Controllability-Gramian Submatrices for a Network Consensus Model

Sandip Roy and Mengran Xue

Abstract—Principal submatrices of the controllability Gramian and their inverses are examined, for a network-consensus model with inputs at a subset of network nodes. Several properties of the Gramian submatrices and their inverses – including dominant eigenvalues and eigenvectors, diagonal entries, and sign patterns – are determined by exploiting the special doubly-nonnegative structure of the matrices. In addition, majorizations for these properties are obtained in terms of cutsets in the network’s graph. The asymptotic (long time horizon) structure of the controllability Gramian is also analyzed. The results on the Gramian are used to study metrics for target control of the network-consensus model.

I. INTRODUCTION

Dynamical models for consensus or synchronization in networks have been exhaustively studied [1]–[3]. One focus of this effort has been on open-loop control of the dynamics using inputs at a subset of the network’s nodes [4]–[9]. In particular, graph-theoretic necessary or sufficient conditions for controllability have been obtained, and some characterizations of the required control energy have also been obtained using analyses of the controllability Gramian. Recently, researchers have begun to study target control of network models, wherein inputs are designed to manipulate a group of target nodes rather than the whole network [10], [11], [13]–[15]. The target-control problem is of practical interest in several application domains, in which stakeholders need to use limited actuation capabilities to guide a few key nodes’ states. In parallel with the general controllability analysis for networks, the effort on target controllability has also yielded graph-theoretic conditions and analyses of metrics. These studies demonstrate that limited-energy target control of network processes is sometimes possible even when full-state control is prohibitively costly or impossible.

Target control for network models can be analyzed in terms of principal submatrices of the controllability Gramian [10], [11]. Precisely, target controllability resolves to invertibility of a principal submatrix of the Gramian, while the minimum energy required to achieve a desired target state and/or guide certain state projections can be found in terms of quadratic forms of the Gramian-submatrix inverses. Thus, the study of target controllability motivates analysis of the Gramian matrix and its principal submatrices for network consensus/synchronization models.

Because of the relevance of Gramian matrices to network controllability as well as dual observability/estimation problems, some structural and graph-theoretic results on the full Gramian have been developed for canonical network-consensus models, as well as for other dynamical network processes [9], [11], [16], [17], [20]. In addition, explicit formulae for the Gramian inverse in terms of the network model’s spectrum have been developed, using Cauchy matrix properties [9]. These explicit computations give insight into the relationship between the network model’s spectrum and the required control energy.

The purpose of this study is to develop new characterizations of the Gramian and its principal submatrices in the context of a canonical discrete-time network consensus model, with the goal of assessing target control metrics. Relative to the earlier studies on the Gramian of network models, the main contributions of this work are to: 1) characterize principal submatrices of the Gramian and their inverse, in addition to the full Gramian matrix; 2) give new insights into the sign patterns and eigenvalues of Gramian- and Gramian-submatrix inverses; 3) develop graph-theoretic majorizations on the Gramian’s entries; and 4) assess target control metrics and optimal inputs using the results on Gramian submatrices. The analyses primarily draw on the diffusive structure of the network model, which imposes a spatial pattern on the input response of the network. A main result is that Gramian submatrix inverses exhibit a special sign pattern as well as a dependence on cutsets of the network graph, which allows majorization of the control energy and analysis of minimum-energy inputs.

Although our focus here is on target control, the analyses of Gramian submatrices are germane to the control and estimation of various network dynamical processes with a diffusive or nonnegative structure. In particular, the analyses are relevant to the controllability analysis of other discrete-time models with nonnegative state matrices and continuous-time models with Metzler state matrices (e.g., models for infection spread, economic systems, etc) [18], [19]. The results also inform other problems in network estimation and control which require consideration of Gramians and Markov parameters, including observability analysis and model reduction [21], [22].

The article is organized as follows. The network consensus model is presented in Section II, and analysis of target control metrics in terms of the Gramian is reviewed in Section III. The main results on the Gramian, and their implications on target control, are developed in Section IV. Finally, an example is presented in Section V. Proofs are suppressed to save space.

