Path Integral Approach to the Nonextensive Canonical Density Matrix

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(December 4, 2017)

Abstract

Feynman’s path integral is herein generalized to the nonextensive canonical density matrix based on Tsallis entropy. This generalization is done in two ways by using unnormalized and normalized constraints. Firstly, we consider the path integral formulation with unnormalized constraints, and this generalization is worked out through two different ways, which are shown to be equivalent. These formulations with unnormalized constraints are solutions to two generalized Bloch equations proposed in this work. The first form of the generalized Bloch equation is linear, but with a temperature-dependent effective Hamiltonian; the second form is nonlinear and resembles the anomalous correlated diffusion equation (porous medium equation). Furthermore, we can extend these results to the prescription of field theory using integral representations. The second development is dedicated to analyzing the path integral formulation with normalized constraints. To illustrate the methods introduced here, we analyze the free particle case and a non-interacting scalar
field. The results herein obtained are expected to be useful in the discussion of generic nonextensive contexts.

PACS number(s): 05.30.–d, 00., 11.10.–z, 31.15.Kb
I. INTRODUCTION

Path integral techniques constitute a powerful tool, employed to analyze countless physical situations, for instance, in standard statistical mechanics (Boltzmann-Gibbs). On the other hand, there is nonextensive behavior common in many branches of physics, for example, systems with long-range (gravitational) interactions, long-time memory, and fractally structured space-time. This nonextensive behavior indicates that the standard statistical mechanics and thermodynamics need some enlargement. Consequently, the path integral employed in standard statistical mechanics also needs some generalizations in order to accommodate the nonextensive effects.

Recently, a theoretical tool based on nonextensive entropy (Tsallis entropy) [1] has successfully been applied in many situations, for example, Lévy-type anomalous superdiffusion [2], Euler turbulence [3], self-gravitating systems [4], cosmic background radiation [5], peculiar velocities of galaxies [6], anomalous relaxation through electron-phonon interaction [7], ferrofluid-like systems [8], and nonlinear dissipative dynamical systems [9]. In this context, it is very important to understand the main properties of the generalized statistical mechanics based on Tsallis entropy more deeply. In particular, it is interesting to develop basic tools for calculations such as the semi-classical approximation [10], perturbation [11,12] and variational methods [11–13], linear response theory [14], Green functions [15,16], non-Gaussian integrals [17], and path integral, which is the subject of this work.

We develop two path integral formalisms and Bloch equations based on Tsallis statistics with unnormalized constraints in Section II. The first form of the generalized Bloch equation is linear, but with an effective Hamiltonian, which has a temperature dependence, and the other equation resembles the anomalous diffusion equation (porous medium equation). The path integral solutions to these two different but equivalent generalized Bloch equations are also obtained in Sections II as well as the extension to the field theory. The path integral is first formulated for effective Hamiltonian case. In the other case, we use integral representations to formulate the path integral. To illustrate theses methods, the free particle
case for $q > 1$ and $q < 1$ and the free scalar field bosonic are studied. Section III is dedicated to analyzing the path integral formulation obtained with normalized constraints and its extension to quantum field theory, where an example is exhibited. Conclusions are presented in Section IV.

II. PATH INTEGRAL BASED IN UNNORMALIZED CONSTRAINTS

Let us start this section with some remarks about Bloch equation and nonextensive Tsallis statistics in order to introduce the generalized Bloch equations. After this presentation, we obtain the path integral formulation for the solutions of these Bloch equations, giving as example the free particle case. Furthermore, we apply these developments for a brief discussion about quantum scalar fields.

A. BLOCH EQUATION

The usual density matrix

$$\hat{\rho}_1(\beta) = e^{-\beta \hat{H}} / Z_1 \quad (\text{Tr} \hat{\rho}_1(\beta) = 1)$$

($Z_1 = \text{Tr} e^{-\beta \hat{H}}$ is the usual partition function) can be obtained directly from a maximum entropy principle [18], and the unnormalized matrix

$$\hat{\rho}_1(\beta) = \exp(-\beta \hat{H})$$

obeys the Bloch equation [19] (see also Ref. [20] p. 48)

$$- \frac{\partial \hat{\rho}_1}{\partial \beta} = \hat{H} \hat{\rho}_1$$

with the initial condition $\hat{\rho}_1(\beta = 0) = 1$. When $\hat{\rho}_1(\beta)$ is calculated and subsequently $Z_1$, we can obtain the usual free energy $F_1 = -(1/\beta) \ln Z_1$.

