The Weyl – Wigner – Moyal Formalism on a Discrete Phase Space. II. The Photon Wigner Function

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Classical model of light in helicity formalism is presented. Then quantum mechanical point of view at photons – construction and interpretation of two equivalent photon wave functions are proposed. Quantum mechanics of photon is investigated. The Białynicki – Birula scalar product $\langle \Psi_1 | \Psi_2 \rangle_{BB}$ and the generalized Hermitian conjugation $\hat{\gamma}^\dagger$ of linear operator $\hat{\gamma}$ are discussed. Quantum description of light on a phase space based on the Weyl – Wigner – Moyal approach is developed. A photon Wigner function is built. Some explicit photon Wigner functions for different kernels are obtained.

1. Introduction

This paper is the second part of our work devoted to the Weyl – Wigner – Moyal formalism for particles with discrete internal degrees of freedom. In the first part[1] we developed a general approach and then we applied it to a nonrelativistic particle of spin $\frac{1}{2}$ in a homogeneous magnetic field and to analysis of magnetic resonance for a spin $\frac{1}{2}$ uncharged nonrelativistic particle endowed with a magnetic moment.

In the present article we study the Weyl – Wigner – Moyal formalism for a photon and, in particular, we find the photon Wigner function which arises in a natural way within this construction. Our contribution belongs to the same trend as the Klein – Gordon equation and the Dirac equation i.e. we propose quantum mechanical relativistic description of photon but without referring to quantum field theory. This is the case when processes of creation and annihilation are negligible.

Quantum mechanics of photon which we adopt in this paper, was developed by Iwo Białynicki – Birula[2–4] and John E. Sipe[5] (see also [6, 7]). The crucial point in this approach is introducing an acceptable wave function of light. We do that by modification of classical Riemann – Silberstein vector[8,9]. The introduced photon wave function $\Psi(\mathbf{x}, t)$ satisfies an energetic condition of normalisation. Thus some modifications of scalar product and hermicity are required. The ”usual” scalar product is substituted for the Białynicki – Birula scalar product and instead of Hermitian operators one deals with generalized Hermitian operators. We prove that considerations made in language of the Białynicki – Birula scalar product are equivalent to the ones with the original scalar product, when we substitute every state $|\Psi\rangle$ by $\hat{H}^{-1/2}|\Psi\rangle$ and every operator $\hat{A}$ by $\hat{A}_{H} = \hat{H}^{-1/2}\hat{A}\hat{H}^{1/2}$. The symbol $\hat{H}$ denotes the Hamilton operator.

We also indicate problems with establishing a position operator. These obstacles imply energetic condition of normalisation.

Given this quantum mechanics a natural question that can be asked is how one can construct a phase space picture of such quantum mechanics in the Weyl – Wigner – Moyal sense and, in particular, how to define a respective Wigner function. The photon Wigner function within quantum mechanics was proposed by I. Białynicki – Birula in [2, 3]. However, it has been constructed by analogy to the usual nonrelativistic case and not as a consequence of any Weyl – Wigner – Moyal formalism for the photon. The same comment concerns also the photon Wigner function defined in [10] with the use of field operators corresponding to the Riemann – Silberstein vector and its conjugate.

In the present paper we adopt a general theory of the Weyl – Wigner – Moyal formalism of quantum particles with internal degrees of freedom[1,11] (see also references therein) to the photon. Thus one deals with a massless relativistic particle of spin 1 but with an additional constraint saying that the helicity of photon can take only two values: $+1$ or $-1$. So the respective Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3) \otimes C^1$ with the subsidiary condition. The associate phase space is $\Gamma = \mathbb{R}^3 \times \mathbb{R}^3 \times \Gamma^3$, where $\Gamma^3$ is a discrete phase space in form of $3 \times 3$ grid. On that phase space we introduce observables and Wigner functions. We establish a general Weyl correspondence between functions on $\Gamma$ and linear operators acting in the Hilbert space $\mathcal{H}$. The straightforward consequence of the Weyl correspondence is construction of a $\ast$ – multiplication of functions being a counterpart of the product of operators.

The generalized Weyl correspondence is used also to build a Wigner function. We discuss its properties and the time evolution. Especially we want to focus the reader attention to marginal probability distributions originated from it.

It is well known that a correspondence between functions on a phase space and linear operators acting in the respective Hilbert
space is not unique. This relationship is determined by a function called kernel. Following the idea presented in our previous article we define functions \( f(\vec{\alpha}, \pi, k, l) \) built from original objects \( f(\vec{\beta}, \phi, n, m) \) and introduce a product \( \tilde{\otimes} \) such that there exists correspondence between functions and operators \( \vec{f} \rightarrow \vec{\tilde{f}} \), \( \vec{\tilde{g}} \rightarrow \vec{\tilde{g}} \) keeping the mapping \( f(\vec{\alpha}, \pi, k, l) \otimes \tilde{g}(\vec{\beta}, \phi, n, m) \rightarrow \tilde{f}(\vec{\alpha}, \pi, k, l) \otimes g(\vec{\beta}, \phi, n, m) \) independent from the choice of kernels.

The paper is organised as follows. In Section 2 we recall some facts from classical Maxwell electrodynamics in vacuum which are pertinent for further considerations. We use the spinorial and helicity formalisms. The Riemann – Silberstein vector is widely employed as the main object appearing in the Maxwell equations. This vector plays a crucial role in formulating quantum mechanics of photon and in defining a photon wave function.

In Sec. 3 the photon wave function is defined and its probability interpretation is given. We mainly follow the works. We consider also a unitary transformation leading to the spinorial representation of the photon wave function.

Section 4 is devoted to further study of structure of photon quantum mechanics. We investigate the Białynicki – Birula scalar function called kernel. Following the idea presented in our previous article we define functions \( f(\vec{\alpha}, \pi, k, l) \) built from original objects \( f(\vec{\beta}, \phi, n, m) \) and introduce a product \( \tilde{\otimes} \) such that there exists correspondence between functions and operators \( \vec{f} \rightarrow \vec{\tilde{f}} \), \( \vec{\tilde{g}} \rightarrow \vec{\tilde{g}} \) keeping the mapping \( f(\vec{\alpha}, \pi, k, l) \otimes \tilde{g}(\vec{\beta}, \phi, n, m) \rightarrow \tilde{f}(\vec{\alpha}, \pi, k, l) \otimes g(\vec{\beta}, \phi, n, m) \) independent from the choice of kernels.

2. The Classical Maxwell Electromagnetic Field. The Spinorial and Helicity Formalisms

In this section we describe the classical Maxwell electromagnetic field in vacuum within the spinorial and helicity formalisms. We deal with the Minkowski spacetime of signature \((+, +, +, -)\). In the Lorentz coordinate system \((x^1, x^2, x^3, x^4 = ct)\) the line element reads

\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 = g_{\alpha\beta} dx^\alpha{dx^\beta},
\]

where

\[
\left( g_{\mu\nu} \right) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

Define a spinorial 1 – form

\[
g^{\hat{A}\hat{B}} := g_{\mu}^{\hat{A}\hat{B}} dx^\mu, \quad A = 1, 2, \quad \hat{B} = \hat{1}, \hat{2}
\]

with matrices \( g^{\mu} \) given by

\[
\left( g^{\mu} \right) := \begin{pmatrix}
0 & 1 & 0 & -i \\
1 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
-i & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Hence by (2.1)

\[
\left( g^{\mu} \right) := \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Then one introduces symmetric spinorial 2 – forms

\[
S^{\hat{A}\hat{B}} := \frac{1}{2} \epsilon^{CDEB} g^{CAB} \wedge g^{DAB} = S^{BA},
\]

\[
\hat{S}^{\hat{A}\hat{B}} := \frac{1}{2} \epsilon^{CDEB} \wedge g^{DAB} = (S^{AB})^* = \hat{S}^{\hat{A}\hat{B}},
\]

\[
\hat{A}, \hat{B}, C, D = 1, 2; \quad \hat{A}, \hat{B}, \hat{C}, \hat{D} = \hat{1}, \hat{2}
\]

where \( \epsilon^{CDE} = \epsilon^{CDE} \) and \( \epsilon^{CDE} = \epsilon^{CDE} \) are antisymmetric spinors (the Levi – Civita symbols)

\[
\epsilon^{CDE} = \epsilon^{CDE} = (0, 1, 0) = (\epsilon^{\hat{A}\hat{B}}), \quad (\epsilon^{\hat{A}\hat{B}})
\]

and the asterix “*” stands for the complex conjugation. [An explicit expression for 2 – form \( S^{\hat{A}\hat{B}} \) is

\[
S^{\hat{A}\hat{B}} = \frac{1}{2} \left( S^{\mu} \right) dx^\mu \wedge dx^\nu
\]

\[
= \begin{pmatrix}
0 & -i \\
-i & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} \wedge dx^1 \wedge dx^2 + \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \wedge dx^1 \wedge dx^3
\]

\[
+ \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \wedge dx^2 \wedge dx^3
\]

\[
= \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \wedge dx^1 \wedge dx^4 + \begin{pmatrix}
0 & -1 \\
-1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} \wedge dx^3 \wedge dx^4.
\]

\( \hat{S}^{\hat{A}\hat{B}} \) is automatically generated from \( S^{\hat{A}\hat{B}} \) via (2.5b).

Spinorial indices are manipulated with the use of \( \epsilon^{\hat{A}\hat{B}} \) and \( \epsilon^{\hat{A}\hat{B}} \) as follows

\[
\chi_i = \epsilon^{\hat{A}\hat{B}} x_i, \quad \hat{\chi}_i = \epsilon^{\hat{B}\hat{A}} x_i
\]

and analogously one defines the lowering and rising indices for undotted spinorial indices with the use of \( \epsilon^{\hat{A}\hat{B}} \) and \( \epsilon^{\hat{A}\hat{B}} \).

Equations (2.5) are equivalent to the formula

\[
g^{\hat{A}\hat{B}} \wedge g^{\hat{C}\hat{D}} = \epsilon^{\hat{B}\hat{D}} S^{\hat{A}\hat{C}} + \epsilon^{\hat{A}\hat{C}} \hat{S}^{\hat{B}\hat{D}}.
\]
We adapt the definition of the Hodge star operation (the Hodge – ∗) to 2 – forms as 
\[ \omega = \frac{1}{2} \omega_{\mu \nu} dx^\mu \wedge dx^\nu \implies \ast \omega := \frac{1}{2} \left( -\frac{i}{2} \epsilon_{\mu \rho \nu} \omega^{\rho \mu} dx^\nu \wedge dx^\rho \right) \] 
(2.9)
with \( \epsilon_{\mu \rho \nu} \) being the Levi – Civita totally antisymmetric symbol \( \epsilon_{123} = 1 \). Indices of \( \omega_{\mu \nu} \) have been raised according to the rule \( \omega^{\mu \nu} = \eta^{\mu \eta} \eta^{\nu \nu} \omega_{\eta \eta} \).

One quickly finds that
\[ ** \omega = \omega \] 
(2.10)
and
\[ * S^{AB} = S_{AB}, \quad * S^\bigwedge = - S^\bigwedge. \] 
(2.11)
Therefore \( S^{AB} \) are self – dual 2 – forms and \( S^\bigwedge \) are anti – self – dual 2 – forms. In fact the system of three 2 – forms \( \{ S^{12}, S^{23}, S^{31} \} \) constitutes a basis of vector space of self – dual 2 – forms and the system \( \{ S^{12}, S^{23}, S^{31} \} \) is a basis of linear space of anti – self – dual 2 – forms.

Let
\[ A = A_\mu dx^\mu, \quad A_\mu = (\vec{A}, - \Phi) \] 
(2.12)
be an electromagnetic potential 1 – form and
\[ F = dA = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu, \]
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \] 
(2.13)
the electromagnetic field 2 – form. Using definition (2.12) of \( A_\mu \) one easily gets
\[ (F_{\mu \nu}) = \begin{pmatrix} 0 & B_3 & -B_1 & \mathcal{E}_1 \\ -B_3 & 0 & B_1 & \mathcal{E}_2 \\ B_1 & -B_2 & 0 & \mathcal{E}_3 \\ -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 & 0 \end{pmatrix} \] 
(2.14)
where \( \mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) \) and \( \vec{B} = (B_1, B_2, B_3) \) are vectors of electric and magnetic fields respectively.

