FAITHFULNESS OF SIMPLE 2-REPRESENTATIONS OF $\mathfrak{sl}_2$

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ABSTRACT. Let $\mathcal{U}$ be the 2-category associated with $\mathfrak{sl}_2$. We prove that a complex of 1-morphisms of $\mathcal{U}$ is null-homotopic if and only if its image in every simple 2-representation is null-homotopic. Under mild boundedness assumptions, we prove that it actually suffices for the image in the simple 2-representations to be acyclic. We apply this result to the study of the Rickard complex $\Theta$ categorifying the action of the simple reflection of $\text{SL}_2$. We prove that $\Theta$ is invertible in the homotopy category of $\mathcal{U}$, and that there is a homotopy equivalence $\Theta E \simeq F \Theta[-1]$.

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1. INTRODUCTION

Consider the Lie algebra $\mathfrak{sl}_2$ of traceless complex $2 \times 2$ matrices, and $\mathcal{U}$ its enveloping algebra or associated quantum group. The category of finite dimensional $\mathcal{U}$-modules is well understood. It is semi-simple and its simple objects are the $L(n)$ for $n \in \mathbb{N}$, where $L(n)$ denotes the simple $\mathcal{U}$-module of dimension $n + 1$. A well-known result is that the collection $(L(n))_{n \in \mathbb{N}}$ of $\mathcal{U}$-modules is faithful. That is, if an element $z$ of $\mathcal{U}$ acts by zero on $L(n)$ for all $n \in \mathbb{N}$, then $z = 0$. In this paper, we prove a categorification of this result.

The current theory of categorifications of quantized enveloping algebras and their representations began with the seminal work of Chuang and Rouquier [CR08]. In their paper, Chuang-Rouquier introduce a notion of $\mathfrak{sl}_2$-categorification, or action of $\mathfrak{sl}_2$ on a category. Roughly speaking, an $\mathfrak{sl}_2$-categorification is the data of a category $\mathcal{V}$, an adjoint pair $(E, F)$ of endofunctors of $\mathcal{V}$ and some natural transformations subject to certain relations. These relations imply in particular that the Grothendieck group of $\mathcal{V}$ inherits an action of $\mathfrak{sl}_2$, with the functors $E$ and $F$ corresponding to the matrices $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. The finite-dimensional simple $\mathcal{U}$-modules admit categorical analogues in this context, called minimal categorifications. Unlike the representation theory of $\mathfrak{sl}_2$, the resulting theory is not semi-simple. However, minimal categorifications do form the elementary building blocks of all $\mathfrak{sl}_2$-categorifications, in the sense that every $\mathfrak{sl}_2$-categorification admits a filtration whose quotients are minimal categorifications. This notably enables Chuang-Rouquier to prove strong structural results about $\mathfrak{sl}_2$-categorifications, with deep consequences such as a proof of Broué’s abelian defect group conjecture for symmetric groups.

This theory was later expanded in [KL11], [KL10], [Lau10], [Rou08], [Rou12]. These papers define a 2-category $\mathcal{U}$ associated with $\mathfrak{sl}_2$ (or more generally associated with any Kac-Moody algebra, but we will only work in type $A_1$ in this paper). This 2-category categorifies the integral idempotent form of $\mathcal{U}$. The $\mathfrak{sl}_2$-categorifications from [CR08] are precisely the 2-representations of $\mathcal{U}$, that is 2-functors from $\mathcal{U}$ to the 2-category of categories. Note that the definitions of Khovanov-Lauda and Rouquier, despite being different, are actually equivalent by work of Brundan [Bru16].

The minimal categorifications from [CR08] have an additive version introduced in [Rou08], called simple 2-representations and denoted $\mathcal{L}(n)$ for $n \in \mathbb{N}$. They form the building blocks of integrable 2-representations of $\mathcal{U}$. For instance, a result of particular interest is [Rou08, Lemma 5.17], which says
that if a complex of 1-morphisms of \( \mathcal{U} \) is null-homotopic in all simple 2-representations, then it is null-homotopic in all integrable 2-representations. Our main theorem is an extension of this result, showing that we can lift the null-homotopy to the 2-category \( \mathcal{U} \) itself.

**Theorem 1.1** (Theorem 4.3).  
(1) Let \( C \) be a complex of 1-morphisms of \( \mathcal{U} \). If the image of \( C \) in \( \mathcal{L}(n) \) is null-homotopic for all \( n \in \mathbb{N} \), then \( C \) is null-homotopic.

(2) Let \( f \) be a morphism between complexes of 1-morphisms of \( \mathcal{U} \). If the image of \( f \) in \( \mathcal{L}(n) \) is a homotopy equivalence for all \( n \in \mathbb{N} \), then \( f \) is a homotopy equivalence.

Note that the second point results from the first, by considering the cone of \( f \). We now give a short overview of the proof of this Theorem. Our proof relies on Lauda’s determination of the indecomposable 1-morphisms of \( \mathcal{U} \), as well as bases for 2-morphism spaces [Lau10]. We prove that the image of an indecomposable 1-morphism of \( \mathcal{U} \) in \( \mathcal{L}(n) \) is indecomposable if \( n \) is large enough and of suitable parity. We also prove that the images of two non-isomorphic indecomposable 1-morphisms of \( \mathcal{U} \) in \( \mathcal{L}(n) \) are non-isomorphic if \( n \) is large enough and of suitable parity. Once these results are established, Theorem 1.1 follows from general facts about Krull-Schmidt categories.

Our approach is similar to that of [Lau10], which uses 2-representations of \( \mathcal{U} \) coming from the geometry of Grassmannians to determine the indecomposable 1-morphisms and bases of 2-morphism spaces of \( \mathcal{U} \). Another related work is [BL14], which proves that \( \mathcal{U} \) can be realized as an inverse limit of some integrable 2-representations. In particular, this shows that \( \mathcal{U} \) is determined by its integrable 2-representation theory.

Using results and methods of [CR08], we can prove that under mild boundedness assumptions, it suffices for the image of a complex of 1-morphisms of \( \mathcal{U} \) in the simple 2-representations to be acyclic to conclude that the complex is null-homotopic.

**Theorem 1.2** (Theorem 4.14).  
(1) Let \( C \) be a complex of 1-morphisms of \( \mathcal{U} \) such that for any object \( M \) of an integrable 2-representation, the complex \( C(M) \) is bounded. If the image of \( C \) in \( \mathcal{L}(n) \) is acyclic for all \( n \in \mathbb{N} \), then \( C \) is null-homotopic.

(2) Let \( f \) be a morphism between complexes of 1-morphisms of \( \mathcal{U} \) that both satisfy the boundedness assumption of (1). If the image of \( f \) in \( \mathcal{L}(n) \) is a quasi-isomorphism for all \( n \in \mathbb{N} \), then \( f \) is a homotopy equivalence.

Categorifications of simple modules exist for all Kac-Moody algebras, see [Rou08] for an abstract approach, and [KK12], [Web17] for a concrete approach using cyclotomic quiver Hecke algebras. It would be interesting to generalize Theorems 1.1 and 1.2 to arbitrary types. Unfortunately, our proof does not carry over since we do not have an explicit form for the indecomposable 1-morphisms of \( \mathcal{U} \) in general. It is known that the indecomposable 1-morphism of \( \mathcal{U} \) decategorify to the canonical basis of \( \mathcal{U} \) (see [Rou12], [VV11]), but the canonical basis does not have an explicit expression in general.

Our main motivation for Theorems 1.1 and 1.2 is that performing explicit computations in the homotopy category of \( \mathcal{U} \) is in general difficult. Our results provide a strategy to approach such computations. The simple 2-representations have an explicit form: they are categories of free modules over certain polynomial rings. In such a setting, proving that a complex is acyclic is a reasonable task in general.

We give a concrete example of this strategy to study the Rickard complex \( \Theta \). To understand our results, let us first recall the decategorified picture. The simple reflection \( s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) of \( SL_2 \) acts on integrable representations of \( \mathcal{U} \), providing isomorphisms between weight spaces of opposite weights. The action of \( s \) and the Chevalley generators \( e, f \) of \( \mathcal{U} \) are related by the relation \( se = -fs \). The Rickard complex \( \Theta \) is a complex of 1-morphisms of \( \mathcal{U} \) which decategorifies to \( s \). Chuang and Rouquier proved in [CR08] that \( \Theta \) provides derived equivalences on integrable 2-representations of \( \mathcal{U} \). The complex \( \Theta \) was also studied in [CK12], [CKL15], with a more geometric framework for 2-representations. In this paper, we prove a categorification of the fact that \( s \) is invertible and of the relation \( se = -fs \).

**Theorem 1.3** (Theorems 5.13 and 5.17). The complex \( \Theta \) is invertible up to homotopy and there is a homotopy equivalence \( \Theta E \simeq F \Theta [-1] \).

Here, \([-1]\) denotes a homological grading shift. To prove these results, we use Theorem 1.2. More precisely, we define a morphism of complexes \( \Theta E \to F \Theta [-1] \), and we check that it is a quasi-isomorphism in every simple 2-representation. Along the way, we revisit some of the results of [CR08] regarding \( \Theta \), in the setup of simple 2-representations rather than minimal categorifications. Namely, we prove that in every simple 2-representation, the cohomology of \( \Theta \) is concentrated in top degree, and that the top cohomology is invertible. The fact that the top cohomology is invertible implies, thanks to Theorem 1.2, that
\( \Theta \) is invertible up to homotopy. Our approach to these results is slightly different from that of Chuang and Rouquier, and is based on constructing explicit bases for the terms of \( \Theta \) on simple 2-representations.

We now give an overview of the structure of our paper. Section 2 introduces the notation used throughout the paper, and Section 3 defines the 2-category \( \mathcal{U} \) and its 2-representations, and gives some of its properties, following [Rou08] and [Lau10]. In Section 4, we recall the definition of the simple 2-representations, and we prove Theorems 1.1 and 1.2. Finally, Section 5 is devoted to the application of our results to the study of the Rickard complex and the proof of Theorem 1.3

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2. Notation and definitions

For the rest of this paper, we fix a field \( K \). We will use the symbol \( \otimes \) to mean \( \otimes_K \).

**Graded categories.** A graded category is a \( K \)-linear category \( \mathcal{C} \) together with an auto-equivalence \( \mathcal{C} \to \mathcal{C} \) called the shift functor. We will use the notation \( M \mapsto qM \) to denote the shift functor on objects. For instance, the category \( \text{Vect}_K \) of graded \( K \)-vector spaces with homogeneous linear maps is graded. In that case the shift functor is as follows: for \( V = \bigoplus_{k \in \mathbb{Z}} V_k \in \text{Vect}_K \), \( qV \) is the graded \( K \)-vector space defined by \( (qV)_k = V_{k-1} \). More generally given a graded \( K \)-algebra \( A \), the category \( A\text{-mod} \) of finitely generated graded modules and its full subcategory \( A\text{-proj} \) of graded projective modules are graded. Their shift functor is the restriction of that of \( \text{Vect}_K \). Given a Laurent polynomial \( p = \sum_{i \in \mathbb{Z}} p_iq^i \in \mathbb{N}[q, q^{-1}] \) and \( M \) an object of a graded category \( \mathcal{C} \), we put

\[ pM = \bigoplus_{i \in \mathbb{Z}} q^i M^{|p_i|}. \]

For two objects \( M, N \) of a graded category \( \mathcal{C} \) and \( k \in \mathbb{Z} \), we define \( \text{Hom}^k_{\mathcal{C}}(M, N) = \text{Hom}_{\mathcal{C}}(M, q^{-k}N) \). The elements of \( \text{Hom}^k_{\mathcal{C}}(M, N) \) are said to be morphisms of degree \( k \) from \( M \) to \( N \). We can then define a graded \( K \)-vector space \( \text{Hom}^\bullet_{\mathcal{C}}(M, N) \) by

\[ \text{Hom}^\bullet_{\mathcal{C}}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}^k_{\mathcal{C}}(M, N). \]

We have isomorphisms of graded vector spaces \( \text{Hom}^\bullet_{\mathcal{C}}(M, qN) \simeq q \text{Hom}^\bullet_{\mathcal{C}}(M, N) \) and \( \text{Hom}^\bullet_{\mathcal{C}}(qM, N) \simeq q^{-1} \text{Hom}^\bullet_{\mathcal{C}}(M, N) \).

**Graded dimensions.** A graded \( K \)-vector space \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) is said to be locally finite if each \( V_k \) is finite dimensional. In this case, its graded dimension is the formal series defined by

\[ \text{grdim}(V) = \sum_{k \in \mathbb{Z}} \text{dim}(V_k)q^k \in \mathbb{N}((q, q^{-1})). \]

We have \( \text{grdim}(qV) = q \text{grdim}(V) \). If \( M, N \) are two objects of a graded category \( \mathcal{C} \) and \( \text{Hom}^\bullet_{\mathcal{C}}(M, N) \) is locally finite, we put

\[ \langle M, N \rangle = \text{grdim}( \text{Hom}^\bullet_{\mathcal{C}}(M, N) ). \]

Then we have \( \langle M, qN \rangle = q \langle M, N \rangle \) and \( \langle qM, N \rangle = q^{-1} \langle M, N \rangle \). If \( M = N \), we put \( |M| = \langle M, M \rangle \).

**Krull-Schmidt categories.** A \( K \)-linear category \( \mathcal{C} \) is called Krull-Schmidt if every object of \( \mathcal{C} \) is isomorphic to a finite direct sum of objects having local endomorphism algebras. If \( C \) is Krull-Schmidt, then the indecomposable objects are precisely the objects having local endomorphism algebras, and every object decomposes uniquely as a finite direct sum of indecomposable objects up to reordering of the terms. We say that a 2-category \( \mathcal{C} \) is Krull-Schmidt if for all objects \( \lambda, \mu \) of \( \mathcal{C} \) the category \( \text{End}_{\mathcal{C}}(\lambda, \mu) \) is Krull-Schmidt. In other words, \( \mathcal{C} \) is Krull-Schmidt if every 1-morphism of \( \mathcal{C} \) decomposes as a finite direct sum of 1-morphisms having local endomorphism algebras.
Homological algebra. Let $C$ be a $K$-linear category. We denote by $\text{Comp}(C)$ the category of (cochain) complexes of $C$. That is, the objects of $\text{Comp}(C)$ are of the form

$$M = (\cdots \to M^r \xrightarrow{d^r_M} M^{r+1} \xrightarrow{d^{r+1}_M} M^{r+2} \to \cdots)$$

with $d^{r+1}_M d^r_M = 0$ for all $r \in \mathbb{Z}$. If $M \in \text{Comp}(C)$, its homological shift $M[1]$ is defined by $(M[1])^r = M^{-r}$ with differential $d^r_{M[1]} = -d^r_M$ for all $r \in \mathbb{Z}$. Note that when $C$ is graded, $\text{Comp}(C)$ has two compatible gradings: the one coming from $C$ and the homological grading.

