Adaptive Minimax Testing for Circular Convolution

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Abstract—Given observations from a circular random variable contaminated by an additive measurement error, we consider the problem of minimax optimal goodness-of-fit testing in a non-asymptotic framework. We propose direct and indirect testing procedures using a projection approach. The structure of the optimal tests depends on regularity and ill-posedness parameters of the model, which are unknown in practice. Therefore, adaptive testing strategies that perform optimally over a wide range of regularity and ill-posedness classes simultaneously are investigated. Considering a multiple testing procedure, we obtain adaptive i.e. assumption-free procedures and analyse their performance. Compared with the non-adaptive tests, their radii of testing face a deterioration by a log-factor. We show that for testing of uniformity this loss is unavoidable by providing a lower bound. The results are illustrated considering Sobolev spaces and ordinary or super smooth error densities.

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1. INTRODUCTION

The statistical model. We consider a circular convolution model where a random variable that takes values on the circle is observed contaminated by an additive error. Identifying the circle with the unit interval $[0, 1)$, the observable random variable can be expressed as

$$Y = X + \varepsilon - \lfloor X + \varepsilon \rfloor,$$

(1.1)

where $X$ and $\varepsilon$ are independent random variables supported on the interval $[0, 1)$ and $\lfloor \cdot \rfloor$ denotes the floor-function. The aim of this paper is to investigate adaptive testing procedures for the circular density $f$ of $X$ given a sample of independent and identically distributed (iid) copies of $Y$. If $\varphi$ denotes the density of the error $\varepsilon$, then the observable random variable $Y$ admits a density $g = f \circledast \varphi$, where $\circledast$ denotes circular convolution defined by

$$f \circledast \varphi(y) = \int_{[0,1]} f(y - s - \lfloor y - s \rfloor) \varphi(s) ds, \quad y \in [0, 1).$$

Hence, making inference on $f$ based on observations from $g$ is a deconvolution problem. Circular, wrapped (around the circumference of the unit circle), spherical or directional data appear in various applications. We briefly mention two popular fields. Circular models are used for data with a temporal or periodic structure, where the circle is identified e.g., with a clock face (cp. [1]). Moreover, identifying the circle with a compass rose, directional data can also be represented by a circular model. For many more examples of circular data we refer the reader to [8, 25, 26], [17, 19], for instance, investigated a circular model with multiplicative error. Nonparametric estimation in the additive error model (1.1)

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has amongst others been considered in [6, 7, 16]. In the circular model (1.1) the densities \( f, \phi \) and \( g \) have a representation as (discrete) Fourier series, which are formally introduced below and exploited in our testing procedure. In contrast, in a deconvolution model on the real line one usually works with (continuous) Fourier transforms of the densities, thus requiring different proof techniques. For a broad overview of deconvolution models we refer to the monograph [29].

Testing task. For a prescribed density \( f^0 \) we test the null hypothesis \( \{ f = f^0 \} \) against the alternative \( \{ f \neq f^0 \} \) based only on observations of \( Y \). We separate the null hypothesis and the alternative to make them distinguishable. Consider the Hilbert space \( L^2 := L^2([0, 1]) \) of square-integrable complex-valued functions on \([0, 1] \) equipped with its usual norm \( \| \cdot \|_{L^2} \). We assume throughout this paper that both \( f \) and \( \phi \) (and, hence, \( g \)) belong to the subset of real probability densities \( D \) in \( L^2 \). Let \( \{ Y_j \}_{j=1}^n \) be \( n \) independent and identically distributed copies of \( Y \), i.e., the observations are given by

\[
\{ Y_j \}_{j=1}^n \overset{\text{iid}}{\sim} g = f \oplus \phi.
\]

(1.2)

Denote by \( \mathbb{P}_f \) and \( \mathbb{E}_f \) the probability distribution and the expectation associated with the data (1.2), respectively. For a separation radius \( \rho \in \mathbb{R}_+ \), let us define the energy set \( L^2_\rho := \{ f \in L^2 : \| f \|_{L^2} \geq \rho \} \).

For a nonparametric class of functions \( \mathcal{E} \), capturing the regularity of the alternative, the testing task can be written as

\[
H_0 : f = f^0 \text{ against } H_1^\rho : f - f^0 \in L^2_\rho \cap \mathcal{E}, \quad f \in D.
\]

(1.3)

In the literature there exist several definitions of rates and radii of testing in an asymptotic and nonasymptotic sense. The classical definition of an asymptotic rate of testing for nonparametric alternatives is essentially introduced in the series of papers [12–14]. An overview of many results in this setting can be found in [15]. For a fixed sample size, two alternative definitions of a nonasymptotic radius of testing are typically considered. For prescribed error probabilities \( \alpha, \beta \in (0, 1) \), [1, 24, 28], amongst others, define a nonasymptotic radius of testing as the smallest separation radius \( \rho \) such that there is an \( \alpha \)-test with maximal type II error probability over the \( \rho \)-separated alternative smaller than \( \beta \). The connection between the asymptotic and nonasymptotic approach is explained in [27]. The (nonasymptotic) definition we use in this paper—which is based on the sum of both error probabilities—is adapted e.g., from [5]. We measure the accuracy of a test \( \Delta \), i.e., a measurable function \( \Delta : \mathbb{R}^n \to \{0, 1\} \), by its maximal risk de

\[
\mathcal{R}(\Delta|\mathcal{E}, \rho) := \mathbb{P}_{f^0}(\Delta = 1) + \sup_{f \in D : f - f^0 \in L^2_\rho \cap \mathcal{E}} \mathbb{P}_f(\Delta = 0).
\]

We are particularly interested in the smallest possible value of \( \rho \) for which the null and the \( \rho \)-separated alternative are still distinguishable. A value \( \rho^2(\mathcal{E}) \) is called an upper bound for the radius of testing for a family of tests \( \{ \Delta, \gamma \in (0, 1) \} \) over the alternative \( \mathcal{E} \), if for all \( \gamma \in (0, 1) \) there exists a constant \( A_\gamma \in \mathbb{R}_+ \) such that

\[
\text{for all } A \geq A_\gamma \text{, we have } \mathcal{R}(\Delta, \gamma|\mathcal{E}, A \rho(\mathcal{E})) \leq \gamma. \quad \text{(upper bound)}
\]

The difficulty of the testing task can be characterised by the minimax risk

\[
\mathcal{R}(\mathcal{E}, \rho) := \inf_{\Delta} \mathcal{R}(\Delta|\mathcal{E}, \rho),
\]

where the infimum is taken over all possible tests. The value \( \rho^2(\mathcal{E}) \) is called minimax radius of testing, if in addition for all \( \gamma \in (0, 1) \) there exists a constant \( A_\gamma \in \mathbb{R}_+ \) such that

\[
\text{for all } A \leq A_\gamma \text{, we have } \mathcal{R}(\mathcal{E}, A \rho(\mathcal{E})) \geq 1 - \gamma. \quad \text{(lower bound)}
\]

and the family \( \{ \Delta, \gamma \in (0, 1) \} \) is then called minimax optimal. Roughly speaking, in the upper bound we want the minimax risk to be small, while it should be close to one in the lower bound case. Hence, in both the upper and the lower bound a small parameter \( \gamma \) represents the relevant situation.

Direct and indirect testing procedures. Considering the density \( f^0 = \mathbb{1}_{[0,1]} \) of a uniform distribution only, minimax radii of testing in the circular model (1.1) are for example derived in [31] for
nonparametric alternatives $\mathcal{E}$, covering Sobolev spaces and ordinary or super smooth error densities. The authors consider a test that is based on a projection estimator of the quantity $||f - f^0||_{L^2}^2$, depending on a dimension parameter. Estimation of squared $L^2$-distances or functionals of densities have extensively been treated in the literature, we refer for instance to [21] or [20] in a direct Gaussian sequence space model, to [3] in an inverse model and to [2] for deconvolution on the real line. The connection between functional estimation and testing is amongst others explored in [5, 18]. The test considered in [30], roughly speaking, compares the estimator to a multiple of its standard deviation. Choosing the dimension parameter optimally, they show that this test is minimax optimal, i.e., it achieves the minimax radius of testing given by a typical bias–variance trade-off. However, estimating $||f - f^0||_{L^2}^2$ based on observations (1.2) in deconvolution models is an inverse problem, since it requires an inversion of the convolution transformation. This inversion introduces additional instability in deconvolution problems, caused by its ill-posedness. To circumvent this problem, in an inverse Gaussian sequence space model [23] argue for a direct testing procedure, which is based on the estimation of the energy in the image space of the operator. Let us explain this idea in our setting. Instead of the direct testing task (1.3) we examine the indirect testing task of the null hypothesis $\{g^0 = g\}$ with $g^0 := f^0 \odot \varphi$ against the alternative $\{g^0 \neq g\}$, where we have direct access to observations from $g$. Using similar arguments as [31], it is possible to derive the testing radii of a direct test, based on a projection estimator of the direct quantity $||g - g^0||_{L^2}^2$. For both the direct and indirect testing procedure the radii of testing, depending on the dimension parameter, are essentially determined by a bias–variance trade-off. As usual the optimal choice of the dimension parameter depends on both the smoothness of the alternative and the ill-posedness of the model, which are unknown in practice. This motivates the study of adaptive testing procedures, which we investigate in this paper.

**Adaptive testing.** In the literature adaptive, i.e., assumption-free, testing strategies have been studied in both an asymptotic and a nonasymptotic framework. In an asymptotic framework, e.g., [32] considers adaptive testing strategies in a sequence space model with Besov-type alternatives, showing that asymptotic adaptation comes with an unavoidable cost of a log–factor. In a nonasymptotic setting, [22] consider adaptive testing in a Gaussian regression model, [9] deal with a density model. Papers [2, 4] determine adaptive rates of testing in a convolution model on the real line using kernel estimators of the $L^2$-distance to the null. The proposed tests have as a common feature that they are based on estimators of the distance to the null, which only depend on the (unknown) smoothness through a tuning parameter (e.g., a bandwidth, a threshold or a dimension parameter). By aggregating the estimators over different tuning parameters into one test statistic—a multiple testing approach—the authors obtain tests, which perform optimally over a wide range of alternatives. Since they no longer depend on the unknown regularity of the alternative, they are assumption-free. To formalise this idea, let us introduce a collection $\mathcal{A}$ of regularity parameters that characterise a family of alternatives $\{\mathcal{E}_{a_\cdot} \subset \mathcal{A}\}$ with corresponding radii $\{\rho_{a_\cdot}(n) := \rho(\mathcal{E}_{a_\cdot}), a_\cdot \in \mathcal{A}\}$, where we now explicitly emphasise the dependence on the regularity parameter $a_\cdot \in \mathcal{A}$ and the number of observations $n$ in the notation. In general, adaptation without a loss is impossible (cp. [32]). To characterise the cost to pay for adaptation, we introduce the effective sample size $\delta_{\cdot,n}$. The factor $\delta_{\cdot,n} \in [0,1]$ shrinks the sample size $n$ and, hence, evaluating the radius at $\delta_{\cdot,n}$ deteriorates the radius of testing. In fact, the value $\delta_{\cdot,n}^{-1}$ is called adaptive factor for the family of tests $\{\Delta_{\gamma}, \gamma \in (0,1)\}$ over the family of alternatives $\{\mathcal{E}_{a_\cdot}, a_\cdot \in \mathcal{A}\}$, if for all $\gamma \in (0,1)$ there exists a constant $\underline{A}_{\cdot,\gamma} \in \mathbb{R}_+$ such that

$$\sup_{\Delta_{\gamma}} \mathcal{R}(\Delta_{\gamma} | \mathcal{E}_{a_\cdot}), A \rho_{a_\cdot}(\delta_{\cdot,n})) \leq \gamma,$$

where $\rho_{a_\cdot}(n)$ denotes a radius of testing of the family $\Delta_{\gamma}, \gamma \in (0,1)$. We shall emphasise that the testing risk now has to be bounded uniformly for all alternatives $\mathcal{E}_{a_\cdot}$. We call $\delta_{\cdot,n}^{-1}$ minimal adaptive factor if for all $\gamma \in (0,1)$ there exists a constant $\underline{A}_{\cdot,\gamma} \in \mathbb{R}_+$ such that

$$\inf_{\Delta} \sup_{\mathcal{E}_{a_\cdot} \in \mathcal{A}} \mathcal{R}(\Delta | \mathcal{E}_{a_\cdot}), A \rho_{a_\cdot}(\delta_{\cdot,n})) \geq 1 - \gamma.$$

**Aggregation procedure.** Let us come back to the circular deconvolution problem and the indirect and direct tests discussed above. In this paper we aggregate both testing procedures over a family $\mathcal{K} \subseteq \mathbb{N}$ of dimension parameters using a classical Bonferroni method, where for a given level $\alpha \in (0,1)$ each of

AGGREGATION PROCEDURE.

Let us come back to the circular deconvolution problem and the indirect and direct tests discussed above. In this paper we aggregate both testing procedures over a family $\mathcal{K} \subseteq \mathbb{N}$ of dimension parameters using a classical Bonferroni method, where for a given level $\alpha \in (0,1)$ each of
the tests in the family has level $\alpha / m$. The aggregated testing procedure rejects the null hypothesis as soon as one test in the collection rejects. It is straight-forward to see that a Bonferroni aggregation of the indirect test proposed in [31] leads to an adaptive factor of order $|\mathcal{K}|$. The choice of the family $\mathcal{K}$ reflects the collection of alternatives, over which the aggregated test performs optimally. If the alternatives characterise ordinary smoothness of the circular density, the size of $\mathcal{K}$ is typically chosen to be of order $\log n$ (cp. [9, 32]). Then the aggregated test will feature a deterioration by an adaptive factor of order $\log n$. However, we show in this paper that generally the minimal adaptive factor is smaller. In order to do so, we first derive sharper bounds for the quantiles of the direct and indirect test statistics using exponential bounds for $U$-statistics and a Bernstein inequality. This allows to define a new version of an indirect and a direct test, for which we derive radii of testing. Aggregating these tests via the Bonferroni method, we obtain an adaptive factor for adaptation with respect to smoothness of order $\sqrt{\log \log n}$. Interestingly, for testing for uniformity, i.e., $f^0 = \mathbb{1}_{[0,1]}$, the aggregated direct test does not depend on the noise density $\varphi$ and is, thus, also adaptive with respect to the ill-posedness of the model. Moreover, in this situation, we derive a lower bound for the adaptive factor, which provides conditions under which it is minimal.

**Outline of the paper.** The upper bounds for the radius of testing via an indirect and a direct testing procedure are derived in Sections 2 and 4, respectively. Sections 3 and 5 are devoted to adaptive indirect and direct testing strategies. We provide lower bounds in Sections 6. Technical derivations are deferred to the Appendices A–C.

### 2. UPPER BOUND VIA AN INDIRECT TESTING PROCEDURE

**Notation.** The inner product on $L^2$ that induces the norm $\| \cdot \|_{L^2}$ is given by $\langle \xi, \zeta \rangle_{L^2} = \int_{[0,1]} \xi(x)\overline{\zeta(x)}dx$, where $\overline{\zeta(x)}$ denotes the complex conjugate of $\zeta(x)$. Consider the family of exponential functions $\{e_j\}_{j \in \mathbb{Z}}$ with $e_j(x) := \exp(-2\pi jx)$ for $x \in [0, 1]$ and $j \in \mathbb{Z}$, which is an orthonormal basis of $L^2$. For each $\zeta \in L^2$ and $j \in \mathbb{Z}$ we define the Fourier coefficient $\zeta_j := \langle \zeta, e_j \rangle_{L^2}$ and thus $\zeta$ admits an expansion as a discrete Fourier series $\zeta = \sum_{j \in \mathbb{Z}} \zeta_j e_j$. Here and subsequently, we refer to any sequence $\zeta_j = (\zeta_j)_{j \in \mathbb{Z}}$ by adding a dot as an index. By Parseval’s identity the sequence of Fourier coefficients $\zeta_j := (\zeta_j)_{j \in \mathbb{Z}}$ of an $L^2$-function $\zeta$ is square summable. In fact, setting $||\zeta||_{L^p} := (\sum_{j \in \mathbb{Z}} |\zeta_j|^p)^{1/p}$ for $p \geq 1$ we have $||\zeta||_{L^2} = ||\zeta||_{L^2}$. For a density $g$ let us further denote by $L^2_{\mathbb{R}}(g)$ the set of all real-valued (Borel-measurable) functions $h$ satisfying $\int_{[0,1]} h^2(x)g(x)dx < \infty$, in particular, $L^2 := L^2_{\mathbb{R}}(\mathbb{1}_{[0,1]})$.

