Non-compact RCD(0, N) Spaces with Linear Volume Growth

Xian-Tao Huang

Received: 13 June 2016 / Published online: 4 May 2017
© Mathematica Josephina, Inc. 2017

Abstract Since non-compact RCD(0, N) spaces have at least linear volume growth, we study non-compact RCD(0, N) spaces with linear volume growth in this paper. One of the main results is that the diameter of level sets of a Busemann function grows at most linearly on a non-compact RCD(0, N) space satisfying the linear volume growth condition. Another main result in this paper is a rigidity theorem at the non-compact end for a RCD(0, N) space with strongly minimal volume growth. These results generalize some theorems on non-compact manifolds with non-negative Ricci curvature to non-smooth settings.

Keywords RCD(0, N) · MCP(0, N) · Busemann function · Linear volume growth

Mathematics Subject Classification 53C23 · 51Fxx

1 Introduction

Cheeger and Gromoll’s splitting theorem (see [22]) is one of the classical theorems on manifolds with non-negative Ricci curvature. In their proof, the Busemann function plays an important role. There are some papers studying the Busemann function on manifolds, see [45,46] etc. Note that the definition of Busemann functions only concerns with the metric notion. Combining with the notion of curvature bound (especially the Ricci curvature lower bound), we obtain many interesting properties and applications.
Recently, people are more and more interested in the study of non-smooth objects. In the framework of metric measure spaces, there are lots of researches of Ricci curvature lower bounds.

A notion of ‘Ricci bounded from below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty)$’ for general metric measure spaces is the so-called CD($K,N$) condition, which is introduced independently by Lott and Villani [38] and by Sturm [47, 48]. For $N \in [1, \infty)$, the CD($K,N$) condition is compatible with the Riemannian case and the class of CD($K,N$) spaces is stable under the measured-Gromov–Hausdorff convergence. In particular, they include Ricci-limit spaces (see [19–21]). Note that the Busemann function plays an important role in the proof of Theorem 1.1. We note that the Busemann function is introduced by Gigli [30, 31].

There are many other results on RCD($K,N$) spaces, see [2, 3, 24, 28] etc. Indeed, an RCD($K,N$) space is a metric measure space which is both a CD($K,N$) space and CD($K,N$), introduced by Bacher and Sturm [10]. A notion of ‘Ricci bounded from below by $K$ and dimension bounded above by $N$’ for general metric measure spaces is the so-called CD($K,N$) condition, denoted by CD($K,N$), introduced by Bacher and Sturm [10]. For the case $K = 0$, CD($0,N$) and CD($0,N$) conditions are equivalent. There is another version of Ricci curvature lower bound, called MCP($K,N$) condition, see Sturm [48], Ohta [41].

Recently, Ambrosio et al. [6] introduced the notion of RCD($K,\infty$) spaces (see also [7] for the simplified axiomatization), which rules out Finsler geometries. The concepts of RCD($K,N$) space and RCD($K,N$) spaces are considered by many authors, see [2, 3, 24, 28] etc. Indeed, an RCD($K,N$) space (resp. RCD($K,N$)) is a metric measure space which is both a CD($K,N$) (resp. CD($K,N$)) and an RCD($K,\infty$) space. Ricci-limit spaces are examples of RCD($K,N$) spaces, and splitting theorem holds for Ricci-limit spaces by Cheeger and Colding [18]. Alexandrov spaces are also examples of RCD($K,N$) spaces. An isometric splitting for Alexandrov spaces with non-negative Ricci curvature is established by Zhang and Zhu in [51], where ‘non-negative Ricci curvature’ for Alexandrov spaces is defined by the authors in the same paper. The splitting theorem on general RCD($0,N$) spaces is proved by Gigli:

**Theorem 1.1** (Gigli [30, 31]) Let $(X, d, m)$ be an RCD($0,N$) space containing a line. Then $(X, d, m)$ is isomorphic to the product of the Euclidean line $(\mathbb{R}, d_{\text{Eucl}}, L^1)$ and another space $(X', d', m')$. Moreover,

1. if $N \geq 2$, then $(X', d', m')$ is a RCD($0,N-1$) space;
2. if $N \in [1, 2)$, then $X'$ is just a point.

We note that the Busemann function plays an important role in the proof of Theorem 1.1.

There are many other results on RCD($K,N$) spaces, see [23, 26, 33, 35–37, 40, 52], and so on.

Because RCD($K,N$) spaces satisfy Bishop–Gromov volume comparison estimate, it is not hard to prove that a non-compact RCD($0,N$) space $(X, d, m)$ has at least linear volume growth, i.e., $m(B_p(r)) \geq Cr$ for some positive constant $C$, see Proposition 2.8. In fact, the proof of Proposition 2.8 only makes use of Bishop–Gromov volume comparison, thus similar conclusion also holds on metric measure spaces with weaker notions of ‘non-negative Ricci curvature’ such as MCP($0,N$), CD($0,N$). This generalizes the famous theorem of Calabi [11] and Yau [50] on non-compact manifolds with non-negative Ricci curvature.
It is interesting to study non-compact metric measure spaces with ‘non-negative Ricci curvature’ and linear volume growth, which is the aim of this paper.

The following theorem is one of our main results:

**Theorem 1.2** Suppose \((X, d, m)\) is a non-compact \(\text{RCD}(0, N)\) space with \(m(B_p(r)) \leq Cr\) for some point \(p\) and positive constant \(C\), and \(b\) is the Busemann function associated with some geodesic ray \(\gamma\). Then the diameter of \(b^{-1}(r)\) grows at most linearly. More precisely, we have

\[
\limsup_{r \to +\infty} \frac{\text{diam}(b^{-1}(r))}{r} \leq C_0 \leq 2, \tag{1.1}
\]

where the diameter of \(b^{-1}(r)\) is computed with respect to the distance \(d\). In particular, \(b^{-1}(r)\) is compact.

Theorem 1.2 generalizes Theorem 19 in Sormani’s paper [46], and part of the proof here is inspired by [46]. However, the non-smoothness of the \(\text{RCD}(0, N)\) space brings technical challenges in the proof. To overcome these difficulties, we make use of the properties of \(L^1\)-optimal transportation under \(\text{RCD}^*(K, N)\) condition.

The aforementioned \(L^1\)-optimal transportation theory is mainly based on the work of Bianchini and Cavalletti (see [8]) and some further development (see [12,14] etc.). Recently, there are many other applications of this theory. For example, it plays an important role in Cavalletti and Mondino’s proof of Lévy–Gromov isoperimetric inequality and other sharp geometric and functional inequalities on metric measure spaces with Ricci curvature lower bounds (see [14,15]).

On a non-compact proper geodesic space, Busemann function \(b\) associated with some geodesic ray \(\gamma\) exists and it is a special 1-Lipschitz function. In Sect. 3, Busemann functions are studied in detail, especially we find their relation to optimal transportation. In particular, when \((X, d, m)\) satisfies the \(\text{RCD}(0, N)\) condition, we adapt the theory developed in [8] (see also [12,14] etc.) to prove that, up to an \(m\)-negligible set, the whole space \(X\) decomposes into geodesic rays. Each such geodesic ray coincides with a so-called Busemann ray, and the restriction of the Busemann function \(b\) on any of such geodesic ray gives parametrization by arc length. In some sense, this construction gives ‘the flow of the gradient field’ of the Busemann function \(b\), even though the concept of ‘the flow of the gradient field’ in non-smooth setting is in general not available. We can then move a compact set \(K\) by the flow, and denote by \(\Xi(K) := \bigcup_{y \in K} R(y), \Xi_s(K) := \Xi(K) \cap b^{-1}(s), \) where \(R(y)\) consists of all points on the aforementioned geodesic ray passing through \(y\). By the theory in [8] (see also [12,14]), we can further obtain the monotonicity of the ‘codimension one volume’ \(m_{-1}(\Xi_s(K))\), see Proposition 3.21 and Corollary 3.22. Proposition 3.21 generalizes Theorem 5 of [46] to non-smooth setting. Here we remark that a priori we do not know the level sets \(b^{-1}(s)\) are compact and have finite ‘area’; we only have information about ‘area’ of the sets \(\Xi_s(K)\).

In Sect. 4, we finish the proof of Theorem 1.2 by making use of the volume comparison properties. The argument in Sect. 4 is a bit refinement of [46], thus we give a better bound \(C_0 \leq 2\) (from the proof of Theorem 19 in [46], we get the bound \(C_0 \leq 6\)).
remark that in Theorem 1 of [45], Sormani proved that for complete manifolds with non-negative Ricci curvature and linear volume growth, the $C_0$ in (1.1) can be chosen to be 0. However, the proof of Theorem 1 in [45] makes use of Cheeger–Colding’s almost rigidity theory, which is not available for RCD*($K$, $N$) spaces at present. It is an interesting question whether we can generalize Theorem 1 in [45] to RCD(0, $N$) spaces with linear volume growth.

We remark that, by making use of the recent results of Cavalletti and Mondino [16], in Theorem 1.2 the assumption that $(X, d, m)$ is an RCD(0, $N$) space can be weakened. In fact we can prove the following result:

**Corollary 1.3** Suppose $(X, d, m)$ is a non-compact essentially non-branching MCP (0, $N$) space with $m(B_p(r)) \leq Cr$ for some point $p$ and positive constant $C$, and $b$ is the Busemann function associated with a geodesic ray $\gamma$, then the diameter of $b^{-1}(r)$ satisfies (1.1). In particular, $b^{-1}(r)$ is compact.

Now we are going to our next main result of this paper. Similar to [45], we introduce a notion of strongly minimal volume growth, which says the asymptotic volume growth of an essentially non-branching MCP (0, $N$) space is `$\lim_{r\to \infty} \frac{m(B_p(r))}{r}$' equals to the `codimension one volume' $m_{-1}(\Xi_{r_0}(K))$ for some $r_0 \in \mathbb{R}$ and compact set $K$, see Definition 5.1 for details. Then we obtain the following result, which generalizes Corollary 10 in [45] to non-smooth setting.

**Theorem 1.4** Suppose $(X, d, m)$ is a non-compact RCD(0, $N$) space and satisfies the strongly minimal volume growth (see Definition 5.1), then $(X, d, m)$ has only one end and there is some metric measure space $(Z, d', m')$ such that one of the following holds:

1. if $Z$ has exactly one point, then $(b^{-1}((r_0, \infty)), d, m)$ is isomorphic to $((r_0, \infty), d_{\text{Eucl}}, cL^1)$ with $c = m(b^{-1}([r_0, r_0 + 1]))$. Furthermore, in this case $(X, d)$ is isometric to some $([\tilde{r}, \infty), d_{\text{Eucl}})$.
2. if $Z$ has more than one point, then $N \geq 2$, and $(Z, d', m')$ is a compact connected RCD(0, $N - 1$) space, and $(b^{-1}((r_0, \infty)), d, m)$ is locally isomorphic to $(Z \times (r_0, \infty), d' \times d_{\text{Eucl}}, m' \otimes L^1)$. Furthermore, let $r_1 = r_0 + \frac{\text{diam}'(Z)}{2}$, then $(b^{-1}((r_1, \infty)), d, m)$ is isomorphic to $(Z \times (r_1, \infty), d' \times d_{\text{Eucl}}, m' \otimes L^1)$. Here $\text{diam}'(Z)$ is the diameter of $Z$ with respect to the distance $d'$.

The following examples show that some conclusions in Theorem 1.4 cannot be strengthened.

**Example 1.5** Suppose $(X, d, m)$ is isomorphic to $([-1, \infty), d_{\text{Eucl}}, fL^1)$, where $f : [-1, \infty) \to (0, 1]$ is any concave function with $f(x) = 1$ for $x \geq 0$. It can be proved that $(X, d, m)$ is an RCD(0, $N$) space (see [24,48,49] etc.), and it is easy to see $(X, d, m)$ satisfies the strongly minimal volume growth condition. Thus in (1) of Theorem 1.4, we cannot prove that the whole space $(X, d, m)$ is isomorphic to $([-1, \infty), d_{\text{Eucl}}, cL^1)$ for some constant $c$.

**Example 1.6** Let $S^2 \hookrightarrow \mathbb{R}^3$ be the unit sphere equipped with the round metric $g_{\text{round}}$, and $\tau : S^2 \to S^2$ be the standard antipodal map given by $\tau(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$, which is an isometry of $(S^2, g_{\text{round}})$. Suppose $(S^2 \times \mathbb{R}, g_{\text{cyl}} = \mathbb{R})^\times \mathbb{R}, g_{\text{cyl}} = \mathbb{R}$.
$g_{\text{ground}} + dr^2$ is the standard metric on the cylinder, then $\sigma : (S^2 \times \mathbb{R}, g_{\text{cyl}}) \to (S^2 \times \mathbb{R}, g_{\text{cyl}}), \sigma((x, r)) = (r(x), -r)$ is obviously an isometry. Denote the quotient space with respect to $\sigma$ by $(X, g_X)$, and we denote $[(x, r)] \in X$ the quotient point of $(x, r)$ and $\sigma((x, r))$. It is easy to see that $b : X \to \mathbb{R}^+, b([(x, r)]) := |r|$ is a well-defined Busemann function. Let $r_0 = \frac{1}{10}$. Note that $X$ inherits a metric measure structure $(X, d_X, m_X)$ from $(X, g_X)$, and $b^{-1}((r_0, \infty))$ inherits the metric measure structure $(b^{-1}((r_0, \infty)), d_X, m_X)$ as a subspace of $X$. Similarly, $S^2 \times (r_0, \infty)$ inherits a metric measure structure $(S^2 \times (r_0, \infty), d_{\text{cyl}}, m_{\text{cyl}})$ from $(S^2 \times (0, \infty), g_{\text{cyl}})$. It is easy to see $(b^{-1}((r_0, \infty)), d_X, m_X)$ satisfies the strongly minimal volume growth condition. Obviously, $(b^{-1}((r_0, \infty)), g_X)$ is isometric to the half cylinder $(S^2 \times (r_0, \infty), g_{\text{cyl}})$ as Riemannian manifolds, but this only induces a local isometry between $(b^{-1}((r_0, \infty)), d_X)$ and $(S^2 \times (r_0, \infty), d_{\text{cyl}})$. In fact we have $d_X([(x, \frac{1}{2})], [(\tau(x), \frac{1}{2})]) = \frac{2}{5} < \pi = d_{\text{cyl}}((x, \frac{1}{2}), (\tau(x), \frac{1}{2}))$ for any $x \in S^2$. Thus in (2) of Theorem 1.4, we cannot expect $(b^{-1}((r_0, \infty)), d, m)$ and $(Z \times (r_0, \infty), d' \times d_{\text{Eucl}}, m' \otimes L^1)$ to be isomorphic.

We remark that recently De Philippis and Gigli have proved ‘volume cone implies metric cone’ in the setting of RCD($K$, $N$) spaces in [23]. The proof is very involved and lengthy, and it relies on the tools and results recently developed in [28–30, 32] etc. As is indicated in [23], the techniques in [23] can be adapted to other setting. In Sect. 5, we will prove that the strongly minimal volume growth condition implies that there is a measure-preserving map from $(b^{-1}((r_0, \infty)), m)$ to $(Z \times \mathbb{R}^+, m' \otimes L^1)$, where $Z = b^{-1}(r')$ for some $r' > r_0$. Hence Theorem 1.4 is essentially a ‘volume cone implies metric cone’-type theorem in the setting of non-compact RCD(0, $N$) spaces. Most of our remaining argument in the proof of Theorem 1.4 follows the strategy in [23] closely. Since [23] has provided a complete and clear proof for the ‘volume cone implies metric cone’-type theorem, in this paper we just highlight some of the calculations due to different backgrounds and we will concentrate on the geometric outcome for the case we are dealing with; the readers can refer to [23] for more details of the proof. Since Theorem 1.4 is a kind of local version of splitting theorem on RCD(0, $N$) spaces, some of the calculations here share the similarity to those in [30, 31].

2 Preliminaries

2.1 Metric Measure Space

Throughout this paper, we will always assume the metric measure space $(X, d, m)$ we consider satisfies the following: $(X, d)$ is a complete separable locally compact geodesic space, and $m$ is a non-negative Radon measure with respect to $d$ and finite on bounded sets, supp($m$) = $X$.

A curve $\gamma : [0, T] \to X$ is called a geodesic provided $d(\gamma_s, \gamma_t) = L(\gamma|_{[s, t]})$ for every $[s, t] \subset [0, T]$, where $L(\gamma)$ means the length of the curve $\gamma$. $(X, d)$ is called a geodesic space if every two points $x, y \in X$ are connected by a geodesic $\gamma$. 

Springer
\((X, d)\) is called non-branching if for any two geodesics \(\gamma^1, \gamma^2 : [0, 1] \to X\) satisfy \(\gamma^1|_I = \gamma^2|_I\) for some interval \(I \subset [0, 1]\), then \(\gamma^1 \equiv \gamma^2\) on \([0, 1]\).

Let \(\text{Geo}(X) \subset \text{Lip}([0, 1], X)\) be the set of all geodesics with domain \([0, 1]\). We equip \(\text{Lip}([0, 1], X)\) with the uniform topology. For \(t \in [0, 1]\), define the evaluation map \(e_t : \text{Geo}(X) \to X\) by \(e_t(\gamma) = \gamma(t)\). Obviously, \(e_t(\gamma)\) is continuous.

A map \(\gamma : [0, \infty) \to X\) is called a geodesic ray if for any \(T > 0\) its restriction to \([0, T]\) is a geodesic. A map \(\gamma : \mathbb{R} \to X\) is called a line if for any \(s, t \in \mathbb{R}\), \(d(\gamma_s, \gamma_t) = |s - t|\). We will always assume that geodesic rays and lines are parametrized by unit speed.

Two metric measure spaces \((X_1, d_1, m_1), (X_2, d_2, m_2)\) with \(\text{supp}(m_1) = X_1\), \(\text{supp}(m_2) = X_2\), are said to be isomorphic provided there exists an isometry \(T : (X_1, d_1) \to (X_2, d_2)\) such that \(T_*m_1 = m_2\).

We denote by \(\mathcal{B}(X)\) the space of all Borel sets in \(X\). Denote by \(\mathcal{P}(X)\) the space of Borel probability measures on \(X\), and \(\mathcal{P}_2(X) \subset \mathcal{P}(X)\) the space of Borel probability measures \(\xi\) satisfying \(\int_X d^2(x, y)\xi(dy) < \infty\) for some (and hence all) \(x \in X\).

### 2.2 Calculus on Metric Measure Spaces

Given a function \(f \in C(X)\), the pointwise Lipschitz constant of \(f\) at \(x\) is defined as

\[
\text{lip}(f)(x) := \lim \sup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)} \in [0, +\infty]
\]

if \(x\) is not isolated, and put \(\text{lip}(f)(x) = 0\) if \(x\) is isolated.

**Definition 2.1** Let \(\pi \in \mathcal{P}(C([0, 1], X))\). We say that \(\pi\) is a test plan if

1. there exists a constant \(C > 0\) such that \((e_t)_*\pi \leq Cm\), \(\forall t \in [0, 1]\),
2. \(\int \int_0^1 |\dot{\gamma}|^2 dt d\pi(\gamma) < \infty\).

We adopt the convention that \(\int_0^1 |\dot{\gamma}|^2 dt = +\infty\) provided \(\gamma\) is not absolutely continuous, so any test plan must be concentrated on absolutely continuous curves.

**Definition 2.2** (see [5]). The Sobolev class \(S^2(X)\) is the space of all Borel functions \(f : X \to \mathbb{R}\) for which there is a non-negative function \(G \in L^2(X)\) such that

\[
\int |f(\gamma_t) - f(\gamma_0)| d\pi(\gamma) \leq \int \int_0^1 G(\gamma_t)|\dot{\gamma}_t| dt d\pi(\gamma)
\]

for every test plan \(\pi\). Any such \(G\) is called weak upper gradient for \(f\). For \(f \in S^2(X)\) there exists a minimal \(G\) in the \(m\)-a.e. sense, which is called minimal weak upper gradient and will be denoted by \(|Df|\).

