The Composite Cosine Transform on the Stiefel Manifold and Generalized Zeta Integrals

E. Ournycheva and B. Rubin

Abstract. The $\lambda$-cosine transform on the unit sphere is defined by

$$(T^{\lambda}f)(u) = \int_{S^{n-1}} f(v)|v \cdot u|^\lambda dv, \quad u \in S^{n-1},$$

and has many applications. We introduce a new integral transform $T^{\lambda}f, \lambda \in \mathbb{C}^m$, which generalizes the previous one for functions on the Stiefel and Grassmann manifolds. We call it the composite cosine transform, by taking into account that its kernel agrees with the composite power function of the cone of positive definite symmetric matrices. Our aim is to describe the set of all $\lambda \in \mathbb{C}^m$ for which $T^{\lambda}$ is injective on the space of integrable functions. We obtain the precise description of this set in some important cases, in particular, for $\lambda$-cosine transforms on Grassmann manifolds. The main tools are the classical Fourier analysis of functions of matrix argument and the relevant zeta integrals.

1. Introduction

The classical cosine transform (also known as the Blaschke-Levy transform) on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ is defined by

$$(Tf)(u) = \int_{S^{n-1}} f(v)|v \cdot u| dv, \quad u \in S^{n-1},$$

where $f$ is an integrable even function on $S^{n-1}$, and $v \cdot u$ is the usual inner product. This transform and its generalization

$$(T^{\lambda}f)(u) = \int_{S^{n-1}} f(v)|v \cdot u|^\lambda dv,$$

arise in diverse areas of mathematics, in particular, in PDE, the Fourier analysis, integral geometry, and the Banach space theory; see [Es, Ga, GH1, GG, Ko, Ru1, Ru3, Sa, Schn, Sc]. Operators \cite{2} have been investigated in detail.

2000 Mathematics Subject Classification. Primary 42B10; Secondary 52A22.

Key words and phrases. the composite cosine transforms, matrix spaces, the Fourier transform, zeta integrals, composite power functions.

The work was supported in part by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany), and the Abraham and Sarah Gelbart Research Institute for Mathematical Sciences.
using the following two approaches. The first one employs the Fourier transform technique \[ Se, Ko, Ru1 \], and relies on the formula

\[
\int_{\mathbb{R}^n} \frac{f(\frac{x}{y})}{|x|^{n+\lambda}} e^{ix \cdot y} dx = c_{\lambda,n} |y|^\lambda (T^\lambda f)(\frac{y}{|y|}), \quad c_{\lambda,n} = \text{const},
\]

which should be properly interpreted. The second approach is based on decomposition in spherical harmonics. Namely, by the Funk-Hecke formula,

\[
(1.4) \quad T^\lambda P_k = c\mu_k(\lambda) P_k,
\]

\[
(1.5) \quad c = 2\pi^{(n-1)/2} (-1)^{k/2}, \quad \mu_k(\lambda) = \frac{\Gamma\left(\frac{\lambda+1}{2}\right) \Gamma\left(\frac{k-\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right) \Gamma\left(\frac{\lambda+k+n}{2}\right)},
\]

for each homogeneous harmonic polynomial \( P_k(x) \) of even degree \( k \) restricted to the unit sphere; see \[ Ru1 \]-\[ Ru3 \], \[ Sa \]. The Fourier-Laplace multiplier \( \mu_k(\lambda) \) provides complete information about properties of \( T^\lambda \).

In the last two decades a considerable attention was attracted to generalizations of \( T \) and \( T^\lambda \) for functions on the Grassmann manifold \( G_{n,m} \) of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \). We recall, that if \( \eta \in G_{n,m}, \xi \in G_{n,l}, l \geq m, \) and \( [\eta|\xi] \) is the \( m \)-dimensional volume of the parallelepiped spanned by the orthogonal projection of a generic orthonormal coordinate frame in \( \eta \) onto \( \xi \), then, by definition,

\[
(1.6) \quad (T^\lambda f)(\xi) = \int_{G_{n,m}} f(\eta) [\eta|\xi]^\lambda d\eta
\]

(we use the same notation as in \[ GR \]). For \( l > m \), the operator \( (1.6) \) represents the composition of the similar one over \( G_{n,l} \) and the corresponding Radon transform acting from \( G_{n,m} \) to \( G_{n,l} \) (see, e.g., \[ A \], \[ GR \]). Thus injectivity results for \( T^\lambda \) in this case can be easily derived from those for the Radon transform (see \[ GR \] and references therein) and the case \( l = m \). Owing to this, in the following we assume \( l = m \). This case bears the basic features of the operator family \( (1.6) \).

The investigation of operators \( (1.6) \) for \( \lambda = 1 \) was initiated in stochastic geometry (processes of flats) by Matheron \[ Mat1, Mat2 \] who conjectured that \( T^1 \) is injective as well as its prototype \( (1.1) \). Matheron’s conjecture was disproved by Goodey and Howard \[ GH1 \] who used the idea of Gluck and Warner \[ GW \] to interpret the Grassmann manifold \( G_{4,2} \) as the direct product \( S^2 \times S^2 \) of 2-spheres. For higher dimensions, the result then follows by induction. Operators \( T^\lambda \) for \( \lambda = 0, 1, 2, \ldots \) were studied in \[ GH2 \] p. 117, where, by using reduction to \( G_{4,2} \), it was proved that \( T^\lambda \) is non-injective for such \( \lambda \); see also \[ GRH, Sp1, Sp2, Gr \]. The range of the \( \lambda \)-cosine transform was studied by Alesker and Bernstein \[ AB \] for \( \lambda = 1 \) and by Alesker \[ A \] for all complex \( \lambda \), who invoked deep results from the representation theory.

In the present article we develop a new approach to operators \( T^\lambda \). Our argument differs essentially from that in the cited papers. For technical reasons, we prefer to deal with \( O(m) \) right-invariant functions on the Stiefel manifold \( V_{n,m} \) of orthonormal \( m \)-frames rather than with functions on \( G_{n,m} \). This does not change
the essence of the matter and leads to the following equivalent definition:

\[(T^\lambda f)(u) = \int_{V_{n,m}} f(v)|\det(v'u)|^\lambda dv, \quad u \in V_{n,m},\]

where “\(^{\prime}\)” stands for the transposed matrix, and the product \(v'u\) is understood in the sense of matrix multiplication. Then we regard \(1.7\) as a member of the more general analytic family

\[(T^\lambda f)(u) = \int_{V_{n,m}} f(v) (u'vv'u)^\lambda dv, \quad u \in V_{n,m},\]

where \(\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m\) and \((\cdot)^\lambda\) denotes the composite power function of the cone of positive definite \(m \times m\) matrices; see Section 2.2. We call \(T^\lambda f\) the composite cosine transform of a function \(f\) on \(V_{n,m}\). The general intention is to obtain a higher-rank analog of the formula \(1.3\), evaluate the corresponding multiplier \(\mu_k(\lambda)\) explicitly in terms of gamma functions, and use it for examination of \(T^\lambda\). The particular case \(\lambda = (1, \ldots, 1)\) corresponds to Matheron’s operator. We do not realize this project in full generality here and leave this work for future publications. However, our approach enables us to obtain the precise description of those \(\lambda\) for which \(T^\lambda\) is injective in the following important cases (a) \(2m \leq n\), \(\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m\), and (b) \(\lambda_1 = \cdots = \lambda_m = \lambda \in \mathbb{C}\), provided that \(T^\lambda f\) and \(T^\lambda f\) exist in the usual Lebesgue sense as absolutely convergent integrals.

The essence of our approach is that we apply the classical Fourier transform technique to obtain higher-rank analogs of \(1.3\) and \(1.4\). In the first case we assume \(f\) to be an arbitrary integrable function on \(V_{n,m}\), and in the second one \(P_k\) stands for the restriction to \(V_{n,m}\) of the corresponding \(O(m)\) right-invariant determinantly homogeneous harmonic polynomial on the space of \(n \times m\) matrices. Different aspects of harmonic analysis based on implementation of such polynomials were studied in [Herz, Str, TT], and in a series of publications related to group representations. In the present article we obtain a higher-rank copy of \(1.4\) with the multiplier \(\mu_k(\lambda)\) explicitly expressed in terms of the gamma functions associated with the cone of positive definite \(m \times m\) matrices. In the particular case \(\lambda_1 = \cdots = \lambda_m\) the main result reads as follows.

**Theorem 1.1.** Let \(n > m \geq 2\), \(f \in L^1(G_{n,m})\). Then \((T^\lambda f)(\xi)\) is finite for almost all \(\xi \in G_{n,m}\) if and only if \(\Re\lambda > -1\). For such \(\lambda\), the operator \(T^\lambda\) is injective on \(L^1(G_{n,m})\) if and only if \(\lambda \neq 0, 1, 2, \ldots\).

In particular, we show that if \(\lambda\) is a non-negative integer and \(2m < n\), then \(T^\lambda\), having been written in the form \(1.7\), annihilates all \(O(m)\) right-invariant determinantly homogeneous harmonic polynomials \(P_k(x)\) of degree \(k > \Re\lambda + m - 1\).

A by-product of our investigation is a functional equation for the generalized zeta integrals with additional “angle component” \(f(v), \ v \in V_{n,m}\); see \(\ref{1.1}, \ref{1.2}\). This result is of independent interest. The main references related to zeta integrals can be found in [B, BSZ, FK, Ki2, Ru4]. The equation obtained below is, in fact, far-reaching higher-rank modifications of \(1.3\) in the language of Schwartz distributions. In cited papers they were obtained for \(f \equiv 1\). The case \(f = P_k\) was considered in [Cl] in the context of Jordan algebras. The proof presented below is
much simpler than that in \[\text{[Ci]}\] (adapted to our case) and employs the idea from \[\text{[Kh2]}\] to derive the result for \(\lambda \in \mathbb{C}\) from the more general one for \(\lambda \in \mathbb{C}^m\).