**Definition of the test statistic.** We expand the densities $f, f^0 \in \mathcal{D} \subset L^2$ appearing in the testing task (1.3) in the exponential basis. Therefore we have $||f - f^0||_{L^2}^2 = \sum_{j \in \mathbb{N}} |f_j - f_j^0|^2 = 2 \sum_{j \in \mathbb{N}} |f_j - f_j|^2$ due to Parseval’s identity, since $f_0 = 1 = f_0^0$, $f_j = f_{-j}$ and $f_j^0 = f_{-j}^0$ for all $j \in \mathbb{Z}$. Additionally, exploiting the circular convolution theorem, the density $g = f \circ \varphi$ of the observations (1.2) admits Fourier coefficients $g_j = f_j \cdot \varphi_j$ for all $j \in \mathbb{Z}$. Keeping $g^0 = f \circ \varphi$ and $Y \sim g = f \circ \varphi$ in mind and assuming from here on $|\varphi_j| > 0$ for all $j \in \mathbb{Z}$, we have

$$q^2(f - f^0) := \sum_{|j| \in \mathbb{N}} |f_j - f_j^0|^2 = \sum_{|j| \in \mathbb{N}} \frac{|g_j - g_j^0|^2}{|\varphi_j|^2} \quad \text{with} \quad g_j = \mathbb{E}_f(e_j(-Y)) \quad \text{for all} \quad j \in \mathbb{Z}.$$  

For $k \in \mathbb{N}$ and $[k] := [1, k] \cap \mathbb{N}$ let us define an unbiased estimator $q_k^2$ of the truncated version

$$q_k^2(f - f^0) := \sum_{|j| \in [k]} |f_j - f_j^0|^2 = \sum_{|j| \in [k]} \frac{|g_j|^2}{|\varphi_j|^2} - 2 \sum_{|j| \in [k]} \frac{g_j g_j}{|\varphi_j|^2} + \sum_{|j| \in [k]} |f_j|^2,$$  

using that $\sum_{|j| \in [k]} |\varphi_j|^{-2} g_j^2 \tilde{g}_j$ is real-valued. Replacing the unknown Fourier coefficients by empirical counterparts based on observations $\{Y_j\}_{j=1}^n$ we consider the test statistic

$$q_k^2 := \hat{T}_k - 2\bar{S}_k + q_k^2(f^0) \quad \text{with} \quad \hat{T}_k := \frac{1}{n(n - 1)} \sum_{|j| \in [k]} \sum_{l \in [n]} \sum_{m \in [n]} \frac{e_j(Y_l)e_j(Y_m)}{|\varphi_j|^2},$$

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and \( \hat{S}_k := \frac{1}{n} \sum_{j \in [k]} \sum_{i \neq m} g^2_j e_j(Y_i) \left| \varphi_j \right|^2. \) (2.2)

Note that \( q_k^2(f^0) \) is known, \( \hat{T}_k \) is a U-statistic and \( \hat{S}_k \) is a linear statistic. We shall emphasise, if \( \{Y_j\}_{j=1}^n \overset{iid}{\sim} g \) as in (1.2) then \( q_k^2 \) is an unbiased estimator of \( q_k^2(f - f^0) \) for each \( k \in \mathbb{N} \). Below we construct a test that, roughly speaking, compares the estimator \( q_k^2 \) to a multiple of its standard deviation.

**Decomposition of the test statistic.** The key element to analyse quantiles of the test statistic \( q_k^2 \) in (2.2) is the following decomposition

\[
U_n := \frac{1}{n(n-1)} \sum_{j \in [k] \setminus \{i\}, l \neq m} \frac{(e_j(-Y_l) - g_j)(e_j(Y_m) - g_j)}{\left| \varphi_j \right|^2} \quad \text{and} \quad V_n := \frac{1}{n} \sum_{j \in [k]} \sum_{l \neq m} \frac{(g_j - g^0_j)(e_j(Y_l) - g_j)}{\left| \varphi_j \right|^2},
\]

(2.3)

where \( q_k^2(f - f^0) \) is a separation term, \( U_n \) is a canonical U-statistic, and \( V_n \) is a centred linear statistic.

**Definition of the threshold.** The next proposition provides bounds for the quantiles of the test statistic \( q_k^2 \). Let \( L_x := (1 - \log x)^{1/2} \in (1, \infty) \) for \( x \in (0, 1) \). Define for \( k \in \mathbb{N} \) the quantities

\[
\nu_k := \left( \sum_{j \in [k]} \left| \varphi_j \right|^{-4} \right)^{1/4} \quad \text{and} \quad m_k := \max_{j \in [k]} \left| \varphi_j \right|^{-1}.
\]

(2.4)

For \( \alpha \in (0,1) \) with \( c_1 := 799\|g^0\|_{L^2} + 1372, c_2 := 52\|g^0\|_{L^1} \) consider the threshold

\[
\tau_k(\alpha) := c_1 \left( 1 + \frac{L^2}{\alpha n^{1/2}} \sqrt{\frac{\nu^2_k}{n}} \right) L^2 \frac{\nu^2_k}{n} + c_2 L^2 \frac{m^2_k}{n}.
\]

(2.5)

Note that \( \|g^0\|_{L^1} \leq \|f^0\|_{L^2} \leq \varphi \|_{L^2} < \infty \) due to the Cauchy-Schwarz inequality and Parseval’s identity.

**Proposition 2.1.** For \( f^0, f, \varphi \in L^2 \) and \( n \in \mathbb{N}, n \geq 2 \), consider \( \{Y_j\}_{j=1}^n \overset{iid}{\sim} g = f \oplus \varphi \) with joint distribution \( \mathbb{P}_f \) and \( g^0 = f^0 \oplus \varphi \). Let \( \alpha, \beta \in (0,1) \) and for \( k \in \mathbb{N} \) consider \( \hat{q}_k^2 \) and \( \tau_k(\alpha) \) as in (2.2) and (2.5), respectively.

(i) If \( L^2(\mathbb{R})^2 = \mathbb{R}^2 \), then \( \mathbb{P}_{f^0}(\hat{q}_k^2 \geq \tau_k(\alpha)) \leq \alpha. \)

(ii) If \( c_3 := 8\|g^0\|_{L^1} + 826\|\varphi^0\|_{L^2} + 1372 \) and

\[
\hat{q}_k^2(f - f^0) \geq 2 \left( \tau_k(\alpha) + c_3 L^4 \left( 1 + \frac{\nu^2_k}{n} \right) \frac{\nu^2_k}{n} \right),
\]

(2.6)

then \( \mathbb{P}_f(\hat{q}_k^2 < \tau_k(\alpha)) \leq \beta. \)

**Proof of Proposition 2.1.** Firstly, consider (i). If \( f = f^0 \) and, hence \( g = g^0 \), the decomposition (2.3) simplifies to \( \hat{q}_k^2 = U_n \), where \( U_n \) is a canonical U-statistic. Applying Proposition A.1 given in the Appendix, a concentration inequality for canonical U-statistics of order 2, with \( x = L^2 \geq 1 \) and quantities \( A - D \) satisfying (A.2) we obtain

\[
\mathbb{P}_{f^0}(\hat{q}_k^2 \geq 8Cn^{-1}L^2 + 13Dn^{-1}L^2 + 261Bn^{-3/2}L^2 + 343An^{-2}L^4) \leq \alpha.
\]

(2.7)

Consider the quantities \( A - C \) in (A.6) and \( D \) in (A.7), which satisfy (A.2) under the additional condition \( L^2(\mathbb{R})^2 = \mathbb{R}^2 \) due to Lemma A.3. We have \( 8Cn^{-1}L^2 + 13Dn^{-1}L^2 + 261Bn^{-3/2}L^2 + 343An^{-2}L^4 \leq \tau_k(\alpha) \). This together with (2.7) shows (i). Secondly, consider (ii). Keeping the decomposition (2.3) in mind we control the deviations of the U-statistic \( U_n \) and the linear statistic \( V_n \) by applying
Proposition A.1 and Lemma A.4, respectively. In fact, the quantities $A-D$ given in (A.6) of Lemma A.3 fulfil (recall $L_{β/2} ≥ 1$)

\[
8Cn^{-1}L_{β/2} + 13Dn^{-1}L_{β/2}^2 + 261Bn^{-3/2}L_{β/2}^3 + 343An^{-2}L_{β/2}^4
\leq L_{β/2}(285||g_●||_{ℓ^2} + 1372)(1 ∨ (ν_k^2n^{-1}))ν_k^2n^{-1} =: τ_1.
\]

Consequently, the event $Ω_1 := \{U_n ≤ −τ_1\}$ satisfies $P_f(Ω_1) ≤ β/2$ due to Proposition A.1 (with the usual symmetry argument). Define further the event $Ω_2 := \{2V_n ≤ −τ_2 − 1/2q_k^2(f − f^0)\}$ with $τ_2 := L_{β/2}(2)||g_●||_{ℓ^2} + ||ϕ_●||_{ℓ^2}(1 ∨ (m_k^2n^{-1}))m_k^2n^{-1}$, then $P_f(Ω_2) ≤ 1/2β ≤ β/2$ due to Lemma A.4 with $x = L_{β/2} ≥ 1$, which is an application of a Bernstein inequality. Since $τ_1 + τ_2 ≤ L_{β/2}c_3(1 ∨ (ν_k^2n^{-1}))ν_k^2n^{-1}$ with $c_3 = 8||g_●||_{ℓ^2} + 826||ϕ_●||_{ℓ^2} + 1372$ due to $m_k^2 ≤ ν_k^2, 1 ≤ L_{β/2}$ and $||g_●||_{ℓ^α} ≤ ||ϕ_●||_{ℓ^α}$ the assumption (2.6) yields $1/2ν_k^2(f − f^0) ≥ τ_k(α) + τ_1 + τ_2$. Thus, the decomposition (2.3) implies

\[
P_f(Ω_1) + P_f(2V_n + q_k^2(f − f^0) < τ_k(α) + τ_1) ≤ β/2 + P_f(Ω_2) ≤ β,
\]

which shows (ii) and completes the proof. □

**Remark 2.2.** The technical assumption $L_{k}^2(g^0) = L_k^2$ in Proposition 2.1 allows us to express elements of $L_k^2(g^0)$ in their Fourier expansion. It is immediately satisfied for $f^0 = e_0 = 1_{[0,1]}$ and if $g^0$ is bounded away from 0 and infinity. If it is not satisfied, we can obtain a similar result to Proposition 2.1 as follows. We redefine our threshold to be

\[
\tilde{τ}_k(α) := c_1\left(1 ∨ L_{α}^2 ν_k \nabla \nabla ν_k \nabla + L_{α}^2 ν_k \nabla + c_2L_{α}^2 ν_k \nabla \right)
\]

i.e., compared to (2.5) the term $m_k^2$ is replaced by $ν_k^2$. We can then follow the proof of Proposition 2.1 using the bound (A.6) in Lemma A.3 instead of (A.7). However, replacing $m_k$ by $ν_k$ results in a larger adaptive factor in Section 3, i.e., Corollary 3.3 (best-case adaptive factor) no longer holds, but Corollary 3.2 (worst-case adaptive factor) does.

**Definition of the test.** For $k ∈ N$ and $α ∈ (0,1)$ using the test statistic $q_k^2$ and the threshold $τ_k(α)$ given in (2.2) and (2.5), respectively, we consider the test

\[
\Delta_{k,α} := 1\{q_k^2 ≥ τ_k(α)\}.
\]

From (i) in Proposition 2.1 it immediately follows that $Δ_{k,α}$ is a level-$α$-test for all $k ∈ N$. To analyse its power over the alternative, we introduce a regularity constraint, i.e., a nonparametric class of functions $E$, which is formulated in terms of Fourier coefficients. Let $R > 0$ and let $a_α := (a_j)_{j ∈ N}$ be a strictly positive, monotonically non-increasing sequence that is bounded by 1. We assume that the difference $f − f^0$ belongs to the $L^2$-ellipsoid

\[
E_{α,∗}^R = \left\{ \zeta ∈ L^2 : 2 \sum_{j ∈ N} a_j^{-2}|ζ_j|^2 ≤ R^2 \right\}.
\]

Note that $f − f^0 ∈ E_{α,∗}^R$ imposes conditions on all coefficients $f_j, j ∈ Z$, since $|f_j|^2 = |f_{−j}|^2, j ∈ N$, for all real-valued functions and, additionally, $f_0 = 1$ for all densities. The definition (2.9) is general enough to cover classes of ordinary and super smooth functions. Proposition 2.1 (ii) allows to characterise elements in $E_{α,∗}^R$ for which $Δ_{k,α}$ is powerful. Exploiting these results, in the next proposition we derive an upper bound for the radius of testing of $Δ_{k,α}$ in terms of $ν_k$ as in (2.4) and the regularity parameter $a_α$.

that is, we define

\[
ρ_{k,α}^2 := ρ_{k,α}(n) := a_k^2 ∨ ν_k^2/n.
\]

In the next proposition we show that both the type I and maximal type II error probability of a test $Δ_{k,γ/2}$ are bounded by $γ/2$ and, thus, their sum by $γ$, by applying Proposition 2.1 with $α = β = γ/2$. 

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Proposition 2.3. Under the assumptions of Proposition 2.1 for $\alpha \in (0, 1)$ define
\[ \overline{A}_\gamma := R^2 + 2(8R||\varphi_s||_2 + 826||\varphi_s||_2^2 + 859||g_0^s||_2 + 2744) L_{1/4}^4. \] (2.10)
For all $A \geq \overline{A}_\gamma$ and for all $n, k \in \mathbb{N}$ with $n^2 \leq n$ we have $R(\Delta_{k, \gamma/2} | E_{A}, A_{P_{k,a_*}}) \leq \gamma$.

Proof of Proposition 2.3. The result follows from Proposition 2.1 with $\alpha = \beta = \gamma/2$ and the definition of the risk $R(\Delta_{k, \gamma/2} | E_{A}, A_{P_{k,a_*}}) = P_{f} (\Delta_{k, \gamma/2} = 1) + \sup f \in E_{A} \cap \mathcal{L}_2 \ {P}_{f} (\Delta_{k, \gamma/2} = 0) \leq \gamma/2 + \gamma/2 = \gamma$. Since the assumption of Proposition 2.1 (i) is fulfilled the test $\Delta_{k, \gamma/2}$ is a level-$\gamma/2$-test. Hence, in order to apply Proposition 2.1 (ii) with $\beta = \gamma/2$ for each density $f \in \mathcal{L}_2$ with $f - f^o \in \mathcal{E}_{A}$ and $f - f^o \in \mathcal{E}_{\overline{A}}$, it remains to verify condition (2.6) which states $q_k^2 (f - f^o) \geq 2\left( \tau_k (\gamma/2) + c_3 L_{1/4}^4 (1 \vee \nu_k n^{-1}) \nu_k n^{-1} \right)$ with $\tau_k (\gamma/2)$ as in (2.5). Indeed, in this situation we have $\sum_{|j| > k} |f_j - f_j^o|^2 \leq A_{P_{k,a_*}}^2 R^2$, which implies
\[ q_k^2 (f - f^o) = ||f - f^o||_2^2 - \sum_{|j| > k} |f_j - f_j^o|^2 \geq \overline{A}_\gamma^2 P_{k,a_*}^2 - A_{P_{k,a_*}}^2 R^2 \]
\[ \geq 2(8R||\varphi_s||_2 + 826||\varphi_s||_2^2 + 859||g_0^s||_2 + 2744) L_{1/4}^4 \nu_k n^{-1} \]
\[ \geq 2(8||g_0^s||_2 + 826||\varphi_s||_2 + 851||g_0^s||_2 + 2744) L_{1/4}^4 \nu_k n^{-1}, \] (2.11)
using the triangular inequality $||g_0^s||_2 \leq ||g_0^s - g_0^s||_2 + ||g_0^s||_2$ and the Cauchy–Schwarz inequality $||g_0^s - g_0^s||_2 \leq ||f_0^s - f_0^s||_2 ||\varphi_s||_2 \leq R ||\varphi_s||_2$ by $||f_0^s - f_0^s||_2 \leq R$. The condition (2.6) follows from (2.11) by exploiting further $1 \leq L_{\gamma/2} \leq L_{1/4}$, $||g_0^s||_2 \leq ||g_0^s||_2$ and $m_k^2 \leq \nu_k^2 \leq n$, which completes the proof.

Let us introduce a dimension that realises an optimal bias–variance trade-off and the corresponding radius
\[ k_{a_*} := \arg \min \min \{ k \in \mathbb{N} : \rho_{k,a_*}^2 \leq \rho_{l,a_*}^2 \text{ for all } l \in \mathbb{N} \} \text{ and } \]
\[ \rho_{a_*}^2 := \rho_{a_*}^2(n) := \min \rho_{k,a_*}^2(n) = \min \{ a_k^2 \vee \nu_k^2 n^{-1} \}. \] (2.12)

Corollary 2.4. Under the assumptions of Proposition 2.1 let $\gamma \in (0, 1)$ and $\overline{A}_\gamma$ as in (2.10), then $R(\Delta_{k_{a_*}, \gamma/2} | E_{A}, A_{P_{a_*}}) \leq \gamma$ for all $A \geq \overline{A}_\gamma$ and $n \geq \sqrt{2} |\varphi_s|^2$.

Proof of Corollary 2.4. The result follows immediately from Proposition 2.3, since $\nu_{k_{a_*}}^2 \leq n$ for all $n \geq \sqrt{2} |\varphi_s|^2$. Indeed, $n \geq \sqrt{2} |\varphi_s|^2$ implies $1 \geq \rho_{k,a_*}^2 \geq \rho_{a_*}^2 \geq \nu_{k_{a_*}}^2 n^{-1}$.

We shall emphasise that in the case $f^o = e_0 = ||1||_{(0,1)}$ the radius of testing $\rho_{a_*}$ is known to be minimax ([31]) and, hence, the test $\Delta_{k_{a_*}, \gamma/2}$ is minimax optimal.

Illustration 2.5. Throughout the paper we illustrate the order of the radii of testing under typical regularity and ill-posedness assumptions. For two real-valued sequences $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ we write $x_j \lesssim y_j$ if there exists a constant $c > 0$ such that $x_j \leq cy_j$ for all $j \in \mathbb{N}$. We write $x_j \sim y_j$ if both $x_j \lesssim y_j$ and $y_j \lesssim x_j$. Concerning the class $E_{A}$ we distinguish two behaviours of the sequence $a_*$, namely the ordinary smooth case $a_j \sim j^{-s}$ for $s > 1/2$, corresponding to a Sobolev ellipsoid, and the super smooth case $a_j \sim \exp(-j^s)$ for $s > 0$, corresponding to a class of analytic functions. We also distinguish two cases for the regularity of the error density $\varphi$. For $p > 1/2$ we consider a mildly ill-posed model $|\varphi_j| \sim |j|^{-p}$ and for $p > 0$ a severely ill-posed model $|\varphi_j| \sim \exp(-|j|^p)$. Many examples of circular densities can be found in Chapter 3 of [26]. Table 1 presents the order of the dimension $k_{a_*}$ and the upper bound $\rho_{a_*}^2$ for the radius of testing (see the appendix of [31] for the calculations).
3. ADAPTIVE INDIRECT TESTING PROCEDURE

For an arbitrary regularity parameter $a_\bullet$, the test $\Delta_{k_{a_\bullet}, \gamma/2}$ in Corollary 2.4 achieves the minimax radius of testing $\rho^2_{a_\bullet}$. However, it relies via the dimension parameter $k_{a_\bullet}$ on the regularity class $\mathcal{E}^{R}_{a_\bullet}$ and, thus, the testing procedure is not adaptive, i.e., assumption-free. Ideally, a test should perform optimally for a wide range of regularity parameters. In this section we therefore propose an adaptive testing procedure by aggregating the test in (2.8) over various dimension parameters.

