MIXED FROBENIUS STRUCTURE AND LOCAL A-MODEL

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Abstract. We define the notion of mixed Frobenius structure which is a generalization of the structure of a Frobenius manifold. We construct a mixed Frobenius structure on the cohomology of weak Fano toric surfaces and that of the three dimensional projective space using local Gromov–Witten invariants. This is an analogue of the Frobenius manifold associated to the quantum cohomology in the local Calabi–Yau setting.

1. Introduction

The purpose of this paper is to introduce the notion of mixed Frobenius structure. It is a generalization of the structure of a Frobenius manifold, which plays an important role in the study of mirror symmetry. Our motivation for introducing it comes from local mirror symmetry. The main results, Theorems 8.7 and 9.3, show that local Gromov–Witten invariants in the local A-model (i.e. the A-model side of local mirror symmetry) give rise to mixed Frobenius structures.

In the rest of the introduction, we first explain the contents of the paper. Then we explain the motivation from local mirror symmetry. Throughout the paper, an algebra is an associative commutative algebra with unit over \( \mathbb{C} \) of finite dimension. We denote by \( \circ \) the multiplication of an algebra.

1.1. A generalization of Frobenius algebra. Recall that a Frobenius algebra is a pair of an algebra \( A \) and a nondegenerate bilinear form \( \langle , \rangle : A \times A \to \mathbb{C} \) satisfying the compatibility condition

\[
\langle x \circ y, z \rangle = \langle x, y \circ z \rangle \quad (x, y, z \in A).
\]

For example, the even part of the cohomology ring of a compact oriented manifold and the intersection form is a Frobenius algebra (see, e.g. [15]).

We generalize this notion as follows ([2]). Let \( A \) be an algebra. A Frobenius filtration on \( A \) is a pair of an increasing sequence of ideals \( I_k \subset A \) and a set of nondegenerate symmetric bilinear forms \( (\ ,\ )_k \) on graded quotients \( I_k/I_{k-1} \) satisfying the condition similar to (1.1):

\[
(x, a \circ y)_k = (a \circ x, y)_k \quad (x, y \in I_k/I_{k-1}, \ a \in A/I_{k-1}).
\]

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Given a Frobenius algebra $A$ with a nilpotent element $n$, there are constructions of the Frobenius filtration in $A$ and the quotient algebra $A/(n)$ where $(n)$ is the ideal generated by $n$. We call them the nilpotent construction and the quotient construction respectively.

To illustrate these constructions, we give two types of examples. First examples are the cohomology rings of compact Kähler manifolds. Second examples are Chen–Ruan’s cohomology rings of some non-compact orbifolds.

1.2. The mixed Frobenius structure. Recall that a Frobenius structure on a manifold $M$ consists of a structure of the Frobenius algebra on the tangent bundle $TM$ and a vector field $E$ on $M$ called the Euler vector field which satisfy certain compatibility conditions (see Definition 6.1). It was defined by Dubrovin but before that K. Saito found this structure in the singularity theory (see also [25]). An example of the Frobenius structure is the quantum cohomology ring (i.e. the cohomology with the quantum cup product) of a compact symplectic manifold. For other examples, see e.g. [22] and references therein.

We generalize this notion and define the mixed Frobenius structure (MFS). A mixed Frobenius structure (MFS) on a complex manifold $M$ consists of a structure of an algebra with a Frobenius filtration, a torsion-free flat connection $\nabla$ on $TM$, and an Euler vector field $E$ which satisfy compatibility conditions (see Definition 6.2). We extend the nilpotent and the quotient constructions to the MFS (§7). If given a Frobenius manifold $M$ with a vector field $n$ which is nilpotent with respect to the multiplication (and satisfies certain compatibility conditions with the metric and the Euler vector field), then we have a MFS on $M$ and one on a certain submanifold (see Theorem 7.2 and Corollary 7.3).

Finally, we apply the quotient construction to the quantum cohomology ring of the projective compactification $\mathbb{P}(K_S \oplus O_S)$ where $S$ is a weak Fano toric surface and $K_S$ is its canonical bundle, and obtain a MFS on $H^*(S, \mathbb{C})$ (§8). We also apply the quotient construction to $\mathbb{P}(K_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3})$ and obtain a MFS on $H^*(\mathbb{P}^3, \mathbb{C})$ (§9).

In the appendix (§A), we give a definition of the deformed connection and show that it is flat. We also write down the deformed flat coordinates of the above MFS’s.

1.3. Motivation from local mirror symmetry. For a Calabi–Yau threefold $X$, two Frobenius structures on $H^*(X, \mathbb{C})$ are known. The one is given by the quantum cohomology ring and the intersection form (the A-model). The other is constructed by Barannikov–Kontsevich which is closely related to the variation of Hodge structures on $H^3(X, \mathbb{C})$ (the B-model). If Calabi–Yau threefolds $X$ and $Y$ are mirror partners, it
is conjectured that the former Frobenius structure on \( H^*(X, \mathbb{C}) \) is isomorphic to the latter Frobenius structure on \( H^*(Y, \mathbb{C}) \). This conjecture was proved when \( X \) is a complete intersection in a projective space [2].

In [7], local mirror symmetry for weak Fano toric surfaces \( S \) was derived from mirror symmetry of toric Calabi–Yau hypersurfaces containing \( S \) as a smooth divisor. Therefore, looking at the A-model side, it is expected that \( H^*(S, \mathbb{C}) \) should inherit a Frobenius structure from the quantum cohomology of the corresponding Calabi–Yau threefold.

However, the above expectation turns out to be too naive because it seems that there is no natural way to obtain a nondegenerate bilinear form on \( H^*(S, \mathbb{C}) \) from the intersection form on the Calabi–Yau threefold. Therefore we have to abandon the nondegenerate pairing and have to generalize the notion of Frobenius algebra.Hints on how come from looking at the B-model sides. In mirror symmetry, the B-model for Calabi–Yau manifolds is about the Hodge structure whereas in local mirror symmetry it is about the mixed Hodge structure, which has one more extra datum called the weight filtration. This leads us to introduce the notion of Frobenius filtration.

We believe that the MFS on \( H^*(S, \mathbb{C}) \) constructed in §8 is what would be obtained from the Frobenius structure of the corresponding Calabi–Yau threefold by the following two reasons. The first reason is that in the case \( S = \mathbb{P}^2 \), if the quotient construction is applied to the quantum cohomology of the corresponding Calabi–Yau threefold \( X \), the resulting MFS is the same as this one. The second reason is that the Frobenius filtration on \( H^*(S, \mathbb{C}) \) agrees with the weight filtration on the local B-model side (see Remark 8.8). It would be very interesting if we can construct a MFS on the local B-model side and see whether it is isomorphic to the one constructed here. These are left as future problems.

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2. Frobenius ideal and Frobenius filtration

Definition 2.1. Let \( A \) be an algebra. A Frobenius ideal of \( A \) is a pair \((I, (\cdot, \cdot))\) of an ideal \( I \) of \( A \) and a nondegenerate symmetric bilinear form on \( I \) which satisfies the condition

\[
(x, a \circ y) = (a \circ x, y) \quad (x, y \in I, \ a \in A).
\]

Definition 2.2. An increasing filtration of an algebra \( A \) by ideals

\[
I_* : 0 \subset \cdots \subset I_k \subset I_{k+1} \subset \cdots \subset A \quad (k \in \mathbb{Z})
\]
together with bilinear forms \((\ , \ )_k\) on \(I_k/I_{k-1}\) is a Frobenius filtration if \(I_*\) is exhaustive and \((I_k/I_{k-1}, \ (\ , \ )_k)\) is a Frobenius ideal of \(A/I_{k-1}\).

For the later purpose, we state the following

**Lemma 2.3.** If \((I_*, \ (\ , \ )_*)\) is a Frobenius filtration on an algebra \(A\), then \((I_*/I_k, \ (\ , \ )_*)\) is a Frobenius filtration on the quotient algebra \(A/I_k\) for any \(k \in \mathbb{Z}\).

### 3. Constructions of Frobenius filtration by a nilpotent element

#### 3.1. Frobenius filtration defined by a nilpotent element

Let \((A, \ (\ , \ ))\) be a Frobenius algebra. Assume that there exists a nilpotent element \(n \in A\) of order \(d\) (i.e. \(n^d = 0, n^{d-1} \neq 0\)). For \(0 \leq k \leq d\), we set \(J_k := \{x \in A \mid x \circ n^k = 0\}\). Then we have a filtration

\[
J_0 = 0 \subset J_1 \subset \cdots \subset J_{d-1} \subset J_d = A
\]

of ideals on \(A\). Let \(I = (n)\) be the ideal generated by \(n\), and let \(I_k := I + J_k \ (0 \leq k \leq d)\). Then we have a filtration

\[
I_{-1} := 0 \subset I_0 = I \subset I_1 \subset \cdots \subset I_{d-1} \subset I_d = A
\]

of ideals in \(A\). Next we define a pairing \((\ , \ )_k\) on each graded quotient \(I_k/I_{k-1}\). Note that \(I_k/I_{k-1} \cong J_k/(J_{k-1} + I \cap J_k)\) for \(k > 0\).

**Definition 3.1.** (i) For \(x, y \in I_0\), there are \(\tilde{x}, \tilde{y} \in A\) such that \(x = n \circ \tilde{x}\) and \(y = n \circ \tilde{y}\). We define the pairing \((\ , \ )_0\) on \(I_0\) by

\[
(x, y)_0 = \langle \tilde{x}, \tilde{y} \circ n \rangle.
\]

(ii) For \(x, y \in I_k/I_{k-1} \ (k > 0)\), we take representatives \(\tilde{x}, \tilde{y} \in J_k\) and set \((x, y)_k := \langle \tilde{x}, \tilde{y} \circ n^{k-1} \rangle\).

It is easy to check that the pairing \((\ , \ )_k\) is well-defined.

**Lemma 3.2.** (i) The pairing \((\ , \ )_k\) on \(I_k/I_{k-1}\) is symmetric and satisfies

\[
(a \circ x, y)_k = (x, a \circ y)_k
\]

for any \(x, y \in I_k/I_{k-1}\) and \(a \in A/I_{k-1}\).

(ii) The pairing \((\ , \ )_k\) is nondegenerate.

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1 An increasing filtration \(I_*\) on a finite dimensional vector space \(A\) is exhaustive if there exists \(k, l\) such that \(I_k = \{0\}\) and \(I_l = A\).
Proof. The first statement follows from the Frobenius property (1.1) of $(\cdot, \cdot)$. For the second, we first consider the case $k = 0$. The pairing $(\cdot, \cdot)_0$ is nondegenerate since
\[(x, y)_0 = 0 \quad \forall y \in I_0 \iff \langle \bar{x}, n \circ \bar{y} \rangle = 0 \quad \forall \bar{y} \in A \iff x = n \circ \bar{x} = 0 .\]

Next we consider the case $k > 0$. Consider the pairing $(\cdot, \cdot)_k$ on $A$ defined by $(a, b)_k := \langle a, b \circ n^{k-1} \rangle$. Let $I^\perp := \{a \in A \mid (a, b)_k = 0, \forall b \in I\}$. Then it follows that $I^\perp = J_k$, since
\[(a, b \circ n^{k-1}) = 0, \quad \forall b \in I \iff \langle a, \tilde{b} \circ n^k \rangle = 0, \quad \forall \bar{b} \in A \iff a \in J_k .\]

The pairing $(\cdot, \cdot)_k$ is degenerate precisely along $J_{k-1}$. Then the orthogonal of $I/(I \cap J_{k-1})$ in $A/J_{k-1}$ with respect to this pairing is $J_k/J_{k-1}$. Therefore the orthogonal $K_k$ of $J_k$ in $A$ is $K_k = I + J_{k-1}$. It follows that $(\cdot, \cdot)_k$ induces a nondegenerate pairing on $J_k/(K_k \cap J_k) = J_k/(J_{k-1} + I \cap J_k)$. It is clear that the induced pairing coincides with $(\cdot, \cdot)_k$ defined above. Thus the pairing $(\cdot, \cdot)_k$ on $I_k/I_{k-1}$ is nondegenerate. \qed

To summarize, we have obtained the following

**Proposition 3.3.** If $(A, (\cdot, \cdot))$ is a Frobenius algebra with a nilpotent element $n$, $(I_\bullet, (\cdot, \cdot)_\bullet)$ defined in (3.1) and Definition 3.1 is a Frobenius filtration on $A$.

**Remark 3.4.** If a linear endomorphism $n$ on a vector space $A$ is nilpotent of order $d$, then we have a filtration on $A$ called the monodromy weight filtration
\[(3.2) \quad 0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2d-1} \subset W_{2d-2} = A \]
uniquely determined by the conditions [5, p.93]:
\[n(W_i) \subset W_{i-2} , \quad n^j : W_{d+j-1}/W_{d+j-2} \sim W_{d-j-1}/W_{d-j-2} \quad (0 \leq j < d) .\]

The filtration (3.1) is related to the monodromy weight filtration by
\[I_0 = \text{Ker } n , \quad I_k = \text{Ker } n + W_{k+d-2} \quad (0 < k \leq d) .\]

This filtration agrees with the filtration $(N_\bullet A)_\bullet$ defined in [14, §3.4] for the filtration $A_\bullet : 0 = A_0 \subset A_1 = A$.

3.2. **Frobenius filtration by quotient construction.** As in [3.1] let $(A, (\cdot, \cdot))$ be a Frobenius algebra, $n \in A$ a nilpotent element of order $d$. Let $J_k = \{x \in A \mid x \circ n^k = 0\}$ and $I = (n)$.

By Lemma 2.3, the Frobenius filtration $(I_\bullet, (\cdot, \cdot)_\bullet)$ constructed in (3.1) induces a Frobenius filtration on the quotient algebra $A' = A/I$. Explicitly, the induced sequence is
\[(3.3) \quad I'_0 \subset I'_1 \subset \cdots \subset I'_d = A' \quad (I'_k := (I + J_k)/I) .\]
The induced bilinear forms $(\cdot, \cdot)'_k$ on $I'_k/I'_{k-1}$ are given by

\begin{equation}
(a', b')'_k = (a, b)_k
\end{equation}

where $a, b \in J_k$ are those such that the image under $J_k \hookrightarrow J_k + I \rightarrow I'_k$ are $a', b' \in I'_k$.