The paper is organized as follows. Section 2 contains the necessary background material. In Section 3 we introduce the generalized zeta integrals. Section 4 plays the central role in the article and establishes connection between zeta integrals and composite cosine transforms. The results of Section 4 are applied in Section 5 to study injectivity of the composite cosine transform.

**Acknowledgement.** We are thankful to Dr. S.P. Khekalo for sharing with us his results \[\text{[Kh2]}\].

---

### 2. Preliminaries

We establish our notation and recall basic facts that will be used throughout the paper. The main references are \[\text{[FK]}, \text{[G]}, \text{[OR]}, \text{[T]}\].

#### 2.1. Notation

Let \(\mathfrak{M}_{n,m}\) be the space of real matrices \(x = (x_{i,j})\) having \(n\) rows and \(m\) columns. We identify \(\mathfrak{M}_{n,m}\) with the real Euclidean space \(\mathbb{R}^{nm}\) and set \(\text{d}x = \prod_{i=1}^{n} \prod_{j=1}^{m} \text{d}x_{i,j}\) for the Lebesgue measure on \(\mathfrak{M}_{n,m}\). If \(n \geq m\), then \(\mathfrak{M}_{n,m}^0\) stands for the set of all matrices \(x \in \mathfrak{M}_{n,m}\) of rank \(m\). This set has a full measure in \(\mathfrak{M}_{n,m}\). In the following, \(x'\) denotes the transpose of \(x\), \(I_m\) is the identity \(m \times m\) matrix, \(0\) stands for zero entries. Given a square matrix \(a\), we denote by \(|a|\) the absolute value of the determinant of \(a\), and by \(\text{tr}(a)\) the trace of \(a\), respectively.

Let \(\Omega = P_m\) be the cone of positive definite symmetric matrices \(r = (r_{i,j})_{m \times m}\) with the elementary volume \(\text{d}r = \prod_{i<j} d r_{i,j}\), and let \(\overline{\Omega}\) be the closure of \(\Omega\), that is the set of all positive semi-definite \(m \times m\) matrices. For \(r \in \Omega\) \((r \in \overline{\Omega})\) we write \(r > 0\) \((r \geq 0)\). Given \(s_1\) and \(s_2\) in \(\overline{\Omega}\), the inequality \(s_1 > s_2\) means \(s_1 - s_2 \in \Omega\). If \(a \in \overline{\Omega}\) and \(b \in \Omega\), then \(\int_a^b f(s) \text{d}s\) denotes the integral over the set \(\{s : s \in \Omega, a < s < b\} = \{s : s - a \in \Omega, b - s \in \Omega\}\).

The group \(G = GL(m, \mathbb{R})\) of real non-singular \(m \times m\) matrices \(g\) acts on \(\Omega\) by the rule \(r \mapsto \tau_g(r) = g r g'\) so that \(\tau_{g_1} \tau_{g_2} = \tau_{g_1 g_2}\); \(g_1, g_2 \in G\). The corresponding \(G\)-invariant measure on \(\Omega\) is

\[
d_{\ast} r = |r|^{-(m+1)/2} \text{d}r, \quad |r| = \det(r),
\]

\[\text{[T]}\] p. 18]. The group \(G\) is transitive on \(\Omega\) but not simply transitive. The transitivity retains if we restrict to the subgroup \(T_m\) of upper triangular matrices with positive diagonal elements. This subgroup is simply transitive. The subgroup of lower triangular matrices with positive diagonal elements also acts simply transitively on \(\Omega\). Each \(r \in \Omega\) has a unique representation \(r = t' t\), \(t = (t_{i,j}) \in T_m\), so that

\[
\text{d}r = 2^m \prod_{j=1}^{m} t_{j,j}^{m-j+1} \text{d}t_{j,j} \text{d}t_{\ast}, \quad \text{d}t_{\ast} = \prod_{i<j} \text{d}t_{i,j};
\]

\[\text{[T]}\] p. 39]. An alternative representation reads \(r = \tau \tau', \tau \in T_m\). To connect both representations, let

\[
r_{\ast} = \omega r \omega, \quad \omega = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 \end{bmatrix}, \quad \omega^2 = I_m.
\]
If \( r \in \Omega \) and \( r_\ast = t't, \ t \in T_m \), then \( r = \tau \tau' \), where \( \tau = \omega t' \omega \in T_m \).

We use a standard notation \( O(n) \) and \( SO(n) \) for the group of real orthogonal \( n \times n \) matrices and its connected component of the identity, respectively. The invariant measure on \( SO(n) \) is normalized to be of total mass 1. The Schwartz space \( S = S(\mathfrak{M}_{n,m}) \) is identified with the respective space on \( \mathbb{R}^{nm} \).

The Fourier transform of a function \( f \in L^1(\mathfrak{M}_{n,m}) \) is defined by

\[
(\mathcal{F} f)(y) = \int_{\mathfrak{M}_{n,m}} e^{i\pi(iy'x)} f(x) dx, \quad y \in \mathfrak{M}_{n,m}.
\]

This is the usual Fourier transform on \( \mathbb{R}^{nm} \), and the relevant Parseval equality reads

\[
(\mathcal{F} f, \mathcal{F} \varphi) = (2\pi)^{nm} (f, \varphi),
\]

where

\[
(f, \varphi) = \int_{\mathfrak{M}_{n,m}} f(x)\overline{\varphi(x)} dx.
\]

**Lemma 2.1.** (see, e.g., [Mu] pp. 57–59).

(i) If \( x = ayb \), where \( y \in \mathfrak{M}_{n,m}, \ a \in GL(n, \mathbb{R}), \) and \( b \in GL(m, \mathbb{R}) \), then \( dx = |a|^{n}|b|^m dy \).

(ii) If \( r = qsq' \), where \( s \in \mathcal{P}_m \) and \( q \in GL(m, \mathbb{R}) \), then \( dr = |q|^{m+1} ds \).

(iii) If \( r = s^{-1}, \ s \in \Omega \), then \( r \in \Omega \) and \( dr = |s|^{-m-1} ds \).

**2.2. The composite power function.** Given \( r = (r_{i,j}) \in \Omega \), let \( \Delta_0(r) = 1, \ \Delta_1(r) = r_{1,1}, \ \Delta_2(r), \ldots, \ \Delta_m(r) = |r| \) be the corresponding principal minors which are strictly positive [Mu] p. 586]. For \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \), the composite power function of the cone \( \Omega \) is defined by

\[
(\mathcal{F} f)(y) = \int_{\mathfrak{M}_{n,m}} e^{i\pi(iy'x)} f(x) dx, \quad y \in \mathfrak{M}_{n,m}.
\]

\[
(f, \varphi) = \int_{\mathfrak{M}_{n,m}} f(x)\overline{\varphi(x)} dx.
\]

**Remark 2.2.** In the case \( m = 1 \), the function \( (2.0) \) becomes \( r^{\lambda/2} \). This notational confusion is easily resolved if one takes into account the equality \( (2.9) \) below.

The composite power functions associated to homogeneous cones were introduced by S. Gindikin [G1]. We also refer to [FK], [Kh2], and [T], where \( (2.0) \) is written in different notation.

We denote

\[
|\lambda| = \lambda_1 + \cdots + \lambda_m.
\]

If \( \lambda_1/2, \ldots, \lambda_m/2 \) are integers and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \), then \( r^\lambda \) is a polynomial of degree \( |\lambda|/2 \). In the special case \( \lambda_1 = \cdots = \lambda_m = \lambda \) we add the subscript 0 so that

\[
\lambda_0 = (\lambda, \ldots, \lambda) \ (\in \mathbb{C}^m)
\]

and

\[
r^{\lambda_0} = |r|^{\lambda/2}.
\]
If \( r = t't \), \( t \in T_m \), then \( \Delta_i(r) = |\Delta_i(t)|^2 = \prod_{j=1}^{m} t_{j,j}^2 \), and therefore,

\[
(2.9) \quad r^\lambda = \prod_{j=1}^{m} t_{j,j}^\lambda = \pi(t).
\]

The expression \( \pi(t) \) is a multiplicative character of \( T_m \), so that

\[
(2.10) \quad \pi(t_1 t_2) = \pi(t_1) \pi(t_2), \quad \pi(t^{-1}) = \pi^{-}(t).
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_m) \), let \( \lambda_* = (\lambda_m, \ldots, \lambda_1) \) be the reverse vector; \( (\lambda_*)_j = \lambda_{m-j+1} \). To each \( r \in \Omega \) we associate the matrix \( r_* = \omega r \omega \) (see (2.10)) with the components \( (r_*)_{i,j} = r_{m-j+1, m-i+1} \).

**Lemma 2.3.** Let \( \lambda, \mu \in \mathbb{C}^m, r \in \Omega \). Then

\[
(2.11) \quad r^\lambda + \mu = r^\lambda r^\mu, \quad r^{\lambda + \alpha_0} = r^{|r|^2/2} \alpha_0 = (\alpha, \ldots, \alpha);
\]

\[
(2.12) \quad (t't')^\lambda = (t't)^\lambda r^\lambda, \quad t \in T_m;
\]

\[
(2.13) \quad r^\lambda_* = (r^{-1})_*^\lambda, \quad (r^{-1})^\lambda_* = r_*^{-\lambda};
\]

\[
(2.14) \quad (cr)^\lambda_* = c^{\lambda/|r|^2} r_*^\lambda, \quad c > 0.
\]

**Proof.** These statements (up to notation) may be found in different sources [Gl], [FK], [Kl2], [Sk]. For the sake of completeness, we outline the proof. The property (2.11) is clear in view of (2.9) and (2.8). To prove (2.12), it suffices to set \( r \) by (2.9), by taking into account the equalities

\[
\text{second equality is a consequence of the first one and (2.13)}.
\]

The first equality in (2.14) follows immediately from (2.6). The statements (up to notation) may be found in different sources.

