Comments on compositeness in the SU(2) linear $\sigma$ model

M. D. Scadron*
Department of Physics,
University of Tasmania,
Hobart, Australia, 7005

Abstract

First we summarize the quark-level linear $\sigma$ model compositeness conditions and verify that indeed $m_\sigma = 2m_q$ when $m_\pi = 0$ and $N_c = 3$, rather than in the $N_c \to \infty$ limit, as is sometimes suggested. Later we show that this compositeness picture also predicts a chiral symmetry restoration temperature $T_c = 2f_\pi$, where $f_\pi$ is the pion decay constant. We contrast this self-consistent $Z = 0$ compositeness analysis with prior studies of the compositeness problem.

PACS # 11.15.Pg, 11.30.Rd To appear in PRD

* Permanent address: Physics Department, University of Arizona, Tucson, AZ 85721, USA
Now that the scalar $\sigma$ meson has been reinstated in the 1996 particle data group tables [1], it is appropriate to take seriously the various theoretical implications of a quark-level linear $\sigma$ model (L$\sigma$M) field theory. The original spontaneously broken L$\sigma$M theory [2] was recently dynamically generated [3] at the quark level in the spirit of Nambu-Jona-Lasinio [4]. In this note we summarize the color number $N_c$ and compositeness properties of the above SU(2) quark-level L$\sigma$M and comment on the recent L$\sigma$M analysis of compositeness given by Lurie and Tupper [5].

First we display the interacting part of the standard L$\sigma$M [2] (quark-level) lagrangian density shifted around the true vacuum $\langle \vec{\pi} \rangle = \langle \sigma \rangle = 0$:

$$L_{\text{int}} = g \bar{\psi}(\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})\psi + g'(\sigma^2 + \vec{\pi}^2)\sigma - (\lambda/4)(\sigma^2 + \vec{\pi}^2)^2 - f_\pi g \bar{\psi}\psi, \quad (1a)$$

with (spontaneously broken) chiral couplings

$$g = m_q/f_\pi, \quad g' = m_\sigma^2/2f_\pi = \lambda f_\pi. \quad (1b)$$

Once the L$\sigma$M scalar field is shifted to $\langle \sigma \rangle = 0$, giving rise to the interacting but chiral-broken L$\sigma$M lagrangian (1), the Lee null-tadpole condition [6] depicted in fig. 1 must be valid. Following ref. [3] which exploits the dimensional regularization [7] characterization of these quadratic divergent tadpole graphs in fig. 1 as $\int d^4p(p^2 - m^2)^{-1} \sim m^2$, one expresses the Lee condition as

$$0 = -4m_q N_f N_c g \cdot m_q^2 + 0 + 3g' \cdot m_\sigma^2, \quad (2a)$$

where the zero on the rhs of (2a) corresponds to $m_\pi^2 = 0$ in the chiral limit. Upon using eqs. (1b), this Lee null-tadpole condition (2a) becomes

$$\frac{1}{2} N_f N_c (2m_q)^4 = 3m_\sigma^4. \quad (2b)$$

Clearly if the NJL relation [4]

$$m_\sigma = 2m_q \quad (3)$$

is valid, then (2b) requires

$$N_f N_c = 6, \quad (4)$$

or $N_c = 3$ when $N_f = 2$, the latter being an input in the SU(2) L$\sigma$M.

It is well known that for $\pi^0 \rightarrow 2\gamma$ decay, the $N_f = 2$ quark triangle empirically suggests $N_c = 3$ (also a L$\sigma$M result). Moreover eq.(4) also follows from
“anomaly matching” [8,9]. However we shall not invoke here the stronger (but consistent) constraints due to dynamically generating the (quark-level) LσM as they follow from comparing quadratic and logarithmically divergent integrals using (compatible) regularization schemes [3].

Thus the condition (4) depends on the NJL relation (3) being also true in the LσM. The latter assertion follows when one dynamically generates [3] the entire LσM lagrangian (1) starting from a simpler chiral quark model (CQM) lagrangian, as well as dynamically generating the two additional equations

\[ m_\sigma = 2m_q, \quad g = 2\pi/\sqrt{N_c}. \] (5)

For \( N_c = 3 \), the latter pion-quark coupling in (5) is \( g = 2\pi/\sqrt{3} \approx 3.63 \), near the anticipated value found from the \( \pi NN \) coupling \( g_{\pi NN} \sim 13.4 \) so that \( g \approx g_{\pi NN}/3g_A \sim 3.5 \). Then the nonstrange constituent quark mass is \( m_q = f_\pi 2\pi/\sqrt{3} \approx 326 \text{ MeV} \), near \( M_N/3 \) as expected. But rather than repeating ref. [3] in detail, we offer an easier derivation of \( m_\sigma = 2m_q \) following only from the quark loops induced by the CQM lagrangian. This naturally leads to the notion of “compositeness”.

