GROTHENDIECK DUALITY AND TRANSITIVITY I: FORMAL SCHEMES

SURESH NAYAK AND PRAMATHANATH SASTRY

Abstract. For a proper map \( f : X \to Y \) of noetherian ordinary schemes, one has a well-known natural transformation, \( \mathbf{L} f^*(-) \otimes^L f^! O_Y \to f^! \), obtained via the projection formula, which extends, using Nagata's compactification, to the case where \( f \) is separated and of finite type. In this paper we extend this transformation to the situation where \( f \) is a pseudo-finite-type map of noetherian formal schemes which is a composite of compactifiable maps, and show it is compatible with the pseudofunctorial structures involved. This natural transformation has implications for the abstract theory of residues and traces, giving Fubini type results for iterated maps. These abstractions are rendered concrete in a sequel to this paper.

All schemes are assumed to be noetherian. They could be formal or ordinary. For any formal scheme \( \mathcal{X} \), \( \mathcal{A}(\mathcal{X}) \) is the category of \( O_X \)-modules and \( D(\mathcal{X}) \) its derived category. The torsion functor \( \Gamma'_{\mathcal{X}} \) on \( O_X \)-modules is defined by the formula

\[
\Gamma'_{\mathcal{X}} := \lim_{\longrightarrow} \mathcal{H}om_{O_X}(O_X/I^n, -)
\]

where \( I \) is any ideal of definition of the formal scheme \( \mathcal{X} \). A torsion module \( \mathcal{F} \) is an object in \( \mathcal{A}(\mathcal{X}) \) such that \( \Gamma'_{\mathcal{X}} \mathcal{F} = \mathcal{F} \). The category \( \mathcal{A}_c(\mathcal{X}) \) (resp. \( \mathcal{A}_q(\mathcal{X}) \), resp. \( \mathcal{A}_{qct}(\mathcal{X}) \)) is the category of coherent (resp. direct limit of coherent, resp. quasi-coherent and torsion) \( O_X \)-modules. \( D_c(\mathcal{X}) \) (resp. \( D_q(\mathcal{X}) \), resp. \( D_{qct}(\mathcal{X}) \)) is the subcategory of \( D(\mathcal{X}) \) of complexes having homology in \( \mathcal{A}_c(\mathcal{X}) \) (resp. \( \mathcal{A}_q(\mathcal{X}) \), resp. \( \mathcal{A}_{qct}(\mathcal{X}) \)), while \( D^+_c(\mathcal{X}) \) (resp. \( D^+_q(\mathcal{X}) \), resp. \( D^+_{qct}(\mathcal{X}) \)) for \( * \) in \( \{ b, +, - \} \) denotes the corresponding full subcategory whose homology is additionally bounded, or bounded below, or bounded above, accordingly.

We will be using the notions of pseudo-proper, pseudo-finite-type, pseudo-finite maps used in [AJL2]. For example if \( f : \mathcal{X} \to \mathcal{Y} \) is a map of formal schemes, it is pseudo-proper if the map of ordinary schemes \( f_0 : X \to Y \) obtained by quotienting the structure sheaves \( O_\mathcal{X} \) and \( O_\mathcal{Y} \) by ideals of definition for \( \mathcal{X} \) and \( \mathcal{Y} \), is proper. If this is true for one pair of defining ideals (\( \mathcal{I}, \mathcal{J} \)) with \( \mathcal{I} \subset O_\mathcal{Y} \) and \( \mathcal{J} O_\mathcal{X} \subset \mathcal{I} \subset O_\mathcal{X} \), then it is true for all such pairs. The definitions of pseudo-finite-type, pseudo-finite are analogous.

We assume familiarity with the viewpoint of duality that was initiated by Deligne in [D1], especially as laid out by Lipman in [L4, Chapter 4]. In particular, we assume familiarity with the main properties of the inverse-image pseudofunctor \( (-)^! \). This is a \( D_{qc}^+ \)-valued contravariant pseudofunctor on the category of schemes and

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Proposition 3.9.4

If $h$ is separated essentially finite-type maps.

1. Introduction

We begin by recalling the notion of a (contravariant) pseudofunctor. A pseudofunctor $(-)^\Delta$ on a category $\mathcal{C}$ is an assignment of categories $X^\Delta$, one for each $X \in \mathcal{C}$, such that for each map $f: X \to Y$ in $\mathcal{C}$, we have a functor $f^\Delta: Y^\Delta \to X^\Delta$, for each $X \in \mathcal{C}$, an isomorphism $\eta^\Delta_X: 1_X^\Delta \xrightarrow{\sim} 1_{X^\Delta}$, for each pair of composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, an isomorphism of functors $C^\Delta_{f,g}: (gf)^\Delta \xrightarrow{\sim} f^\Delta g^\Delta$, such that for a third map $h: Z \to S$ in $\mathcal{C}$, “associativity” holds, i.e., the following diagram commutes,

$$
\begin{array}{c}
\begin{array}{c}
\bigtriangleup \\
\bigcirc \\
\bigtriangledown
\end{array}
\end{array}
\xymatrix{ f^\Delta g^\Delta h^\Delta 
\ar[r]^-{(gf)^\Delta h^\Delta} 
\ar[d]_-{C^\Delta_{f,g}} & \\
(hg)^\Delta 
\ar[r]_-{(hg)^\Delta} & 
}
$$

and for the compositions $1_Y f = f = f 1_X$, the isomorphisms $\eta^\Delta$ and $C^\Delta_{-,-}$ are compatible in the obvious way. However, as we shall see, we may need to relax the condition that $\eta^\Delta_X: 1_X^\Delta \xrightarrow{\sim} 1_{X^\Delta}$ be an isomorphism for objects $X \in \mathcal{C}$. In such a case, if other conditions are satisfied, $(-)^\Delta$ is called a pre-pseudofunctor. For simplicity, we may often call these naturally occurring pre-pseudofunctors as pseudofunctors.

1.1. The principal aim of this paper is to extend results in [L4, §4.9] to the situation of formal schemes. An example of the kind of result in loc.cit. that would interest us is the existence of a map

$$(1.1.1) \quad L f^* g^! \mathcal{O}_Z \otimes_{\mathcal{O}_X} f^! \mathcal{O}_Y \to (gf)^! \mathcal{O}_Z$$

for a pair of finite-type separated maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. This is closely related to the results in [LS]. Let us discuss (1.1.1) in some detail to orient the reader to the kind of questions we are interested in as well as the difficulties involved in answering them for formal schemes.

Recall that if $h: V \to W$ is a proper map of schemes, then the twisted inverse image functor $h^! : D^{qc}_c(W) \to D^{qc}_c(V)$ is a right adjoint to $R h_* : D^{qc}_c(V) \to D^{qc}_c(W)$. We therefore have the co-adjoint unit

$$(1.1.2) \quad \text{Tr}_h : R h_* h^! \to 1_{D^{qc}_c(W)},$$

the so-called trace map for $h$, such that the map

$$\text{Hom}_{D^{qc}_c(V)}(\mathcal{F}, h^! \mathcal{G}) \to \text{Hom}_{D^{qc}_c(W)}(R h_* \mathcal{F}, \mathcal{G})$$

given by $\varphi \mapsto \text{Tr}_h(\mathcal{F}) \circ R h_* (\varphi)$ is an isomorphism [L4, p.204, Theorem 4.8.1 (i)]. Next, if $h$ is étale, by part (ii) of loc.cit., we have $h^! = h^*$. Note that if $\mathcal{F} \in D(X)$ and $\mathcal{G} \in D^{qc}_c(Y)$, we have the bifunctorial projection isomorphism of [L4, p.139, Proposition 3.9.4]

$$(1.1.3) \quad \mathcal{G} \otimes_{\mathcal{O}_Y} R h_* \mathcal{F} \xrightarrow{\sim} R h_* (L h^* \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

Finally, recall that if $h$ is separated and of finite type, by a famous theorem of Nagata [N] one can find a compactification of $h$, i.e., a factorization $h = \bar{h} \circ i$ with

separated essentially finite-type maps.
an open immersion and \( \bar{h} \) a proper map. Thus, after choosing such a compactification, one could define \( h^! \) for such maps, by the formula \( h^! = i^* \bar{h}^! \). That this is independent of the compactification chosen is proven in [D1]. The technical difficulties encountered carrying out this program are formidable, and form the content of [D1], [D2], [D2'], and [L4, Chapter 4].

The map (1.1.1) is described as follows. First suppose \( f \) and \( g \) are proper. One has a natural map

\[
\begin{align*}
\mathbb{R}(gf)_*(L f^* g^! O_Z \otimes \mathcal{O}_Y) & \longrightarrow \mathcal{O}_Z \\
\mathbb{R}(gf)_*(L f^* g^! O_Z \otimes \mathcal{O}_X f^! \mathcal{O}_Y) & \longrightarrow \mathbb{R} g_* \mathbb{R} f_*(L f^* g^! O_Z \otimes \mathcal{O}_X f^! \mathcal{O}_Y) \\
\mathbb{R} g_* (1 \otimes \text{Tr}_f) & \longrightarrow \mathbb{R} g_* g^! \mathcal{O}_Z \\
\text{Tr}_g & \longrightarrow \mathcal{O}_Z.
\end{align*}
\]

Since \((gf)^!\) is right adjoint to \(\mathbb{R}(gf)_*\), the map (1.1.4) gives rise to (1.1.1) as the unique map such that \(\text{Tr}_{gf}(\mathcal{O}_Z) \circ (gf)_*(1 \otimes \text{Tr}_f) = (1.1.4)\).

If \( f \) or \( g \) is not proper, one can find a compactification of \( gf \), say \( gf = F \circ j \), such that \( F = \bar{g} \circ \bar{f} \), with \( \bar{f} \), \( \bar{g} \) proper maps, and where these maps embed into a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \bar{X} \\
\downarrow f \quad & & \downarrow \quad \bar{f} \\
Y \quad & \longrightarrow & \bar{Y} \\
\downarrow s \quad & & \downarrow \quad \bar{g} \\
Z & & \\
\end{array}
\]

with all horizontal arrows open immersions and all south-west pointing arrows proper, with the composite of the two horizontal arrows on the top row being \( j \). The map (1.1.1) for the pair \((f, g)\) can be defined by “restricting” the corresponding map for \((\bar{f}, \bar{g})\) to \( X \). The map (1.1.1) is independent of the choice of such diagrams — this is the essential content of [L4, pp. 231–232, Lemma 4.9.2]. We provide a proof in this paper for formal schemes in Proposition 7.2.4 below.

The map (1.1.1) is to be regarded as an abstract form of certain transitivity results for differential forms which are important for duality, e.g., property (R4) of residues stated in [RD, p.198]. Briefly, if \( h: V \to W \) is a smooth map of relative dimension \( n \), then one can show that there is an isomorphism

\[
(1.1.6) \quad \omega_h[n] \cong h^! \mathcal{O}_W,
\]

where \( \omega_h = \Omega^n_{V/W} \) is the \( n \)th exterior power of the \( \mathcal{O}_V \)-module of relative differential forms \( \Omega^1_{V/W} \) for the map \( h \). There are many descriptions of such isomorphisms (see [V, p. 397, Theorem 3], [HK2, p. 84, Duality theorem] for projective maps with the base which is free of embedded points, [HS, p. 750, Duality Theorem] for maps where the base is free of embedded points). Thus if \( f \) and \( g \) are smooth of relative
dimensions, say, \( m \) and \( n \) respectively, then, upon taking homology in degree \( m+n \), \( (1.1.1) \) induces a map of coherent \( \mathcal{O}_X \)-modules,

\[
f^* \omega_y \otimes_{\mathcal{O}_X} \omega_f \to \omega_{gf}.
\]

The above map is an isomorphism since \( (1.1.1) \) is in this special case, \( f \) being a perfect map. How this compares with the usual isomorphism between \( f^* \omega_y \otimes_{\mathcal{O}_X} \omega_f \) and \( \omega_{gf} \) depends on the choice of \( (1.1.6) \) which in turn depends on the choice of concrete trace maps of the form \( R h \omega_h[n] \to \mathcal{O}_W \). For the one implicit in [HS], the problem is studied in [LS] (see also the correction). In a sequel to this paper we will show that when \( (1.1.6) \) is chosen to be the isomorphism given by Verdier in [V, p. 397, Thm. 3], the map \( (1.1.7) \) is the map given locally by \( f^*(\mu) \otimes \nu \mapsto \nu \wedge f^*(\mu) \) where the notation is self-explanatory.

The problem of finding the concrete expression for \( (1.1.7) \) given \( (1.1.6) \) is best attacked by expanding the scope of our study from ordinary schemes to formal schemes. One reason for this is that maps such as \( (1.1.1) \) are compatible (in a precise sense) with completions along closed subschemes in \( X, Y, \) and \( Z \). This allows for far greater flexibility than the method of compactifying and restricting. From this larger point of view, property (R4) of residues in [RD, p. 198] is a concrete manifestation of \( (1.1.1) \) for maps between formal schemes.

In this paper we lay the foundations for all of this. The generalization of parts of §4.9 of [L4] (in particular of the map \( (1.1.1) \) above) is carried out in section Section 7, especially in Proposition 7.2.4. As for property (R4) for residues, an abstract form of it for Cohen-Macaulay maps is proved by us in Proposition 8.3.1 below. We draw the reader’s attention especially to formulas \( (8.3.2) \) and \( (8.3.3) \) which follow from \textit{loc.cit.}

If we move to formal schemes which are not necessarily ordinary, the main theorem of [AJL2] assures us of the existence of a right adjoint to \( Rf_*: \mathbf{D}_{qc}(\mathcal{X}) \to \mathbf{D}_{qc}(\mathcal{Y}) \) for a pseudo-proper map (and more) \( f: \mathcal{X} \to \mathcal{Y} \) and this right adjoint is denoted \( f! \). However, given a general separated pseudo-finite-type map \( f \), we are no longer assured that \( f \) has a compactification, i.e., we are no longer assured that we have a factorization \( f = \bar{f} \circ i \) with \( i \) an open immersion, and \( \bar{f} \) pseudo-proper.

We are therefore forced to work in the category \( \mathcal{G} \) whose objects are formal schemes and whose morphisms are composites of “compactifiable” maps. In [Nay] the first author shows that we have a pre-pseudofunctor \( (-)^! \) on \( \mathcal{G} \), which generalises what we have for the category of finite-type separated maps on ordinary schemes (see Section 3).

If \( k \) is a field and \( A = k[[X_1, \ldots, X_d]] \) the ring of power series over \( k \) in \( d \) analytically independent variables, \( m \) the maximal ideal of \( A \), \( \mathcal{X} \) the formal spectrum of \( A \) with its \( m \)-adic topology, and \( \mathcal{Y} \) the spectrum of \( k \), then the natural map \( f: \mathcal{X} \to \mathcal{Y} \) is pseudo-proper and \( f^! \mathcal{O}_\mathcal{Y} \) is the torsion sheaf obtained by sheafifying the \( A \)-module \( \mathbb{H}_m^d(\omega_A) \) where \( \omega_A \) is the" canonical module of \( A \). From here to recovering local duality for the complete local ring \( A \) requires a more careful examination of the relationship between \( f^! \mathcal{O}_\mathcal{Y} \) and the canonical module \( \omega_A \). As it turns out, \( \omega_A \), a finitely generated \( A \)-module, can be recovered from the torsion module associated to \( f^! \mathcal{O}_\mathcal{Y} \). More generally, for a pseudo-proper map \( f: \mathcal{X} \to \mathcal{Y} \) between formal schemes and an object \( \mathcal{G} \in \mathbf{D}^+_c(\mathcal{Y}) \), there is a deep relationship between

\[\text{On the other hand, we do not have an example of a separated pseudo-finite-type map which does not have a compactification.}\]
The pseudofunctor \( f^\# \in D_{\text{qct}}(\mathcal{X}) \) and an associated object in \( D_{\text{qct}}^+(\mathcal{X}) \), of which the above mentioned relationship between \( H^d(\omega_A) \) and \( \omega_A \) is an example. This necessitates the development of a second twisted inverse image functor \( f^\# \) related to \( f^! \). The twisted inverse image \( f^\# \) was introduced by Alonso, Jéremías, and Lipman in [AJL2], and the relationship between \( f^! \) and \( f^\# \) is one of the many important portions of that work.

This paper is mainly concerned with a pre-pseudofunctor \((-)^\#\) on the category \( G \) such that for \( f \) pseudo-proper, \( f^\# \) is the functor mentioned in the last paragraph. The pseudofunctor \((-)^\#\) is one of two ways that the twisted inverse image pseudofunctor \((-)^!\) on the category of ordinary schemes and separated finite-type maps generalizes to \( G \), the other being the pre-pseudofunctor \((-)^!\) on \( G \) discussed above. The map \((\ref{1.1})\) can be defined for formal schemes with \( f^! \) and \( g^! \) replaced by \( f^\# \) and \( g^\# \). The principal technical issue which creates complications is the lack of diagrams like \((\ref{1.1})\) into which a pair of maps \( \mathcal{X} \to \mathcal{Y} \) embed. In the next subsection we give brief introduction to \((-)^!\) and \((-)^\#\) on \( G \).

### 1.2. Two twisted inverse images.

The duality pseudofunctors \((-)^\#\) and \((-)^!\) on \( G \) are explained in Section 3 of this paper, but for the purposes of this introduction we say a few quick words. For a pseudo-proper map \( f: \mathcal{X} \to \mathcal{Y} \), the functor \( f^!: D_{\text{qct}}^+(\mathcal{Y}) \to D_{\text{qct}}^+(\mathcal{X}) \) is right adjoint to \( Rf_!: D_{\text{qct}}^+(\mathcal{X}) \to D_{\text{qct}}^+(\mathcal{Y}) \). In fact \( f^! \) extends to a larger category \( D_{\text{qc}}^+(\mathcal{Y}) \) which contains \( D_{\text{qct}}^+(\mathcal{Y}) \), namely the full subcategory of \( D(\mathcal{Y}) \) of objects \( \mathcal{F} \) such that \( R\Gamma^\#_{\mathcal{Y}}(\mathcal{F}) \in D_{\text{qct}}^+(\mathcal{Y}) \), the extended functor being \( f^!: D_{\text{qc}}^+(\mathcal{Y}) \to D_{\text{qc}}^+(\mathcal{Y}) \).

There is another duality functor associated with the pseudo-proper map \( f \), namely \( f^*!: D_{\text{qct}}^+(\mathcal{Y}) \to D_{\text{qct}}^+(\mathcal{X}) \), which is right adjoint to \( Rf_*R\Gamma_{\mathcal{Y}}^*: D_{\text{qct}}^+(\mathcal{X}) \to D_{\text{qct}}^+(\mathcal{X}) \).

The functors \( f^\# \) and \( f^! \) are related via the formulas \( f^! \cong R\Gamma_{\mathcal{Y}}^f f^\# \) and \( f^\# \cong \Lambda_{\mathcal{X}} f^! \), where \( \Lambda_{\mathcal{X}} (\cdot) = R\mathcal{H}om(R\Gamma_{\mathcal{Y}}^f \mathcal{O}_{\mathcal{X}}, \cdot) \).

As an example, if \( k \) is a field, \( \mathcal{X} \) the formal spectrum of the power series ring \( A = k[[X_1, \ldots, X_d]] \) (given the m-adic topology, where \( m \) is the maximal ideal of \( A \)), \( \mathcal{Y} = \text{Spec} k \), and \( f: \mathcal{X} \to \mathcal{Y} \) the map of formal schemes corresponding to the obvious k-algebra map \( k \to k[[X_1, \ldots, X_d]] \), then \( f \) is pseudo-proper. Identifying \( A \)-modules with their associated sheaves on \( \mathcal{X} \), and writing \( \Omega_{\mathcal{X}/k}^d \) for the universally finite module of d-forms for the algebra \( A/k \), we have \( f^!(k) = H^d_{\mathcal{X}/k}(\Omega_{\mathcal{X}/k}^d) \) and \( f^*(k) = \Omega_{\mathcal{X}/k}^d \).

This is for pseudo-proper maps, the original setting for defining \( f^! \) in [AJL2]. However, in [Nay], the first author was able to show that \( f^!: D_{\text{qct}}^+(\mathcal{Y}) \to D_{\text{qct}}^+(\mathcal{X}) \) can be defined when \( f: \mathcal{X} \to \mathcal{Y} \) is in \( G \), even when it is not pseudo-proper, in such a way that (a) when \( f \) is an open immersion, \( f^! \cong f^! R\Gamma_{\mathcal{Y}}^f f^* \), and (b) such that the resulting variance theory \((-)^! \) is a pre-pseudofunctor. In fact, \( 1_{\mathcal{Y}}^{\text{qct}} \cong R\Gamma_{\mathcal{Y}}^f \), and the latter functor is not isomorphic to the identity functor.

For a map \( f: \mathcal{X} \to \mathcal{Y} \) in \( G \), we set \( f^! = \Lambda_{\mathcal{Y}} f^! \). The source of \( f^* \) is \( D_{\text{qct}}^+(\mathcal{Y}) \) and its target is \( D_{\text{qct}}^+(\mathcal{X}) \), so that \( f^*!: D_{\text{qct}}^+(\mathcal{Y}) \to D_{\text{qct}}^+(\mathcal{X}) \). If \( f \) is pseudo-proper this definition of \( f^* \) agrees with the earlier one (as the functor which is right adjoint to \( Rf_*R\Gamma_{\mathcal{Y}}^f \)). The variance theory \((-)^\# \) is a pre-pseudofunctor with \( 1_{\mathcal{Y}}^{\text{qct}} \cong \Lambda_{\mathcal{Y}} f^! \). Our \( f^\# \) agrees with the one in [AJL2] only when \( f \) is pseudo-proper.

The paper is organized as follows. The definitions of \((-)^! \) and \((-)^\# \) and their first properties are given in Section 3. Sections 4 and 5 deal with Cohen-Macaulay...
maps. The meat of the paper is in Sections 7 and 8. There is an appendix which contains a number of results useful in the main body of the text. We have placed these results in the appendix so that the main narrative is not broken into disconnected bits.

2. Notations and basics on formal schemes

2.1. We discuss some basic matters on formal schemes and the derived categories of complexes on them. Most of what we say here can be found with more details in \[AJL2\]. Also, for the basic conventions on derived functors we refer to \[L4\].

For any formal scheme \(\mathcal{X}\) and any coherent ideal \(\mathcal{I}\) in \(\mathcal{O}_\mathcal{X}\), \(\Gamma_\mathcal{X}\) denotes the functor that assigns to any \(\mathcal{O}_\mathcal{X}\)-module \(\mathcal{F}\), the submodule of sections of \(\mathcal{F}\) annihilated locally by some power of \(\mathcal{I}\). The torsion functor \(\Gamma_\mathcal{X}'\) is the one that assigns to any \(\mathcal{F}\) the submodule of sections of \(\mathcal{F}\) annihilated locally by some open ideal in \(\mathcal{O}_\mathcal{X}\). Thus for any defining ideal \(\mathcal{I}\) for \(\mathcal{X}\), \(\Gamma_\mathcal{X} = \Gamma_\mathcal{X}'\). A torsion module is a module \(\mathcal{F}\) satisfying \(\Gamma_\mathcal{X}'\mathcal{F} = \mathcal{F}\).

For any formal scheme \(\mathcal{X}\), the abelian categories \(\mathcal{A}_c(\mathcal{X})\) for \(c, c, qc, qct\) are defined as in the beginning of this paper and the same applies to the definition of derived categories \(\mathcal{D}^+_c(\mathcal{X})\).

We use the notation \(\mathcal{R}\mathcal{F}\) (resp. \(\mathcal{L}\mathcal{F}\)) to denote the right (resp. left) derived functor associated to any (triangulated) functor \(\mathcal{F}\) between derived categories, and for the derived tensor product we use \(\otimes\).

Let \(\mathcal{D}_{qc}(\mathcal{X})\) denote the triangulated full subcategory of \(\mathcal{D}(\mathcal{X})\) whose objects consist of complexes \(\mathcal{F}\) such that \(\mathcal{R}\Gamma_\mathcal{X}^+\mathcal{F} \in \mathcal{D}_{qc}(\mathcal{X})\) (and hence \(\mathcal{R}\Gamma_\mathcal{X}^+\mathcal{F} \in \mathcal{D}_{qct}(\mathcal{X})\)). Similarly, \(\mathcal{D}_{qc}(\mathcal{X})\) denotes the subcategory consisting of complexes \(\mathcal{F}\) such that \(\mathcal{R}\Gamma_\mathcal{X}^+\mathcal{F} \in \mathcal{D}_{qc}(\mathcal{X})\). Thus there are full subcategories \(\mathcal{D}_{qc}(\mathcal{X}) \subset \mathcal{D}_{qc}(\mathcal{X})\) and \(\mathcal{D}_{qc}(\mathcal{X})\) which are all equal when \(\mathcal{X}\) is an ordinary scheme.

The functor \(\mathcal{R}\Gamma_\mathcal{X}^+ : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X})\) has a right adjoint given by

\[
\Lambda_\mathcal{X}(-) = \mathcal{R}\mathcal{A}\mathcal{t}(\mathcal{R}\Gamma_\mathcal{X}^+ \mathcal{O}_\mathcal{X}, -).
\]

Via canonical maps \(\mathcal{R}\Gamma_\mathcal{X}^+ \to 1 \to \Lambda_\mathcal{X}\), both \(\mathcal{R}\Gamma_\mathcal{X}^+\) and \(\Lambda_\mathcal{X}\) are idempotent functors and in fact there are natural isomorphisms

\[
\mathcal{R}\Gamma_\mathcal{X}^+ \mathcal{R}\Gamma_\mathcal{X}^+ \xrightarrow{\sim} \mathcal{R}\Gamma_\mathcal{X}^+ \xrightarrow{\sim} \mathcal{R}\Gamma_\mathcal{X}^+ \Lambda_\mathcal{X}, \quad \Lambda_\mathcal{X} \mathcal{R}\Gamma_\mathcal{X}^+ \xrightarrow{\sim} \Lambda_\mathcal{X} \xrightarrow{\sim} \Lambda_\mathcal{X} \Lambda_\mathcal{X}.
\]

In particular, \(\Lambda_\mathcal{X}(\mathcal{D}_{qc}(\mathcal{X})) \subset \mathcal{D}_{qc}(\mathcal{X})\) and therefore, the restriction of \(\mathcal{R}\Gamma_\mathcal{X}^+\) to \(\mathcal{D}_{qc}(\mathcal{X})\) and of \(\Lambda_\mathcal{X}\) to \(\mathcal{D}_{qc}(\mathcal{X})\) also constitute an adjoint pair.

For \(\mathcal{F} \in \mathcal{D}_{qc}(\mathcal{X})\), the canonical map \(\mathcal{F} \to \Lambda_\mathcal{X}\mathcal{F}\) is an isomorphism by Greenlees–May duality \[AJL2\] Prop 6.2.1. More generally, if \(\mathcal{D}(\mathcal{X})\) is the subcategory of \(\mathcal{D}_{qc}(\mathcal{X})\) consisting of complexes whose homology sheaves are direct limits of coherent ones, then the restriction of \(\Lambda_\mathcal{X}\) to \(\mathcal{D}_c(\mathcal{X})\) is isomorphic to the left-derived functor of the completion functor \(\Lambda_\mathcal{X}\) which assigns to any sheaf \(\mathcal{F}\), the inverse limit \(\lim_{\leftarrow n} \mathcal{F}/\mathcal{I}_{\mathcal{X}}^n\mathcal{F}\) where \(\mathcal{I}\) is any defining ideal in \(\mathcal{O}_\mathcal{X}\). In contrast, note that \(\mathcal{R}\Gamma_\mathcal{X}^+\) does not preserve coherence of homology in general.

Let \(f : \mathcal{X} \to \mathcal{Y}\) be a map of noetherian formal schemes. Then there are natural isomorphisms (see \[AJL2\] Proposition 5.2.8)

\[
(2.1.1) \quad \mathcal{R}\Gamma_\mathcal{X}^+ \mathcal{L}f^* \mathcal{R}\Gamma_\mathcal{Y}^+ \xrightarrow{\sim} \mathcal{R}\Gamma_\mathcal{X}^+ \mathcal{L}f^* \xrightarrow{\sim} \mathcal{R}\Gamma_\mathcal{Y}^+ \mathcal{L}f^* \Lambda_\mathcal{X}, \\
\Lambda_\mathcal{Y} \mathcal{L}f^* \mathcal{R}\Gamma_\mathcal{X}^+ \xrightarrow{\sim} \Lambda_\mathcal{Y} \mathcal{L}f^* \xrightarrow{\sim} \Lambda_\mathcal{Y} \mathcal{L}f^* \Lambda_\mathcal{X}.
\]
Here the first isomorphism in the first line follows easily from the fact that for any coherent ideal \( \mathcal{I} \subset \mathcal{O}_X \), we have \( Lf^* R\Gamma_X^* \cong R\Gamma_{f* \mathcal{O}_X} Lf^* \), a fact which can be checked locally using stable Koszul complexes, see [C.5.2] in Appendix below for instance. The remaining isomorphisms in (2.1.1) result from the first one by pre-composing with \( \Lambda_{\mathcal{X}} \) or post-composing with \( \Lambda_{\mathcal{Y}} \).

For \( f : \mathcal{X} \rightarrow \mathcal{Y} \) as above, \( f^* \) sends torsion \( \mathcal{O}_Y \)-modules to torsion \( \mathcal{O}_X \)-modules and hence we have \( Lf^* (D_{qct}(\mathcal{Y})) \subset D_{qct}(\mathcal{X}) \) (in addition to the usual inclusions \( Lf^*(D_{qc}(\mathcal{Y})) \subset D_{qc}(\mathcal{X}), Lf^*(D_{c}(\mathcal{Y})) \subset D_{c}(\mathcal{X}) \)). By (2.1.1) we also deduce that \( Lf^*(D_{qc}(\mathcal{Y})) \subset D_{qc}(\mathcal{X}) \).

Unlike the case of ordinary (noetherian) schemes, \( Rf_* \) does not map \( D_{qct}(\mathcal{X}) \) (or even \( D_{qc}(\mathcal{X}) \)) inside \( D_{qct}(\mathcal{Y}) \) in general. Under additional torsion conditions we do get the desired behaviour. Thus we have \( Rf_*(D_{qct}(\mathcal{X})) \subset D_{qct}(\mathcal{Y}) \) and therefore \( Rf_* Rf_*(D_{qc}(\mathcal{X})) \subset D_{qct}(\mathcal{Y}) \).

For any morphism \( f : \mathcal{X} \rightarrow \mathcal{Y} \) as above and for any closed subset \( Z \subset \mathcal{X} \), we set \( Rf_* := Rf_*(\mathcal{X}) \). Likewise we set \( Rf_* := Rf_*(\mathcal{X}) \). For any integer \( r \) we use \( Rf_* := H^r Rf_* \), and \( Rf_* := H^r Rf_* \).

3. The duality pseudofunctors over formal schemes

We mainly work with the category \( \mathbb{G} \) of composites of open immersions and pseudo-proper maps between noetherian formal schemes. By Nagata’s compactification theorem, every separated finite-type map of ordinary schemes lies in \( \mathbb{G} \).

3.1. The results in [AJL2] and [Nay] extend the theory of \((-)^!\) over ordinary schemes to that over \( \mathbb{G} \). Thus, there is a contravariant pseudofunctor \((-)^!\) on \( \mathbb{G} \) with values in \( D_{qct}(\mathcal{X}) \) for any formal scheme \( \mathcal{X} \), such that if \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is pseudo-proper, there exists a functorial map \( t_f : Rf_* f^! \rightarrow 1_{D_{qct}(\mathcal{Y})} \) such that \( (f^! , t_f) \) is a right adjoint to \( Rf_* : D_{qct}(\mathcal{X}) \rightarrow D_{qct}(\mathcal{Y}) \) while if \( f \) is an open immersion or more generally, if \( f \) is adic étale, then \( f^! = f^* \) pseudofunctorially. For a formally étale map \( f \) in \( \mathbb{G} \), (e.g., a completion map \( \mathcal{X} \rightarrow X \) where \( X \) is an ordinary scheme and \( \mathcal{X} \) its completion along some coherent ideal in \( \mathcal{O}_X \)), we have \( f^! \cong Rf_! f^* \) (again pseudofunctorially), see [Nay] Theorem 7.1.6]. Note that \( f^! \) does not preserve coherence of homology in general.

There is an extension of \((-)^!\) that we find convenient to use. For any \( f : \mathcal{X} \rightarrow \mathcal{Y} \) in \( \mathbb{G} \) the composite \( f^! R\Gamma^! \) sends \( D_{qct}(\mathcal{Y}) \) to \( D_{qct}(\mathcal{X}) \) and is isomorphic to \( f^! \) when restricted to \( D_{qct}(\mathcal{Y}) \). The extended functor is also denoted as \( f^! \).

However, this extension \((-)^!\) only forms a pre-pseudofunctor (see beginning of [1]). Thus, for \( \mathcal{X} \rightarrow \mathcal{U} \rightarrow \mathcal{Y} \) in \( \mathbb{G} \), the comparison isomorphism \( C^!_{\mathcal{U}} : (gf)^! \cong f^! g^! \) is induced by the usual one over \( D_{qct}(\mathcal{U}) \) and by the isomorphisms

\[
(gf)^! \cong f^! g^! \Rightarrow f^! g^! Rf_! g^! Rf_! f^!
\]

The associativity condition for \( C^!_{\mathcal{U}} \) vis-à-vis composition of 3 maps easily results from the corresponding one over \( D_{qct}(\mathcal{U}) \). For the identity map \( 1_{\mathcal{X}} \) on \( \mathcal{X} \) we have a natural map \( 1_{\mathcal{X}} = Rf^! \rightarrow 1 \) and the comparison isomorphisms corresponding to composing \( f \) on the left or right by identity are the canonical ones \( f^! \cong f^! Rf^! f^! \) and \( f^! \rightarrow Rf^! f^! \).

