Abstract

We previously showed that the real eigenvalues of $3 \times 3$ octonionic Hermitian matrices form two separate families, each containing 3 eigenvalues, and each leading to an orthonormal decomposition of the identity matrix, which would normally correspond to an orthonormal basis. We show here that it nevertheless takes both families in order to decompose an arbitrary vector into components, each of which is an eigenvector of the original matrix; each vector therefore has 6 components, rather than 3.

1 Introduction

In previous work [1, 2, 3] we considered the real eigenvalue problem for $3 \times 3$ octonionic Hermitian matrices. \footnote{There are also eigenvalues which are not real \footnote{\cite{4, 5}}, whose intriguing physical properties were discussed in \footnote{\cite{6, 7}}. A related eigenvalue problem, which admits only real eigenvalues, was discussed in \footnote{\cite{8}}.} For this case, there are 6, rather than 3, real eigenvalues \footnote{Supported in part by U. S. Department of Energy Grant No. DE-F02-ER 40685.}. We showed that these come in 2 independent families, each consisting of 3 real eigenvalues, which satisfy a modified characteristic equation rather than the usual one. Furthermore, the corresponding eigenvectors are not orthogonal in the usual sense, but do satisfy a generalized notion of
orthogonality. Finally, all such matrices admit a decomposition in terms of (the “squares” of) orthonormal eigenvectors, although these matrices are not idempotents (matrices which square to themselves).

It is the purpose of this paper to extend these results by showing how to project an arbitrary 3-component octonionic vector into the eigenspaces determined by the real eigenvalues of a given $3 \times 3$ octonionic Hermitian matrix. Somewhat surprisingly, this turns out to require all 6 eigenvalues; generic 3-component octonionic vectors therefore have 6 vector components, rather than 3.

\section{Octonionic Eigenvectors}

In this section, we summarize the main results of our previous work \cite{1, 2, 3}.

The (real) octonions $\mathbb{O}$ are the noncommutative, nonassociative division algebra generated by adjoining 7 anticommuting square roots of $-1$, with a particular multiplication table, to the real numbers $\mathbb{R}$. This generalizes both the complex numbers $\mathbb{C}$ and Hamilton’s quaternions $\mathbb{H}$; further details can be found in \cite{1} or \cite{10, 11}. We refer to a matrix as \textit{complex} or \textit{quaternionic} if its components lie in some complex or quaternionic subalgebra of $\mathbb{O}$, respectively.

Let $\mathbf{A}$ be a $3 \times 3$ octonionic Hermitian matrix. We consider solutions of the eigenvalue equation

$$\mathbf{A} \mathbf{v} = \lambda \mathbf{v} \quad (1)$$

where $\mathbf{v} \in \mathbb{O}^3$ and $\lambda \in \mathbb{R}$. Then $\lambda$ satisfies a modified characteristic equation of the form

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^3 - (\text{tr} \mathbf{A}) \lambda^2 + \sigma(\mathbf{A}) \lambda - \det \mathbf{A} = r \quad (2)$$

where the determinant is defined in terms of the Freudenthal product and where

$$\sigma(\mathbf{A}) = \frac{1}{2} \left( (\text{tr} \mathbf{A})^2 - \text{tr} (\mathbf{A}^2) \right) \quad (3)$$

The real number $r$ is a root of the quadratic equation

$$r^2 + 4 \Phi r - |\alpha|^2 = 0 \quad (4)$$

where both the associative 3-form $\Phi$ and the associator $\alpha$ are totally antisymmetric functions of the non-real components of $\mathbf{A}$, which are given explicitly in Section \S. There are 3 real solutions for $\lambda$ corresponding to each solution for $r$, which labels the families.

If the roots $r_1$ and $r_2$ of (4) are the same, then, over the real octonions, $\Phi = 0 = \alpha$, which forces $\mathbf{A}$ to be quaternionic (and $r_1 = 0 = r_2$). Unless $\mathbf{A}$ is complex, there are still two families of eigenvectors, but $r$ can not be used to label them. Rather, as described in the next section, one family consists of the usual quaternionic eigenvectors, while the other eigenvectors are purely octonionic, in the sense that their components are orthogonal (as

\footnote{This construction can be performed over any field $K$, although this will not in general yield a division algebra. We restrict ourselves here to the real octonions, as was done in \cite{1, 2}. As was explicitly verified in \cite{3}, most of these results hold over an arbitrary field, since division is only used in constructing normalized eigenvectors.}
vectors in \( \mathbb{R}^8 \) to the quaternionic subalgebra containing the components of \( \mathcal{A} \). Finally, if \( \mathcal{A} \) is complex, there is only one family of eigenvectors, obtained by (right) multiplying the usual complex eigenvectors by arbitrary octonions.

