A Note on the Global Attractivity of a Discrete Model of Nicholson’s Blowflies*

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In this paper, we further study the global attractivity of the positive equilibrium of the discrete Nicholson’s blowflies model

\[ N_{n+1} - N_n = -\delta N_n + pN_{n-k}e^{-aN_{n-\tau}} \quad n = 0, 1, 2, \ldots \]

We obtain a new criterion for the positive equilibrium \( N^* \) to be a global attractor, which improve the corresponding results obtained by So and Yu (J. Math. Anal. Appl. 193 (1995), 233–244).

Keywords: Attractivity, Positive equilibrium, Discrete Nicholson’s blowflies model

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I. INTRODUCTION

The delay difference equation

\[ N_{n+1} - N_n = -\delta N_n + pN_{n-k}e^{-aN_{n-\tau}} \quad n = 0, 1, 2, \ldots \]  

is a discrete analogue of the delay differential equation

\[ N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)} \quad t \geq 0, \]
which has been used in describing the dynamics of Nicholson’s blowflies [2,4–6].

By the biology consideration, we assume that \( \delta \in (0, 1), \ p, a \in (0, +\infty), \) and \( k \in \mathbb{N} = \{0, 1, 2, \ldots \}. \)

The initial condition is

\[ N_j = \varphi_j \geq 0, \quad j \in \{-k, -k+1, \ldots, 0\}, \quad (2) \]

and \( \varphi_j > 0, \) for some \( j \in \{-k, -k+1, \ldots, 0\}. \)

By a solution of (1) and (2) we mean a sequence \( \{N_n\} \) which satisfies (1) for \( n = 0, 1, 2, \ldots \) as well as the initial condition (2). Clearly, the unique solution \( \{N_n\} \) of the above initial value problem is positive for all large \( n \) [1].

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If $p > p$, then Eq. (1) has a unique positive equilibrium $N^p$ and

$$N^p = \frac{1}{a} \ln \left( \frac{p}{\delta} \right).$$

(3)

The global attractivity of $N^p$ was studied by Kocic and Lada [3] and So and Yu [1] respectively. The recent result is the following [1].

**THEOREM A** Assume that $p > p$ and that

$$[(1 - p)^{-k - 1} - 1] \ln \left( \frac{p}{\delta} \right) \leq 1.$$  

(4)

Then any nontrivial solution $N_n$ of (1) and (2) satisfies

$$\lim_{n \to \infty} N_n = N^p.$$  

In this note, our purpose is to improve condition (4). Exactly speaking, we will show some conditions for the global attractivity of $N^p$ when (4) does not hold. Our results are discrete analogues of the results in [2].

To prove our main results, we need some known results.

**LEMMA 1** [1] Let $\{N_n\}$ be a solution of (1) and (2). Then

$$\limsup_{n \to \infty} N_n \leq \frac{p}{ae^6}.$$  

(5)

As in [2], the following system of inequalities

$$(y + \ln(1 + (y/aN^p))) \leq M(e^{-x} - 1),$$

$$x + \ln(1 + (x/aN^p)) \geq M(e^{-y} - 1),$$

(6)

play an important role in our analysis, where $M = aN^p[(1 - p)^{-k - 1} - 1] = [(1 - p)^{-k - 1} - 1] \ln(p/\delta).$

Let

$$D = \{(x, y) : -aN^p < x \leq 0 \leq y < \infty\}.$$  

(7)

**LEMMA 2** [2] If one of the following conditions holds:

(i) $M \leq 1$;
(ii) $M < 1 + (1/aN^p)$ and $aN^p \geq (\sqrt{5} - 1)/2$;
(iii) $M \leq 1 + (1/aN^p)$ and

$$aN^p > (\sqrt{1 + 4\sqrt{3} - 1}/2),$$

then (6) has a unique solution $x = y = 0$ in $D$.

**II. MAIN RESULTS**

The following theorem provides a new sufficient condition for the equilibrium $N^p = (1/a)\ln(p/\delta)$ to be a global attractor.

