THE ZAK TRANSFORM AND WIENER ESTIMATES
ON GELFAND-SHILOV AND MODULATION SPACES

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ABSTRACT. We characterise modulation spaces by suitable Wiener
estimates on the short-time Fourier transforms of the involved func-
tions and distributions. We use the results to characterise modula-
tion spaces by suitable estimates on the short-time Fourier trans-
form of their Zak transforms.

We also characterize Gelfand-Shilov spaces and their distribu-
tion spaces in terms of estimates of their Zak transforms.

0. INTRODUCTION

In the paper we characterise Gelfand-Shilov spaces of functions and
distributions, modulation spaces and Gevrey classes in background of
various kinds of Wiener estimates, and mapping properties of the Zak
transforms. Especially we perform such investigations for periodic and
quasi-periodic functions and distributions.

The Zak transforms are whimsical in several ways. They appear in
natural ways when dealing with Gabor frame operators in the cases of
"critical sampling", where the Gabor theory cease to work properly.
This ought to be the reason why the transform possess several exciting
and almost magical properties, useful in Gabor theory.

For example, in critical sampling cases, the Zak transform $Z$, adapted
to the sampling parameters, takes the Gabor frame operator $S_{\varphi,\psi}$ into
the multiplication operator $F \mapsto c \cdot Z\varphi \cdot Z\psi \cdot F$
for some constant $c$ which depends on the sampling parameters. (See
and Section 1 for notations.) We remark that this property is
heavily used when showing that Gabor atoms and their canonical dual
atoms often belong to the same function classes. (See [3, 4, 25].)

An other example concerns the fact that if $Zf$ is continuous, then it
has zeros. This property is important when deducing various kinds of
Balian-Low theorems, which are essential when finding limitations for
bases and Gabor frames in Gabor analysis (see Theorem 8.4.1 and its
consequences in [25]).

Before entering the Gabor theory, Zak transforms were first intro-
duced and used in a problem in differential equation by Gelfand in [20].
Subsequently, the transforms were rediscovered in various contexts, especially in solid state physics by Zak in [49] and in differential equations by Brezin in [2].

In these considerations it is essential to understand various kinds of mapping properties of the Zak transform. The transform takes suitable functions, defined on the configuration space $\mathbb{R}^d$ into quasi-periodic functions on the phase space $\mathbb{R}^{2d}$. Hence, in similar ways as for periodic functions, the Zak transformed functions are completely described by their behavior on suitable rectangles.

For example, the (standard) Zak transform is given by

$$(Z_1 f)(x, \xi) \equiv \sum_{j \in \mathbb{Z}^d} f(x - j)e^{i\langle j, \xi \rangle},$$

when $f$ is a suitable function or distribution (see (1.20) for the general definition of the Zak transform). By the definition it follows that if $F = Z_1 f$ and $Q_{d,r}$ is the cube $[0, r]^d$, then $F$ is quasi-periodic (with respect to $Q_{d,1} \times Q_{d,2\pi}$). That is,

$$F(x+k, \xi) = e^{i\langle k, \xi \rangle}F(x, \xi) \quad \text{and} \quad F(x, \xi + 2\pi \kappa) = F(x, \xi), \quad k, \kappa \in \mathbb{Z}^d.$$

It follows from these equalities that $F$ is completely reconstructable from its data on $Q_{d,1} \times Q_{d,2\pi}$.

It is well-known that $Z_1$ is bijective from $L^2(\mathbb{R}^d)$ to the set of quasi-periodic elements in $L^2(Q_{d,1} \times Q_{d,2\pi})$. Furthermore, by a straightforward application of Parseval’s formula we have

$$\|Z_1 f\|_{L^2(Q_{d,1} \times Q_{d,2\pi})} = (2\pi)^{d/2}\|f\|_{L^2}, \quad f \in L^2(\mathbb{R}^d). \quad (0.1)$$

(Cf. e. g. [25, Theorem 8.2.3].) Consequently, $L^2(\mathbb{R}^d)$ can be characterized in a convenient way by mapping properties of the Zak transform.

An other space that can be characterized by related mapping properties concerns the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In fact, it is proved in [29] by Janssen that $Z_1$ is continuous and bijective from $\mathcal{S}(\mathbb{R}^d)$ to the set of quasi-periodic elements in $C^\infty(\mathbb{R}^{2d})$.

In [41,42], Heil and Tinaztepe deduce some important mapping properties for the Zak transform on modulation spaces, and apply these results to deduce Balian-Low properties for such spaces. On the other hand, these mapping properties on modulation spaces seems not to be (complete) characterizations, because of absence of bijectivity. In fact, apart from the spaces $L^2(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$, it seems that the whole theory lacks characterizations of essential function and distribution spaces via the Zak transform (cf. Subsection 8.2 (f) in [25]).

In Section 3 we make this part more complete and furnish the theory with various kinds of characterizations. Especially we characterize modulation and Lebesgue spaces by suitable Lebesgue estimates of short-time Fourier transforms of the Zak transforms of the involved
functions. We also characterize the dual \( S' (\mathbb{R}^d) \) of \( S (\mathbb{R}^d) \), the (standard) Gelfand-Shilov spaces and their distribution spaces by their images under the Zak transform.

For example we prove that \( Z_1 \) is continuous and bijective from \( S' (\mathbb{R}^d) \) to the set of all quasi-periodic distributions on \( Q_{d,1} \times Q_{d,2\pi} \). (See Theorem 3.1.) For similar characterizations of Gelfand-Shilov spaces and their distribution spaces, see Theorems 3.2 and 3.3.) An other consequence of our results is that \( Z_1 \) maps the modulation space \( M^p (\mathbb{R}^d) \) continuously and bijectively to the set of all elements in \( M^\infty, p \), \( Q_{d,1} \times Q_{d,2\pi} ^{2\pi} \). (See Theorem 3.9 and Corollary 3.11.)

Our investigations also include some extensions in [48] concerning characterizations of periodic elements in modulation spaces. In fact, it follows from [48] that if \( q \in (0, \infty] \) and \( f \) is a \( 2\pi \)-periodic Gelfand-Shilov distribution on \( \mathbb{R}^d \) with Fourier coefficients \( c(f, \alpha), \alpha \in \mathbb{Z}^d \), then

\[
\{ c(f, \alpha) \}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff f \in M^{\infty, q}.
\]  

Here \( M^{\infty, q} \) is the (unweighted) modulation spaces with Lebesgue parameters \( \infty \) and \( q \).

We note that a proof of (0.3) in the case \( q \in [1, \infty] \) can be found in e.g. [37], and with some extensions in [35].

In [48] related equivalences to (0.3) which involve short-time Fourier transforms were deduced. In fact, let \( p, q \in (0, \infty) \), \( \phi \in S^s (\mathbb{R}^d) \setminus 0 \) and \( f \) be \( 2\pi \)-periodic. Then it is proved in [48] that

\[
\{ c(f, \alpha) \}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \| V_\phi f (\cdot, \xi) \|_{L^\infty (\mathbb{R}^d)} \in L^q.
\]  

By observing that periodicity of \( f \) induce the same periodicity for \( x \mapsto | V_\phi f (x, \xi) | \), it follows that (0.3)’ is the same as

\[
\{ c(f, \alpha) \}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \| V_\phi f (\cdot, \xi) \|_{L^\infty ([0, 2\pi]^d)} \in L^q.
\]  

In Section 2 we show that the latter equivalence hold true with \( L^r ((0, 2\pi]^d) \) norm in place of \( L^\infty ([0, 2\pi]^d) \) norm for every \( r \in (0, \infty) \). That is, we extend (0.3)’ into

\[
\{ c(f, \alpha) \}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \| V_\phi f (\cdot, \xi) \|_{L^r ([0, 2\pi]^d)} \in L^q.
\]  

In particular, if \( q < \infty \) and choosing \( r = q \), then we obtain

\[
\sum_{\alpha \in \mathbb{Z}^d} | c(f, \alpha) |^q < \infty \iff \int \int_{[0, 2\pi]^d \times \mathbb{R}^d} | V_\phi f (x, \xi) |^q \ dx \ d\xi < \infty.
\]  

The improved equivalence (0.3)’’” may under the additional assumptions, \( q, r \geq 1 \) be obtained from (0.3)’” by a suitable combination of
Hölder’s and Young’s inequalities and the inequality
\[ F(X) \lesssim \int_\Omega \Phi(X - Y) F(Y) \, dY, \quad X \in \Omega = [0, 2\pi]^d \times \mathbb{R}^d, \quad (0.5) \]
where
\[ \Phi(x, \xi) = \sum_{k \in \mathbb{Z}^d} |V_{\phi}(x - 2\pi k, \xi)| \quad \text{and} \quad F(X) = |V_{\phi}f(X)|, \]
which follows from Lemma 1.3.3 in [25] for 2\pi-periodic distributions \( f \).

It follows that this case can be handled by straight-forward modifications of the methods that are used when establishing basic results for classical modulation spaces in [11] and in Chapter 11 in [25].

In our situation, the parameters \( q \) and \( r \) are, more generally, allowed to belong to the full interval \( (0, \infty) \) instead of \( [1, \infty] \). The classical approaches in [11, 13, 14, 25] are then insufficient because they require convex structures in the topology of the involved vector spaces. This convexity is absent when \( q < 1 \) or \( r < 1 \).

We manage our more general situation by using techniques based on ideas in [18, 33, 34, 44] and which can handle Lebesgue and Wiener spaces which are quasi-Banach spaces but may fail to be Banach spaces. Especially we shall follow a main idea in [18,44] and replace the usual convolution, used in [11,13,14,25], by a semi-continuous version which is less sensitive when convexity is lacking in the topological structures. For the semi-continuous convolution we deduce in Section 2 various types of characterizations of modulation spaces in terms of Wiener norm estimates on the short-time Fourier transforms of the functions and (ultra-)distributions under considerations. For example, as special case of Propositions [1.17] after Proposition 2.4, we have for \( p, q, r \in (0, \infty] \) that
\[ \|f\|_{M^{p,q}} \asymp \|a\|_{\ell^{p,q}} \quad \text{when} \quad a(j) = \|V_{\phi}f\|_{L^r(j+[0,1]^d)}. \quad (0.6) \]
Similar facts hold true for those Wiener amalgam spaces which are Fourier images of modulation spaces of the form \( M^{p,q} \). In particular our results can be used to deduce certain invariance properties concerning the choice of local component in the Wiener amalgam quasi-norm. (See also Proposition 2.6) Here we remark that for Wiener amalgam spaces which at the same time are Banach spaces, the approaches are often less complicated and there are several examples on other Banach spaces (e.g. suitable modulation spaces) to furnish the local component in the Wiener amalgam norms. (See e.g. [15,16] and the references therein.)

We also present some applications on periodic elements which gives \((0.3)''\) and \((0.4)\) as special cases. (See Propositions 2.7 and 1.20.)

The Wiener spaces under considerations can also be described in terms of coorbit spaces, whose general theory was founded by Feichtinger and Gröchenig in [13,14] and further developed in different
ways, e.g. by Rauhut in [33, 34]. Since our investigations in Section 2 concern quasi-Banach spaces which may fail to be Banach spaces, our investigations are especially linked to Rauhut’s analysis in [33, 34]. In this context, a part of our analysis on modulation spaces can be formulated as coorbit norm estimates of short-time Fourier transforms with local component in $L^r$-spaces with $r \in (0, \infty]$ and global component in other Lebesgue spaces. Proposition 1.17 in Section 2 then shows that different choices of $r$ give rise to equivalent norm estimates on short-time Fourier transforms. Again we remark that if $r$ belongs to the subset $[1, \infty]$ of $(0, \infty]$ and that all involved spaces are Banach spaces, then our results can be obtained in other less complicated ways, given e.g. in Chapters 11 and 12 in [25].

As explained above, our results on Wiener spaces are essential for our results on periodicity and the Zak transform. For example, the analysis behind (0.6) also leads to that (0.2) can be reformulated as $f \in M^p(\mathbb{R}^d) \iff V_\Phi(Z_1 f) \in L^p(Q_{d,1} \times Q_{d,2\pi} \times \mathbb{R}^d \times \mathbb{R}^d)$. (0.7) If $p = 2$, then an application of Parseval’s formula implies that (0.7) is the same as $f \in M^2(\mathbb{R}^d) \iff Z_1 f \in L^2(Q_{d,1} \times Q_{d,2\pi})$, which is a slightly weaker form of (0.1).

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1. **Preliminaries**

In this section we recall some basic facts. We start by discussing Gelfand-Shilov spaces and their properties. Thereafter we recall some properties of modulation spaces and discuss different aspects of periodic distributions.

1.1. **Gelfand-Shilov spaces and Gevrey classes.** Let $0 < s, \sigma \in \mathbb{R}$ be fixed. Then the Gelfand-Shilov space $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ ($\Sigma_s^\sigma(\mathbb{R}^d)$) of Roumieu type (Beurling type) with parameters $s$ and $\sigma$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{\mathcal{S}_\sigma^s} \equiv \sup_{h>0} \frac{|x^\alpha \partial^\beta f(x)|}{h^{\alpha+\beta}|\alpha!\beta!\sigma!}}(1.1)
$$

is finite for some $h > 0$ (for every $h > 0$). Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. We equip $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ ($\Sigma_s^\sigma(\mathbb{R}^d)$) by the canonical inductive limit topology (projective limit topology) with respect to $h > 0$, induced by the semi-norms in (1.1).
The Gelfand-Shilov distribution spaces \( (S^\sigma_\sigma)'(R^d) \) and \( (\Sigma^\sigma_\sigma)'(R^d) \) are the dual spaces of \( S^\sigma_\sigma(R^d) \) and \( \Sigma^\sigma_\sigma(R^d) \), respectively. As for the Gelfand-Shilov spaces there is a canonical projective limit topology (inductive limit topology) for \( (S^\sigma_\sigma)'(R^d) \) \((\Sigma^\sigma_\sigma)'(R^d)\). (Cf. \cite{21,30,32}.) For convenience we set

\[
S_s = S^\sigma_\sigma, \quad S'_s = (S^\sigma_\sigma)'_s, \quad \Sigma_s = \Sigma^\sigma_\sigma, \quad \Sigma'_s = (\Sigma^\sigma_\sigma)'_s.
\]

From now on we let \( \mathcal{F} \) be the Fourier transform which takes the form

\[
(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{R^d} f(x) e^{-i(x,\xi)} \, dx
\]

when \( f \in L^1(R^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( R^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{F}'(R^d) \), from \( (S^\sigma_\sigma)'(R^d) \) to \( (S^\sigma_\sigma)'(R^d) \) and from \( (\Sigma^\sigma_\sigma)'(R^d) \) to \( (\Sigma^\sigma_\sigma)'(R^d) \). Furthermore, \( \mathcal{F} \) restricts to homeomorphisms on \( \mathcal{F}(R^d) \), from \( S^\sigma_\sigma(R^d) \) to \( S^\sigma_\sigma(R^d) \) and from \( \Sigma^\sigma_\sigma(R^d) \) to \( \Sigma^\sigma_\sigma(R^d) \), and to a unitary operator on \( L^2(R^d) \).

Next we consider a more general class of Gelfand-Shilov spaces and their distribution spaces. Let \( 0 \leq s_1, s_2, \sigma_1, \sigma_2 \in R \) be fixed. Then the Gelfand-Shilov space \( S^\sigma_{s_1,s_2}(R^{d_1+d_2}) \) \((\Sigma^\sigma_{s_1,s_2}(R^{d_1+d_2}))\) of Roumieu type (Beurling type) with parameters \( s_1, s_2, \sigma_1 \) and \( \sigma_2 \) consists of all \( f \in C^\infty(R^{d_1+d_2}) \) such that

\[
\|f\|_{S^\sigma_{s_1,s_2},h} = \sup_{h} \left| \frac{x_1^{\alpha_1}x_2^{\alpha_2} \partial_1^{\beta_1} \partial_2^{\beta_2} f(x_1, x_2)}{h^{\alpha_1+\alpha_2+\beta_1+\beta_2} |\alpha_1|! |\alpha_2|! |\beta_1|! |\beta_2|!} \right|
\]

is finite for some \( h > 0 \) (for every \( h > 0 \)). Here the supremum should be taken over all \( \alpha_j, \beta_j \in N^{d_j} \) and \( x_j \in R^{d_j}, j = 1, 2 \). We equip \( S^\sigma_{s_1,s_2}(R^{d_1+d_2}) \) \((\Sigma^\sigma_{s_1,s_2}(R^{d_1+d_2}))\) by the canonical inductive limit topology (projective limit topology) with respect to \( h > 0 \), induced by the semi-norms in \( (1.2) \).

The space \( \Sigma^\sigma_{s_1,s_2}(R^{d_1+d_2}) \) is a Fréchet space when the topology is induced by the seminorms \( \| \cdot \|_{S^\sigma_{s_1,s_2},h}, h > 0 \).

