Free parameters in quantum field theories: an analysis of the variational approach

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Abstract

The usual renormalization procedure for the variational approximation with a trial Gaussian ansatz for the $\lambda \phi^4$ model in 3+1 dimensions is re-analysed as a departing framework for the investigation of the parameters of the model. The so-called asymmetric phase of the model (where $<\text{vac}|\phi|\text{vac}> \neq 0$) is considered for the search of privileged values of these parameters (mass and coupling constant) and possible conditions they may be expected to satisfy. This also may yield a suitable approach for the investigation of the reliability and stability of the approximation. The extremization of the renormalized energy density with relation to the renormalized mass, coupling and $\bar{\phi}$ is done. The minimizations of the renormalized energy with relation to the mass and $\bar{\phi}$ provide different expressions from the ones obtained by the usual variational procedure for the regularized theory. Sort of “energy scale” invariances in expressions for the renormalized mass and coupling constant are found. A different view on the restoration of symmetry issue is discussed. The transcendental character of the GAP equation may be reduced or even eliminated by placing some variables in the complex plane.

Key words: Mass, coupling constant, variational method, symmetry restoration, vacuum, non perturbative method, quantum field theory, spontaneous symmetry breaking, many body quantum theory, Gaussian.

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1 Introduction

The most developed approach to solve interacting field (and many body) theories is perturbation theory which only works well for very small coupling constants as it occurs in Quantum Electrodynamics. Also in this approach there is a systematic and direct way of dealing with ultraviolet (UV) divergences, i.e., one knows precisely how to renormalize the parameters and to make the theory finite [1, 2, 3]. There are many motivations for the development of non perturbative methods in Quantum Field and Many Body Theories such as for the description of strong interacting systems (with or without spontaneous symmetry breaking(s) (SSB)), bound states and phase transitions. One method which has been quite extensively investigated is the variational approximation which, with the use of Gaussian wave functional, has been showed to be useful in a wide variety of situations. It corresponds to a summation of “cactus” type loop diagrams [4, 5, 6, 7, 10]. It is equivalent to the Hartree Bogoliubov approach [8] and also to the leading order large N approximation [9, 11]. In this approach the ground state of the system is determined by equations for the variational parameters, which are chosen to be a mass and the classical expected field characteristic from a SSB state (< ϕ >≡ ¯ϕ, which will be referred to as condensate [8]). These equations are derived by the minimization of the (regularized) averaged energy density with respect to variational parameters of the trial wave functional. The subtraction of the (equations of the) theory with < ϕ >= 0 from the ones of the theory in which < ϕ >≠ 0 provides a general and consistent elimination of the ultraviolet divergences. Limitations pointed out and discussed in [12] have been rediscussed, leading to extensions and higher order calculations for static and time dependent formulations [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 11]. For the sake of conciseness they will not be discussed here although the main ideas of the present work apply to complementary approaches.

It is usually highly desirable to predict the values of the free parameters of a physical theory, such as masses and couplings, from the theory itself before comparisons to experimental observations. For this there may be possible to predict values, either exact ones or ”privileged” range. These ”privileged” values may also be associated to the validity of the approximation method or even about applicability of the model. The main aim of the present work is to suggest and investigate one reasoning according to which values (or range of values) for these parameters could be found. Eventually this may constraint them in a more specific theory. Basic ideas are to search for renormalized couplings and masses which extremize (minimize/maximize) the renormalized energy density. This procedure can be considered as complementary to the renormalization group method [28]. Another procedure will be to consider
parameters in the complex plane to introduce auxiliary (imaginary) variables which are to be eliminated.

The $\lambda \phi^4$ model has been extensively studied for different reasons among which to shed light on nonperturbative effects in quantum field and many body theories (QFT, QMBT). It corresponds to one of the simplest self interacting model whose structure is expected to be (partially) present in several more elaborated theories and it presents interesting features \cite{29,30,7,28,31,24,33,34}. It has also been considered for the study of cosmological models \cite{35} and of the Higgs particle in the standard model, for example in \cite{32}. It also shares several properties with the linear sigma model (LSM) which is an effective model for low energy QCD. Although it strongly seems to possess asymptotic freedom in the asymmetric phase \cite{30,7,33}, the model is “trivial” in the symmetric phase \cite{34,28}.

In the present work the usual renormalization scheme of the Gaussian approach as carried out, for example, in \cite{36} for the $\lambda \phi^4$ model is used as starting point for further investigation. It is proposed the extremization of the renormalized energy density with relation to the renormalized parameters (coupling constant and mass). Besides that some variables are placed in the complex plane to search suitable (physical) values and eventual conditions for these parameters. The work is organized as follows. In the next section the Gaussian approximation is summarized: the GAP equation (transcendental) is derived, obtained from the regularized theory (with a cutoff). The renormalization procedure of the mass and coupling constant as proposed in \cite{36} is considered. In sections 3, 4 and 5 values of the renormalized mass, condensate and coupling constant which extremize the energy density are searched and analysed. They also could yield privileged values of the parameters with which the approximation may be more appropriated. In some cases instabilities are found for values of the parameters. In section 6 a mathematical trick is used to search non transcendental solutions or/and an expression which constrains further the parameters. This is done by allowing some parameters to be complex such that the imaginary part must in fact disappear in the end of the calculation. In the last section the results are summarized.

