Role of time scales and topology on the dynamics of complex networks

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The interplay between time scales and structural properties of complex networks of nonlinear oscillators can generate many interesting phenomena, like amplitude death, cluster synchronization, frequency synchronization etc. We study the emergence of such phenomena and their transitions by considering a complex network of dynamical systems in which a fraction of systems evolves on a slower time scale on the network. We report the transition to amplitude death for the whole network and the scaling near the transitions as the connectivity pattern changes. We also discuss the suppression and recovery of oscillations and the cross over behavior as the number of slow systems increases. By considering a scale free network, we study the role of heterogeneity in link structure on dynamical properties and the consequent critical behaviors. In this case with hubs made slow, our main results are the escape time statistics for loss of complete synchrony as the slowness spreads on the network and the self-organization of the whole network to a new frequency synchronized state. Our results have potential applications in biological, physical, and engineering networks consisting of heterogeneous oscillators.

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Complexity of real world systems are studied mainly in terms of the nonlinearity in the intrinsic dynamics of their sub systems and the complex interaction patterns among them. In such systems, the variability and heterogeneity of the interacting sub systems can add a further level of complexity. In this context heterogeneity arising from differing dynamical time scales offers several challenges and has applications in diverse fields, ranging from biology, economy, sociology to physics and engineering. In our work, we study the interesting cooperative dynamics in interacting nonlinear systems of differing time scales using the frame work of complex networks.

I. INTRODUCTION

Multiple-timescale phenomena are ubiquitous in Nature and their in-depth understanding brings in several novel challenges. Some of the examples of such phenomena in real world systems are neuronal electrical activity¹⁻³, hormonal regulation⁴⁻⁷, chemical reactions⁸⁻¹⁰, turbulent flows¹¹⁻¹³ and population dynamics¹⁴⁻¹⁶ etc. Although there have been isolated studies addressing its various aspects, there are still many interesting questions that demand multidisciplinary approaches. Several modelling frameworks have proposed methods to understand dual time scale phenomena in single systems, like dynamical models for neuronal dynamics¹⁷. However, studies on collective behavior of connected systems that differ in their intrinsic time scales, are very minimal with many open questions. In this context, the framework of complex networks provides a promising tool to study nonlinear multiple time scale dynamics.

The emergence of synchronization in interacting dynamical units is important for the functionality of many systems and coupled oscillator networks are often studied to understand their dynamics¹⁸,¹⁹. Several types of synchronization phenomena like complete, phase and generalized²⁰⁻²⁷, have been studied in various contexts. However, frequency synchronization is of recent interest and has relevance in many realistic situations ranging from neuronal systems to power grids²⁸⁻³¹, where the individual oscillators can have non-identical natural frequencies. Similarly suppression of oscillations or amplitude death³²⁻³⁸ is also an emergent phenomenon that has interesting implications.

Interestingly, heterogeneity of interacting systems plays an important role in the diversity and organization in many complex systems³⁹⁻⁴². The dynamical processes underlying their complexity often display phase transitions and analysis and prediction of such transitions are crucial for their proper functioning. It is known that functional hierarchy is achieved among neurons not only by special connections but also by time scales of neuronal activity⁴³. In biochemical networks, the fastest time scales correspond to the chemical equilibrium between metabolites while the slower ones relate to more physiologically relevant transformations⁴⁴. So also, widely
different time scales are common in systems of chemical reactions. Heterogeneity can also arise from the nature of connectivity among the systems. Thus, in scale-free networks, the heterogeneity is mostly from the broad distribution of node degrees such that there exists a few nodes with very high degrees. Most of the reported research in similar contexts use phase oscillators and hence do not bring out the amplitude information of the oscillators involved.