We express 2 – form \( F \) in terms of \( S^{AB} \) and \( S^\bigwedge \) as
\[ F = \frac{1}{2} \left( f_{AB} S^{AB} + f_{AB} S^\bigwedge \right) \] 
(2.15)
with the obvious rules for raising and lowering indices
\[ S^{\mu \nu}_{AB} = \eta^{\mu \eta} \eta^{\nu \nu} e_{AC} e_{BD} S_{\eta \eta}, \quad S^{\mu \nu}_{\bigwedge} = \eta^{\mu \eta} \eta^{\nu \nu} e_{\bigwedge C} e_{\bigwedge D} S_{\bigwedge \bigwedge}. \] 
(2.17)
One can check easily the following identities
\[ S^{\mu \nu}_{AB} S^{\nu \gamma}_{CD} = 0 = S^{\mu \mu}_{\bigwedge} S^{\nu \nu}_{CD}, \] 
(2.18a)
\[ S^{\mu \nu}_{AB} S^{\nu \gamma}_{CD} = 4 (\delta^A_C \delta^B_D + \delta^A_D \delta^B_C), \] 
(2.18b)
\[ S^{\mu \nu}_{AB} S^{\nu \gamma}_{CD} = 4 (\delta^A_C \delta^B_D + \delta^A_D \delta^B_C). \] 
(2.18c)
Applying (2.18) to (2.15) we obtain
\[ f_{AB} = \frac{1}{4} F_{\mu \nu} S^{\mu \nu}_{AB}, \quad f_{\bigwedge} = \frac{1}{4} F_{\mu \nu} S^{\mu \nu}_{\bigwedge}. \] 
(2.19)
Straightforward calculations lead to relations
\[ f_{11} = \frac{1}{\sqrt{2}} (F_1 - i F_3) = (f_{11})^*, \]
\[ f_{12} = f_{21} = - \frac{1}{\sqrt{2}} F_1 = (f_{12})^*, \]
\[ f_{22} = - \frac{1}{\sqrt{2}} (F_1 + i F_3) = (f_{22})^*. \] 
(2.20)
where \( (F_1, F_2, F_3) = \vec{F} \) is the Riemann – Silberstein vector\(^\text{[2,3,8,9]}\)
\[ \vec{F} := \frac{1}{\sqrt{2}} (\vec{E} + i \vec{B}). \] 
(2.21)
Equations (2.20) can be rewritten in a compact form as
\[ f_{AB} = i \Phi_{AB}^\bigwedge, f_j = (f_{1j})^*, \] 
(2.22)
with \( \Phi_{AB}^\bigwedge \) defined by 2 × 2 matrices
\[ (\Phi_{AB}^\bigwedge) := \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, (\Phi_{ij}^\bigwedge) := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \] 
(2.23)
Rising spinorial indices according to the rule (2.7) we obtain that
\[ (\Phi_{AB}^{\bigwedge}) := \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, (\Phi_{ijk}^{\bigwedge}) := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \] 
(2.24)
Thus one immediately finds that
\[ \Phi_{AB}^\bigwedge, \Phi_{AB}^{\bigwedge} = \delta^A_B, \quad j, k = 1, 2, 3. \] 
(2.25)
From (2.22) and (2.25) we get

\[ F_j = -i \Phi^{j \lambda}_{A B} f_{A B}^\lambda \]  

(2.26)

under the convention that the Latin indices \( j, k = 1, 2, 3 \) are to be manipulated with the use of the Kronecker delta so \( \Phi^{j \lambda}_{A B} = \Phi_{A B}^{j \lambda} \).

Inserting Equation (2.26) back into (2.22) one quickly finds the relation

\[ \Phi_j^{\lambda A} \Phi^{\mu B}_{jCD} = \frac{1}{2} \left( \Phi^{j \lambda}_{C D} + \Phi^{j \lambda}_{D C} \right). \]  

(2.27)

Observe that the scalar product

\[ \vec{F} \cdot \vec{F} = \frac{1}{2} (\vec{e}^2 - \vec{B}^2 + 2i \vec{e} \cdot \vec{B}) \]  

(2.28)

is invariant under the proper orthochronous Lorentz group \( L_1^+ \). Consequently, if \((\Lambda^\nu_j) \in L_1^+\) defines a transformation from an inertial system of frames \( K \) to another inertial system \( K' \) and \((\bar{L}^\nu_j), -\bar{L}^\nu_j \in SL(2, C)\) are two possible representatives of transformation \((\Lambda^\nu_j)\) in the spinor representation, then we have

\[ F'_j = -i \Phi^{j \lambda}_{A B} f'^{\lambda A}_{j} = -i \Phi^{j \lambda}_{A B} \bar{f}_{A B}^\lambda \]  

(2.29)

and

\[ F'^2_1 + F'^2_2 + F'^2_3 = F^2_1 + F^2_2 + F^2_3 = \vec{F} \cdot \vec{F} \]  

(2.30)

where the prime “*” corresponds to objects in the inertial system \( K' \). From (2.30) it follows that there exists an orthogonal complex \( 3 \times 3 \) matrix \((t^\nu_j) \in O(3; C)\) such that

\[ F'^\nu_j = t^\nu_j F^\nu_j; \quad F'^\nu_j \equiv F_j, \quad F^\nu_j \equiv F_j. \]  

(2.31)

Substituting (2.22) into the right – hand side and (2.31) into the left – hand side of (2.29) one obtains

\[ t^\nu_j = \Phi^{j \lambda}_{A B} \bar{t}^\nu_\lambda A B \Phi_{A B}^{j \lambda}. \]  

(2.32)

Employing then (2.27) we get the inverse relation

\[ \frac{1}{2} \left( \bar{t}^\nu_\lambda A B + t^\nu_\lambda A B \right) = \Phi^{j \lambda}_{A B} \bar{t}^\nu_j A B \Phi_{A B}^{j \lambda}. \]  

(2.33)

From (2.32), using also (2.23) and (2.24), one shows that

\[ \det(t^\nu_j) = 1 \implies (t^\nu_j) \in SO(3; C). \]  

(2.34)

Moreover, a simple analysis of Equations (2.32) and (2.33) leads to conclusion that matrices \((t^\nu_\lambda A B)\) and \(-\bar{t}^\nu_\lambda A B\) define the same matrix \((t^\nu_j)\) and conversely, every matrix \((t^\nu_j)\) defines the \( SL(2; C) \) matrix \((\bar{t}^\nu_\lambda A B)\) up to the sign. Therefore, formulae (2.32) and (2.33) realise a group isomorphism

\[ SL(2; C)/Z_2 \cong SO(3; C) \]  

(2.35)

where \( Z_2 = \{+1, -1\} \) is the cyclic group.

Since \( SL(2; C)/Z_2 \cong L_1^+ \) one obtains an isomorphism

\[ L_1^+ \cong SO(3; C). \]  

(2.36)

Thus we arrive at a principal bundle isomorphism

\[ (M_4 \times L_1^+, M_4, \Pi_1) \cong (M_4 \times SO(3; C), M_4, \Pi_2) \]  

(2.37)

where \( M_4 \) is the Minkowski spacetime and

\[ \Pi_1 : M_4 \times L_1^+ \to M_4, \quad \Pi_2 : M_4 \times SO(3; C) \to M_4 \]

are natural projections on \( M_4 \). Principal fibre bundle \((M_4 \times SO(3; C), M_4, \Pi_2)\) can be considered as a reduced bundle of all linear bases in vector space \( C^3 \) over \( M_4 \) given by a reduction of Lie group \( GL(3; C) \) to \( SO(3; C) \). With such an identification one can construct vector bundles associated with principle bundle \((M_4 \times SO(3; C), M_4, \Pi_2)\). Sections of those vector bundles are vector and tensor fields on \( M_4 \) which transform according to a suitable representation of \( SO(3; C) \). From (2.22) and (2.26) one concludes that any vector bundle of \( SO(3; C) \) vectors is isomorphic to the vector bundle of dotted symmetric spinors of the second rank. Of course, \textit{mutatis mutandis}, similar construction can be done for the undotted symmetric spinors of the second rank. In this construction Equations (2.22) and (2.26) take the form

\[ f_\nu_{A B} = -i \Phi_{A B}^{j \nu} f_j^\nu \]  

(2.38)

and

\[ f_\nu^* = i \Phi_{A B}^{j \nu} f_j^\nu \]  

(2.39)

where

\[ \Phi_{A B}^{j \nu} := \left( \Phi_{A B}^{j \nu} \right)^*. \]  

(2.40)

According to the Plebanski’s terminology\(^{[13]}\) the formalism described above as founded on the \( SO(3; C) \) group we call the \textit{helicity formalism}\(^{[1]}\).

From relations (2.22) or (2.26) one quickly infers that \( \Phi_{A B}^{j \nu} \) is a \textit{spin} – \( SO(3; C) \) vector.

In spinorial formalism the Maxwell equations in empty space read

\[ \partial^\alpha f_{\nu \beta} = 0 \iff \partial^\beta f_{\nu \beta} = 0 \]  

(2.41)

where

\[ \partial^\alpha := g^{\mu \alpha \beta} \partial_\mu = \left( \partial^\alpha \right)^* \]  

(2.42)

Under convention used in the present paper it is convenient to deal with the form of Maxwell equations as written on the left – hand side of (2.41).

Inserting relations (2.42) with (2.4) into the Maxwell equations and employing symmetry of \( f_{\nu \beta} \) one concludes that the Maxwell equations split into the evolution matrix equation

\[ \partial f = -c \left( \vec{S} \cdot \vec{\nu} \right) f \]  

(2.43)

\(^{[1]}\) In his notes\(^{[13]}\) Jerzy Plebanski developed the helicity formalism in detail for all real and complex Riemannian 4 – dimensional structures.
where

\[ f := \begin{pmatrix} f_1; \sqrt{2}f_{12} \end{pmatrix}, \quad \tilde{\nu} = (\partial_1, \partial_2, \partial_3) \]

and \( S^\prime = (S_1', S_2', S_3') \) with

\[ S_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad S_3' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \]

and the constraint equation

\[ \partial_\epsilon \left( \Phi^\Lambda f_{\mu\nu} \right) = 0. \]

Writing the Riemann–Silberstein vector \( \mathbf{F} \) in a matrix form

\[ \mathbf{F} := \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \]

we can rewrite relations (2.22) and (2.26) in the form

\[ \mathbf{f} = \mathbf{U} \cdot \mathbf{F} \quad \text{and} \quad \mathbf{F} = \mathbf{U}^\dagger \cdot \mathbf{f} \]

and

\[ S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

In a compact form matrices \( S_j \) can be written with the use of the 3 - D Levi – Civita symbol

\[ (S)_{ji} = -ie_{jkl}, \quad j, k, l = 1, 2, 3. \]

Then comparing (2.45) with (2.26) one quickly finds that the constraint equation (2.45) in terms of \( \mathbf{F} \) reads

\[ \sum_{j=1}^{3} \partial_j F_j = 0. \]

Matrices \( S_j, j = 1, 2, 3 \) are the well known spin – 1 matrices. They satisfy the following commutation relations

\[ S_j \cdot S_k - S_k \cdot S_j = i\epsilon_{jkl} S_l. \]

From (2.51) we infer that the system of matrices \( \{S_1, S_2, S_3\} \) is unitary equivalent to the system \( \{S_1', S_2', S_3'\} \). Hence matrices \( S_j' \)

are also the spin – 1 matrices in another representation. They fulfill the same commutation relation (2.54) as matrices \( S_j \) i.e.

