Non-SUSY Lepton Flavor Model with 3HDM

Yukimura Izawa \(^1\), Yusuke Shimizu \(^{1,2}\)†, Hironori Takei \(^{1} \dagger\)

\(^1\)Physics Program, Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

\(^2\)Core of Research for the Energetic Universe, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

Abstract

We propose a simple non-supersymmetric lepton flavor model with \(A_4\) symmetry. The \(A_4\) group is a minimal one which includes triplet irreducible representation. We introduce three Higgs doublets which are assigned as triplet of the \(A_4\) symmetry. It is natural that there are three generations of the Higgs fields as same as the standard model fermions. We analyse the potential and we get the vacuum expectation values for the local minimum. In the vacuum expectation values, we obtain the charged lepton, Dirac neutrino, and right-handed Majorana neutrino mass matrices. By using type-I seesaw mechanism, we get the left-handed Majorana neutrino mass matrix. In the NuFIT 5.1 data, we predict the Dirac CP phase and the Majorana phases for the only inverted neutrino mass hierarchy. Especially, the Dirac CP phase and lepton mixing angle \(\theta_{23}\) are strongly correlated. If the \(\theta_{23}\) is more precise measured, the Dirac CP phase is more precise predicted, and vice versa. We also predict the effective mass for neutrino-less double beta decay \(m_{ee} \simeq 47.1 \ [\text{meV}]\) and the lightest neutrino mass \(m_3 \simeq 0.789 - 1.43 \ [\text{meV}]\). It is testable for our model in the near future neutrino experiments.

\*izawa-yukimura@hiroshima-u.ac.jp
\†yu-shimizu@hiroshima-u.ac.jp
\‡t-hironori@hiroshima-u.ac.jp
1 Introduction

The standard model (SM) is the successful one with the discovery of the Higgs boson. In the particle physics, the gauge theory is applied and tested by the electroweak precision measurements for the SM. However there are still mysterious puzzles, e.g. the origin of the generations which are differences of the mixing angles and masses for quark and lepton sectors. Actually, the Yukawa couplings are completely free parameters so that the mixing angles and masses cannot be predicted in the SM. In addition, the neutrinos are massless for renormalizable operators. One of the attractive phenomena to solve the puzzles is the neutrino oscillation which provides us the useful informations such as three lepton mixing angles and two neutrino mass squared differences which mean that the neutrinos have non-zero masses. The T2K and NOνA experiments have confirmed the neutrino oscillation in the $\nu_\mu \rightarrow \nu_e$ appearance events [1–3], which are one of the clues of the new physics beyond the SM such as the Dirac CP violating phase for the lepton sector by combining the data of the reactor neutrino experiments [4–5]. The KamLAND-Zen [6,7], GERDA [8,9], and CUORE [10,11] experiments also provide us the significant informations which are whether the neutrinos are Dirac or Majorana particles, the lepton number violation, and Majorana phases if the neutrinos are Majorana particles. Thus the neutrino oscillation experiments go into a new phase of the precise determinations of the lepton mixing angles, the neutrino mass squared differences, and the CP violating phases.

The SM particles obey the gauge theory. After spontaneous symmetry breaking (SSB), the gauge bosons and fermions get the masses through the Higgs mechanism. However the Yukawa couplings cannot be controlled by the gauge symmetry, the Yukawa couplings are completely free parameters in the SM. The flavor symmetry can apply to the generations. The Froggatt-Nielsen mechanism which was proposed by C. D. Froggatt and H. B. Nielsen are introduced global $U(1)$FN symmetry [12]. Thanks for the $U(1)$FN symmetry, it is natural to explain the fermion mass hierarchies. On the other hand, the non-Abelian discrete symmetry (See for the review [13]- [17].) can naturally explain the lepton mixing angles so-called “tri-bimaximal mixing (TBM)” [18,19] before the reactor experiments reported the non-zero reactor angle $\theta_{13}$ [4,5]. Actually, many authors have studied the breaking or deviation from the TBM [20]- [42] or other patterns of the lepton mixing angles, e.g. tri-bimaximal-Cabibbo mixing [43,44]. One of the successful flavor models was proposed by G. Altarelli and F. Feruglio [45,46]. The Altarelli and Feruglio (AF) model show the TBM by using the non-Abelian discrete symmetry $A_4$. They introduced two SM gauge singlet scalar fields so-called “flavons” and taking the vacuum expectation value (VEV) alignments of the $A_4$ triplets as $(1,0,0)$ and $(1,1,1)$, which is naturally explained that the charged lepton is diagonal and neutrino mixing is TBM, respectively. However the corrected VEV alignments cannot be driven from the potential analysis. Then, they applied to the supersymmetry (SUSY) and introduced the “driving” fields. There are so many scalar fields in addition to the no evidence of the SUSY particles for the accelerator experiments, e.g. Large Hadron Collider experiment.

