NONLINEAR CHOQUARD EQUATIONS WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENTS

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Abstract. We consider the following Choquard equation
\[
\begin{cases}
-(a+\varepsilon\int_{\Omega}|\nabla u|^2)\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2^*_{\mu}}}{|x-y|^\mu} dy\right)|u|^{2^*_{\mu}-2}u + \lambda f(x)|u|^{q-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where $\lambda$ is a real parameter, $2^*_{\mu} = \frac{2N-\mu}{N-2}$ ($0 < \mu < N$) is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Under some suitable assumptions on $\lambda$, $\mu$, via the constrained minimizer method and concentration compactness principle, we prove that this system has multiple of solutions, and one of which is a positive ground state solution. Moreover, by using an abstract result due to K.-C Chang, we admit infinitely many pairs of distinct solutions. In addition, we prove the nonexistence result by Pohozaev identity when $\lambda < 0$. The main results extend and complement the earlier works in the literature.

1. Introduction. In this paper, we study the following Choquard equation
\[
\begin{cases}
-(a+\varepsilon\int_{\Omega}|\nabla u|^2)\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2^*_{\mu}}}{|x-y|^\mu} dy\right)|u|^{2^*_{\mu}-2}u + \lambda f(x)|u|^{q-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is smooth bounded domain, $0 < \mu < N$, $a > 0$, $1 < q < 2$, $\varepsilon > 0$ small enough, $\lambda$ is a real parameter, $f \in L^\infty(\Omega)$ is nonnegative. $2^*_{\mu} = \frac{2N-\mu}{N-2}$ is called the upper critical exponent, the exponent is with respect to Hardy-Littlewood-Sobolev inequality. Since problem (1.1) contains Hartree-type and Kirchhoff two nonlinearities, the problem (1.1) is regarded as being nonlocal in many physical applications.

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In this context, equation (1.1) is known as the stationary nonlinear Choquard equation. This type of nonlocal elliptic equation is relevant to the following nonlinear Choquard equation

\[-\Delta u + V(x)u = \left(\frac{1}{|x|^\mu} \ast |u|^p\right) |u|^{p-2}u, \quad u \in H^1_0(\Omega). \quad (1.2)\]

In 1954, it was S. Pekar [25] who described the quantum theory of a polaron at rest if \(\mu = 1, p = 2\) in (1.2). Later, in order to obtain a certain approximation to Hartree-Fock theory of one component plasma, P. Choquard [18] introduced the modeling of an electron trapped in its own hole in 1976. For more details and further mathematical and physical applications, we refer to [21] and the references therein.

The problem (1.1) is a general version of a model without Hartree term, which was firstly proposed in 1883 by Kirchhoff. Recently, Goel and Sreenadh established a steady-state Kirchhoff-Choquard model in [13]. For the related problems about this topic, we refer to [4, 17] for latest literature.

The elliptic equations involving subcritical term have been studied extensively recently, Alves et al. [1] combined mountain pass theorem with the penalization method to consider the following Choquard equation

\[-\Delta u + V(x)u = (I_{\mu} \ast F(u))f(u), \quad (1.3)\]

and they obtained the existence of a nontrivial solution to (1.3). In [5], Battaglia and Schaftingen got a nontrivial ground state solution to (1.3) depending on a scaling trick due to Jeanjean. Inspired by both Berestycki and Lions, Moroz and Van Schaftingen [22] took full advantage of Jeanjean’s method and concentration compactness principle to get a nontrivial solution, then used Pohožaev identity and Brézis-Kato lemma to prove the existence of a ground state solution. If \(V(x) = 1\) and the nonlinearity \(g\) satisfies the general subcritical growth conditions, replace \((I_{\mu} \ast F(u))f(u)\) with \(I_{\mu} \ast |u|^{p-2}u + g(x)\) in (1.3), Li and Tang [15] took into full consideration radial, non-radial two cases and proved the existence of a positive ground state solution.

Critical exponent problems, which have received significant attention for a few decades since the cerebrated work due to Brézis and Nirenberg [6], are very interesting and challenging in terms of mathematics. Under the assumptions on \(\mu\), Cassani et al. [7] proved the existence and multiplicity of solutions to the problem (1.3) with respect to the lower critical growth problems by Nehari manifold approach. Clapp and Salazar [8] investigated the existence of a positive solution and multiple sign-changing solutions to the problem (1.2). When \(\liminf_{|x| \to \infty} (1 - V(x))|x|^2 > \frac{N^2(N-2)}{4(N+1)}\), Moroz and Schaftingen [23] proved the equation (1.2) has a nontrivial solution. In [2], the authors established the existence and concentration of ground states to (1.2) in the light of Trudinger-Moser inequalities with \(\mathbb{R}^2\). If \(a = 1, \varepsilon = 0\), substitute \(f(x)|u|^q - 2u\) for \(\lambda u\), (1.1) reduces to the following equation

\[-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2}u + \lambda u \quad \text{in} \quad \Omega. \quad (1.4)\]

When \(N \geq 4\), Gao and Yang [12] applied mountain pass lemma to obtain a nontrivial solution and discussed the nonexistence of solutions to (1.4) by using the Pohožaev identity.
Recently, another topic which has increasingly received interest is the existence and non-existence of solutions to Kirchhoff-Choquard equations like (1.1). For $N = 3$,

$$ - (a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = (I_\alpha * F(u))f(u) \ u \in H^1(\mathbb{R}^3), $$

(1.5)

under appropriate assumptions on $V$ and $f$, Chen et al. [10] studied the existence of ground state solutions for (1.5) by using the Nehari-Pohožaev manifold and Jeanjean’s monotonicity trick. Arora et al. [4], via mountain pass theorem, Moser-Trudinger and singular Adams-Moser inequalities, proved the existence of solutions. As the equation involves convex-concave nonlinearities, they took advantage of Nehari manifold and minimization approach to get multiple results. If $f$ is a sign-changing weight function, replace $a + \varepsilon \int_{\Omega} |\nabla u|^2$ with $a + \varepsilon^{p} (\int_{\Omega} |\nabla u|^2)^{\theta-1}$ in (1.1), Goel et al. [13] used Nehari manifold and concentration compactness principle to obtain positive solutions to the Kirchhoff-Choquard problem with critical exponential growth nonlinearity. Xiang, Rădulescu and Zhang in [28] studied a critical fractional Kirchhoff-Choquard problem with magnetic field and established the existence of nontrivial radial solutions in non-degenerate and degenerate cases. Combining Nehari manifold with other techniques (concentration compactness principle, Pohožaev inequality, genus theory, mountain pass theorem), the researchers obtained nontrivial weak solutions, mountain pass solutions, ground state solutions, sign-changing solutions.

To the best of our knowledge, there are a few results in the literature on the Choquard equation with Kirchhoff term. Taking advantage of the minimization argument on the sign-changing Nehari manifold and a quantitative deformation lemma, Li et al. [16] considered the existence and the concentration of sign-changing solutions to Kirchhoff type system with Hartree nonlinearity like the problem (1.5). In the case of superlinear and sublinear growth, Pucci et al. [26] studied the equation involving the fractional p-Laplacian with critical exponent, via the mountain pass theorem and Ekeland variation principle, and proved the existence of nonnegative solutions. However, there seems to be few papers to deal with infinitely many pairs of distinct solutions and nonexistence of solutions to problem (1.1).

Motivated by the above works and the fact that several interesting questions originate from searching of ground state solutions, the main purpose of this paper is to investigate more complicated Kirchhoff-Choquard equation, which involves Hardy-Littlewood-Sobolev critical exponent. We prove the existence of the ground state solution, infinitely many pairs of distinct solutions and nonexistence results. Now the main results can be described as follows.

**Theorem 1.1.** Assume $N = 3$, $a > 0$, $1 < q < 2$, $0 < \mu < 3$, $\varepsilon > 0$ sufficiently small and $f \in L^\infty(\Omega)$ is nonnegative, then there exists $\lambda_\ast > 0$, such that for every $\lambda \in (0, \lambda_\ast)$, the problem (1.1) has a local minimum solution, a positive mountain-pass type solution and a positive ground state solution.

**Theorem 1.2.** Assume $a > 0$, $1 < q < 2.4 < \mu < N$, $\varepsilon > 0$ sufficiently small and $f \in L^\infty$ is nonnegative, for every $\lambda > 0$, then the problem (1.1) has infinitely many pairs of distinct solutions.

**Theorem 1.3.** Assume $N \geq 3$, $a > 0$, $1 < q < 2$, $0 < \mu < \min\{4, N\}$, $\varepsilon > 0$ sufficiently small and $f \in L^\infty$ is positive, for every $\lambda < 0$ and $\Omega$ is a strictly star shaped bounded domain (with respect to origin) with $C^{1,1}$ boundary, then problem (1.1) cannot have a nontrivial nonnegative solution.
Remark 1. Compared with and different from [12, 14], we get a positive ground solution, infinitely many pairs of distinct solutions and the nonexistence of a nontrivial nonnegative solution. So our results can be regarded as the complementary work of above literature.

We emphasize that the problem (1.1) includes two nonlocal terms. In order to obtain the main results, we have to encounter some difficulties. Firstly, the main difficulty lies in the compactness of\( (PS)_c \) sequences for \( I_{\lambda,\mu} \). On the one hand, we need to identify a positive critical level \( c \), where one can find \( (PS)_c \) sequences. That is, there exists a sequence \( \{u_n\} \subset H^1_0(\Omega) \) such that \( I_{\lambda,\mu}(u_n) \to c \) and \( I'_{\lambda,\mu}(u_n) \to 0 \). Then we prove that the sequence \( \{u_n\} \subset H^1_0(\Omega) \) has a convergent subsequence [27]. In order to find a \( (PS)_c \) sequence, we will utilize the energy estimate methods to estimate special mountain pass value \( c \) (see Section 3), where all paths are required to be uniformly bounded with respect to the parameter \( \lambda, \mu \). That pulls the energy level down below critical level to recover \( (PS) \) compactness. This indicates that \( c \) is less strict than the critical energy value \( \frac{N+2-\mu}{2(N-\mu)}(aS_{H,L})\frac{2^*}{2^*+\mu} - D\lambda \frac{2^*}{2^*+\mu} \) (see Lemma 3.4). It is worth noting that the Young inequality and convexity play a key role in the proof of Lemma 3.5. In particular, we also need to make a precise estimate on the nonlocal term \( \int_{\Omega} \frac{|u(y)|^{2^*}}{|x-y|^\mu} dy \) with the help of the cut-off function. Indeed, both Riesz potential and Hardy-Littlewood-Sobolev inequality will be used frequently. On the other hand, with the presence of the nonlocal terms \( \int_{\Omega} |\nabla u|^2 \) and \( \int_{\Omega} \frac{|u(y)|^{2^*}}{|x-y|^\mu} dy \), it is hard to show that the nonlocal terms are weakly continuous in \( H^1_0(\Omega) \). Precisely speaking, if \( u_n \rightharpoonup u \), we don’t know whether holds \( \int_{\Omega} |\nabla u_n|^2 \to \int_{\Omega} |\nabla u|^2 \) or \( \int_{\Omega} \frac{|u(y)|^{2^*}}{|x-y|^\mu} dy \) \( u_n \to (\int_{\Omega} \frac{|u(y)|^{2^*}}{|x-y|^\mu} dy) |u|^{2^*} \) in general for \( u \in H^1_0(\Omega) \). To this end, we apply Brézis-Lieb Lemma and the concentration compactness principle [20], which have the most impact on studying a ground state solution to the problem (1.1).

Secondly, the being of critical exponent brings about some obstacles. It is well known that Hardy-Littlewood-Sobolev critical exponent plays an important role for existence and properties of the solutions. The existence and multiplicity of solutions to the problem (1.1) depends heavily on the space dimension \( N \). If \( N > 3 \), the energy functional associated to (1.1) satisfies global \( (PS) \) condition [9] in \( H^1_0(\Omega) \) and if \( N = 3 \), it fails. That is one of the reasons for the Theorem 1.1 concerns the dimension \( N = 3 \). In addition, the critical exponent is related to the parameter \( \mu \). When \( \mu \in (0,3) \), the problem (1.1) admits at least two positive solutions, we’ll also show that infinitely many pairs of distinct solutions and no nontrivial solutions to the problem (1.1) do exist when \( \mu \in (0, \min\{4,N\}) \). With regard to this, we cannot use the variational methods in a standard way. It is natural to adopt some new techniques [9, 21] for the energy function \( I_{\lambda,\mu} \) and a more careful analysis are required for the proof given next.

