Selective separability and $q^+$ on maximal spaces

Ramiro de la Vega, Javier Murgas, Carlos Uzcátegui

January 22, 2020

Abstract

Given a hereditarily meager ideal $\mathcal{I}$ on a countable set $X$ we use Martin’s axiom for countable posets to produce a zero-dimensional maximal topology $\tau^\mathcal{I}$ on $X$ such that $\tau^\mathcal{I} \cap \mathcal{I} = \{\emptyset\}$ and, moreover, if $\mathcal{I}$ is $p^+$ then $\tau^\mathcal{I}$ is selectively separable (SS) and if $\mathcal{I}$ is $q^+$, so is $\tau^\mathcal{I}$. In particular, we obtain regular maximal spaces satisfying all boolean combinations of the properties SS and $q^+$.

Keywords: maximal countable spaces; selective separability; ideals on countable sets; $p^+$; $q^+$.
MSC: 54G05, 54A35, 03E57.

1 Introduction

A topological space $X$ is selectively separable (SS), if for any sequence $(D_n)_n$ of dense subsets of $X$ there are finite sets $F_n \subseteq D_n$, for $n \in \omega$, such that $\bigcup_n F_n$ is dense in $X$. This notion was introduced by Scheepers [13] and has received a lot of attention ever since (see for instance [1] [2] [3] [4] [5] [6] [9] [15]). Bella et al. [4] showed that every separable space with countable fan tightness
is SS. On the other hand, Barman and Dow [1] showed that every separable Fréchet space is also SS (see also [6]).

A topological space is maximal if it is a dense-in-itself space such that any strictly finer topology has an isolated point. It was shown by van Douwen [20] that a space is maximal if, and only if, it is extremely disconnected (i.e. the closure of every open set is open), nodec (i.e. every nowhere dense set is closed) and every open set is irresolvable (i.e. if $U$ is open and $D \subseteq U$ is dense in $U$, then $U \setminus D$ not dense in $U$). He constructed a countable maximal regular space.

A countable space $X$ is $q^+$ at a point $x \in X$, if given any collection of finite sets $F_n \subseteq X$ such that $x \in \bigcup_n F_n$, there is $S \subseteq \bigcup_n F_n$ such that $x \in S$ and $S \cap F_n$ has at most one point for each $n$. We say that $X$ is a $q^+$-space if it is $q^+$ at every point. Every countable sequential space is $q^+$ (see [18, Proposition 3.3]). This notion is motivated by the analogous concept of a $q^+$ filter used in Ramsey theory.

The existence of a maximal regular SS space is independent of ZFC. In fact, in ZFC there is a maximal non SS space [1] and it is consistent with ZFC that no countable maximal space is SS [11, 15]. On the other hand, using MA for countable posets, Barman and Dow [1] showed that it is also consistent that there is a maximal, SS regular countable space.

Similar questions have been studied in the context of countable spaces with an analytic topology (i.e. the topology of the space $X$ is an analytic subset of $2^X$ [17, 18]). Maximal topologies are not analytic. In fact, analytic topologies cannot be extremely disconnected or irresolvable, nevertheless there are nodec regular spaces with analytic topology [11, 19].

Nodec regular spaces are not easy to construct, the most common examples are the maximal spaces. In [11] were constructed several examples of nodec regular spaces with analytic topology that are neither SS nor $q^+$. However, it was left open whether there are nodec regular spaces satisfying other boolean combinations of the properties $q^+$ and SS. Using MA for countable posets, we will construct such nodec (in fact maximal) regular spaces. We do not know if such spaces can be found with analytic topology. We notice, that analogously to what happens with $Q$-points, it is consistent that there are no maximal $q^+$-spaces.

We actually show that (under MA for countable posets) for a given hereditarily meager ideal $\mathcal{I}$ on $\omega$ there is a maximal regular topology $\tau^\mathcal{I}$ on $\omega$ which is crowded in a strong sense, namely, $\tau^\mathcal{I} \cap \mathcal{I} = \{\emptyset\}$ and, moreover, if $\mathcal{I}$ is $p^+$ then $\tau^\mathcal{I}$ is SS and if $\mathcal{I}$ is $q^+$, so is $\tau^\mathcal{I}$. 

2


2 Preliminaries

For us space will always refer to a countable $T_1$ topological space. $X$ will denote an infinite countable set. Note that we can use characteristic functions to view any collection $A$ of subsets of $X$ as a subset of $2^X$. So we will say that such a collection $A$ is closed, $F_\sigma$, analytic, etc., if $A$ has the corresponding property when it is viewed as a subspace of $2^X$ with the usual product topology. The collection $A$ is said to be hereditary if $B \subseteq A \in A$ implies $B \in A$.

An ideal $I$ on $X$ is a hereditary collection of subsets of $X$ which is also closed under taking finite unions. We denote by $I^+$ the $I$-positive subsets of $X$ (i.e. $I^+ = \mathcal{P}(X) \setminus I$). Throughout this article, unless explicitly stated, we will always assume our ideals to be proper (i.e. $X \notin I$) and free (i.e. $[X]^{<\omega} \subseteq I$ where $[X]^{<\omega}$ denote the collection of finite subsets of $X$).

We say that an ideal $I$ is hereditarily meager (HM) if for any $A \in I^+$ the ideal $I|A = I \cap \mathcal{P}(A)$ is meager as a subset of $2^A$. Any analytic or co-analytic ideal (in particular any Borel ideal) is hereditarily meager.

Given a finite $F \subseteq X$ we write $F^\uparrow = \{A \subseteq X : F \subseteq A\}$. Note that if $C$ is a closed and hereditary collection of subsets of $X$ and $A \subseteq X$ is not in $C$ then there is a finite $F \subseteq A$ such that $F^\uparrow \cap C = \emptyset$.

Following [14] we define for an ideal $I$ on $X$ the set

$$C(I) = \{C \subseteq \mathcal{P}(X) : C \text{ is closed, hereditary and such that } \forall A \in I \exists F \in [X]^{<\omega} A \setminus F \in C\}.$$ 

The following is essentially proved in [14] Lemma 2.7:

Lemma 2.1. If $I$ is a HM ideal on $X$ and $A \in I^+$ then there is a $C \in C(I)$ such that $A \setminus F \notin C$ for any $F \in [X]^{<\omega}$.

We write $A \subseteq^* B$ if $A \setminus B$ is finite. An ideal $I$ is $p^+$, if for every decreasing sequence $(A_n)_n$ of sets in $I^+$, there is $A \in I^+$ such that $A \subseteq^* A_n$ for all $n \in \omega$. We say that $I$ is $p^-$, if for every decreasing sequence $(A_n)_n$ of sets in $I^+$ such that $A_n \setminus A_{n+1} \in I$, there is $B \in I^+$ such that $B \subseteq^* A_n$ for all $n$. An ideal $I$ is $q^+$, if for every $A \in I^+$ and every partition $(F_n)_n$ of $A$ into finite sets, there is $S \in I^+$ such that $S \subseteq A$ and $S \cap F_n$ has at most one element for each $n$.