\[ S_j' \cdot S_k' - S_k' \cdot S_j' = i\epsilon_{jkl} S_l'. \]

One can show that the constraint equation (2.53) is equivalent to expression \[3,16]\]

\[ \left( \tilde{\mathbf{S}} \cdot \tilde{\mathbf{V}} \right) \cdot \mathbf{F} = \delta_j \mathbf{F} \]

for any \( j \). Acting on both sides of (2.56) with \( \partial_j \) and summing over \( j \) we get \[3,16]\]

\[ \left( \tilde{\mathbf{S}} \cdot \tilde{\mathbf{V}} \right) \cdot \mathbf{F} = \Delta \mathbf{F}. \]

Of course, Equations (2.56) and (2.57) can be equivalently rewritten in terms of matrix \( \mathbf{f} \) as

\[ \left( \tilde{\mathbf{S}}' \cdot \tilde{\mathbf{V}} \right) \cdot \mathbf{f} = \partial_j \mathbf{f} \]

and

\[ \left( \tilde{\mathbf{S}}' \cdot \tilde{\mathbf{V}} \right) \cdot \mathbf{f} = \Delta \mathbf{f}. \]

Any solution of Equation (2.50) can be expanded in the plane wave solutions

\[ \mathbf{F}(\tilde{\mathbf{x}}, t) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \left[ a_\epsilon(\tilde{k}) \exp\{i(\tilde{k} \cdot \tilde{x} - \omega_k t)\} + a_\epsilon^*(\tilde{k}) \exp\{-i(\tilde{k} \cdot \tilde{x} - \omega_k t)\} \right], \]

with

\[ S^\prime = (S_1', S_2', S_3') := U^\dagger S' U \]
where
\[
\omega_k = c \tilde{k}, \quad \mathbf{a}_\pm(\tilde{k}) = \begin{pmatrix} a_{1\pm} \tilde{k} \\ a_{2\pm} \tilde{k} \end{pmatrix}.
\]

Inserting (2.60) into (2.50) one arrives at conclusions
\[
\left( \mathbf{S} \cdot \tilde{k} \right) \mathbf{a}_\pm(\tilde{k}) = |\tilde{k}| \mathbf{a}_\pm(\tilde{k})
\]

or, by (2.52), to the system of three conditions
\[
\sum_{l,m} i \epsilon_{lm} k_l a_m \pm(\tilde{k}) = |\tilde{k}| a_\pm(\tilde{k}).
\]

Then constraint (2.53) under (2.52) gives
\[
\sum_{l,m} k_l a_m \pm(\tilde{k}) = 0.
\]

From (2.55) (or from (2.54) with (2.51)) we find
\[
a_\mp(\tilde{k}) \cdot \mathbf{a}_\pm(\tilde{k}) = 0 = a_\mp(\tilde{k}) \cdot \mathbf{a}_\pm(\tilde{k}).
\]

Elementary analysis of Equations (2.63) and (2.64) leads to the conclusion that \( a_\pm(\tilde{k}) \) can be written in the form
\[
a_\pm(\tilde{k}) = \alpha_\pm(\tilde{k}) \mathbf{e}(\tilde{k}).
\]

where \( \alpha_\pm(\tilde{k}) \) are some functions of \( \tilde{k} \) and
\[
\mathbf{e}(\tilde{k}) = \begin{pmatrix} e_1(\tilde{k}) \\ e_2(\tilde{k}) \\ e_3(\tilde{k}) \end{pmatrix}
\]
is a solution of the system of equations
\[
\left( \mathbf{S} \cdot \tilde{k} \right) \mathbf{e}(\tilde{k}) = |\tilde{k}| \mathbf{e}(\tilde{k}),
\]

\[
\sum_{j=1}^3 k_j e_j(\tilde{k}) = 0
\]
fulfilling the normalisation condition
\[
\mathbf{e}^*(\tilde{k}) \cdot \mathbf{e}(\tilde{k}) = 1.
\]

By (2.65) one has
\[
\mathbf{e}^*(\tilde{k}) \cdot \mathbf{e}(\tilde{k}) = 0.
\]

Therefore \( \mathbf{e}(\tilde{k}) \) can be written as
\[
\mathbf{e}(\tilde{k}) = \frac{1}{\sqrt{2}} \left( \mathbf{m}(\tilde{k}) + \mathbf{n}(\tilde{k}) \right).
\]
\[ \mathcal{M} = \frac{1}{i\hbar} \int d^3x \bar{x} \times (\vec{p}^2 \times \vec{F}). \]  

(2.77c)

\[ \mathcal{N} = \int d^3x \bar{x} (\vec{p}^2 \cdot \vec{F}). \]  

(2.77d)

### 3. The Wave Function of Photon

It is a trivial statement that photon is, *par excellence*, a relativistic particle. Consequently, a natural theory describing any system of photons is relativistic quantum field theory. Nevertheless several authors\(^1\) in analogy to the case of the Dirac equation which provides quantum mechanical but not quantum field description of electron, are trying to introduce a wave function of photon and to construct quantum mechanics of this particle.

In the present section we consider a version of quantum mechanical approach to a single photon system developed by I. Bialynicki – Birula\(^2\) and J. E. Sipe\(^3\) in their distinguished works.

First, let us multiply both sides of (2.50) by \( i\hbar \). This leads to

\[ i\hbar \partial_t \mathcal{F} = c \left( \vec{S} \cdot \vec{p} \right) \mathcal{F}. \]  

(3.1)

where \( \vec{p} = -i\hbar \vec{\mathcal{V}} = (-i\hbar \partial_1, -i\hbar \partial_2, -i\hbar \partial_3) \) is the momentum operator.

In the same way from (2.43) one obtains

\[ i\hbar \partial_t \mathcal{F} = c \left( \vec{S}^\prime \cdot \vec{p} \right) \mathcal{F}. \]  

(3.2)

At first glance Equation (3.1) is similar to the Dirac equation with an obvious additional assumption that now we are dealing not with an electron but with a photon which is a spin – 1 massless particle. This observation suggests that Equation (3.1) is the right quantum mechanical relativistic equation for a photon and \( \mathcal{F} = \mathcal{F}(\vec{x}, t) \) subject to additional condition (2.53) is the photon wave function.

However, this conclusion is incorrect since Equation (3.1) considered as a quantum evolution equation admits solutions with negative energy

\[ \mathcal{F}(\vec{k}, t) \text{exp}\left[-i(\vec{k} \cdot \vec{x} - \omega t)\right], \quad \vec{k} = \frac{\vec{p}}{\hbar} \]  

(see (2.67) and (2.73)). In the case of the Dirac equation the solutions with negative energies are interpreted as the ones representing the antiparticle. However the antiparticle of photon is the photon itself. Therefore in the photon quantum mechanics the solutions with negative energies are unphysical. It means that some modification of formula (3.1) is required.

First note that the *helicity operator of photon* in the momentum representation reads

\[ \vec{\Sigma} = \frac{\vec{S} \cdot \vec{p}}{|p|} = \frac{\vec{S} \cdot \vec{k}}{|k|}, \quad \vec{p} = \hbar \vec{k} \]  

(3.3)

and its eigenvalues are \( \lambda = \pm 1 \). A general wave function is a superposition of (+1) – helicity states and (−1) – helicity states. Quick look at Equations (2.67), (2.72) and (2.73) with taking into account that only states of positive energy have physical meaning, lead to conclusion that the photon wave function has a form

\[ \Psi(\vec{x}, t) = \sqrt{\frac{\hbar}{2\pi}} \int \frac{d^3k}{(2\pi)^3} \left[ e^{i\vec{k} \cdot \vec{x} - \omega t} \Psi_+(\vec{k}, t) + e^{-i\vec{k} \cdot \vec{x} - \omega t} \Psi_-(\vec{k}, t) \right] \]  

\[ \times \exp\left[i(\vec{k} \cdot \vec{x} - \omega t)\right] \]  

(3.4)

where \( \Psi_+(\vec{k}, t) \) and \( \Psi_-(\vec{k}, t) \) satisfy (2.67b), (2.72b) and (2.73b) with taking into account that only states of positive energy have physical meaning, lead to conclusion that the photon wave function has a form

\[ \Psi_+(\vec{x}, t) = \Psi_{\sigma_+}(\vec{x}, t) + \Psi_{\sigma_-}(\vec{x}, t), \]  

(3.5)

where \( \sigma = (\sigma_+, \sigma_-) \) are superpositions of (1) – helicity states and (−1) – helicity states respectively.

One easily finds that \( \Psi_{\sigma_+}(\vec{x}, t) \) and \( \Psi_{\sigma_-}(\vec{x}, t) \) fulfill the following equations

\[ i\hbar \partial_t \Psi_{\sigma_+} = c \left( \vec{S} \cdot \vec{p} \right) \Psi_{\sigma_+}. \]  

(3.7a)

\[ i\hbar \partial_t \Psi_{\sigma_-} = -c \left( \vec{S} \cdot \vec{p} \right) \Psi_{\sigma_-}. \]  

(3.7b)

Let \( \hat{\Pi}_+ \) and \( \hat{\Pi}_- \) denote projection operators

\[ \hat{\Pi}_+ \Psi = \Psi_{\sigma_+}, \quad \hat{\Pi}_- \Psi = \Psi_{\sigma_-}. \]  

(3.8)

By adding Equations (3.7a) and (3.7b) we can see that the photon wave function satisfies the Schrödinger – like evolution equation

\[ i\hbar \partial_t \Psi = \hat{H} \Psi \]  

(3.9)

where \( \hat{H} \) is the Hamilton operator

\[ \hat{H} := c \left( \vec{S} \cdot \vec{p} \right) \left( \hat{\Pi}_+ - \hat{\Pi}_- \right). \]  

(3.10)

\[ \Psi \]  

(3.11)

and by (2.67b) and (2.72b) is subject to the constraint equation

\[ \sum_{\sigma = \pm} \partial_t \Psi_{\sigma} = 0. \]  

(3.12)
Therefore the photon wave function components satisfy the following systems of differential equations

\[
\begin{cases}
\frac{\partial \Psi_{+1}}{\partial t} = -ic \left( \frac{\partial \Psi_{+1}}{\partial x^1} - \frac{\partial \Psi_{+2}}{\partial x^2} \right), \\
\frac{\partial \Psi_{+2}}{\partial t} = -ic \left( \frac{\partial \Psi_{+1}}{\partial x^2} + \frac{\partial \Psi_{+2}}{\partial x^1} \right), \\
\frac{\partial \Psi_{-1}}{\partial t} = ic \left( \frac{\partial \Psi_{-1}}{\partial x^1} - \frac{\partial \Psi_{-2}}{\partial x^2} \right), \\
\frac{\partial \Psi_{-2}}{\partial t} = ic \left( \frac{\partial \Psi_{-1}}{\partial x^2} + \frac{\partial \Psi_{-2}}{\partial x^1} \right),
\end{cases}
\]

(3.13a)

\[
\begin{cases}
\frac{\partial \Psi_{+1}}{\partial t} = ic \left( \frac{\partial \Psi_{+1}}{\partial x^1} - \frac{\partial \Psi_{+2}}{\partial x^2} \right), \\
\frac{\partial \Psi_{+2}}{\partial t} = ic \left( \frac{\partial \Psi_{+1}}{\partial x^2} + \frac{\partial \Psi_{+2}}{\partial x^1} \right), \\
\frac{\partial \Psi_{-1}}{\partial t} = -ic \left( \frac{\partial \Psi_{-1}}{\partial x^1} - \frac{\partial \Psi_{-2}}{\partial x^2} \right), \\
\frac{\partial \Psi_{-2}}{\partial t} = -ic \left( \frac{\partial \Psi_{-1}}{\partial x^2} + \frac{\partial \Psi_{-2}}{\partial x^1} \right).
\end{cases}
\]

(3.13b)

A natural question arises whether there exists any probabilistic interpretation of the photon wave function \( \Psi(x, t) \). In the Dirac notation one can write \( |\Psi(t)\rangle = \langle \tilde{x} | \Psi(t) \rangle \) with

\[
|\Psi(t)\rangle = \begin{pmatrix} |\Psi_1(t)\rangle \\ |\Psi_2(t)\rangle \\ |\Psi_3(t)\rangle \end{pmatrix}.
\]

Therefore \( \Psi(x, t) \) represents a state of photon in the position representation. However, the Hermitian operator \( \tilde{x} \) cannot be considered as a photon position operator or any other photon observable since \( \tilde{x} \Psi(x, t) \) does not satisfy the constraint condition (3.12) although \( \Psi(x, t) \) does. Consequently, quantity

\[
\int_V d^3x \Psi^\dagger(x, t) \Psi(x, t)
\]

cannot be interpreted as the probability of finding the photon in the domain \( V \) at instant \( t \) like in “usual” quantum mechanics.