**Corollary 3.5.** If $(A, \langle \cdot, \cdot \rangle)$ is a Frobenius algebra with a nilpotent element $n$, $(I'_*, (\cdot, \cdot)_*)$ defined in (3.3), (3.4) is a Frobenius filtration on $A/I$.

### 4. Examples I: cohomology of compact Kähler manifolds

#### 4.1. Examples

Let $X$ be a compact Kähler manifold of complex dimension $d - 1$. Let $A = \oplus_{i=0}^{d-1} H^{2i}(X, \mathbb{C})$ be the even part of the cohomology of $X$. Then $A$ is a Frobenius algebra with respect to the cup product $\cup$ and the usual intersection pairing

$$\langle x, y \rangle = \int_X x \cup y \quad (x, y \in H^*(X, \mathbb{C})).$$

Let $n \in H^2(X, \mathbb{C})$ be a class such that either $n$ or $-n$ is a Kähler class. Then $n$ is nilpotent of order $d$. By the construction in §3.1, we get a Frobenius filtration

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_{d-1} \subset I_d = A.$$

**Example 4.1.** Let $X = \mathbb{P}^{d-1}$, $L = \mathcal{O}_{\mathbb{P}^{d-1}}(m)$ ($m \in \mathbb{Z}$, $m \neq 0$) and $n = c_1(L)$. Then $A := H^*(X, \mathbb{C}) \cong \mathbb{C}[n]/(n^d)$ and the intersection form is given by $\langle n^i, n^j \rangle = m^{i+j}\delta_{i+j,d-1}$.

The Frobenius filtration on $A$ defined by $n$ is given as follows. The filtration is

$$I_0 = \bigoplus_{i=1}^{d-1} \mathbb{C}n^i, \quad I_k = I_0 \quad (1 \leq k \leq d - 1), \quad I_d = I_0 \oplus \mathbb{C}1.$$

If we set $e_i := n^i/m^i$ ($1 \leq i \leq d - 1$) and $e_d := 1$, the pairings are

$$(e_i, e_j)_0 = \frac{1}{m}\delta_{i+j,d}, \quad (e_d, e_d)_d = m^{d-1}.$$ 

**Example 4.2.** Let $(X, L)$ be a polarized algebraic surface and $n = c_1(L)$. Then by the Lefschetz decomposition, we have

$$H^2(X, \mathbb{C}) = H^2_{\text{prim}}(X, \mathbb{C}) \oplus \mathbb{C}n,$$

where $H^2_{\text{prim}}(X, \mathbb{C})$ is the kernel of $n \cup : H^2(X, \mathbb{C}) \rightarrow H^4(X, \mathbb{C})$. The Frobenius filtration on $A = H^{\text{even}}(X, \mathbb{C})$ defined by $n$ is given as follows. The filtration is:

\begin{align*}
I_0 &= \mathbb{C}n \oplus \mathbb{C}n^2, \\
I_1 &= I_0 \oplus H^2_{\text{prim}}(X, \mathbb{C}), \\
I_2 &= I_1, \\
I_3 &= I_1 \oplus \mathbb{C}1.
\end{align*}
The pairings are:

\[(n, n^2) = K, \quad (n, n)^0 = (n^2, n^2) = 0, \]

\[(x, y) = \int_X x \cup y, \quad (1, 1)_3 = K, \]

where \(K = \int_X n^2\).

4.2. Remarks on a mixed Hodge structure for Example [4.2]. In this subsection, we explain that the filtration \(I_\bullet\) in Example [4.2] can be regarded as a weight filtration of an \(\mathbb{R}\)-mixed Hodge structure. See e.g. [23, §3.1] for a definition of the \(\mathbb{R}\)-mixed Hodge structure. Although the result of this subsection is not used in the rest of the paper, it is relevant to local mirror symmetry. See Remark [8.8]

Let \(c \in \sqrt{-1} \mathbb{R}\) be a nonzero purely imaginary number. Let \(X\) and \(n\) be as in Example [4.2]. We set

\[A = \{x \cup e^{cn} | x \in H^{\text{even}}(X, \mathbb{R})\}\]

\[= \mathbb{R}\left(1 + cn + \frac{c^2}{2} n^2\right) \oplus \mathbb{R}(n + cn^2) \oplus H^2_{\text{prim}}(X, \mathbb{R}) \oplus \mathbb{R}n^2.\]

We consider the increasing filtration \(W_\bullet\) on \(A\) given by

\[0 \subset W_1 = \mathbb{R}(n + cn^2) \oplus \mathbb{R}n^2 \subset W_2 = W_1 \oplus H^2_{\text{prim}}(X, \mathbb{R}) = W_3 \subset W_4 = A.\]

Notice that \(W_k \otimes \mathbb{C} = I_{k-1}\) holds. We also consider the decreasing filtration \(F^\bullet\) on \(A \otimes \mathbb{C} = A\) given by the degree of the cohomology:

\[0 \subset F^2 = H^0(X, \mathbb{C}) \subset F^1 = H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \subset F^0 = A.\]

Then it is not difficult to check the following

**Proposition 4.3.** (i) \((W_\bullet, F^\bullet)\) is an \(\mathbb{R}\)-mixed Hodge structure on \(A\).

(ii) If we set

\[A^{p, q} = F^p(\text{Gr}^W_k A) \cap F^q(\text{Gr}^W_k A) \quad (p + q = k),\]

then the Hodge decomposition \(A \cong \bigoplus_{1 \leq k \leq 4} \oplus_{p+q=k} A^{p, q}\) is given by the following table.

| \(A^{p,k-p}\) | \(p = 2\) | 1 | 0 |
|----------------|-----------|---|---|
| \(k = 1\)     | \(\mathbb{C}n\) | \(\mathbb{C}(n^2 + \frac{1}{2\alpha} n)\) | \(0\) |
| 2              | 0         | 0 | 0 |
| 3              | 0         | 0 | 0 |
| 4              | \(\mathbb{C}1\) | 0 | 0 |
Next we explain that the pairings $(\ , \ )\star$ induce a polarization on the above $\mathbb{R}$-mixed Hodge structure. We assume the following conditions:

\begin{align*}
\begin{cases}
    h^{2,0}(X) = h^{0,2}(X) = 0, \\
b_1(X) \text{ is even,} \\
    \int_X n^2 = K > 0.
\end{cases}
\end{align*}

Then the pairing $(\ , \ )$ is negative definite on $H^2_{\text{prim}}(X, \mathbb{R})$. See e.g. \cite{M} §IV, Corollary 2.15.

For simplicity, we set $c = \sqrt{-1}$. Let us define $(-1)^k$-symmetric bilinear form $Q_k : \text{Gr}^{W}_k A_{\mathbb{R}} \times \text{Gr}^{W}_k A_{\mathbb{R}} \to \mathbb{R}$ by

$$Q_k(x, y) = \frac{(\sqrt{-1})^k}{2} \{ (Cx, y)_{k-1} + (-1)^k (x, Cy)_{k-1} \},$$

where $C|_{A^p,q} = (\sqrt{-1})^{p-q}$ is the Weil operator. Explicitly, $Q_k$ is given as follows:

$$Q_1(n^2, n + \sqrt{-1}n^2) = K, \quad Q_1(n + \sqrt{-1}n^2, n + \sqrt{-1}n^2) = Q_1(n^2, n^2) = 0,$$

$$Q_2(x, y) = -\int_X x \cup y,$$

$$Q_4(1, 1) = K.$$

Then the Hermitian forms $H_k(x, y) := Q_k(Cx, y)$ are positive definite, and we have the following

**Proposition 4.4.** Under the conditions (4.1), $(\mathcal{W}_\star, \mathcal{F}_\bullet, Q_\bullet)$ is a graded polarized $\mathbb{R}$-mixed Hodge structure on $A_{\mathbb{R}}$.

5. Examples II: cohomology of some non-compact orbifolds

Let $\mathcal{X}$ be an algebraic orbifold. We denote by $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ the orbifold cohomology ring of $\mathcal{X}$ introduced by Chen–Ruan \cite{C}. If $\mathcal{X}$ is compact, $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ has the Poincaré duality pairing $(\ , \ )$ given by

$$\langle x, y \rangle = \int_{\mathcal{X}} x \cup_{\text{orb}} y \quad (x, y \in H^*_\text{orb}(\mathcal{X}, \mathbb{C})), $$

where $\cup_{\text{orb}}$ denotes Chen–Ruan’s product. It makes $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ into a Frobenius algebra. See \cite{C} Theorem 4.6.6. The following examples can be computed by the results of \cite{C, H, L}.

**Example 5.1.** Let $\mathcal{X} = [\mathbb{C}^d/\mathbb{Z}_d]$ be the quotient orbifold of type $\frac{1}{d}(1, \ldots, 1)$ (in the notation of \cite{K}) and $\overline{\mathcal{X}} = \mathbb{P}(1, \ldots, 1, d)$ be the $d$-dimensional weighted projective space of weight $(1, \ldots, 1, d)$. The latter is a compactification of the former. The divisor $D = \overline{\mathcal{X}} \setminus \mathcal{X}$ is Poincaré dual to $dH \in H^2_{\text{orb}}(\overline{\mathcal{X}}, \mathbb{C})$, where $H := c_1(\mathcal{O}(1, \ldots, 1, d)(1))$. If we denote by
$E \in H^2(\mathcal{X}, \mathbb{C})$ the image of the unit class on the twisted sector $\mathbb{P}(d)$ of age 1 of the inertia orbifold of $\mathcal{X}$, then $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ is isomorphic to $\mathbb{C}[H, E]/(H^d - E^d, HE)$ as an algebra. The pairings on $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ are determined by

\[
\langle H^i, H^j \rangle = \langle E^i, E^j \rangle = \frac{1}{d} \delta_{i+j,d} \quad (0 \leq i, j \leq d),
\]

\[
\langle H^i, E^j \rangle = 0 \quad (1 \leq i, j \leq d).
\]

(i) First, we apply the nilpotent construction to $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ with the nilpotent element $n := dH$ of order $d + 1$. Then we obtain the following Frobenius filtration on $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$. The filtration $I_\bullet$ of ideals is

\[
I_0 = (H) \subset I_1 = \cdots \subset I_d = (H, E) \subset I_{d+1} = H^*_\text{orb}(\mathcal{X}, \mathbb{C}).
\]

The pairings $(\cdot, \cdot)_\bullet$ on graded quotients are

\[
(H^i, H^j)_0 = \frac{1}{d^2} \delta_{i+j,d+1}, \quad (E^i, E^j)_1 = \frac{1}{d} \delta_{i+j,d}, \quad (1, 1)_{d+1} = d^{d-1}.
\]

(ii) Next, we take the quotient of $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ by the ideal $I_0 = (n)$. The quotient algebra is isomorphic to $H^*_\text{orb}(\mathcal{X}, \mathbb{C}) \cong \mathbb{C}[E]/(E^d)$. The induced Frobenius filtration $(I_\bullet, (\cdot, \cdot)_\bullet)$ on the quotient is identical to the one on $H^*(\mathbb{P}^{d-1})$ given in Example 4.1 with $m = d$.

**Example 5.2.** Let $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_4]$ be the quotient orbifold of type $\frac{1}{4}(1, 1, 2)$ and $\mathcal{X}^\vee$ is the Poincaré dual of $4H \in H^2(\mathcal{X}, \mathbb{C})$. If we denote by $E_1$ (resp. $E_2$) $\in H^2(\mathcal{X}, \mathbb{C})$ the image of the unit class on the twisted sector $\mathbb{P}(4)$ (resp. $\mathbb{P}(2, 4)$) of age 1 of the inertia orbifold of $\mathcal{X}$, then $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ is isomorphic to $\mathbb{C}[H, E_1, E_2]/(H^2 - E_2^3, HE_1, 2HE_2 = E_1^2)$ as an algebra. The pairings on $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ are determined by

\[
\langle H^i, H^j \rangle = \frac{1}{8} \delta_{i+j,3}
\]

together with the relations in the algebra and the Frobenius property $(\bullet \bullet)$.

(i) Applying the nilpotent construction to $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ with the nilpotent element $n := 4H$ of order 4, we obtain the following Frobenius filtration on $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$. The filtration $I_\bullet$ of ideals is

\[
I_0 = (H) \subset I_1 = (H, E_1) \subset I_2 = I_3 = (H, E_1, E_2) \subset I_4 = H^*_\text{orb}(\mathcal{X}, \mathbb{C}).
\]

The pairings $(\cdot, \cdot)_\bullet$ on graded quotients are

\[
(H^i, H^j)_0 = \frac{1}{32} \delta_{i+j,4}, \quad (E_1, E_1E_2)_1 = \frac{1}{4}, \quad (E_2, E_2)_2 = \frac{1}{2}, \quad (1, 1)_4 = 8.
\]

(ii) Taking the quotient of $H^*_\text{orb}(\mathcal{X}, \mathbb{C})$ by the ideal $I_0 = (n)$, we obtain the following Frobenius filtration on $H^*_\text{orb}(\mathcal{X}, \mathbb{C}) \cong \mathbb{C}[E_1, E_2]/(E_1^2, E_2^3)$:

\[
I'_1 = \mathbb{C}E_1 \oplus \mathbb{C}E_1E_2 \subset I'_2 = I'_3 = I'_4 = H^*_\text{orb}(\mathcal{X}, \mathbb{C}).
\]
6. The mixed Frobenius structure

In this section, a manifold is a complex manifold. The holomorphic tangent bundle (resp. the holomorphic cotangent bundle) of a manifold $M$ is denoted by $TM$ (resp. $T^*M$). All vector bundles on $M$ are assumed to be holomorphic. The space of holomorphic sections on an open set $U \subset M$ of a vector bundle $E \to M$ is denoted by $\Gamma(U, E)$. We mean by $\Gamma(E) = \Gamma(U, E)$ for some open subset $U \subset M$. The dual vector bundle of $E$ is denoted by $E^\vee$.

6.1. Preliminary. We say that a subbundle $I \subset TM$ is $E$-closed for a given global vector field $E$ on $M$ if

$$[E, x] \in \Gamma(I) \quad (x \in \Gamma(I)).$$

Notice that when this holds, the Lie bracket $[E, *]$ induces a derivation on the quotient bundle $TM/I$.

