Primary currents and Riemannian geometry in $\mathcal{W}$-algebras

G. BANDELLONI and S. LAZZARINI

$^a$ Dipartimento di Fisica dell’Università di Genova, Via Dodecaneso 33, I-16146 GENOVA, Italy
and
Istituto Nazionale di Fisica Nucleare, INFN, Sezione di Genova via Dodecaneso 33, I-16146 GENOVA, Italy

$^b$ Centre de Physique Théorique, CNRS Luminy, Case 907, F-13288 MARSEILLE Cedex, France

Abstract

It is proved that general consistency requirements of stability under complex analytic change of charts show that primary currents in finite chiral $\mathcal{W}$-algebras are described in terms of pure gravitational variables.

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1 Introduction

The description of a physical phenomenon is independent of any parametrization used for its representation. Without this belief any physical law loses its credibility.

This degree of freedom helps us in many complex sets of circumstances where symmetries or work simplification conditions are present. Mathematically speaking, any switch of parametrization is realized by the change of charts procedure, that is a finite (or infinitesimal) reparametrization procedure which modifies the frame leaving the physics unchanged. This recipe defines so to speak a "geometrical stability" for all of kinds of symmetries which are locally realized. This stability requirement must not be confused with the algebra of stability for the redefinitions of the physical parameters. In this paper we discuss the former aspect in the framework of the $\mathcal{W}$ algebras [1], realized over Riemann surfaces [2]-[4]. The intricacies encountered are well known and can be found in the literature [3]-[1], and the outstanding mathematical books [10] as well.

Even if the $\mathcal{W}$ algebras arise from many sources [11]-[18], the proper definition of all the $\mathcal{W}$ observables over the whole Riemann surface is not a simple affair. For sake of truth this problem occurs in the usual conformal models [19], but in the context of $\mathcal{W}$ algebras it becomes relevant [20]-[28]. The capital role played by a $\mathcal{W}$ symmetry has been pointed out in many physical fields, such as integrable models [29]-[31], string theory [32] and supersymmetry [33]-[35]. Many efforts have been made in that direction and important results have been reached [36]-[46]. The physical symmetry constituents are currents whose O.P.E (or different recipes), generate extensions of the conformal symmetry [47]-[49]: for this reason they are usually defined as "primary fields". Their equations of motion define the dynamics of the system and, since they are defined on a Riemann surface [50]-[52], they must be promoted to the status of geometrical objects. Since any physical investigation cannot be separated from any geometrical consideration, the same kind of symmetry arises in the context of writing down welldefined differential equations on Riemann surfaces [20]-[22].

The main difficulty lies in the definition of differential operators on Riemann surfaces [3]; in particular their equipment in a "conformal covariant" frame has been widely discussed in the literature [3],[10]. In particular the more general question of classifying all holomorphically covariant differentials operators amounts to the study of operators which are covariant with respect to projective transformations. At the end, the most general holomorphic covariant operators become a superposition of ordinary derivative operators with functions coefficients, whose reparametrization properties ensure the full covariantization of the operator. The latter (or some of their combinations), within a well defined mathematical framework, become generators of $\mathcal{W}$-algebras [1],[3],[27],[28],[29].

A natural question which is now to be asked concerns the link between these geometrical coefficients and the physical "primary fields" currents arising in $\mathcal{W}$-symmetry. There are other way of controlling the intricacies of covariance and associativity of the symmetry in $\mathcal{W}$ geometry by the use of appropriate holomorphic bundles, such as the jet bundles [41], the Drinfeld-Sokolov...
bundle [42, 43], and the Toda bundle [44]. However, we believe that \( \mathcal{W} \)-algebras describe a local symmetry, for this reason we have recently proposed [43, 44, 45] an approach to \( \mathcal{W} \)-algebras in a symplectic framework. First, the chiral \( \mathcal{W}_\infty \)-algebra [69] emerges from the conspiration of the infinitesimal action of symplectomorphisms together with suitable reparametrizations of the two dimensional local complex \((z, \overline{z})\) coordinates over a compact Riemann surface [63]. In this approach the complex structure plays, as usual, an essential role [64], and the geometry of the outstanding of finite \( \mathcal{W} \)-algebras can be studied [65].

The \( \mathcal{W} \)-algebras govern [63, 64] a hierarchy of smooth changes of local complex coordinates \((z, \overline{z}) \rightarrow (Z_0^r(z, \overline{z}), \overline{Z}_0^r(z, \overline{z}))\), \( r = 1, \cdots n \) from the \((z, \overline{z})\) background local complex coordinates to a sequence of suitable \((Z_0^r(z, \overline{z}), \overline{Z}_0^r(z, \overline{z}))\) ones through a symplectic scenario. We shall follow a B.R.S approach in order to fully exploit the locality of the theory. Accordingly, instead of considering non local commutators (or Poisson brackets) between some primary fields, we shall analyse the B.R.S. transformations of the Fadeev-Popov ghosts.

