$W$-algebras and higher analogs of the Kniznik-Zamolodchikov equations.\textsuperscript{*}

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\textbf{Аннотация}

The key role in the derivation of the Knizhnik-Zamolodchikov equations in the $WZW$-theory is played by the energy-momentum tensor, that is constructed from a central Casimir element of the second order in a universal enveloping algebra of a corresponding Lie algebra. In the paper a possibility of construction of analogs of Knizhnik-Zamolodchikov equations using higher order central elements is investigated. The Gelfand elements of the third order for a simple Lie algebra of series $A$ and Capelli elements of the fourth order for the a simple Lie algebra of series $B$, $D$ are considered. In the first case the construction is not possible a the second case the desired equation is derived.

1 Introduction

In the $WZW$ theory, associated with the Lie algebra $\mathfrak{g}$ it is proved that the correlation functions of $WZW$-primary fields satisfy a system of differential equations that is called the Knizhnik-Zamolodchikov equations. One obtains this system when one equates two representation of the action of the Virasoro operator $L_{-1}$: one as a differential operator and the other as a matrix operator. The operator $L_{-1}$ is defined by an expansion of the energy-momentum tensor

\begin{equation}
T(z) = \frac{1}{2(k + g)} \sum_{\alpha} (J^\alpha J^\alpha)(z) = \sum_{k} L_k z^{-k-1},
\end{equation}

where $J^\alpha$ is an orthonomal base of $\mathfrak{g}$ with respect to the Killing form and $J^\alpha(z)$ is a current corresponding to $J^\alpha$. Also $g$ is a dual Coxeter number, and

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$k$ is a constant from the $WZW$-action. In the derivation of the Knizhnik-Zamolodchikov equation a key role is played by the fact that the element $\sum_\alpha J^\alpha \in U(\mathfrak{g})$ is central.

A natural question arises: is it possible to obtain new equations using analogs of the energy-momentum tensor that are constructed from central elements of higher orders?

As a first step in the derivation of an analog of the Knizhnik-Zamolodchikov equation in the case of a central element $W = \sum d^{\alpha_1,\ldots,\alpha_n} J^{\alpha_1} \ldots J^{\alpha_n}$ of the higher order one constructs a field, that is a higher analog of the energy-momentum tensor

$$W(z) = \sum d^{\alpha_1,\ldots,\alpha_n} (J^{\alpha_1} \ldots J^{\alpha_n})(z) = \sum_k W_k z^{-n-k}.$$ 

An algebra generates by elements $1, L_n, W_m$ is called the $W$-algebra [6].

As the second step a class of considered fields is fixed and the action of $W_{-1}$ on these fields is represented as a differential operator. For this an operator product expansion of $W(z)$ and the considered field $\varphi(w)$ is investigated. To able to represent the action of $W_{-1}$ onto $\varphi(w)$ as a differential operator this expansion must be of type

$$W(z)\varphi(w) = \frac{\text{const}\varphi(w)}{(z-w)^n} + \frac{\mathcal{D}\varphi(w)}{(z-w)^{n-1}} + \ldots,$$ \hspace{1cm} (2)

where $\mathcal{D}$ is a differential operator.

In the case of the energy-momentum tensor $T(z)$ of the second order is postulated by the Ward identity that has geometric origin.

In the present paper instead of $WZW$-primary fields the currents $J^\alpha(z)$ are used. Note that if $k = 0$ then currents $J^\alpha(z)$ are $WZW$-primary fields. Then formulas for the operator expansions of $T(z)$ and $J^\alpha(w)$ is obtained from the relation of the operator algebra.

This algebraic approach is used for the construction of the OPE of the energy-momentum tensor constructed using a Casimir element of the higher order $W(z)$ and a considered field $\varphi(w)$ which is constructed from currents.

As the third step the action of $W_{-1}$ is represented as an algebraic operator. But this can be easily done.

Below we try to realize this construction using two central elements of higher orders: the Gelfand element of the third order in the case when $\mathfrak{g}$ is a simple Lie algebra that belongs to the series $A$, and the Capelli element of the fourth order in the case when $\mathfrak{g}$ belongs to the series $B$, or $D$.

\hspace{1cm} 1In the paper only singular terms in the OPE are written
The case of the energy-momentum tensor associated with the Gelfand element of the third order $W = d^{\alpha,\beta,\gamma} J^\alpha J^\beta J^\gamma$ for the series $A$ is considered in Section 4. It is shown that it is not possible to construct the Knizhnik-Zamolodchikov equations.

The following notation is made. The OPE of the current $J^\alpha(z)$ and the field $W(w) = d^{\alpha,\beta,\gamma}(J^\alpha(J^\beta J^\gamma))(w)$ is of type

$$J^\alpha(z)W(w) = \frac{1}{(z-w)^3}d^{\alpha,\beta,\gamma}(J^\beta J^\gamma)(w)^2 (3)$$

Thus it is not an OPE of type (2): when one takes the OPE the field $J^\alpha(z)$ removes $J^\alpha$ from the normal order product $(J^\alpha(J^\beta J^\gamma))(w)$ and one obtains $(J^\beta J^\gamma)(w)$.

From this notation the following conjecture arises. To obtain the OPE of type (2), it is necessary to take a field $\varphi^\alpha(z)$ and a higher analog of the energy momentum tensor of type $W(w) = \sum_{\alpha}(\varphi^\alpha\varphi^\alpha)(w)$. In particular the corresponding central element is a sum of squares $\sum_{\alpha}\varphi^\alpha\varphi^\alpha$.

Such central elements there exist in the universal enveloping algebra of the orthogonal algebra. These are the Capelli elements, which are sums of squares of noncommutative pfaffians.

This conjecture is verified in Section 7. As $W$ the Capelli element of the fourth order is taken, which is a sum of squares of noncommutative pfaffians and as $\varphi^\alpha(z)$ a noncommutative pfaffian. It is shown that in these settings the construction of an analog of the Knizhnik-Zamolodchikov equation is possible.

1.1 The content of the paper

In Section 2 we give some preliminary facts about affine algebras, construction of primary fields in the $WZW$-theory, extensions of the Virasoro algebra, higher order central elements in the universal enveloping algebra of a simple Lie algebra.

In Section 4 we investigate the question of the possibility of the construction of an analog of the Knizhnik-Zamolodchikov equation using higher order energy-momentum tensor that is associated with the Gelfand element of the third element for the series $A$. The answer is negative.

In Sections 7, 8 it is shown that if one takes the energy-momentum tensor that is associated with the Capelli element of the fourth order the answer is negative.

Some technical details can be found in Appendix 10.

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2 A summation over repeating indices is suggested.
2 Preliminaries

2.1 Affine algebras. Currents

Let $\mathfrak{g}$ be a simple Lie algebra, denote as $\hat{\mathfrak{g}}$ the corresponding non-twisted affine Lie algebra. If $J^a$ is a base of $\mathfrak{g}$, then the base of the untwisted affine Lie algebra $J^a_n$, $K$, $n \in \mathbb{Z}$.

Define a power series

$$J^a(z) = \sum_n J^a_n z^{-n-1}. \quad (4)$$

Then one obtains that

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[[t^{-1}, t]]) \oplus \mathbb{C} K. \quad (5)$$

The defining commutations relation on the language of power series are written as follows

$$[a(t) \oplus nK, b(t) \oplus mK] = [a(t), b(t)] \oplus \omega(a, b)K, \quad (6)$$

where $a(t), b(t) \in \mathfrak{g} \otimes \mathbb{C}[[t^{-1}, t]]$, and $\omega$ is the Killing form. In particular the element $K$ is central. The power series $a(t) \in \mathfrak{g} \otimes \mathbb{C}[[t^{-1}, t]]$ is called a current.

We call a field an arbitrary power series $U(\mathfrak{g}) \otimes \mathbb{C}[[t^{-1}, t]]$, that is a power series with coefficients in $U(\mathfrak{g})$.

Write a product of power series $a(z)$ and $b(w)$ as follows

$$a(z)b(w) = \sum_k \frac{(ab)_k(w)}{(z - w)^k}. \quad (7)$$

The coefficient

$$(ab)_0(w) =: (ab)(w)$$

is called a normal ordered product.

2.2 Currents and $WZW$-primary fields

Below the following explicit construction of $WZW$-primary fields is used.

Let $\hat{\mathfrak{g}} =< J^a_n, K >$ be a nontwisted affine algebra and $\hat{\hat{\mathfrak{g}}} =< \varphi^a_n >$ – be a loop algebra. The projection $\pi : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, that maps $J^a_n$ to $\varphi^a_n$, and $K$ into zero, is a homomorphism of Lie algebras. Thus the algebra $\mathfrak{g}$ act onto $\hat{\mathfrak{g}}$ in an adjoint way with the level 0.

Put
\[ J^a(z) = \sum_n J^a_n z^{-n-1}, \quad \varphi^a(z) = \sum_n \varphi^a_n z^{-n-1}. \]

Then the following OPE takes place

\[ J^a(z) \varphi^b(w) = f_{a,b,c} \varphi^c(w) \frac{1}{(z-w)}. \]

Thus \( \varphi^b(w) \) are primary fields that transform in the adjoint representation. Using fields \( \varphi^b(w) \) one easily obtains fields that transform in the tensor power of the adjoint representation and then as an irreducible components of this representation.