The idea of the proof of Theorem 1.1 is to combine constraint minimization method with the classical mountain pass theorem. First, we obtain the first solution \( u_* \) as a local minimum by using constraint minimizer method, then we obtain a mountain-pass type solution to the problem (1.1), and distinguish the two solutions. What’s more, we will show the existence of a positive ground state solution to (1.1). Next, we will adopt the idea used in [9] to prove the Theorem 1.2, at the same time, we find recent applications in [17], where the Young inequality plays vital role.
Finally, we make full use of Pohožaev identity to prove nonexistence of nonnegative solutions.

The rest of this paper is organized as follows. In Section 2, we set the variational framework for system (1.1) and give some preliminary results. In Section 3-5, via the constraint minimization method, we prove Theorem 1.1. In Section 6 and 7, we devote to proving Theorem 1.2 and Theorem 1.3 respectively.

2. Notations and preliminaries. In this section, we present a variational framework and some preliminary results. Now we give several definitions and some notations.

- Let \( H^1_0(\Omega) \) be the Hilbert space with the norm \(||u|| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}\), the norm in \( L^p(\Omega) \) is denoted by \(| \cdot |_p\).
- Let \( u^\pm = \max\{u, 0\} \), \( F(x, t) = \int_0^t f(x, \tau)d\tau \).
- Set \( S_r \) (respectively \( B_r \)) the sphere (respectively, the closed ball) of center zero and radius \( r \), i.e. \( S_r = \{u \in H^1_0(\Omega) : \|u\| = r\}, B_r = \{u \in H^1_0(\Omega) : \|u\| \leq r\}. \)

It is well known that the nontrivial positive solutions to problem (1.1) are equivalent to the nonzero positive critical points of the energy functional \( I_{t, N, \mu, r} \) given by

\[
I_{t, N, \mu, r}(u) = \frac{a}{2}||u||^2 + \frac{\varepsilon}{4}||u||^4 - \frac{1}{2} \cdot \frac{2}{\mu} \int_\Omega \int_\Omega \frac{|u(x)|^2|u(y)|^{2\mu}}{|x-y|^\mu}dxdy - \lambda \int_\Omega f(x)|u|^q,
\]

and for every \( \varphi \in H^1_0(\Omega) \), then

\[
\langle I'_{t, N, \mu, r}(u), \varphi \rangle = (a + \varepsilon||u||^2) \int_\Omega \nabla u \nabla \varphi - \int_\Omega \int_\Omega \frac{|u(x)|^2|u(y)|^{2\mu-2}u(y)\varphi(y)}{|x-y|^\mu}dxdy - \lambda \int_\Omega f(x)|u|^{q-1}\varphi.
\]

At the beginning of the variation approach to the problem (1.1), we state the well-known Hardy-Littlewood-Sobolev inequality (see [19, Theorem 4.3]).

**Lemma 2.1** (Hardy-Littlewood-Sobolev inequality). Let \( t, r > 1 \) and \( 0 < \mu < N \) with \( \frac{1}{r} + \frac{\mu}{N} + \frac{1}{2} = 2, f \in L^t(\mathbb{R}^N) \) and \( h \in L^r(\mathbb{R}^N) \). There exists a sharp constant \( C(t, N, \mu, r) \) independent of \( f, h \), such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}dxdy \leq C(t, N, \mu, r)||f||_t||h||_r.
\]

If \( t = r = \frac{2N}{2N - \mu} \), then

\[
C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) \frac{\Gamma\left(\frac{N}{2} - \frac{\mu}{2}\right)}{\Gamma\left(\frac{N}{2} - \frac{\mu}{2}\right)} \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{1 - \frac{\mu}{N}}.
\]

In this case, the above equality holds if and only if \( f \equiv Ch \) and

\[
h(x) = A(\gamma^2 + |x - a|^2)^{-\frac{2N-\mu}{4}}
\]

for some \( A \in \mathbb{C}, \gamma \in \mathbb{R} \setminus \{0\} \) and \( a \in \mathbb{R}^N \).

Let \( f = h = |u|^q \), then the following integral

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dxdy
\]
is well defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying
\[
\frac{2}{t} + \frac{\mu}{N} = 2.
\]
Thus, for $u \in H^1(\mathbb{R}^N)$, by Sobolev embedding theorems, we know
\[
2 \leq tq \leq \frac{2N}{N-2},
\]
which implies that
\[
\frac{2N - \mu}{N} \leq q \leq \frac{2N - \mu}{N - 2}.
\]
Hence, $\frac{2N - \mu}{N}$ (or $2^*_\mu = \frac{2N - \mu}{N - 2}$) is called the lower (or upper) critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Of course, the different critical exponent make the existence of solutions different. For $u \in D^{1,2}(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality, one has
\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} \frac{dx dy}{|x - y|^\mu} \right)^{\frac{1}{2^*_\mu}} \leq C(N, \mu) \frac{1}{2^*_\mu} |u|_{2^*_\mu}^{2^*_\mu}.
\]
Let $S_{H,L}$ be the best constant
\[
S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} \frac{dx dy}{|x - y|^\mu} \right)^{\frac{1}{2^*_\mu}}}. \tag{2.2}
\]

Lemma 2.2 ([12]). The constant $S_{H,L}$ is achieved if and only if
\[
u = C \left( \frac{b}{b^2 + |x - a|^2} \right) \frac{N-2}{2},
\]
where $C > 0$, $a \in \mathbb{R}^N$, $b \in (0, \infty)$, therefore
\[
S_{H,L} = \frac{S}{C(N, \mu)^{\frac{2}{2^*_\mu}}},
\]
where $S$ is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_\mu}(\mathbb{R}^N)$
\[
S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{2^*_\mu} \right)^{\frac{1}{2^*_\mu}}}. \tag{2.3}
\]
It is well known that
\[
U(x) = \frac{[N(N - 2)]^{\frac{N-2}{2}}}{(1 + |x|^2)^{\frac{N-2}{2}}}
\]
is a solution of $-\Delta U = U^{2^*-1}$ in $\mathbb{R}^N$, and $S^{\frac{2}{2^*_\mu}} \frac{1}{\pi^{\frac{N-2}{2}}}$ is a minimizer for $S$, then
\[
\tilde{U}(x) = S^{\frac{(N-\mu)(2-N)}{N-N-\mu}} C(N, \mu) \frac{2^*_\mu - N}{2^*_\mu} U(x)
\]
\[
= S^{\frac{(N-\mu)(2-N)}{N-N-\mu}} C(N, \mu) \frac{2^*_\mu - N}{2^*_\mu} \frac{[N(N - 2)]^{\frac{N-2}{2}}}{(1 + |x|^2)^{\frac{N-2}{2}}}
\]
is the unique minimizer for $S_{H,L}$ and satisfies
$$-\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*}}{|x-y|^\mu} \, dy \right) |u|^{2^*-2} u \text{ in } \mathbb{R}^N,$$
moreover
$$\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\bar{U}(x)|^2 |\bar{U}(y)|^2}{|x-y|^\mu} \, dx \, dy = S_{H,L}^{2N-\mu}. $$

**Lemma 2.3** ([12]). Let $N \geq 3$, for all open subset $\Omega$ of $\mathbb{R}^N$,
$$S_{H,L}(\Omega) := \inf_{u \in D^{1,2}_a(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{(\int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} \, dx \, dy)^{\frac{1}{2^*}}} = S_{H,L}, \quad (2.4)$$
where $S_{H,L}(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$.

3. **Proof of Theorem 1.1.**

3.1. **Existence of the first solution to (1.1).**

**Lemma 3.1.** Assume $a > 0$, $0 < q < 2$, $0 < \mu < 3$, $\varepsilon > 0$ sufficiently small and $f \in L^\infty(\Omega)$ is nonnegative. Then there exist $\lambda_0 > 0$ and $R, \rho > 0$ such that for any $\lambda \in (0, \lambda_0)$, we have
$$I_{\lambda,\mu}(u)|_{u \in S_R} \geq \rho, \quad \text{and} \quad \inf_{u \in B_\rho} I_{\lambda,\mu}(u) < 0. \quad (3.1)$$

**Proof.** By the Hölder inequality, (2.3) and (2.4), we have
$$I_{\lambda,\mu}(u) = \frac{a}{2} \|u\|^2 + \frac{\varepsilon}{4} \|u\|^4 - \frac{1}{2} \cdot \frac{2}{2^*} \int_{\Omega} \int_{\Omega} |u^+(x)|^{2^*} |u^+(y)|^{2^*} \, dx \, dy - \frac{\lambda}{q} \int_{\Omega} f(x)(u^+)^q$$
$$\geq \frac{a}{2} \|u\|^2 + \frac{\varepsilon}{4} \|u\|^4 - \frac{1}{2} \cdot \frac{2}{2^*} S_{H,L}^{2^*} \|u^+\|^{22^*} - \lambda C \|u^+\|^q$$
$$\geq \frac{a}{2} \|u\|^2 - \frac{1}{2} \cdot \frac{2}{2^*} S_{H,L}^{2^*} \|u^+\|^{22^*} - C \lambda \|u^+\|^q$$
$$= \|u\|^q \left( \frac{a}{2} \|u\|^{2-q} - \frac{1}{2} \cdot \frac{2}{2^*} S_{H,L}^{2^*} \|u^+\|^{22^*} - C \lambda \right). \quad (3.2)$$

Set
$$g(t) = \frac{a}{2} t^{2-q} - \frac{1}{2} \cdot \frac{2}{2^*} S_{H,L}^{2^*} t^{22^*} - q, \quad t \geq 0,$$
in fact
$$g'(t) = 0, \quad t = \left( \frac{a(2-q)2^* \mu S_{H,L}}{(2 \cdot 2^* - q)} \right)^{\frac{1}{22^*-q}} := R.$$
Thus, $g'(t) > 0$ for $t \in (0, R)$ and $g'(t) < 0$ for $t \in (R, \infty)$. Then $g(t)$ achieves its maximum at $R$. Moreover
$$\max_{t \geq 0} g(t) = g(R) = \frac{a}{2} R^{2-q} - \frac{1}{2} \cdot \frac{2}{2^*} S_{H,L}^{2^*} R^{22^*} - q > 0.$$
Let
$$g(R) = C \lambda = 0,$$
so \( \lambda = \frac{9(R)}{L^2} \). Set \( \lambda_0 := \frac{9(R)}{L^2} \), then for every \( \lambda \in (0, \lambda_0) \), there exists \( \rho > 0 \) such that \( I_{\lambda, \mu}(u)|_{S_R} \geq \rho \). Choosing \( u \in B_R \) with \( \|u\| \) sufficiently small, one has
\[
m_1 := \inf_{u \in B_R} I_{\lambda, \mu}(u) < 0. \tag{3.3}
\]
The proof of the lemma is completed.

**Lemma 3.2.** Let \( a > 0, 1 < q < 2, 0 < \mu < 3, \varepsilon > 0 \) sufficiently small and \( f \in L^\infty(\Omega) \) is nonnegative. Assume that \( \lambda \in (0, \lambda_0) \), then problem (1.11) has a positive solution \( u_0 \in H^1_0(\Omega) \), satisfying \( I_{\lambda, \mu}(u_0) < 0 \).

**Proof.** Firstly, we'll show that there exists \( u_0 \in B_R \), such that \( I_{\lambda, \mu}(u_0) = m_1 < 0 \).

By (3.1), (3.3), we derive that
\[
\frac{a}{2}\|u\|^2 + \frac{\varepsilon}{4}\|u\|^4 - \frac{1}{2 \cdot 2^* \mu} \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*} |u^+(y)|^{2^*}}{|x-y|^{\mu}} dxdy \geq \rho, \text{ for all } u \in S_R, \tag{3.4}
\]
and
\[
\frac{a}{2}\|u\|^2 + \frac{\varepsilon}{4}\|u\|^4 - \frac{1}{2 \cdot 2^* \mu} \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*} |u^+(y)|^{2^*}}{|x-y|^{\mu}} dxdy \geq 0, \text{ for all } u \in B_R. \tag{3.5}
\]
According to the definition of \( m_1 \), then there exists a minimizing sequence \( \{u_n\} \subset B_R \) such that \( \lim_{n \to \infty} I_{\lambda, \mu}(u_n) = m_1 < 0 \). Clearly, \( \{u_n\} \) is bounded on \( H^1_0(\Omega) \). Since \( I_{\lambda, \mu}(u) = I_{\lambda, \mu}(|u|) \), without loss of generality, we may assume \( u_n \geq 0 \). Going if necessary to a subsequence, still denoted by \( \{u_n\} \), there exists \( u_0 \in H^1_0(\Omega) \) with \( u_0 \geq 0 \) such that
\[
\begin{cases}
  u_n \to u_0, & \text{in } H^1_0(\Omega), \\
  u_n \rightharpoonup u_0, & \text{in } L^p(\Omega), 1 \leq p < 2^*, \\
  u_n(x) \to u_0(x), & \text{a.e. in } \Omega,
\end{cases} \tag{3.6}
\]
as \( n \to \infty \). Notice that \( \|u_0\| \leq \inf_{u \in B_R} \|u\| \leq R \), so \( u_0 \in B_R \).

