Quantum uncertainties in sequential measurements under prediction and retrodiction

Le Bin Ho\textsuperscript{1,2,}\textsuperscript{*}

\textsuperscript{1}Frontier Research Institute for Interdisciplinary Sciences, Tohoku University, Sendai 980-8578, Japan
\textsuperscript{2}Department of Applied Physics, Graduate School of Engineering, Tohoku University, Sendai 980-8579, Japan

(Dated: April 5, 2022)

In sequential measurements, we consider the prediction is as an inference of the subsequent observational data from the prior measurements, while the retrodiction is as an inference of the prior observational data from the subsequent measurements. We theoretically study the impact of the quantum backaction (QBA) in the sequential measurements on the inferred observational data from the prediction and retrodiction. The QBA of the prediction behaves like the disturbance caused by prior measurements affecting subsequent measurements, whereas the QBA of the retrodiction behaves through the disturbance caused by the subsequent measurements affecting the prior measurements. In particular, we consider the sequential measurements of two observables $A$ and $B$, one after another, i.e., $A$ first and $B$ later, and then we theoretically formulate the quantum uncertainties in the values of these observables (for both prediction and retrodiction) as figures of merit for the disturbances. These results are illustrated in spin systems, where we present the dependence of the uncertainty in subsequent measurement on the prior measurement and vice versa. We further find a hint for beyond the stronger uncertainty relation. The work is finally extended to $N$-sequential measurements.

I. INTRODUCTION AND PRELIMINARY DEFINITIONS

The theory of quantum measurement lies at the heart of quantum physics in the scene that it reveals all hidden properties of the being measured system [1, 2]. Yet, characterizing quantum system’s quantities through measurements has faced enormous challenges, wherein the quantum backaction (QBA) of measurements is the one in those. It inevitably destroys the initial system state and thus disturbs the system. This impact becomes manifest in sequential measurements [3, 4]: later measurements will be altered by the QBA of former ones.

In sequential measurements, let us define the prediction as an inference of the subsequent observational data from the prior measurements, and the retrodiction is an inference of the prior observational data from the subsequent measurements. For example, in the sequential measurements of two observables $A$ and $B$ ($B$ after $A$), we can predict the obtained probability of the subsequent $B$-measurement conditional on the presence of the prior $A$-measurement, and vice versa, we can retrodict the obtained probability of the prior $A$-measurement conditional on the presence of the subsequent $B$-measurement.

To further examine the QBAs in sequential measurements, we define them as follows. The QBA of prediction is defined by the disturbance caused by prior measurements affecting subsequent measurements, whereas the QBA of retrodiction is expressed through the disturbance caused by the subsequent measurements affecting the prior measurements. The former has been widely being discussed and explored recently, which include sequential weak measurements [5–9], reducing the QBA effect [10–13], and manipulating qubits [14]. The concepts of violating noncontextual realism [15] and the maximal extracting knowledge [16] through sequential measurements are also worth mentioning in this context. Yet, the study on the QBA of retrodiction is still quite humble [17, 18], despite the role of retrodiction on quantum trajectory has been studying so far in continuous measurements [19–22].

In this work, we theoretically examine the QBA of both the prediction and retrodiction in sequential measurements. In the prediction, we evaluate the impact (QBA) of a prior measurement on a subsequent measurement via the change in the corresponding quantum uncertainty in the value of the measured observable. Likewise, in the retrodiction, we evaluate the impact (QBA) of the subsequent measurement on the prior measurement. To do that, we consider the sequential measurements of two observables in two different models: joint measurement model and conditional measurement model (see Fig. 1). In the joint measurement model, the system jointly couples to both two measuring pointers and the joint state in the pointers is measured afterwards. We found that in this model, only the prior measurement causally disturbs the subsequent measurement (only prediction). Whereas, in the conditional measurement model, the subsequent measurement is conditional on the prior measurement and vice versa. It is thus not only the prediction but the retrodiction is also presented, such that the subsequent measurement can noncausally disturb back the prior measurement. We first theoretically formulate such a general mutual influence of the prediction and retrodiction and then illustrate them in spin systems. We also found a hint for beyond the stronger uncertainty relation, which was proposed by Maccone and Pati [23]. We finally provide a general framework for the sequential $N$ measurements.

This paper is organized as follows. In Sec. II, we formulate the sequential measurements framework where both the joint measurement model and conditional measure-
ment model are given. The illustrative results are presented in Sec. III, and the general of N-sequential measurements is provided in Sec. IV. We summarize the work in Sec. V.

II. SEQUENTIAL MEASUREMENTS

For a given system $S$, we consider a measurement of $A$ following by an incompatible measurement of $B$. These measurements are conducted by coupling system $S$ with two corresponding pointers $P_1$ and $P_2$, respectively. The measurement scheme is schematically shown in Fig. 1. The interactions are given by

$$U_1 = e^{-iA \otimes P_1}, \quad U_2 = e^{-iB \otimes P_2},$$

where $p_1$ and $p_2$ are momentum operators for $P_1$ and $P_2$, respectively. The initial state for each pointer is described by a Gaussian wave function such that $|\psi_i(x)\rangle = \int \psi_i(x)|x\rangle \, dx$, where $\psi_i(x) = \left(\frac{1}{2\pi \sigma_i^2}\right)^{1/4} \exp\left(-\frac{x^2}{4\sigma_i^2}\right).$

is the quadrature distribution of the input pointer state, where $i = 1, 2$ which corresponds to pointers $P_1$ and $P_2$, respectively, $|x\rangle$ is a continuous eigenstate of the pointer position $x$, $\sigma_i$ is the width of the measurement pointer $P_i$, which stands for the measurement strength: $\sigma_i \to 0$ is a strong measurement, while $\sigma_i \to \infty$ is a weak measurement.

A. Joint measurement model (prediction)

We first consider a joint measurement approach in the sequential measurements where the later measurements are causally affected by the former measurements. Assume the initial joint state of $S \otimes P_1 \otimes P_2$ is

$$\rho_{\text{fin}} = \rho_0 \otimes |\psi_1(x)\rangle\langle\psi_1(x)| \otimes |\psi_2(x)\rangle\langle\psi_2(x)|.$$  

Subsequent to the sequential measurements, the joint state will transform to $\rho_{\text{fin}} = U_2 U_1 \rho_0 U_1^\dagger U_2^\dagger$. Insert the identity operators $I_1 = \int |m\rangle\langle m| \, dm$ and $I_2 = \int |n\rangle\langle n| \, dn$ for the pointers, we obtain (see App. A)

$$\rho_{\text{fin}} = \int N_n M_m \rho_0 M_m^\dagger N_n^\dagger \otimes |m\rangle\langle m| \otimes |n\rangle\langle n| \, d^4 t,$$

where $t$ represents the 4-tuple $(m, m', n, n')$ and $d^4 t$ is the 4-dimensional volume differential. The operators $M, N$ are the positive-operator valued measure (POVM) for the first and second measurement, respectively,

$$M_m = \int \psi_1(x - a)|a\rangle\langle a| \, da,$$  

$$N_n = \int \psi_2(x - b)|b\rangle\langle b| \, db,$$

where $|a\rangle$ is an eigenstate of $A$ in accordance with an eigenvalue $a$, and similar for $B$.

