Static phantom wormholes of finite size

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In this paper we derive new static phantom traversable wormholes by assuming a shape function with a quadratic dependence on the radial coordinate $r$. We mainly focus our study on wormholes sustained by exotic matter with positive energy density (as seen by any static observer) and a variable equation of state $p/\rho < -1$, dubbed phantom matter. Among phantom wormhole spacetimes extending to infinity, we show that a quadratic shape function allows us to construct static spacetimes of finite size, composed by a phantom wormhole connected to an anisotropic spherically symmetric distribution of dark energy. The wormhole part of the full spacetime does not fulfill the dominant energy condition, while the dark energy part does.

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I. INTRODUCTION

The accelerated expansion of the universe is one of the most exciting and significant discoveries in modern cosmology. In the framework of general relativity, dark energy, which has an equation of state satisfying the relation $-1 < p/\rho < -1/3$, is the most accepted hypothesis to explain the observed acceleration. However, there are observational evidences that the cosmic fluid leading to the acceleration of the universe may satisfy also an equation of state of the form $p/\rho < -1$, with positive energy density. A cosmic fluid characterized by such an equation of state is dubbed phantom energy, and has received increased attention among theorists in cosmology, since if this type of source dominates the cosmic expansion, the universe may end in a Big Rip singularity (the positive energy density becomes infinite in finite time, as well as the pressure). This phantom energy has a very strong negative pressure and violates the dominant energy condition (DEC), which can be written as $\rho \geq 0$ and $-\rho \leq p_i \leq \rho$, where $p_i$ are the radial and lateral pressures.

The study of phantom wormholes involves mainly asymptotically flat phantom wormhole solutions, which extend from the throat to infinity. Non asymptotically flat phantom wormholes also have been studied. In Ref. [3] such wormholes are considered, and spacetimes extend from the throat to infinity, so they are glued to the external Schwarzschild solution. Asymptotically and non asymptotically flat phantom wormholes are also discussed in Ref. [4]. Non asymptotically flat phantom wormholes also have been studied in three dimensions [5]. All these spacetimes also extend to spatial infinity.

In Ref. [6] the notion of phantom energy is also extended to inhomogeneous and anisotropic spherically symmetric spacetimes. The author finds an exact wormhole solution and shows that a spatial distribution of the phantom energy is mainly limited to the vicinity of the

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wormhole throat.

It is interesting to note that evolving wormholes supported by phantom energy also have been studied in the presence of a cosmological constant \[7\] and without it \[8\]. In both cases the equation of state of the radial pressure has the form \(\omega_r = p_r(t, r)/\rho(t, r) < -1\), with constant state parameter \(\omega_r\) (see also Ref. [9] for a slight generalization of dynamical phantom equation of state).

This paper presents phantom traversable wormholes by resorting to a quadratic shape function. For constructing them we use the conventional approach of Morris and Thorne based on the assumption of particular forms of the shape function \(b(r)\), and the redshift function \(\phi(r)\), in the metric \[10\]

\[
ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{1 - \frac{b(r)}{r^3}} - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \tag{1}
\]

We assume that the matter source threading the wormhole is described by a single anisotropic fluid characterized by \(T^\mu_\nu = \text{diag}(\rho, -p_r, -p_l, -p_l)\). Therefore, the Einstein field equations are given by

\[
\kappa_\rho(r) = \frac{b^\prime}{r^2}, \tag{2}
\]

\[
\kappa p_r(r) = 2 \left( 1 - \frac{b}{r} \right) \frac{\phi^\prime}{r} - \frac{b}{r^3}, \tag{3}
\]

\[
\kappa p_l(r) = \left( 1 - \frac{b}{r} \right) \times \left[ \phi'' + \phi'^2 - \frac{b^\prime r + b - 2r}{2r(r - b)} \phi' - \frac{b^\prime r - b}{2r(r - b)} \right], \tag{4}
\]

where \(\rho\) is the energy density, \(p_r\) and \(p_l\) are the radial and lateral pressures respectively, and the prime denotes the derivative \(d/dr\).

The paper is organized as follows. In Sec. II we study Morris-Thorne wormholes characterized by a quadratic shape function. In Sec. III we discuss energy conditions and the positivity of energy density. In Sec. IV we construct phantom wormholes of finite size. In Sec. V we conclude with some remarks.