**Adaptation procedure via Bonferroni aggregation.** Let $\mathcal{K} \subseteq \mathbb{N}$ be a finite collection of dimension parameters. For $k \in \mathcal{K}$ and the level $\alpha/|\mathcal{K}| \in (0,1)$ recall the collection of tests $(\Delta_{k, \alpha/|\mathcal{K}|})_{k \in \mathcal{K}} = (\{\tilde{q}_k^2 > \tau_k(\alpha/|\mathcal{K}|)\})_{k \in \mathcal{K}}$ defined in (2.8). We consider the max-test

$$\Delta_{K, \alpha} := \{\tilde{Q}_{K, \alpha} > 0\} \quad \text{with} \quad \tilde{Q}_{K, \alpha} := \max_{k \in \mathcal{K}} \left( \frac{\tilde{q}_k^2}{\tilde{q}_k} - \tau_k \left( \frac{\alpha}{|\mathcal{K}|} \right) \right), \quad (3.1)$$

i.e., the test rejects the null hypothesis as soon as one of the tests in the collection does. Under the null hypothesis, we bound the type I error probability of the max-test by the sum of the error probabilities of the individual tests,

$$\mathbb{P}_f(\Delta_{K, \alpha} = 1) = \mathbb{P}_f(\tilde{Q}_{K, \alpha} > 0) \leq \sum_{k \in \mathcal{K}} \mathbb{P}_f(\Delta_{k, \alpha/|\mathcal{K}|} = 1) \leq \sum_{k \in \mathcal{K}} \frac{\alpha}{|\mathcal{K}|} = \alpha. \quad (3.2)$$

Hence, $\Delta_{K, \alpha}$ is a level-$\alpha$-test, since $\Delta_{k, \alpha/|\mathcal{K}|}$ is a level-$\alpha/|\mathcal{K}|$-test for each $k \in \mathcal{K}$ due to Proposition 2.1 (i). Under the alternative, we can bound the type II error probability by the error probability of any of the individual tests,

$$\mathbb{P}_f(\Delta_{K, \alpha} = 0) = \mathbb{P}_f(\tilde{Q}_{K, \alpha} \leq 0) \leq \min_{k \in \mathcal{K}} \mathbb{P}_f(\Delta_{k, \alpha/|\mathcal{K}|} = 0). \quad (3.3)$$

Therefore, $\Delta_{K, \alpha}$ has the maximal power achievable by a test in the collection. The bounds (3.2) and (3.3) have opposing effects on the choice of the collection $\mathcal{K}$. On the one hand, it should be as small as possible to keep the type I error probability small. On the other hand, it must be large enough to contain an optimal dimension parameter $k_{a_\bullet}$ for a wide range of regularity parameters $a_\bullet$ that we want to adapt to. In this paper we consider a classical Bonferroni choice of an error level $\alpha/|\mathcal{K}|$. For other aggregation choices, e.g., a Monte-Carlo quantile and a Monte-Carlo threshold method we refer to [9, 22]. Although the Bonferroni choice is a more conservative method, we show its optimality, which is shared with the other methods.

**Testing radius of the indirect max-test.** Denote by $A \subseteq \mathbb{R}_{>0}^N$ a set of strictly positive, monotonically non-increasing sequences bounded by 1. The set $A$ characterises the collection of regularity classes $\{\mathcal{E}_a^R : a \in A\}$, for which the power of the testing procedure is analysed simultaneously. The max-test $\Delta_{K, \alpha}$ in (3.1) only aggregates over a finite set $\mathcal{K} \subseteq \mathbb{N}$. For each $a_\bullet \in A$ we define a minimal achievable radius of testing $\rho^2_{K, a_\bullet}(n)$ over the set $\mathcal{K}$ as

$$\rho^2_{K, a_\bullet}(x) := \min_{k \in \mathcal{K}} \rho^2_{k, a_\bullet}(x) \quad \text{with} \quad \rho^2_{k, a_\bullet}(x) := a_k^2 \vee \frac{\rho_k^2}{x} \quad \text{for all} \quad x \in \mathbb{R}_+. \quad (3.4)$$

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $k_{a_\bullet}$ | $\rho^2_{a_\bullet}$ |
|-------------------|-----------------------------|----------------|---------------------|
| $j^{-s}$          | $|j|^{-p}$                  | $n^{\frac{2}{4p+4s+1}}$ | $n^{-\frac{2}{4p+4s+1}}$ |
| $j^{-s}$          | $e^{-|j|^p}$                | $(\log n)^{\frac{1}{p}}$ | $(\log n)^{-\frac{2}{p}}$ |
| $e^{-j^s}$        | $|j|^{-p}$                  | $(\log n)^{\frac{1}{2}}$ | $n^{-1}(\log n)^{2p+1/2}$ |

**Table 1**

Order of the optimal dimension $k_{a_\bullet}$ and the radius $\rho^2_{a_\bullet}$. 

**References:**

[9, 22]
Since $\rho^2_{a^*}(n) = \rho^2_{R_0,a^*}(n)$ in (2.12) is defined as the minimum taken over $\mathcal{K}$ instead of $\mathcal{K}_f$, for $n \in \mathbb{N}$ we always have $\rho^2_{K,a^*}(n) \geq \rho^2_{a^*}(n)$. Moreover, replacing $\nu_k$ by $m_k$ as in (2.4), let us define a remainder radius $r^2_{K,a^*}(n)$, typically negligible compared to $\rho^2_{K,a^*}(n)$, as follows

$$r^2_{K,a^*}(x) := \min_{k \in \mathcal{K}} r^2_{K,a^*}(x) \quad \text{with} \quad r^2_{K,a^*}(x) := a_k^2 \vee \frac{m_k^2}{x} \quad \text{for all} \quad x \in \mathbb{R}_+.$$  \hspace{1cm} (3.4)

In the next proposition we show for each $a^* \in \mathcal{A}$ that both the type I and maximal type II error probability of a test $\Delta_{K,n}/\gamma/2$ are bounded by $\gamma/2$ and, thus, their sum by $\gamma$, by applying Proposition 2.1 with $\alpha = \beta = \gamma/2$.

**Proposition 3.1 (uniform radius of testing over $\mathcal{A}$).** Under the assumptions of Proposition 2.1 let $\gamma \in (0,1)$ and consider $\overline{A}_{\gamma}$ as in (2.10). Then, for all $R \geq \overline{A}_{\gamma}$ and $n \in \mathbb{N}$

$$\sup_{a^* \in \mathcal{A}} \mathcal{R}(\overline{A}_{\gamma}/n | R, \mathcal{A}_{R}, (\delta^2 n) \vee \rho^2_{K,a^*}(\delta n))(1 \vee \delta^{-3/2} \rho^2_{K,a^*}(\delta n)) \leq \gamma$$

with $\delta = (1 + \log |\mathcal{K}|)^{-1/2}$.

**Proof of Proposition 3.1.** Under the null hypothesis, for each $a^* \in \mathcal{A}$ the claim follows from (3.2) together with Proposition 2.1 (i) and $\sum_{k \in \mathcal{K}} \frac{\nu_k}{\delta^n} = \gamma/2$. Under the alternative, let $f \in \mathcal{L}^2$ with $f - f^0 \in \mathcal{E}_{a^*}^R$ satisfy

$$||f - f^0||^2_{L^2} \geq \overline{A}_{\gamma}^2(\alpha_k^2 \vee \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m_k^2}{\delta^2 n}\right)) \quad \text{for all} \quad k \in \mathcal{K}.$$  \hspace{1cm} (3.6)

It is sufficient to use the elementary bound (3.3) together with the observation that

(i) $\mathbb{P}_f(q_k^2 < \tau_k(\overline{A}_{\gamma}/n)) \leq \gamma/2$ holds for all $k \in \mathcal{K}$ whenever $f \in \mathcal{L}^2$ satisfies $f - f^0 \in \mathcal{E}_{a^*}^R$ and

$$||f - f^0||^2_{L^2} \geq \overline{A}_{\gamma}^2(\alpha_k^2 \vee \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m_k^2}{\delta^2 n}\right)).$$ \hspace{1cm} (3.5)

(ii) under (3.5) there exists $k \in \mathcal{K}$ such that (3.6) is fulfilled.

As a consequence, we have $\mathbb{P}_f(\Delta_{K,n}/\gamma/2 = 0) \leq \gamma/2$ for all $f \in \mathcal{L}^2$ satisfying $f - f^0 \in \mathcal{E}_{a^*}^R$ (and (3.5)), and thus the maximal type II error probability is also bounded by $\gamma/2$. It remains to show (i) and (ii). The claim (i) follows from Proposition 2.1 (ii) (with $\beta = \gamma/2$) since (3.6) implies the condition (2.6), which states $q_k^2(f - f^0) \geq \left(\tau_k(\overline{A}_{\gamma}/n) + c_3 L^4_{\gamma/4} \left(1 \vee \frac{\nu_k}{\delta n} - 1 \right) \right)$ with $\tau_k(\overline{A}_{\gamma}/n)$ as in (2.5). Indeed, exploiting $L^2_{\gamma/2} = \log(2e/\gamma) \geq 1$ and hence $L^2_{\gamma/(2|\mathcal{K}|)} = \log |\mathcal{K}| + L^2_{\gamma/2} \leq L^2_{\gamma/2} (1 + \log |\mathcal{K}|) = L^2_{\gamma/2}/\delta^{-2}$, we have

$$\tau_k(\overline{A}_{\gamma}/n) \leq c_1 L^4_{\gamma/2} \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m^2_k}{\delta^2 n}\right) + \frac{c_2 L^4_{\gamma/2} m_k^2}{\delta^2 n}$$

and

$$\leq (c_1 + c_2) L^4_{\gamma/2} \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m^2_k}{\delta^2 n}\right).$$

Using $L^4_{\gamma/4} \geq L^2_{\gamma/2}$, $1 \geq \delta$ and $\mathcal{A}_{\gamma}^2 - R^2 \geq 2(c_1 + c_2 + c_3) L^4_{\gamma/4}$ by elementary calculations the condition (2.6) holds whenever

$$q_k^2(f - f^0) \geq \left(\mathcal{A}_{\gamma}^2 - R^2 \right) \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m^2_k}{\delta^2 n}\right).$$ \hspace{1cm} (3.7)

Due to $f - f^0 \in \mathcal{E}_{a^*}^R$ and hence $\sum_{|j| > k} |f_j - f^0_j|^2 \leq a_k^2 R^2$, the condition (3.6) implies

$$q_k^2(f - f^0) = ||f - f^0||^2_{L^2} - \sum_{|j| > k} |f_j - f^0_j|^2 \geq \left(\mathcal{A}_{\gamma}^2 - R^2 \right) \left(1 \vee \frac{\nu_k}{\delta^{2n+1/2}} \vee \frac{\nu_k}{\delta^n} \vee \frac{m^2_k}{\delta^2 n}\right).$$

Consequently, if (3.6) is satisfied, then also (3.7) and thus (2.6), which shows the claim (i). Lastly, consider (ii). By Lemma A.1 in [30] we have $r^2_{K,a^*}(\delta^2 n) \vee \rho^2_{K,a^*}(\delta n) = a_k^2 \vee \frac{m_k^2}{\delta^n} \vee \frac{\nu_k^2}{\delta^n} \vee \frac{m^2_k}{\delta^2 n}$ and $\rho^2_{K,a^*}(\delta n) \geq \frac{\nu_k^2}{\delta^n}$.
for at least one \( k \in K \). Hence, there is \( k \in K \) with \( \left( r^2_{K, a_\ast}(\delta^2 n) \lor \rho^2_{K, a_\ast}(\delta n) \right) \left( 1 \lor \frac{\rho_{K, a_\ast}(\delta n)}{\delta} \lor \left( \frac{\nu_k}{\delta^2} \right)^2 \right) \geq a_k^2 \lor \frac{\nu_k^2}{\delta^2 n} \lor \left( \frac{\nu_k^2}{\delta^2 n} \right)^2 \). Since \( 1 \lor \frac{\nu_k^2}{\delta^2 n} \lor 1 \lor \frac{\nu_k^2}{\delta^2 n} \lor \nu_k^2 \), this shows (ii) and completes the proof.

**Corollary 3.2 (worst-case adaptive factor).** Under the assumptions of Proposition 2.1 let \( \gamma \in (0, 1) \) and consider \( \bar{A}_\gamma \) as in (2.10). Then, for all \( A \geq \bar{A}_\gamma \) and \( n \in \mathbb{N} \)

\[
\sup_{a_\ast \in A} R(\Delta_{K, \gamma/2}|E_{a_\ast}, A \rho_{K, a_\ast}(\delta^2 n)(1 \lor \rho_{K, a_\ast}(\delta^2 n))) \leq \gamma
\]

with \( \delta = (1 + \log |K|)^{-1/2} \).

**Proof of Corollary 3.2.** The result may be proved in much the same way as Proposition 3.1. In fact, as in the proof of Proposition 3.1, it is sufficient to show (ii) replacing (3.5) by

\[
||f - f^0||^2_{L^2} \geq \bar{A}_\gamma^2 \rho^2_{K, a_\ast}(\delta^2 n)(1 \lor \rho^2_{K, a_\ast}(\delta^2 n)).
\]

(3.8)

For each \( a_\ast \in A \) under (3.8) the parameter \( k := \arg \min_{k \in K} \rho^2_{K, a_\ast}(\delta^2 n) \) satisfies

\[
\rho^2_{K, a_\ast}(\delta^2 n)(1 \lor \rho^2_{K, a_\ast}(\delta^2 n)) = \rho^2_{K, a_\ast}(\delta^2 n)(1 \lor \rho_{K, a_\ast}(\delta^2 n) \lor \rho^2_{K, a_\ast}(\delta^2 n))
\]

\[
\geq a_k^2 \lor \frac{\nu_k^2}{\delta^2 n} \lor \left( 1 \lor \frac{\nu_k}{\delta^2 n} \lor \frac{\nu_k^2}{\delta^2 n} \right) \geq a_k^2 \lor \frac{\nu_k^2}{\delta^2 n} \lor \frac{\nu_k^2}{\delta^2 n} \lor \left( 1 \lor \frac{\nu_k}{\delta^2 n} \lor \frac{\nu_k^2}{\delta^2 n} \right)
\]

since \( a_k^2 \lor \frac{\nu_k^2}{\delta^2 n} \lor \frac{\nu_k^2}{\delta^2 n} \lor \frac{\nu_k^2}{\delta^2 n} \). This shows (3.6) and, in consequence, (ii). We obtain the assertion proceeding exactly as in the proof of Proposition 3.1.

**Corollary 3.3 (best-case adaptive factor).** Under the assumptions of Proposition 2.1 let \( \gamma \in (0, 1) \) and consider \( \bar{A}_\gamma \) as in (2.10). If there exists a constant \( C \geq 1 \) such that \( r_{K, a_\ast}(\delta^2 n) \leq C \rho_{K, a_\ast}(\delta n) \) and \( \rho_{K, a_\ast}(\delta n) \leq C \delta^{1/2} \) for all \( a_\ast \in A \), then for all \( A \geq C^2 \bar{A}_\gamma \) and \( n \in \mathbb{N} \)

\[
\sup_{a_\ast \in A} R(\Delta_{K, \gamma/2}|E_{a_\ast}, A \rho_{K, a_\ast}(\delta n)) \leq \gamma
\]

with \( \delta = (1 + \log |K|)^{-1/2} \).

**Remark 3.4 (choice of \( K \)).** Ideally, the collection \( K \subset \mathbb{N} \) is chosen such that its elements approximate the optimal parameter \( k_{a_\ast} \) given in (2.12) sufficiently well for each \( a_\ast \in A \). Note that \( k_{a_\ast} \leq n^{2/2} \) for all \( a_\ast \in A \) and for \( n \) reasonably large (precisely \( n \geq \sqrt{2}|\varphi|^{-2} \), which implies \( 1 \geq \rho^2_{K, a_\ast} \geq \rho^2_{K, a_\ast} \geq \nu_k^2 \), \( n \geq (k_{a_\ast})^{1/2} n^{-1} \). Hence, a naive choice is \( K = \lceil n^2/2 \rceil \) with \( |K| = |n^2/2| \), which yields an adaptive factor of order \( (\log n)^{1/2} \). However, in most cases, a minimisation over a geometric grid \( K_g = \{1\} \cup \{2^j : j \in \log_2(n^2/2)\} \) approximates the minimisation over \( \mathbb{N} \) well enough. Since \( |K_g| = \log_2(n^2/2) \) the adaptive factor is then of order \( (\log \log n)^{1/2} \). For some special cases the smaller collection \( K_s = \{1\} \cup \{2^j : j \in [s^{-1} \log_2 \log n]\}, s > 0 \), is still sufficient (see Illustration 3.5 below), resulting in an adaptive factor of order \( (\log \log n)^{1/2} \).

**Illustration 3.5.** For the typical configurations for regularity and ill-posedness introduced in Illustration 2.5 Tables 2 and 3 display the adaptive radii of the max-test \( \Delta_{K, \gamma/2} \) for appropriately chosen grids. In a mildly ill-posed model with ordinary smoothness we choose the geometric grid \( K_g \) and, hence, \( \delta = (1 + \log |K_g|)^{-1/2} \). It is easily seen that the remainder term \( r^2_{K_g, a_\ast}(\delta^2 n) \) is asymptotically negligible compared with \( \rho^2_{K_g, a_\ast}(\delta n) \). Moreover, \( \delta^{-3} \rho^2_{K_g, a_\ast}(\delta n) \) tends to zero.
Table 2

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $r^2_{K_{s,a}^*}(\delta^2 n)$ | $\rho^2_{K_{s,a}^*}(\delta n)$ |
|-------------------|------------------------|-----------------|-----------------|
| $j^{-s}$          | $|j|^{-p}$              | $\left(\frac{n}{\log\log n}\right)^{-\frac{s}{p+s}}$ | $\left(\frac{n}{(\log\log n)^{1/2}}\right)^{-\frac{s}{p+s+1}}$ |

Table 3

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $r^2_{K_{s,a}^*}(\delta^2 n)$ | $\rho^2_{K_{s,a}^*}(\delta n)$ |
|-------------------|------------------------|-----------------|-----------------|
| $e^{-j^a}$        | $|j|^{-p}$              | $\frac{\log\log n}{n} (\log n)^{2p}$ | $\frac{\log\log n}{n} (\log n)^{2p+1/2}$ |

Hence, the upper bound in Proposition 3.1 asymptotically reduces to $\rho^2_{K_{s,a}^*}(\delta n)$ featuring an adaptive factor of order $(\log\log n)^{1/2}$.

