A Group-Theoretic Approach to the WSSUS Pulse Design Problem

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Abstract—We consider the pulse design problem in multicarrier transmission where the pulse shapes are adapted to the second order statistics of the WSSUS channel. Even though the problem has been addressed by many authors analytical insights are rather limited. First we show that the problem is equivalent to the pure state channel fidelity in quantum information theory. Next we present a new approach where the original optimization functional is related to an eigenvalue problem for a pseudo differential operator by utilizing unitary representations of the Weyl–Heisenberg group. A local approximation of the operator–algebraic approach is extended to provide exact solutions for different classes of scattering environments.

I. INTRODUCTION

Pulse shaping in multicarrier transmission is a key ingredient for high rate wireless links. Furthermore it is the standard tool to mitigate the interference caused by doubly dispersive channels. Most multicarrier schemes like conventional OFDM exploiting guard regions (a cyclic prefix), pulse shaped OFDM and OFDM/OQAM can be jointly formulated. Hence we focus on a transmit baseband signal $s(t)$ given as

$$s(t) = \sum_{(mn) \in I} x_{mn} e^{j2\pi mn F t} \gamma(t - n T) = \sum_{(mn) \in I} x_{mn} \gamma_{mn}(t)$$

where $i$ is the imaginary unit and $\gamma_{mn} \equiv S_{(nT, mF)} \gamma$ are time-frequency shifted versions of the transmit pulse $\gamma$, i.e. shifted according to the lattice $TZ \times FZ$. It is also beneficial to consider different lattice structures [1] on which our contribution will apply as well. The time-frequency (or phase space) shift operator $S_{(\tau, \nu)}$ is intimately connected to unitary representations of the Weyl–Heisenberg group as we will elaborate later on. Therefore (1) is also known as Weyl–Heisenberg or Gabor signaling.

The coefficients $x_{mn}$ in (1) are the complex data symbols at time instant $n$ and subcarrier index $m$ with the property $E\{xx^*\} = I$ ($^*$ means conjugate transpose) where $x = (x_1, x_2, \ldots)^T$. The indices $(mn)$ range over some doubly-countable index set $I$, referring to the data burst to be transmitted. We will denote the linear time-variant channel by $H$ and the additive white Gaussian noise process (AWGN) by $n(t)$. The received signal is then

$$r(t) = (Hs)(t) + n(t) = \int \int \Sigma(\tau, \nu)(S_{(\tau, \nu)} \gamma)(t)d\tau d\nu + n(t)$$

with $\Sigma(\tau, \nu)$ being a realization of the "channel spreading function". In practice $\Sigma(\tau, \nu)$ is causal and has finite support. We used here the notion of the WSSUS channel. In the WSSUS assumption the channel is characterized by the second order statistics of $\Sigma(\tau, \nu)$, i.e.

$$E\{\Sigma(\tau, \nu)\Sigma(\tau', \nu')\} = C(\tau, \nu) \delta(\tau - \tau') \delta(\nu - \nu')$$

where $C(\tau, \nu)$ is the scattering function. Without loss of generality we assume $||C||_1 = 1$. To obtain the data symbol $x_{kl}$ the receiver does the projection on $g_{kl} \equiv S_{(IT, kF)} g$, i.e.

$$\hat{x}_{kl} = \langle g_{kl}, r \rangle = \int g_{kl}(t) r(t) dt$$

By introducing the elements $H_{kl, mn} \equiv \langle g_{kl}, H_{\gamma_{mn}} \rangle$ of the channel matrix $H \in C^{T \times I}$, the multicarrier transmission can be formulated as the linear equation $\tilde{x} = Hx + \tilde{n}$, where $\tilde{n}$ is the vector of the projected noise having a power of $\sigma^2$ per component. We assume that the receiver has perfect channel knowledge (given by $\Sigma(\tau, \nu)$), i.e. single carrier based equalization in the absence of noise would be $\tilde{x}_{kl}^\text{eq} = \hat{x}_{kl}/H_{kl, kl}$, with