Given $M, N \in \text{Comp}(C)$ and $f : M \to N$ a morphism of complexes, the cone of $f$ is the object $\text{Cone}(f)$ of $\text{Comp}(C)$ defined by $\text{Cone}(f)^r = M^{r+1} \oplus N^r$ with differential given by

$$d^r_{\text{Cone}(f)} = \begin{bmatrix} -d^{r+1}_M & 0 \\ f & d^r_N \end{bmatrix}.$$

We denote by $K(C)$ the homotopy category of $C$. If $C$ is furthermore abelian, we denote by $D(C)$ its derived category. In that case, we denote by $H^n(M)$ the $n$th cohomology object of a complex $M$.

3. CATEGORIZED QUANTUM $\mathfrak{sl}_2$

3.1. Affine nil Hecke algebras. We start by recalling the essential facts about affine nil Hecke algebras. Let $\mathfrak{S}_n$ be the symmetric group of $n$ letters. It is generated by $s_1, \ldots, s_n$, where $s_i$ denotes the transposition $(i \ i+1)$. The length of an element $\omega \in \mathfrak{S}_n$ is the smallest integer $r$ such that $\omega = s_{i_1} \cdots s_{i_r}$ for some $i_1, \ldots, i_r \in \{1, \ldots, n-1\}$. We denote the length of $\omega$ by $l(\omega)$. For $k \leq \ell$ two integers in $\{1, \ldots, n\}$, we denote by $\mathfrak{S}_{[k,\ell]}$ the subgroup of $\mathfrak{S}_n$ generated by $\{s_i, i \in \{k, \ldots, \ell-1\}\}$. The longest element of $\mathfrak{S}_{[k,\ell]}$ is denoted $\omega_0[n,\ell]$. If $a, b \in \mathbb{N}$ are such that $a + b = n$, we will denote the subgroup $\mathfrak{S}_{[1,a]} \times \mathfrak{S}_{[a+1,n]}$ of $\mathfrak{S}_n$ by $\mathfrak{S}_a \times \mathfrak{S}_b$.

**Definition 3.1.** The affine nil Hecke algebra $H_n$ is the unital $K$-algebra on the generators $x_1, \ldots, x_n$ and $\tau_1, \ldots, \tau_{n-1}$ subject to the following relations for all $i, j \in \{1, \ldots, n\}$ and $k, \ell \in \{1, \ldots, n-1\}$:

1. $x_i x_j = x_j x_i$,
2. $x_i^2 = 0$,
3. $\tau_k x_i - x_{s_k(i)} \tau_k = \delta_{i,k+1} - \delta_{i,k}$,
4. $\tau_k \tau_\ell = \tau_\ell \tau_k$ if $|k - \ell| > 1$,
5. $\tau_{k+1} \tau_k \tau_{k+1} = \tau_k \tau_{k+1} \tau_k$ if $k < n - 1$.

We fix a grading on $H_n$ with $x_1, \ldots, x_n$ in degree 2 and $\tau_1, \ldots, \tau_{n-1}$ in degree $-2$.

Let $P_n = K[x_1, \ldots, x_n]$. We put a grading on $P_n$ with $x_1, \ldots, x_n$ in degree 2. The symmetric group $\mathfrak{S}_n$ acts on $P_n$ by permuting $x_1, \ldots, x_n$. The Demazure operators are the $P_n^{\mathfrak{S}_n}$-linear operators $\partial_1, \ldots, \partial_{n-1}$ on $P_n$ defined by

$$\partial_i(P) = \frac{P - s_i(P)}{x_{i+1} - x_i}.$$

There is an isomorphism of graded $K$-algebras (see [Rou08, Proposition 3.4])

$$\begin{align*}
H_n &\to \text{End}_{P_n^{\mathfrak{S}_n}}(P_n), \\
\tau_k &\mapsto \partial_k, \\
x_k &\mapsto \text{multiplication by } x_k.
\end{align*}$$

(3.1)

Given $\omega = s_{i_1} \cdots s_{i_r}$ with $l(\omega) = r$, the element $\tau_{i_1} \cdots \tau_{i_r} \in H_n$ only depends on $\omega$ and is denoted by $\tau_\omega$.

We define similarly $\partial_\omega = \partial_{i_1} \cdots \partial_{i_r}$.

As a $P_n^{\mathfrak{S}_n}$-module, $P_n$ is free of rank $n!$. The following sets are bases

$$\{x_1^{a_1} \cdots x_n^{a_n}, 0 \leq a_i \leq n - i \},$$

$$\{\partial_\omega (x_2 x_3^2 \cdots x_n^{n-1}), \omega \in \mathfrak{S}_n\}.$$

In particular, $H_n$ is isomorphic the algebra of $(n!) \times (n!)$ matrices with coefficients in $P_n^{\mathfrak{S}_n}$. The $H_n$-module $P_n$ with action given by (3.1) is the unique indecomposable graded projective $H_n$-module up to
isomorphism. Define
\[
e_n = x_2 x_3^2 \ldots x_n^{n-1} \tau_{\omega_0[1,n]},
\]
\[
e_n' = (-1)^{\frac{a(n-1)}{2}} \tau_{\omega_0[1,n]} x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.
\]

These are orthogonal primitive idempotents of \( H_n \). We have isomorphisms of graded \((H_n, P_n^{\mathbb{S}^n})\)-bimodules:
\[
\begin{cases}
P_n \xrightarrow{\sim} q^n \tau_{\omega_0[1,n]}, \quad \text{and} \quad P_n \xrightarrow{\sim} \tau_{\omega_0[1,n]} c_n,
\end{cases}
\]

Hence we have Morita equivalences
\[
\begin{cases}
H_n - \text{mod} \xrightarrow{\sim} P_n^{\mathbb{S}^n} - \text{mod}, \quad \text{and} \quad H_n - \text{mod} \xrightarrow{\sim} c_n - \text{mod}.
\end{cases}
\]

### 3.2. The 2-category \( \mathcal{U} \).

#### 3.2.1. Definitions.
We now define the 2-category associated with \( \mathfrak{sl}_2 \), following \cite{Rou08}. For \( k \in \mathbb{Z} \), we define the quantum integer \([k]\) by
\[
[k] = \frac{q^k - q^{-k}}{q - q^{-1}}.
\]
If \( k, n \in \mathbb{N} \) and \( k \leq n \), we put
\[
[k]! = \prod_{\ell=1}^{k} [\ell], \quad \binom{n}{k} = \frac{[n]!}{[k]![n-k]!}.
\]

The 2-category \( \mathcal{U} \) will be defined as the idempotent completion of a strict 2-category \( \mathcal{U}' \) that we start by defining.

**Definition 3.2.** Define a strict 2-category \( \mathcal{U}' \) by the following data.

- The set of objects of \( \mathcal{U}' \) is \( \mathbb{Z} \).
- Given \( \lambda, \lambda' \in \mathbb{Z} \), the 1-morphisms \( \lambda \to \lambda' \) are direct sums of shifts of words of the form \( E^{n_1} F^{m_1} \ldots E^{n_r} F^{m_r} \), with \( n_i, m_i \in \mathbb{N} \) and \( \lambda' - \lambda = 2 \sum (n_i - m_i) \). Composition of 1-morphisms is given by concatenation of words, and the identity 1-morphism of \( \lambda \) is the empty word, denoted \( 1_{\lambda} \). When we want to specify that a word \( X \) has source \( \lambda \) (resp. target \( \lambda' \)), we will write \( \lambda X \) (resp. \( X \lambda \)).
- The 2-morphisms of \( \mathcal{U}' \) are generated by \( x : F \to F \) of degree 2, \( \tau : F^2 \to F^2 \) of degree \(-2\), \( \eta : 1_{\lambda} \to FE1_{\lambda} \) of degree 1 + \( \lambda \), and \( \varepsilon : EF1_{\lambda} \to 1_{\lambda} \) of degree \( 1 - \lambda \), for all \( \lambda \in \mathbb{Z} \). On these generating 2-morphisms, we impose the following relations:
  1. \( \tau^2 = 0 \),
  2. \( \tau \circ Fx - xF \circ \tau = FX \circ \tau - \tau \circ FX = F^2 \),
  3. \( \tau F \circ \tau F = F \circ \tau F \circ \tau F \),
  4. \( F = FE \circ \eta F \) and \( E = E F \circ E \eta \),
  5. for all \( \lambda \in \mathbb{Z} \), \( \rho_\lambda \) is invertible, where \( \rho_\lambda \) is the 2-morphism defined by

\[
\rho_\lambda = \left\{ \begin{array}{ll}
\sigma & : EF1_{\lambda} \to FE1_{\lambda} \oplus [\lambda] 1_{\lambda} \quad \text{if } \lambda \geq 0, \\
\varepsilon & \\
\varepsilon \circ Ex & \\
\vdots & \\
\varepsilon \circ Ex^{\lambda-1} & \\
\end{array} \right.
\]

with \( \sigma = \varepsilon FE \circ E \tau E \circ EF \eta : FE \to FE \).
Let us give some more comments on this definition. Relations (1), (2) and (3) imply that the affine nil Hecke algebra $H_n$ acts on $F^n$, as follows

\[
\begin{align*}
H_n & \to \text{End}_{U'}(F^n), \\
x_k & \mapsto F^{k-1}x_{\lambda^a-k}, \\
t_k & \mapsto F^{k-1}t_{\lambda^a-k}. 
\end{align*}
\]

Relation (4) says that for all $\lambda \in \mathbb{Z}$, we have adjoint pairs $(E_{1,\lambda}, q^{-1-\lambda}F_{1,\lambda+2})$ with unit and counit of adjunction given by $\eta$ and $\epsilon$. Finally, relation (5) means that there are additional generating 2-morphisms in $U'$, defined to be the inverses of the maps $\rho\lambda$.

**Definition 3.3.** The 2-category $U$ is the idempotent completion of $U'$. This means that $U$ has the same set of objects as $U'$ and that for all $\lambda, \lambda' \in \mathbb{Z}$, the category $\text{Hom}_U(\lambda, \lambda')$ is the idempotent completion of the category $\text{Hom}_{U'}(\lambda, \lambda')$.

In particular, we can define divided powers $F^{(n)}$ and $E^{(n)}$ in $U$ using idempotents 2-morphisms. More precisely, using the idempotent $e_n \in H_n$, we define

\[F^{(n)} = q^{-\frac{n(n-1)}{2}} e_n(F^n).\]

Then we have an isomorphism $F^n \simeq [n]!F^{(n)}$. Furthermore for $a, b \in \mathbb{N}$ we have

\[F^{(a)}F^{(b)} \simeq \left[ \frac{a+b}{a} \right] F^{(a+b)}.\]

Similarly, the adjunction between $E$ and $F$ provides an action of $H_n^{\text{opp}}$ on $E^n$. Using the idempotent $e'_n \in H_n$, we define

\[E^{(n)} = q^{-\frac{n(n-1)}{2}} (E^n)e'_n,\]

where we have written the action of $H_n^{\text{opp}}$ on $E^n$ as a right action of $H_n$. Then there are adjoint pairs $(E^{(n)}1_{\lambda'}, q^{-n(\lambda+n)}F^{(n)}1_{\lambda+2n})$.

Given $\lambda \in \mathbb{Z}$ we define

\[U1_{\lambda} = \bigoplus_{\Lambda' \in \mathbb{Z}} \text{Hom}_U(\lambda, \lambda').\]

It is an additive and idempotent complete category. In $U1_{\lambda}$ there are isomorphisms

\[
E^{(a)}E^{(b)}1_{\lambda} \simeq \bigoplus_{i=0}^{\min\{a,b\}} \left[ \lambda+i-a-b \right] E^{(b-i)}E^{(a-i)}1_{\lambda} \quad \text{if } \lambda \geq b-a, \tag{3.4}
\]

\[
E^{(b)}F^{(a)}1_{\lambda} \simeq \bigoplus_{i=0}^{\min\{a,b\}} \left[ -\lambda+i-a+b \right] E^{(a-i)}F^{(b-i)}1_{\lambda} \quad \text{if } \lambda \leq b-a. \tag{3.5}
\]

They can be constructed using the isomorphisms $\rho\lambda$ (see [Rou08, Lemma 4.14] for details).

### 3.2.2. Indecomposable 1-morphisms.

In [Lau10], Lauda defines a different 2-category. The main differences are that Lauda’s version requires an adjoint pair $(q^{\lambda+1}F_{1,\lambda+2}, E_{1,\lambda})$ and the inverses of the maps $\rho\lambda$ are given explicitly in terms of the unit and counit of this second adjunction. However, Brundan proved in [Bru16] that the 2-categories of Lauda and Rouquier are isomorphic. In particular, the results proved in [Lau10] remain valid in $U$. The main result of interest for this paper is the determination of the indecomposable 1-morphisms of $U$.

**Theorem 3.4 ([Lau10]).** The 2-category $U$ is Krull-Schmidt. A complete set of pairwise non-isomorphic indecomposable 1-morphisms of $U$ is given by

\[
\left\{ q^a E^{(a)}F^{(b)}1_{\lambda}, \lambda \leq b-a, s \in \mathbb{Z} \right\} \cup \left\{ q^a E^{(b)}E^{(a)}1_{\lambda}, \lambda > b-a, s \in \mathbb{Z} \right\}.
\]

Furthermore, if $X$ is an indecomposable 1-morphism of $U$ we have $|X| \in 1 + q\mathbb{N}[q]$. 


