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Quasi-exactly solvable models derived from the quasi-Gaudin algebra

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Abstract

The quasi-Gaudin algebra was introduced to construct integrable systems which are only quasi-exactly solvable. Using a suitable representation of the quasi-Gaudin algebra, we obtain a class of bosonic models which exhibit this curious property. These models have the notable feature that they do not preserve $U(1)$ symmetry, which is typically associated with a non-conservation of particle number. An exact solution for the eigenvalues within the quasi-exactly solvable sector is obtained via the algebraic Bethe ansatz formalism.

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1. Introduction

In [1, 2], Ushveridze proposed a method for studying quasi-exactly solvable (QES) systems [3–5] from the perspective of integrable systems and the quantum inverse scattering method (QISM) [6]. The approach, which is called the partial algebraic Bethe ansatz (ABA), relies on deforming the Yang–Baxter algebra in such a way that it retains most of the features required for the QISM but leads to generating functions of integrable systems which are only QES. This deformation of the Yang–Baxter algebras led to new classes of hitherto unknown algebras. A limiting case is the (rational) quasi-Gaudin algebra which will be the focus of this study.

Exactly solvable models have found many successes in various branches of physics and mathematics. In recent years they have continued to find new applications in diverse fields such as Bose–Einstein condensates (BECs) and degenerate Fermi gases, quantum optics, superconductivity and nuclear pairing among other things e.g. [7–14]. There has also been significant interest in QES models, with new applications of these being found in problems relating to matrix product states [15], and in dissipative systems [16]. However, by comparison the partial ABA approach seems to have received little attention and remains essentially undeveloped. An appealing property of the partial ABA is that it provides us with a constructive algebraic approach for obtaining QES models which have multiple degrees of freedom.

One particular aspect of the ABA which bears some relevance to our present exposition is the study of quantum integrable models which do not preserve $U(1)$ symmetry. Such models
are interesting for a number of reasons. In the context of spin-boson Hamiltonians of the Tavis–Cummings form, these models correspond to physical systems without the rotating wave approximation. Diagonalization of such models is a somewhat complicated affair within the ABA method due to the lack of reference states, often requiring the use of functional Bethe ansatz or Sklyanin’s separation of the variable technique [9, 11]. Non-$U(1)$ preserving models are also relevant to the study of open quantum systems whereby the $U(1)$ symmetry is broken due to coupling to an environment. An example of this is found in the spin-boson Hamiltonian of Leggett et al [17] which has found applications ranging from quantum-state engineering [18] to biomolecular systems [19].

In this paper, we will study QES bosonic models descending from suitable realizations of the quasi-Gaudin algebra. It will be shown that such models correspond to an extension of the $\text{su}(1, 1)$ Dicke Hamiltonian [20, 21] by the addition of $U(1)$ symmetry-breaking terms. The Hamiltonian can be written as

$$H = H_0 + H_1,$$

with $H_0$ and $H_1$ referring to the Dicke Hamiltonian and the $U(1)$ symmetry-breaking component, respectively. Explicitly, they have the form

$$H_0 = wN_b + \sum_{i=1}^{m} 2\epsilon_i S_i^z + g \left( \sum_{i=1}^{m} b_i S_i^+ + b_i^+ S_i^z \right),$$

$$H_1 = g \left( (b + b^+) \left( n + f^z - \sum_{i=1}^{m} S_i^z \right) - b^2 b^2 - (b^+)^2 b \right).$$

Here, $N_b$, $b$, $b^+$ are the standard bosonic operators, $f^z$ is a representation-dependent parameter, $w_0$, $\epsilon_i$, and $g$ are the free parameters, $n$ is an integer and $S_i^z$ are either single-mode or double-mode representations of $\text{su}(1, 1)$ generators (refer to equations (3.4) and (3.5) below). The Hamiltonian $H_1$ may be interpreted as a coupling of the $\text{su}(1, 1)$ Dicke model to an external system.