$$H_{kl, kl} = \langle g_{kl}, H_{\gamma_{kl}} \rangle = \int \int \Sigma(\tau, \nu) g_{kl} S_{(\tau, \nu)} \gamma_{kl} d\tau d\nu$$

$$= \int \int \Sigma(\tau, \nu) e^{-i2\pi(\tau 2F - \nu T)} A_{g\gamma}(\tau, \nu) d\tau d\nu$$

where $A_{g\gamma}(\tau, \nu) = \langle g, S_{(\tau, \nu)} \gamma \rangle$ is the cross ambiguity function of the pulse pair $(\gamma, g)$.

II. PROBLEM STATEMENT

Considering only single carrier equalization, it is natural to require $a \equiv |H_{kl, kl}|^2$ (the channel gain) to be maximal and the interference power $b \equiv \sum_{(kl) \neq (mn)} |H_{kl, mn}|^2$ to be minimal as possible. This addresses the concept of pulse shaping. However to be practicable, the pulses should be adapted to the second order statistics only, given by $C(\tau, \nu)$ and not to a particular channel realization $\Sigma(\tau, \nu)$. Hence, we aim at maximization of

$$\text{SINR} = \frac{E_{\gamma}(a)}{\sigma^2 + E_{\gamma}(b)}$$

by proper design of $\gamma$ and $g$. Up to very few special cases the analytical solution of this global optimization problem (jointly non-convex in $(\gamma, g)$) is unknown. However numerical optimization methods are presented in [2], [3], [4]. Following our previous work [4] we simplify the problem by proposing a
relaxation, which separates the problem into two steps. Upper bounding \( E_{\mathcal{R}}\{b\} \leq B_\gamma - E_{\mathcal{R}}\{a\} \) gives a lower bound on SINR (see [4]), where \( B_\gamma \) is the so called Bessel bound of \( \{\gamma_{mn}\} \) [5]. In this paper we focus on the first step only where \( E_{\mathcal{R}}\{a\} \) should be maximized. This gives the following optimization problem

\[
\{\gamma^{(\text{opt})}, g^{(\text{opt})}\} = \arg \max_{\gamma} E_{\mathcal{R}}\{a\} = \arg \max_{\|\gamma\|=|g|=1} \int |A_{g}\gamma(\tau, \nu)|^2 d\mu
\]  

(3)

where \( d\mu \) is the volume form on the manifold of \( \gamma \) and \( g \). In this context it was first introduced in [6] respectively [7], but similar problems already occurred in radar literature much earlier. In particular for the elliptical symmetry of \( E \) bounding relaxation, which separates the problem into two steps. Upper bounding \( E_{\mathcal{R}}\{b\} \leq B_\gamma - E_{\mathcal{R}}\{a\} \) gives a lower bound on SINR (see [4]), where \( B_\gamma \) is the so called Bessel bound of \( \{\gamma_{mn}\} \) [5]. In this paper we focus on the first step only where \( E_{\mathcal{R}}\{a\} \) should be maximized. This gives the following optimization problem

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where \( d\mu \) is the volume form on the manifold of \( \gamma \) and \( g \). In this context it was first introduced in [6] respectively [7], but similar problems already occurred in radar literature much earlier. In particular for the elliptical symmetry of \( C(\tau, \nu) \) Hermite functions establish local extremal points as found in [7]. The scaling rule for fixed profiles (corresponding to classical bosonic quantum channels) was studied in [7], [8]. Also it is possible to find a close framework to provide exact solutions for the class of Gaussian offsets (or phase space displacement). Then we extent our scattering profiles. Because the underlying theory is partially represented theory of the Weyl–Heisenberg group. Our approach relates the optimal pulses to approximate eigenstates of pseudo differential operators. The procedure naturally embeds the concepts of pulse scaling and optimal time-frequency offsets (or phase space displacement). Then we extend our framework to provide exact solutions for the class of Gaussian scattering profiles. Because the underlying theory is partially not very common in multichannel community we will give a short introduction to the few properties we will need for our investigation. More details can be found in [12].

