BIFURCATION ANALYSIS IN A DELAYED TOXIC-PHYTOPLANKTON AND ZOOPLANKTON ECOSYSTEM WITH MONOD-HALDANE FUNCTIONAL RESPONSE

ZHICHAO JIANG*
School of Liberal Arts and Sciences, North China Institute of Aerospace Engineering
Langfang, 065000, China

ZEXIAN ZHANG
283 Company, Second Research Institute of the CASIC
Beijing, 100089, China

MAOYAN JIE
Aerospace Science and Technology, North China Institute of Aerospace Engineering
Langfang, 065000, China

(Communicated by Hao Wang)

Abstract. We structure a phytoplankton zooplankton interaction system by incorporating (i) Monod-Haldane type functional response function; (ii) two delays accounting, respectively, for the gestation delay $\tau$ of the zooplankton and the time $\tau_1$ required for the maturity of TPP. Firstly, we give the existence of equilibrium and property of solutions. The global convergence to the boundary equilibrium is also derived under a certain criterion. Secondly, in the case without the maturity delay $\tau_1$, the gestation delay $\tau$ may lead to stability switches of the positive equilibrium. Then fixed $\tau$ in stable interval, the effect of $\tau_1$ is investigated and find $\tau_1$ can also cause the oscillation of system. Specially, when $\tau = \tau_1$, under certain conditions, the periodic solution will exist with the wide range as delay away from critical value. To deal with the local stability of the positive equilibrium under a general case with all delays being positive, we use the crossing curve methods, it can obtain the stable changes of positive equilibrium in $(\tau, \tau_1)$ plane. When choosing $\tau$ in the unstable interval, the system still can occur Hopf bifurcation, which extends the crossing curve methods to the system exponentially decayed delay-dependent coefficients. Some numerical simulations are given to indicate the correction of the theoretical analyses.

2020 Mathematics Subject Classification. 34K18, 34K20, 92D25.
Key words and phrases. Double delays, Monod-Haldane functional response, stability, Hopf bifurcation, crossing curve.

This research is supported by National Natural Science Foundation of China (No. 11801014), Natural Science Foundation of Hebei Province from China (No. A2018409004), University Discipline Top Talent Selection and Training Program of Hebei Province from China (No. SLRC2019020) and Graduate Student Demonstration Course Construction of Hebei Province from China (No. KCJSX2020093).

* Corresponding author: jzhsuper@163.com.
1. Introduction. For decades, as plankton is the basis of all food chains and networks and plays an important role in the marine ecology, the dynamics of plankton have become an important research field. Phytoplankton, also known as microalgae, is similar to terrestrial plants because it contains chlorophyll and needs sunlight to survive and grow. Most of phytoplanktons are buoyant, floating on top of the ocean where sunlight penetrates the water. In a balanced ecosystem, they feed a variety of marine life, including whales, shrimp, snails and jellyfish. A striking characteristic associated with many phytoplankton populations is rapid and large-scale bloom formation. These events are characterized by a sharp increase in number, by several orders of magnitude, followed by a sudden decline in the phytoplankton population, which returns to its original low level as if nothing had happened. Zooplankton is the animal that lives in communities of plankton including herbivores and carnivores. Herbivores feed on phytoplankton and are then eaten by carnivorous.

However, a special class of phytoplankton in most aquatic ecosystems has a physiological characteristic of releasing “toxin” or “allelochemicals” that are harmful to the growth of other algae. Algal toxicity has an important effect on the distribution of phytoplankton and zooplankton populations. A viable phytoplankton community usually emerges during the occurrence of harmful algal blooms in riverine ecosystems. The human consequences of harmful algal blooms are high costs for fisheries and tourism. High mortality of fish and other marine animals during red tides sometimes leads to a ban on trade in fish and shellfish for a considerable period thereafter. Further red tides pose a problem for human health, as consumers may die from poisoning after exposure to toxic algae shellfish. Although human deaths are rare, cases of eye irritation, headaches or other diseases can be observed. The economic impact is severe, especially for communities that rely almost entirely on fishing. Tourism has also been affected, as tourists are not allowed to visit the affected areas. Zooplankton is completely dependent on phytoplankton as the most favorable food source, and the change of phytoplankton density has a great influence on the growth of zooplankton. When these harmful species multiply in large numbers, the cumulative effects of all the toxins may lead to a decrease in grazing pressure on zooplankton.

It is important to understand the persistence of algae in the presence of flow and the temporal and spatial variation of algal abundance and toxicity during blooms. The presence of TPP might provide a potential mechanism for the maintenance of the coexistence and biodiversity of many phytoplankton and zooplankton species in a homogeneous environment. In natural waters, the dynamics of phytoplankton and zooplankton is regulated by a huge number of physical and biological factors. Generation of blooms may really be a result of all such factors, making it very difficult to incorporate each and every one in a model. In 2002, Chattopadhyay et al. [5] proposed solely on direct interactions among phytoplankton species structured as TPP, along with the grazer zooplankton, and discussed the role of TPP in harmful algal blooms. The general form of the mathematical model they considered was the following nonlinear coupling of ordinary differential equations

$$\begin{align*}
\frac{dP(t)}{dt} &= rP(t)\left(1 - \frac{P(t)}{L}\right) - \alpha f(P(t))Z(t), \\
\frac{dZ(t)}{dt} &= \beta f(P(t))Z(t) - \mu Z(t) - \theta g(P(t))Z(t),
\end{align*}$$

(1)

where two variables are included: the TPP population $P(t)$ and zooplankton population $Z(t)$. $f(P(t))$ indicates the functional response of zooplankton to phytoplankton and $g(P(t))$ describes the distribution of toxic substances that eventually
kill off zooplankton populations. Since the pioneering work of Chattopadhyay et al, a growing number of biological papers have been published on the TPP-zooplankton model, demonstrating the importance of this interaction [5, 6, 20, 18, 11, 12, 13], and so on.

Some researchers had shown that the toxic substances released by certain phytoplankton species repelled zooplankton, which tried to leave areas of high phytoplankton density. This is similar to the phytoplankton group defense mechanism against zooplankton. The zooplankton can identify TPP because the latter consumes too much and thus kills too many zooplanktons. The zooplankton reduces their consumption through a change in chemotactic sensitivity in a direction opposite to the TPP gradient. This is modeled with a simplified non-monotonic Monod-Haldane type of function response expressed by \( P(t)/(m^2 + P^2(t)) \) [18, 2, 9]. In this paper, we investigate the following system:

\[
\begin{aligned}
\frac{dP(t)}{dt} &= rP(t) \left(1 - \frac{P(t)}{L}\right) - \frac{\alpha P(t)Z(t)}{m^2 + P^2(t)} , \\
\frac{dZ(t)}{dt} &= \frac{\beta e^{-\tau}P(t-\tau)Z(t-\tau)}{m^2 + P^2(t-\tau)} - \mu Z(t) + \frac{\rho P(t-\tau_1)Z(t)}{m^2 + P^2(t-\tau_1)}.
\end{aligned}
\] (2)

The term \( P(t-\tau_1)/(m^2 + P^2(t-\tau_1)) \) represents the distribution of toxic substances that eventually kill off zooplankton populations. The biological senses and units of these parameters in system (2) are shown in Table 1.

### Table 1. Descriptions and units of parameters of system (2)

| Symbol | Parameter Definition                              | Unit   |
|--------|--------------------------------------------------|--------|
| \( r \) | Intrinsic growth rate of TPP                     | day\(^{-1}\) |
| \( L \) | Environmental carrying capacity                   | gCm\(^{-3}\) |
| \( \alpha \) | Grazing efficiency of zooplankton               | gCm\(^{-3}\) |
| \( \beta \) | Growth efficiency of zooplankton                | day\(^{-1}\) |
| \( \mu \) | Natural death rate of zooplankton               | day\(^{-1}\) |
| \( m \) | Half-saturation constant                         | [gCm\(^{-3}\)]\(^2\) |
| \( \rho \) | Toxin-producing rate of TPP                      | gCm\(^{-3}\) |
| \( \tau \) | Gestation delay of zooplankton                  | day\(^{-1}\) |
| \( \tau_1 \) | Delay required for the maturity of TPP          | day\(^{-1}\) |

According to biological senses, the initial conditions are assumed satisfying

\( P(\theta) = \varphi_1(\theta) \geq 0, \ Z(\theta) = \varphi_2(\theta) \geq 0, \ \varphi_i(0) > 0, \ i = 0, 1, \ \theta \in [-\tau_{max}, 0] \), (3)

where \((\varphi_1(\theta), \varphi_2(\theta)) \in C([-\tau_{max}, 0], \mathbb{R}_+^2)\) and \(\tau_{max} = \max\{\tau, \tau_1\}\).

This paper focuses on discussing the effects of \( \tau \) and \( \tau_1 \) to the dynamic behaviors of system (2) and the organization is as follows. In Section 2, we give the existence of equilibrium and property of solutions for system (2). The sufficient conditions ensuring the globally asymptotical stability of the boundary equilibrium are given in Section 3. The conclusions show that the maturity delay of TPP does not effect the stability of the boundary equilibrium while the gestation delay of zooplankton can have the key influence. Then, let \( \tau_1 = 0 \) and dynamic behaviors of system with one delay \( \tau \) are investigated. The stability switches can occur as \( \tau \) varying. Then fixed \( \tau \) in stable interval, using \( \tau_1 \) as parameter, it finds \( \tau_1 \) can also cause the oscillation of system. Specially, when \( \tau = \tau_1 \), the system can still occur the stable switching phenomenon, and, under certain conditions, the periodic solution will exist with the wide range as delay away from critical value. These results
are shown in Section 4. In Section 5, using the crossing curve methods, the stable changes of positive equilibrium in \((\tau, \tau_1)\) plane can be obtained. When choosing \(\tau\) in the unstable interval, the system still can occur Hopf bifurcation as delays varying. These results show that two delays are responsible for the periodic oscillations of the system. Some numerical simulation examples are given to verify the correction of the theoretical analyses in Section 6. At last, some conclusions are given.

2. Equilibrium and property of solutions. The system (2) always exists two boundary equilibria \(E_0(0, 0)\) and \(E_1(L, 0)\). The possible positive equilibrium \(E^*(P^*, Z^*)\) satisfies

\[ Z^* = \frac{r}{\alpha L} \left( m^2 + P^{*2} \right) (L - P^*) \]

and

\[ f(P^*) := \mu P^{*2} - (\beta e^{-\mu \tau} - \rho)P^* + \mu m^2 = 0. \]

Define \( R_\tau = \frac{\beta e^{-\mu \tau} - \rho}{2\mu m} \), some conditions ensuring the existence of \(E^*\) are given as follows.

