TRIGONOMETRIC K-MATRICES FOR FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra and $U_q\hat{\mathfrak{g}}$ the corresponding quantum affine algebra. We prove that every irreducible finite-dimensional $U_q\hat{\mathfrak{g}}$-module gives rise to a family of trigonometric K-matrices, i.e., rational solutions of Cherednik’s generalized reflection equation. The result depends upon the choice of a quantum affine symmetric pair $U_q\mathfrak{k} \subset U_q\hat{\mathfrak{g}}$. It hinges on the construction of universal K-matrices for arbitrary quantum symmetric pairs obtained in [AV22], and relies on proving that every irreducible $U_q\hat{\mathfrak{g}}$-module is generically irreducible under restriction to $U_q\mathfrak{k}$. In the case of small modules and Kirillov-Reshetikhin modules, we obtain new solutions of the standard and the transposed reflection equations.

Contents
1. Introduction 1
2. Quantum affine algebras and their R-matrices 5
3. Quantum affine symmetric pairs 10
4. Spectral K-matrices 15
5. Trigonometric K-matrices 19
6. Generic restricted irreducibility 24
7. Solutions of reflection equations 27
References 33

1. Introduction

1.1. The reflection equation was originally introduced by Cherednik in [Che84] as a consistency condition for factorized scattering on the half-line and was pivotal in the seminal works of Sklyanin [Skl88] and Olshanski [Ols90]. In the more general form introduced in [Che92, Sec. 4], it reads

$$R^{--}(\frac{z}{2})_{21} \cdot id_V \otimes K(w) \cdot R^{-+}(zw) \cdot K(z) \otimes id_V = K(z) \otimes id_V \cdot R^{-+}(zw)_{21} \cdot id_V \otimes K(w) \cdot R^{++}(\frac{z}{2}),$$

where $K(z)$ is an operator on a finite-dimensional vector space $V$, which depends on a parameter $z \in \mathbb{C}^\times$, and $R^{--}(z)$, $R^{-+}(z)$, and $R^{++}(z)$ are operators on $V \otimes V$.

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which satisfy certain mixed Yang-Baxter equations. The operator $K(z)$ is often referred to as a K-matrix.

The two most common variants of the reflection equation are recovered as special cases. Given a solution $R(z)$ of the spectral Yang-Baxter equation on $V$, the choice $R^{\pm}(z) = R(z)$ produces the standard reflection equation from [Che84, Skl88]. On the other hand, if $t_1$ and $t_2$ denote the transpositions with respect to the corresponding tensor factors, the choice $R^{--}(z) = R(z)^{t_2}$, $R^{++}(z) = (R(z)^{-1})^{t_1}$, and $R^{+-}(z) = R(z)$ yields the transposed reflection equation\footnote{This equation is also called the twisted reflection equation, but we opt for this terminology in order to avoid confusion.} from [Ols90].

1.2. The reflection equation soon proved ubiquitous in quantum integrability and in representation theory, with further advances in [FM91, KS92, MN92, GZ94, LM94, JKKMW, BPO96, FSHY, IK97]. In the new millennium, its study has continued to thrive. It is essential in the study of coideal subalgebras in Yangians and quantum affine algebras [MR02, BB10, DLMR, CGM14, GRW17] and connects to affine Hecke and Temperley-Lieb algebras [Doi05, KM06, IO07, DGN08]. Moreover, within the context of quantum integrable systems with boundaries, it plays a key role in the diagonalization of Hamiltonians [DGP04, FNR07, Koj11], in the theory of Baxter Q-operators [FS15, BT18, VW20], and in the theory of quantum KZ equations [ZJ07, SV15, RSV18]. Recently, it arose in connection with Schubert calculus [HKZJ], three-dimensional integrability [KP18, KOY19] and gauge theory [BS19, BS20].

1.3. In this paper, we solve a central problem which has been open since the introduction of the reflection equation: we provide a systematic, universal approach to the construction of trigonometric K-matrices for irreducible finite-dimensional modules over quantum affine algebras. Moreover, we obtain a characterization of trigonometric K-matrices in terms of twisted intertwiners, confirming the expectations from [DG02, DM03].

Our approach requires the extension of the construction of trigonometric R-matrices carried out in [Dri86, FR92, KS95] to the challenging setting of affine quantum symmetric pair (QSP) subalgebras. It crucially relies on our recent work [AV22], where we prove the existence of universal K-matrices for quantum Kac-Moody algebras, building on results by Bao and Wang [BW18] and Balagović and Kolb [BK19]. We now describe our results in more detail.

1.4. The universal K-matrices constructed in [AV22] for the quantum Kac-Moody algebra $U_q\mathfrak{g}_{\text{aff}}$ depends on the following additional data: a QSP subalgebra, i.e., a coideal subalgebra $U_q\mathfrak{t} \subset U_q\mathfrak{g}_{\text{aff}}$ quantizing the fixed-point subalgebra of an involution of the second kind of the affine Lie algebra [Kol14] (see also [RV22]), and a twisting operator $\psi$, i.e., an element of a distinguished class of algebra automorphisms of $U_q\mathfrak{g}_{\text{aff}}$ determined in part by $U_q\mathfrak{t}$ (see Sections 3.5 and 3.6). For any pair $(U_q\mathfrak{t}, \psi)$, we constructed an operator $K$ on category $\mathcal{O}$ integrable $U_q\mathfrak{g}_{\text{aff}}$-modules, which satisfies the twisted reflection equation

\[(R^{\psi})_{21} \cdot 1 \otimes K \cdot R^\psi \cdot K \otimes 1 = K \otimes 1 \cdot (R^\psi)_{21} \cdot 1 \otimes K \cdot R,\]

where $R$ is the universal R-matrix of $U_q\mathfrak{g}_{\text{aff}}$, $R^\psi := (\psi \otimes \text{id})(R)$, and $R^{\psi \psi} := (\psi \otimes \psi)(R)$. Moreover, $K$ satisfies the QSP intertwining equation $K \cdot b = \psi(b) \cdot K$ for all $b \in U_q\mathfrak{t}$.
All universal K-matrices corresponding to a given QSP subalgebra are built out of the quasi-K-matrix, originally introduced in [BW18] as the solution of a certain QSP intertwining equation, which does not depend on $\psi$. In [AV22] we proved that, up to a Cartan correction, the quasi-K-matrix is a solution of a reflection equation of the form (1.2) for a distinguished twisting operator (see Theorem 3.5.1). A plethora of new solutions is now obtained by gauging simultaneously the corrected quasi K-matrix, which we refer to as the standard K-matrix, and the twisting operator (see Section 3.6). In particular, we solve (1.2) for a much wider class of twisting operators than [BK19]. This feature is essential to the purposes of this paper.

1.5. Let $\mathfrak{g}$ be a complex simple Lie algebra and $U_q\widehat{\mathfrak{g}}$ the corresponding quantum affine algebra. By relying on the results from [AV22] for affine type, we adapt the strategy underlying the construction of trigonometric R-matrices to the case of universal K-matrices.

Fix a QSP subalgebra $U_q\mathfrak{k} \subset U_q\widehat{\mathfrak{g}}$ and a twisting operator $\psi$. The corresponding universal K-matrix $K$ acts on integrable category $O$ modules, but not on a finite-dimensional $U_q\widehat{\mathfrak{g}}$-module $V$ in general. To remedy this, we consider the grading shift automorphism $\Sigma_z$ on $U_q\widehat{\mathfrak{g}}$ and the corresponding shifted module $V(\!(z)\!)$. We then show that the shifted universal K-matrix $\Sigma_z(K)$ gives rise to an operator $K_V(z)$ on $V(\!(z)\!)$ and yields the following result (see Theorem 4.2.1).

**Theorem 1.** Let $V$ be a finite-dimensional $U_q\widehat{\mathfrak{g}}$-module and denote by $V^\psi$ is the $\psi$-twisted module $\psi^*(V)$. The universal K-matrix $K$ gives rise to a $U_q\mathfrak{k}$-intertwiner

\[ K_V(z) : V(\!(z)\!) \to V^\psi(\!(z^{-1})\!) , \]

which satisfies the generalized reflection equation (1.1) with respect to the R-matrices $R_{V^\psi}V(z)$, $R_{V^\psi}V(z)$ and $R_{V^\psi}(z)$.

More precisely, Theorem 1 does not hold for every possible choice of $\psi$, somewhat in contrast with the case of the R-matrix. Instead we prove that, for the corresponding universal K-matrix to act on $V(\!(z)\!)$ and produce the intertwiner (1.3), the twisting operator $\psi$ must be carefully chosen. In particular, it has to satisfy $^2\Sigma_z \circ \psi = \psi \circ \Sigma_z^{-1}$. In this case, the twisted reflection equation (1.2) reduces to the desired generalized reflection equation.

1.6. In the case of irreducible modules, we obtain a trigonometric K-matrix as a suitable normalization of $K(z)$ (see Theorem 5.1.2).

**Theorem 2.** Let $V$ be an irreducible finite-dimensional $U_q\widehat{\mathfrak{g}}$-module. Then, there is a scalar-valued formal Laurent series $g_V(z)$ and an operator $K_V(z) \in \text{End}(V)(z)$ such that

\[ K_V(z) = g_V(z) \cdot K_V(z) \]

and $K_V(z)$ is a solution of the generalized reflection equation (1.1) with respect to the trigonometric R-matrices $R_{V^\psi}V(z)$, $R_{V^\psi}V(z)$ and $R_{V^\psi}(z)$.

The proof of Theorem 2 crucially relies on the particular form of the twisting operator and the following result (see Corollary 6.1.2).

**Theorem 3.** Every irreducible finite-dimensional $U_q\widehat{\mathfrak{g}}$-module is generically irreducible as a $U_q\mathfrak{k}$-module.

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\(^2\text{Note that this condition is never satisfied by the twisting operators considered in [BK19].}\)
More generally, we prove that the module $V(\langle z \rangle)$ remains irreducible under restriction to certain subalgebras in $U_q\hat{\mathfrak{g}}$ called modified nilpotent subalgebras (see Theorem 6.1.1). This result may be of independent interest and generalizes results for the standard Borel and nilpotent subalgebras obtained in [CG05, Bow07, HJ12].

1.7. Our results also drastically simplify the approach to the reflection equation developed in [DG02, DM03], where a boundary analogue of Jimbo’s construction of trigonometric R-matrices [Jim86] was proposed. Their approach first assumes the existence of a twisted QSP intertwiner on a given finite-dimensional irreducible $U_q\hat{\mathfrak{g}}$-module. Then, by further assuming that the tensor product is generically irreducible as a $U_q\mathfrak{t}$-module, they proved that the reflection equation holds.

On the other hand, the existence of a twisted QSP intertwiner is guaranteed by Theorem 2 on any irreducible module, its essential uniqueness follows from Theorem 3, and the reflection equation follows independently of the irreducibility of the tensor product.

1.8. Under the additional assumption $V^\psi^2 \simeq V$, we further prove that the trigonometric K-matrix is unitary, i.e., satisfies $K_V(z)^{-1} = K_V(z^{-1})$. The condition $V^\psi^2 \simeq V$ is not restrictive, since the twisting operator $\psi$ can easily be chosen to be an involution (possibly up to a shift). Through unitarity, we establish a direct relation between the poles of $K_V(z)$ and the irreducibility of $V$ under restriction to $U_q\mathfrak{t}$ (see Propositions 5.4.1 and 5.5.1). We further comment on this point in Section 7.9, where we describe the trigonometric K-matrices arising from the q-Onsager algebra on the vector representation of $U_q\hat{\mathfrak{sl}}_2$. Interestingly, the number of poles in this case may vary from two to zero, depending on the chosen embedding of the q-Onsager algebra in $U_q\hat{\mathfrak{sl}}_2$.

1.9. Finally, we prove that our approach produces solutions of the standard and transposed reflection equations on certain distinguished classes of irreducible $U_q\hat{\mathfrak{g}}$-modules. We highlight two special cases: small modules and Kirillov-Reshetikhin modules. We consider only QSP subalgebras whose associated diagrammatic involution $\tau$ is either the identity or the affine extension $\eta_0$ of the opposition involution of the underlying subdiagram of finite type (see Section 7.5). We then prove that there are canonical twisting operators such that the corresponding trigonometric K-matrices are solutions of either the standard reflection equation if $\tau = \eta_0$ or the transposed reflection equation if $\tau = \text{id}$ (see Theorem 7.7.1 for the case of small modules and Theorem 7.8.1 for the case of Kirillov-Reshetikhin modules).

This result produces a large class of new solutions of the standard and the transposed reflection equations. Moreover, it shows that many solutions known in the literature arise from the action of universal K-matrices. In particular, we recover the solutions constructed in [RV16] for the vector representation in types A, B, C and D for QSP subalgebras with $\tau(0) = 0$ (except for type $D_n$ with $n$ even and $\tau \neq \text{id}$) and those recently constructed in [KOW22] for Kirillov-Reshetikhin modules in type A for quasi-split QSP subalgebras with fixed parameters (except for type $A_4$). We expect that the approach used in Section 7 naturally extends beyond the case $\tau \in \{\text{id}, \eta_0\}$.

1.10. Outline. In Section 2, we review the construction of trigonometric R-matrices on irreducible finite-dimensional representations of quantum affine algebras. In Section 3, we outline the construction of universal K-matrices for quantum Kac-Moody
algebras arising from QSP subalgebras obtained in [AV22]. In Section 4, we prove that universal K-matrices for quantum affine algebras give rise to formal spectral operators on finite-dimensional representations. In Section 5, we prove the existence of trigonometric K-matrices on irreducible representations. The latter result relies on generic irreducibility under restriction to a QSP subalgebra, which we prove in Section 6. Finally, we apply our construction to the case of small modules and Kirillov-Reshetikhin modules in Section 7.

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2. Quantum affine algebras and their R-matrices

In this section we recall the definition of (untwisted) quantum affine algebras and basic results on their irreducible finite-dimensional modules. Given a lattice $\Lambda$, we shall denote its non-negative part by $\Lambda_+$.

2.1. Affine Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra defined over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $I := \{1, 2, \ldots, \text{rank}(\mathfrak{g})\}$ be the set of vertices of the corresponding Dynkin diagram, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix, $(\cdot, \cdot)$ the normalized invariant bilinear form on $\mathfrak{g}$, $\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ a basis of simple roots and $\Pi^\vee := \{h_i \mid i \in I\} \subset \mathfrak{h}$ a basis of simple coroots such that $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$. Let $Q := Z\Pi \subset \mathfrak{h}^*$ and $Q^\vee := Z\Pi^\vee \subset \mathfrak{h}$ be the root and coroot lattice, respectively. Let $\Delta_+ \subset Q_+$ be the set of positive roots, $\vartheta = \sum_{i \in I} a_i \alpha_i$ the highest root, and $P := \{\lambda \in \mathfrak{h}^*_+ \mid \lambda(Q^\vee) \subset \mathbb{Z}\}$ the weight lattice.

Let $\hat{\mathfrak{g}}$ be the (untwisted) affine Lie algebra associated to $\mathfrak{g}$ with affine Cartan subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$ [Kac90, Ch. 7]. Set $\hat{\mathfrak{g}} := \tilde{\mathfrak{g}}'$ and $\hat{\mathfrak{h}} := \hat{\mathfrak{g}} \cap \hat{\mathfrak{h}}$. Let $\hat{I} := \{0\} \cup I$ be the set of vertices of the affine Dynkin diagram and $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$ the extended Cartan matrix [Kac90, Table Aff. 1]. We denote by $\hat{Q}^\vee \subset \hat{\mathfrak{h}}$ and $\hat{Q} \subset \hat{\mathfrak{h}}^*$ the affine coroot and root lattices, respectively. Let $\hat{\delta} \in \hat{Q}_+$ and $\hat{c} \in \hat{Q}_+$ be the unique elements such that

$$\{\lambda \in \hat{Q} \mid \forall i \in \hat{I}, \lambda(h_i) = 0\} = \mathbb{Z}\hat{\delta} \quad \text{and} \quad \{h \in \hat{Q}^\vee \mid \forall i \in \hat{I}, \alpha_i(h) = 0\} = \mathbb{Z}\hat{c}$$

In particular, $\delta = \alpha_0 + \vartheta$, $c$ is central in $\hat{\mathfrak{g}}$, and, under the identification $\nu : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$ induced by the bilinear form, one has $\nu(c) = \delta$. The sets of real and imaginary affine positive roots in $\hat{Q}_+$ are described by

$$\hat{\Delta}_+^{\text{re}} = \Delta_+ + \mathbb{Z}_{\geq 0}\hat{\delta} \quad \text{and} \quad \hat{\Delta}_+^{\text{im}} = \mathbb{Z}_{>0}\hat{\delta}.$$

We fix $d \in \hat{\mathfrak{h}}$ such that $\alpha_i(d) = \delta_{i0}$ for any $i \in \hat{I}$. Note that $d$ is defined up to a summand proportional to $c$ and we obtain a natural identification $\hat{\mathfrak{h}} = \hat{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$. In terms of the extended coroot lattice $\hat{Q}^\vee := \hat{Q}^\vee \oplus \mathbb{Z}d \subset \hat{\mathfrak{h}}$ we set

$$\hat{P} := \{\lambda \in \hat{\mathfrak{h}}^*_+ \mid \lambda(\hat{Q}^\vee) \subset \mathbb{Z}\}.$$
Then, the quotient lattice $\hat{\mathcal{P}}/\mathcal{Z} \cong \text{hom}_\mathbb{Z}(\hat{Q}^\vee, \mathbb{Z})$ has a basis given by the images of the fundamental weights in $\hat{\mathcal{P}}$.