In this paper, we propose a simple non-SUSY lepton flavor model with $A_4$ symmetry. The $A_4$ group is a minimal one which includes triplet irreducible representation. We introduce three Higgs doublets which is assigned as triplet of the $A_4$ symmetry. It is natural that there are three generations as same as the SM fermions. We analyse the potential and we get the VEV for the local minimum in the three Higgs doublet model (3HDM) [47] with $A_4$ symmetry [48]- [61]. The left-handed lepton doublets are assigned to triplet and the right-handed charged leptons
are assigned to different singlets of the $A_4$ symmetry, respectively. We introduce the right-handed Majorana neutrinos which are assigned to triplet of the $A_4$ symmetry. In our model, the right-handed Majorana neutrino mass matrix has a simple flavor structure. On the other hand, the Dirac neutrino mass matrix has symmetric and anti-symmetric Yukawa couplings for $A_4$ symmetry. By using the type-I seesaw mechanism [62-66], we obtain the left-handed Majorana neutrino mass matrix. After diagonalizing the charged lepton and left-handed Majorana neutrino mass matrices, we get the lepton mixing matrix which is Pontecorvo-Maki-Nakagawa-Sakata (PMNS) one [67,68]. In the numerical analysis, we use the NuFIT 5.1 data [69,70]. We find that only inverted ordering is acceptable and we cannot find the solutions for normal ordering in the neutrino mass hierarchy. We obtain relevant relations for mixing angles and the effective mass for the neutrino-less double beta ($0\nu\beta\beta$) decay as a function of the lightest neutrino mass. Especially, the Dirac CP phase and lepton mixing angle $\theta_{23}$ are strongly correlated. If the neutrinos are Majorana particles, the effective mass for the neutrino-less double beta decay and the lightest neutrino mass are also predicted in our model.

This paper is organized as follows. In Section 2, we briefly introduce the $A_4$ group. Next, we analyse the potential with $A_4$ symmetry. In Section 3, we present the $A_4$ flavor model and study mass matrices. In Section 4, we show the numerical analysis of our model. Section 5 is devoted to a summary and discussions. We show the relevant multiplication rule of the $A_4$ group in Appendix [A].

2 Potential analysis in the 3HDM with $A_4$ symmetry

In this section, we discuss the 3HDM. First, we briefly introduce the $A_4$ group. Next, we analyse the potential with $A_4$ symmetry in the 3HDM, where we assign the three Higgs doublets as the triplet of the $A_4$ symmetry. It is natural that the Higgs fields are three generations as same as the SM fermions.

Let us analyse the potential in the 3HDM with $A_4$ symmetry. The $A_4$ group which is a minimal including triplet irreducible representation is the symmetric group of a tetrahedron or even permutation of four elements. There are twelve elements and four irreducible representations such as three different singlets $1$, $1'$, $1''$ and triplet 3 in the $A_4$ group, respectively. Also the $A_4$ can be defined as the group generated by two elements $S$ and $T$ which satisfy the following algebraic relations as

$$S^2 = T^3 = (ST)^3 = 1.$$  \hfill (1)

These generators are represented by

$$1 : \quad S = 1, \quad T = 1,$$

$$1' : \quad S = 1, \quad T = e^{\frac{4\pi}{3}} \equiv \omega^2,$$

$$1'' : \quad S = 1, \quad T = e^{\frac{2\pi}{3}} \equiv \omega,$$  \hfill (2)

on the one-dimensional representations. These generators are also represented by

$$3 : \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$  \hfill (3)
on the three-dimensional representation. In these bases Eqs. (2) and (3), we can make the character table and obtain the multiplication rules of the $A_4$ group. The relevant multiplication rule is shown in Appendix A.

We introduce three Higgs doublets $\phi_1, \phi_2, \phi_3$ which are assigned as triplet $\Phi$:

$$\Phi = (\phi_1, \phi_2, \phi_3), \quad (4)$$

of the $A_4$ symmetry. On the other hand, the complex conjugate of the $\Phi$ is considered by the conjugate of the generator $T$ of Eq. (3) as

$$T^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (5)$$

Then, the complex conjugate of the $\Phi$ is given by

$$\Phi^* = (\phi_1^*, \phi_3^*, \phi_2^*), \quad (6)$$

and the multiplication rule of $A_4$ is kept as Eq. (51) in Appendix A. In our model, the Higgs potential is written as

$$V = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \quad (7)$$

This potential is invariant for the $SU(2)_L \times U(1)_Y$ of the SM and $A_4$ symmetry. By using the multiplication rule of $A_4$ in Appendix A, we obtain the Higgs potential as follows:

$$V = -\mu^2 (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2) + \lambda_1 |\phi_1|^2 + 2|\phi_2 \phi_3| + \lambda_2 |\phi_2|^2 + 2|\phi_3 \phi_1|^2 + \lambda_3 |\phi_3|^2 + 2|\phi_1 \phi_2|^2 + \lambda_4 \left[|\phi_1 - \phi_2 \phi_3|^2 + |\phi_2 - \phi_3 \phi_1|^2 + |\phi_3 - \phi_1 \phi_2|^2\right], \quad (8)$$

where we rewrite the coupling $4\lambda_4/9$ as $\lambda_4$ in our convention. We consider the potential minimum conditions as

$$\left(\frac{\partial V}{\partial \phi_1}\right)_{\phi_1=\langle \phi_1 \rangle, \phi_2=\langle \phi_2 \rangle, \phi_3=\langle \phi_3 \rangle} = 0, \quad (9)$$

$$\left(\frac{\partial V}{\partial \phi_2}\right)_{\phi_1=\langle \phi_1 \rangle, \phi_2=\langle \phi_2 \rangle, \phi_3=\langle \phi_3 \rangle} = 0, \quad (10)$$

$$\left(\frac{\partial V}{\partial \phi_3}\right)_{\phi_1=\langle \phi_1 \rangle, \phi_2=\langle \phi_2 \rangle, \phi_3=\langle \phi_3 \rangle} = 0. \quad (11)$$