The main motivation of the present study is to analyze the onset and characterization of interesting collective dynamics or emergent behavior in a network of connected systems with differing time scales. By considering two types of complex networks, we study the effect of heterogeneity in link structure on the dynamical properties and their critical behavior. We also consider the effect of heterogeneity in the natural frequencies on the emergent dynamics by considering systems with different dynamical time scales. Our study uses standard periodic oscillators of Rössler and Landau Stuart type so that it brings out the amplitude variations and their cross over behaviors as slowness factor increases.

We find the difference in time scales and the heterogeneity in connectivity together can drive the whole network to frequency synchronized clusters. Increasing the heterogeneity in time scales by increasing the number of slow systems or the mismatch in time scales, the whole network settles to a state of no oscillations. The transitions to that state as well as recovery to slower oscillations with cross over in amplitudes are some of the interesting results of the study.

We also address the question of what happens if part or even one node of a network of systems suddenly slows down and then how does it affect the performance of the whole network. In this case, the robustness of the network to such changes is studied in terms of the time taken for each node to escape from the synchronized state leading the whole network to desynchronized dynamics. We find this phenomenon of loss of synchrony settles in a time that decreases with the degree of the node that becomes slow first. Consequent to this, the whole network reorganizes to a frequency synchronized state and this self-organization time is characteristic of the difference in time scales.

II. RANDOM NETWORKS OF SLOW AND FAST PERIODIC SYSTEMS

We construct a random network of N nodes where each node represents a dynamical system. In the network, out of N identical systems, m evolve on a slower time scale. The topological connectivity of the network is defined by a parameter p, where p is the probability with which any two nodes of the network are connected. For a random network of N=100 with periodic Rössler system as nodal dynamics, we analyze how the slowness of m of the systems can affect the dynamics of the whole network. The subset of oscillators with slower time scale is taken as S. The equations that govern the dynamics are then

\[
\dot{x}_i = \tau_i (-y_i - z_i) + \tau_i \epsilon \sum_{j=1}^{N} A_{ij} (x_j - x_i) \\
\dot{y}_i = \tau_i (x_i + ay_i) \\
\dot{z}_i = \tau_i (b + z_i (x_i - c))
\]

where i=1, 2, ..., N. With the parameters chosen as \(a=0.1, b=0.1\) and \(c=4\), the intrinsic nodal dynamics is periodic. \(A_{ij}\) represents the adjacency matrix of connections in the network with its elements having values 1, if the nodes i and j are connected and zero otherwise. First, we consider the case of dual time scales, with \(\tau_i = \tau\) if i belongs to the set S and \(\tau_i = 1\) for other nodes. Thus \(\tau\) becomes a parameter whose value indicates the mismatch in time scales between the two sets of oscillators, smaller values of \(\tau\) corresponding to larger mismatch. The system of equations in eqn. (1) are integrated using Adams-Moulton-Bashforth method, with time step 0.01 for 100,000 times and the last 10,000 values of the x-variables are used for calculations in the study.

A. Suppression and recovery of oscillations

In this section, we report the general results on the coupled dynamics of the systems by varying the parameters involved, time scale of slow systems (\(\tau\)), number of slow systems (m) and coupling strength of connections (\(\epsilon\)), keeping the probability of connection of the network \(p = 0.5\). We find that for sufficient time scale mismatch between slow and fast subsets of systems, for strong coupling, for a range m, all the systems go to a synchronized fixed point. This state is generally known as amplitude death (AD) in the context of coupled dynamics. The Fig. 1 shows the state of amplitude death in the random network of slow and fast systems.

![FIG. 1. (colour online) Time series of x variables of periodic Rössler systems in a random network of slow and fast systems showing amplitude death state for m=50, \(\tau=0.35, \epsilon = 0.05\).](image)

We calculate the average difference between global maxima to global minima \((A_{dff})\) from the time series of each oscillator. This averaged over all the N systems in the network serves as an index to identify onset of AD in the network, since \(<A_{dff}>=0\) would correspond to AD state in the whole network. Using this we identify
the region for occurrence of amplitude death for different m, the number of slow systems present in the network, with suitable values chosen for the other parameters, p, \(\tau\), \(\epsilon\). We plot this region for two sets of values of \(\tau\) and \(\epsilon\) in Fig. 2. This shows that a minimum number of slow systems is required for AD to occur, denoted as \(m_1\). Also most interestingly, the network recovers from AD state as \(m\) increases beyond a certain value, \(m_2\). Thus, suppression of dynamics happens as \(m\) reaches a critical minimum value \(m_1\) and recovery to oscillatory state happens beyond the second critical value, \(m_2\); both these values depend on other parameters like \(\tau\), \(p\) and \(\epsilon\) of the system.

