The enclosure method for inverse obstacle scattering problems with dynamical data over a finite time interval: III. Sound-soft obstacle and bistatic data

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Abstract
This paper is concerned with an inverse obstacle problem which employs the dynamical scattering data of an acoustic wave over a finite time interval. The unknown obstacle is assumed to be a sound-soft one. The governing equation of the wave is given by the classical wave equation. The wave is generated by the initial data localized outside the obstacle and observed over a finite time interval at a place which is not necessary the same as the support of the initial data. The observed data are the so-called bistatic data. In this paper, an enclosure method which employs the bistatic data and is based on two main analytical formulae is developed. The first one enables us to extract the maximum spheroid with focal points at the centre of the support of the initial data and that of the observation points whose exterior encloses the unknown obstacle of general shape. The second one, under some technical assumption for the obstacle including convexity as an example, indicates the deviation of the geometry of the boundary of the obstacle and the maximum spheroid at the contact points. Several implications of those two formulae are also given. In particular, a constructive proof of the uniqueness of a spherical obstacle using the bistatic data is given.

1. Introduction
In this paper, we consider an inverse obstacle scattering problem for a sound-soft obstacle with dynamical data over a finite time interval. The governing equation of the wave is the classical wave equation. The wave as the solution is generated by the initial data whose support is localized at the outside of the obstacle and observed over a finite time interval on a different position from the support of the initial data. The observed data are the so-called bistatic data. This is a simple mathematical model of the data collection process using an acoustic wave/electromagnetic wave such as bistatic active sonar, radar, etc. See, e.g., [4]
for the bistatic active sonar. The aim of this paper is to develop an enclosure method which employs the bistatic data.

Let us describe a mathematical formulation of the problem. Let $D$ be a nonempty bounded open subset of $\mathbb{R}^3$ with $C^2$-boundary such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. Let $0 < T < \infty$. Let $f \in L^2(\mathbb{R}^3)$ satisfy $\text{supp} f \cap \overline{D} = \emptyset$. Let $u = u_f(x, t)$ denote the weak solution of the following initial boundary value problem for the classical wave equation:

$$
\begin{align*}
\partial_t^2 u - \Delta u &= 0 \text{ in } (\mathbb{R}^3 \setminus \overline{D}) \times ]0, T[,
\partial_t u(x, 0) &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{D},
\partial_n u(x, 0) &= f(x) \text{ in } \mathbb{R}^3 \setminus \overline{D},
\partial_t u(x, t) &= 0 \text{ on } \partial D \times ]0, T[.
\end{align*}
$$

The boundary condition for $u$ in (1.1) means that $D$ is a sound-soft obstacle. In this paper, $T$ is always fixed. Thus, for our purpose, the weak solution over the bounded interval $]0, T[$ is appropriate. Since the notion of the weak solution for the wave equation is well established, we do not repeat the description here. Instead, we see [5] for the notion and also its use in [9, 11] for inverse obstacle scattering problems with dynamical data over a finite time interval.

In this paper, we consider the following problem.

**Inverse problem.** Let $B$ and $B'$ be two known open balls centred at $p \in \mathbb{R}^3$ and $p' \in \mathbb{R}^3$ with radii $\eta$ and $\eta'$, respectively such that $\overline{B} \cap \overline{D} = \emptyset$ and $\overline{B'} \cap \overline{D} = \emptyset$. Let $\chi_B$ denote the characteristic function of $B$ and set $f = \chi_B$. Assume that $D$ is unknown. Extract information about the location and shape of $D$ from the data $u_f(x, t)$ given at all $x \in B'$ and $t \in ]0, T[$.

As far as the author knows, there is no result to this problem for general configuration of $B$ and $B'$. This is the problem raised in [11] as an open problem related to the enclosure method itself. In particular, the problem contains the case when $\overline{B} \cap \overline{B'} = \emptyset$, which corresponds to the case when the emitter and receiver are placed on different positions at a finite distance from the obstacle. Strictly speaking, we should call the data in this case the bistatic data; however, we include also the case $\overline{B} \cap \overline{B'} \neq \emptyset$.

In this paper, we develop an enclosure method with bistatic data. In short, the enclosure method aims at extracting a domain that encloses an unknown discontinuity, such as cavities, cracks, inclusions or obstacles. The idea of the enclosure method goes back to [7], in which the original enclosure method was developed by considering an inverse boundary value problem governed by the Laplace equation. In [8], an idea for the application of the enclosure method to the dynamical data coming from the heat or wave equations has been introduced. Now we have many applications of this enclosure method to inverse boundary value problems governed by the heat equations in [10, 14, 15], visco-elastic system of equations [13] and inverse obstacle scattering problems governed by the wave equations in [9, 11, 12].

We establish two main analytical formulae. The first one enables us to extract the maximum *spheroid* with focal points at the centre of the support of the initial data and that of the observation points whose exterior encloses the unknown obstacle of general shape. The appearance of the exterior of a spheroid as an enclosing domain is new since previous enclosing domains are a half-plane/space, sphere or its exterior, or cone. The formula shows us an effect of the bistatic data on the obtained information. See theorem 1.1 below. The second one, under some technical assumption for the obstacle including convexity as an example, indicates the deviation of the geometry of the boundary of the obstacle and the maximum spheroid at the contact points. This is also new. See theorem 1.3 below. We also present several implications of those two formulae. In particular, we give a constructive proof of a uniqueness of a *spherical* obstacle using the bistatic data.
1.1. Extracting the first reflection distance and its implication

Let $\tau > 0$. In this paper, given an arbitrary $h \in L^2(\mathbb{R}^3)$, we denote by $v_h$ the unique weak solution $v \in H^1(\mathbb{R}^3)$ of

$$\begin{aligned}
(\Delta - \tau^2)v + h(x) &= 0 \quad \text{in} \quad \mathbb{R}^3, \\
\end{aligned}
$$

$v_h$ has the expression

$$v_h(x) = v_h(x, \tau) = \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-\tau|x-y|} h(y) \, dy. \quad (1.3)$$

Define

$$w_f(x) = w_f(x, \tau) = \int_0^T e^{-\tau t} u_f(x, t) \, dt, \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad \tau > 0.$$  

$w = w_f$ satisfies

$$\begin{aligned}
(\Delta - \tau^2)w + f(x) &= e^{-\tau T} F_f(x, \tau) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \\
w &= 0 \quad \text{on} \quad \partial D. \\
\end{aligned} \quad (1.4)$$

where

$$F_f(x, \tau) = \partial_t u_f(x, T) + \tau u_f(x, T), \quad x \in \mathbb{R}^3 \setminus \overline{D}.$$  

Since $F_f(x, \tau)$ is unknown, it seems that the existence of such a term in (1.4) hides the information about an unknown obstacle. However, the use of the enclosure method presented below does not make it a problem at all and enables us to extract the information about the obstacle provided $T$ is sufficiently large and fixed.

Let $\chi_B$ denote the characteristic function of $B$ and set $g = \chi_B$.

The results of this paper are concerned with the asymptotic behaviour of the indicator function:

$$\tau \mapsto \int_{\mathbb{R}^3 \setminus \overline{D}} (f v_g - w_f g) \, dx = \int_B v_g \, dx - \int_{\overline{B}} w_f \, dx.$$  

For the description of the results, we prepare some notation.

Define

$$\phi(x; y, y') = |y - x| + |x - y'|, \quad (x, y, y') \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3.$$  

This is the length of the broken path connecting $y$ to $x$ and $x$ to $y'$ which plays the central role in this paper.

In this paper, we denote the convex hull of the set $F \subset \mathbb{R}^3$ by $[F]$.

**Theorem 1.1.** Let $[\overline{B} \cup \overline{B'}] \cap \partial D = \emptyset$ and $T$ satisfy

$$T > \min_{x \in \partial D, y \in \partial \overline{B}, y' \in \partial \overline{B}} \phi(x; y, y'). \quad (1.5)$$

Then, there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$,

$$\int_{\mathbb{R}^3 \setminus \overline{D}} (f v_g - w_f g) \, dx > 0$$

and the formula

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log \int_{\mathbb{R}^3 \setminus \overline{D}} (f v_g - w_f g) \, dx = - \min_{x \in \partial D, y \in \partial \overline{B}, y' \in \partial \overline{B}} \phi(x; y, y') \quad (1.6)$$

is valid.
Note that
\[
\min_{x \in \partial D, y \in \partial B} \phi(x; y, y') = \min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta').
\] (1.7)

See the appendix for the proof of (1.7). The quantity \( \min_{x \in \partial D} \phi(x; p, p') \) coincides with the shortest length of the broken paths connecting \( p \) to a point \( q \) on \( \partial D \) and \( q \) to \( p' \), that is, the first reflection distance between \( p \) and \( p' \) by \( D \). Formula (1.6) gives an extraction formula of \( \min_{x \in \partial D} \phi(x; p, p') \) from \( u_f(x, t) \) given at all \( x \in B' \) and \( t \in [0, T] \). This yields a method of extracting the first reflection distance from the waveform mathematically.

Define \([p, p'] = (sp + (1 - s)p') \mid 0 \leq s \leq 1 \). This is the straight line segment connecting the centres of \( B \) and \( B' \) and coincides with \([\{p, p']\} \). Since both \( p \) and \( p' \) are in \( \mathbb{R}^3 \setminus \overline{D} \), \([p, p'] \cap \partial D = \emptyset \) if and only if \([p, p'] \cap \partial D = \emptyset \).

We know that
\begin{itemize}
  \item \( \min_{x \in \partial D} \phi(x; p, p') \geq |p - p'| \);
  \item if \([p, p'] \cap \partial D = \emptyset \), then \( \min_{x \in \partial D} \phi(x; p, p') > |p - p'| \).
\end{itemize}

Given \( c > |p - p'| \), define
\[
E_c(p, p') = \{ x \in \mathbb{R}^3 \mid \phi(x; p, p') = c \}.
\]
This is a spheroid with focal points \( p \) and \( p' \). It is a compact surface of class \( C^\infty \).

Since \( D \) is contained in the exterior of spheroid \( E_c(p, p') \) with \( c = \min_{x \in \partial D} \phi(x; p, p') \), theorem 1.1 gives us the largest spheroid with focal points \( p \) and \( p' \) whose exterior contains \( D \) using dynamical bistatic data \( u_f \) on \( B' \times [0, T] \). The appearance of the spheroid in the enclosure method is new and this is a decisive difference from the previous enclosure method.

Therefore, we obtain the information that there exists a point belonging to \( \partial D \) on the spheroid \( E_c(p, p') \) with \( c = \min_{x \in \partial D} \phi(x; p, p') \) calculated by formula (1.6). Thus, the next problem is: identify all the points belonging to \( \partial D \) on the spheroid. In order to describe the problem precisely, we introduce the following notion.

**Definition 1.1.** Let \( p \) and \( p' \) satisfy \([p, p'] \cap \partial D = \emptyset \). Define
\[
\Lambda_{3D}(p, p') = \{ q \in \partial D \mid \phi(q; p, p') = \min_{x \in \partial D} \phi(x; p, p') \}.
\]
We call this the first reflector between \( p \) and \( p' \). The points in the first reflector are called the first reflection points between \( p \) and \( p' \). Note that \( \Lambda_{3D}(p, p') \) can be an infinite set.

One has the expression
\[
\Lambda_{3D}(p, p') = \partial D \cap E_c(p, p'),
\]
with \( c = \min_{x \in \partial D} \phi(x; p, p') \). Thus, the problem becomes: identify all the first reflection points.

Let \( S^2 \) denote the set of all unit vectors. Given \( \omega \in S^2 \) we denote by \( s(\omega; p, p', c) \) the length of the straight line segment connecting \( p' \) and the unique point on \( E_c(p, p') \cap \{ p' + s\omega \mid s > 0 \} \). We have
\[
s(\omega; p, p', c) = \frac{c^2 - |p - p'|^2}{2(c - \omega \cdot (p - p'))}.
\]
Note that \( \omega \cdot (p - p') < c \) since \( c > |p - p'| \). It is easy to see that the map
\[
S^2 \ni \omega \mapsto p' + s(\omega; p, p', c)\omega \in \mathbb{R}^3
\]
is one-to-one and the image coincides with \( E_c(p, p') \).
Let \( 0 < \eta' < \inf_{q \in \partial D} \tau \phi(x; p, p') \). \( \overline{B} \) is contained in the set of all \( x \) such that \( \phi(x; p, p') < c \), that is, the domain enclosed by \( E_c(p, p') \).

Let \( v = \nu \) denote the unit outward normal to \( D \) on \( \partial D \) at \( x \). The following theorem says that all the first reflection points between \( p \) and \( p' \) together with the tangent planes can be extracted from a single set of the bistatic data. This exceeds the previous enclosure method and suggests that the information contained in the bistatic data is quite rich.

**Theorem 1.2.** Assume that \( c = \min_{x \in \partial D} \phi(x; p, p') \) is known. Let \( \overline{B} \cup \overline{B} \) \( \cap \overline{D} = \emptyset \). Fix \( 0 < \kappa < \eta' \). \( T \) satisfies

\[
T > \sup_{\omega \in S^3} \min_{x \in \partial D} \phi(x; p, p' + \kappa \omega) - (\eta + \eta' - \kappa),
\]

then one can extract all \( q \in \Lambda_{\partial D}(p, p') \) together with \( v_q \) from \( u_f \) on \( B' \times ]0\), \( T[ \) with \( f = \chi_B \).

**Remark 1.1.** Note that

\[
\sup_{\omega \in S^3} \min_{x \in \partial D} \phi(x; p, p' + \kappa \omega) - (\eta + \eta' - \kappa) = \sup_{\omega \in S^3, x \in \partial D, y \in \partial B, y' \in \partial B_{p - \kappa \omega}} \min_{\omega} \phi(x; y, y').
\]

The reason is the same as that of the validity of (1.7). Thus, the constraint on \( T \) is reasonable.

Theorem 1.1 is a direct consequence of the following two estimates: there exist \( \mu_j \in \mathbb{R} \), \( C_j > 0 \) with \( j = 1, 2 \) and \( \tau_0 > 0 \) which are independent of \( \tau \) such that, for all \( \tau \geq \tau_0 \),

\[
e^{\tau \min_{x \in \partial D, y \in \partial B, y' \in \partial B} \phi(x; y, y')} \int_{\mathbb{R}^3 \setminus \overline{D}} (f v_k - w_f g) \, dx \leq C_1 e^{\mu_1}
\]

and

\[
C_2 e^{\mu_2} \leq e^{\tau \min_{x \in \partial D, y \in \partial B, y' \in \partial B} \phi(x; y, y')} \int_{\mathbb{R}^3 \setminus \overline{D}} (f v_k - w_f g) \, dx.
\]

The proof of (1.9) proceeds along the same line as the back-scattering data case \( (B = B') \) and is given in section 2. The point that should be emphasized in the proof of theorem 1.1 is (1.10) which is proved in subsection 3.1.

When \( B' = B \), using the same technique as that for the sound-hard obstacle case in [11], we can prove (1.10) without difficulty. The technique therein does not depend on the boundary condition; however, it heavily depends on the condition \( B = B' \). In this paper, we take another way. It is based on the combination of the maximum principle for the modified Helmholtz equation in the domain \( \mathbb{R}^3 \setminus \overline{D} \) and a reflection across \( \partial D \). It heavily depends on the speciality of the homogeneous Dirichlet boundary condition on \( \partial D \). The idea goes back to the arguments done in the proofs of theorem 3.6 and lemma 3.7 in [18]. Note that, therein, a relationship between the support function and the so-called scattering kernel for a general sound-soft obstacle has been established. They used the arguments to obtain an estimate for the analytic continuation of the Fourier transform of the scattering kernel and then applied the Paley–Wiener theorem. We refer the reader to [19, 22] for several other results using the scattering kernel.