We say that a subbundle $I \subset TM$ is $\nabla$-closed for an affine connection $\nabla$ if

$$\nabla_z x \in \Gamma(I) \quad (x \in \Gamma(I), z \in \Gamma(TM)).$$

Notice that when this holds, $\nabla$ induces a connection on $TM/I$.

6.2. The Mixed Frobenius Structure. Recall the definition of the Frobenius structure [9].

Definition 6.1. A Frobenius structure of charge $D \in \mathbb{C}$ on a manifold $M$ consists of a structure of the Frobenius algebra $(A_t, \langle \ , \ \rangle_t)$ on each tangent space $T_tM \ (t \in M)$ depending complex analytically on $t$, and a globally defined vector field $E$ on $M$ called the Euler vector field satisfying the following conditions.

- The Levi–Civita connection $\nabla$ of the metric $\langle \ , \ \rangle$ is flat and the unit vector field $e$ is $\nabla$-flat (i.e. $\nabla e = 0$).
- The bundle homomorphism $c : TM^{\otimes 3} \to \mathcal{O}_M$ defined by $c(x, y, z) = \langle x, y \circ z \rangle$ satisfies

$$(\nabla_w c)(x, y, z) = (\nabla_z c)(x, y, w),$$

where $\nabla$ is the induced connection on $T^*M^{\otimes 3}$.

This condition together with the Frobenius property [13] implies that $\nabla c : TM^{\otimes 4} \to \mathcal{O}_M$ is symmetric.
The Euler vector field $E$ satisfies $\nabla \nabla E = 0$ (here the leftmost $\nabla$ is the induced connection on $\text{End}(TM)$) and

\begin{align}
[E, x \circ y] - [E, x] \circ y - x \circ [E, y] = x \circ y & \quad (x, y \in \Gamma(TM)) , \\
E(x, y) - \langle [E, x], y \rangle - \langle x, [E, y] \rangle = (2 - D)\langle x, y \rangle & \quad (x, y \in \Gamma(TM)) .
\end{align}

Now we generalize the Frobenius structure to incorporate the Frobenius filtration.

**Definition 6.2.** A mixed Frobenius structure $(\nabla, E, \circ, I_\bullet, (\cdot, \cdot)_\bullet)$ on a manifold $M$ of reference charge $D \in \mathbb{C}$ consists of

- a torsion free, flat connection $\nabla$ on the tangent bundle $TM$,
- a global vector field $E$ satisfying $\nabla \nabla E = 0$ called the Euler vector field,
- a fiberwise multiplication on every tangent space $T_t M$ depending complex analytically on $t$ such that the unit vector field $e$ is $\nabla$-flat,
- a Frobenius filtration $(I_{\bullet, t}, (\cdot, \cdot)_{\bullet, t})$ on every tangent space $T_t M$ depending complex analytically on $t$ such that $I_k$ are $\nabla$-closed and $E$-closed subbundles of $TM$.

These must satisfy the following compatibility conditions. Let $\pi_k : TM \to TM/I_{k-1}$ be the quotient map and let $\circ_k$ be the induced multiplication on the quotient bundle $TM/I_{k-1}$. Let $[E, \ast]_k$ and $\nabla^{(k)}$ be the derivation and the connection on $TM/I_{k-1}$ induced from $[E, \ast]$ and $\nabla$.

- The connection $\nabla$ and the bilinear forms $(\cdot, \cdot)_k$ must be compatible in the sense that

\begin{align}
z(x, y)_k = (\nabla_z^{(k)}x, y)_k + (x, \nabla_z^{(k)}y)_k & \quad (x, y, z \in \Gamma(TM)) .
\end{align}

- The vector bundle homomorphism $c_k : I_k/I_{k-1} \otimes I_k/I_{k-1} \otimes TM \to \mathcal{O}_M$ defined by

\begin{align}
c_k(x, y, z) := (x, \pi_k(z) \circ_k y)_k ,
\end{align}

must satisfy

\begin{align}
(\nabla_{w}^{(k)}c_k)(x, y, z) = (\nabla_{z}^{(k)}c_k)(x, y, w) & \quad (x, y, z, w \in \Gamma(TM)) .
\end{align}

Here $\nabla^{(k)}$ is the induced connection on $(I_k/I_{k-1})^{\vee} \otimes (I_k/I_{k-1})^{\vee} \otimes T^*M$.

- The Euler vector field $E$ must satisfy

\begin{align}
[E, x \circ \pi_k(z)]_k - x \circ_k \pi_k([E, z]) - [E, x]_k \circ_k \pi_k(z) = x \circ_k \pi_k(z) ,
\end{align}

\begin{align}
E(x, y)_k - \langle [E, x], y \rangle_k - \langle x, [E, y] \rangle_k = (2 - D + k)(x, y)_k & \quad (x, y \in \Gamma(I_k/I_{k-1}), z \in \Gamma(TM)) .
\end{align}
6.3. **Flat coordinates.** We write the conditions for the mixed Frobenius structure in a local coordinate expression.

Let \( m_k (k \in \mathbb{Z}) \) be the rank of \( I_k/I_{k-1} \). The flatness, the torsion-free condition for \( \nabla \) and the \( \nabla \)-closedness of \( I_k \)'s imply that there exists on \( M \) a system of local coordinates \( t^{ka} (k \in \mathbb{Z}, 1 \leq a \leq m_k) \) which satisfies the following two conditions:

\[
\nabla \frac{\partial}{\partial t^{ka}} = 0 \quad (k \in \mathbb{Z}, \ 1 \leq a \leq m_k) ,
\]

(6.11)

\[
\left\{ \frac{\partial}{\partial t^l} \mid l \leq k, 1 \leq a \leq m_l \right\} \text{ is a local frame of } I_k.
\]

Now assume that one such system of flat coordinates is fixed. Then we can naturally regard \( \left\{ \frac{\partial}{\partial t^a} \right\}_{1 \leq a \leq m_k} \) as a local frame of \( I_k/I_{k-1} \). We sometimes use the shorthand notation \( \partial_{ka} = \frac{\partial}{\partial t^{ka}} \).

Let \( \eta^{(k)} \) be the matrix representation of \( ( , )_k \):

\[
\eta_{ka,kb} = \left( \frac{\partial}{\partial t^{ka}}, \frac{\partial}{\partial t^{kb}} \right)_k \quad (1 \leq a, b \leq m_k) , \quad \eta^{(k)} := \left( \eta_{ka,kb} \right).
\]

Let \( C^{jc}_{ka,lb} \) be the structure constant of the multiplication:

\[
\frac{\partial}{\partial t^{ka}} \circ \frac{\partial}{\partial t^{lb}} = \sum_{j \in \mathbb{Z}, 1 \leq c \leq m_j} C^{jc}_{ka,lb} \frac{\partial}{\partial t^{jc}} .
\]

Let us write the Euler vector field \( E \) as

\[
E = \sum_{k \in \mathbb{Z}, 1 \leq a \leq m_k} E^{ka} \frac{\partial}{\partial t^{ka}} .
\]

Then the conditions are summarized as follows. (We omit the associativity and the commutativity conditions.)

- \( \eta^{(k)} \) is a symmetric invertible matrix since \( ( , )_k \) is symmetric and nondegenerate. It is a constant matrix by (6.6).
- The condition that \( I_k \) is an ideal is equivalent to \( C^{jc}_{ka,lb} = 0 \) if \( k < j \) or \( l < j \). The compatibility of \( ( , )_k \) and the multiplication (see (2.1)) is equivalent to

\[
\sum_{1 \leq d \leq m_k} C^{kd}_{lc,bb} \eta_{kd,ka} = \sum_{1 \leq d \leq m_k} C^{kd}_{lc,ka} \eta_{kd,kb} .
\]

(6.13)

- The condition (6.8) (with the nondegeneracy of \( \eta^{(k)} \)) is equivalent to

\[
\frac{\partial}{\partial t^{jd}} C^{kb}_{ka,lc} = \frac{\partial}{\partial t^{jc}} C^{kb}_{ka,jd} \quad (l, j \geq k) ,
\]

(6.14)

\[
\frac{\partial}{\partial t^{jd}} C^{kb}_{ka,lc} = 0 \quad (j < k) .
\]
• \(\nabla\nabla E = 0\) and \(E\)-closedness of \(I_k\)'s imply

\[
\frac{\partial^2}{\partial t^a \partial t^b} E^{ka} = 0 \quad (l \geq j, j) \quad \text{and} \quad \frac{\partial}{\partial t^b} E^{ka} = 0 \quad (l < k).
\]

• Eq. (6.9) together with (6.14), (6.15) is equivalent to

\[
\sum_{j \geq k, 1 \leq d \leq m} \partial_{la}(E^{jd}C_{j,d,kb}^{kc}) = \sum_{1 \leq d \leq m} C_{la,kd}^{kd}(\partial_{kb}E^{kd}) = C_{la,kb}^{kc}.
\]

• Eq. (6.10) is equivalent to

\[
\sum_{1 \leq c \leq m} (\eta_{kb,kc}(\partial_{ka}E^{kc}) + \eta_{ka,kc}(\partial_{kb}E^{kc})) = (2 - D + k) \eta_{ka,kb}.
\]

6.4. Mixed Frobenius structure on a transversal slice. Let \((\nabla, E, \circ, I_\bullet, (\ , \ )_\bullet)\) be a mixed Frobenius structure of reference charge \(D\) on \(M\). Let \(m_k\) be the rank of \(I_k/I_{k-1}\). Fix a system of flat coordinates \(t^{ka} (k \in \mathbb{Z}, 1 \leq a \leq m_k)\) on \(M\) (see (6.11)). Since each subbundle \(I_k \subset TM\) is involutive, it defines a foliation on \(M\). Let \(M_k\) be a leaf, i.e. in a neighborhood \(U \subset M\) where the coordinates \(t^{ka}\) are well-defined,

\[
M_k \cap U = \{t^{la} = \text{constant} \mid l > k\}.
\]

This is transversal to the leaf \(M_k\). Then, locally on \(U\), we have the direct sum decomposition \(TM = TM^{(k+1)} \oplus I_k\) and obtain the isomorphism \(TM^{(k+1)} \cong TM/I_k\). Let \(E^{(k+1)}\) be the vector field on \(M^{(k+1)}\) induced from the Euler vector field \(E\): if we use the flat coordinate expression in (6.12),

\[
E^{(k+1)} = \sum_{l \geq k+1, \ 1 \leq a \leq m} E^{la} \frac{\partial}{\partial t^{la}} \quad \text{(just drop the terms } \frac{\partial}{\partial t^{ma}} \text{ for } m \leq k \text{ in } E).
\]

This is a well-defined vector field on \(M^{(k+1)}\) since \(E^{la} (l \geq k + 1)\) is independent of \(t^{mb} (m \leq k)\), see (6.15).

**Lemma 6.3.** \((\nabla^{(k+1)}, E^{(k+1)}, \circ_{k+1}, I_\bullet/I_k, (\ , \ )_\bullet)\) is a mixed Frobenius structure of reference charge \(D\) on \(M^{(k+1)}\).

7. Construction by a nilpotent vector field

7.1. Construction by a nilpotent vector field. Let \((\circ, (\ , \ ), E)\) be a Frobenius structure on a manifold \(M\) of charge \(D\). Let \(\nabla\) be the Levi–Civita connection of the
metric $\langle \, , \, \rangle$. Assume that there exists a global vector field $n$ satisfying the following conditions:

$$
n^d = \underbrace{n \circ n \circ \cdots \circ n}_{\text{d times}} = 0, \quad n^{d-1} \neq 0, \\
\{E, n\} = 0,
$$

(7.1)

$$
\nabla(n \circ x) = n \circ \nabla x, \quad (x \in \Gamma(TM)).
$$

(7.2)

The condition (7.2) implies that the map $n \circ : TM \rightarrow TM$ is a flat bundle homomorphism. So the ranks of the kernel and the image of $n \circ$ are constant. As in (3.1), we define

$$
I = \text{Im} \ (n \circ), \quad J_k = \text{Ker} \ (n^k \circ) \quad (1 \leq k \leq d).
$$

Let $I_0 = I$, $I_k = I + J_k$.

**Lemma 7.1.** $I, J_k, I_k$ are $\nabla$-closed and $E$-closed.

**Proof.** 1. $I$ is $\nabla$-closed: for $x = n \circ \tilde{x} \in \Gamma(I)$ and any vector field $y$,

$$
\nabla_y x = \nabla_y (n \circ \tilde{x}) \quad \text{by (7.2)}
$$

$$
\nabla_y x \in \Gamma(I).
$$

2. $J_k$ is $\nabla$-closed:

$$
x \in \Gamma(J_k) \iff n^k \circ x = 0 \Rightarrow n^k \circ \nabla_y x \quad \text{by (7.2)}
$$

$$
\nabla_y (n^k \circ x) = 0 \iff \nabla_y x \in \Gamma(J_k).
$$

3. $I$ is $E$-closed: for $x = n \circ \tilde{x} \in \Gamma(I)$, we have

$$
[E, x] = [E, n \circ \tilde{x}] \quad \text{by (7.1)}
$$

$$
= [E, n] \circ \tilde{x} + n \circ [E, \tilde{x}] + n \circ \tilde{x} \quad \text{by (7.1)}
$$

$$
= n \circ ([E, \tilde{x}] + \tilde{x}) \in \Gamma(I).
$$

4. $J_k$ is $E$-closed: by (6.1), (7.1) and the induction on $k$, we can show that

$$
[E, n^k] = (k - 1)n^k \quad (1 \leq k \leq d).
$$

(7.3)

We have

$$
x \in \Gamma(J_k) \iff n^k \circ x = 0 \Rightarrow n^k \circ [E, x] \quad \text{by (6.1)}
$$

$$
= [E, n^k \circ x] - [E, n^k] \circ x - n^k \circ x \quad \text{by (6.3)}
$$

$$
= 0.
$$

\[\square\]

Define the bilinear form $(\, , \,)_k$ on $I_k/I_{k-1}$ by (3.1). Then we have a Frobenius filtration of subbundles $(I_\bullet, (\, , \,)_\bullet)$ on $TM$.