Given 2-representations $\mathcal{V}, \mathcal{W}$, a morphism of 2-representations $D : \mathcal{V} \to \mathcal{W}$ is the data of:

- functors $D : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ for all $\lambda \in \mathbb{Z}$,
- natural isomorphisms $\alpha : DE \cong ED$ and $\beta : DF \cong FD$,

such that $\alpha$ and $\beta$ are compatible with the 2-morphisms $x, \tau, \epsilon$ of $\mathcal{U}$. This compatibility condition means that the following diagrams commute:

\[
\begin{array}{ccc}
DF & \xrightarrow{Dx} & DF \\
\downarrow{\beta} & & \downarrow{\beta} \\
FD & \xrightarrow{sD} & FD
\end{array}
\quad
\begin{array}{ccc}
DF^2 & \xrightarrow{DF^2} & DF^2 \\
\downarrow{\tau} & & \downarrow{\tau} \\
FD^2 & \xrightarrow{\tau D} & DF^2
\end{array}
\quad
\begin{array}{ccc}
DEF & \xrightarrow{Dx} & D \\
\downarrow{\epsilon D} & & \downarrow{\epsilon D} \\
EFD & \xrightarrow{\epsilon D} & DFED
\end{array}
\]

Then, given $D$ a morphism of 2-representations and $\mathcal{C}$ a complex of 1-morphisms of $\mathcal{U}$, we have a canonical isomorphism of complexes of functors $DC \simeq CD$.

The following computational result will be useful. It is a slightly different presentation of [Lau10, Proposition 9.8].

**Proposition 3.5.** Let $\mathcal{V}$ be a 2-representation of $\mathcal{U}$ and denote by $\mathcal{W}$ the category of endofunctors of $\mathcal{V}$. Let $\lambda \in \mathbb{Z}$ and denote by $\Phi : U_{1\lambda} \to \mathcal{W}$ the functor induced by the structure of 2-representation on $\mathcal{V}$. Let $a, b, c, d \in \mathbb{N}$ be such that $a - b = c - d$. If $\lambda \geq b - a$, there is an isomorphism

\[
\text{Hom}_W^*\left(\Phi \left(\begin{array}{c}
E(b)E(a)1_\lambda
\end{array}\right), \Phi \left(\begin{array}{c}
E(d)E(c)1_\lambda
\end{array}\right)\right) \simeq \\
\bigoplus_{i=0}^{\min(a,c)} q^{(a+c-i)(\lambda+a+c-i)} \left[\begin{array}{c}
\lambda+a+c-i
\end{array}\right] \left[\begin{array}{c}
b+i
\end{array}\right] \left[\begin{array}{c}
a+d-i
\end{array}\right] \text{End}_W^* \left(\Phi \left(\begin{array}{c}
E(d-a)^{a+d-1}\lambda+2(a+c-i)
\end{array}\right)\right).
\]

If $\lambda \leq b - a$, there is an isomorphism

\[
\text{Hom}_W^*\left(\Phi \left(\begin{array}{c}
E(a)E(b)1_\lambda
\end{array}\right), \Phi \left(\begin{array}{c}
E(c)E(d)1_\lambda
\end{array}\right)\right) \simeq \\
\bigoplus_{i=0}^{\min(b,d)} q^{(b+d-i)(b+d-i-\lambda)} \left[\begin{array}{c}
b+d-i-\lambda
\end{array}\right] \left[\begin{array}{c}
b+d-i
\end{array}\right] \left[\begin{array}{c}
\lambda-2(b+d-i)
\end{array}\right] \text{End}_W^* \left(\Phi \left(\begin{array}{c}
E(d-a)^{a+d-1}\lambda-2(b+d-i)
\end{array}\right)\right).
\]

**Proof.** We prove the first isomorphism, the second one being proved similarly. There is an adjoint pair $(q^{a(\lambda+a)}E(a)1_{\lambda+2a}, E(a)1_{\lambda})$ providing an isomorphism

\[
\text{Hom}_W^*\left(\begin{array}{c}
E(b)E(a)1_\lambda
\end{array}, \begin{array}{c}
E(d)E(c)1_\lambda
\end{array}\right) \simeq q^{a(\lambda+a)} \text{Hom}_W^* \left(\begin{array}{c}
E(b)^{a+2a}1_{\lambda+2a}, \begin{array}{c}
E(d)E(c)1_\lambda
\end{array}\right).
\]

3.2. **2-representations.** A 2-representation of $\mathcal{U}$ is a strict 2-functor $\mathcal{U} \to \mathcal{L}_{\mathbb{K}}$, where $\mathcal{L}_{\mathbb{K}}$ denotes the strict 2-category of $K$-linear, graded and idempotent complete categories. More explicitly, a 2-representation $\mathcal{V}$ of $\mathcal{U}$ is the data of:

- categories $\mathcal{V}_\lambda \in \mathcal{L}_{\mathbb{K}}$ for each $\lambda \in \mathbb{Z}$,
- functors $E : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+2}$ and $F : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-2}$ for each $\lambda \in \mathbb{Z}$,
- natural transformations $x : F \to F$ of degree 2, $\tau : F^2 \to F^2$ of degree $-2$, $\epsilon : EF1_\lambda \to 1_\lambda$ of degree $1 - \lambda$, and $\eta : 1_\lambda \to FE1_\lambda$ of degree $1 + \lambda$,

such that the relations of Definition 3.2 are satisfied. An abelian 2-representation is a 2-representation $\mathcal{V}$ such that for all $\lambda \in \mathbb{Z}$, the category $\mathcal{V}_\lambda$ is abelian. Note that since the functors $E, F$ are both left and right adjoints, they are exact on $\mathcal{V}$. A 2-representation $\mathcal{V}$ is said to be integrable if for every $M \in \mathcal{V}$, we have $E_i(M) = F_i(M) = 0$ for some $i \in \mathbb{N}$.
Assume $\lambda \geq b - a = d - c$. In particular, we have $\lambda + a + c \geq 0$. Thus by (3.4) we have an isomorphism

$$E^{(c)}E^{(a)}1_{\lambda + 2a} \cong \bigoplus_{i=0}^{\min\{a,c\}} [\lambda + a + c] E^{(a-i)}E^{(c-i)}1_{\lambda + 2a}.$$  

Using the adjoint pair $\left( E^{(c-i)}1_{\lambda + 2a}, q^{-1}(c-i)(\lambda + 2a + c - i)E^{(c-i)}1_{\lambda + 2(a+c-i)} \right)$, we deduce that there is an isomorphism

$$\text{Hom}^*_W \left( F^{(b)}E^{(d)}1_\lambda, F^{(d)}E^{(c)}1_\lambda \right) \cong \bigoplus_{i=0}^{\min\{a,c\}} q^{(a+c-i)(\lambda + a + c - i)} \text{Hom}^*_W \left( F^{(b)}E^{(c-i)}1_{\lambda + 2(a+c-i)}, F^{(d)}E^{(d-i)}1_{\lambda + 2(a+c-i)} \right).$$

Using the isomorphisms (3.3), we conclude that

$$\text{Hom}^*_W \left( F^{(b)}E^{(d)}1_\lambda, F^{(d)}E^{(c)}1_\lambda \right) \cong \bigoplus_{i=0}^{\min\{a,c\}} q^{(a+c-i)(\lambda + a + c - i)} [\lambda + a + c] \left[ \frac{b+c-i}{b} \right] \left[ \frac{a+d-i}{a} \right] \text{End}^*_W \left( F^{(a+d-i)}1_{\lambda + 2(a+c-i)} \right).$$

\[\square]\]

4. Faithfulness of Simple 2-representations

4.1. Simple 2-representations. We now define the simple 2-representations of $\mathcal{U}$, following [Rou08]. For $n \in \mathbb{N}$, and $k \in \{0, \ldots, n\}$, we denote by $H_{k,n}$ the subalgebra of $H_n$ generated by $F_n^{S_n}$, $\tau_1, \ldots, \tau_{k-1}$ and $x_1, \ldots, x_k$. We have an isomorphism of algebras $H_{k,n} \cong H_k \otimes p_{n-k}^{S_n}$, and a tower of algebras

$$p_n^{S_n} = H_{0,n} \subseteq H_{1,n} \subseteq \ldots \subseteq H_{n,n} = H_n.$$  

The 2-representation $\mathcal{L}(n)$ of $\mathcal{U}$ is defined as follows:

- for $k \in \{0, \ldots, n\}$, we define $\mathcal{L}(n)_{-n+2k} = H_{k,n}$-proj, the category of finitely generated, projective and graded $H_{k,n}$-modules,
- the functor $E : \mathcal{L}(n)_{-n+2k} \rightarrow \mathcal{L}(n)_{-n+2(k+1)}$ is $\text{ind}_{H_{k,n}}$ and the functor $F : \mathcal{L}(n)_{-n+2k} \rightarrow \mathcal{L}(n)_{-n+2(k-1)}$ is $q^{2k-n-1}\text{res}_{H_{k,n}}$,
- the map $\chi : F \rightarrow F$ is left multiplication by $\chi_k$ on $\text{res}_{H_{k,n}}$ and the map $\tau : F^2 \rightarrow F^2$ is left multiplication by $\tau_{k-1}$ on $\text{res}_{H_{k,n}}$,
- the maps $\varepsilon$ and $\eta$ are the counit and unit of the canonical adjunction between induction and restriction.

**Proposition 4.1** ([Rou08]). The above data defines a 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)$.

Define

$$\mathcal{L}(n) - \text{bim} = \bigoplus_{0 \leq k, n \leq n} (H_{k,n}, H_{n,k}) - \text{bim}$$

where $(H_{k,n}, H_{n,k}) - \text{bim}$ is the category of finitely generated graded $(H_{k,n}, H_{n,n})$-bimodules. The structure of 2-representation of $\mathcal{U}$ on $\mathcal{L}(n)$ induces functors $\Phi_n : \mathcal{U}1_\lambda \rightarrow \mathcal{L}(n) - \text{bim}$ for all $\lambda \in \mathbb{Z}$. For $k \in \{0, \ldots, n\}$ and $a \in \{0, \ldots, k\}$ we have

$$\Phi_n \left( F^n1_{-n+2k} \right) = q^{(2k-n-a)}H_{k,n} \text{ as } (H_{k-n,a}, H_{k,n}) \text{-bimodules},$$

$$\Phi_n \left( E^n1_{-n+2(k-a)} \right) = H_{k,n} \text{ as } (H_{n,n}, H_{k-n}) \text{-bimodules}.$$
Let us now describe explicitly the images of the divided powers under $\Phi_n$. Given two integers $r < \ell \in \{1, \ldots, n\}$, we put
\[
x_{[r, \ell]} = x_{r+1} x^2_{r+2} \cdots x_{\ell-r}^\ell \in P_n,
\]
\[
x_{[r, \ell]}' = (-1)^{\binom{\ell-r}{2}} x_{r+1} x_{r+2}^2 \cdots x_{\ell-1}^\ell \in P_n,
\]
\[
e_{[r, \ell]} = x_{[r, \ell]} \tau_{u_0[r, \ell]} \in H_{\ell,n},
\]
\[
e_{[r, \ell]}' = \tau_{u_0[r, \ell]} x_{[r, \ell]}' \in H_{\ell,n}.
\]
Note that $e_{[r, \ell]}$ and $e_{[r, \ell]}'$ are orthogonal idempotents of $H_{\ell,n}$. Then we have
\[
\Phi_n \left( E^{(a)}_{1-n+2k} \right) = q^{a(2k-n-a)} e^a_{[k-a+1,k]} H_{k,n} \text{ as } (H_{k-n,a}, H_{k,n})\text{-bimodules},
\]
\[
\Phi_n \left( E^{(a)}_{1-n+2k} \right) = q^{-a(2k-n-a)} H_{k,n} e^a_{[k-a+1,k]} \text{ as } (H_{k-n,a}, H_{k,n})\text{-bimodules}.
\]

A crucial fact about the simple 2-representations is the following universal property.

**Theorem 4.2.** ([Rou08, Proposition 5.15]) Let $V$ be an integrable 2-representation, and let $M \in V_n$ be such that $E(M) = 0$. Then there is a unique morphism of 2-representations $R_M : \mathcal{L}(n) \to V$ such that $R_M(P_n) = M$, where $P_n$ is seen as an object of $\mathcal{L}(n)_n$, with the action of $H_n$ being the polynomial representation.

### 4.2. Faithfulness

The goal of this subsection is to prove the following faithfulness result.

**Theorem 4.3.**
1. Let $C$ be a complex of 1-morphisms of $\mathcal{U}$. If $\Phi_n(C)$ is null-homotopic for all $n \in \mathbb{N}$, then $C$ is null-homotopic.
2. Let $f$ be a morphism between complexes of 1-morphisms of $\mathcal{U}$. If $\Phi_n(f)$ is a homotopy equivalence for all $n \in \mathbb{N}$, then $f$ is a homotopy equivalence.

#### 4.2.1. Generalities about Krull-Schmidt categories

**Theorem 4.3** will be a consequence of the following general result.

**Theorem 4.4.** Let $C, (D_j)_{j \in I}$ be Krull-Schmidt categories, and let $(\Phi_i : C \to D_i)_{i \in I}$ be linear functors. Assume that for any finite collection $X_1, \ldots, X_n$ of indecomposable and pairwise non-isomorphic objects of $C$, there exists $i \in I$ such that:
1. $\Phi_i(X_1), \ldots, \Phi_i(X_n)$ are indecomposable and pairwise non-isomorphic,
2. for all $j \in \{1, \ldots, n\}$, the morphism of $K$-algebras $\text{End}_C(X_j) \to \text{End}_{D_i}(\Phi_i(X_j))$ induced by $\Phi_i$ is local.

Let $C \in \text{Comp}(C)$. Then $C$ is null-homotopic if and only if for all $i \in I$, $\Phi_i(C)$ is null-homotopic.

Let us introduce some terminology that we will use throughout the proof. With the notations of **Theorem 4.4**, given a finite collection $X_1, \ldots, X_n$ of indecomposable and pairwise non-isomorphic objects of $C$, we say that an element $i \in I$ is adapted to $X_1, \ldots, X_n$ if conditions (1) and (2) of **Theorem 4.4** are satisfied. Similarly, given an object $M$ of $C$, we say that $i \in I$ is adapted to $M$ if it is adapted to the indecomposable summands of $M$. Given an arrow $f : M \to N$ of $C$, we say that $i \in I$ is adapted to $f$ if it is adapted to $M \oplus N$.