By (2.12),

\[
r^\lambda_* = \prod_{j=1}^{m} t_{j,j}^{\lambda_{m-j+1}}.
\]

On the other hand, since \( r^{-1} = t^{-1}(t')^{-1} \), and \( (r^{-1})_* = \omega r^{-1} \omega = (\omega t^{-1} \omega)(\omega t^{-1} \omega)' \), by taking into account the equalities

\[
t^{-1} = \begin{bmatrix} t_{1,1}^{-1} & * \\ 0 & t_{m,m}^{-1} \end{bmatrix}, \quad \omega t^{-1} \omega = \begin{bmatrix} t_{m,m}^{-1} & 0 \\ * & t_{1,1}^{-1} \end{bmatrix},
\]

by (2.12),

\[
(r^{-1})_*^\lambda = \left( \begin{bmatrix} t_{m,m} & 0 \\ * & t_{1,1}^{-1} \end{bmatrix} \begin{bmatrix} t_{m,m}^{-1} & * \\ 0 & t_{1,1}^{-1} \end{bmatrix} \right)^{-\lambda} = \prod_{j=1}^{m} t_{j,j}^{\lambda_{m-j+1}} = \prod_{j=1}^{m} t_{j,j}^{\lambda_{m-j+1}}.
\]

This gives \( r^\lambda_* = (r^{-1})_*^\lambda \). Replacing \( r \) by \( r^{-1} \) and \( \lambda \), by \( \lambda_* \), we obtain the second equality in (2.13). The first equality in (2.14) follows immediately from (2.10). The second equality is a consequence of the first one and (2.13):

\[
(cr)^\lambda_* = (c^{-1} r^{-1})_*^\lambda_* = c^{\lambda/|r|^2} (r^{-1})_*^\lambda = c^{\lambda/|r|^2} r_*^\lambda.
\]

\( \square \)
2.3. Gamma function of the cone \( \Omega \). The gamma function of the cone \( \Omega \) is defined by

\[
\Gamma_{\Omega}(\lambda) = \int_{\Omega} r^\lambda e^{-\text{tr}(r)} d_s r.
\]

This integral converges absolutely if and only if \( \text{Re}\lambda_j > j - 1 \), and represents a product of ordinary \( \Gamma \)-functions:

\[
\Gamma_{\Omega}(\lambda) = \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma((\lambda_j - j + 1)/2),
\]

see, e.g., [FK] p. 123.

**Lemma 2.4.** For \( s \in \Omega \) and \( \text{Re}\lambda_j > j - 1 \),

\[
\int_{\Omega} r^\lambda e^{-\text{tr}(rs)} d_s r = \Gamma_{\Omega}(\lambda) s^{-\lambda}.
\]

**Proof.** This equality is known ([Gi] p. 23, [FK] p. 124, [Kh2]). By (2.13), it is equivalent to

\[
\int_{\Omega} r^\lambda e^{-\text{tr}(rs^{-1})} d_s r = \Gamma_{\Omega}(\lambda) s^{\lambda}.
\]

Assuming \( s = t' t \), \( t \in T_m \), and changing variable \( r = t' \rho t \), for the left-hand side of (2.18) we obtain

\[
\int_{\Omega} (t' \rho t)^\lambda e^{-\text{tr}(\rho)} d_s \rho \overset{\text{Lemma} 2.2}{=} \Gamma_{\Omega}(\lambda)(t' t)^\lambda = \Gamma_{\Omega}(\lambda)s^{\lambda}.
\]

An important particular case of (2.14) is the Siegel integral

\[
\Gamma_m(\lambda) = \int_{\Omega} |r|^\lambda e^{-\text{tr}(r)} d_s r = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\lambda - j/2),
\]

which converges absolutely if and only if \( \text{Re}\lambda > (m - 1)/2 \). By (2.18),

\[
\Gamma_{\Omega}(\lambda_0) = \Gamma_m(\lambda/2), \quad \lambda_0 = (\lambda, \ldots, \lambda).
\]

2.4. Stiefel manifolds. For \( n \geq m \), let \( V_{n,m} = \{ v \in \mathbb{M}_{n,m} : v' v = I_m \} \) be the Stiefel manifold of orthonormal \( m \)-frames in \( \mathbb{R}^n \). We fix the invariant measure \( dv \) on \( V_{n,m} \) [Mu, p. 70] normalized by

\[
\sigma_{n,m} = \int_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}
\]

and denote \( d_s v = \sigma_{n,m}^{-1}dv \). The polar decomposition on \( \mathbb{M}_{n,m} \) is defined according to the following lemma; see, e.g., [Mu] pp. 66, 591, [Ma].

**Lemma 2.5.** If \( x \in \mathbb{M}_{n,m} \), \( \text{rank}(x) = m \), \( n \geq m \), then

\[
x = vr^{1/2}, \quad v \in V_{n,m}, \quad r = x' x \in \Omega,
\]

and \( dx = 2^{-m}|r|^{(n-m-1)/2} dr dv \).
A modification of Lemma 2.5 in terms of upper triangular matrices reads as follows.

Lemma 2.6. ([P], [Ru4]) If \( x \in \mathcal{M}_{n,m} \), \( \text{rank}(x) = m \), \( n \geq m \), then
\[
  x = ut, \quad u \in V_{n,m}, \quad t \in T_m,
\]
and
\[
dx = \prod_{j=1}^{m} t_{j,j}^{n-j} dt_{j,j} dt_* dv, \quad dt_* = \prod_{i<j} dt_{i,j}.
\]

3. The generalized zeta integrals

By taking into account the polar decomposition \( x = vr^{1/2} \) (see Lemma 2.6), we introduce the following generalized zeta integrals (or zeta distributions):

\[
Z(\phi, \lambda, f) = \int_{\mathcal{M}_{n,m}} r^\lambda f(v) \overline{\phi(x)} dx = (r^\lambda f, \phi),
\]
(3.1)

\[
Z_*(\phi, \lambda, f) = \int_{\mathcal{M}_{n,m}} r_*^\lambda f(v) \overline{\phi(x)} dx = (r_*^\lambda f, \phi).
\]
(3.2)

Here \( r = x'x, \ v = xr^{-1/2}, \) \( f \) is a fixed integrable function on \( V_{n,m} \), and \( \phi \in S(\mathcal{M}_{n,m}) \) is a test function.

The following particular cases are worth mentioning.

1°. \( \lambda = \lambda_0 = (\lambda, \ldots, \lambda) \). In this case

\[
Z(\phi, \lambda, f) = Z_*(\phi, \lambda, f) = \int_{\mathcal{M}_{n,m}} (\det r)^{\lambda/2} f(v) \overline{\phi(x)} dx.
\]
(3.3)

For \( f \equiv 1 \), zeta integrals of this type were studied in [Ge], [Kh1]; see also [Ru4] and references therein.

2°. \( m = 1 \).

In this case \( r = x'x = |x|^2 \) and (3.3) reads

\[
Z(\phi, \lambda, f) = \int_{\mathbb{R}^n} |x|^\lambda f \left( \frac{x}{|x|} \right) \overline{\phi(x)} dx.
\]
(3.4)

Distributions of this form are well known in analysis and integral geometry; see, e.g., [Sc], [Es], [Ko], [Ru1].

3°. \( \lambda \in \mathbb{C}^m, \ f \equiv 1 \).

This case was explored in [FK] (in the context of Jordan algebras) by invoking the relevant \( K \)-Bessel functions; see also [Ra], [Kh2], [E], [BSZ], and references therein.

4°. \( \lambda \in \mathbb{C}^m, \ f \) is a determinantly homogeneous harmonic polynomial (see Section 4.2).

In this case, integrals (3.1) and (3.2) were studied in [Cl] using the argument close to [FK].

Most of the publications mentioned above were focused on evaluation of the Fourier transform of the corresponding zeta distributions. This transform is realized in the form of the relevant functional equation of the Parseval type.
Our nearest goal is to investigate convergence of integrals (3.1) and (3.2), and their analyticity in the \( \lambda \)-variable. We denote

\[
\Lambda = \{ \lambda \in \mathbb{C}^m : \text{Re} \lambda_j > j - n - 1 \; \forall j = 1, \ldots, m \},
\]

\[
\Lambda_0 = \{ \lambda \in \mathbb{C}^m : \lambda_j = j - n - l \; \text{for some} \; j \in \{1, \ldots, m\}, \; \text{and} \; l \in \{1, 3, 5, \ldots \} \}.
\]

**Lemma 3.1.** The integrals (3.1) and (3.2) are absolutely convergent if and only if \( \lambda \in \Lambda \) and extend as meromorphic functions of \( \lambda \) with the polar set \( \Lambda_0 \). The normalized zeta integrals

\[
Z^0(\phi, \lambda, f) = \frac{Z(\phi, \lambda, f)}{\Gamma(\lambda + n_0)}, \quad Z^0_*(\phi, \lambda, f) = \frac{Z_*(\phi, \lambda, f)}{\Gamma(\lambda + n_0)},
\]

\( n_0 = (n, \ldots, n) \), are entire functions of \( \lambda \).