We will prove that \( I_{\lambda, \mu}(u_0) = \inf_{u \in B_R} I_{\lambda, \mu}(u) \). Actually, we only prove to show that \( u_n \to u_0 \) as \( n \to \infty \) in \( H^1_0(\Omega) \). Set \( v_n = u_n - u_0 \), it follows from the Brézis-Lieb Lemma (see [27]), then
\[
\begin{align*}
  \|u_0\|^2 &= \|v_n\|^2 + \|u_0\|^2 + o(1), \tag{3.7} \\
  \|u_n\|^2 &= \|v_n\|^2 + \|u_0\|^2 + 2\|v_n\|^2 + o(1), \tag{3.8} \\
  \lim_{n \to \infty} \int_\Omega |u_n|^q &= \int_\Omega |u_0|^q, \tag{3.9} \\
  \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy &= \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^*}}{|x-y|^{\mu}} dxdy + o(1). \tag{3.10}
\end{align*}
\]
If \( u_0 \neq 0 \), then \( v_n = u_n \), which means that \( v_n \in B_R \). If \( u_0 \neq 0 \), by (3.5) and \( \inf_{u \in B_R} I_{\lambda, \mu}(u) < 0 \), which implies that there exists \( \varepsilon_0 > 0 \) such that \( \|u_0\| \leq R - \varepsilon_0 \).

Moreover, together with (3.7) and \( u_0 \in B_R \), we conclude that \( v_n \in B_R \) for \( n \) large enough. By (3.5), it is easy to see that
\[
\frac{a}{2}\|v_n\|^2 + \frac{\varepsilon}{4}\|v_n\|^4 - \frac{1}{2 \cdot 2^* \mu} \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy \geq 0. \tag{3.11}
\]
From (3.7)-(3.11), one has
\[ m_1 = \inf_{u \in B_R} I_{\lambda, \mu}(u) = I_{\lambda, \mu}(u_n) + o(1) \]
\[ = \frac{a}{2} \|u_n\|^2 + \frac{\varepsilon}{4} \|u_n\|^4 - \frac{1}{2 \cdot 2^*_m} \int_{\Omega} \int_{\Omega} |u_n^+(x)|^{2^*_m} |u_n^+(y)|^{2^*_m} \frac{dx}{|x-y|^\mu} \]
\[ + \frac{\varepsilon}{2} \|v_n\|^2 \|u_n\|^2 + o(1) \]
\[ \geq I_{\lambda, \mu}(u_n) + \frac{\varepsilon}{4} \|v_n\|^2 \|u_n\|^2 + o(1) \geq I_{\lambda, \mu}(u_n) + o(1), \]
thus, \( m_1 \geq I_{\lambda, \mu}(u_n) \) as \( n \to \infty \). Moreover, by (3.3), we conclude that \( I_{\lambda, \mu}(u_n) = m_1 < 0 \) and \( u_n \neq 0 \), which means that \( u_n \) is a local minimizer of \( I_{\lambda, \mu} \).

Secondly, we show that \( u_n \) is a solution to (1.1) and \( u_n \neq 0 \). According to previous arguments, we conclude that \( u_n \) is a local minimizer of \( I_{\lambda, \mu} \). Then for every \( \phi \in H^1_0(\Omega) \), let \( t > 0 \) sufficiently small, such that \( u_n + t \phi \in B_R \), it follows that
\[ 0 \leq I_{\lambda, \mu}(u_n + t \phi) - I_{\lambda, \mu}(u_n) \]
\[ = \frac{a}{2} \|u_n + t \phi\|^2 + \frac{\varepsilon}{4} \|u_n + t \phi\|^4 - \frac{1}{2 \cdot 2^*_m} \int_{\Omega} \int_{\Omega} |(u_n + t \phi)^+(x)|^{2^*_m} |(u_n + t \phi)^+(y)|^{2^*_m} |x-y|^\mu \]
\[ - \frac{\varepsilon}{2} \|u_n\|^2 \|u_n\|^4 + \frac{1}{2 \cdot 2^*_m} \int_{\Omega} \int_{\Omega} |u_n^+(x)|^{2^*_m} |u_n^+(y)|^{2^*_m} |x-y|^\mu \]
\[ + \frac{\varepsilon}{4} \int_{\Omega} \int_{\Omega} |(u_n + t \phi)^+(x)|^{2^*_m} |(u_n + t \phi)^+(y)|^{2^*_m} |x-y|^\mu \]
\[ - \frac{\varepsilon}{2} \|v_n\|^2 \|u_n\|^2 + o(1) \]
\[ = \frac{a}{2} \|u_n\|^2 + \frac{\varepsilon}{4} \|u_n\|^4 - \frac{1}{2 \cdot 2^*_m} \int_{\Omega} \int_{\Omega} |u_n^+(x)|^{2^*_m} |u_n^+(y)|^{2^*_m} \frac{dx}{|x-y|^\mu} \]
\[ - \frac{\varepsilon}{2} \|v_n\|^2 \|u_n\|^2 + o(1) \geq 0, \]
and
\[ \langle I_{\lambda, \mu}'(u_n), \phi \rangle = \lim_{t \to \infty} \frac{1}{t} [I_{\lambda, \mu}(u_n + t \phi) - I_{\lambda, \mu}(u_n)] \]
\[ = (a + \varepsilon \|u_n\|^2) \int_{\Omega} \nabla u_n \nabla \phi - \lambda \int_{\Omega} f(x) (u_n^+)^{q-1} \phi \]
\[ - \int_{\Omega} \int_{\Omega} |u_n^+(x)|^{2^*_m} |u_n^+(y)|^{2^*_m} |x-y|^\mu \]
\[ \geq 0. \]

By the arbitrariness of \( \phi \), replacing \( \phi \) by \( -\phi \) above inequality, then the above inequality holds, so
\[ (a + \varepsilon \|u_n\|^2) \int_{\Omega} \nabla u_n \nabla \phi - \int_{\Omega} \int_{\Omega} |u_n^+(x)|^{2^*_m} |u_n^+(y)|^{2^*_m} |x-y|^\mu \]
\[ - \lambda \int_{\Omega} f(x) (u_n^+)^{q-1} \phi = 0. \]

Furthermore, \( u_n \) is a solution to (1.1). We'll show that \( u_n \) is a positive solution. On the one hand, taking the test function \( \phi = u_n^{-1} \) in (3.13), then \( \|u_n\| = 0 \), we derive that \( u_n \geq 0 \).

On the other hand, from (3.12), it follows that
\[ \int_{\Omega} \nabla u_n \nabla \phi \geq 0, \forall \phi \in H^1_0(\Omega). \]
Consequently, \( u_* \in H^1_0(\Omega) \) and satisfies
\[
-\Delta u_* \geq 0, \text{ in } \Omega.
\]
Since \( u_* \geq 0, u_* \neq 0 \), using the strong maximum principle, we conclude that \( u_* > 0 \) in \( \Omega \). Hence \( u_* \) is a positive solution to (1.1) with \( I_{\lambda,\mu}(u_*) = m_1 < 0 \). This completes the proof of Lemma 3.2. \( \square \)

4. Existence of the second solution to (1.1).

**Lemma 4.1.** Let \( a > 0, 1 < q < 2, 0 < \mu < 3, \varepsilon > 0 \) and \( f \in L^\infty(\Omega) \) is nonnegative. For \( R, \rho > 0 \), assume that \( \lambda < \lambda_0 \) such that

(i) \( I_{\lambda,\mu}(u) \geq \rho > 0 \) if \( u \in S_R \);

(ii) there exists \( u_{**} \in H^1_0(\Omega) \) such that \( \|u_{**}\| > R \) and \( I_{\lambda,\mu}(u_{**}) < \rho \).

**Proof.** It’s easy to check, and the proof of this lemma can be omitted.

**Lemma 4.2.** Assume \( a > 0, 1 < q < 2, 0 < \mu < 3, \varepsilon > 0 \) and \( f \in L^\infty(\Omega) \) is nonnegative. Let \( \{u_n\} \subset H^1_0(\Omega) \) be a \((PS)_c\) sequence for \( I_{\lambda,\mu} \) with
\[
c < \frac{N + 2 - \mu}{2(2N - \mu)}(aS_{H,L})^{\frac{2}{2q - \mu}} - D\lambda^{\frac{2}{2q - \mu}},
\]
where \( D = \left( \frac{4 - q}{24} |f|_\infty S^{-\frac{2}{3}} \right)^{\frac{2}{3}} \left( \frac{2q}{2q - \mu} \right)^{\frac{2}{3}} \). Then there exists \( u \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \),
\[
\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_a}|u_n(y)|^{2^*_a}}{|x-y|^\mu} dx dy \to \int_\Omega \int_\Omega \frac{|u(x)|^{2^*_a}|u(y)|^{2^*_a}}{|x-y|^\mu} dx dy, \text{ as } n \to \infty.
\]

**Proof.** Assume \( \{u_n\} \subset H^1_0(\Omega) \) be a \((PS)_c\) sequence for \( I_{\lambda,\mu} \), that is
\[
I_{\lambda,\mu}(u_n) \to c, \quad I'_{\lambda,\mu}(u_n) \to 0, \quad \text{as } n \to \infty. \tag{4.1}
\]
We assert that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). From (2.2), (2.3) and (4.1), for \( n \) large enough, we have
\[
1 + c + \|u_n\|^2 \geq I_{\lambda,\mu}(u_n) - \frac{1}{2 \cdot 2^*_\mu}(I'_{\lambda,\mu}(u_n), u_n)
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2 \cdot 2^*_\mu} \right)a\|u_n\|^2 + \left( \frac{1}{4} - \frac{1}{2 \cdot 2^*_\mu} \right)\varepsilon\|u_n\|^4 - \lambda f(x)(u_n^*)^q
\]
\[
\geq \frac{N - \mu + 2}{2(2N - \mu)}a\|u_n\|^2 - \frac{2 \cdot 2^*_\mu - q}{2 \cdot 2^*_\mu q} \lambda|f|_\infty S^{-\frac{2}{3}} \|u_n\|^q.
\]
Since \( 1 < q < 2 \), hence \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Moreover there exist a subsequence (still denoted by \( \{u_n\} \) and \( u \in H^1_0(\Omega) \) such that
\[
\begin{align*}
\{u_n\} & \rightharpoonup u, \quad \text{in } H^1_0(\Omega), \\
\{u_n\} & \longrightarrow u, \quad \text{in } L^p(\Omega), \quad 1 \leq p < 2^*, \\
\{u_n\} & \longrightarrow u(x), \quad \text{a.e. in } \Omega.
\end{align*} \tag{4.2}
\]
Consequently, by the concentration compactness principle [20] and [11, Lemma 2.5], there exist a subsequence (still denoted by \( \{u_n\} \)) and \( u \in H^1_0(\Omega) \) such that
\[
|\nabla u_n|^2 \to d\mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j},
\]
\[
|u_n|^{2^*_a} \to d\nu \geq |u|^{2^*_a} + \sum_{j \in J} \nu_j \delta_{x_j},
\]
By (2.3), we have
\[ \varepsilon > \int J |u_\varepsilon(x)|^2 \mu_j + \sum_{j \in J} w_j \delta_{x_j}, \quad \sum_{j \in J} w_j \delta_{x_j} < \infty, \]
where \( J \) is an at most countable index set, \( \delta_{x_j} \) is the Dirac mass at \( x_j \), and let \( x_j \in \Omega \) in the support of \( \mu, \nu, w \). Moreover
\[ \mu_j, \nu_j, w_j \geq 0, \quad \mu_j \geq S_{\nu, \mu}^{\frac{1}{\nu + 1}}, \quad \mu_j \geq S_{H, L, \mu}^{\frac{1}{\nu + 1}}, \quad C(N, \mu) \frac{\varepsilon}{\varepsilon^{1/4} + \mu_j} \geq w_j^{\frac{1}{2^{*} - 2}}. \] (4.3)