Our paper is structured as follows. In section 2, we will briefly review the partial ABA method of obtaining quasi-exact solutions for models associated with the quasi-Gaudin algebra. In section 3, we will use a suitable representation of the quasi-Gaudin algebra to obtain the integrable bosonic model (1.1). We then derive the partial ABA solution of the Hamiltonian and discuss aspects relating to the quasi-exact solvability. Finally in section 4, we summarize our results and discuss possible future lines of work.

2. Quasi-Gaudin algebra and the Bethe ansatz solution

Let us first introduce the rational (rank 1) Gaudin algebra and the associated abstract, integrable models before defining its quasi-counterpart. The rational Gaudin model is a parameter-dependent infinite-dimensional Lie algebra satisfying the following commutation relations:

$$[\mathcal{S}_-^z(\lambda), \mathcal{S}_+^z(\mu)] = 0,$$

$$[\mathcal{S}_-^z(\lambda), \mathcal{S}_-^z(\mu)] = 0,$$

$$[\mathcal{S}_+^z(\lambda), \mathcal{S}_-^z(\mu)] = \pm \frac{\mathcal{S}_+^z(\lambda) - \mathcal{S}_-^z(\mu)}{\mu - \lambda},$$

$$[\mathcal{S}_-^z(\lambda), \mathcal{S}_+^z(\mu)] = 2^{\mathcal{S}_-(\lambda) - \mathcal{S}_+(\mu)} \frac{\mu - \lambda}{\mu - \lambda},$$

whereby $\lambda$ and $\mu$ are the complex spectral parameters. From these relations, it can be shown that

$$H(\lambda) = \mathcal{S}_+^z(\lambda)\mathcal{S}_-^z(\lambda) - \frac{1}{2} \mathcal{S}_+^z(\lambda)\mathcal{S}_-^z(\lambda) - \frac{1}{2} \mathcal{S}_-^z(\lambda)\mathcal{S}_+^z(\lambda)$$

(2.1)
satisfies the following commutation relations:

\[ [H(\lambda), H(\mu)] = 0 \]  

(2.2)

and therefore acts as a generator of commuting operators in an abstract integrable system. Assuming the existence of a suitable reference state, the spectrum of \( H(\lambda) \) can be obtained via the standard ABA [2].

Analogous to the Gaudin algebra is the so-called quasi-Gaudin algebra. It is defined by the following parameter-dependent set of relations [1, 2]:

\[
S^+_n(\lambda)S^+_m(\mu) - S^+_m(\mu)S^+_n(\lambda) = 0, \\
S^\pm_{n+1}(\lambda)S^\pm_n(\mu) - S^\pm_n(\mu)S^\pm_{n+1}(\lambda) = 0, \\
S^z_{n+1}(\lambda)S^z_n(\mu) - S^z_n(\mu)S^z_{n+1}(\lambda) = \pm \frac{S^+_n(\lambda) - S^+_n(\mu)}{\mu - \lambda}, \\
S^-_{n+1}(\lambda)S^+_n(\mu) - S^+_n(\mu)S^-_{n+1}(\lambda) = 2\frac{S^z_n(\lambda) - S^z_n(\mu)}{\mu - \lambda},
\]  

(2.3)

whereby \( n \) is an integer and \( \lambda, \) and \( \mu \) are the complex parameters. While (2.3) appears to be similar to the Gaudin algebra, we stress that there are important qualitative differences between the two. Importantly, note that (2.3) do not define commutation relations and are therefore not Lie algebraic relations. Despite looking somewhat arbitrary, the quasi-Gaudin algebra can be understood as a grading deformation on the original Gaudin algebra. We refer the reader to [2] for a more detailed discussion.