2 Gaussian approximation for the $\lambda \phi^4$ model

The Lagrangian density for the scalar field $\phi(x)$ with bare mass $m^2_0$ and coupling constant $\lambda$ is given by:

$$\mathcal{L}(x) = \frac{1}{2} \left\{ \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) - m^2_0 \phi^2(x) - \frac{\lambda}{12} \phi^4(x) \right\}$$

(1)

The theory is quantized in the Schrödinger picture \cite{37} being the action of the field and momentum operators over a state $|\Psi[\phi]\rangle$ given respectively by:

$$\hat{\phi}|\Psi\rangle = \phi|\Psi\rangle \quad \hat{\pi} = -i\hbar \frac{\delta}{\delta \phi}|\Psi\rangle$$

(2)
In the static Gaussian approximation at zero temperature the trial ground state wave functional $\Psi$ is parametrized by the Gaussian:

$$\Psi[\phi(x)] = N \exp \left\{ -\frac{1}{4} \int dxdy \delta \phi(x)G^{-1}(x,y)\delta \phi(y) \right\},$$

(3)

Where $\delta \phi(x) = \phi(x) - \bar{\phi}(x)$ is the field shifted by the condensate, the point where the wave function is centered; the normalization factor is $N$, the variational parameters are the (classical) expected value of the field, $\bar{\phi}(x) = \langle \Psi | \phi | \Psi \rangle$, and the quantum fluctuations represented by the two point function, i.e., the width of the Gaussian: $G(x,y) = \langle \Psi | \phi(x)\phi(y) | \Psi \rangle$. In variational calculations the averaged energy calculated with $\Psi[\phi(x)]$ is to be minimized to obtain the GAP equations. In principle it would yield a maximum bound for the ground state (averaged) energy, although ultraviolet divergences make this not necessarily reliable. The minimization of the renormalized theory may be useful for this theoretical bound of the variational principle. Each of these variational parameters represents one component of the scalar field: the expected value in the ground state ("classical" part) and the two-point Green’s function with the mass of the quantum which is decomposed into creation and annihilation operators [8].

The average value of the Hamiltonian is calculated and expressed in terms of the variational parameters by means of expressions (2) and (3). It is given by:

$$\mathcal{H} = \frac{1}{2} \left[ \frac{1}{4} G^{-1}(x,x) - \Delta G(x,x) + m_0^2 G(x,x) + \frac{\lambda}{2} G^2(x,x) + m_0^2 \bar{\phi}^2(x) + (\nabla \bar{\phi}(x))^2 + \frac{\lambda}{2} \bar{\phi}^4(x) + \frac{\lambda}{2} \bar{\phi}^2(x) G(x,x) \right].$$

(4)

Although in this expression the variational parameters were allowed to have spatial dependence they will be assumed to be constant. Variations of the averaged energy density with respect to the variational parameters yield the following GAP and condensate equations which define the ground state of the model:

$$\frac{\delta \mathcal{H}}{\delta G(x,y)} \to 0 = -\frac{1}{8} G^{-2}(x,y) + \frac{\Gamma(x,y)}{2} + \frac{\lambda}{2} \bar{\phi}(x)^2 \quad (i)$$

$$\frac{\delta \mathcal{H}}{\delta \bar{\phi}(x)} \to 0 = \Gamma(x,y) \bar{\phi}(y) + \frac{\lambda}{6} \bar{\phi}^2(x), \quad (ii)$$

(5)

Where $\Gamma(x,y) = -\Delta + \left( m_0^2 + \frac{\lambda}{2} G(x,x) \right) \delta(x-y)$. The Green’s function $G$ may be written from expressions above as:

$$G_0(x,y) = \langle x| \frac{1}{\sqrt{-\Delta + m^2}} |y \rangle$$

(6)

where $m^2$ is given by the self consistent (transcendental) GAP equation (expression (3)):

$$m^2 = m_0^2 + \frac{\lambda}{2} \text{Trace} G(x,x,m^2) + \frac{\lambda}{2} \bar{\phi}^2.$$

(7)
An analogous expression holds for the case in which \( \bar{\phi} = 0 \), i.e.,
\[
\mu^2 = m^2(\bar{\phi} = 0) = m_0^2 + \frac{\lambda}{2} \text{Trace}G(x, x, \mu^2).
\]
Expression (6) is equivalent to the Feynman Green’s function with time integrated and with sign changed in the imaginary part by replacing the self consistent mass by the bare mass \( m_0^2 \). The physical masses in the different phases may assume different values from each other. The condition of minimum for this procedure and its stability was partially investigated in [7] and it corresponds to analysing the second order variation of the energy density with respect to the variational parameters.

From the above expressions it is seen that the non zero solutions for the condensate, \( \bar{\phi} \), can be written as:
\[
\bar{\phi}^2 = -6\frac{m_0^2}{\lambda} - 3G(m^2) = 3\frac{m^2}{\lambda}. \tag{8}
\]
For \( G = 0 \) the tree level value for \( \bar{\phi} \) is obtained in terms of the bare mass.