In the case of a simple Lie algebra of type \( A, B, D \) in such a way one constructs field that transform in an arbitrary tensor representation.

Below the fields \( J^a(z) \) are considered for arbitrary \( k \).

### 2.3 The Virasoro algebra and it’s extensions. The energy momentum tensor

In the WZW-theory the following fact takes place that is called the Sugawara construction \( \text{[7]} \). Suggest that the base \( J^a \) is orthonomal with respect to the Killing form. Consider the energy-momentum tensor

\[ T(z) = \frac{1}{2(k+g)} \sum_{\alpha} (J^\alpha J^\alpha)(z), \quad (8) \]

and consider the expansion

\[ T(z) = \sum_n z^{-n-2} L_n \]

then \( L_n \) form the Virasoro algebra.

In the theory WZW it is natural to consider extensions of the Virasoro algebra - the Casimir W-algebras. One obtains them when adds to the energy-momentum tensor one more field that is constructed from a Casimir element of a higher order. See description of these W-algebras in Sections 4.2, 7.2.

### 2.4 Higher order central elements

Below two central elements in the universal enveloping algebra \( U(\mathfrak{g}) \) of a simple Lie algebra \( \mathfrak{g} \) are defined. They are a Gelfand element of the third order in the case of the series \( A \) and the Capelli element in the case of the series \( B \) and \( D \).
2.4.1 Gelfand element of the third order for the series A.

Consider the algebra $\mathfrak{sl}_N$. Chose as a base the generalized Gell-Mann matrices $\lambda_\alpha$, $\alpha = 1, \ldots, N$, then one has

$$d^{\alpha,\beta,\gamma} = \text{Tr}(\lambda_\alpha \{\lambda_\beta, \lambda_\gamma\})$$

$$\{\lambda_\beta, \lambda_\gamma\} = \lambda_\beta \lambda_\gamma + \lambda_\gamma \lambda_\beta.$$

Define the third order Gelfand element by the formula

$$W = d^{\alpha,\beta,\gamma} \lambda_\alpha \lambda_\beta \lambda_\gamma.$$

2.4.2 Noncommutative pfaffians and Capelli elements

Define central elements in the universal enveloping algebra for $\mathfrak{o}_N$ that is for a simple Lie algebra of type $B$ or $D$. These elements are constructed using noncommutative pfaffians. Let us give their definition.

Let the algebra $\mathfrak{o}_N$ be realized as the algebra of skew-symmetric matrices. In this realization the algebra has a base that consists of matrices $F_{ij} = E_{ij} - E_{ji}$, $i, j = 1, \ldots, n$, $i < j$. Here $E_{ij}$ are matrix units. This base is orthogonal with respect to the Killing form.

Commutation relations between these matrices are the following:

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \delta_{ik} F_{jl} + \delta_{jl} F_{ki}.$$  

Let $\Phi = (\Phi_{ij})$, $i, j = 1, \ldots, 2k$ be a skew-symmetric $2k \times 2k$-matrix, whose matrix elements belong to a noncommutative ring.

**Definition 1.** The noncommutative pfaffian of $\Phi$ is defined by the formula

$$\text{Pf} \Phi = \frac{1}{k!2^k} \sum_{\sigma \in S_{2k}} (-1)^\sigma \Phi_{\sigma(1)} \sigma(2) \ldots \Phi_{\sigma(2k-1)} \sigma(2k),$$

where $\sigma$ is a permutation of the set $\{1, \ldots, 2k\}$.

Define a matrix $F = (F_{ij})$, $i, j = 1, \ldots, N$. Since $F_{ij} = -F_{ji}$, then $F$ is skew symmetric.

Below we consider pfaffians in $U(\mathfrak{o}_N)$ of the matrix $F$ and matrices $F_I$, that are constructed as follows. Let $I \subset \{1, \ldots, N\}$ be a set of indices. Denote as $F_I$ a submatrix in $F$ whose rows and columns are indexed by elements from the set $I$.

The following theorem takes place.

**Theorem 1.** (\cite{12}, § 7.6) Let
\( C_k = \sum_{|I|=k, I \subseteq \{1, \ldots, N\}} (PfF_I)^2, \quad k = 2, 4, \ldots, 2\left\lfloor \frac{N}{2} \right\rfloor. \) 

Then \( C_k \) belong to the center of \( U(\mathfrak{o}_N) \). In the case of odd \( N \) they are algebraically independent and generate the center. In the case of even \( N \) the same is true if one takes instead of \( C_N = (PfF)^2 \) the element \( PfF \).

The elements \( C_k \) are called the Capelli elements.

In the paper \cite{2} the commutation relations between pfaffians \( PfF_I \) and generators were found. To formulate them let us define \( F_{ij}I \). Let \( I = \{i_1, \ldots, i_k\}, \ i_r \in \{1, \ldots, N\} \) be a set of indices. Identify \( i_r \) with the vector \( e_{i_r} \) of the standard representation \( V \) of the algebra \( \mathfrak{o}_N \), identify the set \( I = \{i_1, \ldots, i_k\} \) with the tensor \( e_{i_1} \otimes \ldots \otimes e_{i_k} \in V^\otimes k \). Then \( F_{ij}I \) is defined as the tensor that is obtained when one applies \( F_{ij} \) to the tensor \( I \).

For numbers \( \alpha, \beta \in \mathbb{C} \) define

\[ PfF_{\alpha I + \beta J} := \alpha PfF_I + \beta PfF_J. \]

Then for every \( g \in \mathfrak{o}_N \) the expression \( PfF_{gI} \) is well-defined.

**Proposition 1.** (see \cite{2}) \( [F_{ij}, PfF_I] = PfF_{F_{ij}I} \)

### 3 Nonstandard notations

In the paper some nonstandard notations are widely used. Above one such notation was introduced

\[ PfF_{F_{ij}I}. \]

Let \( I \) be divided into subsets \( I_1 \) and \( I_2 \). Denote as

\( (-1)^{(I_1, I_2)} \)

the sign of the permutation \( I \), which first places the set \( I_1 \), and then the set \( I_2 \).

As above identify the set of indices \( I = \{i_1, \ldots, i_k\} \) with a tensor

\( e_{i_1} \wedge \ldots \wedge e_{i_k} \).

Define

\( I \setminus J \)

as follows. If \( J \subseteq I \), then remove from \( I \) the indices that belong to \( J \), construct the corresponding tensor and take it with the sign \( (-1)^{(J, I \setminus J)} \). If \( J \nsubseteq I \), then put \( I \setminus J = 0 \).
Define
\[ P f F_{I \setminus J} \]
as a pfaffian of the matrix \((F_{i,j})_{i,j \in I \setminus J}\) taken with the sign \((-1)^{|I \setminus J|}\) in the case \(J \subset I\) and put \(P f F_{I \setminus J} = 0\) otherwise.

Let us be given a subset \(J\) in the set \(I\), denote as
\[ P f F_{I \setminus J} \]
the pfaffian of the matrix \((F_{i,j})_{i,j \in I \setminus J}\), multiplied by the sign \((-1)^{|J \setminus I|}\).

For the set \(J = \{j_1, j_2\}\) denote as
\[ F_j \]
the generator \(F_{j_1, j_2} \).

4 The Gelfand element of the third order

Below we investigate the possibility of construction of a Knizhnik-Zamolodchikov type equation associated with a Gelfand central element of the third order.

4.1 The energy-momentum tensor of the third order

In the universal enveloping algebra of a simple Lie algebra of type \(A_{N+1}\) there exist a Casimir element of the third order
\[ W = d^{\alpha, \beta, \gamma} J^\alpha J^\beta J^\gamma, \quad (15) \]
where \(d^{\alpha, \beta, \gamma}\) is an invariant traceless tensor. In \([5]\) the following field is defined
\[ W(z) = A_N(k) d^{\alpha, \beta, \gamma} (J^\alpha (J^\beta J^\gamma))(z), \quad (16) \]
where the normalization constant in the of the algebra \(A_{N+1}\) is defined by the equality
\[ A_N(k) = \sqrt{\frac{N}{18(k + N)^2(N + 2k)(N^2 - 4)}} \quad (17) \]
Since the tensor \(d^{\alpha, \beta, \gamma}\) is traceless, the element does not depend on the placement of brackets in this normal ordered product.

As the element \(T(z)\) the element \(W(z)\) is a Sugawara element in the corresponding untwisted affine Lie algebra \([4]\).

Let us show that there exist no Knizhnik-Zamolodchikov equation associated with this energy-momentum tensor.
4.2 The algebra $WA_2$ and fields $W^\alpha$

Consider the $WZW$-theory, associated with a Lie algebra $A_{N+1}$.

Since the field $W(z)$ has the conformal dimension 3 its decomposition is the following

$$W(z) = \sum_{n} (z - w)^{-n-3}W_n(w). \quad (18)$$

The action of the modes of this decomposition on the field $A(w)$ is defined by the equality

$$W(z)A(w) = \sum_{n} (z - w)^{-n-3}(W_nA)(w). \quad (19)$$

**Definition 2.** Introduce a field $W^\alpha(z)$

$$W^\alpha(z) = \frac{1}{2} \epsilon^\alpha,\beta,\gamma (J^\beta J^\gamma)(z), \quad (20)$$

In [5] it is shown that for the fields $W^\alpha(z)$ the following operator expansions take place

$$T(z)W^a(w) = \frac{1}{(z-w)^2}W^a(w) + \frac{1}{2(z-w)}\partial W^a(w), \quad (21)$$

$$J^a(z)W^b(w) = \frac{(k + \frac{1}{2}N)}{(z-w)^2}d^{a,b,c}J^c(w) + \frac{1}{z-w}(f_{a,b,c}W^c(w)), \quad (22)$$

where $f_{a,b,c}$ are structure constants of the algebra.