This extended pre-pseudofunctorial version of \((-)^!\) is what we will use from now on. It has the following properties. For \( f : \mathcal{X} \rightarrow \mathcal{Y} \) in \( \mathbb{G} \), if \( f \) is pseudoproper,
then \( f^! : \tilde{\mathbf{D}}^+_{\mathcal{P}}(\mathcal{Y}) \to \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \) is right adjoint to \( Rf_* : \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \to \tilde{\mathbf{D}}^+_{\mathcal{P}}(\mathcal{Y}) \), while if \( f \) is formally étale, then \( f^! \) is isomorphic to \( Rf^! : \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \to \mathbf{D}^+_{\mathcal{P}}(\mathcal{Y}) \), (see Theorem 7.1.6 and \( \S 7.2 \) of \cite{Nay}).

For \( \mathbf{D}^+ \)-related questions, it is useful to work with another generalization of \((-)\) from ordinary schemes. For \( f : \mathcal{X} \to \mathcal{Y} \) in \( \mathcal{G} \) we define \( f^# : \mathbf{D}^+_{\mathcal{P}}(\mathcal{Y}) \to \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \) by

\[
(3.1.1) \quad f^# := \Lambda_{\mathcal{X}} f^! .
\]

Since \( f^! \xrightarrow{\sim} Rf^! \mathcal{Y}^* \), therefore \((-)\) and \((-)\) determine each other up to isomorphism.

Note that \((-)\) is also a pre-pseudofunctor. For maps \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z} \) in \( \mathcal{G} \), the comparison isomorphism \( C_{f, g}^- : (gf)^* \xrightarrow{\sim} f^# g^# \) is given by the composite

\[
(gf)^* = \Lambda_{\mathcal{X}} (gf)^! \xrightarrow{\sim} \Lambda_{\mathcal{X}} f^! g^! \xrightarrow{\sim} \Lambda_{\mathcal{X}} f^! \Lambda_{\mathcal{Y}} g^! = f^# g^# .
\]

where the last isomorphism is obtained from composite of the following sequence where we use \( f^! R\mathcal{Y}^* \xrightarrow{\sim} f^! \) in the first and the last step:

\[
f^! g^! \xrightarrow{\sim} f^! R\mathcal{Y}^* g^! \xrightarrow{\sim} f^! R\mathcal{Y}^* \Lambda_{\mathcal{Y}} g^! \xrightarrow{\sim} f^! \Lambda_{\mathcal{Y}} g^! .
\]

The associativity condition for \( C_{f, g}^- \) vis-à-vis composition of 3 maps easily results from the corresponding one for \((-)\). For the identity map \( 1_{\mathcal{X}} \) on \( \mathcal{X} \), there is a map \( 1 \to (1_{\mathcal{X}})^* = \Lambda_{\mathcal{X}} \) and the comparison isomorphisms corresponding to composing \( f \) on the left or right by identity are the canonical ones \( f^* \xrightarrow{\sim} \Lambda_{\mathcal{X}} f^* \) and \( f^# \xrightarrow{\sim} f^# \Lambda_{\mathcal{Y}} \).

If \( f : \mathcal{X} \to \mathcal{Y} \) is a pseudo-proper map (whence a map in \( \mathcal{G} \)), then \( f^# \) is right adjoint to \( Rf_* R\mathcal{Y}^* \) so that we have a co-adjoint unit, the so-called trace map

\[
(3.1.2) \quad \text{Tr}_f : Rf_* R\mathcal{Y}^* f^# \to 1 ,
\]

while if \( f \) is an open immersion (or more generally, if \( f \) is formally étale) then there is a natural isomorphism \( f^# \xrightarrow{\sim} \Lambda_{\mathcal{X}} f^* \). Moreover these isomorphisms are pre-pseudofunctorial over the corresponding full subcategories of \( \mathcal{G} \).

A cautionary remark. In \cite{AJL2} Prop. 6.1.4, \( f^* \) is defined as \( \Lambda_{\mathcal{X}} f^! \) where \( f^! \) is the right adjoint to the restriction of \( Rf_* \) to \( \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \). The functor \( \Lambda_{\mathcal{X}} f^! \) is shown to be a right adjoint to \( Rf_* R\mathcal{Y}^* f^# \) for every \( f \). Our definition of \( f^# \) agrees with that of \cite{AJL2} when \( f \) is pseudo-proper, but not in general.

For any \( f : \mathcal{X} \to \mathcal{Y} \) in \( \mathcal{G} \), we have \( f^!(\mathbf{D}^+_{\mathcal{P}}(\mathcal{Y})) \subset \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \). This can be seen by reducing to the special cases when \( f \) is pseudo-proper or \( f \) is an open immersion; in the latter case one uses that \( \Lambda_{\mathcal{X}} f^! \mathbf{D}_{\mathcal{P}}(\mathcal{X}) \) is isomorphic to the identity functor, so that \( f^! \xrightarrow{\sim} \Lambda_{\mathcal{X}} f^* \xrightarrow{\sim} f^* \) on \( \mathbf{D}_{\mathcal{P}}(\mathcal{X}) \), while the former case is dealt with in \cite{AJL2} p. 89, Proposition 8.3.2]. Thus \((-)\) gives a \( \mathbf{D}^+_{\mathcal{P}} \)-valued pseudofunctor on \( \mathcal{G} \). It also follows that if \( f \) is formally étale and \( \mathcal{F} \in \mathbf{D}^+_{\mathcal{P}}(\mathcal{X}) \), then we have an isomorphism

\[
(3.1.3) \quad f^! \mathcal{F} \xrightarrow{\sim} f^# \mathcal{F}
\]

which is pseudofunctorial over the category of formally étale maps. If \( \mathcal{Y} \) is a formal scheme, \( \mathcal{I} \subset \mathcal{O}_{\mathcal{Y}} \) an open coherent ideal, \( \mathcal{W} := \mathcal{Y} \) the completion of \( \mathcal{Y} \) by \( \mathcal{I} \) and \( \kappa : \mathcal{W} \to \mathcal{Y} \) the corresponding completion map, then \( \kappa \) is both pseudoproper and étale. For any \( \mathcal{F} \in \mathbf{D}^+_{\mathcal{P}}(\mathcal{Y}) \), the isomorphism of \((3.1.3)\) is the same (see Lemma \cite{A1.3} in Appendix below) as the map adjoint to the natural composite

\[
R\kappa_* R\mathcal{Y}^* \kappa^* \xrightarrow{\sim} R\mathcal{Y}^* \xrightarrow{\sim} 1 .
\]
Nevertheless, while working with \((-)^\#\), it is also convenient to work with the larger category \(\tilde{D}_{qc}^+\) in addition to \(D_{qc}^+\) since, some of the functors, such as \(Rf_*\Gamma f^*\), the left adjoint to \(f^*\) when \(f: \mathcal{X} \to \mathcal{Y}\) is pseudoproper, do not preserve coherence of homology in general.

Over ordinary schemes \((-)^\#\) and \((-)^!\) are canonically identified and so we use both interchangeably.

3.2. Suppose we have a cartesian square \(s\) of noetherian formal schemes

\[
\begin{array}{c}
\mathcal{Y} \\
v \downarrow \square \\
g \downarrow f \\
\mathcal{W} \xrightarrow{u} \mathcal{Y}
\end{array}
\]

with \(f\) in \(\mathcal{G}\) and \(u\) flat. The flat-base-change theorem for \((-)^!\) gives an isomorphism for \(F \in \tilde{D}_{qc}^+ (\mathcal{Y}):\)

\[
\beta^! s(F): R \Gamma^! v^* f^! F \xrightarrow{\sim} g^! u^* F
\]

(see [AJL2] p. 77, Theorem 7.4] for the case when \(f\) is pseudoproper and [Nay] p. 261, Theorem 7.14] for the general case and also [Nay] §7.2]). For \(F \in \tilde{D}_{qc}^+ (\mathcal{Y})\)

\[
(3.2.1) \quad \beta^* s(F): \Lambda^! v^* f^# F \xrightarrow{\sim} g^! u^* (F)
\]

is induced by the following sequence of natural isomorphisms of functors

\[
\Lambda^! v^* f^# = \Lambda^! v^* \Lambda^! f^! \xrightarrow{\sim} \Lambda^! v^* f^! \xrightarrow{\sim} \Lambda^! R \Gamma^! v^* f^! \xrightarrow{\sim} \Lambda^! g^! u^* = g^! u^*
\]

where the last isomorphism is induced by \(\beta^!\) (cf. [AJL2] p. 86, Theorem 8.1] for \(f, g\) pseudoproper).

If \(F \in D_{qc}^+ (\mathcal{Y}),\) or if \(u\) is open or if \(\mathcal{Y}\) is an ordinary scheme, we have an isomorphism

\[
(3.2.2) \quad v^* f^# F \xrightarrow{\sim} g^! u^* F,
\]

(see [AJL2] Theorem 8.1, Corollary 8.3.3]).

Further properties of the base-change map are explored in Appendix A.1.

3.3. Let \(f: X \to Y\) be a separated map of finite-type between ordinary schemes, and \(Z\) a closed subscheme of \(X\) which is proper over \(Y\). The completion map \(\kappa: \mathcal{X} \to X\) of \(X\) along \(Z\), is formally étale and affine and the composition \(\hat{f} := f \kappa\) is pseudoproper. We define the trace map for \(f\) along \(Z\)

\[
(3.3.1) \quad \text{Tr}_{f,Z}: R f_* f^# \to 1
\]

to be the composite

\[
R f_* \Gamma^! f^# \xrightarrow{\sim} R f_* \kappa_* \Gamma^! f^# \xrightarrow{\sim} R \hat{f}_* R \Gamma^! f^# \xrightarrow{\sim} R \hat{f}_* R \Gamma^! f \xrightarrow{\text{Tr}_{f,Z}} 1
\]

where the first isomorphism is from the canonical isomorphism \(\Gamma^! f^# \xrightarrow{\sim} \kappa_* \Gamma^! f^#\) while the remaining maps are the obvious natural ones.

There is an alternate description of \(\text{Tr}_{f,Z}\) involving compactifications. Let \((u, \bar{f})\) be a compactification of \(f\), i.e., \(u: X \to \overline{X}\) is an open immersion, \(\bar{f}: \overline{X} \to Y\) a proper map such that \(f = \bar{f} \circ u\). A theorem of Nagata assures us that compactifications
always exist (see [N], [D3], [Lu], and [C2]). According to Lemma [A.3.5], \( \text{Tr}_{f,Z} \) can also be described as the composite

\[
R_{f_*} R_{f^!} f^* \xrightarrow{\sim} R_{f_*} R_{f^!} (f^! f^* f^! f^!) \rightarrow R_{f_*} f^! f^! f^! f^! 1.
\]

4. Cohen-Macaulay Maps

4.1. Recall that a locally finite type map \( f: X \to Y \) between ordinary schemes is said to be Cohen-Macaulay of relative dimension \( r \) if it is a flat map and all the non-empty fibres are Cohen-Macaulay and of pure dimension \( r \). This is equivalent to saying that \( f \) is flat, \( f^! \mathcal{O}_Y \) (which is defined locally if \( f \) is not separated) has homology concentrated in only degree \(-r\), and the resulting \( \mathcal{O}_X \)-module \( \omega^r_\mathcal{O}_X \) obtained by gluing the various local \( H^{-r}(f^! \mathcal{O}_Y) \) is coherent and flat over \( Y \). We make the obvious generalization to formal schemes. First we need the following definition.

**Definition 4.1.1.** A map of formal schemes \( f: \mathcal{X} \to \mathcal{Y} \) is said to be locally in \( \mathcal{G} \) if for every point \( x \in \mathcal{X} \), there exists an open neighbourhood \( U \) of \( x \) such that the restriction \( f_U \) of \( f \) to \( U \) is in \( \mathcal{G} \).

Note that if \( f: \mathcal{X} \to \mathcal{Y} \) is locally in \( \mathcal{G} \) and \( \mathcal{F} \in \mathcal{D}_X^+(\mathcal{Y}) \), then locally on \( \mathcal{X} \), \( f^* \mathcal{F} \) is defined, but it need not be defined globally, even though by pseudo-functoriality of \((-)^! \) over \( \mathcal{D}_X^+ \), these local twisted inverse images are isomorphic on overlaps and these isomorphisms form a descent datum for the Zariski topology. However, in this case, for every integer \( n \), and every \( \mathcal{G} \in \mathcal{D}_X^+(\mathcal{Y}) \) the sheaves \( H^i(f^! (\mathcal{F})) \) do glue to give a coherent sheaf on \( \mathcal{X} \) which we denote as \( H^i(f^* \mathcal{G}) \) (even though there might well be no \( f^* \mathcal{G} \)). We are not aware of any example where \( f^* \mathcal{F} \) is defined locally but not globally.

**Definition 4.1.2.** A map of formal schemes \( f: \mathcal{X} \to \mathcal{Y} \) is called Cohen-Macaulay (CM) of relative dimension \( r \) if it is flat, locally in \( \mathcal{G} \) with \( H^i(f^* \mathcal{O}_Y) = 0 \) for \( i \neq -r \) and \( \omega^r_\mathcal{O}_Y := H^{-r}(f^* \mathcal{O}_Y) \) is flat over \( \mathcal{Y} \). The coherent \( \mathcal{O}_X \)-module \( \omega^r_\mathcal{O}_X \) is called the relative dualizing sheaf for the CM map \( f \). If such a map \( f \) is already in \( \mathcal{G} \), we shall make the identification \( f^* \mathcal{O}_Y = \omega^r_\mathcal{O}_X \).

It is tempting to give alternate definitions for a map to be CM that are more local in nature. An extension of the theory of \((-)^! \) and \((-)^* \) to a larger category containing maps that are essentially of pseudo-finite type (see [NLS] §2.1) would provide a natural setting for proving equivalence between various possible alternate definitions. Since there is no such extension in literature and since we do not need such a result here, we do not pursue this matter further.

A map \( f: \mathcal{X} \to \mathcal{Y} \) is said to be smooth if it is in \( \mathcal{G} \) and is formally smooth. In this case the universally finite module of relative differential forms \( \Omega^1_{\mathcal{X}/\mathcal{Y}} \) is a locally free module of finite rank and if it has constant rank \( r \) we say that \( f \) has relative dimension \( r \), see [NLS] §2.6. We use \( \omega_f \) to denote the top exterior power of the universally finite module of differentials.

5. Traces and Residues for Cohen-Macaulay maps

5.1. **Abstract Trace for Cohen-Macaulay maps.** Suppose \( g: \mathcal{X} \to \mathcal{Y} \) is Cohen-Macaulay of relative dimension \( r \) and is pseudo-proper. Since \( \omega^r_\mathcal{O}_Y = g^* \mathcal{O}_Y \), therefore, as in (3.1.2), we have a trace map

\[
\text{Tr}_g(\mathcal{O}_Y): R_g R_{g^!} \omega^r_\mathcal{O}_Y \to \mathcal{O}_Y.
\]
Definition 5.1.1. Let \( g: \mathcal{X} \to \mathcal{Y} \) be as above (i.e., \( g \) is pseudo-proper and Cohen-Macaulay of relative dimension \( r \)). The abstract trace map on \( R^r_{\mathcal{X}} g_* \omega^\#_g \) (or simply the trace map on \( R^r_{\mathcal{X}} g_* \omega^\#_g \)) is the map
\[
(5.1.2) \quad \text{tr}^\#: R^r_{\mathcal{X}} g_* \omega^\#_g \to \mathcal{O}_\mathcal{Y}.
\]
given by \( \text{tr}^\#: R^r_{\mathcal{X}} g_* \omega^\#_g \to \mathcal{O}_\mathcal{Y} \).

Theorem 5.1.3. Let \( g: \mathcal{X} \to \mathcal{Y} \) be pseudo-proper and Cohen-Macaulay of relative dimension \( r \). Then for any quasi-coherent \( \mathcal{O}_\mathcal{X} \)-module \( \mathcal{F} \) satisfying \( R^j_{\mathcal{X}} g_* \mathcal{F} = 0 \) for \( j > r \), we have a functorial isomorphism
\[
\text{Hom}_\mathcal{X}(\mathcal{F}, \omega^\#_g) \xrightarrow{\sim} \text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* \mathcal{F}, \mathcal{O}_\mathcal{Y}),
\]
which is given by sending \( \theta \in \text{Hom}_\mathcal{X}(\mathcal{F}, \omega^\#_g) \) to the composite
\[
R^r_{\mathcal{X}} g_* \mathcal{F} \xrightarrow{R^r_{\mathcal{X}} g_*(\theta)} R^r_{\mathcal{X}} g_* \omega^\#_g \xrightarrow{\text{tr}^\#} \mathcal{O}_\mathcal{Y}.
\]

Proof. By adjointness we have a natural isomorphism
\[
R\text{Hom}_\mathcal{X}(\mathcal{F}, \omega^\#_g[r]) \xrightarrow{\sim} R\text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* \mathcal{F}, \mathcal{O}_\mathcal{Y}),
\]
hence there are natural isomorphisms
\[
\text{Hom}_\mathcal{X}(\mathcal{F}, \omega^\#_g) = \text{Hom}_{\text{D}(\mathcal{X})}(\mathcal{F}, \omega^\#_g)
= H^{-r} R\text{Hom}_\mathcal{X}(\mathcal{F}, \omega^\#_g[r])
\cong H^{-r} R\text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* \mathcal{F}, \mathcal{O}_\mathcal{Y})
= \text{Hom}_{\text{D}(\mathcal{Y})}(R^r_{\mathcal{X}} g_* \mathcal{F}[r], \mathcal{O}_\mathcal{Y})
\cong \text{Hom}_{\text{D}(\mathcal{Y})}(R^r_{\mathcal{X}} g_* \mathcal{F}, \mathcal{O}_\mathcal{Y}) \quad \text{(see [L4] p. 37, Prop. 1.10.1)}
= \text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* \mathcal{F}, \mathcal{O}_\mathcal{Y}).
\]

In the above proof, we don’t know if \( \omega^\#_g \) satisfies the hypotheses required of \( \mathcal{F} \). We are interested in special cases when this is true. This leads to the following.

Corollary 5.1.4. If \( R^j_{\mathcal{X}} g_* (\mathcal{F}) = 0 \) for every \( j > r \) and every \( \mathcal{F} \in \mathcal{A}_c(\mathcal{X}) \) (resp. \( \mathcal{F} \in \mathcal{A}_c(\mathcal{X}) \)), then \( (\omega^\#_g, \text{tr}^\#_g) \) represents the functor \( \text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* (-), \mathcal{O}_\mathcal{Y}) \) on \( \mathcal{A}_c(\mathcal{X}) \) (resp. \( \mathcal{A}_c(\mathcal{X}) \), resp. \( \mathcal{A}_c(\mathcal{X}) \)). In particular if \( \mathcal{Y} = Y \) is an ordinary scheme, \( f: X \to Y \) a Cohen-Macaulay map of ordinary schemes of relative dimension \( r \), \( Z \) a closed subscheme of \( X \) such that the resulting map \( Z \to Y \) is finite and flat, \( \mathcal{X} = X_{/Z} \) the completion of \( X \) along \( Z \), and \( g: \mathcal{X} \to \mathcal{Y} \) the map induced by \( f \), then \( (\omega^\#_g, \text{tr}^\#_g) \) represents the functor \( \text{Hom}_\mathcal{Y}(R^r_{\mathcal{X}} g_* (-), \mathcal{O}_\mathcal{Y}) \) on \( \mathcal{A}_c(\mathcal{X}) \).

Proof. The first assertion holds since, if \( g \) is Cohen-Macaulay, then \( \omega^\#_g \in \mathcal{A}_c(\mathcal{X}) \). The second assertion follows from the first since, if \( \mathcal{F} \) is a coherent ideal defining \( Z \) in \( X \), \( \mathcal{F} \in \mathcal{A}_c(\mathcal{X}) \) and \( \kappa \) denotes the canonical map \( \mathcal{X} \to X \), then \( \mathcal{F} \xrightarrow{\sim} \kappa^* \mathcal{F} \) for some \( \mathcal{F} \in \mathcal{A}_c(X) \) (see [AJL2] p. 31, Prop. 3.1.1]), and hence \( Rg_* R\Gamma^r_{\mathcal{X}} \mathcal{F} \xrightarrow{\sim} Rf_* \kappa_* R\Gamma^r_{\mathcal{X}} \kappa^* \mathcal{F} \xrightarrow{\sim} Rf_* R\Gamma^r_{\mathcal{Y}} \mathcal{F} \) (see [AJL2] §5). \( \square \)
Remark 5.1.5. In a slightly different direction, Lipman observed the following (private communication). First note that according to [AJL2] p. 39, Prop. 3.4.3], since all our schemes are noetherian, if \( f : \mathcal{X} \to \mathcal{Y} \) is a map of schemes (possibly formal), the functor \( Rf_* \) is bounded above on \( \mathsf{D}_C(\mathcal{X}) \). In other words, there is an integer \( e \geq 0 \) such that if \( \mathcal{H} \in \mathsf{D}_C(\mathcal{X}) \) and \( H^i(\mathcal{H}) = 0 \) for \( i > i_0 \), then \( H^i(Rf_*, \mathcal{H}) = 0 \) for all \( i \geq i_0 + e \). Next, by computing local cohomologies using stable Koszul complexes (see (C.3.2)) on affine open subschemes of \( \mathcal{X} \) and using quasi-compactness of the noetherian scheme \( \mathcal{X} \), we see that there is an integer \( t \) such that \( H^j(\mathcal{R}_f^j, \mathcal{F}) = 0 \) for \( \mathcal{F} \in \mathcal{A}_c(\mathcal{X}) \) and \( j > t \). It is then not hard to see that if \( r = e + t \), and if \( \mathcal{H} \in \mathsf{D}_C(\mathcal{X}) \) is such that \( H^i(\mathcal{H}) = 0 \) for \( i > i_0 \), then \( H^j(\mathcal{R}_f^j, \mathcal{R}_f^j(\mathcal{H})) = 0 \) for \( j > i_0 + r \). Now suppose \( f \) is pseudo-proper. By the argument given in [L4] p. 165, Lemma 4.1.8], we see that if \( \mathcal{G} \in \mathcal{A}_c(\mathcal{X}) \) is such that \( f^*\mathcal{G} \in \mathsf{D}_C(\mathcal{X}) \) then \( H^j f^*\mathcal{G} = 0 \) for every any \( j < -r \). Let \( \omega^*_f = H^{-r}(f^*\mathcal{O}_\mathcal{Y}) \). Then as we argued earlier, \( \omega^*_f \in \mathcal{A}_c(\mathcal{X}) \subset \mathsf{D}_C(\mathcal{X}) \). The proof of Theorem 5.1.3 applies and we have a functorial isomorphism (without any Cohen-Macaulay hypotheses)
\[
\text{Hom}_{\mathcal{X}}(\mathcal{F}, \omega^*_f) \xrightarrow{\sim} \text{Hom}_{\mathcal{Y}}(\mathcal{R}_f^r \mathcal{F}, \mathcal{O}_\mathcal{Y})
\]
for every \( \mathcal{F} \in \mathcal{A}_c(\mathcal{X}) \). We point out that \( \mathcal{R}_f^j|_{\mathcal{A}_c(\mathcal{X})} = 0 \) for \( j > r \). In particular, in this argument, \( \mathcal{R}_f^j \omega^*_f = 0 \) for \( j > r \). We could not guarantee this for the \( r \) used in the Theorem.

5.2. Abstract Residue for Cohen-Macaulay maps. Throughout this subsection
\[
f: X \to Y
\]
is a finite-type Cohen-Macaulay map between ordinary schemes of relative dimension \( r \). Suppose \( Z \to X \) is a closed subscheme of \( X \), proper over \( Y \). Let \( \mathcal{X} = X/Z \) be the formal completion of \( X \) along \( Z \), and \( \kappa: \mathcal{X} \to X \) the completion map. Let \( f : \mathcal{X} \to Y \) be the composite \( f \circ \kappa \). We have \( f^* \sim \kappa^* f^* \sim \kappa^* \mathcal{F} \), whence \( H^i(f^*) \sim H^i(\kappa^*) \). Note that \( \hat{f} \) is pseudo-proper and Cohen-Macaulay of relative dimension \( r \) and therefore \( \mathfrak{tr}_f^* \) is defined. In [S2] p. 742, (3.2)] a residue map along \( Z \) is defined using local compactifications. Here is a reformulation of that definition in terms of \( \kappa \).

Definition 5.2.1. Let \( Z \) and \( f \) be as above. The abstract residue along \( Z \)
\[
(5.2.2) \quad \text{res}^Z_\cdot f \omega^*_f : \mathcal{R}_Z^r f_* \omega^*_f \to \mathcal{O}_Y
\]
is the composite
\[
\mathcal{R}_Z^r f_* \omega^*_f \xrightarrow{\mathfrak{A}_f \cup \mathfrak{M}} \mathcal{R}_Z^r \hat{f}_* \omega^*_f \xrightarrow{\mathfrak{tr}_f^*} \mathcal{O}_Y.
\]

It is worth unravelling the first isomorphism in the above composite a little more. The isomorphism \( \kappa_* \mathcal{R}_Y^r \kappa^* \sim \mathcal{R}_Z^r \) gives rise to isomorphisms (one for every \( j \))
\[
(5.2.3) \quad \mathcal{R}_Z^j f_* \mathcal{F} \sim H^j(\mathcal{R}_Y f_* \mathcal{R}_Y^r \kappa^* \mathcal{F}) = \mathcal{R}_Z^j f_* \kappa^* \mathcal{F}
\]
which are functorial in \( \mathcal{F} \) varying over quasi-coherent \( \mathcal{O}_X \)-modules. In affine terms, if \( X = \text{Spec} \, R \), \( M \) an \( R \)-module, and \( Z \) is given by the ideal \( I \), then writing \( \hat{R} \) for the \( I \)-adic completion of \( R \), and \( J = I \hat{R} \), the above isomorphism is the well-known one
\[
H^j_I(M) \sim H^j_J(M \otimes_R \hat{R}).
\]
The isomorphism $R^r_f \omega_f^g \cong R^{\prime r}_f \hat{\omega}_f^g$ induced by (A.3.1) is the composite of the map (5.2.3), i.e., $R^r_f \omega_f^g \cong R^{\prime r}_f \hat{\omega}_f^g$, and the isomorphism induced by (3.1.3), i.e., $R^{\prime r}_f \hat{\omega}_f^g \cong R^{\prime r}_f \hat{\omega}_f^g$.

In [S2] p.742, (3.2) a different, but equivalent, definition is given of $\text{res}_z^g$. In that situation $f$ is separated, and therefore has a compactification by a result of Nagata, say $u: X \rightarrow \overline{X}$ of $X$ over $Y$. Let $f: \overline{X} \rightarrow Y$ be the structure morphism (by definition of a compactification, a proper map) of $X$. In loc. cit., the residue along $Z$ is defined as the composite

$$R^r_Z f_* \omega_f^g = H^0(R_f \mathcal{R}_Z \omega_f^g[r]) = H^0(R_f \mathcal{R}_Z f^1 \mathcal{O}_Y) \xrightarrow{\sim} H^0(R_{\overline{f}} \mathcal{R}_Z f^1 \mathcal{O}_Y)$$

By Lemma [A.3.3] the two definitions coincide in the situation considered in [S2] and therefore the definition in [S2] p.742, (3.2) is independent of the compactification $(u, f)$. This gives another proof of [S2] p.742, Proposition 3.1.1].

If $f$ is proper, it follows that there is a commutative diagram:

$$\begin{array}{ccc}
R^r_Z f_* \omega_f^g & \xrightarrow{\text{res}_z^g} & R^{\prime r}_f \hat{\omega}_f^g \\
\downarrow \text{tr}_Y & & \downarrow \text{tr}_Y \\
\mathcal{O}_Y & & \mathcal{O}_Y
\end{array}$$

(5.2.4)

**Remark 5.2.5.** In [ILN] p.746, Remark 2.3.4], Iyengar, Lipman, and Neeman give a generalization of the residue map in [S2]. Suppose $f: X \rightarrow Y$ is a separated essentially finite type map of ordinary schemes, $W$ a union of closed subsets of $X$ to each of which the restriction of $f$ is proper. (Note that $W$ need not be closed in $X$.) Then one has an integer $d$ such that $H^{-e}(f^1 \mathcal{O}_Y) = 0$ for all $e > d$, while $\omega_f := H^{-d}(f^1 \mathcal{O}_Y) \neq 0$. Iyengar, Lipman, and Neeman then define a natural map

$$H^d(R_f \mathcal{R}_W \omega_f) \rightarrow \mathcal{O}_Y$$

denote by them as $\int_W$, which generalizes the map denoted $\text{res}_W$ in [S2] §3.1. In greater detail, if $D_{\text{qc}}(X)_W$ denotes the essential image of $R_f \mathcal{R}_W$ in $D_{\text{qc}}(X)$, then in [ILN] p.746, Corollary 2.3.3 it is shown that for $E$ in $D_{\text{qc}}(X)_W$ and $G$ in $D_{\text{qc}}^+(Y)$, we have a functorial isomorphism

$$\text{Hom}_{D(Y)}(R_f E, G) \cong \text{Hom}_{D(X)}(E, R_f \mathcal{R}_W f^1 G).$$

In particular one has a counit

$$\text{R}_f \mathcal{R}_W f^1 \mathcal{O}_Y \rightarrow \mathcal{O}_Y.$$ 

(5.2.7)

The map (5.2.6) is defined as the composite

$$H^d(R_f \mathcal{R}_W \omega_f) = H^d(R_f \mathcal{R}_W f^1 \omega_f[d])$$

$$\rightarrow H^0(R_f \mathcal{R}_W f^1 \mathcal{O}_Y) \xrightarrow{\text{tr}_Y} \mathcal{O}_Y.$$
5.3. Traces for finite Cohen-Macaulay maps. We begin with a global construction. Suppose we have a commutative diagram of ordinary schemes

\[ Z \xrightarrow{i} X \xrightarrow{f} Y \]

with \( f \) Cohen-Macaulay of relative dimension \( r \), \( h \) a finite surjective map, \( i \) a closed immersion, the \( \mathcal{O}_X \)-ideal \( \mathcal{I} \) of \( Z \) generated by \( t_1, \ldots, t_r \in \Gamma(X, \mathcal{O}_X) \) such that \( t = (t_1, \ldots, t_r) \) is \( \mathcal{O}_X \)-regular for every \( z \in Z \subset X \). Note that \( h \) is necessarily flat and is Cohen-Macaulay of relative dimension 0. Define

\[ (5.3.2) \quad \tau^\#_h(= \tau^\#_{h,f,i}) : h^*(i^* \omega_Z^\# \otimes \mathcal{O}_Z (\wedge^r \mathcal{O}_Z \mathcal{I}/\mathcal{I}^2)) \longrightarrow \mathcal{O}_Y \]

as the unique map which fills the dotted arrow to make the diagram below commute where \( \eta^\prime_i \) is induced by (C.2.13).

We would like to show that \( \tau^\#_h \) factors through \( \text{res}^\#_Z : \text{R}^r f_* \omega_f^\# \rightarrow \mathcal{O}_Y \). To that end, we make the following definition. First, as in (C.2.10), let \( i^\#: = \mathbf{L}i^*(-) \otimes \mathcal{O}_Z (\mathcal{N}^r_i[-r]) \). Next, for a quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), let

\[ (5.3.3) \quad \psi = \psi(\mathcal{F}) : h_*(i^* \mathcal{F} \otimes \mathcal{O}_Z (\wedge^r \mathcal{O}_Z \mathcal{I}/\mathcal{I}^2))^* \longrightarrow \text{R}^r f_* \mathcal{F} \]

be defined by applying \( H^0 \) to the composite

\[ (5.3.4) \quad h_{i^\#} \mathcal{F}[r] \xrightarrow{\sim} \text{R} f_* i_* i^\# \mathcal{F}[r] \xrightarrow{\sim} \text{R} f_* i_* i^\# \mathcal{F}[r] \longrightarrow \text{R} f_* \text{R} \Gamma_{\mathcal{F}} \mathcal{F}[r]. \]

The map (5.3.3) is a sheafed version of (C.5.4.1), as (C.5.4.2) shows. Moreover it is functorial in \( \mathcal{F} \).

