The growth of linear perturbations in generic defect models for structure formation

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We study the growth of linear perturbations induced by a generic causal scaling source as a function of the cosmological parameters $h$, $\Omega_m^0$ and $\Omega_\Lambda^0$. We show that for wavenumbers $k \gtrsim 0.01$ h Mpc$^{-1}$ the spectrum of density and velocity perturbations scale in a similar way to that found in inflationary models with primordial perturbations. We show that this result is independent of the more or less incoherent nature of the source, the small scale power spectrum of the source and of deviations from scaling which naturally occur at late times if $\Omega_m^0 \neq 1$.

A. Introduction

At present there are two main competing paradigms for the origin of the large scale structure of the Universe: topological defects and inflation. In defect scenarios density perturbations are causally seeded by defect evolution on increasingly larger scales while in inflationary models the causal structure of the universe is altered and correlations can be established on scales much larger than the Hubble volume. Inflationary perturbations are produced at a remote epoch and their evolution is linear until late times. They are passive and coherent. In contrast the evolution of topological defects is highly nonlinear and can only be fully described resorting to large numerical simulations. Defect perturbations are active and more or less incoherent depending on the source. Despite the differences between both scenarios produce a nearly scale invariant spectrum of density perturbations. Hybrid models featuring both inflationary and defect perturbations have also been proposed.

Recently there has been substantial progress in the understanding of defect models as seeds for the large scale structure and cosmic microwave background anisotropies. The matter and microwave anisotropy power spectrum induced by local and global strings, global monopoles and global textures have been calculated and compared with current observations. Although the results seem to indicate some disagreement between standard defect models and observations (in particular in what concerns the cosmic microwave background anisotropies), some issues still remain to be investigated in more detail. This is particularly true for local strings in which the exact contribution of cosmic string loops and gravitational radiation is still uncertain.

Given that current observational data strongly suggest that the density of the Universe is sub-critical (favoring a non-zero cosmological constant) it is very important that structure formation analysis performed for defect models can be generalized for any realization of the cosmological parameters $h$, $\Omega_m^0$ and $\Omega_\Lambda^0$. This has been done in particular for cosmic strings but the validity of the approximations used have not been investigated for generic defect structure formation scenarios. In this letter we describe the dependence of the power spectrum of density and velocity perturbations, induced by a generic causal scaling source, on the cosmological parameters $h$, $\Omega_m^0$ and $\Omega_\Lambda^0$. We show that for $k \gtrsim 0.01$ h Mpc$^{-1}$ these scale in a very similar way to that found in inflationary models with primordial fluctuations. We also study the deviations from scaling which naturally occur at late times if $\Omega_m^0$ starts deviating from unity. This is modeled by switching off the source when $\Omega_m^0$ reaches a certain fixed critical value $\Omega_m^0 < 1$. We demonstrate that our results hold independently of the more or less incoherent nature of the source, the small scale power spectrum of the source and of deviations from scaling which naturally occur at late times if $\Omega_m^0 \neq 1$. Finally, we generalize our results for an arbitrary red-shift.

B. The growth of linear perturbations

The dynamical equation which describes the evolution of the scale factor $a$ in a Friedmann-Robertson-Walker (FRW) universe containing both cold dark matter (CDM) and radiation fluids can be written as:

\[
\frac{\dot{H}}{H^2} = (\Omega_m^0 a^{-1} + \Omega_\gamma a^{-2} + \Omega_\Lambda) \frac{\dot{\eta}}{\eta} - (\Omega_m^0 + \Omega_\gamma + \Omega_\Lambda) \eta \frac{\dot{\eta}}{\eta},
\]

where $H = \dot{a}/a$ (a dot represents a derivative with respect to $\tau = \eta/\eta_{eq}$ and $\eta$ is the conformal time), $\Omega_m^0$, $\Omega_\gamma$ and $\Omega_\Lambda$ express the densities in CDM, radiation and cosmological constant as fractions of the critical density and $\Omega_k = 1 - \Omega_m^0 - \Omega_\gamma - \Omega_\Lambda$. The subscript ‘0’ means that the quantities are to be evaluated today and the subscript ‘eq’ means that the quantities are to be evaluated at the time of equal matter and radiation. We have also assumed that $a_0 = 1$ and $H(\tau_0) = \eta_{eq}$.

