A recent experiment by the ENS group [1] has provided fairly direct evidence for the mechanism driving the remarkable phenomenon known as Berezinsky-Kosterlitz-Thouless (BKT) transition. The latter was predicted in the 1970s and, in an two-dimensional (2D) interacting Bose gas, it is caused by the proliferation of (thermally excited) bound pairs of vortices and antivortices above a temperature, $T_{BKT}$, which depends on the microscopic parameters of the gas. As the transition takes place in a 2D system, long range order is absent both above and below $T_{BKT}$ because of the enhanced effect of thermal fluctuations. Thus, the BKT transition, albeit continuous, is not characterized by the emergence of an order parameter, and therefore does not fit into the general theory for phase transitions established by Landau in the 1930s.

In the ENS experiment [1] two independent, harmonically trapped, 2D Bose gases were created after loading a (several tens of nK) cold $^{87}\text{Rb}$ gas in the standing wave formed by two counter-propagating laser beams (i.e. a 1D optical lattice). Since each 2D gas is a finite-size pancake, what was observed is more a sharp BKT crossover than a phase transition. However, by direct imaging of the expanded gas clouds after release from the trap, it was possible to show that, at high enough temperature, the 2D Bose gases contain free vortices that leave a distinctive signature in the interference pattern observed in time-of-flight experiments. This provided sufficient direct evidence for the BKT mechanism, which could not be obtained in other systems, such like liquid helium films, where the BKT transition was observed much earlier [2].

Concerning the possibility of observing this transition in trapped ultracold atomic gases, there has been much debate in the literature [7, 8, 9, 10, 11]. One of the issues requiring detailed investigation was that, being the experimental systems harmonically trapped, they are inhomogeneous. This, for instance, causes a non-interacting 2D Bose gas to become a true Bose-Einstein condensate (BEC) [8, 10], something that is not possible for the homogeneous 2D system [9, 10]. However, recent theoretical calculations [11] have clarified this issue showing that, in spite of the inhomogeneity, thermal activation of vortex-anti-vortex pairs, and thus a BKT-like crossover, can take place also in the trapped systems. This was later confirmed by the experiments [1].

As we show in this paper, 1D optical lattices, like the one used in [1], offer the possibility of exploring a much richer kind of phenomena, which lies beyond the standard BKT physics: due to the large degree of tunability of these atomic systems, it is experimentally possible to control the tunneling amplitude between the pancakes that are formed in the deep wells of the 1D lattice. This adds a new dimension to the phase diagram of the system: the Josephson coupling between the 2D gases, which indeed can be more easily controlled than temperature. When the amplitude of this coupling, $t_\perp$, is made large enough so that tunneling of an appreciable number of bosons takes place over the typical duration, $t_{\text{exp}}$, of an experiment (i.e. $\hbar/t_{\text{exp}} \ll t_\perp$), but small enough so that the pancakes remain 2D (i.e. $t_\perp \ll \mu$, $\mu$ being the chemical potential of the 2D gas), we find that the Josephson coupling can drive a transition where phase coherence builds up between neighboring pancakes. The competition between the interplane phase coherence and the thermally excited vortices in the pancakes leads to a transition analogous to the \textit{deconfinement} transition studied by some of us [12, 13, 14] in various types of coupled quantum 1D interacting Bose systems, such like those that are created in anisotropic optical lattices [15]. Universality between classical statistical mechanics in 2D and quantum field theories in 1D allows to use the same techniques [12, 14, 16, 17] to tackle the coupled classical two dimensional systems as well. In addition to their intrinsic interest, the physics of coupled 2D Bose systems is also relevant to the theoretical understanding of recent experiments in the high-$T_c$ cuprate superconductors (see reference therein), for which control over the system parameters is much more limited.

As shown in what follows, when only two pancakes are coupled, the resulting system behaves, below a certain temperature $T_c$ (see lower panel of Fig. 1), as single 2D superfluid. Furthermore, if an infinite number...
of pancakes are coupled, the deconfinement transition yields a true BEC. Here we focus on the experimental consequences of this transition: although in the BEC or superfluid phases the system displays true interference fringes, a measurement of the interference contrast like the one conducted by the ENS group will yield a jump, \( \Delta \alpha = \frac{1}{2} \), in the exponent (\( \alpha \)) that characterizes the scaling behavior of the interference contrast with the imaged area (\( \Delta \alpha = \frac{1}{2} \) for the BKT transition \[1\] \[18\]). However, if the interference pattern serves as probe of the coherence between the pancakes, it is not a probe of the superfluid properties of phase-coherent regime. We show that a direct probe of the superfluidity is provided by measuring the moment of inertia of the system at low rotation frequencies. In the presence of weak Josephson coupling between the pancakes, the rotational response exhibits a behavior that differs from that of a single (i.e. uncoupled) pancake (see upper panel of Fig. 1).

