Deconstruction hierarchies and locality diagrams of conformal models

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Abstract: The relationship between locality graphs and deconstruction hierarchies of conformal models is explained, leading to computationally effective procedures for determining the latter, and the relevant notions are illustrated with several examples.

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1 Introduction

Orbifold deconstruction [3, 5], the procedure aimed at recognizing whether a given conformal model is an orbifold [12, 13] of another one, and if so, to determine both this original model and the relevant twist group, is an interesting tool for the study of 2D CFT and, in a broader context, of discrete gauge symmetries. Indeed, since orbifolding amounts to gauging global symmetries of a conformal model [11], deconstructing the orbifold sheds light on the effect of gauging selected symmetries. This is all the more true if one considers the full hierarchy of deconstructions that results from the possibility of orbifolding by stages: gauging the global symmetries forming a group $G$ may be achieved in steps, by first gauging the symmetries from a normal subgroup $N \triangleleft G$, and then gauging the resulting model by the factor group $G/N$. As a consequence, each orbifold model will have different partial deconstructions corresponding to the different normal subgroups of its twist group, and the hierarchy of these partial deconstructions will be described by the lattice of normal subgroups of the latter. But having such a hierarchy of partial deconstructions means that we can study the effect of gauging layer by layer, leading to a more refined understanding of the whole process. This idea points clearly at the importance of controlling the full deconstruction hierarchy for any given conformal model.

Actually, the situation is a bit more complicated, because in many cases one and the same conformal model might be realized as an orbifold in several fundamentally different ways. This means that there might exist several maximal deconstructions of a given model [3, 5], each realizing it as an orbifold with a possibly different twist group and/or deconstructed model: this phenomenon is already apparent for the simplest case of $\mathbb{Z}_2$-orbifolds, e.g. in the construction of the Moonshine module from the Leech lattice VOA [15]. Having different maximal deconstructions means that the hierarchy of all deconstructions of a given model cannot be described by a lattice, but only by some more general ordered structure, which still has pretty special features, like having all its order ideals isomorphic to the normal subgroup lattice of some finite group.

The way out, as explained in [4], is to consider the hierarchy of deconstructions as embedded in the lattice (ordered by inclusion) of so-called FC sets, i.e. sets of
primary fields closed under the fusion product. That such an embedding is possible follows from the observation that in any orbifold model there is a distinguished set of primaries, the so-called vacuum block, which consists of those primaries of the orbifold (in one-to-one correspondence with the irreducible representations of the twist group) that originate in the vacuum sector of the original model. This vacuum block has very special properties, since it is closed under the fusion product, and all its elements have integer conformal weight and quantum dimension, i.e. it is a so-called twister [3, 5]. Each twister of a given model corresponds to a different deconstruction, realizing it as an orbifold in a different way, and the hierarchy of deconstructions is reflected by the inclusion relation among the twisters. Most importantly for us, the basic features of the deconstruction hierarchy follow directly from the fact that the lattice \( \mathcal{L} \) of FC sets is a modular lattice endowed with an involutive and order reversing self-map [4].

It should transpire from the above that the major task for understanding the deconstruction hierarchy of a given model is to determine the corresponding lattice \( \mathcal{L} \) and the precise location of the different twisters inside it. This is by no means a trivial job, since a brute force approach would have a computational cost growing exponentially with the number of primaries, and it would fail already for models with 20-30 primaries, while really interesting examples usually involve at least hundreds, if not thousands of them. At first sight this could seem to be a major obstacle, but, as we shall explain, there is a way out, exploiting the connection between the lattice \( \mathcal{L} \) and the locality graph of the model. Not only does this give us an effective procedure to determine \( \mathcal{L} \), but it does also explain some striking features of the deconstruction hierarchy that could seem accidental otherwise.

In the next section, we shall recall those results about the lattice \( \mathcal{L} \) that are necessary for understanding the rest of the paper. Then we shall turn to the relation between locality graphs and the lattice of FC sets, and explain how this sheds light on some properties of deconstruction hierarchies not understood before, in particular, why all but two Virasoro minimal models share the same lattice. Finally, we shall comment on possible consequences and applications of the results presented.