Thus the fields $W^a(w)$ are Virasoro primary but they are not in general $WZW$-primary fields.

5 Some OPEs

In this Section some operator expansions are calculated.

As it is shown in [5], [6] in addition to operator expansions (21), (22) the following expansions take place:

$$J^a(z)W^b(w) = \frac{k + N}{(z-w)^2}W^a(w). \quad (23)$$
Let us find the operator expansion \( W(z)W^\gamma(w) \). One has the following formula for the contraction \( W(z)(J^\alpha J^\beta)(w) \):

\[
W(z)(J^\alpha J^\beta)(w) = \frac{1}{2\pi i} \int dx \frac{d}{x-w} W(z)J^\alpha(x)J^\beta(w) + J^\alpha(x)W(z)J^\beta(w) = \\
= \frac{1}{2\pi i} \int dx \frac{(k+N)W^\alpha(z)}{(z-x)^2} J^\beta(w) + J^\alpha(x) \frac{(k+N)W^\beta(z)}{(z-w)^2} + \\
= \frac{1}{2\pi i} \int dx \frac{(k+N)(k+\frac{1}{2}N)d^{\alpha,\beta,c}J^c(z)}{(z-x)^2} + \frac{f_{\beta,\alpha,c} W^c(z)}{z-w} + \\
+ \frac{(k+N)(k+\frac{1}{2}N)d^{\alpha,\beta,c}J^c(z)}{(z-w)^2} \frac{J^\alpha(x)}{x-z} + \\
= (k+N)(2(k+\frac{1}{2}N)d^{\alpha,\beta,c}J^c(z)) + \frac{2f_{\beta,\alpha,c} W^c(z)}{(z-w)^3}. 
\]

From here one obtains

\[
W(z)W^\gamma(w) = \frac{1}{2} d^{\alpha,\beta}(k+N)d^{\alpha,\beta,c}J^c(z) + \frac{f_{\beta,\alpha,c} W^c(z)}{(z-w)^3} = \\
= (k+N)(k+\frac{N}{2}) \frac{2}{N(N^2-4)} \frac{J^\gamma(z)}{(z-w)^4}. 
\]

6 The action of the operators \( W_n \) on the fields \( J^\alpha(z) \), \( W^\alpha(z) \) and their derivatives

From the formula (23) it follows that

\[
W(z)J^\alpha(w) = \frac{k+N}{(z-w)^2} W^\alpha(z), 
\]

take a derivative in \( w \), one obtains

\[
W(z) \partial_w J^\alpha(w) = (-1)^r \frac{(k+N)^{r+2}!}{(z-w)^{2r+4}} W^\alpha(z). 
\]

From (27) one gets the operator expansion
W(z)\partial^r J^a(w) = (-1)^r(k+N)(r+2)! \frac{W^a(w)}{(z-w)^{2+r}} + \frac{1}{k!} \partial^k W^a(w) + \ldots + \frac{1}{(2+r-1)!} \partial^{2+r-1} W^a(w) \left( z-w \right)^r.

(28)

Compare the formulas (19) and (28), one obtains the theorem

\textbf{Theorem 2.} The following formulas take place

\[ W_n(\partial^r J^a(w)) = 0, \quad n > r - 1, \]
\[ W_n(\partial^r J^a(w)) = (-1)^r(k+N)\frac{(r+2)!}{2} \frac{1}{(r-n-1)!} \partial^{r-n-1} W^a(w), \quad n \leq r - 1. \]

(29)

Thus under the action of \( W_n \) the fields \( J^a(w) \) and their derivatives are mapped to \( W^a(w) \) and their derivatives.

From the formula (25) one obtains that

\[ W(z)\partial^r W^\gamma(w) = (-1)^r(k+N)(k+N^2-4)\frac{J^\gamma(z)}{(z-w)^{4+r}}. \]

(30)

From (30) one obtains the operator expansion

\[ W(z)\partial^r W^\gamma(w) = (-1)^r(k+N)(k+N^2-4)\frac{J^\gamma(w)}{(z-w)^{4+r}} + \ldots + \frac{1}{k!} \partial^k J^\gamma(w) + \ldots + \frac{1}{(4+r-1)!} \partial^{4+r-1} J^\gamma(w) \left( z-w \right)^r. \]

(31)

Compare the formulas (19) and (31), one obtains the theorem.

\textbf{Theorem 3.} The following formulas take place

\[ W_n\partial^r W^a(w) = 0, \quad n > r + 1 \]
\[ W_n\partial^r W^a(w) = (-1)^r(k+N)(k+N^2-4)\frac{1}{(r-n+1)!} \partial^{r-n+1} J^\gamma(w). \]

(32)

Thus under the action of \( W_n \) the fields \( W^a(w) \) and their derivatives are mapped to \( J^a(w) \) and their derivatives.
Corollary 1. The operators $W_n$ are not differential operators on the $WZW$-primary fields, those are only the compositions $W_n W_m$.

Thus it is shown explicitly that there exist no higher analogue of the Knizhnik-Zamolodchikov equation associated with the algebra $WA_2$ (that is with the Casimir element of the third order).

7 The energy-momentum tensor associated with the Capelli element of the fourth order

Let us construct an analog of the Knizhnik-Zamolodchikov equation using the Capelli element $C_4$.

7.1 The correlation functions under consideration

Let us change the type of considered correlation functions. for this purpose let us define a field associated with a pfaffian

$$PfF_I(z) = \frac{1}{k!2^k} \sum_{\sigma \in S_{2k}} (-1)^\sigma (F_{\sigma(i_1)\sigma(i_2)}(F_{\sigma(i_3)\sigma(i_4)}(F_{\sigma(i_5)\sigma(i_6)}(F_{\sigma(i_{2k-1})\sigma(i_{2k})})...)...(z),$$

(33)

The classical Knizhnik-Zamolodchikov equation is system of PDE for the correlation function if type

$$< \varphi_1(z_1)...\varphi_r(z_r) >,$$

(34)

where $\varphi_k(z_k)$ are $WZW$ primary fields.

The equations that are constructed below are equations for the correlation functions of the following type. For the set of indices $I = \{i_1, i_2, i_3, i_4\} \subset \{1, ..., N\}$ consider a field $\varphi(z) = PfF_I(z)$. This field is not $WZW$-primary even for $k = 0$. Introduce a correlation function

$$< PfF_I(z_1)\varphi_2(z_2)...\varphi_r(z_r) >,$$

(35)

where $\varphi_k(z_k)$ are $WZW$-primary fields. Below an equation for this correlation function is derived.

This correlation function is a function of variables $z_1, ..., z_r$ that takes values in a $r$-th tensor power of an adjoint representation of the algebra $\mathfrak{o}_N$. 
7.2 A field associated with a Capelli element of the order 4. The algebra $WB_2$

Define fields

**Definition 3.**

$$C_n(z) = \sum_{|I|=n} (P f F_I P f F_I)(z). \quad (36)$$

The algebra generated by $1, L_n, C_4^n$, is a Casimir $W$-algebra, it is denoted $WB_2$.

For an arbitrary field $\varphi(w)$ one has

$$C_4(z)\varphi(w) = \sum_k (C_4^k \varphi)(w)(z - w)^{-k-4}. \quad (37)$$

Consider an action of $C_4^{-1}$ on the field $P f F_I(z_1)$. From one hand let us represent this action as a differential operator. More precise we show that

$$C_4^{-1}P f F_I(z_1) = \text{const}\partial P f F_I(z_1) \quad (38)$$

From the other hand we show that the correlation functions

$$< C_4^{-1}P f F_I(z_1)\varphi_2(z_2)\ldots\varphi_r(z_r) > \quad (39)$$

can be algebraically expressed through the correlation functions of type (34), (35). When one equates two these expressions one obtains a higher analog of the Knizhnik-Zamolodchikov equation.

7.3 Representation of $C_4^{-1}$ as a differential operator

To represent $C_4^{-1}$ as a differential operator it is necessary to find two higher terms of the OPE $C_4(z)P f F_j(w)$.

As a first step let us calculate the contraction $P f F_j(z)P f F_I(x)$.

7.3.1 Calculation of $J^\alpha(z)(J^\beta J^\gamma)(w)$

Let us do the following general calculation. Let $J^\alpha$ be an orthonomal with respect to the Killing form base of a simple Lie algebra. Let $f_{\alpha,\beta,\gamma}$ be structure constants.

Calculate the contraction

$$J^\alpha(z)(J^\beta J^\gamma)(w).$$

The Proposition takes place
Proposition 2.

\[
\mathcal{J}^\gamma(z)(J^\beta J^\gamma)(w) = \frac{k \delta_{\alpha, \beta} J^\gamma(w)}{(z - w)^2} + \frac{k \delta_{\alpha, \gamma} J^\beta(w)}{(z - w)^2} + \frac{f_{a, \beta, \gamma} k}{(z - w)^2} \]

\[
+ \frac{f_{a, \beta, \gamma} f_{c, \gamma, d} J^d(w)}{(z - w)^2} + \frac{f_{a, \beta, c} (J^\gamma J^\beta)(w)}{z - w} + \frac{f_{a, \gamma, c} (J^\beta J^c)(w)}{z - w} \tag{40}
\]

The proof of this Proposition can be found in Appendix in Section 10.1.