In the regime where phase fluctuations are classical, i.e. for \( T \gtrsim \mu \), the Bose field in the \( m \)-th \( (m = 1, \ldots, N) \) pancake of a 1D lattice can be written (away from the vortex cores) as \( \Psi_m(r) = \rho_0^{1/2} e^{i \Theta_m(r)} \) \( (r = (x,y)) \). Thus the 1D lattice is described by the following Hamiltonian for a stack of \( N \) coupled XY models:

\[
H = H_{XY} - \frac{g_J}{\pi \alpha_0^2} \sum_{m=1}^{N-1} \int dr \cos [\Theta_m(r) - \Theta_{m+1}(r)], \tag{1}
\]

where \( H_{XY} = \frac{K}{2\pi} \sum_{m=1}^{N} \int dr \left( \nabla \Theta_m(r) \right)^2 \) is the Hamiltonian of the uncoupled pancakes of area \( A \). We have assumed the pancakes to be identical and, for most of what follows, we shall take them to be infinite (i.e. \( A \to \infty \)). The bare parameters are \( K(0) = \pi \hbar^2 \rho_0/M \), where \( M \) is the boson mass, \( T \) the temperature, and \( \rho_0 = \rho_0(T) = |\Psi_m(r)|^2 \), the quasi-condensate density \[8\] \[9\]; the (dimensionless) strength of the Josephson coupling \( g_J(0) = \pi a_0^2 \rho_0 T \) \( / T \), where \( a_0 \) is the order of the healing length \( \xi = \sqrt{\hbar^2 / M \mu} \). It can be shown that, close to the BKT transition, the partition function of (1) can be obtained from the partition function of a 1D quantum model with the following Hamiltonian \[19\] :

\[
\hat{H} = \hat{H}_{sG} - \frac{g_J}{\pi a_0^2} \sum_{m=1}^{N-1} \int dx \cos [\theta_m - \theta_{m+1}], \tag{2}
\]

\[
\hat{H}_{sG} = \sum_{m=1}^{N} \int dx \left[ \frac{K}{2\pi} (\partial_x \theta_m)^2 + K^{-1} (\partial_x \phi_m)^2 \right] - \frac{2g_v}{a_0^2} \cos 2 \phi_m \]. \tag{3}

The operators \( \theta_m \) and \( \partial_x \phi_m \) are canonically conjugated fields. The coupling \( g_v = 2 \pi e^{-E_C / T} \), where \( E_C \approx 1.6 \pi \hbar^2 \rho_0(T) / M \) is the vortex core energy \[16\] \[20\].

To ascertain the nature of the phases of the 1D optical lattice, we apply the renormalization group (RG) to (2), i.e. we iteratively coarse-grain short-distance degrees of freedom to obtain a simpler description of the system. To analyze the dependence on the number of coupled pancakes, we consider two limiting cases: the double pancake system \( (N = 2) \) and the infinite lattice \( (N = \infty) \). We show that the main features of the transition in both cases are similar. In the calculations, we have exploited that (2) also describes a system of Josephson-coupled quantum 1D Bose gases in a periodic potential at commensurate filling \[12\] \[13\] \[14\]. More details on the RG procedure can be found in \[14\] \[19\].