For any topological space $X$ and $x \in X$, let $I_x$ be the ideal of all subsets $A$ of $X$ such that $x \notin A \setminus \{x\}$. A point $x$ is $p^+$ if $I_x$ is $p^+$. The space $X$ is $p^+$
if every point is $p^+$. We define analogously the notions of $p^-$ and $q^+$ points (spaces).

A space $X$ is discretely generated (DG) if for every $A \subseteq X$ and $x \in \overline{A}$, there is $E \subseteq A$ discrete such that $x \in \overline{E}$. This notion was introduced by Dow et al. in [8]. It is not easy to construct spaces which are not DG, the typical examples are maximal spaces (which are nodec).

Every countable $p^-$ regular space is selectively separable and discretely generated (see [6, 11]). In summary, we have the following implications for countable regular spaces.

- $1^{st}$ Fréchet
- Sequential
- DG
- $q^+$
- $p^+$
- $p^-$
- $q^+$
- SS
- non nodec

Thus, among all these properties, a maximal space can only be $q^+$ or SS.

We denote by $m_c$ the minimal cardinal $\kappa$ for which $\text{MA}(\kappa)$ for countable posets fails. The statement that $m_c = c$ is also denoted $\text{MA}_{\text{ctble}}$. It is known that $m_c \leq \frak{d}$. For a given space $X$, we write $w(X)$ and $\pi w(X)$ for its weight and $\pi$-weight respectively. The following facts are known. If $w(X) < \frak{d}$, then $X$ is DG ([12]), SS ([13] and see also [1]) and resolvable ([7]).

For the sake of completeness we give some known examples of regular topologies to illustrate the properties DG, $q^+$ and SS.

For each ideal $\mathcal{I}$ on $\omega$ consider the following topology $\tau_\mathcal{I}$ over $\omega^{<\omega}$, the collection of finite sequences on $\omega$. A subset $U$ of $\omega^{<\omega}$ is open if and only if $\{n \in \omega : s \upharpoonright n \notin U\} \in \mathcal{I}$ for all $s \in U$. Let $\text{Seq}(\mathcal{I})$ denote the space $\omega^{<\omega}, \tau_\mathcal{I}$. This space is $T_2$, zero dimensional and crowded. Notice that when $\mathcal{I}$ is the ideal of finite sets, then $\text{Seq}(\mathcal{I})$ is homeomorphic to the Arkhangel’skiĭ-Franklin space $\text{Seq}$. When $\mathcal{I}$ is analytic, so is $\tau_\mathcal{I}$. We also know that $\text{Seq}(\mathcal{I})$ is DG and non SS for any $\mathcal{I}$. It is $q^+$ if, and only if, $\mathcal{I}$ is $q^+$ ([6]).

Next we show a space which is SS, DG and not $q^+$ that will be used later in the paper.
Example 2.2. Let $CL(2^{\omega})$ be the collection of all clopen subsets of $2^{\omega}$ as a subspace of $2^{2^{\omega}}$. Then $CL(2^{\omega})$ is $p^+$ (and thus SS and DG) and not $q^+$ (see [6]). We will need later a subspace of $CL(2^{\omega})$ where $q^+$ fails in a somewhat stronger form. Let $2^{<\omega}$ be the collection of finite binary sequences. If $s \in 2^{<\omega}$ and $i \in \{0, 1\}$, $|s|$ denotes its length and $s \hat{i}$ the sequence obtained concatenating $s$ with $i$. For $s \in 2^{<\omega}$ and $\alpha \in 2^{\omega}$, let $s < \alpha$ if $\alpha(i) = s(i)$ for all $i < |s|$ and $[s] = \{\alpha \in 2^{\omega} : s < \alpha\}$. Each $x \in CL(2^{\omega})$ is a finite union of sets of the form $[s]$ for $s \in 2^{<\omega}$. Let $2^{n}$, we say that $s$ and $t$ are linked if there is a sequence $u \in 2^{n-1}$ such that $s = u \hat{i}$ and $t = u \hat{j}$ with $i + j = 1$. For a positive integer, we say that a $x \in X$ is $k$-adequated, if $x$ can be written as $[s_1] \cup \cdots \cup [s_m]$ with each $s_i \in 2^k$ and such that any pair of them are not linked. Let $A_k = \{x \in CL(2^{\omega}) : x$ is $k$-adequated$\}$. Notice that $\bigcup_{k \in \omega} A_{k+1}$ is dense in $CL(2^{\omega})$. Let $X = \bigcup_{k \in \omega} A_{k+1}$. Let $S \subseteq X$ be such that $S \cap A_k$ has at most one element for each $k$. Then $S$ is closed in $X$. This clearly shows that $X$ is not $q^+$.

As we have already mentioned, analytic nodec spaces are hard to define and they are the only examples we know of non DG analytic spaces. In [11] was constructed an analytic regular space $\mathcal{Y}(I_{nd}^{\ast})$ which is nodec, non $q^+$ and non SS. However, we do not know if there is an analytic nodec $q^+$ (or SS) regular space.

In the following table we summarize what we know. One of the main goals of this paper is to construct the maximal spaces mentioned in the last column. Notice that the existence of those maximal spaces (except the one at the bottom row) is not provable in ZFC.

| DG | SS | $q^+$ | Analytic topology | Non definable topology |
|----|----|------|------------------|-----------------------|
| ✔  | ✔  | ✔    | $\mathbb{Q}$     |                       |
| ✔  | ✔  | ×    | $CL(2^{k})$      |                       |
| ✔  | ×   | ✔    | $Seq$            |                       |
| ✔  | ×   | ×    | $Seq(I)$ (with $I$ non $q^+$) |                       |
| ×   | ✔   | ✔    | ??              | (MAcubl) Maximal      |
| ×   | ✔   | ×    | ??              | (MAcubl) Maximal      |
| ×   | ×   | ✔    | ??              | (MAcubl) Maximal      |
| ×   | ×   | ×    | $\mathcal{Y}(I_{nd}^{\ast})$ | Maximal |

5
3 $\mathcal{I}$-crowded topologies

When constructing a crowded topology on a set $X$ we only need to worry not to include any finite set in the topology, that is, to keep the open sets outside of the ideal $[X]<\omega$. In order to get additional properties on the topology it will be useful to follow this idea with other ideals.

**Definition 3.1.** Given a space $(X, \tau)$ and an ideal $\mathcal{I}$ on $X$ we say that:

- The topology $\tau$ is $\mathcal{I}$-crowded if $\tau \cap \mathcal{I} = \{\emptyset\}$.
- A subset $A \subseteq X$ is $(\mathcal{I}, \tau)$-crowded if for every $U \in \tau$ the intersection $A \cap U$ is either empty or belongs to $\mathcal{I}^+$.