For every \( \varepsilon > 0 \), set \( \psi_{\varepsilon, j}(x) \) be a smooth cut-off function such that \( 0 \leq \psi_{\varepsilon, j}(x) \leq 1 \),
\[ \psi_{\varepsilon, j}(x) = 1 \text{ in } B(x_j, \frac{\varepsilon}{2}), \quad \psi_{\varepsilon, j}(x) = 0 \text{ in } \Omega \setminus B(x_j, \varepsilon), \quad |\nabla \psi_{\varepsilon, j}(x)| \leq \frac{4}{\varepsilon}. \]

By (2.3), we have
\[ \left| \int_{\Omega} f(x)|u_n^+|^{q - 1}\psi_{\varepsilon, j}u_n \right| \leq \int_{B(x_j, \varepsilon)} f(x)|u_n^+|^q \leq \int_{B(x_j, \varepsilon)} f(x)|u_n|^q \leq |f|_{\infty} \left( \int_{B(x_j, \varepsilon)} |u_n|^q \frac{1}{\varepsilon^{1 - q}} \right)^{\frac{1}{q}} \left( \int_{B(x_j, \varepsilon)} 1 \right)^{\frac{2^{*} - q}{2^{*} - 2}}. \]

Notice that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \), we have
\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} f(x)|u_n^+|^{q - 1}\psi_{\varepsilon, j}u_n = 0, \]
then
\[ 0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left( I_{\lambda, \mu}^{1, \varepsilon}(u_n), \psi_{\varepsilon, j} \right) \]
\[ = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ (a + \varepsilon ||u_n||^2) \int_{\Omega} \nabla u_n \nabla (\psi_{\varepsilon, j} u_n) \right. \]
\[ - \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^2 |u_n^+(y)|^{2^{*} - 1} u_n(y) \psi_{\varepsilon, j} dxdy} {||x - y||^\mu} \]
\[ \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ (a + \varepsilon ||u_n||^2) \int_{\Omega} (||\nabla u_n||^2 \psi_{\varepsilon, j} + u_n \nabla u_n \nabla \psi_{\varepsilon, j}) \right. \]
\[ - \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^2 |u_n^+(y)|^{2^{*} - 1} \psi_{\varepsilon, j} dxdy} {||x - y||^\mu} \]
\[ \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ (a + \varepsilon ||u_n||^2) \int_{\Omega} (||\nabla u_n||^2 \psi_{\varepsilon, j} + u_n \nabla u_n \nabla \psi_{\varepsilon, j}) \right. \]
\[ - \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^2 |u_n^+(y)|^{2^{*} - 1} \psi_{\varepsilon, j} dxdy} {||x - y||^\mu} \]
\[ = a \mu_j - w_j. \]

By (4.3), we have
\[ \mu_j \geq a \frac{1}{\nu} S_{H, L, \mu}^{\frac{1}{\nu + 1}}, \] or \( \mu_j = 0. \)

Next we show that \( \mu_j \geq a \frac{1}{\nu} S_{H, L, \mu}^{\frac{1}{\nu + 1}} \) is impossible. That is, the set \( J \) is empty. Argue by contradiction, we assume that there exists some \( j_0 \in J \) such that \( \mu_{j_0} \geq \)}
\[ a_{n}^{\frac{1}{\mu^*}} \left( \frac{\lambda}{\mu^*} \right)^{\frac{2}{\mu}} S_{H,L}^{\frac{2}{\mu}} \text{, combining (2.3) with (4.3) and Young inequality, we derive that} \]

\[
c = \lim_{n \to \infty} I_{\lambda, \mu}(u_n) \geq \lim_{n \to \infty} \left( I_{\lambda, \mu}(u_n) - \frac{1}{4} \langle I_{\lambda, \mu}'(u_n), u_n \rangle \right) \geq \left[ \frac{1}{2} - \frac{1}{4} \right] a(\|u\| + \sum_{j \in J} \mu_j) + \left( \frac{1}{4} - \frac{1}{2 \mu} \right) \left( \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} \, dx \, dy + \sum_{j \in J} w_j \right) \tag{4.4} \]

where \( D = \left( \frac{1-q}{4} |f|_{\infty} S^{-\frac{2}{2}} \right)^{\frac{2^*}{2^*}} \left( \frac{2^*}{\mu^*} \right)^{\frac{\mu}{\mu}} \). Thus, together with (4.4), we yield that

\[ \frac{N + 2 - \mu}{2(2N - \mu)} a_{n}^{\frac{2}{\mu^*}} \left( \frac{\lambda}{\mu^*} \right)^{\frac{2}{\mu}} S_{H,L}^{\frac{2}{\mu}} - D \lambda^{\frac{2\mu}{\mu}} \leq c < \frac{N + 2 - \mu}{2(2N - \mu)} a_{n}^{\frac{2}{\mu^*}} \left( \frac{\lambda}{\mu^*} \right)^{\frac{2}{\mu}} S_{H,L}^{\frac{2}{\mu}} - D \lambda^{\frac{2\mu}{\mu}}, \]

which is a contradiction. It implies that \( J \) is empty, so we get

\[ \int_{\Omega} \int_{\Omega} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^\mu} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} \, dx \, dy, \text{ as } n \to \infty. \]

Since \( (I_{\lambda}'(u_n), u_n) \to 0 \) and \( (I_{\lambda}'(u_n), u_n) \to 0 \), let \( t = \lim_{n \to \infty} \|u_n\| \), we have

\[ (a + \varepsilon^{2}) |u|^2 - \lambda \int_{\Omega} f(x) |u|^q - \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} = o(1), \]

\[ (a + \varepsilon \|u_n\|^2) \|u_n\|^2 - \lambda \int_{\Omega} f(x) |u_n|^q - \int_{\Omega} \int_{\Omega} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^\mu} = o(1), \]

so

\[ (a + \varepsilon^{2})^2 - \lambda \int_{\Omega} f(x) |u|^q - \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} = 0, \]

moreover, \( u_n \to u \text{ in } H_0^1(\Omega) \). The proof of this lemma is completed. \( \square \)

For \( \varepsilon > 0 \), let \( \eta \in C_0^{\infty}(\Omega) \) be a cut-off function such that

\[ 0 \leq \eta \leq 1, \quad \eta(x) = \begin{cases} 1, & |x| \leq r, \\ 0, & |x| > 2r, \end{cases} \quad |\nabla \eta| \leq C. \]

Set

\[ u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \]

where

\[ U_\varepsilon(x) = \varepsilon^{\frac{N-2}{2}} U \left( \frac{x}{\varepsilon} \right), \quad U(x) = \frac{(N(N-2))^{\frac{N-2}{2}}}{(1 + |x|^2)^\frac{N-2}{2}}. \]

Moreover

\[ \| u_\varepsilon \|^2 = C(N, \mu) a_{n}^{\frac{1}{\mu^*}} S_{H,L}^{\frac{2}{\mu}} + O(\varepsilon^{N-2}), \tag{4.5} \]
\[
\int_{\Omega} \int_{\Omega} \frac{|u_{x}(x)|^{2}\nabla u_{y}(y)|^{2\mu}}{|x-y|^{\mu}} dxdy \geq C(N, \mu) \frac{\varepsilon^{2N-\mu}}{2} S_{H,L}^{\frac{2N-\mu}{4}} + O(\varepsilon^{N-\frac{\mu}{4}}) .
\]

**Lemma 4.3.** Assume that \( a > 0, 1 < q < 2, 0 < \mu < 3, f \in L^\infty(\Omega) \) is nonnegative, and for every \( \varepsilon > 0 \) small enough, it follows that

\[
\sup_{t \geq 0} I_{\lambda,a}(u_{*} + tu_{e}) < \frac{N + 2 - \mu}{2(2N - \mu)} a \lambda^{2.5} - DL \varepsilon^{\frac{\mu}{4}}.
\]

*Proof.* Recalling that \( u_{*} \) is a positive solution of (1.1), \( I_{\lambda,a}(u_{*}) < 0 \), for \( t \geq 0 \) by convexity and Hölder inequality, we obtain

\[
I_{\lambda,a}(u_{*} + tu_{e}) = \frac{a}{2} \|u_{*}\|^2 + at \int \nabla u_{x} \nabla u_{e} + \frac{at^2}{4} \|u_{*}\|^4 + \varepsilon t^2 \left( \frac{1}{4} \int \nabla u_{x} \nabla u_{e} \right)^2 + \varepsilon t^3 \|u_{*}\|^2 \int \nabla u_{x} \nabla u_{e}
\]

\[
- \frac{1}{2} \cdot 2^{\mu}_{\lambda} \int_{\Omega} \int_{\Omega} [u_{x}(x) + tu_{e}(x)]^{2} |u_{y}(y) + tu_{e}(y)|^{2\mu} dxdy - \lambda \int_{\Omega} f(x)(u_{*} + tu_{e})^q
\]

\[
\leq I_{\lambda,a}(u_{*}) + \frac{at^2}{2} \|u_{*}\|^2 + at \int \nabla u_{x} \nabla u_{e} + \frac{at^2}{4} \|u_{*}\|^4 + \frac{\varepsilon t^2}{2} \|u_{*}\|^2 \|u_{e}\|^2 + \varepsilon t^3 \|u_{*}\|^2 \int \nabla u_{x} \nabla u_{e}
\]

\[
- \frac{1}{2} \cdot 2^{\mu}_{\lambda} \int_{\Omega} \int_{\Omega} [u_{x}(x)]^{2} |u_{y}(y)|^{2\mu} dxdy - Ct^{2.5} - 1 \int_{B_{r}} \int_{B_{r}} |u_{x}(x) - u_{y}(y)|^{2\mu} dxdy,
\]

where \( u_{*}(x) \geq C > 0 \) on \( B_{r} \). By direct calculation, one has

\[
\int_{B_{r}} \int_{B_{r}} |u_{x}(x)|^{2\mu} - 1 |u_{y}(y)|^{2\mu} dxdy
\]

\[
= \int_{B_{r}} \int_{B_{r}} |\varepsilon^{2N} U(\varepsilon x)^{2\mu} - 1| \varepsilon^{2N} U(\varepsilon y)^{2\mu} dxdy
\]

\[
= \int_{B_{r}} \int_{B_{r}} \varepsilon^{2N} \varepsilon^{2N} U(\varepsilon x)^{2N} \varepsilon^{2N} \varepsilon^{2N} U(\varepsilon y)^{2N} |x-y|^{-\mu} dxdy
\]
\[
= \varepsilon^{N+2-\mu/2} \left| N(N-2) \right|^{N-2} \int_{B_r} \int_{B_r} \frac{1}{\left(1 + |t|^2 \right)^{N-\mu/2}} \left(1 + |s|^2 \right)^{2N-\mu} \left| x - y \right|^\mu dxdy
\]

\[
= \varepsilon^{N+2-\mu/2} \left| N(N-2) \right|^{N-2} \times \int_{B_r} \int_{B_r} \frac{1}{\left(1 + |t|^2 \right)^{N-\mu/2}} \left(1 + |s|^2 \right)^{2N-\mu} \left| t - s \right|^\mu dtds
\]

\[
= \varepsilon^{N+2-\mu/2} \left| N(N-2) \right|^{N-2} \int_{B_r} \int_{B_r} \frac{1}{\left(1 + |t|^2 \right)^{N-\mu/2}} \left(1 + |s|^2 \right)^{2N-\mu} \left| t - s \right|^\mu dtds
\]

\[
\geq \varepsilon^{N+2-\mu/2} \left| N(N-2) \right|^{N-2} \int_{B_r} \int_{B_r} \frac{1}{\left(1 + |t|^2 \right)^{N-\mu/2}} \left(1 + |s|^2 \right)^{2N-\mu} \left| t - s \right|^\mu dtds
\]

\[
= O(\varepsilon^{N+2-\mu/2}, \varepsilon < 1),
\]