2 FC sets and their lattice

A set \( g \) of primaries of a conformal model is fusion closed, or an FC set for short [4], if it contains the vacuum primary, and if the fusion product of any two of its elements contains only primaries from \( g \). In other words, if \( p \) and \( q \) are elements of an FC set \( g \), and \( N_{pq}^r > 0 \) for some primary \( r \), then \( r \) is also an element of \( g \). Note that an FC set contains automatically the charge conjugate of all its elements. The collection \( \mathcal{L} \) of FC sets of a given conformal model is partially ordered by inclusion, with minimal element the trivial FC set that consists of the vacuum solely, and maximal element the set of all primaries. Since the intersection of two FC sets is clearly an FC set again, \( \mathcal{L} \) is actually a lattice [16].

As it turns out, \( \mathcal{L} \) is a lattice of a very special kind [4]: it is a self-dual lattice admitting a type I embedding into a partition lattice [17], and in particular, it is a modular lattice [16]. This last property is fundamental in relation with orbifold deconstruction, since it guarantees that the set of partial deconstructions of an orbifold
is isomorphic to the normal subgroup lattice of the twist group. Self-duality of \(L\) refers to the fact that to each \(g \in L\) one can associate its trivial class \(g^\perp\), the collection of all those primaries that are mutually local with every primary of \(g\), which is itself an FC set, and the assignment \(g \mapsto g^\perp\) is an involutive and order-reversing map of \(L\) onto itself. If all elements of an FC set \(g\) have integer conformal weight, i.e. \(g\) is a twister, there is a corresponding realization of the conformal model as an orbifold, and the trivial class \(g^\perp\) consists of the primaries contained in the untwisted sector.

Local FC sets, i.e. those contained in their trivial class, play a special role [4], for one may show that all local FC sets are either twisters, in which case they provide a realization of the given model as an orbifold, or \(\mathbb{Z}_2\)-extensions of twisters, in which case a suitable generalization of the deconstruction procedure leads to a fermionic extension a la Runkel-Watts [19].

An interesting class of FC sets is formed by those whose elements have integer quantum dimension. Such integral FC sets include the local ones [4], and form a sublattice of \(L\), with maximal element the set of all those primaries whose quantum dimension is an integer. An interesting aspect of integral FC sets is that it is possible to associate to them a ‘character table’ which shares many non-trivial properties of character tables of finite groups. This is no surprise for twisters, since these correspond to some orbifold realization of the model, and the resulting table is just the ordinary character table of the twist group, but many deep analogies persist even in cases when one can explicitly show (by excluding case-by-case all potential candidates) that there is no suitable group with corresponding character table. These analogies allow to generalize to integral FC sets many notions from group theory, like nilpotency and solubility [4], and suggest that integral FC sets might be related to some kind of ‘generalized group structure’.

There is one more important property of the lattice \(L\) that should be mentioned. To any collection \(X\) of primary fields one can associate the sum

\[
\mu(X) = \sum_{p \in X} d_p^2
\]

where \(d_p\) denotes the quantum dimension of the primary \(p\). When \(g \in L\) happens to be a twister corresponding to a realization of the conformal model as an orbifold, \(\mu(g)\) equals the order of the relevant twist group [3]. It is pretty clear that

\[
\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y)
\]

and one has \(\mu(X) \leq \mu(Y)\) for a subset \(X \subseteq Y\). The point is that the product \(\mu(g)\mu(g^\perp)\) is the same [4] for all \(g \in L\). As we shall see later, this leads to non-trivial restrictions that exclude some possibilities that would look completely healthy otherwise.

3 The locality diagram of a conformal model

Two primary fields of a conformal model are mutually local if their OPE coefficients are single-valued functions of their separation. Using conformal invariance, this translates into the requirement that, denoting by \(h_p\) the conformal weight of a primary \(p\), the
primaries $p$ and $q$ are mutually local if the difference $h_r - h_p - h_q$ is an integer for any primary $r$ such that the fusion rule coefficient $N^r_{pq}$ is positive.

Clearly, mutual locality of primaries is a symmetric binary relation, which can be represented by an undirected graph (with possible loops), whose vertices are the primary fields, with two of its vertices adjacent whenever the corresponding primaries are mutually local. We shall call this graph $G_{loc}$ the locality graph of the given conformal model\(^1\). As we shall see, it is the key to understanding the structure of $L$, but to explain how this comes about, we have to take first a look at some results about undirected graphs [2, 7].