**Lemma 2.1.** (i) If \( R_\tau > \frac{L^2 + m^2}{2mL} \), then system (2) exists a uniquely positive equilibrium \(E^*(P^*_+, Z^*_+)\);

(ii) If \( m < L \), then

(a) system (2) exists two positive equilibria \(E^*_+ (P^*_+, Z^*_+)\) when \( 1 < R_\tau < \frac{L^2 + m^2}{2mL} \);

(b) system (2) exists a uniquely positive equilibrium \(E^*_+ (P^*_-, Z^*_-)\) when \( R_\tau = 1 \)

or \( R_\tau \geq \frac{L^2 + m^2}{2mL} \);

(c) otherwise, system (2) has no positive equilibrium, where

\[
\begin{aligned}
P^*_\pm &= m \left( R_\tau \pm \sqrt{R_\tau^2 - 1} \right), \\
Z^*_\pm &= \frac{r}{\alpha L} \left( m^2 + P^{*2}_\pm \right) (L - P^*_\pm).
\end{aligned}
\]

**Remark 1.** From Lemma 2.1, it knows that the Half-saturation constant \( m \) and environmental carrying capacity \( L \) play an important role in the existence of the positive equilibrium. If \( m \geq L \), then the bifurcation is forward when \( R_\tau = \frac{L^2 + m^2}{2mL} \) (see Figure 1). If \( m < L \), the bifurcation is backward when \( R_\tau = \frac{L^2 + m^2}{2mL} \) (see Figure 2).

![Figure 1](image1.png)

**Figure 1.** The plots of TPP and zooplankton at equilibrium versus \( R_\tau \) when \( r = 0.8, m = 10, \alpha = 5 \) and \( L = 6 \). There is a forward bifurcation from the zooplankton free equilibrium at \( R_\tau = 1.1333 \).

For convenience, when the positive equilibrium exists, we always assume that Lemma 2.1 (i) holds. Other cases are similar to discuss. Especially, when \( R_0 \) >
the proof of (a) the first equation of system (2) that is the maximum internal of the solution and $\sigma$ is the maximum internal of the solution and $\sigma = 0$. Integrating the first equation of system (2) gives $t > 0$. The boundedness of the solutions, we firstly see that it follows from the first equation of system (2) that $P(t) \leq r P(t)(1 - P(t)/L)$, which implies that $t \geq 0, \forall t \in [0, t^*)$. Taking $t^*$ to the second equation of system (2),

$$\frac{dZ(t)}{dt} \bigg|_{t=t^*} = \frac{\beta e^{-\alpha t} P(t^* - \tau) Z(t^* - \tau) - \mu Z(t^*) - \rho P(t^* - \tau) Z(t^*)}{m^2 + P^2(t^* - \tau)} > 0,$$

which leads to a contradiction. Therefore, $Z(t) > 0$ for all $t \in [0, \sigma]$. This completes the proof of (a).

To prove the boundedness of the solutions, we firstly see that it follows from the first equation of system (2) that $P(t) \leq r P(t)(1 - P(t)/L)$, which implies that $t \geq 0, \forall t \in [0, t^*)$. Taking $t^*$ to the second equation of system (2),

$$W(t) = P(t) + \frac{\alpha e^{\alpha t}}{\beta} Z(t + \tau)$$
whose derivative with respect to } t \text{ yields}
\[
\frac{dW(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{L}\right) - \frac{\mu ae^{-\rho t}}{\beta} Z(t + \tau) - \frac{\alpha e^{\alpha t + \rho t} P(t + \tau)}{\beta} + \frac{\alpha e^{\alpha t + \rho t} P(t - \tau)}{\beta} - \frac{\alpha e^{\alpha t + \rho t} P(t - \tau)}{\beta}.
\]
Applying the theorem of differential inequality, it has
\[
0 < W(t) \leq \frac{L(r + \mu)^2}{4\mu}(1 - e^{-\mu t}) + W(0)e^{-\mu t}.
\]
By the positivity of } P(t), it holds that \( \lim sup_{t \to \infty} Z(t) \leq \frac{L(r + \mu)^2}{4\mu} = M \). This completes the proof of (b).

From the first equation of system (2), we take notice of
\[
\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{L}\right) - \frac{\alpha P(t)Z(t)}{M + P(t)} \\
\geq rP(t)\left(1 - \frac{P(t)}{L}\right) - \frac{\alpha M}{mL} P(t) \\
= P(t)\left(r - \frac{\alpha M}{mL} - \frac{\alpha}{\beta} P(t)\right)
\]
which implies that \( \lim inf_{t \to \infty} P(t) \geq L(1 - \frac{\alpha M}{rmL}) = m_0 \text{ if } rm^2 > \alpha M \). This completes the proof of (c).

We appeal to the theory developed in [10] where existence and uniqueness and positivity are treated simultaneously, it has the following theorem.

**Theorem 2.3.** The solution of system (2) with the initial condition (3) is existent, unique, positive and bounded on } [0, +\infty) \text{ and } \Gamma = \{(\varphi_1(\theta), \varphi_2(\theta)) \in C| m_0 \leq \varphi_1(\theta) \leq L, 0 \leq \varphi_2(\theta) \leq M\} \text{ is positively invariant set for system (2).}

3. **Stability of boundary equilibrium.** That’s clear that } E_0 \text{ is always an unstable saddle point. The following result is concerned with the global dynamics of } E_1.

**Theorem 3.1.** If } R_\tau > \frac{L^2 + m^2}{2mL} \text{ (i.e., } 0 \leq \tau < \bar{\tau}) \text{, then } E_1 \text{ is unstable and locally asymptotically stable for } R_\tau < \frac{L^2 + m^2}{2mL} \text{ (i.e., } \tau > \bar{\tau}) \text{. Furthermore, if } R_\tau < \frac{1}{2}\left(\frac{\rho}{\mu m} - \frac{\rho}{\mu L}\right) \text{ (i.e., } \tau > \tilde{\tau} := \frac{1}{2}\mu \ln \frac{\beta L}{\mu m^2}) \text{, then } E_1 \text{ is globally asymptotically stable (GAS), where } \bar{\tau} < \tilde{\tau}.

**Proof.** The characteristic equation of system (2) at } E_1 \text{ is}
\[
(\lambda + r)\left[\lambda + \mu + \frac{\rho L}{m^2 + L^2} - e^{-(\lambda + \mu)\tau - \frac{\beta L}{m^2 + L^2}}\right] = 0.
\]
Equation (4) has one root } \lambda = -r < 0 \text{ and the other roots satisfy}
\[
\lambda + \mu + \frac{\rho L}{m^2 + L^2} e^{(\lambda + \mu)\tau} = \frac{\beta L}{m^2 + L^2}.
\]
Define the function } H(\lambda) := \left[\lambda + \mu + \frac{\rho L}{m^2 + L^2}\right] e^{(\lambda + \mu)\tau}, \text{ then } H(0) = (\mu + \frac{\rho L}{m^2 + L^2}) e^{\mu \tau} > 0, H'(\lambda) > 0 \text{ and } H(+\infty) = +\infty. \text{ Since } \tau > \bar{\tau}, \text{ by intermediate value theorem, (5) exists a uniquely positive root } \lambda(\tau). \text{ It means that (4) has at least one positive root, then } E_1 \text{ is unstable when } \tau < \bar{\tau}.

Next, it will prove } E_1 \text{ is GAS. Let } \zeta (\zeta > 0) \text{ be root of (5), then } \zeta \text{ satisfies}
\[
\zeta^2 = \left[\frac{(\beta e^{-\mu t} + \rho) L}{m^2 + L^2} + \mu\right] \left[\frac{(\beta e^{-\mu t} + \rho) L}{m^2 + L^2} - \mu\right].
\]
Remark 2. From the above results, the maturity delay of TPP has no affect to $\tau > \bar{\tau}$. If $\tau > \bar{\tau}$, then the above equation has no any positive root, i.e., (5) has no purely imaginary roots. Hence $E_1$ is LAS if $\tau > \bar{\tau}$.

Furthermore, define the Lyapunov functional

$$V(P, Z) = P - L - L \ln \frac{P}{L} + \frac{\alpha}{\beta e^{-\mu \tau}} Z + \alpha \int_{t-\tau}^{t} \frac{PZ}{m^2 + P^2} dt$$

whose time derivative along solution of system (2) is

$$\frac{dV}{dt} = \frac{P(t) - L}{P(t)} \left[ rP(t)(1 - \frac{P(t)}{L}) - \frac{\alpha P(t)Z(t)}{m^2 + P^2(t)} + \frac{\alpha}{\beta e^{-\mu \tau}} \left[ \frac{\beta e^{-\mu \tau} P(t-\tau)Z(t-\tau)}{m^2 + P^2(t-\tau)} - \mu Z(t) \right] \right]$$

$$- \frac{m^2}{m^2 + P^2(t-\tau)Z(t)} + \frac{\alpha P(t)Z(t)}{m^2 + P^2(t)} - \frac{\alpha P(t-\tau)Z(t-\tau)}{m^2 + P^2(t-\tau)} - \frac{\alpha \rho P(t-\tau)Z(t)}{m^2 + P^2(t-\tau)}$$

$$\leq - \frac{\mu}{\beta e^{-\mu \tau}} \left[ \frac{L}{m^2} - \frac{\mu}{\beta e^{-\mu \tau}} \right] Z(t).$$

If $\tau > \bar{\tau}$, then $\frac{dV}{dt} \leq 0$. Let $\Phi$ be largest invariant subset of $\frac{dV}{dt} = 0$, then for each element in $\Phi$, it has $P(t) = L$ and $Z(t) = 0$. By Lasalle invariance principle, $E_1$ is a global attractor. Together with the local stability, implies that $E_1$ is GAS. This completes the proof.

4. Stability of positive equilibrium. In this section, it always assumes that $R_0 > \frac{L^2 + m^2}{2mL}$ and $\tau \in [0, \bar{\tau})$ for the existence of $E^*$ and considers the local stability of $E^*$. At this situation, $E_1$ is unstable.