II. MODEL

A network with \( n \) nodes, labeled \( 1, \ldots, n \), is considered. Each node \( i \) has a scalar state \( x_i[k] \) which evolves in discrete time \( (k \in \mathbb{Z}^+) \). A set \( S \) containing \( m \) nodes, which we call the source nodes, are amenable to external actuation. The
nodes’ states evolve according to:
\[ x[k + 1] = Ax[k] + Bu[k], \]  
(1)
where the state vector is \( x[k] = \begin{bmatrix} x_1[k] \\ \vdots \\ x_n[k] \end{bmatrix} \), \( A = [a_{ij}] \) is a row-stochastic matrix \( (a_{ij} \geq 0, \sum_{j=1}^{n} a_{ij} = 1 \) for each \( i \)), and \( u[k] = \begin{bmatrix} u_1[k] \\ \vdots \\ u_m[k] \end{bmatrix} \) specifies the input (actuation) signals at the \( m \) source nodes, and \( B \) is an \( n \times m \) matrix whose columns are \( 0 - 1 \) indicators of the source nodes in \( S \).

The manipulation of the states of a set \( T \) containing \( p \) target nodes is of interest. The target state vector \( y[k] \), defined as containing the states of the \( p \) target nodes at time \( k \), can be expressed as:
\[ y[k] = Cx[k], \]  
(2)
where \( C \) is a \( p \times m \) matrix whose rows are \( 0 - 1 \) indicators of the target nodes in \( T \). We refer to the model as a whole as the input-output network consensus model or simply the network model.

A weighted digraph \( \Gamma \) is defined to represent the topology of nodal interactions in the network model. Specifically, \( \Gamma \) is defined to have \( n \) nodes labeled \( 1, \ldots, n \), which correspond to the \( n \) vertices. A directed edge is drawn from vertex \( i \) to vertex \( j \) if and only if \( a_{ij} > 0 \), and the weight of the edge is set to \( a_{ij} \). We note that the graph may include self-loops (i.e., edges from vertices to themselves). The sum of the weights of the incoming edges to each vertex is 1. An edge in \( \Gamma \) from node \( i \) to node \( j \) indicates that node \( j \)’s state at time \( k + 1 \) is directly influenced by the node \( i \)’s state at time \( k \). The vertices corresponding to the source and target nodes are referred to as source and target vertices, respectively.

Throughout the article, we assume that the matrix \( A \) is irreducible and aperiodic, or equivalently that the graph \( \Gamma \) is ergodic. Under this assumption, the unactuated model reaches consensus, i.e., the manifold where all nodes’ states are identical is globally asymptotically stable.

### III. Preliminaries: Target Control and the Gramian

Target control of the network model is primarily concerned with two questions: 1) deciding whether the input \( u[k] \) can be designed to guide the target state vector \( y[k] \) to a desired goal (i.e., analyzing target controllability); and 2) determining how much actuation energy or effort is needed to do so (i.e., assessing target control metrics). Our primary focus here is on assessing target control metrics.

Formally, target controllability is defined as follows:

**Definition 1:** The input-output network-consensus model is said to be target controllable over \([0, k_f]\) if, for any goal \( \bar{y} \in \mathbb{R}^p \), an input signal \( u[0], \ldots, u[k_f-1] \) can be designed to drive the network model from a relaxed initial state \( x[0] = 0 \) to the target state \( y[k_f] = \bar{y} \).

In the case that the network model is target controllable over an interval \([0, k_f]\), the minimum input energy required to achieve each goal state can be assessed. This notion is formalized in the following definition:

**Definition 2:** The target-control energy for an interval \([0, k_f]\) and goal state \( \bar{y} \) is defined as \( E(k_f, \bar{y}) = \min_{u[0], \ldots, u[k_f-1]} \sum_{i=0}^{k_f-1} u^T[i] u[i], \) subject to the constraint that the input sequence drives the system from a relaxed state to \( y[k_f] = \bar{y} \). We refer to an argument (input sequence) that achieves the minimum as an **optimal target-control input**.