Let us start with the two pancake system. We introduce symmetric and anti-symmetric field combinations \[12\]: \( \phi_\pm = (\phi_1 \pm \phi_2) / \sqrt{2} \) and \( \theta_\pm = (\theta_1 \pm \theta_2) / \sqrt{2} \). The action for these fields depends on the parameters \( K_\pm, g_v \) and \( g_J \), which flow under the RG transformations as described by the following set of equations \[12\] \[17\] :

\[
\frac{dg_v}{d\ell} = \left[ 2 - \frac{K_+ + K_-}{2} \right] g_v, \quad \frac{dg_J}{d\ell} = \left( 2 - 1 \right) g_J. \tag{4}
\]

The above equations describe the crossover between two strong coupling RG fixed points: when the bare couplings determined by \( T, \rho_0, T_\perp \), and \( M \), are such that \( g_v \) becomes of order one first (and the perturbative method employed to obtain \[11\] breaks down), vortex unbinding takes place before phase coherence between the two pancakes is established. In this phase the two pancakes are decoupled and behave as normal Bose gases. On the other hand, when the bare couplings are such that \( g_J \) becomes of order one first, the Josephson coupling term \( \propto g_J \) causes the relative phase \( \theta_- \) of the two pancakes to lock to the same value (modulo \( 2\pi \)). In this (superfluid) phase, only the symmetric field \( \theta_+ \) can fluctuate at large distances. This causes the same decay of correlations in both pancakes: \( \langle \psi_m(r) \psi_m(0) \rangle \sim |r|^{-1/2K_+} \) \( (m = 1, 2) \). We note that \( K_+ \) is related to the superfluid density, \( \rho_\phi(T) \). The relationship is obtained from the transformation of (2) under a phase twist where \( \partial_x \theta_m \to \partial_x \theta_m + \alpha / \sqrt{2} \) but \( \theta_- \to \theta_- \). Thus, \( K_+ = \pi h^2 \rho_\phi(T) / MT \).

The case of large number of pancakes \( N \to \infty \) can be treated by a similar application of the RG to (2). This leads the following set of flow equations \[12\] \[14\] \[16\] :

\[
\frac{dK}{d\ell} = 2g_v^2 - K^2 g_v, \quad \frac{dK}{d\ell} = -g_v^2 K^2, \tag{5}
\]

\[
\frac{dg_v}{d\ell} = \left( 2 - K \right) g_v, \quad \frac{dg_J}{d\ell} = \left( 2 - 1 / 4K \right) g_J.
\]

The physics described by these equations is similar: as we coarse grain the short distance degrees of freedom of the system, \( g_v \) and \( g_J \) get renormalized. For given values of \( T, T_\perp, \rho_0 \), and \( M \), whichever \( g_v \) or \( g_J \) reaches a value of order one first, determines the phase of the system. When the coupling \( g_v \) dominates, vortex unbinding
destroys phase coherence within the pancakes and therefore suppresses Josephson tunnelling at long distances: the pancakes decouple and an array of normal gases is obtained. On the other hand, if $g_f$ dominates over $g_e$, phase coherence builds up and a BEC forms. The resulting system is an anisotropic 3D BEC, whose properties can be computed by using Landau-Ginzburg theory. By the same ‘phase-twist’ argument given above, it is found that $K_s/\pi = h^2 \rho_s(T)/MT$.

Using the above RG equations we can compute the observable consequences of the deconfinement transition. Let us first discuss the interference contrast that was measured by the ENS group. It was argued in Ref. \cite{18} that interference between two independent identical systems provides a way to probe correlations in each system. This idea was employed by the ENS group to observe the BKT transition by creating a double-pancake system in a 1D optical lattice and making the atom hopping between them negligible over the time scale of the duration of the experiment. However, as discussed above, when hopping between pancakes is increased, it leads to a phase where they are not independent anymore. Thus, the expansion images taken after release of the atom clouds from the traps will exhibit true interference fringes. For instance, for two pancakes in the superfluid phase, the imaged (integrated) density distribution of the expanded coulds will exhibit oscillations that do not wash out after averaging over many experimental runs. On the other hand, in the normal gas phase (as well as in the superfluid phase at $t_\perp = 0$) where the pancakes are independent, these oscillations average to zero. However, if while seeking for signatures of the transition between these two phases, one insists in measuring the interference contrast, we show below that the latter also provides information about the phase correlations between neighboring 2D gases.