7.3.2 Calculation of \( F_{j_1, j_2}(z)(F_{i_1, i_2} F_{i_3, i_4})(w) \)

In the case of the orthogonal algebra the formula from the Proposition 2 looks as follows.

Proposition 3.

\[
\mathcal{F}_{j_1, j_2}(z)(F_{i_1, i_2} F_{i_3, i_4})(w) = \frac{k \delta_{(j_1, j_2), (i_1, i_2)} F_{i_3, i_4}}{(z - w)^2} + \frac{k \delta_{(j_1, j_2), (i_3, i_4)} F_{i_1, i_2}}{(z - w)^2} + \]

\[
+ \frac{F_{\mathcal{F}_{j_1, j_2}(z)(F_{i_1, i_2} F_{i_3, i_4})(w)}{(z - w)^2}}{(z - w)^2} + \frac{F_{\mathcal{F}_{j_1, j_2}(z)(F_{i_1, i_2} F_{i_3, i_4})(w)}}{(z - w)^2} + \frac{F_{i_1, i_2} F_{\mathcal{F}_{j_1, j_2}(z)(F_{i_3, i_4})}}{(z - w)}. \tag{41}
\]

7.3.3 Calculation of \( F_{j_1, j_2}(z)PfF_i(w) \) in the case \(|I| = 4\)

Let us find the contraction \( \mathcal{F}_{j_1, j_2}(z)(PfF_i)(w) \) in the case \(|I| = 4\).

Let

\[
I = \{i_1, i_2, i_3, i_4\}.
\]

One has

\[
\mathcal{F}_{j_1, j_2}(z)(PfF_i)(w) = \frac{k(-1)_{(j_1, j_2), (i_1, i_2)} F_{i_3, i_4}}{(z - w)^2} + \]

\[
+ \frac{PF_{F_{i_1, j_2}}(w)}{(z - w)^2} + \frac{(PfF_{F_{i_1, j_2}})(w)}{(z - w)}. \tag{42}
\]

After simplification one gets

Proposition 4.

\[
\mathcal{F}_{j_1, j_2}(z)(PfF_i)(w) = \frac{(k + 2) PF_{F_{i_1, j_2}}(w)}{(z - w)^2} + \frac{(PfF_{F_{i_1, j_2}})(w)}{(z - w)}. \tag{43}
\]
7.3.4 The calculation of a contraction of two pfaffians

Let $|I| = |J| = 4$. One has

**Proposition 5.**

\[
P f F_j(z) P f F_l(x) = \frac{1}{2} \sum_{I = I_1 \sqcup I_2, |I_1| = |I_2| = 2} (-1)^{(I_1, I_2)} \frac{2(k + 2) k P f F_{J \setminus I_1, I_2}(z)}{(z - x)^4} - \frac{2(k + 2) P f F_{F_{I_1, I_2}}(z)}{(z - x)^3} + \frac{(k + 2)(F_{I_2} P f F_{I_1}) (z)}{(z - x)^2} - \frac{(F_{I_2} P f F_{I_1}) (z)}{(z - x)} + \frac{(k + 2)(F_{I_1} P f F_{I_2}) (z)}{(z - x)^2} - \frac{(k + 2)(F_{I_1} P f F_{I_2}) (z)}{(z - x)} \tag{44}
\]

where $(-1)^{(I_1, I_2)}$ is a sign of a permutation of the set $I$, that firstly puts the set $I_1$, and then the set $I_2$.

Доказательство. The following formula takes place which is a particular case of the minor summation formula [2]

\[
P f F_I = \frac{1}{2} \sum_{I = I_1 \sqcup I_2, |I_1| = |I_2| = 2} (-1)^{(I_1, I_2)} F_{I_1} F_{I_2}. \tag{45}
\]

Using it one gets

\[
P f F_j(z) P f F_l(x) = \frac{1}{2} \sum_{I = I_1 \sqcup I_2, |I_1| = |I_2| = 2} (-1)^{(I_1, I_2)} \frac{1}{2\pi i} \oint \frac{dx_1}{(x_1 - x)} \frac{P f F_j(z)}{F_{I_1}(x_1)} F_{I_2}(x) + F_{I_1}(x_1) P f F_j(z) F_{I_2}(x) \tag{46}
\]

Substitute this into this expression the formula for the contraction of a pfaffian and a current that was found in Proposition 4. One gets
\[
\frac{1}{2} \sum_{I = I_1 \cup I_2, |I_1| = |I_2| = 2} (-1)^{|I_1, I_2|} \int \frac{dx_1}{2\pi i} \frac{(k + 2)PfF_{I_1 \setminus I_2}(z)}{x_1 - x} 
- PfF_{I_1, I_2}(z)F_{I_1}(x) + F_{I_1}(x_1)(k + 2)PfF_{I_1 \setminus I_2}(z) - PfF_{I_2, I_1}(z) = 
\]

\[
= \frac{1}{2} \sum_{I = I_1 \cup I_2, |I_1| = |I_2| = 2} (-1)^{|I_1, I_2|} \int \frac{dx_1}{2\pi i} \frac{(k + 2)PfF_{I_1 \setminus I_2}(z)}{x_1 - x} 
- \frac{1}{(z - x)PfF_{I_1, I_2}(z)} + \frac{(k + 2)}{(z - x)^2} F_{I_1}(x_1)PfF_{I_2}(z) - \frac{1}{(z - x)F_{I_1}(x_1)PfF_{I_2, I_1}(z)}. \tag{47}
\]

After integration and application of the formula from Proposition 4 one gets

\[
\frac{1}{2} \sum_{I = I_1 \cup I_2, |I_1| = |I_2| = 2} (-1)^{|I_1, I_2|} \frac{(k + 2)PfF_{I_1 \setminus I_2}(z)}{(z - x)^4} - \frac{(k + 2)}{(z - x)^3} + \frac{(k + 2)(F_{I_2, I_1})(z)}{(z - x)^2} - \frac{(k + 2)(F_{I_2, I_1})(z)}{(z - x)} + \frac{PfF_{I_1, I_2}(z)}{(z - x)^2} - \frac{(F_{I_2, I_1})(z)}{(z - x)} 
+ \frac{(k + 2)(F_{I_1, I_2})(z)}{(z - x)^2} - \frac{(k + 2)(F_{I_1, I_2})(z)}{(z - x)} + \frac{PfF_{I_1, I_2}(z)}{(z - x)^2} - \frac{(F_{I_1, I_2})(z)}{(z - x)}. \tag{48}
\]

After simplification one gets

\[
\frac{1}{2} \sum_{I = I_1 \cup I_2, |I_1| = |I_2| = 2} (-1)^{|I_1, I_2|} \frac{2(k + 2)PfF_{I_1 \setminus I_2}(z)}{(z - x)^4} - \frac{2(k + 2)}{(z - x)^3} + \frac{(k + 2)(F_{I_2, I_1})(z)}{(z - x)^2} - \frac{(k + 2)}{(z - x)} + \frac{(k + 2)(F_{I_1, I_2})(z)}{(z - x)^2} - \frac{(k + 2)}{(z - x)} \tag{49}
\]
7.3.5 Calculation of the contraction of $PfF_j(z)$ and $C_4(w)$

Let us prove the Theorem

**Theorem 4.**

$$PfF_j(z)_4(w) = (6(k + 2)k + 12(N - 4) - (k + 2)((N - 2)(N - 3)(N - 4) + 6))\frac{PfF_j(w)}{(z - w)^4} +$$

$$+ (12(N - 4) - (k + 2)((N - 2)(N - 3)(N - 4) + 6))\frac{4\partial PfF_j(w)}{(z - w)^3} + l.o.t. \tag{50}$$

Доказательство. One has

$$PfF_j(z)(PfF_I PfF_I)(w) = \frac{1}{2\pi i} \oint \frac{dx}{x - w} PfF_j(z)PfF_I(x)PfF_I(w) + PfF_I(x)PfF_j(z)PfF_I(w). \tag{51}$$

This expression consists of two terms. Let us show that they are equal. Thus the expression (51) equals to

$$2 \frac{1}{2\pi i} \oint \frac{dx}{x - w} PfF_j(z)PfF_I(x)PfF_I(w). \tag{52}$$

Indeed, let us introduce notations $a(z) = PfF_j(z), b(x) = PfF_I(x), c(w) = PfF_I(w).$ Proposition 3 gives us an expression of type

$$a(z)b(w) = \sum_m (ab)_m(z)(z - w)^m. \tag{53}$$

Then (51) can be written as follows

$$\frac{1}{2\pi i} \oint \frac{dx}{x - w} a(z)b(x)c(w) + c(x)a(z)b(w) =$$

$$= \sum_m (ab)_m(z)c(w)(z - w)^m + c(w)(ab)_m(z)(z - w)^m. \tag{54}$$

However $(ab)_m(z)c(w)(z - w)^m = c(w)(ab)_m(z - w)^m$, that is why the two summands in (51) are equal.