From Eqs. (8)-(11) we obtain the following conditions:

$$0 = -2\mu^2 v_1 + 4\lambda_1 v_1 (v_1^2 + 2v_2v_3) + 4\lambda_2 v_3 (v_2^2 + 2v_3v_1) + 4\lambda_3 v_2 (v_3^2 + 2v_1v_2) + 4\lambda_4 \left[ v_1 (v_1^2 - v_2v_3) - \frac{1}{2} v_2 (v_2^2 - v_1v_2) - \frac{1}{2} v_3 (v_3^2 - v_3v_1) \right], \quad (12)$$

$$0 = -2\mu^2 v_2 + 4\lambda_1 v_3 (v_1^2 + 2v_2v_3) + 4\lambda_2 v_2 (v_2^2 + 2v_3v_1) + 4\lambda_3 v_1 (v_3^2 + 2v_1v_2) + 4\lambda_4 \left[ v_2 (v_2^2 - v_3v_1) - \frac{1}{2} v_1 (v_1^2 - v_1v_2) - \frac{1}{2} v_3 (v_3^2 - v_2v_3) \right], \quad (13)$$

$$0 = -2\mu^2 v_3 + 4\lambda_1 v_2 (v_1^2 + 2v_2v_3) + 4\lambda_2 v_1 (v_2^2 + 2v_3v_1) + 4\lambda_3 v_3 (v_3^2 + 2v_1v_2) + 4\lambda_4 \left[ v_3 (v_3^2 - v_1v_2) - \frac{1}{2} v_1 (v_1^2 - v_3v_1) - \frac{1}{2} v_2 (v_2^2 - v_2v_3) \right]. \quad (14)$$
We sum the conditions Eqs. (12)-(14) and we obtain a equation as follows:

\[
0 = (v_1 + v_2 + v_3) \left[ -2\mu^2 + 4\lambda_1 (v_1^2 + 2v_1v_2) + 4\lambda_2 (v_2^2 + 2v_2v_3) + 4\lambda_3 (v_3^2 + 2v_3v_1) \\
+ 4\lambda_4 (v_1^2 + v_2^2 + v_3^2 - v_1v_2 - v_2v_3 - v_3v_1) \right]
\]

\[
= 4(v_1 + v_2 + v_3) \left[ (\lambda_1 + \lambda_4)v_1^2 + \{(2\lambda_3 - \lambda_4)v_2 + (2\lambda_2 - \lambda_4)v_3\}v_1 \\
+ (\lambda_2 + \lambda_4)v_2^2 + (\lambda_3 + \lambda_4)v_3^2 + (2\lambda_1 - \lambda_4)v_2v_3 - \frac{\mu^2}{2}\right].
\]  

(15)

When \( v_1 + v_2 + v_3 \neq 0 \) is satisfied, we obtain

\[
(\lambda_1 + \lambda_4)v_1^2 + \{(2\lambda_3 - \lambda_4)v_2 + (2\lambda_2 - \lambda_4)v_3\}v_1 \\
+ (\lambda_2 + \lambda_4)v_2^2 + (\lambda_3 + \lambda_4)v_3^2 + (2\lambda_1 - \lambda_4)v_2v_3 - \frac{\mu^2}{2} = 0.
\]  

(16)

If \( \lambda_1 + \lambda_4 \) which is the coefficient of \( v_1^2 \) in Eq. (16) holds zero, we find the solution \((1, 1, 1)\). In this case we cannot realize the current experimental data. When \( \lambda_1 + \lambda_4 \neq 0 \), \( \lambda_3 = \lambda_2 \), and \( 2\lambda_1 + 2\lambda_2 + \lambda_4 \neq 0 \) hold, we obtain the following solutions:

\[
\langle \phi_1 \rangle = v_1, \\
\langle \phi_2 \rangle = v_2
\]

\[
= -\frac{2\lambda_2 - \lambda_4}{2\lambda_1 + 2\lambda_2 + \lambda_4}v_1 \\
\pm \sqrt{\left\{-2\lambda_1^2 + 2\lambda_2^2 - 2\lambda_2(\lambda_1 - 3\lambda_4) - 3\lambda_1 \lambda_4\right\}v_1^2 + \frac{1}{2}(2\lambda_1 + 2\lambda_2 + \lambda_4)\mu^2 \over 2\lambda_1 + 2\lambda_2 + \lambda_4}
\]

\[
\langle \phi_3 \rangle = v_3
\]

\[
= -\frac{2\lambda_2 - \lambda_4}{2\lambda_1 + 2\lambda_2 + \lambda_4}v_1 \\
\pm \sqrt{\left\{-2\lambda_1^2 + 2\lambda_2^2 - 2\lambda_2(\lambda_1 - 3\lambda_4) - 3\lambda_1 \lambda_4\right\}v_1^2 + \frac{1}{2}(2\lambda_1 + 2\lambda_2 + \lambda_4)\mu^2 \over 2\lambda_1 + 2\lambda_2 + \lambda_4}
\]  

(19)

These VEVs can be written as

\[
v_1 = v \cos \beta, \quad v_2 = \frac{v}{\sqrt{2}} \sin \beta, \quad v_3 = \frac{v}{\sqrt{2}} \sin \beta,
\]

(20)

where, the range of \( \lambda \) are \( \lambda_1 < \lambda_2 \) and \( \lambda_1 < \lambda_4 \). These \( \lambda \) conditions are derived from minimum conditions of Higgs potential.