The above expression for the Gaussian width (6) (and its inverse \( G^{-1} \)) can be calculated in the momentum space with a regulator \( \Lambda \) (cutoff) yielding (for \( \Lambda >> m^2 \)):
\[
G(m^2) = \frac{1}{8\pi^2} \left( \Lambda^2 - m^2Ln\left( \frac{2\Lambda}{\sqrt{e}m} \right) \right),
\]
\[
G^{-1}(m^2) = \frac{1}{8\pi^2} \left( 2\Lambda^4 + 2m^2\Lambda^2 - \frac{m^4}{4} - m^4Ln\left( \frac{2\Lambda}{\sqrt{e}m} \right) \right), \tag{9}
\]
where \( d = 2/\sqrt{e} \). In the (local) limit of infinite cutoff the average energy and observables diverge and the divergences must be eliminated. The renormalization procedure has been performed in three dimensions for example in [36, 3, 30, 7].

### 2.1 Renormalized parameters

The renormalization procedure of the parameters of the model is done as follows. The energy density of the symmetric phase, as well as its GAP equation (7), is subtracted from the corresponding expression of the asymmetric phase. The GAP equation as defined in expression (7) can be rewritten as:
\[
\mu^2 = m^2 + g_R\left( \bar{\phi}^2 + \frac{m^2}{8\pi^2}Ln\left( \frac{m}{\mu} \right) \right), \tag{10}
\]
where the renormalized parameters were defined as:
\[
\mu^2 = m_R^2 = \frac{m_0^2 + \lambda\Lambda^2}{1 + \frac{\lambda}{10\pi\log\left( \frac{\Lambda}{\mu} \right)}},
\]
\[
g_R = \frac{-\frac{\lambda}{2}}{1 + \frac{\lambda}{10\pi\log\left( \frac{\Lambda}{\mu} \right)}}. \tag{11}
\]
In the first of these expressions \( m_R^2 \equiv \mu^2 \) was chosen to produce the usual effective potential [5, 36]. It is seen from the second of these expressions that in the limit of \( \Lambda \to \infty \) the bare coupling constant would go to zero in order to keep \( g_R \) finite if \( \mu \) is kept constant. This is the “triviality problem.

The resulting subtracted energy density, \( H_{sub} = \mathcal{H}(\bar{\phi}) - \mathcal{H}(\bar{\phi} = 0) \), is re-written in terms of the renormalized mass, coupling constant and the mass scale eliminating the cutoff. It is given by:

\[
H_{sub} = \frac{m^2}{2} \bar{\phi}^2 + \frac{1}{4g_R} \left( m^2 - \mu^2 \right)^2 + \frac{1}{128\pi^2} \left( m^4 \ln \left( \frac{m^4}{\mu^4} \right) - m^4 + \mu^4 \right). \tag{12}
\]

The mass scale \( \mu^2 \) is not a free parameter in fact, it can be considered to be a function of the mass \( m^2 \) and the coupling \( g_R \) by the GAP expression (10). Other approaches may be of interest for investigating the variational method in the Schrodinger picture [3]. In the ground state the parameters \( \bar{\phi}, m^2, \mu^2 \) (for a given \( g_R \)) are related by the GAP and condensate expressions shown above. Any deviation of the respective numerical values from the ones related by these expressions induce temporal evolution [8].

It is possible to verify whether the renormalized GAP equation obtained from the regularized energy density given by expression (10) still is a GAP equation for the energy density given by expression (12) in two ways. The minimization of expression (12) with relation to \( m^2 \) is done in the next section. However the integration of the GAP equation with relation to \( m_2 \) should result in an expression equal to (12) if the order of performing renormalization and extracting the ground state does not change results. The integral of the GAP equation is given by:

\[
\int \left( -\mu^2 + m^2 + g_R \left( \bar{\phi}^2 + \frac{m^2}{8\pi^2} \ln \left( \frac{m^4}{\mu^4} \right) \right) \right) dm^2 =
-\mu^2 m^2 + \frac{m^4}{2} + g_R \bar{\phi}^2 m^2 + \frac{g_R}{16\pi^2} \left( \frac{m^4}{2} \ln \left( \frac{m^2}{\mu^2} \right) - \frac{m^6}{6\mu^2} \right) + C(\bar{\phi}, \mu^2), \tag{13}
\]

where \( C(\bar{\phi}, \mu^2) \) does not depend on \( m^2 \). This expression contains terms very different from the renormalized expression (12). This means either that the minimization of the regularized energy is not equivalent to the minimization of the renormalized one or/and that the renormalization procedure has to be improved to make both procedures coincident - if this is possible or desirable. This will be discussed below with the minimization of the energy density with respect to \( \bar{\phi} \).

### 3 Energy density and renormalized mass

In this section the renormalized energy density \( H_{sub} \) is extremized with relation to the renormalized (physical) mass:

\[
\frac{\partial H_{sub}}{\partial m} = 0. \tag{14}
\]
The roots of the resulting expression were calculated by considering that $\bar{\phi}^2$ is in fact dependent on $m^2$ by expression (10). The resulting expression is given by:

$$0 = m^3 \left[ \ln^2 \left( \frac{m}{\mu} \right) a_1 + \ln \left( \frac{m}{\mu} \right) a_2 + a_3 \right],$$

(15)

where $a_i$ can be given in terms of

$$J = 1 - \frac{gR}{(8\pi^2)^2} = 1 - G_R,$$

by:

$$a_1 = \frac{1}{gR} J^2 + \frac{1}{32\pi^2},$$

$$a_2 = \frac{2}{gR} \left( -1 + J + \frac{J^2}{(32\pi^2)} \right) + \frac{1}{128\pi^2} \left( 1 + \frac{2J^2}{(8\pi^2)} \right),$$

$$a_3 = \frac{1}{32\pi^2} \left( 1 + \frac{gR}{32\pi^2} \right).$$

(16)