Within each family, many of the usual properties hold. Two eigenvectors \( u, v \) corresponding to different eigenvalues in the same family are orthogonal, but in the generalized sense that

\[
(uu^\dagger)v = 0 = (vv^\dagger)u
\]

Even in the case of repeated eigenvalues, 3 normalized, orthogonal eigenvectors \( v_m \), with eigenvalues \( \lambda_m \), can be found in each family. Orthonormality can be expressed in the form

\[
\mathcal{I} = \sum_{m=1}^{3} v_m v_m^\dagger
\]

where \( \mathcal{I} \) denotes the identity matrix, and these eigenvectors can be used to decompose \( \mathcal{A} \) in the form

\[
\mathcal{A} = \sum_{m=1}^{3} \lambda_m v_m v_m^\dagger
\]

### 3 Quaternionic Projections

In this section, we consider the much easier case of quaternionic matrices. While we are primarily interested in the \( 3 \times 3 \) case, all results in this section apply to \( n \times n \) quaternionic Hermitian matrices.

It is easily shown \( \Box \) that the right eigenvalue problem

\[
\mathcal{A} v = v \lambda
\]

for quaternionic Hermitian matrices \( \mathcal{A} \) admits solutions only for \( \lambda \in \mathbb{R} \). It is straightforward to find an orthonormal basis \( \{v_n\} \) of (quaternionic) eigenvectors, which satisfies

\[
\mathcal{A} v_m = \lambda_m v_m
\]

\[
v_m^\dagger v_n = \delta_{mn}
\]

Given any quaternionic vectors \( v, x \), the projection of \( x \) along \( v \) is simply \( v(v^\dagger x) \), so that any quaternionic vector has vector components of the form

\[
x = \sum v_m (v_m^\dagger x)
\]

which makes sense, since (the parentheses are not needed and) orthonormality can be written as

\[
\mathcal{I} = \sum v_m v_m^\dagger
\]

We now ask about the octonionic properties of the quaternionic matrix \( \mathcal{A} \). To do this, we view the octonions through the Cayley-Dickson process as

\[
\mathbb{O} = \mathbb{H} \oplus \ell \mathbb{H}
\]
where \( \ell \) is any octonionic unit orthogonal (as vectors in \( \mathbb{R}^8 \)) to all elements of \( \mathbb{H} \), the quaternionic subalgebra of \( \mathbb{O} \) containing the components of \( \mathbf{A} \). 

What are the octonionic eigenvectors of \( \mathbf{A} \)? As discussed in [1], if the components of \( \mathbf{A} \) and \( \mathbf{v} \) lie in the (same) quaternionic subalgebra \( \mathbb{H} \), then we have

\[
\mathbf{A}(\ell \mathbf{v}) = \ell (\overline{\mathbf{A}} \mathbf{v})
\]

Thus, if \( \mathbf{u} \) is an eigenvector of \( \mathbf{A} \), then \( \ell \mathbf{u} \) is an eigenvector of \( \mathbf{A} \) with the same eigenvalue! This implies that, over \( \mathbb{O} \), the real eigenvalues of \( \mathbf{A} \) consist of its eigenvalues over \( \mathbb{H} \) together with the eigenvalues of \( \mathbf{A} \) over \( \mathbb{H} \).

As expected, there are two families of (octonionic) eigenvectors of \( \mathbf{A} \), the usual family from the quaternionic eigenvalue problem, together with an “extra” family, namely \( \{ \ell \mathbf{u}_m \} \) where \( \{ \mathbf{u}_m \} \) are the (quaternionic) eigenvectors of \( \overline{\mathbf{A}} \), with real eigenvalues \( \mu_m \). This latter family also leads to a decomposition of \( \mathbf{A} \). To see this, note that if \( \mathbf{u} \) is quaternionic then

\[
(\ell \mathbf{u})(\ell \mathbf{u})^\dagger = \ell (\overline{\mathbf{u} \mathbf{u}}) = \ell \mathbf{u} \mathbf{u}^\dagger
\]

from which it follows that for any quaternionic \( \mathbf{u} \), \( \mathbf{x} \)

\[
[(\ell \mathbf{u})(\ell \mathbf{u})^\dagger](\ell \mathbf{x}) = \ell \mathbf{u} \mathbf{u}^\dagger(\ell \mathbf{x}) = \ell (\mathbf{u} \mathbf{u}^\dagger \mathbf{x})
\]

