A New Filled Function for Non-smooth Global Optimization and Its Applications

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Abstract. This paper aims to discover a global optima for non-smooth optimal issues by designing a novel filled function, which, regarded as a bridge delivering one minimizer to another minimizer, contains only one parameter for easy adjustment. Theoretically, the natures of the considered filled function are conducted as well as a new algorithm is presented. Finally, certain experimental calculations are presented to illustrate the performance of the constructed algorithm.

Keywords: Non-smooth global optimization; Filled function algorithm; Global optimum.

1. Introduction

With the advancement rapidly in artificial intelligence, engineering design, financial economy, and national defence industry, etc., researches about solving nonlinear programming problem $(P): \min_{x \in \mathbb{R}^n} f(x)$ are a concern field with extremely significance. Up to now, lots of theoretical and computational contributions are performed in\cite{1,2,3}. Especially, the conception of filled function initially mentioned in \cite{4} has been further discussed in \cite{5,7,8,9,10}, which is considered as an utility and effective method to address global optimization issue.

The filled function approach uses the considered filled function as a bridge to deliver the local optima to get one better solution. The implementation of the filled function technique involves two stages. The first stage is to optimize original objective function before the corresponding filled function is considered in the second step to execute the minimization process. The filled functions mentioned in \cite{5,7,8,9,10} have some shortcomings. The filled function presented in \cite{5} with an exponential term, its value varying quickly, may fail to calculation. The filled function constructed by \cite{5} is an improved version of the one proposed in \cite{4}, and the major disadvantage is that the execution of filled function algorithm runs beyond the search space of problem, but lying in a line linking the newly discovered local minima and one point located at certain neighbourhood of the next undetected optimal solution. It is a significant difficulty of computation to discover such the direction. The primitively purpose of the filled functions discussed in \cite{7,8,9,10} is to solve problems of smooth global optimization. However, numerous cases in practical scenario are essentially presented as non-smooth global optimization model. Our work mainly is devoted to expand filled function approaches to solving non-smooth optimal issues, which is more suitable for practical needs.
Usually the settlement of global optimization covers two critical issues: one is how to avoid the local minima to discover a global solution, and the other is the confirmation that whether the current optimum solution is global. This article mainly aims to solve the former problem.

The organization of the article is in this order: Section 2 constructs a new filled function with single parameter on non-smooth issues, theoretically analysis of which is studied before Section 3 derives a new filled function approach. Section 4 includes numerical experiments for nonlinear optimization cases conducted by using the proposed algorithm and conclusions of this work is summarised in Section 5.

2. One Single Parameter Filled Function and Its Properties

A global minimization problem with box constraint condition is discussed in this paper, and the mathematical formula is $(P)\colon \min_{x\in X} f(x)$, where $X$ indicates the box constraint.

For convenience, $(LP)\subset X$ and $(GP)\subset X$ represent the locally minimum solution set and the global best solutions of $\min_{x\in X} f(x)$ respectively. Two hypotheses about natures of function $f(x)$ are given before define a filled function with respect to problem $(P)$. Initially, throughout this paper, hypotheses are assumed below:

**Hypotheses 1.** The function $f(x)$ is Lipschitz continuous on the box set $X$ with a rank $L > 0$.

**Hypotheses 2.** The problem $(P)$ has at least one global optimal solution with limited number of distinct optima values.

Most of filled function algorithms mainly utilizes Clark generalized gradient, the more information of which can be seen in[13], to solve non-smooth global optimization issue. Let $x' \in LP$, and we define the concept of filled function with regard to problem $(P)$.

**Definition.** A function $F(x, x')$ is identified as a filled function of problem $(P)$ at $x' \in LP$, if and only if it meets below conditions:

1. $x'$ represents an absolute maximum of $F(x, x')$ in the box set $X$.

2. For two unequal points $x', x \in X$ and satisfying that $f(x') \leq f(x)$, $0 \notin \partial F(x, x')$ holds.

3. If $x'$ does not indicate a best optimal solution, so one point $x_0 \in S_x = \{x \in X : f(x') > f(x)\}$ represents locally optimal solution to the function $F(x, x')$.