A correct interpretation seems to be the one given by I. Białynicki–Birula and J. E. Sipe. Assume that \( \Psi(x, t) \) is normalised as

\[
\int_{\mathbb{R}^3} d^3x \Psi^\dagger(x, t) \Psi(x, t) = \langle E \rangle(t),
\]

(3.15)

where \( \langle E \rangle(t) \) denotes the average energy of photon at instant \( t \). Condition (3.15) is a counterpart of classical formula (2.77a). Inserting (3.4) into (3.15) one receives

\[
\langle E \rangle(t) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3|k|} \omega_{\tilde{k}} \left( |\alpha(k, +1)|^2 + |\alpha(k, -1)|^2 \right) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3|k|} \omega_{\tilde{k}} \langle \tilde{k} | \Psi(\tilde{k}, t) \rangle \langle \tilde{k} | \Psi(\tilde{k}, t) \rangle,
\]

(3.16)

where

\[
\omega_{\tilde{k}} := \frac{1}{\sqrt{\hbar c}} \int_{\mathbb{R}^3} d^3x \Psi(\tilde{x}, 0) \exp \left\{ -i\tilde{k} \cdot \tilde{x} \right\} = e^{\tilde{k}} \alpha(\tilde{k}, +1) + e^{\tilde{k}} \alpha(\tilde{k}, -1).
\]

(3.17)
and

\[ \mathcal{G}(\vec{k} \in \Omega) = \frac{\int_{\Omega} \frac{d^3k}{(2\pi)^3} \vec{\Psi}^\dagger(\vec{k})\vec{\Psi}(\vec{k})}{\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \vec{\Psi}^\dagger(\vec{k})\vec{\Psi}(\vec{k})} \]  \tag{3.25}

respectively.

In Sec. 2 the electromagnetic field was represented by the Riemann – Silberstein matrix \( F \) and equivalently by matrix \( f \). These two representations are related by a unitary transformation (2.47). In quantum model of photon a wave function \( \Psi(\vec{x}, t) \) based on the Riemann – Silberstein matrix has been built.

Let us consider now a photon wave function obtained from wave function \( \Psi(\vec{x}, t) \) by a unitary transformation defined in classical electrodynamics by (2.47) with (2.48). So one puts

\[ \Psi'(\vec{x}, t) = U \Psi(\vec{x}, t). \]  \tag{3.26}

Inserting (3.4) into (3.26) and applying (2.48) we receive

\[ \Psi'(\vec{x}, t) = \sqrt{\hbar c} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \left[ \left( \begin{array}{c} e_{11}^\dagger(\vec{k}) \\ e_{12}^\dagger(\vec{k}) \\ \end{array} \right) a(\vec{k}, +1) + U \cdot U^\dagger \left( \begin{array}{c} e_{11}^\dagger(\vec{\ell}) \\ e_{12}^\dagger(\vec{\ell}) \\ \end{array} \right)^* a(\vec{\ell}, -1) \right] \exp[i(\vec{k} \cdot \vec{x} - \omega_k t)] \times \exp[i(\vec{\ell} \cdot \vec{x} - \omega_\ell t)] \]  \tag{3.27}

where \( e_{1\alpha}^\dagger(\vec{k}) \) and \( e_{1\beta}^\dagger(\vec{\ell}) \) are symmetric spinors defined as

\[ e_{1\alpha}^\dagger(\vec{k}) := i \Phi_{\alpha\beta}^j e_j(\vec{k}), \quad e_{1\beta}^\dagger(\vec{\ell}) := \left( i \Phi_{\alpha\beta}^j e_j(\vec{\ell}) \right)^* \]  \tag{3.28}

(see (2.22), (2.23) and (2.24)).

Photon wave function \( \Psi'(\vec{x}, t) \) satisfies the evolution Schrödinger – like equation

\[ i\hbar \partial_t \Psi' = \hat{H} \Psi', \]  \tag{3.29}

where the Hamilton operator

\[ \hat{H} = c \left( \hat{S} \cdot \hat{\mathcal{F}} \right) \cdot \left( \hat{H}_- - \hat{H}_+ \right) = \hat{H}, \]  \tag{3.30}

\[ \hat{H}_\pm = U \hat{H}_\pm U^\dagger \]

and with \( \hat{\mathcal{S}'} \) given by (2.44). Comparing (3.27) with (3.5) and (3.6) one concludes that \( \Psi'(\vec{x}, t) \) can be expressed as

\[ \Psi'(\vec{x}, t) = \Psi_+(\vec{x}, t) + \Psi'_-(\vec{x}, t). \]  \tag{3.31}

where

\[ \Psi_+(\vec{x}, t) = \sqrt{\hbar c} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \left( \begin{array}{c} e_{11}^\dagger(\vec{k}) \\ e_{12}^\dagger(\vec{k}) \\ \end{array} \right) a(\vec{k}, +1) \exp[i(\vec{k} \cdot \vec{x} - \omega_k t)] \]  \tag{3.32}

consists of states of helicity +1

\[ \frac{\hat{S}^I \cdot \hat{\mathcal{P}}}{|\mathcal{P}|} \Psi_+(\vec{x}, t) = +1 \cdot \Psi'_+(\vec{x}, t) \]  \tag{3.33}

and

\[ \Psi'_-(\vec{x}, t) = \sqrt{\hbar c} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \left( \begin{array}{c} e_{11}^\dagger(\vec{k}) \\ e_{12}^\dagger(\vec{k}) \\ \end{array} \right) a(\vec{k}, -1) \exp[i(\vec{k} \cdot \vec{x} - \omega_k t)] \times \exp[i(\vec{k} \cdot \vec{x} - \omega_k t)] = \hat{H} \Psi'_-(\vec{x}, t) \]  \tag{3.34}

is a superposition of states of helicity −1

\[ \frac{\hat{S}^I \cdot \hat{\mathcal{P}}}{|\mathcal{P}|} \Psi'_-(\vec{x}, t) = -1 \cdot \Psi'_-(\vec{x}, t). \]  \tag{3.35}

We end this section with a formula giving relation between \( \langle \hat{k} | \Psi(t) \rangle \) and \( \Psi(\vec{k}, t) \). We have

\[ \langle \hat{k} | \Psi(t) \rangle = \int_{\mathbb{R}^3} d^3x \langle \hat{k} | \vec{x} \rangle \langle \vec{x} | \Psi(t) \rangle = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3\mathbf{x} \Psi(\mathbf{x}, t) \times \exp \left\{ -i(\mathbf{k} \cdot \mathbf{x}) \right\} = \sqrt{\frac{\hbar c}{(2\pi)^3}} \Psi(\vec{k}, t) \]  \tag{3.36}

where Equation (3.18) has been used.

4. Scalar Product, Generalized Hermitian Operators and Observables. The Density Operator

One quickly sees that the formula (3.20) leads to the obvious normalization of \( \Psi(\vec{x}, t) \)

\[ \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \bar{\Psi}(\vec{k})\Psi(\vec{k}) = 1 \iff \int_{\mathbb{R}^3} d^3x \Psi(\vec{x}, t) \hat{H}^{-1} \Psi(\vec{x}, t) = 1. \]  \tag{4.1}

In the Dirac notation we write

\[ \langle \Psi(\vec{x}, t) | \hat{H}^{-1} | \Psi(\vec{x}, t) \rangle = 1. \]  \tag{4.2}
Due to the facts that energy and momentum are constants of motion, formulæ (3.15) or (3.16) and (3.21) take forms
\[
\langle E \rangle = \langle \Psi | \hat{H}^{-1} \hat{H} | \Psi \rangle \tag{4.3}
\]
and
\[
\langle \hat{p} \rangle = \langle \Psi | \hat{H}^{-1} \hat{p} | \Psi \rangle \tag{4.4}
\]
respectively. The last two formulæ suggest that it is reasonable to introduce a new scalar product which will be called the Bialynicki–Birula scalar product\(^{1,2,4}\)
\[
\langle \Psi_1 | \Psi_2 \rangle_{BB} := \langle \Psi_1 | \hat{H}^{-1} | \Psi_2 \rangle. \tag{4.5}
\]
Straightforward calculations show that\(^{1}\)
\[
\langle \Psi_1 | \Psi_2 \rangle_{BB} = \int \frac{d^3k}{(2\pi)^3} \frac{\psi_1^*(\mathbf{k}) \psi_2(\mathbf{k})}{|\mathbf{k}|} = \sum_{\lambda=-1}^1 \int \frac{d^3k}{(2\pi)^3} \frac{\alpha^*_\lambda(\mathbf{k}) \alpha_\lambda(\mathbf{k})}{|\mathbf{k}|} = \frac{1}{2\pi^2\hbar c} \int_{\mathbb{R}^3} d^3x d^3x' |\mathbf{x} - \mathbf{x}'|^2 \psi_1^*(\mathbf{x}) \psi_2(\mathbf{x'}). \tag{4.6}
\]
If \(\hat{\gamma}\) is a linear operator then we define the generalized Hermitian conjugation of \(\hat{\gamma}\) as a linear operator \(\hat{\gamma}^*\) such that
\[
\langle \Psi_1 | \hat{\gamma} \Psi_2 \rangle_{BB}^* = \langle \hat{\gamma}^* \hat{\gamma} \Psi_2 | \Psi_1 \rangle_{BB}, \tag{4.7}
\]
(compare with [25,26]).
From (4.5) one gets
\[
\langle \Psi_1 | \hat{\gamma} \Psi_2 \rangle_{BB}^* = \langle \Psi_1 | \hat{H}^{-1} \hat{\gamma} \hat{H}^{-1} | \Psi_2 \rangle = \langle \Psi_1 | \hat{\gamma} \hat{H}^{-1} \hat{\gamma} \hat{H}^{-1} | \Psi_2 \rangle_{BB}. \tag{4.8}
\]
Comparing (4.8) with (4.7) we find
\[
\hat{\gamma}^* = \hat{H} \hat{\gamma} \hat{H}^{-1}. \tag{4.9}
\]
Hence
\[
\langle \Psi | \hat{\gamma} \Psi \rangle_{BB} = \langle \Psi | \hat{H} \hat{\gamma} \hat{H}^{-1} | \Psi \rangle_{BB}. \tag{4.10}
\]
for any \(|\Psi\rangle\) if and only if
\[
\hat{\gamma}^* = \hat{\gamma} \iff \hat{\gamma}^* = \hat{H}^{-1} \hat{\gamma} \hat{H}. \tag{4.11}
\]
Condition (4.11) is called the generalized hermicity condition and a linear operator satisfying (4.11) will be called the generalized Hermitian operator. One easily shows that
\[
\hat{\gamma}^* = \hat{\gamma}, \quad \hat{H}^* = \hat{H}, \quad \hat{x}^* \neq \hat{x}. \tag{4.12}
\]
From (4.3), (4.4) and (4.5) we conclude that if \(\hat{\theta}\) is a photon observable and \(|\Psi\rangle\) is the photon state normalized according to (4.2)
\[
\langle \Psi | \Psi \rangle_{BB} = 1 \quad \text{then the average (expected) value of this observable in state } |\Psi\rangle \text{ reads}
\]
\[
\langle \hat{\theta} \rangle = \langle \Psi | \hat{\theta} | \Psi \rangle_{BB} = \langle \Psi | \hat{H}^{-1} \hat{\theta} | \Psi \rangle. \tag{4.13}
\]
Since \(\langle \hat{\theta} \rangle\) must be real for arbitrary \(|\Psi\rangle\), operator \(\hat{\theta}\) has to be a generalized Hermitian operator. So if \(\hat{\theta}\) is any photon observable then necessary
\[
\hat{\theta}^* = \hat{\theta} \iff \hat{\theta}^* = \hat{H}^{-1} \hat{\theta} \hat{H}. \tag{4.14}
\]
Obviously, in the general case, when \(|\Psi\rangle\) is not normalized as above, one has
\[
\langle \hat{\theta} \rangle = \langle \Psi | \hat{H}^{-1} \hat{\theta} | \Psi \rangle \tag{4.15}
\]
We rewrite (4.15) in the form
\[
\langle \hat{\theta} \rangle = \langle \Psi | \hat{H}^{-1/2} \hat{\theta} \hat{H}^{-1/2} | \Psi \rangle \tag{4.16}
\]
Hence one infers that the average value of \(\hat{\theta}\) in the state \(|\Psi\rangle\) is equal to the average value of the operator
\[
\hat{\theta}_H := \hat{H}^{-1/2} \hat{\theta} \hat{H}^{1/2} \tag{4.17}
\]
in the state \(\hat{H}^{-1/2} |\Psi\rangle\), calculated in accordance with the usual scalar product \(\langle \cdot | \cdot \rangle\).
From (4.14) and (4.17) we find that
\[
\hat{\theta}^* = \hat{\theta} \iff \hat{\theta}^*_H = \hat{\theta}_H. \tag{4.18}
\]
If \(|\theta\rangle\) is an eigenvector of a photon observable \(\hat{\theta}\) corresponding to the eigenvalue \(\hat{\theta}\)
\[
\hat{\theta} |\theta\rangle = \theta |\theta\rangle, \quad \hat{\theta}^* = \hat{\theta} \tag{4.19}
\]
then \(\hat{H}^{-1/2} |\theta\rangle\) is an eigenvector of the Hermitian operator \(\hat{\theta}_H\) given by (4.17) corresponding to the same eigenvalue \(\hat{\theta}\)
\[
\hat{\theta}_H |\theta\rangle = \theta |\theta\rangle. \tag{4.20}
\]
Therefore, keeping also in mind formulæ (4.16), we can conclude that having given a photon in state \(|\Psi\rangle\) one can equivalently consider it as a quantum particle in state \(\hat{H}^{-1/2} |\Psi\rangle\), with observables defined by transformation
\[
\hat{\theta} \iff \hat{\theta}_H = \hat{H}^{-1/2} \hat{\theta} \hat{H}^{1/2}. \tag{4.21}
\]
However the scalar product used now is the usual scalar product \(\langle \cdot | \cdot \rangle\).
Let \(|\Psi\rangle\) be a photon state vector normalized to 1 with respect to the Bialynicki-Birula scalar product (4.5), \(\langle \Psi | \Psi \rangle_{BB} = 1\), and let \(\hat{\theta}\) be a photon observable satisfying the generalized hermicity condition (4.14). Then by (4.15) the average value of \(\hat{\theta}\) in state \(|\Psi\rangle\)
can be written as
\[
\langle \hat{\theta} \rangle = \langle \Psi | \hat{T}^{-1/2} \hat{\theta} | \Psi \rangle = \text{Tr} \left\{ \hat{T}^{-1/2} \hat{\theta} \hat{T}^{-1/2} | \Psi \rangle \langle \Psi | \right\}
\]
\[
\quad = \text{Tr} \left\{ \hat{T}^{-1/2} \hat{\theta} \hat{T}^{-1} | \Psi \rangle \langle \Psi | \right\}.
\] (4.21)