**Proof.** Let us consider the integral (3.1). We set \( x = vt \), \( v \in V_{n,m} \), \( t \in T_m \), and make use of Lemma 2.6. By taking into account that \( x' = t' \) and \( t(t')^{-1/2} \in O(m) \), owing to (2.9), we obtain

\[
Z(\phi, \lambda, f) = \int_{\mathbb{R}^m} F(t_{1,1}, \ldots, t_{m,m}) \prod_{j=1}^{m} t_{j,j}^{\lambda_j + n - j} \, dt_{j,j},
\]

where

\[
F(t_{1,1}, \ldots, t_{m,m}) = \int_{\mathbb{R}^{m(m-1)/2}} dt_* \int_{V_{n,m}} f(v) \overline{\phi(vt)} \, dv, \quad dt_* = \prod_{i<j} dt_{i,j}.
\]

Since \( F \) extends as an even Schwartz function in each argument, it can be written as

\[
F(t_{1,1}, \ldots, t_{m,m}) = F_0(t_{1,1}^2, \ldots, t_{m,m}^2),
\]

where \( F_0 \in S(\mathbb{R}^m) \) (use, e.g., Lemma 5.4 from [15], p. 56). Replacing \( t_{j,j}^2 \) by \( s_{j,j} \), we represent (3.3) as a direct product of one-dimensional distributions

\[
Z(\phi, \lambda, f) = \left( \prod_{j=1}^{m} (s_{j,j})^{\lambda_j + n - j - 1}/2, F_0(s_{1,1}, \ldots, s_{m,m}) \right).
\]

It follows that the integral (3.1) is absolutely convergent provided \( \text{Re} \lambda_j > j - n - 1 \), i.e., \( \lambda \in \Lambda \). The condition \( \lambda \in \Lambda \) is strict. This claim becomes clear if we choose \( f \equiv 1 \) and \( \phi(x) = e^{-\text{tr}(x'x)} \) which give

\[
Z(\phi, \lambda, f) = \int_{\mathcal{M}_{n,m}} (x'x)^\lambda e^{-\text{tr}(x'x)} \, dx = 2^{-m} \sigma_{n,m} \Gamma(\lambda + n_0).
\]

Furthermore, since \( (s_{j,j})^{\lambda_j + n - j - 1}/2 \) extends as a meromorphic distribution with the only poles \( \lambda_j = j - n - 1, j - n - 3, \ldots \), then, by the fundamental Hartogs theorem [16], the function \( \lambda \rightarrow Z(\phi, \lambda, f) \) extends as a meromorphic function with the polar set \( \Lambda_0 \). By the same reason, a direct product of the normalized distributions \( (s_{j,j})^{\lambda_j + n - j - 1}/2 \Gamma((\lambda_j + n - j + 1)/2) \) is an entire function of \( \lambda \).
Let us consider the integral (3.2). By changing variable $x = y \omega$ where $\omega$ is the matrix (2.3), we obtain $r_\ast = (x'x)_\ast = (\omega y'y \omega)_\ast = y'y$. Hence (set $y = u \tau$, $u \in V_{n,m}$, $\tau \in T_m$),

$$Z_\ast(\phi, \lambda, f) = \int_{\mathcal{M}_{n,m}} (y'y)^\lambda f(y(\omega y'y \omega)^{-1/2}) \overline{\phi(y \omega)} dy$$

$$= \int_{\mathbb{R}^m_+} \Phi(\tau_1, \ldots, \tau_{m,m}) \prod_{j=1}^m \tau_{j,j}^{\lambda_j+n-j-1/2} d\tau_{j,j}$$

where, as above (note that $\tau \omega (\omega \tau' \omega)^{-1/2} \in O(m)$),

$$\Phi(\tau_1, \ldots, \tau_{m,m}) = \int_{\mathbb{R}^{m(m-1)/2}} \int_{V_{n,m}} f(u) \overline{\phi(u \tau \omega)} du$$

$$= \Phi_0(\tau_{1,1}^2, \ldots, \tau_{m,m}^2), \quad \Phi_0 \in S(\mathbb{R}^m).$$

This gives

$$Z_\ast(\phi, \lambda, f) = \prod_{j=1}^m (s_{j,j})_{+}^{(\lambda_j+n-j-1)/2}, \quad \Phi_0(s_{1,1}, \ldots, s_{m,m})),$$

and the result follows as in the previous case. □

4. Connection between zeta integrals and the composite cosine transform

4.1. The composite cosine transform. Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, $f$ be an integrable function on $V_{n,m}$. We introduce the following family of intertwining operators

$$(T^\lambda f)(u) = \int_{V_{n,m}} f(v) (u'v'v'\omega^\lambda) dv, \quad u \in V_{n,m}, \quad n > m,$$

which commute with the left action of the orthogonal group $O(n)$. We call $(T^\lambda f)(u)$ the composite cosine transform of $f$. In the particular case $\lambda_1 = \ldots = \lambda_m = \lambda$, the integral (4.1) represents the usual $\lambda$-cosine transform

$$(T^\lambda f)(u) = \int_{V_{n,m}} f(v) |\det(v'\omega)|^\lambda dv.$$

The function $(T^\lambda f)(u)$ extends to all matrices $y \in \mathfrak{M}_{n,m}$ of rank $m$ if we set $y = ut$, $u \in V_{n,m}$, $t \in T_m$. Indeed, by (4.2),

$$(T^\lambda f)(y) = r^\lambda(T^\lambda f)(u)$$

where $r^\lambda = (t't)^\lambda = (y'y)^\lambda$ is “the radial part” of $(T^\lambda f)(y)$.

**Theorem 4.1.** For $f \in L^1(V_{n,m})$, the integral $(T^\lambda f)(u)$ converges absolutely for almost all $u \in V_{n,m}$ if and only if $\Re \lambda_j > j - m - 1$ for all $j = 1, 2, \ldots, m$ and represents an analytic function of $\lambda$ in this domain. For such $\lambda$, the linear operator $T^\lambda$ is bounded on $L^1(V_{n,m})$.

This statement follows immediately from the following lemma which is of independent interest.
Lemma 4.2. Let \( u, v \in V_{n,m} \). Then

\[
I \equiv \int_{V_{n,m}} (u'vv'u)^{\lambda} \, du = \int_{V_{n,m}} (u'vv'u)^{\lambda} \, dv = \frac{2^m \pi^{nm/2} \Gamma_{\Omega}(\lambda + m_0)}{\Gamma_m(m/2) \Gamma_{\Omega}(\lambda + m_0)}.
\]

This integral converges absolutely if and only if \( \Re \lambda_j > j - m - 1 \) for all \( j = 1, 2, \ldots, m \).

Proof. The first equality in (4.4) becomes clear if we write both integrals as those over the group \( SO(n) \) (see Lemma 2.4 in [GR]). These integrals are, in fact, constant with respect to the corresponding exterior variables. Thus, one can write

\[
I = \int_{V_{n,m}} (u'vv'u)^{\lambda} \, du, \quad v_0 = \begin{bmatrix} I_m & 0 \end{bmatrix} \in V_{n,m}.
\]

To evaluate \( I \) we introduce an auxiliary integral

\[
A = \int_{M_{m,m}} (x'v_0'v_0x)^{\lambda} e^{-\text{tr}(x'x)} \, dx,
\]

and transform it in two different ways. Let first

\[
x = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a \in M_{m,m}, \quad b \in M_{n-m,m}.
\]

Then \( v'_0x = a, x'x = a'a' + b'b' \), and we have

\[
A = A_1A_2, \quad A_1 = \int_{M_{m,m}} (a'a)^{\lambda} e^{-\text{tr}(a'a)} \, da, \quad A_2 = \int_{M_{n-m,m}} e^{-\text{tr}(b'b)} \, db.
\]

By Lemma 2.5, 2.11, 2.15, and (2.21), we obtain

\[
A_1 = 2^{-m} \sigma_{m,m} \int_{\Omega} e^{\lambda + m_0} e^{-\text{tr}(r)} \, d\rho_r = \frac{\pi^{m/2} \Gamma_{\Omega}(\lambda + m_0)}{\Gamma_m(m/2)}
\]

provided \( \Re \lambda_j > j - m - 1, \ j = 1, 2, \ldots, m \). The last condition is sharp and provides the “only if” part in the lemma and in the Theorem 4.1. For \( A_2 \) we have

\[
A_2 = \left( \int_{-\infty}^{\infty} e^{-s^2} \, ds \right)^{m(n-m)} = \pi^{m(n-m)/2}.
\]

Thus

\[
A = \frac{\pi^{nm/2} \Gamma_{\Omega}(\lambda + m_0)}{\Gamma_m(m/2)}, \quad \Re \lambda_j > j - m - 1.
\]

On the other hand, by setting \( x = ut, u \in V_{n,m}, t \in T_m \), owing to Lemma 2.6 we obtain

\[
A = \int_{T_m} e^{-\text{tr}(tt')} \, d\mu(t) \int_{V_{n,m}} (t'v_0'v_0'ut)^{\lambda} \, du,
\]

\[
d\mu(t) = \prod_{j=1}^{m} t_{j,j}^{n-j} \, dt_{j,j} \, dt_*, \quad dt_* = \prod_{i<j} dt_{i,j}.
\]
By (2.12), one can write

\[(4.10) \quad A = BI,\]

where \(I\) is our integral \((4.5)\) and

\[
B = \int_{T_m} (t^t)^\lambda e^{-tr(t^t)} \, d\mu(t) = \\
= \prod_{j=1}^m \int_0^\infty t_{j,j}^{\lambda_j+n-j} e^{-t_{j,j}} \, dt_{j,j} \times \prod_{i<j} \int_{-\infty}^\infty e^{-t_{i,j}} \, dt_{i,j} \\
= 2^{-m} \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\left(\frac{\lambda_j + n - j + 1}{2}\right) \\
= 2^{-m} \Gamma_\Omega(\lambda + n_0), \quad \text{Re}\lambda > j - n - 1.
\]

Combining this with (4.9) and (4.10), we obtain

\[
2^{-m} \Gamma_\Omega(\lambda + n_0) I = \frac{\pi^{nm/2} \Gamma_\Omega(\lambda + n_0)}{\Gamma(m/2)} \\
\text{provided that } \text{Re}\lambda_j > j - m - 1 \text{ for all } j = 1, 2, \ldots, m. \text{ This gives (4.4).} \quad \square
\]