(4.7)

where, by the Hardy-Littlewood-Sobolev inequality, we have

\[
\int_{B_r} \int_{B_r} \frac{1}{\left(1 + |t|^2 \right)^{N+2-\mu/2}} \left(1 + |s|^2 \right)^{2N-\mu} \left| t - s \right|^\mu dtds \leq \int_{B_r} \int_{B_r} \frac{1}{\left| t - s \right|^{\mu}} dtds < \infty.
\]

By Hölder inequality, Young inequality and the boundedness of \(u_\ast(u_\ast \in B_r)\), we deduce that

\[
J(tu_\ast)
\]

\[
= \frac{at^2}{2} \left\| u_\ast \right\|^2 + at\left\| u_\ast \right\|^1 \left\| u_\ast \right\|^1 + \frac{\varepsilon}{4} t^4 \left\| u_\ast \right\|^4 + \varepsilon t^3 \left\| u_\ast \right\|^3 \left\| u_\ast \right\|^3 + \frac{3}{2} \varepsilon t^2 \left\| u_\ast \right\|^2 \left\| u_\ast \right\|^2 + \varepsilon t \left\| u_\ast \right\|^3 \left\| u_\ast \right\|^3
\]

\[
- \frac{t^2}{2 \cdot 2^\ast\mu} \int_{\Omega} \int_{\Omega} \frac{|u_\ast(x)|^{2^\ast\mu} |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy - Ct^{2^\ast\mu} \int_{B_r} \int_{B_r} \frac{|u_\ast(x)|^{2^\ast\mu} - 1 |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy
\]

\[
\leq \frac{at^2}{2} \left\| u_\ast \right\|^2 + at\left\| u_\ast \right\|^1 \left\| u_\ast \right\|^1 + \frac{\varepsilon}{4} t^4 \left\| u_\ast \right\|^4 + \varepsilon t^3 \left\| u_\ast \right\|^3 \left\| u_\ast \right\|^3 + \frac{3}{2} \varepsilon t^2 \left\| u_\ast \right\|^2 \left\| u_\ast \right\|^2 + \varepsilon t \left\| u_\ast \right\|^3 \left\| u_\ast \right\|^3
\]

\[
- \frac{t^2}{2 \cdot 2^\ast\mu} \int_{\Omega} \int_{\Omega} \frac{|u_\ast(x)|^{2^\ast\mu} |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy - Ct^{2^\ast\mu} \int_{B_r} \int_{B_r} \frac{|u_\ast(x)|^{2^\ast\mu} - 1 |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy
\]

\[
\leq \frac{at^2}{2} \left\| u_\ast \right\|^2 + \frac{3\varepsilon t^4}{4} \left\| u_\ast \right\|^4 + \frac{at^4}{4} \left\| u_\ast \right\|^4 + \frac{13\varepsilon t^2}{4} \left\| u_\ast \right\|^2 + \frac{3}{4} \left\| u_\ast \right\|^3
\]

\[
- \frac{t^2}{2} \cdot 2^\ast\mu \int_{\Omega} \int_{\Omega} \frac{|u_\ast(x)|^{2^\ast\mu} |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy - Ct^{2^\ast\mu} \int_{B_r} \int_{B_r} \frac{|u_\ast(x)|^{2^\ast\mu} - 1 |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy.
\]

Thus, it is sufficient to show that

\[
\sup_{t \geq 0} J(tu_\ast) = \frac{N + 2 - \mu}{2(2N - \mu)} \frac{\varepsilon^2}{\varepsilon_0^2} 2^\ast\mu \int_{B_r} \int_{B_r} \frac{|u_\ast(x)|^{2^\ast\mu} |u_\ast(y)|^{2^\ast\mu}}{|x - y|^\mu} dxdy - D\lambda \pi^2.
\]
We claim that there exists \( t_\varepsilon > 0 \), such that \( \sup_{t \geq 0} h(t) = h(t_\varepsilon) \), and there exist two positive constants \( t_1, t_2 \) independent on \( \varepsilon, \lambda \), such that

\[
0 < t_1 \leq t_\varepsilon \leq t_2 < \infty.
\]  

(4.8)

It is easy to see that \( \lim_{t \to +\infty} h(t) = -\infty \), there exists \( t_\varepsilon > 0 \), such that

\[
h(t_\varepsilon) = \sup_{t \geq 0} h(t_\varepsilon), \quad \frac{dh(t)}{dt} \bigg|_{t=t_\varepsilon} = 0.
\]  

(4.9)

By (4.9), we obtain

\[
t_\varepsilon a \| u_\varepsilon \|^2 + 6t_\varepsilon^3 \| u_\varepsilon \|^4 + a t_\varepsilon^3 \| u_\varepsilon \|^4 - t_\varepsilon^{2^*_\mu - 1} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy

- C(2 \cdot 2^*_\mu - 1) t_\varepsilon^{2^*_\mu - 2} \int \int_{B_r} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu - 1}}{|x-y|^{\mu}} \, dx \, dy = 0,
\]  

(4.10)

and

\[
a \| u_\varepsilon \|^2 + 3(6\varepsilon + a) t_\varepsilon^2 \| u_\varepsilon \|^4 - (2 \cdot 2^*_\mu - 1) t_\varepsilon^{2^*_\mu - 2} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy

- C(2 \cdot 2^*_\mu - 1)(2 \cdot 2^*_\mu - 2) t_\varepsilon^{2^*_\mu - 3} \int \int_{B_r} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu - 1}}{|x-y|^{\mu}} \, dx \, dy < 0.
\]  

(4.11)

On the one hand, from (4.11), we get \( t_\varepsilon \) is bounded below, so there exists a positive constant \( t_1 > 0 \), such that \( 0 < t_1 \leq t_\varepsilon \). On the other hand, by (4.10), which implies that \( t_\varepsilon \) is bounded above for any \( \varepsilon > 0 \) small enough, so there exists a positive constant \( t_2 > 0 \) such that \( t_2 > t_\varepsilon \). Thus (4.8) holds. Set

\[
h_1(t_\varepsilon) = \frac{a t_\varepsilon^2}{2} \| u_\varepsilon \|^2 - \frac{t_\varepsilon^{2^*_\mu}}{2 \cdot 2^*_\mu} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy,
\]

\[
h_2(t_\varepsilon) = \frac{3}{2} \varepsilon t_\varepsilon^2 \| u_\varepsilon \|^4 + \frac{at_\varepsilon^4}{4} \| u_\varepsilon \|^4 + 3 \frac{a}{4} t_\varepsilon^{2^*_\mu - 1} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu - 1}}{|x-y|^{\mu}} \, dx \, dy,
\]

\[
h_1'(t_\varepsilon) = at_\varepsilon \| u_\varepsilon \|^2 - t_\varepsilon^{2^*_\mu - 1} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy = 0,
\]

and

\[
t_\varepsilon = \left( \frac{a \| u_\varepsilon \|^2}{\int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy} \right)^{\frac{1}{2^*_\mu - 2}}.
\]

So

\[
h_1(t_\varepsilon)
\]

\[
= \frac{aq^2}{2} \| u_\varepsilon \|^2 - \frac{t_\varepsilon^{2^*_\mu}}{2 \cdot 2^*_\mu} \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy
\]

\[
= \left( 1 - \frac{1}{2 \cdot 2^*_\mu} a \| u_\varepsilon \|^2 \left( \int \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\mu} |u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}} \, dx \, dy \right)^{\frac{1}{2^*_\mu - 2}} \right)^{-1}
\]
\[
\begin{align*}
&= \left( \frac{1}{2} - \frac{1}{2 \cdot 2N} \right) \frac{(a\|u_\varepsilon\|^2)^{\frac{2\varepsilon}{\sqrt{2}}} \cdot \varepsilon^{\frac{2\varepsilon}{\sqrt{2} - 2}}}{\|u_\varepsilon\|^2 S_{H,L}^{-1} \cdot \varepsilon^{\frac{2\varepsilon}{\sqrt{2} - 2}}} = \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2}}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1}. \\
h_2(t_\varepsilon) &= \frac{3}{2} t_\varepsilon^4 \|u_\varepsilon\|^4 + \frac{at_\varepsilon^4}{4} \|u_\varepsilon\|^4 + \frac{3}{4} a t_\varepsilon^4 - C t_\varepsilon^{2\varepsilon - 1} - \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^{\frac{2\varepsilon}{\sqrt{2} - 2}}}{|x - y|^\mu} dx dy \\
&\leq \frac{3}{2} t_\varepsilon^4 \|u_\varepsilon\|^4 + \frac{at_\varepsilon^4}{4} \|u_\varepsilon\|^4 + \frac{3}{4} a t_\varepsilon^4 - C t_\varepsilon^{2\varepsilon - 1} - \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^{\frac{2\varepsilon}{\sqrt{2} - 2}}}{|x - y|^\mu} dx dy \\
&\leq C \varepsilon \left( C(N, \mu) \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N - 2}) \right)^2 + C \left( C(N, \mu) \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N - 2}) \right)^2 C \varepsilon \frac{\varepsilon}{\sqrt{2} - 2} \\
&\leq C \varepsilon + C(\varepsilon^{N - 2}) - C \varepsilon \frac{\varepsilon}{\sqrt{2} - 2} \leq C_1 \varepsilon - C_2 \varepsilon \frac{\varepsilon}{\sqrt{2} - 2}.
\end{align*}
\]

Since \( N = 3 \), hence \( h_2(t_\varepsilon) \leq C_1 \varepsilon - C_2 \varepsilon \frac{\varepsilon}{\sqrt{2} - 2} \). By (4.5)-(4.7), one has

\[
\sup_{t \geq 0} I_{\lambda, \mu}(u_* + tu_\varepsilon) \leq \sup_{t \geq 0} J(tu_\varepsilon) \leq \sup_{t \geq 0} h(t_\varepsilon) + \frac{13t_\varepsilon^4}{4} \\
\leq \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2} - 2}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1} + \frac{3t_\varepsilon^4}{4} \left( C(N, \mu) \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N - 2}) \right)^2 \\
+ \frac{at_\varepsilon^4}{4} \left( C(N, \mu) \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N - 2}) \right)^2 - C t_\varepsilon^{2\varepsilon - 1} - \frac{13t_\varepsilon^4}{4} \\
\leq \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2} - 2}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1} + C(1 + \varepsilon)(1 + O(\varepsilon^{N - 2})) + C\varepsilon - C \varepsilon \frac{\varepsilon}{\sqrt{2} - 2} \\
\leq \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2} - 2}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1} + \varepsilon - C \varepsilon \frac{\varepsilon}{\sqrt{2} - 2}.
\]

Let \( \varepsilon = \lambda^\frac{\varepsilon}{\sqrt{2} - 2} \), for \( \lambda \in (0, \lambda_1) \) with \( \lambda_1 = \left( \frac{C_1 + C_2}{C_2} \right) \frac{\sqrt{2} - 2}{\sqrt{2} - 2} \), and \( \varepsilon \) small enough, we derive that

\[
C_1 \varepsilon - C_2 \varepsilon \frac{\varepsilon}{\sqrt{2} - 2} = C_1 \lambda^\frac{\varepsilon}{\sqrt{2} - 2} - C_2 \lambda^\frac{\varepsilon}{\sqrt{2} - 2} = \lambda^\frac{\varepsilon}{\sqrt{2} - 2} \left( C_1 - C_2 \lambda^\frac{\varepsilon}{\sqrt{2} - 2} \right) \\
\leq \lambda^\frac{\varepsilon}{\sqrt{2} - 2} \left( C_1 - C_2 \lambda^\frac{\varepsilon}{\sqrt{2} - 2} \right) = -D \lambda^\frac{\varepsilon}{\sqrt{2} - 2},
\]

moreover

\[
\sup_{t \geq 0} J(tu_\varepsilon) < \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2} - 2}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1} - D \lambda^\frac{\varepsilon}{\sqrt{2} - 2}.
\]

Hence

\[
\sup_{t \geq 0} I_{\lambda, \mu}(u_* + tu_\varepsilon) < \frac{N + 2 - \mu}{2(2N - \mu)} a^{\frac{\varepsilon}{\sqrt{2} - 2}} \frac{2\varepsilon}{\sqrt{2} - 2} S_{H,L}^{-1} - D \lambda^\frac{\varepsilon}{\sqrt{2} - 2}.
\]

This completes the proof of Lemma 4.3.