II. Wormholes with quadratic shape functions

Now we will study Morris-Thorne wormholes by using a quadratic shape function in the form

\[
b(r) = a_1 r^2 + a_2 r + a_3 \tag{5}
\]

where \(a_1, a_2\) and \(a_3\) are constant parameters. In order to have a wormhole we must impose the Morris-Thorne constraints on the shape function, so \(a_1, a_2\) and \(a_3\) are not all free parameters, and they must satisfy specific constraints which we will now discuss.

First of all, the wormhole must have a minimum radius \(r = r_0\), where the wormhole throat is located. This requirement is expressed by \[10, 11\]

\[
b(r_0) = r_0, \tag{6}
\]

and \(r_0\) is the minimum value of the radial coordinate \(r\). On the other hand, the shape function must satisfy the condition

\[
\frac{b(r)}{r} \leq 1, \tag{7}
\]

in order to the metric \[11\] remains Lorentzian \((g_{rr} \leq 0)\).

Evaluating the shape function \[15\] at the throat, i.e. imposing the fulfillment requirement \[6\], we obtain

\[
b(r) = (r - r_0) \left( a_1 r - \frac{a_3}{r_0} \right) + r, \tag{8}
\]

and the metric \[11\] takes the form

\[
ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{1 - \frac{b(r)}{r^3}} - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \tag{9}
\]

It becomes clear that traversable versions of Schwarzschild wormholes are obtained if \(a_1 = 0\) and \(a_3 = r_0\).

Now for having a wormhole geometry the shape function must satisfy the flare-out condition, which is given by \[10\]

\[
\frac{b - b^\prime r}{2b^2} > 0. \tag{10}
\]

From this equation we obtain

\[
a_3 - a_1 r^2 > 0. \tag{11}
\]

This condition allows us to classify and construct three classes of wormhole solutions. Namely:

**Case 1:**

\[
a_1 < 0, a_3 > 0, \tag{11}
\]

and wormhole exists for \(0 < r_0 < r < \infty\).

**Case 2:**

\[
a_1 < 0, a_3 < 0, \tag{12}
\]

and wormhole exists for

\[
\sqrt{\frac{a_3}{a_1}} < r_0 < r < \infty. \tag{13}
\]

**Case 3:**

\[
a_1 > 0, a_3 > 0, \tag{14}
\]

and wormhole exists for

\[
0 < r_0 < r < \sqrt{\frac{a_3}{a_1}}. \tag{15}
\]
For $a_1 > 0$ and $a_3 < 0$ we have static spherically symmetric gravitational configurations which are not wormholes.

It should be noted that these constraints should be compatible with the condition (7), which is expressed in the form $(1 - \frac{r_a}{r}) \left( \frac{a_3}{r_0} - a_1 r \right) > 0$, as we can see from the radial metric component of Eq. (6). Since we have that $r \geq r_0$ for the radial coordinate in a wormhole geometry, we conclude that also it is necessary to satisfy the inequation

$$\frac{a_3}{r_0} - a_1 r > 0.$$  \hspace{1cm} (16)

For the case 1 this constraint is satisfied automatically. For the case 2 we obtain that Eq. (16) implies that

$$r > \frac{a_3}{a_1 r_0}.$$  \hspace{1cm} (17)

while for the case 3 we obtain that

$$r < \frac{a_3}{a_1 r_0}.$$  \hspace{1cm} (18)

It is interesting to note that, in principle, we can make that ranges allowed by the flare-out condition coincide with ranges imposed by Eq. (16). This can be performed by requiring that $\sqrt{\frac{a_3}{a_1}} = \frac{a_3}{a_1 r_0}$. Then we have that

$$a_3 = a_1 r_0^2.$$  \hspace{1cm} (19)

If we put this expression into the radial metric component of the line element (3) we obtain that $g_{rr} = a_1(r - r_0)^2 r^{-1}$. This implies that the relation (19) may be applied only for the case 2, since for the case 3 we have that $a_1 > 0$ and the line element (3) becomes non-Lorentzian.