Consider now the expression (52). Let us prove that when one calculates contraction in the numerator there appear no singular terms in (52) of order greater then $(z - w)^{-4}$, the term of order $(z - w)^{-4}$ is proportional
Lemma 2. \( Pf F_J(w)(z-w)^{-4} \), the term of order \((z-w)^{-3}\) is proportional to \( Pf F_J(w)(z-w)^{-3} \).

To prove this let us substitute the expression \((51)\) into \((52)\) and then consider the terms separately. Thus it is necessary to investigate the following expressions.

1. \( \sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{P f F_{I \setminus I_1 I_2}(z)}{(z-x)^3} P f F_I(w), \)
2. \( \sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{P f F_{I \setminus I_1 I_2}(z)}{(z-x)^4} P f F_I(w), \)
3. \( \sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{F_{I_2} P f F_{I \setminus I_1}(z)}{(z-x)^3} P f F_I(w), \)
4. \( \sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{F_{I_2} P f F_{I \setminus I_1}(z)}{(z-x)^4} P f F_I(w), \)
5. \( \sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{F_{I_2} P f F_{I \setminus I_1}(z)}{(z-x)^3} P f F_I(w). \)

The summation \( \sum I \) is taken over all subsets \( I \subset \{1, \ldots, N\} \) that consist of four elements and the summation \( \sum_{I=I_1 \cup I_2} \) is taken over all partitions of \( I \) into two subsets that consist of two elements.

Consider cases in further five lemmas.

**Lemma 1.**

\[
\sum I \sum_{I_1, I_2} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{(k+2)kP f F_{I \setminus I_1 I_2}(z)}{(z-x)^4} P f_{I_1}(w) = \frac{6(k+2)kP f F_J(w)}{(z-w)^4}.
\] (55)

This Lemma is obvious.

**Lemma 2.**

\[
\sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{P f F_{I \setminus I_1 I_2}(z)}{(z-x)^3} P f F_I(w) =
\]
\[
= (N-4)(12P f F_J(w)) \frac{(z-w)^4}{(z-w)^4} + 2\partial P f F_J(w) \frac{(z-w)^4}{(z-w)^4} + l.o.t.
\] (56)

The proof of the Lemma \(2\) can be found in Application \(10.2\).

**Lemma 3.** In the OPE

\[
\sum I \sum_{I_1, I_2} (-1)^{|I_1, I_2|} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{(F_{I_2} P f F_{I \setminus I_1}(z))}{(z-x)^2} P f F_I(w)
\] (57)

there are no singular terms of order greater than \((z-w)^{-2}\).

\(^4\)Everywhere below we shall use these notations.
The proof of this Lemma can be found in Application 10.3

**Lemma 4.** In the OPE

\[
\sum_I \sum_{I=I_1\cup I_2} (-1)^{(I_1,I_2)} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{(F_{I_2}PfF_{I_1})_1(z)}{(z-x)^2} PfF_I(w)
\]

there are no singular terms of order greater than \((z-w)^{-2}\).

This lemma is proved analogously to Lemma 57.

**Lemma 5.**

\[
\sum_I \sum_{I=I_1\cup I_2} (-1)^{(I_1,I_2)} \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{(F_{I_1}PfF_{I_2})_1(z)PfF_I(w)}{(z-x)} =
\]

\[
= -(k+2)((N-2)(N-3)(N-4)+6)\left(\frac{PfF_I(w)}{(z-w)^4} + \frac{\partial PfF_I(w)}{(z-w)^3} + l.o.t.\right)
\]

This Lemma is proved in Application 10.3.

We have obtained that

\[
\text{PfF}_I(z)_4(w) = (6(k+2)k + 12(N-4) - (k+2)((N-2)(N-3)(N-4)+6))\frac{PfF_I(w)}{(z-w)^4} +
\]

\[
+ (12(N-4) - (k+2)((N-2)(N-3)(N-4)+6))\frac{\partial PfF_I(w)}{(z-w)^3} + l.o.t.
\]

The theorem is proved.

If one changes the fields one obtains the OPE

\[
\text{PfF}_{I_1}(w) = (6(k+2)k + 12(N-4) + (k+2)((N-2)(N-3)(N-4)-6))\frac{PfF_I(w)}{(z-w)^4} +
\]

\[
+ (6(k+2)k)\frac{\partial PfF_I(w)}{(z-w)^3} + l.o.t.
\]

Thus one has

**Corrolary 2.**

\[
C^{-1}_4 PfF_I(w) = 6(k+2)k\partial PfF_I(w).
\]
7.4 Representation of $C_4^{-1}$ as an algebraic operator

In this Section an algebraic operator $A$ is constructed that expresses the correlation function

$$< C_4^{-1} P f F_I(z_1) \phi_2(z_2) \ldots \phi_r(z_r) >,$$

through the correlation functions of type (34), (35).

Consider the field $P f F_I(z)$. For it the following OPE takes place

$$F_j(z) P f F_I(w) = \sum_m (F_j P f F_I)(w)(z - w)^{-m-1} =$$

$$= \frac{(k + 2) P f F_{I,j}(w)}{(z - w)^2} + P F F J I(w).$$

(64)

From this equality one gets

$$F_j^n P f F_I(w) = 0, \ n > 1,$$

$$F_j^1 P f F_I(w) = (k + 2) P f F_{I,j}(w),$$

$$F_j^0 P f F_I(w) = P F F J I(w).$$

(65)

Let use the general fact

$$(A B)_m = \sum_{n \leq h_A} A_n B_{m-n} + \sum_{n > h_A} B_{m-n} A_n.$$  

(66)

Then one has

$$C_4^{-1} = (P f F_I)_2 (P f F_J)_{-3} + (P f F_I)_1 (P f F_J)_{-2} + ...$$

$$+ (P f F_I)_{-4} (P f F_J)_3 + (P f F_I)_{-5} (P f F_J)_4 + ...,$$

(67)

where

$$(P f F_J)_m = \frac{1}{2} \sum_{J_1 \cup J_2} (-1)^{J_1 J_2} F_{J_1}^{J_2} F_{J_2}^{J_1} F_{J_2}^{-m-1} + F_{J_1}^{J_2} F_{J_2}^{J_1} F_{J_2}^{-m-2} F_{J_2}^{J_2} F_{J_2}^{J_3} F_{J_3}^{J_3} + ...$$

(68)

From here one gets

$$(P f F_J)_m P f F_I = 0, \ m > 2.$$  

(69)
Thus using (65), (69), one gets that

$$C_4^{-1} Pf F_1(w) = \frac{1}{4} \sum_J \sum_{J_1 \cup J_2} \sum_{k,l,p,q} (-1)^{(J_1,J_2)} F_{J_1}^k F_{J_2}^l F_p^q P f F_1(w),$$

(70)

where the following sets of indices \((k, l, p, q)\) are admissible

\[(0, 1, -1, -1), (0, 1, 0, -2), (0, 0, 0, -1),\]

(71)

and also all other collections that are obtained from these by any permutation of elements.

Let us prove the theorem

**Theorem 5.**

$$< C_4^{-1} Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >=
= A( < Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >, < \varphi(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >),$$

(72)

where the operator \(A\) is algebraic.

**Доказательство.** Let us use the formula (70). Prove that the summand corresponding to every collection of indices \((k, l, p, q)\) is represented as a result of an application of an algebraic operator to correlation functions of type (34), (35). Let us write explicitely the formulas for the action of operators \(F_{-1}^J, F_{-2}^J, F_0^J, F_1^J\).

Let us consider first the simplest case. One has

$$< F_0^J Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >= F_{j(1)}^J < Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >,$$

(73)

where \(F_{j(1)}^k\) is the operator \(F_j\), acting on the \(k\)-th tensor power (the correlation function takes values in the \(m\)-th tensor power of the adjoint representation of \(\mathfrak{o}_N\) - see Section 7.1).

Standard calculation in the WZW theory show that

$$< F_{-1}^J Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >= - \sum_{j \geq 2} \frac{F_{j(1)}^j}{(z_1 - z_j)} < Pf F_1(z_1) \varphi_2(z_2) ... \varphi_r(z_r) >.$$

(74)

Analogously one gets
Using the Corollary 2 and Theorem 5 one obtains the equation

$$< F_j^{-2} \mathcal{P} F_j(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > = - \sum_{j \geq 2} \frac{F_j^{(j)}}{(z_1 - z_j)^2} < \mathcal{P} F_j(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > .$$

(75)

Finally by (65) one gets

$$< F_j^{(j)} \mathcal{P} F_j(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) >= (k + 2) < \mathcal{P} F_{I\setminus J}(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > ,$$

(76)

note that the field $\mathcal{P} F_{I\setminus J}(z_1) = F_{I\setminus J}(z_1)$ is not primary.

