MULTIPLE NONSMOOTH SOLUTIONS FOR NONCONVEX VARIATIONAL BOUNDARY VALUE PROBLEMS IN $\mathbb{R}^n$

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Abstract. This paper presents a set of complete solutions of a nonconvex variational problem with a double-well potential. Based on the canonical duality-traility theory, the associated nonlinear differential equation with either Dirichlet/Neumann or mixed boundary conditions can be converted into an algebraic equation, which can be solved analytically to obtain all solutions in the dual space. Both global and local extremality conditions are identified by the triality theory. In the application part, typical mechanical models with specific sources and boundary conditions in $\mathbb{R}^2$ are exhibited.

1. Problem formulation and motivations

Our goal of this paper is to solve the following nonconvex variational problem

$$ (P_n) : \min_{u \in U} \left\{ P_n(u) := \int_{\Omega} W(\nabla u) \, dx - \int_{\Omega} f \, u \, dx - \int_{\Gamma_t} t \, u \, d\Gamma \right\}, $$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded and simply connected domain with sufficiently smooth boundary $\partial \Omega = \Gamma = \Gamma_u \cup \Gamma_t$; On $\Gamma_t$, the Neumann force (or surface traction) $t$ is given, while on $\Gamma_u$, the Dirichlet boundary condition is prescribed; the source term $f$ can be viewed as the distributed defects in phase transitions. The nonconvex function $W$ is a fourth-order polynomial defined as

$$ W(\overrightarrow{y}) := \nu/2 \left( 1/2 |\overrightarrow{y}|^2 - \lambda \right)^2, $$

where $\overrightarrow{y} \in \mathbb{R}^n$, $\nu, \lambda > 0$ are given constants. This function is the so-called double-well potential in phase transitions, or the Mexican-hat in quantum mechanics. The double-well potential was first studied by Van der Waals in 1893 for a compressible fluid whose free energy at constant temperature depends not only on the density, but also on the density gradient (see [23]). Since then, this nonconvex function has been found extensive applications in nonlinear sciences. For examples, in phase transitions of Ericksen’s bar [5], or the mathematical theory of super-conductivity [15, 18], $W$ is the

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where partial differential equation in $\Omega$ and natural boundary condition on $\Gamma$ according to the trace theory in [21]. In economics, many existence results for Nash Equilibrium (NE) points of non-cooperative games are concerned with this kind of nonconvex problems. It was discovered in the context of post-buckling analysis [6, 7] that the stored potential energy of a large deformed beam model is also a double-well function, where each potential-well represents a possible buckled beam state, and the local maximizer is corresponding to the unbuckled state. Additionally, the nonconvex function $W$ also plays fundamental roles in cosmology [20], mathematical economics [3, 4, 5], chaotic dynamics [24], finite deformation mechanics [8, 14, 15], and much more (see review articles [7, 11]).

Our goal is to find all possible analytical solutions of the nonconvex variational problem (1). We let $p, q, \beta \in [2, \infty]$ and much more (see review articles [7, 11]).

For our purpose, we restrict $p, q, \beta \in [2, \infty]$ and choose the subset $\mathcal{U}_0$ of the function space $\mathcal{U}$ as the feasible function space (let $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$),

$$\mathcal{U}_0 := \left\{ u \in \mathcal{U} \mid \Delta u \in L^{2p'/(2-p')}(\Omega), \ (u, \frac{\partial u}{\partial n})|_{\Gamma_t} \in L^q(\Gamma_t \times L^\beta(\Gamma_t)) \right\}.$$ 

Here, $\Delta u \in L^\infty(\Omega)$ when $p = 2$. It is evident that the above subset is not empty since $C^m(\Omega) \subset \mathcal{U}_0$, $m = 2, 3, \ldots$.

