Partial Breaking in Rigid Limit of $\mathcal{N} = 2$ Gauged Supergravity

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Abstract

Using a new manner to rescale fields in $\mathcal{N} = 2$ gauged supergravity with $n_V$ vector multiplets and $n_H$ hypermultiplets, we develop the explicit derivation of the rigid limit of quaternionic isometry Ward identities agreeing with known results. We show that the rigid limit can be achieved, amongst others, by performing two successive transformations on the covariantly holomorphic sections $V^M(z, \bar{z})$ of the special Kahler manifold: a particular symplectic change followed by a particular Kahler transformation. We also give a geometric interpretation of the $\eta_i$ parameters used in arXiv:1508.01474 to deal with the expansion of the holomorphic prepotential $F(z)$ of the $\mathcal{N} = 2$ theory. We give as well a D-brane realisation of gauged quaternionic isometries and an interpretation of the embedding tensor $\vartheta^u_M$ in terms of type IIA/IIB mirror symmetry. Moreover, we construct explicit metrics for a new family of 4$r$- dimensional quaternionic manifolds $M_{QK}^{(n_H)}$ classified by ADE Lie algebras generalising the $SO(1, 4)/SO(4)$ geometry which corresponds to $A_1 \sim su(2)$. The conditions of the partial breaking of $\mathcal{N} = 2$ supersymmetry in the rigid limit are also derived for both the observable and the hidden sectors. Other features are also studied.

1 Introduction

It has been shown since a long time that the no-go theorem of [1–3], which forbids the partial breaking of extended supersymmetries, can be overcome by turning on appropriate fluxes [4–14], or by using non linear realisations of extended supersymmetry [15–17], or also
by taking rigid limits of extended gauged supergravities \[18, 19\]; see also [20–29] for the local case. For the 4d $\mathcal{N} = 2$ extended supersymmetric theories in rigid limit, the study partial breaking involves two breaking scales $\Lambda_{\text{susy}}$ and $\Lambda'_{\text{susy}}$, one for each supersymmetry ($\Lambda_{\text{susy}} < \Lambda'_{\text{susy}}$), and leads to interesting phenomenological implications as well as formal ones. If the two $\Lambda_{\text{susy}}$’s are widely separated as proposed in [13], the fermion chirality needed by phenomenology is recovered and we are with a new window for the building of quasi-realistic particle models and for the study of effective low energy stringy inspired prototypes. Also, the infrared of partial breaking in the rigid limit of $\mathcal{N} = 2$ supergravity gives a way to think about the supergravitational origin of the BI theory [30,31]. Recall as well that in $\mathcal{N} = 2$ partial breaking, the no-go theorem is understood as a problem which concerns only the $\mathcal{N} = 2$ superalgebra of supersymmetric charges $Q^A_\alpha$, $\bar{Q}^A_\dot{\alpha}$, and can be avoided by using the $\mathcal{N} = 2$ supersymmetric current algebra generated by local supercurrents $J^A_\mu$, $\bar{J}^\dot{\alpha}_\mu$ which allows, in addition to a diagonal energy momentum tensor $\delta^A_B T_{\mu\nu}$, the introduction of a non-diagonal deformation $C^A_B$ of the scalar potential. This deformation can be different from zero if one introduces magnetic Fayet-Iliopoulos (FI) terms besides the electric ones with a particular $SU(2)_R$ rotation of each of them. In the local theory, the partial breaking of $\mathcal{N} = 2$ of supersymmetry is due to the negative contribution of the two gravitini to the scalar potential $V_{\text{sugra}}$ which allows to bypass naturally the no-go theorem.

As first shown in [20], the global supersymmetry can be derived as a rigid limit of the electrically gauged supergravity for which the avoiding of the no-go theorem restricts the rigid limit to be performed in a particular symplectic frame where the holomorphic prepotential $F$ of the special Kahler manifold $M_{SK}$ vanishes. In this frame, the non-diagonal deformation in the rigid theory corresponds to the contribution of the hidden sector of the $\mathcal{N} = 2$ gauged supergravity to the induced scalar potential $V_{\text{sugra}}$ given by the underlying Ward identities [32,33].

Recently, a new approach has been done in [34] to deal with the rigid Ward identities of $\mathcal{N} = 2$ global supersymmetry. In this study, FI couplings are implemented into an $SP(2n_V)$ symplectic system of $SU(2)_R$ isotriplets $(P^n)^M = P^nM$, interpreted in terms of moment maps living on the hypermultiplet manifold, and partial breaking of supersymmetry is induced by the symplectic invariant isovector $\zeta_a \sim \varepsilon_{abc} P^M b C_{MN} P^N c$ pointing into a particular direction of the $SU(2)_R$ isospin space. The supergravity origin of this generalized partially broken $\mathcal{N} = 2$ global supersymmetric model has been derived in [30] where $\zeta_a$ is interpreted as an anomaly of the $\mathcal{N} = 2$ supercurrent algebra in 4-dimensions coming from the hidden sector of the parent $\mathcal{N} = 2$ gauged supergravity. The authors of [30] showed also that the restriction to the symplectic frame in which the prepotential of the special Kahler manifold vanishes, can be overcome by generalizing the electric gauging of [20] to a dyonic gauging nicely encoded in the embedding tensor $\vartheta^u_M = (\theta^u_\Lambda, \tilde{\theta}^u_{\Lambda})$ of [35] with electric
and magnetic \( \tilde{\theta}^u \Lambda \) components which can be interpreted as fluxes from higher energy theories [36–40].

In this paper, we contribute to the study of partial breaking of \( \mathcal{N} = 2 \) supersymmetry along the approach of [30,34]. We develop the explicit derivation of the rigid limit of Ward identities of \( \mathcal{N} = 2 \) gauged supergravity by focusing on gauging isometries of particular class of quaternionic manifolds \( M_{QK}^{(n_H)} \) in the hypermultiplet branch. We show, amongst others, that the splitting of the graviphoton and Coulomb branch contributions of the covariantly holomorphic symplectic sections \( V^M (z, \bar{z}) \) can be achieved by performing two successive transformations; a particular symplectic frame change followed by a particular Kahler transformation. We also give another manner to rescale the fields of the \( \mathcal{N} = 2 \) theory leading to a natural interpretation of the components \( \theta^u_\Lambda, \tilde{\theta}^{u\Lambda} \) of the embedding tensor as FI couplings scaling like mass\(^2\). We give as well a geometric interpretation of the \( \eta_i \) parameters of [30] as the leading term of the \( \frac{m_{3/2}}{M_{pl}} \) expansion of the 1- form Kahler connection on \( M_{SK} \); we also complete partial results and we recover other ones of literature as particular cases. Moreover, we construct the metrics of a new class of quaternionic manifolds \( M_{QK}^{(ADE)} \) classified by the finite dimensional ADE Lie algebras; the rank \( r \) of the ADE algebras is given by the number \( n_H \) of hypermultiplets where the leading \( M_{QK}^{(su_2)} \) is precisely the \( SO (1, 4) / SO (4) \) quaternionic manifold. The conditions for partial breaking of \( \mathcal{N} = 2 \) supersymmetry in observable and hidden sectors as well as the breaking of \( SU(2)_R \) down to \( U(1)_R \) are also studied.

The organisation of this paper is as follows: In section 2, we review some useful aspects on \( \mathcal{N} = 2 \) gauged supergravity in 4- dimensions. In section 3, we study the rigid limit of symplectic sections on the special Kahler manifold of the Coulomb branch. We derive also the rigid limit of other basic quantities like the period matrix \( \mathcal{N}_{\Lambda \Sigma} \) and the rank two symplectic \( U^{MN} \) of the local special Kahler geometry involved in gauge and scalar sectors. In section 4, we study the description of the quaternionic Kahler manifold in the case \( n_H = 1 \) and give a new generalisation for \( n_H > 1 \) inspired from ADE Toda field theories in two dimensions. We also study the gauging of abelian quaternionic isometries by focussing on \( SO (1, 4) / SO (4) \). In sections 5 and 6, we use \( \mathcal{N} = 2 \) supersymmetric representations and a property TAUB-NUT hyperKahler metric to motivate another manner to rescale the fields of gauged \( \mathcal{N} = 2 \) supergravity; fields in gravity and matter supermultiplets are rescaled by Planck mass \( M_{pl} \) while gauge fields in vector supermultiplets are rescaled by \( \Lambda \sim m_{3/2} \), the mass of the gravitino after partial breaking of supersymmetry. We also derive the model of [34] as a rigid limit and we discuss the partial breaking in the rigid theory. Section 7 is devoted to the conclusion and discussions. The last sections
are devoted to appendices: In Appendix A, we recall results on $\mathcal{N} = 2$ ungauged and gauged supergravities in 4d as well as on the general constraints obeyed by the embedding tensor. In Appendix B, we describe basic tools of special Kahler and special quaternionic manifolds; the proof of some results, that have been used in sections 3 and 4, are also described in this appendix. In Appendix C, we describe the rigid limit of the coupling matrices $\mathcal{N}_{\Lambda \Sigma}$ and $\mathcal{U}^{MN}$ and in Appendix D, we study the Ward identities and give details on their solutions for the case of gauging of abelian quaternionic isometries. Appendix C is devoted to rigid limit of obtained Ward identities.

2 Gauged $\mathcal{N} = 2$ supergravity

In this section, we describe some useful tools for studying the coupling of vector and matter supermultiplets in $\mathcal{N} = 2$ gauged supergravity in 4d. Of particular interest for us here is the issue concerning the gauging of isometries of the scalar manifold $M_{\text{scal}} = M_{SK} \times M_{QK}$ of this theory, the underlying Ward identities and the induced scalar potential to be used later for the study of partial supersymmetry breaking. For explicit details regarding this section, see appendix A of this paper on 4d $\mathcal{N} = 2$ ungauged and gauged extended supergravities; further details can be also found in a recent review given in [40].

2.1 $\mathcal{N} = 2$ supergravity and isometries

We first describe the on shell degrees of freedom of $\mathcal{N} = 2$ supergravity; then we give the component field lagrangian $L^{\mathcal{N}=2}_{\text{sugra}}$ and Ward identities of gauged isometries of scalar manifold $M_{\text{scal}}$ of this theory.

2.1.1 Component fields

$\mathcal{N} = 2$ supergravity in four space time dimensions involves three basic $\mathcal{N} = 2$ supersymmetric representations carrying as well quantum numbers under SU(2) $R$- symmetry of the underlying supersymmetric algebra; these are the $\mathcal{N} = 2$ gravity supermultiplet, the $\mathcal{N} = 2$ vector supermultiplet and the $\mathcal{N} = 2$ matter supermultiplet respectively denoted as follows

$$G_{\mathcal{N}=2}, \quad V_{\mathcal{N}=2}, \quad H_{\mathcal{N}=2}$$ (2.1)

In building 4D $\mathcal{N} = 2$ local models, we use one $G_{\mathcal{N}=2}$ supermultiplet; but we may have several copies of vector supermultiplets $\{V_{\mathcal{N}=2}^i\}_{1 \leq i \leq n_V}$, whose number $n_V$ define the rank of the compact gauge symmetry $G$ of the model. We may also have several hypermultiplets $\{H_{\mathcal{N}=2}^i\}_{1 \leq i \leq n_H}$ that interact with $G_{\mathcal{N}=2}$ and $V_{\mathcal{N}=2}^i$. The interaction between $V_{\mathcal{N}=2}^i$ and $H_{\mathcal{N}=2}^i$ requires hypermultiplets carrying charges under the gauge symmetry group $G$; this
means that hypermultiplets $H_{N=2}^{I}$ have to transform in some non trivial representations of the gauge symmetry to be taken in this study as just $G = U(1)^n$. However, in dealing with the scalar component fields in $H_{N=2}$ that capture information on the underlying geometry of the quaternionic Kahler submanifold $M_{QK}$ of the scalar manifold $M_{scal}$ of the $\mathcal{N} = 2$ supergravity model, we have to worry about the SU(2) R-symmetry representations hosting the real $4n_H$ scalars of $H_{N=2}^{I}$ namely

$$ (s_1, s_2, s_3, s_4)^I = (s_1^I, s_2^I, s_3^I, s_4^I), \quad I = 1, ..., n_H $$

Indeed, recall that from SU(2)$_R$ view, the four scalar components in a given hypermultiplet $H_{N=2}$ can be described in different manners; in particular by the two following ones to be used later on: $(i)$ either by using a complex field doublet $f^A$ and its complex conjugate $\bar{f}_A$ expressed in terms of the $s_k$’s as follows.

$$ f^A = \begin{pmatrix} s_1 + is_2 \\ s_3 + is_4 \end{pmatrix}, \quad \bar{f}_A = (s_1 - is_2, s_3 - is_4) $$

or $(ii)$ a real isotriplet $\vec{\phi}$ and a real isosinglet $\varphi$ given by

$$ \phi^a = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \varphi = s_4 $$

In this study, we will mainly use (2.4) described by real SU(2) representations to parameterise the scalars of hypermultiplets $H_{N=2}^{I}$; the action of the gauge symmetry $G = U(1)^n$ on these hypermultiplets will be realised in terms of local isometries of the quaternionic Kahler submanifold $M_{QK}$ parameterised by scalar fields

$$ Q^{uI} = (\varphi^I, \phi^{aI}), \quad u = 1, 2, 3, 4, \quad I = 1, ..., n_H $$

As we have also scalar fields coming from $V_{N=2}^{I}$, let us recall rapidly here below the component fields of the supersymmetric multiplets (2.1) and their quantum numbers under SU(2)$_R$.

(1) Gravity supermultiplet $G_{N=2}$ : the spins of its on shell degrees of freedom are as \((2, \frac{3}{2}, 1)\); in addition to the spin 2 graviton $e^m_{\mu}$, it has a graviphoton $A^0_{\mu}$, that plays an important role in present study, and two spin $\frac{3}{2}$ gravitini $\psi^{\hat{a}1}_{\mu}$ and $\psi^{\hat{a}2}_{\mu}$ forming an SU(2) doublet described by

$$ \psi^{\hat{a}A}_{\mu} = \begin{pmatrix} \psi^{\alpha A}_{\mu} \\ \bar{\psi}^{\hat{a}\dot{\alpha}A}_{\mu} \end{pmatrix}, \quad \alpha, \dot{\alpha} = 1, 2 $$
As far as $\mathcal{N} = 2$ supersymmetry is preserved, these two gravitini are massless; but if it is partially broken, half of the degrees of freedom in (2.6) become massive with some mass $m_{3/2}$ thought of below as the scale of partial supersymmetry breaking.

(2) Vector supermultiplet $V_{\mathcal{N}=2}$: its on shell degrees of freedom are as $(1,\frac{1}{2},0^2)$; in addition to the gauge field $A_\mu$ which will also play an important role in the present analysis, it has two scalars $z$ and $\bar{z}$ parameterising $M_{SK}$ as well as two Majorana gauginos $\lambda^{\hat{A}}, \bar{\lambda}^{\hat{A}}$ forming a doublet as

$$\lambda^{\hat{A}} = \begin{pmatrix} \lambda^{\alpha A} \\ \bar{\lambda}_{\dot{\alpha}A} \end{pmatrix} \quad (2.7)$$

In the $U(1)^{n_V}$ Coulomb branch, we have a set of $n_V$ vector supermultiplets indexed like $V^i_{\mathcal{N}=2}$ with on shell degrees described by the component fields

$$A^i_\mu, \quad z^i; \quad \lambda^{\hat{A} i} \quad (2.8)$$

The gauginos are organised into $n_V$ doublets of $SU(2)_R$; and the real $2n_V$ scalars are described by $n_V$ complex scalars $z^i$ and their complex conjugates $\bar{z}^i$; they are singlets under the global $SU(2)_R$ and the local $U(1)^{n_V}$. Notice that along with the $n_V$ abelian potential vectors $A^i_\mu$ of the Coulomb branch, we moreover have the $A^0_\mu$ graviphoton of the $G_{\mathcal{N}=2}$ multiplet which is also an abelian gauge field; these electric gauge potentials will be denoted collectively as $A^A_\mu = (A^0_\mu, A^i_\mu)$ with an upper index $A$ running from 0 to $n_V$. Thus, the total electric abelian gauge group of the low energy theory is $U(1)^{1+n_V}$. By implementing the magnetic duals of the electric vector potentials $A^A_\mu$, which for convenience we denote them like $A_{\mu A} = (A_{\mu 0}, A_{\mu i})$ with index $A$ down stairs, the resulting $\mathcal{N} = 2$ dyonic theory has now an $SP(2n+2)$ symplectic symmetry and still a $U(1)^{1+n_V}$ gauge symmetry. As was first argued in [41,42], the low energy properties of the electric- magnetic theory are described by one function, the prepotential $F$, which is a holomorphic function of the scalar field $F = F(z)$; more details on this description are reported in section 3.

(3) Matter hypermultiplet $H_{\mathcal{N}=2}$: its on shell degrees of freedom are given by $(1^2,0^4)$; it has four real scalars $(s_1, s_2, s_3, s_4)$ and two complex Fermi fields which can be hosted

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1 This description has a nice interpretation in type IIB string compactified on Calabi-Yau threefolds (CY3) with symplectic homology three cycle basis $(\alpha^\Lambda, \beta^\Lambda)$. The 4-form gauge potential $C_4$ of the dyonic D3 brane gets reduced after compactification down to the space-time 1-form gauge potentials $A_\mu^A = A_\mu^A dx^\mu$ and $A_{1A} = A_{\mu A} dx^\mu$ related to the 4-form potential like $A^A_4 = \int_{\alpha^\Lambda} C_4$ and $A_{1A} = \int_{\beta^\Lambda} C_4$ and to each other by the symplectic symmetry inherited from $(\alpha^\Lambda, \beta^\Lambda)$.
by different representations of SU(2)\(_R\); in particular as follows

| \(H_{N=2}\) | scalars | fermions |
|------------|---------|---------|
| Rep I      | complex : \(f^A = (f^1, f^2)\) | Dirac : \(\Psi^a = (\psi^a, \chi_\alpha)\) |
| Rep II     | real : \(Q^a = (\varphi, \phi^a)\) | Majorana : \(\xi^{aA} = (\xi^{aA}, \xi_{\dot{\alpha}})\) |

(2.9)

In the first representation (Rep I), the \(f^A\) is a complex isodoublet of scalar fields as in (2.3), and \(\Psi^a\) is a Dirac spinor; while in Rep II, the hyperini \(\xi^{aA}\) form an isodoublet of Majorana fermions. The scalar field doublet \(f^A\) has the usual canonical dimension, that is scaling as mass; and a priori \(Q^a\) should be treated in a similar manner. However this is not the case in present study since we will use quantities like \(e^{2\varphi}\) as in eq(4.4). In subsubsection 5.2.1, we will comment on this issue and give a proposal for the scaling dimensions of Rep II inspired from the structure of TAUB-NUT metric preserving manifestly the SU(2) R- symmetry; see eqs(5.16-5.19) and (5.22-5.23).

- A comment on scalars of \(H_{N=2}\)

The two manners Rep I and Rep II in describing the four real scalars of the hypermultiplet \(H_{N=2}\) given by (2.9) are very remarkable and suggestive. The point is that for the complex representation, the SU(2) doublet \(f^A\) and its complex conjugate \(\bar{f}_A\) allow U(1) phase change preserving SU(2) R- symmetry like

\[
\begin{align*}
  f^A &\rightarrow f'^A = e^{iqw} f^A, & J_0 f^A = q f^A \\
  \bar{f}_A &\rightarrow \bar{f}'_A = e^{-iqw} \bar{f}_A, & J_0 \bar{f}_A = -q \bar{f}_A
\end{align*}
\]

(2.10)

where the real \(w\) stands for the parameter of the abelian U(1) group and the hermitian \(J_0\) is its generator which can be thought of as

\[
J_0 = q \left( f^A \frac{\partial}{\partial f^A} - \bar{f}_A \frac{\partial}{\partial \bar{f}_A} \right), \quad \left( \frac{\partial}{\partial f^A} \right)^\dagger = -\frac{\partial}{\partial f^A}
\]

(2.11)

The same transformations are valid for the gradient \(\partial_\mu f^A\) of the field doublet which transform like \(\partial_\mu f^A \rightarrow e^{iqw} \partial_\mu f^A\) for global parameter \(w\). By gauging the phase change, \(w = w(x)\), one needs implementation of an abelian gauge field \(A^0_\mu\) with gauge transformation \(A^0_\mu \rightarrow A^0_\mu - \partial_\mu w\); and replace \(\partial_\mu f^A\) by the gauge covariant derivative

\[
D_\mu f^A = (\partial_\mu + iA^0_\mu J_0) f^A
\]

(2.12)

in order to have covariance \(D_\mu f^A \rightarrow e^{iqw} D_\mu f^A\). Such abelian phase change is not possible for the real representation \(Q^a = (\varphi, \phi^a)\) since the fields are real scalars. To engineer an
abelian phase symmetry, we have to complexify the fields $\varphi$ and $\vec{\phi} = (\phi^a)$ by doubling the number of degrees of freedom which in general corresponds to having an even number of hypermultiplets. However, one can still engineer abelian symmetries by using field translations like
\begin{align*}
\varphi \to \varphi' &= \varphi + c_0 , \\
\vec{\phi} \to \vec{\phi}' &= \vec{\phi} + \vec{c}
\end{align*}
respectively generated by
\begin{align*}
T_0 &= \frac{\partial}{\partial \varphi} , \\
T_a &= \frac{\partial}{\partial \phi^a}
\end{align*}
These kind of translations will be encountered later on when studying gauging abelian quaternionic isometries of the scalar manifold $M_{QK}$ associated with the hypermultiplet sector. There, we will give more details on the gauging of these translations.

Notice by the way that for the generic case where there are $n_H$ hypermultiplets $H^{A_\mathcal{N}=2}$, the component fields of the hypermultiplets carry also the index $r = 1, \ldots, n_H$; for the example of the real representation we have $Q^{ur} = (\varphi^r, \phi^{ar})$ for scalars and $\xi^{Ar}_{\hat{\alpha}}$ for Majorana fermions. A similar indexing is valid as well for the complex scalars $f^{Ar}$ and their Dirac spinor partners $\Psi^{\hat{\alpha}r} = (\psi^{\hat{\alpha}r}, \bar{\chi}^{r\dot{\alpha}})$.

To conclude this paragraph, observe that scalar fields in $\mathcal{N} = 2$ supergravity come from two sectors: (i) these are the $n_V$ complex scalars $z^i$ parameterising a complex $n_V$ dimension special Kahler manifold $M_{SK}$; and (ii) the $4n_H$ real scalars $Q^{ur}$ parameterising a quaternionic Kahler manifold $M_{QK}$. Altogether, these scalars parameterise a manifold $M_{scal} = M_{SK} \times M_{QK}$ with real $2n_V + 4n_H$ dimensions and a block diagonal metric of the form
\begin{equation}
\tilde{G} = \begin{pmatrix}
G_{ij} & 0 \\
0 & h_{\pi\pi'}
\end{pmatrix}
\end{equation}
with $G_{ij}$ is the metric of $M_{SK}$ and $h_{\pi\pi'}$ is the metric of $M_{QK}$ with $4n_H$ field coordinates $q^\pi \equiv Q^{ur}$. The special Kahler and the quaternionic Kahler submanifolds of $M_{scal}$ are given by the coset group manifolds
\begin{equation}
M_{SK} = \frac{SO(2,n_V)}{SO(2) \times SO(n_V)}
\end{equation}
and
\begin{equation}
M_{QK} = \frac{SO(4,n_H)}{SO(4) \times SO(n_H)}
\end{equation}
In what follows, we shall set $n_V \equiv n$, and keep the notation $n_H$ for the number of hypermultiplets with scalars as $q^\pi$. Later on, we will also consider the particular and interesting case where $n_H = 1$ by using the real representation $Q^n = (\varphi, \phi^n)$.

\footnote{In the embedding of this $\mathcal{N} = 2$ QFT$_4$ in type II string theory on CY3, we have two extra scalars parameterising the particular Kahler submanifold $SU(1,1)/U(1)$.}
2.1.2 Gauging isometries and Lagrangian density

The only known way to introduce a scalar potential in the $\mathcal{N} = 2$ supergravity theory which is compatible with $\mathcal{N} = 2$ supersymmetry is through the procedure of gauging isometries of the scalar manifold of the theory [40, 43]. This procedure, to be detailed on a particular example in subsection 4.3, is nicely described by using the embedding tensor $\vartheta^{\gamma}_{M}$, carrying two different kinds of indices, with components as

\[ \vartheta^{\gamma}_{M} = (\theta^{\gamma}_{A}, \tilde{\theta}^{\gamma}_{A}) \]

with $A = 1, \ldots, n_{V}$ and $\gamma = 1, \ldots, n_{H}$ respectively indexing the number of gauge and matter multiplets. This symplectic object has been first introduced in [35], and can be of two types: (i) either having only electric components $\vartheta^{\gamma}_{M} = (\theta^{\gamma}_{A}, 0)$; in this case the gauging is called electric gauging; or (ii) dyonic gauging where $\vartheta^{\gamma}_{M}$ has both electric and magnetic components $(\theta^{\gamma}_{A}, \tilde{\theta}^{\gamma}_{A})$ [35, 44]. In this regards, notice that the rigid limit of an electric gauged $\mathcal{N} = 2$ supergravity was constructed a long time ago in [20]; there the authors showed that to have, in the rigid theory, a partially broken $\mathcal{N} = 2$ supersymmetry one must choose a symplectic frame in which the prepotential $F$ of the special Kahler geometry does not exist. This constraint is overcome in dyonic $\mathcal{N} = 2$ gauged supergravity; there the rigid limit of dyonic $\mathcal{N} = 2$ theory has been recently constructed in [30] where the limit of $\mathcal{N} = 2$ supergravity is achieved in any symplectic frame and more importantly in a frame in which the prepotential $\mathcal{F}$ of the special Kahler geometry exists.

- Gauging isometries

Following [30, 35], the gauging of a subgroup $H$ of the global isometry group $G$ of the scalar manifold $M_{\text{scal}} = M_{SK} \times M_{QK}$ of the $\mathcal{N} = 2$ gauged supergravity is nicely encoded in the embedding tensor $\vartheta^{\gamma}_{M}$ of (2.18). This tensor allows to express objects valued in the Lie algebra of $H$, like the gauge field $C_{\mu}$, in two equivalent manners: (1) either as $C_{\mu} = A^{M}_{\mu} X_{M}$ where the $X_{M}$'s are the generators of $H$ and where the $A^{M}_{\mu}$'s are given by $(A^{A}_{\mu}, 0)$ transforming as an $\text{SP}(2n_{V} + 2, \mathbb{R})$ symplectic vector; or (2) like $C_{\mu} = A^{M}_{\mu} \vartheta^{\gamma}_{M} T_{\gamma}$ where now the $T_{\gamma}$'s are the generators of $G$. To fix ideas, let us consider the general picture regarding the gauging of isometries in both $M_{SK}$ and $M_{QK}$ factors of $M_{\text{scal}}$; the restriction to the gauging of isometries in $M_{QK}$ is directly obtained by dropping out the part concerning $M_{SK}$; particular examples concerning the values of the $n_{V}$ integer will be also considered in section 4. To that purpose, let us denote by $T_{\gamma} = (T_{A}, T_{m})$ the set of generators of $G$ with $T_{A}$'s refering to isometry generators in $M_{SK}$ and $T_{m}$'s to those in $M_{QK}$. The electric gauging of a part of $G$ is achieved by considering a subgroup $H$ generated by some linear combinations of $T_{A}$ and $T_{m}$ that we define as $X_{A} = \theta^{A}_{\Lambda} T_{A} + \theta^{m}_{\Lambda} T_{m}$.
and which can be rewritten in a condensed manner like $X_\Lambda = \theta^\gamma_\Lambda T_\gamma$, where the coefficients $\theta^\gamma_\Lambda$ are as in the first entry block of eq(2.18). To these electric $X_\Lambda$ generators, it is associated the $A^\mu_\Lambda$ electric gauge fields of the supergravity theory and so we can think about the gauge field valued in the Lie algebra of $H$ as $C^\mu_\mu = A^\Lambda_\mu X_\Lambda$. By substituting $X_\Lambda$ in terms of $T_\gamma$ we can also imagine this vector potential as valued in $G$ like $C^\mu_\mu = C^\gamma_\mu T_\gamma$ but with components as $C^\gamma_\mu = \theta^\gamma_\Lambda A^\Lambda_\mu$. In the dyonic gauging, one has in addition to the electric $X_\Lambda$, magnetic duals defined as $X^\Lambda = \tilde{\theta}^\gamma T_\gamma$ with $\tilde{\theta}^\gamma$ given by the second entry block of (2.18). In this case, the previous expression of the $C^\gamma_\mu$ gauge potentials gets promoted to a general one involving the $\tilde{A}^\mu_\Lambda$’s as well; so we have $C^\gamma_\mu = \theta^\gamma_\Lambda A^\Lambda_\mu + \tilde{\theta}^\gamma A^\mu_\Lambda$. The symplectic structure of these kinds of dyonic quantities can be manifestly exhibited by using $SP(2n + 2, \mathbb{R})$ symplectic representations. For the case of the gauge field, this may be done as follows: first consider the potential form $C^\gamma P N = (X^\mu M)_P^N \phi^\mu M$ which expands explicitly like

$$X_M = \phi^a_M T^{a} + \phi^m_M T^{m}$$

(2.19)

In the case where the gauging of isometries of the scalar manifold is restricted to quaternionic isometries in $M_{QK}$, the embedding tensor has no $\phi^a_M$ components, and so the above expression reduces to $X_M = \phi^m_M T^{m}$. Notice also that consistency of the gauging of the $H$ subgroup of $G$ requires a set of constraint relations; in particular the following ones:

(i) conditions on the embedding tensor components

$$\partial^a_M \phi^b_N f^c_{ab} + (X_M)_P^P \phi^c_P = 0$$

$$\partial^m_M \phi^m_N f^m_{mn} + (X_M)_P^P \phi^m_P = 0$$

(2.20)

and

$$\partial^a_M C^{MN} \phi^b_N = \partial^a_M C^{MN} \phi^m_N = \partial^m_M C^{MN} \phi^m_N = 0$$

(2.21)

where $f^a_{ab}$, $f^m_{mn}$ are structure constants of the underlying Lie algebra of $G$, $(X_M)_P^M$ the matrix elements of $X_M$; and where the antisymmetric $C^{MN}$ is the $Sp(2n + 2)$ invariant matrix given by

$$C^{MN} = \begin{pmatrix} 0 & T_{n+1} \\ -T_{n+1} & 0 \end{pmatrix}$$

(2.22)

(ii) the dualisation of the part of the four real scalars $Q^u$ of the matter sector, involved in the gauging, in terms of rank 2- antisymmetric tensors. This dualisation is needed in order to avoid the introduction of new degrees of freedom in the theory associated with the gauge fields $A^\mu_\Lambda = (A^0_\mu, A^i_\mu)$ [30,35,45–48]. If referring to the part of scalars in question scalars as $\phi^m$ and to the antisymmetric tensors like $B^m_{mn}$, the dualisation can be described.
by help of the coupling \( L_d \sim \varepsilon^{\mu \nu \rho \sigma} \partial_\mu \phi^m H_{\nu \rho \sigma} + \frac{1}{2} H_{\nu \rho \sigma} H^{\nu \rho \sigma} \) with \( H_{\nu \rho \sigma} \) standing for the field strengths of \( B_{\mu \nu}^m \) given by

\[
H_{\nu \rho \sigma} = \frac{1}{3!} \partial_{[\nu} B_{\rho \sigma]}^m
\]

(2.23)

After the dualisation, one can interpret the gauge fields \( A_{\mu}^m \) as the degrees of freedom needed for the antisymmetric \( B_{\mu \nu}^m \) to become a massive tensor field \([49, 50]\). Recall that in a generic \( d \)-dimensional space time, a massless gauge field \( A_{\mu} \) has \((d - 2)\) degrees of freedom while a massless \( B_{\mu \nu} \) has \((d - 2)(d - 3)/2\); in our case \( d = 4 \). By eating \( A_{\mu} \), the number of the degrees of freedom of the resulting antisymmetric field \( \tilde{B}_{\mu \nu} \) is given by the sum of degrees of \( B_{\mu \nu} \) and degrees of \( A_{\mu} \) namely

\[
\frac{1}{2} (d - 2) (d - 3) + (d - 2) = (d - 2) (d - 1)
\]

(2.24)

This number is exactly the total degrees of freedom of a massive rank-2 antisymmetric field \( \tilde{B}_{\mu \nu} \).

(iii) To preserve \( \mathcal{N} = 2 \) supersymmetry, one must add a scalar potential \( \mathcal{V}_{scal}^{\mathcal{N}=2} = \mathcal{V}(z, \bar{z}, Q) \) to the Lagrangian density. We notice that after the dualisation, the \( Q^{\mu} \) scalars of the hypermultiplet split as follows

\[
Q^{\mu} = (\phi^{\hat{a}}, B_{\mu \nu}^m)
\]

(2.25)

and the metric \( h_{\mu \nu} \) of \( \mathbf{M}_{QK} \) becomes \([30, 47]\)

\[
h_{\mu \nu} = \begin{pmatrix}
g_{ab} A_{\hat{a}}^m \\
A_{\hat{a}}^m M_{mn}
\end{pmatrix}
\]

(2.26)

Notice that for \( n_H = 1 \), the indices \( \hat{a} \) and \( m \) take respectively the values \( \hat{a} = 1, 2 \) and \( m = 3, 4 \).