**Lemma 4.4.** Assume that \( a > 0, 1 < q < 2 \), for any \( \varepsilon > 0 \) small enough, there exists \( \lambda_* > 0 \) such that for \( \lambda \in (0, \lambda_*) \), problem (1.1) possesses a positive solution \( \tilde{u} \) with \( I_{\lambda,m}(\tilde{u}) > 0 \).
Proof. Let $u_\varepsilon \in H^1_0(\Omega)$, $u_\varepsilon \neq 0$, then for all $t \in \mathbb{R}$ with $t > 0$, we get
\[
I_{\lambda,\mu}(tu_\varepsilon) = \frac{at^2}{2} \|u_\varepsilon\|^2 + \frac{\varepsilon t^4}{4} \|u_\varepsilon\|^4 - \frac{t^q}{q} \int_{\Omega} f(x) |u_\varepsilon|^q - \frac{t^{2\mu}}{2\mu} \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2\mu} |u_\varepsilon(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy
\]
\[
\leq \frac{at^2}{2} \|u_\varepsilon\|^2 + \frac{\varepsilon t^4}{4} \|u_\varepsilon\|^4 - \frac{t^{2\mu}}{2\mu} \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2\mu} |u_\varepsilon(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy,
\]
which implies that there exists $t_0 > 0$ such that $I_{\lambda,\mu}(t_0 u_\varepsilon) < 0$ and $\|t_0 u_\varepsilon\| > R$. Let $\lambda_* = \min\{\lambda_0, \lambda_1\}$, then Lemmas 4.1-4.3 hold for $0 < \lambda < \lambda_*$. By Lemma 4.1, $I_{\lambda,\mu}$ satisfies the geometry of the mountain pass, using a general mountain pass principle for locating and classifying critical points [3], there exists a sequence $\{u_n\} \subset H^1_0(\Omega)$, such that
\[
I_{\lambda,\mu}(u_n) \to c > \rho, \quad \text{and} \quad I'_{\lambda,\mu}(u_n) \to 0,
\]
where
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\mu}(\gamma(t)),
\]
and
\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = u_*, \quad \gamma(1) = u_* + t_0 u_\varepsilon \}.
\]
From Lemma 4.1 and Lemma 4.3, we have
\[
0 < \rho < c \leq \max_{t \in [0,1]} I_{\lambda,\mu}(u_* + tu_\varepsilon) \leq \sup_{t \geq 0} I_{\lambda,\mu}(u_* + tu_\varepsilon)
\]
\[
\leq \frac{N + 2 - \mu}{2(2N - \mu)} \|u_*\|^2_\ast \|\nabla^\mu u_*\|^2_\ast - D \lambda^\frac{2}{2N - \mu}.
\]
By Lemma 4.2, it is easy to know that $\{u_n\} \subset H^1_0(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, we assume that $u_n \to \bar{u}$ in $H^1_0(\Omega)$ as $n \to \infty$. From (4.12),(4.13) we get
\[
I_{\lambda,\mu}(\bar{u}) = \lim_{n \to \infty} I_{\lambda,\mu}(u_n) = c > \rho > 0.
\]
By (4.14), we derive that $\bar{u} \neq 0$. According to the continuity of $I'_{\lambda,\mu}$, we infer that $\bar{u}$ is a solution to (1.1), that is
\[
(a + \varepsilon \|\bar{u}\|^2) \int_{\Omega} \nabla \bar{u} \nabla \varphi - \int_{\Omega} \int_{\Omega} \frac{|(\bar{u}(x))^+|^{2\mu} |(\bar{u}(y))^+|^{2\mu} - 2(\bar{u}(y))^+ \varphi(y)}{|x-y|^\mu} \, dx \, dy
\]
\[
- \lambda \int_{\Omega} f(x)(\bar{u}(x))^q \varphi = 0
\]
for all $\varphi \in H^1_0(\Omega)$. Taking the test function $\varphi = \bar{u}^-$ in (4.15), then $\|\bar{u}^-\|^2 = 0$, so $\bar{u} \geq 0$ and $\bar{u} \neq 0$. Using the strong maximum principle we conclude that $\bar{u}$ is a positive solution to (1.1). \qed

5. Existence of a positive ground state solution to (1.1). Set
\[
d = \inf_{u \in K} I_{\lambda,\mu}(u),
\]
where
\[
K = \{ u \in H^1_0(\Omega) \mid u \neq 0, \quad I'_{\lambda,\mu}(u) = 0 \text{ in } (H^1_0(\Omega))^\ast \}.
\]
According to the definition of $d$, there exists $\{u_n\} \subset H^1_0(\Omega)$ such that $u_n \neq 0$, it follows that
\[
I_{\lambda,\mu}(u_n) \to d, \quad I'_{\lambda,\mu}(u_n) = 0, \quad (n \to \infty).
\]
Consequently, by (5.1), it is easy to see that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Up to a subsequence, we assume that

\[
\begin{aligned}
&u_n \to u_0, & \text{in} & & H^1_0(\Omega), \\
u_n \to u_0, & \text{in} & & L^p(\Omega), & 1 \leq p < 2^*, \\
u_n(x) \to u_0(x), & \text{a.e.} & & \text{in} & & \Omega.
\end{aligned}
\]  

(5.2)

Now, we'll show \( u_0 \) is a solution to problem (1.1). Since \( I_{\lambda, \mu}'(u_n) = 0 \) as \( n \to \infty \), then

\[
(a + \varepsilon \|u_n\|^2) \int_{\Omega} \nabla u_n \varphi + \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^{2^*_\mu}|u_n^+(y)|^{2^*_\mu} - 2u_n^+(y) \varphi(y)}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)(u_n^+)^{q-1} \varphi = o(1).
\]

Let \( \lim_{n \to \infty} \|u_n\| = l \), we have

\[
(a + \varepsilon l^2) \int_{\Omega} \nabla u_0 \varphi - \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu} - 2u_0^+(y) \varphi(y)}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)(u_0^+)^{q-1} \varphi = 0.
\]  

(5.3)

Taking the test function \( \varphi = u_0 \) in (5.3) infers that

\[
(a + \varepsilon \|u_n\|^2)\|u_0\|^2 - \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu}}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)(u_0^+)^q = 0.
\]  

(5.4)

Noting that

\[
(I_{\lambda, \mu}'(u_n), u_n) \to 0, \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^{2^*_\mu}|u_n^+(y)|^{2^*_\mu}}{|x-y|} dxdy \to \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu}}{|x-y|} dxdy,
\]

one has

\[
(a + \varepsilon \|u_n\|^2)\|u_0\|^2 - \int_{\Omega} \int_{\Omega} \frac{|u_n^+(x)|^{2^*_\mu}|u_n^+(y)|^{2^*_\mu}}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)(u_0^+)^q = o(1).
\]

By Lemma 4.2, one has

\[
(a + \varepsilon l^2)l^2 - \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu}}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)u_0^+ = 0.
\]  

(5.5)

From (5.4) and (5.5), we have \( \|u_0\| = l \), thus \( u_n \to u_0 \) in \( H^1_0(\Omega) \), and \( u_0 \) is a solution to problem (1.1), that is

\[
(a + \varepsilon \|u_0\|^2) \int_{\Omega} \nabla u_0 \nabla \varphi - \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu} - 2u_0^+(y) \varphi(y)}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)|u_0^+|^{q-1} \varphi = 0
\]  

(5.6)

for all \( \varphi \in H^1_0(\Omega) \). Taking the test function \( \varphi = u_0^- \) in (5.6), we get \( \|u_0^-\| = 0 \), thus \( u_0 \geq 0 \), which implies that \( I_{\lambda, \mu}(u_0) \geq 0 \). Now, we'll show that \( d \geq I_{\lambda, \mu}(u_0) \). Since \( I_{\lambda, \mu}'(u_0) = 0 \), we have

\[
I_{\lambda, \mu}(u_0) = \frac{a}{2} \|u_0\|^2 + \frac{c}{4} \|u_0\|^4 - \frac{1}{2 \cdot 2^*_\mu} \int_{\Omega} \int_{\Omega} \frac{|u_0^+(x)|^{2^*_\mu}|u_0^+(y)|^{2^*_\mu}}{|x-y|} dxdy - \lambda \int_{\Omega} f(x)(u_0^+)^q
\]
Proof of Theorem 1.2.

Coupling with Lemma 3.2 and 4.4, this completes the proof of Theorem 1.1.

Furthermore, by (5.1), one has

$$d + o(1) = I_{\lambda,\mu}(u_n) - \frac{1}{2 \cdot 2^\mu} \left( a\|u_n\|^2 - \varepsilon\|u_n\|^4 \right)$$

$$- \int_\Omega \int_\Omega \frac{|u_n^+(x)|^2 |u_n^+(y)|^2}{|x-y|^\mu} \, dx \, dy - \lambda \int_\Omega f(x)(u_n^+)^q \right)$$

$$= \left( \frac{1}{2} - \frac{1}{2 \cdot 2^\mu} \right) a\|u_n\|^2 + \left( \frac{1}{4} - \frac{1}{2 \cdot 2^\mu} \right) \varepsilon\|u_n\|^4 - \left( \frac{1}{q} - \frac{1}{2 \cdot 2^\mu} \right) \lambda \int_\Omega f(x)(u_n^+)^q.$$

By the Fatou’s Lemma, one has

$$d \geq \left( \frac{1}{2} - \frac{1}{2 \cdot 2^\mu} \right) a\|u_0\|^2 + \left( \frac{1}{4} - \frac{1}{2 \cdot 2^\mu} \right) \varepsilon\|u_0\|^4 - \left( \frac{1}{q} - \frac{1}{2 \cdot 2^\mu} \right) \lambda \int_\Omega f(x)(u_0^+)^q = I_{\lambda,\mu}(u_0),$$

which implies that $u_0 \neq 0$ satisfies $I'_{\lambda,\mu}(u_0) = 0$ and $I_{\lambda,\mu}(u_0) = d$. Consequently, using the strong maximum principle, $u_0 > 0$ in $\Omega$. Moreover, $u_0$ is a ground state solution to problem (1.1). □

6. Proof of Theorem 1.2. In order to obtain the infinitely many pairs of distinct solutions, it follows that

**Lemma 6.1.** Let $X$ be a Banach space, and $I \in C^1(X, \mathbb{R})$ be an even function satisfying the (PS) condition. Assume that $\alpha < \beta$ and either $I(0) < \alpha$ or $I(0) > \beta$. If further,

1. there are an $m$-dimensional linear subspace $E$ and $\rho > 0$ such that $\sup_{x \in \mathcal{E} \cap B_\rho(0)} I(x) \leq \beta,$

2. there is a $j$-dimensional linear subspace $F$ such that $\inf_{x \in F} I(x) > \alpha,$

3. $m > j,$

then $I$ has at least $m - j$ pairs of distinct critical points.

**Lemma 6.2.** Assume that $a > 0, 1 < q < 2, 4 < \mu < N, \varepsilon$ small enough, the functional $I_{\lambda,\mu}$ satisfies the (PS) condition in $H_0^1(\Omega)$.