The Gelfand-Shilov distribution spaces \( (S^\sigma_{s_1,s_2})'(R^{d_1+d_2}) \) and \( (\Sigma^\sigma_{s_1,s_2})'(R^{d_1+d_2}) \) are the dual spaces of \( S^\sigma_{s_1,s_2}(R^{d_1+d_2}) \) and \( \Sigma^\sigma_{s_1,s_2}(R^{d_1+d_2}) \), respectively. Evidently, \( S^\sigma_{s_1,s_2}(R^{d_1+d_2}), \Sigma^\sigma_{s_1,s_2}(R^{d_1+d_2}) \) and their duals possess similar topological properties as \( S^\sigma_\sigma(R^d) \), \( \Sigma^\sigma_\sigma(R^d) \) and their duals. By \( (j+k)! \leq 2^{j+k}j!k! \) when \( j, k \geq 0 \) are integers we get \( S^\sigma_{s,s} = S^\sigma_\sigma \) and \( \Sigma^\sigma_{s,s} = \Sigma^\sigma_\sigma \).

For any \( s_j, \sigma_j, s_0, j > 0 \) such that \( s_j > s_0, \sigma_j > \sigma_0 \), we have

\[
S^\sigma_{s_0,s_0}(R^{d_1+d_2}) \hookrightarrow S^\sigma_{s_1,s_2}(R^{d_1+d_2}) \hookrightarrow S^\sigma_{s_1,s_2}(R^{d_1+d_2})
\]

\[
\hookrightarrow \mathcal{F}(R^{d_1+d_2}) \hookrightarrow \mathcal{F}'(R^{d_1+d_2}) \hookrightarrow (S^\sigma_{s_1,s_2})'(R^{d_1+d_2})
\]

\[
\hookrightarrow (\Sigma^\sigma_{s_1,s_2})'(R^{d_1+d_2}) \hookrightarrow (S^\sigma_{s_0,s_0})'(R^{d_1+d_2}),
\]

with dense embeddings, provided the parameters have been chosen such that all spaces are non-trivial. Here and in what follows we use the
notation $A \hookrightarrow B$ when the topological spaces $A$ and $B$ satisfy $A \subseteq B$ with continuous embeddings. The space $\Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2})$ is a Fréchet space with seminorms $\| \cdot \|_{\Sigma_{s_1,s_2}^1, h}^1$, $h > 0$. Moreover, $\Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2}) \neq \{0\}$, if and only if $s_j + \sigma_j \geq 1$ and $(s_j, \sigma_j) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, 2$, and $\Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2}) \neq \{0\}$, if and only if $s_j + \sigma_j \geq 1$, $j = 1, 2$.

Let $\mathcal{F}_j f$ denote the partial Fourier transform of $f(x_1, x_2) \in \mathcal{S}(\mathbb{R}^{d_1+d_2})$ with respect to $x_j$, $j = 1, 2$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ extend uniquely to homeomorphisms

$$\mathcal{F}_1 : (\Sigma_{s_1,s_2}^1)'(\mathbb{R}^{d_1+d_2}) \to (\Sigma_{s_1,s_2}^1)'(\mathbb{R}^{d_1+d_2})$$

$$\mathcal{F}_2 : (\Sigma_{s_1,s_2}^1)'(\mathbb{R}^{d_1+d_2}) \to (\Sigma_{s_1,s_2}^1)'(\mathbb{R}^{d_1+d_2}),$$

respectively, and restricts to homeomorphisms

$$\mathcal{F}_1 : \Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2}) \to \Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2})$$

$$\mathcal{F}_2 : \Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2}) \to \Sigma_{s_1,s_2}^1(\mathbb{R}^{d_1+d_2}),$$

The same holds true after each Gelfand-Shilov function or distribution space of Roumieu type have been replaced by corresponding Beurling type space.

Gelfand-Shilov spaces can in convenient ways be characterized in terms of estimates of the involved functions and their Fourier transforms. More precisely, in $[5, 9]$ it is proved that if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $s, \sigma > 0$, then $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ ($f \in \Sigma_s^\sigma(\mathbb{R}^d)$), if and only if

$$|f(x)| \lesssim e^{-r|x|^\frac{1}{2}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-r|\xi|^\frac{1}{2}}$$

(1.4)

for some $r > 0$ (for every $r > 0$). Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of $\theta$ in the domain of $f$ and $g$. We also set $f(\theta) \asymp g(\theta)$ when $f(\theta) \lesssim g(\theta)$ and $g(\theta) \lesssim f(\theta)$. More generally, it follows from $[5]$ that if $f \in \mathcal{S}'(R_1^{d_1+d_2})$ and $s_1, s_2, \sigma_1, \sigma_2 > 0$, then $f \in \Sigma_{s_1,s_2}(\mathbb{R}^{d_1+d_2})$ ($f \in \Sigma_{s_1,s_2}(\mathbb{R}^{d_1+d_2})$), if and only if

$$|f(x_1, x_2)| \lesssim e^{-r(|x_1|^\frac{1}{2} + |x_2|^\frac{1}{2})} \quad \text{and} \quad |\hat{f}(\xi_1, \xi_2)| \lesssim e^{-r(|\xi_1|^\frac{1}{2} + |\xi_2|^\frac{1}{2})},$$

(1.3)

for some $r > 0$ (for every $r > 0$).

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates of short-time Fourier transforms, (see e.g. $[27, 45]$). More precisely, let $\phi \in \mathcal{S}_s(\mathbb{R}^d)$ be fixed. Then the short-time Fourier transform $V_\phi f$ of $f \in \mathcal{S}_s'(\mathbb{R}^d)$ with respect to the window function $\phi$ is the Gelfand-Shilov distribution on $\mathbb{R}^{2d}$, defined by

$$V_\phi f(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi).$$
If \( f, \phi \in \mathcal{S}_c(\mathbb{R}^d) \), then it follows that
\[
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int f(y) \overline{\phi(y-x)} e^{-i(y, \xi)} \, dy.
\]

By [43, Theorem 2.3] it follows that the map \((f, \phi) \mapsto V_\phi f\) from \(\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)\) to \(\mathcal{S}(\mathbb{R}^d)\) is uniquely extendable to a continuous map from \((\mathcal{S}^*_s)'(\mathbb{R}^d) \times (\mathcal{S}^*_s)'(\mathbb{R}^d)\) to \((\mathcal{S}^*_s)'(\mathbb{R}^d)\), and restricts to a continuous map from \(S^*_s(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)\) to \(S^*_s(\mathbb{R}^d)\).

The same conclusion holds with \(\Sigma_s, \Sigma_{s, s}\) and \(\Sigma_{s, s}\) in place of \(\Sigma_s\) and \(\Sigma_{s, s}\), respectively, at each place.

The following properties characterize Gelfand-Shilov spaces and their distribution spaces in terms of estimates of short-time Fourier transform.

**Proposition 1.1.** Let \( s_j, \sigma_j > 0 \) be such that \( s_j + \sigma_j \geq 1, j = 1, 2 \). Also let \( \phi \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \setminus \{0\} \) (\( \phi \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \setminus \{0\} \)) and \( f \) be a Gelfand-Shilov distribution on \( \mathbb{R}^{d_1 + d_2} \). Then \( f \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \) (\( f \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \)), if and only if
\[
|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{-r(|x_1|^s + |x_2|^s + |\xi_1|^s + |\xi_2|^s)},
\]
for some \( r > 0 \) (for every \( r > 0 \)).

**Proposition 1.2.** Let \( s_j, \sigma_j > 0 \) be such that \( s_j + \sigma_j \geq 1, j = 1, 2 \). Also let \( \phi \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \setminus \{0\} \) (\( \phi \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \setminus \{0\} \)) and \( f \) be a Gelfand-Shilov distribution on \( \mathbb{R}^{d_1 + d_2} \). Then \( f \in (\mathcal{S}^{s_1, s_2})'(\mathbb{R}^{d_1 + d_2}) \) (\( f \in (\mathcal{S}^{s_1, s_2})'(\mathbb{R}^{d_1 + d_2}) \)), if and only if
\[
|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{r(|x_1|^s + |x_2|^s + |\xi_1|^s + |\xi_2|^s)},
\]
for every \( r > 0 \) (for some \( r > 0 \)).

We note that if \( s_j = \sigma_j = \frac{1}{2} \) for some \( j \) in Propositions 1.1 and 1.2, then it is not possible to find any \( \phi \in \mathcal{S}^{s_1, s_2}(\mathbb{R}^{d_1 + d_2}) \setminus \{0\} \). Hence, these results give no information in the Beurling case for such choices of \( s_j \) and \( \sigma_j \).

A proof of Proposition 1.1 can be found in e.g. [27] (cf. [27, Theorem 2.7]) and a proof of Proposition 1.2 in the case \( d_2 = 0 \) can be found in [45]. The general case of Proposition 1.2 follows by similar arguments as in [45] and is left for the reader. See also [6] for related results. In [48, Theorem 2.4] analogous characterizations for periodic functions and distributions are obtained.

Next we consider Gevrey classes on \( \mathbb{R}^d \). Let \( \sigma \geq 0 \). For any compact set \( K \subseteq \mathbb{R}^d, h > 0 \) and \( f \in C^\infty(K) \) let
\[
\|f\|_{K, h, \sigma} \equiv \sup_{\alpha \in \mathbb{N}^d} \left( \frac{\|D^\alpha f\|_{L^\infty(K)}}{h^{\alpha(1+\sigma)}} \right).
\]
The Gevrey class $\mathcal{E}_\sigma(K)$ ($\mathcal{E}_{0,\sigma}(K)$) of order $\sigma$ and of Roumieu type (of Beurling type) is the set of all $f \in C^\infty(K)$ such that (1.7) is finite for some (for every) $h > 0$. We equip $\mathcal{E}_\sigma(K)$ ($\mathcal{E}_{0,\sigma}(K)$) by the inductive (projective) limit topology with respect to $h > 0$, supplied by the seminorms in (1.7). Finally if $\{K_j\}_{j \geq 1}$ is an exhausted sets of compact subsets of $\mathbb{R}^d$, then let

$$\mathcal{E}_\sigma(\mathbb{R}^d) = \text{proj lim } \mathcal{E}_\sigma(K_j) \quad \text{and} \quad \mathcal{E}_{0,\sigma}(\mathbb{R}^d) = \text{proj lim } \mathcal{E}_{0,\sigma}(K_j).$$

In particular,

$$\mathcal{E}_\sigma(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_\sigma(K_j) \quad \text{and} \quad \mathcal{E}_{0,\sigma}(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_{0,\sigma}(K_j).$$

It is clear that $\mathcal{E}_{0,0}(\mathbb{R}^d)$ contains all constant functions on $\mathbb{R}^d$, and that $\mathcal{E}_0(\mathbb{R}^d) \setminus \mathcal{E}_{0,0}(\mathbb{R}^d)$ contains all non-constant trigonometric polynomials.

1.2. **Ordered, dual and phase split bases.** Our discussions involving Zak transforms, periodicity, modulation spaces and Wiener spaces are done in terms of suitable bases.

**Definition 1.3.** Let $E = \{e_1, \ldots, e_d\}$ be an ordered basis of $\mathbb{R}^d$. Then $E'$ denotes the basis of $e'_1, \ldots, e'_d$ in $\mathbb{R}^d$ which satisfies

$$\langle e_j, e'_k \rangle = 2\pi \delta_{jk} \quad \text{for every} \quad j, k = 1, \ldots, d.$$ 

The corresponding lattices are given by

$$\Lambda_E = \{ n_1e_1 + \cdots + n_de_d ; (n_1, \ldots, n_d) \in \mathbb{Z}^d \},$$

and

$$\Lambda'_E = \Lambda_{E'} = \{ \nu_1e'_1 + \cdots + \nu_de'_d ; (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d \}.$$ 

The sets $E'$ and $\Lambda'_E$ are called the dual basis and dual lattice of $E$ and $\Lambda_E$, respectively. If $E_1, E_2$ are ordered bases of $\mathbb{R}^d$ such that a permutation of $E_2$ is the dual basis for $E_1$, then the pair $(E_1, E_2)$ are called permuted dual bases (to each others on $\mathbb{R}^d$).

**Remark 1.4.** Evidently, if $E$ is the same as in Definition 1.3 then there is a matrix $T_E$ with $E$ as the image of the standard basis in $\mathbb{R}^d$. Then $E'$ is the image of the standard basis under the map $T_{E'} = 2\pi (T_E^{-1})^t$.

Two ordered bases on $\mathbb{R}^d$ can be used to construct a uniquely defined ordered basis for $\mathbb{R}^{2d}$ as in the following definition.

**Definition 1.5.** Let $E_1, E_2$ be ordered bases of $\mathbb{R}^d$,

$$V_1 = \{ (x, 0) \in \mathbb{R}^{2d} ; x \in \mathbb{R}^d \}, \quad V_2 = \{ (0, \xi) \in \mathbb{R}^{2d} ; \xi \in \mathbb{R}^d \}$$

and let $\pi_j$ from $\mathbb{R}^{2d}$ to $\mathbb{R}^d$, $j = 1, 2$, be the projections

$$\pi_1(x, \xi) = x \quad \text{and} \quad \pi_2(x, \xi) = \xi.$$
Then $E_1 \times E_2$ is the ordered basis $\{e_1, \ldots, e_{2d}\}$ of $\mathbb{R}^{2d}$ such that

$$
\{e_1, \ldots, e_d\} \subseteq V_1, \quad E_1 = \{\pi_1(e_1), \ldots, \pi_1(e_d)\},
$$

$$
\{e_{d+1}, \ldots, e_{2d}\} \subseteq V_2 \quad \text{and} \quad E_2 = \{\pi_2(e_{d+1}), \ldots, \pi_2(e_{2d})\}.
$$

In the phase space it is convenient to consider phase split bases, which are defined as follows.

**Definition 1.6.** Let $V_1, V_2$, $\pi_1$ and $\pi_2$ be as in Definition 1.5. $E$ be an ordered basis of the phase space $\mathbb{R}^{2d}$ and let $E_0 \subseteq E$. Then $E$ is called *phase split* (with respect to $E_0$), if the following is true:

1. the span of $E_0$ and $E \setminus E_0$ equal $V_1$ and $V_2$, respectively;
2. let $E_1 = \pi_1(E_0)$ and $E_2 = \pi_2(E \setminus E_0)$ be the bases in $\mathbb{R}^d$ which preserves the orders from $E_0$ and $E \setminus E_0$. Then $(E_1, E_2)$ are permuted dual bases.

If $E$ is a phase split basis with respect to $E_0$ and that $E_0$ consists of the first $d$ vectors in $E$, then $E$ is called *strongly phase split* (with respect to $E_0$).

In Definition 1.6 it is understood that the orderings of $E_0$ and $E \setminus E_0$ are inherited from the ordering in $E$.

**Remark 1.7.** Let $E$ and $E_j$, $j = 0, 1, 2$ be the same as in Definition 1.5. It is evident that $E_0$ and $E \setminus E_0$ consist of $d$ elements, and that $E_1$ and $E_2$ are uniquely defined. The pair $(E_1, E_2)$ is called the pair of permuted dual bases, induced by $E$ and $E_0$.

On the other hand, suppose that $(E_1, E_2)$ is a pair of permuted dual bases to each others on $\mathbb{R}^d$. Then it is clear that for $E_1 \times E_2 = \{e_1, \ldots, e_{2d}\}$ in Definition 1.5 and $E_0 = \{e_1, \ldots, e_d\}$, we have that $E_0$ and $E$ fullfils all properties in Definition 1.6. In this case, $E_1 \times E_2$ is the phase split basis (of $\mathbb{R}^{2d}$) induced by $(E_1, E_2)$.

It follows that if $E'$, $E'_1$ and $E'_2$ are the dual bases of $E$, $E_1$ and $E_2$, respectively, then $E' = E'_1 \times E'_2$.

1.3. **Invariant quasi-Banach spaces and spaces of mixed quasi-normed spaces of Lebesgue types.** We recall that a quasi-norm $\| \cdot \|_\mathcal{B}$ of order $r \in (0, 1]$ on the vector-space $\mathcal{B}$ over $\mathbb{C}$ is a nonnegative functional on $\mathcal{B}$ which satisfies

$$
\|f + g\|_\mathcal{B} \leq 2^{r^{-1}}(\|f\|_\mathcal{B} + \|g\|_\mathcal{B}), \quad f, g \in \mathcal{B},
$$

$$
\|\alpha \cdot f\|_\mathcal{B} = |\alpha| \cdot \|f\|_\mathcal{B}, \quad \alpha \in \mathbb{C}, \quad f \in \mathcal{B}
$$

and

$$
\|f\|_\mathcal{B} = 0 \quad \iff \quad f = 0.
$$

The space $\mathcal{B}$ is then called a quasi-norm space. A complete quasi-norm space is called a quasi-Banach space. If $\mathcal{B}$ is a quasi-Banach space with
quasi-norm satisfying \((1.8)\) then by \([1, 36]\) there is an equivalent quasi-norm to \(\| \cdot \|_\mathcal{B}\) which additionally satisfies
\[
\|f + g\|_\mathcal{B} \leq \|f\|_\mathcal{B} + \|g\|_\mathcal{B}, \quad f, g \in \mathcal{B}.
\]
From now on we always assume that the quasi-norm of the quasi-Banach space \(\mathcal{B}\) is chosen in such way that both \((1.8)\) and \((1.9)\) hold.