### III. THE POSITIVITY OF ENERGY DENSITY AND ENERGY CONDITIONS

Since we are interested in finding wormholes supported by phantom energy, we need to require the positivity of energy density. Physically, this requirement implies that everywhere any static observer will measure a positive energy density. Therefore, we shall study conditions which must satisfy the relevant parameters $a_1$ and $a_3$ in order to have a positive energy density. For the considered shape function (5) the energy density is given by

$$\rho = \frac{a_1 r_0 (2 r - r_0) + r_0 - a_3}{r_0 r^2}.$$  \hspace{1cm} (20)

**Case 1:** We consider first the case $a_1 < 0$ and $a_3 > 0$ Note that for large values of radial coordinate we have that $\rho \approx \frac{2 a_3}{r}$, so if $r \to \infty$ then $\rho \to 0$. The expression (20) vanishes at $r_3 = r_0 - a_3 + a_1 |a_3|^2 r_0^{-2}$. In such a way, if $r_3 > r_0$, then the energy density is positive for $r_0 < r < r_3$, while $\rho \leq 0$ for $r \leq r_3$. The energy density is everywhere negative for $r \geq r_0$ if $a_3 > r_0 - |a_1| r_0^2$, and vanishes at $r_0$ if $a_3 = r_0 - |a_1| r_0^2$.

**Case 2:** Now, Eq. (20) implies that if $a_1 < 0$ and $a_3 < 0$ we have two possibilities to be considered: if

$$|a_3| \leq r_0^2 |a_1| - r_0$$  \hspace{1cm} (21)

then $\rho(r_0) \leq 0$ and the energy density is negative everywhere for $r > r_0$, while if

$$|a_3| > r_0^2 |a_1| - r_0$$  \hspace{1cm} (22)

then $\rho(r_0) > 0$, and we obtain $\rho(r) > 0$ for $r_0 < r < r_1$, and $\rho(r) \leq 0$ for $r \geq r_1$, where $r_1 = \frac{|a_3| + r_0 + r_0^2 |a_1|}{2 |a_3| |a_1|}$. 

**Case 3:** Lastly, for $a_1 > 0$ and $a_3 > 0$ we have also two possibilities to be considered: if

$$a_3 \leq r_0^2 a_1 + r_0$$  \hspace{1cm} (23)

then $\rho(r_0) \geq 0$ and the energy density $\rho(r) > 0$ for $r > r_0$, while if

$$a_3 > r_0^2 a_1 + r_0$$  \hspace{1cm} (24)

then $\rho(r_0) < 0$ and we obtain $\rho(r) < 0$ for $r_0 < r < r_2$, and $\rho \geq 0$ for $r \geq r_2$, where $r_2 = \frac{a_1 r_0^2 + a_3 r_0 - r_0}{2 a_3 r_0}$. In conclusion, phantom wormholes may be constructed for all cases $a_1 < 0$, $a_3 > 0$; $a_1 < 0$, $a_3 < 0$ and $a_1 > 0$, $a_3 > 0$. In Figs. 1 and 2 we show the qualitative behavior of the energy density for these cases.

Note that the energy density (20) may be rewritten in the form $\rho = \frac{2 a_3}{r} - \frac{a_3 + a_1 r_0^2 - r_0}{r^2 r_0^2}$. From this expression it...
The figure shows the qualitative behavior of energy density for $a_1 < 0$ and $a_3 < 0$. The throat is located at $r_0$. The dotted line describes the case $|a_2| = r_0^2|a_3| - r_0$ for which $\rho(r_0) = 0$. The solid and dashed lines represent the cases $|a_3| \leq r_0^2|a_3| - r_0$ and $|a_3| \geq r_0^2|a_3| - r_0$, for which we have at the throat $\rho(r_0) < 0$ and $\rho(r_0) > 0$, respectively. We can see that in the case of solid line the energy density is everywhere negative, while for the dashed line the energy density is positive for $r_0 \leq r < r_1$ and becomes negative for $r > r_1$.

The dotted line represents the case $a_3 = r_0^2a_1 + r_0$ for which $\rho(r_0) = 0$. The solid and dashed lines represent the cases $a_3 \leq r_0^2a_1 + r_0$ and $a_3 \geq r_0^2a_1 + r_0$, for which we have at the throat $\rho(r_0) > 0$ and $\rho(r_0) < 0$, respectively. We can see that in the case of solid line the energy density is everywhere positive, while for the dashed line the energy density is negative for $r_0 \leq r < r_2$ and becomes positive for $r > r_2$.

becomes clear that for positive $a_1$ and $a_3$ we have always a positive energy density by requiring

$$a_3 + a_1 r_0^2 - r_0 \leq 0.$$  

If this inequation is not satisfied then the energy density vanishes at some $r$, changing its sign, as described by Eq. (24).