The so-called chiral B.R.S. ghost [19], \( \mathcal{K}^r \) associated to the hierarchy of smooth changes of complex coordinates can be decomposed into a sum of the other ghost fields \( C^j(z, \overline{z}) \) pertaining to the infinite \( \mathcal{W} \)-symmetry and whose coefficients are fixed by the geometrical space content. In our scheme all the elements of the algebra acquire a well defined geometrical meaning, and each of the ghosts \( C^j(z, \overline{z}) \) behaves as a \((-r, 0)\)-differential. So this approach includes together both a hierarchy of smooth changes of complex coordinates and the \( \mathcal{W} \) transformations. We have pointed out in [65] that there are two different physical situations: the first one occurs when the hierarchy realizes a physical symmetry, the second one when this symmetry is broken, but the associativity requirements are preserved. In this set of circumstances arise the primary fields, and the broken symmetry can be ruled by means of a constant Faddeev-Popov field \( \theta \) which, notwithstanding the breaking terms, secures the validity of the Jacobi identity [66]. So the whole symplectic space must be doubled in a 0 component (where the physics lives) and a \( \theta \) partner (which guarantees the algebra closure). The coordinate transformations will be written as:

\[
\mathcal{S}_\mathcal{W} Z_0^r(z, \overline{z}) = \mathcal{K}_0^r(z, \overline{z}) \partial Z_0^r(z, \overline{z}) + Z_\theta^r(z, \overline{z})
\]

\[
\mathcal{S}_\mathcal{W} Z_\theta^r(z, \overline{z}) = \mathcal{K}_0^r(z, \overline{z}) \partial Z_\theta^r(z, \overline{z}) - \mathcal{K}_\theta^r(z, \overline{z}) \partial Z_0^r(z, \overline{z})
\]

(1.1)

with the holomorphic ghosts transformations:

\[
\mathcal{S}_\mathcal{W} (\mathcal{K}_0^r(z, \overline{z}) + \theta \mathcal{K}_\theta^r(z, \overline{z})) = \left( \mathcal{K}_0^r(z, \overline{z}) + \partial \mathcal{K}_\theta^r(z, \overline{z}) \right) \partial \left( \mathcal{K}_0^r(z, \overline{z}) + \theta \mathcal{K}_\theta^r(z, \overline{z}) \right),
\]

(1.2)

and \( \mathcal{S}_\mathcal{W} \theta = -1 \).

In our symplectic formalism all the quantities acquire a well defined geometric meaning. In particular, the holomorphic ghosts give an origin to a decomposition which generate the finite \( \mathcal{W} \)-algebra generators \( \mathcal{C}^j(z, \overline{z}) \) and \( \mathcal{A}^j(z, \overline{z}) \), by setting
\[ \mathcal{K}_0^{(r)}(z, \overline{z}) + \theta \mathcal{K}_0(0, \overline{z}) = \sum_{j=1}^{r} \left( \omega_{(j-1),0}^{(r)}(z, \overline{z}) + \theta \omega_{(j-1),0}(z, \overline{z}) \right) \]

(1.3)

where the intertwining coefficients \( \omega_{(j-1),0}^{(r)}(z, \overline{z}) \) and \( \omega_{(j-1),0}(z, \overline{z}) \) can be recursively expressed in terms of the coordinates as

\[ \omega_{(j-1),0}^{(r)}(z, \overline{z}) = j! \prod_{i=1}^{m_j} \left[ \frac{\partial Z_{(p_i)}^{(r)}(z, \overline{z})}{a_i! \partial Z_0^{(r)}(z, \overline{z})} \right]_{\{ \Sigma_i a_i = j, \Sigma_i a_ip_i = r \}} \]

(1.4)

and in turn are completely fixed by the geometry and behave as true tensors.

At the end the symplectic B.R.S. transformations imply that the variations of the \( \mathcal{C}^{(i)}(z, \overline{z}) \) ghost fields describe the chiral part of \( \mathcal{W} \)-algebras:

\[ S_W \mathcal{C}^{(i)}(z, \overline{z}) = \sum_{s=1}^{l} s \mathcal{C}^{(s)}(z, \overline{z}) \partial \mathcal{C}^{(l-s+1)}(z, \overline{z}) + \mathcal{X}^{(i)}(z, \overline{z}), \quad l = 1, \ldots, n, \]

(1.5)

where the \( \mathcal{X}^{(i)}(z, \overline{z}) \) breaking terms represent, in the B.R.S approach, the contribution to the finite \( \mathcal{W} \)-algebras coming from the O.P.E. between primary fields, and indeed break the chiral representation invariance, but still preserve associativity. In the finite \( \mathcal{W}_3 \) instance we have:

\[ S_{W_3} \mathcal{C}^{(1)}(z, \overline{z}) = \mathcal{C}^{(1)}(z, \overline{z}) \partial \mathcal{C}^{(1)}(z, \overline{z}) - \frac{16}{3} \mathcal{T}(z, \overline{z}) \mathcal{C}^{(2)}(z, \overline{z}) \partial \mathcal{C}^{(2)}(z, \overline{z}) + \left( \partial \mathcal{C}^{(2)}(z, \overline{z}) \partial^2 \mathcal{C}^{(2)}(z, \overline{z}) - \frac{2}{3} \mathcal{C}^{(2)}(z, \overline{z}) \partial^2 \mathcal{C}^{(2)}(z, \overline{z}) \right) \]

(1.6)

\[ S_{W_3} \mathcal{C}^{(2)}(z, \overline{z}) = \mathcal{C}^{(1)}(z, \overline{z}) \partial \mathcal{C}^{(2)}(z, \overline{z}) + 2\mathcal{C}^{(2)}(z, \overline{z}) \partial \mathcal{C}^{(1)}(z, \overline{z}) \]

In order to insure the nilpotency of the BRS operation, \( \mathcal{X}^{(i)}(z, \overline{z}) \) in Eq(1.5) must transform as:

\[ S_W \mathcal{X}^{(i)}(z, \overline{z}) = \sum_{s=1}^{l} \left( s \mathcal{C}^{(s)}(z, \overline{z}) \partial \mathcal{X}^{(l-s+1)}(z, \overline{z}) \right) - s \mathcal{X}^{(s)}(z, \overline{z}) \partial \mathcal{C}^{(l-s+1)}(z, \overline{z}). \]

(1.7)