- Lagrangian density

The component field Lagrangian \( \sqrt{g} \mathcal{L}_{\mathcal{N}=2} \) of the 4d \( \mathcal{N} = 2 \) gauged supergravity depends on the geometry of \( \mathbf{M}_{scal} \); the explicit expression of its bosonic part, describing the interacting dynamics of the above mentioned degrees of freedom, is given by \([30, 45, 47]\)

\[
\mathcal{L}_{\mathcal{N}=2} = \mathcal{R} + \mathcal{L} - \mathcal{V}_{scal}^{\mathcal{N}=2}
\]

(2.27)

with

\[
\mathcal{L} = G_{ij} \partial^i z^i \partial_\mu z^j + g_{ab} \partial_\mu \phi^{\hat{a}} \partial^\mu \phi^b + 2 \varepsilon^{\mu \nu \rho \sigma} A_{\hat{a}}^m H_{\nu \rho \sigma} \partial_\mu \phi^{\hat{a}}
\]

\[
- 2 \varepsilon^{\mu \nu \lambda \sigma} B_{\mu \nu}^m \partial_\lambda \left( \tilde{F}_\rho^\Lambda - \partial_\rho \Lambda B_{\rho \sigma} \right)
\]

\[
+ i \left( N_{\Lambda \Sigma} \tilde{F}_{\mu \nu}^\Lambda \tilde{F}^\Sigma_{\mu \nu} - N_{\Sigma \Lambda} \tilde{F}_{\mu \nu}^\Sigma \tilde{F}^\Lambda_{\mu \nu} \right)
\]

\[
+ 6 M_{mn} \left( H_{\nu \rho \sigma} H^{\nu \rho \sigma} \right)
\]

(2.28)
where $\mathcal{F}_{\mu\nu}^A$ are the field strengths of the vectors $A_{\mu}^A$ defined as
\[ \hat{\mathcal{F}}_{\mu\nu}^A = \mathcal{F}_{\mu\nu}^A + 2\theta^A \lambda^m B_{m\mu\nu} \] (2.29)
and
\[ \mathcal{F}_{\mu\nu}^{\pm A} = \frac{1}{2} (\mathcal{F}_{\mu\nu}^A \pm \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma}^A) \] (2.30)

In (2.28), the $N_{\Sigma}$ is the period matrix that we will define later on, see (C.4); the metric components $g_{ab}$, $A_{a}^m$, $M_{mn}^a$ are as in (2.26) and the $H_{m\rho\sigma}^a$'s are the field strength tensors of the $B_{m\rho\sigma}^a$'s given by (2.23). As we will show in section 5, the scalar potential reads [30]
\[ V_{scal}^{N=2} = (U_i^M G^{ij} U_j^M) \mathcal{P}_M^a \mathcal{P}_N^a - V^M V^N \mathcal{P}_M^a \mathcal{P}_N^a \] (2.31)
where $V^M$ is the covariantly holomorphic section on $M_{SK}$ with metric $G_{ij}$, $U_i^M$ is the covariant derivative of $V^M$ and $\mathcal{P}_M^a$ are moment maps on $M_{QK}$; all these objects and their rigid expressions will described with details later on.

2.2 Isometries and Ward identities

Under the gauging of isometries of the scalar manifold $M_{scal} = M_{SK} \times M_{QK}$ of $\mathcal{N} = 2$ supergravity, the usual supersymmetric transformations $\delta^{(0)}_\epsilon \chi$ of the fermion fields $\chi = (\lambda^i A, \zeta^i, \psi_{A\mu})$ of the theory get extra contributions $\delta^{(\theta)}_\epsilon \chi$ required by supersymmetric invariance. These extra transformations read in terms of the supersymmetric parameters $\epsilon^B$ and auxiliary functional fields $(W^i)_B^A$, $N^\gamma_A$ and $S^A_B$ as follows
\[ \begin{align*}
\delta^{(\theta)}_\epsilon \lambda_{iA}^j &= (W^i)_B^A \epsilon^B \\
\delta^{(\theta)}_\epsilon \zeta^i &= N^\gamma_A \epsilon^A \\
\delta^{(\theta)}_\epsilon \psi_{A\mu} &= i\gamma_\mu \epsilon^B S^A_B
\end{align*} \] (2.32)

The various fermion shift matrices $(W^i)_B^A$, $N^\gamma_A$ and $S^A_B$ involved in these transformations are not arbitrary; they are related to each others by Ward identities given by the $2 \times 2$ matrix equation [3, 51]
\[ G_{ij} (W^i)_C^B (\bar{W}^j)_C^B + 2\bar{N}^A_{\alpha} N^\alpha_B - 12 \bar{S}^A_C S^C_B = \delta^A_B V_{sugra} \] (2.33)
where $V_{sugra}$ is the scalar potential of the supergravity theory; this formula may be naively compared to the usual expression of the scalar potential $|F|^2 + \frac{1}{2} D^2$ of global supersymmetry where F and D are the auxiliary fields of the $\mathcal{N} = 2$ U(1) vector multiplet in the Wess-Zumino gauge. Because of the particular form of the right hand side of above matrix equation, these identities can be split by help of SU(2) R-symmetry representations as follows
\[ \begin{align*}
Tr (\tau^a G_{ij} (W^i)_C^B (\bar{W}^j)_C^B + 2\bar{N}^A_{\alpha} N^\alpha_B - 12 \bar{S}^A_C S^C_B) &= 0 \\
Tr (\left[ G_{ij} (W^i)_C^B (\bar{W}^j)_C^B + 2\bar{N}^A_{\alpha} N^\alpha_B - 12 \bar{S}^A_C S^C_B \right]) &= 2 V_{sugra}
\end{align*} \] (2.34)
The projection along the isosinglet dimension gives the induced scalar potential \( V_{\text{sugra}} \) in \( \mathcal{N} = 2 \) gauged supergravity and whose rigid limit is given subsection 5.2. If we restrict ourselves to the case of gauging only isometries of the quaternionic Kahler manifold \( M_{QK} \subset M_{\text{scal}} \), the constraint eqs (2.34) are solved in terms of Killing vectors \( \kappa^u_M \), isovectors of moment maps \( P^a_M = (P^x_M, P^y_M, P^z_M) \) and quaternionic vielbeins \( E^A_{u\alpha} \) as follows [52]

\[
(W^i)^A_B = -i G^{ij} \ddot{U}_j^M (P^a_M \tau^A_a)_B \\
N^A_\alpha = 2 \xi^A_{u\alpha} \kappa^u_M \bar{V}^M \\
S_{AB} = \frac{i}{2} (\tau_a)^C_A \varepsilon_{BC} P^a_M V^M
\]

with complex conjugates as

\[
(W^j)_{BC} = i (\tau_a)^D_B \varepsilon_{DC} P^a_M G^{ij} U^j_i \\
\bar{N}^\alpha_A = -2 \xi^\alpha_{aA} \kappa^a_M V^M \\
\bar{S}^{AB} = \frac{i}{2} (\tau_a)^C_B \varepsilon_{CB} P^a_M \bar{V}^M
\]

In this solution, \( E^A_{u\alpha} \) is the vielbein of the quaternionic Kahler manifold \( M_{QK} \); see eq (4.16) and eqs (4.20-4.17) to fix the ideas.

### 3 Kahler sector in \( \mathcal{N} = 2 \) supergravity

In this section, we develop the study of the rigid limit of \( \mathcal{N} = 2 \) supergravity and completes partial results obtained in [30]. In particular, we give the explicit derivation of the rigid limits of the symplectic sections \( \Omega^M, V^M \) and \( U^M_i \), living in the Kahler sector of \( \mathcal{N} = 2 \) supergravity, as well as the \( \mathcal{N}_{A\Sigma} \) and \( \mathcal{U}^{MN} \) tensors involved in eqs (2.27, 2.28) and (2.31). These rigid limits, which are needed for the computation of the scalar potential \( V_{\text{kah}}^{\mathcal{N}=2} \) and the matrix anomaly \( \tilde{C}^A_B \) in the observable sector of the gauged supergravity, were first considered in [20] for the case of one \( \mathcal{N} = 2 \) vector supermultiplet \( V_{\mathcal{N}=2} \); an extension of [20] to an arbitrary number \( n_V \) of \( \mathcal{N} = 2 \) vector supermultiplets \( \{V^i_{\mathcal{N}=2}\}_{1 \leq i \leq n_V} \) was studied in [53], and a new extension, using extra parameters \( \eta_i \) interpreted as charges associated with the gauging procedure, has been proposed recently in [30].

In the present study, the aforementioned \( \eta_i \)’s will be also given a geometric interpretation; they are the rigid limit of the gauge connection components \( \omega^0_i (z, \bar{z}) \) of the \( U(1) \) bundle on the special Kahler manifold \( M_{SK} \). We also show that the splitting of the \( V^M \) covariantly holomorphic sections on the special Kahler manifold as the sum \( V^M_{\text{grav}} + V^M_{\text{rigid}} \) can be achieved by using two successive and particular symmetry transformations namely a particular symplectic change mapping \( V^M \) to \( V^M \) followed by a particular Kahler transformation mapping \( V^M \) to \( V^M \). In both these transformations, the \( \eta_i \) parameters play an important role.
3.1 $\frac{1}{\mu}$-expansion of symplectic sections

We begin by recalling that in special Kahler geometry of the Coulomb branch of $\mathcal{N} = 2$ supergravity, one distinguishes three kinds of symplectic sections $\Omega^M, V^M$ and $U^M_i$ related to each others: First, we have the holomorphic $\Omega^M$ satisfying $\frac{\partial \Omega^M}{\partial \bar{z}} = 0$ with complex field coordinates $z^i$ as in eq(2.8). Second, we also have the covariantly holomorphic sections $V^M$ related to $\Omega^M$ as follows

$$V^M = e^{\frac{i}{2} K} \Omega^M, \quad \bar{\nabla}_i V^M = 0$$

(3.1)

with $K = K(z, \bar{z})$ standing for the special Kahler potential obeying the usual Kahler transformation symmetry

$$K'(z, \bar{z}) = K(z, \bar{z}) - f(z) - \bar{f}(\bar{z})$$

(3.2)

where $f(z)$ is an arbitrary holomorphic function. $\bar{\nabla}_i$ is the covariant derivative induced by the Kahler transformation, it is given

$$\bar{\nabla}_i = \frac{\partial}{\partial \bar{z}^k} + \frac{1}{2} \frac{\partial K}{\partial z^k}$$

(3.3)

Our objective here is to determine the $\frac{1}{\mu}$-expansions of these sections; for that purpose, we need to introduce extra tools like the Kahler 1-form $\omega_1^0$ and the holomorphic prepotential $\mathcal{F}$ as described below.

3.1.1 Kahler 1-form $\omega_1^0$ and holomorphic prepotential $\mathcal{F}$

The covariant derivatives $\nabla_i$ and $\bar{\nabla}_i$ in eqs(3.1,3.3) are associated with the 1-form gauge connection

$$\omega_i^0 = \frac{i}{2} \left( \partial K - \bar{\partial} K \right)$$

(3.4)

of the $\text{U}(1)$ bundle living on $M_{SK}$; here $\partial = dz^k \frac{\partial}{\partial z^k}$ and $\bar{\partial} = d\bar{z}^k \frac{\partial}{\partial \bar{z}^k}$. This hermitian 1-form is an isosinglet of $\text{SU}(2)_R$; and the index 0 on the hermitian $\omega_1^0$ is to distinguish it from an analogous isotriplet of 1-forms denoted like

$$\omega^a_1 = \begin{pmatrix} \omega^x_1 \\ \omega^y_1 \\ \omega^z_1 \end{pmatrix}$$

(3.5)

and which will be introduced later on when studying the geometry of the quaternionic Kahler manifold $M_{QK}$ of hypermultiplets. By expressing the 1-form $\omega_1^0$ as

$$\omega_1^0 = i \left( dz^k \omega_k^0 - d\bar{z}^k \bar{\omega}_k^0 \right)$$

(3.6)
we have
\[ \omega^0_k = \frac{1}{2} \partial K \frac{\partial}{\partial z^k}, \quad \bar{\omega}^0_{\bar{k}} = \frac{1}{2} \partial K \frac{\partial}{\partial \bar{z}^k} \] (3.7)

Notice that under the Kahler transformation (3.2), the \( \omega^0_k \) and \( \bar{\omega}^0_{\bar{k}} \) transform like
\[ \omega'^0_k = \omega^0_k - \frac{1}{2} \frac{\partial f}{\partial z^k}, \quad \bar{\omega}'^0_{\bar{k}} = \bar{\omega}^0_{\bar{k}} - \frac{1}{2} \frac{\partial \bar{f}}{\partial \bar{z}^k} \] (3.8)

Following [30], the rigid limit of \( \mathcal{N} = 2 \) supergravity, in particular of \( \Omega^M, V^M, U_i^M \) and \( \omega^0_1 \), can be approached by considering the holomorphic prepotential \( F(z) \) of the \( \mathcal{N} = 2 \) theory and using perturbation techniques with a perturbation parameter \( \lambda = \frac{1}{\mu} \). The general procedure is as summarised below:

- Introduce a real expansion parameter \( \frac{1}{\mu} \ll 1 \) where the dimensionless \( \mu \) is given by the reduced mass scale \( \mu = \frac{M_{pl}}{\Lambda} \); with \( M_{pl} \) is the Planck scale and \( \Lambda \) a mass scale of order of the supersymmetric breaking scale; say the mass \( m_{3/2} \) of a \( \mathcal{N} = 1 \) massive gravitino supermultiplet \( G^{N=1}_{3/2} \) with spins as
\[ G^{N=1}_{3/2} = \left( \frac{3}{2}, 1^2, 1^2 \right) \] (3.9)

The use of this supermultiplet is motivated by partial supersymmetry breaking assumed to take place at scale \( \Lambda \).

- Expand the usual holomorphic prepotential \( F(X) \) of the \( U(1)^n \) Coulomb branch in power series of \( \frac{1}{\mu} \). In this regards, recall that the holomorphic \( F(X) \) is homogeneous function of degree 2; it can be factorised like
\[ F(X) = -i (X^0)^2 F \left( \frac{X^I}{X^0} \right) \] (3.10)

Thinking of the \( X \)- homogenous coordinates as \( X^\Lambda = (X^0, X^I) \) and working in the special frame where \( \frac{X^I}{X^0} = \delta^I_j z^j \) (conveniently by setting \( X^0 = 1 \), a general form of the expansion of \( F(X^I/X^0) \) into \( \frac{1}{\mu} \) power series reads, up to third order, as follows
\[ F(z) = \frac{1}{4} + \frac{1}{2\mu} \sum_{i=1}^{n} \eta_i z^i + \frac{1}{2\mu^2} \Phi(z) + \mathcal{O} \left( \frac{1}{\mu^3} \right) \] (3.11)

This parametrisation choice of \( F(z) \) reproduces the model of Antoniadis- Partouche-Taylor; a simple version of \( F(z) \) was first introduced by Ferrara- Girardelo- Poratti for a single vector multiplet [20] and generalized to the form (3.11) by Andranopoli et al for the case of several multiplets. In this development, the constant parameters \( \eta_1, ..., \eta_n \) are real numbers interpreted in [30] as charges of gauged abelian isometries of the scalar
manifold of $\mathcal{N} = 2$ supergravity; the reality condition of the $\eta_i$'s is a requirement of special Kahler geometry. For later use, notice that the terms $\eta_i \times z^i$ in above expansion is scale invariant as it behaves like the leading constant $\frac{1}{4}$ in (3.11); so the $\eta_i$ parameters can be also interpreted as a kind of dual parameters to $z^i$ in a similar way as space time coordinates $x^\mu$ and energy momentum vector $P_\mu$. Notice also that in addition to the complex field variables $z^i$, the prepotential $F(z)$ depends as well on the $\eta_i$'s and the perturbation parameter $\frac{1}{\mu}$; i.e: $F(z) \equiv F(\{z, \eta_i, \frac{1}{\mu}\})$. These extra parameters will be often hidden below.

### 3.1.2 Special Kahler potential $K$

Using the above $\frac{1}{\mu}$- expansion (3.11) of the holomorphic prepotential $F(X)$, one can write down the $\frac{1}{\mu}$- development of the holomorphic sections $\Omega^M = \Omega^M(z)$ living on the special Kahler subspace $M_{SK}$ of the scalar manifold $M_{SK} \times M_{QK}$ of $\mathcal{N} = 2$ supergravity. By using (3.10), we can express the symplectic vector $\Omega^M$ as follows

$$\Omega^M = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}$$

(3.12)

with

$$X^\Lambda = \begin{pmatrix} 1 \\ z^k \end{pmatrix}, \quad F_\Lambda = \begin{pmatrix} F_0 \\ F_k \end{pmatrix}$$

(3.13)

and

$$F_0 = \frac{1}{\mu} \left(2F - z^k \partial_k F\right)$$

$$F_k = \frac{1}{\mu} \partial_k F$$

(3.14)

then using (3.11), we obtain the following behaviour up to third order in $\frac{1}{\mu}$,

$$F_0 = \frac{1}{\mu} \left[ \frac{1}{2} + \frac{1}{2\mu} \eta_k z^k + \frac{1}{\mu^2} \left( \Phi - \frac{1}{2} z^k \partial_k \Phi \right) \right] + O\left(\frac{1}{\mu^3}\right)$$

$$F_k = \frac{1}{\mu} \left( \frac{2}{2\mu} + \frac{1}{2\mu^2} \partial_k \Phi \right) + O\left(\frac{1}{\mu^3}\right)$$

(3.15)

Substituting these expansions back into (3.12), we can split the holomorphic $\Omega^M$ as a sum of a power series of $\frac{1}{\mu}$ with leading term corresponding to the limit $\frac{1}{\mu} \to 0$. Expressing this expansion like

$$\Omega^M = \Omega^M_{(0)} + \frac{1}{\mu} \Omega^M_{(1)} + \frac{1}{\mu^2} \Omega^M_{(2)} + O\left(\frac{1}{\mu^3}\right)$$

(3.16)

we learn that

$$\Omega^M_{(0)} = \begin{pmatrix} 1 \\ z^i \\ \frac{1}{2i} \\ 0 \end{pmatrix}, \quad \Omega^M_{(1)} = \begin{pmatrix} 0 \\ 0 \\ \eta_i \frac{z^i}{2i} \\ \eta_i \frac{1}{2i} \end{pmatrix}, \quad \Omega^M_{(2)} = \begin{pmatrix} 0 \\ 0 \\ \frac{2\Phi - z^i \partial_k \Phi}{2i} \\ \frac{1}{2i} \end{pmatrix}$$

(3.17)
But this splitting is not interesting for our analysis as it does not separate the pure gravity sector, associated with graviphoton direction, from the rigid limit of the supergravity theory. To overcome this difficulty, we use the symmetries of the supergravity theory; in particular the symplectic and Kahler invariance, to sit in a particular coordinate frame on $M_{SK}$ where gravity and the rigid limit appear as two ”orthogonal” sectors at the first order of the limit $\frac{1}{\mu} \to 0$. Indeed, as the holomorphic section $\Omega^M$ is not unique since it is defined up to a symplectic transformation

$$\Omega^M \to \Omega'^M = S_N^M \Omega^N$$ (3.18)

one can work with $\Omega'^M$ instead of $\Omega^M$ without affecting physical properties. In this regards, recall that the above symplectic transformation leaves invariant the special Kahler potential of the $M_{SK}$

$$\mathcal{K} = - \ln \left[ -i \Omega^M C_{MN} \bar{\Omega}^N \right]$$ (3.19)

Recall also that the matrix $S_N^M$ in (3.18) is a real symplectic $(2n+2) \times (2n+2)$ matrix satisfying the usual property $S_N^M C^{PQ} S_Q^N = C^{MN}$ and can be defined in terms of four submatrices as follows

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ (3.20)

Here the antisymmetric $C^{MN}$ is the symplectic metric of $\text{SP}(2n+2)$, and the $A$, $B$, $C$ and $D$ are $(n+1) \times (n+1)$ submatrices related to each others like

$$A^T D - C^T B = 1_{n+1} , \quad A^T C = C^T A , \quad B^T D = D^T B$$ (3.21)

By using (3.12) and substituting the expressions (3.13-3.15) back into the special Kahler potential $\mathcal{K}$ given by

$$\mathcal{K} = - \ln \left( i \left[ \bar{X}^A F_\lambda - X^A \bar{F}_\lambda \right] \right)$$ (3.22)

we get the following $\frac{1}{\mu}$- development of the special Kahler potential

$$\mathcal{K} = - \ln \left[ 1 + \frac{1}{\mu} \eta_i (z^i + \bar{z}^i) + \frac{1}{\mu^2} (\Phi + \bar{\Phi}) - \frac{1}{2\mu^2} (z - \bar{z})^i (\partial_i \Phi - \bar{\partial}_i \bar{\Phi}) + \mathcal{D} \left( \frac{1}{\mu^3} \right) \right]$$ (3.23)

Expanding the logarithm up to third order in $\frac{1}{\mu}$, we end with the following expression

$$\mathcal{K} = - \frac{1}{\mu} \eta_i (z^i + \bar{z}^i) - \frac{1}{\mu^2} \left[ (\Phi + \bar{\Phi}) - \frac{1}{2} (z - \bar{z})^i (\partial_i \Phi - \bar{\partial}_i \bar{\Phi}) - \frac{1}{2} (\eta_i z^i + \eta_i \bar{z}^i) \right] + \mathcal{D} \left( \frac{1}{\mu^3} \right)$$ (3.24)
from which we learn several information; in particular the $\frac{1}{\mu}$- expansion of the $U(1)$ Kahler gauge components $\omega^0_i = \frac{1}{2} \partial_i K$ and $\bar{\omega}^0_j = \frac{1}{2} \partial_j \bar{K}$ as well as the special Kahler metric $G_{ij} = \partial_i \partial_j K$. For the gauge components we have

$$
\omega^0_i = -\frac{i}{2\mu} \eta_i - \frac{i}{2\mu} a_i + O \left( \frac{1}{\mu^3} \right)
$$

with

$$
a_i = \frac{1}{2} \partial_i \Phi - \frac{1}{2} (z - \bar{z})^k \partial_i \partial_k \Phi - \eta_i \left( \eta_k z^k + \eta_k \bar{z}^k \right)
$$

$$
\bar{a}_j = \frac{1}{2} \partial_j \bar{\Phi} + \frac{1}{2} (z - \bar{z})^k \partial_j \partial_k \bar{\Phi} - \eta_j \left( \eta_k z^k + \eta_k \bar{z}^k \right)
$$

From the expansions (3.25), we observe in the limit $\mu \to \infty$, the Kahler 1-form $\omega^0_1$ (3.6) vanishes; and that the constants $\frac{1}{2} \eta_k$ appear as just the leading components of the $U(1)$ Kahler gauge fields $\omega^0_k$ and $\bar{\omega}^0_k$ in the $\frac{1}{\mu}$ expansion; a property which can be stated like

$$
\eta_k = 2 \lim_{\mu \to \infty} \left( \mu \omega^0_k \right), \quad \bar{\eta}_k = 2 \lim_{\mu \to \infty} \left( \mu \bar{\omega}^0_k \right)
$$

So the $\eta_k$'s can be interpreted as just the zero modes in the expansion of the components of the scaled 1-form connection $(\mu \omega^0_1)$ on $M_{SK}$,

$$
\mu \omega^0_1 = -\frac{i}{2} \eta_k \left( dz^k - d\bar{z}^k \right) + O \left( \frac{1}{\mu} \right)
$$

More comments on these $\eta_k$'s will be given later on; see eqs(3.53). Regarding the Kahler metric, we have

$$
G_{ij} = \frac{1}{\mu^2} \hat{G}_{ij} + O \left( \frac{1}{\mu^3} \right), \quad \hat{G}_{ij} = \eta_i \eta_j - \frac{1}{2} (\partial_i \partial_j \Phi + cc)
$$

where $\hat{G}_{ij}$ is the metric of the rigid special Kahler manifold $M_{SK}$; thus

$$
\hat{G}_{ij} = \lim_{\mu \to \infty} \left( \mu^2 G_{ij} \right)
$$

Following [54–56], the metric $G_{ij}$ is related to the covariant gradients $U^M_i = \nabla_i V^M$ and $\bar{U}^M_i = \bar{\nabla}_i \bar{V}^M$ as

$$
G_{kj} = -i U^M_k C_{MN} U^N_i
$$

with the metric $C_{MN}$ as before. In the rigid theory, we also have

$$
\hat{G}_{ki} = -i \bar{U}^M_k C_{MN} \bar{U}^N_i
$$

where $C_{MN}$ is now the metric of the SP(2n) symmetry along the Coulomb branch directions. Its underlined indices $M, N$ run from 1 to $n$; and where

$$
\bar{U}^M_k = \delta^M_M \frac{\partial \Omega^M}{\partial z^k}
$$
with $\hat{\Omega}^M$ the holomorphic section of the rigid theory; see eq(3.42) for its explicit expression.

Moreover, by thinking of $\hat{G}_{kj}$ of (3.31) in terms of the approximation (3.29), we learn that the $\hat{U}_k^M$ is nothing but the leading term in the $\frac{1}{\mu}$ expansion of $U_k^M$. Notice that in eq(3.32), we have used the convention notation $\Delta^M = (\delta^i, \bar{\delta}_i)$ for SP(2n) vectors with symplectic metric

$$C_{MN} = \delta^M_M C_{MN} \delta^N_N$$

(3.34)

Recall as well that for SP(2n + 2) of the special Kahler geometry of $M_{SK}$, the metric is denoted like $\hat{C}_{MN}$ and vectors

$$\Delta^M = (\delta^\Lambda, \bar{\delta}_\Lambda)$$

(3.35)

have $2n + 2$ components with index $\Lambda = (0, i)$ and $i = 1, \ldots, n$. To complete our notations, we denote by

$$\Delta^0 = (\delta^0, \bar{\delta}_0)$$

(3.36)

the SP(2) vectors along the graviphoton direction with 2x2 symplectic metric as $\hat{C}_{\tau\sigma} = \delta^M_M C_{MN} \delta^N_N$. With these notations, an SP(2n + 2) vector like the holomorphic section $\Omega^M$ can be written as

$$\Omega^M = \delta^M_\tau \Omega^\tau + \delta^M_N \Omega^N$$

(3.37)

with

$$\delta^M_\tau \Omega^\tau = \begin{pmatrix} 1 \\ \bar{z}^k \\ F_0 \\ 0_k \end{pmatrix}, \quad \delta^M_N \Omega^N = \begin{pmatrix} 0 \\ z^k \\ 0 \\ F_k \end{pmatrix}$$

(3.38)

### 3.2 Rigid limit of symplectic sections

Here we give the rigid limits of several useful quantities; in particular the holomorphic sections $\Omega^M$ on the special Kahler manifold $M_{SK}$, the corresponding covariantly holomorphic $V^M$ and their covariant gradient $U_i^M = (\partial_i + \frac{1}{2} \partial_i K) V^M$. We also study the splitting of pure gravity contribution of these quantities from their rigid limits viewed from the side of the Coulomb branch.

- **Rigid limit of holomorphic section $\Omega^M$**

First notice that in eq(3.29), the rigid metric $\hat{G}_{ij}$ appears as the leading term of the $\frac{1}{\mu}$ expansion of the special Kahler metric $G_{ij}$ of $M_{SK}$ subspace of the scalar manifold of $N = 2$ supergravity. The limit $\hat{G}_{ij}$ can be thought of as descending from the following "rigid" holomorphic prepotential

$$\hat{F} = \frac{i}{4} \left[ (\eta_i z^i)^2 - 2\Phi \right]$$

(3.39)
By calculating the second derivative of above expression, we have

\[ \partial_i \partial_j \hat{F} = \frac{i}{2} (\eta_i \eta_j - \partial_j \Phi) \]  

(3.40)

from which we read the real and imaginary parts which are given by

\[ \text{Im } \partial_i \partial_j \hat{F} = \frac{i}{2} \hat{G}_{ij} \]
\[ \text{Re } \partial_i \partial_j \hat{F} = -\frac{i}{4} (\partial_i \partial_j \Phi - cc) \]  

(3.41)

The holomorphic section \( \hat{\Omega}^M \) of the rigid limit of the \( \mathcal{N} = 2 \) supergravity theory can be then imagined as given by

\[ \hat{\Omega}^M = \begin{pmatrix} 0 \\ z^i \\ 0 \\ \frac{i}{2} (\eta_i \eta_j z^j - \partial_i \Phi) \end{pmatrix} \]  

(3.42)

with no \( \hat{X}^0 \) nor \( \hat{F}_0 \) components compared to (3.12); so the 2n components \( \hat{\Omega}^M \)'s behave as an \( SP(2n) \) symplectic vector \( \Omega^M \) as in (3.38) in contrast to the \( \Omega^M \) of eq(3.12) which form a vector of \( SP(2n + 2) \). In other words, we have \( \hat{\Omega}^M = \delta^M_M \hat{\Omega}^M \) with

\[ \hat{\Omega}^M = \begin{pmatrix} z^i \\ \frac{i}{2} (\eta_i \eta_j z^j - \partial_i \Phi) \end{pmatrix}, \quad i = 1, \ldots, n \]  

(3.43)

This reduction of the symplectic symmetry in the rigid limit of \( \mathcal{N} = 2 \) supergravity is a remarkable property; it will be studied in detail in a moment; before that notice that

\[ \hat{U}^M_k = \delta^M_M \hat{U}^M_k \]  

(3.44)

reads as

\[ \hat{U}^M_k = \frac{\partial \hat{\Omega}^M}{\partial z^k} \]  

(3.45)

showing that \( \hat{U}^M_k \) has no components along the graviphoton direction; that is:

\[ \delta^M_M \hat{U}^M_k = 0 \]  

(3.46)

- **Rigid limit of \( V^M \) and \( U^M_i \)**

The reduction of the \( SP(2n + 2) \) symplectic symmetry down to the subsymmetry \( SP(2n) \) of the rigid limit is an obvious property as in the supergravity theory there are \( n + 1 \) electric
charges and \( n+1 \) magnetic ones; two of them are associated with the graviphoton \( A^0_\mu \) living in the pure gravity sector; the other \( 2n \) ones have origin in the Coulomb branch with gauge field potentials \( A^i_\mu \). Though expected, the derivation of this reduction property in the rigid limit is not a trivial feature. As we will see below, it needs using symmetry properties of the covariantly holomorphic section \( V^M = e^{\frac{K}{\mu}} \Omega^M \) on \( M_{SK} \) which allow to put \( V^M \) into the form

\[
V^M = V^M_{grav} + \frac{1}{\mu} \hat{\Omega}^M + \mathcal{O}\left(\frac{1}{\mu^2}\right)
\]

with \( V^M_{grav} \) given by

\[
V^M_{grav} = \begin{pmatrix}
X^0 \\
0 \\
-\frac{1}{2} X^0 \\
0
\end{pmatrix}
\]

appearing as the zero mode of the \( \frac{1}{\mu} \) expansion and \( \hat{\Omega}^M \), which is given by (3.42), as the first order. In other words

\[
\lim_{\mu \to \infty} V^M = V^M_{grav}, \quad \lim_{\mu \to \infty} \left[ \mu \left( V^M - V^M_{grav} \right) \right] = \hat{\Omega}^M
\]

The splitting (3.47) can be derived by performing some manipulations using symmetry properties of \( V^M \) and rigid limit approximation as follows:

First, we start from the holomorphic \( \Omega^M \) and perform two (commuting) successive transformations: (i) a particular symplectic change \( \tilde{\Omega}^M = S^M_N \Omega^N \) with symplectic matrix \( S^M_N \) as in eq(3.20) leaving special Kahler potential invariant; that is \( \tilde{K} = K \); and (ii) another particular Kahler transformation on \( \tilde{\Omega}^M \) given by

\[
\tilde{\Omega}^M = e^{f_0} \tilde{\Omega}^M
\]

with holomorphic parameter \( f_0 \) taken as

\[
f_0 = -\frac{1}{\mu} \eta_i \bar{z}^i
\]

The above particular Kahler transformation acts as well non trivially on the \( K \) potential and on the components \( \omega^0_k, \bar{\omega}^0_k \) of the 1-form gauge connection \( \omega_1 \) like

\[
\begin{align*}
K' &= K + \frac{1}{\mu} (\eta_i \bar{z}^i + \eta_i \bar{z}^i) \\
\omega^0_k &= \omega^0_k + \frac{1}{2\mu} \eta_k \\
\bar{\omega}^0_k &= \bar{\omega}^0_k + \frac{1}{2\mu} \eta_k
\end{align*}
\]

which can be also rewritten like

\[
\begin{align*}
\frac{1}{\mu} (\eta_i \bar{z}^i + \eta_i \bar{z}^i) &= K' - K \\
\frac{1}{2\mu} \eta_k &= \omega^0_k - \omega^0_k \\
\frac{1}{2\mu} \eta_k &= \bar{\omega}^0_k - \bar{\omega}^0_k
\end{align*}
\]
giving another way to think of the $\eta_k$ parameters and on the quantity $\frac{1}{\mu} (\eta_i z^i + \eta_i \bar{z}^i)$. The particular Kahler change leads therefore to the following new quantities

$$\mathcal{K}' = -\frac{1}{\mu^2} \left[ (\Phi + \bar{\Phi}) - \frac{1}{2} (z - \bar{z})^i \left( \partial_i \Phi - \bar{\partial}_i \Phi \right) - \frac{1}{2} (\eta_i z^i + \eta_i \bar{z}^i)^2 \right] + \mathcal{O} \left( \frac{1}{\mu^3} \right) \quad (3.54)$$

and

$$\omega^0_i = -\frac{1}{2\mu^2} a_i + \mathcal{O} \left( \frac{1}{\mu^4} \right)$$
$$\tilde{\omega}^0_j = -\frac{1}{2\mu^2} \tilde{a}_j + \mathcal{O} \left( \frac{1}{\mu^4} \right) \quad (3.55)$$

showing that $\mathcal{K}'$ and $\omega^0_i$, $\tilde{\omega}^0_j$ are of the order $\frac{1}{\mu^2}$. Observe that $\mathcal{K}'$ can be also expressed like

$$\mathcal{K}' = \frac{1}{\mu^2} \hat{\mathcal{K}} + \mathcal{O} \left( \frac{1}{\mu^3} \right) \quad (3.56)$$

with leading term $\hat{\mathcal{K}}$ in the $\frac{1}{\mu}$ expansion as

$$\hat{\mathcal{K}} = (\Phi + \bar{\Phi}) - \frac{1}{2} (z - \bar{z})^i \left( \partial_i \Phi - \bar{\partial}_i \Phi \right) - \frac{1}{2} (\eta_i z^i + \eta_i \bar{z}^i)^2 \quad (3.57)$$

This term $\hat{\mathcal{K}}$ is nothing but the special Kahler potential of the rigid theory.

