INVARIANT METRICS ON THE COMPLEX ELLIPSOID

GUNHEE CHO

ABSTRACT. We provide a class of geometric convex domains on which the Carathéodory-Reiffen metric, the Bergman metric, the complete Kähler-Einstein metric of negative scalar curvature are uniformly equivalent, but not proportional to each other. As an application, we provide a necessary and sufficient condition for a taut complex manifold to be biholomorphic to a geometric convex domain.

1. INTRODUCTION AND RESULTS

Our paper is related to the very recent paper \cite{24} in which D. Wu and S.T. Yau show that for a simply connected complete Kähler manifold \((M, \omega)\) with negative Riemmanian sectional curvature range, the base Kähler metric \(\omega\) is uniformly equivalent to the Bergman metric, the complete Kähler-Einstein metric and the Kobayashi metric. Based on this result, it is reasonable to ask whether such \((M, \omega)\) must be biholomorphic to a bounded strongly pseudoconvex domain in \(\mathbb{C}^n\), since it is known that the metrics are uniformly equivalent to each other for these domains (see, for example, \cite{4}, \cite{6}, \cite{9}, \cite{11}, \cite{19}, \cite{26} and references therein). We will show that there are bounded weakly pseudoconvex domains in \(\mathbb{C}^n\) such that metrics are uniformly equivalent but never proportional to each other.

Let \(B_n\) be the unit disk in \(\mathbb{C}^n\). Denote by \(\gamma_{B_n}, \chi_{B_n}, g^B_{B_n}, g^{KE}_{B_n}\) the Carathéodory-Reiffen metric, the Kobayashi-Royden metric, the Bergman metric, and the complete Kähler-Einstein metric respectively. These metrics are all invariant under biholomorphism and we have

\[
\gamma_{B_n}(a; v) = \chi_{B_n}(a; v) < \sqrt{g^B_{B_n}((a; v), (a; v))} = \sqrt{g^{KE}_{B_n}((a; v), (a; v))} = c\gamma_{B_n}(a; v)
\]

(1.1)

for any non-zero tangent vector \((a; v)\) where \(c = c(n) > 0\). Hence if instead of the unit disk in \(\mathbb{C}^n\) one considers more general complex manifolds, the four metrics provide characterizations into several classes. For a class of pseudoconvex domains in \(\mathbb{C}^n\), one should expect that the relation among intrinsic metrics is different from (1.1) but that some common phenomenon can be captured.

One might regard convex domains in \(\mathbb{C}^n\) as generalizations of the unit disk in \(\mathbb{C}^n\). The classical Lempert’s theorem tell us that on a geometric convex domain \(\Omega\) in \(\mathbb{C}^n\), the Carathéodory-Reiffen metric \(\gamma_\Omega\) and the Kobayashi-Royden metric \(\chi_\Omega\) are equal and an analogous conclusion holds for the Carathéodory-distance \(c_\Omega\) and the Kobayashi-distance \(k_\Omega\) (e.g, see \cite{19} and \cite{25}). Hence for geometric convex domains,
the Bergman metric and the Kähler-Einstein metric in (1.1) can provide the information for weakly pseudoconvex domains. To study this situation, we will consider the complex ellipsoid \( E = E(m, n, p) := \{ (z, w) \in \mathbb{C}^m \times \mathbb{C}^n : |z|^2 + |w|^{2p} < 1 \} \).

In [22], B. Wong proved that any bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with noncompact automorphism group is biholomorphic to the unit ball \( B_n \) in \( \mathbb{C}^n \). For the weakly pseudoconvex case, E. Bedford and S. Pinchuk proved in [2] that any bounded pseudoconvex domain with real analytic boundary in \( \mathbb{C}^2 \) with noncompact automorphism group is biholomorphic to a complex ellipsoid \( E = E(1, 1, p) := \{ (z, w) \in \mathbb{C} \times \mathbb{C}^1 : |z|^2 + |w|^{2p} < 1 \} \). The geometric convexity of \( E \) when \( p > 1/2 \) implies that the Carathéodory-Reiffen metric and the Kobayashi-Royden metric are the same. However, \( E \) is not biholomorphic to the unit disk \( B_2 \) in \( \mathbb{C}^2 \), and \( E \) is \( C^2 \)-weakly pseudoconvex when \( p > 1 \) because of the special boundary points \( |z| = 1 \). Hence one may naturally ask for information on the Bergman metric and the complete Kähler-Einstein metric on weakly pseudoconvex domains \( E \).

In [10], S. Fu and B. Wong showed that for a simply-connected strictly pseudoconvex domain in \( \mathbb{C}^2 \) with smooth boundary, if the Bergman metric is Kähler-Einstein, then the domain must biholomorphic to the ball. This is also claimed to be true for \( \mathbb{C}^n \) (see, for example, [13]). Since the complex ellipsoid \( E = E(m, n, p) \) with \( p > 1 \) is a weakly pseudoconvex domain, those results do not imply the certain information of the Bergman metric and the complete Kähler-Einstein metric on \( E \). Thus it is natural to ask whether or not the Kobayashi-Royden metric, the Bergman metric and the Kähler-Einstein metric on \( E \) are uniformly equivalent or even equal to each other.

For any \( \lambda > 0 \), denote the complete Kähler-Einstein metric of the Ricci eigenvalue \( -\lambda \), the Bergman metric and the Kobayashi-Royden metric on \( E = E(m, n, p) \) by \( g_{KE}^E \), \( g_B^E \) and \( \chi_E \) respectively. In the next theorem results from [24] are applied to show that even though \( E(m, n, p) \) with \( p \geq 1 \) is a weakly pseudoconvex domain, the intrinsic metrics are uniformly equivalent to each other. Here are the results on the complex ellipsoid:

**Theorem 1.** Let \( E = E(m, n, p) \) for any \( m, n \in \mathbb{N} \) and \( p \geq 1 \). Then there exists \( C > 0 \) such that \( g_{KE}^E \), \( g_B^E \) and \( \chi_E \) are uniformly equivalent to each other by \( C > 0 \).