Let $x(t) = P(t) - P^*$ and $y(t) = Z(t) - Z^*$, then system (2) becomes

$$\begin{cases} 
\frac{dx(t)}{dt} = A_1 x(t) - \alpha A_2 y(t) + O(2), \\
\frac{dy(t)}{dt} = \beta e^{-\mu \tau} A_3 Z^* x(t-\tau) + \beta e^{-\mu \tau} A_2 y(t-\tau) - \rho A_3 Z^* x(t-\tau_1) - (\mu + \rho A_2) y(t) + O(2),
\end{cases}$$

where

$$A_1 = -\frac{L^*}{m^2 + P^2}, \quad A_2 = \frac{L^*}{m^2 + P^2}, \quad A_3 = \frac{m^2 - P^2}{m^2 + P^2} > 0.$$

The associated characteristic equation of the linearization of system (6) is

$$D(\lambda, \tau, \tau_1) := \lambda^2 + (\mu + \rho A_2 - A_1)\lambda - A_1(\mu + \rho A_2)$$

$$+ [\beta e^{-\mu \tau} A_2 (\alpha A_3 Z^* + A_1 - \lambda)] e^{-\lambda \tau_1} - \alpha \rho A_2 A_3 Z^* e^{-\lambda \tau_1} = 0 \quad (7)$$

and

$$D(\lambda, 0, 0) := \lambda^2 + (\mu + \rho A_2 - A_1 - \beta A_2)\lambda - A_1(\mu + \rho A_2 - \beta A_2)$$

$$+ \alpha (\beta - \rho) A_2 A_3 Z^* = 0. \quad (8)$$

The Routh-Hurwitz criterion implies that all solutions of (8) have negative real parts if and only if

$$\mu + (\rho - \beta) A_2 > A_1 \quad \text{and} \quad \alpha (\beta - \rho) A_2 A_3 Z^* > [\mu + (\rho - \beta) A_2] A_1. \quad (9)$$

Since system (2) includes two delays, the linearized system at the positive equilibrium produces a characteristic equation with delay-related parameters. Firstly, we follow the original method in [3] to establish a parallel result to deal with the problem which contains only one delay.
4.1. Case 1: $\tau_1 = 0, \tau > 0$. In this subsection, we investigate the effect of the delay $\tau$. Let $\tau_1 = 0$, then system (6) becomes
\[
\begin{align*}
\frac{dx(t)}{dt} &= A_1 x(t) - \alpha A_2 y(t) + O(2), \\
\frac{dy(t)}{dt} &= -\rho A_3 Z^* x(t) - (\mu + \rho A_2) y(t) + \beta e^{-\mu \tau} A_3 Z^* x(t - \tau) + \beta e^{-\mu \tau} A_2 y(t - \tau) + O(2),
\end{align*}
\] (10)
and in this case, (7) is equivalent to
\[
\Delta_1(\lambda, \tau) + \Delta_2(\lambda, \tau) e^{-\lambda \tau} = 0,
\] (11)
where
\[
\begin{align*}
\Delta_1(\lambda, \tau) &= \lambda^2 + p\lambda + q, \\
\Delta_2(\lambda, \tau) &= r_1 + s\lambda,
\end{align*}
\]
and
\[
\begin{align*}
p &= \mu + \rho A_2 - A_1, \\
q &= -A_1(\mu + \rho A_2) - \alpha \rho A_2 A_3 Z^*, \\
r_1 &= -s(\alpha A_3 Z^* + A_1), \\
s &= -\beta e^{-\mu \tau} A_2.
\end{align*}
\]
Next, it investigates the existence of purely imaginary roots $\lambda = i\varpi$ ($\varpi = \varpi(\tau) > 0$). It easily can verify the following relations:
(i) $\Delta_1(0, \tau) + \Delta_2(0, \tau) \neq 0$;
(ii) $\Delta_1(i\varpi, \tau) + \Delta_2(i\varpi, \tau) \neq 0$;
(iii) $\limsup \left\{ \frac{\Delta_2(\lambda, \tau)}{\Delta_1(\lambda, \tau)} : |\lambda| \to \infty, \Re \lambda \geq 0 \right\} < 1$;
(iv) Let $F(\varpi, \tau) = |\Delta_1(i\varpi, \tau)|^2 - |\Delta_2(i\varpi, \tau)|^2$, then it has finite roots;
(v) If $\varpi > 0$ exists satisfying $F(\varpi, \tau) = 0$, then it is continuous and differentiable in $\tau$.

Substitute $\lambda = i\varpi$ into (11), it can yield
\[
\begin{align*}
\begin{cases}
r_1 \cos \varpi \tau + s\varpi \sin \varpi \tau = \varpi^2 - q, \\
s\varpi \cos \varpi \tau - r_1 \sin \varpi \tau = -p\varpi.
\end{cases}
\end{align*}
\] (12)
Furthermore,
\[
\begin{align*}
\sin \varpi \tau &= \frac{(\varpi^2 - q)s\varpi + \varpi p r_1}{\varpi^2 + p^2 r_1^2}, \\
\cos \varpi \tau &= \frac{-(q - \varpi^2)r_1 + \varpi^2 ps}{\varpi^2 + p^2 r_1^2},
\end{align*}
\] (13)
which yields
\[
F(\varpi, \tau) = \varpi^4 + 3_1(\tau) \varpi^2 + 3_2(\tau) = 0
\] (14)
whose roots are given by
\[
\varpi^2_\pm = \frac{1}{2} \left[ -3_1(\tau) \pm \sqrt{\Delta} \right],
\] (15)
where
\[
3_1(\tau) = p^2 - 2q - s^2, \\
3_2(\tau) = q^2 - r_1^2 \\
\Delta = 3_1^2(\tau) - 43_2(\tau).
\]
Since $3_1(\tau) = 2\alpha \rho A_2 A_3 Z^* > 0$ and $q + r_1 = \mu \alpha A_3 Z^* > 0$, (14) has a uniquely positive real root $\varpi_+$ if and only if $q < r_1$.

Let
\[
I_\tau = \left\{ \tau \in [0, \bar{\tau}) : q < r_1 \right\}.
\]
Then for all $\tau \in I_\tau$, $\varpi$ satisfies (15) and $\varpi$ is not defined if $\tau \notin I_\tau$.

For $\tau \in I_\tau$, let $\theta_+(\tau) \in [0, 2\pi)$ be defined by
\[
\begin{align*}
\sin \theta_+(\tau) &= \frac{(\varpi^2 - q)s\varpi_+ + \varpi_+ p r_1}{\varpi_+^2 + p^2 r_1^2}, \\
\cos \theta_+(\tau) &= \frac{-(q - \varpi^2_+)r_1 + \varpi_+^2 ps}{\varpi_+^2 + p^2 r_1^2},
\end{align*}
\]
Theorem 4.2. Let \( \tau \in \mathbb{R}_+ \) defined by
\[
\tau_n(\tau) := \frac{\theta_n(\tau) + 2n\pi}{w_n(\tau)}, \quad n \in \mathbb{N}.
\]
Furthermore, it can introduce the continuous and differentiable functions \( S_n(\tau) \):
\[
S_n(\tau) := \tau - \tau_n(\tau), \quad \tau \in I_\tau, \quad n \in \mathbb{N}.
\]

**Theorem 4.1.** The equation (11) has a pair of simply imaginary roots \( \lambda = \pm i\varpi_+ \), \( \varpi_+ \) is real for \( \tau \in I_\tau \), and at some \( \tau^* \in I_\tau \),
\[
S_n(\tau^*) = 0 \quad \text{for some } n \in \mathbb{N}.
\]
This pair roots cross the imaginary axis from left (right) to right (left) if \( \delta_+^{(\tau^*)} > 0 \) \((< 0)\), where
\[
\delta_+^{(\tau^*)} := \text{Sign}\left\{ \frac{d\text{Re}\lambda}{d\tau} \bigg|_{\lambda = i\varpi_+} \right\} = \text{Sign}\left\{ \frac{dS_n(\tau)}{d\tau} \bigg|_{\tau = \tau^*} \right\}.
\]

Hence, the following theorem holds for system (10).

**Theorem 4.2.** Assume that \( \tau \in [0, \tau] \), \( R_0 > \frac{L^2 + m^2}{2mL} \) and (9) are satisfied. System (10) has the following dynamic properties.
(i) If \( I_\tau \) is empty or not empty set, but \( S_n(\tau) = 0 \) has no positive root in \( I_\tau \), then \( E^* \) is LAS for all \( \tau \in [0, \tau] \);
(ii) If \( I_\tau \) is non-empty, \( S_n(\tau) = 0 \) has positive roots in \( I_\tau \) and \( \delta_+^{(\tau^*)} \neq 0 \), for some \( n \in \mathbb{N} \), let \( \tau^0 = \min\{\tau : S_n(\tau) = 0\} \) and \( \tau^1 = \max\{\tau : S_n(\tau) = 0\} \), then \( E^* \) is LAS for \( \tau \in [0, \tau^0) \cup (\tau^1, \tau) \) and unstable for \( \tau \in (\tau^0, \tau^1) \). Here \( \tau^0 \) and \( \tau^1 \) are the Hopf bifurcation values.

4.2. **Case 2:** \( \tau > 0 \), \( \tau_1 > 0 \). Next, we fix \( \tau = \tau^* \) in the stable interval \( \mathbb{U} \) of \( E^* \) and increase the value of \( \tau_1 \) from zero to see possible bifurcation. In this case, the equation (7) becomes
\[
\mathbf{D}(\lambda, \tau^*, \tau_1) = \lambda^2 + (\mu + \rho A_2 - A_1)\lambda - A_1(\mu + \rho A_2)
+ [\beta e^{-\mu \tau} A_2(\alpha Z^* + A_1 - \lambda)]e^{-\lambda \tau^*} - \alpha \rho A_2 A_3 Z^* e^{-\lambda \tau_1} = 0,
\]
where
\[
q_1 = -A_1(\mu + \rho A_2) \quad \text{and} \quad q_2 = -\alpha \rho A_2 A_3 Z^*.
\]
By computing, it has
\[
q_1 + q_2 + r_1 = \alpha A_2 A_3 Z^*(\beta e^{-\mu \tau} \pm \rho) > 0.
\]
Clearly, for each fixed value of delay \( \tau^* \), we have a characteristic equation depending on \( \tau_1 \). Then stability switches may occur when a purely imaginary root exists and crosses the imaginary axis as the value of \( \tau_1 \) increases. Let \( iw \ (w > 0) \) be the root of (16), separating the real part from the imaginary part, then \( w \) satisfies
\[
\begin{cases}
q_2 \cos w \tau_1 = w^2 - q_1 - r_1 \cos w \tau^* - ws \sin w \tau^* := C_w, \\
q_2 \sin w \tau_1 = wp + ws \cos w \tau^* - r_1 \sin w \tau^* := S_w.
\end{cases}
\]
Furthermore,
\[
\tilde{F}(w) := C_w^2 + S_w^2 - q_2^2 = 0
\]
with \( \tilde{F}(0) = (q_1 + r_1)^2 - q_2^2 > 0 \) and \( \tilde{F}(+\infty) = +\infty \). Note that (18) has at most finite roots denoted by \( \{w_\kappa\}_{\kappa=1}^N \) if they exist. Substituting \( \{w_\kappa\}_{\kappa=1}^N \) into (18), for
When Case 3:

4.3. Using the same method as Theorem 3.1, it has

\( E \) and \( E \) then

\[ \frac{1}{w_c} \{ \arccos \left( \frac{C_{ww}}{q_2} \right) + 2j\pi \}, \quad S_{w_c} \leq 0, \]

\[ \frac{1}{w_c} \{ 2\pi - \arccos \left( \frac{C_{ww}}{q_2} \right) + 2j\pi \}, \quad S_{w_c} > 0, \]

Let \( \tau^0_1 = \min \{ \tau_{1}^{\ast} \} \) and \( \pm i\omega_0 \) are the roots of (16) for \( \tau_1 = \tau^0_1 \).