Before giving the definition of \(v\)-invariant spaces, we recall some facts on weight functions.

A weight or weight function on \(\mathbb{R}^d\) is a positive function \(\omega \in L^\infty_{loc}(\mathbb{R}^d)\) such that \(1/\omega \in L^\infty_{loc}(\mathbb{R}^d)\). The weight \(\omega\) is called moderate, if there is a positive weight \(v\) on \(\mathbb{R}^d\) such that
\[
\omega(x + y) \leq \omega(x)v(y), \quad x, y \in \mathbb{R}^d.
\]
If \(\omega\) and \(v\) are weights on \(\mathbb{R}^d\) such that \((1.10)\) holds, then \(\omega\) is also called \(v\)-moderate. We note that \((1.10)\) implies that \(\omega\) fulfills the estimates
\[
v(-x)^{-1} \leq \omega(x) \leq v(x), \quad x \in \mathbb{R}^d.
\]
We let \(\mathcal{P}_E(\mathbb{R}^d)\) be the set of all moderate weights on \(\mathbb{R}^d\).

It can be proved that if \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\), then \(\omega\) is \(v\)-moderate for some \(v(x) = e^{r|x|}\), provided the positive constant \(r\) is large enough (cf. \([26]\)). In particular, \((1.11)\) shows that for any \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\), there is a constant \(r > 0\) such that
\[
e^{-r|x|} \leq \omega(x) \leq e^{r|x|}, \quad x \in \mathbb{R}^d.
\]
We say that \(v\) is submultiplicative if \(v\) is even and \((1.10)\) holds with \(\omega = v\). In the sequel, \(v\) and \(v_j\) for \(j \geq 0\), always stand for submultiplicative weights if nothing else is stated. The next definition is similar to \([13\text{, Section 3}]\) in the Banach space case.

**Definition 1.8.** Let \(r \in (0, 1], \, v \in \mathcal{P}_E(\mathbb{R}^d)\) and let \(\mathcal{B} = \mathcal{B}(\mathbb{R}^d) \subseteq L^r_{loc}(\mathbb{R}^d)\) be a quasi-Banach space such that \(\Sigma_1(\mathbb{R}^d) \subseteq \mathcal{B}(\mathbb{R}^d)\). Then \(\mathcal{B}\) is called \(v\)-invariant on \(\mathbb{R}^d\) if the following is true:

1. \(x \mapsto f(x + y)\) belongs to \(\mathcal{B}\) for every \(f \in \mathcal{B}\) and \(y \in \mathbb{R}^d\).
2. There is a constant \(C > 0\) such that \(\|f_1\|_\mathcal{B} \leq C\|f_2\|_\mathcal{B}\) when \(f_1, f_2 \in \mathcal{B}\) are such that \(|f_1| \leq |f_2|\). Moreover,
   \[\|f(\cdot + y)\|_\mathcal{B} \lesssim \|f\|_\mathcal{B}v(y), \quad f \in \mathcal{B}, \; y \in \mathbb{R}^d.\]

Let \(\mathcal{B}\) be as in Definition \([13]\), \(E\) be a basis for \(\mathbb{R}^d\) and let \(\kappa(E)\) be the closed parallelepiped spanned by \(E\). The discrete version, \(\ell_{\mathcal{B}, E} = \ell_{\mathcal{B}, E}(\Lambda_E)\), of \(\mathcal{B}\) with respect to \(E\) is the set of all \(a \in \ell_0(\Lambda_E)\) such that
\[
\|a\|_{\ell_{\mathcal{B}, E}} \equiv \left\| \sum_{j \in \Lambda_E} a(j)\chi_{j+\kappa(E)} \right\|_{\mathcal{B}}
\]
is finite.

An important example on \(v\)-invariant spaces concerns mixed quasi-norm spaces of Lebesgue type, given in the following definition.
**Definition 1.9.** Let $E = \{e_1, \ldots, e_d\}$ be an ordered basis of $\mathbb{R}^d$, $\kappa(E)$ be the parallelepiped spanned by $E$, $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, $q = (q_1, \ldots, q_d) \in (0, \infty]^d$ and $r = \min(1, q)$. If $f \in L^r_{\text{loc}}(\mathbb{R}^d)$, then

$$\|f\|_{L^q_{E,(\omega)}} \equiv \|g_d\|_{L^q(\mathbb{R})}$$

where $g_k : \mathbb{R}^{d-k} \to \mathbb{R}$, $k = 0, \ldots, d-1$, are inductively defined as

$$g_0(x_1, \ldots, x_d) \equiv |f(x_1e_1 + \cdots + x_de_d)\omega(x_1e_1 + \cdots + x_de_d)|,$$

and

$$g_k(z_k) \equiv \|g_{k-1}(\cdot, z_k)\|_{L^q(\mathbb{R})}, \quad z_k \in \mathbb{R}^{d-k}, \quad k = 1, \ldots, d-1.$$

If $\Omega \subseteq \mathbb{R}^d$ is measurable, then $L^q_{E,(\omega)}(\Omega)$ consists of all $f \in L^r_{\text{loc}}(\Omega)$ with finite quasi-norm

$$\|f\|_{L^q_{E,(\omega)}(\Omega)} \equiv \|f\|_{L^q_{E,(\omega)}(\mathbb{R}^d)}, \quad f_\Omega(x) \equiv \begin{cases} f(x), & \text{when } x \in \Omega \\ 0, & \text{when } x \notin \Omega. \end{cases}$$

The space $L^q_{E,(\omega)}(\Omega)$ is called $E$-split Lebesgue space (with respect to $\omega$, $q$ and $\Omega$).

We let $\ell^p_{E,(\omega)}(\Lambda_E)$ be the discrete version of $\mathcal{B} = L^p_{E,(\omega)}(\mathbb{R}^d)$ when $p \in (0, \infty]^d$.

Suppose that $E$ and $\Lambda$ are the same as in Definition 1.9. Then we let $(\ell^0_{E})'(\Lambda)$ be the set of all formal sequences $\{a(j)\}_{j \in \Lambda}$, and we let $\ell^0_{E}(\Lambda)$ be the set of all such sequences such that at most finite numbers of $a(j)$ are non-zero.

**Remark 1.10.** Evidently, $L^q_{E,(\omega)}(\Omega)$ and $\ell^q_{E,(\omega)}(\Lambda)$ in Definition 1.9 are quasi-Banach spaces of order $\min(p, 1)$. We set

$$L^q = L^q_{E,(\omega)} \quad \text{and} \quad \ell^q = \ell^q_{E,(\omega)}$$

when $\omega = 1$. For convenience we identify $q = (q_1, \ldots, q) \in (0, \infty]^d$ with $q \in (0, \infty]$ when considering spaces involving Lebesgue exponents. In particular,

$$L^q_{E,(\omega)} = L^q_{E,(\omega)}; \quad L^q_E = L^q_E; \quad \ell^q_{E,(\omega)} = \ell^q_{E,(\omega)} \quad \text{and} \quad \ell^q_E = \ell^q_E$$

for such $q$, and notice that these spaces agree with

$$L^q(\omega), \quad L^q, \quad \ell^q(\omega) \quad \text{and} \quad \ell^q,$$

respectively, with equivalent quasi-norms.
1.4. Modulation and Wiener spaces. We consider a general class of modulation spaces given in the following definition (cf. [12]).

**Definition 1.11.** Let $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $\mathcal{B}$ be a $v$-invariant quasi-Banach space on $\mathbb{R}^{2d}$, and let $\phi \in \mathcal{S}_1(\mathbb{R}^d) \setminus 0$. Then the modulation space $M(\omega, \mathcal{B})$ consists of all $f \in \mathcal{S}_1(\mathbb{R}^d)$ such that

$$\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f \cdot \omega\|_{\mathcal{B}}$$

is finite.

An important family of modulation spaces which contains the classical modulation spaces, introduced by Feichtinger in [11], is given next.

**Definition 1.12.** Let $p, q \in (0, \infty]^d$, $E_1$ and $E_2$ be ordered bases of $\mathbb{R}^d$, $E = E_1 \times E_2$, $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$ and let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$. For any $f \in \Sigma_1(\mathbb{R}^d)$ set

$$\|f\|_{M_{E,(\omega)}^{p,q}} \equiv \|H_{1,f,E_1,p}\omega\|_{L^p_{E_1}},$$

where $H_{1,f,E_1,p}\omega(\xi) \equiv \|V_\phi f(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p_{E_1}}$ and

$$\|f\|_{W_{E,(\omega)}^{p,q}} \equiv \|H_{2,f,E_2,q}\omega\|_{L^p_{E_2}},$$

where $H_{2,f,E_2,q}\omega(x) \equiv \|V_\phi f(x, \cdot)\omega(x, \cdot)\|_{L^p_{E_2}}$.

The modulation space $M_{E,(\omega)}^{p,q}(\mathbb{R}^d)$ ($W_{E,(\omega)}^{p,q}(\mathbb{R}^d)$) consist of all $f \in \Sigma_1(\mathbb{R}^d)$ such that $\|f\|_{M_{E,(\omega)}^{p,q}}$ ($\|f\|_{W_{E,(\omega)}^{p,q}}$) is finite.

The theory of modulation spaces has developed in different ways since they were introduced in [11] by Feichtinger. (Cf. e.g. [12, 18, 25, 41].) For example, let $p, q, E, \omega$ and $v$ be the same as in Definition 1.11 and 1.12 and let $\mathcal{B} = L^p_E(\mathbb{R}^{2d})$ and $r = \min(1, p, q)$. Then $M(\omega, \mathcal{B}) = M_{E,(\omega)}^{p,q}(\mathbb{R}^d)$ is a quasi-Banach space. Moreover, $f \in M_{E,(\omega)}^{p,q}(\mathbb{R}^d)$ if and only if $V_\phi f \cdot \omega \in L^p_E(\mathbb{R}^{2d})$, and different choices of $\phi$ give rise to equivalent quasi-norms in Definition 1.12. We also note that for any such $\mathcal{B}$, then

$$\Sigma_1(\mathbb{R}^d) \subseteq M_{E,(\omega)}^{p,q}(\mathbb{R}^d) \subseteq \Sigma_1(\mathbb{R}^d).$$

Similar facts hold for the space $W_{E,(\omega)}^{p,q}(\mathbb{R}^d)$. (Cf. [18, 41].)

We shall consider various kinds of Wiener spaces involved later on when finding different characterizations of modulation spaces. The following type of Wiener spaces can essentially be found in e.g. [13, 18, 25], and is related to coorbit spaces of Lebesgue spaces.

**Definition 1.13.** Let $r \in (0, \infty]^d$, $\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$, $E \subseteq \mathbb{R}^d$ be an ordered basis, and let $\kappa(E)$ be the closed parallelepiped spanned by $E$. Also let $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_0 = $
\( \mathcal{B}_0(\mathbb{R}^d) \) be invariant QBF-spaces on \( \mathbb{R}^d \), \( f \) and \( F \) be measurable on \( \mathbb{R}^d \) respective \( \mathbb{R}^{2d} \), \( F_\omega = F \cdot \omega \), and let \( \ell_{\mathcal{B},E}(\Lambda_E) \) be the discrete version of \( \mathcal{B} \) with respect to \( E \).

(1) Then \( \| f \|_{W^q_E(\omega_0, \ell_{\mathcal{B},E})} \) is given by
\[
\| f \|_{W^q_E(\omega_0, \ell_{\mathcal{B},E})} = \| h_{E, \omega_0} f \|_{\ell_{\mathcal{B},E}(\Lambda_E)},
\]
where
\[
h_{E, \omega_0} f(j) = \| f \|_{L^2_E(j + \kappa(E)) \omega_0(j)}, \quad j \in \Lambda_E.
\]

The set \( W^q_E(\omega, \ell_{\mathcal{B},E}) \) consists of all measurable \( f \) on \( \mathbb{R}^d \) such that \( \| f \|_{W^q_E(\omega, \ell_{\mathcal{B},E})} < \infty \);

(2) Then \( \| F \|_{W^q_{1,E}(\omega_0, \ell_{\mathcal{B},E}, \mathcal{B}_0)} \), \( k = 1, 2 \), are given by
\[
\| F \|_{W^q_{1,E}(\omega_0, \ell_{\mathcal{B},E}, \mathcal{B}_0)} = \| \varphi_{F, \omega, r, E} \|_{\mathcal{B}_0}, \quad \varphi_{F, \omega, r, E}(\xi) = \| F_\omega(\cdot, \xi) \|_{W^q_E(1, \ell_{\mathcal{B},E})},
\]
and
\[
\| F \|_{W^q_{k,E}(\omega_0, \ell_{\mathcal{B},E}, \mathcal{B}_0)} = \| \psi_{F, \omega, 0} \|_{W^q_E(1, \ell_{\mathcal{B},E})}, \quad \psi_{F, \omega, 0}(x) = \| F_\omega(x, \cdot) \|_{\mathcal{B}_0}.
\]

The set \( W^q_{k,E}(\omega, \ell_{\mathcal{B},E}, \mathcal{B}_0) \) consists of all measurable \( F \) on \( \mathbb{R}^{2d} \) such that \( \| F \|_{W^q_{k,E}(\omega, \ell_{\mathcal{B},E}, \mathcal{B}_0)} < \infty \), \( k = 1, 2 \).

The space \( W^q_E(\omega_0, \ell_{\mathcal{B},E}) \) in Definition 1.13 is essentially a Wiener amalgam space with \( L^2_E \) as local (quasi-)norm and \( \mathcal{B} \) or \( \ell_{\mathcal{B},E}(\Lambda_E) \) as global component. They are also related to coorbit spaces. (See \[10,13,15,33,34\].)

In fact, \( W^\infty(\omega_0, r) \) in Definition 1.13 (i.e. the case \( r = (\infty, \ldots, \infty) \) and \( E \) is the standard basis) is the coorbit space of \( L^p(\mathbb{R}^d) \) with weight \( \omega_0 \), and is sometimes denoted by

\( \text{Co}(L^p_{(\omega_0)}(\mathbb{R}^d)) \) or \( W(L^p_{(\omega_0)}) = W(L^p_{(\omega)}(\mathbb{R}^d)), \)

in the literature (cf. \[25,33,34\]).

Remark 1.14. Let \( p, \omega_0, \omega, E, \mathcal{B}, \mathcal{B}_0, f \) and \( F \) be the same as in Definition 1.13. Evidently, by using the fact that \( \omega_0 \) is \( v_0 \)-moderate for some \( v_0 \), it follows that
\[
\| f \cdot \omega_0 \|_{W^q_{1,E}(1, \ell_{\mathcal{B},E})} \asymp \| f \|_{W^q_E(\omega_0, \ell_{\mathcal{B},E})}
\]
and
\[
\| F \cdot \omega \|_{W^q_{k,E}(1, \ell_{\mathcal{B},E}, \mathcal{B}_0)} = \| F \|_{W^q_{k,E}(\omega_0, \ell_{\mathcal{B},E}, \mathcal{B}_0)}
\]
for \( k = 1, 2 \). Furthermore,
\[
W^q_{1,E}(\omega, \ell_{\mathcal{B},E}, \mathcal{B}_0) = \omega^{-1} \cdot W^q_{1,E}(1, \ell_{\mathcal{B},E}; \mathcal{B}_0)
\]
and
\[
W^q_{2,E}(\omega, \ell_{\mathcal{B},E}, \mathcal{B}_0) = \omega^{-1} \cdot \mathcal{B}_0(\mathbb{R}^d; W^q_E(1, \ell_{\mathcal{B},E})).
\]
Here and in what follows, \( \mathcal{B}(\mathbb{R}^d; \mathcal{B}_0) = \mathcal{B}(\mathbb{R}^d; \mathcal{B}_0(\mathbb{R}^{d_0})) \) is the set of all functions \( g \) in \( \mathcal{B} \) with values in \( \mathcal{B}_0 \), which are equipped with the quasi-norm

\[
\|g\|_{\mathcal{B}(\mathbb{R}^d; \mathcal{B}_0)} \equiv \|g_0\|_{\mathcal{B}}, \quad g_0(x) \equiv \|g(x)\|_{\mathcal{B}_0},
\]

when \( \mathcal{B}(\mathbb{R}^d) \) and \( \mathcal{B}_0(\mathbb{R}^{d_0}) \) are invariant QBF-spaces.

Later on we discuss periodicity in the framework of certain modulation spaces which are related to spaces which are defined by imposing \( L^\infty \)-conditions on the configuration variable of corresponding short-time Fourier transforms.

