Existence and stability of periodic solutions of quasi-linear Korteweg – de Vries equation

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Abstract. We consider the scalar nonlinear differential-difference equation with two delays, which models electrical activity of a neuron. Under some additional suppositions for this equation well known method of quasi-normal forms can be applied. Its essence lies in the formal normalization of the Poincare – Dulac obtaining quasi-normal form and the subsequent application of the theorems of conformity. In this case, the result of the application of quasi-normal forms is a countable system of differential-difference equations, which can be turned into a boundary value problem of the Korteweg – de Vries equation. The investigation of this boundary value problem allows us to draw a conclusion about the behaviour of the original equation. Namely, for a suitable choice of parameters in the framework of this equation is implemented buffer phenomenon consisting in the presence of the bifurcation mechanism for the birth of an arbitrarily large number of stable cycles.

1. General formulation of the problem

We consider the scalar non-linear differential equation with two delays

\[ \dot{u} = \lambda \left((a + 1)f(u(t - h_1)) - a - b g(u(t - h_2))\right)u \] (1)

which is used to model the impulse activity of a neuron (see [1, 2, 3, 4]). Here \( u(t) > 0 \) is the normalized membrane potential of the neuron, the sufficiently large parameter \( \lambda \) characterizes the rate of the electrical processes in the system, the parameters \( h_1, h_2, a, b \) are positive. The functions \( f(u) \), \( g(u) \) are from \( C^\infty(R_+) \), where \( R_+ = \{ u \in R : u \geq 0 \} \), and satisfy (see [2]) the following conditions.

Condition 1. We assume that \( f'(u) < 0 \), \( g'(u) > 0 \) \( \forall u \in R_+ \) and \( f(0) = 1, g(0) = 0, \lim_{u \to +\infty} f(u) = 0, \lim_{u \to +\infty} g(u) = 1 \).

The monotony of functions \( f(u) \) and \( g(u) \) ensures the existence and uniqueness of solution \( u = u_0(a, b) > 0 \) of the equation \( (a + 1)f(u) - a - b g(u) = 0 \).

Condition 2. Let the parameter \( a > 0 \) be fixed. The equation

\[ \psi(b) \equiv (a + 1)f'(u)|_{u=u_0(a,b)} + bg'(u)|_{u=u_0(a,b)} = 0 \]

has the unique solution \( b = b_* > 0 \) on the half-line \( b \in R_+ \) such that \( \psi'(b_*) > 0 \).

For the last condition we need the Taylor expansions of the following functions at the point \( v = 0 \):

\[ V_1(v, b) = (f(u(1 + v)) - f(u) - f'(u)uv)/(2f'(u)uv)|_{u=u_0(a,b)} \] and \( V_2(v, b) = \]
\(-b(g(u(1+v)) - g(u) - g'(u)uv)/(2(a+1)f'(u)u)|_{u=u_0(a,b)}\). The expansions begin with quadratic terms:

\[
V_j(v, b_*) = a_j^2v^2 + a_j^3v^3 + O(v^4), \quad j = 1, 2.
\]  

(2)

**Condition 3.** We assume that \(d \equiv 4(a_2^2 - a_{22}^2) + 2(a_{23} - a_{13}) < 0\).

Under the above restrictions we consider the question about the self-oscillating modes of the equation (1), which bifurcates from the equilibrium state \(u = u_0(a, b)\) as the parameters \(b, h_1, h_2\) are changed.

First of all, choose the parameters. As in [2] we are interested in the singular perturbed case: \(h_2/h_1 = \varepsilon h, \ h = \text{const} > 0, \ \varepsilon = \lambda^{-1}, \ 0 < \varepsilon \ll 1\). We choose the parameter \(h\) later and assume that \(b\) is close to \(b = b_*\), from condition 2. Then we make the problem suitable for use in the method of quasi-normal forms. Assume

\[
b = b_* + \alpha(\mu), \ |\mu| \ll 1,
\]  

(3)

where \(\mu\) is small enough parameter, the function \(\alpha(\mu), \ \alpha(0) = 0\), is defined from equation

\[
\psi(\alpha(\mu)) = (1 + \alpha) f'(u)/((b_* + \alpha)g'(u))^{-1}|_{u=u_0(a,b_*)} + (1 - 2\mu)(1 + 2\mu)^{-1} = 0.
\]