The full completion of the last equation (1.7) amounts to introducing together with the \( \mathcal{W} \)-variations all the set of the primary fields (including fields not appearing in \( \mathcal{X}^{(i)}(z, \overline{z}) \) ) which
belong to the algebra. In the $\mathcal{W}_3$ case we obtain:

$$S_{\mathcal{W}_3}\mathcal{T}(z,\bar{z}) = C(z,\bar{z})\partial\mathcal{T}(z,\bar{z}) + 2\mathcal{T}(z,\bar{z})\partial C(z,\bar{z}) - \mathcal{W}(z,\bar{z})\partial C^{(2)}(z,\bar{z})$$

$$- \frac{2}{3}C^{(2)}(z,\bar{z})\partial\mathcal{W}(z,\bar{z}) + \partial^3 C(z,\bar{z}), \quad (1.8)$$

which force to introduce a cubic differential $\mathcal{W}(z,\bar{z})$ as a field not involved in $\mathcal{X}^{(1)}(z,\bar{z})$; the minimal nilpotency chain closes by:

$$S_{\mathcal{W}_3}\mathcal{W}(z,\bar{z}) = C(z,\bar{z})\partial\mathcal{W}(z,\bar{z}) + 3\partial C(z,\bar{z})\mathcal{W}(z,\bar{z}) + 16\mathcal{T}(z,\bar{z})\partial \left(C^{(2)}(z,\bar{z})\mathcal{T}(z,\bar{z})\right)$$

$$+ \left(\partial^5 C(z,\bar{z}) + 2C^{(2)}(z,\bar{z})\partial^3\mathcal{T}(z,\bar{z}) + 10\mathcal{T}(z,\bar{z})\partial^3 C^{(2)}(z,\bar{z})\right)$$

$$+ 15\partial\mathcal{T}(z,\bar{z})\partial^2 C^{(2)}(z,\bar{z}) + 9\partial^2\mathcal{T}(z,\bar{z})\partial C^{(2)}(z,\bar{z})\right), \quad (1.9)$$

The structure of Eqs(1.5)(1.7) fixes a hierarchy such that the lowest orders are fixed by the higher ones; so the highest breaking term $\mathcal{X}^{(n)}(z,\bar{z})$ is so to speak related to the lowest order terms.

In the paper we aim to study the properties of the $\mathcal{X}^{(l)}(z,\bar{z})$ quantities under finite and infinitesimal change of charts in order to guarantee a global definition of the finite $\mathcal{W}$-algebra over a Riemann surface. Our investigation will lead to the result that primary fields are related to the function coefficients required to build up the conformal covariant derivatives and as such they describe gravitational degrees of freedom. This will be shown first in Section 2 through a finite change of charts. Anyhow this procedure does not fully exploit the locality values. To utilize them we further use the infinitesimal local change of charts in a B.R.S. way. The derived consistency requirements in this approach generate a connection between functions containing primary fields and the coefficients necessary to define the action of the holomorphic derivative in an intrinsic way. This link proves the inverse of the result, previously cited, found by Refs [7], [8], [27], [28], [29], and it is obtained within a fully general covariance calculations. As a concluding remark we stress that this result greatly takes some benefits from the local approach to $\mathcal{W}$ algebras we have recently provided in Refs [63], [64], [65].

### 2 The stability of $\mathcal{W}$-algebra under holomorphic change of charts

The purpose of this Section is to investigate the consequences of the background holomorphic change of coordinates

$$(w,\bar{w}) \rightarrow (z,\bar{z}), \quad w = w(z) \quad (2.1)$$

for the two equations (1.5)(1.7). To hold its validity over the whole surface, each member of each equation must verify the same transformation law. It is obvious that changes of charts and $\mathcal{W}$-symmetry will commute. Since the fields $C^{(l)}(z,\bar{z})$ behave as $(-r,0)$-differentials
\[ C^{(l)}(w, \overline{w}) = (w')^l C^{(l)}(z, \overline{z}) \]  

(2.2)

each term of Eqs (2.3) and (2.4) must transform in the same way.

Thanks to the ghost property, a simple calculation shows that the chiral summand

\[ \sum_{s=1}^{l} s C^{(s)}(z, \overline{z}) \partial C^{(l-s+1)}(z, \overline{z}) \]

behaves under finite change of charts as a \((-l,0)-\)differential, so that problems come from the presence of the \(X^{(l)}(z, \overline{z})\) breaking term which by covariance must transform also as a \((-l,0)-\)differential

\[ X^{(l)}(w, \overline{w}) = (w')^{-1+l} X^{(l)}(z, \overline{z}). \]  

(2.3)

According to the Faddeev-Popov grading and for \(l = 1, \ldots, n\) the \(X^{(l)}\) breaking term is decomposed over the ghost monomials but \(C^{(1)}\) as

\[ X^{(l)}(z, \overline{z}) = \frac{1}{2} \sum_{r,s \geq 0, p,q=2} \partial^r C^{(p)}(z, \overline{z}) \partial^s C^{(q)}(z, \overline{z}) \Lambda_{p+q-r-s-l}^{(l)} (r,p|s,q|l)(z, \overline{z}), \]  

(2.4)

where the zero graded coefficient functions \(\Lambda_{p+q-r-s-l}^{(l)} (r,p|s,q|l)\) is skewsymmetric under the permutation of the pairs \((r,p) \leftrightarrow (s,q)\) and contain primary fields.