For $N = 2$, following Ref. \cite{18}, we consider the integrated interference contrast,

$$\langle |C_Q|^2 \rangle = (L_x L_y)^{-2} \int_{L_x \times L_y} dr_1 dr_2 F_{12}(r_1, r_2),$$

where $F_{12}(r_1, r_2) = \langle \Psi_2^\dagger(r_1) \Psi_1(r_1) \Psi_2^\dagger(r_2) \Psi_2(r_2) \rangle$; $\langle |C_Q|^2 \rangle$ is the square of the amplitude of the oscillations in the interference pattern with wave number $Q = m d/t$, normalized to (the square of) the imaging length $L_x L_y$ (see Eqs. (6,8)). In terms of symmetric and antisymmetric fields, $F_{12}(r_1, r_2) \propto \langle e^{i\sqrt{2\theta_-}(r_1)} e^{-i\sqrt{2\theta_-}(r_2)} \rangle$. In the normal Bose gas phase, the relative phase $\theta_-$ undergoes wild fluctuations because vortex-anti-vortex pairs are unbound. Thus correlations of $\theta_-$ asymptotically decay to zero exponentially. This implies that $\langle |C_Q|^2 \rangle \sim L_x^{-2\alpha}$ with $\alpha = \frac{1}{2}$ \cite{18}. On the other hand, when the Josephson coupling dominates, long-distance behavior of the system, the energy of the system is minimized by locking $\theta_-$ into one of the minima of the cosine in the second term of (2). Hence, in the $|r_1 - r_2| \to \infty$ limit, $\langle e^{i\sqrt{2\theta_-}(r_1)} e^{-i\sqrt{2\theta_-}(r_2)} \rangle \to \langle e^{i\sqrt{2\theta_-}} \rangle \langle e^{-i\sqrt{2\theta_-}} \rangle = \text{const.}$ Therefore, in the superfluid phase $\alpha = 0$. Thus, in going from the superfluid phase to a phase of decoupled 2D normal Bose gases by changing $T$ (or, easier, $t_\perp/\mu$), we predict a stronger change in the scaling behavior of the interference contrast: $\langle |C_Q|^2 \rangle \sim L_x^{-2\alpha}$, where $\alpha$ changes from 0 in the superfluid to $\frac{1}{2}$ for the normal gas phase. Notice that this is a much larger jump than the one predicted \cite{18} (and observed \cite{1}) for the standard BKT transition in a 2D Bose gas, where $\alpha$ jumps from $\frac{1}{2}$ at $T < T_{KT}$ to $\frac{1}{2}$ at $T > T_{KT}$.

For $N = \infty$, the interference pattern receives contributions from an infinite number pancakes. Thus, the correlation function of the integrated density after expansion

![Diagram](image-url)
reads:
\[
\langle \rho(z_1) \rho(z_2) \rangle = \sum_p e^{ipQ(z_2-z_1)} N^2 (L_x L_y)^2 \langle |C_Q|^2 \rangle,
\] (7)
where \( z \) is the coordinate of along the axis normal to the pancakes and the \( p \)-th contrast is
\[
\langle |C_Q|^2 \rangle = \frac{(L_x L_y)^{-2}}{N^2} \sum_{mn} \int_{L_x \times L_y} \alpha \beta dr_1 dr_2 e^{ipQ(m-n)d} \\
(\Psi_{m-p}^\dagger (r_1) \Psi_m (r_1) \Psi_n^\dagger (r_2) \Psi_{n+p} (r_2)).
\] (8)

Nevertheless, in the (superfluid) BEC phase, the above correlation functions also decay to a constant asymptotically for \( |r_1 - r_2| \rightarrow \infty \). Therefore, \( \langle |C_Q|^2 \rangle \sim L_x^{2\alpha} \) with \( \alpha = 0 \). In the normal gas phase, the pancakes decouple, and one finds \( \langle |C_Q|^2 \rangle \sim L_x^\alpha \) with \( \alpha = \frac{1}{2} \). Thus the jump in \( \alpha \) as for \( N = 2 \) should be observed.