**Lemma 4.3.** If (18) has roots, for \( \tau = \tau^* \in U \), then the root distributions of (16) are as follows.

(i) If (18) has no positively real roots, then all roots of (16) have negative real parts for any \( \tau_1 \geq 0 \);

(ii) If (18) has finite real roots denoted by \( \{ w_n \}_{n=1}^n \), then all roots of (16) have negative real parts for \( \tau_1 \in [0, \tau^0_1] \).

Let \( \lambda(\tau_1) = \alpha_1(\tau_1) + i\lambda_2(\tau_1) \) be the root of (16) satisfying \( \alpha_1(\tau^0_1) = 0 \) and \( \alpha_2(\tau^0_1) = 0 \). Derivating to \( \tau_1 \) at \( \tau_1 = \tau^0_1 \) in two sides of (16), it has

\[ \frac{d\lambda}{d\tau_1}(\lambda = i\omega_0) = \frac{q_2 \lambda e^{-\lambda \tau_1}}{(2\lambda + p + se^{-\lambda \tau_1} - \tau'(\lambda + s\lambda)e^{-\lambda \tau_1} - q_2 \tau^0_1 e^{-\lambda \tau_1}}. \]

Furthermore,

\[ \frac{d\lambda}{d\tau_1}(\lambda = i\omega_0) = \frac{q_2 \lambda e^{-\lambda \tau_1}}{(2\lambda + p + se^{-\lambda \tau_1} - \tau'(\lambda + s\lambda)e^{-\lambda \tau_1} - q_2 \tau^0_1 e^{-\lambda \tau_1}} \]

where

\[ \tilde{A}_1 = q_2 \omega_0 \sin \omega_0 \tau^0_1, \quad \tilde{A}_2 = q_2 \omega_0 \cos \omega_0 \tau^0_1, \]

\[ \tilde{A}_3 = p - \tau'(s \omega_0 \sin \omega_0 \tau^* + r \cos \omega_0 \tau^*) + \tau' \omega_0 \cos \omega_0 \tau^* - \tau^0_1 q_2 \cos \omega_0 \tau^0_1, \]

\[ \tilde{A}_4 = 2 \omega_0 - \tau'(s \omega_0 \cos \omega_0 \tau^* - r \sin \omega_0 \tau^*) - \tau' \omega_0 \sin \omega_0 \tau^* + \tau^0_1 q_2 \sin \omega_0 \tau^0_1. \]

Hence, it has the following result.

**Lemma 4.4.** \( \text{Sign} \left\{ \frac{d\alpha_1(\tau_1)}{d\tau_1} \right\}_{\tau_1 = \tau^0_1} = \text{Sign} \{ \tilde{A}_1 \tilde{A}_3 + \tilde{A}_2 \tilde{A}_4 \} \).

**Theorem 4.5.** Suppose that \( \tau = \tau^* \in U \) is satisfied.

(i) If (18) has no positive root, then \( E^* \) is LAS for any \( \tau_1 \geq 0 \);

(ii) If (18) has positive roots denoted by \( \{ \omega_k \}_{k=1}^n \) and \( \text{Sign} \left\{ \frac{d\alpha_1(\tau_1)}{d\tau_1} \right\}_{\tau_1 = \tau^0_1} \neq 0 \), then \( E^* \) is LAS for \( \tau_1 \in [0, \tau^0_1] \), where \( \tau^0_1 \) is the Hopf bifurcation value.

4.3. Case 3: \( \tau_1 = \tau > 0 \). In this subsection, it will investigate local Hopf bifurcation when \( \tau_1 = \tau = \nu \). In this situation, system (6) becomes

\[
\begin{align*}
\frac{dx(t)}{dt} &= A_1 x(t) - \alpha A_2 y(t) + O(2), \\
\frac{dy(t)}{dt} &= \left[ \beta e^{-\mu \nu} A_3 Z^* - \rho A_3 Z^* \right] x(t - \nu) + \beta e^{-\mu \nu} A_2 y(t - \nu), \\
&\quad - (\mu + \rho A_2) y(t) + O(2),
\end{align*}
\]

The system (19) has the same equilibria as system (2) when \( \tau_1 = \tau, \) i.e., \( E_0(0, 0), E_1(L, 0) \) and \( E^* (\mathcal{P}(\nu), \mathcal{Z}(\nu)). \) \( E^* \) exists when \( R_0 > \frac{L^2 + m^2}{2mL} \) and \( 0 \leq \nu < \nu := \frac{1}{\rho} \ln \left( \frac{\beta L}{\rho + \beta L + m^2} \right). \)

As for \( E_1, \) the characteristic equation at \( E_1 \) of system (6) is given by

\[
(\lambda + \nu) \left[ \lambda + \mu + \frac{\rho L}{m^2 + L^2} - e^{(\lambda + \nu)} \frac{\beta L}{m^2 + L^2} \right] = 0.
\]

Using the same method as Theorem 3.1, it has
\textbf{Theorem 4.6.} If $0 \leq \nu < \tilde{\nu}$, then $E_1$ is unstable. If $\nu > \hat{\nu} := \frac{1}{\mu} \ln \frac{\beta L}{\mu m^2}$, then $E_1$ is GAS, where $\tilde{\nu} < \hat{\nu}$.

Equation (7) has one root $\lambda = -r < 0$, while the other roots satisfy the following equation
\[ \mathcal{P}_1(\lambda, \nu) + \mathcal{Q}_1(\lambda, \nu)e^{-\lambda \nu} = 0, \] (21)
where
\[ \mathcal{P}_1(\lambda, \nu) = \lambda + \mu + \frac{\rho L}{m^2 + L^2} \quad \text{and} \quad \mathcal{Q}_1(\lambda, \nu) = -e^{-\mu \nu} \frac{\beta L}{m^2 + L^2}. \]
with
\[ \mathcal{P}_1(0, \nu) + \mathcal{Q}_1(0, \nu) \neq 0 \quad \text{and} \quad \mathcal{P}_1(i \omega, \nu) + \mathcal{Q}_1(i \omega, \nu) \neq 0. \]

Let $\lambda = \pm i \omega(\nu)$ ($\omega(\nu) > 0$) be the roots of (21), then stability switches may occur at $\nu^*_{n+}(\nu)$ values:
\[ \nu^*_{n+}(\nu) = \frac{\omega^0_n(\nu) + 2n \pi}{\omega^0_n(\nu)}, \quad n \in \mathbb{N}, \]
where
\[ \omega^0_n(\nu) = \sqrt{\frac{\beta^2 L^2 e^{-2 \omega^0_n(\nu)} - (\mu + \frac{\rho L}{m^2 + L^2})^2}{(m^2 + L^2)^2}} \]
and $\delta^0_n(\nu) \in [0, 2\pi)$ satisfies
\[ \begin{cases} \sin \delta^0_n(\nu) = -\frac{\omega^0_n(\nu) (m^2 + L^2) e^{\nu \omega}}{\beta L}, \\ \cos \delta^0_n(\nu) = \frac{\mu(m^2 + L^2) + \rho L e^{\nu \omega}}{\beta L}. \end{cases} \]

Define
\[ Z_n(\tau) := \nu - \nu^*_{n+}(\nu), \quad n \in \mathbb{N} \]
and
\[ J^0 = \left\{ \nu_j : \nu_j \in [0, \nu) \quad \text{and} \quad Z_n(\nu_j) = 0 \right\}, \]
where
\[ \nu^*_{n+}(\nu) = \frac{1}{\omega^0_n(\nu)} \left\{ 2 \pi - \arccos \frac{\mu(m^2 + L^2) + \rho L e^{\nu \omega}}{\beta L} + 2n \pi \right\}. \]

\textbf{Theorem 4.7.} If $0 \leq \nu < \tilde{\nu}$, then $\pm i \omega^0_n(\nu)$ are the roots of (10). For some $\nu_\ast \in [0, \nu)$ and $n \in \mathbb{N}$, $Z_n(\nu_\ast) = 0$. This pair of roots traverse the imaginary axis from left (right) to right (left) if $\delta^0_+(\nu_\ast) > 0 \,(< 0)$, where
\[ \delta^0_+(\nu_\ast) := \text{Sign} \left\{ \frac{d \text{Re} \lambda}{d \nu} \bigg|_{\lambda = i \omega^0_n(\nu_\ast)} \right\} = \text{Sign} \left\{ \frac{d Z_n(\nu)}{d \nu} \bigg|_{\nu = \nu_\ast} \right\}. \]

If $J^0$ is not empty. For all $\nu_j \in J^0$, if $\frac{d Z_n(\nu)}{d \nu} \neq 0$ holds, then system (19) undergoes Hopf bifurcations at $E_1$ when $\nu = \nu_j$.

In the following, it investigates the existence of local Hopf bifurcation at $E^\ast$. When $\tau_1 = \tau = \nu$, the characteristic equation (7) becomes
\[ \Delta^1(\lambda, \nu) + \Delta^2(\lambda, \nu)e^{-\lambda \nu} = 0, \] (22)
where
\[ \Delta^1(\lambda, \nu) = \lambda^2 + p \lambda + q_1 \quad \text{and} \quad \Delta^2(\lambda, \nu) = r_1 + q_2 + s \lambda. \]

Let $\lambda = i \omega$ ($\omega = \omega(\nu) > 0$) be the root of (22) and use similar arguments to those in Section 3. The following relations hold.
(i) $\Delta^1(0, \nu) + \Delta^2(0, \nu) \neq 0$;
(ii) $\Delta^1(i \omega, \nu) + \Delta^2(i \omega, \nu) \neq 0$;
(iii) $\lim \sup \left\{ \left| \frac{\Delta^1(\lambda, \nu)}{\Delta^2(\lambda, \nu)} \right| : |\lambda| \to \infty, \text{Re} \lambda \geq 0 \right\} < 1$;
(iv) Define $F(\omega, \nu) = |\Delta^1(i \omega, \nu)|^2 - |\Delta^2(i \omega, \nu)|^2$, then it has finite roots;
(v) If there exists $\omega > 0$ satisfying $F(\omega, \nu) = 0$, then it is continuous and differentiable in $\nu$.