### 2.1 Behaviour under finite holomorphic change of complex coordinates

As is well known troubles come from the derivative operators: in fact the \(r\)-th derivative of a \(p\)-th order tensor is no more a tensor and transforms in a convoluted way. For this reason we shall use ‘normalized’ covariant Bol operators \(L_r\):

\[ L_r(z, \overline{z}) = \sum_{j=0}^{r} a_j^{(r)}(z, \overline{z}) \partial^{r-j}, \quad a_0^{(r)}(z, \overline{z}) = 1, \quad a_1^{(r)}(z, \overline{z}) = 0 \]  

(2.5)

which under any change of holomorphic charts transform as

\[ L_r(w, \overline{w}) = (w')^{-\frac{1}{2}+r} L_r(z, \overline{z})(w')^{\frac{1}{2}-r}. \]  

(2.6)

\(L_r\) transforms functions of conformal weight \(\frac{1}{2}-r\) into ones of weight \(\frac{1}{2}+r\). In particular the kernel of \(L_r(w, \overline{w})\) is \(r\) dimensional linear space which is stable under any holomorphic change of charts \((w, \overline{w}) \rightarrow (z, \overline{z})\).
This covariance property enables a (non unique) construction of the $a^{(r)}_j(z,\bar{z})$ coefficients within a coordinate description. How this construction can be carried out in our coordinate scheme is discussed in the Appendix;

The explicit expressions in terms of the coordinates for the coefficients $a^{(2)}_2(z,\bar{z})$, $a^{(3)}_2(z,\bar{z})$ and $a^{(3)}_3(z,\bar{z})$ (the latter will be useful to discuss the $\mathcal{W}_3$-algebra) are:

\[
a^{(2)}_2(z,\bar{z}) = \partial^2 \ln \partial Z(z,\bar{z}) - \frac{1}{2} (\partial \ln \partial Z(z,\bar{z}))^2
\]
\[
a^{(3)}_2(z,\bar{z}) = \partial^2 \ln w(z,\bar{z}) - \frac{1}{3} (\partial \ln w(z,\bar{z}))^2 + \frac{v(z,\bar{z})}{w(z,\bar{z})}
\]
\[
a^{(3)}_3(z,\bar{z}) = \frac{1}{3} \partial^3 \ln w(z,\bar{z}) + \frac{2}{27} (\partial \ln w(z,\bar{z}))^3 + \frac{v(z,\bar{z})}{3w(z,\bar{z})} \partial \ln w(z,\bar{z})
\]

where we have set the determinants

\[
w(z,\bar{z}) = \left| \begin{array}{cc} \partial Z(z,\bar{z}) & \partial Z^2(z,\bar{z}) \\ \partial^2 Z(z,\bar{z}) & \partial^2 Z^2(z,\bar{z}) \end{array} \right|, \quad v(z,\bar{z}) = \left| \begin{array}{ccc} \partial^2 Z(z,\bar{z}) & \partial^2 Z^2(z,\bar{z}) \\ \partial^3 Z(z,\bar{z}) & \partial^3 Z^2(z,\bar{z}) \end{array} \right|.
\]

Turning back to the glueing problem of the breaking term $X^{(l)}$, without any loss of generality, (2.4) rewrites for $l = 1, \ldots, n$

\[
X^{(l)}(z,\bar{z}) = \frac{1}{2} \sum_{r,s \geq 0} \sum_{p,q=2}^n L_r C^{(p)}(z,\bar{z}) L_s C^{(q)}(z,\bar{z}) A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(z,\bar{z}),
\]

where $A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)$ is of zero ghost grading, skewsymmetric under the permutation of the pairs $(r,p) \leftrightarrow (s,q)$ and constructed over some primary fields.

It is easy to see that the derivative troubles can be overcome by inverting equation (2.5) iteratively,

\[
\partial^r = \sum_{j=0}^r b_j^{(r)}(z,\bar{z}) L_{r-j}(z,\bar{z}), \quad b_j^{(r)}(z,\bar{z}) = 1, \quad b_1^{(r)}(z,\bar{z}) = 0,
\]

and where each of the $b_j^{(r)}$ for $j \geq 2$ is a polynomial expression in the $a^{(s)}_k$, namely for

\[
j \geq 2, \quad J_k := \sum_{m=0}^{k-1} j_m, \quad J_1 = 0, \quad b_j^{(r)} = \sum_{i=1}^r (-1)^i \left[ \prod_{\ell=1}^i \left( \sum_{j=1}^{J_\ell} a^{(r-J_\ell)}_j \right) \delta_{j,J_{i+1}} \right].
\]

Equation (2.11) thus writes:

\[
\sum_{r,s \geq 0} \sum_{p,q=2}^n L_r(w,\bar{w}) C^{(p)}(w,\bar{w}) L_s(w,\bar{w}) C^{(q)}(w,\bar{w}) A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(w,\bar{w})
\]
\[
= (w')^l \sum_{r,s \geq 0} \sum_{p,q=2}^n L_r(z,\overline{z})C^{(p)}(z,\overline{z})L_s(z,\overline{z})C^{(q)}(z,\overline{z})A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(z,\overline{z})
\]

\[
\equiv \sum_{r,s \geq 0} \sum_{p,q=2}^n (w')^{-(\frac{1+s}{2})} L_r(z,\overline{z})(w')^{(\frac{1+s}{2})} + pC^{(p)}(z,\overline{z}) L_s(z,\overline{z})(w')^{(\frac{1-s}{2})} + qC^{(q)}(z,\overline{z}) A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(w,\overline{w}) \tag{2.12}
\]

Using successively (2.5), Leibniz rule and (2.10) one gets after some combinatoric manipulations

\[
(w')^{-(\frac{1+s}{2})} L_r(z,\overline{z}) (w')^{(\frac{1-r}{2})} + pC^{(p)}(z,\overline{z}) = \sum_{m=0}^r \alpha(r,m,p)(z,\overline{z})L_mC^{(p)}(z,\overline{z}), \tag{2.13}
\]

with the polynomial in the \(a_k^{(s)}\)

\[
\alpha(r,m,p)(z,\overline{z}) = \sum_{j=0}^{r-m} \sum_{k=m}^{r-j} \binom{r-j}{k} (w')^{-(\frac{j+k}{2})} a_j^{(r)}(z,\overline{z}) \partial^{r-j-k}(w')^{(\frac{j+k}{2})} + p. \tag{2.14}
\]