Additionally, the energy required to drive projections of the target state to a unit value may be important to characterize, as an indication of the the manipulability or security of key output statistics. This notion is formalized as follows:

**Definition 3:** The projection-manipulation energy for an interval \([0, k_f]\) and projection vector \( \alpha \) is defined as \( F(k_f, \alpha) = \min_{u[0], \ldots, u[k_f-1]} \sum_{i=0}^{k_f-1} u^T[i] u[i], \) subject to the constraint that the input sequence drives the system from a relaxed state to \( \alpha^T y[k_f] = 1 \). We refer to an input sequence that achieves the minimum as an **optimal projection-manipulation input**.

Often, it is of interest to characterize extremal values of the target-control energy across goal states with a particular norm, or dually of the projection-manipulation energy across projection vectors of a certain norm. In particular, the minimum values of the two metrics across goal states and projection vectors, respectively, are indications of the security of the network model to manipulation. These global security notions are formalized in the following definitions:

**Definition 4:** The minimum of the target-control energy over unit-two-norm goal states, i.e., \( E_{\min}(k_f) = \min_{y \text{ s.t. } ||y||_2=1} E(k_f, \bar{y}) \), is referred to as the target security of the network model. A goal state \( \bar{y} \) that achieves the minimum is denoted as \( \bar{y}_{\min} \), and is termed a **minimally-secure goal**.

**Definition 5:** The minimum of the projection-manipulation energy over projection vectors with unit one-norm, i.e. \( F_{\min}(k_f) = \min_{\alpha \text{ s.t. } ||\alpha||_1=1} F(k_f, \alpha) \), is referred to as the **projection security** of the network model. A projection vector \( \bar{\alpha} \) that achieves the minimum is denoted as \( \bar{\alpha}_{\min} \), and is termed a minimally-secure projection.

Target controllability and the target-control metrics can readily be characterized in terms of principal submatrices of the controllability Gramian of the network model. The controllability Gramian for the network model over the interval \([0, k_f]\) is defined by:
\[ W(k_f) = \sum_{i=0}^{k_f-1} (A^T B)(A^T B)^T. \]
We define principal submatrices of the controllability Gramian using a set \( B \) which lists a subset of the nodes \( 1, \ldots, n \) in the network. The \( B \)-controllability Gramian \( W(B, k_f) \) is defined as the principal submatrix of \( W(k_f) \) in which the rows and columns indicated in \( B \) are maintained.

The following lemma provides characterizations of target controllability and the target control metrics in terms of the \( T \)-controllability Gramian (i.e. the principal submatrix of controllability Gramian associated with the target nodes \( T \)). These results follow directly from standard analyses of output controllability [23], [24], hence the proof is omitted.
Lemma 1: The input-output network-consensus model is target controllable if and only if the $\mathcal{T}$-controllability Gramian $W(\mathcal{T}, k_f)$ is invertible.

If the network model is target controllable, then the target-control energy is given by: $E(k_f, y) = y^T W(\mathcal{T}, k_f)^{-1} y$, and an optimal target-control input is

$$\hat{u}[i] = (CA^{k_f-1}B) W(\mathcal{T}, k_f)^{-1} y$$

for $i = 0, \ldots, k_f - 1$. Further, the target security is given by:

$$E_{\text{min}}(k_f) = \frac{1}{\lambda_{\text{max}}(W(\mathcal{T}, k_f))}$$

where $\lambda_{\text{max}}(W(\mathcal{T}, k_f))$ refers to the largest eigenvalue of matrix $W(\mathcal{T}, k_f)$. The minimally secure goal is given by $\bar{y}_{\text{min}} = \nu_{\text{max}}(W(\mathcal{T}, k_f))$, where $\nu_{\text{max}}(W(\mathcal{T}, k_f))$ is the right eigenvector of $W(\mathcal{T}, k_f)$ associated with the largest eigenvalue.

The projection manipulation energy is given by:

$$E_{\text{min}}(k_f, \alpha) = \frac{1}{\alpha W(\mathcal{T}, k_f)\nu_{\text{min}}}.$$  

The projection security is given by $E_{\text{min}}(k_f) = \frac{1}{\nu_{\text{min}}(W(\mathcal{T}, k_f))}$.

The minimally-secure projection is given by $\nu_{\text{min}} = e_j$, where $j = \arg \max_i |W(\mathcal{T}, k_f)|_{i,i}$.