Now some words about the energy conditions. It is well known that the violation of the DEC

$$\rho \geq 0, \rho + p_r \geq 0, \rho + p_t \geq 0$$  

is a necessary condition for a static wormhole to exist. It is interesting to note that for the Morris-Thorne metric (1), if $\phi(r) = \text{const}$, the strong energy condition $\rho + p_{\text{total}} \geq 0$ is satisfied, since the relation $\rho + p_r + 2p_t = 0$ is everywhere valid.

In order to discuss DEC, for simplicity, we shall consider the zero-tidal-force wormhole version of these quadratic wormholes (i.e. $\phi(r) = \text{const}$). For a such wormhole the energy density is defined by Eq. (20), and the pressures are given by

$$p_r = \frac{(r - r_0)(a_1 r - a_2 r_0)}{r^3},$$  

$$p_t = \frac{a_3 - a_1 r_0^2}{2r^3}.$$  

By rewriting the radial pressure as $p_r = \frac{a_1}{r_0} - \frac{a_3}{r_0} + \frac{a_3 + a_1 r_0^2 - r_0}{r_0 r^2}$ we conclude that for positive $a_1$ and $a_3$, if Eq. (24) is fulfilled, the radial pressure is everywhere negative. The lateral pressure vanishes at $r = \sqrt{a_3/a_1}$, and $p_t > 0$ for $r_0 \leq r < \sqrt{a_3/a_1}$, while $p_t < 0$ for $r \geq \sqrt{a_3/a_1}$.

Let us now consider the behavior of $\rho + p_r$ and $\rho + p_t$. For the first expression we have that

$$\rho + p_r = \frac{a_1 r^2 - a_3}{r^3},$$

At the throat this relation gives $\rho + p_r = \frac{1}{r_0} \left( a_1 - \frac{a_3}{r_0} \right)$. It should be noted that the expression (29) vanishes at $r = \sqrt{a_3/a_1}$. For negative values of $a_1$ and $a_3$ we obtain that $\rho + p_r \leq 0$ for $r \geq r_0$. Now, for positive $a_1$ and $a_3$, from Eq. (15) we have that $\sqrt{a_3/a_1} > r_0$, implying that $a_1 < a_3/r_0^2$, and then at the throat $\rho + p_r \leq 0$. Since the expression (29) vanishes at $r = \sqrt{a_3/a_1}$, the weak energy condition is violated at $r_0 \leq r < \sqrt{a_3/a_1}$ as we should expect. For the range $\sqrt{a_3/a_1} \leq r \leq a_3/a_1$ we have that $\rho + p_r \geq 0$, so DEC may be fulfilled in this range.

For the expression $\rho + p_t$ we have that

$$\rho + p_t = \frac{3a_3}{2r} + \frac{a_3}{2r^3} - \frac{a_3 + a_1 r_0^2 - r_0}{r_0^2},$$  

and at the throat $\rho + p_t = \frac{a_1 r_0^2 - a_3 + 2a_3 r_0}{2r_0^2}$, which in general can be positive as well as negative. However, for positive $a_1$ and $a_3$ if Eq. (23) is fulfilled then the expression in Eq. (30) is everywhere positive.
IV. PHANTOM WORMHOLES OF FINITE SIZE AND THEIR EMBEDDINGS

Now let us consider embeddings diagrams of the studied wormholes. The embedding of two dimensional slices \( t = \text{const}, \theta = \frac{\pi}{2} \) of the metric \( [9] \) is performed by using the embedding function \( z(r) \) in equation

\[
\frac{dz(r)}{dr} = \left( \frac{r}{b(r)} - 1 \right)^{-1/2}, \tag{31}
\]

which takes the form

\[
\frac{dz(r)}{dr} = \left( \frac{r - r_0}{(a_1 r - a_3/r_0) + r} \right)^{1/2} \tag{32}
\]

It becomes clear that everywhere the embedding exists in a Euclidean space if the expression under the square root is positive. The Eqs. \([10-13] \) imply that the denominator is positive, so we must require the positivity of the numerator, i.e. \( b(r) > 0 \). The shape function \( [8] \) is quadratic in \( r \), so the numerator under the square root in Eq. \([32] \) may have two roots, one root, or no roots. The roots of Eq. \([8] \) are given by

\[
r_{\pm} = 1 \frac{a_3 - a_1 r_0^2 - r_0}{2 a_1}, \tag{33}
\]

where \( \Delta = (a_1 r_0 + a_3/r_0 - 1)^2 - 4 a_1 a_3 \). The existence of real roots depends on values of \( \Delta \), therefore the existence of a wormhole embedding in the Euclidean space depends on values of \( \Delta \).