To this end, we invoke the log-divergent gap equation from fig. 2

\[ 1 = -i4\frac{1}{2}N_fN_c g^2 \int d^4 p (p^2 - m_q^2)^{-2}, \] (6)

where \( d^4 p = (2\pi)^{-4}d^4p \). Equation (6) is the chiral-limiting one-loop nonperturbative expression of the pion decay constant \( f_\pi = m_q/g \) with the quark mass \( m_q \) cancelling out. This LσM log-divergent gap equation (6) also holds in the context of the four-quark NJL model [10]. Then the one-loop-order \( g_{\sigma\pi\pi} \) coupling depicted in fig. 3 is

\[ g_{\sigma\pi\pi} = 2gm_q \left[ -i4\frac{1}{2}N_fN_c g^2 \int d^4 p (p^2 - m_q^2)^{-2} \right] = 2gm_q \] . (7)

The one-loop \( g_{\sigma\pi\pi} \) in (7) “shrinks” to the tree-order meson-meson coupling in (1b), \( g' = m_\sigma^2/2f_\pi \), only if \( m_\sigma = 2m_q \) is valid along with the GTR \( f_\pi g = m_q \). This is a \( Z = 0 \) compositeness condition [11], stating that the loosely bound \( \sigma \) meson could be treated either as a \( \bar{q}q \) bound state (as in the NJL picture) or as an elementary particle as in the LσM framework of fig.3. But in either case \( m_\sigma = 2m_q \) must hold and therefore the additional LσM Lee condition (2) requires \( N_c = 3 \) when \( N_f = 2 \) in (4).
It is also possible to appreciate the one-loop order $Z = 0$ compositeness condition in the context of the LσM [3] in a different manner. Our version of the $Z = 0$ compositeness condition is that the log-divergent gap equation (6) can be expressed in terms of a four-dimensional UV cutoff as

$$1 = \ln(1 + \Lambda^2/m^2_q) - (1 + m^2_q/\Lambda^2)^{-1},$$

where we have substituted only $g = 2\pi/\sqrt{N_c}$ and $N_f = 2$ into (6) in order to deduce (8). The numerical solution of (8) is the dimensionless ratio $\Lambda/m_q \approx 2.3$, which is slightly larger than the NJL ratio in (3) or in (5), $m_\sigma/m_q = 2$. Introducing the above dynamically generated quark mass of 326 MeV, the UV cutoff inferred from (8) (i.e. from (6)) is $\Lambda \approx 2.3m_q \approx 750$ MeV. This 750 MeV cutoff in turn suggests (in the LσM) that lighter masses signal elementary particles, such as $m_\pi = 0$, $m_q \approx 325$ MeV, $m_\sigma = 2m_q \approx 650$ MeV. Heavier meson masses than 750 MeV signal $\bar{q}q$ bound states, such as $\rho(770), \omega(783), A_1(1260)$, etc. This is the essence of the $Z = 0$ compositeness conditions of refs. [11].

Given the above eqs. (3)-(8), we are now prepared to comment in detail on the LσM compositeness analysis of ref. [5]. Again using the log-divergent cutoff condition (8), the LσM renormalization constant $Z_3$ computed in eq. (3) of ref. [5] can be expressed as

$$Z_3 = 1 - \frac{N_c g^2}{4\pi^2}. \quad (9)$$

Then the dynamically generated LσM meson-quark coupling in (5) indeed corresponds to $Z_3 = 0$ from (9), as anticipated.