**Remark 1.1.** Indeed, the above $\mathcal{U}$ is a special Sobolev space equipped with the norm

$$\|u\|_{\mathcal{U}} := \|u\|_{L^p} + \left( \sum_{i=1}^{n} \|u_{x_i}\|_{L^4}^2 \right)^{1/2}.$$ 

Clearly, $\mathcal{U}$ is a reflexive Banach space when $p \in (1, \infty)$. If $p = 4$, then $\mathcal{U}$ is equivalent to $W^{1,4}(\Omega)$ (see [1]). Moreover, when $p \in [1, 4]$, $L^4(\Omega)$ is continuously embedded in $L^p(\Omega)$ since $\Omega$ is bounded. The smoothness of $\Gamma_u \cup \Gamma_t$ assures the regularity on the boundary. For instance, when $p = 4$, $\Gamma_u \cup \Gamma_t \in C^1$, then the trace $\gamma u \in W^{3/4,4}(\Gamma_u \cup \Gamma_t)$ according to the trace theory in [21].

Therefore, the criticality condition $DP_n(u) = 0$ leads to the following nonlinear partial differential equation in $\Omega$ and natural boundary condition on $\Gamma_t$:

$$\begin{align*}
\nabla \cdot \mathbf{\sigma}(\nabla u) &= f \text{ in } \Omega, \\
\mathbf{n} \cdot \mathbf{\sigma}(\nabla u) &= t \text{ on } \Gamma_t,
\end{align*}$$

where

$$\mathbf{\sigma} = (\sigma_1, \ldots, \sigma_n) = \nu(1/2|\mathbf{\gamma}|^2 - \lambda)\mathbf{\gamma},$$

and $\mathbf{n}$ stands for the unit outward normal vector. Actually,

$$\mathbf{\sigma} \in \left\{ \mathbf{\gamma} \in (L^{4/3}(\Omega))^n \mid \nabla \cdot \mathbf{\gamma} \in L^{p'}(\Omega), \ \mathbf{\gamma} \cdot \mathbf{n} \in L^{q'}(\Gamma_t) \right\}.$$
Due to the nonconvexity of $W(\nabla u)$, for given parameter $\lambda > 0$, the source $f \in L^p(\Omega)$ and boundary term $t \in L^q(\Gamma_t)$, the nonlinear differential equation (4) may have multiple solutions at each material point $x \in \Omega$. Since the domain $\Omega$ is continuous, therefore, the boundary value problem (4) could have infinite number of solutions. Each of these solutions is a critical point of $P_n(u)$, i.e. it could be either an extremum or a saddle point of the total potential. This phenomenon has been verified by Ericksen who proved that many local solutions are metastable and may have arbitrary number of phase interfaces. Compared with convex problems, a fundamentally different issue in nonconvex analysis is that the solutions of the boundary-value problem is not equivalent to the associated minimum variational problem. By the fact that the second-order condition $\delta^2 P_n(\bar{u}) \geq 0$ is only a necessary condition for identifying the global minimal solutions, it is well-known that traditional direct approaches for solving the nonconvex variational problem ($P_n$) are fundamentally difficult. Actually, even in the case of finite dimensional space, many nonconvex global optimization problems are considered to be NP-hard.

The purpose of this paper is to solve the challenging nonconvex minimization problem ($P_n$) by using the canonical duality theory. This is a methodological theory which can be used for solving a large class of nonconvex/nonsmooth/discrete problems in multidisciplinary fields, including mathematical physics, global optimization, computational science, industrial and systems engineering, etc. [11, 12, 16]. The canonical duality theory has been used successfully by D. Y. Gao and R. W. Ogden for 1D problems in finite deformation mechanics [13] and phase transitions of the Ericksen bar [14]. Their work showed that by using the canonical duality theory, the nonlinear ordinary differential equations can be converted into algebraic equations which can be solved completely to obtain all possible solutions. Both global and local extrema can be identified by the triality theory. They discovered that for certain given external loads, the global minimizer is nonsmooth and cannot be determined by any Newton-type numerical methods.

The rest of the paper is organized as follows. In Section 2, we apply the nonlinear canonical dual transformation to establish the perfect dual problem and a pure complementary energy principle for ($P_n$). The triality theory provides both global and local extremality conditions for the nonconvex problem. A set of complete solutions for ($P_n$) is given and the existence of analytical solutions of the corresponding boundary value problems is also discussed. Finally, applications in 2D are illustrated in Section 3.

