A KLEIMAN CRITERION FOR GIT STACK QUOTIENTS

MARK SHOEMAKER

In memory of Bumsig Kim

ABSTRACT. Kleiman’s criterion states that, for $X$ a projective scheme, a divisor $D$ is ample if and only if it pairs positively with every non-zero element of the closure of the cone of curves. In other words, the cone of ample divisors in $N^1(X)$ is the interior of the nef cone. In this paper we present an analogous statement for a variety $X$ acted on by a reductive group $G$ with a choice of $G$-linearization $L \to X$. In this new context, the ample cone of $X$ is replaced by a cell in the variation of GIT decomposition of the $G$-ample cone, and curves in $X$ are replaced by quasimaps to $[X/G]$.

1. INTRODUCTION

Let $X$ be a projective scheme over an algebraically closed field. Let $N^1(X)_\mathbb{R}$ denote the group of $\mathbb{R}$-divisors modulo numerical equivalence and denote by $\text{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone spanned by ample divisors. Let $N_1(X)_{\mathbb{R}}$ be the group of 1-cycles in $X$ up to numerical equivalence. Define the cone of curves $\text{NE}(X)$ to be the convex cone spanned by effective curve classes in $N_1(X)_{\mathbb{R}}$.

It is a remarkable fact that the ampleness of a divisor can be determined solely based on its intersection with 1-cycles.

Theorem 1.1 (Kleiman’s criterion [9]). A divisor $D$ on $X$ is ample if and only if
\[ D \cdot \gamma > 0 \]
for all $\gamma \in \overline{\text{NE}(X)} \setminus \mathbb{R}$. In other words, the ample cone is the interior of the dual of the cone of curves:
\[ \text{Amp}(X) = \text{int}(\overline{\text{NE}(X)}^\vee). \]

The goal of this paper is to obtain an analogous statement in the setting of variation of GIT.

1.1. variation of GIT. Let $G$ be a reductive group acting on $X$. As is now well-understood [10, 3, 12], there is not a canonical algebraic variety representing the “quotient” of $X$ by $G$; one must first fix a linearization, i.e., a $G$-equivariant line bundle $L \to X$. Given such a choice of $L$, one can construct the GIT quotient $X \sslash_L G$, which is a categorical quotient of the $L$-semi-stable locus $X^{ss}(L)$, an open $G$-invariant subset of $X$. 
Fix an ample $G$-equivariant line bundle $L \to X$ such that the semi-stable locus $X^{ss}(L)$ is nonempty. We will think of the triple of data of $(X, G, L)$ as a $G$-linearized variety $X$. We would like to understand if there is an analogue of Theorem 1.1 for $(X, G, L)$.

If two $G$-equivariant ample line bundles $L, L' \to X$ have the same semi-stable locus we say $L$ and $L'$ are GIT equivalent. In this case, the GIT quotients $X \sslash L$ and $X \sslash L'$ are isomorphic. Denote by $\text{NS}_G^G(X)$ the group of $G$-equivariant line bundles up to $G$-algebraic equivalence (see Section 2.2) and let $\text{Amp}_G^G(X) \subset \text{NS}_G^G(X)_Q$ denote the cone generated by ample line bundles. Define $C^o(L)$ to be the cone of classes of $G$-ample line bundles which are GIT equivalent to $L$:

$$C^o(L) = \{ [L'] \in \text{Amp}_G^G(X) | X^{ss}(L) = X^{ss}(L') \}.$$

Whenever $[L'] \in C^o(L)$, the line bundle $L' \to X$ descends to an ample line bundle on the GIT quotient $X \sslash L$. In fact, under sufficiently nice conditions the cone $C^o(L)$ may be identified with $\text{Amp}(X \sslash L)$. It is reasonable, therefore, to view $C^o(L)$ as the analogue of the ample cone for the triple $(X, G, L)$.

On the other hand, in general the equivalence classes

$$\{ C^o(L) \}_{L \in \text{Amp}_G^G(X)}$$

carry more information than just the isomorphism class of the GIT quotient $X \sslash L$, as illustrated by the following example.

**Example 1.2.** For simplicity we give a toric example. Let $T = G_m^2$ act on $V = \mathbb{A}^3$ with charge matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$ 

Consider the characters $\theta_+ , \theta_- \in \chi(T)$ with the following weights

$$\theta_+ = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \theta_- = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$ 

The characters $\theta_+$ and $\theta_-$ define $T$-linearizations of the trivial line bundle $\mathcal{O}_V$, which we denote by $\mathcal{O}_V(\theta_+)$ and $\mathcal{O}_V(\theta_-)$. Although $\theta_+$ and $\theta_-$ are not GIT equivalent, the respective GIT quotients $V \sslash_{\theta_+} T$ and $V \sslash_{\theta_-} T$ are both isomorphic to $\mathbb{P}^1$. Furthermore, the $T$-equivariant line bundles $\mathcal{O}_V(\theta_+)$ and $\mathcal{O}_V(\theta_-)$ both descend to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the GIT quotient.