The next step is to consider the new covariantly holomorphic section $\tilde{V}'^M = e^{\frac{1}{\mu} \tilde{\mathcal{K}}'} \tilde{\Omega}^M$ and expand it in $\frac{1}{\mu}$ power terms like

$$\tilde{V}'^M = \tilde{V}_{(0)}^M + \frac{1}{\mu} \tilde{V}_{(1)}^M + \mathcal{O} \left( \frac{1}{\mu^2} \right) \quad (3.58)$$

To obtain the expression of $\tilde{V}'^M$, we need to know the factors $S_N^M$ and $\tilde{\Omega}^M$; see that the expression of $\mathcal{K}'$ is given by (3.56). Let us first determine $S_N^M$, and turn after to $\tilde{\Omega}^M$. By using eqs(3.29-3.32), we learn that the leading term in the $\frac{1}{\mu}$ expansion of $U_i^M$ (3.31) should, up to a symplectic transformation $\tilde{U}_i^M = S_N^M U_i^N$, have the typical form

$$U_i^M = \frac{1}{\mu} \tilde{U}_i^M + \mathcal{O} \left( \frac{1}{\mu^2} \right) \quad (3.59)$$

This means that there exists a symplectic frame on $M_{SK}$ where we have

$$S_N^M U_i^N = \frac{1}{\mu} \tilde{U}_i^M + \mathcal{O} \left( \frac{1}{\mu^2} \right)$$
$$= \frac{1}{\mu} S_M^N \tilde{U}_i^M + \mathcal{O} \left( \frac{1}{\mu^2} \right) \quad (3.60)$$

with $\tilde{U}_i^M$ as in (3.45). To obtain the explicit expression of the symplectic matrix $S_N^M$ at first order in the expansion parameter $\frac{1}{\mu}$, we solve the constraint equation $S_N^M U_i^N = \frac{1}{\mu} \tilde{U}_i^M$. Using eq(3.45) and the following $\frac{1}{\mu}$- expansion of $U_k^M$

$$U_k^M = \begin{pmatrix}
\frac{-1}{\mu} \eta_k \\
1 - \frac{1}{2\mu} (\eta_i z^i + \eta_i \bar{z}^i) - \frac{1}{2\mu} (z^i \eta_k + \eta_k z^i) \\
\frac{-i}{2\mu} [\eta_i \eta_k - \partial_i \partial_k \Phi] + \cdots
\end{pmatrix} \quad (3.61)$$
we find that $S_{N}^{M}$ is given by (3.20) with the particular matrices

$$A = \begin{pmatrix} 1 & \frac{1}{\mu} \eta_I \\ 0 & \frac{1}{\mu} I_{n \times n} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ -\eta_I & \mu I_{n \times n} \end{pmatrix}$$

(3.62)

and vanishing $(n+1) \times (n+1)$ matrices $B$ and $C$. Having obtained $S_{N}^{M}$, we can now determine $\tilde{\Omega}^{M}$. First by using the particular symplectic change $\tilde{\Omega}^{M} = S_{N}^{M} \Omega^{M}$, we have the new components of the holomorphic sections

$$\begin{pmatrix} \tilde{X}^0 \\ \tilde{X}^I \\ \tilde{F}_0 \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\mu} \eta_I & 0 & 0 \\ 0 & \frac{1}{\mu} I_{n \times n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\eta_I & \mu I_{n \times n} \end{pmatrix} \begin{pmatrix} X^0 \\ X^I \\ F_0 \\ F_I \end{pmatrix}$$

(3.63)

that read explicitly as follows

$$\tilde{\Omega}^{M} = \begin{pmatrix} X^0 + \frac{1}{\mu} X^0 \eta_{J} z^j \\ \frac{1}{\mu} X^0 z^j \\ -\frac{i}{2} X^0 - \frac{i}{2 \mu} \eta_I z^i X^0 + \mathcal{O}(\frac{1}{\mu^2}) \\ \frac{i}{2 \mu} X^0 (\eta_I \eta_J z^i - \partial_i \phi) + \mathcal{O}(\frac{1}{\mu^2}) \end{pmatrix}$$

(3.64)

Then, by performing the particular Kahler transformation (3.50-3.54), we obtain

$$\tilde{\Omega}^{M} = \begin{pmatrix} X^0 \\ \frac{1}{\mu} X^I \\ -\frac{i}{2} X^0 \\ \frac{i}{2 \mu} (\eta_I \eta_J X^J - \partial_i \phi) \end{pmatrix} + \mathcal{O}(\frac{1}{\mu^2})$$

(3.65)

which can be splitted like

$$\tilde{\Omega}^{M} = \tilde{\Omega}^{M}_{(0)} + \frac{1}{\mu} \tilde{\Omega}^{M}_{(1)} + \mathcal{O}(\frac{1}{\mu^2})$$

(3.66)

with

$$\tilde{\Omega}^{M}_{(0)} = X^0 \begin{pmatrix} 1 \\ 0 \\ -\frac{i}{2} \\ 0 \end{pmatrix}, \quad \tilde{\Omega}^{M}_{(1)} = \begin{pmatrix} 0 \\ X^I \\ 0 \\ \frac{i}{2} (\eta_I \eta_J X^J - \partial_i \phi) \end{pmatrix}$$

(3.67)

where $\tilde{\Omega}^{M}_{(0)}$ is the canonical holomorphic section of pure supergravity [57]; and where $\tilde{\Omega}^{M}_{(1)}$ is nothing but the rigid $\hat{\Omega}^{M}$ of eq(3.42) in the special coordinate frame $X^0 = 1$. Notice that $\tilde{\Omega}^{M}$ can be also expanded like

$$\tilde{\Omega}^{M} = \tilde{\Omega}^{M}_{\text{grav}} + \frac{1}{\mu} \tilde{\Omega}^{M}_{(1)} + \mathcal{O}(\frac{1}{\mu^2})$$

(3.68)
but, because of the form of $K'$ which is given by $-\frac{1}{\mu^2} \tilde{K} + \mathcal{O}(\frac{1}{\mu^3})$ as in (3.56), the $\frac{1}{\mu}$ expansion of $\tilde{V}^M = e^{\frac{K'}{2}} \tilde{\Omega}^M$ expands as follows

$$e^{\frac{K'}{2}} \tilde{\Omega}^M = \left( \tilde{\Omega}^M_{(0)} + \frac{1}{\mu} \tilde{\Omega}^M_{(1)} \right) \left[ 1 + \mathcal{O}\left( \frac{1}{\mu^2} \right) \right]$$

(3.69)

which, up to $\frac{1}{\mu^2}$ terms, coincides exactly with the expansion of $\tilde{\Omega}^M$; that is

$$\tilde{V}^M_{\text{grav}} = \tilde{\Omega}^M_{(0)} \quad , \quad \tilde{V}^M_{(1)} = \tilde{\Omega}^M.$$  

(3.70)

Notice moreover that, under the symplectic transformation, the special Kahler potential (3.22) remains invariant and the prepotential $\mathcal{F}(X)$ (3.10) gets expressed in terms of the new variables like

$$\hat{\mathcal{F}} \left( \hat{X} \right) = -i \left( \hat{X}^0 \right)^2 \tilde{\mathcal{F}} \left( \hat{X} \right)$$

(3.71)

If substituting the $\hat{X}^\Lambda$ components by their expressions $\hat{X}(X)$ in terms of the old $X^\Lambda$'s as in (3.64), the prepotential $\hat{\mathcal{F}} \left( \hat{X} \right)$ takes a fat form like

$$\hat{\mathcal{F}} \left( \hat{X} \right) = -i \left( X^0 + \frac{1}{\mu} \eta_I X^I \right)^2 \tilde{\mathcal{F}} \left[ \hat{X}(X) \right]$$

(3.72)

The prepotential remains invariant ($\hat{\mathcal{F}} \left( \hat{X} \right) = \mathcal{F}(X)$) since due to homogeneity and the particular form of $S^M_N$ where $B = C = 0$ and $D^T A = I_{id}$, we have

$$\hat{\mathcal{F}} \left( \hat{X} \right) = \frac{1}{2} F^\Lambda D^T A X^\Lambda = \frac{1}{2} X^\Lambda F_\Lambda = \mathcal{F}(X)$$

(3.73)

For an explicit study of the rigid limits of the coupling matrices $\mathcal{N}_{\Lambda \Sigma}$ and $\mathcal{U}^{MN}$, see appendix C.

### 4 Isometries of quaternionic Kahler manifold

In this section, we focus on the gauging quaternionic isometries in $\mathcal{N} = 2$ gauged supergravity and develop two things: First, we build a new family of real $4r$-dimensional quaternionic Kahler manifolds $M^{(ADE)}_{QR}$ classified by rank $r$ finite dimensional $ADE$ Lie algebras where the integer $r$ stands for the number of hypermultiplets. Second, we give a D-brane realisation of gauged quaternionic isometries and an interpretation of the embedding tensor $\vartheta^\Lambda_M$ in terms of type IIA/IIB mirror symmetry.

Recall that the gauging of two abelian quaternionic isometries offers a manner to break $\mathcal{N} = 2$ supersymmetry partially [18, 27]; so, we first consider the example of the quaternionic geometry $SO(1,4) / SO(4)$ associated with one hypermultiplet ($n_H = 1$). This matter supermultiplet has four real scalar fields $Q^u$ ($u = 1, 2, 3, 4$) hosted by two $SU(2)_R$
representations namely a real isosinglet $\varphi$ and a real isotriplet $\vec{\phi} = (\phi^a)$ with $a = 1, 2, 3$.

$$Q^u \equiv \varphi \oplus \vec{\varphi} \quad (4.1)$$

Then, we develop the study of a new $M^{(ADE)}_{QK}$ manifolds, generalising $SO(1, 4)/SO(4)$, and associated with a matter sector having $n_H = r$ hypermultiplets where $r$ stands for the ranks of ADE. After that, we turn to the study of the gauging of abelian quaternionic isometries of this family of quaternionic manifolds by focussing the simplest $M^{(A_1)}_{QK}$ with $A_1$ the Lie algebra of $SU(2)$.

### 4.1 Quaternionic Kahler manifold $M^{(n_H)}_{QK}$: case $n_H = 1$

In $\mathcal{N} = 2$ supergravity theory with one hypermultiplet $H_{N=2}$, whose four real scalars $Q^u$ parameterised in terms of $SU(2)_R$ singlet $\varphi$ and triplet $\phi^a$ as in (4.1), the real 4-dimensional matter manifold $M_{QK}$ has an $SU(2) \times SP(2, \mathbb{R})$ holonomy group and is given by the coset space

$$M^{(n_H=1)}_{QK} = \frac{SO(1, 4)}{SO(4)} \quad , \quad n_H = 1 \quad (4.2)$$

In what follows, we describe some relevant properties of this manifold which are useful for the study of: (i) global isometries of the hyperKahler metric and hyperKahler 2-form as well as gaugings. (ii) the ADE-type generalisation of $SO(1, 4)/SO(4)$; and (iii) the partial supersymmetry breaking.

#### 4.1.1 HyperKahler metric of $M^{(n_H)}_{QK}$ and global isometries

Using the group homomorphism $SP(2, \mathbb{R}) \sim SU(2)'$, the holonomy group of the $M^{(n_H=1)}_{QK}$ manifold becomes $SU(2) \times SU(2)'$ and is homomorphic to $SO(4, \mathbb{R})$. By identifying the two $SU(2)$ factors making this orthogonal group, we are left with one $SU(2)$ symmetry which is nothing but the $SU(2)$ R-symmetry of $\mathcal{N} = 2$ supersymmetry algebra. Under this identification, the four real scalar fields $Q^u$ of the hypermultiplet are hosted by the two real irreducible representations of $SU(2)$ mentioned above namely the isosinglet $\varphi$ and the isotriplet $\phi^a$ like

$$\varphi = Q^u \delta^0_u \quad , \quad \phi^a = Q^u \delta^a_u \quad (4.3)$$

A typical form of the quaternionic Kahler metric $ds^2 = h_{uv}dQ^udQ^v$ of the 4-dimensional space $SO(1, 4)/SO(4)$, parameterised with these real coordinate variables, is given by [30]

$$ds^2 = \frac{1}{2} d\varphi^2 + \frac{e^{2\varphi}}{2} (d\phi^a \delta_{ab} d\phi^b) \quad (4.4)$$

From this relation, we read the expression of the local metric

$$h_{uv} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\varphi} \delta_{ab} \end{pmatrix} \quad , \quad h^{uv} = 2 \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\varphi} \delta^{ab} \end{pmatrix} \quad (4.5)$$

25
it is diagonal and depends only on the isosinglet component \( \varphi \), but not on the isotriplet \( \phi^a \). Moreover, the dependence on \( \varphi \) is very special in the sense that it is a function of the remarkable form \( e^{2\varphi} \). The metric \( ds^2 \) can be also expressed like

\[
ds^2 = \frac{1}{2} \left( d\varphi^2 + \vec{\omega}_1 \cdot \vec{\omega}_1 \right)
\]

where the real 1-form isovector \( \vec{\omega}_1 = (\omega_1^a) \) is given by \( \omega_1^a = e^\varphi d\phi^a \). The normal quadratic expression (4.6) reflects the property that the metric can be factorised in terms of vielbeins as in eq(4.17) given below. The scalar product \( \vec{\omega}_1 \cdot \vec{\omega}_1 \) in above relation can be also thought of as following from the trace \( tr_{SU(2)} (\omega_1 \otimes \omega_1) \) over quadratic product of Pauli matrices \( \tau_a \) involved in the matrix expansion \( \omega_1 = \tau_a \omega_1^a \); and \( ds^2 = \frac{1}{4} tr_{SU(2)} (\omega_1 \otimes \omega_1) \) with

\[
\omega_1 = e^{\varphi} \tau_a d\phi^a \quad , \quad \vec{\omega}_1 = d\varphi \tau_0 + e^{\varphi} \tau_a d\phi^a
\]

where \( \tau_0 \) stands for the \( 2 \times 2 \) matrix identity.

- **Global isometries**
  The metric \( ds^2 \) has two special subsets of global isometries that we describe in what follows; some of these isometries will be used later on when we study the gauging of field translations.

  (i) **SO(3) symmetry group**
  The metric (4.6) is manifestly preserved by global rotations of the three component fields of the isotriplet \( \phi^a \) of hypermultiplet; but living invariant the isosinglet \( \varphi \). Explicitly,

\[
\begin{align*}
\phi_a & \rightarrow \phi'_a = R^b_a \phi_b \\
\varphi & \rightarrow \varphi' = \varphi
\end{align*}
\]

where \( R^b_a = e^{w^a J_a} \) is a \( 3 \times 3 \) orthogonal matrix with global parameters \( w^a \). The underlying three dimensional Lie algebra of this \( SO(3) \) invariance is generated by three generators \( J_a \); it is a non abelian symmetry isomorphic to the Lie algebra of the \( SU(2) \) R- symmetry with the usual Lie bracket

\[
[J_a, J_b] = i\varepsilon_{abc} J_c
\]

(ii) **Translation symmetry group \( T_3 \)**
  The metric \( ds^2 \) is as well manifestly invariant under global translations of the three \( \phi^a \) field variables as follows

\[
\begin{align*}
\phi_a & \rightarrow \phi'_a = \phi_a + c_a \\
\varphi & \rightarrow \varphi' = \varphi
\end{align*}
\]

where \( c_a = \partial_\mu c^a = 0 \)

The three dimensional Lie algebra of \( T_3 \) is abelian; it is generated by three commuting translation generators \( T_a = \frac{\partial}{\partial \phi^a} \).
• **Beyond SO(3) × T₃ isometry**

The combination of the two above global symmetries of (4.6) is therefore given by

\[
\begin{align*}
\phi_a & \rightarrow \phi'_a = R^b_a \phi_b + c_a \\
\varphi & \rightarrow \varphi' = \varphi
\end{align*}
\]

(4.11)

with \((R^b_a, c_a)\) belonging to the non abelian Euclidean group \(SO(3) \ltimes T₃\) symmetry with non abelian factor \(SO(3) \simeq SU(2)_R\). This is a 6-dim Lie group with three non commuting rotation generators \(J_a\) and three commuting translations \(T_a = \partial / \partial \phi^a\); the \(T_a\)'s transform like a 3-vector under rotations.

Notice that under global shift of the isosinglet like \(\varphi \rightarrow \varphi + c_0\) with \(dc_0 = 0\), the invariance of the above hyperkahler metric \(h_{uv}\) requires a real rescaling of the isotriplet like

\[
\begin{align*}
\varphi' &= \varphi + c_0 \\
\phi'_a &= e^{-c_0} \phi_a + c_a, \quad dc_0 = 0
\end{align*}
\]

(4.12)

But this invariance leads to a non abelian Lie algebra and, like the \(SO(3)\) rotations, it is not relevant in the study of abelian gauging of the quaternionic isometries. Non commutativity of the scale transformations can be viewed by figuring out the generators of infinitesimal transformations of the coordinate change (4.12) namely

\[
\delta_G \varphi = c_0, \quad \delta_G \phi_a = c_a - c_0 \phi_a
\]

(4.13)

The four \(T_u\) generators of this infinitesimal coordinate change are given by

\[
\begin{align*}
T_a &= \frac{\partial}{\partial \phi^a}, \quad T_0 = \phi^a \frac{\partial}{\partial \phi^a} + \phi_a \frac{\partial}{\partial \phi^a}
\end{align*}
\]

(4.14)

obeying the non abelian commutation relations algebra

\[
[T_a, T_0] = T_a, \quad [T_a, T_b] = 0
\]

(4.15)

### 4.1.2 HyperKahler 2-forms of \(M_{QK}^{(n_H)}\): case \(n_H = 1\)

The hyperKahler metric \(h_{uv}\) of the 4-dimensional quaternionic Kahler manifold \(SO(1, 4)/SO(4)\) can be also expressed in different form by using vielbein formalism. In this approach, vielbein 1-forms \(E^A_A\) carry \((\frac{1}{2}, \frac{1}{2})\) quantum numbers under SO(4) and expand in the local real \(Q^u\)-coordinates of the manifold as follows

\[
E^A_A = E^A_A[u] dQ^u
\]

(4.16)
with \( \mathcal{E}^A_{\bar{A}[u} \) the components along the \( dQ^u \) directions. Using this method, the metric \( h_{uv} \) and the antisymmetric components \( K^{a}_{uv} \) of the isotriplet of Kahler 2-forms \( K^a_2 = K^a_{uv}dQ^u \wedge dQ^v \) (hyperKahler 2-form) of the quaternionic Kahler \( SO(1,4)/SO(4) \) read as follows

\[
\begin{align*}
    h_{uv} &= \mathcal{E}^A_{\bar{A}[u} \mathcal{E}^B_{\bar{B}[v} \varepsilon_{AB} \varepsilon^{\bar{A} \bar{B}} \\
    K^a_{uv} &= (\tau^a \varepsilon)_{AB} \mathcal{E}^A_{\bar{A}[u} \mathcal{E}^B_{\bar{B}[v} \varepsilon^{\bar{A} \bar{B}}
\end{align*}
\]  

(4.17)

The non vanishing components \( K^a_{0b} \) and \( K^a_{bc} \) of the isotriplets of the hyperKahler 2-form are respectively proportional to \( \delta^a_b \) and \( \varepsilon^{abc} \). The explicit expressions of 1-form vielbein quartet (4.16), that splits in terms of the isosinglet \( d\varphi \) and isotriplet \( d\phi^a \) like

\[
\mathcal{E}^A_{\bar{A}} = \mathcal{E}^A_{\bar{A}0} d\varphi + \mathcal{E}^A_{\bar{A}a} d\phi^a
\]

(4.18)

are obtained by using eq(4.5) and solving the following relation

\[
\begin{align*}
    h_{uv} &= \mathcal{E}^A_{\bar{A}[u} \mathcal{E}^B_{\bar{B}[v} \varepsilon_{AB} \varepsilon^{\bar{A} \bar{B}}
\end{align*}
\]

(4.19)

We find

\[
\begin{align*}
    \mathcal{E}^A_{\bar{A}0} &= \frac{1}{2} \varepsilon^{AB} \delta_{BA} \\
    \mathcal{E}^A_{\bar{A}a} &= -\frac{i}{2} \varepsilon_B^a \delta_{BA} (\tau_a)_{BA}
\end{align*}
\]

(4.20)

where one recognises the term \( \mathcal{E}^A_{\bar{A}a} d\phi^a \) as just \( \frac{i}{2} \delta^a_B (\omega_1)^B_A \) with 1-form matrix \( (\omega_1)^B_A = (\tau_a)_{BA} \omega_1^a \) precisely as in eq(4.6). By using these expressions, we learn that we can express \( \mathcal{E}^A_A \) like \( \delta^B_B \mathcal{E}^A_A \) with

\[
\begin{align*}
    \mathcal{E}^A_B &= \delta^A_B d\varphi - i e^a (\tau_a)^B_A d\phi^a \\
    &= \delta^A_B d\varphi - i \omega_1^a (\tau_a)^B_A
\end{align*}
\]

(4.21)

Furthermore, thinking of the isotriplet of 1-forms \( \omega_1^a \) as an SU(2) gauge connection \( \omega_1^a = \omega_u^a dQ^u \) on the quaternionic manifold \( SO(1,4)/SO(4) \) and of the hyperKahler 2-forms \( K^a_2 = K^a_{uv} dQ^u \wedge dQ^v \) as proportional to the SU(2) gauge curvature

\[
\Omega_2^a = d\omega_1^a + \frac{1}{2} \varepsilon^{abc} \omega_1^b \wedge \omega_1^c
\]

(4.22)

of the connection \( \omega_1^a \), we can write down a relation between the \( \Omega_{us}^a \) components of the curvature and the metric \( h_{uv} \) namely

\[
\begin{align*}
    h^{st} \Omega_{us}^a \Omega_{te}^b &= -\delta^a_b h_{uv} - \varepsilon^{abc} \Omega_{uv}^c
\end{align*}
\]

(4.23)

This relationship follows from properties of the three complex structures \( \mathcal{J}^a \) living on quaternionic Kahler manifold obeying

\[
(\mathcal{J}^a)^r_s (\mathcal{J}^b)^t_s = -\delta^a_b \delta^r_s + \varepsilon^{abc} (\mathcal{J}^c)^r_s
\]

(4.24)
By taking the trace over the isotriplet indices in (4.23), we can express the quaternionic Kahler metric like

\[ h_{uv} = -\frac{1}{3} h_{st} \Omega^a_{us} \delta_{ab} \Omega^b_{tv}. \]

By using eq(4.4) we learn as well that the 1-form \( \omega^a \) is indeed given by \( e^{\varphi} d\varphi^a \). This property can be checked by substituting \( \omega^a \) in the curvature 2-forms, we obtain

\[ \Omega^a_2 = e^{\varphi} d\varphi \wedge d\varphi^a + \frac{e^{2\varphi}}{2} \varepsilon^{abc} d\varphi^b \wedge d\varphi^c \]  

from which we read the curvature components

\[ \Omega^a_{0u} = \frac{1}{2} \delta^a_u e^{\varphi}, \quad \Omega^a_{bc} = \frac{e^{2\varphi}}{2} \varepsilon^{abc}. \]  

4.2 Building metric for \( M_{QK}^{(n_H)} \) : case \( n_H > 1 \)

In this subsection, we use known results on 2d- integrable models and the geometric engineering method of 4d QFTs to reach two things: First show that there is a correspondence between the 4- dimensional quaternionic geometry of \( SO(1,4)/SO(4) \) manifold (\( n_H = 1 \)) and the rank \( r = 1 \) of a hidden SU(2) symmetry. Second, use this correspondence to study a class of 4r- dimensional quaternionic geometries describing a matter sector of the \( \mathcal{N} = 2 \) supergravity with \( n_H = r \) hypermultiplets. Concretely, we give an extension of the analysis done above, for the \( SO(1,4)/SO(4) \) geometry of the \( \mathcal{N} = 2 \) supergravity with \( n_H = 1 \), to the case of a 4r quaternionic geometry involving several hypermultiplets; say \( n_H = r > 1 \). This extension has been motivated by a formal similarity with two well established approaches successfully used in QFT literature namely: (i) integrable 2d-Toda theories generalising Liouville theory and classified by ADE Lie algebras; and (ii) the geometric engineering of \( \mathcal{N} = 2 \) supersymmetric QFT in 4d describing the Coulomb branch of type II string compactified on local Calabi-Yau threefolds (CY3) classified as well by ADE Lie algebras. To make a general idea about the method, let us first give a very brief comment on the two above mentioned approaches and turn after to present our extension.

For the 2d- integrable Toda models, which are classified by Lie algebras and whose solvability is known to be due to existence of rich symmetries, the integrability of the non linear 2d- Toda field equations of motion having the form,

\[ \frac{\partial^2 u_i}{\partial z^+ \partial z^-} - \kappa \exp \sum_{j=1}^{r} (K_{ij} u_j) = 0, \quad r \geq 1 \]

may be also understood in terms of existence of a way to linearise these equations like \( \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \). In these relations, the \( u_i = u_i (\tau, \sigma) \) are the 2d- Toda fields, \( K_{ij} \) is the Cartan matrix of the underlying ADE Lie algebra with rank \( r \) and \( A_{\pm} = A_{\pm} [u] \).
related to the $u_i$-fields as in footnote$^3$. This linearised equation is remarkably interpreted as the flatness condition of the curvature $F_{+-}$ of two component gauge fields given by $A_{\pm}$. In this gauge language with a Lax pair $(A_+, A_-)$ valued in the Lie algebra of ADE, the curvature $F_{+-}$ is given by the commutator $[\partial_+ + A_+, \partial_- + A_-]$ and the flatness condition $F_{+-} = 0$ leads precisely the above Toda field equations $[58, 59]$ recovering the Liouville theory as just the leading $r = 1$ case.

For the case of the geometric engineering of $\mathcal{N} = 2$ supersymmetric QFT$_4$’s, there is also a classification based on Lie algebras. Though the apparent picture looks different from the Toda theory as here one works with algebraic geometry techniques, the key idea is formally similar in the sense it uses as well particular properties of the ADE Lie algebras. In this method, a class of $\mathcal{N} = 2$ supersymmetric QFT$_4$ are engineered as low energy effective field theories resulting from type IIB string compactified on local Calabi-Yau threefolds preserving eight supersymmetric charges $[60–62]$. The local CY3s are generally speaking realised as ALE surfaces with an ADE singularity fibered on complex projective line. The resulting $\mathcal{N} = 2$ supersymmetric field theory has a gauge symmetry dictated by the type of the singularity. For the example of the particular local complex surface $S$ defined by $z_1 z_2 = z_3^{r+1}$ where $z_1$, $z_2$, $z_3$ are three complex variables; there is an $A_r$ singularity at the origin of $S$ and then an SU$(r+1)$ gauge symmetry at the level of the 4d field theory. The maximal deformation of this singularity is given by

$$z_1 z_2 = z_3^{r+1} + \sum_{i=1}^{r} a_i z_i^r , \quad r \geq 1$$

where the $a_i$'s are $r$ complex moduli capturing information on the resulting $\mathcal{N} = 2$ supersymmetric QFT$_4$. Under this complex deformation, the singularity is lifted and the SU$(r+1)$ gauge symmetry of the $\mathcal{N} = 2$ super QFT$_4$ gets broken down to the abelian U$(1)^r$ gauge symmetry. The same picture holds for the case of generic ADE geometries; all of them contain the SU$(2)$ theory as corresponding to the leading term of the family; i.e: $r = 1$ case.

In our way of dealing with building quaternionic Kahler geometries associated to a number of hypermultiplets $n_H > 1$, we borrow the idea behind the construction of Toda theories and ADE geometries by thinking of the $SO(1, 4)/SO(4)$ geometry as corresponding to the leading $r = 1$ case of a family of spaces $M_{QK}^{(ADE)}$ indexed by ADE rank with $r \geq 1$. In this view, the number $n_H$ of hypermultiplets is therefore linked with the rank $r$ of finite

---

$^3$ The Lax pair leading to the Liouville equation are given by $A_+ = \frac{\partial u}{\partial z} h + \kappa E_+$ and $A_- = e^{u} E_-$ where $h, E_\pm$ generate an su$(2)$ Lie algebra with commutators $[h, E_\pm] = \pm 2E_\pm$ and $[E_+, E_-] = h$. For 2d Toda theories with ADE type, the corresponding Lax pair leading the 2d Toda field equations are given by $A_+ = \sum_{i=1}^{r} h_i \frac{\partial u}{\partial z} + \kappa \sum_{i=1}^{r} E_{+i}$ and $A_- = \sum_{i=1}^{r} E_{-i} \exp \left( \sum_{j=1}^{r} K_{ij} u_j \right)$ where $h_i, E_{\pm i}$ obey the ADE commutation relations $[h_i, E_{\pm j}] = \pm K_{ij} E_{\pm j}$ and $[E_{+i}, E_{-j}] = \delta_{ij} h_j$. 

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dimensional ADE Lie algebras
\[ n_H = \text{rank}(ADE) \] (4.27)

As such, one obtains a family of \( 4n_H \)-dimensional geometries characterised by the rank of ADE all of them containing \( SO(1,4)/SO(4) \) as the leading \( n_H = 1 \) which in turns is associated with the \( su(2) \) Lie algebra having \( r = 1 \).

4.2.1 Quaternionic Kahler ADE type models

A generalisation of the \( SO(1,4)/SO(4) \) analysis given above for \( n_H = 1 \) to a higher number of hypermultiplets may be achieved by mimicking the construction of 2d- Toda field theories [63–65] which, in turns, is obtained by extending the well known 2d- Liouville theory [59,66,67]. Recall that 2d Liouville theory is associated with a hidden \( su(2) \equiv A_1 \) behind its integrability; the generalisation of this particular integrable 2d- theory to higher finite dimensional dimensional ADE Lie algebras is given by the integrable 2d- Toda theories having hidden ADE symmetries. Our main motivation for the formal correspondence between the \( SO(1,4)/SO(4) \) metric and 2d- Liouville equation
\[ \frac{d^2 u}{dz \bar{z}} - \kappa e^{2u} = 0, \]
with a real 2d- field \( u = u(z, \bar{z}) \), is of the factor \( e^{2\varphi} \) in the quaternionic metric
\[ ds^2 = \frac{1}{2} \left( d\varphi^2 + e^{2\varphi} d\phi^a d\phi^a \right) \] (4.28)

From the view of 2d- Liouville theory, \( e^{2u} \) has an interpretation in terms of underlying \( su(2) \) symmetry that captures the solvability of Liouville equation formulated in terms of a vanishing curvature of an \( su(2) \) valued gauge connection. Extending this observation to the \( SO(1,4)/SO(4) \) metric, the presence of the factor \( e^{2\varphi} \) in (4.28) is then very suggestive; it can be imagined as
\[ e^{2\varphi} = e^{K_{11}\varphi_1}, \quad \varphi_1 \equiv \varphi \] (4.29)
with \( K_{11} = 2 \) referring to the usual ” 1 × 1” Cartan matrix of \( su(2) = A_1 \) Lie algebra which we write as
\[ K_{A_1} = 2 \] (4.30)

In other words, the metric (4.28) may be formally put in 1 to 1 with rank of the \( su(2) \) Lie algebra and therefore we have the following correspondence
\[ n_H = 1 \quad \longleftrightarrow \quad \text{rank}(su_2) = 1 \] (4.31)

---

4 Notice that for the particular case where the u- field has no \( \sigma \)-dependence; that is \( u = u(\tau) \), the field equation \( \frac{d^2 u}{dz \bar{z}} - \kappa e^{2u} = 0 \) reduces to the 1- dim \( \frac{d^2 u}{d\tau^2} - \kappa e^{2u} = 0 \) and follows from the variation of the lagrangian \( L = \frac{1}{2} \left( \frac{\partial u}{\partial \tau} \right)^2 + \kappa e^{2u} \).