To evaluate the uncertainty of each measurement, we consider the (unnormalized) density state of the two pointers after the sequential measurements $\eta = \text{Tr}_S[\rho_{\text{fin}}]$, where $\text{Tr}_S[\cdot]$ is a partial trace w.r.t. system $S$. We explicitly derive (see App. B)

$$\eta = \iiint \langle b \rangle \langle a | \rho_0 | a' \rangle \langle a' | b \rangle \times$$

$$|\psi_1(x - a)\rangle\langle\psi_1(x - a')| \otimes |\psi_2(x - b)\rangle\langle\psi_2(x - b)| \, da \, db \, da',$$

Here, it represents a physical insightful result that the system’s state is collapsed into the eigenstates of the two observables $A$ and $B$, while the pointers’ states are shifted according to their eigenvalues.

The expectation value of the position operator $x_A$ in
pointer $\mathcal{P}_1$ is given by
\[
\langle x_1 \otimes I_2 \rangle = \frac{\text{Tr}_{\mathcal{P}_1,\mathcal{P}_2}[(x_1 \otimes I_2)\eta]}{\text{Tr}_{\mathcal{P}_1,\mathcal{P}_2}[\eta]} = \text{Tr}_{\mathcal{S}}[A\rho_0].
\] (8)

Then, the uncertainty of the first measurement is given through the variance of $x_1$ (see App. C)
\[
\Delta^2(x_1) = \langle (I_1 \otimes I_2)^2 \rangle - \langle I_1 \otimes I_2 \rangle^2 = \sigma_1^2 + \Delta^2(A),
\] (9)
where $\Delta^2(A) = \text{Tr}_{\mathcal{S}}[A^2\rho_0] - (\text{Tr}_{\mathcal{S}}[A\rho_0])^2$. It is causal that the variance of the first measurement does not depend on the second measurement, and it is given via the measurement strength $\sigma_1$. Inversely, one can infer the uncertainty in the value of $\mathcal{A}$ in system $\mathcal{S}$ from the variance of $x_1$ in pointer $\mathcal{P}_1$, i.e., $\Delta^2(\mathcal{A}) = \Delta^2(x_1) - \sigma_1^2$, (see also [18]). Here, the difference amount is $\sigma_1^2$, which stands for the shot noise fluctuations.

Next, to evaluate the backaction effect (QBA) of the first measurement on system $\mathcal{S}$, we examine the uncertainty of the second measurement as follows
\[
\Delta^2(x_2) = \langle (I_1 \otimes x_2)^2 \rangle - \langle I_1 \otimes x_2 \rangle^2 = \sigma_2^2 + \Delta^2(B'),
\] (10)
where the expectation $\langle I_1 \otimes x_2 \rangle = \frac{\text{Tr}_{\mathcal{P}_1,\mathcal{P}_2}[(I_1 \otimes x_2)\eta]}{\text{Tr}_{\mathcal{P}_1,\mathcal{P}_2}[\eta]}$, and
\[
\Delta^2(B') = \text{Tr}_{\mathcal{S}}[B^2\rho_1] - (\text{Tr}_{\mathcal{S}}[B\rho_1])^2,
\] (11)
is the uncertainty in the value of $B$ that associates with the system’s state $\rho_1$ after the first measurement and can be inferred from $\Delta^2(x_2) - \sigma_2^2$. Here, $\rho_1 = \text{Tr}_{\mathcal{P}_2}[U_1(\rho_0 \otimes |\psi_1(x)\rangle\langle\psi_1(x)|)U_1^\dagger]$ is the system’s state after the first measurement. Noting that in this approach, we do not constrain the outcomes of the measurements, and thus $\rho_1$ is normalized. Straightforwardly, we have
\[
\rho_1 = \int \int |a\rangle\langle a|\rho_0\langle a' | \langle a' | e^{-\frac{(a-a')^2}{2\sigma_1^2}} da'da',
\] (12)
See detailed calculation in App. C. Obviously, if the initial state $\rho_0$ is an eigenstate of $A$, i.e., $\rho_0 = |a_i\rangle\langle a_i|$, then $\rho_1 = \rho_0$, which implies that system $\mathcal{S}$ will not be disturbed after the first measurement [12]. In general, however, system $\mathcal{S}$ is disturbed through the first measurement and thus $\rho_1 \neq \rho_0$. Hence, Eq. (11) explicitly gives
\[
\Delta^2(B') = \int b^2(b|\rho_1|b) \ db - \left[ \int b(b|\rho_1|b) \ db \right]^2,
\] (13)
which significantly depends on the first measurement.

In general, the uncertainty in the value of $B$ after the first measurement is different from that one before the first measurement due to the QBA, such that $\Delta^2(B') \neq \Delta^2(B)$, where $\Delta^2(B) = \text{Tr}_{\mathcal{S}}[B^2\rho_0] - (\text{Tr}_{\mathcal{S}}[B\rho_0])^2$ is the uncertainty in the value of $B$ that associates with the system’s state $\rho_0$ before the first measurement. The dependence of $\Delta^2(B')$ on the first measurement is a manifestation of the in situ causal effect, whereby the later measurements will be affected by the former measurements, but not vice versa.

Furthermore, when the first measurement is weak, $\sigma_1 \to \infty$, the variance of the second measurement is simplified to $\Delta^2(x_2) = \sigma_2^2 + \Delta^2(B)$, which does not depend on the first measurement, or in other words, the QBA is eliminated [24, 25]. The cost to be paid is that, however, the fluctuation of the first measurement will be increased, i.e., $\Delta^2(x_1)$ is large [24, 26]. Discussing this issue is out of the scope here, where we rather focus on the mutual influence in the sequential measurements.

### B. Conditional measurement model (prediction and retrodiction)

In this approach, assume that we measure $x_1$ in pointer $\mathcal{P}_1$ and get an outcome $x_1$. After the first measurement, the initial system state $\rho_0$ will transform to an unnormalized density state
\[
\rho_1 = M_{x_1\rho_0}M_{x_1}^\dagger,
\]
\[
= \int \int |a\rangle\langle a|\rho_0\langle a' | \langle a' | \psi_1(x_1-a)\psi_1(x_1-a') da'da',
\] (14)
Here, we use $\rho_1$ to specify that the system’s state is given according to the outcome of the measurement (see, for example, Chap. II in Ref. [27]).