With \(m\) chosen from the region of AD in Fig. 2 we isolate the region of AD in \((\tau, \epsilon)\) plane, for a chosen \(p = 0.5\), and this is shown as region 1 in (Fig. 3).

**FIG. 2.** (colour online) Variation of average \(A_{diff}\) with \(m\). Here \(\tau=0.35\) and \(\epsilon=0.01\) for red curve and \(\tau=0.35\) and \(\epsilon=0.05\) for green curve. N=100, \(p=0.5\).

When coupling is strong and time scale mismatch is small, all the systems in the network settle to an organized state with oscillations of differing amplitudes but same frequency. This state of frequency synchronization is seen in region 2 in the Fig. 3. This is identified by calculating the frequency of each oscillator from its \(x\)-time series using equation,

\[
\omega = \frac{1}{K} \sum_{k=1}^{K} \frac{2\pi}{(t_{k+1} - t_k)}
\]

where \(t_k\) is the time of the \(k^{th}\) zero crossing point in the time series of the oscillator and \(K\) is the total number of intervals for which the zero crossings are counted. In this state, the oscillations of slow systems are relatively closer in phase and so are fast oscillators among themselves but the phase difference between slow and fast sets is relatively large. Below region 2, with low coupling strength the oscillators show a two-frequency state and as time scale mismatch increases they become periodic with two separate time scales. For very high coupling strength in the region marked as 3, network becomes unstable but before this there is region for low \(\tau\) (region 4), where the systems are in a transient state, diverging from AD (Fig. 3).

**C. Crossover phenomena in the emergent dynamics for large \(m\)**

When the systems are in the state of frequency synchronized oscillations, corresponding to region 2 in Fig. 3 the amplitudes of slow and fast sets of systems vary from each other. In general, for low \(m\), we observe that amplitudes of slow systems are smaller than those of fast systems, while for higher \(m\) this behavior gets reversed with the slow set having larger amplitudes than the fast one. Thus, we observe a novel phenomenon of crossover behavior in the amplitudes as \(m\) is varied. To show this explicitly, we study the average amplitude of all the slow systems and that of all the fast systems as \(m\) is varied keeping the values of \(\tau\) and \(\epsilon\) in frequency synchronization state. We find that at a critical value of \(m\), the amplitude of slow and fast systems undergoes a reversal as shown in Fig. 4.

In the state of frequency synchronization, we also observe a similar cross over in the synchronized frequency, which is high for low \(m\) and very low for high \(m\). The synchronized frequency calculated using eqn. (2) for all the oscillators is plotted as function of \(m\) in Fig. 5. From each particular \(\tau\) the frequency of the intrinsic slow and fast frequencies is also shown (black lines). Thus, when the synchronized frequency for any given slow and fast frequencies is also shown (black lines). Thus, when the synchronized frequency for any given \(\tau\) crosses the mean and decreases below that with larger \(m\), we say frequency suppression occurs. The value of \(m\) for which this happens is noted as the crossover point for the emergent frequency of the oscillators.