The Sobolev space \(W^{1,2}(X)\) is defined as \(L^2(X) \cap S^2(X)\) and is equipped with the norm

\[
\|f\|_{W^{1,2}} := \|f\|_{L^2}^2 + \|Df\|_{L^2}^2
\]
We define \( S^2_{\text{loc}}(X) \) (resp. \( W^{1,2}_{\text{loc}}(X) \)) to be the space of functions locally equal to some function in \( S^2(X) \) (resp. \( W^{1,2}(X) \)). For \( U \subset X \) open, we define \( S^2(U) \) (resp. \( W^{1,2}(U) \)) to be the space of functions locally in \( U \) equal to some function in \( S^2(X) \) (resp. \( W^{1,2}(X) \)) such that \( |Df| \in L^2(U) \) (resp. \( f, |Df| \in L^2(U) \)). We can also define the spaces \( S^2_{\text{loc}}(U) \) or \( W^{1,2}_{\text{loc}}(U) \).

Note that every Lipschitz function \( f \) belongs to \( W^{1,2}_{\text{loc}}(X) \) and satisfies

\[
|Df| \leq \text{lip}(f), \quad m\text{-a.e.}
\]

The equality may not hold for general metric measure space. However, by the results in [17] and [5], if \((X, d, m)\) support a local doubling property and a weak local \((1, 1)\)-Poincaré inequality, then for any Lipschitz function \( f \), \( |Df| = \text{lip}(f) \) holds \( m\text{-a.e.} \).

The Dirichlet energy \( \mathcal{E} : L^2(X) \to [0, \infty] \) is defined to be

\[
\mathcal{E}(f) := \left\{ \begin{array}{ll}
\frac{1}{2} \int |Df|^2 \, dm, & \text{if } f \in W^{1,2}(X); \\
+\infty, & \text{otherwise.}
\end{array} \right.
\]  

(2.1)

**Definition 2.3** \((X, d, m)\) is called an infinitesimally Hilbertian space if \( W^{1,2}(X) \) is an Hilbert space.

On an infinitesimally Hilbertian space \((X, d, m)\), for any \( f, g \in S^2_{\text{loc}}(X) \), the map \( \langle \nabla f, \nabla g \rangle : X \to \mathbb{R} \) is \( m\text{-a.e.} \) defined to be

\[
\langle \nabla f, \nabla g \rangle := \inf_{\epsilon > 0} \frac{|D(g + \epsilon f)|^2 - |Dg|^2}{2\epsilon},
\]

where the infimum is in \( m\)-essential sense. Obviously, \( \langle \nabla f, \nabla f \rangle = |Df|^2 \). The infinitesimal Hilbertianity makes the map \( S^2(X) \ni f, g \mapsto \langle \nabla f, \nabla g \rangle \in L^1(X) \) bilinear and symmetric. Furthermore, \( \langle \nabla f, \nabla g \rangle \) satisfies the chain rule and Leibniz rule (see [28]).

If \((X, d, m)\) is infinitesimally Hilbertian, given an open set \( U \subset X \), \( D(\Delta, U) \subset W^{1,2}_{\text{loc}}(U) \) is the space of Borel functions \( f \in W^{1,2}_{\text{loc}}(U) \) such that there exists a signed Radon measures \( \mu \) on \( U \) such that

\[
\int g \, d\mu = -\int \langle \nabla f, \nabla g \rangle \, dm
\]

(2.2)

holds for any \( g : X \to \mathbb{R} \) Lipschitz with \( \text{supp}(g) \subset U \). \( \mu \) is uniquely characterized and we denote it by \( \Delta f \big|_U \). In case \( U = X \) we simply write \( g \in D(\Delta) \) and \( \mu = \Delta g \).

The space \( D(\Delta) \subset W^{1,2}(X) \) is the space of functions \( f \) for which there is a function in \( L^2(X) \), called the Laplacian of \( f \) and denoted by \( \Delta f \), such that

\[
\int g \Delta f \, dm = -\int \langle \nabla f, \nabla g \rangle \, dm, \quad \forall g \in W^{1,2}(X).
\]

(2.3)
If $(X, d, m)$ is a proper infinitesimally Hilbertian space, then by Proposition 4.24 in [28], for any $f \in W^{1,2}(X)$, $f \in D(\Delta)$ if and only if $f \in D(\Delta)$ with $\Delta f = hm$ for some $h \in L^2(X, m)$. Furthermore, if this holds then we have $h = \Delta f$.

In [29], the notions of tangent and cotangent modules of a metric measure space $(X, d, m)$ are introduced. Denote the tangent and cotangent modules by $L^2(TX)$ and $L^2(T^*X)$, respectively. The pointwise norm on both spaces will be denoted by $| \cdot |$. The differential of a function $f \in W^{1,2}(X)$ is an element $df \in L^2(T^*X)$ defined in Sect. 2.2.2 of [29]. The differential operator $d$ satisfies the locality property, chain rule, and Leibniz rule. For $f \in W^{1,2}(X)$, $|df| = |Df|$ holds $m$-a.e.. In case $X$ is infinitesimally Hilbertian, the gradient $\nabla f \in L^2(TX)$ of $f \in W^{1,2}(X)$ is the unique element associated with the differential $df$ via the Riesz isomorphism for modules.

The notion of divergence of vector fields was introduced in [29] (see [34] for related previous work). The space $D(\text{div}) \subset L^2(TX)$ is the set of all vector fields $V$ for which there exists $f \in L^2(X)$ such that

$$
\int fgdm = -\int dg(V)dm, \quad \forall g \in W^{1,2}(X).
$$

$f$ is uniquely characterized, we call it the divergence of $V$ and denote it by $\text{div}(V)$. The Leibniz rule holds for the divergence. Suppose $(X, d, m)$ is infinitesimally Hilbertian, $f \in W^{1,2}(X)$, then $\nabla f \in D(\text{div})$ iff $f \in D(\Delta)$, and in this case, we have $\text{div}(\nabla f) = \Delta f$. See Sect. 2.3.3 in [29] for more details about the divergence.

### 2.3 Optimal Transport

Let $c : X \times X \to \mathbb{R}$ be the function $c(x, y) = \frac{d^2(x,y)}{2}$. For $\mu, \nu \in \mathcal{P}_2(X)$, consider their Wasserstein distance $W^2_2(\mu, \nu)$ defined by

$$W^2_2(\mu, \nu) = \min_{\eta \in \Gamma(\mu, \nu)} \int_{X \times X} d^2(x, y) d\eta(x, y), \quad (2.4)$$

where $\Gamma(\mu, \nu)$ is the set of Borel probability measures $\eta$ on $X \times X$ satisfying $\eta(A \times X) = \mu(A), \eta(X \times A) = \nu(A)$ for every Borel set $A \subset X$. We call a plan $\eta$ that minimizes (2.4) an optimal plan.

Since $(X, d)$ is a geodesic space, $W_2$ can be equivalently characterized as

$$W^2_2(\mu, \nu) = \min \int \int_0^1 |\dot{\gamma}|^2 dt d\pi(\gamma), \quad (2.5)$$

where the minimum is taken among all $\pi \in \mathcal{P}(C([0, 1], X))$ such that $(e_0)_*\pi = \mu$ and $(e_1)_*\pi = \nu$. The set of optimal dynamical plans realizing the minimum in (2.5) is denoted by $\text{OptGeo}(\mu, \nu)$.

Given a map $\varphi : X \to \mathbb{R} \cup \{-\infty\}$, its $c$-transform $\varphi^c : X \to \mathbb{R} \cup \{-\infty\}$ is defined to be

$\mathcal{S}$ Springer
A function \( \varphi : X \to \mathbb{R} \cup \{ -\infty \} \) is called \( c \)-concave provided it is not identically \(-\infty\) and it holds \( \varphi = \psi^c \) for some \( \psi \).

For a general function \( \varphi : X \to \mathbb{R} \cup \{ -\infty \} \) it always holds \( \varphi^{cc} \geq \varphi \). (2.6)

It turns out that \( \varphi \) is \( c \)-concave if and only if \( \varphi^{cc} = \varphi \). The \( c \)-superdifferential \( \partial^c \varphi \) of the \( c \)-concave function \( \varphi \) is defined by

\[
\partial^c \varphi := \left\{ (x, y) \in X \times X | \varphi(x) + \varphi^c(y) = \frac{d^2(x, y)}{2} \right\}.
\]

The set \( \partial^c \varphi(x) \subset X \) is the set of those \( y \)'s such that \( (x, y) \in \partial^c \varphi \).

A subset \( \Upsilon \subset X \times X \) is said to be \( c \)-cyclically monotone if for any \( N \in \mathbb{N} \) and \( \{(x_i, y_i)\}_{i \leq N} \subset \Upsilon \), we have (with the convention \( y_{N+1} = y_1 \))

\[
\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}). \quad (2.7)
\]

It is well known that the concepts of optimal plan, \( c \)-concave function, and \( c \)-cyclically monotone set are closely related, see e.g., Theorem 5.10 in [49] or Theorem 2.13 in [1].

### 2.4 The Curvature-Dimension Conditions

For \( N \in (1, \infty) \) we define the functional \( \mathcal{U}_N : \mathcal{P}_2(X) \to [-\infty, 0] \) to be

\[
\mathcal{U}_N(\mu) := -\int \rho^{1-\frac{1}{N}} dm, \quad \mu = \rho m + \mu^s, \mu^s \perp m,
\]

and for \( N = 1 \), we define \( \mathcal{U}_1 : \mathcal{P}_2(X) \to [-\infty, 0] \) to be

\[
\mathcal{U}_1(\mu) := -m(\{ \rho > 0 \}), \quad \mu = \rho m + \mu^s, \mu^s \perp m.
\]

For \( N \in [1, \infty) \), \( K \in \mathbb{R} \), the distortion coefficients \( \tau^{(\nu)}_{K, N}(\theta) \) are functions \( [0, 1] \times [0, \infty) \ni (t, \theta) \mapsto \tau^{(\nu)}_{K, N}(\theta) \in [0, +\infty] \) defined by
We say a metric measure space $(X, d, m)$ satisfies the curvature-dimension condition $CD(K, N)$ if and only if for any $\mu_0, \mu_1 \in P_2(X)$ with bounded support and $\mu_0 = \rho_0 m$, $\mu_1 = \rho_1 m$, there exists a geodesic $\{\mu_t\}_{t \in [0, 1]} \subset P_2(X)$ connecting $\mu_0, \mu_1$ and inducing an optimal coupling $\pi$ of $\mu_0, \mu_1$ such that

$$U_{N'}(\mu_t) \leq - \int_{X \times X} \left[ \tau_{K, N'}^{(1-t)}(d(x_0, x_1)) \rho_0^{\frac{1}{N'}}(x_0) + \tau_{K, N'}^{(t)}(d(x_0, x_1)) \rho_1^{\frac{1}{N'}}(x_1) \right] d\pi(x_0, x_1)$$

for all $t \in [0, 1]$ and $N' \in [N, \infty)$.

Let $\sigma_{K, N-1}^{(t)}(\theta) := \left[ \tau_{K, N}^{(t)}(\theta) \right]^{\frac{N}{N-1}}$.

**Definition 2.5** We say $(X, d, m)$ satisfies the reduced curvature-dimension condition $CD^*(K, N)$ if we replace in Definition 2.4 the coefficients $\tau_{K, N}^{(t)}(\theta)$ by $\sigma_{K, N}^{(t)}(\theta)$.

**Definition 2.6** A $CD^*(K, N)$ space which is also infinitesimally Hilbertian is called an $RCD^*(K, N)$ space.

Obviously, $CD(0, N)$ and $CD^*(0, N)$ conditions are equivalent. In this paper, we use $RCD(0, N)$ instead of $RCD^*(0, N)$.

**Definition 2.7** We say a metric measure space $(X, d, m)$ satisfies the $(K, N)$-measure contraction property $(MCP(K, N))$ if for every point $x \in X$ and $m$-measurable set $A \subset X$ with $m(A) \in (0, \infty)$, there exists $\Xi \in \text{OptGeo} \left( \frac{1}{m(A)} m|_A, \delta_x \right)$, such that for every $t \in [0, 1]$,

$$m \geq (e_t)_* \left( \left( \tau_{K, N}^{(1-t)}(d(y_0, x)) \right)^N m(A) \Xi \right).$$

(2.10)

An $RCD^*(K, N)$ space always satisfies the $MCP(K, N)$ condition (see [33]).

The Bishop–Gromov volume comparison estimate holds on $RCD^*(K, N)$, in particular, for an $RCD(0, N)$ space, we have

$$\frac{m(B_x(r))}{m(B_x(R))} \geq \frac{r^N}{R^N}, \quad \forall x \in X, 0 \leq r \leq R.$$

(2.11)
From (2.11), we can easily obtain
\[
\frac{m(B_x(R))}{R^N} \geq \frac{m(B_x(R)) - m(B_x(r))}{R^N - r^N}, \quad \forall x \in X, 0 \leq r \leq R. \tag{2.12}
\]

An interesting application of (2.12) is Proposition 2.8. Proposition 2.8 generalizes the famous theorem on non-compact manifolds with non-negative Ricci curvature proved independently by Calabi [11] and Yau [50].

**Proposition 2.8** Suppose \((X, d, m)\) is a non-compact RCD\((0, N)\) space with \(N \geq 1\), then \(X\) has at least linear volume growth. More precisely, for every \(p \in X\), there exists a constant \(C\) depending only on \(m(B_p(1))\) and \(N\) such that
\[
m(B_p(r)) \geq Cr.
\]

**Proof** Let \(x \in \partial B_p(1 + r)\), then we have \(B_p(1) \subset B_x(2 + r) \setminus B_x(r)\) and \(B_x(2 + r) \subset B_p(3 + 2r)\). Thus we have
\[
m(B_p(1)) \leq m(B_x(2 + r) - m(B_x(r)), \tag{2.13}
m(B_x(2 + r)) \leq m(B_p(3 + 2r)). \tag{2.14}
\]

On the other hand, by (2.12), we have
\[
m(B_x(2 + r)) - m(B_x(r)) \leq m(B_x(2 + r))(2 + r)^N - r^N \tag{2.15}
\]

Combining (2.13), (2.14), and (2.15), we obtain
\[
m(B_p(3 + 2r)) \geq m(B_p(1))(2 + r)^N - r^N \geq Cr
\]
and finish the proof. \(\square\)

We remark that in the proof of Proposition 2.8, we only use the Bishop–Gromov volume comparison estimate (2.11), thus similar property also holds for non-compact MCP\((0, N)\) or CD\((0, N)\) spaces.

In the following, we review some results on RCD\((0, N)\) spaces that will be used in this paper. In fact, similar results are valid for general RCD\(^*\)(\(K, N\)) spaces, but we state them for RCD\((0, N)\) spaces for simplicity.

An RCD\((0, N)\) space always has the Sobolev-to-Lipschitz property, i.e., any \(f \in W^{1,2}(X)\) with \(|Df| \leq 1\) \(m\)-a.e. admits a 1-Lipschitz representative (see [6, 30]).

On an RCD\((0, N)\) space, we consider the following space of test functions:
\[
\text{Test}(X) := \{ f \in D(\Delta) \mid f, |Df| \in L^\infty(X), \Delta f \in W^{1,2}(X) \}.
\]

Test\((X)\) is dense in \(W^{1,2}(X)\).
On an RCD\((0, N)\) space \((X, d, m)\), \(\mathcal{E}\) is a quadratic form. By the theory of gradient flows of convex and lower semicontinuous functions on Hilbert spaces (see e.g., [4] for a comprehensive presentation), the heat flow \(H_t : L^2(X) \to L^2(X), \ t \geq 0\) is the unique family of maps such that for any \(f \in L^2(X)\) the curve \(t \mapsto H_t(f) \in L^2(X)\) is continuous on \([0, \infty)\), locally absolutely continuous on \((0, \infty)\), and fulfills \(H_0(f) = f, H_t(f) \in D(\Delta)\) for every \(t > 0\), and

\[
\frac{d}{dt} H_t(f) = \Delta H_t(f), \quad \mathcal{L}^1\text{-a.e.}t > 0.
\]

Some classical results are

\[
\frac{d}{dt} \| H_t(f) \|_{L^2}^2 = -4\mathcal{E}(H_t(f)), \quad \forall t > 0;
\]

\[
\| H_t(f) \|_{L^2} \leq \| f \|_{L^2}, \quad \forall t \geq 0;
\]

\[
\mathcal{E}(H_t(f)) \leq \frac{1}{4t} \| f \|_{L^2}^2, \quad \forall t > 0.
\]

On RCD\((0, N)\) spaces, the following Bochner inequality holds (see [3, 24]): for all \(f, g \in \text{Test}(X)\) with \(g \geq 0\), we have

\[
\frac{1}{2} \int \Delta g |Df|^2 dm \geq \int g \left[ \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle \right] dm.
\]

The following lemma can be found in [23] (see [2, 40] for related results), which provides cut-off functions with quantitative estimates on RCD\((0, N)\).

**Lemma 2.9** Suppose \((X, d, m)\) is an RCD\((0, N)\) space. For every \(r > 0\) there exists a constant \(C(r) > 0\) such that the following holds. Given \(K \subset U \subset X\) with \(K\) compact and \(U\) open such that \(\inf_{x \in K, y \in U} d(x, y) \geq r\), there exists a test function \(\chi\) with values in \([0, 1]\), which is 1 on \(K\), with \(\text{supp}(\chi) \subset \subset U\) and satisfying

\[
\text{Lip}(\chi) + \| \Delta \chi \|_{L^\infty} \leq C(r).
\]

For an RCD\((0, N)\) space \((X, d, m)\), the notion of second-order Sobolev space \(W^{2, 2}(X)\) can be introduced as in [29]. Since \(L^2(T^*X)\) is a Hilbert module, we can define the Hilbert tensor product \(L^2((T^*)^2 X)\). See Sect. 1.5 in [29] for related notions.

**Definition 2.10** A function \(f \in W^{1, 2}(X)\) belongs to \(W^{2, 2}(X)\) provided there is an element of \(L^2((T^*)^2 X)\), called the Hessian of \(f\) and denoted by \(\text{Hess}(f)\), such that for any \(g_1, g_2, h \in \text{Test}(X)\) it holds

\[
2 \int h \text{Hess}(f)(\nabla g_1, \nabla g_2) = \int \left[ -\langle \nabla f, \nabla g_1 \rangle \text{div}(h \nabla g_2) - \langle \nabla f, \nabla g_2 \rangle \text{div}(h \nabla g_1) \\
- h \langle \nabla f, \nabla (\langle \nabla g_1, \nabla g_2 \rangle) \rangle \right] dm.
\]
$W^{2,2}(X)$ is equipped with the norm
\[ \| f \|_{W^{2,2}}^2 = \int \left( |f|^2 + |Df|^2 + |\text{Hess}(f)|^2 \right) dm, \]
which makes it a separable Hilbert space.

Hessian satisfies the chain rule and Leibniz rule as well as the locality property, see Propositions 3.3.20 to 3.3.24 in [29] for precise statements.

By the locality property of Hess, given an open subset $U$, we define $W^{1,2}_{\text{loc}}(U)$ as the subspace of $W^{1,2}(X)$ consisting of functions $f$ for which there is $\text{Hess}(f) \in L^2((T^*)^2 X)$ such that (2.20) holds for any $g_1, g_2, h \in \text{Test}(X)$ with support in $U$.

Finally, let us recall a useful approximation lemma, which is well known to experts:

**Lemma 2.11** Suppose $(X, d, m)$ is an $\text{RCD}^*(K, N)$ space. If $f \in W^{1,2}(X)$ satisfies $\text{supp}(f) \subset U$, where $U$ is an open set, then there exists a sequence of functions $f_i \in \text{Test}(X)$ such that $\text{supp}(f_i) \subset \subset U$, $f_i \in L^\infty(X)$ and $f_i$ converging to $f$ in $W^{1,2}(X)$.