5 Notice also that if thinking of the 4d space time fields \( \varphi(t, \bar{x}) \) and \( \phi^a(t, \bar{x}) \) as having no \( \bar{x} \) space dependence and moreover related to the Liouville field \( u(t) \) like \( \varphi = u(t) \) and \( \phi^a = tM^a\sqrt{2} \) with some constants \( M^a \); then eq(4.28) reads as \( ds^2 = 2Ldt^2 \) where \( L \) is precisely the lagrangian of footnote 3 with \( \kappa = M^aM^a \).
This correspondence valid for \( n_H = 1 \) allows to ask whether this construction can be extended to higher \( n_H \)'s and higher ranks of finite dimensional ADE Lie algebras. An example of such extension is given by the \( su(1 + n_H) = A_r \) Lie algebra with rank \( r = n_H \). Below, we construct the particular extension associated with \( su(1 + n_H) \) Lie algebra; a similar construction can be done for generic ADE Lie algebras. For the case of \( su(1 + n_H) \), the Cartan matrix \( K_{A_r} \) generalising \( K_{A_r} \) (4.30) is given by

\[
K_{A_r} = \begin{pmatrix}
2 & -1 \\
-1 & 2 \\
& & \ddots \\
& & & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}
\]  

(4.32)

The simplest example coming after (4.28-4.29) is given by the 8- dimensional manifold based on the \( su(3) = A_2 \) Lie algebras with Cartan matrix as

\[
K_{A_2} = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\]  

(4.33)

The geometry of this real 8d manifold involves two hypermultiplets \( n_H = 2 \) with local coordinate fields described by 4+4 real scalars \( Q_u^i \) and \( Q_u^{a} \) containing two \( SU(2)_R \) isosinglets denoted as \( \varphi_i = (\varphi_1, \varphi_2) \); and two \( SU(2)_R \) isotriplets \( \phi_i^a = (\phi_1^a, \phi_2^a) \),

\[
\varphi_i = Q_u^{a} \delta_i^a \\
\phi_i^a = Q_u^{a} \delta_i^a
\]  

(4.34)

In the generic \( su(1 + n_H) = A_{n_H} \) model, the generalisation of the four scalars \( (\varphi, \phi^a) \) for one multiplet to the case of \( n_H \) hypermultiplets is given by the coordinate system \( (\varphi^r, \phi_i^a) \) with index \( r = 1, \ldots, n_H \). The quadratic \( d\varphi d\varphi \) and \( d\phi^a d\phi^a \) in the metric (4.28) can be extended like

\[
\begin{array}{c|c}
\text{su}(2) & su(1 + n_H) \\
\hline
d\varphi d\varphi & \frac{1}{2} \sum_{r,s=1}^{n_H} K_{rs} d\varphi^r d\varphi^s \\
\hline
d\phi^a d\phi^a & \frac{1}{2} \sum_{r,s=1}^{n_H} K_{rs} d\phi^a_r d\phi^a_s
\end{array}
\]  

(4.35)

The new real coordinates \( (\varphi^r, \phi_i^a) \) carry a vector index \( r \) transforming in the vector representation of \( SO(n_H, \mathbb{R}) \) isotropy group. The appearance of Cartan matrix \( K_{rs} \) can be exhibited explicitly by using simple roots \( \vec{\alpha}_r \) of the \( su(1 + n_H) \) Lie algebra related to the Cartan matrix as follows:

\[
\vec{\alpha}_r . \vec{\alpha}_s = K_{rs}
\]  

(4.36)
This feature allows to think about the local coordinates field variables of the quaternionic Kahler manifold $M_Q^{(n_H)}$ as follows

$$\vec{\phi} = \vec{\alpha}_r \phi^r$$

$$\vec{\phi}^a = \vec{\alpha}_r \phi^a$$

and generally for the matter supermultiplets

$$\vec{H}^{N=2} = \vec{\alpha}_r H^r_{N=2}$$

So similar expressions to (4.37) can be also written down for the fermionic partners $\xi^A_r$, the hyperini (2.9); we have $\xi^A_r = \vec{\alpha}_r \xi^A_r$. By using the $n_H$ fundamental weights $\vec{\lambda}_r$ of the $su(1 + n_H)$ Lie algebra satisfying the property $\vec{\lambda}_r.\vec{\alpha}_s = \delta_{rs}$, we have

$$\phi^r = \vec{\lambda}_r.\vec{\phi}$$

$$\phi^a = \vec{\lambda}_r.\vec{\phi}^a$$

$$\xi^A_r = \vec{\lambda}_r.\vec{\xi}^A$$

and $H^{N=2} = \vec{\alpha}_r H^{N=2}$ for hypermultiplets.

### 4.2.2 Metrics and hyperKahler 2-forms

Clearly, the generalisation (4.35) is a particular extension of $SO(1,4)/SO(4)$ manifold in the sense it depends on the real $n_H$-dimensional vectors $\vec{\alpha}_r$ whose intersection matrix $\vec{\alpha}_r.\vec{\alpha}_s$ is the $n_H \times n_H$ symmetric matrix (4.32). To build an extension of the $SO(1,4)/SO(4)$ metric (4.28) to geometries with $n_H$ hypermultiplets classified by the intersection matrix $K_{rs}$ of the $su(1 + n_H)$ Lie algebra, we extend the $e^{2\varphi}$ factor in (4.28) like

$$e^{2\varphi} \equiv e^{\vec{\alpha}.\vec{\varphi}} \rightarrow \sum_{i=1}^{n_H} e^{\vec{\alpha}_i.\vec{\varphi}}$$

(4.40)

For the case $n_H = 1$, we have one simple root $\vec{\alpha}_1 \equiv \vec{\alpha}$ and (4.37) reduces to $\vec{\phi} = \vec{\alpha}_r \varphi^r$ and $\vec{\phi}^a = \vec{\alpha}_r \phi^a$. Moreover, because of the property $\vec{\alpha}.\vec{\alpha} = 2$, we have $\vec{\alpha}_r.\vec{\varphi} = 2 \varphi$ and then $e^{2\varphi} = e^{\vec{\alpha}.\vec{\varphi}}$. Therefore, a generalisation of (4.28) to the case of several $n_H$ hypermultiplets is obtained by using eqs(4.35-4.40); it reads as follows

$$ds^2_{n_H} = \frac{1}{4} \left( d\vec{\varphi}.d\vec{\varphi} + f(\varphi) d\vec{\phi}.d\vec{\phi} \right)$$

(4.41)

with $f(\varphi)$, depending on the number $n_H$, given by

$$f(\varphi) = \sum_{i=1}^{n_H} e^{\vec{\alpha}_i.\vec{\varphi}}$$

(4.42)

The metric $ds^2_{n_H}$ has global isometries generalising those of $SO(1,4)/SO(4)$; in particular the following abelian ones

$$\phi^r_a \rightarrow \phi^r_a' = \phi^r_a + c^r_a$$

$$\varphi^r \rightarrow \varphi^r' = \varphi^r$$

$$\partial_\mu c^r_a = 0$$

(4.43)
extending eq(4.10). Moreover, using positivity of $e^{\vec{a},\vec{b}}$, one can rewrite the above generalised metric in the following way

$$ds^2 = \frac{1}{4} (d\vec{\varphi}.d\vec{\varphi} + \vec{\omega}_1^a \vec{\omega}_1^a)$$

with 1-form $\vec{\omega}_1^a = \sum_{r=1}^{n_H} \vec{\alpha}_r \omega_1^{ar}$ like

$$\vec{\omega}_1^a = d\vec{\varphi} \sqrt{f(\varphi)}$$

$$\omega_1^{ar} = d\varphi^r \sqrt{f(\varphi)}$$

Such construction done for $su(n_H + 1)$ Cartan matrices $K_{ij}^{(A_n)}$ given by the particular family (4.32) extends straightforwardly to Cartan matrices $K_{ij}^{(ADE)}$ of all finite dimensional ADE Lie algebras. Using the 1-form vielbein formalism on $M_{QK}^{(n_H)}$, 

$$\mathcal{E}^{A\bar{A}} = \mathcal{E}_{us}^{A\bar{A}} dQ^{us}$$

with

$$\mathcal{E}_{A|R}^{A|0} = \frac{1}{2} \alpha_i^s \varepsilon^{AB} \delta_{B\bar{A}}$$

$$\mathcal{E}_{A|rs}^{A|0} = - \frac{1}{2} \alpha_i^s f(\varphi) \delta_{B\bar{A}} (\tau_a)_{\bar{A}}$$

with the $\alpha_i^s$ the components of the simple root vectors $\vec{\alpha}_i$, we can also write down the metric of $M_{QK}^{(n_H)}$, 

$$ds^2 = h_{urvs}^{(n_H)} dQ^{ur} dQ^{vs}$$

and the components $(K^{(n_H)})^{a}_{urvs}$ of the hyperKahler 2-forms

$$K_2^{(n_H)} = (K^{(n_H)})^{a}_{urvs} dQ^{ur} \wedge dQ^{vs}$$

For the metrics $h_{urvs}^{(n_H)}$, we have the following factorisation

$$h_{urvs}^{(n_H)} = \mathcal{E}_{A|ur}^{Ar} \delta_{r's'} \mathcal{E}_{B|us}^{Bs'} \varepsilon_{AB} \varepsilon^{\bar{A}\bar{B}}$$

and for $(K^{(n_H)})^{a}_{urvs}$, we have

$$(K^{(n_H)})^{a}_{urvs} = (\tau^a)_{AB} \mathcal{E}_{A|ur}^{Ar} \delta_{r's'} \mathcal{E}_{B|us}^{Bs'} \varepsilon^{\bar{A}\bar{B}}$$

More comments regarding these generalized manifolds and the ADE classification will be given in the section devoted to the conclusion and discussions.

### 4.3 Gauging quaternionic isometries

Following [18], partial breaking of $\mathcal{N} = 2$ supersymmetry in supergravity theory can be nicely realised by gauging two abelian quaternionic isometries of the scalar manifold. This scenario requires therefore a hypermatter sector that couples to gauge and gravity branches. In this subsection, we study the gauging of quaternionic isometries of $\mathcal{N} = 2$ supergravity with $n_V$ vector supermultiplets ($n_V = n$) and $n_H$ hypermultiplets. We also give a realisation of this system in terms of D-branes wrapping cycles in type II strings compactified on Calabi-Yau threefolds.
4.3.1 Gauging translations of $M_{QK}^{(n_H)}$: case $n_H = 1$

We begin by considering the four real scalar fields $Q^u = (\varphi, \phi^a)$ parameterising the real 4-dim quaternionic Kahler manifold $SO(1, 4)/SO(4)$ with metric (4.4), and study the gauging of the three global translations $\delta_{G}^{O^a} = c^a$ under which the metric $ds^2 = h_{uv}dQ^udQ^v$ and the corresponding hyperKahler 2-form remain invariant. This gauging of abelian quaternionic isometries is accompanied with three abelian gauge fields $C^a_{\mu} = (C^x_{\mu}, C^y_{\mu}, C^z_{\mu})$ related to the gauge fields $A^M_{\mu}$ of gravity and Coulomb sectors by the so called embedding tensor $\vartheta^a_M$ introduced in section 2 and which will described with details later on. But before going into the gauging process, let recall some useful features for present analysis: First, the three global translations (4.10) preserve (4.4) and are generated by three commuting operators

$$T_a = \frac{\partial}{\partial \phi^a}, \quad T_a = \delta_u^a T_u, \quad T_u = \frac{\partial}{\partial Q^u}$$

(4.52)

Second, the differential of the field shifts $\delta_{G}Q^u = c^a \delta_u^a$ obey the natural property

$$d (\delta_{G}Q^u) = 0$$

(4.53)

due to

$$dc^a = dx^\mu \partial_\mu c^a = 0$$

(4.54)

The real constants $c^a$ are just shifts of the origin of the $Q^u$- coordinate frame; and alike the $Q^u$’s which, in type II string on CY3, have an interpretation in terms of the geometric and stringy moduli, the gauging of two of these three $c^a$’s can be given as well an interpretation in type II strings on CY3 with a homological basis of symplectic 3-cycles given by

$$\Pi^M_3 = (\mathfrak{A}^A_3, \mathfrak{B}^A_3)$$

(4.55)

The $c^a$’s can be interpreted in terms of turning on fluxes $\pi^M$ of an external field strength 3-forms, say $H_3 = dB_2$; the fluxes $\pi^M$ are through the 3-cycles $\Pi^M_3$ and are related to $c^a$ as follows

$$c^a = \vartheta^a_M \pi^M$$

(4.56)

with

$$\pi^M = \int_{\Pi^M_3} H_3, \quad \pi^M = (p^A, q_A)$$

(4.57)

In (4.56), the real quantity $\vartheta^a_M$ is the embedding tensor carrying two indices $(a, M)$; it transforms in the bi-fundamental of $SO(3) \times SP(2n + 2)$; other comments on this remarkable tensor will be given below; and more general properties can be found in [35].

By gauging the scalar field translations, the above three parameters $c^a$ are no longer constants; they are coordinate dependent and, to avoid confusion, they will be denoted here below as follows

$$c^a \rightarrow -\xi^a (x)$$

(4.58)
So the local field translations in the quaternionic Kahler manifold $SO(1, 4)/SO(4)$ read as

$$
\delta_G Q^u = -\xi^a \delta_a^u, \quad \partial_\mu \xi^a = 0 \quad (4.59)
$$

By exhibiting the local gauge parameters $\xi^a$, the variation operator $\delta_G$ can be expressed in terms of the Killing generators $T_a$ like $\delta_G = \xi^a T_a$; and the action of these Killing operators on the field coordinates $Q^u$ is given by $T_a Q^u = \kappa^u_a$ with Killing vectors $\kappa^u_a$ as follows

$$
\kappa^u_a = -\delta^u_a \quad (4.60)
$$

Notice that under gauging of the translations, the differential the coordinate field variations $d(\delta_G Q^u)$ is no longer vanishing; by using (4.59), we get

$$
d(\delta_G Q^u) = -d\xi^a \delta^u_a \quad (4.61)
$$

To have the property (4.53) valid even for local field translations, we need to introduce a triplet $C^a_1 = C^a_1 \delta_a^u$ of 1-form connections living on $SO(1, 4)/SO(4)$ with abelian gauge transformation given by the opposite of (4.61) namely

$$
\delta_G C^a_1 = d\xi^a \delta^u_a \quad (4.62)
$$

By help of these three gauge connections, the previous global constraint $d(\delta_G Q^u) = 0$ gets now mapped to $\delta_G (\mathcal{D} Q^u) = 0$ with gauge covariant differential operator $\mathcal{D}$ as

$$
\mathcal{D} Q^u = (d + C_1) Q^u \quad (4.63)
$$

In the above expression, we have used the convention notation $C_1 = C^a_1 T_a$ describing a 1-form gauge potential valued in the abelian Lie algebra of the translation group $T_3$ generated by the three $T_a$’s. In terms of the space time coordinates, we have $C^a_1 = C^a_\mu dx^\mu$ with gauge transformations as $\delta_G C^a_\mu = \partial_\mu \xi^a$. The space time gauge covariant derivative $\mathcal{D}_\mu Q^u$ is then defined as

$$
\mathcal{D}_\mu Q^u = (\partial_\mu + C^a_\mu T_a) Q^u \quad (4.64)
$$

It is manifestly invariant under local translations $\delta_G Q^u = -\xi^a \delta^u_a$. Notice that by substituting $T_a Q^u = \delta^u_a$, the covariant derivative $\mathcal{D}_\mu Q^u$ can be also defined like

$$
\mathcal{D}_\mu Q^u = \partial_\mu Q^u + C^a_\mu \delta^u_a \quad (4.65)
$$

* Implementing symplectic structure

The abelian 1-form gauge connections $C^a_1 = C^a_\mu dx^\mu$ introduced above form a real isotriplet of $SO(3) \simeq SU(2)_R$ as follows

$$
C^a_\mu = \begin{pmatrix}
C^x_\mu \\
C^y_\mu \\
C^z_\mu
\end{pmatrix} \quad (4.66)
$$
To interpret these $C^a_\mu$ as gauge fields of the $\mathcal{N} = 2$ supergravity theory, we have to think about them as given by some linear combinations of the graviphoton $A^0_\mu$ and the gauge fields $A^i_\mu$ of the Coulomb branch as well as their magnetic counterparts $\tilde{A}^0_\mu$ and $\tilde{A}^i_\mu$. The relation between $C^a_\mu$ and $A^M_\mu$ is captured by the embedding tensor introduced previously; it reads as

$$C^a_\mu = \vartheta^a_M A^M_\mu$$ (4.67)

Using the electric and magnetic components of the embedding tensor $\vartheta^a_M = \left( \vartheta^a_\Lambda, \tilde{\vartheta}^a_\Lambda \right)$ and those of the gauge potential fields $A^M_\mu = \left( A^\Lambda_\mu, \tilde{A}_{\mu\Lambda} \right)$, the relation (4.67) expands as

$$C^a_\mu = \left( \vartheta^0_a A^0_\mu + \tilde{\vartheta}^0_a \tilde{A}_{\mu0} \right) + \left( \vartheta^a_i A^i_\mu + \tilde{\vartheta}^{ai} \tilde{A}_{\mu i} \right)$$ (4.68)

However, to break local supersymmetry partially, we need two massive gauge fields as shown on the following decomposition of the $\mathcal{N} = 2$ supergravity multiplets in terms of the $\mathcal{N} = 1$ ones [18,21,68]

| $\mathcal{N} = 2$ massless repres | $\mathcal{N} = 1$ repres |
|-----------------------------------|---------------------------|
| $(2,\frac{3}{2},1) \oplus (1,\frac{1}{2},0^2) \oplus (\frac{1}{2},0^4)$ | $(2,\frac{3}{2}) \oplus (\frac{3}{2},1,1) \oplus (\frac{1}{2},0^2)^2$ |

where $(\frac{3}{2},1,1)\frac{1}{2}$) is a massive $\mathcal{N} = 1$ supersymmetric gravitino multiplet having two massive gauge fields. This property is implemented by setting to zero one of the three components in eq(4.66); for instance by taking $C^z_\mu = 0$; so we have

$$C^a_\mu = \left( \begin{array}{c} C^z_\mu \\ C^0_\mu \\ C^i_\mu \\ 0 \end{array} \right) \quad \rightarrow \quad C^m_\mu = \left( \begin{array}{c} C^i_\mu \\ C^0_\mu \\ C^z_\mu \end{array} \right)$$ (4.70)

Notice that by requiring $C^z_\mu = 0$ in eq(4.70), the $SO(3) \simeq SU(2)_R$ rotation symmetry of $C^a_\mu$ breaks down to $SO(2) \simeq U(1)_R$ rotating the two components of $C^m_\mu$. This is an important feature of partial breaking of $\mathcal{N} = 2$ local supersymmetry since the symmetry reduction

$$SO(3) \rightarrow SO(2) \quad \sim \quad SU(2)_R \rightarrow U(1)_R$$ (4.71)

is a necessary condition for partial supersymmetry breaking in $\mathcal{N} = 2$ supersymmetric theory.

* Brane interpretation

The 1-form gauge potential doublet $C^m_\mu = C^m_\mu dx^\mu$ can be given a nice interpretation in
the class of $\mathcal{N} = 2$ theories embedded into type IIA string on CY3$_{IIA}$. There, these $C^m_1$’s descend from the gauge potential $C_3$ of a D2- brane wrapping 2-cycles $\mathfrak{A}^m_2$ in CY3$_{IIA}$ as follows

$$C^m_1 = \int_{\mathfrak{A}^m_2} C_3$$  \hfill (4.72)

Observe that $SO(2)$ rotation symmetry of $C^m_\mu$ requires a CY3$_{IIA}$ with at least two 2-cycles $\mathfrak{A}^m_2$. Observe as well that to implement the $SP(2n_H)$ holonomy symmetry group of the quaternionic Kahler sector, we have to add two other gauge potentials $\tilde{C}^m_1$ namely the magnetic duals of the $C^m_1$ ones given by eq(4.72). These $\tilde{C}^m_1$’s descend from the 4-form potential $\tilde{C}_5$ of a D4- brane wrapping two 4- cycles $\mathfrak{V}^m_4$ of the CY3$_{IIA}$

$$\tilde{C}^m_1 = \int_{\mathfrak{V}^m_4} \tilde{C}_5$$  \hfill (4.73)

Therefore, the implementation of the $SP(2n_H)$ symplectic symmetry in the hypermatter sector requires a gauge field $C^{m\alpha}_\mu$ carrying two indices: the index $m = 1, 2$ for the $SO(2) \sim U(1)_R$ symmetry and the index $\alpha = 1, 2$ for $SP(2n_H)$ with one hypermultiplet

$$C^{m\alpha}_\mu = \begin{pmatrix} C^m_1 \\ \tilde{C}^m_1 \end{pmatrix}$$  \hfill (4.74)

By taking into account the magnetic sector, the embedding relation (4.67) gets extended like

$$C^{m\alpha}_\mu = \Theta^{m\alpha}_M A^M_\mu$$  \hfill (4.75)

with generalised embedding tensor $\Theta^{m\alpha}_M$ given by

$$\Theta^{m\alpha}_M = \begin{pmatrix} \vartheta^m_M \\ \tilde{\vartheta}^m_M \end{pmatrix}$$  \hfill (4.76)

Notice finally the in the above D- brane interpretation of the gauging process, we have used type IIA strings on CY3$_{IIA}$ picture; however the special Kahler construction of section 3 is embedded in type IIB string on CY3$_{IIB}$. To overcome this apparent difficulty, we use type II strings duality as described below.

* Mirror symmetry

By using mirror symmetry between type IIA string on CY3$_{IIA}$ and type IIB string on CY3$_{IIB}$, the two gauge field potentials $C^m_\mu$ can be also interpreted in terms of linear combinations of the gauge potentials $A^\Lambda_\mu$ and their magnetic duals $\tilde{A}_{\mu\Lambda}$ living in the gravity and Coulomb branches; a similar thing can be done for $\tilde{C}_{\mu m}$; it will be understood here below.
Recall that in type IIB string on CY3, the gauge potentials $A_\mu^\Lambda$ and $\tilde{A}_{\mu\Lambda}$ constitute together a $SP(2n+2)$ symplectic vector denoted as

$$A_\mu^M = \begin{pmatrix} A_\mu^\Lambda \\ \tilde{A}_{\mu\Lambda} \end{pmatrix}, \quad \Lambda = 0, 1, ..., n \tag{4.77}$$

The gauge potentials $A_1^\Lambda = A_\mu^\Lambda dx^\mu$ and $\tilde{A}_{1\Lambda} = \tilde{A}_{\mu\Lambda} dx^\mu$ descend from the 4-form gauge potential $\mathcal{C}_4$ of a D3-brane wrapping 3-cycles $\mathfrak{A}_3^\Lambda$ and $\mathfrak{B}_3\Lambda$ in the CY3 as follows

$$A_1^\Lambda = \int_{\mathfrak{A}_3^\Lambda} \mathcal{C}_4, \quad \tilde{A}_{1\Lambda} = \int_{\mathfrak{B}_3\Lambda} \mathcal{C}_4 \tag{4.78}$$

The relationship between the $C_m^\mu$'s and $A_\mu^M$ can be written as in (4.67) where $\vartheta_M^m$ is the embedding tensor; the last quantity is in the bi-fundamental of

$$SO(2) \times SP(2n+2) \tag{4.79}$$

where $n = n_V$ the number of vector multiplets; it has two index legs; one leg m in the sector type IIA string on CY3 and the other leg M in the sector of type string IIB on CY3. The tensor $\vartheta_M^m$ can be imagined as an object realising mirror symmetry which exchanges Kahler and complex structures of CY3 and CY3.

* a comment on extension to $n_H$ hypermultiplets

Here we briefly describe the extension of the analysis done for $n_H = 1$ to the general case of several hypermultiplets within the ADE geometry picture; this generalisation will be used in the discussion section - see eqs(7.4-7.6) - to motivate the structure of the rigid limits of the scalar potential and the anomaly for ADE geometries. Of particular interest for us is the form of the generalised embedding tensor and the corresponding moment maps. To that purpose, notice that for the case of a matter sector having $n_H$ hypermultiplets with $4n_H$ scalars described by the r quartets $Q^{ar}$, eqs(4.65,4.66,4.67) concerning the gauge fields $C_\mu^a$ and the embedding tensor $\vartheta_M^a$ extend like

$$\mathcal{D}_\mu Q^{ar} = \partial_\mu Q^{ar} + \delta^a_b C_\mu^{ar}, \quad C_\mu^{ar} = \vartheta_M^a A_\mu^M \tag{4.80}$$

where the generalised gauge fields $C_\mu^{ar}$ carry an extra index r running from 1 to $n_H$,

$$C_\mu^{ar} = \begin{pmatrix} C_\mu^{xr} \\ C_\mu^{yr} \\ C_{\mu r} \end{pmatrix}, \quad r = 1, ..., n_H \tag{4.81}$$
and where the new embedding tensor $\vartheta_M^{ar}$ carry as well the $SO(n_H)$ quantum number $r$ as follows

$$\vartheta_M^{ar} = \begin{pmatrix} \theta_M^{ar} \\ \tilde{\theta}_m^{aA_r} \end{pmatrix} , \quad r = 1, \ldots, n_H$$  \hspace{1cm} (4.82)

The electric $\theta^{ar}_\Lambda$ an magnetic $\tilde{\theta}^{aA_r}$ components given by

$$\theta^{ar}_\Lambda = \begin{pmatrix} \theta^{0r}_\Lambda \\ \theta^{ar}_i \end{pmatrix} , \quad \tilde{\theta}^{aA_r} = \begin{pmatrix} \tilde{\theta}^{a0r} \\ \tilde{\theta}^{air} \end{pmatrix}$$  \hspace{1cm} (4.83)

with $\theta^{0r}_0, \tilde{\theta}^{a0r}$ are the components along the graviphoton direction and $\theta^{ar}_i, \tilde{\theta}^{air}$ associated with the Coulomb branch dimensions.

### 4.3.2 More on the embedding tensor $\vartheta_M^m$: case $n_H = 1$

To make contact between the gauge potentials $C^m_\mu$ of eq(4.74) and the $A^M_\mu$’s of eq(4.77), we use the embedding tensor $\vartheta_M^m$ of (4.56) to relate them as in (4.67). The $\vartheta_M^m$ carries two indices: the $SO(2)$ orthogonal $m = 1, 2$ and the $SP(2n+2)$ symplectic $M = 0, \ldots, 2n+2$; it splits into electric and magnetic components like

$$\vartheta_M^m = \begin{pmatrix} \theta^m_\Lambda \\ \tilde{\theta}^m_{\Lambda m} \end{pmatrix} , \quad \vartheta^m_\Lambda = \begin{pmatrix} -\tilde{\theta}^m_{\Lambda m} \\ \theta^m_\Lambda \end{pmatrix}$$  \hspace{1cm} (4.84)

By using the components of $C^m_\mu$ eq(4.74), we have

$$C^m_\mu = \theta^m_\Lambda A^\Lambda_\mu + \tilde{\theta}^m_{\Lambda m} \tilde{A}_{\mu\Lambda}$$  \hspace{1cm} (4.85)

Recall that in a generic $\mathcal{N} = 2$ abelian gauge theory with electric/magnetic duality, one has $2n+2$ abelian gauge fields: $(n+1)$ gauge fields of electric type

$$A^\Lambda_\mu = \begin{pmatrix} A^0_\mu \\ A^i_\mu \end{pmatrix}$$  \hspace{1cm} (4.86)

and $(n+1)$ magnetic duals

$$\tilde{A}_{\mu\Lambda} = \begin{pmatrix} \tilde{A}_{\mu0} \\ \tilde{A}_{\mu i} \end{pmatrix}$$  \hspace{1cm} (4.87)

These $(n+1)+(n+1)$ gauge fields combine together into the $SP(2n+2)$ symplectic vector like in (4.77).

Substituting $C^m_\mu$ in terms of $A^M_\mu$ back into the gauge covariant (4.64), we have

$$\mathcal{D}_\mu Q^u = (\partial_\mu + A^M_\mu \kappa_M) Q^u$$  \hspace{1cm} (4.88)
with generators given by
\[ \kappa_M = \vartheta_M^m T_m , \quad Y^\Lambda = \theta^\Lambda m T_m , \quad \tilde{Y}_\Lambda = \tilde{\theta}_\Lambda^m T_m \] (4.89)

and where the embedding tensor \( \vartheta_M^m \) appears as the gauge coupling constants of the \( A_M^\mu \)'s to the \( Q^u \)'s. The embedding tensor \( \vartheta_M^m \) satisfies a set of constraint relations \([30, 35]\); one of them is the following one which is required to keep the charges mutually local,

\[ \vartheta_M^m \vartheta_N^n = 0 \] (4.90)

To solve this constraint, it is interesting to express it into a manifestly symplectic invariant manner like \( \vartheta_M^m \vartheta^{MN} \vartheta_N^n = 0 \); by expanding the symplectic trace \( \vartheta_M^m \vartheta^{nM} = 0 \), we can rewrite (4.90) like

\[ \theta^\Lambda \tilde{\theta}^\Lambda_n - \tilde{\theta}^\Lambda m \theta^\Lambda_m = 0 \quad , \quad m, n = 1, 2 \] (4.91)

with

\[ \theta^\Lambda \tilde{\theta}^\Lambda_n = \theta^\Lambda_n \tilde{\theta}^\Lambda + \tilde{\theta}^\Lambda m \tilde{\theta}^m_n \] (4.92)

A class of solutions of the constraint eq.(4.91) is obtained by constraining each of the two terms to vanish, that is \( \theta^\Lambda_n \tilde{\theta}^\Lambda = \theta^\Lambda_m \tilde{\theta}^m = 0 \). This constraint can be equivalently stated by splitting \( \theta^\Lambda_n = (\theta^\Lambda_n^0, \theta^\Lambda_n^i) \); this leads to

\[ \theta^\Lambda_n \tilde{\theta}^\Lambda = 0 \quad , \quad m, n = 1, 2 \] (4.93)

The above constraint relation can be solved in two basic manners as follows: either by constraining each one of the two terms of the above sum to vanish identically as

\[ \theta^\Lambda_n \tilde{\theta}^\Lambda = \theta^\Lambda m \tilde{\theta}^m = 0 \] (4.94)

or by compensation of the two terms like

\[ \theta^\Lambda_n \tilde{\theta}^\Lambda = -\tilde{\theta}^\Lambda m \tilde{\theta}^m \neq 0 \] (4.95)

Let us describe briefly the solutions of these two sets of conditions:

- **Solution of eq.(4.94)**

Eq(4.94) can be solved in four manners as follows:

|   | \( \theta^\Lambda_n \) | \( \tilde{\theta}^\Lambda_n \) | \( \theta^\Lambda m \) | \( \tilde{\theta}^m \) |
|---|---|---|---|---|
| (i) | \( e^m \) | 0 | \( g^m \tilde{\zeta}_i \) | 0 |
| (ii) | \( e^m \) | 0 | 0 | \( \tilde{g}^m \tilde{\zeta}_i \) |
| (iii) | 0 | \( \tilde{e}^m \) | \( g^m \tilde{\zeta}_i \) | 0 |
| (iv) | 0 | \( \tilde{e}^m \) | 0 | \( \tilde{g}^m \tilde{\zeta}_i \) |
Under the symplectic transformation (3.62), we have \( \vartheta^m_M \rightarrow \vartheta^m_N \left( S^{-1} \right)_M^N \). So these solution are mapped into the following ones still preserving the constraint relation (4.90)

\[
\begin{array}{cccc}
\theta^m_0 & \tilde{\theta}^0m & \theta'^m_i & \tilde{\tilde{\theta}}^0m \\
(i) & e^m & 0 & \mu g^m \zeta_i - \eta_i e^m & 0 \\
(ii) & e^m & \frac{1}{\mu} \eta_i \tilde{\zeta}^i g^m & -\eta_i e^m & \frac{1}{\mu} \tilde{g}^m \zeta_i \\
(iii) & 0 & \tilde{e}^m & \mu g^m \zeta_i & 0 \\
(iv) & 0 & \tilde{e}^m + \frac{1}{\mu} \eta_i \tilde{g}^m \zeta_i & 0 & \frac{1}{\mu} \tilde{g}^m \zeta_i \\
\end{array}
\]

(4.97)

Notice that the above transformed solutions satisfy eq(4.90); the solutions (i), (iii) and (iv) are somehow particular; they carry either electric or magnetic charges in each of the graviphoton and Coulomb branch sectors. However, the solution (ii) is dyonic; and the constraint (4.90) is ensured by compensation like \( \tilde{\theta}^0m \theta^m_0 = -\tilde{\theta}^0m \theta'^m_i \).

