Surgeries of the Gieseking hyperbolic ideal simplex manifold *

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Abstract

In our Novi Sad conference paper (1999) we described Dehn type surgeries of the famous Gieseking (1912) hyperbolic ideal simplex manifold $S$, leading to compact fundamental domain $S(k)$, $k = 2, 3, \ldots$ with singularity geodesics of rotation order $k$, but as later turned out with cone angle $2(k-1)/k$. We computed also the volume of $S(k)$, tending to zero if $k$ goes to infinity. That time we naively thought that we obtained orbifolds with the above surprising property.

As the reviewer of Math. Rev., Kevin P. Scannell (MR1770996 (2001g:57030)) rightly remarked, “this is in conflict with the well-known theorem of D. A. Kazhdan and G. A. Margulis (1968) and with the work of Thurston, describing the geometric convergence of orbifolds under large Dehn fillings”.

In this paper we refresh our previous publication. Correctly, we obtained cone manifolds (for $k > 2$), as A. D. Mednykh and V. S. Petrov (2006) kindly pointed out. We complete our discussion and derive

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the above cone manifold series (Gies.1 and Gies.2) in two geometrically equivalent form, by the half turn symmetry of any ideal simplex. Moreover we obtain a second orbifold series (Gies.3 and 4), tending to the regular ideal simplex as the original Gieseking manifold.

1 Introduction

The famous non-orientable Gieseking manifold (1912) is the regular simplex with face angles $\pi/3$ in the Bolyai-Lobachevskian hyperbolic space $\mathbb{H}^3$ with ideal vertices forming a so-called cusp at the absolute, equipped by face pairing isometries $z_1, z_2$ as horospherical glide reflections (Fig. 1). These induce one equivalence class of the 6 edges and so a ball-like neighbourhood for any point of an edge, and so for any point of the identified simplex $\tilde{S}$. If we vary the face angles $\alpha_1, \alpha_2, \alpha_3$ at the opposite edges of $\tilde{S}$, then it will be no more a manifold, but for special angles there exists a natural parameter $1 < k \in \mathbb{N}$, such that $\tilde{S}(k)$ seems to represent a compact hyperbolic orbifold with a closed singularity geodesics of rotation order $k$. We shall use the Poincaré half space model of $\mathbb{H}^3$, with the complex projective line $\mathbb{C}_\infty$ for the absolute and with $(w, \zeta), |z_1 - w||w - z_2| = \zeta \cdot \zeta$ for an interior point of $\mathbb{H}^3$, over $w \in \mathbb{C}_\infty$ and third coordinate $\zeta$ on the half circle $z_1z_2$ (see Fig. 1.b and Fig. 5-6). Moreover, We compute the volume $V(k)$ of $\tilde{S}(k)$ as well by means of the Lobachevski function $\Lambda(x)$ in (2.12). It maybe surprising that $V(k) \to 0$ if $k \to \infty$ in the conflict mentioned in the abstract. Our result implies similar consequence for the double orientable cover of the Gieseking manifold, i.e. for the figure-eight-knot manifold examined also by Thurston [13]. This paper is refreshing [9] as byproduct of [6-12]. A. D. Mednykh and V. S. Petrov [5] cited our [9, 10] and noticed our deeper mistake, as it will be improved here. See also [1] with N. V. Abrosimov, showing some new phenomena and interpretations as well.

After some preliminaries in Section 2 we derive our general surgery equation (2.9) and our previous series in [9] by equation (2.11): an orbifold for $k = 2$ (uniformly for our series Fig. 3.b) and cone manifolds for $k > 2$ (Fig. 3.a and figures in the corresponding sections). Our exact figures and Table 2.1 show the first computer results (agreed with A. Mednykh and J. Weeks by different implementations). (2.9) yields also our further analogous results in Sections 3-5 with Theorems 5.1-2, Remark 5.3.

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ful discussion and friendly help, just in the COVID-19 pandemic time.

