The five-dimensional Kepler Problem
as an SU(2) Gauge System:
Algebraic Constraint Quantization

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Abstract

Starting from the structural similarity between the quantum theory of gauge systems and that of the Kepler problem, an SU(2) gauge description of the five-dimensional Kepler problem is given. This non-abelian gauge system is used as a testing ground for the application of an algebraic constraint quantization scheme which can be formulated entirely in terms of observable quantities. For the quantum mechanical reduction only the quadratic Casimir of the constraint algebra, interpreted as an observable, is needed.
1 Introduction

In this paper we present an example for the quantization and reduction of a non-abelian gauge system along the guidelines of an algebraic constraint quantization scheme which can be formulated entirely in terms of observable quantities. This algebraic scheme does not make use of the individual constraints as projectors onto the physical subspace of an extended Hilbert space. Rather, it treats the intrinsically defined observable content of the constraint algebra, comprised in its Casimir elements, as representation conditions to determine the physical representations of the algebra of observables. A heuristic formulation of the procedure, starting from a classical first class constrained system, is as follows.

- Define the algebra of observables $\mathcal{O}$ as the strong Poisson commutant of the constraint algebra $\mathcal{C}$. By definition of $\mathcal{O}$, the Casimirs of $\mathcal{C}$ must be addressed as observables. At the same time they are also Casimirs of $\mathcal{O}$ and especially of the observable symmetry or invariance algebra $\mathcal{S}$ of the system.

- In general there exist functional relations between the Casimirs of $\mathcal{C}$ and those of $\mathcal{O}$ ($\mathcal{S}$), which allow to express the Casimirs of $\mathcal{C}$ as functions of the generators of $\mathcal{O}$ ($\mathcal{S}$).

- As a first step towards the quantization of the system, choose two subalgebras of $\mathcal{O}$: $i$) the symmetry algebra $\mathcal{S}$, which contains a good deal of the structural information about the system, and, $ii$) an algebra $\tilde{\mathcal{O}}$ which should allow to generate $\mathcal{O}$ and to reproduce the dynamical content of the system. Of course, (part of) $\mathcal{S}$ can be contained in $\tilde{\mathcal{O}}$ to form a dynamical or non-invariance algebra for the system. $\mathcal{S}$ as well as $\tilde{\mathcal{O}}$ are characterized by the Poisson commutation relations of and the functional dependencies between their generators.

- The next step consists in constructing the quantum analogs $\mathcal{Q}\mathcal{S}$ and $\mathcal{Q}\tilde{\mathcal{O}}$ of $\mathcal{S}$ and $\tilde{\mathcal{O}}$, i.e. the commutator algebras which correspond to them. As the connection between the classical and quantum theories is closest on the algebraic level, we shall require that $\mathcal{Q}\mathcal{S}$, as a commutator algebra, be isomorphic to $\mathcal{S}$. As to the relationship between $\tilde{\mathcal{O}}$ and $\mathcal{Q}\tilde{\mathcal{O}}$, we cannot a priori require them to be isomorphic. This is possible only in special cases, depending on the physical interpretation: For example if $\tilde{\mathcal{O}}$ can be chosen to consist of globally defined canonical coordinates on the reduced phase space or if it forms a dynamical Lie algebra. Nevertheless, the algebraic structure of $\tilde{\mathcal{O}}$ cannot be drastically changed, and that of $\mathcal{Q}\tilde{\mathcal{O}}$ should be as close as possible to that of $\tilde{\mathcal{O}}$. In any case the covariant transformation properties of its generators with respect to $\mathcal{S}$ should be preserved. Also, for the theory to possess the correct classical limit, in leading order in $\hbar$ (as far as explicit
powers of $\hbar$ are concerned, cf. the next item) the commutator algebra must be isomorphic to the Poisson bracket algebra.

- The crucial step is the determination of the functional relations between the Casimirs of $\mathcal{C}$, $\mathcal{QS}$ and $\mathcal{Q}\tilde{O}$, and thereby of the expressions for the Casimirs of $\mathcal{C}$ in terms of the generators of $\mathcal{QS}$ or $\mathcal{Q}\tilde{O}$. As the generators of $\mathcal{QS}$ and $\mathcal{Q}\tilde{O}$, and therefore their Casimirs, are to carry a definite gradation in terms of physical dimensions, possible correction terms, which must be formed from Casimirs and carry explicit powers of $\hbar$, can be largely restricted by a dimensional analysis. In leading (i.e. zeroth) order in (explicit powers of) $\hbar$ the classical expressions must be reproduced.

- The final step is the construction of the irreducible representations of $\mathcal{QS}$ and $\mathcal{Q}\tilde{O}$ on a suitably chosen Hilbert space $\mathcal{H}$. The system being constrained means that $\mathcal{H}$ cannot be irreducible with respect to $\mathcal{Q}\tilde{O}$ and that it contains unphysical representations of $\mathcal{Q}\tilde{O}$ and $\mathcal{QS}$. The “vanishing” of the constraints has two consequences. It implies the projection onto the zero–eigenspaces of the Casimirs of $\mathcal{C}$, and, via the functional dependencies between the Casimirs of $\mathcal{C}$ and those of $\mathcal{Q}\tilde{O}$ and $\mathcal{QS}$, it induces relations which must be satisfied by the Casimirs of $\mathcal{Q}\tilde{O}$ or $\mathcal{QS}$ respectively.

- The selection of the physical representations of $\mathcal{Q}\tilde{O}$ and $\mathcal{QS}$ is achieved by means of the eigenvalues of and the relations between their Casimir operators. That is, the physical Hilbert space is spanned by those irreducible representations of $\mathcal{Q}\tilde{O}$ and $\mathcal{QS}$, in which the Casimirs of $\mathcal{C}$ vanish and the induced relations are satisfied. In many cases, to determine the physical Hilbert space, it will be sufficient and more favourable to restrict the procedure to the symmetry algebra alone, because its representations are often well studied and easier to handle.

Note that the application of the algebraic scheme does not require the quantization of the unphysical constraint algebra (although it may facilitate the analysis if we can consistently quantize it).

If the above reasoning is correct, it should be a common feature of the quantum theory of gauge systems, that their physical Hilbert space carries only a restricted class of the irreducible representations of the symmetry algebra of the system, which is determined by the characteristic identities satisfied by the Casimirs of the symmetry algebra. This property is shared by the quantum theory of the Kepler problem (KP) in any number of dimensions $\geq 3$. This structural similarity suggests that it should be possible to describe the KP as the reduced form of a higher-dimensional gauge system. In ref. [1] this has been done successfully for the three-dimensional KP with SO(2) as the gauge group. There it has been shown that the application of the algebraic constraint quantization scheme to the
corresponding gauge system yields the well known quantum theory of the hydrogen atom. Of course, for an abelian constraint algebra the Casimirs are the constraints themselves, and there are as many Casimirs as there are constraints. But in the case of a non-abelian constraint algebra the number of Casimirs is less than the number of constraints, and it has to be shown that the information contained in the Casimirs is sufficient for the reduction of the system.

In the present paper we will give a description of the five-dimensional Kepler problem (KP$_5$) as an SU(2) gauge system by application of the so-called Hurwitz transformation. It will be shown that the reduction of the resulting gauge system according to the algebraic constraint quantization scheme, which uses only the quadratic Casimir of the constraint algebra su(2), yields the quantum theory of the KP$_5$.

The organization of the paper is as follows. In section 2 we briefly describe the symmetry algebra of the classical KP$_5$ and the characteristic relations obeyed by its Casimirs. In section 3 the Hurwitz transformation is introduced and applied to the KP$_5$ in section 4. In section 5 we present the quantum theory of the KP$_5$ and determine the representations of its symmetry algebra. The quantization and reduction of the corresponding SU(2) gauge system is carried out in section 6, and the resulting quantum theory is compared to the quantum theory of the KP$_5$. The appendix contains a brief characterization of the unitary irreducible representations of the groups SO(6), E(5) and SO(5,1).
2 The symmetry algebra of the five-dimensional Kepler problem

The five-dimensional Kepler problem (KP\(_5\)) is the dynamical system \((P,\omega,H)\), with phase space \(P = T^*\mathbb{R}^5 = \mathbb{R}^5 \times \dot{\mathbb{R}}^5\) \((\dot{\mathbb{R}}^5 = \mathbb{R}^5 \setminus \{0\})\), symplectic form \(\omega = dq \wedge dp\), and Hamiltonian

\[
H = \frac{1}{2} p^2 - \frac{k}{q}, \quad k > 0, \quad q = \sqrt{q^2}. \tag{1}
\]

The constants of motion are the components of the angular momentum tensor

\[
L_{ij} = q_i p_j - q_j p_i, \quad 1 \leq i, j \leq 5 \tag{2}
\]

and of the Lenz–Runge vector

\[
M = (\frac{1}{2} p^2 + H) q - (q \cdot p) p. \tag{3}
\]

The Poisson commutation relations of \(H, L_{ij}\) and \(M_i\)

\[
\{H, L_{ij}\} = 0 = \{H, M_i\} \quad \{L_{ij}, L_{ik}\} = L_{jk} \tag{4}
\]

\[
\{L_{ij}, M_i\} = M_j \quad \{M_i, M_j\} = -2HL_{ij} \tag{5}
\]
define an abstract algebra \(S\). The algebra \(S\) possesses three algebraically independent Casimir invariants, which can be obtained in the following way. Let

\[
l_{ij} = \sqrt{-2H} L_{ij}, \quad l_{i6} = -l_{6i} = M_i, \tag{6}
\]

then, for constant \(H \neq 0, l_{\mu\nu}, 1 \leq \mu, \nu \leq 6\), generate a deformation of the Lie algebra so(6), which is characterized by the commutation relations

\[
\{l_{\mu\nu}, l_{\mu\rho}\} = \sqrt{-2H} l_{\nu\rho}. \tag{7}
\]

Irrespective of the sign of the square root, the Casimir invariants of this algebra are also Casimirs of the algebra \(S\), even if \(H\) is not kept constant. They are built in the same way as the Casimir invariants of so(6) and can be chosen as

\[
\tilde{C}_2 = \frac{1}{2} l_{\mu\nu} l_{\mu\nu} = M^2 - 2HL^2 \quad (L^2 = \frac{1}{2} L_{ij} L_{ij}) \tag{8}
\]

\[
\tilde{C}_3 = \frac{1}{-2H} \varepsilon_{\mu\nu\rho\sigma\tau} l_{\mu\nu} l_{\rho\sigma} l_{\tau\nu} = 6 \varepsilon_{ijklm} L_{ij} L_{kl} M_m \tag{9}
\]

\[
\tilde{C}_4 = \frac{1}{2} l_{\mu\nu} l_{\nu\rho} l_{\rho\sigma} l_{\sigma\mu} = 2H^2 L_{ij} L_{jk} L_{ki} + 4HM_i M_j L_{jk} L_{ki} + (M^2)^2. \tag{10}
\]
For $H = 0$ the right hand sides of these expressions are still well defined, but $\tilde{C}_4$ becomes algebraically dependent on $\tilde{C}_2$, and we will use instead

$$\tilde{C}_4^{(0)} = M^2 L^2 - \frac{1}{4} (M_i L_{ij} + L_{ij} M_i)(M_k L_{kj} + L_{kj} M_k) = M^2 L^2 + M_i M_k L_{kj} L_{ji}$$

as the third Casimir invariant. In the realization of $S$ by the dynamical quantities (1)-(3) of the KP$_5$ the Casimirs are not functionally independent. Rather, they obey the relations

$$\tilde{C}_2 \equiv k^2, \quad \tilde{C}_3 \equiv 0, \quad \tilde{C}_4 \equiv \tilde{C}_2^2, \quad \tilde{C}_4^{(0)} \equiv 0$$

which, at the same time, completely fix their values. The algebra $S$, together with the relations (12), will be referred to as the symmetry algebra of the classical KP$_5$.