Recently, the density matrix (1) was generalized in the nonextensive context [1,21,22], i.e.,
\[ \hat{\rho}_q(\beta) = [1 - (1 - q)\beta \hat{H}]^{1/(1-q)} / Z_q , \]  

(4)

where \( Z_q = \text{Tr}[1 - (1 - q)\beta \hat{H}]^{1/(1-q)} \) is the generalized partition function. In Eq. (4) we are supposing that \( 1 - (1 - q)\beta E_n \geq 0 \), where \( E_n \) are the eigenvalues of \( \hat{H} \). When this condition is not satisfied, we employ \( p(E_n) = 0 \), i.e., a cut-off is used so that \( p_q(E_n) \) remains positive. By using \( Z_q \), we can obtain the generalized free energy

\[ F_q = -(1/\beta)(Z_q^{1-q} - 1) / (1 - q) . \]

These generalizations were obtained by employing the maximum entropy principle with the Tsallis entropy,

\[ S_q = -k \text{Tr} \frac{\hat{\rho} - \hat{\rho}^q}{1 - q} , \]

(5)

and the constraints \( \langle \hat{H} \rangle_q = \text{Tr} \hat{\rho}_q \hat{H} \) and \( \text{Tr} \hat{\rho}_q = 1 \) (\( k \) is a positive constant). The parameter \( q \in \mathbb{R} \) gives the degree of nonextensivity, and in the limit \( q \to 1 \) the usual statistical mechanics is recovered. Note that the constraint of the energy is unnormalized, since \( \langle 1 \rangle_q = \text{Tr} \hat{\rho}_q^q \neq 1 \). We return to this point at Section IV (see Ref. [23] for a complete discussion about the choice of the constraints).

Following the previous discussion about the Bloch equation, we employ the unnormalized matrix

\[ \hat{\rho}_q(\beta) = [1 - (1 - q)\beta \hat{H}]^{1/(1-q)} . \]

(6)

This matrix satisfies alternatively the equations

\[ -\frac{\partial \hat{\rho}_q}{\partial \beta} = \frac{\hat{H}}{1 - (1 - q)\beta \hat{H}} \hat{\rho}_q , \]

(7)

\[ -\frac{\partial \hat{\rho}_q}{\partial \beta} = \hat{H} \hat{\rho}_q^q , \]

(8)

with the initial condition \( \hat{\rho}_q(\beta = 0) = 1 \). These equations generalize the Bloch equation in the nonextensive Tsallis context. In fact, the generalized Bloch equation (7) is linear in \( \hat{\rho}_q \) and has the same form of the Eq. (3), if we employ the effective Hamiltonian \( \hat{H}_{\text{eff}} = \hat{H}[1 - (1 - q)\beta \hat{H}]^{-1} \). On the other hand, the generalized Bloch equation (8) has some formal resemblance to the anomalous correlated diffusion equation [24]. In the following, we develop the path integral solution for both generalized Bloch equations.
B. PATH INTEGRAL FORMULATION WITH $H_{\text{eff}}$

Since Eqs. (3) and (7) have the same form, the path integral representation for $\hat{\rho}_q$ has the same structure of the usual path integral. Thus, the path integral for $\hat{\rho}_q$ can be written as

$$\rho_q(x, x'; \beta) = \langle x | \hat{\rho}_q(\beta) | x' \rangle$$

$$= \lim_{N \to \infty} \int \ldots \int \left[ \prod_{n=1}^{N-1} dx_n \right] \left[ \prod_{n=1}^{N} \frac{dp_n}{2\pi \hbar} \right] \exp \left\{ \frac{1}{\hbar} \sum_{n=1}^{N} [i p_n (x_n - x_{n-1}) - \epsilon H_{\text{eff}}(p_n, x_n, n\epsilon)] \right\}$$

$$= \int \mathcal{D}x \mathcal{D}p \exp \left[ \frac{1}{\hbar} \int_{0}^{\beta} (i p \dot{x} - H_{\text{eff}}) \, d\tau \right] ,$$