2 Gieseking Manifold and its Surgeries

We start with the ideal simplex of $H^3$ in the half-space model, where its ideal vertices at infinity are represented by

$$0, 1, z, \infty \in \mathbb{C} \cup \{\infty\} =: \mathbb{C}_\infty$$

of the complex projective line. This will be an identified ideal simplex $\tilde{S}$ with face pairing mappings $z_1$ and $z_2$ as “horospherical glide reflections”

$$z_1 : \infty z_1 \mapsto z_1^{-1}$$

and

$$z_2 : 0 \mapsto 0 \quad 1 \mapsto 1 \quad z \mapsto z$$

As usual (e.g. in [14, 15]), we extend the actions of the transformations into the upper half space by half-circles and half spheres orthogonally to the boundary plane, represented by $\mathbb{C}_\infty$. Circles and spheres through the infinity $\infty$ will be orthogonal half lines and half planes, respectively. Thus we can describe the lines and the planes of the model space of $H^3$, moreover its congruence transformations.

Going round e.g. the edge $\infty z$ from the starting identity simplex, we meet first the face $z_1^{-1}$, then follows, on the other side, the image face $[z_1]z_1^{-1}$ at the edge $(\infty 1)\tilde{z}_1^{-1}$ in the $z_1^{-1}$-image simplex. Then the image face $[z_1]z_1^{-1}$ and, on the other side, the face $[z_1]z_1^{-1}z_1^{-1}$ come at edge $(\infty 0)\tilde{z}_1^{-1}$ in the $z_1^{-1}z_1^{-1}$-image simplex. Then we meet the image face $[z_2]z_2^{-1}z_1^{-1}$ and, on the other side, the image simplex $z_2^{-1}z_1^{-1}z_1^{-1}$ by the conjugate mapping $z_1z_1z_2^{-1}z_1^{-1}z_1^{-1}$ of $z_2^{-1}$. Thus [6], we obtain the cycle transformation $z_2z_1z_2^{-1}z_1^{-1}z_1^{-1}$ and we prescribe the trivial rotation order $\nu = 1$ for the unique edge class containing 6 edges. Finally we get the cycle relation

$$z_1z_1z_2z_2^{-1}z_1^{-1}z_2^{-1} = 1$$

(2.3)
in equivalent form, in conformity with the fact that the dihedral angles of a regular ideal simplex are $\pi/3$, $6 \cdot (\pi/3) = 2\pi$ will guarantee ball-like neighbourhood of any point at simplex edges. However, the relation (2.3) with (2.2) - by careful computations - leads to equation

$$|z - 1|^2 = |z|$$

(2.4)

with more general ideal simplex, not necessarily the regular one.

Now, we turn to the ideal vertex class forming a cusp (Fig. 2–3). This will be represented by gluing; corresponding to images of the 4 vertex domains to that of $\infty$. The side face pairing of $\tilde{S}$ induces the pairing of the sides of a 2-dimensional polygon, denoted by $\tilde{s}$ in Fig. 2–3, say, on a horosphere centred in $\infty$. This is represented in our half-space model by a Euclidean plane parallel to the absolute, and it can also be described on the absolute by $\mathbb{C}_\infty$. Topologically, the polygon $\tilde{s}$ is a Klein-bottle with fundamental group

![Figure 1](image)

Figure 1: The Gieseking simplex: a. with its Schlegel diagram; b. in half space model

equivariant to the Euclidean crystallographic plane group 4. pg. This group, as the stabilizer $G_\infty$ of $\infty$, is determined by the starting group $G(z_1, z_2)$ in formulae (2.2). Fig. 3 exactly (for $k = 3$ and $k = 2$, respectively) shows the
Figure 2: The Gieseking regular ideal simplex tiling in the “touching plane” at $\infty$, as $\mathbb{C}_\infty$

more general situation that $\mathcal{G}_\infty$ is generated by pairing of $s$:

$$z_1 : [z_1^{-1}] \to [z_1] \text{ “glide reflection” as before; then}$$

$$p : [z_1^{-1}]^* := [z_1^{-1}]z_2^{-1} \to [z_1]^* := [z_1]z_2^{-1}z_2^{-1}$$

i.e. $p = z_2z_1z_2^{-1}z_2^{-1} = z_1z_1 : u \to (u - z)/|z|$

a “translation” as a central similarity in $\mathbb{C}_\infty$, (2.5)