**THEOREM 1** Assume that $p > p$ and the assumption in Lemma 2 holds. Then any nontrivial solution $\{N_n\}$ of (1) and (2) satisfies

$$\lim_{n \to \infty} N_n = N^p.$$  

**Proof** Let

$$N_n = N^p + \frac{1}{a} x_n.$$  

Then $\{x_n\}$ is a solution of the equation

$$x_{n+1} - x_n + \delta x_n + aN^p(1 - e^{-x_n})$$

$$- \delta x_n e^{-x_n} = 0, \quad n = 0, 1, 2, \ldots.$$  

(8)

Since $N_n > 0$ for all large $n$, it follows that $x_n > -aN^p$ for all large $n$.

To prove this theorem, it is sufficient to prove

$$\lim_{n \to \infty} x_n = 0.$$  

Lemma 1 implies that $\{x_n\}$ is bounded above. Let

$$\mu = \limsup_{n \to \infty} x_n \quad \text{and} \quad \lambda = \liminf_{n \to \infty} x_n.$$  

(9)

Then $-aN^p \leq \lambda \leq \mu < \infty$. We claim that $\lambda = \mu = 0$.

For the case $\{x_n\}$ is eventually nonnegative or eventually nonpositive, this has been proved in the proof of Theorem 2 in [3]. Therefore it is sufficient to consider the case that $\{x_n\}$ is an oscillatory solution of (8).

Our purpose is to prove that $\lambda = \mu = 0$ under the assumptions. There are four possible cases:

1. $\lambda = \mu = 0$;
2. $\mu > 0$ and $\lambda = 0$;
(3) \( \mu = 0 \) and \( \lambda < 0 \);
(4) \( \mu > 0 \) and \( \lambda < 0 \).

The cases 2 and 3 can be considered to be special cases of case 4. Now we consider case 4.

In this case, there exists a sequence \( \{n_i\} \) of positive integers such that

\[
k < n_1 < n_2 < \cdots < n_i < n_{i+1} \to \infty \text{ as } i \to \infty.
\]

and \( x_{n_i} < 0 \) and \( x_{n_{i+1}} \geq 0 \), for \( i = 1, 2, \ldots \),

and for each \( i = 1, 2, \ldots \), the terms of the finite sequence \( x_j \) for \( n_i < j < n_{i+1} \) assume both positive and negative values. Let \( m_i \) and \( M_i \) be integers in \((n_i, n_{i+1})\) such that for \( i = 1, 2, \ldots \)

\[
x_{M_i} = \max \{ x_j : n_i < j < n_{i+1} \},
\]

and

\[
x_{m_i} = \min \{ x_j : n_i < j < n_{i+1} \}.
\]

We can assume without loss of generality that for \( i = 1, 2, \ldots \)

\[
x_{M_i} > 0, \quad x_{M_i} - x_{M_{i-1}} \geq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{M_i} = \mu > 0,
\]

while

\[
x_{m_i} < 0, \quad x_{m_i} - x_{m_{i-1}} \leq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{m_i} = \lambda < 0.
\]

Then there exist subsequence \( \{q_i\} \) of \( \{m_i\} \) and subsequence \( \{Q_i\} \) of \( \{M_i\} \) such that

\[
x_{Q_i} > 0, \quad x_{Q_i} - x_{Q_{i-1}} \geq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{Q_i} = \mu > 0,
\]

while

\[
x_{q_i} < 0, \quad x_{q_i} - x_{q_{i-1}} \leq 0 \quad \text{and} \quad \lim_{i \to \infty} x_{q_i} = \lambda < 0.
\]

It follows from (8) and (10) that

\[
x_{Q_{i+1}} + aN^* \leq x_{Q_{i-k+1}} + aN^* e^{-x_{Q_{i-k+1}}},
\]

thus

\[
x_{Q_i} + aN^* = (1 - \delta)(x_{Q_{i-k+1}} + aN^*) + \delta(x_{Q_{i-k+1}} + aN^*) e^{-x_{Q_{i-k+1}}}
\]

\[
\leq (1 - \delta)(x_{Q_{i-k+1}} + aN^*) e^{-x_{Q_{i-k+1}}}
\]

\[
+ \delta(x_{Q_{i-k+1}} + aN^*) e^{-x_{Q_{i-k+1}}}
\]

\[
= x_{Q_{i-k+1}} + aN^* e^{-x_{Q_{i-k+1}}}
\]

that is

\[
x_{Q_i} + aN^* \leq (x_{Q_{i-k+1}} + aN^*) e^{-x_{Q_{i-k+1}}}.
\]