Define

$$F(x, x', \delta) = \min \left\{ (f(x) - f(x'))^\delta / \delta, -\|x - x'\| - \delta(f(x) - f(x')) \right\},$$

where $\delta > 0$ is a fixed small scalar. In the next, we will give the process of proving that $F(x, x', \delta)$ represents a filled function.

**Theorem 1.** Assume $x' \in LP$, provided that $\delta > 0$ is a fixed small scalar, so that $x'$ indicates one strictly local maximum of $F(x, x', r)$.

Proof: Due to $x'$ belonging to $LP$, it is means that some neighbourhood $O(x', \sigma)$ at $x'$ with $\sigma > 0$ exists, that is to say $f(x') \leq f(x)$ for $\forall x \in O(x', \sigma)$. For any $x \neq x'$, $x \in O(x', \sigma)$, applying the mean-value theorem, we can obtain

$$F(x, x', \delta) - F(x', x^*, \delta) \in -\|x - x'\| - \delta f(x')(x - x') \leq \|x - x'\|(-1 + \delta L) < 0.$$
**Theorem 2.** Provided that $x^* \neq x$ meeting $f(x^*) \leq f(x)$, and the fixed small scalar $\delta < 0$ satisfies $\delta L < 1$, $0 \notin \partial F(x,x^*,\delta)$ holds.

Proof: According to the definition, one has $F(x,x^*,\delta) = -\delta f(x) - \|x-x^*\| + \delta f(x^*)$. Thus, We obtain

$$\langle x-x^*, \partial F(x,x^*,\delta) \rangle \leq -\|x-x^*\| - \delta \langle \partial f(x), x-x^* \rangle \leq \|x-x^*\| (\delta L - 1) < 0.$$ (3)

Therefore, one obtains $0 \notin \partial F(x,x^*,\delta)$.

**Theorem 3.** If $x^* \in L(P)$ but $x^* \notin G(P)$, one point $x_0$ is some local optima of $F(x,x^*,\delta)$ satisfying $f(x_0) < f(x^*)$.

Proof: Assume $x_0 \in G(P), f(x^*) - f(x_0) = 2\varepsilon$. Obviously, there has one neighbourhood $O(x_0,\sigma)$ at the point $x_0$, with $f(x) < f(x_0) + \varepsilon, \forall x \in O(x_0,\sigma)$. Then, one achieves

$$f(x^*) - f(x) \geq 2\varepsilon + f(x_0) - (f(x_0) - \varepsilon) = \varepsilon > 0, \forall x \in O(x_0,\sigma).$$ (4)

Provided that

$$0 < \delta < \min \left\{ 1, \frac{\varepsilon^5}{K + KL} \right\},$$ (5)

for any $x \in O(x_0,\sigma)$, where $K = \max_{x,y: x \neq y} \|x-y\|$.

Then, one has

$$-\|x-x^*\| - \delta f(x) + \delta f(x^*) \geq -(1+\delta L)\|x-x^*\| \geq -K - \delta L K \geq -K - KL$$

$$\geq -\frac{\varepsilon^5}{\delta} \geq (f(x) - f(x^*))^5.$$ (6)

Based on the aforementioned inequality, one derives

$$F(x,x^*,\delta) = \left( \frac{f(x) - f(x^*)}{\delta} \right)^5, \forall x \in O(x_0,\sigma).$$ (7)

Due to $f(x_0) < f(x), \forall x \in O(x_0,\sigma)$, we obtain

$$F(x,x^*,\delta) = \left( \frac{f(x) - f(x^*)}{\delta} \right)^5 \geq \left( \frac{f(x_0) - f(x^*)}{\delta} \right)^5 = F(x_0,x^*,\delta),$$ (8)

for any $x \in O(x_0,\sigma)$, which implies that $x_0 = x_0$ with $f(x_0) < f(x^*)$ represents one local optima of $F(x,x^*,\delta)$.