A. The Weyl-Heisenberg Group and Pseudo differential Operators

The two families of shift operators \( S_{(\tau,0)} \) and \( S_{(0,\nu)} \) are unitary representations of the group corresponding to the real line \( \mathbb{R} \) with addition as group operation. The extension to \( \mathbb{R}^2 \) in the sense of

\[
S_{(\alpha,\beta)} \cdot S_{(\gamma,\delta)} = e^{-i2\pi\alpha\delta} S_{(\alpha+\gamma,\beta+\delta)}
\]  

(4)

is not closed because of the phase factor. Closeness is achieved by introducing the torus \( \mathbb{T} \) as the third variable, i.e.

\[
e^{i2\pi\phi} S_{(\alpha,\beta)} , e^{i2\pi\psi} S_{(\gamma,\delta)} = e^{i2\pi(\phi+\psi-\alpha\delta)} S_{(\alpha+\gamma,\beta+\delta)}
\]

The corresponding group \( \mathbb{H} = \mathbb{R} \times \mathbb{R} \times \mathbb{T} \) with the group law \((\alpha,\beta,\phi)(\gamma,\delta,\psi) = (\alpha + \gamma, \beta + \delta, \phi + \psi - \alpha\delta)\) is called the (reduced) polarized Heisenberg group (HG). The HG can be represented as a group of upper triangular matrices by the group homomorphism

\[
(\alpha, \beta, \phi) \rightarrow H(\alpha, \beta, \phi) = \begin{pmatrix} 1 & \alpha & \phi \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}
\]

where the group action is matrix multiplication. The matrices \( h(\alpha, \beta, \phi) = H(\alpha, \beta, \phi) - 1 \) written with \( \alpha \) as \((1,0,0), x = (0,1,0) \) and \( e = (0,0,1) \) as \( h(\alpha, \beta, \phi) = \alpha h(d) + \beta h(x) + \phi h(e) \) are clearly isomorphic to \( \mathbb{R}^3 \) and with the matrix commutator they turn into a Lie algebra. The Lie bracket in this case is \([([\alpha, \beta, \phi], \gamma, \delta, \psi)] = (0,0,\alpha\delta - \beta\gamma)\). Due to the bilinearity of the Lie bracket this can be short written as the Heisenberg Commutation Relations, i.e. \([d, x] = e \quad [x, e] = 0 \quad [d, e] = 0\). That is this exactly is the Heisenberg algebra connected to the HG follows from \( h(\alpha, \beta, \phi)^2 = h(0,0,\alpha\beta) \) and \( h(\alpha, \beta, \phi)^n = 0 \) for \( n > 2 \). The exponential map of the matrix \( h(\alpha, \beta, \phi) \) is then
given as

\[
e^{h(\alpha,\beta,\phi)} = \sum_{n=0}^{\infty} \frac{h(\alpha,\beta,\phi)^n}{n!} = 1 + h(\alpha, \beta, \phi) + \frac{1}{2}h(0,0,\alpha\beta)
\]  

(5)

Thus, it maps the Heisenberg algebra to the unpolarized HG. The series expansion is finite (the elements \( h(\alpha, \beta, \phi) \) are nilpotent endomorphisms). Returning to the polarized Heisenberg group we transform finally \( H(\alpha, \beta, \phi) = H(0,0, -\frac{1}{2}\alpha\beta)\).