**Theorem 7.2.** If $(\circ, \langle \, , \, \rangle, E)$ is a Frobenius structure on $M$ of charge $D$ with a nilpotent global vector field $n$ satisfying (7.1) and (7.2), then the Levi-Civita connection $\nabla$, the Euler vector field $E$, the multiplication $\circ$ and the Frobenius filtration $(I_\bullet, (\, , \,)_\bullet)$ (defined in Eq. (3.1) and Definition (3.1)) form a mixed Frobenius structure on $M$ of reference charge $D + 1$. 
Proof. We have to check the conditions (6.6), (6.8), (6.9) and (6.10).

We first show the case $k \geq 1$. For $x, y \in \Gamma(I_k/I_{k-1})$ ($k \geq 1$), take representatives $\bar{x}, \bar{y} \in \Gamma(J_k)$. Let $z, w$ be vector fields on $M$.

Eq. (6.6):

$$z(x, y)_k = z(\bar{x}, \bar{y} \circ n^{k-1})$$

$$= \langle \nabla_z \bar{x}, \bar{y} \circ n^{k-1} \rangle + \langle \bar{x}, \nabla_z (\bar{y} \circ n^{k-1}) \rangle$$

$$= \langle \nabla_z \bar{x}, \bar{y} \circ n^{k-1} \rangle + \langle \bar{x}, (\nabla_z \bar{y}) \circ n^{k-1} \rangle$$

$$= (\nabla_z^{(k)} x, y)_k + (x, \nabla_z^{(k)} y)_k .$$

Eq. (6.8):

$$(\nabla_w c_k)(x, y, z) = w(x, y \circ_k \pi_k(z))_k - (\nabla_w^{(k)} x, y \circ_k \pi_k(z))_k$$

$$= w(\bar{x}, \bar{y} \circ z \circ n^{k-1}) - \langle \nabla_w \bar{x}, \bar{y} \circ n^{k-1} \circ z \rangle$$

$$= w(\bar{x}, \bar{y} \circ z \circ n^{k-1}) - \langle \bar{x}, \bar{y} \circ n^{k-1} \circ \nabla_w z \rangle$$

$$= (\nabla_w c)(\bar{x}, n^{k-1} \circ \bar{y}, z) - (\nabla_z c)(\bar{x}, n^{k-1} \circ \bar{y}, w)$$

$$= (\nabla_z c_k)(x, y, w) .$$

Eq. (6.9): by (6.4), we have

$$[E, \bar{x} \circ z] - \bar{x} \circ [E, z] - [E, \bar{x}] \circ z = \bar{x} \circ z .$$

Applying the projection $\pi_k : TM \rightarrow TM/I_{k-1}$ to the both sides, we obtain (6.9).

Eq. (6.10):

$$E(\bar{x}, \bar{y} \circ n^{k-1}) - \langle [E, \bar{x}], \bar{y} \circ n^{k-1} \rangle - \langle x, [E, \bar{y}] \circ n^{k-1} \rangle$$

$$= E(\bar{x}, \bar{y} \circ n^{k-1}) - \langle [E, \bar{x}], \bar{y} \circ n^{k-1} \rangle - \langle \bar{x}, [E, n^{k-1} \circ \bar{y}] - \bar{y} \circ [E, n^{k-1}] - \bar{y} \circ n^{k-1} \rangle$$

$$= (2 - D)(\bar{x}, \bar{y} \circ n^{k-1}) + \langle \bar{x}, \bar{y} \circ [E, n^{k-1}] \rangle + \langle \bar{x}, \bar{y} \circ n^{k-1} \rangle$$

$$= (1 - D + k)(\bar{x}, \bar{y} \circ n^{k-1}) .$$

This implies

$$E(x, y)_k - ([E, x]_k, y)_k - (x, [E, y]_k)_k = (1 - D + k)(x, y)_k .$$

Next we show the case $k = 0$. Let $x = n \circ \bar{x}, y \in \Gamma(I)$. Recall that $(x, y)_0 = (\bar{x}, y)$.

Eq. (6.6): since $\nabla_z x = \nabla_z (n \circ \bar{x}) = n \circ \nabla_z \bar{x}$ by (7.2), we have

$$(\nabla_z^{(0)} x, y)_0 + (x, \nabla_z^{(0)} y)_0 = (\nabla_z\bar{x}, y) + (\bar{x}, \nabla_z y) = z(\bar{x}, y) = z(x, y)_0 .$$
Eq. (6.8) : since \( \nabla_z x = n \circ \nabla_z \tilde{x} \),

\[
(\nabla_w c_0)(x, y, z) = w(x, y \circ z) - (\nabla_w x, y \circ z)_0 - (x, z \circ \nabla_w y)_0 - (x, y \circ \nabla_w z)_0
\]

\[
= w(x, y \circ z) - \langle \nabla_w \tilde{x}, y \circ z \rangle - \langle \tilde{x}, z \circ \nabla_w y \rangle - \langle \tilde{x}, y \circ \nabla_w z \rangle
\]

\[
= (\nabla_w c)(\tilde{x}, y, z) \overset{\text{def}}{=} (\nabla_z c)(\tilde{x}, y, w)
\]

\[
= (\nabla_z c_0)(x, y, z).
\]

Eq. (6.9) follows immediately from (6.4).

Eq. (6.10) : notice that \( [E, x] = n \circ ([E, \tilde{x}] + \tilde{x}) \) holds by (6.4). Therefore

\[
E(x, y)_0 - ([E, x], y)_0 - (x, [E, y])_0 = E(\tilde{x}, y) - ([E, \tilde{x}] + \tilde{x}, y) - \langle \tilde{x}, [E, y] \rangle
\]

\[
\overset{\text{def}}{=} (2 - D)(\tilde{x}, y) - \langle \tilde{x}, y \rangle
\]

\[
= (1 - D)(x, y)_0.
\]

7.2. Quotient construction. Next we apply Lemma 6.3 to the mixed Frobenius structure obtained in Theorem 7.2 with \( k = 0 \).

Corollary 7.3. If \( (\circ, \langle , \rangle, E) \) is a Frobenius structure on \( M \) of charge \( D \) with a nilpotent global vector field \( n \) satisfying (7.1) and (7.2), then \( (\nabla^{(1)}, E^{(1)}, \circ_1, I_*/I_0, \langle , \rangle_\bullet) \) is a mixed Frobenius structure of reference charge \( D + 1 \) on \( M^{(1)} \).

Examples of this construction can be found in §8 and §9.

8. Local quantum cohomology of weak Fano toric surfaces

Let \( S \) be a weak Fano toric surface. We define the local quantum cup product on the cohomology \( H^*(S, \mathbb{C}) \) using genus zero local Gromov–Witten invariants and construct a mixed Frobenius structure.

First we recall basic facts about the quantum cohomology in §8.1. Then we state the results in §8.2. In §8.3 and §8.4 we explain that they can be obtained from the quantum cohomology of the projective compactification \( V \) of the canonical bundle \( K_S \).

8.1. Quantum cohomology. Let \( V \) be a nonsingular projective variety. \( \mathcal{M}_{g,n}(V, \beta) \) denotes the moduli stack of genus \( g \), \( n \)-pointed stable maps to \( V \) of degree \( \beta \in H_2(V, \mathbb{Z}) \). Its virtual dimension is

\[
(1 - g)(\dim V - 3) - \int_\beta c_1(K_V) + n,
\]

where \( K_V \) is the canonical bundle of \( V \). Let \( ev_i : \mathcal{M}_{g,n}(V, \beta) \to V \) be the evaluation map at the \( i \)-th marked point.
Fix a basis of the even part $H^{\text{even}}(V, \mathbb{Q})$ of the cohomology $H^*(V, \mathbb{Q})$:

$$\Gamma_0 = 1 \ , \ \Gamma_1, \ldots, \Gamma_r \ , \ \Gamma_{r+1}, \ldots, \Gamma_s \ .$$

The dual basis with respect to the intersection form is denoted $\{ \Gamma^\vee_i \}$, i.e.

$$\int_V \Gamma_i \cup \Gamma^\vee_j = \delta_{i,j} \ .$$

Let $t^0, \ldots, t^s$ be the coordinates of $H^{\text{even}}(V, \mathbb{Q})$ associated to (8.2).

**Definition 8.1.** The genus zero Gromov–Witten potential of $V$ is defined by

$$\Phi(t,q) = \sum_{n=0}^{\infty} \sum_{\beta \in H_2(V,\mathbb{Z})} \frac{1}{n!} \left( \int_{[M_{0,n}(V,\beta)]^{\text{vir}}} \prod_{i=1}^{n} eV_i t^i \right) q^\beta,$$

(8.3)

$$t = \sum_{i=0}^{s} t^i \Gamma_i \ .$$

Here $q$ is the parameter associated to $H_2(V,\mathbb{Z})$ and $[M_{0,n}(V,\beta)]^{\text{vir}}$ is the virtual fundamental class.

Recall that the contribution $\Phi_{cl}$ from $\beta = 0$ in $\Phi(t,q)$ is given by the triple intersection because of the point mapping axiom (see [18, §2], also [8, Chapter 8], for axioms of Gromov–Witten invariants):

$$\Phi_{cl} = \sum_{i,j,k=0}^{s} \frac{t^i t^j t^k}{3!} \int_V \Gamma_i \cup \Gamma_j \cup \Gamma_k \ .$$

**Definition 8.2.** The quantum cup product $\circ_t$ is defined by

$$(8.5) \quad \Gamma_i \circ_t \Gamma_j = \sum_{k=0}^{s} \frac{\partial^3 \Phi}{\partial t^i \partial t^j \partial t^k} \Gamma^\vee_k \quad (0 \leq i,j \leq s) \ .$$

We call $(H^{\text{even}}(V), \circ_t)$ the quantum cohomology of $V$.

The intersection form $\langle , \rangle$ is defined by

$$(8.6) \quad \langle \Gamma_i, \Gamma_j \rangle = \int_V \Gamma_i \cup \Gamma_j \ .$$

Then $(H^{\text{even}}(V), \circ_t, \langle , \rangle)$ is a Frobenius algebra.

Define the vector field $E$ on $M = H^*(V,\mathbb{C})$ by

$$(8.7) \quad E = \sum_{i=0}^{s} \frac{2 - \deg \Gamma_i}{2} t^i \frac{\partial}{\partial t^i} + \sum_{i=1}^{r} \xi_i \frac{\partial}{\partial t^i} \ .$$

---

3 Usually the quantum cohomology refers to a superalgebra structure on $H^*(V)$ and the quantum cohomology considered here is its subalgebra. For our purpose, this is sufficient because $H^*(V) = H^{\text{even}}(V)$ for nonsingular toric varieties $V$ which we deal with in [8, §9].
Here the numbers $\xi_i$ are coefficients of $\Gamma_i$ in $-c_1(K_V)$:

$$-c_1(K_V) = \sum_{i=1}^r \xi_i \Gamma_i .$$

**Theorem 8.3.** $(c, \langle , , \rangle, E)$ defined in (8.5), (8.6) and (8.7) is a Frobenius structure on $M = H^{\text{even}}(V, \mathbb{C})$ of charge $\dim V$ (18, see also 22).

**Remark 8.4.** When $V$ is a smooth projective toric variety, the convergence of the Gromov–Witten potential (8.3) was proved by Iritani [12, Theorem 1.3]. We only deal with such cases in §8 and §9.

8.2. Results. Let $S$ be a weak Fano toric surface. Let $\gamma_{r+1} \in H^4(S, \mathbb{Z})$ be the Poincaré dual of the point class and $\gamma_0 = 1 \in H^0(S)$. Let $\gamma_1, \ldots , \gamma_r$ be a basis of $H^2(S, \mathbb{Z})$. Define the integers $c_{ij}, b_i, b_i^\vee, \kappa$ ($1 \leq i, j \leq r$) by

$$c_{ij} = \int_S \gamma_i \cup \gamma_j , \quad -c_1(K_S) = \sum_{i=1}^r b_i \gamma_i ,$$

$$b_i^\vee = \sum_{j=1}^r b_j c_{ji} , \quad \kappa = \int_S c_1(K_S)^2 = \sum_{i,j=1}^r b_i b_j c_{ij} = \sum_{i=1}^r b_i b_i^\vee ,$$

where $c_1(K_S)$ is the first Chern class of the canonical bundle $K_S$. Notice that the matrix $(c_{ij})$ is invertible. Notice also that $\kappa > 0$ holds for weak Fano toric surfaces.

Consider the filtration\(^4\) on $H^*(S, \mathbb{C})$ by subspaces

$$0 = I_0 \subset I_1 = \mathbb{C} c_1(K_S) \oplus H^4(S) \subset I_2 = H^2(S) \oplus H^4(S) = I_3 \subset I_4 = H^*(S) ,$$

and the following bilinear forms on $I_k/I_{k-1}$:

$$(c_1(K_S), \gamma_{r+1})_1 = 1 , \quad (c_1(K_S), c_1(K_S))_1 = (\gamma_{r+1}, \gamma_{r+1})_1 = 0 ,$$

$$(\gamma_i, \gamma_j)_2 = c_{ij} - \frac{b_i^\vee b_j^\vee}{\kappa} (1 \leq i, j \leq r) ,$$

$$(\gamma_0, \gamma_0)_4 = \kappa .$$

Next we define the local quantum cup product. Let $\overline{M}_{g,n}(S, \beta)$ be the moduli stack of $n$-pointed genus $g$ stable maps to $S$ of degree $\beta \in H_2(S, \mathbb{Z})$, $ev_i : \overline{M}_{g,n}(S, \beta) \to S$ be the evaluation map at the $i$-th marked point, $\mu : \overline{M}_{g,1}(S, \beta) \to \overline{M}_{g,0}(S, \beta)$ be the forgetful map.

**Definition 8.5.** For an effective class $\beta \in H_2(S, \mathbb{Z})$ satisfying $-\int_S c_1(K_S) > 0$, we define

$$N_\beta = \int_{\overline{M}_{0,0}(S, \beta)^{\text{vir}}} e(R^1 \mu_* ev_1^* K_S) .$$

\(^4\)When $S$ is Fano, (8.8) and (8.9) agree with those in Example 12 with $n = K_S$ if the filtration is shifted by one.
Here $e$ stands for the Euler class. For other $\beta \in H_2(S, \mathbb{Z})$, we just set $N_\beta = 0$. We call $N_\beta$ the genus zero local Gromov–Witten invariants of degree $\beta$ of $S$.

These numbers can be found in [7] for some $S$.

Let $\{C_1, \ldots, C_r\}$ be the basis of $H_2(S, \mathbb{Z})$ dual to $\gamma_1, \ldots, \gamma_r$.