**Definition 1.15.** Let \( E, r, \mathcal{B}_0 \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be the same as in Definition 1.13 and let \( \phi \in \Sigma^1_1(\mathbb{R}^d) \setminus 0 \). Then \( \mathcal{M}_E^r(\omega, \mathcal{B}_0) \) and \( \mathcal{W}_E^r(\omega, \mathcal{B}_0) \) are the sets of all \( f \in \Sigma^1_1(\mathbb{R}^d) \) such that

\[
\|f\|_{\mathcal{M}_E^r(\omega, \mathcal{B}_0)} \equiv \|V_{\phi} f\|_{\mathcal{W}_{E, \mathcal{B}}^r(\omega, \mathcal{B}_0)}
\]

respectively

\[
\|f\|_{\mathcal{W}_E^r(\omega, \mathcal{B}_0)} \equiv \|V_{\phi} f\|_{\mathcal{W}_{E, \mathcal{B}}^r(\omega, \mathcal{B}_0)}
\]

are finite.

**Remark 1.16.** For the spaces in Definition 1.13 we set \( \mathcal{W}^{r_0, r_0} = \mathcal{W}^r \), when

\[
\rho_0 = (r_1, \ldots, r_d), \quad \text{and} \quad \rho = (q_0, \ldots, q_0, r_1, \ldots, r_d) \in (0, \infty)^{2d},
\]

and similarly for other types of exponents and for the spaces in Definitions 1.12 and 1.15. (See also Remark 1.10.) We also set

\[
\mathcal{M}_E^{\infty, q}(\omega) = \mathcal{M}_E^{q, \infty}(\omega) \quad \text{and} \quad \mathcal{W}_E^{\infty, q}(\omega) = \mathcal{W}_E^{q, \infty}(\omega)
\]

when \( E_1, E_2 \) are ordered bases of \( \mathbb{R}^d \) and \( E = E_1 \times E_2 \), for spaces in Definition 1.12 since these spaces are independent of \( E_1 \).

In Section 2 we prove that if \( \mathcal{B}_0 \) is an \( E \)-split Lebesgue space on \( \mathbb{R}^d \) and \( \omega(x, \xi) \in \mathcal{P}_E(\mathbb{R}^{2d}) \) which is constant with respect to the \( x \) variable, then \( \mathcal{M}_E^r(\omega, \mathcal{B}_0) \) and \( \mathcal{W}_E^r(\omega, \mathcal{B}_0) \) are independent of \( \rho \) and agree with modulation spaces of the form in Definition 1.11 (cf. Proposition 2.6).

The next result is a reformulation of [44, Proposition 3.4], and indicates how Wiener spaces are connected to modulation spaces. The proof is therefore omitted. Here, let

\[
(\Theta_{\rho, \nu})(x, \xi) = \nu(x, \xi) \langle x, \xi \rangle^\rho, \quad \text{where} \quad \rho \geq 2d \left( \frac{1}{r} - 1 \right), \quad (1.13)
\]

for any submultiplicative \( \nu \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( r \in (0, 1] \). It follows that \( \mathcal{L}^1_{(\Theta_{\rho, \nu})}(\mathbb{R}^d) \) is continuously embedded in \( \mathcal{L}^r_{(\nu)}(\mathbb{R}^{2d}) \), giving that \( \mathcal{M}^\Lambda_{(\Theta_{\rho, \nu})}(\mathbb{R}^d) \subseteq \mathcal{M}^r_{(\nu)}(\mathbb{R}^d) \). Hence if \( \phi \in \mathcal{M}^1_{(\Theta_{\rho, \nu})} \setminus 0, \varepsilon_0 \) is chosen such that \( S^\Lambda_{\phi, \nu} \) is invertible on \( \mathcal{M}^1_{(\Theta_{\rho, \nu})}(\mathbb{R}^d) \) for every \( \Lambda = \varepsilon \Lambda_E, \varepsilon \in (0, \varepsilon_0] \), it
follows that both $\phi$ and its canonical dual with respect to $\Lambda$ belong to $\mathcal{M}^p_{(\nu)}(\mathbb{R}^d)$. Notice that such $\varepsilon_0 > 0$ exists in view of [24, Theorem 5].

**Proposition 1.17.** Let $E$ be a phase split basis for $\mathbb{R}^d$, $p \in (0, \infty]^d$, $r = \min(1, p)$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate, $\rho$ and $\Theta_{\rho, v}$ be as in (1.13) with strict inequality when $r < 1$, and let $\phi_1, \phi_2 \in \mathcal{M}^1_{(\Theta_{\rho, v})}(\mathbb{R}^d) \setminus \{0\}$. Then

$$
\|f\|_{\mathcal{M}_{E, (\omega)}^p} \equiv \|V_{\phi_1} f\|_{L^p_{E, (\omega)}} \equiv \|V_{\phi_2} f\|_{W_E^p(\omega, \ell_E^p)}, \quad f \in \mathcal{S}_{1/2}(\mathbb{R}^d).
$$

In particular, if $f \in \mathcal{S}_{1/2}(\mathbb{R}^d)$, then

$$
f \in \mathcal{M}_{E, (\omega)}^p(\mathbb{R}^d) \iff V_{\phi_1} f \in L^p_{E, (\omega)}(\mathbb{R}^d) \iff V_{\phi_2} f \in W_E^p(\omega, \ell_E^p(\Lambda_E)).
$$

In Section 2 we extend this result in such way that we may replace $W_E^p(\omega, \ell_E^p)$ by $W_E^p(\omega, \ell_E^p(\Lambda_E))$ for any $r > 0$.

1.5. **Classes of periodic elements.** We consider spaces of periodic Gevrey functions and their duals.

Let $s, \sigma \in \mathbb{R}_+$ be such that $s + t \geq 1$, $f \in (\mathcal{S}_s^t)'(\mathbb{R}^d)$, $E$ be a basis of $\mathbb{R}^d$ and let $E_0 \subseteq E$. Then $f$ is called $E_0$-periodic if $f(x + y) = f(x)$ for every $x \in \mathbb{R}^d$ and $y \in E_0$.

We note that for any $\Lambda_E$-periodic function $f \in C^\infty(\mathbb{R}^d)$, we have

$$
f = \sum_{\alpha \in \mathcal{N}_E} c(f, \alpha)e^{i(\cdot, \alpha)}, \quad (1.14)
$$

where $c(f, \alpha)$ are the Fourier coefficients given by

$$
c(f, \alpha) \equiv |\kappa(E)|^{-1}(f, e^{i(\cdot, \alpha)})_{L^2(\mathbb{R}^d)}.
$$

For any $s \geq 0$ and basis $E \subseteq \mathbb{R}^d$ we let $\mathcal{E}_{0, \sigma}^E(\mathbb{R}^d)$ and $\mathcal{E}_0^E(\mathbb{R}^d)$ be the sets of all $E$-periodic elements in $\mathcal{E}_{0, \sigma}^E(\mathbb{R}^d)$ and in $\mathcal{E}_0^E(\mathbb{R}^d)$, respectively. Evidently,

$$
\mathcal{E}_0^E(\mathbb{R}^d) \simeq \mathcal{E}_\sigma(\mathbb{R}^d / \Lambda_E) \quad \text{and} \quad \mathcal{E}_{0, \sigma}^E(\mathbb{R}^d) \simeq \mathcal{E}_{0, \sigma}(\mathbb{R}^d / \Lambda_E),
$$

which is a common approach in the literature.

**Remark 1.18.** Let $E$ be an ordered basis on $\mathbb{R}^d$ and $V$ be a topological space of functions or (ultra-)distributions on $\mathbb{R}^d$. Then we use the convention that $V^E$ ($E$ as upper case index) denotes the $E$ periodic elements in $V$, while $V_E$ ($E$ as lower case index) is the space analogous to $V$ when $E$ is used as basis.

Let $s, s_0, \sigma, \sigma_0 > 0$ be such that $s + \sigma \geq 1$, $s_0 + \sigma_0 \geq 1$ and $(s_0, \sigma_0) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$. Then we recall that the duals $(\mathcal{E}_\sigma^E)'(\mathbb{R}^d)$ and $(\mathcal{E}_{0, \sigma_0}^E)'(\mathbb{R}^d)$ of $\mathcal{E}_\sigma^E(\mathbb{R}^d)$ and $\mathcal{E}_{0, \sigma_0}^E(\mathbb{R}^d)$, respectively, can be identified with the $E$-periodic
elements in \((S_s')^r(\mathbb{R}^d)\) and \((\Sigma_{s_0}^0)'(\mathbb{R}^d)\) respectively via unique extension of the form
\[
(f, \phi)_E = \sum_{\alpha \in \Lambda_E} c(f, \alpha)\overline{c(\phi, \alpha)}
\]
on \mathcal{E}^E_{0,\sigma_0}(\mathbb{R}^d) \times \mathcal{E}^E_{0,\sigma_0}(\mathbb{R}^d).\] We also let \((\mathcal{E}^E_0)'(\mathbb{R}^d)\) be the set of all formal expansions in (1.14) and \(\mathcal{E}^E_0(\mathbb{R}^d)\) be the set of all formal expansions in (1.14) such that at most finite numbers of \(c(f, \alpha)\) are non-zero (cf. [48]). We refer to [31,48] for more characterizations of \(\mathcal{E}^E_\sigma, \mathcal{E}^E_{0,\sigma}\) and their duals.

The following definition takes care of spaces of formal expansions (1.14) with coefficients obeying specific quasi-norm estimates.

**Definition 1.19.** Let \(E\) be a basis of \(\mathbb{R}^d\), \(\mathcal{B}\) be a quasi-Banach space continuously embedded in \(\ell_0'(\mathcal{N}_E)\) and let \(\omega_0\) be a weight on \(\mathbb{R}^d\). Then \(\mathcal{L}^E(\omega_0, \mathcal{B})\) consists of all \(f \in (\mathcal{E}^E_0)'(\mathbb{R}^d)\) such that
\[
\|f\|_{\mathcal{L}^E(\omega_0, \mathcal{B})} \equiv \|\{c(f, \alpha)\omega_0(\alpha)\}_{\alpha \in \Lambda_E}\|_{\mathcal{B}}
\]
is finite.

If \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)\) and \(\omega(x, \xi) = \omega_0(\xi)\), then
\[
\|f\|_{\mathcal{N}^E_{\omega_0}(\omega, \mathcal{B})} = \|\gamma_0\|_{\mathcal{B}},
\]
when \(g(\xi) = \|\nu_0 f(\cdot, \xi)\|_{L^E_{\kappa}(\mathcal{N}_E)}, \ f \in (\mathcal{E}^E_0)'(\mathbb{R}^d)\), (1.15)
and
\[
\|f\|_{\mathcal{W}^E_{\omega_0}(\omega, \mathcal{B})} = \|h\|_{L^E_{\kappa}(\mathcal{N}_E))},
\]
when \(h(x) = \|\nu_0 f(x, \cdot)\omega_0\|_{\mathcal{B}}, \ f \in (\mathcal{E}^E_0)'(\mathbb{R}^d),\) (1.16)
because the \(E\)-periodicity of \(x \mapsto |\nu_0 f(x, \xi)|\) when \(f\) is \(E\) periodic gives
\[
g(\xi) = \|\nu_0 f(\cdot, \xi)\|_{L^E_{\kappa}(\mathcal{N}_E)} = \|\nu_0 f(\cdot, \xi)\|_{L^E_{\kappa}(x+\kappa(\mathcal{E}))},
\]
\[
\|h\|_{L^E_{\kappa}(\mathcal{N}_E))} = \|h\|_{L^E_{\kappa}(x+\kappa(\mathcal{E}))}, \quad x \in \mathbb{R}^d.
\]
**Proposition 1.20.** Let \(E\) be a basis of \(\mathbb{R}^d\), \(r \in (0, 1]\), \(\mathcal{B} \subseteq L^r_{loc}(\mathbb{R}^d)\) be an \(E'\)-split Lebesgue space, \(\ell_0'(\mathcal{N}_E)\) be its discrete version, \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)\) and let \(\omega(x, \xi) = \omega_0(\xi)\) when \(x, \xi \in \mathbb{R}^d\). Then
\[
\mathcal{L}^E(\omega_0, \ell_0'(\mathcal{B})) = \mathcal{M}^\infty(\omega, \mathcal{B}) \bigcap (\mathcal{E}^E_0)'(\mathbb{R}^d) = \mathcal{W}^\infty(\omega, \mathcal{B}) \bigcap (\mathcal{E}^E_0)'(\mathbb{R}^d).
\]
When proving that \(\mathcal{W}^r_E(\omega, \ell_0'(\mathcal{B}))\) is independent of \(r \in (0, \infty)\) in Section 2 as announced earlier, it will at the same time follow that if \(\mathcal{B}\) is a suitable quasi-norm space of Lebesgue type, then
\[
\mathcal{M}^{r_1}_E(\omega, \mathcal{B}) = \mathcal{W}^{r_2}_E(\omega, \mathcal{B}) \quad \text{when} \quad \omega \in \mathcal{P}_E(\mathbb{R}^{2d})
\]
for every \(r_1, r_2 \in (0, \infty]\).
Remark 1.21. The link between periodic Gelfand-Shilov distributions and formal Fourier series expansions is given by the formula
\[ \langle f, \phi \rangle = (2\pi)^{\frac{d}{2}} \sum_{\alpha \in \Lambda_E^*} c(f, \alpha) \hat{\phi}(-\alpha). \] (1.19)

1.6. The Zak transform. For any ordered basis \( E \) of \( \mathbb{R}^d \) and \( f \in \mathcal{S}(\mathbb{R}^d) \), the Zak transform is defined by
\[ (Z_E f)(x, \xi) \equiv \sum_{j \in \Lambda_E} f(x - j) e^{i(j, \xi)} \] (1.20)

Several properties for the Zak transform can be found in [25]. For example, by the definition it follows that \( Z_E \) is continuous from \( \mathcal{S}(\mathbb{R}^d) \) to the set of all smooth functions on \( \mathbb{R}^{2d} \) which are bounded together with all their derivatives. It is also clear that \( Z_E f \) is quasi-periodic of order \( E \). Here, if \( F \) is a function or an ultra-distribution, then \( F \) is called quasi-periodic of order \( E \), when
\[ F(x + k, \xi) = e^{i(k, \xi)} F(x, \xi) \quad \text{and} \quad F(x, \xi + \kappa) = F(x, \xi), \] \[ k \in \Lambda_E, \ \kappa \in \Lambda_E^*. \] (1.21)

It follows by similar arguments as in Section 7.2 in [28] that if \( F \) above is a distribution (Gevrey distribution), then \( F \) is a tempered distribution (Gelfand-Shilov) distribution. For convenience we set \( Z_1 = Z_E \) when \( E \) is the standard basis for \( \mathbb{R}^d \).

For the Zak transform we recall the following important mapping properties on \( L^2(\mathbb{R}^d) \) and \( \mathcal{S}(\mathbb{R}^d) \).

Proposition 1.22. Let \( E \) be an ordered basis of \( \mathbb{R}^d \). Then the following is true:

1. The operator \( Z_E \) is homeomorphic from \( L^2(\mathbb{R}^d) \) to the set of all quasi-periodic elements of order \( E \) in \( L^2_{\text{loc}}(\mathbb{R}^{2d}) \). Furthermore,
   \[ \|Z_E f\|_{L^2(\mathbb{R}^{2d})} = |\kappa(E')|^\frac{d}{2} \|f\|_{L^2}, \quad f \in L^2(\mathbb{R}^d). \] (0.1)
   holds;

2. The operator \( Z_E \) restricts to a homeomorphism from \( \mathcal{S}(\mathbb{R}^d) \) to the set of all quasi-periodic elements of order \( E \) in \( C^\infty(\mathbb{R}^{2d}) \).

Proof. Let \( T_E \) be as in Remark 1.4. By straightforward computations it follows that
\[ Z_E f(x, \xi) = (Z_1 f_E)(T_E^{-1} x, T_{E^*} \xi), \quad f_E = f \circ T_E. \] (1.22)

The assertion (1) now follows from (0.1), (1.22) and suitable changes of variables in the involved integrals. The details are left for the reader. The assertion (2) follows from (1.22) and [25, Theorem 8.2.5]. \( \square \)
2. Estimates on Wiener spaces and periodic elements in modulation spaces

In this section we deduce equivalences between various Wiener (quasi-)norm estimates on short-time Fourier transforms. Especially we prove that ([18]) holds for every \( r_1, r_2 \in (0, \infty]^d \).

2.1. Estimates of Wiener spaces. We begin with the following inclusions between the different Wiener spaces in the previous section.

**Proposition 2.1.** Let \((E_1, E_2)\) be permuted dual bases of \(R^d\), \(E = E_1 \times E_2\), \(p, q, r \in (0, \infty]^d\) \(r_1 \in (0, \min(p, q, r)]\), \(r_2 \in (0, \min(q)]\), and let \(\omega_1, \omega_2 \in \mathcal{P}_E(R^{2d})\) be such that \(\omega_1(x, \xi) = \omega_2(\xi, x)\), \(x, \xi \in R^d\).