2.2. Drinfeld-Jimbo presentation of the quantum affine algebra. Let $q$ be an indeterminate\textsuperscript{3} and set $\mathbb{F} := \mathbb{C}(q)$. Fix non-negative integers $\{c_i \mid i \in \hat{I}\}$ such that the matrix $(c_i a_{ij})_{i,j \in \hat{I}}$ is symmetric and set $q_i := q^{c_i}$. The quantum Kac-Moody algebra associated to $\hat{\mathfrak{g}}$ is the unital associative algebra $U_q \hat{\mathfrak{g}}$ defined over $\mathbb{F}$ with generators $E_i$ and $F_i$ ($i \in \hat{I}$), and $K_h$ ($h \in \hat{Q}^\vee$) subject to the following defining relations:

$$K_h K_{h'} = K_{h+h'}, \quad K_0 = 1,$$

$$K_h E_i = q^{\alpha_i(h)} E_i K_h, \quad K_h F_i = q^{-\alpha_i(h)} F_i K_h,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\text{Serre}_{ij}(E_i, E_j) = 0 = \text{Serre}_{ij}(F_i, F_j) \quad (i \neq j)$$

for any $i, j \in \hat{I}$ and $h, h' \in \hat{Q}^\vee$, where $K_i^{\pm 1} := K_{\pm x_i h}$, and the last line amounts to the usual q-deformed Serre relations. We denote by $U_q \mathfrak{g}$ the subalgebra generated by $E_i, F_i$, $i \in I$, and $K_h$, $h \in \hat{Q}^\vee$, a quantum Kac-Moody algebra of finite type.

We consider the Hopf algebra structure determined by the coproduct formulae

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_h) = K_h \otimes K_h,$$

for any $i \in \hat{I}$ and $h \in \hat{Q}^\vee$. Let $\omega : U_q \hat{\mathfrak{g}} \to (U_q \hat{\mathfrak{g}})^{op}$ be the Chevalley involution, i.e., the isomorphism of Hopf algebras defined by

$$\omega(K_h) = K_{-h}, \quad \omega(E_i) = -F_i, \quad \omega(F_i) = -E_i,$$

for any $i \in \hat{I}$ and $h \in \hat{Q}^\vee$.

We denote by $U_q \hat{\mathfrak{h}}, U_q \hat{\mathfrak{n}}^+, \text{ and } U_q \hat{\mathfrak{n}}^-$ the subalgebras generated, respectively, by the elements $\{K_i^{\pm 1} \mid i \in \hat{I}\}, \{E_i \mid i \in \hat{I}\}$ and $\{F_i \mid i \in \hat{I}\}$, respectively, and by $U_q \mathfrak{h}, U_q \mathfrak{n}^\pm \subset U_q \hat{\mathfrak{g}}$ their finite-type analogues, with $\hat{I}$ replaced by $I$. We set $U_q \hat{\mathfrak{h}}^\pm := U_q \hat{\mathfrak{h}}^\pm U_q \hat{\mathfrak{n}}$, where $U_q \hat{\mathfrak{h}}$ is the commutative subalgebra generated by $K_h$, $h \in \hat{Q}^\vee$.

The quantum affine algebra is the subalgebra $U_q \hat{\mathfrak{g}} := U_q \hat{\mathfrak{n}}^+ U_q \hat{\mathfrak{h}} U_q \hat{\mathfrak{n}}^-$, while the quantum loop algebra $U_q L \mathfrak{g}$ is the quotient of $U_q \hat{\mathfrak{g}}$ by the ideal generated by $K_c - 1$. Note that the Hopf algebra structure and the Chevalley involution descend to $U_q \hat{\mathfrak{g}}$ and $U_q L \mathfrak{g}$.

2.3. Category $\mathcal{O}$ and finite-dimensional modules. It is well-known that the categories of finite-dimensional modules and integrable category $\mathcal{O}$ modules coincide for finite-type quantum groups, while they are strikingly different in the affine type.

Recall that $V \in \text{Mod}(U_q \hat{\mathfrak{g}})$ is

(1) a (type 1) weight module if $V = \bigoplus_{\mu \in \hat{\mathcal{P}}} V_\mu$, where

$$V_\mu := \{v \in V \mid \forall h \in \hat{Q}^\vee, K_h \cdot v = q^{\ell(h)} v\};$$

\textsuperscript{3}The results of this paper are also valid in the case $\mathbb{F} = \mathbb{C}$ and $q \in \mathbb{C}^\times$ not a root of unity.
(2) an integrable module if it is a weight module and the action of the elements 
\{E_i, F_i \mid i \in \hat{I}\} is locally nilpotent;

(3) a category $\mathcal{O}$ module if it is a weight module and the action of $U_q\hat{\mathfrak{g}}^+$ is locally finite.

It is well-known that the full subcategory of integrable category $\mathcal{O}$ modules is semisimple, its nontrivial irreducible modules are infinite-dimensional and classified on the tower of algebras

\[ \text{completions}. \]

2.4. Completions. In the following, we consider suitable completions of the algebras $U_q\hat{\mathfrak{g}}$ and $U_qL\mathfrak{g}$, which contain certain distinguished operators acting on integrable category $\mathcal{O}$ modules and on $\text{Mod}_{\text{id}}(U_qL\mathfrak{g})$, respectively.

Let $\text{Mod}_{\text{id}}(U_qL\mathfrak{g})$ denote the category of finite-dimensional (type 1) weight modules over the quantum affine algebra $U_q\hat{\mathfrak{g}}$. The notation is justified since the central element $K_e$ acts as 1 on any finite-dimensional $U_q\hat{\mathfrak{g}}$-module $V$. Note also that any $V \in \text{Mod}_{\text{id}}(U_qL\mathfrak{g})$ admits a weight decomposition over the quotient lattice $\hat{\mathfrak{g}}/\mathfrak{z}\mathfrak{g}$. The category $\text{Mod}_{\text{id}}(U_qL\mathfrak{g})$ is not semisimple and its irreducible representations are classified by $\text{rank}(\mathfrak{g})$-tuples of monic polynomials over $\mathbb{F}$, see, e.g., [CP95, Thm. 12.2.6]. Moreover, $\text{Mod}_{\text{id}}(U_qL\mathfrak{g})$ is monoidal, but it is not braided.

2.5. The universal $R$-matrix. The Hopf algebra $U_q\hat{\mathfrak{g}}$ is quasitriangular, i.e., it admits a universal $R$-matrix $R \in (U_q\hat{\mathfrak{g}}^\otimes 2)^{\text{Id}}$, satisfying the intertwining equation

\[ R \Delta(x) = \Delta^{\text{op}}(x) R \]

for any $x \in U_q\hat{\mathfrak{g}}$ and the coproduct identities

\[ \Delta \otimes \text{id}(R) = R_{13} R_{23} \quad \text{and} \quad \text{id} \otimes \Delta(R) = R_{13} R_{12}. \]

In particular, it follows that $R$ satisfies the Yang-Baxter equation

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \]

As a consequence, the tensor category $\mathcal{O}$ is braided.

The element $R$ arises from the Drinfeld double construction of $U_q\hat{\mathfrak{g}}$ as the canonical tensor of a Hopf pairing between $U_q\hat{\mathfrak{g}}^-$ and $U_q\hat{\mathfrak{g}}^+$ [Dri86, Lus94]. More specifically, let $\{u_i\}, \{u^i\} \subset \mathfrak{h}$ be dual bases and set

\[ \Omega_0 := \sum_i u_i \otimes u^i, \quad \hat{\Omega}_0 := m(c \otimes d + d \otimes c) + \Omega_0, \]
where \( m = 1, 2, 3 \) if \( \mathfrak{g} \) is of type ADE, BCF, or \( \mathcal{G} \), respectively. Then the R-matrix of \( U_q\mathfrak{g} \) lies in the completion of \( U_q\mathfrak{b}^- \otimes U_q\mathfrak{b}^+ \) with respect to the \( \mathbb{Q}^* \)-grading and has the form

\[
R = q^{\hat{\Theta}_0} \cdot \sum_{\mu > 0} \Xi_\mu,
\]

Here \( \Xi_\mu \in U_q\mathfrak{h}^- \otimes U_q\mathfrak{h}^+ \) is the \( \mu \)-component of the canonical tensor and the factor \( q^{\hat{\Theta}_0} \) acts on tensor products of weight vectors as \( q^{\hat{\Theta}_0} \cdot v \otimes w = q^{(\omega(v),\omega(w))} \cdot v \otimes w \). Finally, note that \( \omega \otimes \omega(R) = R_{21} \). Hence the Chevalley involution is an isomorphism of the quasitriangular Hopf algebras \( U_q\mathfrak{g} \) and \( U_q\mathfrak{g}^{\text{cop}} \).

2.6. The spectral R-matrix. The universal R-matrix of \( U_q\mathfrak{g} \) does not immediately act on finite-dimensional \( U_q\mathfrak{L}_\mathfrak{g} \)-modules. The first obstacle is given by the operator \( q^{\alpha(b(x+y)-a(x+y))} \). However, this is easily solved by observing that, since the central element \( K_c \) acts by 1 if and only if \( c \) acts by zero, that factor can be ignored. The second obstacle is given by the fact that the operator \( \Xi := \sum_{\mu > 0} \Xi_\mu \) is not necessarily defined on finite-dimensional representations over \( U_q\mathfrak{L}_\mathfrak{g} \). To this end, set

\[
U_q\mathfrak{g}[z,z^{-1}] := U_q\mathfrak{g} \otimes \mathbb{F}[z,z^{-1}]
\]

and consider the homogeneous grading shift automorphism

\[
\Sigma_z : U_q\mathfrak{g} \to U_q\mathfrak{g}[z,z^{-1}]
\]

given by \( \Sigma_z(K_c) := K_c, \Sigma_z(E_i) := z^{\delta_{i0}}E_i, \) and \( \Sigma_z(F_i) := z^{-\delta_{i0}}F_i \). Note that, by specializing \( z \) in \( \mathbb{F} \), we obtain a one-parameter family of automorphism of \( U_q\mathfrak{g} \). Then, let

\[
\Delta_z, \Delta_z^{\text{op}} : U_q\mathfrak{g} \to (U_q\mathfrak{g} \otimes U_q\mathfrak{g})[z,z^{-1}]
\]

be the shifted coproducts defined by

\[
\Delta_z(x) := \text{id} \otimes \Sigma_z(\Delta(x)), \quad \Delta_z^{\text{op}}(x) := \text{id} \otimes \Sigma_z(\Delta^{\text{op}}(x)).
\]

The grading shift is clearly well-defined on \( U_q\mathfrak{L}_\mathfrak{g} \). For any \( V \in \text{Mod}_{\mathfrak{L}}(U_q\mathfrak{L}_\mathfrak{g}) \) with action \( \pi_V : U_q\mathfrak{L}_\mathfrak{g} \to \text{End}(V) \) acts by \( \text{id} \otimes \Sigma_z(\Delta^{\text{op}}(x)) \). The grading shift is clearly well-defined on \( U_q\mathfrak{L}_\mathfrak{g} \). For any \( V \in \text{Mod}_{\mathfrak{L}}(U_q\mathfrak{L}_\mathfrak{g}) \) with action \( \pi_V : U_q\mathfrak{L}_\mathfrak{g} \to \text{End}(V) \) acts by \( \text{id} \otimes \Sigma_z(\Delta^{\text{op}}(x)) \). By considering the projection of the formal series \( \text{id} \otimes \Sigma_z(R) \in (U_q\mathfrak{g} \otimes \mathbb{F}[z]) \) on the quantum loop algebra, one obtains the following theorem (see \cite{Dri86, FR92, Her19}).

**Theorem 2.6.1.**

1. The quantum loop algebra \( U_q\mathfrak{L}_\mathfrak{g} \) has a universal spectral R-matrix, i.e., a distinguished element \( R(z) \in (U_q\mathfrak{L}_\mathfrak{g} \otimes \mathbb{F}[z]) \) such that \( \Sigma_a \otimes \Sigma_b(R(z)) = R(z) \delta_{ab} \) and the following identities are satisfied:

\[
R(z)\Delta_z(x) = \Delta_z^{\text{op}}(x)R(z) \quad \text{for all } x \in U_q\mathfrak{L}_\mathfrak{g},
\]

\[
\Delta_z \otimes \text{id}(R(zw)) = R(zw)_{13}R(w)_{12}, \quad \text{id} \otimes \Delta_w(R(z)) = R(z)_{13}R(zw)_{12}.
\]

In particular, the spectral Yang-Baxter equation holds:

\[
R(z)_{12}R(zw)_{13}R(w)_{23} = R(w)_{23}R(zw)_{13}R(z)_{12}. \tag{2.3}
\]
For any \( V, W \in \text{Mod}_{\mathfrak{d}}(U_q \mathfrak{g}) \), the operator
\[
R_{VW}(z) := \pi_V \otimes \pi_W(R(z)) \in \text{End}(V \otimes W)[[z]]
\]
is well-defined and yields an intertwiner
\[
\tilde{R}_{VW}(z) := (12) \circ R_{VW}(z) : V \otimes W((z)) \to W((z)) \otimes V.
\]

**Remark 2.6.2.** The specialization of \( z \) in \( \mathbb{F}^\times \) yields the notion of a shifted finite-dimensional \( U_q \mathfrak{g} \)-module. Namely, for any \( a \in \mathbb{F}^\times \) and \( V \in \text{Mod}_{\mathfrak{d}}(U_q \mathfrak{g}) \) with action map \( \pi_V : U_q \mathfrak{g} \to \text{End}(V) \), the shifted \( U_q \mathfrak{g} \)-module \( V(a) \) is the vector space \( V \) equipped with the action \( \pi_V \circ \Sigma_a \).

### 2.7. Trigonometric R-matrices.

In the case of irreducible modules the operator \( R_{VW}(z) \) has the following rationality property (see, e.g., [Dri86, FR92, KS95] and cf. [Jim86]).

**Theorem 2.7.1.** Let \( V, W \in \text{Mod}_{\mathfrak{d}}(U_q \mathfrak{g}) \) be two irreducible representations. There exists a canonical scalar-valued formal Laurent series \( f_{VW}(z) \in \mathbb{F}(z) \) such that
\[
R_{VW}(z) := f_{VW}(z)^{-1}R(z) \in \text{End}(V \otimes W)((z))
\]
is rational, satisfies the spectral Yang-Baxter equation (2.3) and satisfies the unitarity relation
\[
R_{VW}(z)^{-1} = (12) \circ R_{WV}(z^{-1}) \circ (12).
\]
In particular, \( \tilde{R}_{VW}(z) := (12) \circ R_{VW}(z) \) is an intertwiner \( V \otimes W(z) \to W(z) \otimes V \).

The proof of the theorem relies on the *generic irreducibility* of the tensor product \( V \otimes W \), i.e., on the irreducibility of \( V \otimes W((z)) \) over \( U_q \mathfrak{g}((z)) \) (see [KS95, Sec. 4.2] or [Cha02, Thm. 3]). Note that the function \( f_{VW}(z) \) is uniquely determined by the condition \( R(z)(v_0 \otimes w_0) = v_0 \otimes w_0 \), where \( v_0 \in V \) and \( w_0 \in W \) are \( \ell \)-highest weight vectors\(^4\).

**Remarks 2.7.2.**

1. If \( q \in \mathbb{C}^\times \) is not a root of unity, \( R_{VW}(z) \) is a meromorphic operator for any finite-dimensional modules \( V \) and \( W \) [EM02, FR92, KS95]. Relying on the crossing symmetry, i.e., the functional relation between \( R_{VW}(z) \) and \( R_{WV}(z) \), one proves that the operator \( R(z) \) is analytic near zero and therefore meromorphic on \( \mathbb{C} \).

2. A finite-dimensional irreducible \( U_q \mathfrak{g} \)-module \( V \) is called *real* if \( V \otimes V \) is still irreducible. Thus, in this case, one has \( \tilde{R}_{VW}(1) = \text{id}_{V \otimes V} \) (see also [FHR21, Lemma 10.2 and the following discussion]). \( \checkmark \)

**Example 2.7.3.** Let \( V_1 = \mathbb{F}^2 \) be the fundamental representation of \( U_q \mathfrak{sl}(2) \). For any \( a \in \mathbb{F}^\times \), we consider the evaluation representation \( V_1(a) = \mathbb{F}^2 \) with action of \( U_q \mathfrak{l}(2) \) given by
\[
\pi(K_0) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} = \pi(K_1)^{-1},
\]

\(^4\)By [CP95, Cor. 12.2.5], every finite-dimensional irreducible module \( V \) is generated by an \( \ell \)-highest weight vector, i.e., a weight vector \( v_0 \), which is annihilated by \( F_0 \) and \( E_i \) (\( i \in I \)) and is a simultaneous eigenvector for all imaginary root vectors.
\[
\pi(E_0) = \begin{pmatrix} 0 & 0 \\ q^{-1}a & 0 \end{pmatrix} = q^{-1}a \pi(F_1) \quad \text{and} \quad \pi(F_0) = \begin{pmatrix} 0 & qa^{-1} \\ 0 & 0 \end{pmatrix} = qa^{-1} \pi(E_1).
\]

In the case of \( V_1(a) \otimes V_1(b) \), the rational function \( R_{ab}(z) := R_{V_1(a) V_1(b)}(z) \) is easily computed (see, e.g., [CP95, 12.5.7], [Jim86]). Set \( \lambda := b/a \). Then

\[
R_{ab}(z) := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{q^{1-\lambda_2}}{q^{1-\lambda_2}} & \frac{\lambda z (q^2 - 1)}{q^{1-\lambda_2}} & 0 \\
0 & \frac{\lambda z (q^2 - 1)}{q^{1-\lambda_2}} & \frac{q^{1-\lambda_2}}{q^{1-\lambda_2}} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Note that, if \( \lambda = q^2 \), \( R_{ab}(z) \) has a pole at \( z = 1 \), while, if \( \lambda = q^{-2} \), \( R_{ab}(z) \) is not invertible at \( z = 1 \). It is well-known that \( V_1(a) \otimes V_1(b) \) fails to be irreducible precisely when \( \lambda = q^{\pm 2} \). Finally, note that \( \tilde{R}_{ab}(1) \) is the identity. \( \nabla \)

3. Quantum affine symmetric pairs

3.1. Generalized Satake diagrams. Classical and quantum Kac-Moody algebras are defined in terms of combinatorial datum encoded by the Dynkin diagram and the Cartan matrix. Similarly, classical and quantum symmetric pairs [Let02, Kol14] arise from a refinement of such datum.