Besides the symmetry algebra there is a well known dynamical or non-invariance algebra (cf. [2] and refs. therein), spanned by the functions $H_{ij} = L_{ij}$ and

$$H_{i6} = \frac{1}{2} q_i p^2 - (q \cdot p) p_i, \quad H_{i7} = q_i, \quad H_{i8} = q p_i$$

$$H_{68} = \frac{1}{2} q p^2, \quad H_{78} = q, \quad H_{67} = q \cdot p.$$

The Poisson algebra of the functions $H_{ab} = -H_{ba}$, $1 \leq a, b \leq 8$, is isomorphic to the Lie algebra so(6,2), as can be read off the Poisson brackets of the formal linear combinations (formal, because the terms have different dimensions) $G_{ij} = H_{ij}$, $G_{i6} = H_{i6} - \frac{1}{2} H_{i7}$, $G_{i7} = H_{i6} + \frac{1}{2} H_{i7}$, $G_{i8} = H_{i8}$, $G_{67} = H_{67}$, $G_{68} = H_{68} - \frac{1}{2} H_{78}$ and $G_{78} = H_{68} + \frac{1}{2} H_{78}$:

$$\{G_{ab}, G_{ac}\} = g_{aa} G_{bc}, \quad g_{ab} = \text{diag}(++ + + + + - -).$$

3 The Hurwitz transformation

The Hurwitz transformation (HT) ([3] and refs. therein) is a surjective map

$$h : \tilde{\mathbb{R}}^8 \longrightarrow \tilde{\mathbb{R}}^5 \quad u \longmapsto q.$$  

We will alternatively view $\tilde{\mathbb{R}}^8 = \mathbb{R}^8 \setminus \{0\}$ as the space $\tilde{\mathbb{H}}^2$ of nonvanishing bi-quaternions and represent quaternions $S = s_1 + i s_2 + j s_3 + k s_4$ as $2 \times 2$ complex matrices by putting $1 \rightarrow E_2$, $i \rightarrow -i \sigma_1$, $j \rightarrow -i \sigma_2$, $k \rightarrow -i \sigma_3$, where $E_2$ is the $2 \times 2$ unit matrix and $\sigma_i$ are the Pauli matrices. The multiplication of quaternions is then realized by matrix multiplication, the conjugation $S \rightarrow \bar{S}$ by hermitian conjugation, the real part of $S$ is $\text{Re} S = \frac{1}{2} (S + \bar{S})$ and the imaginary part $\text{Im} S = \frac{1}{2} (S - \bar{S})$. 

5
Upon the identification $\mathbb{R}^8 = \mathbb{H}^2$, the components of a vector $u = (u_1, \ldots, u_8)$ are connected to the components of a bi-quaternion $U = (U_1, U_2)$ via

$$U_1 = \begin{pmatrix} u_1 - iu_4 & -u_3 - iu_2 \\ u_3 - iu_2 & u_1 + iu_4 \end{pmatrix} \quad U_2 = \begin{pmatrix} u_5 - iu_8 & -u_7 - iu_6 \\ u_7 - iu_6 & u_5 + iu_8 \end{pmatrix}. \quad (16)$$

The Euclidean scalar product and norm on $\mathbb{R}^8$ appear as

$$u' \cdot u = \langle U' | U \rangle = \frac{1}{2} \text{tr} (U_1^\dagger U_1 + U_2^\dagger U_2) \quad (17)$$

and

$$u^2 := u \cdot u = U^2 := \langle U | U \rangle = \det U_1 + \det U_2. \quad (18)$$

In terms of bi-quaternions the HT explicitly reads

$$Q_1 := \begin{pmatrix} q_1 - iq_4 & -q_3 - iq_2 \\ q_3 - iq_2 & q_1 + iq_4 \end{pmatrix} = 2U_2 U_1 \quad \text{and} \quad Q_2 := q_5 E_2 = U_1^\dagger U_1 - U_2^\dagger U_2. \quad (19)$$

As can be seen from the identity $q = \sqrt{q^2} = u^2$, the HT maps spheres of radius $R$ in $\hat{\mathbb{R}}^8$ onto spheres of radius $R^2$ in $\mathbb{R}^8$. Geometrically the HT represents a realization of the extension of the Hopf bundle $SU(2) \longrightarrow S^7 \longrightarrow S^4$ to the bundle $SU(2) \longrightarrow \mathbb{R}^8 \longrightarrow \hat{\mathbb{R}}^5$. The fibers are 3-spheres.

The fact that the space of unit quaternions is isomorphic to the group $SU(2)$, $\hat{\mathbb{H}} = \mathbb{R}^+ \times SU(2)$, can be used to introduce Euler coordinates in $\hat{\mathbb{H}}^2$, in terms of which the HT becomes more transparent (cf. [1]). Let

$$U_1 = |u| \cos \frac{\chi}{2} \tilde{U}(\varphi_1, \vartheta_1, \psi_1) \quad U_2 = |u| \sin \frac{\chi}{2} \tilde{U}(\varphi_2, \vartheta_2, \psi_2) \quad (20)$$

$$\tilde{U}(\varphi, \vartheta, \psi) = \begin{pmatrix} \cos \frac{\vartheta}{2} e^{-i \frac{\varphi + \psi}{2}} & -i \sin \frac{\vartheta}{2} e^{-i \frac{\varphi - \psi}{2}} \\ -i \sin \frac{\vartheta}{2} e^{i \frac{\varphi - \psi}{2}} & \cos \frac{\vartheta}{2} e^{i \frac{\varphi + \psi}{2}} \end{pmatrix} \quad (21)$$

$$\text{with} \quad 0 \leq \chi < \pi, \quad 0 \leq \vartheta_i < \pi, \quad 0 \leq \varphi_i < 2\pi, \quad -2\pi \leq \psi < 2\pi$$

then

$$Q_1 = q \sin \chi \tilde{U}(\varphi, \vartheta, \psi) \quad Q_2 = q \cos \chi E_2 \quad (22)$$

$(q = u^2)$ where $\varphi, \vartheta, \psi$ are expressed in terms of $\varphi_1, \vartheta_1, \psi_1$ and $\varphi_2, \vartheta_2, \psi_2$ by the addition theorem for Euler angles (see e.g. [3]).

Finally, the HT can be extended to a nonbijective “canonical” transformation

$$\tilde{h} : T^*\hat{\mathbb{R}}^8 \longrightarrow T^*\hat{\mathbb{R}}^5 \quad (u, v) \longmapsto (q, p).$$

The explicit expression for $p(u, v)$, identifying $T^*\hat{\mathbb{R}}^8$ with $\mathbb{H}^2 \times \hat{\mathbb{H}}^2$, reads

$$P_1 := \begin{pmatrix} p_1 - ip_4 & -p_3 - ip_2 \\ p_3 - ip_2 & p_1 + ip_4 \end{pmatrix} = \frac{1}{2U^2} (V_2 U_1 + U_2 V_1) \quad (23)$$

$$P_2 := p_5 E_2 = \frac{1}{2U^2} \text{Re}(V_1^\dagger U_1 - U_2 V_1^\dagger). \quad (24)$$
4 The Hurwitz-Kepler problem

By application of the HT the KP can be described as the reduced form of a singular Hamiltonian system on $T^* \mathbb{R}^8$, with SU(2) acting as a gauge group. In the sequel the singular system will be called the Hurwitz-Kepler problem (HKP).

As a starting point we will take the Lagrangian $L(u, \dot{u})$, which can be obtained from the Lagrangian $L(q, \dot{q})$ of the KP

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 + \frac{k}{q} \quad (25)$$

by expressing $q$ and $\dot{q}$ as functions of $u$ and $\dot{u}$ by means of the HT: $q = q(u)$, $\dot{q}(u, \dot{u}) = (q(u)) \dot{u}$. $L(u, \dot{u})$ reads explicitly

$$L(u, \dot{u}) = \frac{1}{2} S_{ab}(u) \dot{u}_a \dot{u}_b + \frac{k}{u^2}, \quad 1 \leq a, b \leq 8 \quad (26)$$

where

$$S_{ab}(u) = 4 (u^2 \delta_{ab} - \sum_{l=1}^{3} w^{(l)}_a w^{(l)}_b) \quad (27)$$

$$w^{(1)} = (-u_2, u_1, -u_4, u_3, u_6, -u_5, -u_8, u_7)$$
$$w^{(2)} = (-u_3, u_4, u_1, -u_2, u_7, u_8, -u_5, -u_6)$$
$$w^{(3)} = (-u_4, -u_3, u_2, u_1, u_8, -u_7, u_6, -u_5). \quad (28)$$

The matrix $S_{ab}$ being singular

$$\det \left| \frac{\partial^2 L}{\partial \dot{u}_a \partial \dot{u}_b} \right| = \det |S_{ab}(u)| = 0, \quad \text{rank} S_{ab} = 5, \quad (29)$$

only five of the canonical momenta

$$v_a = \frac{\partial L}{\partial \dot{u}_a} = S_{ab}(u) \dot{u}_b \quad (30)$$

are independent functions of the velocities $\dot{u}_a$ and there are three primary constraints $K_i$ [6]. Observing that $v \cdot w^{(i)} = 0$, the constraints can be identified as

$$K_i = \frac{1}{2} w^{(i)} \cdot v \approx 0. \quad (31)$$

Their Poisson brackets with respect to the canonical symplectic form $\omega_8 = du \wedge dv$ generate an su(2) algebra

$$\{K_i, K_j\} = \varepsilon_{ijk} K_k. \quad (32)$$

Following Dirac [6], we have to pass to the total Hamiltonian $H_T = v_a \dot{u}_a - L(u, \dot{u}) + \mu_i K_i$. Making use of the arbitrariness of the Lagrange multipliers $\mu_i$, $H_T$ can be brought into the form

$$H_T = \frac{1}{8 u^2} v^2 - \frac{k}{u^2} + \lambda_i K_i =: H + \lambda_i K_i \quad (33)$$
where \( \lambda_i \) are still arbitrary functions. The Poisson brackets of \( H_T \) with the constraints vanish weakly, so that there are no secondary constraints and the \( K_i \) are first class [6].