2. Canonical duality techniques and main results

By the fact that the linear operator $\nabla$ cannot change the nonconvexity of the double-well function $W(\nabla u)$, instead, we use the following geometrically nonlinear measure [13]

$$\xi := \Lambda(u) = 1/2|\nabla u|^2 : U \rightarrow E \subset L^2(\Omega),$$

where

$$E := \{\xi \in L^2(\Omega) \mid \xi \geq 0\}. $$
Thus, in terms of this nonlinear measure, the nonconvex function \( W(\nabla u) \) can be written in the so-called canonical form \( W(\nabla u) = U(\Lambda(u)) \), where
\[
U(\xi) := \nu / 2(\xi - \lambda)^2,
\]
which is a convex function with respect to \( \xi \). Therefore, the canonical dual stress
\[
\zeta = DU(\xi) = \nu(\xi - \lambda)
\]
is well defined and belongs to
\[
E := \left\{ \zeta \in L^2(\Omega) \mid \zeta \geq -\nu \lambda, \nabla \zeta \in (L^{3\nu/(4-\nu)})(\Omega) \right\},
\]
where
\[
L^{3\nu/(4-\nu)}(\Omega) := L^{\beta(3\nu/(4-\nu))}(\Omega).
\]
By the Legendre transformation, the complementary energy function \( U^*(\zeta) \) can be obtained by
\[
U^*(\zeta) = \xi \zeta - U(\xi) = \zeta^2/(2\nu) + \lambda \zeta.
\]
Replacing \( W(\nabla u) = U(\Lambda(u)) \) in \((P_n)\) by \( \Lambda(u)\zeta - U^*(\zeta) \), we obtain the Gao-Strang total complementary energy \( \Xi(u, \zeta) \) in the form
\[
\Xi(u, \zeta) := \int_{\Omega} \{\Lambda(u)\zeta - U^*(\zeta) - fu\} \, dx - \int_{\Gamma_t} tu \, d\Gamma.
\]
Next, we introduce the following criticality condition.

**Definition 2.1.** \((\bar{u}, \bar{\zeta}) \in U_0 \times E\) is said to be a critical point of \( \Xi(u, \zeta) \) if and only if
\[
D_u \Xi(\bar{u}, \bar{\zeta}) = 0
\]
and
\[
D_\zeta \Xi(\bar{u}, \bar{\zeta}) = 0,
\]
where \(D_u, D_\zeta\) denote the partial Gâteaux derivatives, respectively.