In fact, the GIT equivalence classes carry not just the information of the GIT quotient, they also distinguish the various GIT stack quotients. In this case, the GIT stack quotients $[V^{ss}(\theta_+)/T]$ and $[V^{ss}(\theta_-)/T]$ are not isomorphic. Over $\mathbb{C}$ the first is the “football” $[\mathbb{P}^1_C/\mu_2]$ while the second is the “teardrop” $\mathbb{W}P_C(2,1)$. 
1.2. Quasimaps. If the GIT equivalence class $C^\circ(L)$ plays the role of the ample cone for the triple $(X, G, L)$, what is the correct analogue of the cone of curves? From Example 1.2, we see that it is not sufficient to simply consider the cone of curves of the GIT quotient, $\text{NE}(X \sslash L \sigma)$, because $T$ cannot distinguish between $\mathcal{O}_V(\theta_+)$ and $\mathcal{O}_V(\theta_-)$.

We propose that the correct replacement for curves in this context is given by $L$-quasimaps. An $L$-quasimap is a morphism $f : C \to [X/G]$ from a smooth curve $C$ to the stack quotient $[X/G]$ such that the preimage of the $L$-semi-stable locus $f^{-1}(X_{ss}(L)/G)$ is a dense open subset of $C$.

There is a notion of degree for a morphism $f : C \to [X/G]$, which naturally gives an element $\deg(f) \in \text{Hom}(\text{NS}^G(X), \mathbb{Z})$. Define $\text{NE}(L) \subset \text{Hom}(\text{NS}^G(X), \mathbb{Q})$ to be the cone generated by the degrees of $L$-quasimaps. This plays the role of the cone of curves for $(X, G, L)$.

In Section 4.1, we show that a certain class of quasimaps is closely related to the Hilbert–Mumford criterion. This is the main tool of the paper.

1.3. Results. Let $G$ be a reductive group acting on a normal projective variety $X$ over an algebraically closed field $k$. Fix an ample $G$-equivariant line bundle $L \to X$ such that the semi-stable locus $X_{ss}(L)$ is nonempty. Our main result is the following analogue of Kleiman’s criterion.

**Theorem 1.3** (Theorem 4.4). The following cones are equal
\begin{equation}
C^\circ(L) = \text{relint}(\text{NE}(L)^\vee) \cap \text{Amp}^G(X),
\end{equation}
where relint denotes the relative interior.

In particular, one can test whether a $G$-ample line bundle $L' \to X$ is GIT equivalent to $L$ by intersecting it with classes in $\text{NE}(L)$. The relative interior is necessary, as the cone $C^\circ(L)$ may be contained in a proper subspace of $\text{NS}^G(X)\mathbb{Q}$.

If we restrict our attention to linearizations of a fixed ample line bundle $P \to X$, the statement takes a form even closer to (1.0.1). Let $\text{NS}^G_P(X)$ denote the group of all linearizations of all powers of $P$.

**Theorem 1.4** (Theorem 4.10). Let $L \in \text{NS}^G_P(X)$ be a linearization of a positive power of $P$. Let $A_P(L) = C^\circ(L) \cap \text{NS}^G_P(X)$ be the cone of all linearizations of powers of $P$ which are GIT equivalent to $L$ and let $\text{NE}_P(L)$ be the cone of degrees of $L$-quasimaps on $\text{NS}^G_P(X)$. If $X_{ss}(L)$ is nonempty, then
\begin{equation}
A_P(L) = \text{relint}(\text{NE}_P(L)^\vee).
\end{equation}

**Corollary 1.5.** If $\text{NS}(X)\mathbb{Q} \cong \mathbb{Q}$, then
\begin{equation}
C^\circ(L) = \text{relint}(\text{NE}(L)^\vee).
\end{equation}
Next, we look at what can be said beyond the ample cone. If Pic\((X)\) is torsion, then \(NS^G(X)_Q = \text{Pic}^G(X)_Q\). In this case GIT equivalence classes are well defined on all of \(\text{Pic}^G(X)\) and not just in the ample cone \(\text{Amp}^G(X)\). Define
\[
A(L) := \{ L' \in \text{Pic}^G(X)_Q | X^{ss}(L') = X^{ss}(L) \}
\]
to be the GIT equivalence class of \(L\) in \(\text{Pic}^G(X)_Q\).

We obtain results in two different contexts. The first is for generalized flag varieties.

**Theorem 1.6** (Proposition 4.7). If \(X = H/P\) is a generalized flag variety, then
\[
A(L) = \overline{\text{NE}(L)}^\vee.
\]

The closure is necessary here, as the equivalence class \(A(L)\) may contain points of its boundary. We expect that \((1.3.2)\) holds more generally.

