R) \land t_1)^{-1} + \rho_1(h_1^{-1}g_1)^{Q_1 + \varepsilon}} \frac{(\ell_2(R) \land t_2)^\varepsilon (\ell_2(R)t_2)^{-\varepsilon}}{(\ell_2(R) \land t_2)^{-1} + \rho_1(h_2^{-1}g_2)^{Q_2 + \varepsilon}}. \]

Here $\mu_{j,t}$ is the least decreasing biradial majorant for all the functions $\tilde{\varphi}_R \ast \psi_t(h^{-1})$ when $h \in R \in \mathcal{R}(G)$, and $a \land b$ denotes the minimum of $a$ and $b$. Recall that the “sidelengths” of the cubes making the rectangle $R \in \mathcal{R}(G)$ are $\kappa^{j_1}$ and $\kappa^{j_2}$. Then, by also using (22) and (3), we see that

\begin{equation}
|f \ast \psi_t(g)|^\theta \leq \left( \sum_{j \in Z^2} \sum_{R \in \mathcal{R}(G)} |R| \min_{g \in R} \mathcal{M}_{\varphi,0}(f)(g) |\tilde{\varphi}_R \ast \psi_t(g)|^\theta \right)^{1/\theta} \leq \left( \sum_{j \in Z^2} \sum_{R \in \mathcal{R}(G)} |R|^{\theta} \left( \min_{g \in R} \mathcal{M}_{\varphi,0}(f)(g) \right)^\theta |\tilde{\varphi}_R \ast \psi_t(g)|^\theta \right)^{1/\theta} = \left( \sum_{j \in Z^2} \sum_{R \in \mathcal{R}(G)} |R|^{\theta-1} \int_R \left( \min_{g \in R} \mathcal{M}_{\varphi,0}(f)(g) \right)^\theta |\tilde{\varphi}_R \ast \psi_t(g)|^\theta \, dh \right)^{1/\theta} \sim \left( \sum_{j \in Z^2} (\kappa_1^{j_1} \kappa_2^{j_2})^{\theta-1} \int_G \left( \mathcal{M}_{\varphi,0}(f)(h) \right)^\theta \mu_{j,t}(h^{-1}g)^\theta \, dh \right)^{1/\theta} \leq \left( \sum_{j \in Z^2} (\kappa_1^{j_1} \kappa_2^{j_2})^{\theta-1} \left\| \mu_{j,t} \right\|_{L^1(G)}^\theta M_S \left( \mathcal{M}_{\varphi,0}(f) \right)^\theta (g) \right)^{1/\theta}.
\end{equation}

If $\max\{Q_1/(Q_1 + \varepsilon), Q_2/(Q_2 + \varepsilon)\} < \theta < 1$, then computation shows that

\[ \sum_{j \in Z^2} (\kappa_1^{j_1} \kappa_2^{j_2})^{\theta-1} \left\| \mu_{j,t} \right\|_{L^1(G)}^\theta < \infty. \]

Thus the right-hand side of (23) is bounded by a multiple of

\[ (M_S\left( |M_{\varphi,0}(f)|^\theta \right))(g), \]

which implies (21) and proves our claim.
Claim 2: for θ less than but close to 1,
\[ |f * ζ_t(g)| \lesssim_θ \|ζ_1\|_{M_0(G_1)} \left( M_S \left( |M_{φ,0}(f)|^θ \right)(g) \right)^{1/θ} \]
for all \( f \in L^1(G) \), all \( t \in T \), where \( ζ = ψ_1 \otimes φ_2 \); here \( ψ_1 \in M_0(G_1) \).

