ON THE GODSIL – HIGMAN NECESSARY CONDITION FOR EQUITABLE PARTITIONS OF ASSOCIATION SCHEMES

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1 Introduction

Equitable partitions of association schemes are often related to important and interesting combinatorial and geometric objects such as combinatorial designs, orthogonal arrays and the Cameron – Liebler line classes in projective geometries. Thereby the question of existence of these structures in association schemes is very difficult in general.

In this short note, we show that a necessary condition for equitable partitions recently proposed by C. Godsil in his monograph [6] is not stronger than the well-known Lloyd theorem (see Theorem 1 in Section 2). In the next section, we recall some basic definitions and notions. Section 3 contains the proof of our result (Theorem 2).

2 Association schemes and their equitable partitions

Let $V$ be a finite set of size $v$ and $C^{V \times V}$ be the set of matrices over $C$ with rows and columns indexed by $V$. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of non-empty subsets of $V \times V$. For $i = 0, \ldots, d$, let $A_i \in C^{V \times V}$ be the adjacency matrix of the graph $(V, R_i)$. The pair $\mathcal{A} = (V, \mathcal{R})$ is said to be an association scheme with $d$ classes and vertex set $V$ if the following properties hold:

(1) $A_0 = I$, the identity matrix,

(2) $\sum_{i=0}^{d} A_i = J$, where $J$ is the all ones matrix,

(3) $A_i^T \in \{A_0, \ldots, A_d\}$, for every $i = 0, \ldots, d$,

(4) $A_i A_j$ is a linear combination of $A_0, \ldots, A_d$, for all $i, j = 0, \ldots, d$. 

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* e-mail: alexander.gavriliouk@gmail.com
† e-mail: ivmog84@gmail.com
The matrix algebra $\mathbb{C}[\mathcal{A}]$ over $\mathbb{C}$ generated by $A_0, \ldots, A_d$ is called the Bose–Mesner algebra of $\mathcal{A}$. It now follows from properties (1)-(4) that $\mathbb{C}[\mathcal{A}]$ has a basis consisting of the matrices $A_0, \ldots, A_d$ and its dimension is $d + 1$. We say that $\mathcal{A}$ is commutative if $\mathbb{C}[\mathcal{A}]$ is commutative, and that $\mathcal{A}$ is symmetric if the matrices $A_i$ are symmetric. Clearly, a symmetric association scheme is commutative.

In the paper, we assume that $\mathcal{A}$ is a symmetric association scheme with $d$ classes.

Since $\mathcal{A}$ is commutative, it follows that the matrices $A_0, A_1, \ldots, A_d$ are simultaneously diagonalized by an appropriate unitary matrix. This means that $\mathbb{C}^V$ is decomposed as an orthogonal direct sum of $d + 1$ maximal common eigenspaces of $A_0, A_1, \ldots, A_d$:

$$\mathbb{C}^V = W_0 \oplus W_1 \oplus \ldots W_d,$$

and, for every $0 \leq j \leq d$, define $E_j \in \mathbb{C}^{V \times V}$ to be the orthogonal projection onto $W_j$. Let $\{ \overline{w}_{j\ell} | 1 \leq \ell \leq \dim(W_j) \}$ be an orthonormal basis of $W_j$, $0 \leq j \leq d$.

Note [2] that the matrices $E_j$ form another basis for $\mathcal{A}$ consisting of the primitive idempotents of $\mathcal{A}$, i.e., $E_i E_j = \delta_{ij} E_i$ and $\sum_{j=0}^d E_j = I$.

For the two basises $A_0, \ldots, A_d$ and $E_0, \ldots, E_d$ of $\mathbb{C}[\mathcal{A}]$, the change-of-basis matrices $P$ and $Q$ are defined by

$$A_i = \sum_{j=0}^d P_{ji} E_j, \quad E_i = \frac{1}{v} \sum_{j=0}^d Q_{ji} A_j,$$

where, in fact, $P_{ji}$ is an eigenvalue of $A_i$ on the eigenspace $W_j$. It now follows that

$$PQ = vI.$$

The numbers $v_i = P_{0i}$, $0 \leq i \leq d$, are called the valencies of the scheme. The numbers $f_j = \text{tr}(E_j) = \text{rank}(E_j) = \dim(W_j)$, $0 \leq j \leq d$, are called the multiplicities of the scheme. By [2] Lemma 2.2.1(iv)] the following relation holds:

$$\frac{Q_{ji}}{f_j} = \frac{P_{ji}}{v_i}. \quad (1)$$

The distance-regular graphs provide important but not only examples of symmetric association schemes. For more results and background on distance-regular graphs and association schemes, we refer the reader to [1], [2], [3].