Expression (15) is not equal to the GAP (15) obtained from the minimization of the regularized energy density with relation to $G(m^2)$. There are therefore five solutions for the renormalized mass $m^2$ which can be written in the following form:

$$m^3 = 0,$$

$$m^\pm = \mu \exp(H^\pm),$$

(17)

where:

$$H^\pm = -a_2 \pm \sqrt{a_2^2 - 4a_1a_0} \over 2a_1.$$  

(18)

These solutions for $m^\pm$ can be viewed as having corrections for the value of $\mu$ due to the self interaction through the parameters $H^\pm$ due to the appearance of $\bar{\phi} \neq 0$. It is noted that there is a sort of “energy scale” invariance in these expressions for $m^\pm$ with simultaneous changes in the mass renormalization parameter $\mu$.

The particular case of $m^2 = \mu^2$, for which the GAP equation is not necessarily valid because $\bar{\phi} \rightarrow 0$, is found for

$$a_0 = 0, \quad \rightarrow \quad gR = -32\pi^2.$$  

(19)

This point may correspond to a restoration of the symmetry.

The zero mass solutions correspond to a saddle point, they are not minima neither maxima of the energy density. If the others solutions are minima is checked via the positiveness of the second derivative:

$$\frac{\partial^2 H_{sub}}{\partial m^2} = \frac{m^2}{(8\pi^2)^2} \left( 2\ln \left( \frac{m}{\mu} \right) a_1 + a_2 \right) > 0.$$  

(20)

For the derivation of these expressions the complete self consistency of the Gaussian equations was not considered. There has been used a truncation on the dependence on $\mu$, i.e., the dependence of $\ln(\mu/m)$
on $\mu$ (self consistency) was considered only for $\mu$ not very different from $m$, i.e. $\mu^2 = m^2 + \delta$ where $\delta << m^2$. Out of this range the above solutions are not expected to be valid.

In Figures 1a and 1b the solutions of the above equations ($m^\pm/\mu$ from (17)) are shown as a function of $G_R = g_R/(32\pi^2)$. All the solutions of figure 1a, for $m^+$, correspond to stable solutions ($d^2H/dm^2 > 0$). The solutions of figure 1b, for $m^-$, are stable for $G_R$ nearly equal or smaller than $-1.45$ or equal or greater than nearly $1.25$. The point $g_R = 0$ is not plotted. Values between $-1 < G_R < 0$ do not correspond to physical stable values of the condensate as it will be shown below, in expression (24). In the limits of $g_R \to \pm \infty$ we obtain analytically that either $m = \mu$ or $m = 0$. For the case $\mu \to \infty$ the renormalized coupling constant $G_R \to 0$. While the solution $m^-_R$ in the weak coupling regime can be identified to the renormalization point usually considered (for $\mu >> m$ and/or the cutoff going to infinite) there is another stable solution $m^+_R$ for which $\mu \simeq m^+$. 

4 The condensate: $\bar{\phi}$

The variational equation for the condensate (expression (3 (ii))) is obtained from the regularized energy density $H_{reg}$. The minimization of the renormalized energy density is done in the following:

$$\frac{\partial H_{sub}}{\partial \bar{\phi}} = 0.$$  \hspace{1cm} (21)

For this derivation the GAP equation provides the dependence of the mass on the condensate, i.e., $m^2(\bar{\phi})$ and $\mu^2 \equiv m^2(\bar{\phi} = 0)$ is kept constant. It yields the following expressions:

$$\bar{\phi} = 0,$$

$$\bar{\phi}^2 = m^2 - \frac{m^2}{g_R} \left( 1 + \frac{1}{8\pi^2} \ln \left( \frac{m}{\mu} \right) \right).  \hspace{1cm} (22)$$

This last expression may coincide with the expression of $\bar{\phi}_0$ obtained from the minimization of the regularized energy density (expression (8)) depending on the relation between $\lambda$ and mass scale $\mu$ as it will be shown below. However it is not completely consistent with the GAP equation (10) which is obtained from the minimization of the regularized energy density with respect to the mass $m$ in the asymmetric phase and then renormalized. To make these expressions compatible it would be necessary to consider the following alternatives for these expressions:

$$\mu^2 \neq m^2_R, \quad \text{or} \quad \mu^2 = (g_R - 1) \frac{m^2}{8\pi^2} \ln \left( \frac{m}{\mu} \right),$$  \hspace{1cm} (23)

where $m^2_R$ is the one of expression (11). It is not clear whether these identifications are reasonable or if they imply a meaningful loss of generality. The limit of $g_R = 1$ does not seem to be reasonable.
Therefore the two minimization procedures (of the regularized and the renormalized energy densities with respect to the regularized and renormalized parameters respectively) do not seem to yield necessarily the same expressions for the parameters in the ground state. Nevertheless it is worth to remember that renormalization is performed basically from the regularized GAP equation.

