Area spectrum and thermodynamics of KS black holes in Hořava gravity

Abstract We investigate the area spectrum of Kehagias-Sfetsos black hole in Hořava-Lifshitz gravity via modified adiabatic invariant $I = \oint p_i dq_i$ and Bohr-Sommerfeld quantization rule. We find that the area spectrum is equally spaced with a spacing of $\Delta A = 4\pi l^2_p$. We have also studied the thermodynamic behavior of KS black hole by deriving different thermodynamic quantities.

Keywords Hořava gravity · Black holes · Adiabatic invariant · Entropy spectrum · Area spectrum

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1 Introduction

Hořava proposed a renormalizable theory of gravity in four dimensions. This is a non relativistic theory of gravity, which can be considered as a candidate for General Relativity at UV scale [3–5]. Various aspects and solutions of this theory have been studied by many. By introducing a dynamical parameter $\lambda$ in asymptotically Lifshitz spacetimes, a spherically symmetric black hole solution was first given by Lü, Mei, and Pope [6]. Cai, Cao, and Ohta studied the general topological black holes in [7]. In addition, Kehagias and

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Sfetsos have found a black hole solution in asymptotically flat spacetimes by choosing the dynamical parameter $\lambda = 1$ and introducing another parameter $\omega$ \cite{8}. Park obtained a black hole solution for arbitrary values of the parameter $\omega$ and the cosmological constant $\Lambda_w$, with $\lambda = 1$ \cite{9}. There are several studies on the thermodynamic and cosmological properties of black holes in Horava-Lifshitz (HL) gravity theory \cite{9,10,11,12,13,14,15,16,18,19}. In this paper, we investigate the area spectrum of a black hole in HL gravity.

It is widely believed that horizon area of a black hole is to be quantized. In 1974 Bekenstein proposed that the black hole area is equally spaced \cite{1} and the discrete spectrum was obtained as

$$A_n = 8\pi n l_p^2,$$  \hspace{1cm} (1)

where $l_p$ is the Planck length and $n = 1, 2, \ldots$. Bekenstein also proved \cite{2} that the black hole horizon area spectrum has a minimal spacing of $8\pi l_p^2$. A method to quantize the horizon area was put forward by Hod \cite{21,22} in which the quasinormal modes (QNMs) were used. Considering both quasinormal mode frequency and Bohr’s correspondence principle he found that the area spectrum is related to the real part of QNM. Later Kunstatter \cite{23,24} furthered this idea to find, the action $I = \int \frac{dE}{\omega_R}$ is an adiabatic invariant, where $\omega_R$ is the real part of QNM frequencies, which leads to an equally spaced area spectrum. Maggiore provided a new interpretation \cite{25,26,27} in which he proved that the physical frequency of QNM is determined by its real and imaginary parts and he derived the area spectrum which is in consistence with that of Bekenstein. Using these ideas, there are several studies on the area spectrum of different kinds of black holes \cite{28,29,30}.

Recently Majhi and Vagenas \cite{31} proposed another method to quantize the horizon area without QNMs. In their work, adiabatic invariant quantity connects to the Hamiltonian of the black hole and using both Hamiltonian and Bohr-Sommerfeld quantization rule, they derived the equally spaced entropy spectrum of a spherically symmetric black hole. According to Jiang and Han \cite{32}, the adiabatic invariant quantity $\int p_i dq_i$, used in \cite{31} is not canonically invariant and they suggested that by using the adiabatic invariant quantity of the covariant form $I = \oint p_i dq_i$, one can quantize the horizon area of a spherically symmetric black hole.

In this paper we adopt the methods suggested by Majhi and Vagenas and Jiang and Han to quantize the horizon area. Therefore, by introducing the adiabatic invariant and using Bohr-Sommerfeld quantization rule, we determine the entropy spectrum and the area spectrum of a Kehagias-Sfetsos (KS) black hole in HL gravity.
The rest of this paper is organized as follows. In Sect. 2, we investigate the thermodynamics of KS black hole in HL gravity. The phase transition and stability of the black hole are discussed. In Sect. 3, the entropy and horizon area quantization via adiabatic invariant and Bohr-Sommerfeld quantization rule are studied. Finally, Sect. 4 ends up with a brief discussion and conclusion.