Similar to the Gaudin algebra, there exists a generating function of commuting operators for the quasi-Gaudin algebra. It has the form

\[
H_n(\lambda) = S^z_n(\lambda)S^+_n(\lambda) - \frac{1}{2}S^z_{n+1}(\lambda)S^+_n(\lambda) - \frac{1}{2}S^+_n(\lambda)S^z_n(\lambda)
\]  

(2.4)

and can be shown to form a commutative family with respect to the spectral parameters, i.e.

\[
[H_n(\lambda), H_m(\mu)] = 0.
\]  

(2.5)

Note that the commutation relation (2.5) does not extend to the general case where \( H_n(\lambda) \) and \( H_m(\mu) \) have different integer values of \( n \) and \( m \). This is due to the lack of a defining relations between elements of the algebra with arbitrary integer indexes. The ABA solution for the generating function \( H_n(\lambda) \) of the quasi-Gaudin algebra has been obtained in [1, 2]. As with the standard Gaudin algebra, the ABA diagonalization of \( H_n(\lambda) \) works if the representation of (2.3) supports a reference state \( |0\rangle \), namely:

\[
S^+_0(\lambda)|0\rangle = f(\lambda)|0\rangle, \quad S^z_0(\lambda)|0\rangle = 0.
\]  

(2.6)

The Bethe vector is given by

\[
\psi(\mu_1, \ldots, \mu_n) = S^+_{n-1}(\mu_\nu)S^+_{n-2}(\mu_{\nu-1}) \cdots S^+_0(\mu_1)|0\rangle.
\]  

(2.7)

By successively applying the following relation:

\[
H_n(\lambda)S^+_1(\mu_\nu) = S^+_1(\mu_\nu)H_n-1(\lambda) + 2\frac{S^+_{n-1}(\mu_\nu)S^+_{n-2}(\lambda) - S^z_{n-1}(\lambda)S^+_{n-2}(\mu_\nu)}{\lambda - \mu_n},
\]  

(2.8)

we can shift the operator \( H_n(\lambda) \) towards the right of the product of \( S^+_1(\mu_\nu+1) \) operators on the right-hand side of (2.7), so that we finally have \( H_n(\lambda) \) acting on the reference state. After having completed this procedure, we perform the same operation for the various \( S^z_1(\mu_\nu) \), \( S^+_1(\mu_\nu+1) \) that were generated as a byproduct of shifting the \( H_n(\lambda) \) through the product of the \( S^+_1(\mu_\nu+1) \). The final form is given by

\[
H(\lambda)\psi(\mu_1, \ldots, \mu_n) = A(\lambda)\psi(\mu_1, \ldots, \mu_n) \\
+ 2 \sum_{i=1}^n B(\mu_i)\psi(\mu_1, \ldots, \mu_{i-1}, \lambda, \mu_{i+1}, \ldots, \mu_n),
\]  

(2.9)
whereby

\[ A(\lambda) = f(\lambda)^2 + f'(\lambda) + 2 \sum_{i=1}^{n} \frac{f(\lambda)}{\lambda - \mu_i} + 2 \sum_{i=1}^{n} \frac{1}{\lambda - \mu_i} \sum_{j \neq i}^{n} \frac{1}{\mu_i - \mu_j}, \]

\[ B(\mu_i) = f(\mu_i) + \sum_{j \neq i}^{n} \frac{1}{\mu_i - \mu_j}. \]

By requiring that the unwanted terms vanish, we obtain the following Bethe ansatz equations:

\[ \sum_{k=1, k \neq i}^{n} \frac{1}{\mu_i - \mu_k} + f(\mu_i) = 0, \quad i = 1, 2, \ldots, n, \]

with the eigenvalue for \( H_n(\lambda) \) given by

\[ E_n(\lambda) = f(\lambda)^2 + f'(\lambda) + 2 \sum_{i=1}^{n} \frac{f(\lambda) - f(\mu_i)}{\lambda - \mu_i}. \]

As a proof of existence, an explicit representation for (2.3) is provided in [1, 2]:

\[ S_n(\lambda) = S^-(\lambda) + \frac{f^- - S^- + n}{\lambda - c}, \]

\[ S_0^0(\lambda) = S^0(\lambda) + \frac{f^- - S^0 + n + d}{\lambda - c}, \]

\[ S_n^+(\lambda) = S^+(\lambda) + \frac{f^- - S^+ + n + 2d}{\lambda - c}, \]

with \( c \) and \( d \) as free parameters, \( S^\pm, f^\pm(\lambda) \) are generators of the Gaudin algebra, and \( S^- \) and \( f^- \) are defined as

\[ S^- = \lim_{\lambda \to \infty} \lambda S^-(\lambda), \quad S^- |0\rangle = f^- |0\rangle. \]