Note that saying that $\tau$ is $\mathcal{I}$-crowded is equivalent to say that $X$ is $(\mathcal{I}, \tau)$-crowded. For future reference we collect a few simple observations in the following remark which the reader can easily corroborate.

**Remark 3.2.**

1. A topology $\tau$ is crowded if and only if it is $[X]<\omega$-crowded if and only if it is $\mathcal{I}$-crowded for some $\mathcal{I}$.

2. If $\tau$ has a $\pi$-net (in particular if it has a base or a $\pi$-base) contained in $\mathcal{I}^+$ then $\tau$ is $\mathcal{I}$-crowded.

3. The union of any $\subseteq$-chain of $\mathcal{I}$-crowded topologies on $X$ generates an $\mathcal{I}$-crowded topology on $X$.

4. If $A$ is $(\mathcal{I}, \tau)$-crowded and $A \subseteq B \subseteq \text{cl}_\tau(A)$ then $B$ is also $(\mathcal{I}, \tau)$-crowded.

The following lemma allows us to add certain subsets of $X$ to a given $\mathcal{I}$-crowded topology on $X$ while keeping the topology $\mathcal{I}$-crowded.

**Lemma 3.3.** Let $(X, \tau)$ be a zero-dimensional $\mathcal{I}$-crowded space and $A \subseteq X$. Suppose that $A \subseteq X$ admits a partition $A = \bigcup_{m \in \omega} A_m$ such that $\overline{A}_m = \overline{A}$ and $A_m$ is $(\mathcal{I}, \tau)$-crowded for all $m \in \omega$. Then there is a zero-dimensional $\mathcal{I}$-crowded topology $\tau' \supseteq \tau$ on $X$ such that $w(\tau') \leq w(\tau)$ and $A \in \tau'$. Moreover $\text{cl}_{\tau'}(A) = \text{cl}_\tau(A)$. 

6
Proof. Fix a base $B$ of clopen subsets for $\tau$, closed under finite intersections and with $|B| = w(\tau)$. Define 

$$B' = B \cup \{A_m : m \in \omega\} \cup \{X \setminus A_m : m \in \omega\}.$$ 

Since $B \subseteq B'$ and $|B'| = |B|$ we have that $B'$ is a subbase for a topology $\tau' \supseteq \tau$ with $w(\tau') \leq w(\tau)$. Clearly all the elements of $B'$ are $\tau'$-clopen and therefore $\tau'$ is zero-dimensional. Note that $A \in \tau'$ being a union of elements of $B'$. To show that $\tau'$ is $I$-crowded, note that if $V$ is a non-empty finite intersection of elements of $B'$ then there is $U \in B$ such that $V$ is of the form $V = U \cap A_m$ for some $m \in \omega$ or $V = U \cap \bigcap\{(X \setminus A_j) : j \in J\}$ for some finite $J \subseteq \omega$. If $U \cap A = \emptyset$ then $V = U$ which is in $\mathcal{I}^+$ since $\tau$ is $\mathcal{I}$-crowded. If $U \cap A \neq \emptyset$ then by assumption $U \cap A_i$ is non-empty and belongs to $\mathcal{I}^+$ for every $i \in \omega$ and therefore $V \in \mathcal{I}^+$ since it contains at least one of these. To see that $cl_{\tau'}(A) = cl_\tau(A)$ note that if $x \in cl_\tau(A)$ and $V$ is a basic $\tau'$-neighborhood of $x$, the previous argument shows in particular that $V \cap A \neq \emptyset$ and hence $x \in cl_{\tau'}(A)$. 

Perhaps it is well known that a crowded space of weight smaller than $m_c$ is $\omega$-resolvable (i.e. it can be partitioned into countably many disjoint dense subsets). Next we prove a stronger result.

**Lemma 3.4.** Let $\mathcal{I}$ be a HM ideal on $X$ and $\tau$ an $\mathcal{I}$-crowded topology on $X$ with $w(\tau) < m_c$. Suppose that $A \subseteq X$ is $(\mathcal{I}, \tau)$-crowded. Then there is a partition $A = \bigcup_{m \in \omega} A_m$ such that each $A_m$ is $(\mathcal{I}, \tau)$-crowded and dense in $A$.

**Proof.** Let $\mathbb{P}$ be the set of all finite partial functions $p : A \to \omega$ ordered by reverse inclusion and fix $B$ a base for $\tau$ with $|B| < m_c$. For every $U \in B$ for which $A \cap U$ is non-empty we use Lemma 2.1 to fix $\mathcal{C}_U \subseteq C(\mathcal{I})$ such that $(A \cap U) \setminus F \notin \mathcal{C}_U$ for any finite $F \subseteq X$. Now for every such $U$, every $F \in [X]^{<\omega}$ and every $m \in \omega$, the set

$$\mathbb{D}(U, F, m) = \{p \in \mathbb{P} : (p^{-1}(m) \cap U \setminus F) \uparrow \cap \mathcal{C}_U = \emptyset\}$$

is dense in $\mathbb{P}$.

To see this, fix $U, F, m$ as before and $q \in \mathbb{P}$. Since $A \cap U \setminus (F \cup \text{dom } q) \notin \mathcal{C}_U$, using that $\mathcal{C}_U$ is closed and hereditary, we can find a finite $E \subseteq A \cap U \setminus (F \cup \text{dom } q)$ such that $E \uparrow \cap \mathcal{C}_U = \emptyset$. But now $p = q \cup (E \times \{m\})$ is an extension of $q$ that belongs to $\mathbb{D}(U, F, m)$. 

7
Since \( P \) is countable and \(|B| < m_c\), there exists a filter \( G \subseteq P \) intersecting all the \( D(U,F,m) \) and also intersecting the dense sets \( \{ p \in P : a \in \text{dom} \ p \} \) for \( a \in A \). Thus \( \bigcup G : A \to \omega \) is a total function and \( A_m = (\bigcup G)^{-1}(m) \) for \( m \in \omega \) defines a partition of \( A \) as desired.

To see that \( A_m \) is \((I,\tau)\)-crowded and dense in \( A \) let \( U \in B \) such that \( A \cap U \neq \emptyset \). We want to show that \( A_m \cap U \in I^{+} \), so fix a finite \( F \subseteq X \) and choose a \( p \in G \cap D(U,F,m) \). Then \( (p^{-1}(m) \cap F) \uparrow \cap C_U = \emptyset \) and therefore \( A_m \cap U \setminus F \notin C_U \). Since this is true for any finite \( F \) and \( C_U \) belongs to \( C(I) \), it follows that \( A_m \cap U \notin I \).

