Some Additional Solutions of Conformal Turbulence

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Abstract

We made a careful study of Polyakov’s Diofantian equations for 2D turbulence and found several additional CFTs which meet his criterion. This fact implies that we need further conditions for CFT in order to determine the exponent of the energy spectrum function.
In these few years, the conformal field theory (CFT) began to find its new applications to various areas in the theoretical physics. One of such example is the two dimensional quantum gravity, or the massive solvable system near the critical point. Recently, Polyakov[1] made a proposal—its new application to the two dimensional turbulence. He argued that the exact exponent of the energy spectrum function of the two dimensional fluid can be computed by the knowledge of the minimal models of CFT.

The two dimensional turbulence is notorious for its subtleties compared with three dimensional analogues. In 3D case, a scenario that there occur cascades from the large vortices to the smaller ones works very well and as the consequence Kolgomonov’s celebrated -5/3 law can explain the basic feature of the system. In two dimensions, there is a similar proposal, the enstrophy cascade.[2] It predicts that the energy spectrum function behaves as $k^{-3}$. In the numerical simulation, however, the turbulent fluid is not so stable and the prediction to the exponent is drifting between $-3$ and $-4$.[2]

Polyakov combines the idea of the constant enstrophy flow and CFT. He derived a few conditions on a primary field that describes the velocity field of the two dimensional fluid. He found a candidate that satisfies his criterion and his prediction of the exponent is $-25/7 = -3.571428571$, which is beautifully located between $-3$ and $-4$. However, even in his own paper, he admitted that his solution might not be the unique one. In this report, we make a more careful study of his proposal and we find that this is indeed the case. Although the Polyakov’s solution is the simplest one, there seems to be at least more than 20 admissible models within the range $p, q < 250$. 

We first give a brief review of Polyakov’s argument. The Navier-Stokes equation in two dimensions is,

$$\dot{\omega} + \epsilon_{\alpha\beta}\partial_{\alpha}\psi\partial_{\beta}\psi = \nu\partial^2\psi,$$

where $\psi$ is the stream function, $\omega = \partial^2\psi$ the vortex function, and $u_\alpha = \epsilon_{\alpha\beta}\partial\psi$ are the velocity fields. The energy spectrum function ($E(k)$) and the enstrophy ($\Omega$) are,

$$E = \frac{1}{2} \int d^2x u^2(x) \equiv \int dk E(k),$$

$$\Omega = \frac{1}{2} \int d^2x \omega^2(x) = \int dk k^2 E(k).$$

(1)
Direct computations from the Navier-Stokes equation lead to the energy and enstrophy diffusion due to the viscosity,

\[ \frac{dE}{dt} = -2\nu\Omega \quad \frac{d\Omega}{dt} = -2\nu P. \quad (2) \]

\( P \) is so-called palinstrophy. The nonlinear term in the Navier-Stokes equation does not contribute to these equations. It is physically natural since the kinetic term should not cause any diffusions.

The characteristic difference between three and two dimensional turbulence is following. In the limit \( \nu \rightarrow 0 \), there are infinite numbers of conservation laws in the two dimensional case. Any powers of the vortex function are conserved by the equation of motion. In three dimensional case, there is constant energy flow from vortices of large scale to those of smaller scale. On the other hand, in two dimensional cases, the energy flow itself vanishes as the viscosity approaches zero. However, there does occur constant enstrophy flow, \( \frac{d\Omega}{dt} \rightarrow \text{constant} \), due Eq.(2). A simple dimensional counting gives that \( E(k) \sim k^{-3} \) as the function of wave number \( k \).