Note that there are other two solution forms in Eq. (20). When \( \lambda_3 = \lambda_1, \lambda_2 + \lambda_4 \neq 0 \) and \( 2\lambda_1 + 2\lambda_2 + \lambda_4 \neq 0 \) hold, we obtain

\[
v_1 = \frac{v}{\sqrt{2}} \sin \beta, \quad v_2 = v \cos \beta, \quad v_3 = \frac{v}{\sqrt{2}} \sin \beta.
\]  

(21)

On the other hand, when \( \lambda_2 = \lambda_1, \lambda_3 + \lambda_4 \neq 0 \) and \( 2\lambda_1 + 2\lambda_3 + \lambda_4 \neq 0 \) hold, we obtain

\[
v_1 = \frac{v}{\sqrt{2}} \sin \beta, \quad v_2 = \frac{v}{\sqrt{2}} \sin \beta, \quad v_3 = v \cos \beta.
\]  

(22)

In the next section, we present our \( A_4 \) model and calculate the mass matrices.
3 Lepton flavor model in the $A_4$ symmetry

In this section, we present a non-SUSY lepton flavor model in the $A_4$ symmetry. The left-handed lepton doublets are assigned to triplet and the right-handed charged leptons are assigned to different singlets as $1$, $1''$, and $1'$ of the $A_4$ symmetry, respectively. We introduce the right-handed Majorana neutrinos which are assigned to triplet of the $A_4$ symmetry. We also introduce three Higgs doublets which are assigned as triplet of the $A_4$ symmetry as discussed in Section 2.

In Table 1, we summarize the particle assignments of $SU(2)_L$ and $A_4$ symmetry.

| $SU(2)_L$ | $\ell = (\ell_e, \ell_\mu, \ell_\tau)$ | $e_R$ | $\mu_R$ | $\tau_R$ | $\nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})$ | $\Phi = (\phi_1, \phi_2, \phi_3)$ |
|-----------|-----------------|-------|--------|----------|-----------------|-----------------|
| $A_4$     | 3               | 1     | 1''    | 1'       | 2               | 3               |

Table 1: The charge assignments of $SU(2)_L \times A_4$ symmetry in our model.

We can write down the Lagrangian for Yukawa interactions and Majorana mass term in our model. The $SU(2)_L \times A_4$ invariant Lagrangian is written as

$$L_Y = L_\ell + L_D + L_M + h.c.,$$

where

$$L_\ell = y_e \bar{\ell} e \Phi e + y_\mu \bar{\ell} e \mu + y_\tau \bar{\ell} e \tau,$$

$$L_D = y_D \bar{\ell} \Phi \nu,$$

$$L_M = \frac{1}{2} M \bar{\nu} \nu.$$

Note that $y_e, y_\mu, y_\tau,$ and $y_D$ are Yukawa couplings and $M$ is the right-handed Majorana neutrino mass. After the SSB, three Higgs doublets have VEVs as $\langle \Phi \rangle = (v_1, v_2, v_3)$. In the charged lepton sector Eq. (24), the Yukawa interactions are rewritten as

$$y_e \bar{\ell} e \Phi e = y_e (\bar{\ell}_e \phi_1 + \bar{\ell}_\mu \phi_3 + \bar{\ell}_\tau \phi_2) e_R,$$

$$y_\mu \bar{\ell} e \mu = y_\mu (\bar{\ell}_e \phi_1 + \bar{\ell}_\mu \phi_3 + \bar{\ell}_\tau \phi_2) \tau_R,$$

$$y_\tau \bar{\ell} e \tau = y_\tau (\bar{\ell}_e \phi_1 + \bar{\ell}_\mu \phi_3 + \bar{\ell}_\tau \phi_2) \mu_R.$$

Then, the charged lepton mass matrix $M_\ell$ is obtained as

$$M_\ell = \begin{pmatrix} y_e v_1 & y_\mu v_2 & y_\tau v_3 \\ y_\mu v_3 & y_\mu v_1 & y_\tau v_2 \\ y_\tau v_2 & y_\mu v_3 & y_\tau v_1 \end{pmatrix}_{LR}.$$
Here and hereafter we take the left-right basis in the mass matrices. In order to obtain the left-side unitary mixing matrix $U_L$, we consider $M_\ell M_\ell^\dagger$ as

$$M_\ell M_\ell^\dagger = \begin{pmatrix} |y_e|^2 v_1^2 & |y_\mu|^2 v_2^2 & |y_\tau|^2 v_3^2 \\ |y_e|^2 v_1 v_3 + |y_\mu|^2 v_1 v_2 + |y_\tau|^2 v_2 v_3 & |y_e|^2 v_1 v_2 + |y_\mu|^2 v_2 v_3 + |y_\tau|^2 v_1 v_3 \\ \vdots & \vdots & \vdots \\ |y_e|^2 v_3^2 & |y_\mu|^2 v_2^2 & |y_\tau|^2 v_1^2 \end{pmatrix}. $$

Then, the elements of the mass matrix Eq. (31) are real and we can numerically calculate the charged lepton masses in Section 4. Next, we derive the Dirac neutrino mass matrix from Eq. (25).