The first step to prove **Theorem 4.4** is to prove a lifting result for split injections and split surjections. This will be based on the following Lemma.

**Lemma 4.5.** Let $C$ be a Krull-Schmidt category, and let $f : M \to N, g : N \to L$ be arrows in $C$. Assume given a decomposition $N = \bigoplus N_j$ in indecomposable objects. Denote by $i_j : N_j \to N, p_j : N \to N_j$ the inclusion and projection morphisms associated to this decomposition.
1. Assume that $L$ is indecomposable and that $g f$ is a split surjection. Then there exists $j$ such that $g i_j$ is an isomorphism and $p_i f$ is a split surjection.
2. Assume that $M$ is indecomposable and that $g f$ is a split injection. Then there exists $j$ such that $g i_j$ is a split injection and $p_j f$ is an isomorphism.

**Proof.** We prove the first statement, the second one being similar. Fix a right inverse $u$ of $g f$. Then we have:
\[
\sum_j g i_j p_j f u = \text{id}_L.
\]
Since $\text{End}_C(L)$ is a local $K$-algebra, there exists $j$ such that $g_i p_j f u$ is invertible. Thus $g_i$ is a split surjection. Since $N_i$ is indecomposable, it follows that $g_i$ is an isomorphism. Hence $p_j f u$ is invertible, so $p_j f$ is a split surjection.

**Proposition 4.6.** Let $C, D$ be Krull-Schmidt categories, and let $\Phi : C \to D$ be a functor such that:

1. for any indecomposable object $X \in C$, $\Phi(X)$ is indecomposable and the morphism of $K$-algebras $\text{End}_C(X) \to \text{End}_D(\Phi(X))$ induced by $\Phi$ is local,
2. for any indecomposable objects $X, Y \in C$, $X \simeq Y$ if and only if $\Phi(X) \simeq \Phi(Y)$.

Let $f : M \to N$ be a morphism in $C$. Then $f$ is a split surjection (resp. injection) if and only if $\Phi(f)$ is a split surjection (resp. injection).

**Proof.** We treat the case of split surjections, the case of split injections being similar. If $f$ is a split surjection, it is clear that $\Phi(f)$ is a split surjection. Conversely, assume that $\Phi(f)$ is a split surjection. We proceed by induction on the number of isomorphism classes of indecomposable summands of $N$. The result is clear if $N = 0$.

Assume that $N \neq 0$ and fix a decomposition $N = N_1 \oplus N'$ with $N_1$ indecomposable. Denote by $p : N \to N_1$ the projection associated with this decomposition. We also fix a decomposition $M = \oplus_{j=1}^m M_j$ in indecomposable objects, and we denote by $i_j : M_j \to M$, $p_j : M \to M_j$ the inclusion and projection morphisms associated with this decomposition.

The composition $\Phi(M) \xrightarrow{\text{id}} \Phi(M) \xrightarrow{\Phi(p_1)} \Phi(N_1)$ is a split surjection and $\Phi(N_1), (\Phi(M_j))_j$ are indecomposable by assumption (1). Thus by Lemma 4.5, there exists $j$ such that $\Phi(p f_i)$ is an isomorphism. Without loss of generality, we can assume that $j = 1$. Then $\Phi(N_1)$ and $\Phi(M_1)$ are isomorphic, so by assumption (2) $N_1$ and $M_1$ are isomorphic. Fix an isomorphism $g : N_1 \xrightarrow{\sim} M_1$. The morphism $\text{End}_C(N_1) \to \text{End}_D(\Phi(N_1))$ induced by $\Phi$ is local and $\Phi(p f_1 g)$ is invertible. Thus $p f_1 g$ is invertible, and $p f_1 i_1$ is invertible as well.

Hence, if we let $M' = \oplus_{j>1} M_j$, in the decompositions $M = M_1 \oplus M'$ and $N = N_1 \oplus N'$ we can write $f$ as a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a = p f_1 i_1$. Since $a$ is an isomorphism, there exist automorphisms $u$ of $N$ and $v$ of $M$ such that

$$u f v = \begin{pmatrix} a & 0 \\ 0 & f' \end{pmatrix}$$

for some morphism $f' : M' \to N'$. Since $\Phi(f)$ is a split surjection and $u, v$ are invertible, $\Phi(u f v)$ is a split surjection as well. Thus, $\Phi(f')$ is a split surjection. The number of indecomposable summands of $N'$ is one less than that of $N$. By induction, we deduce that $f'$ is a split surjection. It follows that $u f v$ is a split surjection. So $f$ is a split surjection.

**Corollary 4.7.** Let $C, (D_i)_{i \in I}$ be Krull-Schmidt categories, and let $(\Phi_i : C \to D_i)_{i \in I}$ be functors satisfying the same assumptions as in Theorem 1. Let $f : M \to N$ be a morphism in $C$. The following are equivalent:

1. $f$ is a split surjection (resp. injection).
2. there exists $i \in I$ adapted to $f$ such that $\Phi_i(f)$ is a split surjection (resp. injection).
3. for all $i \in I$, $\Phi_i(f)$ is a split surjection (resp. injection).

**Proof.** Denote by $C_f$ the smallest full subcategory of $C$ containing $M \oplus N$ and closed under direct sums and direct summands. This is a Krull-Schmidt category with finitely many isomorphism classes of indecomposable objects. By assumption, we can fix $i \in I$ adapted to $f$. Then the functor $C_f \to D_i$ induced by $\Phi_i$ satisfies the assumptions of Proposition 4.6. The result then follows from Proposition 4.6.

We can now prove Theorem 4.4 for complexes that are bounded above or below.

**Proposition 4.8.** Let $C, (D_i)_{i \in I}$ be Krull-Schmidt categories, and let $(\Phi_i : C \to D_i)_{i \in I}$ be functors satisfying the same assumptions as in Theorem 1. Let $C \in \text{Comp}(C)$ be bounded above or below. Then $C$ is null-homotopic if and only if for all $i \in I$, $\Phi_i(C)$ is null-homotopic.
Proof. We treat the case where $C$ is bounded above, the bounded below case being similar. If $C$ is null-homotopic, it is clear that for all $i \in I$, $\Phi_i(C)$ is null-homotopic. Conversely, assume that for all $i \in I$, $\Phi_i(C)$ is null-homotopic. Since $C$ is bounded above, it has the form
\[
\cdots \to C^i \xrightarrow{d^i} C^{i+1} \to \cdots \to C^{\ell-1} \xrightarrow{d^{\ell-1}} C^{\ell} \to 0,
\]
with $\ell$ such that $C^j = 0$ for all $j > \ell$. We inductively construct maps $h_j \in \text{Hom}_C(C^j, C^{j-1})$ for $j \leq \ell$, such that $h_{j+1}d^j + d^{j-1}h_j = 1_{C^j}$.

By assumption, $\Phi_i(d^{-1})$ is a split surjection for all $i \in I$. Hence $d^{-1}$ is a split surjection since $\Phi_i(C)$ is null-homotopic. Thus, $h$ is a right inverse to $d$, and define $h_{\ell-1} = (0 \ 1) : \text{im}(e) \oplus \text{im}(1-e) \to C^{\ell-2}$. By construction, we have $h_jd^j + d^{j-1}h_{j-1} = 1_{C^j}$, which completes the induction.

To finish the proof of Theorem 4.4, we show how to extend this result to unbounded complexes. We will need the following standard Lemma.

**Lemma 4.9 (Gaussian elimination).**

1. Let $C$ be an additive category, and let $C \in \text{Comp}(C)$. Assume that $C$ has the form:

\[
C = \cdots \to C^{i-1} \to X \oplus Y \xrightarrow{(a \ b)} Z \oplus W \to C^i \to \cdots
\]

with $a$ an isomorphism. Then $C$ is homotopy equivalent to a complex of the form:

\[
\cdots \to C^{i-1} \to Y \to W \to C^i \to \cdots
\]

2. Let $C$ be an additive, idempotent complete category, and let $C \in \text{Comp}(C)$. Assume that $C$ has the form:

\[
C = \cdots \to C^{i-1} \to C^i \xrightarrow{(\ell)} Z \oplus W \to C^i \to \cdots
\]

with $\ell$ a split surjection. Then $C$ is homotopy equivalent to a complex of the form:

\[
\cdots \to C^{i-1} \to Y \to W \to C^i \to \cdots
\]

where $Y$ is an object of $C$ such that $C^i \cong Y \oplus Z$.

3. Let $C$ be an additive, idempotent complete category, and let $C \in \text{Comp}(C)$. Assume that $C$ has the form:

\[
C = \cdots \to C^{i-1} \to X \oplus Y \xrightarrow{(a \ b)} C^i+1 \to C^i+2 \to \cdots
\]

with $a$ a split injection. Then $C$ is homotopy equivalent to a complex of the form:

\[
\cdots \to C^{i-1} \to Y \to W \to C^i+2 \to \cdots
\]

where $W$ is an object of $C$ such that $C^{i+1} \cong X \oplus W$.

**Proof.** Since $a$ is an isomorphism, there exists an automorphism $p$ of $X \oplus Y$, an automorphism $q$ of $Y \oplus Z$ and a morphism $d' : Y \to W$ such that

\[
q \begin{pmatrix} a & b \\ c & d \end{pmatrix} p = \begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix}.
\]

Hence $C$ is isomorphic to

\[
\cdots \to C^{i-1} \to X \oplus Y \xrightarrow{(a \ b)} Z \oplus W \to C^i \to \cdots
\]

Since $a$ is an isomorphism, the compositions $C^{i-1} \to X \oplus Y \to X$ and $X \hookrightarrow X \oplus Y \to C^{i+1}$ are zero. Hence $C$ is isomorphic to a direct sum

\[
\left( \cdots \to C^{i-1} \to Y \xrightarrow{d'} W \to C^i \to \cdots \right) \oplus \left( 0 \to X \xrightarrow{d} Z \to 0 \right).
\]
The complex $0 \to X \to Z \to 0$ is homotopy equivalent to zero since $a$ is an isomorphism, so the result follows.

(2) Fix a right inverse $a'$ of $a$. Then $a'a$ is an idempotent of $\text{End}_C(C')$. Put $X = \text{im}(a'a)$ and $Y = \text{im}(1 - a'a)$. We have a decomposition $C' = X \oplus Y$ and $a$ induces an isomorphism $X \to Z$. The result then follows from (1).

(3) The proof is similar to (2).

Proof of Theorem 4.4. For $C \in \text{Comp}(C')$, we denote by $n(C)$ the number of summands in a decomposition of $C^0$ into indecomposable components. Assume that for all $i \in I$, $\Phi_i(C)$ is null-homotopic. We start by proving that $d_C^{-1} = 0$ or $C$ is homotopy equivalent to a complex $C'$ with $n(C') = n(C) - 1$.

Fix $i \in I$ adapted to $d_C^{-1}$ and $d_C^0$. If $\Phi_i(d_C^{-1}) = 0$, then $\Phi_i(d_C^{-1})$ is a split surjection since $\Phi_i(C)$ is null-homotopic. By Corollary 4.7, we deduce that $d_C^{-1}$ is a split surjection. Thus, $d_C^0 = 0$. Assume now that $\Phi_i(d_C^0) \neq 0$. Fix a decomposition into indecomposable summands $C^0 = \oplus_i M_i$. Since $\Phi_i(C)$ is null-homotopic, there exists a split injection $u : M \to \Phi_i(C^0)$ with $M \in D_i$ indecomposable such that $\Phi_i(d_C^0)u$ is a split injection. By Lemma 4.5, there exists $j$ such that $\Phi_i(d_C^0j)$ is a split injection, where $i : M_j \to C^0$ is the inclusion associated with the decomposition of $C^0$. By Corollary 4.7, we deduce that $d_C^0u_j$ is a split injection. Thus, by Lemma 4.9, $C$ is homotopy equivalent to a complex $C'$ with $n(C') = n(C) - 1$.

Thus, if $\Phi_i(C)$ is null-homotopic for all $i \in I$, we have proved that the $d_C^0 = 0$, or $C$ is homotopy equivalent to a complex $C'$ with $n(C') = n(C) - 1$. By induction, it follows that $C$ is homotopy equivalent to a complex $C'$ such that $d_C^0 = 0$. Hence, $C$ is homotopy equivalent to a direct sum $C_1 \oplus C_2$, with $C_1$ bounded above and $C_2$ bounded below. For all $i \in I$, $\Phi_i(C_1)$ and $\Phi_i(C_2)$ are null-homotopic since $\Phi_i(C)$ is as well. So by Proposition 4.8, $C_1$ and $C_2$ are null-homotopic. Conversely, if $C$ is null-homotopic, it is clear that for all $i \in I$, $\Phi_i(C)$ is null-homotopic.

Corollary 4.10. Let $C_i (D_i)_{i \in I}$ be Krull-Schmidt categories, and let $(\Phi_i : C \to D_i)_{i \in I}$ be functors satisfying the same assumptions as in Theorem 4.4. Let $f : C \to D$ be a morphism between complexes of objects of $C$. Then $f$ is a homotopy equivalence if and only if for all $i \in I$, $\Phi_i(f)$ is a homotopy equivalence.

Proof. It suffices to apply Theorem 4.4 to the complex $\text{Cone}(f)$.

4.2.2. Application to simple 2-representations. To prove our faithfulness result, we show that the collection of simple 2-representations of $U$ satisfies the assumptions of Theorem 4.4. The first step is to study the image of the indecomposable 1-morphisms of $U$ in the simple 2-representations. This will be based on the following observation.

Lemma 4.11. Let $k, \ell, r \in \mathbb{N}$ and put $n = k + \ell + r$. Then the multiplication map

$$P_n^{S_{k+\ell} \times S_r} \otimes P_n^{S_k \times S_{\ell+r}} \to P_n^{S_k \times S_{\ell+r} \times S_r}$$

is surjective.