**Corollary 4.3.** Let \(\lambda_1 = \ldots = \lambda_m = \lambda, \quad \text{Re}\lambda > -1.\) Then

\[
(4.11) \quad \int_{V_{n,m}} |\det(v'u)|^\lambda \, dv = \frac{2^m \pi^{nm/2} \Gamma_m\left(\frac{m+\lambda}{2}\right)}{\Gamma_m\left(\frac{m}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}.
\]

If we set \(\lambda = 1\) in (4.11) and replace \(dv\) by the normalized measure \(d_*v\) then the resulting formula bears an important geometrical meaning. Namely, let \(\xi\) and \(\eta\) be \(m\)-dimensional linear subspaces of \(\mathbb{R}^n\), i.e., \(\xi, \eta \in G_{n,m}\), and let \(u\) and \(v\) be coordinate frames of \(\xi\) and \(\eta\), respectively. Then \(|\det(v'u)| = [\xi|\eta]\) is the volume of the projection onto \(\eta\) of the parallelepiped spanned by \(u\). The corresponding averaged volume is valuated as

\[
(4.12) \quad \int_{V_{n,m}} |\det(v'u)| \, d_*v = \int_{G_{n,m}} [\xi|\eta] \, d\eta = \frac{\Gamma_m\left(\frac{n}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{m}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}.
\]

**4.2. The basic functional equation.** Below we establish connection between zeta integrals and composite cosine transforms. This can be done in the form of a functional equation which is, in fact, the usual Parseval equality in the framework of the corresponding Fourier analysis. We start with the following

**Lemma 4.4.** Let \(f\) be an integrable \(O(m)\) right-invariant function on \(V_{n,m}; \)
\(x = vr^{1/2}, \quad \psi(x) = f(v)|r^{(m-n)/2}. \) For \(s \in \Omega\) and \(\phi \in S(\mathfrak{M}_{n,m}),\)

\[
(4.13) \quad \frac{|s|^{-m/2}}{\sigma_{m,m}} \int_{\mathfrak{M}_{n,m}} (\mathfrak{F}\phi)(y) \, dy \int_{V_{n,m}} f(v) e^{-tr(\pi v' y s^{-1} y' v)} \, dv \\
= (2\pi)^{n-m} \int_{\mathfrak{M}_{n,m}} \psi(x) e^{-tr(x x'/4\pi)} \phi(x) \, dx.
\]
Proof. By the Parseval equality (2.5), it suffices to find the Fourier transform of the function \( \psi_s(x) = \psi(x) e^{-\text{tr}(xz^2/4\pi)} \). In polar coordinates we have

\[
(F\psi_s)(y) = 2^{-m} \int_{V_{n,m}} f(v) dv \int \Omega \left| r \right|^{-1/2} e^{i\text{tr}(iy'_v r_{1/2} - r_{1/2} v_{1/2})/4\pi} dr
\]

(replace \( v \) by \( v_\gamma, \gamma \in O(m) \), and integrate in \( \gamma \))

\[
= \frac{1}{\sigma_{m,m}} \int_{V_{n,m}} f(v) dv \int_{\mathbb{M}_{m,m}} e^{i\text{tr}(iy'_v vz - wz'/4\pi)} dz.
\]

The inner integral can be evaluated by the formula

\[
\int_{\mathbb{M}_{m,m}} e^{i\text{tr}(i\zeta'z)} e^{-i\text{tr}(ss'/4\pi)} dz = (2\pi)^{m^2/2} e^{i\text{tr}(-\pi s's')},
\]

where \( \zeta = v'y, \) [Hez, p. 481]. This gives

\[
(F\psi_s)(y) = \frac{(2\pi)^{m^2/2} \sigma_{m,m}}{\sigma_{m,m}} \int_{V_{n,m}} f(v) e^{-i\text{tr}(\pi v'y_s - s'y')} dv,
\]

and (4.13) follows. \( \square \)

Let us compute the Fourier transform of the function

\[
(\mathcal{F}\varphi_\lambda)(y) = \frac{c_\lambda}{\Gamma(m + m_0)} (T^\lambda f)(y), \quad c_\lambda = 2^{-|\lambda|} \sigma_{m,m},
\]

in the sense of \( S' \)-distributions, i.e.

\[
\frac{c_\lambda}{\Gamma(m + m_0)} (T^\lambda f, \mathcal{F}\phi) = (2\pi)^{nm} \langle \varphi_\lambda, \phi \rangle = (2\pi)^{nm} \mathcal{Z}(\phi, -\lambda_s - n_0, f)
\]

for each \( \phi \in S(\mathbb{M}_{n,m}) \).

Proof. The equality (4.17) demonstrates an intimate interrelation between zeta integrals and cosine transforms. To prove (4.17), we multiply (4.13) by \( s^{\lambda + m_0} \) and integrate against \( ds \). We obtain

\[
\frac{1}{\sigma_{m,m}} \int_{\mathbb{M}_{n,m}} (\mathcal{F}\phi)(y) dy \int_{V_{n,m}} f(v) A_1(v'y) dv
\]

\[
= (2\pi)^{(n-m)m} \int_{\mathbb{M}_{n,m}} \psi(x) A_2(x) \phi(x) dx,
\]

THEOREM 4.5. Let \( f \) be an integrable \( O(m) \) right-invariant function on \( V_{n,m} \), \( \lambda \in \mathbb{C}^m \). Then

(4.16) \( (F\varphi_\lambda)(y) = \frac{c_\lambda}{\Gamma(m + m_0)} (T^\lambda f)(y), \quad c_\lambda = 2^{-|\lambda|} \sigma_{m,m}, \)

in the sense of \( S' \)-distributions, i.e.

(4.17) \( \frac{c_\lambda}{\Gamma(m + m_0)} (T^\lambda f, \mathcal{F}\phi) = (2\pi)^{nm} \langle \varphi_\lambda, \phi \rangle = (2\pi)^{nm} \mathcal{Z}(\phi, -\lambda_s - n_0, f) \)

for each \( \phi \in S(\mathbb{M}_{n,m}) \).

Proof. The equality (4.17) demonstrates an intimate interrelation between zeta integrals and cosine transforms. To prove (4.17), we multiply (4.13) by \( s^{\lambda + m_0} \) and integrate against \( ds \). We obtain

(4.18) \( \frac{1}{\sigma_{m,m}} \int_{\mathbb{M}_{n,m}} (\mathcal{F}\phi)(y) dy \int_{V_{n,m}} f(v) A_1(v'y) dv \)

\[
= (2\pi)^{(n-m)m} \int_{\mathbb{M}_{n,m}} \psi(x) A_2(x) \phi(x) dx,
\]
where $\psi(x) = f(v)|r|^{(m-n)/2}$,
\[
A_1(\zeta) = \int_{\Omega} s^{\lambda+\mathbf{m}_0}|s|^{-m/2}e^{-tr(\pi \zeta s^{-1} \zeta^s)}ds, \quad \zeta \in \mathfrak{M}_{m,m},
\]
\[
A_2(x) = \int_{\Omega} s^{\lambda+\mathbf{m}_0}e^{-tr(xsx'/4\pi)}ds, \quad x \in \mathfrak{M}_{n,m}.
\]

Let us compute $A_1$ and $A_2$. By (2.11), setting $a = \pi \zeta'$, $r = s^{-1}$, we obtain
\[
A_1(a) = \int_{\Omega} (r^{-1})^{\lambda}e^{-tr(ar)}dr
\]
\[
= \int_{\Omega} r_s^{-\lambda}e^{-tr(\omega r \omega)}dr \quad (r_s = \omega r; \text{ see (2.3)})
\]
\[
= \int_{\Omega} r^{-\lambda}e^{-tr(\omega r \omega)}dr
\]
\[
= \int_{\Omega} r^{-\lambda}e^{-tr(a \cdot r)}dr \quad \Gamma_{\Omega}(-\lambda) a^\lambda
\]

provided $Re(\lambda_s)_j = Re(\lambda_s)_{j+1} < 1 - j$, or $Re \lambda_j < j - m$. Finally, by (2.14), we have
\[
A_1(\zeta) = \pi^{\lambda/2}\Gamma_{\Omega}(-\lambda_s)(\zeta')^\lambda, \quad Re \lambda_j < j - m.
\]

Furthermore, by (2.17) and (2.14),
\[
A_2(x) = (4\pi)^{(\lambda |m|^2)/2}\Gamma_{\Omega}(\lambda + \mathbf{m}_0)(x'x)^{-\lambda_s - \mathbf{m}_0}, \quad Re \lambda_j > j - m - 1.
\]

Hence, (4.18) reads
\[
\pi^{\lambda/2} \frac{\Gamma_{\Omega}(\lambda + \mathbf{m}_0)}{\Gamma_{\Omega}(-\lambda_s)} \int_{\mathfrak{M}_{n,m}} (T^{\lambda} f)(y) dy
\]
\[
= \frac{(2\pi)^{(n-m)m}(4\pi)^{(\lambda |m|^2)/2}}{\Gamma_{\Omega}(-\lambda_s)} \int_{\mathfrak{M}_{n,m}} r_s^{-\lambda_s - \mathbf{m}_0} f(r \phi(x)) dx,
\]
\[
= (2\pi)^{(n-m)m}(4\pi)^{(\lambda |m|^2)/2} \mathcal{Z}_\nu(\phi_s - \lambda_s - \mathbf{m}_0, f),
\]
\[
(4.22)
\]

where $\psi(x) = f(v)|r|^{(m-n)/2}$,
\[
A_1(\zeta) = \int_{\Omega} s^{\lambda+\mathbf{m}_0}|s|^{-m/2}e^{-tr(\pi \zeta s^{-1} \zeta^s)}ds, \quad \zeta \in \mathfrak{M}_{m,m},
\]
\[
A_2(x) = \int_{\Omega} s^{\lambda+\mathbf{m}_0}e^{-tr(xsx'/4\pi)}ds, \quad x \in \mathfrak{M}_{n,m}.
\]

By Lemma 3.1, the expression (4.21) extends as an entire function of $\lambda$. Hence, it can be regarded as analytic continuation of the integral (4.20) outside of the domain (4.22), and the result follows. 