Inserting twice (2.13) into (2.12) we finally deduce a set of algebraic equations

\[
A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(z,\overline{z}) = \sum_{m \geq r} \sum_{m' \geq s} w'^{-l} \alpha(m,r,p)(z,\overline{z}) \alpha(m',s,q)(z,\overline{z})
\]

\[
A^{(l)}_{p+q-r-s-l}(m,p|m',q|l)(w,\overline{w}) \tag{2.15}
\]

In particular, this system gives for each value of \(p,q,r,s,l\) the glueing properties of the function \(A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)\). Restricting the system (2.15) to the scalar sector, that is for \(p + q - r - s - l = 0\) and

\[
A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(z,\overline{z}) |_{r+s-p-q-l=0} = A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)(w,\overline{w}) |_{r+s-p-q-l=0}
\]

the scalar component of \(A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)\) cancels out and the system reduces to an homogeneous one containing \(A^{(l)}_{r+s-m-m'}(m,p|m',q|l)\), \(m + m' > r + s\) with therefore negative lower indices. The finite degree \(n\) of the algebra guarantees a finite number of equations \(l = 1, \ldots, n\), and the determinant of the system will be non zero in general leading to \(A^{(l)}_{h} \equiv 0\) for \(h < 0\) and \(l = 1, \ldots, n\). We reach the conclusion:

**Theorem 1** The stability under change of charts requires that the only non vanishing \(A^{(l)}_{p+q-r-s-l}(r,p|s,q|l)\) have a positive lower indices content, \(p + q - r - s - l \geq 0\).

Anyhow in the more general cases, the way out to get fully local relations in only one argument, is to use the infinitesimal holomorphic change of coordinates.
2.2 Infinitesimal holomorphic change of complex coordinates

Under infinitesimal holomorphic change of coordinates \( z \rightarrow z - \epsilon(z) \) and following the notation used in [2] we have:

\[
\delta \mathcal{C}^{(m)}(z, \bar{z}) = X_{-m}(z) \mathcal{C}^{(m)}(z, \bar{z}), \quad X_m(z) = \epsilon(z) \partial + m \partial \epsilon(z)
\]  
(2.16)

At the infinitesimal level, (2.3) reduces to

\[
\delta \epsilon X^{(l)}(z, \bar{z}) = X_{-l}(z) X^{(l)}(z, \bar{z})
\]  
(2.17)

while the conformal covariant derivative transforms as [2, 8]

\[
\left[ \delta \epsilon, L_r \right] = X_{(1+r)} L_r - L_r X_{(1-r)}
\]  
(2.18)

and in particular the case \( r = 1 \) gives \([\delta \epsilon, \partial] = 0\. Accordingly the \( a_j^{(k)} \) coefficients satisfy

\[
\delta \epsilon a_j^{(k)}(z, \bar{z}) = X_j(z) a_j^{(k)}(z, \bar{z}) + \frac{1}{2} \left( \begin{array}{c} k+1 \\ k-j \end{array} \right) (j-1) \partial^{j+1} \epsilon(z)
\]

\[
- \sum_{l=2}^{j-1} \left\{ \left( \begin{array}{c} k-l \\ j-l+1 \end{array} \right) - \frac{k-1}{2} \left( \begin{array}{c} k-l \\ j-l \end{array} \right) \right\} a_l^{(k)}(z, \bar{z}) \partial^{j-l+1} \epsilon(z)
\]  
(2.19)

for \( j \geq 2 \). Note that (2.19) is linear in \( a_j^{(k)} \) and depends on its first derivative only. Turning the holomorphic \((-1,0)\)-differential \( \epsilon(z) \) a Faddeev-Popov ghost (and accordingly \( X_l \) as well), so that \( \delta \epsilon^2 = 0 \), one has:

\[
\delta \epsilon \epsilon(z) = \frac{1}{2} X_{-1} \epsilon(z) = \epsilon(z) \partial \epsilon(z)
\]  
(2.20)

and using (2.18) for \( r = 1 \), one finds

\[
\left\{ \delta \epsilon, X_l(z) \right\} = \epsilon(z) \left[ \partial, X_l(z) \right]
\]  
(2.21)

so that the operator \( \delta \epsilon - X_l \) is nilpotent for any \( l \) in turn.

The independence of the transformations assure that the \( \mathcal{W} \)-transformations anticommute with the \( \delta \epsilon \) operator:

\[
\mathcal{S}_{\mathcal{W}} \epsilon(z) = 0, \quad \{\delta \epsilon, \mathcal{S}_{\mathcal{W}}\} = 0.
\]  
(2.22)

It could be noted that the underlying algebra of \( \delta \epsilon \) is equivalent to the \( \mathcal{W} \)-algebra where the chiral ghosts \( \mathcal{C}^{(l)}(z, \bar{z}) \) \( l \geq 2 \) are set to be zero which is nothing but the usual conformal
algebra found in [13]. In our approach the \( \mathcal{W} \)-algebras come out from the action of smooth cotangent diffeomorphisms on a chain of smooth changes of local complex coordinates on the Riemann surface. In turn, all the \( Z_0^{(r)}(z, \bar{z}) \), \( Z_0^{(r)}(z, \bar{z}) \) complex coordinates have well defined \( \mathcal{W} \)-transformations, and the B.R.S. algebra is kept nilpotent by performing a \( \theta \) doubling in order to obey the Jacobi identities. This doubling trick amounts to studying the true structure of the \( Z_0^{(r)}(z, \bar{z}) \) coordinates. The question will be partially solved (this will be enough for the purpose) by looking at the properties of the infinitesimal change of charts of \( Z_0^{(r)}(z, \bar{z}) \) and their B.R.S. \( \mathcal{W} \)-transformations as well.