**Definition 8.6.** The local quantum cup product on $H^*(S, \mathbb{C})$ is the following family of multiplications $\circ_t$ parameterized by $t = (t^1, \ldots, t^r)$:

$$
\begin{align*}
\gamma_0 \circ_t \gamma_i &= \gamma_i, & \gamma_0 \circ_t \gamma_{r+1} &= \gamma_{r+1}, & \gamma_i \circ_t \gamma_{r+1} &= \gamma_{r+1} \circ_t \gamma_{r+1} &= 0, \\
\gamma_i \circ_t \gamma_j &= \left( c_{ij} - \sum_{\beta = \sum_i \beta_i C_i} \beta_i \beta_j (b \cdot \beta) N_{\beta} e_{\beta \cdot t} \right) \gamma_{r+1} & (1 \leq i, j \leq r),
\end{align*}
$$

(8.11)

where $\beta \cdot t = \sum_{i=1}^r \beta_i t^i$ and $b \cdot \beta = \sum_{i=1}^r b_i \beta_i$.

Let $M = H^*(S, \mathbb{C})$ and $t^0, t^1, \ldots, t^{r+1}$ be the coordinates associated to the basis $\gamma_0, \gamma_1, \ldots, \gamma_{r+1}$. Define a vector field $E$ on $M$ by

$$
E = t^0 \frac{\partial}{\partial t^0} - t^{r+1} \frac{\partial}{\partial t^{r+1}}.
$$

Regard the multiplication (8.11), the filtration (8.8) and the bilinear forms (8.9) on those on the tangent space $T_1 M$ by identifying $T_1 M$ with $H^*(S, \mathbb{C})$.

**Theorem 8.7.** The trivial connection, the above $E$, the multiplication (8.11) and the Frobenius filtration (8.8) and (8.9) form a mixed Frobenius structure on $M = H^*(S, \mathbb{C})$ of reference charge four.

The proof of Theorem 8.7 will be given in §8.4, by applying the quotient construction (Corollary 7.3) to the quantum cohomology ring of the projective compactification of $K_S$.

**Remark 8.8.** In this example, the operator $\mathcal{V} = \nabla E - \frac{2-D}{2} = -\frac{\deg}{2} + 2$ is diagonalizable and eigenvalues are integers. Therefore other than the Frobenius filtration $I_\bullet$, we can consider the decreasing filtration $\mathcal{F}^\bullet$ on $H^*(S, \mathbb{C})$ defined by

$$
\mathcal{F}^p = \bigoplus_{p' \geq p} \text{the eigenspace of } \mathcal{V} \text{ with eigenvalue } p'.
$$

This $\mathcal{F}^\bullet$ is the same as the one in [112]. Moreover $I_k$ (here) $= W_k \otimes \mathbb{C}$ with $n = c_1(K_S)$. They agree with the Hodge filtration and the weight filtration of the mixed Hodge structure for the corresponding local B-model under an appropriate vector space isomorphism (see [17] Theorem 4.2 and references therein).
8.3. **Quantum cohomology of** \( V = \mathbb{P}(\mathcal{O}_S \oplus K_S) \). Let \( V = \mathbb{P}(\mathcal{O}_S \oplus K_S) \) be the projective compactification of the canonical bundle \( K_S \) of a weak Fano toric surface \( S \) and \( \text{pr} : V \to S \) be the projection. Let \( \Gamma_i = \text{pr}^* \gamma_i \) \((0 \leq i \leq r + 1)\) and let \( \Delta_0 \) be the Poincaré dual of the infinity section. Set \( \Delta_i = \Gamma_i \cup \Delta_0 \) \((1 \leq i \leq r)\). We have the following basis of \( H^*(V; \mathbb{C}) \):

\[
\begin{align*}
\Gamma_0, & \quad \Delta_0, \quad \Gamma_1, \ldots, \Gamma_r, \\
& \quad \Delta_r, \quad \Gamma_{r+1}, \Delta_1, \ldots, \Delta_r \\
& \quad \Delta_{r+1}, \quad \Gamma_{r+1}. \\
\end{align*}
\]

Let \( t_i, s_i \) \((0 \leq i \leq r + 1)\) be the coordinates of \( H^*(V; \mathbb{C}) \) associated to this basis.

The cup product \( \cup \) can be calculated by the intersection theory of toric varieties. The unit is \( \Gamma_0, \Gamma_i \cup \Delta_0 = \Delta_i \) by definition \((1 \leq i \leq r + 1)\) and

\[
\begin{align*}
\Gamma_i \cup \Gamma_j &= c_{ij} \Gamma_{r+1}, \quad \Gamma_i \cup \Gamma_{r+1} = 0, \\
\Gamma_i \cup \Delta_j &= c_{ij} \Delta_{r+1}, \\
\Delta_i &= \sum_{i=1}^{r} b_i \Delta_i, \quad \Delta_0 \cup \Delta_j = b_j^\vee \Delta_{r+1} \quad (1 \leq i, j \leq r).
\end{align*}
\]

Other products vanish by the degree reason. The intersection form \( \langle \ , \ \rangle \) can be obtained from \( \int_V \Delta_{r+1} = 1 \) and the above cup product. Explicitly pairings which do not vanish are:

\[
\begin{align*}
\langle \Gamma_0, \Delta_{r+1} \rangle &= 1, \quad \langle \Delta_0, \Gamma_{r+1} \rangle = 1, \\
\langle \Gamma_i, \Delta_j \rangle &= c_{ij}, \quad \langle \Delta_0, \Delta_j \rangle = b_j^\vee \quad (1 \leq i, j \leq r).
\end{align*}
\]

Let \( A = (a_{ij}) \) be the inverse of the matrix \( (c_{ij})_{1 \leq i, j \leq r} \). The dual basis of \((8.12)\) is given by the following.

\[
\begin{align*}
\Gamma_0^\vee &= \Delta_{r+1}, \quad \Gamma_i^\vee = \sum_{j=1}^{r} a_{ij} \Delta_j - b_i \Gamma_{r+1}, \quad \Delta_0^\vee = \Gamma_{r+1}, \\
\Gamma_{r+1}^\vee &= \Delta_0 - \sum_{k=1}^{r} b_k \Gamma_k, \quad \Delta_i^\vee = \sum_{j=1}^{r} a_{ij} \Gamma_j, \quad \Delta_{r+1}^\vee = \Gamma_0.
\end{align*}
\]

Now consider the Gromov–Witten potential \( \Phi(t, s; q) \) (see \((8.3)\)) and the quantum cup product \((8.5)\) of \( V \). To be concrete, let us fix a basis of \( H_2(V, \mathbb{Z}) \) as follows. Let \( \iota : S \to V \) be the inclusion as the zero section of \( K_S \subset V \) and \( C'_i = \iota_* C_i \) \((1 \leq i \leq r)\). Let \( C'_0 \) be the fiber class of \( \text{pr} : V \to S \). Then \( \{C'_0, C'_1, \ldots, C'_r\} \) is a base of \( H_2(V, \mathbb{Z}) \) which is dual to the basis \( \{\Delta_0, \Gamma_1, \ldots, \Gamma_r\} \) of \( H^2(V, \mathbb{Z}) \). Let \( q_i \) \((0 \leq i \leq r)\) be parameters associated to \( C'_0, C'_1, \ldots, C'_r \). The parameter \( q^\beta \) in \((8.3)\) is written as

\[
q^\beta = \prod_{i=0}^{r} q_i^{\beta_i} \quad \text{for} \quad \beta = \sum_{i=0}^{r} \beta_i C'_i.
\]
Lemma 8.9. We have

$$
\Phi(t, s; q) = \Phi_{cl} + \Phi_0 + \Phi_1,
$$

$$
\Phi_{cl} = \frac{1}{2} t_0^2 s_{r+1} + t_0 \left( \sum_{i,j=1}^{r} c_{ij} t_i s_j + t_{r+1} s_0 + \sum_{i=1}^{r} b_i^\gamma s_0 s_i \right) + \frac{1}{2} \sum_{i,j=1}^{r} c_{ij} t_i s_j + \frac{1}{2} \sum_{i=1}^{r} b_i^\gamma s_0^2,
$$

(8.16)

$$
\Phi_0 = \sum_{\beta = \beta_1 c_1 + \cdots + \beta_r c_r \neq 0} N^V_{\beta} e^{\beta \cdot t} q^\beta,
\quad \text{where } N^V_{\beta} = \int_{[\overline{M}_{0,0}(V,\beta)]^{vir}} 1,
$$

$$
\Phi_1 = \mathcal{O}(q_0).
$$

Proof. We decompose the Gromov–Witten potential Φ into three parts: Φ_{cl} which is the contribution of $\beta = 0$, Φ_{0} which is the contribution of homology classes $\beta \neq 0$ with $\beta_0 = 0$, and the remaining part Φ_{1} which is the contribution of $\beta$ with $\beta_0 \neq 0$.

Φ_{cl} can be computed by (8.4).

In Φ_{1}, the terms with $\beta_0 < 0$ vanish because such $\beta$ are not effective and the moduli stack $\overline{M}_{0,n}(V,\beta)$ is empty. Therefore $\Phi_{1} = \mathcal{O}(q_0)$.

Notice that since $-c_1(K_V) = 2\Delta_0$, the virtual dimension (8.1) of $\overline{M}_{0,n}(V,\beta)$ is $2\beta_0 + n$.

If $\beta_0 = 0$,

$$
\int_{[\overline{M}_{0,n}(V,\beta)]^{vir}} \prod_{i=1}^{n} ev_i^* t = \int_{[\overline{M}_{0,n}(V,\beta)]^{vir}} \prod_{i=1}^{n} ev_i^* t' \quad (t' = \sum_{i=1}^{r} t_i \gamma_i + s_0 \Delta_0)
$$

$$
= \left( \sum_{i=1}^{r} \beta_i t_i \right)^n \int_{[\overline{M}_{0,0}(V,\beta)]^{vir}} 1.
$$

Here the first equality follows from the degree consideration and the fundamental class axiom, the second equality follows from the divisor axiom. This proves the equation for $\Phi_0$.

We consider the specialization $q_0 = 0$, and set $q_1 = \cdots = q_r = 1$. Then by Lemma 8.9 we see that the quantum cup product (8.5) reduces to

$$
\Gamma_i \circ_t \Gamma_j = c_{ij} \Gamma_{r+1} + \sum_{k=1}^{r} \left( \sum_{\beta = \beta_1 c_1 + \cdots + \beta_r c_r \neq 0} \beta_i \beta_j \beta_k N^V_{\beta} e^{\beta \cdot t} \right) \Gamma_k^\vee \quad (1 \leq i, j \leq r),
$$

(8.17)

$$
\Delta_i \circ_t * = \Delta_i \cup *.
$$

5 Since $q_i$ ($1 \leq i \leq r$) appears in $\Phi_0$ always in the combination $e^{t_i} q_i$, one can always recover $q_1, \ldots, q_r$ in $\Phi_0$. 
Lemma 8.10. The multiplication (8.17), the intersection form (8.14) (regarded as the multiplication on $T_tH^*(V,\mathbb{C})$ and the metric by the canonical isomorphism $H^*(V) \sim T_tH^*(V)$) and the vector field

\begin{equation}
E = t^0 \frac{\partial}{\partial t^0} + 2 \frac{\partial}{\partial s^0} - t^{r+1} \frac{\partial}{\partial t^{r+1}} - \sum_{i=1}^r s^i \frac{\partial}{\partial s^i} - 2s^{r+1} \frac{\partial}{\partial s^{r+1}}
\end{equation}

form a Frobenius structure of charge three on $\tilde{M} = H^*(V,\mathbb{C})$.

Proof. The quantum cup product on $V$ is a power series in $t^0, e^{t^i}q_i (1 \leq i \leq r), e^{s^0}q_0, t^{r+1}, s^i (1 \leq i \leq r+1)$ and it is convergent [12]. By Lemma 8.9, it becomes as follows.

\begin{equation}
\Gamma_i \circ_t \Gamma_j = c_{ij} \Gamma_{r+1} + \sum_{k=1}^r \left( \sum_{\beta_1 + \cdots + \beta_r = 0} \beta_i \beta_j \beta_k N_{\beta} e^{\beta \cdot t} \right) \Gamma_k^\vee + O(e^{s^0}q_0) \quad (1 \leq i, j \leq r),
\end{equation}

\begin{equation}
\Delta_i \circ_t \ast = \Delta_i \cup \ast + O(e^{s^0}q_0).
\end{equation}

The above multiplication (8.17) is just the terms of degree zero in $e^{s^0}q_0$ of this product. Therefore the associativity and the commutativity of (8.17) follow from those of this quantum cup product.

The Euler vector field (8.7) for the quantum cohomology of $V$ is given by (8.18) since $-c_1(K_V) = 2\Delta_0$. The compatibility (6.4) with the multiplication (8.17) follows from that of the quantum cohomology.

The symmetry (6.3) of $\nabla c$ holds because of the same reason. \hfill \Box

8.4. Proof of Theorem 8.7. We first apply Theorem 7.2 to the Frobenius structure on $\tilde{M} = H^*(V,\mathbb{C})$ obtained in Lemma 8.10 with the nilpotent vector field $\frac{\partial}{\partial s^0}$.