### 2.5 Disintegration

**Definition 2.12** Suppose $(X, \Omega, \mu)$ and $(Y, \Sigma, \nu)$ are measure spaces, and $f : X \to Y$ is a measurable map. A disintegration of $\mu$ over $\nu$ consistent with $f$ is a map $\rho : \Omega \times Y \to [0, \infty]$ such that

1. $\rho_y(\cdot)$ is a measure on $(X, \Omega)$ for every $y \in Y$,
2. $\rho(\cdot, B)$ is $\nu$-measurable for all $B \in \Sigma$,
3. the consistency condition
\[ \mu(B \cap f^{-1}(C)) = \int_C \rho_y(B) \nu(dy) \]
holds for all $B \in \Omega, C \in \Sigma$.

We say that the disintegration is unique if $\rho_1, \rho_2$ are two consistent disintegrations then $\rho_{1,y}(\cdot) = \rho_{2,y}(\cdot)$ for $\nu$-a.e. $y \in Y$.

A disintegration is strongly consistent with $f$ if $\rho_y(X \setminus f^{-1}([y])) = 0$ holds for $\nu$-a.e. $y$.

We recall the following version of the disintegration theorem which can be found in [25] (see 452G and 452I) or [9] (see Theorem A.7).

**Theorem 2.13** (Disintegration of measures) Suppose $(X, \Omega, \mu)$ and $(Y, \Sigma, \nu)$ are measure spaces such that $(X, \Omega, \mu)$ is a countably generated Radon measure space and $(Y, \Sigma)$ is countably separated. Suppose there is an inverse-measure-preserving map $f : X \to Y$, then there exists a unique disintegration $y \mapsto \rho_y$ over $\nu$ strongly consistent with $f$. 

© Springer
We recall that a measurable space \((X, \Omega)\) is countably separated if there is a countable set \(\Upsilon \subset \Omega\) such that for any distinct \(x, y \in X\) there is some \(E \in \Upsilon\) such that \(x \in E\) and \(y \notin E\).

Let \((X, \Omega, \mu)\) and \((Y, \Sigma, \nu)\) be measure spaces, a map \(f : X \to Y\) is called inverse-measure-preserving provided for any \(C \in \Sigma\), it holds \(f^{-1}(C) \in \Omega\) and \(\nu(C) = \mu(f^{-1}(C))\).

Note that the \(\rho_y\) in Theorem 2.13 are probability measures for \(\nu\)-a.e. \(y\).

### 3 Busemann Function and Disintegration Revisited

#### 3.1 General Properties of Busemann Functions

Let \((X, d)\) be a complete non-compact separable locally compact geodesic space. Firstly, let’s recall some classical notions about Busemann functions.

Given a geodesic ray \(\gamma\) emitted from \(p\), for any \(t \geq 0\), denote by \(b_t(x) := t - d(x, \gamma_t)\). By the triangle inequality, it is easy to see

1. given any \(x \in X\), the function \(t \mapsto b_t(x)\) is non-decreasing;
2. \(b_t(x) \leq d(x, p)\) for all \(t \geq 0\).

We define the Busemann function associated with \(\gamma\) as

\[
b(x) := \lim_{t \to +\infty} b_t(x).
\]

Note that the convergence is uniform on any given compact set.

Since \(b_t(x)\) are all 1-Lipschitz functions, \(b\) is also 1-Lipschitz.

For any given \(x \in X\), let \(\gamma^{t,x} : [0, d(x, \gamma_t)] \to X\) be a unit speed geodesic connecting \(x\) to \(\gamma_t\), where \(t \geq 0\). By the properness of \(X\), there is a sequence \(\{t_n\}\), with \(t_n \to \infty\), such that \(\gamma^{t_n,x}\) converge on compact sets to a geodesic ray \(\gamma^x : [0, \infty) \to X\) with \(\gamma^x(0) = x\). Such a ray \(\gamma^x\) is called a Busemann ray associated with \(\gamma\). We note that different choices of sequences \(\{t_n\}\) may give different Busemann rays.

**Lemma 3.1** For every \(t \geq 0\), we have

\[
b(\gamma^x(t)) = b(x) + t.
\]

**Proof** Suppose \(\gamma^{t_n,x} : [0, d(x, \gamma_{t_n})] \to X\) converge uniformly on compact sets to \(\gamma^x : [0, \infty) \to X\). Then \(\gamma^{t_n,x}(t)\) converge to \(\gamma^x(t)\). Thus

\[
b(\gamma^x(t)) = \lim_{n \to \infty} (t_n - d(\gamma^x(t), \gamma(t_n)))
\]

\[
= \lim_{n \to \infty} [t_n - d(\gamma^{t_n,x}(t), \gamma(t_n)) + (d(\gamma^{t_n,x}(t), \gamma(t_n)) - d(\gamma^x(t), \gamma(t_n)))]
\]

\[
= \lim_{n \to \infty} (t_n - d(x, \gamma(t_n))) + t = b(x) + t,
\]

and we finish the proof. \(\square\)
For any \( s \in \mathbb{R} \), we denote \( \Omega_s := \{ x \in X \mid b(x) > s \} \). For \( c < d \), we denote the set \( \{ x \in X \mid b(x) \in [c, d] \} \) by \( \Omega_{[c,d]} \).

**Lemma 3.2** For any \( s \in \mathbb{R} \), \( \Omega_s \) is path-connected. More precisely, any two points \( x, y \in \Omega_s \) can be connected by a Lipschitz curve \( \sigma \subset \Omega_s \).

**Proof** We only need to prove that any \( x \in \Omega_s \) can be connected to \( \gamma_{s+1} \) by a Lipschitz curve \( \sigma \subset \Omega_s \). Firstly, \( \gamma^x(t) \subset \Omega_s \) connects \( x \) and \( y = \gamma^x(1) \), with \( b(y) \geq s + 1 \). Since \( b(y) = \lim_{t \to -\infty} (t - d(y, \gamma_t)) \), we find \( t \) sufficiently large such that \( b_t(y) = t - d(y, \gamma_t) > s + \frac{1}{2} \). Let \( \sigma : [0, d(y, \gamma_{s+1})] \to X \) be a unit speed geodesic connecting \( y \) to \( \gamma_{s+1} \). Then for any \( r \in [0, d(y, \gamma_{s+1})] \), we have \( b_t(\sigma(r)) = b_t(y) + r \). Thus \( b_t(\sigma(r)) \geq b_t(\sigma(t)) > s + \frac{1}{2} \) for any \( r \in [0, d(y, \gamma_{s+1})] \). So \( \sigma \subset \Omega_s \) connects \( y \) and \( \gamma_{s+1} \). Glue the above geodesics to form a Lipschitz curve and we complete the proof. \( \Box \)

**Lemma 3.3** Suppose \( s_1 < s_2 \) and \( x_1 \in b^{-1}(s_1) \), \( x_2 \in b^{-1}(s_2) \), then \( d(x_1, x_2) \geq s_2 - s_1 \).

**Proof** Suppose on the contrary, \( d(x_1, x_2) < s_2 - s_1 \). Since \( b \) is 1-Lipschitz, we have \( s_2 - s_1 = b(x_2) - b(x_1) \leq d(x_1, x_2) < s_2 - s_1 \), which is a contradiction. \( \Box \)

On the other hand, suppose \( s_1 < s_2 \) and \( x \in b^{-1}(s_1) \), since \( \gamma^x([0, \infty)) \cap b^{-1}(s_2) = \gamma^x(s_2 - s_1) \) by Lemma 3.1, we have \( \text{dist}(b^{-1}(s_1), b^{-1}(s_2)) \leq s_2 - s_1 \). Combining this with Lemma 3.3, we obtain

**Corollary 3.4** Suppose \( s_1 < s_2 \) and \( b^{-1}(s_1) \neq \emptyset \), then \( \text{dist}(b^{-1}(s_1), b^{-1}(s_2)) = s_2 - s_1 \).

**Corollary 3.5** Suppose \( s_1 < s_2 \), \( b^{-1}(s_1) \neq \emptyset \) and \( \text{diam}(b^{-1}(s_2)) \leq T \), then \( \text{diam}(b^{-1}(s_1)) \leq T + 2(s_2 - s_1) \) and \( \text{diam}(\Omega_{[s_1, s_2]}) \leq T + 2(s_2 - s_1) \).

In Lemma 5.18 of [28], it is proved that for a Busemann function \( b \), \( -b \) is \( c \)-concave, where \( c = \frac{d^2}{2} \). We extend this result to the following proposition:

**Proposition 3.6** For any \( t > 0 \), the function \( -tb \) is \( c \)-concave, where \( c = \frac{d^2}{2} \). Furthermore, for any \( x, \bar{x} \in X \) satisfying \( b(\bar{x}) - b(x) = d(x, \bar{x}) = t \), it holds \( (x, \bar{x}) \in \partial^c(-tb) \).

**Proof** For any \( x, \bar{x} \in X \) satisfying \( b(\bar{x}) - b(x) = d(x, \bar{x}) = t \), we have

\[
(-tb)^c(x) = \inf_{y \in X} \left[ \frac{d^2(x, y)}{2} - (-tb)^c(y) \right] \\
\leq \frac{d^2(x, \bar{x})}{2} - (-tb)^c(\bar{x}) \\
= \frac{t^2}{2} - \inf_{z \in X} \left[ \frac{d^2(\bar{x}, z)}{2} + tb(z) \right] \\
\leq \frac{t^2}{2} - \inf_{z \in X} \left[ \frac{d^2(\bar{x}, z)}{2} + tb(\bar{x}) - td(\bar{x}, z) \right]
\]
\[
= \frac{t^2}{2} - \inf_{z \in X} \left[ \frac{d^2(\tilde{x}, z)}{2} + tb(x) + t^2 - td(\tilde{x}, z) \right]
\]

\[
= -tb(x) - \inf_{z \in X} \left[ \frac{d^2(\tilde{x}, z)}{2} - td(\tilde{x}, z) + \frac{t^2}{2} \right]
\]

\[
= -tb(x) - \frac{1}{2} \inf_{z \in X} [d(\tilde{x}, z) - t]^2
\]

\[\leq -tb(x).\] (3.1)

On the other hand, by (2.6), \((-tb)^{cc} \geq -tb\) always holds, thus \((-tb)^{cc}(x) = -tb(x)\), and all the inequalities in (3.1) must be equalities. In particular, we have

\[-tb(x) = \frac{d^2(x, \tilde{x})}{2} - (-tb)^c(\tilde{x}),\]

i.e., \((x, \tilde{x}) \in \partial^c(-tb)\). \(\square\)

The main result of this section is that we have a monotonic property on the area of level sets of a Busemann function under suitable conditions, see Proposition 3.21, Corollary 3.22 and similar properties in Sect. 3.3. The tool we use is the general results established in [8] and [12]. In the following, we will present how the notions and results in [8] and [12] can be modified to our setting, and obtain some special properties on the transport set of Busemann function.

**Definition 3.7** The set of couples moved by \(b\) is defined to be

\[\Gamma := \{(x, y) \in X \times X \mid b(y) - b(x) = d(x, y)\}.\] (3.2)

**Lemma 3.8** Suppose \((x, y), (y, z) \in \Gamma\), then \((x, z) \in \Gamma\) and \(d(x, z) = d(x, y) + d(y, z)\).

**Proof** Because \(d(x, z) \leq d(x, y) + d(y, z) = b(z) - b(x) \leq d(x, y)\). \(\square\)

**Lemma 3.9** Let \((x, y) \in X \times X\) be an element of \(\Gamma\). Let \(\sigma \in \text{Geo}(X)\) be such that \(\sigma_0 = x\) and \(\sigma_1 = y\). Then \((\sigma_s, \sigma_t) \in \Gamma\) for all \(0 \leq s \leq t \leq 1\).

**Proof** For \(0 \leq s \leq t \leq 1\), we have

\[d(\sigma_s, \sigma_t) \geq b(\sigma_t) - b(\sigma_s) = (b(\sigma_1) - b(\sigma_0)) + (b(\sigma_t) - b(\sigma_1)) + (b(\sigma_0) - b(\sigma_s)) \geq d(\sigma_0, \sigma_1) - d(\sigma_t, \sigma_1) - d(\sigma_0, \sigma_s) = d(\sigma_s, \sigma_t),\]

the claim follows. \(\square\)

It is then natural to consider the set of geodesics \(G := \{\sigma \in \text{Geo}(X) \mid (\sigma_0, \sigma_1) \in \Gamma\}\).

For any \(x\) and a Busemann ray \(\gamma^x : [0, \infty) \to X\), by Lemma 3.1 we have \((\gamma^x_s, \gamma^x_t) \in \Gamma\) for any \(s \leq t\).
**Definition 3.10** We define the set of transport rays by

\[ R = \Gamma \cup \Gamma^{-1}, \]

where \( \Gamma^{-1} := \{(x, y) \in X \times X \mid (y, x) \in \Gamma\} \). For fixed \( x \), we use \( \Gamma(x) \) to denote \( \{y \in X \mid (y, x) \in \Gamma\} \), \( \Gamma^{-1}(x) \) to denote \( \{y \in X \mid (y, x) \in \Gamma\} \). Set \( R(x) = \Gamma(x) \cup \Gamma^{-1}(x) \).

**Remark 3.11** Since \( b \) is 1-Lipschitz, \( \Gamma, \Gamma^{-1} \), and \( R \) are all closed. Moreover \( \Gamma, \Gamma^{-1} \), and \( R \) are all \( \sigma \)-compact because \( X \) is proper.

### 3.2 RCD(0,N) Case

In this subsection, we use the cost function \( c = \frac{d^2}{2} \). Firstly, we recall the following result from [33]:

**Theorem 3.12** (Theorem 1.3 in [33]) Suppose \( (X, d, m) \) is an \( RCD^*(K, N) \) space and \( \varphi : X \to \mathbb{R} \) a c-concave function. Then for \( m \)-a.e. \( x \in X \) there exists exactly one geodesic \( \eta \) such that \( \eta_0 = x \) and \( \eta_1 \in \partial c \varphi(x) \).

Theorem 3.12 is equivalent to the fact that on an \( RCD^*(K, N) \) space \( (X, d, m) \), for every \( \mu, \nu \in \mathcal{P}_2(X) \) with \( \mu \ll m \), there exists a unique plan \( \pi \in \text{OptGeo}(\mu, \nu) \) and this \( \pi \) is induced by a map and concentrated on a set of non-branching geodesics. See [33]. This generalizes the results in [27, 44].

In this subsection, we always assume \( (X, d, m) \) is non-compact and satisfies the \( RCD^*(K, N) \) condition.

By Proposition 3.6, for any \( n \in \mathbb{N} \), the function \( -nb \) is \( c \)-concave. Now we apply Theorem 3.12 to the function \( -nb \) and obtain that, for any \( n \in \mathbb{N} \), there exists a Borel set \( B_n \subset X \) satisfying the following: \( m(B_n) = 0 \), and if we denote by \( A_n = X \setminus B_n \), then for any \( x \in A_n \), there exists exactly one point \( y \in X \) such that \( b(y) - b(x) = d(x, y) = n \) and there is only one geodesic \( \eta^{(x)} : [0, n] \to X \) such that \( \eta^{(x)}(0) = x \) and \( \eta^{(x)}(n) = y \).

Let \( T := \bigcap_{n=1}^{\infty} A_n \), then

**Proposition 3.13** \( m(X \setminus T) = 0 \), and for any \( x \in T \), there exists exactly one geodesic \( \eta^{(x)} : [0, \infty) \to T \) such that \( \eta^{(x)}_0 = x \) and \( b(\eta^{(x)}_t) - b(x) = t \) for any \( t \geq 0 \). Furthermore, if \( \eta^{(x)} : [0, \infty) \to T \) and \( \eta^{(y)} : [0, \infty) \to T \) satisfies \( \eta^{(x)}(s) = \eta^{(y)}(t) \) for some \( s \leq t \), then \( \eta^{(x)}(t) \) must coincide with \( \eta^{(y)}|_{[t-s, \infty)} \).

**Proof** For any \( x \in T \), let \( \gamma^x : [0, \infty) \to X \) be one of the Busemann rays, if we can prove that for any \( t > 0 \), \( \gamma^x(t) \in T \), then by the definition of \( T \), it is easy to check that \( \gamma^x \) is the only Busemann rays starting from \( x \), and it is the \( \eta^{(x)} \) we want to find. Suppose \( \gamma^x(m) \notin T \) for some \( m > 0 \), then \( \gamma^x(m) \notin A_n \) for some \( n \), thus there are two different geodesics \( \sigma_1, \sigma_2 : [0, n] \to X \) such that \( \sigma_1(0) = \sigma_2(0) = \gamma^x(m), b(\sigma_1(n)) - b(\sigma_1(0)) = d(\sigma_1(0), \sigma_1(n)) = n, i = 1, 2 \). By Lemmas 3.1, 3.8, 3.9, and Proposition 3.6, \( \gamma^x|_{[0, m]} \) and \( \sigma_i \) can be glued to form a longer geodesic \( \tilde{\sigma}_i : [0, m + n] \to X \), with \( (x, \tilde{\sigma}_i(m + n)) \in \partial c(-m - n)b \). This contradicts \( x \in T \). \( \square \)
By Proposition 3.13, it is easy to see $R$ is an equivalent relation on $T$, and for all $x \in T$, $R(x) \cap T$ forms a single geodesic ray. In addition, for distinct $x, y \in T$, either $R(x) = R(y)$, or $R(x) \cap R(y) \cap T = \emptyset$.

Making use of the continuity and local compactness of geodesics as well as a special form of selection principle (see Corollary 2.7 in [8]), one can prove the following:

**Proposition 3.14** (See Proposition 4.4 in [8]) There exists an $m$-measurable section $f : T \to T$ for the equivalence relation $R$. More precisely, there exists a saturated set $Z \subset T$ such that $Z \in \mathcal{A}(X)$, $T \setminus Z$ is $m$-negligible, and the section $f$ restricted on $Z$ is $\mathcal{A}(X)$-measurable.

Here $\mathcal{A}(X)$ denotes the $\sigma$-algebra generated by all analytic subsets in $X$.

The proof of proposition 3.14 is the same as Proposition 4.4 in [8].

Here we recall that a set $A \subset X$ is said to be saturated for the equivalence relation $E \subset X \times X$ if $A = \bigcup_{x \in A} E(x)$. A map $f : X \to X$ is called a section of an equivalence relation $E$ if for any $x, y \in X$ it holds
\[
(x, f(x)) \in E, \quad \text{and} \quad (x, y) \in E \Rightarrow f(x) = f(y).
\]

A cross section of the equivalence relation $E$ is a set $S \subset E$ such that the intersection of $S$ with each equivalence class is a singleton.

The set $D := f(T) = \{x \in T \mid d(x, f(x)) = 0\}$, is obviously a cross section. $D$ inherit the subspace topology from $T$. In particular, $(D, \mathcal{B}(D))$ is countably separated. It is not hard to prove that there exists a $\sigma$-compact set $\tilde{S} \subset D$ such that $m(T \setminus f^{-1}(\tilde{S})) = 0$, and $f$ restricted on $f^{-1}(\tilde{S})$ is Borel.

Since $m$ is $\sigma$-finite, we fix a partition $\{\Gamma_n\}_{n \geq 1}$ of $X$ into Borel sets of finite measure. Let $\{\lambda_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $\sum_{n \geq 1} \lambda_n m(\Gamma_n) = 1$, and take
\[
\tilde{m}(B) = \sum_{n \geq 1} \lambda_n m(\Gamma_n \setminus B) \tag{3.3}
\]
for $B \in \mathcal{B}(X)$. Clearly, $\tilde{m}$ is a probability measure; $m$ and $\tilde{m}$ are absolutely continuous with respect to each other. In particular, $m(B) = 0 \iff \tilde{m}(B) = 0$ for $B \in \mathcal{B}(X)$. Now the push forward measure $\tilde{v} = f_\ast \tilde{m}$ is well defined. Obviously, $\tilde{v}$ concentrates on $D$, and $\tilde{v}(D \setminus S) = 0$.