In the Theorem 1, \( C > 0 \) only depends on \( m, n \) and the negative holomorphic sectional curvature range of the W. Yin’s complete invariant Kähler metric in [29]. For the special case of \( E = E(m, 1, p) \), one can say that \( C > 0 \) only depends on \( m \) and the negative Riemmanian sectional curvature range of the Kähler-Einstein metric on \( E \). In [29], it was also shown that the there exists \( C > 0 \) satisfying \( \sqrt{g_B^E(v, v)} \leq C\chi_E(v) \) for any \( v \in TE \).

**Remark 1.** After the author submitted the paper on ArXiv, Prof. Yunhui Wu pointed out the Theorem 1 can be proved by an alternative approach, which combines Theorem 1 in [15] and Theorem 7.2 in [18] or Theorem 2 in [27]. On the other hand, Prof. Damin Wu informed me that the proof of Theorem 1 in this paper also achieve the following further consequence: for any closed complex submanifold \( S \) in \( E = E(m, n, p) \),
we restrict W. Yin’s complete invariant Kähler metric from $E$ to $S$, which still has the negative pinched holomorphic sectional curvature range on $S$. Hence by Theorem 2 and Theorem 3 in [24], we can extend the Theorem 1 from $E$ to $S$.

**Corollary 1.** Let $E = E(m, n, p)$ for any $m, n \in \mathbb{N}$ and $p \geq 1$. Then for any closed complex submanifold $S$ in $E$, there exists $C > 0$ such that $g_{S}^{KE}$ and $\chi_{S}$ are uniformly equivalent by $C > 0$.

Next two theorems show that invariant metrics are not proportional to each other.

**Theorem 2.** Let $E = E(1, 1, p) = \{(z, w) \in \mathbb{C}^{1} \times \mathbb{C}^{1} : |z|^{2} + |w|^{2p} < 1\}$ for any $p > 0$ but $p \neq 1$. Then $g_{E}^{B} \neq g_{E}^{KE}$.

For the proofs of the Theorem 1 and Theorem 2, the explicit Bergman kernel on $E$ is used which was obtained by J.P. D’Angelo (See [8]).

**Theorem 3.** Let $E = E(m, n, p)$ for any $m, n, p \in \mathbb{N}$ but $p \neq 1$. Denote $\gamma_{E}$ be the Carathéodory-Reiffen metric. Then

$$\gamma_{E}(a; v) < \sqrt{\lambda} \sqrt{g_{E}^{KE}((a; v), (a; v))}$$

for some nonzero tangent vector $(a; v)$.

Notice that

$$\gamma_{E}(a; v) \leq \sqrt{\lambda} \sqrt{g_{E}^{KE}((a; v), (a; v))}$$

for all nonzero tangent vector $(a; v)$ (1.2) is the consequence of the generalized Schwarz lemma (see [28]).

Combining the Theorems 1, 2, 3, we obtain:

**Corollary 2.** On the complex ellipsoid $E = E(1, 1, p) = \{(z, w) \in \mathbb{C}^{1} \times \mathbb{C}^{1} : |z|^{2} + |w|^{2p} < 1\}$, $1 \neq p \in \mathbb{N}$, there exists $C > 0$ such that the followings hold:

$$\gamma_{E}(a; v) = \chi_{E}(a; v)$$

for all tangent vectors $(a; v)$.

$$\frac{1}{C} \sqrt{g_{E}^{B}} < \gamma_{E} < C \sqrt{g_{E}^{B}}.$$  

$$\frac{1}{C} \sqrt{g_{E}^{KE}} < \gamma_{E} < C \sqrt{g_{E}^{KE}}.$$  

$$\frac{1}{C} g_{E}^{KE} < g_{E}^{B} < C g_{E}^{KE}.$$  

$$\gamma_{E}(a; v) < \sqrt{g_{E}^{B}((a; v), (a; v))}$$

for all nonzero tangent vectors $(a; v)$. (1.3)

$$\gamma_{E}(a; v) \neq \sqrt{\lambda} \sqrt{g_{E}^{B}((a; v), (a; v))}$$

for some nonzero tangent vector $(a; v)$. (1.4)

$$\gamma_{E}(a; v) \leq \sqrt{\lambda} \sqrt{g_{E}^{KE}((a; v), (a; v))}$$

for all nonzero tangent vector $(a; v)$.

$$\gamma_{E}(a; v) < \sqrt{\lambda} \sqrt{g_{E}^{KE}((a; v), (a; v))}$$

for some nonzero tangent vector $(a; v)$.

$$g_{E}^{B}((a; v), (a; v)) \neq g_{E}^{KE}((a; v), (a; v))$$

for some nonzero tangent vector $(a; v)$.
It was known that (1.3) holds for any bounded domains in \( \mathbb{C}^n \) (e.g., see [14]). The proof of (1.4) is given within the proof of Theorem 3.

The above consequence yields the following corollary:

**Corollary 3.** There is a simply connected, weakly (but not strongly) pseudoconvex domain in \( \mathbb{C}^2 \) with negative Riemannian sectional curvature range with respect to the Kähler-Einstein metric such that the Bergman metric, the Kähler-Einstein metric, the Kobayashi metric are uniformly equivalent but never proportional to each other.

**Remark 2.** Based on our result, it is still meaningful to ask the following question: given the simply connected complete Kähler manifold \((M, \omega)\) with negative Riemannian sectional curvature range, can we show that \((M, \omega)\) must be biholomorphic to a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \)? In this case, it is possible that the Carathéodory-Reiffen metric may be equivalent to other intrinsic metrics. However, the existence of the non-constant holomorphic function \(f: M \to \mathbb{D}\) is not yet proven. To address this issue, one might consider the \(L^2\)-estimate on complex manifolds (e.g., see [5].)

We are also interested in asking under what assumptions a complex manifold is biholomorphic to the complex ellipsoid. In the following theorem, we borrowed the same assumption on \(G\) from the theorem in [12], but we assumed the Carathéodory-isometry at one point instead of the Kobayashi-isometry.

**Theorem 4.** Let \(M^n\) be a connected taut complex manifold and let \(G\) be a bounded balanced pseudoconvex domain in \(\mathbb{C}^n\) with continuous Minkowski function. Assume that there exist a finite number of complex hyperplanes \(H_i\) through the origin such that every point of \(\partial G - \bigcup_i H_i\) is an extreme point for \(\partial G\). Let \(f: M \to G\) be a holomorphic map. Let \(a\) be a point of \(M\). Assume that \(f(a) = 0\) and that \(\gamma_M(a; v) = \gamma_G(f(a); df(v))\) for all \((a; v)\). Then \(M\) is biholomorphic to \(G\) and \(G\) is geometric convex if and only if \(\gamma_M(a; v) = \chi_M(a; v)\) for all \((a; v)\).