**B. Frequency synchronized dynamics**

In this section we present the possible dynamical states outside the region of AD in the parameter plane \((\tau, \epsilon)\). When coupling is strong and time scale mismatch is
D. Transition to amplitude death and connectivity of the network

The topology or structure of the random network used, depends on its connectivity which is decided by the probability of connections $p$. As $p$ is increased from 0 to 1, the topology goes from sparsely randomly connected to fully connected network. In order to understand the role of topology or connectivity of the network in the transition to amplitude death, we study the collective behavior of all the systems by varying $p$, for an $m$ value that lies in the AD region of Fig. 2 and values of $\tau$ and $\epsilon$ from the AD region in the parameter plane (Fig 3). For each value of $p$, we take 100 realizations of the network and check what fraction of them goes to an amplitude death state for the whole network. This fraction of the realizations $f$ gives the probability of transitions. We plot $f$ for different values of $m$, to get the corresponding transition curves. We observe that as $m$ increases, the transition to AD occurs at lower values of $p$, till it reaches a minimum and with further increase of $m$, the transitions move to higher values of $p$. This is clear from Fig 4. The threshold value for the transition, where half of the realizations go to amplitude death, is taken as $p_t$.

For each value of $m$ used, we get the width of the transition curve as $\delta$ and normalize the transition curves by replacing $p$ with $(p - p_t)/\delta$. Then we find all the transition curves fall on top of each other revealing a universal behavior. This data crunched curve is shown in the Fig. 6(b). Moreover, the threshold value $p_t$ varies with $m$ as shown in Fig. 6(c). For the random network of 100 periodic Rössler systems, $p_t$ is observed to be minimum when the number of slow systems is around 40.

1. Scaling with size of network

We repeat the above analysis and obtain the transition curves for different network sizes, with $N=100, 150, 200, 300, 500, 600$, keeping $m/N$ ratio fixed at 0.5. We notice the larger the size of the network lower the value of $p$ at which transition takes place. To study the scaling properties of these transitions, with system size, we fit each transition curve with the functional form

$$f = (p - p_t)^\alpha.$$  

Here the value of $p_t$ is chosen as the one where the function gives best fit, and then the corresponding value of the scaling index $\alpha$ is calculated for each transition curve. Our results indicate that the index $\alpha$ varies with the network size $N$. To get the value of $\alpha$ in the large size limit, ie. as $N$ approaches infinity, we plot the calculated $\alpha$ vs $1/N$ and take the asymptote as $1/N$ goes to zero. This comes out to be 0.68, which within numerical errors, can be taken as $2/3$. (Fig. 7)
As is well known, scale-free networks are inherently more heterogeneous than random networks, with broad distribution of node degrees and a few nodes with very high degrees, called hubs. Hence the emergent dynamics due to interactions among slow and fast dynamical systems on such a scale free network will be interesting. For this we generate several realizations of scale free networks using Barabási-Albert algorithm, and consider the dynamics on each node of the network, as that of periodic Rössler systems. The equations for the dynamics on such a network will be the same as eqn (1) with all parameters taken the same way. But the adjacency matrix $A_{ij}$ is taken as per the scale free network topology obtained from the Barabási-Albert algorithm.

In a typical calculation, we take a network of size 100, with a set of $m$ nodes evolving at the slower time scale. Since in a scale free network, hubs play the role of control nodes, we mostly concentrate on cases where hubs follow slower dynamics. Hence in this case the number of slow systems required for AD to occur is much smaller. Thus on a scale free network of 100 systems even with eight of the higher degree nodes or hubs having a time scale mismatch of $\tau$, we find the dynamics of all the systems can be suppressed to AD state. We isolate the region of AD in $(\tau, \epsilon)$ plane as the region where the difference between the global maxima and global minima of all the oscillators goes to zero. This is shown in Fig. 14.