$$z_2^* : [z_2^{-1}]^* := [z_2^{-1}]z_1^{-1}z_2^{-1}z_2^{-1} \to [z_2]^* := [z_2]z_1^{-1}z_2^{-1}z_2^{-1},$$

i.e. $z_2^* = z_2z_2z_1z_2^{-1}z_1z_2^{-1}z_2^{-1}$ a “glide reflection”

again, it is conjugated to $z_2$. We see that $p$ is a “translation”, it is $z_2$-
conjugated to $z_1z_2^{-1}$. This group $\mathcal{G}_\infty$ is 4. $\mathbb{P}g$ itself (on the Euclidean plane
represented by $\mathbb{C}_\infty$) if $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then Fig. 2 shows the exact situation. We
have obtained the Gieseking manifold with one cusp. Other $z$, as a complex
parameter, makes the stabilizer $\mathcal{G}_\infty$ to a conformal group with fixed points

$$\infty \text{ and } v = \frac{z}{1 - |z|}.$$  

This line $v\infty$ will not be covered by the $\mathcal{G}_\infty$-images of the simplex $\tilde{S}$ in
$
\mathbb{H}^3$. In the model half-space the translations of $\mathcal{G}_\infty$ in (2.5) by (2.2) will be
similarities of $\mathbb{C}_\infty$ with fixed points $v, \infty$. E.g. $z_1$ in (2.2) and $z_2$ in (2.5)
Figure 3: a. The topological Klein bottle group $\mathbf{pg}$ in $\infty$ of $\mathbb{C}_\infty$ glued by fundamental domain $S$; $z_1z_2^*$ is a rotation through $-2\pi/k = 2\pi(k-1)/k$ (mod $2\pi$) for $k = 3$; b. The orbifold for $k = 2$.
are similarity-reflections indicated in Fig. 2–5. The simple ratio on 01 is \( u = 1/(1 + |z - 1|) \). For \( z_2^* \) in (2.5) we can write by (2.2)

\[
z_2^* : (u, 1) \rightarrow (u, 1) \begin{pmatrix} 1 & 1 \\ 0 & |1 - z|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & \overline{z} - 1 \end{pmatrix}.
\]

By the tricky use of (2.4), as

\[
|z| = |z - 1|^2 = (z - 1)(\overline{z} - 1) = |z|^2 - z - \overline{z} + 1,
\]

we obtain

\[
z_1z_2^* : (u, 1) \rightarrow (u, 1) \begin{pmatrix} z(1 - \overline{z})^2 \\ |z| + 1 \end{pmatrix} \begin{pmatrix} 0 \\ |z|^2 - z^2 \end{pmatrix} \begin{pmatrix} \overline{z}(1 - z)^2 \\ \overline{z}(1 - z) \end{pmatrix},
\]

fixing \( \infty \) and \( v \), of course. We see by (2.4) that \( z_1z_2^* \) describes a rotation of the model half-space about the line \( \infty v \) with angle

\[
\phi := \text{arg} \left( \frac{z(1 - \overline{z})^2}{\overline{z}(1 - z)^2} \right) = 2 \text{arg} z - 4 \text{arg}(1 - z) \pmod{2\pi},
\]

i.e. \( \phi = 2 \cdot \alpha_1 + 4 \cdot \alpha_3 \pmod{2\pi} \).
If we require the stabilizer \( G_\infty \) to act discontinuously on the model half-space, then this angle necessarily will be \( \pm 2\pi/k \) (mod \( 2\pi \)), i.e.

\[
\frac{z}{(1-z)^2} = \pm e^{\pm i\pi/k} \quad \text{with} \quad 1 < |z-1| < |z|
\]

or \(|z| < |z-1| < 1\) and \( \Im z > 0 \), \( k = 2,3,\ldots \)

(2.9)

can be assumed. Then \( k \) is the periodicity of the rotation \( z_1 z_2^* \) and we get 4 root series for the fundamental simplices, called Gies.1–Gies.4. The first one is chosen

\[
z = 1 + \frac{1}{2} e^{i\pi/k} \left( 1 + \sqrt{1 + 4e^{-i\pi/k}} \right), \quad k = 2,3,\ldots
\]