Finally, we consider the slightly different setting of a quotient of a normal affine variety \(V\). In this case, \(\text{Pic}^G(V) = \chi(G)\). We prove:

**Theorem 1.7** (Theorem 5.3). If \(V^{ss}(\theta)\) is nonempty, then
\[
A(\theta) = \text{relint} (\overline{\text{NE}(\theta)}^\vee).
\]

**Example 1.8.** Let us illustrate the theorem in a simple example. Let \(T = \mathbb{G}_m\) act on \(V = \mathbb{A}^1\) by scaling, and let \(\theta = \text{id} \in \chi(T)\) be the identity character. The GIT quotient \(V/\theta T\) is a point. In this case the cone \(A(\theta)\) consists of positive rational multiples of \(\theta\). After identifying \(\chi(T) \otimes \mathbb{Q}\) with \(\mathbb{Q}\), \(A(\theta)\) is the cone \(\mathbb{Q}_{\geq 0}\).

For any \(d \in \mathbb{Z}_{\geq 0}\), consider the \(\theta\)-quasimap \(f : \mathbb{P}^1 \to [V/T]\) given in coordinates by
\[
[s : t] \mapsto [s^d].
\]
It follows that the cone \(\text{NE}(\theta)\) contains the ray
\[
\mathbb{Q}_{\geq 0} \subset \mathbb{Q} \cong \text{Hom}(\chi(T), \mathbb{Q}).
\]
From here one can check that \(\text{NE}(\theta)\) in fact equals \(\mathbb{Q}_{\geq 0}\). We conclude that
\[
A(\theta) = \text{relint} (\overline{\text{NE}(\theta)}^\vee),
\]
in agreement with Theorem 1.7.

1.4. **Acknowledgements.** I am grateful to Andres Fernandez Herrero and Victoria Hoskins for helpful correspondences and conversations, and to Jeff Achter and the anonymous referee for their valuable comments and suggestions on earlier drafts. This work was partially supported by NSF grant DMS-1708104 and Simons Foundation Travel Grant 958189.

This paper is dedicated to the memory of Prof. Bumsig Kim. Through the beauty of his mathematics and the generosity with which he shared both his time and ideas, he had a great influence on me and so many others. I am grateful for our time together.
2. GIT Setup

Fix $G$ a reductive group acting on a normal projective algebraic variety $X$, both defined over an algebraically closed field $k$. Let $\pi : L \to X$ be a $G$-linearized line bundle, i.e. a line bundle with a $G$-action for which $\pi$ is $G$-equivariant. We recall the definition of semi-stable and stable points of $X$ with respect to $L$.

**Definition 2.1.** A geometric point $x$ in $X$ is $L$-semi-stable if, for some $m \geq 0$, there exists a section $\sigma \in \Gamma(X, L^\otimes m)^G$ such that $\sigma(x) \neq 0$. We denote the associated open subset of $X$ by $X_{\text{ss}}(L)$. A geometric point $x$ in $X$ is $L$-stable if $x$ is $L$-semi-stable, the orbit $G \cdot x$ is closed in $X_{\text{ss}}(L)$, and $|G_x| < \infty$. We denote the associated open subset by $X_{\text{s}}(L)$. The $L$-unstable locus is $X_{\text{us}}(L) := X \setminus X_{\text{ss}}(L)$. When $L$ is fixed we will sometimes refer to these as simply the semi-stable, stable, and unstable loci respectively.

**Definition 2.2.** Define the GIT quotient of $X$ (with respect to $L$) to be:

$$X //_L G := \text{Proj} \left( \bigoplus_{r \geq 0} H^0(X, L^\otimes r)^G \right).$$

Let $[X/G]$ denote the stack quotient of $X$ by $G$. Define the GIT stack quotient to be

$$[X //_L G] := [X_{\text{ss}}(L)/G].$$

The map $X_{\text{ss}}(L) \to X //_L G$ is a categorical quotient, and consequently $X //_L G$ depends only on $X_{\text{ss}}(L)$ and not $L$ itself. This motivates the following definition.

**Definition 2.3.** Two $G$-linearized line bundles $L$ and $L'$ on $X$ are said to be GIT equivalent if

$$X_{\text{ss}}(L) = X_{\text{ss}}(L').$$

By the work of Dolgachev–Hu [3], there are only a finite number of GIT equivalence classes within the ample cone.

2.1. A test for stability. A one parameter subgroup (1-PS) of $G$ is a non-trivial group homomorphism $\lambda : G_m \hookrightarrow G$. Given a 1-PS $\lambda$ and a point $x \in X$, by properness of $X$ the map

$$\lambda_x : G_m \to X$$

$$t \mapsto \lambda(t) \cdot x$$

extends uniquely to a morphism $\bar{\lambda}_x : \mathbb{A}^1 \to X$. Let $x_0 = \bar{\lambda}_x(0)$. By construction, $x_0$ is fixed by the action of $G_m$, so $G_m$ acts on $L|_{x_0}$ with some weight $\rho$.

