Picard-Fuchs Equation and Prepotential of Five Dimensional SUSY Gauge Theory Compactified on a Circle

Hiroaki Kanno
Department of Mathematics
Hiroshima University
Higashi-Hiroshima 739, Japan

and

Yūji Ohta
Research Institute for Mathematical Sciences
Kyoto University
Sakyoku, Kyoto 606, Japan

Abstract

Five dimensional supersymmetric gauge theory compactified on a circle defines an effective $N = 2$ supersymmetric theory for massless fields in four dimensions. Based on the relativistic Toda chain Hamiltonian proposed by Nekrasov, we derive the Picard-Fuchs equation on the moduli space of the Coulomb branch of $SU(2)$ gauge theory. Our Picard-Fuchs equation agrees with those from other approaches; the spectral curve of XXZ spin chain and supersymmetric cycle in compactified $M$ theory. By making use of a relation to the Picard-Fuchs equation of $SU(2)$ Seiberg-Witten theory, we obtain the prepotential and the effective coupling constant that incorporate both a perturbative effect of Kaluza-Klein modes and a non-perturbative one of four dimensional instantons. In the weak coupling regime we check that the prepotential exhibits a consistent behavior in large and small radius limits of the circle.

PACS: 11.15.Tk, 12.60.Jv, 02.30.Hq.
Keywords: Picard-Fuchs equation, prepotential, Seiberg-Witten theory, five dimensions, integrable system, $M$ theory.
1 Introduction

The seminal work of Seiberg-Witten [1, 2] has produced a striking progress in our understanding of non-perturbative dynamics of $N = 2$ supersymmetric (SUSY) gauge theory in four dimensions. The holomorphy and the electro-magnetic duality give constraints on the geometry of the moduli space of the Coulomb branch. Complex curve of hyperelliptic type and a specific differential, the Seiberg-Witten form, on it provide natural tools for this moduli geometry called rigid special geometry [3]. The deep meaning of these objects, which appeared auxiliary, has been made clear from the viewpoint of Calabi-Yau compactification of type II strings [4] and also from $M$ theory [5].

Recent developments in dualities of superstrings and $M$ theory have made it more practical and important to consider higher dimensional SUSY gauge theories that are perturbatively non-renormalizable. Among them in this article we will bring into focus the five dimensional (5D) theory compactified on a circle [6]. There are several reasons why this theory is interesting. For example, it is a SUSY gauge theory directly related to the Calabi-Yau compactification of $M$ theory and its type IIA limit. It also gives us various tests on the idea of the low energy effective action of SUSY gauge theories. Moreover, 5D gauge theory is linked to topological field theory and current algebra in four dimensions [6, 7, 8].

There is an intriguing connection of the Seiberg-Witten theory to the theory of integrable systems. The hyperelliptic curve of pure gauge theory can be identified as the spectral curve of the periodic Toda system [4, 11]. Furthermore, the Seiberg-Witten form is given by the canonical one-form ‘$pdq$’ of the symplectic dynamics. Finally the prepotential is related to the tau function of the integrable system. The compactified 5D SUSY gauge theory seems to be accommodated to this idea. Nekrasov proposed a relativistic generalization of the periodic Toda system as an underlying integrable system for 5D theory [6]. More generally the periodic spin chain is to be associated with five and six dimensional theories compactified to four dimensions [11, 12, 13].

It has been argued that the 5D SUSY gauge theory has no instanton corrections [4, 5]. However, it is no longer true, if we compactify the theory to an effective four dimensional one. In addition to the usual four dimensional one-loop corrections we expect one-loop contributions from an infinite tower of the Kaluza-Klein modes. There should be also instanton corrections. Both effects are supposed to be encoded in the complex curve for the periods and the prepotential.

In this paper we will work out the Picard-Fuchs equation for the periods of 5D SUSY $SU(2)$ gauge theory. Though the curve itself looks quite similar to the one for four dimen-
sional theory, the identification of moduli parameter is different. The $S^1$ compactification introduces the radius $R$ of the circle. This is an additional dimensionful parameter to the dynamical scale parameter $\Lambda$ and the combination $\zeta = \Lambda R$ is dimensionless. Due to these changes the analysis of the Picard-Fuchs equation becomes more complicated in five dimensional theory. The appearance of dimensionless parameter makes the solution more involved and the limit $R \to 0$ more subtle.

In section 2, the Picard-Fuchs equation is derived from the spectral curve of the relativistic Toda system. We check that the perturbative part of the effective coupling derived from our Picard-Fuchs equation incorporates the contribution from the Kaluza-Klein modes as expected. We also show that the Picard-Fuchs equations obtained from curves in other approaches; the XXZ spin chain and M theory viewpoint, are the same as ours. The periods are obtained by solving the Picard-Fuchs equation in section 3. We express both vector multiplet $A$ and its dual $A_D$ as power series in the gauge invariant moduli parameter $U$ at $U \to \infty$. In section 4 we establish non-linear differential equations for the moduli parameter $U$ and find the prepotential $F_5$ in the weak coupling. We try to identify the instanton contribution in the non-perturbative part of the prepotential by comparing it with the prepotential $F_4$ in four dimensions. The final section is devoted to miscellaneous topics. We briefly report the prepotential in strong coupling region and give a preliminary result on the $SU(3)$ case. Many technical informations are collected in appendices.

## 2 Curve and Picard-Fuchs equation

### 2.1 Elliptic curve

Nekrasov’s proposal [3] for the description of the moduli space of supersymmetric Yang-Mills gauge theory in five dimensions [14] with one compactified direction was to use an elliptic curve arising from the Hamiltonian of the relativistic Toda chain. For $SU(2)$ the curve is given by

$$y^2 = (x^2 - \Lambda^4) \left[ x - \frac{1}{2} \left( R^2 U^2 - \frac{1}{R^2} \right) \right], \quad (2.1)$$

where $R$ is the radius of a circle and $U$ is the moduli parameter of the five dimensional theory. In the semi-classical limit $U$ has an asymptotic behavior $U \sim R^{-2} \cosh 2RA$ [8], where $A$ is a vector multiplet whose scalar component has expectation value on the Coulomb branch. The prepotential $F_5(A)$ that characterizes the low energy effective
action is locally a function of $A$. This curve is the same as the usual $SU(2)$ Seiberg-Witten elliptic curve \[\text{[1]}\] (see also appendix A), if the moduli $u$ of the four dimensional theory is related with $U$ by
\[
u = \frac{1}{2} \left( R^2 U^2 - \frac{1}{R^2} \right). \tag{2.2}\]
Both the moduli parameters $u$ and $U$ have mass dimension two.

The elliptic curve (2.1) has four branching points
\[
x_1 = -\Lambda^2, \quad x_2 = \Lambda^2, \quad x_3 = -\frac{1}{2R^2} + \frac{R^2 U^2}{2}, \quad x_4 = \infty. \tag{2.3}\]
With the identification \((2.2)\) these branching points correspond to those of the Seiberg-Witten curve. Thus the $\alpha$- and $\beta$-cycles are taken as loops going around counterclockwise as follows
\[
\alpha\text{-cycle} : \quad x_1 \longrightarrow x_2, \quad \beta\text{-cycle} : \quad x_2 \longrightarrow x_3. \tag{2.4}\]
Then the intersection number is $\alpha \cap \beta = +1$.

The periods are defined by the integral over these cycles of the holomorphic one form $dx/y$ on the curve. According to Seiberg and Witten \[\text{[1, 2]}\] we should identify the periods as
\[
\frac{dA}{dU} := \frac{\sqrt{2}}{8\pi} \oint_{\alpha} \frac{dx}{y}, \quad \frac{dA_D}{dU} := \frac{\sqrt{2}}{8\pi} \oint_{\beta} \frac{dx}{y}, \tag{2.5}\]
where $A_D$ is a dual vector multiplet. We have chosen the numerical factor so that these periods may be identified with those of the four dimensional theory after the replacement \((2.2)\). Namely, we observe the following basic relation to the pure $SU(2)$ Seiberg-Witten theory
\[
\frac{dA}{dU} = \frac{da}{du}, \quad \frac{dA_D}{dU} = \frac{da_D}{du}. \tag{2.6}\]
Though $A$ and $A_D$ are obtained in principle from integration over $U$ of \((2.3)\) or contour integral of $\lambda_{SWN}$ to be introduced right now, \((2.6)\) is useful for later discussion.

The five dimensional analogue of the Seiberg-Witten one-form can be obtained from integration of $dx/y$ over $U$. Up to a total derivative term, this is given by
\[
\lambda_{SWN} = \frac{i}{4\pi R} \log[R^2 U + \sqrt{-1 + R^4 U^2 - 2R^2 x}] dx, \tag{2.7}\]
which we call the Seiberg-Witten-Nekrasov one-form. It is also a characteristic one-form in the Hamiltonian dynamics of the relativistic Toda system \[\text{[1]}\]. We can easily check that
\[
\frac{d\lambda_{SWN}}{dU} = \frac{\sqrt{2}}{8\pi} \sqrt{\frac{dx}{(x^2 - \Lambda^4)[x - (R^2 U^2 - 1/R^2)/2]}}. \tag{2.8}\]
We also note that
\[
\frac{d\lambda_{SW}(U)}{dU} = \frac{d\lambda_{SW}(u)}{du},
\]  
if we use (2.2).