### 1.2. Leading term of the indicator function and its implication

Estimates (1.9) and (1.10) suggest that the following integral as \( \tau \longrightarrow \infty \)

\[
e^{\tau \min_{x \in \partial D, y \in \partial B, y' \in \partial B} \phi(x; y, y')} \int_{\mathbb{R}^3 \setminus \overline{D}} (f v_k - w_f g) \, dx
\]

may behave as some power of \( \tau \) multiplied by a positive constant. The constant may contain some information about the geometry of the boundary of the obstacle at the points on \( \partial D \) that attain \( \min_{x \in \partial D, y \in \partial B, y' \in \partial B} \phi(x; y, y') \), i.e., the first reflection points between \( p \) and \( p' \).

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If \( q \in \Lambda_{ad}(p, p') \), then \( q \in E_c(p, p') \) with \( c = \min_{x \in \partial D} \phi(x; p, p') \) and the two tangent planes at \( q \) of \( \partial D \) and \( E_c(p, p') \) coincide (see lemma 4.3). We denote by \( S_q(\partial D) \) and \( S_q(E_c(p, p')) \) the shape operators (or the Weingarten maps) at \( q \) with respect to \( v_q \). Those are symmetric linear operators on the common tangent space at \( q \) with respect to \( v_q \). It is easy to see that \( S_q(E_c(p, p')) = S_q(\partial D) \geq 0 \) as the quadratic form on the same tangent space at \( q \) (see (4.21)).

Let \( Z_f \in H^1(\mathbb{R}^3 \setminus \overline{D}) \) solve
\[
(\Delta - \tau^2)Z_f = F_f(x, \tau) \text{ in } \mathbb{R}^3 \setminus \overline{D},
\]
Z_f = 0 on \( \partial D \).

(1.11)

It follows from (1.2) for \( h = f \) and (1.4) that \( w_f \) has the form
\[
w_f = v_f + \epsilon_f^0 + e^{-\tau T}Z_f,
\]
where \( \epsilon_f^0 \) satisfies
\[
(\Delta - \tau^2)\epsilon_f^0 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D},
\]
\[
\epsilon_f^0 = -v_f \text{ on } \partial D.
\]

(1.13)

Note that since \( \text{supp } f \cap \partial \overline{D} = \emptyset \), \( v_f \) is smooth in a neighbourhood of \( \partial D \) and thus, by elliptic regularity, we see that \( \epsilon_f^0 \) is smooth for \( x \in \mathbb{R}^3 \setminus D \). Moreover, note that \( \epsilon_f^0(x) \longrightarrow 0 \) as \( |x| \longrightarrow \infty \) rapidly and uniformly with respect to \( x/|x| \). This is a combination of the uniqueness of the weak solution of (1.13) and a potential theoretic construction of the solution; see, e.g., [3, 20] for the approach and [15] for an application to an inverse problem for the heat equation.

Given \( x \in \mathbb{R}^3 \), define \( d_{ad}(x) = \inf_{y \in \partial D} |y - x| \). It is well known that there exists a positive constant \( \delta_0 \) such that given \( x \in \partial D \), \( d_{ad}(x) < 2\delta_0 \), there exists a unique \( q = q(x) \) be the boundary point on \( \partial D \) such that \( x = q + d_{ad}(x)v_q \) [6]. If \( \partial D \) is \( C^3 \), then one may assume that both \( d_{ad}(x) \) and \( q(x) \) are \( C^2 \) for \( x \in \partial D \), \( d_{ad}(x) < 2\delta_0 \); \( x \in \mathbb{R}^3 \setminus D \), \( d_{ad}(x) < 2\delta_0 \), \( \partial D \) (lemma 1 of appendix in [6]). Note that \( v_q \) is the unit outer normal to \( \partial D \) at \( q \). For \( x \) with \( d_{ad}(x) < 2\delta_0 \), define \( x' = 2q(x) - x \).

Before describing our third result, we introduce a restriction on a class of obstacles which is satisfied with all convex obstacles.

**Definition 1.2.** We say that \( D \) is admissible if there exist positive constants \( C, \delta' (\leq 2\delta_0) \) and \( \tau_0 \) such that, for all \( y \in D \), \( d_{ad}(y) < \delta' \) and \( \tau \geq \tau_0 \).

\[
|\epsilon_f^0(y')| \leq C \int g e^{-\tau|x-x'|} \, dx.
\]

The following theorem gives an answer to the question raised above.

**Theorem 1.3.** Let \( B \) and \( B' \) satisfy \( \overline{B} \cup \overline{B'} \cap \partial \overline{D} = \emptyset \). Let \( f = \chi_B \) and \( g = \chi_{B'} \). Let \( c = \min_{x \in \partial D} \phi(x; p, p') \). Let \( T \) satisfy (1.5).

Assume that \( D \) is admissible and \( \partial D \) is \( C^3 \). If \( \Lambda_{ad}(p, p') \) is finite and for all \( q \in \Lambda_{ad}(p, p') \),
\[
\text{det } (S_q(E_c(p, p')) - S_q(\partial D)) > 0,
\]
then we have
\[
\lim_{\tau \to \infty} \tau^4 e^{\tau \min_{x \in \partial D, y \in \partial D'} \phi(x; y, y')} \int_{\mathbb{R}^3 \setminus \overline{B}} (f v_q - w_f) \, dx = \frac{\pi}{2} \sum_{q \in \Lambda_{ad}(p, p')} \left( \frac{\text{diam} B}{2|q - p|} \right) \left( \frac{\text{diam} B'}{2|q - p'|} \right) \frac{1}{\sqrt{\text{det } (S_q(E_c(p, p')) - S_q(\partial D))}}.
\]

(1.15)
Some remarks are in order.

- The right-hand side of (1.15) is symmetric with respect to the replacements $p \to p'$ and $p' \to p$. This is a kind of reciprocity.
- The quantity \( \det (S_q(E_c(p, p'))) = S_q(\partial D) \) gives some kind of information about the difference or deviation of the geometry between \( \partial D \) and \( E_c(p, p') \) at \( q \in \Lambda_{\partial D}(p, p') \).

The following proposition says that theorem 1.3 can cover convex obstacles.

**Proposition 1.1.** (i) If \( D \) is convex, then \( D \) is admissible and \( \Lambda_{\partial D}(p, p') \) consists of a single point. (ii) Let \( q \in \Lambda_{\partial D}(p, p') \) and assume that \( \partial D \) is contained in the half-space \( (x - q) \cdot v_q \leq 0 \); then (1.14) at \( q \) is satisfied.

For the proof, see the appendix. Thus, as a corollary, we obtain the following result.

**Corollary 1.1.** Let \( B \) and \( B' \) satisfy \( [\bar{B} \cup \bar{B}] \cap \bar{D} = \emptyset \). Let \( f = \chi_B \) and \( g = \chi_{B'} \). Let \( c = \min_{x \in \partial D} \phi(x; p, p') \). Let \( T \) satisfy (1.5). If \( D \) is convex and \( \partial D \) is \( C^3 \), then (1.15) whose right-hand side consists of a single term is valid.

Note that, for the back-scattering case \( B = B' \), using completely the same argument as in [12] in a bounded domain, we obtain

\[
\lim_{\tau \to \infty} \tau^4 e^{2\text{dist}(\partial D, B)} \int_B (v_f - w_f) \, dx = \frac{\pi}{2} \left( \frac{\text{diam}B}{2d_{\partial D}(p)} \right)^2 \sum_{x \in \partial D, |x - p| = d_{\partial D}(p)} \frac{1}{\sqrt{P_{\partial D}(1/d_{\partial D}(p); x)}},
\]

where \( d_{\partial D}(p) = \inf_{\phi \in \partial D} |x - p| \); \( P_{\partial D}(\lambda; q) = (\lambda - k_1(q))(\lambda - k_2(q)) \) and \( k_1(q) \) and \( k_2(q) \) denote the principle curvatures of \( \partial D \) at \( q \) with respect to \( v_q \) (see appendix in [6]). Note that the Gauss curvature \( K_{\partial D}(q) \) and mean curvature \( H_{\partial D}(q) \) at \( q \) with respect to \( v_q \) are given by \( k_1(q)k_2(q) \) and \( (k_1(q) + k_2(q))/2 \), respectively.

The assumptions therein are

- \( \partial D \) is \( C^3 \);
- \( T > 2\text{dist}(D, B) \);
- the set of all points \( x \in \partial D \) with \( |x - p| = d_{\partial D}(p) \) is finite and each point \( p \) in the set satisfies \( P_{\partial D}(1/d_{\partial D}(p); q) > 0 \).

It is not assumed that \( D \) is admissible in (1.16) unlike (1.15).

The quantity \( P_{\partial D}(1/d_{\partial D}(p); q) \) at \( q \in \partial D \) with \( |q - p| = d_{\partial D}(p) \) denotes a ‘deflection’ of the surface \( \partial D \) at \( q \) from the sphere \( |x - p| = d_{\partial D}(p) \) since we know from, e.g., proposition 4.2 in this paper that \( P_{\partial D}(q; 1/d_{\partial D}(p)) = \det (S_q(\partial B_{d_{\partial D}(p)}(p))) = S_q(\partial D) \), where \( B_{d_{\partial D}(p)}(p) = \{ x \in \mathbb{R}^3 \mid |x - p| < d_{\partial D}(p) \} \). Thus, formula (1.15) of theorem 1.3 can be considered as an extension of (1.16) to the bistatic data case. See also remark 5.2 for a comparison.

After having theorem 1.3, everyone wishes to extract the geometry of \( \partial D \) at all the first reflection points. The complete answer for general obstacle is not known; however, under the admissibility of \( D \), one can obtain the following result.

**Theorem 1.4.** Let \( q \in \Lambda_{\partial D}(p, p') \) be known. Let \( T \) satisfy (1.5). Let \( B \) and \( B' \) satisfy \( \bar{B} \cup \bar{B} \cap \bar{D} = \emptyset \). If \( D \) is admissible and \( \partial D \) is \( C^3 \), then one can extract

\[
H_{\partial D}(q) = \frac{S_q(\partial D)(A_q(p) \times A_q(p')) \cdot (A_q(p) \times A_q(p'))}{2(1 + A_q(p) \cdot A_q(p'))}
\]

and

\[ K_{\partial D}(q) \]

from \( u_f \) on \( B' \times ]0, T[ \) with \( f = \chi_{B'} \), where

\[ A_q(x) = \frac{q - x}{|q - x|}. \]
Note that $A_q(p) \times A_q(p')$ belongs to the tangent space of $\partial D$ at $q$. For this, see lemma 4.3 in section 4.

This theorem may suggest the following.

- If one wishes to know the mean curvature at a first reflection point precisely, then one should make the transmitter and the receiver approach as much as possible. It is because $A_q(p) \times A_q(p')$ will disappear approximately at this time and thus the correction term in (1.17) can be ignored.
- On the other hand, the Gauss curvature at the first reflection point can be extracted regardless of the position of a transmitter and a receiver at any time except for the condition $[\overline{B} \cup \overline{B}] \cap \overline{D} = \emptyset$.

As a corollary of theorems 1.1, 1.2 and 1.4 we have the following result.

**Corollary 1.2.** Assume that $D$ is an open ball. Let $T$ satisfy (1.8). Let $B$ and $B'$ satisfy $[\overline{B} \cup \overline{B}] \cap \overline{D} = \emptyset$. Then, one can extract $D$ itself from $u_f$ on $B' \times ]0, T[$ with $f = \chi_B$.

The steps to reconstruct an unknown open ball $D$ are as follows.

1. **Step 1.** Determine $c = \min_{x \in \partial D} \phi(x; p, p')$ via theorem 1.1.
2. **Step 2.** Determine the unique point $q$ in $\Lambda_{\partial D}(p, p')$ together with $v_q$ via theorem 1.2.
3. **Step 3.** Determine $K_{\partial D}(q)$ via theorem 1.4.

Then, the radius and centre of $D$ are given by $1/\sqrt{K_{\partial D}(q)}$ and $q - (1/\sqrt{K_{\partial D}(q)})v_q$, respectively.

The reconstruction problem of a spherical obstacle has also been considered in the frequency domain. For example, see [1], which employs a spherical wave as an incident wave and uses a low frequency limit for the reconstruction.

The four steps described above give a constructive proof of a uniqueness theorem in an inverse obstacle problem in the sense that it does not make use of the uniqueness of the continuation of the solution of the governing equation of the wave. The following uniqueness result employs the bistatic data over a finite time interval and itself seems to be new.

**Theorem 1.5.** Let $D_1$ and $D_2$ be open balls. Let $u_j'$ be the solution of (1.1) with $f = \chi_B$ and $D = D_j$. Let $T$ satisfy (1.8). Let $B$ and $B'$ satisfy $[\overline{B} \cup \overline{B}] \cap \overline{D} = \emptyset$ for $j = 1, 2$. If $u_j' = u_2'$ on $B' \times ]0, T[$, then $D_1 = D_2$.

We refer the readers to [16, 17, 23, 24] for various uniqueness theorems for inverse obstacle problems for hyperbolic equations over a finite time interval.

Another corollary from theorem 1.4 is concerned with the determination of the directions of principle curvatures at a point on $\partial D$.

Assume that $D$ is convex and $\partial D$ is $C^3$. From proposition 1.1, we know that $\Lambda_{\partial D}(p, p')$ consists of a single point. We denote the point by $q(p, p')$. We denote by $p(\theta)$ and $p(\theta)$ the points rotated around the line directed $v_q$ at $q = q(p, p')$ counterclockwise with the rotation angle $\theta \in [0, 2\pi]$ of $p$ and $p'$. Thus $p(0) = p$ and $p'(0) = p'$.

Then, for all $\theta \in [0, 2\pi]$ we know that $\Lambda_{\partial D}(p(\theta), p'(\theta))$, $A_q(p(\theta)) \cdot A_q(p'(\theta))$, $[A_q(p(\theta)) \times A_q(p'(\theta))]$, $\phi(q(\theta); p(\theta), p'(\theta))$ and $v_q$ at $q = q(p(\theta), p'(\theta))$ are invariant with respect to $\theta$.

Let $B(\theta)$ denote the open ball centred at $p(\theta)$ with radius $\eta$ and $B'(\theta)$ the open ball centred at $p'(\theta)$ with radius $\eta$. We have $[\overline{B}(\theta) \cup \overline{B}(\theta)] \cap \overline{D} = \emptyset$ provided $[\overline{B} \cup \overline{B}] \cap \overline{D} = \emptyset$ and $D$ is convex.
Then, from (1.17) in theorem 1.4 applied to \( f = f(\theta) \) and \( B = B(\theta) \) and \( B' = B'(\theta) \), we obtain the function of \( \theta \):

\[
\theta \mapsto \tilde{H}_{\partial D}(q; p(\theta), p'(\theta)) \equiv H_{\partial D}(q) - \frac{1}{2} \sqrt{\frac{1 - A_q(p) \cdot A_q(p')}{1 + A_q(p) \cdot A_q(p')}} S_q(\partial D)(V(\theta)) \cdot V(\theta),
\]

where \( V(\theta) \) denotes the unit vector directed to \( A_q(p(\theta)) \times A_q(p'(\theta)) \).

Now assume that \( A_q(p) \times A_q(p') \neq 0 \). Then, \( V(\theta) \) attains all the tangent vectors at \( q \) of \( \partial D \) and thus from the behaviour of \( \tilde{H}_{\partial D}(q; p(\theta), p'(\theta)) \) as a function of \( \theta \) one can determine all the directions of principle curvatures, say, \( V(\theta_1) \) and \( V(\theta_2) \) with some \( \theta_1 \) and \( \theta_2 \). Then, we have

\[
\frac{\tilde{H}_{\partial D}(q; p(\theta_1), p'(\theta_1)) + \tilde{H}_{\partial D}(q; p(\theta_2), p'(\theta_2))}{2} = \left\{ \begin{array}{ll} 1 & \frac{1 - A_q(p) \cdot A_q(p')}{1 + A_q(p) \cdot A_q(p')} \\ \end{array} \right\} H_{\partial D}(q).
\]

Thus, we obtain \( H_{\partial D}(q) \).

Summing up, we have obtained the following result.