There are different sign conventions for $\rho_{(\lambda, x)}^L$ based on whether one defines the line bundle associated to an invertible sheaf $\mathcal{L}$ to be $\text{Spec} (\text{Sym}^\bullet \mathcal{L})$ or $\text{Spec} (\text{Sym}^\bullet \mathcal{L}^\vee)$. We take the latter convention.
for all \( v \in L|_{x_0} \) and \( t \in G_m \).

The celebrated Hilbert–Mumford criterion asserts that one can test the (semi-)stability of a point \( x \in X \) using the asymptotics of all 1-PS's.

**Theorem 2.4 (Hilbert–Mumford criterion)** \cite{10}. If \( L \to X \) is an ample \( G \)-linearized line bundle, then

\[
\begin{align*}
x \in X^{ss}(L) & \iff \rho^L_{(\lambda,x)} \geq 0 \text{ for all 1-PS's } \lambda : G_m \hookrightarrow G; \\
x \in X^s(L) & \iff \rho^L_{(\lambda,x)} > 0 \text{ for all 1-PS's } \lambda : G_m \hookrightarrow G.
\end{align*}
\]

**2.2. Variation of GIT.**

**Notation 2.5.** Let \( W \) be a finite-dimensional vector space over \( Q \), and let

\[ W^\vee = \text{Hom}(W, Q) \]

denote the dual vector space. For \( C \subset W \) a convex cone, define the dual cone \( C^\vee \subset W^\vee \) by

\[ C^\vee = \{ f \in W^\vee | f(c) \geq 0 \text{ for all } c \in C \}. \]

Denote by \( \text{relint}(C) \) the relative interior of \( C \).

Let \( \text{Pic}^G(X) \) denote the Picard group of \( G \)-linearized isomorphism classes of line bundles on \( X \). We say \( L_1, L_2 \in \text{Pic}^G(X) \) are \( G \)-algebraically equivalent \cite{12} if there is a connected variety \( T \), points \( t_1, t_2 \in T \), and a \( G \)-linearized line bundle \( L \to T \times X \) (where the action of \( G \) on \( T \times X \) is induced by the action on \( X \)), such that

\[ L|_{t_1 \times X} \cong L_1, \quad L|_{t_2 \times X} \cong L_2. \]

Define \( \text{NS}^G(X) \) to be the set of \( G \)-algebraic equivalence classes in \( \text{Pic}^G(X) \). By \cite{12} Proposition 2.1, after tensoring with \( Q \) we have an exact sequence

\[ 0 \to \chi(G)_Q \to \text{NS}^G(X)_Q \to \text{NS}(X)_Q \to 0, \]

where \( \chi(G) \) is the group of characters of \( G \) and the last map forgets the \( G \)-linearization. Consequently, \( \text{NS}^G(X)_Q \) is a finitely generated abelian group. By the Hilbert–Mumford criterion, if two ample \( G \)-linearized line bundles \( L \) and \( L' \) define the same element in \( \text{NS}^G(X) \), then they are GIT equivalent.

Denote by \( \text{Amp}^G(X) \subset \text{NS}^G(X)_Q \) the convex cone spanned by the classes of \( G \)-linearized ample line bundles. Denote by \( \text{C}^G(X) \subset \text{Amp}^G(X) \) the cone of \( G \)-effective ample \( G \)-linearized line bundles, i.e. those \([L] \in \text{Amp}^G(X)\) for which \( X^{ss}(L) \) is nonempty. Such classes are called \( G \)-ample.

**Definition 2.6.** For \([L]\) a class in \( \text{C}^G(X) \), define

\[
C(L) = \{ [L'] \in \text{C}^G(X) | X^{ss}(L) \subset X^{ss}(L') \}.
\]

Let \( C^\circ(L) \) denote the cone of \( G \)-ample line bundles which are GIT equivalent to \( L \):

\[
C^\circ(L) = \{ [L'] \in C^G(X) | X^{ss}(L) = X^{ss}(L') \}.
\]
Following the work of Thaddeus [12] and Dolgachev–Hu [3], Ressayre proved the following:

**Theorem 2.7 ([11]).** The sets $C(L)$ are closed convex rational polyhedral cones in $\text{Amp}^G(X)$ which form a fan covering $C^G(X)$. The GIT equivalence class $C^\circ(L)$ is the relative interior $\text{relint}(C(L))$.

### 3. Quasimaps

Fix a $G$-ample line bundle $L \to X$ as above.

**Definition 3.1.** An $L$-quasimap $f : C \to [X/G]$ is a morphism from a smooth curve $C$ to the stack $[X/G]$ for which $f^{-1}([X^{ss}(L)/G])$ is a dense open subset.

A $G$-linearized line bundle $N \to X$ induces a line bundle on the stack quotient $[X/G]$ which we will denote by $[N/G] \to [X/G]$. Given a map $f : C \to [X/G]$ one can pull back $[N/G]$ to $C$ and take degree to obtain an integer $\deg(f^*[N/G])$.