Proof. For any $c \in \mathbb{R}$, suppose that $\{u_n\} \subset H_0^1(\Omega)$ is a (PS)$_c$ sequence of $I_{\lambda,\mu}$, that is

$$I_{\lambda,\mu}(u_n) \to c, \quad I'_{\lambda,\mu}(u_n) \to 0, \quad n \to \infty.$$

(6.1)

Since $1 < q < 2$, by (2.2), (2.3), (6.1) and the H"older inequality, for $n$ large enough, we yield

$$1 + c + \|u_n\|^2 \geq I_{\lambda,\mu}(u_n) - \frac{1}{2 \cdot 2^\mu} (I'_{\lambda,\mu}(u_n), u_n)$$

$$\geq \left( \frac{1}{2} - \frac{1}{2 \cdot 2^\mu} \right) a\|u_n\|^2 + \left( \frac{1}{4} - \frac{1}{2 \cdot 2^\mu} \right) \varepsilon\|u_n\|^4 - \left( \frac{1}{q} - \frac{1}{2 \cdot 2^\mu} \right) \lambda \int_\Omega f(x)(u_n^+)^q$$

$$\geq \frac{N - \mu + 2}{2(2N - \mu)} a\|u_n\|^2 - \frac{2 \cdot 2^\mu - q}{2 \cdot 2^\mu q} \lambda \|f\|_{L^q} S^{-\frac{q}{2}} \|u_n\|^q.$$
thus \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Going if necessary to a subsequence, still denoted by \( \{u_n\} \), there exists \( u^* \in H^1_0(\Omega) \) such that

\[
\begin{aligned}
&u_n \rightharpoonup u^* \quad \text{in} \quad H^1_0(\Omega), \\
u_n \rightarrow u^* \quad \text{in} \quad L^p(\Omega), \\
u_n(x) \rightarrow u^*(x), &\quad \text{a.e. in} \quad \Omega.
\end{aligned}
\]

(6.2)

We will show that \( u_n \rightharpoonup u^* \) as \( n \to \infty \) in \( H^1_0(\Omega) \). Let \( v_n = u_n - u^* \), we only need to prove that \( \|v_n\| \to 0 \) as \( n \to \infty \). Without loss of generality, set \( \lim_{n \to \infty} \|v_n\| = l \), by Brézis-Lieb Lemma, we have

\[
\begin{aligned}
&\lim_{n \to \infty} \int_\Omega |u_n|^q = \int_\Omega |u^*|^q, \\
&\|u_n\|^2 = \|v_n\|^2 + \|u^*\|^2 + o(1), \\
&\|u_n\|^4 = \|v_n\|^4 + \|u^*\|^4 + 2\|v_n\|^2\|u^*\|^2 + o(1), \\
&\int_\Omega \int_\Omega \frac{|u_n(x)|^2\|u_n(y)|^2}{|x-y|^\mu} dxdy = \int_\Omega \int_\Omega \frac{|u^*(x)|^2\|u(y)|^2}{|x-y|^\mu} dxdy + \int_\Omega \int_\Omega \frac{|v_n(x)|^2\|v(y)|^2}{|x-y|^\mu} dxdy + o(1).
\end{aligned}
\]

(6.3)

Since \( u_n \rightharpoonup u^* \) in \( H^1_0(\Omega) \), so \( u_n \rightharpoonup u^* \) in \( L^\infty(\Omega) \) as \( n \to \infty \). Then

\[ |u_n|^{2^*_\mu} \to |u^*|^{2^*_\mu} \text{ in } L^{\frac{2N}{N-\mu}}(\Omega), \quad n \to \infty. \]

According to the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N-\mu}}(\Omega) \) to \( L^{\frac{2N}{N-\mu}}(\Omega) \), one has

\[
\int_\Omega \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \to \int_\Omega \frac{|u^*(y)|^{2^*_\mu}}{|x-y|^\mu} \quad \text{in } L^{\frac{2N}{N-\mu}}(\Omega), \quad n \to \infty.
\]

Noting that

\[ |u_n|^{2^*_\mu-2}u_n \to |u^*|^{2^*_\mu-2}u^* \text{ in } L^{\frac{2N}{N-\mu}}(\Omega), \quad n \to \infty, \]

which implies that

\[
\left( \int_\Omega \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) u_n(x) \to \left( \int_\Omega \frac{|u^*(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) u^*(x) \quad \text{in } L^{\frac{2N}{N-\mu}}(\Omega),
\]

as \( n \to \infty \). So, for every \( \varphi \in H^1_0(\Omega) \), one has

\[
\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\mu}u_n(y)^{2^*_\mu-2}u_n(y)\varphi(y)}{|x-y|^\mu} dxdy \to \int_\Omega \int_\Omega \frac{|u^*(x)|^{2^*_\mu}u^*(y)^{2^*_\mu-2}u^*(y)\varphi(y)}{|x-y|^\mu} dxdy.
\]

Moreover, for every \( \varphi \in H^1_0(\Omega) \), we have

\[
(a + \varepsilon \|u_n\|^2) \int_\Omega \nabla u_n \nabla \varphi - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\mu}u_n(y)^{2^*_\mu-2}u_n(y)\varphi(y)}{|x-y|^\mu} dxdy
\]

\[
- \lambda \int_\Omega f(x)|u_n|^q \varphi \to 0.
\]

By (6.1), we obtain

\[
\langle I_{\lambda,n}(u_n), u_n \rangle
\]

\[
= (a + \varepsilon \|u_n\|^2) \int_\Omega |\nabla u_n|^2 - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\mu}u_n(y)^{2^*_\mu}}{|x-y|^\mu} - \lambda \int_\Omega f(x)|u_n|^q = o(1),
\]
and
\[ \langle I_{\lambda,\mu}(u_n), u^* \rangle = a \int_{\Omega} \nabla u_n \nabla u^* + \varepsilon \|u^*\|^2 + \varepsilon \|u_n - u^*\|^2 \int_{\Omega} \nabla u_n \nabla u^* \] (6.4)
\[
- \int_{\Omega} \int_{\Omega} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^* - 2} u_n(y) u^*(y)}{|x-y|^\mu} - \lambda \int_{\Omega} f(x)|u_n|^{q-1} u^*,
\]
then passing to the limit as \( n \to \infty \), we conclude
\[ a\|u^*\|^2 + \varepsilon \|u^*\|^4 + \varepsilon l^2\|u^*\|^2 - \lambda \int_{\Omega} f(x)|u^*|^q - \int_{\Omega} \int_{\Omega} \frac{|u^*(x)|^{2_\mu^*} |u^*(y)|^{2_\mu^*}}{|x-y|^\mu} = 0. \] (6.5)

On the other hand
\[ \langle I_{\lambda,\mu}(u_n), u_n \rangle = a\|u_n - u^*\|^2 + \varepsilon \|u_n - u^*\|^4 + 2\varepsilon \|u_n - u^*\|^2 \int_{\Omega} |u^*|^{2_\mu^* - 2} + \varepsilon \|u^*\|^2 - \lambda \int_{\Omega} f(x)|u_n|^q \]
\[ - \int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u^*(x)|^{2_\mu^*} |u_n(x) - u^*(y)|^{2_\mu^*}}{|x-y|^\mu} - \int_{\Omega} \int_{\Omega} \frac{|u^*(x)|^{2_\mu^*} |u^*(y)|^{2_\mu^*}}{|x-y|^\mu} = o(1). \]

Combining (6.5) with (6.6), we get
\[ (a + \varepsilon \|v_n\|^2)\|v_n\|^2 + \varepsilon \|v_n\|^2 \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_\mu^*} |v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dy = o(1). \] (6.7)

From (2.3), (2.4), one has
\[ a l^2 + \varepsilon l^4 \leq S_{H,L}^{2_\mu^*} l^{2_\mu^*}, \]
so
\[ a l^2 + \varepsilon l^4 \leq S_{H,L}^{2_\mu^*} l^{2_\mu^*}, \]
using the Young inequality
\[ a l^2 + b l^4 \leq S_{H,L}^{2_\mu^*} l^{4(2_\mu^* - 1)} \left( \frac{\varepsilon}{2_\mu^* - 1} \right)^{2_\mu^* - 1} \]
\[ \leq (2 - 2_\mu^*) \left[ S_{H,L}^{2_\mu^*} l^{4(2_\mu^* - 1)} \left( \frac{2_\mu^* - 1}{\varepsilon} \right)^{2_\mu^* - 1} \right] \frac{1}{2_\mu^*} + (2_\mu^* - 1) \left[ l^{4(2_\mu^* - 1)} \left( \frac{\varepsilon}{2_\mu^* - 1} \right) \right] \frac{1}{2_\mu^*}, \]
that is
\[ \left\{ a - (2 - 2_\mu^*) \left[ S_{H,L}^{2_\mu^*} \left( \frac{2_\mu^* - 1}{\varepsilon} \right)^{2_\mu^* - 1} \right] \frac{1}{2_\mu^*} \right\} l^2 \leq 0, \]
so \( l = 0 \) for \( a > (2 - 2_\mu^*) \left[ S_{H,L}^{2_\mu^*} \left( \frac{2_\mu^* - 1}{\varepsilon} \right)^{2_\mu^* - 1} \right] \frac{1}{2_\mu^*} \), moreover, \( u_n \to u^* \) as \( n \to \infty \) in \( H_0^1(\Omega) \). The proof of the lemma is completed.

Proof of Theorem 1.2. Noting that the function \( I_{\lambda,\mu} \in C^1(H_0^1(\Omega), \mathbb{R}) \) is an even function. It is easy to see that the energy functional \( I_{\lambda,\mu} \) satisfies the conditions of Lemma 6.1. By Lemma 6.2, \( I_{\lambda,\mu} \) satisfies the (PS) condition. Recalling that
1 < p < 2, 4 < \mu < N$, choosing $E = H^1_0(\Omega)$ and $F = \emptyset$, from Lemma 6.2, then there exists $\rho > 0$ such that

$$\sup_{u \in E \cap \partial B_{\rho}(0)} I_{\lambda, \mu}(u) \leq \beta < 0 = I_{\lambda, \mu}(0),$$

where $\partial B_{\rho}(0) = \{u \in H^1_0(\Omega)||u|| = \rho\}$, and

$$\inf_{u \in F_\rho} I_{\lambda, \mu}(u) > -\infty.$$ Consequently, $I_{\lambda, \mu}$ satisfies the conditions of Lemma 6.1, which implies that $I_{\lambda, \mu}$ has infinitely many pairs of distinct critical points on $H^1_0(\Omega)$, therefore, problem (1.1) has infinitely many pairs of distinct solutions. This completes the proof of Theorem 1.2. \qed

7. **Proof of Theorem 1.3.** In this section, we will, by using Pohožaev identity, prove a nonexistence result for (1.1) with $\lambda \leq 0$ when $\Omega$ is a star shaped domain. In order to prove the Pohožaev identity, we’ll show that some regularity.

**Lemma 7.1.** Let $0 \leq u \in H^1_0(\Omega)$ be a weak solution of (1.1), then $u \in L^\infty(\Omega)$.

**Lemma 7.2.** If $N \geq 3, \lambda < 0$, and $u \in H^1_0(\Omega)$ is a solution to problem (1.1), it follows that

$$\left(\alpha + \varepsilon \int_{\Omega} |\nabla u|^2 \right) \int_{\partial \Omega} \frac{|\nabla u|^2}{2} x_{x'} ds + \frac{N - 2}{2} (\alpha + \varepsilon \int_{\Omega} |\nabla u|^2) \int_{\Omega} |\nabla u|^2$$

$$= \frac{2N - \mu}{2 \cdot 2^\mu} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x - y|^\mu} \, dx \, dy - \frac{N \lambda}{q} \int_{\Omega} |f(x)|^q, \quad \nu \text{ denotes the unit outward normal to } \partial \Omega.$$}

**Proof.** Let $\psi \in C^1_0(\Omega)$ and $\psi = 1$ on $B_1$. By testing the equation against the function $v_\tau \in H^1_0(\Omega)$ defined for $\tau \in (0, \infty)$ and $x \in \Omega$ by

$$v_\tau = \psi(\tau x) x \cdot \nabla u,$$

we have

$$- \int_{\Omega} (\psi(\tau x) x \cdot \nabla u)(\alpha + \varepsilon \int_{\Omega} |\nabla u|^2) \Delta u$$

$$= \int_{\Omega} (\psi(\tau x) x \cdot \nabla u) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x - y|^\mu} \, dy \right) |u|^{2^*_\mu - 2} u \, dx + \int_{\Omega} (\psi(\tau x) x \cdot \nabla u) \lambda f(x) |u|^{q - 2} u.$$ For every $\tau > 0$, we have

$$(\psi(\tau x) x \cdot \nabla u) \Delta u = \text{div} (\nabla u \psi(\tau x) \nabla u) - \nabla (\nabla u \psi(\tau x)) \nabla u,$$

where

$$\nabla (\psi(\tau x) \nabla u) \nabla u = \nabla (\psi(\tau x) \nabla u) \nabla u + \psi(\tau x) (\nabla (\nabla u)) \nabla u,$$

and

$$\nabla (x \nabla u) \nabla u = \nabla \left( \sum_{i=1}^{n} x_i u_{x_i} \right) \nabla u$$

$$= \frac{\partial u}{\partial x_1} \frac{\partial}{\partial x_1} \left( \sum_{i=1}^{n} x_i u_{x_i} \right) + \frac{\partial u}{\partial x_2} \frac{\partial}{\partial x_2} \left( \sum_{i=1}^{n} x_i u_{x_i} \right) + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial}{\partial x_n} \left( \sum_{i=1}^{n} x_i u_{x_i} \right)$$
By Lebesgue dominate convergence theorem, one has

\[ \int |\nabla u|^2 + \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_j u_{j,i} = |\nabla u|^2 + \frac{1}{2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} (\sum_{i=1}^{n} |u_i|^2) \]

so

\[ \nabla (\psi(\tau x) \nabla u(x)) \nabla u = \nabla \psi(\tau x) \nabla u \nabla u + \psi(\tau x) (\nabla (\nabla u) \nabla u) \]

\[ = \nabla \psi(\tau x) x |\nabla u|^2 + \frac{1}{2} |\nabla u|^2 x \psi(\tau x) + \psi(\tau x) |\nabla u|^2, \]

moreover

\[ \Delta u \psi(\tau x) x \nabla u(x) \]

\[ = \text{div}(\nabla u \psi(\tau x) x \nabla u(x)) - \nabla \left( \frac{|\nabla u|^2}{2} \psi(\tau x) x - |\nabla u|^2 \psi(\tau x) x - |\nabla u|^2 \psi(\tau x) x \right) \]

\[ = \text{div} \left( \nabla u \psi(\tau x) x \nabla u(x) - \psi(\tau x) x \frac{|\nabla u|^2}{2} \right) + \frac{N - 2}{2} \psi(\tau x) |\nabla u|^2 - \frac{1}{2} \psi(\tau x) x |\nabla u|^2. \]

