Spontaneous magnetization of the integrable chiral Potts model

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Abstract
We show how $Z$-invariance in the chiral Potts model provides a strategy to calculate the pair correlation in the general integrable chiral Potts model using only the superintegrable eigenvectors. When the distance between the two spins in the correlation function becomes infinite it becomes the square of the order parameter. In this way, we show that the spontaneous magnetization can be expressed in terms of the inner products of the eigenvectors of the $N$ asymptotically degenerate maximum eigenvalues. Using our previous results on these eigenvectors, we are able to obtain the order parameter as a sum almost identical to the one given by Baxter. This gives the known spontaneous magnetization of the chiral Potts model by an entirely different approach.

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1. Introduction

In 1988, Albertini \textit{et al} \cite{1} conjectured a simple formula for the spontaneous magnetization $\mathcal{M}_r$ of the chiral Potts model. It took many years to find a proof of this conjecture, as the usual corner transfer matrix technique \cite{4} could not be used, because the variables on the rapidity lines of the chiral Potts model \cite{2, 3} live on a higher genus curve and thus do not satisfy the typical difference property. This conjecture was finally proven by Baxter in 2005 \cite{5, 6} using functional equations and the ‘broken rapidity line’ technique of Jimbo \textit{et al} \cite{7}, invoking two rather mild analyticity assumptions of the type commonly used in the field of Yang–Baxter integrable models. Most recently, in a series of papers \cite{8–13}, an algebraic (Ising-like) way of obtaining $\mathcal{M}_r$ has been given, providing more insight into the algebraic structure.

It now looks very plausible that the pair correlation function is also calculable. In this paper, we outline a strategy to attack this problem. As a first application we derive the order parameter in a new way.
1.1. Baxter’s Z-invariance for correlation functions

The integrable chiral Potts model [3] is Z-invariant by definition—its transfer matrices commute with one another and one can permute them without changing the partition function $Z$. Furthermore, Z-invariance also implies that in the thermodynamic limit the pair correlation function $\langle s_{A}^{n} s_{B}^{s-n} \rangle$ in the bulk is given by a set of universal functions $g_{2m}^{(n)}$, which can only depend on the $2m$ rapidities that pass in the same direction between the two spins at A and B [14, 15]. This may require us to flip the direction of some rapidities by the automorphism $q \rightarrow Rq$, given by $xq \rightarrow yq, yq \rightarrow \omega xq$, to make all the $2m$ lines point in the same direction. Each spin $s$ can take the $N$ values $s = \omega^{\sigma}$, with $\sigma = 0, \ldots, N - 1$, $\omega \equiv e^{2\pi i/N}$. Shown in figure 1 are three examples with universal functions $g_{6}^{(n)}$ and $g_{8}^{(n)}$.

The Z-invariance property means that we can calculate the correlation functions of a general integrable chiral Potts model on an infinite lattice from special correlations in the much simpler superintegrable case [16–19]. We can take an infinite square lattice with special vertical rapidities $p$ and $p'$ alternating, but with more general horizontal rapidities $q_{1}, q_{2}, \ldots, q_{2m}$. Choose two spins within faces in the same vertical column. As such a pair correlation function only depends on the modulus $k$ and the rapidities passing between the two spins, it is independent of $p$ and $p'$, see also the example shown in figure 1 as $\langle s_{A}^{n} s_{B}^{s-n} \rangle$.

1.2. Superintegrable chiral Potts model

If we assume alternating vertical rapidities $x_{p'} = y_{p}, y_{p'} = x_{p}, \mu_{p'} = 1/\mu_{p}$, the model becomes ‘superintegrable’ [16–19], as it has Ising-like properties. We consider here only the case with $p = p'$, so that $\mu_{p} = 1$. This case both obeys Yang–Baxter integrability [3] and has an underlying Onsager algebra [19], whence the name superintegrable [1]. Its spin chain Hamiltonian was first constructed by von Gehlen and Rittenberg [18].

To calculate pair correlations of the superintegrable spin chain in a ground state, we only need correlations in a horizontal row of figure 1, with uniform vertical rapidities satisfying $x_{p} = y_{p}$. This pair correlation is the expectation value of the spin pair in that ground state. From our previous work [20–23] we know the $N$ ground state eigenvectors in the commensurate phase that are needed.
Several partial results on the pair correlations in the superintegrable quantum spin chain already exist, including some results for the leading asymptotic long separation behavior from finite-size calculations and conformal field theory [24, 25], a few terms in series expansions in the coupling constant [26–30], results using the density matrix renormalization group technique [31] and a few exact results for very small chains [32]. Using exact knowledge of the eigenvectors much more can be done for the superintegrable spin chain, but in this paper we rather go in the vertical direction in figure 1, as this will allow us to describe what is needed for the more general integrable chiral Potts model and to express the order parameters in terms of inner products of ground state vectors.

### 1.3. Correlation functions and order parameters

First we define the row-to-row transfer matrices with periodic boundary conditions in the usual way [4], i.e.

\[
T_j = T(x, y, \sigma) = \prod_{j=1}^{L} W_{p_q}(\sigma_j - \sigma_j', \omega_{p_q} (\sigma_j' - \sigma_{j+1})),
\]

\[
\tilde{T}_j = \tilde{T}(x, y, \sigma) = \prod_{j=1}^{L} W_{p_q}(\sigma_j' - \sigma_{j+1}'), \omega_{p_q}(\sigma_j - \sigma_{j+1}'),
\]

where the chiral Potts Boltzmann weights depend on spin differences modulo \(N\).

Next, we define the vertical pair correlation functions, between spins \(\sigma_1\) and \(\tilde{\sigma}_1\) in the first column separated by \(2\ell\) horizontal rapidities, as

\[
s_{2\ell}^{(r)}(k; q_1, \ldots, q_{2\ell}) = \frac{1}{Z} \sum_{[\sigma]} \omega^r(\sigma_1 - \tilde{\sigma}_1) \prod_{all bonds} W_{p_q}(\sigma - \sigma') \omega_{p_q}(\sigma - \sigma''),
\]

where \(r = 1, \ldots, N - 1\). In the Ising case \(N = 2\) we only have \(r = 1\). Assuming periodic boundary conditions in both directions, we can write

\[
s_{2\ell}^{(r)}(k; q_1, \ldots, q_{2\ell}) = \frac{1}{Z} \text{Tr}_{[\sigma]} \left[ \hat{Z}_1^r \left( \prod_{j=1}^{\ell} T_{q_{2j-1}} \hat{T}_{q_{2j}} \right) \hat{Z}_1^{\omega} \left( \prod_{j=\ell+1}^{M} T_{q_{2j-1}} \hat{T}_{q_{2j}} \right) \right],
\]

with \(Z = \text{Tr}_{[\sigma]} \left[ \prod_{j=1}^{L} T_{q_{2j-1}} \hat{T}_{q_{2j}} \right]\). Once the thermodynamic limit \(L, M \to \infty\) is taken, this should give the most general pair correlations in integrable chiral Potts models on infinite \(Z\)-invariant lattices.

As in [9], the operators \(\hat{Z}\) and \(\hat{X}\) are defined as \(\hat{Z}[\sigma] = \omega^r[\sigma]\) and \(\hat{X}[\sigma] = [\sigma + 1] \) and \(L\) pairs of copies \(\hat{Z}_j\) and \(\hat{X}_j\), acting in the \(j\)th column \((j = 1, \ldots, L)\), are introduced. They are different from \(Z_j\) and \(X_j\) of [23] which act on the edge variables \(n_j = \sigma_j - \sigma_{j+1}\), instead of the spins \(\sigma\). The transfer matrices are invariant under the spin shift operator

\[
\mathcal{X} \equiv \prod_{j=1}^{L} \hat{X}_j, \quad \mathcal{X}[\sigma_1, \ldots, \sigma_L] = [\sigma_1+1, \ldots, \sigma_L].
\]

In fact, the transfer matrix elements only depend on differences \(n_j = \sigma_j - \sigma_{j+1}\) in both rows and \(m \equiv \sigma_1 - \sigma_1^\prime\) in the first column, i.e.

\[
\langle \sigma_1^\prime, \{n_j^\prime\} | T | \sigma_1, \{n_j\} \rangle = T(\sigma_1 - \sigma_1^\prime, \{n_j^\prime\}, \{n_j\}) = T(m, \{n_j^\prime\}, \{n_j\}),
\]

writing

\[
[\sigma_1, \ldots, \sigma_L] \equiv [\{\sigma_j\}] \equiv [\sigma_1, \{n_j\}], \quad n_j = \sigma_j - \sigma_{j+1}, \quad \sum_{j=1}^{L} n_j = 0.
\]
We define a new (Fourier-transformed) basis \[20\]

\[|Q; \{n_j\} \rangle \equiv N^{-1/2} \sum_{\sigma_1=0}^{N-1} \omega^{-Q\sigma_1} |\sigma_1, \{n_j\} \rangle. \tag{1.7}\]