Let \( \text{Aut}(\hat{A}) \) be the group of diagram automorphisms of the affine Cartan matrix, i.e., the group of bijections \( \tau : \hat{I} \rightarrow \hat{I} \) such that \( a_{ij} = a_{\tau(i)\tau(j)} \). Let \( X \subset \hat{I} \) be a proper subset of indices. Thus, the corresponding Cartan matrix \( A_X \) is necessarily of finite type. We denote by \( \mathfrak{g}_X \subset \mathfrak{g} \) the corresponding Lie subalgebra and by \( \text{oi}_X \in \text{Aut}(A_X) \) the opposition involution of \( X \), i.e., the involutive diagram automorphism of \( X \) induced by the action of the longest element \( w_X \) of the Weyl group \( W_X \) on \( Q_X \).

**Definition 3.1.1** ([RV20, RV22]). A generalized (affine) Satake diagram is a pair \((X, \tau)\) where \( X \subset \hat{I} \) and \( \tau \) is an involutive diagram automorphism stabilizing \( X \) such that

1. \( \tau |_X = \text{oi}_X \);
2. for any \( i \in \hat{I} \setminus X \) such that \( \tau(i) = i \), the connected component of \( X \cup \{i\} \) containing \( i \) is not of type \( A_2 \).

The set of all such diagrams is denoted by \( \text{GSat}(\hat{A}) \). \( \nabla \)

A classification of generalized Satake diagrams of affine type is provided in [RV22, App. A, Tables 5, 6 and 7]. Henceforth, we fix \((X, \tau) \in \text{GSat}(\hat{A})\).

**Example 3.1.2.** Consider the affine Lie algebra \( \hat{sl}_2 \) and set \( \hat{I} = \{0,1\} \). There are four generalized Satake diagrams given by

\( (\emptyset, \text{id}) \), \( (\emptyset, (01)) \), \( (\{0\}, \text{id}) \), \( (\{1\}, \text{id}) \)

where \( (01) \) denotes the nontrivial diagram automorphism. \( \nabla \)
3.2. **Pseudo-involutions.** The diagram automorphism \( \tau \in \text{Aut}(\hat{A}) \) extends canonically to an automorphism of \( \hat{g} \), given on the generators by \( \tau(h_i) = h_{r(i)} \), \( \tau(e_i) = e_{r(i)} \), and \( \tau(f_i) = f_{r(i)} \). The pair \((X, \tau)\) is then associated to a the Lie algebra automorphism \( \theta: \hat{g} \to \hat{g} \), given by

\[
\theta := \text{Ad}(w_X) \circ \omega \circ \tau
\]

where \( \omega \) denotes the Chevalley involution on \( \hat{g} \). Note that \( \tau(c) = c \) and \( \theta(c) = -c \). Hence, \( \theta \) descends to an automorphism of \( Lg \).

**Remarks 3.2.1.**

1. In [KW92, Sec. 4.9], Kac and Wang defined a canonical procedure (for arbitrary generalized Cartan matrices) to extend a diagram automorphism from \( \hat{h} \) to \( \hat{h} \). Thus, \( \theta \) extends to an automorphism of \( \hat{g} \), which is of the second kind (see [KW92, 4.6]) and is an involution on \( \hat{h} \). Following [RV22], we shall refer to \( \theta \) as a pseudo-involution of \( \hat{g} \) of the second kind. Note that \( \hat{h}^\theta \subseteq \hat{h} \), see [AV22, Sec. 6.2].

2. Since \((X, \tau)\) and therefore \( \theta \) are fixed, in the following we will use the subscript \( \theta \) even in the case of objects explicitly defined in terms of \((X, \tau)\). Note that the datum \((X, \tau)\) can be recovered from \( \theta \) since \( X = \{i \in \hat{I} | \theta(h_i) = h_i\} \) and \( \tau = \omega \circ \text{Ad}(w_X)^{-1} \circ \theta \).

3. The map on \( \hat{h}^\ast \) dual to \( \theta \) is denoted by the same symbol and preserves \( \hat{Q} \).

Because \( \theta(c) = -c \), we have \( \theta(\delta) = -\delta \) and hence \( Z\delta \subseteq \hat{Q}^{-\delta} \). Moreover, \( \hat{Q}_+ = Q_+ \oplus \mathbb{Z}_{\geq 0} \delta \) and thus \( \hat{Q}_+^{-\delta} = Q_+^{-\delta} \oplus \mathbb{Z}_{\geq 0} \delta \). The restricted rank of \( \theta \) is the rank of \( \hat{Q}_+^{-\delta} \), given by the number of \( \tau \)-orbits in \( \hat{I} \setminus X \) (see e.g. [RV22, Sec. 4] and [AV22, Sec. 8.10]). In particular, \( \theta \) has restricted rank one if and only if \( \hat{Q}_+^{-\delta} = Z\delta \).

3.3. **Quantum pseudo-involutions.** We shall consider a distinguished lift of the pseudo-involution \( \theta \) to an algebra automorphism \( \theta_q \) of \( U_q\hat{g} \) and \( U_qLg \). This is obtained by choosing a suitable lift for each of the three factors in \( \theta \). First, we consider the standard Chevalley involution on \( U_q\hat{g} \) given by (2.1). The diagram automorphism \( \tau \) extends canonically to an automorphism of \( U_q\hat{g} \) given on the generators by \( \tau(E_i) = E_{r(i)} \), \( \tau(F_i) = F_{r(i)} \), and \( \tau(K_i) = K_{r(i)} \).

The action of the Weyl group operator \( w_X \in W_X \) is lifted to \( U_q\hat{g} \) as follows. Let \( S_X \) be the braid group operator on integrable category \( \mathcal{O} \) modules corresponding to \( w_X \), cf. [Lus94, Sec. 5] (see also [AV22, Sec. 5]). More precisely, given a reduced expression \( s_{i_1} \cdots s_{i_t} \) of \( w_X \) in terms of fundamental reflections, one sets \( S_X := S_{i_1} \cdots S_{i_t} \), where \( S_j = T_{j,1}^\nu \) in the notation from [Lus94, 5.2.1]. It follows from the braid relations that \( S_X \) is independent of the chosen reduced expression. We shall consider a Cartan correction of \( S_X \) given by

\[
S_\theta := \xi_\theta \cdot S_X,
\]

where \( \xi_\theta \) is the Cartan operator defined on any weight vector of weight \( \lambda \) as the multiplication by \( q^{\theta(\lambda,\lambda)/2 + (\lambda,\rho_X)} \) and \( \rho_X \) is the half-sum of the positive roots in \( \Delta_X \) (cf. [AV22, Sec. 4.9]). By [AV22, Lemma 4.3 (iii)], \( T_\theta := \text{Ad} S_\theta \) yields an algebra automorphism of \( U_q\hat{g} \).
The quantum pseudo-involution is the automorphism \( \theta_q : U_q \mathfrak{g} \to U_q \mathfrak{g} \) given by
\[
\theta_q := T_\theta \circ \omega \circ \tau.
\]
Note that, as in the classical case, \( \theta_q \) is independent of the order of the three factors. We have the basic properties
\[
\theta_q|_{U_q \mathfrak{x}} = \text{id}_{U_q \mathfrak{x}}, \quad \theta_q(K_h) = K_{\theta(h)}, \quad \theta_q((U_q \mathfrak{g})\lambda) = (U_q \mathfrak{g})\theta(\lambda),
\]
for any \( h \in \mathfrak{q}^\ast \) and \( \lambda \in \mathfrak{q} \), see, e.g., [AV22, Lem. 6.10]. Note that \( \theta_q \) descends to an automorphism of the quantum loop algebra \( U_q L\mathfrak{g} \).

**Remark 3.3.1.** Further to Remark 3.2.1, note that the Kac-Wang extension of \( \tau \) to \( \hat{\mathfrak{h}} \) does not necessarily preserve the extended coroot lattice \( \hat{\mathfrak{q}}^\ast \subset \hat{\mathfrak{h}} \), and therefore does not automatically extend to an automorphism of \( U_q \hat{\mathfrak{g}} \). To remedy this, one can modify the lattice itself by replacing the standard derivation \( d \in \hat{\mathfrak{h}} \) with any \( d' \in \hat{\mathfrak{h}} \) such that \( \alpha_i(d') = \alpha_{\tau(i)}(d) \), see [Kol14, Sec. 2.6]. However, for the purposes of this paper, it is sufficient to regard \( \theta_q \) as an automorphism of \( U_q \mathfrak{g} \).

3.4. QSP subalgebras. By [AV22, Sec. 6.2], \( \hat{\mathfrak{b}}^\theta \subset \hat{\mathfrak{h}} \) and we get \( (U_q \hat{\mathfrak{b}})^{\theta_\mathfrak{n}} = U_q \hat{\mathfrak{b}}^\theta \subset U_q \mathfrak{g} \). Associated to the pseudo-involution \( \theta \) there is a family of coideal subalgebras \( U_q \hat{\mathfrak{k}} \subset U_q \hat{\mathfrak{g}} \), see [AV22, Def. 6.11], which is parametrized by two sets \( \Gamma \subset (\mathbb{R}^\ast)^I \) and \( \Sigma \subset \mathbb{F}^I \) which we define below. The subalgebras thus defined coincide with the subalgebras considered in [Let02, Kol14] up to reparametrization, see [AV22, Rmk. 6.13].

**Definition 3.4.1.** The QSP subalgebra of \( U_q \mathfrak{g} \) (corresponding to \( \theta \)) with parameters \( (\gamma, \sigma) \in \Gamma \times \Sigma \) is the subalgebra \( U_q \hat{\mathfrak{k}} \) generated by \( U_q \hat{\mathfrak{k}} \), \( (U_q \hat{\mathfrak{b}})^{\theta_\mathfrak{n}} \), and \( B_i (i \in \hat{I}) \), where the elements \( B_i \in U_q \mathfrak{g} \) are given by
\[
B_i := \begin{cases} 
F_i & \text{if } i \in \hat{X}, \\
F_i + \gamma_i \theta_q(F_i) + \sigma_i K_i^{-1} & \text{if } i \notin \hat{X}.
\end{cases}
\]

The parameter sets \( \Gamma \) and \( \Sigma \) are defined as follows, see [Kol14, Eqns. (5.9) and (5.11)]. First, we fix a subset \( \hat{X} \subset \hat{I} \setminus X \) containing a representative for every \( \tau \)-orbit in \( \hat{I} \setminus X \). We set
\[
\hat{I}_{\text{diff}} := \{ i \in \hat{I} \mid \tau(i) \neq i \text{ and } \exists j \in X \cup \{ \tau(i) \} \text{ such that } a_{ij} \neq 0 \}, \\
\hat{I}_{\text{ms}} := \{ i \in \hat{I} \mid \tau(i) = i \text{ and } a_{ij} = 0 \text{ for all } j \in X \}.
\]

Then, \( \Gamma \) is the set of tuples \( \gamma \in (\mathbb{R}^\ast)^I \) such that \( \gamma_i = 1 \) if \( i \in X \) and \( \gamma_i = \gamma_{\tau(i)} \) if \( \{i, \tau(i)\} \cap \hat{I}_{\text{diff}} = \emptyset \), while \( \Sigma \) is the set of tuples \( \sigma \in \mathbb{F}^I \) such that \( \sigma_i = 0 \) if \( i \in \hat{I} \setminus \hat{I}_{\text{ms}} \) and, for all \( (i, j) \in \hat{I}_{\text{ms}}^2 \), \( a_{ij} \in 2\mathbb{Z} \) or \( \sigma_j = 0 \). Note that \( \Gamma \) and \( \Sigma \) do not depend on the choice of \( \hat{I}^\ast \).

The constraints given by the parameter sets \( \Gamma \) and \( \Sigma \) are motivated by Proposition 3.4.2 below, proved in [Kol14, Props. 5.2 and 6.2] for Satake diagrams.

**Proposition 3.4.2.** The subalgebra \( U_q \hat{\mathfrak{k}} \) is a right coideal in \( U_q \mathfrak{g} \) and has minimal intersection with \( U_q \hat{\mathfrak{b}} \), i.e.,
\[
\Delta(U_q \mathfrak{k}) \subset U_q \mathfrak{k} \otimes U_q \mathfrak{g} \quad \text{and} \quad U_q \mathfrak{k} \cap U_q \hat{\mathfrak{b}} = (U_q \hat{\mathfrak{b}})^{\theta_\mathfrak{n}}.
\]
Remark 3.4.3. Following [AV22, Sec. 7.4], we shall regard the tuple $\gamma$ as a diagonal operator on weight representations. Namely, we fix a group homomorphism $\gamma : \hat{P} \to F^\times$ such that $\gamma(\alpha_i) := \gamma_i (i \in \hat{I})$. Then, $\gamma$ acts on any weight vector of weight $\lambda$ as multiplication by $\gamma(\lambda)$. \hfill \nabla

Example 3.4.4. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $(X, \tau) = (\emptyset, \text{id})$ (cf. Example 3.1.2). The corresponding QSP subalgebra is the q-Onsager algebra (see, e.g., [BK05] and references therein). In our conventions, given a choice of parameters $\gamma_0, \gamma_1 \in F^\times$ and $\sigma_0, \sigma_1 \in F$, this is the coideal subalgebra of $U_q \hat{\mathfrak{g}}_2$ generated by the elements

$$B_i := F_i - q^{-1} \gamma_i E_i K_i^{-1} + \sigma_i K_i^{-1}$$

with $i = 0, 1$. Note that the parameters $\sigma_i$ are allowed to be nonzero, since in this case $\hat{I}_{rs} = \{0, 1\}$. \hfill \nabla

3.5. The standard K-matrix. In [AV22, Thms. 8.11-8.12], we proved that the QSP subalgebra $U_q \mathfrak{k}$ gives rise to a discrete family of K-matrices in $U_q \hat{\mathfrak{g}}$, indexed by affine generalized Satake diagrams. The result crucially relies on the construction of the quasi-K-matrix, due to Bao and Wang [BW18] and generalized in [BK19, AV22].

We first recall the definition of the standard K-matrix.

Theorem 3.5.1. There exists a unique series $\Upsilon_\theta := \sum_{\mu \in \hat{Q}_-^+} \Upsilon_{\theta, \mu} \in U_q \hat{\mathfrak{g}}$ with $\Upsilon_{\theta, 0} = 1$ and the operator

$$K_\theta := \gamma^{-1} \cdot \Upsilon_\theta \in (U_q \hat{\mathfrak{g}})^{\text{can}}$$

satisfies the intertwining identity

$$K_\theta \cdot b = \theta_q^{-1}(b) \cdot K_\theta \quad (b \in U_q \mathfrak{k})$$

and the coproduct identity

$$\Delta(K_\theta) = R_{\theta}^{-1} \cdot K_\theta \cdot \Delta(\theta) \cdot K_\theta \otimes 1,$$

where $R$ is the R-matrix of $U_q \hat{\mathfrak{g}}$, $R_{\theta}^{-1} := (\theta_q^{-1} \otimes \text{id})(R)$, and $R_{\theta} := (S_\theta \otimes S_\theta)^{-1} \cdot \Delta(S_\theta)$. Moreover, the following (parameter-independent) generalized reflection equation holds:

$$R_{21}^{\theta-1} \cdot 1 \otimes K_\theta \cdot R_{21}^{\theta^{-1}} \cdot K_\theta \otimes 1 = K_\theta \otimes 1 \cdot (R_{21}^{\theta^{-1}})_{21} \cdot 1 \otimes K_\theta \cdot R.$$

Remarks 3.5.2.

(1) We shall refer to the operator $\Upsilon_\theta$ as the quasi-K-matrix of $U_q \mathfrak{k}$. This terminology is used in [BW18, BK19] for the operator $\hat{x}_\theta = \Upsilon_\theta$. In [AV22], we provide a more general construction of the operator $\Upsilon_\theta$, which does not rely on the existence of the QSP involution. In fact, $\Upsilon_\theta$ can be used to define the bar involution on $U_q \mathfrak{k}$, as later observed in [Kol22].