By using the $A_4$ multiplication rule (See appendix A), we can make the singlet term. When we take the complex conjugate, we need to take care $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_3, \tilde{\phi}_2)$ because of the complex conjugate for the generator $T$ in Eq. (5). Since Eq. (25) is $3 \otimes 3 \otimes 3$ form in $A_4$ symmetry, we first make $3_S \otimes 3_A \in 3 \otimes 3$ as

$$y_D \ell \tilde{\Phi} = \frac{y_{DS}}{3} \begin{pmatrix} \ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2} - \ell_{\tilde{\phi}_3} \\ \ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3} - \ell_{\tilde{\phi}_1} \\ \ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1} - \ell_{\tilde{\phi}_2} \end{pmatrix}_{3S} + \frac{y_{DA}}{2} \begin{pmatrix} \ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3} \\ \ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1} \\ \ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2} \end{pmatrix}_{3A},$$

where $y_{DS}$ and $y_{DA}$ are the symmetric and anti-symmetric Dirac Yukawa couplings, respectively. Then, the Dirac neutrino Yukawa interaction in Eq. (25) can be written as follows:

$$y_D \ell \tilde{\Phi} \nu_R = \frac{y_{DS}}{3} \begin{pmatrix} \ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2} - \ell_{\tilde{\phi}_3} \\ \ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3} - \ell_{\tilde{\phi}_1} \\ \ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1} - \ell_{\tilde{\phi}_2} \end{pmatrix}_{3S} \otimes \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} + \frac{y_{DA}}{2} \begin{pmatrix} \ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3} \\ \ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1} \\ \ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2} \end{pmatrix}_{3A} \otimes \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix}$$

$$= \frac{y_{DS}}{3} \left[ (2 \ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2} - \ell_{\tilde{\phi}_3}) \nu_{R1} + (2 \ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3} - \ell_{\tilde{\phi}_1}) \nu_{R2} + (2 \ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1} - \ell_{\tilde{\phi}_2}) \nu_{R3} \right] + \frac{y_{DA}}{2} \left[ (\ell_{\tilde{\phi}_2} - \bar{\ell}_{\tilde{\phi}_3}) \nu_{R1} + (\ell_{\tilde{\phi}_3} - \bar{\ell}_{\tilde{\phi}_1}) \nu_{R2} + (\ell_{\tilde{\phi}_1} - \bar{\ell}_{\tilde{\phi}_2}) \nu_{R3} \right],$$

where $\tilde{\Phi} = -i\sigma_2 \Phi^*$ and the VEVs of $\tilde{\phi}_i$ are $\langle \tilde{\phi}_i \rangle = v_i$. Therefore, the Dirac neutrino mass matrix $M_D$ is obtained as

$$M_D = \frac{y_{DS}}{3} \begin{pmatrix} 2v_1 & -v_2 & -v_3 \\ -v_2 & 2v_3 & -v_1 \\ -v_3 & -v_1 & 2v_2 \end{pmatrix}_{LR} + \frac{y_{DA}}{2} \begin{pmatrix} 0 & -v_2 & v_3 \\ v_2 & 0 & -v_1 \\ -v_3 & v_1 & 0 \end{pmatrix}_{LR}. $$

Next we discuss the right-handed Majorana neutrino mass matrix. In Eq. (26), the right-handed Majorana neutrino mass term is decomposed as follows:

$$M \nu_R^c \nu_R = M(\bar{\nu}_{R1}^c \nu_{R1} + \bar{\nu}_{R2}^c \nu_{R3} + \bar{\nu}_{R3}^c \nu_{R2}).$$

Then, the right-handed Majorana neutrino mass matrix $M_R$ is

$$M_R = M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. $$

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By using the type-I seesaw mechanism \(^{62-66}\), the left-handed Majorana neutrino mass matrix \(m_\nu\) is written as

\[
m_\nu = -M_DM_R^{-1}M_D^T
\]  

where we redefine the Dirac Yukawa couplings \(y_{DS}/3 \to y_{DS}\) and \(y_{DA}/2 \to y_{DA}\) for simplicity. In our model, the Dirac neutrino mass matrix has symmetric and anti-symmetric Yukawa couplings for the \(A_4\) symmetry in Eq. \(35\). On the other hand, in Eq. \(37\), the right-handed Majorana neutrino mass matrix has a simple flavor structure, c.f., in the AF Model the Dirac neutrino mass matrix is simple. On the other hand, the right-handed Majorana neutrino mass matrix is the structure which derives the TBM in their model. In the next section, we show the numerical analysis.

### 4 Numerical analysis

In this section, we show the numerical analysis such as the lepton flavor mixing angles, Dirac CP phase, Majorana phases, and the effective mass for the \(0\nu\beta\beta\) decay. We discuss what is our model verifiable in the near future experiments.

In section 2, we have analyzed the Higgs potential and found the following solution\(^2\) which was derived from Higgs potential minimization as

\[
(v_1, v_2, v_3) = (v \cos \beta, \frac{v}{\sqrt{2}} \sin \beta, \frac{v}{\sqrt{2}} \sin \beta).
\]

Since charged lepton mass matrix Eq. \(31\) is only depend on three Yukawa couplings and Higgs VEVs. Then once we fix the Higgs VEVs, we can obtain charged lepton Yukawa couplings by solving the following equations and the unitary matrix which diagonalizes charged lepton mass matrix.

\[
\begin{align*}
\text{Tr}(M_\ell M_\ell^\dagger) &= m_e^2 + m_\mu^2 + m_\tau^2, \\
\det(M_\ell M_\ell^\dagger) &= m_e^2 m_\mu^2 m_\tau^2, \\
\left(\text{Tr}M_\ell M_\ell^\dagger\right)^2 - \text{Tr}\left[(M_\ell M_\ell^\dagger)^2\right] &= 2(m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2),
\end{align*}
\]