These are eigenvectors of the spin shift operator, as

\[\mathbf{X} |\sigma_1, \{n_j\} \rangle = |\sigma_1 + 1, \{n_j\} \rangle, \quad \text{implies} \quad \mathbf{X} |Q; \{n_j\} \rangle = \omega^Q |Q; \{n_j\} \rangle. \tag{1.8}\]

In this new basis, the transfer matrix elements become

\[
\langle Q; \{n_j'\} | T | Q; \{n_j\} \rangle = \frac{1}{N} \sum_{\sigma_i^0=0,\sigma_i=0}^{N-1, N-1} \omega^{Q(\sigma_i^0-\sigma_i)} \langle \sigma_i^0, \{n_j'\} | T | \sigma_i, \{n_j\} \rangle, \tag{1.9}\]

or

\[
\langle Q; \{n_j'\} | T | Q; \{n_j\} \rangle = \sum_{m=0}^{N-1} \omega^{-mQ} T(m, \{n_j'\}, \{n_j\}) \equiv T_Q(\{n_j'\}, \{n_j\}). \tag{1.10}\]

Thus we find that in the spin-shift \(Q\) sector the transfer matrices only depend on bond (link, or edge) variables.

The inverse relation is

\[
\langle |\sigma_i'\rangle | T | |\sigma_i\rangle \rangle = \frac{1}{N} \sum_{Q=0}^{N-1} \omega^{Q(\sigma_i' - \sigma_i)} T_Q(\{n_j'\}, \{n_j\}). \tag{1.11}\]

Matrix products can also be rewritten in the \(Q\) basis:

\[
\langle |\sigma_i'\rangle | T \hat{T} | |\sigma_i\rangle \rangle = \frac{1}{N^2} \sum_{\sigma_i^0=0, \sigma_i=0}^{N-1, N-1} \sum_{\sigma_i^0=0, \sigma_i=0}^{N-1, N-1} \omega^{Q(\sigma_i^0 - \sigma_i)} T_Q(\{n_j'\}, \{n_j^0\}) \omega^{Q(\sigma_i^0 - \sigma_i)} T_{\hat{T}}(\{n_j^0\}, \{n_j\})
\]

\[
= \frac{1}{N} \sum_{\sigma_i^0=0, \sigma_i=0}^{N-1, N-1} \sum_{Q=0}^{N-1} \omega^{Q(\sigma_i^0 - \sigma_i)} \delta_{Q,0} T_Q(\{n_j'\}, \{n_j^0\}) \hat{T}_{\hat{T}}(\{n_j^0\}, \{n_j\})
\]

\[
= \frac{1}{N} \sum_{Q=0}^{N-1} \omega^{Q(\sigma_i^0 - \sigma_i)} \langle |n_j'\rangle \rangle T_Q \hat{T}_{\hat{T}} | |n_j\rangle \rangle. \tag{1.12}\]

The pair correlation function can be worked out similarly. In the special case of equal horizontal rapidities (\(q_\ell \equiv q\)) and using \(\langle |\sigma\rangle | \tilde{Z}_\ell (|\sigma\rangle \rangle = \omega^{Q\sigma}\), we find

\[
\delta_{2\ell}^{(0)}(k; q, \ldots, q) = \frac{1}{Z} \text{Tr} \left[ Z_{\ell} \left( \prod_{j=1}^{\ell} T_{\ell+1} \right) \hat{T}_{\ell} \left( \prod_{j=\ell+1}^{M} T_{\ell+1} \right) \right]
\]

\[
= \frac{1}{N} \sum_{Q=0}^{N-1} \text{Tr} \left[ [T(x_q, y_q) \hat{T}(x_q, y_q) \hat{T}(x_{q+i}, y_{q+i})]^\ell \hat{T}(x_q, y_q) \hat{T}(x_{q+i}, y_{q+i}) \right]^M, \tag{1.13}\]

with \(P \equiv Q - r \mod N\). Let the eigenvectors of the transfer matrices be given by

\[
T_Q(x_q, y_q) \hat{T}_Q(x_q, y_q) |\gamma_{ij}^Q\rangle = \left( \Delta_{ij}^Q \right)^\ell |\gamma_{ij}^Q\rangle, \quad |\gamma_{ij}^Q\rangle |\gamma_{ij}^Q\rangle = \delta_{ij}. \tag{1.14}\]
where \( \Delta^0_j \) denotes the \( j \)th eigenvalue, and let \( \Delta^0_{\text{max}} \) be the maximum eigenvalue of the transfer matrix \( T_Q \). Then, in the limit of an infinite number \( 2M \) of rows, the partition function becomes

\[
Z = \sum_{Q=0}^{N-1} (\Delta^0_{\text{max}})^{2M} \to N(\Delta^0_{\text{max}})^{2M}, \quad \text{as} \quad L, M \to \infty, \quad (1.15)
\]
as the \( \Delta^0_{\text{max}} \) for \( 0 \leq Q \leq N-1 \) are asymptotically degenerate as \( L \to \infty \). Therefore,

\[
g_{2M}^{(r)}(k; q, \ldots, q) = \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{j=1}^{J} \left[ \frac{\Delta^0_j}{\Delta^0_{\text{max}}} \right]^{2L} \langle \gamma^0_{\text{max}} | \gamma^p_j | \gamma^0_{\text{max}} \rangle, \quad (1.16)
\]

where \( J = N^{L-1}, L \to \infty \).

In the limit \( \ell \to \infty \), the pair correlation becomes the product of order parameters,

\[
\langle \omega^{\alpha n} \rangle \langle \omega^{\alpha' n} \rangle = \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{j=1}^{J} \langle \gamma^0_{\text{max}} | \gamma^p_j | \gamma^{p'}_j | \gamma^0_{\text{max}} \rangle, \quad P = Q = r \mod N. \quad (1.17)
\]

For the Ising case \( (N = 2) \), this reduces to the simple formula for the spontaneous magnetization \( m_{\text{sp}} = \left| \langle \gamma^0_{\text{max}}|\gamma^1_{\text{max}} \rangle \right| \) in the ordered phase.

### 1.4. Baxter’s approach

Recognizing that the \( N \)-state superintegrable model resembles the 2D Ising model [1, 17, 21, 23, 33–35], and that the two parts of its Hamiltonian generate the Onsager algebra [18, 33], Baxter concluded that there must be an algebraic route to calculate the order parameter. He started [8] with a new algebraic approach to calculate \( M_\ell \) for the Ising model, representing it first as a square root of an \( L \) by \( L \) determinant, which he then reduced to an \( m \) by \( m \) determinant [8], where \( m \leq L/2 \). Having these new Ising results Baxter tried to guess their extensions to the \( N \)-state superintegrable model. Thus, he conjectured [9, 10] that the special combination \( D_{PQ} = M_\ell(Z_P Z_Q)^{1/2} \), with \( Z_Q \) denoting the partition function with fixed boundary condition of type \( Q \), is both a determinant and also given by the sum

\[
D_{PQ} = \sum_s \sum_{s'} \sum_{y_1}^s \sum_{y_2}^{s'} \cdots \sum_{y_m}^{s'} \left( \frac{A_s}{C_s D_s} \right) y_1^{y_2} y_2^{y_3} \cdots y_m^{y_m}. \quad (1.18)
\]

Here, \( m = m_P \) and \( m' = m_Q \), with \( m_Q = \lceil (N - 1) L/N - Q/N \rceil \) and \( m_P \) given similarly:

\[
s = [s_1, s_2, \ldots, s_m], \quad s' = [s'_1, s'_2, \ldots, s'_m], \quad s, s' = 0, 1, \quad (1.19)
\]

and

\[
A_{s,x} = \prod_{i \in W} \prod_{j \in V} (c_i - c'_j), \quad B_{s,x} = \prod_{i \in W} \prod_{j \in W'} (c_i - c'_j),
\]

\[
C_s = \prod_{i \in W} \prod_{j \in V} (c_j - c_i), \quad D_s = \prod_{i \in V} \prod_{j \in W'} (c'_j - c'_i), \quad \quad (1.20)
\]
in which for a given set \( s, s' \) the set of integers \( s \) such that \( s_0 = 0 \) and \( W \) such that \( s_1 = 1 \), while \( W' \) are defined similarly for the set \( s' \). In (1.18) the number of elements in \( W \) equals the number in \( W' \). The variables \( c_i \) are related to the roots \( w_i \) of \( P_r(w) \) and \( c'_i \) to the roots \( w'_i \) of \( P_Q(w) \), with both polynomials defined by

\[
P_Q(w) = \frac{1}{2} \sum_{n=0}^{N-1} \omega^{nQ} \left[ \frac{(1 - i^n)}{(1 - i \omega^n)} \right]^L \quad (1.21)
\]

and the \( c_i \) and \( c'_i \) by the relation \( c = (w + 1)/(w - 1) \). This form (1.18) was recently proven by Iorgov et al [13]. Finally, in [12], Baxter showed that this sum \( D_{PQ} \) is the conjectured determinant, which he had already evaluated in the thermodynamic limit in [11]. Thus, he completed an algebraic proof of the order parameter.
1.5. Approach of Iorgov et al

Very recently, Iorgov et al [13] derived matrix elements of the spin operator in the finite superintegrable chiral Potts quantum chain in factorized form. They also gave another method to derive the order parameters in the thermodynamic limit, performing the sum without going to the determinant formula of Baxter.