Let $\pi$ be a partition of $V$ with $t$ cells $C_1, C_2, \ldots, C_t$. The characteristic matrix $H$ of $\pi$ is the $v \times t$-matrix whose columns are the characteristic vectors of $\mathbb{C}^V$ of the cells of $\pi$.

We say that a partition $\pi$ of $\mathcal{A}$ is equitable [4] if the column space of $H$ is $\mathbb{C}[\mathcal{A}]$-invariant, i.e., for every $A \in \mathbb{C}[\mathcal{A}]$, there exists a $t \times t$-matrix $B$ usually called the quotient matrix of $A$ such that

$$AH = HB.$$

One can give the following equivalent combinatorial definition of equitable partition: $\pi$ is equitable if, for all $0 \leq i \leq d$, $1 \leq j, k \leq t$ and every vertex $x \in C_k$, there exist exactly $n_{ij}^k$ vertices $y \in C_j$ such that $(x, y) \in R_i$. We define the $t \times t$-matrices $N_i$, $0 \leq i \leq d$, by

$$(N_i)_{k,j} = n_{ij}^k,$$

and it is easy to see that, in fact, $N_i$ is the quotient matrix of $A_i$, i.e., $A_i H = H N_i$. 

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Let $C$ be a subset of the vertex set of a graph $\Gamma$. The covering radius $\rho_C$ of $C$ is defined to be $\rho_C := \max\{d(x, C) \mid x \in \Gamma\}$, where $d(x, C) := \min\{d(x, y) \mid y \in C\}$, and $d(x, y)$ is the usual graph distance. For $i = 0, \ldots, \rho_C$, define $\Gamma_i(C)$ to be the set of vertices that are at distance $i$ from $C$. The partition $\{C = \Gamma_0(C), \Gamma_1(C), \ldots, \Gamma_{\rho_C}(C)\}$ is referred to as the distance partition of the vertex set of $\Gamma$ with respect to $C$.

For an association scheme $\mathcal{A} = (V, R)$, assume that a graph $\Gamma = (V, R_i)$ is distance-regular for some $i$. An important type of equitable partitions is provided by completely regular codes. A vertex subset $C$ of $\Gamma$ is called a completely regular code if the distance partition with respect to $C$ is equitable. Clearly, a single vertex of the graph $\Gamma$ is a completely regular code.

For instance, the completely regular codes of the Hamming graphs give rise to orthogonal arrays, of the Johnson graphs — to combinatorial designs [4], [8], and of the Grassmann graphs of diameter 2 — to the Cameron – Liebler line classes in the projective geometry of dimension 3 [9].

Another interesting type of equitable partitions is provided by the partition of the vertex set of antipodal distance-regular graphs into antipodal classes [8].

The following result is well known in Algebraic Combinatorics and is sometimes referred to as Lloyd’s theorem [5].

**Theorem 1.** If $\pi$ is an equitable partition of symmetric association scheme $\mathcal{A}$, then, for any matrix $A \in \mathcal{A}$, the characteristic polynomial of its quotient matrix $B$ divides the characteristic polynomial of $A$.

In other words, Theorem 1 states that every eigenvalue of the quotient matrix $B$ is an eigenvalue of $A$, and its multiplicity as an eigenvalue of $B$ is not greater than its multiplicity as an eigenvalue of $A$.

### 3 Main result

Following Godsil [4], let us define a complex inner product on $\mathbb{C}^{V \times V}$ by

$$\langle M, N \rangle = \text{tr}(M^*N) = \sum \text{sum}(M \circ N),$$

where $\text{sum}(M)$ denotes the sum of the entries of $M$, and $\circ$ denotes the Schur multiplication of matrices. Then the basis $A_0, A_1, \ldots, A_d$ of the Bose – Mesner algebra $\mathbb{C}[\mathcal{A}]$ is orthogonal with respect to the inner product.