The evolution of radiation and CDM density fluctuations ($\delta^r = \delta \rho^r / \rho^r$ and $\delta^m = \delta \rho^m / \rho^m$ respectively) in the synchronous gauge is given by

\[
\dot{\delta}^r_k + H \delta^r_k - \frac{3}{2} H^2 (\Omega_m^0 \delta^r_k + 2 \Omega_\gamma \delta^m_k) = D(k, \tau),
\]

where $D(k, \tau)$ is

\[
D(k, \tau) = \frac{H_0^2}{2} \left( \frac{k^2}{a^2} \right) \left( \frac{\Omega_m^0}{H^2} \right) \frac{\dot{\eta}}{\eta}.
\]
\[
\dot{S}_k^c + \frac{1}{3} k^2 S_k^c - \frac{4}{3} \dot{S}_k^m = 0,
\]
where \(S_k^c = \delta_k^c k^4, \) \(\delta_k^m\) is the Fourier transform of \(\delta_k, \) \(k = |\mathbf{k}|\) is given in units of \((c^3_{\text{eq}})^{-1},\) \(D(k, \tau) = 4\pi k^3 G \Theta_+ + \Theta_+ = \Theta_{00} + \Theta_{0i} \) and \(\Theta_{0q}\) is the energy-momentum tensor of the external source.

At early times when \(a \ll \min((\Omega^m_0/\Omega^\Lambda_0)^\frac{1}{2}, \Omega^m_0/\Omega^k_0)\) the scale factor is a quadratic function of the conformal time:
\[
a(\tau) = a_{eq} \left(2(\sqrt{2} - 1)\tau + (3 - 2\sqrt{2})\tau^2 \right),
\]
and
\[
\Omega^m = \frac{a}{a + a_{eq}}, \quad \Omega^\Lambda = \frac{a_{eq}}{a + a_{eq}},
\]
with \(a_{eq} \propto (\Omega^m_0)^{-1}.\) For a scaling source no other length scale apart from the horizon scale is involved and so
\[
D(k, \tau) = \phi_0^2 k^3 \tau^{-1/2} F^2(k, \tau),
\]
with
\[
\langle D(k, \tau)^2 \rangle = \phi_0^4 k^3 \tau^{-1} \mathcal{F}_{av}(k\tau),
\]
due to homogeneity and isotropy. Here \(\phi_0\) is the defect symmetry breaking scale.

We can see from equations (1) that for a scaling source the evolution of density perturbations on a wavenumber \(k\) up to a conformal time \(\tau_f\) satisfying \(a(\tau_f) \ll \min((\Omega^m_0/\Omega^\Lambda_0)^{\frac{1}{2}}, \Omega^m_0/\Omega^k_0)\) does not depend on the cosmological parameters \(\Omega_0^m, \) \(\Omega_0^\Lambda\) and \(\Omega_0^k\) (note that \(k\) is given in units of \((c^3_{\text{eq}})^{-1} \propto \Omega^m_0 h^2 \text{Mpc}^{-1}\) and \(\tau = \eta/\eta_{\text{eq}}).\)

