Anytime Proximity Moving Horizon Estimation: Stability and Regret

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Abstract—In this article, we present a novel Anytime Proximity moving horizon estimation (pMHE) approach. This approach solves the underlying optimization problem with a fixed horizon at each time instant, while allowing for a flexible design via rather general convex approximation schemes. The main contribution of this article is the introduction of a new proximity-based formulation that computes an estimate from the state of a dynamical system by using a finite number of the most recent measurements. More specifically, a suitable optimization problem is solved to compute the optimal estimate at each time instant, and the horizon of measurements is shifted forward in time whenever a new measurement becomes available. Various MHE formulations have been proposed and investigated for stability and are by now well established in state estimation area [1]–[7]. Practical issues related to the online solution of MHE have drawn special attention since the underlying optimization problem has to be solved online at each time instant. In order to overcome this computational burden, fast optimization strategies based on interior-point methods [8], [9] are proposed, however, with no theoretical guarantees. In [10], approximation schemes are considered, in which suboptimal solutions for minimizing quadratic cost functions with a given accuracy are allowed, and upper bounds on the estimation errors are derived under observability assumptions. However, no optimization algorithm is specified. A similar convergence analysis is carried in [11] for the MHE algorithm presented in [12], where a nominal background problem is solved based on predicted future measurements, and when the true measurement arrives, the actual state is computed using a fast online correction step. To show that the generated estimation errors remain bounded, the associated approximate cost and resulting suboptimality are taken into account in the analysis. Particularly interesting are works that explicitly consider the dynamics of the optimization algorithm in the convergence analysis [13], [14]. In [13], a fast MHE implementation is achieved by performing single or multiple iterations of gradient or Newton methods to minimize least-squares cost functions. For linear systems, global exponential stability (GES) of the estimation errors is shown based on an explicit representation of the error dynamics. However, the required observability assumption restricts the choice of the horizon length and implies that it has to be greater than the state dimension. Moreover, variants of the so-called real-time iteration scheme [15], which performs a single Gauss-Newton iteration per time instant, are proposed. The local convergence results derived in [16] are established for the unconstrained case, i.e., no inequality constraints are considered, and hold under the assumptions of observability and a sufficiently small initial estimation error. Real-time implementations of MHE are also successfully carried out in real-world applications, such as structural vibration applications [17], induction machines [18], and industrial separation processes [19]. However, theoretical studies that consider both stability as well as performance of MHE schemes under rather mild assumptions are to the best of our knowledge and rarely addressed in the literature.

Statement of contributions: In this article, we present a novel MHE iteration scheme for constrained linear discrete-time systems, which is based on the idea of proximity MHE (pMHE), recently introduced in [20] and [21]. The pMHE framework exploits the advantages of a stabilizing a priori estimate from which stability can provably be inherited for any horizon length while allowing for a flexible design via rather general convex stage costs. In [20], stability is investigated under the assumption that a solution of the optimization problem is available at each time instant. In [21], an unconstrained MHE problem in which the inequality constraints are incorporated into the cost function by means of so-called relaxed barrier functions is considered. Although only a limited number of iterations are executed at each time instant, the derived stability conditions tailored to this relaxed barrier function-based formulation are rather conservative. The contribution of this article is fourfold. First, we present...
a pMHE iteration scheme where at each time instant, a limited number of optimization iterations are carried out and a state estimate is delivered in real-time. The underlying optimization algorithm consists of a proximal point algorithm [22] and is warm started by a stabilizing a priori estimate constructed based on the Luenberger observer. Second, we establish GES of the underlying estimation errors under minimal assumptions and by means of a Lyapunov analysis. In particular, the iteration scheme can be considered as an anytime algorithm in which stability is guaranteed after any number of optimization algorithm iterations, including the case of a single iteration per time instant. Third, in contrast to the pMHE scheme in [20], the a priori estimate is only used to warm start the proposed algorithm. Nevertheless, stability is inherited from it, despite its stabilizing effect is fading away with each iteration. Forth, we study the performance of the pMHE iteration scheme by using the notion of regret, which is widely used in the field of online convex optimization to characterize performance [23]–[25], and adapting it to our setting. More specifically, we define the regret as the difference of the accumulated costs generated by the iteration scheme relative to a comparator sequence and show that this regret can be upper bounded. Furthermore, we prove that, for any given comparator sequence, this bound can be rendered smaller by increasing the number of optimization iterations, and that a constant regret bound can be derived for the special case of exponentially stable comparator sequences. Overall, we present a novel anytime pMHE iteration scheme that is designed based on rather general convex stage cost functions, ensures stability after each iteration, as well as for any horizon length, and for which performance guarantees are provided and characterized in terms of rigorously derived regret bounds.

**Organization:** The rest of this article is organized as follows. The constrained MHE problem for discrete-time linear systems is stated in Section II. The proposed pMHE iteration scheme is described in details in Section III, and its stability properties are established in Section IV. In Section V, the focus is on the performance properties of the iteration scheme, which are reflected by the derived regret upper bounds. A simulation example that illustrates both the stability and performance properties is presented in Section VI. Finally, Section VII concludes this article.

**Notations:** Let \( \mathbb{N}_+ \) denote the set of positive natural numbers, \( \mathbb{R}_+ \) and \( \mathbb{R}_{++} \) the sets of nonnegative real and positive real numbers, respectively, and \( \mathbb{S}_+^n \) and \( \mathbb{S}_{++}^n \) the sets of symmetric positive semidefinite and positive-definite matrices of dimension \( n \in \mathbb{N}_+ \), respectively. For a vector \( v \in \mathbb{R}^n \), let \( \|v\|_P := \sqrt{v^T P v} \) for any \( P \in \mathbb{S}_+^n \). Moreover, let \( \mathbf{0} := \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^T \).

**II. PROBLEM SETUP AND PRELIMINARIES**

We consider the following discrete-time linear time-invariant (LTI) system:

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k & (1a) \\
y_k &= Cx_k & (1b)
\end{align*}
\]

where \( x_k \in \mathbb{R}^n \) denotes the state vector, \( u_k \in \mathbb{R}^m \) denotes the input vector, and \( y_k \in \mathbb{R}^p \) denotes the measurement vector. We assume that the pair \((A,C)\) is detectable and that the state satisfies polytopic constraints

\[
x_k \in \mathcal{X} := \{x \in \mathbb{R}^n : Cx \leq d_x\} (2)
\]

Algorithm 1: pMHE according to [20].

1. **Initialize:** Choose \( \hat{x}_0 \) and set \( z_0 = \hat{x}_0 \)
2. **for** \( k = 1, 2, \ldots \) **do**
   3. \[
   \hat{z}_k = \arg \min_{\hat{z}_k \in S_k} \left\{ f_k(\hat{z}_k) + D \psi(\hat{z}_k, \bar{z}_k) \right\} (7)
   \]
   4. obtain \( \hat{x}_k \) according to (9)
   5. \( \hat{z}_{k+1} = \Phi_k(\hat{z}_k) \)
6. **end for**

where \( C \in \mathbb{R}^{q_n \times n} \) and \( d_x \in \mathbb{R}^q \) with \( q_n \in \mathbb{N}_+ \). We aim to compute an estimate of the state \( x_k \) based on an MHE scheme. More specifically, at each time instant \( k \), given the last \( N \) measurements \( \{y_{k-N}, \ldots, y_{k-1}\} \) and inputs \( \{u_{k-N}, \ldots, u_{k-1}\} \), our goal is to find a solution to the following optimization problem:

\[
\begin{align}
\min_{\hat{x}_{k-N}, \hat{v}, \hat{w}} & \quad \sum_{i=k-N}^{k-1} r(\hat{v}_i) + q(\hat{w}_i) \\
\text{s.t.} & \\
\hat{x}_i &= A \hat{x}_{i-1} + Bu_{i-1} + \hat{w}_{i-1} & (3b) \\
y_i &= C \hat{x}_i + \hat{v}_i & (3c) \\
\hat{x}_i &\in \mathcal{X}, \quad i = k - N, \ldots, k - 1 (3d)
\end{align}
\]

where \( \hat{v} = \{\hat{v}_{k-N}, \ldots, \hat{v}_1\} \) and \( \hat{w} = \{\hat{w}_{k-N}, \ldots, \hat{w}_{k-1}\} \) denote the output residual and the model residual sequences over the estimation horizon with length \( N \in \mathbb{N}_+ \), respectively. In (3a), the stage cost \( r : \mathbb{R}^p \to \mathbb{R} \) is a convex function, which penalizes the output residual \( \hat{v}_i \in \mathbb{R}^p \), and the stage cost \( q : \mathbb{R}^n \to \mathbb{R} \) is a convex function, which penalizes the model residual \( \hat{w}_i \in \mathbb{R}^n \). By using the system dynamics (3b) and (3c), we can express each output residual \( \hat{v}_i \) in terms of the remaining decision variables \( \{\hat{x}_{k-N}, \hat{w}\} \), which we collect in the vector

\[
\tilde{z}_k := \begin{bmatrix} \hat{x}_{k-N} \\ \hat{w}_{k-N} \\ \vdots \\ \hat{w}_{k-1} \end{bmatrix} \in \mathbb{R}^{(N+1)n} (4)
\]

and use it to reformulate problem (3) as

\[
\begin{align}
\min_{\tilde{z}_k} & \quad f_k(\tilde{z}_k) \\
\text{s.t.} & \quad \tilde{z}_k \in S_k.
\end{align}
\]

Here, the convex function \( f_k : \mathbb{R}^{(N+1)n} \to \mathbb{R} \) denotes the sum of stage costs and the convex set \( S_k \subset \mathbb{R}^{(N+1)n} \) represents the (stacked) state constraints given by

\[
S_k = \left\{ z = \begin{bmatrix} x \\ w \end{bmatrix}, x \in \mathbb{R}^n, w \in \mathbb{R}^N : Gx + Fw \leq E_k \right\} \quad (6)
\]

Note that \( S_k \) is time-dependent due to the changing input sequence \( \{u_{k-N}, \ldots, u_{k-1}\} \) that enters \( E_k \) over time. The matrices \( G \) and \( F \) and the vector \( E_k \) as well as more details on the reformulation of the estimation problem (3) to (5) can be found in Appendix A. Within the proximity-based formulation, as introduced in [20] and [26] and related to [4], we solve a regularized form of (5) in which we add to the cost function (5a) a proximity measure to a stabilizing a priori estimate, which we refer to as \( \tilde{z}_k \in \mathbb{R}^{(N+1)n} \). A corresponding pseudocode is given in Algorithm 1.
In (7), the overall cost function is strictly convex and
\[ D_\psi : \mathbb{R}^{(N+1)n} \times \mathbb{R}^{(N+1)n} \to \mathbb{R} \]
denotes the Bregman distance induced from a continuously differentiable and strongly convex function \( \psi : \mathbb{R}^{(N+1)n} \to \mathbb{R} \) as
\[ D_\psi(z_1, z_2) = \psi(z_1) - \psi(z_2) - (z_1 - z_2)^\top \nabla \psi(z_2). \] (8)

More detail on Bregman distances as well as some of their central properties can be found in Appendix B. Based on the resulting pMHE solution \( \hat{z}_k^* \), the state estimate \( \hat{x}_k \) is obtained via a forward prediction of the dynamics (3b)
\[ \hat{x}_k = A^N \hat{x}_k^* + \sum_{j=k-N}^{k-1} A^{k-j} (B u_j + \bar{w}_j^*) \] (9)

and the stabilizing a priori estimate \( \bar{z}_{k+1} \) is computed using the operator \( \Phi_k : \mathbb{R}^{(N+1)n} \to \mathbb{R}^{(N+1)n} \). While the performance of pMHE can be enforced with rather general convex stage costs \( r \) and \( q \), stability can be ensured for any horizon length \( N \geq 1 \) with an appropriate choice of the a priori estimate operator \( \Phi_k \) and the Bregman distance [20]. Furthermore, among many interesting properties, Bregman distances can adapt to the problem at hand and act as a barrier for the constraint set, in particular with so-called relaxed barrier functions [21].