From the expression (22) the following conditions to obtain real values of $\tilde{\phi}$ can be considered:

\[
\begin{align*}
\text{if } g_R > 0 & \rightarrow \ln \left( \frac{m}{\mu} \right) < -8\pi^2, \\
\text{if } g_R < 0 & \rightarrow \ln \left( \frac{m}{\mu} \right) > -8\pi^2.
\end{align*}
\] (24)

The energy density is expected be stable for the condensate values found in expression (22). This minimum is verified by calculating the second derivative of the energy density with relation to $\tilde{\phi}$, i.e.: $\partial^2 H_{\text{sub}} / \partial \tilde{\phi}^2 > 0$. Its positiveness corresponds to the condition:

\[
g_R \left( 1 + \frac{g_R}{32\pi^2} \right) > 0.
\] (25)

From this it is seen that for positive coupling constant $g_R$, it can assume any value (from this stability criterium) whereas if $g_R < 0$ one would have to consider $g_R < -32\pi^2$. Expressions (24) and (25) may correspond to constraints for the values that the renormalized coupling may assume in order to yield stable real ground states.

Expression (22) can be written as:

\[
g_R \tilde{\phi}^2 = -m^2 \left( 1 + \frac{1}{8\pi^2} \ln \left( \frac{m}{\mu} \right) \right).
\] (26)

When $\mu = m \exp(8\pi^2)$ it follows that either $\tilde{\phi} = 0$ or $g_R = 0$ in the asymmetric phase of the potential. This may correspond to the so called symmetry restoration when the condensate disappears at a particularly high excitation energy, i.e., the symmetry is restored. A different solution for the particular limit of $\tilde{\phi} = 0$ was found in expression (19) where the energy density is minimum with relation to the mass for $m^2 = \mu^2$.

The above expression for the condensate (22) can be equated to the previous (regularized) one (8). Taking into account the expression of the renormalized coupling constant in terms of the bare one (expression (11)) this can be written as:

\[
\lambda = \frac{16\pi^2}{\ln \left( \frac{\Lambda d}{\mu} \right)} \left( -1 + \frac{3}{2 \left( 1 + \frac{1}{8\pi^2} \ln \left( \frac{m}{\mu} \right) \right)} \right).
\] (27)
If the cutoff is sent to infinite the bare coupling constant assumes different values depending on the ratio of $\mu/m$. For example, there is a case in which $\lambda = 0$ if either $\Lambda \to \infty$ for finite $\mu$ or:

$$\frac{m}{\mu} = \exp(4\pi^2),$$

being therefore $m^2 >> \mu^2$. Varying $\mu$ together with $\Lambda$ there may have non zero $\lambda$ solutions. For $\Lambda/\mu$ finite, the coupling $\lambda$ may even diverge when:

$$\frac{m}{\mu} = \exp(-8\pi^2).$$

This is the same point found above (for expression (24) for the possible restoration of the symmetry.

It is worth emphasizing that it has been assumed, as it usually is, that the minimum of the effective potential with relation to the condensate coincides necessarily with its minimum in respect to the physical mass $m^2_R$ in the regularized theory.

5 Analysis of the renormalized coupling constant

Analogously to what was done for the renormalized mass in the preceeding section the extremization of the renormalized energy density with respect to the renormalized coupling constant is done in this section. Moreover one relevant subject for any approximation method is the understanding of the range of values of the parameters of the model (as mass and mainly coupling constants) for which the approximation is more appropriated. The extremization is found from:

$$\frac{\partial \mathcal{H}_{sub}}{\partial g_R} = 0.$$

It is considered, in the following, a truncation of the self consistency of the GAP equations. This is done by taking the scale parameter to be close to the mass $\mu^2 = m^2 + \delta$, where $\delta << m^2$ is determined from the GAP equation self consistently. From the renormalized GAP equation (expression (10)) it follows that:

$$\delta = \frac{g_R \bar{\phi}^2}{1 + \frac{g_R}{16\pi^2}}.$$

Since $\delta << m^2$ either $\bar{\phi}$ is large or $g_R$ is very large for positive coupling $g_R$. The minimization of the renormalized energy yields the following third order algebraic expression:

$$\left( G'_R \right)^3 + \left( G'_R \right)^2 \left( 3 + (1 + H) \frac{1}{16\pi^2} \right) + G'_R \left( 3 + (1 + H) \frac{3}{2} + (1 + H) \frac{1}{32\pi^2} \right) + 1 + \frac{(1 + H)^2}{2} = 0,$$
where
\[ H = \frac{\ln \left( \frac{m}{\mu} \right)}{(8\pi^2)}, \quad G'_R = \frac{g_R}{(16\pi^2)}. \]

In figures 2a, 2b and 2c the solutions of expression (31) are showed as function of a limited range of \( H \), i.e., \( \ln(m/\mu) \). Figures 2b and 2c exhibit the same behavior. It is plotted only the region in which the above truncation scheme of the self consistency may be expected to be reliable. The values for \( g_R \) are large and obtained for \( m^2 \sim \mu^2 \) which cannot be simultaneously compatible with the results of Figures 1a and 1b (from the minimization of the energy with relation to the mass \( m^2 \)) - in this limit of \( \delta << m^2 \). The point \( H = -0.001 \) corresponds to \( m/\mu = 0.985 \) which is not obtained for the larger values of \( g_R \) from figures 1a and 1b.

These solutions (31) may nevertheless correspond to minima or maxima of the solutions of the above equation is also verified. For this the positiveness of the second derivative is calculated: \( \frac{\partial^2 \mathcal{H}_{sub}}{\partial g'_R^2} > 0 \). All the solutions have a negative second derivative corresponding to maxima, instead of minima, of \( \mathcal{H}_{sub}(g_R) \).