## 2.2 The Picard-Fuchs equation

The most useful tool to study the moduli space of the Coulomb branch would be the Picard-Fuchs equation for the periods. In four dimensions, this method was widely used for many cases, e.g., for the $SU(2), SU(3), E_6, G_2$ gauge theories \cite{16, 17, 18, 19} and some algorithms to get a closed form of Picard-Fuchs equation were developed in \cite{20, 21}. This method is also available for five dimensional case. The Picard-Fuchs equation to be derived below is the first example for five dimensional SUSY gauge theory.

We can easily find that the period integrals in (2.5) satisfy the following Picard-Fuchs equation
\[
d^3\Pi dU^3 + \left[ \frac{4R^4U}{\Delta(U)}(1 - R^4U^2) - \frac{1}{U} \right]d^2\Pi dU^2 - \frac{R^8U^2}{\Delta(U)} d\Pi dU = 0, \tag{2.10}
\]
where $\Pi = \oint_\gamma \lambda_{SW N}$ for a one-cycle $\gamma$.

\[
\Delta(U) = -1 + 4\zeta^4 + 2R^4U^2 - R^8U^4 \tag{2.11}
\]
is the discriminant of (2.1) and hereafter we will often use a dimensionless combination $\zeta = \Lambda R$. Note that (2.10) is invariant under the sign reflection $U \rightarrow -U$ \cite{3}. By (2.2) $\Delta(U)$ can be transformed into
\[
\Delta(U(u)) = 4R^4(\Lambda^4 - u^2), \tag{2.12}
\]
which is just the discriminant of the Seiberg-Witten curve. Similarly we can see a relation between (2.10) and the Picard-Fuchs equation in four dimensions. In fact, with the help of (2.6) we find that by the change of variables $(U, A) \rightarrow (u, a)$, (2.10) is reduced to
\[
\frac{d}{du} \left[ 4(\Lambda^4 - u^2)\frac{d^2\Pi}{du^2} - \Pi \right] = 0. \tag{2.13}
\]
Then we can integrate once to obtain the $SU(2)$ Picard-Fuchs equation in four dimensions. We note that in terms of the moduli $U$ of five dimensional theory (2.10) is never total derivative and it cannot be reduced to the second order equation for $\Pi$. 
We can see that the Picard-Fuchs equation (2.10) incorporates the contribution of the (perturbative) Kaluza-Klein modes of the $S^1$ compactification by examining the perturbative effective coupling $\tau_5^{\text{pert}}$. The effective coupling is given by the second derivative of the prepotential

$$\tau_5 = \frac{\partial^2 F_5(A)}{\partial A^2} = \frac{dA_D}{dA} = \frac{dA_D/dU}{dA/dU}.$$  

Since this is the ratio of two independent solutions of (2.10), which is second order in $d\Pi/dU$, it should satisfy the Schwarzian differential equation (see appendix C),

$$\{\tau_5, U\} = -\frac{1}{2} P^2 - \frac{dP}{dU} + 2Q,$$  

where $\{\ast, \ast\}$ is called Schwarzian derivative and

$$P = \frac{4R^4U}{\Delta(U)}(1 - R^4U^2) - \frac{1}{U}, \quad Q = -\frac{R^8U^2}{\Delta(U)},$$  

are the coefficients of (2.10). To estimate the perturbative part, which is independent of the dynamical scale $\Lambda$, we will formally take the limit $\Lambda \to 0$, which makes the discriminant degenerate,

$$\Delta(U) = -(1 - R^4U^2)^2.$$  

Then the differential equation for the perturbative effective coupling $\tau_5^{\text{pert}}$ is

$$\{\tau_5^{\text{pert}}, U\} = \frac{2R^8U^2}{(R^4U^2 - 1)^2} - \frac{3}{2} \frac{1}{U^2}.$$  

It is easy to see that

$$\tau_5^{\text{pert}} = \log(R^4U^2 - 1),$$  

is a solution. The asymptotic behavior $UR^2 \sim \cosh 2RA$ \cite{footnote}, which is valid in semi-classical region, implies

$$\tau_5^{\text{pert}}(A) \sim \log(\sinh^2 2RA).$$  

By the infinite product expansion of sinh we recognize the contribution of an infinite tower of the Kaluza-Klein excitations;

$$\tau_5^{KK} = \sum_{n=1}^{\infty} \log \left[ A^2 + \left( \frac{n\pi}{2R} \right)^2 \right],$$  

to the perturbative effective coupling $\tau_5^{\text{pert}}$. We have discarded an infinite term in this manipulation. Such a divergence could appear, since we formally turned off the scale parameter $\Lambda$ that plays a role of a cut off in perturbative calculation.
2.3 Curves from the integrable system and $M$ theory

We have derived the Picard-Fuchs equation (2.10) from the elliptic curve (2.1) and its periods, but the same differential equation can be obtained from the spectral curve and associated one-form of integrable system; XXZ spin chain which includes a relativistic Toda chain as a particular limit [12, 13, 22, 23, 24]. Moreover, the curves of the same type have been shown to be obtained from the SUSY cycles in compactified $M$ theory [25]. For integrable (periodic) spin chain with $N$-sites the spectral curve is defined by

$$\det (T_N(λ) - z) = 0,$$

(2.22)

where $T_N(λ)$ is the $2 \times 2$ transfer (monodromy) matrix. Thus, we obtain a complex curve

$$z^2 - 2P_N(λ)z + Q_{2N}(λ) = 0,$$

(2.23)

with $2P_N(λ) = \text{tr} T_N(λ)$ and $Q_{2N} = \det T_N(λ)$. According to [12, 13], pure gauge theory should be regarded as a degenerating limit of the model with massive matter described by the XXZ spin chain. In our notation and normalization the spectral curve for $SU(N_c)$ pure gauge theory is

$$z^2 + 2zx^{N_c/2} \prod_{i=1}^{N_c} \frac{1}{R'} \sinh R'(v - A_i) + \Lambda^{2N_c} = 0,$$

(2.24)

where $R' = R/\sqrt{2}$ and $x = e^{2R'v}$. Defining

$$z + \frac{\Lambda^{2N_c}}{z} = -\frac{2}{(2R')^{N_c}} P_{N_c}, \quad z - \frac{\Lambda^{2N_c}}{z} = -\frac{2}{(2R')^{N_c}} y,$$

(2.25)

with

$$P_{N_c} = (x^{N_c} + \cdots + (-1)^{N_c}),$$

(2.26)

we obtain a hyperelliptic curve;

$$y^2 = P_{N_c}^2 - (\sqrt{2}ζ)^{2N_c}.$$

(2.27)

In the case of $N_c = 2$, this is elliptic;

$$y^2 = (x^2 + sx + 1)^2 - 4ζ^4,$$

(2.28)

where $s = -2 \cosh 2R'A$. For the $SU(N_c)$ theory coupled with massless $N_f$ matter hypermultiplets, the curve takes the form

$$z^2 + 2zx^{N_c/2} \prod_{i=1}^{N_c} \frac{1}{R'} \sinh R'(v - A_i) + \Lambda^{2N_c-N_f}x^{N_f/2}\sinh^{N_f} R'v = 0.$$

(2.29)
As we have remarked these curves also arise as the SUSY cycles in \( M \) theory compactification.

For a period integral \( d\tilde{\Pi}/ds = \oint dx/y \) along a 1-cycle \( \gamma \), the Picard-Fuchs equation for (2.28) is easily deduced to be

\[
\frac{d^3\tilde{\Pi}}{ds^3} + \left[ \frac{1}{\Delta(s)}(-16s + 4s^3) - \frac{1}{s} \right] \frac{d^2\tilde{\Pi}}{ds^2} + \frac{s^2}{\Delta(s)} \frac{d\tilde{\Pi}}{ds} = 0,
\]

(2.30)

where

\[
\tilde{\Delta}(s) = 16 - 64\zeta^4 - 8s^2 + s^4.
\]

(2.31)

(2.30) is identical to (2.10) under the identification

\[
s = -2R^2U.
\]

(2.32)

Therefore, both of the complex curves based on the spin chain model and on \( M \) theory give the same Picard-Fuchs equation as the one we have derived in sect 2.2.