For a predictive process, we consider the situation that the obtained outcome of the second measurement on $x_2$ is affected by the outcome of the first measurement. The conditional probability is given by (see App. D)
\[
\text{Pr}(x_2|x_1) = \frac{\text{Tr}[N_{x_2}\rho_1 N_{x_2}^\dagger]}{\text{Tr}[N_{x_2}\rho_1 N_{x_2}^\dagger]} \ dx_2 = \int \frac{\psi_2^\dagger(x_2-b)(b|\rho_1|b) \ db}{\int \psi_2^\dagger(x_2-b)(b|\rho_1|b) \ db} \ dx_2',
\] (15)
where $x_2$ is the outcome of the second measurement. Then, the conditional expectation value is given by
\[
\langle x_2 \rangle_{x_1} = \int_{-\infty}^{\infty} x_2 \text{Pr}(x_2|x_1) \ dx_2.
\] (16)
In general, this expectation value will be affected by the QBA of the first measurement. The conditional uncertainty of this second measurement thus is
\[
\Delta^2(x_2|x_1) = \langle x_2^2 \rangle_{x_1} - \left[ \langle x_2 \rangle_{x_1} \right]^2 = \sigma_2^2 + \Delta^2(B|A).
\] (17)
Here, $\Delta^2(B|A)$ is the uncertainty in the value of $B$ condition on the prior measurement of $A$, which can be inferred from $\Delta^2(x_2|x_1) - \sigma_2^2$, and in general, it depends on the measurement's strength of the first measurement. We will illustrate such dependence in Sec. III below.

Likewise, for a retrodictive process, where the first measurement was noncausally affected by the second measurement. To evaluate such kind of noncausal back-action, we consider the conditional probability

$$
\text{Pr}(x_1|x_2) = \frac{\text{Tr} \left[ N_{x_2} \tilde{\rho}_1 N_{x_2}^T \right]}{\text{Tr} \left[ N_{x_2} \tilde{\rho}_1 N_{x_2}^T \right] dx_1'},
$$

$$
= \frac{\int \psi_2^* (x_2 - b) (b | \tilde{\rho}_1 | b) db}{\int \psi_2^* (x_2 - b) (b | \tilde{\rho}_1 | b) db} dx_1'.
$$

The expectation and conditional variance are given by $\langle x_1 \rangle_{x_2} = \int x_1 \text{Pr}(x_1|x_2) dx_1$ and $\Delta^2(x_1|x_2) = \langle x_1^2 \rangle_{x_2} - [\langle x_1 \rangle_{x_2}]^2$. Similarly, we can infer the uncertainty in the value of $A$ condition on the subsequent measurement of $B$ by

$$
\Delta^2(A|B) = \Delta^2(x_1|x_2) - \sigma_1^2.
$$

Different from the joint measurement approach, here, in general, $\Delta^2(A|B) \neq \Delta^2(A)$, as a result of the retrodictive effect caused by the second measurement, which behaves as a noncausal effect. Equations (15, 18) explicitly show the impacts of the prior measurement of $A$ on the substantial measurement of $B$ (prediction), and vice versa (retrodiction), respectively.

C. Beyond stronger uncertainty relations

One foundation issue in measurements of incompatible observables is that it is infeasible to attain the uncertainties with arbitrary precision, neither the measurements are separable [28, 29], simultaneous, nor sequential [24]. The Heisenberg-Robertson uncertainty relation [28, 29] for two incompatible observables $A$ and $B$ can be given in terms of the commutator $\Delta^2(A)\Delta^2(B) \geq \frac{1}{2} [\langle A, B \rangle]^2$, which is a very first investment on this issue. The lower bound in this relation, however, can be trivial, such that it can reach zero for the system's state is an eigenstate of either $A$ or $B$. Stronger (nontrivial) uncertainty relations were later proposed by Maccone and Pati [23] in terms of the summation of variances

$$
\Delta^2(A) + \Delta^2(B) \geq \max(\mathcal{R}_a, \mathcal{R}_b),
$$

which we name as Maccone-Pati uncertainty relation (MPUR), where

$$
\mathcal{R}_a = \pm i \langle [A, B] \rangle + |\langle \psi | A \pm i B | \psi \rangle|^2,
$$

$$
\mathcal{R}_b = \frac{1}{2} |\langle \psi \pm i A + B | A + B | \psi \rangle|^2.
$$

Here, $|\psi\rangle$ is an arbitrary orthogonal state, and the sign should be chosen so that $\pm i \langle [A, B] \rangle$ is positive, and $|\psi \pm i A + B \rangle \propto (A + B - [A + B]|\psi\rangle$. It is said that the MPUR is nontrivial, i.e., the lower bound is nonzero. Such the MPUR has been experimentally verified in Ref. [30].

In our current framework, we can examine the MPUR in terms of the conditional variances where the summation $\Delta^2(A) + \Delta^2(B)$ is replaced by $\Delta^2(A|B) + \Delta^2(B|A)$. In the following section, we illustrate the beyond of the MPUR (20), i.e., $\min(\Delta^2(A|B) + \Delta^2(B|A)) < \max(\mathcal{R}_a, \mathcal{R}_b)$, in an appropriate illustration.

III. ILLUSTRATION IN SPIN SYSTEMS

We consider the sequential measurements of a spin system by light meters in an atom-light interaction scheme [18, 31–40]. For simplicity, we consider the system is a single spin-$1/2$. We choose $A = S_z = \sigma_z/2$ and $B = S_x = \sigma_x/2$, where $\sigma_i, (i = x, y, z)$ is a Pauli matrix. We fixed the initial system state as $\rho_0 = |+\rangle \langle +|$, where $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $S_z$. For this choice, we have $\Delta^2(S_z) = 1/4$ and $\Delta^2(S_x) = 0$. Furthermore, we expand operators $S_z$ and $S_x$ into their eigenvalues and eigenstates as following

$$
S_z = \sum_i a_i |a_i\rangle\langle a_i| = \frac{1}{2} |\uparrow\rangle\langle \uparrow| - \frac{1}{2} |\downarrow\rangle\langle \downarrow|,
$$

$$
S_x = \sum_i b_i |b_i\rangle\langle b_i| = \frac{1}{2} |+\rangle\langle +| - \frac{1}{2} |-\rangle\langle -|.
$$

Noting that in this illustration, operators $S_z$ and $S_x$ have discrete spectra, we thus can replace the integrals w.r.t $da$ and $db$ by the corresponding summation over the discrete eigenstates.