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**III. SCALE FREE NETWORKS OF SLOW AND FAST SYSTEMS**

In the study presented in previous sections, the probability of connections for generating the random network $p$ is kept the same for slow and fast nodes. We now consider a much more heterogeneous case of random networks, generated with three different probabilities and study the effect of slow and fast dynamics on it. This is done by taking the probability with which a slow system connects to another slow system as $p_1$, while a fast system connects with another fast system with $p_2$ and a slow system connects with a fast system with $p_3$. We compute the fraction of realizations resulting in amplitude death in this random network of slow and fast systems by varying $p_3$ for different sets of values of $p_1$ and $p_2$

It is interesting to note that amplitude death happens even in a bipartite network, with $p_1 = 0$ and $p_2 = 0$ but non-zero $p_3$. However, having non-zero values for $p_1$ and $p_2$ helps the network to reach amplitude death state at lower values of probability $p_3$ and the minimum $p_3$ for this transition becomes smaller with increasing $p_1$ and $p_2$ (Fig. 3). We also study the special cases when with $p_1 = 0$, $p_3$ is varied keeping $p_2$ fixed as well as $p_2 = 0$ and $p_3$ is varied with $p_1$ fixed. The results shown are for $p_1 = 0.8$ and $p_2 = 0.8$ in the respective cases and $p_3$ is varied (Fig. 3, b).

**E. Transitions to amplitude death in random networks with non-uniform probabilities of connections**

![Figure 7](image1)

**FIG. 7. (colour online)** Fraction of realizations $f$ for the transition to AD plotted with the probability $p$ for $N=100$ (red, plus), 150 (green, cross), 200 (blue, star), 300 (magenta, square), 500 (cyan, solid square), 600 (black, circle), $\tau = 0.35$, $\epsilon = 0.01$. a) Normalized transition curves $p - p_1/\delta$ for different $N$, b) Variation of the scaling index with $1/N$ with error bar shown in red.

![Figure 8](image2)

**FIG. 8. (colour online)** Fraction of realizations for transition to AD for random network of heterogeneous probabilities for varying $p_3$ a) $p_1=0, p_2=0.8$, b) $p_1=0.3, p_2=0.3$. Here $\tau = 0.35$, $\epsilon = 0.01$. $m=30$ (red, plus), 40 (green, cross), 50 (blue, star), 60 (magenta, square), 70 (cyan, solid square), 80 (black, circle).

![Figure 9](image3)

**FIG. 9. (colour online)** Fraction of realizations for transition to AD for a random network of heterogeneous probabilities for varying $p_3$ a) $p_1=0, p_2=0.8$, b) $p_1=0.3, p_2=0.3$. Here $\tau = 0.35$, $\epsilon = 0.01$. $m=30$ (red, plus), 40 (green, cross), 50 (blue, star), 60 (magenta, square), 70 (cyan, solid square), 80 (black, circle).

The equations for the dynamics on each node of the network, as that of periodic Rössler systems. The equations for the dynamics on such a network will be the same as eqn (1) with all parameters taken the same way. But the adjacency matrix $A_{ij}$ is taken as per the scale free network topology obtained from the Barabási-Albert algorithm. In a typical calculation, we take a network of size 100, with a set of $m$ nodes evolving at the slower time scale. Since in a scale free network, hubs play the role of control nodes, we mostly concentrate on cases where hubs follow slower dynamics. Hence in this case the number of slow systems required for AD to occur, is much smaller. Thus on a scale free network of 100 systems even with eight of the higher degree nodes or hubs having a time scale mismatch of $\tau$, we find the dynamics of all the systems can be suppressed to AD state. We isolate the region of AD in $(\tau, \epsilon)$ plane as the region where the difference between the global maxima and global minima of all the oscillators goes to zero. This is shown in Fig. 14.

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required to induce AD starting with the highest degree as slow and increasing the number one by one. For each case, the average amplitude differences of all the oscillators is calculated. The plot of this averaged amplitude (\( A_{diff} \)) with the number of slow hubs \( m \) gives this as the value of \( m \) at which (\( A_{diff} \)) becomes zero. This is repeated for different realizations and shown in Fig. 12. Here the values of \( \tau \) and \( \epsilon \) are chosen from the amplitude death region in Fig. 11.