Now let us prove

\[
x_{Q_{i-k+1}} < 0,
\]

assume the contrary, then \( x_{Q_{i-k+1}} = 0 \) or \( x_{Q_{i-k+1}} > 0 \). If \( x_{Q_{i-k+1}} = 0 \), then \( x_{Q_i} \leq 0 \), which contradicts (10). If \( x_{Q_{i-k+1}} > 0 \), then \( x_{Q_{i-k+1}} > x_{Q_i} \),

\[
\lim_{i \to \infty} x_{Q_{i-k+1}} \geq \lim_{i \to \infty} x_{Q_i} = \mu,
\]

on the other hand, we have

\[
\lim_{i \to \infty} x_{Q_{i-k+1}} \leq \lim_{i \to \infty} x_{M_i} = \mu,
\]

so we get

\[
\lim_{i \to \infty} x_{Q_{i-k+1}} = \mu,
\]

then taking the limit in (12), we obtain

\[
\mu + aN^* \leq (\mu + aN^*) e^{-\mu},
\]

which implies \( \mu \leq 0 \) that contradicts (10), so (13) holds.

From (12) and (13), we have

\[
x_{Q_i} + aN^* < aN^* e^{-x_{Q_{i-k+1}}}.
\]
therefore
\[ x_{Q_i-k-1} < -\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right). \] (15)

For given \( \varepsilon > 0 \), by (9), there exists a positive integer \( n^* \) such that
\[ \lambda - \varepsilon < x_n < \mu + \varepsilon, \quad \text{for} \ n \geq n^* - k, \]
this induce \( x_{n-k}e^{-x_{n-k}} < \mu + \varepsilon \), for \( n \geq n^* \).

Rewriting Eq. (8) into the following form:
\[
(1 - \delta)^{-n-1}x_{n+1} - (1 - \delta)^{-n}x_n
+ a\delta N^*(1 - \delta)^{-n-1}(1 - e^{-x_{n-k}})
- \delta(1 - \delta)^{-n-1}x_{n-k}e^{-x_{n-k}} = 0. \] (16)

Now summing (16) up from \( n = Q_i - k - 1 \) (assuming \( Q_i - k - 1 \geq n^* \)) to \( n = Q_i - 1 \). we have
\[
(1 - \delta)^{-Q_i}x_{Q_i} = (1 - \delta)^{-Q_i+k+1}x_{Q_i-k-1} - a\delta N^*
\times \sum_{n=Q_i-k-1}^{Q_i-k-1} (1 - \delta)^{-n-1}(1 - e^{-x_{n-k}})
+ \delta \sum_{n=Q_i-k-1}^{Q_i-k-1} (1 - \delta)^{-n-1}x_{n-k}e^{-x_{n-k}}
< (1 - \delta)^{-Q_i+k+1}x_{Q_i-k-1} + a\delta N^*
\times \sum_{n=Q_i-k-1}^{Q_i-k-1} (1 - \delta)^{-n-1}(e^{-\lambda + \varepsilon} - 1)
+ \delta \sum_{n=Q_i-k-1}^{Q_i-k-1} (1 - \delta)^{-n-1}(\mu + \varepsilon)
= (1 - \delta)^{-Q_i+k+1}x_{Q_i-k-1}
+ [(\mu + \varepsilon) + aN^*(e^{-\lambda + \varepsilon} - 1)]
\times (1 - \delta)^{-Q_i}[1 - (1 - \delta)^k]. \]