Motivated by the respective formula for the average value of any observable in nonrelativistic quantum mechanics we define the \textit{density operator for a photon pure state} \(|\Psi\rangle\), \(|\Psi\rangle_\text{BB} = 1\), as
\[
\hat{\rho} = |\Psi\rangle \langle \Psi | \hat{T}^{-1}.
\] (4.22)

This operator fulfills the following properties
\begin{enumerate}[(i)]
\item \(\hat{\rho}^* = \hat{\rho}\),
\item \(\text{Tr}(\hat{\rho}) = 1\),
\item \(\langle \chi | \hat{\rho} | \chi \rangle \geq 0 \quad \text{for every } |\chi\rangle\),
\item \(\hat{\rho}^2 = \hat{\rho}\).
\end{enumerate}
(i) \(\hat{\rho}^* = \hat{\rho}\),
(ii) \(\text{Tr}(\hat{\rho}) = 1\),
(iii) \(\langle \chi | \hat{\rho} | \chi \rangle \geq 0 \quad \text{for every } |\chi\rangle\),
(iv) \(\hat{\rho}^2 = \hat{\rho}\).
(4.23)

By analogy to (4.17) one introduces an operator
\[
\hat{\rho}_H = \hat{T}^{-1/2} \hat{\rho} \hat{T}^{1/2} = \hat{T}^{-1/2} |\Psi\rangle \langle \Psi | \hat{T}^{-1/2}.
\] (4.24)

It satisfies
\begin{enumerate}[(i')]
\item \(\hat{\rho}^*_{H} = \hat{\rho}_{H}\),
\item \(\text{Tr}(\hat{\rho}_{H}) = 1\),
\item \(\langle \chi | \hat{\rho}_{H} | \chi \rangle \geq 0 \quad \text{for every } |\chi\rangle\),
\item \(\hat{\rho}_{H}^2 = \hat{\rho}_{H}\).
\end{enumerate}
(i') \(\hat{\rho}^*_{H} = \hat{\rho}_{H}\),
(ii') \(\text{Tr}(\hat{\rho}_{H}) = 1\),
(iii') \(\langle \chi | \hat{\rho}_{H} | \chi \rangle \geq 0 \quad \text{for every } |\chi\rangle\),
(iv') \(\hat{\rho}_{H}^2 = \hat{\rho}_{H}\).
(4.25)

Employing the above results concerning the density operator for a pure state of photon we assume that in the general case of pure or mixed photon state this state is represented by operator \(\hat{\rho}\) fulfilling the conditions (i), (ii) and (iii) of (4.23). This operator is called the \textit{density operator for the photon state}. Density operator \(\hat{\rho}\) defines uniquely operator \(\hat{\rho}^*_{H}\) according to the first equality of (4.24). Operator \(\hat{\rho}^*_{H}\) satisfies the conditions (i'), (ii') and (iii') of (4.25).

The average (expected) value of any observable \(\hat{\theta}\) reads
\[
\langle \hat{\theta} \rangle = \text{Tr} \left\{ \hat{\theta} \hat{\rho} \right\} = \text{Tr} \left\{ \hat{\theta} \hat{\rho}^*_{H} \right\}
\] (4.26)
where \(\hat{\theta}^*_{H}\) is given by (4.17).

Finally, a photon state is pure iff \(\hat{\rho}\) fulfills the condition (iv) of (4.23) or, equivalently, \(\hat{\rho}^*_{H}\) fulfills (iv') of (4.25). The density operator satisfies the Liouville – von Neumann evolution equation
\[
\frac{i}{\hbar} \frac{\partial \hat{\theta}}{\partial t} = \left[ \hat{H}, \hat{\theta} \right].
\] (4.27)

Analogously
\[
\frac{i}{\hbar} \frac{\partial \hat{\theta}^*_{H}}{\partial t} = \left[ \hat{H}, \hat{\theta}^*_{H} \right].
\] (4.28)

All the formulæ can be easily written down in the representation defined by the \(U\) – transformation (see (3.26)).

5. The Weyl – Wigner – Moyal Formalism and the Wigner Function for Photon

Now we have at our disposal all elements required to develop the Weyl – Wigner – Moyal formalism for photon. Our aim is to construct this formalism in close analogy to that considered in our previous works\(^\text{[1,11]}\).

First we construct the photon phase space. One starts with the Hilbert space
\[
\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^3.
\] (5.1)

As it has been shown in [1, 11], the associated phase space is
\[
\Gamma = \{ (\mathbf{p}, \mathbf{x}, \phi_m, n) \} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3,
\] (5.2)
where \(\mathbb{R}^3\) is a 3 x 3 grid, i.e. \(\mathbb{R}^3 = \{(m, n)\} \) with \(m, n = 0, 1, 2\) and \(\phi_m = \frac{2\pi}{3} m\). (In the current paper the position vector is denoted by \(\mathbf{x}\) and not by \(q\) as in [1]).

As we remember, a function on a phase space associated to an observable is real. However, the observables considered in the previous cases were Hermitian operators. On the contrary, the photon observables analysed in Section 4 are generalized Hermitian operators. So now one should define a correspondence between operators in \(\mathcal{H}\) and functions in \(\Gamma\) in such a way that the functions corresponding to the generalized Hermitian operators are real. To this end we proceed as follows. Given unitary operators (see Equations (2.9), (2.12), (2.13) and (2.15) in our previous paper\(^1\) for \(s + 1 = 3\))
\[
\hat{D}(k, l) = \exp \left\{ \frac{i\pi}{3} \sum_{m=0}^{2} \exp \left\{ \frac{2\pi k m}{3} \right\} |\phi_m\rangle \langle \phi_m| \right\} = \exp \left\{ \frac{i\pi}{3} \sum_{m=0}^{2} \exp \left\{ \frac{2\pi m n}{3} \right\} \langle n+k \text{ mod } 3 \rangle \right\}
\] (5.3)
for \(k, l = 0, 1, 2\) and
\[
\hat{U}(\tilde{\lambda}, \tilde{\mu}) = \exp \{ i(\tilde{\lambda} \cdot \tilde{\mu} + \tilde{\mu} \cdot \tilde{x}) \}
\]
\[
= \int_{\mathbb{R}^3} d^3x \exp \{ i\tilde{\mu} \cdot \tilde{x} \} \left| \tilde{x} - \frac{h\tilde{\lambda}}{2} \right| \left( \tilde{x} + \frac{h\tilde{\lambda}}{2} \right)
\]
\[
= \int_{\mathbb{R}^3} d^3p \exp \{ i\tilde{\mu} \cdot \tilde{p} \} \left| \tilde{p} + \frac{h\tilde{\mu}}{2} \right| \left( \tilde{p} - \frac{h\tilde{\mu}}{2} \right)
\] (5.4)
we define two \textit{generalized unitary operators}
\[
\hat{D}(k, l) := \hat{T}^{-1/2} \hat{D}(k, l) \hat{T}^{-1/2} = \hat{D}(k, l),
\] (5.5)
\[
\hat{U}(\tilde{\lambda}, \tilde{\mu}) := \hat{T}^{-1/2} \hat{U}(\tilde{\lambda}, \tilde{\mu}) \hat{T}^{-1/2}.
\]

They satisfy the following properties
\[
\hat{D}^\dagger(k, l) = \hat{D}^{-1}(k, l) = \hat{D}(-k, -l),
\] (5.6a)
From (5.9c) we infer that

$$\text{Tr} \left\{ \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}, \hat{x}, \phi_m, n) \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}', \hat{x}', \phi_m', n') \right\} = (2\pi \hbar)^3 \delta(\hat{p} - \hat{p}') \delta(\hat{x} - \hat{x}') \delta_{m', n'} \delta_{m, n} \tag{5.10}$$

for

$$|P| = 1 \text{ and } |\mathcal{K}| = 1. \tag{5.11}$$

Of course the relation (5.9a) says that the operator \(\hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}, \hat{x}, \phi_m, n)\) is a generalized Hermitian operator for every \((\hat{p}, \hat{x}, \phi_m, n) \in \Gamma\).