Further, for a matrix $M \in \mathbb{C}^{V \times V}$, let $\hat{M}$ denote its orthogonal projection onto $\mathbb{C}[\mathcal{A}]$, i.e.,

$$\hat{M} = \sum_{i=0}^d \frac{\langle A_i, M \rangle}{\langle A_i, A_i \rangle} A_i = \sum_{j=0}^d \frac{\langle E_j, M \rangle}{\langle E_j, E_j \rangle} E_j.$$

It follows from [6, Theorem 2.2.1] that

$$\hat{M} = \sum_{i=0}^d \frac{\langle A_i, M \rangle}{\nu_i} A_i = \sum_{j=0}^d \frac{\langle E_j, M \rangle}{m_j} E_j. \quad (2)$$

Let $P$ be a $v \times v$ permutation matrix. Then $P$ is an automorphism of $\mathcal{A}$ if it commutes with each matrix of $\mathbb{C}[\mathcal{A}]$. G. Higman derived the following necessary condition for $P$ to be an automorphism of $\mathcal{A}$, see also [3].
Let $P$ be an automorphism of $\mathcal{A}$ and $\sigma$ denote the corresponding permutation associated with $P$. Define $\alpha_i(\sigma)$ to be the number of vertices $x$ of the vertex set of $\mathcal{A}$ such that $(x, \sigma(x)) \in R_i$. Note that $\alpha_i(\sigma) = \langle A_i, P \rangle$. Then using equality (2): 

$$
\hat{P} = \sum_{i=0}^{d} \frac{\alpha_i(\sigma)}{v_i} A_i = \sum_{i=0}^{d} \frac{\langle P, E_i \rangle}{f_i} E_i,
$$

and, further, exploiting this equation, one can show (see [6]) that a number 

$$
\langle P, E_j \rangle = \frac{f_j}{v} \sum_{i=0}^{d} \frac{P_{ji}}{v_i} \alpha_i(\sigma)
$$

must be an algebraic integer.

This condition is widely used in a study of feasible automorphisms of distance-regular graphs [7].

Now let $F$ be a projection matrix (i.e., $F^2 = F$) that commutes with $C[A]$. In his monograph [6], C. Godsil notices that the arguments that prove the Higman condition also yield that 

$$
\langle F, E_j \rangle = \frac{f_j}{v} \sum_{i=0}^{d} \frac{P_{ji}}{v_i} \langle F, A_i \rangle
$$

must be a non-negative integer and this observation “could be used to show that certain equitable partitions do not exist”.

Let us define a projection matrix related to an equitable partition. Let $\pi$ be a partition of $\mathcal{A}$ with characteristic matrix $H$ and the cells $C_1, \ldots, C_t$. Then $D = H^T H$ is the diagonal matrix such that $D_{i,i} = |C_i|$. Now the matrix $F = HD^{-1}(HD^{-1})^T$ represents an orthogonal projection onto the column space of $H$. For an appropriate ordering of $X$, one can see that $F$ has the following block form:

$$
F = \begin{pmatrix}
\frac{1}{|C_1|} J_{|C_1|} & O_{|C_1| \times |C_2|} & \cdots & O_{|C_1| \times |C_t|} \\
O_{|C_2| \times |C_1|} & \frac{1}{|C_2|} J_{|C_2|} & \cdots & O_{|C_2| \times |C_t|} \\
\vdots & \vdots & \ddots & \vdots \\
O_{|C_t| \times |C_1|} & O_{|C_t| \times |C_{t-1}|} & \cdots & \frac{1}{|C_t|} J_{|C_t|}
\end{pmatrix},
$$

where $J$ is the all ones square matrix of proper order. Note that $\pi$ is equitable if and only if $F$ commutes with $C[A]$.