We note that this conclusion still holds in the more realistic case in which the right hand side of equation (1) is multiplied by a function of \(\tau\) to account for the deviation from scaling which naturally occurs during the radiation-matter transition. Consequently, we conclude that in order to study the dependence of the normalization of the power spectrum of density fluctuations, induced by a scaling source, on the cosmological parameters \(\Omega_0^m, \) \(\Omega_0^\Lambda\) and \(\Omega_0^k\) we only need to study the evolution of perturbations from a conformal time \(\tau_i\) onwards (with \(\Omega_0^m/\Omega_0^k \ll a(\tau_i) \ll \min((\Omega^m_0/\Omega^\Lambda_0)^{\frac{1}{2}}, \Omega^m_0/\Omega^k_0)\)). In this case equation (2) reduces to:
\[
\ddot{S}_k^m + \mathcal{H} \dot{S}_k^m - \frac{3}{2} \eta_{\text{eq}} \Omega^m_0 \frac{S_k^m}{a} = D(k, \tau),
\]
We solve this equation numerically assuming a simple form for the scaling source:
\[
F(k, \tau) = F(k\tau) = C \phi_0^4 \Theta(\beta k \tau - 1) \Theta(1 - \beta k \tau),
\]
with \(\beta \beta_s \geq 1, C = \text{constant}\) and \(\Theta(x)\) is the step function \(\Theta(x) = 0\) if \(x < 0\) and \(\Theta(x) = 1\) if \(x \geq 0\). The initial conditions in equation (8) are \(S_k^m(\tau_i) = 0\) and \(S_k^m(\tau_i) = 0\) for \(\beta k \tau \leq 1\). This source incorporates the most important features of causal scaling models relevant to our study. The parameter \(\beta\) is related to the compensation scale while \(\beta_s\) is a small scale cut-off which models the deviation of the power spectrum of the source from a white-noise spectrum on small scales. For the moment we are assuming that the source is coherent but in the next section we will see that this does not affect our results.

Different values of \(\Omega^m_0\) and \(\Omega^\Lambda_0\) lead to different linear growth factors from early times to the present. For primordial perturbations the quantity \(\Omega^m_0 h^2 g(\Omega^m_0, \Omega^\Lambda_0)\)
\[
g(\Omega^m_0, \Omega^\Lambda_0) = \frac{5\Omega^m_0/2}{(\Omega^m_0)^{4/7} - (1 + \Omega^m_0/2)(1 + \Omega^\Lambda_0/70)},
\]
provides a very good fit to the growth factor of density perturbations from early times \((\Omega^m_0/\Omega^\Lambda_0) \ll \min((\Omega^m_0/\Omega^\Lambda_0)^{\frac{1}{2}}, \Omega^m_0/\Omega^k_0))\) to the present [21] (see also [22]). This fit was shown to be good to a few percent for \(0.1 \leq \Omega^m_0 \leq 1\) and \(0 \leq \Omega^\Lambda_0 \leq 1\) and is normalized to unity for \(\Omega^m_0 = 1\) and \(\Omega^\Lambda_0 = 0.\)

A quantity directly related to the amplitude of matter perturbations is their rate of growth which may be described by:
\[
f = \frac{d \ln \delta}{d \ln a}.
\]
For primordial perturbations a very accurate approximation to the present rate of growth of matter perturbations is given by [23]
\[
f(\Omega^m_0, \Omega^\Lambda_0) = (\Omega^m_0)^{0.6} + \frac{\Omega^\Lambda_0}{70} \left(1 + \frac{\Omega^m_0}{2}\right).
\]
In the next section we will show how these results can be generalized for generic defect models of structure formation.

C. Results and discussion

The power spectrum induced by a scaling source for arbitrary values of the cosmological parameters \(\Omega_0^m, \) \(\Omega_0^\Lambda\) and \(\Omega_0^k\) can be written as
\[
S_k(h, \Omega_0^m, \Omega_0^\Lambda) = A \cdot g_0^2(\Omega_0^m, \Omega_0^\Lambda) \cdot (\Omega_0^m h^2)^2 \cdot S_k(1, 1, 0),
\]
where
\[
S_k(h, \Omega_0^m, \Omega_0^\Lambda) = \langle |\mathcal{S}_k(h, \Omega_0^m, \Omega_0^\Lambda, \tau)|^2 \rangle,
\]
and
\[
A = \phi_0^4(\Omega_0^m, \Omega_0^\Lambda)/\phi_0^4(1,0)\text{ and } k \text{ is given in units of } (c^3_{\text{eq}})^{-1}\text{ or equivalently, in units of } \Omega_0^m h^2 \text{Mpc}^{-1}\).
factor $A(\Omega_0^m, \Omega_0^\Lambda)$ can be normalized using, for example, the cosmic microwave background observations.

It is also useful to define

$$f_k \equiv \frac{d \ln |S_k|}{d \ln a},$$

in a similar way to equation (11).

The functions

$$\epsilon_g(k) \equiv |1 - g_k/g| \times 100,$$

and

$$\epsilon_f(k) \equiv |1 - f_k/f| \times 100,$$

measure the deviation of $f_k$ and $g_k$ from $f$ and $g$, given respectively by equations (10) and (12), evaluated at the present time.