A. Spreading of slowness and self organization on scale free networks

When there is no time scale mismatch in the dynamics of systems, all the systems on a scale free network, can be completely synchronized with a sufficiently strong coupling strength. Starting with such a state, after giving sufficient time so that all the oscillators settle to complete synchronization, we make one of the nodes, called source node, slower in its dynamics. Clearly this can disrupt the dynamics of all other nodes as the slowness spreads over the network. Consequently, all oscillators will then move away from the state of complete synchronization.

Due to the heterogeneity of connections in the scale free network, the time taken by each oscillator to move away from synchronization will not be the same. We analyze this scenario in terms of the degree of the node and shortest path from the source node, in the following two ways. We calculate the change in the variance of all oscillators in time. When they are completely synchronized, the variance would be zero as shown in Fig. 13. When one node is made slow, the variation of each oscillator from the mean of anticipated synchronized oscillations (the synchronized oscillation they would have followed if this node was not made slow), is nonzero indicating onset of desynchronization. From the Fig. 13 it is evident that for each oscillator the time taken for the variance to go to a non zero value \( \pm \nu \) is different, with the source node taking the least time obviously. This time, \( t_{\nu} \), for each oscillator to reach a specific value \( \pm \nu \) (typically -0.01 or 0.01) for its variation is plotted as a function of the degree of the nodes. It is easy to see that \( t_{\nu} \) increases as the shortest path of that node from the source node increases. We repeat this for several nodes as sources, including hubs and low degree nodes. Fig. 14 shows the plot of \( t_{\nu} \) against degree of nodes for the two cases with a hub as the source node and a low degree node as the source node for a typical realization. In the case where a hub is the source of slowness, we see most of the nodes move away in much shorter times since the shortest path from the source node is small.

We repeat the study for different realizations of the network and calculate the number of systems that get
1. Self organization of the network to frequency synchronized state

Once synchrony is disturbed as discussed above due to a single node going slow, de-synchronization sets in characteristic times depending on the degrees of nodes in the network. Subsequent to this, given sufficient time, all the oscillators are found to reorganise themselves into a frequency synchronized state. (Fig. 17). This is an interesting and novel phenomenon of self-organization, where the network goes from a collective behavior of complete synchronization to another less ordered but coherent emergent state of frequency synchronization by re-adjusting the dynamics of all the nodes, after the network is perturbed by making one node slow.

To characterize this process, we calculate the frequency of each oscillator using eqn(2) and plot them with time (Fig. 18). The figure shows the synchronized frequencies in the beginning, the de-synchronized frequencies just after one node is made slow at t = 500, and finally the re-adjusted lower frequency after self-organization to frequency synchronized state. We also study the time taken for self-organization, called self organization time, \( t_{so} \), averaged over several realizations for \( \tau = 0.3 \) and \( \epsilon = 0.03 \).

B. Scale free network with multiple time scales

In this section, we study the collective dynamics of nonlinear systems on a scale free network where the time scale of each node varies with its degree following the

\[ T(k) = (a/k) + b \]

with \( a = 180 \) and \( b = 18 \).

FIG. 15. (colour online) Number of systems that move away from synchrony \( (N_s) \) in a range of time is plotted with time. The results shown are averaged over six realisations. Here a) corresponds to the highest hub as source of slowness for each realisation and b) corresponds to the lowest degree node being made slow. In this case also, \( \tau = 0.3 \) for the node and \( \epsilon = 0.03 \).

FIG. 14. (colour online) Time taken for each oscillator to move away from synchrony is plotted with its degree \( (k_i) \) when one source node becomes slow. This is shown for a particular realisation of the network of 100 systems where a) the highest degree hub is made slow with degree 47 and in b) the lowest degree node is made slow with degree 2. Different colors represent different shortest path lengths from the source node with shortest path 1(red, plus), 2(green, cross), 3(blue, star). Here \( \tau = 0.3 \) for the source node and \( \epsilon = 0.03 \).