Lemma 1 is the starting point for the main graph-theoretic and structural analyses developed in the paper.

IV. MAIN RESULTS

Per Lemma 1, target controllability and the target control metrics are tied to properties of the controllability Gramian. Our focus here is to develop structural and graph-theoretic results on the Gramian of the network model, with the aim of giving insights into target control. Because a number of graph-theoretic results have already been developed for the binary question of target controllability [10]-[12], we will primarily focus on the target-control metrics.

First, we identify some matrix-theoretic properties of Gramian submatrices and their inverses for the network model. These properties depend on the diffusive structure of the network-consensus model, but not on the specifics of the network’s topology.

Theorem 1: Consider any principal submatrix of the Gramian for the network model, say $Q = W(\mathcal{B}, k_f)$. Also, for invertible $Q$, consider $R = Q^{-1}$. The matrices $Q$ and $R$ have the following properties:

1) $Q$ is doubly nonnegative, i.e. it is symmetric, positive semi-definite, and entry-wise nonnegative. For sufficiently large $k_f$, $Q$ is entry-wise strictly positive.

2) The eigenvalues of $Q$ are real, nonnegative, and non-defective. For sufficiently large $k_f$, the largest eigenvalue $\lambda_{\text{max}}(Q)$ has algebraic multiplicity of 1, and its associated eigenvector $\nu_{\text{max}}(Q)$ is strictly positive (within a scale factor).

3) $\lambda_{\text{max}}(Q) \leq \lambda_{\text{max}}(W[k_f])$. Furthermore, for sufficiently large $k_f$, the inequality is strict.

4) The matrix $R$ is symmetric and positive definite. For sufficiently large $k_f$, $R$ is irreducible.

5) Consider any permutation $T = PRP^{-1}$ of the matrix $R$, and consider any block-partition of $T$ as $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, where $T_{11}$ is square. Assuming that $k_f$ is sufficiently large, the matrix $T_{12}$ has at least one negative entry.

6) For the special case that the cardinality of the set $\mathcal{B}$ (denoted $|\mathcal{B}|$) is 2, the matrix $R$ is a nonsingular $M$ matrix.

Remark: The characterizations in Theorem 1 crucially depend on the doubly-nonnegative structure of the Gramian, which is a consequence of the network-consensus model being nonnegative. Doubly-nonnegative matrices also arise in e.g. semi-definite programming and covariance-matrix analysis [28], [29].

Item 5 of Theorem 1 indicates that inverses of Gramian submatrices have a sophisticated sign pattern. While the diagonal entries of the inverse are positive, the off-diagonal entries may be of either sign. Item 5 indicates, however, that some of the off-diagonal entries must be negative. The pattern of nonnegative off-diagonal entries can be given a graph-theoretic interpretation, which helps to give insight into the target-control metrics. To formalize this interpretation, it is helpful to define a graph that represents the sign pattern of the inverse of a Gramian submatrix. Specifically, we define the negative-inverse graph for the invertible Gramian submatrix $W(\mathcal{B}, k_f)$ as an (unweighted, undirected) graph on $|\mathcal{B}|$ vertices labeled $1, \ldots, |\mathcal{B}|$, where the notation $|\mathcal{B}|$ indicates the cardinality of the set. An edge is drawn between vertex $i$ and $j$ if $[R^{-1}]_{ij}$ is negative. The following corollary is an immediate consequence of Item 5 of Theorem 1:

Corollary 1: The negative-inverse graph for any invertible Gramian submatrix $W(\mathcal{B}, k_f)$ is connected.

The matrix-theoretic properties of the Gramian’s principal submatrices developed in Theorem 1 allow characterization of the defined target-control metrics and associated optimal input signals. Several results on the target-control metrics are listed in the following theorem, and then interpreted in the following discussion:

Theorem 2: Assume that the input-output network-consensus model is target controllable. The target-control metrics and associated optimal input signals have the following properties, for all sufficiently large $k_f$:

1) The network model has a minimally-secure goal $\bar{y}_{\text{min}}$ which is strictly positive. Additionally, the optimal input signal for the minimally-secure goal is nonnegative for $k = 0, \ldots, k_f - 1$.