**Example 4.6.** Let $m \geq 1$, $\lambda_1 = \ldots = \lambda_m = \lambda$, $|x|_m = \det(x'x)^{1/2}$. Then
\[
\varphi_\lambda(x) = \frac{|x|_m^{-\lambda - n}}{\Gamma_m(-\lambda/2)} f(x'x)^{-1/2}, \quad (T^{\lambda} f)(y) = \int_{V_{n,m}} f(v)|\det(v'y)|^{\lambda} dv,
\]
and we have

\begin{equation}
(\mathcal{F} \varphi(x))(y) = \frac{2^{-\lambda m} \pi^{m^2/2}}{\sigma_{m,m} \Gamma_m((\lambda + m)/2)} (T^\lambda f)(y).
\end{equation}

**Example 4.7.** Let \( m = 1 \). Then for \( x, y \in \mathbb{R}^n \setminus \{0\} \),

\[ \varphi(x) = \frac{|x|^{-\lambda - n}}{\Gamma(-\lambda/2)} f \left( \frac{x}{|x|} \right), \quad (T^\lambda f)(y) = \int_{S^{n-1}} f(v) |v \cdot y|^\lambda \, dv. \]

In this case

\begin{equation}
(\mathcal{F} \varphi(x))(y) = \frac{2^{1-\lambda m^1/2}}{\Gamma((\lambda + 1)/2)} (T^\lambda f)(y).
\end{equation}

### 4.3. The case of homogeneous polynomials.

Let \( P_k(x) \) be a polynomial on \( \mathbb{M}_{n,m} \), which is harmonic (as a function on \( \mathbb{R}^{nm} \)) and determinantly homogeneous of degree \( k \), so that

\begin{equation}
P_k(xg) = \det(g)^k P_k(x), \quad \forall g \in GL(m, \mathbb{R}), \quad x \in \mathbb{M}_{n,m}.
\end{equation}

The latter means that \( P_k \) is a usual homogeneous harmonic polynomial of degree \( km \) on \( \mathbb{R}^{nm} \). Theorem 4.3 can be essentially strengthened if we choose \( f \) to be the restriction of \( P_k(x) \) onto \( V_{n,m} \). According to (4.15), for \( x = v^{1/2} \), we denote

\begin{equation}
\varphi_{\lambda, k}(x) = \frac{r^{\lambda} - n_0}{\Gamma_0(-\lambda_s)} P_k(v).
\end{equation}

**Theorem 4.8.** Let \( y = v^{1/2} \), \( v \in V_{n,m} \), \( r \in \Omega \). For all \( \lambda \in \mathbb{C}^m \),

\begin{equation}
(\mathcal{F} \varphi_{\lambda, k})(y) = \frac{d_\lambda \Gamma_0(k_0 - \lambda_s)}{\Gamma_0(-\lambda_s) \Gamma_0(\lambda + k_0 + n_0)} P_k(v) r^\lambda,
\end{equation}

\[ d_\lambda = 2^{-|\lambda|} \pi^{nm/2} i^{km}, \]

in the sense of \( S' \)-distributions, i.e.,

\[ \frac{d_\lambda \Gamma_0(k_0 - \lambda_s)}{\Gamma_0(\lambda + k_0 + n_0)} (P_k(v) r^\lambda, \mathcal{F} \phi) = (2\pi)^{nm} (\varphi_{\lambda, k}, \phi) \]

\[ = (2\pi)^{nm} Z_s^0(\phi, -\lambda_s - n_0, P_k) \]

for each \( \phi \in \mathcal{S}(\mathbb{M}_{n,m}) \).

**Proof.** By the Hecke identity [St, p. 87],

\begin{equation}
\int_{\mathbb{M}_{n,m}} P_k(x) e^{-\text{tr}(\pi x' x)} e^{\text{tr}(2\pi iy' x)} \, dx = i^{km} P_k(y) e^{-\text{tr}(\pi y' y)},
\end{equation}

or (replace \( x \) by \( xs^{1/2}/2\pi \) and \( y \) by \( ys^{-1/2} \), \( s \in \Omega \))

\[ \int_{\mathbb{M}_{n,m}} P_k(x) e^{-\text{tr}(sx' x)/4\pi} e^{\text{tr}(iy' x)} \, dx = (2\pi)^{(k+n)m} i^{km} |s|^{-k-n/2} P_k(y) e^{-\text{tr}(\pi ys^{-1} y')} \]
By the Parseval equality,
\begin{equation}
|s|^{-k-n/2} \int_{\mathbb{M}_{n,m}} P_k(y)e^{-\text{tr}(\pi y s^{-1} y')} \overline{\phi(y)} \, dy = (2\pi i)^{-k} \int_{\mathbb{M}_{n,m}} P_k(x)e^{-\text{tr}(xx' / 4 \pi)} \phi(x) \, dx.
\end{equation}

We multiply by $s^{\lambda+n_0+k_0}$ and integrate against $d_s$. This gives
\begin{equation}
\int_{\mathbb{M}_{n,m}} P_k(y)I_1(y) \overline{\phi(y)} \, dy = (2\pi i)^{-k} \int_{\mathbb{M}_{n,m}} P_k(x)I_2(x) \phi(x) \, dx,
\end{equation}

where
\begin{align*}
I_1(y) &= \int_{\Omega} s^{\lambda-k_0} e^{-\text{tr}(\pi y s^{-1} y')} d_s, \\
I_2(x) &= \int_{\Omega} s^{\lambda+n_0+k_0} e^{-\text{tr}(xx' / 4 \pi)} d_s.
\end{align*}

As in the proof of Theorem 4.5 we have
\begin{align*}
I_1(y) &= \pi^{(|\lambda|-km)/2} \Gamma_\Omega(k_0 - \lambda_*)(y'/y)^{\lambda-k_0}, \quad \text{Re} \, \lambda_j < j + k - m, \\
I_2(x) &= (4\pi)^{(|\lambda|+nm+km)/2} \Gamma_\Omega(\lambda + n_0 + k_0)(x'/x)^{-\lambda-n_0-k_0}, \\
& \quad \text{Re} \, \lambda_j > j - n - k - 1.
\end{align*}

Hence, if $j - n - k - 1 < \text{Re} \, \lambda_j < j + k - m$, then
\begin{align}
\Gamma_\Omega(k_0 - \lambda_*) \int_{\mathbb{M}_{n,m}} P_k(y) (y'/y)^{\lambda-k_0} \overline{\phi(y)} \, dy & = c_\lambda \Gamma_\Omega(\lambda + k_0 + n_0) \int_{\mathbb{M}_{n,m}} P_k(x) (x'/x)^{-\lambda-n_0-k_0} \phi(x) \, dx,
\end{align}

where $c_\lambda = i^{-km} 2^{n+|\lambda|} |\pi|^{nm/2}$. By taking into account homogeneity of $P_k$ (i.e., $P_k(v r^{1/2}) = P_k(v) |r|^{k/2}$), and setting $\psi_{\lambda,k}(x) = P_k(v) x^\lambda$, $x = v r^{1/2}$, this can be written as
\begin{align}
d_\lambda \frac{\Gamma_\Omega(k_0 - \lambda_*)}{\Gamma_\Omega(-\lambda_*) \Gamma_\Omega(\lambda + k_0 + n_0)} (\psi_{\lambda,k}, \phi) &= (2\pi)^{nm} (\varphi_{\lambda,k}, \phi) \\
& = (2\pi)^{nm} Z_v^0(\phi, -\lambda_* - n_0, P_k),
\end{align}

since the domain (4.32) is not void for all $k = 0, 1, 2, \ldots$, and the normalized zeta integral (4.31) is an entire function of $\lambda$ (use Lemma 3.1), the result follows.

Example 4.9. Let $m \geq 1$, $\lambda_1 = \ldots = \lambda_m = \lambda$,
\begin{equation}
\varphi_{\lambda,k}(x) = \frac{\langle x \rangle^{\lambda-n}}{\Gamma_m(-\lambda/2)} P_k(x x')^{-1/2}.
\end{equation}
Then

\[
(4.33) \quad (F\varphi_{\lambda,k})(y) = \frac{d_{\lambda}}{\Gamma(-\lambda/2) \Gamma((\lambda + k + n)/2)} |y|^{\lambda} P_k(y(y')^{-1/2}),
\]

\[
d_{\lambda} = 2^{-\lambda m} \pi^{m/2} \pi^{km}.
\]

**Example 4.10.** Let \( m = 1, \)

\[
\varphi_{\lambda,k}(x) = \frac{|x|^{-\lambda-n}}{\Gamma(-\lambda/2)} P_k\left(\frac{x}{|x|}\right).
\]

Then

\[
(4.34) \quad (F\varphi_{\lambda,k})(y) = \frac{d_{\lambda}}{\Gamma(-\lambda/2) \Gamma((\lambda + k + n)/2)} |y|^{\lambda} P_k\left(\frac{y}{|y|}\right),
\]

\[
d_{\lambda} = 2^{-\lambda} \pi^{n/2} \pi^{km}.
\]

5. **Injectivity of the composite cosine transform**

We denote by \( \mathfrak{L} \) the set of all \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \) satisfying \( \text{Re}\lambda_j > j - m - 1 \) for all \( j = 1, \ldots, m. \) By Theorem 4.1, \( (T^\lambda f)(v) \) exists a.e. on \( V_{n,m} \) if and only if \( \lambda \in \mathfrak{L}. \) In the following we focus only on this domain, although other \( \lambda \) can also be treated using analytic continuation. We also restrict our consideration to the space \( L^2(V_{n,m}) \) which consists of \( O(m) \) right-invariant integrable functions on \( V_{n,m}. \) This space is isomorphic to the space \( L^1(G_{n,m}) \) of integrable functions on the Grassmann manifold \( G_{n,m}. \)

**Theorem 5.1.** Let \( n > m \) and \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathfrak{L}. \) If

\[
(5.1) \quad \lambda_j + m - j \neq 0, 2, 4, \ldots \quad \text{for all} \quad j = 1, \ldots, m,
\]

then the operator

\[
(T^\lambda f)(u) = \int_{V_{n,m}} f(v) (u'vv')^\lambda dv
\]

is injective on \( L^2(V_{n,m}). \) If \( 2m \leq n \) and (5.1) fails then \( T^\lambda \) is non-injective.

