Kinetic modeling of wakefield generation in ultrahigh intensity laser-plasma interaction

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Abstract. Quasistatic model of wakefield generation in ultrahigh intensity laser-plasma interaction is presented. The model implies that the plasma wake is slowly changed in the laser pulse frame. 2D axially symmetrical quasistatic PIC code based on this model is developed. The code provides stable numerical computation in ultrahigh intensity regime and much shorter computational time than that of standard PIC codes. The numerical modeling of the plasma wakefield generated by high power laser pulse is presented. A comparison with simulation results obtained by fully 3D electromagnetic PIC code is done.

1. Introduction
Ultrahigh intensity laser-plasma interaction is a fundamental phenomenon, which has many important applications: inertial confinement fusion [1], new schemes of charged particle accelerators [2], advanced radiation sources [3] etc. Recent dramatic progress in laser system technology makes ultrahigh intensity regime of laser-plasma interaction available in the laboratory experiments. The interaction is a strongly nonlinear phenomenon, which is rich of complex processes, so its theoretical analysis is very difficult [4]. One of the most powerful tools to study laser-plasma interaction as well as to understand experimental data is a numerical simulation.

Among various numerical schemes, particle in cell (PIC) codes are the most efficient now tool to simulate laser-plasma interaction [5]. However, PIC codes require a lot of computer resources for modeling of full-scale experiments. Several codes based on so-called quasistatic approximation and PIC technique have been developed to accelerate computer calculations [6, 7]. Our simulations show that the numerical schemes, on which the codes are based, become unstable in the ultrahigh intensity regime of laser-plasma interaction. In this paper, a numerical scheme providing low-noise stable simulation of ultrahigh intensity regime is presented.

2. Mathematical model of quasistatic PIC code
Quasistatic PIC codes are based on simpler mathematical model than that of standard PIC codes. The quasistatic model implies that the laser pulse and plasma response are not significantly modified during the time of electron transition through the pulse and the wake. The model is valid if the contribution of fast electrons co-moving with the laser pulse in the plasma wake formation is negligible. We assume that the ions are immobile because the plasma wake size in ultrahigh intensity regime is less than the ion response length \( \rho_\text{ion} = \frac{c}{\omega_\text{pl}} \), where \( \omega_\text{pl} = \sqrt{4\pi e^2 n_0 / M} \) is the ion plasma frequency and \( M \) is the ion mass, \( c \) is the speed of light, \( e \) is the electron charge, \( n_0 \) is the...
background plasma density. For simplicity, it is also assumed that the background plasma is cold and homogeneous, the laser pulse evolution and the electron trapping are neglected. The last assumption is valid for small interaction time.

To derive the equations of electromagnetic field in quasistatic model we start from Maxwell’s equations written in terms of potentials

\[
\Delta \Phi = -\rho - \frac{1}{2} \left( \frac{\partial j}{\partial t} + \frac{\partial}{\partial x} \left( \nabla \cdot A \right) + \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial j}{\partial t} - \frac{\partial}{\partial x} \Phi \right) \right),
\]

\[
\nabla \times \nabla \times A + j + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \Phi \right) = 0,
\]

where “wake” potential \( \Phi = \phi - A_z \) is introduced instead of the scalar one, \( A \) and \( \phi \) are the vector and scalar potentials, respectively, \( \rho \) and \( j \) are the electron number density and the electron current density, respectively. Here we use the gauge \( A_z = -\phi \), and dimensionless units, normalizing the time to \( \omega_{pe}^{-1} \), the lengths to \( c/\omega_{pe} \), the velocity to \( c \), the electromagnetic fields to \( mc\omega_{pe}/|e| \), and the electron density, \( n \), to the background density \( n_0 \), where \( \omega_{pe} = (4\pi e^2 n_0/m)^{1/2} \) is the electron plasma frequency and \( m \) is the electron mass.