**Corollary 1.3.** Let \( B \) and \( B' \) satisfy \( \overline{B \cup B'} \cap \overline{D} = \emptyset \). Let \( T \) satisfy (1.5). Assume that \( D \) is convex and \( \partial D \) is \( C^3 \); \( q = q(p, p') \) is known; \( A_q(p) \times A_q(p') \neq 0 \). Then, one can extract all the directions of principle curvatures, mean and Gauss curvatures, in other words, the shape operator at \( q \) of \( \partial D \) from \( u_f(\theta) \) over \( B(\theta) \times [0, 1] \) for all \( \theta \in [0, 2\pi] \), where \( f(\theta) \) denotes the characteristic function of \( B(\theta) \).

A brief outline of this paper is as follows. Theorem 1.1 is proved in sections 2 and 3. As described above, the key point of the proof is to derive (1.9) and (1.10) and those are proved in sections 2 and 3, respectively.

Theorem 1.2 is proved in subsection 5.1. The proof contains an explicit characterization of the first reflector in terms of the bistatic data. See remark 5.1 for the resulting procedure to determine all the first reflection points.

Theorem 1.3 is proved in section 4. The key point in the proof of (1.15) as well as (1.16) is to identify the term which contains the leading term of the indicator function. See (4.1) for the term. We found that one of two reflection arguments developed in [18] works for the purpose. It is based on the reflection across \( \partial D \) and a pointwise estimate of \( \varepsilon^2 \) near \( \partial D \), that is, the use of the admissibility of \( D \). The argument is presented in the proof of lemma 4.2 in section 4. Note that another reflection argument used in the proof of (1.16) is free from the admissibility assumption; however, it cannot be applied to the case when \( f \neq g \).

Theorem 1.4 is proved in subsection 5.2. The proof is based on an asymptotic formula which is a consequence of theorem 1.3 and an explicit formula of the determinant of the difference of two shape operators at \( q \in A_{\partial D}(p, p') \) as derived in subsection A.4 in the appendix.

In the final section, we give a conclusion of this paper and comments on further problems.

### 2. An upper bound of the indicator function

Define

\[
J(\tau; f, g) = \int_D (\nabla v_f \cdot \nabla v_g + \tau^2 v_f v_g) \, dx. \tag{2.1}
\]

We have the following expression of the indicator function.
Proposition 2.1. It follows that
\[
\int_{\mathbb{R}^d \setminus D} (fv_g - w_f g) \, dx = J(\tau; f, g) + \int_{\mathbb{R}^d \setminus D} (\nabla \varepsilon^0_f \cdot \nabla \varepsilon^0_g + \tau^2 \varepsilon^0_f \varepsilon^0_g) \, dx \\
- e^{-\tau T} \int_{\mathbb{R}^d \setminus D} (\nabla Z_f \cdot \nabla v_g + \tau^2 Z_f v_g) \, dx.
\] (2.2)

Proof. From (1.2) and (1.4), we have
\[
\int_{\mathbb{R}^d \setminus D} (fv_g - w_f g) \, dx = \int_{\partial D} \frac{\partial w_f}{\partial \nu} v_g \, dS + e^{-\tau T} \int_{\mathbb{R}^d \setminus D} F_f(x, \tau) v_g \, dx.
\] (2.3)
Rewrite
\[
\int_{\partial D} \frac{\partial w_f}{\partial \nu} v_g \, dS = \int_{\partial D} \frac{\partial w_f}{\partial v} v_g \, dS + \int_{\partial D} \frac{\partial \varepsilon^0_f}{\partial v} v_g \, dS + e^{-\tau T} \int_{\partial D} \frac{\partial Z_f}{\partial v} v_g \, dS.
\] (2.4)
Integration by parts yields
\[
\int_{\partial D} \frac{\partial w_f}{\partial v} v_g \, dS = J(\tau; f, g).
\] (2.5)
On the other hand, (1.13) yields
\[
\int_{\partial D} \frac{\partial \varepsilon^0_f}{\partial v} v_g \, dS = - \int_{\partial D} \frac{\partial \varepsilon^0_f}{\partial \nu} v_g \, dS = \int_{\mathbb{R}^d \setminus D} (\nabla \varepsilon^0_f \cdot \nabla \varepsilon^0_g + \tau^2 \varepsilon^0_f \varepsilon^0_g) \, dx.
\] (2.6)
Furthermore, it follows from (1.2) and (1.11) that
\[
- \int_{\partial D} \frac{\partial Z_f}{\partial v} v_g \, dS = \int_{\mathbb{R}^d \setminus D} (\nabla Z_f \cdot \nabla v_g + \tau^2 Z_f v_g + F_f v_g) \, dx.
\]
Now from this together with (2.3)–(2.6) we obtain (2.2). □

Lemma 2.1. As \( \tau \to \infty \),
\[
\left| \int_{\mathbb{R}^d \setminus D} (\nabla Z_f \cdot \nabla v_g + \tau^2 Z_f v_g) \, dx \right| = O(\tau^{-1}).
\] (2.7)

Proof. From (1.2), we obtain
\[
\int_{\mathbb{R}^d} \left( |\nabla v_g|^2 + \tau^2 |v_g - \frac{g}{2\tau^2}|^2 \right) \, dx = \frac{1}{4\tau^2} \int_{B} |g|^2 \, dx.
\]
Since
\[
|v_g - \frac{g}{2\tau^2}|^2 \geq \frac{1}{2} |v_g|^2 - \frac{|g|^2}{4\tau^2},
\]
from this we obtain
\[
\frac{1}{2} \int_{\mathbb{R}^d} (|\nabla v_g|^2 + \tau^2 |v_g|^2) \, dx \leq \frac{1}{2\tau^2} \int_{B} |g|^2 \, dx
\]
and thus
\[
\int_{\mathbb{R}^d} (|\nabla v_g|^2 + \tau^2 |v_g|^2) \, dx \leq \frac{1}{\tau^2} \int_{B} |g|^2 \, dx.
\] (2.8)
Similarly, it follows from (1.11) that
\[
\int_{\mathbb{R}^d \setminus D} (|\nabla Z_f|^2 + \tau^2 |Z_f|^2) \, dx \leq \frac{1}{\tau^2} \int_{\mathbb{R}^d \setminus D} |F_f|^2 \, dx.
\] (2.9)
A combination of (2.8) and (2.9) and the estimate \( \|F_j\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(\tau) \) yields (2.7).

Thus, a combination of (2.2) and (2.7) gives
\[
\int_{\mathbb{R}^3 \setminus \mathcal{D}} (fv - wg) \, dx = J(\tau; f, g) + \int_{\mathbb{R}^3 \setminus \mathcal{D}} (\nabla \varepsilon^0_j \cdot \nabla \varepsilon + \tau^2 \varepsilon^0_j \varepsilon^0_j) \, dx + O(\tau^{-1} e^{-\tau T}). \tag{2.10}
\]
For the second term on the right-hand side, we have the following estimate.

**Lemma 2.2.** As \( \tau \to \infty \),
\[
\int_{\mathbb{R}^3 \setminus \mathcal{D}} \left( |\nabla \varepsilon^0_j|^2 + \tau^2 |\varepsilon^0_j|^2 \right) \, dx = O(\tau^2 J(\tau; f, f)). \tag{2.11}
\]

**Proof.** It is an application of the trace theorem twice and integration by parts. More precisely, choose \( \tilde{v} \in H^1(\mathbb{R}^3 \setminus \mathcal{D}) \) in such a way that
\[
\|\tilde{v}\|_{H^1(\mathbb{R}^3 \setminus \mathcal{D})} \leq C \|v\|_{L^2(\partial \mathcal{D})}, \tag{2.12}
\]
where \( C > 0 \) and is independent of \( v \). Integration by parts (or the weak formulation of (1.13)) yields
\[
\int_{\partial \mathcal{D}} \frac{\partial \epsilon^0_j}{\partial v} \, dv = -\int_{\mathbb{R}^3 \setminus \mathcal{D}} (\nabla \varepsilon^0_j \cdot \nabla \tilde{v} + \tau^2 \varepsilon^0_j \tilde{v}) \, dx. \tag{2.13}
\]
A combination of (2.12) and (2.13) gives
\[
\left| \int_{\partial \mathcal{D}} \frac{\partial \epsilon^0_j}{\partial v} \, dv \right| \leq C(|\nabla \epsilon^0_j|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} + \tau^2 |\epsilon^0_j|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})}) \|v\|_{L^2(\partial \mathcal{D})} \|v\|_{H^1(\partial \mathcal{D})}. \tag{2.14}
\]
Since
\[
\max (|\nabla \epsilon^0_j|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})}, \tau |\epsilon^0_j|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})}) \leq \left( \int_{\mathbb{R}^3 \setminus \mathcal{D}} (|\nabla \epsilon^0_j|^2 + \tau^2 |\epsilon^0_j|^2) \, dx \right)^{1/2},
\]
it follows from (2.14) that
\[
\left| \int_{\partial \mathcal{D}} \frac{\partial \epsilon^0_j}{\partial v} \, dv \right| \leq C(1 + \tau) \left( \int_{\mathbb{R}^3 \setminus \mathcal{D}} (|\nabla \epsilon^0_j|^2 + \tau^2 |\epsilon^0_j|^2) \, dx \right)^{1/2} \|v\|_{L^2(\partial \mathcal{D})} \|v\|_{H^1(\partial \mathcal{D})}.
\]
From this together with (2.6) for \( f = g \), we obtain
\[
\int_{\mathbb{R}^3 \setminus \mathcal{D}} (|\nabla \epsilon^0_j|^2 + \tau^2 |\epsilon^0_j|^2) \, dx \leq C^2 (1 + \tau)^2 \|v\|_{L^2(\partial \mathcal{D})}^2 \|v\|_{H^1(\partial \mathcal{D})}^2. \tag{2.15}
\]
By the trace theorem, we have
\[
\|v\|_{L^2(\partial \mathcal{D})} \leq C' (|\nabla v|_{L^2(\mathcal{D})} + \|v\|_{L^2(\mathcal{D})}^2),
\]
where \( C' > 0 \) is independent of \( v \). Now this together with (2.15) and the trivial estimates
\[
\max (\|\nabla v\|_{L^2(\mathcal{D})}^2, \tau^2 \|v\|_{L^2(\mathcal{D})}^2) \leq J(\tau; f, f),
\]
yields (2.11).

Therefore, (1.9) with \( \mu_1 = 2 \) follows from (2.10) and (2.11) together with the following estimate.

**Lemma 2.3.** We have, as \( \tau \to \infty \),
\[
J(\tau; f, g) = O(\tau e^{-\tau \min_{\text{admissible}} \phi(\mu_1, y)}, \tag{2.16}
\]
\[\phi(\mu_1, y)).\]
3.1. A reduction to a convex obstacle and the proof of (1.10).

Proof. It follows from (1.3) and (2.5) that
\[ J(\tau; f, g) = \left( \frac{1}{4\pi} \right)^2 \int_{\partial D} dS \int_{B \times B} k_\tau(x, y, y') dy dy', \]
where
\[ k_\tau(x, y, y') = \left( \frac{1}{|x - y|} + \tau \right) \frac{(y - x) \cdot \nu_x}{|x - y|^2|x - y'|} e^{-\tau|x-y|+|x-y'|}, \quad (x, y, y') \in \partial D \times B \times B'. \]

Since we have
\[ \inf_{x \in \partial D, y, y' \in B} \phi(x; y, y') = \min_{x \in \partial D \times \partial B, y \in \partial B} \phi(x; y, y') \]
and \( \overline{B} \cap \overline{D} = \overline{B} \cap \overline{D} = \emptyset \), from (2.17) and (2.18), we obtain (2.16).

3. A lower bound of the indicator function

3.1. A reduction to a convex obstacle and the proof of (1.10).

Rewriting the second term on the right-hand side of (2.10) with (2.6), one has
\[ \int_{\mathbb{R}^3 \setminus \overline{D}} (f u - w_j g) dx = J(\tau; f, g) + \int_{\partial D} \frac{\partial e^0_j}{\partial v} v_x dS + O(\tau^{-1} e^{-\tau T}). \]  
(3.1)

In general, we do not know the signature of the second term on the right-hand side of (3.1); however, we know that the function under integral is non-negative at a special point on \( \partial D \) by virtue of the following lemma which is an application of the maximum principle for differential operator \( \Delta - \tau^2 \) and a reflection argument in [18]. It corresponds to lemma 3.7 in [18] in which \( v_j \) in (1.13) is replaced with \( -e^{-\tau x \cdot \omega} \) for a \( \omega \in S^2 \).

Lemma 3.1. Let \( q \in \partial D \) be a point of support of \( D \), i.e., \( D \) is contained in the half-space \( x \cdot n_q < q \cdot n_q \). We have
\[ \frac{\partial e^0_j}{\partial v}(q) \geq 0. \]  
(3.2)

Proof. First, we prove that, for all \( x \in \mathbb{R}^3 \setminus \overline{D} \),
\[ e^0_j(x) \geq -v_j(x). \]  
(3.3)

Define \( w^j_j = e^0_j + v_j \in H^1(\mathbb{R}^3 \setminus \overline{D}) \). It holds that
\( (\Delta - \tau^2) w^j_j = -f \) in \( \mathbb{R}^3 \setminus \overline{D}, \)
\[ w^j_j = 0 \text{ on } \partial D. \]

Since \( (\Delta - \tau^2) w^j_j \leq 0 \) in \( \mathbb{R}^3 \setminus \overline{D} \), from the weak maximum principle for operators of divergence form (theorem 8.1 in [6]), we obtain
\[ \inf_{(R^3 \setminus \overline{D}) \cap B_R} w^j_j \geq \min_{(R^3 \setminus \overline{D}) \cap S_R} (\inf w^j_j, 0), \]  
(3.4)

where \( B_R \) denotes an arbitrary open ball centred at the origin such that \( \overline{D} \cup \supp f \subset B_R \) and \( S_R = \partial B_R \). Since \( e^0_j \) decays as \( |x| \to \infty \) uniformly with respect to \( x/|x| \), we have \( \inf_{S_R} w^j_j \to 0 \) as \( R \to \infty \), and thus, from (3.4), we obtain \( w^j_j(x) \geq 0 \) for all \( x \in \mathbb{R}^3 \setminus \overline{D} \). This completes the proof of (3.3).
The equality in (3.3) holds for \( x = q \). This implies the following inequality for the normal derivatives:
\[
\frac{\partial \epsilon^0}{\partial v}(q) \geq - \frac{\partial v_f}{\partial v}(q). \tag{3.5}
\]

Second, we prove that, for all points that satisfy \( x \cdot n_q \geq q \cdot n_q \),
\[
\epsilon^0_f(x) \geq -v_f(x'), \tag{3.6}
\]
where \( x' \) is the image of \( x \) under reflection across the plane \( x \cdot n_q = q \cdot n_q \). Since \( x = x' \) on the plane \( x \cdot n_q = q \cdot n_q \), (3.3) shows that (3.6) is satisfied there. Note that also \( v_f'(x) \equiv v_f(x') \) satisfies \((\Delta - \tau^2)v_f' = -f(x') \leq 0\). Applying the weak maximum principle to \((u^0) = \epsilon^0 + v_f'\) in the half-space \( x \cdot n_q > q \cdot n_q \), one obtains as before (3.6) holds throughout the half-space. This completes the proof of (3.6).

Since the equality in (3.6) holds for \( x = q \), it follows as before that
\[
\frac{\partial \epsilon^0_f}{\partial v}(q) \geq - \frac{\partial v_f}{\partial v}[v_f(x')]_{x=q} = \frac{\partial v_f}{\partial v}(q). \tag{3.7}
\]
Now a combination of (3.5) and (3.7) yields (3.2). \( \square \)

The following lemma is an easy consequence of the \( C^2 \)-regularity of \( \partial \mathcal{D} \) and thus the proof is omitted.

**Lemma 3.2.** Let \( q \in \Lambda_{\partial \mathcal{D}}(p, p') \). Then, there exists an open ball \( \tilde{\mathcal{D}} \) contained in \( \mathcal{D} \) such that \( q \in \Lambda_{\partial \tilde{\mathcal{D}}}(p, p') \) and thus \( \min_{x \in \partial \tilde{\mathcal{D}}} \phi(x; p, p') = \min_{x \in \partial \mathcal{D}} \phi(x; p, p') \).