A. Joint measurement model

First, we calculate $\rho_1$ in Eq. (12):

$$
\rho_1 = \sum_{i,j=\uparrow}^4 |a_i\rangle\langle a_i| + |a_j\rangle\langle a_j| e^{-\frac{(a_j-a_i)^2}{\sigma^2 t^2}}
$$

$$
= \frac{1}{2} \left( \frac{1}{e^{-\frac{1}{\sigma^2 t^2}}} \right) e^{-\frac{1}{\sigma^2 t^2}}.
$$

Substituting Eq. (24) into Eq. (13), we obtain

$$
\Delta^2(S_x') = \sum_{i=+}^4 b_i^2 \langle b_i|\rho_1|b_i\rangle - \left[ \sum_{i=+}^4 b_i \langle b_i|\rho_1|b_i\rangle \right]^2
$$

$$
= \frac{1}{4} \left( 1 - e^{-\frac{1}{\sigma^2 t^2}} \right).
$$

In Fig. 2, we show $\Delta^2(S_x')$ as a function of $\sigma_1$ for a spin-$1/2$ system $S$. For strong measurement, $\sigma_1 \ll 1$, $\Delta^2(S_x')$
reaches the maximum of 1/4. For weak measurement \( \sigma_1 \to \infty \), we have \( \Delta^2(S_x') \to 0 \), which implies that if the first measurement is weak, then it will not affect the second one.

**B. Conditional measurement model**

First, let us calculate \( \bar{\rho}_1 \) in Eq. (14):

\[
\bar{\rho}_1 = \sum_{i,j=1}^{s} |a_i\rangle\langle a_i| \psi_1(x_1 - a_i) \psi_1^*(x_1 - a_j) = \frac{1}{2} \left( \frac{1}{2\pi \sigma_1^2} \right)^{1/2} e^{-\frac{x_1^2}{2\sigma_1^2}} \left( e^{-\frac{x_1^2}{2\sigma_1^2}} \right) \tag{26}
\]

This state significantly depends on the outcome \( x_1 \) and the interaction strength \( \sigma_1 \) of the first measurement.

For the predictive process, we first determine the conditional probability \( \Pr(x_2|x_1) \) for obtaining the outcome \( x_2 \) conditioned on the outcome \( x_1 \). The conditional expectation value Eq. (16) explicitly gives

\[
\langle x_2 \rangle \big|_{x_1} = \frac{e^{-\frac{x_1^2}{2\sigma_1^2}}}{1 + e^{-\frac{x_1^2}{\sigma_2^2}}} , \text{ and } \langle x_2^2 \rangle \big|_{x_1} = \frac{1}{4} + \sigma_2^2 \tag{27}
\]

and the conditional variance of the second measurement is

\[
\Delta^2(x_2|x_1) = \frac{1}{4} \tanh \left( \frac{x_1}{2\sigma_1^2} \right)^2 + \sigma_2^2 \tag{28}
\]

Inversely, we can obtain the uncertainty in the value of \( S_z \) condition on \( S_z \) as

\[
\Delta^2(S_z|S_z) = \frac{1}{4} \tanh \left( \frac{x_1}{2\sigma_1^2} \right)^2 \tag{29}
\]

We show the result in Fig. 3. It can be seen that, when the first measurement is strong, \( \sigma_1 \to 0 \), the conditional uncertainty reaches the maximum of 1/4 for \( x_1 \neq 0 \), which implies that the system is maximum disturbed by the first measurement as a result of the QBA. In other words, before the first measurement, the variance \( \Delta^2(S_z) = 0 \), and after the measurement it reaches the maximum \( \Delta^2(S_z|S_z) = 1/4 \). When the first measurement is weak \( \sigma_1 \to \infty \), it will less disturb the system and thus \( \Delta^2(S_z|S_z) \) will reduce to \( \Delta^2(S_z) \). This observation is the same as the joint measurement approach above.

Next, for the retrodictive process, we obtain

\[
\Delta^2(S_z|S_z) = \frac{1}{4} \left( \frac{s_1 - 1}{s_1} \right) s_2 - 2 , \tag{30}
\]

where \( s_1 = 1 + \exp \left( \frac{1}{8\sigma_1^2} \right) \) and \( s_2 = 1 + \exp \left( \frac{24}{\sigma_2^2} \right) \). It can be seen that there is an interplay between \( \sigma_1 \) and \( \sigma_2 \). When \( \sigma_1 \to 0 \) (the first measurement is strong), i.e., \( s_1 \to \infty \), it will dominate the result, and thus the uncertainty \( \Delta^2(S_z|S_z) = 1/4 = \Delta^2(S_z) \), regardless \( s_2 \). In this case, the second measurement will not disturb the first one, as we have \( \Delta^2(S_z|S_z) = \Delta^2(S_z) \). Inversely, when \( \sigma_1 \to \infty \) (the first measurement is weak), i.e., \( s_1 \approx 2 \), then \( \Delta^2(S_z|S_z) \approx \frac{1}{8} \frac{s_2 - 2}{s_2 - 1} = \frac{1}{8} \left( 1 + e^{-\frac{24}{\sigma_2^2}} \right) \). In this case, the measurement of \( S_z \) will be disturbed by the second measurement via \( x_2 \) and \( \sigma_2 \). This is the QBA of retrodict. Different from the prediction case, here, the uncertainty

**FIG. 2.** The uncertainty in the value of \( S_z \) after the first measurement, \( \Delta^2(S_x') \). The (second) measurement of \( S_z \) is affected by the first measurement via \( \sigma_1 \). When the first measurement is strong, i.e., \( \sigma_1 \to 0 \), the uncertainty \( \Delta^2(S_x') \) is large which implies the large disturbed of the system. When the first measurement is weak, i.e., large \( \sigma_1 \), \( \Delta^2(S_x') \) reduces to zero, which means that the system is not be disturbed, just the same as the uncertainty in the value of \( S_z \) before the first measurement \( \Delta^2(S_x) \).

**FIG. 3.** The conditional uncertainty \( \Delta^2(S_z|S_z) \) as a function of \( x_1 \) and \( \sigma_1 \). When \( \Delta^2(S_z|S_z) = \Delta^2(S_z) = 0 \), there is no disturbance for the system, such as when \( x_1 = 0 \). For small \( \sigma_1 \) and \( x_1 \neq 0 \), \( \Delta^2(S_z|S_z) \) reaches 1/4, the maximum uncertainty, as a result of the QBA. For large \( \sigma_1 \), \( \Delta^2(S_z|S_z) \) reduces to zero, as a result of the weak measurement.
will be strong disturbed, wherein 1 is no disturbance for the system, such as when $x = 0$. For small $\sigma_2$ and $x < 0$, then $\Delta^2(S_x | S_z) > 1/4$, which implies the retrodictive effect of the second measurement affects on the first one. Here, $\Delta^2(S_x | S_z) \to \infty$ when $\sigma_2 \to 0$, however, we show the result up to $\Delta^2(S_z | S_x) = 0.5$. For large $\sigma_2$ and $x < 0$, $\Delta^2(S_z | S_x)$ reduces to $1/4$, as a result of the weak interaction in the second measurement. Likewise, for $x > 0$, the conditional uncertainty $\Delta^2(S_z | S_x)$ ranges from $1/8$ to $1/4$ when $\sigma_2$ goes from strong to weak interaction, respectively.

\[ \Delta^2(S_z | S_x) \]

does not symmetry via the outcome $x_2$. For $x_2 = 0$, the uncertainty reaches $1/4$, which is no disturbance for the first measurement. For $x_2 < 0$, the system will be strong disturbed, wherein $1/4 < \Delta^2(S_z | S_x) < \infty$. Here, $\Delta^2(S_z | S_x) \to \infty$ when $\sigma_2 \to 0$, however, we show the result up to $\Delta^2(S_z | S_x) = 0.5$, as can be seen in Fig. 4, for the sake of illustration. For $x_2 > 0$, the uncertainty is bound in the range $1/8 \leq \Delta^2(S_z | S_x) < 1/4$ when $\sigma_2$ goes from strong to weak interaction.