In terms of this realization, the generating function \( H_n(\lambda) \) takes the form

\[ H_n(\lambda) = S^+(\lambda)S^-(\lambda) - \frac{1}{2} S^-(\lambda)S^+(\lambda) - \frac{1}{2} S^+(\lambda)S^-(\lambda) \]

\[ + \frac{2S^-(\lambda)(n + d + f^- - S^-) - S^-(\lambda)(n + 2d + f^- - S^-) - S^+(\lambda)(n + f^- - S^-)}{\lambda - c} \]

\[ = \frac{1}{4(\lambda - c)^2}. \]

It can be seen that the condition of hermiticity for (2.15) is satisfied when \( d = 1/2 \) and the representation for the Gaudin algebra is unitary, i.e. satisfying the condition

\[ S^+(\lambda)| = S^+(\lambda), \quad S^-(\lambda)| = S^-(\lambda). \]

3. Bosonic representations of the quasi-Gaudin algebra

The quasi-Gaudin algebra of the form (2.13) admits mixed representations, consisting of \( su(1, 1) \) algebras and the Heisenberg algebra, with the following form:

\[ S_n(g) = \frac{2b}{g} + \sum_{i=1}^{m} \frac{S^-_{i}}{\lambda - \epsilon_i} + \frac{f^- - N_b - \sum_{i=0}^{m} S^+_{i}}{\lambda - c}. \]
\[
S_i^-(\lambda) = \frac{w - 2\lambda}{g^2} + \sum_{i=1}^{m} \frac{S_i^-}{\lambda - \epsilon_i} + \frac{f^2 - N_b - \sum_{i=1}^{m} S_i^+ + n + \frac{1}{2}}{\lambda - c},
\]
\[
S_i^+(\lambda) = \frac{2b^i}{g} + \sum_{i=1}^{m} \frac{S_i^+}{\lambda - \epsilon_i} + \frac{f^2 - N_b - \sum_{i=1}^{m} S_i^- + n + 1}{\lambda - c}. \tag{3.1}
\]

The \(S_i^{\pm\pm}\) and \([N_b, b, b^\dagger]\) are respectively the \(su(1, 1)\) and Heisenberg algebras, which obey the commutation relations
\[
[S_i^+, S_j^-] = \pm S_i^\pm \delta_{ij}, \quad [S_i^-, S_j^+] = 2S_i^\pm \delta_{ij},
\]
\[
[N_b, b^\dagger] = b^\dagger, \quad [N_b, b] = -b, \quad [b, b^\dagger] = 1, \tag{3.2}
\]
and \(S^i\) and \(f^i\) are defined as
\[
S^i = \sum_{i=1}^{m} S_i^i + N_b, \quad S^i|0\rangle = f^i|0\rangle. \tag{3.3}
\]

We note here that our definition for \(S^i\) differs from that of (2.14) as the prior definition is divergent for this particular realization.

The \(su(1, 1)\) algebra has two bosonic operator realizations. The first is given by the single-mode representation, whereby
\[
S_i^+ = \frac{a^i a_i}{2} + \frac{1}{4} = \frac{N_a}{2} + \frac{1}{4}, \quad S_i^- = \frac{(a^i)^2}{2}, \quad S_i^0 = \frac{a^2_i}{2}. \tag{3.4}
\]
The second one is given by the two-mode representation,
\[
S_i^+ = \frac{1}{2} (a^i a_i + c^i c_i) + \frac{1}{4} = \frac{(N_a + N_c)}{2} + \frac{1}{4}, \quad S_i^+ = a^i c_i^\dagger, \quad S_i^- = a_i c_i. \tag{3.5}
\]