Interestingly, Theorem 4 tells us that under the Carathéodory-isometry assumption on the one point between a complex manifold \(M\) and a certain bounded balanced pseudoconvex domain, \(\gamma_M = \chi_M\) characterizes the geometric convexity of \(M\). In general however, \(\gamma_M(p; v) = \chi_M(p; v)\) for all \((p; v)\) on a complex manifold \(M\) does not imply that \(M\) is biholomorphic to a convex domain in \(\mathbb{C}^n\); a well-known counterexample is the symmetric bidisk \(G_2 = \{(z + w, zw): (z, w) \in \mathbb{D} \times \mathbb{D}\}\).

In Theorem 4, \(M\) is said to be taut if whenever \(N\) is a complex manifold and \(f_j: N \to M\) is a sequence of holomorphic maps, then either there exists a subsequence which is compactly divergent or a subsequence which converges uniformly on compact subsets to a holomorphic map \(f: N \to M\). If \(M\) is taut, then the infimum of (2.1) is always attained by some holomorphic map \(f\). Also a boundary point \(x\) of \(G\) in \(\mathbb{C}^n\) is said to be an extreme point for \(\overline{G}\) if there is no non-constant holomorphic map \(f: \mathbb{D} \to \overline{G}\) with \(x = f(0)\).

Finally, the complex ellipsoid \(E = E(m, n, p)\) for any \(m, n \in \mathbb{N}\) and \(p > 0\) satisfies the conditions for \(G\) in the Theorem 4. Thus we obtained the answer for our question:
Corollary 4. Let $M^n$ be a connected taut complex manifold. Let $f : M \to E$ be a holomorphic map. Let $a$ be a point of $M$. Assume that $f(a) = 0$ and that $\gamma_M(a; v) = \gamma_E(f(a); df(v))$ for all $(a; v)$. Then $\gamma_M(a; v) = \chi_M(a; v)$ for all $(a; v)$ if and only if $M$ is biholomorphic to $E$ and $p > 1/2$.

2. Necessary ingredients

In this section, we collect the necessary definitions that we use to prove our results.

Let $G$ be a domain in $\mathbb{C}^n$. A pseudometric $F(z, u) : G \times \mathbb{C}^n \to [0, \infty]$ on a domain $G$ in $\mathbb{C}^n$ is called (biholomorphically) invariant if $F(z, \lambda u) = |\lambda| F(z, u)$ for all $\lambda \in \mathbb{C}^n$, and $F(z, u) = F(f(z), f(z)u)$ for any biholomorphism $f : G \to G$. The Carathéodory-Reiffen metric, Kobayashi-Royden metric, Bergman, Kähler-Einstein metric of negative scalar curvature are examples of invariant metrics on bounded weakly pseudoconvex domains in $\mathbb{C}^n$.

Let $D$ denote the open unit disk in $\mathbb{C}$. Let $z \in G$ and $v \in T_z G$ a tangent vector at $z$. Define the Carathéodory-Reiffen metric by

$$\gamma_G(z; v) = \sup \{|df(z)v| : f \in \text{Hol}(G, D)\}.$$ 

The Kobayashi-Royden metric is defined by

$$\chi_G(z; v) = \inf \left\{ \frac{1}{\alpha} : \alpha > 0, f \in \text{Hol}(\mathbb{D}, G), f(0) = z, f'(0) = \alpha v \right\}. \quad (2.1)$$

Let $\rho_D(a, b)$ denotes the distance between two points $a, b \in \mathbb{D}$ with respect to the Poincaré metric of constant holomorphic sectional curvature $-4$.

The Carathéodory pseudo-distance $c_G$ on $G$ is defined by

$$c_G(x, y) := \sup_{f \in \text{Hol}(G, \mathbb{D})} \rho_D(f(x), f(y)).$$

Here, $\rho_D(a, b)$ denotes the integrated Poincare distance on the unit disk $\mathbb{D}$.

The Kobayashi pseudo-distance $k_G$ on $G$ is defined by

$$k_G(x, y) := \inf_{f_i \in \text{Hol}(\mathbb{D}, G)} \left\{ \sum_{i=1}^n \rho_D(a_i, b_i) \right\}$$

where $x = p_0, \ldots, p_n = y, f_i(a_i) = p_{i-1}, f_i(b_i) = p_i$.

The inner-Carathéodory length and the Kobayashi length of a piecewise $C^1$ curve $\sigma : [0, 1] \to G$ are given by

$$l^c(\sigma) := \int_0^1 \gamma_G(\sigma, \sigma')dt,$$

and

$$l^k(\sigma) := \int_0^1 \chi_G(\sigma, \sigma')dt.$$
respectively. The inner-Carathéodory pseudo-distance and the inner-Kobayashi pseudo-
distance on $G$ are defined by
\[ c_G^i(x, y) := \inf \{ l^i(\sigma)(x, y) \} \quad \text{and} \quad k_G^i(x, y) := \inf \{ l^k(\sigma)(x, y) \}, \]
where the infimums are taken over all piece-wise $C^1$ curves in $G$ joining $x$ and $y$.

The following relation is true in general:
\[ 0 \leq c_G \leq c_G^i \leq k_G^i = k_G. \quad (2.2) \]

Note that if $G$ is a bounded domain, then $k_G$, $c_G^i$, $c_G$ are non-degenerate and the
topology induced by these distances is the Euclidean topology.

For a bounded domain $G$ in $\mathbb{C}^n$, denote $A^2(G)$ the holomorphic functions in
$L^2(G)$. Let \{\varphi_j : j \in \mathbb{N}\} be an orthonormal basis for $A^2(G)$ with respect to the $L^2$-inner product. The Bergman kernel $K_G$ associated to $G$ is given by
\[ K_G(z, \bar{z}) = \sum_{j=1}^{\infty} \varphi_j(z) \bar{\varphi}_j(\bar{z}). \]

Note that $K_G$ does not depend on the choice of orthonormal basis, gives rise to an
invariant metric, the Bergman metric on $G$ as follows:
\[ g_B^G(\xi, \xi) = \sum_{\alpha, \beta=1}^{n} \partial^2 \log K_G(z, \bar{z}) \frac{\partial z_\alpha}{\partial \bar{z}_\beta} \frac{\partial z_\beta}{\partial \bar{z}_\alpha}. \]

We say a domain $G \subset \mathbb{C}^n$ is weakly pseudoconvex if $G$ has a continuous plurisubharmonic exhaustion function. In particular, every geometric convex set is a weakly pseudoconvex domain.