Given an undirected graph $G$, one can associate to each vertex $v$ its neighborhood $G(v)$, the collection of all those vertices that are adjacent to it; in case of the locality graph $G_{loc}$, the neighborhood of a primary will consist of those primaries with which it is mutually local. More generally, to any collection $X$ of vertices one can associate their common neighborhood $G(X)$, which consists of all vertices adjacent to each vertex in $X$, i.e. the intersection of the neighborhoods of the vertices in $X$. It is immediate that $X \subseteq Y$ implies $G(Y) \subseteq G(X)$, and that $X \subseteq G^2(X) = G(G(X))$. As a result, the assignment $X \mapsto G^2(X)$ is a closure operation on sets of vertices, hence the collection $L(G)$, ordered by inclusion, of those sets of vertices for which $G^2(X) = X$ (the closed ones) is a finite lattice. What is more, this lattice $L(G)$ comes naturally equipped with a duality map, i.e. an involutive and order-reversing self-map that assigns to each $X \in L(G)$ the set $G(X) \in L(G)$; put differently, the lattice $L(G)$ is self-dual.

The basic result is that the lattice $L(G_{loc})$ associated to the locality graph coincides with the lattice $L$ of FC sets, in such a way that the corresponding duality maps are equal. This is indeed a truly remarkable fact, since it exhibits a close relation between the fusion algebra and the locality graph, while neither of these is completely fixed by the other. As a direct application, one can determine the collection of all FC sets in a conformal model from the mere knowledge of the locality graph, without having to deal with the details of the fusion algebra. Even more is true, since the equality of the corresponding duality maps allows to single out the local FC sets that form the input of the deconstruction algorithm: these will correspond to those closed sets $X \in L(G)$ that are contained in $G(X)$. As a consequence, FC sets corresponding to maximal deconstructions are in one-to-one correspondence with maximal cliques of the locality graph $G_{loc}$ (more precisely, those maximal cliques all of whose vertices are self-adjacent). Finding these is a classical problem of graph theory [9], with many applications ranging from bioinformatics [10] through electrical engineering to social network analysis [20].

While the above ideas already provide a serious improvement in the handling of FC sets, they still have the drawback that, since the size (i.e. number of vertices) of the locality graph equals the number of different primaries, the computational cost of determining the corresponding lattice still grows exponentially with the latter. But one can remediate this situation, as we shall now explain.

Two vertices of an undirected graph $G$ are said to be equilocal if their neighborhoods

\(^1\)Note that every locality graph is connected, since it contains a universal vertex (i.e. one adjacent to all other vertices) corresponding to the vacuum primary.
coincide. Clearly, this is an equivalence relation, whose equivalence classes partition
the set of vertices in a way that is compatible with adjacency. Put another way,
the equilocality classes provide a modular partition of the graph [7]. This allows
to consider the quotient graph $G'$ of $G$ by the equilocality relation\(^2\), whose vertices
correspond to equilocality classes, with two classes being adjacent whenever they
contain adjacent vertices. We shall call the corresponding quotient $G_{10c}'$ of the locality
graph the locality diagram of the conformal model.

It follows from the symmetry of the adjacency relation that any element of $L(G)$ is
actually a union of equilocality classes. As a result, there exists a map sending each
$X \in L(G)$ to the collection of equilocality classes that it contains, and one may show
that this deflation map induces an isomorphism between the lattices $L(G)$ and $L(G')$
that commutes with the respective duality maps. It follows that the structure of the
lattice $L$ of FC sets is completely determined by the collection of equilocality classes
of the locality graph and by the lattice $L(G_{10c}')$ associated to the locality diagram: as
far as one is only interested in the lattice structure of $L$, one may dispense with the
locality graph itself, and concentrate solely on its quotient, the locality diagram.

This last result has many important consequences. In particular, it leads to a
dramatic decrease in the computational resources needed to determine the lattice $L'$,
since there are usually much less equilocality classes than there are primary fields.
This leads to the following effective procedure: first, determine the locality graph $G_{10c}$
of the model (the computational cost being polynomial in the number of primaries),
from which one can read off the equilocality classes and the locality diagram $G_{10c}'$;
the next step (whose cost is exponential in the number of equilocality classes) is to
compute the lattice $L(G_{10c}')$ associated to the locality diagram; finally, one has to use
the deflation isomorphism between $L(G_{10c}')$ and $L(G_{10c}) = L$ to get the result. This
opens the way to perform the necessary analysis for models with several hundreds,
even thousands of primaries.