C. Beyond stronger uncertainty relations

We examine the MPUR in this spin model under prediction and retrodiction. Here, the initial system’s state is $|\psi\rangle = |+\rangle$, an eigenstate of $S_z$. We thus can choose $|\psi^{+}\rangle = (S_z | S_z) |+\rangle / \Delta(S_z) = |\rangle$ [23]. We also obtain $|\psi^{+}_{S_{z}+S_{x}}\rangle = |\rangle$. Then, $\mathcal{R}_{a(b)}$ in Eq. (21) yields

\[ R_a = \frac{1}{4}, \quad \text{and} \quad R_b = \frac{1}{8}. \] (31)

The MPUR for the system before the measurements reads

\[ \Delta^2(S_x) + \Delta^2(S_z) = \frac{1}{4} = \max(R_a, R_b), \] (32)

in which reaches the lower bound [See Eq. (20).]

Next, we consider the MPUR for the conditional uncertainties. Since $\Delta^2(S_z | S_x)$ and $\Delta^2(S_x | S_z)$ can be archived individually, we thus can obtain an inequality $\Delta^2(S_z | S_x) + \Delta^2(S_x | S_z) \geq \frac{1}{8}$. The equality is archivable when $\Delta^2(S_z | S_x) = \frac{1}{8}$ and $\Delta^2(S_x | S_z) = 0$, as we analyzed above. This could be a demonstration for beyond the MPUR in the prediction and retrodiction measurements framework.

IV. N-SEQUENTIAL MEASUREMENTS

We extend the framework into $N$-sequential measurements. Let $U_k = e^{-iA_k \otimes p_k}$ is the evolution operator of the $k^{th}$ measurement, where $A_k = \int a_k |a_k\rangle \langle a_k| da_k$ is the corresponding system’s operator, and $p_k$ the $k^{th}$ pointer’s operator. Let $\Omega_{x_k}$ be the POVM of the $k^{th}$ measurement, we have

\[ \Omega_{x_k} = \int \psi_{x_k}(x_k - a_k) |a_k\rangle \langle a_k| \ da_k. \] (33)

The system’s state after $k$ measurements is given by

\[ \hat{\rho}_k = \Omega_{x_k} p_{k-1} \Omega_{x_{k-1}}^\dagger, \quad \text{for} \quad k = 1 \cdots N. \] (34)

The probability to obtain the outcome $x_k$ at the $k^{th}$ measurement conditional on the rest of the other outcomes is given by

\[ \Pr(x_k | \{x_1, \cdots, x_N\} \backslash x_k) = \frac{\text{Tr}[\hat{\rho}_k E]}{\text{Tr}[\hat{\rho}_k E]} dx_k, \] (35)

where $\{x_1, \cdots, x_N\} \backslash x_k$ stands for the rest of the other outcomes, i.e., a set of all elements from $x_1$ to $x_N$ except $x_k$, and

\[ E = \Omega_{x_{k+1}}^\dagger \Omega_{x_k}^\dagger \cdots \Omega_{x_{N-1}}^\dagger \Omega_{x_N} \Omega_{x_{N-1}} \cdots \Omega_{x_k}. \] (36)

See the proof in App. E. This result is similar to the one is given in Ref. [18]. However, in Ref. [18], the authors approximate the Gaussian distribution of the conditional probability to deduce the corresponding variance and using a hypothesis projective measurement at the $k^{th}$ measurement. Here, we derive the exact variance from the expectation value of the position operator $x_k$ in the pointer $p_k$. At first, the conditional expectation value is given by

\[ \langle x_k | \{x_1, \cdots, x_N\} \backslash x_k \rangle = \int x_k \Pr(x_k | \{x_1, \cdots, x_N\} \backslash x_k) \ dx_k. \] (37)

Then, the conditional uncertainty is expressed as

\[ \Delta^2(x_k | \{x_1, \cdots, x_N\} \backslash x_k) \\
= \langle x_k^2 | \{x_1, \cdots, x_N\} \backslash x_k \rangle - \left[\langle x_k | \{x_1, \cdots, x_N\} \backslash x_k \rangle\right]^2 \\
= \sigma_k^2 + \Delta^2(A_k | \{A_1, \cdots, A_N\} \backslash A_k). \] (38)
We thus can extract the conditional uncertainty in the value of $A_k$ at the $k$-measurement, i.e.,
\[ \Delta^2(A_k \{ \{ A_1, \cdots , A_N \} \setminus A_k \}). \]

For example, in App. E, we provide a useful Mathematica code for calculating the conditional variance $\Delta^2(x_2|x_1,x_3,x_4)$ of 4-sequential measurements: $S_x - S_x - S_x - S_x$.

\[ \text{V. CONCLUSION} \]

We have considered the QBA effect of prediction and retrodiction on the sequential measurements of incompatible operators. We formulated the uncertainties in the sequential measurements through two scenarios of joint measurement and conditional measurement models, wherein the disturbances in the uncertainties present for the QAB effects. In the joint measurement approach, the latter measurements will be affected by former ones, and thus, only the QBA of prediction is presented. Otherwise, in the conditional measurement approach, there is a mutual effect in the sequential measurements and is expressed through the QBA of prediction and retrodiction. We have illustrated the results in spin systems, where we have presented the dependence of the uncertainty in subsequent measurement on the prior measurement and vice versa. We further found a demonstration for beyond the stronger uncertainty relation.

\[ \text{Appendix A: Final joint state in the joint measurement model} \]

In this Appendix, we derive the final state of the sequential measurements:
\[ \rho_{\text{fin}} = U_2 U_1 \rho_{\text{int}} U_1^\dagger U_2^\dagger. \] (A.1)

Insert $I_1 = \int |m\rangle \langle m| \ dm$ and $I_2 = \int |n\rangle \langle n| \ dn$ we have

\[ \rho_{\text{fin}} = \int \int N_m M_n \rho_0 M_m^\dagger N_n^\dagger \otimes \langle m| \langle n| \ dm \ dn. \] (A.2)

Inserting $M_m$ and $N_n$ into Eq. (A.6), we obtain
\[ \rho_{\text{fin}} = \int \int \int N_m M_m \rho_0 M_m^\dagger N_n^\dagger \otimes \langle m| \langle n| \ dm \ dn. \] (A.5)

which is given in Eq. 4 in the main text.

\[ \text{Appendix B: Final joint pointers state in the joint measurement model} \]