In terms of the quaternionic coordinates the action of the gauge group SU(2), generated by the constraints \( K_i \), can easily be integrated. Let \( g = \exp(-i\vec{\alpha} \cdot \vec{\sigma}/2) \in SU(2) \) (\( \vec{\sigma} \) is the vector of Pauli matrices), then

\[
(U_1, U_2) \mapsto (g^* U_1, U_2 g^*) \quad (V_1, V_2) \mapsto (g V_1, V_2 g^*).
\]

Furthermore, the quaternionic coordinates allow for a complete description of the algebra of observables (i.e. strongly gauge invariant functions). As can be seen from (34), the bilinear combinations

\[
U_1^\dagger U_1, V_1^\dagger V_1, U_2 U_2^\dagger, V_2 V_2^\dagger, U_1^\dagger U_1, V_1^\dagger V_1, U_2^\dagger U_2, V_2^\dagger V_2
\]

and their conjugates are invariant under the action of the gauge group SU(2). All observables must be functions of the 28 algebraically independent components of these elements, which can be seen to form a realization of the Lie algebra so(6,2), analogous to that given in (13)-(15), by defining

\[
\begin{pmatrix}
H_{67} - iH_{14} & -H_{13} - iH_{12} \\
H_{13} - iH_{12} & H_{67} + iH_{14}
\end{pmatrix} = \frac{1}{2} (U_1^\dagger V_1 + V_2 U_2^\dagger)
\]

\[
\begin{pmatrix}
H_{58} - iH_{23} & -H_{42} - iH_{34} \\
H_{42} - iH_{34} & H_{58} + iH_{23}
\end{pmatrix} = \frac{1}{2} (V_1^\dagger U_1 - U_2 V_2^\dagger)
\]

\[
\begin{pmatrix}
H_{15} - iH_{45} & -H_{35} - iH_{25} \\
H_{35} - iH_{25} & H_{15} + iH_{45}
\end{pmatrix} = \frac{1}{2} (U_2 V_1 - V_2 U_1)
\]

\[
\begin{pmatrix}
H_{16} - iH_{46} & -H_{36} - iH_{26} \\
H_{36} - iH_{26} & H_{16} + iH_{46}
\end{pmatrix} = -\frac{1}{4} V_2 V_1
\]

\[
H_{56} E_2 = -\frac{1}{8} (V_1^\dagger V_1 - V_2 V_2^\dagger)
\]

\[
\begin{pmatrix}
H_{17} - iH_{47} & -H_{37} - iH_{27} \\
H_{37} - iH_{27} & H_{17} + iH_{47}
\end{pmatrix} = 2 U_2 U_1
\]

\[
H_{57} E_2 = U_1^\dagger U_1 - U_2 U_2^\dagger
\]

\[
\begin{pmatrix}
H_{18} - iH_{48} & -H_{38} - iH_{28} \\
H_{38} - iH_{28} & H_{18} + iH_{48}
\end{pmatrix} = \frac{1}{2} (U_2 V_1 + V_2 U_1)
\]

\[
H_{68} E_2 = \frac{1}{8} (V_1^\dagger V_1 + V_2 V_2^\dagger)
\]

\[
H_{78} E_2 = U_1^\dagger U_1 + U_2 U_2^\dagger.
\]

As this so(6,2) algebra allows to generate the algebra of observables, we shall take it as a dynamical algebra for the HKP.
4.1 The symmetry algebra of the Hurwitz-Kepler problem

The canonical Hamiltonian $H$ possesses 64 integrals of motion

$$F_{ab} = u_av_b - u_bv_a \quad A_{ab} = \frac{1}{2} v_av_b - 4 H u_a u_b$$ (46)

which, together with $H$, generate an algebra $A$ characterized by the commutation relations

$$\{F_{ab}, F_{ac}\} = F_{bc}$$ (47)

$$\{F_{ab}, A_{cd}\} = \delta_{ac} A_{bd} + \delta_{ad} A_{bc} - \delta_{bc} A_{ad} - \delta_{bd} A_{ac}$$ (48)

$$\{A_{ab}, A_{cd}\} = -2H (\delta_{ac} F_{bd} + \delta_{ad} F_{bc} + \delta_{bc} F_{ad} + \delta_{bd} F_{ac})$$ (49)

The observable part of $A$, i.e. the commutant of the constraint algebra $su(2) \subset A$ in $A$, is spanned by $H$ and the manifestly $SU(2)$--invariant functions $L_{ij} = H_{ij}$, $1 \leq i, j \leq 5$, as defined above, and the

$$\begin{pmatrix} M_1 - iM_4 & -M_3 - iM_2 \\ M_3 - iM_2 & M_1 + iM_4 \end{pmatrix} = -\frac{1}{4} V_2 V_1 + 2H U_2 U_1$$ (50)

$$M_5 E_2 = -\frac{1}{8} (V_1 V_1^\dagger - V_2 V_2^\dagger) + H (U_1 U_1^\dagger - U_2 U_2^\dagger)$$ (51)

and is isomorphic to the algebra $S$.

The relations (12), which characterize the symmetry algebra of the KP$_5$, are now changed into relations between the Casimirs of the algebra $S$ and the Casimir of the constraint algebra

$$\tilde{C}_2 \equiv k^2 - 4H \tilde{K}^2 \quad \tilde{C}_3 \equiv 48k \tilde{K}^2$$ (52)

$$\tilde{C}_4 \equiv \tilde{C}_2^2 + 8H \tilde{K}^2 \tilde{C}_2 + 24H^2(\tilde{K}^2)^2 \quad \tilde{C}_4^{(0)} \equiv 2 \tilde{K}^2 \tilde{C}_2.$$ (53)

The symmetry algebra of the classical HKP is the algebra $S$ together with the relations (52) and (53). For $\tilde{K}^2 = 0$ we regain the symmetry algebra of the KP$_5$.

The above relations express the searched for functional dependencies between the the Casimir of the constraint algebra and the Casimirs of the true symmetry algebra $S$, thereby also confirming the observable status of $\tilde{K}^2$.

5 Quantization of the KP$_5$

As we want to compare the result of the quantum mechanical reduction of the HKP to the quantum theory of the KP$_5$, we shall first carry out the quantization

$^1$The expressions for $H$, $M_i$ and $H_{ab}$ are weakly equal to the pull-backs of the corresponding quantities in the KP$_5$ by the transformation $\hbar$ and will be denoted by the same symbols.
of the latter. This will be done in the Schrödinger representation on the Hilbert space $\mathcal{H}_5 = L^2(\mathbb{R}^5, d^5q)$. The operators for position and momentum are

$$ Q_i = q_i \quad P_i = -i\hbar \partial_i = -i\hbar \frac{\partial}{\partial q_i} \quad [Q_i, P_j] = i\hbar \delta_{ij}. $$

(54)

For the operators $H$ and $L_{ij}$ there are no factor ordering problems, the factor ordering for the $M_i$ can be fixed by requiring that the commutator algebra of $H, L_{ij}$ and $M_i$ be isomorphic to the classical Poisson algebra. The so-obtained expressions

$$ H = \frac{1}{2} P^2 - \frac{k}{Q} \quad L_{ij} = Q_i P_j - Q_j P_i $$

(55)

$$ M_i = Q_i \left( \frac{1}{2} P^2 + H \right) - (Q \cdot P) P_i + 2i\hbar P_i $$

(56)

($Q = \sqrt{Q^2} = \sqrt{q^2}$) with the commutation relations

$$ [H, L_{ij}] = 0 = [H, M_i] \quad [L_{ij}, L_{ik}] = i\hbar L_{jk} $$

(57)

$$ [L_{ij}, M_i] = i\hbar M_j \quad [M_i, M_j] = i\hbar (-2H)L_{ij} $$

(58)

are simultaneously hermitian with respect to the measure $d^5q$. The classical identities (12) for the Casimirs of the algebra $\mathcal{S}$ acquire quantum corrections of order $\hbar^2$

$$ \tilde{C}_2 = M^2 - 2HL^2 \equiv k^2 + 8\hbar^2 H \quad \tilde{C}_3 = 6 \varepsilon_{ijklm}L_{ij}L_{kl}M_m \equiv 0 $$

(59)

$$ \tilde{C}_4 = 2H^2L_{ij}L_{jk}L_{kl}L_{li} + \frac{1}{2} \left( M_iM_iM_jM_j + M_iM_jM_jM_i \right) + $$

$$ + H \left( M_iL_{ij}L_{jk}M_k + M_iM_jL_{jk}L_{ki} + L_{ij}M_jM_kL_{ki} + L_{ij}L_{jk}M_kM_i \right) $$

$$ \equiv \tilde{C}_2^2 - 12\hbar^2 H\tilde{C}_2 $$

(60)

$$ \tilde{C}_4^{(0)} = M^2L^2 - \frac{1}{4} (M_iL_{ij} + L_{ij}M_i)(M_kL_{kj} + L_{kj}M_k) \equiv -4\hbar^2 \tilde{C}_2. $$

(61)

Consequently, the symmetry algebra of the quantized KP$_5$ is the algebra $\mathcal{S}$ together with these modified relations. Note that the terms in the defining expression for $\tilde{C}_4$ do not reflect factor ordering ambiguities but are uniquely determined by the ordering of the terms according to the original classical expression $\tilde{C}_4 = \frac{1}{2} l_{\mu \nu} l_{\nu \rho} l_{\rho \sigma} l_{\sigma \mu}$.

As the symmetry algebra does not contain all the dynamical information, we shall in addition require that the dynamical algebra generated by $H_{ab}$ be represented on $\mathcal{H}_5$ as a commutator algebra. By this requirement the (hermitian) expressions for the operators $H_{ab}$ are found to be

$$ H_{i6} = \frac{1}{2} Q_i P^2 - (Q \cdot P) P_i + i\hbar \frac{5}{2} P_i - \hbar^2 \frac{1}{8} \frac{Q_i}{Q^2} \quad H_{i7} = Q_i $$

(62)

\footnote{The five-dimensional generalizations of the operators $L_{ab}$ of ref. [2], which are hermitian with respect to the measure $q^{-1} d^5q$, can be obtained from our $H_{ab}$ (resp. $G_{ab}$, which are formed from $H_{ab}$ as above (13)) by replacing $P_i$ with $\Pi_i = -i\hbar (\partial_i - \frac{\partial}{2q})$. The operators $\Pi_i$ are canonically conjugate to the $Q_j$ and hermitian with respect to $q^{-1} d^5q$.}
They are explicitly given by

\[ H_{i8} = QP_i - i\hbar \frac{1}{2} \frac{Q_i}{Q} \]

\[ H_{67} = Q \cdot P - i\hbar \frac{5}{2} \]

\[ H_{68} = \frac{1}{2} Q P^2 - i\hbar \frac{1}{2} \frac{1}{Q} Q \cdot P - \hbar^2 \frac{7}{8} \frac{1}{Q} \]

\[ H_{78} = Q. \]

Note that a mere symmetrization of the classical expressions would not have resulted in the desired closing algebra.