Then
\[
W^r_1(\omega, \ell^p_1(\Lambda_E)) \hookrightarrow W^r_1(\omega, \ell^p_2(\Lambda_{E_1}), L^q_2(R^d))
\]
and
\[
W^\infty_1(\omega, \ell^p_1(\Lambda_E)) \hookrightarrow W^\infty_1(\omega, \ell^{q,p}_2(\Lambda_{E_1}, \Lambda_{E_2}), L^q_2(R^d))
\]
(2.1)

** Remark 2.2.** For the involved spaces in Proposition 2.1 it follows from Hölder’s inequality that
\[
W^r_1(\omega, \ell^p_1(\Lambda_E), L^q_2(R^d)), \quad W^r_2(\omega, \ell^p_2(\Lambda_E), L^q_2(R^d))
\]
and
\[
W^r_1(\omega, \ell^p_1(\Lambda_E))
\]
increase with respect to \(p\) and decrease with respect to \(r\).

We need the following lemma for the proof of Proposition 2.1.

**Lemma 2.3.** Let \(\omega \in \mathcal{P}_E(R^d)\), \(E\) be an ordered basis of \(R^d\), \(\kappa(E)\) the parallelepiped spanned by \(E\), \(p \in (0, \infty]^d\), \(r \in (0, \min(p)]\) and \(f\) be measurable on \(R^d\). Then
\[
\|a\|_{\ell^p_1(\Lambda_E)} \lesssim \|f\|_{L^p_1(\omega)}, \quad a(j) = \|f\|_{L^p_1(\omega, j+\kappa(E))}\omega(j). \tag{2.3}
\]

**Proof.** Let \(f\) be measurable on \(R^d\), \(g_k\) be the same as in Definition 1.9. \(T_E\) be the linear map which maps the standard basis into \(E\), \(Q_k = [0, 1]^k\); and let \(p_k = (p_{k+1}, \ldots, p_d)\), when \(k \geq 1\). Then
\[
\|f\|_{L^p_1(T_j+\kappa(E))}\omega(j) \asymp \|f \cdot \omega\|_{L^p_1(T_j+\kappa(E))} \asymp \|g_0\|_{L^p_1(T_j+Q_d)}, \quad j \in Z^d.
\]
This reduces the situation to the case that \(E\) is the standard basis, \(\kappa(E) = Q\) and \(\omega = 1\). Moreover, by replacing \(|f|^r\) with \(f\) and \(p_jr\) by \(p_j, j = 1, \ldots, d\), we may assume that \(r = 1\) (and that each \(p_j \geq 1\)).
By induction it suffices to prove that if
\[ a_k(l) = \| g_k \|_{L^1(l+Q_{d-k})}, \quad l \in \mathbb{Z}^{d-k}, \]
then
\[ \| a_k \|_{\ell^{p_k}(\mathbb{Z}^{d-k})} \lesssim \| a_{k+1} \|_{\ell^{p_{k+1}}(\mathbb{Z}^{d-k-1})}, \quad k = 0, \ldots, d-1, \quad (2.4) \]
since \( \| a_0 \|_{\ell^{p_0}(\mathbb{Z}^d)} \) is equal to the left-hand side of \((2.3)\), and \( a_d = \| a_d \|_{\ell^{\infty}(\mathbb{Z}^d)} \) is equal to the right-hand side of \((2.3)\).

Let \( m \in \mathbb{Z}^{d-k-1} \) be fixed. We only prove \((2.4)\) in the case \( p_{k+1} < \infty \). The case \( p_{k+1} = \infty \) will follow by similar arguments and is left for the reader. By first using Minkowski’s inequality and then Hölder’s inequality we get
\[
\| a_k(\cdot, m) \|_{\ell^{p_{k+1}}(\mathbb{Z}^{d-k-1})} = \left( \sum_{l_1 \in \mathbb{Z}} \| g_k \|_{L^1(l_1, m+Q_{d-k})}^{p_{k+1}} \right)^{\frac{1}{p_{k+1}}}
\]
\[
\leq \left( \sum_{l_1 \in \mathbb{Z}} \left( \int_{m+Q_{d-k-1}} g_k(t, y) \, dy \right)^{p_{k+1}} \right)^{\frac{1}{p_{k+1}}}
\]
\[
\leq \left( \int_{m+Q_{d-k-1}} \left( \sum_{l_1 \in \mathbb{Z}} \int_{l_1+Q_1} g_k(t, y)^{p_{k+1}} \, dt \right) \, dy \right)^{\frac{1}{p_{k+1}}}
\]
\[
= \int_{m+Q_{d-k-1}} \left( \int_{\mathbb{R}} g_k(t, y)^{p_{k+1}} \, dt \right) \, dy = \int_{m+Q_{d-k-1}} g_{k+1}(y) \, dy.
\]
Hence,
\[ \| a_k(\cdot, m) \|_{\ell^{p_{k+1}}(\mathbb{Z}^{d-k-1})} \lesssim a_{k+1}(m), \quad m \in \mathbb{Z}^{d-k-1}. \quad (2.5) \]
By applying the \( \ell^{p_{k+1}}(\mathbb{Z}^{d-k-1}) \)-norm on \((2.5)\) we get \((2.4)\), and thereby \((2.3)\).

Proof of Proposition 2.1. Since the map \( F \mapsto F \cdot \omega \) is homeomorphic between the involved spaces and their corresponding non-weighted versions, we may assume that \( \omega_1 = \omega_2 = 1 \). Furthermore, by a linear change of variables, we may assume that \( E_1 \) is the standard basis and \( E_2 = 2\pi E_1 \). Then \( \kappa(E_1) = Q_d \), \( E_1' = E_2 \) and \( E_2' = E_1 \).

Let \( F \) be measurable on \( \mathbb{R}^{2d} \),
\[
f_{1,r}(\xi, j) = \| F(\cdot, \xi) \|_{L^r(j+\kappa(E_1))}, \quad g_{1}(\xi) = \| f_{1,r}(\xi, \cdot) \|_{L^r}
\]
and

\[ G_1(j, t) = \| F \|_{L^{r, \infty}(E_1 \times E_2)} . \]

Then

\[ g_1 \leq g = \sum_{t+\Lambda_{E_2}} \left( \| g \|_{L^{\infty}(t+2\pi Q)} \right) \cdot \chi_{t+2\pi Q} , \]

and

\[ \| F \|_{W^{r,1}(E_1, L^q)} = \| g_1 \|_{L^q} \leq \| g \|_{L^q} = \| G_1 \|_{\ell^p, q} \approx \| F \|_{W^{r,\infty}(E_1, L^q)} . \]

This implies that \( W^{r,\infty}(E_1, L^q) \hookrightarrow W^{r,1}(E_1, L^q) \), and the first inclusion in (2.1) follows.

In order to prove the second inclusion in (2.1), we may assume that \( r_0 < \infty \), since otherwise the result is trivial. Let

\[ \psi(\xi) = \| f_{1, r_0}(\xi, \cdot) \|_{L^p}, \quad a(t) = \| \psi \|_{L^q(t+\Lambda_{E_2})} \]

and

\[ H_1(j, t) = \| f_{1, r_0}(\cdot, j) \|_{L^{r_0}(t+\Lambda_{E_2})} . \]

Then

\[ \| \psi \|_{L^q(\mathbb{R}^d)} = \| F \|_{W^{r_0, q}(E_1, L^q)} \quad \text{and} \quad \| H_1 \|_{\ell^p, q} = \| F \|_{W^{r,\infty}(E_1, L^q)} . \]

By Minkowski’s inequality and the fact that \( \min(p) \geq r_0 \) we get

\[ \| H_1(\cdot, t) \|_{L^p} = \left( \int_{\Lambda_{E_2}} f_{1, r_0}(\xi, \cdot) \cdot d\xi \right)^{\frac{1}{r_0}} \]

\[ = \left( \left( \int_{\Lambda_{E_2}} f_{1, r_0}(\xi, \cdot) \cdot d\xi \right)^{\frac{1}{r_0}} \right) \]

\[ \leq \left( \int_{\Lambda_{E_2}} \| f_{1, r_0}(\xi, \cdot) \|_{L^p} \cdot d\xi \right)^{\frac{1}{r_0}} = a(t) . \]

Hence \( \| H_1 \|_{\ell^p, q} \leq \| a \|_{L^q} \). By Lemma 2.3 it follows that \( \| a \|_{L^q} \leq \| \psi \|_{L^q} \), and the second inclusion of (2.1) follows by combining these relations.

It remains to prove (2.2). Again we may assume that \( r_2 < \infty \), since otherwise the result is trivial. Let

\[ f_{2, q}(x, t) = \| F(x, \cdot) \|_{L^q(t+\Lambda_{E_2})}, \quad f_3(x) = \| F(x, \cdot) \|_{L^q(\mathbb{R}^d)}, \]

\[ H_{2, q_1, q_2}(t, j) = \| f_{2, q_1}(\cdot, t) \|_{L^{q_2}(j+\Lambda_{E_1})}, \quad \text{and} \quad H_{2, q} = H_{2, q, q} . \]
when \( q, q_1, q_2 \in (0, \infty) \]. Then the fact that \( r_2 \leq \min(q) \), Minkowski’s inequality and Lemma 2.3 give

\[
\| H_{2,r_2}(\cdot, j) \|_{\ell^q} \leq \left( \int_{j + \kappa(E_1)} \| f_{2,r_2}(x, \cdot) \|_{\ell^q}^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{j + \kappa(E_1)} \| F(x, \cdot) \|_{\ell^q}^2 \, dx \right)^{\frac{1}{2}}.
\]

By applying the \( \ell^p \) norm on the latter inequality we get

\[
\| F \|_{W^p_{2,E_2}(1,\ell^q,p)} \leq \| F \|_{W^p_{2,E_2}(1,\ell^q,L^q)},
\]

and the second relation in (2.2) follows.

On the other hand, we have

\[
\left( \int_{j + \kappa(E_1)} \| F(x, \cdot) \|_{L^q}(\mathbb{R}^d)}^2 \, dx \right)^{\frac{1}{2}} \lesssim \left( \int_{j + \kappa(E_1)} \| f_{2,\infty}(x, \cdot) \|_{\ell^q}^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \| H_{2,\infty}(\cdot, j) \|_{\ell^q}
\]

Again, by applying the \( \ell^p \) norm with respect to the \( j \) variable, we get

\[
\| F \|_{W^p_{2,E_2}(1,\ell^p,L^q)} \leq \| F \|_{W^p_{2,E_2}(1,\ell^q,p)},
\]

and the first relation in (2.2) follows. \( \square \)

2.2. Wiener estimates on short-time Fourier transforms, and modulation spaces. Essential parts of our analysis are based on Lebesgue estimates of the semi-discrete convolution with respect to (the ordered) basis \( E \) in \( \mathbb{R}^d \), given by

\[
(a *_{[E]} f)(x) \sum_{j \in \Lambda_E} a(j) f(x - j), \quad (2.6)
\]

when \( f \in S'_{1/2}(\mathbb{R}^d) \) and \( a \in \ell_0(\Lambda_E) \).

The next result is an extension of [44 Proposition 2.1] and [18 Lemma 2.6], but a special case of [16 Theorem 2.1]. The proof is therefore omitted. An other special case of [16 Theorem 2.1] will be used in Section 3 when discussing characterizations of the modulation spaces via norm estimates of Zak transforms. The domain of integration is of the form

\[
I = \{ x_1 e_1 + \cdots + x_d e_d ; x_k \in J_k \}, \quad J_k = \begin{cases} [0,1], & e_k \in E_0 \\ \mathbb{R}, & e_k \notin E_0 \end{cases} \quad (2.7)
\]

**Proposition 2.4.** Let \( E \) be an ordered basis of \( \mathbb{R}^d \), \( E_0 \subseteq E \), \( I \) be given by (2.7), \( \omega, v \in \mathcal{P}_E(\mathbb{R}^d) \) be such that \( \omega \) is \( v \)-moderate, and let \( p, r \in (0, \infty]^d \) be such that

\[
r_k \leq \min_{m \leq k}(1, p_m).
\]
Also let $f$ be measurable on $\mathbb{R}^d$ such that $|f|$ is $E_0$-periodic and $f \in L^p_{E,(\omega)}(I)$. Then the map $a \mapsto a[E]$ from $\ell_0(\Lambda_E)$ to $L^p_{E,(\omega)}(I)$ extends uniquely to a linear and continuous map from $\ell^r_{E,(\omega)}(\Lambda_E)$ to $L^p_{E,(\omega)}(I)$, and
\[
\|a \ast [E] f\|_{L^p_{E,(\omega)}(I)} \leq C\|a\|_{\ell^r_{E,(\omega)}(\Lambda_E)} \|f\|_{L^p_{E,(\omega)}(I)}, \tag{2.8}
\]
for some constant $C > 0$ which is independent of $a \in \ell^r_{E,(\omega)}(\Lambda_E)$ and measurable $f$ on $\mathbb{R}^d$ such that $|f|$ is $E_0$-periodic.

We have now the following result, which agrees with Proposition 1.17 when $r = (\infty, \ldots, \infty)$.

**Proposition 1.17.** Let $E$ be a phase split basis for $\mathbb{R}^{2d}$, $p, r \in (0, \infty]^2$, $r \in (0, \min(1,p)]$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $\rho$ and $\Theta_\rho \rho$ $\rho$ be as in (1.13) with strict inequality when $r < 1$, and let $\phi_1, \phi_2 \in M^1(\omega,v)(\mathbb{R}^d) \setminus 0$. Then
\[
\|f\|_{M^p_{E,(\omega)}} \lesssim \|V_{\phi_1} f\|_{L^p_{E,(\omega)}} \lesssim \|V_{\phi_2} f\|_{W^r_{E,(\omega),P^r_E}}, \quad f \in \mathcal{S}'_{1/2}(\mathbb{R}^d).
\]
In particular, if $f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)$, then
\[
f \in M^p_{E,(\omega)}(\mathbb{R}^{2d}) \iff V_{\phi_1} f \in L^p_{E,(\omega)}(\mathbb{R}^{2d}) \iff V_{\phi_2} f \in W^r_{E,(\omega),P^r_E}(\Lambda_E).
\]

We need the following lemma for the proof.

**Lemma 2.5.** Let $p \in (0, \infty]$, $r > 0$, $(x_0, \xi_0) \in \mathbb{R}^{2d}$ be fixed, and let $\phi \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ be a Gaussian. Then
\[
|V_{\phi} f(x_0, \xi_0)| \leq C\|V_{\phi} f\|_{L^p(B_r(x_0,\xi_0))}, \quad f \in \mathcal{S}'_{1/2}(\mathbb{R}^d),
\]
where the constant $C$ is independent of $(x_0, \xi_0)$ and $f$.

When proving Lemma 2.5 we may first reduce ourself to the case that the Gaussian $\phi$ should be centered at origin, by straight-forward arguments involving pullbacks with translations. The result then follows by using the same arguments as in [18, Lemma 2.3] and its proof, based on the fact that
\[
z \mapsto F_{\omega}(z) = e^{c_1|z|^2+c_2(z,\omega)+c_3|\omega|^{3/2}V_{\phi} f(x, \xi)}, \quad z = x + i\xi
\]
is an entire function for some choice of the constant $c_1$ (depending on $\phi$).

**Proof of Proposition 1.17.** Let $F = V_{\phi} f$, $F_0 = V_{\phi_0} f$, $\kappa(E)$ be the (closed) parallelepiped which is spanned by $E = \{e_1, \ldots, e_{2d}\}$, and let
\[
\kappa_M(E) = \{x_1 e_1 + \cdots + x_{2d} e_{2d}; |x_k| \leq 2, k = 1, \ldots, 2d\}.
\]
Also choose $r_0 > 0$ small enough such that
\[
\kappa(E) + B_{r_0}(0,0) \subseteq \kappa_M(E)
\]
The result holds when \( r = (\infty, \ldots, \infty) \), in view of Proposition 1.17. By Hölder’s inequality we also have
\[
\| V_\phi f \|_{W^r_E(\omega, \ell^p_r)} \lesssim \| V_\phi f \|_{W^r_E(\omega, \ell^p_r)}. \tag{2.9}
\]
We need to prove the reversed inequality
\[
\| V_\phi f \|_{W^r_E(\omega, \ell^p_r)} \lesssim \| V_\phi f \|_{W^r_E(\omega, \ell^p_r)}, \tag{2.10}
\]
and it suffices to prove this for \( r = (r, \ldots, r) \) for some \( r \in (0, 1] \) in view of Hölder’s inequality.