However the renormalization constant $Z_4$ in ref. [5] then becomes using (8),

$$Z_4 = 1 + \left[3\lambda - \frac{2N_c g^4}{\lambda}\right] \frac{1}{4\pi^2}. \quad (10)$$

Ignoring for the moment the second term in (10) proportional to $3\lambda$, we note that the log-divergent gap equation (6) requires the $\pi\pi \rightarrow \pi\pi$ quark box (dynamically generated by the CQM lagrangian) to “shrink” (as in eq. (7) and in fig. 3) to a point contact term $\lambda$ provided that [3]

$$\lambda = 2g^2. \quad (11)$$
Equation (11) also follows from both LσM couplings [2] in (1b) combined with $g_{\sigma\pi\pi} = 2g m_q$ from (7). Substituting (11) into the third (quark loop) term in (10), one finds

$$Z_4 = 1 + 0 - \frac{N_c g^2}{4\pi^2},$$  \hspace{1cm} (12)

(where the middle zero term in (12) corresponds to the neglected meson loop in contrast to ref. [5]). Equation (12) parallels the $Z_3$ renormalization constant in (9). In these two cases

$$Z_3 = Z_4 = 1 - \frac{N_c g^2}{4\pi^2},$$ \hspace{1cm} (13)

and then the resulting compositeness conditions $Z_3 = Z_4 = 0$ both reconfirm that $g = 2\pi/\sqrt{N_c}$, as earlier dynamical generated in eqs.(5).

The reason why one must neglect the second meson loop term proportional to $3\lambda$ in (10) is because e.g. $\pi_\alpha \pi_\beta \rightarrow \pi_\gamma \pi_\delta$ scattering has tree level (or one-loop) graphs which must vanish in the strict zero momentum chiral limit. This fact was emphasized on pp 324-327 of the text by de Alfaro et al. [DFFR] in ref. [2]. Specifically the quartic LσM contact term $-\lambda$ is cancelled by the cubic $\sigma$ pole term $2g'^2/m_\sigma^2 \rightarrow \lambda$ by virtue of the Gell-Mann-Lévy LσM meson chiral couplings in (1b). After the (tree-level) lead term cancellation between contact term $\lambda$ and $s, t, u, \sigma$ meson poles in the LσM, DFFR obtain the amplitude

$$T_{\pi\pi} \propto \frac{1}{f_\pi^2} (s\delta_{\alpha\beta}\delta_{\gamma\delta} + t\delta_{\alpha\gamma}\delta_{\beta\delta} + u\delta_{\alpha\delta}\delta_{\beta\gamma}).$$  \hspace{1cm} (14)

Then DFFR in [2] note that (14) above is just the Weinberg $\pi\pi$ amplitude [12] when $m_\pi^2 = 0$, found instead via the model-independent current algebra and PCAC rather than from the linear $\sigma$ model (LσM). Also note that (14) indeed vanishes in the strict zero momentum chiral limit. A similar chiral cancellation of the $3\lambda$ term in (10) also holds in one-loop order.

When computing the one-loop order renormalization constant $Z_4$ as done by ref. [5] leading to eq. (10) above, one must be careful to (a) account for the DFFR-cancellation due to the soft chiral symmetry relation $2g'^2/m_\sigma^2 \rightarrow \lambda$, (b) reorganize the perturbation theory using the log-divergent gap equation (6) shrink quark loops to a contact meson term $\lambda$ with $\lambda = 2g^2$ as found in (11). Then even in one-loop order one must recover the Weinberg form for $\pi\pi$ scattering eq. (14) in a model-independent fashion.
This means that the meson loop graph with quartic couplings proportional to $3\lambda^2$ contributing to $\lambda Z_4$ as $3\lambda^2/4\pi^2$ in (10) will be cancelled by fermion box graphs which are of higher loop order. Although our nonperturbative approach mixes perturbation theory loops of different order, both DFFR and our use of the Gell-Mann-Lévy chiral symmetry meson relation $2g^2/m_\sigma^2 \to \lambda$ has the bonus of our nonperturbative approach retaining the consistent chiral symmetry compositeness condition $Z_3 = Z_4 = 0$ from (13).

Keeping instead the middle term in (10) proportional to $3\lambda$, ref. [5] concludes that the resulting $Z_4 = 0$ (then different) compositeness condition requires that the NJL limit $m_\sigma \to 2m_q$ is recovered only when $N_c \to \infty$. References [13] reach the same conclusion although they are not working with SU(2) chiral mesons ($\sigma, \vec{\pi}$). In our opinion however, the chiral SU(2) L$\sigma$M (1) already has $N_c = 3$ and not $N_c \to \infty$ built in via the Lee condition in eqs. (2) but only when $m_\sigma = 2m_q$ in the chiral limit. We obtain these satisfying results only by cancelling the middle $3\lambda$ meson term in (10) against higher quark loop graphs. Ref. [5] does not account for the above DFFR cancellation.