To establish the connection to \( S(\alpha,\beta) \) considered as operators on \( S(\mathbb{R}) \) (the Schwartz space of rapidly decreasing functions) we have to switch to the so called Schrödinger representation. In this picture the hermitian operators \( X \) and \( D \) with

\[
(X f)(t) \overset{\text{def}}{=} tf(t)
\]

\[
(D f)(t) \overset{\text{def}}{=} \frac{1}{2\pi i} f'(t)
\]

setup a basis representation for the Heisenberg Lie algebra. The skew-hermitian operators \( 2\pi i X \) (generates the frequency shifts), \( 2\pi i D \) (generates the time shifts) and \( 2\pi i E = 2\pi i (E \) is the identity) correspond to \( x, d \) and \( e \). They give again the Heisenberg commutation rules, hence linear combinations of them fulfill the same Lie bracket (the commutator of linear operators) and consequently \((\tau, \nu, s) \rightarrow d\rho(\tau, \nu, s) = 2\pi i (s + \nu X + \tau D)\) is again a Lie algebra isomorphism for the Heisenberg algebra. As in (5) the

1The addition in third component is taken to be mod 1. Otherwise this yields the (full) polarized Heisenberg group with non-compact center.
HG is then given by exponentiation, i.e. the so called Weyl transform is given as
\[ \rho(\tau,\nu,s) = e^{id\rho(\tau,\nu,s)} = e^{2\pi i s (s+\nu \tau + \nu D)} = e^{2\pi i s \nu \tau} \mathcal{S}_\tau(\tau,\nu) \]

With \( \rho(\tau,\nu,0) = \mathbf{S}_{(\tau,\nu)} = e^{\tau i \nu} \rho(-\tau,\nu) = e^{\pi i \nu \tau} e^{2\pi i \nu X - \tau D} \), i.e. integrals over shift operators as in (2) are in fact pseudo differential operators [12] of the following spreading representation (the Weyl transform)
\[ \hat{\rho}(\tau,\nu) = \int dM \hat{\rho}(\tau,\nu) \mathcal{S}(\tau,\nu) d\tau d\nu \]

\( \hat{\rho}(\tau,\nu) \) is the called spreading function (or representing function, i.e. the 2D Fourier transform of the symbol function of the operator \( \sigma(D,X) \)).

**B. The WSSUS Pulse Design Problem**

Straight forward calculation shows now that the squared magnitude of the cross ambiguity function \( |A_{g,G}(\tau,\nu)|^2 \) can be written in the following form
\[
|A_{g,G}(\tau,\nu)|^2 = \langle g, S_{(\tau,\nu)} \rangle \langle \gamma, S_{(\tau,\nu)}^* \gamma \rangle = \text{Tr} G S_{(\tau,\nu)} G S_{(\tau,\nu)}^* \]

where \( G \) (\( \Gamma \)) is the (rank-one) orthogonal projector onto \( g \) (\( \gamma \)). By that transformation we emphasize that \( \Gamma \) undergoes a linear transformations before being projected onto \( g \). This special kind of linear transformation is also called a unitary evolution, which preserve the spectrum of \( \Gamma \) (in our case the rank). This obviously does not hold in generality if taking the sum over different unitary evolution of the same argument. Hence, we collect them together by defining affine maps \( A \) and \( \tilde{A} \) such that
\[
E_T(a) = \text{Tr} G A_{\Gamma} = \text{Tr} G \Gamma A_{\Gamma} = \text{Tr} \Gamma A_{\Gamma} = \text{Tr} \Gamma A_{\tilde{G}}(G)
\]

The main reason for this reformulation is the notion of completely positive maps (CP-maps) [13] which directly apply on the pulse design problem. CP-maps like \( A(\cdot) \) received much attention due to its application in quantum information theory. Before going more in detail, let us define \( T_1 \) as the set of trace class operators. The set
\[ M_1 = \{ z \in T_1, z = z^*, z \geq 0, T \tau = 1 \} \]

is a convex subset of \( T_1 \). With \( Z \) we will denote the extremeal boundary of \( M_1 \), which is the set of all orthogonal rank-one projectors. With the definition of \( \tilde{A} \) in (7) follows that \( \tilde{A} \) is adjoint of \( A \) with respect to the inner product \( \text{Tr} X \cdot Y \). Due to \( || C || = 1 \) both maps are trace preserving \( \text{Tr} A(X) = \text{Tr} X \). Moreover they are hermiticity preserving \( A(X^*) = A(X^*) \) and entropy increasing \( X \succ A(X) \) (\( \succ \) is the partial order due to eigenvalue majorization). The complete positivity and the trace-preserving property is ensured by
\[
\int d\mu S_{(\tau,\nu)}^* S_{(\tau,\nu)} = 1
\]