Substituting $\lambda = i\omega$ into (22), it has

$$\begin{cases}
\omega^2 - q_1 = (r_1 + q_2) \cos \omega \nu + s\omega \sin \omega \nu, \\
\omega = -s\omega \cos \omega \nu + (r_1 + q_2) \sin \omega \nu,
\end{cases}$$

furthermore,

$$\begin{cases}
\sin \omega \nu = \frac{(\omega^2-q_1)\omega + \omega p(r_1+q_2)}{\omega^2 s^2 + (r_1+q_2)^2}, \\
\cos \omega \nu = -\frac{q_1 - \omega^2 p(r_1+q_2) + \omega^2 ps}{\omega^2 s^2 + (r_1+q_2)^2},
\end{cases}$$

which yields

$$F(\omega, \nu) := \omega^4 + \Xi_1(\nu) \omega^2 + \Xi_2(\nu) = 0$$

and its roots are given by

$$\omega^2 = \frac{1}{2} \left[ -\Xi_1(\nu) \pm \sqrt{\Delta} \right],$$

where

$$\Xi_1(\nu) = p^2 - 2q_1 - s^2, \ \Xi_2(\nu) = q_1^2 - (r_1 + q_2)^2, \ \Delta = \Xi_1^2(\nu) - 4\Xi_2(\nu).$$

It is obvious that $\Xi_1(\nu) = A_3^2 > 0$ and $q_1 + r_1 + q_2 = \mu A_3 Z^* > 0$, then (24) has a uniquely positive real root $\omega_+$ if and only if $q_1 < r_1 + q_2$.

Let

$$I_\nu = \{ \nu \in [0, \bar{\nu}) : q_1 < r_1 + q_2 \}.$$

Then, for $\nu \in I_\nu$, $\Theta_+(\nu) \in [0, 2\pi)$ satisfies

$$\begin{cases}
\sin \Theta_+(\nu) = \frac{(\omega^2-q_1)\omega + \omega p(r_1+q_2)}{\omega^2 s^2 + (r_1+q_2)^2}, \\
\cos \Theta_+(\nu) = -\frac{q_1 - \omega^2 p(r_1+q_2) + \omega^2 ps}{\omega^2 s^2 + (r_1+q_2)^2},
\end{cases}$$

with $F(\omega_+, \nu) = 0$ for $\nu \in I_\nu$. Hence, the maps $\nu_n(\nu) : I_\nu \to \mathbb{R}$ are defined by

$$\nu_n(\nu) := \frac{\Theta_+(\nu) + 2n\pi}{\omega_+(\nu)}, \ n \in \mathbb{N}.$$

Furthermore, define the continuous and differentiable functions in $\nu$:

$$T_n(\nu) := \nu - \nu_n(\nu), \ \nu \in I_\nu, \ n \in \mathbb{N}.$$

**Theorem 4.8.** The equation (22) has the roots $\lambda = \pm i\omega(\nu^*)$, and at some $\nu^* \in I_\nu$,

$$T_n(\nu^*) = 0$$

for some $n \in \mathbb{N}$.

This pair of roots cross the imaginary axis from left (right) to right (left) if $\delta_+(\nu^*) > 0$ ($< 0$), where

$$\delta_+(\nu^*) := \text{Sign} \left\{ \left. \frac{d\Re \lambda}{d\nu} \right|_{\lambda = i\omega_+} \right\} = \text{Sign} \left\{ \left. \frac{dT_n(\nu)}{d\nu} \right|_{\nu = \nu^*} \right\}.$$

**Theorem 4.9.** For system (19), it assumes that $0 < \nu < \bar{\nu}$.

(i) If $I_\nu$ is empty set or not empty set, but $T_n(\nu) = 0$ has no positive root in $I_\nu$, then $E^*$ is LAS for any $\tau \in [0, \bar{\nu})$;

(ii) If $I_\nu$ is non-empty set, $T_n(\nu) = 0$ exists positive roots at $I_\nu$ and $\delta_+(\nu^*) \neq 0$, for $n \in \mathbb{N}$, and rearrange these roots as the set $J := \{ \nu^0, \nu^1, ..., \nu^m \}$ with $\nu^j < \nu^{j+1}, j = 0, ..., m - 1$. Then $E^*$ is LAS for $\nu \in [0, \nu^0] \cup (\nu^m, \bar{\nu})$ and unstable for $\nu \in (\nu^0, \nu^m)$. Hopf bifurcations occur at $E^*$ when $\nu = \nu^j$. 

Remark 3. If (7) exists two pairs of purely imaginary roots for some $\tau$ and $\tau_1$, say $\pm i \omega_1$ and $\pm i \omega_2$, and all the other roots have non-zero real parts, then system (2) undergoes a double-Hopf bifurcation with a ratio $k_1 : k_2$, where $w_1 : w_2 = k_1 : k_2$. When $k_1, k_2 \in \mathbb{Z}^+$, it is called $k_1 : k_2$ resonant double-Hopf bifurcation, otherwise, it is called a non-resonant double Hopf bifurcation. More specially, since there are multiple parameters in system (2) except for $\tau$ and $\tau_1$, the bifurcation with codimension greater than 1 can be considered. The interesting researches on this topic can be found in [17, 4, 7, 14], and so on.

4.4. Global Hopf bifurcation of system (19). In Section 4.2, it has known that system (19) undergoes local Hopf bifurcations at $E^*$ when $\nu$ near $\nu^j$, $\nu^j \in J$. In this section, using the methods in [21, 15, 19], it investigates the existence of globally periodic solutions.

The following notations from [19] are given. Denote

$$J_0 = \left\{ \nu \in J : T_0(\nu) = 0 \right\}, \quad J_+ := J - J_0,$$

$$A_j = \begin{cases} \max \{ \nu_j : \nu_j \in J^0, \nu_j < \nu^j \}, & J^0 \neq \emptyset, \\ 0, & \text{else}, \end{cases}$$

$$B_j = \begin{cases} \min \{ \nu_j : \nu_j \in J^0, \nu_j > \nu^j \}, & J^0 \neq \emptyset, \\ \sup J^0, & \text{else}, \end{cases}$$

$$A^j = \max \{ \nu^j : \nu^j \in J_+ \cup J^0, \tau^j < \nu^j \in J_+ \},$$

$$B^j = \min \{ \nu^j : \nu^j \in J_+ \cup J^0, \nu^j > \nu^j \in J_+ \}.$$

Next, it assumes that $J_+ \neq \emptyset$. Using global Hopf bifurcation theorem [21], it will investigate the global existence of periodic solutions bifurcating from $(E^*, \nu^j, \frac{2\pi i}{\omega^j_+})$ for system (19), where $\nu^j \in J^+$, $\omega^j_+ = \omega_+(\nu^j)$ and $\pm i \omega^j_+$ are the roots of (22) when $\nu = \nu^j$. Setting $z_t = (P_t, Z_t)$, system (19) can be rewritten as:

$$\dot{z}(t) = Y(z_t, \nu, p),$$

where $z_t(\theta) = z(t + \theta)$. System (26) has three equilibria $\bar{z}_1 = (0, 0)$, $\bar{z}_2 = (L, 0)$ and $z^* = E^*$.

Define

$$\mathbf{X} = C([-\nu, 0], \mathbb{R}^2_+),$$

$$\Sigma = \text{Cl}\{(z_t, \nu, p) \in \mathbf{X} \times \mathbb{R} \times (0, +\infty) : z_{t+p} = z_t \},$$

$$\mathbf{N} = \{(\bar{z}, \nu, p) : Y(\bar{z}, \nu, p) = 0\},$$

and let $\ell_{(z^*, \nu^j, \frac{2\pi i}{\omega^j_+})}$ which is nonempty be the connected component of $(z^*, \nu^j, \frac{2\pi i}{\omega^j_+})$ in $\Sigma$, and $\text{Proj}_\nu (z^*, \nu^j, \frac{2\pi i}{\omega^j_+})$ be its projection on $\nu$. By Theorem 2.3, it has the following results.

Lemma 4.10. All periodic solutions of system (26) are uniformly bounded in $\mathbb{R}^2_+$.

Lemma 4.11. If $m \geq L$, then system (26) does not exist nonconstant $\nu$-periodic solution.

Proof. Because the orbits of system (26) don’t intersect and $P-Z$ axis are the invariant manifold, which implies that, in the first quadrant, if there exists any
periodic solutions, then there must be $E^*$ in its interior. It will prove by contradiction. It assumes that system (26) has nonconstant $\nu$-periodic solution, then it definitely obtain that the following systems have nontrivial periodic solutions:

$$
\begin{aligned}
\frac{dP(t)}{dt} &= rP(t)\left(1 - \frac{P(t)}{L}\right) - \frac{\alpha P(t)Z(t)}{m^2 + P^2(t)} := M, \\
\frac{dZ(t)}{dt} &= \beta e^{-\mu Z(t)} - \mu Z(t) - \frac{P(t)Z(t)}{m^2 + P^2(t)} = N.
\end{aligned}
$$

(27)

Define Dulac function $Q = \frac{m^2 + Z^2}{P^2}$, then

$$
\frac{\partial (MQ)}{\partial \nu} + \frac{\partial (NQ)}{\partial Z} = \frac{1}{2} \left[-\frac{Z}{P}(m^2 + P^2) + 2P(1 - \frac{P}{L})\right]
= -\frac{m^2 + 2P^2 + (P - L)^2 - L^2}{P^2}. 
$$

If $m \geq L$, then $\frac{\partial (MQ)}{\partial \nu} + \frac{\partial (NQ)}{\partial Z} < 0$. By Dulac criterion, system (27) has no periodic solution in the first quadrant, which is a contradiction. This completes the proof.

\[\square\]

**Theorem 4.12.** Assume that the conditions $m \geq L$, $J_+ \neq \emptyset$, $\nu \in [0, \bar{\nu})$ and $R_0 > \frac{L^2 + m^2}{2mL}$ hold. Then for any $\nu^j \in J_+$, there exists $\nu^j \in J^0 \cup J_+ - \{\nu^j\}$, so that system (26) has at least one positive periodic solution for $\nu$ varies between $\nu^j$ and $\nu^j$.

**Proof.** The characteristic equation of system (26) at some equilibrium $\bar{z}$ is expressed as:

$$
\Delta(z, \nu, p)(\lambda) = \lambda \text{Id} - D\mathbf{Y}(z, \nu, p)(e^\lambda \text{Id}) = 0.
$$

The system (26) has three equilibria $\bar{z}_1, \bar{z}_2$ and $z^*$. It easily knows that $(\bar{z}_1, \nu, p)$ is not a center, while $(\bar{z}_2, \nu, p)$ and $(z^*, \nu, p)$ are isolated centers. By Theorem 4.9 and $\frac{dT_{z^j}(\nu)}{\nu = \nu^j} \neq 0$, there exist $\varepsilon > 0$, $\delta > 0$ and $\lambda : (\nu^j - \delta, \nu^j + \delta) \to C$, such that $\det (\Delta(\lambda(\nu^j))) = 0, |\lambda(\nu^j) - i\omega_1^j| < \varepsilon$ for all $\nu \in [\nu^j - \delta, \nu^j + \delta]$ and $\lambda(\nu^j) = i\omega_1^j$, $d\text{Re} \lambda(\nu^j)/d\nu \neq 0$.