Let \( \tilde{u} = \tilde{u}_f \) denote the weak solution of the following initial boundary value problem:
\[
\begin{align*}
\partial_t^2 \tilde{u} - \Delta \tilde{u} &= 0 \text{ in } (\mathbb{R}^3 \setminus \tilde{\mathcal{D}}) \times [0, T], \\
\tilde{u}(x, 0) &= 0 \text{ in } \mathbb{R}^3 \setminus \tilde{\mathcal{D}}, \\
\partial_t \tilde{u}(x, 0) &= f(x) \text{ in } \mathbb{R}^3 \setminus \tilde{\mathcal{D}}, \\
\tilde{u} &= 0 \text{ on } \partial \mathcal{D} \times [0, T].
\end{align*} \tag{3.8}
\]

Define
\[
\tilde{w}_f(x, \tau) = \int_0^\tau e^{-\tau \tau} \tilde{u}(x, t) \, dt, \quad x \in \mathbb{R}^3 \setminus \tilde{\mathcal{D}}, \quad \tau > 0.
\]

**Lemma 3.3.** We have
\[
\int_{\mathbb{R}^3 \setminus \mathcal{D}} (f \epsilon_k - w_f g) \, dx \geq \int_{\mathbb{R}^3 \setminus \tilde{\mathcal{D}}} (f \epsilon_k - \tilde{w}_f g) \, dx + O(\tau^{-1} e^{-\tau\tau}). \tag{3.9}
\]

**Proof.** Let \( \tilde{Z}_f \in H^1(\mathbb{R}^3 \setminus \tilde{\mathcal{D}}) \) solve
\[
(\Delta - \tau^2) \tilde{Z}_f = \tilde{F}_f(x, \tau) \text{ in } \mathbb{R}^3 \setminus \tilde{\mathcal{D}},
\]
\[
\tilde{Z}_f = 0 \text{ on } \partial \mathcal{D},
\]
where
\[
\tilde{F}_f(x, \tau) = \partial_t \tilde{u}_f(x, T) + \tau \tilde{u}_f(x, T), \quad x \in \mathbb{R}^3 \setminus \tilde{\mathcal{D}}.
\]
Similar to \( Z_f \) which is the solution of (2.2), we have \( \|\tilde{Z}_f\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(\tau^{-1}) \). And, similar to (1.12) for \( w_f, \tilde{w}_f \) has the form
\[
\tilde{w}_f = v_f + \tilde{e}^0_f + e^{-\tau\tau} \tilde{Z}_f,
\]
where $\tilde{\epsilon}_f^0$ satisfies
\[
(\Delta - \tau^2)\tilde{\epsilon}_f^0 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D},
\]
\[
\tilde{\epsilon}_f^0 = -v_f \text{ on } \partial D.
\]
Thus, we have
\[
\tilde{w}_f - w_f = (\tilde{\epsilon}_f^0 - \epsilon_f^0) + e^{-\tau T}(\tilde{Z}_f - Z_f) \text{ in } \mathbb{R}^3 \setminus \overline{D}.
\]
Since $\tilde{v}_f \geq 0$ on $\partial \tilde{D}$ and $\epsilon_f^0(x) \to 0$ as $|x| \to \infty$, by the maximum principle for the modified Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$, we have $-\tilde{\epsilon}_f^0 \leq \tilde{v}_f$ in $\mathbb{R}^3 \setminus \tilde{D}$ and thus $-\epsilon_f^0 \leq \epsilon_f^0$ on $\partial D$. Again by the maximum principle for the modified Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$, we obtain $-\tilde{\epsilon}_f^0 \leq -\epsilon_f^0$ in $\mathbb{R}^3 \setminus D$. Therefore, we obtain
\[
\tilde{w}_f - w_f \geq e^{-\tau T}(\tilde{Z}_f - Z_f) \text{ in } \mathbb{R}^3 \setminus \overline{D}.
\]
(3.10)
Since both supp $g$ and supp $f$ are contained in $\mathbb{R}^3 \setminus \overline{D}$ and thus combining this with (3.10), we obtain
\[
\int_{\mathbb{R}^3 \setminus \overline{D}} (fv_g - w_f g) \, dx \geq \int_{\mathbb{R}^3 \setminus \overline{D}} (fv_g - \tilde{w}_f g) \, dx \geq e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} (\tilde{Z}_f - Z_f) g \, dx.
\]
From the $L^2$-bounds for $Z_f$ and $\tilde{Z}_f$, we see that this right-hand side has the bound $O(\tau^{-1} e^{-\tau T})$. \hfill \square

Since $\tilde{D}$ is convex, every point $q \in \partial \tilde{D}$ is a point of support of $\tilde{D}$ and thus, from (3.1), (3.2) and (3.9), we obtain
\[
\int_{\mathbb{R}^3 \setminus \overline{D}} (fv_g - w_f g) \, dx \geq \tilde{J}(\tau; f, g) + O(\tau^{-1} e^{-\tau T}),
\]
(3.11)
where
\[
\tilde{J}(\tau; f, g) = \int_{\tilde{D}} (\nabla v_f \cdot \nabla v_g + \tau^2 v_f v_g) \, dx.
\]
Now everything is reduced to give a lower estimate for $\tilde{J}(\tau; f, g)$ as $\tau \to \infty$. For this and future use of it in the sound-hard obstacle case, we give the estimate for $J(\tau; f, g)$ for general $D$.

In the following lemma, we do not assume that $D$ is convex.

**Lemma 3.4.** There exist positive constants $C$, $\mu$ and $\tau_0$ such that, for all $\tau \geq \tau_0$,
\[
\tau^{2+\mu} e^{\min_{x \in \partial D} \phi(x, p, f')} e^{-\tau(q + y')} J(\tau; f, g) \geq C.
\]
(3.12)

We give the proof of this lemma in the following subsection.

It follows from (3.11) and (3.12) for $D = \tilde{D}$ that there exist positive constants $C'$ and $\tau_0 > 0$ such that, for all $\tau \geq \tau_0'$,
\[
\tau^{2+\mu} e^{\min_{x \in \partial D} \phi(x, u, f')} \int_{\mathbb{R}^3 \setminus \overline{D}} (fv_g - w_f g) \, dx \geq C'
\]
provided $T$ satisfies (1.5). This completes the proof of (1.10).
3.2. Proof of lemma 3.4.

In this subsection, we never assume that \( D \) is convex. Let \( A(x, \tau) \) be an arbitrary positive function of \( x \in \partial D \) with parameter \( \tau > 0 \). For another function \( B(x, \tau) \), in the following, \( B(x, \tau) = O(A(x, \tau)) \) as \( \tau \to \infty \) and uniformly with respect to \( x \in \partial D \) means that there exist positive constants \( \tau_0 \) and \( C \) independent of \( x \in \partial D \) such that, for all \( \tau \geq \tau_0 \) and \( x \in \partial D \), we have \( |B(x, \tau)| \leq CA(x, \tau) \).

The proof of lemma 3.4 starts by having the following expression.

**Lemma 3.5.** There exists a positive constant \( C \) such that, as \( \tau \to \infty \)

\[
J(\tau; f, g) = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right) \left( \eta' - \frac{1}{\tau} \right) \int_{\partial D} \frac{(p - x) \cdot v_k}{|x - p|} \left( 1 + \frac{1}{\tau |x - p|} \right) e^{-\tau(df(x) + dg(x))} dS_x \\
+ O\left( \tau^{-1} e^{-\tau \inf_{x \in \partial D} (df(x) + dg(x))} \right) (e^{-C \tau \min_{x \in \partial D} df(x)} + e^{-C \tau \min_{x \in \partial D} dg(x)}).
\tag{3.13}
\]

**Proof.** By [12], we have, as \( \tau \to \infty \) and uniformly with respect to \( x \in \partial D \),

\[
v_f(x) = \frac{1}{2\tau} ((J_f^+)_g(x, \tau) + O(e^{-\tau df(x)(1+C)})
\]

and

\[
\nabla v_f(x) = \frac{1}{2} \left( (J_f^+)_f(x, \tau) \frac{p - x}{|x - p|} + O(e^{-\tau df(x)(1+C)}) \right),
\]

where \( C \) is a positive constant,

\[
(J_f^+)_f(x, \tau) = \frac{e^{-\tau df(x)}}{\tau |x - p|} \left( \eta - \frac{1}{\tau} \right) \left( 1 + \frac{1}{\tau |x - p|} \right) \\
+ e^{-\tau df(x)/\tau^2} \left( dB(x) \left( \frac{1 + 2\eta}{d^2(x)} + \frac{1}{\tau} \right) \right)
\]

and

\[
(J_f^+)_g(x, \tau) = \frac{e^{-\tau df(x)}}{\tau \tau |x - p|} \left( \eta' - \frac{1}{\tau} \right) + e^{-\tau df(x)/\tau^2} \left| \frac{1 + 2\eta}{d^2(x)} + \frac{1}{\tau} \right|.
\]

From these and (2.5), we obtain

\[
J(\tau; f, g) = \frac{1}{4\tau^3} \int_{\partial D} \frac{(p - x) \cdot v_k}{|x - p|} ((J_f^+)_g(x, \tau) dS_x \\
+ O\left( \tau^{-1} e^{-\tau \inf_{x \in \partial D} (df(x) + dg(x))} \right) (e^{-C \tau \min_{x \in \partial D} df(x)} + e^{-C \tau \min_{x \in \partial D} dg(x)})).
\]

This yields (3.13) since we have as \( \tau \to \infty \) and uniformly with respect to \( x \in \partial D \),

\[
(J_f^+)_g(x, \tau)(J_f^+)_f(x, \tau) = \frac{e^{-\tau df(x)/\tau^2(|x - p|)}}{\tau^2 |x - p|} \left( \eta - \frac{1}{\tau} \right) \left( \eta' - \frac{1}{\tau} \right) \left( 1 + \frac{1}{\tau |x - p|} \right) \\
+ O\left( \tau^{-3} e^{-\tau df(x)} \left( e^{-\tau df(x)/\tau^{2(\sqrt{2})}} + e^{-\tau df(x)/\tau^{2(\sqrt{2})}} \right) \right) + O(\tau^{-4} e^{-\tau df(x) + dg(x)}(1+C)).
\]

Now we give a lower estimate of \( J(\tau; f, g) \) as \( \tau \to \infty \) by using (3.13).

Define

\[
I_m(\tau) = \int_{\partial D} \frac{(p - x) \cdot v_k}{|x - p|^{m+1}} e^{-\tau \phi(p, p')} \ dS_x,
\]

where \( m = 2, 3 \).
Since $d_B(x) = |x - p| - \eta$ for $x \in \mathbb{R}^3 \setminus B$ and $d_B(x) = |x - p'| - \eta'$ for $x \in \mathbb{R}^3 \setminus B'$, it follows from (3.13) that
\[
e^{-\tau(g + \eta)} J(\tau; f, g) = \frac{1}{4\tau^3} \left( \eta - \frac{1}{\tau} \right) \left( \eta' - \frac{1}{\tau} \right) \left( I_2(\tau) + \frac{1}{\tau} I_3(\tau) \right) + O(\tau^{-1} e^{-\tau \min_{x \in \partial D} \Phi(x; p, p')} (e^{-\tau \min_{x \in \partial D} d_B(x)} + e^{-\tau \min_{x \in \partial D} d_B(x)}). \tag{3.14}
\]

Since
\[
\nabla \cdot \left\{ \frac{(p - x)}{|x - p|^m|x - p'|} \right\} = \frac{(m - 3)}{|x - p|^m|x - p'|} + \frac{(p - x) \cdot (p' - x)}{|x - p|^m|x - p'|^3}
\]
and
\[
(p - x) \cdot \nabla \phi(x; p, p') = \frac{(p - x) \cdot (p' - x)}{|x - p'|},
\]
we have
\[
I_m(\tau) = (m - 3) \int_D \frac{e^{-\tau \phi(x; p, p')}}{|x - p|^m|x - p'|} \, dx + \frac{(p - x) \cdot (p' - x)}{|x - p|^m|x - p'|^3} \int_D e^{-\tau \phi(x; p, p')} \, dx
\]
\[
- \tau \int_D \frac{(p - x) \cdot \nabla \phi(x; p, p')}{|x - p|^m|x - p'|^3} e^{-\tau \phi(x; p, p')} \, dx
\]
\[
= (m - 3) \int_D \frac{e^{-\tau \phi(x; p, p')}}{|x - p|^m|x - p'|} \, dx + \frac{(p - x) \cdot (p' - x)}{|x - p|^m|x - p'|^3} \int_D e^{-\tau \phi(x; p, p')} \, dx
\]
\[
+ \tau \int_D \frac{e^{-\tau \phi(x; p, p')}}{|x - p|^m|x - p'|^3} \, dx + \tau \int_D \frac{(p - x) \cdot (p' - x)}{|x - p|^m|x - p'|^3} e^{-\tau \phi(x; p, p')} \, dx.
\]
This yields
\[
I_2(\tau) + \frac{1}{\tau} I_3(\tau) = \tau \int_D \left\{ 1 + \frac{(p - x) \cdot (p' - x)}{|x - p|^m|x - p'|} \right\} \frac{e^{-\tau \phi(x; p, p')}}{|x - p|^m|x - p'|} \, dx
\]
\[
+ \int_D \left( \frac{1}{|x - p|} + \frac{1}{|x - p'|} \right) \frac{(p - x) \cdot (p' - x)}{|x - p|^2|x - p'|^2} e^{-\tau \phi(x; p, p')} \, dx
\]
\[
+ \frac{1}{\tau} \int_D (p - x) \cdot (p' - x) e^{-\tau \phi(x; p, p')} \, dx. \tag{3.15}
\]

**Lemma 3.6.** Let $D$ be an arbitrary nonempty bounded open set. If $p$ and $p'$ be arbitrary points in $\mathbb{R}^3 \setminus \overline{D}$ such that $[p, p'] \cap \partial D = \emptyset$, then
\[
C_D(p, p') \equiv \inf_{x \in \partial D} \left\{ 1 + \frac{(p - x) \cdot (p' - x)}{|x - p||p' - x|} \right\} > 0. \tag{3.16}
\]

**Proof.** It is easy to see that $0 \leq C_D(p, p') \leq 2$. Assume that $C_D(p, p') = 0$. Since we have the identity
\[
1 + \frac{(p - x) \cdot (p' - x)}{|x - p||p' - x|} = \frac{1}{2} \left( \frac{p - x}{|p - x|} + \frac{p' - x}{|p' - x|} \right)^2, \quad x \neq p, p',
\]
there exists a sequence $\{x_n\}$ in $D$ such that
\[
\frac{|p - x_n|}{|p - x_n|} + \frac{p' - x_n}{|p' - x_n|} \to 0. \tag{3.17}
\]
Since $\overline{D}$ is compact, choosing a subsequence of $\{x_n\}$ if necessary, one may assume that $\{x_n\}$ converges to a point $y \in \overline{D}$. Since $p, p' \neq y$ by assumption, it follows from (3.17) that

$$\frac{p - y}{|p - y|} + \frac{p' - y}{|p' - y|} = 0.$$  

This gives $y \in [p, p']$ and thus $[p, p'] \cap \overline{D} \neq \emptyset$. This is a contradiction. \hfill $\square$

A combination of (3.15) and (3.16) gives

$$I_2(\tau) + \frac{1}{\tau}I_3(\tau) \geq \tau K(\tau) \int_D e^{-\tau \phi(x; p, p')} \, dx,$$

where

$$K(\tau) = \frac{C_D(p, p')}{d_{\partial D}(p)d_{\partial D}(p')} - \left( \frac{1}{d_{\partial D}(p)} + \frac{1}{d_{\partial D}(p')} \right) \frac{1}{d_{\partial D}(p)d_{\partial D}(p')} \frac{1}{\tau} = \frac{1}{d_{\partial D}(p)^2d_{\partial D}(p')^2 \tau^2}.$$  

**Lemma 3.7.** Let $p, p' \in \mathbb{R}^3$. Then, there exists a number $\mu$ such that

$$\liminf_{\tau \to +\infty} \tau^\mu e^{\tau \min_{x \in \partial D} \phi(x; p, p')} \int_D e^{-\tau \phi(x; p, p')} \, dx > 0.$$  

**Proof.** Let $x_0 \in \partial D$ be a point such that $\phi(x_0; p, p') = \min_{x \in \partial D} \phi(x; p, p')$. Since $|p - x| \leq |p - x_0| + |x_0 - x|$ and $|p' - x| \leq |p' - x_0| + |x_0 - x|$, we have

$$\phi(x; p, p') \leq \phi(x_0; p, p') + 2|x_0 - x|, \forall x \in \mathbb{R}^3.$$  

This gives

$$e^{\tau \min_{x \in \partial D} \phi(x; p, p')} \int_D e^{-\tau \phi(x; p, p')} \, dx \geq \int_D e^{-2\tau |x - x_0|} \, dx.$$  

In [15], we already known that

$$\liminf_{\tau \to +\infty} \tau^3 \int_D e^{-2\tau |x - x_0|} \, dx > 0.$$  

Thus, (3.19) is valid for $\mu = 3$. \hfill $\square$

Now it follows from (3.14), (3.18) and (3.19) that (3.12) is valid.