To describe the turbulent behavior of fluid, it is a good idea to interpret the Navier-Stokes equation from the viewpoint of statistical mechanics. Since a power law appears in the expression of the energy spectrum function, there emerges the conformal invariance. Since we are discussing the two dimensional cases, we can apply powerful tools of the conformal field theory. Polyakov identified the stream function as a primary field of the minimal model of the conformal field theory. Once we identified those fields as quantum operators, one needs to define the nonlinear term more carefully due to the short distance effects of those operators. Polyakov shows that

\[ \epsilon_{\alpha\beta} \partial_\alpha \psi \partial_\beta \partial^2 \psi \sim a^{\Delta - 2\Delta \psi} (L_0^2 - L_{-1}^2) \phi, \quad (3) \]

where \( \phi \) is the primary field of the smallest dimension which appears in the OPE of \( \psi \times \psi \). \( a \) is the point splitting parameter. Due to the appearance of singularity in the short distance, there is a shift of dimension of the composite operator that appears in the Navier-Stokes equation. Polyakov’s criterion is

\[ ^1 \text{Unlike the statistical systems where CFT is normally applied to, the critical behavior of the turbulence is achieved by supply of energy (or enstrophy) from outside. In this sense, this is the first example where CFT is applied to the non-equilibrium statistically} \]
that there should be constant enstrophy flow even in this quantum situation. Since the enstrophy flow is described by,

\[ \frac{d}{dt} < \frac{1}{2} \omega(x)^2 > = < (L_{-2}L_{-1} - L_{-2}L_{-1}) \phi \cdot L_{-1}L_{-1} \psi > = \text{constant}, \]

one gets a constraint on the dimension of the primary field,

\[ \Delta_\psi + \Delta_\phi = -3. \quad (4) \]

The another constraint that Polyakov introduces is,

\[ \Delta_\psi < -1. \quad (5) \]

In other words, there should not appear any singularity in the OPE \( \psi \times \psi \).

Since we have a table of CFT minimal models, what is left for us to do is just to find out solutions of Eq.\( (4) \), \( (5) \). The minimal models of the Virasoro algebra are characterized by their central charge and the conformal dimensions,

\[ c_{pq} = 1 - \frac{6(p - q)^2}{pq}, \quad \Delta_{rs} = \frac{(ps - qr)^2 - (p - q)^2}{4pq}, \quad (6) \]

where \( p, q \) are co-prime positive integers and \( 0 < r < p, 0 < s < q \). We pick up one of the primary field that has the dimension given in Eq.\( (6) \) and search for the primary field with the smallest dimension that appears in the OPE of \( \psi \times \psi \). The OPE rule of CFT is well-known by now.

\[ \psi_{rs} \times \psi_{r's'} = \sum_{(tu) \in \Delta} \psi_{tu}. \quad (7) \]

with \( \Delta \) defined by

\[ \begin{align*}
max(-r + r' + 1, r - r' + 1) & \leq t \leq \min(r + r' - 1, 2p - 1 - r - r') \\
max(-s + s' + 1, s - s' + 1) & \leq u \leq \min(s + s' - 1, 2p - 1 - s - s') \quad \text{or} \\
max(-r - r' + 1 - p, r + r' + 1 - p) & \leq t \leq \min(r - r' - 1 + p, -r + r' - 1 + p) \\
max(-s - s' + 1 - p, s + s' + 1 - p) & \leq u \leq \min(s - s' - 1 + p, -s + s' - 1 + p). \\
\end{align*} \]

\( t \) and \( u \) run over even (resp. odd) integers if it is bounded by even (resp. odd) integers.
In his letter, Polyakov gave only one solution that satisfies his criterion, i.e. \((p, q) = (21, 2)\) and \((r, s) = (4, 1)\). In this case, the conformal dimension of \(\psi\) equals \(-\frac{8}{7}\). The energy spectrum function behaves as,

\[
E(k) \sim k^{4\Delta_\psi+1} = k^{-25/7}.
\]  

One of the most important issues is whether this solution is the unique one. We understand that the actual numerical simulations neither pin down the unique value for the exponent nor prove the uniqueness of its universality class.