**Definition 3.2.** We define the degree of $f$ to be the element of $\text{Hom}(\text{NS}^G(X), \mathbb{Z})$ given by $N \mapsto \deg(f^*[N/G])$.

Those classes $\beta \in \text{Hom}(\text{NS}^G(X), \mathbb{Z})$ which are realized as the degree of an $L$-quasimap will be called $L$-quasimap classes. Denote by $\text{NE}(L) \subset \text{Hom}(\text{NS}^G(X), \mathbb{Q})$ the cone generated by $L$-quasimap classes.

**Proposition 3.3.** Given an $L$-quasimap $f : C \to [X/G]$, the degree $\deg(f^*[L/G])$ is non-negative.

**Proof.** This was proven in [2, Lemma 3.2.1] in the case that $X$ is an affine variety over $\mathbb{C}$. We recall the argument here with the necessary modifications.

Assume without loss of generality that $C$ is irreducible. Choose a point $c \in C$ such that $f(c)$ lies in $[X^{ss}(L)/G]$. Such a point exists because $f$ is assumed to be an $L$-quasimap. Let $x \in X^{ss}(L)$ be a point mapping to $f(c)$ and let $\sigma \in \Gamma(X, L^\otimes m)^G$ be a $G$-invariant section of a positive power of $L$ such that $\sigma(x) \neq 0$. The section $\sigma$ descends to a section $\tilde{\sigma} \in \Gamma([X/G], [L^\otimes m/G])$ which is nonzero at $f(c)$. The degree $\deg(f^*[L^\otimes m/G])$ is therefore non-negative because it has a non-vanishing section $f^*(\tilde{\sigma})$. We conclude by noting that $\deg(f^*[L/G]) = \frac{1}{m} \deg(f^*[L^\otimes m/G])$. \qed
Remark 3.4. Proposition \[3.3\] holds with the same proof for \(X\) a quasiprojective variety with an action of \(G\).

**Corollary 3.5.** We have the following inclusion of cones:
\[
C(L) \subset \text{NE}(L)^\vee.
\]

**Proof.** If \([N] \in C(L)\), then \(X^{ss}(L) \subset X^{ss}(N)\), thus any \(L\)-quasimap \(f : C \to [X/G]\) is also an \(N\)-quasimap. By the proposition, we see that \(\deg(f^*[N/G])\) will then be non-negative. \(\square\)

### 4. HILBERT–MUMFORD CRITERION AND QUASIMAPS

#### 4.1. The main construction.

In this section we relate the weights \(\rho^L_{(\lambda, x)}\) to the degrees of certain quasimaps.

**Construction 4.1.** Fix \(\lambda : G_m \hookrightarrow G\) a 1-PS and \(x \in X^{ss}(L)\) a semi-stable point. Following Section 2.1, the morphism \(\lambda_x : G_m \to X\) given by \(t \mapsto \lambda(t) \cdot x\) extends to a morphism
\[
\bar{\lambda}_x : \mathbb{A}^1 \to X.
\]

Define the map \(\hat{\phi}_{(\lambda, x)} : \mathbb{A}^2 \setminus \{0\} \to X\) by
\[
\hat{\phi}_{(\lambda, x)}(s, t) \mapsto \bar{\lambda}_x(t).
\]

Note that when \(t \neq 0\), \((s, t)\) maps to \(X^{ss}(L)\). If we let \(G_m\) act on \(\mathbb{A}^2 \setminus \{0\}\) by scaling and on \(X\) via \(\lambda\), then \(\hat{\phi}_{(\lambda, x)}\) is \(G_m\)-equivariant. It therefore induces an \(L\)-quasimap
\[
\phi_{(\lambda, x)} : \mathbb{P}^1 = [\mathbb{A}^2 \setminus \{0\}/G_m] \to [X/G].
\]

**Lemma 4.2.** Given a \(G\)-linearized line bundle \(N \to X\), the degree of \(\phi^*_{(\lambda, x)}([N/G])\) is \(\rho^N_{(\lambda, x)}\).

**Proof.** The morphism \(\phi_{(\lambda, x)}\) factors through the projection onto the second factor:
\[
\phi_{(\lambda, x)} : [\mathbb{A}^2 \setminus \{0\}/G_m] \xrightarrow{\pi_2} [\mathbb{A}^1/G_m] \xrightarrow{[\bar{\lambda}_x/G_m]} [X/G]
\]

The pullback of \(N\) to \(\mathbb{A}^1\) via \(\bar{\lambda}_x\) is trivial, so the isomorphism class of \([\bar{\lambda}_x/G_m]^*([N/G])\) is determined by the weight of the action of \(G_m\) on \(\bar{\lambda}_x^*(N)|_0\), which is \(\rho^N_{(\lambda, x)}\). \(\square\)

#### 4.2. A Kleiman criterion, and some questions.

Construction 4.1 gives a partial converse to Corollary 3.5.