Whereas the interference contrast provides evidence for the existence or lack of phase coherence between pancakes, it is not a direct measurement of the superfluid density \( \rho_s \). Such a measurement can be obtained from the response of the system to a slow rotation about the axis of the 1D lattice (\( z \)-axis). In the (rotating) reference frame where the gas is stationary, the equilibrium state is found by minimizing the free energy \( F(\Omega) = -T \ln Z[\Omega] \), \( Z[\Omega] = \text{Tr} e^{-(H-\Omega L_z)/T} \) with \( H \) given by (1) and \( L_z = \sum_{s} \epsilon_s \hat{\Omega}^s \hat{e}_s^{\dagger} \hat{p}_{s}^{\dagger} \hat{p}_{s} \) being the \( z \)-component of the total angular momentum operator \( (\hat{r}_s, \hat{p}_s) \) are the position and canonical momentum operators for the \( i \)-th particle, respectively), and \( \epsilon_{xy} = -\epsilon_{yx} = +1 \). By treating the term \( \Omega L_z \) as a perturbation (assuming \( \Omega \) to be small), we can write \( \Omega L_z = \int \hat{r} \cdot \hat{A}^{2}(r) \cdot j(r) \), \( \hat{r}(r) = \frac{\hbar}{2} \sum_{n=1}^{N} \hat{\Psi}_n(r) \nabla \hat{\Psi}_n(r) - \nabla \hat{\Psi}_n(r) \hat{\Psi}_n(r) \) being the momentum density and \( \hat{A}^{2}(r) = \Omega(-y,x) \) an effective transverse term (i.e. \( \nabla \cdot \hat{A}^{2}(r) = 0 \) vector potential. Within linear response theory, the angular momentum of the lattice \( \langle L_z \rangle_{\Omega \neq 0} = \sum_{\alpha} \int \hat{r} \alpha_j \cdot \hat{r}_\alpha (j_\alpha(r))_{\Omega \neq 0} = \sum_{\alpha} \int \hat{r} \cdot \hat{A}^{2}(r) + O(\Omega^2) \). Thus the response to rotation is given by the momentum density correlation function at zero rotation frequency, \( C_{\alpha\beta}(r,r') = \frac{1}{2} \langle j_\alpha(r) j_\beta(r') \rangle_{\Omega = 0} \). The latter can be decomposed into a longitudinal and a transverse part, \( C_{\alpha\beta}(q) = F_{\parallel}(q) \delta_{\alpha\beta} + F_{\perp}(q) (\delta_{\alpha \beta} - \tilde{q}_{\alpha} \tilde{q}_{\beta}) \), in Fourier space, where \( \tilde{q}_{\alpha} = q_{/\alpha}/q \), and \( F_{\parallel}(q) \) and \( F_{\perp}(q) \) are functions of \( q = |q| \) only. Since \( \hat{q} \cdot \hat{A}^{2}(r) = 0 \), only the transverse part of the correlation function, which is associated with the irrotational part of the gas flow caused by the vortices [21], will contribute to the response to rotation. For small \( \Omega \), the vector potential is a sufficiently slow perturbation so that only the \( q = 0 \) component of the correlation function matters, and it can be shown [21] that \( F_{\perp}(0) = \rho_n = \rho - \rho_s \), where \( \rho \) is the gas density. Hence, \( \langle L_z \rangle_{\Omega \neq 0} = I \Omega \simeq (1 - \rho_s/\rho) I\Omega \), where \( I\Omega \) is the classical moment of inertia of the pancake. Since \( \rho_s \) is related to \( K_+ (N = 2) \) or \( K_s (N \rightarrow \infty) \), the behavior of the ratio \( I/I_{cl} = (1 - \rho_s/\rho) \) can be obtained from the RG equations [10]. For \( N = 2 \), in the normal gas phase, \( K_+ \rightarrow 0 \) (\( K_s \rightarrow 0 \), for \( N = \infty \)) and \( I/I_{cl} \rightarrow 1 \). On the other hand, in the superfluid phase, \( K_+ (K_s) \) flows to a finite value and therefore \( I/I_{cl} < 1 \) because the superfluid fraction does not respond to rotation. The transition between these two phases is shown in the upper panel of Fig. [1].

The (abrupt, for \( N = 1 \)) change in \( I/I_{cl} \) between the superfluid (or BEC) and the normal gas phases is rendered smooth(er) by finite-size effects, which are mimicked by stopping the RG flow at a length scale equal to the size of the pancake (\( R = 100 \gamma \) in Fig. [1]). The change may be further smoothen by inhomogeneity effects [10]. Experimental determination of \( I/I_{cl} \) by measuring \( \langle L_z \rangle_{\Omega \neq 0} \) is not an easy task. Nevertheless, obtaining information on \( \langle L_z \rangle_{\Omega \neq 0} \), and hence on \( I/I_{cl} \), is possible by measuring the shape oscillations of the rotating pancakes, the so-called ‘scissors mode’ oscillation caused by a sudden rotation of a quadrupole deformation of the trap [22].

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