To prove this theorem, minor preparation is still needed.

**Definition 5.2.** Following [Herz], we call a polynomial \( P_k(x) \) on \( \mathfrak{M}_{n,m} \) an \( H\)-polynomial of degree \( k \) if it is \( O(m) \) right-invariant, harmonic, and determinantal homogeneous of degree \( k. \) We denote by \( \mathcal{H}_k \) the space of all such polynomials.

**Lemma 5.3.** Let \( P_k \in \mathcal{H}_k, \)

\[
(5.2) \quad \mu_k(\lambda) = \frac{\Gamma(\lambda + m_0) \Gamma(\lambda - \lambda_0)}{\Gamma(\lambda + k_0 + m_0) \Gamma(\lambda - \lambda_0)}, \quad \lambda \in \mathbb{C}^m.
\]

If \( \lambda \) does not belong to the polar set of \( \Gamma(\lambda + m_0), \) then

\[
(5.3) \quad (T^\lambda P_k)(vr^{1/2}) = c \mu_k(\lambda) P_k(v)r^\lambda, \quad c = \pi^{m(n-m)/2} i^{km} \sigma_{m,m}.
\]

in the sense of \( S'\)-distributions.
Theorem 5.1. Let us compare (4.10) and (4.28), assuming \( f \) to be the restriction of \( P_k \) onto \( V_{n,m} \). For all \( \lambda \in \mathbb{C}^m \), we obtain

\[
(5.4) \quad \frac{c_{\lambda}}{\Gamma_{\Omega}(\lambda + m_0)} (T^\lambda P_k)(v r^{1/2}) = \frac{d_{\lambda} \Gamma_{\Omega}(k_0 - \lambda_*)}{\Gamma_{\Omega}(-\lambda_*) \Gamma_{\Omega}(\lambda + k_0 + m_0)} P_k(v) r^\lambda,
\]

\[
c_{\lambda} = 2^{-|\lambda|^2/2} / \sigma_{m,m}, \quad d_{\lambda} = 2^{-|\lambda|^m/2} \lambda^{km},
\]
in the sense of \( S' \)-distributions. If we exclude all \( \lambda \) belonging to the polar set of \( \Gamma_{\Omega}(\lambda + m_0) \), then we get (5.3). □

Corollary 5.4. If \( \lambda \in \mathcal{S} \) and \( P_k \in \mathcal{H}_k \), then for all \( v \in V_{n,m} \),

\[
(5.5) \quad (T^\lambda P_k)(v) = c \mu_k(\lambda) P_k(v),
\]
c and \( \mu_k(\lambda) \) having the same meaning as in (5.3).

Proof. The function

\[
\Gamma_{\Omega}(\lambda + m_0) = \pi^m (m-1)/4 \prod_{j=1}^m \left( \frac{\lambda_j + m - j + 1}{2} \right)
\]

has no poles in the domain \( \mathcal{S} \), and therefore, by (5.3),

\[
(5.6) \quad ((T^\lambda P_0)(v r^{1/2}), \phi) = c \mu_k(\lambda) (P_k(v) r^\lambda, \phi)
\]

for all \( \phi(y) = \phi(v r^{1/2}) \in S(\mathcal{M}_{n,m}) \). Now we choose \( \phi(y) = \chi(r) \psi(v) \) where \( \chi(r) \) is a non-negative \( C^\infty \) cut-off function supported away from the boundary of \( \Omega \) and \( \psi(v) \) is a \( C^\infty \) function on \( V_{n,m} \). By passing to polar coordinates, from (5.6) and (4.3) we obtain

\[
c_\chi \int_{V_{n,m}} [(T^\lambda P_k)(v) - c \mu_k(\lambda) P_k(v)] \psi(v) \, dv = 0, \quad c_\chi = \text{const} \neq 0.
\]

This implies (5.5). □

Remark 5.5. A simple computation shows that for \( k = 0 \), the equality (5.4) coincides with (4.11). Note also that if \( m = 1 \) then \( P_k \in \mathcal{H}_k \) is necessarily of even degree, and we have

\[
(5.7) \quad (T^\lambda P_k)(v) = 2\pi^{(n-1)/2} (-1)^{k/2} \frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{-\lambda}{2})} \frac{\Gamma(\frac{k-\lambda}{2})}{\Gamma(\frac{\lambda+k+n}{2})} P_k(v).
\]

This coincides with the known formula (4.4) for spherical harmonics.

Remark 5.6. An important question is, do there exist \( H \)-polynomials of a given degree \( k \)? Note that for \( n = m \) we have exactly two such polynomials, namely, \( P_0(x) = 1 \) and \( P_1(x) = \det(x) \). The following statement is due to Herz [Herz, p. 484: For \( 2m \leq n \) there exist \( H \)-polynomials of every degree \( k \)]. It is also known [TT, p. 27], that for \( 2m \leq n \), the space \( \mathcal{H}_k \) is spanned by polynomials of the form \( P_k(x) = \det(a'x)^k \) where \( a \) is a complex \( n \times m \) matrix satisfying \( a'a = 0 \).

Proof of Theorem 5.1. To prove the first statement, we consider the equality

\[
(5.8) \quad ((T^\lambda f)(v r^{1/2}), \mathcal{F} \phi) = A(\lambda) \left( r_*^{-\lambda_*-m_0} f(v), \phi \right),
\]

\[
A(\lambda) = \frac{(2\pi)^m \Gamma_{\Omega}(\lambda + m_0)}{c_{\lambda} \Gamma_{\Omega}(-\lambda_*)}.
\]
which follows from (4.17) and (4.15). Suppose that \((T^\lambda f)(v) = 0\) almost everywhere on \(V_{n,m}\) for some \(\lambda \in \mathcal{L}\). Then \((T^\lambda f)(y)\), \(y = vr^{1/2} \in \mathfrak{M}_{n,m}\), is zero for almost all \(y \in \mathfrak{M}_{n,m}\), and yields \(A(\lambda) (r^{-\lambda} - n_o) f(v), \phi) = 0\). The assumption (5.1) together with \(\lambda \in \mathcal{L}\) imply
\[Re \lambda_j > j - m - 1 \quad \text{and} \quad \lambda_j \neq j - m, j - m + 2, \ldots.\]
Hence \(\lambda\) is not a pole of \(\Gamma_\Omega(-\lambda_s)\), and therefore \(A(\lambda) \neq 0\). This gives
\[(\ref{eq:5.9})\]
the left-hand side being understood in the sense of analytic continuation. By choosing \(\phi\) as in the proof of Corollary 5.4, we obtain \(f(v) = 0\) a.e. on \(V_{n,m}\).

Let us prove the second statement of the theorem. We first note that for \(2m < n\), \(H\)-polynomials of every degree \(k\) do exist. Hence we can proceed as follows. We observe that the function \(\Gamma_\Omega(k_0 - \lambda_s)\) in the expression of \(\mu_k(\lambda)\) has the form
\[
\Gamma_\Omega(k_0 - \lambda_s) = \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma\left(\frac{k - (\lambda_s)j - j + 1}{2}\right)
\]
(we remind that \((\lambda_s)_j = Re \lambda_{m-j+1}\). It has no poles in \(\mathcal{L}\) if
\[(5.10)\]
\[k > \lambda_j + m - j \quad \text{for all} \quad j = 1, 2, \ldots, m.\]
Since \(\Gamma_\Omega(\lambda + m_0)\) also has no poles in \(\mathcal{L}\), then, by (5.10), \(T^\lambda f_k = 0\) for all \(k\) satisfying (5.10), provided that \(\lambda\) is a pole of \(\Gamma_\Omega(-\lambda_s)\) (i.e., the condition (5.1) fails). Thus \(T^\lambda\) is non-injective in this case. □

We do not know if the condition (5.1) is necessary for injectivity of \(T^\lambda\) in the case \(2m > n\). We suspect that to answer this question one has to explore \(T^\lambda\) on polynomial representations of the group \(SO(n)\) parametrized by highest weights \((m_1, m_2, \ldots, m_{n/2})\), more general than just \((k, k, \ldots, k)\) corresponding to \(H_k\)-spaces; cf. [Str], [TT]. However, if \(\lambda_1 = \cdots = \lambda_m = \lambda\) then for the \(\lambda\)-cosine transform
\[\langle T^\lambda f \rangle(u) = \int_{V_{n,m}} f(v) |\det(v'u)|^{\lambda} \, dv \quad \text{or} \quad \langle T^\lambda f \rangle(\xi) = \int_{G_{n,m}} f(\eta) |\eta|^{\lambda} \, d\eta,\]
we can give a complete answer.

**Theorem 5.7.** Let \(n > m\) and \(Re \lambda > -1\). Then \(T^\lambda\) is injective on \(L^2(V_{n,m})\) (or on \(L^1(G_{n,m})\)) if and only if \(\lambda \neq 0, 1, 2, \ldots\).

**Proof.** If \(\lambda_1 = \cdots = \lambda_m = \lambda\) then the condition \(\lambda \neq 0, 1, 2, \ldots\) coincides with (5.1). Hence, for \(2m \leq n\), the result follows from Theorem 5.4. If \(2m > n\), i.e., \(2(n - m) < n\), we reduce the problem to the dual one on the manifold \(V_{n,n-m}\). To this end we replace the integral \(T^\lambda f\) by the equivalent expression in terms of \(G_{n,m}\) and write the latter as an integral over the dual Grassmann manifold \(G_{n,n-m}\). This argument can be realized as follows.