In the following section, we present an iteration scheme to pMHE, in which, rather than finding the pMHE solution at each time instant \( k \), we reduce the computation time by executing only a finite number of optimization iterations of a gradient type algorithm.

### III. ANYTIME pMHE ALGORITHM

In this section, we propose a novel pMHE iteration scheme in which, at each time instant \( k \), problem (5) is approximately solved by executing a fixed number \( it(k) \in \mathbb{N}_k \) of optimization algorithm iterations. In more details, at each time \( k \), a suitable warm start \( \bar{z}_k^0 \) is generated from a stabilizing a priori estimate \( \bar{z}_k \) and an iterative optimization update is carried out, from which the sequence \( \{\bar{z}_k^i\} \) with \( i = 1, \ldots, it(k) \) is obtained. The steps of the scheme are given in Algorithm 2 and illustrated in Fig. 1.

Before we explain the proposed algorithm in more detail, and for the sake of clarity, let us first introduce some notations. The index \( k \) denotes the time instant in which we receive a new measurement and it \((k)\) is the number of iterations of the optimization algorithm between time instants \( k \) and \( k + 1 \).

**Algorithm 2:** Anytime pMHE.

1: Initialize: Choose \( \hat{x}_0 \) and set \( z_0 = \hat{x}_0 \)
2: for \( k = 1, 2, \ldots \) do
3: \( \bar{z}_k^0 = \arg \min_{z \in \mathcal{S}_k} D_\psi(z, z_k) \) warm start
4: for \( i = 0, \ldots, it(k) - 1 \) do optimizer update
5: \( \bar{z}_{k+1}^i = \arg \min_{z \in \mathcal{S}_k} \{ \eta_k \nabla f_k (\bar{z}_k^i)^\top z + D_\psi(z, \bar{z}_k^i) \} \)
6: end for
7: \( z_{k+1} = \Phi_k (\bar{z}_{k+1}^i) \)
8: end for

Moreover, we introduce
\[ \bar{z}_k^i := \begin{bmatrix} \bar{x}_{k-N}^i \\ \bar{w}_{k-N}^i \\ \vdots \\ \bar{x}_{k-1}^i \\ \bar{w}_{k-1}^i \\ 0 \end{bmatrix}, \quad \hat{z}_k := \begin{bmatrix} \hat{x}_{k-N} \\ \hat{w}_{k-N} \\ \vdots \\ \hat{x}_{k-1} \\ \hat{w}_{k-1} \\ 0 \end{bmatrix}, \quad \bar{z}_k := \begin{bmatrix} \bar{x}_{k-N} \\ \bar{w}_{k-N} \\ \vdots \\ \bar{x}_{k-1} \\ \bar{w}_{k-1} \\ 0 \end{bmatrix}. \] (10)

With \( \bar{z}_k^i \), we denote the \( i \)th iterate of the optimization algorithm at time \( k \). With \( \bar{z}_k \), we refer to the a priori estimate at time \( k \) and with \( z_k \) to the true state \( x_{k-N} \) with true model residual sequence \( \{0, \ldots, 0\} \).

Upon arrival of a new measurement at time \( k \), the optimization algorithm is initialized based on the a priori estimate \( z_k \). In particular, we compute the warm start \( z_k^0 \) as the Bregman projection of \( z_k \) onto the constraint set \( \mathcal{S}_k \), as formulated in line 3 of Algorithm 2. Then, a fixed number \( it(k) \) of optimization iterations is performed via (13), generating \( \{\bar{z}_k^i, \ldots, \bar{z}_k^{it(k)}\} \).

Here, \( \eta_k > 0 \) denotes the step size employed at the \( i \)th iteration at time \( k \). From this sequence of iterates, an arbitrary iterate \( \bar{z}_k^{it(k)} \) with \( j(k) \in \{0, \ldots, it(k)\} \) can be chosen, based on which the state estimate \( \hat{x}_k \) is obtained using
\[ \hat{x}_k = A^N \bar{x}_{k-N}^{it(k)} + \sum_{j=k-N}^{k-1} A^{k-j} (B u_j + \bar{w}_j^{it(k)}) \] (11)

for \( k > N \) (see Remark 1 for the case where \( 0 < k \leq N \)). Moreover, the a priori estimate \( z_{k+1} \) for the next time instant is computed through the operator \( \Phi_k : \mathbb{R}^{(N+1)n} \to \mathbb{R}^{(N+1)n} \), which will be defined in (12). As abovementioned, the basic idea of the pMHE framework is to use the Bregman distance \( D_\psi \) as a proximity measure to a stabilizing a priori estimate in order to inherit its stability properties. Since the Luenberger observer appears as a simple candidate for constructing the a priori estimates, we require that the operator \( \Phi_k \) incorporates its dynamics as follows:
\[ \Phi_k (\bar{z}_k^{it(k)}) := \begin{bmatrix} A \bar{x}_{k-N}^{it(k)} + B u_{k-N} + L \left( y_{k-N} - C \bar{x}_{k-N}^{it(k)} \right) \\ 0 \end{bmatrix} \] (12)

where \( 0 \in \mathbb{R}^{n} \). Here, the observer gain \( L \) is chosen such that all the eigenvalues of \( A - LC \) are strictly within the unit circle.

In the following, we compare Algorithm 2 with our earlier formulation of pMHE, given in Algorithm 1. Observe that, while a solution of the optimization problem (7) is computed, step 4 in Algorithm 2 employs the so-called mirror descent algorithm [22].
that iterates (13) until a given number of iterations $i(k)$ is achieved. For $D_\psi(z_1, z_2) = \frac{1}{2} \|z_1 - z_2\|^2$ and $S_k = \mathbb{R}^{(N+1)n}$, the optimizer update step (13) corresponds to an iteration step of the classical gradient descent algorithm, and hence, step 4 can be executed very quickly. In the constrained case, (13) can be regarded as a generalization of the projected gradient algorithm [22]. For this reason, we can view Algorithm 2 as a real-time version of the pMHE scheme given in Algorithm 1. Note also that choosing the so-called Kullback–Leiber divergence as Bregman distance yields the efficient entropic descent algorithm if the constraint set is given by the unit simplex [22]. An appealing feature of Algorithm 2 is that, depending on the available computation time between two subsequent time instants $k$ and $k+1$, the user can specify a maximum number of iterations $i(k)$ at which the optimization algorithm at time $k$ has to return a solution.

Another key difference between the two algorithms is that Algorithm 1 is biased by the stabilizing a priori estimate $\hat{z}_k$, while this bias is fading away in Algorithm 2. In other words, the a priori estimate constructed based on the Luenberger observer (12) has less impact at each optimization iteration, which improves the performance of the pMHE iteration scheme. This is due to the fact that $\hat{z}_k$ might degenerate performance in Algorithm 1, since the solution lies in proximity to the a priori estimate. In Algorithm 2, however, the Luenberger observer enters only in the warm start. From this point of view, it is quite surprising that, even though the effect of this stabilizing ingredient is fading away, stability is provably preserved, as we will show in the subsequent section. Thus, this “implicit stabilizing regularization” approach of the a priori estimate is in contrast to the explicit stabilizing regularization proposed in [4] and [20] (see also [27]). Moreover, the proposed MHE algorithm possesses the anytime property. The anytime property refers to the fact that the algorithm will yield stable estimation errors after any number of optimization algorithm iterations. This is similar in spirit to anytime model predictive control (MPC) algorithms, which compute stabilizing control inputs after any optimization iteration [28], [29].

Remark 1: For $0 < k \leq N$, we can employ the steps of Algorithm 2 by setting all the negative indices to zero. More specifically, $\tilde{z}_k = [\tilde{z}_0^\top \tilde{w}_0^\top \cdots \tilde{w}_{k-1}^\top]^\top$ and $\tilde{x}_k = [\tilde{x}_0 \tilde{x}_1 \cdots 0]^\top$, where $\tilde{x}_k, \tilde{z}_k \in \mathbb{R}^{(k+1)n}$. In order to compute the state estimate, (11) has to be explicitly modified to

$$\dot{x}_k = A^k \tilde{x}_k + \sum_{j=0}^{k-1} A^{k-1-j} \left( B u_j + \tilde{w}_j^j \right). \quad (14)$$

In the following, we impose some standard assumptions.

**Assumption 1 (Properties of $S_k$):** The set $S_k$ of constraints is closed and convex with nonempty interior.

**Assumption 2 (Convexity of $f_k$):** The sum of stage costs $f_k$ is continuously differentiable, convex for all $k > 0$, and achieves its minimum at $\hat{z}_k$.

**Assumption 3 (Strong smoothness of $f_k$):** The sum of stage costs $f_k$ is strongly smooth with constant $L_f > 0$, i.e.,

$$f_k(z_2) \leq f_k(z_1) + \nabla f_k(z_1)^\top (z_2 - z_1) + \frac{L_f}{2} \|z_1 - z_2\|^2$$

(15)

for all $z_1, z_2 \in \mathbb{R}^{(N+1)n}$ and $k > 0$.

**Assumption 4 (Strong convexity and smoothness of $D_\psi$):** The function $\psi$ is continuously differentiable, strongly convex with constant $\sigma > 0$, and strongly smooth with constant $\gamma > 0$, which implies the following for the Bregman distance:

$$\frac{\sigma}{2} \|z_1 - z_2\|^2 \leq D_\psi(z_1, z_2) \leq \frac{\gamma}{2} \|z_1 - z_2\|^2$$

(16)

for all $z_1, z_2 \in \mathbb{R}^{(N+1)n}$.