\(^2\)The solution form in Eq. \(21\) can be realised by taking charge assignments such that the left-handed lepton doublets are assigned to triplet \(\bar{\ell} = (\ell_\tau, \ell_\mu, \ell_e)\) and the right-handed charged leptons are assigned to different singlets as \(1, 1',\) and \(1''\) and the right-handed Majorana neutrinos are assigned to triplet as \(\nu_R = (\nu_{R3}, \nu_{R2}, \nu_{R1})\). On the other hand, the solution form in Eq. \(22\) can be realised by taking charge assignments such that the left-handed lepton doublets are assigned to triplet \(\bar{\ell} = (\ell_\mu, \ell_e, \ell_\tau)\) and the right-handed charged leptons are assigned to different singlets as \(1, 1',\) and \(1''\) and the right-handed Majorana neutrinos are assigned to triplet as \(\nu_R = (\nu_{R2}, \nu_{R1}, \nu_{R3})\). In these solution forms we obtain same numerical results in section 4.
where $m_e, m_\mu, m_\tau$ are charged lepton masses. The Dirac neutrino Yukawa couplings similarly obtained by solving equations which are substituted Eq. (38) into Eq. (40), where we use the left-handed Majorana neutrino masses instead of the charged lepton masses in the right-side of Eq. (40). However neutrino masses are only known mass squared differences $\Delta m^2$ in the inverted (normal) ordering of the neutrino mass hierarchy. Then we need to decide the proportional each other and which are satisfied the current experimental data in Table 2. These Yukawa couplings look like in Fig. 1a, we show the symmetric and anti-symmetric Dirac Yukawa couplings for Eq. (35) estimated in Ref. [71]. We analyze our model in normal and inverted neutrino mass orderings. When we assume the normal ordering, there are no realistic parameters which satisfy the current experimental data in Table 2. In our numerical analysis, we simulate by assigning different numerical values to the lightest neutrino mass at the range $m_3 \in [0, 15.9]$ [meV], where the upper limit of the lightest neutrino mass is estimated in Ref. [71]. In our model, the complex phase of the Dirac Yukawa coupling which contribute to the mixing matrix is only $\phi_{DA}$, where $\phi_{DA}$ only appear in 70[°] - 100[°]. In our model, the complex phase of the Dirac Yukawa coupling which contribute to the mixing matrix is only $\phi_{DA}$, then this result has a strong influence to the CP phases.

In the PDG parametrization, we can write the PMNS matrix as

$$U_{\text{PMNS}}^\text{PDG} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{CP}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{CP}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_{CP}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_{CP}} & c_{23} c_{13} \end{pmatrix} \begin{pmatrix} e^{i \eta_1} \\ e^{i \eta_2} \\ 1 \end{pmatrix},$$

where $U_{ai}$ ($\alpha = e, \mu, \tau$, $i = 1, 2, 3$) are the PMNS matrix elements, $c_{ij}$ and $s_{ij}$ denote $\cos \theta_{ij}$ and

| Inverted Ordering | bfp ±1σ | 3σ range |
|-------------------|---------|----------|
| $\sin^2 \theta_{12}$ | 0.304$^{+0.013}_{-0.012}$ | 0.269 → 0.343 |
| $\theta_{12}/^{°}$ | 33.45$^{+0.78}_{-0.75}$ | 31.27 → 35.87 |
| $\sin^2 \theta_{23}$ | 0.570$^{+0.016}_{-0.022}$ | 0.410 → 0.613 |
| $\theta_{23}/^{°}$ | 49.0$^{+0.9}_{-1.3}$ | 39.8 → 51.6 |
| $\sin^2 \theta_{13}$ | 0.02241$^{+0.00074}_{-0.00062}$ | 0.02055 → 0.02457 |
| $\theta_{13}/^{°}$ | 8.61$^{+0.14}_{-0.12}$ | 8.24 → 9.02 |
| $\delta_{CP}/^{°}$ | 278$^{+22}_{-30}$ | 194 → 345 |
| $\Delta m^2_{21}$/[eV$^2$] | 7.42$^{+0.21}_{-0.20}$ | 6.82 → 8.04 |
| $\Delta m^2_{32}$/[eV$^2$] | $-2.490^{+0.026}_{-0.028}$ | $-2.574 → -2.410$ |

Table 2: NuFIT 5.1 data [69, 70] in the inverted neutrino mass ordering, where $\Delta m^2_{ij}$ is mass squared difference between $m_i$ and $m_j$. 

In the PDG parametrization, we can write the PMNS matrix as

$$U_{\text{PMNS}}^\text{PDG} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{CP}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{CP}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_{CP}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_{CP}} & c_{23} c_{13} \end{pmatrix} \begin{pmatrix} e^{i \eta_1} \\ e^{i \eta_2} \\ 1 \end{pmatrix},$$

where $U_{ai}$ ($\alpha = e, \mu, \tau$, $i = 1, 2, 3$) are the PMNS matrix elements, $c_{ij}$ and $s_{ij}$ denote $\cos \theta_{ij}$ and