But this means that \( (\ell \mathbf{u})(\ell \mathbf{u})^\dagger \) has precisely the same projection properties over \( \ell \mathbb{H} \) as \( \mathbf{u} \mathbf{u}^\dagger \) has over \( \mathbb{H} \). Thus, if \( \{ \mathbf{u}_m \} \) are (quaternionic) orthonormal eigenvectors of \( \mathbf{A} \), then \( \{ \ell \mathbf{u}_m \} \) are (octonionic) orthonormal eigenvectors of \( \mathbf{A} \), and in particular

\[
\mathbf{I} = \sum (\ell \mathbf{u}_m)(\ell \mathbf{u}_m^\dagger)
\]

\[
\mathbf{A} = \sum \mu_m (\ell \mathbf{u}_m)(\ell \mathbf{u}_m^\dagger)
\]

Putting this all together, we can decompose an arbitrary octonionic vector \( \mathbf{x} \) in terms of the eigenvectors of \( \mathbf{A} \) and \( \overline{\mathbf{A}} \). First of all, there is a unique decomposition of \( \mathbf{x} \) into a quaternionic piece and a “purely octonionic” piece, given by

\[
\mathbf{x} = \mathbf{x}_1 + \ell \mathbf{x}_2
\]

where both \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are quaternionic. The rest is easy: Expand \( \mathbf{x}_1 \) in terms of the (quaternionic) eigenvectors of \( \mathbf{A} \), and \( \mathbf{x}_2 \) in terms of the (quaternionic) eigenvectors of \( \overline{\mathbf{A}} \). Equivalently, expand \( \mathbf{x}_1 \) in terms of \( \{ \mathbf{v}_m \} \) and \( \ell \mathbf{x}_2 \) in terms of \( \{ \ell \mathbf{u}_m \} \). This leads to

\[
\mathbf{x} = \sum \mathbf{v}_m (\mathbf{v}_m^\dagger \mathbf{x}_1) + \sum (\ell \mathbf{u}_m)(\ell \mathbf{u}_m^\dagger) (\ell \mathbf{x}_2)
\]

\[
= \sum \mathbf{v}_m \mathbf{v}_m^\dagger \mathbf{x}_1 + \ell \sum \mathbf{u}_m \mathbf{u}_m^\dagger \mathbf{x}_2
\]

We have therefore succeeded in projecting \( \mathbf{x} \) into vector components, each of which is an eigenvector of \( \mathbf{A} \). However, there are twice as many components as one would have expected from the purely quaternionic problem. Furthermore, while it is clear that \( \mathbf{v}_m \mathbf{v}_m^\dagger \mathbf{x}_1 \) is an eigenvector of \( \mathbf{A} \) with eigenvalue \( \lambda_m \), since it is a quaternionic right multiple of \( \mathbf{v}_m \), it is rather surprising that \( \ell (\mathbf{u}_m \mathbf{u}_m^\dagger \mathbf{x}_1) \) is an eigenvector of \( \mathbf{A} \), since it is not in general a multiple of \( \ell \mathbf{u}_m \).

We will see in the next section that this entire structure carries over to the fully octonionic setting, but that the identification of the two families is not as easy.

\[\text{If } \mathbf{A} \text{ is complex, the choice of } \mathbb{H} \subset \mathbb{O} \text{ is not fixed. In this case, however, there is only one family of (octonionic) eigenvectors. We assume in what follows that } \mathbf{A} \text{ is not complex.}\]
4 Decompositions of the Octonions

In this section, we summarize and interpret some of the more technical results from [3], which we will need to prove our main theorems in Section 8. These results are interesting in their own right, however, as they lead to the consideration of several intriguing ways of decomposing the octonions into subspaces.