We assume that the laser pulse propagates along \( z \) axis. Let the laser pulse and plasma wake are axially symmetrical. Quasistatic approximation implies that all quantities depend on \( \xi = z - t \) instead of \( z \) and \( t \). The Maxwell’s equations can be reduced to the form

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \Phi}{\partial \xi^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial \xi} \right) - \frac{3}{2} \left( 1 - \rho \right) - \frac{1}{2} j_z,
\]

\[
\frac{\partial^2 \Phi}{\partial \xi^2} + j_z = 0.
\]

Electric and magnetic fields can be expressed through wave potential and vector potential

\[
E_z = -\frac{\partial \Phi}{\partial \xi}, \quad E_r = -\frac{1}{2} \frac{\partial \Phi}{\partial r} + \frac{\partial A}{\partial \xi} + \frac{1}{2} \frac{\partial \Phi}{\partial r}, \quad B_\theta = \frac{\partial A}{\partial \xi} + \frac{1}{2} \frac{\partial \Phi}{\partial r}.
\]

Finally, in our quasistatic model the electromagnetic field equations take the form

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) = j_z + 1 - \rho,
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (rB) = \frac{\partial^2 \Phi}{\partial \xi^2} + j_z.
\]

It should be noted that the continuity equation follows from equations (4) and (6)

\[
\frac{\partial (j_z - \rho)}{\partial \xi} + \frac{1}{r} \frac{\partial}{\partial r} (rj_z) = 0.
\]

The boundary conditions for equations (6) and (7) are

\[
\frac{\partial \Phi}{\partial r} (r = 0) = \Phi (r = \infty) = B (r = 0) = B (r = \infty) = 0.
\]

The numerical solution of equations (6) and (7) is simpler than solution of Maxwell’s equations.

The interaction between the laser pulse and the electron is described through averaged ponderomotive potential in quasistatic approximation. The electron Hamiltonian is

\[
\hat{I} = \sqrt{1 + (P + A)^2 + a_L^2 - \phi},
\]

where \( \hat{I} \) is the electron momentum.
where $\mathbf{P}$ is the canonical momentum of electron, $a_i$ is the laser pulse vector potential amplitude and $a_i^2$ is the corresponding ponderomotive potential. The electron motion is determined by Hamiltonian equations

$$\frac{d\mathbf{P}}{dt} = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{\gamma} \nabla a_i^2,$$

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\gamma},$$

where $\mathbf{v}$ is the electron velocity, $\mathbf{p} = \mathbf{v} \gamma$ is the kinematical momentum of the electron, $\gamma$ is the electron energy. $H - P_i$ is the integral of motion in the quasistatic approximation because Hamiltonian (10) depends on $\xi$. By using the motion integral $H - P_i = \text{const}$, the electron energy and the longitudinal component of electron momentum can be expressed through the radial momentum

$$\gamma = \frac{1 + p_i^2 + 2a_i^2}{2(1 + \Phi)} + \frac{1 + \Phi}{2},$$

$$p_i = \frac{1 + p_i^2 + 2a_i^2}{2(1 + \Phi)} - \frac{1 + \Phi}{2}.$$

Advantage of quasistatic model is that the relations (13) and (14) can be used to calculate $p_i$ instead of numerically solving the differential equation (11) for the longitudinal component of momentum.

One more advantage of quasistatic approximation is the possibility to eliminate $t$. It is convenient to choose $\xi$ as new time. Using the relation

$$\frac{d\xi}{dt} = 1 - v_\xi = \frac{1}{\gamma} (1 + \Phi),$$

equation of motion for radial component takes a form

$$\frac{dp_r}{d\xi} = \frac{1}{1 + \Phi} \left( \gamma \frac{\partial \Phi}{\partial r} - \frac{\partial a_i^2}{\partial r} \right) - B,$$

$$\frac{dr}{d\xi} = \frac{p_r}{1 + \Phi}.$$

Equations (13), (14) and (16), (17) are the main equations of the electron motion in the quasistatic model.