Proof. Let $l$ be the image of the multiplication map. We denote by $e_j$ the elementary symmetric polynomial of degree $j$. To prove the result, it suffices to show that $e_j(x_{k+1}, \ldots, x_{k+\ell}) \in I$ for all $j \geq 1$. For $j = 1$, we have

$$e_1(x_{k+1}, \ldots, x_{k+\ell}) = e_1(x_{1}, \ldots, x_{k+\ell}) - e_1(x_{1}, \ldots, x_{k}) \in I.$$

In general, we have:

$$e_{j+1}(x_{k+1}, \ldots, x_{k+\ell}) = e_{j+1}(x_{1}, \ldots, x_{k+\ell}) - \sum_{i=1}^{j+1} e_i(x_{1}, \ldots, x_{k}) e_{j+1-i}(x_{k+1}, \ldots, x_{k+\ell}),$$

and the result follows by induction on $j$.

Proposition 4.12. Let $X$ be an indecomposable object of $U(\lambda)$, for some $\lambda \in \mathbb{Z}$. If $n$ is large enough and of the same parity as $\lambda$, then $|\Phi_n(X)| \in 1 + q\mathbb{N}[q]$. 

\end{document}
Proof. We start by proving the result for $X = E(a)1_\lambda$ for some $a \in \mathbb{N}$. Let $n > |\lambda| + 2a$ be an integer of the same parity as $\lambda$, and let $k = \frac{|\lambda| + 2a}{n}$. Then $\Phi_n(E(a)1_\lambda) = q^{-\frac{a(a-1)}{2}}H_{a,n}E_{[k+1,k+a]}^d$ as $(H_{a,n}, H_{k,n})$-bimodules. By Morita equivalence, we have an isomorphism of graded $K$-algebras

$$\text{End}^*_{(H_{k,a,n}, H_{a,n})}(H_{k+a,n}E_{[k+1,k+a]}^d) \cong \text{End}^*_{(p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k})}(e_{[k,a]}H_{k+a,n}E_{[k+1,k+a]}^d)^*.$$

There is an isomorphism of $(p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k})$-bimodules

$$\left\{ \begin{array}{l} p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k} \cong e_{[k,a]}H_{k+a,n}E_{[k+1,k+a]}^d \cr P \rightarrow e_{[k,a]}P x_{i,k}^d x_{[k+1,k+a]}^d. \end{array} \right.$$

By Lemma 4.11, $p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k}$ is cyclic generated by 1 as a $(p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k})$-bimodule. It follows that

$$\text{End}^*_{(U(n) - \text{bim})}(\Phi_n(E(a)1_\lambda)) = p_nE_{k+a}^\otimes E_{n-k,a}^\otimes P_nE_{n-k}^\otimes E_{n-k}.$$

Thus, the result holds for $X = E(a)1_\lambda$. The case $X = E(a)1_\lambda$ is similar.

Assume now that $X = q^aE(b)E(a)1_\lambda$ for $s \in \mathbb{Z}$ and $\lambda \geq b - a$. By Proposition 3.5, we have

$$\Phi_n(F(b)E(a)1_\lambda) = \sum_{i=0}^{a} q^{2(a+i)(\lambda + 2a - i)} \frac{1}{[\lambda + 2a]} \frac{1}{[a+b-i]}^2 \left| \Phi_n\left( F(a+b-i)1_{\lambda+2(2a-i)} \right) \right|.$$

Let us find the lowest degree term in that expression. The lowest degree term of $[\frac{\lambda + 2a}{i}]$ is $q^{-i(\lambda + 2a - i)}$, and that of $[\frac{a+b-i}{i}]^2$ is $q^{-2(\lambda + 2a - i)}$. Hence, the lowest degree term of $q^{2(a+i)(\lambda + 2a - i)} \frac{1}{[\lambda + 2a]} \frac{1}{[a+b-i]}^2$ is $q^{-2(\lambda + 2a - i)}$. For $i = a$, this term is 1. For $i < a$, we have $(a - i)(\lambda + 2a - b - i) > 0$ since $\lambda \geq b - a$. Now using the first part of the proof, we know that for $n$ large enough and of the same parity as $\lambda$, the lowest degree term of $\Phi_n\left( F(a+b-i)1_{\lambda+2(2a-i)} \right)$ is 1 for all $i \in \{0, \ldots, a\}$. Putting all this together, we indeed have

$$\Phi_n(F(b)E(a)1_\lambda) \in 1 + q\mathbb{N}[q],$$

when $n$ is large enough and of the same parity as $\lambda$. The case $X = q^aE(a)E(b)1_\lambda$ with $\lambda \leq b - a$ and $s \in \mathbb{Z}$ is proved similarly.

In particular, if $X$ is an indecomposable object of $U1_\lambda$ and $n$ is large enough and of the same parity as $\lambda$, then $\Phi_n(X)$ is indecomposable and the morphism $\text{End}_{U(X)} \rightarrow \text{End}_{(U(n) - \text{bim})}(\Phi_n(X))$ induced by $\Phi_n$ is local (both sides are equal to $K$). The next step is to check that the simple 2-representations distinguish non-isomorphic indecomposable 1-morphisms of $\mathcal{U}$.

**Proposition 4.13.** Let $X, Y$ be indecomposable objects of $U1_\lambda$ for some $\lambda \in \mathbb{Z}$. If $X$ and $Y$ are not isomorphic, then $\Phi_n(X)$ and $\Phi_n(Y)$ are not isomorphic for $n \in \mathbb{N}$ large enough and of the same parity as $\lambda$.

**Proof.** Assume $X = F(b)E(a)1_\lambda$ and $Y = F(d)E(c)1_\lambda$ with $\lambda \geq b - a = d - c$, and $a \neq c$. By Proposition 3.5 we have

$$\langle \Phi_n(X), \Phi_n(Y) \rangle = \sum_{i=0}^{\min(a,c,d)} q^{a+c-i)(\lambda + a+c-i)} \frac{1}{[\lambda + a+c]} \frac{1}{[b+c-i]}^2 \left| \Phi_n\left( F(a+b-i)1_{\lambda+2(a+c-i)} \right) \right|.$$

The lowest degree term of $q^{a+c-i)(\lambda + a+c-i)} \frac{1}{[\lambda + a+c]} \frac{1}{[b+c-i]}^2 \left| \Phi_n\left( F(a+b-i)1_{\lambda+2(a+c-i)} \right) \right|$ is $q^{a+c-i)(\lambda + a+c-i) - i(i+1)(\lambda + a+c-i) - b(c-i) - d(a-i)}$.

Let us rewrite the exponent. We have:

$$(a + c - i)(\lambda + a + c - i) - i(\lambda + a + c - i) - b(c-i) - d(a-i))$$

$$= (a - i)(\lambda + a + c - i) + (c-i)(\lambda + a + c - i) - b(c-i) - d(a-i)$$

$$= (a - i)(\lambda + a + c - i) + (c-i)(\lambda + a + c - i) - d(a-i)$$

$$= (a - i)(\lambda + 2a - b - i) + (c-i)(\lambda + 2c - d - i).$$

Without loss of generality, we can assume that $a > c$. Then using $i \leq a < c$ and $\lambda \geq b - a = d - c$ we see that $(a - i)(\lambda + 2a - b - i) > 0$ and $(c-i)(\lambda + 2c - d - i) \geq 0$. By Proposition 4.12, we have
and Corollary still hold if the null-homotopic assumption in (4.4)

Thus, we can fix \( n \in \mathbb{N} \) such that \( (\Phi_n(X), \Phi_n(Y)) \in q\mathbb{N}[q] \) and \( \Phi_n(X), \Phi_n(Y) \) are indecomposable. Then a composition \( \Phi_n(X) \to \Phi_n(Y) \to \Phi_n(X) \) of homogeneous morphisms of arbitrary degrees is zero or of positive degree. In particular it is not invertible. Hence \( \Phi_n(X) \) and \( \Phi_n(Y) \) are not isomorphic. The same argument holds if \( X = q^sE^{(d)}E^{(a)}1_\lambda \) and \( Y = q^sE^{(d)}E^{(c)}1_\lambda \) for some \( r, s \in \mathbb{Z} \), and the case of \( X = q^sE^{(d)}E^{(a)}1_\lambda \) and \( Y = q^sE^{(c)}E^{(d)}1_\lambda \) is proved similarly.

**Proof of Theorem 4.3.** Note that every complex of 1-morphisms of \( U \) can be written as a direct sum of complexes of objects of \( U1_\lambda \), for \( \lambda \) ranging over \( \mathbb{Z} \). Hence, it suffices to prove the result for complex of objects of \( U1_\lambda \). By Propositions 4.12 and 4.13, the family of functors \( (\Phi_n : U1_\lambda \to \mathcal{L}(n)-\text{bim})_{n \in \mathbb{N}} \) satisfies the assumptions of Theorem 4.4. Thus, the result follows from Theorem 4.4 and Corollary 4.10.

4.3. Extension to the derived category. The goal of this subsection is to show that the conclusions of Theorem 4.3 still hold if the null-homotopic assumption in (1) and the homotopy equivalence assumption in (2) are weakened to acyclic and quasi-isomorphism respectively, modulo some finiteness assumptions.

**Theorem 4.14.** (1) Let \( C \) be a complex of 1-morphisms of \( U \) such that for every integrable 2-representation \( V \) and \( M \in V \), the complex \( C(M) \) is bounded. Assume that for all \( n \in \mathbb{N} \), \( \Phi_n(C) \) is acyclic. Then \( C \) is null-homotopic.

(2) Let \( f \) be a morphism between complexes of 1-morphisms of \( U \) that satisfy the same condition as in (1). Assume that for all \( n \in \mathbb{N} \), \( \Phi_n(f) \) is a quasi-isomorphism. Then \( f \) is a homotopy equivalence.

The first step is the following result from [CR08]. Since this result is not explicitly singled out in [CR08], we provide a proof. However all of the arguments are taken from [CR08], with slight modifications to change from the abelian setup of minimal 2-representations to simple 2-representations.

**Proposition 4.15.** Let \( C \) be a complex of 1-morphisms of \( U \) such that for every integrable 2-representation \( V \) and \( M \in V \), the complex \( C(M) \) is bounded. Assume that for all \( n \in \mathbb{N} \), \( \Phi_n(C) \) is acyclic. Then for any abelian integrable 2-representation \( V \) and \( M \in V \), the complex \( C(M) \) is null-homotopic.

**Proof.** Let \( V \) be an abelian integrable 2-representation of \( U \). The first step is to show that for every highest weight object \( N \in V \) and \( i \geq 0 \), the complex \( C(FP_i) \) is null-homotopic. Denote by \( n \) the weight of \( N \). By Proposition 4.2, we have a morphism of 2-representations \( R_N : \mathcal{L}(n) \to V \) sending \( P_n \) to \( N \). The complex of projective modules \( C(FP_n) \) is bounded and acyclic by assumption. Hence \( C(FP_n) \) is null-homotopic. It follows that \( C(FP_N) \simeq R_N(C(FP_n)) \) is null-homotopic as well.

The second step is to show that for any \( M \in V \), \( C(M) \) is acyclic. Let \( X = C^\vee C(M) \), where \( C^\vee \) denotes the right dual of \( C \). Since \( \text{End}_{\mathcal{D}(V)}(C(M)) \simeq \text{Hom}_{\mathcal{D}(V)}(M, X) \), it suffices to show that \( X \) is acyclic. Assume it is not, and denote by \( j \) the smallest integer such that \( H^j(X) \neq 0 \). Let \( k \) be the maximal integer such that \( E^kH^j(X) \neq 0 \). Put \( N = E^kH^j(X) \simeq H^j(E^kX) \), a highest weight object of \( V \). Then we have

\[
\text{Hom}_{\mathcal{D}(V)}(F^kN, X[-j]) \simeq \text{Hom}_{\mathcal{D}(V)}(N, E^kX[-j]) \neq 0,
\]

the isomorphism begin up to a degree shift. However, by the first step we have

\[
\text{Hom}_{\mathcal{D}(V)}(F^kN, X[-j]) \simeq \text{Hom}_{\mathcal{D}(V)}(C(F^kN), C(M)[-j]) = 0,
\]

which is a contradiction. Thus \( C(M) \) is acyclic.

Finally, we prove that \( C(M) \) is null-homotopic. By [CR08, Corollary 5.33], there exists an algebra \( A \), a 2-representation of \( U \) on \( A-\text{mod} \) restricting to a 2-representation on \( A-\text{proj} \), and a morphism of 2-representations \( S : A-\text{mod} \to V \) such that \( M \) is a direct summand of \( S(A) \). Applying the previous step to the 2-representation \( A-\text{mod} \) and the object \( A \), we obtain that \( C(A) \) is acyclic. Since it is also a bounded complex of projective \( A \)-modules, we conclude that \( C(A) \) is null-homotopic. Hence, \( C(S(A)) \simeq S(C(A)) \) is null-homotopic, from which we deduce that \( C(M) \) is null-homotopic. □
Proof of Theorem 4.14. Let \( n \in \mathbb{N} \). The 2-representation of \( \mathcal{U} \) on \( \mathcal{L}(n) \) induces a 2-representation of \( \mathcal{U} \) on \( \mathcal{L}(n) \)–bim by tensoring on the left. Consider the object

\[
M_n = \bigoplus_{k=0}^{n} H_{k,n} \in \mathcal{L}(n) \text{–bim},
\]

where each summand \( H_{k,n} \) is seen as a \((H_{k,n}, H_{k,n})\)-bimodule with left and right actions given by multiplication. If \( C \) is a complex of 1-morphisms of \( \mathcal{U} \) then we have \( C(M_n) = \Phi_n(C) \) as objects of \( \mathcal{L}(n) \)–bim.

Assume now that for every integrable 2-representation \( \mathcal{V} \) and \( M \in \mathcal{V} \), the complex \( C(M) \) is bounded, and that \( \Phi_n(C) \) is acyclic for all \( n \in \mathbb{N} \). By Proposition 4.15, the complex \( C(M_n) \) is null-homotopic. Thus \( \Phi_n(C) \) is null-homotopic. The first statement then follows from Theorem 4.3(1). We deduce the second statement by applying the first one to \( C = \text{Cone}(f) \).

5. Application to the Rickard complex

One of the important features of Chuang and Rouquier’s approach to categorification of representations of \( sl_2 \) in [CR08] is a categorification of the action of the simple reflection of \( SL_2 \). This takes the form of a complex, whose definition we recall now.

**Definition 5.1.** Given \( \lambda \in \mathbb{Z} \), we define a complex \( \Omega \lambda \) of objects of \( 1_{\mathcal{U}} \mathcal{M} \) as follows.