Besides its computational utility, there is another, more conceptual aspect of the
deflation isomorphism: it does explain why so many, at first sight pretty differently
looking conformal models share the same lattice $L'$. This is due to the fact that,
while their locality graphs are truly different, the corresponding locality diagrams
coincide in many cases, leading to the same lattice structure. A nice example of
this phenomenon is provided by the (unitary) Virasoro minimal models: while their
structure is different, their lattice of FC sets are, except for two of them, all isomorphic
to a generic Virasoro lattice $L_{Vir}$, whose Hasse diagram is depicted on the left of
Fig.1. The reason for this is that, while the locality graphs differ from each other, the
locality diagrams are all isomorphic to the graph shown on the right of Fig.1, with the
exception of the models with respective central charges $c = \frac{7}{10}$ and $\frac{1}{2}$, whose locality
diagrams are shown in Fig.2.

\(^2\)Note that this quotient graph is always irreducible in the sense that no two of its vertices are equilocal.
Figure 1. Hasse diagram of the generic Virasoro lattice $\mathcal{L}_{\text{gen}}$, and the corresponding locality diagram (the labelling of the equilocaly class being compatible with that of Table 1).

Actually, even the two cases that do not fit in this generic pattern may be understood as degenerations of the latter. Indeed, the corresponding models have too few (only 6, resp. 3) primaries to fill each of the 8 equilocaly classes in Fig.1, so some of them have to be left empty, and the corresponding vertices should be left out from the relevant diagram. The primary content of the different equilocaly classes of Virasoro minimal models is summarized in Table 1, showing that for the model with central charge $c = 7/10$ the equilocaly classes labeled $\mathcal{E}_2$ and $\mathcal{E}_7$ are empty, hence the corresponding vertices are missing from the relevant locality diagram, while for central charge $c = 1/2$ the classes labeled $\mathcal{E}_1$, $\mathcal{E}_3$ and $\mathcal{E}_5$ are also void, so the relevant vertices have to be left out as well. The resulting locality diagrams are depicted in Fig.2.

$$
\begin{array}{|c|c|c|c|}
\hline
\text{Class} & c = 1/2 & c = 7/10 & c = 4/5 & c = 6/7 \\
\hline
\mathcal{E}_0 & (1, 1) & (1, 1) & (1, 1) & (1, 1) \\
\mathcal{E}_1 & (2, 2) & (2, 2), (3, 2) & (2, 2), (4, 2), (4, 3), (4, 4) & \\
\mathcal{E}_2 & (4, 3) & & (3, 1) & \\
\mathcal{E}_3 & (3, 3) & (2, 1) & (5, 3), (5, 5) & \\
\mathcal{E}_4 & (2, 2) & (2, 1), (4, 2), (4, 4) & (2, 1), (4, 1) & \\
\mathcal{E}_5 & (3, 2) & (3, 1) & (5, 2), (5, 4) & \\
\mathcal{E}_6 & (2, 1) & (3, 1) & (4, 1) & (5, 1) \\
\mathcal{E}_7 & (3, 3) & (3, 2) & (3, 3) & \\
\hline
\end{array}
$$

Table 1. Kac labels of the primaries filling the different equilocaly classes of the first few Virasoro minimal models.

Similar results hold for other classes of conformal models, like Wess-Zumino, superconformal, parafermionic, etc., but usually the pattern is more complicated, with several classes of ‘generic’ diagrams and their different degenerations. In case of
Figure 2. Locality diagrams of the Virasoro minimal models of respective central charges $c = 7/10$ and $1/2$.

Figure 3. Locality diagram of $N = 2$ superconformal minimal models of central charge $c = 3 - \frac{6}{p}$, with $p$ an odd prime.

$N = 2$ superconformal minimal models, the locality diagram for models of central charge $c$ seems to be determined by the primary decomposition (as a product of prime powers) of the integer $\frac{6}{3-c}$, the trilobite-like diagram depicted in Fig.3 corresponding to the case when this last number is actually an odd prime. Similar patterns can be observed for parafermionic and Ashkin-Teller models (i.e. $\mathbb{Z}_2$ orbifolds of a free boson compactified on a circle of suitable radius).