First we consider the case when \( \phi = \phi_0 \). If \( r > 0 \) is small enough and \( j \in \Lambda_E \), then Lemma 2.5 gives for some \( (x_j, \xi_j) \in j + \kappa(E) \) that
\[
\| V_{\phi_0} f \|_{L^r(j + \kappa(E))} = | V_{\phi_0} f(x_j, \xi_j) | \lesssim \| V_{\phi_0} f \|_{L^r(B_r(x_j, \xi_j))} \leq \| V_{\phi_0} f \|_{L^r(j + \kappa(E))} \tag{2.12}
\]
Hence,
\[
\| V_{\phi_0} f \|_{W^r_E(\omega, \ell^p_r)} = \{ \| V_{\phi_0} f \|_{L^r(j + \kappa(E))}\omega(j) \}_{j \in \Lambda_E} \| p_E(\Lambda_E) \]
\[
\lesssim \{ \| V_{\phi_0} f \|_{L^r(j + \kappa(E))}\omega(j) \}_{j \in \Lambda_E} \| p_E(\Lambda_E) \]
\[
\asymp \{ \| V_{\phi_0} f \|_{L^r(j + \kappa(E))\omega(j)} \}_{j \in \Lambda_E} \| p_E(\Lambda_E) \}
\]
and (2.10) holds for \( \phi = \phi_0 \).

Next suppose that \( \phi \) is arbitrary and let \( n \geq 1 \) be a large enough integer such that if
\[
E_n = \frac{1}{n} \cdot E \equiv \left\{ \frac{e_1}{n}, \ldots, \frac{e_d}{n} \right\} \quad \text{and} \quad \Lambda = \frac{1}{n} \Lambda_E = \Lambda_{E_n},
\]
then
\[
\{ \phi(\cdot - k)e^{i(\cdot, \kappa)} \}_{(k, \kappa) \in \Lambda}
\]
is a frame. Since \( \phi \in M^1(\Theta, \nu) \), it follows that its canonical dual \( \psi \) also belongs to \( M^1(\Theta, \nu) \) (cf. [24, Theorem S]). Consequently, any \( f \) possess the expansions
\[
f = \sum_{(k, \kappa) \in \Lambda} V_\phi f(k, \kappa) \psi(\cdot - k)e^{i(\cdot, \kappa)}
\]
\[
= \sum_{(k, \kappa) \in \Lambda} V_\phi f(k, \kappa) \phi(\cdot - k)e^{i(\cdot, \kappa)} \tag{2.11}
\]
with suitable interpretation of convergences.

Let
\[
F_0 = | V_{\phi_0} f \| \cdot \omega, \quad F = | V_{\phi} f \| \cdot \omega, \quad \text{and} \quad a(k) = | V_{\psi} \phi_0(\cdot - k) |.
\]
As in the proofs of [13, Theorem 3.1] and [24, Proposition 3.1] we use the fact that
\[
| V_{\phi_0} f | \leq (2\pi)^{-\frac{d}{2}} a \ast_{|E_n|} | V_{\phi} f |, \tag{2.12}
\]
which follows from
\[ |V_{\phi_0}f(x, \xi)| = (2\pi)^{-\frac{d}{2}} |(f, e^{i(x, \cdot)\phi_0(\cdot - x)})| \]
\[ \leq (2\pi)^{-\frac{d}{2}} \sum_{(k, \kappa) \in \Lambda} |(V_{\phi_0})_{(k, \kappa)}|(f, e^{i(x, \cdot)\kappa}\phi(\cdot - x - k))| \]
\[ = (2\pi)^{-\frac{d}{2}} \sum_{(k, \kappa) \in \Lambda} |(V_{\phi_0})_{(k, \kappa)}||V_\phi f(x + k, \xi + \kappa)| \]
\[ = (\ast_{E^\ell_n}) |V_\phi f|(x, \xi). \]
Here we have used (2.11) with \( \phi_0 \) in place of \( f \), in the inequality. By using that
\[ \omega(x, \xi) \lesssim v(k, \kappa)\omega(x + k, \xi + \kappa), \]
(2.12) gives
\[ F_0 \lesssim (a \cdot v)_{E^\ell_n} F. \] (2.13)
If we set
\[ b_0(j) = \int_{j + \kappa(E)} |F_0(X)|^r dX \quad \text{and} \quad b(j) = \int_{j + \kappa(E)} |F(X)|^r dX, \quad j \in \Lambda, \]
integrate (2.13) and use the fact that \( r \leq 1 \), we get for \( j \in \Lambda \) that
\[ b_0(j) \lesssim \int_{j + \kappa(E)} \left( \sum_{k \in \Lambda} a(k)v(k)|F(X - k)| \right)^r dX \]
\[ \lesssim \sum_{k \in \Lambda} (a(k)v(k))^r \int_{j + \kappa(E)} |F(X - k)|^r dX = ((a \cdot v)^r \ast b)(j), \]
where \( \ast \) is the discrete convolution with respect to the lattice \( \Lambda \).
Let \( q = p/r \). Then \( \min(q) \geq 1 \), and Young’s inequality applied on the last inequality gives
\[ \|F_0\|_{W^{1,1}(\ell_n^p, \ell_n^p)} = \|b_0^\frac{1}{p}\|_{\ell_n^p(\Lambda_E)} \lesssim \|(a \cdot v)^r \ast b\|_{\ell_n^p(\Lambda_E)} \]
\[ \leq \|(a \cdot v)^r \ast b\|_{\ell_n^p(\Lambda)} \leq \left( \|(a \cdot v)^r\|_{\ell^q(\Lambda)} \|b\|_{\ell^p(\Lambda)} \right)^\frac{1}{r} \]
\[ \lesssim \|a\|_{\ell_{\min}(\Theta^{\nu})} \|b\|_{\ell_n^p(\Lambda)} \lesssim \|a\|_{\ell_{\min}(\Theta^{\nu})} \|b\|_{\ell_n^p(\Lambda)} \]
\[ \lesssim \|\phi\|_{\Lambda_{\min}(\Theta^{\nu})} \|b\|_{\ell_n^p(\Lambda)}. \] (2.14)
In the last steps we have used Hölder’s inequality and
\[ \|V_{\phi_0}\|_{L^{1}(\Theta^{\nu})} \asymp \|\{\|V_{\phi_0}\|_{L^{\infty}(j + \kappa(E))}(\Theta_{\rho^j}(\cdot))(j) \}_{j \in \Lambda_E} \|_{\ell^1(\Lambda_E)} \asymp \|\phi\|_{\Lambda_{\min}(\Theta^{\nu})}. \]
We have
\[ \|b^\frac{1}{p}\|_{\ell_n^p(\Lambda)} = \|\{\|F\|_{L^r(j + \kappa(E))}\}_{j \in \Lambda} \|_{\ell_n^p(\Lambda)}, \]
We get the following.

By combining Proposition \ref{prop:1.20} and Remark \ref{rem:2.2}, we get the following.

By combining Proposition \ref{prop:1.17} with Proposition \ref{prop:2.6} and Remark \ref{rem:2.2}, we get the following.

Proposition \ref{prop:2.6}. Let $E_0$ be a basis for $\mathbb{R}^d$, $E_0'$ be its dual basis, $E = E_0 \times E_0'$, $q, r \in (0, \infty]^d$, $\omega, v_0 \in \mathcal{R}_E(\mathbb{R}^d)$ be such that $\omega$ is $v_0$-moderate, $\omega(x, \xi) = \omega_0(\xi)$, $v(x, \xi) = v_0(\xi)$, $\Theta_p v$ be as in \eqref{eq:1.13} with strict inequality when $r < 1$, and let $\phi \in M^1_{(\Theta_p v)}(\mathbb{R}^d) \setminus 0$. Then

$$M^{\infty, q}_{E_0}(\omega, L^q_{E_0}(\mathbb{R}^d)) = M^{\infty, q}_{E_0}(\omega, L^q_{E_0}(\mathbb{R}^d)),$$

and

$$\|f\|_{M^{\infty, q}_{E_0}(\omega)} \asymp \|V_0 f\|_{\mathcal{W}_{E_0}(\omega, L^q_{E_0}(L^{q_0}_{E_0}))), \quad \|f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega)} \asymp \|V_0 f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega, L^q_{E_0}(L^{q_0}_{E_0})).$$

2.3 Periodic elements in modulation spaces. By a straight-forward combination of Propositions \ref{prop:1.20} and \ref{prop:2.6}, we get the following. The details are left for the reader.

Proposition \ref{prop:2.7}. Let $E_0$ be a basis for $\mathbb{R}^d$, $E_0'$ be its dual basis, $E = E_0 \times E_0'$, $q, r \in (0, \infty]^d$, $\omega, v_0 \in \mathcal{R}_E(\mathbb{R}^d)$ be such that $\omega$ is $v_0$-moderate, $\omega(x, \xi) = \omega_0(\xi)$, $v(x, \xi) = v_0(\xi)$, $\Theta_p v$ be as in \eqref{eq:1.13} with strict inequality when $r < 1$, and let $\phi \in M^1_{(\Theta_p v)}(\mathbb{R}^d) \setminus 0$. Then

$$\|f\|_{M^{\infty, q}_{E_0}(\omega)} \asymp \|f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega)} \asymp \|f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega, L^q_{E_0}(L^{q_0}_{E_0}))), \quad \|f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega)} \asymp \|f\|_{\mathcal{W}^{\infty, q}_{E_0}(\omega, L^q_{E_0}(L^{q_0}_{E_0}))), \quad f \in \mathcal{E}_{0}^{F}(\mathbb{R}^d). \quad (2.15)$$

As an immediate consequence of the previous result we get the following extension of Proposition \ref{prop:1.20}. The details are left for the reader.
Proposition 2.10. Let $E$ be a basis of $\mathbb{R}^d$, $r \in (0, \infty]^d$, $r \in (0, 1]$, $\mathcal{B} \subseteq L^r_{loc}(\mathbb{R}^d)$ be an $E'$-split Lebesgue space, $\ell_{\mathcal{B}, E}(\Lambda_E)$ its discrete version, and let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$. Then

$$
L^E(\omega, \ell_{\mathcal{B}, E}) = \mathcal{M}^E(\omega, \mathcal{B}) \cap (E^E)'(\mathbb{R}^d) = \mathcal{W}^E(\omega, \mathcal{B}) \cap (E^E)'(\mathbb{R}^d)
$$

Remark 2.8. Let

$$
E_0 = \{e_1, \ldots, e_d\}, \quad E'_0 = \{e_1', \ldots, e_d'\}, \quad q = (q_1, \ldots, q_d), \quad r = (r_1, \ldots, r_d),
$$

$\omega$, $v$ and $\phi$ be the same as in Proposition 2.7 and let $r_0 \leq \min(r)$ and $f, f' \in (E^E)'(\mathbb{R}^d)$ with Fourier series expansion (1.14). Then (1.15) and (2.15) imply that

$$
\|V_\phi f \cdot \omega\|_{L^r_{E}(\kappa(E_0) \times \mathbb{R}^d)} \leq \|V_\phi f \cdot \omega\|_{L^q_{E}(\mathbb{R}^d \times \kappa(E_0))} \leq \|c(f, \cdot)\|_{r_0^*_{E_0, (\omega_0)}}.
$$

(2.16)

Let $\| \cdot \|$ be the quasi-norm on the left-hand side of (2.16), after the orders of the involved $L^q_{E_0}(\mathbb{R})$ and $L^q_{E_0}(\kappa(e_k))$ quasi-norms have been permuted in such way that the internal order of the hitting $L^q_{E_0}(\mathbb{R})$ quasi-norms remains the same. Then

$$
\|F\|_{L^q_{E_0}(\kappa(E_0) \times \mathbb{R}^d)} \leq \|F\|_{L^q_{E_0}(\mathbb{R}^d \times \kappa(E_0))},
$$

(2.17)

by repeated application of Hölder's inequality. A combination of (2.16) and (2.17) give

$$
\|V_\phi f \cdot \omega\|_{L^q_{E_0}(\kappa(E_0) \times \mathbb{R}^d)} \leq \|c(f, \cdot)\|_{r_0^*_{E_0, (\omega_0)}}.
$$

(2.18)

In particular, if $e_j$ are the same as in Remark 1.7, $E_*$ is the ordered basis $\{e_1, e_{d+1}, \ldots, e_d, e_{2d}\}$ of $\mathbb{R}^{2d}$,

$$
\Omega = \{ y_1e_1 + \cdots + y_{2d}e_{2d} : 0 \leq y_j \leq 1 \text{ and } y_{d+j} \in \mathbb{R}, \ j = 1, \ldots, d \}
$$

and $q_0 = (q_1, q_1, q_2, \ldots, 0, 0) \in (0, \infty]^{2d}$, then

$$
\|V_\phi f \cdot \omega\|_{L^{q_0}_{E_*, (\omega_0)}} \leq \|c(f, \cdot)\|_{r_0^*_{E_0, (\omega_0)}}.
$$

(2.19)

Remark 2.9. With the same notation as in the previous remark, we note that if $E'_0$ is the standar basis of $\mathbb{R}^d$, $X_j = (x_j, \xi_j)$, $j = 1, \ldots, d$, $I = \mathbb{R} \times [0, 2\pi]$ and $\max(q) < \infty$, then (2.19) is the same as

$$
\left( \int_I \left( \cdots \left( \int_I |V_\phi f(x, \xi)\omega_0(\xi)|^{q_1} dX_1 \right)^{\frac{q_2}{q_1}} \cdots \left( \int_I \right)^{\frac{q_d}{p_d - 1}} dX_d \right)^{\frac{1}{p_d}} \right.
$$

$$
\times \left( \sum_{a_d \in \mathbb{Z}} \left( \cdots \left( \sum_{a_1 \in \mathbb{Z}} |c(f, \alpha)\omega_0(\alpha)|^{q_1} \right)^{\frac{q_2}{q_1}} \cdots \left( \int_I \right)^{\frac{q_d}{p_d - 1}} \right)^{\frac{1}{p_d}} \right)
$$

(2.19)
3. ZAK TRANSFORM ON GELFAND-SHILOV SPACES, LEBESGUE SPACES AND MODULATION SPACES

In this section we deduce characterizations of Lebesgue spaces, modulation spaces, and Gelfand-Shilov spaces and their distribution spaces in terms of suitable estimates of the Zak transforms of the involved elements. The characterizations on modulation spaces are related to results given in [41, 42].

3.1. The Zak transform on test function spaces and their distribution spaces. For the classical spaces \( \mathcal{S}(\mathbb{R}^d) \) and its distribution space \( \mathcal{S}'(\mathbb{R}^d) \) we have the following

**Theorem 3.1.** Let \( E \) be an ordered basis of \( \mathbb{R}^d \). Then the following is true:

1. The operator \( Z_E \) is a homeomorphism from \( \mathcal{S}(\mathbb{R}^d) \) to the set of quasi-periodic elements of order \( E \) in \( C^\infty(\mathbb{R}^{2d}) \);
2. The operator \( Z_E \) from \( \mathcal{S}(\mathbb{R}^d) \) to \( C^\infty(\mathbb{R}^{2d}) \) is uniquely to a homeomorphism from \( \mathcal{S}'(\mathbb{R}^d) \) to the set of quasi-periodic elements of order \( E \) in \( \mathcal{S}'(\mathbb{R}^{2d}) \).

The assertion (1) in Theorem 3.1 is essentially the same as [25, Theorem 8.2.5], and (2) in the same theorem follows by similar arguments as in the proof of Theorem 3.3 below. The verifications of Theorem 3.1 are therefore left for the reader.

The analogous result of the (1) previous theorem for Gelfand-Shilov functions is the following.

**Theorem 3.2.** Let \( s, \sigma > 0 \) and \( E \) be an ordered basis. Then the operator \( Z_E \) from \( \mathcal{S}(\mathbb{R}^d) \) to \( C^\infty(\mathbb{R}^{2d}) \) restricts to a homeomorphism from \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) to the set of quasi-periodic elements of order \( \rho \) in \( \mathcal{E}_{s,\sigma}(\mathbb{R}^{2d}) \).

The same holds true with \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) and \( \mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d}) \) in place of \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) and \( \mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d}) \) at each occurrence.

Before the proof of the previous result we present the analogous result for corresponding distribution spaces.

**Theorem 3.3.** Let \( s, \sigma > 0 \) and \( E \) be an ordered basis. Then the operator \( Z_E \) from \( \mathcal{S}(\mathbb{R}^d) \) to \( C^\infty(\mathbb{R}^{2d}) \) extends uniquely to a homeomorphism from \( (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \) to the set of quasi-periodic elements of order \( E \) in \( (\mathcal{S}_{s,\sigma}^\sigma)'(\mathbb{R}^{2d}) \).

The same holds true with \( (\mathcal{E}_s^\sigma)'(\mathbb{R}^d) \) and \( (\mathcal{E}_{s,\sigma}^\sigma)'(\mathbb{R}^{2d}) \) in place of \( (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \) and \( (\mathcal{S}_{s,\sigma}^\sigma)'(\mathbb{R}^{2d}) \) at each occurrence.