- The \( \ell \)th component of \( \Omega \lambda \) is \( \Theta \lambda \) for \( \ell \leq 0 \).
- The differential of \( \Omega \lambda \) is given by the composition of \( F(\lambda + r) \) for \( r \leq 0 \).

This complex has \( k + 1 \) non-zero terms, and the last non-zero term is in cohomological degree \( n - k \).

Let \( n \in \mathbb{N} \). For all \( \lambda \in \mathbb{Z} \), the complex \( \Phi_n(\Omega \lambda) \) is bounded. Let us describe these complexes explicitly. Let \( k \in \{0, \ldots, n\} \). Then as complexes of graded \((H_{n-k,n}, H_{k,n})\)-bimodules we have

\[
q^{(2k-n)(2k-n-1)} \Phi_n(\Omega \lambda_{n+2k}) = 0 \rightarrow H_{n-k,n}e_{k+1,n-k} \rightarrow \cdots \rightarrow e_{n-k+1,n}H_{n,n}e'_{k+1,n} \rightarrow \cdots \rightarrow e_{n-k+1,n}H_{n,n}e'_{k+1,n} \rightarrow 0. \quad (5.1)
\]

This complex has \( n-k+1 \) non-zero terms, the last one being in cohomological degree \( n-k \). In both cases, the differential is the composition of the inclusion with multiplication by the suitable idempotents. The main result regarding these complexes is the following.

**Theorem 5.2 ([CR08]).** Let \( n \in \mathbb{N} \) and let \( k \in \{0, \ldots, n\} \). The cohomology of \( \Phi_n(\Omega \lambda_{n+2k}) \) is concentrated in top degree \( n-k \), and \( H^{n-k}(\Phi_n(\Omega \lambda_{n+2k})) \) is an equivalence \( \mathcal{L}(n) \)–bim.

In [CR08], Chuang and Rouquier prove this theorem for the minimal 2-representations rather than the simple 2-representations, and use it to prove that \( \mathcal{O} \) provides derived equivalences on all integrable 2-representations of \( \mathcal{U} \). In subsection 5.1, we give another proof of Theorem 5.2 in the setting of simple 2-representations. Our proof is based on finding explicit bases for the terms of \( \Phi_n(\Omega \lambda_{n+2k}) \) and computing the action of the differential in these bases. In subsections 5.2 and 5.3, we apply our main result on faithfulness of simple 2-representations, Theorem 4.14, to prove that \( \Theta \) is invertible up to homotopy and that there are homotopy equivalences \( \Theta \Omega \lambda \simeq q^{k/2} \Theta \Omega \lambda [-1] \) for all \( \lambda \in \mathbb{Z} \).
5.1. Action of $\Theta$ on simple 2-representations. In this subsection, we study the complexes (5.1) and (5.2) and give the proof of Theorem 5.2. To do so, we will need to compute various graded dimensions, so we start by introducing notation and recalling some elementary results.

The graded $k$-algebras $P_n$ and $P_n^{S_n}$ are locally finite and have graded dimensions given by
\[
\text{grdim} (P_n) = \frac{1}{\left(1-q^2\right)^n}, \quad \text{grdim} \left(P_n^{S_n}\right) = \frac{1}{\left(1-q^2\right) \ldots \left(1-q^{2n}\right)}.
\]
The following notation will be convenient: given $k \geq 0$, we define a variant of the quantum integer $\{k\}$ and the quantum factorial $\{k\}!$ by
\[
\{k\} = \frac{q^{2k} - 1}{q^2 - 1}, \quad \{k\}! = \prod_{\ell=1}^{k} \{\ell\}.
\]
With these definitions, we have $\text{grdim}(P_n) = \text{grdim} \left(P_n^{S_n}\right) \{n\}!$. More generally, given two integers $k < \ell \in \{1, \ldots, n\}$, we have
\[
\text{grdim}(P_n) = \text{grdim} \left(P_n^{S_n}\right) \{\ell - k + 1\}!.
\]
From $H_n \simeq \text{End}_{P_n^{S_n}}(P_n)$, we deduce that the graded dimension of $H_n$ is given by
\[
\text{grdim}(H_n) = \text{grdim} \left(P_n^{S_n}\right) \{n\}! \{n\}!,
\]
where $\Theta$ refers to the automorphism switching $q$ and $q^{-1}$ on formal series. If $M$ is a finitely generated graded $H_n$-module, then $M$ is locally finite as a $k$-vector space, and we have $M \simeq \{n\}! e_n(M) \simeq \{n\}! e'_n(M)$ as graded $P_n^{S_n}$-modules. In particular, we have
\[
\text{grdim}(M) = \{n\}! \text{grdim}(e_nM) = \{n\}! \text{grdim}(e'_nM).
\]
Finally, we will need the two following formulas:
\[
\sum_{\omega \in S_n} q^{2(\omega)} = \{n\}!,
\]
\[
\sum_{0 \leq u_1 < \ldots < u_r \leq n} q^{2(u_1 + \ldots + u_r)} = \frac{q^{r(r-1)}}{(r)!} \frac{\{n\}!}{\{n-r\}!}.
\]
The first one can be proved easily by induction on $n$, see [KC02, Theorem 6.1] for a proof of the second one.

5.1.1. Bases. Fix an integer $n \in \mathbb{N}$. Let $k, \ell, m$ be integers such that $k, \ell \leq m \leq n$. We construct a basis for $e_{(\ell,m)}H_{m,n} e'_{[k,m]}$ as a left $P_n^{S_n}$-module, which is the general form of a term of $\Phi_n(\Theta)$. Put
\[
X_{\ell,m} = \{ (a_\ell, \ldots, a_m), 0 \leq a_i \leq n - i \},
\]
\[
Y_{\ell,m} = \{ (a_\ell, \ldots, a_m) \in X_{\ell,m}, a_\ell > \ldots > a_m \}.
\]
Note that these sets actually depend on $n$, that we assume fixed for the rest of this subsection. We denote by $S_{k,m}$ the set of minimal length representatives of left cosets of $S_{[k,m]}$ in $S_m$. For $a \in Y_{\ell,m}$ and $\omega \in S_{k,m}$ we define
\[
b_m(a, \omega) = e_{(\ell,m)}x^a r_\omega e'_{[k,m]} \in e_{(\ell,m)}H_{m,n} e'_{[k,m]},
\]
where $x^a = x_\ell^{a_\ell} \ldots x_m^{a_m}$. Our goal is to prove the following result.

**Theorem 5.3.** The set $\{b_m(a, \omega), a \in Y_{\ell,m}, \omega \in S_{k,m}\}$ is a basis of $e_{(\ell,m)}H_{m,n} e'_{[k,m]}$ as a left $P_n^{S_n}$-module.

**Example 5.4.** Let us give an explicit example to illustrate the result. Assume $n = m = 3$, $\ell = 1$ and $k = 2$. Then we have $Y_{1,3} = \{(2, 1, 0)\}$, $S_{2,3} = \{1, s_1, s_2 s_1\}$ and we obtain the following basis after simplification:
\[
\left\{ e_{(1,3)}x_1^2 x_2, -e_{(1,3)}x_1 x_2, e_{(1,3)}x_2 \right\}.
\]
We will need the following Lemma in the proof of Theorem 5.3.
Lemma 5.5. The set \( \{ \partial_{\omega} | \ell, m \}(x^a), a \in Y_{\ell, m} \) is a basis of \( P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \) as a \( P_n^{E[\ell, m]} \)-module.

Proof. It is well-known that the set \( \{ x^a, a \in X_{\ell, m} \} \) is a basis of \( P_n^{E[\ell, m]} \) over \( P_n^{E[\ell, m]} \). Since the \( P_n^{E[\ell, m]} \)-linear map \( \partial_{\omega} : P_n^{E[\ell, m]} \rightarrow P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \) is surjective, the set \( \{ \partial_{\omega} | \ell, m \}(x^a), a \in X_{\ell, m} \} \) generates \( P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \) as a \( P_n^{E[\ell, m]} \)-module. Furthermore, we have \( \partial_{\omega} | \ell, m \}(x^a) = -\partial_{\omega} | \ell, m \}(x^a) \) for \( i \in \{ \ell, \ldots, m - 1 \} \) and \( \partial_{\omega} | \ell, m \}(x^a) = 0 \) if there exists two indexes \( i \neq j \) such that \( a_i = a_j \). Thus if \( a \in X_{\ell, m} \), the element \( \partial_{\omega} | \ell, m \}(x^a) \) is a multiple of \( \partial_{\omega} | \ell, m \}(x^b) \) for some \( b \in Y_{\ell, m} \). It follows that the set \( \{ \partial_{\omega} | \ell, m \}(x^a), a \in Y_{\ell, m} \} \) generates \( P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \) as a \( P_n^{E[\ell, m]} \)-module.

To conclude, we check that the graded dimensions match. We have

\[
\text{grdim} \left( \bigoplus_{a \in Y_{\ell, m}} P_n^{E[\ell, m]} \partial_{\omega} | \ell, m \}(x^a) \right) = \text{grdim} \left( P_n^{E[\ell, m]} \sum_{a \in Y_{\ell, m}} q^{2(a_1 + \ldots + a_m) - (m - \ell)(m - \ell + 1)} \right)
\]

\[
= \text{grdim} \left( P_n^{E[\ell, m]} \frac{(n - \ell + 1)!}{(m - \ell + 1)!(n - m)!} \right)
\]

\[
= \text{grdim} \left( P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \right),
\]

the second equality following from (5.7) and the third one from (5.3). Hence the result is proved. \( \square \)

Proof of Theorem 5.3. Let us start by proving that the set \( \{ b_m(a, \omega), a \in Y_{\ell, m}, \omega \in S_{k, m} \} \) is free over \( P_n^{E[\ell, m]} \). Let \( a \in Y_{\ell, m} \). In \( H_{m, n} \) we have a decomposition of the form

\[
\tau_{\omega} | \ell, m \}(x^a) \in \partial_{\omega} | \ell, m \}(x^a) + \sum_{z > 1} P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \tau_z.
\]

Hence for \( \omega \in S_{k, m} \) we have

\[
e^{[\ell, m]} x^a \tau_e^{[k, m]} \in x^a \partial_{\omega} | \ell, m \}(x^a) \tau_e^{[k, m]} + \sum_{z > 1} P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \tau_z e^{[k, m]}.
\]

Using Lemma 5.5 and the fact that the set \( \{ \tau_z e^{[k, m]}, z \in S_{k, m} \} \) is free over \( P_n \), we obtain that the set \( \{ b_m(a, \omega), a \in Y_{\ell, m}, \omega \in S_{k, m} \} \) is free over \( P_n^{E[\ell, m]} \). To conclude, it suffices to compute graded dimensions. On the one hand, by (5.4) and (5.5) we have

\[
\text{grdim} \left( e^{[\ell, m]} H_{m, n} e^{[k, m]} \right) = \text{grdim} \left( P_n^{E[\ell, m] \otimes \mathfrak{S}[m + 1, n]} \frac{m! [m]!}{(m - \ell + 1)! (m - k + 1)!} \right).
\]

On the other hand, the left \( P_n^{E[\ell, m]} \)-submodule spanned by the set \( \{ b_m(a, \omega), a \in Y_{\ell, m}, \omega \in S_{k, m} \} \) has graded dimension

\[
\text{grdim} \left( P_n^{E[\ell, m]} \left( \sum_{a \in Y_{\ell, m}} q^{2(a_1 + \ldots + a_m)} \right) \right) \left( \sum_{\omega \in S_{k, m}} q^{-2l(\omega)} \right).
\]

By (5.7) we have

\[
\sum_{a \in Y_{\ell, m}} q^{2(a_1 + \ldots + a_m)} = q^{(m - \ell)(m - \ell + 1)} \frac{(n - \ell + 1)!}{(m - \ell + 1)! (n - m)!} = \frac{(n - \ell + 1)!}{(m - \ell + 1)! (n - m)!}.
\]

By (5.6) we have

\[
\sum_{\omega \in S_{k, m}} q^{-2l(\omega)} = \frac{m!}{(m - k)!}.
\]
Hence the graded dimension in equation (5.8) is equal to
\[
gr\dim \left( P_\nu^{\Theta_\nu} \right) = \frac{(n-\ell+1)!}{\{n-m\}!} \cdot \frac{\{m\}!}{\{m-k+1\}!} = \frac{\{m\}!}{\{m-k+1\}!}
\]
\[
gr\dim \left( P_\nu^{\Theta_\nu(\nu)} \right) = \frac{\{m\}!}{\{m-k+1\}!} = \frac{\{m\}!}{\{m-k+1\}!}
\]
and the proof is complete.

5.1.2. Combinatorics of the differential. We now study the effect of the differential of \( \Phi_\nu(\Theta) \) on the bases given by Theorem 5.3. On a general term of \( \Phi_\nu(\Theta) \), the differential has the form
\[
d_m : \begin{cases} 
\ell_{\nu,m} \to \ell_{\nu,m+1} \to \ell_{\nu,m+1} \to \ell_{\nu,m+1} 
\end{cases}
\]
where the integers \( k, \ell, m \) depend on the weight and the cohomological degree. Remark that \( d_m \) is a \( P_\nu^{\Theta_\nu(\nu)} \) linear map. We will explicitly determine the kernel and image of \( d_m \) in terms of the bases of Theorem 5.3. To do so, we prove that either \( d_m(b_m(a,\omega)) = 0 \) or \( d_m(b_m(a,\omega)) = b_{m+1}(\phi(a,\omega)) \), where \( \phi \) is an injective map that we define below. Hence the study of the map \( d_m \) can be done by studying the combinatorics of the sets \( Y_{\nu,m} \) and \( S_{\nu,m} \) and the map \( \phi \).

We now introduce the necessary notation to define the map \( \phi \). Given \( r \in \{0, \ldots, m-\ell\} \), we define
\[
Y_{\nu,m}^r = \{ a \in Y_{\nu,m} \mid \forall i \leq r, a_{m-i} = i \text{ and } a_{m-r-1} > r + 1 \}.
\]
The subsets \( (Y_{\nu,m}^r)^r \) are disjoint and we denote by \( Y_{\nu,m}^r \) the complement of their union. More explicitly, the elements of \( Y_{\nu,m}^r \) are the sequences \( a \in Y_{\nu,m} \) such that \( a_m > 0 \). Given \( u \in \{1, \ldots, m\} \), we denote by \( S_{\nu,m}^u \) (resp. \( S_{\nu,m}^{\leq u}, S_{\nu,m}^{> u} \)) the subset of \( S_{\nu,m} \) consisting of the elements \( \omega \) such that \( \omega(m) = u \) (resp. \( \omega(m) \geq u, \omega(m) < u \).