Note that in particular this last lemma tells us that \( X \) can be partitioned into countably many dense subsets, so \((X,\tau)\) is \( \omega \)-resolvable in a strong sense: the dense subsets can be chosen outside of the ideal \( I \). For the ideal \( I = [X]^{<\omega} \) we just get:

**Corollary 3.5.** Any crowded space of weight smaller than \( m_c \) is \( \omega \)-resolvable.

We can also improve Lemma 3.3 for spaces of small weight. The next corollary will be key in our main construction.

**Corollary 3.6.** Let \( I \) be a HM ideal on \( X \) and \( \tau \) a zero-dimensional \( I \)-crowded topology on \( X \) with \( w(\tau) < m_c \). Suppose that \( A \subseteq X \) is \((I,\tau)\)-crowded. Then there is a zero-dimensional \( I \)-crowded topology \( \tau' \supseteq \tau \) on \( X \) such that \( w(\tau') \leq w(\tau) \) and \( A \in \tau' \). Moreover \( cl_{\tau'}(A) = cl_{\tau}(A) \).

**Proof.** It follows immediately from Lemmas 3.3 and 3.4.

The last two observations on Remark 3.2 and a repeated application of the previous result allows us to show:

**Lemma 3.7.** Let \( I \) be an HM ideal on \( X \) and \( \tau \) a zero-dimensional \( I \)-crowded topology on \( X \) with \( w(\tau) < m_c \). Suppose that \( D \) is an \((I,\tau)\)-crowded subset of \( X \). Then there is a zero-dimensional \( I \)-crowded topology \( \tau' \supseteq \tau \) on \( X \) such that \( w(\tau') \leq w(\tau) \), \( cl_{\tau'}(D) \in \tau' \) and \( cl_{\tau'}(D) \setminus D \) is \( \tau' \)-discrete.

**Proof.** Suppose that \( cl_{\tau}(D) \setminus D \) is infinite (otherwise we can just let \( \tau' \) be the topology given by Corollary 3.6 applied to \( \tau \) and \( cl_{\tau}(D) \)) and fix an enumeration \( cl_{\tau}(D) \setminus D = \{ a_n : n \in \omega \} \). Let \( \tau_0 = \tau \) and given \( \tau_n \) let \( \tau_{n+1} \) be the topology given by Corollary 3.6 applied to the space \((X,\tau_n)\) and the subset \( D \cup \{ a_i : i \leq n \} \). It is clear that all the \( \tau_n \)'s are \( I \)-crowded, zero-dimensional and \( w(\tau_n) \leq w(\tau) \). So if we let \( \tau' \) be the topology generated
by $\bigcup_{n \in \omega} \tau_n$ we also get a zero-dimensional $\mathcal{I}$-crowded topology with $w(\tau') \leq w(\tau)$. Moreover, since the set $D \cup \{a_i : i \leq n\}$ is $\tau'$-open for each $n \in \omega$, $cl_{\tau}(D) \setminus D$ is $\tau'$-discrete and $cl_{\tau}(D)$ is $\tau'$-open.

Note that the conclusion of the previous result guarantees that $cl_{\tau}(D) = cl_{\tau'}(D)$ for any crowded topology $\tau' \supseteq \tau$.

The following two lemmas generalize the fact that any space of weight smaller than $m_c$ is both selectively separable and $q^+$.

**Lemma 3.8.** Let $\mathcal{I}$ be a $p^+$ and HM ideal on $X$ and $\tau$ an $\mathcal{I}$-crowded topology on $X$ with $\pi w(\tau) < m_c$. Suppose that $\langle D_i : i \in \omega \rangle$ is a decreasing sequence of $(\mathcal{I}, \tau)$-crowded dense subsets of $X$. Then there are finite sets $K_i \subseteq D_i$ for $i \in \omega$ such that $\bigcup_{i \in \omega} K_i$ is dense and $(\mathcal{I}, \tau)$-crowded.

**Proof.** Let $D = \bigcup_{i \in \omega} D_i$ and let $\mathbb{P}$ be the set of all finite partial functions $p : \omega \to [D]^{<\omega}$ such that $p(i) \subseteq D_i$ for all $i \in \text{dom } p$. We order $\mathbb{P}$ by reverse inclusion.

Fix $\mathcal{B}$ a $\pi$-base for $\tau$ with $|\mathcal{B}| < m_c$. Since each $D_i$ is $(\mathcal{I}, \tau)$-crowded and dense, using that $\mathcal{I}$ is a $p^+$-ideal, we can find for each $U \in \mathcal{B}$ a set $D_U \in \mathcal{I}^+$ such that $D_U \subseteq^* D_i \cap U$ for all $i \in \omega$. Using Lemma 2.1 we can fix $C_U \in C(\mathcal{I})$ such that $D_U \setminus F \notin C_U$ for any finite $F \subseteq X$.

For each $U \in \mathcal{B}$ and each finite $F \subseteq X$, the set

$$\mathbb{D}(U, F) = \{p \in \mathbb{P} : \exists i \in \text{dom } p \ (p(i) \cap U \setminus F) \uparrow \cap C_U = \emptyset\}$$

is dense in $\mathbb{P}$. To see this, fix $U \in \mathcal{B}$, $F \in [X]^{<\omega}$ and $q \in \mathbb{P}$. Choose $i \in \omega \setminus \text{dom } q$. Since $D_U \setminus (D_i \cap U)$ is finite we have that $D_U \cap D_i \cap U \setminus F \notin C_U$. Using that $C_U$ is closed and hereditary we can find a finite $E \subseteq D_i \cap U \setminus F$ such that $E \uparrow \cap C_U = \emptyset$. But now $p = q \cup \{(i, E)\}$ is an extension of $q$ that belongs to $\mathbb{D}(U, F)$.

Since $\mathbb{P}$ is countable and $|\mathcal{B}| < m_c$, there exists a filter $G \subseteq \mathbb{P}$ intersecting all the $\mathbb{D}(U, F)$ and also intersecting the dense sets $\{p \in \mathbb{P} : i \in \text{dom } p\}$ for $i \in \omega$. Thus $\bigcup G : \omega \to [D]^{<\omega}$ is a total function and the sets $K_i = (\bigcup G)(i)$ for $i \in \omega$ satisfy the conclusion of the lemma, since one can see that $U \cap \bigcup_{i \in \omega} K_i \setminus F \notin C_U$ for all $F \in [X]^{<\omega}$ and all $U \in \mathcal{B}$.

Using the previous result with the ideal $\mathcal{I} = [X]^{<\omega}$ we get:

**Corollary 3.9.** Any crowded space of $\pi$-weight smaller than $m_c$ is selectively separable.
Now we show an analogous result concerning the property $q^+$.  