In order to determine the energy spectrum and a basis of \( \mathcal{H}_5 \) which is well adapted to the representations of the symmetry algebra, we shall separate the Schrödinger equation in the Euler coordinates defined by (22). In these coordinates the wave functions are the simultaneous eigenfunctions of the complete set of commuting observables \( H, L^2, J^2 = 2(J_1^2 + J_2^2) \), \( J_{13} \) and \( J_{23} \), where the angular momentum vectors

\[ \vec{J}_1 = \frac{1}{2} \begin{pmatrix} L_{12} + L_{34} \\ L_{13} + L_{42} \\ L_{14} + L_{23} \end{pmatrix} \]

\[ \vec{J}_2 = \frac{1}{2} \begin{pmatrix} L_{12} - L_{34} \\ L_{13} - L_{42} \\ L_{14} - L_{23} \end{pmatrix} \]

generate an SO(4) subgroup of SO(5). The angular wave functions \( Y^l_{jm_1 m_2}(\chi, \varphi, \theta, \psi) \) are labelled according to the group chain \( \text{SO}(5) \supset \text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2) \supset \text{U}(1) \times \text{U}(1) \), and satisfy the eigenvalue equations

\[ J_{13} Y^l_{jm_1 m_2} = -i\hbar \partial_\varphi Y^l_{jm_1 m_2} = -\hbar m_1 Y^l_{jm_1 m_2} \]

\[ J_{23} Y^l_{jm_1 m_2} = -i\hbar \partial_\psi Y^l_{jm_1 m_2} = -\hbar m_2 Y^l_{jm_1 m_2} \]

\[ J^2 Y^l_{jm_1 m_2} = -4\hbar^2 (\partial^2_\varphi + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} (\partial^2_\theta + \partial^2_\psi - 2 \cos \theta \partial_\varphi \partial_\psi)) Y^l_{jm_1 m_2} \]

\[ = 4\hbar^2 j(j+1) Y^l_{jm_1 m_2} \]

\[ L^2 Y^l_{jm_1 m_2} = -\hbar^2 (\partial^2_\chi + 3 \cot \chi \partial_\chi - \frac{1}{\sin^2 \chi} \frac{1}{\hbar^2} J^2) Y^l_{jm_1 m_2} \]

\[ = \hbar^2 l(l+3) Y^l_{jm_1 m_2}. \]

They are explicitly given by

\[ Y^l_{jm_1 m_2}(\chi, \varphi, \theta, \psi) = N^l_j \sin^{2j} \chi C^{2j+\frac{1}{2}}_{i-2j}(\cos \chi) D^l_{m_1 m_2}(\varphi, \theta, \psi) \]

where \( N^l_j \) is a normalization constant, \( C^{2j+\frac{1}{2}}_{i-2j}(z) \) are Gegenbauer polynomials and

\[ D^l_{m_1 m_2}(\varphi, \theta, \psi) = e^{-im_1 \varphi} P^l_{m_1 m_2}(\cos \theta) e^{-im_2 \psi} \]

is the \((m_1, m_2)\)-matrix element in the representation \( D^l \) of the element \( g \in \text{SU}(2) \), which is described by the Euler angles \((\varphi, \theta, \psi)\) (cf. \[n\]).

The energy spectrum and the radial wave functions \( R_{N_l}(q) \) are determined by the radial Schrödinger equation

\[ \left( -\frac{\hbar^2}{2} \left( \frac{\partial^2_q}{q^2} + \frac{4}{q} \frac{\partial_q}{q} - \frac{l(l+3)}{q^2} \right) - \frac{k}{q} \right) R_{N_l}(q) = E_N R_{N_l}(q). \]
The regular (as \( q \to 0 \)) solutions of this equation can be obtained in the same way as in the three-dimensional case (cf. [8]). For \( E < 0, = 0 \) and \( > 0 \) they are proportional to Laguerre polynomials, Bessel functions and confluent hypergeometric functions respectively (see [7])

\[
R_{nl}(q) = N_{nl} e^{-\frac{\rho}{2}} \rho^l L_{n-l-1}^{2l+3}(\rho) \quad \rho = \frac{2k}{\hbar^2(n+1)} q \quad (73)
\]

\[
R_{0l}(q) = N_{0l} \rho^{-\frac{3}{2}} J_{2l+3}(\sqrt{\rho}) \quad \rho = \frac{8k}{\hbar^2} q \quad (74)
\]

\[
R_{\nu l}(q) = N_{\nu l} e^{-\frac{\rho}{2}} \rho^l {_1F_1}(l + 2 + i\nu, 2l + 4; \rho) \quad \rho = i \frac{2k}{\hbar^2 \nu} q. \quad (75)
\]

The \( N_{Nl} \) are normalization constants. Of course, for \( E \geq 0 \), the wave functions are improper eigenstates of \( H \) and can only be normalized to delta functions. The energy spectrum is of the form

\[
E_n = -\frac{k^2}{2\hbar^2 (n+1)^2} \quad E_0 = 0 \quad E_{\nu} = \frac{k^2}{2\hbar^2 \nu^2}. \quad (76)
\]

The range of the quantum numbers \( N = (n, 0, \nu), l, j, m_1 \) and \( m_2 \), needed to uniquely label the states, is as follows

\[
0 < n \in \mathbb{N}, \quad 0 < \nu \in \mathbb{R} \quad (77)
\]

\[
0 \leq l \leq n - 1 \quad (E < 0), \quad 0 \leq l \quad (E \geq 0) \quad (78)
\]

\[
0 \leq 2j \leq l, \quad -j \leq m_1, m_2 \leq j. \quad (79)
\]

\( l \) is integer, \( m_1 \) and \( m_2 \) are, simultaneously with \( j \), both integers or both half integers. With respect to the measure \( d^5q \) the states are orthogonal in all quantum numbers.

### 5.1 Group theoretical considerations

The Hamiltonian \( H \) of the KP5 possesses an obvious symmetry under the canonical action of the group \( SO(5) \) of rotations in \( \mathbb{R}^5 \). This means, that its eigenvalues cannot depend on the quantum numbers \( j, m_1 \) and \( m_2 \), which label the states within an irreducible representation of \( SO(5) \). But the energy eigenvalues are also degenerate in the angular momentum quantum number \( l \) and the eigenspaces of \( H \) are not irreducible with respect to \( SO(5) \). This “accidental” degeneracy is due to the higher symmetry of the KP5, which reflects itself in the existence of the additional conserved vector \( M \), and can be explained by an invariance of the Hamiltonian under the groups \( SO(6), E(5) \) and \( SO(5,1) \) for \( H < 0, H = \)
0 or \( H > 0 \) respectively (cf. [1]). Consequently, the eigenspaces of \( H \) carry irreducible representations of the three groups. However, not all the irreducible representations of the three groups do occur as energy eigenspaces. The physically realized representations are selected by the relations between the Casimir operators of the symmetry algebra, which at the same time fix their eigenvalues and allow to express the spectrum of \( H \) by the group quantum numbers. In the following we shall determine the relevant unitary irreducible representations (UIR) of the above groups, which simultaneously furnish the irreducible representations of the symmetry algebra of the KP\(_5\).

As \( H \) is central in \( S \), it must be represented by a multiple of unity in the irreducible representations of \( S \). In the eigenspaces of \( H \), i.e. for \( H = E = \text{const.} \), the algebra \( S \) can be reduced to a trivial central extension of the Lie algebras so(6), e(5) and so(5,1) by replacing \( M_i \) with

\[
\tilde{M}_i = L_{i6} = -L_{6i} = \begin{cases} 
(\frac{-2H}{\sqrt{2}})^\frac{1}{2} M_i & E < 0 \\
M_i & E = 0. \\
(\frac{2H}{\sqrt{2}})^\frac{1}{2} M_i & E > 0 
\end{cases}
\]  

(80)

The commutation relations of the operators \( L_{\mu\nu} \) \((1 \leq \mu, \nu \leq 6)\) read

\[
[L_{\mu\nu}, L_{\rho\sigma}] = i\hbar \, g_{\mu\rho} L_{\nu\sigma} \quad (81)
\]

\[
g_{\mu\nu} = \text{diag}(1, 1, 1, 1, 1, \varepsilon), \quad \varepsilon = -\text{sign}(E). \quad (82)
\]

Upon this reduction the identities (59)-(61), which hold for the Casimirs of \( S \), induce relations for the Casimirs of the three Lie algebras. Therefore, the hermitian irreducible representations of the symmetry algebra of the KP\(_5\) are uniquely determined by those irreducible representations of the Lie algebras so(6), e(5) and so(5,1), in which these induced relations are satisfied, and which correspond to UIR of the groups SO(6), E(5) and SO(5,1).

The three cases will be treated separately. For the UIR of the groups SO(6), E(5) and SO(5,1) and the eigenvalues of their Casimir operators consult the appendix.

- \( E < 0 \)

For negative energies we have the relations

\[
C_2 = \frac{1}{2\hbar^2} \, L_{\mu\nu} L_{\mu\nu} = \frac{1}{\hbar^2} \, (L^2 + \tilde{M}^2) \equiv -\frac{1}{2\hbar^2} \, (k^2 + 8\hbar^2H) \quad (83)
\]

\[
C_3 = \frac{1}{\hbar^2} \, \varepsilon_{\mu\rho\sigma\tau\nu} L_{\mu\nu} L_{\rho\sigma} L_{\tau\nu} = \frac{6}{\hbar^2} \, \varepsilon_{ijklm} L_{ij} L_{kl} \tilde{M}_m \equiv 0 \quad (84)
\]

\[
C_4 = \frac{1}{2\hbar^4} \, L_{\mu\nu} L_{\nu\rho} L_{\rho\sigma} L_{\sigma\mu} \equiv C_2^2 + 6 \, C_2. \quad (85)
\]

\(^3\text{E(5) is the group of motions in five-dimensional Euclidean space, also called the inhomogeneous rotation group ISO(5).}\)
The last two identities require that the SO(6) quantum numbers $\mu_2$ and $\mu_3$ (cf. (A 15)-(A 17)) be zero, the first one allows to express the energy by the eigenvalue of the Casimir $C_2$

$$H = E = -\frac{k^2}{2\hbar^2(C_2 + 4)} = -\frac{k^2}{2\hbar^2(\mu_1 + 2)^2}. \quad (86)$$

This means that the energy quantum number $n$ is connected to the SO(6) quantum number $\mu_1$ via $n = \mu_1 + 1$. The $H$–eigenspace belonging to energy $E_n$ carries the representation $D(n,0,0)$ of SO(6) and the branching rules for $SO(6) \supset SO(5)$

$$\mu_1 \geq l \geq 0 \iff n - 1 \geq l \geq 0 \quad (87)$$

(see (A 3), $m_{5,1} = \mu_1$, $m_{5,2} = \mu_2 = 0$, $m_{4,1} = l$) correctly account for the "accidental" degeneracy.

- $E = 0$

For $E = 0$ the relations

$$C_2 = \frac{1}{k^2} L_{i6} L_{i6} = \frac{1}{k^2} \tilde{M}^2 \equiv 1 \quad (88)$$

$$C_3 = \frac{1}{k^2} \varepsilon_{\mu\nu\rho\tau\upsilon} L_{\mu\nu} L_{\rho\sigma} L_{\tau\upsilon} = \frac{6}{k^2} \varepsilon_{ijklm} L_{ij} L_{kl} \tilde{M}_m \equiv 0 \quad (89)$$

$$C_4^{(0)} = \frac{1}{k^2 \hbar^2} (L^2 \tilde{M}^2 - \frac{1}{4} (\tilde{M}_i L_{ij} + L_{ij} \tilde{M}_i)(\tilde{M}_k L_{kj} + L_{kj} \tilde{M}_k)) \equiv -4 C_2 \quad (90)$$

can only be satisfied by putting $\sigma = 1$, $\mu_2 = \mu_3 = 0$ (cf. (A 21)-(A 23)). The 0–eigenspace of $H$ carries the representation $D(1,0,0)$ of $E(5)$, and the accidental degeneracy is explained by the branching rules for $E(5) \supset SO(5)$:

$$l \geq 0 \quad (91)$$

($m_{5,1} = \infty$, $m_{5,2} = \mu_2 = 0$, $m_{4,1} = l$ in (A 3)).