We apply Theorem 2.13 to $(f^{-1}(S), \mathcal{B}(f^{-1}(S)), \tilde{m})$, and $(S, \mathcal{B}(S), \tilde{v})$ and obtain a unique disintegration $y \mapsto \tilde{\rho}_y$ of $\tilde{m}$ over $\tilde{v}$ strongly consistent with $f$, i.e.,
\[
\tilde{m}(B \cap f^{-1}(C)) = \int_C \tilde{\rho}_y(B) \tilde{v}(dy) \tag{3.4}
\]
holds for every $B \in \mathcal{B}(f^{-1}(S)), C \in \mathcal{B}(S)$. 

$\square$ Springer
We define $\rho : \mathcal{B}(f^{-1}(S)) \times S \to [0, \infty]$ by

$$\rho_y(B) = \sum_{n \geq 1} \lambda_n^{-1} \hat{\rho}_y(\Gamma_n \cap B)$$

for every $B \in \mathcal{B}(f^{-1}(S))$ and $y \in S$, then we have

$$m(B \cap f^{-1}(C)) = \sum_{n \geq 1} \lambda_n^{-1} \tilde{m}(\Gamma_n \cap B \cap f^{-1}(C)) = \sum_{n \geq 1} \lambda_n^{-1} \int_C \tilde{\rho}_y(\Gamma_n \cap B) \tilde{\nu}(dy) = \int_C \rho_y(B) \tilde{\nu}(dy).$$

Thus $y \mapsto \rho_y$ is a disintegration of $m$ over $\tilde{\nu}$ consistent with $f$. It is easy to check $y \mapsto \rho_y$ is the unique strongly consistent disintegration w.r.t. $f$ over $\tilde{\nu}$.

Since $\tilde{\nu}(D \setminus S) = m(T \setminus f^{-1}(S)) = 0$, for any $B \subset T$ $m$-measurable and $C \subset D$ $\tilde{\nu}$-measurable, it holds

$$m(B \cap f^{-1}(C)) = \int_C \rho_y(B) \tilde{\nu}(dy), \quad (3.5)$$

and for $\tilde{\nu}$-a.e. $y \in D$, it holds

$$\rho_y(f^{-1}(D \setminus \{y\})) = 0. \quad (3.6)$$

We define the ray map as follows:

**Definition 3.15** Define the ray map $g : \text{Dom}(g) \subset D \times \mathbb{R} \to T$ by the formula

$$\text{graph}(g) := \{(y, t, x) \in D \times \mathbb{R} \times T \mid y \in D, t \in \mathbb{R}, x \in R(y), b(x) = t\}.$$ 

Since $R(y) \cap T$ is the single geodesic for $y \in D$, and the restriction of $b$ on $R(y)$ is a strictly monotonic function, the set in the above definition is clearly the graph of some map $g$.

From the definition of the ray map, we immediately obtain the following properties:

**Proposition 3.16** The following properties hold:

1. The ray map $g$ restricted on $\text{Dom}(g) \cap S \times \mathbb{R}$ is Borel.
2. For every fixed $y \in D$, the map $t \mapsto g(y, t)$ is 1-Lipschitz $\Gamma$-order preserving.
3. $(y, t) \mapsto g(y, t)$ is bijective on $T$, and its inverse is

$$x \mapsto g^{-1}(x) = (f(x), b(x)),$$

where $f$ is the quotient map defined in Proposition 3.14.
We remark that in Sect. 4 of [8], there is a definition of ray map, and it has similar properties as in Proposition 3.16.

**Definition 3.17** The ray map $g$ defines a flow on $\mathcal{T}$: for $t \in \mathbb{R}^+$, we define

$$F_t(x) := g(y, s + t) \tag{3.7}$$

for any $x = g(y, s) \in \mathcal{T}$.

It is not hard to check $F_t$ is Borel for any $t \in \mathbb{R}^+$, and for every $t, s \in \mathbb{R}^+$ and $x \in \mathcal{T}$, it holds

$$F_{t+s}(x) = F_t(F_s(x)), \quad d(F_s(x), F_t(x)) = |s - t|. \tag{3.8}$$

$$d(F_s(x), F_t(x)) = |s - t|. \tag{3.9}$$

Now we recall an important estimate from [8,12]. Let $\bar{\mathcal{A}}$ be a compact set contained in $\mathcal{T}$ with $m(\bar{\mathcal{A}}) > 0$, without loss of generality, we assume $\max_{x \in \bar{\mathcal{A}}} b(x) = 0$. For any $\delta > 0$, define a Borel transport map $T$ on $\bar{\mathcal{A}}$ by

$$\bar{T} \ni x \mapsto T(x) = F_{\delta}(x),$$

where $s(x) = \delta - b(x)$.

Denote by $\mu = \frac{1}{m(\bar{\mathcal{A}})}m|_{\bar{\mathcal{A}}}$, $\nu = T_s \mu$. Then $\omega = (I_d, T)_s \mu$ be a transport plan between $\mu$ and $\nu$, and $\omega$ is concentrated on the set $\mathcal{Y} = \{(x, T(x))|x \in \bar{\mathcal{A}}\} \subset \Gamma$. For any $\{(x_i, y_i)\}_{i \leq N} \subset \mathcal{Y}$, we have

$$\sum_{i=1}^N d^2(x_i, y_i) = \sum_{i=1}^N |b(x_i) - b(y_i)|^2 \leq \sum_{i=1}^N |b(x_i) - b(y_{i+1})|^2 \leq \sum_{i=1}^N d^2(x_i, y_{i+1}), \tag{3.10}$$

that is, $\mathcal{Y}$ is $c$-cyclically monotone. Thus $\omega$ is an optimal transportation between $\mu$ and $\nu$. By Theorem 1.1 in [33], $\omega$ is the unique optimal transportation, and the curve $t \mapsto (T_t)_s \mu$, $t \in [0, 1]$, is the unique geodesic connecting $\mu$ and $\nu$, where $T_t : \bar{\mathcal{A}} \rightarrow X$ is defined by $T_t(x) = F_{t_s}(x)$ for $x \in \bar{\mathcal{A}}$. Denote by $\tilde{T}_t := \{T_t(x) | x \in \bar{\mathcal{A}}\}$. We can prove the following estimate:

$$m(\tilde{T}_t) \geq (1 - t)m(\tilde{\mathcal{A}}) \min_{x \in \bar{\mathcal{A}}} \left\{ \frac{s_K((1 - t)d(x, T(x))/\sqrt{N - 1})}{s_K(d(x, T(x))/\sqrt{N - 1})} \right\}^{N-1}, \quad \forall t \in [0, 1]. \tag{3.11}$$

Here the function $s_K : [0, +\infty) \rightarrow \mathbb{R}$ (on $[0, \pi/\sqrt{K})$ if $K > 0$) is defined to be

$$s_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$
The proof of (3.11) can be modified from Sect. 9 in [8] or Sect. 4 in [13] (see also Proposition 6.4 in [12]). We remark that among the many properties of RCD\(^*\)(K, N) spaces, the keys we use to obtain estimate (3.11) are: (X, d, m) satisfies MCP(K, N) condition; the \(L^2\)-optimal transport plans on it are unique (i.e., Theorem 1.1 in [33]).

By the inner regularity of Radon measures, it is not hard to see that the estimate (3.11) still holds if \(\tilde{A}\) is only assumed to be Borel and \(m(\tilde{A}) \in (0, \infty)\).

Under the assumption that (X, d, m) is an RCD\(^*\)(K, N) space, we can prove the absolute continuity of conditional measures:

**Theorem 3.18** For \(\tilde{v}\)-a.e. \(y \in S\), the conditional measures \(\rho_y\) are absolutely continuous w.r.t. \(g(y, \cdot)_*L^1\). More precisely, there is some function \(q(\cdot, \cdot) : \text{Dom}(g) \cap S \times \mathbb{R} \to [0, \infty)\) such that

\[
m = g_* (q \tilde{v} \otimes L^1) \tag{3.12}
\]

and

\[
\rho_y = g(y, \cdot)_*(q(y, \cdot)L^1) \tag{3.13}
\]

for \(\tilde{v}\)-a.e. \(y \in S\).

The proof of Theorem 3.18 needs the estimate (3.11), see Sect. 6.1 in [12] for details.

Furthermore we can obtain the following estimate for the function \(q\):

**Theorem 3.19** If (X, d, m) is a non-compact RCD\(^*\)(K, N) space, then

\[
\left[ \frac{s_K((\sigma_+ - t)/\sqrt{N - 1})}{s_K((\sigma_+ - s)/\sqrt{N - 1})} \right]^{N-1} \leq \frac{q(y, t)}{q(y, s)} \leq \left[ \frac{s_K((t - \sigma_-)/\sqrt{N - 1})}{s_K((s - \sigma_-)/\sqrt{N - 1})} \right]^{N-1},
\]

holds for \(\tilde{v}\)-a.e. \(y \in S\) and \(\sigma_- < s \leq t < \sigma_+\) such that \((\sigma_-, \sigma_+,) \subset \text{Dom}(g(y, \cdot))\).

Theorem 3.19 can be proved by using the disintegration formula (3.5) to localize estimates of the form (3.11). See Sect. 9 in [8] or Appendix in [12] for more details of the proof. Note that in the statement of Theorem 3.18, the function \(q\) is just a measurable function, while in the statement of Theorem 3.19, the \(q\) means one of its representative. In the following, \(q\) will always be a representative satisfying (3.14). Note that for \(\tilde{v}\)-a.e. \(y \in S\), \(t \mapsto q(y, t)\) is continuous in the \(t\) direction.

In the remaining of this subsection, we assume (X, d, m) is a non-compact RCD(0, N) space. Note that in (3.14), \(\sigma_+\) can be taken to be any large number eventually converging to \(+\infty\), and

\[
\lim_{\sigma_+ \to +\infty} \frac{s_0((\sigma_+ - t)/\sqrt{N - 1})}{s_0((\sigma_+ - s)/\sqrt{N - 1})} = \lim_{\sigma_+ \to +\infty} \frac{\sigma_+ - t}{\sigma_+ - s} = 1,
\]

thus we obtain
Corollary 3.20 If \((X, d, m)\) is a non-compact RCD\(0, N\) space, then
\[
q(y, t) \geq q(y, s)
\]
holds for \(\tilde{\nu}\)-a.e. \(y\) and \(s \leq t\) such that \([s, t] \subset \text{Dom}(g(y, \cdot))\).

Let \(K \subset b^{-1}((\neg\infty, r_0])\) be a compact set. Denote by
\[
\Xi(K) := \bigcup_{y \in K} R(y),
\]
\[
\Xi_{[s, t]}(K) := \Xi(K) \cap b^{-1}([s, t]),
\]
\[
\Xi_{s}(K) := \Xi(K) \cap b^{-1}(s),
\]
\[
\mathcal{S}(K) := \Xi(K) \cap \mathcal{S}.
\]

It is easy to check that \(\Xi(K)\), \(\Xi_{[s, t]}(K)\), and \(\Xi_{s}(K)\) are all closed sets provided \(s \geq r_0\).

Let \(K \subset b^{-1}((\neg\infty, r_0])\) be a compact set contained in \(f^{-1}(\mathcal{S})\). By the construction of \(T\) and \(\mathcal{S}\), (3.5) and Theorem 3.18, for any \(r_2 > r_1 \geq r_0\), we have
\[
m(\Xi_{[r_1, r_2]}(K)) = \int_{\mathcal{S}(K)} \left( \int_{r_1}^{r_2} q(y, s) ds \right) \tilde{\nu}(dy).
\]

By (3.22), \(m_{-1}(\Xi_{s}(K))\) is well defined for \(s \in [r_0, \infty)\). By Theorem 3.19, the map \(s \mapsto \int_{\mathcal{S}(K)} q(y, s) \tilde{\nu}(dy)\) is continuous. By Fubini’s Theorem, we have
\[
m_{-1}(\Xi_{s}(K)) = \int_{\mathcal{S}(K)} q(y, s) \tilde{\nu}(dy).
\]

By Proposition 3.21, we have:
Corollary 3.22 Suppose $K \subset b^{-1}((-\infty, r_0])$ is a compact set, then
\[
m_{-1}(\Xi r_1(K)) \leq \frac{m(\Xi [r_1,r_2](K))}{r_2 - r_1} \leq m_{-1}(\Xi r_2(K))
\] holds for any $r_2 > r_1 \geq r_0$.

We can also define the ‘codimension 1’ volume of $b^{-1}(s)$ to be
\[
m_{-1}(b^{-1}(s)) := \sup_K m_{-1}(\Xi s(K)),
\]
where the supremum is taken with respect to all compact sets $K \subset b^{-1}((-\infty, s])$.

3.3 Essentially Non-branching MCP(0,N) Case

In this subsection, we remark that all the properties in Sect. 3.2 still hold for any non-compact essentially non-branching MCP($0$, $N$) space $(X, d, m)$.

Firstly, we recall that in Sect. 3.2, there are two occasions where we use RCD($0$, $N$) (or RCD$^*$($K$, $N$)) condition. The first is the proof of Proposition 3.13, where we use Theorem 3.12. The second is the proof of the estimate (3.11), where we use two corollaries of the RCD$^*$($K$, $N$) condition, i.e., the MCP($K$, $N$) condition and the uniqueness of $L^2$-optimal transport plans.

In a recent paper [16], Cavalletti and Mondino prove that on an essentially non-branching MCP($K$, $N$) space $(X, d, m)$, for every $\mu, \nu \in P_2(X)$ with $\mu \ll m$, there exists a unique optimal dynamical plan $\pi \in \text{OptGeo}(\mu, \nu)$ and this $\pi$ is induced by a map. Equivalently, the conclusion of Theorem 3.12 holds for any essentially non-branching MCP($K$, $N$) space.

Thus with the help of results in [16], all the constructions and conclusions in Sect. 3.2 are valid for non-compact essentially non-branching MCP($0$, $N$) spaces.

Remark 3.23 On an essentially non-branching MCP($K$, $N$) space $(X, d, m)$, estimates more general than (3.11) have been obtained in Theorem 1.1 of [16] (see (1.2) in [16]).

Remark 3.24 We thank a referee for informing us of the paper [16].

4 Non-compact RCD(0,N) Spaces with Minimal Volume Growth

Recall that a non-compact RCD($0$, $N$) space has at least linear volume growth. It is a natural problem to investigate those non-compact RCD($0$, $N$) spaces $(X, d, m)$ satisfying
\[
\limsup_{R \to \infty} \frac{m(B_p(R))}{R} = V_0 < \infty.
\] We call $X$ has minimal volume growth if (4.1) holds.
In this section, the metric measure space \((X, d, m)\) is assumed to be a non-compact RCD\((0, N)\) space satisfying (4.1).

**Proposition 4.1** Suppose \((X, d, m)\) is a non-compact RCD\((0, N)\) space satisfying (4.1), then the level sets \(b^{-1}(r)\) have finite ‘codimension 1’ volume \(m_{-1}\). Furthermore, for any \(r_2 > r_1\), we have
\[
m_{-1}(b^{-1}(r_1)) \leq m_{-1}(b^{-1}(r_2)) \leq V_0. \quad (4.2)
\]

**Proof** Let \(K \subset b^{-1}((−\infty, r_1))\) be any compact set, denoted by \(\bar{r} = \max_{x \in \Xi_{[r_1, r_2]}(K)} d(p, x)\). By Corollary 3.4, \(\Xi_{[r_1, r_2]}(K) \subset m(B_p(\bar{r} + r_2 - r_1))\), and then by (3.23)
\[
m_{-1}(\Xi_{r_1}(K)) \leq \frac{m(\Xi_{[r_1, r_2]}(K))}{r_2 - r_1} \leq \frac{m(B_p(\bar{r} + r_2 - r_1))}{r_2 - r_1}, \quad (4.3)
\]
holds for every \(r_2 > r_1\). Let \(r_2 \to \infty\), by (4.1), we obtain
\[
m_{-1}(\Xi_{r_1}(K)) \leq V_0. \quad (4.4)
\]
By the arbitrariness of \(K \subset b^{-1}((−\infty, r_1))\),
\[
m_{-1}(b^{-1}(r_1)) \leq V_0 \quad (4.5)
\]
holds for any \(r_1\). By (3.23) again, we have
\[
m_{-1}(b^{-1}(r_1)) \leq m_{-1}(b^{-1}(r_2)) \quad (4.6)
\]
for any \(r_2 > r_1\). \(\square\)

By (4.2), the limit
\[
V_\infty := \lim_{r \to \infty} m_{-1}(b^{-1}(r))
\]
exists and \(0 < V_\infty \leq V_0\).

The following theorem generalizes Theorem 19 of [46] to RCD\((0, N)\) spaces, and our argument is adapted from that of [46].

**Theorem 4.2** (=Theorem 1.2) Suppose \((X, d, m)\) is a non-compact RCD\((0, N)\) space satisfying the minimal volume growth condition (4.1), and \(b\) is the Busemann function associated with a geodesic ray \(γ\). Then the diameter of the level sets \(b^{-1}(r)\) grows at most linearly. More precisely, we have
\[
\limsup_{r \to +\infty} \frac{\text{diam}(b^{-1}(r))}{r} \leq C_0 \leq 2, \quad (4.7)
\]
where the diameter of \(b^{-1}(r)\) is computed with respect to the distance \(d\). In particular, \(b^{-1}(r)\) is compact.

Springer
Proof. We argue by contradiction. Suppose (4.7) does not hold, then there is a constant $c \in (0, \frac{1}{10})$ such that

$$\limsup_{R \to +\infty} \frac{\text{diam}(b^{-1}(R))}{R} \geq 2 + 8c. \quad (4.8)$$

By Proposition 4.1 and the definition of $V_\infty$, we can find $r_0$ and a compact set $K \subset b^{-1}((-\infty, r_0]) \cap T$ such that

$$\frac{V_\infty}{V_{r_0}} \leq \left(\frac{c}{2 + 5c}\right)^N + 1, \quad (4.9)$$

where

$$V_{r_0} := m_{-1}(\Xi_{r_0}(K)).$$

Fix an $x \in \Xi_{r_0}(K)$. For any $r > 0$, take $R = (1 + 2c)r + \text{diam}(\Xi_{r_0}(K))$. We claim that there exists $T > 0$ such that

$$b^{-1}(r + r_0) \subset B_{x}(R) \quad (4.10)$$

for every $r > T$. It is easy to see that if the claim holds, then it contradicts (4.8) and thus (4.7) is proved. Furthermore, by Corollary 3.5, the diameter of $b^{-1}(s)$ is finite provided $\text{diam}(b^{-1}(t)) < \infty$ for some $t > s$, thus for any $r$, $\text{diam}(b^{-1}(r))$ is finite and hence $b^{-1}(r)$ is a compact subset of $X$.

Suppose (4.10) does not hold, then there exist sequences of $\{r_i\} \subset \mathbb{R}^+, \{q_i\} \subset X$ such that $r_i \to \infty$, $b(q_i) = r_0 + r_i$, $d(x, q_i) > (1 + 2c)r_i + \text{diam}(\Xi_{r_0}(K))$. By Corollary 3.4, $\Xi_{[r_0+(1-c)r_i, r_0+(1+c)r_i]}(K) \subset B_{x}(R - cr)$ holds for any $r > 0$, hence $d(q_i, \Xi_{[r_0+(1-c)r_i, r_0+(1+c)r_i]}(K)) > cr_i$. On the other hand, by $b(q_i) = r_0 + r_i$ and Corollary 3.4, $d(q_i, \Xi(K) \setminus \Xi_{[r_0+(1-c)r_i, r_0+(1+c)r_i]}(K)) > cr_i$ also holds. Thus $d(q_i, \Xi(K)) > cr_i$ for every $i$.