For a fixed \( \zeta \in E \), (6) leads to the equilibrium equation
\[
\begin{cases}
\nabla \cdot (\zeta \nabla \bar{u}) + f = 0 & \text{in } \Omega, \\
(\zeta \nabla \bar{u}) \cdot \vec{n} = t & \text{on } \Gamma_t.
\end{cases}
\]
In particular, according to Hölder’s inequality, \( \|\zeta \nabla u\|_{L^{3\nu/(4-\nu)}(\Omega)} \leq \|\zeta\|_{L^2(\Omega)} \|\nabla u\|_{L^4(\Omega)} \). While for a fixed \( u \in U_0 \), (7) is consistent with the constitutive law
\[
\Lambda(u) = DU^*(\bar{\zeta}).
\]
Next we consider the pure complementary energy functional
\[
P_n^d(\zeta) := \Xi(\bar{u}, \zeta),
\]
where $\bar{u}$ is a solution of BVP (8). By applying Green’s formula, $\Xi(u, \zeta)$ can be rewritten as
\begin{equation}
\Xi(u, \zeta) = \int_{\Omega} \left\{ (1/2|\nabla u|^2 - \lambda)\zeta - U^*(\zeta) - fu \right\} dx - \int_{\Gamma_t} t u d\Gamma
\end{equation}
\begin{align}
&= \int_{\Omega} \left\{ |\nabla u|^2 \zeta - \lambda \zeta - U^*(\zeta) - fu \right\} dx - \int_{\Omega} 1/2|\nabla u|^2 \zeta dx - \int_{\Gamma_t} t u d\Gamma \\
&= \int_{\Gamma_t} \left\{ \sigma \cdot \bar{n} - t \right\} u d\Gamma - \int_{\Omega} \left\{ \nabla \cdot f + \nu \right\} u dx - \int_{\Omega} \left\{ 1/2|\nabla u|^2 \zeta + \lambda \zeta + U^*(\zeta) \right\} dx,
\end{align}
where $\sigma = \zeta \nabla u$. Therefore, if $\bar{u}$ solves BVP (8), then the pure complementary energy functional is in fact
\begin{equation}
P_n^d(\zeta) = -1/2 \int_{\Omega} \left( |\sigma|^2 / \zeta + 2\lambda \zeta + \zeta^2 / \nu \right) dx,
\end{equation}
where $\sigma$ is a solution of the BVP (4). From the constitutive principle, it is clear that $|\sigma|^2 = o(\zeta)$. The variation of $P_n^d$ with respect to $\zeta$ leads to the dual algebraic equation (DAE), namely,
\begin{equation}
|\sigma|^2 = 2\zeta^2(\lambda + \zeta / \nu).
\end{equation}
For given parameters $\nu$, $\lambda$ and $\sigma$, the three complex solutions of the cubic DAE (13) are listed below,
\begin{align}
\zeta_1 &= 1/3 \left( -\nu \lambda + \sqrt[3]{4\nu^2 \lambda^2 \omega^{-1}(\nu, \lambda, \sigma)} + \omega(\nu, \lambda, \sigma) \right), \\
\zeta_2 &= -\nu \lambda / 3 - 3^{-1/3} \cdot 2^{-1/3} \left( 1 - i \sqrt{3} \right) \nu^2 \lambda^2 \omega^{-1}(\nu, \lambda, \sigma) - 6^{-1/3} \cdot 4^{-1/3} \left( 1 + i \sqrt{3} \right) \omega(\nu, \lambda, \sigma), \\
\zeta_3 &= -\nu \lambda / 3 - 3^{-1/3} \cdot 2^{-1/3} \left( 1 + i \sqrt{3} \right) \nu^2 \lambda^2 \omega^{-1}(\nu, \lambda, \sigma) - 6^{-1/3} \cdot 4^{-1/3} \left( 1 - i \sqrt{3} \right) \omega(\nu, \lambda, \sigma),
\end{align}
where
\begin{equation}
\omega(\nu, \lambda, \sigma) := \left( -4\nu^3 \lambda^3 + 27\nu |\sigma|^2 + 3\sqrt{3} \sqrt{-8\nu^4 \lambda^4 |\sigma|^2 + 27\nu^2 |\sigma|^4} \right)^{1/3}.
\end{equation}

**Lemma 2.2.** From (13)-(16), we know that $|\sigma|^2$ has a maximum $8\lambda^3 \nu^2 / 27$ at $\zeta = -2\lambda \nu / 3$ and minimum $0$ at $0$. If $|\sigma|^2 \in (8\lambda^3 \nu^2 / 27, \infty)$, then there exists only one real root $\zeta > 0$ of the polynomial (13). If $|\sigma|^2 \in (0, 8\lambda^3 \nu^2 / 27)$, then there exist three real roots $\zeta_1 > \zeta_2 > \zeta_3$. While when $|\sigma|^2 = 8\lambda^3 \nu^2 / 27$, there exist two real roots.

**Proof.** It suffices to prove the fact $\zeta_1 > \zeta_2 > \zeta_3$ when $|\sigma|^2 \in (0, 8\lambda^3 \nu^2 / 27)$. Actually, $\omega = \sqrt[3]{16\nu^2 \lambda^2}$. From complex analysis, it is reasonable to set $\omega = \sqrt[3]{4\nu \lambda (\cos \theta + i \sin \theta)}$, $\theta \in (0, \pi / 3)$. Through simple calculation, one knows immediately that
\begin{equation}
\zeta_1 = 1/3 \nu \lambda (2 \cos \theta - 1) > 0;
\end{equation}
−2νλ/3 < ζ_2 = −νλ/3(1 + cos θ − √3 sin θ) < 0;
−νλ < ζ_3 = −νλ/3(1 + cos θ + √3 sin θ) < −2νλ/3.