2 HL gravity and Thermodynamics

Hořava has used the ADM formalism in which the four dimensional metric of general relativity is parametrized as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt),$$  \hspace{1cm} (2)

where $N$ and $N^i$ denote the lapse and shift functions, respectively. The Einstein-Hilbert action is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \, N \left\{ (K_{ij}K^{ij} - K^2) + R - 2\Lambda \right\},$$ \hspace{1cm} (3)

where $G$ is Newton’s gravitational constant, $R$ is the curvature scalar and $K_{ij}$ is the extrinsic curvature that takes the form,

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i),$$ \hspace{1cm} (4)

here the dot denotes differentiation with respect to the time coordinate $t$. The action of non-relativistic renormalizable gravitational theory proposed by Hořava can be written as [3]

$$I = \int dt \, d^3x \sqrt{g} \, N \left( \mathcal{L}_0 + \mathcal{L}_1 \right),$$ \hspace{1cm} (5)

where

$$\mathcal{L}_0 = \left\{ \frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (AR - 3A^2)}{8 (1 - 3\lambda)} \right\}$$ \hspace{1cm} (6)

and

$$\mathcal{L}_1 = \left\{ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32 (1 - 3\lambda)} R^2 - \frac{\kappa^2}{2\omega^4} Z_{ij}Z^{ij} \right\}.$$ \hspace{1cm} (7)

Here,

$$Z_{ij} = C_{ij} - \frac{\mu}{2} \frac{\omega^2}{2} R_{ij},$$ \hspace{1cm} (8)

in which $C_{ij}$ is the Cotton tensor and it has the form

$$C^{ij} = \epsilon^{ikl} \nabla_k \left( R^l_i - \frac{1}{4} R \delta^l_i \right) = \epsilon^{ikl} \nabla_k R^l_i - \frac{1}{4} \epsilon^{ikl} \partial_k R,$$ \hspace{1cm} (9)

where $\kappa^2$, $\mu$, $\omega$, $\lambda$ and $A$ are constants. For spherically symmetric solution of HL gravity, let us consider the line element
\[ ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \] (10)

Substituting this metric ansatz in Eq.(5) and after angular integration, the Lagrangian will reduce to

\[ \tilde{\mathcal{L}} = \frac{\kappa^2 \mu^2 N}{8(1-3\lambda)} \sqrt{f} \left\{ \frac{\lambda - 1}{2} f'^2 + \frac{(2\lambda - 1)(f - 1)^2}{r^2} - \frac{2\lambda(f - 1)}{r} f' - 2\omega (1 - f - rf') \right\}, \] (11)

where

\[ \omega = \frac{8\mu^2 (3\lambda - 1)}{\kappa^2}. \] (12)

By giving \( \lambda = 1 \) and solving the field equations obtained from Eq.(11), we can arrive at the KS solution \[ f_{KS} = \frac{N_{KS}^2}{N} = 1 + \omega r - \sqrt{r(\omega^2 r^3 + 4\omega M)}. \] (13)

Now we will investigate the thermodynamic aspects of KS black hole. In this section all quantities are expressed in the Planck units \( (c = \bar{G} = \hbar = 1) \). From the condition \( f_{KS}(r_{\pm}) = 0 \), the outer and inner horizons are given by,

\[ r_{\pm} = M \pm \sqrt{M^2 - \frac{1}{2\omega}}. \] (14)

By considering \( r_{+} \) from (14) we can establish a connection between mass of the black hole and its horizon radius as,

\[ M = \frac{r_{+}}{2} + \frac{1}{4\omega r_{+}}. \] (15)

From Bekenstein-Hawking area law, we can write

\[ S = \frac{A}{4} = \pi r_{+}^2. \] (16)

Hence, the horizon radius \( r_{+} \) can be written in terms of entropy as,

\[ r_{+} = \sqrt{\frac{S}{\pi}}. \] (17)

Therefore, we can rewrite the mass-horizon radius (15) as

\[ M = \frac{1}{4\omega} \sqrt{\frac{\pi}{S}} + \frac{1}{2} \sqrt{\frac{S}{\pi}}. \] (18)