8 Higher Knizhnik-Zamolodchikov equations

Using the Corollary 2 and Theorem 5 one obtains the equation

$$6(k + 2)k \partial_2 \varphi < \mathcal{P} F_j(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > =$$

$$= \frac{1}{4} \sum_{I = I_1 \cup I_2} \sum_{I' = I_1' \cup I_2'} (-1)^{(I_1, I_2)} (-1)^{(I_1', I_2')}

2 \sum_{j \geq 2} \frac{F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2}}{(z_1 - z_j)^2} < \mathcal{P} F_j(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > +$$

$$+ (k + 2)(8 \sum_{i \leq j \geq 2} \frac{F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2}}{(z_1 - z_j)(z_1 - z_1)}$$

$$- 4 \sum_{j \geq 2} \frac{F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2}}{(z_1 - z_1)^2} < \mathcal{P} F_{I\setminus I_1}(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) > +$$

$$+ (k + 2)(8 \sum_{i \leq j \geq 2} \frac{F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2}}{(z_1 - z_1)(z_1 - z_1)}$$

$$- 4 \sum_{j \geq 2} \frac{F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2} + F_{I_1}^{(i)} F_{I_2}^{(j)} I_{I_1} I_{I_2}}{(z_1 - z_1)^2} < \mathcal{P} F_{I\setminus I_1}(z_1) \varphi_2(z_2) \ldots \varphi_r(z_r) >$$

(77)

The summation $\sum_I$ is taken over all subsets $I \subset \{1, \ldots, N\}$ that consist of four elements. The summation $\sum_{I = I_1 \cup I_2, I' = I_1' \cup I_2'}$ is taken over all partitions.
\( I = I_1 \sqcup I_2, I' = I_1' \sqcup I_2' \) of the set \( I \) into two two-element subsets. The operator \( F_I^{(k)} \) is the operator \( F_I \), acting onto the \( k \)-th tensor component of the correlation function.

To simplify the equation we use the fact that for \( I_1 \cap I_2 = \emptyset, I_1' \cap I_2' = \emptyset \), one has \([F_{I_1}^{(k)}; F_{I_2}^{(l)}] = [F_{I_1'}^{(k)}; F_{I_2'}^{(l)}] = 0\).

Thus the sum that corresponds to the set \((0, 0, 0, -1)\) and its permutations can be written as follows

\[
\sum_{j \geq 2} F_{I_1}^{(1)} F_{I_2}^{(1)} F_{I_1'}^{(j)} F_{I_2'}^{(1)} + F_{I_1}^{(1)} F_{I_2}^{(j)} F_{I_1'}^{(1)} F_{I_2'}^{(1)} + F_{I_1}^{(1)} F_{I_2}^{(1)} F_{I_1'}^{(j)} F_{I_2'}^{(1)} + F_{I_1}^{(j)} F_{I_2}^{(1)} F_{I_1'}^{(1)} F_{I_2'}^{(1)} =
\]

\[
= 2 \sum_{j \geq 2} F_{I_1}^{(1)} F_{I_2}^{(1)} F_{I_1'}^{(j)} F_{I_2'}^{(1)} + F_{I_1}^{(1)} F_{I_2}^{(j)} F_{I_1'}^{(1)} F_{I_2'}^{(1)}.
\]

(78)

Analogously one can simplify the sums to the sets \((0, 1, 0, -2)), (0, 1, -1, -1)\). As the result one obtains the equation written above.

## 9 Conclusion

We investigate the possibility of construction of Knizhnik-Zamolodchikov type in the \( WZW \)-theory using higher order Casimir elements.

We consider the \( W \)-algebra that is generated by modes of the field constructed from a Gelfand element of the third order. In this case the construction of the equation is impossible.

Also we consider the \( W \)-algebra that is generated by modes of the field constructed from a Capelli element. Here we manage to express the action of the operator \( W_{-1} \) through a differential operator. Hence it is possible to construct a Knizhnik-Zamolodchikov type equation.

Considerations are done in the case of fields that transform in the adjoint representation. Analogous construction in the case of arbitrary tensor representation can be obtained by taking a tensor power of the adjoint representation and passing to irreducible components.

Note that there is a well-known relation between the Knizhnik-Zamolodchikov equation and Gaudin hamiltonians. Gaudin hamiltonians are right sides of Knizhnik-Zamolodchikov equations. At the same time there exist higher Gaudin hamiltonians that are actually higher Sugawara elements [9]. Higher Gaudin hamiltonians commute with Gaudin hamiltonians and thus have the same eigenfunctions. This indicates that the higher Gaudin hamiltonians are candidates for right sides of higher Knizhnik-Zamolodchikov equations.
But actually it is impossible to obtain such equation. The reason is that it is impossible to express the action of the higher Gaudin hamiltonians as differential operators (see for example [10], [11]).

10 Appendix

10.1 Proof of Proposition

\[ J^a(z)(J^\beta J^\gamma)(w) = \frac{1}{2\pi i} \oint \frac{dx}{x-w} J^a(z) J^\beta(x) J^\gamma(w) + J^\beta(x) J^a(z) J^\gamma(w). \]  

(79)

The integration is taken along a contour that pass around \( w \) the point \( z \) is not contained inside the contour.

One has

\[ J^a(z) J^\beta(x) = \frac{k\delta_{a,\beta}}{(z-x)^2} + \sum_c f_{a,\beta,c} J^c(x) \]  

\[ J^a(z) J^\gamma(w) = \frac{k\delta_{a,\gamma}}{(z-w)^2} + \sum_c f_{a,\gamma,c} J^c(w) \]  

(80)

Thus one gets

\[ \frac{1}{2\pi i} \oint \frac{dx}{x-w} J^a(z) J^\beta(x) J^\gamma(w) + J^\beta(x) J^a(z) J^\gamma(w) = \]

\[ = \frac{1}{2\pi i} \oint \frac{dx}{x-w} \left( \frac{k\delta_{a,\beta}}{(z-x)^2} + \sum_c f_{a,\beta,c} J^c(x) \right) J^\gamma(w) + J^\beta(x) \left( \frac{k\delta_{a,\gamma}}{(z-w)^2} + \sum_c f_{a,\gamma,c} J^c(w) \right) \]

(81)

The following expansions take place

\[ \frac{1}{(z-x)^2} = \frac{1}{(z-w)-(x-w)^2} = \frac{1}{(z-w)^2(1-\frac{x-w}{z-w})^2} = \]

\[ = \frac{1}{(z-w)^2} \left( 1 + \frac{(x-w)}{z-w} + \frac{(x-w)^2}{(z-w)^2} + \frac{(x-w)^3}{(z-w)^3} + \ldots \right)^2 = \]

\[ = \frac{1}{(z-w)^2} \left( 1 + 2 \frac{(x-w)}{z-w} + 3 \frac{(x-w)^2}{(z-w)^2} + 4 \frac{(x-w)^3}{(z-w)^3} + \ldots \right). \]  

(82)
Using these expansions one gets

\[ \frac{1}{2\pi i} \oint \frac{dx}{x-w} \left( \frac{k\delta_{\alpha,\beta}}{(z-w)^2} J^\gamma(w) + \frac{k\delta_{\alpha,\gamma}}{(z-w)^2} J^\beta(x) \right) = \]

\[ = \frac{k\delta_{\alpha,\beta}}{(z-w)^2} J^\gamma(w) + \frac{k\delta_{\alpha,\gamma}}{(z-w)^2} J^\beta(w). \tag{83} \]

After substitution of (83) into (81) one gets

\[ \frac{k\delta_{\alpha,\beta} J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma} J^\beta(w)}{(z-w)^2} + \sum_c \frac{1}{2\pi i} \oint \frac{dx}{x-w} f_{a,\beta,c} J^c(x) J^\gamma(w) = \]

\[ = \frac{k\delta_{\alpha,\beta} J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma} J^\beta(w)}{(z-w)^2} + \sum_c \frac{1}{2\pi i} \oint \frac{dx}{x-w} f_{a,\beta,c} J^c(x) J^\gamma(w) + \frac{f_{a,\alpha,c} J^3(x) J^c(w)}{(z-w)} \tag{84} \]

One has the expansions

\[ J(x) J^\gamma(w) = \frac{k\delta_{c,\gamma}}{(x-w)^2} + \sum_c \frac{f_{c,\gamma,d} J^d(w)}{(x-w)} + (J^c J^\gamma)(w) + \ldots \tag{85} \]

\[ J^\beta(x) J^c(w) = \frac{k\delta_{\beta,\gamma}}{(x-w)^2} + \sum_c \frac{f_{\beta,c,d} J^d(w)}{(x-w)} + (J^\beta J^c)(w) + \ldots \]

After substitution of (85) into (84) one gets

\[ \frac{k\delta_{\alpha,\beta} J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma} J^\beta(w)}{(z-w)^2} + \sum_c \frac{1}{2\pi i} \oint \frac{dx}{x-w} f_{a,\beta,c} J^c(x) J^\gamma(w) = \]

\[ = \frac{k\delta_{\alpha,\beta} J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma} J^\beta(w)}{(z-w)^2} + \]

\[ + \sum_c \frac{1}{2\pi i} \oint \frac{dx}{x-w} f_{a,\beta,c} J^c(x) J^\gamma(w) + \frac{f_{a,\alpha,c} J^3(x) J^c(w)}{(z-w)} + \frac{f_{a,\gamma,c} J^3(x) J^c(w)}{(z-w)} \]

\[ + \frac{1}{2\pi i} \oint \frac{dx}{x-w} f_{a,\alpha,\gamma} \frac{k\delta_{\alpha,\beta}}{(x-w)^2} + \sum_c \frac{f_{a,\beta,c} J^c(w)}{(x-w)} + (J^\beta J^c)(w) + \ldots \tag{86} \]