In the following we will focus our attention on the study of wormholes with finite size (case 3), therefore we shall consider wormholes with positive \( a_1 \) and \( a_3 \).

Case \( \Delta = 0 \): Let us first consider the case where \( \Delta = 0 \) (i.e. there exists a unique root, and \( b(r) \geq 0 \)). This condition gives \( a_3 = r_0 + a_1 r_0^2 + 2 r_0 \sqrt{a_1 r_0} \) obtaining two branches for the root \([33] \): \( r_+ = r_- = r_0 + \sqrt{r_0/a_1} \).

In case, the shape function is given by \( b(r) = r_0 + (r^2 + r_0^2 - 2 r_0 a_1) - 2 (r - r_0) \sqrt{r_0 a_1} \) and the wormhole extends from \( r_0 \) to \( r_{\text{max}} = \sqrt{r_0 + a_1 r_0^2 + 2 r_0 \sqrt{a_1 r_0}} \). Notice that, for the negative branch we have that

\[
r_0 < \sqrt{\frac{r_0 + a_1 r_0^2 - 2 r_0 \sqrt{a_1 r_0}}{a_1}} \tag{34}
\]

if \( a_1 < 1/(4 r_0) \), while for the positive branch

\[
r_0 < \sqrt{\frac{r_0 + a_1 r_0^2 + 2 r_0 \sqrt{a_1 r_0}}{a_1}} \tag{35}
\]

for any \( a_1 > 0 \).

Case \( \Delta < 0 \): In this case there are no roots, and \( b(r) > 0 \). Since \( a_1 > 0 \), the requirement \( \Delta < 0 \) implies that the embedding always exist if the parameter \( a_3 \) satisfies

\[
r_0 \left( 1 + a_1 r_0 - 2 \sqrt{a_1 r_0} \right) < a_3 < r_0 \left( 1 + a_1 r_0 + 2 \sqrt{a_1 r_0} \right). \tag{36}
\]

In other words, if positive \( a_1 \) and \( a_3 \) satisfy Eq. \([36] \), then the obtained finite size wormhole can be entirely embedded into the Euclidean space: i.e. the wormhole spacetime, and simultaneously its embedding in a Euclidean space, extend from \( r_0 \) to \( r_{\text{max}} = \frac{a_3}{a_1 r_0} \).

Case \( \Delta > 0 \): If condition \([30] \) is not satisfied, then roots \( r_+ \) and \( r_- \) of Eq. \([33] \) are real, and due to the positivity of \( a_1 \), the shape function is positive in the intervals \((-\infty, r-) \cup (r_+, +\infty) \). Therefore, the embedding of a constructed finite size wormhole may partially exists in a Euclidean space. Specifically, an equatorial slice \( \theta = \pi/2 \) can be embedded into the Euclidean space in those ranges obtained from the intersection of intervals \((-\infty, r-) \cup (r_+, +\infty) \) with the extension of the wormhole spacetime \([r_0, a_3/r_0] \).

A. Constructing wormholes

For constructing explicit examples of phantom wormholes of finite size we shall use the condition \([29] \) discussed above. As we have shown, for positive \( a_1 \) and \( a_3 \) the fulfilment of the condition \([25] \) ensures that the energy density and \( p + \rho_4 \) are everywhere positive, while the radial pressure \( p_r \) is everywhere negative. By locating the throat at \( r_0 = 1 \) the condition \([25] \) becomes \( a_1 + a_3 \leq 1 \). For simplicity, we shall use the equality \( a_1 + a_3 = 1 \).

Wormhole with positive \( \rho \): Let us first consider the parameter set \( a_1 = 1/5 \) and \( a_3 = 4/5 \). From Eq. \([15] \) we conclude that the wormhole extends from \( r_0 = 1 \) to \( r_{\text{max}} = 2 \). The energy density and pressures are given by

\[
\rho = \frac{2}{5 r}, \quad p_r = -\frac{r^2 + 4}{5 r^3}, \quad p_l = \frac{4 - r^2}{10 r^3}. \tag{37}
\]