Let $\Omega_{\varepsilon, \frac{2\pi}{\omega_1^j}}, \Delta(z^*, \nu, p)(\eta + \frac{2\pi}{\omega_1^j} i) = 0$ iff $\eta = 0, \nu = \nu^j, p = \frac{2\pi}{\omega_1^j}$. Furthermore, define

$$
\mathcal{R}^\pm(z^*, \nu^j, \frac{2\pi}{\omega_1^j})(\eta, p) = \Delta(z^*, \nu^j \pm \delta, p)(\eta + \frac{2\pi}{\omega_1^j} i),
$$

then the crossing number of $(z^*, \nu^j, \frac{2\pi}{\omega_1^j})$ is given by

$$
\mathcal{R}(z^*, \nu^j, \frac{2\pi}{\omega_1^j}) = \text{deg}_B(\mathcal{R}^-(z^*, \nu^j, \frac{2\pi}{\omega_1^j}), \Omega_{\varepsilon, \frac{2\pi}{\omega_1^j}}) - \text{deg}_B(\mathcal{R}^+(z^*, \nu^j, \frac{2\pi}{\omega_1^j}), \Omega_{\varepsilon, \frac{2\pi}{\omega_1^j}})
= \begin{cases} 
-1, & \frac{dT_{z^j}(\nu^j)}{\nu = \nu^j} > 0, \\
1, & \frac{dT_{z^j}(\nu^j)}{\nu = \nu^j} < 0.
\end{cases}
$$

Furthermore,

$$
\sum_{(\bar{z}, \nu, p) \in \mathcal{E}(z^*, \nu^j, \frac{2\pi}{\omega_1^j})} \mathcal{R}(\bar{z}, \nu, p) \neq 0,
$$

where $\bar{z}$ is either $z^*$ or $\bar{z}_2$. Hence, the connected component $\ell_{(z^*, \nu^j, \frac{2\pi}{\omega_1^j})}$ through $(z^*, \nu^j, \frac{2\pi}{\omega_1^j})$ in $\Sigma$ is unbounded.
5. Crossing curve methods. The results of Theorem 4.9 show the stability of the equilibrium \( E^s \) when \( \tau \) and \( \tau_1 \) change. Clearly, there is a Hopf bifurcation at \( \tau_0 \) and there may be multiple stable switches. If we leave \( \tau \) in an unstable region, \( \tau_0 \) may not exist such that \( E^s \) is unstable in \( \tau_1 \in [0, \tau_0) \), it is stable in \( \tau_1 > \tau_0 \).

However, this is not entirely satisfactory because it does not allow for a bifurcated analysis of \( (\tau, \tau_1) \). That is, no information is given about \( (\tau, \tau_1) \) that generates stable or unstable stable states. For this purpose, Gu et al. [8] provided crossing curve method which are defined as the curves that separate the stable and unstable regions in the \( (\tau, \tau_1) \) plane. However, the system referenced in [8] does not depend on the delay in the coefficients, so these results cannot be used directly (6). Recently, An et al. [1] improved these results in [8] to the system with the coefficients depending on the delay. In the following, we will use the methods in [1, 16] to give the couples \( (\tau, \tau_1) \) that generate a stable or an unstable equilibrium. Define \( \mathbf{I} = [0, \bar{\tau}] \). We rewrite (6) as

\[
\begin{align*}
\frac{dx(t)}{dt} &= A_1 x(t) - \alpha A_2 y(t) + o(2), \\
\frac{dy(t)}{dt} &= \beta e^{-\mu \tau} A_2 z^* x(t - \tau) + \beta e^{-\mu \tau} A_2 y(t - \tau) - \rho A_2 z^* x(t - \tau_1) - (\mu + \rho A_2) y(t) + o(2)
\end{align*}
\]

(28)

whose characteristic equation is

\[
D(\lambda, \tau, \tau_1) := \Sigma_0(\lambda, \tau) + \Sigma_1(\lambda, \tau) e^{-\lambda \tau} + \Sigma_2(\lambda, \tau) e^{-\lambda \tau_1} = 0,
\]

(29)

where

\[
\begin{align*}
\Sigma_0(\lambda, \tau) &= \lambda^2 + (\mu + \rho A_2 - A_1) \lambda - A_1 (\mu + \rho A_2), \\
\Sigma_1(\lambda, \tau) &= \beta e^{-\mu \tau} A_2 (\alpha A_2 z^* + A_1 - \lambda) \quad \text{and} \quad \Sigma_2(\lambda, \tau) = -\alpha \rho A_2 A_3 z^*
\end{align*}
\]

satisfying

\[
\deg(\Sigma_0(\lambda, \tau)) \geq \max\{\deg(\Sigma_1(\lambda, \tau)), \deg(\Sigma_2(\lambda, \tau))\}.
\]

It is easily to verify \( \Sigma_s(\lambda, \tau) \) \((s = 0, 1, 2)\) satisfying the following conditions [1]:

1. \( \Sigma_0(0, \tau) + \Sigma_1(0, \tau) + \Sigma_2(0, \tau) \neq 0 \);
2. \( \lim_{Re(\lambda) \to \infty, \tau \to I} \sup_{\tau \in I} \left( \left| \frac{\Sigma_1(\lambda, \tau)}{\Sigma_0(\lambda, \tau)} \right| + \left| \frac{\Sigma_2(\lambda, \tau)}{\Sigma_0(\lambda, \tau)} \right| \right) < 1 \);
3. \( \Sigma_s(\omega, \tau) \neq 0 \) \((s = 0, 1, 2)\) for any \( \tau \in I \) and \( \omega > 0 \);
4. For any \( \omega > 0 \), at least one of \( |\Sigma_s(\omega, \tau)| \) \((s = 0, 1, 2)\) tends to infinity as \( \tau \) tends to -\( \infty \).
5.1. Crossing curve. In this subsection, we will determine the feasible values of $(\tau, \tau_1) \in I \times [0, +\infty)$ so that $\lambda = \omega \ (\omega > 0)$ is a root of (29). The curve formed by all these points is called “crossing curve”. Let

$$
\begin{align*}
\beta_1(\omega, \tau) &= \frac{\Sigma_1(\omega, \tau)}{\Sigma_0(\omega, \tau)}, \\
\beta_2(\omega, \tau) &= \frac{\Sigma_2(\omega, \tau)}{\Sigma_0(\omega, \tau)},
\end{align*}
$$

(30)

where

$$
\Sigma_0(\omega, \tau) = -\omega^2 - A_1(\mu + \rho A_2) + \omega(\mu + \rho A_2 - A_1),
$$

$$
\Sigma_1(\omega, \tau) = \beta e^{-\tau} A_2(\alpha A_3 Z^* + A_1 - \omega) \text{ and } \Sigma_2(\omega, \tau) = -\alpha \rho A_2 A Z^*.
$$

Hence, $(\omega, \tau, \tau_1)$ is the zeros of (29) iff

$$
D(\omega, \tau, \tau_1) := 1 + \beta_1(\omega, \tau)e^{-i\omega \tau} + \beta_2(\omega, \tau)e^{-i\omega \tau_1} = 0.
$$

(31)

It assumes that $(\omega, \tau, \tau_1)$ is the zero of (29), then the left side of (31) must form a triangle on the complex plane. Since $|\Sigma_0(\omega, \tau)| + |\Sigma_1(\omega, \tau)| \geq |\Sigma_2(\omega, \tau)|$ is always true, therefore, the feasible region $\Omega$ can be obtained.

**Lemma 5.1.** For $\tau \in I$, the feasible region $\Omega$ for $(\omega, \tau)$ satisfies

$$
\Omega = \left\{ (\omega, \tau) \in (0, +\infty) \times I : f_1(\omega, \tau) \geq 0, \ f_2(\omega, \tau) \geq 0 \right\},
$$

where $f_1(\omega, \tau) = |\Sigma_1(\omega, \tau)| + |\Sigma_2(\omega, \tau)| - |\Sigma_0(\omega, \tau)|$ and $f_2(\omega, \tau) = |\Sigma_0(\omega, \tau)| + |\Sigma_2(\omega, \tau)| - |\Sigma_1(\omega, \tau)|$.

Therefore, the feasible region is surrounded by the line $\tau = \hat{\tau}$, $\tau$-axis and $\omega$-axis, the curves $f_1(\omega, \tau) = 0$ and $f_2(\omega, \tau) = 0$. For every connected region $\Omega_k$, the feasible range for $\omega$ is denoted by $I_k = [\omega^k_1, \omega^k_2], k = 1, 2, ..., N$. Furthermore, for each $\omega \in I_k$, there exists the intervals $I^k_l = [\tau^k_{\omega, l}, \tau^k_{\omega, r}] \subseteq I$, on which two inequalities hold in $\Omega$.

Let the angles formed by 1 and $\beta_1(\omega, \tau)e^{-i\omega \tau}$ be $\theta_1(\omega, \tau)$, and the angles formed by 1 and $\beta_2(\omega, \tau)e^{-i\omega \tau_1}$ be $\theta_2(\omega, \tau)$, by the law of cosine, it has

$$
\begin{align*}
\theta_1(\omega, \tau) &= \arccos \left( \frac{1 + |\beta_1(\omega, \tau)|^2 - |\beta_2(\omega, \tau)|^2}{2|\beta_1(\omega, \tau)|} \right), \\
\theta_2(\omega, \tau) &= \arccos \left( \frac{1 + |\beta_2(\omega, \tau)|^2 - |\beta_1(\omega, \tau)|^2}{2|\beta_2(\omega, \tau)|} \right),
\end{align*}
$$

where

$$
|\beta_1(\omega, \tau)| = \left| \frac{\Sigma_1(\omega, \tau)}{\Sigma_0(\omega, \tau)} \right| \text{ and } |\beta_2(\omega, \tau)| = \left| \frac{\Sigma_2(\omega, \tau)}{\Sigma_0(\omega, \tau)} \right|.
$$

For each fixed $\omega \in I_k$, it can note that $\text{Im}(\beta_1(\omega, \tau)e^{-i\omega \tau}) = 0$ iff $\theta_1(\omega, \tau) = 0$ or $\pi$, which is equivalent to $\tau = \tau_1^{k, \omega, l}$ or $\tau = \tau_1^{k, \omega, r}$. Therefore, $\text{Im}(\beta_1(\omega, \tau)e^{-i\omega \tau})$ cannot change sign for $\tau \in \text{Int}I^k_l$ which represents the interior of $I^k_l$. In the following, two feasible cases are considered: (i) $\text{Im}(\beta_1(\omega, \tau)e^{-i\omega \tau}) > 0$;

From the triangle, it can obtain

$$
\arg(\beta_1(\omega, \tau)e^{-i\omega \tau}) = \pi - \theta_1(\omega, \tau) \text{ and } \arg(\beta_2(\omega, \tau_1)e^{-i\omega \tau_1}) = \theta_2(\omega, \tau) - \pi.
$$

Then there exists one $n \in \mathbb{Z}$ such that

$$
\arg(\beta_1(\omega, \tau)) + \theta_1(\omega, \tau) + (2n - 1)\pi = \omega \tau,
$$

(32)

and

$$
\tau_1 = \frac{1}{2} \left[ \arg(\beta_2(\omega, \tau)) - \theta_2(\omega, \tau) + (2j + 1)\pi \right], \text{ for some } j \in \mathbb{Z},
$$

(33)

where $j \geq j^+_0$ and $j^+_0$ is the smallest integer such that the right side of (33) is positive.
(ii) \( \text{Im}(\beta_1(\omega, \tau)e^{-\omega \tau}) < 0 \):

In this case, the triangle formed is the mirror image of the case (i) about the real axis. We have

\[
\arg(\beta_1(\omega, \tau)) - \theta_1(\omega, \tau) + (2n - 1)\pi = \omega \tau, \quad \text{for some } n \in \mathbb{Z},
\]

and

\[
\tau_1 = \frac{1}{2} \left[ \arg(\beta_2(\omega, \tau)) + \theta_2(\omega, \tau) + (2j + 1)\pi \right], \quad \text{for some } j \in \mathbb{Z},
\]

where \( j \geq j_0^\pm \) and \( j_0^\pm \) is the smallest integer such that the right side of (35) is positive.