where $x_n = (x_{n+1} - x_n)/2$, $x = x_0$, $x' = x_N$ and $\epsilon = \beta/N$. In this path integral representation, we are considering that the Hamiltonian $\hat{H}$ as well as $\hat{H}_{\text{eff}}$ are Weyl ordered (if this is not the case, the discussion has to be redone along the present lines), and consequently the midpoint prescription was employed [25]. In general, this path integral is not easy to calculate because the Hamiltonian $\hat{H}_{\text{eff}}(= \hat{H}[1 - (1 - q) \beta \hat{H}]^{-1})$ has an unusual nonlinear dependence on the moments and coordinates. However, as we shall see, it is possible to calculate $\hat{\rho}_q$ for the free particle case. Simple textbook examples, such as free particle, the harmonic oscillator, the non-interacting scalar field, etc., do not exhibit any kind of nonextensive behavior. It is not necessary to introduce a generalized statistical mechanics to deal with these systems. However, these simple examples are very useful in order to illustrate how the formalism works. And they are even more instructive when they provide exactly solvable cases. In this case, considering the free particle case, we first perform the integration in the coordinates,

$$\rho_q(x, x'; \beta) = \int \frac{dp}{2\pi \hbar} \exp \left[ \frac{ip}{\hbar} (x - x') \right] \lim_{N \to \infty} \int \ldots \int \left[ \prod_{n=1}^{N-1} dp_n \delta(p_n - p_{n-1}) \right] \times$$

$$\times \exp \left\{ -\epsilon \sum_{n=1}^{N} \left[ \frac{p_n^2/2m}{1 - \epsilon n (1 - q)p_n^2/2m} \right] \right\} .$$

By using the identity (see appendix A)

$$\ln(1 + x) = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{x}{N} \right) \frac{1}{1 + \frac{nx}{N}} ,$$

and considering the case $q > 1$ with $\beta > 0$, we verified that Eq. (10) reduces to
\[ \rho_q(x, x'; \beta) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp \left[ \frac{ip}{\hbar} (x - x') \right] \left[ 1 - (1 - q)\beta \frac{p^2}{2m} \right]^{1/(1-q)} \]

\[ = \frac{1}{\Gamma \left( \frac{1}{q-1} \right)} \left[ \frac{2m}{\pi(q-1)\beta^2} \right]^{1/2} \left[ \frac{m(x - x')^2}{2(q-1)\beta\hbar^2} \right]^{1/2} \times \]

\[ \times K_{1/(q-1)-1/2} \left[ \frac{2m(x - x')^2}{(q-1)\beta\hbar^2} \right]^{1/2} . \] (12)

In the last step, we used a known integral (for instance, see Ref. [26] p. 321) and relation between the Wittaker function \( W_{0, \nu} \) and the modified Bessel function of second kind \( K_{\nu} \).

In the case \( q < 1 \) with \( \beta > 0 \), that has a cut-off, the integration limits of the moments are limited and the path integral (10) reduces to the first part of Eq. (12) with \( p \) satisfying the condition \( 1 - (1 - q)\beta p^2/2m \geq 0 \). Thus, when the remaining integration is performed (see Ref. [26] p. 321), we obtain

\[ \rho_q(x, x'; \beta) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp \left[ \frac{ip}{\hbar} (x - x') \right] \left[ 1 - (1 - q)\beta \frac{p^2}{2m} \right]^{1/(1-q)} \]

\[ = \Gamma \left( \frac{1}{1-q} + 1 \right) \left[ \frac{m}{2\pi(1-q)\beta^2} \right]^{1/2} \left[ \frac{2(1-q)\beta\hbar^2}{m(x - x')^2} \right]^{1/2} \times \]

\[ \times J_{1/(1-q)+1/2} \left[ \frac{2m(x - x')^2}{(1-q)\beta^2} \right]^{1/2} . \] (13)