As it is depicted in Fig.1, the two horizons of the black hole merge at the point \( r = r_{e} = 0.7 \) (for \( \omega = 1 \)) [17]. The same behavior is repeated for other
values of $\omega$ with slight changes in $r_e$ values. Thermodynamic quantities such as temperature and specific heat are defined respectively as,

$$T = \left(\frac{\partial M}{\partial S}\right)$$ \hspace{1em} (19)

$$C = T \left(\frac{\partial S}{\partial T}\right).$$ \hspace{1em} (20)

Then, from these equations, we can have the black hole temperature as

$$T = \frac{1}{4\sqrt{\pi S}} - \frac{\sqrt{\pi}}{8\omega S^2}$$ \hspace{1em} (21)

and the heat capacity of the black hole as,

$$C = -\left(\frac{4\omega S^2 - 2\pi S}{2\omega S - 3\pi}\right).$$ \hspace{1em} (22)
In Fig 2, variation of temperature with respect to entropy is plotted while in Fig 3, the variation of heat capacity with respect to entropy for the different values of coupling parameter $\omega$ is plotted. In Fig 3, there is a discontinuity in the plot, which shows that black hole may undergo a phase transition. Heat capacity is an important thermodynamic quantity because from that we can tell about the stability of the black hole. From Eq. (22) it is evident that the heat capacity is positive for a range $\frac{\pi}{2}\omega < S < \frac{3\pi}{2}\omega$. Hence, a KS black hole is stable for this range of values of $S$.

### 3 Area spectrum of KS black hole

In this part of our work, entropy and horizon area of the KS black hole are quantized via the adiabatic invariance and the Bohr-Sommerfeld quantization rule. By considering the properties of black hole such as adiabaticity and oscillating velocity of black hole horizon, we can write the action as

$$I = \oint p_i dq_i = \int_{q_i^{\text{out}}}^{q_i^{\text{in}}} p_i^{\text{in}} dq_i + \int_{q_i^{\text{out}}}^{q_i^{\text{in}}} p_i^{\text{out}} dq_i. \quad (23)$$

Here $p_i^{\text{in}}$ or $p_i^{\text{out}}$ is the conjugate momentum corresponding to the coordinate $q_i^{\text{in}}$ or $q_i^{\text{out}}$, respectively, and $i = 0, 1, 2...$ It is also to be considered that $q_i^{\text{in}} = r_i^{\text{in}} (q_i^{\text{out}} = r_i^{\text{out}})$ and $q_0^{\text{in}} (q_0^{\text{out}}) = \tau$ where $r_i$ is the horizon radius and $\tau$ is the Euclidean time with a periodicity $\frac{2\pi}{\kappa}$ in which $\kappa$ is the surface gravity which is given by,

$$\kappa = \frac{1}{2} \frac{df_{KS}}{dr} |_{r_h}. \quad (24)$$

Considering the Hamilton equation $\dot{q}_i = \frac{dH}{dp_i}$, where $H$ is the Hamiltonian of the system, the integral given by Eq. (23), adiabatic covariant action can be evaluated by considering the contour integration over a closed path from $q_i^{\text{out}}$ (outside the event horizon) to $q_i^{\text{in}}$ (inside the event horizon).

The action given by Eq. (23) can be written as,

$$\int_{q_i^{\text{out}}}^{q_i^{\text{in}}} p_i^{\text{out}} dq_i = \int_{r_{\text{out}}}^{r_{\text{in}}} \int_0^H dH' dr + \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h = 2 \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h . \quad (25)$$

where $r_{\text{out}}$ and $r_{\text{in}}$ denote the horizon location before and after shrinking and $\dot{r}_h = \frac{dr_h}{d\tau}$ is the oscillating velocity of black hole horizon. From the tunneling picture, it is evident that when a particle tunnels in or out, the black hole horizon will shrink or expand due to the loss or gain of black hole mass. Since tunneling and oscillation take place at the same time we can write,

$$\dot{r}_h = - \dot{r} . \quad (26)$$

where $\dot{r}$ is the velocity of tunneling particle. Since the two contour integrals in the Eq. (25) are equal we can write it as,

$$\oint p_i dq_i = 4 \int_{r_{\text{out}}}^{r_{\text{in}}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h . \quad (27)$$
To evaluate this adiabatic invariant quantity for the black hole in this discussion, let us consider the static spherically symmetric spacetime given by the line element (10).