(2.10)

All data can be computed from (2.10), especially the face angles of \( \tilde{S} \), equal
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at the opposite edges (Fig. 1.a – 3.b)

\[
\begin{align*}
&\{ \frac{\infty}{z_1} \} : \alpha_1 = \arg z; \\
&\{ \frac{\infty}{z_2} \} : \alpha_2 = \arg \frac{z - 1}{z}; \\
&\{ \frac{\infty}{z_0} \} : \alpha_3 = \arg \frac{1}{1 - z}.
\end{align*}
\]

(2.11)

However, the computer gives more guarantees. In Table 2.1 we have computed by Maple the volume of \( \tilde{S} \) as well for some values of \( k \). We know [14, 15] that the Lobachevski function

\[
\Lambda(x) = - \int_0^x \ln |2 \sin \xi| d\xi \quad \text{with} \quad \text{Vol} \tilde{S} = \Lambda(\alpha_1) + \Lambda(\alpha_2) + \Lambda(\alpha_3) \quad (2.12)
\]

provides the volume of the ideal simplex with the above angles. The formal monodromy group \( \mathcal{G}(z_1, z_2, k) \) above has a unified “presentation”

\[
\mathcal{G}(k) = \left( z_1, z_2, -1, 1 = z_1 z_1 z_2 z_2 z_1^{-1} z_2^{-1} = (z_1 z_2 z_1 z_2 z_1^{-1} z_2^{-2})^k_{k-1} \right). \quad (2.13)
\]

For \( k = 2, 3, \ldots \) we sketchily indicate by Fig. 1.b – 6 how to construct a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{A sketchy compact fundamental domain for \( \mathcal{G}(k) \)}
\end{figure}

compact fundamental domain \( \tilde{F}_{\mathcal{G}(k)} \) (in Fig. 6) by deforming an ideal vertex
Figure 7: A combinatorial fundamental domain for the Gieseking manifold and cone manifold to defining relations $\Rightarrow z_1 z_2 z_2^{-1} z_2^{-1}$ and $\Rightarrow (z_1 z_2 z_1 z_2 z_2^{-1} z_2^{-2})_{k-1}$ in (2.13)

domain to a compact one. We introduce an edge $(e)$ on the line $\infty v$ and its $z_1$-image with

$$(e) = (EE'), \quad e^{z_1} = e^{z_2^{-1}} = (E'E''), \quad (EE') \cap (E'E'') = \emptyset.$$ 

Then we choose a point $Z_1$ in the simplex $\hat{S} = \infty 01z$ and consider the segments

$$(EZ_1), \quad (EZ_1)^{z_1} = (E'Z'_1), \quad (EZ_1)^{z_1 z_2} = (E''Z'_2).$$

Similarly take $Z_2$, as the $z_1^{-1}z_2^{-1}z_2^{-1}$-image of $Z_1$ at the cusp gluing

$$(EZ_2), \quad (EZ_2)^{z_2^{-1}} = (E'Z'_2), \quad (EZ_2)^{z_2^{-1}z_2^{-1}} = (E''Z'_2).$$

Then the corresponding curved (bent) surfaces $[q^{-1}]$ and its $z_1^{-1}z_2^{-1}z_2^{-1}$-image $[q]$ will be constructed, transversally to the edges of $\hat{S}$ (see Fig. 2, 3, 5, 6) and also [6]. Finally, in Fig. 6 we get a compact fundamental domain $\hat{F}$, with piecewise linear bent faces, equipped by a pairing $\mathcal{I}(z_1, z_2, p, q)$ and defining relations to the corresponding edge classes:

$$z_1 z_1 p^{-1} = z_2^* z_2^* p = q z_2^*^{-1} q^{-1} p q^{-1} p^{-1} = z_1 q q p^{-1} q^{-1} = (z_1 z_2)^k = 1.$$ (2.14)
In Fig. 7 we have only pictured the very economic presentation with its combinatorial fundamental domain:

\[ \mathcal{G}(k) = (z_1, z_2, -1, 1 = z_1^2 z_2 z_1^{-1} z_2^{-1} = (z_1 z_2^2 z_1 z_2 z_1^{-1} z_2^{-2})_k^{-1}) \]  