2. Our approach

In our previous papers [21, 23] we have given the $2^{m_Q}$ eigenvector pairs in the ground state sectors of the transfer matrices $T_Q$ and $\hat{T}_Q$. Using the notation of Baxter [10, 11], but without going to Baxter’s reduced representation of the vector space, we label these eigenvectors $|\chi_s^Q\rangle$ and $|\chi_s^{Q'}\rangle$, with $s'$ chosen as in (1.19). In fact, the eigenvectors are given in [21, 23] as

$$|\chi_s^Q\rangle = \prod_{j=1}^{m_Q} R_{j,Q} \prod_{i \in W'_n} E_{s_i}^+ \Omega, \quad |\chi_s^{Q'}\rangle = \prod_{j=1}^{m_Q} S_{j,Q} \prod_{i \in W'_n} E_{s_i}^+ \Omega, \quad (2.1)$$

where $W'_n$ is a subset of $\{1, 2, \ldots, m_Q\}$ containing $n$ integers. They satisfy the eigenvalue equations

$$T_Q(x_q, y_q) |\chi_s^Q\rangle = \Delta_s^Q |\chi_s^Q\rangle, \quad \hat{T}_Q(x_q, y_q) |\chi_s^{Q'}\rangle = \Delta_s^{Q'} |\chi_s^{Q'}\rangle, \quad (2.2)$$

Particularly, we use the ground state eigenvectors

$$|\chi_0^Q\rangle = |\chi_0^{Q'}\rangle = \prod_{j=1}^{m_Q} S_{j,\emptyset} \Omega, \quad |\chi_0^{Q'}\rangle = |\Omega\rangle \prod_{j=1}^{m_Q} S_{j,\emptyset}^{-1}, \quad (2.3)$$

for which $W_{\emptyset} = \emptyset$ is the empty set. The ‘rotation’ $S = \prod_j S_j$ is given by (IV.124) or (IV.150) with

$$S_{j,p} = \frac{1}{2}(s_{11}^{j,p} + s_{22}^{j,p}) 1 + \frac{1}{2}(s_{11}^{j,p} - s_{22}^{j,p}) H_j, p + s_{12}^{j,p} E_{j,p}^+ + s_{21}^{j,p} E_{j,p}^-, \quad (2.4)$$

$$S_{j,p}^{-1} = \frac{1}{2}(s_{22}^{j,p} + s_{11}^{j,p}) 1 + \frac{1}{2}(s_{22}^{j,p} - s_{11}^{j,p}) H_j, p - s_{12}^{j,p} E_{j,p}^+ - s_{21}^{j,p} E_{j,p}^-, \quad (2.5)$$

where $E_{j,p}^\pm$ and $H_{j,p}$ are generators of $\mathfrak{sl}_2$ algebras. Furthermore, we have shown in [21, 23] that for the state $|\Omega\rangle = |\{n_j = 0\}\rangle$ we have $E_{m,Q}^- |\Omega\rangle = 0, H_{m,Q} |\Omega\rangle = -|\Omega\rangle$, $\langle \Omega| E_{l,Q}^- = 0, \langle \Omega| H_{l,Q} = -|\Omega\rangle$ and

$$\langle \Omega| E_{l,Q}^+ = -\frac{\beta_{l,0}^Q - \beta_{l,0}^{Q'} - \alpha l}{\Lambda_0} \sum_{|m_G| \leq n-1 \atop \Lambda = \alpha l, \kappa = \kappa'} \langle \{n_j\}| \omega^{\sum_j |m_j^G|} G_Q(\{n_j\}, z_{l,Q}), \quad (2.6)$$

$$E_{l,Q}^+ |\Omega\rangle = \frac{\beta_{l,0}^Q - \beta_{l,0}^{Q'} - \alpha l}{\Lambda_0} \sum_{|m_G| \leq n-1 \atop \Lambda = \alpha l, \kappa = \kappa'} \omega^{\sum_j |m_j^G|} G_Q(\{n_j\}, z_{l,Q}) |\{n_j\}\rangle, \quad (2.6)$$

see (IV.66) and (IV.67). The polynomials here are given in (III.16) as

$$G_m(\{n\}, z) = \sum_{n=0}^{m-1} K_{n+Q}(\{n\}) z^n, \quad G_m(\{n\}, z) = \sum_{n=0}^{m-1} K_{n+P}(\{n\}) z^n, \quad (2.7)$$

$3$ Equations in our papers [20–23] are denoted here by respectively prefacing I, II, III or IV to the equation number.
with \( m \equiv m_p, m' \equiv m_Q \) and coefficients given by (III.7) and (III.8) as the sums
\[
K_\ell((n_j)) = \sum_{\{0 \leq n_i \leq N-1\}} \prod_{j=1}^{L} \binom{n_j + n'_j}{n_j'} \omega^{n_j}n_j', \quad N_j = \sum_{\ell} n_j, \\
\tilde{K}_\ell((n_j)) = \sum_{\{0 \leq n_i \leq N-1\}} \prod_{j=1}^{L} \binom{n_j + n'_j}{n_j'} \omega^{n_j}n_j', \quad \tilde{N}_j = \sum_{\ell+j+1} n_j.
\] (2.8)

2.1. Form factors

From (2.3)–(2.5) we obtain
\[
\langle \chi^Q_{\text{max}} | \chi^p \rangle = \bigl( \Omega \bigr) \prod_{j=1}^{m_Q} (s_{j\ell}^Q 1 - s_{j\ell}^Q E_{j,p}) \prod_{j=1}^{m_p} (s_{j\ell}^p 1 + s_{j\ell}^p E_{j,p}^+) |\Omega\rangle
\]
\[
= \prod_{j=1}^{m'} s_{j1}^p \prod_{j=1}^{m} s_{j2}^p \prod_{j=1}^{m'} (1 + u_j E_{j,p}^+) \prod_{j=1}^{m} (1 + u_j E_{j,p}) |\Omega\rangle,
\] (2.9)
where \( m = m_p, m' = m_Q \), and
\[
u_j = s_{12}^j / s_{22}^j = T_{21}^j E_{12}^j, \quad u_j' = -s_{12}^j / s_{11}^j = -s_{21}^j / s_{11}^j = -T_{12}^j / T_{11}^j,
\] (2.10)
see (II.C.2), (II.C.10) and (IV.152).