3. Numerical scheme

Differential equations (6), (7), (16) and (17) are solved using a finite difference scheme, in which a grid is set up in both axial coordinate $\xi$ and radial coordinate $r$. Like in PIC codes plasma electrons are modelled by macroparticles. In cylindrical geometry the macroparticle is the ring of radius $r$ and width $\Delta r$, which is equal to the step in $r$. Each macroparticle is determined by longitudinal coordinate, $\xi$, transversal coordinate, $r$, momentum components, $p_r$ and $p_z$, mass $m$, charge $q$. The radius $r$ as well as the momentum components $p_r$ and $p_z$ are functions of time $\xi$. The macroparticles are Langragian particles, which move along trajectory line determined by the initial value of transversal coordinate, $r_0$. The initial distribution of electron density is assumed to be inhomogeneous. The number of the macroparticles is taken to be equal in all cells at the initial moment of time while $m$ and $q$ are proportional to $r_0$ because the cylindrical geometry. Information propagates from the head of laser pulse ($\xi = 0$) to negative values of $\xi$ in the speed of light frame. Accordingly, we
integrate equation of motion slice by slice from $\xi = 0$ to $\xi = -L$ with step $\Delta \xi$, where $L$ is the length of the simulation region.

Following [6, 7] in order to provide numerical stability it is convenient to rewrite equations (6) and (7) in the form

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) = f_i, \quad f_i = j_i + 1 - \rho - \Phi,
$$

(18)

$$
\frac{1}{r} \frac{\partial}{\partial r} (r B) + B = f_2, \quad f_2 = \frac{\partial^2 \Phi}{\partial \xi^2} + j_i + B.
$$

(19)

The solution to equations (18) and (19) can be presented in the form

$$
\Phi(\xi, r) = -K_0(r) \int_0^r r l_0(r) f_1(\xi, r) - l_0(r) \int_r^\infty r K_0(r) f_1(\xi, r),
$$

(20)

$$
r B(\xi, r) = e^{-i \xi} \int_0^r e^i r f_2(\xi, r),
$$

(21)

where $K_0(r)$ and $I_0(r)$ are the modified Bessel functions of zero order. $\Phi$ and $B$ in the sources $f_i$ and $f_2$ are taken at the previous $\xi$-layer. Note that these solutions fulfill the boundary conditions at $r = \infty$ while the latter is difficult to provide for the solutions of equations (6) and (7).

The charge assignment to the grid is similar to the standard PIC codes. In the quasistatic model the contribution of the macroparticle to charge density and the current density on the grid in the frame $r - \xi$ co-moving with the laser pulse is proportional to the amount of time spent by the macroparticle in the corresponding cell:

$$
\rho(r) = \sum_n \frac{q_n}{1 - v_{n,x}} S_i(r, r_n),
$$

(22)

$$
\dot{j}_i(r) = \sum_n v_{n,x} q_n S_i(r, r_n),
$$

(23)

$$
\dot{j}_i(r) - \rho(r) = \sum_n v_{n,y} q_n S_i(r, r_n),
$$

(24)

where $S_i(x, y)$ is the first-order form-factor defining charge assignment [8,9]. The quantities $a$, $\Phi$, $\rho$ and $j_i$ are defined on $r$- and $\xi$-grids. Magnetic field is defined on $\xi$-grid points and halfway between two neighboring $r$-grid points. Macroparticle quantities are defined on $\xi$-grid points and everywhere on $r$-axis. The radial component of macroparticle momentum is defined everywhere on $r$-axis and halfway between two neighboring $\xi$-grid points.

The laser pulse evolution has been neglected in the presented model. However it can be included in the model [6]. The characteristic time of the laser pulse modification is much longer than the plasma response time. Therefore, first, the plasma response can be calculated for the given laser pulse as described above. Further, the evolution of the laser pulse can be found for the dielectric constant defined by the calculated electron density distribution. Then plasma response can be calculated for the new shape of the pulse. Other effects: ion dynamics [6], electron acceleration [7], generation of the electromagnetic radiation [3], ionization etc. can be also included in the quasistatic PIC codes.