Here, we derived the joint state of the pointers $\mathcal{P}_1 \oplus \mathcal{P}_2$ by partial tracing out the system space
\[ \eta = \text{Tr}_S[\rho_{\text{fin}}]. \] (B.1)

We first consider the cyclic property of the trace as $\text{Tr}_S[N_m M_m \rho_0 M_m^\dagger N_n^\dagger] = \text{Tr}_S[M_m \rho_0 M_m^\dagger N_n^\dagger N_n^\dagger]$, where
\[ N_n^\dagger N_n = \int \int \int \int \psi_2(n-b)b\langle b| \ psi_2(n-b)\langle b| dbdb. \] (B.2)
and
\[ M_{m_1 m_2}^\dagger = \int \psi_1(m-a)|a\rangle \langle a| \, da' \times \rho_0 \]
\[ \times \int \psi_1(m' - a')|a'\rangle \langle a'| \, da'' \]
\[ = \int |a\rangle \langle a| \rho_0 |a'\rangle \langle a'| \times \]
\[ \psi_1(m-a)\psi_1(m' - a') \, da da'. \]  
\[ \text{(B.3)} \]

From Eqs. (B.2, B.3), we have
\[ \text{Tr}_S \left[ M_{m_1 m_2}^\dagger N_{m_1}^\dagger N_m \right] \]
\[ = \int \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| \psi_1(m-a)\psi_1(m' - a') \times \]
\[ \psi_2(n-b)\psi_2(n' - b) \, da da' db. \]  
\[ \text{(B.4)} \]

Submitting it into Eq. (A.6) then Eq. (B.1) explicit gives
\[ \eta = \int \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| b \rangle \psi_1(m-a)\psi_1(m' - a') \times \]
\[ \psi_2(n-b)\psi_2(n' - b) \, da da' db, \]  
\[ \text{(B.5)} \]

where we have used \( \int \psi_1(m-a)|a\rangle \, dm = |\psi(m-a)| \) and similarly for these others. Noting that this formula does not explicitly depend on \( m \), we thus can replace \( m \) by \( x \), and recast Eq. (B.5) by
\[ \eta = \int \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| b \rangle \psi_1(x-a)\psi_1(x' - a') \times \]
\[ \psi_2(x-b)\psi_2(x' - b) \, da da' db, \]  
\[ \text{(B.6)} \]

which are shown in the main text.

\section*{Appendix C: Variances in the joint measurement model}

\subsection*{1. Derivative of the variance \( \Delta^2(x_1) \)}

We first calculate the expectation value of the first measurement which is given in Eq. (8):
\[ \langle x_1 \otimes I_2 \rangle = \text{Tr}_{P_1 P_2 \{ x_1 \otimes I_2 \}} [\eta], \]  
\[ \text{(C.1)} \]

The numerator explicitly gives
\[ \text{Tr}_{P_1 P_2 \{ x_1 \otimes I_2 \}} [\eta] = \int \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| b \rangle \psi_1(x-a)\psi_1(x' - a') \times \]
\[ \psi_1(x'-a')|x_1| \psi_1(x-a) \, da da' db, \]  
\[ \text{(C.2)} \]

where
\[ \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| b \rangle \, db = \text{Tr}_S \left[ |a\rangle \langle a| \rho_0 |a'\rangle \langle a'| \right] \]
\[ = \text{Tr}_S \left[ |a\rangle \langle a| \rho_0 \right], \]  
\[ \text{(C.3)} \]

where we have used the cyclic property of the trace then only \( a = a' \) is non-vanish, and
\[ \langle \psi_1(x-a')|x_1| \psi_1(x-a) \rangle \]
\[ = \left[ \int \left( \frac{1}{2\pi \sigma_1^2} \right)^{1/4} e^{-\frac{(x-a')^2}{4\sigma_1^2}} \, dx' \right] \]
\[ \times \left[ \int x_1|x_1| \, dx_1 \right] \]
\[ \times \left[ \int \left( \frac{1}{2\pi \sigma_1^2} \right)^{1/4} e^{-\frac{(x-a)^2}{4\sigma_1^2}} \, dx \right] \]
\[ = a + \frac{a'}{2} e^{-\frac{(a-a')^2}{8\sigma_1^2}}. \]  
\[ \text{(C.4)} \]

Then, the numerator is recast as
\[ \text{Tr}_{P_1 P_2 \{ x_1 \otimes I_2 \}} [\eta] = \int a \text{Tr}_S \left[ |a\rangle \langle a| \rho_0 \right] \, da \]
\[ = \text{Tr}_S [A \rho_0]. \]  
\[ \text{(C.5)} \]

Similarly in the denominator, we have
\[ \text{Tr}_{P_1 P_2 \{ \eta \}} = \int \int \langle b|a\rangle \langle a| \rho_0 |a\rangle \langle a'| b \rangle \times \]
\[ \langle \psi_1(x-a')| \psi_1(x-a) \rangle \, da da' db, \]
\[ = 1. \]  
\[ \text{(C.6)} \]

Submitting Eqs. (C.5, C.6) into Eq. (C.1), we have
\[ \langle x_1 \otimes I_2 \rangle = \text{Tr}_S [A \rho_0]. \]  
\[ \text{(C.7)} \]

Obviously, the expectation value obtained from the first measurement does not depend on the second measurement.

We next calculate \( \langle (x_1 \otimes I_2)^2 \rangle \) for the non-vanish terms, i.e., \( a = a' \). We first evaluate
\[ \langle \psi_1(x-a)|x_1^2| \psi_1(x-a) \rangle \]
\[ = \left[ \int \left( \frac{1}{2\pi \sigma_1^2} \right)^{1/2} e^{-\frac{(x'-a)^2}{4\sigma_1^2}} \, dx' \right] \]
\[ \times \left[ \int x_1^2|x_1| \, dx_1 \right] \]
\[ \times \left[ \int \left( \frac{1}{2\pi \sigma_1^2} \right)^{1/2} e^{-\frac{(x-a)^2}{4\sigma_1^2}} \, dx \right] \]
\[ = \sigma_1^2 + a^2. \]  
\[ \text{(C.8)} \]

Combining with Eq. (C.3), we obtain
\[ \langle (x_1 \otimes I_2)^2 \rangle = \int (\sigma_1^2 + a^2) \, \text{Tr}_S \left[ |a\rangle \langle a| \rho_0 \right] \, da \]
\[ = \sigma_1^2 + \text{Tr}_S [A^2 \rho_0]. \]  
\[ \text{(C.9)} \]

Then, the variance is
\[ \Delta^2(x_1) = \langle (x_1 \otimes I_2)^2 \rangle - \langle x_1 \otimes I_2 \rangle^2 \]
\[ = \sigma_1^2 + \Delta^2(A), \]  
\[ \text{(C.10)} \]

where \( \Delta^2(A) = \text{Tr}_S [A^2 \rho_0] - (\text{Tr}_S [A \rho_0])^2 \). This result is casually trivial, where the first measurement will not be affected by the second one.
2. Derivative of the variance $\Delta^2(x_2)$