The existence of the complete Kähler-Einstein metric on a bounded pseudoconvex
domain was given in the main theorem in [20]. Based on this result, we can always find the unique complete Kähler-Einstein metric of the Ricci eigenvalue $-\lambda$ on a bounded weakly pseudoconvex domain $G$ for any $\lambda > 0$. i.e., $g^KE_G$ satisfies $g^KE_G = -\lambda \text{Ric}_{g^KE_G}$ as a two tensor.

Finally, the Minkowski function $h_G : \mathbb{C}^n \to \mathbb{R} > 0$ on the balanced domain $G$ in $\mathbb{C}^n$ is defined by
\[ h_G(X) := \inf \{ t > 0 : X/t \in G \}. \]
With $h_G$, $G$ can be written by
\[ G = \{ z \in \mathbb{C}^n : h_G(z) < 1 \}. \]
Note that $G$ is a geometric convex domain if and only if $h_G$ is a semi-norm. i.e.,
h_G(X + Y) \leq h_G(X) + h_G(Y)$ for any $X, Y \in \mathbb{C}^n$. If a $G$ is a balanced pseudoconvex
domain in $\mathbb{C}^n$, then $h_G$ is the plurisubharmonic function satisfying $h_G(\lambda z) = |\lambda|h_G(z)$ for $\lambda \in \mathbb{C}^n$.

**Remark 3.** For $E = \{(z, w) \in \mathbb{C}^2||z|^2 + |w|^{2p} < 1\}$, $p > 1/2$. In [1], $\chi_E(0; )$ is explicitly obtained:
\[ \chi_E(0; (u, v)) = \frac{1}{\mu}, \]
where \( \mu \) is the unique solution of \( \mu^2|u|^2 + \mu^2p|v|^{2p} = 1 \). Although \( h_E(.) = \gamma_E(0; .) = \chi_E(0; .) \) is also a semi-norm because of the geometric convexity of \( E \), the formula of \( \gamma_E(0; (u, v)) \) can be quite different from \( \gamma_{B_n}(0; (u, v)) = |(u, v)| \) in general. For instance, when \( p = 2 \), we can easily obtain the explicit Minkowski function:

\[
\gamma_E(0; (u, v)) = \chi_E(0; (u, v)) = h_E((u, v)) = \sqrt{\frac{2|v|^4}{(|u|^4 + 4|v|^4)^{1/2} - |u|^2}}.
\]

### 3. Proof of Theorem 1

**Proof of Theorem 1.** In the case of \( E = E(m, 1, p) \), It was shown that for \( p \geq 1 \), the Kähler-Einstein metric on \( E \) has Riemannian sectional curvature negative range (See the Theorem 4-(a) in [3]). From this information, we can immediately conclude that the Kähler-Einstein metric is uniformly equivalent to the Kobayashi-Royden metric and the Bergman metric by the Corollary 7 in [24]. We will consider the different approach to cover the cases of \( E = E(m, n, p) \) with \( n \neq 1 \). For this purpose, W. Yin’s complete invariant Kähler metric \( Y \) will be used. (see [29]).

From [8], the Bergman kernel for \( E = E(m, n, p) \) can be written as the following function:

\[
K_E((z,w),(z,w)) = F(X)(1 - |z|^2)^{-N},
\]

where \( X = X(z,w) = |w|^2(1 - |z|^2)^{-1/p}, \; N_1 = (n + 1)p + m, \; N = N_1/p, \; F(X) = \sum_{j=0}^{n+1} b_j(1 - X)^{-(m+j)}. \) Then it was shown that

\[
K((z,w),(z,w)) = (1 - X)^{-\lambda}(1 - |z|^2)^{-N}
\]

generates the complete invariant Kähler metric \( Y \) on \( E = E(m, n, p) \) satisfying \( Y \geq g_E^B \) if \( \lambda \) satisfies \( \lambda \geq \max\{m_1, m_2\} \), where

\[
m_1 = \max_{0 \leq X \leq 1} \frac{F'(X)(1-X)}{F(X)},
\]
\[
m_2 = \max_{0 \leq X \leq 1} \frac{|F(X)F''(X) - F(X)^2(1-X)^2|}{F(X)^2}.
\]

From the same paper, it was also shown that there exists \( C > 0 \) such that the holomorphic sectional curvature of \( Y \) on \( E \) is bounded above by \(-C\). Then by the generalized Schwarz lemma and the geometric convexity of \( E \),

\[
\sqrt{g_E^B(v)} \leq \sqrt{Y(v,v)} \leq C\chi_E(v) = C\gamma_E(v) \leq C\sqrt{g_E^B(v,v)} \tag{3.1}
\]

for any vector \( v \in TE \). We will show that there exists \( D > 0 \) such that the holomorphic sectional curvature of \( Y \) is bounded below by \(-D\). Then by the Theorem 3 in [24], \( Y \) must be equivalent to the \( g_E^{KE} \) and we get the desired result by (3.1).