**Proposition 4.3.** If \(L\) and \(N\) are ample \(G\)-linearized line bundles such that \([N] \) is not contained in \(C(L)\) then there exists an \(L\)-quasimap \(f : C \to [X/G]\) such that \(\deg(f^*[N/G]) < 0\). In other words,
\[
\text{Amp}^G(X) \setminus C(L) \subset (\text{NE}(L)^\vee)^c.
\]
Proof. If \([N]\) is not contained in \(C(L)\), then there exists a point 
\[x \in X^{ss}(L) \setminus X^{ss}(N).\]
By the Hilbert–Mumford criterion, there must exist a 1-PS \(\lambda : \mathbb{G}_m \to G\) for which \(\rho^N_{(\lambda, x)} < 0\). The associated quasimap \(\varphi_{(\lambda, x)}\) then has the desired properties. \(\square\)

Combining Theorem 2.7, Corollary 3.5, and Proposition 4.3, we obtain the following analogue of Kleiman’s criterion.

**Theorem 4.4.** Fix a \(G\)-linearized normal projective variety \((X, G, L)\), with \(L \to X\) a \(G\)-ample line bundle. Then

\[
C^\circ(L) = \text{relint} \left( \text{NE}(L) \vee \right) \cap \text{Amp}^G(X).
\]

In other words, a \(G\)-linearized ample line bundle \(L' \to X\) is GIT equivalent to \(L\) if and only if

\[
L \cdot \gamma \text{ is } \begin{cases} > 0 \text{ for all } \gamma \in \text{NE}(L) \text{ such that } -\gamma \notin \text{NE}(L) \\ = 0 \text{ for all } \gamma \in \text{NE}(L) \text{ such that } -\gamma \in \text{NE}(L) \end{cases}.
\]

Assume \(\text{Pic}(X)_0\) is torsion, so \(\text{NS}^G(X)_\mathbb{Q} = \text{Pic}^G(X)_\mathbb{Q}\). This guarantees that GIT equivalence classes are well-defined on all of \(\text{Pic}^G(X)_\mathbb{Q}\) and not just on the ample cone \(\text{Amp}^G(X)\). Under this assumption, let 
\[A(L) := \{L' \in \text{Pic}^G(X)_\mathbb{Q} | X^{ss}(L') = X^{ss}(L)\}\]
denote the GIT equivalence class of \(L\). If \(L\) is \(G\)-ample, then 
\[A(L) \cap \text{Amp}^G(X) = C^\circ(L).\]

In this case, Theorem 4.4 may be rewritten as:

\[
A(L) \cap \text{Amp}^G(X) = \text{relint} \left( \text{NE}(L) \vee \right) \cap \text{Amp}^G(X).
\]

Theorem 4.4 and equation (4.2.2) may be viewed as an analog of Kleiman’s criterion on the ample cone. Here \(A(L)\) plays the role of the ample cone with respect to the linearization \(L\), and \(\text{NE}(L)\) plays the role of the cone of curves. It would be nice to understand when this relationship between cones extends beyond \(\text{Amp}^G(X)\).

**Question 4.5.** Assume \(\text{Pic}(X)_0\) is torsion. For which triples \((X, G, L)\) does

\[
A(L) = \text{relint} \left( \text{NE}(L) \vee \right)?
\]

It is not hard to construct examples where \(A(L)\) contains some points of its boundary. (Consider, for instance, the action of \(G = \mathbb{G}_m\) on \(\mathbb{P}^1 \times \mathbb{P}^1\) which scales one of the \(\mathbb{P}^1\) factors.) We are hopeful, however, that the following weaker condition is more common:

**Question 4.6.** Find conditions on \(X, G,\) and \(L\), such that

\[
\overline{A(L)} = \text{NE}(L) \vee.
\]
We remark in passing that if $L$ is not ample, Question 4.6 is already interesting when the group $G$ is trivial.

One case where (4.2.4) holds is when $X$ is a generalized flag variety.

**Proposition 4.7.** Suppose $X = H/P$ where $P$ is a parabolic subgroup of a reductive group $H$. If $L \to X$ is a $G$-ample line bundle then

$$\overline{A(L)} = \overline{\text{NE}(L)^\vee}.$$ 

**Proof.** The proof of Corollary 3.5 shows that $\overline{A(L)} \subset \overline{\text{NE}(L)^\vee}$. Thus by (4.2.2), it suffices to show $\overline{\text{NE}(L)^\vee} \subset \overline{\text{Amp}^G(X)}$. Suppose

$$N \in \text{Pic}^G(X)_\mathbb{Q} \setminus \overline{\text{Amp}^G(X)}.$$ 

We will construct an $L$-quasimap $f$ such that $\text{deg}(f^*[N/G]) < 0$.