Next we expand the products in (2.9). Since
\[
\prod_{j=1}^{L} E_{\ell_j,p}^+ |\Omega\rangle \in \bigoplus_{\{\langle n_j \rangle\}, \sum_{j} n_j = kN} \bigl( \Omega \bigr) \prod_{\ell=1}^{L} E_{\ell_j,p}^+ \in \bigoplus_{\{\langle n'_j \rangle\}, \sum_{j} n'_j = lN} \bigl( \langle n'_j \rangle \bigr),
\] (2.11)
we find that the only non-vanishing terms in this expansion are those with equal numbers of creation and annihilation operators. Consequently,
\[
\langle \chi^Q_{\ell} | \chi^p \rangle = C \biggl[ 1 + \sum_{j=1}^{m'} \sum_{\ell=1}^{m} u_j' u_{\ell} \langle \Omega | E_{j,p}^+ E_{\ell,p}^+ |\Omega\rangle + \cdots \\
+ \sum_{1 \leq \ell_1, \cdots, \ell_n \leq m'} \sum_{1 \leq \ell_1, \cdots, \ell_n \leq m} \sum_{1 \leq \ell_1, \cdots, \ell_n \leq m'} (u_{\ell_1} \cdots u_{\ell_n}) (u'_{\ell_1} \cdots u'_{\ell_n}) \langle \Omega | \prod_{j=1}^{n} E_{j,p}^+ \prod_{\ell=1}^{n} E_{\ell,p}^+ |\Omega\rangle \\
+ \cdots + (u_1 \cdots u_m) (u'_{1} \cdots u'_{m}) \langle \Omega | \prod_{j=1}^{m'} \prod_{\ell=1}^{m} E_{\ell,j,p}^+ E_{\ell,j,p}^+ |\Omega\rangle \biggr],
\] (2.12)
with
\[
C \equiv \prod_{j=1}^{m'} s_{j1}^p \prod_{j=1}^{m} s_{j2}^p.
\] (2.13)
The last term in (2.12) is identically zero unless \( m = m' \). Let \( \lambda_p = \mu_{j,p}^N = 1 \) in (III.149) and (II.C.5), and denote \( z_j = z_{j,p}, z'_j = z_{j,Q}, \theta_j = \theta_{j,p} \) and \( \theta'_j = \theta_{j,Q} \) as in Baxter’s papers. Then, (2.10) becomes
\[
u_j = \frac{1 - k'}{e^{2\pi i} - k'}, \quad u_j' = \frac{z'_j (1 - k')} {e^{2\pi i} - k'};
\] (2.14)
compare (III.124) and (III.126) with \( \lambda_p = 1 \). The \( \theta_j \) in these equations are not the same as those in Baxter’s papers [9, 10], but are related to our \( z_j \) by (II.C.5), i.e.

\[
e^{2\lambda_j} + e^{-2\lambda_j} = k^2 + 1/k^2 - (1 - k^2)^2 z_j/k^2.
\]

Solving for \( e^{2\lambda_j} \) while using Baxter’s notations (BaxII.3.16) and (BaxII.2.18)\(^4\)

\[
\lambda_j \equiv \frac{1 + k^2 + 2k^2 + z_j}{1 - z_j} \quad \text{or} \quad z_j = \frac{\lambda_j^2 - (1 + k')^2}{\lambda_j^2 - (1 - k')^2}.
\]

identifying \( w_j \equiv 1/z_j \) and \( c_j \equiv \cos (\theta_j^B) = -(1 + z_j)/(1 - z_j) \) (where we use \( \theta_j^B \) to denote \( \theta_j \) in Baxter’s paper), we have

\[
e^{2\lambda_j} = \frac{\lambda_j + 1 - k'}{\lambda_j + 1 + k'}.
\]

Consequently (2.14) becomes

\[
u_j = \frac{\lambda_j - 1 + k'}{\lambda_j + 1 + k'}, \quad u_j' = -\frac{\lambda_j' - 1 - k'}{\lambda_j' + 1 - k'},
\]

where the second equation in (2.16) is used to get rid of the factor \( z_j' \). Comparing with (BaxIV.4.13) and (BaxIV.4.14), we find \( u_j' \rightarrow -y_j \) and \( u_j \rightarrow -y_j' \). The minus signs cancel out upon multiplication. We show later that this gives the correct result, due to the identification \( z_j = 1/w_j \).

Adopting the notations of Baxter as shown in (1.19) and (1.20), we find \( W_n = \{\ell_1, \ldots, \ell_n\} \) and \( W'_n = \{j_1, \ldots, j_n\} \), so that (2.12) becomes

\[
\langle \lambda_j^P | \lambda_j^Q \rangle = \mathcal{C} \sum_{i_1} \sum_{i_2} \cdots \sum_{i_m} u_1^i u_2^i \cdots u_m^i \langle \Omega \rangle \prod_{i \in W_n} E_{i,j}\langle \Omega \rangle \prod_{j \in W_n} E_{i,j}\langle \Omega \rangle u_1^i u_2^i \cdots u_m^i.
\]

Similarly, for \( P \leftrightarrow Q \), we have

\[
\langle \lambda_j^P | \lambda_j^Q \rangle = \mathcal{C} \sum_{i_1} \sum_{i_2} \cdots \sum_{i_m} \hat{u}_1^i \hat{u}_2^i \cdots \hat{u}_m^i \langle \Omega \rangle \prod_{i \in W_n} E_{i,j}\langle \Omega \rangle \prod_{j \in W_n} E_{i,j}\langle \Omega \rangle \hat{u}_1^i \hat{u}_2^i \cdots \hat{u}_m^i,
\]

where, instead of (2.10) and (2.13),

\[
\hat{u}_j = s_{12}^j, \quad \hat{u}_j = -s_{21}^j,
\]

\[
\mathcal{C} = \prod_{j=1}^m s_{11}^j \prod_{j=1}^m s_{22}^j.
\]

It is easily seen from (2.10), (2.14) and (2.21), followed by (2.18) and (2.16), that

\[
\hat{u}_j = -z_j u_j = \frac{\lambda_j - 1 - k'}{\lambda_j + 1 - k'}, \quad \hat{u}_j = -u_j' = \frac{\lambda_j' - 1 + k'}{\lambda_j' + 1 + k'}.
\]

Finally, from (II.C.6) and (2.17), we have

\[
s_{11} s_{22}^j = \frac{e^{2\lambda_j} - k'}{2 \sinh 2\theta_j} = \frac{(\lambda_j + 1)^2 - k^2}{4\lambda_j},
\]

which relates to the inverse of \( Z_p \) in (BaxIV.3.7), as it should.

Looking at (2.19) we now need to evaluate the expectation value in the state \( \langle \Omega \rangle \) of the product of \( n \) creation operators \( E_{j}^\dagger \) and \( n \) annihilation operators \( E_{j} \), which all operate on the edge variables.

\(^4\) Equations in Baxter’s papers [8–12] are denoted here by respectively prefacing BaxI, BaxII, BaxIII, BaxIV or BaxV to the equation number. Our transfer matrices and \( \mathbf{X} \) are transposes of the ones of Baxter, see footnote to (IV.107). Therefore, to compare with Baxter, we must replace \( Q \rightarrow N - Q \) or \( z_j \rightarrow 1/z_j = w_j \).
3. Proposition

Comparing (2.19) with the sum $D_{pq}$ of Baxter [9, 10] given here in (1.18), we propose the following identity:

$$
\langle \Omega | \prod_{j \in W_\ell} E^-_{j,0} \prod_{\ell \in W_\ell} E^+_{\ell,0} | \Omega \rangle = \frac{\tilde{A}_{s,t} \tilde{B}_{s,t}}{C_s D_s} \times \frac{\tilde{A}_{s,t} \tilde{B}_{s,t}}{C_s D_s},
$$

(3.1)

where $A$, $B$, $C$ and $D$ are defined in (1.20) (or (BaxIII.3.44)), while

$$
\tilde{A}_{s,t} = \prod_{i \in W_a, j \in V_b} (z_i - z_j), \quad \tilde{B}_{s,t} = \prod_{i \in W_b, j \in V_a} (z_i' - z_j),
$$

$$
\tilde{C}_s = \prod_{i \in W_a} (z_i - z_j), \quad \tilde{D}_s = \prod_{i \in W_b} (z_i' - z_j).
$$

(2.2)

The difference between $A$ and $D$ of Baxter and $\tilde{A}$ and $\tilde{D}$ here is the replacement of $c = (z + 1)/(z - 1)$ by $z$. In $\tilde{B}$ and $\tilde{C}$, we have also flipped the signs to make them more symmetric.

We first consider the simplest case, with $n = 1$, and then prove (3.1) by induction.