By Lemma 3.2, for every $i$ there exists a curve $\sigma_i : [0, 1] \to b^{-1}([r_0 + r_i, \infty))$ such that $\sigma_i(0) = q_i$ and $\sigma_i(1) \in \Xi_{r_0+r_i}(K)$. By

$$d(\sigma_i(0), \Xi(K)) - c(b(\sigma_i(0)) - r_0) > 0,$$

and

$$d(\sigma_i(1), \Xi(K)) - c(b(\sigma_i(1)) - r_0) = -cr_i < 0,$$

there exists $p_i = \sigma_i(t_i)$ with $t_i \in (0, 1)$ such that

$$d(p_i, \Xi(K)) = c(b(p_i) - r_0). \quad (4.11)$$

Denote by $h_i = b(p_i) - r_0$. Obviously $h_i \to \infty$. 

\[\Box\] Springer
Denote by \( \bar{r} = \inf_{y \in K} b(y) \). Take \( \bar{R}_i = (2 + 3c)h_i + \text{diam}(K) + 2(r_0 - \bar{r}) \), then it is easy to check \( \Xi_{[b(p_i) - ch_i, b(p_i) + ch_i]}(K) \subseteq B_{p_i}(\bar{R}_i) \). By the volume comparison properties, we have

\[
m(B_{p_i}(ch_i)) \geq \left( \frac{ch_i}{\bar{R}_i} \right)^N m(B_{p_i}(\bar{R}_i)) \geq \left( \frac{ch_i}{\bar{R}_i} \right)^N m(\Xi_{[b(p_i) - ch_i, b(p_i) + ch_i]}(K))
\]

\[
\geq \left( \frac{ch_i}{(2 + 3c)h_i + \text{diam}(K) + 2(r_0 - \bar{r})} \right)^N 2ch_i V_{r_0}, \tag{4.12}
\]

where we use (2.11) in the first inequality and use Corollary 3.22 in the last inequality. Hence for \( i \) large enough, we have

\[
m(B_{p_i}(ch_i)) \geq \left( \frac{c}{2 + 4c} \right)^N 2ch_i V_{r_0} > 0. \tag{4.13}
\]

It is easy to see that \( B_{p_i}(ch_i) \subset b^{-1}([b(p_i) - ch_i, b(p_i) + ch_i]). \)

Thus for any \( t > 0 \), we have

\[
m(\Xi_{[b(p_i) + ch_i, b(p_i) + ch_i + t]}(B_{p_i}(ch_i))) \geq \frac{t}{2ch_i} m(\Xi_{[b(p_i) - ch_i, b(p_i) + ch_i]}(B_{p_i}(ch_i)))
\]

\[
\geq \frac{t}{2ch_i} m(B_{p_i}(ch_i)) \geq \left( \frac{c}{2 + 4c} \right)^N tV_{r_0}, \tag{4.14}
\]

where we use (3.21) in the first inequality, and use (4.13) in the last inequality.

By (4.11), we have \( B_{p_i}(ch_i) \cap \Xi(K) = \emptyset \), hence \( \Xi(B_{p_i}(ch_i)) \cap \Xi(K) \) is contained in the \( m \)-negligible set \( X \setminus T \). Thus for any \( t > 0 \), we have

\[
m(b^{-1}([b(p_i) + ch_i, b(p_i) + ch_i + t]))
\]

\[
\geq m(\Xi_{[b(p_i) + ch_i, b(p_i) + ch_i + t]}(B_{p_i}(ch_i))) + m(\Xi_{[b(p_i) + ch_i, b(p_i) + ch_i + t]}(K))
\]

\[
\geq \left( \frac{c}{2 + 4c} \right)^N tV_{r_0} + tV_{r_0}. \tag{4.15}
\]

Multiply both sides of (4.15) by \( \frac{1}{t} \), and use Corollary 3.22, we have

\[
m_{-1}(b^{-1}(b(p_i) + ch_i + t)) \geq \left( \frac{c}{2 + 4c} \right)^N V_{r_0} + V_{r_0}. \tag{4.16}
\]

Let \( t \to \infty \), by the definition of \( V_\infty \), we have

\[
V_\infty \geq \left( \frac{c}{2 + 4c} \right)^N V_{r_0} + V_{r_0}, \tag{4.17}
\]

which contradicts (4.9). Thus we have completed the proof. \( \Box \)
Remark 4.3  Note that the proof of Theorem 1.2 only makes use of (4.1), (2.11), Proposition 3.21, and Corollary 3.22. Since all the above hold on non-compact essentially non-branching MCP(0, N) space, Corollary 1.3 can be proved by the same argument.

5 Non-compact RCD(0,N) Spaces with Strongly Minimal Volume Growth

5.1 Strongly Minimal Volume Growth

Definition 5.1  Suppose (X, d, m) is a non-compact RCD(0, N) space, b is the Busemann function associated with a geodesic ray γ. We say (X, d, m) has strongly minimal volume growth if

\[ \lim_{R \to \infty} \frac{m(B_p(R))}{R} = m_{-1}(\mathcal{E}_r_0(K)) < \infty, \]  

(5.1)

holds for some \( r_0 \in \mathbb{R} \) and a compact set \( K \subset b^{-1}((-\infty, r_0]) \). (See Sect. 3 for the definition of \( \mathcal{E}_r_0(K) \).)

Note that if (5.1) holds, then by Theorem 1.2, every level set \( b^{-1}(r) \) is compact.

In the remaining part of this paper, we will prove Theorem 1.4. From now on, the metric measure space \( (X, d, m) \) is always assumed to be non-compact and satisfying the RCD(0, N) condition as well as (5.1). We assume \( r_0 = 0 \) in (5.1) for convenience.

Proposition 5.2  Suppose \((X, d, m)\) is a non-compact RCD(0, N) space and satisfies (5.1), then \( b^{-1}((0, \infty)) \subset \mathcal{E}(K) \) and

\[ \frac{m(b^{-1}([r_1, r_2]))}{r_2 - r_1} = m_{-1}(\mathcal{E}_0(K)) \]  

(5.2)

for any \( r_2 > r_1 \geq 0 \).

Proof  We first prove \( b^{-1}((0, \infty)) \subset \mathcal{E}(K) \). Suppose on the contrary, there is some \( z \in b^{-1}((0, \infty)) \setminus \mathcal{E}(K) \). Since \( \mathcal{E}(K) \) is a closed set and \( b^{-1}((0, \infty)) \) is open, there is a small \( \delta > 0 \) such that \( B_z(\delta) \subset b^{-1}((0, \infty)) \setminus \mathcal{E}(K) \). Denote by \( t = \min_{y \in K} \{ b(y) \} \) and \( r_1 = b(z) + \delta \). For any \( r_2 > r_1 + 1 \), take \( R = \max \{ d(z, p) + r_2 - b(z) + \delta, d(p, K) + \text{diam}(K) + r_2 - r \} \). It is easy to see \( B_z(\delta) \subset b^{-1}((-\infty, r_1]) \) and \( \mathcal{E}_{[r_1, r_2]}(B_z(\delta)) \cup \mathcal{E}_{[r_1, r_2]}(K) \subset B_p(R) \).

By Proposition 3.21 and Corollary 3.22, we have

\[ m(\mathcal{E}_{[r_1, r_2]}(B_z(\delta))) \geq (r_2 - r_1)m(\mathcal{E}_{[r_1, r_1+1]}(B_z(\delta))) > 0, \]  

(5.3)

and

\[ m(\mathcal{E}_{[r_1, r_2]}(K)) \geq (r_2 - r_1)m_{-1}(\mathcal{E}_{r_1}(K)) \geq (r_2 - r_1)m_{-1}(\mathcal{E}_0(K)). \]  

(5.4)

On the other hand, since \( B_z(\delta) \subset b^{-1}((0, \infty)) \setminus \mathcal{E}(K) \), it is easy to see \( \mathcal{E}(B_z(\delta)) \cap \mathcal{E}(K) \cap b^{-1}((0, \infty)) \subset X \setminus T \), which is \( m \)-negligible. Therefore,
\[
\frac{m(B_p(R))}{R} \geq \frac{m(\Xi_{[r_1,r_2]}(B_\delta) \cup \Xi_{[r_1,r_2]}(K))}{R} \\
\geq \frac{(r_2 - r_1)}{R} \left[ m(\Xi_{[r_1,r_1+1]}(B_\delta)) + m_{-1}(\Xi_0(K)) \right].
\]

(5.5)

Let \( r_2 \to \infty \), we obtain
\[
\lim_{R \to \infty} \frac{m(B_p(R))}{R} > m_{-1}(\Xi_0(K)),
\]

which contradicts (5.1). This proves \( b^{-1}((0, \infty)) \subset \Xi(K) \).

Now we prove (5.2). Suppose (5.2) does not hold for some \( r_2 > r_1 \geq 0 \), then \( m_{-1}(b^{-1}(r_2)) > m_{-1}(\Xi_0(K)) \). For any \( r_3 > r_2 \), take \( R = d(p, b^{-1}(r_2)) + \text{diam}(b^{-1}(r_2)) + r_3 - r_2 \), then we have \( \Xi_{[r_2,r_3]}(b^{-1}(r_2)) \subset B_p(R) \).

By Corollary 3.22, we have
\[
\frac{m(B_p(R))}{R} \geq \frac{m(\Xi_{[r_2,r_3]}(b^{-1}(r_2)))}{R} \geq \frac{(r_3 - r_2)}{R} m_{-1}(b^{-1}(r_2)).
\]

Let \( r_3 \to \infty \), we obtain
\[
\lim_{R \to \infty} \frac{m(B_p(R))}{R} \geq m_{-1}(b^{-1}(r_2)) > m_{-1}(\Xi_0(K)),
\]

which contradicts (5.1). The proof is completed. \( \Box \)

By Proposition 5.2, Corollary 3.20, and (3.20), it is easy to see \((0, \infty) \subset \text{Dom}(g(y, \cdot))\) and \( q(y, r) = q(y, 1) \) hold for \( \tilde{v} \)-a.e. \( y \in S \) and \( r > 0 \). If we take \( dv'(y) = q(y, 1)dv(y) \), and endow a measure \( \mu := v' \otimes L^1 \) on \( S \times (0, \infty) \), then by Theorem 3.18, we have \( g_* \mu = m \).

Recall the flow \( F_t \) in Definition 3.17. From the discussion above, there is an \( m \)-negligible set \( \mathcal{N} \subset X \setminus T \) such that for all \( t \geq 0 \), \( F_t : b^{-1}((0, \infty)) \setminus \mathcal{N} \to b^{-1}((t, \infty)) \setminus \mathcal{N} \) is a bijection, whose inverse is denoted by \( F_{-t} := F_{t}^{-1} \), a map from \( b^{-1}((t, \infty)) \setminus \mathcal{N} \) to \( b^{-1}((0, \infty)) \setminus \mathcal{N} \). We can redefine the value of \( F_t \) on \( \mathcal{N} \) to make them Borel on \( b^{-1}((0, \infty)) \), similarly for \( F_{-t} \), then \( F_{-t} \circ F_t = 1d \) \( m \)-a.e. on \( b^{-1}((0, \infty)) \), \( F_t \circ F_{-t} = 1d \) \( m \)-a.e. on \( b^{-1}((t, \infty)) \).

Suppose \( t \geq 0 \), for any \( \varphi \in C_c(b^{-1}((t, \infty))) \), we have
\[
\int_{b^{-1}((t, \infty))} \varphi \circ d(F_t)_m
\]
\[
= \int_S \int_0^\infty \varphi(F_t \circ g(x', s))dv'(x') \otimes dL^1(s)
\]
\[
= \int_S \int_t^\infty \varphi(g(x', s + t))dv'(x') \otimes dL^1(s)
\]
\[
= \int_S \int_t^\infty \varphi(g(x', s))dv'(x') \otimes dL^1(s)
\]
\[
= \int_{b^{-1}((t, \infty))} \varphi dm.
\]

(5.6)
Theorem 5.4 Thus $F_t : (b^{-1}((0, \infty)), m) \rightarrow (b^{-1}((t, \infty)), m)$ is measure-preserving. Similarly, $F_{-t}$ is also measure-preserving on $b^{-1}((t, \infty))$.

In conclusion, we have the following proposition:

**Proposition 5.3** Suppose $(X, d, m)$ is a non-compact RCD$(0, N)$ space and satisfies (5.1), then the ray map $g$ is measure-preserving when viewed as a map from $(\mathcal{S} \times (0, \infty), \mu = \nu \otimes \mathcal{L}^1)$ to $(b^{-1}((0, \infty)), m)$, i.e.,

$$g_* \mu = m. \quad (5.7)$$

For any $t \geq 0$, both $F_t : (b^{-1}((0, \infty)), m) \rightarrow (b^{-1}((t, \infty)), m)$ and $F_{-t} : (b^{-1}((t, \infty)), m) \rightarrow (b^{-1}((0, \infty)), m)$ are measure-preserving.

In the following, we associate each level set $b^{-1}(r)$ with a natural measure.

Consider the Busemann function $b$ as a map from $b^{-1}((0, \infty))$ to $\mathbb{R}^+$. By (5.2), this map is measure-preserving: $b_* (m|_{b^{-1}[r_1, r_2]}) = c \mathcal{L}^1([r_1, r_2])$ for any $r_2 \geq r_1 > 0$, where $c = m_{-1}(\mathcal{E}_0(K))$. Now we apply Theorem 2.13 to $(b^{-1}((0, \infty)), \mathcal{B}(b^{-1}((0, \infty))), m)$ and $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \mathcal{L}^1)$, then we obtain a unique disintegration $r \mapsto \tilde{m}_r$ of $m$ over $c \mathcal{L}^1$ strongly consistent with $b$. Hence

$$\int_{b^{-1}((0, \infty))} \varphi dm = c \int_0^\infty \int \varphi d\tilde{m}_r dr, \quad \forall \varphi \in C_c(b^{-1}((0, \infty))). \quad (5.8)$$

Note that a priori the existence and uniqueness of the $\tilde{m}_r$'s only hold in a.e. $r \in \mathbb{R}^+$ sense. However, in this case, we have

**Lemma 5.4** The $\tilde{m}_r$ can be chosen to be a weakly continuous family, i.e., for any $\varphi \in C_c(b^{-1}((0, \infty)))$, the map $r \mapsto I_\varphi(r) := \int \varphi d\tilde{m}_r$ is continuous. Furthermore, for every $t \geq 0$ $(F_t)_* \tilde{m}_r = \tilde{m}_{r+t}$ holds for a.e. $r \in \mathbb{R}^+$.

The proof of Lemma 5.4 is adapted from Corollary 3.8 in [23]; we provide some details here.

**Proof** Firstly, for every $t \geq 0$, by (5.6) and (5.8), for any $\varphi \in C_c(b^{-1}((t, \infty)))$, on one hand we have

$$\int_{b^{-1}((t, \infty))} \varphi d(F_t)_* m = \int_{b^{-1}((0, \infty))} \varphi \circ F_t dm = c \int_0^\infty \int \varphi \circ F_t d\tilde{m}_r dr \quad (5.9)$$

on the other hand,

$$\int_{b^{-1}((t, \infty))} \varphi d(F_t)_* m = \int_{b^{-1}((t, \infty))} \varphi dm = c \int_t^\infty \int \varphi d\tilde{m}_r dr = c \int_0^\infty \int \varphi d\tilde{m}_{r+t} dr, \quad (5.10)$$

hence for every $t \geq 0$ $(F_t)_* \tilde{m}_r = \tilde{m}_{r+t}$ holds for a.e. $r \in \mathbb{R}^+$. 

\[ \square \]
We claim that for any Lipschitz function \( \varphi \) with \( \text{supp} \varphi \subset b^{-1}(0, \infty) \), the map \( r \mapsto I_\varphi(r) \) admits a Lipschitz representative. Since \( C_c(b^{-1}(0, \infty)) \) has a countable dense subset consisting of Lipschitz functions with bounded support, by an easy density argument one can check that the conclusion of the lemma follows from this claim.

Now for any fixed Lipschitz function \( \varphi \) with \( \text{supp} \varphi \subset b^{-1}(0, \infty) \), and for a.e. \( r \in \mathbb{R}^+ \),

\[
|I_\varphi(r + t) - I_\varphi(r)| = \left| \int \varphi d\tilde{m}_{r+t} - \int \varphi d\tilde{m}_r \right| = \left| \int (\varphi \circ F_t - \varphi) d\tilde{m}_r \right| \leq t \text{Lip}(\varphi),
\]

(5.11)

where we use (3.9) in the last inequality. By (5.11), it is easy to check that the distributional derivative of \( r \mapsto I_\varphi(r) \) is bounded by \( \text{Lip}(\varphi) \), thus \( r \mapsto I_\varphi(r) \) admits a Lipschitz representative. \( \square \)

In the remaining part of this paper, we will fix some \( r' > 0 \) and denote by \( Z = b^{-1}(r') \).

We first prove that under the assumptions of Theorem 1.4, \((X, d, m)\) has exactly one end.

We claim that the Busemann function \( b \) obtains a global minimum on \( X \). If the claim holds, then by Theorem 1.2 and Corollary 3.5 it is easy to see \( b \) is a proper function on \( X \), and then by Lemma 3.2 \((X, d, m)\) has only one end. Suppose on the contrary the claim does not hold, then there is a sequence of points \( \{x_i\} \subset X \) such that \( b(x_i) = -i \). From each \( x_i \), there is a geodesic ray \( \eta^i : [-i, \infty) \to X \) such that \( \eta^i(-i) = x_i \), \( \eta^i(r') \in Z \), and \( (x_i, \eta^i(t)) \in \Gamma, \forall t \geq -i \). Recall that \( Z \) is a compact set. Hence up to a subsequence, \( \eta^i \) converge to a line \( \eta : (-\infty, \infty) \to X \) with \( \eta(r') \in Z \). Now by Gigli’s splitting theorem on \( \text{RCD}(0, N) \) (see Theorem 1.1), \((X, d, m)\) is isomorphic to the product of \((\mathbb{R}, d_{\text{Eucl}}, L^1)\) and some \((X', d', m')\), where \((X', d', m')\) is either a point (when \( N \in (1, 2) \)) or a \( \text{RCD}(0, N-1) \) space (when \( N \geq 2 \)). In the case of \( N \geq 2 \), if \( X' \) is non-compact, then by Proposition 2.8 it has infinite volume, thus \((X, d, m)\) cannot have linear volume growth and we get a contradiction. Hence \( X' \) must be compact. From the compactness of \( X \), it is easy to check that the Busemann function \( \hat{b} \) associated with \( \eta \) must coincide with \( b \), and

\[
\lim_{R \to \infty} \frac{m(B_p(R))}{R} = 2m_{-1}(\overline{\mathbb{S}}_0(K)),
\]

(5.12)

which contradicts (5.1). Thus the claim holds.

The remaining arguments in Theorem 1.4 is to prove a ‘volume cone implies metric cone’-type property on \((X, d, m)\). We follow the strategy in [23].

### 5.2 Basic Properties of \( b \)

In this subsection, we obtain some basic properties of \( b \).

From the argument at the end of last subsection, we have obtained
Lemma 5.5  $b$ is a proper function on $X$.

By Lemma 3.1, it is easy to see $\text{lip}(b) \equiv 1$ on $X$. Since volume doubling property and a weak local $(1, 1)$-Poincaré inequality hold on the RCD($0, N$) space $(X, d, m)$ (see [39,42,43]), we have

Lemma 5.6  $|Db(x)| = \text{lip}(b)(x) = 1 \text{ m-a.e.}$.

We recall the following proposition, which is a consequence of the Laplacian comparison estimates for the distance function [28].

Proposition 5.7  (Proposition 5.19 in [28]). Let $(X, d, m)$ be an RCD($0, N$) space. Let $\gamma$ be a geodesic ray and $b$ the Busemann function associated with it. Then $b \in D(\Delta)$ and $\Delta b \geq 0$.

Combining Propositions 5.3, 5.7, and Lemma 5.6, we can prove the following result:

Proposition 5.8  $\Delta b = 0$ on $b^{-1}((0, \infty))$.