Write the components of the octonionic Hermitian matrix $A$ as

$$A = \begin{pmatrix} d & a & \overline{b} \\ \overline{a} & e & c \\ b & \overline{c} & f \end{pmatrix}$$

(21)

with $d, e, f \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$. With these conventions, the associative 3-form [12] is given by

$$\Phi = \text{Re}(a \times b \times c) = \frac{1}{2} \text{Re}(a\overline{bc} - c\overline{ba})$$

(22)

and the associator is given by

$$\alpha = [a, b, c] = (ab)c - a(bc)$$

(23)

Viewing the octonions as the real vector space $\mathbb{R}^8$, we introduce the subspace $T \subset \mathbb{O}$ spanned by the elements of $A$. In other words, $T$ is the real vector space consisting of all (real) linear combinations of the components of $A$, that is

$$T = \langle 1, a, b, c \rangle$$

(24)

Since the quaternionic case was covered in the previous section, we will assume throughout the remainder of the paper that $\alpha \neq 0$, which in turn implies that $T$ has dimension 4 (as a real vector space).

Define the characteristic operator $K$ associated with $A$ by

$$K[x] = A(A(Ax)) - (\text{tr } A)(A(Ax)) + \sigma(A)(Ax) - (\det A)x$$

(25)

where $x \in \mathbb{O}^3$. As we may readily verify, the operator $K$ is diagonal, and can thus be reinterpreted as an operator acting on single octonions. This operator, also called $K$, can be written in the form

$$K[p] = c(b(ap)) + \overline{a}(\overline{b}(\overline{c}p)) - \left(c(ba) + \overline{a}(\overline{b}\overline{c})\right)p$$

(26)

for any $p \in \mathbb{O}$. In this form, $K$ is the same as the operator introduced in [3], where it was shown to satisfy the same quadratic equation as $r$, that is (compare (4))

$$K^2 + 4\Phi K - |\alpha|^2 = 0$$

(27)

In fact, comparing (25) with (2), we see that if $v$ is an eigenvector of $A$, so that (1) holds, then

$$K[v] = rv$$

(28)
with $r$ a solution of (4). Thus, the operator $K$ can be used to distinguish the 2 families of eigenvectors, labeled by the 2 possible values of $r$. \[4\]

Using results from [3], we can show that $K$ takes a particularly simple form on $T$, namely

$$K[t] = t\alpha \quad (t \in T)$$

(29)

Since

$$T^\perp \equiv T\alpha$$

(30)

it follows using (27) that

$$K[u] = -u(\alpha + 4\Phi) \quad (u \in T^\perp)$$

(31)

An intriguing property of the decomposition $O = T \oplus T\alpha$

(32)

is that it is almost of the Cayley-Dickson form. Direct computation establishes the multiplication table (for $t_1, t_2 \in T$)

$$t_1(t_2\alpha) = (t_2t_1)\alpha$$

(33)

$$(t_1\alpha)t_2 = (t_1t_2)\alpha$$

(34)

$$(t_1\alpha)(t_2\alpha) = -\overline{t_2t_1}|\alpha|^2$$

(35)

which is of the Cayley-Dickson form. However, this is not a true Cayley-Dickson extension of $T$, since $T$ is not closed under multiplication.

Using these ideas, it can be shown (Proposition 3.1 of [3]) that the eigenspace of $K$ with eigenvalue $r$ is precisely $T(r + 4\Phi + \alpha)$, that is

$$K[q] = m \iff \exists t \in T : q = t(r_m + 4\Phi + \alpha)$$

(36)

where the solutions of (4) are denoted $r_m$, with $m = 1, 2$. If we define $\tilde{s}_m = \frac{r_m + 4\Phi + \alpha}{2(r_m + 2\Phi)}$

(37)

then the decomposition of $O$ into its eigenspaces under $K$ takes the form

$$O = T\tilde{s}_1 \oplus T\tilde{s}_2$$

(38)

and the normalization is chosen such that

$$s_1 + s_2 = 1$$

(39)

\[4\] Recall that, over the real octonions, the two solutions of (4) are distinct unless $\Phi = 0 = \alpha$, in which case $A$ is quaternionic and can be handled by the methods of the previous section, and $K$ is not needed to distinguish the two families.

\[5\] Since $r_1 + r_2 + 4\Phi = 0$, the denominator can only vanish if $r_1 = r_2$, or equivalently $\tau = 0$ in the notation of [3]; this cannot happen here since we are assuming $\alpha \neq 0$. 