It is convenient to extend the transformation (4.17) on an arbitrary linear operator. So given an operator \(\hat{\gamma}\) we define \(\hat{\gamma}_{kl}\) as

$$\hat{\gamma}_{kl} := \hat{\gamma}^{-1/2} \hat{\gamma} \hat{\gamma}^{1/2}. \tag{5.12}$$

Comparing (5.11) with (5.5) and (5.8) one gets

$$\hat{\Delta}_{kl} = \hat{\Delta}, \quad \hat{\gamma}_{kl} = \hat{\gamma}.$$ \(\tag{5.13}\)

Now we are in position to define the generalized Weyl correspondence between the functions on phase space \(\Gamma\) and operators in the Hilbert space \(\mathcal{H}\).

According to this correspondence for a function \(f = f(\hat{p}, \hat{x}, \phi_m, n)\) on \(\Gamma\) we assign an operator \(\hat{f}\) in \(\mathcal{H}\) given by

$$\hat{f} = \frac{1}{(2\pi \hbar)^3} \sum_{m,n=0}^{\infty} \int_{\mathbb{R}^2} d\hat{p} d\hat{x} \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}, \hat{x}, \phi_m, n) \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}', \hat{x}', \phi_m', n') \tag{5.14}$$

(compare with (2.32b) of Ref. [1].) Multiplying both sides of (5.14) by \(\hat{\Delta}_{kl}\) and \(\hat{\gamma}_{kl}\), taking the trace and using also (5.9c) one gets the formula inverse to (5.14) as

$$f(\hat{p}, \hat{x}, \phi_m, n) = \frac{1}{9(2\pi)^6} \sum_{k,l,m,n=0}^{\infty} \int_{\mathbb{R}^4} d^4 \lambda d^4 \mu d^4 p' d^4 x'$$

$$\times \left[ \hat{\Delta}_{kl} \hat{\gamma}_{kl} \right]^{-2} \exp \left( i \frac{\hat{\lambda} \cdot (\hat{p} - \hat{p}') + \hat{\mu} \cdot (\hat{x} - \hat{x}') \right) \exp \left( i k \cdot (\phi_m - \phi_m') + \phi_l \cdot (n - n') \right) \times \text{Tr} \left\{ \int \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}', \hat{x}', \phi_m', n') \right\} \tag{5.15}$$

(compare with (2.39) in [1].)

One easily finds that if (5.11) holds true then Equation (5.15) simplifies considerably and it reads then

$$f(\hat{p}, \hat{x}, \phi_m, n) = \text{Tr} \left\{ \hat{f} \hat{\Omega}[\mathcal{P}, \mathcal{K}](\hat{p}, \hat{x}, \phi_m, n) \right\}. \tag{5.16}$$
Thus formulae (5.14) and (5.15) give a one – to – one correspondence between functions on phase space $\Gamma$ and operators in the Hilbert space $H$. Employing (5.9a) we conclude that operator $\hat{f}$ is a generalized Hermitian operator, $\hat{f}^+ = \hat{f}$, iff $f = \langle \hat{p}, \vec{x}, \phi_m, n \rangle$ is a real function.

Observe that Equations (5.14), (5.15) and (5.16) can be equivalently rewritten by the substitutions:

$$\hat{f} \mapsto \hat{f}_{\|} = \hat{H}^{-1/2}\hat{f}\hat{H}^{1/2} \quad \text{and} \quad \hat{\Omega}[P, K] \mapsto \hat{\Omega}(P, K),$$

where $\hat{\Omega}[P, K]$ is the Stratonovich – Weyl quantizer introduced in Ref. [1].

The notion of star product can be introduced in the same way as it is done in nonrelativistic case. Namely, if $f = f(\hat{p}, \vec{x}, \phi_m, n)$ and $g = g(\hat{p}, \vec{x}, \phi_m, n)$ are functions on $\Gamma$, and $\hat{f}$ and $\hat{g}$ are their respective operators in $H$ then the function corresponding to the product $\hat{f} \cdot \hat{g}$ is denoted by $f \ast g$ and according to (5.15) it reads

$$\langle f \ast g \rangle(\hat{p}, \vec{x}, \phi_m, n) = \frac{1}{(2\pi)^{3/2}} \sum_{k,l,m,n=0}^{2} \int_{\mathbb{R}^{12}} d^3\mathbf{\lambda} d^3\mu d^3p' d^3x'$$

$$\left| P\left(\frac{\mathbf{\hbar} \cdot \mathbf{\mu}}{2}\right) K\left(\frac{\pi k}{3}\right) \right|^{-2} \exp\left\{i\mathbf{\lambda} \cdot (\hat{p} - \hat{p}') + \mathbf{\mu} \cdot (\vec{x} - \vec{x}')\right\} \times \exp\left\{i[k \cdot (\phi_m - \phi_0) + \phi_l \cdot (n - n')\right\} \times \text{Tr}\left\{\hat{f} \cdot \hat{g} \hat{\Omega}[P, K] \left[\vec{p}', \vec{x}', \phi_{m'}, n'\right]\right\}. \quad (5.17)$$

Inserting into (5.17) $\hat{f}$ and $\hat{g}$ in accordance with (5.14) one gets

$$\langle f \ast g \rangle(\hat{p}, \vec{x}, \phi_m, n) = \frac{1}{81\hbar^3(2\pi)^{15/2}} \times \sum_{k,l,m,n=0}^{2} \int_{\mathbb{R}^{12}} d^3\mathbf{\lambda} d^3\mu d^3p' d^3x' d^3\mathbf{p}'' d^3\mathbf{x}'' d^3\mathbf{p}''' d^3\mathbf{x}'''$$

$$\left| P\left(\frac{\mathbf{\hbar} \cdot \mathbf{\mu}}{2}\right) K\left(\frac{\pi k}{3}\right) \right|^{-2} \exp\left\{i\mathbf{\lambda} \cdot (\hat{p} - \hat{p}') + \mathbf{\mu} \cdot (\vec{x} - \vec{x}')\right\} \times \exp\left\{i[k \cdot (\phi_m - \phi_0) + \phi_l \cdot (n - n')\right\} \times f(\hat{p}'', \vec{x}'', \phi_{m''}, n'')g(\hat{p}', \vec{x}', \phi_{m'}, n') \times \text{Tr}\left\{\hat{\Omega}[P, K] \left[\vec{p}'', \vec{x}'', \phi_{m''}, n''\right]\right\} \times \text{Tr}\left\{\hat{\Omega}[P, K] \left[\vec{p}', \vec{x}', \phi_{m'}, n'\right]\right\}. \quad (5.18)$$

Using the relation between $\hat{\Omega}[P, K]$ and $\hat{\Omega}(P, K)$ given by (5.8) we quickly conclude that in (5.18) one can equivalently put $\hat{\Omega}[P, K]$ instead of $\hat{\Omega}[P, K]$. Therefore the star product given by (5.18) is exactly the same as the star product defined in our previous work[1] (see Equation (3.2) in [1]) for $(s + 1) = 3$. An explicit expression of product (5.18) is presented in Appendix A.

If the kernels $P$ and $K$ fulfill the conditions (5.11) then $f \ast g$ can be written in a simpler form

$$\langle f \ast g \rangle(\hat{p}, \vec{x}, \phi_m, n) = \frac{1}{9(2\pi\hbar)^{6}} \times \sum_{m,n,n',n''=0}^{2} \int_{\mathbb{R}^{12}} d^3\mathbf{\lambda} d^3\mu d^3p' d^3x' d^3\mathbf{p}'' d^3\mathbf{x}'' d^3\mathbf{p}''' d^3\mathbf{x}''' f(\hat{p}, \vec{x}', \phi_{m'}, n')$$

$$\times \text{Tr}\left\{\hat{\Omega}[P, K] \left[\vec{p}', \vec{x}', \phi_{m'}, n'\right]\hat{\Omega}[P, K] \left[\vec{p}'', \vec{x}'', \phi_{m''}, n''\right] g(\hat{p}'', \vec{x}'', \phi_{m''}, n'')\right\}. \quad (5.19)$$

where due to the above comment we put $\hat{\Omega}[P, K]$ in place of $\hat{\Omega}(P, K)$.

We then define a photon Wigner function. Assume that operator $\hat{f}$ satisfying $\hat{f}^+ = \hat{f}$, represents a photon observable. The average value of this observable in the photon state $\hat{\rho}$ is determined by the formula (4.26) with $\hat{\rho}$ substituted by $\hat{f}$.

Using also (5.14) one finally gets

$$\langle \hat{f} \rangle = \text{Tr}\left\{\hat{f} \hat{\rho}\right\} = \text{Tr}\left\{\hat{f}_{\|} \hat{\rho}_{\|}\right\} = \frac{1}{3(2\pi\hbar)^{3}} \sum_{m,n,n'=0}^{2} \int_{\mathbb{R}^{12}} d^3\mathbf{\lambda} d^3\mu d^3p' d^3x' f(\hat{p}, \vec{x}', \phi_{m'}, n')$$

$$\times \text{Tr}\left\{\hat{\Omega}[P, K] \left[\vec{p}', \vec{x}', \phi_{m'}, n'\right]\hat{\Omega}[P, K] \left[\vec{p}'', \vec{x}'', \phi_{m''}, n''\right] g(\hat{p}'', \vec{x}'', \phi_{m''}, n'')\right\}. \quad (5.20)$$

In close analogy to the previous works we define the Wigner function of photon in state $\hat{\rho}$ for the kernels $(P, K)$ as

$$\rho_w[P, K](\hat{p}, \vec{x}, \phi_m, n) := \frac{1}{3(2\pi\hbar)^{3}} \text{Tr}\left\{\hat{\rho} \hat{\Omega}[P, K] \left[\vec{p}, \vec{x}, \phi_m, n\right]\right\}$$

$$= \frac{1}{3(2\pi\hbar)^{3}} \text{Tr}\left\{\hat{\rho}_{\|} \hat{\Omega}[P, K] \left[\vec{p}, \vec{x}, \phi_m, n\right]\right\}. \quad (5.21)$$

Notice that the Wigner function $\rho_w[P, K](\hat{p}, \vec{x}, \phi_m, n)$ is the same for the quantum state representations: $\hat{\rho}$ and $\hat{\rho}_{\|}$.

A formula inverse to (5.21) reads

$$\hat{\rho} = \frac{1}{9(2\pi\hbar)^{6}} \sum_{k,l,m,n,m',n'=0}^{2} \int_{\mathbb{R}^{12}} d^3\mathbf{\lambda} d^3\mu d^3p' d^3x' d^3\mathbf{p}' d^3\mathbf{x}'$$

$$\left| P\left(\frac{\mathbf{\hbar} \cdot \mathbf{\mu}}{2}\right) K\left(\frac{\pi k}{3}\right) \right|^{-2} \exp\left\{i\mathbf{\lambda} \cdot (\hat{p} - \hat{p}') + \mathbf{\mu} \cdot (\vec{x} - \vec{x}')\right\} \times \exp\left\{i[k \cdot (\phi_m - \phi_0) + \phi_l \cdot (n - n')\right\} \times \text{Tr}\left\{\hat{\Omega}[P, K] \left[\vec{p}', \vec{x}', \phi_{m'}, n'\right]\hat{\Omega}[P, K] \left[\vec{p}'', \vec{x}'', \phi_{m''}, n''\right]\right\}. \quad (5.22)$$
Equation (5.22) simplifies considerably when the conditions (5.11) are fulfilled. In that case we have
\[
\hat{\rho} = \sum_{m,n=0}^{\infty} \int d^3 p d^3 x \rho_{w}[P, K][\vec{p}, \vec{x}, \phi_m, n] \Omega(P, K)[\vec{p}, \vec{x}, \phi_m, n].
\] (5.23)

One quickly finds that the Wigner function (5.21) has the following properties
\[
\rho_w[P, K] = \rho_w[P, K], \quad \sum_{m,n=0}^{\infty} \int d^3 p d^3 x \rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n] = \text{Tr}[\hat{\rho}] = 1, \quad (5.24a)
\]
\[
\sum_{m,n=0}^{\infty} \int d^3 p d^3 x \rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n] = \text{Tr}[\hat{\rho} \hat{F}^{1/2} \hat{F}^{-1/2}] = 0, \quad (5.24b)
\]
\[
\sum_{m,n=0}^{\infty} \int d^3 p d^3 x \rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n] = \text{Tr}[\hat{\rho} \hat{F}^{1/2} \hat{F}^{-1/2}] = 0, \quad (5.24c)
\]
\[
\sum_{m,n=0}^{\infty} \int d^3 p d^3 x \rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n] = \text{Tr}[\hat{\rho} \hat{F}^{1/2} \hat{F}^{-1/2}] = 0, \quad (5.24d)
\]

where \(\hat{\rho}_{ij}\) is defined by (4.24). We assume that the formula (5.25) is valid for any state (pure or mixed) \(\hat{\rho}\).