Put (3) to (1) and introduce a new variable \(v\): let \(u = u_\alpha(\mu)(1 + v)\) where \(u_\alpha(\mu) = u_0(a, b_* + \alpha(\mu))\). As a result of normalization \(\varepsilon(\chi_\alpha(\mu))^{-1} \rightarrow \varepsilon, \ h\chi_\alpha(\mu) \rightarrow h\), where \(\chi_\alpha(\mu) \equiv -2(1 + \alpha)f'(u_\alpha(\mu))u_\alpha(\mu)(1 - 2\mu)^{-1} > 0\), the equation (1) is transformed to

\[
\varepsilon \dot{v} = -\left(\frac{1}{2} - \mu\right)v(t - \varepsilon h) + \frac{1}{2} + \mu)v(t - 1) + \Delta_1(v(t - \varepsilon h), \mu) + \Delta_2(v(t - 1), \mu)\right)(1 + v)
\]  

(4)

where \(\Delta_j(v, \mu) = (1 - 2\mu)V_j(v, b_* + \alpha(\mu)), \ j = 1, 2\).

Now choose the parameter \(h\). Consider the characteristic equation

\[
2\varepsilon\lambda + (1 - 2\mu)e^{-\varepsilon h\lambda} + (1 + 2\mu)e^{-\varepsilon h\lambda} = 0
\]  

(5)

corresponding to zero equilibrium state of the equation (4). The equation (5) is analysed in [5] where in case \(h < \pi\) all its roots are divided into two groups. The first group contains the roots from the left complex half plane \(\{\lambda : \text{Re} \lambda < 0\}\). Besides, there are no roots from the group close to the imaginary axis as \(\varepsilon, \ \mu \rightarrow 0\). All other roots \(\lambda_n(\varepsilon, \mu), \ \bar{\lambda}_n(\varepsilon, \mu), \ n \in \mathbb{N}\), belong to the second group. As \(\varepsilon = \mu = 0\) the second group roots become the roots \(\lambda_n = i\omega_n, \ \omega_n = \pi(2n - 1), \ n \in \mathbb{N}\), of the equation \(e^{-\omega_n} = -1\). In the article [5] the roots are defined by equations

\[
\lambda_n(\varepsilon, \mu) = i\omega_n(1 + \varepsilon(h - 2) + \varepsilon^2(h - 2)^2) - 2\varepsilon^2\omega_n^2(1 - h) + 4\mu + O(\varepsilon^3 + \varepsilon\mu), \ \varepsilon, \mu \rightarrow 0.
\]  

(6)

The last equations allow to define the smallness order of \(\mu\) and \(\varepsilon\). Let below

\[
h = 1 - h_0\varepsilon, \ \mu = \mu_0\varepsilon^3, \ h_0, \mu_0 = \text{const} > 0.
\]  

(7)

Note that with this choice of parameters the order of \(\text{Re} \lambda_n(\varepsilon, \mu)\) is \(\varepsilon^3\).

**2. Application of the quasi-normal form method**

In this case, the algorithmic part of the quasi-normal form method is as follows. Put (7) to (4) and make change of time

\[
\tau = (1 + \sigma)t, \ s = \varepsilon^3t, \ \sigma = \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \varepsilon^3\sigma_3, \ \sigma_1 = -1, \ \sigma_2 = 1 - h_0, \ \sigma_3 = 2h_0 - 1
\]  

(8)

where \(\sigma_1, \ \sigma_2, \ \sigma_3\) are from (6). We find the main asymptotic behavior of self-oscillating mode as the series

\[
v = \sum_{k=0}^{\infty} v_k(s, \tau)\varepsilon^{(k+3)/2}.
\]  