In a severely ill-posed model with ordinary smoothness, we have seen in Illustration 2.5 that the order of the optimal dimension parameter does not depend on the smoothness parameter. Hence, the test $\Delta_k$ is automatically adaptive with respect to ordinary smoothness. In a mildly ill-posed model with super smoothness, we choose the smaller geometric grid $K_s$ for adaptation to smoothness $s \leq s^*$ and, hence, $\delta = (1 + \log|K_s|)^{-1/2}$. It is easily seen that the remainder term $r^2_{K_{s,a}^*}(\delta^2 n)$ is asymptotically negligible compared with $\rho^2_{K_{s,a}^*}(\delta n)$. Moreover, $\delta^{-3} \rho^2_{K_{s,a}^*}(\delta n)$ tends to zero. Hence, the upper bound in Proposition 3.1 asymptotically reduces to $\rho^2_{K_{s,a}^*}(\delta n)$ featuring an adaptive factor of order $(\log\log n)^{1/2}$.

Finally, we point out that the results presented in these illustrations are all of an asymptotic nature, e.g., the adaptive factors are only larger than 1 for very large $n$.

Remark 3.6 (adaptation to the radius $R$ of the regularity class). In this paper the parameter $R > 0$ is unknown but assumed to be fixed and we consider adaptation to a collection of alternatives $\{E^R_a : a \in A\}$ only. From Corollary 2.4 (and the definition of $A^R$ therein) it follows immediately that adaptation to $\{E^R_a : R \in (0, R^*)\}$ is achieved without a loss. Indeed, replacing $R$ by $R^*$ in the definition of $A^R$, we readily obtain a result similar to Corollary 2.4 with an additional supremum taken over $R \in (0, R^*)$. However, adaptation to $\{E^R_a : R \in (0, \infty)\}$ is not possible without a loss, for an explanation we refer to Section 6.3 of [1] for a similar observation in the Gaussian sequence space model.

4. UPPER BOUND VIA A DIRECT TESTING PROCEDURE

Definition of the test statistic. In this section we consider a test that is based on an estimation of the quantity

$$q^2(g - g^0) = \sum_{|j| \in \mathbb{N}} |g_j - g_j^0|^2 = \sum_{|j| \in \mathbb{N}} |f_j - f_j^0|^2 |\varphi_j|^2 \quad \text{with} \quad g_j = E(f(Y_j)) \quad \text{for all} \quad j \in \mathbb{Z}.$$
For $k \in \mathbb{N}$ we define an unbiased estimator $\tilde{q}_k^2$ of the truncated version
\begin{equation}
q_k^2(g - g^o) = \sum_{|j| \in [k]} |g_j - g_j^o|^2 = \sum_{|j| \in [k]} |g_j|^2 - 2 \sum_{|j| \in [k]} g_j^o \bar{g}_j + \sum_{|j| \in [k]} |g_j^o|^2,
\end{equation}
using again that $\sum_{|j| \in [k]} g_j^o \bar{g}_j$ is real-valued. Replacing the unknown Fourier coefficients by empirical counterparts based on observations $\{Y_i\}_{i=1}^n$, we consider the test statistic
\begin{equation}
\tilde{q}_k^2 := \tilde{T}_k - 2\tilde{S}_k + \tilde{q}_k^2(g^o) \quad \text{with} \quad \tilde{T}_k := \frac{1}{n(n-1)} \sum_{|i| \in [k]} \sum_{l,m \in [n], l \neq m} e_j(-Y_l)e_j(Y_m)
\end{equation}
and \begin{equation}
\tilde{S}_k := \frac{1}{n} \sum_{|j| \in [k]} \sum_{l \in [n]} g_j^o e_j(Y_l).
\end{equation}
Note that $q_k^2(g^o)$ is known, $\tilde{T}_k$ is a U-statistic and $\tilde{S}_k$ is a linear statistic. Moreover, if $\{Y_j\}_{j=1}^n \overset{\text{iid}}{\sim} g$ as in (1.2) then $\tilde{q}_k^2$ is an unbiased projection estimator of $q_k^2(g - g^o)$ for each $k \in \mathbb{N}$.

**Decomposition of the test statistic.** We analyse quantiles of the test statistic $\tilde{q}_k^2$ in (4.2) using the decomposition
\begin{equation}
\tilde{q}_k^2 = u_n^d + 2v_n^d + q_k^2(g - g^o)
\end{equation}
with
\begin{equation}
u_n^d := \frac{1}{n(n-1)} \sum_{|i| \in [k]} \sum_{l,m \in [n], l \neq m} (e_j(-Y_l) - g_j)(e_j(Y_m) - \bar{g}_j)
\end{equation}
and
\begin{equation}
v_n^d := \frac{1}{n} \sum_{|j| \in [k]} \sum_{l \in [n]} (g_j^o - g^o_j)(e_j(Y_l) - \bar{g}_j),
\end{equation}
where $q_k^2(g - g^o)$ is a separation term, $u_n^d$ is a canonical U-statistic and $v_n^d$ is a centred linear statistic.

**Definition of the threshold.** The next proposition provides bounds for the quantiles of the test statistic $\tilde{q}_k^2$. Recall that $L_x = (1 - \log x)^{1/2} > 1$ for $x \in (0, 1)$. For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ with $c_1 := 799 \|g^o\|_1^2 + 1372$, $c_2 := 52 \|g^o\|_1^2$, we define the threshold
\begin{equation}
\tau_k^d(\alpha) := c_1 \left( 1 \lor \frac{L_{\alpha}^2(2k)^{1/4}}{n^{1/2}} \lor \frac{L_{\alpha}^3((2k)^{1/2}}{n} \right) \frac{L_{\alpha}(2k)^{1/2}}{n} + c_2 L_{\alpha}^2 \frac{1}{n}.
\end{equation}

**Proposition 4.1.** For $f^o, f, \varphi \in L^2$ and $n \in \mathbb{N}$, $n \geq 2$, consider $\{Y_j\}_{j=1}^n \overset{\text{iid}}{\sim} g = f \circ \varphi$ with joint distribution $P_f$ and $g^o = f^o \circ \varphi$. Let $\alpha, \beta \in (0, 1)$ and for $k \in \mathbb{N}$ consider $\tilde{q}_k^2$ and $\tau_k^d(\alpha)$ as in (4.2) and (4.4), respectively.

(i) If $L_{\alpha}^2(g^o) = L_{\alpha}^2$, then $P_{f^o}(\tilde{q}_k^2 \geq \tau_k^d(\alpha)) \leq \alpha$.
(ii) If $c_3 := 837 \|\varphi\|_1$ and $\tau_k^d(\alpha) + c_3 L_{\beta/2}^2 \left( 1 \lor \frac{(2k)^{1/2}}{n} \right) \frac{(2k)^{1/2}}{n}$, then $P_{f}(\tilde{q}_k^2 < \tau_k^d(\alpha)) \leq \beta$.

**Proof of Proposition 4.1.** The result (i) may be proved in much the same way as Proposition 2.1 (i) using the decomposition (4.3) rather than (2.3) and applying Proposition A.1 in the Appendix A, a concentration inequality for canonical U-statistics of order 2, together with Corollary A.5 rather than Lemma A.3. Secondly, consider (ii). Keeping the decomposition (4.3) in mind we control the deviations of the U-statistic $u_n^d$ and the linear statistic $v_n^d$ applying Proposition A.1 and Lemma A.6 in the Appendix, respectively. In fact, the proof is similar to the proof of Proposition 2.1 (ii) using Corollary A.5 and Lemma A.6 rather than Lemmas A.3 and A.4, and we omit the details.

**Definition of the test.** For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ using the test statistic $\tilde{q}_k^2$ and the threshold $\tau_k^d(\alpha)$ given in (4.3) and (4.4), respectively, we consider the test
\begin{equation}
\Delta_{k,\alpha}^d := \mathbb{I}\{\tilde{q}_k^2 \geq \tau_k^d(\alpha)\}.
\end{equation}
From (i) in Proposition 4.1 it immediately follows that $\Delta_{k,\alpha}^d$ is a level-$\alpha$-test for all $k \in \mathbb{N}$. Moreover, Proposition 4.1 (ii) allows to characterize elements in $E^{R}_{\alpha}$ (defined in (2.9)) for which $\Delta_{k,\alpha}^d$ is powerful. Exploiting these results, in the next proposition we derive an upper bound for the radius of testing of $\Delta_{k,\alpha}^d$ in terms of $m_k^2 = \max_{j \in [k]} |\varphi_j|^{-2}$ as in (2.4) and the regularity parameter $\alpha_\bullet \in \mathcal{A}$. Thus, we define

$$(\rho_{k,\alpha_\bullet}^d)^2 := (\rho_{k,\alpha_\bullet}^d(n)) := a_k^2 \cdot (2k)^{1/2} \cdot \frac{1}{n} \cdot m_k^2.$$ 

Next, similar to Proposition 2.3 we show that both the type I and maximal type II error probability of a test $\Delta_{k,\gamma/2}^d$ are bounded by $\gamma/2$ and, thus, their sum by $\gamma$, by applying Proposition 4.1 with $\alpha = \beta = \gamma/2$.

**Proposition 4.2.** Under the assumptions of Proposition 4.1 for $\gamma \in (0,1)$ define

$$\overline{A}_\gamma := R^2 + 2(837||\varphi_\bullet||_\ell^2 + 851||g_\bullet^0||_\ell + 2745)L_{\gamma/4}^4.$$ 

For all $A \geq \overline{A}_\gamma$ and for all $n, k \in \mathbb{N}$ with $2k \leq n^2$ we have $\mathcal{R}(\Delta_{k,\gamma/2}^d|E^{R}_{\alpha_\bullet}, A_{\rho_{k,\alpha_\bullet}^d}) \leq \gamma$.

**Proof of Proposition 4.2.** The result follows from Proposition 4.1 with $\alpha = \beta = \gamma/2$ and the definition of the risk $\mathcal{R}(\Delta_{k,\gamma/2}^d|E^{R}_{\alpha_\bullet}, A_{\rho_{k,\alpha_\bullet}^d})$. Indeed, since the assumption of Proposition 4.1 (i) is fulfilled the test $\Delta_{k,\gamma/2}^d$ is a level-$\gamma/2$-test. Hence, in order to apply Proposition 4.1 (ii) (with $\beta = \gamma/2$) for each density $f \in \mathcal{L}$ with $||f - f^0||_2^2 \geq \overline{A}_{\gamma}(\rho_{k,\alpha_\bullet}^d)^2$ and $f - f^0 \in E^{R}_{\alpha_\bullet}$ it remains to verify condition (4.5) which states $a_k^2(g - g^0) \geq 2 (\tau_k^d(\gamma/2) + c_2 L_{\gamma/4}^4 (1 \vee (2k)^{1/2} - n^{-1}))(2k)^{1/2}n^{-1}$ with $\tau_k^d(\gamma/2)$ as in (4.4).

Since $\sum_{|j| > k} |f_j - f_j^0|^2 \leq a_k^2 R^2$ it follows

$$q_k^2(f - f^0) = ||f - f^0||_2^2 - \sum_{|j| > k} |f_j - f_j^0|^2 \geq \overline{A}_{\gamma}(\rho_{k,\alpha_\bullet}^d)^2 - a_k^2 R^2$$

$$\geq 2(837||\varphi_\bullet||_\ell^2 + 799||g_\bullet^0||_\ell^2 + 52||g_\bullet^0||_\ell + 2745)L_{\gamma/4}^4 \frac{(2k)^{1/2}}{n} \cdot m_k^2,$$

by using $||g_\bullet^0||_\ell^2 \leq ||g_\bullet^0||_\ell$. The condition (4.5) follows from (4.7) together with $a_k^2(g - g^0)$ by exploiting further $1 \leq L_{\gamma/2} \leq L_{\gamma/4}$ and $2k \leq n^2$, which completes the proof.

The upper bound $(\rho_{k,\alpha_\bullet}^d)^2$ for the radius of testing of $\Delta_{k,\gamma/2}^d$ depends on the dimension parameter $k$.

Let us introduce a dimension that realises an optimal bias-variance trade-off, and the corresponding radius

$$k_{\alpha_\bullet}^d := \operatorname{arg min} \rho_{k,\alpha_\bullet}^d \in \mathbb{N}: \rho_{k,\alpha_\bullet}^d \leq \rho_{l,\alpha_\bullet}^d \text{ for all } l \in \mathbb{N}$$

and

$$\rho_{\alpha_\bullet}^d := \operatorname{min} (\rho_{k,\alpha_\bullet}^d)^2 = \operatorname{min} \left\{ a_k^2 \cdot (\frac{2k)^{1/2}}{n} \cdot m_k^2 \right\}.$$ 

**Corollary 4.3.** Under the assumptions of Proposition 4.1 let $\gamma \in (0,1)$ and $\overline{A}_\gamma$ as in (4.6), then $\mathcal{R}(\Delta_{k_{\alpha_\bullet}^d,\gamma/2}^d|E^{R}_{\alpha_\bullet}, A_{\rho_{\alpha_\bullet}^d}) \leq \gamma$ for all $A \geq \overline{A}_\gamma$ and for all $n \geq \sqrt{2}||\varphi_1||^{-2}$.

**Proof of Corollary 4.3.** The result follows immediately from Proposition 4.2, since $2k_{\alpha_\bullet}^d \leq n^2$ for all $n \geq \sqrt{2}||\varphi_1||^{-2}$. Indeed, $n \geq \sqrt{2}||\varphi_1||^{-2}$ implies $1 \geq (\rho_{\alpha_\bullet}^d)^2 \geq (\rho_{k,\alpha_\bullet}^d)^2 \geq (2k_{\alpha_\bullet}^d)^{1/2}n^{-1}$. \(\square\)

**Remark 4.4 (optimality of the direct testing procedure).** Let us compare the upper bound $(\rho_{\alpha_\bullet}^d)^2 = \operatorname{min}_{k \in \mathbb{N}} (a_k^2 \cdot (2k)^{1/2} \cdot m_k^2 \cdot n^{-1})$ for the direct testing procedure with the minimax radius of testing $\rho_{\alpha_\bullet}^d = \operatorname{min}_{k \in \mathbb{N}} (a_k^2 \cdot (2k)^{1/2} \cdot n^{-1})$. If there exists a constant $c > 0$ such that

$$\nu_k^2 = \left( \sum_{|j| \in [k]} |\varphi_j|^4 \right)^{1/2} \leq (\max_{|j| \in [k]} |\varphi_j|^{-2})(2k)^{1/2} = m_k^2 (2k)^{1/2} \leq c \nu_k^2,$$ 

with the constant $c = \frac{2 m_k^2}{(2k)^{1/2}} \leq 8 \nu_k^2$.
then \( \rho_{d,0} \) and \( \rho_{d,0} \) are of the same order and, thus, the direct testing procedure is minimax optimal. Condition (4.9) is for instance satisfied for a mildly ill-posed model, i.e., if \(|\varphi_j|\)\(\in\mathbb{N}\) decays polynomially. Note, however, that (4.9) is a sufficient but not a necessary condition. For a severely ill-posed model, i.e., if \(|\varphi_j|\)\(\in\mathbb{N}\) decays exponentially, (4.9) is not fulfilled. Nevertheless, the direct testing procedure still performs optimally (see Illustration 4.5 below).

Illustration 4.5. In Table 4 we illustrate the order of the upper bound for the radius of testing of the direct test \( \Delta^d_{k,\alpha} \) under the regularity and ill-posedness assumptions introduced in Illustration 2.5. Comparing the resulting upper bounds \( \rho^d_{\alpha} \) with the radii \( \rho^d_{\alpha} \), we conclude that the direct test performs as well as the indirect test in all three cases.

In a severely ill-posed model with ordinary smoothness, the order of the optimal dimension parameter does not depend on the smoothness parameter. Hence, the test \( \Delta^d_{k,\alpha} \) is automatically adaptive with respect to ordinary smoothness and no aggregation procedure is needed.

5. ADAPTIVE DIRECT TESTING PROCEDURE

Adaptation procedure via Bonferroni aggregation. The choice of the optimal dimension parameter used in the test in Corollary 4.3 requires the knowledge of the regularity parameter \( \alpha \) belonging to a set \( \mathcal{A} \) of strictly positive, monotonically non-increasing sequences bounded by 1. Let \( \mathcal{K} \subseteq \mathbb{N} \) be a finite collection of dimension parameters. Applying the Bonferroni aggregation method described in Section 3 to the collection of direct tests \( \{\Delta^d_{k,\alpha}\}_{k \in \mathcal{K}} \) we introduce the direct max-test with Bonferroni choice of the error level, i.e.,

\[
\Delta^d_{k,\alpha} := \{\tilde{Q}_{k,\alpha} > 0\} \quad \text{with} \quad \tilde{Q}_{k,\alpha} := \max_{k \in \mathcal{K}} \left( \tilde{\alpha}^2_k - \tau^d_k \left( \frac{\alpha}{|\mathcal{K}|} \right) \right).
\]

Testing radius of the indirect max-test. We define the minimal achievable radius of testing \( \rho^d_{k,\alpha}(n) \) over the set \( \mathcal{K} \) in terms of \( m_k \) as in (2.4) and the regularity parameter \( \alpha \in \mathcal{A} \) by

\[
\rho^d_{k,\alpha}(x) := \min_{k \in \mathcal{K}} \rho^d_{k,\alpha}(x) \quad \text{with} \quad \left( \rho^d_{k,\alpha}(x) \right)^2 := a^2_k \vee (2k)^{1/2} m_k^2 x^{-1} \quad \text{for all} \quad x \in \mathbb{R}_+.
\]

Since \( \rho^d_{\alpha}(n) = \rho^d_{\alpha}(n) \) in (4.8) is defined as minimum taken over \( \mathbb{N} \) instead of \( \mathcal{K} \), for all \( n \in \mathbb{N} \) we always have \( \rho^d_{k,\alpha}(n) \geq \rho^d_{\alpha}(n) \). Let us furthermore recall the remainder radius \( r^2_{k,\alpha}(x) = \min_{k \in \mathcal{K}} \{a^2_k \vee m_k^2 x^{-1}\} \) defined in (3.4) for \( x \in \mathbb{R}_+ \). Next, similar to Proposition 3.1 we show for each \( \alpha \in \mathcal{A} \) that both the type I and maximal type II error probability of a test \( \Delta^d_{k,\alpha} \) are bounded by \( \gamma / 2 \) and, thus, their sum by \( \gamma \), by applying Proposition 4.1 with \( \alpha = \beta = \gamma / 2 \).