Finally we extend the above zero temperature ($T = 0$) chiral symmetry absence of quartic meson loops in eqs. (10), (12), (14) to finite temperature. Again following ref. [5] we write the tadpole equation in mean field approximation at high temperatures for the quark-level SU(2) L$\sigma$M as

$$v \left[ (3 + N_f^2 - 1)\lambda T^2/12 + N_f N_c g^2 T^2/12 + \lambda(v^2 - f_\pi^2) \right] = 0 \quad (15)$$

for flavor $N_f = 2$ and $v = v(T)$ with $v(0) = f_\pi \sim 90$MeV in the chiral limit. The first two terms in (15) represent quartic $\sigma$ and $\vec{\pi}$ loops, while the third term involving $N_c$ is the $u$ and $d$ quark bubble loop. The temperature factors of $T^2/12$ in (15) were originally obtained from finite temperature field theory Feynman rules [14].

Now in fact there should be no quartic meson loop contributions surviving in (15) due to the above DFFR-type argument or the resulting Weinberg $\pi\pi$ amplitude in (14), even at finite temperatures. So the nontrivial solution of (15) at the chiral symmetry restoration temperature $T_c$ (where $v(T_c) = 0$) is for $N_f = 2$, $N_c = 3$ and $\lambda = 2g^2$, with the first two meson loop terms in (15)
proportional to \((3 + N_f^2 - 1)\lambda\) consequently omitted,

\[ T_c = 2f_\pi \sim 180\text{MeV} \]  

(16)

While this predicted temperature scale in (16) had been obtained earlier [15,16], ref. [5] also noted (16) above but rejected it because of the meson loop contributions in (15).

We in turn claim that the first two \(\sigma\) and \(\vec{\pi}\) loop terms in (15) (and the middle term in (10) proportional to \(3\lambda\)) are all zero due to chiral cancellations as in DFFR [2]. Then (15) reduces to the nontrivial solution \(N_c g^2 T_c^2 / 6 = \lambda f_\pi^2\), (leading to \(T_c = 2f_\pi\)) or to a quark box loop shrinking to a meson-meson quartic point [3] due to the log-divergent gap equation (6), itself a version of the \(Z = 0\) compositeness condition.

Although we concur with ref. [5]'s choice of the finite temperature quark bubble sign in eq. (15) (as opposed to the studies in ref. [15]), there is an easier way to deduce \(T_c = 2f_\pi\) by studying the single fermion loop propagator dynamically generating the quark mass [3]. Then, with no sign ambiguity arising at finite temperature one finds [17]

\[ m_q(T) = m_q + \frac{8N_c g^2 m_q T^2}{-m_\sigma^2} \frac{T^2}{24}, \]  

(17)

where the \(-m_\sigma^2\) factor in (17) indicates the \(\sigma\) meson tadpole propagator generating the quark mass. When \(T = T_c\) the quark mass “melts”, \(m_q(T_c) = 0\), and (17) reduces to

\[ m_\sigma^2 = g^2 T_c^2 \quad \text{or} \quad T_c = 2f_\pi \]  

(18)

provided that \(N_c = 3\) and \(m_\sigma = 2m_q = 2f_\pi g\).

We believe it significant that recent numerical simulations of lattice gauge theories find [18] \(T_c = 150 \pm 30\text{MeV}\), consistent with (16) and (18). In fact the zero temperature quark-level \(L\sigma\text{M}\) theory in ref. [3] is likewise compatible with the reinstated scalar \(\sigma\) in the PDG tables [1] or in ref. [19], the latter deducing a broad nonstrange \(\sigma\) scalar as \(f_0\) (400–900) with mean mass \(m_\sigma \approx 650\text{ MeV}\). This latter scale is in fact predicted in ref. [3] as \(m_\sigma = 2f_\pi \frac{2\pi}{\sqrt{3}} \approx 650\text{ MeV}\).
Rather than starting at $T = 0$, an alternative approach to generating a realistic low energy chiral field theory begins at the chiral restoration temperature (with $m_q(T_c) = 0$) involving bosons $\pi$ and $\sigma$ alone [20] and later adds in the fundamental meson-quark interaction in (1). Only then does one deduce the quark-level linear $\sigma$ model (L$\sigma$M) field theory [21]. While issues of $N_c = 3$ and compositeness are then postponed, the resulting L$\sigma$M theory in ref. [21] starting at $T = T_c \sim 200 MeV$ with $\lambda \sim 20$ appears quite similar to the $T = 0$ L$\sigma$M field theory in refs. [2,3] with $\lambda \approx 26$ from (11) and $T_c \approx 180 MeV$ from (16). In effect, what goes around comes around . . . .