By Kleiman’s criterion (Theorem 1.1), there exists an irreducible curve $C$ with $N \cdot [C] < 0$. We can move $C$ using the action of $H$ on $X$ so that it intersects the $L$-semi-stable locus. In particular, there exists an element $h \in H$ and a curve $C'$ which intersects $X^s(L)$ such that $C' = h \cdot C$. Let $\tilde{C} \to C'$ be the normalization, and define $f : \tilde{C} \to [X/G]$ to be the composition

$$\tilde{C} \to C' \hookrightarrow X \to [X/G].$$

Then $\text{deg}(f^*[N/G])$ is a positive multiple of $N \cdot [C']$ which is negative. By construction of $C'$, $f$ is an $L$-quasimap. $\square$

We conclude this section by mentioning another line of inquiry which might be of interest. The definition of the semi-stable locus (Definition 2.1) naturally extends to the setting of $X$ an Artin stack with a fixed line bundle $L \to X$ [1]. On the other hand, the $L$-quasimaps $\phi(\lambda, x)$ of Construction 4.1 are closely related to $\Theta$-stability for Artin stacks as defined in [6] and $L$-stability of [7]. Questions 4.5 and 4.6 above can therefore be extended to Artin stacks, and Theorem 4.4 may be viewed as concerning the special case that $X = [X/G]$ is the stack quotient of a projective variety by a reductive group.

**Question 4.8.** Let $X$ be an Artin stack and let $L \to X$ be a line bundle. Find conditions on $X$ and $L$ such that

$$\overline{A(L)} = \overline{\text{NE}(L)^\vee}.$$ 

4.3. **The case of a fixed ample line bundle.** For general $X$, the result is cleaner if we restrict our attention to powers of a fixed ample line bundle $P \to X$ and allow the linearization to vary.

Let $P \to X$ be an ample line bundle without a choice of $G$-linearization.

**Definition 4.9.** Let $\text{NS}^G_P(X)_\mathbb{Q}$ denote the subspace of $\text{NS}^G(X)_\mathbb{Q}$ spanned by all $G$-linearizations of powers of $P$. Define

$$A_P(L) := C^G(L) \cap \text{NS}^G_P(X)_\mathbb{Q};$$

$$C_P^G(X) := C^G(X) \cap \text{NS}^G_P(X)_\mathbb{Q}.$$
Elements of Hom(\(NS^G(X), \mathbb{Q}\)) naturally restrict to \(NS^G_P(X)\). Let \(NE_P(L)\) denote the image of \(NE(L)\) in Hom(\(NS^G_P(X), \mathbb{Q}\)).

**Theorem 4.10.** In \(NS^G_P(X)\), if \(L \in C^G_P(X)\), then
\[
A_P(L) = \text{relint } (NE_P(L)^\vee)\,.
\]

**Proof.** By intersecting (4.2.1) with \(NS^G_P(X)\), we have that
\[
A_P(L) = \text{relint } (NE_P(L)^\vee) \cap \text{Amp}^G(X).
\]

Thus, it suffices to show that if \(L \in C^G_P(X)\) and \(0 \neq N \in NS^G_P(X) \setminus \text{Amp}^G(X)\), then there exists an \(L\)-quasimap \(f\) for which \(\deg(f^*[N/G]) < 0\).

If \(N\) is not ample then after forgetting the linearization it is a non-positive power of \(P\) and therefore has non-positive degree. If \(\deg(N) < 0\), let \(X \hookrightarrow \mathbb{P}^r\) be the projective embedding of \(X\) determined by a very ample power of \(P\). Intersecting \(X\) with a general linear subspace of the correct dimension, we obtain a curve \(C \subset X\) which intersects the semi-stable locus \(X^{ss}(L)\). If \(C\) is reducible, choose an irreducible component \(C'\) which intersects the semi-stable locus \(X^{ss}(L)\). If \(C'\) is not smooth, let \(\hat{C} \to C'\) be a resolution. Let \(f : \hat{C} \to [X/G]\) denote the composition
\[
\hat{C} \longrightarrow C' \longleftarrow X \longrightarrow [X/G]\,.
\]
As in the proof of Proposition 4.7, \(f\) is an \(L\)-quasimap and \(\deg(f^*[N/G]) < 0\).

If \(N\) has degree zero then it is \(O_X(\theta)\) for some non-trivial character \(\theta \in \chi(G)\). Choose a 1-parameter subgroup \(\lambda : G_m \hookrightarrow G\) such that \(\theta \circ \lambda\) has weight \(-d\) for some \(d > 0\). Choose \(x \in X^{ss}(L)\) and consider \(\phi_{(\lambda,x)}\) as in Construction 4.1. Then \(\phi_{(\lambda,x)}\) is an \(L\)-quasimap, and \(\deg(\phi_{(\lambda,x)}^*[N/G]) = -d < 0\).

This immediately implies the following special case.