Let us construct the Frobenius filtration. Using the canonical isomorphism $T_t\tilde{M} \cong H^*(V,\mathbb{C})$, we write it down as that of $H^*(V,\mathbb{C})$. Let $I$ be the ideal generated by $\Delta_0$:

$$I = \mathbb{C}\Delta_0 \oplus \bigoplus_{i=1}^r \mathbb{C}\Delta_i \oplus \mathbb{C}\Delta_{r+1}.$$ 

Then following the construction in 8.1 we compute $J_k = \text{Ker} \Delta_k^k$:

$$J_1 = \mathbb{C}\Gamma^\vee_{r+1} \oplus \bigoplus_{i=1}^r \mathbb{C}\Gamma^\vee_i \oplus \mathbb{C}\Delta_{r+1},$$

$$J_2 = \bigoplus_{i=1}^r \mathbb{C}(\Gamma_i - \frac{b^i}{K}\Delta_0) \oplus \mathbb{C}\Gamma_{r+1} \oplus \bigoplus_{i=1}^r \mathbb{C}\Delta_i \oplus \mathbb{C}\Delta_{r+1},$$

$$J_3 = \bigoplus_{i=1}^r \mathbb{C}\Gamma_i \oplus \mathbb{C}\Delta_0 \oplus \mathbb{C}\Gamma_{r+1} \bigoplus_{i=1}^r \mathbb{C}\Delta_i \oplus \mathbb{C}\Delta_{r+1} = H^{\geq 2}(V) \supset I,$$

$$J_4 = \mathbb{C}\Gamma_0 \oplus J_3 = H^*(V).$$
So $I_k = I + J_k \ (k = 1, 2, 3, 4)$ are

\[
I_0 = I , \\
I_1 = \mathbb{C}\Gamma_r^\vee \oplus \mathbb{C}\Delta_0 \oplus \mathbb{C}\Gamma_{r+1} \oplus \bigoplus_{i=1}^r \mathbb{C}\Delta_i \oplus \mathbb{C}\Delta_{r+1} , \\
I_2 = I_3 = H^{\geq 2}(V) , \\
I_4 = H^*(V) .
\]

(8.19)

The induced bilinear forms $(\ , \ )_k$ on $I_k/I_{k-1}$ are as follows.

\[
k = 0 \quad (\Delta_0, \Delta_{r+1})_0 = \langle \Delta_0, \Gamma_{r+1} \rangle = 1 , \\
(\Delta_i, \Delta_j)_0 = \langle \Delta_i, \Gamma_j \rangle = c_{ij} \quad (1 \leq i, j \leq r) , \\
k = 1 \quad ([\Gamma_r^\vee], [\Gamma_{r+1}])_1 = \langle \Gamma_r^\vee, \Gamma_{r+1} \rangle = 1 , \\
([\Gamma_r^\vee], [\Gamma_{r+1}])_1 = ([\Gamma_{r+1}], [\Gamma_{r+1}])_1 = 0 , \\
(8.20) \\
k = 2 \quad ([\Gamma_i], [\Gamma_j])_2 = \left\langle \left( \Gamma_i - \frac{b_i^\vee}{\kappa}\Delta_0 \right) \circ_t \left( \Gamma_j - \frac{b_j^\vee}{\kappa}\Delta_0 \right) , \Delta_0 \right\rangle \\
\quad = c_{ij} - \frac{b_i^\vee b_j^\vee}{\kappa} \quad (1 \leq i, j \leq r) , \\
k = 4 \quad ([\Gamma_0], [\Gamma_0])_4 = \langle \Gamma_0 \cup \Gamma_0, \Delta_0^3 \rangle = \kappa .
\]

(8.20)

Thus by Theorem 7.2 we have the following lemma.

Lemma 8.11. The trivial connection, the vector field $E$ (8.18), the multiplication (8.17), the filtration (8.19) and the bilinear forms (8.20) form a MFS of reference charge four on $\tilde{M} = H^*(V, \mathbb{C})$.

Next we apply Corollary 7.3. Since $I$ is the kernel of the pullback $\iota^* : H^*(V, \mathbb{C}) \to H^*(S, \mathbb{C})$ by the inclusion $\iota : S \hookrightarrow K_S \subset V$, if we set

\[
\tilde{M}^{(1)} = \{ s^0 = s^1 = \cdots = s^{r+1} = 0 \} \subset \tilde{M} ,
\]

then it is naturally isomorphic to $H^*(S, \mathbb{C})$. Theorem 8.7 follows from Corollary 7.3 and Lemma 8.10.

9. LOCAL QUANTUM COHOMOLOGY OF $\mathbb{P}^3$

In this section, we construct a mixed Frobenius structure on the cohomology of the projective space $\mathbb{P}^3$ similar to the one in 8.
9.1. **Results.** Take the following basis of the cohomology $H^*(\mathbb{P}^3, \mathbb{C})$:

$$
\gamma_0 = 1, \quad \gamma_1 = c_1(\mathcal{O}_{\mathbb{P}^3}(1)), \quad \gamma_2 = \gamma_1 \cup \gamma_1, \quad \gamma_3 = \gamma_1 \cup \gamma_2.
$$

Let $t^0, t^1, t^2, t^3$ be the associated coordinates. We identify $H_2(\mathbb{P}^3, \mathbb{Z})$ with $\mathbb{Z}$ by $\beta \mapsto \int_{\beta} \gamma_1$.

We consider the following filtration on $H^*(\mathbb{P}^3)$.

$$
I_0 = 0, \\
I_1 = \cdots = I_4 = H^{\geq 2}(\mathbb{P}^3), \\
I_5 = H^*(\mathbb{P}^3).
$$

On the graded quotients, we consider the bilinear forms:

$$
I_1/I_0 : (\gamma_k, \gamma_l)_1 = \begin{cases} 
-\frac{1}{4} & (k + l = 4) \\
0 & (k + l \neq 4)
\end{cases}, \\
I_5/I_4 : (1, 1)_5 = 4^3.
$$

Next we define the local quantum cup product. Let $\overline{M}_{g,n}(\mathbb{P}^3, \beta)$ be the moduli stack of $n$-pointed genus $g$ stable maps to $\mathbb{P}^3$ of degree $\beta \in H_2(\mathbb{P}^3, \mathbb{Z}) \cong \mathbb{Z}$, $ev_i : \overline{M}_{g,n}(\mathbb{P}^3, \beta) \to \mathbb{P}^3$ be the evaluation map at the $i$-th marked point and $\mu : \overline{M}_{g,2}(\mathbb{P}^3, \beta) \to \overline{M}_{g,1}(\mathbb{P}^3, \beta)$ be the forgetful map.

**Definition 9.1.** For $\beta \neq 0$, we define the genus zero local Gromov–Witten invariant $N_\beta \in \mathbb{Q}$ of degree $\beta \in H_2(\mathbb{P}^3, \mathbb{Z})$ by

$$
N_\beta := \int_{[\overline{M}_{0,1}(\mathbb{P}^3, \beta)]^\text{vir}} ev_1^* \gamma_2 \cup e(R^1\mu_* ev_2^* \mathcal{O}_{\mathbb{P}^3}(-4)).
$$

These numbers are computed in [21, Table 1].

**Definition 9.2.** Let

$$
\Phi_{\text{qu}}(t^1, t^2) := t^2 \sum_{\beta > 0} N_\beta e^{\beta t^1}.
$$

The local quantum cup product $\circ_t$ on $H^*(\mathbb{P}^3, \mathbb{C})$ is the family of multiplications given by

$$
\gamma_i \circ_t \gamma_j = \gamma_i \cup \gamma_j - 4 \sum_{k=1}^{2} \frac{\partial^3 \Phi_{\text{qu}}}{\partial t^i \partial t^j \partial t^k} \gamma_{4-k}.
$$

\[\text{Note:} \Phi_{\text{qu}}(t^1, t^2) \text{ and } \Phi_{\text{qu}}(t^1, t^2) \text{ agree with those in Example 4.1 with } m = -4 \text{ if the filtration is shifted by one.}\]
More explicitly, the local quantum cup product is given by

\[
\gamma_0 \circ_t \gamma_0 = \gamma_0 \cup \gamma_0 , \\
\gamma_3 \circ_t \gamma_1 = \gamma_2 - 4 \left( \sum_{\beta > 0} \beta^2 N_\beta e^{\beta t} \right) \gamma_2 , \\
\gamma_1 \circ_t \gamma_1 = \gamma_2 - 4 \left( \sum_{\beta > 0} \beta^2 N_\beta e^{\beta t} \right) \gamma_2 , \\
\gamma_3 \circ_t \gamma_3 = \gamma_3 - 4 \left( \sum_{\beta > 0} \beta^2 N_\beta e^{\beta t} \right) \gamma_3 , \\
\gamma_2 \circ_t \gamma_2 = 0 ,
\]

(9.6)

Regard \( M = H^* (\mathbb{P}^3, \mathbb{C}) \) as a manifold. We identify each tangent space \( T_t M \) with \( H^* (\mathbb{P}^3, \mathbb{C}) \) by \( \frac{\partial}{\partial t} \mapsto \gamma_i \). Let \( \nabla \) be the trivial connection on \( TM \) such that \( \frac{\partial}{\partial t} \) are flat sections. We set

\[
E = t^0 \frac{\partial}{\partial t^0} - t^2 \frac{\partial}{\partial t^2} - 2t^3 \frac{\partial}{\partial t^3} .
\]

(9.7)

**Theorem 9.3.** The trivial connection \( \nabla \), the vector field \( E \), the local quantum cup product (9.6), the filtration (9.1) and the bilinear forms (9.2) form a mixed Frobenius structure on \( M \) of reference charge five.

A proof will be given in §9.3.

9.2. **Quantum cohomology of** \( V = \mathbb{P}(O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-4)) \). Let \( V = \mathbb{P}(O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-4)) \) be the projective compactification of the line bundle \( O(-4) \to \mathbb{P}^3 \). Let \( pr : V \to \mathbb{P}^3 \) be the projection and let \( \Gamma_i = pr^* \gamma_i \) (\( 0 \leq i \leq 3 \)). Let \( \Delta_0 \in H^2(V, \mathbb{C}) \) be the Poincaré dual of the zero section of \( O_{\mathbb{P}^3} \subset V \) and \( \Delta_i = \Delta_0 \cup \Gamma_i \in H^{2(i+1)}(V, \mathbb{C}) \). The cohomology \( H^*(V, \mathbb{C}) \) is spanned by the basis

\[
\Gamma_0 = 1 , \quad \Gamma_1 , \Delta_0 , \quad \Gamma_2 , \Delta_1 , \quad \Gamma_3 , \Delta_2 , \quad \Delta_3 .
\]

(9.8)

The cup product is given as follows.

\[
\Gamma_i \cup \Gamma_j = \begin{cases} 
\Gamma_{i+j} & (i + j \leq 3) \\
0 & (i + j > 3)
\end{cases} ,
\]

(9.9)

\[
\Delta_i \cup \Gamma_j = \begin{cases} 
\Delta_{i+j} & (i + j \leq 3) \\
0 & (i + j > 3)
\end{cases} ,
\]

\[
\Delta_i \cup \Delta_j = \begin{cases} 
4\Delta_{i+j+1} & (i + j < 3) \\
0 & (i + j \geq 3)
\end{cases} .
\]
The intersection form $\langle \ , \ \rangle$ is computed from the cup product and $\int_V \Delta_3 = 1$. Explicitly, pairings which do not vanish are

$$\langle \Gamma_0, \Delta_3 \rangle = 1, \quad \langle \Gamma_k, \Delta_{3-k} \rangle = 1,$$

$$\langle \Delta_{k-1}, \Delta_{3-k} \rangle = 4 \quad (1 \leq k \leq 3).$$

The dual basis of (9.8) is given by

$$\Gamma_0^\vee = \Delta_3, \quad \Delta_3^\vee = \Gamma_0,$$

$$\Gamma_k^\vee = \Delta_{3-k} - 4\Delta_{4-k}, \quad \Delta_{k-1}^\vee = \Gamma_{4-k} \quad (1 \leq k \leq 3).$$

Now we consider the quantum cup product. First we fix a basis of $H^2(V, \mathbb{Z})$. Let $\iota : \mathbb{P}^3 \to V$ be the inclusion as the zero section of $\mathcal{O}_{\mathbb{P}^3}(-4) \subset V$ and $C'_1 = \iota^*\mathbb{P}^1$. Let $C'_0$ be the fiber class of the $\mathbb{P}^1$-bundle $\text{pr} : V \to \mathbb{P}^3$. Then $\{C'_0, C'_1\}$ is a basis of $H_2(V, \mathbb{Z})$ which is dual to the basis $\{\Delta_0, \Gamma_1\}$ of $H^2(V, \mathbb{C})$. Let $t^0, t^1, s^0, t^2, s^1, t^3, s^2, s^3$ be the coordinates of $H^*(V)$ associated with the basis (9.8). Let $q_0, q_1$ be parameters associated to $C'_0, C'_1$.

**Lemma 9.4.** The genus zero Gromov–Witten potential $\Phi(t, s ; q)$ of $V$ becomes as follows.

$$\Phi(t, s ; q) = \Phi_{cl} + \Phi_{qu} + \Phi_1,$$

$$\Phi_{cl}(t, s) = \frac{1}{2} \sum_{0 \leq k, l \leq 3, k+l \leq 3} t^k t^l s^{3-k-l} + \frac{4}{2} \sum_{0 \leq k, l \leq 2, k+l \leq 2} t^{2-k-l} s^k s^l,$$

$$\Phi_1 = \mathcal{O}(q_0).$$

Here $\Phi_{qu}$ is the function in $t^1, t^2$ defined in (9.4).

**Proof.** Notice that the class $\beta \in H_2(V, \mathbb{Z})$ of an effective curve can be written as $\beta = \beta_0 C'_0 + \beta_1 C'_1 \neq 0$ with $\beta_0, \beta_1 \geq 0$. Since the moduli of the stable maps $\overline{M}_{0,n}(V, \beta)$ ($\beta \neq 0$) is empty if $\beta$ is not effective, we can decompose the Gromov–Witten potential $\Phi$ into three parts: $\Phi_{cl}$ which is the contribution of $\beta = 0$, $\Phi_0$ which is the contribution of homology classes $\beta$ such that $\beta = \beta_1 C'_1$ ($\beta_1 > 0$), and the remaining part $\Phi_1$ which is the contribution of $\beta = \beta_0 C'_0 + \beta_1 C'_1$ ($\beta_0, \beta_1 \geq 0$) with $\beta_0 \neq 0$.

$\Phi_{cl}$ can be computed by the triple intersection (see (8.4)).