The eigenspaces \( T_m := T s_m \) are orthogonal, since \( K \) is self-adjoint. Explicitly, the inner product in \( \mathbb{R}^8 \) takes the form

\[
2p \cdot q = p\overline{q} + q\overline{p} = \overline{pq} + \overline{qp} \quad (p, q \in \mathbb{O})
\]  

and \( K \) satisfies

\[
K[p] \cdot q = p \cdot K[q]
\]  

It is now straightforward to define projection operators

\[
K_m = \frac{K + r_m + 4\Phi}{2(r_m + 2\Phi)}
\]  

from \( \mathbb{O} \) to \( T_m \), which satisfy

\[
K_1 + K_2 = 1 \quad K_1K_2 = 0 = K_2K_1 \quad K_1^2 = K_1 \quad K_2^2 = K_2
\]  

Finally, since \( s_1s_2 \) is proportional to \( \alpha \), we also have

\[
T_2 \equiv T_1\alpha
\]  

The following two results are key to understanding the family structure. First, assertions (v) and (vi) of Proposition 3.2 of [3] imply

\[
p, q \in T_m \implies p\overline{q} \in T
\]  

Second, as we also see from Proposition 4.1 of [3], for any \( p_1, p_2 \in T \) and \( q \in T_m \), we have

\[
[p_1, p_2, q] = p_3q
\]  

where \( p_3 \in \mathbb{O} \) depends on \( p_1, p_2 \) (and \( m \)), but does not depend on \( q \). These two results are the primary ingredients in the proofs in Section 8 of our main results.

5 Octonionic Projections

We now return to the discussion of octonionic eigenvectors, showing how to project any vector into the 6 eigenspaces they determine. We begin with the following result.

**Theorem 1:** Let a \( 3 \times 3 \) octonionic Hermitian matrix \( A \) be given, and suppose that \( v \in \mathbb{O}^3 \) is an eigenvector of \( A \), that is, \( Av = \lambda v \) with \( \lambda \in \mathbb{R} \). Suppose further that \( K[v] = rv \), with \( K \) as in (25), and let \( y \in \mathbb{O}^3 \) be any other vector satisfying \( K[y] = ry \) (for the same \( r \)). Then

\[
(v\dagger)(v\dagger)y = (v^\dagger v)((v\dagger)y)
\]  

**Proof:** See Section 8.

In other words, \((v\dagger)y\) is an eigenvector of a new \( 3 \times 3 \) octonionic Hermitian matrix, namely \( v\dagger \); if \( v \) is normalized, the eigenvalue is 1. Note, however, that \((v\dagger)y\) is not necessarily a multiple of \( v \). If \( u \) is another eigenvector of \( A \) in the same family, so that \( K[u] = ru \) and \( Au = \mu u \) and where we assume \( \lambda \neq \mu \in \mathbb{R} \), then \( u \) and \( v \) are orthogonal (in the
sense (3)). This in turn means that \( u \) and \((vv^\dagger)y\) are eigenvectors of the matrix \( vv^\dagger \) with different eigenvalues, and hence must themselves be orthogonal, that is

\[ uu^\dagger((vv^\dagger)y) = 0 \] (48)

This shows that, restricted to the appropriate \( K \) eigenspace, \( uu^\dagger \) and \( vv^\dagger \) are projection operators! Extending this to a family \( \{u, v, w\} \) of orthonormal eigenvectors, we obtain using (3)

\[ A((vv^\dagger)y) = \lambda((vv^\dagger)y) \] (49)

Furthermore, (3) leads to

\[ y = (uu^\dagger)y + (vv^\dagger)y + (ww^\dagger)y \] (50)

which expresses \( y \) as the sum of its projections along the eigenvectors.

Given an arbitrary vector \( x \in \mathbb{O}^3 \), we can use the projection operators \( K_m \) to project \( x \) into the eigenspaces of \( K \), by defining

\[ x_n = K_n[x] \] (51)

and noting that

\[ x = x_1 + x_2 \] (52)

We can then use (50) to expand each of \( x_m \) along the appropriate eigenvectors. Putting this all together, any vector \( x \) can be expanded into six terms, consisting first of the projections into the 2 eigenspaces of \( K \), then decomposing each of these separately into 3 eigenspaces of \( A \).