Substituting in (5.25)
\[
\hat{\rho}_i := \frac{\hat{F}^{1/2} \hat{\rho} \hat{F}^{-1/2}}{\text{Tr}[\hat{F}^{1/2} \hat{\rho} \hat{F}^{-1/2}]} \Rightarrow \hat{\rho}_{iH} = \frac{\hat{F}^{1/2} \hat{\rho}_{ij} \hat{F}^{-1/2}}{\text{Tr}[\hat{F}^{1/2} \hat{\rho}_{ij} \hat{F}^{-1/2}]} \quad (5.26)
\]
in the place of \(\hat{\rho}\) and \(\hat{\rho}_{ij}\), respectively, one gets
\[
\varphi(\vec{x} \in V) = \frac{\int d^3 x \text{Tr}[\hat{\rho} \hat{H}(\vec{x})]}{\text{Tr}[\hat{\rho} \hat{H}]} = \frac{\int d^3 x \text{Tr}[\hat{\rho} \hat{H}(\vec{x})]}{\text{Tr}[\hat{\rho} \hat{H}]} = \varphi(\vec{x} \in V). \quad (5.27)
\]

This last formula provides us with simple interpretation for the first of Equations (5.24c). Namely, \(\text{Tr}[\hat{\rho}_{ij}(\vec{x})]\) is the density of probability to find the photon energy localized at point \(\vec{x}\) for state \(\hat{\rho}\), given by (5.26).

Interpretation of the second formula of Equations (5.24c) is much simpler and clear. Indeed, the function \(\text{Tr}[\hat{\rho}(\vec{p})(\vec{p})]\) defines the probability distribution for the momentum of photon \(\vec{p} = \hbar \vec{k}\) in state \(\hat{\rho}\).

Sense of the last two properties of (5.24c) is rather unclear. Although operators \(|n\rangle\langle n|, n = 0, 1, 2, \) and \(|\phi_m\rangle\langle \phi_m|, m = 0, 1, 2,\) are the generalized Hermitian operators, they are not photon observables in general. So we are not able to give any reasonable explanation of these two formulae.

Now we are going to find an evolution equation for the Wigner function \(\rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n, t]\). From (5.21), using also the Liouville – von Neumann equation (4.27) one gets
\[
\frac{\partial \rho_w[P, K]}{\partial t} + \frac{1}{3(2\pi \hbar)^2} \text{Tr}\left\{ \frac{1}{i\hbar} [\hat{\rho}, \hat{H}] \bar{\rho}_{[P, K]} \right\} = 0. \quad (5.28)
\]

Employing (5.14), (5.22), (5.18) and defining
\[
R_w[P, K][\vec{p}', \vec{x}', \phi_m', n'; t] := \frac{1}{9(2\pi \hbar)^2} \sum_{k,l,m,n=0} \int d^3 \lambda d^3 \mu d^3 p d^3 x \left| P\left(\frac{\hbar \vec{k} \cdot \vec{\mu}}{2}\right)\left(\frac{\pi \hbar^2}{3}\right)^2 \right|^2 \times \exp\left\{ i[\vec{k} \cdot (\vec{p} - \vec{p}') + \vec{\mu} \cdot (\vec{x} - \vec{x}')]\right\} \times \exp\left\{ i[k \cdot (\phi_m - \phi_m') + \phi_1 \cdot (n - n')]\right\} \times \rho_w[P, K][\vec{p}, \vec{x}, \phi_m, n, t]. \quad (5.29)
\]
we can rewrite Equation (5.28) as
\[
\frac{\partial \rho_w[P, K]}{\partial t} + \frac{1}{3i\hbar} (R_w[P, K] * H - H * R_w[P, K]) = 0. \quad (5.30)
\]

Equation (5.30) will be called the Liouville – von Neumann – Wigner equation. It is rather involved, but when the kernels \((P, K)\) satisfy conditions (5.11) then \(R_w[P, K] = \rho_w[P, K]\) and Equa-
A relation between\( \mu = \frac{x}{y} \)

Then one finds that if \( \widetilde{f} \rightarrow \hat{f} \) and \( \widetilde{g} \rightarrow \hat{g} \) then \( \widetilde{f} \otimes \widetilde{g} \rightarrow \hat{f} \cdot \hat{g} \).

Similarly as \( \widetilde{f} \) in formula (5.33), we define \( \tilde{\rho}_w \)

Differentiating \( \tilde{\rho}_w(\lambda, \mu, k, l; t) \) over \( t \), using then the Liouville–von Neumann equation (4.27) and the definition of the \( \otimes \) – product (5.35) one finds the evolution equation for \( \tilde{\rho}_w \)

This equation we also call the Liouville–von Neumann–Wigner equation. Given any solution of Equation (5.37) one finds the respective Wigner function \( \rho_w(P, K)[\tilde{p}, \tilde{x}, \phi_m, n; t] \) from the formula

6. Some Explicit Forms of the Photon Wigner Functions for Specific Kernels

This section contains expressions for quantities appearing in the phase space description of photons. They result from considerations presented in the previous section.

Inserting (5.8) with (5.3) and (5.4) into (5.21) one can express the Wigner function in the form

\[
\rho_w(P, K)[\tilde{p}, \tilde{x}, \phi_m, n; t] = \frac{1}{(2\pi\hbar)^3} \int d^3\lambda d^3\mu d^3\nu d^3\zeta \left( \frac{\hbar}{2} \right)^6 \exp \left\{ i[k\phi_m + \phi(n)] \right\} \left( \frac{\kappa}{3} \right)^3 \exp \left\{ i\kappa \right\}
\]

\[
= \frac{1}{(2\pi\hbar)^3} \int d^3\lambda d^3\mu d^3\nu d^3\zeta \left( \frac{\hbar}{2} \right)^6 \exp \left\{ i[k\phi_m + \phi(n)] \right\} \left( \frac{\kappa}{3} \right)^3 \exp \left\{ i\kappa \right\}
\]

\[
= \frac{1}{(2\pi\hbar)^3} \int d^3\lambda d^3\mu d^3\nu d^3\zeta \left( \frac{\hbar}{2} \right)^6 \exp \left\{ i[k\phi_m + \phi(n)] \right\} \left( \frac{\kappa}{3} \right)^3 \exp \left\{ i\kappa \right\}
\]

\[
\rho_w(P, K)[\tilde{p}, \tilde{x}, \phi_m, n; t] = \frac{1}{(2\pi\hbar)^3} \int d^3\lambda d^3\mu d^3\nu d^3\zeta \left( \frac{\hbar}{2} \right)^6 \exp \left\{ i[k\phi_m + \phi(n)] \right\} \left( \frac{\kappa}{3} \right)^3 \exp \left\{ i\kappa \right\}
\]
Equivalently, we can write

\[
\rho_w[P, \mathbf{K}](\vec{p}, \vec{x}, \phi_m, n) = \frac{1}{9(2\pi)^9} \sum_{k,l,m=0}^2 \int_{\mathbb{R}^3} d^3 \lambda d^3 \mu d^3 \nu \rho \left( \frac{\hbar \lambda \cdot \mu}{2} \right) \kappa \left( \frac{\pi kl}{3} \right) \\
\times \exp \left\{ -i \left[ \vec{\lambda} \cdot \vec{\mu} + \vec{\mu} \cdot (\vec{\nu} - \vec{\lambda}) \right] \right\} \\
\times \exp \left\{ -i [k(\phi_m - \phi_{m^+}) + \phi_{n^+}] \right\} \\
\times \left( \vec{x}' + \frac{\hbar \mu}{2}, \phi_{n^+} \mid \vec{p}' + \frac{\hbar \mu}{2}, \phi_{m^+} \right). \tag{6.2}
\]

Assume now that the photon is in a pure state \(|\Psi\rangle\), \langle\Psi|\Psi\rangle_{\mathbb{R}^3} = 1. Then \(\hat{\rho}_{ij}\) is given by (4.24). Substituting this \(\hat{\rho}_{ij}\) into the last formula of (6.1), having also in mind that

\[
\langle \vec{p} \mid \Psi \rangle = \frac{1}{\hbar^{1/2}} \langle k \mid \Psi \rangle = \frac{1}{\hbar} \sqrt{\frac{e}{(2\pi)^3}} \langle k \mid \Psi \rangle,
\tag{6.3}
\]

where

\[
\tilde{\Psi}(\vec{k}) = \begin{pmatrix} \tilde{\Psi}_1(\vec{k}) \\ \tilde{\Psi}_2(\vec{k}) \\ \tilde{\Psi}_3(\vec{k}) \end{pmatrix}, \tag{6.4}
\]

is defined by (3.17), one gets the Wigner function as

\[
\rho_w[P, \mathbf{K}](\vec{p}, \vec{x}, \phi_m, n) \nonumber
\]

\[
= \frac{1}{9(2\pi)^9} \sum_{k,l,m=0}^2 \int_{\mathbb{R}^3} d^3 \lambda d^3 \mu d^3 \nu \rho \left( \frac{\hbar \lambda \cdot \mu}{2} \right) \kappa \left( \frac{\pi kl}{3} \right) \\
\times \exp \left\{ -i \left[ \vec{\lambda} \cdot \vec{\mu} + \vec{\mu} \cdot (\vec{\nu} - \vec{\lambda}) \right] \right\} \\
\times \exp \left\{ -i [k(\phi_m - \phi_{m^+}) + \phi_{n^+}] \right\} \\
\times \tilde{\Psi}_{[\nu + [k,l,m] \ mod \ 3]+1} \left( \vec{\nu} + \frac{\hbar \mu}{2}, \phi_{n+1} \right). \tag{6.5}
\]

Employing then Equation (3.18) giving the time evolution of photon wave function \(\tilde{\Psi}(\vec{k}, t)\) we find the time evolution of Wigner function (6.5)

\[
\rho_w[P, \mathbf{K}](\vec{p}, \vec{x}, \phi_m, n; t) \nonumber
\]

\[
= \frac{1}{9(2\pi)^9} \sum_{k,l,m=0}^2 \int_{\mathbb{R}^3} d^3 \lambda d^3 \mu d^3 \nu \rho \left( \frac{\hbar \lambda \cdot \mu}{2} \right) \kappa \left( \frac{\pi kl}{3} \right) \\
\times \exp \left\{ -i \left[ \vec{\lambda} \cdot \vec{\mu} + \vec{\mu} \cdot (\vec{\nu} - \vec{\lambda}) \right] \right\} \\
\times \exp \left\{ -i [k(\phi_m - \phi_{m^+}) + \phi_{n^+}] \right\} \\
\times \tilde{\Psi}_{[\nu + [k,l,m] \ mod \ 3]+1} \left( \vec{\nu} + \frac{\hbar \mu}{2}, \phi_{n+1} \right). \tag{6.6}
\]

Let us apply formula (6.6) to the case when

\[
P \left( \frac{\hbar \lambda \cdot \mu}{2} \right) = 1, \quad \kappa \left( \frac{\pi kl}{3} \right) = (-1)^{\hat{\delta} i}. \tag{6.7}
\]