We next examine the backaction effect of the first measurement affects the second measurement. We start from the expectation value:

$$\langle I_1 \otimes x_2 \rangle = \frac{\text{Tr}_{p_1 p_2}[(I_1 \otimes x_2) \eta]}{\text{Tr}_{p_1 p_2} \eta}, \quad (C.11)$$

where the denominator $\text{Tr}_{p_1 p_2} \eta = 1$. Then, we have

$$\langle I_1 \otimes x_2 \rangle = \text{Tr}_{p_1 p_2}[(I_1 \otimes x_2) \eta] = \iint \langle b | \langle a | \rho_0 | a' \rangle \langle a' | b \rangle \times \langle \psi_1(x-a') | \psi_1(x-a) \rangle \times \langle \psi_2(x-b) | x_2 | \psi_2(x-b) \rangle \rangle da'db. \quad (C.12)$$

The middle term is

$$\langle \psi_1(x-a') | \psi_1(x-a) \rangle = e^{-\frac{(a-a')^2}{2\sigma_x^2}}. \quad (C.13)$$

The last term is (similar as (C.4))

$$\langle \psi_2(x-b) | x_2 | \psi_2(x-b) \rangle = b. \quad (C.14)$$

Then, Eq. (C.12) explicitly gives

$$\langle I_1 \otimes x_2 \rangle = \iint b | \langle a | \rho_0 | a' \rangle \langle a' | b \rangle \times e^{-\frac{(a-a')^2}{2\sigma_x^2}} \rangle da'db = \text{Tr}_S[B \rho_1], \quad (C.15)$$

where

$$\rho_1 = \text{Tr}_{p_1} e^{-iA \otimes p_1} \left( \rho_0 \otimes | \psi_1(x) \rangle \langle \psi_1(x) | \right) e^{iA \otimes p_1} = \iint | a \rangle \langle a | \rho_0 | a' \rangle \langle a' | \times e^{-\frac{(a-a')^2}{2\sigma_x^2}} \rangle da'db \quad (C.16)$$

is the system state after the first interaction. For weak measurement, $\sigma_1 \gg | a - a' |$, i.e., $e^{-\frac{(a-a')^2}{2\sigma_x^2}} \approx 1$, we have

$$\langle I_1 \otimes x_2 \rangle \approx \text{Tr}_S[B \rho_1].$$

We next calculate the term $\langle I_1 \otimes x_2 \rangle^2$ and obtain

$$\langle I_1 \otimes x_2 \rangle^2 = \sigma_1^2 + \text{Tr}_S[B^2 \rho_1] \quad (C.17)$$

Finally, the variance is given by

$$\Delta^2 x_2 = \sigma_1^2 + \Delta^2(B'), \quad (C.18)$$

where $\Delta^2(B') = \text{Tr}_S[B^2 \rho_1] - (\text{Tr}_S[B \rho_1])^2$ and particularly gives

$$\Delta^2(B') = \int b^2 | \langle a | \rho_1 | b \rangle | db - \left[ \int b | \langle a | \rho_1 | b \rangle | \right]^2. \quad (C.19)$$

For weak measurement, $\Delta^2(B') = \Delta^2(B)$, which implies that a weak measurement will not affect the later measurement results.

Appendix D: Conditional measurement model

We calculate the conditional probability

$$\text{Pr}(x_2 | x_1) = \frac{\text{Tr} \left[ N_{x_2} | \hat{p}_1 N_{x_2}^\dagger \right]}{\int \text{Tr} \left[ N_{x_2} | \hat{p}_1 N_{x_2}^\dagger \right] dx_2'}, \quad (D.1)$$

We have

$$\text{Tr} \left[ N_{x_2} | \hat{p}_1 N_{x_2}^\dagger \right] = \text{Tr} \left[ \int \psi_2(x_2 - b) | b \rangle \langle b | db \times \hat{p}_1 \times \int \psi_2(x_2 - b') | b' \rangle \langle b' | db' \right]$$

$$= \int \psi_2^2(x_2 - b) | b \rangle \langle b | db. \quad (D.2)$$

Submitting to Eq. (D.3), we obtain

$$\text{Pr}(x_2 | x_1) = \frac{\int \psi_2^2(x_2 - b) | b \rangle \langle b | db}{\int \int \psi_2^2(x_2 - b) | b \rangle \langle b | db \times \hat{p}_1 \times \int \psi_2(x_2 - b') | b' \rangle \langle b' | db' \right] dx_2'}, \quad (D.3)$$

as shown in Eq. (15) in the main text.

Appendix E: $N$-sequential measurements

Here, we prove Eq. (35) in the main text. Under $N$-sequential measurements, the joint probability is given by

$$\text{Pr}(x_1, \cdots, x_N) = \text{Tr}_S \left[ \Omega_N \cdots \Omega_{k+1} \hat{p}_k \Omega_k^\dagger \cdots \Omega_N^\dagger \right]$$

$$= \text{Tr}_S \left[ \hat{p}_k \Omega_{k+1}^\dagger \cdots \Omega_N^\dagger \right]$$

$$= \text{Tr}_S \left[ \hat{p}_k E \right]. \quad (E.1)$$

Here, we have used $\hat{p}_k = \Omega_k \hat{p}_{k-1} \Omega_k^\dagger$, and $E = \Omega_{k+1}^\dagger \cdots \Omega_N^\dagger \Omega_{k} \cdots \Omega_{k+1}$. Then, the conditional probability is defined as

$$\text{Pr}(x_k | x_1, \cdots, x_N) = \text{Pr}(x_1, \cdots, x_N) \int \frac{\text{Pr}(x_1, \cdots, x_N) dx_k'}{\text{Tr}_S \left[ \hat{p}_k E \right]}.$$  \quad (E.2)$$

Next, substituting $\Omega_k = \int \psi_k(x - a_k) | a_k \rangle \langle a_k | da_k$ into $\hat{p}_k$, we obtain

$$\hat{p}_k = \left[ \int \psi_k(x - a_k) | a_k \rangle \langle a_k | da_k \right] \times \hat{p}_{k-1} \times \left[ \int \psi_k(x - a'_{k}) | a'_{k} \rangle \langle a'_{k} | da'_k \right],$$

$$= \int \left[ \psi_k(x - a_k) \psi_k(x - a'_k) | a_k \rangle \langle a'_{k} | da_k \right] da_k. \quad (E.3)$$

Submitting Eq. (E.3) into Eq. (E.2), we have
\[
\Pr(x_k \, | \, x_1, \ldots, x_N) = \frac{\int \psi_k(x_k - a_k) \psi_k(x_k - a_k') \langle a_k | \rho_k - 1 | a_k \rangle \langle a_k' | E | a_k \rangle \, da_k \, da_k'}{\int \int \psi_k(x_k - a_k) \psi_k(x_k - a_k') \langle a_k | \rho_k - 1 | a_k \rangle \langle a_k' | E | a_k \rangle \, da_k \, da_k' \, dx_k},
\]

(E.4)

as shown in the main text.