| Inverted Ordering | bfp ±1σ | 3σ range |
|-------------------|---------|----------|
| $\sin^2 \theta_{12}$ | 0.304$^{+0.013}_{-0.012}$ | 0.269 → 0.343 |
| $\theta_{12}/^{°}$ | 33.45$^{+0.78}_{-0.75}$ | 31.27 → 35.87 |
| $\sin^2 \theta_{23}$ | 0.570$^{+0.016}_{-0.022}$ | 0.410 → 0.613 |
| $\theta_{23}/^{°}$ | 49.0$^{+0.9}_{-1.3}$ | 39.8 → 51.6 |
| $\sin^2 \theta_{13}$ | 0.02241$^{+0.00074}_{-0.00062}$ | 0.02055 → 0.02457 |
| $\theta_{13}/^{°}$ | 8.61$^{+0.14}_{-0.12}$ | 8.24 → 9.02 |
| $\delta_{CP}/^{°}$ | 278$^{+22}_{-30}$ | 194 → 345 |
| $\Delta m^2_{21}$/[eV$^2$] | 7.42$^{+0.21}_{-0.20}$ | 6.82 → 8.04 |
| $\Delta m^2_{32}$/[eV$^2$] | $-2.490^{+0.026}_{-0.028}$ | $-2.574 → -2.410$ |
Figure 1: (a) The relation between the Dirac neutrino Yukawa couplings $y_{DS}$ and $y_{DA}$. We simulate by assigning different numerical values to the Dirac neutrino Yukawa couplings at $O(1)$. (b) Higgs VEV ratio $\tan \beta$ and complex phase of the Dirac Yukawa coupling $\phi_{DA}$.

$\sin \theta_{ij}, \delta_{CP}$ is the Dirac CP phase, and $\eta_1$ and $\eta_2$ are Majorana phases, respectively [72]. The lepton mixing angles are obtained as follows:

$$
\sin \theta_{13} = |U_{e3}|, \quad \tan \theta_{12} = \left| \frac{U_{e2}}{U_{e1}} \right|, \quad \tan \theta_{23} = \left| \frac{U_{\mu 3}}{U_{\tau 3}} \right|. \tag{42}
$$

In addition, the Dirac CP phase is determined by one of the Jarlskog Invariants [73] as

$$
J_{\text{CP}} = \text{Im} \left[ U_{e1} U_{e2}^* U_{\mu 2}^* U_{\mu 1} \right], \tag{43}
$$

and

$$
J_{\text{CP}} = \sin \theta_{12} \cos \theta_{12} \sin \theta_{23} \cos \theta_{23} \sin \theta_{13} \cos^2 \theta_{13} \sin \delta_{CP}, \tag{44}
$$

in the PDG parametrization in Ref. [72]. The $\delta_{CP}$ is also determined by one of the absolute values for PMNS mixing matrix elements:

$$
|U_{\tau 1}|^2 = \sin^2 \theta_{12} \sin^2 \theta_{23} + \cos^2 \theta_{12} \cos^2 \theta_{23} \sin^2 \theta_{13}
- 2 \sin \theta_{12} \sin \theta_{23} \cos \theta_{12} \cos \theta_{23} \sin \theta_{13} \cos \delta_{CP}. \tag{45}
$$

Then, we can determine the $\delta_{CP}$. In Fig. 2, we show the allowed region for the $\sin^2 \theta_{23}$ and $\delta_{CP}$ within $3\sigma$ standard deviation of the Table 2. The gray area is outside of $3\sigma$ standard deviation of $\delta_{CP}$ for NuFIT 5.1 data in Ref. [69, 70]. The relation between $\sin^2 \theta_{23}$ and Dirac CP phase $\delta_{CP}$ has strong correlation. Especially, in $\sin^2 \theta_{23} \in [0.41, 0.52]$, this relation has one to one correspondence because the complex phase comes from one Yukawa coupling phase $\phi_{DA}$. Then if the $\theta_{23}$ is more precise measured by the future neutrino oscillation experiments, the Dirac CP phase is more precise predicted, and vice versa.

We can diagonalize the $m_\nu m_\nu^T$ in Eq. (38) by using the unitary matrix $U_\nu$. We can also diagonalize the complex symmetric matrix $m_\nu$ by using $U_\nu$ as follows:

$$
U_\nu^T m_\nu U_\nu^* = \text{diag}(m_1 e^{i\xi_1}, m_2 e^{i\xi_2}, m_3 e^{i\xi_3}). \tag{46}
$$

In order to remove these phases from mass diagonal matrix, we need to multiply phase diagonal matrix $P_\nu = \text{diag}(e^{i\xi_1/2}, e^{i\xi_2/2}, e^{i\xi_3/2})$ on both sides of Eq (46),

$$
P_\nu^T (U_\nu^T m_\nu U_\nu^*) P_\nu^* = \text{diag}(m_1, m_2, m_3). \tag{47}
$$
Then, the unitary matrix $U_\nu P_\nu$ makes the mass matrix real diagonal. Similarly, we diagonalize $M_\ell M_\ell^\dagger$ in Eq. (31) by using the unitary matrix $U_\ell$. Therefore we can calculate the PMNS matrix in our model as follows:

$$U_{\text{PMNS}}^{\text{mod}} = U_\ell^\dagger U_\nu P_\nu = \begin{pmatrix} U_{\text{mod}}^{\text{e}1} & U_{\text{mod}}^{\text{e}2} & U_{\text{mod}}^{\text{e}3} \\ U_{\text{mod}}^{\mu 1} & U_{\text{mod}}^{\mu 2} & U_{\text{mod}}^{\mu 3} \\ U_{\text{mod}}^{\tau 1} & U_{\text{mod}}^{\tau 2} & U_{\text{mod}}^{\tau 3} \end{pmatrix}.$$  \hspace{1cm} (48)