With this framework we can write now the optimization problem as
\[
\max_{G,\Gamma \in \mathbb{Z}} \text{Tr} G A(\Gamma)
\]

where \( \Gamma \) represent the transmitter and the CP-map \( A(\cdot) \) represent the "averaged" action of the channel and \( G \) is the receiver. This formulation is similar to the channel fidelity in quantum information processing. In fact - the problems are equivalent if considering so called pure states. The initial preparation of a pure quantum state (the symbol to transmit) is represented by a so called rank-one density operator (in our case \( G \)). The quantum channel is represented by a CP-map \( A(\cdot) \) having again a density operator as its output. The measurement (the detection of the transmitted symbol) is performed in our case with \( G \). Obviously either \( G \) or \( \Gamma \) can be dropped in the optimization, i.e.
\[
\max_{G,\Gamma \in \mathbb{Z}} \text{Tr} G A(\Gamma) = \max_{G \in \mathbb{Z}} || A(\Gamma) || = \max_{G \in \mathbb{Z}} || \hat{A}(G) ||\]

The physical meaning is that (8) is invariant with respect to common time–frequency shifts of \( G \) and \( \Gamma \). A trivial but important conclusion is that Weyl-Heisenberg (Gabor) signaling is a reasonable scheme, which guarantees the same performance on all lattice points. Alternatively it can be viewed in the quantum picture as symbol alphabet of pure states achieving all the same fidelity. From (10) follows furthermore that different maps \( A_1 \) and \( A_2 \) commute, i.e. \( A_1 \circ A_2 = A_2 \circ A_1 \). Coming back to the formulation of pulse design problem in (9) we can finally relax the constraint set from \( Z \) to \( M_1 \) which gives
\[
\max_{\Gamma \in M_1} || A(\Gamma) || = \max_{G \in \mathbb{Z}} || \hat{A}(G) ||
\]

provided by the convexity of \( || \cdot \cdot || \) and linearity of \( A(\cdot) \). To the authors knowledge this reformulation of the pulse design criterion as a convex maximization problem seems to be new. Without further investigations of the analytical structure of \( A(\cdot) \) such global-type optimization problems are in general difficult to solve. Therefore we will emphasize in the following more on the Heisenberg group structure contained in \( A(\cdot) \).

**C. The Schrödinger Representation**

The connection between Weyl operators (the unitary representations of the Weyl-Heisenberg group in the Schrödinger picture) and \( S_{(\tau,\nu)} \) will reveal the fundamental role of Gaussians in WSSUS signaling. We will show this first in a simpler lower bound analysis which mainly admits the same maximizer as the original problem (given in the appendix). Thus, coming back now to (3) and let \( (\tau_0,\nu_0) \) be an arbitrary
offset between \(g\) and \(\gamma\) in the time-frequency plane, hence we define \(\tilde{\gamma} = \mathcal{S}(\tau_0, \nu_0)\).

\[
\mathbf{E}_\nu \{ A \} = \int [A_{g,\gamma}(\tau, \nu)]^2 d\mu = \int [(g, \mathcal{S}(\tau_0, \nu_0)\tilde{\gamma})]^2 d\mu
\]

In the latter we used Jensen’s inequality\(^2\) (if \(\mu = \|C\|_1 = 1\), see also [4]). We will now use (12) for further analytical studies. The bound becomes sharp iff \(\xi \mathcal{A}(\tau, \nu) \in \mathbb{R}\) is constant on \(\text{supp} C\) for some \(\xi \in \mathbb{T}\), hence is well suited for underspread channels. The operator \(\mathcal{L}\) is a pseudo differential operator with spreading function \(\mathcal{S}(\tau, \nu) - \mathcal{C}(\tau_0 - \nu + \nu_0)\).