This completed the proof.

3. Filled Function Algorithm

According to the above-mentioned theoretical analysis of the natures with respect to presented $P(x,x^*,r)$, and then this section describe a novel filled function approach for address optimal issues with non-smooth function.

**Filled function algorithm**

*Initial Stage:*
\( \delta \) represents the infimum of parameter \( \delta \), \( x_i \) indicates a starting point in the problem domain, and \( e_1, e_2, \ldots, e_n \) are the directions of unit coordinate with \( n \) positive and negative direction. Let \( k = 1 \), and then get into main stage.

**Main Stage:**

Step 1: Beginning from selected point \( x_1 \), utilize one suitable optimal algorithm for non-smooth local optimization to discover a local optima \( x_1^* \) with respect to primal problem \( (P) \), and then get into step 2.

Step 2: Take \( \delta = 1 \).

Step 3: Establish the filled function \( F(x_1, x_1^*, \delta) \) based on formula (1) and then get into step 4.

Step 4: If \( k > 2n \), step 7 is performed; if not, let \( x = x_1^* + 0.1e_1 \), and utilize \( x \) to discover one local optimal solution \( x_k \) with respect to considered filled function: \( \min_{y \in X} F(y, x_1^*, \delta) \).

Step 5: If \( x_k \notin X \), let \( k = k + 1 \), then get into step 4; If not, step 6 is selected.

Step 6: If \( f(x_1^*) \geq f(x_k) \), (1) let \( x = x_k, k = 1 \). (II) \( x \) is utilized to address the original problem \( (P) \) for discover one new local optima \( x_2^* \) satisfying \( f(x_2^*) < f(x_1^*) \). (III) Take \( x_1^* = x_2^* \) and goto2; Else if \( f(x_1^*) \leq f(x_k) \), and get into step 7.

Step 7: Lessen \( \delta \) through utilizing \( \delta = 0.1 \delta \). If \( \delta > \delta_1 \), let \( k = 1 \), step 3 is selected; if \( \delta \leq \delta_1 \), \( x_1^* \) indicates the best solution with respect to the problem \( (P) \). Therefore, the iterative loop terminates.

**Notes:**

1. The given method according to the discussed filled function is suitable for smooth cases, as well.
2. The iterative loop of proposed filled function method includes two phases (as shown Figure 1): a local minimization phase and a global drop-off phase. A local optima \( x^* \) could be detected by the means of arbitrary suitable optimal methods \([11, 14]\) in the former phase. The second stage aims to solve the considered function \( F(x, x^*, \delta) \) to get the optima. During the procedure of minimizing \( F(x, x^*, \delta) \), discover one local optima \( x_k \) satisfying \( f(x_k) < f(x^*) \), and then the second phase terminates and transfers to the first phase to detect one better local optima of \( f(x) \). Perform both stages repeatedly until no new local solution cannot be detected. Then the current local solution can approximate the global minimum.

![Figure 1](image-url)  
**Figure 1.** Illustration of the minimization process in one iterative loop.

### 4. Numerical Experiment

Some numerical experiments, including one case applying the filled function algorithm to address NPC issue, are conducted by the presented filled function method. All experiments are executed in Fortran 95. In order to discover a local optimal solution for optimal problem, for non-smooth scenario, we adopt the optimal methods mentioned in \([6, 9]\), however, for smooth scenario, the penalty method and conjugate gradient method are selected.

Example 1\([6]\):
\[
\begin{align*}
\min f(x) &= -\exp(0.5(\cos(2\pi x_i) + \cos(2\pi x_j))) - 20(\exp(-0.02|x_i|)|x_j|) - 1) \\
\text{s.t. } x_i^2 + x_j^2 &\leq 300, \quad x_i + 0.5x_j \leq 2, \quad -30 \leq x_i \leq 30, \quad i = 1, 2
\end{align*}
\]

The presented method obtained the best results \((x_1^*, x_2^*) = (0, 0)\) and the corresponding function optimal value \(f^* = -2.7183\). Table 1 illustrates the experimental calculations.