The following expansion takes place

\[ \frac{1}{z-x} = \frac{1}{(z-w) - (x-w)} = \frac{1}{z-w} \frac{1}{1 - \frac{x-w}{z-w}} = \frac{1}{z-w} + \frac{x-w}{(z-w)^2} + \frac{(x-w)^2}{(z-w)^3} + \ldots \tag{87} \]
Substitute (87) into (86), one gets

\[
\frac{k\delta_{\alpha,\beta}J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma}J^\beta(w)}{(z-w)^2} + \sum_c \frac{1}{2\pi i} \oint \frac{dx}{x-w} \frac{f_{a,\beta,c}k\delta_{\epsilon,\gamma}}{(z-w)^3} + \sum_c \frac{f_{a,\beta,c}f_{c,\gamma,d}J^d(w)}{(z-w)^2} + \\
+ f_{a,\beta,c}(J^\epsilon J^\gamma)(w) + \ldots + \frac{f_{a,\gamma,c}}{(z-w)^2} + \sum_c \frac{f_{c,\epsilon,d}J^d(w)}{(x-w)} + (J^\beta J^\epsilon)(w) + \ldots = \\
\frac{k\delta_{\alpha,\beta}J^\gamma(w)}{(z-w)^2} + \frac{k\delta_{\alpha,\gamma}J^\beta(w)}{(z-w)^2} + \frac{f_{a,\beta,\gamma}k}{(z-w)^3} + \\
+ \frac{f_{a,\beta,c}f_{c,\gamma,d}J^d(w)}{(z-w)^2} + \frac{f_{a,\beta,c}(J^\epsilon J^\gamma)(w)}{z-w} + \frac{f_{a,\gamma,c}(J^\beta J^\epsilon)(w)}{z-w}.
\]

(88)

### 10.2 Proof of Lemma 2

One has

\[
\sum_I \sum_{I=I_1 \sqcup I_2} \frac{1}{2\pi i} \oint \frac{dx}{x-w} P f_{F_{I_2}(J \setminus I_1)}(z)P f_{F_I}(w) = \sum_I \sum_{I=I_1 \sqcup I_2} P f_{F_{I_2}(J \setminus I_1)}(z)P f_{F_I}(w).
\]

(89)

Let us consider the expression

\[
\sum_I \sum_{I=I_1 \sqcup I_2} \frac{P f_{F_{I_2}(J \setminus I_1)}(z)P f_{F_I}(w)}{(z-w)^3}.
\]

(90)

Using the Proposition 4 we obtain that the considered product equals

\[
\sum_I \sum_{I=I_1 \sqcup I_2} \frac{P f_{F_{I_2}(J \setminus I_1)}(w)}{(z-w)^3} + \frac{P f_{F_{I_2}(J \setminus I_1)}I(w)}{(z-w)^4} + \frac{(F_{I_2}(J \setminus I_1)P f_{F_I}(w)}{(z-w)^3} + \text{l.o.t}
\]

(91)

Let us prove that the term of the highest order in (91) is equal to zero.

Prove the following Proposition.

**Proposition 6.** Let $|I| = 4$ and let us be given a partition $I = I_1 \sqcup I_2$, $|I_1| = |I_2| = 2$. Let us also be given another set of indices $J$, such that $|J| = 4$. Then

\[
F_{I_2}(J \setminus I_1)
\]

26
does not equal to zero and does not contain repetitions if and only if \( I_1 \subset J \), one of the elements of \( I_2 \) is contained in \( J \), and the other is not contained in \( J \). In particular \( I \) and \( J \) differ only by one element.

Доказательство. Indeed if \( I_1 \not\subset J \) then \( J \setminus I_1 = 0 \). If \( I_1 \subset J \) then two cases are possible. In the first case \( I_2 \subset J \setminus I_1 \) (that is actually \( I = J \)). Then \( F_{I_2}(J \setminus I_1) \) contains repetitions. In the second case \( I_2 \not\subset J \setminus I_1 \), then \( J \setminus I_1 \) contains an one element that does not belong to \( I \). Then \( F_{I_2}(J \setminus I_1) \) also contains one such an element.

As a corollary one get that either \( F_{I_2}(J \setminus I_1) \not\subset I \) or \( F_{I_2}(J \setminus I_1) \) equals to zero, or contains repetitions. In all cases \( P f_{F_{I_2}(J \setminus I_1)}(w) = 0 \).

Thus the terms of the highest order in (91) equals to zero.

Consider the next term in (91)

\[
\sum_i \sum_{I = I_1 \sqcup I_2} \frac{P f_{F_{I_2}(J \setminus I_1)}(w)}{(z - w)^4} (92)
\]

According to Proposition 6, \( F_{I_2}(J \setminus I_1) \) is not equal to zero and does not contain repetitions if and only if the sets \( I \) and \( J \) differ by exactly one element, that is the following equalities take place

\[
J = \{a, b, c, e\}, \quad I = \{a, b, c, d\} \text{ or } \{a, b, d, e\} \text{ or } \{a, d, c, e\} \text{ or } \{d, b, c, e\}. (93)
\]

Consider possible choices for partitions \( I = I_1 \sqcup I_2 \) and calculate \( F_{I_2}(J \subset I_1) \).

If \( I = \{a, b, c, d\} \) then

\[
I_1 = \{a, b\}, \quad I_2 = \{c, d\}, \quad (J \setminus I_1) = \{c, e\}, \quad F_{I_2}(J \setminus I_1) = -\{d, e\}
\]

\[
I_1 = -\{a, c\}, \quad I_2 = -\{b, d\}, \quad (J \setminus I_1) = -\{b, e\}, \quad F_{I_2}(J \setminus I_1) = -\{d, e\}
\]

\[
I_1 = \{b, c\}, \quad I_2 = \{a, d\}, \quad (J \setminus I_1) = \{a, e\}, \quad F_{I_2}(J \setminus I_1) = -\{d, e\}
\]

(94)

If \( I = \{a, b, d, e\} \) then

\[
I_1 = \{a, b\}, \quad I_2 = \{d, e\}, \quad (J \setminus I_1) = \{c, e\}, \quad F_{I_2}(J \setminus I_1) = \{c, d\}
\]

\[
I_1 = -\{b, e\}, \quad I_2 = -\{a, d\}, \quad (J \setminus I_1) = -\{a, c\}, \quad F_{I_2}(J \setminus I_1) = -\{d, c\}
\]

\[
I_1 = \{a, e\}, \quad I_2 = \{b, d\}, \quad (J \setminus I_1) = \{b, c\}, \quad F_{I_2}(J \setminus I_1) = -\{d, c\}
\]

(95)
If $I = \{a, d, c, e\}$ then

$I_1 = \{c, e\}, \ I_2 = \{a, d\}, \ (J \setminus I_1) = \{a, b\}, \ F_{I_2}(J \setminus I_1) = -\{d, b\}$

$I_1 = \{-a, c\}, \ I_2 = \{-d, e\}, \ (J \setminus I_1) = \{-b, e\}, \ F_{I_2}(J \setminus I_1) = \{b, d\} \quad (96)$

$I_1 = \{a, e\}, \ I_2 = \{d, c\}, \ (J \setminus I_1) = \{b, c\}, \ F_{I_2}(J \setminus I_1) = \{b, d\}$

If $I = \{d, b, c, e\}$ then

$I_1 = \{c, e\}, \ I_2 = \{a, d\}, \ (J \setminus I_1) = \{a, b\}, \ F_{I_2}(J \setminus I_1) = \{d, e\}$

$I_1 = \{-b, e\}, \ I_2 = \{-a, c\}, \ (J \setminus I_1) = \{-a, c\}, \ F_{I_2}(J \setminus I_1) = \{a, d\} \quad (97)$

$I_1 = \{b, c\}, \ I_2 = \{d, e\}, \ (J \setminus I_1) = \{a, e\}, \ F_{I_2}(J \setminus I_1) = \{a, d\}$

Thus in the case $I = \{a, b, c, d\}$ one has

$$\frac{P f F_{\{a,b,c,d\}}(w)}{(z - w)^4} = \frac{P f F_{\{a,b,c,e\}}(w)}{(z - w)^4} = \frac{P f F_{J}(w)}{(z - w)^4} \quad (98)$$

In the other cases considerations are similar. Thus (92) equals to

$$12(N - 4)\frac{P f F_{J}(w)}{(z - w)^4}. \quad (99)$$

Here the factor 12 in the numerator is a product of the factor 3 that corresponds to the choice of one of four cases described above and the factor 4 that corresponds to the choice of the partition $I = I_1 \cup I_2$. The factor $(N - 4)$ corresponds to the choice an element $d \in I$, that does not belong to $J$.