We have $\Phi_1 = \mathcal{O}(q_0)$. 
Now we consider $\Phi_0$. Notice that since $-c_1(K_V) = 2\Delta_0$, the virtual dimension (8.1) of $\overline{M}_{0,n}(V, \beta)$ is $1 + n$ if $\beta_0 = 0$. Therefore we have

$$\int_{[\overline{M}_{0,n}(V, \beta)]^{vir}} \prod_{i=1}^{n} ev_i^* t = \sum_{j=1}^{n} \int_{[\overline{M}_{0,n}(V, \beta)]^{vir}} ev_j^* t' \prod_{1 \leq i \leq n; i \neq j} ev_i^* t'$$

$$= n(\beta_1 t_1)^{n-1} \int_{[\overline{M}_{0,1}(V, \beta)]^{vir}} ev_1^* t''$$

$$= n(\beta_1 t_1)^{n-1} t_2 \int_{[\overline{M}_{0,1}(V, \beta)]^{vir}} ev_1^* \Gamma_2$$

$$= n(\beta_1 t_1)^{n-1} t_2 N_{\beta_1} ,$$

$$\left( t' = t_1^1 \Gamma_1 + s^0 \Delta_0 \right) \quad \left( t'' = t_2^2 \Gamma_2 + s^1 \Delta_1 \right).$$

Here $N_{\beta_1}$ is the genus zero local Gromov–Witten invariant of $\mathbb{P}^3$ defined in (9.3). The first equality follows from the degree consideration and the fundamental class axiom, the second equality follows from the divisor axiom. The third and fourth equalities follow from the next lemma. This shows that $\Phi_0 = \Phi_{qu}$.

**Lemma 9.5.** For $\beta_1 > 0$,

$$\int_{[\overline{M}_{0,1}(V, \beta_1 C_1)]^{vir}} ev_1^* \Delta_1 = 0,$$

$$\int_{[\overline{M}_{0,1}(V, \beta_1 C_1)]^{vir}} ev_1^* \Gamma_2 = N_{\beta_1} .$$

**Proof.** Let us consider the $\mathbb{C}^*$-action in the fiber direction of $V$ and do the localization calculation [11]. The fixed point loci is

$$\overline{M}_{0,1}(V, \beta_1 C_1)^{\mathbb{C}^*} = \iota(\overline{M}_{0,1}(\mathbb{P}^3, \beta_1 \mathbb{P}^1)) .$$

Here we use $\iota$ also as the map $\overline{M}_{0,1}(\mathbb{P}^3, \beta_1 \mathbb{P}^1) \rightarrow \overline{M}_{0,1}(V, \beta_1 C_1)$ induced from the inclusion $\iota : \mathbb{P}^3 \rightarrow V$ as the zero section of $\mathcal{O}(-4)$. Therefore we have

$$\int_{[\overline{M}_{0,1}(V, \beta_1 C_1)]^{vir}} ev_1^* \Delta_1 = \int_{[\overline{M}_{0,1}(\mathbb{P}^3, \beta_1 \mathbb{P}^1)]^{vir}} \iota^* (ev_1^* \Delta_1) \cup e(R^1 \mu_* ev_2^* \mathcal{O}_{\mathbb{P}^3}(-4)) .$$

Here $e(R^1 \mu_* ev_2^* \mathcal{O}_{\mathbb{P}^3}(-4))$ is the contribution of the normal bundle of the fixed loci (see [16, Proposition 2.2]). Since the commutativity of the diagram

$$\overline{M}_{0,1}(\mathbb{P}^3, \beta_1 \mathbb{P}^1) \xrightarrow{\iota} \overline{M}_{0,1}(V, \beta_1 C_1)$$

$$\xrightarrow{ev_1} \mathbb{P}^3 \xrightarrow{ev_1} V$$

implies $\iota^* ev_1^* \Delta_1 = ev_1^* \iota^* \Delta_1 = 0$, we obtain the first statement. The proof of the second statement is similar. \qed
Let us consider the specialization $q_0 = 0$, $q_1 = 1$. Then by Lemma 9.4, the quantum cup product reduces to the following:

$$\Gamma_k \cup \Gamma_l = \Gamma_{k+l} + \sum_{j=1}^{2} \frac{\partial^3 \Phi_{q_0}}{\partial t^k \partial t^l \partial t^j} (\Gamma_j)^\vee,$$

(9.11)

$$\Delta_i \circ = \Delta_i \cup.$$

**Lemma 9.6.** The multiplication (9.11), the intersection form (9.10) (regarded as the multiplication on $T_t \tilde{M}$ and the metric by the canonical isomorphism $H^*(V) \cong T_t \tilde{M}$) and the vector field

$$(9.12)\quad E = t^0 \frac{\partial}{\partial t^0} - \sum_{k=2}^{3} (k-1) \left( t^k \frac{\partial}{\partial t^k} + s^{k-1} \frac{\partial}{\partial s^{k-1}} \right) - 3s^3 \frac{\partial}{\partial s^3} + 2 \frac{\partial}{\partial s^0}$$

form a Frobenius structure of charge four on $\tilde{M} = H^*(V, \mathbb{C})$.

The proof of the lemma is similar to that of Lemma 8.10 and omitted.

9.3. **Proof of Theorem 9.3.** We first apply Theorem 7.2 to the Frobenius structure on $\tilde{M} = H^*(V, \mathbb{C})$ in Lemma 9.6 with the nilpotent vector field $\frac{\partial}{\partial \varphi^0}$.

Let us construct the Frobenius filtration. Using the canonical isomorphism $T_t \tilde{M} \cong H^*(V, \mathbb{C})$, we write it down as that of $H^*(V, \mathbb{C})$. Let $I$ be the ideal generated by $\Delta_0$:

$$I = \mathbb{C}\Delta_0 \oplus \mathbb{C}\Delta_1 \oplus \mathbb{C}\Delta_2 \oplus \mathbb{C}\Delta_3 \subset H^*(V).$$

$J_k = \text{Ker}(\Delta_0 \cup)$ are as follows.

$$J_1 = \mathbb{C}\Gamma_3^{\vee} \oplus \mathbb{C}\Gamma_2^{\vee} \oplus \mathbb{C}\Gamma_1^{\vee} \oplus \mathbb{C}\Delta_3,$$

$$J_2 = J_1 + (\mathbb{C}\Delta_2 \oplus \mathbb{C}\Delta_3) = \mathbb{C}\Gamma_3^{\vee} \oplus \mathbb{C}\Gamma_2^{\vee} \oplus H^6(V),$$

(9.13)

$$J_3 = \mathbb{C}\Gamma_3^{\vee} \oplus H^4(V),$$

$$J_4 = H^2(V),$$

$$J_5 = H^*(V).$$

Therefore the filtration $I_\bullet$ defined in (3.1) on $H^*(V, \mathbb{C})$ is given by

$$I_0 = I, \quad I_k = H^{2k}(V) \quad (1 \leq k \leq 4), \quad I_5 = H^*(V).$$

(9.14)
The bilinear forms on $I_k/I_{k-1}$ (see Definition (3.1)) are

$$k = 0 \quad (\Delta_k, \Delta_l)_0 = \langle \Delta_k, \Gamma_l \rangle = \delta_{k+l,3},$$

$$k = 1 \quad ([\Gamma_k], [\Gamma_l])_1 = \langle \Gamma_k - \frac{1}{4} \Delta_{k-1}, \Gamma_l - \frac{1}{4} \Delta_{l-1} \rangle$$

(9.15)

$$= \begin{cases} -\frac{1}{4} \quad & (k + l = 4) \\ 0 \quad & (k + l \neq 4) \end{cases},$$

$$k = 5 \quad (1, 1)_5 = \langle 1, \Delta_0^4 \rangle = 4^3.$$  

By Theorem 7.2, we have the following

**Lemma 9.7.** The trivial connection, the vector field $E$ (9.12), the multiplication (9.11), the filtration (9.14) and the bilinear forms (9.15) form a MFS of reference charge five on $\tilde{M} = H^*(V, \mathbb{C})$.

Next we apply Corollary 7.3. Since $I$ is the kernel of the pullback $\iota^* : H^*(V, \mathbb{C}) \to H^*(\mathbb{P}^3, \mathbb{C})$ by the inclusion $\iota : \mathbb{P}^3 \hookrightarrow \mathcal{O}(-4) \subset V$, if we set

$$\tilde{M}^{(1)} = \{ s^0 = s^1 = s^2 = s^3 = 0 \} \subset \tilde{M},$$

then it is naturally isomorphic to $H^*(\mathbb{P}^3, \mathbb{C})$. Theorem 9.3 follows from Corollary 7.3 and Lemma 9.6.

**Appendix A. Deformed connection**

In this appendix, we define an analogue of the deformed connection of the Frobenius structure [10] for the MFS. Let $(\nabla, E, \circ, I_\bullet, (\ , \ )_\bullet)$ be a MFS on $M$ of reference charge $D$ and let $t^{ka} (k \in \mathbb{Z}, \ 1 \leq a \leq m_k)$ be a system of local flat coordinates satisfying (6.11).

**A.1. Operators.** Define endomorphisms $U, V : TM \to TM$ by

$$U(x) = E \circ x, \quad V(x) = \nabla_x E = -\frac{2 - D}{2} x.$$  

(3.1)

**Lemma A.1.** If $x \in \Gamma(I_k)$, then $U(x), V(x) \in \Gamma(I_k)$.

**Proof.** If $x \in \Gamma(I_k)$, $U(x) = E \circ x \in \Gamma(I_k)$ since $I_k$ is an ideal.

If $x \in \Gamma(I_k)$, we have

$$\nabla_x E = [E, x] - \nabla_E x \in \Gamma(I_k),$$

by the torsion free condition for $\nabla$ and the assumptions that $I_k$ is $E$-closed and $\nabla$-closed.  

$\square$
The above lemma implies that $\mathcal{U}$, $\mathcal{V}$ induce endomorphisms $\mathcal{U}^{(k)}$, $\mathcal{V}^{(k)}$ on $I_k/I_{k-1}$. In the local flat coordinate expression,

$$\mathcal{U}(\partial_{ka}) = \sum_{l \in \mathbb{Z}} \sum_{1 \leq b \leq m_l} \sum_{1 \leq c \leq m_j} E^{lb} C^{jc}_{ka,lb} \partial_{jc},$$

$$\mathcal{U}^{(k)}(\partial_{ka}) = \sum_{l \geq k} \sum_{1 \leq b \leq m_l} \sum_{1 \leq c \leq m_k} E^{lb} C^{kc}_{ka,lb} \partial_{kc},$$

$$\mathcal{V}(\partial_{ka}) = \sum_{l \leq k} \sum_{1 \leq b \leq m_l} (\partial_{ka} E^{lb}) \partial_{lb} - \frac{2}{\hbar} \partial_{ka},$$

$$\mathcal{V}^{(k)}(\partial_{ka}) = \sum_{1 \leq b \leq m_k} (\partial_{ka} E^{kb}) \partial_{kb} - \frac{2}{\hbar} \partial_{ka}.$$

**Remark A.2.** The assumption $\nabla \nabla E = 0$ implies $\nabla \mathcal{V} = 0$. In other words, the matrix representations of $\mathcal{V}$ and $\mathcal{V}^{(k)}$ with respect to the flat basis $\{\partial_{ka}\}$ are constant matrices. Notice also that the condition (6.10) is equivalent to

$$\mathcal{V}^{(k)}(x), y) + (x, \mathcal{V}^{(k)}(y)) = k(x, y).$$

**A.2. Deformed connection.** Let $\tilde{M} = M \times \mathbb{C}^*$ and let $\hbar$ be the coordinate of $\mathbb{C}^*$. For a holomorphic vector bundle $\mathcal{E} \to \tilde{M}$, $\tilde{\Gamma}(\mathcal{E})$ denotes the space of holomorphic sections of $\mathcal{E}$ on some open subset $\tilde{U} \subset \tilde{M}$.

Recall that $\pi_k : TM \to TM/I_{k-1}$ is the projection and that $\nabla^{(k)}$ and $\phi_k$ are the connection and the multiplication on $TM/I_{k-1}$ induced from the connection $\nabla$ and the multiplication $\circ$.

**Definition A.3.** Define a connection $\tilde{\nabla}^{(k)}$ on $I_k/I_{k-1} \times T \mathbb{C}^* \to \tilde{M}$ by

$$\tilde{\nabla}^{(k)}_x y = \nabla^{(k)}_x y + \hbar \pi_k(x) \phi_k y \quad (x \in \tilde{\Gamma}(TM), \ y \in \tilde{\Gamma}(I_k/I_{k-1})).$$

$$\tilde{\nabla}^{(k)}_h y = \partial_h y + \mathcal{U}^{(k)}(y) + \frac{1}{\hbar} (\mathcal{V}^{(k)}(y) - k y),$$

$$\tilde{\nabla}^{(k)}_x (\partial_{h}) = \tilde{\nabla}^{(k)}_h (\partial_{h}) = 0.$$

We write $\tilde{\nabla}^{(k)}_{la}$ for $\tilde{\nabla}^{(k)}_x$ with $x = \partial_{la}$. In the local flat coordinate expression,

$$\tilde{\nabla}^{(k)}_h (\partial_{ka}) = \hbar \sum_{1 \leq c \leq m_k} C^{kc}_{ka,lb} \partial_{kc} \quad \text{(this is zero if } l < k \text{ by (6.14)}),$$

$$\tilde{\nabla}^{(k)}_h (\partial_{ka}) = \sum_{l \geq k} \sum_{1 \leq b \leq m_l} (E^{lb} C^{kc}_{ka,lb}) \partial_{kc}$$

$$+ \frac{1}{\hbar} \left( \sum_{1 \leq b \leq m_k} (\partial_{ka} E^{kb}) \partial_{kc} - \frac{2}{\hbar} \partial_{ka} \right).$$
Proposition A.4. The deformed connection $\tilde{\nabla}^{(k)}$ is flat.

Proof. Let $\tilde{\Omega}^{(k)}$ be the curvature of $\tilde{\nabla}^{(k)}$. We first show that $\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{jb}) = \frac{1}{2}(\tilde{\nabla}^{(k)}_{la} \tilde{\nabla}^{(k)}_{jb} - \tilde{\nabla}^{(k)}_{jb} \tilde{\nabla}^{(k)}_{la}) = 0$. By the first equation in (A.4), it is immediate to check that $\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{jb})(\partial_{kc}) = 0$ if $l, j < k$. If $l < k$ and $j \geq k$,

$$\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{jb})(\partial_{kc}) = \tilde{\nabla}^{(k)}_{la} \left( \sum_{1 \leq d \leq m_k} C^{kd}_{jb, kc} \partial_{kd} \right) = \sum_{1 \leq d \leq m_k} (\partial_{la} C^{kd}_{jb, kc}) \partial_{kd} \overset{(6.14)}{=} 0 .$$

If $l, j \geq k$,

$$\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{jb})(\partial_{kc}) = \tilde{\nabla}^{(k)}_{la} \left( \sum_{1 \leq d \leq m_k} C^{kd}_{jb, kc} \partial_{kd} \right) - (l a \leftrightarrow j b)$$

$$= \sum_{1 \leq d \leq m_k} (\partial_{la} C^{kd}_{jb, kc} - \partial_{jb} C^{kd}_{la, kc}) \partial_{kd}$$

$$+ \hbar \sum_{1 \leq d, f \leq m_k} (C^{kd}_{jb, kc} C^{kf}_{la, kd} - C^{kd}_{la, kc} C^{kf}_{jb, kd}) \partial_{kf} = 0 ,$$

by (6.14) and the associativity.