2) When the network has two target nodes, the target control energy satisfies $E(k_f, y) \leq E(k_f, \bar{y})$ for any goal state $y$.

3) The projection-manipulation energy satisfies $E_{\text{min}}(k_f, \alpha) \leq E_{\text{min}}(k_f, \alpha)$ for any projection vector $\alpha$. Further, the optimal input signal for manipulation of any projection is nonnegative.

4) The global security metrics satisfy the inequality: $E_{\text{min,full}}(k_f) < E_{\text{min}}(k_f) < E_{\text{min}}(k_f, \alpha)$, where $E_{\text{min,full}}(k_f)$ refers to the target-security metric when the set of target nodes $\mathcal{T}$ contains all nodes in the network.

Item 1 of Theorem 2 indicates that the network model always has a minimally-secure goal (the unit-norm goal that
takes the minimum energy to reach) in the positive orthant. Further, the lowest-energy input needed to reach this goal is nonnegative. It is worth noting, however, that the minimum-energy input needed to reach other positively-valued goals need not be positive.

Because of the diffusive structure of the network dynamics, one might postulate that goal states in the positive orthant (i.e., goals with identically-signed entries) require less energy to achieve. Item 2 in the theorem demonstrates that goal states in the positive orthant indeed require less energy to reach compared to their sign-reversed versions, when there are only two target nodes. In other words, it is easier to move any pair of nodes’ states to a more synchronized goal (with both nodes’ goal states having the same sign), than to a comparable differentially-signed goal. The low-energy characteristic of the positive orthorant results specifically form the \( M \)-matrix structure of the inverse Gramian submatrix in the two-target case. However, the result does not generalize to models with more than two target nodes: as Theorem 1 indicates, the inverse Gramian submatrix has a complicated sign pattern when the target set has more than two nodes, which means that mixed-sign goals may sometimes require less energy to reach than their positive orthant counterparts. However, the connectedness of the negative-sign graph (Corollary 1) does indicate that low-energy control is possible for many goal states in the nonnegative orthant.

Per Item 3, projections defined by vectors in the first quadrant are easier to manipulate than their sign-reversed counterparts, regardless of the number of target states. This characteristic is a direct consequence of the nonnegative form of the Gramian submatrix.

Finally, Item 4 provides a comparison of different global security metrics. The inequalities follow immediately from the positive definiteness of the Gramian and its submatrices, however the fact that they are strict is a consequence of the doubly-nonnegative structure of the Gramian.

**Remark:** All goal states in the positive orthant require less energy to reach than their sign-reversed counterparts if the corresponding inverse Gramian submatrix is an \( M \)-matrix. The class of nonnegative matrices whose inverses are \( M \)-matrices has been characterized algebraically in the linear-algebra literature (see e.g. [30]).

Next, we study how the graph topology of the input-output network consensus model constrains the associated Gramian submatrix and its inverse. The main outcome of this analysis is that the magnitudes of the entries in the Gramian submatrix are small if the target nodes are far from the source. In fact, the entries decrease monotonically as the target nodes are moved further away from the source nodes, in a certain sense (related to cutsets of the network graph). Conversely, metrics related to the inverse Gramian are large if the target nodes are far from the source nodes. To formalize these notions, we find it convenient to consider vertex-cutsets in the network graph that separate the source and target vertices. Formally, a set of vertices \( C \) in the graph \( \Gamma \) (equivalently, nodes in the network) is referred to as a separating cutset, if all directed paths between source and target nodes in \( \Gamma \) pass through a vertex in \( C \).

The following theorem characterizes the principal submatrix of the Gramian associated with the network model (i.e. the \( T \)-Gramian submatrix), and its inverse, in terms of a separating cutset:

**Theorem 3:** Consider the Gramian submatrix \( W(T,k_f) \) for the input-output network-consensus model. Also, let \( C \) be a separating cutset of the network graph \( \Gamma \). Then the following inequalities hold:

1. \[ W(T,k_f)_{ij} \leq \max_i W(C,k_f)_{ii} \]
   That is, all entries in the Gramian submatrix associated with the target nodes are smaller than at least one of the diagonal entries of the Gramian matrix corresponding to the separating cutset nodes.
2. Consider any vector \( \alpha \) such that \( |\alpha|^T1 = 1 \). Then \( \alpha^TW(T,k_f)\alpha \leq \max_i W(C,k_f)_{ii} \)
3. \[ \lambda_{\max}(W(T,k_f)) \leq p \max_i W(C,k_f)_{ii} \]
   where \( p \) is the number of target nodes.