As is known from Refs. [2, 11] (see also the references therein) such a choice of kernels is acceptable for \(s + 1 = 3\). Inserting (6.7) into (6.6) one gets (we omit the symbol \([P, \mathbf{K}]\) at \(\rho_w\))

\[
\rho_w(\vec{p}, \vec{x}, \phi_m, n; t) = \frac{1}{3(2\pi)^3 \hbar^3} \left\{ \int_{\mathbb{R}^3} d^3 \mu \right\} \left( \frac{\hbar \lambda \cdot \mu}{2} \right) \\
\times \exp \left\{ -i \left[ \vec{\lambda} \cdot \vec{\mu} + \vec{\mu} \cdot (\vec{\nu} - \vec{\lambda}) \right] \right\} \\
\times \exp \left\{ -i [k(\phi_m - \phi_{n^+}) + \phi_{n^+}] \right\} \\
\times \tilde{\Psi}_{[\nu + [k,l,m] \ mod \ 3]+1} \left( \vec{\nu} + \frac{\hbar \mu}{2}, \phi_{n+1} \right) + 2 \exp \left\{ -i \phi_{m^+} \tilde{\Psi}_{[\nu + 2] \ mod \ 3]+1} \right\} \\
\times \left( \vec{\nu} + \frac{\hbar \mu}{2}, \phi_{n+1} \right). \tag{6.8}
\]

Note that the kernels given by (6.7) fulfill the conditions (5.11). The next example concerns the choice of kernels \([P, \mathbf{K}]\)

\[
P \left( \frac{\hbar \lambda \cdot \mu}{2} \right) = 1, \quad \kappa \left( \frac{\pi kl}{3} \right) = \cos \left( \frac{\pi kl}{3} \right). \tag{6.9}
\]

Substituting (6.9) into (6.6) and performing straightforward calculations one gets (we omit the symbol \([P, \mathbf{K}]\) at \(\rho_w\))

\[
\rho_w(\vec{p}, \vec{x}, \phi_m, n; t) = \frac{1}{3(2\pi)^3 \hbar^3} \left\{ \int_{\mathbb{R}^3} d^3 \mu \right\} \left( \frac{\hbar \lambda \cdot \mu}{2} \right) \\
\times \exp \left\{ -i \left[ \vec{\lambda} \cdot \vec{\mu} + \vec{\mu} \cdot (\vec{\nu} - \vec{\lambda}) \right] \right\} \\
\times \exp \left\{ -i [k(\phi_m - \phi_{n^+}) + \phi_{n^+}] \right\} \\
\times \tilde{\Psi}_{[\nu + [k,l,m] \ mod \ 3]+1} \left( \vec{\nu} + \frac{\hbar \mu}{2}, \phi_{n+1} \right). \tag{6.10}
\]
The constraint equation (3.12), which is equivalent to the condition
\[ \sum_{j=1}^{3} k_j \Psi_0 (\vec{k}) = 0 \quad (6.11) \]
leads to the following constraint equation for \( \rho_W \) given by (6.10)
\[ \sum_{n=0}^{2} \left( k_{n+1} - \frac{1}{2} i \frac{\partial}{\partial \varphi_{n+1}} \right) \Delta_W (\vec{p}, \vec{x}, 0; n; t) = 0, \]
where
\[ \Delta_W (\vec{p}, \vec{x}, \phi_m; n; t) := \frac{1}{(2\pi)^3} \hbar^3 \sum_{n'=0}^{2} \int_{\mathbb{R}^3} \frac{d^3 \mu}{(k - \frac{\mu}{2} + \frac{\mu}{2})} \frac{\exp \left\{-i \vec{p} \cdot \vec{x} \right\} \times \exp \left\{-i \vec{p} \cdot \vec{x} \right\} \times \exp \left\{-i \vec{p} \cdot \vec{x} \right\}}{\mu_0} \]

It is obvious that
\[ \rho_W = \Re (\Delta_W). \quad (6.13) \]

### 7. Concluding Remarks

The representation of creation and annihilation of photons can be done exclusively in frames of quantum field theory. However, having in mind that in several problems related to relativistic quantum phenomena the Klein – Gordon equation or the Dirac equation are sufficient, we decided to propose an analogous quantum mechanical description for photons. Thus in the first part of our paper we have shown that such a program can be easily implemented within quantum mechanics for light developed by I. Białynicki – Birula[2–4] and J. E. Sipe.[5]

Yet another goal was to set up a phase space formulation of the photon quantum mechanics. In order to achieve it we establish the Weyl – Wigner – Moyal formalism for quantum mechanics of photon and to define the respective photon Wigner function within this formalism. We do that with the use of the continuous – discrete Weyl – Wigner – Moyal formalism[10,11] (see also the wide bibliography therein).

The results that were found, reinforce our belief that one can successfully apply this formalism to relativistic particles in searching for their relativistic Wigner functions.[10,27–31] On the other hand the problems with interpretation of the vector \( \vec{x} \) are to be expected since for relativistic particles the operator \( \hat{x} \) does not represent the position observable.[16,24,26,34–39] So the natural question arises if one can construct the Weyl – Wigner – Moyal formalism for photon employing the position operator introduced by Margaret Hawton.[38]

### Appendix A: Star Product On the Phase Space

The \(*\) product on phase space \( \Gamma^3 \) consists of two parts: a product on the space \( \mathbb{R}^6 \) and a multiplication on the grid \( \Gamma^1 \). Its general form is
\[ (f * g)(\vec{p}, \vec{x}, \phi_m; n) = \sum_{k,l=0}^{2} \int_{\mathbb{R}^6} d^3 \mu d^3 \nu \left( \frac{\hbar}{\nu} \cdot \frac{\mu}{2} \right) \left( \frac{\pi k l}{3} \right)^{-1} \times \exp \left\{ i(\vec{p} \cdot \vec{\mu} + \vec{\nu} \cdot \vec{\nu}) \right\} \times \exp \left\{ \frac{i 2\pi}{3} (k \phi_m + \phi_n) \right\} \langle \vec{f} \circ \mathbb{G} \rangle (\vec{x}, \vec{\mu}, k, l), \quad (A.1) \]
where the \( \circ \) – product does not depend on the kernels \( P \) and \( \mathbb{K} \).

Auxiliary functions \( \vec{f} (\vec{x}, \vec{\mu}, k, l) \) one calculates according to the rule
\[ \vec{f} (\vec{x}, \vec{\mu}, k, l) = \frac{1}{(2\pi)^6} \frac{1}{9 \nu} \left( \frac{\hbar}{\nu} \cdot \frac{\mu}{2} \right) \mathbb{K} (\frac{\pi k l}{3}) \]
\[ \times \int_{\mathbb{R}^6} d^3 \mu d^3 \nu \times \exp \left\{ -i(\vec{p} \cdot \vec{\nu} + \vec{\mu} \cdot \vec{\nu}) \right\} \]
\[ \times \exp \left\{ -i \frac{2\pi}{3} (k \phi_m + \phi_n) \right\} \vec{f} (\vec{p}, \vec{x}, \phi_m; n) \quad (A.2) \]

being a straightforward consequence of formula (5.34).

The \( \circ \) – multiplication on phase space \( \mathbb{R}^6 \) is defined as (compare\[1\])
\[ \langle \vec{f} \circ \mathbb{G} \rangle (\vec{x}, \vec{\mu}) = \int_{\mathbb{R}^3} d^3 \lambda \cdot d^3 \mu \cdot d^3 \lambda' \cdot d^3 \mu' \cdot d^3 \lambda'' \cdot d^3 \mu'' \langle \vec{f} (\vec{\lambda}, \vec{\mu}) \rangle \]
\[ \times \exp \left\{ \frac{i}{2} (\vec{\lambda} \cdot \vec{\mu} - \vec{\lambda'} \cdot \vec{\mu'}) \right\} \]
\[ \times \delta(\vec{\lambda} + \vec{\lambda} - \vec{\lambda'} + \vec{\mu} + \vec{\mu} - \vec{\mu}' - \vec{\mu}'') \langle \vec{f} (\vec{\lambda''}, \vec{\mu''}) \rangle \quad (A.3) \]

An explicit expression for the \( \circ \) – product (5.35) on the phase space \( \Gamma^3 \) consists of the following set of terms
\[ \langle \vec{f} \circ \mathbb{G} \rangle (0, 0) = \langle \vec{f} (0, 0) \rangle \langle 0, 0 \rangle + \langle \vec{f} (0, 0) \rangle \langle 0, 1 \rangle + \langle \vec{f} (0, 1) \rangle \langle 0, 2 \rangle \]
\[ + \langle \vec{f} (2, 0) \rangle \langle 1, 1 \rangle + \langle \vec{f} (2, 1) \rangle \langle 1, 2 \rangle + \langle \vec{f} (2, 1) \rangle \langle 2, 0 \rangle \]
\[ + \langle \vec{f} (1, 2) \rangle \langle 2, 1 \rangle - \langle \vec{f} (1, 1) \rangle \langle 2, 2 \rangle, \]
\[ \langle \vec{f} \circ \mathbb{G} \rangle (0, 1) = \langle \vec{f} (0, 1) \rangle \langle 0, 0 \rangle + \langle \vec{f} (0, 0) \rangle \langle 0, 1 \rangle + \langle \vec{f} (0, 2) \rangle \langle 0, 2 \rangle \]
\[ + \exp \left\{ 2i\pi \frac{3}{3} \right\} \langle \vec{f} (2, 1) \rangle \langle 1, 0 \rangle - \exp \left\{ 2i\pi \frac{3}{3} \right\} \langle \vec{f} (2, 0) \rangle \langle 1, 1 \rangle \]
\[ + \exp \left\{ 2i\pi \frac{3}{3} \right\} \langle \vec{f} (2, 2) \rangle \langle 1, 2 \rangle + \exp \left\{ i\pi \frac{3}{3} \right\} \langle \vec{f} (1, 1) \rangle \langle 2, 0 \rangle \]
\[-\exp\left(\frac{i\pi}{3}\right)\tilde{f}(1,0)\tilde{g}(2,1) + \exp\left(-\frac{i\pi}{3}\right)\tilde{f}(1,2)\tilde{g}(2,2).\]

\[\left(\tilde{f} \otimes \tilde{g}\right)(0,2) = \tilde{f}(0,2)\tilde{g}(0,0) + \tilde{f}(0,1)\tilde{g}(0,1) + \tilde{f}(0,0)\tilde{g}(0,2)\]

\[+ \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(2,2)\tilde{g}(1,0) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(2,1)\tilde{g}(1,1) + \tilde{f}(0,0)\tilde{g}(2,0)\]

\[+ \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(0,2)\tilde{g}(2,0) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(0,0)\tilde{g}(2,2).\]

\[\left(\tilde{f} \otimes \tilde{g}\right)(1,0) = \tilde{f}(1,0)\tilde{g}(0,0) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(1,1)\tilde{g}(0,1) + \exp\left(-\frac{i\pi}{3}\right)\tilde{f}(1,0)\tilde{g}(2,2)\]

\[+ \exp\left(-\frac{i\pi}{3}\right)\tilde{f}(1,1)\tilde{g}(0,0) + \exp\left(-\frac{i\pi}{3}\right)\tilde{f}(0,1)\tilde{g}(1,0) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(0,2)\tilde{g}(2,0)\]

\[+ \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(2,2)\tilde{g}(2,0) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(2,1)\tilde{g}(2,1) + \exp\left(-\frac{i\pi}{3}\right)\tilde{f}(2,0)\tilde{g}(2,2).\]

\[\left(\tilde{f} \otimes \tilde{g}\right)(2,2) = \tilde{f}(2,2)\tilde{g}(0,0) + \exp\left(\frac{2i\pi}{3}\right)\tilde{f}(2,1)\tilde{g}(0,1) + \exp\left(-\frac{2i\pi}{3}\right)\tilde{f}(2,2)\tilde{g}(0,2) + \exp\left(\frac{2i\pi}{3}\right)\tilde{f}(2,0)\tilde{g}(2,2).\]

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Conflict of Interest

The authors have declared no conflict of interest.

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