(2) Theorem 3.5.1 shows that, up to a small Cartan correction, the quasi-K-matrix $\Upsilon_\theta$ is already a universal solution of a generalized reflection equation. In [AV22], we exploit this observation to produce the abovementioned discrete family of K-matrices. In Section 3.6, we provide an even more general construction, which justifies the terminology standard K-matrix. \hfill \nabla
3.6. Gauging K-matrices. Solutions of other generalized reflection equations are easily obtained from the standard K-matrix by acting simultaneously on the K-matrix $K_\theta$, the operator $\theta^{-1}_q$ and the R-matrix $R_\theta$. Let $\mathcal{G}$ be the group of invertible elements $g \in (U_q\tilde{\mathfrak{g}})^{\text{int}}$ such that $\text{Ad}(g)$ preserves $U_q\tilde{\mathfrak{g}} \subseteq (U_q\tilde{\mathfrak{g}})^{\text{int}}$. We have the following

**Corollary 3.6.1.** For any $g \in \mathcal{G}$, set

\[ \psi := \text{Ad}(g) \circ \theta^{-1}_q, \quad R_\psi := g \otimes g \cdot R_\theta \cdot \Delta(g)^{-1}. \]

The element

\[ K_\psi := g \cdot K_\theta \]

in $(U_q\tilde{\mathfrak{g}})^{\text{int}}$ satisfies the intertwining identity

\[ K_\psi \cdot b = \psi(b) \cdot K_\psi \quad (b \in U_q\mathfrak{k}) \]

and the coproduct identity

\[ \Delta(K_\psi) = R_\psi^{-1} \cdot 1 \otimes K_\psi \cdot R_\psi \cdot K_\psi \otimes 1, \]

where $R$ is the R-matrix of $U_q\tilde{\mathfrak{g}}$ and $R_\psi := (\psi \otimes \text{id})(R)$. Moreover, the following (parameter-independent) generalized reflection equation holds:

\[ R^{\psi \psi}_{21} \cdot 1 \otimes K_\psi \cdot R_\psi \cdot K_\psi \otimes 1 = K_\psi \otimes 1 \cdot (R_\psi)_{21} \cdot 1 \otimes K_\psi \cdot R. \]

**Proof.** It is enough to observe that, since $K_\psi = g \cdot K_\theta$, the equations (3.3), (3.4), and (3.5) reduce to their analogues for the standard K-matrix $K_\theta$ in Theorem 3.5.1. $\square$

The evaluation of $K_\psi$ on $V \in \mathcal{O}^{\text{int}}$ yields a QSP intertwiner

\[ K_{\psi, V} : V \to V^{\psi}, \]

where $V^{\psi}$ denotes the pullback of $V$ through $\psi$. We shall refer to the automorphism $\psi$ as the twisting operator of the reflection equation.

The coproduct identity (3.4) and the generalized reflection equation (3.5) both admit a similar representation-theoretic interpretation. First note that, for any weight $U_q\tilde{\mathfrak{g}}$-module $M$ and $W \in \mathcal{O}$, the operator $R_{MW}$ is always well-defined, whilst this is not true for $R_{W,M}$. Recall that any algebra automorphism of $U_q\tilde{\mathfrak{g}}$ preserves $U_q\tilde{\mathfrak{g}}$ and thus its pullback preserves weight modules. Therefore, $V^{\psi}$ is still a weight module and $R_{V^{\psi}, W}$ is well-defined. Moreover, we have $R^{\psi \psi}_{21} = R_{\psi, 21} \cdot R \cdot R^{-1}_\psi$. Thus, the operator $R_{V^{\psi}, W}$ is also well-defined and (3.5) holds.

**Remark 3.6.2.** In terms of braid group actions, the generalized reflection equation yields an action of a cylindrical ribbon braid groupoid, since as we mentioned above the operators involved cannot be arbitrarily composed. However, in special cases, this action does yields a representation of the cylindrical braid group, e.g., if $V^{\psi} = V$ and $W^{\psi} = W$ (see Section 7).

The previous remark explains the relevance of the gauge for applications (e.g., [AP22]), since the generalized reflection equation (3.5) directly depends on the choice of $\psi$ (and therefore of $g$).

**Example 3.6.3.** We describe some distinguished examples of $g$ and $\psi$. 

\[ \psi := \text{Ad}(g) \circ \theta^{-1}_q, \quad R_\psi := g \otimes g \cdot R_\theta \cdot \Delta(g)^{-1}. \]
(1) We may consider “diagonal” modifications by gauging by $g = \beta$, where $\beta$ is any map $\hat{\mathcal{P}} \to \mathbb{F}^\times$. 

(2) For $g = S_\theta$, the twisting operator reduces to the involution $\psi = \omega \circ \tau$. The corresponding K-matrix, which we refer to as the semi-standard K-matrix, was also considered (up to conventions) in [BW21, Thm. 3.15].

(3) Let $(Y, \eta) \in \text{GSat}(\hat{A})$ be a generalized Satake diagram and $\zeta$ the corresponding pseudo-involution. For $g = S^{-1}_\theta$, one recovers the combinatorial family of universal K-matrices constructed in [AV22]. In particular, the standard K-matrix corresponds to the special case $\zeta = \theta$. The finite-type K-matrix constructed by Balagović and Kolb in [BK19] corresponds instead to the choice $S^{5}(Y, \eta) = (I, \omega l_\tau)$, whose associated pseudo-involution is $\zeta = \text{id}$. In analogy with this, in Section 7, we will discuss the case $Y = \hat{\Gamma}\{0, \tau(0)\}$, $\eta(0) = \tau(0)$, and $\eta|_Y = \omega l_Y$. 

4. SPECTRAL K-MATRICES

In this section, we present the first main result of the paper. Let $(X, \tau) \in \text{GSat}(\hat{A})$ with pseudo-involution $\theta$, $(\gamma, \sigma) \in \Gamma \times \Sigma$, and $U_q \mathfrak{g} \subset U_q \hat{\mathfrak{g}}$ the corresponding QSP subalgebra. We prove that, under mild assumptions, the universal K-matrices constructed in [AV22] and in Corollary 3.6.1 specialize to spectral operators on finite-dimensional $U_q L \mathfrak{g}$-modules.

4.1. $\tau$-minimal grading shifts. We shall need to replace the homogeneous grading shift (2.2), most commonly used in the context of quantum loop algebras, with a distinguished $\tau$-invariant grading shift. Note that for any group homomorphism $s: \hat{Q} \to \mathbb{Z}$, there is a grading shift $\Sigma^s_z: U_q \mathfrak{g} \to U_q \mathfrak{g}[z, z^{-1}]$ defined by

$$\Sigma^s_z(E_i) = z^{s(\al_i)}E_i, \quad \Sigma^s_z(F_i) = z^{-s(\al_i)}F_i, \quad \Sigma^s_z(K_h) = K_h$$

for $i \in \hat{I}$, $h \in \hat{Q}^\vee$. Then, $\Sigma^s_z$ is $\tau$-invariant if $s \circ \tau = s$ and therefore $\Sigma^s_z \circ \tau = \tau \circ \Sigma^s_z$. Note that $\Sigma^s_z$ is $\tau$-invariant if and only if, as a function on the set of affine simple roots, $s$ is the characteristic function of a union of $\tau$-orbits. For instance, the principal grading shift $\Sigma^1_z$ corresponding to $s(\al_i) = 1$ ($i \in \hat{I}$) is always $\tau$-invariant. The $\tau$-minimal grading shift $\Sigma^x_z$ corresponds instead to the characteristic function $s_\tau$ of the $\tau$-orbit of the affine node 0, i.e.,

$$s_\tau(\al_i) = \begin{cases} 
1 & \text{if } i \in \{0, \tau(0)\}, \\
0 & \text{otherwise}.
\end{cases}$$

Note that $\Sigma^x_z = \Sigma^\text{hom}_Z$ if and only if $\tau(0) = 0$. Moreover, the analogue of Theorem 2.6.1 holds if we replace $R(z)$ by $\text{id} \otimes \Sigma^x_z(R)$ and $\Delta_z$ by $\text{id} \otimes \Sigma^x_z \circ \Delta$.

Henceforth we shall only use the $\tau$-minimal grading shift $\Sigma^x_z$ and drop the upper index $s_\tau$ unless needed. In particular, we shall denote by $\pi_{V, z}$ the action on $V$ shifted by $\Sigma^x_z$.

Note that $(I, \omega l_{\tau})$ can be regarded as a degenerate case, since the corresponding QSP subalgebra is the full quantum group $U_q \mathfrak{g}$. This distinguished choice is only available in finite type.
4.2. **Spectral K-matrices.** Let $\mathcal{G}_{\theta, \gamma} \subset \mathcal{G}$ be the subset of gauge transformations $g \in \mathcal{G}$ of the form $g := S_Y^{-1} S_X \beta^{-1}$, where

1. $Y \subset \hat{\mathcal{T}}$ is any proper subdiagram such that $s(\alpha_i) = 0$ for any $i \in Y$;
2. $\beta : \hat{\mathcal{T}} \to \mathbb{F}^\times$ is any map such that $\gamma(\delta)\beta(\delta) = 1$.

The definition of $\mathcal{G}_{\theta, \gamma}$ is motivated by the results of the next section, cf. Remark 4.3.1. We refer to a twisting operator of the form

$$\psi = \text{Ad}(g) \circ \theta_q^{-1}$$

with $g \in \mathcal{G}_{\theta, \gamma}$ as a QSP-admissible twisting operator. Henceforth we assume that $\psi$ is a QSP-admissible twisting operator. We shall prove the following spectral analogue of Theorem 4.2.1.

**Theorem 4.2.1.** The quantum loop algebra $U_q L \mathfrak{g}$ has a $\mathcal{G}_{\theta, \gamma}$-family of universal spectral K-matrices relative to the QSP subalgebra $U_q \mathfrak{k}$. More precisely, for any $g \in \mathcal{G}_{\theta, \gamma}$, set

$$\psi := \text{Ad}(g) \circ \theta_q^{-1} \quad \text{and} \quad R_\psi := (S_\psi \otimes S_\psi)^{-1} \cdot \Delta(S_\psi)$$

where $S_\psi := S_g \cdot g^{-1}$. There is a canonical Laurent series $K_\psi(z) \in (U_q L \mathfrak{g})^{\id}([z])$ such that $\Sigma_o(K_\psi(z)) = K_\psi(az)$ ($a \in \mathbb{F}^\times$) and the following properties hold.

1. For any $b \in U_q \mathfrak{k}$,

   $$K_\psi(z) \cdot \Sigma_z(b) = \psi(\Sigma_{1/z}(b)) \cdot K_\psi(z).$$

2. Set $R(z)^\psi := \psi \otimes \id(R(z)).$ Then,

   $$\Delta_{w/z}(K_\psi(z)) = R_{\psi}^{-1} \cdot 1 \otimes K_\psi(w) \cdot R(zw)^{\psi} \cdot K_\psi(z) \otimes 1.$$

Moreover, $K_\psi(z)$ is a solution of the generalized reflection equation

$$R(w/z)^{\psi}_{21} \cdot 1 \otimes K_\psi(w) \cdot R(zw)^{\psi} \cdot K_\psi(z) \otimes 1 = K_\psi(z) \otimes 1 \cdot R(zw)^{\psi}_{21} \cdot 1 \otimes K_\psi(w) \cdot R(w/z),$$

where $R(z)^{\psi}_{21} := \psi \otimes \id(R(z))_{21}$.

**Remark 4.2.2.**

1. The identities (4.1) and (4.2) hold in $(U_q L \mathfrak{g})^{\id}((w/z, z))$, where $\mathbb{F}((w/z, z)) := \text{Frac}(\mathbb{F}[[w/z, z]])$. Following [Che84, Eq. (10)], it may be convenient to use an adapted set of coordinates, given by $u = w/z$ and $v = z$. Then, (4.1) and (4.2) read

   $$\Delta_u(K_\psi(v)) = R_{\psi}^{-1} \cdot 1 \otimes K_\psi(uv) \cdot R(uv^2)^{\psi} \cdot K_\psi(v) \otimes 1,$$

   and

   $$R(u)^{\psi}_{21} \cdot 1 \otimes K_\psi(v) \cdot R(u^2)^{\psi} \cdot K_\psi(uv) \otimes 1 = K_\psi(uv) \otimes 1 \cdot R(u^2)^{\psi}_{21} \cdot 1 \otimes K_\psi(v) \cdot R(u),$$

   in $(U_q L \mathfrak{g})^{\id}((u, v))$. 

By definition, every element in \( G_{\theta, \gamma} \) has the form \( g = S^{-1}_Y S_X \beta^{-1} \) for some \( Y \subset \hat{I} \) such that \( \Sigma_z(SY) = SY \) and \( \beta: p \to \mathbb{F}^\times \) (cf. Section 4.2). Therefore \( g \) is shift-invariant if and only if \( S_X \) is shift-invariant. In this case, the K-matrix \( K_\psi(z) \) is a formal series in \( (U_q L\mathfrak{g})^{fd}[z] \).

(3) The QSP subalgebra of \( U_q L\mathfrak{sl}_2 \) with \( (X, \tau) = (\mathcal{Q}, (01)) \), see Example 3.1.2, is known as the augmented \( q \)-Onsager algebra. In [BT18, Sec. 4.1.2] so-called generic K-operators in a completion of \( U_q \mathfrak{h}(z) \) are considered. It would be interesting to establish a relation with \( K_{\psi^{-1}}(z) \).

\[ \nabla \]

In analogy with the case of the R-matrix, the spectral K-matrix is obtained by applying the shift operator to the universal K-matrix \( K_\psi \) from Theorem 3.5.1, i.e., \( K_\psi(z) := \Sigma_z(K_\psi) \). The identities (4.1), (4.2), and (4.3) are then recovered from their analogues (3.3), (3.4), and (3.5), respectively, by applying the shift operator \( \Sigma_z \otimes \Sigma_w \). Clearly, since the operator \( K_\psi(z) \) is valued in \( (U_q L\mathfrak{g})^{fd} \), the statements above are to be interpreted as operators on finite-dimensional modules in \( \text{Mod}_{id}(U_q L\mathfrak{g}) \) and it is therefore necessary to prove that \( K_\psi(z) \) gives rise to a well-defined element \( K_{\psi, V}(z) \in \text{End}(V)(\!(z)\!) \) for any \( V \in \text{Mod}_{id}(U_q L\mathfrak{g}) \). The proof is carried out in Sections 4.3-4.4.

### 4.3. Descent to finite-dimensional modules

The first step in the proof of Theorem 4.2.1 amounts to proving that, for any \( V \in \text{Mod}_{id}(U_q L\mathfrak{g}) \) with action \( \pi_V : U_q L\mathfrak{g} \to \text{End}(V) \), we obtain a well-defined operator

\[ K_{\psi, V}(z) := \pi_{V,z}(K_\psi) = \pi_V \circ \Sigma_z(K_\psi) \in \text{End}(V)(\!(z)\!). \]

More precisely, we shall prove that each of the operators involved in the definition of the universal K-matrix (3.2) and in the coproduct identity (4.2) descends to one on any finite-dimensional \( U_q L\mathfrak{g} \)-module.

**Proposition 4.3.1.** The following operators in \( (U_q \hat{\mathfrak{g}})^{\text{gln}}_\gamma \) descend to operators in \( (U_q L\mathfrak{g})^{fd} \):

1. The operator \( \xi_\theta \), defined on any weight vector \( v \) of weight \( \lambda \) by

\[ \xi_\theta(v) = q^{(\theta(\lambda), \lambda)/2 + \rho_X \cdot \lambda} \cdot v, \]

where \( \rho_X \) is the half-sum of the positive roots in \( \Delta_X \) (cf. Section 3.3);

2. For any \( i \in \hat{I} \), the braid group operator \( S_i \);

3. For any map \( \psi : \hat{P} \to \mathbb{F}^\times \) such that \( \psi(\delta) = 1 \), the diagonal operator defined on any weight vector \( v \) of weight \( \lambda \) by \( \psi(v) := \psi(\lambda) \cdot v \).

Moreover,

4. For any \( \Psi \in \bigoplus_{\mu \in Q_+} U_q \hat{\mathfrak{g}} \), the shifted operator \( \Sigma_z(\Psi) \) descends to \( (U_q L\mathfrak{g})^{fd}[z] \).

**Proof.** (1) In analogy with the operator \( \varphi^{\hat{Q}_0} \) from 2.5, one checks easily that

\[ \xi_\theta = q^{\sum_{\mu \in \mathcal{Q}_+} \theta(\mu) n_{\mu} + m(\theta(c) d + \theta(d) c) - \rho_X}, \]

where \( m = 1, 2, 3 \) if \( \mathfrak{g} \) is of type \( \text{ADE}, \text{BCF}, \) or \( \mathfrak{g} \), respectively. Since \( \theta(c) = -c \), the term \( m(\theta(c) d + \theta(d) c) = m(\theta(d) - d) c \) acts as 0 and can be ignored. Therefore, as in the case of the operator \( \varphi^{\hat{Q}_0} \) in 2.6, \( \xi_\theta \) descends to an operator in \( (U_q L\mathfrak{g})^{fd} \). Note
that $\xi_\theta(\delta) = 1$.

(2) By restriction, $V$ is a finite-dimensional representation of $U_q\tilde{\Theta}_{[i]}$ and therefore integrable. In particular, the action of $S_i$ on $V$ is well-defined.