It becomes clear that everywhere the energy density is positive and the radial pressure is negative, while the lateral pressure is negative for \( 1 < r < 2 \) and positive for \( r > 2 \). From Eq. \([37] \) we have that \( \rho + p_r = \frac{2^2}{5 r^2} \) and \( \rho + p_l = \frac{4^2}{10 r^4} \), therefore in the range \( 1 < r < 2 \) we have that \( \rho + p_r < 0 \) and DEC is violated. Notice that the metric is given by

\[
ds^2 = dt^2 - \frac{dr^2}{(1 - \frac{1}{7})(\frac{1}{2} - \frac{1}{7})} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{38}
\]

For \( r > 4 \) the spacetime ceases to be Lorentzian, then the spacetime extends from the throat at \( r_0 = 1 \) to \( r = 4 \). So, the whole spacetime is of finite size, characterized by a wormhole part connected to a dark energy distribution.
which extends from $r = 2$ to $r = 4$. In this case the shape function is given by $b(r) = (r - 1)(r/3 - 4/3) + r$, and it is positive for any value of radial coordinate $r$, implying that the embedding exists for the whole spacetime as shown in Fig. 4. Notice that the spacetime in the range $2 < r < 4$ is supported by a dark energy distribution, which satisfies DEC. It is interesting to discuss the behavior of the variable equation of state $p_r/r$. For $1 < r < 2$ we have the phantom behavior $-2.5 < p_r/r < -1$, while for $2 < r < 4$ we have that $-1 < p_r/r < -0.625$, as we would expect since the wormhole is connected to a distribution of dark energy (see Figs. 5-6).

Microscopic wormhole: It is relevant to note that the wormhole part of the spacetime can be made arbitrarily small. For doing this the parameters $a_1 > 0$ and $a_3 > 0$ must be chosen in such a way that $\sqrt{a_3/a_1} \approx r_0$. By using the condition (25) we may construct microscopic wormholes by imposing the equality $a_3 + a_1 r_0^2 - r_0 = 0$, implying that the relations $a_1 \approx \frac{1}{2} r_0$ and $a_3 \approx \frac{1}{2} r_0^2$ must be required. As an example, let us consider the case $r_0 = 1$. Then we can construct an arbitrarily small wormhole by making $a_1 = \frac{1}{2} r_0 - \delta$, $a_3 = \frac{1}{2} r_0^2 + \delta$, where $\delta \approx 0$. Let us put $\delta = 0.01$. Then the wormhole extends from $r_0 = 1$ to $r = 1.0202$, and the whole spacetime to $r = 1.04082$. For energy density and pressures we have that

$$\rho = \frac{0.98}{r^3}, p_r = -\frac{0.51 + 0.49 r^2}{r^3}, p_t = \frac{0.51 - 0.49 r^2}{2r^3}.$$  (39)

Clearly, $\rho > 0$ and $p_r < 0$ everywhere. On the other hand, $\rho + p_r = -\frac{0.51 + 0.49 r^2}{r^3}, \rho + p_t = \frac{0.51 - 1.49 r^2}{2r^3} > 0$.

$-g_{rr}^{-1} = (1 - 1/r)(0.51 - 0.49 r) \geq 0$ at $1 \leq r \leq 1.04082$, and $b(r) = (r - 1)(0.49 r - 0.51) + r > 0$ for $r \geq 1$. The relation $\rho + p_r$ is negative at $1 < r < 1.0202$, while $\rho + p_r > 0$ at $1.0202 \leq r \leq 1.04082$. In this case, for the wormhole part we have that $-1.0204 \leq p_r/\rho \leq -1$, and for the dark energy distribution part we have $-1 \leq p_r/\rho \leq -0.9804$.