- $E > 0$

For $E > 0$ we have ($\tilde{L}_{ij} = L_{ij}$, $\tilde{L}_{i6} = -iL_{i6} = -i\tilde{M}_i$)

$$C_2 = \frac{1}{2\hbar^2} \tilde{L}_{\mu\nu} \tilde{L}_{\mu\nu} = \frac{1}{\hbar^2} (L^2 - \tilde{M}^2) \equiv -\frac{1}{2\hbar^2} (k^2 + 8\hbar^2 H) \quad (92)$$

$$C_3 = \frac{i}{\hbar^3} \varepsilon_{\mu\nu\rho\sigma\tau\upsilon} \tilde{L}_{\mu\nu} \tilde{L}_{\rho\sigma} \tilde{L}_{\tau\upsilon} = \frac{6}{\hbar^3} \varepsilon_{ijklm} L_{ij} L_{kl} \tilde{M}_m \equiv 0 \quad (93)$$

$$C_4 = \frac{1}{2\hbar^4} \tilde{L}_{\mu\nu} \tilde{L}_{\upsilon\rho} \tilde{L}_{\sigma\mu} \equiv C_2^2 + 6 C_2. \quad (94)$$
The requirement $H > 0$, together with the identity (92), restricts us to the representations of the principal series. Within the principal series the relations (92)-(94) are only compatible with $\mu_1 = \mu_2 = -1$, $\mu_3 = i\tau$, $\tau > 0$ (cf. (A 15)-(A 17) and the remark below). Solving equation (92) for $H$, we obtain
\[ H = E = -\frac{k^2}{2\hbar^2(C_2 + 4)} = \frac{k^2}{2\hbar^2 \tau^2}. \] (95)

The energy quantum number $\nu$ is equal to the SO(5,1) quantum number $\tau$ and the eigenspace of $H$ corresponding to energy $E_\nu$ carries the representation $D(p; -1, -1, i\nu)$ of SO(5,1). Again, the “accidental” degeneracy is correctly reproduced by the branching rules for SO(5,1) $\supset$ SO(5), which require that $l \geq 0$.

As the range of the quantum numbers $j$, $m_1$ and $m_2$ is in accordance with the branching rules for the subgroup chain SO(5) $\supset$ SO(4) $\cong$ SU(2) $\times$ SU(2) $\supset$ U(1) $\times$ U(1) in the representation $D(l,0)$ of SO(5) (cf. the appendix), we see that the quantum numbers $N$, $l$, $j$, $m_1$ and $m_2$ uniquely label the states within the above UIR of the groups SO(6), E(5) and SO(5,1).

6 Quantization of the Hurwitz-Kepler problem

In this section we will perform the quantization of the HKP and its reduction according to the algebraic constraint quantization scheme. The first step is the quantization of the extended system, without imposing the constraints, and the determination of the irreducible representations of the symmetry algebra which span the Hilbert space. The second step consists in the identification of the physical representations and of the physical Hilbert space. The resulting quantum theory will be compared to that of the KP$_5$.

6.1 Hilbert space and observables

The quantization of the HKP will be performed in the Schrödinger representation
\[ u_a \rightarrow u_a \quad v_a \rightarrow -i\hbar \partial_a = -i\hbar \frac{\partial}{\partial u_a} \] (96)
on the Hilbert space $\mathcal{H}_8 = L^2(\mathbb{R}^8, d\mu(u))$, which represents the appropriate kinematical setting corresponding to the symplectic structure with respect to which the classical HKP is defined. The measure $d\mu(u)$ will be determined later on by physical requirements.

As the fundamental subalgebras of the algebra of observables to be quantized we shall choose the symmetry algebra $\mathcal{S}$ and the dynamical algebra generated by
\( H_{ab} \). As the latter linearly encodes the structure of the algebra of observables, we shall require that the corresponding commutator algebras be isomorphic to the classical Poisson algebras. For the symmetry algebra this can be achieved by simply putting all \( v \)'s to the right of all \( u \)'s in the classical expressions and replacing them with operators, for the \( H_{ab} \) we obtain the more complicated expressions

\[
\begin{pmatrix}
H_{16} - iH_{46} & -H_{36} - iH_{26} \\
H_{36} - iH_{26} & H_{16} + iH_{46}
\end{pmatrix} = -\frac{1}{4} (-i\hbar)^2 D_2 D_1 + i\hbar \left( \frac{-i\hbar}{4U^2} (U_2 D_1 + D_2 U_1) - \hbar^2 \frac{1}{4U^4} U_2 U_1 \right)
\]

(97)

\[
\begin{pmatrix}
H_{17} - iH_{47} & -H_{37} - iH_{27} \\
H_{37} - iH_{27} & H_{17} + iH_{47}
\end{pmatrix} = 2U_2 U_1
\]

(98)

\[
\begin{pmatrix}
H_{18} - iH_{48} & -H_{38} - iH_{28} \\
H_{38} - iH_{28} & H_{18} + iH_{48}
\end{pmatrix} = \frac{1}{2} (-i\hbar) (U_2 D_1 + D_2 U_1) - i\hbar \frac{1}{U^2} U_2 U_1
\]

(99)

\[
H_{56} E_2 = -\frac{1}{8} (-i\hbar)^2 (D_1^\dagger D_1 - D_2^\dagger D_2) + i\hbar \left( \frac{-i\hbar}{4U^2} \text{Re} (U_1^\dagger D_1 - U_2 D_2^\dagger) - \hbar^2 \frac{1}{8U^4} (U_1^\dagger U_1 - U_2 U_2^\dagger) \right)
\]

(100)

\[
H_{57} E_2 = U_1^\dagger U_1 - U_2 U_2^\dagger
\]

(101)

\[
H_{58} E_2 = \frac{1}{2} (-i\hbar) \text{Re} (U_1^\dagger D_1 - U_2 D_2^\dagger) - i\hbar \frac{1}{2U^2} (U_1^\dagger U_1 - U_2 U_2^\dagger)
\]

(102)

\[
H_{67} E_2 = \frac{1}{2} (-i\hbar) \text{Re} (U_1^\dagger D_1 + U_2 D_2^\dagger) - i\hbar \frac{5}{2} E_2
\]

(103)

\[
H_{68} E_2 = \frac{1}{8} (-i\hbar)^2 (D_1^\dagger D_1 + D_2^\dagger D_2) - i\hbar \left( \frac{-i\hbar}{4U^2} \text{Re} (U_1^\dagger D_1 + U_2 D_2^\dagger) - \hbar^2 \frac{7}{8U^2} E_2 \right)
\]

(104)

\[
H_{78} E_2 = U_1^\dagger U_1 + U_2 U_2^\dagger
\]

(105)

where

\[
D_1 = \begin{pmatrix}
\partial_1 - i\partial_4 & -\partial_3 - i\partial_2 \\
-\partial_3 - i\partial_2 & \partial_1 + i\partial_4
\end{pmatrix}, \quad D_2 = \begin{pmatrix}
\partial_5 - i\partial_8 & -\partial_7 - i\partial_6 \\
-\partial_7 - i\partial_6 & \partial_5 + i\partial_8
\end{pmatrix}.
\]

(106)

Although it is possible to consistently quantize the constraint algebra in the same way as the symmetry algebra, we will not make use of it except for comparative purposes.

In order to find the operator corresponding to the observable \( \vec{K}^2 \), the Casimir of the constraint algebra, we start from the classical identity \( \mathcal{C}_3 = 48k \vec{K}^2 \). For the theory to possess the correct classical limit, in leading order in \( \hbar \) this identity must be reproduced. The possible correction terms must contain explicit powers
of \( \bar{h} \) and be polynomials in \( H, \tilde{C}_2, \tilde{K}^2 \) and \( k \). The only polynomial combination of these elements with the correct dimensions is the term \( \bar{h}^2 k \). As can easily be seen from the defining expression for \( \tilde{C}_3 \), such a term cannot occur in it. Therefore we conclude that the relation between \( \tilde{C}_3 \) and \( \tilde{K}^2 \) remains unaltered. Using the so-obtained expression for \( \tilde{K}^2 \), which coincides with the one that can be inferred from the quantization of the constraint algebra, the quantum analogs of the relations (107)-(108) for the Casimirs of the symmetry algebra are found to be (the defining expressions are the same as in (59)-(61))

\[
\begin{align*}
\tilde{C}_2 &\equiv k^2 - 4H\tilde{K}^2 + 8\bar{h}^2H = \tilde{C}_3 \equiv 48k\tilde{K}^2 \\
\tilde{C}_4 &\equiv \tilde{C}_2^2 + 8H\tilde{K}^2\tilde{C}_2 + 24H^2(\tilde{K}^2)^2 - 12\bar{h}^2H\tilde{C}_2 - 48\bar{h}^2H^2\tilde{K}^2 \\
\tilde{C}_4^{(0)} &\equiv 2\tilde{K}^2\tilde{C}_2 - 4\bar{h}^2\tilde{C}_2.
\end{align*}
\]

The symmetry algebra of the quantized HKP is the algebra \( \mathcal{S} \), supplemented by these relations. For \( \tilde{K}^2 = 0 \) it becomes isomorphic to the symmetry algebra of the quantized KP.

The measure \( d\mu(u) \) can be determined by two physically motivated requirements: i) it should be invariant under the action of the linear (in \( V \)) part of the symmetry algebra, generated by \( L_{ij} \); ii) the observables should be hermitian with respect to it. The first requirement can be satisfied by putting

\[
d\mu(u) = c |u|^n d^8u, \quad c = \text{const.},
\]

the second can then only be fulfilled if \( n = 2 \). Thus we are led to choose the space \( \mathcal{H}_8 = L^2(\mathbb{R}^8, cu^2 d^8u) \) as the extended Hilbert space for the HKP.