Our proof is concluded. □

By comparing (4) with (8), we deduce that, for \(i, j = 1, \ldots, n\), in order to give an integral form of the solution \(u\), the following compatibility condition has to be satisfied

\[
\Phi_ζ(σ_i, σ_j) := \begin{vmatrix}
\partial x_i & \partial x_j \\
σ_iζ^{-1} & σ_jζ^{-1}
\end{vmatrix} = 0.
\]

Let us define the subregion \(S\) as

\[
S := \{x \in Ω | \Phi_ζ(σ_i, σ_j) = 0, \ i, j = 1, \ldots, n\}.
\]

Evidently, the compatibility condition (17) guarantees the path independence of the integral for \(\vec{σ}ζ^{-1}\) in \(S\). By replacing \(\vec{σ}\) in (4) by \(ν(1/2|∇u|^2 − λ)∇u\), then \((P_n)\) is equivalent to the following BVP,

\[
\begin{align*}
\nabla \cdot (ν(1/2|∇u|^2 − λ)∇u) + f &= 0 \text{ in } Ω, \\
(ν(1/2|∇u|^2 − λ)∇u) \cdot \vec{n} &= t \text{ on } Γ_i.
\end{align*}
\]

In \(S\), the analytical solutions of BVP (19) can be given by the path integral

\[
u(x) = \int_{x_0}^x \vec{σ}ζ^{-1}ds + u(x_0),
\]

where \(x, x_0 \in S\). Summarizing the above discussion, we obtain the theorem below.

**Theorem 2.3.** For a given source \(f(x)\) and boundary condition \(t\) such that \(\vec{σ}(x)\) is determined by BVP (4), then DAE (13) has at most three real roots \(ζ_i(x), i = 1, 2, 3\), given by (14)-(16) and ordered as

\[
ζ_1(x) ≥ 0 ≥ ζ_2(x) ≥ −2νλ/3 ≥ ζ_3(x) ≥ −νλ.
\]

For \(i = 1, 2, 3\), the functions defined in \(S\) by

\[
\vec{u}_i(x) = \int_{x_0}^x \vec{σ}(s)ζ_i^{-1}(s)ds + u(x_0)
\]

are solutions of BVP (19). Furthermore,

\[
P_n(\vec{u}_i) = P^d_n(ζ_i), i = 1, 2, 3.
\]

**Proof.** The relation (23) is obtained by direct calculation from the representations of \(P_n(u)\) and \(P^d_n(ζ)\), respectively. □

Theorem 2.3 demonstrates that the pure complementary energy functional \(P^d_n(ζ)\) is canonically dual to the total potential energy functional \(P_n(u)\). The equation (23) indicates there is no duality gap between the primal and dual variational problems. In the following, we apply the triality theory to obtain the extremality conditions for these critical points.
Theorem 2.4. Suppose that the source term $f$ and boundary condition $t$ are given and $\mathcal{F}(x)$ satisfies the divergence equation (4). Then, if $|\mathcal{F}(x)|^2 \in (8\lambda^3 \nu^2/27, \infty), \forall x \in S$, then DAE (13) has a unique solution $\tilde{\zeta} > 0$, which is a global maximizer of $P_n^d(\zeta)$ over $\mathcal{E}$, and the corresponding solution $\tilde{u}$ in the form of (20) is a global minimizer of $P_n(u)$ over $U_0$,

$$\tag{24} P_n(\tilde{u}) = \min_{u \in U_0} P_n(u) = \max_{\zeta \in \mathcal{E}} P_n^d(\zeta) = P_n^d(\tilde{\zeta}).$$

If $|\mathcal{F}(x)|^2 \in (0, 8\lambda^3 \nu^2/27), \forall x \in S$, then DAE (13) has three real roots ordered as in Theorem 2.3. Furthermore, $\tilde{\zeta}_1$ is a local maximizer of $P_n^d(\zeta)$ over $\zeta > 0$, the corresponding solution $\tilde{u}_1$ is a local minimizer of $P_n(u)$ over $U_1$,

$$\tag{25} P_n(\tilde{u}_1) = \min_{u \in U_1} P_n(u) = \max_{\zeta > 0} P_n^d(\zeta) = P_n^d(\tilde{\zeta}_1),$$

where $U_1$ is a neighborhood of $\tilde{u}_1$. While for the local maximizer $\tilde{\zeta}_3$, the corresponding solution $\tilde{u}_3$ is a local maximizer of $P_n(u)$,

$$\tag{26} P_n(\tilde{u}_3) = \max_{u \in U_3} P_n(u) = \max_{-\nu \lambda < \zeta < -2\nu \lambda/3} P_n^d(\zeta) = P_n^d(\tilde{\zeta}_3),$$