We note that (2.28) can be converted to the hyperelliptic curve in four dimensions

\[
\tilde{y}^2 = (\tilde{x}^2 - u)^2 - \Lambda^4
\]

(2.33)

by the transformation

\[
\sqrt{2}R\tilde{x} = x + \frac{s}{2}, \quad y = 2R^2\tilde{y}, \quad 2R^2u = \frac{s^2}{4} - 1,
\]

(2.34)

where the third equation with (2.32) implies (2.2). However, this isomorphism breaks down when hypermultiplets are introduced. For example, let us consider the \( SU(2) \) theory with a massless hypermultiplet (the flavor index \( N_f = 1 \)). Then the hyperelliptic curve derived in the \( M \) theory approach can be written in the form [23]

\[
y^2 = (x^2 + sx + 1)^2 - (\Lambda R)^3(x - 1),
\]

(2.35)

where we are not careful with the numerical factor for \((\Lambda R)^3\) because it can be absorbed by rescaling of variables. However, this curve cannot be reduced to the corresponding massless \( SU(2) \) curve in four dimensions even if we apply a transformation like (2.34). In this sense, the isomorphism via the simple transformation of variables between the curves of the five and four dimensional theories is valid only for pure gauge theories. For the models including hypermultiplets, we have to compare physics from spectral curves of integrable systems and hyperelliptic curves of \( M \) theory further. This would tell us which curve we should use.
Finally, we comment a little more on (2.33). If (2.33) is rewritten as

\[ y^2 = (\tilde{x}^2 - u)^2 - \zeta^3\left(\tilde{x} - \frac{s}{2} - 1\right) \]  

(2.36)

by the transformation

\[ \tilde{x} = x + \frac{s}{2}, \quad u = \frac{s^2}{4} - 1, \]  

(2.37)

(2.36) is reminiscent of the curve of massive \( SU(2) \) \( N_f = 1 \) theory in four dimensions \[34\] with a “mass” of \(-(1 + s/2)\) and the “QCD scale parameter” \( \zeta \). Of course, since the mass of the hypermultiplet and the moduli are different object, we cannot actually interpret (2.36) as the massive curve in four dimensions. However, the reader might recall that there was a similar phenomenon in the \( G_2 \) gauge theory in four dimensions \[26\]. There, the \( G_2 \) curve looked like the \( SU(3) \) curve coupled with two massive hypermultiplets with equal mass. We do not know what it means, but it may imply that the moduli spaces of such different theories can be connected in a sense.

## 3 Periods

Since the Picard-Fuchs equation (2.10) is essentially a second order differential equation, if we solve it for \( d\Pi/dU \), (2.10) can be easily solved by the Frobenius' method. In order to compare the results of the five dimensional theory with those of the corresponding four dimensional theory, we want to solve it in the weak coupling limit. As is easy to see, from the relation (2.2), the weak coupling regime of \( u \), i.e., \( u = \infty \), corresponds to \( U = \infty \). Therefore, we have the following two solutions around \( U = \infty \) (\( z = 1/U \)):

\[ \rho_1(z) = z \sum_{i=0}^{\infty} a_i z^i, \]

\[ \rho_2(z) = \rho_1(z) \log z + z \sum_{i=1}^{\infty} b_i z^i, \]  

(3.1)

where the first several expansion coefficients are

\[ a_0 = 1, \]

\[ a_2 = \frac{1}{2R^2}, \]

\[ a_4 = \frac{3(1 + 2\zeta^4)}{8R^8}, \]

\[ a_6 = \frac{5(1 + 6\zeta^4)}{16R^{12}}, \]
\[ a_8 = \frac{35(1 + 12\zeta^4 + 6\zeta^8)}{128R^{16}}, \]
\[ a_{10} = \frac{63(1 + 20\zeta^4 + 30\zeta^8)}{256R^{20}}, \]
\[ a_{12} = \frac{231(1 + 30\zeta^4 + 90\zeta^8 + 20\zeta^{12})}{1024R^{24}}, \]
\[ a_{14} = \frac{429(1 + 42\zeta^4 + 210\zeta^8 + 140\zeta^{12})}{2048R^{28}} \] (3.2)

and

\[ b_2 = \frac{1}{2R^4}, \]
\[ b_4 = \frac{4 + 5\zeta^4}{8R^8}, \]
\[ b_6 = \frac{23 + 93\zeta^4}{48R^{12}}, \]
\[ b_8 = \frac{352 + 2964\zeta^4 + 1167\zeta^8}{768R^{16}}, \]
\[ b_{10} = \frac{1126 + 16220\zeta^4 + 19605\zeta^8}{2560R^{20}}, \]
\[ b_{12} = \frac{13016 + 286530\zeta^4 + 703665\zeta^8 + 133270\zeta^{12}}{30720R^{24}}, \]
\[ b_{14} = \frac{176138 + 5505906\zeta^4 + 22799805\zeta^8 + 13097770\zeta^{12}}{430080R^{28}}. \] (3.3)

We note that \( a_{2n+1} = b_{2n+1} = 0 \) due to the invariance under \( U \to -U \).

The period integrals (2.5) are given by some linear combinations of the fundamental solutions \( \rho_1 \) and \( \rho_2 \). Lower order expansion fixes the combination to be

\[ \frac{dA}{dU} = \frac{1}{2R}\rho_1, \quad \frac{dA_D}{dU} = i\frac{2}{\pi R} \log \left( \frac{2R}{\Lambda} \right) \rho_1 - \frac{i}{\pi R} \rho_2. \] (3.4)

We must further integrate by \( U \) in order to obtain \( A \) and \( A_D \). First, let us consider \( A \). Integration over \( U \) gives

\[ A = \text{integration const.} + \int \frac{\rho_1}{2R} dU \]
\[ = \frac{1}{2R} \log 2R^2 + \frac{1}{2R} \log U - \frac{1}{8R^2 U^2} - \frac{1}{U^4} \left( \frac{3}{64R^9} + \frac{3\Lambda^4}{32R^5} \right) - \frac{1}{U^6} \left( \frac{5}{192R^{13}} + \frac{5\Lambda^4}{32R^9} \right) \]
\[ - \frac{1}{U^8} \left( \frac{35}{2048R^{17}} + \frac{105\Lambda^4}{512R^{13}} + \frac{105\Lambda^8}{1024R^9} \right) - \cdots, \] (3.5)

where we have identified the integration constant with \((\log 2R^2)/(2R)\), because the direct calculation of \( A \) reads

\[ A = \oint \lambda_{SWN} \]

10
\[
\frac{i}{2\pi R} \int_{-\Lambda^2}^{\Lambda^2} \log\left[ R^2 U + \sqrt{1 + R^4 U^2 - 2 R^2 x} \right] dx
= \frac{1}{2R} \log 2R^2 + \frac{1}{2R} \log U - \frac{1}{8R^6 U^2} - \cdots. \tag{3.6}
\]

It is easy to find that \( \Lambda \)-independent terms in (3.3) are originated from the expansion of \((1/2R) \log(R^2 U + \sqrt{R^4 U^2 - 1})\) for large \( U \). Therefore, it follows that

\[
A = \frac{1}{2R} \log(R^2 U + \sqrt{R^4 U^2 - 1}) - \frac{3\Lambda^4}{32 R^6 U^4} - \frac{5\Lambda^4}{32 R^8 U^6} - \frac{1}{U^8} \left( \frac{105\Lambda^4}{512 R^{13}} + \frac{105\Lambda^8}{1024 R^9} \right) - \cdots. \tag{3.7}
\]

We expect that the remaining \( \Lambda \)-dependent terms may be deduced from expansion of some simple functions, but we could not find such expressions.

Similarly, we can find the asymptotic behavior of \( dA_D/dU \)

\[
\frac{dA_D}{dU} \sim i \frac{R}{\pi \sqrt{R^4 U^2 - 1}} \log \frac{4(R^4 U^2 - 1)}{\zeta^2}. \tag{3.8}
\]

Integration over \( U \) gives the leading part of \( A_D \) and it is represented by using di-logarithm (see appendix D), but the explicit form is not so simple and elegant, so we do not give it here. We do not need to include a integration constant in the expansion of \( A_D \), because it has no effect on prepotential or monodromy. Note that the integration constant term corresponds to a linear term in the prepotential.

Since the periods are determined, their monodromy can be now obtained by leading terms

\[
A = \frac{1}{2R} \log 2R^2 + \frac{1}{2R} \log U - \cdots,
A_D = i \frac{4}{\pi} A \log \frac{2RU}{A} - i \frac{2}{\pi R} \log(2R^2) \log U - i \frac{1}{\pi R} \log^2 U + \cdots. \tag{3.9}
\]

Then the monodromy \( U \to e^{2\pi i} \cdot U \) around \( U = \infty \) transforms these periods into

\[
A \to A + \frac{i}{R},
A_D \to A_D - 8A - i \frac{4\pi}{R} + \frac{4}{R} \log \zeta. \tag{3.10}
\]

Thus the monodromy matrix is given by

\[
\begin{pmatrix}
A_D \\
A \\
1/R
\end{pmatrix}
\to
\begin{pmatrix}
1 - 8m \\
0 1 \ i\pi \\
0 0 1
\end{pmatrix}
\begin{pmatrix}
A_D \\
A \\
1/R
\end{pmatrix}, \tag{3.11}
\]
where \( m = -4\pi i + 4\log \zeta \).