For the particular case of one vector multiplet in the Coulomb branch, the embedding tensor \( \vartheta^m_M \) takes the following form in terms of the electric and magnetic coupling constants

\[
\vartheta^m_M = \begin{pmatrix}
\theta^1_0 & \theta^2_0 \\
\theta^1_1 & \theta^1_2 \\
\theta^2_1 & \theta^2_2 \\
\end{pmatrix} = \begin{pmatrix}
e_1 & e_2 \\
g_1 & g_2 \\
\tilde{e}_1 & \tilde{e}_2 \\
\tilde{g}_1 & \tilde{g}_2 \\
\end{pmatrix}
\]

(4.98)

Two particular examples are respectively given by the embedding tensors \( (\vartheta^m_M)_{\text{elect}} \) and \( (\vartheta^m_M)_{\text{dyonic}} \) of the electric and dyonic gauging studied in [20, 30]

\[
(\vartheta^m_M)_{\text{elect}} = \begin{pmatrix}
e_1 & e_2 \\
g_1 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad (\vartheta^m_M)_{\text{dyonic}} = \begin{pmatrix}
e_1 & e_2 \\
0 & 0 \\
0 & 0 \\
\tilde{g}_1 & 0 \\
\end{pmatrix}
\]

(4.99)

Putting (4.97) back into (4.85), we have the following relations between \( C^m_\mu \) and the gauge fields associated with the special Kahler geometry

\[
\begin{align*}
(i) & : C^m_\mu = e^m A^0_\mu + g^m \zeta_i A^i_\mu \\
(ii) & : C^m_\mu = e^m A^0_\mu + \tilde{g}^m \tilde{\zeta}_i \tilde{A}^i_\mu \\
(iii) & : C^m_\mu = \tilde{e}^m \tilde{A}_\mu^0 + g^m \zeta_i A^i_\mu \\
(iv) & : C^m_\mu = \tilde{e}^m \tilde{A}_\mu^0 + \tilde{g}^m \tilde{\zeta}_i \tilde{A}^i_\mu \\
\end{align*}
\]

(4.100)

where the first (fourth) combination is purely electric (magnetic) while the second and the third are dyonic. By setting

\[
\tilde{A}_\mu = \tilde{\zeta}_i \tilde{A}^i_\mu, \quad A_\mu = \zeta_i A^i_\mu
\]

(4.101)
the above relations read as

\[(i) \quad C^m_{\mu} = e^m A^0_{\mu} + g^m A_{\mu} \]
\[(ii) \quad C^m_{\mu} = e^m A^0_{\mu} + \tilde{g}^m \tilde{A}_{\mu} \]
\[(iii) \quad C^m_{\mu} = \tilde{e}^m \tilde{A}_{\mu 0} + g^m A_{\mu} \]
\[(iv) \quad C^m_{\mu} = \tilde{e}^m \tilde{A}_{\mu 0} + \tilde{g}^m \tilde{A}_{\mu} \] (4.102)

- **Solution of eq(4.95)**

By using the factorisation \( \theta^m_i = g^m \zeta_i \) and \( \tilde{\theta}^m = \tilde{g}^m \tilde{\zeta}^i \) and by setting \( \zeta_i \tilde{\zeta}^i = -\epsilon |\zeta|^2 \) with \( \epsilon = \pm 1 \), eq(4.95) can be put into the following form

\[\theta^m_0 \tilde{\theta}^0_n = \epsilon |\zeta|^2 g^m \tilde{g}^n \] (4.103)

whose a solution is given by

\[\theta^m_0 = \epsilon_1 |\zeta| g^m \quad , \quad \tilde{\theta}^0_n = \epsilon_2 |\zeta| \tilde{g}^n \] (4.104)

with \( \epsilon_1 \epsilon_2 = \epsilon \) solved like

\[
\begin{array}{cccccc}
\epsilon_1 & = & + & + & - & - \\
\epsilon_2 & = & + & - & + & - \\
\epsilon & = & + & - & - & + \\
\end{array}
\] (4.105)

By substituting (4.104) back into \( C^m_{\mu} = \theta^m_{\Lambda} A^\Lambda_{\mu} + \tilde{\theta}^m_{\Lambda} \tilde{A}_{\mu \Lambda} \) and using the change (4.101), we obtain

\[C^m_{\mu} = g^m \left( \epsilon_1 |\zeta| A^0_{\mu} + A_{\mu} \right) + \tilde{g}^m \left( \epsilon_2 |\zeta| \tilde{A}_{\mu 0} + \tilde{A}_{\mu} \right) \] (4.106)

which can be put into the form

\[C^m_{\mu} = g^m G_{\mu} + \tilde{g}^m \tilde{G}_{\mu} \] (4.107)

with \( \tilde{G}_{\mu} \) and \( G_{\mu} \) given by

\[G_{\mu} = A_{\mu} + \epsilon_1 |\zeta| A^0_{\mu} \quad , \quad \tilde{G}_{\mu} = \tilde{A}_{\mu} + \epsilon_2 |\zeta| \tilde{A}_{\mu 0} \] (4.108)

### 4.3.3 Symplectic moment maps \( P^a_M \)

Because of the factorisation \( M_{SK} \times M_{QK} \) of the scalar manifold in \( N = 2 \) supergravity, the symplectic moment maps on the target space are of two types: 

- **(i)** isosinglet moment maps \( P^0_M \) transforming as a symplectic vector associated with the gauging of isometries of the special Kahler manifold \( M_{SK} \); and
- **(ii)** isotriplets \( P^a_M \) transforming as well as a symplectic vector; but associated with the quaternionic Kahler manifold \( M_{QK} \). Here, we are interested into
those $\mathcal{P}_M^a$’s induced by the gauging of the translations\(^6\) on the 4-dimensional quaternionic manifold namely

$$\delta_G Q^u = -\xi^a \delta_a^u \quad (4.109)$$

These $\mathcal{P}_M^a$’s have a quite similar structure as the embedding tensor $\vartheta_M^a$ in the sense they also carry two vector indices of different kinds; the index $M$ for the $SP(2n + 2)$ symplectic group in $M_{SK}$ and the index $a = 1, 2, 3$ for $SO(3) \simeq SU(2)_R$. Our interest into these $\mathcal{P}_M^a$’s is because they play an important role in the study of the scalar potential in $\mathcal{N} = 2$ supergravity namely

$$\mathcal{V}_{\text{sugra}}^{\mathcal{N} = 2} = G_{ij} W^{ai} \bar{W}^{aj} + 2 N^a \bar{N}^a - 12 S^a \bar{S}^a \quad (4.110)$$

For example, we have $W^{ai} \sim \mathcal{P}_M^a G_{ij} \bar{U}_M^j$. They are also important for the study of the central anomaly term

$$C_{BA} = C_a (\tau^a)^A_B \quad (4.111)$$

appearing in the $\mathcal{N} = 2$ supercurrent algebra with typical anticommutation relations as follows [4, 8, 20]

$$\{ J^{0A} (x), \bar{J}^\theta_B (y) \} = \delta_3 (x - y) H_A^B \quad (4.112)$$

with 2×2 matrix operator $H_A^B$ as

$$H_A^B = \delta_B^A \sigma_\mu T^{\mu 0} + C_a (\tau^a)^A_B \quad (4.113)$$

where $J^{0A} (x)$, $\bar{J}^\theta_B (x)$ and $T^{0\mu}_\nu (x)$ are the time components of the supersymmetric current densities $J^{\nu A}$, $\bar{J}^{\theta B}_\nu$, and $T^\nu_{\mu}$. The time component densities in the current superalgebra (4.113) are related to the usual $Q^A$, $\bar{Q}^{\dot{A}}$, and $P_\mu$ charges of the $\mathcal{N} = 2$ supersymmetric QFT\(_4\) as follows

$$Q_{\alpha A} = \int d^3 x J^{0A}_{\alpha A} \quad , \quad \bar{Q}_{\dot{\alpha} A} = \int d^3 x \bar{J}^{\theta B}_\nu \quad , \quad P_\mu = \int d^3 x T^{0\mu}_\nu \quad (4.114)$$

To obtain the explicit expression of the $\mathcal{P}_M^a$’s, we start form the Killing vector fields

$$T_a = T_a^u \frac{\partial}{\partial Q^u} \quad , \quad T_a^u = -\delta_a^u \quad (4.115)$$

and use the embedding tensor $\vartheta_M^a$ to map them into a symplectic Killing vector fields $\kappa_M^a$ as follows \(^6\)

$$\kappa_M^a = \vartheta_M^a T_a \quad , \quad \kappa_M^a = -\vartheta_M^a \quad (4.116)$$

\(^6\) for convenience we shall the keep manifest the $SU(2)_R$ symmetry by making an analysis as if we three gauged translations; later on we impose the condition $\xi_3 = 0$. 

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By using the vector field language \( \mathcal{L}_\kappa M^a \equiv \mathcal{L}_M Q^a \) with Lie derivative \( \mathcal{L}_M \), reading, in terms of exterior differential \( d \) and contraction \( i_M \) operators like \( \mathcal{L}_M = d \circ i_M + i_M \circ d \), the \( \mathcal{P}_a^a \)'s can be defined up to compensator shifts \( W^a_M \) as
\[
\mathcal{P}_a^a = -i_M (\omega^a_1) = -\partial^a_M \omega^a_u \quad (4.117)
\]
with \( \omega^a_1 = \omega^a_0 dQ^a \). The differential form is the 1-form SU(2) connection of the quaternionic Kahler geometry given by eqs(4.6,4.21); by substituting \( \omega^a_1 = e^\varphi d\phi^a \), we obtain the following relation between the moment maps and the embedding tensor
\[
\mathcal{P}_a^a = e^\varphi \partial^a_M \quad (4.118)
\]
Notice that in the case of several hypermultiplets, the moment maps carry as well an SO(\( n_H \)) index \( r = 1, ..., n_H \), and the above relation extends like \( \mathcal{P}_a^a = \sqrt{f(\varphi)} \partial^a_M \).

5 Rescalings and rigid limit of \( \mathcal{N} = 2 \) supergravity

In this section, we first use supersymmetry and the structure of the 4d- TAUB-NUT hyperKahler metric to propose another way to rescale fields of \( \mathcal{N} = 2 \) gauged supergravity. Then, we use the results of sections 3 and 4 to study the rigid limit of the scalar potential of gauged supergravity, the rigid limit of Ward identities underlying the gauging and the matrix anomaly of the \( \mathcal{N} = 2 \) supercurrent algebra.

To that purpose, we shall proceed progressively as follows: (i) we use rescaling properties of the fields of the \( \mathcal{N} = 2 \) supergravity in order to determine the leading terms in \( \frac{1}{\mu} \) expansions. (ii) we compute the value of observables in the rigid limit \( \frac{1}{\mu} \to 0 \); the parameter \( \mu \) of the development is given by the ratio \( \frac{M_{pl}}{\Lambda} \) with \( \Lambda \) standing for the scale of partial supersymmetry breaking; say the \( m_{3/2} \) mass of the \( \mathcal{N} = 1 \) massive gravitino multiplet \((\frac{3}{2}, 1, 1, \frac{1}{2})\) eq(4.69).

5.1 Rescaling of fields in \( \mathcal{N} = 2 \) gauged supergravity

To determine the rescaling dimensions of the component fields involved in \( \mathcal{N} = 2 \) supergravity, we will use supersymmetric representations to think about rescalings as a general property shared by all those field components belonging to the same \( \mathcal{N} = 2 \) supermultiplet. In other words, fields in the same \( \mathcal{N} = 2 \) supersymmetric representation share the same rescaling behaviour with respect to the scales \( \Lambda \) and \( M_{pl} \). To that end, let us use the language of ”\( \mathcal{N} = 2 \) superfields” \( \Phi(x^\mu, \theta^A_\alpha, \bar{\theta}^{\dot{A}}_{\dot{\alpha}}) \) to think of these supersymmetric representations; this allows to deduce directly the rescaling properties of the various component fields of \( \Phi^{\mathcal{N}=2} \) knowing those of the supercoordinates \( x^\mu, \theta^A_\alpha, \bar{\theta}^{\dot{A}}_{\dot{\alpha}} \). So the first thing to begin
with is the rescaling of the local supercoordinate variables $Z^{\mathcal{N}=2} = (x^\mu, \theta^A, \bar{\theta}^\dot{A})$ of the $\mathcal{N} = 2$ superspace. These rescalings are given by the canonical dimensions expressed in terms of Planck scale $M_{pl}$ as follows

$$
\left( x^\mu, \theta^A, \bar{\theta}^\dot{A} \right) \rightarrow \left( x^\mu M_{pl}, \theta^A \sqrt{M_{pl}}, \bar{\theta}^\dot{A} \sqrt{M_{pl}} \right)
$$

(5.1)

The two supersymmetric parameters $\epsilon^A_\alpha$ are then rescaled in the same manner as $\theta^A$ as they are just shifts of the Grassman variables: $\epsilon^A_\alpha \rightarrow \epsilon^A_\alpha \sqrt{M_{pl}}$. Similar expressions can be written down for the superspace derivatives.

Regarding the superfields $\Phi^{\mathcal{N}=2}$, their rescaling property depend on the type of the $\mathcal{N} = 2$ supermultiplet we are interested in namely the gravity multiplet $G^{\mathcal{N}=2} \sim (2, \frac{3}{2}, 1)$, the vector supermultiplet $V^{\mathcal{N}=2} \sim (1, \frac{1}{2}, 0^2)$ and the hypermultiplet $H^{\mathcal{N}=2} \sim (\frac{1}{2}, 0^4)$. The rescaling properties of these supersymmetric representations are described in what follows.

### 5.1.1 Case of $\mathcal{N} = 2$ gravity and vector multiplets

As there is no simple off shell description of $\mathcal{N} = 2$ supersymmetric gauge theory preserving manifestly the SU(2) R-symmetry, we shall consider only those field components contributing to the study of the gauging isometries of the scalar manifold, to Ward identities and to the induced scalar potential.

- **$\mathcal{N} = 2$ gravity multiplet**

The component fields of the gravity multiplet carrying physical degrees of freedom are the graviton vierbein $e^m_\mu$, the graviphoton $A^0_\mu$ and the two gravitini $\psi^{\alpha A}_\mu$; they are respectively rescaled as follows

| dimensionless fields | rescaled fields |
|----------------------|-----------------|
| $e^m_\mu$            | $\frac{1}{M_{pl}} e^m_\mu$ |
| $A^0_\mu$            | $\frac{1}{M_{pl}} A^0_\mu$ |
| $\psi^{\alpha A}_\mu$| $\frac{1}{M_{pl}} \psi^{\alpha A}_\mu$ |

(5.2)

These rescalings involve only the Planck scale $M_{pl}$ but no scale $\Lambda$; in the rigid limit $\frac{1}{M_{pl}} \rightarrow 0$, this supermultiplet decouples.

- **$\mathcal{N} = 2$ vector multiplet**

The basic component fields making $\mathcal{N} = 2$ vector multiplets $V^{\mathcal{N}=2}$ the in Wess-Zumino gauge are as follows

$$
V^{\mathcal{N}=2} \equiv z^i, A^i_\mu, \lambda_\alpha^{Ai}; \quad \left( D^i \right)^A_B
$$

(5.3)
with \( z^i \) standing for the complex scalar fields and the Majorana spinors \( \lambda^A_i = \left( \lambda^A_i, \bar{\lambda}^A_i \right) \) referring to the two gauginos. The extra \( 2 \times 2 \) extra matrices \( (D^i)^A_B \) expands in terms of isotriplets \( D^i_a \) like

\[
(D^i)^A_B = \sum_{a=1}^{3} D^i_a (\tau^a)^B_A
\]

with \( \tau^a \) the usual Pauli matrices. The \( D^i_a \)'s refer to the auxiliary fields which scale as mass\(^2\); they contribute to the scalar potential \( V^{N=2}_{\text{scal}} \) scaling as mass\(^4\). The rescaling properties of the fields of (5.3) depend on the supersymmetric breaking scale \( \Lambda \); they are given by

| dimensionless fields | rescaled fields |
|----------------------|----------------|
| \( z^i \) | \( \frac{1}{\Lambda} z^i \) |
| \( A^i_\mu \) | \( \frac{1}{\Lambda} A^i_\mu \) |
| \( \chi^i_A \) | \( \frac{1}{\sqrt{M_{pl}}} \chi^i_A \) |
| \( (D^i)^A_B \) | \( \frac{1}{\sqrt{M_{pl}}} (D^i)^A_B \) |

These rescalings are different from those used in [30] where \( z^i \) was rescaled like \( \frac{1}{M_{pl}} z^i \). In our way of doing, all fields of the vector supermultiplets \( V^i_{\mathcal{N}=2} \) have the same rescaling with respect to \( \Lambda \); that is

\[
V^i_{\mathcal{N}=2} \rightarrow \frac{\Lambda}{M_{pl}} V^i_{\mathcal{N}=2}
\]

Notice that in the language of \( \mathcal{N} = 1 \) superfields, the \( D^i_a \) isotriplets of auxiliary fields can be imagined as the complex \( F^i \) auxiliary fields of \( \mathcal{N} = 1 \) chiral multiplets and the real \( \bar{D}^i \) auxiliary fields of \( \mathcal{N} = 1 \) vector ones

\[
D^i_a \sim F^i, \bar{F}^i, D^i
\]

The rescaling property of these auxiliary fields is useful for determining the rescalings of the Fayet-Iliopoulos coupling constants \( \nu^i_a = (\nu^i_1, \nu^i_2, \bar{\nu}^i_0) \) contributing to the \( \mathcal{N} = 1 \) supersymmetric scalar potential \( V^{N=1}_{\text{scal}} \). Notice also that by using the reduced scale parameter \( \mu = \frac{M_{pl}}{\Lambda} \), the field rescalings (5.5) read as follows

\[
\begin{align*}
 z^i & \rightarrow \frac{\mu}{M_{pl}} z^i \\
 A^i_\mu & \rightarrow \frac{\mu}{M_{pl}} A^i_\mu \\
 \chi^i_A & \rightarrow \frac{\mu}{M_{pl}} \chi^i_A \\
 D^i_a & \rightarrow \frac{\mu}{M_{pl}} D^i_a
\end{align*}
\]

From these rescalings of the fields in the Coulomb branch sector, we deduce other rescaling behaviours; for example the metric \( g_{ij} = \frac{\partial^2 K}{\partial z^i \partial z^j} \) of the special Kahler manifold which is rescaled like

\[
g_{ij} \rightarrow \frac{M^2_{pl}}{\mu^2} g_{ij}
\]
This is because the derivatives of the Kahler potential $K$ are not affected by the rescalings since $K$ is given by the logarithm of a volume, i.e: $K \sim \ln(V_{vol})$; the rescaling shifts then $K$ by a constant. Moreover, using the space time property $\partial_\mu \to \frac{1}{M_{pl}}\partial_\mu$, it follows that gauge covariant derivative $D_\mu = \partial_\mu + A^M_\mu \kappa_M$ (4.88) having the explicit form

$$D_\mu = \partial_\mu + \left(A^0_\mu Y_0 + \tilde{A}_\mu \tilde{Y}^0\right) + \left(A^i_\mu Y_i + \tilde{A}_\mu \tilde{Y}^i\right) \tag{5.10}$$

is rescaled in same manner like $\partial_\mu$, thus

$$D_\mu \to \frac{1}{M_{pl}}D_\mu \tag{5.11}$$

By help of eqs(5.2-5.8), we learn that the $\tilde{Y}_0$ and $\tilde{Y}_i$ generators should transform differently like

$$\tilde{Y}_0 \to \tilde{Y}_0, \quad \tilde{Y}_i \to \frac{1}{\mu}\tilde{Y}_i \tag{5.12}$$

Similar relations may be derived as well for the isotriplets of Fayet-Iliopoulos coupling constants $\nu^a_i$; by using the fact that the scalar potential $V_{scal}^{N=2}$ of the theory is related to variations of the fermions $\delta_B \lambda^{iA}_a$ induced by the gauged translations like

$$V_{scal}^{N=2} \sim G_{ij} \left(\delta_A \lambda^{iB} \right) \left(\delta^B \lambda^i_A\right) + \ldots \tag{5.13}$$

we have

$$V_{scal}^{N=2} \to \frac{1}{M_{pl}^2}V_{scal}^{N=2} \tag{5.14}$$

Moreover, knowing that generic scalar potentials $V_{scal}^{N=2}$ in $N = 2$ supersymmetric theory involve FI terms of type $\nu^a_i D^i_a$ as well as quadratic terms type $G_{ij} D^i_a \delta^{ab} D^j_b$, and using the rescaling property $D^i_a \to \frac{\mu}{M_{pl}^2} D^i_a$, we conclude that FI coupling constants $\nu^a_i$ rescale like

$$\nu^a_i \to \frac{1}{\mu M_{pl}^2} \nu^a_i \tag{5.15}$$

### 5.1.2 Case of hypermultiplets

The rescaling properties of the component fields of the hypermatter superfield $H_{N=2}$ may be obtained in similar manner as the gravity multiplet; except that now the field variables parameterising the metric $ds^2 = \frac{1}{2} \left(d\varphi^2 + e^{2\varphi} d\phi^a d\phi^a\right)$ of the quaternionic Kahler manifold (4.28) have a different scaling mass with respect to the canonical one. The point is that because the factor $e^{2\varphi}$ is dimensionless, the isosinglet variable $\varphi$ should be a dimensionless field variable; the same thing should hold for the isotriplet $\phi^a$ due to consistency since $d\phi^a d\phi^a$ should have same scaling as $d\varphi^2$. Let us give some details on this issue; in particular on how $(\varphi, \phi^a)$ can be rescaled.
• Rescaling the coordinate fields \((\varphi, \phi^a)\)

If denoting by \((h^0, h^a)\) the real four scalars the hypermultiplet having the right canonical dimensions \((h^0, h^a \sim \text{mass})\), and by \((\zeta_A^\alpha, \bar{\zeta}_A^\alpha)\) the hyperini partners (with dimension \(\text{mass}^{3/2}\)); then one can write down a relationship between the supermultiplet \((\varphi, \phi^a; \xi_A^\alpha, \bar{\xi}_A^\alpha)\) of eq(2.9), used in the metric building, and the \((h^0, h^a; \zeta_A^\alpha, \bar{\zeta}_A^\alpha)\) by using a massive parameter \(M\). Thinking of this parameter \(M\) as given by the Planck mass \(M_{pl}\), we then have

\[
\begin{align*}
\varphi &= \frac{h^0}{M_{pl}}, & \phi^a &= \frac{h^a}{M_{pl}} \\
\xi_A^\alpha &= \frac{\zeta_A^\alpha}{M_{pl}}, & \bar{\xi}_A^\alpha &= \frac{\bar{\zeta}_A^\alpha}{M_{pl}}
\end{align*}
\] (5.16)

However, this is one way to think about the rescaling as the real four fields \((h^0, h^a)\) are not the unique way to describe the scalars of the hypermultiplet. In fact the scalars of the hypermultiplet can be also described in terms of a complex field doublet \(f_A^\alpha\) and its complex conjugate \(\bar{f}_A\); this is an interesting option especially that the partial breaking of \(\mathcal{N} = 2\) supersymmetry requires complex scalar fields; this option is also interesting because it allows to recover the right scaling dimensions of the Fayet-Iliopoulos coupling constants. Therefore, we propose to think about the real four variables \(Q^u = (\varphi, \phi^a)\) and their fermionic partners \((\xi_A^\alpha, \bar{\xi}_A^\alpha)\) as composite fields respecting the SU(2) representation group properties

\[
2 \otimes \bar{2} = 1 \oplus 3, \quad 2 \otimes 1 = 2
\] (5.17)

Focussing on scalars and using \(2 \otimes \bar{2} = 1 \oplus 3\), the real four field variables \(\varphi\) and \(\phi^a\) are given by composites of a complex scalar isodoublets \(f_A^\alpha\) and \(\bar{f}_A\) as follows

\[
\varphi \equiv f \bar{f} \sim 1, \quad \phi^a \equiv f \tau^a \bar{f} \sim 3
\] (5.18)

where here \(f_A^\alpha\) is assumed a dimensionless complex field doublet \((f_A^\alpha \sim 2)\); the right dimension will be restored later by performing a rescaling by using Planck mass \((f_A^\alpha = \frac{1}{M_{pl}} f'^\alpha\) with \(f'^\alpha\) scaling as mass). Similar relations to the scalars (5.18) can be written down for the fermionic fields \(\xi_A^\alpha\) and \(\bar{\xi}_A^\alpha\); they are also given by composites involving the scalar field doublets \(f_A^\alpha\) and \(\bar{f}_A\) and a Dirac spinor \((\psi_\alpha, \bar{\chi}_\alpha)\) transforming as an isosinglet of SU(2) as in eq(2.9). We have

\[
\begin{align*}
\xi_A^\alpha &= f_A^\alpha \psi_\alpha + \varepsilon^{AB} \bar{f}_B \chi_\alpha, & \bar{\xi}_A^\alpha &= \bar{f}_A^\alpha \psi_\alpha - \varepsilon^{AB} f_B^\alpha \bar{\chi}_\alpha
\end{align*}
\] (5.19)

By exhibiting explicitly the SU(2) index in eq(5.18), we have

\[
\begin{align*}
\varphi &= f_A^\alpha \bar{f}_A, \\
\phi^a &= f_A^\alpha (\tau^a)_A^B \bar{f}_B
\end{align*}
\] (5.20)
Rescaling the complex field doublet $f^A$ by their canonical mass dimension like $f^A \rightarrow \frac{1}{M_{pl}} f^A$, the component fields $(\varphi, \phi^a)$ of the hypermultiplet get rescaled by a factor $\frac{1}{M_{pl}^2}$ as follows

| Dimensionless fields | Fields with dimensions |
|----------------------|------------------------|
| $\varphi$            | $\frac{1}{M_{pl}} \varphi$ |
| $\phi^a$             | $\frac{1}{M_{pl}^2} \phi^a$ |

Typical hyperKahler metrics with a parameter $\lambda = \frac{1}{M_{pl}^2}$ scaling as mass$^{-2}$ have been constructed in hyperKahler manifolds literature in terms of self interacting hypermultiplets by using the $\mathcal{N} = 2$ harmonic superspace method [69–71]. A simple example of such metrics is given by the $\mathcal{N} = 2$ 4d- TAUB-NUT model whose bosonic Lagrangian density reads as follows

$$L_{TB}(f, \bar{f}) = h^B_A \partial_\mu f^A \partial^\mu \bar{f}_B + \bar{g}_{AB} \partial_\mu f^A \partial^\mu \bar{f}^B + g^{AB} \partial_\mu \bar{f}_A \partial^\mu \bar{f}_B$$

(5.22)

with metric components $h^B_A, g^{AB}, \bar{g}_{AB}$ given by

\[
\begin{align*}
    h^B_A &= \delta^B_A \left(1 + \lambda f \bar{f}\right) - \frac{2 + \lambda f \bar{f}}{2(1 + \lambda f \bar{f})} \left(\lambda f^A \bar{f}_B\right) \\
    g^{AB} &= \frac{2 + \lambda f \bar{f}}{2(1 + \lambda f \bar{f})} \left(\lambda f^A f^B\right) \\
    \bar{g}_{AB} &= \frac{2 + \lambda f \bar{f}}{2(1 + \lambda f \bar{f})} \left(\lambda \bar{f}_A \bar{f}_B\right)
\end{align*}
\]

(5.23)

where $M = \frac{1}{2\sqrt{\lambda}}$ is the mass of the Taub-NUT black hole with horizon at $r = M$. In terms of the scalar fields, this horizon is associated with the limit $f \bar{f} = 0$ and corresponds to the vanishing of the relation $f \bar{f} = 2M (r - M) > 0$. For other explicit metrics generalising (5.22) to higher 4r-dimensional hyperKahler manifolds; see [70]. Our interest in giving (5.22) is just to motivate the rescaling (5.21) by thinking of the dimensionless fields $(\varphi, \phi^a)$ like

$$\varphi = \lambda f \bar{f}, \quad \phi^a = \lambda f \tau^a \bar{f}$$

(5.24)

In what follows, we shall use the rescaling picture (5.21) to deal with the real scalar fields $Q^a = (\varphi, \phi^a)$ of hypermultiplets; this way of doing constitutes another difference with the approach of [30].

- **Rescaling of embedding tensor $\vartheta^m_M$**

Because of the structure of the covariant derivative $D_\mu = \partial_\mu + \mathcal{C}_\mu^m T_m$, following from eq(4.64) with rescaling property $D_\mu \rightarrow \frac{1}{M_{pl}} D_\mu$, and due to the relation $T_m \sim \frac{\partial}{\partial \phi^m}$ which rescales as the inverse of the field variable $\phi^m$, that is $T_m \rightarrow M_{pl}^2 T_m$, then the gauge fields $\mathcal{C}_\mu^m$ should be rescaled like

$$\mathcal{C}_\mu^a \rightarrow \frac{1}{M_{pl}^2} \mathcal{C}_\mu^a$$

(5.25)
Moreover, using the relationship between $C^m_\mu$ and the gauge fields $A^M_\mu$ namely $C^m_\mu = \vartheta^m_M A^M_\mu$ we can determine the rescaling of the components of the embedding tensor $\vartheta^m_M$. Since fields in pure gravity sector and Coulomb branch have different scalings as shown by (5.2) and (5.5), it is interesting to split $\vartheta^m_M A^M_\mu$ as follows

$$\vartheta^m_\mu = \theta^m_\mu A^0_\mu + \tilde{\theta}^m_\mu \tilde{A}^0_\mu = \left( \theta^m_\mu A^0_\mu + \tilde{\theta}^m_\mu \tilde{A}^0_\mu \right)$$

From this expansion, we deduce the following rescaling properties of the components of the embedding tensor

$$\theta^m_0 \to \frac{1}{M^2} \theta^m_0, \quad \theta^0_\mu \to \frac{1}{M^2} \theta^0_\mu, \quad \theta^\mu i \to \frac{1}{\mu M^2} \theta^\mu i (5.27)$$

The different behaviours of the rescalings of the $0$- components $\tilde{\vartheta}^m_\mu = \left( \theta^m_0, \tilde{\theta}^0_\mu \right)$ along the graviphoton dimension and the $M$-th components $\vartheta^m_M = \left( \theta^m_i, \tilde{\theta}^m_i \right)$ along the Coulomb branch directions show that the rescaled embedding tensor $\hat{\vartheta}^m_M$ can be split in terms of the $\frac{1}{\mu}$ parameter like

$$\vartheta^m_M \to \hat{\vartheta}^m_M = \frac{1}{M^2} \left( \tilde{\vartheta}^m_\mu \vartheta^0_M + \frac{1}{\mu} \tilde{\vartheta}^m_M \vartheta^0_M \right) (5.28)$$

where the pure gravity contribution $\tilde{\vartheta}^m_\mu$ appears as the leading term and the Coulomb branch one $\vartheta^m_M$ as the next term. In matrix notation, we have

$$\hat{\vartheta}^m_M = \vartheta^m_M + \frac{1}{\mu} \tilde{\vartheta}^m_M (5.29)$$

with

$$\vartheta^m_M = \frac{1}{M^2} \left( \begin{array}{c} \theta^m_0 \\ \tilde{\theta}^m_0 \\ 0 \\ 0 \end{array} \right), \quad \tilde{\vartheta}^m_M = \frac{1}{M^2} \left( \begin{array}{c} 0 \\ \theta^m_i \\ 0 \\ \tilde{\theta}^m_i \end{array} \right) (5.30)$$

Notice that the rescaling of $\tilde{\theta}^m_i$ and $\tilde{\theta}^m_i$ may be compared with the rescalings of the FI coupling constants $\nu^a_i$ given by (5.15). Furthermore, using the expression of the Killing vector field $\kappa_M = \vartheta^m_M T_m$ with components

$$\kappa_M = \left( \begin{array}{c} Y_\Lambda \\ \bar{Y}_\Lambda \end{array} \right), \quad Y_\Lambda = \left( \begin{array}{c} Y_0 \\ Y_i \end{array} \right), \quad \bar{Y}^\Lambda = \left( \begin{array}{c} \bar{Y}^0 \\ \bar{Y}_i \end{array} \right) (5.31)$$
and

\[
Y_0 = \theta^{m_0}_m T_m, \quad \tilde{Y}^0 = \tilde{\theta}^{0m}_m T_m
\]

\[
Y_i = \theta^{m_i}_m T_m, \quad \tilde{Y}^i = \tilde{\theta}^{im}_m T_m
\]

we deduce from (5.27) the following rescalings

\[
Y_0 \rightarrow Y_0, \quad \tilde{Y}^0 \rightarrow \tilde{Y}^0
\]

\[
Y_i \rightarrow \frac{1}{\mu} Y_i, \quad \tilde{Y}^i \rightarrow \frac{1}{\mu} \tilde{Y}^i
\]

Before proceeding, let us collect in the two following tables the rescaling behaviours of useful objects; they are needed later on

| components fields | $z^i$ | $(A^i, \tilde{A}_{i\mu})$ | $(A_\mu^0, \tilde{A}_{\mu0})$ | $Q^a$ | $C^a_\mu$ | $G_{ij}$ |
|-------------------|-------|-------------------------|-------------------------|-------|----------|--------|
| rescaling factors | $\mu M_{pl}$ | $\mu M_{pl}$ | $\frac{1}{M_{pl}}$ | $\frac{1}{M_{pl}}$ | $\frac{1}{M_{pl}^2}$ | $\frac{M_{pl}^2}{\mu^2}$ |

and

| embedding tensor | $(\theta^{m_0}, \tilde{\theta}^0_i)$ | $(\theta^{m_0}, \tilde{\theta}^0_i)$ | $(Y^i, \tilde{Y}_i)$ | $(Y^0, \tilde{Y}_0)$ | $T_m$ |
|------------------|---------------------------------|---------------------------------|-----------------|-----------------|------|
| rescaling factors | $\frac{1}{\mu M_{pl}^2}$ | $\frac{1}{M_{pl}^2}$ | $\frac{1}{\mu}$ | 1 | $M_{pl}^2$ |

Using the vielbein $E^{A\tilde{A}} = E^{A\tilde{A}}_u d^u$ on the quaternionic Kahler manifold (4.16), we determine the rescaling of $E^{A\tilde{A}}_u$; it is given by $E^{A\tilde{A}} \rightarrow M_{pl}^2 E^{A\tilde{A}}_u$.

### 5.2 Rigid limit of Ward identities

In this subsection we study the Ward identities of gauging quaternionic isometries of the scalar manifold of $\mathcal{N} = 2$ supergravity, their rigid limit and the induced scalar potential $\hat{\mathcal{V}}_{\text{kah}}$. These limits will be used in next subsection for the derivation of partial supersymmetry breaking conditions.

#### 5.2.1 Ward identities and induced scalar potential

$\mathcal{N} = 2$ local supersymmetry can be broken partially by gauging two abelian quaternionic isometries of the scalar manifold of $\mathcal{N} = 2$ supergravity theory. To tier $\mathcal{N} = 2$ supersymmetry and the quaternionic gauging isometries, we have to modify the usual supersymmetric transformations of the fermions of the $\mathcal{N} = 2$ supergravity theory by adding extra terms proportional to embedding tensor $\theta^m_M$ as in eq(5.36) reported below.

Following [30, 52, 56, 72, 73], the gauging of abelian quaternionic isometries affects the underlying supersymmetric properties of $\mathcal{N} = 2$ supergravity; it requires modifying the usual
supersymmetry transformations $\delta^{(0)}_\epsilon \chi_\alpha$ of the fermions $\chi_\alpha$ of the $\mathcal{N} = 2$ local theory, namely the gauginos $\chi^{Ai}_\alpha$, the hyperini $\zeta^{A}_\alpha$ and the gravitini $\psi^{A}_\mu$, like

$$
\delta^{(0)}_\epsilon \chi^{Ai}_\alpha = \delta^{(0)}_\epsilon \chi^{Ai}_\alpha + \delta^{(0)}_\epsilon \chi^{Ai}_\alpha \\
\delta^{(0)}_\epsilon \zeta^{A}_\alpha = \delta^{(0)}_\epsilon \zeta^{A}_\alpha + \delta^{(0)}_\epsilon \zeta^{A}_\alpha \\
\delta^{(0)}_\epsilon \psi^{A}_\mu = \delta^{(0)}_\epsilon \psi^{A}_\mu + \delta^{(0)}_\epsilon \psi^{A}_\mu
$$

(5.36)

where spinor indices $\hat{\alpha} = (\alpha, \hat{\alpha})$ have been dropped out for simplicity. The extra $\delta^{(0)}_\epsilon \chi$'s in above relations depend on the embedding tensor $\vartheta^M_a$ of (4.84) and encode data on the induced scalar potential $\mathcal{V}_{\text{sugra}}^\mathcal{N}=2$ of the $\mathcal{N} = 2$ supergravity whose structure will be described in a moment; see eqs (2.33-5.40). The extra terms $\delta^{(0)}_\epsilon \chi$ in (5.36) depend on the Killing vectors $\kappa^a_M = -i\vartheta^a_M \delta^a_\alpha$ (4.116) and moment maps $\mathcal{P}^a_M$ (4.118) of the quaternionic Kahler manifold $\mathcal{M}_{\mathbb{Q}K}$ and have the typical form (2.32).