Proposition 5.1 (Uniform radius of testing over \( \mathcal{A} \)). Under the assumptions of Proposition 4.1 let \( \gamma \in (0, 1) \) and consider \( \overline{\mathcal{A}}_\gamma \) as in (2.10). Then, for all \( \mathcal{A} \geq \overline{\mathcal{A}}_\gamma \) and for all \( n \in \mathbb{N} \)

\[
\sup_{\alpha \in \mathcal{A}} \mathcal{R}(\Delta^d_{k,\gamma / 2} | \mathcal{A}^R, A(r_{k,\alpha}(\delta^2 n) \vee \rho^d_{k,\alpha}(\delta n))(1 \vee \delta^{-3/2} \rho^d_{k,\alpha}(\delta n))) \leq \gamma
\]

Table 4

| \( a_j \) |
| --- |
| \( \varphi_j \) |
| \( k^d_{\alpha} \) |
| \( \rho^d_{\alpha} \) |

(smoothness) (ill-posedness)
with \( \delta = (1 + \log |K|)^{-1/2} \).

**Proof of Proposition 5.1.** The proof follows along the lines of the proof of Proposition 3.1 making use of Proposition 4.1 rather than Proposition 2.1. Similarly to (3.2) together with Proposition 4.1 (i) and (ii) it follows that the type I error probability is bounded by \( \gamma/2 \). Under the alternative, let \( f \in \mathcal{L}^2 \) with \( f - f^o \in \mathcal{E}^R \) satisfy
\[
||f - f^o||_{\mathcal{L}^2}^2 \geq \overline{A}_\gamma^2 (r^2_{K^\gamma}(\delta^2 n) \vee (\rho^d_{K^\gamma}(\delta n))^2)(1 \vee \delta^3 (\rho^d_{K^\gamma}(\delta n))^2).
\]
Equation (5.1)

It is sufficient to use the elementary bound (3.3) together with the observation that

(i) \( \mathbb{P}_f (q^2_k < \tau^d_k (2|K|)) \leq \frac{1}{2} \)

holds for all \( k \in K \) whenever \( f \in \mathcal{L}^2 \) satisfies \( f - f^o \in \mathcal{E}^R \) and
\[
||f - f^o||_{\mathcal{L}^2}^2 \geq \overline{A}_\gamma^2 \left( a^2_k \vee \left( 1 \vee \frac{(2k)^{1/4}}{\delta^2 n^{1/2}} \vee \frac{(2k)^{1/2}}{\delta^3 n} \right) \frac{(2k)^{1/2}m^2_k}{\delta n} \vee \frac{m^2_k}{\delta^2 n} \right); \quad (5.2)
\]

(ii) under (5.1) there exists \( k \in K \) such that also (5.2) is fulfilled.

Consequently, we have \( \mathbb{P}_f (A^d_{1/2} / K = 0) \leq \gamma/2 \) for all \( f \in \mathcal{L}^2 \) satisfying \( f - f^o \in \mathcal{E}^R \) and (5.1). Since both the type I and the maximal type II error probability are bounded by \( \gamma/2 \) Proposition 5.1 follows immediately from the definition of the risk. It remains to show (i) and (ii). The claim (i) follows from Proposition 4.1 (ii) (with \( \beta = \gamma/2 \)) since (5.2) implies the condition (4.5), which states \( q^2_k(g-g^o) \geq 2 \left( \tau^d_k (2|K|) + c_3 \overline{L}^{d/4}_{1/2} (1 \vee (2k)^{1/2} n^{-1}) (2k)^{1/2} n^{-1} \right) \) with \( \tau^d_k (2|K|) \) as in (4.4). Indeed, we have
\[
\tau^d_k (2|K|) \leq (c_1 + c_2) \overline{L}^{d/4}_{1/2} \left( 1 \vee \frac{(2k)^{1/4}}{\delta^2 n^{1/2}} \vee \frac{(2k)^{1/2}}{\delta^3 n} \right) \frac{(2k)^{1/2} m^2_k}{\delta n} \vee \frac{m^2_k}{\delta^2 n}. \]

Using \( \overline{A}_\gamma^2 - R^2 \geq 2(c_1 + c_2 + c_3) \overline{L}^{d/4}_{1/2} \) elementary calculations show that (4.5) holds whenever
\[
q^2_k(g-g^o) \geq \left( \overline{A}_\gamma^2 - R^2 \right) \left( 1 \vee \frac{(2k)^{1/4}}{\delta^2 n^{1/2}} \vee \frac{(2k)^{1/2}}{\delta^3 n} \right) \frac{(2k)^{1/2} m^2_k}{\delta n} \vee \frac{m^2_k}{\delta^2 n}. \quad (5.3)
\]

Due to \( f - f^o \in \mathcal{E}^R \) and hence \( \sum_{j > k} |f_j - f_j^o|^2 \leq a^2_k R^2 \), the condition (5.2) implies
\[
q^2_k(f-f^o) = ||f - f^o||_{\mathcal{L}^2}^2 - \sum_{j > k} |f_j - f_j^o|^2 \geq m^2_k \left( \overline{A}_\gamma^2 - R^2 \right) \left( 1 \vee \frac{(2k)^{1/4}}{\delta^2 n^{1/2}} \vee \frac{(2k)^{1/2}}{\delta^3 n} \right) \frac{(2k)^{1/2} m^2_k}{\delta n} \vee \frac{m^2_k}{\delta^2 n}. \]

As a consequence, if (5.2) is satisfied then due to \( m^2_k q^2_k(g-g^o) \geq q^2_k(f-f^o) \) also (5.3), and thus (4.5), which shows the claim (i). Lastly, consider (ii). By Lemma A.1 in [30] we have \( r^2_{K^\gamma}(\delta^2 n) \vee (\rho^d_{K^\gamma}(\delta n))^2 = a_k \vee \frac{m^2_k}{\delta n} \vee \frac{(2k)^{1/2} m^2_k}{\delta n} \) and \( (\rho^d_{K^\gamma}(\delta n))^2 \geq (2k)^{1/2} \delta^{-1} n^{-1} \) for at least one \( k \in K \). Hence, there is \( k \in K \) with \( \left( r^2_{K^\gamma}(\delta^2 n) \vee (\rho^d_{K^\gamma}(\delta n))^2 \right) \left( 1 \vee \frac{m^2_k}{\delta^2 n} \right) \geq a_k \vee \frac{m^2_k}{\delta n} \vee \frac{(2k)^{1/2} m^2_k}{\delta n} \left( 1 \vee \frac{(2k)^{1/2}}{\delta^4 n} \right) \). Since \( 1 \vee \frac{(2k)^{1/2}}{\delta^4 n} \geq 1 \vee \frac{(2k)^{1/2}}{\delta^2 n^{1/2}} \vee \frac{(2k)^{1/2}}{\delta^4 n} \), this shows (ii) and completes the proof. \( \square \)

**Corollary 5.2 (worst-case adaptive factor).** Under the assumptions of Proposition 4.1 let \( \gamma \in (0, 1) \) and consider \( \overline{A}_\gamma \) as in (4.6). Then for all \( A \geq \overline{A}_\gamma \), and \( n \in \mathbb{N} \)
\[
\sup_{a^d_{K^\gamma}} \mathcal{R}(A_{K^\gamma}/|K|) \leq \gamma \overline{A}_\gamma \left( \rho^d_{K^\gamma}(\delta^2 n)(1 \vee \rho^d_{K^\gamma}(\delta^2 n)) \right) \leq \gamma
\]

with \( \delta = (1 + \log |K|)^{-1/2} \).
**Proof of Corollary 5.2.** The result may be proved in much the same way as Proposition 5.1. In fact, as in the proof of Proposition 5.1, it is sufficient to show (ii) replacing (5.1) by
\[
||f - f'||^2_{L^2} \geq \overline{A}_2^2(\rho_{K,a_s}(\delta^2 n))^2(1 \lor (\rho_{K,a_s}(\delta^2 n))^2).
\]
For each \(a_s \in \mathcal{A}\) under (5.4) the dimension parameter \(k := \arg \min_{K' \in \mathcal{K}}(\rho_{K,a_s}(\delta^2 n))^2\) satisfies
\[
(\rho_{K,a_s}(\delta^2 n))^2(1 \lor (\rho_{K,a_s}(\delta^2 n))^2) = (\rho_{K,a_s}(\delta^2 n))^2(1 \lor \rho_{K,a_s}(\delta^2 n) \lor (\rho_{K,a_s}(\delta^2 n))^2)
\]
\[
\geq a_k^2 \lor \frac{m_k^2(2k)^{1/2}}{\delta^2 n}(1 \lor \frac{2k^{1/4}}{\delta n^{1/2}} \lor \frac{2k^{1/2}}{\delta^2 n}) \geq a_k^2 \lor \frac{m_k^2(2k)^{1/2}}{\delta^2 n}(1 \lor \frac{2k^{1/4}}{\delta n^{1/2}} \lor \frac{2k^{1/2}}{\delta^3 n})
\]
since \(a_k^2 \lor \frac{m_k^2(2k)^{1/2}}{\delta^2 n} = (\rho_{K,a_s}(\delta^2 n))^2\) and \(\delta \leq 1\). This shows (5.2) and, in consequence, (ii). We obtain the assertion proceeding exactly as in the proof of Proposition 5.1. \( \square \)

By Corollary 5.2, \((\rho_{K,a_s}(\delta^2 n))^2\) is an upper bound for the radius of testing of the direct max-test if \((\rho_{K,a_s}(\delta^2 n))^2 \leq 1\). The latter is satisfied for an arbitrary regularity parameter \(a_s \in \mathcal{A}\), if \(1 \in \mathcal{K}\) and \(n \geq \sqrt{2}\varphi_1^{-2}(1 + \log |\mathcal{K}|)\). The next corollary establishes \((\rho_{K,a_s}(\delta n))^2\) as a sharper upper bound for the radius of testing of the direct max-test under additional conditions, which are satisfied in all the examples considered in Illustration 5.4 below. The result follows immediately from Proposition 5.1 and we omit its proof.

**Corollary 5.3 (best-case adaptive factor).** Under the assumptions of Proposition 4.1 let \(\gamma \in (0, 1)\) and consider \(\overline{A}_3\) as in (4.6). If there exists a constant \(C \geq 1\) such that \(r_{K,a_s}(\delta^2 n) \leq Cr_{K,a_s}(\delta n)\) and \(\rho_{K,a_s}(\delta n) \leq C\delta^{5/2}\) for all \(a_s \in \mathcal{A}\), then for all \(A \geq C^2\overline{A}_3\), and \(n \in \mathbb{N}\)
\[
\sup_{a_s \in \mathcal{A}} \mathcal{R}(\Delta^d_{K,a_s}(\delta n)) \leq \gamma
\]
with \(\delta = (1 + \log |\mathcal{K}|)^{-1/2}\).

Concerning the choice of the collection \(\mathcal{K}\) of dimensions we refer to Remark 3.4.

**Illustration 5.4.** For the typical configurations for regularity and ill-posedness introduced in Illustration 2.5 Tables 5, 6 display the adaptive radii of the direct max-test \(\Delta^d_{K,a_s}(\delta^2 n)\) for appropriately chosen grids. In a mildly ill-posed model with ordinary smoothness we choose the geometric grid \(K_g\) and hence consider \(\delta = (1 + \log |\mathcal{K}_g|)^{-1/2}\). It is easily seen that the remainder term \(r_{K,g,a_s}(\delta^2 n)\) is asymptotically negligible compared with \((\rho_{K,g,a_s}(\delta n))^2\). Moreover, \(\delta^{-3}(\rho_{K,g,a_s}(\delta n))^2\) tends to zero. Hence, the upper bound in Proposition 5.1 asymptotically reduces to \((\rho_{K,g,a_s}(\delta n))^2\) featuring an adaptive factor of order \((\log \log n)^{1/2}\) as can be seen in Table 5. In a severely ill-posed model we have seen in Illustration 4.5 that a direct test with optimal dimension is automatically adaptive with respect to ordinary smoothness. However, the optimal dimension depends on the ill-posedness parameter. Therefore, we consider here the max-test \(\Delta^d_{K,a_s}(\delta^2 n)\) also in this situation. Since \(\delta^{-3}(\rho_{K,g,a_s}(\delta n))^2\) tends to zero, and both the remainder term \(r_{K,g,a_s}(\delta^2 n)\) and \((\rho_{K,g,a_s}(\delta n))^2\) are of the same order as \((\rho_{K,g,a_s}(n))^2\), the upper bound in Proposition 5.1 asymptotically reduces to \((\rho_{K,g,a_s}(n))^2\).

In a mildly ill-posed model with super smoothness, we choose the smaller geometric grid \(K_{s*}\) for adaptation to smoothness \(s \geq s_*\) and hence \(\delta = (1 + \log |K_{s*}|)^{-1/2}\). It is easily seen that the remainder term \(r_{K,s*,a_s}(\delta^2 n)\) is asymptotically negligible compared with \((\rho_{K,s*,a_s}(\delta n))^2\) and \(\delta^{-3}(\rho_{K,s*,a_s}(\delta n))^2\) tends to zero. Hence, the upper bound in Proposition 5.1 asymptotically reduces to \((\rho_{K,s*,a_s}(\delta n))^2\) featuring an adaptive factor of order \((\log \log \log n)^{1/2}\).

We conclude that in all the cases considered in this illustration the direct max-test achieves a testing radius of the same order as the indirect max-test (cp. Illustration 3.5).
We emphasise that in the case $f^0 = \mathbb{I}_{[0,1]}$ the direct max-test in contrast to the indirect max-test does not require any knowledge about the error density $\varphi$. Indeed, neither the test statistic $\tilde{\tau}_k^d$ in (4.2) nor the threshold $\tau_k^d(\alpha)$ in (4.4) depends on characteristics of the error density $\varphi$. However, in a mildly ill-posed model with ordinary smoothness both the direct and the indirect max-test feature an adaptive factor of order $(\log \log n)^{1/2}$ which we show below is unavoidable when testing for uniformity. Moreover, considering only super smooth densities in a mildly ill-posed model both the direct and the indirect max-test share an adaptive factor of order $(\log \log n)^{1/2}$ which we show below also is unavoidable when testing for uniformity. It is important to note that without any prior knowledge about the ill-posedness of the model the direct max-test for uniformity attains the optimal testing radius simultaneously in a mildly and severely ill-posed model with ordinary smoothness.

### Table 5

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $\frac{\hat{r}_k^2}{\hat{d}_k}((\delta^2 n))$ | $\frac{(\delta^2 n)}{\hat{d}_k}$ |
|---|---|---|---|
| $j^{-s}$ | $|j|^{-p}$ | $\left(\frac{n}{\log \log n}\right)^{-\frac{2p}{2+p}}$ | $\left(\frac{n}{\log \log n}\right)^{-\frac{2p}{2+p}}$ |
| $j^{-s}$ | $e^{-|j|^p}$ | $(\log n)^{-\frac{2p}{2+p}}$ | $(\log n)^{-\frac{2p}{2+p}}$ |

### Table 6

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $\frac{\hat{r}_k^2}{\hat{d}_k}((\delta^2 n))$ | $\frac{\hat{d}_k}{\hat{d}_k}$ |
|---|---|---|---|
| $e^{-j^s}$ | $|j|^{-p}$ | $\frac{\log \log n}{n} \frac{(\log n)^{2p}}{2p}$ | $\frac{(\log \log n)^{1/2}}{n} \frac{(\log n)^{2p+1/2}}{2p+1/2}$ |

6. LOWER BOUND

Throughout this section we consider testing for uniformity, i.e., $f^0 = \mathbb{I}_{[0,1]}$. The next proposition states general conditions on the class $A$ under which an adaptive factor $\delta^{-1}$ is an unavoidable cost to pay for adaptation over $A$. The proof of Proposition 6.1 makes use of Lemma C.2 in the Appendix C, which provides a bound on the $\chi^2$-divergence between the null and a mixture over several alternative classes. Inspired by Assouad’s cube technique the candidate densities, i.e., the vertices of the hypercubes, are constructed such that, roughly speaking, they are statistically indistinguishable from the null $f^0$ while having largest possible $L^2$-distance.

**Proposition 6.1 (adaptive lower bound).** Let $\gamma \in (0,1)$ and $\delta \in (0,1]$. Assume a collection of $N \in \mathbb{N}$ regularity parameters $\{a_m \in A : m \in [N]\}$, where we abbreviate $\rho_m^d := \rho^d_m(\delta n)$ with associated $k_m := k_{a_m}$ for $m \in [N]$ as in (2.12), satisfies the following four conditions:

1. $k_m \leq k^l$ and $\rho_m \leq \delta^l$ whenever $m < l$ and $l, m \in [N],$
2. There is a finite constant $c_\gamma > 0$ such that $\exp(c_\gamma \delta^{-2}) \leq N \gamma^2,$
3. There is a finite constant $a > 0$ such that $2 \max_{m \in [N]} \|a_m\|^2_{L^2(N)} \leq a,$
(C4) there is a constant \( \eta \in (0, 1] \) such that
\[
\eta \leq \min_{m \in [N]} \left( \frac{(a_k^m)^2 \land (\Delta n)^{-1} \nu_k^{2m}}{(a_k^m)^2 \lor (\Delta n)^{-1} \nu_k^{2m}} \right) = \min_{m \in [N]} \frac{(a_k^m)^2 \land (\Delta n)^{-1} \nu_k^{2m}}{(\rho^m)^2}.
\] (6.1)

Then, with \( A^2 := \eta \sqrt{\log(1 + \gamma^2)} \land a^{-1} \land \sqrt{\gamma} \), we obtain for all \( A \in [0, A^2] \)
\[
\inf_{\Delta} \sup_{\alpha} \mathcal{R}(A | \mathcal{E}_{\alpha}^R, \rho_\alpha(\Delta n)) \geq 1 - \gamma.
\] (6.2)

**Remark 6.2 (conditions of Proposition 6.1).** Let us briefly discuss the conditions of Proposition 6.1. Under (C1) the collection of regularity parameters \( A \) is rich enough to make a factor \( \delta^{-1} \) for adaptation unavoidable, i.e., it contains distinguishable elements resulting in significantly different radii. (C2) is a bound for the maximal size of an unavoidable adaptive factor. (C3) guarantees that the candidates constructed in the reduction scheme of the proof are indeed densities. The condition (C4) relates the behaviour of the sequences \( \varphi \) and \( a^m, m \in [N] \), and essentially guarantees an optimal balance of the bias and the variance term in the dimension \( k^m \) uniformly over \( m \in [N] \). Moreover, for all regularity and ill-posedness examples considered in Illustration 6.5 (C4) holds uniformly for all \( n \in \mathbb{N} \). We shall emphasise that the optimal dimensions \( k^m \) and the corresponding radii \( \rho^m \) are determined in terms of the effective sample size \( \Delta n \).