For completeness, let us consider the strong coupling region. The strong coupling region in four dimensions is located at \( u = \pm \Lambda^2 \) where a magnetically charged particle and a dyon become massless, respectively, but in this five dimensional theory the corresponding singularities in the moduli space are splitted into

\[
U_\pm^+ := \pm \frac{1}{R^2} \sqrt{1 + 2\zeta^2}, \quad U_\pm^- := \pm \frac{1}{R^2} \sqrt{1 - 2\zeta^2},
\]

respectively (see also figure 1).

Let us consider a singularity corresponding to \( u = +\Lambda^2 \). In this case, we may take either \( U_+^+ \) or \( U_-^- \), but we take the former. By a similar discussion to the weak coupling case, the periods in the strong coupling regime can be easily found. However, since the solutions to the Picard-Fuchs equation are represented by a series with lengthy expansion coefficients, we consider equivalent expressions which can be deduced from the periods in four dimensional theory in the strong coupling regime (recall that the periods are related in the sense of (2.6)). The result is

\[
\tilde{z} = U - \sqrt{1 + 2\zeta^2}/R^2:
\]

\[
\frac{dA_D}{dU} = i \frac{\Lambda}{2} \sum_{n=0}^{\infty} \tilde{a}_n (n + 1) \left[ \frac{(\zeta/\Lambda)^2 \tilde{z}^2 + 2\tilde{z} \sqrt{1 + 2\zeta^2}}{4\Lambda^2} \right]^n,
\]

\[
\frac{dA}{dU} = i \frac{dA_D}{2\pi} \left[ \log \frac{(\zeta/\Lambda)^2 \tilde{z}^2 + 2\tilde{z} \sqrt{1 + 2\zeta^2}}{4\Lambda^2} - 1 - \log 2 \right]
- \frac{\Lambda}{4\pi} \sum_{n=0}^{\infty} \left[ \tilde{a}_n + \tilde{b}_n (n + 1) \right] \left[ \frac{(\zeta/\Lambda)^2 \tilde{z}^2 + 2\tilde{z} \sqrt{1 + 2\zeta^2}}{4\Lambda^2} \right]^n,
\]

where

\[
\tilde{a}_n = (-1)^n \frac{(1/2)_n^2}{n!(2)_n},
\]

\[
\tilde{b}_n = \tilde{a}_n \left[ 2 \left[ \psi \left( n + \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) \right] + \psi(1) - \psi(n + 1) + \psi(2) - \psi(n + 2) \right].
\]

Furthermore, \((*)_n\) is the Pochhammer’s symbol and \(\psi(*)\) is the polygamma function. Precisely speaking, we should expand these expressions around \( \tilde{z} = 0 \).

Integration over \( U \) provides the periods in the strong coupling regime and then the monodromy at \( U = U_+^+ \) is calculated from the asymptotic property

\[
A_D = \frac{i}{2\Lambda} (U - U_+^+) + \cdots,
\]

\[
A = \frac{i}{2\pi} A_D \log (U - U_+^+) + \cdots.
\]
Thus the monodromy matrix is given by

\[
\begin{pmatrix}
A_D \\
A \\
1/R
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
A_D \\
A \\
1/R
\end{pmatrix}.
\]

(3.16)

We note that in contrast with the weak coupling case, $1/R$ does not mix with the monodromy of $A$ and $A_D$ in the strong coupling.

4 Prepotential in weak coupling limit

4.1 The prepotential

By the basic relation of special geometry;

\[ A_D = \frac{\partial F_5}{\partial A}, \quad \tau_5 = \frac{\partial^2 F_5}{\partial A^2}, \]

(4.1)

the prepotential $F_5$ in five dimensions can be obtained by integrating either $A_D$ or $\tau_5$ over $A$ after expressing them as a function of $A$. Hence, our first task is to find an inversion of the relation $A = A(U)$ obtained in the last section. This inversion requires some technical care and we take the following manner. Recall that in the four dimensional theory similar inversion of period was proceeded by solving a differential equation for the moduli [27, 28]. Also for the case at hand, we can find the required relation $U = U(A)$ by solving a non-linear differential equation for $U$ as follows.

Let

\[ u = u(a), \quad U = U(A), \]

(4.2)

then by the relation of period integrals (2.6) we have

\[ \dot{u} = U', \]

(4.3)

where $\dot{=} = d/da$ and $\dot{} = d/dA$. The second and third order derivatives are then given by

\[ U'' = R^2 \ddot{u}, \quad U''' = R^2 (\dddot{u} + R^2 U^2 \ddot{u}). \]

(4.4)

We also have the following relation;

\[ \frac{dA}{dU} = (U')^{-1}, \quad \frac{d^2 A}{dU^2} = -(U')^{-3} U'', \quad \frac{d^3 A}{dU^3} = 3(U')^{-5} U'^2 - (U')^{-4} U'''', \]

(4.5)
Substituting \((4.2), (4.3)\) and \((4.5)\) into the Picard-Fuchs equation for \(A\), we get

\[
U'''U' - 3U''^2 + \left[ \frac{4R^4U}{\Delta(U)} (1 - R^4U^2) - \frac{1}{U} \right] U'^2 U'' + \frac{R^8U^2}{\Delta(U)} U'^4 = 0, \tag{4.6}
\]

which can be rewritten by using \((4.4)\) as

\[
\frac{d}{da} \left[ \tilde{\Delta}(u) \tilde{u} + au^3 \right] = 0, \tag{4.7}
\]

provided

\[
\Delta(U) = R^4\tilde{\Delta}(u) = 4R^4(\Lambda^4 - u^2). \tag{4.8}
\]

\((4.7)\) is nothing but the (differentiation of) Matone’s differential equation for \(u\) found in four dimensions \([27, 28]\).

We can now solve \((4.6)\), but we must take into account of an “initial condition”. Note that from \((3.7)\) \(A\) is asymptotically

\[
A \sim \frac{1}{2R} \log(R^2U + \sqrt{R^4U^2} - 1), \tag{4.9}
\]

or

\[
U \sim \frac{1}{R^2} \cosh 2RA. \tag{4.10}
\]

This means that we should solve \((4.6)\) by assuming

\[
U = \frac{1}{2R^2} \left( q + \frac{1}{q} \right) + \frac{1}{R^2} \sum_{n=1}^{\infty} c_n(q)\zeta^{4n}, \quad (\zeta = \Lambda R), \tag{4.11}
\]

where \(q = e^{2RA}\). In fact, substituting \((4.11)\) into \((4.6)\), we can obtain differential equations for \(c_n\) by equating the coefficients of powers in \(\zeta\) to zero. For higher \(c_n\), such differential equations are very complicated in general, but we can solve it for \(c_1\). The result is

\[
U = \frac{1}{R^2} \cosh 2RA + \frac{\Lambda^4R^2}{2} \frac{3q^2 - 1}{q(1 - q^2)^2} + O(\Lambda^8). \tag{4.12}
\]

The prepotential can be calculated by using \(A_D\) (or \(\tau_5\)) and \((4.12)\). For example, for large \(R\), the prepotential is given by

\[
\mathcal{F}_5 = i\frac{4R}{3\pi} A^3 - i\frac{2\log \zeta}{\pi} A^2 - i\frac{1}{4\pi R^2} Li_3(q^{-2})
- \frac{\Lambda^4R^2}{\pi} \left( \frac{7}{16q^4} + \frac{13}{18q^6} + \frac{63}{64q^8} + \frac{31}{25q^{10}} + \frac{215}{144q^{12}} + \cdots \right) - O(\Lambda^8), \tag{4.13}
\]

where \(Li_3\) is a special function called tri-logarithm (see \([29, 30]\) and also appendix D). In this derivation, we have suppressed linear terms in \(A\). Note that the first and the second
terms in (4.13) are independent of \( q \). Furthermore, though in the large radius limit \( Li_3(q^{-2}) \) vanishes exponentially, we can see the small radius behavior of it by considering analytic property of tri-logarithm (see below). Therefore, the \( SU(2) \) prepotential in the five dimensional theory can be generally written as

\[
F_5 = i \frac{4R}{3\pi} A^3 - i \frac{2 \log \zeta}{\pi} A^2 - i \frac{1}{4\pi R^2} Li_3(q^{-2}) + \sum_{n=1}^{\infty} A^{4n} F_{5,n}(q, R),
\]

where \( F_{5,n} \) is a function of \( q \) and \( R \), in particular, we have

\[
F_{5,1}|_{R \to \infty} = -i \frac{R^2}{\pi} \left( \frac{7}{16q^4} + \frac{13}{18q^6} + \frac{63}{64q^8} + \frac{31}{25q^{10}} + \frac{215}{144q^{12}} + \cdots \right),
\]

although we could not find the exact function \( F_{5,1} \) itself (but see appendix E).