**Corollary 4.11.** If \(NS(X)_{\mathbb{Q}} \cong \mathbb{Q}\), and \(L \in C^G(X)\), then
\[
C^o(L) = \text{relint } (NE(L)^\vee)\,.
\]

### 5. Quotients of Affine Varieties

In this section we extend Theorem 4.4 to the affine setting. Let \(G\) be a reductive group acting linearly on a normal affine variety \(V\). In this case \(Pic^G(V) = \chi(G)\). With this identification we will use \(\theta\) to denote both a character of \(G\) as well as the associated \(G\)-linearized line bundle on \(V\).

Given \(\theta \in \chi(G)\), define a \(\theta\)-quasimap quasimap
\[
f : C \to [V/G]\]
Theorem 5.1. The sets $C_V(\theta)$ are closed convex rational polyhedral cones which form a fan in $\chi(G)_Q$. The GIT equivalence class $A(\theta)$ is the relative interior of $C_V(\theta)$.

Next we state the analogue of the Hilbert–Mumford criterion for affine varieties, which was obtained by King in [8]. As with the Hilbert–Mumford criterion for projective varieties (Theorem 2.4), this is a consequence of the fact that when $G$ is a reductive group stability can be tested using 1-PS's. In this case, for a 1-PS $\lambda : G_m \to G$, there is no guarantee that the map $\lambda : G_m \to V$ given by $t \mapsto \lambda(t) \cdot v$ extends to a morphism $\lambda_v : A^1 \to V$. It turns out that one can effectively ignore those 1-PS’s for which $\lambda_v$ does not extend when testing for stability. When $\lambda_v : A^1 \to V$ exists, the weight $\rho^\theta_{(\lambda,v)}$ is defined as in Section 2.1.

Theorem 5.2 ([8],[4]). A point $v \in V$ is $\theta$-semi-stable if and only if $\rho^\theta_{(\lambda,v)} \geq 0$ for all 1-PS's $\lambda : G_m \to G$ such that $\lambda_v : G_m \to V$ extends to $A^1$.

We conclude with the following analogue of Theorem 4.4.

Theorem 5.3. Let $G$ be a reductive group acting linearly on a normal affine variety $V$. Then for each $\theta \in \chi(G)$ with $V^{ss}(\theta) \neq \emptyset$, we have

$$A(\theta) = \text{relint}(\text{NE}(\theta)^\vee).$$

Proof. The argument follows the same steps as the proof of Theorem 4.4. By [2] Lemma 3.2.1 (or Remark 3.4), if a morphism $f : C \to [V/G]$ is a $\theta$-quasimap, then $\text{deg}(f^*[O_V(\theta)/G]) \geq 0$. Therefore,

$$C_V(\theta) \subset \text{NE}(\theta)^\vee.$$

Next, assume $\kappa \notin C_V(\theta)$. Then there exists a point $v \in V^{ss}(\theta) \setminus V^{ss}(\kappa)$. By Theorem 5.2 there exists a 1-PS $\lambda : G_m \to G$ such that $\lambda_v : A^1 \to V$ exists and $\rho^\kappa_{(\lambda,v)} < 0$. Using $\lambda_v$, we can define $\phi_{(\lambda,v)} : P^1 \to [V/G]$ as in Construction 4.1. Then $\phi_{(\lambda,v)}$ is a $\theta$-quasimap such that $\text{deg}(\phi_{(\lambda,v)}^*[O_V(\kappa)/G]) < 0$. Consequently,

$$C_V(\theta)^c \subset (\text{NE}(\theta)^\vee)^c,$$
and therefore
\[ C_V(\theta) = \text{NE}(\theta)^\vee. \]
By Corollary 5.1, \( A(\theta) = \text{relint}(C_V(\theta)) \). This proves the theorem. □

REFERENCES

[1] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.
[2] Ionuț Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014.
[3] Igor V. Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998. With an appendix by Nicolas Ressayre.
[4] Martin G. Gulbrandsen, Lars H. Halle, and Klaus Hulek. A relative Hilbert-Mumford criterion. *Manuscripta Math.*, 148(3-4):283–301, 2015.
[5] Mihai Halic. Quotients of affine spaces for actions of reductive groups. https://arxiv.org/abs/math/0412278, 2004.
[6] Daniel Halpern-Leistner. On the structure of instability in moduli theory. https://arxiv.org/abs/1411.0627, 2014.
[7] Jochen Heinloth. Hilbert-Mumford stability on algebraic stacks and applications to G-bundles on curves. *Épijournal Géom. Algèbrique*, 1:Art. 11, 37, 2017.
[8] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.
[9] Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–344, 1966.
[10] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994.
[11] N. Ressayre. The GIT-equivalence for G-line bundles. *Geom. Dedicata*, 81(1-3):295–324, 2000.
[12] Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.

MARK SHOEMAKER
COLORADO STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
1874 CAMPUS DELIVERY
FORT COLLINS, CO, USA, 80523-1874
EMAIL: mark.shoemaker@colostate.edu