**Proof of Proposition 6.1. Reduction step.** Let \( P_1 := N^{-1} \sum_{m \in [N]} P_{a^m} \) with mixture \( P_{a^m} \) over the alternative \( \mathcal{E}_{a^m} \cap \mathcal{L}_{A^2} \rho^m \) to be specified below, and set \( P_0 := P_{f^0} \). Introducing the \( \chi^2 \)-divergence \( \chi^2(P_0, P_1) \) the risk is lower bounded due to a classical reduction argument as follows
\[
\inf_{\Delta} \sup_{\alpha} \mathcal{R}(A | \mathcal{E}_{\alpha}^R, A, \rho_\alpha(\Delta n)) \geq \inf_{\Delta} \sup_{\alpha} \mathcal{R}(A | \mathcal{E}_{\alpha}^R, A, \rho_\alpha(\Delta n)) \geq \inf_{\Delta} \sup_{\alpha} \mathcal{R}(P_0, P_1)
\]
\[
\geq 1 - \sqrt{\frac{\chi^2(P_1, P_0)}{2}}.
\]
The last inequality follows e.g., from Lemma 2.5 combined with (2.7) in [33].

**Definition of the mixture.** For \( m \in [N] \) define \( \theta^m_j := A \rho^m \nu_k^{2m} |\varphi_j|^{-2} \) for all \( j \in [k^m] \) and \( \theta^m_j := 0 \) otherwise. (6.3)

We first collect elementary properties of the \( \ell^2(\mathbb{N}) \)-sequences \( \theta^m_j, \theta^m_j/\ell_j^m := (\theta^m_j / \ell_j^m)_{j \in \mathbb{N}} \) and \( \theta^m_j \theta^m_j|\varphi_j|^2 := (\theta^m_j \varphi_j|^2)_{j \in \mathbb{N}} \). Exploiting (6.1) shows by direct calculations
\[
2 \|\theta^m_j\|_{\ell^2(\mathbb{N})} \leq A \rho^m \nu_k^{2m} \text{ for all } j \in [k^m] \quad \text{and} \quad \theta^m_j := 0 \text{ otherwise.}
\] (6.3)

As a consequence, since \( 2 \|\theta^m_j\|_{\ell^2(\mathbb{N})} \leq a \) by (C3) and the definition of \( A \gamma \), we conclude that
\[
4 \|\theta^m_j\|_{\ell^2(\mathbb{N})} \leq 2 \|\theta^m_j/\ell^m_j\|_{\ell^2(\mathbb{N})} \leq \eta^{-1} A \gamma \quad \text{and} \quad 2 \|\theta^m_j/\ell^m_j\|_{\ell^2(\mathbb{N})} \leq \eta^{-1} A \gamma \leq R^2.
\] (6.5)

Let \( m \leq m \), then \( \nu_k^{2m} \leq \nu_k^{2m} \) by (C1). Due to (6.4) combined with (6.1) and (C1) we have
\[
n^2 \|\theta^m_j \varphi_j|\varphi_j|^2\|_{\ell^2(\mathbb{N})} \leq A \gamma \frac{(\rho^m)^2 \rho^2}{\nu_k^{2m}} \leq A \gamma \eta^{-2} \rho^2 \nu_k^{2m} \leq A \gamma \eta^{-2} \leq 2 \log(1 + a^2). \] (6.6)

Let \( m \leq l \), then from (6.4) exploiting (6.1) we obtain
\[
n^2 \|\theta^m_j \varphi_j|\varphi_j|^2\|_{\ell^2(\mathbb{N})} = A \gamma \frac{(\rho^m)^2 \rho^4}{\nu_k^{2m}} \leq A \gamma \eta^{-2} \delta^2 \leq c_\gamma \delta^2.
\] (6.7)
Combining (6.6) and (6.7) from (C2) we conclude
\[
\frac{1}{N^2} \sum_{l,m \in [N]} \exp \left( n^2 \| \theta_\theta^m \| \varphi_\tau^2 \|_2^2 \right) \leq \frac{1}{N} \exp \left( c_\gamma \delta - 2 \right) + \frac{(N - 1)}{N} (1 + \gamma^2) \leq 2 \gamma^2 + 1. \tag{6.8}
\]
For each \( m \in [N] \), \( \theta_\theta^m \) as in (6.3) and \( \tau \in \{ \pm \}^{k_m} \), define a density
\[
f^{m,\tau} := c_0 + \sum_{|j| \in [k_m]} \tau_j \theta_\theta^m e_j \in D.
\]
Indeed, by construction \( f^{m,\tau} \) belongs to \( \mathcal{L}^2 \), integrates to one, is real-valued and positive, since \( 2\| \theta_\theta^m \|_{e^1} \leq 1 \) by (6.5). Moreover, we have
\[
f^{m,\tau} - f^0 = \sum_{|j| \in [k_m]} \tau_j \theta_\theta^m e_j \in \mathcal{E}_\theta^R \cap \mathcal{L}_\Delta^2 \rho^m,
\]
exploiting \( 2 \sum_{j \in [k_m]} (a_j^m)^{-2} \| j \| \theta_\theta^m \|_2^2 = 2 \| \theta_\theta^m / a_j^m \|_2^2 (N) \leq R^2 \) by (6.5) and \( \| f^{m,\tau} - f^0 \|_2^2 = 2 \| \theta_\theta^m \|_2^2 (N) = A_2^2 (\rho^m) \) due to (6.4). As a consequence \( \mathbb{P}_{a_j^m} := 2^{-k_m} \sum_{\tau \in \{ \pm \}^{k_m}} \mathbb{P}_{f^{m,\tau}} \) is a mixture on the alternative.

**Bound for the \( \chi^2 \)-divergence.** From (6.8) by applying Lemma C.2 we obtain
\[
\chi^2(\mathbb{P}_0, \mathbb{P}_1) \leq \frac{1}{N^2} \sum_{l,m \in [N]} \exp \left( n^2 \| \theta_\theta^m \| \varphi_\tau^2 \|_2^2 (N) \right) - 1 \leq 2 \gamma^2.
\]
Combining this inequality with the reduction step yields the assertion (6.2).

**Adaptive lower bounds in specific situations.** In the sequel we apply Proposition 6.1 to two specific classes of alternatives \( \{ \mathcal{E}_\theta^R : a_\tau \in A \} \). We consider a set \( A \) which is non-trivial with respect to either polynomial or exponential decay, that is, \( \{ (j^{-s}) : j \in \mathbb{N} : s \in [s_*, s^*] \} \subseteq A \) or \( \{ (\exp(-j^p)) : j \in \mathbb{N} : s \in [s_*, s^*] \} \subseteq A \) for \( s_* < s^* \) and \( s_*, s^* > 0 \).

**Theorem 6.3 (adaptive factor for ordinary smoothness and mild ill-posedness).** Let \( A \) be non-trivial with respect to polynomial decay for \( 1/2 < s_* < s^* \). Let \( |j| \sim j^{-p} \) for \( p > 1/2 \). For \( \gamma \in (0, 1) \) there exists an \( n_0 \in \mathbb{N} \) and \( A_\gamma \in (0, \infty) \) such that for all \( n \geq n_0 \) and \( A \in [0, A_\gamma] \)
\[
\inf_{A_\theta} \sup_{a_\theta \in A} \mathcal{R}(\Delta | \mathcal{E}_\theta^R, A_{\theta} (\delta_n)) \geq 1 - \gamma
\]
with \( \delta = (\log \log n + 1)^{-1/2} \), i.e., \( \delta^{-1} \) is a lower bound for the minimal adaptive factor over \( A \).

**Proof of Theorem 6.3.** We intend to apply Proposition 6.1. To do so, we construct a collection of regularity parameters \( A_N := \{ a_\theta^m : \theta_\theta^m \in A, m \in [N] \} \subseteq A \), that satisfies (C1)–(C4).

**Definition of the collection.** By Illustration 2.5 the minimax radius of testing is of order \( \rho_{a_\theta^m}^2 \sim n^{-\epsilon(s)} \) with exponent \( \epsilon(s) := \frac{4s}{3s + 4p + 1} \) for \( s \in [s_*, s^*] \). Since \( A \) is non-trivial with respect to polynomial decay, it contains the corresponding regularity sequences to the exponents in the interval \( [e_*, e^*] := [e(s_*) \} \subseteq A \) for \( s_* < s^* \). We define a grid of regularity \( N \) (specified below) on \( A \) by placing a linear grid on the interval of exponents. Indeed, for \( d := e_* - e_*^* \) we define \( A_N := \{ a_\theta^m := (j^{-s_m}) : j \in \mathbb{N} : e(s_m) \} \) for \( s_m = s^* - (m - 1)d, m \in [N] \) \( \subseteq A \).

**Verification of the conditions (C1)–(C4).** We define \( N := \left[ \frac{e_* - e_*^*}{4s} \log(\delta_n) \right] \). Tiedous but straightforward calculations show that (C1) is satisfied for \( n \) large enough. Moreover, it is easily seen that \( \delta^2 \log N \rightarrow 1 \) for \( n \rightarrow \infty \). Hence, \( \log N - \delta^2 \rightarrow \infty \) and, thus, also (C2) is satisfied for \( n \) large enough and \( c_\gamma := 1/2 \). Concerning (C3) we observe that \( \sup_{m \in [N]} \sum_{j \in \mathbb{N}} (a_j^m)^2 \leq \sup_{s \in [s_*, s^*]} \sum_{j \in \mathbb{N}} j^{-2s} \leq \frac{2}{\sqrt{2e_*}} =: a/2 \). Again, for \( n \) large enough the existence of a constant \( \eta \) satisfying (C4) uniformly over \( n \) follows, because for \( a_\theta \sim (j^{-s}) \) with \( s \in [s_*, s^*] \) the terms \( a_{\theta}^2 \) and \( (\delta_n)^{-1} \) are of the same order.
Theorem 6.4 (adaptive factor for super smoothness and mild ill-posedness). Let $A$ be non-trivial with respect to exponential decay for $0 < s_* < s^*$. Let $|\varphi_j| \sim j^{-p}$ for $p > 1/2$. For $\gamma \in (0, 1)$ there exists an $n_0 \in \mathbb{N}$ and $A_0 \in (0, \infty)$ such that for all $n \geq n_0$ and $A \in [0, A_0]$

$$
\inf_{\Delta} \sup_{\alpha_* \in A} R(\Delta, C_{\alpha_*}, \alpha_*^*(\delta_n)) \geq 1 - \gamma
$$

with $\delta = (\log \log n \vee 1)^{-1/2}$, i.e., $\delta^{-1}$ is a lower bound for the minimal adaptive factor over $A$.

Proof of Theorem 6.4. We intend to apply Proposition 6.1. To do so, we construct a collection of regularity parameters $A_N := \{a^m \in A, m \in [N]\} \subseteq A$, that satisfies (C1)–(C4).

Definition of the collection. By Illustration 2.5 the minimax radius of testing is of order $n^{-1}(\log n)^{\epsilon(s)}$ with exponent $e(s) := \frac{2p+1/2}{s}$ for $s \in [s_*, s^*]$. Since $A$ is non-trivial with respect to exponential decay, it contains the corresponding regularity sequences to the exponents in the interval $[e_*, e^*] := [e(s^*), e(s_*)]$. We define a grid of size $N$ (specified below) on $A$ by placing a linear grid on the interval of exponents. Indeed, for $d := \frac{e^*-e_*}{N}$ we define $A_N := \{a^m := (e^{-j\delta/m})_{j \in \mathbb{N}} : e(s_m) = e_* + (m-1)d, m \in [N]\} \subseteq A$.

Verification of the conditions (C1)–(C4). We define $N := \left\lfloor \frac{e^*-e_* \log \log (\delta_n)}{4|\log(\delta)|} \right\rfloor$. Tedious but straight-forward calculations show that (C1) is satisfied for $n$ large enough. Moreover, it is easily seen that $\delta^2 \log n \to 1$ for $n \to \infty$. Hence, $\log N - \delta/2 \to \infty$ and, thus, also (C2) is satisfied for $n$ large enough and $c_{\gamma} = 1/2$. Concerning (C3) we observe that $\sup_{s \in [s_*, s^*]} \sum_{j \in \mathbb{N}} |a^m_j|^2 \leq \sup_{s \in [s_*, s^*]} \sum_{j \in \mathbb{N}} e^{-2s} \leq (1/2)^{1/s_1} \Gamma(1/s_1 + 1) =: a_2/2$, where $\Gamma$ denotes the Gamma-function. Again, for $n$ large enough the existence of a constant $\eta$ satisfying (C4) uniformly over $n$ follows, because for $a_* \sim (e^{-j\delta/m})_{j \in \mathbb{N}}$ with $s \in [s_*, s^*]$ the terms $a^2_{\eta s_*}$ and $(\delta_n^{-1})^{s_1}$ are of the same order.

Comparing Theorems 6.3 and 6.4 with Illustration 3.5 (for the indirect test) and Illustration 5.4 (for the direct test) shows that the adaptive factors are minimal. Indeed, in the ordinary smooth—mildly ill-posed model both the direct and the indirect max-test face a deterioration by a $\log \log n$-factor, which Theorem 6.3 shows to be unavoidable. In the more restrictive setting of super smoothness and mild ill-posedness both tests feature a $\log \log \log n$-factor, which is unavoidable due to Theorem 6.4. In the ordinary smooth—severely ill-posed model there is no loss for adaptation visible in the testing radius.

APPENDIX A

AUXILIARY RESULTS USED IN SECTIONS 2 AND 4

The next two assertions, a concentration inequality for canonical U-statistics and a Bernstein inequality, provide our key arguments in order to control the deviation of the test statistics. The first assertion is a reformulation of Theorem 3.4.8 in [11].

Proposition A.1. Let $\{Y_j\}_{j=1}^n$ be independent and identically distributed $[0, 1]$-valued random variables, and let $h : [0, 1]^2 \to \mathbb{R}$ be a bounded symmetric kernel, i.e., $h(y, y) = h(\tilde{y}, \tilde{y})$ for all $y, \tilde{y} \in [0, 1)$, fulfilling in addition

$$
\mathbb{E}(h(Y_1, y_2)) = 0 \quad \forall y_2 \in [0, 1).
$$

(A.1)

Then there are finite constants $A, B, C$, and $d$ such that

$$
\sup_{y_1, y_2 \in [0, 1]} |h(y_1, y_2)| \leq A, \quad \sup_{y_2 \in [0, 1]} \mathbb{E} h^2(Y_1, y_2) \leq B^2, \quad \mathbb{E} h^2(Y_1, Y_2) \leq C^2, \quad \text{and}
$$

$$
\sup_{y \in [0, 1]} \mathbb{E}(h(Y_1, Y_2) \xi(Y_1) \xi(Y_2)), \mathbb{E} \xi^2(Y_1) \leq 1, \mathbb{E} \xi^2(Y_2) \leq 1 \leq d
$$

(A.2)

and for all $n \geq 2$ the real-valued canonical U-statistic

$$
U_n = \frac{1}{n(n-1)} \sum_{l, m \in [n], l \neq m} h(Y_l, Y_m)
$$

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satisfies for all $x \geq 0$

$$
\mathbb{P}(U_n \geq 8Cn^{-1}x^{1/2} + 13Dn^{-1}x + 261Bn^{-3/2}x^{3/2} + 343An^{-2}x^2) \leq \exp(1 - x).
$$

The following version of Bernstein’s inequality can directly be deduced from Theorem 3.1.7 in [11].