**Local approximation:** The nilpotent property with respect to the moment product celebrated in (5) unfortunately does not translate into the Schrödinger picture, so that

\[
\mathcal{S}(\tau, \nu) = e^{i \pi \tau \nu} \rho(\tau, \nu)
\]

\[
\approx e^{i \pi \tau \nu}[1 + \frac{1}{2} d\rho(\tau, \nu)^2] + o(2) \quad (13)
\]

with the hermitian operator \(K \equiv \nu X - \tau D\) holds only as an approximation (for \(\tau\) and \(\nu\) being small), i.e. gives a local approximation of \(\mathcal{L}\) which is

\[
\mathcal{L} = \int \mathcal{L}(\tau - \nu + \nu_0) e^{2\pi i (\nu X + \tau D)} d\tau d\nu
\]

where \(\mathcal{L} = \int \mathcal{L}(\tau, \nu) (\tau_0 - \nu + \nu_0)\mathcal{C}\) are the moments of the scattering function around \((\tau_0, \nu_0)\). Because \(X\) and \(D\) are hermitian operators, \(\mathcal{L}\) is hermitian too if \(C_{mn} \equiv m^{+n} \in \mathbb{R}\) for \(m, n = 0, 1, 2\). In this case the optimization problem is an eigenvalue problem. Moreover then it follows that \(g = \alpha L \gamma\) for some \(\alpha \in \mathbb{C}\), because only in this case equality in \((g, \mathcal{L} \gamma) \leq \|g\|_2 \|\mathcal{L} \gamma\|_2\) is achieved. \(\mathcal{L}\) can be made hermitian if we choose \(\tau_0 = \|\mathcal{L} C\|_1\) and \(\nu_0 = \|\nu C\|_1\), so that \(C_{10} = C_{01} = 0\). Thus we have

\[
\mathcal{L} = \mathcal{L}(\tau_0 - \nu + \nu_0) e^{2\pi i (\nu X + \tau D)} d\tau d\nu
\]

which is an hermitian differential operator of second order. With

\[
(d_{\alpha} f)(t) \equiv \frac{1}{\sqrt{\alpha}} f(t/\alpha)
\]

we define now dilated functions \(g_{\alpha} \equiv d_{\alpha} g\) and \(g_{\alpha} \equiv d_{\alpha} \tilde{\gamma}\) and the dilated operator \(\mathcal{L}_{\alpha} = d_{\alpha} L d_{1/\alpha}\). Using furthermore that

\[
d_{\alpha} X d_{1/\alpha} = \frac{1}{\alpha} X
\]

\[
d_{\alpha} D d_{1/\alpha} = \alpha D
\]

we get

\[
\langle g_{\alpha}, \mathcal{L}_{\alpha} \tilde{\gamma}_{\alpha} \rangle = \langle g_{\alpha}, \mathcal{C}_{\alpha} - 2\pi^2 \left(\frac{C_{20}}{\alpha^2} X^2 + C_{20} X^2 D^2 - C_{11} (X D + D X) \right) \rangle
\]

\[
\text{and with } \alpha^4 = C_{20}/C_{20} \text{ the phase space symmetric version}
\]

\[
\mathcal{L}_{\alpha} = 2\pi^2 \sqrt{C_{20}/C_{20}} \left\{ [X^2 + D^2 - C_{11} (X D + D X)] \right\}
\]

\[
\text{where the constant is } \kappa = \frac{C_{20}}{2\pi^2 \sqrt{C_{20}/C_{20}}}
\]

\[
\text{(with our WSSUS assumptions follows also } C_{00} = 1). \text{ For simplicity let us assume that the shifted scattering function is separable yielding}
\]

\[
\mathcal{L}_{\alpha} = 0 \quad C_{00} = \pi \sqrt{C_{20}/C_{20}} (2n + 1) \quad h_n\]