Integrating by parts, we obtain

\[ \int_{\Omega} \Delta u \psi(\tau x) x \nabla u(x) \]

\[ = \int_{\partial \Omega} \left( \nabla u \psi(\tau x) x \nabla u(x) - \psi(\tau x) x \frac{|\nabla u|^2}{2} \right) \nu ds \]

\[ + \frac{N - 2}{2} \int_{\Omega} \psi(\tau x) |\nabla u|^2 - \int_{\Omega} \psi(\tau x) x \frac{|\nabla u|^2}{2} \]

\[ = \int_{\partial \Omega} \psi(\tau x) \frac{|\nabla u|^2}{2} x \nu ds + \frac{N - 2}{2} \int_{\Omega} \psi(\tau x) |\nabla u|^2 - \int_{\Omega} \psi(\tau x) x \frac{|\nabla u|^2}{2}. \]

By Lebesgue dominate convergence theorem, one has

\[ \lim_{\tau \to 0} \left( \int_{\partial \Omega} \psi(\tau x) \frac{|\nabla u|^2}{2} x \nu ds + \frac{N - 2}{2} \int_{\Omega} \psi(\tau x) |\nabla u|^2 - \frac{1}{2} \int_{\Omega} \psi(\tau x) x \frac{|\nabla u|^2}{2} \right) \]

\[ = \int_{\partial \Omega} \frac{|\nabla u|^2}{2} x \nu ds + \frac{N - 2}{2} \int_{\Omega} |\nabla u|^2. \]

Moreover

\[ - \int_{\Omega} \left( a + \epsilon \int_{\Omega} |\nabla u|^2 \right) \Delta u \psi(\tau x) x \nabla u(x) \]

\[ = - \frac{1}{2} \int_{\partial \Omega} \left( a + \epsilon \int_{\Omega} |\nabla u|^2 \right) |\nabla u|^2 x \nu ds - \frac{N - 2}{2} \int_{\Omega} \left( a + \epsilon \int_{\Omega} |\nabla u|^2 \right) |\nabla u|^2. \quad (7.1) \]

Calculate the term

\[ \left( \int_{\Omega} \frac{|u(y)|^{2n}}{|x - y|^\mu} dy \right) |u(x)|^{2n-1} \psi(\tau x) x \nabla u(x) \]

\[ = \left( \int_{\Omega} \frac{|u(y)|^{2n}}{|x - y|^\mu} dy \right) \psi(\tau x) x \nabla \frac{|u(x)|^{2n}}{2n-1} \]

\[ = \text{div} \left( \int_{\Omega} \frac{|u(y)|^{2n}}{|x - y|^\mu} dy \right) \psi(\tau x) x \nabla \frac{|u(x)|^{2n}}{2n-1} \]

\[ - \text{div} \left( \frac{\int_{\Omega} |u(y)|^{2n}}{|x - y|^\mu} dy \right) \psi(\tau x) |u(x)|^{2n} \]

\[ \frac{2n}{2n-1}. \]
Furthermore

\[- \nabla \left( \int_{\Omega} |u(y)|^{2*} \frac{\psi(\tau x)}{|x-y|^\mu} dy \right) \frac{x|u(x)|^{2*}}{2^*_\mu} \]

\[= \text{div} \left( \int_{\Omega} |u(y)|^{2*} \frac{\psi(\tau x) x|u(x)|^{2*}}{2^*_\mu} dy \right) - N \left( \int_{\Omega} |u(y)|^{2*} \frac{\psi(\tau x) x|u(x)|^{2*}}{2^*_\mu} dy \right) \]

\[+ \mu \left( \int_{\Omega} |u(y)|^{2*} \frac{x-y}{|x-y|^\mu} \psi(\tau x) x|u(x)|^{2*} \right) - \left( \int_{\Omega} |u(y)|^{2*} \frac{x-y}{|x-y|^\mu} \psi(\tau x) x|u(x)|^{2*} \right) \]

Similarly

\[\int_{\Omega} |u(x)|^{2*} \frac{\psi(\tau y) y\nabla u(y)}{|x-y|^\mu} dx \]

\[= \int_{\Omega} |u(x)|^{2*} \frac{\psi(\tau y) y\nabla \left( \frac{|u(y)|^{2*}}{2^*_\mu} \right)}{|x-y|^\mu} dx \]

\[= \text{div} \left( \int_{\Omega} |u(x)|^{2*} \frac{\psi(\tau y) y\frac{|u(y)|^{2*}}{2^*_\mu}}{|x-y|^\mu} dx \right) - N \left( \int_{\Omega} \frac{|u(x)|^{2*}}{2^*_\mu} dx \right) \psi(\tau y) \frac{|u(y)|^{2*}}{2^*_\mu} \]

\[- \mu \left( \int_{\Omega} \frac{|u(x)|^{2*}}{2^*_\mu} dx \frac{x-y}{|x-y|^\mu} \psi(\tau y) \frac{|u(y)|^{2*}}{2^*_\mu} \right) - \left( \int_{\Omega} \frac{|u(x)|^{2*}}{2^*_\mu} dx \frac{x-y}{|x-y|^\mu} \psi(\tau y) \frac{|u(y)|^{2*}}{2^*_\mu} \right) \]

Furthermore

\[\int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} dy \int_{\Omega} |u(x)|^{2*} \psi(\tau x) x\nabla u(x) \]

\[= \left( \int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} dy \int_{\Omega} \psi(\tau x) x\nabla \left( \frac{|u(x)|^{2*}}{2^*_\mu} \right) dx \right) \]

\[= \frac{1}{2} \left[ -N \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} \psi(\tau x) \frac{|u(x)|^{2*}}{2^*_\mu} dxdy \right.

\[+ \mu \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} \frac{x-y}{|x-y|^\mu} \psi(\tau x) x \frac{|u(x)|^{2*}}{2^*_\mu} dxdy \]

\[- \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} \nabla \psi(\tau x) x \frac{|u(x)|^{2*}}{2^*_\mu} dxdy - N \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2*}}{|x-y|^\mu} \psi(\tau y) \frac{|u(y)|^{2*}}{2^*_\mu} dxdy \]

\[- \mu \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2*}}{|x-y|^\mu} \frac{x-y}{|x-y|^\mu} \psi(\tau y) y \frac{|u(y)|^{2*}}{2^*_\mu} dxdy \]

\[- \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2*}}{|x-y|^\mu} \nabla \psi(\tau y) y \frac{|u(y)|^{2*}}{2^*_\mu} dxdy \]

\[= -N \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{2*}}{|x-y|^\mu} \psi(\tau x) x \frac{|u(x)|^{2*}}{2^*_\mu} dxdy \]
The proof of this lemma is completed.

From (7.1)-(7.3), we have

\[
\int_{\Omega} |(x-y)(\psi(\tau x)x - \psi(\tau y)y) | u(x) |^2 \, dx dy
\]

\[
\int_{\Omega} |x-y|^\mu \nabla \psi(\tau x) | x | u(x) |^2 \, dx dy
\]

\[
= - \int_{\Omega} \int_{\Omega} \frac{|u(y)|^2}{|x-y|^\mu} (N\psi(\tau x) + \nabla \psi(\tau x)x) \frac{|u(x)|^2}{2\mu} \, dx dy
\]

\[
+ \frac{\mu}{2} \int_{\Omega} \int_{\Omega} \frac{|u(y)|^2}{|x-y|^\mu} (x-y)(\psi(\tau x)x - \psi(\tau y)y) \frac{|u(x)|^2}{2\mu} \, dx dy
\]

For every \( \tau > 0 \), then

\[
\left| \frac{(x-y)(x\psi(\tau x) - y\psi(\tau y))}{|x-y|^2} \right| = \left( \frac{\tau x - \tau y}{\tau x - \tau y} \right) \left( \frac{x\psi(\tau x) - y\psi(\tau y)}{|x-y|^2} \right) \leq \sup_{z,w \in \Omega} \left| \frac{(w-z)(w\psi(w) - z\psi(z))}{|w-z|^2} \right| < +\infty.
\]

By Lebesgue’s dominated convergence theorem, we have

\[
\lim_{\tau \to 0} \int_{\Omega} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu-1} v}{|x-y|^\mu} \, dx dy = \frac{-2N+\mu}{2^\mu} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu}}{|x-y|^\mu} \, dx dy. \quad (7.2)
\]

Consider the term

\[
|f(x)| |u|^q - 2uv_x,
\]

\[
= f(x)|u|^q - 2uv_x \psi(\tau x) x \nabla u(x) = f(x)\psi(\tau x) x \nabla \left( \frac{|u|^q}{q} \right)
\]

\[
= \text{div} \left( f(x)\psi(\tau x) x \frac{|u|^q}{q} \right) - NF(x)\psi(\tau x) \frac{|u|^q}{q} - (\nabla f(x)\psi(\tau x) + f(x)\nabla \psi(\tau x)) x \frac{|u|^q}{q},
\]

hence

\[
\int_{\Omega} f(x)|u|^q - 2uv_x
\]

\[
= \int_{\partial \Omega} \left( f(x)\psi(\tau x) \frac{|u|^q}{q} \right) x \nu ds - N \int_{\Omega} f(x)\psi(\tau x) \frac{|u|^q}{q}
\]

\[
- \int_{\Omega} (\nabla f(x)\psi(\tau x) + f(x)\nabla \psi(\tau x)) x \frac{|u|^q}{q}
\]

\[
= - N \int_{\Omega} f(x)\psi(\tau x) \frac{|u|^q}{q} - \int_{\Omega} |\nabla f(x)\psi(\tau x) + f(x)\nabla \psi(\tau x)| x \frac{|u|^q}{q}.
\]

By Lebesgue’s dominated convergence theorem

\[
\lim_{\tau \to 0} \int_{\Omega} f(x)|u|^q - 2uv_x = - \frac{N}{q} \int_{\Omega} f(x)|u|^q. \quad (7.3)
\]

From (7.1)-(7.3), we have

\[
(a + \varepsilon) \int_{\Omega} |\nabla u|^2 \int_{\partial \Omega} \frac{|\nabla u|^2}{2} x \nu ds + \frac{N-2}{2} \int_{\Omega} |\nabla u|^2 \int_{\Omega} |\nabla u|^2
\]

\[
= \frac{2N-\mu}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu}}{|x-y|^\mu} \, dx dy + \frac{N}{q} \int_{\Omega} f(x)|u|^q.
\]

The proof of this lemma is completed. \( \square \)
Proof of Theorem 1.3. Let $u \geq 0$ be a nontrivial weak solution to problem (1.1), then by Lemma 7.1, $u \in L^\infty(\Omega)$. So
\[
a \|u\|^2 + \varepsilon \|u\|^4 = \int \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x - y|^{\mu}} dxdy + \lambda \int_{\Omega} f(x)|u|^q.
\]
Using Lemma 7.2, we have
\[
\lambda \int_{\Omega} f(x)|u|^q = \frac{1}{2} \left( \frac{2q}{2 - q} N + 2q \int_{\partial\Omega} \left( a + \varepsilon \int_{\Omega} |\nabla u|^2 \right) |\nabla u|^2 xds. \right.
\]
But, since $\Omega$ is star shaped with respect to origin in $\mathbb{R}^N$, so $x \cdot \nu > 0$. Since $\lambda < 0, f > 0$, by (7.4), so $u \equiv 0$, which is a contradiction. This completes the proof.

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