### 4. Asymptotic behaviour of the indicator function

First, we claim that

$$\int_{\mathbb{R}^3 \setminus \overline{D}} (f v - w f) \, dx = 2J(\tau; f, g)(1 + O(\tau^{-1/2})) + O(\tau^{-1} e^{-\tau^2}).$$  

(4.1)

This is a consequence of the following asymptotic formula and (2.10).

**Proposition 4.1.** As $\tau \to +\infty$,

$$\int_{\mathbb{R}^3 \setminus \overline{B}} (\nabla \epsilon_x^0 \cdot \nabla \epsilon_x^0 + \tau^2 \epsilon_x^0 \epsilon_x^0) \, dx = J(\tau; f, g)(1 + O(\tau^{-1/2})).$$

Proposition 4.1 is a direct consequence of the following lemmas.
Lemma 4.1. Let $D$ be an arbitrary nonempty bounded open set. If $B$ and $B'$ be arbitrary open balls such that $[\bar{B} \cup \bar{B}'] \cap \bar{D} = \emptyset$, then there exist positive constants $C$ and $\tau_0$ independent of $\tau$ such that, for all $\tau \geq \tau_0$, $J(\tau; f, g) > 0$ and

$$J(\tau; f, g) \geq C \tau^2 \int_{D} dx \int_{B \times B'} e^{-\tau \phi(x, y')} dy'. \quad (4.2)$$

Proof. From (1.3) and (2.1), we have

$$J(\tau; f, g) = \left(\frac{1}{4\pi}\right)^2 \int_{D} dx \int_{B \times B'} dy dy' \frac{K_r(x, y, y')}{|x - y'||x - y|} e^{-\tau \phi(x, y')}, \quad (4.2)$$

where

$$K_r(x, y, y') = \left(\frac{1}{|x - y|} + \tau\right) \left(\frac{1}{|x - y'|} + \tau\right) \frac{y - x}{|x - y|} \cdot \frac{y' - x}{|x - y'|} + \tau^2.$$ 

Since $\bar{B} \cap \bar{D} = \bar{B} \cap \bar{D} = \emptyset$, there exist positive constants $C_1$ and $C_2$ such that

$$K_r(x, y, y') \geq \tau^2 \left(1 + \frac{(y - x)}{|x - y|} \cdot \frac{(y' - x)}{|x - y'|}\right) - C_1 \tau - C_2, \forall (x, y, y') \in D \times B \times B'. \quad (4.3)$$

Here, we claim the following estimate:

$$C_D(B, B') \equiv \inf_{(x, y, y') \in D \times B \times B'} \left(1 + \frac{(y - x)}{|x - y|} \cdot \frac{(y' - x)}{|x - y'|}\right) > 0. \quad (4.4)$$

This is proved as follows.

It is easy to see that $0 \leq C_D(B, B') \leq 2$. Assume that $C_D(B, B') = 0$. Since we have the identity

$$1 + \frac{y - x}{|y - x|} \cdot \frac{y' - x}{|y' - x|} = \frac{1}{2} \left|\frac{y - x}{|y - x|} + \frac{y' - x}{|y' - x|}\right|^2, \quad \forall x \neq y, y',$$

there exist sequences $\{x_n\}$ in $D$, $\{y_n\}$ in $B$ and $\{y'_n\}$ in $B'$ such that

$$\left|\frac{y_n - x_n}{|y_n - x_n|} + \frac{y'_n - x_n}{|y'_n - x_n|}\right|^2 \rightarrow 0. \quad (4.5)$$

Since $\bar{D} \times \bar{B} \times \bar{B}'$ is compact, choosing a subsequence of $\{(x_n, y_n, y'_n)\}$ if necessary, one may assume that $\{(x_n, y_n, y'_n)\}$ converges to a point $(x_*, y_*, y'_*) \in \bar{D} \times \bar{B} \times \bar{B}'$. Since $y_*, y'_* \neq x_*$ by assumption, it follows from (4.5) that

$$\frac{y_n - x_n}{|y_n - x_n|} + \frac{y'_n - x_n}{|y'_n - x_n|} = 0.$$

This gives $x_* \in \{y_n + (1 - s)y'_n | 0 < s < 1\}$ and thus $[\bar{B} \cup \bar{B}'] \cap \bar{D} \neq \emptyset$. This is a contradiction.

Thus, applying (4.4) to the right-hand side of (4.3), we obtain, for a sufficiently large $\tau_0 > 0$,

$$\inf_{(x, y, y') \in D \times B \times B'} K_r(x, y, y') \geq \frac{\tau^2}{2} C_D(B, B'), \forall \tau \geq \tau_0.$$ 

A combination of this and (4.2) ensures the validity of lemma 4.1 with

$$C = \left(\frac{1}{4\pi}\right)^2 \frac{\tau^2 C_D(B, B')}{2 \text{dist}(D, B) \text{dist}(D, B')}.$$ 

□

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Lemma 4.2. Assume that $D$ is admissible and its boundary is $C^3$. Then, there exist positive constants $C$ and $\tau_0$ independent of $\tau$ such that, for all $\tau \geq \tau_0$,

$$\left| \int_{\mathbb{R}^3 \setminus D} (\nabla e_j^0 \cdot \nabla e_j^0 + \tau^2 e_j^0 e_j^0) \, dx - J(\tau; f, g) \right| \leq C \tau^{3/2} \int_{D} \int_{B \times B} e^{-\rho(x,y)\alpha} \, dy \, dx.'$$

Proof. Let $0 < \delta < \delta_0$. Let $\phi = \phi_0$ be a smooth cut-off function, $0 \leq \phi(x) \leq 1$, and such that $\phi(x) = 1$ if $d_{ad}(x) < \delta$ and $\phi(x) = 0$ if $d_{ad}(x) > 2\delta$; $|\nabla \phi(x)| \leq C\delta^{-1}$; $|\Delta \phi(x)| \leq C\delta^{-2}$.

For $\ast = f, g$, define $(e_*^0)'(x) = \phi(x)e_*^0(x')$ for $x \in D$ and $v_*'(x) = \phi(x) v_*(x')$ for $x \in \mathbb{R}^3 \setminus D$.

The Lax–Phillips reflection argument starts with the following expression:

$$\int_{\mathbb{R}^3 \setminus D} (\nabla e_j^0 \cdot \nabla e_j^0 + \tau^2 e_j^0 e_j^0) \, dx = J(\tau; f, g) + \int_{\mathbb{R}^3 \setminus D} e_j^0 (\Delta - \tau^2) v_j' \, dx. \quad (4.6)$$

In the proof, the following relationship between $v_*'$ and $v_*$ and the boundary condition for $e_*^0$ are essential:

$$\frac{\partial v_*'}{\partial v} = -\frac{\partial v_*}{\partial v} \text{ on } \partial D,$$

$$v_*' = v_* = -e_*^0 \text{ on } \partial D.$$

Another device is the following differential identity which is a consequence of (4.15) in [18] (see also appendix 1 in [12]):

$$(\Delta - \tau^2)(v_j') = \phi(x) \sum_{i,j} a_{ij}(x)(\partial_{ij} v_j)(x') + \sum_{j} b_j(x)(\partial_j v_j)(x') + (\Delta \phi(x) v_j(x'),$$

where $a_{ij}(x), i, j = 1, 2, 3$ are $C^1$ in a neighbourhood of $\partial D$, independent of $\phi$ and $v_j$ and satisfy

$$\exists C > 0 \forall x \in \mathbb{R}^3 |a_{ij}(x)| \leq C_{d_{ad}}(x); \quad (4.7)$$

each $b_j(x)$ has the form

$$b_j(x) = \sum_{jk} b_{jk}(x) \partial_k \phi(x) + d_j(x) \phi(x), \quad (4.8)$$

with $b_{jk}(x)$ and $d_j(x)$ which are $C^1$ and $C^0$ in a neighbourhood of $\partial D$, respectively, and independent of $\phi$ and $v_j$.

This together with the change of variables $x = y'$ yields

$$\int_{\mathbb{R}^3 \setminus D} e_j^0(x)(\Delta - \tau^2) v_j'(x) \, dx = \int_{\mathbb{R}^3 \setminus D} e_j^0(x) \left\{ \phi(x) \sum_{i,j} a_{ij}(x)(\partial_{ij} v_j)(x') + \sum_{j} b_j(x)(\partial_j v_j)(x') + (\Delta \phi)(x) v_j(x') \right\} \, dx$$

$$= \int_{D} e_j^0(y') \left\{ \phi(y') \sum_{i,j} a_{ij}(y')(\partial_{ij} v_j)(y) + \sum_{j} b_j(y')(\partial_j v_j)(y) + (\Delta \phi)(y') v_j(y) \right\} J(y) \, dy, \quad (4.9)$$

where $J(y)$ denotes the Jacobian.
Hereafter, we give an estimation for each term on the right-hand side of (4.9) pointwisely, without making use of integration by parts further. This idea is exactly the same as the proof of lemma 3.3 in [18]. This is different from the back-scattering case; see also the proof of lemma 4.2 in [18] and appendix 1 in [12] for comparison.

Since $D$ is admissible we have, for all $y \in D$ with $d_{\partial D}(y) < \delta'$ and all $\tau > \tau_0$,

$$|e_j^0(y)| \leq C \int_{\partial B} e^{-\tau|y-x|} \, dx,$$

where $C$ is independent of $y$ and $\tau$.

Choosing of lemma 3.3 in [18]. This is different from the back-scattering case; see also the proof without making use of integration by parts further. This idea is exactly the same as the proof of proposition 4.2.

From this, (4.10), (4.7) and (4.8) and the choice of $\phi$ we obtain, for all $y \in D$,

$$\tau^{-2} |\partial_i \partial_j v_y(y)| + \tau^{-1} |\partial_i v_y(y)| + |v_y(y)| \leq C \int_{\partial B} e^{-\tau|y-x|} \, dx.'$$

From this, (4.10), (4.7) and (4.8) and the choice of $\phi$ we obtain, for all $y \in D$,

$$\left| e_j^0(y') \phi(y') \sum_{i,j} a_{ij}(y') (\partial_i \partial_j v_y)(y) + \sum_j b_j(y') (\partial_j v_y)(y) + (\Delta \phi)(y') v_y(y) \right| \leq C (\delta \tau^2 + \delta^2 \tau + \delta^2) \int_{\partial B} e^{-\tau \phi(y;x,x')} \, dx \, dx'.$$

Choosing $\delta = \tau^{-1/2}$, we have $\delta \tau^2 + \delta^2 \tau + \delta^2 = O(\tau^{1/2})$ as $\tau \rightarrow \infty$. Now from (4.9) and (4.11), we obtain the desired conclusion of lemma 4.2.

Thus, everything is reduced to studying the asymptotic behaviour of $J(\tau; f, g)$ as $\tau \rightarrow \infty$. For this purpose, we employ the asymptotic formula (3.13). Note that in section 3 we made use of the formula to give a lower estimate of $J(\tau; f, g)$. Here using the formula, we determine its leading term as $\tau \rightarrow \infty$.

From (3.13), we see that the asymptotic behaviour of the following integral is the key:

$$\int_{\partial D} \frac{(p-x) \cdot v_x}{|x-p|^2 |x-p|} \left( 1 + \frac{1}{\tau |x-p|} \right) e^{-\tau \phi(x; p, p')} \, dS_x.$$

**Proposition 4.2.** Let $c = \min_{x \in \partial D} \phi(x; p, p')$. Assume that $\Lambda_{\partial D}(p, p')$ is finite and

$$\det (S_q(E_c(p, p')) - S_q(\partial D)) > 0 \forall q \in \Lambda_{\partial D}(p, p').$$

Then, we have

$$\lim_{\tau \rightarrow \infty} \tau^{\pi} \int_{\partial D} \frac{(p-x) \cdot v_x}{|x-p|^2 |x-p|} \left( 1 + \frac{1}{\tau |x-p|} \right) e^{-\tau \phi(x; p, p')} \, dS_x \pi \frac{1}{|q - p||q - p|^{\pi} \sqrt{\det (S_q(E_c(p, p')) - S_q(\partial D))}}.$$

Theorem 1.3 is a direct consequence of this together with (3.13) and (4.1).

In the following subsection, we describe the proof of proposition 4.2.

**4.1. Proof of proposition 4.2**

We employ the Laplace method and so one has to compute the Hessian of the real phase function $\partial D \ni x \mapsto \phi(x; p, p')$ at all the points on $\partial D$ where it takes the minimum value.

Let $q \in \Lambda_{\partial D}(p, p')$. One can choose a local coordinates system $\sigma = (\sigma_1, \sigma_2)$ around $q$ on $\partial D$ in such a way that $x \in \partial D$ around $q$ has the form

$$x = x_0(\sigma) = q + \sigma_1 e_1 + \sigma_2 e_2 + f(\sigma)v_q, \quad |\sigma| < \delta.$$
where $\delta$ is a sufficiently small positive number independent of $q$; $e_1$ and $e_2$ are two unit tangent vectors at $q$ to $\partial D$ which are perpendicular to each other and $e_1 \times e_2 = v_q$; $f = f_q \in C^2_0(\mathbb{R}^3)$ and satisfies $f(0) = 0$, $\nabla f(0) = 0$; $v_i$ takes the form

$$v_i = \frac{-f_i e_1 - f_j e_2 + v_q}{\sqrt{1 + f_i^2 + f_j^2}},$$

where $f_1 = \frac{\partial f}{\partial \sigma_1}$ and $f_2 = \frac{\partial f}{\partial \sigma_2}$; $dS = \sqrt{1 + f_1^2 + f_2^2} \, d\sigma$.

We have, for $z = p$, $p'$ and $j = 1, 2$,

$$\frac{\partial}{\partial \sigma_j} x_q(\sigma) - z = \frac{1}{|x_q(\sigma) - z|} [(x_q(\sigma) - z) \cdot e_j + f_j(\sigma) (x_q(\sigma) - z) \cdot v_q]$$

and thus

$$\frac{\partial^2}{\partial \sigma_k \partial \sigma_j} [x_q(\sigma) - z] = - \frac{1}{|x_q(\sigma) - z|^3} [(x_q(\sigma) - z) \cdot e_j + f_j(\sigma) (x_q(\sigma) - z) \cdot v_q]$$

$$+ \frac{1}{|x_q(\sigma) - z|} \left( \delta_{kj} + \frac{\partial^2 f}{\partial \sigma_k \partial \sigma_j}(\sigma) (x_q(\sigma) - z) \cdot v_q + f_j(\sigma) f_k(\sigma) \right).$$

This yields

$$\frac{\partial^2}{\partial \sigma_k \partial \sigma_j} \phi(x_q(\sigma); p, p')|_{\sigma = 0} = \lambda(q; p, p')\delta_{kj} - a_{kj}(q; p, p'),$$

where

$$\lambda(q; p, p') = \frac{1}{|p - q|} + \frac{1}{|p' - q|},$$

$$a_{kj}(q; p, p') = \frac{1}{|p - q|} A \cdot e_k A \cdot e_j + \frac{1}{|p' - q|} A' \cdot e_k A' \cdot e_j - \frac{\partial^2 f}{\partial \sigma_k \partial \sigma_j}(0) (A \cdot v_q + A' \cdot v_q)$$

and

$$A = A_q(p), A' = A_q(p').$$

Note that $\lambda(q; p, p') = \lambda(q; p', p)$ and $a_{kj}(q; p, p') = a_{kj}(q; p', p)$.