For the proof of Theorem 3.3 we need the following lemma on tensor product of Gelfand-Shilov distributions.
Lemma 3.4. Let \( s_j, \sigma_j > 0 \) and \( f_j \in (\mathcal{S}_{s_j}^{\sigma_j})'(\mathbb{R}^d) \), \( j = 1, 2 \). Then there is a unique \( f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \) such that
\[
\langle f, \varphi \otimes \varphi_2 \rangle = \langle f_1, \varphi_1 \rangle, \quad \varphi_j \in \mathcal{S}_{s_j}^{\sigma_j}(\mathbb{R}^d), \quad j = 1, 2.
\]
Moreover, if \( \varphi \in \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2}) \),
\[
\varphi_1(x_1) = \langle f_2, \varphi(x_1, \cdot) \rangle \quad \text{and} \quad \varphi_2(x_2) = \langle f_1, \varphi(\cdot, x_2) \rangle,
\]
then
\[
\langle f, \varphi \rangle = \langle f_1, \varphi_1 \rangle = \langle f_2, \varphi_2 \rangle.
\]
The same holds true with \( \Sigma_{s_j}^{\sigma_j}, \Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}, (\Sigma_{s_j}^{\sigma_j})' \) and \((\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'\) in place of \( \mathcal{S}_{s_j}^{\sigma_j}, \mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2}, (\mathcal{S}_{s_j}^{\sigma_j})' \) and \((\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'\), respectively, \( j = 1, 2 \).

Lemma 3.4 is essentially a restatement of Theorem 2.4 in [47]. The proof is therefore omitted.

Remark 3.5. We notice that the uniqueness assertions in Lemma 3.4 is an immediate consequence of [47], Lemma 2.3 which asserts that if \( f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \) (\( f \in (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \)) satisfies
\[
\langle f, \varphi_1 \otimes \varphi_2 \rangle = 0
\]
for every \( \varphi_j \in \mathcal{S}_{s_j}^{\sigma_j}(\mathbb{R}^d) \) \( (\varphi_j \in \Sigma_{s_j}^{\sigma_j}(\mathbb{R}^d)) \), then \( f = 0 \) (as an element in \( (\mathcal{S}_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \) \( (\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2}) \)).

Proof of Theorem 3.3. Let \( T_E \) be the same as in (1.22). Then the map \( F(x, \xi) \mapsto F(T_E^{-1}x, T_E^*\xi) \) maps quasi-periodic elements of order \( E \) to quasi-periodic elements with respect to the standard basis. Since \( f \mapsto f \circ T \) maps \( E \)-periodic elements to \( 1 \)-periodic functions, it follows from these observations and (1.22) that it suffices to prove the result when \( E \) is the standard basis.

We begin to prove (2). Let \( \Phi \in \mathcal{S}_{s,\sigma}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d}) \). Then
\[
\mathcal{F}_2^{-1}\Phi \in \mathcal{S}_s^{\sigma_1}(\mathbb{R}^{2d}), \quad \mathcal{F}_1(\mathcal{F}_2^{-1}\Phi) \in \mathcal{S}_{s,s}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d}),
\]
\[
|\mathcal{F}_2^{-1}\Phi(x,y)| \lesssim e^{-r_0(|x|+|y|)} \quad \text{and} \quad |\mathcal{F}_1\Phi(\xi,\eta)| \lesssim e^{-r_0(|\xi|+|\eta|)}
\]
If \( f \in \mathcal{S}(\mathbb{R}^{d}) \), then
\[
\langle Z_1f, \Phi \rangle = \int_{\mathbb{R}^{2d}} \left( \sum_{j \in \mathbb{Z}^d} f(x-j)e^{i(j,\xi)} \right) \Phi(x, \xi) \, dx \, d\xi
\]
\[
= \sum_{j \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^{d}} f(x)(\mathcal{F}_2^{-1}\Phi)(x+j) \, dx \right)
\]
\[
= \sum_{j \in \mathbb{Z}^d} (\hat{f} \ast \Phi_2(\cdot, j))(j), \quad (3.1)
\]
where \( \Phi_2 = \mathcal{F}_2^{-1}\Phi \) and \( \hat{f}(x) = f(-x) \).
Assume instead that \( f \in (S^s_\sigma)'(\mathbb{R}^d) \) is arbitrary. We claim that the series on the right-hand side of (3.1) converges absolutely for every \( \Phi \) as above.

In fact, since \( f \in (S^s_\sigma)'(\mathbb{R}^d) \), we have
\[
|\langle f, \phi \rangle| \lesssim \|e^{r|\cdot|} \phi \|_{L^\infty} + \|e^{r|\cdot|} \hat{\phi} \|_{L^\infty}
\]
for every \( r > 0 \), giving that for some \( r_0 > 0 \) and \( c \geq 1 \) we have
\[
|\langle \hat{f} \ast \Phi_2(\cdot, y) \rangle(x)| \lesssim \|e^{r|\cdot|} \Phi_2(x - \cdot, y) \|_{L^\infty} + \|e^{r|\cdot|} \hat{\Phi}(\cdot, y) \|_{L^\infty}
\]
\[
\lesssim e^{r|x|^\frac{1}{2} - 2r_0|y|^\frac{1}{2}/c}.
\]
Hence, if \( r \) is chosen small enough and \( x = y = j \), then
\[
|\langle \hat{f} \ast \Phi_2(\cdot, 2\pi\rho j) \rangle(\rho j)| \lesssim e^{-r_0|j|^\frac{1}{2}/c},
\]
(3.2)
when \((S^s_\sigma)'(\mathbb{R}^d)\). The absolutely convergence of the series of the right-hand side of (3.1) now follows from (3.2).

If \( f \in (S^s_\sigma)'(\mathbb{R}^d) \), then \( Z_1f \) is defined as the element in \((S^s_\sigma)'(\mathbb{R}^{2d})\), given by the right-hand side of (3.1). The previous estimates show that this definition makes sense, and that the map \( f \mapsto Z_1f \) is continuous from \((S^s_\sigma)'(\mathbb{R}^d)\) to the set of all quasi-periodic elements of order 1 in \((S^s_\sigma)'(\mathbb{R}^{2d})\). By approximating elements in \((S^s_\sigma)'(\mathbb{R}^d)\) by sequences of elements in \(\mathcal{S}(\mathbb{R}^d)\), it also follows that the continuous extension of \(Z_1\) to such distribution is unique.

We need to prove that any quasi-periodic element of order 1 in \((S^s_\sigma)'(\mathbb{R}^{2d})\) is the Zak transform of an element in \((S^s_\sigma)'(\mathbb{R}^d)\). Therefore, let \( \varphi \in S^s_\sigma(\mathbb{R}^d) \), \( F \) be a quasi-periodic elements of order 1 in \((S^s_\sigma)'(\mathbb{R}^{2d})\), and let \( g_\varphi \in (S^s_\sigma)'(\mathbb{R}^d) \) be defined by
\[
\langle g_\varphi, \psi \rangle = \langle F, \varphi \otimes \psi \rangle, \quad \psi \in S^s_\sigma(\mathbb{R}^d).
\]
Then \( g_\varphi \) is 2\pi-periodic, and it follows from Remark 2.3 and Proposition 2.5 in [15] that if \( \phi \in S^s_\sigma(\mathbb{R}^d) \setminus 0 \), then
\[
g_\varphi = \sum_{k \in \mathbb{Z}^d} c(g_\varphi, k) e^{i(k, \xi)},
\]
where the series converges in \((S^s_\sigma)'(\mathbb{R}^d)\), and
\[
c(g_\varphi, 0) = \frac{1}{(2\pi)^d \|\phi\|_{L^2}^2} \int_{Q_{d,2\pi}} \left( \int_{\mathbb{R}^d} (V_\phi g_\varphi)(\eta, y) \hat{\phi}(-y) e^{i(y, \eta)} \, dy \right) \, d\eta
\]
and
\[
c(g_\varphi, k) = c(g_\varphi e^{-i(k, \cdot)}, 0)
\]
By straight-forward computations we get
\[
c(g_\varphi, 0) = \frac{1}{(2\pi)^d \|\phi\|_{L^2}^2} \int_{Q_{d,2\pi}} \left( \int_{\mathbb{R}^d} \langle F, \varphi \otimes (\phi(\cdot - \eta) e^{-i(y, \cdot)}) \rangle \hat{\phi}(-y) e^{i(y, \eta)} \, dy \right) \, d\eta,
\]
30
and it is clear that the map which takes $\varphi$ into the right-hand side defines a continuous linear form on $\mathcal{S}_\sigma'(\mathbb{R}^d)$. Hence
\[ c(g_\varphi, 0) = \langle f, \varphi \rangle, \quad \varphi \in \mathcal{S}_\sigma(\mathbb{R}^d), \]
for some $f \in (\mathcal{S}_\sigma')(\mathbb{R}^d)$.

It now follows from the quasi-periodicity of $F$ that
\[ c(g_\varphi, k) = c(g_\varphi e^{-i(k \cdot \cdot )}, 0) \]
\[ = c(g_\varphi( \cdot + k), 0) = \langle f, \varphi( \cdot + k) \rangle = \langle f( \cdot - k), \varphi \rangle. \]
Hence, if $F_0 = F - Z_1 f$, then
\[ \langle F_0, \varphi \otimes \psi \rangle = 0 \]
when $\varphi \in \mathcal{S}_\sigma(\mathbb{R}^d)$ and $\psi \in \mathcal{S}_\sigma(\mathbb{R}^d)$. By Remark 3.5 it now follows that $F = Z_1 f$, which gives the result. □

We need the following lemma for the proof of Theorem 3.2.

**Lemma 3.6.** Let $r, s > 0$. Then there is a constant $h > 0$ such that
\[ |t|^d e^{-r|t|^d / 2} \lesssim h^\beta \beta^! s, \quad t \in \mathbb{R}, \beta \in \mathbb{N}. \]

**Proof.** By reasons of symmetry and Stirling’s formula, the result follows if we prove
\[ t^r e^{-rt^d / 2} \lesssim h^\tau \tau^s, \quad 0 \leq t, \tau \in \mathbb{R} \]
for some $h > 0$. By taking the logarithm it follows that we need to prove that for some constant $C > 0$,
\[ g(\tau) = -rt^d + \tau \log t - C\tau - s\tau \log \tau \]
is bounded from above by a constant which is independent of $t, \tau > 0$.

By differentiation and checking the sign of $g'(\tau)$, it follows that $g(\tau)$ has a global maximum for
\[ \tau_0 = e^{-1 - C / s} t^{1 / 2} \]
with value
\[ g(\tau_0) = t^{1 / 2} (-r + seh^{-1 / 2}) \]
By choosing
\[ h > \left( \frac{r}{se} \right)^s \]
it follows that $g(\tau)$ is negative, giving the result. □

**Proof of Theorem 3.2.** By similar arguments as in the proof of Theorem 3.3 we may assume that $E$ is the standard basis for $\mathbb{R}^d$.

The assertion (1) is the same as Theorem 8.2.5 in [25].

In order to prove (2) we shall follow the proof of Theorem 8.2.5 in [25]. In fact, assume first that $f \in \mathcal{S}_\sigma(\mathbb{R}^d)$, $x \in k_0 + Q_{d,1}$ for some fixed $k_0 \in \mathbb{Z}^d$, and let $F = Z_1 f$. Then
\[ |\partial_x^\alpha f(x - k)| \leq C h^\alpha e^{-2r(|k_1|^d + \ldots + |k_d|^d)} \alpha!^s, \quad x \in k_0 + Q_{d,1}, \]
for some positive constant $C$ which only depends on $k_0$ and some positive constants $h > 0$ and $r > 0$ which are independent of $x$, $k_0$, $k$ and $\alpha$.

The series in (1.20) is absolutely convergent together with all its derivatives. This gives

$$|(\partial^\alpha \partial^\beta_x F)(x, \xi)| \leq \sum_{k \in \mathbb{Z}^d} |f^{(\alpha)}(x - k)| h^\beta$$

$$\lesssim h_1^{\alpha+\beta} \sum_{k \in \mathbb{Z}^d} e^{-2r(|k_1|^{\frac{1}{s}} + \cdots + |k_d|^{\frac{1}{s}})} |k^\beta|$$

$$= h_1^{\alpha+\beta} \prod_{j=1}^d \left( \sum_{k_j \in \mathbb{Z}} e^{-r|k_j|^{\frac{1}{s}} \left( e^{-r|k_j|^{\frac{1}{s}} |k_j|^{\beta_j}} \right) } \right),$$

for some constant $h_1 > 0$. By Lemma 3.6 it follows that

$$e^{-r|k_j|^{\frac{1}{s}} |k_j|^{\beta_j}} \lesssim h^{\beta_j} j_1^{\beta_j}$$

for some constant $h > 0$. A combination of these estimates give

$$|(\partial^\alpha \partial^\beta_x F)(x, \xi)| \lesssim h^{\alpha+\beta} \alpha! \beta! s^s,$$

and it follows that $F \in \mathcal{E}_{\sigma,s}(\mathbb{R}^{2d})$. This shows that $Z_1$ is continuous from $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ to the set of all quasi-periodic elements of order 1 in $\mathcal{E}_{\sigma,s}(\mathbb{R}^{2d})$.

Next we show that any quasi-periodic element $F$ of order 1 in $\mathcal{E}_{\sigma,s}(\mathbb{R}^{2d})$ is the Zak transform of an element in $\mathcal{S}_s^\sigma(\mathbb{R}^d)$. By Theorem 8.2.5 in [25] it follows that $F = Z_1 f$ when

$$f(x) = \int_{Q_{d,2\pi}} F(x, \xi) \, d\xi.$$

We need to prove that $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$.

Since $k \mapsto f(x-k)$ is the Fourier coefficient of order $k$ for the function $\xi \mapsto F(x, \xi)$, we have

$$f(x-k) = \int_{Q_{d,2\pi}} F(x, \xi) e^{-i(k,\xi)} \, d\xi.$$

By applying the operator $k^\alpha \partial^\beta_x$ and integrating by parts we get

$$k^\alpha(\partial^\beta_x f)(x-k) = \int_{Q_{d,2\pi}} (\partial^\beta_x F)(x, \xi) k^\alpha e^{-i(k,\xi)} \, d\xi$$

$$= (-1)^{|\beta|} \alpha! \int_{Q_{d,2\pi}} (\partial^\beta_x \partial^\alpha_x F)(x, \xi) e^{-i(k,\xi)} \, d\xi.$$
This gives
\[
\sup_{x \in \mathbb{R}^d} |x^\alpha f^{(\beta)}(x)| \leq \sup_{k \in \mathbb{Z}^d} \sup_{x \in Q_{d,\rho}} |k^\alpha f^{(\beta)}(x - k)|
\]
\[
\lesssim \|\partial_x^\beta \partial_x^\alpha F\|_{L^\infty(Q_{d,\rho} \times Q_{d,2\rho})} \lesssim h^{\alpha + \beta} \alpha! \beta! \sigma,
\]
which is the same as \( f \in S^\sigma_s(\mathbb{R}^d) \). This gives (2). The assertion (3) follows by similar arguments and is left for the reader. \( \square \)

For completeness we also show that all quasi-periodic distribution are Gelfand-Shilov distributions. (Cf. [28, Section 7.2].)

**Proposition 3.7.** Let \( s, \sigma > 1 \) and \( E \) be an ordered basis of \( \mathbb{R}^d \). Then the following is true:

1. The set of all quasi-periodic elements of order \( E \) in \( \mathcal{D}'(\mathbb{R}^{2d}) \) are contained in \( \mathcal{S}'(\mathbb{R}^{2d}) \);
2. The set of all quasi-periodic elements of order \( E \) in \( \mathcal{D}'_{s,\sigma}(\mathbb{R}^{2d}) \) are contained in \( (S^\sigma_s)(\mathbb{R}^{2d}) \);
3. The set of all quasi-periodic elements of order \( E \) in \( \mathcal{D}'_{0,\sigma,s}(\mathbb{R}^{2d}) \) are contained in \( (\Sigma^\sigma_{s,s})(\mathbb{R}^{2d}) \).

**Proof.** We only prove (2). The other assertions follow by similar arguments and are left for the reader.

Let \( F \in C^\infty(\mathbb{R}^{2d}) \) be quasi-periodic of order \( E \), \( \Phi \in S^\sigma_s(\mathbb{R}^{2d}) \) and let \( \chi \in C^\infty_0(\mathbb{R}^{2d}) \cap E_{s,s}(\mathbb{R}^{2d}) \) be such that
\[
\sum_{k \in \Lambda_E} \sum_{\kappa \in \Lambda'_E} \chi(x + k, \xi + \kappa) = 1.
\]
If \( \Phi \in S^\sigma_s(\mathbb{R}^{2d}) \), then it follows by the quasi-periodicity of \( F \) and some straight-forward computations that
\[
\langle F, \Phi \rangle = \langle F, T_\chi \Phi \rangle,
\]
where
\[
(T_\chi \Phi)(x, \xi) = \sum_{k \in \Lambda_E} \sum_{\kappa \in \Lambda'_E} e^{-i(k, \xi)} \Phi(x - k, \xi - \kappa) \chi(x, \xi),
\]
and that \( T_\chi \) in \( (3.3) \) is continuous from \( S^\sigma_s(\mathbb{R}^{2d}) \) to \( C^\infty_0(\mathbb{R}^{2d}) \cap E_{s,s}(\mathbb{R}^{2d}) \).