4. Results of modelling and discussion

Similar quasistatic PIC code has been developed in [6] (code “WAKE”). Equations (16) and (17) for macroparticle motion are identical to that used in code WAKE while the equation (6) for wake
potential is slightly different from that used in WAKE. Making use of the continuity equation (8) equation (6) can be rewritten as follows
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) - \frac{\partial^2 \Phi}{\partial \xi^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial j_r}{\partial r} - rB \right) + j_z.
\]  
(25)
The obtained equation is identical to the equation used in [6].

The quasistatic PIC code LCODE has been designed to study the interaction between intense electron beam and plasma [7]. The radial component of electric field instead of the wake potential is used in code LCODE to describe electromagnetic field. The equation for radial electric field used in [7] can be obtained from equation (6) by making use of equations (5) and (8):
\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = \frac{\partial \rho}{\partial r} - \frac{\partial j_r}{\partial r} - E_r.
\]  
(26)

We point out the distinction between the electromagnetic field equations used in our code, in code WAKE and in code LCODE. The source in the equations of electromagnetic field is defined by \(\rho\) and \(j\) in our model, by \(j\) and \(j\) in model used in [6], by \(j\) and \(\rho\) in model used in [7]. Our simulation results show that the model based on equations (6) and (7) provides stable low-noise numerical modeling of the plasma wake generation in ultrahigh intensity regime \(a_\parallel > 1\). It is in contrast to the numerical schemes based on equation (25) or equation (26). These schemes demonstrate development of numerical instabilities in the limit \(a_\parallel > 1\).

![Figure 1](image)

**Figure 1.** The electron density distribution calculated by (a) 2D axially symmetrical quasistatic PIC code and (b) by VLPL 3D PIC code [5]. The darker is gray color, the higher is electron density. The coordinates are in \(c / \omega_p\) units.

The presented quasistatic PIC has been benchmarked against fully 3D relativistic PIC codes which are free from quasistatic approximation. First the simulation results obtained by our quasistatic axially symmetrical 2D PIC code are compared with that obtained by 3D PIC code VLPL [5]. The incident laser pulse of wavelength \(\lambda = 0.82\) µm is circularly polarized and has Gaussian envelope shape \(a_t = a_\parallel \exp \left( -r^2 / r_t^2 - \xi^2 / L^2 \right) \). The parameters of the laser pulse are as follows: \(n = 5\), \(L = 2\), \(a_\parallel = 10\). The pulse propagates in plasma of density \(n_0 = 10^{20}\) cm\(^{-3}\). The electron density distributions calculated by the quasistatic PIC code and by VLPL code are shown in figure 1. Higher gray level corresponds to higher electron density. It is seen from figure 1 that electron density distribution obtained by our code is close to that obtained by VLPL code. There is slight electron density modulation. The characteristic length of the modulation is of the order of the laser wavelength in the front of the plasma cavity obtained by VLPL code (see figure 1(b)). This modulation is absent in the electron density distribution
obtained by quasistatic PIC code (see figure 1(a)). This is due to the fact that the quasistatic model implies averaged over one period description of the laser ponderomotive effect.

The electron density distribution obtained by our code has been also compared with that calculated by 3D PIC code OSIRIS [10]. The power of the Gaussian laser pulse is 200 TW and the pulse duration is 30 fs. The pulse propagates in plasma with the density \( n_0 = 1.5 \times 10^{18} \text{ cm}^{-3} \). The parameters of two simulation results are \( a_0 = 4, W_0 = 4 \) and \( a_0 = 5.3, W_0 = 3 \), where \( W_0 \) is the laser spot size. The electron density distribution calculated by the quasistatic PIC code is shown in figure 2. It is seen from the figure that electron density distribution around the first plasma cavity obtained by our code is close to that obtained by OSIRIS code.

In conclusion, a quasistatic model of wakefield generation in ultrahigh intensity laser-plasma interaction is developed. 2D axially symmetrical quasistatic PIC code based on this model is designed. The code provides numerically stable simulation of ultrahigh intensity laser-plasma interaction while it requires much less computer resources than that of standard PIC codes. The simulation results are in good agreement with that obtained by 3D PIC codes, which are free from quasistatic approximation.

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