Motion of gas in highly rarefied space

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Abstract. A model describing a motion of gas in a highly rarefied space received an unlucky number 13 in the list of the basic models of the motion of gas in the three-dimensional space obtained by L.V. Ovsyannikov. For a given initial pressure distribution, a special choice of mass Lagrangian variables leads to the system describing this motion for which the number of independent variables is less by one. Hence, there is a foliation of a highly rarefied gas with respect to pressure. In a strongly rarefied space for each given initial pressure distribution, all gas particles are localized on a two-dimensional surface that moves with time in this space. We found some exact solutions of the obtained system that describe the processes taking place inside of the tornado. For this system we found all nontrivial conservation laws of the first order. In addition to the classical conservation laws the system has another conservation law, which generalizes the energy conservation law. With the additional condition we found another one generalized energy conservation law.

1. Introduction
The basic models of gas motion in three-dimensional space were obtained in [1]. A model describing the motion of a gas in a highly sparse space got an unlucky No 13 in the list of these models. The study of these model was begun in [2] and was continued in [3, 4]. In this report we will use materials of these articles.

The model of the thermal motion of a gas in a three-dimensional rarefied space is determined by the following system of differential equations [2–4]

\[
\begin{align*}
\mathbf{u}_t + \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} + \frac{1}{\rho} \nabla p &= \mathbf{0}, \\
\rho_t + \mathbf{u} \cdot \nabla \rho + \rho \text{div} \mathbf{u} &= 0, \\
p_t + \mathbf{u} \cdot \nabla p &= 0
\end{align*}
\]

where \( t \) is a time, \( \mathbf{x} = \mathbf{x}(x,y,z) \in R^3 \), \( \mathbf{u} = \mathbf{u}(t,\mathbf{x}) \in R^3 \) is a velocity vector, \( \rho = \rho(t,\mathbf{x}) \) is a density, \( p = p(t,\mathbf{x}) \) is a pressure.

Suppose the variables \( \xi = (\xi,\eta,\zeta) \) are initial values of variables \( \mathbf{x} = \mathbf{x}(t,\xi) = (x,y,z) \), i.e. \( \mathbf{x}(0,\xi) = \xi \).

With the help of these variables, the system (1) is written in the form of a vector equation:

\[
\begin{align*}
\left| \frac{\partial \mathbf{x}}{\partial \xi} \right|^{-1} \left( \frac{\partial \mathbf{x}}{\partial \xi} \right)^T \mathbf{x}_{tt} + \frac{1}{\rho_0} p_\xi &= \mathbf{0}, \\
\rho &= \rho_0 \left| \frac{\partial \mathbf{x}}{\partial \xi} \right|^{-1}, \\
p_t &= 0
\end{align*}
\]

where \( \rho_0 = \rho_0(\xi) \) is an initial density distribution. A solution of the last equation of the system (2) is \( p = p(\xi) \), where \( p(\xi) \) is a given function that determines the initial pressure distribution. Since the pressure is stored in the particle, it is Lagrange variable.
After the change of variables

\[ \xi' = \xi'(\xi), \eta' = \eta'(\xi), \zeta' = p(\xi) \]  

(3)

such that

\[ \frac{\partial(\xi', \eta', p)}{\partial(\zeta, \eta, \xi)} = \rho_0(\xi) \]  

(4)

the system (2) is transformed to the equivalent vector equation (the strokes are omitted)

\[ x_{tt} + x_{\xi} \times x_{\eta} = 0 \]  

(5)

The velocity vector and density are determined by the formulae

\[ u = x_t, \quad \frac{1}{\rho} = \left| \frac{\partial x}{\partial \xi} \right| \]

The equation (5) does not contain the variable \( \zeta' \) and the derivatives \( \partial_{\zeta'} x \). Thus, the special choice of the Lagrange mass variables, by the formulae (3) and (4), makes it possible to transform the system (1) to vector equation (5) containing only three independent variables. The variable \( \zeta' = p \) is a parameter that, according to (3) and (4) defines the variables of equation (5). It means that there is a stratification of a highly rarefied gas with respect to pressure. Namely, in a strongly rarefied space for each given initial pressure distribution, at each instant of time all the gas particles are localized on the two–dimensional surface \( S_t \), defined by equation \( x = x(t, \xi, \eta) \). The surface \( S_t \) moves with time in this space. At each point of the surface \( S_t \), the acceleration vector is collinear to the normal vector to this surface.