For these purposes, we study Polyakov’s criterion carefully and find many other minimal models that meet his criterion. Although we could not find general solutions, we surveyed CFT up to \(p, q \leq 250\) and found 22 solutions. In the table below, we show the \((p,q)\) of the minimal model in Eq.(6), OPE (see Eq.7), exponent and the central charge. Since we need to have at least one primary field with negative dimension, all solutions are described by non-unitary representations. However, there are at least four models whose exponents are located between -3 and -4, i.e., \((21,2)\), \((55,6)\), \((217,23)\), \((234,25)\). Since the exponents for these models are quite close with each other, it will be very hard to tell which is the solution by using the numerical simulation. Clearly, although Polyakov’s solution is the simplest one, we need to find more constraints on CFT if we want to determine the exponent uniquely. The other option is that there might be several universality classes and our solutions correspond to each of them.

Since we do not have a strong reason we restrict ourselves to the minimal models of the Virasoro algebra, we can also apply Polyakov’s criterion to other known representation of (extended) Virasoro algebra. One of the most interesting example is that of \(A_1^{(1)}\) Kac-Moody Algebra,[5] that is characterized by,

\[
\begin{align*}
c &= \frac{3k}{k+2} \\
h_{rs} &= \frac{j_{rs}(j_{rs} + 1)}{k+2}, \\
\dot{j}_{rs} &= \frac{r-st-1}{2} \quad (1 \leq r < p, 1 \leq s < q), \quad t \equiv k+2 = p/q.
\end{align*}
\]  

The OPE rule for this model can be found in [8]. In this case, however, we can not find any solution that meets the criteria within the range \(p, q < 200\). It may be interesting to do similar survey for the other models, especially the W-algebra[4].
To conclude this paper, we would like to point out some of the future issues that should be clarified.

- What is the role of primary fields besides $\psi$ and $\phi$ in the 2D turbulence?
- How can we measure the central charge of the Virasoro algebra?
- How can we find the general solution for Polyakov’s criterion? Is there any systematic methods analogous to the construction of W-algebra?

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Table 1: Solutions for Polyakov’s Diofantian equation

| (p,q)  | OPE          | exponent       | cpq       |
|--------|--------------|----------------|-----------|
| (21,2) | (4,1)→(7,1)  | -3.571429      | -50.571429|
| (25,3) | (11,1)→(9,1) | -4.600000      | -37.72    |
| (26,3) | (5,1)→(9,1)  | -4.230769      | -39.692308|
| (55,6) | (14,1)→(9,1) | -3.727273      | -42.654545|
| (62,7) | (13,1)→(9,1) | -4.032258      | -40.820276|
| (67,8) | (28,3)→(25,3)| -4.507463      | -37.966418|
| (71,9) | (32,4)→(55,7)| -4.990610      | -35.093897|
| (87,11)| (16,2)→(23,3)| -5.031348      | -35.213166|
| (91,11)| (14,2)→(25,3)| -4.610390      | -37.361638|
| (93,11)| (20,2)→(25,3)| -4.442815      | -38.43695 |
| (111,14)| (8,1)→(7,1)  | -5.054054      | -35.328185|
| (115,14)| (6,1)→(9,1)  | -4.739130      | -37.016149|
| (135,16)| (56,7)→(59,7)| -4.444444      | -38.336111|
| (166,21)| (31,4)→(55,7)| -4.982788      | -35.187608|
| (179,22)| (10,1)→(9,1) | -4.832402      | -36.55612 |
| (197,25)| (87,11)→(71,9)| -4.993909    | -35.041121|
| (205,26)| (39,5)→(71,9)| -4.988743      | -35.068668|
| (213,26)| (76,9)→(41,5)| -4.685807      | -36.886241|
| (217,23)| (62,6)→(85,9)| -3.460028      | -44.2464  |
| (223,26)| (12,1)→(9,1) | -4.327354      | -39.16109 |
| (229,27)| (88,10)→(161,19)| -4.403202| -38.596312|
| (234,25)| (79,9)→(103,11)| -3.533333| -43.801026|
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