(3) By definition, any type $1$ $U_qLg$-module $V$ admits a weight decomposition over the quotient lattice $\tilde{\mathbf{P}}/\mathbb{Z}\delta$, i.e., $V = \bigoplus_{\lambda \in \tilde{\mathbf{P}}/\mathbb{Z}\delta} V_\lambda$. Therefore, the operator $\psi$ acts on $V$ if and only if it factors through $\tilde{\mathbf{P}}/\mathbb{Z}\delta$, i.e., if $\psi(\delta) = 1$.

(4) Set $\Psi = \sum_{\mu \in \mathbb{Q}_+} \Psi_\mu$, with $\Psi_\mu \in U_q\hat{\mathfrak{g}}_\mu^+$. Then, $\Sigma_z(\Psi) = \sum_{n \geq 0} \Psi(n) z^n$ where $\Psi(n) := \sum_{\mu \in s^{-1}(n)} \Psi_\mu$. Fix $n \geq 0$. We shall prove that $\Psi(n)$ is a well-defined operator on $V$. Note that, in the case of the principal grading shift (cf. Section 4.1), $s^{-1}(n)$ is a finite set and the result is clear. Assume without loss of generality that $s(\alpha_0) \neq 0$. Any $\mu \in s^{-1}(n)$ has the form $\mu = m\alpha_0 + \lambda$, where $m \leq n$ and $\lambda \in \mathbb{Q}_+$. However, the action of $U_qn^+ := \langle \{E_i\}_{i \in I} \rangle$ is locally finite on $V$ and the number of occurrences of $E_0$ is bounded by $n$. Therefore, the action of $\Psi_\mu$ on $V$ is non-zero only for finitely many $\mu \in s^{-1}(n)$ and $\Psi(n)$ is well-defined on $V$. The same argument applies if $0$ is replaced by any other node $i \in I$ such that $s(\alpha_i) \neq 0$. □

Recall that, by Theorem 3.5.1 and (3.2), we have $K_\psi = g \cdot \gamma^{-1}.Y_\theta$, where $g \in \mathcal{G}_{\theta, \gamma}$ is a gauge, $\gamma$ is the parameter operator, and $Y_\theta$ is the quasi-K-matrix. Note that, by definition of $\mathcal{G}_{\theta, \gamma}$, $\Sigma_z(g \cdot \gamma)$ is an element of $(U_qLg)^{fd}[z, z^{-1}]$. Therefore, by the result above, $K_{\psi}(z) := \Sigma_z(K_\psi) \in (U_qLg)^{fd}(\mathbb{C})$, i.e., for any $V \in \text{Mod}_{fd}(U_qLg)$, $K_{\psi,v}(z) := \pi_{\psi,v}(K_{\psi})$ is a well-defined operator in $\text{End}(V)(\mathbb{C})$.

Remark 4.3.2. We did not use yet the fact that the grading shift is $\tau$-invariant and $g \cdot S_\theta$ is shift-invariant. These assumptions become crucial in the next step. □

4.4. Spectral K-matrices on finite-dimensional modules. We complete the proof of Theorem 4.2.1 by showing that the relations (4.1), (4.2), and (4.3) hold. Note that, since the grading shift is $\tau$-invariant and $g \in \mathcal{G}_{\theta, \gamma}$ is shift-invariant, we have $\tau \circ \Sigma_z = \Sigma_z \circ \tau$ and $\text{Ad}(g \cdot S_\theta) \circ \Sigma_z = \Sigma_z \circ \text{Ad}(g \cdot S_\theta)$. Finally, since $\omega \circ \Sigma_z = \Sigma_1/\omega \circ \psi$ and $\psi = \text{Ad}(g \cdot S_\theta) \circ \omega \circ \tau$, we have

$$\psi \circ \Sigma_z = \Sigma_1/\psi.$$

Let $V \in \text{Mod}_{fd}(U_qLg)$. The action on the shifted representation $V^\psi(\langle 1/z \rangle)$ is, by definition, given by

$$\pi_{V^\psi, 1/z}(x) = \pi_{V^\psi}(\Sigma_{1/z}(x)) = \pi_V(\psi \circ \Sigma_{1/z}(x)).$$

Since $\pi_V(\psi \circ \Sigma_{1/z}(x)) = \pi_V(\Sigma_z \circ \psi(x))$, one has $V^\psi(\langle 1/z \rangle) = V(\mathbb{C})^\psi$. Therefore, the evaluation of the intertwining identity (3.3) on $V(\mathbb{C})$ through $\pi_{V,z}$ yields

$$K_{\psi,v}(z) \pi_{V,z}(b) = \pi_{V^\psi, 1/z}(b) K_{\psi,v}(z).$$

It follows that $K_{\psi,v}(z)$ is a QSP intertwiner $V(\mathbb{C}) \rightarrow V^\psi(\langle 1/z \rangle)$, which is equivalent to (4.1).

Set $K_{\psi,v,w}(z, w) := \pi_{V,z} \otimes \pi_{V,w}(\Delta(K_{\psi,v})).$ Then, the evaluation of the coproduct identity (3.4) on $V(\mathbb{C}) \otimes W(\mathbb{C})$ through $\pi_{V,z} \otimes \pi_{V,w}$ yields

$$K_{\psi,v,w}(z, w) = R_{\psi,v,w}^{-1} \cdot \text{id} \otimes K_{\psi,w}(w) \cdot R_{v \rightarrow w}(zw) \cdot K_{\psi,v}(z) \otimes \text{id},$$

where $R_{v \rightarrow w}$ is the minimal solution to the Yang-Baxter equation.

$\Delta$ denotes the coproduct on $Lg$.

$\text{id}$ is the identity operator on $V$.

$\Psi$ is a well-defined operator on $V$.

$\tau$ is an automorphism of $V$.

$\psi$ is a well-defined operator on $V$.

$\theta$ is a grading shift on $V$.

$\omega$ is a weight function on $V$.

$\gamma$ is a parameter on $V$.

$\mathcal{G}_{\theta, \gamma}$ is a set of gauges on $V$.

$U_qLg$ is a quantum Lie algebra on $V$.

$\pi_V$ is a representation of $U_qLg$ on $V$.

$\Sigma_z$ is a grading shift on $V$.

$\text{Ad}$ is the adjoint action on $V$.

$\pi_{V,z}$ is a representation of $U_qLg$ on $V(\mathbb{C})$.

$\pi_{V,w}$ is a representation of $U_qLg$ on $V(\mathbb{C})$.

$\Delta$ is the coproduct on $U_qLg$.

$\text{id}$ is the identity operator on $V(\mathbb{C})$.
By (4.4), this is equivalent to (4.2). Finally, the evaluation of the generalized reflection equation (3.5) on $V((z)) \otimes W((w))$ through $\pi_{V,z} \otimes \pi_{W,w}$ yields

$$R_{W^\psi V^\psi(w/z)_{21}} \cdot \text{id} \otimes K_{\psi,W}(w) \cdot R_{V^\psi W}(zw) \cdot K_{\psi,V}(z) \otimes \text{id} = K_{\psi,V}(z) \otimes \text{id} \cdot R_{W^\psi V}(zw)_{21} \cdot \text{id} \otimes K_{\psi,W}(w) \cdot R_{VW}(w/z),$$

where $R_{WV}(z)_{21} := (12) \circ R_{WV}(z) \circ (12)$. As before, this is equivalent to (4.3).

5. TRIGONOMETRIC K-MATRICES

In this section, we prove that the spectral K-matrix constructed in Theorem 4.2.1 gives rise to trigonometric solutions (i.e., operator-valued rational functions in $z$) of the generalized reflection equation on irreducible $U_qL\mathfrak{g}$-modules.

5.1. Trigonometric K-matrices. The construction of the spectral K-matrix from Theorem 4.2.1 immediately implies the existence of a trigonometric QSP intertwiner on any finite-dimensional $U_qL\mathfrak{g}$-module.

Lemma 5.1.1. Let $V \in \text{Mod}_d(U_qL\mathfrak{g})$. There exists a QSP intertwiner $K_{\psi,V}(z) : V(z) \to V^p((1/z))$ in $\text{End}(V)(z)$.

Proof. By Theorems 4.2.1 and 4.4, for any $V \in \text{Mod}_d(U_qL\mathfrak{g})$, the element $K_{\psi}(z)$ provides an intertwiner

$$K_{\psi,V}(z) : V((z)) \to V^p((1/z)).$$

This is equivalent to the existence of a solution $K_{\psi,V}(z) \in \text{End}(V)((z))$ of a finite system of linear equations, namely

$$K_{\psi,V}(z) \cdot \pi_{V,z}(b) = \pi_{V^p,1/z}(b) \cdot K_{\psi,V}(z),$$

where $b \in U_q\mathfrak{g}$ runs over any finite set of generators of $U_q\mathfrak{g}$. Since the system is consistent and defined over $\mathbb{F}(z)$, it admits a solution in $\text{End}(V)(z)$.

It follows that, in the case of irreducible representations, the spectral K-matrix $K_{\psi,V}(z)$ is rational up to a non-zero scalar in $\mathbb{F}(z)$.

Theorem 5.1.2. Let $V, W \in \text{Mod}_d(U_qL\mathfrak{g})$ be irreducible modules.

1. The space of QSP intertwiners $V((z)) \to V^p((1/z))$ is one-dimensional.

2. There exist a formal Laurent series $g_V(z) \in \mathbb{F}(z)$ and a non-vanishing operator-valued polynomial $K_{\psi,V}(z) \in \text{End}(V)[z]$, uniquely defined up to a scalar in $\mathbb{F}^\times$, such that

$$K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z).$$

3. The operators $K_{\psi,V}(z)$ and $K_{\psi,W}(w)$ satisfy the generalized reflection equation in $\text{End}(V \otimes W))(z,w)$

$$R_{W^\psi V^\psi((z,w)_{21}} \cdot 1 \otimes K_{\psi,W}(w) \cdot R_{V^\psi W}(zw) \cdot K_{\psi,V}(z) \otimes 1 = K_{\psi,V}(z) \otimes 1 \cdot R_{W^\psi V}(zw)_{21} \cdot 1 \otimes K_{\psi,W}(w) \cdot R_{VW}(z,w),$$

where $R_{VW}(z)$ is the trigonometric R-matrix (see 2.7), and

$$R_{WV}(z)_{21} := (12) \circ R_{WV}(z) \circ (12).$$

We shall refer to the operator $K_{\psi,V}(z)$ as a polynomial trigonometric K-matrix.

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6We are grateful to V. Toledano Laredo for pointing out to us this argument.
The proof of Theorem 5.1.2 is carried out in Sections 5.2-5.3. Note that, from Remark 4.2.2 (2), we get the following

**Corollary 5.1.3.** If $\Sigma_z(g) = g$, the universal $K$-matrix descends on any irreducible $U_q \mathfrak{g}$-module $V$ to a formal series operator $K_{\psi,V}(z) \in \text{End}(V)[[z]]$, endowed with a unique factorization

$$K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z),$$

where $g_V(z) \in \mathbb{F}[[z]]$ such that $g_V(0) = 1$ and $K_{\psi,V}(z) \in \text{End}(V)[z]$ is non-vanishing.

**5.2. Proof of Theorem 5.1.2: part (1).** We say that an irreducible $U_q \mathfrak{g}$-module $V$ is generically QSP irreducible if $V(\langle z \rangle)$ is irreducible as a representation over $U_q \mathfrak{g}(\langle z \rangle)$. Such a condition is the natural counterpart of the generic irreducibility of the tensor product $V \otimes W(\langle z \rangle)$ in Theorem 2.7.1, which holds for any pair of irreducible representations $V, W$.

The condition of generic irreducibility may depend on the choice of a grading shift. In Corollary 6.1.2, we prove that every irreducible $U_q \mathfrak{g}$-representation is generically QSP irreducible with respect to the principal grading shift. It is then clear that, in this case, (1) follows by Schur’s lemma. Namely, let $K_1, K_2 \in \text{End}(V)(\langle z \rangle)$ be two solutions of (5.1). Let

$$\mathbb{F}\{z\} := \bigcup_{n>0} \mathbb{F}(\langle z^{1/n} \rangle)$$

be the field of Puiseux series over $\mathbb{F}$, i.e., the algebraic closure of $\mathbb{F}(\langle z \rangle)$, and set $V(\{z\}) := V \otimes \mathbb{F}\{z\}$. The composition $K_2^{-1} K_1 : V(\{z\}) \to V(\{z\})$ is an intertwiner. Therefore, by generic QSP irreducibility and Schur’s lemma, there exists $g(z) \in \mathbb{F}\{z\}$ such that $K_1 = g(z) K_2$. Clearly, since both operators are defined over $\mathbb{F}(\langle z \rangle)$, one has $g(z) \in \mathbb{F}(\langle z \rangle)$.

The irreducibility result does not immediately carry over to the case of the $\tau$-minimal grading shift. Instead, we generalize (1) by proving that the above result on the one-dimensionality of the space of QSP intertwiners (5.1) for the principal grading implies the one-dimensionality of the space of QSP intertwiners for the $\tau$-minimal grading shift.

Let $\text{pr} : \hat{\mathbb{Q}} \to \mathbb{Z}$ be the group homomorphism defined by $\alpha_i \mapsto 1$ for all $i \in \hat{I}$ so that, in the notation of Section 4.1, $\Sigma_\text{pr}$ denotes the principal grading shift. Consider any extension of $\text{pr}|\mathbb{Q}$ and $s_\tau|\mathbb{Q}$ to group homomorphisms from $\mathbb{P} \to \hat{\mathbb{Q}}$, also denoted $\text{pr}$ and $s_\tau$, respectively. Note that the extended $\text{pr}$ and $s_\tau$ will in fact take images in $\frac{1}{m}\mathbb{Z}$ for some positive integer $m$. We shall regard $V$ as a $\mathbb{P}$-graded vector space. Let $M_V(z)$ be the linear operator on $V(\langle z^{1/m} \rangle) \subset V(\{z\})$ given by $M_V(z) v_\lambda = z^{\text{pr}(\lambda)} v_\lambda$ for any weight vector $v_\lambda$ ($\lambda \in \mathbb{P}$). Let $h$ be the Coxeter number of $\mathfrak{g}$ and define $h_\tau \in \frac{1}{m}\mathbb{Z}$ by

$$h_\tau := \begin{cases} h & \text{if } \tau(0) = 0, \\ h + 1 \frac{1}{m} & \text{if } \tau(0) \neq 0. \end{cases}$$

Since $h = \text{ht}(\delta)$, it follows that $M_V(z)$ intertwines between the principal shifted action and the $\tau$-minimal shifted action, i.e., $\text{Ad}(M_V(z)) \circ \pi_{z^{h_\tau},V}^{\text{pr}} = \pi_{z^{h_\tau},V}^{\text{pr}}$. Finally,
we observe that the linear map sending $R^\psi_{\psi,V}(z)$ to
\[ K^\psi_{\psi,V}(z) := M_{V^\psi}(z^{-1})^{-1} \cdot K^\psi_{\psi,V}(z) \cdot M_V(z) \]
is an isomorphism between the spaces of principal and $\tau$-minimal QSP intertwiners. Thus, the result follows from Corollary 6.1.2.

5.3. **Proof of Theorem 5.1.2: part (2) and (3).** Part (2) follows immediately from part (1) and Lemma 5.1.1. Namely, let $K_{\psi}(z) \in \text{End}(V)(z)$ be any rational solution of (5.1), whose existence is guaranteed by Lemma 5.1.1. By (1), there exists $g_V(z) \in \mathbb{F}(z)$ such that the identity $K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z)$ holds. Then, by factoring out every pole and common zero of $K_{\psi,V}(z)$, we obtain the desired factorization $K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z)$, where $K_{\psi,V}(z)$ is a non-vanishing operator-valued polynomial.

It remains to prove part (3). Let $R_{V,W}(z) = f_{V,W}(z)^{-1} R_{V,W}(z)$ be the trigonometric R-matrix (cf. Thm. 2.7.1). Then, (2) reduces to prove that
\[ f_{V^\psi V^\psi}(\frac{z}{z}) \cdot g_V(z) \cdot f_{V^\psi W}(zw) \cdot g_W(w) = g_V(z) \cdot f_{V^\psi V}(zw) \cdot g_W(w) \cdot f_{V,W}(\frac{z}{z}) \]
Note that, by [AV22, Thm. 8.10], we have $R_{V,W}^{\psi \psi} = R_\psi \cdot R_{V,W}^{-1}$. Therefore,
\[ R_{V^\psi V^\psi}(\frac{z}{z}) = R_\psi \cdot R_{V,W}(\frac{z}{z}) \cdot R_{V,W}^{-1} = f_{V,W}(\frac{z}{z}) R_{V^\psi V^\psi}(\frac{z}{z}) \]
\[ i.e., f_{V^\psi V^\psi}(\frac{z}{z}) = f_{V,W}(\frac{z}{z}) \]
Similarly, replacing $V$ with $V^\psi$, we get $f_{V^\psi W}(zw) = f_{V^\psi V^\psi}(zw)$. As $g \in \mathcal{G}_{\theta,\gamma}$ it follows that $\psi^2$ is the identity on $U_q\hat{\mathfrak{h}}$ and hence the pullback by $\psi^2$ has no impact on the normalization, i.e.,
\[ f_{V^\psi W}(zw) = f_{V^\psi V^\psi}(zw) = f_{V^\psi V}(zw) \]
The result follows.