A basis of \( \mathcal{H}_8 \) which is well suited for the explicit construction of the representations of the symmetry algebra and which facilitates the comparison of the quantum theory of the HKP to that of the KP can be obtained by separating the Schrödinger equation in the bi-Euler coordinates (20). In these coordinates the Laplace operator has the form

\[
\Delta^{(8)} = \partial^2_v + \frac{7}{v} \partial_v + \frac{4}{v^2} \left( \partial^2_\chi + 3 \cot \chi \partial_\chi + \frac{1}{\cos^2 \chi} \frac{1}{2} \Delta_{\partial_1 \varphi_1 \psi_1} + \frac{1}{\sin^2 \chi} \frac{1}{2} \Delta_{\partial_2 \varphi_2 \psi_2} \right)
\]

\[
= \partial^2_v + \frac{7}{v} \partial_v - \frac{1}{v^2} F^2
\]

\[
\Delta_{\partial_\varphi \psi} = \partial^2_\varphi + \cot \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \left( \partial^2_\varphi + \partial^2_\psi - 2 \cos \vartheta \partial_\varphi \partial_\psi \right)
\]

\((v = |u|)\). Besides the energy quantum number, there are seven more quantum numbers needed to label the states, corresponding to the group chain \( \text{SO}(8) \supset \text{SO}(4) \times \text{SO}(4) \supset \text{U}(1) \times \text{U}(1) \times \text{U}(1) \times \text{U}(1) \), where \( \text{SO}(4) \times \text{SO}(4) \) is generated by the angular momentum vectors \( (F_{ab} = -i\hbar (u_a \partial_b - u_b \partial_a)) \)

\[
\begin{align*}
\vec{I}_1 &= \frac{1}{2} \begin{pmatrix} F_{12} + F_{34} \\ F_{13} + F_{42} \\ F_{14} + F_{23} \end{pmatrix} \\
\vec{I}_2 &= \frac{1}{2} \begin{pmatrix} F_{12} - F_{34} \\ F_{13} - F_{42} \\ F_{14} - F_{23} \end{pmatrix}
\end{align*}
\]

\(17\)
\[
\tilde{N}_1 = \frac{1}{2} \begin{pmatrix} F_{56} + F_{78} \\ F_{57} + F_{86} \\ F_{58} + F_{67} \end{pmatrix}, \quad \tilde{N}_2 = \frac{1}{2} \begin{pmatrix} F_{56} - F_{78} \\ F_{57} - F_{86} \\ F_{58} - F_{67} \end{pmatrix}.
\]

The angular wave functions \(Y_{j_1 m_1 j_2 m_2}^f(\chi, \varphi_1, \vartheta_1, \psi_1, \varphi_2, \vartheta_2, \psi_2)\) are the simultaneous eigenfunctions of the mutually commuting operators \(F^2, I^2 = 2 (\tilde{I}_1^2 + \tilde{I}_2^2)\), \(N^2 = 2 (\tilde{N}_1^2 + \tilde{N}_2^2)\), \(I_{13}, I_{23}, N_{13}\) and \(N_{23}\) with the eigenvalues

\[
\begin{align*}
F^2 Y^f &= h^2 f(f + 6) Y^f \\
I^2 Y^f &= -4 h^2 \Delta_{\varphi_1 \varphi_1'} Y^f = 4 h^2 j_1(j_1 + 1) Y^f \\
N^2 Y^f &= -4 h^2 \Delta_{\vartheta_2 \vartheta_2'} Y^f = 4 h^2 j_2(j_2 + 1) Y^f \\
I_{13} Y^f &= -i h \partial_{\varphi_1} Y^f = -\hbar m_{11} Y^f \\
I_{23} Y^f &= -i h \partial_{\psi_1} Y^f = -\hbar m_{12} Y^f \\
N_{13} Y^f &= -i h \partial_{\varphi_2} Y^f = -\hbar m_{21} Y^f \\
N_{23} Y^f &= -i h \partial_{\psi_2} Y^f = -\hbar m_{22} Y^f.
\end{align*}
\]

Their explicit form is

\[
Y_{j_1 m_1 j_2 m_2}^f = \frac{N_{j_1 j_2}}{j_1 j_2} (1 + \cos \chi)^{j_1} (1 - \cos \chi)^{j_2} \left( \frac{P_{2 j_2 + 1}^{(2 j_2 + 1, 2 j_1 + 1)}}{2 - j_1 - j_2} \right) \times
\]

\[(\cos \chi) \left( \frac{\partial^2_{\varphi_1} + \frac{4}{q} \partial_q - \frac{1}{q^2} \frac{f(\hat{f} f)}{2} + 3} {\frac{k}{q}} \right) R_{N_{4 \frac{f}{2}}} = E_{N} R_{N_{4 \frac{f}{2}}} (q)
\]

with \(l\) replaced by \(\frac{l}{2}\). Correspondingly, the spectrum and the radial wave functions coincide with those of the KP$_5$, but for negative energies the quantum number \(n\) can now take on also half integer values. Simultaneously with \(2n\), \(f\) has to be even or odd. The quantum numbers \(N = (n, 0, \nu)\), \(f, j_1, j_2, m_{11}, m_{12}, m_{21}\) and \(m_{22}\) obey the inequalities

\[
2 \leq 2n \in \mathbb{N}, \quad 0 < \nu \in \mathbb{R}
\]

\[
\begin{aligned}
0 &\leq f \leq 2n - 2 & \text{\(n\) integer, \(f\) even} \\
1 &\leq f \leq 2n - 1 & \text{\(n\) half integer, \(f\) odd}
\end{aligned}
\] \(H < 0\), \quad 0 \leq f \in \mathbb{N} \quad (H \geq 0)

\[
\begin{aligned}
0 &\leq j_1 + j_2 \leq f & \text{\(f\) even, \(j_1 + j_2\) integer} \\
\frac{1}{2} &\leq j_1 + j_2 \leq \frac{f}{2} & \text{\(f\) odd, \(j_1 + j_2\) half integer}
\end{aligned}
\]

\(j_i \leq m_{1i}, m_{i2} \leq j_i\).  \(j_i\) (126)

With respect to the measure \(u^2 d^8 u\) the wave functions are orthogonal in all quantum numbers.
6.2 Representations of the symmetry algebra

Just as in the case of the KP, in the eigenspaces of $H$ the symmetry algebra of the HKP can be reduced to a trivial central extension of the Lie algebras so(6), e(5) or so(5,1), and its irreducible representations can be inferred from the UIR of the groups SO(6), E(5) and SO(5,1). The relevant representations, from which the Hilbert space $\mathcal{H}_8$ is made up, and the spectrum of $\vec{K}^2$ are determined by the relations (107)-(109) between the Casimirs of the symmetry algebra and the Casimir of the constraint algebra.

In order to obtain the spectrum of $\vec{K}^2$, we start from the identities for the Casimirs of the Lie algebra e(5) which are induced by the relations (107)-(109) (the definitions of the generators and the Casimirs of the three Lie algebras are the same as in the case of the KP)

$$C_2 \equiv 1 \quad C_3 \equiv \frac{48}{\hbar^2} \vec{K}^2 \quad C_4^{(0)} \equiv \frac{2}{\hbar^2} \vec{K}^2 C_2 - 4 C_2.$$  \hspace{1cm} (127)

The first relation requires that $\sigma = 1$, the second and third that $\mu_2 = \mu_3 =: K$ (cf. (A 21)-(A 23)). Therefore, from the second relation, the spectrum of $\vec{K}^2$ is of the form $\hbar^2 K (K + 1)$, $2K \in \mathbb{N}$, in accordance with the spectrum of the Casimir of the constraint algebra su(2). The zero-eigenspace $\mathcal{H}_0$ of $H$ decomposes into a direct sum of UIR of E(5) according to

$$\mathcal{H}_0 = \bigoplus_{2K=0}^{\infty} D(1; K, K).$$  \hspace{1cm} (128)

Similarly, for negative and positive eigenvalues of $H$ the eigenspaces $\mathcal{H}_N$ decompose into direct sums of UIR of the groups SO(6) and SO(5,1), and the spectrum can be expressed by the group quantum numbers. We shall indicate below the induced identities for the Casimirs of the Lie algebras so(6) and so(5,1), the form of the energy spectrum, and the representations which are contained in the corresponding energy eigenspaces.

- $E < 0$ : $(\vec{K}^2 = \frac{1}{\hbar^2} \vec{K}^2)$

$$C_2 \equiv - \frac{1}{2 \hbar^2 H} (k^2 - 4 H \vec{K}^2 + 8 \hbar^2 H)$$  \hspace{1cm} (129)

$$C_3 \equiv \frac{48 k}{\hbar^3 \sqrt{-2 H}} \vec{K}^2$$  \hspace{1cm} (130)

$$C_4 \equiv C_2^2 + 6 C_2 - 4 \vec{K}^2 C_2 - 12 \vec{K}^2 + 6 (\vec{K}^2)^2$$  \hspace{1cm} (131)

$$H = E = - \frac{k^2}{2 \hbar^2 (C_2 + 4 - 2 \vec{K}^2)} = - \frac{k^2}{2 \hbar^2 (\mu_1 + 2)^2}$$  \hspace{1cm} (132)

$$\mathcal{H}_n = \bigoplus_{K=0}^{n-1} D(n-1, K, K) \quad \mu_1 = n - 1 \text{ integer}$$  \hspace{1cm} (133)
\[ \mathcal{H}_n = \bigoplus_{K = \frac{1}{2}, \ldots, n-1} D(n-1, K, K) \quad \mu_1 = n - 1 \text{ half integer} \quad (134) \]

- \( E > 0 \)

\[ C_2 \equiv -\frac{1}{2 \hbar^2 H} (k^2 - 4 \hbar \vec{K}^2 + 8 \hbar^2 H) \quad (135) \]
\[ C_3 \equiv \frac{48 k}{\hbar^4 \sqrt{2} H} \vec{K}^2 \quad (136) \]
\[ C_4 \equiv C_2^0 + 6 \vec{K}^2 C_2 - 12 \vec{K}^2 + 6 (\vec{K}^2)^2 \quad (137) \]
\[ H = E = -\frac{k^2}{2 \hbar^2(C_2 + 4 - 2 \vec{K}^2)} = \frac{k^2}{2 \hbar^2 \tau^2} \quad (138) \]
\[ \mathcal{H}_\nu = \bigoplus_{2K = 0} D(p; -1 + K, -1 + K, \nu) \quad \tau = \nu, \tau > 0. \quad (139) \]

As can be seen from the above decomposition of the eigenspaces of \( H \), the Hilbert space \( \mathcal{H}_8 \) is the direct sum (integral) of all those UIR of the groups SO(6), E(5) and SO(5,1), which are compatible with the relations (107)-(109), i.e. of all hermitian irreducible representations of the symmetry algebra of the HKP. The basis states for the corresponding representation spaces, which diagonalize the mutually commuting operators \( H, L^2, J^2 = 2 (J_1^2 + J_2^2), \vec{J}^2 = 8 (\vec{J}_1^2 - \vec{J}_2^2), J_{13}, J_{23}, \vec{K}^2 \) and \( K_3 \) (the vectors \( \vec{J}_i \) are defined as in (134)), can easily be constructed from the basis of \( \mathcal{H}_8 \) as given in the previous paragraph.

First of all, observe that the vectors \( \vec{J}_1 \) and \( \vec{K} \) can be expressed as linear combinations of the vectors \( \vec{I}_i \) and \( \vec{N}_i \)

\[ \vec{J}_1 = \vec{N}_1, \quad \vec{J}_2 = \vec{I}_2, \quad \vec{K} = \vec{I}_1 - \vec{N}_2. \quad (140) \]

Therefore, and because of the identities \( \vec{I}_1^2 = \vec{I}_2^2, \vec{N}_1^2 = \vec{N}_2^2 \), the angular wave functions \( Y_{j_1 m_{11} j_2 m_{22} m_{22}} \) are already eigenfunctions of the operators \( J^2, \vec{J}_2, J_{13}, J_{23} \) and \( K_3 \) with the eigenvalues \( 2 \hbar^2(j_1(j_1 + 1) + j_2(j_2 + 1)), 8 \hbar^2(j_1(j_1 + 1) - j_2(j_2 + 1)), -\hbar m_{21}, -\hbar m_{12} \) and \( -\hbar (m_{11} - m_{22}) \). To get the eigenfunctions of \( \vec{K}^2 \), we simply have to apply the rules for the addition of angular momenta. They can be written as the Clebsch–Gordan series

\[ Y_{j_1 m_1 j_2 m_2 K M}^l := \sum_{m_{11} = -j_1}^{j_1} \sum_{m_{22} = -j_2}^{j_2} \langle j_1 m_{11} j_2 (-m_{22}) | K M \rangle Y_{j_1 m_1 j_2 m_2 m_{22}}^{2l} \quad (141) \]

\((f = 2l, m_1 = m_{21}, m_2 = m_{12}, M = m_{11} - m_{22}, |j_1 - j_2| \leq K \leq j_1 + j_2)\) and satisfy the eigenvalue equations

\[ \vec{K}^2 Y_{j_1 m_1 j_2 m_2 K M}^l = \hbar^2 K(K + 1) Y_{j_1 m_1 j_2 m_2 K M}^l \quad (142) \]
\[ K_3 Y_{j_1 m_1 j_2 m_2 K M}^l = -\hbar M Y_{j_1 m_1 j_2 m_2 K M}^l. \quad (143) \]
Because of the identity
\[ F^2 \equiv 4 (L^2 - \vec{K}^2), \tag{144} \]
these states are also eigenstates of \( L^2 \) with eigenvalue \( \hbar^2 (l(l + 3) + K(K + 1)) \).
Thus, the desired basis states are \( R_{Nl}(q) Y_{j_1 m_1 j_2 m_2}^{lM}(\chi; \varphi_1, \vartheta_1, \psi_1, \varphi_2, \vartheta_2, \psi_2) \).