In the cylindrical variables \( r = \sqrt{x^2 + y^2}, \varphi = \arctan \frac{y}{x}, z \) the system (5) is written as

\[ r_{tt} - r_{\varphi t}^2 = r (z_{\xi} \varphi_\eta - z_\eta \varphi_\xi), \quad (r^2 \varphi_t)_t = r (r_{\xi} z_\eta - r_\eta z_\xi), \quad z_{tt} = r (\varphi_\xi r_\eta - \varphi_\eta r_\xi) \]  

(6)

In further we assume that the initial pressure distribution in a highly rarefied gas is given.

2. Exact solutions describing the processes taking place inside of the tornado

2.1.

The solution of the system (6) for which \( z = \pm r \) is defined by the formulas

\[ r = t \alpha(\varphi) + \beta(\varphi), \quad \varphi = \varphi(\xi, \eta), \quad z = \pm r \]  

(7)

where \( \varphi(\xi, \eta), \alpha(\varphi) \) and \( \beta(\varphi) \) are arbitrary functions.

The solution (7) describes the following dynamic process: each gas particle is arranged on a circular cone \( z = \pm \sqrt{x^2 + y^2} \) and move along its generator with a constant velocity

\[ u = \alpha(\varphi)e_r + 0 \cdot e_\varphi \pm \alpha(\varphi)e_z \]

where \( e_r, e_\varphi, e_z \) are the orts of cylindrical coordinate system.

If \( \alpha < 0 \), then each particle of the gas moves toward the apex of the cone, and if \( \alpha > 0 \), then each gas particle moves in a direction from apex of the cone.
2.2. Let $k$ and $m$ are arbitrary constants, $k > 0$. System (6) has only two solutions, for which $\varphi = kt + m$. The first solution is

$$r = f(\alpha) \exp(kt) + \frac{c}{f(\alpha)} \exp(-kt), \quad \varphi = kt + m, \quad z = \alpha t + g(\alpha) + \frac{2k^2 f(\alpha)}{f'(\alpha)} h(\xi, \eta)$$  \hspace{1cm} (8)

and the second solution is

$$r = f(\alpha) \exp(-kt), \quad \varphi = kt + m, \quad z = \alpha t + g(\alpha) - \frac{2k^2 f(\alpha)}{f'(\alpha)} h(\xi, \eta)$$  \hspace{1cm} (9)

where $f(\alpha) > 0$, $g(\alpha)$, $\alpha = \alpha(\xi, \eta)$ are arbitrary functions, $c$ is an arbitrary positive constant,

$$g(\alpha) = \begin{cases} \int \frac{d\xi}{\alpha(\eta)_{\alpha=\text{const}}}, & \text{for } \alpha_\eta \neq 0, \\ \int \frac{d\alpha}{\alpha(\xi)_{\alpha=\text{const}}}, & \text{for } \alpha_{\xi} \neq 0. \end{cases}$$

Vector of velocity $u = u_r e_r + u_\varphi e_\varphi + u_z e_z$ and vector of acceleration of each particle of gas for these solutions are defined by the formulas:

for the first solution

$$u_r = k \left( f(\alpha) e^{kt} - \frac{c}{f(\alpha)} e^{-kt} \right), \quad u_\varphi = k \left( f(\alpha) e^{kt} + \frac{c}{f(\alpha)} e^{-kt} \right), \quad u_z = \alpha(\xi, \eta),$$

$$a_r = 0, \quad a_\varphi = 2k^2 \left( f(\alpha) e^{kt} - \frac{c}{f(\alpha)} e^{-kt} \right), \quad a_z = 0,$$