The Majorana phases $\eta_1$ and $\eta_2$ are determined by using PMNS matrix in Eq. (48) as follows:

$$\eta_1 = \arg \left[ \frac{U_{\text{mod}}^{\text{e}1} U_{\text{mod}}^{\text{e}3\ast}}{\cos \theta_{12} \cos \theta_{13} \sin \theta_{13} e^{i\delta_{\text{CP}}}} \right], \quad \eta_2 = \arg \left[ \frac{U_{\text{mod}}^{\text{e}2} U_{\text{mod}}^{\text{e}3\ast}}{\sin \theta_{12} \cos \theta_{13} \sin \theta_{13} e^{i\delta_{\text{CP}}}} \right].$$  \hspace{1cm} (49)

The $0\nu\beta\beta$ decay is determined by the magnitude of the lightest neutrino mass and neutrino mass ordering. In Figs. 3a and 3b, we show the prediction of the effective mass $m_{ee}$ for the $0\nu\beta\beta$ decay as

$$m_{ee} = \left| \sum_{i=1}^{3} m_i U_{ei}^{\text{mod}2} \right| = \left| m_1 U_{e1}^{\text{mod}2} + m_2 U_{e2}^{\text{mod}2} + m_3 U_{e3}^{\text{mod}2} \right|. \hspace{1cm} (50)$$

Note that only inverted ordering is acceptable then the lightest neutrino mass is $m_3$. The lightest neutrino mass $m_3$ appears in 0.789 - 1.43 [meV] which is the within the Planck data [71] and the effective mass takes very restricted region $m_{ee} \simeq 47.1$ [meV]. This value is in upper limit on the effective mass of 36 - 156 [meV] at 90% C.L. in Ref. [7]. Then this model can be verified in the near future neutrino experiments, e.g. the KamLAND-Zen [6, 7], GERDA [8, 9], and CUORE [10, 11] experiments. In Fig. 3c, we show the relation among Majorana phases $\eta_1$ and $\eta_2$. Since the phase of our model parameter is only $\phi_{DA}$, then the Majorana phases are strongly correlated.

## 5 Summary and Discussions

We have proposed the simple non-SUSY lepton flavor model with $A_4$ symmetry. The $A_4$ group is a minimal one which includes triplet irreducible representation. We have introduced three Higgs
Figure 3: (a) The relation between the effective mass for $0\nu\beta\beta$ decay $m_{ee}$ and lightest neutrino mass. The red area and blue area are model independent analyses for the inverted ordering and normal ordering of the neutrino mass hierarchies, respectively. The gray area is upper limit on the effective mass of 36 - 156 [meV] at 90% C.L. in Ref. [7]. The yellow area is upper limit on the lightest neutrino mass $m_3 \simeq 15.9$ [meV] which is estimated in Ref. [71]. (b) The enlargement of figure (a). (c) The relation between Majorana phases $\eta_1$ and $\eta_2$.

doublets which are assigned as triplet of the $A_4$ symmetry. It is natural that there are three generations as same as the SM fermions. First, we have analysed the potential and we have got the VEV for the local minimum. Next, we have presented our $A_4$ model. The left-handed lepton doublets are assigned to triplet and the right-handed leptons are assigned to different singlets of the $A_4$ symmetry, respectively. We have introduced the right-handed Majorana neutrinos which are assigned to triplet of the $A_4$ symmetry. In our model, the right-handed Majorana neutrino mass matrix has a simple flavor structure. On the other hand, the Dirac neutrino mass matrix has symmetric and anti-symmetric Yukawa couplings for $A_4$ symmetry. By using the type-I seesaw mechanism, we have obtained the left-handed Majorana neutrino mass matrix. After diagonalizing the charged lepton and left-handed Majorana neutrino mass matrices, we have got the PMNS mixing matrix. In our numerical analyses, we have used the NuFIT 5.1 data. We found that only inverted ordering is acceptable and we could not find the solutions for normal ordering in the neutrino mass hierarchy. We have obtained relevant relations for mixing angles and neutrino effective mass $m_{ee}$ as a function of the lightest neutrino mass. Especially, the Dirac
CP phase and lepton mixing angle $\theta_{23}$ are strongly correlated. If the $\theta_{23}$ is more precise measured, the Dirac CP phase is more precise predicted, and vice versa. In this model, the effective mass for the $0\nu\beta\beta$ decay can be predicted as $m_{ee} \simeq 47.1$ [meV] and the lightest neutrino mass can be also predicted as $m_3 \simeq 0.789 - 1.43$ [meV]. It is testable for near future neutrino experiments.

The flavor symmetry also apply to the quark sector. In the same assignments as the charged leptons, the elements of the mass matrices are real. Then, we take quarks different assignments, e.g. left-handed quark doublets are assigned to the different singlets and right-handed up and down-type quarks are assigned to the triplets of the $A_4$ symmetry, respectively. The more details are in the future work. In our model, the right-handed Majorana mass matrix is very simple and masses are degenerate because the right-handed Majorana neutrino is $A_4$ triplet. Then, we cannot apply to the leptogenesis. Fortunately, there are three Higgs doublets, we can discuss the electroweak baryogenesis which are also in the future work.

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**Appendix**

**A Multiplication rule of $A_4$ group**

We show the multiplication rule of the $A_4$ triplets as follows:

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}_3 \otimes \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}_3 = (a_1b_1 + a_2b_3 + a_3b_2)_1 \oplus (a_3b_3 + a_1b_2 + a_2b_1)_1',
\]

\[
\oplus (a_2b_2 + a_1b_3 + a_3b_1)_1'' \oplus \frac{1}{3} \left( 2a_1b_1 - a_2b_3 - a_3b_2 ight)_3 \oplus \frac{1}{2} \left( a_2b_3 - a_3b_2 ight)_3.
\]

More details are shown in the review [14,15].

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