In Assumption 2, we can ensure that $f_k$ achieves its minimum at $z_k$ by designing the stage costs $r(f) \cdot q(f)$ such that their corresponding minimum is achieved at zero. Requiring strong smoothness of $f_k$ in Assumption 3 is rather customary in the analysis of first-order optimization algorithms [30] and will prove central in the subsequent theoretical studies of the pMHE iteration scheme. Note that Assumption 4 restricts the class of employed Bregman distances to functions, which can be quadratically lower and upper bounded. Obviously, this includes the important special case of quadratic distances which are widely used as prior weighting in the MHE literature in order to ensure stability of the estimation error. In addition, we can use any Bregman distance $D_\psi$ constructed based on the function $\psi(z) = \frac{1}{2} \|z\|^2 + B(z)$, where $B : \mathbb{R}^{(N+1)n} \rightarrow \mathbb{R}$ is a convex and strongly smooth function, i.e., a convex function whose gradient is Lipschitz continuous.

### IV. Stability Analysis

In this section, we analyze the stability properties of the proposed pMHE iteration scheme (Algorithm 2). More specifically, we derive sufficient conditions on the Bregman distance $D_\psi$ as well as on the step sizes $\eta_k^j$ for the GES of the estimation error

$$e_{k-N} := x_{k-N} - \hat{x}_k$$

(17)

for any $j \in \{0, \ldots, i(k)\}$. The following key result establishes the stability properties of the pMHE iteration scheme.

**Theorem 1:** Consider Algorithm 2 and suppose that Assumptions 1–4 hold true. If we choose the Bregman distance $D_\psi$ such that

$$D_\psi(\Phi_k(z), \Phi_k(\bar{z})) - D_\psi(\bar{z}, \tilde{z}) \leq -c \|z - \bar{z}\|^2$$

(18)

is satisfied for all $z, \tilde{z} \in \mathbb{R}^{(N+1)n}$, where $\Phi_k(\cdot)$ is defined in (12), and if the step size at the $i$th iteration and time instant $k$ satisfies

$$\eta_k^j \leq \frac{\sigma}{L_f}$$

(19)

then the estimation error (17) is GES.

To prove this theorem, we require the following result.

**Lemma 1:** Consider Algorithm 2 and suppose Assumptions 1–4 hold true. Then, for two consecutive iterates $\tilde{z}_k^i$ and $\tilde{z}_k^{i+1}$ at any time instant $k > 0$, we obtain

$$D_\psi(\tilde{z}_k^i, \tilde{z}_k^{i+1}) \leq D_\psi(\tilde{z}_k^i, \tilde{z}_k^{i+1}) + \frac{1}{2} (\eta_k^j L_f - \sigma) \|\tilde{z}_k^{i+1} - \tilde{z}_k^i\|^2$$

(20)

where $i \in \{0, \ldots, i(k)\}$ and $\tilde{z}_k$ is defined in (10). Moreover, we have that for any $j \in \{0, \ldots, i(k)\}$

$$D_\psi(\tilde{z}_k^j, \tilde{z}_k^{j+1}) \leq D_\psi(\tilde{z}_k^j, \tilde{z}_k^{j+1}) + \frac{1}{2} \sum_{i=0}^{j-1} (\eta_k^i L_f - \sigma) \|\tilde{z}_k^{i+1} - \tilde{z}_k^i\|^2 \cdot (21)$$

The proof of Lemma 1 can be found in Appendix C. We are now in a position to prove the stability result for the proposed pMHE scheme.

**Proof of Theorem 1:** We first prove GES of the estimation error (17) with $j = j(k)$. Let $V$ be a candidate Lyapunov function

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chosen as the Bregman distance in (13), i.e.,

\[ V(z_k, \hat{z}_k^{(k)}) = D_\psi(z_k, \hat{z}_k^{(k)}) \]. \quad (22) \]

Here, \( z_k \) denotes the true state with zero model residual, as defined in (10), and \( \hat{z}_k^{(k)} \) the selected pMHE iterate at time instant \( k \). In the following, we show that \( V \) satisfies the following conditions:

\[ \alpha_1 \| z_k - \hat{z}_k^{(k)} \|^2 \leq V(z_k, \hat{z}_k^{(k)}) = \alpha_2 \| z_k - \hat{z}_k^{(k)} \|^2 \quad (23a) \]

and

\[ \Delta V := V(z_{k+1}, \hat{z}_{k+1}^{(k+1)}) - V(z_k, \hat{z}_k^{(k)}) \leq -\alpha_3 \| z_k - \hat{z}_k^{(k)} \|^2 \quad (23b) \]

for some positive constants \( \alpha_1, \alpha_2, \) and \( \alpha_3 \).

Note that, in view of (10), the error generated by the pMHE iteration scheme at time \( k \) is given by

\[ z_k - \hat{z}_k^{(k)} = \left[ x_{k-N} \right] \left[ 0 \right]_{N \times 1} = \left[ \bar{z}_k^{(k)} \right] - \left[ \bar{w}_k^{(k)} \right] \]

where \( \bar{w}_k^{(k)} := ([u_k^{(k)}] \ldots [u_{k-1}^{(k)}])\top \).

By Assumption 4, (23a) follows with \( \alpha_1 = \frac{\gamma}{2} \) and \( \alpha_2 = \frac{\gamma}{2} \). Furthermore, by (21) in Lemma 1, we have

\[ \Delta V = D_\psi(z_{k+1}, \hat{z}_{k+1}^{(k+1)}) - D_\psi(z_k, \hat{z}_k^{(k)}) \leq D_\psi(z_{k+1}, z_{k+1}^{(k+1)}) - D_\psi(z_k, \hat{z}_k^{(k)}) + \frac{1}{2} \sum_{i=0}^{j(k(k)+1)-1} (\eta_i L_f - \sigma) \| \hat{z}_{i+1}^{(k+1)} - \hat{z}_{i+1}^{(k)} \|^2. \quad (25) \]

The condition on the step sizes given in (19) implies that \( \eta_i L_f - \sigma \leq 0 \), and hence,

\[ \frac{1}{2} \sum_{i=0}^{j(k(k)+1)-1} (\eta_i L_f - \sigma) \| \hat{z}_{i+1}^{(k+1)} - \hat{z}_{i+1}^{(k)} \|^2 \leq 0. \quad (26) \]

Moreover, since \( \tilde{z}_{k+1} = \Phi_k(z_k) \),

\[ \Phi_k(z_k) = \begin{bmatrix} A x_{k-N} + B u_{k-N} + L (y_{k-N} - C x_{k-N}) \\ 0 \end{bmatrix} \]

\[ = \begin{bmatrix} A x_{k-N} + B u_{k-N} \\ 0 \end{bmatrix} = z_{k+1} \quad (27) \]

in view of (12), we have

\[ \Delta V \leq D_\psi(z_{k+1}, z_{k+1}^{(k)}) - D_\psi(z_k, \hat{z}_k^{(k)}) = D_\psi(\Phi_k(z_k), \Phi_k(\hat{z}_k^{(k)})) - D_\psi(z_k, \hat{z}_k^{(k)}). \quad (28) \]

Given that the Bregman distance satisfies (18), we obtain

\[ \Delta V \leq -c \| z_k - \hat{z}_k^{(k)} \|^2 = -c \| \hat{z}_{k-N} \|^2 - c \| \bar{w}_k^{(k)} \|^2. \quad (29) \]

Hence, the candidate Lyapunov function satisfies (23b) with \( \alpha_3 = c \) and the estimation error (17) with \( j = j(k) \) is GES.

In the following, we show that GES holds also for any \( j \in \{0, \ldots, \mu(k)\} \). Based on the previous Lyapunov analysis, we have that

\[ D_\psi(z_k, \hat{z}_k^{(k)}) - D_\psi(z_{k-1}, \hat{z}_{k-1}^{(k-1)}) \leq -c \| z_{k-1} - \hat{z}_{k-1}^{(k-1)} \|^2 \leq -\frac{2c}{\gamma} D_\psi(z_{k-1}, \hat{z}_{k-1}^{(k-1)}) \quad (30) \]

where the last inequality holds by the strong smoothness of the Bregman distance. By defining \( \beta_k := 1 - \frac{c}{\gamma} \)

\[ 0 \leq D_\psi(z_k, \hat{z}_k^{(k)}) \leq \beta_e D_\psi(z_{k-1}, \hat{z}_{k-1}^{(k-1)}) \quad (31) \]

where \( \beta_e \in [0, 1) \) since \( D_\psi(z_0, z_0) \) is nonnegative and \( \frac{c}{\gamma} > 0 \). Hence,

\[ D_\psi(z_k, \hat{z}_k^{(k)}) \leq \beta_e D_\psi(z_0, z_0). \quad (32) \]

We consider the difference \( D_\psi(z_k, \hat{z}_k^{(k)}) - D_\psi(z_{k+1}, \hat{z}_{k+1}^{(k)}) \) for any \( j \in \{0, \ldots, \mu(k+1)\} \). By (21), we have

\[ D_\psi(z_k, \hat{z}_k^{(k)}) - D_\psi(z_{k+1}, \hat{z}_{k+1}^{(k)}) \leq D_\psi(z_{k+1}, z_{k+1}^{(k+1)}) - D_\psi(z_k, \hat{z}_k^{(k)}) \leq -c \| z_k - \hat{z}_k^{(k)} \|^2 \quad (33) \]

where the last inequality holds in view of (28) and (29). Hence, by using \( \beta_e \) again, we have for any \( j \in \{0, \ldots, \mu(k+1)\} \)

\[ D_\psi(z_{k+1}, \hat{z}_{k+1}^{(k)}) \leq \beta_e D_\psi(z_k, \hat{z}_k^{(k)}). \quad (34) \]

By (32), we obtain \( \forall k > 0 \)

\[ D_\psi(z_{k+1}, \hat{z}_{k+1}^{(k)}) \leq \beta_k D_\psi(z_0, z_0). \quad (35) \]

By the strong smoothness and convexity of \( D_\psi \), we, therefore, get the following GES property of the estimation error:

\[ \| z_k - \hat{z}_k^{(k)} \|^2 \leq \gamma \beta_k \| z_0 - z_0 \|^2. \quad (36) \]

The theorem implies that the stability of the estimation error is guaranteed for any iterate and is independent of which iterate \( \hat{z}_k^{(k)} \) is picked from the sequence \( \{\hat{z}_k^{(k)} \ldots \hat{z}_k^{(k)}\} \) in Step 7 in Algorithm 2, as well as of the number of iterations \( \mu(k) \).

Hence, the algorithm generates convergent estimates after each optimizer update step and can be, therefore, considered as an anytime MHE algorithm. Moreover, selecting \( \eta_k = \frac{1}{L_f} \) for all \( i \) and \( k \), is sufficient for ensuring GES. In addition, this guarantee holds independently of the choice of the horizon length \( N \in \mathbb{N}_+ \) and for any convex stage cost satisfying Assumptions 2 and 3.