Note that the holomorphic sectional curvature is invariant under the biholomorphic maps, and for any \((z,w) \in E \), there exists an automorphism \( f \) on \( E \) such that \( f(z,w) = (0,w') \). Thus it suffices to compute the holomorphic sectional curvature.
when \( z = 0 \). The formula of the holomorphic sectional curvature \( \omega([z,w],d(z,w)]_{z=0} \) of \( Y \) is explicitly given in [29]: for any \( D > 0 \),

\[
\omega_1([z,w],d(z,w)]_{z=0} - D - \frac{\omega_1([z,w],d(z,w)]}{[p^{-1}(XW' + N_1)|dz|^2 + W'|dw|^2 + W''|\bar{w}dw|^2]^2},
\]

where \( W' = \lambda(1 - X)^{-1}, \ W'' = \lambda(1 - X)^{-2}, \)

\[
\omega_1([z,w],d(z,w)] = P_1^*|\bar{w}dw|^4 + P_{12}^*|w\bar{w}|^2|dw|^2 + P_2^*|dw|^4
\]
\[+ Q_1^*|dz|^2|dw|^2 + + Q_2^*|w\bar{w}|^2|dz|^2 + R^*|dz|^4,
\]

\[
P_1^* = aN_1(1 - X)^{-4}(2 - DaN_1),
\]

\[
P_{12}^* = 2aN_1(1 - X)^{-3}(2 - DaN_1),
\]

\[
P_2^* = aN_1(1 - X)^{-2}(2 - DaN_1),
\]

\[
Q_1^* = 2p^{-1}aN_1(1 - X)^{-1}[(2 - DaN_1)(1 - X)^{-1} - DN_1(1 - a)],
\]

\[
Q_2^* = 2p^{-1}(XW' + N_1)^{-1}(1 - X)^{-2}aN_2[(1 - X)^{-2}(2a - Da^2N_1)
\]
\[+ (1 - X)^{-1}[4(1 - a) - 2DaN_1(1 - a)] - D(1 - a)^2N_1],
\]

\[
R^* = p^{-2}N_1a[(1 - X)^{-2}(2 - DaN_1) + (1 - X)^{-1}(p - 1)
\]
\[+ DN_1(a - 1)] + (1 - 1/a)[-2p - D(a - 1)N_1]
\]

with \( \lambda = aN_1 \). Thus if we can prove \( P_1^*, P_{12}^*, P_2^*, Q_1^*, Q_2^* \) and \( R^* \) are all non positive for some \( D > 0 \) then we obtain

\[
\omega([z,w],d(z,w)]_{z=0} \geq -D.
\]

Let \( y = (1 - X)^{-1} \), then \( X \in [0,1) \) implies \( y \in [1,\infty] \). Hence for \( P_1^*, P_{12}^* \) and \( P_2^* \), take \( D > 2(aN_1)^{-1} \).

For \( Q_1^* \), consider \( f(y) = (2 - DaN_1)y - DN_1(1 - a) \) from (3.2). If \( D \geq 2(aN_1)^{-1} \), \( f(y) \) is decreasing which has the maximum \( f(1) \) on \([1,\infty] \). Since \( f(1) \leq 0 \) if \( D \geq 2(N_1)^{-1} \), so take \( D \geq \max\{2(aN_1)^{-1}, 2(N_1)^{-1}\} \).

For \( Q_2^* \), consider \( f(y) = (2a - Da^2N_1)y^2 - 2(1 - a)(2 - DaN_1)y - D(1 - a^2)N_1 \) from (3.3). If \( D > 2(aN_1)^{-1} \), then \( f(y) \) has a maximum value \( f(y_0) = -2a^{-1}(a - 1)^2 < 0 \), where \( y_0 = 1 - 1/a \).

Finally, for \( R^* \), consider \( f(y) = (2 - DaN_1)y^2 + 2(p - 1 + DN_1(a - 1))y + (1 - 1/a)(-2p - D(a - 1)N_1) \). Then \( f'(y) = 2(2 - DaN_1)y + 2(p - 1 + DaN_1 - DN_1) \), thus \( f'(y) \) is decreasing on \([1,\infty] \) if \( D \geq 2(aN_1)^{-1} \). \( f'(1) = 2(p + 1 - DN_1) \leq 0 \) if \( D \geq (p + 1)/N_1 \), and \( f(1) = (1/a)(2p - DN_1) \leq 0 \) if \( D \geq (2p/N_1) \). Hence if \( D \geq \max\{2(aN_1)^{-1}, (p + 1)/N_1, (2p/N_1)\} \) then \( f(y) \leq 0 \).

In all, if \( D \geq \max\{2(aN_1)^{-1}, 2(N_1)^{-1}, (p + 1)/N_1, (2p/N_1)\} \), then \( P_1^*, P_{12}^*, P_2^*, Q_1^*, Q_2^* \) and \( R^* \) are all non positive and the proof is over.
Remark 4. We actually proved here is more stronger than we claimed in the theorem 1. We showed that $g_E^{K_E}, g^B_E, \chi_E$ and Yin’s complete invariant Kähler metric $Y$ are uniformly equivalent. In the case of $E = E(m, 1, p)$, by the Corollary 7 in [24], we have the same conclusion based on the negative range of the Riemannian sectional curvature of $g_E^{K_E}$ and the relation

$$\sqrt{g^B_E(v, v)} \leq \sqrt{Y(v, v)} \leq C \chi_E(v) = C g_E(v, v) \leq C \sqrt{g^B_E(v, v)}$$

for any vector $v \in TE$.

4. Proof of Theorem 2

Proof of Theorem 2. With the global coordinate $(z, w) \in E$ in $\mathbb{C}^2$, let $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\}$ be the basis on $T_{0}^{1,0}E$. In [8], the explicit Bergman kernel $K_E$ on $E$ is known:

$$K_E((z, w), (z, w)) = c_1 \frac{(1 - |z|^2)^{-2+\frac{2}{p}}}{((1 - |z|^2)^{1/p} - |w|^2)^2} + c_2 \frac{(1 - |z|^2)^{-2+\frac{2}{p}}}{((1 - |z|^2)^{1/p} - |w|^2)^3},$$

(4.1)

where $c_2, c_1$ are given by

$$c_1 = \frac{2}{p\pi^2}(p - 1), c_2 = \frac{2}{p\pi^2}.$$

To prove $g^B_E \neq g_E^{K_E}$, it is enough to show that $g^B_E$ is not Einstein, thus if there is no constant $c \in \mathbb{R}$ satisfying $
abla g_E^B((\frac{\partial}{\partial z}, \frac{\partial}{\partial w})) = c$ for any $((z, w), (z, w)) \in E$, then we are done. To show this, assume that

$$\frac{\text{Ric}(g^B_E)(\frac{\partial}{\partial z}, \frac{\partial}{\partial w})}{g^B_E((\frac{\partial}{\partial z}, \frac{\partial}{\partial w}))} = c$$