### 4.2 Large and small radius limit

Firstly, we take the large radius limit of (4.13). At \( R \to \infty \), divided by the length of \( S^1 \), i.e., \( 2\pi R \), \( \log \zeta / R \) and \( Li_3(q^{-2}) \) vanish because of \( q^{-1} \sim 0 \), thus only the cubic term of (4.13) survives in the decompactification limit;

\[
F_5 \bigg|_{R \to \infty} = i \frac{2}{3\pi^2} A^3.
\]

More generally the cubic term in the prepotential of (uncompactified) five dimensional SUSY gauge theory is identified as the intersection forms of Calabi-Yau manifolds in the Calabi-Yau compactification of \( M \) theory \[14, 15\].

The small radius limit is obtained as follows; Let us denote the leading part of \( F_5 \) by

\[
F_{5,\text{led}} = i \frac{4R}{3\pi} A^3 - i \frac{2 \log \zeta}{\pi} A^2 - i \frac{1}{4\pi R^2} Li_3(q^{-2}).
\]

Then from the asymptotic property of \( Li_3 \) we have

\[
F_{5,\text{led}} \big|_{R \to 0} = i \frac{A^2}{\pi} \left[ \log \left( \frac{A}{L} \right)^2 + 4 \log 2 - 3 \right]
\]

for \( R \to 0 \), where we have suppressed \( O(A) \) terms because they are irrelevant to prepotential. (4.18) coincide with the leading part of \( F_4 \) if \( A \) is identified with \( a \) (see below).

In the above small radius limit, we keep only the leading terms that give rise to the one-loop prepotential in four dimensions. The non-perturbative part of the prepotential in small radius limit is not clear in the naive limit, since we have \( R \)-dependence in \( q = e^{2RA} \). We will discuss the instanton contributions by a slightly different approach in the next part.
4.3 Instanton contributions to the prepotential

We want to see how instantons contribute to the prepotential in the five dimensional theory. However, we do not know an exact expression of \( F_{5,n} \) in (4.14). To proceed we will take the following way. Recall that we have normalized the relation between four and five dimensional theories such that their effective coupling constants are equivalent by (2.6). Therefore, even if \( \tau_5 \) is written by \( A \), it is equivalent to \( \tau_4 \), so the four and five dimensional theories are equivalent in this sense because \( \tau_5 \) is defined by

\[
\tau_5 = \frac{dA_D}{dU} \left/ \frac{dA}{dU} \right. = \frac{da_D}{du} \left/ \frac{da}{du} \right. = \tau_4,
\]

(4.19)

where \( \tau_4 \) is the effective coupling constant in four dimensions. Moreover, \( \tau_4 \) as a function of \( a \) is obtained from differentiation of \( F_4 \)

\[
\tau_4 = \frac{d^2 F_4}{da^2} = \frac{i}{8} \log 2 + \frac{i}{\pi} \log \left( \frac{a}{\Lambda} \right)^2 + \frac{i}{\pi} \sum_{k=1}^{\infty} (2k - 1)(4k - 1) F_k \left( \frac{\Lambda}{a} \right)^{4k}.
\]

(4.20)

Therefore, if \( a \) can be expressed by \( A \), then twice integration over \( A \) will generate a prepotential of the five dimensional theory. Then this prepotential will include the instanton expansion coefficients \( F_k \) in four dimensions.

Based on this observation, we can calculate the prepotential as follows; Note that the transformation of variables between \( a \) and \( A \) is found by

\[
\frac{da}{dA} = \frac{du}{dU} = R^2 U
\]

(4.21)

with (2.6) and (2.2). If we integrate (4.21) after substitution of (1.12), \( a \) is represented by \( A \). Thus, we find

\[
a = \frac{1}{2R} \sinh 2RA - \frac{\Lambda^4 R^3}{4q} \frac{1}{q^2 - 1} + O(\Lambda^8),
\]

(4.22)

where the integration constant has been set to zero by the following reason. Assume that the leading part of (4.22) is given by

\[
a \sim c \quad (= \text{integration const.}) + \frac{1}{4R} \left( q - \frac{1}{q} \right)
\]

\[
= c + \frac{1}{2R} \sinh 2RA.
\]

(4.23)
Then, (4.23) means
\[ R^4 U^2 = \cosh^2 2RA = 1 + 4R^2(a - c)^2. \] (4.24)

But the leading part of \( u \) as a function of \( a \) is known to be \( 2a^2 \) and from (2.2) we conclude that \( c = 0 \).

By expanding \( \tau_4 \) by \( q \) after substitution of (4.22) into (4.20), we obtain \( \tau_4 \) as a function of \( A \). In view of (4.19) twice integration of \( \tau_4(A) \) over \( A \) gives the prepotential
\[
\hat{F}_5 = \frac{4R}{3\pi} A^3 - i \frac{2 \log \zeta}{\pi} A^2 - i \frac{1}{4\pi R^2} L_{i3}(q^{-2})
\]
\[ + \frac{\Lambda^4 R^2}{\pi} \left[ \frac{1}{q^4} \left( -\frac{1}{16} + 24F_1 \right) + \frac{1}{q^6} \left( -\frac{1}{18} + \frac{128}{3} F_1 \right) + \frac{1}{q^8} \left( -\frac{3}{64} + 60F_1 \right) \right.
\]
\[ + \frac{1}{q^{10}} \left( -\frac{1}{25} + \frac{384}{5} F_1 \right) + \frac{1}{q^{12}} \left( -\frac{5}{144} + \frac{280}{3} F_1 \right) + \cdots \right] - O\left( \Lambda^8 \right). \] (4.25)

Using explicit numerical data (A.17) of instanton expansion coefficients, we find that
\[
\hat{F}_5 = \frac{4R}{3\pi} A^3 - i \frac{2 \log \zeta}{\pi} A^2 - i \frac{1}{4\pi R^2} L_{i3}(q^{-2})
\]
\[ - i \frac{\Lambda^4 R^2}{\pi} \left( \frac{7}{16q^4} + \frac{13}{18q^6} + \frac{63}{64q^8} + \frac{31}{25q^{10}} + \frac{215}{144q^{12}} + \cdots \right) - O\left( \Lambda^8 \right). \] (4.26)

This prepotential \( \hat{F}_5 \) coincides with (4.13)! Since we have calculated the prepotential by two essentially independent ways and checked they are in agreement, we can conclude that the prepotential is correctly determined.

### 4.4 Relation between \( F_5 \) and \( F_4 \)

We have shown how the instanton corrections arise in the prepotential in five dimensions by comparing with \( F_4 \). In this subsection, we give an argument to establish that \( F_5 \) reduces to \( F_4 \) (in the weak coupling regime), provided the radius of \( S^1 \) vanishes.

Let us recall that \( \tau_4(a) = \tau_5(A) \). Since \( F_5 \) is a twice integrated quantity over \( A \), it follows that
\[
F_5 = \int \int (\tau_5 dA) dA
= \int \int \left( \tau_4 \frac{dA}{da} da \right) \frac{dA}{da} da. \] (4.27)
If \( dA/da \) can be written by \( a \), \( F_5 \) will be a function of \( a \). Furthermore, if we take the limit \( R \to 0 \), we will obtain some relation between \( F_5 \) and \( F_4 \). For this purpose, however, the inversion of \( a = a(A) \) is required. From (4.22), we observe that

\[
\frac{1}{2R} \sinh 2RA = a + Y,
\]

where

\[
Y = \frac{\Lambda^4 R^3}{4q} \frac{1}{q^2 - 1} + O(\Lambda^8).
\]

Thus we find

\[
A = \frac{1}{2R} \sinh^{-1}(2Ra + 2RY).
\]

Solving (4.30) recursively in \( A \) and expanding around \( R = 0 \), we can obtain

\[
A = a + \left( -\frac{2}{3}a^3 + \frac{\Lambda^4}{16a} \right) R^2 - \frac{\Lambda^4}{4} R^3 + \cdots.
\]

Therefore,

\[
\frac{dA}{da} = 1 + Z,
\]

where we have denoted miscellaneous terms depending on \( R \) by \( Z \), which is manifestly \( Z \to 0 \) as \( R \to 0 \). It is obvious that \( A \sim a \) as \( R \to 0 \).