There are multiple reference states for both bosonic realizations. For the single-mode realization, there are finitely many of them. We can express them as
\[
|0, \{l\} \rangle = \prod_{i=1}^{m} (a_i^\dagger)^{l_i} |0\rangle, \quad l_i = 0 \text{ or } 1, \tag{3.6}
\]
where \(|l\rangle\) is a shorthand notation for the set \(\{l_1, \ldots, l_m\}\) and
\[
S^i|0, \{l\} \rangle = f^i|0, \{l\} \rangle = \left( \sum_{i=1}^{m} \frac{L_i}{2} + \frac{1}{4} \right) |0, \{l\} \rangle. \tag{3.7}
\]
For the two-mode realization, there are infinitely many reference states. Without loss of generality we can write them as
\[
|0, \{l\} \rangle = \prod_{i=1}^{m} (a_i^\dagger)^{l_i} |0\rangle, \quad l_i = 0, 1, 2, \ldots, \tag{3.8}
\]
with
\[
S^i|0, \{l\} \rangle = f^i|0, \{l\} \rangle = \left( \sum_{i=1}^{m} \frac{L_i}{2} + \frac{1}{2} \right) |0, \{l\} \rangle. \tag{3.9}
\]

It can be seen that each reference state corresponds to a distinct eigenfunction of the Casimir operators for the \(su(1, 1)\) generators \(S_i^{\pm\pm}\). As the \(su(1, 1)\) Casimir operator acts as a central element with respect to (3.4) and (3.5), we can use Schur’s lemma to deduce that each reference state gives rise to a distinct irreducible representation.
4. Quasi-exactly solvable Hamiltonians

We now consider the generating function $H_n(\lambda)$ of the quasi-Gaudin algebra obtained from the representation (3.1). Assuming $\epsilon_i \neq \epsilon_j$, it can be seen that

$$H_n(\lambda) = -\frac{4}{g^2} \left( n + f^2 + \frac{1}{2} \right) + \frac{1}{g^2} (w - 2\lambda)^2 - \frac{2}{g^2} \left( \frac{H_c}{\lambda - \epsilon_i} + \sum_{j=1}^{m} \frac{H_j}{\lambda - \epsilon_j} \right)$$

$$+ \sum_{i=1}^{m} \frac{K_i}{(\lambda - \epsilon_i)^2} - \frac{1}{4(\lambda - \epsilon_i)^2},$$

with

$$H_j = (2\epsilon_i - w)S_j^z + g(b_i^0 S_j^- + b_i S_j^-) + \sum_{i \neq j}^{m} \frac{1}{\epsilon_j - \epsilon_i} (2S_j^z S_j^0 - S_j^z S_j^- - S_i^z S_j^-)$$

$$+ \frac{g^2}{\epsilon_j - \epsilon_i} (n + \frac{1}{2} + f^2 - \sum_{i=1}^{m} S_i^z - N_b) - \frac{1}{2} S_j^z (n + f^2 - S_j^z) - \frac{1}{2} S_j^z (n + 1 + f^2 - S_j^z)$$

$$H_c = g^2 \sum_{i=1}^{m} S_i^z (n + \frac{1}{2} + f^2 - \sum_{i=1}^{m} S_i^z - N_b) - \frac{1}{2} S_i^z (n + f^2 - S_i^z) - \frac{1}{2} S_i^z (n + 1 + f^2 - S_i^z)$$

$$+ (2c - w) \left( n + \frac{1}{2} + f^2 - \sum_{i=1}^{m} S_i^z - N_b \right)$$

$$+ gb \left( n - \sum_{i=1}^{m} S_i^z - N_b \right) + gb \left( n + 1 - \sum_{i=1}^{m} S_i^z - N_b \right),$$

$$K_i = S_i^z S_i^z - \frac{1}{2} (S_i^z S_i^+ + S_i^z S_i^-).$$

(4.2)