The formal-scheme version is as follows. Let \( \kappa : \mathcal{X} \rightarrow X \) be the completion of \( X \) along \( Z \), \( j : Z \rightarrow \mathcal{X} \) the natural closed immersion (so that \( \kappa \circ j = i \)) and \( \hat{f} = f \circ \kappa \). As before, let \( j^\# \) be as in (C.2.10). For \( \mathcal{G} \in \mathcal{A}_c(\mathcal{X}) \) we have a map

\[ (5.3.5) \quad \hat{\psi} : h_*(j^* \mathcal{G} \otimes \mathcal{O}_Z \mathcal{N}_j^r) \longrightarrow \text{R}^r f_* \hat{\mathcal{F}} \mathcal{G} \]

defined by applying \( H^0 \) to the composite

\[ (5.3.6) \quad h_{j^\#} \mathcal{G}[r] \xrightarrow{\sim} \text{R} f_* j_* j^\# \mathcal{G}[r] \xrightarrow{\sim} \text{R} f_* j_* j^\# \mathcal{G}[r] \xrightarrow{\text{Tr}_f} \text{R} f_* \text{R} \Gamma_{\mathcal{G}} \mathcal{G}[r]. \]
Since the composite $i_*i^! \to R\Gamma_Z \to 1$ is “evaluation at one”, i.e., it is the trace map (if one identifies $i^!$ with $i^!$), it is easy to see that the diagram

$$h_*i^*\mathcal{F}[r] \xrightarrow{\tau} Rf_*R\Gamma_Z\mathcal{F}[r]$$

(5.3.7) \hspace{1cm} h_*j^*\kappa^*\mathcal{F}[r] \xrightarrow{\tau} Rf_*\kappa_*R\Gamma_Z\mathcal{F}[r]

commutes where the upward arrow on the right is induced by the isomorphism $\kappa_*R\Gamma_Z\mathcal{F}[r] \xrightarrow{\sim} R\Gamma_Z$.

**Theorem 5.3.8.** In the situation of (5.3.1), the following diagram commutes.

$$h_*(i^\# \otimes \theta_Z (\wedge_{\mathcal{F}/\mathcal{F}^2})^*) \xrightarrow{\tau^*} \mathcal{O}_Y$$

$$\psi(\omega_f^\#) \downarrow \downarrow$$

$$R\hat{f}_*\omega_f^\# \xrightarrow{\text{res}^*} \mathcal{O}_Y$$

Proof. The diagram in the statement of the theorem can be realized as the transpose of the border of the following one.

$$h_*(i^\# \otimes \mathcal{N}_f^*) \xrightarrow{\psi} R\hat{f}_*\omega_f^\#$$

$$h_*(j^\# \otimes \mathcal{N}_f^*) \xrightarrow{\psi} R\hat{f}_*\omega_f^\#$$

We have to show the above diagram commutes. Applying $H^0$ to (5.3.7), with $\mathcal{F} = \omega_f^\#$, and using the isomorphism $\kappa^*\omega_f^\# \xrightarrow{\sim} \omega_f^\#$, we see that the upper trapezium commutes. The triangle on the right commutes by definition of $\text{res}^*$ (see Definition 5.2.1), while the one on the left corresponds to the natural isomorphisms $i^\#f^\# \xrightarrow{\sim} j^\#f^\# \xrightarrow{\sim} h^\#$ (after applying $H^0$ and $h_*$). Finally, the lower trapezium corresponds to $H^0$ of the outer border of the following diagram.

$$h_*j^\#f^\#\mathcal{O}_Y \xrightarrow{\sim} \hat{f}_*j^\#f^\#\mathcal{O}_Y$$

The upper rectangle commutes trivially while the lower one results from the identification of the adjoint $h^\#$ with the composition of the adjoints $j^\#f^\#$. \qed
Remark 5.3.9. The map $\psi$ in (5.3.3) is compatible with open immersions in $X$ containing $Z$. In greater detail, suppose $i$ factors as $$Z \xrightarrow{u} U \xrightarrow{x} X$$ with $x: U \to X$ an open immersion, and $u$ (necessarily) a closed immersion. Then $$h^*_u x^* \mathcal{F} \otimes \mathcal{N}^r_u \xrightarrow{(5.3.3)} R_Z^r f_* \mathcal{F}$$ commutes. We leave the verification to the reader, but point out that one method is to move to formal schemes, using (5.3.7), noting that the completion of $X$ along $Z$ is the same as the completion of $U$ along $Z$. This means $\tau^h_u$ is unaffected if $X$ is replaced by $U$.

5.4. A residue formula for Cohen-Macaulay maps. Consider again Diagram (5.3.1). Suppose now that $X$, $Y$, and $Z$ are affine, say $X = \text{Spec} \ R$, $Y = \text{Spec} \ A$, and $Z = \text{Spec} \ B$. In other words $A \to R$ is a finite-type map of rings which is Cohen-Macaulay of relative dimension $r$, we have an ideal $I$ in $R$ generated by a quasi-regular sequence $t = (t_1, \ldots, t_r)$ in $R$, and $B = R/I$. Assume as before that $h$ is a finite (and hence flat) surjective map.

Let us write $\omega^R_{R/A} = \Gamma(X, \omega^f_R)$, $\omega^B_{B/A} = \Gamma(Z, \omega^h_R)$, $\text{tr}^\#_{B/A} = \Gamma(Y, \text{tr}^h_R)$. The global sections of $\tau^h_u$ give us an $A$-linear map

$$\tau^h_{B/A} = (\tau^h_{B/A,R}): \omega^R_{R/A} \otimes_R (\mathcal{N}^r_B/I^2)^* \rightarrow A$$

such that the following diagram commutes

$$\begin{array}{ccc}
\omega^R_{R/A} \otimes_R (\mathcal{N}^r_B/I^2)^* & \xrightarrow{\sim} & \omega^B_{B/A} \\
\tau^h_{B/A} \downarrow & & \text{tr}^h_{B/A} \\
A & = & A
\end{array}$$

where the horizontal isomorphism on the top row is the global sections of the composite $h_*(i^* \omega^f_I \otimes_{\mathcal{O}_X} (\mathcal{N}^r_{\mathcal{F}/\mathcal{F}^2})^*) \xrightarrow{\eta'_f} h_* \mathcal{H}^0(i^! f^! \mathcal{O}_Y) = \mathcal{H}^0 h_*(i^! f^! \mathcal{O}_Y) \xrightarrow{\eta'_f} \mathcal{H}^0(h_* h^! \mathcal{O}_Y)$.

5.4.2. Notation. In an obvious extension of our notational philosophy, we should use the symbol $\text{res}^h_I$ for the global sections of the residue map $\text{res}^h_I$ in (5.3.2). However, for psychological reasons we will continue to use the symbol $\text{res}^h_I$ to denote this map. Thus we have $\text{res}^h_I: H^r(I^*)_{(\omega^R_{R/A})} \rightarrow A$.

In what follows, elements of $H^r(I^*)_{(\omega^R_{R/A})}$ are denoted by generalized fractions

$$\left[ \frac{\nu}{t_1, \ldots, t_r} \right]$$

as in Subsection C.5 (see especially (C.5.2) and (C.5.3) and the discussions around them).
Finally, define
\[\frac{1}{t} \in (\wedge_B I/I^2)^*\]
as the element which sends \((t_1 + I^2) \wedge \cdots \wedge (t_r + I^2) \in \wedge_B I/I^2\) to 1.

**Proposition 5.4.4.** With the above notations, for any \(\nu \in \omega^\#_{R/A}\) we have
\[\text{res}_{Z}^\# \left[ \nu \frac{1}{t_1}, \ldots, \frac{1}{t_r} \right] = \tau_{B/A}^\# \left( \nu \otimes \frac{1}{t} \right)\]
where \(\frac{1}{t}\) is as in (5.4.3).

**Proof.** According to Theorem 5.3.8, the following diagram commutes.
\[
\begin{array}{ccc}
\omega^\#_{R/A} \otimes_R (\wedge_B I/I^2)^* & \xrightarrow{(5.4.3)} & H_f^r(\omega^\#_{R/A}) \\
\tau^\#_{B/A} & \downarrow & \\
\ast & \downarrow & \\
\text{res}_{Z}^\# & \end{array}
\]
The Proposition then follows from Lemma C.5.4.4.

**Theorem 5.4.5.** Suppose \(J\) is another ideal in \(R\) such that \(I \subset J\) and \(J\) is generated by a quasi-regular sequence \(g = (g_1, \ldots, g_r)\). Let \(t_i = \sum_j u_{ij} g_j, u_{ij} \in R\). Let \(W = \text{Spec } R/J\). Then, for any \(\nu \in \omega^\#_{R/A}\)
\[\text{res}_{Z}^\# \left[ \det(u_{ij}) \nu \frac{1}{t_1}, \ldots, \frac{1}{t_r} \right] = \text{res}_{W}^\# \left[ \nu g_1, \ldots, g_r \right]\]

**Proof.** This is an immediate consequence of Theorem C.7.2 and Proposition 5.4.4.

6. **Base change for residues**

**6.1. Hypotheses.** Throughout this section (i.e., §6), we fix a commutative diagram of ordinary schemes
\[(6.1.1)\]
with \(f\) separated Cohen-Macaulay of relative dimension \(r\), the rectangles cartesian, \(i: Z \to X\) a closed immersion such that \(h = f \circ i: Z \to Y\) is finite and the quasi-coherent ideal sheaf \(\mathcal{I}\) of \(Z\) is generated by global sections \(t_1, \ldots, t_r \in \Gamma(X, \mathcal{O}_X)\) with the property that \(t = (t_1, \ldots, t_r)\) is \(\mathcal{O}_{X,z}\)-regular for every \(z \in Z \subset X\).
(Note that \(Z \to Y\) is flat by [EGA 0IV, 15.1.16].) We also use the following additional notations: \(\mathcal{J} = v^* \mathcal{I}\) is the ideal sheaf of \(Z'\), \(\mathcal{N} = (\wedge_{v^2}^r \mathcal{I} / \mathcal{I}^2)^*\), and \(\mathcal{N}' = (\wedge_{v^2}^r \mathcal{I} / \mathcal{J}^2)^* = w^* \mathcal{N}\).
6.2. Base change for direct image with supports. Since $f$ is Cohen-Macaulay of relative dimension $r$, therefore, according to \cite[Theorem 2.3.5 (a)]{S2}, we have a base-change isomorphism

$$\theta^f_u: v^*\omega^g \cong \omega^g.$$\label{6.2.1}

The principal aim of this section is to show that the composite

$$u^*h_*((i^*\omega^g \otimes_{\mathcal{O}_Z} \mathcal{N}) \cong h'_*((j^*v^*\omega^g \otimes_{\mathcal{O}_{Z'}} \mathcal{N'})) \cong h'_*((j^*\omega^g \otimes_{\mathcal{O}_{Z'}} \mathcal{N'})) \xrightarrow{\theta^f_{u,k}} \mathcal{O}_{Y'}$$

is $u^*\tau^g_k$, i.e., speaking informally, $\tau^g_k$ is stable under base change (here the first isomorphism results from the fact that $h$ is an affine map). We would also like to show that the result in Theorem \ref{5.3.8} is stable under base change. Indeed, that is how we will prove that $\tau^g_k$ is stable under base change. To set things up, we now discuss, very briefly, base change for cohomology with supports, at least for the situation we are in.

In our situation, we have base-change maps (see, for example, \cite[p. 768, (A.5)]{S2}), one for each $k$.

$$b(u, f) = b(u, f, k): u^*R^k_Zf_* \longrightarrow R^k_Zg_*v^*.$$\label{6.2.2}

These are natural transformation of functors on quasi-coherent sheaves on $X$. In the event $u$ is flat, $b(u, f)$ is an isomorphism. In fact, in this case, $b(u, f, k)$ is $H^k(-)$ applied to the composite of natural isomorphisms

$$u^*Rf_\circ R\Gamma_Z \cong Rg_*v^* \circ R\Gamma_Z \cong Rg_*R\Gamma_{Z'},v^*.$$\label{6.2.3}

It is useful for us to recast $b(u, f)$ in terms of the formal completions of $X$ and $X'$. To that end, let $\kappa: \mathcal{X} \to X$ (resp. $\kappa': \mathcal{X}' \to X'$) be the completion of $X$ along $Z$ (resp. of $X'$ along $Z'$) and let $\alpha: Z \to \mathcal{X}$, $\beta: Z' \to \mathcal{X}'$ be the natural closed immersions, so that $i = \kappa \circ \alpha$ and $j = \kappa' \circ \beta$. Let $\bar{f} = f \circ \kappa$, $\bar{g} = g \circ \kappa'$, and finally let $\bar{v}: \mathcal{X}' \to \mathcal{X}$ be the natural map induced by $v$, so that the following diagram is cartesian:

$$\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\bar{v}} & \mathcal{X} \\
\downarrow \bar{g} \downarrow \square \downarrow \bar{f} \\
Y' & \underset{u}{\longrightarrow} & Y
\end{array}$$\label{6.2.4}

The maps $b(u, f, k)$ in \eqref{6.2.1} give rise, in a natural way, maps

$$b(u, \bar{f}) = b(u, \bar{f}, k): u^*R^k_{\bar{f}}f_* \longrightarrow R^k_{\bar{f}}g_*\bar{v}^*$$\label{6.2.5}

induced by \eqref{6.2.4} applied to $\kappa$ and to $\kappa'$. In the event $u$ is flat, then as in the case of ordinary schemes, $b(u, \bar{f}, k)$ is an isomorphism and is, in fact, $H^k(-)$ applied to the natural composite

$$u^*R\bar{f}_\circ R\Gamma_{\bar{f}} \cong R\bar{g}\bar{v}^* \circ R\Gamma_{\bar{f}} \cong R\bar{g}\circ R\Gamma_{\bar{f}},\bar{v}^*.$$\label{6.2.6}

We will in fact show that when $k = r$, the map $b(u, f, k)$ is an isomorphism even when $u$ is not flat.
Proposition 6.2.4. Suppose \( u \) is a flat map and \( \mathcal{F} \in \mathcal{D}_{qc}(X) \). Then the following diagram commutes, where the unlabelled arrows arise from the natural maps (“evaluation at 1”) \( i_*i^\# \rightarrow R\Gamma_Z \) and \( j_*j^\# \rightarrow R\Gamma_Z^\cdot \):

\[
\begin{array}{c}
\xymatrix{ 
& & & j_*j^\# v^* \mathcal{F} \ar[rr]^\sim \ar[dd] & & j_*j^\# v^* \mathcal{F} \ar[rr]_{R\Gamma_Z v^* \mathcal{F}} \ar[dd] & & R\Gamma_Z v^* \mathcal{F} \\
& & & v^* i_* i^\# \mathcal{F} \ar[rr]^\sim & & v^* i_* i^\# \mathcal{F} \ar[rr]_{v^* (i_* i^\# \mathcal{F})} & & v^* R\Gamma_Z \mathcal{F}
}
\end{array}
\]

Remark: The maps \((C.2.11)\) make sense for \( \mathcal{F} \in \mathcal{D}_{qc}(X) \) because \( X \) is an ordinary scheme (see discussion in Subsection C.2.15). Flat base change works in this case for all \( \mathcal{F} \in \mathcal{D}_{qc}(X) \) without boundedness hypotheses because \( i_* \) takes perfect complexes to perfect complexes (see [4] p. 197, Thm. 4.7.4).

Proof. Let \( Tr_1^\#: i_*i^\# \rightarrow 1 \) be as in \((B.1.1)\), i.e., \( Tr_1^\# \) is the composite \( i_*i^\# \rightarrow R\Gamma_Z \rightarrow 1 \) (see the discussion in Subsection B.1 with \( Z = Z \) and \( X' = X \)). Then \( Tr_1^\# \) is simply evaluation at 1, and hence equals the composite

\[
i_*i^\# \sim \rightarrow i_*i^\#_1 Tr_1 \rightarrow 1.
\]

The two maps, \( i_*i^\# \rightarrow R\Gamma_Z \) and \( Tr_1^\# \), determine each other and hence we have show that the diagram

\[
\begin{array}{c}
\xymatrix{ 
& & & j_*j^\# v^* \ar[rr]^\sim \ar[dd] & & j_*j^\# v^* \ar[rr]_{Tr_j^\#} \ar[dd] & & v^*
}
\end{array}
\]

commutes.

The composite \( i_*i^\# \rightarrow i_*i^\#_1 Tr_1 \rightarrow 1 \) is clearly the same as the composite \( i_*i^\# \rightarrow i_*i^\#_1 Tr_1 \rightarrow 1 \). We will denote the common value by

\[
Tr_1^\#: i_*i^\# \rightarrow 1.
\]

We have to show that

\[
v^* \circ Tr_1^\# = Tr_j^\# \circ v^*.
\]

The question in local on \( X \) and \( X' \) and hence we assume that \( X = \text{Spec} R, Z = \text{Spec} A, X' = \text{Spec} R', Z' = \text{Spec} A' \) where \( A' = A \otimes_R R' \). We write \( I \) for the ideal in \( R \) generated by \( t \), \( J \) for its extension to \( R' \), \( N \) for the \( A \)-module \((\wedge^p A/J^2)^* \), and \( N' \) for \( N \otimes_A A' = (\wedge^p A/J^2)^* \). Finally, let \( Tr_{A/R}^\# \) and \( Tr_{A'/R'}^\# \) be the maps in \( D(\text{Mod}_R) \) and \( D(\text{Mod}_{R'}) \) whose “sheafied” versions are \( Tr_1^\# \) and \( Tr_1^\# \) respectively. The discussions in Remark (C.2.14) and in Subsection C.2.15 apply. In particular, from the commutative diagram \((C.2.14.1)\) we only have to show:

\[
(*) \quad Tr_{A/R}^\# (R) \otimes_R R' = Tr_{A'/R'}^\# (R').
\]

This follows from the explicit description of \( Tr_{A/R}^\# (R) \) in \((C.2.15.1)\), for the maps \( \varphi_t \) and \( \pi_t \) occurring in loc.cit. are compatible with base change. In greater detail, if \( t'_i \) are the images in \( R' \) of \( t_i \) and \( t' = (t'_1, \ldots, t'_m) \), then \( \varphi_t \otimes_R R' = \varphi_{t'} \) and \( \pi_t \otimes_R R' = \pi_{t'} \). Since \( Tr_{A/R}^\# (R) = \pi_t \circ \varphi_t^{-1} \) and \( Tr_{A'/R'}^\# (R') = \pi_{t'} \circ \varphi_{t'}^{-1} \), the relation asserted in \((*)\) is true.\( \square \)
Remark 6.2.5. Formula (∗) in the above proof is true in greater generality. Suppose \( Z = \text{Spec} A, X = \text{Spec} R, \) and we have a regular immersion \( i: Z \to X \) given by an \( R \)-sequence \( t = (t_1, \ldots, t_r) \). Then (∗) remains valid without the assumption that \( X \) be Cohen-Macaulay over another scheme. In fact, by checking locally one easily deduces that if \( i: Z \hookrightarrow X \) is a regular immersion (not necessarily of affine schemes, and not necessarily given globally by the vanishing of a sequence of the form \( t \)) and we have a cartesian diagram

\[
\begin{array}{c}
Z' \xrightarrow{w} Z \\
\downarrow \quad \downarrow \\
X' \xrightarrow{\Theta} X
\end{array}
\]

with \( u \) flat and if \( \Theta: w^*i^! \mathcal{O}_X \xrightarrow{\sim} j^! \mathcal{O}_{X'} \) is the flat base change isomorphism, then the following diagram commutes:

\[
\begin{array}{c}
w^*N_i^r[-r] \xrightarrow{\varphi_r} N_j^r[-r] \\
\downarrow \quad \downarrow \\
w^*i^! \mathcal{O}_X \xrightarrow{\Theta} j^! \mathcal{O}_{X'}
\end{array}
\]

The flatness hypothesis on \( v \) can be relaxed, since (∗) works even when \( R' \) is not flat over \( R \), but for now, we leave matters as they are.

6.3. Base-change theorems. We now prove that \( \tau_h^\# \) is stable under arbitrary base change. We embed that result in a larger set of base-change results, namely in Theorem 6.3.1.

**Theorem 6.3.1.** With the hypotheses as in Subsection 6.1 we have:

(a) For a coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), the diagram

\[
\begin{array}{c}
h_!(j^*v^* \mathcal{F} \otimes_{\mathcal{O}_{Z'}} \mathcal{N'}) \xrightarrow{\text{TR}} u^*h_!(i^* \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N}) \\
\downarrow \quad \downarrow \\
R'_Z v^* \mathcal{F} \xleftarrow{b(u, f)} u^*R'_Z \mathcal{F}
\end{array}
\]

commutes.

(b) The map

\[
b(u, f): u^*R'_Z f_* \to R'_Z g_*v^*
\]

is an isomorphism.

(c) The diagram

\[
\begin{array}{c}
h_!(j^*(\omega_y^\#) \otimes_{\mathcal{O}_{Z'}} \mathcal{N'}) \xrightarrow{\text{TR}} h_!(j^*v^*(\omega_y^\#) \otimes_{\mathcal{O}_{Z'}} \mathcal{N}) \\
\downarrow \quad \downarrow \\
\theta'_{\mathcal{N'}} \quad h_!w^*(i^*(\omega_y^\#) \otimes_{\mathcal{O}_{Z'}} \mathcal{N}) \\
\downarrow \quad \downarrow \\
\mathcal{O}_{Y'} \xleftarrow{u^*(\tau_h^* \mathcal{F})} u^*h_!(i^*(\omega_y^\#) \otimes_{\mathcal{O}_Z} \mathcal{N})
\end{array}
\]
commutes, where \( \theta_u^f : v^* \omega_f^g \to \omega_y^g \) is the base-change isomorphism of \([S2]
 p. 740, Theorem 2.3.5 (a)]\).

(d) The diagram

\[
\begin{array}{ccc}
R_{Z'}^f g_* v^* \omega_f^g & \xrightarrow{\sim} & R_{Z'} g_* v^* \omega_f^g \\
\downarrow \text{res}_{Z'}^f & & \downarrow R_{Z'} g_* (\theta_u^f) \\
\mathcal{O}_{Y'} & \xrightarrow{\text{res}_{Z'}^f} & R_{Z'} g_* \omega_y^g
\end{array}
\]

commutes.

**Proof.** From the definitions, we may, without loss of generality, assume that \( Y = \text{Spec} \ A \) and \( Y' = \text{Spec} \ A' \). Consider the composite natural transform of functors of quasi-coherent \( \mathcal{O}_X \)-modules:

\[
\text{Ext}_{A'}^r (\mathcal{O}_Z, -) \to \text{Ext}_{A}^r (\mathcal{O}_Z, v_* v^*(-)) \to \text{Ext}_{A'}^r (\mathcal{O}_{Z'}, v^*(-))
\]
giving a base change map

\[
(6.3.1.1) \quad A' \otimes_A \text{Ext}_{A}^r (\mathcal{O}_Z, -) \to \text{Ext}_{A'}^r (\mathcal{O}_{Z'}, v^*(-))
\]

From the definition of \([6.2.1]\) it is easy to see that

\[
(6.3.1.2) \quad A' \otimes_A \text{Ext}_{A}^r (\mathcal{O}_Z, -) \xrightarrow{(6.3.1.1)} \text{Ext}_{A'}^r (\mathcal{O}_{Z'}, v^*(-))
\]

commutes.

Let \( \mathcal{E}xt^i_f (\mathcal{O}_Z, -) \) be the \( i \)th right derived functor of \( f_* \mathcal{H}om_X (\mathcal{O}_Z, -) \). Since \( Z \) is affine and \( \mathcal{E}xt^i_f (\mathcal{O}_Z, -) \) is supported on \( Z \), this is simply \( h_* \text{Ext}_X^i (\mathcal{O}_Z, -) \). Similarly, one defines \( \mathcal{E}xt^i_{f'} (\mathcal{O}_{Z'}, -) \). Using this, and computing \( \mathcal{E}xt^i_{f'} (\mathcal{O}_{Z}, -) \) and \( \mathcal{E}xt^i_{f'} (\mathcal{O}_{Z'}, -) \) via the Koszul resolutions on \( t \) of \( \mathcal{O}_Z \) and \( \mathcal{O}_{Z'} \), we get see easily that the fundamental local isomorphisms \([C.2.7]\) is compatible with \((6.3.1.1)\). In other words, the following diagram of functors of coherent \( \mathcal{O}_X \)-modules commutes:

\[
\begin{array}{ccc}
h_* (j^* v^*(-) \otimes \mathcal{O}_Z, \mathcal{N}) & \xrightarrow{(6.3.1.3)} & u^* h_* (i^*(-) \otimes \mathcal{O}_Z, \mathcal{N}) \\
\downarrow \mathcal{E}xt^i_{f'}(\mathcal{O}_Z, v^*(-)) & & \downarrow u^* \mathcal{E}xt^i_{f'}(\mathcal{O}_Z, -) \\
\mathcal{E}xt^i_f (\mathcal{O}_{Z'}, v^*(-)) & & \mathcal{E}xt^i_{f'} (\mathcal{O}_Z, -)
\end{array}
\]

This together with \((6.3.1.2)\) gives part (a). In particular, applied to coherent \( \mathcal{O}_X \)-modules, \([6.3.1.1]\) is an isomorphism.

Applying the fact that \((6.3.1.1)\) is an isomorphism to the closed schemes \( Z_0 \) of \( X \) defined by \( t^n_1, \ldots, t^n_r \), and taking the direct limit as \( n \to \infty \) we get (b) from \((6.3.1.2)\).

According to \([S2]\) pp. 755–756, Prop. 6.2.2 (b) and (c)], part (d) is true when either \( u \) is flat or when \( Z \to X \) is a good immersion for \( f \), i.e., it satisfies:

- There is an affine open covering \( \mathcal{U} = \{ U_\alpha = \text{Spec} \ A_\alpha \} \) of \( Y \), and for each index \( \alpha \) an affine open scheme \( V_\alpha = \text{Spec} \ R_\alpha \) of \( f^{-1}(U_\alpha) \) such that \( Z \cap f^{-1}(U_\alpha) \) is contained in \( V_\alpha \).
• The closed immersion $i$ is given in $V_\alpha$ by a quasi-regular $R_\alpha$-sequence.
• $Z \to Y$ is finite.

(See also [S2] p. 744, Def. 3.1.4] and [HK1] pp. 77–78, Assumptions 4.3.)

Let $p$ be a prime ideal of $A$, $y$ the point in $Y$ corresponding to $p$, $\overline{Y}$ the completion of the local ring $A_p$, and $\overline{Y}'$ the completion of $A'_p$ with respect to the ideal $pA'_p$. We then have a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow g & & \downarrow f \\
\overline{Y}' & \xrightarrow{\overline{\sigma}} & \overline{Y} \\
\downarrow \sigma & & \downarrow s \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

All the lateral faces are cartesian, however the top and bottom faces need not be. We set $\overline{Z} = t^{-1}(Z)$ and $\overline{Z}' = \vartheta^{-1}(Z')$.

One checks easily that

\[
(\star) \quad b(\sigma, g) \circ \sigma^* b(u, f) = b(u \circ \sigma, f) = b(\overline{u}, \overline{f}) \circ \overline{u}^* b(s, f)
\]

and according to [S2] p. 747, Remark 3.3.2, we have

\[
(\dagger) \quad \theta^f_{u} \circ \overline{\vartheta}^f_{s} = \theta^f_{u} = \theta^g_{\sigma} \circ \vartheta^g_{s}.
\]

We remark that Cohen-Macaulay maps of relative dimension $r$ are, in the terminology of *ibid*, locally $r$-compactifiable.

From our observations about the compatibility of residues with certain base changes, (d) is true for the left, right and front faces of (6.3.1.4). Indeed, $s$ and $\sigma$ are flat, whereas $\overline{Z}$ is a good immersion for $\overline{f}$. We therefore have:

\[
\begin{align*}
\text{res}^\# \circ R^r_{\overline{Z}; \overline{g}}(\theta^f_{u}) \circ b(s, f) &= s^*(\text{res}^\#) \\
\text{res}^\# \circ R^r_{\overline{Z}; \overline{g}}(\theta^g_{\sigma}) \circ b(\sigma, g) &= \sigma^*(\text{res}^\#)
\end{align*}
\]

The formulas (\star), (\dagger), and (\ddagger) say that the diagram in part (d) of the statement of the theorem commutes after applying $\sigma^*$. Now use part (a), which we have proven, to see that the diagram in (c) commutes after applying $\sigma^*$. Since all the sheaves involved in the diagram are coherent, this means the diagram in (c) commutes in a Zariski open neighbourhood of $u^{-1}(y)$. This proves (c) since $y \in Y$ is arbitrary.

Part (d) now follows by replacing $Z$ by $Z_n$ as before, where $Z_n$ is defined by $t_1^n, \ldots, t_r^n$, applying (c) to $Z_n$, and taking direct limits.

\[\square\]

7. Iterated traces

An important formula concerning residues is a Fubini like statement for iterated residues (see [RD] p.198, (R4)). To establish this via our approach to residues, i.e., via Verdier’s isomorphism, we have to understand iterated traces (for a composite
of pseudo-proper maps) in various ways. That is the main thrust of this section. The circle of ideas is sometimes referred to as “transitivity” (cf. [LS]). In somewhat greater detail suppose

\[ \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{X} \]

is a pair of pseudo-proper maps. Recall that \( \text{Tr}_f : R_f R_\mathcal{X}^* f^# \to 1_{\text{D}(\mathcal{Y})} \) factors through the natural map \( R_\mathcal{X}^* f^# \to 1_{\text{D}(\mathcal{Y})} \). Moreover, we abuse notation and write \( \text{Tr}_f : R_f R_\mathcal{X}^* f^# \to R_\mathcal{X}^* f^# \) for the missing factor in the just mentioned factorization of \( \text{Tr}_f : R_f R_\mathcal{X}^* f^# \to 1_{\text{D}(\mathcal{Y})} \). Given \( \mathcal{F}, \mathcal{G} \in \mathcal{D}_\text{qc}(\mathcal{Y}) \) the torsion version of the projection isomorphism, which we shall denote as \( p^!_{\mathcal{F}, \mathcal{G}} \), is the following composition

\[
\mathcal{F}^ L \otimes_{\mathcal{O}_\mathcal{Y}} R_f R_\mathcal{X}^* \mathcal{G} \xrightarrow{\sim} R_f (L f^* \mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} R_\mathcal{X}^* \mathcal{G}) \xrightarrow{\sim} R_f R_\mathcal{X}^* (L f^* \mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{G})
\]

where the first isomorphism is induced by projection. In this situation, we have the following iterated trace on \( R((gf)_* R_\mathcal{X}^* (L f^* g^# \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y})) \) where the map labelled \( p \) is the natural one induced by \( (p^!_{f})^{-1} \) while the one labelled \( T \) is induced by \( \text{Tr}_f \):

\[
R((gf)_* R_\mathcal{X}^* (L f^* g^# \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y})) \xrightarrow{\sim} R_{g*} R_f R_\mathcal{X}^* (L f^* g^# \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y}) \xrightarrow{T} R_{g*} R_f R_\mathcal{X}^* (L f^* g^# \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y}) \xrightarrow{\sim} R_f R_\mathcal{X}^* (L f^* g^# \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y}) \xrightarrow{\text{Tr}_f} \mathcal{O}_\mathcal{Y}.
\]

By adjointness, this gives us a map

\[ \chi_{[g,f]} : L f^* (g^# \mathcal{O}_\mathcal{X}) \otimes f^* \mathcal{O}_\mathcal{Y} \to (gf)^# \mathcal{O}_\mathcal{X}. \]

In fact one does not need \( f \) and \( g \) to be pseudo-proper to define \( \chi_{[g,f]} \). Our definition below works under the assumption that each of them is a composite of compactifiable maps.

Part of the theme of transitivity is to work out a concrete formula for \( \chi_{[g,f]} \) when \( f \) and \( g \) are smooth, and when \( g^* \mathcal{O}_\mathcal{X}, f^* \mathcal{O}_\mathcal{Y}, \) and \( (gf)^* \mathcal{O}_\mathcal{X} \) are substituted with suitable differential forms (placed in the appropriate degree) via Verdier’s isomorphism [V]. That is done in a later paper based on the work done here.

7.1. **Traces in affine terms.** If \( A \to B \) is a pseudo-finite-type map of adic rings, \( I \subset A \) and \( J \subset B \) defining ideals for the adic rings \( A \) and \( B \) respectively, and \( f : \text{Spf}(B, J) \to \text{Spf}(A, I) \) the resulting map of formal schemes, then the complex \( f^* \mathcal{O}_{\text{Spf}B} \) can be represented by a bounded-below complex

\[ \omega^*_{(B,J)/(A,I)} = \omega^*_{B/A} \in D^+(\text{Mod}_B) \]

which has finitely generated cohomology modules, where the more elaborate notation \( \omega^*_{(B,J)/(A,I)} \) is used only when the role of the adic structures needs to be emphasised. To simplify notation further, we shall use \( \omega^*_{B/A} \) in place of \( \omega^*_{B/A} \) from now on.

It then follows that if \( f \) is Cohen-Macaulay then \( \omega^*_{B/A} = \omega^*_{B/A}[d] \).

Regarding the affine version of traces there are two related situations which we wish to discuss.
A. Suppose $A \to B/J$ is finite. Recall that the trace map

$$\text{Tr}_f : \mathbb{R}^\Gamma_{\text{Spf}(B/J)} f^\# \mathcal{O}_{\text{Spf}(A,I)} \to \mathcal{O}_{\text{Spf}(A,I)[0]}$$

factors through the natural map $\mathbb{R}^\Gamma_{\text{Spf}(B/J)} f^\# \mathcal{O}_{\text{Spf}(A,I)} \to \mathbb{R}^\Gamma_{\text{Spf}(A,I)} \mathcal{O}_{\text{Spf}(A,I)[0]}$ and that the map $\mathbb{R}^\Gamma_{\text{Spf}(B/J)} f^\# \mathcal{O}_{\text{Spf}(A,I)} \to \mathbb{R}^\Gamma_{\text{Spf}(A,I)} \mathcal{O}_{\text{Spf}(A,I)[0]}$ inducing this trace map is also called the trace map, and is also denoted $\text{Tr}_f$. Corresponding to these maps $\text{Tr}_f$ we have, at the affine level, two maps, again denoted by the same symbol $\text{Tr}_B/A$.