The \( Z_0^{(r)} \) complex coordinates are scalars under change of charts:

\[
\delta_{\epsilon} Z_0^{(r)}(z, \bar{z}) = \epsilon(z) \partial Z_0^{(r)}(z, \bar{z})
\]  
(2.23)

while in their \( \mathcal{W} \)-transformations, see [1.1], the most general form of \( Z_0^{(r)} \) can be made explicit in terms of the chiral ghosts [65],

\[
S_W Z_0^{(r)}(z, \bar{z}) = \mathcal{K}_0^{(r)}(z, \bar{z}) \partial Z_0^{(r)}(z, \bar{z}) + \sum_{s \geq 0} \sum_{p=2}^n L_s(z, \bar{z}) \mathcal{C}^{(p)}(z, \bar{z}) B_{p-s}^{(r)}(p|s|r)(z, \bar{z}).
\]  
(2.24)

The anticommutativity \( 2.22 \) between the two operations on \( Z_0^{(r)} \) and the use of \( 2.18 \) yield

\[
(\delta_{\epsilon} - X_{p-s}(z)) B_{p-s}^{(r)}(p|s|r)(z, \bar{z}) = N_{p-s}^{(r)}(p|s|r|a, \mathcal{B})(z, \bar{z})
\]  
(2.25)

where \( N_{p-s}^{(r)} \) is linear in the \( B \)s and of grading one with respect to \( \epsilon \) of the form

\[
N_{p-s}^{(r)}(p|s|r|a, \mathcal{B})(z, \bar{z}) = \sum_{k \geq 2} \partial^k \epsilon(z) \sum_{m \geq s+k+1} N_{m+1-k-s}(p|s|k|m;a)(z, \bar{z}) B_{p-m}^{(r)}(z, \bar{z})
\]  
(2.26)

where \( N_{m+1-k-s}(p|s|k|m;a) \) is a known polynomial in the \( a_k^{(s)} \) coefficients,

\[
N_{m+1-k-s}(p|s|k|m;a) = \sum_{j=0}^{m+1-s-k} \left( \frac{1 - m}{2} + p \right) \binom{m - j}{m + 1 - k - j} a_j^{(m)} b_{m+1-k-j-s}^{(m+1-k-j)}.
\]  
(2.27)

By nilpotency, one gets

\[
(\delta_{\epsilon} - X_{p-s}) N_{p-s}^{(r)}(p|s|r|a, \mathcal{B}) = 0.
\]  
(2.28)

Owing to \( 2.19 \) this expression is still algebraic in the coefficients \( a_k^{(s)} \) and their first order \( z \)-derivatives. Decomposing over the monomials \( \partial^k \epsilon \partial^l \epsilon \) leads to a homogeneous linear system in the \( B \)s that can be inverted. Hence expressing algebraically the \( B \)s in terms of the \( a_k^{(s)} \) coefficients and their first order \( z \)-derivatives implies that \( Z_0^{(s)} \) is expressible by means of both \( Z_0^{(r)} \) and \( C^{(l)} \) and their \( z \)-derivatives, see the appendix.

Turning back to the study of the behaviour of \( A \) under an infinitesimal holomorphic change of complex coordinates. Let us now derive the properties of \( A_{p+q-r-s-l}^{(l)}(r, p|s|q|l) \) from \( 2.19 \) and \( 2.17 \). After some calculation we get:

\[
(\delta_{\epsilon} - X_{p+q-r-s-l}(z)) A_{p+q-r-s-l}^{(l)}(r, p|s|q|l)(z, \bar{z}) = \Omega_{p+q-r-s-l}^{(l)}(r, p|s|q|l)(z, \bar{z})
\]  
(2.29)
where $\Omega_{p+q-r-s-l}^{(l)}(r, p|s, q|l)$ is the following linear expression in the $A^{(l)}$ for

$$\Omega_{p+q-r-s-l}^{(l)}(r, p|s, q|l) = \sum_{k \geq 2} \partial^k \epsilon(z) \times$$

$$\left( \sum_{m \geq r+k+1} N_{m+1-k-r}(p|k|m; a)(z, \overline{z}) A_{p+q-m-s-l}^{(l)}(m, p|s, q|l)(z, \overline{z}) + \sum_{m \geq s+k+1} N_{m+1-k-s}(p|k|m; a)(z, \overline{z}) A_{p+q-r-m-l}^{(l)}(r, p|m, q|l)(z, \overline{z}) \right), \quad (2.30)$$

where $N$ has already been defined in (2.27).

Once more by nilpotency, one has

$$\left( \delta^2_\epsilon - X_{p+q-r-s-l}(z) \right) \Omega_{p+q-r-s-l}^{(l)}(r, p|s, q|l) = 0. \quad (2.31)$$

Solving this condition insures the proper definiteness of the whole theory over a Riemann surface. As before decomposing over the monomials $\partial^k \epsilon \partial^\ell \epsilon$ leads for each sector a homogeneous linear system in the $A$s in finite dimension thanks to Theorem (I) and the finite degree of $W$-algebra that can be inverted.

This shows that trivial solutions $A^{(l)}$ of Eq(2.31) are algebraic expressions in the coefficients $a_{j}^{(l)}(z, \overline{z})$ and their first order $z$-derivatives. Therefore they are related to the complex coordinates $Z_0^{(r)}$. This drastically changes the physical scenario, namely, if we rely on the idea that primary fields describe a conformal matter we have to change our mind and relate them to gravitational degrees of freedom.