Now it can define \( \Gamma^\omega \) as the interval of \( \omega \) and \( \Gamma^\omega_\tau \) the feasible values of \( \tau \) for every fixed \( \omega \in \Gamma^\omega \). Fixed \( \omega \in \Gamma^\omega \), it can define the following functions:

\[
S_n^\omega(\omega, \tau) = \tau - \frac{1}{\omega} \left[ \arg(\beta_1(\omega, \tau)) \pm \theta_1(\omega, \tau) + (2n - 1)\pi \right], \quad \text{for some } n \in \mathbb{Z}.
\]

If (36) has zeros written as \( \hat{\tau}^{i\pm}(\omega), i = 1, 2, \cdots \). Furthermore, it can obtain \( \tau_1 \) values as follows:

\[
\hat{\tau}_1^{i\pm,j\pm}(\omega) = \frac{1}{2} \left[ \arg(\beta_2(\omega, \hat{\tau}^{i\pm}(\omega))) \mp \theta_2(\omega, \hat{\tau}^{i\pm}(\omega)) + (2j + 1)\pi \right],
\]

where \( j_{\pm} \geq j_{0}^{\pm} \) and \( j_{0}^{\pm} \) is the smallest integer such that \( \hat{\tau}_1^{i\pm,j\pm}(\omega) > 0 \).

If \( \omega \) takes all the values in the whole interval \( \Gamma^\omega \), then it can obtain the following curve on \( \Omega \)

\[
C := \left\{ (\omega, \hat{\tau}^{i\pm}(\omega)) : \omega \in \Gamma^\omega, \ S_n(\omega, \hat{\tau}^{i\pm}(\omega)) = 0 \right\},
\]

and the crossing curves on \( (\tau, \tau_1) \) plane

\[
T := \left\{ (\hat{\tau}^{i\pm}(\omega), \hat{\tau}_1^{i\pm,j\pm}(\omega)) \in \Gamma^\omega : \omega \in \Gamma^\omega, \ S_n(\omega, \hat{\tau}^{i\pm}(\omega)) = 0 \right\}.
\]

Note that \( \tau_\omega = \tau_1^{k,l} \neq 0 \) or \( \tau_\omega = \tau_1^{k,r} \neq \infty \) must satisfying one of the following equations:

\[
\begin{cases}
|\beta_1(\omega, \tau_\omega)| + |\beta_2(\omega, \tau_\omega)| = 1, \\
|\beta_1(\omega, \tau_\omega)| - |\beta_2(\omega, \tau_\omega)| = 1, \\
|\beta_1(\omega, \tau_\omega)| - |\beta_2(\omega, \tau_\omega)| = -1.
\end{cases}
\]

The following results directly come from [1].

**Lemma 5.2.** (i) If (40) or (41) holds for \( \tau_\omega = \tau_1^{k,l} \neq 0 \) or \( \tau_\omega = \tau_1^{k,r} \neq \infty \), then \( \theta_1(\omega, \tau_\omega) = 0 \) and \( S_n^+ (\omega, \tau_\omega) = S_n^- (\omega, \tau_\omega) \);

(ii) If (42) holds for \( \tau_\omega = \tau_1^{k,l} \neq 0 \) or \( \tau_\omega = \tau_1^{k,r} \neq \infty \), then \( \theta_1(\omega, \tau_\omega) = \pi \) and \( S_n^+ (\omega, \tau_\omega) = S_n^- (\omega, \tau_\omega) + 1 \).

For each fixed \( \omega \in I_k \), it can classify the interval \( I_\omega^k \) into 4 types: Type 1: \( \theta_1(\omega, \tau_1^{k,l}) = \theta_1(\omega, \tau_1^{k,r}) \); Type 2: \( \theta_1(\omega, \tau_1^{k,l}) \neq \theta_1(\omega, \tau_1^{k,r}) \); Type 3: \( \tau_1^{k,l} = 0 \) and \( \tau_1^{k,r} \neq \infty \); Type 4: \( \tau_1^{k,l} = \infty \). On each Type, the plots of \( S_n^\pm (\omega, \tau) \) are different.

Furthermore, it assumes that

\[
(\text{C5}) \quad \partial S_n^\pm (\omega, \tau) / \partial \tau \neq 0 \text{ for any } (\omega, \tau) \in C.
\]

By (C5), the endpoints \( (\omega_{\tau}, \hat{\tau}^{i\pm}(\omega_{\tau})) \) of every component of curve \( C \) must locate the boundary of \( \Omega \) and can be divided into 3 types for \( (\omega, \tau_\omega) = (\omega_{\tau}, \hat{\tau}^{i\pm}(\omega_{\tau})) \):

1. Type A: (40) holds. 2. Type B: (41) holds. 3. Type C: (42) holds.
For convenience, Type AA indicates that both end points of the component are Type A. Depending on the type of endpoint, each component of the curve \( C \) falls into the following six categories: AA, BB, CC, AB, AC, and BC.

**Lemma 5.3.** If the condition (C5) holds, then any two components of curve \( C \) will not intersect in the interior of \( \Omega \).

**Theorem 5.4.** Under the conditions (C1)-(C5), the crossing curve on \((\tau, \tau_1)\)-plane consists of one or several curves in the following types:

(a) A series of open-ended curves along \( \tau_1 \)-axis;
(b) A series of closed curves along \( \tau \)-axis;
(c) A series of spiral-like curves along \( \tau_1 \)-axis; and each of these curves approaching \( \infty \) in the direction of \( \tau_1 \)-axis;
(d) Truncated curves of one of the above 3 cases.

### 5.2. Crossing directions.

Assume that \((\tau^*, \tau_1^*) \in T\), then there is an \( \omega^* > 0 \) such that \((i\omega^*, \tau^*, \tau_1^*)\) is the zero of (29). If \( \partial D/\partial \lambda \neq 0 \), then \( \lambda(\tau, \tau_1) = \gamma_1(\tau, \tau_1) + i\gamma_2(\tau, \tau_1) \) is the simple root of (29), which satisfies \( \gamma_1(\tau^*, \tau_1^*) = 0 \) and \( \gamma_2(\tau^*, \tau_1^*) = \omega^* \) in the neighborhood of \((\tau^*, \tau_1^*)\). In this section, we can calculate the crossing direction of the crossing curves (39). Define

\[
R(\tau, \tau_1) = \text{Re}\left\{ \frac{\partial D(\lambda, \tau, \tau_1)}{\partial \tau} \right\}, \quad I(\tau, \tau_1) = \text{Im}\left\{ \frac{\partial D(\lambda, \tau, \tau_1)}{\partial \tau} \right\},
\]

\[
R_1(\tau, \tau_1) = \text{Re}\left\{ \frac{\partial D(\lambda, \tau, \tau_1)}{\partial \tau_1} \right\}, \quad I_1(\tau, \tau_1) = \text{Im}\left\{ \frac{\partial D(\lambda, \tau, \tau_1)}{\partial \tau_1} \right\}.
\]

Furthermore, it can calculate \( R_1 - R_1 = -\text{Im}\left\{ \frac{\partial D}{\partial \tau} \cdot \frac{\partial D}{\partial \tau_1} \right\} \).

If \( R_1 - R_1 > 0 \), then the pair characteristic roots \( \gamma_1(\tau, \tau_1) \pm i\gamma_2(\tau, \tau_1) \) of (29) cross the imaginary axis to the right half plane when \((\tau, \tau_1)\) crosses the crossing curve to the right region. If the inequality is reversed, the crossing direction is opposite. From (29), it has

\[
\frac{\partial D}{\partial \tau}(i\omega^*, \tau^*, \tau_1^*) = \Sigma_0^\tau + \Sigma_1^\tau e^{-i\omega^* \tau^*} + \Sigma_2^\tau e^{-i\omega^* \tau_1^*} + \Sigma_3^\tau e^{-i\omega^* \tau_1^*},
\]

where \( \Sigma_0^\tau = \Sigma_s(\omega^*, \tau^*) \) and \( \Sigma_2^\tau = \partial \Sigma_2(\omega^*, \tau^*)/\partial \tau, \ s = 0, 1, 2 \).

Furthermore,

\[
-\text{Im}\left\{ \frac{\partial D}{\partial \tau} \cdot \frac{\partial D}{\partial \tau_1} \right\} = -\omega^* \text{Re}\left\{ \left[ \Sigma_0^\tau e^{i\omega^* \tau_1^*} + (\Sigma_1^\tau - i\omega^* \Sigma_4^\tau) e^{i\omega^*(\tau_1^* - \tau^*)} + \Sigma_2^\tau \right] \Sigma_2^\tau \right\}.
\]

**Theorem 5.5.** If \( \delta(\tau^*, \tau_1^*) > 0 \) (< 0), then the pair imaginary roots crosses the imaginary axis from left to right as \((\tau, \tau_1)\) passes through the crossing curve to the right (left) region, where

\[
\delta(\tau^*, \tau_1^*) = -\text{Re}\left\{ \left[ \Sigma_0^\tau e^{i\omega^* \tau_1^*} + (\Sigma_1^\tau - i\omega^* \Sigma_4^\tau) e^{i\omega^*(\tau_1^* - \tau^*)} + \Sigma_2^\tau \right] \Sigma_2^\tau \right\}.
\]

### 6. Numerical simulations.

In this part, we will carry out some numerical simulations. Firstly, choosing the following parameters in system (2):

\[
r = 5, \quad L = 8, \quad m = 10, \quad \mu = 0.005, \quad \alpha = 0.8, \quad \beta = 0.6, \quad \rho = 0.02.
\]

When \( \tau_1 = 0 \), \( \tau \) can obtained as 317.7637. Under the parameters in (43), it can obtain the plots of \((\tau, \Sigma_n(\tau))\) (see Figure 3). From the plots, it knows that \( \Sigma_0(\tau) = 0 \) has two roots \( \tau^0 = 54.4 \) and \( \tau^1 = 165.247 \). Therefore, choosing \( \tau = 30 < \tau^0, \tau = 54.4 \rightarrow \tau^0, \tau = 200 > \tau^1 \), respectively, the results are shown in Figures 4-6 and verify the correction of theoretical analyses.
Figure 3. \((\tau, S_n(\tau)) \; (n = 0, 1)\) plots.