where $U_3$ is a neighborhood of $\tilde{u}_3$. As for $\tilde{\zeta}_2$, in the case of 1D, the corresponding solution $\tilde{u}_2$ is a local minimizer of $P_n(u)$,

$$\tag{27} P_n(\tilde{u}_2) = \min_{u \in U_2} P_n(u) = \min_{-2\nu \lambda/3 < \zeta < 0} P_n^d(\zeta) = P_n^d(\tilde{\zeta}_2),$$

where $U_2$ is a neighborhood of $\tilde{u}_2$. It is worth noticing that, when $n \geq 2$, $\tilde{u}_2$ is not necessarily a local minimizer.

Proof. First, we recall the second variation formula for both $P_n(u)$ and $P_n^d(\zeta)$. On the one hand, for $\forall \zeta \in U_1 := \{ u \in U_0 \mid \nabla u \neq 0 \}$,

$$\delta_\zeta^2 P_n(u) = \int_\Omega \frac{d^2}{dt^2} \left\{ \nu/2 \left(1/2|\nabla (u + t\zeta)|^2 - \lambda \right) \right\}_{t=0}^2 dx$$

$$= \nu \int_\Omega \left\{ |\nabla u \cdot \nabla \zeta|^2 + \left(1/2|\nabla u|^2 - \lambda \right)|\nabla \zeta|^2 \right\} dx.$$ 

On the other hand, for $\forall \eta \neq 0 \in \mathcal{E}$,

$$\delta_\eta^2 P_n^d(\zeta) = -1/2 \int_\Omega \frac{d^2}{dt^2} \left\{ |\mathcal{F}|^2/(\zeta + t\eta) + 2\lambda(\zeta + t\eta) + (\zeta + t\eta)^2/\nu \right\}_{t=0}^2 dx$$

$$= -\int_\Omega \left\{ |\mathcal{F}|^2/\zeta^2 + 1/\nu \right\} \eta^2 dx.$$ 

If $\zeta > 0$, according to the definition of $\zeta = \nu(1/2|\nabla u|^2 - \lambda)$, one knows immediately that

$$\delta_\zeta^2 P_n(u) > 0, \quad \delta_\eta^2 P_n^d(\zeta) < 0.$$ 

Then (24) and (25) are concluded. Actually, when $|\mathcal{F}|^2 \in (8\lambda^3 \nu^2/27, \infty)$, then by applying the definition of $\mathcal{F}$, one has

$$\lambda^3 < 27/8(1/2|\nabla u|^2 - \lambda)^2 |\nabla u|^2.$$
Solving this inequality, we obtain \( \lambda \in (0, 3/8|\nabla u|^2) \). Now we consider the negative 
\( \tilde{\zeta}_i(x), i = 2, 3 \). When \( \zeta < -2/3\nu\lambda \), then \( \delta_n^2 P_n^d(\zeta) < 0 \) and \( \lambda > 3/2|\nabla u|^2 \). In this case,
\[
\delta_n^2 P_n(u) \leq \nu \int_{\Omega} \left\{ |\nabla u|^2 |\nabla \zeta|^2 + \left( \frac{1}{2} |\nabla u|^2 - \lambda \right) |\nabla \zeta|^2 \right\} dx \\
= \nu \int_{\Omega} \left( \frac{3}{2} |\nabla u|^2 - \lambda \right) |\nabla \zeta|^2 dx < 0.
\]

Then (26) is proved. It remains to consider (27). For \( \zeta > -\sqrt{3\sigma^2\nu} \), the second variation \( \delta_n^2 P_n^d(\zeta) > 0 \). In addition, \(-2\nu\lambda/3 < \zeta < 0 \) indicates \( \lambda \in (1/2|u_+|^2, 3/2|u_+|^2) \). In fact, \(-2/3\nu\lambda < \zeta < 0 \) gives \( \delta_n^2 P_n^d(\zeta) > 0 \), which indicates that \( \bar{u}_2 \) is a local minimizer.