(9)
We call the boundary value problem (11) quasi-normal form of (4). Here \( s, \) to Taylor series by delays \( \tilde{\tau}, \tilde{\sigma} \) then after normalization \( s/\varepsilon \) for all \( \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 = 1 - h_0 \varepsilon_0, \tilde{\tau} = \varepsilon (1 - h_0 \varepsilon_0)(1 + \tilde{\sigma}) \), equate the coefficients of the same powers of \( \varepsilon \) in the left and right parts of the \( v_k, k = 0, 1, 2, \ldots \) equation. As a result, we get equations

\[
(v_k(s, \tau) + v_k(s, \tau - 1))/2 = F_k(s, \tau), \quad k = 0, 1, 2, \ldots
\]

where functions \( F_k(s, \tau) \) are 2-periodic by \( \tau \). For the analysis (see [5]) of the equations (10) divide the function \( F_k(s, \tau) \) to two summands \( F_k^{(1)}(s, \tau) + F_k^{(2)}(s, \tau) \) such that \( F_k^{(1)}(s, \tau + 1) = -F_k^{(1)}(s, \tau), F_k^{(2)}(s, \tau + 1) \equiv F_k^{(2)}(s, \tau) \) where \( F_k^{(1)}(s, \tau) = \frac{(F_k(s, \tau) - F_k(s, \tau - 1))/2, F_k^{(2)}(s, \tau) = (F_k(s, \tau) + F_k(s, \tau - 1))/2} \), the system (10) has 2-periodic solution if \( F_k^{(1)}(s, \tau) \equiv 0 \). In this case the solution of the system is \( v_k(s, \tau) = F_k^{(2)}(s, \tau) \).

The choice of \( \sigma_1, \sigma_2, \sigma_3 \) ensures that the equation (10) has 2-periodic solution with numbers from \( k = 1 \) to 5. So like above we find \( v_1(s, \tau) \equiv 0, v_2(s, \tau) \equiv 0, v_3(s, \tau) = -a_2 + 2a_2 \xi \), \( v_4(s, \tau) \equiv 0, v_5(s, \tau) = (4a_1 + 1)\xi \).

When \( k = 6 \) from the solvability condition we have the boundary value problem for \( \xi(s, \tau) \)

\[
\xi_s = 2h_0 \xi_{\tau\tau} + \frac{1}{3} \xi_{\tau\tau\tau} + 4\mu_0 \xi + (4(a_{12}^2 - a_{22}^2) + 2(a_{23} - a_{13}))\xi^3, \quad \xi(s, \tau + 1) = -\xi(s, \tau).
\]

We call the boundary value problem (11) quasi-normal form of (4).

Consider the statement of the correspondence between the self-oscillating modes of quasi-normal form (11) and equation (4). As the phase space of problem (11) we take the Sobolev space \( W^3_2[0,1] \) of anti-periodic functions. And let the phase space of equation (4) be \( C[-1,0] \).

**Theorem 1.** Let \( \mu \) and \( h \) satisfy (7) and quasi-normal form (11) has the periodic traveling-wave solution

\[
\xi = \xi_0(y), \quad y = s + \tau, \quad \xi_0(y + 1) = -\xi_0(y), \quad \alpha_0 = \text{const},
\]

which is orbitally exponentially stable or dichotomous. Then, there exists \( \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon \leq \varepsilon_0 \) solution (12) corresponds to a cycle of equation (4) with the same stability properties. The main asymptotic of the cycle can be found from the part of above constructed series (9) using the formulas (12) and (8).

The proof of this fact is based on results from works [6], [7]. It is clear from the theorem 1 that we should be interested in periodic solutions of (11).

### 3. Secondary quasi-normalization

We suppose that \( \mu_0 = \varepsilon/12, h_0 = \varepsilon \kappa/6 \) as \( 0 < \varepsilon \ll 1 \) is a new small parameter, \( \kappa = \text{const} > 0 \), then after normalization \( s/3 \rightarrow s, -\tau \rightarrow \tau, \sqrt{-d} \xi \rightarrow \xi \) as \( d \) is from condition 3, the problem (11) is transformed to

\[
\xi_s + \xi_{\tau\tau\tau} = \varepsilon \kappa \xi_{\tau\tau} + \varepsilon \xi - \xi^3, \quad \xi(s, \tau + 1) = -\xi(s, \tau).
\]

Here \( s \) serves as a time, \( \tau \) — as space variable. The phase space is the Sobolev space \( W^3_2 \) of anti-periodic function.