(7.1.1) \[ \text{Tr}_{B/A} = \text{Tr}_{(B,J)/(A,I)} : \mathbb{R}^\Gamma j\omega^\bullet_{B/A} \to \mathbb{R}^\Gamma I A[0]. \]

and

(7.1.2) \[ \text{Tr}_{B/A} = \text{Tr}_{(B,J)/(A,I)} : \mathbb{R}^\Gamma j\omega^\bullet_{B/A} \to A[0]. \]

Note that the two uses of the symbol $\text{Tr}_{B/A}$ occur in the following commutative diagram:

\[ \begin{CD}
\mathbb{R}^\Gamma j\omega^\bullet_{B/A} @>>> \mathbb{R}^\Gamma I A[0] \\
\text{Tr}_{B/A} @VVV \text{Tr}_{B/A} V \\
\mathbb{R}^\Gamma I A[0] @>>> A[0]
\end{CD} \]

B. Next suppose $A$ and $B$ both have discrete topology, and we have a finite-type map $A \to B$. Suppose $J$ is an ideal in $B$ such that $A \to B/J$ is finite. Let $\widehat{B}$ be the completion of $B$ with respect to $J$. Note that if $\kappa : \text{Spf}(\widehat{B}, J\widehat{B}) \to \text{Spec} B$ is the completion map, then the canonical isomorphism $\kappa^* f^* \cong (f\kappa)^*$ results in a canonical isomorphism $\omega^\bullet_{B/A} \otimes_B \widehat{B} \cong \omega^\bullet_{\widehat{B}/A}$. Define

(7.1.3) \[ \text{Tr}_J : \mathbb{R}^\Gamma j\omega^\bullet_{B/A} \to A[0] \]

as the composite

\[ \begin{CD}
\mathbb{R}^\Gamma j\omega^\bullet_{B/A} @>>> \mathbb{R}^\Gamma j\widehat{B} (\omega^\bullet_{B/A} \otimes_B \widehat{B}) \\
@VVV \mathbb{R}^\Gamma j\widehat{B} \omega^\bullet_{B/A} @VVV \mathbb{R}^\Gamma j\widehat{B} \omega^\bullet_{B/A} @>>> A[0] \\
\text{Tr}_{B/A} @AAA \text{Tr}_{B/A} @AAA
\end{CD} \]

7.1.4. There is potential for confusion over the symbol $\omega^\bullet_{B/A}$ in a situation we will be in and we would like to clarify the issues here. Let $(A, I)$ and $(B, J)$ be adic rings. Let $L = IB + J \subset B$, and assume further that $B$ is also $L$-adically complete. Suppose there is a ring homomorphism $A \to B$ such that the induced map $A \to B/J$ is finite. Then $A \to B/L$ is also finite and the formal-scheme maps $\text{Spf}(B, J) \xrightarrow{p} \text{Spec} A$ and $\text{Spf}(B, L) \xrightarrow{f} \text{Spf}(A, I)$ are both pseudo-finite. Moreover we have a cartesian square as follows.

\[ \begin{CD}
\text{Spf}(B, L) @>{\kappa_L}>> \text{Spf}(B, J) \\
| f V | | V | \\
\text{Spf}(A, I) @>{\kappa_I}>> \text{Spec} A
\end{CD} \]
Since $\kappa_i$ is flat, we have $\kappa_i^* p^! \mathcal{O}_{\text{Spec } A} \xrightarrow{\sim} f^! \kappa_i^* \mathcal{O}_{\text{Spec } A} = f^! \mathcal{O}_{\text{Spec } (A, I)}$. This means we can, and we will, identify $\omega^*_1(B, L)/(A, I)$ and $\omega^*_1(B, J)/(A, I)$. Therefore, denoting the common complex $\omega^*_1(B, A)$ in this situation causes no confusion. Thus,

$$\omega^*_1(B, A) = \omega^*_1(B, L)/(A, I) = \omega^*_1(B, J)/(A, I).$$

We have two maps $\text{Tr}_L: \text{R} \Gamma_L \omega^*_1(B, A) \rightarrow \text{R} \Gamma_I(A[0])$ and $\text{Tr}_J: \text{R} \Gamma_J \omega^*_1(B, A) \rightarrow A[0]$ corresponding to $\text{Tr}_I$ (cf. (7.1.1)) and $\text{Tr}_P$ (cf. (7.1.2)) respectively. In these circumstances, according to Proposition \[A.2.4\] in the appendix, the following diagram commutes:

$$
\begin{array}{ccc}
\text{R} \Gamma_I \text{R} \Gamma_J(\omega^*_1(B, A)) & \xrightarrow{\sim} & \text{R} \Gamma_L(\omega^*_1(B, A)) \\
\downarrow \text{Tr}_I & & \downarrow \text{Tr}_J \\
\text{R} \Gamma_I A[0] & = & \text{R} \Gamma_I A[0]
\end{array}
$$

### 7.2. Abstract Transitivity

This section is a digression on setting up a suitable bifunctor for every morphism in $\mathcal{G}$ which will then be used to define an abstract transitivity relation.

For a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{G}$, and complexes $\mathcal{F}, \mathcal{G} \in \text{D}^+_{\text{qc}}(\mathcal{Y})$, we shall now define a bifunctorial map

$$
\chi^f(\mathcal{F}, \mathcal{G}):\text{L} f^* \mathcal{F} \otimes_{\mathcal{O}_Y} f^! \mathcal{G} \rightarrow f^!(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})
$$

which, a-priori, will depend on the choice of a factorization $f = f_n f_{n-1} \cdots f_1$ where each $f_i$ is either an open immersion or a pseudoproper map. In these two special cases, there is a simple version of this bifunctorial map and the general case is handled by putting together these special ones. In Proposition \[7.2.4\] below we prove that $\chi^f(\cdot, \cdot)$ is independent of the choice of the factorization.

Since $(-)^!$ is only a pre-pseudofunctor, even for $f$ any identity map, say $f = 1_{\mathcal{X}}$, some non-trivial considerations arise. For $\mathcal{F}, \mathcal{G} \in \text{D}^+_{\text{qc}}(\mathcal{X})$, we define

$$q_{\mathcal{X}}(\mathcal{F}, \mathcal{G}):\mathcal{F} \otimes_{\mathcal{O}_X} \Delta^X \mathcal{G} \rightarrow \Delta^X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

to be the map, which, via right adjointness of $\Delta^X$ to $\text{R} \Gamma^X$, corresponds to the composite of natural maps

$$\text{R} \Gamma^X(\mathcal{F} \otimes_{\mathcal{O}_X} \Delta^X \mathcal{G}) \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \text{R} \Gamma^X \Delta^X \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}.$$

Below we shall define $\chi^X$ to be $q_{\mathcal{X}}$. For now, we collect a few properties of $q_{\mathcal{X}}$ that we shall use.

The natural map $1 \rightarrow \Delta^X$ on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ factors through $q_{\mathcal{X}}(\mathcal{F}, \mathcal{G})$:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{q_{\mathcal{X}}(\mathcal{F}, \mathcal{G})} \Delta^X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Note that $q_{\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is an isomorphism if both $\mathcal{G}$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ have coherent homology or if $\mathcal{F}$ is perfect, i.e., locally isomorphic to bounded complex of finite-rank locally free modules. Also note that $\text{R} \Gamma^X q_{\mathcal{X}}$ is an isomorphism and hence $\Delta^X q_{\mathcal{X}}$, which is isomorphic to $\Delta^X \text{R} \Gamma^X q_{\mathcal{X}}$, is also an isomorphism, i.e., the natural map is an isomorphism

$$\Delta^X q_{\mathcal{X}}(\mathcal{F}, \mathcal{G}):\Delta^X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \rightarrow \Delta^X(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}).$$
Via natural identifications, \(q_F(\mathcal{O}_X, \mathcal{G})\) identifies with the identity map on \(\Lambda_X \mathcal{G}\). Finally, to get a more explicit description of \(q_F\), if we choose the adjoint pair \((\Lambda_X, \varepsilon)\) to \(R_F^*\) to be \(\Lambda_X : \mathcal{M} = R\mathcal{H}om_X(RF^* \mathcal{O}_X, \mathcal{M})\) with \(\varepsilon\) being the following composition on canonical maps

\[
R_F^* \mathcal{H}om_X(RF^* \mathcal{O}_X, \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_X(RF^* \mathcal{O}_X, \mathcal{M}) \otimes_X RF^* \mathcal{O}_X \xrightarrow{\text{eval}} \mathcal{M},
\]

then \(q_F(\mathcal{F}, \mathcal{G})\) can be described as the canonical map

\[
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_X(RF^* \mathcal{O}_X, \mathcal{G}) \rightarrow \mathcal{H}om_X(RF^* \mathcal{O}_X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}).
\]

We will now set up some notation that will be useful for keeping track of the numerous issues that arise out handling factorizations of maps into open immersions and pseudoproper maps.

Let \(f: \mathcal{X} \rightarrow \mathcal{Y}\) be a morphism in \(\mathcal{G}\) and \(f = f_n f_{n-1} \cdots f_1\) a factorization where each \(f_i: \mathcal{X}_i \rightarrow \mathcal{X}_{i+1}\) is an open immersion or a pseudoproper map with \(\mathcal{X}_1:=\mathcal{X}\) and \(\mathcal{X}_{n+1}:=\mathcal{Y}\). Let us assign to each \(f_i\) a label \(\lambda_i\), with \(\lambda_i\) being one of either \(\mathcal{O}\) or \(\mathcal{P}\) (where \(\mathcal{O}\) = open immersions and \(\mathcal{P}\) = pseudoproper maps), together with the requirement that each \(f_i\) lies in the subcategory corresponding to \(\lambda_i\). We shall denote the labelled map as \(f_\lambda\) and the above factorization together with the assigned labels will be called a labelled factorization (of \(f\)). The corresponding sequence \(F = (f_{\lambda_1}, \ldots, f_{\lambda_n})\) will be called a labelled sequence of length \(n\) and \(|F|\) shall denote the composite \(f\). To ease notation, the labels shall often be suppressed and we shall spell them out only when it is necessary. Thus we shall often denote a labelled map \(f_\lambda\) by the underlying map \(f\) itself. If \(F = (f_1, \ldots, f_n)\) and \(G = (g_1, \ldots, g_m)\) are labelled sequences, and if \(g f_n\) makes sense, then we denote the composite labelled sequence \((f_1, \ldots, f_n, g_1, \ldots, g_m)\) as \((F, G)\), which is evidently a labelled factorization of \(|(F, G)| = |G||F|\).

For a labelled sequence \(F = (f_1, \ldots, f_n)\), set

\[
F^* := Lf_1 f_2^* \cdots f_n^*, \quad F^\# := f_1^* f_2^* \cdots f_n^*.
\]

With \(|F| = f\), there are canonical pseudofunctorial isomorphisms \(F^* \rightarrow Lf^*\) and \(F^\# \rightarrow f^\#\). If \(F, G\) are labelled sequences such that the composite \((F, G)\) exists, then \((F, G)^\# = F^\# G^\#\) and \((F, G)^* = F^* G^*.\)

For a labelled sequence \(F = (f_1, \ldots, f_n)\) factoring \(f: \mathcal{X} \rightarrow \mathcal{Y}\) and for complexes \(\mathcal{F}, \mathcal{G} \in D^+_q(\mathcal{X})\), we recursively define

\[
\chi_F(\mathcal{F}, \mathcal{G}): F^* \mathcal{F} \otimes_{\mathcal{O}_X} F^\# \mathcal{G} \rightarrow F^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})
\]

as follows. If \(n = 1\), then \(F = f^O\) or \(F = f^P\) and moreover \(F^* = Lf^* = f^*, F^\# = f^\#\). If \(F = f^O\), so that \(f\) is an open immersion, then using \(f^\# = \Lambda_X f^* \Rightarrow f^* \Lambda_X\), we take \(\chi_F(\mathcal{F}, \mathcal{G})\) to be the composite along the top row of the following commutative diagram

\[
\begin{array}{ccc}
\text{f}^* \mathcal{F} \otimes_{\mathcal{O}_X} \Lambda_X \text{f}^* \mathcal{G} & \xrightarrow{q^*} & \Lambda_X (\text{f}^* \mathcal{F} \otimes_{\mathcal{O}_X} \text{f}^* \mathcal{G}) \\
\end{array}
\]

(7.2.2)

\[
\begin{array}{ccc}
\text{f}^* \mathcal{F} \otimes_{\mathcal{O}_X} \Lambda_X \text{f}^* \mathcal{G} & \xrightarrow{f^* q^*} & f^* (\mathcal{F} \otimes_{\mathcal{O}_X} \Lambda_X \mathcal{G}) \\
\end{array}
\]

where \(q^*\) is defined above. (The commutativity of this diagram, which will only be used later, follows easily from the explicit description of \(q^*\) above.) If \(F = f^P\), so
that \( f \) is pseudoproper, we set \( \chi_F(\mathcal{F}, \mathcal{G}) \) to be the map adjoint to the composite

\[
\mathbf{R}f_* \mathbf{R}i'_*(\mathbf{L}f^* \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}}^L f^*\mathcal{G}) \xrightarrow{\text{via} (p'_1)^{-1}} \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}}^L \mathbf{R}f_* \mathbf{R}i'_*(f^*\mathcal{G}) \xrightarrow{\text{via} \text{Tr}_f} \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}}^L \mathcal{G}
\]

where \( p'_1 \) is the torsion projection isomorphism as defined in the beginning of §7. In general, if \( n > 1 \), we decompose \( F \) as \( \mathcal{X} \xrightarrow{f_1} \mathcal{X}_2 \xrightarrow{f_2} \mathcal{Y} \) where \( G = (f_2, \ldots, f_n) \) gives a labelled factorization of \( g \) while \( f_1 \) is naturally a labelled sequence of length 1. Assuming \( \chi_G(\mathcal{F}, \mathcal{G}) \) is already defined, we define \( \chi_F(\mathcal{F}, \mathcal{G}) \) to be the composite (with \( \otimes_{\mathcal{X}} = \otimes_{\mathcal{O}_{\mathcal{X}}}, \otimes_{\mathcal{Y}} = \otimes_{\mathcal{O}_{\mathcal{Y}}} \))

\[
F^* \mathcal{F} \otimes_{\mathcal{X}}^L F^*\mathcal{G} = f_1^*G^* \mathcal{F} \otimes_{\mathcal{X}}^L f_1^*G^* \mathcal{G} & \xrightarrow{\chi_{f_1}(G^* \mathcal{F}, G^* \mathcal{G})} f_1^*(G^* \mathcal{F}) \otimes_{\mathcal{X}_2} G^* \mathcal{G} \\
& \xrightarrow{f_1^*\chi_G(\mathcal{F}, \mathcal{G})} f_1^*(\mathcal{F} \otimes_{\mathcal{Y}}^L \mathcal{G}) = F^*(\mathcal{F} \otimes_{\mathcal{Y}}^L \mathcal{G}).
\]

In more concise terms, we may write that if \( F = (f_1, G) \), then \( \chi_F = (f_1^* \chi_G) \circ \chi_{f_1} \). It follows from the recursive nature of the definition that for any decomposition \( F = (F_1, F_2) \) we have \( \chi_F = \chi_{(F_1, F_2)} = (F_1^* \chi_{F_2}) \circ \chi_{F_1} \).

For \( \mathcal{F} = \mathcal{O}_{\mathcal{X}} \), via the obvious natural identifications \( F^*\mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} F^*\mathcal{G} \) and \( F^*(\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{Y}} \mathcal{G}) \xrightarrow{\sim} F^*\mathcal{G} \) we see that \( \chi_F(\mathcal{O}_{\mathcal{X}}, \mathcal{G}) \) identifies with the identity map on \( F^*\mathcal{G} \).

The identity map \( 1_{\mathcal{X}} \) for any formal scheme \( \mathcal{X} \), being in both \( \mathcal{O} \) and \( \mathcal{P} \), forms a factorization of length 1 of itself for any of the two labels. With either label, we see that \( \chi_{1_{\mathcal{X}}} \) equals \( q_{\mathcal{X}} \) defined above.

More generally, \( f: \mathcal{X} \to \mathcal{Y} \) is in \( \mathcal{O} \) and \( \mathcal{P} \) iff \( f \) is an isomorphism of \( \mathcal{X} \) onto a connected component of \( \mathcal{Y} \) and so any such \( f \) is a length-one factorization of itself with either label.

**Lemma 7.2.3.** If \( f: \mathcal{X} \to \mathcal{Y} \) is an isomorphism, then \( \chi_{f^*} = \chi_{f_*} \).

**Proof.** In this case, \( f^* = \mathbf{L}f^* \) and \( f_* = \mathbf{R}f_* \) are both left and right adjoint to each other and the unit/counit maps for either adjoint pair is given by the canonical isomorphisms \( f_* f^* \cong 1 \), \( f^* f_* \cong 1 \). Since \( f^* = \mathbf{L}f^* \), the result follows from the commutativity of the outer border of the following diagram for \( \mathcal{F}, \mathcal{G} \in \mathbf{D}^{+}_{qc}(\mathcal{Y}) \), where to reduce clutter, the derived functors are denoted by their non-derived
counterparts and moreover \( \Lambda = \Lambda_{\mathcal{X}}, \Gamma' = R\Gamma'_{\mathcal{Y}} \).

\[
(f^* \mathcal{F} \otimes \Lambda f^* \mathcal{G})
\]

The bottom triangle is seen to commute by unravelling the definition of the projection isomorphism. The remaining parts commute trivially. \(\square\)

If \( f : \mathcal{X} \to \mathcal{Y} \) is in \( \mathcal{G} \) and \( F \) is a labelled factorization of \( f \), for \( \mathcal{F}, \mathcal{G} \in \widehat{\mathcal{D}}_{qc}^+(\mathcal{Y}) \) we set \( \chi^f_f(\mathcal{F}, \mathcal{G}) \) to be the composite

\[
Lf^* \mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} f^* \mathcal{G} \xrightarrow{\chi^f} F^* \mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} F^* \mathcal{G} \xrightarrow{\chi^f} F^* (\mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} \mathcal{G}) \xrightarrow{\sim} f^* (\mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} \mathcal{G}).
\]

If \( f = 1_{\mathcal{X}} \), then \( \chi^f_f = q_{\mathcal{X}} \) for either label as mentioned before.

**Proposition-Definition 7.2.4.**

(i) If \( F_1 \) and \( F_2 \) are two labelled factorizations of a map \( f : \mathcal{X} \to \mathcal{Y} \) in \( \mathcal{G} \), then \( \chi^f_{F_1}(-,-) = \chi^f_{F_2}(-,-) \). We thus define \( \chi^f_{\mathcal{F}}(-,-) \) in (7.2.1) to be \( \chi^f_{F}(-,-) \) for any choice of a labelled factorization \( F \) of \( f \).

(ii) If \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \) are maps in \( \mathcal{G} \), then for any complexes \( \mathcal{F}, \mathcal{G} \in \widehat{\mathcal{D}}_{qc}^+(\mathcal{Z}) \), the following diagram commutes.

\[
L(gf)^* \mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} (gf)^* \mathcal{G} \xrightarrow{\chi^{gf}} (gf)^* (\mathcal{F} \overset{L}{\otimes} \otimes_{\Theta_z} \mathcal{G})
\]

Part (ii) of Proposition 7.2.4 is the transitivity property for \( \chi \). It is an easy consequence of part (i) and the relation \( \chi_{(F,G)} = (F^* \chi_G) \circ \chi_F \) where \( F, G \) are labelled factorizations of \( f, g \) respectively so that \( (F,G) \) can be chosen as a labelled factorization of \( fg \).

The proof of Proposition 7.2.4(i) is somewhat long and proceeds via several special cases of both parts (i) and (ii) first. We tackle these in the next few lemmas. In all these proofs, to reduce clutter in numerous diagrams, we shall use the following shorthand notation where \( f \) is a generic name for a map and \( \mathcal{X} \) for a formal scheme.

\[
f^* = Lf^*, \quad \otimes = \otimes, \quad \Gamma'_{\mathcal{X}} = R\Gamma'_{\mathcal{Y}}, \quad f_*^* = Rf_* R\Gamma'_{\mathcal{X}}
\]
Lemma 7.2.5. Let $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ be a map in $\mathcal{G}$.

(i) If $F_1, F_2$ are labelled factorizations of $f$ such that $\chi^f_{F_1} = \chi^f_{F_2}$, then for any maps $\mathcal{W} \xrightarrow{h} \mathcal{X}$ and $\mathcal{Y} \xrightarrow{g} \mathcal{Z}$ in $\mathcal{G}$ and labelled factorizations $G, H$ of $g, h$ respectively, we have $\chi^{fg}_{(F_1,G)} = \chi^{fg}_{(F_2,G)}$ and $\chi^{fh}_{(H,F_1)} = \chi^{fh}_{(H,F_2)}$.

(ii) For any labelled factorization $F$ of $f$ we have $\chi^{f}_{(1_{\mathcal{X}},F)} = \chi^f_{F}$.

Proof. (i) To prove that $\chi^{fg}_{(F_1,G)} = \chi^{fg}_{(F_2,G)}$ it suffices to check that for any complexes $\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_q(\mathcal{G})$ the outer border of the following diagram commutes.

\[
\begin{array}{ccc}
F_1^*G^* \mathcal{F} \otimes F_1^*G^*\mathcal{G} & \longrightarrow & F_1^*(G^* \mathcal{F} \otimes G^*\mathcal{G}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
f^*g^* \mathcal{F} \otimes f^*g^*\mathcal{G} & \longrightarrow & f^*(G^* \mathcal{F} \otimes G^*\mathcal{G}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
F_2^*G^* \mathcal{F} \otimes F_2^*G^*\mathcal{G} & \longrightarrow & F_2^*(G^* \mathcal{F} \otimes G^*\mathcal{G}) \\
\end{array}
\]

Along the leftmost and the rightmost columns, the composite of maps remains unchanged if in the objects of the middle row, $g^*, g^*$ are replaced by $G^*, G^*$ respectively. Thus the left half commutes if $\chi^{fg}_{(F_1,G)} = \chi^{fg}_{(F_2,G)}$ while the right one commutes for functorial reasons. The proof of the other relation is similar.

(ii) For $\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_q(\mathcal{G})$, the following diagram, where $1 = 1_{\mathcal{X}}$, is easily seen to commute keeping in mind the isomorphisms $F^* \xrightarrow{\cong} \Lambda_{\mathcal{F}} F^* = 1^* F^*$.

\[
\begin{array}{ccc}
1^*F^* \mathcal{F} \otimes 1^*F^*\mathcal{G} & \longrightarrow & 1^*(F^* \mathcal{F} \otimes F^*\mathcal{G}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
F^* \mathcal{F} \otimes F^*\mathcal{G} & \longrightarrow & F^*(\mathcal{F} \otimes \mathcal{G}) \\
\end{array}
\]

From the outer border we get that $\chi^{f}_{(1_{\mathcal{X}},F)} = \chi^f_{F}$. The other relation is proved similarly. \hfill \square

Lemma 7.2.6. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map in $\mathcal{G}$ and $F = (f_1, \ldots, f_n)$ a labelled sequence factoring $f$ such that all the $f_i$'s have the same label, say $\lambda$, so that $f$ is also in $\lambda$. Then $\chi^f_{F} = \chi^f_{\lambda}$.

Proof. It suffices to prove the case $n = 2$ for then, by Lemma 7.2.5(i), in the general case we have $\chi^f_{(f_1,\ldots,f_n)} = \chi^f_{(f_2f_1,\ldots,f_n)}$, whence the result follows by induction.

In effect, we have reduced to proving Proposition 7.2.4 (ii), with $\chi^f_{f}, \chi^g_{g}, \chi^g_{h}$ replaced by $\chi^f_{\lambda}, \chi^g_{\lambda}, \chi^g_{\lambda}$ respectively. For the rest of the proof, we use the notation from there. Let $\mathcal{F}, \mathcal{G} \in \mathcal{D}^+_q(\mathcal{G})$.

If $\lambda = 0$, then the result follows from the outer border of the following commutative diagram where (1) commutes by (7.2.2) and the remaining parts commute.
for trivial reasons.

\[ f^* g^* \mathcal{F} \otimes \Lambda_x f^* \Lambda_y g^* \mathcal{G} \xrightarrow{\Delta_x} f^* (f^* g^* \mathcal{F} \otimes f^* \Lambda_y g^* \mathcal{G}) \rightarrow \Delta_x f^* (g^* \mathcal{F} \otimes \Lambda_y g^* \mathcal{G}) \]

\[ f^* g^* \mathcal{F} \otimes \Lambda_x f^* g^* \mathcal{G} \xrightarrow{\Delta_x} f^* (f^* g^* \mathcal{F} \otimes f^* g^* \mathcal{G}) \rightarrow \Delta_x f^* \Lambda_y g^* (\mathcal{F} \otimes \mathcal{G}) \]

\[ (g f)^* \mathcal{F} \otimes \Lambda_x (g f)^* \mathcal{G} \xrightarrow{\Delta_x} f^* (f^* \mathcal{F} \otimes f^* g^* \mathcal{G}) \rightarrow \Delta_x f^* (g f)^* \mathcal{G} \]

\[ \Lambda_x ((g f)^* \mathcal{F} \otimes (g f)^* \mathcal{G}) \rightarrow \Delta_x f^* (g f)^* \mathcal{G} \]

If \( \lambda = P \), then by adjointness it suffices to check that the outer border of the following diagram commutes where \( p_1 \) is the torsion projection isomorphism as defined in the beginning of [4].

\[ g_1^* f_1^* (f^* g^* \mathcal{F} \otimes f^* g^* \mathcal{G}) \xrightarrow{(p_1^*)^{-1}} g_1^* (g^* \mathcal{F} \otimes f_1^* f^* g^* \mathcal{G}) \rightarrow g_1^* (g^* \mathcal{F} \otimes g^* \mathcal{G}) \]

\[ g_1^* (f^* g^* \mathcal{F} \otimes f^* g^* \mathcal{G}) \xrightarrow{(p_1^*)^{-1}} \mathcal{F} \otimes g_1^* f_1^* g^* \mathcal{G} \rightarrow \mathcal{F} \otimes g_1^* \mathcal{G} \]

\[ (g f)_1^* ((g f)^* \mathcal{F} \otimes (g f)^* \mathcal{G}) \xrightarrow{(p_1')^{-1}} \mathcal{F} \otimes (g f)_1^* (g f)^* \mathcal{G} \rightarrow \mathcal{F} \otimes \mathcal{G} \]

The upper rectangle on the right commutes trivially, while the lower one on the right commutes because of the way the composite of adjoints is identified as an adjoint pseudofunctorially. Commutativity of the diagram on the left follows easily from the outer border of the following one with obvious natural maps where we use \( \mathcal{E} = f^* g^* \mathcal{G} \).

\[ g_1^* f_1^* (f^* g^* \mathcal{F} \otimes \mathcal{E}) \rightarrow g_1^* f_1^* (f^* g^* \mathcal{F} \otimes \Gamma'_x \mathcal{E}) \rightarrow g_1^* (g^* \mathcal{F} \otimes f_* \Gamma'_y \mathcal{E}) \]

\[ g_1^* (f^* g^* \mathcal{F} \otimes \mathcal{E}) \rightarrow g_1^* (g^* \mathcal{F} \otimes f_* \Gamma'_y \mathcal{E}) \rightarrow g_1^* (g^* \mathcal{F} \otimes f_* \Gamma'_y \mathcal{E}) \]

\[ g_1^* (f^* g^* \mathcal{F} \otimes \Gamma'_y \mathcal{E}) \rightarrow g_1^* (g^* \mathcal{F} \otimes f_* \Gamma'_y \mathcal{E}) \rightarrow g_1^* (g^* \mathcal{F} \otimes f_* \Gamma'_y \mathcal{E}) \]

\[ (g f)_1^* ((g f)^* \mathcal{F} \otimes \mathcal{E}) \rightarrow (g f)_1^* ((g f)^* \mathcal{F} \otimes \Gamma'_y \mathcal{E}) \rightarrow (g f)_1^* ((g f)^* \mathcal{F} \otimes \Gamma'_y \mathcal{E}) \]

Here (†) commutes by [4] p. 125, Prop. 3.7.1. Commutativity of the remaining parts is obvious. \( \square \)
Consider a cartesian diagram in $\mathcal{G}$ as follows.

$$\begin{array}{ccc}
\mathcal{W} & \xrightarrow{v} & \mathcal{V} \\
\downarrow{g} & & \downarrow{f} \\
\mathcal{Z} & \xrightarrow{u} & \mathcal{Y}
\end{array}$$

(7.2.7)

Pick labelled factorizations $F = (f_1, \ldots, f_n)$ and $U = (u_1, \ldots, u_m)$ of $f, u$ respectively so that these in turn, induce corresponding ones $G, V$ of $g, v$ by base change in the obvious manner. Thus the composite map $h = fv = ug$ admits two labelled factorizations, namely, $(V, F)$ and $(G, U)$.

**Lemma 7.2.8.** In the above setup, $\chi^h_{(V,F)} = \chi^h_{(G,U)}$.

**Proof.** By decomposing the factorizations $F$ and $U$, let us first reduce to the case $m = n = 1$. For instance, a decomposition $U = (U', U'')$, induces a corresponding one $V = (V', V'')$ and we have a horizontally decomposed cartesian diagram as follows.

$$\begin{array}{ccc}
\mathcal{W} & \xrightarrow{v'} & \mathcal{W}' & \xrightarrow{v''} & \mathcal{X} \\
\downarrow{g'} & & \downarrow{g''} & & \downarrow{f} \\
\mathcal{Z} & \xrightarrow{u'} & \mathcal{Z}' & \xrightarrow{u''} & \mathcal{Y}
\end{array}$$

Set $h' := u'g = g'v'$ and $h'' := u''g' = f'v''$. If $G'$ is the induced factorization of $g'$, then by Lemma 7.2.5(i), it suffices to prove that $\chi_{(G',U')}$ is the adjoint of $(7.2.8.1)$. The maps are natural ones induced by isomorphisms $\chi^h_{(V,F)} = \chi^h_{(V',F)}$. Thus we inductively reduce to the case $m = 1$. A similar argument further reduces it to $n = 1$. Moreover, after assuming $m = n = 1$, by Lemma 7.2.6 it suffices to consider the case when $f, g$ have label $P$ while $u, v$ have label $O$.

For this special case, using the identifications $u^* = \Lambda_\mathcal{W} u^*$, $v^* = \Lambda_\mathcal{W} v^*$, proving the relation $\chi_{(v,f)}^h = \chi_{(g,u)}^h$ amounts to proving that the following diagram commutes for $\mathcal{F}, \mathcal{G} \in \mathcal{D}_{\mathcal{W}}(\mathcal{Y})$.

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{f^*} & \mathcal{F} & \xrightarrow{\Lambda_\mathcal{W} v^* f^*} & \mathcal{G} \\
\downarrow{g^*} & & \downarrow{\Lambda_\mathcal{W} u^*} & & \downarrow{g^*} \\
\mathcal{Y} & \xrightarrow{u^*} & \mathcal{G} & \xrightarrow{h^*} & \mathcal{F}
\end{array}$$

(7.2.8.1)

The composite along each of the two rows is induced by the composite isomorphism $v^* f^* \sim h^* u^*$ which in fact, identifies with the base-change isomorphism $\beta^*: \Lambda_\mathcal{W} v^* f^* \sim g^* \Lambda_\mathcal{W} u^*$. Using the adjointness of $g^*$ to $g^* = g_\Lambda^t \Gamma_{\mathcal{W}}$, we consider the adjoint of (7.2.8.1). The adjoint diagram is expanded below, which, for convenience, is broken into two parts. Thus the rightmost column of (7.2.8.2) is the same as the left column of (7.2.8.3) and the outer border of the conjoined diagram is the adjoint of (7.2.8.1). The maps are natural ones induced by isomorphisms $\Gamma_t (\mathcal{M} \otimes N) \sim \Gamma_t (\mathcal{N} \otimes \mathcal{M})$, $\Lambda \sim \Lambda$, $\Gamma_t u^* \sim u^* \Gamma_t$, $\Gamma_{\mathcal{W}} v^* \sim v^* \Gamma_{\mathcal{W}}$.
and also the isomorphisms $\Gamma'_y f^* \xrightarrow{\sim} f'_1 \Gamma'_y$, $\Gamma'_y g^* \xrightarrow{\sim} g'_1 \Gamma'_y$.

(7.2.8.2)

\[
g'_1(v^* f^* \mathcal{F} \otimes \Lambda_x v^* f^* \mathcal{G}) \xrightarrow{g'_1(v^* f^* \mathcal{F} \otimes \mathcal{G})} g'_1(g^* u^* \mathcal{F} \otimes g^* \Lambda_x u^* \mathcal{G})
\]

For any labelled factorization $F$ of the identity map $1_{\mathcal{F}}$, we have $\chi^1_F = \chi^1_{\mathcal{F}} = q_{\mathcal{F}}$.

**Proof.** First we consider the special case where the length of $F$ is 2, say $F = (f_1, f_2)$, and where the label of $f_1$ is $P$. If the label of $f_2$ is also $P$ then the result follows from Lemma 7.2.6 while if the label is $O$, then $f_2$, which is necessarily surjective (as $f_2f_1 = 1_{\mathcal{F}}$), is an isomorphism. By Lemma 7.2.3, $\chi^1_F$ does not change if we replace the label of $f_2$ by $P$, and upon doing so, the result follows from Lemma 7.2.6.