**Lemma 3.10.** Let $\mathcal{I}$ be a $q^+$ and HM ideal on $X$ and $\tau$ an $\mathcal{I}$-crowded topology on $X$ with $w(\tau) < m_c$. If $\langle F_i : i \in \omega \rangle$ is a sequence of pairwise disjoint finite subsets of $X$ such that $\bigcup_{i \in \omega} F_i$ is $(\mathcal{I}, \tau)$-crowded, then there is an $(\mathcal{I}, \tau)$-crowded $S \subseteq \bigcup_{i \in \omega} F_i$ such that $|S \cap F_i| \leq 1$ for each $i \in \omega$ and $S = \bigcup_{i \in \omega} F_i$.

**Proof.** Just for the purpose of this proof let us say that a subset of $X$ is a **selector** if it intersects each $F_i$ in at most one point. Let $\mathbb{P}$ be the set of all finite selectors $p \subseteq \bigcup_{i \in \omega} F_i$ ordered by reverse inclusion. Also fix $\mathcal{B}$ a base for $\tau$ with $|\mathcal{B}| < m_c$.

For each $U \in \mathcal{B}$ which intersects $\bigcup_{i \in \omega} F_i$ we have that $(\bigcup_{i \in \omega} F_i) \cap U \in \mathcal{I}^+$ and since $\mathcal{I}$ is a $q^+$-ideal there is a selector $S_U \subseteq (\bigcup_{i \in \omega} F_i) \cap U$ with $S_U \in \mathcal{I}^+$. Using Lemma 2.1 we can fix $\mathcal{C}_U \in C(\mathcal{I})$ such that $S_U \setminus F \notin \mathcal{C}_U$ for any finite $F \subseteq X$. Now for each $F \in [X]^{<\omega}$ the set

$$\mathbb{D}(U, F) = \{ p \in \mathbb{P} : (p \cap U \setminus F) \uparrow \cap \mathcal{C}_U = \emptyset \}$$

is dense in $\mathbb{P}$.

To see this, fix $U, F$ as before and $q \in \mathbb{P}$. Since $S_U \setminus F \notin \mathcal{C}_U$ and $\mathcal{C}_U$ is closed and hereditary, we can find a finite $s \subseteq S_U \setminus F \setminus q$ such that $s \uparrow \cap \mathcal{C}_U = \emptyset$. But now $p = q \cup s$ is an extension of $q$ that belongs to $\mathbb{D}(U, F)$.

Since $\mathbb{P}$ is countable and $|\mathcal{B}| < m_c$, there exists a filter $G \subseteq \mathbb{P}$ intersecting all the $\mathbb{D}(U, F)$ and we can let $S = \bigcup G$. Clearly $S$ is a selector.

To see that $S$ is $(\mathcal{I}, \tau)$-crowded and $S = \bigcup_{i \in \omega} F_i$, fix $U \in \mathcal{B}$ which intersects $\bigcup_{i \in \omega} F_i$ and a finite $F \subseteq X$. Then there is a $p \in G$ such that $(p \cap U \setminus F) \uparrow \cap \mathcal{C}_U = \emptyset$ and therefore $S \cap U \setminus F \notin \mathcal{C}_U$. Since this is true for any finite $F$ and $\mathcal{C}_U$ belongs to $C(\mathcal{I})$, it follows that $S \cap U \notin \mathcal{I}$.

Using $\mathcal{I} = [X]^{<\omega}$ again, we obtain:

**Corollary 3.11.** Any space is $q^+$ at each point of character smaller than $m_c$.

**Proof.** Let $Y$ be a space, $y \in Y$, $\mathcal{B}$ a local base at $y$ with $|\mathcal{B}| < m_c$ and suppose that $y \in \bigcup_{n \in \omega} F_n \setminus \bigcup_{n \in \omega} F_n$ where each $F_n$ is a finite subset of $Y$. Use the previous lemma with the space $X = \{ y \} \cup \bigcup_{n \in \omega} F_n$ with the topology generated by $\{ U \cap Y : U \in \mathcal{B} \} \cup \{ \bigcup_{n \in \omega} F_n \}$ and the ideal $[X]^{<\omega}$. 

\[ \square \]
4 Maximal $\mathcal{I}$-crowded topologies

Depending on the ideal $\mathcal{I}$, there might or might not exist a nice $\mathcal{I}$-crowded topology on $X$. If $\mathcal{I} = \{\emptyset\}$ then any topology on $X$ is $\mathcal{I}$-crowded, but if $\mathcal{I} = \mathcal{P}(X)$ then only the trivial topology is $\mathcal{I}$-crowded. For $\mathcal{I} = [X]^{<\omega}$, a topology on $X$ is $\mathcal{I}$-crowded if and only if it is crowded. If $\mathcal{I}$ is a prime ideal then $\mathcal{I}^+ \cup \{\emptyset\}$ is the finest $\mathcal{I}$-crowded topology on $X$ and it is not Hausdorff. Finally, for meager ideals we have the following:

**Theorem 4.1.** Let $\mathcal{I}$ be a meager ideal on $X$ and let $(Y, \rho)$ be a crowded space with $\pi w(\rho) < m_c$. Then there exists an $\mathcal{I}$-crowded topology $\tau$ on $X$ such that $(X, \tau)$ is homeomorphic to $(Y, \rho)$.

**Proof.** By the Jalali-Naini–Talagrand Theorem (see, for instance, [16, p. 33]), we can fix a partition $X = \bigcup_{n \in \omega} F_n$ into finite subsets such that no element of $\mathcal{I}$ contains infinitely many of the $F_n$’s. Let’s also fix a $\pi$-base $\mathcal{B}$ for $\rho$ with $|\mathcal{B}| < m_c$.

Let $\mathbb{P}$ be the set of all finite partial injective functions $p : X \to Y$ ordered by reverse inclusion. For each $U \in \mathcal{B}$ and $k \in \omega$ the set

$$D(U, k) = \{ p \in \mathbb{P} : F_n \subseteq p^{-1}(U) \text{ for some } n \geq k \}$$

is dense in $\mathbb{P}$. To see this fix $U \in \mathcal{B}$, $k \in \omega$ and $q \in \mathbb{P}$. Find $n \geq k$ such that $F_n \cap \text{dom} \, q = \emptyset$ and choose any $E \subseteq U \setminus \text{rng} \, q$ with $|E| = |F_n|$ and a bijection $r : F_n \to E$. Now $p = q \cup r$ is an extension of $q$ that belongs to $D(U, k)$.