Consequently, the prepotential will have the form

\[
F_5 = \int \int [\tau_4(1 + Z)da] (1 + Z)da
= F_4 + \int \int (\tau_4 da)Zda + \int \int (\tau_4 Z da)da + \int \int (\tau_4 Z da)Z da.
\]

This expression shows that in the weak coupling regime, \( F_5 \) reduces to \( F_4 \) if the radius of \( S^1 \) vanishes;

\[
F_5|_{R \to 0} = F_4.
\]

5 Miscellany

In the last sections the expansion in the weak coupling has been worked out. The calculation in the strong coupling regime is slightly easy in contrast with the weak coupling case. Repeating a similar construction as in the preceding sections by interchanging a role of \( A \) and \( A_D \), we obtain the dual prepotential by using the inversion of \( A_D \) in (3.13):

\[
\bar{z} = -2i\Lambda A_D - \frac{1}{4} \sqrt{1 + 2\zeta^2 A_D^2} + i \frac{(-3 + 10\zeta^2)}{96\Lambda} A_D^3 - \frac{\sqrt{1 + 2\zeta^2(-15 + 26\zeta^2)}}{1536\Lambda^2} A_D^4
\]
\[-i\frac{-495 + 420\zeta^2 + 1028\zeta^4}{122880\Lambda^3} A_D^5 \]
\[+ \frac{\sqrt{1 + 2\zeta^2(-2835 + 5460\zeta^2 + 116\zeta^4)}}{1474560\Lambda^4} A_D^4 + \ldots.\] (5.1)

Also in this case we can check (5.1) from (4.6) by exchanging $A$ and $A_D$. Then the dual prepotential is found to be

\[\mathcal{F}_{D_5} = A_D^2 \left[ \frac{i}{8\pi} \log \left( 1 + 2\zeta^2 \right) \right] \left( \frac{A_D}{\Lambda} \right)^2 + \frac{i\Lambda^2}{\pi} (1 + 2\zeta^2) \sum_{k=3}^{\infty} \mathcal{F}_{D_5,k} \tilde{A}_D^k,\] (5.2)

where $\mathcal{F}_{D_5,2} = 1/8 - i3/(8\pi) - i \log 2/(4\pi)$,

\[\tilde{A}_D = i\frac{A_D}{\Lambda\sqrt{1 + 2\zeta^2}}\] (5.3)

and the first several dual “instanton” expansion coefficients are given by

\[
\begin{align*}
\mathcal{F}_{D_5,3} &= \frac{-3 + 2\zeta^2}{96}, \\
\mathcal{F}_{D_5,4} &= \frac{-45 - 68\zeta^2 + 236\zeta^4}{9216}, \\
\mathcal{F}_{D_5,5} &= \frac{-165 - 390\zeta^2 + 292\zeta^4 + 1848\zeta^6}{122880}, \\
\mathcal{F}_{D_5,6} &= \frac{-42525 - 138600\zeta^2 + 43592\zeta^4 + 415328\zeta^6 + 596528\zeta^8}{88473600}, \\
\mathcal{F}_{D_5,7} &= \frac{-33201 - 137970\zeta^2 - 25704\zeta^4 + 502000\zeta^6 + 475568\zeta^8 + 275808\zeta^{10}}{165150720}.
\end{align*}\] (5.4)

One can see that this dual prepotential coincides with the dual prepotential in four dimensions [16, 31] for $R \rightarrow 0$ if $A_D$ is identified with $a_D$.

In this paper we have studied the five dimensional $SU(2)$ theory, but we can consider higher rank gauge theories as well. For example, the $SU(3)$ curve in five dimensions takes the form

\[y^2 = (x^3 + s_1 x^2 + s_2 x - 1)^2 - (\Lambda R)^6,\] (5.5)

where $s_1$ and $s_2$ are moduli [23]. Under the identification

\[s_2 - \frac{s_1^2}{3} = -R^2 u, \quad 1 - \frac{2s_1^3}{27} + \frac{s_1 s_2}{3} = -R^3 v,\] (5.6)

(5.5) is shown to be equivalent to the curve of four dimensional theory [16, 32, 33, 34, 35, 36] given by

\[y^2 = (x^3 - u x - v)^2 - \Lambda^6,\] (5.7)
where \( u \) and \( v \) are moduli in four dimensions. Then the discriminant of (5.5) is given by

\[
\Delta_{SU(3)} = \left[ 27(1 - \zeta^3)^2 + (1 - \zeta^3)(-4s_1^3 + 18s_1s_2) - s_1^2s_2^2 + 4s_2^3 \right] \\
\times \left[ 27(1 + \zeta^3)^2 + (1 + \zeta^3)(-4s_1^3 + 18s_1s_2) - s_1^2s_2^2 + 4s_2^3 \right],
\]

which can be transformed into

\[
\Delta_{SU(3)} = R^{12} \left[ 4u^3 - 27(\Lambda^3 - v)^2 \right] \left[ 4u^3 - 27(\Lambda^3 + v)^2 \right].
\]

This is the discriminant of (5.7) (up to overall constant). We do not write down the \( SU(3) \) Picard-Fuchs equations in five dimensions because it is too lengthy and complicated, but it would be interesting to further discuss and compare with the results of four dimensions.

One of the authors (H.K.) would like to thank N. Nekrasov for sending a copy of his thesis. He also thanks Katsushi Ito and S.-K. Yang for continual discussions on the Seiberg-Witten theory.

**Appendix A. The \( SU(2) \) theory in four dimensions**

In this appendix, we briefly summarize the \( N = 2 \) \( SU(2) \) supersymmetric Yang-Mills theory in four dimensions. For details, see [1, 2, 16].

The Seiberg-Witten elliptic curve and the meromorphic 1-form are given by

\[
y^2 = (x^2 - \Lambda^4)(x - u)
\]

and

\[
\lambda_{SW} = \frac{\sqrt{2}}{4\pi} \sqrt{\frac{x - u}{x^2 - \Lambda^4}} dx,
\]

respectively. We choose the \( \alpha \)- and \( \beta \)-cycles as

\[
\alpha \text{- cycle} : -\Lambda^2 \rightarrow \Lambda^2, \ \beta \text{- cycle} : \Lambda^2 \rightarrow u.
\]

For these cycles, the periods are defined by

\[
a = \oint_{\alpha} \lambda_{SW}, \ a_D = \oint_{\beta} \lambda_{SW}.
\]
Using the formulae for hypergeometric function (see Appendix B), we find that these periods can be represented by

\[
\frac{da}{du} = \frac{\sqrt{2}}{4\sqrt{u + \Lambda^2}} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{2\Lambda^2}{\Lambda^2 + u}\right)
= \frac{1}{4} \sqrt{\frac{2}{u}} \, _2F_1\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{\Lambda^4}{u^2}\right),
\]

\[
\frac{da_D}{du} = \frac{i}{2\Lambda} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{\Lambda^2 - u}{2\Lambda^2}\right). \quad (A.5)
\]

For \(da_D/du\), it is useful to note the analytic continuation formula

\[
_2F_1\left(\frac{1}{4}, \frac{1}{4}, 1; \frac{\Lambda^2 - u}{2\Lambda^2}\right) = \frac{\Gamma(1/2)}{\Gamma(3/4)^2} \, _2F_1\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{u^2}{\Lambda^4}\right) + \frac{\Gamma(-1/2)}{\Gamma(1/4)^2 \Lambda^2} \, _2F_1\left(\frac{3}{4}, \frac{3}{4}, 3; \frac{u^2}{\Lambda^4}\right).
\]

Both \(a\) and \(a_D\) satisfy the following Picard-Fuchs equation

\[
4(\Lambda^4 - u^2) \frac{d^2\Pi}{du^2} - \Pi = 0. \quad (A.7)
\]

This differential equation can be translated into the hypergeometric differential equation of type \(_2F_1[1/4, 1/4, 1/2; z]\)

\[
z(1 - z) \frac{d^2\Pi}{dz^2} + \left(\frac{1}{2} - \frac{z}{2}\right) \frac{d\Pi}{dz} - \frac{\Pi}{16} = 0, \quad (A.8)
\]

where

\[
z = \frac{u^2}{\Lambda^4}. \quad (A.9)
\]

The solution around \(u = \infty\) are given by

\[
\rho_1 = z^{1/4} \sum_{i=0}^{\infty} \frac{a_i}{z^i}, \quad \rho_2 = \rho_1 \log \frac{1}{z} + z^{1/4} \sum_{i=1}^{\infty} \frac{b_i}{z^i}, \quad (A.10)
\]

where

\[
a_0 = 1, \quad a_1 = -\frac{1}{16}, \quad a_2 = -\frac{1}{1024}, \quad a_3 = -\frac{1}{16384}.
\]
\[ a_4 = -\frac{15015}{4194304}, \quad a_5 = -\frac{67108864}{6789783}, \quad a_6 = -\frac{4294967296}{79676025}, \quad a_7 = -\frac{68719476736}{62386327575}, \quad a_8 = -\frac{70368744177664}{787916211225}, \quad a_9 = -\frac{112589906842624}{40814059741455}, \quad a_{10} = -\frac{72057594037927936}{70368744177664} \]  

(A.11)

and

\[ b_1 = \frac{1}{8}, \quad b_2 = \frac{13}{1024}, \quad b_3 = \frac{49152}{163}, \quad b_4 = \frac{31183}{25165824}, \quad b_5 = \frac{134217728}{74791}, \quad b_6 = \frac{7190449}{25769803776}, \quad b_7 = \frac{161098027539}{429352037}, \quad b_8 = \frac{1970324836974592}{2886218022912}, \quad b_9 = \frac{12747539619133}{1283726776524341248}, \quad b_{10} = \frac{307142061004141}{12970366926827028480}. \]  

(A.12)

These expansion coefficients can be also obtained from the Pochhammer symbols of associated hypergeometric functions, but we write explicitly for convenience.