A useful Mathematica code for calculating the conditional variance in the pointers.

Listing 1. Example code

\begin{verbatim}
(* Eigenstates and eigenvectors *)
ez1 = 0.5;
ex2 = -0.5;
MatrixForm[ez1 = {{1}, {0}}];
MatrixForm[ex2 = {{0}, {1}}];
ex1 = 0.5;
ex2 = -0.5;
MatrixForm[ez1 = 1/Sqrt[2]*{{1}, {1}}];
MatrixForm[ez2 = 1/Sqrt[2]*{{1}, {-1}}];
(* POVMs *)
\end{verbatim}

\[
\psi_{x,a} = \frac{1}{2\pi*\sigma^2} \frac{1}{4} \exp\left[-\frac{(x-a)^2}{4\sigma^2}\right];
\]

[1] J. A. Wheeler and W. H. Zurek, Quantum theory and measurement (Princeton University Press, 2014).

[2] D. Bohm, Quantum theory (Dover, New York, 1989).

[3] P. Busch, G. Cassinelli, and P. J. Lahti, Foundations of Physics 20, 757 (1990).

[4] S. Gudder and G. Nagy, Journal of Mathematical Physics 42, 5212 (2001).

[5] G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A 76, 062105 (2007).

[6] J.-S. Chen, M.-J. Hu, X.-M. Hu, B.-H. Liu, Y.-F. Huang, C.-F. Li, C.-G. Guo, and Y.-S. Zhang, Opt. Express 27, 6089 (2019).

[7] M. Pfender, P. Wang, H. Sumiya, S. Onoda, W. Yang, D. B. R. Dasari, P. Neumann, X.-Y. Pan, J. Isoya, R.-B. Liu, and J. Wrachtrup, Nature Communications 10, 594 (2019).

[8] A. Brodutch and E. Cohen, Quantum Studies: Mathematics and Foundations 4, 13 (2017).

[9] W.-L. Ma, P. Wang, W.-H. Leong, and R.-B. Liu, Phys. Rev. A 98, 012117 (2018).

[10] H. M. Wiseman, Phys. Rev. A 51, 2450 (1995).

[11] K.-D. Wu, Y. Yuan, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and M. Perarnau-Llobet, Science Advances 5 (2019), 10.1126/sciadv.aav4944.

[12] K.-D. Wu, E. Bäumer, J.-F. Tang, K. V. Hovhannisyan, M. Perarnau-Llobet, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. 125, 210401 (2020).

[13] I. Shimroni, L. Qiu, D. Malz, A. Nunnenkamp, and T. J. Kippelin, Nature Communications 10, 2086 (2019).

[14] M. S. Blök, C. Bonato, M. L. Markham, D. J. Twitchen, V. V. Dobrovitski, and R. Hanson, Nature Physics 10, 189 (2014).

[15] J. Larsson, M. Kleinmann, O. Gühne, and A. Cabello, AIP Conference Proceedings 1327, 401 (2011).

[16] E. Nagali, S. Felicetti, P.-L. de Assis, V. D’Ambrosio, R. Filip, and F. Sciarrino, Scientific Reports 2, 443 (2012).

[17] J. Thompson, A. J. P. Garner, J. R. Mahoney, J. P. Crutchfield, V. Vedral, and M. Gu, Phys. Rev. X 8, 031013 (2018).

[18] H. Bao, S. Jin, J. Duan, S. Jia, K. Molmer, H. Shen, and Y. Xiao, Nature Communications 11, 5658 (2020).

[19] J. Zhang and K. Molmer, Phys. Rev. A 96, 062131 (2017).

[20] D. Tan, S. J. Weber, I. Siddiqi, K. Melmer, and K. W. Murch, Phys. Rev. Lett. 114, 090403 (2015).

[21] L. B. Ho, Quantum Information Processing 18, 206 (2019).

[22] P. Campagne-Ibarcq, L. Brethouex, E. Flurin, A. Auffèves, F. Mallet, and B. Huard, Phys. Rev. Lett. 112, 180402 (2014).

[23] L. Maccone and A. K. Pati, Phys. Rev. Lett. 113, 260401 (2014).

[24] M. Sindelka and N. Moisseyev, Phys. Rev. A 97, 012122 (2018).

[25] A. A. Abbott, R. Silva, J. Wechs, N. Brunner, and C. Branciard, Quantum 3, 194 (2019).

[26] Y. Aharonov and L. Vaidman, in Time in Quantum Mechanics - Vol. 2, edited by M. Gonzalez, R. Andreass, and D. C. Adollo (Springer-Verlag Berlin Heidelberg, 2009) pp. 369–412.

[27] M. A. Nielsen and I. L. Chuang, “Introduction to quantum mechanics,” in Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2010) p. 60–119.

[28] H. P. Robertson, Phys. Rev. 34, 163 (1929).

[29] E. Arthurs and M. S. Goodman, Phys. Rev. Lett. 60, 2447 (1988).
[30] K. Wang, X. Zhan, Z. Bian, J. Li, Y. Zhang, and P. Xue, Phys. Rev. A 93, 052108 (2016).
[31] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Rev. Mod. Phys. 82, 1041 (2010).
[32] B. Julsgaard, J. Sherson, J. L. Sørensen, and E. S. Polzik, Journal of Optics B: Quantum and Semiclassical Optics 6, 5 (2003).
[33] G. Colangelo, F. M. Ciurana, L. C. Bianchet, R. J. Sewell, and M. W. Mitchell, Nature 543, 525 (2017).
[34] M. Jasperse, M. J. Kewming, S. N. Fischer, P. Pakkiam, R. P. Anderson, and L. D. Turner, Phys. Rev. A 96, 063402 (2017).
[35] M. Atatüre, J. Dreiser, A. Badolato, and A. Imamoglu, Nature Physics 3, 101 (2007).
[36] D. V. Vasilyev, K. Hammerer, N. Korolev, and A. S. Sørensen, Journal of Physics B: Atomic, Molecular and Optical Physics 45, 124007 (2012).
[37] J. L. Robb, Y. Chen, A. Timmons, K. C. Hall, O. B. Shchekin, and D. G. Deppe, Applied Physics Letters 90, 153118 (2007).
[38] M. H. Muñoz Arias, P. M. Poggi, P. S. Jessen, and I. H. Deutsch, Phys. Rev. Lett. 124, 110503 (2020).
[39] L. B. Madsen and K. Mølmer, Phys. Rev. A 70, 052324 (2004).
[40] Y. Liu, S. Jung, S. E. Maxwell, L. D. Turner, E. Tiesinga, and P. D. Lett, Phys. Rev. Lett. 102, 125301 (2009).