(2.15)

Of course, (2.15) equivalent with (2.13) and with (2.14) if

\[ p = z_1^2, \quad z_2^* = (z_2 z_1) z_2 (z_1^{-1} z_2^{-2}), \quad q = z_1^{-1} z_2^{-1}. \]  

(2.16)

**Remark 2.1** Observe that our compactification procedure works also for \( k = 1 \) as Fig. 4 indicates. The cusp of our ideal simplex \( \tilde{S} \) as a Klein-bottle can be glued by a “solid Klein-bottle” \( K \). Then the splitting effect occurs. The cusp of \( \tilde{S} \) will be cut along a Klein-bottle surface to get a boundary. Then we glue to this boundary the boundary of \( K \), considered as \( S^2 \times \mathbb{R} \)-manifold with boundary, as follows

\[ K := S^2 \times \mathbb{R} / \langle g \rangle, \quad g : (P, r) \mapsto (P^m; r + \frac{1}{2}), \quad (P, r) \in S^2 \times \mathbb{R}. \]  

(2.17)

The generator \( g \) is a product of a reflection \( m \), say, in the equator of \( S^2 \), combined by a 1/2-translation in \( \mathbb{R} \). The boundary Klein-bottle can be obtained by cutting out one half-sphere, say, at longitudes 0 and \( \pi \), of \( S^2 \) with complete \( \mathbb{R} \)-fibers.
Table 2.1, Cone manifold surgeries of Gieseking manifold

| $k/(k-1)$ | $z$ | Angles | Volume |
|-----------|-----|--------|--------|
| $1/2$     | $1.624810533844 + i \cdot 1.300242590220$ | $0.674888845586 \approx 38.67^\circ$ | $0.696701139104$ |
| $2/3$     | $2.121964426952 + i \cdot 1.053755774241$ | $0.460919465741 \approx 26.41^\circ$ | $0.486617604149$ |
| $3/4$     | $2.327485420368 + i \cdot 0.844915596541$ | $0.218587372551 \approx 12.52^\circ$ | $0.370676286965$ |
| $8/9$     | $2.558212860705 + i \cdot 0.401960317976$ | $0.155851202654 \approx 8.93^\circ$ | $0.167339803689$ |
| $49/50$   | $2.616076631698 + i \cdot 0.073525294232$ | $0.017367036846 \approx 1.00^\circ$ | $0.030231732869$ |
| $k \to \infty$ | $z \rightarrow (3 + \sqrt{5})/2 \approx 2.6180339$ | $\alpha_1 \rightarrow 0^\circ$ | $Vol \rightarrow 0$ |

3 The second variant of our cone manifold series

The requirements in (2.9) provide the second root series

$$z = 1 + \frac{1}{2} e^{-i\pi/k} \left(1 - \sqrt{1 + 4 e^{i\pi/k}}\right), \quad k = 2, 3, \ldots$$

and the cone manifold series Gies.2 for $k > 2$. Our Fig. 8 and Table 3.1 show these, surprisingly a little bit.