The partition function can be obtained for \( q > 1 \) and \( q < 1 \) by taking the trace of \( \rho_q(x, x'; \beta) \). In fact, for \( q > 1 \) we obtain

\[ Z_q = \int_{-L/2}^{L/2} dx \ \rho_q(x, x; \beta) \]

\[ = L \left[ \frac{m}{2\pi(q-1)\beta^2} \right]^{1/2} \frac{\Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{q-1} \right)} , \] (14)

and for \( q < 1 \) we have

\[ Z_q = L \left[ \frac{m}{2\pi(1-q)\beta^2} \right]^{1/2} \frac{\Gamma \left( \frac{1}{1-q} + 1 \right)}{\Gamma \left( \frac{1}{1-q} + \frac{3}{2} \right)} , \] (15)

where \( L \) is the length of the integration region (box size). Because \( L \) is large the expressions (14) and (13) reduce to the generalized classical partition function.
C. PATH INTEGRAL FORMULATION WITH INTEGRAL REPRESENTATIONS

Let us consider the Hilhorst’s identity (private communication to Tsallis [27])

\[
[1 - (1 - q)\beta\hat{H}]^{1/(1-q)} = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty dv \, v^{\frac{2-q}{1-q} - 1} e^{-v} \exp(-v(q-1)\beta\hat{H}) .
\]  

(16)

This operator identity is essentially a direct application of the usual integral representation of the gamma function. Indeed, if both sides of the Eq. (16) are applied on an eigenstate of \(\hat{H}\) with \(q > 1\), \(\beta > 0\), and \(E_n > 0\) (\(E_n\) is an arbitrary eigenvalue of \(\hat{H}\)) the identity (16) reduces to the usual integral representation of the gamma function. From Eq. (16) the path integral representation for \(\hat{\rho}_q(\beta)\) can be obtained. In fact, it is necessary to employ only the path integral representation for \(\hat{\rho}_1(v(q-1)\beta)\) in the last part of Eq. (16). Thus,

\[
\rho_q(x, x'; \beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty dv \, v^{\frac{2-q}{1-q} - 1} e^{-v} \rho_1(x, x'; v(q-1)\beta) ,
\]  

(17)

where \(\rho_1(x, x'; v(q-1)\beta)\) must be replaced by its usual (or holomorphic) path integral representation. Furthermore, if we take the trace of Eq. (17), we obtain the connection between the generalized and usual partition functions [27].

As illustration, let us apply Eq. (17) to obtain the generalized density matrix for a free particle in the one-dimensional case. For this example, the usual density matrix in the coordinate representation is (for instance, see Ref. [20] p. 49)

\[
\rho_1(x, x'; \beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \exp\left[-\left(\frac{m}{2\hbar^2\beta}\right)(x - x')^2\right] .
\]  

(18)

When this expression is replaced in Eq. (17) and the integral is calculated (see Ref. [26] p. 935), we obtain, as expected, Eq. (12).

The integral transformation (16) can not be applied for \(q < 1\), because the integral is not convergent. However, for \(q < 1\) the Prato’s identity

\[
[1 - (1 - q)\beta\hat{H}]^{1/(1-q)} = \frac{i}{2\pi} \Gamma\left(\frac{2-q}{1-q}\right) \int_C dv \, (-v)^{-\frac{2-q}{1-q} - 1} e^{-v} \exp(v(1-q)\beta\hat{H})
\]  

(19)

can be employed [28], except for some particular values of \(q\), \((2-q)/(1-q) = integer\). As in the case \(q > 1\), this integral representation is a direct consequence of another gamma
function representation (for instance, see Ref. [26] p. 935). The contour integration can be chosen in order to eliminate the condition \((2 - q)/(1 - q) \neq \text{integer}\), see for instance, Refs. [12] and [29]. Thus, we can formulate a path integral for \(q < 1\) using Eq. (19). In fact, it is sufficient to replace \(\exp(v(1 - q)\beta\hat{H})\) by its path integral representation in Eq. (19) to obtain the desirable path integral representation for \(\rho_q(x, x'; \beta)\) with \(q < 1\).