Now let $\pi$ be a putative equitable partition of $\mathcal{A}$ with quotient matrices $N_0, \ldots, N_d$, characteristic matrix $H$, and $F$ be the corresponding projection matrix. For a left eigenvector $\overline{w}_{ji}$ of $A_i$ with eigenvalue $P_{ji}$, we have $\overline{w}_{ji} A_i H = P_{ji} \overline{w}_{ji} H = \overline{w}_{ji} H N_i$ so that $\overline{w}_{ji} H$ is a left eigenvector of $N_i$ with the same eigenvalue $P_{ji}$ if $\overline{w}_{ji} H \neq 0$. Thus, if a space $W_j H$ is non-zero then it is a subspace of an eigenspace of $N_i$. Since $H$ has rank $t$, it follows that there are exactly $t$ linearly independent vectors of type $\overline{w}_{ji} H$ for some $0 \leq j \leq d$, which are left eigenvectors of $N_i$.

Let $m_j$ denote the dimension of $W_j H$, $1 \leq j \leq d$. It is now easily seen that, for each $k = 0, \ldots, t$, the spectrum of $N_k$ consists of the numbers $P_{0k}, \ldots, P_{dk}$ (some of them might be equal) with multiplicities $m_0, m_1, \ldots, m_d$ respectively.

**Theorem 2.** The following equality holds:

$$
\langle F, E_j \rangle = m_j,
$$

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where $m_j$ is the multiplicity of the eigenvalue $P_{ji}$ as an eigenvalue of $N_i$ on the eigenspace $W_j H$, for every $0 \leq i \leq d$.

**Proof.** First of all, we note that, for $0 \leq i \leq d$,

$$\langle F, A_i \rangle = \text{tr}(FA_i) = n_{i1}^1 + n_{i2}^2 + \ldots + n_{it}^t = \text{tr}(N_i),$$

and therefore $\langle F, A_i \rangle$ is equal to the sum of all eigenvalues of $N_i$.

We now have

$$\langle F, E_j \rangle = f_j v \sum_{i=0}^d P_{ji} \langle F, A_i \rangle = f_j v \sum_{i=0}^d P_{ji} \text{tr}(N_i) = f_j v \sum_{i=0}^d P_{ji} \sum_{k=0}^d m_k P_{ki}.$$

Using equality (1), we obtain

$$\langle F, E_j \rangle = \frac{1}{v} \sum_{i=0}^d Q_{ij} \sum_{k=0}^d P_{ki} m_k = \sum_{k=0}^d m_k \sum_{i=0}^d \frac{Q_{ij} P_{ki}}{v} = \sum_{k=0}^d m_k \delta_{j,k} = m_j,$$

which proves the theorem.

Therefore it follows from Theorem 2 that, for a putative equitable partition of association scheme, condition (3) cannot say more than Theorem 1. Moreover, the following example shows that the condition may be even weaker than an obvious condition of integrality of the elements of quotient matrix.

Consider a perfect matching $M = \{(i, i') : i \in \{0, \ldots, 4\}\}$. Let $C$ be a labeled cycle with vertex set $\{0, \ldots, 4\}$, and $C'$ be its complement with vertex set $\{0', \ldots, 4'\}$ so that $i' \sim_{C'} j'$ if and only if $i \not\sim_C j$.

Define a graph $\Gamma$ with vertex set $\{i : i \in \{0, \ldots, 4\}\} \cup \{i' : i \in \{0, \ldots, 4\}\}$ and edge set consisting of edges $C$, $C'$, and $M$. It is easy to see that $\Gamma$ is the Petersen graph, which is known to be *distance-regular* with diameter 2, i.e., the distance relations on its vertex set form an association scheme with two classes.

Let us consider a 5-partition consisting of a cell with a pair of adjacent vertices $\Gamma$ (say, $C_1 = \{0, 0'\}$) and of four cells of pairs of non-adjacent vertices (say, $C_2 = \{1, 2'\}$, $C_3 = \{2, 1'\}$, $C_4 = \{3, 4'\}$, $C_5 = \{4, 3'\}$). Clearly, the partition is not equitable, because, for example, the vertex $2'$ has a neighbour in $C_4$, whereas the vertex 1 does not.

However, the partition is feasible with respect to (3). It is easy to see that $\langle F, A_0 \rangle$ is always the number of cells in the partition, i.e.

$$\langle F, A_0 \rangle = 5,$$

and, further,

$$\langle F, A_1 \rangle = 1, \langle F, A_2 \rangle = 4.$$

Using equality (3), we obtain:

$$\langle F, E_0 \rangle = 1, \langle F, E_1 \rangle = 2, \langle F, E_2 \rangle = 2,$$

which completes our example.
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