Example 5.6. Assume \( n = 4, m = 3, k = 2 \) and \( \ell = 1 \). Then we have
\[
Y_{1,4} = \{(2,1,0), (3,1,0), (3,2,0), (3,2,1)\}, \quad S_{2,3} = \{1,s_1,s_2s_1\},
\]
and
\[
Y_{1,4}^0 = \{(3,2,0)\}, \quad Y_{1,4}^1 = \{(3,1,0)\},
\]
\[
Y_{1,4}^2 = \{(2,1,0)\}, \quad Y_{1,4}^3 = \{(3,2,1)\},
\]
\[
S_{2,3}^0 = \{1,s_1\}, \quad S_{2,3}^1 = \{s_2s_1\}.\]

We now define a map \( \phi_Y : Y_{\nu,m} \to Y_{\nu,m+1} \). If \( a \in Y_{\nu,m}^r \) we put \( \phi_Y(a) = (a,0) \). If \( a \in Y_{\nu,m}^r \) we put \( \phi_Y(a) = (a_r, \ldots, a_{m-r-1}, r+1, r, \ldots) \) (so we have increased the entries \( a_{m-r}, \ldots, a_m \) by 1 and added a 0 as the last entry). We can now define the map \( \phi \) as follows
\[
\phi : \begin{cases} 
(Y_{\nu,m}^r \times S_{\nu,m}) \cup \bigcup_{r=0}^{m-\ell} (Y_{\nu,m}^r \times S_{\nu,m}^{r-m-r}) & \to Y_{\nu,m+1} \times S_{\nu,m+1} \\
(a,\omega) & \to \begin{cases} 
(\phi_Y(a),\omega) & \text{if } a \in Y_{\nu,m}^r, \\
(\phi_Y(a),s_m \ldots s_m\omega) & \text{if } a \in Y_{\nu,m}^{r+1}. 
\end{cases}
\end{cases}
\]

Proposition 5.7. The map \( \phi \) is injective and has image
\[
\text{im}(\phi) = \bigcup_{r=0}^{m+1-\ell} Y_{\nu,m+1}^{r+1} \times S_{\nu,m+1}^{r+1-\ell}.
\]
Proof. The map $\varphi$ sends $Y_{l,m}^+ \times S_{k,m}$ to $Y_{l,m+1} \times S_{k,m+1}^{m+1}$ and $Y_{l,m}^+ \times S_{k,m}^{<m-r}$ to $Y_{l,m+1} \times S_{k,m+1}^{m-r}$. Since the $S_{k,m+1}^{u}$ are disjoint for distinct values of $u$, it suffices to prove that $\varphi|_{Y_{l,m}^+ \times S_{k,m}}$ and $\varphi|_{Y_{l,m}^+ \times S_{k,m}^{<m-r}}$ are injective. However it is clear that the maps $(S_{k,m} \to S_{k,m+1}, \omega \mapsto \omega)$ and $(S_{k,m}^{<m-r} \to S_{k,m+1}, \omega \mapsto s_m \ldots s_m \omega)$ are injective. Hence $\varphi$ is injective.

We now compute $\text{im}(\varphi)$. We have

$$\varphi\left(Y_{l,m}^+ \times S_{k,m}\right) = \bigcup_{r'=0}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m+1},$$

and

$$\varphi\left(Y_{l,m}^+ \times S_{k,m}^{<m-r}\right) = \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r}.$$ 

Hence

$$\text{im}(\varphi) = \left( \bigcup_{r'=0}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m+1} \right) \bigcup \left( \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r} \right).$$

Switching the order of the two unions in the second term gives

$$\text{im}(\varphi) = \left( \bigcup_{r'=0}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m+1} \right) \bigcup \left( \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r} \right).$$

We can isolate the term $r' = 0$ in the first union and merge the two remaining unions to obtain

$$\text{im}(\varphi) = \left( \bigcup_{r'=0}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m+1} \right) \bigcup \left( \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r} \right) \bigcup \left( \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r} \right) \bigcup \left( \bigcup_{r'=r+1}^{m+1-\ell} Y_{l,m+1} \times S_{k,m+1}^{m-r} \right).$$

Proposition 5.8. Let $(a, \omega) \in Y_{l,m} \times S_{k,m}.$ Then we have

$$d_m(b_m(a, \omega)) = \begin{cases} 0 & \text{if } (a, \omega) \in \bigcup_{r=0}^{m-\ell} Y_{l,m} \times S_{k,m}^{<m-r}, \\ b_{m+1}(\varphi(a, \omega)) & \text{otherwise}. \end{cases}$$

Proof. We have $d_m(b_m(a, \omega)) = e_{[l+1, m]}x^{a}x^d \tau_{\omega}e_{[k, m+1]}$. When $a \in Y_{l,m}^+$, it is clear that $d_m(b_m(a, \omega)) = b_{m+1}(\varphi(a, \omega))$. Assume now that $a \in Y_{l,m}^+$. In $P_n$, we have

$$d_m \ldots d_{m-r} (x_{m-r+1} \ldots x_m) = (-1)^{r+1}x_{m-r+1}^{-1} \ldots x_{m-1}. $$

Furthermore, if $w \in \mathfrak{S}_{[l, m+1]}$ and $P \in P_n$ we have $\tau_{\omega}[l, m+1]P \tau_{\omega} = (-1)^{|w|} \tau_{\omega[l, m+1]} \tau_{\omega[l, m+1]} \tau_{w^{-1}}(P)$. It follows that

$$e_{[l+1, m]}x^{a}x^d = e_{[l+1, m]}x^{a}x^{m-r}x_{m-r+1} \ldots x_m \tau_{m-r} \tau_{m-r+1} \ldots \tau_m.$$

Hence

$$d_m(b_m(a, \omega)) = e_{[l+1, m]}x^{a}x^{m-r} \ldots x_m \tau_{m-r} \tau_{m-r+1} \ldots \tau_m.$$ 

If $\omega(m) \geq m-r$, then $s_m \ldots s_m \omega$ does not have minimal length in its left coset modulo $\mathfrak{S}_{[k+1, m]}$. In that case, $\tau_{m-r} \ldots \tau_m \tau_{m}\omega e_{[k+1, m+1]} = 0$, and we conclude that $d_m(b_m(a, \omega)) = 0$. Otherwise if $\omega(m) < m-r$, then $s_m \ldots s_m \omega$ has minimal length in its left coset modulo $\mathfrak{S}_{[k+1, m+1]}$ and $\tau_{m-r} \ldots \tau_m \tau_{m} = \tau_{m-r} \ldots s_m \omega$. If follows that $d_m(b_m(a, \omega)) = b_{m+1}(\varphi(a, \omega))$. 

From Propositions 5.7 and 5.8, we obtain bases for $\text{ker}(d_m)$ and $\text{im}(d_m)$ and deduce that the cohomology of $\Phi_n(\Theta)$ is concentrated in top degree, which is the first part of Theorem 5.2.
Corollary 5.9. The sets
\[ \left\{ d_m(a, \omega), (a, \omega) \in \bigcup_{r=0}^{m-\ell} Y_{r,m} \times S_{k,m}^{\geq m-r} \right\} \]
\[ \left\{ d_{m+1}(a, \omega), (a, \omega) \in \bigcup_{r=0}^{m+1-\ell} Y_{r,m+1} \times S_{k,m+1}^{\geq m+1-r} \right\} , \]
are bases of \( \ker(d_m) \), \( \text{im}(d_m) \) respectively, as left \( P_n^{[n]} \)-modules.

Corollary 5.10. For all \( n \in \mathbb{N}, k \in \{0, \ldots, n\} \) and \( \ell \neq n-k \) we have:
\[ H^\ell (\Phi_n (\Theta n+2k)) = 0. \]

Proof. By Corollary 5.9 we have \( \text{im}(d_m) = \ker(d_{m+1}) \) if \( m+1 < n \). The result follows. \( \square \)

5.2. Invertibility of \( \Theta \).

5.2.1. Top cohomology. Let \( n \in \mathbb{N} \) and \( k \in \{0, \ldots, n\} \). We give a description of \( H^{n-k} (\Phi_n (\Theta 1-n+2k)) \) as \( (H_{n-k, H_k}) \)-bimodule. Using Corollary 5.9, we can give a basis for \( H^{n-k} (\Phi_n (\Theta 1-n+2k)) \) as a left \( P_n^{[n-k+1,n]} \)-module. With the notations introduced previously, we have \( Y_{n-k+1,n} = \{(k-1, \ldots, 0)\} \) and \( S_{k, n}^{\geq n-k+1} = \{\sigma_k, \omega \in \mathcal{E}_k\} \), where \( \sigma_k \) is given by the element \( \sigma_k \) sending \( i \) to \( i + n - k \) if \( i \leq k \), and to \( i - k \) otherwise. The picture below depicts the element \( \sigma_k \) as a strand diagram.

\[
\sigma_k =
\]

Then there is a decomposition as left \( P_n^{[n-k+1,n]} \)-module:
\[ q^{(2k-2)n(n-k-1)} H^{n-k} (\Phi_n (\Theta 1-n+2k)) = \bigoplus_{\omega \in \mathcal{E}_k} P_n^{[n-k+1,n]} e_{[n+1-k,n]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]} \tag{5.9} \]

To describe the bimodule structure, it is more easier to work in terms of \( (P_n^{[n-k+1,n]} \times \mathcal{E}_k, P_n^{[n-k+1,n]} \times \mathcal{E}_k^{-1}) \)-bimodules using the Morita equivalences from (3.2).

Theorem 5.11. There is an integer \( d \) and an isomorphism of graded \( (P_n^{[n-k+1,n]} \times \mathcal{E}_k, P_n^{[n-k+1,n]} \times \mathcal{E}_k^{-1}) \)-bimodules
\[ e_{[1,n-k]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]}' \simeq q^d P_n^{[n-k+1,n]} \times \mathcal{E}_k, \]
where the left action of \( P_n^{[n-k+1,n]} \times \mathcal{E}_k \) is given by multiplication and the right action of \( P_n^{[n-k+1,n]} \times \mathcal{E}_k^{-1} \) is given by \( \sigma_k \).

Proof. Define
\[ b' = x_{[1,n-k]} x_{[n-k+1,n]} \tau_{\omega} x_{[1,k]} x_{[k+1,n]} e_{[1,k]}' e_{[1,n-k]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]}'. \]
This is an element of \( e_{[1,n-k]} \Phi_n (\Theta 1-n+2k) e_{[1,k]}' \). Let \( b \) be the class of \( b' \) in \( e_{[1,n-k]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]}' \). Consider the \( P_n^{[n-k+1,n]} \)-linear map defined by
\[ f : \begin{cases} P_n^{[n-k+1,n]} \times \mathcal{E}_k & \to & e_{[1,n-k]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]}' \\ 1 & \mapsto & b. \end{cases} \]
Let us show that \( f \) is surjective. If \( \omega \in \mathcal{E}_k \setminus \{1\} \), we have \( \tau_{\omega} e_{[1,k]}' = 0 \). Hence decomposition (5.9) yields
\[ e_{[1,n-k]} H^{n-k} (\Phi_n (\Theta 1-n+2k)) e_{[1,k]}' = e_{[1,n-k]} P_n^{[n-k+1,n]} e_{[n-k+1,n]} x_{[1,k]} x_{[k+1,n]} \tau_{\omega} e_{[1,k]}' e_{[1,k]}'. \]
Hence we deduce
\[ \text{grdim} \left( H^{n-k} \left( \Phi_n \left( \Theta 1_{n+2k} \right) \right) \right) = \text{grdim} \left( P_n^{\mathcal{O}_{n-k+1,n}} \right) q^{k(1-k) - 2k(n-k)} \]
Hence we deduce
\[ \text{grdim} \left( e^n_{1,n-k} H^{n-k} \left( \Phi_n \left( \Theta 1_{n+2k} \right) \right) e^n_{1,k} \right) = \text{grdim} \left( P_n^{\mathcal{O}_{n-k+1,n}} \right) q^{k(1-k) - 2k(n-k)} \]
So \( f \) is an isomorphism of left \( P_n^{\mathcal{O}_{n-k+1,n}} \)-modules. All that remains to prove is the compatibility of \( f \) with the right \( P_n^{\mathcal{O}_{n-k+1,n}} \)-module structure, which we do in Lemma 5.12 below.

**Lemma 5.12.** Keep the notations of the proof of Theorem 5.11. For all \( P \in P_n^{\mathcal{O}_{n-k+1,n}} \), we have \( bP = \sigma_b(P)b \).

**Proof.** We prove that \( bP - \sigma_b(P)b \) is a coboundary of \( \Phi_n(\Theta 1_{n+2k}) \). Using the relations of the affine nil Hecke algebra, we have
\[ bP - \sigma_b(P)b \in \sum_{\omega \in \mathcal{O}_{n-k+1,n}} P_n \omega e^n_{1,k} e^n_{1,k} \]
Since \( b' = e^n_{1,k+1,n}b' \), we actually have
\[ bP - \sigma_b(P)b \in \sum_{\omega \in \mathcal{O}_{n-k+1,n}} e^n_{1,k+1,n} P_n \omega e^n_{1,k} e^n_{1,k} \]
Let us explain why the right-hand side of (5.10) is contained in the image of the differential of \( \Phi_n(\Theta 1_{n+2k}) \). Since \( \{ \omega \in \mathcal{O}_{n-k+1,n} \} \) is a basis for \( P_n \) as a \( P_n^{\mathcal{O}_{n-k+1,n}} \)-module, we have:
\[ e^n_{1,k+1,n} P_n = \sum_{\omega \in \mathcal{O}_{n-k+1,n}} P_n^{\mathcal{O}_{n-k+1,n}} e^n_{1,k+1,n} \]
Hence the right-hand side of (5.10) has the form
\[ \sum_{\omega \in \mathcal{O}_{n-k+1,n}} P_n^{\mathcal{O}_{n-k+1,n}} e^n_{1,k+1,n} \]
For \( \omega \in \mathcal{O}_{n-k+1,n} \), the condition \( \omega < \sigma_b \) is equivalent to \( \omega(n) > n-k \). If furthermore \( \omega' \in \mathcal{O}_{n-k+1,n} \), we have \( \omega' \omega(n) > n-k \). Hence on the basis of Theorem 5.3, the right-hand side of (5.10) decomposes as
\[ \sum_{\omega \in \mathcal{O}_{n-k+1,n}} P_n^{\mathcal{O}_{n-k+1,n}} e^n_{1,k+1,n} \]
By Corollary 5.9, this is contained in the image of the differential of \( \Phi_n(\Theta 1_{n+2k}) \). Hence \( bP - \sigma_b(P)b \) is a coboundary, and the result follows.