In general, fix an integer $n \geq 2$ and let $F = (f_1, \ldots, f_n)$ be a factorization of $1_{\mathcal{F}}$ so that with $f_i : X_i \to X_{i+1}$ we have $X_1 = \mathcal{F} = X_{n+1}$. Let $r(F)$ be the largest integer between 1 and $n$ such that if $1 \leq i \leq r$ then $f_i$ has label $P$. We prove the result by descending induction on $r(F)$. If $r(F) = n$, then the result follows by...
Lemma 7.2.6. Let \( r(F) = k \) and assume that the result is true for any complex for which \( r > k \). Consider the following diagram containing a fibered square

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{Y} \\
\downarrow p_1 & & \downarrow g \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

where \( g = f_{n-1} \cdots f_1 \) is pseudoproper, \( \Delta \) is the diagonal map and \( p_i \) are the usual projections. For the map \( g \) which is drawn in parallel to \( p_2 \) we choose the factorization \( G = (f_1, \ldots, f_{n-1}) \) while for the other one we choose the length 1 factorization \( g \) itself with label \( P \). The parallel edges pick up corresponding labelled factorizations by base change: for \( p_2 \) we denote it as \( G' = (f'_1, \ldots, f'_{n-1}) \) while for \( p_1 \) it is \( p_1 \) itself with label \( P \). Finally, we assign to \( \Delta \), the label \( P \).

By the special case considered in the first para, \( \chi_{1x}^{(\Delta, p_1, G, f_n)} = \chi_{1x}^{(1x, G, f_n)} \). Therefore, by Lemma 7.2.5 and Lemma 7.2.8

\[
\chi_{1x}^{f} = \chi_{1x}^{f} = \chi_{1x}^{f} = \chi_{1x}^{f} = \chi_{1x}^{f}.
\]

\[\square\]

Using the above lemmas, Proposition 7.2.4 is proved as follows.

Proof of 7.2.4(i). Consider the following diagram with a fibered square.

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{Y} \\
\downarrow p_1 & & \downarrow f \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

We choose for the map \( f \) which is drawn parallel to \( p_1 \), the factorization \( F_1 \), and denote the induced factorization of \( p_1 \) by \( F'_1 \), while for the other \( f \) we choose \( F_2 \) as a factorization and denote the induced one on \( p_2 \) by \( F'_2 \). Assigning to \( \Delta \), the label \( P \), by Lemma 7.2.5, Lemma 7.2.8 and Lemma 7.2.9 we have

\[
\chi_{F_2}^f = \chi_{(1x, F_2)}^f = \chi_{(\Delta, F'_1, F_2)}^f = \chi_{(\Delta, F'_2, F_1)}^f = \chi_{(1x, F_1)}^f = \chi_{F_1}^f.
\]

\[\square\]

Proof of 7.2.4(ii). Let us pick labelled factorizations \( F, G \) of \( f, g \) respectively so that \( (F, G) \) is a factorization for \( gf \). It suffices to prove that the outer border of
the following diagram of obvious natural maps commutes.

\[
\begin{array}{cccc}
  f^* g^* \mathcal{F} \otimes f^* g^* \mathcal{G} & \rightarrow & f^* (g^* \mathcal{F} \otimes g^* \mathcal{G}) & \rightarrow & f^* g^* (\mathcal{F} \otimes \mathcal{G}) \\
  \downarrow \square_f & & \downarrow \square_g & & \downarrow \square_{gf} \\
  F^* g^* \mathcal{F} \otimes F^* g^* \mathcal{G} & \rightarrow & F^* (g^* \mathcal{F} \otimes g^* \mathcal{G}) & \rightarrow & F^* g^* (\mathcal{F} \otimes \mathcal{G}) \\
  \downarrow \square_g & & \downarrow \square_{gf} & & \downarrow \square_{gf} \\
  F^* G^* \mathcal{F} \otimes F^* G^* \mathcal{G} & \rightarrow & F^* (G^* \mathcal{F} \otimes G^* \mathcal{G}) & \rightarrow & F^* G^* (\mathcal{F} \otimes \mathcal{G}) \\
  \downarrow \square_{gf} & & \downarrow \square_{gf} & & \downarrow \square_{gf} \\
  (gf)^* \mathcal{F} \otimes (gf)^* \mathcal{G} & \rightarrow & (gf)^* (\mathcal{F} \otimes \mathcal{G}) & \rightarrow & (gf)^* (\mathcal{F} \otimes \mathcal{G}) \\
\end{array}
\]

Here \(\square_f, \square_g, \square_{gf}\) commute by definition of \(\chi_f, \chi_G, \chi_{(F,G)}\) respectively while the rest of the diagram commutes trivially. \(\square\)

Here are some additional properties of \(\chi\) that we need below. We begin with compatibility with flat base change. For simplicity we shall assume that the complexes have coherent homology.

**Proposition 7.2.10.** Suppose \(\sigma\) is a cartesian square as follows

\[
\begin{array}{ccc}
  \mathcal{V} & \xrightarrow{u} & \mathcal{X} \\
  \downarrow g & & \downarrow f \\
  \mathcal{U} & \xrightarrow{u} & \mathcal{Y}
\end{array}
\]

with \(f\) and \(g\) in \(\mathcal{G}\) and \(u\) flat. Then for any \(\mathcal{F}, \mathcal{G} \in \mathbf{D}_c^+(\mathcal{Y})\) the diagram \(D_\sigma\) given as follows, commutes:

\[
\begin{array}{cccc}
  v^*(f^* \mathcal{F} \otimes f^* \mathcal{G}) & \rightarrow & v^* \chi_f ^* \mathcal{F} \otimes v^* \chi_f ^* \mathcal{G} & \rightarrow & v^* f^* (\mathcal{F} \otimes \mathcal{G}) \\
  \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
  g^* u^* \mathcal{F} \otimes v^* g^* \mathcal{G} & \rightarrow & g^* u^* (\mathcal{F} \otimes \mathcal{G}) & \rightarrow & g^* u^* \mathcal{F} \otimes g^* u^* \mathcal{G} \\
  \downarrow \cong & & \downarrow \cong & & \downarrow \chi_g \\
  g^* u^* \mathcal{F} \otimes g^* u^* \mathcal{G} & \rightarrow & g^* u^* \mathcal{F} \otimes g^* u^* \mathcal{G} & \rightarrow & g^* u^* (\mathcal{F} \otimes \mathcal{G}) \\
\end{array}
\]

where the two isomorphisms displayed are the ones arising from the flat base change isomorphism \(\beta^*_u : v^* f^* \rightarrow g^* u^*\).

**Proof.** If \(f\) is an open immersion, the base-change isomorphism \(\beta^*_u\) is induced, via canonical identifications, by the pseudofunctoriality of \((-)^*\) and the same is true of \(\chi_f, \chi_g\), hence the result is obvious in this case. If \(f\) is pseudoproper, the result follows from the proof of commutativity of (7.2.8.1). In general, suppose \(f = f_2f_1\)
where \( f_i \in \mathcal{G} \), so that \( \sigma \) can be correspondingly expanded into a diagram as follows.

```
\[ \begin{array}{ccc}
V & \xrightarrow{v} & X \\
\downarrow g_1 & & \downarrow f_1 \\
W & \xrightarrow{w} & X \\
\downarrow g_2 & & \downarrow f_2 \\
U & \xrightarrow{u} & Y
\end{array} \]
```

Then checking commutativity of \( D_{\sigma} \) reduces to checking that of the outer border of the following diagram of obvious natural maps.

```
\[ \begin{array}{ccc}
v^*f_1^*f_2^* & \xrightarrow{v^*(f_1^*f_2^* \mathcal{F} \otimes f_1^*f_2^* \mathcal{G})} & v^*f_1^*(f_2^* \mathcal{F} \otimes f_2^* \mathcal{G}) \\
\downarrow & & \downarrow \\
g_1^*w^*f_2^* & \xrightarrow{g_1^*(w^*f_2^* \mathcal{F} \otimes w^*f_2^* \mathcal{G})} & v^*f_1^*(f_2^* \mathcal{F} \otimes f_2^* \mathcal{G}) \\
\downarrow & & \downarrow \\
g_1^*g_2^*u^* & \xrightarrow{g_1^*g_2^*(u^* \mathcal{F} \otimes u^* \mathcal{G})} & g_1^*g_2^*((u^* \mathcal{F} \otimes u^* \mathcal{G})) \\
\downarrow & & \downarrow \\
g_1^*(g_2^*u^* \mathcal{F} \otimes g_2^*u^* \mathcal{G}) & \xrightarrow{g_1^*(g_2^*u^* \mathcal{F} \otimes g_2^*u^* \mathcal{G})} & g_1^*g_2^*((u^* \mathcal{F} \otimes u^* \mathcal{G}))
\end{array} \]
```

Since the unlabelled parts commute trivially, we reduce to checking commutativity of \( D_{\sigma_1}, D_{\sigma_2} \). Thus if we fix a labelled factorization of \( f \) then proceeding inductively we reduce to the already-resolved case of when the length of the factorization is 1.

\[ \square \]

**Lemma 7.2.11.** Let \( \mathcal{Y} \) be a formal scheme and \( \mathcal{I} \) a coherent open \( \mathcal{O}_{\mathcal{Y}} \)-ideal. Let \( \kappa: \mathcal{X} \rightarrow \mathcal{Y} \) be the completion of \( \mathcal{X} \) with respect to \( \mathcal{I} \). Let \( \mathcal{F}, \mathcal{G} \in \mathcal{D}_c^+(\mathcal{Y}) \). Then the following diagram commutes where the vertically drawn maps are induced by the natural isomorphisms \( \kappa^* \xrightarrow{\sim} \kappa^* \) on \( \mathcal{D}_c^+(\mathcal{Y}) \) while the map in the top row is the obvious isomorphism.

```
\[ \begin{array}{ccc}
\kappa^* & \xrightarrow{\mathcal{L}} & \kappa^* \\
\downarrow & & \downarrow \\
\kappa^* & \xrightarrow{\mathcal{L}} & \kappa^*
\end{array} \]
```

**Proof.** By adjointness of \( \kappa^* \) to \( \kappa_* \mathcal{R}^!_{\mathcal{Y}} \), the assertion reduces to checking commutativity of the corresponding adjoint diagram which appears as the outer border of
For a map $f: \mathcal{X} \to \mathcal{Y}$ in $\mathcal{G}$, and $\mathcal{F}, \mathcal{G} \in \widehat{\mathcal{D}}^+_\Omega(\mathcal{Y})$, we define a conjugate version of $\chi^f(\mathcal{F}, \mathcal{G})$, denoted as $\chi^f(\mathcal{F}, \mathcal{G})$, to be the following composite of obvious natural maps

\[
(7.2.12) \quad f^* \mathcal{F} \otimes_{\mathcal{Y}} Lf^* \mathcal{G} \xrightarrow{\sim} Lf^* \mathcal{F} \otimes_{\mathcal{Y}} f^* \mathcal{G} \xrightarrow{\chi^f} f^* (\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{G}) \xrightarrow{\sim} f^*(\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{G}).
\]

7.2.13. **Transitivity, completions and traces.** We apply the abstract results of the previous subsection to relative dualizing modules.

**Lemma 7.2.14.** Suppose

\[
\mathcal{X} \xrightarrow{f_1} \mathcal{Y}_1 \xrightarrow{\varphi_2} \mathcal{Y}_2 \xrightarrow{\varphi_3} \mathcal{X}
\]

are maps in $\mathcal{G}$ with $\kappa$ a completion map with respect to an ideal $\mathcal{I}$ of $\mathcal{O}_{\mathcal{Y}_2}$. Then the following diagram commutes where the unlabelled arrows are the obvious natural
isomorphisms.

\[
\begin{array}{cccc}
\text{L}f^*\kappa^\# g^\# \mathcal{O}_X \otimes \mathcal{O}_X & \xrightarrow{f^*} & \text{L}f^* (g\kappa)^\# \mathcal{O}_X \otimes \mathcal{O}_X & \xrightarrow{\chi f} & f^* (g\kappa)^\# \mathcal{O}_X \\
\kappa^* \cong \kappa^* & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{L}f^*\kappa^\# g^\# \mathcal{O}_X \otimes \mathcal{O}_X & \xrightarrow{f^*} & \text{L}(\kappa f)^* g^\# \mathcal{O}_X \otimes \mathcal{O}_X & \xrightarrow{\chi_{(\kappa f)} f} & (\kappa f)^* g^\# \mathcal{O}_X
\end{array}
\]

Proof. It suffices to prove that the following diagram commutes since the outer border gives us the required commutativity. As before, to simplify notation, we use \( f^* \) instead of \( \text{L}f^* \) and drop the subscripts to \( \otimes \). For the definition of \( \chi \) we refer to [7.2.12].

The unlabelled parts commute for functorial reasons. Both \( \triangle \), \( \triangledown \) commute by Proposition 7.2.11(ii), namely transitivity of \( \chi \) (which also implies transitivity of \( \chi \)). Finally for \( \Delta \) we use the outer border of the following diagram where \( \mathcal{G} = g^\# \mathcal{O}_X \) and where \( \theta \) denotes the canonical isomorphism \( \mathcal{M} \otimes \mathcal{N} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{M} \).

---

Proposition 7.2.15. Suppose
is a commutative diagram of formal schemes with \( \kappa_1 \) and \( \kappa_2 \) being completions with respect to open coherent ideals of \( \mathcal{O}_Y \) and \( \mathcal{O}_X \) respectively. Then, making the identifications \( \kappa^* = \kappa_1^* \), \( \hat{f}^* \mathcal{O}_X = \hat{f}^* \mathcal{O}_Y = \hat{f}^* \mathcal{O}_X \), the following diagram commutes, with the map labelled \( \alpha \) being the isomorphism arising from \( \kappa_2^* f^* = \kappa_2^* f^* \leadsto \hat{f}^* \kappa_1^* = \hat{f}^* \kappa_1^* \).

\[
\begin{align*}
\kappa_2^*(f^* g^* \mathcal{O}_X \otimes f^* \mathcal{O}_Y) & \xrightarrow{\alpha^* \chi_f} \kappa_2^* f^* g^* \mathcal{O}_X \\
\kappa_2^* f^* g^* \mathcal{O}_X & \xrightarrow{\kappa_2^* f^* g^* \mathcal{O}_Y} \kappa_2^* (gf)^* \mathcal{O}_X
\end{align*}
\]

Proof: This follows from the outer border of the following diagram where the unlabelled parts commute trivially.

\[
\begin{align*}
\kappa_2^*(f^* g^* \mathcal{O}_X \otimes f^* \mathcal{O}_Y) & \xrightarrow{\alpha^* \chi_f} \kappa_2^* f^* g^* \mathcal{O}_X \\

\kappa_2^* f^* g^* \mathcal{O}_X & \xrightarrow{\kappa_2^* f^* g^* \mathcal{O}_Y} \kappa_2^* (gf)^* \mathcal{O}_X
\end{align*}
\]

Definition 7.2.16. Let \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \) be maps in \( \mathcal{G} \). We define \( \chi_{\langle f, g \rangle} \) to the composite of the following natural maps:

\[
\begin{align*}
\mathbf{L} f^*(g^* \mathcal{O}_X) \otimes \mathbf{L} f^* \mathcal{O}_Y & \xrightarrow{\chi_f} f^*(g^* \mathcal{O}_X \otimes g^* \mathcal{O}_Y) \\
\mathbf{L} f^* \mathcal{O}_X & \xrightarrow{\kappa_1^* + \kappa_2^*} \mathbf{L} f^* \mathcal{O}_X \xrightarrow{\kappa_2^* \chi_f} (gf)^* \mathcal{O}_Y
\end{align*}
\]

When, \( f, g \) are both pseudoproper, this definition agrees with the one given in the beginning of \( \S \).

Definition 7.2.17. A \( \xrightarrow{f} R \xrightarrow{g} S \) be pseudo-finite maps between adic rings. Let \( f : \text{Spf}(S) \to \text{Spf}(R) \) and \( g : \text{Spf}(R) \to \text{Spf}(A) \) denote the resulting maps of formal
schemes. We define

\[ \chi_{\{S/R/A\}} : \omega_{R/A}^L \otimes_R \omega_{S/R}^L \to \omega_{S/A}^L \]

to be the map corresponding to \( \chi_{\{g,f\}} \) of 7.2.16, where \( \omega^L \) is defined as in the beginning of §7.1.

**Proposition 7.2.18.** Let \( A \to R \to S \) be a pair of maps of rings, \( I \subset R, J \subset S \) ideals, such that \( A \to R/I \) and \( R \to S/J \) are finite. Let \( L = IS + J \).

(a) Suppose \( R \) is complete in the \( I \)-adic topology and \( S \) is complete in the \( L \)-adic topology (so that \( S \) is then complete in the \( J \)-adic topology too). The following diagram commutes, with \( \chi = \chi_{\{S/R/A\}} \):

\[
\begin{array}{ccc}
R \Gamma_L (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{R \Gamma_L (\chi)} & R \Gamma_L \omega_{S/A}^L \\
\downarrow & & \downarrow \text{Tr}_{S/A} \\
R \Gamma_I R \Gamma_J (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{R \Gamma_I (1 \otimes \text{Tr}_{S/J})} & R \Gamma_I \omega_{R/A}^L \\
\downarrow & & \downarrow \text{Tr}_{R/A} \\
R \Gamma_I (\omega_{R/A}^L \otimes_R R \Gamma_J (\omega_{S/R}^L)) & \xrightarrow{R \Gamma_I (1 \otimes \text{Tr}_{J})} & R \Gamma_I \omega_{R/A}^L
\end{array}
\]

(b) Suppose the topology on \( A, R, \) and \( S \) are discrete, and \( A \to R \), \( R \to S \) are of finite type. Then the following diagram commutes with \( \chi = \chi_{\{S/R/A\}} \):

\[
\begin{array}{ccc}
R \Gamma_L (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{R \Gamma_L (\chi)} & R \Gamma_L \omega_{S/A}^L \\
\downarrow & & \downarrow \text{Tr}_L \\
R \Gamma_I R \Gamma_J (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{R \Gamma_I (1 \otimes \text{Tr}_{J})} & R \Gamma_I \omega_{R/A}^L \\
\downarrow & & \downarrow \text{Tr}_I \\
R \Gamma_I (\omega_{R/A}^L \otimes_R R \Gamma_J (\omega_{S/R}^L)) & \xrightarrow{R \Gamma_I (1 \otimes \text{Tr}_{J})} & R \Gamma_I \omega_{R/A}^L
\end{array}
\]

**Proof.** For part (a), first note that the diagram below commutes:

\[
\begin{array}{ccc}
R \Gamma_L (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{\sim} & \omega_{R/A}^L \otimes_R R \Gamma_L (\omega_{S/R}^L) \\
\downarrow & & \downarrow \\
R \Gamma_I R \Gamma_J (\omega_{R/A}^L \otimes_R \omega_{S/R}^L) & \xrightarrow{\sim} & \omega_{R/A}^L \otimes_R R \Gamma_I R \Gamma_J (\omega_{S/R}^L) \\
\downarrow & & \\
R \Gamma_I (\omega_{R/A}^L \otimes_R R \Gamma_J (\omega_{S/R}^L))
\end{array}
\]

The assertion now follows from the commutativity of (7.1.4.3) and the definition of \( \chi_{\{S/R/A\}} \).
For part (b), let \( \hat{R} \) be the completion of \( R \) with respect to \( I \), \( S' \) the completion of \( S \) with respect to \( J \) and \( \hat{S} \) the completion of \( S \) with respect to \( L \). Let \( \bar{I} = I \hat{R} \), \( J' = J S' \), \( L' = L S' \), \( \hat{L} = L \hat{S} \). Let \( \mathcal{X} = \text{Spf}(\hat{S}, \hat{L}) \), \( X' = \text{Spf}(S', J') \), \( X = \text{Spec} S \), \( \mathcal{Y} = \text{Spf}(\hat{R}, \bar{I}) \), \( Y = \text{Spec} R \), and \( Z = \text{Spec} A \). The various natural relations between the adic rings can be represented by a commutative diagram of formal schemes:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\kappa_1} & X' \\
\downarrow{\bar{f}} & & \downarrow{f'} \\
\mathcal{Y} & \xrightarrow{\kappa_2} & X \\
\downarrow{g} & & \downarrow{f} \\
\mathcal{Y} & \xrightarrow{\kappa_3} & Y \\
\downarrow{\bar{g}} & & \downarrow{g} \\
Z & & Z
\end{array}
\]

We have:

\[
\kappa_1^* f^* \mathcal{O}_Y \cong (\kappa_1 f')^* \mathcal{O}_{X'} \cong \bar{f}^* \kappa_2^* \mathcal{O}_X = \bar{f}^* \mathcal{O}_\mathcal{Y}.
\]

Moreover, \( \kappa_2^* f^* \mathcal{O}_Y \cong f^* \mathcal{O}_Y \). We may thus make the following identifications:

\[
\omega_{S/R}^* \otimes_S \hat{S} = \omega_{S/R}^* \otimes_S \hat{S} = \omega_{S/R}^*, \quad \text{and} \quad \omega_{S'/R}^* = \omega_{S'/R}^* \otimes_S S'.
\]

The natural isomorphism \( \kappa_3^* \mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Z \) allows us to make the identification \( \omega_{\hat{R}/A}^* = \omega_{R/A}^* \otimes_R \hat{R} \).

Let us write \( \chi = \chi_{[S/R/A]} \) and \( \bar{\chi} = \chi_{[\hat{S}/R/A]} \). The above identifications and Proposition \( \text{[7.2.15]} \) gives \( \chi \otimes \hat{S} = \bar{\chi} \), whence the following diagram commutes:

\[
\begin{array}{ccc}
\text{R}\Gamma_L(\omega_{R/A}^* \otimes_R \omega_{S/R}^*) & \xrightarrow{\chi} & \text{R}\Gamma_L \omega_{S/A}^* \\
\downarrow & & \downarrow \\
\text{R}\Gamma_L(\omega_{\hat{R}/A}^* \otimes_R \omega_{\hat{S}/R}^*) & \xrightarrow{\bar{\chi}} & \text{R}\Gamma_L \bar{\omega}_{S/A}^*
\end{array}
\]

By part (a), it is therefore enough to prove that the diagram below commutes:

\[
\begin{array}{ccc}
\text{R}\Gamma_L(\omega_{R/A}^* \otimes_R \omega_{S/R}^*) & \xrightarrow{\sim} & \text{R}\Gamma_L(\omega_{R/A}^* \otimes_R \omega_{\hat{S}/\hat{R}}^*) \\
\downarrow & & \downarrow \\
\text{R}\Gamma_I \text{R}\Gamma_J(\omega_{R/A}^* \otimes_R \omega_{S/R}^*) & \xrightarrow{\sim} & \text{R}\Gamma_I \text{R}\Gamma_J(\omega_{R/A}^* \otimes_R \omega_{\hat{S}/\hat{R}}^*) \\
\downarrow & & \downarrow \\
\text{R}\Gamma_I(\omega_{R/A}^* \otimes_R \text{R}\Gamma_J(\omega_{S/R}^*)) & \xrightarrow{\sim} & \text{R}\Gamma_I(\omega_{R/A}^* \otimes_R \text{R}\Gamma_J(\omega_{\hat{S}/\hat{R}}^*)) \\
\downarrow & & \downarrow \\
\text{R}\Gamma_I(\omega_{R/A}^*) & \xrightarrow{\sim} & \text{R}\Gamma_I(\omega_{R/A}^*).
\end{array}
\]
The proof of the commutativity of (**) is as follows. Suppose \( F \) is a bounded-below complex of \( R \)-modules with finitely generated cohomology and \( G \) is a bounded-below complex of finitely generated \( S \)-modules, we have a bifunctorial commutative diagram (with \( \tilde{F} = F \otimes_R \tilde{R}, G' = G \otimes_S S' \), and \( \tilde{G} = G \otimes_S \tilde{S} \)):

\[
\begin{array}{c}
\text{R} \Gamma_L(F \otimes_R G) \xrightarrow{\sim} \text{R} \Gamma_L(F \otimes_R G') \xrightarrow{\sim} \text{R} \Gamma_L(\tilde{F} \otimes_{\tilde{R}} \tilde{G}) \\
\text{R} \Gamma_J(\tilde{F} \otimes_{\tilde{R}} \tilde{G}) \xrightarrow{\sim} \text{R} \Gamma_J(F \otimes_R G') \xrightarrow{\sim} \text{R} \Gamma_J(\tilde{F} \otimes_{\tilde{R}} \tilde{G}) \\
\text{R} \Gamma_J(F \otimes_R \text{R} \Gamma_J G) \xrightarrow{\sim} \text{R} \Gamma_J(F \otimes_R \text{R} \Gamma_J G') \xrightarrow{\sim} \text{R} \Gamma_J(\tilde{F} \otimes_{\tilde{R}} \tilde{G})
\end{array}
\]

This shows that the top two rectangles in (**) commute. For the rest of (**) it is enough to show that the following diagram commutes:

\[
\begin{array}{c}
\text{R} \Gamma_I(\omega^*_{R/A} \otimes_R \text{R} \Gamma_J(\omega^*_{S/R})) \xrightarrow{\text{Tr}_J} \text{R} \Gamma_I(\omega^*_{R/A}) \\
\text{R} \Gamma_I(\omega^*_{R/A} \otimes_R \text{R} \Gamma_J(\omega^*_{S'/R})) \xrightarrow{\text{Tr}_{S'/R}} \text{R} \Gamma_I(\omega^*_{R/A}) \xrightarrow{A[0]} A[0] \\
\text{R} \Gamma_I(\omega^*_{R/A} \otimes_R \text{R} \Gamma_J(\omega^*_{S/\tilde{R}})) \xrightarrow{\text{Tr}_{\tilde{R}/R}} \text{R} \Gamma_I(\omega^*_{R/A})
\end{array}
\]

The rectangle on the top commutes by definition of \( \text{Tr}_J \). The triangle on the right end of the diagram commutes by definition of \( \text{Tr}_I \). The rectangle at the bottom commutes by flat base change, since the following diagram is cartesian:

\[
\begin{array}{c}
\text{Spf}(\tilde{S}, \tilde{J}) \xrightarrow{\square} \text{Spf}(S', J') \\
\text{Spf}(\tilde{R}, \tilde{I}) \xrightarrow{\square} \text{Spec } R
\end{array}
\]

\[\square\]

8. Iterated residues

8.1. Comment on Translations. This is more of an orienting remark. Suppose \( M \) and \( N \) are \( \mathcal{O}_X \)-modules on a ringed space \( (X, \mathcal{O}_X) \), and \( d, e \) are integers. According to [La 4, pp.28–29, (1.5.4)] the functor \( F_{N[d]} = (-) \otimes N[d] \) on the homotopy category of complexes of \( \mathcal{O}_X \)-modules is triangle preserving with the isomorphism \( \mathcal{A}^*[1] \otimes N[d] \rightarrow (\mathcal{A}^* \otimes N[d])[1] \) being the identity map ("without the intervention of signs" in the language of [C1]). Signs do intervene if the first argument in the tensor product is fixed and the second varies. However, if the fixed first argument is an \( \mathcal{O}_X \)-module, i.e., a complex concentrated in the 0th-spot, then signs do not intervene. More precisely, \( G_M = M \otimes (-) \) is triangle preserving, for the identity
isomorphism $M \otimes (B^\bullet[1]) \simto (M \otimes B^\bullet)[1]$. The same sign conventions apply for the derived tensor product on the derived category, see [L4 pp.62–63, (2.5.7)].

For complexes of $\mathcal{O}_X$-modules $A^\bullet$ and $B^\bullet$, let

$$\theta^j: (A^\bullet[i]) \otimes (B^\bullet[j]) \simto (A^\bullet \otimes B^\bullet)[i+j]$$

be as in [L4 pp.28–29, (1.5.4)]. Then the following composite is a composite of identity maps and hence is the identity map.

$$\overset{(8.1.1)}{\xymatrix{M[e] \otimes N[d] \ar[r]_{\theta^{d+e}} & (M \otimes N)[d][e] \ar[r]_{\theta^{d+e}} & (M \otimes N)[d+e] \ar[r] & (M \otimes N)[d+e].}}$$

(Strictly speaking, (8.1.1) is the identity map when the tensor product is in the ordinary category of complexes; over the derived category, the induced map on the homology in degree $-(d+e)$ canonically identifies with the identity map. In particular, if either of $M, N$ is flat as $\mathcal{O}_X$-modules, then (8.1.1), viewed as a derived-category map, also canonically identifies with identity.)

Thus, given a map $\bar{\psi}: M \otimes N \rightarrow T$ of $\mathcal{O}_X$-modules, we get a map in $D(X)$

$$\overset{(8.1.2)}{\xymatrix{\psi: M[e] \otimes N[d] \ar[r] & T[d+e]}}$$

given by $(\bar{\psi}[d+e]) \circ (8.1.1)$. The maps $\bar{\psi}$ and $\psi$ determine each other. Indeed, $\psi = H^{-(d+e)}(\bar{\psi})$.

### 8.2. Iterated generalized fractions.

Let $R$ be a (noetherian) ring, $I \subset R$ an ideal generated by $u = (u_1, \ldots, u_d)$. For any $R$-module $M$ we have a map of complexes

$$\overset{(8.2.1)}{\xymatrix{M[d] \otimes_R K^\bullet_\infty(u) \ar[r] & H^d_I(M)[0].}}$$

defined on 0-cochains by

$$\overset{m}{m \otimes \frac{1}{u_1^{\alpha_1} \cdots u_d^{\alpha_d}}} \rightarrow (-1)^d \left[\begin{array}{c} m \\ u_1^{\alpha_1} \ldots u_d^{\alpha_d} \end{array}\right].$$

This is a map of complexes since every 0-cochain of $M[d] \otimes_R K^\bullet_\infty(u)$ (and of $H^d_I(M)[0]$) is a 0-cocycle and because $H^d_I(M)[0]$ is a complex concentrated only in degree 0. In the event $M$ is a free $R$-module and $u$ is a quasi-regular sequence (or if $u$ is locally an $M$-sequence), (8.2.1) is a quasi-isomorphism. The map (8.2.1) is functorial in $M$.

The above is a map of complexes, i.e., a morphism in the category $C(\text{Mod}_R)$. There is an analogous map in $D(\text{Mod}_R)$ described as follows. Since $H^j_I(M) = 0$ for $j > d$, there is a canonical map in $D(\text{Mod}_R)$,

$$\overset{(8.2.2)}{\xymatrix{\phi_{R,I}(M) \colon R\Gamma_I(M[d]) \ar[r] & H^d_I(M)[0]}}$$

such that $H^0(\phi_{R,I}(M))$ is the identity map on $H^d_I(M)$. One checks, using the definition of the generalized fraction $[u_1^{\alpha_1} \cdots u_d^{\alpha_d}]$, that the following diagram commutes in $D(\text{Mod}_R)$

$$\overset{(8.2.3)}{\xymatrix{M[d] \otimes_R K^\bullet_\infty(u) \ar[r] \ar[d]_{\phi_{R,I}} & \ar[d]^{\phi_{R,I}} R\Gamma_I(M[d]) \ar[r] & \ar[d] R\Gamma_I(M[d]) \ar[r]_{\phi_{R,I}} & H^d_I(M)[0].}}$$
Next, suppose $S$ is an $R$-algebra and $J \subset S$ is an $S$-ideal generated by $v = (v_1, \ldots, v_e)$. Suppose $N$ is an $R$-module. We have an isomorphism

$$(8.2.4) \quad H_{IS+J}^{d+e}(M \otimes_R N) \xrightarrow{\sim} H_{I}^{d}(M \otimes_R H_{J}^{e}(N))$$

given by

$$(8.2.5) \quad \left[ \begin{array}{c} m \otimes n \\ v^{\beta_1}_1, \ldots, v^{\beta_e}_e, u^{\alpha_1}_1, \ldots, u^{\alpha_d}_d \\ \end{array} \right] \mapsto \left[ \begin{array}{c} m \otimes n \\ v^{\beta_1}_1, \ldots, v^{\beta_e}_e, u^{\alpha_1}_1, \ldots, u^{\alpha_d}_d \\ \end{array} \right]$$

We claim that the following diagram commutes where we identify $M[d] \otimes_R N[e]$ with $(M \otimes_R N)[d + e]$ as in (8.1.1):

$$(8.2.6) \quad M[d] \otimes_R N[e] \otimes_S K^{\bullet}_\infty(v, u, S) \xrightarrow{\text{S.3.1}} H_{IS+J}^{d+e}(M \otimes N)[0]$$

Indeed, consider a 0-cocycle $m \otimes n \otimes \frac{1}{v^{\beta_1}_1 \ldots v^{\beta_e}_e u^{\alpha_1}_1 \ldots u^{\alpha_d}_d}$ of $M[d] \otimes N[e] \otimes K^{\bullet}_\infty(v, u, S)$. Its image along either possible route (east-followed-by-south or south-followed-by-east) is $(-1)^{d+e}[m \otimes n \otimes \frac{1}{v^{\beta_1}_1 \ldots v^{\beta_e}_e u^{\alpha_1}_1 \ldots u^{\alpha_d}_d}]$. This proves that (8.2.3) commutes.