The closure under of the B.R.S. $W$-transformations of the $Z_0^{(r)}$ fields secures that all of the necessary primary fields, not involved in the breaking terms $X^{(l)}$, will be also differential polynomials (with higher order $z$-derivatives) in the coefficients $a_{j}^{(l)}$. They accordingly turn to be also functions of the $Z_0^{(r)}(z, \overline{z})$ coefficients and their derivatives. Collecting together these results yields the Statement:

**Statement 1** All primary fields involved in a given finite $W_n$-algebra are local functions of the $Z_0^{(r)}$ complex coordinates and their $z$-derivatives.

The latter completely modifies the physical point of view of this scenario.

The identifications of the old primary fields in terms of monomials of $a_{j}^{(l)}(z, \overline{z})$ and their derivatives was performed in Ref [8]: our treatment gives the exactly inverse of the proof given there: while in this reference it was proved that the $a_{j}^{(l)}$ can be expressed in terms of the generators $w$ of a given $W$-algebra. Here, we state the converse, namely, that the primary fields
of this algebra can be expressed with the \( a_j^{(l)} \) and their \( z \)-derivatives. For the \( W_3 \) instance, the currents are related to the \( a_j^{(l)}(z, \bar{z}) \) coefficients from the solutions of the equation [8]

\[
L_3 f_i(z, \bar{z}) = 0 \quad i = 1, 2, 3
\]

\[
\sum_{i,j,k} c^{i,j,k} \partial^2 f_i(z, \bar{z}) \partial f_j(z, \bar{z}) f_k(z, \bar{z}) = 0
\]  

(2.32)

by

\[
T(z, \bar{z}) = \frac{a_2^{(3)}(z, \bar{z})}{2}, \quad \text{and} \quad W(z, \bar{z}) = \frac{1}{8} \left( \frac{1}{2} \partial a_2^{(3)}(z, \bar{z}) - a_3^{(3)}(z, \bar{z}) \right)
\] 

(2.33)

With the help of (2.8) the cubic differential \( W \) current reads

\[
W(z, \bar{z}) = \frac{1}{24} \left( \frac{1}{2} \partial^3 \ln w(z, \bar{z}) - \partial^2 \ln w(z, \bar{z}) \partial \ln w(z, \bar{z}) \right) - \frac{2}{9} (\partial \ln w(z, \bar{z}))^3
\]

\[+ \frac{1}{16} \left( \frac{\partial v(z, \bar{z})}{w(z, \bar{z})} - \frac{5v(z, \bar{z})}{3w(z, \bar{z})} \partial \ln w(z, \bar{z}) \right)
\]  

(2.34)

and is nothing but a Laguerre invariant while \( a_2^{(3)}(z, \bar{z}) \) and the combination \( 9(\partial a_2^{(3)}(z, \bar{z}) - 3a_3^{(3)}(z, \bar{z})) \) are both the so-called Painlevé invariants [77] as pointed out in [71].

By the way from Eq (2.33) (1.8) (1.9) is easy to derive [70]:

\[
S f_i(z, \bar{z}) = C(z, \bar{z}) \partial f_i(z, \bar{z}) - f_i(z, \bar{z}) \partial C(z, \bar{z}) + 2C^{(2)}(z, \bar{z}) \partial^2 f_i(z, \bar{z}) - \partial C^{(2)} \partial f_i(z, \bar{z})
\]

\[+ \frac{1}{3} f_i(z, \bar{z}) \partial C^{(2)}(z, \bar{z}) + \frac{8}{3} T(z, \bar{z}) f_i(z, \bar{z}) C^{(2)}(z, \bar{z})
\]  

(2.35)

The coordinates are defined [88] as the scalars obtained from the previous solutions; for example \( Z_0 = f_2/f_1 \) and \( Z_0^{(2)} = f_3/f_1 \). By quotients the variations of the former turn out to be

\[
SZ_0(z, \bar{z}) = C \partial Z_0(z, \bar{z}) + 2C^{(2)}(z, \bar{z}) \partial^2 Z_0(z, \bar{z})
\]

\[- \partial C^{(2)}(z, \bar{z}) \partial Z_0(z, \bar{z}) - \frac{4}{3} C^{(2)}(z, \bar{z}) \partial Z_0(z, \bar{z}) \partial \ln \omega(z, \bar{z})
\]  

(2.36)

\[
SZ_0^{(2)}(z, \bar{z}) = C(z, \bar{z}) \partial Z_0^{(2)}(z, \bar{z}) + 2C^{(2)}(z, \bar{z}) \partial^2 Z_0^{(2)}(z, \bar{z})
\]

\[- \partial C^{(2)}(z, \bar{z}) \partial Z_0^{(2)}(z, \bar{z}) - \frac{4}{3} C^{(2)}(z, \bar{z}) \partial Z_0^{(2)}(z, \bar{z}) \partial \ln \omega(z, \bar{z})
\]  

(2.37)

where \( \omega(z, \bar{z}) \) has been introduced in Eq (2.8)

3 Conclusions

Our results show how general principles may clarify the physical landscape, in the sense that the quantum extension of models which satisfy \( W \)-symmetry generated by primary fields, should help the quantum description of gravitational degrees of freedom. This extension greatly complicates this research topic, but, on an other hand, is extremely provocative.
Also the Laguerre-Forsyth construction offers the possibility of expressing primary current fields directly in terms of “matter fields” considered as solutions of linear differential equations with algebraic integrals of the type \( L f = 0 \) where \( L \) is the most general covariant operators. This type of equations has been previously obtained in a systematic way [70] through vanishing curvature conditions (a useful trick already used in [12, 11]). Even if there exist few Lagrangian models giving rise to field equations containing Bol’s operators [41], a general construction of such Lagrangians is still missing. Furthermore, the intimate link between these solutions and a hierarchy of complex coordinates \( Z_0^{(r)} \) coming from the symplectic scenario raises the question whether which of either coordinates or matter fields are the most important at the physical level. Expressing \( Z_0^{(r)} \) in terms of the solutions of \( L_n f = 0 \) as in (4.42) generates the breaking from \( W\infty \) to finite \( W_n \)-algebras where \( n \) is exactly the order of the generalized Bol operator \( L_n \) and may give a partial answer to this question. Note also that the \( \theta \)-’trick’ used in [65] for introducing the chiral breaking terms \( \mathcal{X} \) seems to be in relation with the restriction to \( \ker L_n \).