Wormhole with negative $\rho$: We can construct also a finite size wormhole with a negative energy density. In order to do this we can require that the energy density vanishes at $r = \sqrt{a_3/a_1}$. This implies that $\frac{a_3 + a_1 r_0^2 - r_0}{2r_0^2 a_1} = \frac{a_3}{a_1}$. This condition is satisfied by requiring that $a_1 = $
\[
\frac{a_3+r_0\pm\sqrt{a_3}}{r_0^3}. \tag{40}
\]
This allows us to write the energy density and pressures as (for positive and negative branches)
\[
\rho = \frac{-2\left(\sqrt{a_3} \pm \sqrt{a_3} \right) \left(r_0\sqrt{a_3} - (\sqrt{a_3} \pm \sqrt{a_3} \right)r_0^2}{r_0^2},
\]
\[
p_r = -\frac{\left(\sqrt{a_3} \pm \sqrt{a_3} \right) r + r_0\sqrt{a_3}^2}{r_0^3}, \tag{41}
\]
\[
p_l = \frac{-\left(\sqrt{a_3} \pm \sqrt{a_3} \right)^2 r^2 + r_0^2 a_3}{2r_0^2 a_3}. \tag{42}
\]
These relations give for the radial state parameter
\[
\frac{p_r}{\rho} = -\frac{\sqrt{a_3} \pm \sqrt{a_3} \right)r + r_0\sqrt{a_3}^2}{2r_0\sqrt{a_3}}. \tag{43}
\]
For an explicit example we take \( r_0 = 1, a_1 = 1, a_3 = 4. \) From expressions of the negative branch we get that
\[
\rho = \frac{2r - 4}{r^2}, p_r = -\frac{(r - 2)^2}{r^3}, p_l = \frac{4 - r^2}{2r^3}, \tag{44}
\]
while \( \rho + p_r = \frac{r^2 - 4}{r^4}, \rho + p_l = \frac{(3r - 2)(r - 2)}{2r^3}. \) The general behavior of these relevant physical magnitudes are shown in Fig. 7. It becomes clear that due to \( \rho < 0 \) at \( 1 \leq r < 2, \) DEC is violated where the wormhole is located. At \( r = 2 \) this wormhole is connected to an anisotropic spherically symmetric distribution respecting DEC. In this case for the radial equation of state we have that \( 0 \leq p_r/\rho \leq -1/4 \) for \( 2 \leq r < 4 \) (see Fig. 8).

**Wormhole with mixed energy dependence:** Now we are interested in construction of finite wormholes with an energy density exhibiting a mixed dependence, in the sense that the energy density changes its sign at some sphere of radius \( r_0 < r_2 \leq a_3/a_1, \) where \( \rho(r_2) = 0, \) and \( r_2 = a_3^2 + r_0 a_1 - r_0. \) We have discussed above that if energy density vanishes at some \( r, \) then at the throat always \( \rho(r_0) < 0. \) Therefore, if \( r_0 < r_2 < \sqrt{a_3/a_1} \) we have that \( \rho(r) < 0 \) in this range, and for \( r_2 < r < a_3/a_1 \) the energy density becomes positive. In this way, the requirement
\[
r_0 < \frac{a_3 + r_0^2 a_1}{2a_1 r_0} < \sqrt{a_3/a_1} \implies \text{that the parameter } a_1 \text{ satisfies}
\]
\[
0 < a_1 < \frac{1}{4} \left(1 + \sqrt{17}\right)
\]
while the parameter \( a_3 \) the condition
\[
-\frac{a_1^2 + a_1 + 2}{a_1} < a_3 < \frac{-a_1^2 + a_1 + 2}{a_1} + 2 \sqrt{-\frac{a_1^2 - a_1^3 - 1}{a_1^3}}.
\]
For example, we may construct such a wormhole for \( r_0 = 1, a_1 = 1 \) and \( a_3 = 3, \) obtaining for the relevant quantities
\[
\rho = \frac{2r - 3}{r^2}, p_r = -\frac{r^2 - 3r - 3}{r^3}, p_l = \frac{3 - r^2}{2r^3},
\]
and \( \rho + p_r = \frac{r^2 - 3}{r^4}, \rho + p_l = \frac{3(r^2 - 2r + 1)}{2r^4}. \)

The change of sign of the energy density may also occur for a some radius between \( \sqrt{a_3/a_1} \) and \( a_3/a_1. \) In this case we must require that \( a_3 > a_1 + 2\sqrt{a_1} + 1 \) for any positive \( a_1. \) This implies that for the wormhole structure the exotic energy density is always negative, while for the anisotropic spherically symmetric distribution of matter respecting DEC, the energy density changes its sign.

Lastly, we may construct finite wormhole solutions for which the energy density is negative for the whole spacetime, i.e. at \( r_0 \leq r \leq a_3/a_1. \) By requiring that the energy density vanishes at \( a_3/a_1 \) we assure that \( \rho(r) < 0 \)
everywhere. From the condition $\frac{a_3 + r_0^2 a_1 - r_0}{2a_1 r_0} = \frac{a_2}{a_3}$, we obtain that if $r_0 = \frac{1}{2}$ we must require $a_1 = 2$ and $a_3 > 2$. If the throat is located at $0 < r_0 < \frac{1}{2}$ we must require that $0 < a_1 < \frac{r_0}{a_3 - 2 a_1}$, while for $\frac{1}{2} < r_0 \leq a_3/a_1$ (with $r_0 \neq 1$) the condition $a_1 > \frac{r_0}{r_0 - 2 r_0 + 1}$ must be satisfied. In the last two cases we have that $a_3 = \frac{a_2 r_0^2 - r_0}{2r_0 - 1}$. Notice that for $r_0 = 1$ is not possible to construct such a spacetime with $\rho(a_3/a_1) = 0$.