Let \(\xi \in G_{n,m}\) and \(\eta \in G_{n,m}\) be \(m\)-dimensional subspaces of \(\mathbb{R}^n\) spanned \(u \in V_{n,m}\) and \(v \in V_{n,m}\), respectively. We denote by \(\xi^\perp \in G_{n,n-m}\) and \(\eta^\perp \in G_{n,n-m}\) the
corresponding orthogonal complements, and choose coordinate frames $u_{\perp} \in V_{n,n-m}$ in $\xi_{\perp}$ and $v_{\perp} \in V_{n,n-m}$ in $\eta_{\perp}$. By using the same notation as in (1.6), we have

\begin{equation}
(\mathbf{5.11}) \quad \mathbf{[\eta]} = [\mathbf{\eta}_{\perp}][\mathbf{\xi}_{\perp}].
\end{equation}

Indeed (we recall that $| \cdot |$ denotes the determinant of the matrix inside)

\[
|\eta[\xi]|^2 = |\det(v'u')|^2 = |u'vv'u| = |u'(I_n - v'u'u)v'_u| = |I_n - u'v'u'u'v'\mid = |v'u'u'u'u'u'\mid = |\det(v'u'u')|^2 = |[\eta_{\perp}][\xi_{\perp}]|^2.
\]

Now we successively define the functions $F(\eta)$ on $G_{n,m}$, $F_{\perp}(\eta_{\perp})$ on $G_{n,n-m}$, and $f_{\perp}(v_{\perp})$ on $V_{n,n-m}$ by

\[
F(\eta) = f(u), \quad F_{\perp}(\eta_{\perp}) = F([\eta_{\perp}],) \quad f_{\perp}(v_{\perp}) = F_{\perp}(\eta_{\perp}).
\]

Then

\[
(T^\lambda f)(u) = \int_{G_{n,m}} F(\eta) |\eta[\xi]|^\lambda d\eta
\]

\[
= \int_{G_{n,n-m}} F_{\perp}(\eta_{\perp}) [\mathbf{\eta}_{\perp}][\mathbf{\xi}_{\perp}]^\lambda d\eta_{\perp}
\]

\[
= \int_{V_{n,n-m}} f_{\perp}(v_{\perp}) |\det(v'u'u')|^\lambda dv_{\perp}
\]

\[
= (T^\lambda_{\perp} f_{\perp})(u_{\perp}).
\]

Thus $T^\lambda$ can be expressed as the similar operator $T^\lambda_{\perp}$ on the dual manifold $V_{n,n-m}$. By above, the latter is non-injective for $2(n-m) < n$, and we are done.

\[\square\]

References

[A] S. Alesker, The $\alpha$-cosine transform and intertwining integrals, Preprint, 2003.

[AB] S. Alesker, and J. Bernstein, Range characterization of the cosine transform on higher Grassmannians, Advances in Math., 184 (2004), 367–379.

[B] L. Barchini, Zeta distributions and boundary values of Poisson transforms, J. Funct. Anal., 216 (2004), 47–70.

[BSZ] L. Barchini, M. Sepanski, and R. Zierau, Positivity of Zeta distributions and small representations, Preprint.

[Cl] J.-L. Clerc, Zeta distributions associated to a representation of a Jordan algebra, Math. Z. 239 (2002), 263–276.

[Es] G. I. Eskin, Boundary value problems for elliptic pseudodifferential equations, Amer. Math. Soc., Providence, R.I., 1981.

[FK] J. Faraut, and A. Korányi, Analysis on symmetric cones, Clarendon Press, Oxford, (1994).

[Ga] R.J. Gardner, Geometric tomography, Cambridge University Press, New York, 1995.

[GG] R. J. Gardner, and A. A. Giannopoulos, $p$-cross-section bodies, Indiana Univ. Math. J., 48 (1999), 593–61.

[Ge] S.S. Gelbart, Fourier analysis on matrix space, Memoirs of the Amer. Math. Soc., No. 108, AMS, Providence, RI, 1971.

[GSh] I. M. Gel’fand, and G. E. Shilov, Generalized functions, Vol. 1. Properties and operations, Academic Press, New York-London, 1964.

[Gi] S.G. Gindikin, Analysis on homogeneous domains, Russian Math. Surveys, 19 (1964), No. 4, 1–89.
[GW] H. Gluck, and F.W. Warner, *Great circle fibrations of the three-sphere*, Duke Math. J., 50 (1983), 107–132.

[GH1] P. Goodey, and R. Howard, *Processes of flats induced by higher-dimensional processes*, Adv. in Math., 80 (1) (1990), 92–109.

[GH2] P. Goodey, and R. Howard, *Processes of flats induced by higher-dimensional processes. II. Integral geometry and tomography* (Arcata, CA, 1989), 111–119, Contemp. Math., 113, Amer. Math. Soc., Providence, RI, 1990.

[GHR] P. Goodey, R. Howard, and M. Reeder, *Processes of flats induced by higher-dimensional processes. III*, Geom. Dedicata, 61 (1996), 257–269.

[Gr] E.L. Grinberg, *Cosine and Radon transforms on Grassmannians*, Preprint, 2000.

[GR] E. Grinberg, and B. Rubin, *Radon inversion on Grassmannians via Garding-Gindikin fractional integrals*, Annals of Math., 159 (2004), 809–843.

[Herz] C. Herz, *Bessel functions of matrix argument*, Ann. of Math., 61 (1955), 474–523.

[Kh1] S.P. Khekalo, *Riesz potentials in the space of rectangular matrices and iso-Huygens deformations of the Cayley-Laplace operator*, Doklady Mathematics, 63, No. 1 (2001), 35-37.

[Kh2] _, *The Igusa zeta function associated with a composite power function on the space of rectangular matrices*, Preprint POMI RAN, 10 (2004), 1–20.

[Ko] A. Koldobsky, *Inverse formula for the Blaschke-Levy representation*, Houston J. Math., 23 (1997), 95-107.

[Ma1] A.M. Mathai, *Jacobians of matrix transformations and functions of matrix argument*, World Sci. Publ. Co. Pte. Ltd, Singapore, 1997.

[Ma2] G. Matheron, *Un théorème d’unicité pour les hyperplans poissoniens* (French), J. Appl. Probability, 11 (1974), 184–189.

[Mat] R.J. Muirhead, *Aspects of multivariate statistical theory*, John Wiley & Sons, New York-London-Sydney, 1982.

[OR] E. Ournycheva, and B. Rubin, *Radon transform of functions of matrix argument*, Preprint, 2004 (math.FA/0406573).

[P] E.E. Petrov, *The Radon transform in spaces of matrices*, Trudy seminara po vektornomu i tenzornomu analizu, M.G.U., Moscow, 15 (1970), 279–315 (Russian).

[Ra] M. Raïs, *Distributions homogènes sur des espaces de matrices*, Bull. Soc. math. France, Mem., 30, (1972), 3–109.

[Ru1] B. Rubin, *Inversion of fractional integrals related to the spherical Radon transform*, Journal of Functional Analysis, 157 (1998), 470–487.

[Ru2] _, *Fractional integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere*, Israel Journal of Mathematics, 114 (1999), 1–27.

[Ru3] _, *Notes on Radon transforms in integral geometry*, Fractional Calculus and Applied Analysis, 6 (2003), 25-72.

[Ru4] _, *Zeta distributions and integral geometry on the space of rectangular matrices*, Preprint 2004, Math. FA/0406289.

[Sa] S.G. Samko, *Singular integrals over a sphere and the construction of the characteristic from the symbol*, Soviet Math. (Iz. VUZ), 27 (1983), No. 4, 35–52.

[Se] V.I. Semyanistyi, *Some integral transformations and integral geometry in an elliptic space*, Trudy Sem. Vektor. Tenzor. Anal., 12 (1963), 397–411 (in Russian).

[Sh] B. V. Shabat, *Introduction to Complex Analysis: Functions of Several Variables: Part II*, Translations of Mathematical Monographs, Vol. 110, American Mathematical Society, 1992.

[Schn] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, Cambridge Univ. Press, 1993.

[Sp1] E. Spodarev, *On the rose of intersections of stationary flat processes*, Adv. in Appl. Probab., 33 (2001), 584-599.

[Sp2] _, *Cauchy-Kubota-type integral formulae for the generalized cosine transforms*, Izv. Nats. Akad. Nauk Armenii Mat., 37 (2002), no. 1, 52–69 (2003); translation in J. Contemp. Math. Anal., 37 (2002), no. 1, 47–63.

[St] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ. 1970.

[Str] R.S. Strichartz, *The explicit Fourier decomposition of $L^2(SO(n)/SO(n-m))$*, Can. J. Math., 27 (1975), 294–310.
[T] A. Terras, *Harmonic analysis on symmetric spaces and applications*, Vol. II, Springer, Berlin, 1988.

[TT] T. Ton-That, *Lie group representations and harmonic polynomials of a matrix variable*, Trans. Amer. Math. Soc. 216 (1976), 1–46.

[Tr] J. F. Treves, *Lectures on linear partial differential equations with constant coefficients*, Notas de Matemática, No. 27 Instituto de Matemática Pura e Aplicada do Conselho Nacional de Pesquisas, Rio de Janeiro, 1961.

[VK] N. Ja. Vilenkin, and A. V. Klimyk, *Representations of Lie groups and Special functions*, Vol. 2, Kluwer Academic publishers, Dordrecht, 1993.

Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel

E-mail address: ounyche@macs.biu.ac.il

Department of Mathematics, Louisiana State University, Baton Rouge, LA, 70803, USA,

Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel

E-mail address: borisr@math.lsu.edu