Capillary–Gravity Water Waves with Discontinuous Vorticity: Existence and Regularity Results

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Abstract: In this paper we construct periodic capillarity–gravity water waves with an arbitrary bounded vorticity distribution. This is achieved by re-expressing, in the height function formulation of the water wave problem, the boundary condition obtained from Bernoulli’s principle as a nonlocal differential equation. This enables us to establish the existence of weak solutions of the problem by using elliptic estimates and bifurcation theory. Secondly, we investigate the a priori regularity of these weak solutions and prove that they are in fact strong solutions of the problem, describing waves with a real-analytic free surface. Moreover, assuming merely integrability of the vorticity function, we show that any weak solution corresponds to flows having real-analytic streamlines.

1. Introduction

This paper is concerned with periodic capillarity–gravity water waves traveling over a homogeneous fluid and having an arbitrary bounded vorticity distribution. Our study is motivated by the physical setting of wind generated waves which possess a thin layer of high vorticity [39], or even high vorticity regions beneath the wave crests [37]. On the other hand, in the near-bed region there may exist strong tidal currents which interact with the water waves and contribute to the transportation of sediments [38]. The plethora of phenomena resulting from the wave-current interactions makes the study of rotational water waves so interesting, cf. [4,25,42]. Indeed, for irrotational waves in the absence of an underlying current, the fluid velocity, the pressure, and the particle paths in the flow present very regular features that can be described qualitatively even for waves of large amplitude (see [2,3,43]). However, already within the setting of irrotational steady waves with an underlying uniform current one encounters new particle path patterns, cf. [9,21], while the behavior of the velocity field and of the pressure is considerably altered by an underlying current of constant non-zero vorticity. For rotational waves, the most dramatic changes (in the form of critical layers) are triggered by the presence of stagnation points in the flow but even in the absence of stagnation points significant changes occur (see
A discontinuous vorticity enhances these departures from features that hold within the irrotational regime, as indicated by the numerical simulations in [26,27].

On the basis of a rigorous theory, exact periodic gravity water waves with a discontinuous vorticity have been shown to exist in [10] by making use of a weak formulation of the water wave problem. Subsequently, capillary–gravity water waves interacting with several vertically superposed and linearly sheared currents of different vorticities have been constructed in [31], by regarding the height function formulation of the hydrodynamical problem as a diffraction problem. We develop herein a rigorous existence theory for capillary–gravity water waves with a bounded general vorticity function, some of the analysis in [31] serving as a preliminary step.

The existence of exact capillary–gravity water waves was first established in the irrotational setting [22–24,40], the existence theory for rotational waves being developed more recently in the setting of waves with constant vorticity, stagnation points, and possibly with overhanging profiles [30] (see also [11,32]), or for waves with a general Hölder continuous vorticity distribution [44]. Many papers are also dedicated to the study of the properties of capillary–gravity water waves and of the flow beneath them, such as the regularity of the wave profile and that of the streamlines [18,19,28,33,45], or the description of the particle paths [17].

The first goal of this paper is to establish the existence of two-dimensional capillary–gravity water waves with an arbitrary bounded vorticity and without stagnation points. This is achieved by using the height function formulation of the water wave problem and by defining a suitable notion of weak solution for this problem. Re-expressing the boundary condition obtained from Bernoulli’s law as a nonlocal boundary condition, we obtain a new equivalent formulation of the problem which enables us to consider the existence problem of weak solutions in an abstract bifurcation setting. Using elliptic theory [16] and local bifurcation tools [12], we then establish the existence of infinitely many bifurcation branches consisting of non-laminar weak solutions of the hydrodynamical problem. Our second goal is to determine the a priori regularity properties of the weak solutions in the case when the vorticity function is merely integrable. This problem is in the setting of rotational waves very recent [7,13], but its implications are very important when studying the symmetry properties of water waves. More precisely, in view of [34, Theorem 3.1 and Remark 3.2] and [14, Corollary 1.2] the following statement holds true:

Within the set of all periodic gravity waves without stagnation points the symmetric waves with one crest and trough per period are characterized by the property that all the streamlines have a global minimum on the same vertical line.

We emphasize that the gravity waves with only one crest and trough per period are symmetric waves [5,6,36]. The availability of Schauder estimates for the new formulation of the problem stands at the basis of our regularity result where we show that the streamlines and the wave profiles corresponding to such weak solutions are real-analytic graphs. As a particular case, we establish the real-analyticity of the streamlines also for pure capillary water waves, generalizing previous results [20,28]. Our regularity result could serve as a tool when studying the symmetry of waves with capillary effects, the symmetry problem being in this setting still open. Besides, the additional regularity properties of the weak solutions help us to prove that the weak solutions that we have found are in fact strong solutions, and even classical if the vorticity function is continuous.

The outline of the paper is as follows: we start Sect. 2 by presenting the mathematical model and state at the end the main existence result Theorem 2.1. In Sect. 3 we derive a new formulation of the water wave problem, which we recast in Sect. 4 as a nonlinear