**Remark 5.3.1.** The definition of the set $\mathcal{G}_{\theta,\gamma}$ given in Section 4.2 may appear at first quite ad hoc. However, for the proofs of Theorems 4.2.1 and 5.1.2 to work, a gauge transformation $g \in \mathcal{G}$ is required to satisfy the following three conditions:

\begin{enumerate}
\item[(G1)] $g \cdot S^{-1}_\psi$ is shift-invariant;
\item[(G2)] $g \cdot \gamma^{-1}$ descends to an operator on shifted representations;
\item[(G3)] $\psi^2$ is the identity on $U_q\hat{\mathfrak{h}}$, where $\psi = \text{Ad}(g) \circ \theta^{-1}$.
\end{enumerate}

By observing the action of $g$ on the Cartan subalgebra at $q = 1$, one checks that the conditions (G1), (G2), (G3) essentially determine $\mathcal{G}_{\theta,\gamma}$.

5.4. **Unitary K-matrices for $\psi$-involutive modules.** Let $V, W \in \text{Mod}_{\text{id}}(U_qL_\theta)$ be irreducible representations. By Theorem 2.7.1, the trigonometric R-matrix $R_{V,W}(z)$ satisfies the unitarity condition $R_{V,W}(z)^{-1} = (1,2) \circ R_{W,V}(z^{-1}) \circ (1,2)$. Due to the lack of a canonical eigenvector, the analogue result for K-matrices requires a different approach. In this section we discuss the case of $U_qL_\theta$-modules such that $V^{\psi^2} = V$, while in Section 5.5 we discuss the special case $V^\psi = V$.\[\n\]
Proposition 5.4.1. Let $V \in \text{Mod}_d(U_q L_0)$ be an irreducible module such that $V^{\psi^2} = V$. There exist non-vanishing trigonometric $K$-matrices $K_{\psi, V}(z), K_{\psi, V}(z) \in \text{End}(V)(z)$ such that
\[ (U1) \quad K_{\psi, V}(z)^{-1} = K_{\psi, V}(z^{-1}) ; \]
\[ (U2) \quad \text{if } V(\zeta) \text{ is QSP irreducible for some } \zeta \in \mathbb{F}^x, \text{ then } K_{\psi, V}(\zeta) \text{ is well-defined and invertible.} \]

Proof. Let $K_{\psi, V}(z)$ and $K_{\psi, V}(z)$ be any two polynomial trigonometric $K$-matrices for $V$ and $V^{\psi}$, respectively. The composition
\[ V(z) \xrightarrow{K_{\psi, V}(z)} V^{\psi}(z^{-1}) \xrightarrow{K_{\psi, V}(z^{-1})} V^{\psi^2}(z) = V(z) \]
is a QSP intertwiner. Therefore, by Theorem 5.1.2, we must have
\[ K_{\psi, V}(z^{-1})K_{\psi, V}(z) = f(z)\text{id}_V(z), \]
for some nonzero Laurent polynomial $f(z) \in \mathbb{F}[z, z^{-1}]$. Thus (U1) follows by rescaling either $K_{\psi, V}(z)$ or $K_{\psi, V}(z^{-1})$. Note that the rescaled operators are non-vanishing $\text{End}(V)$-valued rational functions satisfying the unitarity condition $K_{\psi, V}(z)^{-1} = K_{\psi, V}(z^{-1})$.

To prove (U2), let $m \geq 0$ be the order of the pole of $K_{\psi, V}(z)$ at $z = \zeta$. The map
\[ \lim_{z \to \zeta} (z - \zeta)^m K_{\psi, V}(z) \]
is a non-zero intertwiner $V(\zeta) \to V^{\psi}(\zeta^{-1})$ and therefore invertible. By unitarity, this implies that $K_{\psi, V}(z^{-1})$ must have a zero of order $m$ at $z = \zeta$. Since the latter is non-vanishing, we conclude that $m = 0$ and (U2) follows. □

Remark 5.4.2. Recall the twist $\psi = \omega \circ \tau$ in the semi-standard case, see Remark 3.6.3 (2). Since $\psi$ is involutive, the condition $V^{\psi^2} = V$ is satisfied for any $V \in \text{Mod}_d(U_q L_0)$. In general, however, some further adjustment is required. For instance, we show in Section 7 that the distinguished twisting operator $\psi_0$ defined by (7.3) acts (up to shift) as an involution on a certain class of representations, generalizing the case of $U_q \hat{\mathfrak{sl}}_2$ discussed in [AV22, Sec. 9]. ▽

5.5. Unitary $K$-matrices for $\psi$-fixed modules. In the special case $V^{\psi} = V$ we have the following refinement of Proposition 5.4.1, in the spirit of Remark 2.7.2 (2). The proof follows closely [RV16, Sec. 6.3].

Proposition 5.5.1. Let $V \in \text{Mod}_d U_q L_0$ be an irreducible representation such that $V^{\psi} = V$. Then there exists a non-vanishing trigonometric $K$-matrix $K_{\psi, V}(z) \in \text{End}(V)(z)$ such that
\[ (U1') \quad K_{\psi, V}(z)^{-1} = K_{\psi, V}(z^{-1}) ; \]
\[ (U2') \quad \text{if } V(\zeta) \text{ is QSP irreducible for some } \zeta \in \mathbb{F}^x, \text{ then } K_{\psi, V}(\zeta) \text{ is well-defined and invertible ;} \]
\[ (U3') \quad \text{if } V(\pm 1) \text{ is QSP irreducible, then } K_{\psi, V}(\pm 1) = \text{id}_V . \]

Proof. Proceeding as in the proof of Proposition 5.4.1, we consider any polynomial trigonometric $K$-matrix $K_{\psi, V}(z)$ for $V$ satisfying the identity
\[ K_{\psi, V}(z^{-1})K_{\psi, V}(z) = f(z)\text{id}_V(z), \]
for some nonzero Laurent polynomial \( f(z) \in \mathbb{F}[z, z^{-1}] \). Conjugation by \( K_{\psi, V}(z^{-1}) \) then yields \( f(z) = f(z^{-1}) \). As a consequence, the roots of \( f \) are nonzero and, counting multiplicities, the set of roots of \( f \) is invariant under inversion. Therefore, \( f(z) = g(z)g(z^{-1}) \) for some polynomial \( g(z) \in \mathbb{F}[z] \). By replacing \( K_{\psi, V}(z) \) with \( g(z)^{-1}K_{\psi, V}(z) \), we get \((U1')\). Part \((U2')\) follows as in the proof of Proposition 5.4.1.

It remains to prove \((U3')\). Assume first that \( V(\sigma) \) is QSP irreducible, while \( V(-\sigma) \) is not, for some \( \sigma \in \{\pm 1\} \). Then, \( K_{\psi, V}(\sigma)^2 = \text{id}_V \) and therefore, by QSP irreducibility, \( K_{\psi, V}(\sigma) = \epsilon \cdot \text{id}_V \) with \( \epsilon^2 = 1 \). By replacing \( K_{\psi, V}(z) \) with \( \epsilon \cdot K_{\psi, V}(z) \), we get \((U3')\). Assume now that \( V(1) \) and \( V(-1) \) are both QSP irreducible. As in the previous case, we get \( K_{\psi, V}(\pm 1) = \epsilon_\pm \cdot \text{id}_V \) with \( \epsilon^2_\pm = 1 \). By replacing \( K_{\psi, V}(z) \) with \( \epsilon_\pm + z^{-1} \cdot K_{\psi, V}(z) \), we get \((U3')\). □

5.6. Abelian quasi-K-matrices and diagonal K-matrices. The normalizations introduced in Propositions 5.4.1 and 5.5.1 become canonical in presence of a distinguished eigenvector. This is the case for restricted rank one QSP subalgebras (cf. Remark 3.2.1 (3)).

Indeed, for \( \theta \) of restricted rank one, the quasi-K-matrix defined in Theorem 3.5.1 is given purely in terms of imaginary root vectors, since it has the form \( \Upsilon_{\theta} = 1 + \sum_{\mu \in \mathbb{Z}_{\geq 0}} \Upsilon_{\theta, \mu} \) with \( \Upsilon_{\theta, \mu} \in \mathbb{U}_q \mathfrak{n}^+ \). In particular, it preserves the weight spaces of any finite-dimensional \( \mathbb{U}_q \mathfrak{g} \)-module.

**Proposition 5.6.1.** Assume that \( \theta \) has restricted rank one and \( \Sigma_z(\mathfrak{g}) = \mathfrak{g} \). Let \( V \) be an irreducible finite-dimensional \( \mathbb{U}_q \mathfrak{g} \)-module with \( \ell \)-highest weight vector \( v_0 \in V \). There is a unique non-vanishing trigonometric K-matrix \( K_{\psi, V}(z) \in \text{End}(V)(z) \) regular at \( z = 0 \) such that

\[
K_{\psi, V}(z)v_0 = \mathfrak{g} \cdot v_0.
\]

**Proof.** Note that, since \( \Sigma_z(\mathfrak{g}) = \mathfrak{g} \), the quasi-K-matrix \( \Upsilon_{\theta} \) descends to the only spectral component of \( K_{\psi, V}(z) \). In particular, there exists \( g_V(z) \in \mathbb{F}[z]^{\times} \) such that \( K_{\psi, V}(z)v_0 = g_V(z)v_0 \), where \( K_{\psi, V}(z) := \pi_{V, z}(K_{\theta}) \). It follows that \( K_{\psi, V}(z) := g_V(z)^{-1}K_{\psi, V}(z) \) is the unique intertwiner \( V((z)) \rightarrow V((z^{-1})) \) such that \( K_{\psi, V}(z)v_0 = \mathfrak{g} \cdot v_0 \). Therefore, \( K_{\psi, V}(z) \in \text{End}(V)(z) \) is a non-vanishing rational operator regular at \( z = 0 \). □

**Remarks 5.6.2.**

1. If \( \mathfrak{g} \) is a weight zero vector, the \( K_{\psi, V}(z) \) preserves the weight spaces of \( V \). If the weight spaces are all one-dimensional, this yields a diagonal trigonometric K-matrix.

2. A similar result can be achieved through normalization with respect to any vector which is a simultaneous eigenvector for the imaginary root vectors. In particular, one can consider any extremal vector (cf. [Kas02, Cha02]). □

**Corollary 5.6.3.** Assume that \( \theta \) has restricted rank one, \( \mathfrak{g} \) is weight zero and \( \Sigma_z(\mathfrak{g}) = \mathfrak{g} \).
(1) Let $V \in \text{Mod}_d(U_qL\mathfrak{g})$ be such that $V^{\psi^2} = V$ and consider the trigonometric $K$-matrices normalized by the conditions

$$K_{\psi,V}(z)v_0 = v_0 = K_{\psi,V}(z)v_0.$$ 

Then $K_{\psi,V}(z)^{-1} = K_{\psi,V}(z^{-1})$ holds. Moreover, if $V(\zeta)$ is QSP irreducible for some $\zeta \in \mathbb{F}^\times$, then $K_{\psi,V}(\zeta)$ is well-defined and invertible.

(2) Let $V \in \text{Mod}_d(U_qL\mathfrak{g})$ be such that $V^\psi = V$ and consider the diagonal trigonometric $K$-matrix normalized by the condition $K_{\psi,V}(z)v_0 = v_0$. Then, if $V(\pm 1)$ is QSP irreducible, $K_{\psi,V}(\pm 1) = \text{id}_V$.

Proof. It suffices to note that, since $\mathfrak{g}$ is weight zero, then there exists $g_V(z) \in \mathbb{F}[z]^\times$ such that $K_{\psi,V}(z)v_0 = g_V(z)v_0$, therefore $K_{\psi,V}(z) = g_V(z)^{-1}K_{\psi,V}(z)$. Moreover, $v_0 \in V^\psi$ is still a simultaneous eigenvector for the imaginary root vectors. Then the results follow as in Propositions 5.4.1 and 5.6.1. 

6. Generic restricted irreducibility

In this stand-alone section, we discuss various problems of restricted irreducibility for finite-dimensional $U_qL\mathfrak{g}$-modules. More precisely, we consider certain proper subalgebras $\mathcal{A} \subset U_qL\mathfrak{g}$ and we describe the irreducible representations which are generically irreducible under restriction, i.e., those $V \in \text{Mod}_d(U_qL\mathfrak{g})$ such that $V(z)$ is irreducible as a module over $\mathcal{A}(z) := \mathcal{A} \otimes \mathbb{F}(z)$. Recall that the definition of $V(z)$ depends on the choice of a grading shift. For a class of subalgebras, which we refer to as modified nilpotent, we prove that every irreducible $U_qL\mathfrak{g}$-module remains, with respect to the principal grading shift, generically irreducible under restriction (Theorem 6.1.1). As a corollary, we prove that every irreducible $U_qL\mathfrak{g}$-module is generically QSP irreducible (Corollary 6.1.2).

In this section we shall use the shorthand notations $\hat{U}^+$, $\hat{U}^0$ and $\hat{U}^-$ to denote the subalgebras of $U_qL\mathfrak{g}$ generated by $E_i$, $K_i^\pm 1$, and $F_i$, respectively, for all $i \in \hat{I}$. Set $\hat{U}^{\geq 0} := \hat{U}^+ \cup \hat{U}^0$. We shall consider only the principal grading shift $\Sigma^\sigma : U_qL\mathfrak{g} \to U_qL\mathfrak{g}[z, z^{-1}]$, see Section 4.1. As before, for any finite-dimensional $U_qL\mathfrak{g}$-module $V$, we denote by $V(z) = V \otimes \mathbb{F}(z)$ the corresponding shifted representation.

6.1. Generic irreducibility for modified nilpotent subalgebras. A subalgebra of $U_qL\mathfrak{g}$ is a modified (lower) nilpotent subalgebra if it is generated elements $\hat{F}_i$ satisfying $\hat{F}_i = F_i \in \hat{U}^{\geq 0}$ for all $i \in \hat{I}$. The main result of this section is the following statement, which we prove in Section 6.3.

**Theorem 6.1.1.** Every irreducible finite-dimensional $U_qL\mathfrak{g}$-module is generically irreducible under restriction to a modified nilpotent subalgebra.

As in Section 3, let $\theta$ be a pseudo-involution of $\hat{\mathfrak{h}}$ of the second kind with associated generalized Satake diagram $(X, \tau)$. Let $(\gamma, \sigma) \in \Gamma \times \Sigma$ and set $\hat{F}_i = B_i$. From $\theta_p(F_i) \in \hat{U}^+ \cdot K^{-1}$ for all $i \notin X$ and $B_i = F_i$ if $i \in X$ we obtain $\hat{F}_i - F_i \in \hat{U}^{\geq 0}$. Thus, every QSP subalgebra contains a modified nilpotent subalgebra. Hence Theorem 6.1.1 yields the following result of generic QSP irreducibility.

**Corollary 6.1.2.** Every irreducible finite-dimensional $U_qL\mathfrak{g}$-module is generically irreducible under restriction to a QSP subalgebra.
6.2. Restricted irreducibility for nilpotent subalgebras. The proof of Theorem 6.1.1 crucially relies on the following result: every finite-dimensional irreducible $U_qL\mathfrak{g}$-module remains irreducible under restriction to $\hat{U}^\epsilon$. To our knowledge, this result is not explicitly stated in the literature. However, the analogous result for $U_q^\epsilon$ is known as stated below.

**Proposition 6.2.1** ([HJ12, Prop. 3.5] or [Bow07, Thm. 2.3 (ii) for $\epsilon = (1)^n$]). Every irreducible finite-dimensional $U_qL\mathfrak{g}$-module remains irreducible under restriction to $\hat{U}^\epsilon$.

In fact, the proof of [HJ12, Prop. 3.5], which relies on the Drinfeld presentation of $U_qL\mathfrak{g}$, can be easily strengthened to show that every finite-dimensional irreducible $U_qL\mathfrak{g}$-module remains irreducible under restriction to $\hat{U}^\epsilon$. Nonetheless, in order to be consistent with the rest of the paper, we provide a proof in terms of the Drinfeld-Jimbo presentation.

Let $V$ be a fixed finite-dimensional irreducible $U_qL\mathfrak{g}$-module. By restriction to $U_q\mathfrak{g}$, $V$ decomposes into irreducible $U_q\mathfrak{g}$-modules, yielding a weight decomposition $V = \bigoplus_{\lambda \in \Phi} V_\lambda$ where

$$V_\lambda := \{ v \in V \mid \forall i \in I, K_i \cdot v = q_i^{\lambda(h_i)} v \}.$$  

Note that $K_0$ acts on each $V_\lambda$ as multiplication by $q_i^{\lambda(h_0)} = q_{\lambda, \delta}$. Let $\tau: \hat{Q} \to Q$ be the unique $\mathbb{Z}$-linear projection such that $\tau_\lambda = \alpha_i (i \in I)$ and $\delta = 0$. For $i \in I$ and $\lambda \in P$, we have

$$E_i \cdot V_\lambda \subseteq V_{\lambda+\tau_\lambda}$$  
  and  
  $$F_i \cdot V_\lambda \subseteq V_{\lambda-\tau_\lambda}.$$  

Set $\text{Supp}(V) := \{ \lambda \in P \mid V_\lambda \neq \{0\} \}$. From Proposition 6.2.1 we deduce some useful technical results, the first of which directly generalizes [CG05, Cor. 2.7].