The proof of this claim involves use of a reproducing formula that involves the first variable only, namely,
\[ f^{[1]}_1 *_1 ψ^{[1]}_1 (g_1) = \sum_{Q ∈ ℂ(G_1)} |Q| f^{[1]}_1 *_1 [φ^{[1]}_1]_Q (σ(Q)) \tilde{φ}^{[1]}_Q * ψ^{[1]}_1 (g_1) \quad ∀g_1 \in G_1 \]
where \( f^{[1]}_1 \in L^1(G_1) \). This implies that
\[ f * ζ_t(g) = \sum_{Q ∈ ℂ(G_1)} |Q| f * ([φ^{[1]}_1]_Q \otimes φ^{[2]}_2) (σ(Q), g_2) \tilde{φ}^{[1]}_Q * ψ^{[1]}_1 (g_1) \quad ∀g \in G, \]
when \( f \in L^1(G) \); a similar argument to that for Claim 1 may be used. We see that
\[ |φ_R * ζ_t(g)|^θ ≤ \left( \sum_{j_1 \in ℤ} \sum_{Q \in ℂ^1(G_1)} |Q| \min_{g_1 ∈ Q} M_{φ,0}(f)(g_1, g_2) \left| \tilde{φ}^{[1]}_Q * ψ^{[1]}_1 (g_1) \right| \right)^θ \]
\[ ≤ \left( \sum_{j_1 \in ℤ} (n^{[2]}_{j_1})^{θ-1} \left\| \left( M^{[1]}_{j_1} \right)^{θ} \right\|_{L^1(G_1)} \left( M_1 \otimes I_2 \right) \left( M_{φ,0}(f)^{θ} \right)(g) \right) \]
\[ ≤ M_S \left( M_{φ,0}(f)^{θ} \right)(g), \]
as claimed; here \( I_2 \) denotes the identity operator acting on functions on \( G_2 \).

Claim 3: for θ less than but close to 1,
\[ |f * ψ_t(g)| \lesssim_θ \|ψ_2\|_{M_0(G_2)} \left( M_S \left( |M_{φ,0}(f)|^θ \right)(g) \right)^{1/θ} \]
for all \( f \in L^1(G) \), all \( t \in T \), where now \( ψ_1 = φ_1 \) while \( ψ_2 \in M_0(G_2) \). The proof of this is a very minor modification of that of Claim 2.

To finish the proof, we must estimate \( M_{ζ,0}f \), where \( ζ_1 \in M(G_1) \) and \( ζ_2 \in M(G_2) \). We write \( ζ_1 = c_1φ_1 + ψ_1 \), where \( ψ \in M_0(G_1) \) and \( c_1 \) is chosen to make the integrals of both sides equal, and we decompose \( ζ_2 \) analogously. Then \( M_{ζ,0}f(g) \) is dominated by a sum of four terms, each of which is bounded pointwise by \( (M_S(M_{φ,0}f)^θ)(g) \). This proves the desired inequality and hence Proposition \ref{5.3}.

6. Concluding remarks
Many of our results can be proved in greater generality. For example, Proposition \ref{5.3} should be true on much more general spaces of homogeneous type. Other results require the structure of stratified group that we have used here. These include Theorem \ref{1.1} and Proposition \ref{5.2}. Indeed, the first relies on the Christ–Geller singular integral characterisation of the Hardy space on stratified groups, and the second on various properties of the Poisson kernel. It is an interesting challenge to extend either of these to a more substantial class of nilpotent Lie groups, let alone to general spaces of homogeneous type.