### 6 Families of Eigenvectors

Is the family structure a property of the matrix \( A \), or of its eigenvectors? We show in this section that it is the latter.

Decompose \( A \) in terms of an orthonormal eigenbasis as in (7), yielding

\[ A = \lambda uu^\dagger + \mu vv^\dagger + \nu ww^\dagger \] (53)

Then, by the results of the last section, all of \( u, v, w \) have the same eigenvalue under \( K \), namely

\[ r = \lambda\mu\nu - \det A \] (54)

Note that this result holds regardless of the values of \( \lambda, \mu, \nu \); it is a property of the orthonormal “basis” \( \{u, v, w\} \), not of a particular matrix \( A \).

Consider the special case where \( A = vv^\dagger \), with \( v^\dagger v = 1 \). Then \( v \) is an eigenvector of \( A \) with eigenvalue 1, and

\[ K[v] = -(\det A)v \] (55)

since

\[ \text{tr} \ A = 1 \quad \sigma(A) = 0 \] (56)

In this case, (47) simplifies to

\[ K[y] = -(\det vv^\dagger)y \iff vv^\dagger((vv^\dagger)y) = (vv^\dagger)y \] (57)
since the converse follows immediately from the definition of $K$ (with $A = vv^\dagger$). Special cases of vectors $u$ satisfying this condition are $u$ orthogonal to $v$, so that

$$A u = (vv^\dagger) u = 0$$

or

$$uu^\dagger = s vv^\dagger$$

with $s \in \mathbb{R}$, in which case

$$A u = (vv^\dagger) u = \frac{1}{s} (uu^\dagger) u = \frac{1}{s} u(u^\dagger u) = u$$

The decomposition (50) shows that any $y$ satisfying

$$K[y] = -(\det A) y$$

can be written as the sum of terms of one of these two types.

If $vv^\dagger$ is complex, then $v$ determines a single “family” containing all vectors. Furthermore, $v$ must then be a (right) octonionic multiple of a complex vector $v_0$; the “family” structure in this case corresponds to completing $v_0$ as usual to a complex orthonormal basis. However, there are also other families containing $v$. (If $v$ is a real eigenvector of $A$, then $A$ must be real or complex, and there is then no nontrivial family containing $v$.)

If $vv^\dagger$ is not complex, there is an 8-parameter family of vectors orthogonal to $v$, and a 4-parameter family of vectors satisfying (59). We therefore see that $v$ alone determines a 12-parameter family of vectors satisfying both (57) and (59). This family consists precisely of the (real) linear combinations of a family of eigenvectors of any matrix of the form (53). In other words, a family of eigenvectors is just an orthonormal “basis”, but each such basis spans only half of $O_3$.

We can complete this picture by showing that each vector $v$ (with $vv^\dagger$ not complex) lies in exactly one such family. To do this, we first remove the requirement that $v$ be an eigenvector from (57). Specifically, we have

**Theorem 2:** Let $K$ be the operator constructed via (23) from a given $3 \times 3$ octonionic Hermitian matrix $A$, and let $y, z \in O_3$. Suppose $K[y] = ry$ and $K[z] = rz$ with $r \in \mathbb{R}$. Then

$$(yy^\dagger) ((yy^\dagger)z) = (y^\dagger y) ((yy^\dagger)z)$$

**Proof:** See Section 8.

This finally allows us to argue that if $y$ is in the family defined by $v$, by virtue of satisfying either (and hence both) of (17) and (21), then $v$ is also in the family defined by $y$. This follows immediately from Theorem 2, since (21) is enough to satisfy the condition of the theorem, and (22) (with $z = v$) ensures that $v$ is in the family defined by $y$.

More generally, Theorem 2 shows that $u, w$ belong to the same family (with $u$ normalized by $u^\dagger u = 1$) if and only if

$$uu^\dagger (uu^\dagger)w = ((uu^\dagger)w)$$

For, (23) shows that $w$ is in the family defined by $u$. If $u$ is in the family defined by $v$, say, then the previous argument shows that $v$ is also in the family defined by $u$. But then $v$ and $w$ are both in the family defined by $u$, and the theorem shows that they must each be in the family defined by the other. This shows that any family which contains $u$ must also contain $w$. The converse, namely that $u, w$ in the same family implies (23), follows immediately from the theorem.
7 Discussion

As made clear in Section 4, the family structure, which was originally discovered on \( O^3 \) arising from the eigenvectors of \( A \), can be viewed as a property of the octonions themselves. From this point of view, the only purpose of \( A \) is in determining the octonions \( a, b, c \) used to define \( T \).