3.1. Proof of (3.1) for $n = 1$

From (2.6), we find

$$
\langle \Omega | E^-_{j,0} E^+_{\ell,0} | \Omega \rangle = \frac{\rho_j^Q \rho_{\ell,0}^P z^{-j,\ell}}{A_0^Q A_0^P} \sum_{[0 \leq n, \ell \leq N-1]} G_Q([n], z_{j,0}) G_P([\ell], z_{\ell,0}).
$$

(3.3)

Similar to (II.72) or (III.45), we introduce the polynomial

$$
h_{j,\ell}^Q(z', z) = \sum_{[0 \leq n, \ell \leq N-1]} G_Q([n], z_{j,0}) G_P([\ell], z).$

(3.4)

Substituting (2.7) into this equation, we find

$$
h_{j,\ell}^Q(z', z) \equiv \sum_{m=0}^{m-1} \sum_{m=0}^{m-1} z_j^I z_{\ell}^I G_{\ell+j,\ell+j}. 
$$

(3.5)

where

$$
G_{\ell, j} = \sum_{[0 \leq n, \ell \leq N-1]} K_{\ell}([n]) K_{j}([n]).
$$

(3.6)

We can generalize identity (III.37) or (III.44) to $P \neq Q$ cases. For $P \geq Q$ we have

$$
G_{\ell+j, \ell+j} = G_{\ell+j, \ell+j},
$$

(3.7)

which becomes (III.37) for $P = Q$ after replacing $n \rightarrow j - n$. The proof of (3.7) is much harder, however, and in the appendix we show why. We have been able to prove the identity using a different approach, which is presented in detail elsewhere [36]. Due to the symmetry $G_{\ell+k} = G_{\ell+k}$ given in (3.7), we find

$$
h_{j,\ell}^Q(z', z) = \sum_{m=0}^{m-1} \sum_{m=0}^{m-1} z_j^I z_{\ell}^I G_{\ell+j, \ell+j} = h_{j, \ell}^Q(z', z).
$$

(3.8)
Inserting (3.7), and interchanging the order of summations over \( j \) and \( n \), we rewrite (3.5) as
\[
h_{0.P}(z', z) = \sum_{\ell=0}^{m'} \sum_{n=0}^{m' + \ell - 1} z^{\ell+n} \begin{bmatrix} (j - n + 1) & n \end{bmatrix}_{\Lambda_1} - (n - \ell) \Lambda_1 A_n^{\ell,j} \right] z^{n+\ell}, \tag{3.9}
\]
where the summation over \( j \) is changed to \( i = j - n + 1 \). Since \( m' = m \) or \( m' = m + 1 \), and also \( \Lambda_0 = 0 \) for \( n > m' \) and \( \Lambda_0^P = 0 \) for \( n > m \), we may extend the intervals of summation to \( 0 \leq \ell, n \leq m' \) (also noting \( \sum_{i=0}^{k-1} a_i = - \sum_{i=k+1}^{k} a_i \) for \( l < k \)), so that
\[
h_{0.P}(z', z) = \sum_{\ell=0}^{m'} \sum_{n=0}^{m'} \sum_{i=0}^{m + \ell - n} \left( \frac{z^i}{z} \right)^{\ell+n} \begin{bmatrix} (i - \ell) & n \end{bmatrix}_{\Lambda_1} - (n - \ell) \Lambda_1 A_n^{\ell,i} \right] z^{n+\ell}. \tag{3.10}
\]
To evaluate this we enlarge the summation interval for \( i \) from \( \ell + 1 \leq i \leq m + \ell - n \) to \( 0 \leq i \leq m' \) and subtract the contributions of \( 0 \leq i \leq \ell \) and \( m + \ell - n + 1 \leq i \leq m' \). More precisely, defining
\[
\alpha(z) = \sum_{n=0}^{m'} \sum_{i=0}^{m'} \left( \frac{z^i}{z} \right)^{\ell+n} \begin{bmatrix} (i - \ell) & n \end{bmatrix}_{\Lambda_1} - (n - \ell) \Lambda_1 A_n^{\ell,i} \right] z^{n+\ell}, \tag{3.11}
\]
\[
\beta(z) = \sum_{n=0}^{m'} \sum_{i=0}^{\ell} \left( \frac{z^i}{z} \right)^{\ell+n} \begin{bmatrix} (i - \ell) & n \end{bmatrix}_{\Lambda_1} - (n - \ell) \Lambda_1 A_n^{\ell,i} \right] z^{n+\ell}, \tag{3.12}
\]
and
\[
\gamma(z) = \sum_{n=0}^{m'} \sum_{i=m + \ell - n + 1}^{m' + \ell - n} \left[ (i - \ell) \Lambda_0^P \Lambda_n - (n - \ell) \Lambda_0^P A_n^{\ell} \right] z^{n+\ell}. \tag{3.13}
\]
we have
\[
h_{0.P}(z', z) = \sum_{\ell=0}^{m'} \left( \frac{z^\ell}{z} \right) \left[ \alpha(z) - \beta(z) - \gamma(z) \right]. \tag{3.14}
\]
It is easily seen that \( \alpha(z) = 0 \) by interchanging \( n \leftrightarrow i \) in the sum of the second term of the summand. Similarly, as in [22], we can show that for \( n \leq \ell \), so that \( i \geq m + 1, m' \), the sum over \( i \) vanishes. This leaves
\[
\gamma(z) = \sum_{n=m + \ell - n + 1}^{m'} \sum_{i=m + \ell - n + 1}^{m'} \left[ (i - \ell) \Lambda_0^P \Lambda_n - (n - \ell) \Lambda_0^P A_n^{\ell} \right] z^{n+\ell} = 0, \tag{3.15}
\]
which is identically zero when we interchange the order of the summations over \( i \) and \( n \) for the first term in the summand (while noting again \( \Lambda_0^P = 0 \) if \( m' = m + 1 \)) and make the interchange \( n \leftrightarrow i \) for the second term.

Consequently, the only nonvanishing term is
\[
h_{0.P}(z', z) = - \sum_{\ell=0}^{m'} \left( \frac{z^\ell}{z} \right) \beta(z)
\]
\[
= \sum_{n=0}^{m'} \sum_{i=0}^{\ell} z^{n+\ell} \left[ (n \Lambda_0^P A_n - i \Lambda_0^P A_n^{\ell}) \sum_{\ell=0}^{m'} \left( \frac{z^\ell}{z} \right) \beta(z)
\]
\[
= \left( \Lambda_1^P \Lambda_n - \Lambda_0^P A_n^{\ell} \right) \sum_{\ell=0}^{m'} \left( \frac{z^\ell}{z} \right) \beta(z). \tag{3.16}
\]
Using
\[
\sum_{\ell=0}^{m'} u^\ell = \frac{u^{m'+1} - u^j}{u - 1}, \quad \sum_{\ell=0}^{m'} \xi u^\ell = \frac{d}{du} \sum_{\ell=0}^{m'} u^\ell,
\] (3.17)
and
\[
\sum_{j=0}^{m} \Lambda_j P_j = P_P(u), \quad \sum_{j=0}^{m} j \Lambda_j Q_j = P_Q(u), \quad P_Q(z_\ell) = 0,
\] (3.18)
we find
\[
h^{0,p}(z', z) = \frac{z' P_P(z) }{z' - z} + \frac{z' P_Q(z) }{(z' - z)^2} P_P(z).
\] (3.19)
Substituting (3.4) and (3.19) into (3.3), we find
\[
\langle \Omega | E_{j,0}^{-} E_{i,p}^{+} | \Omega \rangle = -\beta_j^{0} \Lambda_j \Lambda_o \frac{h^{0,p}(z', z)}{\Lambda_0}.
\] (3.20)
From (III.17) and (IV.68) we have
\[
P_P(z) = \Lambda_m \prod_{k=1}^{m} (z - z_k), \quad \beta_{i,0} = -\frac{\Lambda_0}{\Lambda_m z_i} \prod_{k=1, k \neq i}^{m} \frac{1}{(z_i - z_k)}.
\] (3.21)
Substituting these into (3.20) we obtain
\[
\langle \Omega | E_{j,0}^{-} E_{i,p}^{+} | \Omega \rangle = \prod_{j=1, j \neq i}^{m} \frac{(z_j - z_i)}{(z_j - z_i)} \prod_{j=1, j \neq i}^{m} \frac{(z_j - z_i)}{(z_j - z_i)}.
\] (3.22)
This is exactly the form in (3.1) with $W_1 = \{ \ell \}$ and $W_1' = \{ j \}$.

Because of symmetry (3.8), we find from (3.3) that
\[
\langle \Omega | E_{j,0}^{-} E_{i,p}^{+} | \Omega \rangle = -\beta_j^{0} \Lambda_j \Lambda_o \beta_{i,0} \frac{h^{0,p}(z', z)}{\Lambda_0}.
\] (3.23)

### 3.2. Proof by induction

Now we prove the proposition by induction. The idea of the proof is inspired by reading paper [12] by Baxter. Denote
\[
\psi_n(W, W') = \langle \Omega | \prod_{j \in W} E_{j,0}^{+} \prod_{j \in W'} E_{j,p}^{+} | \Omega \rangle,
\]
\[
\tilde{\psi}_n(W, W') = \langle \Omega | \prod_{j \in W} E_{j,0}^{+} \prod_{j \in W'} E_{j,p}^{+} | \Omega \rangle.
\] (3.24)
We have so far shown that for $n = 1$, the following holds:
\[
\psi_n(W, W') = \frac{\tilde{A}_{i,x} \tilde{B}_{i,x}}{C_i D_i},
\] (3.25)
\[
\tilde{\psi}_n(W, W') = \psi_n(W, W') \prod_{j \in W} z_j^{-1} \prod_{j \in W'} z_j'.
\] (3.26)
where $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ are given in (3.2). Substituting these we can summarize (3.24) as

$$\psi_n(W_n, W'_n) = \prod_{i\in W_n} \prod_{j\in V_n} (z_i - z_j') \prod_{k\in W_n} \frac{\phi(z_k)}{\phi(z_k')},$$

choosing $\phi(z) = 1$ for $\psi_n(W_n, W'_n)$ and $\phi(z) = 1/z$ for $\tilde{\psi}_n(W_n, W'_n)$.