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REFERENCES

[1] Auscher, P., McIntosh, A., Russ, E.: Hardy spaces of differential forms on Riemannian manifolds, J. Geom. Anal. 18, 192–248 (2008).
[2] Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Stratiﬁed Lie Groups and Potential Theory for their Sub-Laplacians. Springer Monographs in Mathematics. Berlin–Heidelberg–New York (2007).
[3] Chang, S.-Y. A., Fefferman, R.: A continuous version of duality of H^1 with BMO on the bidisc, Ann. of Math. 112, 179–201 (1980).
[4] Chen, P., Cowling, M. G., Lee, M., Li, J., Ottazzi, A., Flag Hardy space theory on Heisenberg groups and applications. Available at arXiv:2102.07371 (2021).
[5] Chen, P., Duong, X. T., Li, J., Ward, L. A., Yan, L., Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type, Math. Z. 282, 1033–1065 (2016).
[6] Coifman, R. R., and Weiss, G.: Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math. 242. Springer Verlag, Berlin–Heidelberg–New York (1971).
[7] Coifman, R. R., and Weiss, G.: Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83, 569–645 (1977).
[8] Cordoba, A., Fefferman, R.: A geometric proof of the strong maximal theorem, Ann. of Math. 102, 95–100 (1975).
[9] Coulhon, T., Duong, X.T.: Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss, Adv. Diff. Equat. 5, 343–368 (2000).
[10] Christ, M.: A T(1) theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 61, 601–628 (1990).
[11] Christ, M., Geller, D.: Singular integral characterisations of Hardy spaces on homogeneous groups, Duke. Math. J. 61, 601–628 (1990).
[12] Fefferman, C., Stein, E. M.: H^p spaces of several variables, Acta Math. 129, 137–193 (1972).
[13] Fefferman, R.: Harmonic Analysis on product spaces, Ann. of Math. 126, 109–130 (1987).
[14] Fefferman, R., Stein, E. M.: Singular integrals on product spaces, Adv. Math. 45, 117–143 (1982).
[15] Folland, G. B.: Subelliptic estimates and function spaces on nilpotent Lie groups, Arkiv för Mat 13, 161–207 (1975).
[16] Ferguson, S. H., Lacey, M. T.: A characterization of product BMO by commutators, Acta Math. 189, 143–160 (2002).
[17] Folland, G. B., Stein, E. M.: Hardy Spaces on Homogeneous Groups. Princeton University Press, Princeton, N. J. (1982).
[18] Franchi, B., Tchou, N., Tesi, M. C., Div-curl type theorem, H-convergence and Stokes formula in the Heisenberg group, Comm. Contemp. Math. 8, 67–99 (2006).
[19] Geller, D, Mayeli, A.: Continuous wavelets and frames on stratiﬁed Lie groups, I, J. Fourier Anal. Appl. 12, 543–579 (2006).
[20] Grafakos, L.: Classical Fourier Analysis. Third Edition. Graduate Texts in Mathematics 249, Springer (2014).
[21] Gundy, R. F., Stein, E. M.: H^p theory for the poly-disc, Proc. Nat. Acad. Sci. U. S. A. 76, 1026–1029 (1979).
[22] Han, Y., Li, J., Lu, G.: Multiparameter Hardy space theory on Carnot–Carathéodory spaces and product spaces of homogeneous type, Trans. Amer. Math. Soc. 365, 319–360 (2013).
[23] Han, Y., Li, J., Pereyra, M. C., and Ward, L. A.: Atomic decomposition of product Hardy spaces via wavelet bases on spaces of homogeneous type, New York J. Math. 27, 1173–1239 (2021).
[24] Han, Y., Li, J., Ward, L. A.: Hardy space theory on spaces of homogeneous type via orthonormal wavelet bases, Appl. Comp. Harm. Anal. 45, 120–169 (2018).
[25] Hebisch, W. Sikora, A.: A smooth subadditive homogeneous norm on a homogeneous group, Studia Math. XCVI, 231–236 (1990).
[26] Hulanicki, A.: A functional calculus for Rockland operators on nilpotent Lie groups, Studia Math. LXXVIII, 253–266 (1984).
[27] Hytönen, T., Kairema, A.: Systems of pseudodyadic cubes in a doubling metric space, Colloq. Math. 126, 1–33 (2012).
[28] Hytönen, T., Martikainen, H.: Non-homogeneous T1 theorem for bi-parameter singular integrals, Adv. Math. 261, 220–273 (2014).
[29] Journé, J.-L.: Calderón–Zygmund operators on product spaces, Rev. Mat. Iberoam. 1, 55–91 (1985).
[30] Lacey, M. T., Petermichl, S., Pipher, J., Wick, B. D.: Multiparameter Riesz commutators, Amer. J. Math. 131, 731–769 (2009).
[31] Merryfield, K. G.: On the area integral, Carleson measures and $H^p$ in the polydisc, Indiana Univ. Math. J. 34, 663–685 (1985).
[32] Ou, Y.: A $T(b)$ theorem on product spaces, Trans. Amer. Math. Soc. 367, 6159–6197 (2015).
[33] Ou, Y., Petermichl, S., Strouse, E.: Higher order Journé commutators and characterizations of multi-parameter BMO, Adv. Math. 291, 24–58 (2016).
[34] Pipher, J.: Journé’s covering lemma and its extension to higher dimensions, Duke Math. J. 53, 683–690 (1986).
[35] Sawyer, E., Wheeden, R.: Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, Amer. J. Math. 114, 813–874 (1992).
[36] Stein, E. M.: Harmonic Analysis Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series 43. Princeton University Press, Princeton, N. J. (1993).
[37] Stein, E. M., Weiss, G.: On the theory of harmonic functions of several variables, I. The theory of $H^p$ spaces, Acta Math. 103, 25–62 (1960).
[38] Uchiyama, A.: The factorization of $H^p$ on the space of homogeneous type, Pacific J. Math. 92, 453–468 (1981).
[39] Varopoulos, N. Th., Saloff-Coste, L., Coulhon, T.: Analysis and Geometry on Groups. Cambridge Tracts in Mathematics 100. Cambridge University Press, Cambridge (1992).

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