Proof  Let $\varphi : \mathbb{R}^+ \to [0, 1]$ be a Lipschitz function with $\text{supp}(\varphi) \subset \subset (0, \infty)$, we have

$$0 = \int S \left( \int_0^\infty \varphi'(s)ds \right) v'(dy)$$

$$= \int_{b^{-1}((0, \infty))} \varphi'(b(x)) dm$$

$$= \int_{b^{-1}((0, \infty))} \varphi'(b(x)) |Db(x)|^2 dm$$

$$= \int_{b^{-1}((0, \infty))} \langle \nabla \varphi(b(x)), \nabla b(x) \rangle dm$$

$$= - \int_{b^{-1}((0, \infty))} \varphi(b(x)) d\Delta b \leq 0.$$

Hence $\int_{b^{-1}((0, \infty))} \varphi(b(x)) d\Delta b = 0$. By $\Delta b \geq 0$ and the arbitrariness of $\varphi$, we obtain $\Delta b = 0$ on $b^{-1}((0, \infty))$. \hfill $\Box$

Evidently, $b$ is not in $W^{1,2}(X)$ but only in $W^{1,2}_{\text{loc}}(X)$. Given any $\tilde{R} > \tilde{r} > 0$, let $\tilde{\varphi} \in C^\infty(\mathbb{R})$ be a smooth function satisfying $\tilde{\varphi} \equiv 0$ on $(-\infty, \frac{\tilde{r}}{2}] \cup [2\tilde{R}, \infty)$, $\tilde{\varphi}(x) = x$ on $[\tilde{r}, \tilde{R}]$. Define $\tilde{b} : X \to \mathbb{R}$ to be

$$\tilde{b}(x) = \tilde{\varphi} \circ b(x).$$

Proposition 5.9  $\tilde{b} \in \text{Test}(X)$ and $\Delta \tilde{b} \in W^{1,2}(X) \cap L^\infty(X)$.

Proof  Evidently $\tilde{b}$ is Lipschitz with $\text{supp}(\tilde{b}) \subset b^{-1}([\frac{\tilde{r}}{2}, 2\tilde{R}])$. Recall the facts that $|Db|^2 = 1 \text{ m-a.e. and } \Delta b = 0$ on $b^{-1}((0, \infty))$, then by the chain rule for the distributional Laplacian (see Proposition 4.11 in [28]), we have

$$\Delta \tilde{b} = \tilde{\varphi}'' \circ b |Db|^2 m + \tilde{\varphi}' \circ b \Delta b = \tilde{\varphi}'' \circ bm.$$
Obviously \( \tilde{\varphi}'' \circ b \in W^{1,2}(X) \cap L^\infty(X) \), hence \( \tilde{b} \in D(\Delta) \) with \( \Delta \tilde{b} \in W^{1,2}(X) \cap L^\infty(X) \). \( \square \)

**Proposition 5.10** (Euler equation for \( b \)) Let \( f, g \in \text{Test}(X) \) with \( \text{supp}(f) \subset B^{-1}((0, \infty)) \). Then

\[
\int \Delta f \langle \nabla b, \nabla g \rangle dm = \int f \langle \nabla b, \nabla \Delta g \rangle dm. \tag{5.13}
\]

**Proof** Evidently to conclude we only need to prove the case \( f \geq 0 \). Choose \( \tilde{R} > \tilde{r} > 0 \) such that \( \text{supp}(f) \subset B^{-1}([\tilde{r}, \tilde{R}]) \), and construct \( \tilde{b} \) as in the previous paragraphs. Let \( \epsilon \in \mathbb{R} \), since \( f, \tilde{b} + \epsilon g \in \text{Test}(X) \) and \( f \geq 0 \), we can apply the Bochner inequality \( (2.19) \):

\[
\frac{1}{2} \int \Delta f |D(\tilde{b} + \epsilon g)|^2 dm \geq \int f \left[ (\Delta(\tilde{b} + \epsilon g))^2 \right] + \langle \nabla (\tilde{b} + \epsilon g), \nabla \Delta (\tilde{b} + \epsilon g) \rangle dm. \tag{5.14}
\]

Using the facts that \( |D\tilde{b}|^2 = 1 \) and \( \Delta \tilde{b} = 0 \) m.a.e. on \( B^{-1}([\tilde{r}, \tilde{R}]) \), it is easy to see the equality in \( (5.14) \) holds when \( \epsilon = 0 \). Expanding \( (5.14) \), we obtain

\[
\int \Delta f \left( \epsilon \langle \nabla g, \nabla \tilde{b} \rangle + \frac{\epsilon^2}{2} |Dg|^2 \right) dm \geq \int f \left( \epsilon \langle \nabla \tilde{b}, \nabla g \rangle + \epsilon^2 \left( \frac{\Delta g}{N} \right) + \langle \nabla g, \nabla \Delta g \rangle \right) dm.
\]

Divide by \( \epsilon > 0 \) (resp. \( \epsilon < 0 \)) and let \( \epsilon \downarrow 0 \) (resp. \( \epsilon \uparrow 0 \)) we obtain \( (5.13) \). \( \square \)

Using the Euler equation \( (5.13) \), we obtain the information on Hess\((b)\), which was indicated formally in Remark 4.15 of [30].

**Lemma 5.11** Hess\((b)\) = 0 m.a.e. on \( B^{-1}((0, \infty)) \).

**Proof** Let \( g, h \in \text{Test}(X) \) with compact support such that \( \text{supp}(g), \text{supp}(h) \subset B^{-1}([\tilde{r}, \tilde{R}]) \) for some \( \tilde{R} > \tilde{r} > 0 \). Construct \( \tilde{b} \) as in the previous paragraphs. Using the Euler equation \( (5.13) \), the Leibniz rule for the divergence, the facts that \( |D\tilde{b}|^2 = 1 \) and \( \Delta \tilde{b} = 0 \) m.a.e. on \( B^{-1}([\tilde{r}, \tilde{R}]) \), as well as some integration by parts we obtain:

\[
\int \text{div}(h \nabla g) \langle \nabla \tilde{b}, \nabla g \rangle dm
\]

\[
= \int \Delta hg \langle \nabla \tilde{b}, \nabla g \rangle dm - \int g \Delta h \langle \nabla \tilde{b}, \nabla g \rangle dm - \int \langle \nabla h, \nabla g \rangle \langle \nabla \tilde{b}, \nabla g \rangle dm
\]

\[
= \int gh \langle \nabla \tilde{b}, \nabla \Delta g \rangle dm - \int \Delta h \langle \nabla \tilde{b}, \nabla \Delta g \rangle dm - \int \langle \nabla h, \nabla g \rangle \langle \nabla \tilde{b}, \nabla g \rangle dm
\]

\[
= \int \langle \nabla \tilde{b}, \nabla (gh \Delta g) \rangle dm - \int \Delta g \langle \nabla \tilde{b}, \nabla (gh) \rangle dm - \int \langle \nabla h, \nabla g \rangle \langle \nabla \tilde{b}, \nabla g \rangle dm
\]
\[ -\int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm \]
\[ = -\int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm - \int h \Delta g \langle \nabla \tilde{b}, \nabla g \rangle \, dm - \int \langle \nabla h, \nabla g \rangle \langle \nabla \tilde{b}, \nabla g \rangle \, dm \]
\[ -\int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm \]
\[ = -\int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm - \int \text{div}(h \nabla g) \langle \nabla \tilde{b}, \nabla g \rangle \, dm - \int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm. \]

Hence,
\[ \int \text{div}(h \nabla g) \langle \nabla \tilde{b}, \nabla g \rangle \, dm = -\frac{1}{2} \int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm - \frac{1}{2} \int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm. \] (5.15)

On the other hand,
\[ \int h \langle \nabla \tilde{b}, \nabla \left( \frac{\|g\|^2}{2} \right) \rangle \, dm \]
\[ = \int \langle \nabla \tilde{b}, \nabla \left( h \frac{\|g\|^2}{2} \right) \rangle \, dm - \int \frac{\|g\|^2}{2} \langle \nabla \tilde{b}, \nabla h \rangle \, dm \]
\[ = -\int \Delta \left( \frac{g}{2} \right) \langle \nabla \tilde{b}, \nabla h \rangle \, dm + \int \frac{\|g\|^2}{2} \langle \nabla \tilde{b}, \nabla h \rangle \, dm + \int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm \]
\[ = -\int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm + \int \frac{\|g\|^2}{2} \langle \nabla \tilde{b}, \nabla h \rangle \, dm + \int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm + \int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm. \]

Hence
\[ \int h \langle \nabla \tilde{b}, \nabla \left( \frac{\|g\|^2}{2} \right) \rangle \, dm = \frac{1}{2} \int \Delta h \langle \nabla \tilde{b}, \nabla \left( \frac{g}{2} \right) \rangle \, dm + \frac{1}{2} \int g \Delta g \langle \nabla \tilde{b}, \nabla h \rangle \, dm. \] (5.16)

By (2.20) and polarization, we conclude \( \text{Hess}(b) = 0 \) m.a.e. on \( b^{-1}(0, \infty) \).

5.3 Effect of \( F_t \) on the Dirichlet Energy and Metric

Because we only know information on \( b^{-1}(0, \infty) \), we need some cut-off argument. Let \( \tilde{r} \) be a fixed positive number. Let \( \tilde{\varphi} \in C^\infty(\mathbb{R}) \) be a smooth increasing function satisfying \( \tilde{\varphi} \equiv 0 \) on \( (-\infty, \frac{\tilde{r}}{2}] \), \( \tilde{\varphi}(x) = x \) on \( [\tilde{r}, \infty) \). Define \( \tilde{b} : X \to \mathbb{R} \) to be
\[ \tilde{b}(x) = \tilde{\varphi} \circ b(x). \]
Note that \( \tilde{b} \) is Lipschitz, with support in \( b^{-1}((\frac{\tilde{F}}{2}, \infty)) \) and equal to \( b \) on \( b^{-1}([\tilde{F}, \infty)) \).
We define the reparametrization function \( \text{rep} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \), \( \text{rep}_t(r) = \text{rep}(t, r) \) by requiring it satisfying \( \text{rep}_0(r) = 0 \) and
\[
\partial_t \text{rep}_t(r) = \tilde{\varphi}'(r + \text{rep}_t(r)) \tag{5.17}
\]
for every \( t \geq 0 \). It is easy to see \( \text{rep}_t(r) = 0 \) for \( (t, r) \in \mathbb{R}^+ \times (\infty, \frac{\tilde{F}}{2}] \) and \( \text{rep}_t(r) = t \) on \( (t, r) \in \mathbb{R}^+ \times [\tilde{F}, \infty) \). We then define the flow \( \tilde{F} : \mathbb{R}^+ \times (X \setminus \mathcal{N}) \to X \setminus \mathcal{N} \) to be
\[
\tilde{F}_t(x) = \text{F}_{\text{rep}_t(b(x))}(x).
\]
\( \tilde{F}_t : X \setminus \mathcal{N} \to X \setminus \mathcal{N} \) is invertible for every \( t \geq 0 \). Thus for \( t > 0 \) the map \( \tilde{F}_{-t} = (\tilde{F}_t)^{-1} : X \setminus \mathcal{N} \to X \setminus \mathcal{N} \) is well defined. We then suitably define \( \tilde{F}_t \) and \( \tilde{F}_{-t} \) on \( \mathcal{N} \) to make them Borel maps on \( X \).

**Lemma 5.12** The following properties hold:

(i) \( \tilde{F}_t \) is the identity on \( b^{-1}((\infty, \frac{\tilde{F}}{2})) \setminus \mathcal{N} \), \( \tilde{F}_t \) sends \( b^{-1}((0, \infty)) \setminus \mathcal{N} \) into itself for every \( t \geq 0 \).

(ii) \( \tilde{F}_t \) coincides with \( F_t \) in \( b^{-1}([\tilde{F}, \infty)) \setminus \mathcal{N} \).

(iii) For every \( x \in X \setminus \mathcal{N} \) the curve \( [0, \infty) \ni t \mapsto \eta_t(x) := \tilde{F}_t(x) \) satisfies
\[
\tilde{b}(\eta_t(x)) = \tilde{b}(\eta_s(x)) + \frac{1}{2} \int_s^t \left[ |\dot{\eta}_r(x)|^2 + \text{lip}(\tilde{b})^2(\eta_r(x)) \right] dr, \quad \forall 0 \leq s \leq t. \tag{5.18}
\]

In particular, the speed of \( \eta_t(x) \) is equal to \( \text{lip}(\tilde{b}) (\tilde{F}_t(x)) \) for every \( t \), thus granting that \( t \to \tilde{F}_t(x) \) is \( \text{Lip}(\tilde{b}) \)-Lipschitz for every \( x \in X \setminus \mathcal{N} \).

(iv) For every \( t \geq 0 \),
\[
c(t)m \leq (\tilde{F}_t)_* m \leq C(t)m, \tag{5.19}
\]
where \( c, C : \mathbb{R}^+ \to (0, \infty) \) are continuous functions.

(v) The maps \( \tilde{F}_t \) form a group, i.e., \( \tilde{F}_0 = \text{Id} \) m-a.e. and
\[
\tilde{F}_{t+s} = \tilde{F}_t \circ \tilde{F}_s \quad \text{m-a.e.}
\]
for every \( t, s \in \mathbb{R} \).

**Proof** Properties (i), (ii), (v) follow from the definition and the obvious properties of the reparametrization function \( \text{rep} \) and the flow \( F_t \).

Now we prove (iii). Note that for \( x \in X \setminus \mathcal{N} \),
\[
b(\eta_t(x)) = b(\tilde{F}_t(x)) = b(F_{\text{rep}_t(b(x))}(x)) = b(x) + \text{rep}_t(b(x)).
\]
Recall that \( \tilde{b} = \tilde{\varphi} \circ b \), we have
\[
\text{lip}(\tilde{b}) (\eta_t(x)) = |\tilde{\varphi}'(b(x))| \text{lip}(b) (\eta_t(x)) = |\tilde{\varphi}'(b(x) + \text{rep}_t(b(x)))|.
\]
On the other hand, $|\dot{\eta}_{t}^{(s)}| = |\dot{\sigma}_{s}^{(x)}||\partial r \text{rep}_{t}(b(x))| = |\partial r \text{rep}_{t}(b(x))|$, where $[0, \infty) \ni s \mapsto \sigma_{s}^{(x)} := F_{s}(x)$ is the geodesic ray emitting from $x$. Hence from (5.17), we have $|\dot{\eta}_{t}^{(s)}| = \text{lip}(\tilde{b})(\eta_{t}^{(s)})$. Note that

$$
\frac{d}{dt} \tilde{b} \left( \eta_{t}^{(s)} \right) = \dot{\varphi} \left( \tilde{b} \left( \eta_{t}^{(s)} \right) \right) \frac{d}{dt} \tilde{b} \left( \eta_{t}^{(s)} \right) = \dot{\varphi} \left( b \left( \eta_{t}^{(s)} \right) \right) \frac{\partial}{\partial t} \text{rep}_{t} \left( b \left( x \right) \right)
$$

$$
= \frac{1}{2} \left[ \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip} \left( \tilde{b} \right)^{2} \left( \eta_{t}^{(s)} \right) \right],
$$

integrate over $[s, t] \subset [0, \infty)$ and we obtain (5.18).

Now we prove (iv). Since $\bar{F}_{t}$ is identity on $b^{-1}(]-\infty, \frac{r}{2}]) \setminus \mathcal{N}$, we have

$$
\left( \bar{F}_{t} \right)_{*} m \bigg|_{(-\infty, \frac{r}{2}]}(s, t) = m \bigg|_{(-\infty, \frac{r}{2}]}(s, t), \quad \forall t \geq 0.
$$

In the following, we consider the behavior of $\bar{F}_{t}$ on $b^{-1}(0, \infty)$.

For $t \geq 0$, denote by $f_{t}(s) := s + \text{rep}_{t}(s)$. By definition, we have

$$
\bar{F}_{t}(y) = g(x', s + \text{rep}_{t}(s)) = g(x', f_{t}(s))
$$

for any $y = g(x', s) \in b^{-1}(0, \infty)$.

Let $\varphi$ be any Borel function with $\text{supp}(\varphi) \subset b^{-1}(0, \infty)$, then

$$
\int_{b^{-1}(0, \infty)} \varphi d \left( \bar{F}_{t} \right)_{*} m
$$

$$
= \int_{\mathcal{S}} \int_{0}^{\infty} \varphi \left( \bar{F}_{t} \circ g \left( x', s \right) \right) d
$$

$$
\nu' \left( x' \right) \otimes d \mathcal{L}^{1} \left( s \right)
$$

$$
= \int_{\mathcal{S}} \int_{0}^{\infty} \varphi \left( g \left( x', f_{t} \left( s \right) \right) \right) d \nu' \left( x' \right) \otimes d \mathcal{L}^{1} \left( s \right)
$$

$$
= \int_{\mathcal{S}} \int_{0}^{\infty} \varphi \left( g \left( x', s \right) \right) \left( \partial_{s} f_{t}^{-1} \left( s \right) \right) d \nu' \left( x' \right) \otimes d \mathcal{L}^{1} \left( s \right).
$$

Thus to complete the proof of (iv) we only need to prove that there are functions $c(t), C(t) : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ such that $\forall r \in \mathbb{R}^{+}$ we have

$$
c(t) \leq \partial_{r} f_{t}^{-1}(r) \leq C(t). \quad (5.20)
$$

Denote by $h_{t}(r) := \partial_{r} \text{rep}_{t}(r) = \partial_{r} f_{t}(r) - 1$. Differentiate (5.17) in $r$ to deduce that $h_{t}(r)$ solves the equation

$$
\partial_{r} h_{t}(r) = \tilde{\varphi}''(r + \text{rep}_{t}(r))(1 + h_{t}(r))
$$

with the initial condition $h_{0}(r) = 0$. Note that the function identically $-1$ is a solution of (5.21), also note that $\tilde{\varphi}''$ has compact support, by comparison principle we can prove

$$
\varphi
$$

$$
\geq \text{lip}(\tilde{b})(\eta_{t}^{(s)})
$$

$$
\geq \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)})
$$

$$
= \frac{1}{2} \left[ \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right],
$$

$$
\geq \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$

$$
= \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$

$$
= \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$

$$
= \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$

$$
= \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$

$$
= \frac{1}{2} \left( \left| \dot{\eta}_{t}^{(s)} \right|^{2} + \text{lip}(\tilde{b})^{2}(\eta_{t}^{(s)}) \right).
$$
for a function $\tilde{c}(t) > 0$. On the other hand, since $\text{rep}_r(t) = t$ for $r \geq \bar{r}$, we know $h(t) = 0$ for all $r \geq \bar{r}$. Also note that $h(0) = 0$. Then by the smoothness of $h(t)$, we have $h(t) = \tilde{C}(t)$ for any $r \in \mathbb{R}_+$. Hence

$$\frac{1}{1 + \tilde{C}(t)} \leq \partial_r f_t^{-1}(r) \leq \frac{1}{\tilde{c}(t)}$$

for any $r \in \mathbb{R}_+$. This completes the proof of (iv). \[ \square \]

The properties in Lemma 5.12 are similar to the properties a)- f) in Proposition 3.9 of [23]. Using these properties we can establish the following lemma, which is very similar to Lemma 3.11 in [23].

**Lemma 5.13** If $f \in L^p(X), p < \infty$, then $t \mapsto f \circ \bar{F}_t \in L^p(X)$ is continuous.

If $f \in W^{1,2}(X)$, then $t \mapsto f \circ \bar{F}_t \in L^2(X)$ is $C^1$ and its derivative is given by

$$\frac{d}{dt} f \circ \bar{F}_t = \langle \nabla f, \nabla \bar{b} \rangle \circ \bar{F}_t. \quad (5.22)$$

We omit the detailed proof of Lemma 5.13, and the reader can refer to [23] for the very same argument.

Our next goal is to study the effect of $\bar{F}_t$ on the Dirichlet energy $\mathcal{E}$, see Theorem 5.16. The main tool is the heat flow, which is the $L^2$-gradient flow of $\mathcal{E}$. A problem arises in the proof of Theorem 5.16 is that the support of $H_s(f)$ will not stay in $b^{-1}(\bar{r}, \infty)$ even if the support of $f$ does. Thus we need suitable cut-off functions and controlling the error terms. Our argument follows [23] closely.