4 Appendix

The aim of this appendix is to give a coordinate description of the coefficients \( a_j^{(s)}(z, \overline{z}) \forall s = 2 \cdots n \): in terms of the coordinates \( Z_0^{(r)}(z, \overline{z}) \), \( r = 1 \cdots n \).

To do this we use the Di Francesco-Itzykson-Zuber [8] within our construction [63].

Consider the space \( \mathcal{V}_{1-z} \) of the functions \( f_i(z, \overline{z}) \) with holomorphic weight \( d = 1- \frac{s}{2} \) solutions of the equation:

\[
L_s f_i(z, \overline{z}) = 0 \quad i = 1 \cdots s
\]  

such that:

\[
\begin{vmatrix}
\partial^{(s-1)} f_1(z, \overline{z}) & \ldots & \partial^{(s-1)} f_s(z, \overline{z}) \\
\vdots & \ddots & \vdots \\
\partial f_1(z, \overline{z}) & \ldots & \partial f_s(z, \overline{z}) \\
f_1(z, \overline{z}) & \ldots & f_s(z, \overline{z})
\end{vmatrix} = 1
\]  

(4.39)

So for a whatever other function \( f(z, \overline{z}) \in \mathcal{V}_{1-z} \) the action \( L_s f(z, \overline{z}) \) can be defined by:

\[
L_s f(z, \overline{z}) = \begin{vmatrix}
\partial^{(s)} f(z, \overline{z}) & \partial^{(s)} f_1(z, \overline{z}) & \ldots & \partial^{(s)} f_s(z, \overline{z}) \\
\partial^{(s-1)} f(z, \overline{z}) & \partial^{(s-1)} f_1(z, \overline{z}) & \ldots & \partial^{(s-1)} f_s(z, \overline{z}) \\
\vdots & \vdots & \ddots & \vdots \\
\partial f(z, \overline{z}) & \partial f_1(z, \overline{z}) & \ldots & \partial f_s(z, \overline{z}) \\
f(z, \overline{z}) & f_1(z, \overline{z}) & \ldots & f_s(z, \overline{z})
\end{vmatrix}
\]  

(4.40)

which is a quantity with weight \( d = -\frac{1-s}{2} \). So the \( a_j^{(s)}(z, \overline{z}) \) can be identified with minors of this determinant.
\[ a_j^{(s)}(z, \bar{z}) = \begin{vmatrix}
\partial^{(s)} f(z, \bar{z}) & \partial^{(s)} f_1(z, \bar{z}) & \cdots & \partial^{(s)} f_{(s)}(z, \bar{z}) \\
\vdots & \vdots & \ddots & \vdots \\
\partial^{(j+1)} f(z, \bar{z}) & \partial^{(j+1)} f_1(z, \bar{z}) & \cdots & \partial^{(j+1)} f_{(s)}(z, \bar{z}) \\
\vdots & \vdots & \ddots & \vdots \\
\partial f(z, \bar{z}) & \partial f_1(z, \bar{z}) & \cdots & \partial f_{(s)}(z, \bar{z}) \\
f(z, \bar{z}) & f_1(z, \bar{z}) & \cdots & f_{(s)}(z, \bar{z})
\end{vmatrix} \quad (4.41) \]

From the \( f_i(z, \bar{z}) \quad i = 1 \cdots s \) functions we can define \( s - 1 \) scalars:

\[ Z^{(j-1)}(z, \bar{z}) = \frac{f_j(z, \bar{z})}{f_1(z, \bar{z})}, \quad j = 2 \cdots s \quad (4.42) \]

which provide a local \( s - 1 \) dimensional system of coordinates. Assuming that this system coincides with the coordinates previously introduced we invert the procedure to construct a system of \( f_i(z, \bar{z}) \) functions:

First we define the quantity:

\[ \omega(z, \bar{z}) = \begin{vmatrix}
\partial^{(s-1)} Z^{(1)}(z, \bar{z}) & \cdots & \partial^{(s-1)} Z^{(s-1)}(z, \bar{z}) \\
\vdots & \ddots & \vdots \\
\partial Z^{(1)}(z, \bar{z}) & \cdots & \partial Z^{(s-1)}(z, \bar{z})
\end{vmatrix} \quad (4.43) \]

is an object with weight \( \frac{s(s-1)}{2} \), so we can build the \( s \) linearly independent functions with weight \( d = \frac{1-s}{2} \):

\[ f_1(z, \bar{z}) = \omega^{-\frac{1}{2}}(z, \bar{z}) \]
\[ f_2(z, \bar{z}) = Z^{(1)}(z, \bar{z})\omega^{-\frac{1}{2}}(z, \bar{z}) \]
\[ \vdots \]
\[ f_s(z, \bar{z}) = Z^{(s-1)}(z, \bar{z})\omega^{-\frac{1}{2}}(z, \bar{z}) \quad (4.44) \]

which provide a system of solutions for the Eq(4.38), with the coefficients \( a_j^{(s)}(z, \bar{z}) \) given in Eq(4.41).

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