Hence by letting \( \langle F, \Phi \rangle \) be defined by the right-hand side of \( (3.3) \) when \( F \in \mathcal{D}'_{s,\sigma}(\mathbb{R}^{2d}) \) and \( \Phi \in S^\sigma_s(\mathbb{R}^{2d}) \), it follows that \( \Phi \mapsto \langle F, T_\chi \Phi \rangle \) in \( (3.3) \) defines a linear and continuous form on \( S^\sigma_s(\mathbb{R}^{2d}) \) which agree with the usual distribution action, \( \Phi \mapsto \langle F, \Phi \rangle \) when \( \Phi \in C^\infty_0(\mathbb{R}^{2d}) \cap E_{s,s}(\mathbb{R}^{2d}) \). \( \square \)
3.2. The Zak transform on Lebesgue and modulation spaces.

When investigating mapping properties of the Zak transform on modulation spaces, we need to deduce various kinds of estimates on short-time Fourier transforms and partial short-time Fourier transforms of Zak transforms. Especially we search suitable estimates on 

\[
(V_\Phi(Z_E f))(x, \xi, \eta) = \left(V_\Phi(Z_E f(\cdot, \xi))(x, \eta)\right)
\]

and

\[
(V_\Phi(Z_{E,\phi}^2 f))(x, \xi, y) = \left(V_{\Phi_2}(Z_E f(x, \cdot))(\xi, y)\right),
\]

which are compositions of the Zak transform and the partial short-time Fourier transforms with respect to the first and second variable, respectively.

From the previous section it is clear that there is a one-to-one correspondence between quasi-periodic functions and distributions, and Zak transforms of functions and distributions. For a quasi-periodic function or distribution \(F\) on \(\mathbb{R}^d\) which satisfies (1.21), and a suitable function or distribution \(\Phi\) on \(\mathbb{R}^d\), we have

\[
\begin{align*}
(V_\Phi F)(x + k, \xi, \eta, y) &= e^{-i(k, \eta)}(V_\Phi F)(x, \xi, \eta, y - k), \quad k \in \Lambda_E, \\
(V_\Phi F)(x, \xi + \kappa, \eta, y) &= e^{-i(y, \kappa)}(V_\Phi F)(x, \xi, \eta, y), \quad \kappa \in \Lambda'_E,
\end{align*}
\]

which follows by straightforward computations. We remark that functions and distributions which satisfy conditions given in (3.7) are special cases of so-called echo-periodic functions and distributions, given in [46].

First we have the following result concerning identifying Lebesgue spaces via estimates of corresponding Zak transforms.

**Theorem 3.8.** Let \(E\) be an ordered basis of \(\mathbb{R}^d\), \(p, r \in (0, \infty]\), \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\), \(\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0\), and let \(f\) be a Gelfand-Shilov distribution on \(\mathbb{R}^d\). Then

\[
\|f\|_{L_p^\omega} \asymp \|G_{E, r, \omega, f}\|_{L_p(\kappa(E) \times \mathbb{R}^d)},
\]

where

\[
G_{E, r, \omega, f}(x, y) \equiv \|(ZV_{E,\phi}^2 f)(x, \cdot, y)\omega(-y)\|_{L_q(\kappa(E'))}.
\]

In particular,

\[
\|f\|_{L_p} \asymp \|ZV_{E,\phi}^2 f\|_{L_p(\kappa(E) \times E' \times \mathbb{R}^d)}.
\]

**Proof.** We only prove the result for \(p < \infty\). The case \(p = \infty\) follows by similar arguments and is left for the reader.
The distribution $\xi \mapsto Z_E f(x, \xi)$ is $E'$-periodic, and it follows from (2.16) that

$$\left( \sum_{j \in \Lambda_E} |f(x - j)\omega(x - j)|^p \right)^\frac{1}{p} \cong \left( \sum_{j \in \Lambda_E} |f(x - j)\omega(-j)|^p \right)^\frac{1}{p} \cong \|G_{E,r,\omega,f}(x, \cdot)\|_{L^p(\mathbb{R}^d)}, \quad x \in \kappa(E).$$

The result now follows by applying the $L^p(\kappa(E))$ quasi-norm with respect to the $x$-variable. □

In the same way we may identify modulation spaces by using the Zak transform as in the next result. Here we let

$$\mathcal{R}\{e_1, \ldots, e_{d+1}, \ldots, e_{2d}\} = \{e_{d+1}, \ldots, e_{2d}, e_1, \ldots, e_d\}, \quad (3.11)$$

We also recall Definition 1.12 and Remark 1.16 for definitions and notions concerning the Wiener amalgam space $W^{p,E}_{p,\omega}(\mathbb{R}^d)$.

**Theorem 3.9.** Let $E, E_0$ be an ordered bases of $\mathbb{R}^d$, $\mathcal{R}$ be as in (3.11), $E_1 = E \times E_0'$, $E_2 = \mathcal{RE}_1$, $p \in (0, \infty]^{2d}$, $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $\omega \in \mathcal{P}_E(\mathbb{R}^{4d})$ be such that

$$\omega(x, \xi, \eta, y) = \omega_0(x - y, \eta).$$

Then $Z_E$ from $\Sigma_1(\mathbb{R}^d)$ to $C^\infty(\mathbb{R}^{2d})$ is uniquely extendable to a homeomorphism from $M^{p,E_1,\omega_0}_{p,\omega}(\mathbb{R}^{2d})$ to the set of quasi-periodic elements of order $E$ in $W^{\infty,p}_{E_2,\omega}(\mathbb{R}^{2d})$, and

$$\|f\|_{M^{p,E_1,\omega_0}_{p,\omega}} \cong \|Z_E f\|_{W^{\infty,p}_{E_2,\omega}}, \quad f \in \Sigma_1(\mathbb{R}^d). \quad (3.12)$$

**Proof.** Since $\omega(x, \xi, \eta, y)$ is constant with respect to the $\xi$-variable, we identify $\omega(x, \xi, \eta, y)$ with $\omega(x, \eta, y)$.

Let $\Phi = \phi_1 \otimes \phi_2$ with $\phi_1, \phi_2 \in \Sigma_1(\mathbb{R}^d) \setminus 0$. By straightforward computations we get

$$(Z_{V^{(1)}_{1,\phi_1}} f)(x, \xi, \eta) = \sum_{j \in \Lambda_E} \left( (V_{\phi_1} f)(x - j, \eta) e^{-i(j, \cdot)} \right) e^{i(j, \xi)}. \quad (3.13)$$

Let

$$p_1 = (p_1, \ldots, p_d) \quad \text{and} \quad p_2 = (p_{d+1}, \ldots, p_{2d})$$

when

$$p = (p_1, \ldots, p_{2d}),$$
and consider the functions
\[ F(x, \eta) = \|(V_{\psi,f})(x - \cdot, \eta)\omega(x, \cdot, \eta)\|_{L^p_0(\Lambda_E)}; \]
\[ g_0(x) = \|F(x, \cdot)\|_{L^p_0(\mathbb{R}^d)} \]
\[ G_0(x, \xi, \eta, y) \equiv |V_\psi(\Lambda_E f)(x, \xi, \eta, y)|, \]
\[ G(x, \xi, \eta) \equiv \|G_0(x, \xi, \eta, \cdot)\|_{L^p_0(\mathbb{R}^d)}, \]
\[ H(x, \xi) = \|G(x, \xi, \cdot)\|_{L^p_0(\mathbb{R}^d)}, \]
and
\[ h_0(x) = h_{0,r_0}(x) = \|H(x, \cdot)\|_{L^p_0(\kappa(E_0))}, \quad r_0 \in (0, \infty]. \]

Since \( \xi \mapsto (Z^{(1)}_{\chi,\psi,f})(x, \xi, \eta) \) is \( E' \)-periodic with Fourier coefficients
\[ j \mapsto (V_{\psi,f})(x - j, \eta)e^{-i(j, \eta)} \]
(cf. \( (3.13) \)), and the (partial) short-time Fourier transform of that distribution equals \( V_\psi(\Lambda_E f) \), it follows from \( (2.16) \) that
\[ F(x, \eta) \asymp \|G(x, \cdot, \eta)\|_{L^p_0(\kappa(E'))}, \quad r \in (0, \infty]. \quad (3.14) \]

First let \( r_0 \leq \min(p) \). If we apply the \( L^p_{E_0} \) norm on \( (3.14) \) with respect to the \( \eta \) variable and using Hölder’s inequality we get
\[ g_0(x) = \|F(x, \cdot)\|_{L^p_{E_0}(\mathbb{R}^d)} \asymp \|G(x, \cdot)\|_{L^p_{E_0}(\kappa(E') \times \mathbb{R}^d)}, \]
\[ \lesssim \|H(x, \cdot)\|_{L^p_{E_0}(\kappa(E'))} = h_{0,r_0}(x). \quad (3.15) \]

If
\[ g_1(x) = \|F_1(x, \cdot)\|_{L^p_{E_0}} \quad \text{with} \quad F_1(x, t) = \|F(x, t)\|_{L^p_{E_0}(\kappa(E))}, \]
then the fact that \( r_0 \leq \min(p) \) and Jensen’s inequality give \( g_1 \lesssim g_0 \).

By applying the \( L^p_{E_0}(\kappa(E)) \) norm on the latter inequality, using the fact that
\[ \omega_0(x - y, \eta) \asymp \omega_0(-y, \eta), \quad x \in \kappa(E) \]
and Jensen’s inequality again we obtain
\[ \|a_{1,f}\|_{L^p_{E_1}} \lesssim \|g_0\|_{L^p(\kappa(E))}, \quad \text{where} \quad a_{1,f}(j, t) = \|V_{\psi,1} f \cdot \omega_0\|_{L^p_{E_1}(\kappa(E))}, \]
That is,
\[ \|V_{\psi,1} f\|_{W^p_{E_0}(\omega_0, L^p_{E_1}(\kappa(E_1)))} \lesssim \|g_0\|_{L^p(\kappa(E))}, \]
which is the same as
\[ \|f\|_{M^p_{E_1}(\omega_0)} \lesssim \|g_0\|_{L^p(\kappa(E))} \quad (3.16) \]
in view of Proposition 1.17.
In order to estimate $h_{0,r_0}$ we apply (3.11) to get

\[ |V_\Phi(Z_E f)(x + j, \xi + \iota, \eta, y)\omega_0(x + j - y, \eta)| = |V_\Phi(Z_E f)(x, \xi, \eta, y - j)\omega_0(x - y + j, \eta)|, \quad (j, \iota) \in \Lambda_{E \times E'}.
\]

By first applying the $L^p_E(\mathbb{R}^d)$ norm with respect to the $y$ variable and then the $L^p_{E_0}(\mathbb{R}^d)$ norm with respect to the $\eta$ variable we get

\[ H(x + j, \xi + \iota) = H(x, \xi), \quad (j, \iota) \in \Lambda_{E \times E'}.
\]

Hence, by applying the $L^p_{E_0}(\kappa(E))$ norm on $h_{0,r_0}$ and using Hölder’s and Jensen’s inequalities we get

\[
\|h_{0,r_0}\|_{L^p_{E_0}(\kappa(E))} = \|H\|_{L^p_{E \times E'}(\kappa(E \times E'))} \leq \|H\|_{L^p_{E \times E'}(\kappa(E \times E'))} = \sup_{(j, \iota) \in \Lambda_{E \times E'}} \left(\|H\|_{L^p_{E \times E'}((j, \iota) + \kappa(E \times E'))}\right), \quad (3.17)
\]

A combination of (3.15)–(3.17) and Proposition 1.17 now gives

\[
\|f\|_{M^p_{E_1}(-\omega_0)} \lesssim \|Z_E f\|_{W^{\infty,p}_{E_2}(-\omega)} = \sum_{\omega' \in \Sigma_r(\mathbb{R}^d)}\|Z_{E'} f\|_{W^{\infty,p}_{E_2}(-\omega')}, \quad f \in \Sigma_r(\mathbb{R}^d).
\] (3.18)

In order to get the reverse estimate we again apply the $L^p_{E_0}$ norm on (3.14) with respect to the $\eta$ variable and use Hölder’s inequality to get

\[ g_0(x) = \|F(x, \cdot)\|_{L^p_{E_0}(\mathbb{R}^d)} \gtrsim \|G(x, \cdot)\|_{L^p_{E_0}(\kappa(E) \times \mathbb{R}^d)} \gtrsim \|H(x, \cdot)\|_{L^p_{E_0}(\kappa(E'))} = h_{0,\infty}(x). \quad (3.19)
\]

If

\[ g_2(x) = \|F_2(x, \cdot)\|_{L^p_{E_0}(\mathbb{R}^d)} \quad \text{with} \quad F_2(x, \iota) = \|F(x, \cdot)\|_{L^p_{E_0}((j, \iota) + \kappa(E_1))},
\]

then Jensen’s inequality give $g_0 \lesssim g_2$.

By applying the $L^p_{E_0}(\kappa(E))$ norm on the latter inequality and using Jensen’s inequality again we obtain

\[ \|a_{2,j}\|_{\rho_{E_1}^p} \gtrsim \|g_0\|_{L^\infty(\kappa(E))}, \quad \text{where} \quad a_{2,j}(j, \iota) = \|V_{0,j} f \cdot \omega_0\|_{L^\infty((j, \iota) + \kappa(E_1))}.
\]

That is,

\[ \|V_{0,j} f\|_{W^{\infty,p}_{E_0}(\omega_0, \rho_{E_1}^p(\Lambda_{E_1}))} \gtrsim \|g_0\|_{L^\infty(\kappa(E))},
\]

which is the same as

\[ \|f\|_{M^p_{E_1}(-\omega_0)} \gtrsim \|g_0\|_{L^\infty(\kappa(E))} \quad (3.20)
\]

in view of Proposition 1.17.

By applying the $L^p_{E_0}(\kappa(E))$ norm on $h_{0,\infty}$ and using (3.17) we get

\[ \|h_{0,\infty}\|_{L^p_{E_0}(\kappa(E))} = \|H\|_{L^p_{E \times E'}(\kappa(E \times E'))} \approx \|Z_E f\|_{W^{\infty,p}_{E_2}(-\omega)}, \quad (3.21)
\]
where the last relation follows from Proposition 1.17. A combination of (3.19), (3.20) and (3.21) now gives
\[ \| f \|_{M_{E_1}^p(\omega_0)} \lesssim \| Z_E f \|_{W_{E_2}^p(\omega)}, \quad f \in \Sigma'_1(\mathbb{R}^d), \quad (3.22) \]
and the result follows by combining (3.22) with \(3.18\).

If \(p, E, E_0, E_1, E_2\) and \(\omega\) are the same as in Theorem 3.9, and \(f \in \Sigma'_1(\mathbb{R}^d)\) and \(\Phi \in \Sigma_1(\mathbb{R}^{2d}) \setminus 0\), then it follows from the echo-periodicity (3.7) that
\[ H_{f,\omega,E,E_0}(x, \xi) \equiv \| (V_\Phi(Z_E f))(x, \xi, \cdot) \omega(x, \xi, \cdot) \|_{L_p^{\kappa(E \times E')}} \quad (3.23) \]
is \(E \times E'\) periodic. The following result now follows from Proposition 1.17, Theorem 3.9 and the previous observation. The details are left for the reader.

**Theorem 3.10.** Let \(p, E, E_0, E_1, E_2, \omega, \omega_0\) and \(H_{f,\omega,E,E_0}\) be the same as in Theorem 3.9 and (3.23), and let \(f \in \Sigma'_1(\mathbb{R}^d), \Phi \in \Sigma_1(\mathbb{R}^{2d}) \setminus 0\) and \(r \in (0, \infty)^{2d}\). Then
\[ f \in M_{E_1}^p(\omega_0)(\mathbb{R}^d) \iff H_{f,\omega,E,E_0} \in L_r^{\kappa(E \times E')} \quad (3.24) \]
and
\[ \| f \|_{M_{E_1}^p(\omega_0)} \asymp \| H_{f,\omega,E,E_0} \|_{L_r^{\kappa(E \times E')}} \quad (3.25) \]

As a special case of the previous result we have the following.

**Corollary 3.11.** Let \(E\) and \(\Phi\) be the same as in Theorem 3.10, and let \(p \in (0, \infty)\). Then
\[ \| f \|_{M^p} \asymp \| (V_\Phi(Z_E f)) \|_{L_p^{(\kappa(E \times E') \times \mathbb{R}^{2d})}}, \quad f \in \Sigma'_1(\mathbb{R}^d). \]

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