**Proposition A.2.** Let $\{Z_j\}_{j=1}^n$ be independent with $|Z_j| \leq b$ almost surely and $\mathbb{E}(|Z_j|^2) \leq v$ for all $j \in [n]$, then for all $x > 0$ and $n \geq 1$, we have

$$
\mathbb{P} \left( \frac{1}{n} \sum_{j \in [n]} (Z_j - \mathbb{E}Z_j) \geq \sqrt{\frac{2vx}{n} + \frac{bx}{3n}} \right) \leq \exp(-x).
$$

**Preliminaries.** We eventually calculate first A, B, and C satisfying (A.2) and exploit that $D := C$ automatically also fulfills (A.2), which we briefly justify next. Throughout this section we assume that $\{Y_j\}_{j=1}^n$ are independent and identically distributed with Lebesgue-density $g = f \oplus \varphi \in L^2$. We denote by $L^2(\mathbb{R})$ the set of (Borel-measurable) functions $\zeta : [0, 1) \to \mathbb{R}$ with $||\zeta||^2_{L^2(\mathbb{R})} := \int_0^1 \zeta^2(x)g(x)dx < \infty$. Let $h : [0, 1)^2 \to \mathbb{R}$ be a bounded kernel, i.e., $||h||_{L^\infty} := \sup_{y_1, y_2 \in [0, 1)} |h(y_1, y_2)| < \infty$, and define the integral operator $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $\zeta \mapsto H\zeta$ and $H(\zeta) := \int_{[0, 1)} h(t, s)\zeta(t)g(t)dt$ for all $s \in [0, 1)$. Then $H$ is linear and bounded, i.e., $||H||_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} := \sup\{|H\zeta, \zeta\}_{L^2(\mathbb{R})} : ||\zeta||_{L^2(\mathbb{R})} \leq 1 \leq \mathbb{E}h^2(Y_1, Y_2) \leq C^2$ due to the Cauchy–Schwarz inequality. This shows the claim since its operator norm satisfies

$$
||H||_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \sup\{\mathbb{E}(h(Y_1, Y_2)\zeta(Y_1)\zeta(Y_2)), \mathbb{E}\zeta^2(Y_1) \leq 1, \mathbb{E}\zeta^2(Y_2) \leq 1\}.
$$

However, an additional assumption allows us to determine a slightly different quantity $D$. For a symmetric kernel the operator $H$ is self-adjoint, i.e., $\langle H\zeta, \zeta\rangle_{L^2(\mathbb{R})} = \langle \zeta, H\zeta\rangle_{L^2(\mathbb{R})}$ for all $\zeta, \xi \in L^2(\mathbb{R})$ using the inner product $\langle \zeta, \xi\rangle_{L^2(\mathbb{R})} := \int_{[0, 1)} \zeta(s)\xi(s)ds$, and we have

$$
||H||_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \sup\{|(H\zeta, \zeta)_{L^2(\mathbb{R})} : ||\zeta||_{L^2(\mathbb{R})} \leq 1\}.
$$

The last identity can further be reformulated in terms of a discrete convolution, which we briefly recall next. For $p \geq 1$ we denote by $\ell^p(\mathbb{Z})$ the Banach space of complex sequences over $\mathbb{Z}$ endowed with its usual $\ell^p$-norm given by $||a||_{\ell^p} := \left( \sum_{j \in \mathbb{Z}} |a_j|^p \right)^{1/p}$ for $a := (a_j)_{j \in \mathbb{Z}} \subset \ell^p$. In the case $p = 2$, $\ell^2$ is a Hilbert space and the $\ell^2$-norm is induced by the inner product $\langle a, b\rangle_{\ell^2} := \sum_{j \in \mathbb{Z}} a_j \overline{b_j}$ for all $a, b \in \ell^2$. For each sequence $\alpha \in \ell^1$ the discrete convolution operator $\alpha * : \ell^2 \to \ell^2$ with $b \mapsto \alpha * b := \sum_{j \in \mathbb{Z}} \alpha_j b_{j+1}$ and $(\alpha * b)_j := \sum_{k \in \mathbb{Z}} \alpha_{j-k} b_k$ for all $j \in \mathbb{Z}$, is linear and bounded by $||\alpha * b||_{\ell^2}$, i.e., $||\alpha * ||_{\ell^1 \to \ell^2} := \sup\{||\alpha * b||_{\ell^2} : ||b||_{\ell^2} \leq 1\} \leq ||\alpha||_{\ell^1}$. Particularly, it holds

$$
\sum_{i \in \mathbb{Z}} b_{-i} \sum_{j \in \mathbb{Z}} a_{j-i} = ||\alpha * b||_{\ell^2} \leq ||a||_{\ell^2} ||b||_{\ell^2} \quad \text{for all } \alpha \in \ell^1 \text{ and } b \in \ell^2.
$$

Note that the adjoint of $\alpha *$ is a discrete convolution operator $\alpha^{\ast *}$ with $\alpha_{-j} := \overline{\alpha_j}$ for all $j \in \mathbb{Z}$. Hence, if in addition $a_j = \overline{a_{-j}}$ for all $j \in \mathbb{Z}$, then $\alpha *$ is self-adjoint. Recall that the real density $\varphi \in \ell^2$ admits Fourier coefficients $g_j = \langle g, e_j \rangle_{\ell^2}$. The coefficients belong to both $\ell^2$ by Parseval’s identity, i.e., $||g||_{\ell^2} = ||g||_{\ell^2}$, and also to $\ell^1$ due to the convolution theorem. Indeed, $(g_j = f \otimes \varphi)_j \in \ell^2$ implies $||g||_{\ell^1} \leq ||f||_{\ell^1} ||\varphi||_{\ell^2} < \infty$ due to the Cauchy–Schwarz inequality. Consequently, the discrete convolution $g_{*} : \ell^2 \to \ell^2$ is linear, bounded and self-adjoint. Moreover, for all $\zeta \in \ell^2$ with $||\zeta||_{\ell^2}$ and Fourier coefficients $\zeta_j \in \ell^2$ we have $\langle \zeta, \zeta \rangle_{\ell^2} = \sum_{j \in \mathbb{Z}} \zeta_j \sum_{i \in \mathbb{Z}} \zeta_{-i} = \sum_{j \in \mathbb{Z}} \zeta_j \sum_{i \in \mathbb{Z}} \mathbb{E}(e_i(Y) e_j(-Y)) \zeta_i = \mathbb{E}||\zeta||^2_{\ell^2} \geq 0$. Thereby, if for all $\zeta \in \ell^2$, then $g_{*}$ is non-negative. As a result, there is a non-negative operator $(g_{*})^{1/2}$ with $||g_{*}||^{1/2}_{\ell^2} = ||g||^{1/2}_{\ell^2}$ and $||\zeta||^2_{\ell^2} = ||\zeta||^2_{\ell^2}$ for all $\zeta \in \ell^2$, which we use frequently in the proofs below.
A.1 Auxiliary Results Used in the Proof of Proposition 2.1

Lemma A.3. Consider \( \{Y_i\}_{i=1}^n \overset{iid}{\sim} g \in \mathcal{L}^2 \) and for \( k \in \mathbb{N} \) the kernel \( h : [0,1)^2 \to \mathbb{R} \) given by

\[
h(y_1, y_2) = \sum_{|j| \in [k]} \frac{\langle e_j - g_j, (e_{j}(y_2) - \bar{g}_{j}) \rangle}{|\varphi_j|^2}, \quad \forall y_1, y_2 \in [0,1),
\]

which is real-valued, bounded, symmetric and fulfills (A.1). Let \( \nu_k \) and \( m_k \) as in (2.4) then

\[
A = 4\nu_k^4, \quad B = 3\|g\_\|_{\ell^2} \nu_k^2 \quad \text{and} \quad D = 2\|g\_\|_{\ell^1} \nu_k^2
\]

(A.6) satisfy the condition (A.2) in Proposition A.1. If, in addition, \( \mathcal{L}^2_{\mathbb{R}}(g) = \mathcal{L}^2_{\mathbb{R}} \), then

\[
D = 4\|g\_\|_{\ell^2} \nu_k^2
\]

(A.7) also satisfies the condition (A.2) in Proposition A.1.

Proof of Lemma A.3. We first calculate quantities \( A, B, \) and \( C \) satisfying (A.2), then by the above discussion \( D = C \) also satisfies (A.2). First, consider \( A \). From

\[
\| \langle e_j - g_j, (e_{j} - \bar{g}_{j}) \rangle \|_{\mathcal{L}^\infty} \leq 4 \quad \text{and} \quad |\varphi_j| \leq 1 \quad \text{for all} \quad j, l \in \mathbb{Z}
\]

(A.8) we immediately conclude that \( |\| h \|_{\mathcal{L}^\infty} \| \leq 4 \sum_{|j| \in [k]} |\varphi_j|^{-2} \leq 4\nu_k^4 = A \). Next, consider \( B \). Since \( E\langle e_j(-Y_1)e_l(Y_1) \rangle = g_{j-l} \) for all \( j, l \in \mathbb{Z} \), we deduce for arbitrary \( y_2 \in [0,1) \) that

\[
\mathbb{E} \left( |h(Y_1, y_2)|^2 \right) = \mathbb{V} \mathbb{A} \left( \sum_{|j| \in [k]} \frac{e_j(-Y_1)(e_j(y_2) - \bar{g}_{j})}{|\varphi_j|^2} \right)^2 \leq \mathbb{E} \left( \sum_{|j| \in [k]} \frac{e_j(-Y_1)(e_j(y_2) - \bar{g}_{j})}{|\varphi_j|^2} \right)^2 = \mathbb{E} \left( \sum_{|j| \in [k]} \frac{e_j(y_2) - \bar{g}_{j}}{|\varphi_j|^2} \right)^2 = \mathbb{E} \left( \sum_{|j| \in [k]} \frac{g_{j-l}(e_j(y_2) - \bar{g}_{j})}{|\varphi_j|^2} \right)^2 = \langle a_{\bullet} \ast b_{\bullet}, b_{\bullet} \rangle_{\ell^2},
\]

where \( a_l := |g_l| \|\{l\} \in [2k]\| \} \) and \( b_l := |\varphi_l|^{-2} \|\{l\} \in [k]\| \) for all \( l \in \mathbb{Z} \). Making use of (A.5), (A.8), \( \|g\_\|_{\ell^2} \geq 1 \) and \( (2k)^{1/2} \leq \nu_k \) it follows

\[
\langle a_{\bullet} \ast b_{\bullet}, b_{\bullet} \rangle_{\ell^2} \leq \left( \sum_{|j| \in [2k]} |g_{j}| \right) \sum_{|j| \in [k]} \frac{|e_j(y_2) - \bar{g}_{j}|^2}{|\varphi_j|^4} \leq (4k)^{1/2} \left( \sum_{|j| \in [2k]} |g_{j}|^2 \right) \nu_k^4 \leq 9\|g\_\|_{\ell^2} \nu_k^4 = B^2.
\]

Combining the last bound and (A.9) we see that \( \sup_{y_2 \in [0,1)} \mathbb{E}|h(Y_1, y_2)|^2 \leq B^2 \). Next, consider \( C \). Since \( \mathbb{E}\langle e_j(-Y_1) - g_j, (e_l(Y_1) - \bar{g}_{l}) \rangle = g_{j-l} - g_j \bar{g}_l \) for all \( j, l \in \mathbb{Z} \), applying the Cauchy–Schwarz inequality we obtain

\[
\mathbb{E}|h(Y_1, Y_2)|^2 = \sum_{|j| \in [k]} |\varphi_j|^2 \sum_{|l| \in [k]} \frac{|g_{j-l} - g_j \bar{g}_l|^2}{|\varphi_l|^2} \leq 2\langle a_{\bullet} \ast b_{\bullet}, b_{\bullet} \rangle_{\ell^2} + 2\nu_k^4 \|g\_\|_{\ell^2}^2,
\]

(A.10) where \( a_l := |g_l| \} \|\{l\} \in [2k]\| \} \) and \( b_l := |\varphi_l|^{-2} \} \|\{l\} \in [k]\| \} \) for all \( l \in \mathbb{Z} \). Moreover, from \( \langle a_{\bullet} \ast b_{\bullet}, b_{\bullet} \rangle_{\ell^2} \leq \|a_{\bullet}\|_{\ell^2} \|b_{\bullet}\|_{\ell^2} \leq \|g\_\|_{\ell^2} \nu_k^2 \) due to (A.5) we conclude that (A.10) and \( |g_{j}| \leq 1, j \in \mathbb{Z} \), together imply \( \mathbb{E}|h(Y_1, Y_2)|^2 \leq 4\nu_k^4 |\|g\_\|_{\ell^2}^2 = C^2 \). Finally, consider \( \mathcal{D} \) and assume in addition \( \mathcal{L}^2_{\mathbb{R}}(g) = \mathcal{L}^2_{\mathbb{R}} \) which allows us to use the identities (A.3) and (A.4) formulated in terms of an operator \( H \). Let \( \zeta \in \mathcal{L}^2_{\mathbb{R}}(g) \), which implies \( \zeta = \sum_{j \in \mathbb{Z}} \zeta_j e_j \in \mathcal{L}^2 \). Exploiting \( \mathbb{E}\langle e_j(-Y_1) - g_j, \zeta_j \rangle = (g_{\bullet} \ast \zeta)_{\bullet} e_j \) and \( |g_j| \leq 1 \) for all \( j \in \mathbb{Z} \) straightforward calculations show

\[
|\langle H \zeta, \zeta \rangle_{\mathcal{L}^2_{\mathbb{R}}(g)}| = \sum_{|j| \in [k]} \frac{1}{|\varphi_j|^2} \mathbb{E}|\langle e_j(-Y_1) - g_j, \zeta \rangle|^2 \leq m_k^2 \sum_{|j| \in [k]} \|g_{\bullet} \ast \zeta_{\bullet} - g_j \mathbb{E}\zeta(Y_1)\|^2 \leq 2m_k^2 \left( \|g_{\bullet} \ast \zeta_{\bullet}\|_{\ell^2}^2 + \|g_{\bullet}\|_{\ell^1} \|\zeta\|_{\mathcal{L}^2_{\mathbb{R}}(g)}^2 \right).
\]

(A.11)
Using the properties of the discrete convolution recalled above it follows
\[ ||g_\bullet \ast \zeta||^2_{l_2} \leq ||(g_\bullet \ast)^{1/2}||_{l_2} ||(g_\bullet \ast)^{1/2} \zeta||^2_{l_2} = ||g_\bullet \ast||_{l_2} \leq ||g_\bullet \ast||_{l_1} ||\zeta||^2_{L^2_\mathbb{R}(g)}, \]
which together with (A.11) implies \(|<H \zeta, \zeta>_{L^2_\mathbb{R}(g)}| \leq 4m_k^2 ||g_\bullet \ast||_{l_1} ||\zeta||^2_{L^2_\mathbb{R}(g)}\) for all \( \zeta \in L^2_\mathbb{R}(g) \). We conclude from (A.4) that \(|<H||L^2_\mathbb{R}(g)\to L^2_\mathbb{R}(g)| \leq 4m_k^2 ||g_\bullet \ast||_{l_1} = D, \) and finally that \( D \) satisfies (A.2), by (A.4), which completes the proof.

\[ \Box \]

**Lemma A.4.** Let \( \{Y_i\}_{i=1}^\infty \overset{iid}{\sim} g = f \odot \varphi \in L^2 \) with joint distribution \( \mathbb{P}_f \) and \( g^0 = f^0 \odot \varphi \in L^2 \). For each \( k \in \mathbb{N} \) consider \( q_k^2(f-f^0) \) and \( m_k \) as in (2.1) and (2.4), respectively. Then the linear centred statistic \( V_n \) defined in (2.3) satisfies for all \( x \geq 1 \) and \( n \geq 1 \)
\[ \mathbb{P}_f(2V_n \leq -cx^2(1 \vee m_k^2 n^{-1})m_k^2 n^{-1} - \frac{1}{2}q_k^2(f-f^0)) \leq \exp(-x), \]
where \( c = 8||g_\bullet \ast||_{l_1} + ||\varphi||^2_{l_2} \).

**Proof of Lemma A.4.** Introducing the real function \( \psi := \sum_{|j| \in [k]} (g_j - g_j^0) |\varphi_j|^2 e_j \) and independent and identically distributed random variables \( Z_j := 2\psi(Y_j), j \in [n] \), we intend to apply Proposition A.2 to \( V_n = \frac{1}{n} \sum_{j=1}^{n} (Z_j - \mathbb{E}_f(Z_j)) \). Therefore, we compute the required quantities \( v \) and \( b \). First consider \( b \). Using subsequently the identity \( g_l - g_l^0 = (f_l - f_l^0) \varphi_l, l \in \mathbb{Z} \), and the Cauchy–Schwarz inequality we deduce that
\[ |Z_1| \leq 2||\psi||_{L^\infty} \leq 2m_k^2 \sum_{|l| \in [k]} |g_l - g_l^0| \leq 2m_k^2 q_k^2(f-f^0)||\varphi_\bullet||_{l_2} =: b. \tag{A.12} \]
Secondly, consider \( v \). Since \( \mathbb{E}_f(e_j(-Y_1)e_l(Y_1)) = g_{l-j} \) for all \( j, l \in \mathbb{Z} \), we see that
\[ \mathbb{E}_f|Z_1|^2 = 4\mathbb{E}_f|\psi(Y_1)|^2 = 4 \sum_{|j| \in [k]} \bar{g}_j - \bar{g}_j^0 \sum_{|l| \in [k]} g_j - g_j^0 |||\varphi_j||^2_2 = 4\langle a_\bullet \ast b_\bullet, b_\bullet \rangle_{l_2}, \]
where \( a_l := g_l \mathbb{1}_{|l| \in [2k]} \) and \( b_l := (g_l - g_l^0) |\varphi_l|^{-2} \mathbb{1}_{|l| \in [k]} \) for all \( l \in \mathbb{Z} \). Successively exploiting further (A.5) and the identity \( g_l - g_l^0 = (f_l - f_l^0) \varphi_l, l \in \mathbb{Z} \), we conclude that
\[ \mathbb{E}_f|Z_1|^2 \leq 4||b_\bullet||^2_{l_2} ||a_\bullet||_{l_1} = 4 \sum_{|j| \in [k]} |\varphi_j|^{-4} |g_j - g_j^0|^2 \sum_{|j| \in [k]} |g_j| \leq 4m_k^2 q_k^2(f-f^0)||g_\bullet||_{l_1} =: v. \tag{A.13} \]
The claim of Lemma A.4 now follows from Proposition A.2 with \( b \) and \( v \) as in (A.12) and (A.13), respectively. Indeed, making use of \( 2ac \leq \frac{a^2}{\epsilon} + \epsilon c^2 \) for any \( \epsilon, a, c > 0 \), we have
\[ \frac{bx}{3n} \leq \epsilon_1 q_k^2(f-f^0) + \frac{x^2}{9\epsilon_1} ||\varphi_\bullet||^2_{l_2} m_k^4 \] and
\[ \frac{2vx}{n} \leq \epsilon_2 q_k^2(f-f^0) + \frac{2x m_k^2}{\epsilon_2} ||g_\bullet||_{l_1}. \]
Combining both bounds (with \( \epsilon_1 = \epsilon_2 = \frac{1}{4} \)) yields for all \( x \geq 1 \)
\[ \sqrt{\frac{2vx}{n} + \frac{bx}{3n}} \leq \frac{1}{2}q_k^2(f-f^0) + cx^2 \left( 1 \vee \frac{m_k^2}{n} \right) \frac{m_k^2}{n} \] with \( c = 8||g_\bullet||_{l_1} + ||\varphi||^2_{l_2} \).
Hence, the assertion follows from Proposition A.2 by the usual symmetry argument. \( \Box \)

**A.2 Auxiliary Results Used in the Proof of Proposition 4.1**

**Corollary A.5.** Consider \( \{Y_i\}_{i=1}^n \overset{iid}{\sim} g \in L^2 \) and for \( k \in \mathbb{N} \) the kernel \( h : [0,1]^2 \to \mathbb{R} \) given by
\[ h(y_1, y_2) = \sum_{|j| \in [k]} (e_j(-y_1) - g_j)(e_j(y_2) - \bar{g}_j), \quad \forall y_1, y_2 \in [0,1], \]
which is real-valued, bounded, symmetric and fulfills (A.1). Then the quantities

\[ A = 8k, \quad B = 3\|g\|_{\ell^2(2k)^{3/4},} \quad \text{and} \quad D = C = 2\|g\|_{\ell^2(2k)^{1/2}} \]

satisfy the condition (A.2) in Proposition A.1. If, in addition, \( \mathcal{L}^2_R(g) = \mathcal{L}^2_R \), then

\[ D = 4\|g\|_{\ell^1} \]

also satisfies the condition (A.2) in Proposition A.1.