(4.2)

for some constant $c \in \mathbb{R}$. From

$$\frac{\partial}{\partial z} \frac{\partial}{\partial w} \text{det} g^B_E = -\frac{\partial^2 \log \text{det} g^B_E}{\partial z \partial \bar{z}}$$

$$= (\text{det} g^B_E)^{-1} \frac{\partial \text{det} g^B_E}{\partial z} \frac{\partial \text{det} g^B_E}{\partial \bar{z}} - (\text{det} g^B_E)^{-1} \frac{\partial^2 \text{det} g^B_E}{\partial z \partial \bar{z}},$$

(4.2) becomes

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \text{det} g^B_E - \text{det} g^B_E \frac{\partial^2 \text{det} g^B_E}{\partial z \partial \bar{z}} = c(\text{det} g^B_E)^2 \{K^{-1}_E \frac{\partial^2 K_E}{\partial z \partial \bar{z}} - K^{-2}_E \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial \bar{z}} \}.$$  

(4.3)

From

$$g^B_E = \left( \begin{array}{cc} K^{-1}_E \frac{\partial^2 K_E}{\partial z \partial \bar{z}} - K^{-2}_E \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial \bar{z}} & K^{-1}_E \frac{\partial^2 K_E}{\partial z \partial \bar{w}} - K^{-2}_E \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial \bar{w}} \\ K^{-1}_E \frac{\partial^2 K_E}{\partial w \partial \bar{z}} - K^{-2}_E \frac{\partial K_E}{\partial w} \frac{\partial K_E}{\partial \bar{z}} & K^{-1}_E \frac{\partial^2 K_E}{\partial w \partial \bar{w}} - K^{-2}_E \frac{\partial K_E}{\partial w} \frac{\partial K_E}{\partial \bar{w}} \end{array} \right),$$

we have

$$\text{det} g^B_E = (K^{-1}_E \frac{\partial^2 K_E}{\partial z \partial \bar{z}} - K^{-2}_E \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial \bar{z}})(K^{-1}_E \frac{\partial^2 K_E}{\partial z \partial \bar{w}} - K^{-2}_E \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial \bar{w}}).$$


\[-(K_E^{-1} \frac{\partial^2 K_E}{\partial z \partial \overline{w}} - K_E^{-2} \frac{\partial K_E}{\partial z} \frac{\partial K_E}{\partial w}) (K_E^{-1} \frac{\partial^2 K_E}{\partial w \partial \overline{z}} - K_E^{-2} \frac{\partial K_E}{\partial w} \frac{\partial K_E}{\partial \overline{z}})\]

Let us fix the element \((z, w), (\overline{z}, \overline{w})\) in \(E\) satisfying \(z = \overline{z}\) and \(w = \overline{w} = 0\). For the computation of \(\det g_E^B\), one can notice that the following terms are vanished because of the assumption \(w = \overline{w} = 0\):

\[
\frac{\partial^2 K_E}{\partial w \partial \overline{z}} = 0, \quad \frac{\partial K_E}{\partial w} = 0, \quad \frac{\partial K_E}{\partial \overline{w}} = 0, \quad \frac{\partial^2 K_E}{\partial z \partial \overline{w}} = 0,
\]

which gives a remedy for the ugly formula of derivatives of \(\det g_E^B\). From

\[K_E = (c_1 + c_2)(1 - z\overline{z})^{-2 - \frac{1}{p}},\]

the right hand side of (4.3) becomes

\[c(\det g_E^B)^2 (2 + \frac{1}{p})|z|^2 (1 - |z|^2)^{-2}. \quad (4.4)\]

Also one can compute

\[\det g_E^B = \frac{2c_1 + 3c_2}{c_1 + c_2} (2 + \frac{1}{p}) \{(1 - z\overline{z})^{3 - \frac{2}{p}} + z\overline{z}(1 - z\overline{z})^{-\frac{1}{p}}\}. \quad (4.5)\]

Hence one can see that (4.4) consist of following terms: \(z\overline{z}(1 - z\overline{z})^{4 - \frac{2}{p}}\), \((z\overline{z})^2 (1 - z\overline{z})^{-\frac{2}{p}}\) and \((z\overline{z})^3 (1 - z\overline{z})^{3 - \frac{2}{p}}\).

On the other hand, in the left hand side of (4.3), one can check \(\frac{\partial \det g_E^B}{\partial z} \frac{\partial \det g_E^B}{\partial \overline{z}}\) in (4.3) only contains \(z\overline{z}(1 - z\overline{z})^{4 - \frac{2}{p}}\), \((z\overline{z})^2 (1 - z\overline{z})^{3 - \frac{2}{p}}\) and \((z\overline{z})^3 (1 - z\overline{z})^{3 - \frac{2}{p}}\).

For \(\det g_E^B \frac{\partial^2 \det g_E^B}{\partial z \partial \overline{z}}\), the second order derivative contains the different term compare to previous two cases: \(\det g_E^B \frac{\partial^2 \det g_E^B}{\partial z \partial \overline{z}}\) contains \(z\overline{z}(1 - z\overline{z})^{4 - \frac{2}{p}}\), \((z\overline{z})^2 (1 - z\overline{z})^{3 - \frac{2}{p}}\) and \((z\overline{z})^3 (1 - z\overline{z})^{3 - \frac{2}{p}}\), but also contains the additional term \((1 - z\overline{z})^5 - \frac{2}{p}\) because \(\frac{\partial^2 \det g_E^B}{\partial z \partial \overline{z}}\) generates \((1 - z\overline{z})^5 - \frac{2}{p}\) by the second order derivative. One can check that the coefficient of \((1 - z\overline{z})^5 - \frac{2}{p}\) is given by \(-\frac{2c_1 + 3c_2}{c_1 + c_2}(2 + \frac{1}{p})^3\) which is non-zero. Thus by dividing each side of (4.3) by \(-\frac{2c_1 + 3c_2}{c_1 + c_2}(2 + \frac{1}{p})^3\) \((1 - z\overline{z})^5 - \frac{2}{p}\) and letting \(z \to 1\), then (4.3) becomes \(1 = \pm\infty\), which is a contradiction and the proof is over.

\[\square\]

Remark 5. One can see that it is possible to get the explicit formula of (4.3) for arbitrary \((z, w), (\overline{z}, \overline{w})\) in \(E\) by using (4.1) to get the conclusion, but this approach might be hard to enlighten the mind.
5. Proof of Theorem 3

Proof of Theorem 3. We will proceed with the two different proofs. The first proof is by using the Theorem 2 in [22]: for $G$ be a simply-connected bounded domain in $\mathbb{C}^n$, assume that $G$ is complete hyperbolic and $\gamma_G = \chi_G$, and they are smooth hermitian metrics in the usual sense of differential geometry. Then $G$ is biholomorphic to the unit ball. Hence if we assume $\gamma_E(a, \cdot) = \sqrt{\lambda} \sqrt{g_E^{KE}(\cdot, \cdot)}$, then $E$ must be biholomorphic to the unit disk, which is impossible when $p \neq 1$. Also, the same argument works for the $g^P_E$, the proof of (1.4) is also established. We want to consider the second proof because this alternative argument would be useful for many other purposes. The similar argument below was considered to get the distance comparison on bounded weakly pseudoconvex domains but not geometric convex, see [7].