So suppose \( a, b, c \in O \) are given, and further assume that \( \alpha = [a, b, c] \neq 0 \). Does this decomposition of \( O \) depend on the choice of \( a, b, c \), or merely on \( T \)? Suppose \( A, B, C \in T \), so that

\[
A = A_0 + A_1a + A_2b + A_3c
\]

with \( A_m \in \mathbb{R} \), and similarly for \( B \) and \( C \). If we write \( \vec{A} \) for \( (A_1, A_2, A_3) \in \mathbb{R}^3 \), etc., then

\[
\begin{align*}
[A, B, C] &= (\vec{A}, \vec{B}, \vec{C}) [a, b, c] \\
\Phi(A, B, C) &= (\vec{A}, \vec{B}, \vec{C}) \Phi(a, b, c)
\end{align*}
\]

where \( (\vec{A}, \vec{B}, \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \) is the vector triple product, assumed to be nonzero. Since both \( \alpha \) and \( \Phi \) scale with the same factor, so does \( r \) — which means that the \( s_m \) are unchanged! The decomposition of \( O \) thus depends only on the subspace \( T \).

In terms of matrices, this means that any matrix with non-real components \( A, B, C \in T \) satisfying \( (\vec{A}, \vec{B}, \vec{C}) \neq 0 \) determines the same family structure as the original matrix \( A \). The 2 families determined by \( A \) are just

\[
O^3 = F_1 \oplus F_2 := (T_1)^3 \oplus (T_2)^3
\]

An essential ingredient in this picture is the result (45), which ensures that, for any vector \( v \in F_m \), the components of \( vv^\dagger \) are in \( T \).

We conclude by emphasizing that the subspaces \( T_m \) have many of the properties of the quaternions: They are 4-dimensional, they lead to an almost Cayley-Dickson multiplication table for the octonions, and, our main result, they have just enough associativity to enable arbitrary vectors to be projected into eigenspaces.
8 Proofs of the Theorems

We give here the proofs of our two theorems. This section can be skipped on first reading; all results have also been independently derived using Mathematica.

**Theorem 1:** Let a $3 \times 3$ octonionic Hermitian matrix $A$ be given, and suppose that $v \in O^3$ is an eigenvector of $A$, that is, $Av = \lambda v$ with $\lambda \in \mathbb{R}$. Suppose further that $K[v] = rv$, with $K$ as in (23), and let $y \in O^3$ be any other vector satisfying $K[y] = ry$ (for the same $r$). Then

$$((vv^\dagger)y) = (v^\dagger v) ((vv^\dagger)y)$$

**Proof:** By (36), $v, y \in (T_m)^3$, and by (45) the components of $vv^\dagger$ are in $T$. Thus, introducing a natural matrix associator, (46) implies that

$$((vv^\dagger)y) = ((vv^\dagger)(vv^\dagger)) y + [vv^\dagger, vv^\dagger, y] = \mathcal{V} y$$

(68)

where $\mathcal{V}$ does not depend on $y$! But alternativity implies that the vector associator satisfies

$$[v, v^\dagger, v] \equiv 0$$

(69)

which in turn implies that

$$((vv^\dagger))((vv^\dagger)v) = (v^\dagger v) ((vv^\dagger)v)$$

(70)

(and the right-hand-side simplifies even further). We want to show that

$$\mathcal{V} = (v^\dagger v)(vv^\dagger)$$

but (70) in fact only shows that

$$\mathcal{V} = (v^\dagger v)(vv^\dagger) + \mathcal{V}_1; \quad \mathcal{V}_1 v = 0$$

(72)

However, if $u$ is a different eigenvector of $A$ in the same family, that is, if

$$A u = \mu u \quad K[u] = r_m u$$

(73)

(74)

where we assume without loss of generality that $\mu \neq \lambda$, then the orthogonality (5) of $u$ and $v$ implies that

$$((vv^\dagger)u) = (v^\dagger v) ((vv^\dagger)u)$$

(75)