The solutions for the coupling constant of expression (31) depend only on the ratio \( m/\mu \) and not on the absolute values of these parameters. This may be seen as a sort of energy scale invariance for different physical processes (eventually in different systems) at different energy scales with different physical masses.

5.1 Fixing the energy density

For the analysis of the system with an energy density given by \( \mathcal{H}_{sub} \) and a given mass scale, \( \mu \), (acceptable) values for the renormalized coupling constant and mass may be suggested such that the values remain in the physically allowed part of the phase space of the model [27]. This corresponds to fix the renormalized mass and energy density \( \mathcal{H}_{sub} \) and to calculate the resulting physical coupling constants for the process involved at a scale \( \mu \). A third degree algebraic equation is obtained and which can be written as:

\[
g_R^3 \frac{H^2}{128\pi^2} + g_R^2 \left( \frac{H^2}{4} - \frac{2H^2}{x} \right) + g_R \left( -\frac{H(H+1)}{2} + \frac{1}{x} \left( \frac{H}{4} - 1 + H^2 \right) - \frac{\mathcal{H}_{sub}}{m^4} \right) - \frac{1}{4} + \frac{H^2}{4} = 0. \tag{32}
\]

where \( x = 128\pi^2 \) and \( H = \ln(m/\mu)/(8\pi^2) \). This expression also presents a sort of “energy scale” invariance for the parameters \( m/\mu \) unless for the term which depends on the total energy density, if \( \mathcal{H}_{sub} \) scales differently from \( m^4 \).

In figures 3a, 3b and 3c the solutions of this algebraic equation are shown as functions of \( H \) for a fixed energy density \( \mathcal{H} = (100MeV)^{-4} \) and \( m_R = 100MeV \). The coupling \( g_R \) can be strong in the region
of $\mu \simeq m$. In particular in the limit of $\mu = m$ an unique value is found, it is given by:

$$g_R = -\frac{1}{4 \left( \frac{\mu^2}{m^4} + \frac{1}{128\pi^2} \right)}.$$  \hfill (33)

For $H \to -\infty$, which is equivalent to $\mu/m \to \infty$, it follows $g_R \to 0$ as seen in figure 3a. However solutions of figures 3b and 3c do not correspond to $g_R \to 0$, but to a finite (quite strong) value close to 10.

6 The transcendental character of the GAP equation

In this section an heuristic trick is used to extract analytical non transcendental solutions from the GAP equation or to provide possible further relations among the parameters. Firstly it is considered that the the mass scale and renormalized mass may have imaginary parts:

$$\mu^2 \to \nu^2 = re^{i\theta}, \quad m^2 \to \tau = te^{i\omega},$$  \hfill (34)

where $r, s, \theta, \omega$ are respectively modulus and phases. With these parametrizations the GAP equation (10) can be written as:

$$\left( r \cos \theta - tc\omega - g_R \bar{\phi}^2 - D(t \cos \omega \ln(t/r) - (\omega - \theta)t\sin\omega) \right) +$$

$$+ i \left[ t \sin \omega - r \sin \theta + D(\omega t \cos \omega - t\theta \cos \omega + t \sin \omega \ln(t/r)) \right] = 0,$$  \hfill (35)

where $D = \frac{g_R}{8\pi}$. Both the real and the imaginary parts in this expression have basically the same structure of the usual GAP equation. It is worth emphasizing that requiring the imaginary part to disappear in the GAP equation implies real mass parameters if only one of them is placed in the imaginary plane, i.e., either $\theta = 0$ or $\omega = 0$. In these cases the parametrization in the complex plane is just a trick to reduce the transcendental character of the GAP equation (10).

Requiring the GAP equation (35) to have only real component (this is considered to be a stable system) the imaginary part is set to zero. The expression still is quite complicated but the analysis of some particular cases will be very useful. For $\omega = 0$ it follows that:

$$rsin\theta = -Dt\theta,$$  \hfill (36)

This may be written as:

$$\cos \theta = \sqrt{1 - \frac{B^2g^2}{r^2}}.$$  \hfill (37)

In this case the self consistent character of the GAP equation remains strong. The real part of expression (35) keeps the same form of expression (10) basically with the mass parameters $m^2, \mu^2$ replaced by $r, t$.\hfill 11
For $\theta = 0$ (and $\omega \neq 0$) the resulting expressions for the real part of the GAP equation and its imaginary part (to be equated to zero) can be obtained from expression (35). They can be written as:

$$
\begin{align*}
& r - t \cos \omega - g_R \phi^2 - D(t \cos \omega \ln(t/r) - \omega t \sin \omega) = 0, \\
& i[t \sin \omega + D(\omega t \cos \omega + t \ln(t/r) \sin \omega)] = 0.
\end{align*}
$$

(38)

It does not provide simpler solutions and therefore they are not shown. The resulting number of free parameters is not reduced because although there is one more expression ($\Im(GAP)$) there also is one extra variable ($\omega$).

Since the phases are auxiliary parameters it is reasonable to assume they are very small without (great) loss of generality for the results. The expression for the imaginary part of the GAP equation in the limit when $\sin(\theta) \sim \theta$ and $\sin(\omega) \sim \omega$ is given by:

$$
\omega t \left( 1 + D + DLn \left( \frac{t}{r} \right) \right) = \theta (r + Dt).
$$

(39)

This expression can be regarded as fixing the ratio $\theta/\omega$. Several particular cases are analyzed below although the more interesting case is obtained for $\omega, \theta$ non-zero and very small.