**D. EXTENSION TO THE FIELD THEORY FORMALISM**

We can use the previous path integral prescription to study some aspects of the Tsallis’ nonextensive thermostatistics as applied to field theory. Let us start our discussion about nonextensive statistical field theory by considering the scalar field. In this example, we obtain the generalized partition function and the pressure attributed to the ground state of a free scalar field. The Lagrangian density is

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 ,
\]

with \(c = 1, \hbar = 1\), and consequently the Hamiltonian density is

\[
\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 .
\]

By taking the trace of the Eq. (19), the generalized partition function for the free field, we get

\[
Z_q = \int_C dv \ K_q^{(1)}(v) \int \cdots \int D\phi D\pi \ e^{\int_0^\beta^* d\tau \int d^3 x (\phi \partial_{\tau} \phi - \mathcal{H}(\phi, \pi))} ,
\]

with \(\beta^* = (1 - q)(-v)\beta\) and \(K_q^{(1)} = \frac{1}{2\pi} \Gamma \left(\frac{2-q}{1-q}\right)[(1-q)^{-\frac{2-q}{1-q}}] e^{-v}\). This partition function, after the momentum integration reduces to

\[
Z_q = \int_C dv \ K_q^{(1)}(v) \int \cdots \int D\phi \ e^{\int_0^\beta^* d\tau \int d^3 x \mathcal{L}(\phi, \partial_{\tau} \phi)} .
\]

By substituting the Lagrangian density and following the usual calculation [30], we obtain for the generalized partition function the following expression
where $\omega = (\mathbf{p}^2 + m^2)^{1/2}$ and $\omega_n = 2\pi n/\beta^*$. Performing the sum over $n$, the above expression can be rewritten as

$$Z_q = \int_C dv \, K_q^{(1)}(v) \exp \left( -\frac{1}{2} (\beta^*)^2 \sum_n \sum_p (\omega_n^2 + \omega^2) \right), \quad (24)$$

By taking $m \to 0$ this expression recovers essentially the generalized partition function for the blackbody radiation \[29\]. In the following, as illustration, we calculate the generalized pressure attributed to the ground state. In this case, we use the definition of the generalized pressure \[29,31\]

$$P_q = \frac{1}{\beta} \frac{\partial}{\partial V} \left( \frac{Z_q^{1-q} - 1}{q - 1} \right), \quad (26)$$

In this way, by using Eqs. (25) and (26) with $T = 1/\beta$ sufficiently small (a detail discussion of this limit is presented in Ref. \[12\]), we obtain

$$P_q = \frac{1}{Z_q} \int_C dv \, K_q^{(2)}(v) \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{2} \omega + \frac{1}{\beta} \ln \left( 1 - e^{\beta(1-q)(-v)\omega} \right) \right] \times \exp \left\{ -V \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{2} \beta(1-q)(-v)\omega + \ln \left( 1 - e^{\beta(1-q)(-v)\omega} \right) \right] \right\} \times \frac{d^3 p}{(2\pi)^3} \omega, \quad (27)$$

where

$$K_q^{(2)}(v) = \Gamma \left( \frac{1}{1-q} \right) e^{-v(-v)^{-1/(1-q)}}, \quad (28)$$

At this point, we remark that the result for the generalized pressure of the ground state coincides with the same $q = 1$ case \[31\]. In general, the ground state properties can be obtained by using the small temperature limit of the generalized statistics independently of the $q$ value.
III. PATH INTEGRAL BASED ON NORMALIZED CONSTRAINTS

As in the above discussion, we can formulate the path integral for nonextensive systems using another version of the generalized statistics with different constraints (a detailed discussion of the choice of the constraints is presented in Ref. [23]). Considering the case with the normalized constraint \( \langle \langle \hat{H} \rangle \rangle_q = \text{Tr} \hat{\rho}_q^q \hat{H} / \text{Tr} \hat{\rho}_q^q \), instead of \( \langle \hat{H} \rangle_q = \text{Tr} \hat{\rho}_q^q \hat{H} \), we obtain that

\[
\hat{p}_q = \frac{\hat{\rho}_q}{Z_q}, \tag{29}
\]

with \( \hat{\rho}_q = (1 - (1 - q)\bar{\beta}(\hat{H} - \langle \langle \hat{H} \rangle \rangle_q))^{1/(1-q)} \), \( \bar{Z}_q = \text{Tr} \hat{\rho}_q \), and \( \bar{\beta} = \beta/\text{Tr} \hat{\rho}_q^q \). In the same way of the unnormalized constraint case, we can express Eq. (29) through the path integral,

\[
\tilde{\rho}_q(\phi, \phi', \beta) = \int_C d\bar{K}_q^{(1)}(v) \int \cdots \int D\phi D\pi \exp \left[ \int_0^\beta d\tau_1 \int d^3x (\phi \cdot \pi - \mathcal{H}(\phi, \pi)) \right], \tag{30}
\]

where \( \bar{\beta} = (1 - q)\bar{\beta}(-v) \) and \( \bar{K}_q^{(1)}(v) = K_q^{(1)}(v)e^{v(1-q)\bar{\beta}(\langle \hat{H} \rangle)_q} \).