(A) Wave plots.  \(B\) Phase plot.

Figure 4. \(E^*\) is stable when \(\tau = 30\).

(A) Wave plots.  \(B\) Phase plot.

Figure 5. \(E^*\) is unstable when \(\tau = 54.4\) and there exists a stable periodic solution.
Furthermore, fixed $\tau = 30 \in [0, \tau^0)$ and choosing $\tau_1$ as parameter, it can obtain $\tau^0_1 = 100.29$. Let $\tau_1 = 5 < \tau^0_1$, the results of the numerical simulations are sown in Figure 7.

Specially, when $\tau = \tau_1 = \nu$, the following parameters are chosen

$$r = 5, \quad L = 8, \quad m = 6, \quad \mu = 0.005, \quad \alpha = 0.8, \quad \beta = 0.7, \quad \rho = 0.01.$$  

(44)

It can obtain $\bar{\nu} = 453.4988$ under the parameters (44). The plots of $(\nu, T_n(\nu))$ can be obtained (see Figure 8). From the plots, it knows that $T_n(\nu) = 0$ ($n = 0, 1, 2$) has four roots $\nu^0 = 7.5, \nu^1 = 73.5, \nu^2 = 381.26$ and $\nu^3 = 425.266$. Therefore, choosing $\nu = 1 < \nu^0, \tau = 7.5 = \nu^0, \tau = 430 > \nu^3$, respectively, the results are shown in Figure 9 (A)-(C). All global Hopf bifurcations of periodic solutions emanating from the Hopf bifurcation points are depicted in Figure 9 (D). It is seen from Figure 9 (D) that these branches are all bounded and each branch connects exactly a pair of bifurcation values $(\nu^0, \nu^3)$ and $(\nu^1, \nu^2)$, which makes the coexistence of multiple periodic solutions possible. These are consistent with theoretical results.

In the following, it also obtain the crossing curves in $(\tau, \tau_1)$ plane using the parameters in (43). Under the parameters (43), it can obtain $I_1 = [0.04, 0.044]$ and $I_2 = [0, 0.04]$, furthermore, $I^1_1 = [0, 15.0495]$ and $I^2_1 = [0, 222.425]$. It can know that both $I^1_2 = [0, 15.0495]$ and $I^2_2 = [0, 222.425]$ belong to Type 1. The curve $C$ and
Figure 8. \((\nu, T_n(\nu))\) plots \((n = 0, 1, 2)\).

(A) \(\nu = 1\).  
(B) \(\nu = 7.5\).  
(C) \(\nu = 430\).  
(D) Bifurcation graphs.

Figure 9. \(E^*\) is stable for (A) and (C). \(E^*\) is unstable and there exists a stable periodic solution for (B). The bifurcation diagram showing stability switches at \(E^*\) and all global Hopf bifurcations shown in (D).
crossing curves are shown in Figures 10-11. It can obtain the feasible region $\Omega$ that is surrounded by $\tau$-axis, $\omega$-axis and the blue curves in Figure 10. The curves $C$, which is composed by the feasible values of $(\tau, \omega)$ (see the color loops between the two blue curves in Figure 10). Furthermore, we can see that the two closed loop curves $C$ on $\Omega$ with Type AB lead to two series of spiral-like crossing curves on $(\tau, \tau_1)$ plane along $\tau_1$-axis shown in Figure 11. Finally, according to Theorem 5.5, the crossing direction is calculated and the final result is given in Figure 11, that is, when $(\tau, \tau_1)$ changes along the arrow direction, the characteristic root passes through the imaginary axis from left to right. Fixed $\tau_1 = 10$ and choosing $\tau = 20, 80, 180$, the stable switches phenomena can be seen (see Figures 12-14). It shows that Hopf bifurcation can still occur when $\tau$ locates the unstable interval.

![Figure 10. Feasible region and curve C in $(\tau, \omega)$ plane.](image1)

![Figure 11. Crossing curves and crossing directions.](image2)
7. Conclusions. In this paper, we study a planktonic ecosystem (2) with two delays and Monod-Haldane type functional response. We focus on the effects of two delays on the system. In Theorem 2.3, we obtain the positively invariant set.
for system (2). The system (2) always exists two boundary equilibria $E_0(0,0)$ and $E_1(L,0)$ while $E_0(0,0)$ is unstable and the stability of $E_1(L,0)$ is shown in Theorem 3.1. When the maturation rate $\tau$ of zooplankton is more than some value, zooplankton population will dies out in the end. When the maturation rate of zooplankton $\tau$ are restricted to a certain interval, the system (2) exists a uniquely positive equilibrium $E^*(P^*,Z^*)$. When the toxin delay $\tau_1$ does not exist, Theorem 4.2 shows the stability of $E^*$ and system (2) may occur the stability switches when the maturation rate $\tau$ of zooplankton changes which depends on the set $I_\tau$ and the roots of $S^+_{\tau_1}(\tau) = 0$. When $\tau_1$ exists, it still can obtain the conditions which the system occurs Hopf bifurcation shown in Theorem 4.5. Furthermore, in Section 5, using the crossing curve method, when $\tau$ is chosen in unstable interval it also can obtain the stability of $E^*$. By these researches, it finds the important effects of two delays for the production of red tides. Nevertheless, these results have been drawn from the analysis of mathematical models applicable to situations where the species are supposed to experience spatial homogeneity. However, in natural waters, the species of phytoplankton and zooplankton are distributed over a considerably large spatial regime and are prone to spatial movements due to several physical and biological reasons. Species interaction in space and time is a well-known mechanism for the emergence of spatial structure. We leave the generalization of this model in spatial heterogeneous environment for future investigation.

Acknowledgments. We would like to thank the anonymous referees for many constructive suggestions.

REFERENCES

[1] Q. An, E. Beretta, Y. Kuang, C. Wang and H. Wang, Geometric stability switch criteria in delay differential equations with two delays and delay dependent parameters, *J. Differential Equations*, 266 (2019), 7073–7100.
[2] M. Banerjee and E. Venturino, A phytoplankton-toxic phytoplankton-zooplankton model, *Ecol. Complex.*, 8 (2011), 239–248.
[3] E. Beretta and Y. Kuang, Geometric, stability switch criteria in delay differential systems with delay dependent parameters, *SIAM J. Math. Anal.*, 33 (2002), 1144–1165.
[4] P. Bi and S. Ruan, Bifurcations in delay differential equations and applications to tumor and immune system interaction models, *SIAM J. Applied Dynamical Systems*, 12 (2013), 1847–1888.
[5] J. Chattopadhyay, R. Sarkar and S. Mandal, Toxin producing plankton may act as a biological control for planktonic blooms: A field study and mathematical modelling, *J. Theor. Biol.*, 215 (2002), 333–344.
[6] J. Chattopadhyay, R. Sarkar and AE Abdllaoui, A delay differential equation model on harmful algal blooms in the presence of toxic substances, *IMA J. Appl. Math.*, 19 (2002), 137–161.
[7] Y. Ding, W. Jiang and P. Yu, Double Hopf bifurcation in delayed vander pol-duffing equation, *Internat. J. Bifur. Chaos*, 23 (2013), 1350014, 15 pages.
[8] K. Gu, S. Niculescu and J. Chen, On stability crossing curves for general systems with two delays, *J. Math. Anal. Appl.*, 311 (2005), 231–253.
[9] R. Han and B. Dai, Cross-diffusion induced Turing instability and amplitude equation for a toxic-phytoplankton-zooplankton model with nonmonotonic functional response, *Internat. J. Bifur. Chaos*, 27 (2017), 1750088, 24 pages.
[10] J. Hale and S. Lunel, *Introduction to functional differential Equations*, Springer-Verlag, New York, 1993.
[11] Z. Jiang and T. Zhang, Dynamical analysis of a reaction-diffusion phytoplankton-zooplankton system with delay, *Chaos, Solitons Fractals*, 104 (2017), 693–704.
[12] Z. Jiang, W. Zhang, J. Zhang and T. Zhang, Dynamical analysis of a phytoplankton-zooplankton system with harvesting term and Holling III functional response, *Internat. J. Bifur. Chaos.*, 28 (2018), 1850162, 23 pages.
BIFURCATION ANALYSIS IN A DELAYED P-Z ECOSYSTEM

[13] Z. Jiang, J. Dai and T. Zhang, Bifurcation analysis of phytoplankton and zooplankton interaction system with two delays, Internat. J. Bifur. Chaos, 30 (2020), 2050039, 21 pages.

[14] H. Jiang and Y. Song, Normal forms of non-resonance and weak resonance double Hopf bifurcation in the retarded functional differential equations and applications, Appl. Math. Comput., 266 (2015), 1102–1126.

[15] Z. Jiang and L. Wang, Global Hopf bifurcation for a predator-prey system with three delays, Internat. J. Bifur. Chaos, 27 (2017), 1750108, 15 pages.

[16] Z. Jiang and Y. Guo, Hopf bifurcation and stability crossing curve in a planktonic resource-consumer system with double delays, Internat. J. Bifur. Chaos, 30 (2020), 2050190, 20 pages.

[17] S. Ma, Q. Lu and Z. Feng, Double Hopf bifurcation for van der pol-duffing oscillator with parametric delay feedback control, J. Math. Anal. Appl., 338 (2008), 993–1007.

[18] R. Pal, D. Basu and M. Banerjee, Modelling of phytoplankton allelopathy with Monod-Haldanetype functional response—A mathematical study, Biosystems, 95 (2009), 243–253.

[19] Y. Qu, J. Wei and S. Ruan, Stability and bifurcation analysis in hematopoietic stem cell dynamics with multiple delays, Physica D., 239 (2010), 2011–2024.

[20] S. Roy, S. Bhattacharya, P. Das and J. Chattopadhyay, Interaction among non-toxic phytoplankton, toxic phytoplankton and zooplankton: Inferences from field observations, J. Biol. Phys., 33 (2007), 1–17.

[21] J. Wu, Symmetric functional differential equations and neural networks with memory, Trans. Amer. Math. Soc., 350 (1998), 4799–4838.

Received September 2020; revised January 2021.

E-mail address: jzhauper@163.com
E-mail address: htzhangzx@163.com
E-mail address: jmy970907@163.com