Next we show $\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{h}) = \frac{1}{2}(\tilde{\nabla}^{(k)}_{la} \tilde{\nabla}^{(k)}_{h} - \tilde{\nabla}^{(k)}_{h} \tilde{\nabla}^{(k)}_{la}) = 0$. By the second equation in (A.4), (6.15) and (6.14), if $l < k$, we have

$$\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{h})(\partial_{kc}) = \sum_{j \geq k, 1 \leq b \leq m_j} \sum_{1 \leq d \leq m_k} (\partial_{la}(E^{jb} C^{kd}_{kc, jb})) \partial_{kd} = 0 .$$

If $l \geq k$, we have

$$\tilde{\nabla}^{(k)}_{la} \tilde{\nabla}^{(k)}_{h} (\partial_{kc}) = \sum_{1 \leq d \leq m_k} \left( \sum_{j \geq k, 1 \leq b \leq m_j} \partial_{la}(E^{jb} C^{kd}_{kc, jb}) + \sum_{1 \leq b \leq m_k} C^{kd}_{la, kb} \partial_{kc} E^{kb} - \frac{2 - D + k}{2} C^{kd}_{la, kc} \right) \partial_{kd}$$

$$+ \hbar \sum_{1 \leq f \leq m_k} \left( \sum_{1 \leq d \leq m_k} \sum_{j \geq k, 1 \leq b \leq m_j} E^{jb} C^{kd}_{kc, jb} C^{kf}_{la, kd} \right) \partial_{kf} ,$$

$$\tilde{\nabla}^{(k)}_{h} \tilde{\nabla}^{(k)}_{la} (\partial_{kc}) = \sum_{1 \leq d \leq m_k} \left( C^{kd}_{la, kc} + \sum_{1 \leq b \leq m_k} C^{kb}_{la, kc} \partial_{kb} E^{kd} - \frac{2 - D + k}{2} C^{kd}_{la, kc} \right) \partial_{kd}$$

$$+ \hbar \sum_{1 \leq f \leq m_k} \left( \sum_{1 \leq d \leq m_k} \sum_{j \geq k, 1 \leq b \leq m_j} E^{jb} C^{kd}_{la, kc} C^{kf}_{kd, jb} \right) \partial_{kf} .$$

Thus by (6.16) and the associativity, we obtain

$$\tilde{\Omega}^{(k)}(\partial_{la}, \partial_{h})(\partial_{kc}) = \tilde{\nabla}^{(k)}_{la} \tilde{\nabla}^{(k)}_{h} (\partial_{kc}) - \tilde{\nabla}^{(k)}_{h} \tilde{\nabla}^{(k)}_{la} (\partial_{kc}) = 0 .$$

$\square$
A.3. Deformed flat coordinates.

Proposition A.5. There exist (local) holomorphic functions $\tilde{t}^{ka}(t, h)$ ($k \in \mathbb{Z}, 1 \leq a \leq m_k$) on $\tilde{M}$ such that $h, \tilde{t}^{ka}(t, h)$ are a system of local coordinates on $\tilde{M}$ satisfying the following conditions:

$$\left\{ \frac{\partial}{\partial \tilde{t}^{ia}} \mid l \leq k, 1 \leq a \leq m_l \right\} \text{ is a local frame of } I_k,$$

(A.5) $\tilde{\nabla}^{(k)} \pi_k \left( \frac{\partial}{\partial \tilde{t}^{ka}} \right) = 0 .

We call $\tilde{t}^{ka}(t, h)$ deformed flat coordinates.

Proof. Let $\text{Ann}_k \subset T^*M$ be the annihilator of $I_k$:

$$\text{Ann}_k := \{ x \in T^*M \mid x(y) = 0, \forall y \in I_k \} .$$

Its local frame is given by $dt^{la}$ ($l > k$). Notice that the dual bundle of $I_k/I_{k-1}$ is isomorphic to $\text{Ann}_{k-1}/\text{Ann}_k$. We use the same notation $\nabla^{(k)}$ for the induced dual connection on $\text{Ann}_{k-1}/\text{Ann}_k \times T^*\mathbb{C}^* \to \tilde{M}$.

The $\nabla^{(k)}$-flatness condition $\tilde{\nabla}^{(k)} \xi = 0$ for a section $\xi = \sum_{1 \leq a \leq m_k} \xi_{ka} dt^{ka} \in \tilde{\Gamma}(\text{Ann}_{k-1}/\text{Ann}_k)$ is equivalent to

$$\partial_{lb}(\xi_{ka}) = h \sum_{1 \leq c \leq m_k} C^{kc}_{lb, ka} \xi_{kc} \quad \text{and}$$

(A.6) $$\partial_h(\xi_{ka}) = \sum_{l \geq k, 1 \leq c \leq m_k} \sum_{1 \leq c \leq m_k} E^{lc} C^{kc}_{lb, ka} \xi_{kc} + \frac{1}{h} \left( \sum_{1 \leq c \leq m_k} (\partial_{ka} E^{kc}) \xi_{kc} - \frac{2 - D + k}{2} \xi_{ka} \right) .$$

The first equation in (A.6) implies that $\partial_{lb} \xi_{ka} = \partial_{ka} \xi_{kb}$ and $\partial_{lb} \xi_{ka} = 0$ if $l < k$. Therefore if $\tilde{\nabla}^{(k)} \xi = 0$, there exists a local function $\tilde{t} = \tilde{t}(t, h)$ on $\tilde{M}$ satisfying $\partial_{ka} \tilde{t} = \xi_{ka}$ and $\partial_{ia} \tilde{t} = 0$ ($l < k$). In other words, there exists $\tilde{t}(t, h)$ such that

$$d\tilde{t} = \xi + \text{ terms involving } dt^{lb} (l > k) \text{ and } dh .$$

Since $\tilde{\nabla}^{(k)}$ is flat, there exists a local frame $\{ p^{ka}(t, h) \mid 1 \leq a \leq m_k \}$ of $\text{Ann}_{k-1}/\text{Ann}_k \otimes T^*\mathbb{C}^* \to \tilde{M}$ such that $\tilde{\nabla}^{(k)} p^{ka} = 0$. From the argument in the previous paragraph, we see that there exist local functions $\tilde{t}^{ka}(t, h)$ ($k \in \mathbb{Z}, 1 \leq a \leq m_k$) satisfying the following two conditions:

$$\{ d\tilde{t}^{(la)} \mid l \geq k, 1 \leq a \leq m_l \} \text{ is a local frame of } \text{Ann}_{k-1},$$

$$p^{(ka)} = d\tilde{t}^{(ka)} \mod dt^{lb} (l > k), dh .$$

These $\tilde{t}^{(ka)}(t, h)$ satisfy the conditions in the above proposition. $\square$
A.4. **Deformed flat coordinates for weak Fano toric surfaces.** The deformed flat coordinates for the MFS in Theorem 8.7 is written as follows. Assume that $b^r_\nu \neq 0$. We take the following flat coordinates on $M = H^*(S, \mathbb{C})$ so that the condition (6.11) is satisfied:

\[
t^{r+1}, \quad u^r = -\frac{1}{\kappa} \sum_{k=1}^{r} b^r_k t^k, \\
u^k = \frac{1}{\kappa \cdot b^r_k} \left\{ \sum_{j \neq k, 1 \leq j \leq r} b_j b^r_j t^k - b_k \sum_{j \neq k, 1 \leq j \leq r} b^r_j t^j \right\} \quad (1 \leq k \leq r - 1),
\]

(A.7)

Solving the flatness equation for $\tilde{\nabla}^{(k)} (1 \leq k \leq 4)$, we obtain the following deformed flat coordinates.

\[
\tilde{t}^{r+1} = e^{ht^0} \left\{ \frac{1}{\sqrt{h}} t^{r+1} + \sqrt{h} \left( \frac{\kappa}{2} u^r_2 - \sum_{\beta \neq 0} (b \cdot \beta) N_{\beta} e^{\beta t_1} \right) \right\}, \\
\tilde{u}^r = \sqrt{h} e^{ht^0} u^r, \\
\tilde{u}^k = e^{ht^0} u^k \quad (1 \leq k \leq r - 1), \\
\tilde{t}^0 = e^{ht^0}.
\]

(A.8)

A.5. **Deformed flat coordinates for $\mathbb{P}^3$.** The deformed flat coordinates for the MFS in Theorem 9.3 is written as follows.

\[
\tilde{t}^0 = \frac{1}{\hbar} e^{ht^0}, \\
\tilde{t}^1 = \hbar e^{ht^0} t^1, \\
\tilde{t}^2 = e^{ht^0} \left\{ t^2 + \hbar \left( \frac{(t^1)^2}{2} - 4 \sum_{\beta > 0} N_{\beta} e^{\beta t_1} \right) \right\}, \\
\tilde{t}^3 = e^{ht^0} \left\{ \frac{t^3}{\hbar} + t^1 t^2 - 4t^2 \sum_{\beta > 0} \beta N_{\beta} e^{\beta t_1} \\
+ \frac{\hbar}{2} \left[ \frac{(t^1)^3}{3} - 8 \sum_{\beta > 0} \left( t^1 - \frac{1}{\beta} \right) N_{\beta} e^{\beta t_1} + 16 \sum_{\beta, \gamma > 0} \frac{\beta \gamma}{\beta + \gamma} N_{\beta} N_{\gamma} e^{(\beta + \gamma) t_1} \right] \right\}. 
\]

(A.9)

**References**

[1] Barannikov, Sergey; Kontsevich, Maxim, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices 1998, no. 4, 201–215.

[2] Barannikov, Serguei, *Non-commutative periods and mirror symmetry in higher dimensions*, Comm. Math. Phys. 228 (2002), no. 2, 281–325.

\[\text{We omit the detail of the calculation for the following results. It can be found in the first version of this paper at the arXiv.}\]
[3] Barth, Wolf P.; Hulek, Klaus; Peters, Chris A. M.; Van de Ven, Antonius, Compact complex surfaces, Second Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 4, Springer-Verlag, Berlin, 2004.

[4] Borisov, Lev A.; Chen, Linda; Smith, Gregory G., The orbifold Chow ring of toric Deligne–Mumford stacks, J. Amer. Math. Soc. 18 (2005), no. 1, 193–215.

[5] Cattani, Edouardo H., Mixed Hodge Structures, Compactifications and Monodromy Weight Filtration, in Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), 75–100, Ann. of Math. Stud. 106, Princeton Univ. Press, Princeton, NJ, 1984.

[6] Chen, Weimin; Ruan, Yongbin, A new cohomology theory of orbifold, Comm. Math. Phys. 248 (2004), no. 1, 1–31.

[7] Chiang, T.-M.; Klemm, A.; Yau, S.-T.; Zaslow, E., Local mirror symmetry: calculations and interpretations, Adv. Theor. Math. Phys. 3 (1999), no. 3, 495–565.

[8] Cox, David A.; Katz, Sheldon, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs 68, American Mathematical Society, Providence, RI, 1999.

[9] Dubrovin, Boris, Geometry of 2D topological field theories, in Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math. 1620, Springer, Berlin, 1996.

[10] ______, Painlevé transcendents in two-dimensional topological field theory, in The Painlevé property, 287–412, CRM Ser. Math. Phys., Springer, New York, 1999.

[11] Graber, T.; Pandharipande, R. Localization of virtual classes, Invent. Math. 135 (1999), no. 2, 487–518.

[12] Iritani, Hiroshi, Convergence of quantum cohomology by quantum Lefschetz, J. Reine Angew. Math. 610 (2007), 29–69.

[13] Jiang, Yunfeng, The Chen–Ruan cohomology of weighted projective spaces, Canad. J. Math. 59 (2007), no. 5, 981–1007.

[14] Kashiwara, Masaki, A study of variation of mixed Hodge structure, Publ. Res. Inst. Math. Sci. 22 (1986), no. 5, 991–1024.

[15] Kock, Joachim, Frobenius algebras and 2D topological quantum field theories, London Mathematical Society Student Texts, 59. Cambridge University Press, Cambridge, 2004.

[16] Konishi, Yukiko; Minabe, Satoshi, Local Gromov–Witten invariants of cubic surfaces via nef toric degeneration, Ark. Mat. 47 (2009), no. 2, 345–360.

[17] ______, Local B-model and mixed Hodge structure, Adv. Theor. Math. Phys. 14 (2010), no. 4, 1089–1145.

[18] Kontsevich, M.; Manin, Yu., Gromov–Witten classes, quantum cohomology, and enumerative geometry, Comm. Math. Phys. 164 (1994), no. 3, 525–562.

[19] Mann, Etienne, Orbifold quantum cohomology of weighted projective spaces, J. Algebraic Geom. 17 (2008), no. 1, 137–166.

[20] Reid, Miles, Young person’s guide to canonical singularities, in Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[21] Klemm, A.; Pandharipande, R., Enumerative geometry of Calabi–Yau 4-folds, Comm. Math. Phys. 281 (2008), no. 3, 621–653.

[22] Manin, Yu. I., Three constructions of Frobenius manifolds: a comparative study, in Surveys in differential geometry, 497–554, Surv. Differ. Geom. VII, Int. Press, Somerville, MA, 2000.
[23] Peters, Chris A. M.; Steenbrink, Joseph H. M., *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 52, Springer-Verlag, Berlin, 2008.

[24] Saito, Kyoji, *Period mapping associated to a primitive form*, Publ. Res. Inst. Math. Sci. 19 (1983), no. 3, 1231–1264.

[25] Saito, Kyoji; Takahashi, Atsushi, *From primitive forms to Frobenius manifolds*, in From Hodge theory to integrability and TQFT tt*-geometry, 31–48, Proc. Sympos. Pure Math. 78, Amer. Math. Soc., Providence, RI, 2008.

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