Substituting $\kappa^a_M = -i\vartheta^a_M \delta^a_\alpha$ and using the vielbein expression $\mathcal{E}^A_{aB} = \frac{1}{2} \delta^A_a \delta^0_u - i \epsilon^a \delta^a_\alpha (\tau_a)^A_B$ as well as $\mathcal{P}^a_M = e^{\varphi} \vartheta^a_M$ (4.118); then expanding the traceless matrices $(W^i)^A_B$, $N^a_B$ and $S^a_B$ as

$$
(W^i)^A_B = W^{ia} (\tau_a)^{AB} \\
N^A_B = N^a (\tau_a)^A_B \\
S^a_B = S^a (\tau_a)^A_B
$$

(5.37)

eqs (2.35) can be brought to the form of complex isotriplet equations

$$
W^{ia} = -i \mathcal{P}^a_M \mathcal{G}^U_j \bar{U}^M_j \\
N^a = i \mathcal{P}^a_M \bar{V}^M \\
S^a = \frac{1}{2} \mathcal{P}^a_M V^M
$$

(5.38)

Putting these relations back into (2.33), we obtain the explicit expression of the induced potential $\mathcal{V}_{\text{sugra}}^\mathcal{N}=2$ in terms of moment maps and geometric objects of the scalar manifold of the $\mathcal{N} = 2$ supergravity. By taking the trace over SU(2) R-symmetry indices, we obtain

$$
\mathcal{V}_{\text{sugra}}^\mathcal{N}=2 = \frac{1}{2} \text{Tr} \left(-12 \mathcal{G}^A_C S^C_B + 2 \bar{N}_C^A N_B^C + \sum_{i,j=1}^{n_v} \mathcal{G}_{ij} (W^i)^A_C (\bar{W}^j)^B_C \right)
$$

(5.39)

which reads also as

$$
\mathcal{V}_{\text{sugra}}^\mathcal{N}=2 = -12 \mathcal{G}^a S^a + 2 \bar{N}^a N^a + \sum_{i,j=1}^{n_v} \mathcal{G}_{ij} W^{ia} \bar{W}^j a
$$

(5.40)

By using eqs (5.38), we can bring the above expression of the scalar potential first to the form

$$
\mathcal{V}_{\text{sugra}}^\mathcal{N}=2 = \left[ \mathcal{G}^i_k \bar{U}^M_k \bar{U}^N_i \right] \mathcal{P}^a_M \mathcal{P}^a_N - \mathcal{P}^a_M \mathcal{P}^a_N V^M \bar{V}^N
$$

(5.41)

and then to

$$
\mathcal{V}_{\text{sugra}}^\mathcal{N}=2 = - \frac{1}{2} \mathcal{M}^M_N \mathcal{P}^a_M \mathcal{P}^a_N - 2 V^M \bar{V}^N \mathcal{P}^a_M \mathcal{P}^a_N
$$

(5.42)

with $U^{MN} = U^M_i \mathcal{G}^{ij} \bar{U}_j^N$ and the symmetric $\mathcal{M}^{MN}$ as in eq (C.15).
5.2.2 Rigid limit of Ward identities

Here we study the rigid limits of the Ward identities, the scalar potential and the matrix anomaly of the $\mathcal{N} = 2$ supercurrent algebra. These limits are obtained by working out the $\mu$- expansions of the fermion shifts $W^a_i$, $N^a$ and $S^a$, given in the Appendix E, and then of (2.33) and (5.40). Before going into details let us first restore the scaling dimensions of the various quantities involved in the computations. By implementing the canonical dimension of $\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2}$ using Planck mass, the left hand of eq(5.40) can be rewritten as follows

\[
\frac{1}{M_{pl}} \mathcal{V}_{\text{sugra}}^{\mathcal{N}=2} = -12 \bar{S}^a S^a + 2 \bar{N}^a N^a + \sum_{i,j=1}^{n_\nu} g_{ij} W^a_i \bar{W}^j a
\]

where now $\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2}$ scales as mass$^4$. Multiplying both sides of this equation by $M_{pl}^4$, the above relation can be then brought to the form

\[
\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2} = -12 \bar{S}'^a S'^a + 2 \bar{N}'^a N'^a + g_{ij} W'^a_i \bar{W}'^j a
\]

where the primed $W'^a_i$, $N'^a$ and $S'^a$ scale as mass$^2$; they are related to the old dimensionless $W^a_i$, $N^a$ and $S^a$ like

\[
\begin{align*}
W'^a_i &= M_{pl}^2 W^a_i \\
N'^a &= M_{pl}^2 N^a \\
S'^a &= M_{pl}^2 S^a
\end{align*}
\]

Below, we shall drop out the prime indices in (5.44) and think about $W^a_i$, $N^a$ and $S^a$ as well as on the $\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2}$ and the anomaly matrix $C^B_A$ as dimensionful objects. To deal with the rigid limit of (5.44), we use the fermionic transformations (2.32) to think of $\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2}$ as given by the sum of three contributions $\mathcal{V}_{\text{kah}}^{\mathcal{N}=2}$, $\mathcal{V}_{\text{hyper}}^{\mathcal{N}=2}$, $\mathcal{V}_{\text{gra}}^{\mathcal{N}=2}$ respectively associated with the transformations $\delta, \lambda^A_i$, $\delta, \zeta^A$, $\delta, \psi^A_\mu$ of fermions belonging in the three supersymmetric field representations $V_i^{\mathcal{N}=2}$, $H_{\mathcal{N}=2}$ and $G_{\mathcal{N}=2}$ of eq(2.1) involved in the $\mathcal{N} = 2$ supergravity.

So, we have,

\[
\mathcal{V}_{\text{sugra}}^{\mathcal{N}=2} = \mathcal{V}_{\text{kah}}^{\mathcal{N}=2} + \mathcal{V}_{\text{hyper}}^{\mathcal{N}=2} - \mathcal{V}_{\text{gra}}^{\mathcal{N}=2}
\]

where we have set

\[
\begin{align*}
\mathcal{V}_{\text{kah}}^{\mathcal{N}=2} &= g_{ij} W'^a_i \delta_{ab} \bar{W}'^j a \\
\mathcal{V}_{\text{hyper}}^{\mathcal{N}=2} &= 2 \bar{N}'^a \delta_{ab} N'^b \\
\mathcal{V}_{\text{gra}}^{\mathcal{N}=2} &= 12 \bar{S}'^a \delta_{ab} S'^b
\end{align*}
\]

with $W'^a_i, N'^a, S'^a$ scaling as mass$^2$; but functions of the dimensionless expansion parameter $\mu$ as described in what follows.

**b) Rigid limit of Ward identities**

To determine the rigid limit of Ward identities (2.33), we use the $\mu$ expansion of the scalar
potential $\mathcal{V}_\text{sugra}^{\mathcal{N}=2}$, given in the Appendix E, and the development $\mathcal{V}_\text{sugra}^{\mathcal{N}=2} = \sum_n \mu^{-n} \mathcal{V}_\text{sugra}^{(n)}$ to determine the rigid limit of the Ward identities that we re-express as follows

$$G_{ij}^A(W_i)^A_B C(W_j)^C_B = \delta^A_B \mathcal{V}_\text{sugra}^{\mathcal{N}=2} - 2\hat{N}_C^A \hat{N}_B^C + 12\hat{S}_B^C \hat{S}_B^C$$

(5.48)

The left hand side of above relation gives the contribution coming from the gauged supersymmetric transformations $\delta^{(g)} \lambda_0^A$ of the two gauginos (2.32); its trace gives the contribution to the scalar potential in the Coulomb branch.

**Rigid Ward identities**

These Ward identities are obtained by taking the rigid limit of the supergravity ones (5.48); they are given by

$$\sum_{i,j=1}^{n_v} \hat{G}_{ij}^A(W_i)^A_B C(W_j)^C_B = \delta^A_B \mathcal{V}_\text{kah}^{\mathcal{N}=2} + \hat{C}_B^A$$

(5.49)

with $\mathcal{V}_\text{kah}^{\mathcal{N}=2}$ like in (E.16). The extra non diagonal $\hat{C}_B^A$, which corresponds to the rigid limit of the anomalous term of the $\mathcal{N}=2$ supercurrent algebra, is an interesting term as it captures crucial data on $\mathcal{N}=2$ supersymmetry breaking. It scales in same manner as the scalar potential $\mathcal{V}_\text{kah}^{\mathcal{N}=2}$ and reads as follows

$$\hat{C}_B^A = 12\hat{S}_B^A \hat{S}_B^C - 2\hat{N}_C^A \hat{N}_B^C$$

(5.50)

where the derivation of the expressions of $\hat{S}_B^A$ and $\hat{N}_B^C$ is reported in Appendix E. By replacing $\hat{N}_B^A$ and $\hat{S}_B^A$ by their values (E.11,E.12) in terms of the embedding tensor components, this anomaly can be expressed like $\hat{C}_B^A = \sum_{c=1}^3 \zeta_c (\tau^c)_B^A$ with isovector $\zeta_c$ as

$$\zeta_c = \frac{1}{2} \varepsilon_{abc} \left( \tilde{\theta}^0_a \theta^b_0 - \theta^a_0 \tilde{\theta}^b_0 \right)$$

(5.51)

Putting back into (5.49), the explicit form of rigid limit of Ward identities in terms of the embedding tensor read then as follows

$$\hat{G}_{ij}^A(W_i)^A_B C(W_j)^C_B = \delta^A_B \mathcal{V}_\text{kah}^{\mathcal{N}=2} + \frac{1}{2} \varepsilon_{abc} \left( \tilde{\theta}^0_a \theta^b_0 - \theta^a_0 \tilde{\theta}^b_0 \right) (\tau^c)_B^A$$

(5.52)

Notice the three following features concerning this analysis: (i) the antisymmetric tensor $\tilde{\theta}^0_a \theta^b_0 - \theta^b_0 \tilde{\theta}^0_a$ involved in the isovector (5.51) is a remarkable quantity; it can be also expressed like $\tilde{\theta}^0_a \mathcal{C}^a = \mathcal{C}^a$ with $\theta^0_a$ standing for an SP(2,R) vector and $\mathcal{C}^a$ the metric of this symplectic group

$$\mathcal{C}^a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(5.53)
This $\text{SP}(2, \mathbb{R})$ should be thought of as the subgroup of $\text{SP}(2n + 2, \mathbb{R})$ associated with the pure supergravity sector.

(ii) the isotriplet $\zeta_c$ is also invariant under the $\text{SP}(2n, \mathbb{R})$ symplectic symmetry of the rigid theory; a property which may be explicitly exhibited by using the constraint relation

$$\vartheta^a_M C^{MN} \vartheta^b_N = 0$$

which splits like

$$\vartheta^a_M C^{a \underline{a}} \vartheta^b_{\underline{a}} = - \vartheta^a_M C^{MN} \vartheta^a_N.$$ 

So the above $\text{SP}(2, \mathbb{R})$ invariant isovector $\zeta_c = \frac{1}{2} \varepsilon_{abc} \vartheta^a_M C^{abc} \vartheta^c_N$ can be as well written like

$$\zeta_c = - \frac{1}{2} \varepsilon_{abc} \vartheta^a_M C^{MN} \vartheta^a_N$$

which coincides with the $\text{SP}(2n, \mathbb{R})$ invariant isovector of the rigid theory of [34]. The two ways (5.50) and (5.54) of expressing the anomaly show that the partial breaking of the rigid theory can be derived either by using the $\text{SP}(2n, \mathbb{R})$ symplectic structure of the vector multiplet of the observable sector as in [34]; or by using the $\text{SP}(2, \mathbb{R})$ symplectic structure associated with the graviphoton of the hidden sector. The two equivalent expressions show as well that in the rigid limit, the $\text{SP}(2n + 2, \mathbb{R})$ symplectic symmetry of the $\mathcal{N} = 2$ supergravity theory gets broken down to $\text{SP}(2, \mathbb{R}) \times \text{SP}(2, \mathbb{R}).$

(iii) Eq (5.50) is a general relation as it is directly expressed in terms of the embedding tensor components. From this formula, we can recover the expression of the traceless matrix $\bar{C}^A_B$ obtained in literature. By choosing the components of the embedding tensor like

$$\bar{\theta}^{a0} = \eta_i m^a, \quad \bar{\theta}^{ia} = m^a, \quad \theta^a_0 = e^a \quad (5.55)$$

where the parameter $m^a, e^a \text{ and } \eta_i$ are as in [30], we end with the following moment maps

$$\mathcal{P}_M^a = \left( - \frac{e^a}{m^a} \right), \quad \mathcal{P}_{\underline{a}}^a = \left( \frac{e^a}{m^a} \right)$$

(5.56)

with $e^a_i \equiv \eta_i e^a$ and $m^a \equiv \eta_i m^a$. The above $\mathcal{P}_M^a$ is precisely the moment maps obtained in the rigid theory of [34]. Notice that in the old frame, that is before performing the symplectic transformation, the choice (5.55) corresponds to an embedding tensor of the form,

$$\vartheta^a_M = \begin{pmatrix} e^a \\ 0 \\ 0 \\ \mu m^a \end{pmatrix}$$

and should be compared with the solution (ii) of (4.96).
6 Partial supersymmetry breaking

In this section, we study the conditions for partial supersymmetry breaking in the rigid limit of \( \mathcal{N} = 2 \) gauged supergravity. By taking this limit, one can distinguish two sectors: (i) the observable sector characterised by the contribution of the vector multiplet to the Ward identities; and (ii) the hidden sector with contribution to Ward identities coming from gravity and hypermatter. To study the breaking we will consider below these two sectors; first, we focus on the partial breaking of \( \mathcal{N} = 2 \) supersymmetry in the observable sector by deriving the scalar potential and the anomaly in the rigid limit. Then, we study of the partial breaking in the hidden sector by following the same approach.

6.1 Observable sector

With the analysis on the \( \frac{1}{\mu} \) expansion of Ward identities of \( \mathcal{N} = 2 \) gauged supergravity given in subsubsection 5.2.2, we can now study the conditions for partial breaking of \( \mathcal{N} = 2 \) supersymmetry in the rigid limit. From eq(5.52), it follows that the rigid limit of the Ward identities of the gauging of two abelian isometries of quaternionic manifold \( M_{QK} \) can be put into the the following hermitian 2 \( \times \) 2 matrix form

\[
\sum_{i,j=1}^{n_v} \mathcal{G}_{ij}(\tilde{W}^i)_{A}^{C}(\tilde{W}^j)^{B}_{C} = H^{A}_{B} \tag{6.1}
\]

with

\[
H^{A}_{B} = \begin{pmatrix}
\hat{\mathcal{V}}_{\text{kah}}^{\mathcal{N}=2} + \zeta_3 & \zeta_1 - i\zeta_2 \\
\zeta_1 + i\zeta_2 & \hat{\mathcal{V}}_{\text{kah}}^{\mathcal{N}=2} - \zeta_3
\end{pmatrix} \tag{6.2}
\]

By help of a unitary transformation \( U \) on the matrices \( (\tilde{W}^i)_{A}^{C} \), the hermitian matrix \( H \) in the right hand side of (6.1) can be put into a diagonal form \( \hat{H} = (U H U^\dagger) \) given by

\[
\hat{H}^{A}_{B} = \begin{pmatrix}
\hat{\mathcal{V}}_{\text{kah}}^{\mathcal{N}=2} + \sqrt{\left|\zeta\right|^2} & 0 \\
0 & \hat{\mathcal{V}}_{\text{kah}}^{\mathcal{N}=2} - \sqrt{\left|\zeta\right|^2}
\end{pmatrix} \tag{6.3}
\]

where

\[
\left|\zeta\right|^2 \equiv (\zeta_1)^2 + (\zeta_2)^2 + (\zeta_3)^2 \tag{6.4}
\]

standing for the norm of the real isovector \( \zeta_a \). Notice that the gauging of two abelian quaternionic isometries requires an embedding tensor \( \vartheta_M^a = (\vartheta_M^m, \vartheta_M^3) \) of the form

\[
\vartheta_M^m = \begin{pmatrix}
\theta_{\Lambda}^m \\
\bar{\theta}^{m\Lambda}
\end{pmatrix}, \quad m = 1, 2, \quad \Lambda = 0, i = 1, ..., n \tag{6.5}
\]

and

\[
\vartheta_M^3 = 0 \tag{6.6}
\]

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Therefore the isovector $\zeta_c$ in (5.51) and $|\zeta|^2$ should be read as

$$\zeta_3 = \frac{1}{2} \varepsilon_{3mn} \left( \bar{\theta}^{m0} \theta^n_0 - \theta^n_0 \bar{\theta}_0^m \right), \quad |\zeta|^2 = (\zeta_3)^2 \quad (6.7)$$

This feature, required by the gauging of two abelian quaternionic isometries, means that the isovector $\zeta_c$ has in fact one component as follows

$$\zeta_c = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \bar{\theta}_0^{10} - \theta^1_0 \bar{\theta}^{20} \end{pmatrix} \quad (6.8)$$

It breaks the SU(2) $R$-symmetry down to U(1)$_R$ in agreement with partial breaking of $\mathcal{N} = 2$ supersymmetry. So the matrix eq (6.1) may be interpreted like

$$\sum_{i,j=1}^{n_v} \hat{g}_{ij} (\tilde{W}^n)^A_C (W^j)^B = \begin{pmatrix} \hat{\nu}^N=2_{kah} + |\zeta_3| & 0 \\ 0 & \hat{\nu}^{N=2}_{kah} - |\zeta_3| \end{pmatrix} \quad (6.9)$$

with $(\tilde{W}^n)^A_C = U^A_B (\tilde{W}^i)^B_C$. The $\mathcal{N} = 2$ supersymmetry in the rigid limit of the gauged supergravity theory is then partially broken if the following condition holds

$$\hat{\nu}^N_{kah} = |\zeta_3| \quad , \quad |\zeta_3| \neq 0 \quad (6.10)$$

it is recovered if $|\zeta_3| \to 0$. This condition agrees with known results in this matter as recently done in [34].

### 6.2 Hidden sector

In the hidden sector of $\mathcal{N} = 2$ gauged supergravity concerning the gravity and matter supermultiplets, the contribution of gauging quaternionic isometries to the rigid limit of the Ward identities can be expressed like

$$\sum_{i,j=1}^{n_v} \hat{g}_{ij} (\tilde{W}^n)^A_C (W^j)^B = \delta_B^A \nu_{\text{sugra}} - \sum_{i,j=1}^{n_v} G_{ij} (\tilde{W}^i)^A_C (\tilde{W}^j)^B \quad (6.11)$$

This quantity reads in terms of the moment maps and the covariantly covariant sections along the graviphoton direction as follows

$$2 \tilde{N}^A_C \tilde{N}^C_B - 12 \tilde{S}^A_C \tilde{S}^C_B = \delta_B^A \nu_{\text{sugra}} - \sum_{i,j=1}^{n_v} G_{ij} (\tilde{W}^i)^A_C (\tilde{W}^j)^B \quad (6.12)$$

Substituting the $\hat{P}^{a_l}$’s and the $V\omega$’s by their expressions, we can put the above expression into the form

$$2 \tilde{N}^A_C \tilde{N}^C_B - 12 \tilde{S}^A_C \tilde{S}^C_B = 2 \delta_B^A \Delta \tilde{V} - \tilde{C}^A_B \quad (6.13)$$

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where the rigid anomaly $\tilde{C}_B^A$ is as in eq(5.50) and where $\Delta \hat{\mathcal{V}}$ is given by (E.20). Thinking of $2\delta_B^A \Delta \hat{\mathcal{V}} - \tilde{C}_B^A$ as a hermitian $2 \times 2$ matrix $\mathbb{H}_B^A$ as

$$\mathbb{H}_B^A = \begin{pmatrix} 2\Delta \hat{\mathcal{V}} - \zeta_3 & -\zeta_1 + i\zeta_2 \\ -\zeta_1 - i\zeta_2 & 2\Delta \hat{\mathcal{V}} + \zeta_3 \end{pmatrix}$$

We can diagonalise it by help of a unitary transformation to bring it to the following form

$$\hat{\mathbb{H}}_B^A = \begin{pmatrix} 2\Delta \hat{\mathcal{V}} + \sqrt{\zeta_3^2} & 0 \\ 0 & 2\Delta \hat{\mathcal{V}} - \sqrt{\zeta_3^2} \end{pmatrix}$$

where $|\zeta_3|^2$ is as in (6.4). Like for the observable sector and because of the gauging of two abelian isometries, which breaks SU(2)$_R$ symmetry down to U(1)$_R$, the above diagonal matrix reduces to

$$\hat{\mathbb{H}}_B^A = \begin{pmatrix} 2\Delta \hat{\mathcal{V}} + |\zeta_3| & 0 \\ 0 & 2\Delta \hat{\mathcal{V}} - |\zeta_3| \end{pmatrix}$$

Thus, $\mathcal{N} = 2$ supersymmetry in the hidden supergravity sector is partially broken if the following conditions hold

$$-2\Delta \hat{\mathcal{V}} = \pm |\zeta_3| , \quad |\zeta_3| \neq 0$$

Replacing $\Delta \hat{\mathcal{V}}$ and $|\zeta_3|$ by their respective expressions as in eqs(E.20) and (6.7), we obtain

$$\sum_{m=1,2} \left( \tilde{\theta}_0^m \tilde{\theta}_m^0 + 4\theta_0^m \theta_0^m \right) = \pm \sum_{m=1,2} \varepsilon_{3mn} \left( \tilde{\theta}_0^m \theta_0^n - \theta_0^m \tilde{\theta}_0^n \right)$$

and leads to the following conditions on the electric $\theta_0^m$ and the magnetic $\tilde{\theta}_0^m$ components of the embedding tensor

$$\left( \tilde{\theta}_0^{10} \right)^2 + \left( \tilde{\theta}_0^{20} \right)^2 + 4 \left( \theta_0^1 \right)^2 + 4 \left( \theta_0^2 \right)^2 \mp \left( \tilde{\theta}_0^{10} \theta_0^{20} - \tilde{\theta}_0^{20} \theta_0^{10} \right) = 0$$

Comparing these conditions with the analysis of [30], where the particular choice $\theta_0^1 = \tilde{\theta}_0^{20} = 0$ has been used for the values of the embedding tensor, the above constraint relations reduce to

$$\left( \theta_0^0 \right)^2 \pm \tilde{\theta}_0^{10} \theta_0^0 + \frac{1}{4} \left( \theta_0^{10} \right)^2 = 0$$

These constraints factorise like $\left( \theta_0^0 \mp \frac{1}{2} \theta_0^{10} \right)^2 = 0$ and are solved like $\tilde{\theta}_0^{10} = \pm 2\theta_0^2$ in perfect agreement with the result of [30].
7 Conclusion and discussions

Using a novel manner of rescaling fields in $\mathcal{N} = 2$ gauged supergravity, we have developed in this paper the explicit derivation of the rigid limit of Ward identities of the gauging of two abelian quaternionic isometries agreeing with known results in literature; the two abelian gaugings are necessary for partial supersymmetry breaking. This explicit analysis allowed us to figure out the $\frac{1}{\mu}$ expansion ($\frac{1}{\mu}$ of order $\frac{m^3/2}{M_{pl}} << 1$) of the basic quantities of this $\mathcal{N} = 2$ theory such as the symplectic sections of special Kahler geometry $M_{SK}$; the embedding tensor $\vartheta^m_M$ and the moment maps $P^m_M$ as well as Ward identities. It also allowed us to determine the rigid limit of the induced scalar potential $V_{kah}$ in the observable and hidden sectors; and the rigid limit of the central anomaly $C^A_B = \zeta_a (\tau^a)^A_B$ of the $\mathcal{N} = 2$ supercurrent algebra as well as the conditions for partial supersymmetry breaking. Known results in literature have been recovered by making particular choices of the embedding tensor components; and new features have been also obtained.

To study the $\frac{1}{\mu}$- expansion of Ward identities of $\mathcal{N} = 2$ gauged supergravity and their rigid limit, we have proceeded as follows: (a) We have used the approach of [30] based on working with a special holomorphic prepotential $F$ depending, in addition to the usual complex holomorphic $X^A (z)$ of the Coulomb branch, on $n_V$ parameters $\eta_i$ whose geometric interpretation has been given in present study. (b) We have organised the presentation of our analysis into five complementary sections in order to exhibit explicitly the contributions of each one of the three sectors of the theory namely gravity, gauge and matter. Our motivation in doing so is to take the opportunity of this explicit analysis in order to (i) complete partial results in literature, (ii) shed more light on some crucial steps in the derivation of the rigid limit; and (iii) give some generalisations like the ADE classification developed in subsection 4.2. The main lines of the rigid limit study performed in present paper may be summarised in three principal steps as follows:

First, we started by introducing some basic tools on the scalar manifold $M_{scal}$ of $\mathcal{N} = 2$ supergravity which factorises as $M_{SK} \times M_{QK}$ with $M_{SK}$ standing for the special Kahler submanifold, associated with the Coulomb branch, and $M_{QK}$ referring to the submanifold associated with the matter sector. We have also recalled useful relations on Ward identities in $\mathcal{N} = 2$ gauged supergravity and on solutions of the fermionic shifts $(W^i)^A_B$, $N^A_a$ and $S^A_B$, induced by the gauging of quaternionic isometries, in terms of the covariantly holomorphic sections $V^M (z, \bar{z})$ living on $M_{SK}$ as well as the vielbein $E^A_{\alpha \dot{\alpha}}$ and the moment maps $P^m_M$ of the special quaternionic $M_{QK}$ given by eqs(2.35).

In the second step, we have studied the $\frac{1}{\mu}$ expansion of the various $SP (2n_V + 2)$ symplectic sections of $M_{SK}$ with expansion parameter $\frac{1}{\mu} = \frac{\Lambda}{M_{pl}}$ where the massive scale $\Lambda << M_{pl}$
is of order of \( m^{3/2} \), the mass of gravitino multiplet generated by partial supersymmetry breaking as sketched by eq\((4.69)\). These expansions are needed for the determination of the rigid limit of Ward identities \( \sum_t \gamma_t \delta A_t \chi^C_t \delta C^B_t = \delta^B_A \chi^N = 2 \) as well as the derivation of the rigid limit of the induced scalar potential \((E.16)\) and the rigid limit of the anomalous central charge matrix \( (5.50, 5.51, 5.54) \) of the \( \mathcal{N} = 2 \) supercurrent algebra \( (4.112-4.97) \).

After that we have studied the gauging of abelian isometries of a family of special quaternionic manifolds \( M_{QK}^{(n_H=r)} \) whose first element is given by \( M_{QK}^{(n_H=1)} = SO(1,4)/SO(4) \) involving one hypermultiplet. Next, we have addressed the solving of the embedding tensor constraint equation \((4.90)\). We derived different solutions for \((4.90)\) which include the interesting dyonic solution as shown by eqs\((4.96, 4.97)\). We have also given an interpretation of the embedding tensor \( \vartheta^u_M \) in terms of D-branes of type II strings on Calabi-Yau threefolds and type IIA/IIB mirror symmetry.

In the third step, we have first described the rescalings of the various fields of the \( \mathcal{N} = 2 \) gauged supergravity. In this regards, we have proposed another way to implement the scale dimensions that leads to the right mass dimensions of the components of the embedding tensor which can be interpreted in terms of FI couplings scaling as mass\(^2\) as in eq\((5.15)\); see also eqs\((5.34-5.35)\). In our approach fields belonging to the same \( \mathcal{N} = 2 \) representation have the same dependence on the scale \( \Lambda \) as shown on eqs\((5.2, 5.5)\) and \((5.16, 5.21)\). Then, we have derived the rigid limit of Ward identities of gauged quaternionic isometries of the special quaternionic manifold \( M_{QK}^{(n_H=1)} \). As a result in this direction, we have shown that the rigid limit of the induced scalar potential is given by eqs\((E.13-E.16)\). We have also derived the explicit expression of the rigid limit \( \tilde{C}^A_B \) of the anomalous central extension matrix \( C^A_B \) of the supercurrent algebra in \( \mathcal{N} = 2 \) supergravity \( (4.112-4.113) \) and determined the conditions for partial supersymmetry breaking both in observable and hidden sectors; see eq\((6.10, 6.17-6.19)\). In this context, known results in literature have been rederived by making particular choices of the embedding tensor components.

We end this study by making a rough discussion regarding the gauging of abelian quaternionic isometries in \( \mathcal{N} = 2 \) supergravity with several hypermultiplets parameterising \( M_{QK}^{(ADE)} \), a rigorous analysis requires the derivation of exact the solutions of Ward identities extending eqs\((2.35)\). Here, we use general arguments to derive the structure of the rigid limits of the induced scalar potential \( \tilde{V}_{kah}^{(ADE)} \) and the matrix anomaly \( (C^{ADE})^A_B \) in the observable sector.

We begin by making a comment on the extension and describing the motivation behind the ADE generalisation. As a comment, notice that in our ADE model to deal with the matter sector of \( \mathcal{N} = 2 \) gauged supergravity, the number \( n_H \) of hypermultiplets is interpreted as the rank \( r \) of a finite dimensional ADE Lie algebra; the simplest model corresponds
therefore to \(su(2)\) algebra; it concerns the \(SO(1,4)/SO(4)\) geometry extensively studied in the present paper. The ADE geometry for several hypermultiplets is motivated from a remarkable property of the 4- dimensional real manifold \(SO(1,4)/SO(4)\) of the \(\mathcal{N} = 2\) gauged supergravity with \(n_H = 1\). For this simple model, the factor \(e^{2\varphi}\), involved in the metric (4.28) of the coset \(SO(1,4)/SO(4)\), may be put in correspondence with the \(e^{2u}\) term appearing in the 2d- Liouville theory with integrable field equation given by

\[
\frac{\partial^2 u}{\partial \bar{z} \partial z} + \kappa e^{2u} = 0 , \quad u = u(z, \bar{z}) \tag{7.1}
\]

The integrability of this 2d- field equation is known to be captured by a Lax pair \(L_\pm\) valued in a hidden \(su(2) \simeq A_1\) algebra having a flat curvature. By using the formal correspondence between the exponential factors appearing in the two theories \((e^{2u} \leftrightarrow e^{2\varphi})\), the metric (4.28) can be conjectured to have as well a hidden \(su(2)\) symmetry where the number \(n_H = 1\) is interpreted as the rank of \(su(2)\). Within this view, we have built in subsection 4.2 a family of 4r- dimensional manifolds \(M_{QK}^{(ADE)}\) with metric as in (4.41) generalising the (4.28) of the 4- dimensional \(M_{QK}^{(A_1)} \sim SO(1,4)/SO(4)\). These ADE geometries can be put in turns into correspondence with integrable ADE field equations of 2d- Toda field theories based on finite dimensional ADE Lie algebras

| Integrable 2d- QFT | hypermultiplet geometry | hidden symmetry |
|-------------------|-------------------------|----------------|
| 2d- Liouville      | \(M_{QK}^{(A_1)}\)     | \(su(2)\)      |
| ↓                 | ↓                       | ↓              |
| 2d- Toda          | \(M_{QK}^{(ADE)}\)     | ADE            |

Within this generalisation, we can extend the analysis done for \(M_{QK}^{(A_1)}\) to \(M_{QK}^{(ADE)}\); in particular the question concerning the explicit derivation of the expressions of the rigid limits of the induced scalar potential (E.16) and the anomaly (5.54) in the observable sector. Recall that for \(n_H = 1\), the rigid limit of the induced scalar potential is given by

\[
\hat{V}_{\mathcal{N}=2}^{kah} = \frac{1}{2} \hat{v}_M^{a} \hat{M}^{MN} \hat{v}_N^{a} \tag{7.3}
\]

with \(SP(2n_V)\) symplectic tensor \(\hat{M}^{MN}\) living on \(M_{SK}\), and so independent on the \(M_{QK}^{(A_1)}\) quaternionic geometry as shown by (C.22). The dependence on properties of \(M_{QK}^{(A_1)}\) is captured by the embedding tensor \(\hat{v}_M^{a}\) which, according to (4.82-4.83), has the generic form \(\hat{v}_M^{as}\) with an extra index running as \(s = 1,...n_H\). By help of these arguments, the natural candidate extending \(\hat{V}_{\mathcal{N}=2}^{kah}\) for the case of ADE geometries reads as follows

\[
\hat{V}_{\mathcal{N}=2}^{(ADE)}^{kah} = \frac{1}{4} K_{rs} \hat{v}_M^{ar} \hat{v}_N^{as} \hat{M}^{MN} \tag{7.4}
\]
where the symmetric $K_{rs}$ is the $n_H \times n_H$ Cartan matrix of ADE Lie algebras. For the case $n_H = 1$, we recover the expression of $\hat{\psi}_{kah}^{N=2}$. The structure of the matrix anomaly $(C^{ADE})^A_B$ associated with the $M^{ADE}_{QK}$ geometries can be derived by using the same arguments as for the induced scalar potential. Expression $(C^{ADE})^A_B$ in terms of the development $\zeta^{ADE}_c (\tau^c)^A_B$, we can write down the generalisation of $\zeta^{(A_1)}_c = \frac{1}{2} \epsilon^{abc} (\tilde{\theta}^a_0 \theta^b_r - \theta^a_0 \tilde{\theta}^b_r)$ of eq(5.51). It reads as

$$\zeta^{(ADE)}_c = \frac{1}{4} \epsilon_{abc} K_{rs} \left( \hat{\theta}^{a0s} \hat{\theta}^{b0r} - \theta^a_0 \tilde{\theta}^{b0r} \right) \quad (7.5)$$

where $\theta^a_0$ and $\tilde{\theta}^{a0s}$ are the electric and magnetic components of the embedding tensors $\vartheta^m_\Lambda$ and $\tilde{\vartheta}^{a\Lambda s}$ along the graviphoton direction as in (4.83). By using the constraint eqs(4.90), which generalises for the case of ADE geometries like $K_{rs} \vartheta^m_M C^{MN} \vartheta^n_N = 0$, we can re-express the above isovector $\zeta^{(ADE)}_c$ in terms of the electric $\vartheta^a_i$ and magnetic $\tilde{\vartheta}^{a_is}$ along the Coulomb branch directions. We have

$$\zeta^{(ADE)}_c = -\frac{1}{4} \epsilon_{abc} K_{rs} \vartheta^a_M C^{MN} \vartheta^{bs}_N \quad (7.6)$$

where $\vartheta^a_M = \delta^a_N \vartheta^a_R$ and where the components of the embedding tensors $\vartheta^a_M$ are as in eq(4.82). Here, the antisymmetric $C^{MN}$ is the metric of the $\text{SP}(2n_V)$ symplectic symmetry of the rigid theory, and the symmetric $n_H \times n_H$ matrix $K_{rs}$ is the Cartan matrix of finite dimensional ADE Lie algebras. The condition for partial breaking of $N = 2$ supersymmetry is the same as in (6.10). It would be interesting to check explicitly the above relations for $\hat{\psi}_{kah}^{ADE}$ and $\zeta^{(ADE)}_c$; this needs working out the exact solutions of Ward identities for the generalised geometries; progress in this direction will be reported in a future occasion.