### 6.3 Algebraic reduction and physical Hilbert space

According to the algebraic constraint quantization scheme outlined in the introduction, the physical Hilbert space \( \mathcal{H}_{\text{phys}} \) is the direct sum of (the representation spaces of) those UIR of the symmetry algebra of the HKP, in which the Casimir of the constraint algebra has the value zero. This definition is consistent because \( \vec{K}^2 \) is a Casimir of the algebra of observables, and can therefore be represented by a multiple of unity, and because zero is contained in its spectrum. For \( \vec{K}^2 = 0 \) the relations (107)-(109) between the Casimirs of the algebra \( S \) and the Casimir of the constraint algebra induce relations which must be satisfied by the Casimirs of \( S \). Consequently, the physical Hilbert space is spanned by those irreducible representations of \( S \), in which these relations hold.

Using the material of the previous section, the physical representations can easily be found. For \( \vec{K}^2 = 0 \) we have \( K = 0 \), and we are left with the following representations (identifying the representations with their carrier spaces)

\[
\begin{align*}
\mathcal{H}_{n, \text{phys}} & = D(n - 1, 0, 0) & E_n & = - \frac{k^2}{2 \hbar^2 (n + 1)^2} & n \text{ integer} \\
\mathcal{H}_{0, \text{phys}} & = D(1; 0, 0) & E_0 & = 0 \\
\mathcal{H}_{\nu, \text{phys}} & = D(p; -1, -1, i\nu) & E_{\nu} & = \frac{k^2}{2 \hbar^2 \nu^2}
\end{align*}
\tag{145}
\]

which coincide with the representations of the symmetry algebra of the KP\(_5\) as given in section 5.1.

The physical Hilbert space is spanned by the basis states of the above representations, i.e. by the states \( R_{Nl}(q) Y_{j_1 m_1 j_2 m_2}^{l0} \) \((K = M = 0, j_1 = j_2 =: j)\). Clearly, these states also constitute a basis of the space of SU(2)-invariant states. Furthermore, \( \mathcal{H}_{\text{phys}} \) can isomorphically be mapped onto the Hilbert space \( L^2(\mathbb{R}^5, dq^5) \) of the KP\(_5\), as can be seen as follows.

First observe that for \( j_1 = j_2 = j \) the Jacobi polynomials are proportional to Gegenbauer polynomials \( \mathcal{C} \) and the \( \chi \)-dependent functions become

\[
(1 + \cos \chi)^j(1 - \cos \chi)^j P_{l-2j}^{(2j+1,2j+1)}(\cos \chi) = \frac{(4j + 2)!(l + 1)!}{(2j + 1)!(2j + l + 2)!} \sin^{2j} \chi \times C_{l-2j}^{2j+\frac{2}{2}}(\cos \chi). \tag{146}
\]
Then, from the properties of Clebsch–Gordan coefficients, \( \langle jm_1, j (-m_2) | 00 \rangle \sim \delta_{m_1, m_2} \), and the group representation property of Wigner functions

\[
\sum_{m=-j}^{j} D_{m_1 m}^j (\varphi_1, \vartheta_1, \psi_1) D_{m_2 m}^j (\varphi_2, \vartheta_2, \psi_2) = D_{m_1 m_2}^j (\varphi, \vartheta, \psi)
\]

(\( \varphi, \vartheta, \psi \) being connected to \( \varphi_1, \vartheta_1, \psi_1, \varphi_2, \vartheta_2, \psi_2 \) via the addition theorem for Euler angles) we see that the Clebsch-Gordan series (141) yields

\[
Y_{jm_1,jm_200} = N_j \sin^{2j} \chi C_{l-2j}^{2j+4} (\cos \chi) D_{m_1 m_2}^j (\varphi, \vartheta, \psi).
\]

(148)

Thus, the gauge invariant states depend only on the gauge invariant variables \( q_i \) (22), and equations (70) and (123) show that they coincide with the eigenfunctions of the KP_5 in Euler coordinates. Therefore, the bases of the spaces \( \mathcal{H}_{\text{phys}} \) and \( L^2(\mathbb{R}^5, d^5q) \) can be mapped onto one another in a one-to-one manner.

Furthermore, the induced measure on \( \mathcal{H}_{\text{phys}} \) can be shown to be proportional to \( d^5q \). For this purpose we use the 1-forms

\[
dq_i = \frac{\partial q_i}{\partial u_a} du_a, \quad \kappa_i = 2 w_a^{(i)} du_a.
\]

(149)

The 1-forms \( \kappa_i \) are “dual” to the constraint vector fields

\[
X_i = \frac{1}{2} w_a^{(i)} \frac{\partial}{\partial u_a}, \quad \kappa_i (X_j) = u^2 \delta_{ij}.
\]

(150)

In terms of \( dq_i \) and \( \kappa_i \) the measure \( c u^2 d^8u \) can be written as

\[
c u^2 d^8u = \frac{c}{28} d^5q \wedge \kappa_1 \wedge \kappa_2 \wedge \kappa_3,
\]

(151)

where \( \kappa_1 \wedge \kappa_2 \wedge \kappa_3 \) is the volume form on the gauge group \( SU(2) \). The induced measure on \( \mathcal{H}_{\text{phys}} \) is obtained by integrating over the gauge group, and, with \( c = \frac{27}{\pi^2} \), is equal to \( d^5q \), showing that \( \mathcal{H}_{\text{phys}} \) and \( L^2(\mathbb{R}^5, d^5q) \) are isomorphic as Hilbert spaces. The integration over the group can also be interpreted as the projector onto the gauge invariant states.

### 6.4 Reduction of observables

In order to fully establish the equivalence of the quantum theories of the HKP and of the KP_5, we still have to prove that the representations of the fundamental observables on \( \mathcal{H}_{\text{phys}} \) and on \( L^2(\mathbb{R}^5, d^5q) \) are equivalent. This will be done by showing that the generators of the symmetry algebra and of the dynamical algebra act in the same way on the basis states of the two spaces. For the generators \( H, L_{ij} \)
and \( M_i \) of the symmetry algebra this is clear from the group theoretical treatment. For the generators of the dynamical algebra it is proved if we can show it for the operators \( H_{a8} \), because all other elements of the algebra can be obtained from them by means of the commutation relations. The operators \( H_{a8} \) can be expressed in terms of the operators \( \vec{K}^2 \), \( Q_i = Q_i(u) = q_i(u) \) and \( P_i \), where

\[
\begin{pmatrix}
    P_1 - iP_4 & -P_3 - iP_2 \\
    P_3 - iP_2 & P_1 + iP_4
\end{pmatrix} = \frac{(-i\hbar)}{2U^2} (U_2D_1 + D_2U_1)
\]

\[P_5E_2 = \frac{(-i\hbar)}{2U^2} \text{Re} (U_1^\dagger D_1 - U_2D_2^\dagger),\]

as

\[
H_{i8} = QP_i - i\hbar \frac{1}{2} \frac{Q_i}{Q}, \quad H_{78} = Q = q
\]

\[
H_{68} = \frac{1}{2} Q \mathbf{P}^2 + \frac{1}{2Q} \vec{K}^2 - i\hbar \frac{1}{2} \frac{1}{Q} \mathbf{Q} \cdot \mathbf{P} - \hbar^2 \frac{7}{8Q^2}
\]

Therefore, because of (63) and (64), it suffices to show that the action of \( Q \) and \( P \) on \( \mathcal{H}_{\text{phys}} \) coincides with that on \( L^2(\mathbb{R}^5, d^5q) \). For the operators \( Q_i \) this follows immediately from (22), because they have the same form on both spaces in Euler coordinates. For the \( P_i \) it can be seen from the commutation relations of \( Q \) and \( P \), \([Q_i, P_j] = i\hbar \delta_{ij}\), and the fact that the Hamiltonian can be written as

\[H = \frac{1}{2} \mathbf{P}^2 - \frac{k}{Q} + \frac{1}{2Q^2} \vec{K}^2,\]

which allows to represent \( P_i \) on \( \mathcal{H}_{\text{phys}} \) as

\[P_i = \frac{i}{\hbar} [H, Q_i]\]

and to infer the action of \( P_i \) from that of \( H \) and \( Q_i \).

### 7 Conclusions

As we have demonstrated, the algebraic approach to the quantization of constrained systems provides a powerful and elegant tool for the quantization and reduction of the HKP. The interpretation of (the vanishing of) the Casimir of the constraint algebra as a representation condition on the physical representations of the symmetry algebra could very effectively be used for the construction of the physical Hilbert space. The quantum expression for the Casimir of the constraint algebra and its spectrum could be determined intrinsically, without having to quantize the constraint algebra, thereby also confirming its interpretation as an observable.
To summarize, it may be said that, by laying emphasis on observable quantities, the algebraic method is closer to the physical interpretation of gauge systems than is the more kinematical method of implementing the individual constraints as projectors onto the physical states, and that the difficulties connected with the representation of the unphysical constraint algebra (Dirac’s bit of luck \cite{6}) can be completely avoided.
Appendix: The UIR of SO(6), SO(5,1) and E(5)

In this appendix we give a brief characterization of the unitary irreducible representations (UIR) of the groups $G = \text{SO}(6)$, $\text{SO}(5,1)$, $\text{E}(5)$ ($=\text{ISO}(5)$) by means of the eigenvalues of their Casimir invariants and the branching rules for the restriction of the representations according to the subgroup chain $G \supset \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \supset \text{SO}(2)$. For the details see the cited literature.