This includes, for instance, quadratic functions and the Huber penalty function, which allows to handle the important case of measurement outliers.

It is worth pointing out that detectability of the pair \( (A, C) \) implies that we can find suitable choices of the Bregman distance \( D_\psi \) that fulfill condition (18). In particular, let

\[ D_\psi(z_1, z_2) = \frac{1}{2} \| x_1 - x_2 \|^2 + \frac{1}{2} \| w_1 - w_2 \|^2 \quad (37) \]

with \( z_1 = [x_1 \top \ w_1 \top] \), \( z_2 = [x_2 \top \ w_2 \top] \), \( x_1, x_2 \in \mathbb{R}^n \), \( w_1, w_2 \in \mathbb{R}^{N_1} \), and \( P \in S_+^{N_2}, W \in S_+^{N_2} \). Using (12) and a simple algebraic manipulation, we have that

\[ D_\psi(\Phi_k(z_1), \Phi_k(z_2)) = D_\psi(z_1, z_2) \]

\[ = \frac{1}{2} \| (A - LC)(x_1 - x_2) \|^2 + \frac{1}{2} \| w_1 - w_2 \|^2 \quad (38) \]
Hence, satisfying (18) amounts to designing the weight matrix $P \in \mathbb{S}^{n}_{++}$ such that the linear matrix inequality (LMI)
\[(A - LC)^{\top} P (A - LC) - P < -Q\] holds for some $Q \in \mathbb{S}^{n}_{++}$. This is because when (39) holds, (38) yields
\[
D_{\psi}(\Phi_{k}(z_1), \Phi_{k}(z_2)) - D_{\psi}(z_1, z_2) \\
\leq -\frac{\lambda_{\min}(Q)}{2} \|x_1 - x_2\|^2 - \frac{\lambda_{\min}(W)}{2} \|w_1 - w_2\|^2 \\
\leq -c \|z_1 - z_2\|^2
\] where $c = \frac{1}{2} \min\{\lambda_{\min}(Q), \lambda_{\min}(W)\}$.

V. REGRET ANALYSIS

In this section, we study the performance of the proposed anytime pMHE iteration scheme. Recall the performance criterion of the original estimation problem (5), which is to minimize at each time instant $k$ the sum of stage costs $f_k$. In order to characterize the overall performance of Algorithm 2, we investigate the accumulation of losses $f_k$ over the considered simulation time $T \in \mathbb{N}_+$ given by
\[
\sum_{k=1}^{T} \min_{0 \leq i \leq \alpha(k)} f_k(\hat{z}_k^i).
\] Note that the min operator in (41) follows from the fact that the generated sequence of iterates $\{\hat{z}_k^0, \ldots, \hat{z}_k^{\alpha(k)}\}$ does not necessarily produce $f_k(\hat{z}_k^0) \geq \cdots \geq f_k(\hat{z}_k^{\alpha(k)})$. Hence, given $\{\hat{z}_k^0, \ldots, \hat{z}_k^{\alpha(k)}\}$, we have to choose a suitable $\hat{z}_k^i$ whose function value is then used in the performance analysis. Following the literature on mirror descent algorithms [22], we select the iterate with the minimal cost as our estimate, i.e., $\hat{z}_k^i = \min_{0 \leq i \leq \alpha(k)} f_k(\hat{z}_k^i)$. One advantage of this selection is that it allows us to adapt many tools used in the convergence proof of the mirror descent algorithm to the regret analysis. Further, we choose $\alpha(k) = \alpha(k)$ in Algorithm 2. Any other choice is in principle possible, but one has to adapt the subsequent analysis accordingly.

Our goal is to ensure that (41) is not much larger than the total loss $\sum_{k=1}^{T} f_k(\hat{z}_k^i)$ incurred by any comparator sequence $\{\hat{z}_k^1, \hat{z}_k^2, \ldots, \hat{z}_k^T\}$ satisfying $\hat{z}_k^i \in S_k$. In other words, we aim to obtain a low regret, which we define as
\[
R(T) := \sum_{k=1}^{T} \min_{0 \leq i \leq \alpha(k)} f_k(\hat{z}_k^i) - \sum_{k=1}^{T} f_k(z_k^i).
\] By computing an upper bound for the regret, we can design suitable step sizes that yield a sublinear regret, i.e., the regret bound $\mathcal{O}(\sqrt{T})$. This is a meaningful regret bound and well known in the context of online convex optimization since it implies that the average regret $R(T)/T$ tends to zero for $T \to \infty$ and, hence, that the proposed algorithm performs well, on average as well as the comparator [30]. This property of the algorithm is especially desirable when the regret is used to evaluate how well the pMHE iteration scheme performs compared to an estimation scheme that knows the optimal solutions $\{\hat{z}_1^*, \hat{z}_2^*, \ldots, \hat{z}_T^*\}$. Hence, we measure the real-time regret of our algorithm that carries out only finitely many optimization iterations (due to limited hardware resources and/or minimum required sampling rate) relative to a comparator algorithm that gets instantaneously an optimal solution from some oracle.

A. Regret With Respect to Arbitrary Comparator Sequences

In this section, we establish bounds on the regret generated by Algorithm 2. Similar to [25], we derive regret bounds that depend on the variation of the comparator sequence with respect to the dynamics $\Phi_k$ defined in (12)
\[
C_T(z_1^T, \ldots, z_T^T) := \sum_{k=1}^{T} \|z_{k+1}^c - \Phi_k(z_k^c)\|.\]
Moreover, we define the following notations:
\[
G_f := \max_{z \in S_k, k > 0} \|\nabla f_k(z)\|, \quad M_1 := \max_{z \in S_k, k > 0} \|\nabla \psi(\Phi(z))\|, \quad M_2 := \max_{z \in S_k, k > 0} \|\nabla \psi(\Phi(z))\|
\]
\[
M := M_1 + M_2, \quad D_{\max} := \max_{z_1, z_2 \in S_k, k > 0} D_{\psi}(z_1, z_2)
\]
where we assume that the maximum in each definition is well defined. Our first main result is stated next.

Theorem 2: Consider Algorithm 2 with $j(k) = \alpha(k)$ and any comparator sequence $\{\hat{z}_1^*, \hat{z}_2^*, \ldots, \hat{z}_T^*\}$ with $\hat{z}_k^i \in S_k$. Let Assumptions 1, 2, and 4 hold true. If we choose the Bregman distance $D_{\psi}$ such that
\[
D_{\psi}(\Phi_k(z), \hat{\Phi}_k(z)) - D_{\psi}(z, \hat{z}) \leq 0
\]
and employ nonincreasing sequences
\[
\sum_{i=0}^{\alpha(k)-1} \eta_{k+1}^i \leq \sum_{i=0}^{\alpha(k)-1} \eta_{k}^i
\]
then Algorithm 2 gives the following regret bound:
\[
R(T) \leq D_{\max} \sum_{i=0}^{\alpha(k)-1} \eta_{k+1}^i + \frac{G_f}{2\sigma} \sum_{k=1}^{T} \sum_{i=0}^{\alpha(k)-1} \eta_{k}^i \|\nabla \psi(\Phi_k(z_k^i))\|^2 + M \sum_{k=1}^{T} \|z_{k+1}^c - \Phi_k(z_k^c)\|^2.
\] The proof of this result relies on the next lemma.

Lemma 2: Consider Algorithm 2 with $j(k) = \alpha(k)$ and any comparator sequence $\{\hat{z}_1^*, \hat{z}_2^*, \ldots, \hat{z}_T^*\}$ with $\hat{z}_k^i \in S_k$. Suppose Assumptions 1, 2, and 4 hold. Then, for a given iteration step $i$ and a time instant $k > 0$, we have that
\[
\eta_{k}^i (f_k(\hat{z}_k^i) - f_k(z_k^i))
\]
\[
\leq D_{\psi}(z_k^i, \hat{z}_k^i) - D_{\psi}(z_k^{i+1}, \hat{z}_k^{i+1}) + \frac{\eta_{k}^2}{2\sigma} \|\nabla f_k (\hat{z}_k^i)\|^2.
\] Moreover, if we choose the Bregman distance $D_{\psi}$ such that
\[
D_{\psi}(\Phi_k(z), \hat{\Phi}_k(z)) - D_{\psi}(z, \hat{z}) \leq 0
\]
then
\[
\min_{0 \leq i \leq \alpha(k)} f_k(\hat{z}_k^i) - f_k(z_k^i)
\]
\[
\leq \frac{1}{\sum_{i=0}^{\alpha(k)-1} \eta_{k}^i} \left( D_{\psi}(z_k^i, \hat{z}_k^i) - D_{\psi}(z_k^{i+1}, \hat{z}_k^{i+1}) + \frac{G_f}{2\sigma} \sum_{i=0}^{\alpha(k)-1} (\eta_{k}^i)^2 + M \|z_{k+1}^c - \Phi_k(z_k^c)\|^2 \right).
\]
The proof of Lemma 2 can be found in Appendix D. We are now in a position to prove the theorem.

Proof of Theorem 2: The proof is similar to the proof of [25, Th. 4], which derives a regret upper bound for the dynamic mirror descent in the context of online convex optimization. For ease of notation, we employ \( \sum \eta_k^i \) to refer to the sum of all the step sizes used within the time instant \( k \), i.e., to \( \sum_{i=0}^{\mathbf{u}(k)-1} \eta_k^i \).

By Lemma 2, (49) holds true. Summing (49) over \( k = 1, \ldots, T \) yields

\[
R(T) = \sum_{k=1}^{T} \min_{0 \leq i \leq \mathbf{u}(k)} f_k(\tilde{z}_k^i) - \sum_{k=1}^{T} f_k(z_k^i) \\
\leq \sum_{k=1}^{T} \frac{1}{\eta_k} \left( D_\psi \left( z_k^i, \tilde{z}_k^i \right) - D_\psi \left( z_k^{c+1}, \tilde{z}_k^{c+1} \right) \right) + \frac{G_0^2}{2\eta} \sum_{k=1}^{T} \left( \eta_k^i \right)^2 + M \left\| z_{k+1}^{c+1} - \Phi_k(z_k^{c+1}) \right\|.
\]

(50)

Using (45), i.e., the fact that \( \sum \eta_{k+1}^i \leq \sum \eta_k^i \), we have

\[
\sum_{k=1}^{T} \frac{1}{\sum \eta_k^i} \left( D_\psi \left( z_k^i, \tilde{z}_k^i \right) - D_\psi \left( z_k^{c+1}, \tilde{z}_k^{c+1} \right) \right) = D_\psi \left( \Phi_k(z), \Phi_k(z) \right) - D_\psi \left( \hat{z}_k(z), \hat{z}_k(z) \right) \leq -c \left\| z - \hat{z} \right\|^2
\]

and that \( \mathbf{u}(k+1) \leq \mathbf{u}(k) \).

\[
\eta_k^i = \frac{\sigma}{L_f \sqrt{k}}
\]

(57)

for all \( i = 0, \ldots, \mathbf{u}(k) - 1 \) and \( k > 0 \). Then, the estimation error is GEO and we have that

\[
R(T) \leq \sqrt{T} \frac{L_f}{\mathbf{u}(T)} \left( D_{\max} + M \sum_{k=1}^{T} \left\| z_{k+1}^{c+1} - \Phi_k(z_k^{c+1}) \right\| \right).
\]

(58)

The proof of this result relies on the next lemma.