Firstly we recall the following lemma from [23]:

**Lemma 5.14** (Lemma 3.14 in [23]) Suppose $(X, d, m)$ is an RCD$(0, N)$ space. For every $r > 0$ there is a constant $C(r) > 0$ such that the following holds. Given $K \subset U$ with $K$ compact and $U$ open such that $\inf_{x \in K, y \in U \setminus K} d(x, y) \geq r$. If $f \in L^2(X)$ with $\text{supp}(f) \subset K$, then for every $t > 0$ the quantities

$$\int_{U \setminus K} |H_t(f)|^2 dm, \quad \int_0^t \int_{U \setminus K} |H_s(f)|^2 dm ds, \quad \int_0^t \int_{U \setminus K} |DH_t(f)|^2 dm ds,$$

$$\int_0^t \int_{U \setminus K} |\Delta H_s(f)|^2 dm ds, \quad \int_0^t \int_{U \setminus K} |D\Delta H_s(f)|^2 dm ds \quad (5.23)$$

are all bounded from above by $C(r)^2 \| f \|_{L^2}^2$.

Applying Lemma 5.14, we can prove the following estimate.

**Proposition 5.15** Given any $r > 0$ there is a constant $C(r)$ such that for any $f \in L^2(X)$ with $\text{supp}(f) \subset b^{-1}(\bar{r} + r, \infty)$ and any $s \in (0, 1)$ we have

$$\left| \int (\nabla H_{2s}(f), \nabla \bar{b}) f dm \right| \leq s^2 C(r) \| f \|_{L^2}^2. \quad (5.24)$$
Proof By (2.18), it is easy to check that the term $\int \langle \nabla H_{2s}(f), \nabla \tilde{b} \rangle f dm$ varies continuously as $f$ varies in $L^2(X)$. Hence by Lemma 2.11 and a simple density argument we only need to prove that the conclusion of the proposition holds for $f \in \text{Test}(X)$ with support contained in a compact set $K$ with $K \subset b^{-1}((\bar{r} + r, R))$. Denote by $U = b^{-1}((\bar{r} + \frac{r}{2}, 2R))$ and note that $\inf_{x \in K, y \in U} d(x, y) \geq \frac{\bar{r}}{2}$.

Since the function $t \mapsto \int \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle H_{s-t}(f) dm$ is absolutely continuous on $[0, s]$, we have

$$\int \langle \nabla H_{2s}(f), \nabla \tilde{b} \rangle f dm = \int \langle \nabla H_s(f), \nabla \tilde{b} \rangle H_s(f) dm + \int_0^s \frac{d}{dt} \int \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle H_{s-t}(f) dmdt. \quad (5.25)$$

Note that

$$\int \langle \nabla H_s(f), \nabla \tilde{b} \rangle H_s(f) dm = \int \left( \nabla \left( \frac{H_s(f)^2}{2} \right), \nabla \tilde{b} \right) dm \leq \|
abla \tilde{b} \|_{L^\infty} \int_{U^c} \frac{|H_s(f)|^2}{2} dm \leq C(r)s^2 \| \nabla \tilde{b} \|_{L^\infty} \| f \|_{L^2}^2, \quad (5.26)$$

where we use Lemma 5.14 in the last inequality.

Let $\chi$ be the cut-off function given by Lemma 2.9 relative to the compact set $b^{-1}((\bar{r} + \frac{r}{2}, 2R))$ and the open set $b^{-1}((\bar{r}, 4R))$, then

$$\int_0^s \frac{d}{dt} \int \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle H_{s-t}(f) dmdt$$

$$= \int_0^s \int \left( \langle \nabla \Delta H_{s+t}(f), \nabla \tilde{b} \rangle H_{s-t}(f) \right) + \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle \Delta H_{s-t}(f) \right) dmdt$$

$$= \int_0^s \int \left( \langle \nabla \Delta H_{s+t}(f), \nabla \tilde{b} \rangle (\chi H_{s-t}(f)) \right) + \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle \Delta (\chi H_{s-t}(f)) dmdt$$

$$+ \int_0^s \int \left( \langle \nabla \Delta H_{s+t}(f), \nabla \tilde{b} \rangle (1 - \chi) H_{s-t}(f) \right)$$

$$- \langle \nabla H_{s+t}(f), \nabla \tilde{b} \rangle \Delta ((1 - \chi) H_{s-t}(f)) dmdt$$

$$\leq S^2 \int_0^s \int_{U^c} (|H_{s-t}(f)| |D\Delta H_{s+t}(f)| + |D H_{s+t}(f)|||H_{s-t}(f)| + |D H_{s-t}(f)|$$

$$+ |\Delta H_{s-t}(f)||) dmdt$$

$$\leq C(r)s^2 \| f \|_{L^2}^2, \quad (5.27)$$

where $S := \max\{1, \text{Lip}(\tilde{b}), \text{Lip}(\chi), \| \Delta \chi \|_{L^\infty}\}$, with $S \leq C(r)$ by Lemma 2.9. In (5.27) we use the fact that $\text{supp}(\chi H_{s-t}(f)) \subset b^{-1}((\bar{r}, \infty))$ as well as Proposition 5.10 to conclude that the first integral in the third equality vanishes, and use Lemma 5.14 in the last inequality.

Combining (5.25), (5.26), and (5.27), we obtain the estimate (5.24). □
Theorem 5.16 Suppose $f \in L^2(X)$ satisfies $\text{supp}(f) \subset b^{-1}((\bar{r} + T, \infty))$ for some $T \geq 0$. Then

$$\mathcal{E}(f \circ \tilde{F}_t) = \mathcal{E}(f), \quad \forall t \leq T. \quad (5.28)$$

In particular, $f \in W^{1,2}(X)$ if and only if $f \circ \tilde{F}_t \in W^{1,2}(X)$.

Proof Let $f_t := f \circ \tilde{F}_t$ and notice that $\text{supp}(f_t) \subset b^{-1}((\bar{r} + T - t, \infty)) \subset b^{-1}((\bar{r}, \infty))$ for every $t \leq T$. By Proposition 5.3 we have $\int |f_t|^2 dm = \int |f|^2 dm$ and hence $\frac{d}{dt} \int |f_t|^2 dm = 0$ for any $t \leq T$.

Recall that by Lemma 5.13, $t \mapsto f_t \in L^2(X)$ is continuous. By (2.17) and (2.18), we have

$$\|H_s(f_t_1) - H_s(f_t_0)\|_{W^{1,2}} \leq 2\sqrt{2s} \|f_t_1 - f_t_0\|_{L^2},$$

for any $s \in (0, \frac{1}{2})$. Thus by Lemma 5.13 again, for any fixed $s \in (0, \frac{1}{2})$, $t \mapsto H_s(f_t) \in L^2(X)$ is $C^1$. Hence for $t \leq T$ we have

$${\frac{1}{2}} \frac{d}{dt} \int |H_s(f_t)|^2 dm = \lim_{h \to 0} \int H_s(f_t) \frac{H_s(f_t \circ \tilde{F}_h) - H_s(f_t)}{h} dm$$

$$= \lim_{h \to 0} \int H_{2s}(f_t) \frac{f_t \circ \tilde{F}_h - f_t}{h} dm$$

$$= \lim_{h \to 0} \int f_t \frac{H_{2s}(f_t) \circ \tilde{F}_h - H_{2s}(f_t)}{h} dm$$

$$= - \int f_t \langle \nabla H_{2s}(f_t), \nabla \tilde{b} \rangle dm, \quad (5.29)$$

where we use (5.22) in the last equality.

Denote by

$$G(t, s) := \int \frac{|f_t|^2 - |H_s(f_t)|^2}{4s} dm.$$

For any $s \in (0, \frac{1}{2})$ the map $t \mapsto G(t, s) \in L^2(X)$ is $C^1$ with

$$\frac{d}{dt} G(t, s) = \frac{1}{2s} \int f_t \langle \nabla H_{2s}(f_t), \nabla \tilde{b} \rangle dm \quad (5.30)$$

for $t \leq T$. By Proposition 5.15 we have

$$\left| \int \langle \nabla H_{2s}(f_t), \nabla \tilde{b} \rangle f_t dm \right| \leq Cs^2 \|f_t\|_{L^2}^2, \quad (5.31)$$
hence for any $s \in (0, \frac{1}{2})$, we have

$$\left| \frac{d}{dt} G(t, s) \right| \leq C s \lVert f_t \rVert_{L^2}^2 = C s \lVert f \rVert_{L^2}^2,$$

and thus

$$|G(t + \delta, s) - G(t, s)| \leq C s \delta \lVert f \rVert_{L^2}^2$$

(5.32)

for any $\delta \geq 0$ with $t + \delta \leq T$.

Suppose $f \in W^{1,2}(X)$. Note that $G(t, s) \uparrow \mathcal{E}(f_t)$ as $s \downarrow 0$. Let $s \downarrow 0$ in (5.32), we have

$$\mathcal{E}(f_t) = \mathcal{E}(f) < \infty$$

for every $t \leq T$. Hence (5.28) is proved.

Suppose $f \notin W^{1,2}(X)$, i.e., $\mathcal{E}(f) = +\infty$, then for any $t \leq T$, $f_t \notin W^{1,2}(X)$. Otherwise change the role of $f$ and $f_t$ in the above argument, we have $\mathcal{E}(f) = \mathcal{E}(f_t) < \infty$, which is a contradiction. Thus we have completed the proof. $\Box$

Theorem 5.16 can be ‘localized’ as follows:

**Corollary 5.17** Suppose $f \in L^2(X)$ with $\text{supp}(f) \subset b^{-1}((T, \infty))$ with $T \geq 0$, then $f \in W^{1,2}(X)$ if and only if $f \circ F_t \in W^{1,2}(X)$ for any $t \leq T$ and in this case

$$|D(f \circ F_t)| = |Df| \circ F_t \text{ m-a.e.}$$

(5.33)

**Sketch of the proof** Choose $\bar{\rho} > 0$ such that $\text{supp}(f) \subset b^{-1}((\bar{\rho} + T, \infty))$ and then build a function $\bar{b}$ and the flow $\bar{F}_t$ with respect to $\bar{\rho}$ as we have done in this subsection. Obviously, $f \circ \bar{F}_t = f \circ F_t$ m-a.e., and by the locality property of minimal weak upper gradient, we have $|D(f \circ \bar{F}_t)| = |D(f \circ F_t)|$ m-a.e.. By Theorem 5.16 we deduce that $f \in W^{1,2}(X)$ if and only if $f \circ \bar{F}_t \in W^{1,2}(X)$. Thus we only need to prove $|D(f \circ \bar{F}_t)| = |Df| \circ \bar{F}_t$ m-a.e. for $f \in W^{1,2}(X)$ with $\text{supp}(f) \subset b^{-1}((T, \infty))$. The later can be obtained by the same proof of Corollary 3.17 in [23] or Lemma 4.17 in [30]. $\Box$

The notion of Sobolev-to-Lipschitz property is a key to deduce metric information from the study of Sobolev functions. See Proposition 4.20 in [30].

In Corollary 5.17, (5.33) holds only for $f \in W^{1,2}(X)$ with $\text{supp}(f) \subset b^{-1}((T, \infty))$, we obtain that $F_t$’s are local isometry instead of isometry, see Theorem 5.18. The arguments here follow that of Theorem 3.18 in [23].

**Theorem 5.18** If we define the map $F : \mathbb{R}^+ \times b^{-1}((0, \infty)) \rightarrow b^{-1}((0, \infty))$ to be $F(t, x) = F_t(x)$, then $F$ admits a locally Lipschitz representative w.r.t. the measure $\mathcal{L}^1 \times m$. We still denote such representative by $F$. Furthermore, $F$ satisfies the following:
(1) for every $t \in \mathbb{R}^+$, $F_t$ is an invertible locally isometry from $b^{-1}((0, \infty))$ to $b^{-1}((t, \infty))$;

(2) for every $t, s \in \mathbb{R}^+$ and $x \in b^{-1}((0, \infty))$, we have

\[
F_{t+s}(x) = F_t(F_s(x)),
\]

\[
d(F_s(x), F_t(x)) = |s - t|; \tag{5.34}
\]

(3) for any curve $\eta : [0, 1] \ni s \mapsto b^{-1}((0, \infty))$, putting $\tilde{\eta} := F_t \circ \eta$ then $\tilde{\eta}$ is absolutely continuous if and only if $\tilde{\eta}$ is and in this case $|\dot{\tilde{\eta}}_s| = |\dot{\eta}_s|$ for a.e. $s \in [0, 1]$.

**Sketch of the proof** (2) (3) and the conclusion that $F$ admits a locally Lipschitz representative are easy to prove once we can prove $F_t$ has a locally isometric representative for every $t \geq 0$. For any $t \in \mathbb{R}^+$ and $x_0 \in b^{-1}((t, \infty))$, choose $r > 0$ such that $B_{x_0}(5r) \subset b^{-1}((t, \infty))$. Let $\Lambda$ be a countable dense subset of the set of $1$-Lipschitz functions with support in $B_{x_0}(5r)$. We can prove $d(y_0, y_1) = \sup_{f \in \Lambda} |f(y_1) - f(y_0)|$, $\forall y_0, y_1 \in B_{x_0}(r)$, as in [23]. By Corollary 5.17, for any $f \in \Lambda$, $|D(f \circ F_t)| = |Df| \circ F_t \leq 1$ a.e., then by the Sobolev-to-Lipschitz property of $X$, $f \circ F_t$ has a $1$-Lipschitz representative. Since $\Lambda$ is countable, there exists an $m$-negligible Borel set $N$ such that the restriction of $f \circ F_t$ to $X \setminus N$ is $1$-Lipschitz for every $f \in \Lambda$. Hence for any $x_1, x_2 \in F_t^{-1}(B_{x_0}(r) \setminus N)$, we have $d(F_t(x_1), F_t(x_2)) = \sup_{f \in \Lambda} |f(F_t(x_1)) - f(F_t(x_2))| \leq d(x_1, x_2)$. Since $b^{-1}((t, \infty))$ is $\sigma$-compact, by an easy covering argument, $F_t$ is locally $1$-Lipschitz on $b^{-1}((0, \infty)) \setminus N'$ for some $m$-negligible set $N'$. The same argument can be applied to $F_t^{-1} : b^{-1}((t, \infty)) \rightarrow b^{-1}((0, \infty))$. Hence $F_t$ has an invertible locally isometric representative.

\[\square\]

### 5.4 Equipped $Z$ with the Induced Distance and Measure

Now we consider the cross section $Z = b^{-1}(r')$. Recall that $Z$ is compact by Theorem 1.2.

Suppose $Z$ consists of exactly one point, i.e., $Z = \{x\}$, it is easy to see in this case $X = R(x)$. From Theorem 5.18, it is easy to see $b^{-1}((0, \infty))$ consists of a geodesic ray emitting from $b^{-1}(0)$. Then by Lemma 5.4, we know $(b^{-1}((0, \infty)), d, m)$ is isomorphic to $(\mathbb{R}^+, d_{\text{Euc}}, cL^1)$ with $c = m(b^{-1}([0, 1]))$.

We are going to prove that $b$ is an injective function on $X$. If this holds, then combining with the fact that $b$ is a proper function, we know $(X, d)$ is isometric to a half-line. Suppose $b$ is not injective on $X$, then we can find three distinct points $x_0, x_1, x_2 \in X$ such that $b(x_1) = b(x_2) < b(x_0) \leq 0$, and $b^{-1}((b(x_0), \infty))$ is isometric to a half-line. It is easy to see $d(x, x_1) = d(x, x_2)$ for any $x$ with $b(x) \geq b(x_0)$. Let $\mu = \frac{1}{m(b^{-1}([0, 1]))}m|b^{-1}([0, 1])$, $\nu = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$. It is easy to check that the plan $\mu \times \nu$ is optimal, but it is not induced by a map. This contradicts Theorem 1.1 in [33].

In conclusion, we have
Lemma 5.19 If $Z$ consists of exactly one point, then $(b^{-1}([0, \infty)), d, m)$ is isomorphic to $(\mathbb{R}^+, d_{\text{Eucl}}, c\mathcal{L}^1)$, where $c = m(b^{-1}([0, 1]))$. Furthermore, $(X, d)$ is isometric to some $([\bar{r}, \infty), d_{\text{Eucl}})$ with $\bar{r} \leq 0$.

From now on, we will always assume $Z$ contains at least 2 points. We define the projection map $P_j : b^{-1}((0, \infty)) \to Z$ to be

$$P_j(x) = F_{r-b(x)}(x).$$

By Theorem 5.18 the map $P_j$ is well defined and locally Lipschitz.

Suppose $x', y' \in Z$, by Lemma 3.2, there is a Lipschitz curve $\eta : [0, 1] \ni t \mapsto \eta_t \in b^{-1}([r', \infty])$ with $\eta(0) = x', \eta(1) = y'$. By Theorem 5.18, the curve $\sigma = P_j(\eta)$ is a Lipschitz curve connecting $x'$ and $y'$ in $Z$. In conclusion, we have

Corollary 5.20 If $Z$ consists of more than one point, then for every $x', y' \in Z$ there is a Lipschitz curve $\sigma : [0, 1] \to Z$ with $\sigma_0 = x', \sigma_1 = y'$. In particular, $Z$ is a path-connected space with Hausdorff dimension at least 1.

We define a new distance $d'$ on $Z$:

$$d'(x', y') := \inf_{\sigma} \left( \int_0^1 |\dot{\sigma}_t|^2 dt \right)^{\frac{1}{2}}$$

for $x', y' \in Z$, where the infimum is taken among all Lipschitz curves $\sigma : [0, 1] \to Z \subset X$ and the metric speed is computed w.r.t. the distance $d$.

By Corollary 5.20, $d'$ is always finite and is a distance on $Z$. It is easy to prove that a curve $\sigma : [0, 1] \to Z \subset X$ is absolutely continuous w.r.t. $d'$ if and only if it is absolutely continuous w.r.t. $d$ and in this case the metric speeds computed w.r.t. the two distances are the same, so we use $|\dot{\sigma}_t|$ to denote both of these metric speeds if there is no ambiguity. We can easily check that the map $P_j$ is still locally Lipschitz even if $Z$ equipped with the new distance $d'$.

The measure $m'$ on $Z$ is chosen to be $m' := c\tilde{m}_{r'}$ with $c = m(b^{-1}([0, 1]))$. By Lemma 5.4, it is easy to see

$$P_j_*(m b^{-1}([s,t])) = (t-s)m'.$$ \hspace{1cm} (5.36)

holds for any $t > s \geq 0$.

Combining the facts that $X$ is doubling, that $P_j$ is locally Lipschitz, and Lemma 5.4, we can derive that $(Z, d', m')$ is also doubling, see Proposition 3.26 in [23] for similar proof.

Denote by $Y = Z \times (0, \infty)$, we endow $Y$ with the product measure $m' \otimes \mathcal{L}^1$ and the product distance $d' \times d_{\text{Eucl}}$ defined by

$$d'(x', t), (y', s) : = \sqrt{d'(x', y')^2 + |t - s|^2}.$$

We denote $(Y, d' \times d_{\text{Eucl}}, m' \otimes \mathcal{L}^1)$ by $(Y, dy, m_Y)$ for simplicity.
For $0 \leq c < d \leq \infty$, we denote by $Y_{(c,d)} := Z \times (c, d)$ for simplicity, similar notations are $Y_{[c,d]}$ etc.

We introduce the maps $T : Y \to b^{-1}((0, \infty))$ and $S : b^{-1}((0, \infty)) \to Y$ to be

$$T(x', t) := F_{t-r'}(x'),$$

$$S(x) := (P_j(x), b(x)).$$

It is easy to see that $S \circ T = \text{Id}|_Y$ and $T \circ S = \text{Id}|_{b^{-1}((0,\infty))}$. From Lemma 5.4, it is easy to check

$$T_*m_Y = m|_{b^{-1}((0,\infty))} \quad \text{and} \quad S_*m|_{b^{-1}((0,\infty))} = m_Y. \quad (5.37)$$

By Theorem 5.18, both $S$ and $T$ are locally Lipschitz.

5.5 Estimate on the Speed of the Projection

As in Sect. 3.6.2 of [23], we also need some suitable cut-off function and reparametrization function.