**Lemma 6.2.2.** Let $\lambda \in \text{Supp}(V)$ and $0 \neq v_\lambda \in V_\lambda$.

1. $V = \hat{U}^\epsilon \cdot v_\lambda$.

2. For any $\mu \in \text{Supp}(V)$, there exist $\ell \in \mathbb{Z}_{\geq 0}$ and $i \in \hat{I}$ such that $E_{i_1} \cdots E_i \cdot v_\lambda$ is nonzero and has weight $\mu$.

**Proof.** Part (1) follows from Proposition 6.2.1, the decomposition $\hat{U}^\epsilon = \hat{U}^+ \cdot \hat{U}^\epsilon$ and the fact that $\hat{U}^\epsilon \cdot v_\lambda = \oplus_{\lambda \in \text{Supp}(V)} v_\lambda$. To prove part (2), note that (1) implies the existence of $x \in \hat{U}^\epsilon$ such that $0 \neq x \cdot v_\lambda \in V_\mu$. Since each monomial in the $E_i (i \in \hat{I})$ sends $v_\lambda$ into a weight space, we may assume without loss of generality that all monomials occurring in $x$ map $v$ to $V_\mu$. Since at least one of these monomials does not annihilate $v_\lambda$, we obtain the result.

For any $v \in V$, let $\text{Supp}(v) \subseteq \text{Supp}(V)$ be defined by the property $v = \sum_{\lambda \in \text{Supp}(v)} v_\lambda$ for some non-zero $v_\lambda \in V_\lambda$. Note that $|\text{Supp}(v)| = 1$ if and only if $v$ is a weight vector. For any $i \in \hat{I}$ we have

$(6.1)$ \hspace{1cm}$\text{Supp}(E_i \cdot v) \subseteq \text{Supp}(v) + \tau_i$ \hspace{1cm}$\text{and}$ \hspace{1cm}$\text{Supp}(F_i \cdot v) \subseteq \text{Supp}(v) - \tau_i$.

Hence,

$(6.2)$ \hspace{1cm}|$\text{Supp}(E_i \cdot v)$|, $|\text{Supp}(F_i \cdot v)$| $\leq$ $|\text{Supp}(v)$|.

By [CP94a, Sec. 3], also see [CG05, Sec. 2.3], for a given irreducible finite-dimensional $U_qL\mathfrak{g}$ module-V there exists $\lambda_0 \in P$ such that $\text{Supp}(V)$ is contained in
\[
\lambda_0 - Q_+ , E_i \cdot V_{\lambda_0} = \{0\} \text{ for all } i \in I \text{ and } F_0 \cdot V_{\lambda_0} = \{0\}; \text{ moreover } \dim(V_{\lambda_0}) = 1. \]

We define the depth of a weight vector as follows. For any \( \lambda \in \text{Supp}(V) \) and \( 0 \neq v_\lambda \in V_\lambda \), we set
\[
d^+(v_\lambda) := \min \{ \ell \in \mathbb{Z}_{\geq 0} | \exists i \in \hat{I} \text{ such that } 0 \neq E_{i_1} \cdots E_{i_\ell} \cdot v_\lambda \in V_{\lambda_0} \}.
\]

Note that, by Lemma 6.2.2 (2), \( d^+(v_\lambda) \in \mathbb{Z}_{\geq 0} \) is well-defined. More generally, for any \( 0 \neq v \in V \) with \( v = \sum_{\lambda \in \text{Supp}(v)} v_\lambda \), we set \( d^+(v) := \min \{ d^+(v_\lambda) | \lambda \in \text{Supp}(v) \} \). Note that \( d^+(v) = 0 \) if and only if \( \lambda_0 \in \text{Supp}(v) \).

**Lemma 6.2.3.** Let \( 0 \neq v \in V \). If \( d^+(v) > 0 \), there exists \( i \in \hat{I} \) such that \( E_i \cdot v \neq 0 \) and \( d^+(E_i \cdot v) < d^+(v) \).

**Proof.** Set \( \ell := d^+(v) \in \mathbb{Z}_{>0} \). By definition, \( \lambda_0 \notin \text{Supp}(v) \) and there exists \( \lambda \in \text{Supp}(v) \) such that \( d^+(v_\lambda) = \ell \). By Lemma 6.2.2 (2), there exists \( i_\ell \in \hat{I} \) such that \( 0 \neq E_{i_1} \cdots E_{i_\ell} \cdot v_\lambda \in V_{\lambda_0} \). Set \( i = i_\ell \). Then \( d^+(E_i \cdot v_\lambda) = \ell - 1 \). The relation (6.1) and the definition of \( d^+ \) yield \( d^+(E_i \cdot v) < \ell \). \( \square \)

Finally, we obtain the following result of restricted irreducibility.

**Theorem 6.2.4.** Every irreducible finite-dimensional representation of \( U_q \mathfrak{g} \) remains irreducible under restriction to \( \hat{U}^+ \).

**Proof.** We will show that \( \hat{U}^+ \cdot v = V \) for all \( v \in V \setminus \{0\} \). By Lemma 6.2.2 (1), it suffices to show that \( \hat{U}^+ \cdot v \) contains a weight vector. We proceed by induction on \( s := |\text{Supp}(v)| \in \mathbb{Z}_{>0} \). For \( s = 1 \), it is clear. For \( s > 1 \), suppose that there exists \( v \in V \) such that \( |\text{Supp}(v)| = s \) and \( \hat{U}^+ \cdot v \) does not contain a weight vector. Choose one such \( v \) with minimal \( d^+(v) \).

1. Suppose \( d^+(v) = 0 \). Then \( \lambda_0 \in \text{Supp}(v) \). Since \( |\text{Supp}(v)| > 1 \), we may choose \( \mu \in \text{Supp}(v) \) with \( \mu \neq \lambda_0 \). By Lemma 6.2.2 (2) there exists \( i \in \hat{I} \) with \( \ell = d^+(\mu) \) such that \( 0 \neq E_{i_1} \cdots E_{i_\ell} \cdot v_\mu \in V_{\lambda_0} \). Thus, we must have \( \mu + \alpha_{i_1} + \cdots + \alpha_{i_\ell} = \lambda_0 \). Since \( \text{Supp}(V) \subset \lambda_0 - Q_+ \), we obtain \( \alpha_{i_1} + \cdots + \alpha_{i_\ell} \in Q_+ \). Since \( \lambda_0 + \alpha_{i_1} + \cdots + \alpha_{i_\ell} \in \lambda_0 + Q_+ \), we obtain \( \lambda_0 + \alpha_{i_1} + \cdots + \alpha_{i_\ell} \notin \text{Supp}(V) \). Hence \( E_{i_1} \cdots E_{i_\ell} \cdot v_{\lambda_0} = 0 \). Therefore, \( |\text{Supp}(E_{i_1} \cdots E_{i_\ell} \cdot v)| < |\text{Supp}(v)| \).

2. Suppose \( d^+(v) > 0 \). By Lemma 6.2.3 there exists \( i \in \hat{I} \) such that \( E_i \cdot v \neq 0 \) and \( d^+(E_i \cdot v) < d^+(v) \). From the minimality of \( v \) and (6.2) we deduce that \( |\text{Supp}(E_i \cdot v)| < |\text{Supp}(v)| \).

In either case, we obtain an inequality of the form \( |\text{Supp}(E_{i_1} \cdots E_{i_\ell} \cdot v)| < |\text{Supp}(v)| \).

Noting that \( \hat{U}^+ \cdot (E_{i_1} \cdots E_{i_\ell} \cdot v) \leq \hat{U}^+ \cdot v \), we obtain a contradiction. Thus, \( \hat{U}^+ \cdot v \) necessarily contains a weight vector and the result follows. \( \square \)

By applying the Chevalley involution, we get the negative counterpart of the previous result.

**Corollary 6.2.5.** Every irreducible finite-dimensional representation of \( U_q \mathfrak{g} \) remains irreducible under restriction to \( \hat{U}^- \).

6.3. **Proof of Theorem 6.1.1.** We now proceed with the proof of the main result of this section, which relies on the following basic fact.
Lemma 6.3.1. Let $z$ be an indeterminate and $V$ a finite-dimensional $\mathbb{F}$-linear space. Let $\mathcal{M}(z) := \{M_i(z)\}_{i \in \hat{I}}$ be a finite set of endomorphisms of $V(z) := V \otimes \mathbb{F}(z)$ depending polynomially on $z$ and set $\mathcal{M} := \{M_i(0)\}_{i \in \hat{I}} \subset \text{End}(V)$. If $V$ has no nontrivial proper $\mathcal{M}$-invariant subspace, then $V(z)$ has no nontrivial proper $\mathcal{M}(z)$-invariant subspace.

Proof. Let $\text{ev}_0 : V[z] := V \otimes \mathbb{F}[z] \to V$ be the evaluation at $0$. Let $S(z) \neq \{0\}$ be an $\mathcal{M}(z)$-invariant subspace of $V(z)$ and set $S[z] := S(z) \cap V[z]$. For any $s(z) \in S[z]$ and $i \in \hat{I}$, we have $M_i(z) \cdot s(z) \in S[z]$. It follows that $\text{ev}_0(S[z])$ is a nontrivial $\mathcal{M}$-invariant subspace of $V$ and therefore $\text{ev}_0(S[z]) = V$. By extension of scalars, we get $S(z) = V(z)$. \qed

We can now complete the proof of Theorem 6.1.1. Since $\tilde{F}_i - F_i \in \tilde{U}^{\geq 0} \langle i \in \hat{I} \rangle$, the action of $z\tilde{F}_i \in U_qLg(z)$ on $V(z)$ is polynomial in $z$. Let $M_i(z)$ denote this action and note that $M_i(0)$ is equal to the action of $zF_i$, which descends to an action on $V$. By Corollary 6.2.5 and Lemma 6.3.1, it follows that $V(z)$ has no nontrivial proper $\{M_i(z)\}_{i \in \hat{I}}$-invariant subspace. Since $z$ is invertible in $\mathbb{F}(z)$, we replace $z\tilde{F}_i$ by $\tilde{F}_i$ and the result follows.

7. Solutions of reflection equations

In this section, we apply the results from Section 5 to produce new solutions of the two most common forms of the generalized reflection equation, the standard and transposed reflection equation. These solutions arise from distinguished trigonometric K-matrices on small modules and Kirillov-Reshetikhin modules.

7.1. The standard and the transposed reflection equations. Let $V$ and $W$ be two irreducible finite-dimensional $U_qLg$-modules and denote by $R_{VW}(z)$ the trigonometric R-matrix from Theorem 2.7.1. On the tensor product $V(z) \otimes W(w)$ we consider the standard reflection equation

$$R_{VW}(\frac{z}{z})_{21} \cdot \text{id}_V \otimes K_W(w) \cdot R_{VW}(zw) \cdot K_V(z) \otimes \text{id}_W =$$

$$= K_V(z) \otimes \text{id}_W \cdot R_{VW}(zw)_{21} \cdot \text{id}_V \otimes K_W(w) \cdot R_{VW}(\frac{w}{z})$$

(7.1)

where $K_V(z)$ and $K_W(w)$ are unknown operator-valued rational functions, and the transposed reflection equation

$$R_{VW}(\frac{z}{z})_{21}^{1} \cdot \text{id}_V \otimes \tilde{K}_W(w) \cdot (R_{VW}(zw)_{21}^{1})^{1} \cdot \tilde{K}_V(z) \otimes \text{id}_W =$$

$$= \tilde{K}_V(z) \otimes \text{id}_W \cdot (R_{VW}(zw)_{21}^{1})^{1} \cdot \text{id}_V \otimes \tilde{K}_W(w) \cdot R_{VW}(\frac{w}{z})$$

(7.2)

where $\tilde{K}_V(z)$ and $\tilde{K}_W(w)$ are unknown operator-valued rational functions and $t_V$ and $t_W$ denote the transposition on the corresponding component.

7.2. Small modules. A finite-dimensional $U_qLg$-module is small if it remains irreducible under restriction to $U_qg \subset U_qLg$, i.e., it is an irreducible finite-dimensional $U_qg$-module, whose action extends to one of $U_qLg$. In type $A$, a $U_qLsl_N$-module is small if and only if it is an evaluation representation, cf. [CP94b]. In other types, an exhaustive description of small modules is not known. However, the case of fundamental representations is studied in detail in [CP94a, Thm. 6.8]. Indeed, in this case we have the following sufficient condition. Recall that $a_i$ indicates the multiplicity of the simple root $\alpha_i$ in the highest root $\vartheta$ of $g$. 

Lemma 7.2.1. Let \( i \in I \). If \( \alpha_i = 1 \) or \( \alpha_i = (\vartheta, \vartheta) / (\alpha_i, \alpha_i) \), the action of \( U_q \mathfrak{g} \) on the fundamental representation \( V_{\omega_i} \) extends to an action of \( U_q \mathfrak{g} \).

Proof. By [CP95, Prop. 12.1.17], under the same hypothesis the action of \( \mathfrak{g} \) on its fundamental representation extends to an action of the Yangian. By [GTL16], the same result applies to fundamental representations of \( U_q \mathfrak{g} \).

For classical types, this result applies to every fundamental representation in type \( A \) and \( C \), the vector and the spin representations in types \( B \) and \( D \). For exceptional types, in terms of the labelling conventions of [Bon68], it applies to the fundamental representations \( V_{\omega_i} \) and \( \nu_v \) for \( E_6 \), \( V_{\omega_7} \) for \( E_7 \), \( V_{\omega_4} \) for \( F_4 \), and \( V_{\omega_2} \) for \( G_2 \).

7.3. Amenable representations. Let \( V \) be a small finite-dimensional \( U_q \mathfrak{g} \)-module. By Jacobson’s density theorem, the defining data of \( V \) as a representation over \( U_q \mathfrak{g} \) can be expressed entirely in terms of \( U_q \mathfrak{g} \). Thus, there exist \( u_0^\pm \in U_q \mathfrak{g} \) such that \( \pi_V(x_0^\pm) = \pi_V(u_0^\pm) \), where \( x_0 := E_0 \) and \( x_0 := F_0 \). In the following we shall consider small representations where the elements \( u_0^\pm \) have a prescribed expression.

Let \( x_0^\pm \in U_q \mathfrak{h}^\pm \) be a choice of root vectors corresponding to the highest root \( \vartheta \) such that \( \omega(x_0^\pm) = -x_0^\pm \). Note that \( \omega_i(x_0^\pm) = \epsilon x_0^\pm \) for some \( \epsilon \in \{ \pm 1 \} \).

Definition 7.3.1. A small finite-dimensional \( U_q \mathfrak{g} \)-module \( V \) is amenable if

\[
\pi_V(x_0^\pm) = \lambda_\pm \pi_V(x_0^\mp)
\]

for some \( \lambda_\pm \in \mathbb{F}^\times \).

Remark 7.3.2. In type \( A \), every evaluation representation is amenable. In the other classical types, one checks by direct inspection that the vector representation is amenable. More generally, we expect that every small finite-dimensional \( U_q \mathfrak{g} \)-module is amenable.

7.4. The twisting operator \( \psi_0 \). Let \( (X, \tau) \in \text{GSat}(\hat{A}) \) be an affine generalized Satake diagram with pseudo-involution \( \theta, (\gamma, \sigma) \in \Gamma \times \Sigma \), and \( U_q \mathfrak{t} \subset U_q \mathfrak{g} \) the corresponding QSP subalgebra. We consider the subdiagram \( Y_0 \subset \hat{I} \) and the automorphism \( \eta_0 : \hat{I} \to \hat{I} \) given by

\[
Y_0 := \hat{I} \setminus \{ 0, \tau(0) \}, \quad \eta_0(0) = \tau(0), \quad \text{and} \quad \eta_0|_{Y_0} = \omega_0|_{Y_0}.
\]

By casework, \( \eta_0 \) is a diagram automorphism and \( (Y_0, \eta_0) \) is a generalized Satake diagram. Note also that \( \eta_0 \) and \( \tau \) commute. Finally, let \( \zeta_0 \) be the pseudo-involution corresponding to \( (Y_0, \eta_0) \) and choose a QSP admissible twisting operator (cf. Section 4.2)

\[
\psi_0 = \text{Ad}(g_0) \circ \theta_q^{-1}
\]

where \( g_0 \in G_{\theta, \gamma} \) has the form \( g_0 := S_{\gamma_0}^{-1} S_\theta \beta^{-1} \) for some \( \beta : \mathbb{P} \to \mathbb{F}^\times \).

7.5. \( \tau \)-restrictable QSP subalgebras. An affine QSP subalgebra \( U_q \mathfrak{t} \subset U_q \mathfrak{g} \) is \( \tau \)-restrictable if \( \tau(0) = 0 \). In this case, the \( \tau \)-minimal grading shift coincides with the homogeneous grading shift and \( Y_0 \) and \( \eta_0 \) are independent of \( \tau \), i.e., \( Y_0 = I \) and \( \eta_0 \) is the unique affine extension of the opposition involution \( \omega_i \) on the finite Dynkin diagram. Moreover, in (7.3) we choose \( \beta \) such that \( \beta(\alpha_i) = 1 \) if \( i \neq 0 \) and \( \beta(\alpha_0) = \gamma^{-1}(\delta) \).
Lemma 7.5.1. Let $U_q \mathfrak{k} \subset U_q \mathfrak{g}$ be a $\tau$-restrictable affine QSP subalgebra and $V$ a small amenable finite-dimensional $U_q \mathfrak{g}$-module. There exists unique $c_V \in \mathbb{F}^X$ such that $V^{\psi_0} = V^{\eta_0 \tau}(c_V)$.