This will be geometrically equivalent to Gies.1 by the half-turn symmetry of ideal simplex: $0 \leftrightarrow \infty$, $1 \leftrightarrow z$. 
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Figure 8: Gies.2 series by cone manifold for $k = 3$
| $(k - 1)/k$ | $z$ | $\text{Angles}$ | $\text{Volume}$ |
|-----------|------|----------------|---------------|
| 1/2 orbifold | 0.375 189 466 155 + $i \cdot 0.300 242 902 218$ | $0.674 888 845 586 \approx 38.67^\circ$ | 0.696 701 139 104 |
| 2/3 | 0.378 035 573 048 + $i \cdot 0.187 730 370 454$ | $0.460 919 465 741 \approx 26.41^\circ$ | 0.486 617 604 149 |
| 3/4 | 0.379 621 360 824 + $i \cdot 0.137 808 815 354$ | $0.348 223 418 295 \approx 19.95^\circ$ | 0.370 676 286 965 |
| 8/9 | 0.381 479 760 078 + $i \cdot 0.059 940 174 652$ | $0.155 851 202 654 \approx 8.93^\circ$ | 0.167 339 803 689 |
| 49/50 | 0.381 950 096 732 + $i \cdot 0.010 734 774 696$ | $0.017 367 036 846 \approx 1.00^\circ$ | 0.030 231 732 869 |
| $k \to \infty$ | $z \to (3 - \sqrt{5})/2 \approx 0.381 966 012$ | $\alpha_1 \to 0^\circ$ | $\text{Vol} \to 0$ |
| $v = \frac{1}{\sqrt{\alpha_1}} \to \frac{3 - \sqrt{5}}{\sqrt{\alpha_1 - 1}} \approx 0.618 033 989$ | $\alpha_2 \to 180^\circ$ | $\alpha_3 \to 0^\circ$ |

### 4 Gies.3-4 tend to the regular ideal simplex manifold

The requirements in (2.9) provide the third root series

$$z = 1 - \frac{1}{2}e^{-i\pi/k}\left(1 + \sqrt{1 - 4e^{i\pi/k}}\right), \quad k = 2, 3, \ldots$$ (4.1)

and the orbifold series Gies.3. Our Fig. 9 show the case $k = 3$ and 9. Table 4.1 gives computer results. The fourth root series will be

$$z = 1 - \frac{1}{2}e^{i\pi/k}\left(1 - \sqrt{1 - 4e^{-i\pi/k}}\right), \quad k = 2, 3, \ldots$$ (4.2)

and the orbifold series Gies.4. Both last series tend to the Gieseking regular ideal simplex manifold, as J. R. Weeks predicted for us in our discussions.
Figure 9: Gies.3 series is represented by orbifolds a. \(k = 3\); b. \(k = 9\)
We do not give here illustration to Gies.4 series, equivalent to the previous one by half-turn symmetry again.

As we see in Table 4.1 our orbifold volumes (more important, the half of them) are large enough. T. H. Marshall and G. J. Martin [4] determined the exact lower bound $\approx 0.0390$, the next is $\approx 0.0408$ for orientable orbifolds in dimension three. Compare that the half of the Coxeter orthoscheme $(5,3,5) \approx 0.0467$ and $(3,5,3) \approx 0.01953$, two-times less than the optimal one, but this orthoscheme has also reflection, as the authors noticed as well. In higher dimensions the problem is open, in general.

## 5 Summary

Now we summarize our results.

**Theorem 5.1** The surgery procedure of Gieseking manifold leads essentially to two different series. For rotation parameter $k = 2$ we get an orbifold. For $k > 2$ the surgery yields compact nonorientable hyperbolic cone manifolds in the first case Gies.1-2, with underlying Gieseking manifold before, where a closed geodesic line exists with cone angle $2\pi(k-1)/k$. This can be realized by a deformed ideal simplex $S(k)$ by (2.1–2) with complex parameter $z(k)$ by
that uniquely determines all metric data in our figures and tables. The volume of \( S(k) \) tends to 0 if \( k \to \infty \). □

**Theorem 5.2** The second series in cases Gies.3-4 leads to orbifolds also for \( k \geq 2 \), tending to the original Gieseking manifold if \( k \) goes to infinity.

**Remark 5.3** The orientable double cover of Gieseking manifold, known as Thurston manifold (or the complement of the figure-eight-knot) has a “manifold surgery” of volume 0.9813688289..., which is known as second minimal one. But the above construction leads to cone manifold surgeries whose volumes tend to zero. to the first series and tend to the original manifold to the second series.

The minimal volume orientable manifold of Fomenko-Matveev-Weeks (with volume 0.94270736...) can also be obtained by surgery. The occasionally possible (?) orbifold surgery is not examined yet (?).

We found in [10] the third non-orientable double-ideal-regular-simplex-manifold by computer. This has similar surgery phenomena that will be published again, because of its actuality. We plan to discuss the general surgery situations by refreshing [10] as well.

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