Since $\mathbb{P}$ is countable and $|\mathcal{B}| < m_c$, there is a filter $G \subseteq \mathbb{P}$ intersecting all the $D(U, k)$ and also intersecting the dense sets $\{ p \in \mathbb{P} : x \in \text{dom} \, p \}$ for $x \in X$ and $\{ p \in \mathbb{P} : y \in \text{rng} \, p \}$ for $y \in Y$. Then $f = \bigcup G : X \to Y$ is a bijection and for each $U \in \mathcal{B}$ we have $f^{-1}(U) \in \mathcal{I}^+$. Therefore if we let $\tau = \{ f^{-1}(V) : V \in \rho \}$ we get an $\mathcal{I}$-crowded topology on $X$ and $f$ is a homeomorphism between $(X, \tau)$ and $(Y, \rho)$. 

**Question 4.2.** Can we remove the hypothesis on the $\pi$-weight of $\rho$?

We need one more lemma before we state and prove our main result.

**Lemma 4.3.** Let $X$ be a maximal space, $A \subseteq X$ and $x \in X$. If $x \in \overline{A} \setminus A$, then there is $B \subseteq A$ without isolated points such that $x \in \overline{B}$.

**Proof.** Let $D = \{ y \in A : y$ is an isolated point of $A \}$. Since $D$ is discrete and $X$ is maximal, then $D$ is closed. Let $B = A \setminus D$. We have $x \in \overline{A} = (A \setminus D) \cup \overline{D} = \overline{B} \cup D$. Thus $x \in \overline{B}$. It is easy to see that $B$ has no isolated points.
Now we are ready for the main result of this paper.

**Theorem 4.4** \((m_c = c)\). For any HM ideal \(\mathcal{I}\) on \(X\) there exists a maximal, zero-dimensional and \(\mathcal{I}\)-crowded topology \(\tau^\mathcal{I}\) on \(X\) such that:

(i) If \(\mathcal{I}\) is \(p^+\) then \(\tau^\mathcal{I}\) is selectively separable.

(ii) If \(\mathcal{I}\) is \(q^+\) then \(\tau^\mathcal{I}\) is \(q^+\).

**Proof.** By Theorem 4.1 there is an \(\mathcal{I}\)-crowded topology \(\tau_0\) on \(X\) such that \((X, \tau_0)\) is homeomorphic to the rational numbers with their usual topology.

Now we fix an enumeration \(\{\langle A_\alpha^i : i \in \omega \rangle \}_\alpha \subseteq c\) of all sequences of subsets of \(X\) and construct inductively an increasing sequence \(\langle \tau_\alpha : \alpha \leq c \rangle\) of topologies on \(X\) with the following properties:

1. Each \(\tau_\alpha\) is zero-dimensional, \(\mathcal{I}\)-crowded and \(w(\tau_\alpha) \leq |\alpha| + \aleph_0 < c\).

2. If \(A_0^\alpha = \emptyset\) and \(A_1^\alpha\) is \((\mathcal{I}, \tau_\alpha)\)-crowded then \(A_1^\alpha \in \tau_{\alpha+1}\).

3. If \(A_0^\alpha = \emptyset\) and \(A_1^\alpha\) is infinite but not \((\mathcal{I}, \tau_\alpha)\)-crowded then \(A_1^\alpha\) has \(\tau_{\alpha+1}\)-isolated points.

4. If \(A_i^\alpha\) is \(\tau_\alpha\)-dense for all \(i \in \omega\) and there is a sequence \(\langle K_i \rangle \subseteq \prod_i [A_i^\alpha]^{<\omega}\) such that \(\bigcup_{i \in \omega} K_i\) is \((\mathcal{I}, \tau)\)-crowded and \(\tau_\alpha\)-dense, then \(X \setminus \bigcup_{i \in \omega} K_i\) is \(\tau_{\alpha+1}\)-discrete for some such \(\langle K_i \rangle_i\).

5. If \(\langle A_i^\alpha \rangle_i\) is a sequence of pairwise disjoint finite sets and there is an \(S \subseteq \bigcup_i A_i^\alpha\) which is \((\mathcal{I}, \tau_\alpha)\)-crowded, \(\tau_\alpha\)-dense in \(\bigcup_i A_i^\alpha\) and \(|S \cap A_i^\alpha| \leq 1\) for all \(i \in \omega\), then \(cl_{\tau_\alpha}(S) \in \tau_{\alpha+1}\) and \(cl_{\tau_\alpha}(S) \setminus S\) is \(\tau_{\alpha+1}\)-discrete for some such \(S\).

Observe that at a limit ordinal \(\lambda \leq c\) we can just let \(\tau_\lambda\) be the topology generated by \(\bigcup_{\alpha \in \lambda} \tau_\alpha\). Only condition 1 needs to be checked but this is easy. What we do at a successor ordinal \(\alpha + 1\) depends on which of the hypothesis on conditions 2, 3, 4 and 5 is satisfied by \(\langle A_i^\alpha : i \in \omega \rangle\) (note that these conditions are mutually exclusive and if neither is satisfied we just let \(\tau_{\alpha+1} = \tau_\alpha\)):

- If \(A_0^\alpha = \emptyset\) and \(A_1^\alpha\) is \((\mathcal{I}, \tau_\alpha)\)-crowded we let \(\tau_{\alpha+1}\) be the topology given by Lemma 3.6 for \(\tau_\alpha\) and \(A_1^\alpha\).
• If $A^0_\omega = \emptyset$ and $A^0_1$ is infinite but not $(\mathcal{I}, \tau_\alpha)$-crowded then there exists $U \in \tau_\alpha$ such that $A^0_1 \cap U \in \mathcal{I}\setminus\{\emptyset\}$. Fix $x \in A^0_1 \cap U$ and let $A = X \setminus (A^0_1 \cap U) \cup \{x\}$. Note that for any $V \in \tau_\alpha$ we have $A \cap V \supseteq V \setminus (A^0_1 \cap U) \in \mathcal{I}^+$ (since $\tau_\alpha$ is $\mathcal{I}$-crowded) and hence $A$ is $(\mathcal{I}, \tau_\alpha)$-crowded. Thus we can let $\tau_{\alpha+1}$ be the topology given by Corollary 3.9 for $\tau_\alpha$ and $A$. Now since $A$ and $U$ are both $\tau_{\alpha+1}$-open, we get that $x$ is $\tau_{\alpha+1}$-isolated in $A^0_1$.

• If $A^0_\omega$ is $\tau_\alpha$-dense for all $i \in \omega$ and there is a sequence $\langle K_i \rangle_i \subseteq \prod_i [A^0_\omega]^{\omega}$ such that $\bigcup_{i \in \omega} K_i$ is $(\mathcal{I}, \tau)$-crowded and $\tau_\alpha$-dense, we can let $\tau_{\alpha+1}$ be the topology given by Lemma 3.7 for $\tau_\alpha$ and $\bigcup_{i \in \omega} K_i$. Since $\bigcup_{i \in \omega} K_i$ is $\tau_\alpha$-dense, we get that $X \setminus \bigcup_{i \in \omega} K_i$ is $\tau_{\alpha+1}$-discrete.