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**Appendices**

In this appendix, which is organised into five sections, we give some general results on $\mathcal{N} = 2$ supergravity which are useful for our study; but which have not been explicitied in the core of the paper; we also give some details regarding sections 3 and 5. In Appendix A, we recall some results on the Lagrangian of $\mathcal{N} = 2$ supergravity for both ungauged and gauged versions by focusing on the bosonic part of the degrees of freedom of the theory described in section 2. We also give the general expressions of the constraints on the embedding tensor $\vartheta^a_M$. In Appendix B, we collect useful tools on the special Kahler and the quaternionic Kahler manifolds, and also give some explicit computations; certain of
the geometrical tools are standard ones; but for completeness of our study, we have judged interesting to recall them here for direct access. In Appendix C, we study the rigid limit of the coupling matrices $N_{\Lambda \Sigma}$ and $U^{MN}$. In Appendix D, we describe the solutions of the Ward identities in $\mathcal{N} = 2$ gauged supergravity by focussing on abelian quaternionic isometries and in Appendix E, we give the rigid limit of the scalar potential.

A Appendix A: $\mathcal{N} = 2$ supergravities in 4d

First, we describe the bosonic part of the Lagrangian density of 4d $\mathcal{N} = 2$ ungauged supergravity in presence of $n_V$ abelian vector supermultiplets, with gauge fields $A_\mu^i$, and $n_H$ hypermultiplets. Then, we give its gauged extension with gauging isometries in the scalar manifold $M_{SK} \times M_{QK}$.

A.1 Ungauged supergravity

First recall that the bosonic fields of the 4d $\mathcal{N} = 2$ ungauged supergravity as appearing in the three supermultiplets (2.1) are given by: (i) the space time metric $g_{\mu \nu} = e^m_\mu n_{mn} e^m_\nu$ with vierbein $e^m_\mu$, (ii) the complex scalars $z^i$ of the $n_V$ vector supermultiplets; and (iii) the real $4n_H$ scalars $Q^u$ of the hypermultiplets. The list of the abelian gauge fields is given by $A_\mu^\Lambda = (A_\mu^0, A_\mu^i)$ with $A_\mu^0$ the graviphoton.

Following [40, 43], the bosonic part of the Lagrangian density of the 4d $\mathcal{N} = 2$ ungauged supergravity is given by

$$e^{-1} \mathcal{L}^{\text{bos ungauged}} = -\frac{R}{2} + \frac{1}{2} G_{ij} \partial_\mu z^i \partial_\mu \bar{z}^j + \frac{1}{2} h_{uv} \partial_\mu Q^u \partial_\mu Q^v + \frac{1}{4} I_{\Sigma \Lambda} F^{\Sigma}_{\mu \nu} F_{\mu \nu} + \frac{1}{8} R_{\Sigma \Lambda} \varepsilon^{\mu \nu \rho \sigma} F^{\Sigma}_{\mu \nu} F^{\Lambda}_{\rho \sigma}$$

(A.1)

where $e = \det (e^m_\mu)$ and where

$$F^{\Lambda}_{\mu \nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$$

(A.2)

is the field strength of the vector fields $A^\Lambda_\mu$. The local gauge metrics $I_{\Sigma \Lambda} = \text{Im} N_{\Sigma \Lambda}$ and $R_{\Sigma \Lambda} = \text{Re} N_{\Sigma \Lambda}$ generalize respectively the usual $-1/g^2$ and the theta-term of the Yang-Mills kinetic terms with gauge coupling $g$. Moreover, $G_{ij} = G_{ij}(z, \bar{z})$ is the metric of the complex n- dimensional special Kahler manifold $M_{SK}$ parameterised by the complex scalars $z^i$ of the Coulomb branch. The local field matrix $h_{uv} = h_{uv}(Q)$ is the hyperkahler metric of the real $4n_H$ quaternionic manifold $M_{QK}$. Notice that in eq(A.1), there is no scalar potential term; it can be generated, without breaking of the supersymmetry, only through the gauging isometries procedure.
A.2 Dyonic gauged $\mathcal{N} = 2$ supergravity

Following [30, 35], the dyonic gauging of a subgroup $H$ of the global isometry group $G$ of the scalar manifold of $\mathcal{N} = 2$ gauged supergravity

$$M_{\text{scal}} = M_{SK} \times M_{QK}$$

(A.3)

can be encoded in the embedding tensor $\vartheta^u_M$ having two indices (two legs), one in the Coulomb branch and the other in the matter one. $\vartheta^u_M$ is not an arbitrary quantity, it obeys constraint relations. Let us first give properties of this tensor and turns after to the Lagrangian density.

- **Constraints on $\vartheta^u_M$**

Denoting by $\mathcal{T}_a = (t_a, t_m)$ the set of generators of global isometry group $G$ of (A.3) and by $X_M = (X_A, X^A)$ a generators basis of a subgroup $H \subset G$, the embedding can be stated as $X_M = \vartheta^a_M T_a$; and reads explicitly, by exhibiting the electric $t_a$ and magnetic $t_m$, as follows

$$X_M = \vartheta^a_M t_a + \vartheta^m_M t_m$$

(A.4)

with action of the matrix representation $(X_M)^P_N \equiv X^P_{MN}$ like

$$(X_M)^P_N = \vartheta^a_M (T^P_a)_N$$

(A.5)

Consistency of the gauging of the subgroup $H$ is guaranteed by the following set of linear and quadratic constraints relations on the embedding tensor

$$\vartheta^a_M \vartheta^b_N f^c_{ab} + (X_M)^P_N \vartheta^c_P = 0$$
$$\vartheta^m_M \vartheta^m_N f^m_{mn} + (X_M)^P_N \vartheta^m_P = 0$$

(A.6)

and

$$X_{(MNP)} \equiv C_{(PQ)} X_{MN}^Q = 0$$

(A.7)

as well as

$$\vartheta^a_M \vartheta^b_N f^c_{ab} + (X_M)^P_N \vartheta^c_P = 0$$
$$\vartheta^m_M \vartheta^m_N f^m_{mn} + (X_M)^P_N \vartheta^m_P = 0$$

(A.8)

and

$$\vartheta^a_M C^{MN} \vartheta^b_N = 0$$
$$\vartheta^a_M C^{MN} \vartheta^m_N = 0$$
$$\vartheta^m_M C^{MN} \vartheta^m_N = 0$$

(A.9)

The $f^a_{ab}$ and $f^m_{mn}$ in (A.8) are the structure constants respectively associated with $t_a, t_m$.

The conditions (A.8) are closure constraints, they follow from

$$[X_M, X_N] = -X^P_{MN} X_P$$

(A.10)
Moreover, the first two equalities in (A.9) descend from (A.7) and (A.8) while the third one has to be imposed independently [30, 35]. By using the embedding tensor, we can define gauged quantities $\kappa^i_M$ out of ungauged counterparts $\kappa^i_\alpha$; the examples the gauged Killing vectors $(\kappa^i_M, \kappa^u_M)$ on $M_{SK} \times M_{QK}$ and the moment maps $(\mathcal{P}^0_M, \mathcal{P}^a_M)$, we have

\[
\begin{align*}
\kappa^i_M &= \frac{\partial^a_M}{\partial^a_M} \kappa^i_a \\
\kappa^u_M &= \frac{\partial^m_M}{\partial^m_M} \kappa^u_m
\end{align*}
\]

with $(\kappa^i_a, \kappa^u_m)$ ungauged Killing vectors, and

\[
\begin{align*}
\mathcal{P}^0_M &= \frac{\partial^a_M}{\partial^a_M} P_a \\
\mathcal{P}^a_M &= \frac{\partial^m_M}{\partial^m_M} P^a_m
\end{align*}
\]

with $(P_a, P^a_m)$ ungauged moments. These quantities satisfy the following algebras [30]

\[
iG_{ij} \kappa^i_M \kappa^j_N = \frac{1}{2} X^P_{MN} \mathcal{P}_P^0
\]

and

\[
2 K^a_{uv} \kappa^u_M \kappa^v_N + \varepsilon^{abc} P^b_M \mathcal{P}_N^c = X^P_{MN} \mathcal{P}_P^a
\]

where $K^a_{uv}$ is the hyperkahler 2-forms.

- $\mathcal{N} = 2$ gauged supergravity Lagrangian

The bosonic part of the lagrangian density of the $\mathcal{N} = 2$ gauged supergravity, with bosonic degrees of freedom as described previously, is obtained by covariantizing space time derivatives as follows [40]

\[
e^{-1} \mathcal{L}^{\text{bos}}_{\text{gauged}} = -\frac{1}{2} R + G_{ij} \mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^j + h_{uv} \mathcal{D}_\mu Q^u \mathcal{D}^\mu Q^v + X^P_{MN} \mathcal{D}_\mu \mathcal{P}_N^c \mathcal{P}_P^c
\]

Here, $\mathcal{L}_{\text{top}}$ and $\mathcal{L}_{\text{CS}}$ are topological terms required to maintain the gauge invariance; they are given by

\[
\mathcal{L}_{\text{top}} = -\frac{1}{8e} \varepsilon^{\mu \nu \rho \sigma} \mathcal{D}_\mu A_{\rho \sigma} + X_{MNA} A^M_\rho A^N_\sigma - \frac{1}{4} \vartheta^a_{\Lambda} B_{\mu \nu \alpha}
\]

and

\[
\mathcal{L}_{\text{CS}} = -\frac{1}{6e} \varepsilon^{\mu \nu \rho \sigma} X_{MNA} A^M_\mu A^N_\nu (\partial_\rho A^\Lambda_\sigma + \frac{1}{4} X^P_{\rho \sigma} A^P_\Lambda A^Q_\sigma) - \frac{1}{6e} \varepsilon^{\mu \nu \rho \sigma} X^P_{MN} A^M_\mu A^N_\nu (\partial_\rho A_{\sigma \Lambda} + \frac{1}{4} X_{PQA} A^P_\rho A^Q_\sigma)
\]

with

\[
\begin{align*}
\mathcal{D}_\mu z^i &= \partial_\mu z^i + \vartheta^a_{M \mu} A^M_\mu k^i_a \\
\mathcal{D}_\mu Q^u &= \partial_\mu Q^u + \vartheta^m_{M \mu} A^M_\mu k^u_m
\end{align*}
\]
and
\[ A^M = \begin{pmatrix} A^\Lambda \\ A_{\mu}^{\Lambda} \end{pmatrix}, \quad \Lambda = 0, I, \quad I = 1, \ldots, n \vspace{1em} \]

We also have
\[
\mathcal{F}_{\mu
u}^\Lambda = \partial_\mu A^\Lambda_{\nu} - \partial_\nu A^\Lambda_{\mu} \\
\tilde{\mathcal{F}}_{\mu
u}^\Lambda = \mathcal{F}_{\mu
u}^\Lambda + \frac{1}{2} \theta^\Lambda_{\mu
u} B_{\mu\nu} \mu
\]

The $B_{\mu\nu}$ are massless antisymmetric tensor fields which must be introduced in order to construct gauge covariant field strengths $\tilde{\mathcal{F}}_{\mu
u}^\Lambda$. The scalar potential $V(z, \bar{z}, Q)$ is given by
\[
V(z, \bar{z}, Q) = (G_{ij} K^i_M K^j_N + 4 h_{uv} K^u_M K^v_N) V^M V^N + (U^{MN} - 3 V^M V^N) \mathcal{P}_{M}^a \mathcal{P}_N^a \]

where $V^M$ is the covariantly holomorphic section of the special Kahler manifold $M_{SK}$

\[ U^{MN} = U_i^M G^i_j U_j^N \] (A.21)

with $U_i^M$ is holomorphic section of $M_{SK}$.

\section*{B Appendix B: Scalar manifold in $\mathcal{N} = 2$ gauged supergravity}

In this appendix, we collect useful geometrical features on the scalar manifold $M_{scal}$ in $\mathcal{N} = 2$ gauged supergravity. As this manifold factorises like $M_{scal} = M_{SK} \times M_{QK}$, we first describe properties of the special Kähler $M_{SK}$, and turn after to those of the quaternionic Kahler $M_{QK}$.

\subsection*{B.1 Special Kahler Manifolds $M_{SK}$}

A special Kähler manifold $M_{SK}$ is a Hodge- Kahler manifold endowed with a holomorphic flat vector bundle with structure group $SP(2n, \mathbb{R})$ satisfying special properties [30, 40, 52, 54–56, 74–79]. In what follows, we describe properties of sections on $M_{SK}$ and the decomposition of eq(A.21) as in (C.15).

\begin{itemize}
  \item holomorphic section $\Omega^M(z)$ and Kahler metric
\end{itemize}

The holomorphic section of $M_{SK}$ is given by eq(3.12) that we recall hereafter

\[ \Omega^M(z) = \begin{pmatrix} X^\Lambda(z) \\ F_\Lambda(z) \end{pmatrix}, \quad \Lambda = 0, \ldots, n \] (B.1)

\footnote{The $B_{\mu\nu}$'s can be also interpreted as the dualized scalars used in gauging isometries as in sec 2.}
It transforms as a global $SP(2n+2,\mathbb{R})$ symplectic vector; holomorphy and symplectic symmetries act on $\Omega^M(z)$ like

$$\Omega^M_{(\alpha)} = e^{i f(\alpha,\beta)} \left[ R_{(\alpha,\beta)}^N \Omega^N_{(\beta)} \right]$$

where: (i) the $f(\alpha,\beta) = f(z^{(\alpha)}, \bar{z}^{(\beta)})$ is a holomorphic transition function connecting two overlapping patches $U_{(\alpha)}$ and $U_{(\beta)}$ on the special Kahler manifold $M_{SK}$ with local coordinates $z^{(\alpha)}$ and $z^{(\beta)}$ ($U_{(\alpha)} \cap U_{(\beta)} \neq \emptyset$); and (ii) $[ R_{(\alpha,\beta)}^N ]$ is a constant $SP(2n+2,\mathbb{R})$ matrix.

From the holomorphic section $\Omega^M$, one can define other objects like the Kähler potential $K = K(z, \bar{z})$, the covariantly holomorphic section $V^M = V^M(z, \bar{z})$ and its covariant derivatives $U_i^M = D_i V^M(z, \bar{z})$. For the Kahler potential $K$, it is related to $\Omega^M$ and its complex conjugate like

$$K = -\log[i \Omega^M C_{MN}\Omega^N]$$

where $C_{MN}$ is the invariant $SP(2n+2,\mathbb{R})$ metric

$$C_{MN} = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix} .$$

With eq(B.3), we can also build the metric $G_{ij}$ of $M_{SK}$ and the closed Kahler 2-form $K_2$ as given hereafter

$$G_{ij} = \partial_i \partial_j K , \quad K = K(z, \bar{z})$$

which is invariant under Kahler transformation

$$K_{(\alpha)} = K_{(\beta)} - f_{(\alpha,\beta)} - \bar{f}_{(\alpha,\beta)}$$

and

$$K_2 = i G_{ij} dz^i \wedge d\bar{z}^j , \quad dK_2 = 0$$

From $dK_2 = 0$, we learn that

$$K_2 = d\omega_1^0$$

where $\omega_1^0$ is the $U(1)_R$ Kahler connection 1-form

$$\omega_1^0 = -\frac{i}{2} \left[ \partial K - \bar{\partial} K \right] , \quad \partial K = \partial_i K dz^i$$

• Sections $V^M$ and $U^M_i$

The covariantly holomorphic section is defined as $V^M = e^{\frac{K}{2}} \Omega^M$ with symplectic components like

$$V^M = \begin{pmatrix} \Upsilon^\Lambda \\ \Gamma_\Lambda \end{pmatrix} , \quad V^M = V^M(z, \bar{z})$$

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it satisfies the properties
\[ D_i V^M \equiv \left( \partial_i - \frac{\partial_i K}{2} \right) V^M = 0 \] (B.11)
and
\[ -i V^M C_{MN} \bar{V}^N = 1 , \quad i \bar{V}^M C_{MN} V^N = 1 , \quad V^M C_{MN} V^N = 0 \] (B.12)
Eqs.(B.12) can be obtained from the definition of \( V^M \) and (B.3). By using (B.2) and (B.6), we obtain the transformation of \( V^M \) under the U(1) Kahler and SP(2n + 2, R) symmetries namely
\[ V_{(\alpha)} = e^{i \text{Im} f(\alpha, \beta)} \left[ \mathcal{R}_{(\alpha, \beta)} \right]_N^M V_{(\beta)} \] (B.13)
Regarding the \( U(1) \)-covariant derivatives \( U_i = D_i V \); it reads explicitly like
\[ U_i = \left( \partial_i + \frac{\partial_i K}{2} \right) V , \quad \partial_i = \frac{\partial}{\partial z^i} \] (B.14)
In a special Kahler manifold, the section \( V^M \) and its covariant derivative \( U^M_i \) need to satisfy the following properties
\[ D_i U^M_j \equiv \partial_i U^M_j + \frac{\partial_i K}{2} U^M_j - \Gamma^M_{ij} U^M_k = 0 \] (B.15)
and
\[ D_i \bar{U}^M_j = \mathcal{G}_{ij} \bar{U}^M_j \]
\[ V^M C_{MN} U_i^N = 0 \]
\[ V^M C_{MN} \bar{U}_k^N = 0 \] (B.16)
From above equation, one has [75]
\[ \mathcal{G}_{kl} = i U^M_k C_{MN} \bar{U}_l^N , \quad U^{iM} C_{MN} \bar{U}_j^N = -i \delta^i_j , \quad \bar{U}^{\bar{j}M} C_{MN} U_i^N = i \delta^\bar{j}_i \] (B.17)

- **Decomposing the factor \( U^{MN} \)**
The symplectic tensor, given by
\[ U^{MN} = \mathcal{G}^{ij} U_i^M U_j^N , \] (B.18)
appears in the study of Ward identities and the induced scalar potential (2.31); its reducibility is interesting in the derivation of its rigid limit. Following, [30], by using \( V^M \) and \( U^M_i \), we can construct a \((2n + 2) \times (2n + 2)\) matrix \( \mathbb{L} = \mathbb{L}(z, \bar{z}) \) whose entries as follows
\[ \mathbb{L} = \begin{pmatrix} V^M & \bar{U}^M_I & V^M \bar{U}^M_I' \\ \bar{U}^{\bar{i}M}_I & V^M & \bar{U}^{\bar{i}M}_I' \\ \bar{U}^{\bar{i}M}_I' & \bar{U}^{\bar{i}M}_I & \bar{U}^{\bar{i}M}_I' \end{pmatrix} , \quad \mathbb{L} = (\mathbb{L}^M_{N'}) \] (B.19)
where \( M, N' = 0, I' \) are indices of \( \text{SP}(2n + 2, \mathbb{R}) \) and \( \text{USP}(n + 1, n + 1) \) respectively, and
\[ U^M_i = e^i_I U^M_i , \quad \bar{U}^{\bar{i}M}_I = e^{\bar{i}}_{\bar{I}} \bar{U}^{\bar{i}M}_I \] (B.20)
where \( e^j_i \) is the (inverse) vielbein on the \( M_{SK} \) related to the Kahler metric \( G_{ij} \) as

\[
G_{ij} = \sum_{i'=1}^{nv} \bar{e}^i_{i'} \bar{e}^{i'}_j \quad , \quad e^i_{i'} \bar{e}^{i'}_j = \delta^i_j \quad , \quad e^i_{i'} \bar{e}^{i'}_j = \delta^j_i
\]  

(B.21)

Eqs(B.16,B.17) imply the matrix \( L \) has remarkable properties; in particular its transforms in the bifundamental of \( SP(2n+2) \times USP(n+1,n+1) \) and obeys [30, 80]

\[
LL^\dagger C = \varpi
\]

\[
L\varpi L^\dagger = C
\]  

(B.22)

with

\[
\varpi = -i \begin{pmatrix} \mathcal{I}_{n+1} & 0 \\ 0 & -\mathcal{I}_{n+1} \end{pmatrix}
\]  

(B.23)

and where \( C \) as in (B.4). Explicitly, we have

\[
(L^\dagger)^{M'}_M (C)^M_N (L)^N_{N'} = (\varpi)^{M'}_{N'}
\]

\[
(L)^N_{N'} (\varpi)^{N'}_{M'} (L^\dagger)^{M'}_M = C^N_M
\]  

(B.24)

Notice that the index \( N' \) of \( USP(n+1,n+1) \) can be brought into an index of \( SP(2n+2,R) \) by using the following \((2n+2) \times (2n+2)\) Cayley matrix \( A = (A^N_N) \) giben by [55, 56]

\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I}_{n+1} & i\mathcal{I}_{n+1} \\ i\mathcal{I}_{n+1} & -\mathcal{I}_{n+1} \end{pmatrix}
\]

(B.25)

with the properties

\[
AA^\dagger = \mathcal{I}_{2n+2} \quad , \quad \varpi = AC\ A^\dagger \quad , \quad C = A^\dagger \varpi \ A
\]  

(B.26)

Multiplying the matrix \( L^M_{N'} \) from the right by \( A^N_N \), we can bring \( L^M_{N'} \) to a symplectic matrix as follows

\[
S = LL^\dagger \quad , \quad S^M_N = L^M_{N'} A^N_N
\]

\[
L = SA^\dagger \quad , \quad L^M_{N'} = S^M_N (A^\dagger)^N_{N'}
\]  

(B.27)

Moreover, multiplying both sides of eq(B.22) on left by \( A \) and on right by \( A^\dagger \), and using the relation \( C = A^\dagger \varpi \ A \), we obtain

\[
(A^\dagger L^\dagger) C (LA) = C
\]

(B.28)

showing that \( S = LL^\dagger \) is indeed a symplectic matrix. In other words

\[
L^\dagger CL = \varpi \quad \iff \quad S^\dagger CS = C
\]  

(B.29)
Notice also that $S$ is a real matrix; by using eqs(B.24-B.25), we have

$$S^M_N = \frac{1}{\sqrt{2}} \left[ V^M + \bar{V}^M, \bar{U}_K^M + U_K^M, i \left( V^M - \bar{V}^M \right), i \left( \bar{U}_K^M - U_K^M \right) \right] \text{ (B.30)}$$

With the matrix $L$, its adjoint $L^\dagger$ and the metric $C$, we can build a hermitian, symmetric matrix $\mathcal{M} = \mathcal{M}(z, \bar{z})$ encoding all information about the coupling of the vector fields to the scalars; this matrix defined like

$$\mathcal{M} = C L L^\dagger C, \quad \mathcal{M}^{-1} = -L L^\dagger C \text{ (B.31)}$$

has several remarkable properties that we describe below: First, given $\mathcal{M} = C L L^\dagger C$, one can directly check that $\mathcal{M}^{-1} = -L L^\dagger$. Indeed, let us compute the product $\mathcal{M} \mathcal{M}^{-1}$ explicitly; we have after substitution

$$\mathcal{M} \mathcal{M}^{-1} = C L L^\dagger C (-L L^\dagger) = -C L L^\dagger C L L^\dagger \text{ (B.32)}$$

By help of the first relation of eq(B.22), we also have

$$\mathcal{M} \mathcal{M}^{-1} = -C L L^\dagger C = -C^2 = I \text{ (B.33)}$$

Moreover, using the factorisation $L = S A^\dagger$ (B.27), the matrices $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ can be expressed in terms of the symplectic matrix $S$ as follows

$$\mathcal{M} = C S S^T C, \quad \mathcal{M}^{-1} = -S S^T \text{ (B.34)}$$

Another property of the matrix $\mathcal{M}$ is that it is a symplectic matrix satisfying the usual property

$$\mathcal{M} C \mathcal{M} = C \text{ (B.35)}$$

This feature can be checked by substituting $\mathcal{M} = C L L^\dagger C$ and using the properties of $\mathcal{L}$. In doing so, we first have

$$\mathcal{M} C \mathcal{M} = -C L L^\dagger C L L^\dagger C$$

where we have used $C^2 = -I$; then using $L^\dagger C L = \varpi$, we can reduce the expression of $\mathcal{M} C \mathcal{M}$ down to $-C L \varpi L^\dagger C$ which, by help of $L \varpi L^\dagger = C$ given by eq(B.22), can be further reduced to (B.35).

The third property concerns the change of the local matrix $\mathcal{M}(z, \bar{z})$ with respect to Kahler $U(1)$ and symplectic $\text{SP}(2n + 2, \mathbb{R})$ symmetries on $M_{SK}$. Under a symmetry transformation, the matrix $\mathcal{M}(z, \bar{z})$ transforms into $\mathcal{M}^{(g)}(z', \bar{z}')$ given by

$$\mathcal{M}^{(g)}(z', \bar{z}') = \left[ \mathcal{R}^{(g)} \right]^T \mathcal{M}(z, \bar{z}) \mathcal{R}^{(g)} \text{ (B.37)}$$

where $z' = z'(z)$ a holomorphic transformation and $\mathcal{R}^{(g)}$ a symplectic representation as in eq(B.2). The last feature we give deals with the relation between $\mathcal{M}^{MN}$ and the $U^{MN}$ of eq(B.18). We have $\mathcal{M}^{MN} = -2U^{MN} - i C^{MN} - 2\bar{V}^M V^N$ which reads also like [30]

$$U^{MN} = -\frac{1}{2} \mathcal{M}^{MN} - \frac{i}{2} C^{MN} - \bar{V}^M V^N \text{ (B.38)}$$
B.2 Quaternionic Kahler Manifolds

Here we give properties of the quaternionic Kahler manifold $M_{QK}$ useful in the study of the $\mathcal{N} = 2$ gauged supergravity $[30,40,52,56,81–84]$; this is a real $4n_H$-dimensional Riemannian variety having an $SU(2)_R \times G'$ holonomy group with $G'$ contained in a symplectic group as follows

$$G' \subset SP(2n_H, \mathbb{R})$$ (B.39)

Notice that the combination of the $SU(2)_R$ symmetry of $M_{QK}$ with the $U(1)_R$ Kahler group of $M_{SK}$ define the usual $U(2)_R = U(1)_R \times SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ supersymmetric algebra

$$\{ Q^A, Q_B \} \sim \delta^A_B \sigma^\mu P_\mu$$ (B.40)

- hyperKahler metric and hyperKahler 2-form

On the quaternionic Kahler manifold $M_{QK}$ of the matter sector, the metric (infinitesimal length) $ds_H^2$ and the hermitian Kahler 2-form triplet $K^a_2$ can be expressed in terms of covariantly constant 1-form vielbeins $\mathcal{E}^{AM}$ as follows

$$ds_H^2 = \sum_{A,B=1}^{2} \varepsilon_{AB} \left( \sum_{M=1}^{2n_H} \mathcal{E}^{AM} \otimes \mathcal{E}^{BN} \right)$$ (B.41)

and

$$K^a_2 = \sum_{A,B=1}^{2} \left( \tau^a \varepsilon \right)_{AB} \left( \sum_{M=1}^{2n_H} \mathcal{E}^{AM} \wedge \mathcal{E}^{BN} \varepsilon_{CMN} \right)$$ (B.42)

where the three $\tau^a$'s are the usual $2 \times 2$ Pauli matrices with the symmetric property $(\tau^a \varepsilon)_{AB} = (\tau^a \varepsilon)_{BA}$. The $ds_H^2$ is invariant under $SU(2)_R \times SP(2n_H)$ isotropy group with metric $\varepsilon_{AB} \varepsilon_{CMN}$; the hyperkahler 2-form $K^a_2$ is invariant under $SP(2n_H)$ but behaves as a triplet under $SU(2)_R$. By using the real $4n_H$ local coordinates field variables $Q^u$, we can express the 1-form vielbeins $\mathcal{E}^{AM}$ in above equations like

$$\mathcal{E}^{AM} = \mathcal{E}_u^{AM} dQ^u$$ (B.43)

with $\mathcal{E}_u^{AM}$ satisfying the reality condition $(\mathcal{E}_u^{AM})^\dagger = \mathcal{E}_{uAM}$ with

$$\mathcal{E}_{uAM} = \varepsilon_{AB} \varepsilon_{CMN} \mathcal{E}_u^{BN}$$ (B.44)

By substituting (B.44) back into eqs(3.51B.42), we can bring the hyperkahler metric and the hyperKahler 2-form to

$$ds^2 = h_{uv} dQ^u dQ^v , \quad K^a_2 = K^a_{uv} dQ^u \wedge dQ^v$$ (B.45)
with symmetric $h_{uv} = h_{vu}$ and antisymmetric $K^a_{uv} = -K^a_{vu}$ as follows

$$h_{uv} = \varepsilon_{AB} E^{AM}_u E^{BN}_v$$

$$K^a_{uv} = -i (\tau^a)_{AB} E^{AM}_u E^{BN}_v$$  \hfill (B.46)

- **three complex structures**

In the above relations, the $SP(2n_H)$ symplectic invariance of the metric $h_{uv}$ and the closed hyperkahler 2-forms $K^a_{uv}$ is manifestly exhibited. By using the hermiticity property of the vielbeins (B.44) expressing $E_{uAM}$ as $\varepsilon_{AB} C_{MN} E^B_M$, we can rewrite the hermitian metric like $h_{uv} = E^{AM} E_{vAM}$ and the hyperkahler 2-form like $K^a_{uv} = i E^{AM} (\tau^a)^B_A E_{vBM}$. From the SU(2) $R$-symmetry view, the metric is nothing but the trace of the matrix $E^{AM} E_{vBM}$. The last object can in general be decomposed as the sum of a singlet and a triplet as follows

$$E^{AM} E_{vBM} = \frac{1}{2} \delta^A_B h_{uv} - \frac{i}{2} K^a_{uv} (\tau^a)^A_B$$  \hfill (B.47)

Notice also that the quaternionic Kahler $M_{QK}$ has three complex structures $J_1, J_2, J_3$, generating $SU(2)_R$, acting on the tangent space like $(J^a)^u_v$ and satisfying the quaternionic algebra

$$J^a J^b = -\delta^{ab} + \varepsilon^{abc} J^c$$  \hfill (B.48)

In terms of the quaternionic structure, the three $4n_H \times 4n_H$ matrices $K^a_{uv}$ are related to the hyperkahler metric $h_{uv}$ as follows

$$K^a_{uv} = h_{uw} (J^a)^w_v$$  \hfill (B.49)

Notice moreover that the hyperkahler 2-forms $K^a_2$ obey other special features; in particular the following ones: First, by using $h_{uw}$, the inverse of the metric, we can reexpress $(J^a)^u_v$ like $h^{uw} K^a_{uw}$. Putting it back into eq(B.48) we obtain the property

$$K^a_{uw} h^{ws} K^b_{sv} = -\delta^{ab} h_{uw} + \varepsilon^{abc} K^c_{uw}$$  \hfill (B.50)

By multiplying both sides of this relation by $h^{tu}$ and setting $(K^b)^{tw} = h^{tv} h^{us} K^b_{sv}$, we have

$$K^a_{uw} (K^b)^{tw} = -\delta^{ab} \delta^t_u + \varepsilon^{abc} K^c_{uw} h^{tv}$$  \hfill (B.51)

from which we deduce that for a given value of index a, $(K^a)^{wt}$ is the inverse of $K^a_{uw}$; i.e:

$$K^a_{uw} (K^a)^{wt} = \delta^t_u$$  \hfill (B.52)

Second, the 2-forms $K^a_2$ are covariantly constant $\nabla K^a_2 = 0$ with respect to the $SU(2)_R$ connection given by the three 1-forms $\omega^t_i = \omega^t_u dQ^u$ living in the cotangent space bundle of the scalar manifold. This condition reads explicitly as follows

$$\nabla K^a_2 = dK^a_2 + \varepsilon^{abc} \omega^b_1 K^c_2 = 0$$  \hfill (B.53)

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This is a remarkable condition as it allows to get more insight into the properties of \( K_2^a \). Indeed, the gauge curvature 2-forms \( \Omega_2^a = d\omega_1^a + \frac{1}{2} \varepsilon^{abc} \omega_b^i \omega_c^i \) of the SU(2) connection \( \omega_1^a \) obey a similar constraint following from the Jacobi identity of the SU(2) bracket namely \( d\Omega_2^a + \varepsilon^{abc} \omega_b^i \Omega_2^c = 0 \). By comparing this relation with (B.53), it follows that the 2-forms \( K_2^a \) and \( \Omega_2^a \) should be proportional like

\[
\Omega_2^a = \lambda K_2^a
\]

where \( \lambda \) is a real coefficient, depending on the normalization of the metric, taken here \( \lambda = -1 \).