The following lemma corresponds to Snell’s law in geometrical optics.

**Lemma 4.3.** Let $p$ and $p'$ be arbitrary points in $\mathbb{R}^3 \setminus \partial D$ such that $[p, p'] \cap \partial D = \emptyset$. Let $q \in A_\sigma(p, p')$. Then, we have: (i) the vector $A + A'$ is parallel to $v_q$; (ii) $A \cdot v_q = A' \cdot v_q \neq 0$. In addition, if $p$ and $p'$ are in $\mathbb{R}^3 \setminus \overline{D}$, then we have

$$v_q = -\frac{1}{\sqrt{2(1 + A \cdot A')}} (A + A'),$$

that is, the unit inward normal to $E_\sigma(p, p')$ at $q$ with $c = \phi(q; p, p')$ (see (A.1) in the appendix) coincides with the unit outward normal to $\partial D$ at the same point.

**Proof.** Since the function $\sigma \mapsto \phi(x_q(\sigma); p, p')$ takes its minimum at $\sigma = 0$, it follows from (4.12) that

$$A \cdot e_1 + A' \cdot e_1 = 0$$

$$A \cdot e_2 + A' \cdot e_2 = 0.$$
Write
\[ A = \alpha e_1 + \beta e_2 + \gamma v_q, \]
\[ A' = \alpha' e_1 + \beta' e_2 + \gamma' v_q. \] (4.16)
Equation (4.15) is equivalent to \( \alpha + \alpha' = 0 \) and \( \beta + \beta' = 0 \). Then, we have \( \gamma^2 = \gamma'^2 \), that is, \( \gamma = \gamma' \) or \( \gamma = -\gamma' \). Assume that \( \gamma = -\gamma' \). Then, from (4.16) we have \( A + A' = 0 \), that is,
\[ \frac{q - p}{|q - p|} + \frac{q - p'}{|q - p'|} = 0. \]
This means that \( q \in [p, p'] \) and contradicts the condition \( [p, p'] \cap \partial D = \emptyset \). Thus, \( \gamma = \gamma' \) and we have
\[ A + A' = 2\gamma v_q \] (4.17)
and
\[ \gamma = \pm \frac{\sqrt{1 + A \cdot A'}}{\sqrt{2}}. \]
Since \( |A + A'|^2 = 2(1 + A \cdot A') \), from the argument above we have \( 1 + A \cdot A' > 0 \). Thus, from (4.17), one obtains \( A \cdot v_q \neq 0 \) and \( A' \cdot v_q \neq 0 \).

Note that if \( p \) and \( p' \) are in \( \mathbb{R}^3 \setminus \partial D \), \( \gamma \) has to be negative. The reason is the following. Assume that \( \gamma > 0 \). Then, \( -(A + A') \) is directed to \(-v_q \). Since \( \partial D \) is \( C^2 \), one can find a sufficiently small \( s > 0 \) such that \( q' \equiv q - s(A + A') \in D \). Since \( p \in \mathbb{R}^3 \setminus \partial D \), one can find a point \( p'' \in \partial D \) on the segment with endpoints \( q' \) and \( p \). Note that \( -(A + A') \) is directed to the unit inward normal to \( E_c(p, p') \). This together with the condition \( c > |p - p'| \) ensures that both \( q' \) and \( p \) are in the domain enclosed by \( E_c(p, p') \) and thus by the convexity of the domain, we have \( \phi(p''; p, p') < c \). However, since \( p'' \in \partial D \), we have \( \phi(p''; p, p') \geq c \). This is a contradiction. Therefore, one obtains \( \gamma < 0 \) and now it is easy to see that all the conclusions are valid.

It follows from (i) in lemma 4.3 that
\[ A \cdot e_k A \cdot e_j = A' \cdot e_k A' \cdot e_j. \]
Thus, we have
\[ \frac{1}{|p - q|} A \cdot e_k A \cdot e_j = \frac{1}{|p - q|} + \frac{1}{|p' - q|} A \cdot e_k A \cdot e_j = \lambda(q; p, p') A \cdot e_k A \cdot e_j - \frac{1}{|p' - q|} A' \cdot e_k A' \cdot e_j \]
and hence
\[ \frac{1}{|p - q|} A \cdot e_k A \cdot e_j + \frac{1}{|p' - q|} A' \cdot e_k A' \cdot e_j = \lambda(q; p, p') A \cdot e_k A \cdot e_j. \]
Changing the role of \( p \) and \( p' \), we also have
\[ \frac{1}{|p - q|} A \cdot e_k A \cdot e_j + \frac{1}{|p' - q|} A' \cdot e_k A' \cdot e_j = \lambda(q; p, p') A' \cdot e_k A' \cdot e_j. \]
From these and (4.14), we obtain another expression for \( a_{jk}(q; p, p') \):
\[ a_{jk}(q; p, p') = \frac{\lambda(q; p, p')}{2} (A \cdot e_k A \cdot e_j + A' \cdot e_k A' \cdot e_j) \]
\[ + \sqrt{2(1 + A \cdot A')} \frac{\partial^2 f}{\partial \sigma_k \partial \sigma_j} (0). \]
This together with (4.13) implies that 
\[ \frac{\partial^2}{\partial \sigma_i \partial \sigma_j} \phi(x_q(\sigma); p, p')|_{\sigma=0} = \sqrt{2(1 + A \cdot A')} \times \left\{ \frac{\lambda(q; p, p')}{\sqrt{2(1 + A \cdot A')}} \left( \delta_{jk} - \frac{1}{2} (A \cdot e_k A \cdot e_j + A' \cdot e_k A' \cdot e_j) \right) - \frac{\partial^2 f}{\partial \sigma_i \partial \sigma_j}(0) \right\}. \] (4.18)

From (A.2), we have
\[ \left( \begin{array}{c} e_1^T \\ e_2^T \end{array} \right) S_q(E_c(p, p'))(e_1, e_2) = \frac{\lambda(q; p, p')}{\sqrt{2(1 + A \cdot A')}} \left( \delta_{jk} - \frac{1}{2} (A \cdot e_k A \cdot e_j + A' \cdot e_k A' \cdot e_j) \right)_{j=1,2, k=1,2} \]
and we know
\[ \left( \begin{array}{c} e_1^T \\ e_2^T \end{array} \right) S_q(\partial D)(e_1, e_2) = \left( \frac{\partial^2 f}{\partial \sigma_j \partial \sigma_k}(0) \right)_{j=1,2, k=1,2}. \] (4.19)
Thus, we obtain the following formula which gives the geometrical meaning of the Hessian of \( \phi(x_q(\sigma); p, p') \) at \( \sigma = 0 \).

**Lemma 4.4.** Let \( p \) and \( p' \) be in \( \mathbb{R}^3 \setminus \bar{D} \) such that \( [p, p'] \cap \partial D = \emptyset \). Let \( q \in \Lambda_{\partial D}(p, p') \) and \( c = \phi(q; p, p') \). Then, \( c > |p - p'| \) and we have
\[ \nabla_c^2 \phi(x_q(\sigma); p, p')|_{\sigma=0} = \sqrt{2(1 + A \cdot A')} \left( \begin{array}{c} e_1^T \\ e_2^T \end{array} \right) \left( S_q(E_c(p, p')) - S_q(\partial D) \right)(e_1, e_2). \] (4.20)

Since \( \phi(x_q(\sigma); p, p') \) takes the minimum at \( \sigma = 0 \), from (4.20) one concludes that, for all tangent vectors \( \nu \) at \( q \),
\[ (S_q(E_c(p, p')) - S_q(\partial D)) \nu \cdot \nu \geq 0. \] (4.21)
Thus, if (1.14) is satisfied, then from (4.21) one knows that \( S_q(E_c(p, p')) - S_q(\partial D) \) is positive definite on the tangent space at \( q \) and from (4.20)
\[ \det \nabla_c^2 \phi(x_q(\sigma); p, p')|_{\sigma=0} = 2(1 + A \cdot A') \det (S_q(E_c(p, p')) - S_q(\partial D)) > 0. \]
Also from (4.14), we have
\[ (p - q) \cdot v_q = \frac{|q - p|}{\sqrt{2}} \sqrt{1 + A \cdot A'}. \]

Now proposition 4.2 is a direct consequence of the Laplace method [2].

5. **Proof of theorems 1.2 and 1.4**

5.1. **Extracting the first reflector: proof of theorem 1.2**

Let \( c > |p - p'| \). Theorem 1.2 is based on the following proposition which gives a characterization of a first reflection point \( q \) between \( p \) and \( p' \) in terms of the minimum length of the broken path connecting \( p \) to \( q \) and \( q \) to \( p' + s(q - p')/|q - p'| \) with a fixed small \( s > 0 \).

**Proposition 5.1.** Fix \( 0 < s < \eta' \). Let \( c = \min_{q \in \partial D} \phi(x; p, p') \). We have:
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(1) if $p' + s(\omega; p, p', c) \omega$ belongs to $\partial D$, then
\[
\min_{x \in \partial D} \phi(x; p, p' + s \omega) = c - s;
\]

(2) if $p' + s(\omega; p, p', c) \omega$ does not belong to $\partial D$, then
\[
\min_{x \in \partial D} \phi(x; p, p' + s \omega) > c - s.
\]

Thus, one has the following characterization of the first reflector:
\[
\Lambda_{AD}(p, p') = \{p' + s(\omega; p, p', c) \omega \mid \min_{x \in \partial D} \phi(x; p, p' + s \omega) = c - s, \omega \in S^2\}.
\]

**Proof.** Set $p''(\omega) = p' + s \omega$. Let $x \in \partial D$. We have
\[
\phi(x; p, p''(\omega)) = |p - x| + |x - p''(\omega)|
\]
\[
= |p - x| + |(x - p') + (p' - p''(\omega))| = |p - x| + |x - p'| - |p' - p''(\omega)| = \phi(x; p, p') - s. \quad (5.1)
\]

This gives
\[
\min_{x \in \partial D} \phi(x; p, p''(\omega)) \geq c - s. \quad (5.2)
\]

Now we describe the proof of (i). Noting that $s < s(\omega; p, p', c)$ and $p' + s(\omega; p, p', c) \omega \in E_c(p, p')$, we have
\[
\phi(p'' + s(\omega; p, p', c) \omega; p, p''(\omega))
\]
\[
= |p - (p'' + s(\omega; p, p', c) \omega)| + |(p' + s(\omega; p, p', c) \omega) - p''(\omega)|
\]
\[
= |p - (p' + s(\omega; p, p', c) \omega)| + |(s(\omega; p, p', c) - s)
\]
\[
= |p - (p'' + s(\omega; p, p', c) \omega)| + |(p' + s(\omega; p, p', c) \omega) - p'| - s
\]
\[
= \phi(p' + s(\omega; p, p', c) \omega; p, p') - s = c - s.
\]

Thus, if $p' + s(\omega; p, p', c) \omega$ belongs to $\partial D$, then the inequality in (5.2)
\[
\min_{x \in \partial D} \phi(x; p, p''(\omega)) > c - s
\]
never occurs. Thus, it must hold that $\min_{x \in \partial D} \phi(x; p, p''(\omega)) = c - s$.

Since we always have (5.2), the statement of (ii) is equivalent to the following: if $\min_{x \in \partial D} \phi(x; p, p''(\omega)) = c - s$, then $p' + s(\omega; p, p', c) \omega \in \partial D$.

Assume that $\min_{x \in \partial D} \phi(x; p, p''(\omega)) = c - s$. Choose a point $x \in \partial D$ such that $\phi(x; p, p''(\omega)) = c - s$. Then, from (5.1) we have $c \geq \phi(x; p, p')$. Since $c = \min_{x \in \partial D} \phi(x; p, p')$, from this we obtain $\phi(x; p, p') = c$. Thus, we have $\phi(x; p, p''(\omega)) = \phi(x; p, p') - s$. This is equivalent to
\[
|x - p''(\omega)| = |x - p'| - s. \quad (5.3)
\]

Since $x$ is outside the open ball $B'$ centred at $p'$ with radius $s$, one can find the unique point $x'$ on $\partial B'$ such that $|x - x'| = \min_{y \in \partial B'} |y - x|$. Since we have $|x - p'| = |x - x'| + s$, from (5.3), we obtain $|x - p''(\omega)| = |x - x'|$. Since $p''(\omega) \in \partial B''$, it must hold that $p''(\omega) = x'$ and thus $x = p' + s(\omega; p, p', c) \omega$. This gives $p' + s(\omega; p, p', c) \omega \in \partial D$. \hfill \Box

Now we are ready to describe the proof of theorem 1.2.

In what follows, we denote the open ball centred at a point $z$ and with radius $\rho$ by $B_\rho(z)$.

Since $B_{\eta - \rho}(p' + s \omega)$ is contained in $B'$, from $u_f$ on $B' \times [0, T]$ with $f = \chi_B$, one obtains $u_f$ on $B_{\eta - \rho}(p' + s \omega) \times [0, T]$. By theorem 1.1, we obtain $\inf_{x \in \partial D} \phi(x; p, p' + s \omega)$ for each $\omega \in S^2$. 24
Thus, from proposition 5.1 we obtain $\Lambda_{\partial D}(p, p')$ itself. From formula (4.14), one obtains $v_q$ at given $q \in \Lambda_{\partial D}(p, p')$.

Thus, one can completely determine the first reflectors between $p$ and $p'$ using the bistatic data $u_f$ on $B' \times [0, T]$ for $f = \chi_B$ and sufficiently large and fixed $T$. In particular, note that $B$ is fixed.

**Remark 5.1.** We summarize how to detect the points in $\Lambda_{\partial D}(p, p') \subset E_c(p, p')$.

1. **Step 1.** Compute $w_f$ on $B'$ with $f = \chi_B$ from the data $u_f$ on $B' \times [0, T]$.
2. **Step 2.** Fix $s$ with $0 < s < \eta'$. Choose a direction $\omega \in S^2$.
3. **Step 3.** Choose a direction $\omega \in S^2$.
4. **Step 4.** Compute the following integral with $g = \chi_{B_{\varepsilon_{\omega}}(p' + \omega \omega)}$:
   \[ \int_B v_g \, dx - \int_{B_{\varepsilon_{\omega}}(p' + \omega \omega)} w_f \, dx. \]
5. **Step 5.** Compute the following quantity:
   \[ \lim_{\tau \to \infty} \frac{1}{\tau} \log \left( \int_B v_g \, dx - \int_{B_{\varepsilon_{\omega}}(p' + \omega \omega)} w_f \, dx \right) - \eta - \eta' + s. \]
6. **Step 6.** If the computed quantity in step 5 is equal to $c - s$, then $p' + s(\omega; p, p', c) \omega \in \Lambda_{\partial D}(p, p')$. If not so, then choose another $\omega$ and go to step 4.

5.2. **Extracting the geometry of $\partial D$ at a first reflection point: proof of theorem 1.4**

In this subsection, we consider the case when a point $q \in \Lambda_{\partial D}(p, p')$ is known. We use the notation $A$ and $A'$ instead of $A_q(p)$ and $A_q(p')$, respectively, for simplicity of description. The aim of this subsection is to extract the geometry of $\partial D$ at $q$. The main idea is to replace $p'$ in $\Lambda_{\partial D}(p, p')$ with $p' + sA'$ with a small $s > 0$. The advantage of this idea is described in the following proposition.