Substituting (15) into the above inequality, we get
\[
(1 - \delta)^{-Q_i}x_{Q_i} < -(1 - \delta)^{-Q_i+k+1} \ln\left(1 + \frac{x_{Q_i}}{aN^*}\right)
+ [(\mu + \varepsilon) + aN^*(e^{-\lambda + \varepsilon} - 1)]
\times (1 - \delta)^{-Q_i}[1 - (1 - \delta)^k]. \]

and
\[
x_{Q_i} + (1 - \delta)^{k+1} \ln\left(1 + \frac{x_{Q_i}}{aN^*}\right)
< [\mu + \varepsilon] + aN^*(e^{-\lambda + \varepsilon} - 1)[1 - (1 - \delta)^k]. \]

let \( i \to \infty, \varepsilon \to 0 \), we get
\[
\mu + (1 - \delta)^{k+1} \ln\left(1 + \frac{\mu}{aN^*}\right)
\leq [\mu + aN^*(e^{-\lambda} - 1)][1 - (1 - \delta)^k]. \]

We rewrite the above inequality:
\[
\mu + \ln\left(1 + \frac{\mu}{aN^*}\right) \leq M(e^{-\lambda} - 1). \] (17)

In a similar way, we have
\[
\lambda + \ln\left(1 + \frac{\lambda}{aN^*}\right) \geq M(e^{-\mu} - 1). \] (18)

Then we establish the following system of inequalities:
\[
\begin{cases}
\mu + \ln\left(1 + (\mu/aN^*)\right) \leq M(e^{-\lambda} - 1), \\
\lambda + \ln\left(1 + (\lambda/aN^*)\right) \geq M(e^{-\mu} - 1).
\end{cases} \] (19)

For case 2, the system of inequalities corresponding to (19) is
\[
\begin{cases}
\mu + \ln\left(1 + (\mu/aN^*)\right) \leq M(e^{-\lambda} - 1), \\
\lambda = 0.
\end{cases} \] (20)

It is obvious that (20) holds iff \( \lambda = \mu = 0 \).

For case 3, the system of inequalities corresponding to (19) is
\[
\begin{cases}
\mu = 0, \\
\lambda + \ln\left(1 + (\lambda/aN^*)\right) \geq M(e^{-\mu} - 1).
\end{cases} \] (21)

Similarly, (21) holds iff \( \lambda = \mu = 0 \).

Thus it will suffice to consider case 4, for (19) in
case 4, by Lemma 2, we get \( \lambda = \mu = 0 \). So the proof is complete.
Remark 1 In cases 2 and 3 in Theorem 1, we add some reasonable conditions to \( aN^* \). We know

\[
M = aN^*[(1 - \delta)^{-k-1} - 1] \leq 1 + \frac{1}{aN^*},
\]

on the right side of which there is nothing to do with \( \delta \) and \( k \). While \( 1 + (1/aN^*) \to \infty \) as \( aN^* \to 0^+ \), properly choosing the values of \( [(1 - \delta)^{-k-1} - 1] \), we can let \( M \) equal or infinitely tend to the value of \( 1 + (1/aN^*) \), then \( M \) can be changed to arbitrarily large. Obviously this is not reasonable.

Remark 2 Theorem 4.1 in [1] only applies to the case \( M \leq 1 \), while Theorem 1 in this paper not only applies to \( M \leq 1 \) but also to \( M > 1 \). So the results in this paper improve those in [1].

Example Consider the delay difference equation

\[
N_{n+1} - N_n = -\frac{1}{4}N_n + \frac{1}{4}e^{(\sqrt{5} - 1)/2}N_{n-3}e^{-2N_{n-3}},
\]

then we can calculate

\[
aN^* = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad [(1 - \delta)^{-k-1} - 1] = \frac{175}{81},
\]

thus,

\[
M \approx 1.335 \quad \text{and} \quad 1 + \frac{1}{aN^*} = \frac{\sqrt{5} + 3}{2} \approx 2.618.
\]

The conditions in Theorem 1 are satisfied. Thus

\[
N^* = \frac{\sqrt{5} - 1}{4}
\]

is a global attractor or (22). But Theorem 4.1 in [1] cannot apply to this case.

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