3. Applications

Now we apply Theorem 2.4 to practical problems in 2D domain. Let us consider the open annulus \( \Omega = \mathbb{R}_+^{R_2} \) bounded by two concentric circles with radii \( R_1 \) and \( R_2 \) (\( R_1 < R_2 \)). Let \( f \) and \( t \) be given, \( \Gamma_t := \Gamma_1 \cup \Gamma_2 \),
\[
f = -r := -\sqrt{x^2 + y^2}, \quad t = R_2^2/3 \text{ on } \Gamma_1, \quad t = -R_2^2/3 \text{ on } \Gamma_2.
\]
It is evident that \( f \in L^\infty(\Omega) \) and \( t \in L^\infty(\Gamma_1 \cup \Gamma_2) \). In the case of \( \nu = 1, \lambda = 1 \), the primal problem is of the form
\[
(P_2) : \min_{u \in \mathcal{U}_0} \left\{ P_2(u) = \int_{\Omega} \left[ \frac{1}{2} \left( \frac{1}{2} |\nabla u|^2 - 1 \right)^2 - ru \right] dx + \int_{\Gamma_2} u R_2^2/3 d\Gamma - \int_{\Gamma_1} u R_1^2/3 d\Gamma \right\}.
\]

![Figure 1](image.png)

**Figure 1.** Dual solution \( \tilde{\zeta}_1 \) and primal solution \( \bar{u}_1, r \in [0.500, 1.277] \)

We consider the radially symmetric solution of (19). Without any confusion, we denote the radially symmetric functions \( u(r) := u(x, y) \) and \( \zeta(r) := \zeta(x, y) \). For (4), we have the unique solution in the form of \( \sigma = (-rx/3, -ry/3), r \in (R_1, R_2) \). Then the singular algebraic polynomial (13) is given as
\[
r^4/9 = 2\zeta^2(1 + \zeta).
\]

From Theorem 2.3, one knows immediately that when \( r^4 \in (8/3, \infty) \), (28) has a unique real root; while when \( r^4 \in (0, 8/3) \), (28) has three real roots (see Figures 1-3(a)(b)). For instance,
(a) $\bar{\zeta}_2$ with respect to $r$  
(b) $\bar{\zeta}_2$ with respect to $x$ and $y$  
(c) $\bar{u}_2(r) - \bar{u}_2(0.500)$ with respect to $r$

**Figure 2.** Dual solution $\bar{\zeta}_2$ and primal solution $\bar{u}_2$, $r \in [0.500, 1.277]$

(a) $\bar{\zeta}_3$ with respect to $r$  
(b) $\bar{\zeta}_3$ with respect to $x$ and $y$  
(c) $\bar{u}_3(r) - \bar{u}_3(0.500)$ with respect to $r$

**Figure 3.** Dual solution $\bar{\zeta}_3$ and primal solution $\bar{u}_3$, $r \in [0.500, 1.277]$

- if we set $r = 2$, then $\bar{\zeta}_1 = 0.719078$, $\bar{\zeta}_2 = -0.859539 - 0.705226i$, $\bar{\zeta}_3 = -0.859539 + 0.705226i$;
- if we set $r = 1$, then $\bar{\zeta}_1 = 0.213928$, $\bar{\zeta}_2 = -0.277249$, $\bar{\zeta}_3 = -0.936679$;
- if we set $r = 0.5$, then $\bar{\zeta}_1 = 0.0573064$, $\bar{\zeta}_2 = -0.0608031$, $\bar{\zeta}_3 = -0.996503$.

Since $\bar{\zeta}_i$ is radially symmetric, as a result,

$$\mathcal{F}/\bar{\zeta}_i = (-r x/(3\bar{\zeta}_i), -ry/(3\bar{\zeta}_i))$$

satisfies the compatibility condition (17) on $\Omega$, $i = 1, 2, 3$. According to Theorem 2.4, we have corresponding analytical solutions of the form for $i = 1, 2, 3$, respectively,

$$\bar{u}_i(x, y) = -\int_{(x_0, y_0)}^{(x, y)} \frac{y}{3\bar{\zeta}_i(\sqrt{x^2+y^2})} dy - \int_{(x_0, y_0)}^{(x, y)} \frac{x}{3\bar{\zeta}_i(\sqrt{x^2+y^2})} dx + \bar{u}_i(x_0, y_0),$$

where $(x_0, y_0), (x, y)$ on $\Omega$ (see Figures 1-3(c)).

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