**Proposition 5.2.** Let $c = \min_{x \in \partial D} \phi(x; p, p')$ and satisfy $c > |p - p'|$. Fix $0 < s < \eta'$. If $q \in \Lambda_{\partial D}(p, p')$, then $\Lambda_{\partial D}(p, p' + sA') = \{q\}$. Moreover, if $s < \frac{1}{2}(c - |p - p'|)$, then the operator $S_q(E_{\varepsilon_{\omega}}(p, p' + sA')) - S_q(\partial D)$ is positive definite on the common tangent space $T_q(\partial D) = T_q(E_{\varepsilon_{\omega}}(p, p' + sA')) = T_q(E_c(p, p'))$ at $q$.

**Proof.** From the definition of $\Lambda_{\partial D}(p, p')$, we can write

\[ q = p' + s(A; p, p', c)A'. \]

Then, we have $\phi(q; p, p') = c - s$, where $p'' = p' + sA'$. Since $\phi(x; p, p'') \geq c - s$ for all $x \in \partial D$, this means $c - s = \min_{x \in \partial D} \phi(x; p, p'')$ and thus

\[ q \in \Lambda_{\partial D}(p, p' + sA'). \]

Next let

\[ x \in \Lambda_{\partial D}(p, p' + sA'). \]

We have $\phi(x; p, p'') = \min_{x \in \partial D} \phi(x; p, p'') = c - s$. Then, we have

\[
\begin{align*}
  c - s &= \phi(x; p, p'') = |p - x| + |x - p''| \\
  &= |p - x| + |x - p' + p''| \\
  &\geq |p - x| + |x - p' - p''| = \phi(x; p, p') - s.
\end{align*}
\]
Thus, $\phi(x; p, p') \leq c$ and hence $c = \phi(x; p, p')$. From these, we have $\phi(x; p, p') - s = \phi(x; p, p'')$. This is equivalent to
\begin{equation}
|x - p'| = |x - p'| - s. \tag{5.4}
\end{equation}
Since $x$ is outside the open ball $B'$ centred at $p'$ with radius $s$, one can find the unique point $x'$ on $\partial B'$ such that $|x - x'| = \min_{y \in \partial B} |y - x|$. Since we have $|x - p'| = |x - x'| + s$, from (5.4), we obtain $|x - p'| = |x - x'|$. Since $p'' \in \partial B'''$, it must hold that $p'' = x'$ and thus $x = p' + s(A'; p', c) = q$.

The last statement is based on the fact that the eigenvectors for both shape operators are common and $\lambda' > \lambda$, where
\begin{equation}
\lambda = \lambda(q; p, p') = \frac{1}{|q - p|} + \frac{1}{|q - p'|}, \quad \lambda' = \lambda(q; p, p' + sA') = \frac{1}{|q - p|} + \frac{1}{|q - p' - s|}.
\end{equation}
See the appendix for these. Thus, one concludes that the operator $S_q(E_{p-s}(p, p' + sA')) - S_q(E_{p-s}(p, p'))$ is positive definite. Since $S_q(E_{p-s}(p, p' + sA')) - S_q(\partial D) > 0$, from these one obtains the desired conclusion. Note that the condition $s < \frac{1}{2}(c - |p - p'|)$ is just for ensuring that $|p - (p' + sA')| < c - s$.
\[ \square \]

We have
\begin{equation}
\min_{x \in D \setminus \chi_B} \phi(x; y, y) = \min_{x \in D} \phi(x; p, p' + sA') - (\eta + \eta' - s)
\end{equation}
and, by (i) of proposition 5.1, $\min_{x \in \partial D} \phi(x; p, p' + sA') = c - s$ if $q \in \Lambda_3D(p, p')$.

Thus, the condition
\begin{equation}
T > \min_{x \in \partial D \setminus \chi_B} \phi(x; y, y)
\end{equation}
is equivalent to (1.5). Therefore, the following proposition is a direct consequence of theorem 1.3 together with proposition 5.2.

**Proposition 5.3.** Fix $0 < s < \min (\eta', (c - |p - p'|)/2)$. Let $B$ and $B'$ satisfy $[\overline{B} \cup B'] \cap \overline{D} = \emptyset$. Let $f = \chi_B$ and $g = \chi_{B_{p-s}(p' + sA')}$. Let $T$ satisfy (1.5). Let $q \in \Lambda_3D(p, p')$ and set $c = \phi(q; p, p')$.

If $D$ is admissible and $\partial D$ is $C^3$, then we have
\begin{equation}
\lim_{r \to \infty} r^4 e^{r(c-\eta'-\eta)} \left( \int_B v_f \, dx - \int_{B_{p-s}(p' + sA')} w_f \, dx \right) = \frac{\pi}{2} \sqrt{\frac{\eta' - s}{|q - p'| - s}} \frac{1}{\sqrt{\det (S_q(E_{p-s}(p, p' + sA')) - S_q(\partial D))}}.
\end{equation}

It is quite interesting to know the quantities contained in $\det (S_q(E_{p-s}(p, p' + sA')) - S_q(\partial D))$. The following lemma, whose proof is given in the appendix, clarifies them.

**Lemma 5.1.** Let $0 \leq s < \min (\eta', (c - |p - p'|)/2)$, $q \in \Lambda_3D(p, p')$ and set $c = \phi(q, p, p')$. One has
\begin{align}
\det(S_q(E_{p-s}(p, p' + sA')) - S_q(\partial D)) &= \frac{1}{4} \left( 2 \lambda(q; p, p' + sA')^{2} \right) \\
&= \frac{\sqrt{1 + A \cdot A} \lambda(q; p, p' + sA')}{2} \left( H_{\partial D}(q) - \frac{S_q(\partial D)(A \times A') \cdot (A \times A')}{2(1 + A \cdot A')} \right) + K_{\partial D}(q) \tag{5.6}
\end{align}
Now we are ready to describe the proof of theorem 1.4.

Choose \(0 < s_1 < s_2 < \min(n_1, (c - |p - p'|)/2)\). Let \(s = s_1, s_2\). By (5.5) in proposition 5.3, one obtains \(\det (S_q(E_{c,s_1}(p, p')) - S_q(\partial D))\) for \(s = s_1, s_2\). From (5.6), we obtain the system

\[
\begin{pmatrix}
-\sqrt{\frac{2}{1 + A \cdot A}} \lambda(q; p, p' + s_1 A') \\
-\sqrt{\frac{2}{1 + A \cdot A}} \lambda(q; p, p' + s_2 A') \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix}
= F,
\]

where

\[
X = \left( \frac{H_{\partial D}(q) - S_q(\partial D)(A \times A') \cdot (A \times A')}{2(1 + A \cdot A')} \right)
\]

and

\[
F = \begin{pmatrix}
\det (S_q(E_{c,s_1}(p, p' + s_1 A')) - S_q(\partial D)) - \frac{\lambda(q; p, p' + s_1 A')^2}{4} \\
\det (S_q(E_{c,s_2}(p, p' + s_2 A')) - S_q(\partial D)) - \frac{\lambda(q; p, p' + s_2 A')^2}{4}
\end{pmatrix}.
\]

Since \(\lambda(q; p, p' + s_1 A') < \lambda(q; p, p' + s_2 A')\), (5.7) is uniquely solvable with respect to \(X\).

This completes the proof of theorem 1.4.

**Remark 5.2.** It follows from (5.6) that

\[
\det (S_q(E_c(p, p')) - S_q(\partial D)) - P_{\partial D}(\mu; q)
\]

\[
= -\mu \left( \sqrt{\frac{2}{1 + A \cdot A}} - 1 \right) 2H_{\partial D}(q) - \sqrt{\frac{2}{1 + A \cdot A'}} \frac{S_q(\partial D)(A \times A') \cdot (A \times A')}{(1 + A \cdot A')} ,
\]

where

\[
\mu = \frac{\lambda(q; p, p')}{2} = \frac{1}{2} \left( \frac{1}{|q - p|} + \frac{1}{|q - p'|} \right) .
\]

Since the right-hand side has a bound \(O(|A \times A'|^2)\), this formula indicates an effect of the bistatic data on \(\det (S_q(E_c(p, p')) - S_q(\partial D))\).

6. Summary and some open problems

This paper is concerned with an inverse obstacle problem which employs the dynamical scattering data of an acoustic wave over a finite time interval. The unknown obstacle \(D\) is assumed to be a sound-soft one. The governing equation of the wave is given by the classical wave equation. The wave is generated by the initial data which is a characteristic function of an open ball \(B\) centred at \(p\) and observed over a finite time interval on a different ball \(B'\) centred at \(p'\). It is assumed that \([\overline{B} \cup \overline{F}] \cap \overline{D} = \emptyset\). The observed data are the so-called bistatic data. This is a simple mathematical model of the data collection process using an acoustic wave/electromagnetic wave such as bistatic active sonar, radar, etc. This paper aims to develop an enclosure method which employs the bistatic data.

It is shown that from the data with some additional assumptions on the lower bound of \(T\) one can extract:

(i) the first arrival time in the geometrical optics sense, that is, the shortest length of the broken paths connecting \(p\) to a point \(q \in \partial D\) and \(q\) to \(p'\).
(ii) the first reflection points between \( p \) and \( p' \), that is, all the points \( q \in \partial D \) that minimize the length of the broken paths connecting \( p \) to \( q \) and \( q \) to \( p' \).

(iii) the tangent planes of \( \partial D \) at all the first reflection points.

It is also shown that, under the admissibility condition for \( D \), one can extract the Gauss curvature at an arbitrary first reflection point and the mean curvature with an additional term which depends on the positions of \( p, p' \) and the first reflection point. As a byproduct, for an example, a constructive proof of a uniqueness theorem for a spherical obstacle using the bistatic data is also given.

We think that the problem taken up in this paper is a prototype of various other interesting problems. It would be interesting to see whether the approach presented here can be applied to them or whether it is necessary to develop a modification. Here, we mention some of them.

- Consider the sound-hard obstacle case or the obstacles with a dissipative boundary condition (cf [21]). It is also quite important to consider the corresponding problem for the Maxwell system. These remain open.
- It would be interesting to consider also the case when obstacles are embedded in one of two layers with different known propagation speeds, and both the source and receivers are placed in another layer.
- Perhaps the most interesting problem is that of extracting geometrical information about an unknown obstacle \( D \) behind a known obstacle \( D_0 \) from monostatic or bistatic data over a finite time interval. \( B \) and \( B' \) satisfy \([B \cup B'] \cap (D_0 \cup D) = \emptyset\) and \( D \) is optically invisible from \( B \) and \( B' \) because of the existence of \( D_0 \).

- It would be interesting to consider the case when \( B' \) is placed in the shadow region of \( D \) with respect to \( B \). It means that \( B' \) is invisible from \( B \) because of the existence of \( D \) between them. This is the case when \([B \cup B'] \cap \partial D \neq \emptyset\). What information about \( D \) can one extract from \( u_f \) on \( B' \times ]0, T[ \)?

Although the author’s interest is in the pursuit of the possibility of the enclosure method itself, research by other approaches to the problem taken up in this paper is also expected. Someone may think about the use of geometrical optics in the time domain as Majda did in [19] for the problem considered in [18]. See also pages 440–447 in [25] for geometrical optics in the time domain and [22] for Majda’s approach. His approach depends heavily on the hyperbolic nature of the governing equation in contrast to our approach and it should be noted that the existing results by our approach can cover inverse problems for different types of equations, such as elliptic, parabolic or hyperbolic ones. Since this paper is not a comparative study of the various approaches, we leave this to other opportunities.

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Appendix

A.1. Proof of (1.7)

First we prove that

\[
\min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta') \geq \min_{x \in \partial D, y \in \partial B, y' \in \partial B} \phi(x; y, y').
\]
Choose \( q \in \partial D \) such that
\[
\phi(q; p, p') = \min_{x \in \partial D} \phi(x; p, p').
\]
One can find \( y_0 \in \partial B \cap [q, p] \) and \( y'_0 \in \partial B' \cap [q, p'] \). Since \( |y_0 - q| = |p - q| - \eta \) and \( |y'_0 - q| = |p' - q| - \eta' \), we have
\[
\phi(q; y_0, y'_0) = \phi(q; p, p') - (\eta + \eta').
\]
This yields
\[
\phi(q; p, p') - (\eta + \eta') \geq \min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y').
\]
Next we prove that
\[
\min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta') \leq \min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y').
\]
Choose \( q \in \partial D, y_0 \in \partial B \) and \( y'_0 \in \partial B' \) such that
\[
\phi(q; y_0, y'_0) = \min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y').
\]
Let \( y(s) \) be an arbitrary curve on \( \partial B \) such that \( y(0) = y_0 \). Since \( \phi(q; y(s), y'_0) \) takes its minimum value at \( s = 0 \), we have
\[
\frac{d}{ds} \phi(q; y(s), y'_0)|_{s=0} = 0.
\]
This gives
\[
q - y_0 \mid_{q - y_0} \cdot \frac{dy}{ds}(0) = 0.
\]
Since \( dy/ds(0) \) can be an arbitrary vector perpendicular to the normal vector at \( y_0 \in \partial B \) and \( q \) is outside of \( B \), we have
\[
\frac{q - y_0}{|q - y_0|} = \frac{y_0 - p}{|y_0 - p|}.
\]
The yields \( y_0 \in [q, p] \) and similarly we have \( y'_0 \in [q, p'] \). Thus, we have \( |y_0 - q| = |p - q| - \eta \) and \( |y'_0 - q| = |p' - q| - \eta' \). This yields
\[
\phi(q; y_0, y'_0) = \phi(q; p, p') - (\eta + \eta')
\]
and thus
\[
\phi(q; y_0, y'_0) \geq \min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta').
\]

A.2. Proof of proposition 1.1.

Proof of (i). Since every point on \( \partial D \) is a point of support of \( D \), we have (3.6). And from (3.6) we have \( \epsilon^0_j(y') \geq -v_f(y) \) for \( d_{\partial D}(y) \ll 1 \) and \( y \in D \). Since \( v_f \geq 0 \) and \( \epsilon^0_j < 0 \) (maximum principle), we have, for \( d_{\partial D}(y) \ll 1 \) and \( y \in D \), \( |\epsilon^0_j(y')| \leq v_f(y) \). This yields the desired estimate.

Let \( c = \min_{x \in \partial D} \phi(x; p, p') \). Assume that there exist two distinct points \( q \) and \( q' \) on \( \Lambda_{\partial D}(p, p') \). Since \( D \) is convex, \( \overline{D} \) also becomes convex. Thus, every point on \([q, q']\) is in \( \overline{D} \). Choose an arbitrary point \( q'' \) on \([q, q'] \backslash \{q, q'\} \). We have \( q'' \in \overline{D} \). Since both \( q \) and \( q' \) are on \( E_\epsilon(p, p') \) with \( c = \min_{x \in \partial D} \phi(x; p, p') \), it is clear that every point \( y \) on \([q, q'] \backslash \{q, q'\} \) satisfies \( \phi(y; p, p') \leq c \) and thus \( \phi(q''; p, p') \leq c \). Since \( \phi(x; p, p') \leq c \) for all \( x \in \partial D \) and
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$q'' \in D \cup \partial D$, it must hold that $q'' \in D$. Since $p$ in $\mathbb{R}^3 \setminus \overline{D}$, there exists a point $q''$ on $[q'' \in D \setminus \partial D$. Then, it is clear we have $\phi(q'''; p, p') \leq \phi(q'''; p, p')$ and thus $\phi(q'''; p, p') < c$. However, since $q'' \in \partial D$, we have $\phi(q'''; p, p') \geq c$. This is a contradiction.

Proof of (ii). Since $\partial D$ is in the half-space $(x - q) \cdot v_q \leq 0$, it is easy to see that $S_q(\partial D) \leq 0$ as the quadratic form on $T_q(\partial D)$. On the other hand, we have $S_q(\partial E_c(p, p'))$ positive definite as the quadratic form on $T_q(\partial E_c(p, p')) = T_q(\partial D)$ (see section A.3). Thus, $S_q(\partial E_c(p, p'))$ is positive definite as the quadratic form on the common tangent space and this yields (1.14).