The above is diagram in the category of complexes $C(\text{Mod}_R)$. This can be upgraded to the following:

**Proposition 8.2.7.** Suppose $S$-modules $N$ and $H_{J}^{e}(N)$ are flat over the ring $R$, so that $- \otimes_R N[e] = - \otimes_R N[e]$ and $- \otimes_R H_{J}^{e}(N)[0] = - \otimes_R H_{J}^{e}(N)$. Then the following diagram commutes in $D(\text{Mod}_R)$:

$$\begin{array}{c}
\text{R} \Gamma_{IS+J}(M[d] \otimes_R N[e]) \xrightarrow{\phi_{IS+J}} H_{IS+K}^{d+e}(M \otimes N)[0] \\
\text{R} \Gamma_{I}(M[d] \otimes_R \text{R} \Gamma_{J} N[e]) \\
\text{R} \Gamma_{I}(M[d] \otimes_R \text{H}^{e}_{J}(N)[0]) \xrightarrow{\phi_{R,I}} H_{I}^{d}(M \otimes_R \text{H}^{e}_{J}(N))[0]
\end{array}$$

**Proof.** This is a straightforward re-interpretation of the commutativity of (8.2.6) using the commutativity of (8.2.3). □

**Remarks 8.2.8.** (1) The assumptions on the flatness of $N$ and $H_{J}^{e}(N)$ over $R$ are perhaps not necessary in view of (8.2.6), but since we are dealing now with
objects and functors in the derived category \( \mathbf{D}(\text{Mod}_R) \), we have to make sure that the arrows in the diagram in the proposition make sense. It was not clear to us that the second and third arrows in the column on the left are meaningful in the derived category without our assumptions.

(2) Proposition 8.2.7 can be interpreted as saying 8.2.4 is the isomorphism given by the Leray spectral sequence for the composite functor \( \Gamma_{IS+J} = \Gamma_I \circ \Gamma_J \).

See [LS, Proposition (3.3.1)] as well as the correction by the second author.

8.3. Cohom-Macaulay maps and iterated residues. Suppose \( X = \text{Spec} \, S \), \( Y = \text{Spec} \, R \), \( Z = \text{Spec} \, A \) are affine schemes, and \( f: X \to Y \) is Cohen-Macaulay of relative dimension \( e \), \( g: Y \to Z \) is Cohen-Macaulay of relative dimension \( d \). Note that we have finite type maps of rings \( A \to R \) and \( R \to S \). Suppose \( I \subset R \) and \( J \subset S \) are as in Subsection 8.2 with the added condition that the given generators of \( I \) and \( J \), namely \( u = (u_1, \ldots, u_d) \) and \( v = (v_1, \ldots, v_e) \) respectively are quasi-regular, and that \( A \to R/I \) and \( R \to S/J \) are finite.

Since \( A \to R \) and \( R \to S \) are flat with Cohen-Macaulay fibres, under our hypotheses, \( A \to R/I \) and \( R \to S/J \) are finite and flat, i.e., Cohen-Macaulay of relative dimension 0. Let \( L = IS + J \), \( W_1 = \text{Spec} \, S/J \hookrightarrow X \), \( W_2 = \text{Spec} \, R/I \hookrightarrow Y \), and \( W = W_1 \cap f^{-1}(W_2) = \text{Spec} \, S/L \hookrightarrow X \).

In what follows, for \( M \in \text{Mod}_R \) and \( N \in \text{Mod}_S \) we make the standard identifications, \( H^j_M = H^j_{W_2}(Y, \overline{M}) \), \( H^j_N = H^j_{W_1}(X, \overline{N}) \), and \( H^{d+e}_N = H^{d+e}(\overline{N}) \). We remind the reader that \( \omega^*_R/\mathfrak{a} = \omega^*_R[\mathfrak{a}] \), \( \omega^*_S/\mathfrak{a} = \omega^*_S[\mathfrak{a}] \), and \( \omega^*_S/\mathfrak{a} = \omega^*_S[\mathfrak{a}] \).

Finally, let \( \hat{R} \) be the \( I \)-adic completion of \( R \), \( \hat{S} \) the \( L \)-adic completion of \( S \), and \( \hat{S}^* \) the \( J \)-adic completion of \( S \). Let \( \hat{J} = \hat{I} \hat{S} \), \( \hat{L} = \hat{L} \), and \( \hat{I} = I \hat{R} \). Let \( \hat{R} \to \hat{S} / \hat{J} \) is finite. Let \( \omega^*_S \otimes \hat{R} \), \( \omega^*_S \otimes \hat{R} \), \( \omega^*_S \otimes \hat{R} \), and \( \omega^*_R \otimes \hat{R} \) give rise, on applying the cohomology functor \( \hat{H}^i(-) \) to maps \( \Pi_\hat{S} \): \( H^{d+e}_L(\omega^*_S/\mathfrak{a}) \to A \), \( \Pi_\hat{R} : H^{d+e}_L(\omega^*_S/\mathfrak{a}) \to \hat{R} \), and \( \Pi_\hat{R} : H^{d+e}_L(\omega^*_S/\mathfrak{a}) \to A \).

Proposition 8.3.1. Let notations be as above.

(a) The following diagram commutes, with \( \chi = \chi_{\hat{g}, \hat{f}} \):

\[
\begin{array}{ccc}
H^{d+e}_L(\omega^*_R/\mathfrak{a} \otimes_R \omega^*_S/\mathfrak{a}) & \xrightarrow{H^{d+e}_L(\chi)} & H^{d+e}_L(\omega^*_S/\mathfrak{a}) \\
\downarrow \text{res}^*_R & & \downarrow \text{res}^*_R \\
H^d_L(\omega^*_R/\mathfrak{a} \otimes_R \omega^*_S/\mathfrak{a}) & & A
\end{array}
\]
Proof. Part (a) is mainly a re-statement of Proposition 7.2.18 (b), with Prop. 8.2.7 explaining how (8.2.4) enters into the picture. Before giving more details, we make some observations. First, let $J_n$ denote the $S$-ideal generated by $(v_1^\alpha, \ldots, v_l^\alpha)$. Then $S/J_n$ is finite and flat over $R$, and hence is Cohen-Macaulay of relative dimension 0 over $R$. This means that the relative dualizing module for the algebra $R \to S/J_n$, i.e., $\omega_{S/R}^\sharp \otimes_S \bigotimes_{J_n}^\sharp$, is flat over $R$, whence so its direct limit over $n$, namely $H^e_J(\omega_{S/R}^\sharp)$.

Since $f$, $g$ and $gf$ are Cohen-Macaulay of relative dimensions $e$, $d$, and $d + e$ respectively, the maps $\phi_{S,J}(\omega_{S/R}^\sharp)$, $\phi_{R,I}(\omega_{R/A}^\sharp)$, and $\phi_{S,L}(\omega_{S/A}^\sharp)$ are all isomorphisms. Moreover, since $H_J^e(\omega_{S/R}^\sharp)$ is flat, the map $\phi_{R,I}(\omega_{R/A}^\sharp \otimes_R H_J^e(\omega_{S/R}^\sharp))$ is also an isomorphism.

Since $\omega_{S/R}^\sharp$ and $H_J^e(\omega_{S/R}^\sharp)$ are both flat over $R$, we can apply Proposition 8.2.7 with $M = \omega_{R/A}^\sharp$, $N = \omega_{S/R}^\sharp$. Using the isomorphisms $\phi_{S,J}(\omega_{S/R}^\sharp)$, $\phi_{R,I}(\omega_{R/A}^\sharp)$, and $\phi_{S,L}(\omega_{S/A}^\sharp) = \phi_{R,J}(\omega_{R/A}^\sharp \otimes_R H_J^e(\omega_{S/R}^\sharp))$, and applying Proposition 8.2.7, our assertion is equivalent to the commutativity of the diagram in part (b) of Proposition 7.2.18, this proves (a).

The proof of (b) is identical, with part (a) of Proposition 7.2.18 replacing part (b) of loc.cit. □

Proposition 8.3.1 gives rise to two related residue formulas. The following is a consequence of part (a) of the proposition and the formula for the map (8.2.4) given in (8.2.3). For $\mu \in \omega_{R/A}^\sharp$ and $\nu \in \omega_{S/R}^\sharp$ and for integers $\alpha_l > 0$, $\beta_k > 0$, $l \in \{1, \ldots, d\}$, $k \in \{1, \ldots, e\}$, we have

$$ \text{res}^*_{w_2} \left[ \text{res}^*_{\omega_{1}} \left[ \frac{v_1^{\beta_1}, \ldots, v_d^{\beta_d}}{u_1^{\alpha_1}, \ldots, u_d^{\alpha_d}} \right] \mu \right]_{\nu} = \text{res}^*_{w} \left[ \frac{\chi_{(S/R/A)}(\mu \otimes \nu)}{v_1^{\alpha_1}, \ldots, v_e^{\alpha_e}, u_1^{\alpha_1}, \ldots, u_d^{\alpha_d}} \right] $$

Similarly, for $\mu \in \omega_{R/A}^\sharp$ and $\nu \in \omega_{S/R}^\sharp$ and $\alpha_l$, $\beta_k$ as above, we have by part (b) of the proposition, and the formula for the map (8.2.4) given in (8.2.5),

$$ \text{tr}^*_{R/A} \left[ \text{tr}^*_{S/R} \left[ \frac{v_1^{\beta_1}, \ldots, v_d^{\beta_d}}{u_1^{\alpha_1}, \ldots, u_d^{\alpha_d}} \right] \mu \right]_{\nu} = \text{tr}^*_{S/A} \left[ \frac{\chi_{(S/R/A)}(\mu \otimes \nu)}{v_1^{\alpha_1}, \ldots, v_e^{\alpha_e}, u_1^{\alpha_1}, \ldots, u_d^{\alpha_d}} \right] $$

Remark 8.3.4. We will apply part (b) of the Proposition 8.3.1 in a later paper in the following situation. Let $R = A[u_1, \ldots, u_d]$, $S = R[v_1, \ldots, v_e]$ where $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_e)$ are algebraically independent variables over $A$
and \( R \) respectively. Let \( I = uR \), and \( J = vS \). Then \( \hat{R} = A[[u]] \) and \( \hat{S} = R[[v]] = A[[u, v]] \).

**APPENDIX A. BASE CHANGE AND COMPLETIONS**

**A.1.** We gather a few basic properties of the flat base-change map of (3.2.1). By default, we work with complexes in \( \widetilde{D}_c^{+} \).

Consider a cartesian square

\[
\begin{array}{c}
\mathcal{V} \\
v \\
g \\
\mathcal{W}
\end{array} \quad \begin{array}{c}
\mathcal{K} \\
f \\
\mathcal{X}
\end{array} \quad \begin{array}{c}
\mathcal{Y}
\end{array}
\]

with \( f \) in \( \mathcal{G} \) and \( u \) flat so that we have a flat-base-change isomorphism

\[ \beta_s^\#: \Lambda_{\mathcal{V}} v^* f^# \sim \sim g^# u^* \]

as in (3.2.1). If \( f \) (and hence \( g \)) is pseudoproper, then another description of \( \beta_s^\# \) is that it is the map adjoint to the composite of the following natural maps (cf. [AJL2, Theorem 8.1, p. 86]).

\[
Rg_* R\Gamma_{\mathcal{V}}' v^* f^# \sim \sim Rg_* R\Gamma_{\mathcal{V}}' v^* f^# \rightarrow Rg_* v^* R\Gamma_{\mathcal{V}}' f^# \sim \sim u^* Rf_* R\Gamma_{\mathcal{X}}' f^# \rightarrow u^*
\]

If \( f, g \) are formally étale, then we have natural isomorphisms \( f^# \sim \sim \Lambda_{\mathcal{V}} f^* \) and \( g^# \sim \sim \Lambda_{\mathcal{Y}} g^* \) induced by the corresponding ones for \((-)^! \), \( f^! \sim \sim R\Gamma_{\mathcal{Y}}' f^* \) and \( g^! \sim \sim R\Gamma_{\mathcal{Y}}' g^* \) respectively. In this case, the base-change map \( \beta_s^! \) for \((-)^! \) is induced by the composite of the canonical isomorphisms

\[
R\Gamma_{\mathcal{Y}}' v^* f^! \sim \sim R\Gamma_{\mathcal{Y}}' v^* R\Gamma_{\mathcal{Y}}' f^* \sim \sim R\Gamma_{\mathcal{Y}}' v^* f^* \rightarrow R\Gamma_{\mathcal{Y}}' g^* u^* \sim \sim g^! u^*.
\]

Hence another description of \( \beta_s^* \) is that it is given by the composite of the following isomorphisms

\[
\Lambda_{\mathcal{V}} v^* f^# \sim \sim \Lambda_{\mathcal{Y}} v^* \Lambda_{\mathcal{X}} f^* \sim \sim \Lambda_{\mathcal{Y}} v^* f^* \sim \sim \Lambda_{\mathcal{Y}} g^* u^* \sim \sim g^# u^*.
\]

In particular, if \( \mathcal{F} \in D_c^+(\mathcal{Y}) \), or if \( u \) is open or if \( \mathcal{V} \) is an ordinary scheme, then \( \beta_s^!(\mathcal{F}) \) is given by the natural composite

\[ v^* f^# \mathcal{F} \sim \sim v^* f^* \mathcal{F} \sim \sim g^* u^* \mathcal{F} \sim \sim g^# u^* \mathcal{F}. \]

Next we look at transitivity properties of \( \beta^# \) vis-à-vis extension of the square \( s \) horizontally or vertically. These are also proved by reducing to the corresponding property for \( \beta^! \) (see [Nay Theorem 2.3.2(i)]).

**Proposition A.1.1.**

(i) Consider cartesian squares \( s_1, s_2 \) as follows

\[
\begin{array}{c}
\mathcal{V}_2 \\
v_2 \\
h \end{array} \quad \begin{array}{c}
\mathcal{V}_1 \\
v_1 \\
g \\
\mathcal{W}_1 \\
u_1 \\
f \\
\mathcal{W}_2 \\
u_2 \\
\mathcal{X}
\end{array} \quad \begin{array}{c}
\mathcal{Y}
\end{array}
\]

...
where \( f, g, h \) are in \( \mathbb{G} \) and \( u_i, v_i \) are flat. Let \( u = u_1 u_2 \) and \( v = v_1 v_2 \) and let \( s \) denote the composite cartesian diagram. Then the following diagram of isomorphisms commutes.

\[
\begin{array}{cccccc}
\Lambda v_2^* v_1^* f^* & \rightarrow & \Lambda v_2^* v_1^* f^* & \rightarrow & \Lambda v_2^* g^* u_1^* & \rightarrow & h^* u_1^* \\
\Lambda v_2^* f^* & \rightarrow & \Lambda v_2^* f^* & \rightarrow & g^* u_1^* & \rightarrow & h^* u_1^*
\end{array}
\]

(ii) Consider cartesian squares \( s_1, s_2 \) as follows

\[
\begin{array}{cccccc}
\gamma_2 & \overset{w}{\rightarrow} & \mathcal{D}_2 \\
\gamma_2 & \downarrow & \gamma_2 \\
\gamma_1 & \overset{v}{\rightarrow} & \mathcal{D}_1 \\
\gamma_1 & \downarrow & \gamma_1 \\
\mathcal{W} & \overset{u}{\rightarrow} & \mathcal{W}
\end{array}
\]

where \( f_i, g_i \) are in \( \mathbb{G} \) and \( u, v, w \) are flat. Let \( f = f_1 f_2 \) and \( g = g_1 g_2 \) and let \( s \) denote the composite cartesian diagram. Then the following diagram of isomorphisms commutes.

\[
\begin{array}{cccccc}
\Lambda w^* f_1 f_2^* & \rightarrow & \Lambda g_2^* g_1^* f_1^* & \rightarrow & \Lambda g_2^* g_1^* f_1^* & \rightarrow & g_1^* u_1^* \\
\Lambda w^* f^* & \rightarrow & \Lambda w^* f_1 f_2^* & \rightarrow & g_1^* u_1^* & \rightarrow & g_1^* u_1^*
\end{array}
\]

**Proof.** (i). For convenience we shall consider the transposed version of the diagram in question. Using the definitions \( f^* = \Lambda f, g^* = \Lambda g, h^* = \Lambda h \) and the isomorphisms in (2.1.1) we reduce to checking that the outer border of the following diagram of isomorphisms commutes where to reduce clutter we have dropped the \( R \)'s.

\[
\begin{array}{cccccc}
\Lambda v_2^* v_1^* f^1 & \rightarrow & \Lambda v_2^* v_1^* f^1 & \rightarrow & \Lambda v_2^* v_1^* f^1 & \rightarrow & \Lambda v_2^* v_1^* f^1 \\
\Lambda v_2^* f^1 & \rightarrow & \Lambda v_2^* f^1 & \rightarrow & \Lambda v_2^* f^1 & \rightarrow & \Lambda v_2^* f^1 \\
\Lambda v_2^* g^1 u_1^1 & \rightarrow & \Lambda v_2^* g^1 u_1^1 & \rightarrow & \Lambda v_2^* g^1 u_1^1 & \rightarrow & \Lambda v_2^* g^1 u_1^1 \\
\Lambda g^1 & \rightarrow & \Lambda g^1 & \rightarrow & \Lambda g^1 & \rightarrow & \Lambda g^1
\end{array}
\]

The unlabelled arrows are obvious natural maps. The rectangle \( \square \) commutes by the transitivity of base-change for \((-)^1 \). Commutativity of the remaining parts is obvious.
(ii). Once again we consider the transpose of the diagram under consideration. Using the isomorphisms in (2.1.1) we reduce to checking that the outer border of the following diagram of isomorphisms commutes.

\[
\begin{array}{ccc}
\Lambda_{y_2}w^*\Lambda_{x_2}f_2^1\Lambda_{y_1}f_1 & \longrightarrow & \Lambda_{y_2}w^*\Lambda_{x_2}f_2^1 \\
\Lambda_{y_2}\Gamma_{y_2}w^*f_2^1\Lambda_{x_1}f_1 & \longrightarrow & \Lambda_{y_2}\Gamma_{y_2}w^*f_2^1 \\
\Lambda_{y_2}g_2^1v^*\Lambda_{x_1}f_1 & \longrightarrow & \Lambda_{y_2}g_2^1v^*f_1 \\
\Lambda_{y_2}g_2^1\Gamma_{y_1}v^*f_1 & \longrightarrow & \Lambda_{y_2}g_2^1\Gamma_{y_1}v^*f_1 \\
\Lambda_{y_2}g_2^1\Lambda_{y_1}g_1^1u^* & \longrightarrow & \Lambda_{y_2}g_2^1\Lambda_{y_1}g_1^1u^* \\
\Lambda_{y_2}g_2^1\Lambda_{y_1}g_1^1u^* & \longrightarrow & \Lambda_{y_2}g_2^1\Lambda_{y_1}g_1^1u^*
\end{array}
\]

The rectangle \(\boxplus\) commutes by transitivity of base-change for \((-)^!\) while the other rectangles commute for obvious reasons.

Completion maps, being pseudo-proper, formally étale, and flat, give rise to additional compatibility issues. Now we consider some special situations involving completion maps.

Let \(\mathcal{X}\) be a formal scheme and \(\mathcal{I} \subset \mathcal{O}_\mathcal{X}\) an open coherent ideal. Let \(\mathcal{W} := \hat{\mathcal{X}}\) be the completion of \(\mathcal{X}\) along \(\mathcal{I}\) and \(\kappa: \mathcal{W} \to \mathcal{X}\) the corresponding completion map. Then there are canonical isomorphisms (see proof of [AJL3 Lemma 4.1], and of [AJL2 Proposition 5.2.4])

\[
(A.1.2) \quad \kappa_*R\Gamma^w_\mathcal{W} \kappa_*^* \cong \kappa_*\kappa_*^* R\Gamma_\mathcal{X} \cong R\Gamma_\mathcal{X} \kappa_* \kappa_*^* \Leftarrow \kappa_* R\Gamma_\mathcal{X}.
\]

For the next two results regarding \((-)^\kappa\) for completion maps, we will first need to look at the corresponding results for \((-)^!\). For that purpose we recall that in [Nay], \((-)^!\) is obtained by gluing the pseudofunctor \((-)^\kappa\) over pseudoproper maps in \(\mathcal{G}\) given by

\[
f^\kappa = \text{right adjoint to } Rf_*, \quad (f \text{ pseudoproper}),
\]

with the pseudofunctor \((-)^!*\) over étale maps in \(\mathcal{G}\) given by

\[
f \mapsto R\Gamma^w_f f_*, \quad (f: \mathcal{X} \to \mathcal{Y} \text{ étale}),
\]

and this gluing utilizes, among other things, the étale base-change isomorphisms associated to cartesian squares involving étale base change of a pseudoproper map (see [Nay] Theorem 7.1.6, §7.2.7]).

**Lemma A.1.3.** For a completion map \(\kappa: \mathcal{W} \to \mathcal{X}\) by an open coherent ideal \(\mathcal{I} \subset \mathcal{O}_\mathcal{X}\) as above and for \(\mathcal{F} \in D^+_c(\mathcal{X})\), the isomorphism \(\kappa^* \mathcal{F} \to \kappa^* \mathcal{F}\) of (3.1.3) is also the map adjoint to the composite \(\psi\) given by

\[
\kappa_* R\Gamma^w_\mathcal{W} \kappa_*^* \cong R\Gamma_\mathcal{X} \to 1.
\]
Sketch Proof. It suffices to prove that the corresponding property for $\kappa^!$ holds, i.e., the canonical isomorphism $\phi = \phi_\kappa : R\Gamma^*_Y \kappa^* \xrightarrow{\sim} \kappa^!$ is the map adjoint to $\psi$. Indeed, as per the proof of [Nav] Theorem 7.1.6, the isomorphism $\phi$ equals $(\beta^!)^{-1}$ where $\beta^! : 1^! \kappa^! \xrightarrow{\sim} 1^! \kappa^* = R\Gamma^*_Y \kappa^*$ is the base-change isomorphism associated to the cartesian square in the following diagram.

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{1} & \mathcal{W} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{\kappa} & X
\end{array}
\]

Therefore, $\phi = \alpha_1 \alpha_2^{-1}$ for $\alpha_i$ as given in the commutative diagram below, where $\alpha_1$ is the canonical map $\kappa^* \kappa_* \to 1$, (which is an isomorphism over $D_{qc}(\mathcal{W})$) while $\alpha_2$ results from the fact that the trace $\text{Tr}^i_\kappa : \kappa_* \kappa^! \to 1$ factors through $R\Gamma^*_Y \to 1$.

The adjointness of $\phi$ and $\psi$ amounts to proving that $\text{Tr}^i_\kappa (\phi) = \psi$, which results from the commutativity of the following.

\[
\begin{array}{ccc}
\kappa_* \kappa^! & \xrightarrow{\kappa_* \alpha_1} & \kappa_* \kappa^* \kappa_* \kappa^! \\
\downarrow & & \downarrow \\
1 & \to & \kappa_* \kappa^*
\end{array}
\]

Lemma A.1.4. Consider a cartesian diagram in $\mathcal{G}$ as follows

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\pi} & \mathcal{X} \\
\downarrow & \square & \downarrow \\
\mathcal{Y} & \xrightarrow{\kappa} & \mathcal{Y}
\end{array}
\]

where $\kappa, \bar{\kappa}$ are completion maps by open coherent ideal sheaves. Let $\mathcal{F} \in D^+_c(\mathcal{Y})$.

(i) The following diagram of obvious natural isomorphisms commutes.

\[
\begin{array}{ccc}
\tilde{\kappa}^* f^* \mathcal{F} & \xrightarrow{\tilde{\kappa}^* g^* \kappa^*} & g^* \kappa^* \mathcal{F} \\
\downarrow & \color{red} \text{(3.2.1)} & \downarrow \\
\bar{\kappa}^* f^* \mathcal{F} & \xrightarrow{\bar{\kappa}^* g^* \kappa^*} & g^* \kappa^* \mathcal{F}
\end{array}
\]

(ii) If $f$ is flat then the following diagram of obvious natural isomorphisms commutes.

\[
\begin{array}{ccc}
g^* \kappa^* \mathcal{F} & \xrightarrow{g^* \kappa^*} & \tilde{\kappa}^* f^* \mathcal{F} \\
\downarrow & \color{red} \text{(3.2.1)} & \downarrow \\
g^* \kappa^* \mathcal{F} & \xrightarrow{g^* \kappa^*} & \bar{\kappa}^* f^* \mathcal{F}
\end{array}
\]
Sketch Proof. As a consequence of the gluing result in [Nay, Theorem 7.1.6], via the canonical isomorphisms $\phi_\kappa: \tilde{\kappa} \xrightarrow{\sim} R\Gamma^\kappa_\kappa$ and $\phi_{\kappa'}: \kappa' \xrightarrow{\sim} R\Gamma^\kappa_\kappa$, for the situation in (i), we have a commutative diagram of isomorphisms

$$
\begin{array}{c}
\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa f^*
\end{array}
\xrightarrow{\beta^!}
\begin{array}{c}
g^! \Gamma^\kappa_\kappa \\
\downarrow \\
g^! \kappa
\end{array}
\begin{array}{c}
\kappa f^* \\
\downarrow \\
\kappa f^*
\end{array}
\xrightarrow{\beta^!}
\begin{array}{c}
g^! \kappa \\
\downarrow \\
g^! \kappa
\end{array}
$$

reflecting the compatibility of $\beta$ with the pseudofunctorial structure of $(-)^!$, while for the one in (ii), we have a commutative diagram of isomorphisms as follows,

$$
\begin{array}{c}
\Gamma^\kappa_\kappa f^* \Gamma^\kappa_\kappa \\
\downarrow \\
\kappa f^*
\end{array}
\xrightarrow{\beta^!}
\begin{array}{c}
g^! \Gamma^\kappa_\kappa \\
\downarrow \\
g^! \kappa
\end{array}
\begin{array}{c}
\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa f^*
\end{array}
\xrightarrow{\beta^!}
\begin{array}{c}
g^! \kappa \\
\downarrow \\
g^! \kappa
\end{array}
$$

reflecting the compatibility of $\beta$ with the pseudofunctorial structure of $(-)^!$ over étale maps. The result now follows by applying $A$’s appropriately in each diagram and using the pre-pseudofunctorial properties of $(-)^!$. □

A.2. Let $f: \mathcal{X} \to \mathcal{Y}$ be a pseudo-proper map and let $f'$ be an ideal of definition of $\mathcal{X}$. Suppose $\mathcal{I}$ is an open coherent ideal in $\mathcal{O}_\mathcal{X}$ and $\kappa: \mathcal{Y} \to \mathcal{Y}$ is the completion of $\mathcal{Y}$ with respect to $\mathcal{I}$. Let $\mathcal{V} = \mathcal{X} \times_\mathcal{Y} \mathcal{U}$ and $\kappa': \mathcal{V} \to \mathcal{X}$, $g: \mathcal{V} \to \mathcal{U}$ the projection maps. Note that $\mathcal{V}$ is the completion of $\mathcal{X}$ with respect to the $\mathcal{O}_\mathcal{X}$-ideal $\mathcal{I} \mathcal{O}_\mathcal{X} + f'$, and $\kappa'$ is the completion map. We thus have a cartesian square:

$$
\begin{array}{c}
\mathcal{V} \\
\downarrow f' \square \downarrow f
\end{array}
\begin{array}{c}
\mathcal{X} \\
\downarrow \kappa \downarrow \kappa'
\end{array}
\begin{array}{c}
\mathcal{U} \\
\downarrow \kappa \downarrow \kappa'
\end{array}
$$

By [A.1.2], we have $\kappa R\Gamma^\kappa_\kappa \xrightarrow{\sim} \Gamma\mathcal{X}$ and $\kappa' R\Gamma^{\kappa'}_\kappa \xrightarrow{\sim} \Gamma\mathcal{X}$.

Proposition A.2.1. The following diagram commutes:

$$
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa (f')^* f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa (f')^* f^* \quad \Gamma^\kappa_\kappa (f')^* f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \\
\downarrow \\
\kappa Rf_*(\kappa') R\Gamma^\kappa_\kappa f^* \quad \Gamma^\kappa_\kappa f^*
where $\gamma$ is induced by applying the functor $\mathbf{R}f_0^*\kappa_0^*\mathbf{R}I_y^*\kappa^*$ to the natural isomorphism $\kappa^* f^\# \sim \sim f^# \kappa^*$ and the upward pointing arrow on the southwest corner is the isomorphism of \cite{AJL2 Proposition 5.2.8 (d)].

**Proof.** Consider the diagram in the proposition. Let $\alpha: \mathbf{R}f_0^*\kappa_0^*\mathbf{R}I_y^*\kappa^* f^\# \rightarrow \mathbf{R}I_y^*$ be the map obtained by composing maps along the route in the diagram which starts at the northwest corner, travelling south and then east. Let $\beta: \mathbf{R}f_0^*\kappa_0^*\mathbf{R}I_y^*\kappa^* f^\# \rightarrow \mathbf{R}I_y^*$ be the composition which starts in the easterly direction and then moves south. Let $\psi: \mathbf{R}I_y^* \rightarrow 1_{\mathbf{D}(\mathcal{X})}$ be the natural map. We have to show that $\alpha = \beta$. This is equivalent to showing

\begin{equation}
\psi \circ \alpha = \psi \circ \beta.
\end{equation}

We now proceed to prove \cite{A2.2}. In what follows we identify $\kappa^*$ with $\kappa^*$ and $\kappa^*$ with $\kappa^*$. Recall that the isomorphism $\kappa^*: f^\# \sim \sim f^# \kappa^*$ mentioned in the theorem can be interpreted in two ways, and the two interpretations agree: (a) as a base change isomorphism, and (b) as the composite

\begin{equation}
\kappa^* f^\# = \kappa^* f^\# \sim \sim (f \kappa')^\# = (\kappa f')^\# \sim \sim f'^\# \kappa^* = f'^# \kappa^*.
\end{equation}

We point out the trace map $\text{Tr}_\kappa: \kappa_0^*\mathbf{R}I_y^*\kappa^* \rightarrow \mathbf{D}(\mathcal{X})$ under the identification $\kappa^* = \kappa^*$ is the composite $\kappa_0^*\mathbf{R}I_y^* \kappa^* \sim \sim \mathbf{R}I_y^* \rightarrow 1_{\mathbf{D}(\mathcal{X})}$. Similarly, $\text{Tr}_{\kappa^*}: \kappa_0^*\mathbf{R}I_y^*\kappa^* \rightarrow \mathbf{R}I_y^*$ is the composite $\kappa_0^*\mathbf{R}I_y^* \kappa^* \sim \sim \mathbf{R}I_y^* \rightarrow \mathbf{R}f_0^*\mathbf{R}I_y^* = \mathbf{R}I_y^*$.

From the definition of the isomorphism in \cite{A2.3} it follows that the following diagram commutes:

\[\begin{array}{ccc}
\mathbf{R}f_0^*\kappa_0^*\mathbf{R}I_y^*\kappa^* f^\# & \sim \sim & \kappa_0^*\mathbf{R}f_0^*\mathbf{R}I_y^* (f')^\# \kappa^* \\
\text{via}\quad \text{Tr}_{f_0^*} & & \kappa_0^* \text{Tr}_f
\\
\mathbf{R}f_0^*\mathbf{R}I_y^* f^\# & \sim \sim & \mathbf{R}I_y^* \\
\text{Tr}_f & & \\
\mathbf{R}f_0^*\mathbf{R}I_y^* f^\# & \sim \sim & \mathbf{R}I_y^* \\
\psi & & \text{Tr}_f
\\
\end{array}\]

Let $\theta: \mathbf{R}f_0^*\kappa_0^*\mathbf{R}I_y^*\kappa^* f^\# \rightarrow 1_{\mathbf{D}(\mathcal{X})}$ be the map obtained from taking any route from the top left corner to the bottom right corner in the above commutative diagram. Note that $\theta = \psi \circ \beta$. It is therefore enough to show that $\theta = \psi \circ \alpha$. Consider the following diagram where the arrow in the top row and the second map in the second row arise from the natural maps $\mathbf{R}I_y \rightarrow \mathbf{R}I_y$ and $\mathbf{R}I_y \rightarrow 1_{\mathbf{D}(\mathcal{X})}$.
respectively:

\[
\begin{array}{c}
\xymatrix{
Rf_\ast R\Gamma'_f \ar[r]^{f^\#} & Rf_\ast R\Gamma_f^{\#} \\
Rf_\ast R\Gamma_f' \ar[u] \ar[r]^{f^\#} & Rf_\ast R\Gamma_f \ar[u] \ar[r]^{f^\#} & Rf_\ast R\Gamma'_f \ar[u]
}
\end{array}
\]

We claim this diagram commutes. The sub-rectangle on the top clearly commutes. According to \[A_{JL2}, \text{Proposition 5.2.8 (d)}\], the composite of the two arrows in the second row is the natural map arising from \(\psi: R\Gamma_f \to 1_{D(\mathfrak{F})}\). It follows that the rectangle at the bottom also commutes, whence the whole diagram commutes. This proves that \(\theta = \psi \circ \alpha\). Thus \(\psi \circ \alpha = \theta = \psi \circ \beta\), establishing \(A_{2.2}\).