The graph-theoretic analyses of Gramian submatrix properties in Theorem 3 immediately yield graph-theoretic bounds on the defined target control metrics. In particular, the target control metrics can be majorized in terms of the energy required to manipulate the nodes on any separating cutset of the network graph. To present these comparisons, it is helpful to explicitly define control energy metrics for the nodes on a separating cutset. Specifically, first consider any vertex \( c \) contained in a separating cutset \( C \) of the network graph. We refer to the minimum input energy required to move the state of the corresponding network node \( c \) to a unity value over the interval \([0,k_f]\) (assuming that the network is initially relaxed) as \( E_c(k_f) \). In analogy with Definition 2, \( E_c(k_f) \) can be formally defined as \( E_c(k_f) = \min_{u[0],...,u[k_f-1]} \sum_{i=0}^{k_f-1} u[i]u[i]^T \), subject to the constraint that the input sequence drives the system from a relaxed state to \( x_c[k_f] = 1 \). We then define the cutset-control energy as \( E_C(k_f) = \min_{c \in cutC} E_c(k_f) \). The target-control metrics can be majorized in terms of the cutset-control energy, as follows:

**Theorem 4:** Consider the input-output network consensus model. The following inequalities hold for the target-control metrics, for any separating cutset \( C \) of the network graph:

1. The projection-manipulation energy satisfies \( F(k_f,\alpha) \geq E_C(k_f) \) for any \( \alpha \) such that \( |\alpha|^T1 = 1 \).
2. The projection security satisfies \( F_{\min}(k_f) \geq E_C(k_f) \).
3. The target security satisfies \( E_{\min}(k_f) \geq \frac{1}{p} E_C(k_f) \).

Theorem 4 demonstrates that the target control metrics follow a spatial majorization, with respect to cutsets in the network graph away from the source nodes. Specifically, the energy required to manipulate any projection of the target state is larger than the energy required to manipulate at least one of the nodes on a separating cutset. Thus, state projections become more secure (harder to manipulate) away from the source nodes. A similar result also holds for the target security metric, but with a scale factor related to the number of nodes being manipulated.
The values of the target-control metrics for long time horizons (i.e., in the limit of large $k_f$) are of interest, since they serve as lower bounds on energy requirements for arbitrary horizons. Because the network-consensus process naturally asymptotes to a synchronized state, one might expect that manipulation of the dynamics to a desired synchronized state can be achieved with a vanishingly-small energy requirement, given a long time horizon. This notion can be formalized by characterizing Gramian submatrices and their spectra for large $k_f$. This characterization of Gramian submatrices, and consequent analyses of the target-control metrics, are formalized in the following theorem:

**Theorem 5:** Consider any principal submatrix of the Gramian for the network model, say $Q = W(B, k_f)$. The matrices $Q$ has the following properties:

1) $Q$ can be written as $Q = (k_f \sum_{i \in S} w_i^2)11^T + H(k_f)$, where the absolute values of the entries in the matrix $H(k_f)$ have an upper bound that is independent of $k_f$. In the expression, $w_i$ is $i$th entry in the left eigenvector of $A$ associated with its unity eigenvalue, where the eigenvector has been normalized so that it's entries sum to 1. Also, 1 represents a vector with all entries equal to 1, of appropriate dimension.

2) The dominant eigenvalue of the matrix $Q$ is given by $\lambda_{\text{max}}(Q) = |B|k_f \sum_{i \in S} w_i^2 + \lambda(k_f)|$, where $|\lambda(k_f)|$ has an upper-bound that is independent of $k_f$. The corresponding dominant eigenvector is given by $v = 1 + c(k_f)$, where each entry in $|c(k_f)|$ is upper bounded by an asymptotically-vanishing function of $k_f$.