Acknowledgements:

This research was partially supported by the Australian Research Council. M.D.S. appreciates hospitality of the University of Western Ontario and the University of Tasmania. He also is grateful to V. Elias, D. McKeon, R. Mendel, V. Miransky and especially R. Delbourgo for insightful comments.

Figure Captions

Fig. 1 Quark and meson tadpole loops summing to zero.

Fig. 2 Quark loops for the axial current matrix element $\langle 0|A_\mu|\pi \rangle$.

Fig. 3 Chiral quark model loops for $\sigma \rightarrow \pi \pi$. 

8
References

1. Particle Data Group, R. M. Barnett et al., Phys. Rev. D54, 1 (1996).

2. M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960); also see V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, Chap. 5 in “Currents in Hadron Physics” (North Holland, Amsterdam, 1973).

3. R. Delbourgo and M. D. Scadron, Mod. Phys. Lett. A10, 251 (1995) regularize the $L\sigma M$ using dimensional regularization. This was recently extended to analytic and Pauli-Villars regularization by R. Delbourgo, A. Rawlinson and M. Scadron, submitted for publication.

4. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

5. D. Lurie and G. B. Tupper, Phys. Rev. D 47, 3580 (1993).

6. B.W. Lee, Chiral Dynamics (Gordon and Breach, 1972) p. 12.

7. See e.g. G. ’tHooft and M. Veltman, Nucl. Phys. B44, 189 (1972) and review by R. Delbourgo, Repts. Prog. Phys. 39, 345 (1976).

8. S. Adler, Phys. Rev. 177, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969). See also J. Schwinger, Phys. Rev. 82, 664 (1951).

9. G. ’t Hooft, in Recent developments in gauge theories, ed. G. ’t Hooft et al. (Plenum, NY 1980); Y. Frishman, A. Schwimmer, T. Banks and S. Yankielowicz, Nucl. Phys. B177, 157 (1981); S. Coleman, and B. Grossman, ibid B203, 205 (1982). Also see K. Huang in Quarks, Leptons and Gauge Fields (World Scientific, Singapore, 1992).

10. See e.g. V. Dmitrasinovic, H. Schulze, R. Tegen and R. Lemmer, Phys. Rev. D52, 2855 (1995). Combine their equs. (8) and (15) to obtain our gap equation (6).

11. A. Salam, Nuovo Cimento 25, 224 (1962); S. Weinberg, Phys. Rev. 130, 776 (1963).

12. S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).
13. K. Akama, Phys. Rev. Lett. 76, 184 (1996) and J. Zinn Justin, Nucl. Phys. B367, 105 (1991) consider a four-quark $U(1)_L \times U(1)_R$ simple NJL model finding $\lambda = g^2$ and $m_\sigma \to 2m_q$ when $N_c \to \infty$. But this does not correspond to the SU(2) LσM with $m_\sigma = 2m_q$ when $N_c = 3$.

14. See e.g. L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974).

15. D. Bailin, J. Cleymans and M. D. Scadron, Phys. Rev. D31, 164 (1985); J. Cleymans, A. Kocić and M. D. Scadron, ibid D32, 323 (1989).

16. See e.g. review by T. Hasuda, Nucl. Phys. A544, 27 (1992); also see M. Asaka and K. Yazaki, ibid. A504, 668 (1989); A. Barducci, R. Casalbuoni, Phys. Rev. D41, 1610 (1990).

17. N. Bilić, J. Cleymans and M. D. Scadron, Int. J. Mod. Phys. A10, 1169 (1995).

18. See e.g. reviews by B. Petersson, Nucl. Phys. (Proc. Supp) B30, 66 (1993); K. M. Bitar et al. ibid. B30, 315 (1993).

19. N. Tornqvist and M. Roos, Phys. Rev. Lett. 76, 1575 (1996).

20. K. Rajagopal and F. Wilczek, Nucl. Phys. B 404, 577 (1993); M. Asakawa, Z. Huang and X. -N. Wang, Phys. Rev. Lett. 74, 3126 (1995); J. Randrup, ibid. 77, 1226 (1996).

21. L. P. Csernai and I. N. Mishustin, Phys. Rev. Lett. 74, 5005 (1995).