Assuming that (3.1), or more explicitly (3.27), holds up to $n$, we prove that it also holds for $n+1$.

### 3.3. Proof for $n+1$

Consider

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = \langle \Omega | \prod_{j \in \tilde{W}_n} E^{-}_{0, j, Q} E^{+}_{0, j, P} \prod_{j \in W_n} E^{+}_{j, P} | \Omega \rangle,$$

with $W_{n+1} = \{W_n, \ell\}$ and $W'_{n+1} = \{W'_n, \ell\}$, so that $V_{n+1} = V_n/\{k\}$ and $V'_{n+1} = V'_n/\{\ell\}$, where $V/U = \{j \in V, j \notin U\}$. The operators $E^{+}_{j, Q}$ and $E^{-}_{j, Q}$ were defined for $Q = 0$ in [21] and for general $Q$ in [23]. For fixed $Q$ they satisfy the commutation relations in (II.15) and generate a direct sum of spin-1/2 representations of $sl_2$ algebras, see also the appendix of [21].

Because this implies $(E^{+}_{j, P})^2 = 0$ and $(E^{-}_{0, j, Q})^2 = 0$, we find that

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = 0 \quad \text{if} \quad \ell \in W'_n \quad \text{or} \quad k \in W_n.$$

From (2.6), we see that with each of the roots $z'_j$ of the polynomial $P_Q(z)$, we associate an operator $E^{-}_{j, Q}$. Thus, $\ell \in W'_n$ is equivalent to $z'_\ell = z_j$ for some $j \in W'_n$. Similarly, $k \in W_n$ corresponds to $z_k = z_j$ for some $j \in W_n$.

If we let $E^{-}_{i, Q} \to E^{-}_{i, P}$, (or $Q = P, i = \ell$), then we must have $z'_\ell = z_j$. There are three possibilities: if $i \notin W_{n+1}$, or equivalently $i \in V_{n+1}$, then $\psi_{n+1}(W_{n+1}, W'_{n+1}) = 0$, due to the commutation relation $[E^{+}_{j, P}, E^{-}_{k, P}] = \delta_{k, j} H_{\ell, P}$. If, however, $E^{-}_{i, Q} \to E^{-}_{i, P}$, making $z'_\ell = z_k$, then

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = \langle \Omega | \prod_{j \in \tilde{W}_n} E^{-}_{0, j, Q} E^{+}_{0, j, P} \prod_{j \in W_n} E^{+}_{j, P} | \Omega \rangle = \psi_n(W_n, W'_n),$$

as $H_{\ell, P}|\Omega\rangle = -|\Omega\rangle$. The third case $i = j \in W_n$ is similar making $z'_\ell = z_j$ leading to a reduction to $\psi_n(W_n, W'_n)$, with $W_n = W'_n/\{\ell\}$.

Likewise, if $E^{+}_{i, P} \to E^{+}_{i, Q}$, or $z'_\ell = z'_j$, for $i \in V'_{n+1}$, then $\psi_{n+1}(W_{n+1}, W'_{n+1}) = 0$. For $z_k = z'_j$ we have $\psi_{n+1}(W_{n+1}, W'_{n+1}) = \psi_n(W_n, W'_n)$, whereas for $i = j \in W'_n$ we have another reduction to $\psi_n(W_n, W'_n)$, with $W'_n = W'_n/\{k\}$.

To satisfy all these conditions, we must have

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = \prod_{j \in \tilde{W}_n} z'_\ell - z'_j \prod_{j \in W_n} \frac{z'_\ell - z_j}{z_k - z_j} \prod_{j \in V_{n+1}} \frac{z'_\ell - z_j}{z_k - z_j} \prod_{j \in V_{n+1}} \frac{z'_\ell - z'_j}{z_k - z'_j} \phi(z_k) \psi_n(W_n, W'_n),$$

(3.31)

where $\phi(z)$ is a function to be determined. Note that (3.31) has all the desired zeros and that all fractions in it equal 1 for $z'_k = z_k$. Because of the reduction to (3.22) for $n = 1$, there can only be simple zeros. Setting $n=0$, noting that $\psi_0(\theta, \bar{\theta}) = 1$ is the normalization of the state $|\Omega\rangle$, we would at first have found the more general

$$\psi_1(W_1, W'_1) = \prod_{j \neq k} z'_k - z'_j \prod_{j \neq \ell} z'_\ell - z'_j \Phi(z_k, z'_k),$$

(3.32)
instead of (3.31) for this case. But different reductions from \( \psi_n \)'s with \( n > 1 \) down to \( n = 0 \) lead to consistency conditions that are satisfied when \( \Phi(z_i', z_j') = \phi(z_i)/\phi(z_j) \).

So far in this subsection, the reasoning works for both \( \psi_n \) and \( \tilde{\psi}_n \), so that (3.31) is valid for both. However, we still have to compare with (3.22) and (3.23). This leads to the conclusion that we must have \( \phi(z) = 1 \) for \( \psi_n \) and \( \phi(z) = 1/z \) for \( \tilde{\psi}_n \).

Because (3.27) is assumed to hold for \( n \), we substitute it into the above equation (3.31).

Using \( \mathcal{V}_n' = \{V_{n+1}, \ell \} \), \( \mathcal{W}_{n+1} = \{W_n, k \} \), we have

\[
\prod_{i \in \mathcal{W}_n'} \prod_{j \in \mathcal{V}_n'} (z_i - z_j') \prod_{j \in \mathcal{V}_n'} (z_i - z_j) = \prod_{i \in \mathcal{W}_n} \prod_{j \in \mathcal{V}_n} (z_i - z_j' - z_j),
\]

and we can write six similar relations. Thus, we find

\[
\psi_{n+1}(\mathcal{W}_{n+1}, \mathcal{W}_{n+1}') = \prod_{i \in \mathcal{W}_{n+1}} \prod_{j \in \mathcal{V}_{n+1}} (z_i - z_j) \prod_{j \in \mathcal{V}_{n+1}} (z_i - z_j') \prod_{i \in \mathcal{W}_{n+1}} \phi(z_i).
\]

This then completes the proof by induction, establishing (3.25) and (3.26).

### 3.4. Inner products

Using (3.1), (3.24), (3.26) and (2.23), we find that (2.19) and (2.20) become

\[
\langle \psi_n' | \psi_n' \rangle = \mathcal{C} \mathcal{D}_{PQ}, \quad \langle \psi_n^p | \psi_n^p \rangle = \mathcal{C} \mathcal{D}_{QP}, \quad \mathcal{D}_{QP} = \mathcal{D}_{PQ},
\]

where

\[
\mathcal{D}_{PQ} = \sum_{s} \sum_{s'} u_1' u_2' \cdots u_n' \frac{\tilde{\mathcal{A}}_{s,s'} \tilde{\mathcal{B}}_{s,s'} v_1' \cdots v_n'}{\mathcal{C}_{s} \mathcal{D}_{s'}}.
\]

### 3.5. Comparison with Baxter’s sum

Since \( c = (z + 1)/(z - 1) \), we find

\[
z_i - z_j' = \frac{-2(c_i - c_j')}{(c_i - 1)(c_j' - 1)}, \quad z_i' - z_j = \frac{-2(c_i' - c_j)}{(c_i' - 1)(c_j - 1)},
\]

and similar expressions for \( z_i - z_j \) and \( z_i' - z_j' \), so that

\[
\frac{\tilde{\mathcal{A}}_{s,s'}}{\tilde{\mathcal{C}}_s \mathcal{D}_{s'}} = \frac{(-2)^{n(m-n)} \tilde{A}_{s,s'}}{(2)^{n(m-n)} \tilde{B}_{s,s'}} \frac{(2)^{n(m-n)} \mathcal{C}_s \mathcal{D}_s}{(-2)^{n(m-n)} \tilde{D}_{s,s'}}
\]

and

\[
\frac{\tilde{\mathcal{A}}_{s,s'} \tilde{\mathcal{B}}_{s,s'}}{\tilde{\mathcal{C}}_s \mathcal{D}_{s'}} = \frac{A_{s,s'} B_{s,s'} v_1' \cdots v_n'}{C_{s} \mathcal{D}_{s'}} \prod_{i \in \mathcal{W}} (c_i' - 1)^{m-n} \prod_{j \in \mathcal{V}} (c_j - 1)^{m-n}.
\]
For \( m = m' \), the \( c_j - 1 \) and \( c'_j - 1 \) factors in (3.39) cancel out. For \( m' = m + 1 \), we find from (2.16) and (2.18) that

\[
\frac{c_j - 1}{z_j - 1} = \frac{2}{2k'} = \frac{(1 - k')^2 - \lambda_j^2}{2k'},
\]