Lemma 3: Consider Algorithm 2 with \( j(k) = \mathbf{u}(k) \) and any comparator sequence \( \{z_1, z_2, \ldots, z_T\} \) with \( z_k^{c+1} \in S_k \). Let Assumptions 1–4 hold true. Then, we choose the step size \( \eta_k^i \) in (46).

\[
\eta_k^i = \sigma \left( \sum \eta_k^i \right)^{-1}
\]

(59)

for a given iteration step \( i \) and time \( k > 0 \), we have that

\[
\eta_k^i \left( f_k(\tilde{z}_k^{c+1}) - f_k(z_k^{c+1}) \right) \leq D_\psi \left( z_k^{c+1}, \hat{z}_k^{c+1} \right) - D_\psi \left( z_k^{c+1}, \hat{z}_k^{c+1} \right) \]

(60)

Moreover, if we choose the Bregman distance \( D_\psi \) such that

\[
D_\psi \left( \Phi_k(z), \Phi_k(z) \right) - D_\psi \left( \hat{z}_k(z), \hat{z}_k(z) \right) \leq 0
\]

(61)

then

\[
\min_{0 \leq i \leq \mathbf{u}(k)} f_k(\tilde{z}_k^i) - f_k(z_k^i) \leq \frac{1}{\sum_{i=0}^{\mathbf{u}(k)-1} \eta_k^i} \left( D_\psi \left( z_k^{c+1}, \tilde{z}_k^{c+1} \right) - D_\psi \left( z_k^{c+1}, \tilde{z}_k^{c+1} \right) \right) + M \left\| z_{k+1}^{c+1} - \Phi_k(z_k^{c+1}) \right\|^2.
\]

(62)

The proof of Lemma 3 can be found in Appendix D. We are now in a position to prove Theorem 3.
Proof of Theorem 3: GES of the estimation error follows, since \( \eta^k_i \) in (57) satisfies (19), i.e., \( \eta^k_i \leq \frac{n}{2} \). Note that, by Lemma 3, i.e., (62)

\[
R(T) = \sum_{k=1}^{T} \min_{0 \leq i \leq \tau(k)} f_k(\hat{z}^0_k) - \sum_{k=1}^{T} f_k(z^c_k)
\]

\[
\leq \sum_{k=1}^{T} \frac{1}{\eta^k_i} \left( D_{\psi}(z^c_k, z^0_k) - D_{\psi}(z^c_{k+1}, z^0_{k+1}) + M\|z^c_{k+1} - \Phi_k(z^c_k)\| \right).
\]

Since \( it(k + 1) \leq it(k) \), we have that

\[
\sum_{i=0}^{it(k) - 1} \frac{1}{\eta^k_i} \leq \frac{\sigma}{L_f} \sqrt{T} = \frac{\sigma}{L_f} \sqrt{T}.
\]

Hence, (45) holds, and as a consequence, we can derive an upper bound, similar to (51), to obtain

\[
\sum_{k=1}^{T} \frac{1}{\eta^k_i} \|z^c_{k+1} - \Phi_k(z^c_k)\|
\]

\[
\leq \frac{D_{\max} L_f \sqrt{T}}{\eta^T_i} = \frac{D_{\max} L_f \sqrt{T}}{\eta^T_i}.
\]

Moreover, given (64), we have that

\[
\sum_{k=1}^{T} \frac{M}{\eta^T_i} \|z^c_{k+1} - \Phi_k(z^c_k)\|
\]

\[
\leq \frac{M \sqrt{T}}{\eta^T_i} \|z^c_1 - \Phi_0(z^c_0)\|
\]

\[
= \frac{M \sqrt{T}}{\eta^T_i} \|z^c_1 - \Phi_0(z^c_0)\|.
\]

Combining (65) and (66) completes the proof.

A direct consequence of Theorem 3 is that fixing the number of optimization iterations \( it(k) = it(k + 1) =: it \) and increasing it leads to a smaller regret bound. This allows for a tradeoff between computational effort and performance. In fact, if the comparator sequence \( \{z^1_i, z^2_i, \ldots, z^n_i\} \) follows the dynamics described by the a priori estimate operator \( \Phi_k \), closely, and if we let it \( \to \infty \), then the bound in Theorem 3 vanishes and we obtain an algorithm with zero regret, i.e., \( \lim_{k \to \infty} R(T) = 0 \).

We also remark that the condition \( it(k + 1) \leq it(k) \) requires that we employ a smaller or equal number of optimization iterations each time we receive a new measurement. This condition is in line with the intuitive observation that it is preferable to execute more iterations at the beginning of the pMHE iteration scheme, since our regret measure is aggregated over time and, thus, memorizes initially poor estimates.

B. Regret With Respect to Exponentially Stable Comparator Sequences

As we mentioned before, in general, there is no requirement that the comparator sequence converges to the true state. This being said, it is reasonable to restrict the class of comparator sequences to sequences that converge exponentially fast to the true state. We study this case in this section by imposing the following additional assumption.

Assumption 5 (Exponentially stable comparator sequence): The comparator sequence \( \{z^1_i, z^2_i, \ldots, z^n_i\} \) with initial guess \( z^0_0 \) is generated from a state estimator that yields GES error dynamics. More specifically, there exists positive constants \( \alpha_c \geq 1 \) and \( 0 \leq \beta_c < 1 \) such that

\[
\|z_k - z^c_k\| \leq \alpha_c \beta_c^k \|z_0 - z^c_0\|
\]

holds for each \( 0 < k \leq T \). Here, \( z^c_k = \left[ x^c_k, \mathbf{w}_k \right] \).

Notably, when the comparator sequence satisfies the exponential stability assumption, Algorithm 2 leads to constant regret, as our next result shows.

Theorem 4: Consider Algorithm 2 and let Assumptions 1–4 hold true. Suppose that a comparator sequence \( \{z^1_i, z^2_i, \ldots, z^n_i\} \) is generated from a GES estimator with initial guess \( z^0_0 \), as in Assumption 5. If the Bregman distance \( D_{\psi} \) satisfies

\[
D_{\psi}(\Phi_k(z), \Phi_k(\hat{z})) - D_{\psi}(z, \hat{z}) \leq -c \|z - \hat{z}\|^2
\]

for all \( z, \hat{z} \in \mathbb{R}^{(N+1)n} \) and \( \eta^k_i \leq \frac{n}{2} \), then the estimation error is GES and

\[
R(T) \leq \frac{L_f}{2} \frac{\alpha^2 \beta^2}{1 - \beta^2} \|z_0 - z^c_0\|^2 + \frac{L_f}{2} \frac{\alpha^2 \beta^2}{1 - \beta^2} \|z_0 - z^c_0\|^2
\]

with \( \beta := \sqrt{1 - \frac{2c}{\gamma}} \in [0, 1) \) and \( \alpha := \sqrt{\gamma} / \sigma \).

Proof: In view of Theorem 1, GES holds since the Lyapunov function \( V(z_k, \hat{z}^{(k)}_k) = D_{\psi}(z_k, \hat{z}^{(k)}_k) \) satisfies

\[
\Delta V = V(z_k, \hat{z}^{(k)}_k) - V(z_{k-1}, \hat{z}^{(k-1)}_{k-1}) \leq -c \|z_{k-1} - \hat{z}^{(k-1)}_{k-1}\|^2.
\]

In particular, this implies based on (32) that

\[
D_{\psi}(z_k, \hat{z}^{(k)}_k) \leq \beta_c \ D_{\psi}(z_0, \hat{z}_0)
\]

where \( \beta_c := 1 - \frac{2c}{\gamma} \in [0, 1) \). Given that \( D_{\psi} \) is strongly convex and strongly smooth, we have that

\[
\|z_k - \hat{z}^{(k)}_k\|^2 \leq \frac{2}{\sigma} \beta_c^k \ D_{\psi}(z_0, \hat{z}_0) \leq \frac{2}{\sigma} \beta_c^k \|z_0 - \hat{z}_0\|^2.
\]

With \( \beta := \sqrt{1 - 2c/\gamma} \in [0, 1) \) and \( \alpha := \sqrt{\gamma} / \sigma \geq 1 \), we obtain that

\[
\|z_k - \hat{z}^{(k)}_k\| \leq \alpha \beta^k \|z_0 - \hat{z}_0\|.
\]

The regret can be upper bounded as follows:

\[
R(T) = \sum_{k=1}^{T} \min_{0 \leq i \leq \tau(k)} f_k(\hat{z}^i_k) - \sum_{k=1}^{T} f_k(z^c_k)
\]

\[
\leq \sum_{k=1}^{T} f_k \left( \hat{z}^{(k)}_k \right) - \sum_{k=1}^{T} f_k(z^c_k).
\]

Furthermore,

\[
f_k(\hat{z}^{(k)}_k) - f_k(z^c_k) = f_k \left( \hat{z}^{(k)}_k - f_k(z^c_k) + f_k(z^c_k) \right)
\]

\[
- f_k(z^c_k) \leq f_k \left( \hat{z}^{(k)}_k \right) - f_k(z^c_k) \leq f_k(z^c_k) + \left| f_k(\hat{z}^{(k)}_k) - f_k(z^c_k) \right|
\]

\[
+ f_k(z^c_k) - f_k(z_k) \leq f_k(z^c_k) + \left| f_k(\hat{z}^{(k)}_k) - f_k(z_k) \right|.
\]
By Assumption 3, we have that for any $z \in \mathbb{R}^{(N+1)n}$

$$f_k(z) \leq f_k(z_k) + \nabla f_k(z_k)^T(z-z_k) + L_f \frac{1}{2} \|z_k - z\|^2.$$  \hspace{1cm} (76)

Since $f_k$ achieves its minimal value at $z_k$ by Assumption 2, $\nabla f_k(z_k) = 0$ and we obtain in (76) for $z = \hat{z}_k^{(k)}$

$$0 \leq f_k(\hat{z}_k^{(k)}) - f_k(z_k) \leq \frac{L_f}{2} \|z_k - \hat{z}_k^{(k)}\|^2.$$  \hspace{1cm} (77)

Similarly, we have for $z = z_k^c$ in (76)

$$0 \leq f_k(z_k^c) - f_k(z_k) \leq \frac{L_f}{2} \|z_k - \hat{z}_k^{(k)}\|^2.$$  \hspace{1cm} (78)

Substituting the latter two inequalities into (75) yields

$$f_k(\hat{z}_k^{(k)}) - f_k(z_k) \leq \frac{L_f}{2} \|z_k - \hat{z}_k^{(k)}\|^2 + \frac{L_f}{2} \|z_k - z_k^c\|^2.$$  \hspace{1cm} (79)

By Assumption 5 and (73), we obtain

$$f_k(\hat{z}_k^{(k)}) - f_k(z_k) \leq \frac{L_f}{2} \alpha^2 \beta k \|z_0 - \hat{z}_0\|^2 + \frac{L_f}{2} \alpha^2 \beta^2 k \|z_0 - \hat{z}_0\|^2.$$  \hspace{1cm} (80)

Hence,

$$R(T) \leq \sum_{k=1}^{T} \frac{L_f}{2} \alpha^2 \beta k \|z_0 - \hat{z}_0\|^2 + \frac{L_f}{2} \alpha^2 \beta^2 k \|z_0 - \hat{z}_0\|^2.$$  \hspace{1cm} (81)

Therefore, we have that

$$\sum_{k=1}^{T} (\beta^2 k) = \frac{\beta^2 - \beta^2(T+1)}{1 - \beta^2} \leq \frac{\beta^2}{1 - \beta^2}.$$  \hspace{1cm} (82)

By carrying out a similar analysis for the second sum in (81), the desired regret upper bound can be obtained.