For any fixed $\hat{r}, \hat{R}$ with $\hat{R} > r' > \hat{r} > 0$, pick a function $\hat{\phi} \in C^\infty(\mathbb{R})$ with support in $([\hat{r}, \hat{R}]$ so that

$$\hat{\phi}(z) = (z - r')^2 \quad \text{on } [\hat{r}, \hat{R}].$$

Define $\hat{b} : X \to \mathbb{R}$ as $\hat{b}(x) = \hat{\phi} \circ b(x)$. Define the reparametrization function $\hat{\text{rep}} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\hat{\text{rep}}_t(r) = \text{rep}(t, r)$ satisfying $\text{rep}_0(r) = 0$ and

$$\partial_t \hat{\text{rep}}_t(r) = \hat{\phi}'(r - \hat{\text{rep}}_t(r)) \quad (5.38)$$

for any $r$. We then define the flow $\hat{F} : \mathbb{R}^+ \times X \to X$ to be

$$\hat{F}_t(x) = F_{-\hat{\text{rep}}_t(\hat{b}(x))}(x).$$

Obviously, $\hat{F}_t$ is the identity on $b^{-1}((-\infty, \hat{r})] \cup b^{-1}([\hat{R}, \infty))$, $\hat{F}_t$ sends $b^{-1}([\hat{r}, \hat{R}])$ into itself for every $t \geq 0$; $\hat{F}_t : X \to X$ is invertible for every $t \geq 0$, and for $t > 0$ we define $\hat{F}_{-t} := (\hat{F}_t)^{-1}$. For every $x \in X$, let $\eta^{(x)} : [0, \infty) \mapsto X$ be the curve given by $\eta^{(x)} := \hat{F}_t(x)$. Properties for $\hat{F}_t$ similar to b) d) e) in Sect. 3.6.2 of [23] can be easily established. Furthermore, it is easy to obtain

$$\frac{d}{dt} \hat{b}(\eta^{(x)}) = -(\hat{\phi}' \circ b)^2(\eta^{(x)}) = -4\hat{b}(\eta^{(x)}), \quad \text{if } x \in b^{-1}([\hat{r}, \hat{R}]),$$

thus $|b(\eta^{(x)}) - r'|$ decreases exponentially as $t \to +\infty$, hence $|\hat{F}_t(x) - P_j(x)| \to 0$ uniformly for $x \in b^{-1}([\hat{r}, \hat{R}])$ as $t \to +\infty$. Similar to f) in Sect. 3.6.2 of [23], we
can derive that there exist positive constants $C$ and $\delta$ such that $\text{Lip}(\hat{F}_s) \leq 1 + |s|C$ for every $s \in [-\delta, \delta]$.

On the other hand, from the chain rule for Laplacian it is easy to see that $\hat{b} \in \text{Test}(X)$ and $\Delta \hat{b} \in L^\infty(X)$. Furthermore, recall that by Lemma 5.11, $\text{Hess}(b) = 0$ on $(b^{-1}(0, \infty))$, hence by the chain rules for Hessian, we have

$$\text{Hess}(\hat{b}) = \hat{\phi}' \circ b \text{Hess}(b) + \hat{\phi}'' \circ b \nabla b \otimes \nabla b = 2\nabla b \otimes \nabla b,$$

holds on $b^{-1}(\hat{\mathcal{F}}, \hat{\mathcal{R}})$.

By the very similar arguments in Proposition 3.31 of [23], using the properties of $\hat{b}$ and $\hat{F}$, here, we can establish the following proposition:

**Proposition 5.21** Let $v \in L^2(TX)$ and put $v_s := d\hat{F}_s(v)$. Then the map $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F}_s \in L^1(X)$ is $C^1$ and its derivative is given by

$$\frac{1}{2} \frac{d}{ds} |v_s|^2 \circ \hat{F}_s = -\text{Hess}(\hat{b})(v_s, v_s) \circ \hat{F}_s,$$

the incremental ratios being converging both in $L^1(X)$ and $m$-a.e..

We have the following corollary of Proposition 5.21:

**Corollary 5.22** Let $v \in L^2(TX)$ be concentrated on $b^{-1}(\hat{\mathcal{F}}, \hat{\mathcal{R}})$ and put $v_s := d\hat{F}_s(v)$. Then for every $s_2 > s_1 \geq 0$ we have

$$|v_{s_2}|^2 \circ \hat{F}_{s_2} \leq |v_{s_1}|^2 \circ \hat{F}_{s_1}, \quad m$-a.e.. (5.41)

**Proof** By Proposition 5.21, the map $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F}_s \in L^1(X)$ is $C^1$, and its derivative is given by $\frac{d}{ds}|v_s|^2 \circ \hat{F}_s = -2\text{Hess}(\hat{b})(v_s, v_s) \circ \hat{F}_s$. Note that for $s \geq 0$, $\frac{1}{2}|v_s|^2 \circ \hat{F}_s$ is $m$-a.e. on $X \setminus b^{-1}(\hat{\mathcal{F}}, \hat{\mathcal{R}})$. Furthermore by (5.39), $\text{Hess}(\hat{b})(v_s, v_s) \circ \hat{F}_s = 2(\nabla b, v)^2$ on $b^{-1}(\hat{\mathcal{F}}, \hat{\mathcal{R}})$. Hence $\frac{d}{ds}|v_s|^2 \circ \hat{F}_s \leq 0$ holds for $s \geq 0$ and we conclude. □

The following proposition can be proved by repeating verbatim the proof of Proposition 3.33 in [23], Corollary 5.22 is used in place of Corollary 3.32 in [23] in the argument.

**Proposition 5.23** Let $[c, d] \subset (0, \infty)$ and $\pi$ be a test plan on $X$ such that $b(\eta_t) \in [c, d]$ for every $t \in [0, 1]$ and $\pi$-a.e. $\eta$. Then for $\pi$-a.e. $\eta$ the curve $\hat{\eta} := P_j \circ \eta$ is absolutely continuous and satisfies

$$|\dot{\hat{\eta}}_t| \leq |\dot{\eta}_t|, \quad \text{for a.e. } t \in [0, 1].$$

**5.6 Sobolev Spaces on Y and X**

By Proposition 5.23, we can understand the Sobolev Spaces on $Z$ well.
Proposition 5.24  Let \([c, d] \subset (0, \infty), h \in \text{Lip}(\mathbb{R})\) with \(\text{supp}(h) \subset \subset (0, \infty)\) and identically 1 on \([c, d]\). Suppose \(f \in L^2(X)\) is of the form \(f(x) = g(P_j(x))h(b(x))\) for some \(g \in L^2(\mathbb{Z})\). Then \(f \in W^{1,2}(X)\) if and only if \(g \in W^{1,2}(\mathbb{Z})\) and in this case we have

\[
|Df|_X(x) = |Dg|_Z(P_j(x))
\]  

(5.43)

for \(m\)-a.e. \(x\) such that \(b(x) \in [c, d]\).

The proof can be carried out by an easy adaption from Theorem 3.24 of [23], \(|Df|_X(x) \geq |Dg|_Z(P_j(x))\) can be obtained directly from definitions and Theorem 5.18, while the proof of \(|Df|_X(x) \leq |Dg|_Z(P_j(x))\) need to use Proposition 5.23 in place of Proposition 3.33 in [23].

In [32], the authors introduce the notion of ‘measured-length space,’ and prove that if a metric measure space is locally doubling and measured-length, then it has the Sobolev-to-Lipschitz property (see Proposition 3.18 in [32]).

Proposition 5.25  \((Z, d', m')\) is infinitesimally Hilbertian and a measured-length space.

The infinitesimal Hilbertianity of \((Z, d', m')\) is a direct consequence of Proposition 5.24. To check that \((Z, d', m')\) is a measured-length space, we just follow the lines in the proof of Proposition 3.26 in [23], whose idea is to use the good properties of optimal transport on \(X\) (see [33, 44]), and use \(P_j\) to map test plans on \(X\) to test plans on \(Z\). Besides the good properties of optimal transport on \(X\), the fundamental properties in Lemma 5.4, Proposition 5.23 and that \(P_j\) is locally Lipschitz are also used in the proof; we omit the details here.

Now we can apply the results of Sect. 3 in [32] to conclude that \(Y\) is infinitesimally Hilbertian and has the Sobolev-to-Lipschitz property.

Now we can compare the Sobolev spaces of \(X\) and \(Y\). Given an open set \(U \subset X\), we denote by \(W^{1,2}_0(U)\) the \(W^{1,2}(X)\)-completion of the space of functions in \(W^{1,2}(X)\) whose support is compact and contained in \(U\). Similar notation is used for \(W^{1,2}_0(V)\) with open set \(V \subset Y\).

Theorem 5.26  Suppose \(0 < c < d\), then \(f \in W^{1,2}_0(Y(c,d))\) if and only if \(f \circ S \in W^{1,2}_0(b^{-1}((c, d)))\) and in this case

\[
|Df|_Y \circ S = |D(f \circ S)|_X, \ m\text{-a.e. on } b^{-1}((c, d)).
\]  

(5.44)

Theorem 5.26 can be proved by the very same arguments in [30] or [23].

5.7 Back to the Metric Properties and Conclusion

Proof of Theorem 1.4  We will handle Case (2) here. We know \(T\) and \(S\) are measure-preserving in (5.37). Combining Theorem 5.26 with the fact that both \(X\) and \(Y\) have the Sobolev-to-Lipschitz property, the same argument as in the proof of Theorem 5.18
can be applied to show that $T$ and $S$ are local isometries. We omit the details here. Now we will derive the additional conclusions.

For any $x_0, x_1 \in b^{-1}(0, \infty)$, note that $(Y, d_Y)$ is a geodesic space, thus there is always a shortest geodesic $\sigma : [0, 1] \to Y$ with $\sigma(0) = S(x_0), \sigma(1) = S(x_1)$. Since $T$ is a local isometry, $T \circ \sigma : [0, 1] \to X$ is a Lipschitz curve of length $d_Y(S(x_0), S(x_1))$ connecting $x_0$ and $x_1$, thus $d(x_0, x_1) \leq d_Y(S(x_0), S(x_1))$.

Now let $r_1 = \frac{\text{diam}(Z)}{2}$, and suppose $x_0, x_1 \in b^{-1}((r_1, \infty))$. Let $\eta : [0, 1] \to X$ be a shortest geodesic with $\eta(0) = x_0, \eta(1) = x_1$. $\eta$ has to be contained in $b^{-1}((0, \infty))$, for otherwise there is a point $z = \eta_l \in b^{-1}((\infty, 0])$, then

\[ d(x_0, x_1) = d(z, x_0) + d(z, x_1) \geq b(x_0) + b(x_1) > |b(x_1) - b(x_0)| + \text{diam}'(Z) \geq d_Y(S(x_0), S(x_1)), \]

which is a contradiction. Now $S \circ \eta : [0, 1] \to Y$ is a Lipschitz curve of length $d(x_0, x_1)$, and hence $d(x_0, x_1) \geq d_Y(S(x_0), S(x_1))$. Thus $S : (b^{-1}((r_1, \infty)), d) \to (Z \times (r_1, \infty), d_Y)$ is an isometry.

Note that for any $x_0, x_1 \in b^{-1}((r_1, \infty))$, suppose $\sigma : [0, 1] \to Z \times (r_1, \infty)$ is a shortest geodesic connecting $S(x_0)$ and $S(x_1)$, then $T \circ \sigma$ is a shortest geodesic connecting $x_0$ and $x_1$, hence $b^{-1}((r_1, \infty))$ is a geodesic space. Thus by Proposition 7.7 of [2], $(b^{-1}((r_1, \infty)), d, m)$ is RCD$(0, N)$. By the isomorphism $S : (b^{-1}((r_1, \infty)), d, m) \to (Z \times (r_1, \infty), d_Y, m_Y)$ and a natural isomorphism between $(Z \times (r_1, \infty), d_Y, m_Y)$ and $(Y, d_Y, m_Y)$, we know $(Y, d_Y, m_Y)$ is also an RCD$(0, N)$ space. By Corollary 2.5 in [48], the Hausdorff dimension of $(Y, d_Y, m_Y)$ is at most $N$. On the other hand, by Corollary 5.20, $Y$ has Hausdorff dimension at least 2, hence $N \geq 2$.

Since $(Y, d_Y, m_Y)$ is an RCD$(0, N)$ space, and $(Y, d_Y, m_Y)$ is the product of $(Z, d', m')$ and $(\mathbb{R}^+, d_{\text{Eucl}}, L^1)$, the argument in Corollary 5.30 and Theorem 7.4 of [30] shows that $(Z, d', m')$ is an RCD$(0, N-1)$ space.

The proof is completed. \hfill \Box

Acknowledgements The author would like to thank Prof. X.P. Zhu, B.L. Chen, and H.C. Zhang for their encouragements and helpful discussions. The author is grateful to the anonymous referees for careful reading and giving many valuable suggestions.

References

1. Ambrosio, L., Gigli, N.: Lecture Notes in Mathematics. A User’s Guide to Optimal Transport. Modelling and Optimisation of Flows on Networks. Springer, Heidelberg (2011)
2. Ambrosio, L., Mondino, A., Savaré, G.: On the Bakry-Émery condition, the gradient estimates and the local-to-global property of RCD$^*$(K, N) metric measure spaces. J. Geom. Anal. 26, 24–56 (2016)
3. Ambrosio, L., Mondino, A., Savaré, G.: Nonlinear diffusion equations and curvature conditions in metric measure spaces. arXiv:1509.07273 (2015) (to appear in Mem. Am. Math. Soc.)
4. Ambrosio, L., Gigli, N., Savaré, G.: Lectures in Mathematics. Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. ETH Zürich, Birkhäuser, Basel (2008)
5. Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. 195, 289–391 (2014)
6. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. 163, 1405–1490 (2014)
7. Ambrosio, L., Gigli, N., Mondino, A., Rajala, T.: Riemannian Ricci curvature lower bounds in metric spaces with $\sigma$-finite measure. Trans. Am. Math. Soc. 367, 4661–4701 (2015)
8. Bianchini, S., Cavalletti, F.: The Monge problem for distance cost in geodesic spaces. Commun. Math. Phys. 318, 615–673 (2013)
9. Bianchini, S., Caravenna, L.: On the extremality, uniqueness and optimality of transference plans. Bull. Inst. Math. Acad. Sin. (N.S.) 4, 353–454 (2009)
10. Bacher, K., Sturm, K.-T.: Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. J. Funct. Anal. 259, 28–56 (2010)
11. Calabi, E.: On manifolds with non-negative Ricci-curvature II. Not. Am. Math. Soc. 22, A205 (1975)
12. Cavalletti, F.: Monge problem in metric measure spaces with Riemannian curvature-dimension condition. Nonlinear Anal. 99, 136–151 (2014)
13. Cavalletti, F., Huesmann, M.: Existence and uniqueness of optimal transport maps. Ann. Inst. H. Poincaré Anal. Non Linéaire 32, 1367–1377 (2015)
14. Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. (2016). doi: 10.1007/s00222-016-0700-6
15. Cavalletti, F., Mondino, A.: Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. Geom. Topol. 21, 603–645 (2017)
16. Cavalletti, F., Mondino, A.: Optimal maps in essentially non-branching spaces. Commun. Contemp. Math. (2016). doi:10.1142/S0219199717500079
17. Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. 9, 428–517 (1999)
18. Cheeger, J., Colding, T.H.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. Math. 144, 189–237 (1996)
19. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. I. J. Differ. Geom. 46, 406–480 (1997)
20. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. II. J. Differ. Geom. 54, 13–35 (2000)
21. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. III. J. Differ. Geom. 54, 37–74 (2000)
22. Cheeger, J., Gromoll, D.: The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differ. Geom. 6, 119–128 (1971/72)
23. De Philippis, G., Gigli, N.: From volume cone to metric cone in the nonsmooth setting. Geom. Funct. Anal. 26, 1526–1587 (2016)
24. Erbar, M., Kuwada, K., Sturm, K.T.: On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure space. Invent. Math. 201, 993–1071 (2015)
25. Fremlin, D.H.: Measure Theory, vol. 4. Torres Fremlin, Colchester (2002)
26. Garofalo, N., Mondino, A.: Li-Yau and Harnack type inequalities in RCD$^*$($K$, $N$) metric measure spaces. Nonlinear Analysis: Theory, Methods & Applications 95, 721–734 (2014)
27. Gigli, N.: Optimal maps in non branching spaces with Ricci curvature bounded from below. Geom. Funct. Anal. 22, 990–999 (2012)
28. Gigli, N.: On the Differential Structure of Metric Measure Spaces and Applications, vol. 236. Memoirs of the American Mathematical Society, Providence (2015)
29. Gigli, N.: Nonsmooth differential geometry-an approach tailored for spaces with Ricci curvature bounded from below. arXiv:1407.0809 (2014) (to appear in Mem. Am. Math. Soc)
30. Gigli, N.: The splitting theorem in non-smooth context. arXiv:1302.5555 (2013)
31. Gigli, N.: An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature. Anal. Geom. Metr. Spaces 2, 169–213 (2014)
32. Gigli, N., Han, B.: Sobolev spaces on warped products. arXiv:1512.03177v1 (2015)
33. Gigli, N., Rajala, T., Sturm, K.-T.: Optimal maps and exponentiation on finite-dimensional spaces with Ricci curvature bounded from below. J. Geom. Anal. 26, 2914–2929 (2016)
34. Gigli, N., Mondino, A.: A PDE approach to nonlinear potential theory in metric measure spaces. J. Math. Pures Appl. 100, 505–534 (2013)
35. Gigli, N., Mondino, A., Rajala, T.: Euclidean spaces as weak tangents of infinetesimally Hilbertian metric measure spaces with Ricci curvature bounded below. J. Reine Angew. Math. 705, 233–244 (2015)
36. Jiang, Y., Zhang, H.C.: Sharp spectral gaps on metric measure spaces. Calc. Var. PDE 55, 14 (2016)
37. Ketterer, C.: Cones over metric measure spaces and the maximal diameter theorem. J. Math. Pures Appl. 103, 1228–1275 (2015)
38. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. Math. (2) 169, 903–991 (2009)
39. Lott, J., Villani, C.: Weak curvature conditions and functional inequalities. J. Funct. Anal. 245, 311–333 (2007)
40. Mondino, A., Naber, A.: Structure theory of metric-measure spaces with lower Ricci curvature bounds I. arXiv:1405.2222 (2014)
41. Ohta, Shin-ichi: On the measure contraction property of metric measure spaces. Comment. Math. Helv. 82, 805–828 (2007)
42. Rajala, T.: Local Poincaré inequalities from stable curvature conditions on metric spaces. Calc. Var. PDE. 44, 477–494 (2012)
43. Rajala, T.: Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm. J. Funct. Anal. 263, 896–924 (2012)
44. Rajala, T., Sturm, K.T.: Non-branching geodesics and optimal maps in strong CD(K, ∞)-spaces. Calc. Var. PDE. 50, 831–846 (2014)
45. Sormani, C.: The almost rigidity of manifolds with lower bounds on Ricci curvature and minimal volume growth. Commun. Anal. Geom. 8, 159–212 (2000)
46. Sormani, C.: Busemann functions on manifolds with lower bounds on Ricci curvature and minimal volume growth. J. Differ. Geom. 48, 557–585 (1998)
47. Sturm, K.T.: On the geometry of metric measure spaces I. Acta Math. 196, 65–131 (2006)
48. Sturm, K.T.: On the geometry of metric measure spaces II. Acta Math. 196, 133–177 (2006)
49. Villani, C.: Optimal transport, Old and New. Grundlehren der Mathematischen Wissenschaften, vol. 338. Springer, Berlin (2009)
50. Yau, S.-T.: Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. Indiana Math. J. 25, 659–670 (1976)
51. Zhang, H.C., Zhu, X.P.: Ricci curvature on Alexandrov spaces and rigidity theorems. Commun. Anal. Geom. 18, 503–553 (2010)
52. Zhang, H.C., Zhu, X.P.: Local Li-Yau’s estimates on RCD*(K, N) metric measure spaces. Calc. Var. PDE 55, 93 (2016)