Primaries that belong to the same equilocality class share many properties. For example, if the quantum dimension of a primary equals 1 (resp. is an integer), then
the same is true for all primaries in the same equilocality class. More generally, one may show that the number field generated by the quantum dimension of a primary is the same for all elements of its equilocality class. In particular, self-local (i.e. self-adjacent) equilocality classes are integral in the sense that the quantum dimension of all their elements are rational integers, and their conformal weights are either integers or half-integers.

An interesting aspect of equilocality classes is related to orbifold deconstruction. Indeed, consider a twister \( g \) corresponding to a realization of the given conformal model as an orbifold. According to general principles of orbifold deconstruction \([3, 5]\), the elements of the twister \( g \) correspond to irreducible representations of the twist group, and two elements of \( g \) are equilocal precisely when the kernels of the associated representations coincide. This means that to each equilocality class contained in a twister is associated a normal subgroup of the corresponding twist group. Conversely, to each normal subgroup of the twist group there corresponds a partial deconstruction whose vacuum block \( h \) is a twister contained in \( g \), and the normal subgroup can be recovered as the intersection of the normal subgroups associated to the different equilocality classes contained in \( h \).

While locality diagrams are helpful in describing the lattice of FC sets, we should note that not every irreducible graph is the locality diagram of some conformal model. An obvious property follows from the existence of the vacuum primary, which is mutually local with all the primaries, and forms an equilocality class in itself, hence every locality diagram has a universal vertex that is adjacent to all vertices (including itself). A more subtle requirement comes from the lattice \( \mathcal{L} \) admitting a type I embedding into a partition lattice, since there are undirected graphs for which the associated lattice is not even modular, an example being shown in Fig. 4.

![Figure 4: Hasse diagram of a non-modular lattice, and the corresponding graph.](image)

Actually, the situation is a bit more complicated, as there are unoriented graphs that satisfy all the above criteria, namely that they are irreducible, have a universal vertex, and their associated lattice admits a type I embedding into a partition lattice, but cannot show up as the locality diagram of a conformal model. The reason can be traced back to the fact mentioned at the end of Section 2, namely that the product \( \mu(g)\mu(g^\perp) \) is the same for every FC set \( g \in \mathcal{L} \). Taking into account Eq.(2.2), this requirement can be translated into a set of quadratic equations to be satisfied by the values assigned to the different equilocality classes, and these equations must have a positive solution for the graph to be realizable as the locality diagram of some
conformal model. That this is not automatic is illustrated by the graph shown on Fig. 5, whose associated lattice is identical to that of the Virasoro minimal model of central charge $c = 7/10$, but nevertheless cannot be the locality diagram of a conformal model because it does not satisfy this last requirement.

Figure 5. An irreducible graph that does not correspond to a locality diagram, while its associated lattice is modular.

4 Summary

As we tried to explain in the previous sections, the full deconstruction hierarchy, and the closely related lattice $\mathcal{L}$ of FC sets can be determined from the sole knowledge of the locality graph, or even better, from the locality diagram and the primary content of the individual equilocality classes. Not only does this simplify dramatically actual computations, but it does also provide new insights into the structure of conformal models, and in particular, it brings to the forefront the notion of equilocality classes, and the relationship between primaries in the same class.

Actually, the existence of generic locality diagrams makes it possible to compare not only primaries of a given model, but also primaries coming from different models (provided the relevant locality diagrams are isomorphic, or at least degenerations of a common graph), establishing some kind of ‘kinship’ between them. For example, the equilocality class labeled $E_6$ on Fig. 1 is present in all Virasoro minimal models, and it contains a single primary field having analogous properties in each of them. Such relations between primaries of different (although somehow related) models could prove helpful, e.g. in classification attempts.

Finally, it should be pointed out that, while our exposition was formulated using notions of 2D CFT [6, 14], most of the ideas and results presented could be directly applied to such related fields as the theory of Vertex Operator Algebras [8, 15, 18] or that of Modular Tensor Categories [1, 21], providing potential new insights. In particular, one may speculate about their application in the analysis of topological order [22]. From a more general perspective, application of techniques from graph theory to the study of QFT seems a most interesting possibility.

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