Proof. First, we consider the case $\gamma(\delta) = 1$. Since $U_q \mathfrak{k}$ is $\tau$-restrictable, we have $Y_0 = I$ and $\eta_0|_I = \text{oi}_I$. Therefore, $\zeta_{0,q}$ is the identity on $U_q \mathfrak{g}$ and $V^{\psi_0} = V^{\eta_0 \tau}$ as $U_q \mathfrak{g}$-modules. Since $V$ is small, there exist $u_0^\pm \in U_q \mathfrak{g}$ such that $\pi_V(x_0^\pm) = \pi_V(u_0^\pm)$, where as before $x_0^+ := E_0$ and $x_0^- := F_0$. By construction, $\psi_0 = \text{Ad}(S_{c_0}^{-1}) \circ \omega \circ \tau$ with $S_{c_0} = \xi_{c_0} \cdot S_{Y_0}$. Since $S_{Y_0}$ is supported on $U_q \mathfrak{g}$ and $V$ is a $P$-weight module, it follows that

$$\pi_V(\psi_0(x_0^\pm)) = -\pi_V(S_{c_0}^{-1}) \cdot \pi_V(x_0^\pm) \cdot \pi_V(S_{c_0}) = -\pi_V(\Ad(S_{c_0}^{-1})(u_0^\pm)).$$

Since $\text{Ad}(S_{c_0}^{-1})$ acts on $U_q \mathfrak{g}$ as $\omega \circ \text{oi}_I$, we obtain $\pi_V(\psi_0(x_0^\pm)) = -\pi_V((\omega \circ \text{oi}_I)(u_0^\pm))$. Since $V$ is amenable, $u_0^\pm = \lambda_{\pm} x_0^\pm$ for some $\lambda_{\pm} \in \mathbb{F}^X$. Therefore, by setting $c := e\lambda_- / \lambda_+$, we get $-(\omega \circ \text{oi}_I)(u_0^\pm) = c^{\pm 1} u_0^\pm$. Finally this yields $\pi_V(\psi_0(x_0^\pm)) = c^{\pm 1} \pi_V(x_0^\pm)$ and therefore $V^{\psi_0} = V^{\eta_0 \tau}(c)$. Indeed, since the composition $\eta_0 \tau$ fixes the node 0, the identification holds for $E_0$ and $F_0$. The proof for $K_0$ is analogous and relies on the fact that $\pi_V(K_0) = \pi_V(K_{0}^{-1})$. The case $\gamma(\delta) \neq 1$ is similar and its proof is omitted for simplicity.

Remarks 7.5.2.

1. The factor $c_V \in \mathbb{F}^X$ determined by the identity $V^{\psi_0} = V^{\eta_0 \tau}(c_V)$ can be removed through a suitable shift of the representation $V$. Namely, proceeding as in [AV22, Sec. 9], one shows that there exists a unique $c \in \mathbb{F}^X$ (up to a sign) such that the shifted representation $V_c := V(c)$ satisfies $V_c^{\psi_0} = V^{\eta_0 \tau}$. Therefore, up to a uniquely determined shift, for any small amenable finite-dimensional $U_q \mathfrak{g}$-module $V$, one has $V^{\psi_0} = V^{\eta_0 \tau}.$

2. An affine QSP subalgebra $U_q \mathfrak{k} \subset U_q \mathfrak{g}$ is restrictable if it is $\tau$-restrictable and $0 \notin X$. By Corollary 5.1.3 the universal K-matrix associated to $\psi_0$ descends on any finite-dimensional $U_q \mathfrak{g}$-module $V$ to a formal operator $K_{\psi_0,V}(z) \in \text{End}(V)[[z]]$. Moreover, in analogy with the case of the R-matrix, it follows by the uniqueness of the quasi-K-matrix that $K_{\psi_0,V}(0)$ corresponds to the action on $V$ of the universal K-matrix associated to $U_q \mathfrak{g}$ considered in [BK19].

7.6. Solutions of the diagrammatic reflection equation. In the case of $\tau$-restrictable QSP subalgebras, the trigonometric K-matrices constructed in Theorem 5.1.2 on small amenable modules are solutions of a diagrammatic reflection equation.

Theorem 7.6.1. Let $U_q \mathfrak{k} \subset U_q \mathfrak{g}$ be a $\tau$-restrictable affine QSP subalgebra.

1. Let $V$ be a small amenable finite-dimensional $U_q \mathfrak{g}$-module. Up to a uniquely determined shift in $V$ (cf. Remark 7.5.2 (1)), there exists (up to a scalar multiple) a unique QSP intertwiner

$$K_V(z) : V(z) \to V^{\eta_0 \tau}(z^{-1})$$

(7.4)

2. Let $V,W$ be two small amenable finite-dimensional $U_q \mathfrak{g}$-module. The operators $K_V(z)$, $K_W(w)$ satisfy the diagrammatic reflection equation

$$R_{WW}(\frac{z}{w})_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}^{\eta_0 \tau}(zw) \cdot K_V(z) \otimes \text{id} = K_V(z) \otimes \text{id} \cdot R_{WV}^{\eta_0 \tau}(zw)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{WW}(\frac{w}{z}).$$

(7.5)
Proof. It is enough to observe that, by Lemma 7.5.1 and Remark 7.5.2 (1), $V^\psi_0 = V^\eta_0 \tau$ (up to a suitable shift in $V$). Then the trigonometric K-matrix $K_{\psi_0, V}(z)$ constructed in Theorem 5.1.2, provides the desired intertwiner. Moreover, the generalized reflection equation satisfied by $K_{\psi_0, V}(z)$ and $K_{\psi_0, W}(w)$ on $V(z) \otimes W(w)$ reduces to (7.5). □

Remark 7.6.2. The same result applies for any irreducible finite-dimensional representation $V$ (not necessary small) equipped with a distinguished isomorphism $V^\psi_0 \cong V^\eta_0 \tau$.

In 7.7-7.8, we shall discuss several refinements of Theorem 7.6.1.

7.7. Solutions of the standard and the transposed reflection equations. Solutions of (7.1) and (7.2) on amenable modules are obtained as an immediate consequence of Theorem 7.6.1.

Theorem 7.7.1. Let $U_q \alpha \subset U_q L g$ be an affine $\tau$-restrictable QSP subalgebra and $V, W$ small amenable finite-dimensional $U_q L g$-modules.

(1) If $\tau = \eta_0$, up to a uniquely determined shift in $V$, there exists (up to a scalar multiple) a unique QSP intertwiner

$$K_V(z) : V(z) \rightarrow V(z^{-1})$$

Moreover, $K_V(z)$ is a solution of the standard reflection equation (7.1).

(2) If $\tau = \text{id}$, up to a uniquely determined shift in $V$, there exists (up to a scalar multiple) a unique QSP intertwiner

$$K_V(z) : V(z) \rightarrow V^*(z^{-1})$$

Moreover, under the identification of $V$ and $V^*$ as vector spaces, $K_V(z)$ is a solution of the transposed reflection equation (7.2).

Proof. (1) The result follows immediately from Theorem 7.6.1 and equation (7.5).

(2) By [Cha02, Sec. 2] there exists an integer $c \in \mathbb{Z}$ depending only on $g$ such that, for any irreducible finite-dimensional $U_q L g$-module $W$, one has $W^\psi_0 \cong W^*(q^c)$. Then, proceeding as in Theorem 7.6.1 and relying on the latter identification, we obtain a QSP intertwiner $K_V(z) : V(z) \rightarrow V^*(z^{-1})$. Since in any quasitriangular Hopf algebra with universal R-matrix $R$ one has $S \otimes \text{id}(R) = R^{-1}$, the equation (7.5) reduces to the transposed reflection equation (7.2) through the identification of $V^*$ and $V$ as vector spaces. □

From the classification of generalized Satake diagrams of affine type in [RV22, App. A, Tables 5, 6 and 7] it follows that a QSP subalgebra is $\tau$-restrictable if and only if $\tau$ is either the identity or $\eta_0$, except in type $D_n^{(1)}$ with $n$ even (where $\eta_0 = \text{id}$ but there exist nontrivial involutive diagram automorphisms fixing 0). In [RV16], solutions of the standard and transposed reflection equations on the vector representation of quantum loop algebras of classical Lie type are computed in terms of QSP intertwiners. Hence Theorem 7.7.1 implies the following result.

Corollary 7.7.2. Every explicit solution constructed in [RV16] for $\tau$-restrictable QSP subalgebras (except in type $D_n^{(1)}$ with $n$ even and $\tau \neq \text{id}$) arises from the action of a universal K-matrix.
7.8. Trigonometric K-matrices on Kirillov-Reshetikhin modules. Kirillov-Reshetikhin modules \([\text{KR87}]\) are minimal affinizations of irreducible \(U_q\mathfrak{g}\)-modules of highest weight a multiple of a fundamental weight, see, e.g., \([\text{Her06}]\) and references therein. More precisely, for any \(i \in I, k \in \mathbb{Z}_{\geq 0}, \) and \(a \in \mathbb{C}^\times, \) the Kirillov-Reshetikhin module \(W^{(i)}_{k,a}\) is the unique irreducible \(U_q\mathfrak{g}\)-module whose Drinfeld polynomials are all trivial except for the node \(i, \) where the roots are given by a \(q_i\)-string of length \(k\) starting at \(a, \) cf. \([\text{CP94a}]\). For these modules, we obtain the following analogue of Theorems 7.6.1 and 7.7.1.

**Theorem 7.8.1.** Let \(U_q\mathfrak{t} \subset U_q\mathfrak{g}\) be an affine \(\tau\)-restrictable QSP subalgebra and \(W\) a Kirillov-Reshetikhin \(U_q\mathfrak{g}\)-module.

1. Up to a uniquely determined shift in \(W, \) there exists (up to a scalar multiple) a unique QSP intertwiner

\[ K_W(z) : W(z) \to W^{\omega \tau}(z^{-1}) \]

Moreover, \(K_W(z)\) is a solution of the diagrammatic reflection equation (7.5).

2. If \(\tau = \eta_0,\) up to a uniquely determined shift in \(W, \) there exists (up to a scalar multiple) a unique QSP intertwiner

\[ K_W(z) : W(z) \to W(z^{-1}) \]

Moreover, \(K_W(z)\) is a solution of the standard reflection equation (7.1).

3. If \(\tau = \text{id}, \) up to a uniquely determined shift in \(W, \) there exists (up to a scalar multiple) a unique QSP intertwiner

\[ K_W(z) : W(z) \to W^*(z^{-1}) \]

Moreover, under the identification of \(W\) and \(W^*\) as vector spaces, \(K_W(z)\) is a solution of the transposed reflection equation (7.2).

**Proof.** (2) and (3) follow from (1) as in Theorem 7.7.1. For (1), it is enough to observe that, if \(W = W^{(i)}_{k,a},\) then

\[ W^\omega \cong W^{m_0}(a^{-2}q_i^{-2(k-1)}) \]

By Theorem 5.1.2, the semi-standard K-matrix (cf. Example 3.6.3 (2)) provides a QSP intertwiner \(W(z) \to W^{\omega \tau}(z^{-1}),\) which solves the generalized reflection equation. We compose it with the isomorphism (7.6) and proceed as in Theorem 7.6.1 to obtain the desired QSP intertwiner \(K_W(z)\) satisfying (7.5). \(\square\)

In the case of quasi-split QSP subalgebras of affine type A where \(\tau = \text{id}\) (type A.1 in the classification of \([\text{RV16}]\), \(\tau\) is the half-turn rotation of the Dynkin diagram (type A.4) or \(\tau = \eta_0\) (types A.3a and A.3b), a combinatorial formula for the trigonometric K-matrices (2) and (3) above has been obtained in \([\text{KOW22}]\). Theorem 7.8.1 now implies the following result.

**Corollary 7.8.2.** Every explicit solution in \([\text{KOW22}]\) for QSP subalgebras of type A.1 and A.3 arises from the action of a universal K-matrix.
7.9. The q-Onsager algebra and the vector representation. We conclude with the explicit formula of the trigonometric K-matrix $K(z)$ arising from the q-Onsager algebra $U_q\mathfrak{sl}_2$ on the vector representation $V$ of $U_q\mathfrak{sl}_2$, which first appeared in [DVGR]. In particular, we observe that the poles of $K(z)$ completely detect the irreducibility of $V$ under restriction to $U_q\mathfrak{sl}_2$, only for a generic choice of the QSP parameters (cf. Propositions 5.4.1, 5.5.1). More specifically, the denominator in $K(z)$ on $V$ is generically a polynomial of degree 2, but for QSP parameters satisfying specific closed conditions it reduces to a linear and even constant polynomial. In the latter cases, one checks that there are values $\zeta \in \mathbb{F}^\times$ such that $V(\zeta)$ is reducible and the operator $K(\zeta)$ is well-defined.

Let $V := V_{\zeta_1}(q)$ be the 2-dimensional representation with action given by

$$
\pi(K_0) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} = \pi(K_1)^{-1},
$$

$$
\pi(E_0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \pi(F_1) \quad \text{and} \quad \pi(F_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \pi(E_1),
$$

cf. Remark 2.7.2. Let $U_q\mathfrak{sl}_2 \subset U_q\hat{\mathfrak{sl}}_2$ be the q-Onsager algebra from Example 3.4.4 with QSP parameters $\gamma_0, \gamma_1 \in \mathbb{F}^\times$, $\sigma_0, \sigma_1 \in \mathbb{F}$. Since $V$ is amenable and $\tau = \text{id} = \eta_0$, Theorem 7.6.1 applies and yields a rational QSP intertwiner

$$
K(z) : V(\kappa z) \to V(\kappa/z)
$$

where $\kappa^{-2} = \gamma_0 \gamma_1$ (see also [AV22, Cor. 9.2]). This satisfies the equation (7.5), which in this particular case reduces to the standard reflection equation (7.1).

Up to a scalar, $K(z)$ can be obtained by computing directly a polynomial non-vanishing solution of the intertwining equation (7.4), relying on the explicit formulas of the QSP generators (3.1), and by further imposing the unitarity condition $K(z)^{-1} = K(z^{-1})$. For convenience, we replace the QSP parameters by choosing $\lambda, \mu, \nu \in \mathbb{F}^\times$ such that

$$
\gamma_0 = \kappa^{-2} \nu^2, \quad \gamma_1 = \nu^{-2}, \quad \sigma_0 = \kappa^{-1} \nu \frac{\mu + \mu^{-1}}{q - q^{-1}}, \quad \sigma_1 = -\nu^{-1} \lambda + \lambda^{-1} q - q^{-1}.
$$

(1) If $\lambda \mu^{\pm 1} \neq \pm 1$, we get

$$
K(z) = \frac{\lambda \mu}{(\lambda \mu + z)(\lambda + \mu z)} \begin{pmatrix}
\mu & \frac{1}{\mu} \\
\frac{1}{\mu} & \mu
\end{pmatrix} z + \left(\lambda + \frac{1}{\lambda}\right) z^2
\begin{pmatrix}
-\frac{1}{\nu} & (z^2 - 1) \\
\nu & (z^2 - 1)
\end{pmatrix}
\begin{pmatrix}
\mu & \frac{1}{\mu} \\
\frac{1}{\mu} & \mu
\end{pmatrix} z + \lambda + \frac{1}{\lambda}
\end{pmatrix}
\begin{pmatrix}
\mu & \frac{1}{\mu} \\
\frac{1}{\mu} & \mu
\end{pmatrix} z + \lambda + \frac{1}{\lambda}
$$

In particular, $K(\pm 1) = \text{id}$.

(2) If $\lambda \neq \pm i$ and $\lambda \mu^{\pm 1} = \varepsilon$ with $\varepsilon \in \{\pm 1\}$, then

$$
K(z) = \frac{1}{\varepsilon \lambda + \frac{1}{\lambda} z} \begin{pmatrix}
(\lambda + \frac{1}{\lambda}) & -\frac{1}{\nu} (z - \varepsilon) \\
\nu (z - \varepsilon) & \varepsilon (\lambda + \frac{1}{\lambda})
\end{pmatrix}
\begin{pmatrix}
(\lambda + \frac{1}{\lambda}) & -\frac{1}{\nu} (z - \varepsilon) \\
\nu (z - \varepsilon) & \varepsilon (\lambda + \frac{1}{\lambda})
\end{pmatrix}
$$
In particular, $K(\varepsilon) = \text{id}$. Note however that $V(-\varepsilon \kappa)$ is reducible. If $\lambda = \pm 1$, $K(z)$ has a pole at $z = -\varepsilon$, which therefore detects the reducibility of $V(-\varepsilon \kappa)$. On the other hand, if $\lambda \neq \pm 1$, $K(-\varepsilon)$ is still well-defined.

(3) Finally, if $\lambda = \pm i$ and $\lambda \mu^{\pm 1} = \varepsilon$ with $\varepsilon \in \{\pm 1\}$, then

$$K(z) = \begin{pmatrix} 0 & -\frac{1}{\mu} \\ \nu & 0 \end{pmatrix}$$

In this case $\sigma_0 = 0 = \sigma_1$ and $V(\pm \kappa)$ is reducible. However, $K(z)$ is constant and always well-defined.

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