Let us consider the question of the existence and the stability of invariant toruses of the system (13). We find self-oscillating solution of the boundary value problem (13) as the sum

\[
\xi = \sum_{k=0}^{\infty} \xi_k(s, \theta, \tau)e^{i(2k+1)/2},
\]
where \( \xi_0(s, \theta, \tau) = \sum_{k=1}^{\infty} (z_k(\theta)e^{iy_k} + \bar{z}_k(\theta)e^{-iy_k}) \), \( y_k = \nu_k^3 s + \nu_k \tau, \nu_k = (2k - 1)\pi, \theta = \epsilon s \). \( z_k(\theta) \) are unknown, and will be defined. The function \( \xi_0(s, \theta, \tau) \) satisfies by \( s, \tau \) linear homogeneous equation

\[
\xi_s + \xi_{\tau\tau} = 0.
\]

We find the function \( \xi_1 \) as the trigonometrical series by \( y_k, k \in \mathbb{N} \). Put (14) to (13) and equate the coefficients of the same powers of \( z \) complex amplitudes \( \xi \) are unknown, and will be defined. The function \( \xi_0(s, \theta, \tau) \) satisfies by \( s, \tau \) linear homogeneous equation

\[
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\[
\xi_s + \xi_{\tau\tau} = 0.
\]
Now we study the stability of the equilibrium state (20). To this end, we denote the right-hand side in (17) through \( f(\eta) = (f_1(\eta), \ldots, f_p(\eta), \ldots) \), where \( f_k(\eta) = (1 - \kappa \nu^2 - 6 \sum_{l=1, l \neq k}^{\infty} \eta_l - 3 \eta_k) \eta_k \).

Partial derivatives of functions \( f_k(\eta) \) at the point \( \hat{\eta} \) are as follows:
\[
\frac{\partial f_k}{\partial \hat{\eta}_j} \bigg|_{\eta = \hat{\eta}} = \begin{cases} 
-6 \hat{\eta}_{m}, & k = n_m, \ m = 1, \ldots, p, \ j \neq k, \\
-3 \hat{\eta}_{m}, & k = n_m, \ m = 1, \ldots, p, \ j = k, \\
0, & k \neq n_m, \ m = 1, \ldots, p, \ j \neq k, \\
1 - \kappa \nu^2 - 6 \sum_{l=1}^{p} \hat{\eta}_l, & k \neq n_m, \ m = 1, \ldots, p, \ j = k.
\end{cases}
\]

Then the system in variations of the interesting us equilibrium state has the form \( \dot{\eta} = J_{n_1, \ldots, n_p} \hat{\eta} \) where \( J_{n_1, \ldots, n_p} \) is the Jacobi matrix of \( f(\eta) \).

It is convenient to investigate the state of equilibrium in two cases: \( p = 1 \) and \( p \geq 2 \). Firstly, let \( p = 1 \) and fix natural \( n \). Then the solution (20) of the system (17) is transformed to
\[
\eta_n = (1 - \kappa \nu^2) / 3, \ \eta_k = 0 \quad k \neq n.
\] (21)

Besides, it exists if \( \kappa < \nu_n^{-2} \). The operator \( J_n \) has eigenvalues \( \lambda_n = -3\eta_n, \ \lambda_k = 2\kappa \nu^2 - \kappa \nu_n^2 - 1, \ k \neq n \). By requiring that all eigenvalues are negative, we obtain the condition for the stability of the equilibrium state (21):
\[
\kappa < (2\nu_n^2 - \nu_n^4)^{-1}.
\] (22)

Now let \( p \geq 2 \). Consider a finite-dimensional operator
\[
J_{n_1, \ldots, n_p} = \begin{pmatrix}
-3 \eta_{n_1} & -6 \eta_{n_1} & \cdots & -6 \eta_{n_2} \\
-6 \eta_{n_2} & -3 \eta_{n_2} & \cdots & -6 \eta_{n_2} \\
\vdots & \vdots & \ddots & \vdots \\
-6 \eta_{n_p} & -6 \eta_{n_p} & \cdots & -3 \eta_{n_p}
\end{pmatrix}.
\]

Note that the eigenvalues of this operator are the eigenvalues of the operator \( J_{n_1, \ldots, n_p} \). The eigenvalues are found from the characteristic equation
\[
(1 - \sum_{m=1}^{p} \frac{6 \eta_{m}}{3 \eta_{m} - \lambda}) \prod_{l=1}^{p} (3 \eta_{m} - \lambda) = 0.
\] (23)

It follows from the analysis of (23) that the equilibrium state (20) of system (17) with the number of nonzero coordinates \( p \geq 2 \) is unstable. Thus, the equilibrium state (20) is stable with only one non-zero coordinate when \( \kappa \) satisfies (22). Besides, (22) implies that the number of stable equilibrium states increases without bound with decreasing \( \kappa \).