Now consider target control for the input-output network-consensus model. Provided that the model is target controllable, the target security metric is given by: $E_{\text{min}}(k_f) = \frac{1}{p^{k_f}} \sum_{i \in S} w_i^2 + E(k_f)$, where $|E(k_f)|$ is upper-bounded by an asymptotically-vanishing function of $k_f$. The minimally-secure goal is given by $F_{\text{min}}(k_f) = 1 + c(k_f)$, where each entry in $|c(k_f)|$ is upper bounded by an asymptotically-vanishing function of $k_f$.

**Remark:** The graph-theoretic analysis of Gramian submatrices in Theorem 3 is relevant to myriad techniques which use the Gramian, such as several model reduction methods [22, 32].

V. EXAMPLE

The spatial majorization of target-control metrics, as developed in Theorems 3 and 4, is illustrated in an example 50-node network. The graph $\Gamma$ for the example network was constructed by placing vertices randomly in the unit square, and connecting vertices within a certain radius. The edge weights of the incoming edges into each vertex were selected to be identical.

One node in the network was subjected to actuation. The energy required to manipulate each individual node’s state over a time horizon of 200 steps, which is the inverse of the corresponding diagonal entry of the Gramian, was determined. These energy requirements are illustrated on the network graph in Figure 1. Specifically, the actuated node (in the bottom right part of the graph) is shown in red in the plot. Also, for all nodes, the energy required for manipulation is indicated by the size of the disk at the node. It is seen that the nodes that are close to the source (actuated) node in the graph can be manipulated with limited energy, while distant nodes require significant energy to manipulate. This spatial growth in the energy requirement is commensurate with Theorems 3 and 4.

![Figure 1](image_url)

**REFERENCES**

[1] Wu, Chai Wah, and Leon O. Chua. "Application of Kronecker products to the analysis of systems with uniform linear coupling.” IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications 42, no. 10 (1995): 775-778.

[2] Xiao, Lin, Stephen Boyd, and Seung-jean Kim. “Distributed average consensus with least-mean-square deviation.” Journal of Parallel and Distributed Computing 67. 6 (2007): 33-46.

[3] Ren, Wei, Randal W. Beard, and Ella M. Atkins. “Information consensus in multivehicle cooperative control.” IEEE Control Systems Magazine 27, no. 2 (2007): 71-82.

[4] Rahmani, Amirreza, Meng Ji, Mehran Mesbahi, and Magnus Egerstedt. “Controllability of multi-agent systems from a graph-theoretic perspective.” SIAM Journal on Control and Optimization 48, no. 1 (2009): 162-186.

[5] Ji, Zhijian, Hai Lin, and Haihong Yu. “Protocols design and uncontrollable topologies construction for multi-agent networks.” IEEE Transactions on Automatic Control 60. 3 (2015): 781-786.

[6] Wang, Lin, Guanrong Chen, Xiaofan Wang, and Wallace K.S. Tang. “Controllability of networked MIMO systems.” Automatica 60 (2016): 405-409.

[7] Xue, Mengran, and Sandip Roy. “Input-output properties of linearly-coupled dynamical systems: interplay between local dynamics and network interactions.” In 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pp. 487-492. IEEE, 2017.

[8] Pasqualetti, Fabio, Sandro Zampieri, and Francesco Bullo. “Controllability metrics, limitations and algorithms for complex networks.” IEEE Transactions on Control of Network Systems 1, no. 1 (2014): 40-52.

[9] Dhal, Rahul, and Sandip Roy. “Vulnerability of network synchronization processes: a minimum energy perspective.” IEEE Transactions on Automatic Control 61, no. 9 (2016): 2525-2530.
Van Waarde, Henk J., M. Kanat Camlibel, and Harry L. Trentelman. “A distance-based approach to strong target control of dynamical networks.” *IEEE Transactions on Automatic Control* 62, no. 12 (2017): 6266-6277.

Yosugh, Amirhosro, Charles Johnson, Mengran Xue, Sandip Roy, and Sean Warnick. “Target control and source estimation metrics for dynamical networks.” *Automatica* 100 (2019): 412-416.