A.3. Suppose \(f: X \to Y\) is a map of ordinary schemes in \(\mathcal{G}\) and \(Z \to X\) is a closed subscheme such that \(Z \to Y\) is \textit{proper}. Let \(\kappa: X' = X/_{Z} \to X\) be the formal completion of \(X\) along \(Z\) and \(\widehat{f}: X' \to Y\) the composition \(\widehat{f} = f \circ \kappa\). Then \(\widehat{f}\) is \textit{pseudo-proper}. The isomorphism \(\kappa^* \xrightarrow{\sim} \kappa^\#\) of \((3.1.3)\) gives us an isomorphism \(\kappa^* f^\# \xrightarrow{\sim} \widehat{f}^\#\), and hence an isomorphism \(\alpha: Rf_\ast \kappa_\ast R\Gamma'_f \kappa^* f^\# \xrightarrow{\sim} R\widehat{f}_\ast R\Gamma'_{X'} \widehat{f}^\#\). On the other hand we have \(\beta: Rf_\ast \kappa_\ast R\Gamma'_f \kappa^* f^\# \xrightarrow{\sim} R\widehat{f}_\ast R\Gamma'_{X'} \widehat{f}^\#\) induced by \((A_{1.2})\). We thus have an isomorphism

\[
(A.3.1) \quad \alpha \circ \beta^{-1}: Rf_\ast R\Gamma'_Z f^\# \xrightarrow{\sim} R\widehat{f}_\ast R\Gamma'_{X'} \widehat{f}^\#
\]

If \(u: X \to X'\) is an open immersion of finite type \(Y\)-schemes, with \(g: X' \to Y\) the structure map, then the natural isomorphism

\[
Rf_\ast R\Gamma'_Z f^\# \xrightarrow{\sim} Rf_\ast R\Gamma'_Z u^* g^\# = Rg_\ast R\Gamma'_{u(Z)} g^\#
\]

fits into a commutative diagram

\[
\begin{array}{c}
\xymatrix{
Rf_\ast R\Gamma'_Z f^\# \ar[d] \ar[r]_{\sim} & Rf_\ast R\Gamma'_Z u^* g^\# \ar[d] \\
Rg_\ast R\Gamma'_{u(Z)} g^\# \ar[r]_{\sim} & R\widehat{f}_\ast R\Gamma'_{X'} \widehat{f}^\#
}
\end{array}
\]

If \(f\) is \textit{proper}, then the isomorphism \(\kappa^* f^\# \xrightarrow{\sim} \widehat{f}^\#\) is the one adjoint to the composite

\[
Rf_\ast \kappa_\ast R\Gamma'_f \kappa^* f^\# \xrightarrow{Rf_\ast \delta_{(\mathfrak{F})}} Rf_\ast f^\# \to 1
\]

and so the isomorphism \(\kappa^* f^\# \xrightarrow{\sim} \widehat{f}^\#\) is characterised by the commutativity of the following diagram

\[
\begin{array}{c}
\xymatrix{
Rf_\ast R\Gamma'_Z f^\# \ar[r]_{\beta} & Rf_\ast \kappa_\ast R\Gamma'_f \kappa^* f^\# \ar[r]_{\alpha} & R\widehat{f}_\ast R\Gamma'_{X'} \widehat{f}^\# \\
Rf_\ast f^\# \ar[u]_{\text{natural}} \ar[r]_{\text{Tr} f} & 1 \ar[u]_{\text{Tr} f}
}
\end{array}
\]

\[
(A.3.3)
\]
In general, when \( f \) is not necessarily proper, it is still separated (being in \( G \)) and hence we do have a compactification of \( f \), i.e., an open immersion of \( Y \)-schemes \( u: X \to \bar{X} \), such that the structure map \( \bar{f}: \bar{X} \to Y \) is proper. We have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\kappa} & X \\
\downarrow{\bar{f}} & | & \downarrow{f} \\
Y & \rightarrow & \bar{Y}
\end{array}
\]

(A.3.4)

We then have the following lemma.

**Lemma A.3.5.** Under the assumptions and notation of (A.3.4), the following diagram commutes:

\[
\begin{array}{ccc}
Rf_*R\Gamma^q_Zf^# & \xrightarrow{\sim} & \bar{R}f_*\Gamma^q_{\bar{X}}\bar{f}^# \\
\downarrow & & \downarrow{\text{Tr}f} \\
\bar{R}\bar{f}_*R\Gamma_{u(Z)}\bar{f}^# & \xrightarrow{\sim} & \bar{R}\bar{f}_*\bar{f}^# \\
\end{array}
\]

In particular, the composite

\[
Rf_*R\Gamma^q_Zf^# \xrightarrow{\sim} \bar{R}f_*\Gamma^q_{u(Z)}\bar{f}^# \rightarrow R\bar{f}_*\bar{f}^# \xrightarrow{\text{Tr}f} 1
\]

is independent of the compactification \((u, \bar{f})\) of \( f \).

**Proof.** We expand the diagram to

\[
\begin{array}{ccc}
Rf_*R\Gamma^q_Zf^# & \xrightarrow{\sim} & \bar{R}f_*\Gamma^q_{\bar{X}}\bar{f}^# \\
\downarrow & & \downarrow{\text{Tr}f} \\
\bar{R}\bar{f}_*R\Gamma_{u(Z)}\bar{f}^# & \xrightarrow{\sim} & \bar{R}\bar{f}_*\bar{f}^# \\
\end{array}
\]

The triangle on the left commutes by (A.3.2). The parallelogram is simply (A.3.3), for \( \alpha \circ \beta^{-1} = (A.3.1) \).

**Appendix B. Closed immersions and completions**

**B.1.** Let \( i: \mathcal{X} \to \mathcal{Y} \) be a closed immersion of noetherian formal schemes. We use \( \tilde{i} \) to denote the flat map of ringed spaces \( (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \). We define the functor \( \tilde{i}^*: \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X}) \) by

\[
\tilde{i}^*: = \tilde{i}^* \mathcal{R} \mathcal{H} \text{om}^\bullet(i_* \mathcal{O}_\mathcal{Y}, -).
\]

The functor \( \tilde{i}^* \) enjoys the following properties (see [AJL2] Examples 6.1.3(4)).

1) \( \tilde{i}^*(\mathcal{D}_{qc}^+(\mathcal{Y})) \subset \mathcal{D}_{qc}^+(\mathcal{X}) \) and \( \tilde{i}^*(\mathcal{D}_c^+(\mathcal{Y})) \subset \mathcal{D}_c^+(\mathcal{X}) \). This follows from the fact that \( i_* \mathcal{O}_\mathcal{Y} \) is coherent \( \mathcal{O}_\mathcal{X} \)-module.

2) There is a natural isomorphism \( \tilde{i}^* \mathcal{R}\Gamma^q_\mathcal{X} \xrightarrow{\sim} \mathcal{R}\Gamma^q_\mathcal{Y} \tilde{i}^* \) whose composition with the natural map \( \mathcal{R}\Gamma^q_\mathcal{X} \tilde{i}^* \to \tilde{i}^* \) is the natural map \( \tilde{i}^* \mathcal{R}\Gamma^q_\mathcal{Y} \to \tilde{i}^* \).

3) Using 2) we also obtain that \( \tilde{i}^*(\mathcal{D}_{qc}^+(\mathcal{Y})) \subset \mathcal{D}_{qc}^+(\mathcal{X}) \). Hence we also deduce that \( \tilde{i}^*(\tilde{\mathcal{D}}_{qc}^+(\mathcal{X})) \subset \tilde{\mathcal{D}}_{qc}^+(\mathcal{Y}) \).
4) There is a canonical trace map on $\mathbf{D}(\mathscr{X})$, namely

\[(B.1.1) \quad \text{Tr}^\flat_i: i_* i^\flat = R \mathcal{H}om^\bullet_X (i_* \mathcal{O}_X, -) \longrightarrow 1,\]

which is given by “evaluation at 1”, and which induces a natural map of functors from $i^\flat: \mathbf{D}_{qc}^+(\mathscr{X}) \to \mathbf{D}_{qc}^+(\mathscr{Z})$ to the right adjoint $i^\times$ of $i_*: \mathbf{D}_{qc}^+(\mathscr{Z}) \to \mathbf{D}_{qc}^+(\mathscr{X})$. Moreover, this induced map $i^\flat \to i^\times$ is an isomorphism. Keeping in mind that the values of $(-)!$ range in $\mathbf{D}_{qc}^+$, we deduce that for any $\mathcal{F} \in \mathbf{D}_{qc}^+(\mathscr{X})$, there is a natural isomorphism

\[
i^\flat R \Gamma^\prime_{\mathcal{X}} \mathcal{F} \sim \mathcal{R} \Gamma^\prime_{\mathcal{Z}} i^\flat \mathcal{F} \sim \text{via Tr}_i i^\flat \mathcal{F},\]

and hence for $\mathcal{F} \in \mathbf{D}_{qc}^+(\mathscr{X})$, there is also a natural isomorphism

\[
\Lambda_{\mathcal{Z}} i^\flat \mathcal{F} \sim \mathcal{i}^\flat \mathcal{F}
\]

where the corresponding trace map $\text{Tr}_i$ is the natural composite

\[
i_* R\Gamma^\prime_{\mathcal{X}} A_{\mathcal{X}} i^\flat \sim \mathcal{i}_* R\Gamma^\prime_{\mathcal{Z}} i^\flat \mathcal{F} \to i_* i^\flat \sim \text{Tr}_i i^\flat \mathcal{F} \sim 1.
\]

In particular, if $\mathcal{F} \in \mathbf{D}_c^+(\mathscr{X})$, or if $\mathscr{X}$ is an ordinary scheme, then we have a canonical isomorphism

\[(B.1.2) \quad i^\flat \mathcal{F} \sim \mathcal{i}^\flat \mathcal{F}.
\]

B.2. Suppose $X$ is an ordinary scheme, $\mathscr{F}$ a coherent ideal sheaf on $X$, $Z$ the closed subscheme of $X$ defined by $\mathscr{F}$, and $\kappa: \mathcal{X} = X/Z \to X$ the completion of $X$ along $Z$. We then have a commutative diagram with $i$ and $j$ closed immersions:

\[
\begin{array}{ccc}
Z & \mathcal{i} & \mathcal{F} \\
\downarrow & \downarrow \kappa & \\
X & \end{array}
\]

We define $i$ and $\mathcal{i}$ as in (B.1) above, and it follows that if $\mathcal{F}$ is a $j_* \mathcal{O}_Z$-module, then $i^\flat \kappa_* \mathcal{F} = \mathcal{i}^\flat \mathcal{F}$. We also define $i^\flat, j^\flat$ as in (B.1) and in what follows we will drop the symbols $i_*, j_*$ occurring in the definition of $i^\flat, j^\flat$ respectively. Finally note that, since $Z$ is an ordinary scheme so that $\Gamma^\prime_Z$ is the identity functor, $i^\flat$ and $j^\flat$ are right adjoint to $i_*$ and $j_*$ respectively.

The natural map

\[(B.2.1) \quad R \mathcal{H}om^\bullet_X (\mathcal{O}_Z, -) \longrightarrow \kappa_* R \mathcal{H}om^\bullet_X (\mathcal{O}_Z, \kappa^* -)
\]

is an isomorphism, whence we have an isomorphism

\[(B.2.2) \quad i^\flat \sim j^\flat \kappa^*
\]

given by

\[
i^\flat R \mathcal{H}om^\bullet_X (\mathcal{O}_Z, -) \sim \mathcal{i}^\flat \kappa_* R \mathcal{H}om^\bullet_X (\mathcal{O}_Z, \kappa^* -) \sim \mathcal{j}^\flat R \mathcal{H}om^\bullet_X (\mathcal{O}_Z, \kappa^* -).
\]

The essential content of the following lemma is that, (B.2.2) is, up to canonical identifications, the inverse of the canonical isomorphism $j^\flat \kappa^* \sim (\kappa j)^\flat = i^\flat$. 
Lemma B.2.3. The following diagram commutes

$$
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \\
\bar{j}^* \kappa^* & \bar{j}^* \kappa^* & \bar{j}^* \kappa^* & \bar{i}^* \\
\downarrow & \downarrow & \downarrow & \\
\bar{i}^* & \bar{i}^* & \bar{i}^* & \\
\end{array}
$$

where the unlabelled isomorphism $\bar{j}^* \kappa^* \xrightarrow{\sim} \bar{i}^*$ is the canonical one.

Proof. Keeping in mind that the canonical maps $j_* j^\flat \to 1$ and $j_* j^\# \to 1$ factor through $R \Gamma'_X \to 1$ and that the canonical map $i_* \bar{i}^\flat \to 1$ factors through $R \Gamma'_Z \to 1$ we see that the diagram of the lemma corresponds, via adjointness of $\bar{i}^*$ to $i_*$, to the outer border of the following commutative diagram of obvious natural maps.

$$
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \\
\kappa_* j_* j^\flat \kappa^* & \kappa_* j_* j^\# \kappa^* & \kappa_* j_* j^\# \kappa^* & \\
\downarrow & \downarrow & \downarrow & \\
\kappa_* R \mathcal{H}om_X(\mathcal{O}_Z, \kappa^*(-)) & \kappa_* R \mathcal{H}om'_X(\mathcal{O}_Z, \kappa^*) & \kappa_* R \mathcal{H}om'_X(\mathcal{O}_Z, \kappa^*) & \\
\downarrow & \downarrow & \downarrow & \\
i_* \bar{i}^\flat & R \mathcal{H}om_X(\mathcal{O}_Z, -) & R \Gamma'_Z & 1
\end{array}
$$

□

Appendix C. Koszul complexes

Since our goal is to understand Verdier’s isomorphism explicitly, we have to lay out our conventions for maps between complexes, especially the fundamental local isomorphism which is at the heart of explicit formulas for residues, and hence integrals (i.e., traces).

C.1. Our version of Koszul complexes. Let $R$ be a noetherian ring. For $t \in R$, we write $K_*(t)$ for the homology complex

$$
0 \to K_1(t) \to K_0(t) \to 0
$$

where $K_1(t) = K_0(t) = R$ and the arrow between them is multiplication by $t$. For a sequence of elements $t = (t_1, \ldots, t_r)$ in $R$, we set $K_*(t)$ to be the complex:

$$
K_*(t) = K_*(t_1) \otimes_R \cdots \otimes_R K_*(t_r).
$$

For an $R$-module $M$ and an integer $i$, we write $K^i(t, M) = \text{Hom}_R(K_i(t, M))$ and define $\partial^i : K^i(t, M) \to K^{i+1}(t, M)$ to be the transpose of the differential $K_{i+1}(t) \to K_i(t)$, without the intervention of any signs. Then $K_*(t, M)$ together with $\partial^*$ is a cohomology complex, and this is what we will call the Koszul (cohomology) complex on $M$ and $t$. We write $K_*(t)$ for $K_*(t, R)$. We refer the reader to [CT1 pp. 17–18] for a discussion of various versions of Koszul complexes and the relationship between them. Here are three basic properties:

1) $K_*(t, M)$ is bounded by degrees 0 and $r$, with $K^0(t, M) = K^r(t, M) = M$.
2) $K_*(t, M) = M \otimes_R K_*(t)$.
3) \( K^i(t, M) \) is the direct sum of \( \binom{n}{i} \) copies of \( M \).

The reason we use this version is the relationship with a certain Čech complex associated to an affine open cover of \( \text{Spec } R \setminus Z \), where \( Z \) is the closed subscheme defined by the vanishing of the \( t_i \)'s (see Subsection C.3). The homology complex \( H_\bullet(t) \) is also called a Koszul complex, and to distinguish it from \( K_\bullet(t) \), we will call it the Koszul homology complex on \( t \).

There is a well known way in which these Koszul complexes vary with respect to \( t \). Let \( I \) be the ideal generated by \( t \). Let \( J \) be an ideal in \( R \) such that \( I \subset J \), and such that \( J \) is generated by \( g = (g_1, \ldots, g_r) \). Since \( I \subset J \) we have \( u_{ij} \in R \) such that

\[
t_i = \sum_{j=1}^r u_{ij} g_j \quad (i = 1, \ldots, r).
\]

As is well-known, one has a map of homology Koszul complexes

\[
U_\bullet: K_\bullet(t) \to K_\bullet(g)
\]

such that

- \( H_0(U_\bullet): R/I \to R/J \) is the natural surjection.
- \( R = K_0(t) \xrightarrow{U_0} K_0(g) = R \) is the identity map on \( R \).
- \( R = K_n(t) \xrightarrow{U_n} K_n(g) = R \) is the map \( x \mapsto \det(u_{ij}) \cdot x \).

Taking transposes and tensoring with \( M \) we get a map on (cohomology) Koszul complexes:

(C.1.1) \[
U^\bullet = U_M^\bullet: K^\bullet(g, M) \to K^\bullet(t, M)
\]

such that \( U^0 \) is the identity map on \( M \) and

(C.1.2) \[
U^n: M \to M
\]

is the map \( m \mapsto \det(u_{ij}) \cdot m \).

C.2. The Fundamental Local Isomorphism. With \( R \) and \( M \) as above, suppose \( t = (t_1, \ldots, t_r) \) is an \( R \)-sequence, \( I \) the ideal generated by \( \{t_1, \ldots, t_r\} \), and \( A = R/I \). Then

1) The ideal \( I \) is the image of the coboundary map from \( K^{r-1}(t) \) to \( K^r(t) = R \), and the resulting map of complexes \( K^\bullet(t) \to A[-r] \) is a quasi-isomorphism. Thus we have an isomorphism in \( \mathbf{D}({\text{Mod}}_R) \):

(C.2.1) \[
K^\bullet(t) \congto A[-r].
\]

Since \( K^\bullet(t, M) \) is a (bounded) complex of free modules, for every complex \( M^\bullet \) we have an isomorphism in \( \mathbf{D}({\text{Mod}}_R) \)

(C.2.2) \[
M^\bullet \otimes_R K^\bullet(t) \congto M^\bullet \otimes_R (A[-r]) = M^\bullet \otimes_A (A[-r])
\]

where \( M^\bullet = M^\bullet \otimes_R A \).

2) We have \( \text{Hom}_R(A, M) = \ker (K^0(t, M) \to K^1(t, M)) \) where \( M = K^0(t, M) \) and \( \text{Hom}_R(A, M) \) is identified with the submodule of \( I \)-torsion elements of \( M \) namely \( 0(I) \) in the usual way (i.e., by “evaluation at 1”). We thus have a map of complexes \( \text{Hom}_R(A, M) \to K^\bullet(t, M) \). If \( M \) is an injective \( R \)-module then this map is a quasi-isomorphism. It follows that if \( M^\bullet \) is a bounded-below complex, and \( M^\bullet \to E^\bullet \) is an injective resolution with \( E^\bullet \) a bounded-below complex, then
we have quasi-isomorphisms $M^* \otimes_R K^* (t) \to E^* \otimes_R K^* (t)$ and $\text{Hom}_R (A, E^*) \to E^* \otimes_R K^* (t)$ so that in $\mathbf{D} (\text{Mod}_R)$ we have an isomorphism

(C.2.3) $M^* \otimes_R K^* (t) \xrightarrow{\sim} \text{RHom}^*_R (A, M^*)$

fitting into a commutative diagram in $\mathbf{D} (\text{Mod}_R)$ as follows.

$$
\begin{array}{ccc}
M^* \otimes_R K^* (t) & \xrightarrow{\sim} & E^* \otimes_R K^* (t) \\
\downarrow \psi_t & & \downarrow \psi_t \\
\text{RHom}^*_R (A, M^*) & \xrightarrow{\sim} & \text{Hom}_R (A, E^*)
\end{array}
$$

In particular we have an isomorphism

$$\psi_t : M^* \overset{L}{\otimes}_R (A[-r]) \xrightarrow{\sim} \text{RHom}^*_R (A, M^*)$$

where $\psi_t = (C.2.3) \circ (C.2.2)^{-1}$.

3) Let $\frac{1}{t}$ (or $1/t$ for typographical convenience) be the element of $(\wedge^r_A I/I^2)^*$ defined in (5.4.3). Then $(\wedge^r_A I/I^2)^*$ is a free $A$ module of rank one, with $\frac{1}{t}$ as a generator. One therefore has an isomorphism:

(C.2.4) $\lambda_t : A \xrightarrow{\sim} (\wedge^r_A I/I^2)^*$,

given by $1 \mapsto (-1)^r 1/t$. The reason for the sign $(-1)^r$ will be clear later. We thus get an isomorphism,

(C.2.5) $\eta_{R,A}(M^*) : M^* \overset{L}{\otimes}_R ((\wedge^r_A I/I^2)^*[-r]) \xrightarrow{\sim} \text{RHom}^*_R (A, M^*)$

with $\eta_{R,A} = \psi_t \circ (\lambda_t[-r])^{-1}$. The crucial property here is that $\eta_{R,A}$ does not depend on $t$, even though $\psi_t$ and $\lambda_t$ do.

The data above fits into the following commutative diagram

$$
\begin{array}{ccc}
M^* \overset{L}{\otimes}_R (A[-r]) & \xrightarrow{\sim} & M^* \otimes_R K^* (t) & \xrightarrow{\sim} & E^* \otimes_R K^* (t) \\
\downarrow \psi_t & & \downarrow \psi_t & & \downarrow \psi_t \\
M^* \overset{L}{\otimes}_R ((\wedge^r_A I/I^2)^*[-r]) & \xrightarrow{\eta_{R,A}} & \text{RHom}^*_R (A, M^*) & \xrightarrow{\sim} & \text{Hom}_R (A, E^*)
\end{array}
$$

Let $M$ be an $R$-module. Our version of the fundamental local isomorphism is the isomorphism

(C.2.7) $\phi_{R,A}(M) : M \otimes_R (\wedge^r_A I/I^2)^* \xrightarrow{\sim} \text{Ext}^r_R (A, M)$

given by

$$\phi_{R,A}(M) = H^0 (\eta_{R,A} (M[r])).$$

Let us globalize this construction. Let $\mathcal{Z}$ be a formal scheme, and $\mathcal{I}$ a coherent ideal sheaf such that the resulting closed immersion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ is a regular immersion of codimension $r$, i.e., it is given locally by a regular sequence of length $r$.

Let us write $\mathcal{N}_i$ for the normal bundle of $\mathcal{Z}$ in $\mathcal{X}$, i.e. $\mathcal{N}_i = (\mathcal{I} / \mathcal{I}^2)^*$ and set

(C.2.8) $\mathcal{N}_i^* := \wedge^r \mathcal{N}_i = (\wedge^r \mathcal{I} / \mathcal{I}^2)^*$.

There is a natural isomorphism

$$\mathcal{N}_i^* \xrightarrow{\sim} i^* \mathcal{E}xt^r_{\mathcal{O}_\mathcal{X}} (\mathcal{O}_\mathcal{Z}, \mathcal{O}_\mathcal{X}) = H^r i^* \mathcal{O}_\mathcal{X}$$
Applying \( \♭ \) at 1" is respected. In greater detail if \( \text{Tr} \) is the composite
\[
\eta (\mathcal{F}) : i^! \mathcal{F} \rightarrow i^\flat \mathcal{F}
\]
given by the composite
\[
i^! \mathcal{F} = L \text{Li}^* (\mathcal{F}) \otimes_{\mathcal{O}_X} (\mathcal{N}^r_1 [-r]) \rightarrow L \text{Li}^* \mathcal{F} \otimes_{\mathcal{O}_X} i^\flat \mathcal{R} \mathcal{H} \text{om}^\bullet (\mathcal{O}_X, \mathcal{O}_X)
\]
where the first isomorphism is given by \( \text{Li}^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{N}^r_1 [-r] \) while the third one results from the fact that \( i_* \mathcal{O}_X \) is coherent and has finite tor dimension over \( \mathcal{O}_X \).

For \( \mathcal{F} \in \mathcal{D}_c^+ (\mathcal{X}) \), let
\[
\eta_i (\mathcal{F}) = i^! \mathcal{F} \rightarrow i^\flat \mathcal{F}
\]
be the composite \( \eta_i = \eta_1 \circ \eta_i \).

**Remark C.2.14.** In the above, the isomorphism \( i^! \mathcal{O}_X \rightarrow i^\flat \mathcal{O}_X \) in \( (\mathcal{C}.2.9) \) is what drives the isomorphism \( (\mathcal{C}.2.11) \). In slightly greater detail, for \( \mathcal{F} \in \mathcal{D}_c^+ (\mathcal{X}) \), we have (by definition of \( i^! \)):
\[
i^! \mathcal{F} = L \text{Li}^* (\mathcal{F}) \otimes_{\mathcal{O}_X} i^* (\mathcal{O}_X).
\]

We also have an isomorphism (whose inverse is the composite of the last two maps in \( (\mathcal{C}.2.12) \))
\[
\text{Li}^* (\mathcal{F}) \otimes_{\mathcal{O}_X} i^\flat (\mathcal{O}_X) \rightarrow i^\flat (\mathcal{F}).
\]

Applying \( i^! \mathcal{O}_X \rightarrow i^\flat \mathcal{O}_X \) (from \( (\mathcal{C}.2.9) \)) to the two isomorphisms above, we get \( \eta_i (\mathcal{F}) \).

The isomorphism \( \text{Li}^* (\mathcal{F}) \otimes_{\mathcal{O}_X} i^\flat (\mathcal{O}_X) \rightarrow i^\flat (\mathcal{F}) \) above is such that “evaluation at 1” is respected. In greater detail if \( \text{Tr}^i : i_* i^\flat \rightarrow 1 \) is as in \( (\mathcal{B}.1.1) \), then the composite
\[
\mathcal{F} \otimes_{\mathcal{O}_X} i_* i^\flat \mathcal{O}_X \rightarrow i_* (\text{Li}^* (\mathcal{F}) \otimes_{\mathcal{O}_X} i^\flat (\mathcal{O}_X)) \rightarrow i_* (\mathcal{F}) \xrightarrow{\text{Tr}^i} (\mathcal{F})
\]
is equal to \( 1 \otimes \text{Tr}^i (\mathcal{O}_X) \). This means that if \( \text{Tr}^i : i_* i^! \rightarrow 1 \) is defined by the formula
\[
\text{Tr}^i = \text{Tr}^\flat \circ i_* \eta_i,
\]
then the following diagram commutes:

\[
\begin{array}{ccc}
i_* (\text{Li}^* \mathcal{F} \otimes_{\mathcal{O}_X} i^* \mathcal{O}_X) & \rightarrow & i_* i^! \\
\text{projection formula} & & \\
\mathcal{F} \otimes_{\mathcal{O}_X} i_* i^! \mathcal{O}_X & \rightarrow & \text{Tr}^i (\mathcal{F})
\end{array}
\]

\[
(C.2.14.1)
\]
C.2.15. If \( X = X \) is an ordinary scheme, so that \( i^* = i! \), then the maps \( \eta_i(\mathcal{F}) \) and \( \eta'_i(\mathcal{F}) \) above can be extended to isomorphisms for \( \mathcal{F} \in \mathbf{D}_{qc}(X) \), without any boundedness hypotheses on \( \mathcal{F} \). In greater detail, recall that a complex \( \mathcal{F} \) of \( \mathcal{O}_X \)-modules is called perfect if there exist \( a, b \in \mathbb{Z} \), \( a \leq b \), and locally \( \mathcal{F} \) is \( \mathbf{D}(X) \)-isomorphic to a complex \( E \) of finite rank free \( \mathcal{O}_X \)-modules with \( E^n = 0 \) for \( n \notin [a, b] \). The map \( i_* \) takes perfect complexes to perfect complexes (locally use appropriate Koszul complexes!). In other words \( i \) is a quasi-perfect map (see \([L4, \text{p. 192, Definition 4.7.2}]\)). According to a result of Neeman in \([Nc1, \text{Bondal and van den Bergh in BB, since} \ i_* \text{takes perfect complexes to perfect complexes, one has a unique isomorphism (with } Z = \mathcal{F})
\]

\[
\text{Li}^*(\mathcal{F}) \overset{L}{\rightarrow} i^! \mathcal{O}_X \xrightarrow{\sim} i^! \mathcal{F}
\]

such that Diagram (C.2.14.1) commutes with \( i^* \) replaced by \( i^! \), \( \text{Tr}^{\bullet} \) by \( \text{Tr}_i \), the equality on the top row by \( i_* \) of the isomorphism displayed above, and allowing \( \mathcal{F} \) to vary \( \mathbf{D}_{qc}(X) \) rather than in \( \mathbf{D}_c^+(X) \). It is now clear that one can extend \( \eta'_i \) to an isomorphism of functors on \( \mathbf{D}_{qc}(X) \). As for \( \eta_i \), see \([C1, \text{p. 53, (2.5.3)}]\), keeping in mind the differing sign conventions for \( \mathcal{K} \). In fact the isomorphism \( \bar{\iota}^*(\mathcal{F} \otimes\mathcal{O}_X \mathcal{R} \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)) \xrightarrow{\sim} \bar{\iota}^* \mathcal{R} \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \) in (C.2.12) works for \( \mathcal{F} \in \mathbf{D}_{qc}(X) \) when \( X \) is an ordinary scheme.

In view of (C.2.14.1), in order to understand \( \text{Tr}^\bullet_{\mathcal{O}_X} \) it is enough to understand \( \text{Tr}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X) \). We give an explicit representation of \( \text{Tr}^\bullet_{\mathcal{O}_X} \) when \( X = \text{Spec } R, Z = \text{Spec } A, \) and the \( I = \ker R \rightarrow A \) is generated by a quasi-regular sequence \( t = (t_1, \ldots, t_r) \), i.e., the situation we have been with for most of this section. Let \( N = \Gamma(X, \mathcal{M}) \). In this case, the quasi-isomorphism of complexes of \( R \)-modules \( K^\bullet(t) \rightarrow A[-r] \) in (C.2.1), is the map defined by \( K^\bullet(t) = R \xrightarrow{\text{natural}} R/I = (A[-r])^\circ \).

Using the isomorphism \( A \xrightarrow{\sim} N \) given by \( 1 \mapsto 1/t \) we get a quasi-isomorphism \( \varphi_t : K^\bullet(t) \rightarrow N[-r] \), where \( \varphi_t \) is defined by \( \varphi_t : K^\bullet(t) = R \rightarrow N = (N[-r])^\circ \), the arrow \( R \rightarrow N \) being \( 1 \mapsto 1/t \). For a complex of \( R \)-modules \( M^\bullet \), let \( \text{Tr}^\bullet_{A/R}(M^\bullet) : M^\bullet \otimes_R N[-r] \rightarrow M^\bullet \) and \( \text{Tr}^{\bar{\varphi}}_{A/R}(M^\bullet) : \mathcal{R} \mathcal{H}om^\bullet_R(A, M^\bullet) \rightarrow M^\bullet \) be the maps corresponding to \( \text{Tr}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X) \) and \( \text{Tr}^\bullet_{\mathcal{O}_X}(\mathcal{F}) \). By definition of (C.2.11), we have a commutative diagram in the category \( \mathbf{D}(\text{Mod}_R) \) with isomorphisms bordering the triangle on the right:

\[
\begin{array}{ccc}
N[-r] & \xrightarrow{\varphi_t} & K^\bullet(t) \\
\text{Tr}^\bullet_{A/R}(R) & \downarrow{\eta_i} & \mathcal{R} \mathcal{H}om^\bullet_R(A, R) \\
R & \xrightarrow{\text{C.2.3}} & \mathcal{R} \mathcal{H}om^\bullet_R(A, R)
\end{array}
\]

The composite \( \text{Tr}^\bullet_{A/R}(R) \circ \text{C.2.3} \) is the natural projection

\[
\pi_t : K^\bullet(t) \longrightarrow K^0(t) = R
\]

which is a map of complexes, since \( K^\bullet(t) \) has no negative terms. Thus

(C.2.15.1) \( \text{Tr}^\bullet_{A/R}(R) = \pi_t \circ \varphi_t^{-1} \).
C.3. Compatibility with completions. In view of the above, Lemma [B.2.3] has a useful re-interpretation in the special case where the two closed immersions of $Z$ into $X$ and $\mathcal{X}$ are regular immersions of codimension $r$. In greater detail, suppose as in Section [B.2] we have a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{i} & \mathcal{X} \\
\downarrow{i} & & \downarrow{\kappa} \\
\mathcal{X} & \xrightarrow{\eta} & X
\end{array}
$$

with $X$ an ordinary scheme, but with $i$, $j$ regular closed immersions, $\mathcal{X} = X/\mathfrak{Z}$ the completion of $X$ along $Z$, $\kappa$ the completion map, and let $\mathcal{I}$ and $\mathcal{J} = \mathcal{I} \mathcal{O}_{\mathcal{X}}$ be the ideal sheaves for $Z$ in $X$ and $\mathcal{X}$ respectively. Now regarding $\mathcal{I}/\mathcal{I}^2$ and $\mathcal{J}/\mathcal{J}^2$ as invertible sheaves on $Z$, we have an obvious identification $\mathcal{I}/\mathcal{I}^2 = \mathcal{J}/\mathcal{J}^2$, whence the identification $j^* \kappa^* = i^*$. Then the following is an easy corollary to Lemma [B.2.3]

Lemma C.3.1. The following diagram commutes.

$$
\begin{array}{ccc}
j^* \kappa^* & \xrightarrow{\sim} & j^# \kappa^* \\
j^* \kappa^* & \downarrow{\eta_j^*} & j^# \kappa^* \\
i^* & \downarrow{\eta_i^*} & i^#
\end{array}
$$

where the unlabelled isomorphism $j^# \kappa^* \xrightarrow{\sim} i^*$ is the canonical one.