(1) Assuming the phases are equal $\theta = \omega$ expression (39) reduces to:

$$
r - t = Dt \ln \left( \frac{t}{r} \right),
$$

(40)

which fixes the ratio $r/t$ or correspondently $m^2/\mu^2$. This expression is only consistent with the renormalized GAP equation [1] for $\phi = 0$ (which is obtained from the minimization of the regularized energy density). Besides that it was mentioned above that, since $\omega \neq 0$ and $\theta \neq 0$, it is not clear whether $\mu^2$ and $m^2$ remain real although the GAP equation is necessarily real. This may happen because in this case the imaginary part of both parameters may cancel with each other to result a real GAP equation instead of allowing for independent cancelation. On the other hand each angle ($\omega$ or $\theta$) may be set to zero separately.

(2) For $\omega = 0$ it follows:

$$
r \simeq -Dt,
$$

(41)

which also fixes the ratio $m^2/\mu^2$ being a real number only for $g_R < 0$.

(3) For $\theta = 0$, expression (39) is computed up to the order of $O(\omega^2)$ and it reduces to:

$$
\omega^2 = 2D + 2 + 2 \ln \left( \frac{t}{r} \right),
$$

(42)
where it has been assumed that \( \sin \omega \sim \omega \). In this case it is reasonable to consider \( \omega^2 \sim 0 \) leading to the expression:

\[
\frac{t}{r} = \exp(-2 - D).
\] (43)

If \( \omega \neq 0 \) it will appear in the real part of the GAP equation and therefore the number of free parameters in the renormalized equation does not diminish with the new parametrization. Therefore \( \omega = 0 \) would be the only possibly interesting case. This does not happen because of expression (41) which imposes negative coupling \( g_R < 0 \).

The real part of the GAP equation for small angles keeps nearly the form of the original GAP, it can be written as:

\[
t - r + g_R \bar{\phi}^2 + D \left[ t \ln \left( \frac{t}{r} \right) - t \omega (\omega - \theta) \right] = 0,
\] (44)

where either \( r \) or \( t \) can be written as a function of the other by means of the constraints of the imaginary parts from the expressions (40), (41) or (42). In this third case the auxiliary parameter \( \omega \) was not eliminated (although \( \theta = 0 \)). However for very small phases the expression (44) reduces to the usual real GAP equation (10). In this case the real part of the GAP equation is the same as expression (10) written as:

\[
t - r + g_R \bar{\phi}^2 + D t \ln \left( \frac{t}{r} \right) = 0.
\] (45)

Simultaneously the renormalized energy density must be a real number. The imaginary part due to the introduction of parametrization (34) has to disappear. However it is easy to notice from expression (39) that the resulting expression for the imaginary part of \( \mathcal{H}_{\text{sub}} \) will be quite complicated. Below it will be assumed that the phases have small values. This should not impose great limitations in the results because they are auxiliary parameters. With this assumption several simplifications occurs because: \( \sin(\theta) \sim \theta \) and \( \sin(\omega) \sim \omega \). The result for the imaginary part of the energy density, up to first order in the phases, will be given by:

\[
\Im m(\mathcal{H}_{\text{sub}}) = \omega \left( \frac{t \bar{\phi}^2}{2} + 2t^2 A_- + \frac{tr}{2g_R} + \frac{r^2}{64\pi^2} \right) + \theta \left( 2A_+ r^2 - \frac{rt}{2g_R} + \frac{r^2}{32\pi^2} \ln\left( \frac{t}{r} \right) - r^2 \right) \to 0,
\] (46)

Where

\[
A_\pm = \frac{1}{4g_R} \pm \frac{1}{128\pi^2}.
\]

One of these variables \( A_+ \) can be identified with the solution of fixed \( \mathcal{H}_{\text{sub}}(\omega = \theta = 0) \) for \( \mu = m \) given by expression (33):

\[
A_+ = - \frac{\mathcal{H}_{\text{sub}}}{m^4} \bigg|_{\mu=m}.
\] (47)
Expression (46) still is very complicated and it can also be used to fix the ratio $\theta/\omega$ which can be equated to the same ratio obtained from expression (39). However this has been written for $\mathcal{H}_{sub}$ in the form written in (12), which can be written differently by means of the GAP equation for $m^2 = m^2(\mu^2)$. This allows to write an equation of $r$ as a function of $t$ and eliminate one of these variables. The resulting identity reads:

$$
\omega \theta = -2 A r^2 - \frac{tr}{2g_R} + \frac{r^2}{32\pi^2} \ln \left( \frac{\mu}{r} \right) - r^2 - \frac{r}{\bar{\phi}^2} + \frac{D}{1 + D + DLn \left( \frac{\mu}{r} \right)).
$$

In this expression the same parameter is used: $D = g_R/(8\pi^2)$. This (highly non transcendental) expression appears in addition to the usual real part of the GAP equation, expression (45), making a system of two algebraic expressions with two variables $(r, t)$. $g_R$ is another input/free parameter.