Hence in local approximation the maximization problem is solved by \(h_0\), i.e. \(g = d_{1/\alpha} h_0\) and \(\gamma = \mathcal{S}_{\tau_0, \nu_0} d_{1/\alpha} h_0\) which are both scaled and proper separated Gaussians (the ground state of the harmonic oscillator). This is an important (and expected) result for the pulse design problem in WSSUS channels. It includes the concepts of pulse scaling (by \(d_{1/\alpha}\)) and proper phase space displacement (by \(\mathcal{S}_{\tau_0, \nu_0}\)) as natural operations. However this approximations is only valid for \(C_{20}/C_{20} \ll 1\) (underspread channel), such that \((C_{00} - \pi \sqrt{C_{20}/C_{20}} (2n + 1)) > 0\). We obtain the same solutions in the original problem if we apply this approximation (see the appendix). Next we will derive cases where this approximation turns out to be exact.

**Gaussian scattering functions:** Let us assume that after performing proper pulse scaling and separation the scattering function is given by the symmetric Gaussian \(\mathcal{C}(\tau, \nu) = \frac{1}{2\pi} e^{-2\alpha (\tau^2 + \nu^2)}\) where \(0 < \alpha \in \mathbb{R}\). If \(\alpha \gg 1\) the channel is underspread. It can be shown that then \(\mathcal{L}\) essentially self-adjoint, hence the maximum in (12) is again achieved by eigenfunctions of \(\mathcal{L}\). Operators having such scattering functions are contained in the so called oscillator semigroup [15] and for \(\alpha > 1\) they have the representation [12]

\[
\mathcal{L} = e^{-2\pi \alpha \text{arcoth} \alpha (X^2 + D^2)}
\]

Thus we have that \(\mathcal{L} \cdot h_n = e^{-2(2n+1)} (\text{arcoth} \alpha) h_n\), hence \(h_0\) is the optimum of (12). The special case \(\alpha = 1\) can be included by observing that then \(\mathcal{C}(\tau, \nu) \sim \mathcal{A}_{\hat{h}_0 h_0}(\tau, \nu)\). Such pseudo differential operators perform simple projections, in this case onto the span of \(h_0\). Note that for \(\mathcal{S}(\tau, \nu) = \langle \phi, \rho(\tau, \nu) \psi \rangle\) follows

\[
\langle g, \mathcal{S}(D, X) \gamma \rangle = \langle \mathcal{S}, \langle g, \rho(\cdot, \cdot) \rangle \gamma \rangle = \langle \mathcal{S}, \langle g, \rho(\cdot, \cdot) \rangle \rangle \gamma \rangle
\]

Thus \(\mathcal{S}(D, X)\) is a rank one projector (orthogonal in the case \(\psi = \phi\)) if \(\langle \phi, \psi \rangle = 1\).

Finally we conclude that for underspread channels the Gaussian pulse shape is an approximate solution of (12) which
becomes more optimal as the support of $C$ decreases. Furthermore the solution is exact for a Gaussian scattering function. In the quantum channel context the same arguments hold for coherent states (phase space translated Gaussians).

**Appendix**

We will sketch now that the results obtained from the lower bound analysis will hold with minor restrictions in the direct problem. Writing the CP-map $A(\cdot)$ using shift operators gives

$$A(\Gamma) = \int S_{(\tau,\nu)} \hat{S}_{(\tau,\nu)} \mu$$

where we introduced again an arbitrary offset $(\tau_0,\nu_0)$ as already done in (12) for the lower bound analysis, i.e. $\hat{\Gamma} \equiv S_{(\tau_0,\nu_0)} \Gamma S_{(\tau_0,\nu_0)}$. Using again (13) gives

$$A(\Gamma) \approx \int (1 + 2\pi i K - 2\pi^2 K^2) \Gamma (1 - 2\pi i K - 2\pi^2 K^2) d\mu$$

where the self-adjoint operator is now $\hat{K} \equiv (\nu - \nu_0) X - (\tau - \tau_0) D$. Hence we get the following approximation on the optimization functional

$$\text{Tr} \, GA(\Gamma) \approx \text{Tr} \left( \left(1 - 4\pi^2 K^2 \right) \Gamma G + 4\pi^2 \hat{K} T \hat{K} G + 2\pi i \hat{K} \Gamma \right) d\mu + o(2)$$