The UIR of SO(6)

The UIR of SO(6) can be characterized by the components of the corresponding highest weight $(m_{5,1}, m_{5,2}, m_{5,3})$, where the $m_{5,i}$ are all integers or all half integers which satisfy the inequality

$$m_{5,1} \geq m_{5,2} \geq |m_{5,3}|. \quad (A\ 1)$$

The representations will be denoted $D(m_{5,1}, m_{5,2}, m_{5,3})$. The branching rules for the subgroup chain $\text{SO}(6) \supset \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \supset \text{SO}(2)$ can be read off the Gelfand-Zetlin pattern

$$\begin{array}{ccc}
m_{5,1} & m_{5,2} & m_{5,3} \\
m_{4,1} & m_{4,2} & \\
m_{3,1} & m_{3,2} & \\
m_{2,1} & \\
m_{1,1} & \\
\end{array}, \quad (A\ 2)$$

where $m_{i,j}$ denotes the j-th component of the highest weights for the representations of $\text{SO}(i+1)$ which are contained in the representation $D(m_{5,1}, m_{5,2}, m_{5,3})$ upon subsequent restriction to these subgroups. The $m_{i,j}$ are all integers or all half integers and satisfy the inequalities

$$m_{5,1} \geq m_{4,1} \geq m_{5,2} \geq m_{4,2} \geq |m_{5,3}|$$
$$m_{4,1} \geq m_{3,1} \geq m_{4,2} \geq m_{3,2} \geq -m_{4,2}$$
$$m_{3,1} \geq m_{2,1} \geq |m_{3,2}|$$
$$m_{2,1} \geq m_{1,1} \geq -m_{2,1}. \quad (A\ 3)$$

All representations, whose highest weights are compatible with these inequalities, occur exactly once.

The UIR of SO(5,1)

The UIR of SO(5,1) can also be labelled by a Gelfand-Zetlin pattern as in (A 2), but now the range of the numbers $m_{5,i}$ and the branching rules for $\text{SO}(5,1) \supset \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \supset \text{SO}(2)$. For the details see the cited literature.

$$\begin{array}{ccc}
m_{5,1} & m_{5,2} & m_{5,3} \\
m_{4,1} & m_{4,2} & \\
m_{3,1} & m_{3,2} & \\
m_{2,1} & \\
m_{1,1} & \\
\end{array}, \quad (A\ 2)$$

where $m_{i,j}$ denotes the j-th component of the highest weights for the representations of $\text{SO}(i+1)$ which are contained in the representation $D(m_{5,1}, m_{5,2}, m_{5,3})$ upon subsequent restriction to these subgroups. The $m_{i,j}$ are all integers or all half integers and satisfy the inequalities

$$m_{5,1} \geq m_{4,1} \geq m_{5,2} \geq m_{4,2} \geq |m_{5,3}|$$
$$m_{4,1} \geq m_{3,1} \geq m_{4,2} \geq m_{3,2} \geq -m_{4,2}$$
$$m_{3,1} \geq m_{2,1} \geq |m_{3,2}|$$
$$m_{2,1} \geq m_{1,1} \geq -m_{2,1}. \quad (A\ 3)$$

All representations, whose highest weights are compatible with these inequalities, occur exactly once.

The UIR of SO(5,1)

The UIR of SO(5,1) can also be labelled by a Gelfand-Zetlin pattern as in (A 2), but now the range of the numbers $m_{5,i}$ and the branching rules for $\text{SO}(5,1) \supset \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \supset \text{SO}(2)$. For the details see the cited literature.
SO(5) are different. The branching rules for \( \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \supset \text{SO}(2) \) are the same as in the case of \( \text{SO}(6) \). There are three series of UIR (see [11], the numbers \( l_{i,j} \) of Ottoson are connected with the \( m_{i,j} \) by \( l_{2r,k} = m_{2r,k} + r - k + 1 \), \( l_{2r-1,k} = m_{2r-1,k} + r - k \)):

1. The principal series: \( D(p; m_{5,1}, m_{5,2}, i\tau) \)

\[
m_{5,1} \geq m_{5,2} \geq -1, \quad m_{5,3} = i\tau, \quad \tau \in \mathbb{R}
\]

\[
m_{4,1} \geq m_{5,1} + 1 \geq m_{4,2} \geq m_{5,2} + 1.
\]

\( m_{5,i} \) and \( m_{4,i} \) are all integers or all half integers. If \( m_{5,2} = -1 \), then \( \tau > 0 \).

2. The supplementary series: \( D(s; m_{5,1}, -1, 1) \)

\[
m_{5,1} \geq -1, \quad m_{5,2} = -m_{5,3} = -1
\]

\[
m_{4,1} \geq m_{5,1} + 1, \quad m_{4,2} = 0.
\]

Here \( m_{5,1} \) and \( m_{4,1} \) are both integers.

3. The exceptional series: \( D(e; m_{5,1}, -1, m_{5,3}) \)

\[
m_{5,1} \geq -1, \quad m_{5,2} = -1, \quad 0 < m_{5,3} < 1
\]

\[
m_{4,1} \geq m_{5,1} + 1 \geq m_{4,2} \geq 0.
\]

Here \( m_{5,1} \) and \( m_{4,i} \) are all integers.

Again, each of the representations of the subgroups, whose highest weights are compatible with the branching rules, occurs exactly once.

**The UIR of E(5)**

The Gelfand-Zetlin pattern for the UIR of E(5) can be obtained from that of \( \text{SO}(6) \) by making \( m_{5,1} \to \infty \). In addition, one needs a continuous parameter \( \sigma \in \mathbb{R} \) (see [12]). The range of the numbers \( m_{i,j} \), the branching rules, and the multiplicities of the representations of the subgroups are the same as for \( \text{SO}(6) \). The representations will be denoted \( D(\sigma; m_{5,2}, m_{5,3}) \).

**The Casimir operators and their eigenvalues**

The generators of the Lie algebras so(6), so(5,1) and e(5) and their commutation relations can be written in a uniform manner. Let

\[
L_{\mu\nu} = -L_{\nu\mu}, \quad 1 \leq \mu, \nu \leq 6
\]
then
\[ [L_{\mu\nu}, L_{\rho\sigma}] = ig_{\mu\rho} L_{\nu\sigma} \] (A 11)
where \( g_{ii} = 1 \) for \( 1 \leq i \leq 5 \), \( g_{66} = 1, -1, 0 \) for \( \text{so}(6) \), \( \text{so}(5,1) \), \( \text{e}(5) \) resp., and \( g_{\mu\nu} = 0 \) for \( \mu \neq \nu \). Each of the algebras possesses three Casimir operators. For \( \text{so}(6) \) they can be chosen as
\[
C_2 = \frac{1}{2} L_{\mu\nu} L_{\mu\nu} \] (A 12)
\[
C_3 = \varepsilon_{\mu\nu\rho\sigma\tau\upsilon} L_{\mu\nu} L_{\rho\sigma} L_{\tau\upsilon} \] (A 13)
\[
C_4 = \frac{1}{2} L_{\mu\nu} L_{\nu\rho} L_{\rho\sigma} L_{\sigma\mu} \] (A 14)
with the eigenvalues (see [13, 14])
\[
C_2 = \mu_1(\mu_1 + 4) + \mu_2(\mu_2 + 2) + \mu_3^2 \] (A 15)
\[
C_3 = 48 (\mu_1 + 2)(\mu_2 + 1)\mu_3 \] (A 16)
\[
C_4 = (\mu_1(\mu_1 + 4))^2 + 6\mu_1(\mu_1 + 4) + (\mu_2(\mu_2 + 2))^2 + \mu_4^2 - 2\mu_2^2 \] (A 17)
where \( \mu_i = m_{5,i} \) as given above. The Casimirs for \( \text{so}(5,1) \) can be obtained from those of \( \text{so}(6) \) by making the substitution \( L_{i6} \rightarrow -i L_{i6} \) and multiplying the defining expression for \( C_3 \) by \(+i\). The eigenvalues in terms of \( \mu_1, \mu_2, \mu_3 \) are the same, apart from a factor \(-i\) on the right hand side of (A 16).

For \( \text{e}(5) \) the Casimir invariants are [15]
\[
C_2 = L_{i6} L_{i6} \] (A 18)
\[
C_3 = \varepsilon_{\mu\nu\rho\sigma\tau\upsilon} L_{\mu\nu} L_{\rho\sigma} L_{\tau\upsilon} \] (A 19)
\[
C_4 = \frac{1}{2} L_{ij} L_{ji} - \frac{1}{4} (L_{i6} L_{ij} + L_{ij} L_{i6})(L_{k6} L_{kj} + L_{kj} L_{k6}). \] (A 20)
Their eigenvalues are
\[
C_2 = \sigma^2 \] (A 21)
\[
C_3 = 48 \sigma(\mu_2 + 1)\mu_3 \] (A 22)
\[
C_4 = \sigma^2(\mu_2(\mu_2 + 2) + \mu_3^2 - 4) \] (A 23)

The UIR of \( \text{SO}(6) \) and \( \text{E}(5) \) are uniquely determined by the eigenvalues of their Casimir operators, in the case of \( \text{SO}(5,1) \) the additional knowledge of the series or the branching rules for \( \text{SO}(5,1) \supset \text{SO}(5) \) is necessary.

Finally, we want to give the Casimir invariants for the subalgebras \( \text{so}(5) \) and \( \text{so}(4) \) and their eigenvalues. For \( \text{so}(5) \), generated by \( L_{ij} \), \( 1 \leq i, j \leq 5 \), there are two Casimirs
\[
L^2 = \frac{1}{2} L_{ij} L_{ij} \] (A 24)
\[
C_4^{(5)} = \frac{1}{2} L_{ij} L_{jk} L_{ki} L_{ii} \] (A 25)
with the eigenvalues \((\lambda_1 = m_{4,1}, \lambda_2 = m_{4,2})\)

\[
L^2 = \lambda_1(\lambda_1 + 3) + \lambda_2(\lambda_2 + 1) \tag{A 26}
\]

\[
C^{(5)}_4 = (\lambda_1(\lambda_1 + 3))^2 + 3\lambda_1(\lambda_1 + 3) + (\lambda_2(\lambda_2 + 1))^2 - \lambda_2(\lambda_2 + 1). \tag{A 27}
\]

The UIR of SO(5), which are uniquely specified by the numbers \(\lambda_1\) and \(\lambda_2\), will be denoted \(D(\lambda_1, \lambda_2)\).

For \(\text{so}(4) = \text{su}(2) \oplus \text{su}(2)\), generated by the two commuting angular momentum vectors

\[
\vec{J}_1 = \frac{1}{2} \begin{pmatrix} L_{12} + L_{34} \\ L_{13} + L_{42} \\ L_{14} + L_{23} \end{pmatrix} \quad \vec{J}_2 = \frac{1}{2} \begin{pmatrix} L_{12} - L_{34} \\ L_{13} - L_{42} \\ L_{14} - L_{23} \end{pmatrix} \tag{A 28}
\]

the Casimir invariants are \((1 \leq i, j, k, l \leq 4)\)

\[
J^2 = \frac{1}{2} L_{ij} L_{ij} = 2 (\vec{J}_1^2 + \vec{J}_2^2) \tag{A 29}
\]

\[
\vec{J}^2 = \varepsilon_{ijkl} L_{ij} L_{kl} = 8 (\vec{J}_1^2 - \vec{J}_2^2). \tag{A 30}
\]

They have the eigenvalues \((m_{3,1} = j_1 + j_2, m_{3,2} = j_1 - j_2)\)

\[
J^2 = 2 (j_1(j_1 + 1) + j_2(j_2 + 1)) \tag{A 31}
\]

\[
\vec{J}^2 = 8 (j_1(j_1 + 1) - j_2(j_2 + 1)). \tag{A 32}
\]

Again, the UIR are uniquely labelled by \(j_1\) and \(j_2\). They will be denoted \(D^{j_1,j_2}\).
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