### 7 Summary

A further analysis of the usual renormalization scheme for the variational Gaussian approximation was done. The renormalized energy density was extremized with respect to the renormalized mass and coupling and to the condensate. Concerning the extremization with respect to the mass, five solutions were found, two of which which can correspond to stable vacua in specific ranges of the renormalized coupling constant. For this it was considered that the mass scale $\mu$ is close to the physical mass. A sort of “energy scale” invariant algebraic expression was found in this calculation. In other words, changes in the renormalized (physical) mass $m^2$ with corresponding change in the renormalization mass scale parameter $\mu^2$ yield the same solutions.

Values for $\bar{\phi}$, in the vacuum, were also found by minimizing the renormalized energy density with relation to it. The resulting expression is not completely consistent with the renormalized GAP equation unless the expression (11) is modified such that $\mu^2 \neq m^2_R \rightarrow 0$. From this expression it was pointed out that either the “condensate” or $g_R$ disappears when the mass scale (introduced in the renormalization procedure) assumes the value

$$
\bar{\phi} \left( \mu = m \ exp(8\pi^2) \right) = 0.
$$

This can be seen as a restoration of the spontaneous symmetry breaking. With this value for $\mu$, the bare coupling $\lambda$ may also diverge for $\Lambda/\mu$ finite, as shown in expression (27).

Particular values of the renormalized coupling constant which extremize the energy density were also found. The coupling constant may constraint the values of the renormalized mass which yield maxima of the energy density (effective potential). The extremization of the effective potential with respect to the
coupling constant was also performed in the limiting case that the mass scale \( \mu \) is close to the physical mass. This is a way of truncating the self-consistency of the approximation. Another kind of “energy scale” invariant expression was also obtained. No minima of the energy density with relation to \( g_R \) was found (considered without the whole renormalization group equations) within the truncation scheme which was adopted. The renormalized energy was also fixed to provide specific values for the coupling constant as a function of the energy density and the mass, which disappears in the limit of \( \mu = m \).

The masses were allowed to assume complex values to search non-transcendental solutions for the GAP equation and other relations among the parameters reducing the number of free parameters. The imaginary part is required to be zero at the end of the calculation producing another expression which relates the mass, coupling and the renormalization scale parameter. This parametrization for the imaginary part may lead to new relation between the parameters reducing the number of free variables. These imaginary parameters may be required to be very small \( (\sin(\omega) \sim \omega \text{ or } \sin(\theta) \sim \theta) \). The same parametrization is applied to the energy density which also must be a real number. The number of free parameters \( (m^2 \text{ or } \mu^2, \text{ and } g_R) \) is reduced and non transcendental solutions may result such as that of expression (41).

The ground state in the framework of the variational approximation is found by the minimization with respect to the two point function \( G(x, y, m^2) \) (which is a function of the physical mass \( m^2 \) or \( \mu^2 = m^2(\bar{\phi} = 0) \)) and to the condensate \( \bar{\phi} \) - they are the variational parameters (given in expressions (5)). Although they are regarded initially as independent variables, the GAP equation (for ground state) relates them. While the GAP equation is used for the renormalization of the bare parameters in the vacuum, expression (22) was calculated from renormalized expressions for the ground state. However there is nothing really defined about the behavior of the renormalized parameters in excited states. It was shown with sections 2.1, 3 and 4 that the minimizations of the renormalized energy with relation to the mass and \( \bar{\phi} \) yield different ground state (GAP) expressions from the ones obtained by the usual variational procedure for the regularized theory. This may have several meanings. It may not be evident whether these variational parameters are really or completely suitable as independent parameters for the Gaussian approximation and extensions (or leading order large N, Hartree Bogoliubov) or even in the exact ground state, i.e., the energy must be minimum with respect to particular combination(s) of these (or other) (physical?) variables. Notwithstanding the minimization of the regularized energy may not be equivalent to the minimization of the renormalized one because in the regularized theory there still are other (bare) parameters which are eliminated in the renormalization procedure corresponding to a
sort of "hidden dependences" among them. It may also be that the renormalization procedure has to be improved such as to make both ways of obtaining the ground state expressions equivalent. In this case the renormalization procedure would be allowed to be done at any moment independently of the order of the the variation, renormalization and extraction of observables within a certain nonperturbative approach.

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Figure captions

**Figure 1a** - First solution of expression (15) - a new GAP equation - for the ratio of the renormalized mass to the mass scale $\mu$ as a function of $g_R/(8\pi^2)$.

**Figure 1b** - Second solution of expression (15) - a new GAP equation - for the ratio of the renormalized mass to the mass scale $\mu$ as a function of $g_R/(8\pi^2)$.

**Figure 2a** - First solution of expression (31) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.

**Figure 2b** - Second solution of expression (31) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.
\begin{align*}
H &= \ln(m_R/\mu)/(8\pi^2).
\end{align*}

**Figure 2c** - Third solution of expression (31) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.

**Figure 3a** - First solution for the renormalized coupling constant of expression (32) - fixing $\mathcal{H} = (100\text{MeV})^4$ and $m_R = 100\text{MeV}$ - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.

**Figure 3b** - Second solution for the renormalized coupling constant of expression (32) - fixing $\mathcal{H} = (100\text{MeV})^4$ and $m_R = 100\text{MeV}$ - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.

**Figure 3c** - Third solution for the renormalized coupling constant of expression (32) - fixing $\mathcal{H} = (100\text{MeV})^4$ and $m_R = 100\text{MeV}$ - as a function of $H = \ln(m_R/\mu)/(8\pi^2)$.
Figure 2a

Figure 2b

Figure 2c
Figure 3c