We summarize the obtained results of this article in Table I.

### VI. SIMULATION RESULTS

In order to demonstrate the stability and performance properties of the anytime pMHE algorithm, we consider the following discrete-time linear system of the form (1), where:

$$A = \begin{bmatrix} 0.8831 & 0.0078 & 0.0022 \\ 0.1150 & 0.9563 & 0.0028 \\ 0.1178 & 0.0102 & 0.9954 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 32.84 \\ 32.84 \\ 32.84 \end{bmatrix}$$

with $(A, C)$ is detectable. This system is taken from [4], where the nonlinear model of a well mixed, constant volume, isothermal batch reactor is linearized and discretized with a sampling time of $T_s = 0.25$. The associated (continuous time) nonlinear system can be found in [6, Sec. 3]. Given that the states represent concentrations, they are constrained to be nonnegative, i.e., $x_k \geq 0$. We employ the proposed anytime pMHE scheme introduced in Algorithm 2 with the horizon length of $N = 2$ and designed such that the assumptions and conditions of Theorem 3 are fulfilled. For the $a$ priori estimate, we choose $\hat{j}(k) = it(k)$ and design the observer gain $L$ in (12) such that the eigenvalues of $A - LC$ are given by $\lambda = [0.4754, 0.8497, 0.9727]$. Moreover, we only consider the first state in the horizon window $\hat{x}_{k-N}^c$ as decision variable, i.e., we set the stage cost $r$ in (3a) and the model residual $\hat{w}_0$ to be zero. The stage cost $\gamma$ in (3a) is chosen as $\gamma = \frac{1}{2} \|z_k^c\|^2$. With $R = 0.01$. The resulting sum of stage costs at time $k$ is

$$f_k(x) = \frac{1}{2} \sum_{i=k-N}^{k-1} \|y_i - CA^{i-k+N} x\|^2.$$  \hspace{1cm} (85)

Furthermore, we choose the quadratic Bregman distance $D_\psi(x_1, x_2) = \frac{1}{2} \|x_1 - x_2\|^2$. To satisfy the stability condition (56), we design the weight matrix $P$ so that the LMI (39) is satisfied. In addition, we fix the number of iterations $it(k)$, i.e., $it(k) = it(k+1) = 0$. The step sizes are chosen as (57), i.e., $\eta_{it}^k = \frac{\sigma}{\sqrt{it(k)}}$. Here, $\sigma$ denotes the strong convexity parameter of the Bregman distance, which is given by $\sigma = \min(\lambda_i(P))$. The constant $L_f$ is the strong smoothness parameter of $f_k$ defined in (85). It can be computed as

$$L_f = R \sum_{i=k-N}^{k-1} \|CA^{i-k+N}\|^2.$$  \hspace{1cm} (86)

As our estimate, we select at each time $k$ the iterate $\hat{z}_k^{(k)}$ with the minimal cost, i.e., $i_{\min}(k) = \arg \min_{0 \leq i \leq L} f_k(\hat{z}_k^{(k)})$. We compare the obtained stability results with the Luenberger observer designed with the same matrix $L$, as well as with those obtained from Algorithm 1, where the pMHE scheme is based.

---

**TABLE I**

| Theorem | Assumptions | Step size | Result |
|---------|-------------|----------|--------|
| Thm. 1 | $A_1 \cdot A_4$ | $\Delta c_k D_v(x, \hat{x}) \leq -c \|z - \hat{z}\|^2$ | $\eta_k \leq \frac{L_f}{\sqrt{c}}$ |
| Thm. 2 | $A_1, A_2, A_4$ | $\Delta c_k D_v(x, \hat{x}) \leq 0$ | $\sum_{i=1}^{T} \eta_k \leq \sum_{i=1}^{T} \eta_i$ |
| Thm. 3 | $A_1 \cdot A_4$ | $\Delta c_k D_v(x, \hat{x}) \leq -c \|z - \hat{z}\|^2$ | $\eta_k = \frac{L_f}{\sqrt{c}}$ |
| Thm. 4 | $A_1 \cdot A_5$ | $\Delta c_k D_v(x, \hat{x}) \leq -c \|z - \hat{z}\|^2$ | $\eta_k \leq \frac{L_f}{\sqrt{c}}$ |

We employ the following notation for abbreviation: $\Delta c_k D_v(x, \hat{x}) := D_v(x, \hat{x}) - D_v(x, \hat{x})$ and $\sum_{i=1}^{T} \eta_k := \sum_{i=0}^{T} \eta_i$.

---
Evolution of the estimation errors corresponding to the employed estimation strategies over time.

Fig. 2. Evolution of the estimation errors corresponding to the employed estimation strategies over time.

Fig. 3. Evolution of the estimation errors corresponding to anytime pMHE with different number of iterations over time.

on solving (7). For this estimator, we choose the same design parameters of the anytime pMHE iteration scheme given by $N$, $f_k$, $D_0$, and $L$. The resulting estimation errors for each estimation strategy are shown in Fig. 2.

All estimators exhibit GES of the estimation errors. This includes the case where we execute only one iteration of the optimization algorithm per time instant $k$, i.e., $i = 1$. Note that for a small number of iterations, the choice of the observer gain $L$ affects the performance of the estimator. In this case, it is useful to tune $L$ such that a satisfactory performance is attained. Nevertheless, if we perform it $= 200$ iterations, for example, the choice of $L$ does not have much impact on performance and we can observe that the iteration scheme performs even better than Algorithm 1. We illustrate the effect of increasing the number of iterations on the convergence of the estimation error in Fig. 3. We can see that the more we iterate, the faster is the convergence of the estimation error to zero.

We also compare the proposed pMHE iteration scheme with the MHE approach in [13], in which single and multiple iterations of descent methods are performed each time a new measurement becomes available. Given the similarity between the underlying optimization algorithms, we employ the gradient descent for the MHE algorithm in [13], which is referred to as GMHE, with it $= 100$ iterations at each time instant. In the associated cost function, we select the same sum of stage cost (85) as in anytime pMHE. In GMHE, the a priori estimate is set to $\hat{x}_{k-N} = A^{k-N} \hat{x}_{k-N-1}$. Moreover, GES of the estimation error can be ensured through a suitable condition on the step size used in the iteration step of the gradient descent [13, Corollary 1]. However, for this example, and after performing many numerical tests, we were not able to find a suitable value of the step size that satisfies this condition. Nevertheless, we tested the approach for arbitrary values of the step size and observed convergence of the estimation error to zero. For these values, we computed the resulting root-mean-square error (RMSE)

$$\text{RMSE} = \sqrt{\frac{\sum_{k=N}^{T_{\text{sim}}} \|e_k\|^2}{T_{\text{sim}} - N + 1}}$$

(87)

where $T_{\text{sim}}$ denotes the simulation time. For example, if we perform a single iteration per time instant and if the step size is chosen as the pMHE step size, we obtain 1.2694 for GMHE and 1.0913 for anytime pMHE. Note that if we additionally construct the a priori estimate in GMHE based on the Luenberger observer (as is the case in pMHE), we obtain the exact same state estimates. If the step size in GMHE is chosen as 0.01, the RMSE generated by GMHE becomes smaller than that of anytime pMHE and has the value 0.9602. However, if we perform it $= 10$ iterations per time instant, pMHE performs better than GMHE (0.9930 versus 1.0094). This again demonstrates that the bias of the Luenberger observer in anytime pMHE is eventually fading away with each iteration. In GMHE, however, increasing the number of iterations to it $= 10$ does not seem to yield an improved performance (in fact it is worse than that with it $= 1$). Summarizing, although the optimization algorithm in both GMHE and anytime pMHE consists of gradient descent steps, a suitable step size in GMHE that ensures exponential stability of the estimation error is not always easy to compute, which is also remarked in [13]. Nevertheless, we observed in simulations that employing the Newton method instead yields much better results for the approach in [13]. This is due to the fact that the cost function is quadratic, which implies that the MHE problem is solved after one iteration. Moreover, similar to anytime pMHE, the underlying sufficient condition for stability can easily be fulfilled. Since we do not cover the use of the Newton method in the pMHE algorithm, we omit the carried out comparisons due to space constraints.

In the following, we investigate for anytime pMHE the regret (42) with respect to the comparator sequence given by the true states $x_{k-N}$. Note that

$$f_k(x_{k-N}) = \frac{1}{2} \sum_{i=k-N}^{k-1} \|y_i - CAi^{i-k+N} x_{k-N}\|^2_R = 0.$$  

(88)

We employ different number of iterations for each pMHE iteration scheme. After each simulation time $T$, we compute and plot the resulting regrets $R(T)$ as well as the average regrets $R(T)/T$ in Figs. 4 and 5, respectively. Moreover, we plot the regret associated to Algorithm 1 when compared with this optimal sequence of true states.

We can see that anytime pMHE exhibits a sublinear regret in Fig. 4 and that the average regret $R(T)/T$ tends to zero for $T \to \infty$ in Fig. 5. Note that one could also deduce the qualitative behavior of the average regret directly from Fig. 4, since a sublinear regret characterized by the regret bound $O(\sqrt{T})$ implies for the average regret that $\lim_{T \to \infty} R(T)/T \to 0$. Observe also that we can achieve lower regrets by increasing the number of iterations. This observation is in line with the regret upper
Fig. 4. Resulting regrets of anytime pMHE schemes with different number of optimization iterations.