Theorem 5.11 implies in particular that \( H^{n-k} \left( \Phi_n \left( \Theta 1_{n+2k} \right) \right) \) is an invertible bimodule, which proves the second part of Theorem 5.2.
5.2.2. Invertibility of $\Theta$. We denote by $\Theta^\vee$ the right dual of $\Theta$.

**Theorem 5.13.** For all $\lambda \in \mathbb{Z}$, the unit $u_\lambda : 1 \rightarrow \Theta^\vee \Theta_1$ and counit $v_\lambda : \Theta \Theta^\vee 1_{-\lambda} \rightarrow 1_{-\lambda}$ of adjunction are homotopy equivalences. In other words, $\Theta$ is invertible in $K(U)$.

**Proof.** If $M$ is an object of an integrable 2-representation of $U$, we have $E^i(M) = F^i(M) = 0$ for $i$ large enough. Hence the complexes $\Theta \Theta^\vee (M)$ and $\Theta^\vee \Theta (M)$ are bounded. So by Theorem 4.14, it suffices to check that for the $all n \in \mathbb{N}$, $\Phi_n(u_\lambda)$ and $\Phi_n(v_\lambda)$ are quasi-isomorphisms.

Let $n \in \mathbb{N}$. If $\lambda \notin \{ -n + 2k, k \in \{0, \ldots, n\} \}$, then $\Phi_n(1_{-\lambda}) = \Phi_n(1_{-\lambda}) = 0$, so the result is clear. If $k \in \{0, \ldots, n\}$, by Corollary 5.10 there is a quasi-isomorphism

$$\Phi_n(\Theta_1) \approx H^{n-2k}(\Phi_n(\Theta_1(n-2k))) [n-k].$$

By Theorem 5.11, $H^{n-k}(\Phi_n(\Theta_1(n-2k)))$ is invertible as a $(H_{n-k}, H_{n})$-bimodule. It follows that the complex $\Phi_n(\Theta_1(n-2k))$ is invertible in $D(L(n)\text{-}\text{bim})$. Thus $\Phi_n(u_{n-2k})$ and $\Phi_n(v_{n-2k})$ are quasi-isomorphisms, which ends the proof. \hfill $\square$

5.3. Compatibility with Chevalley generators. The goal of this subsection is to prove that for all $\lambda \in \mathbb{Z}$, there is a homotopy equivalence $\Theta E_1 \approx \varphi^{1+2} \Theta \Theta_1[-1]$. To prove this, we start by constructing a map $G_\lambda : \Theta E_1 \rightarrow \varphi^{1+2} \Theta \Theta_1[-1]$. We will then prove that this is a homotopy equivalence using Theorem 4.14.

Given $k, \ell \in \mathbb{N}$, we define an element $G_{k,\ell} \in H_k \otimes H_\ell^{\text{op}}$ of degree $2(\ell - k)$ by

$$G_{k,\ell} = \sum_{r \geq 0} (-1)^r x_1^r x_2 [2, k] \tau_{0 \mu [1, k]} \otimes \epsilon_{\ell - 1 - r}(x_2, \ldots, x_\ell) e_{\ell'}^r,$$

where $\epsilon_j$ denotes the elementary symmetric polynomial of degree $j$. In $H_k \otimes H_\ell^{\text{op}}$, we have $G_{k,\ell} = G_{k,\ell}(e_k \otimes e_{\ell'}^2) = (e_{[2,k]} \otimes e_{\ell'})G_{k,\ell}$. Hence $G_{k,\ell}$ defines a map $G_{k,\ell} : F^{(k)}E^{(\ell-1)} \rightarrow q^{k-\ell} FF^{(k-1)}E^{(\ell)}$.

**Lemma 5.14.** For all $k, \ell \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc}
F^{(k)}E^{(\ell-1)}E & \xrightarrow{d_{k,\ell}^{-1}\epsilon} & F^{(k+1)}E^{(\ell)}E \\
\downarrow G_{k,\ell} & & \downarrow \left[ G_{k+1,\ell+1} \right] \\
FF^{(k-1)}E^{(\ell)} & \xrightarrow{\bar{d}_{k,\ell}^{-1}} & FF^{(k)}E^{(\ell+1)}
\end{array}$$

where $d_{k,\ell} : F^{(k)}E^{(\ell)} \rightarrow F^{(k+1)}E^{(\ell+1)}$ is the composition of the unit of adjunction with the projection on the idempotent $e_{\ell+1} \otimes e_{\ell'}^r$.

**Proof.** This is a straightforward computation. On the one hand, we have

$$Fd_{k-1,\ell} \circ G_{k,\ell} = \left( e_{[2,k+1]} \otimes e_{\ell'}^r \right) \circ F^k \eta E^\ell \circ \sum_{r = 0}^{\ell} (-1)^r x_1^r x_2 [2, k] \tau_{0 \mu [1, k]} \otimes \epsilon_{\ell - 1 - r}(x_2, \ldots, x_\ell) e_{\ell'}^r$$

$$= \left( \sum_{r = 0}^{\ell} (-1)^r e_{[2,k+1]} x_1^r x_2 [2, k] \tau_{0 \mu [1, k]} \otimes \epsilon_{\ell - 1 - r}(x_2, \ldots, x_\ell) e_{\ell'}^r \right) \circ F^k \eta E^\ell.$$
and we can rewrite the above as

\[ G_{k+1,\ell+1} \circ d_{k,\ell-1} E = \left( \sum_{r=0}^{\ell-1} (-1)^r x_1^r x_{2,k+1}^r \tau_{uv_0[1,k+1]} \otimes x_{\ell+1} e_{\ell-\ell+1} (x_2, \ldots, x_\ell) e_{\ell+1} \right) \circ F^k \eta E^\ell. \]

We have \( Fx \circ \eta = xE \circ \eta \). Using this, we can take the \( x_{\ell+1} \) in the first sum to the right left side of the tensor and we obtain

\[ G_{k+1,\ell+1} \circ d_{k,\ell-1} E = \left( \sum_{r=0}^{\ell-1} (-1)^r x_1^r x_{2,k+1}^r \tau_{uv_0[1,k+1]} \otimes x_{\ell-\ell-1} (x_2, \ldots, x_\ell) e_{\ell+1} \right) \circ F^k \eta E^\ell. \]

In \( H_{k+1} \), we have the relation \( \tau_{uv_0[1,k+1]} x_{\ell+1} = x_1 \tau_{uv_0[1,k+1]} \) and this completes the proof. \( \square \)

We can now define the morphism \( G^*_\lambda : \Theta E_1 \lambda \rightarrow q^{\lambda+2} \Theta \Omega_1 [-1] \). On the \( \ell \)th component of \( \Theta E_1 \lambda \), \( G^*_\lambda \) is defined to be the map \( G^*_\lambda + r + 2 \tau + 1 : q^{-r} F^{\lambda+1} E(r+1) \lambda \rightarrow q^{\lambda+1} F F^{\lambda+1} E(r+1) \lambda \). This gives a morphism of complexes by Lemma 5.14. Similarly, we define a morphism of complexes \( T^*_\lambda : q^{\lambda+2} \Theta \Omega_1 [-1] \rightarrow \Theta E_1 \lambda \) using the elements

\[ T_{k,\ell} = \sum_{r=0}^{k-1} (-1)^r e_k e_{k-\ell} (x_2, \ldots, x_k) \otimes \tau_{uv_0[1,\ell]} x_{2,\ell}^r e_{\ell+1} \in H_k \otimes H_{\ell+1}^\text{op}. \]

The fact that these give a morphism of complexes is proved as for the map \( G^*_\lambda \).

**Lemma 5.15.** Let \( k, \ell \in \mathbb{N} \). If \( \ell \leq k \), then \( T_{k,\ell} G_{k,\ell} = e_k \otimes e_{\ell+1} \). If \( k \leq \ell \), then \( G_{k,\ell} T_{k,\ell} = e_{k+1} \otimes e_{\ell+1} \).

**Proof.** We prove the result in the case \( \ell \leq k \), the other one being similar. We have

\[ T_{k,\ell} G_{k,\ell} = \sum_{0 \leq r < k-1} (-1)^{r+1} e_k e_{k-\ell-1} (x_2, \ldots, x_k) x_{2,k}^r \tau_{uv_0[1,k+1]} \otimes e_{\ell-\ell+1} (x_2, \ldots, x_\ell) e_{\ell+1} \tau_{uv_0[1,\ell]} x_{2,\ell}^r e_{\ell+1}. \]

In \( H_k \) we have

\[ e_k e_{k-\ell} (x_2, \ldots, x_k) x_{2,k}^r \tau_{uv_0[1,k+1]} = \partial_{k-1} \ldots \partial_1 (e_{k-\ell+1} (x_2, \ldots, x_k)) e_{k} = (-1)^{\ell} \delta_{u} e_{k}. \]

Hence

\[ T_{k,\ell} G_{k,\ell} = \sum_{u=0}^{\ell-1} (-1)^u e_k \otimes e_{\ell-\ell+1} (x_2, \ldots, x_\ell) \tau_{uv_0[1,\ell]} x_{2,\ell}^r e_{\ell+1} \]

\[ = \sum_{u=0}^{\ell-1} (-1)^u e_k \otimes e_{\ell-\ell+1} (x_2, \ldots, x_\ell) \tau_{uv_0[1,\ell]} x_{2,\ell}^r e_{\ell+1}. \]

Using the polynomial representation, it is easy to check that

\[ \sum_{u=0}^{\ell-1} (-1)^u e_{\ell-\ell+1} (x_2, \ldots, x_\ell) \tau_{uv_0[1,\ell]} x_{2,\ell}^r e_{\ell+1} = e_{\ell+1} \]

and the result follows. \( \square \)

**Corollary 5.16.** If \( \lambda \geq 0 \), then \( T_\lambda G_\lambda = \text{id}_{\Theta E_1 \lambda} \). If \( \lambda \leq 0 \), then \( G_\lambda T_\lambda = \text{id}_{F \Omega_1 [-1]} \).

**Theorem 5.17.** For all \( \lambda \in \mathbb{Z} \), \( G_\lambda : \Theta E_1 \lambda \rightarrow q^{\lambda+2} \Theta \Omega_1 [-1] \) is a homotopy equivalence.
Proof. If $M$ is an object of an integrable 2-representation of $\mathcal{U}$, the complexes $\Theta E(M)$ and $F\Theta(M)$ are bounded. Thus by Theorem 4.14 it suffices to prove that for all $n \in \mathbb{N}$, $\Phi_n(G_{\lambda})$ is a quasi-isomorphism.

Let $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. There is an isomorphism of \((P^e_n)_{\lambda}^{\mathcal{E}_{k+1}, \mathcal{E}_{n-k-1}}, P_n^e \times \mathcal{E}_{n-1})\)-bimodules
\[
e^e_{\{1,k+1\}}(E_{1-n+2k}) e^e_{\{1,k\}} \simeq P_n^e \mathcal{E}_{\{1\}} \times \mathcal{E}_{|k+2n|}.
\]
Hence by Corollary 5.10 and Proposition 5.11 there is an isomorphism of \((P_n^e \mathcal{E}_{n-k-1} \times \mathcal{E}_{k+1}, P^e_n \mathcal{E}_{k} \times \mathcal{E}_{n-k})\)-bimodules
\[
e^e_{\{1,n-k-1\}} H^\ell \left( \Phi_n \left( \Theta E_{1-n+2k} \right) \right) e^e_{\{1,k\}} = \begin{cases} P_n^e \mathcal{E}_{\{1\}} \times \mathcal{E}_{|k+2n|} & \text{if } \ell = n-k-1, \\
0 & \text{otherwise}, \end{cases}
\]
where the left action is given by $e^{-1}_{k+1}$ in the first case. Similarly, there is an isomorphism
\[
e^e_{\{1,n-k-1\}} H^\ell \left( \Phi_n \left( F \Theta_{1-n+2k} \right) \right) e^e_{\{1,k\}} = \begin{cases} P_n^e \mathcal{E}_{\{1,n-k-1\}} \times \mathcal{E}_{|n-k+1,n|} & \text{if } \ell = n-k, \\
0 & \text{otherwise}, \end{cases}
\]
where the right action is given by $e_2$ in the first case. Now remark that there is an isomorphism of \((P^e_n)_{\lambda}^{\mathcal{E}_{n-k-1} \times \mathcal{E}_{k+1}}, P_n^e \mathcal{E}_{k} \times \mathcal{E}_{n-k})\)-bimodules
\[
s_{n-k-1} \cdots s_1 e^e_k : P_n^e \mathcal{E}_{\{1\}} \times \mathcal{E}_{|k+2n|} \simeq P_n^e \mathcal{E}_{\{n-k-1\}} \times \mathcal{E}_{|n-k+1,n|}.
\]
Hence the cohomology bimodules of $\Phi_n(\Theta E_{1,\lambda})$ and $\Phi_n(F \Theta_{1,\lambda}([-1]))$ are isomorphic. Furthermore by Lemma 5.16, for all $\ell$, $H^\ell(\Phi_n(G_{\lambda}))$ is either a split surjection or a split injection. Since the source and target of $H^\ell(\Phi_n(G_{\lambda}))$ are locally finite and have the same graded dimension, we conclude that $H^\ell(\Phi_n(G_{\lambda}))$ is an isomorphism. Hence $\Phi_n(G_{\lambda})$ is a quasi-isomorphism, and the proof is complete. \(\square\)

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