Step 1, we will establish the basic settings for the proof. We will assume that $n = m = 1$ and $p = 2$. For arbitrary $(p, n, m)$, the proof works in the same way. For each $r > 0$, define

$$E_r := \{(z, w) \in \mathbb{C}^1 \times \mathbb{C}^1 : |z|^2 + r|w|^4 < 1\}.$$  

It’s obvious that $E_r$ is biholomorphic to $E$ for any $r > 0$. With the global coordinate $(z_1, z_2) \in E_r$ in $\mathbb{C}^2$, let $\{\frac{\partial}{\partial z_i}^2, i = 1, 2\}$ be the basis on $T^1_0 E_r$. Without loss of generality, we may assume that $\{\frac{\partial}{\partial z_1}^2, i = 1, 2\}$ are orthonormal with respect to the Euclidean metric and orthogonal with respect to $g_{E_r}^{KE}(0)$. From now on, with the usual global coordinates $(z_1, z_2) \in E_r$, denote $\frac{\partial}{\partial z_1} = X_1, \frac{\partial}{\partial z_2} = X_2$. Also, we may assume that $0 < g_{E_r}^{KE}(0)(X_1, \overline{X_1}) \leq g_{E_r}^{KE}(0)(X_2, \overline{X_2})$.

For $z = (z_1, z_2)$, define $g(z) := z_1^2 + rz_2^4 \in \text{Hol}(E_r, \mathbb{D})$. Then by the distance-decreasing property, $c_{\mathbb{D}}(g(0, 0), g(x, x)) \leq c_{G_2}((0, 0), (x, x))$, so we get

$$c_{\mathbb{D}}(0, |x|^2 + r|x|^4) \leq c_{E_r}((0, 0), (x, x)).$$  \hspace{1cm} (5.1)

Step 2, We will see that for each $0 < \delta < 1$, there exists $(x, x) \in E_r$ such that

$$x(x + rx^3) < 1, \delta(x + rx^3) > 2.$$  

for some $r > 0$. These inequalities are equivalent to

$$\frac{2}{\delta} < rx^3 + x < \frac{1}{x}.  \hspace{1cm} (5.2)$$

These inequalities can be obtained since $\delta$ is fixed and we can control $x$ which can go to zero and also $r > 0$ if necessary. With $(x, x) \in E_r$, we will control the extremal map $f : E_r \to \mathbb{D}$ with respect to the Carathéodory-distance such that

$$c_{E_r}((0, 0), (x, x)) = c_{\mathbb{D}}(f(0, 0), f(x, x)).$$  \hspace{1cm} (5.3)

After acting on the unit disk by an automorphism, we may assume that $f(0, 0) = 0$. Also we can replace $f$ by the symmetrization map $\frac{1}{2}(f(x, y) + f(y, x))$. Around the origin, one can write $f$ as a power series $\sum_{i=1}^2 a_i z_i + \sum_{p=m}^\infty P_m(z_1, z_2) = \sum_{i=1}^2 a_i z_i + f_d(x, y)$, where each $P_m(z_1, z_2)$ is a homogeneous polynomial of degree $m$. Since we use $\frac{1}{2}(f(x, y) + f(y, x))$, we can replace $a_1 z_1 + a_2 z_2$ by $a(z_1 + z_2)$ for some real
number $a$ (after acting on the unit disk by an automorphism if necessary). Then by the generalized Schwarz Lemma in [28], $f^*\gamma_\partial \leq \lambda g_{E_r}^{KE}$. In particular, we get

$$a \leq \sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}.$$  

Step 3. Restrict the Carathéodory-extremal map $f$ in step 2 to $\delta \mathbb{D}$ with respect to the first coordinate $z_1$ (i.e., make $w = 0$), gives the map $f : \delta \mathbb{D} \rightarrow \mathbb{D}$, and the origin maps to origin by this restriction map $f$. Then the Classical-Schwarz lemma implies $a \delta \leq 1$ and with (5.2), this implies we have $(x, x) \in E_r$ such that

$$(x + rx^3) > \frac{2}{\delta} \geq 2a.$$  

(5.4)

Step 4, we will show there exists a hypersurface of $E_r$ given by

$$\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(X_1, \overline{X_1})(0)(z_1 + z_2) = 1.}$$

To do so, consider a projection map, i.e., we can define the holomorphic map from $(E_r, g_{E_r}^{KE})$ to $(\mathbb{D}, h)$, where $h$ is the Poincaré metric of the constant Gaussian curvature $-1$ on $\mathbb{D}$. Then by the generalized Schwarz lemma, $h(0) \leq \lambda g_{E_r}^{KE}(0)$. This implies

$$1 \leq \sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1}).}$$  

(5.5)

This implies we can take $(z_1, z_2) \in E_r$ which satisfies $\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(X_1, \overline{X_1}(0)(z_1 + z_2) = 1$, because $z_1, z_2$ can be arbitrary small near the origin in $E_r$.