In figure 1 we plot $\epsilon_g$ and $\epsilon_f$ as a function of $\Omega_0^m$ for open universes (solid lines) and flat universes (dashed lines) in the case with $\beta/\beta_* \rightarrow \infty$ for $k = 0.05, 0.01$ and $0.002\ Mpc^{-1}$ in ascending order. We can see that our results are not greatly influenced by our sharp cut-off on large scales in equation (9) by the more realistic cut-off $1/((k\beta\tau)^2)^2$.

We also investigate the effect of deviations from scaling which may occur at late times by switching off the source for $\Omega_0^m \leq \Omega_0^m < 1$. In figure 2 we plot $\epsilon_g$ and $\epsilon_f$ as a function of $\Omega_0^m$ for open universes in the case with $\beta/\beta_* \rightarrow \infty$ for $k = 0.01\ Mpc^{-1}$ and $\Omega_0^m = 0, 0.5, 0.7$. We can see that our results are not greatly influenced by the late time behavior of the source. This is specially true if $\beta/\beta_* \lesssim 5$ or for flat universes, in which case $\epsilon_g, f \lesssim 0.05$ for $\Omega_0^m \lesssim 0.9$ and $k \gtrsim 0.01\ Mpc^{-1}$.

Using the linearized continuity equation

$$\delta_k = -i \mathbf{k} \cdot \mathbf{v},$$

where $\mathbf{v}$ is the peculiar velocity, it can also be inferred from our results that the velocity spectrum at the present time can also be rescaled for arbitrary values of the cosmological parameters $h$, $\Omega_0^m$ and $\Omega_0^\Lambda$ in a simple and accurate way:

$$\mathbf{v}_k(\Omega_0^m, \Omega_0^\Lambda) = h \cdot f(\Omega_0^m, \Omega_0^\Lambda) \cdot g(\Omega_0^m, \Omega_0^\Lambda) \cdot \mathbf{v}_k(1, 0),$$

for $k \gtrsim 0.01\ Mpc^{-1}$.

We have also verified that our results hold for an incoherent source satisfying:

$$(\Theta_+(\mathbf{k}, \tau)\Theta_+(-\mathbf{k}, \tau + \Delta \tau)) = 0,$$

for $\Delta \tau > \tau_c$ where $\tau_c$ is assumed to be smaller than one Hubble time and
\[
F(k\tau) \propto \int_{-\tau_c}^{\tau_c} (\Theta_+(k, \tau)\Theta_+(-k, \tau + \tau'))d\tau',
\]
(21)
is given by equation (1).

Finally we note that our results may be generalized for an arbitrary red-shift \(z = 1/a - 1\):

\[
S_k(h, \Omega_m, \Omega_\Lambda, z) = S_k(h, \Omega_m^0, \Omega_\Lambda^0) \cdot \left(1 + z\right)^2 \cdot \frac{g^2(\Omega_m^0, \Omega_\Lambda^0)}{g^2(\Omega_m, \Omega_\Lambda)}
\]
(22)
for \(k\tau \gtrsim 0.02\).

Although we have assumed the dark matter to be cold our results are also valid for hot dark matter due to scaling of the neutrino free-streaming length with \(\Omega_\Lambda^0 h^2\) (see for example [25]).

D. Conclusion

In this letter we have shown that the spectrum of density and velocity perturbations induced by a causal scaling source for a particular realization of the cosmological parameters \(h, \Omega_m^0\) and \(\Omega_\Lambda^0\) can be generalized in a very simple and accurate way for any reasonable choice of these parameters on scales \(k \gtrsim 0.01 \, h \, \text{Mpc}^{-1}\). This rescaling greatly simplifies the study of structure formation with topological defects because it allows for an easy generalization, on all scales of cosmological interest, of the results of structure formation simulations induced by causal scaling sources for any reasonable values of the cosmological parameters \(h, \Omega_m^0\) and \(\Omega_\Lambda^0\). We have verified that our results are generic in the sense that they hold independently of the more or less incoherent nature of the source, the small scale power spectrum of the source and of deviations from scaling which may occur at late times.

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