To exemplify the normalized formalism, let us consider the free particle case,

\[
\tilde{\rho}_q(x, x', \beta) = \int_C dv \bar{K}_q^{(1)}(v) \int \cdots \int D x D p \exp \left[ \int_0^\beta d\tau (xp - H(x, p)) \right]. \tag{31}
\]

Here, the calculation resembles the unnormalized case. Thus, we restrict ourselves to presenting the main results. For instance, in the \( q < 1 \) case, after some calculations, the density matrix for one particle is given by

\[
\tilde{\rho}_q(x, x', \beta) = \left[ \frac{m \text{Tr} \tilde{\rho}_q^q}{2\pi(1-q)\beta h^2} \right]^{1/2} \left[ \left( \frac{2(1-q)\beta h^2}{m \text{Tr} \tilde{\rho}_q^q (x - x')^2} \left( 1 + \frac{(1-q)\beta}{\text{Tr} \tilde{\rho}_q^q} \langle \langle \hat{H} \rangle \rangle_q \right) \right)^{1/2} \right]^{1/2}, \tag{32}
\]

and, for the \( q > 1 \), it reduces to

\[
\tilde{\rho}_q(x, x', \beta) = \left[ \frac{2m \text{Tr} \tilde{\rho}_q^q}{\pi(q - 1)\beta h^2} \left( 1 + \frac{(1-q)\beta}{\text{Tr} \tilde{\rho}_q^q} \langle \langle \hat{H} \rangle \rangle_q \right)^{1/2} \right]^{1/2} \left[ \left( \frac{m(x - x')^2 \text{Tr} \tilde{\rho}_q^q}{2(q - 1)\beta h^2} \right)^{1/2} \right]^{1/2}, \tag{33}
\]
Here, $\tilde{Z}_q$ and $\langle\langle \hat{H} \rangle\rangle_q$ for $q < 1$ are

$$
\tilde{Z}_q = \left[ L \left( \frac{2m\pi}{(1-q)\beta\hbar^2} \right)^{1/2} \frac{\Gamma \left( \frac{2-q}{1-q} \right)}{\Gamma \left( \frac{2-q}{1-q} + \frac{1}{2} \right)} (1 + (1 - q)\frac{1}{2})^{1/(1-q)+1/2} \right]^{2/(1+q)}
$$

$$
\langle\langle \hat{H} \rangle\rangle_q = \frac{1}{2\beta} \text{Tr} \tilde{p}_q^q,
$$

and, for the case $q > 1$, are

$$
\tilde{Z}_q = \left[ L \left( \frac{2m\pi}{(q-1)\beta\hbar^2} \right)^{1/2} \frac{\Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{q-1} \right)} (1 + (1 - q)\frac{1}{2})^{1/(1-q)+1/2} \right]^{2/(1+q)}
$$

$$
\langle\langle \hat{H} \rangle\rangle_q = \frac{1}{2\beta} \text{Tr} \tilde{p}_q^q.
$$

(34)

(35)

In these calculations the general relation $\text{Tr} \tilde{p}_q^q = \tilde{Z}_q^{1-q}$ was employed.

A useful guide to calculate thermodynamics quantities in the normalized formulation is based on the relation [23]

$$
\beta^{\text{unnor}} = \frac{\beta^{\text{nor}}}{\text{Tr} \tilde{p}_q^q + (1 - q)\beta^{\text{nor}} \langle\langle \hat{H} \rangle\rangle_q},
$$

(36)

where the $\beta^{\text{unnor}}$ and $\beta^{\text{nor}}$ are Lagrange multipliers associated with unnormalized and normalized constraints, respectively.