If we restrict furthermore $g$ and $\gamma$ to be real it can be shown that this will become

$$\text{Tr} \, GA(\Gamma) \approx \text{Tr} \left( \left(1 - 4\pi^2 K^2 \right) \Gamma G + 4\pi^2 \hat{K} T \hat{K} G + 2\pi i \hat{K} \Gamma \right) d\mu$$

Thus from the latter we can separate the following approximated version $A_1(\cdot)$ of the CP-map $A(\cdot)$

$$A_1(\Gamma) \equiv \int \left( (1 - 4\pi^2 K^2) \Gamma G + 4\pi^2 \hat{K} T \hat{K} G + 2\pi i \hat{K} \Gamma \right) d\mu$$

In the next steps we will perform the integration of the three integrands. Using

$$\int \hat{K}^2 \hat{K} d\mu = C_{20} \hat{D}^2 \hat{K} + C_{02} X^2 \hat{K} - C_{11} (DX \hat{K} + XD \hat{K})$$

$$\int \hat{K} \hat{K} d\mu = C_{20} \hat{D} \hat{K} + C_{02} X \hat{K} - C_{11} (D \hat{K} + X \hat{K}D)$$

$$\int [\hat{K}, \hat{K}] d\mu = C_{10} [D, \hat{K}] - C_{01} [X, \hat{K}]$$

we get finally

$$A_1(\Gamma) = C_{00} - 4\pi^2 \left( C_{20} D [D, \hat{K}] + C_{02} X [X, \hat{K}] \right) + 4\pi^2 \left( C_{11} D [X, \hat{K}] + C_{11} X [D, \hat{K}] \right)$$

$$+ 2\pi i \left( C_{10} [D, \hat{K}] - C_{01} [X, \hat{K}] \right)$$

We choose again $(\tau_0,\nu_0)$ such that $C_{10} = C_{01} = 0$. Furthermore let us assume again for simplicity that the scattering function is separable around $(\tau_0,\nu_0)$ yielding $C_{11} = 0$. Then we will get

$$A_1(\Gamma) = C_{00} - 4\pi^2 \left( C_{20} D [D, \hat{K}] + C_{02} X [X, \hat{K}] \right)$$

Next we apply the same dilation procedure used already for the lower bound analysis. Let $\hat{\Gamma}_\alpha = d_\alpha \hat{\Gamma} d_{1/\alpha}$. Using again (14) gives

$$A_1(\Gamma) = C_{00} - 4\pi^2 \sqrt{C_{20} C_{20}} \left( D^2 [D, \hat{\Gamma}] + X [X, \hat{\Gamma}] \right)$$

This will result in the same pulse scaling rule as from the lower bound analysis. Hence it remains to show that Gaussians are the right pulses to perform the scaling, thus we aim at maximization of $\|A_1(\Gamma)\|_\infty$ which is

$$\max_\Gamma \| \kappa / 2 - D^2 [D, \hat{\Gamma}] - X [X, \hat{\Gamma}] \|_\infty = \max_\Gamma \| \kappa / 2 - (D^2 + X^2) \hat{\Gamma} + \hat{\Gamma} D + X \hat{\Gamma} X \|_\infty$$

where $\kappa = \frac{C_{02}}{2\pi^2 C_{02} C_{20}}$. It can be shown that for time-frequency symmetric $\Gamma$ and $G$ follows $\text{Tr} \, G X \hat{\Gamma} X = \text{Tr} \, G D \hat{\Gamma} D = 0$. With this restriction remains

$$\max_\Gamma \| \kappa / 2 - (D^2 + X^2) \hat{\Gamma} \|_\infty$$

which is maximized by $\hat{\Gamma}$, being a projection onto the eigenspace of $D^2 + X^2$ corresponding to the minimal eigenvalue which is again $h_0$. For Gaussian scattering functions in turn the calculation in [10] suggest that this will hold also for $A(\cdot)$.

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