C.4. Compatibility between the flat-base-change isomorphisms of $-^*$ and of $-^#$. Suppose we have a cartesian diagram $s$ of formal schemes

$$
\begin{array}{ccc}
\mathcal{W}' & \xrightarrow{j} & \mathcal{W} \\
\kappa_0 & \downarrow{\square} & \kappa \\
\mathcal{X}' & \xrightarrow{i} & \mathcal{X}
\end{array}
$$

such that $i$ is a regular immersion (i.e., given locally by the vanishing of a regular sequence) and $\kappa_0$ is the completion of $\mathcal{X}$ with respect to a closed subscheme given by a coherent ideal. By [C.2.13], for any $\mathcal{F} \in \mathbf{D}_c^+ (\mathcal{X})$ and $\mathcal{G} \in \mathbf{D}_c^+ (\mathcal{W})$ there are natural isomorphisms

$$
i^* \mathcal{F} \xrightarrow{\sim} i^# \mathcal{F}, \quad j^* \mathcal{G} \xrightarrow{\sim} j^# \mathcal{G}.
$$

Now, on one hand we have the flat base-change isomorphism

$$
\beta^*_s: \kappa^* t^# \xrightarrow{\sim} j^* \kappa_0^*
$$

of (3.2.2) while on the other we have an isomorphism

$$
\kappa^* t^* \xrightarrow{\sim} j^* \kappa_0^*
$$

(C.4.2)

given by the composite

$$
\kappa^* ((\mathcal{L} t^* (-) \otimes \mathcal{N}_0^*[r])) \xrightarrow{\sim} (\mathcal{L} j^* L \kappa_0^* (-)) \otimes \kappa^* \mathcal{N}_0^*[r]
$$

$$
\xrightarrow{\sim} (\mathcal{L} j^* \kappa_0^* (-)) \otimes \mathcal{N}_0^*[r]
$$
where the second isomorphism is the one that arises from the canonical isomorphism $\kappa^*\mathcal{N} \simto \mathcal{N}$. Fortunately these two flat-base-change isomorphisms are compatible:

**Proposition C.4.3.** For the diagram $\mathcal{s}$ in (C.4.1), for any $\mathcal{F} \in \mathcal{D}_c^+(\mathcal{X})$ the following diagram commutes.

\[ \begin{array}{ccc}
\kappa^* i^* \mathcal{F} & \simto & j^* \kappa_0^* \mathcal{F} \\
\eta_i^j & \downarrow & \eta_j^j \\
\kappa^* i^* \mathcal{F} & \xrightarrow{\beta^*} & j^* \kappa_0^* \mathcal{F}
\end{array} \]

(C.4.3.1)

**Proof.** As per the definition of $\eta_i^j$ in (C.2.13), the diagram in (C.4.3.1) expands as follows

\[ \begin{array}{ccc}
\kappa^* i^* \mathcal{F} & \simto & j^* \kappa_0^* \mathcal{F} \\
\eta_i^j & \downarrow & \eta_j^j \\
\kappa^* i^* \mathcal{F} & \xrightarrow{\beta^*} & j^* \kappa_0^* \mathcal{F}
\end{array} \]

(B.1.2)

where $\beta^*$ is induced by the natural isomorphism

\[ \kappa_0^* \mathcal{R}\mathcal{H}\mathcal{o}m_{\mathcal{O}_X}(i_*\mathcal{O}_{X'}, \mathcal{F}) \simto \mathcal{R}\mathcal{H}\mathcal{o}m_{\mathcal{O}_W}(j_*\mathcal{O}_{W'}, \kappa_0^* \mathcal{F}). \]

It is straightforward to check that the top rectangle commutes. For the bottom one, using the adjointness property of $j^*$, it suffices to check that the outer border of the following diagram commutes where, as before, $\Gamma' = \mathcal{R}\Gamma'$ etc..

\[ \begin{array}{ccc}
j_* \Gamma'_{W'} \kappa^* i^* & \xrightarrow{\beta^*} & j_* \Gamma'_{W'} j^* \kappa_0^* \\
\rotate[90]{\kappa_0^* i_* \Gamma'_W} & \xrightarrow{\beta^*} & j_* \kappa_* i^* \mathcal{F} \\
\rotate[90]{\kappa_0^* i_* \Gamma'_W} & \xrightarrow{\beta^*} & j_* \kappa_* i^* \mathcal{F}
\end{array} \]

Here the maps $\alpha_i$ are induced by the composite of natural maps

\[ j_* \mathcal{R}\Gamma'_{W'}, \kappa_0^* \xrightarrow{\beta^*} j_* \kappa_* \mathcal{R}\Gamma'_{W'}, \simto \kappa_0^* i_* \Gamma'_{W'}. \]

The unlabelled maps are the obvious natural ones. The diagram $\square$ commutes by definition of the map $i^* \simto i^*$ in (B.1.2). Commutativity of the remaining parts is easy to check. \qed
C.5. Stable Koszul complexes and generalized fractions. Let \( R, I, A \) be as above, and let \( t = (t_1, \ldots, t_d) \) be generators for \( I \). Note that, for now, we are not requiring \( t \) to be a quasi-regular sequence. We now recall the relationship between \( K^*(t, M) \) and the local cohomology of \( M \) and relate the above discussion to generalized fractions leading to the explicit formula in Lemma \( \text{[C.5.4]} \) below. For an \( r \)-tuple of positive integers \( \alpha = (\alpha_1, \ldots, \alpha_r) \), let \( t^\alpha = (t_1^{\alpha_1}, \ldots, t_r^{\alpha_r}) \). Let

\[
K^*_\infty(t) := \lim_{\alpha} K^*(t^\alpha)
\]

(\ref{C.5})

\[
K^*_\infty(t, M) := \lim_{\alpha} K^*(t^\alpha, M) = M \otimes_R K^*_\infty(t).
\]

The complex \( K^*_\infty(t, M) \) is called the stable Koszul complex of \( M \) associated to \( t \) and it has a well known relationship with the \( \check{\text{C}} \)ech complex \( C^* = C^*(\Omega, \tilde{M}) \) associated with the open cover \( \Omega = \{ \{t_i \neq 0 \} \mid i = 1, \ldots, r \} \) of the scheme \( U := \text{Spec} R \setminus V(I) \). The relationship is that \( C^i = K^{i+1}_\infty(t, M) \) for \( i \geq 0 \) and in this range and the coboundary maps \( C^i \to C^{i+1} \) and \( K^{i+1}_\infty(t, M) \to K^{i+2}_\infty(t, M) \) are equal. We also note that the natural map \( K^*(t, M) \to K^*_\infty(t, M) = C^r-1 \), is the map \( M \to M_{t_1 \ldots t_r} \) given by \( m \mapsto m/t_1 \ldots t_r \).

We point out that there is an obvious commutative diagram

\[
\begin{array}{ccc}
\lim_{\alpha} \text{Hom}_R(R/t^\alpha, M) & \longrightarrow & K^0_\infty(t, M) \\
\Gamma_I(M) \downarrow & & \downarrow \\
M & \longrightarrow & M
\end{array}
\]

where the horizontal arrow in the top row is the one obtained by applying a direct limit to the map of direct systems \( \text{Hom}_R(R/t^\alpha, M) \to K^0_\infty(t^\alpha, M) \) and the horizontal arrow in the bottom row is the natural inclusion. If, as before, \( M \to E^* \) is an injective resolution of \( M \), we have \( \text{Hom}_R(R/t^\alpha R, E^*) \xrightarrow{\sim} E^* \otimes_R K^*_\infty(t^\alpha) \) whence an isomorphism

\[
\lim_{\alpha} \text{Hom}_R(R/t^\alpha R, E^*) \xrightarrow{\sim} E^* \otimes_R K^*_\infty(t).
\]

We then we have a diagram of isomorphisms in \( \text{D}(\text{Mod}_R) \):

\[
\begin{array}{ccc}
K^*_\infty(t, M) & \longrightarrow & M^* \otimes_R K^*_\infty(t) \\
\Gamma_I(M) \downarrow & & \downarrow \\
E^* \otimes_R K^*_\infty(t) & \longrightarrow & \text{lim}_{\alpha} \text{Hom}_R(R/t^\alpha R, E^*)
\end{array}
\]

Since all solid arrows in this diagram are isomorphisms, we can fill the dotted arrow, i.e., we have a unique isomorphism

\[
K^*_\infty(t, M) \xrightarrow{\sim} \Gamma_I(M)
\]

which fills the dotted arrow to make the diagram commute. Since \( K^2_\infty(t, M) = 0 \) for \( j > r \), we have a surjective map \( M_{t_1 \ldots t_r} = K^r_\infty(t, M) \to \Gamma_I(M) \). The image of \( m/t_1^{\alpha_1} \ldots t_r^{\alpha_r} \in M_{t_1 \ldots t_r} \) is denoted by the generalised fraction \( \{ t_1^{m_1}, \ldots, t_r^{m_r} \} \).
Now, standard excision arguments give us a map $H^{r-1}(U, \tilde{M}) \to H^r_f(M)$ which is an isomorphism when $r \geq 2$ and surjective when $r = 1$. In the Čech complex $C^\bullet$, we have $C^j = 0$ for $j \geq r$. We thus have a composition of surjective maps

$$M_{t_1, \ldots, t_r} = C^{r-1} \to H^{r-1}(U, \tilde{M}) \to H^r_f(U, \tilde{M}) \to H^r_f(M).$$

The image of $\frac{m}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \in M_{t_1, \ldots, t_r}$ is denoted by the generalized fraction $[c_{t_1}^{\alpha_1}, \ldots, c_{t_r}^{\alpha_r}]$. The two generalized fractions are related by the formula

$$(C.5.3) \quad \left[ \frac{m}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \right] = (-1)^r \left\{ \frac{m}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \right\}$$

(see [LNS, p.47, Lemma 4.1.1]).

**Lemma C.5.4.** Suppose the sequence $t$ above is a quasi-regular sequence in $R$. Let $M$ be an $R$-module. Then the composite map (with $\phi_{R,A}(M)$ as in (C.2.7))

$$(C.5.4.1) \quad M \otimes_R (\wedge^r_A I/I^2)^* \xrightarrow{\phi_{R,A}(M)} \text{Ext}^r_R(A, M) \to H^r_f(M)$$

is given by

$$m \otimes \frac{1}{t} \mapsto \left[ \frac{m}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \right] \quad (m \in M)$$

Where $\frac{1}{t}$ is as in (5.4.3).

**Proof.** For an arbitrary bounded complex $M^\bullet$ consider the following commutative diagram

$$\begin{CD}
M^\bullet \otimes_R (A[-r]) @<\lambda_{t}>> M^\bullet \otimes_R K^\bullet(t) \to M^\bullet \otimes_R K^\bullet_{\infty}(t) \\
\downarrow \downarrow \downarrow \downarrow \\
E^\bullet \otimes_R K^\bullet(t) \to E^\bullet \otimes_R K^\bullet_{\infty}(t) \to \text{Hom}_R(A, E^\bullet) \to \Gamma_I(E^\bullet) \\
\downarrow \downarrow \downarrow \downarrow \\
M^\bullet \otimes_R ((\wedge^r_A I/I^2)^*[-r]) \xrightarrow{\eta_{R,A}} \mathbf{R}\text{Hom}_R(A, M^\bullet) \to \mathbf{R}\Gamma_I(M^\bullet)
\end{CD}$$

(C.5.4.2)

Set $M^\bullet = M[r]$ in the above and apply the cohomology functor $H^0(-)$. We get a commutative diagram

$$\begin{CD}
M \otimes_R A \to M/IM \to H^r(M \otimes_R K^\bullet_{\infty}(t)) \\
\downarrow \downarrow \downarrow \\
M \otimes_R (\wedge^r_A I/I^2)^* \xrightarrow{\phi_{R,A}(M)} \text{Ext}^r_R(A, M) \to \text{natural} \to H^r_f(M)
\end{CD}$$

Let us write $[x]$ for the image of $x \in M_{t_1, \ldots, t_r} = M \otimes_R K^\bullet_{\infty}(t)$ in the module $H^r(M \otimes_R K^\bullet_{\infty}(t))$. Then chasing an element $m \otimes (1/t) \in M \otimes_R (\wedge^r_A I/I^2)^*$ by first going north (via $\lambda_{t}^{-1}$) and then east along the above rectangle, we arrive at...
the element \((-1)^r[m/t_1 \ldots t_r] \in H^r(M \otimes_R K^n_\infty(t))\). The assertion follows from \((C.5.3)\).

\[\square\]

C.6. **Duality for composite for closed immersions.** Let \(R\) be a noetherian ring, \(I \subset R\) an ideal, \(A = R/I\) and \(i: \text{Spec } A \hookrightarrow \text{Spec } R\) the closed immersion corresponding to the natural surjection \(R \rightarrow R/I = A\). Let \(M^*\) be a bounded below complex of \(A\)-modules. Consider the “evaluation at 1” map:

\[\text{ev}_I : \text{RHom}^\bullet_R(A, M^*) \longrightarrow M^*.\]

As is well-known (and easy to verify from the definitions) the following diagram commutes

\[
\begin{array}{ccc}
\text{ev}_I & \longrightarrow & \text{Tr}_i \\
\downarrow & & \downarrow \\
\hat{M}^* & \longrightarrow & \hat{\hat{M}}^*
\end{array}
\]

Now suppose \(\bar{L} \subset A\) is an ideal, and \(L \subset R\) the unique \(R\)-ideal such that \(L \supset I\) and \(L/I = \bar{L}\). We then have the standard isomorphism

\[\text{RHom}^\bullet_A(B, \text{RHom}^\bullet_R(A, M^*)) \sim \text{RHom}^\bullet_R(B, M^*)\]

which, after replacing \(M^*\) by a complex of injective modules if necessary, amounts to the observation that elements in an \(R\)-module which are killed by \(I\) and also by \(\bar{L}\) are the exactly elements which are killed by \(L\). The following diagram clearly commutes

\[
\begin{array}{ccc}
\text{RHom}^\bullet_A(B, \text{RHom}^\bullet_R(A, M^*)) & \longrightarrow & \text{RHom}^\bullet_R(B, M^*) \\
\text{ev}_L & \downarrow & \text{ev}_L \\
\text{RHom}^\bullet_R(A, M^*) & \longrightarrow & M^*
\end{array}
\]

This means that the following diagram commutes (with \(j : \text{Spec } B \rightarrow \text{Spec } A\) the natural inclusion):

\[
\begin{array}{ccc}
\text{ev}_I \big|_{\text{ev}_L} & \longrightarrow & (ij) \big|_{\text{ev}_L} \\
\downarrow & \uparrow & \uparrow \\
\text{RHom}^\bullet_A(B, \text{RHom}^\bullet_R(A, M^*)) & \longrightarrow & \text{RHom}^\bullet_R(B, M^*) \\
\text{RHom}^\bullet_R(A, M^*) & \longrightarrow & M^*
\end{array}
\]

Suppose \(I\) is generated by \(t = (t_1, \ldots, t_d)\), \(\bar{L}\) is generated by \(\bar{u} = (\bar{u}_1, \ldots, \bar{u}_e)\), and \(u_i \in L\) are lifts of \(\bar{u}_i\) for \(i = 1, \ldots, e\). Set \(u = (u_1, \ldots, u_e)\). Suppose \((t, u)\) is a quasi-regular sequence in \(R\) (so that \(\bar{u}\) is quasi-regular in \(A\)). The map

\[\text{ev}_t : M^* \otimes_R K^* (t) \longrightarrow M^*\]

corresponding to \(\text{ev}_I\) under the isomorphism \(M^* \otimes K^* (t) \sim \text{RHom}^\bullet_R(A, M^*)\) (cf. \((C.2.3)\)) is the map which in degree \(n\) is

\[(C.6.4) \quad (M^* \otimes_R K^* (t))^n = \bigoplus_{p+q=n} M^p \otimes_R K^q (t) \xrightarrow{\text{projection}} M^n \otimes_R K^0 (t) = M^n.\]
(Note that the map $e_t$ is defined even if $t$ is not quasi-regular, so that in particular $e_u$ makes sense.) The following diagram clearly commutes

\[ \begin{array}{ccc} M^* \otimes_R K^*(t) \otimes_R K^*(u) & \rightarrow & M^* \otimes_R K^*(t, u) \\ e_u \downarrow & & \downarrow e_{(t, u)} \\ M^* \otimes_R K^*(t) & \rightarrow & M^* \end{array} \]

An obvious re-interpretation of this, in our case, is that the following diagram commutes (we are implicitly using the fact that if $N$ is an $A$-module, then $e_u = e_{\bar{u}}$ on $N \otimes_R K^*(u) = N \otimes_A K^*(\bar{u})$):

\[ \begin{array}{ccc} M^* \otimes_R K^*(t, u) & \rightarrow & \tilde{\text{RHom}}_R(B, M^*) \\ \downarrow & & \downarrow \text{(C.2.3)} \\ M^* \otimes_R K^*(t) \otimes_R K^*(u) & \rightarrow & \text{RHom}_R(A, M^*) \otimes_R K^*(u) \\ \downarrow & & \downarrow \text{(C.6.5)} \\ \text{RHom}_R(A, M^*) \otimes_A K^*(\bar{u}) & \rightarrow & \text{RHom}_A(B, \text{RHom}_R(A, M^*)) \end{array} \]

Setting $n = d + e$ we have an isomorphism of rank one free $A$-modules

\[ \alpha: (\wedge_R^d I/I^2)^* \otimes_R (\wedge_A^d L/L^2)^* \xrightarrow{\sim} (\wedge_A^n L/L^2)^* \]

given by $1/t \otimes 1/\bar{u} \mapsto 1/(t, u)$.

The role of the hypotheses on the Tor$(-, \bullet)$ functors in the statement of Proposition C.6.6 below is the following: Suppose $S$ is a ring, $P$, $Q$ $S$-modules such that Tor$^S_i(P, Q) = 0$ for $i \neq 0$. Then $P \otimes_S Q$ is canonically isomorphic to $P \otimes S Q$ and we treat this as an identity, i.e., in this case we write $P \otimes_S Q = P \otimes_S Q$. In particular if $J$ is an $S$-ideal generated by a quasi-regular sequence, and Tor$^S_i(P, S/J) = 0$, then we have $P \otimes_S (\wedge_{S/J}^m J/J^2)^* = P \otimes_S (\wedge_{S/J}^m J/J^2)^* = (P \otimes_S S/J) \otimes_{S/J} (\wedge_{S/J}^m J/J^2)^*$.

In other words, if $\mathcal{F} = \bar{P}$, the quasi-coherent sheaf on $W = \text{Spec} S$ corresponding to $P$, and $u: Z = \text{Spec} S/J \rightarrow W$ the natural closed immersion, we have

\[ u^* \mathcal{F}[m] = L u^* \mathcal{F}[m] \otimes_{\mathcal{O}_Z} (\mathcal{N}_m[-m]) \]
\[ = L u^* \mathcal{F} \otimes_{\mathcal{O}_Z} (\mathcal{N}_m[-m])[m] \]
\[ = (L u^* \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N}_m)[0] \]
\[ = (u^* \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N}_m)[0]. \]

**Proposition C.6.6.** Let $R$, $A$, $B$, $I$, $L$, $t$, $u$, $i$, $j$, be as above with $(t, u)$ being a quasi-regular sequence in $R$. Let $M$ be an $R$-module, $\mathcal{F} = \bar{M}$, the quasi-coherent $\mathcal{O}_{\text{Spec} R}$-module corresponding to $M$. Suppose we have Tor$^R_i(M, A) =$
\[ \text{Tor}^R(M, B) = \text{Tor}^A(M/IM, B) = 0 \text{ for } i \neq 0. \] Then the following diagram commutes:

\[
\begin{array}{ccc}
(j^*i^*(F) \otimes \mathcal{N}_{ij}^n)[0] & \xrightarrow{\sim} & (ij)^*(F[n]) \\
\downarrow \text{via } \alpha & & \downarrow \text{via } \eta'_i \\
(j^*i^*(F) \otimes j^*\mathcal{N}_{i}^d \otimes \mathcal{N}_{j}^e)[0] & \xrightarrow{\sim} & j^*(i^*(F[d])[e]) \\
\downarrow \text{via } \eta'_j \text{ and } \eta'_i & & \downarrow \text{via } \eta'_j \text{ and } \eta'_i \\
j^*(i^*(F[d])[e]) & \xrightarrow{\sim} & j^*(F[n])
\end{array}
\]

**Proof.** For any noetherian ring \( S \) and \( S \)-ideal \( J \) generated by a quasi-regular sequence \( v = (v_1, \ldots, v_m) \), and every \( S \)-module \( P \), we have, with \( S = S/J \), a map of complexes

\[ w_{S,v} = w_{S,v,P} : P[m] \otimes_S K^\bullet(v) \to P \otimes_S \wedge^m(J/J^2)^\bullet[0] \]

defined on 0-cochains by

\[ x \mapsto (-1)^m x \otimes 1/v, \quad x \in P = (P[m] \otimes_S K^\bullet(v))^0. \]

Since \( P[m] \otimes_S K^\bullet(v) \) is a complex which is zero in positive degrees and the complex \( P \otimes_S \wedge^m(J/J^2)^\bullet[0] \) is concentrated in degree 0, the above recipe defines \( w_{S,v} \). Moreover, if \( \text{Tor}^S_j(P, S) = 0 \) for \( j \neq 0 \), then in \( \mathbf{D}(\text{Mod}_S) \) the image of the map \( w_{S,v} \) under the localization functor is the composite

\[
P[m] \otimes_S K^\bullet(v) \xrightarrow{\sim} P[m] \otimes_S S[-m] \xrightarrow{L} P[m] \otimes_S (\wedge^m(J/J^2)^\bullet[-m]) = P[m] \otimes_S (\wedge^m(J/J^2)^\bullet[-m]) = P \otimes_S (\wedge^m(J/J^2)^\bullet)[0]
\]

where the first arrow is (C.2.1) and the second arrow \( 1 \otimes \lambda_v[-m] \). In other words the following diagram in \( \mathbf{D}(\text{Mod}_S) \), consisting of isomorphisms, commutes:

\[
\begin{array}{ccc}
P[m] \otimes_S K^\bullet(v) & \xrightarrow{w_{S,v}} & P \otimes_S (\wedge^m(J/J^2)^\bullet)[0] \\
\downarrow \ & \ & \downarrow \eta_{S,\overline{S}} \ & \rightarrow \ & \rightarrow \\
P \otimes_S (\wedge^m(J/J^2)^\bullet)[0] & \xrightarrow{\sim} & R\text{Hom}_S(S, P[m])
\end{array}
\]

(see (C.2.5) for the definition of \( \eta_{S,\overline{S}} \)). In view of these observations, as well the commutativity of (C.6.2) and (C.6.3), we are done if we show that the following
Using (C.6.2) and (C.2.13) (the latter for \(A\) and (C.7.1) \(\phi\)), diagram commutes where for convenience we use \(N\) to denote \(\wedge^d_A(I/I^2)^*\).

\[
\begin{array}{ccc}
M[n] \otimes_R K^*(t, u) & \xrightarrow{w_{R,(t,u)}} & M \otimes_R (\wedge^n_B L/L^2)^*[0] \\
\downarrow & & \downarrow \{1 \otimes \alpha\}[0] \\
(M[d] \otimes_R K^*(t))[c] \otimes_R K^*(u) & \xrightarrow{w_{R,t}} & (M \otimes_R N)[e] \otimes_A K^*(\bar{u}) \xrightarrow{w_{A,\bar{u}}} M \otimes_R N \otimes_A (\wedge^n_B \bar{L}/\bar{L}^2)^*[0]
\end{array}
\]

Indeed, we only have to check on 0-cochains as we argued earlier. Let \(m \in M\) be an element. Regard it as a 0-cochain of the complex on the northeast corner. Its image in \(M \otimes_R (\wedge^d_A I/I^2)^* \otimes_A (\wedge^n_B L/L^2)^*\) in the southeast corner under the composite \(w_{A,\bar{u}} \circ w_{R,t}\) is \((-1)^n m \otimes 1/\bar{t} \otimes 1/\bar{u}\), and its image in \(M \otimes_R (\wedge^n_B L/L^2)^*\) in the northeast corner (via \(w_{R,(t,u)}\)) is \((-1)^n m \otimes 1/(t, u)\). This proves our assertion. \(\square\)

C.7. This is a slightly different but related exploration of duality for compositions of closed immersions. So, as before, suppose \(R\) is a noetherian ring, \(I, J\) ideals in \(R\), \(I \subset J\), \(I\) (resp. \(J\)) generated by a regular sequence \(\{t_1, \ldots, t_r\}\) (resp. \(\{g_1, \ldots, g_r\}\)). Note that the number of \(t\)'s equals the number of \(g\)'s.

In this set-up, let \(t_i = \sum u_{ij} g_j, A = R/I, B = R/J = A/J\), where \(J = JA\). Let \(i\): Spec \(A \hookrightarrow\) Spec \(R, j\): Spec \(B \hookrightarrow\) Spec \(R\), and \(h\): Spec \(B \hookrightarrow\) Spec \(A\) be the closed immersions corresponding respectively to the surjections \(R \twoheadrightarrow A, R \twoheadrightarrow B\), and \(A \twoheadrightarrow B\). Then \(i \circ h = j\). We have a composite

\[
\phi^j_i : h_* h^! j^! \xrightarrow{\mathcal{H}} h_* j^! \xrightarrow{\mathcal{H}} i^!.
\]

Using (C.6.2) and (C.2.13) (the latter for \(i\) and \(j\)), this corresponds to a map

\[
\phi^h_i : h_* j^* \to i^*.
\]

In particular, for an \(R\)-module \(M\), \(H^n(\phi^h_i(M))\) gives us a map

(C.7.1) \(\phi_h : M \otimes_R (\wedge^n_B J/J^2)^* \to M \otimes_R (\wedge^n_A I/I^2)^*\).

From the definition of \(U^*(M)\) in (C.7.1), it is straightforward that for an \(R\)-module \(M\), the following diagram commutes.

\[
\begin{array}{ccc}
\mathbf{R} \text{Hom}_R(A, M) & \to & \mathbf{R} \text{Hom}_R(B, M) \\
\downarrow & & \downarrow \\text{(C.7.1)} \\
M \otimes_R K^*(t) & \xrightarrow{U^*} & M \otimes_R K^*(s)
\end{array}
\]

commutes. The unlabelled arrow is the one arising from the map \(A \twoheadrightarrow B\). Unwinding all the definitions, we see that \(\phi_h(m \otimes 1/g) = \det(u_{ij})m \otimes 1/t\). Moreover if \(H^*_f(M) \to H^*_f(M)\) is the natural map arising from the inclusion \(\Gamma_J \hookrightarrow \Gamma_I\), the element \([g_1, \ldots, g_r]\) maps to \([\det(u_{ij})]m\].

These are a well-known results (see, e.g., [HK1] §3, pp.71–72 or [L2] Chap. III, §7, pp.59–60]). We record them for completeness. It should be pointed out that (ii) and (iii) in Theorem C.7.2 do not need \(g\) or \(t\) to be regular sequences.

**Theorem C.7.2.** Let \(R, I, J, t, g\) and \(u_{ij}\) be as above.
(i) Let $\phi_h: M \otimes (\wedge^r_B J/J^2)^* \rightarrow M \otimes (\wedge^r_A I/I^2)^*$ be as in (C.7.1). Then

$$\phi_h \left( m \otimes \frac{1}{g} \right) = \det(u_{ij}) m \otimes \frac{1}{t}.$$ 

(ii) If $\psi: H^r_J(M) \rightarrow H^r_I(M)$ is the natural map arising from the inclusion $\Gamma_J \hookrightarrow \Gamma_I$, then

$$\psi \left( \begin{bmatrix} m \\
 g_1, \ldots, g_r \end{bmatrix} \right) = \begin{bmatrix} \det(u_{ij})m \\
 t_1, \ldots, t_r \end{bmatrix}$$

(iii) If $\sqrt{I} = \sqrt{J}$, so that $H^r_I(M) = H^r_J(M)$, then

$$\begin{bmatrix} m \\
 g_1, \ldots, g_r \end{bmatrix} = \begin{bmatrix} \det(u_{ij})m \\
 t_1, \ldots, t_r \end{bmatrix}$$

Proof. Part (i) has been established. We elaborate a bit on (ii) and (ii). Let $Z = \text{Spec } A$ and $W = \text{Spec } B$. The natural maps $i_*h^b \rightarrow R\Gamma_Z$ and $j_*j^b \rightarrow R\Gamma_W$ fit into a commutative diagram, as the reader can readily verify:

Here $Tr_h^b$ is the map in (B.1.1). Parts (ii) and (iii) then follow from (i).

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References

[EGA] A Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique, Publ. Math. IHES, 4, 8, 11, 17, 20, 24, 28, 32, Paris, 1960-7.

[RD] R. Hartshorne, Residues and Duality, Lecture Notes in Math., no. 20, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[AJL1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, Local homology and cohomology on schemes, Ann. Sci. École Norm. Sup. (4) 30, 1997, 1–39; see also correction, in: Collected Papers of Joseph Lipman, vol 2, Queen’s Papers in Pure and Appl. Math., vol 117, Queen’s Univ., Kingston, ON, Canada, 2000, p. 879.

[AJL2], Duality and flat base change on formal schemes. Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes. Contemp. Math., vol. 244, Amer. Math. Soc., Providence, RI, 1999, 3–90.

[AJL3], Greenlees-May Duality on Formal Schemes. Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes. Contemp. Math., vol. 244, Amer. Math. Soc., Providence, RI, 1999, 91–112.

[BB] A. Bondal, M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Moscow Math. J., 3 (2003), pp. 1–36.

[C1] B. Conrad, Grothendieck Duality and Base Change, Lecture Notes in Math., no. 1750, Springer, New York, 2000.
B. Conrad, Deligne’s notes on Nagata’s Compactification, J. Ramanujan Math. Soc., 22, no. 3, 2007, 205–257.

P. Deligne, Cohomologie à support propre et construction du foncteur $f^!$. Appendix to [RD] above, 404–421.

La formule de dualité globale, ibid.

La théorème de plongement de Nagata, Kyoto J. Math., 50, number 4, 2010, 661–670.

R. Hüb and E. Kunz, Integration of differential forms on schemes, J. Reine Angew. Math., 410, 1990, 53–83.

R. Hüb and P. Sastry, Regular differential forms and relative duality for projective morphisms, J. Reine Angew. Math., 115 (4), 1993, 749–787.

S. B. Iyengar, J. Lipman, and A Neeman, Relation between two twisted inverse image pseudofunctors in duality theory, Compositio Math., 151, 2015, 735–764.

S. Kleiman, Relative duality for quasi-coherent sheaves, Compositio Math., 41, 1980, 39–60.

R. Hüb and E. Kunz, Kähler Differentials, Vieweg., Braunschweig-Wiesbaden, 1986.

R. Wald, Regular Differential Forms, Contemp. Math. 79, Amer. Math. Soc., Providence, RI, 1988.

J. Lipman, Double point resolutions of deformations of rational singularities, Compositio Math., 38, 1979, 37–43.

Dualizing sheaves, differentials and residues on algebraic varieties, Asterisque 117, Société Mathématique de France, 1984.

Lectures on local cohomology and duality, Local cohomology and its applications, (Guanajuato, 1999), Lecture Notes in Pure and App. Math., 226, Dekker, New York, 2002, 39–89.

Notes on derived categories and Grothendieck Duality. Foundations of Grothendieck duality for diagrams of schemes. Lecture Notes in Math., 1960, Springer-Verlag, Berlin-New York, 2009, 1–259.

S. Nayak, and P. Sastry, Pseudofunctorial behaviour of Cousin complexes. Variance and Duality for Cousin Complexes on Formal Schemes. Contemp. Math., 375, Amer. Math. Soc., Providence, RI, 2005, 3–133.

Quasi-perfect scheme-maps and boundedness of the twisted inverse image, Illinois J. Math., vol. 51, no. 1, 2007, 209–236.

On the fundamental class of an essentially smooth scheme map. Algebraic Geometry, Foundation Compositio Mathematica, 5, (2), 2018, 151-159. (doi:10.14231/AG-2018-005)

P. Sastry, Regular differentials and equidimensional scheme maps, J. Algebraic Geom., 1, 1992, 101–130; see also correction by P. Sastry in J. Algebraic Geom., 14, 2005, 593–599.

On compactification of schemes, Manuscripta Math., 80:95—111, 1993.

M. Nagata, Imbedding an Abstract Variety in a Complete Variety, J. Math. Kyoto Univ., 2:1—10, 1962.

S. Nayak, Pasting Pseudofunctors. Variance and Duality for Cousin Complexes on Formal Schemes. Contemp. Math., 375, Amer. Math. Soc., Providence, RI, 2005, 195–271.

Grothendieck Duality and Transitivity II: Traces and Residues via Verdier’s Isomorphism, arXiv:1903.01783 [math.AG], 2019.

The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, Jour. of the AMS, vol. 9, no. 1, 1996, 205–236.

An improvement on the base-change theorem and the functor $f^!$, arXiv:1406.7599 [math.AG], 2017.

Traces and Residues, Indiana University Mathematics Journal, Vol. 64, No. 1, 2015, 217–229.
[R] M. Rosenlicht, Differentials of the second kind for algebraic function fields of one variable, *The Annals of Math.*, Ser. 2, **57**, 1953, 517–523.

[S1] P. Sastry, Residues and duality for algebraic schemes, *Compositio Math.*, **101**, 1996, 133–179.

[S2] ———, Base change and duality for Cohen-Macaulay maps, *Compositio Math.*, **140**, 2004, 729–777.

[S3] ———, Duality for Cousin Complexes. *Variance and Duality for Cousin Complexes on Formal Schemes*. Contemp. Math., **375**, Amer. Math. Soc., Providence, RI, 2005, 135–192.

[V] J-L. Verdier, *Base change for twisted inverse image of coherent sheaves*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford University Press, London, 1969, 393–408.

[Vo] P. Vojta, *Nagata embedding theorem*, [arXiv:0706.1907], 2007.

Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore, Karnataka-560059, INDIA

E-mail address: snayak@isibang.ac.in

Chennai Mathematical Institute, SIPCOT IT park, Siruseri, Kanchipuram Dist TN, 603103, INDIA

E-mail address: pramath@cmi.ac.in