Direct calculation of the periods fixes the combination of the solutions to Picard-Fuchs equations, thus

\[ a = \frac{\Lambda}{\sqrt{2}} \rho_1, \quad a_P = i \frac{\Lambda}{\sqrt{2\pi}} (-4 + 6 \log 2) \rho_1 - i \frac{\Lambda}{\sqrt{2\pi}} \rho_2. \]  

(A.13)
The effective coupling constant is then given by

\[ \tau_4 = \frac{d a_P}{d u} / \frac{d a}{d u} \]

\[ = \frac{2}{\pi} \log \frac{8u}{\Lambda^2} - i \frac{5 \Lambda^4}{8\pi u^2} - i \frac{269 \Lambda^8}{1024\pi u^4} - i \frac{1939 \Lambda^{12}}{12288\pi u^6} \]

\[ - i \frac{922253 \Lambda^{16}}{8388608\pi u^8} - i \frac{1394369 \Lambda^{20}}{16777216\pi u^{10}} - \ldots. \] (A.14)

In order to obtain the prepotential, we need the inversion of \( a = a(u) \), which is given by

\[ u = 2a^2 + \frac{\Lambda^4}{16a^2} + \frac{5\Lambda^8}{4096a^6} + \frac{9\Lambda^{12}}{131072a^{10}} + \frac{1469\Lambda^{16}}{268435456a^{14}} + \ldots. \] (A.15)

In this way, we get the prepotential

\[ \mathcal{F}_4 = i \frac{a^2}{\pi} \left[ \log \left( \frac{a}{\Lambda} \right)^2 + 4 \log 2 - 3 + \sum_{k=1}^{\infty} F_k \left( \frac{\Lambda}{a} \right)^{4k} \right], \] (A.16)

where we have omitted the lower order terms than \( O(a) \) and the instanton expansion coefficients are

\[ F_1 = -\frac{1}{64}, \quad F_2 = -\frac{5}{32768}. \] (A.17)

**Appendix B. Hypergeometric function**

Gauss’s hypergeometric function \( F(\alpha, \beta, \gamma; z) \) often denoted by \(_2F_1(\alpha, \beta, \gamma; z)\) satisfies the differential equation

\[ z(1-z) \frac{d^2 F}{dz^2} + [\gamma - (\alpha + \beta + 1)] \frac{d F}{dz} - \alpha \beta F = 0. \] (B.1)

For \(|z| < 1\), it has a convergent series representation

\[ F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n) z^n}{\Gamma(\gamma + n) n!} \]

\[ = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n n!}, \] (B.2)

where \((\ast)_n = \Gamma(\ast + n)/\Gamma(\ast)\) is Pochhammer’s symbol.

Using the Euler integral representation

\[ F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt, \] (B.3)
we can establish the formula
\[ \int_{e_1}^{e_2} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} = \frac{1}{\sqrt{e_3-e_1}} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-h t)}} = \frac{\pi}{\sqrt{e_3-e_1}} F(1/2, 1/2, 1; h), \] (B.4)
where \( h = (e_2-e_1)/(e_3-e_1) \). Note that the elliptic curve
\[ y^2 := (x-e_1)(x-e_2)(x-e_3) \] (B.5)
has branching points located at \( x = e_i \ (i = 1, 2, 3) \) and \( \infty \).

**Appendix C. Schwarzian differential equation**

We summarize the basics of Schwarzian differential equation in this appendix.

For the two independent solutions denoted by \( y_1 \) and \( y_2 \) of the second order differential equation
\[ \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = 0, \] (C.1)
the ratio \( z = y_2/y_1 \) is known to satisfy a non-linear differential equation called Schwarzian differential equation
\[ \{ z, x \} = -\frac{p^2}{2} - \frac{dp}{dx} + 2q, \] (C.2)
where
\[ \{ z, x \} = z''' z' - 3 \frac{z''}{z'}^2 \] (C.3)
is Schwarzian derivative and \( ' = d/dx \).

As important properties, it is well-known that (C.3) satisfies
\[ \{ y, x \} = -\left( \frac{dy}{dx} \right)^2 \{ x, y \} \] (C.4)
and the Cayley’s identity
\[ \{ y, x \} = \left( \frac{dz}{dx} \right)^2 \left( \{ y, z \} - \{ x, z \} \right). \] (C.5)
Furthermore, (C.3) is invariant under the action of \( SL(2, \mathbb{C}) \), i.e.,
\[ \left\{ \frac{Ax + B}{Cx + D}, x \right\} = \{ z, x \}, \] (C.6)
where
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in SL(2, \mathbb{C}). \quad (C.7)
\]

(C.6) implies that
\[
\{z, x\} = 0 \quad (C.8)
\]

has a solution of the form
\[
z = \frac{Ax + B}{Cx + D}. \quad (C.9)
\]

**Appendix D. Polylogarithms**

In this appendix, we summarize on polylogarithms. For more details, see [29, 30].

Polylogarithm \( Li_k(x) \) is defined by
\[
Li_0(x) = \frac{x}{1 - x}, \quad Li_k(x) = \int_0^x \frac{Li_{k-1}(t)}{t} dt. \quad (D.1)
\]

In particular, \( Li_2 \) is usually called as di-logarithm, whereas \( Li_3 \) tri-logarithm. For \( |x| < 1 \), \( Li_k \) can be represented by a series
\[
Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}. \quad (D.2)
\]

It is easy to show that \( Li_k \) \( (k > 0) \) satisfies a differential equation of order \( k + 1 \)
\[
\frac{d}{dx} \left[ (1 - x) \frac{d}{dx} \left( x \frac{d}{dx} \right)^{k-1} \right] y = 0. \quad (D.3)
\]

(D.3) is actually equivalent to the generalized hypergeometric differential equation
\[
\left[ x \frac{d}{dx} \prod_{i=1}^{B} \left( x \frac{d}{dx} + \beta_i - 1 \right) - x \prod_{i=1}^{A} \left( x \frac{d}{dx} + \alpha_i \right) \right] \bar{y} = 0, \quad (D.4)
\]

provided
\[
A = B + 1, \quad B = k, \quad \alpha_i = 1, \quad \beta_i = 2, \quad \bar{y} = \frac{dy}{dx}. \quad (D.5)
\]

Note that (D.4) is not Fuchsian when \( k = 1 \).

Let
\[
(*)_k = (\_,\_\_\_,\ldots\_,\_\_) \quad \underbrace{\_,\_,\ldots\_,\_}_{k \text{ times}}. \quad (D.6)
\]
and let the generalized hypergeometric function be $pFq$. Then we find

$$\tilde{y} = k F_{k-1} [(1)_k; (2)_{k-1}; x]. \quad (D.7)$$

$$\frac{dLi_k}{dx} = \frac{Li_{k-1}}{x} = k F_{k-1} [(1)_k; (2)_{k-1}; x]. \quad (D.8)$$

Thus,

$$Li_k(x) = x^{k+1} F_k [(1)_{k+1}; (2)_k; x]. \quad (D.9)$$

The other $k$-independent solutions are given by

$$1, \log x, \cdots, \log^{k-1} x, \quad (D.10)$$

where $\log^i x = (\log x)^i$.

Note that

$$Li_k(1) = \zeta(k) = k + 1 F_k [(1)_{k+1}; (2)_k; 1], \quad (D.11)$$

where $\zeta(k)$ is Riemann’s zeta function.

**Appendix E. Differential equation for prepotential**

In this appendix, we show that the prepotential can be obtained from a differential equation analogous to the scaling relation in four dimensions [38, 39].