V. CONCLUSIONS

In this paper we derived new static spherically symmetric traversable wormholes by assuming a shape function with a quadratic dependence on the radial coordinate $r$, and it is shown that there exist wormhole spacetimes sustained by phantom energy. In order to do this, we specify the equation of state of the radial pressure for the distribution of the energy density threading the wormhole by imposing on it a phantom equation of state of the form $p_r/\rho < -1$. It should be noted that for a quadratic shape function we have an equation of state $p_r/\rho$ with a variable character. We mainly focus our study on wormholes sustained by exotic matter with positive energy density, as seen by any static observer.

An important feature of the wormhole description with a quadratic shape function is that it includes phantom wormhole spacetimes extending to infinity, as well as static spacetimes of finite size, composed by a phantom wormhole connected to an inhomogeneous and anisotropic spherically symmetric distribution of dark energy. For latter wormhole types we can construct solutions with phantom matter confined to a finite region around the throat, which is connected to the dark energy distribution. The wormhole part does not fulfill the dominant energy condition, while the dark energy distribution part does.

Summarizing, in general for finite wormholes ($a_1 > 0$ and $a_3 > 0$) the exotic matter threading the phantom wormhole extends from the throat at $r_0$ to the sphere of radius $r_{max} = \sqrt{a_3/a_1}$, and the whole spacetime extends to the square of this $r_{max}$. The matter source of the gravitational configuration at $\sqrt{a_3/a_1} \leq r \leq a_3/a_1$ is of dark energy, so it satisfies DEC.

Finally, let us note that in general the spacetime (1) is not asymptotically flat. As a result, the matter distribution for wormholes extending to infinity must be cut off at some radius $r = r^* > r_0$ and joined to an exterior asymptotically flat spacetime, such as, for example, the vacuum Schwarzschild spacetime without cosmological constant (note that the studied wormholes (1) satisfy Einstein equations in the absence of cosmological constant). On the other hand, for wormholes with finite dimensions, in which the phantom matter distribution extends from the throat $r_0$ to the radius $r_{max} = \sqrt{a_3/a_1}$, and the dark energy distribution extends from $\sqrt{a_3/a_1}$ to $r = a_3/a_1$, the matching to the exterior vacuum Schwarzschild spacetime can be performed at $r^* = \sqrt{a_3/a_1} > r_0$ or $r^* = a_3/a_1 > r_0$. In other words, the discussed here wormhole spacetimes can be considered as an interior solution, which must be matched to an exterior solution, such as the Schwarzschild geometry, at some radius $r^* > r_0$.

The procedure of construction of traversable wormholes through matching an interior wormhole solution to the exterior Schwarzschild solutions is discussed by authors in Ref. [3]. In order to do this matching one must apply the junction conditions that follow from the theory of general relativity.

Due to the spherical symmetry of the spacetime, the components $g_{\theta \theta}$ and $g_{\phi \phi}$ are already continuous [3], so one needs to impose continuity only on the remaining metric components $g_{tt}$ and $g_{rr}$ at $r = r^*$, i.e.

$$g^W_{tt}(r^*) = g^{Schw}_{tt}(r^*),$$

$$g^W_{rr}(r^*) = g^{Schw}_{rr}(r^*).$$

These requirements, in turn, lead to following restrictions for the redshift and shape functions

$$\phi^W(r^*) = \phi^{Schw}(r^*),$$

$$b^W(r^*) = b^{Schw}(r^*).$$

In such a way, the exterior and interior solutions become identical at the sphere boundary $r = r^*$.

It is interesting to note that for spherically symmetric spacetimes, one can use directly the field equations to perform the match at the boundary $r^*$. Einstein equations allow us to determine the energy density and stresses of the surface $r = r^*$ necessary to have a match between the interior and exterior spacetimes. If there are no surface stress-energy terms at the surface $r^*$, the junction is called a boundary surface. On the other hand, if surface stress-energy terms are present, the junction is called a thin shell (see Lemos et al. [3] for a nice review of this issue).

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