Since there exists a hypersurface of $E_r$ given by $\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(X_1, \overline{X_1}(0)(z_1 + z_2) = 1$, thus we can choose $(u_1, u_2) \in E_r$ such that $u_1 + u_2 = \frac{1}{\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}}$. Define

$$g \in \mathbb{D} \rightarrow E_r by t \mapsto (tu_1, tu_2).$$

Then $f \circ g(0) = 0$ and $(f \circ g)'(0) = 1$. Thus $f \circ g = id_\mathbb{D}$ by the classical Schwarz’s lemma. This gives $f_2(tu_1, tu_2) = 0$. Since $t \in \mathbb{D}$ is arbitrary and with other choices of $(u_1, u_2)$, we can take a small open set in $E_r$ such that $f_2 = 0$ on this open set. Thus by the identity theorem, $f_2 \equiv 0$ on $E_r$. Hence $f(z_1, z_2) = a(z_1 + z_2)$. Then by (5.1) and (5.3),

$$c_\mathbb{D}(0, 2\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}x) = c_{E_r}((0, 0), (x, x)) \geq c_\mathbb{D}(0, x^2 + rx^4).$$

Thus we obtain $x + rx^3 \leq 2\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}$. On the other hand, from (5.4), we also have $x + rx^3 > 2a = 2\sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}$, which is impossible. Hence

$$a < \sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}.$$  

Step 6, $a < \sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_1, \overline{X_1})}$ implies

$$|f'(0)X_i| < \sqrt{\lambda} \sqrt{g_{E_r}^{KE}(0)(X_i, \overline{X_i})}, i = 1, 2.$$  

(5.6)
Since we showed the above inequality for $X_1$ and $X_2$, which are orthogonal to the Euclidean metric and $g^{KE}_E(0)$ at the same time, the same inequality holds for arbitrary tangent vectors $X \in \mathbb{C}^n - \{0\}$. Consequently, this implies the assumption in Lemma 1, thus $c_E(0, (x, x)) < \sqrt{\lambda d^{KE}_E(0, (x, x))}$. Since $E$ and $E_r$ are biholomorphic, we get $c_E(0,.) \neq \sqrt{\lambda d^{KE}_E(0,.)}$.

Step 7, we will finish the proof. Suppose $\gamma_E(a; v) = \sqrt{\lambda} \sqrt{g^{KE}_E((a; v), (a; v))}$ for all nonzero tangent vector $(a; v)$. Then we have

$$c^i_E = \sqrt{\lambda} d^{KE}_E.$$

Since $E$ is geometric convex, the Carathéodory pseudo-distance becomes inner. This implies $c_E(0,.) = c^i_E(0,.) = \sqrt{\lambda d^{KE}_E(0,.)}$, which contradicts the step 6 and the proof is over.

\begin{lemma}
Let $G$ be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$, and $a, b \in G$, $a \neq b$. For any $\lambda > 0$, denote the complete Kähler-Einstein metric on $G$ by $g^{KE}_G$, and let $d^{KE}_G$ be the distance induced by $g^{KE}_G$. Suppose there exists $f \in \text{Hol}(M, \mathbb{D})$ that is extremal for $c_M(a, b)$ such that

$$\gamma_D(f(a), f'(a)X) < \sqrt{\lambda} \sqrt{g^{KE}_G(a)(X, X)}, \text{ for any } X \in \mathbb{C}^n - \{0\}.$$ 

Then

$$c_M(a, b) < \sqrt{\lambda} d^{KE}_M(a, b).$$

\end{lemma}

\begin{proof}
By assumption, continuity of metrics gives $\epsilon, \delta$ such that

$$\gamma_D(f(z), f'(z)X) + \epsilon \|X\| \leq \sqrt{\lambda} \sqrt{g^{KE}_G(z)(X, X)}, \text{ for all } z \in \mathbb{B}(a, \delta) \subseteq G, X \in \mathbb{C}^n.$$ 

On the other hand, by the generalized Schwarz lemma, we have

$$\gamma_D(f(z), f'(z)X) \leq \sqrt{\lambda} \sqrt{g^{KE}_G(z)(X, X)}, \text{ for all } z \in G, X \in \mathbb{C}^n.$$ 

Let $\alpha : [0, 1] \rightarrow G$ be any piecewise $C^1$-curve joining $a$ and $b$. Denote by $t_0$ the maximal $t \in [0, 1]$ such that $\alpha([0, t]) \subset \overline{\mathbb{B}(a, \delta)}$. Denote the arc-length of $\alpha$ with respect to the Kähler-Einstein metric by $L_{KE}(\alpha)$. Then

$$\sqrt{\lambda} L_{KE}(\alpha) = \int_0^{t_0} \sqrt{\lambda} \sqrt{g^{KE}_G(\alpha(t))(\alpha'(t), \alpha'(t))} dt + \int_{t_0}^1 \sqrt{\lambda} \sqrt{g^{KE}_G(\alpha(t))(\alpha'(t), \alpha'(t))} dt$$

$$\geq \int_0^1 \gamma_D(f(\alpha(t)), f'(\alpha(t))dt + \epsilon \int_0^{t_0} \|\alpha'(t)\|dt$$

$$\geq L_{\gamma_D}(f \circ \alpha) + \epsilon \delta \geq \rho_D(f(a), f(b)) + \epsilon \delta = c_G(a, b) + \epsilon \delta.$$ 

Hence $\sqrt{\lambda} d^{KE}_G(a, b) \geq c_G(a, b) + \epsilon \delta$.

\end{proof}
6. Proof of Theorem 4

Proof of Theorem 4. If $\gamma_M(p; v) \neq \chi_M(p; v)$ for some $(p; v)$, it is clear that $M$ can’t be biholomorphic to a convex domain from the Lempert’s theorem. Conversely, suppose that $\gamma_M(p; v) = \chi_M(p; v)$ for all $(p; v)$. Then

$$\gamma_G(0; df(v)) = \gamma_M(p; v) = \chi_M(p; v) \geq \chi_G(0; df(v)).$$

(6.1)

Hence $\chi_M(p; v) = \chi_G(f(p); df(v))$ for all $(p; v)$. By Theorem 1 in [12], $f$ is a biholomorphism and $\gamma_G(0; ) = \chi_G(0; )$. From the proposition 3.5.3 in [14], $\chi_G(0; .) = h_G(.)$, where $h_G(.)$ is the Minkowki function. Since it is clear that $\gamma_G(0; X + Y) \leq \gamma_G(0; X) + \gamma_G(0; Y)$ for any $X, Y \in \mathbb{C}^n$ from the definition of $\gamma_G$, $h_G$ is a semi-norm. This implies that $G$ should be a convex domain and we are done.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, 196 AUDITORIUM ROAD, STORRS, CT 06269-3009, USA

E-mail address: gunhee.cho@uconn.edu