Fig. 5. Resulting average regret of anytime pMHE schemes with different number of optimization iterations.

Fig. 6. Resulting regret for the pMHE scheme with update step (89) and it iterations at each time instant $k$.

bound (58) obtained in Theorem 3. Moreover, we can see that the regret of the iteration scheme with it = 20 is lower than the regret of Algorithm 1, in which the solution of the optimization problem (7) is computed at each time instant. This is due to the novel warm-start strategy in the proposed approach; although stability of the pMHE algorithm is induced from the Luenberger observer, it is only used in the a priori estimate to warm start the optimization algorithm. Hence, its bias is fading away each time we perform the optimization iteration step (13) and an improved performance can be achieved with each iteration. In Algorithm 1, however, we can see that the solution of (7) is designed to lie in proximity of the a priori estimate. This implies that the suboptimal bias of the Luenberger observer is present in each internal iteration of the optimization algorithm used to solve the underlying pMHE problem, which indicates that increasing the number of iterations in this case might not yield to a smaller regret. In fact, in order to validate this observation via simulations and illustrate the impact of Luenberger observer, we also compute the regret $R(T)$ of the pMHE scheme in which, instead of centering the Bregman distance around the previous iterate (see (13)), we use

$$\hat{z}_{k+1} = \arg \min_{z \in S_k} \left\{ \eta_k \nabla f_k(\hat{z}_k) \top z + D_{\psi}(z, \bar{z}_k) \right\}.$$  \hspace{1cm} (89)

In this case, the Bregman distance is always centered around the current a priori estimate $\bar{z}_k$ given by the Luenberger observer. The results are depicted in Fig. 6.

As demonstrated, increasing the number of iterations per time instant in this case does not necessarily yield to lower regrets.

VII. CONCLUSION

In this article, we presented a computationally tractable approach for constrained MHE of discrete-time linear systems. An anytime pMHE iteration scheme is proposed in which a state estimate is computed at each time instant based on an arbitrary number of optimization algorithm iterations. The underlying optimization algorithm consists of a mirror descent-like method, which generalizes the gradient descent and can, therefore, be executed quickly. Under suitable assumptions on the Bregman distance and the step sizes, GES of the estimation errors was established and is ensured after any number of optimization algorithm iterations. In addition, the performance of the iteration scheme was characterized by the resulting real-time regret for which upper bounds were derived. The proposed iteration scheme provides stable estimates after each optimization iteration and possesses a sublinear regret, which can be rendered arbitrarily small by increasing the number of iterations.

The proposed anytime pMHE iteration scheme is conceptually related to the anytime MPC iteration scheme with relaxed barrier functions [28], [29], where stabilizing control inputs is generated after any number of optimization iterations. Our goal in future research is to combine both the MPC and MHE iteration schemes in an overall anytime estimation-based MPC algorithm. Furthermore, comparisons to real-time MHE techniques established in the literature and a further exploration of the computational complexity of the proposed algorithm deserve further research. Moreover, it would be interesting to study the robustness properties of the iteration scheme with respect to process and measurement disturbances.

APPENDIX A

REFORMATION OF THE ESTIMATION PROBLEM

Using the system dynamics (3b) and (3c), we can write each output residual $\hat{v}_i$ in the estimation window in terms of the decision variable $\hat{z}_k$ defined in (4) as follows:

$$\hat{v}_i = y_i - C\hat{x}_i = y_i - O_i \hat{x}_{k-N} - C\bar{u}_i - \sum_{j=k-N}^{i-1} CA^{i-j-1}\hat{w}_j.$$  \hspace{1cm} (90)
with \( O_i := CA^{i-k+N} \) and \( \tilde{u}_i := \sum_{j=k-N}^{i-1} A^{i-j-1} Bu_j \). We obtain the sum of stage costs
\[
f_k(\tilde{z}_k) := \sum_{i=k-N}^{k-1} q(\tilde{w}_i)
+ r \left( y_i - O_i \tilde{x}_{k-N} - C \tilde{u}_i - \sum_{j=k-N}^{i-1} A^{i-j-1} \tilde{w}_j \right).
\]  
(91)

The matrices \( G \) and \( F \) in the constraint set \( S_k \) defined in (6) are given by
\[
G := \begin{bmatrix}
C_x \\
C_x A \\
\vdots \\
C_x A^{N-1}
\end{bmatrix} \in \mathbb{R}^{(N+1)q_k \times n}
\]

\[
F := \begin{bmatrix}
0 & 0 & \ldots & 0 \\
C_x & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_x A^{N-2} & C_x A^{N-3} & \ldots & C_x
\end{bmatrix} \in \mathbb{R}^{(N+1)q_k \times n}
\]

and the vector \( E_k \) is
\[
E_k := \begin{bmatrix}
d_x \\
d_x - C_x \tilde{u}_{k-N+1} \\
\vdots \\
d_x - C_x \tilde{u}_k
\end{bmatrix} \in \mathbb{R}^{(N+1)q_k}.
\]  
(92)

For more details on Bregman distances, we refer the reader to [31].

\section*{Appendix C

\section*{Proof of Lemma 1}

\textit{Proof:} The proof generalizes and follows similar steps as in the proof of [30, Prop. 2], in which the performance of the online gradient descent method is investigated. Convexity of \( f_k \) implies that
\[
f_k(z) \geq f_k(\hat{z}_k) + \nabla f_k(\hat{z}_k)^\top (z - \hat{z}_k)
\]  
(97)

for any \( z \in S_k \), and hence,
\[
f_k(z) \geq f_k(\hat{z}_k) + \nabla f_k(\hat{z}_k)^\top (\hat{z}_k^{i+1} - \hat{z}_k)
\]  
(98)

\[+ \frac{1}{\eta_k} \left( \nabla \psi(\hat{z}_k) - \nabla \psi(\hat{z}_k^{i+1}) \right)^\top (z - \hat{z}_k^{i+1}).
\]  
(99)

Thus, (98) becomes
\[
f_k(z) \geq f_k(\hat{z}_k) + \nabla f_k(\hat{z}_k)^\top (\hat{z}_k^{i+1} - \hat{z}_k)
\]  
(100)

\[+ \frac{1}{\eta_k} \left( \nabla \psi(\hat{z}_k) - \nabla \psi(\hat{z}_k^{i+1}) \right)^\top (z - \hat{z}_k^{i+1}).
\]  
(101)

Since the gradients of \( f_k \) are Lipschitz continuous by Assumption 3, we have that
\[
f_k(\hat{z}_k^{i+1}) \leq f_k(\hat{z}_k) + \nabla f_k(\hat{z}_k)^\top (\hat{z}_k^{i+1} - \hat{z}_k)
\]  
(102)

\[+ \frac{L_f}{2} \| \hat{z}_k^{i+1} - \hat{z}_k \|^2
\]  
(103)

and hence,
\[
f_k(z) \geq f_k(\hat{z}_k) - \frac{L_f}{2} \| \hat{z}_k^{i+1} - \hat{z}_k \|^2
\]  
(104)

In view of the three points identity (95) and the strong convexity of \( D_\psi \), we have
\[
(\nabla \psi(\hat{z}_k) - \nabla \psi(\hat{z}_k^{i+1}))^\top (z - \hat{z}_k^{i+1}) = D_\psi(\bar{z}_k, \hat{z}_k^{i+1}) + D_\psi(\hat{z}_k^{i+1}, \bar{z}_k) - D_\psi(z, \hat{z}_k^{i+1}) \geq D_\psi(z, \hat{z}_k^{i+1}) + \frac{\sigma}{2} \| \hat{z}_k^{i+1} - \hat{z}_k \|^2 - D_\psi(z, \hat{z}_k). \]  
(105)

Therefore, using (102)
\[
f_k(z) \geq f_k(\hat{z}_k^{i+1}) - \frac{L_f}{2} \| \hat{z}_k^{i+1} - \hat{z}_k \|^2
\]  
(106)

\[+ \frac{1}{\eta_k} \left( D_\psi(z, \hat{z}_k^{i+1}) - D_\psi(z, \hat{z}_k) + \frac{\sigma}{2} \| \hat{z}_k^{i+1} - \hat{z}_k \|^2 \right)
\]  
(107)

\[= f_k(\hat{z}_k^{i+1}) + \left( \frac{\sigma}{2} - L_f \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2
\]  
(108)

\[+ \frac{1}{\eta_k} \left( D_\psi(z, \hat{z}_k^{i+1}) - D_\psi(z, \hat{z}_k) \right).
\]  
(109)
We set $z = z_k \in \mathcal{S}_k$, i.e., the true state with zero model residual, and obtain
\[
0 \geq f_k(z_k) - f_k(\hat{z}_k^{i+1}) \geq \frac{1}{2} \left( \frac{\eta_k}{\sigma} - L_f \right) \| \hat{z}_k^{i+1} - z_k \|^2 
\]
\[
+ \frac{1}{\eta_k} \left( D_{\psi}(z_k, \hat{z}_k^{i+1}) - D_{\psi}(z_k, \hat{z}_k) \right). 
\]  
(105)

The inequality $f_k(\hat{z}_k^{i+1}) \geq f_k(z_k)$ holds by Assumption 2, which states that $f_k$ achieves its minimal value at $z_k$. Hence, we get
\[
D_{\psi}(z_k, \hat{z}_k^{i+1}) \leq D_{\psi}(z_k, \hat{z}_k) \leq \frac{\eta_k}{\sigma} \left( L_f - \frac{\sigma}{\eta_k} \right) \| \hat{z}_k^{i+1} - z_k \|^2 
\]
which proves the first statement in Lemma 1. Applying (106) for each two subsequent iterations $i$ and $i+1$ (where $i = 0, \ldots, j$) yields
\[
D_{\psi}(z_k, \hat{z}_k)
\]
\[
\leq D_{\psi}(z_k, \hat{z}_k^{i}) + \frac{1}{2} \left( \frac{\eta_k}{\sigma} L_f - \sigma \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2 
\]
\[
\leq D_{\psi}(z_k, \hat{z}_k^{i-1}) + \frac{1}{2} \left( \frac{\eta_k}{\sigma} L_f - \sigma \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2 
\]
\[
+ \frac{1}{2} \left( \frac{\eta_k}{\sigma} L_f - \sigma \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2 
\]
\[
\leq \cdots 
\]
\[
\leq D_{\psi}(z_k, \hat{z}_k^{0}) + \frac{1}{2} \sum_{i=0}^{j-1} \left( \frac{\eta_k}{\sigma} L_f - \sigma \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2. 
\]  
(107)

Since $\hat{z}_k^{0} = \Pi^{\mathcal{K}}_{\eta_k}(z_k)$ by (94) and $z_k \in \mathcal{S}_k$, in view of (96) in Lemma 5, we have
\[
0 \leq D_{\psi}(z_k^{0}, z_k) \leq D_{\psi}(z_k, \hat{z}_k) - D_{\psi}(z_k, \hat{z}_k). 
\]  
(108)