for the second solution

$$u_r = -kf(\alpha) e^{-kt}, \quad u_\varphi = kf(\alpha) e^{-kt}, \quad u_z = \alpha(\xi, \eta),$$

$$a_r = 0, \quad a_\varphi = -2k^2 f(\alpha) e^{-kt}, \quad a_z = 0.$$

The solution (8) for $c = 0$ describes the movement of the gas particles along helical lines obtained by stretching along an axis $z$ unwinding logarithmic spirals. At that, the components $u_r$, $u_\varphi$ of the velocity vector and component $a_\varphi$ of the acceleration vector tend to infinity when $t \to \infty$.

The solution (9) describes the movement of the gas particles along helical lines obtained by stretching along an axis $z$ twisting logarithmic spirals. At that, the components $u_r$, $u_\varphi$ of the velocity vector and component $a_\varphi$ of the acceleration vector tend to zero when $t \to \infty$.

The solutions (7), (8) and (9) can be used at the study of the processes taking place inside of the tornado.

2.3. The main Lie algebra of the system (5) was obtained in [2]. The following operators

$$X_1 = t \partial_t + \xi \partial_\xi + \eta \partial_\eta, \quad X_2 = t \partial_t - 2x \cdot \partial_x, \quad X_3 = \eta \partial_\xi - \xi \partial_\eta$$

are contained in this algebra. Let $\alpha$ and $\beta$ are arbitrary constants.

The invariant $\langle X_1 + \alpha X_2, \beta X_2 + X_3 \rangle$-solution of the system (5) describes the gas flow, in which the Euler coordinates of each gas particle change according to the law

$$x = (\xi^2 + \eta^2)^{-\alpha} e^{2\beta \arctan \frac{\eta}{\xi}} w(\lambda), \quad \lambda = t (\xi^2 + \eta^2)^{\frac{\alpha+1}{2}} e^{\beta \arctan \frac{\eta}{\xi}}.$$
The function \( w \) is a solution of the vector equation
\[
 w'' + 2\beta \lambda w' \times w = 0
\]

The main mechanical characteristics of this gas flow are:
\[
 |x| = \frac{1}{(\xi^2 + \eta^2)^{\alpha/2}} \left( \frac{c_1 t^2 e^{\beta \arctan \frac{\eta}{\xi}}}{(\xi^2 + \eta^2)^{\alpha/2}} + \frac{c_2 t}{(\xi^2 + \eta^2)^{\alpha/2}} + c_3 \right)^{\frac{1}{2}} e^{\frac{\beta}{2} \arctan \frac{\eta}{\xi}}
\]
\[
 |u| = \frac{c_1 e^{\frac{3\beta}{2} \arctan \frac{\eta}{\xi}}}{(\xi^2 + \eta^2)^{\frac{3\alpha+1}{2}}},
\]
\[
 |a| = \frac{2c_1 |\beta| t}{(\xi^2 + \eta^2)^{\frac{3\alpha+1}{2}}} \left( \frac{c_1 t^2 e^{\beta \arctan \frac{\eta}{\xi}}}{(\xi^2 + \eta^2)^{\alpha+1}} + \frac{c_2 t}{(\xi^2 + \eta^2)^{\alpha+1}} + c_3 \right)^{\frac{1}{2}} e^{\frac{11\beta}{2} \arctan \frac{\eta}{\xi} \sin \theta},
\]
where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

If \( c_2^2 - 4c_1^2 c_3 > 0 \), then this solution exists only in two cases:

(i) either
\[
 t < -\frac{1}{2 c_1} \left( c_2 + (c_2^2 - 4c_1^2 c_3)^{\frac{1}{2}} \right) (\xi^2 + \eta^2)^{\frac{\alpha+1}{2}} e^{-\beta \arctan \frac{\eta}{\xi}},
\]

(ii) either
\[
 t > -\frac{1}{2 c_1} \left( c_2 - (c_2^2 - 4c_1^2 c_3)^{\frac{1}{2}} \right) (\xi^2 + \eta^2)^{\frac{\alpha+1}{2}} e^{-\beta \arctan \frac{\eta}{\xi}}.
\]

If \( c_2^2 - 4c_1^2 c_3 < 0 \), then this solution exists for all \( t, \xi, \eta \).