• If $\langle A^0_i \rangle_i$ is a sequence of pairwise disjoint finite sets and there is an $S \subseteq \bigcup_i A^0_i$ which is $(\mathcal{I}, \tau_\alpha)$-crowded, $\tau_\alpha$-dense in $\bigcup_i A^0_i$ and $|S \cap A^0_i| \leq 1$ for all $i \in \omega$, then we can let $\tau_{\alpha+1}$ be the topology given by Lemma 3.7 for $\tau_\alpha$ and $S$. Then $\text{cl}_{\tau_\alpha}(S) \subseteq \tau_{\alpha+1}$ and $\text{cl}_{\tau_\alpha}(S) \setminus \tau_{\alpha+1}$ is $\tau_{\alpha+1}$-discrete.

Now we show that $\tau^\mathcal{I} = \tau_\epsilon$ has all the properties that we want. It is immediate from the construction that $\tau^\mathcal{I}$ is zero-dimensional and $\mathcal{I}$-crowded.

If $A \subseteq X$ has no $\tau^\mathcal{I}$-isolated points, find $\alpha \in \mathcal{C}$ such that $A^\alpha_\omega = \emptyset$ and $A^\alpha_1 = A$. Since $\tau_{\alpha+1} \subseteq \tau^\mathcal{I}$, $A$ has no $\tau_{\alpha+1}$-isolated points and by condition 3 we conclude that $A$ is $(\mathcal{I}, \tau_\alpha)$-crowded and therefore, by condition 2, $A \in \tau^\mathcal{I}$. This shows that $\tau^\mathcal{I}$ is maximal (note that $\tau^\mathcal{I}$ is crowded by the first observation on Remark 3.2).

To prove (i) suppose that $\mathcal{I}$ is $p^+$ and fix a decreasing sequence $\langle D_i : i \in \omega \rangle$ of $\tau^\mathcal{I}$-dense subsets of $X$. Since dense subsets of a maximal space are necessarily open we have that each $D_i$ is $\tau^\mathcal{I}$-open and therefore $(\mathcal{I}, \tau^\mathcal{I})$-crowded. Thus if we take $\alpha \in \mathcal{C}$ such that $\langle A^\alpha_i : i \in \omega \rangle = \langle D_i : i \in \omega \rangle$, we have that each $A^\alpha_i$ is $(\mathcal{I}, \tau_\alpha)$-crowded and $\tau_\alpha$-dense. Thus Lemma 3.8 tells us that there exist finite sets $K_i \subseteq A^\alpha_i$ for $i \in \omega$ such that $\bigcup_{i \in \omega} K_i$ is $\tau_\alpha$-dense and $(\mathcal{I}, \tau_\alpha)$-crowded. But now condition 4 guarantees that $X \setminus \bigcup_{i \in \omega} K_i$ is $\tau^\mathcal{I}$-discrete for some such $\langle K_i \rangle_i$ and hence $\bigcup_{i \in \omega} K_i$ is dense, showing that $\tau^\mathcal{I}$ is selectively separable.

Finally we prove (ii). Suppose that $\mathcal{I}$ is $q^+$, let $\langle K_i : i \in \omega \rangle$ be a sequence of pairwise disjoint finite subsets of $X$ and suppose that $x \in \overline{\bigcup_{i \in \omega} K_i \setminus \bigcup_{i \in \omega} K_i}$ (here closures are taken with respect to $\tau^\mathcal{I}$). Since $\tau^\mathcal{I}$ is maximal it follows from Lemma 4.3 that there are $F_i \subseteq K_i$ for $i \in \omega$ such that $x \in \overline{\bigcup_{i \in \omega} F_i}$ and $\bigcup_{i \in \omega} F_i$ is $\tau^\mathcal{I}$-open and hence $(\mathcal{I}, \tau^\mathcal{I})$-crowded. Find $\alpha \in \mathcal{C}$ such that $\langle A^\alpha_i : i \in \omega \rangle = \langle F_i : i \in \omega \rangle$. Now we have that $\bigcup_{i \in \omega} A^\alpha_i$ is $(\mathcal{I}, \tau_\alpha)$-crowded so
Lemma 3.10 tells us that there is an \((\mathcal{I}, \tau_\alpha)\)-crowded \(S \subseteq \bigcup_{i \in \omega} A_i^\alpha\) such that \(|S \cap A_i^\alpha| \leq 1\) for each \(i \in \omega\) and \(S\) is \(\tau_\alpha\)-dense in \(\bigcup_{i \in \omega} A_i^\alpha\). But now condition 5 guarantees that \(cl_{\tau_\alpha}(S) \in \tau_{\alpha+1}\) and \(cl_{\tau_\alpha}(S) \setminus S\) is \(\tau_{\alpha+1}\)-discrete for some such \(S\). Since \(\tau_{\alpha+1} \subseteq \tau_I\), it follows that \(x \in cl_{\tau_\alpha}(\bigcup_{i \in \omega} A_i^\alpha) = cl_{\tau_\alpha}(S) = \overline{S}\).

Next we want to apply this result to various specific ideals in order to get maximal topologies satisfying all boolean combinations of the properties \(q^+\) and selective separability. But first we need the following:

**Lemma 4.5.** If \((X, \tau)\) is a maximal \(\mathcal{I}\)-crowded space and \(A \in \mathcal{I}\) then \(A\) is closed with empty interior (i.e. \(X \setminus A\) is dense open).

**Proof.** Since \(\tau \cap \mathcal{I} = \emptyset\) we have that \(A\) has empty interior. Thus \(X \setminus A\) is dense and therefore open since \(\tau\) is maximal.

**Theorem 4.6** (\(m_c = c\)). There exists a maximal zero-dimensional space which is selectively separable and \(q^+\).

**Proof.** Let \(\mathcal{I} = [X]^{<\omega}\). It is well known and easy to see that \(\mathcal{I}\) is \(p^+, q^+\) and HM so Theorem 4.4 gives us a maximal zero-dimensional topology \(\tau^\mathcal{I}\) on \(X\) which is selectively separable and \(q^+\).

**Theorem 4.7** (\(m_c = c\)). There exists a maximal zero-dimensional space which is selectively separable but not \(q^+\).

**Proof.** Fix a partition of \(X\) into finite sets \(X = \bigcup_{n \in \omega} F_n\) such that \(|F_n| = n\). Let \(\mathcal{I} = \bigcup_{m \in \omega} \bigcap_{n \in \omega} \{A \subseteq X : |A \cap F_n| \leq m\}\). Note that \(\mathcal{I}\) is an \(F_\sigma\) ideal and therefore it is HM and \(p^+\) so Theorem 4.4 gives us a maximal zero-dimensional topology \(\tau^\mathcal{I}\) on \(X\) which is selectively separable. Now we show that \(\tau^\mathcal{I}\) is not \(q^+\) at any point. Let \(x \in X\) and note that \(x \in \bigcup_{n \in \omega} G_n \setminus \bigcup_{n \in \omega} G_n\) where \(G_n = F_n \setminus \{x\}\). However if \(S \subseteq \bigcup_{n \in \omega} G_n\) intersects each \(G_n\) in at most one point, we see that \(S \in \mathcal{I}\) and by Lemma 4.3 \(S\) is closed. Thus \(x \notin \overline{S}\), showing that \(\tau^\mathcal{I}\) is not \(q^+\) at \(x\).