(since each side is 0). Since the right-hand-side is $\mathcal{V} u$, we must also have

$$\mathcal{V}_1 u = 0$$

(76)

so that $\mathcal{V}_1 = 0$ when acting on any eigenvector of $A$ (in the same family as $v$). But this means that $A$ and $A + \mathcal{V}_1$ have the same decomposition and are therefore equal, which forces $\mathcal{V}_1 = 0$, and establishes (71).
**Theorem 2:** Let $K$ be the operator constructed via (25) from a given $3 \times 3$ octonionic Hermitian matrix $A$, and let $y, z \in \mathbb{O}^3$. Suppose $K[y] = ry$ and $K[z] = rz$ with $r \in \mathbb{R}$. Then

$$(yy^\dagger)((yy^\dagger)z) = (y^\dagger y)((yy^\dagger)z)$$

**Proof:** As in the proof of the Theorem 1, $y, z \in (\mathbb{T}_m)^3$ by (36), and the components of $yy^\dagger$ are in $\mathbb{T}$. We can therefore write

$$yy^\dagger = \begin{pmatrix} d_1 & t_3 & \overline{t_2} \\
\overline{t_3} & d_2 & t_1 \\
t_2 & \overline{t_1} & d_3 \end{pmatrix}$$  \hspace{1cm} (77)$$

with $d_n \in \mathbb{R}$ and $t_n \in \mathbb{O}$. Introducing the components $y_n, z_n \in \mathbb{T}_m$ of $y, z$, direct computation using alternativity shows that

$$t_2(t_3y_2) = y_3|y_1|^2|y_2|^2 = d_1\overline{t_1}y_2$$  \hspace{1cm} (78)$$

and cyclic permutations, where we have used the fact that

$$d_1 = |t_3|^2$$  \hspace{1cm} (79)$$
$$
$$t_3 = y_1\overline{y_2}$$  \hspace{1cm} (80)$$

and so forth. But (10) implies that

$$t_2(t_3q) = (t_2t_3)q + [t_2, t_3, q] = pq$$  \hspace{1cm} (81)$$

for some $p \in \mathbb{O}$ which is independent of $q \in \mathbb{T}_m$, and we have therefore shown that

$$p = d_1\overline{t_1}$$  \hspace{1cm} (82)$$

In particular, we must have

$$t_2(t_3z_2) = d_1\overline{t_1}z_2$$  \hspace{1cm} (83)$$

and cyclic permutations. A similar argument establishes

$$\overline{t_1}(\overline{t_3}q) = d_2t_2q$$  \hspace{1cm} (84)$$

for any $q \in \mathbb{T}_m$, together with its cyclic permutations.

We are now ready to compute both sides of (62) explicitly. We have first of all

$$(yy^\dagger)z = \begin{pmatrix} d_1z_1 + t_3z_2 + \overline{t_2}z_3 \\
\overline{t_3}z_1 + d_2z_2 + t_1z_3 \\
t_2z_1 + \overline{t_1}z_2 + d_3z_3 \end{pmatrix}$$  \hspace{1cm} (85)$$

Multiply on the left by $yy^\dagger$, and consider for simplicity only the last component, which is

$$t_2(d_1z_1 + t_3z_2 + \overline{t_2}z_3) + \overline{t_1}(\overline{t_3}z_1 + d_2z_2 + t_1z_3) + d_3(t_2z_1 + \overline{t_1}z_2 + d_3z_3)$$

$$= (d_1 + d_3)t_2z_1 + \overline{t_1}(\overline{t_3}z_1) + t_2(t_3z_2) + (d_2 + d_3)\overline{t_1}z_2 + (|t_1|^2 + |t_2|^2 + d_3^2)z_3$$  \hspace{1cm} (86)$$

$$= (d_1 + d_2 + d_3)(t_2z_1 + \overline{t_1}z_2 + d_3z_3)$$
where we have used (79), (80), (83), and (84) in the last equality. Repeating this for the remaining components, and noticing that

\[ d_1 + d_2 + d_3 = \text{tr} (yy^\dagger) = y^\dagger y \]  

(87)

completes the proof.

Theorem 1 clearly follows from Theorem 2. But one can also argue that the reverse is true, thus avoiding the detailed computation given above. The argument goes as follows: \( yy^\dagger \) is itself a Hermitian matrix with components in \( \mathbb{T} \), and, by the argument given in Section 4, it determines the same family structure as the given matrix \( A \), and can therefore be used instead of \( A \) in applying Theorem 1.
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