**Theorem 2 (see [1]).** Any equilibrium state of the amplitude system (17) with \( p \) non-zero positive coordinate corresponds to an invariant \( p \)-dimension torus of the problem (13). The stability of the equilibrium state and the torus is the same.

The following statement implies from the above research and the theorem 2.

**Theorem 3.** For each natural \( n \) there exists a sufficiently small \( \epsilon_n > 0 \) such that for all \( 0 < \epsilon \leq \epsilon_n \) and the parameter \( \kappa \), satisfying the condition (22), the problem (19) has a stable travelling wave, defined by the equality \( \xi = \sqrt{\epsilon \xi_0(y)} + O(\epsilon^{3/2}) \) as \( y = (2n - 1)\pi x + \nu_n(\epsilon) t \), \( \nu_n(\epsilon) = (2n - 1)^3 \pi^3 + O(\epsilon) \). If \( p \geq 2 \), then the \( p \)-dimensional toruses of the problem (13) are unstable.

As follows from the condition (22) the number \( K(\kappa) \) of stable cycles of the boundary value problem (13) satisfies
\[
K(\kappa) \geq \left\lfloor \left( \frac{\sqrt{\pi - 2\kappa^{-1}}}{\sqrt{\pi - 2\kappa^{-1} + 1}} \right)/2 \right\rfloor.
\] (24)
ξ D. S. Glyzin (the method of the fourth-order Runge-Kutta precision increments of $10^{-3}$) for a description of the numerical experiment we fix an arbitrary integer $N = 25$.

5. Numerical experiment

For a description of the numerical experiment we fix an arbitrary integer $N = 25$.

$$\xi_{\tau} = N^{2}(\xi_{k+1} - 2\xi_{k} + \xi_{k-1}), \quad \xi_{\tau\tau} = N^{3}(\xi_{k+2} - 2\xi_{k+1} + 2\xi_{k-1} - \xi_{k-2})/2$$

where $\xi_{k} = \xi(s, \tau)|_{\tau=k/N}$, $k = 0, 1, \ldots, N - 1$. After time normalization $N^{3}/2 \mapsto N$ we get the system for $\xi_{k}(s)$:

$$\begin{align*}
\dot{\xi}_{k} &= -(\xi_{k+2} - 2\xi_{k+1} + 2\xi_{k-1} - \xi_{k-2}) + 2\epsilon\kappa(\xi_{k+1} - 2\xi_{k} + \xi_{k-1})/N + 2\epsilon\xi_{k}/N^{3} - 2\xi_{k}/N^{3}, \\
&= k = 0, 1, \ldots, N - 1, \quad \xi_{N} = -\xi_{1}, \quad \xi_{-1} = -\xi_{N-1}.
\end{align*}$$

Numerical analysis was performed using the software package tracer 3.7 developed by D. S. Glyzin (the method of the fourth-order Runge-Kutta precision increments of $10^{-3}$) for $N = 25$, $\kappa = 10^{-3}$, $\epsilon = 1$ and the initial conditions

$$\xi_{0k} = (2/\sqrt{3}) \cos ((2j + 1)\pi k/N), \quad k = 1, \ldots, N.$$  

Note that the initial conditions (26) are taken from the theoretical analysis, namely the initial condition is determined as the solution of the homogeneous equation corresponding to (13) as $\epsilon = 1$, $s = 0$, $\tau = j/N$, $j = 1, \ldots, N$.

It follows from (24) that the amount $K(\kappa)$ of the coexisting stable cycles of problem (13) satisfies the inequality $K(\kappa) \geq 5$ as $\kappa = 10^{-3}$. The fact is confirmed by the numerical simulations. Indeed, as a result of the computer experiment we get five different stable cycles.

Thus, we are developing the theory that not only works for small values of $\epsilon \ll 1$, but also predicts good result for $\epsilon \sim 1$. In the last case the number of simultaneously existing stable cycles coincides with their number expected from the theory for a small $\epsilon$. Thus, the results for $\epsilon \sim 1$ are qualitatively the same and the problem (13) has the buffering property also as $\epsilon \sim 1$. From the theorem 1 stable solutions of the original problem (4) correspond to exponentially stable solutions of the boundary value problem (13).

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