**Theorem 4.8** (\(m_c = c\)). There exists a maximal zero-dimensional space which is \(q^+\) but not selectively separable.

**Proof.** Fix a partition of \(X\) into infinite sets \(X = \bigcup_{n \in \omega} X_n\) and let \(\mathcal{I}\) be the collection of all subsets of \(X\) whose intersection with all but finitely many \(X_n\)’s is finite. It is easy to see that \(\mathcal{I}\) is \(q^+\) and it is also HM being a Borel ideal. Theorem 4.4 gives us a maximal zero-dimensional topology \(\tau^\mathcal{I}\) on \(X\).
which is $q^+$. Now we show that $\tau^I$ is not selectively separable. For each $i \in \omega$ let $D_i = \bigcup_{n \geq i} X_n$. Note that each $X \setminus D_i \in \mathcal{I}$ so by Lemma 4.5 it is dense. However if $\bigcup_{i \in \omega} F_i \in \mathcal{I}$ and using again Lemma 4.5 we get that $\bigcup_{i \in \omega} F_i$ is closed and therefore not dense.

Suppose $m_c = c$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the ideals on $X$ defined as in the proofs of theorems 4.7 and 4.8 respectively. Now Theorem 4.4 gives us maximal zero-dimensional topologies $\tau^{\mathcal{I}_i}$ and $\tau^{\mathcal{I}_2}$ on $X$ such that $\tau^{\mathcal{I}_1}$ is not $q^+$ and $\tau^{\mathcal{I}_2}$ is not selectively separable. Taking the disjoint union of these two spaces we obtain a space which is neither SS nor $q^+$. We show next that this can be proved without assuming that $m_c = c$.

**Theorem 4.9.** There exists a maximal zero-dimensional space which is neither $q^+$ nor selectively separable.

**Proof.** Barman and Dow [1] have shown that there is a maximal non SS space $Z$. We use the same idea in their proof to show that there is a maximal non $q^+$ regular space. Let $(X, \tau)$ be the space given in Example 2.2. Recall that $X = \bigcup_k A_k$ where each $A_k$ is finite and every selector is closed in $X$. It was observed in [1, Lemma 2.19] that van Douwen [20] implicitly showed that there is a regular topology $\tau'$ on $X$ finer than $\tau$ and a dense subspace $Y$ of $(X, \tau')$ which is maximal. Then $Y = \bigcup_k A_k \cap Y$ and this decomposition shows that $Y$ is not $q^+$. Finally, the disjoint union of $Z$ and $Y$ is the required maximal space. \[\square\]

It is consistent with ZFC that no countable maximal space is SS [1, 15]. The same happens with the $q^+$ property as we show next.

**Theorem 4.10.** It is consistent that there are no maximal $q^+$ spaces.

**Proof.** Let $X$ be a countable maximal space and $x \in X$. Then $U_x = \{ A \subseteq X : x \in A \setminus \{x\} \}$ is an ultrafilter (see [20]). If $X$ is a $q^+$ space, then $U_x$ is a Q-point. Since it is consistent that there are no Q-points (see [10]), we are done. \[\square\]

**Acknowledgments:** The third author thanks La Vicerrectoría de Investigación y Extensión de la Universidad Industrial de Santander for the financial support for this work, which is part of the VIE project #2422.
References

[1] D. Barman and A. Dow. Selective separability and $SS^+$. Topology Proc., 37:181–204, 2011.

[2] D. Barman and A. Dow. Proper forcing axiom and selective separability. Top. and its Appl., 159(3):806 – 813, 2012.

[3] A. Bella, M. Bonanzinga, and M. Matveev. Variations of selective separability. Top. and its Appl., 156(7):1241 – 1252, 2009.

[4] A. Bella, M. Bonanzinga, M. Matveev, and V. Tkachuk. Selective separability: general facts and behavior in countable spaces. Topology Proceedings, 32:15–30, 2008.

[5] A. Bella. When is a Pixley-Roy hyperspace $SS^+$? Top. and its Appl., 160(1):99 – 104, 2013.

[6] J. Camargo and C. Uzcátegui. Selective separability on spaces with an analytic topology. Top. and its Appl., 248(1):176–191, 2018.

[7] Jonathan Cancino-Manríquez, Michael Hrušák, and David Meza-Alcántara. Countable irresolvable spaces and cardinal invariants. Topology Proc., 44:189–196, 2014.

[8] A. Dow, M. G. Tkachenko, V. V. Tkachuk, and R. G. Wilson. Topologies generated by discrete subspaces. Glas. Math. Ser. III, 37(57):187–210, 2002.

[9] G. Gruenhage and M. Sakai. Selective separability and its variations. Top. and its Appl., 158(12):1352 – 1359, 2011.

[10] A. Miller. There are no Q-points in Laver’s model for the Borel conjecture. Proc. Amer. Math. Soc., 78(1):103–106, 1980.

[11] J. Murgas and C. Uzcátegui. Combinatorial properties on nodec countable spaces with analytic topology. doi.org/10.1016/j.topol.2020.107066.

[12] E. Murtinova. On products of discretely generated spaces. Top. and its Appl., 153(18):3402–3408, 2006.
[13] M. Scheepers. Combinatorics of open covers VI: Selectors for sequences of dense sets. *Quaestiones Mathematicae*, 22(1):109–130, 1999.

[14] S. Solecki. Analytic ideals and their applications. *Ann. Pure Appl. Logic*, 99:51–72, 1999.

[15] D. Repovš and L. Zdomskyy. On $M$-separability of countable spaces and function spaces. *Top. and its Appl.*, 157(16):2538–2541, 2010.

[16] S. Todorčević. Topics in Topology. *Lect. Notes Monogr.*, vol. 1652, Springer, 1997.

[17] S. Todorčević and C. Uzcátegui. Analytic topologies over countable sets. *Top. and its Appl.*, 111(3):299–326, 2001.

[18] S. Todorčević and C. Uzcátegui. Analytic $k$-spaces. *Top. and its Appl.*, 146-147:511–526, 2005.

[19] S. Todorčević and C. Uzcátegui. A nodec regular analytic space. *Top. and its Appl.*, 166:85–91, 2014.

[20] E. Van Douwen. Applications of maximal topologies. *Top. and its Appl.*, 51:125-139, 1993.