**Table 1.** Computational results of example 1.

| \(k\) | \(x_i\) | \(x_j^*\) | \(f(x_j^*)\) |
|-------|--------|---------|-----------|
| 1     | (-1.0000, -1.0000) | (-15.0000, 0.0000) | 5.7164     |
| 2     | (-1.0584, 0.5165)  | (-0.0001, -0.2093) | -0.3691    |
| 3     | (0.0007, -0.0434)  | (0.0000, 0.0000)   | -2.7183    |

**Figure 2.** The typical 2D plots of \(f(x)\) and the convergence curve of the presented algorithm.

Now, we give a new example to illustrate how to apply the our new approach for addressing nonlinear equations, which can be expressed as problem \((NE)\): \(G(x) = 0, \quad x \in X\), where \(G(x)\) represents a mapping with continuation: \((f_1(x), f_2(x), \ldots, f_n(x))^T : \mathbb{R}^n \rightarrow \mathbb{R}^n\), and \(X \subset \mathbb{R}^n\) indicates constraint condition. Obviously, \(G(x)\) is transformed into \(f(x) = \sum_{i=1}^{n}|f_i(x)|\), so that the procedure of solving the problem \((NE)\) is equivalent to a global minimization problem \((P)\): \(\min_{x \in X} f(x)\). Therefore, one can address the problem \((P)\) to achieve the solutions of \((NE)\). Specially, assume that the number of problem \((NE)\) roots is not less than one, so that each best solution \(x^*\) to problem \((P)\) satisfying \(f(x^*) = 0\) obtained by our method represents a root of the \((NE)\).

Example 2 [12]:

\[
\begin{align*}
10^{-4} x_1 x_2 - 1 &= 0, \quad e^{-x} + e^{-x} - 1.001 = 0  \\
\text{s.t. } 5.49 \times 10^{-4} \leq x_1 \leq 4.553, 2.196 \times 10^{-3} \leq x_2 \leq 18.21
\end{align*}
\]

The best solution \(x^* = (1.452 \times 10^{-5}, 6.8933045)\) of the above example was obtained successfully by the means of the presented algorithm. And the corresponding computations are described in the following Table 2.

**Table 2.** Computational results of example 2.

| \(k\) | \(x_i^0\) | \(x_i^*\) | \(f(x_i^*)\) | \(G(x_i^*)\) |
|-------|---------|---------|-------------|-------------|
| 1     | \(3.00000000\) | \(0.0001457\) | \(2.39498 \times 10^{-6}\) | \(0.001547446\) |
|       | \(3.00000000\) | \(6.87403875\) | \(1.97215 \times 10^{-5}\) |             |
| 2     | \(1.4523 \times 10^{-5}\) | \(0.00014354\) | \(5.12614 \times 10^{-9}\) | \(0.00080012\) |
|       | \(6.89330451\) | \(6.96670021\) |             | \(-0.0007159\) |
Table 1 and Table 2 report the calculations of the experiment cases, which state that the structured filled function approach can avoid local optima to obtain the best solution with a high accuracy for non-smooth optimization issues. The convergence curve of the presented algorithm is described in Figure 2 to observe the convergence rate of the algorithm. Therefore, the results of this section indicate that the new algorithm has advantages of local optimal solution avoidance along with convergence speed simultaneously during the period of the iterative loop.

5. Conclusion
It has been proven that the filled function approach, as one of the auxiliary function methods, can effectively address the optimization problem to obtain the global optimum solution. A new one single parameter filled function with structure simplicity as well as computation easily is constructed, which is fit for both non-smooth and smooth scenarios of global optimal issues. We studies theoretically the natures with respect to the constructed filled function, and then provide the new algorithm. Some smooth and non-smooth cases are conducted and computational results revealed that the theoretical analysis of presented filled function is correct along with the designed filled function algorithm is applicable and effective with satisfactory performance.

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