A.3. The shape operator for a spheroid

Since $c > |p - p'|$, we have $[p, p'] \cap E_c(p, p') = \emptyset$. Thus, for all $x \in E_c(p, p')$,

$$\frac{x - p}{|x - p|} + \frac{x - p'}{|x - p'|} \neq 0,$$

and hence

$$1 + \frac{x - p}{|x - p|} \frac{x - p'}{|x - p'|} > 0.$$

Since

$$\nabla \phi(x; p, p') = \frac{x - p}{|x - p|} + \frac{x - p'}{|x - p'|},$$

we conclude that $E_c(p, p')$ is a $C^\infty$ surface and clearly compact. Let $v_x$ denote the unit inward normal for $x \in E_c(p, p')$. We have

$$v_x = -\frac{\nabla \phi}{|\nabla \phi|}$$

and note that

$$|\nabla \phi| = \sqrt{2} \sqrt{1 + \frac{x - p}{|x - p|} \frac{x - p'}{|x - p'|}}.$$

These give the expression

$$v_x = \frac{\lambda(x)}{\sqrt{2(1 + \lambda(x) \cdot \mathbf{A}(x))}}, \quad (A.1)$$

where

$$\lambda(x) = \frac{x - p}{|x - p|}, \quad \lambda(x) = \frac{x - p'}{|x - p'|}.$$

Now we are ready to prove the following proposition.

Proposition A.1. Let $S_x$ denote the shape operator at $x \in E_c(p, p')$ with respect to $v_x$ which is the unit inward normal to $E_c(p, p')$. We have, for all $v \in T_i E_c(p, p')$,

$$S_x(v) = \frac{\lambda(x)}{\sqrt{2(1 + \lambda(x) \cdot \mathbf{A}(x))}} \left( I_3 - \frac{1}{2} \mathbf{A}(x) \otimes \mathbf{A}(x) - \frac{1}{2} \mathbf{A}'(x) \otimes \mathbf{A}'(x) \right) v, \quad (A.2)$$

where

$$\lambda(x) = \frac{1}{|x - p|} + \frac{1}{|x - p'|}.$$
Proof. Let \( x = x(\sigma_1, \sigma_2) \) be an equation for \( E_c(p, p') \) around \( q \in E_c(p, p') \). This means that \( x(0, 0) = q \) and \( \phi(x(\sigma_1, \sigma_2)); p, p' = c. \) Since

\[
\frac{\partial}{\partial \sigma_j} | x - p | = \frac{x - p}{| x - p |} \cdot \frac{\partial x}{\partial \sigma_j},
\]

we have

\[
\frac{\partial}{\partial \sigma_j} A(x) = \frac{1}{| x - p |} \frac{\partial x}{\partial \sigma_j} - \frac{x - p}{| x - p |^2} \frac{\partial x}{\partial \sigma_j} - A(x) \left( \frac{\partial x}{\partial \sigma_j} \right) \frac{\partial x}{\partial \sigma_j}.
\]

This together with a corresponding expression for \((\partial / \partial \sigma_j)A'(x)\) gives

\[
\frac{\partial}{\partial \sigma_j} (A(x) + A'(x)) = \left( \dot{c}(x)I_3 - \frac{1}{| x - p |} A(x) \otimes A(x) - \frac{1}{| x - p' |} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}.
\]

Moreover, since

\[
\frac{\partial x}{\partial \sigma_j} \cdot A(x) = -\frac{\partial x}{\partial \sigma_j} \cdot A'(x),
\]

we have

\[
\frac{\partial}{\partial \sigma_j} (A(x) \cdot A'(x)) = \frac{1}{| x - p |} (I_3 - A(x) \otimes A(x)) \frac{\partial x}{\partial \sigma_j} \cdot A'(x) + \frac{1}{| x - p' |} A(x) \cdot \left( I_3 - A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j} \cdot A'(x)
\]

\[
= \frac{1}{| x - p |} \left( A(x) \cdot \frac{\partial x}{\partial \sigma_j} \right) \left( A(x) \cdot A'(x) \right) - \frac{1}{| x - p |} \left( A'(x) \cdot \frac{\partial x}{\partial \sigma_j} \right) \left( A(x) \cdot A'(x) \right)
\]

\[
= -\frac{1}{| x - p |} (1 + A(x) \cdot A'(x)) \left( A(x) \cdot \frac{\partial x}{\partial \sigma_j} \right) - \frac{1}{| x - p |} (1 + A(x) \cdot A'(x)) \left( A'(x) \cdot \frac{\partial x}{\partial \sigma_j} \right)
\]

\[
= -(1 + A(x) \cdot A'(x)) \left( \frac{1}{| x - p |} A(x) \cdot \frac{\partial x}{\partial \sigma_j} + \frac{1}{| x - p' |} A'(x) \cdot \frac{\partial x}{\partial \sigma_j} \right),
\]

and we obtain

\[
\frac{\partial}{\partial \sigma_j} \frac{1}{\sqrt{1 + A(x) \cdot A'(x)}} = -\frac{1}{2} \frac{\partial}{\partial \sigma_j} (A(x) \cdot A'(x))^{\frac{3}{2}}
\]

\[
= \frac{1}{2 (1 + A(x) \cdot A'(x))^{\frac{3}{2}}} \left( \frac{1}{| x - p |} A(x) \cdot \frac{\partial x}{\partial \sigma_j} + \frac{1}{| x - p' |} A'(x) \cdot \frac{\partial x}{\partial \sigma_j} \right).
\]
Thus, one obtains
\[
\frac{\partial}{\partial \sigma_j} \sqrt{1 + A(x) \cdot A'(x)} = \frac{A(x) + A'(x)}{2 \sqrt{1 + A(x) \cdot A'(x)}} \left( \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A(x) + \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A'(x) \right)
\]
\[\quad + \frac{1}{\sqrt{1 + A(x) \cdot A'(x)}} \left( \lambda(x) I_3 - \frac{1}{|x - p|} A(x) \otimes A(x) \right)
\]
\[\quad - \frac{\partial}{\partial \sigma_j} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}
\]
and thus
\[
\sqrt{1 + A(x) \cdot A'(x)} \frac{\partial}{\partial \sigma_j} \sqrt{1 + A(x) \cdot A'(x)} = \frac{A(x) + A'(x)}{2 \sqrt{1 + A(x) \cdot A'(x)}} \left( \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A(x) \right)
\]
\[\quad + \frac{1}{\sqrt{1 + A(x) \cdot A'(x)}} \left( \lambda(x) I_3 - \frac{1}{|x - p|} A(x) \otimes A(x) \right)
\]
\[\quad - \frac{\partial}{\partial \sigma_j} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}
\]
\[\quad - A(x) \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A(x) - A'(x) \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A'(x) + \lambda(x) \frac{\partial x}{\partial \sigma_j}
\]
\[\quad + \frac{1}{2} A'(x) \frac{1}{|x - p|} \frac{\partial x}{\partial \sigma_j} \cdot A'(x) + \lambda(x) \frac{\partial x}{\partial \sigma_j}
\]
\[\quad - \frac{\partial}{\partial \sigma_j} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}
\]
Therefore, we obtain
\[
\frac{\partial}{\partial \sigma_j} \sqrt{1 + A(x) \cdot A'(x)} = \frac{\lambda(x)}{\sqrt{1 + A(x) \cdot A'(x)}} \left( I_3 - \frac{1}{2} A(x) \otimes A(x) - \frac{1}{2} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}
\]
and hence
\[
- \frac{\partial}{\partial \sigma_j} v_x = \frac{\lambda(x)}{\sqrt{2(1 + A(x) \cdot A'(x))}} \left( I_3 - \frac{1}{2} A(x) \otimes A(x) - \frac{1}{2} A'(x) \otimes A'(x) \right) \frac{\partial x}{\partial \sigma_j}
\]
\[
\lambda(x)
\]
Since
\[
S_x \left( \xi_1 \frac{\partial x}{\partial \sigma_1} \mid_{\sigma = 0} + \xi_2 \frac{\partial x}{\partial \sigma_2} \mid_{\sigma = 0} \right) = - \left( \xi_1 \frac{\partial}{\partial \sigma_1} v_x \mid_{\sigma = 0} + \xi_2 \frac{\partial}{\partial \sigma_2} v_x \mid_{\sigma = 0} \right),
\]
we obtain (A.2).
\[
\square
\]
From (A.2), one can compute the principle curvatures at \( x \in E_c (p', p') \).
Consider the case \( A(x) \neq A'(x) \). Since \( A(x) \) and \( A'(x) \) are unit vectors and \( A(x) \neq -A'(x) \), we have \( v = A(x) \times A'(x) \neq 0 \). Since \( A(x) \cdot v = A'(x) \cdot v = 0 \), \( v \) satisfies
\[
S_x (v) = \frac{\lambda(x)}{\sqrt{2(1 + A(x) \cdot A'(x))}} v.
\]
Next choose \( v' = A(x) - A'(x) \). Since \( v_i \) and \( A(x) + A'(x) \) are parallel, \( v' \in T_x E_c(p, p') \). We have

\[
S_i(v') = \frac{\lambda(x)\sqrt{1 + A(x) \cdot A'(x)}}{2\sqrt{2}} v'.
\]

Note also that \( v \) and \( v' \) are perpendicular to each other. Therefore, the eigenvalues of \( S_i \) consist of two real numbers:

\[
k_1(x) = \frac{\lambda(x)}{\sqrt{2(1 + A(x) \cdot A'(x))}}.
\]
\[
k_2(x) = \frac{\lambda(x)\sqrt{1 + A(x) \cdot A'(x)}}{2\sqrt{2}}.
\]

(A.3)

If \( A(x) = A'(x) \), then \( v_i = A(x) \). Since \( v \cdot v_i = 0 \) for all \( v \in T_x E_c(p, p') \), from (A.2) we obtain

\[
S_i(v) = \frac{\lambda(x)}{2} v, \quad \forall v \in T_x E_c(p, p').
\]

Thus, the set of all eigenvalues of \( S_i \) consists of only \( \lambda(x)/2 \). Therefore, \( k_1(x) \), \( k_2(x) \) given by (A.3) also covers this special case. Note that \( k_2(x) \leq k_1(x) \) and \( k_1(x) = k_2(x) \) if and only if \( A(x) = A'(x) \). Therefore, the Gauss curvature \( K(x) \) at \( x \in E_c(p, p') \) and the mean curvature \( H(x) \) with respect to \( v_i \) are

\[
K(x) = k_1(x)k_2(x) = \frac{\lambda(x)^2}{4},
\]
\[
H(x) = \frac{k_1(x) + k_2(x)}{2} = \frac{\lambda(x)}{8} (3 + A(x) \cdot A'(x)).
\]

(A.4).

Thus, one obtains

\[
\text{trace}(M_{kj}) = \frac{\lambda(q; p, p' + sA') \cdot (1 - A \cdot A')}{2\sqrt{2(1 + A \cdot A')}} + 2H_{3D}(q).
\]

For the computation of \( \det(M_{kj}) \), we prepare the following formula.
Proposition A.2. Let $B$ be a $2 \times 2$ matrix, and $c$ and $c'$ be two-dimensional vectors. Let $\gamma$ be a constant. Let

$$M = \gamma (c \otimes c + c' \otimes c') + B. \quad (A.7)$$

We have

$$\det M = \gamma^2 \left( \det \begin{pmatrix} c_1 & c'_1 \\ c_2 & c'_2 \end{pmatrix} \right)^2 + \gamma \left[ B \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \cdot \begin{pmatrix} c'_2 \\ -c'_1 \end{pmatrix} + B \begin{pmatrix} c'_2 \\ -c'_1 \end{pmatrix} \cdot \begin{pmatrix} c'_2 \\ -c'_1 \end{pmatrix} \right] + \det B. \quad (A.8)$$

Proof. We have

$$\det M = (\gamma (c_1^2 + c_2^2) + b_{11})(\gamma (c_1'^2 + c_2'^2) + b_{22})$$

$$- \gamma^2 (c_1^2 + c_1'^2)(c_2^2 + c_2'^2) + \gamma ((c_1^2 + c_1'^2)b_{22} + (c_2^2 + c_2'^2)b_{11})b_{12}b_{21}$$

$$= \gamma^2 (c_1^2 + c_1'^2)(c_2^2 + c_2'^2) - \gamma((c_1c_2 + c_1'c_2')b_{21}) - \gamma((c_1c_2 + c_1'c_2')b_{21} - (c_1c_2 + c_1'c_2')(b_{12} + b_{21}) + \det B$$

$$= \gamma^2 (c_1c_2 - c_1'c_2)^2$$

$$+ \gamma((c_1^2 + c_1'^2)b_{22} + (c_2^2 + c_2'^2)b_{11} - (c_1c_2 + c_1'c_2')(b_{12} + b_{21}) + \det B. \quad \Box$$

Note that $(M_{kj})$ given by (A.5) coincides with (A.7) in the case when $c = (A \cdot e_j)$, $c' = (A' \cdot e_j)$, $B = \nabla^2 f(0)$ and

$$\gamma = \frac{\lambda(q; p, p' + sA')}{2\sqrt{2(1 + A \cdot A')}}. \quad (A.9)$$

From (4.15), we have

$$\det \begin{pmatrix} A \cdot e_1 & A' \cdot e_1 \\ A \cdot e_2 & A' \cdot e_2 \end{pmatrix} = \det \begin{pmatrix} A \cdot e_1 & -A \cdot e_1 \\ A \cdot e_2 & -A \cdot e_2 \end{pmatrix} = 0; \quad (A.8)$$

from (4.19), we have

$$\nabla^2 f(0) \begin{pmatrix} A \cdot e_2 \\ -A \cdot e_1 \end{pmatrix} \cdot \begin{pmatrix} A \cdot e_2 \\ -A \cdot e_1 \end{pmatrix} = S_q(\partial D)((A \cdot e_2)e_1 - (A \cdot e_1)e_2) \cdot ((A \cdot e_2)e_1 - (A \cdot e_1)e_2). \quad (A.9)$$

By lemma 4.3, we have

$$A \times A' = -\sqrt{2(1 + A \cdot A')}(A \cdot e_2)e_1 - (A \cdot e_1)e_2). \quad (A.10)$$

This gives

$$S_q(\partial D)((A \cdot e_2)e_1 - (A \cdot e_1)e_2) \cdot ((A \cdot e_2)e_1 - (A \cdot e_1)e_2)$$

$$= \frac{S_q(\partial D)(A \cdot A') \cdot (A \times A')}{2(1 + A \cdot A')}$$

and thus from (A.9) one obtains

$$\nabla^2 f(0) \begin{pmatrix} A \cdot e_2 \\ -A \cdot e_1 \end{pmatrix} \cdot \begin{pmatrix} A \cdot e_2 \\ -A \cdot e_1 \end{pmatrix} = \frac{S_q(\partial D)(A \times A') \cdot (A \times A')}{2(1 + A \cdot A')}. \quad (A.10)$$

Since

$$(A' \cdot e_2)e_1 - (A' \cdot e_1)e_2 = -(A \cdot e_2)e_1 - (A \cdot e_1)e_2,$$
we also obtain
\[ \nabla^2 f(0) \left( \begin{array}{c} A' \cdot e_2 \\ -A' \cdot e_1 \end{array} \right) \cdot \left( \begin{array}{c} A' \cdot e_2 \\ -A' \cdot e_1 \end{array} \right) = \frac{S_q(\partial D)(A \times A') \cdot (A \times A')}{2(1 + A \cdot A')} \] (A.11)

From proposition A.2, (A.8), (A.10) and (A.11), we obtain
\[ \det(M_{kj}) = \lambda(p', p + sA') \sqrt{2(1 + A \cdot A')}^{3/2} S_q(\partial D)(A \times A') \cdot (A \times A') + K_{qD}(q). \]

Substituting this together with (A.6) into (A.4), we obtain (5.6).

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