(3.40)

\[
\frac{u_i}{c_j - 1} = \frac{-2k'}{(\lambda_j + 1)^2 - k'^2}, \quad u'_j(c'_j - 1) = \frac{(\lambda'_j - 1)^2 - k^2}{2k'}.
\]

(3.41)

Comparing with (BaxIV.A.1), one can see that the results agree. This shows that we have obtained the sum in (BaxIII.3.48) by a completely different route. We can now finish the calculation of the order parameter following Baxter [11, 12] closely, apart from some subtle differences due to different choice of parameters. Therefore, we will not present the details of our calculation in the next subsection.

3.6. Cauchy determinant

In the most recent paper [12], Baxter has proven that the sum in (1.18) is a determinant. In the same way, we may write (3.36) as

\[
\hat{D}_{PQ} = \det[1_m + Y B Y^* B^*],
\]

(3.42)

where we may choose to use either the variables \( c_i, c'_i, y_j \) and \( y'_j \) of Baxter [10] or our variables \( z_j, z'_j, u_j \) and \( u'_j \). With the latter choice we have matrix elements

\[
B_{ij} = \frac{f_i f'_j}{z_i - z'_j}, \quad Y_{ij} = \delta_{ij} u_j, \quad Y'_{ij} = \delta_{ij} u'_j,
\]

(3.43)

with \( u_i \) and \( u'_j \) given in (2.18). The constants \( f_i \) and \( f'_i \) in this case are related to the Drinfeld polynomials via

\[
f_i^2 = \frac{\epsilon a_i}{b_i}, \quad f'_i^2 = -\frac{\epsilon a'_i}{b'_i}, \quad \epsilon = \pm 1,
\]

\[
a_i = \prod_{j=1}^{m} (z_i - z'_j) = P_q(z_i)/\Lambda^Q_m, \quad a'_i = \prod_{j=1}^{m} (z'_i - z_j) = P_p(z'_i)/\Lambda^P_m,
\]

(3.44)

\[
b_i = \prod_{j=1, j \neq i}^{m} (z_i - z_j) = P'_q(z_i)/\Lambda^Q_m, \quad b'_i = \prod_{j=1, j \neq i}^{m} (z'_i - z'_j) = P'_p(z'_i)/\Lambda^Q'_m,
\]

(3.44)

so that \( B \) is orthogonal in the sense that

\[
B^\dagger B = 1_m \quad \text{if} \quad m \geq m', \quad B B^\dagger = 1_m \quad \text{if} \quad m \leq m',
\]

(3.45)

see subsections 6.1 and 6.2 of [9]. The result is not dependent on using \( c \)’s or \( c' \)’s.

Again, comparing with Baxter we have to take account of \( z_j = 1/w_j \), (so that \( c_j \) still lies in the interval \((-1, 1)\)). For \( P = 0 \), both \( z_j \) and \( 1/z_j \) are roots of the polynomial \( P_0(z) \). From (2.6) we can see that to assign \( 1/z_j \) instead of \( z_j \) to \( E_{i,p}^+ \) is merely a change of convention. For \( P > 0 \), if \( z_j \) is a root of \( P_0(z) \), then \( 1/z_j \) is a root of \( P_{N-p}(z) \), and vice versa. The particular choice in (2.6) is no longer arbitrary; it is chosen such that the \( E_{i,p}^\pm \) are to act on the ground state sector of the spin-translation quantum number \( P \). Due to this complication, there are some minus sign differences presented below.

Instead of (BaxIV.4.21), we find for \( m = m' \),

\[
\hat{D}_{PQ} = \frac{\Delta_m(\lambda, \lambda')}{\Delta_m(\lambda^2, \lambda^{2'})} \prod_{i=1}^{m} \frac{2}{(1 + k' + \lambda_i)(1 - k' + \lambda'_i)},
\]

(3.46)
where, as in (BaxIV.2.8),
\[ \Delta_{m,m'}(c, c') = \frac{\prod_{i \leq j \leq m'} (c_i - c_j) \prod_{i \leq j \leq m'} (c'_i - c'_j)}{\prod_{i=1}^{m'} \prod_{j=1}^{m} (c_i - c_j)}. \] (3.47)

Therefore, using (2.24), (2.23) and (3.35), we find for \( m = m' \)
\[ \langle \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p \rangle = \frac{\mathcal{R}(1 - k')}{\mathcal{R}(1 + k')} \prod_{j=1}^{m} \mathcal{R}(\lambda'_j) \mathcal{R}(\lambda_j), \] (3.48)
where, as in (BaxIV.3.8), we define
\[ \mathcal{R}(\lambda) = \prod_{i=1}^{m} (\lambda + \lambda_i)/2. \] (3.49)

However, because \( z_j = 1/w_j \), instead of (BaxIV.5.8) and (BaxIV.5.12), we have here
\[ \mathcal{R}(1 - k') = (1 - k')^{m-m'+(P-Q)/N}, \quad \mathcal{R}(1 + k') = (1 + k')^{(Q-P)/N}. \] (3.51)

As a consequence, we have from (3.48) the spontaneous magnetization given as
\[ (\mathcal{M}_l)^2 = \langle \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p \rangle = (1 - k^2)^{(P-Q)(N-P+Q)/2} = (1 - k^2)^{(N-r)/N^2}. \] (3.52)

For \( m' = m + 1 \), we find the same result as (BaxIV.A.17), namely
\[ \hat{D}_{PQ} = \frac{\Delta_{m,m+1}(\lambda', \lambda)}{\Delta_{m,m+1}(\lambda^2, \lambda^2)} \prod_{i=1}^{m} \frac{2}{(1 + \lambda_i)^2 - k^2}, \] (3.53)
so that
\[ (\mathcal{M}_l)^2 = \langle \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p | \hat{Y}'_p \rangle = C\hat{D}_{PQ}^2 \]
\[ = \frac{1}{\mathcal{R}(1 - k')\mathcal{R}(1 + k')} \prod_{j=1}^{m} \mathcal{R}(\lambda'_j) \mathcal{R}(\lambda_j) = (1 - k^2)^{(N-r)/N^2}, \] (3.54)
which is the same as (3.52). Thus, from (3.52) and (3.54), we see that the \( N \) terms in the sum given in (1.17) are all equal in the thermodynamic limit.

4. Summary and outlook

In this paper, we have discussed two approaches to the pair correlation function in the chiral Potts model. In subsection 1.2, we noted that the ground state eigenvectors of [21] suffice for the calculation of the pair correlation in the superintegrable chiral Potts chain in the commensurate phase. In fact, to calculate its correlation \( \langle \hat{Z}'_l \hat{Z}'_{l+1} \rangle \), we need to evaluate
\[ \langle \Omega \prod_{k=1}^{n} E_{i_k, p}(Z_{i_k} \cdots Z_{i_k}) \prod_{k=1}^{n} E_{i_k, p}^*(Z_{i_k}) \rangle, \] (4.1)
which we do not know how to evaluate yet for arbitrarily large system size. However, for \( N = 3 \) and \( L = 3 \), it yields a result identical to that of Fabricius and McCoy [32]. In subsection 1.1, we explained that the knowledge of the transfer matrix eigenvectors of the superintegrable chiral Potts should suffice for the calculation of the pair correlation in the more general integrable chiral Potts model.
In our previous papers [21, 22], we have determined $2^m \omega$ eigenvectors for each $Q$ sector and they are given here in (2.1). We have learned to calculate inner products of such states chosen from different $Q$ sectors by evaluating the inner product of the state with $n$ creation operators $E_j^+$ acting on $|\Omega\rangle$ with one with $n$ annihilation operators $E_j^-$ acting on $|\Omega\rangle$ given in (3.1). We have used these inner products to calculate the pair correlation function in the large separation limit, which gave us the order parameter. This derivation uses (3.7), which in earlier preprint versions of this paper had to be left as a conjecture, but finally got proved [36] in a rather lengthy manner.