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FIG. 18. (colour online) a) Self-organization of oscillators into frequency synchronized state for the whole network for a typical realization of scale free network after one hub is made slow. On the x axis the time and on the y axis the corresponding frequency are shown. At time=500 one hub is made slow, b) Time taken to organize into the new synchronized state, $t_{so}$, averaged over six realizations is plotted for different $\tau$ values with $\epsilon$ kept at 0. $t_{so}$ falls off with increasing mismatch $\tau$ as a straight line.

FIG. 19. (colour online) $\langle A_{\text{diff}} \rangle$ vs $\epsilon$ in a typical realization of scale free network showing onset of AD due to multiplicity of time scales for periodic Rössler systems with $\tau_i = 2/k_i$ for $N=100$ (red), $500$ (green), $1000$ (blue), to show.

relation

$$\tau_i = 2/k_i \quad (5)$$

This is chosen such that the node with highest degree will have the slowest time scale and the time scale increases as degree decreases. Since the network is scale free, the number of different degrees, $k_i$ and hence the number of different time scales, $\tau_i$, will be less than $N$ but still will have a multiplicity of time scales. Thus, for one typical realisation of 1000 nodes, we get 30 different time scales in the network. We find the presence of multiple time scales, forces the whole network to collapse to a state of AD. The onset of AD, after a threshold coupling strength, is evident from the plot of $\langle A_{\text{diff}} \rangle$, for different values of $\epsilon$, in Fig. 19 with three different sizes $N=100, 500$ and 1000.

For lower values of $\epsilon$, prior to onset of AD, we see oscillations with differing amplitudes. But even with multiple time scales, the connectivity through the network makes the systems organize into three groups, high degree nodes with lower time scales having smaller amplitudes, low degree nodes with faster time scales having larger amplitudes and an intermediate group with amplitudes in between. This is clear from the distribution of amplitudes of all the oscillators in the network from six realizations of the network of size $N=1000$ is shown in Fig. 20.

IV. CONCLUSION

The study reported, in general, addresses the important question of how the mismatch in dynamical time scales of different interacting units can affect the collective performance of a complex system. We consider nonlinear dynamical systems with complex interaction patterns modelled by random and scale free topologies. We are motivated by the fact that one important factor for heterogeneity of interacting systems in real-world complex systems is the diversity in time scales. This makes our study highly relevant in understanding such systems from a dynamical systems perspective.

We study the robustness of coupled oscillator networks which are widely used as models for understanding the dynamics of networked systems in biology, physics, and engineering and see how increasing the heterogeneity in time scales makes them undergo transitions. The emergent dynamics is then characterized in terms of the average amplitude of oscillations and the common frequency of the coupled systems.

The results presented are primarily for periodic Rössler systems, however qualitatively similar results are obtained in the case of random networks, with Landau-Stuart oscillators, chaotic Rössler and Lorenz systems.

When the network of connections is random, we study suppression and recovery of oscillations and cross over in amplitudes and frequency as number of slow systems in-
increases. The transition to AD in terms of the probability of connections p scales with network size, the index of scaling being 2/3.

If the systems are connected on a scale free network with heterogeneity in the degrees, we find hubs can function as control nodes in the emergent dynamics. We study the spread of slowness through the network due to one node being slow and discuss the self-organization of the whole network from completely synchronized state to one of frequency synchronization. The hierarchical structure of this network leads to different patterns in the spread of slowness due to a slow source node. The study of the times involved gives an estimate of minimum time within which corrective measures are to be initiated to restore the network dynamics.

When multiple time scales are introduced on a scale free network, we observe reorganization of the systems into groups of differing amplitudes and onset of AD with increasing coupling strength.

In all cases studied, suppression of dynamics with the whole network settling to a common fixed point, seems to be the most prevalent emergent state. Thus, difference in time scales is established as another mechanism for inducing AD in interacting systems, which is important to suppress unwanted oscillations that hinder certain processes in several contexts. It is interesting that cooperative phenomenon of lower order like frequency synchronization, may still emerge even in the presence of time scale diversity.

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