• **tri-holomorphic moment maps**

On the quaternionic Kahler manifold \( M_{QK} \), the moment maps \( P_n^a (Q) \), associated with the Killing vector fields \( k_n = k_n^a (Q) \frac{\partial}{\partial Q^a} \), carry a quantum number of the SU(2) R-symmetry; they behave as hermitian triplets. To get the relationship between \( P_n^a \) and \( \iota_n (\omega_1^a) \), the contraction of the 1-forms isotriplet \( \omega_1^a \), we consider infinitesimal isometries generated by \( t_m \), whose action on the scalar fields given by the Killing vectors \( k_m \). These generators obey the isometry algebra

\[
[t_m, t_n] = f_{mn}^p t_p \\
[k_m, k_n] = -f_{mn}^p k_p
\]

(B.55)

living invariant the 4-form \( \sum K_2^a \wedge K_2^a [30, 52] \). This property leads to

\[
\mathcal{L}_n K_2^a = \varepsilon^{abc} K_2^b W_n^c
\]

(B.56)

where \( W_n^a \) is an SU(2)_R- compensator. Eq(B.56) is solved as follows [30, 52]

\[
\iota_n (K_2^a) = -\nabla P_n^a
\]

(B.57)

with covariant derivative 1-form as follows

\[
\nabla P_n^a = dP_n^a + \varepsilon^{abc} \omega_1^b P_n^c
\]

(B.58)

which gives the following expression of the moment maps [52, 83]

\[
P_n^a = \lambda^{-1} (\iota_n (\omega_1^a) - W_n^a)
\]

(B.59)

By taking \( \lambda = -1 \), we have

\[
P_n^a = W_n^a - \iota_n (\omega_1^a)
\]

(B.60)

Notice that for those isometries with vanishing compensator, \( W_n^a = 0 \), the moment maps take the simple expression \( P_n^a = -k_n^a \omega_1^a \). In this case, the quaternionic manifold is globally isometric to a solvable Lie group \( e^{Solv} \) generated by a solvable Lie algebra \( Solv [30] \).
C Appendix C: Coupling matrices $\mathcal{N}_{\Lambda\Sigma}$ and $U^{MN}$

In this appendix, we determine the rigid limit of the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ appearing in the supergravity Lagrangian density (2.27, 2.28) as well as the rigid limit $\hat{U}^{MN}$ of the factor

$$U^{MN} = U_j^M G^{\bar{j}} \hat{U}_j^N$$ (C.1)

involved in the structure of the scalar potential (2.31).

C.1 Rigid limit of period matrix $\mathcal{N}_{\Lambda\Sigma}$

Using the $\text{SP}(2n+2)$ symplectic structure on the special Kahler manifold $M_{SK}$, the $2 (n+1)$ components of $V^M$ can be decomposed into a $(n+1)$ electric part and a $(n+1)$ magnetic dual like

$$V^M = \left( \begin{array}{c} \Upsilon^\Lambda \\ \Gamma_\Lambda \end{array} \right)$$ (C.2)

with $\Upsilon^\Lambda = \Upsilon^\Lambda (z, \bar{z})$ and $\Gamma_\Lambda = \Gamma_\Lambda (z, \bar{z})$ splitting in turn in terms of graviphoton components and Coulomb branch ones as follows

$$\Upsilon^\Lambda = \left( \begin{array}{c} \Upsilon_0 \\ \Upsilon_i \end{array} \right), \quad \Gamma_\Lambda = \left( \begin{array}{c} \Gamma_0 \\ \Gamma_i \end{array} \right)$$ (C.3)

where the Coulomb branch index $i$ runs from 1 to $n$. Following [54, 76, 85], the blocks $\Gamma_\Lambda$ and $\Upsilon^\Lambda$ are related like

$$\Gamma_\Lambda = \mathcal{N}_{\Lambda\Sigma} \Upsilon^\Sigma$$ (C.4)

with complex coupling and symmetric $(n+1) \times (n+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}$ defined as follows

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{F}_{\Lambda\Sigma} + \mathcal{N}^{\text{loc}}_{\Lambda\Sigma}$$ (C.5)

The first term in above relation is given by $\mathcal{F}_{\Lambda\Sigma} = \frac{\partial^2 \mathcal{F}}{\partial X^\Lambda \partial X^\Sigma}$ with leading terms in the $\frac{1}{\mu}$-expansion as

$$\mathcal{F}_{\Lambda\Sigma} = \left( \begin{array}{cc} \frac{i}{2} \bar{X} + \mathcal{D}(\frac{1}{\mu^2}) & \frac{i}{2\mu} \bar{\eta}_j + \mathcal{D}(\frac{1}{\mu^2}) \\ \frac{i}{2\mu} \eta_i + \mathcal{D}(\frac{1}{\mu^2}) & \frac{i}{2\mu^2} \partial_j \phi + \mathcal{D}(\frac{1}{\mu^2}) \end{array} \right)$$ (C.6)

The extra matrix term $\mathcal{N}^{\text{loc}}_{\Lambda\Sigma}$ is given by

$$\mathcal{N}^{\text{loc}}_{\Lambda\Sigma} = 2i \frac{[(\text{Im}\mathcal{F})_{\Lambda\Delta} X^\Delta][((\text{Im}\mathcal{F})_{\Sigma\Gamma} X^\Gamma)]}{X^\Lambda (\text{Im}\mathcal{F})_{\Delta\Gamma} X^\Gamma}$$ (C.7)
its leading terms in the $\frac{1}{\mu}$-expansion read as follows

\[
N_{\Lambda\Sigma}^{\text{loc}} = \begin{pmatrix}
-i + \mathcal{O}(\frac{1}{\mu}) & -\frac{i}{\mu} \eta_j + \mathcal{O}(\frac{1}{\mu^2}) \\
-\frac{i}{\mu} \eta_i + \mathcal{O}(\frac{1}{\mu^2}) & -\frac{i}{\mu^2} \eta_i \eta_j + \mathcal{O}(\frac{1}{\mu^3})
\end{pmatrix}
\] (C.8)

Using the symplectic transformation (3.62), one can show that the period matrix $N_{\Lambda\Sigma}$ may be diagonalised by help of the two matrices $A$ and $D$ (3.62) like

\[
\tilde{N}_{\Lambda\Sigma} = (D N A^{-1})_{\Lambda\Sigma}
\] (C.9)

and can as well be split into the following form [76,85–88]

\[
\tilde{N}_{\Lambda\Sigma} = \tilde{F}_{\Lambda\Sigma} + \tilde{N}_{\Lambda\Sigma}^{\text{loc}}
\] (C.10)

By using eqs(3.10-3.11), we get

\[
\tilde{F}_{\Lambda\Sigma} = \begin{pmatrix}
\frac{i}{2} & 0_{1 \times n} \\
0_{n \times 1} & -\frac{i}{2} (\eta_i \eta_j - \partial_{ji} \bar{\phi})
\end{pmatrix} + \mathcal{O} \left( \frac{1}{\mu} \right)
\] (C.11)

\[
\tilde{N}_{\Lambda\Sigma}^{\text{loc}} = \begin{pmatrix}
-i & 0_{1 \times n} \\
0_{n \times 1} & 0_{n \times n}
\end{pmatrix} + \mathcal{O} \left( \frac{1}{\mu} \right)
\]

and then

\[
\tilde{N}_{\Lambda\Sigma} = \begin{pmatrix}
\frac{i}{2} & 0_{1 \times n} \\
0_{n \times 1} & -\frac{i}{2} (\eta_i \eta_j - \partial_{ji} \bar{\phi})
\end{pmatrix} + \mathcal{O} \left( \frac{1}{\mu} \right)
\] (C.12)

We notice that because of this nontrivial form of $\tilde{N}_{\Lambda\Sigma}$; which contains contribution from both the graviphoton and rigid parts having the same zero order of $\frac{1}{\mu}$; the graviphoton will play an important role in the rigid theory as we will see in section 5. From eq(C.12), we learn the expressions of the prepotential $\tilde{F}$ of the rigid theory and the prepotential $F^{\text{grav}}$ associated with the graviphoton part; they read as

\[
\tilde{F} = \frac{i}{4} \left[ \frac{1}{2\mu} (\eta_i \eta_j - \partial_{ji} \bar{\phi})^2 - 2\Phi \right] + \mathcal{O} \left( \frac{1}{\mu} \right)
\]

\[
F^{\text{grav}} = c X^0 - \frac{i}{4} (X^0)^2 + \mathcal{O} \left( \frac{1}{\mu} \right)
\] (C.13)

where $c$ is independent of $X^0$; it can be set to zero by performing the particular Kahler transformation (3.51). So the leading term of the pure supergravity prepotential $F^{\text{grav}}$ in the $\frac{1}{\mu}$-expansion reduces to the following relation which is in agreement with the usual term of the literature [57]

\[
F^{\text{grav}} = -\frac{i}{4} (X^0)^2 + \mathcal{O} \left( \frac{1}{\mu} \right)
\] (C.14)
C.2 Rigid limit of $U^{MN}$

Recall that in $\mathcal{N} = 2$ supergravity theory, we have the following decomposition property of the rank 2 symplectic tensor $U^{MN} = U^j_i \hat{G}^{ij} \hat{U}^N_j$ [30], for its derivation see eq(B.37) also appendix B,

$$U^{MN} = -\frac{1}{2} \mathcal{M}^{MN} - \frac{i}{2} C^{MN} - \bar{V}^M V^N \quad (C.15)$$

where the antisymmetric $C^{MN}$ is the symplectic metric of $\text{SP}(2n + 2)$ and $\mathcal{M}^{MN} = (\mathcal{M}_{MN})^{-1}$, the inverse of the symmetric matrix built of $\text{Re} \mathcal{N}_{\Lambda \Sigma}$ and $\text{Im} \mathcal{N}_{\Lambda \Sigma}$ as follows

$$\mathcal{M}^{MN} = \begin{pmatrix} \text{Im} \mathcal{N} + \text{Re} \mathcal{N} (\text{Im} \mathcal{N})^{-1} \text{Re} \mathcal{N} & -\text{Re} \mathcal{N} (\text{Im} \mathcal{N})^{-1} \\ - (\text{Im} \mathcal{N})^{-1} \text{Re} \mathcal{N} & (\text{Im} \mathcal{N})^{-1} \end{pmatrix} \quad (C.16)$$

with $\mathcal{N}_{\Lambda \Sigma}$ standing for the period matrix of eq(C.4). The rigid limit of $U^{MN}$ is obtained by determining the rigid limit of $\mathcal{M}^{MN}$ and $\bar{V}^M V^N$. To obtain the rigid limit of $\mathcal{M}^{MN}$, we use eq(C.12) which tells us that the rigid theory is described by $\bar{\mathcal{N}}_{ij} = \mathcal{F}_{ij}$ with

$$\mathcal{F}_{ij} = -\frac{i}{2} (\eta_i \eta_j - \partial_{ji} \Phi) + \mathcal{O} \left( \frac{1}{\mu} \right) \quad (C.17)$$

Putting $\text{Im} \bar{\mathcal{N}}_{ij} = -\text{Im} \mathcal{F}_{ij}$ back into (C.16), we get

$$\mathcal{M}^{MN} = -\delta^M_{\mu} \delta^N_{\nu} \hat{\mathcal{M}}^{MN} + \mathcal{O} \left( \frac{1}{\mu} \right) \quad (C.18)$$

with $\mathcal{M}^{MN} \equiv (\hat{\mathcal{M}}_{MN})^{-1}$ and

$$\hat{\mathcal{M}}^{MN} = \begin{pmatrix} \text{Im} \hat{\mathcal{F}} + \text{Re} \hat{\mathcal{F}} (\text{Im} \hat{\mathcal{F}})^{-1} \text{Re} \hat{\mathcal{F}} & -\text{Re} \hat{\mathcal{F}} (\text{Im} \hat{\mathcal{F}})^{-1} \\ - (\text{Im} \hat{\mathcal{F}})^{-1} \text{Re} \hat{\mathcal{F}} & (\text{Im} \hat{\mathcal{F}})^{-1} \end{pmatrix} \quad (C.19)$$

where $\text{Im} \hat{\mathcal{F}} \equiv \text{Im} \hat{F}_{IJ}$ and $\text{Re} \hat{\mathcal{F}} \equiv \text{Re} \hat{F}_{IJ}$. Moreover, we learn from eq(3.47) that the components $\bar{V}^M V^N = \delta^M_{\mu} \delta^N_{\nu} \bar{V}^M V^N$ in eq(C.15) do not contribute; so we have the two following following expressions

$$U^i_M \hat{G}^{ij} \hat{U}^N_j \rightarrow \delta^M_{\mu} \delta^N_{\nu} \left[ \frac{1}{2} \hat{\mathcal{M}}^{MN} - \frac{i}{2} \mathcal{C}^{MN} \right] \quad (C.20)$$

and

$$U^M_i \hat{G}^{ij} \hat{U}^N_j \rightarrow \frac{1}{\mu} \hat{U}^M_i \left( \mu^2 \hat{G}^{ij} \right) \frac{1}{\mu} \hat{U}^N_j \quad (C.21)$$

from which we deduce the relation

$$\hat{U}^M_i \hat{G}^{ij} \hat{U}^N_j = \frac{1}{2} \hat{\mathcal{M}}^{MN} - \frac{i}{2} \mathcal{C}^{MN} \quad (C.22)$$
which coincides with the rigid theory relation given in \([34, 89]\). Notice also that the components associated with the graviphoton direction in \((4.1)\) is trivially satisfied at the \(O(\mu)\) order as shown here below

\[
-\frac{1}{2} \mathcal{M}^{gr}_{grav} - \frac{i}{2} \mathcal{C}^{gr} - \mathcal{V}^{gr}_{grav} = 0_{2 \times 2}
\]  

(C.23)

with \(\mathcal{C}^{gr}\) is the symplectic \(SP(2)\) metric, \(V^{gr}_{grav}\) as in eq\((3.48)\) and

\[
\mathcal{M}^{gr}_{grav} = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}
\]

(C.24)

D Appendix D: Solving Ward identities

In this appendix, we study the Ward identities \((2.34)\) and a class of solutions in terms of the geometric objects of the scalar manifold of \(\mathcal{N} = 2\) supergravity. We also give an explicit check of these solutions.

D.1 Ward identities for quaternionic gauging

First, we recall the Ward identities of gauging isometries; then we give the solution for the particular case of Abelian quaternionic isometries.

- Ward identities

Following \([30]\), the Ward identities of \(\mathcal{N} = 2\) gauged supergravity

\[
-12S_C^A S_B^C + 2N_C^A N_B^C + \sum_{i,j=1}^{n_V} G_{ij} \left(W^i\right)_A^C \left(W^j\right)_B^C = \delta^A_B \mathcal{V}^{\mathcal{N}=2}_{sugra}
\]  

(D.1)

By setting

\[
X_B^A = \sum_{i,j=1}^{n_V} G_{ij} \left(W^i\right)_C^A \left(W^j\right)_B^C \\
Y_B^A = 2N_C^A N_B^C \\
Z_B^A = -12S_C^A S_B^C
\]

(D.2)

the matrix eq\((D.1)\) becomes

\[
X_B^A + Y_B^A + Z_B^A = \delta^A_B \mathcal{V}^{\mathcal{N}=2}_{sugra}
\]  

(D.3)

and splits as

\[
tr \left[ \tau^a \left(X_B^A + Y_B^A + Z_B^A\right) \right] = 0
\]  

(D.4)

and

\[
tr \left(X_B^A + Y_B^A + Z_B^A\right) = 2\mathcal{V}^{\mathcal{N}=2}_{sugra}
\]  

(D.5)
• Solutions of Ward identities

For the gauging of Abelian quaternionic isometries, the \((W^i)^A_B\), \(N^A_B\) and \(S^A_B\) matrices can be expressed like

\[
(W^i)^A_B = W^{ia}(\tau_a)^A_B \\
N^A_B = N^a(\tau_a)^A_B \\
S^A_B = S^a(\tau_a)^A_B
\]  

(D.6)

and the above Ward identities have a solution in terms of the geometric objects of the scalar manifold \(M_{SK} \times M_{QK}\) as follows

\[
W^{ia} = -i\mathcal{P}^a M \mathcal{G}^i \bar{U}^M \bar{j} \\
N^a = i\mathcal{P}^a M \bar{V}^M \\
S^a = i\mathcal{P}^a M V^M
\]  

(D.7)

D.2 Checking the solution of Ward identities

Here, we want to explicitly check that the solution (D.7) satisfies indeed the Ward identities (D.1); the sum of contributions coming from gauge and matter sectors are compensated by a negative contribution coming from gravity sector.

• Computing \(X^A_B\)

Substituting \(W^{iAC}\) by its expression, we have

\[
X^A_B = (\tau_a)^C_B (\tau_b)^C_A \left[ \mathcal{G}^i \bar{U}^M_k \bar{U}^N_k \right] \mathcal{P}^a_M \mathcal{P}^b_N
\]  

(D.8)

Expanding \(\tau_a \tau_b\) as \(\frac{1}{2} \{\tau_a, \tau_b\} + i \frac{1}{2} [\tau_a, \tau_b]\), we find that the 2 \& 2 matrix \(X^A_B\) is a sum of two contributions, an isosinglet, coming from \(\{\tau_a, \tau_b\}\), and an isotriplet, associated with \([\tau_a, \tau_b]\), as shown on the following relation

\[
X^A_B = \delta_{ab} \delta^A_B \left[ \mathcal{G}^i \bar{U}^M_k \bar{U}^N_k \right] \mathcal{P}^a_M \mathcal{P}^b_N + i\varepsilon_{abc} (\tau^c)^A_B \mathcal{P}^a_M \mathcal{P}^b_N \left[ V^M \bar{V}^N \right]
\]  

(D.9)

• Computing \(Y^A_B\)

By using eqs(D.6,D.7), we obtain

\[
Y^A_B = 2 \left[ -i (\tau_a)^A_C \mathcal{P}^a_M V^M \right] \left[ i (\tau_b)^C_B \mathcal{P}^b_N \bar{V}^N \right]
\]  

(D.10)

having as well two contributions as

\[
Y^A_B = 2\delta_{ab} \delta^A_B \mathcal{P}^a_M \mathcal{P}^b_N V^M \bar{V}^N + 2i\varepsilon_{abc} \mathcal{P}^a_M \mathcal{P}^b_N V^M \bar{V}^N (\tau^c)^A_B
\]  

(D.11)
Computing $Z^A_B$

For the gravitini shifts $S_{ABC}$, we have a quite similar expression given by

$$Z^A_B = -12 \left[ -\frac{i}{2} (\tau_a)_C P^a_N \bar{V}^N \right] \left[ \frac{i}{2} (\tau_b)_C P^b_N V^N \right]$$

(D.12)

and splitting as

$$Z^A_B = -3 \delta_{ab} \delta^A_B P^a_M V^N V^N - 3i\varepsilon_{abc} P^a_M P^b_N V^M \bar{V}^N (\tau^c)_B$$

(D.13)

Combining eqs (D.9, D.11, D.13), we obtain

$$X^A_B + Y^A_B + Z^A_B = \delta^A_B \mathcal{V}^{N=2}_{\text{sugra}}$$

(D.14)

with

$$\mathcal{V}^{N=2}_{\text{sugra}} = -3 P^a_M P^a_N V^M \bar{V}^N + 2 P^a_M P^a_N V^M \bar{V}^N + \left[ G^{ik} \tilde{U}_k^M U^N_k \right] P^a_M P^a_N$$

(D.15)

reducing to

$$\mathcal{V}^{N=2}_{\text{sugra}} = \left[ G^{ik} \tilde{U}_k^M U^N_k \right] P^a_M P^a_N - P^a_M P^a_N V^M \bar{V}^N$$

(D.16)

and

$$(\tau^c)_B P^a_M P^b_N V^M \bar{V}^N \left[i\varepsilon_{abc} + 2i\varepsilon_{abc} - 3i\varepsilon_{abc}\right] = 0$$

(D.17)

E Appendix E: Rigid limit of the scalar potential

The aim of this appendix is to compute the rigid limit of the scalar potential $\mathcal{V}^{N=2}_{\text{sugra}}$. To this end, we first give the $\frac{1}{\mu}$-expansion of the fermion shift matrices, namely $W^{ia}$, $N^a$ and $S^a$.

E.1 $\frac{1}{\mu}$-expansion of fermion shift matrices

To derive the rigid limit of (5.44), we have to determine the $\frac{1}{\mu}$-expansions of $W^{ia}$, $N^a$ and $S^a$ by proceeding in three steps as follows: First, we compute the $\frac{1}{\mu}$-expansion of the $W^{ia}$ describing the contribution coming from the Coulomb branch. Then, we determine the expansion of the term $N^a$ of hypermultiplet sector; and after that we turn to the $\frac{1}{\mu}$-development of the factor $S^a$ of gravity branch.

- $\frac{1}{\mu}$-expansion of the term $W^{ia}$

To obtain the $\frac{1}{\mu}$-expansion of $W^{ia}$, we use results derived in previous sections concerning developments in power series of $\frac{1}{\mu}$ of symplectic sections and moment maps on the scalar
manifold $M_{SK} \times M_{QK}$; in particular the following things: (i) eq(3.29) giving the $\frac{1}{\mu}$-expansion of the metric $G_{ij}$; (ii) eq(3.59) determining the development of the $U_1^M$ sections; (iii) eq(4.118) giving the expression of the moment maps $P_M^a$ in terms of the embedding tensor $\vartheta_M^a$; and (iv) eqs(5.29-5.30) regarding the $\frac{1}{\mu}$-expansion of the embedding tensor.

For convenience, we recall these relationships here below

\begin{align*}
G_{ij} &= \frac{1}{\mu^2} \tilde{G}_{ij} + \mathcal{O}\left( \frac{1}{\mu^3} \right) \\
U_i^M &\simeq \frac{1}{\mu} \tilde{U}_i^M + \mathcal{O}\left( \frac{1}{\mu^2} \right) \\
P_M^a &= \delta_M^a \mathcal{P}_a^a + \frac{1}{\mu} \delta_M^a \mathcal{P}_N^a \\
&= e^{-\frac{m}{\mu}} \left( \vartheta_M^a + \frac{1}{\mu} \vartheta_M^a \right) \\
\end{align*}

(E.1)

Substituting these $\frac{1}{\mu}$-developments into

\begin{equation}
W^{ia} = -i G^{ij} \tilde{U}_j^M \mathcal{P}_M^a
\end{equation}

(E.2)

with $\mathcal{G}^{\tilde{ij}}$ the inverse of $G_{ij}$, we have

\begin{equation}
W^{ia} = -i \mu^2 G^{ij} \frac{1}{\mu} \left[ \delta_M^a \tilde{U}_j^M + \mathcal{O}\left( \frac{1}{\mu} \right) \right] \left[ \delta_M^a \mathcal{P}_a^a + \frac{1}{\mu} \delta_M^a \mathcal{P}_N^a \right]
\end{equation}

(E.3)

By expanding the product on right hand side of above expression in a power series of the parameter $\mu$, we obtain the leading contributions

\begin{equation}
W^{ia} = \tilde{W}^{ia} + \mathcal{O}\left( \frac{1}{\mu} \right)
\end{equation}

(E.4)

with

\begin{align*}
\tilde{W}^{ia} &= -i G^{ij} \tilde{U}_j^M \mathcal{P}_M^a \\
&= -i e^{-\frac{m}{\mu}} \tilde{G}^{ik} \tilde{U}_k^M \vartheta_M^m \\
\end{align*}

(E.5)

where $\vartheta_M^m$ and $\tilde{\vartheta}_M^m$ scale as mass$^2$, in same manner as the Fayet-Iliopoulos coupling constants. The rigid limit of $W^{ia}$ has no contribution along the graviphoton direction since the $\tilde{W}^{ia}$’s are proportional to the $\vartheta_M^m$’s having components in the Coulomb branch as given below

\begin{equation}
\vartheta_M^m = \begin{pmatrix} 0 & \vartheta_i^m \\ \vartheta_i^m & 0 \end{pmatrix} \equiv \vartheta_M^m
\end{equation}

(E.6)

Here also the $\theta_i^m$ and $\tilde{\theta}_i^m$ components of the embedding tensor scale as mass$^2$ and can be interpreted in terms of electric and magnetic FI coupling constants.
• $\frac{1}{\mu}$-expansion of the term $N^a$

By using the $\frac{1}{\mu}$-expansion of the covariantly holomorphic section (3.47) namely

\[ V^M = V^M_{\text{grav}} + \frac{1}{\mu} \vec{\Omega}^M + \mathcal{O}(\frac{1}{\mu^2}) \]  

(E.7)

and the expression of $\mathcal{P}^a_M$ in terms of the embedding tensor (5.29)

\[ \mathcal{P}^a_M = e^{\lambda \phi} \left( \partial^a_M + \frac{1}{\mu} \vartheta^a \right) \]  

(E.8)

where we have set

\[ \lambda = \frac{1}{M_{\text{pl}}^2} \]

the $\frac{1}{\mu}$-expansion of the term $N^a = \i \mathcal{P}^a_M \bar{V}^M$ reads as follows

\[ N^a = \i \left[ e^{\lambda \phi} \left( \partial^a_M + \frac{1}{\mu} \vartheta^a \right) \right] \left[ \bar{V}^M_{\text{grav}} + \frac{1}{\mu} \bar{\Omega}^M + \mathcal{O}(\frac{1}{\mu^2}) \right] \]  

(E.9)

with leading $\mu^{-n}$-terms as

\[ N^a = \hat{N}^a + \mathcal{O}(\frac{1}{\mu^2}) \]  

(E.10)

In this relation, $\hat{N}^a$ is the rigid limit whose expression in terms of the components of the embedding tensor reads as $\hat{N}^a = \i \mathcal{P}^a_M \bar{\vartheta}^M$; by substituting $\mathcal{P}^a_M$ and $\bar{\vartheta}^M_{\text{grav}}$ by their expressions, we also have

\[ \hat{N}^a = -\frac{1}{2} e^{\lambda \phi} \left( \bar{\vartheta}^{a0} - 2i\theta^a_0 \right) \]  

(E.11)

The expansion of $N^a$ has a gravity-like contribution in the sense that it involves only the embedding tensor components $\theta^a_0$ and $\bar{\theta}^{a0}$ associated with the graviphoton direction. Notice that this particular dependence only in $\left( \theta^a_0, \bar{\theta}^{a0} \right)$ does not mean that $\hat{N}^a$ is free from $\theta^a_i$ and $\bar{\theta}^{ai}$; this is because $\theta^a_0$ and $\bar{\theta}^{a0}$ are related to $\theta^a_i$ and $\bar{\theta}^{ai}$ through the constraint relations (4.90-4.91).

• $\frac{1}{\mu}$-expansion of $S^a$

The expansion of $S^a$ has a quite similar behaviour as the $N^a$ associated with the transformations of the hyperini (2.32). By using eqs(3.47, 5.29) the leading term $\hat{S}^a$ of the $\frac{1}{\mu}$-expansion of $S^a = \frac{i}{2} \mathcal{P}^a_M V^M$ associated with the transformations of the two gravitini is given by $\hat{S}^a = \frac{i}{2} \mathcal{P}^a_M \bar{V}^M_{\text{grav}}$ and reads in terms of the embedding tensor components as follows

\[ \hat{S}^a = \frac{1}{2} e^{\lambda \phi} \left( \bar{\theta}^{a0} + 2i\theta^a_0 \right) \]  

(E.12)

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8 Notice that in the limit where $M_{\text{pl}}$ is thought of very large with respect to $\Lambda$, ($M_{\text{pl}} \to \infty$); then $\lambda = \frac{1}{M_{\text{pl}}} \to 0$ and therefore $e^{\lambda \phi} \to 1$. 

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Like $\hat{N}^a$, the rigid $\hat{S}^a$ has a gravity-like contribution with no manifest contribution from the embedding tensor components $\tilde{\theta}_i^a$ and $\bar{\tilde{\theta}}^{ai}$ associated with the Coulomb branch dimensions. By using the constraint eqs (4.90-4.91), one can also express $\hat{S}^a$ in terms of $\theta_i^a$ and $\bar{\tilde{\theta}}^{ai}$; see also the comment given in the conclusion regarding this matter.

### E.2 Rigid limit of the supergravity scalar potential

Using eqs (E.4-E.11), we can expand the induced potential $\hat{V}^N_{2 \text{ sugra}}$ as a power series of the parameter $\mu$. The obtained development is as given below

$$\hat{V}^N_{2 \text{ sugra}} = \hat{V}^N_{(0)} + \mathcal{O} \left( \frac{1}{\mu} \right)$$  \hspace{1cm} (E.13)

with

$$\hat{V}^N_{(0)} = \hat{V}^N_{\text{kah}} + \hat{V}^N_{\text{hyper}} - \hat{V}^N_{\text{gra}}$$  \hspace{1cm} (E.14)

where

$$\begin{align*}
\hat{V}^N_{\text{kah}} &= \hat{G}^a_{ij} \hat{W}^{ai} \hat{W}^{aj} \\
\hat{V}^N_{\text{hyper}} &= 2 \hat{N}^a \\
\hat{V}^N_{\text{gra}} &= 12 \hat{S}^a \hat{S}^a
\end{align*}$$  \hspace{1cm} (E.15)

By substituting the expressions of $\hat{W}^{ai}$, $\hat{N}^a$ and $\hat{S}^a$ back into (E.15), we first have $\hat{V}^N_{\text{kah}} = \frac{1}{2} \mathcal{M}^{\text{MN}} \mathcal{P}^a_N \mathcal{P}^a_M$; and by replacing the moment maps by their expressions in terms of the embedding tensor, we obtain

$$\hat{V}^N_{\text{kah}} = \frac{1}{2} \hat{M}^{\text{MN}} \hat{\theta}^a_N \hat{\theta}^a_M$$  \hspace{1cm} (E.16)

with $\hat{M}^{\text{MN}}$ the inverse of the matrix $\hat{M}^{\text{MN}}$ given by eq(C.19). Similarly, we have for the extra term $\Delta \hat{V} = \hat{V}^N_{\text{hyper}} - \hat{V}^N_{\text{gra}}$ in (E.14)

$$\Delta \hat{V} = 2 \hat{N}^a \hat{N}^a - 12 \hat{S}^a \hat{S}^a$$  \hspace{1cm} (E.17)

the following expression

$$\Delta \hat{V} = 2 \left( -i \mathcal{P}^a_{\underline{a}} V_{\underline{a} \text{ grav}}^a \right) \left( i \mathcal{P}^a_{\underline{a}} \tilde{V}_{\underline{a} \text{ grav}}^a \right) - 12 \left( -\frac{i}{2} \mathcal{P}^a_{\underline{a}} \tilde{V}_{\underline{a} \text{ grav}}^a \right) \left( \frac{i}{2} \mathcal{P}^a_{\underline{a}} V_{\underline{a} \text{ grav}}^a \right)$$  \hspace{1cm} (E.18)

By expanding, we have

$$\Delta \hat{V} = - \left( \mathcal{P}^a_{\underline{a}} V_{\underline{a} \text{ grav}}^a \right) \left( \mathcal{P}^a_{\underline{a}} \tilde{V}_{\underline{a} \text{ grav}}^a \right)$$  \hspace{1cm} (E.19)

and by substituting $\mathcal{P}^a_{\underline{a}}$ and $V_{\underline{a} \text{ grav}}^a$ by their values in terms of $\theta_i^a$ and $\bar{\tilde{\theta}}^{ai}$, we end with

$$\Delta \hat{V} = - \frac{1}{4} \left( \theta_{i}^a \bar{\theta}_{i}^{a0} + 4 \theta_{i}^a \bar{\theta}_{i}^{a0} \right)$$  \hspace{1cm} (E.20)
where we have dropped out the factor $e^{\lambda \varphi}$ which, in the limit $M_{pl}$ very large, reduces to 1. By adding (E.16) and (E.20), we obtain the explicit expression of the rigid potential (E.14) in terms of the components of the embedding tensor namely

$$\hat{V}^{N=2}_{(0)} = \frac{1}{2} \hat{M}^{MN} \hat{\phi}^a_M \hat{\phi}^a_N - \frac{1}{4} \left( \hat{\theta}^{a0} \hat{\theta}^{a0} + 4 \hat{\theta}_0^a \hat{\theta}_0^a \right)$$ (E.21)

Notice that eq(E.21) can be derived directly from (5.42) by taking its rigid limit as follows:

$$\hat{V}^{N=2}_{(0)} = \frac{1}{2} \hat{M}^{MN} \hat{P}_a^a M - \frac{1}{2} \hat{M}_{\text{grav}}^{a\sigma} \hat{P}^a_\rho \hat{P}^a_\sigma - 2 \hat{V}^{a\sigma} \hat{P}^a_\rho \hat{P}^a_\sigma$$ (E.22)

where $\hat{M}_{\text{grav}}^{a\sigma}$ as in (C.24). From eq(C.23), we have

$$- \frac{1}{2} \hat{M}_{\text{grav}}^{a\sigma} \hat{P}^a_\rho \hat{P}^a_\sigma = \left( \frac{i}{2} \hat{C}^a_{\rho\sigma} + \hat{V}^{a\rho}_{\text{grav}} \hat{V}^{a\sigma}_{\text{grav}} \right) \hat{P}^a_\rho \hat{P}^a_\sigma$$ (E.23)

which reduces to

$$- \frac{1}{2} \hat{M}_{\text{grav}}^{a\sigma} \hat{P}^a_\rho \hat{P}^a_\sigma = \left( \hat{P}^a_\rho \hat{V}^{a\rho}_{\text{grav}} \right) \left( \hat{P}^a_\sigma \hat{V}^{a\sigma}_{\text{grav}} \right)$$ (E.24)

Putting back into (E.22), we obtain

$$\hat{V}^{N=2}_{(0)} = \frac{1}{2} \hat{M}^{MN} \hat{P}_a^a M - \hat{V}^{a\sigma} \hat{P}^a_\rho \hat{P}^a_\sigma$$ (E.25)

which coincides with eq(E.21).

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