Li, Jingqi, Ximing Chen, Sergio Pequito, George J. Pappas, and Victor M. Preciado. “Structural target controllability of undirected networks.” In 2018 IEEE Conference on Decision and Control (CDC), pp. 6656-6661. IEEE, 2018.

Guan, Yongqiang, and Long Wang. “Target controllability of multiagent systems under fixed and switching topologies.” *International Journal of Robust and Nonlinear Control*.

Vosughi, Amirkhosro, Charles Johnson, Mengran Xue, Sandip Roy, and Sean Warnick. “Target control and source estimation metrics for dynamical networks.” *Automatica* 100 (2019): 412-416.

Li, Jingqi, Ximing Chen, Sergio Pequito, George J. Pappas, and Victor M. Preciado. “Structural target controllability of undirected networks.” In 2018 IEEE Conference on Decision and Control (CDC), pp. 6656-6661. IEEE, 2018.

Klickstein, Isaac, Afroz Sharif, and Francesco Sorrentino. “Energy scaling of targeted optimal control of complex networks.” *Nature Communications* 8 (2017): 15145.

Summers, Tyler H., Fabrizio L. Cortesi, and John Lygeros. “On submodularity and controllability in complex dynamical networks.” *IEEE Transactions on Control of Network Systems* 3, no. 1 (2016): 91-101.

Sahabandu, Dinuka, Jackeline Abad Torres, Rahul Dhal, and Sandip Roy. “Local openand closedloop manipulation of multiagent networks.” *International Journal of Robust and Nonlinear Control* 29, no. 5 (2019): 1339-1360.

Roy, Sandip, Mengran Xue, and Sajal K. Das. “Security and discoverability of spread dynamics in cyber-physical networks.” *IEEE Transactions on Parallel and Distributed Systems* 23, no. 9 (2012): 1694-1707.

Berman, Abraham, and Robert J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. vol. 9. SIAM, 1994.

Zhao, Shiyu, and Fabio Pasqualetti. “Networks with diagonal controllability Gramian: Analysis, graphical conditions, and design algorithms.” *Automatica* 102 (2019): 10-18.

Xue, Mengran, Wei Wang, and Sandip Roy. “Security concepts for the dynamics of autonomous vehicle networks.” *Automatica* 50, no. 3 (2014): 852-857.

Hardin, Hanna, and Jan van Schuppen. “System reduction of nonlinear positive systems by linearization and truncation.” *Positive Systems* (2006): 431-438.

Kreinlindler, E., and P. Sarachik. “On the concepts of controllability and observability of linear systems.” *IEEE Transactions on Automatic Control* 9, no. 2 (1964): 129-136.

Rugh, Wilson J. *Linear system theory*. Vol. 2. Upper Saddle River, NJ: Prentice Hall, 1996.

Gallager, Robert G. *Discrete Stochastic Processes*. vol. 321. Springer Science and Business Media, 2012.

Fiedler, Miroslav. “Old and new about positive definite matrices.” *Linear Algebra and its Applications* 484 (2015): 496-503.

S. Roy and M. Xue, “Sign patterns of inverse doubly nonnegative matrices,” submitted to *Linear Algebra and its Applications*. Also available at https://arxiv.org/abs/1903.04141

Yoshise, Akiko, and Yasuaki Matsukawa. ”On optimization over the doubly nonnegative cone.” In 2010 IEEE International Symposium on Computer-Aided Control System Design, pp. 13-18. IEEE, 2010.

Johnson, Charles R., Brian Lins, and Olivia Walsh. “The critical exponent for continuous conventional powers of doubly nonnegative matrices.” *Linear Algebra and its Applications* 435, no. 9 (2011): 2175-2182.

Willoughby, Ralph A. “The inverse M-matrix problem.” *Linear Algebra and its Applications* 18, no. 1 (1977): 75-94.

Wilkinson, James Hardy. *The Algebraic Eigenvalue Problem*. vol. 662. Clarendon: Oxford, 1965.

Gugercin, Serkan, and Athanasios C. Antoulas. ”A survey of model reduction by balanced truncation and some new results.” *International Journal of Control* 77, no. 8 (2004): 748-766.