First, recall that $A_D$ is a differentiation of $F_5$, i.e., $A_D = dF_5/dA$. Repeating differentiation over $U$, we obtain

$$\frac{dA_D}{dU} = A'_D \frac{dA}{dU} = \frac{F''_5}{U'},$$

$$\frac{d^2A_D}{dU^2} = \frac{1}{U^3} (F''''_5 U' - F'''_5 U''),$$

$$\frac{d^3A_D}{dU^3} = \frac{1}{U'^5} \left[ (F^{(4)}_5 U' - F'''_5 U'') U' - 3U'' (F''''_5 U' - F'''_5 U'') \right], \quad (E.1)$$

where $' = d/dA$. Substituting these into the Picard-Fuchs equation for $A_D$ and using (4.6), we obtain

$$F^{(4)}_5 + \left[ \frac{4R^4 U}{\Delta(U)} (1 - R^4 U^2) - \frac{1}{U} - 3 \frac{U''}{U'^2} \right] U' F'''_5 = 0. \quad (E.2)$$
In this derivation we do not use Wronskian. This method gives a similar equation in the four dimensional theory, but the resulting equation is essentially the scaling relation of the prepotential. In fact, using (2.13) and (4.7) for the four dimensional theory, one can obtain \( \dot{\mathcal{F}}_4 \dot{u} - (a\mathcal{F}_4 - \dot{\mathcal{F}}_4)\ddot{u} = 0 \), \(^{(E.3)}\) which is (twice) integrated to give
\[
\frac{c_1}{2} u + c_2 = \mathcal{F}_4 - \frac{a}{2} \dot{\mathcal{F}}_4, \quad (E.4)
\]
where \( c_i \) are integration constants and for \( SU(2) \) pure gauge theory \( c_1 = -i/\pi \) and \( c_2 = 0 \). Then (E.4) is identified with the scaling relation of the prepotential \( \mathcal{F}_4 \). Note that this derivation of the scaling relation is different from those given in [38, 39].

(E.2) can be easily integrated to give
\[
\mathcal{F}_5''' = c \frac{UU''}{\Delta(U)}, \quad (E.5)
\]
where \( c \) is an integration constant and is determined by the comparison with (4.13). The result is \( c = -iR^6/\pi \). This is the analogue of the scaling relation of the four dimensional theory. We have already calculated \( U \) as a function of \( A \) in section 4.1, so we can determine the prepotential by triple integration over \( A \). In fact, we can check that after suitably choosing integration constants appearing in the triple integration the prepotential obtained in this way agrees to, e.g., the one in the large radius limit (4.13). Of course, we can extract the information on the prepotential in the small radius limit from this relation. Thus the (E.5) is the “global” form of prepotential in the five dimensional theory in a sense.

Remark: Triple integration of (E.3) (with the explicit value of \( c \)) over \( A \) gives the prepotential itself, but \( O(A^2) \)-terms are not fixed by this integration (these terms include integration constants). The cubic term of (4.13) or (4.14) is originated from the integration of the right hand side of (E.3). In this sense, (E.3) gives only the cubic term and the infinite sum in (4.14).

Next, let us consider the relation with the Wronskian of the Picard-Fuchs equation. The Wronskian is defined by
\[
W = \frac{d^2 A}{dU^2} \frac{dA_D}{dU} - \frac{dA}{dU} \frac{d^2 A_D}{dU^2}, \quad (E.6)
\]
if the Picard-Fuchs equation (2.10) is regarded as a second order differential equation for
$d\Pi/dU$. The Picard-Fuchs equation implies that $W$ satisfies the differential equation

$$
\frac{dW}{dU} = - \left[ \frac{4R^4U}{\Delta(U)}(1 - R^4U^2) - \frac{1}{U} \right] W,
$$

(E.7)

which can be integrated to give

$$
W = \frac{U}{\Delta(U)},
$$

(E.8)

up to overall integration constant. This is a very simple relation between the Wronskian and the moduli. From (E.5) (with explicit value of $c$) and (E.8), we get

$$
F''''_5 = -\frac{R^6}{\pi}WU^3.
$$

(E.9)

In the four dimensional theory, integral of Wronskian provided the scaling relation. Unfortunately, the integral of the Wronskian of the five dimensional theory does not give such a simple linear relation between prepotential and moduli, but the relation between Wronskian and the prepotential can be seen from this expression.

Finally, let us consider the relation between $F_5$ and $F_4$ by using (E.5) ($c = -iR^6/\pi$) and (E.3). Recall that the identification of the periods is given by (2.6). Using (2.6) with the differential equation for $u$, (c.f. (4.7)) and the scaling relation (E.4) (with $c_1 = -i/\pi$ and $c_2 = 0$), we can derive the following relation

$$
F''''_5 = R^2U \tilde{F}_4.
$$

(E.10)

As is noted in (4.11), the combination $R^2U$ tends to one for $R \to 0$. Furthermore, in this limit we have $dA \sim da$ (c.f. (4.32)), so we can easily integrate (E.10) to give $F_5 = F_4$ (up to terms involving integration constants)! This supports the result in section 4.4.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B 431 (1994) 484.

[2] N. Seiberg and E. Witten, Nucl. Phys. B 435 (1994) 129.

[3] A. Ceresole, R. D’Auria and S. Ferrara, Phys. Lett. B 339 (1994) 71.

[4] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. B 477 (1996) 746.

[5] E. Witten, Nucl. Phys. B 500 (1997) 3.
[6] N. Nekrasov, Five dimensional gauge theories and relativistic integrable systems, ITEP-TH-26/96, HUTP-96/A023, hep-th/9609219.

[7] A. Losev, G. Moore, N. Nekrasov and S. Shatashvili, Nucl. Phys. B (Proc. Supple.) 46 (1996) 130.

[8] L. Baulieu, A. Losev and N. Nekrasov Chern-Simons and twisted supersymmetry in various dimensions, ITEP-TH-34/97, HUTP-97/A035, LPTHE-9733, hep-th/9707174.

[9] E. Martinec and N. Warner, Nucl. Phys. B 459 (1996) 97.

[10] T. Nakatsu and K. Takasaki, Mod. Phys. Lett. A11 (1996) 157.

[11] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B 380 (1996) 75.

[12] A. Gorsky, S. Gukov and A. Mironov, SUSY field theories, integrable systems and their stringy/brane origin-II, ITEP/TH-57/97, FIAN/TD-14/97, hep-th/970239.

[13] A. Marshakov and A. Mironov, 5d and 6d supersymmetric gauge theories: prepotentials from integrable systems, FIAN/TD-15/97, ITEP/TH-63/97, NBI-HE-97-61, hep-th/9711156.

[14] N. Seiberg, Phys. Lett. B 388 (1996) 753.

[15] K. Intriligator, D.R. Morrison and N. Seiberg, Nucl. Phys. B 497 (1997) 56.

[16] A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A 11 (1996) 1929.

[17] K. Ito, Phys. Lett. B 406 (1997) 54.

[18] A. M. Ghezelbash, A. Shafiekhani and M. R. Abolhasani, On the Picard-Fuchs equations of $N = 2$ supersymmetric $E_6$ Yang-Mills theory, IPM-97-226, hep-th/9708073.

[19] K. Ito and S.-K. Yang, Flat coordinates, topological Landau-Ginzburg models and the Seiberg-Witten period integrals, UTHEP-367, hep-th/9708017.

[20] M. Alishahiha, Phys. Lett. B 398 (1997) 100.

[21] J. M. Isidro, A. Mukherjee, J. P. Nunes and H. J. Schnitzer, Nucl. Phys. B 492 (1997) 647.
[22] A. Marshakov, Seiberg-Witten theory, integrable systems and D-branes, FIAN/TD-12/97, ITEP/TH-39/97, hep-th/9709001.

[23] A. Marshakov, Theor. Math. Phys. 112 (1997) 791.

[24] A. Marshakov, A. Mironov and A. Morozov, More evidence for the WDVV equations in $N = 2$ SUSY Yang-Mills theories, FIAN/TD-15/96, ITEP/TH-46/96, hep-th/9701123.

[25] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen and S. Yankielowicz, On the M-theory approach to (compactified) 5D field theories, TAUP-2449-97, LMU-TPW-97-22, hep-th/9709010.

[26] A. Alishahiha, F. Ardalan and F. Mansouri, Phys. Lett. B 381 (1996) 446.

[27] M. Matone, Phys. Lett. B 357 (1995) 342; Phys. Rev. Lett. 78 (1997) 1412; Phys. Rev. D 53 (1996) 7354.

[28] G. Bonelli, M. Matone and M. Tonin, Phys. Rev. D 55 (1997) 6466.

[29] L. Lewin, Polylogarithms and associated functions (North Holland, 1981).

[30] L. Lewin., ed., Structural properties of polylogarithms (American Mathematical Society, Providence, Rhode Island, 1991).

[31] K. Ito and S.-K. Yang, Picard-Fuchs equations and prepotentials in $N = 2$ supersymmetric QCD, UTHEP-330, hep-th/9603073.

[32] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B 344 (1995) 169.

[33] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, On the monodromies of $N = 2$ supersymmetric Yang-Mills theory, CERN-TH-7538-94, hep-th/9412158.

[34] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931.

[35] A. Hanany and Y. Oz, Nucl. Phys. B 452 (1995) 283.

[36] M. R. Abolhasani, M. Alishahiha and A. M. Ghezelbash, Nucl. Phys. B 480 (1996) 279.

[37] D. Ramakrishnan, Proc. Amer. Math. Soc. 85 (1982) 596.
[38] T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A 11 (1996) 131.

[39] J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. B 367 (1996) 145.
Figure 1: Relation among singularities