From (4.1) and the commutation relation (2.5), it follows that $H_{c,e}$ and $K_i$ form a set of mutually commuting operators. By considering the following linear combination $H = \sum_{i=1}^{m} H_i + H_c$ and setting the coefficient $c = 0$, we obtain the desired bosonic Hamiltonian. For the single-mode representations, we have

$$H = w N_b + \sum_{i=1}^{m} \epsilon_i N_a_i + g \sum_{i=1}^{m} \left( b_i^a c_i^a + b_i^a c_i^a \right)$$

$$+ g \left( n + f^2 \right) (b_i^a + b_i^a) - \left( b_i^a + b_i^a \right) \sum_{i=1}^{m} \frac{N_a_i}{2} - b_i^a b_i^a - \left( b_i^a \right)^2 b_i^a,$$

(4.3)

where

$$\Upsilon = w \left( n + \frac{1}{2} + f^2 \right) - \sum_{i=1}^{m} \frac{\epsilon_i}{2}.$$ 

For the two-mode representations, we obtain

$$H = w N_b + \sum_{i=1}^{m} \epsilon_i \left( N_a_i + N_b_i \right) + g \sum_{i=1}^{m} \left( b_i^a c_i^a + b_i^b c_i^b \right)$$

$$+ g \left( n + f^2 \right) (b_i^a + b_i^a) - \left( b_i^a + b_i^a \right) \sum_{i=1}^{m} \frac{N_a_i}{2} - b_i^a b_i^a - \left( b_i^a \right)^2 b_i^a,$$

(4.4)
whereby we have introduced a grading structure on the Hamiltonian through setting

\[
V = w\left(n + \frac{1}{2} + f^2\right) - \sum_{i=1}^{m} \epsilon_i.
\]

We note that for the case when \( m = 1 \), the models correspond to QES extensions for atom–molecule BEC models contained in [22].

The eigenvalues for the Hamiltonians can be extracted from the Bethe ansatz solution of (4.1):

\[
E_n(\lambda) = f^2(\lambda) + f'(\lambda) + 2 \sum_{i=1}^{n} \frac{f(\lambda) - f(\mu_i)}{\lambda - \mu_i}.
\]

This is done by evaluating the residues of the poles \( \epsilon_i \) and \( c \). Doing so yields

\[
E = V - w\left(\sum_{i=1}^{m} s_i^2 + \frac{1}{2}\right) + \sum_{i=1}^{m} 2\epsilon_i s_i^2 + \frac{s_i^2}{2} \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{s_j^2}{\mu_i - \epsilon_j} + \sum_{i=1}^{n} \frac{1}{2(\mu_i - c)}\right),
\]

whereby \( s_i^2 = (2i + 1)/4 \) for the single-mode representations and \( s_i^2 = (i + 1)/2 \) for the two-mode representations.

We now examine the QES nature of the Hamiltonians in more detail. For the sake of clarity, we shall only consider the Hamiltonian with the single-mode bosonic representation (4.3), as results for the two-mode representation will follow analogously. It is straightforward to see that (4.3) acts on an infinite-dimensional Hilbert space \( V \) span by the following basis states:

\[
V \equiv \text{span}\{b^b|l\rangle \cdots (a^b_m)^{p_m}|0\rangle \} \equiv \text{span}\{|l_0, \ldots, l_m\rangle\}, \quad l_i \in \mathbb{Z}^+.
\]

In order to identify the invariant subspace which characterizes the quasi-exact solvability of the Hamiltonian, let us write the Hamiltonian as

\[
H_x = H_0 + H_+ + H_+,
\]

whereby we have introduced a grading structure on the Hamiltonian through setting

\[
H_0 = wN_b + \sum_{i=1}^{m} \epsilon_i N_a + g \left(\sum_{i=1}^{m} b(a_i^b)^2 + b^b a_i^b + (n + f^2)(b^b + b)\right).
\]

\[
H_+ = (n + f^2)b^b - b^b \sum_{i=1}^{m} \frac{N_a}{2} - (b^b)^2 b,
\]

\[
H_- = (n + f^2)b - b \sum_{i=1}^{m} \frac{N_a}{2} - b^b b^2.
\]