Proof of Corollary A.5. Setting \(|\varphi| = 1\) for all \(|j| \in [k]\) the assertion immediately follows from Lemma A.3.

Lemma A.6. Let \( \{Y_i\}_{i=1}^n \) be independent and identically distributed random variables \( Z_j := 2\psi(Y_j),\ j \in [n] \), we intend to apply Proposition A.2 to \( V_n^d = \frac{1}{n}\sum_{j=0}^n (Z_j - \mathbb{E}_f(Z_j)) \). Therefore, we compute the required quantities v and b. First consider b. Using the Cauchy–Schwarz inequality we see that

\[ |Z_1| \leq 2\|\psi\|_{L^\infty} \leq 2\sum_{|l| \in [k]} |g_l - g_0^l| \leq 2(2k)^{1/2}q_k(g - g^0) =: b. \quad \text{(A.14)} \]

Secondly, consider v. Since \( \mathbb{E}_f(e_j(-Y_1)e_l(Y_1)) = g_{j-l} \) for all \( j, l \in \mathbb{Z} \), we deduce that

\[ \mathbb{E}_f|Z_1|^2 = 4\mathbb{E}_f|\psi(Y_1)|^2 = 4\sum_{|l| \in [k]} (g_l - g_0^l) \sum_{|l| \in [k]} g_{j-l}(g_l - g_0^l) = 4(a\cdot b, b)_{\ell^2}, \]

where \( a_l := g_l \mathbb{I}\{|l| \in [2k]\} \) and \( b_l := (g_l - g_0^l) \mathbb{I}\{|l| \in [k]\} \) for all \( l \in \mathbb{Z} \). Hence, (A.5) shows that

\[ \mathbb{E}_f|Z_1|^2 \leq 4\|b\|_{\ell^2}^2\|a\|_{\ell^1} = 4q_k^2(g - g^0) \sum_{|j| \in [2k]} |g_j| =: v. \quad \text{(A.15)} \]

The claim of Lemma A.6 follows now from Proposition A.2 with b and v as in (A.14) and (A.15), respectively. Indeed, exploiting \( 2ac \leq a^2 + c^2\) for any \( a, c > 0 \) we see that

\[ \frac{bx}{3n} \leq \frac{\varepsilon_1 q_k^2(g - g^0)}{\varepsilon_1} + \frac{x^2 2k}{9\varepsilon_1 n^2} \]

and

\[ \sqrt{\frac{2\varepsilon x}{n}} \leq \frac{\varepsilon_2 q_k^2(g - g^0)}{\varepsilon_2} + \frac{2x}{\varepsilon_2 n} \sum_{|j| \in [2k]} |g_j| \leq \frac{\varepsilon_2 q_k^2(g - g^0)}{\varepsilon_2} + \frac{2x (4k)^{1/2}}{\varepsilon_2 n} \|g\|_{\ell^2}. \]

Combining both bounds (with \( \varepsilon_1 = \varepsilon_2 = \frac{1}{4} \)) we get for all \( x \geq 1 \)

\[ \sqrt{\frac{2\varepsilon x}{n}} + \frac{bx}{3n} \leq \frac{1}{2} q_k^2(g - g^0) + ex^2(1 \vee (2k)^{1/2}n^{-1})(2k)^{1/2}n^{-1} \quad \text{with} \quad c = 12\|g\|_{\ell^2} + 1. \]

Hence, the assertion follows from Proposition A.2 by the usual symmetry argument.
CALCULATIONS FOR THE ILLUSTRATIONS

B.1 Calculations for the Radius Bounds in Illustration 3.5

Firstly, we determine the order of the term $\rho_{K_{s,a}}^2(\delta n)$ by showing that $\rho_{K_{s,a}}^2(\delta n) \sim \rho_{a}^2(n)$ and replacing $n$ with $\delta n$. Indeed, we trivially have $\rho_{K_{s,a}}^2(n) \leq \rho_{K_{s,a}}^2(n)$. By defining $j_* := \left[ \frac{2}{8p+4s+7} \log_2 n \right] \lesssim \log(n^2/2)$ (in the ordinary smooth case) respectively $j_* := \left[ \frac{1}{4} \log_2 \log n \right] \lesssim \frac{1}{s_1} \log \log n$ (in the super smooth case), straightforward calculations then show that $\rho_{K_{s,a}}^2(n) \lesssim \rho_{a}^2(n)$. Next, we determine the order of the remainder term $r_{K_{s,a}}^2(\delta^2 n)$ by first calculating

$\frac{m_2^2}{n}$ showing $r_{K_{s,a}}^2(n) \sim r_{K_{s,a}}^2(n)$ and then replacing $n$ with $\delta^2 n$. The variance term $\frac{m_2^2}{n}$ is of order $\frac{k_{s_1}}{n}$. In the ordinary smooth case the bias term $a^2_k$ is of order $k^{-2s}$. Hence, the minimising $k_*$ satisfies $k_* \sim n^{-\frac{1}{2s_1+2p}}$, which yields $r_{K_{s,a}}^2(n) \sim n^{-\frac{1}{2s_1+2p}}$. We define $j_* := \left[ \frac{1}{2p+2s} \log_2 n \right] \lesssim \log(n^2/2)$. Straightforward calculations then show that $\rho_{K_{s,a}}^2(n) \lesssim \rho_{a}^2(n)$. Since, trivially $r_{K_{s,a}}^2(n) \leq r_{K_{s,a}}^2(n)$, we obtain the assertion. In the super smooth case the bias term $a^2_k$ is of order $e^{-2k^2s}$. Hence, the minimising $k_*$ satisfies $k_*) \sim \log(n)^{\frac{1}{3}}$, which yields $r_{K_{s,a}}^2(n) \sim \frac{1}{n} \log(n)^{\frac{2s}{p}}$. We define $j_* := \left[ \frac{1}{2p+2s} \log_2 n \right] \lesssim \log(n^2/2)$. Straightforward calculations then show that $\rho_{K_{s,a}}^2(n) \lesssim \rho_{a}^2(n)$. Since, trivially $r_{K_{s,a}}^2(n) \leq r_{K_{s,a}}^2(n)$, we obtain the assertion.

B.2 Calculations for the Radius Bounds in Illustration 4.5

Ordinary smooth—mildly ill-posed. Since $\frac{(2k)^{1/2}}{n} m_2^2 \sim \frac{1}{n} k^{2p+1/2}$ and $a^2_k \sim k^{-2s}$, the optimal $k_{s_1}$ satisfies $k_{s_1} \sim n^{\frac{1}{2s_1+2p}}$, which yields an upper bound of order $(\rho_{a}^2(n))^{2s} \sim n^{-\frac{4p+4s}{2s_1+2p+1}}$.

Ordinary smooth—severely ill-posed. Since $\frac{(2k)^{1/2}}{n} m_2^2 \sim \frac{1}{n} k^{1/2} e^{2k^2s}$ and $a_k^2 \sim k^{-2s}$, we obtain $k_{s_1} \sim (\log n)^{\frac{1}{3}}$, which yields an upper bound of order $(\rho_{a}^2(n))^{2s} \sim k_k^{\frac{1}{2s_1+2p+1}} \sim \frac{1}{n} \log(n)^{\frac{2s}{p}}$.

Super smooth—mildly ill-posed. Since $\frac{(2k)^{1/2}}{n} m_2^2 \sim \frac{1}{n} k^{2p+1/2}$ and $a^2_k \sim e^{-2k^2s}$, we obtain $k_{s_1} \sim (\log n)^{\frac{1}{3}}$, which yields an upper bound of order $(\rho_{a}^2(n))^{2s} \sim \frac{1}{n} (k_k^{d})^{2p+1/2} \sim \frac{1}{n} (\log n)^{\frac{2p+1/2}{2s_1+2p+1}}$.

B.3 Calculations for the Radius Bounds in Illustration 5.4

Firstly, we determine the order of the terms $(\rho_{K_{s,a}}^2(\delta n))^2$ by showing that $(\rho_{K_{s,a}}^2(\delta n))^2 \sim (\rho_{a}^2(n))^2$ and replacing $n$ with $\delta n$. Indeed, we trivially have $(\rho_{K_{s,a}}^2(\delta n))^2 \leq (\rho_{a}^2(n))^2$. Define $j_* := \left[ \frac{2}{8p+4s+7} \log_2 n \right] \lesssim \log(n^2/2)$ (ordinary smooth—mildly ill-posed case), $j_* := \left[ \frac{1}{4} \log_2 \log n \right] \lesssim \frac{1}{s_1} \log \log n$ (super smooth—mildly ill-posed case) respectively $j_* := \left[ \frac{1}{2p+2s} \log_2 n \right] \lesssim \log(n^2/2)$ (ordinary smooth—severely ill-posed case). Straightforward calculations then show that $\rho_{K_{s,a}}^2(\delta n) \lesssim \rho_{a}^2(n)$. Next, we determine the order of the remainder term $r_{K_{s,a}}^2(\delta^2 n)$ by first calculating $r_{K_{s,a}}^2(\delta^2 n) := \min_{k \in \mathbb{N}} a_k^2 \sqrt{m_2^2} \frac{n}{n}$ and then showing that minimisation over $K_{s_1}$ approximates the minimisation over $\mathbb{N}$ well enough. The calculations in the (ordinary smooth—mildly ill-posed) and (super smooth—mildly ill-posed) cases have already been done in Illustration 3.5. It remains to consider the third case (ordinary smooth—severely ill-posed). Since $\frac{m_2^2}{n} \sim \frac{1}{n} k^{2p}$ and $a_k^2 \sim k^{-2s}$, the minimising $k_*$ satisfies $k_* \sim k^{2p} \sim \frac{1}{p}$, which yields $r_{K_{s,a}}^2(n) \sim (\log n)^{\frac{2s}{p}}$. Next, we show $r_{K_{s,a}}^2(n) \sim r_{K_{s,a}}^2(n)$. We define $j_* := \left[ \frac{1}{p} \log_2 \log n \right] \lesssim \log(n^2/2)$. Straightforward calculations then show that $r_{K_{s,a}}^2(n) \lesssim r_{K_{s,a}}^2(n) \lesssim r_{K_{s,a}}^2(n)$. Since, trivially $r_{K_{s,a}}^2(n) \leq r_{K_{s,a}}^2(n)$, we obtain the assertion by replacing $n$ with $\delta^2 n$.
Calculations for the $\chi^2$-Divergence

In the proof of Lemma C.2 below we apply the following assertion due to [31] (Lemma A.1 in the Appendix A).

**Lemma C.1.** For $k \in \mathbb{N}$ and for each sign vector $\tau \in \{\pm\}^k$ let $J^\tau = (J^\tau_j)_{j \in [k]} \in \mathbb{R}^k$. Then,

$$\frac{1}{2k} \sum_{\tau \in \{\pm\}^k} \prod_{j \in [k]} J^\tau_j = \prod_{j \in [k]} \frac{J^+_j + J^-_j}{2}. $$

**Lemma C.2 ($\chi^2$-divergence for mixtures over hypercubes over several classes).** Let $S$ be an arbitrary index set of finite cardinality $|S| \subset \mathbb{N}$. For each $s \in S$ assume $k^s \in \mathbb{N}$ and $\theta^0_s \in \ell^2(\mathbb{N}) \subseteq \mathbb{R}^N$. For $\tau \in \{\pm\}^{k^s}$ define coefficients $\theta^{s,\tau}_j \in \ell^2(\mathbb{N})$ and functions $g^{s,\tau} \in L^2$ by setting

$$\theta^{s,\tau}_j = \begin{cases} \tau_j \theta^0_j, & j \in [k^s] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g^{s,\tau} = e_0 + \sum_{|j| \in [k^s]} \theta^{s,\tau}_{|j|} e_j. $$

Assuming $g^{s,\tau} \in D$ for each $s \in S$ and $\tau \in \{\pm\}^{k^s}$, we consider the mixture $P_1$ with probability density $\frac{1}{|S|} \sum_{s \in S} \frac{1}{2k^s} \sum_{\tau \in \{\pm\}^{k^s}} \prod_{j \in [n]} g^{s,\tau}(z_j)$, $z_j \in [0,1], j \in [n]$ and denote $P_0 := P_{1[0,1]}$. Then, the $\chi^2$-divergence satisfies

$$\chi^2(P_1, P_0) \leq \frac{1}{|S|^2} \sum_{s, t \in S} \exp \left( 2n^2 \sum_{j \in [k^s \land k_t]} (\theta^{s,\tau}_j \theta^{t,\eta}_j)^2 \right) - 1. $$

**Proof of Lemma C.2.** Recall that $\chi^2(P_1, P_0) = \mathbb{E}_0 \left( \frac{dP_1}{dP_0}(z_{j \in [n]}) \right)^2 - 1$ where $(Z_j)_{j \in [n]}$ are independent with identical marginal density $e_0 = \mathbb{I}_{[0,1)}$ under $P_0$. Let $\zeta_j \in [0,1], j \in [n]$, then

$$\frac{dP_1}{dP_0}(z_{j \in [n]}) = \frac{1}{|S|} \sum_{s \in S} \left( \frac{1}{2k^s} \sum_{\tau \in \{\pm\}^{k^s}} \prod_{j \in [n]} g^{s,\tau}(z_j) \right). $$

Squaring, taking the expectation under $P_0$ and exploiting the independence yields

$$\mathbb{E}_0 \left( \frac{dP_1}{dP_0}(z_{j \in [n]}) \right)^2 = \frac{1}{|S|^2} \sum_{s, t \in S} \frac{1}{2k^s} \frac{1}{2k^t} \sum_{\tau \in \{\pm\}^{k^s}} \sum_{\eta \in \{\pm\}^{k^t}} \prod_{j \in [n]} \mathbb{E}_0(g^{s,\tau}(Z_j)g^{t,\eta}(Z_j))$$

$$= \frac{1}{|S|^2} \sum_{s, t \in S} \frac{1}{2k^s} \frac{1}{2k^t} \sum_{\tau \in \{\pm\}^{k^s}} \sum_{\eta \in \{\pm\}^{k^t}} \left( \mathbb{E}_0(g^{s,\tau}(Z_j)g^{t,\eta}(Z_j)) \right)^2. $$

Exploiting the orthonormality of $(e_j)_{j \in \mathbb{Z}}$ we calculate

$$\mathbb{E}_0(g^{s,\tau}(Z_1)g^{t,\eta}(Z_1)) = \int g^{s,\tau}(z)g^{t,\eta}(z) dz = 1 + 2 \sum_{j \in [k^s \land k^t]} \theta^{s,\tau}_j \theta^{t,\eta}_j. $$

Applying the inequality $1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$ we obtain

$$\mathbb{E}_0(g^{s,\tau}(Z_1)g^{t,\eta}(Z_1)) \leq \exp \left( 2 \sum_{j \in [k^s \land k^t]} \theta^{s,\tau}_j \theta^{t,\eta}_j \right)^2 = \prod_{j \in [k^s \land k^t]} \exp \left( 2 \theta^{s,\tau}_j \theta^{t,\eta}_j \right). $$
Hence,
\[
E_0 \left( \frac{dP_1}{dP_0} (Z_{j,j \in [n]}) \right)^2 \leq \frac{1}{|S|^2} \sum_{s,t \in S} \sum_{\tau \in \{\pm\}^k} \sum_{j_1, j_2 \in [k^s \land k^t]} \prod \exp \left( 2n \theta_{j_1, j_2, \tau} \right),
\]
where we apply Lemma C.1 to the \( \eta \)-summation with \( J^{(\eta)}_j = \exp(2n \theta_{j, \tau} \theta^{(\eta)}_j) \) and obtain
\[
E_0 \left( \frac{dP_1}{dP_0} (Z_{j,j \in [n]}) \right)^2 \leq \frac{1}{|S|^2} \sum_{s,t \in S} \sum_{\tau \in \{\pm\}^k} \prod \exp \left( -2n \theta_{j, \tau} \theta^{(\eta)}_j \right)
\]
Again applying Lemma C.1 to the \( \tau \)-summation with \( J^{(\tau)}_j = \exp(-2n \theta_{j, \tau} \theta^{(\tau)}_j) \) yields
\[
E_0 \left( \frac{dP_1}{dP_0} (Z_{j,j \in [n]}) \right)^2 \leq \frac{1}{|S|^2} \sum_{s,t \in S} \prod \cosh(2n \theta_{j, \tau} \theta^{(\tau)}_j).
\]
Since \( \cosh(x) \leq \exp(x^2/2) \), \( x \in \mathbb{R} \), we obtain
\[
E_0 \left( \frac{dP_1}{dP_0} (Z_{j,j \in [n]}) \right)^2 \leq \frac{1}{|S|^2} \sum_{s,t \in S} \prod \exp \left( 2n^2 \theta_{j, \tau} \theta^{(\tau)}_j \right) \leq \frac{1}{|S|^2} \sum_{s,t \in S} \exp \left( 2n^2 \sum_{j \in [k^s \land k^t]} (\theta_{j, \tau} \theta^{(\tau)}_j)^2 \right),
\]
which completes the proof. \( \square \)

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