\[ ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \] (28)

To euclideanize this metric, we consider the transformation in time coordinate \( t \to -i\tau \). Hence,

\[ ds^2 = N(r)^2 d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \] (29)

Let a photon travel across the black hole horizon, then from the radial null path, i.e., \( ds^2 = d\Omega^2 = 0 \), we can write

\[ \dot{r} = \pm i \sqrt{N(r)^2 f(r)}. \] (30)

From here onwards, our discussion will focus on the outgoing paths. From Eq. (13) it is evident that \( f_{KS} = N_{KS}^2(r) \), and thus, from Eq. (30) we get

\[ \dot{r} = +if_{KS}(r). \] (31)

The adiabatic invariant will be,

\[ \oint p_i dq_i = -4i \int_{r_{in}}^{r_{out}} \int_0^H \frac{dH'}{f_{KS}(r)} dr. \] (32)

Using the near horizon approximation, \( f_{KS}(r) \) can be Taylor expanded to get,

\[ f_{KS}(r) = f_{KS}(r) \bigg|_{r_h} + (r - r_h) \frac{df_{KS}(r)}{dr} \bigg|_{r_h} + \cdots \] (33)

Since there is a pole at horizon \( r_h \), we can consider a contour integral over a half loop going above the pole from right to left. Using the Cauchy’s theorem, we can evaluate the integral in Eq. (32) to get

\[ \oint p_i dq_i = 4\pi H \int_0^H \frac{dH'}{\kappa} = 2\hbar H \int_0^H \frac{dH'}{T}, \] (34)

where we have used the relation \( T = \frac{\hbar \kappa}{2\pi} \) to connect temperature of the black hole with the surface gravity. According to first law of thermodynamics of black hole we can have,

\[ dH' = T dS. \] (35)

Therefore,

\[ \oint p_i dq_i = 2\hbar S. \] (36)

From Bohr-Sommerfeld quantization

\[ \oint p_i dq_i = 2\pi n\hbar, \quad n = 1, 2, 3, \cdots \] (37)
the entropy spectrum is given by
\[ S = n\pi, \quad n = 1, 2, 3, \ldots \] (38)

So the black hole entropy is quantized with a spacing of the entropy spectrum given by
\[ \Delta S = S_{(n+1)} - S_{(n)} = \pi. \] (39)

From Bekenstein-Hawking entropy relation [2], area spectrum can be found to be spaced as
\[ \Delta A = 4\pi l_p^2. \] (40)

Thus, we see that both entropy and area spectra of KS black hole are quantized and are equally spaced and they are independent of the black hole parameters.

4 Discussion and conclusion

In this paper we have studied the quantization of entropy and horizon area of KS black hole in HL gravity using the method put forward by Majhi and Vagenas, which was later modified by Jiang and Han. The entropy and the area spectra are derived using adiabatic invariant and Bohr-Sommerfeld quantization rule and we have showed that both entropy and area spectra are equally spaced with a spacing of \( \Delta S = \pi \) and \( \Delta A = 4\pi l_p^2 \) respectively. Even though the values of equispacing obtained in the present study are different from the values obtained using QNMs approach for LMP black holes [33] and KS black holes [34] in HL theory, the equispaced property is maintained and their order of magnitudes are the same. Majhi [33] has used tunneling mechanism and QNMs to study the entropy spectrum of KS black hole and found that though the entropy is quantized, the magnitude of equispacing obtained is different for the two methods. The results of the present study agrees with the result obtained through tunneling mechanism in [33]. The discrepancy in the results may due to the fact that these methods are semi-classical. It is also found that the area and entropy spectra do not depend on the black hole parameters. We have also studied the thermodynamic aspects of KS black hole and found that they are thermodynamically stable [17] for a certain range of values of the entropy.

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