Thus, $D_{\psi}(z_k, z_k^{0}) \leq D_{\psi}(z_k, z_k)$ and we obtain in (107)
\[
D_{\psi}(z_k, \hat{z}_k)
\]
\[
\leq D_{\psi}(z_k, z_k) + \frac{1}{2} \sum_{i=0}^{j-1} \left( \frac{\eta_k}{\sigma} L_f - \sigma \right) \| \hat{z}_k^{i+1} - \hat{z}_k \|^2. 
\]  
(109)

**APPENDIX D**

**PROOF OF LEMMA 2**

**Proof:** The following analysis is based on the convergence proof of the mirror descent algorithm presented in [22]. Since $z_k \in \mathcal{S}_k$, we can evaluate the optimality condition (99) of $\hat{z}_k^{i+1}$ for $z = z_k$ to obtain
\[
\left( \frac{\eta_k}{\sigma} \nabla f_k(z_k) + \nabla \psi(\hat{z}_k^{i+1}) - \nabla \psi(\hat{z}_k) \right)^{\top} (z_k^{i+1} - z_k^{i+1}) \geq 0. 
\]  
(110)

Given that $f_k$ is convex, we have
\[
\eta_k \left( f_k(z_k^{i+1}) - f_k(z_k) \right) \leq \eta_k \nabla f_k(z_k) \quad (z_k^{i+1} - z_k^{i+1}) 
\]
\[
= s_1 + s_2 + s_3 
\]  
(111a)
where
\[
s_1 := (\nabla \psi(\hat{z}_k^{i+1}) - \nabla \psi(\hat{z}_k^{i+1}))^{\top} (z_k^{i+1} - z_k^{i+1}) 
\]  
(111b)
\[
s_2 := (\nabla \psi(\hat{z}_k^{i+1}) - \nabla \psi(\hat{z}_k^{i+1}))^{\top} (z_k^{i+1} - z_k^{i+1}) 
\]  
(111c)
\[
s_3 := \eta_k \nabla f_k(z_k) \quad (z_k^{i+1} - z_k^{i+1}) 
\]  
(111d)

By (10), $s_1 \leq 0$. Using the three-points identity (95) as well as the strong convexity of $D_{\psi}$ assumed in Assumption 4, we have that
\[
s_2 \leq - (\nabla \psi(\hat{z}_k^{i+1}) - \nabla \psi(\hat{z}_k^{i+1}))^{\top} (z_k^{i+1} - z_k^{i+1}) 
\]
\[
= D_{\psi}(z_k^{c}, \hat{z}_k^{i+1}) - D_{\psi}(z_k^{c}, z_k^{i+1}) - D_{\psi}(z_k^{c}, z_k^{i+1}) 
\]
\[
\leq D_{\psi}(z_k^{c}, \hat{z}_k^{i+1}) - D_{\psi}(z_k^{c}, z_k^{i+1}) - \frac{\sigma}{2} \| z_k^{i+1} - z_k \|^2. 
\]  
(112)

Moreover, by Young’s inequality
\[
s_3 \leq \eta_k \left( \frac{\eta_k}{2} \| \nabla f_k(z_k) \|^2 + \frac{\sigma}{2 \eta_k} \| \hat{z}_k^{i+1} - z_k \|^2 \right). 
\]  
(113)

Hence, we obtain the first statement of Lemma 2 by substituting (112) and (113) into (111). Evaluating (47) for $i = 0$ yields
\[
\eta_k \left( f_k(z_k^{0}) - f_k(z_k^{0}) \right) 
\]
\[
\leq D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - D_{\psi}(z_k^{c}, \hat{z}_k^{0}) + \frac{\eta_k}{2} \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
= D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - D_{\psi}(z_k^{c}, \hat{z}_k^{0}) + \frac{\eta_k}{2} \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
+ T_1 + T_2 
\]  
(114a)
where
\[
T_1 := D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - D_{\psi}(z_k^{c}, \hat{z}_k^{0}) 
\]  
(114b)
\[
T_2 := D_{\psi}(z_k^{0}, \hat{z}_k^{0}) - D_{\psi}(z_k^{0}, \hat{z}_k^{0}) 
\]  
(114c)

We can compute an upper bound for each of these terms as follows. Since $z_k^{i+1} = \Phi_k(z_k^{i+1})$, using (48), we have
\[
T_1 = D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - D_{\psi}(z_k^{c}, \hat{z}_k^{0}) 
\]
\[
\leq D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - D_{\psi}(z_k^{c}, \hat{z}_k^{0}) 
\]  
(115)

Moreover, employing (47) in Lemma 2 (as just proved above) for each iteration step starting from $i = \text{it}(k) - 1$ to $i = 2$ yields
\[
T_1 \leq D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - \frac{\eta_k}{2} \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
+ \eta_k \frac{\text{it}(k) - 1}{2} \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
\leq D_{\psi}(z_k^{c}, \hat{z}_k^{0}) - \frac{\eta_k}{2} \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
+ \eta_k \| \nabla f_k(z_k^{0}) \|^2 
\]
\[
\leq \cdots 
\]
\[
\leq \sum_{i=1}^{\text{it}(k) - 1} \frac{\eta_k}{2} \| \nabla f_k(z_k^{0}) \|^2 + \sum_{i=1}^{\text{it}(k) - 1} \eta_k (z_k^{c}, \hat{z}_k^{0}) 
\]  
(116)
Given that \( \tilde{z}^0_k = \Pi^\perp_{S_k} (\tilde{z}_k) \) and \( z^c_k \in S_k \), by (96) in Lemma 5, we have
\[
0 \leq D_\psi (\tilde{z}^0_k, \tilde{z}_k) \leq D_\psi (z^c_k, \tilde{z}_k) - D_\psi (z^c_k, \tilde{z}^0_k)
\] (117)
for all \( k > 0 \). Hence, \( D_\psi (z^c_k, \tilde{z}^0_k) \leq D_\psi (x^c_k, \tilde{z}_k) \), for all \( k > 0 \), and we obtain
\[
T_2 = D_\psi (\tilde{z}^c_{k+1}, \tilde{z}_k) - D_\psi (\Phi_k (z^c_k), \tilde{z}_{k+1}) \leq D_\psi (\tilde{z}^c_{k+1}, \tilde{z}_k) - D_\psi (\Phi_k (z^c_k), \tilde{z}_{k+1}).
\] (118)

In addition, using the definition of the Bregman distance and the convexity of \( \psi \), we get
\[
T_2 \leq \psi (\tilde{z}^c_{k+1}) - \psi (\tilde{z}_k) + \psi (\Phi_k (z^c_k)) - \psi (\tilde{z}_{k+1}) \leq \psi (\tilde{z}^c_{k+1} - \tilde{z}_k) + \psi (\Phi_k (z^c_k) - \tilde{z}_{k+1}) - D_\psi (\Phi_k (z^c_k), \tilde{z}_{k+1}).
\] (119)

Substituting (116) and (119) into (114) yields
\[
\eta^0_k (f_k (z^0_k) - f_k (z^c_k)) \leq D_\psi (z^0_k, z^c_k) - D_\psi (z^c_k, \tilde{z}^0_k) + \frac{\eta^0_k}{2\sigma} \| \nabla f_k (z^c_k) \|^2 + \sum_{i=1}^{i(k)-1} \left( \eta^0_k \right)^2 \| \nabla f_k (\tilde{z}^i_k) \|^2 + \sum_{i=1}^{i(k)-1} \eta^0_k (f_k (z^c_k) - f_k (\tilde{z}^i_k)) + M \| z^c_k - z^c_k \|. \]
(120)

Rearranging the previous inequality yields
\[
\sum_{i=0}^{i(k)-1} \eta^0_k (f_k (\tilde{z}^i_k) - f_k (z^c_k)) \leq \sum_{i=0}^{i(k)-1} \eta^0_k (f_k (z^c_k)) + \sum_{i=0}^{i(k)-1} \left( \eta^0_k \right)^2 \| \nabla f_k (\tilde{z}^i_k) \|^2 + M \| z^c_k - z^c_k \|. \]
(121)

Since \( \min_{0 \leq i \leq i(k)} f_k (\tilde{z}^i_k) \) \( \sum_{i=0}^{i(k)-1} \eta^0_k \leq \sum_{i=0}^{i(k)-1} \eta^0_k f_k (\tilde{z}^i_k) \), we obtain
\[
\left( \min_{0 \leq i \leq i(k)} f_k (\tilde{z}^i_k) - f_k (z^c_k) \right)^{i(k)-1} \sum_{i=0}^{i(k)-1} \eta^0_k \leq \sum_{i=0}^{i(k)-1} \left( \eta^0_k \right)^2 G^2 + M \| z^c_k - z^c_k \|. \]
(122)

Dividing the latter inequality by \( \sum_{i=0}^{i(k)-1} \eta^0_k \) yields the desired result. □

Note that \( R_2 = T_2 \) in (114c). Hence, by (119), we obtain
\[
\eta_k^0 \left( f_k(\tilde{z}_k^i) - f_k(\tilde{z}_k^0) \right) \leq D_\psi \left( \tilde{z}_k^i, \tilde{z}_k^0 \right) - D_\psi \left( \tilde{z}_k^{i+1}, \tilde{z}_k^{0+1} \right) + \sum_{i=2}^{\text{it}(k)} \eta_k^{i-1} \left( f_k(\tilde{z}_k^i) - f_k(\tilde{z}_k^{i-1}) \right) + M \| z_{i+1} - \Phi_k (z_k^i) \|.
\] (129)

Rearranging the previous inequality yields
\[
\sum_{i=1}^{\text{it}(k)} \eta_k^{i-1} \left( f_k(\tilde{z}_k^i) - f_k(\tilde{z}_k^{i-1}) \right)
\leq \sum_{i=1}^{\text{it}(k)} \eta_k^{i-1} f_k(\tilde{z}_k^i)
\leq \left( \sum_{i=0}^{\text{it}(k)-1} \eta_k^i \right) f_k(\tilde{z}_k^i)
\leq \sum_{i=0}^{\text{it}(k)-1} \eta_k^i f_k(\tilde{z}_k^i)
\] (131)

we obtain
\[
\sum_{i=0}^{\text{it}(k)-1} \eta_k^i f_k(\tilde{z}_k^i) - f_k(\tilde{z}_k)^{i-1}) \leq D_\psi (\tilde{z}_k^i, \tilde{z}_k^{i+1}) - D_\psi (\tilde{z}_k^{i+1}, \tilde{z}_k^{i+2}) + M \| z_{i+1} - \Phi_k (z_k^i) \|. \] (132)

Dividing the latter inequality by \(\sum_{i=0}^{\text{it}(k)-1} \eta_k^i\) yields the desired result. \(\square\)

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