Each particle of gas with the passage of the time moves away from the origin by a nonlinear law, the absolute value of the velocity vector of each particle is constant, and the absolute value of the acceleration vector nonlinearly depends on the time. The acceleration vector is perpendicular to the velocity vector. In particular, such processes are characteristic for the motion of gas particles inside of the tornado [5-7].

3. Conservation laws

The conservation law of the first order for the system (5) is a vector [8–15]
\[
 A = A(t, \xi, \eta, x, x_\xi, x_\eta) = (A_0, A_1, A_2)
\]
such that
\[
 (D \cdot A)_{(5)} = 0,
\]
where \( \{5\} \) is a manifold, which is defined by the equations of the system (5) in the extended space, \( D = (D_t, D_\xi, D_\eta) \) and \( D_t, D_\xi, D_\eta \) are operators of total differentiation by \( t, \xi, \eta \).

In addition to the classical conservation laws (an energy conservation law, a momentum conservation law, a angular momentum conservation law, a conservation law, which determines the law of motion of centre of mass) the system (5) has an additional generalized energy conservation law
\[
 A_0 = t \left( \frac{1}{2} |x_t|^2 + \frac{1}{3} (x_\xi \times x_\eta) \cdot x \right) + \left( \frac{1}{3} x + \frac{5}{6} (\xi x_\xi + \eta x_\eta) \right) \cdot x_t,
\]
\[
 A_2 = -\frac{1}{3} t (x_\xi \times x_t) \cdot x - \frac{5}{12} \eta |x_t|^2.
\]
Other conservation laws are possible under the additional conditions imposed to the nature of the motion of the gas. In particular, let, the motion of the gas is such that at each time in each point \((\xi, \eta)\) the velocity vector is perpendicular to the vector \(\xi x_{\xi} + \eta x_{\eta}\), lying in the plane tangent to the surface \(S_t\). This means that a differential connection
\[
x_t \cdot (\xi x_{\xi} + \eta x_{\eta}) = 0
\]
is added to the system (5). In this case, another one generalized energy conservation law is added to the classical conservation laws and conservation law (10)
\[
A_0 = t^2 \left( \frac{1}{2} |x_t|^2 + \frac{1}{3} (x_{\xi} \times x_{\eta}) \cdot x \right) + \frac{1}{3} t \left( 2x + 5 (\xi x_{\xi} + \eta x_{\eta}) \right) \cdot x_t - \frac{1}{3} |x_t|^2,
\]
\[
A_1 = \frac{1}{3} t^2 (x_{\eta} \times x_t) \cdot x - \frac{5}{6} t \xi |x_t|^2, \quad A_2 = -\frac{1}{3} t^2 (x_{\xi} \times x_t) \cdot x - \frac{5}{6} t \eta |x_t|^2.
\]

4. Conclusion and discussion
The main models of the motion of the gas in three–dimensional space were obtained in [1]. The model describing motion of the gas in the rarefied space, got unlucky number 13 in the list of these models. This model can be used, in particular, in the study of the processes taking place inside of tornado.

For a given initial distribution of the pressure a specific selection of mass Lagrange variables leads to a reduction of the differential equations describing this motion to the system, for which the number of independent variables is less on the unit. Consequently, there is foliation of gas in the rarefied space with respect to the pressure. In the rarefied space for any given initial distribution of the pressure, all the gas particles are localized on the two–dimensional surface. This surface over time moves in this space. At each point of the surface, the vector of acceleration is collinear to the vector of normal to this surface.

We found some exact solutions of the obtained system that describe the processes taking place inside of the tornado. For this system we found all nontrivial conservation laws of the first order. In addition to the classical conservation laws the system has another conservation law, which generalizes the energy conservation law. With the additional condition we found another one generalized energy conservation law.

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