The assigned grading of ±, 0 is determined by the commutation relations of \( H_{\pm,0} \) with the \( U(1) \) charge \( S_i = N_b + \sum_{i=1}^{m} (2N_a_i + 1)/4 \):

\[
[S_i, b^b] = 0, \quad [S_i, H^\pm] = H^\pm, \quad [S_i, H^-] = -H^-.
\]

In light of these relations, we may decompose \( V \) into a direct sum of eigenspace \( V_{i,p} \) of the \( U(1) \) charge \( S_i \) and the Casimir operators of the \( su(1, 1) \) algebra \( K_i = S_i^2 - S_i^+ S_i^- \), i.e.

\[
V = \bigoplus_{i, p} V_{i, p}.
\]

Explicitly, the subspace \( V_{i, p} \) can be written as

\[
V_{i, p} = \text{span}\{(b^b)^{p_i} (a_i^b)^{2l_i + p_i} \cdots (a_m^b)^{2l_m + p_m}|0\rangle\}, \quad \sum_{j=0}^{m} l_j = i, \quad p_i = 0 \text{ or } 1.
\]
It can also be verified that

$$S^2 V_i,\{p\} = \left( i + \sum_{j=1}^{m} \left( \frac{p_j}{2} + \frac{1}{4} \right) \right) V_i,\{p\}, \quad K_j V_i,\{p\} = \left( \frac{p_j}{2} + \frac{1}{4} \right) \left( \frac{p_j}{2} - \frac{3}{4} \right) V_i,\{p\}. \quad (4.13)$$

From the commutation relations (4.10), we therefore have

$$H_{+} V_i,\{p\} \subseteq V_{i+1},\{p\}, \quad H_{0} V_i,\{p\} \subseteq V_i,\{p\}, \quad H_{-} V_i,\{p\} \subseteq V_{i-1},\{p\}. \quad (4.14)$$

The QES property of the Hamiltonian arises from the fact that for a given integer value of $n$ and $f^z = \sum_i (l_i + 1) / 4$, we have

$$H_{+} V_n,\{l\} = [0].$$

As a result, the Hamiltonian leaves the following subspace invariant:

$$V_{\text{QES}} \equiv \bigoplus_{i=0}^{n} V_i,\{l\}. \quad (4.15)$$

We can indeed verify that the Bethe vectors lie within this invariant subspace by expanding the eigenvectors (2.7) explicitly. It would be interesting to examine the possibility of obtaining exact solutions outside of this sector.

5. Conclusion

We have investigated a class of QES, integrable multi-mode bosonic models using the quasi-Gaudin algebra. We see that such models are obtained via a mixed representation consisting of commuting copies of $\mathfrak{su}(1,1)$ algebras and the Heisenberg algebra. Integrable Hamiltonians were extracted from the generating function of commuting operators. A notable feature was that the QES Hamiltonians we obtain do not preserve $U(1)$ symmetry. We identified the QES sector of the Hamiltonian as the direct sum of the eigensubspaces of the $U(1)$ charge with eigenvalues no greater than $n$.

The ABA method leads to partial solutions of the Hamiltonians we have considered. Given the integrability of the Hamiltonian, in the sense that $H_n(\lambda)$ acts as a generator of conserved operators, it would be interesting to explore the possibility of obtaining the entire spectrum via some other techniques. The dominating experience is that integrability and exact solvability go hand in hand. It is not apparent for these Hamiltonians whether the full spectrum is potentially accessible.

Finally, we note that due to the constraint arising from imposing hermiticity on the generating function $H_n(\lambda)$, the quasi-Gaudin formalism is at present limited to cases based on underlying unitary representations of the $\mathfrak{su}(1,1)$ algebra, or the Heisenberg algebra. It would be of interest to obtain representations of the quasi-Gaudin algebra based on non-unitary (in particular finite-dimensional) representations of $\mathfrak{su}(1,1)$, which are also able to accommodate Hermitian Hamiltonians.

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