However, if we want to go further and study the separation dependence of the pair correlation function using (1.16), we will need to know the inner products for all eigenvectors $|Y_j\rangle$ that are not orthogonal to the maximum eigenvectors. One way is to first calculate all eigenvectors of the superintegrable $\tau_2$ model and the related higher spin XXZ model by Bethe ansatz working out the prescription of Tarasov [37–39], and then to make proper linear combinations in each degenerate eigenvalue space [39] as we have done [21, 22] for the ground state sector using Onsager algebra and quantum loop algebra. But it will be very hard to work out all necessary details this way. We believe that there should be a better way using more of the combinatorial structure of the problem, which we may start exploring using the approach proposed in subsection 1.2, for which we have the needed eigenvectors explicitly.

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Appendix. Generating function $G(t, u)$

The generating functions $g$ and $\bar{g}$ are defined in (II.62), (III.14) and (III.15) as

$$g(|n_j\rangle, t) = \sum_{m=0}^{(N-1)L-N} K_m(|n_j\rangle)t^m,$$

$$\bar{g}(|n_j\rangle, t) = \sum_{m=0}^{(N-1)L-N} \bar{K}_m(|n_j\rangle)t^m. \tag{A.1}$$

It was shown in [22] that they have the simple form

$$g(|n_j\rangle, t) = \bar{g}(|n_j\rangle, t) = (1 - t^N)^{L-1} \prod_{j=1}^{L} (1 - t\omega^j)^{-1}. \tag{A.2}$$

We now define the two-variable generating function

$$G(t, u) = \sum_{\{0 \leq n_j \leq N-1\} \atop {n_1 + \cdots + n_L = N}} \tilde{g}(|n_j\rangle, t) g(|n_j\rangle, u). \tag{A.3}$$

Substituting (A.1) into the above equation and comparing with (3.6), we find

$$G(t, u) = \sum_{\ell=0}^{N(r-1)} \sum_{k=0}^{N(r-1)} G_{\ell,k} t^\ell u^k, \quad r = (N-1)L/N. \tag{A.4}$$
From Baxter’s paper [10], we realized that the coefficients $G_{\ell;k}$ must be related to the coefficients of the Drinfeld polynomials (1.21). However, the method that we used in [22] cannot be used, as is shown next.

**A.1. Difficulties for $P \neq Q$**

From (3.6) and (2.8), we may write

$$G_{N+Q,N+P} = \sum_{\mu_i \in P} \sum_{\lambda_i \in Q} \prod_{j=1}^{L} \left[ \frac{n_j + \lambda_j}{n_j} \right] \alpha^{n_j(N'_j + \bar{b}_j)},$$

(A.5)

where $b_j = \sum_{i<j} \lambda_i$, and $N'_j = \sum_{i<j} n'_i$. Let $\mu_i = n_i + n'_i$, so that $\sum \mu_i = (\ell + 1)N + Q$. Then we find

$$G_{N+Q,N+P} = \sum_{\mu_i \in P} \sum_{\lambda_i \in Q} \prod_{j=1}^{L} \left[ \frac{\mu_j}{n_j} \right] \alpha^{n_j(N'_j + \bar{b}_j)},$$

(A.6)

where $a_j = \sum_{i<j} \mu_i$, and $N_j = \sum_{i<j} n_i$ and

$$\mathcal{I}_n([\mu_i]; [\lambda_i]) \equiv \sum_{\mu_i \in P} \prod_{j=1}^{L} \left[ \frac{\mu_j}{n_j} \right] \alpha^{n_j(N'_j + \bar{b}_j)},$$

(A.7)

see (III.22). Using $a_{\ell+1} = \sum \mu_i = (\ell + 1)N + Q$ and $\bar{b}_0 = \sum \lambda_i = jN + P$, we find from (III.28), (III.33), (III.34) and (III.35) the relation

$$\sum_{n=0}^{a_{\ell+1}} (-1)^n \omega^{1/2} \mathcal{I}_n([\mu_i]; [\lambda_i]) t^n = (\omega^{1/2} t; \omega)_N^{-P+Q}(1 + t^N)^{\ell-j} \sum_{n=0}^{\bar{b}_0} (-1)^n \omega^{1/2} \tilde{\mathcal{I}}_n([\lambda_i]; [\mu_i]) t^n,$$

(A.8)

where

$$\tilde{\mathcal{I}}_n([\lambda_i]; [\mu_i]) \equiv \sum_{\mu_i \in P} \prod_{j=1}^{L} \left[ \frac{\lambda_j}{n_j} \right] \alpha^{n_j(N'_j - \bar{b}_j) + \sigma a_0},$$

(A.9)

as in (III.26). By equating the coefficients of $t^n$ on both sides of (A.8), we relate the sums $\mathcal{I}_n$ and $\tilde{\mathcal{I}}_n$. For $P = Q$, this process simplifies as $(\omega^{1/2} t; \omega)_N = 1 + t^N$ in (A.8). For $P \neq Q$, it is more complicated, but we can still use (A.8) to relate $\mathcal{I}_n$ with $\tilde{\mathcal{I}}_n$ and other $\tilde{\mathcal{I}}_n$’s with the $n < N$ coefficients calculated from (III.11). Thus, we obtain

$$G_{N+Q,N+P} = (\ell - j) \Lambda^{Q}_{\ell+1} \Lambda^P_{\ell+1} + G_{(\ell+1)N+Q,(j-1)N+P} + U^{Q,P}_{\ell,j},$$

(A.10)

where

$$U^{Q,P}_{\ell,j} = \sum_{k=0}^{Q} \left[ \frac{N-P+Q}{Q-k} \right] \omega^{k+2P} \sum_{\mu_i \in P} \sum_{\lambda_i \in Q} \mathcal{I}_{P-k}([\mu_i]; [\lambda_i]).$$

(A.11)
We have changed the \( \tilde{I}_\ell \)'s to \( I_\ell \)'s by changing the summation variables to \( \lambda_i = \lambda_i' + n_i \) and \( \mu_j = \mu_j' - n_j \), where the \( n_i \)'s are the summation variables in (A.7) and (A.9).

For \( P = Q = 0 \), the only nonzero term is \( k = Q = P \). Since \( I_0([\mu_l']; [\lambda_m']) = 1 \), we find

\[
\mathcal{U}^Q = \sum_{[\mu_l'] \subseteq \mathbb{C}^k} \sum_{[\lambda_m'] \subseteq \mathbb{C}^k} 1 = \Lambda^Q_{\ell+1} \Lambda^Q_j, \tag{A.12}
\]

so that (A.10) becomes identical to (III.43) as it should be.

For \( P > Q \), we have to keep using (A.8) to express each \( I_\ell ([\mu_j'); [\lambda_m']) \) in (A.11) in terms of \( I_\ell ([\mu_j]; [\lambda_m]) \) with \( \ell \leq k \) and \( n = \sum \lambda_j \leq \sum \lambda'_j \), as we have done once before to arrive at (A.10), continuing until \( n \leq k \), such that \( I_\ell = 0 \). Let

\[
\sum_{n} \frac{1}{\Lambda^{Q}_n} c_{mN+Q} = \Lambda^Q_n. \tag{A.13}
\]

Since \( I_0 = 1 \), we find that it is possible to express \( \mathcal{U}^{P,Q}_{\ell,j} \) as a sum of \( c_{(\ell+1)N+P+M} c_{N-M} \). However, this is very tedious and messy. Using Maple we have discovered (and checked for different \( P, Q \) and \( N \) values and small system sizes \( L \)) that the following equation holds:

\[
\mathcal{U}^{P,Q}_{\ell,j} = \sum_{n=0}^{j} \Lambda^P_{\ell+1+j-n} \Lambda^Q_{\ell+1+j-n} - \sum_{n=0}^{j-1} \Lambda^P_{n} \Lambda^Q_{\ell+1+j-n}, \tag{A.14}
\]

so that

\[
G_{(N+Q)N+P} = \sum_{m=0}^{j} \left( (\ell + j + 2n) \Lambda^Q_{\ell+1+n} \Lambda^P_{\ell+1+n} + \sum_{n=0}^{j-n} \Lambda^Q_{\ell+1+j-n} \Lambda^P_{\ell+1+j-n} - \sum_{m=0}^{j-n-1} \Lambda^Q_{\ell+1+j-m} \Lambda^P_{\ell+1+j-m} \right) \tag{A.15}
\]

which is the identity in (3.7). In the first preprint version [40] of this work, (A.14) had to be left as a conjecture. In the meantime, a lengthy proof of (A.15) has been found, which is presented elsewhere [36].

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