**IV. CONCLUSIONS**

In this work, the path integral formulation for nonextensive systems based on Tsallis entropy was developed, considering both the unnormalized and the normalized constraints. For unnormalized formalism, we deduced, in a unified way, two different but equivalent generalizations of the path integral as well as their related Bloch equations. In the first case, the Bloch equation has the merit of preserving the form when compared with the usual Bloch equation, but containing a temperature-dependent effective Hamiltonian. This temperature dependence makes the evaluation of the respective path integral more difficult. The other generalized Bloch equation has a usual form in the sense that there is no explicit temperature dependence; but on the other hand, it is a nonlinear operator equation (it resembles the
anomalous correlated diffusion equation). The corresponding path integral (one and the same from both generalized Bloch equations) is obtained from integral representations of the gamma function. Thus, the usual path integral \((q = 1)\) is directly used to perform its generalization \((q \neq 1)\). This fact provides a natural way to generalize the main properties of the usual path integral, covering from the usual quantum mechanics to quantum field theory. Consequently, the second path integral formulation, based on integral representations, is more interesting than the first one. In addition, we analyzed the path integral formulation using the normalized constraint in the same way as the unnormalized one, where the free particle case was analyzed and our result are in agreement with the classical context [32] for one particle in one-dimensional space. Furthermore, the free scalar field was considered.

In particular, this work can be useful in the future investigations of macroscopic properties of one-electron states systems with a multifractal internal structure [33]. This idea must be explored in another opportunity. Let us conclude by saying that we expect the above generalizations to be useful in the discussion of generic nonextensive systems, hopefully in a similar way Feynman’s path integral is very useful in many branches of physics. Moreover, the present generalization appears as a kind of extension of the scale-invariant diffusion in space-time (see figure of [20] p. 177), in the sense that it could be anomalous \((q \neq 1)\), instead of the usual Brownian one \((q = 1)\).

ACKNOWLEDGMENTS

We gratefully acknowledge useful remarks by L. Borland and L. R. Evangelista. We also thank partial financial support by CNPq and PRONEX (Brazilian Agencies).

APPENDIX A:

Since we have not found the identity

\[
\ln(1 + x) = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{x}{N} \right)^n \frac{1}{1 + \frac{nx}{N}} \quad (x > -1)
\]  

(A1)
in anywhere else, we establish it in the following. In order to retain the convergence in all
the steps of the subsequent calculation, we employ a new variable \( y \), defined by the relation
\( x = m - 1 + y \), where \( m \geq 1/2 \), and \( |y| < m \). Thus, the identity (A1) can be written as
\[
\ln(m + y) = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{m - 1 + y}{N} \right) \frac{1}{1 + \frac{2(m-1+y)}{N}}.
\] (A2)

After a simple algebra, we obtain
\[
\lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{m - 1 + y}{mN} \right) \sum_{k=0}^{\infty} (-1)^k a^k,
\] (A3)
for r.h.s. of (A2), where \( a = \frac{y}{m} + \frac{m-1}{m} \frac{(n-N)}{N} < 1 \). Furthermore, by using a binomial
expansion for \( a \), in the above expression we have
\[
\lim_{N \to \infty} \sum_{k=0}^{\infty} \sum_{s=0}^{k} (-1)^k \binom{k}{s} \left( \frac{m - 1 + y}{mN} \right) \frac{y^{k-s}(m-1)^s}{(mN)^k} \sum_{n=1}^{N} n^{k-s} (n-N)^s.
\] (A4)
To evaluate the sum in \( n \), it is convenient to employ a further binomial expansion,
\[
(n - N)^s = \sum_{u=0}^{s} \binom{s}{u} n^u (-N)^{s-u}.
\] (A5)
After this, we can sum over \( n \), retaining only the dominating contribution in \( N \), as the other
contributions vanish in the limit \( N \to \infty \). In this way, r.h.s. of (A2) is reduced to
\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \left( \frac{m - 1 + y}{m^{k+1}} \right) \sum_{s=0}^{k} (-1)^s (m-1)^s y^{k-s}.
\] (A6)
Performing the sum over \( s = 0 \), we get for this expression
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left( \frac{y}{m} \right)^{k+1} + \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{m - 1}{m} \right)^k.
\] (A7)
The resultant sums are respectively the series representations for \( \ln \left(1 + \frac{y}{m} \right) \) and \( \ln(m) \),
establishing, therefore, the identity (A2).

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