Consider the third term in (91)

$$\sum_{I} \sum_{I=I_1 \cup I_2} \frac{(P f F_{E_{J}(J \setminus I_1)} P f F_{J}(w))}{(z - w)^3}. \quad (100)$$

According to Proposition 6 a summand in this expression can be nonzero only in the for cases that were described above. After summation of inputs from these cases one gets

$$\frac{1}{(z - w)^4}(- (F_{de} P f F_{\{a,b,c,d\}})(w) + (F_{ad} P f F_{\{a,b,d,e\}})(w) - (F_{db} P f F_{\{a,d,c,e\}})(w) + (F_{ad} P f F_{\{d,b,c,e\}})(w)). \quad (101)$$
The numerator in this expression can be simplified. Let us write the terms of the first and the second pfaffians corresponding to identical permutation and the permutation that changes the first and the third, the second and the fourth indices.

\[-(F_{de}(F_{a,b}F_{c,d}))(w) - (F_{de}(F_{c,d}F_{a,b}))(w) + (F_{c,d}(F_{a,b}F_{d,e}))(w) + (F_{c,d}(F_{d,e}F_{a,b}))(w).\]

Let us simplify the sum of the first and the fourth summands:

\[-(F_{de}(F_{a,b}F_{c,d}))(w) + (F_{c,d}(F_{d,e}F_{a,b}))(w).\]  

Let us transform the first summand. We are going to use the equalities \[7\].

\[((AB)E)(w) - (A(BE))(w) = (A([E, B]))(w) + ([E, A])B)(w) + ([AB], E](w),\]  

\[(BA) = (AB)(w) - \partial(AB)_{-1}(w) + \frac{1}{2!}\partial^2(AB)_{-2}(w) - \frac{1}{3!}\partial^3(AB)_{-3}(w) + \ldots\]

Using \[104\] one gets

\[-(F_{de}(F_{a,b}F_{c,d}))(w) = -(F_{de}F_{a,b}F_{c,d})(w),\]

using \[105\], one gets

\[-((F_{de}F_{a,b})F_{c,d}))(w) = -(F_{c,d}(F_{de}F_{a,b})))(w) + \partial(F_{c,d}(F_{de}F_{a,b})_{-1}(w) - \frac{1}{2!}\partial^2(F_{c,d}(F_{de}F_{a,b}))_{-2}(w) + \ldots\]

Thus \[103\] equals to

\[\partial(F_{c,d}(F_{de}F_{a,b}))_{-1}(w) - \frac{1}{2!}\partial^2(F_{c,d}(F_{de}F_{a,b}))_{-2}(w) + \ldots\]

By symmetry it also equals

\[29\]
\[
\partial(F_{d,e}(F_{a,b}F_{c,d})) - \frac{1}{2!}\partial^2(F_{d,e}(F_{a,b}F_{c,d}))-2(w) + \ldots
\]  

(109)

Take into account other terms of the pfaffian in (101). One gets that (101) equals

\[
\frac{1}{(z-w)^4}(-\partial(F_{d,e}PfF_{\{a,b,c,d\}})(w) + \partial(F_{cd}PfF_{\{a,b,d,e\}})(w) - \\
- \partial(F_{db}PfF_{\{a,d,c,e\}})(w) + \partial(F_{ad}PfF_{\{d,b,c,e\}})(w)).
\]

(110)

Above in the proof of this Lemma it was shown that \(-F_{d,e}PfF_{\{a,b,c,d\}} - k(w) = 0\) for \(k > 1\) and \(-F_{d,e}PfF_{\{a,b,c,d\}} - 1(w) = PfF_{\{a,b,c,d\}}(w)\). Analogously all other pfaffians are considered. Thus the sum of the first and the second pfaffians in (101) equals

\[
12(N-4)\partial PfF_{\{a,b,c,d\}}(w) = 12(N-4)\partial PfF_{f}(w).
\]

(111)

Thus the expression (100) equals

\[
\frac{12(N-4)\partial PfF_{f}(w)}{(z-w)^3}
\]

(112)

10.3 The proof of Lemma 3

It is enough to prove that when one calculates the expansion of the expression

\[
\frac{(F_{I_2}PfF_{J\setminus I_1})(z)PfF_{f}(w)}{(z-w)^2}
\]

(113)

one does not obtain singular terms of orders greater than \((z-w)^{-2}\). The expression (113) is nonzero only if two element from \(I\) belong to \(J\), that is these sets are of type

\[I = \{a, b, c, d\}, \quad J = \{a, b, e, f\}.\]

The numerator in (113) is of type

\[
(F_{cd}F_{ef})(z)PfF_{\{a,b,c,d\}}(w).
\]

(114)
Using notation \( I_1' = \{c, d\} \), \( I_2' = \{e, f\} \), \( I = \{a, b, c, d\} \) let us write (114) as follows

\[
(F_{I_1'} F_{I_2'})(z) P f F_I(w).
\]

Apriory in the calculation of this OPE the terms of the order \((z - w)^{-4}\), \((z - w)^{-3}\), and terms of lower order. Let us show that actually the terms of orders 4 and 3 do not appear.

Indeed, the OPE is calculated using Proposition 4. But since \( J \setminus I_1 \not\supseteq I_1 \), then \( I \neq I_1' \cup I_2' \) and the terms of the highest order in the OPE vanishes.

The second terms corresponds to

\[
F_{I_1'}(I \setminus I_2') = F_{cd}\{a, b\}\delta_{(ab),(cd)} = 0, \quad F_{I_2'}(I \setminus I_1') = F_{ef}\{a, b\} = 0
\]

Thus this terms also vanishes. Thus there are no terms of the order 4 and 3.

10.4 Proof of Lemma 5

The expression on the right hand side in (59) equals

\[
\sum_{I = I_1 \cup I_2, |I_1| = |I_2| = 2} (-1)^{(I_1, I_2)} \frac{(F_{I_1} P f F_{I_2})(z) P f F_I(w)}{(z - w)}.
\]

Let us divide proof of the lemma into several steps. We shall investigate the OPE in the numerator in (117).

**Step 1.** Show that the OPE in the numerator begins with the term of order \((z - w)^{-3}\). Indeed this OPE can be calculated using Proposition 4. Since \(|I| = 4\), than the most singular term that one can apriory obtain is proportional to \((z - w)^{-4}\). But a term of order \((z - w)^{-4}\) can appear only if

\[
I \subset I_1 \cup F_{I_2} J \iff I_2 \subset F_{I_2} J.
\]

This inclusion is impossible. Thus after calculation of the OPE in the numerator in (117), one gets an expansion that begins with \((z - w)^{-3}\).

**Step 2.** Let us find a relations between a coefficient at \((z - w)^{-3}\) and a coefficient at \((z - w)^{-2}\) in the OPE in the numerator in (117). Let us show that the coefficient at \((z - w)^{-2}\) is a derivative of the coefficient of the coefficient at \((z - w)^{-3}\).
For this purpose put \( a(z) = ([P f F_I, P f F_J])(z) \), \( b(w) = P f F_I(w) \). Firstly show that \( (ab)_{-3}(w) \) equals to a coefficient at \( (z - w)^{-4} \), and \( (ab)_{-2}(w) \) equals to a coefficient at \( (z - w)^{-3} \). Then show that \( \partial(ab)_{-3}(w) = (ab)_{-2}(w) \).

Let us use the following fact. In [2] a formula for a commutator of two pfaffians was proved. In the case of two pfaffians of order 4 it looks as follows:

\[
[P f F_I, P f F_J] = P f F_{P f F_J} \sum_{I = I_1 \sqcup I_2} (-1)^{|I_1, I_2|} F_{I_1} P f F_{I_2} J. \tag{118}
\]

Thus one has:

\[
a(z)b(w) = P f F_{P f F_J} J(z) P f F_I(w) + \sum_{I = I_1 \sqcup I_2} (-1)^{|I_1, I_2|} (F_{I_1} P f F_{I_2} J(z) P f F_I(w). \tag{119}
\]

To do the first step one must prove the Proposition

**Proposition 7.** The OPE

\[
P f F_{P f F_J} J(z) P f F_I(w) \tag{120}
\]

begins with the term \( (z - w)^{-1} \).

**Доказательство.** To calculate the OPE in the numerator the formula (119) is used.

In this calculation a terms of order 4 can appear. More precise it appears if

\( P f F_J J = I \).

Note that the number of elements of \( J \), that contain in \( I \), is the same as the number of elements of \( P f F_J J \), that contain in \( I \). Thus the equality written above is possible if and only if \( J = I \). But then \( P f F_J J \) contains repetitions. Thus the terms of order 4 does not appear.

The terms of order 3 appears if

\( F_{I_2} ((P f F_J) \setminus I_1) \)

is nonzero and does not contain repetitions for some partition \( I = I_1 \sqcup I_2 \). Put \( J = P f F_J J \). Above it was shown that \( F_{I_2} (J \setminus I_1) \) has this property if and only if \( J \) and \( I \) differ by exactly one element. If \( P f F_J J \) and \( I \) differ by one element then \( J \) and \( I \) also differ by one element. In this case \( P f F_J J \) contains repetitions. Thus the term of the order 3 does not appear.
The term of the order 2 appears if 

\[(F_{I_2} P F_{P F_{I_1}})(w) \neq 0\]

for some partition \(I = I_1 \sqcup I_2\). As above one can prove that this is impossible. Thus the terms of order 2 do not appear.

Thus the expansion of the expression \(120\) begins with the term \((z - w)^{-1}\).

Let us show that \(\partial(ab)_{-3}(w) = (ab)_{-2}(w)\). In \([3]\) it was shown that the tensor

\[
\sum_{\{i_1, \ldots, i_4\}} (e_{i_1} \wedge \ldots \wedge e_{i_4}) \otimes (e_{i_1} \wedge \ldots \wedge e_{i_4})
\]

\[(121)\]

is invariant under the action of \(o_N\). Here \(i_k = 1, \ldots, N\) and \(e_i\) is a standard base in \(\mathbb{C}^N\).

From the invariance of \(121\) it follows that

\[
([P f F_I, P f F_J])(z) P f F_I(w) + P f F_I(z) ([P f F_I, P f F_J])(w) = 0.
\]

Thus one has

\[
a(z) b(w) + b(z) a(w) = 0.
\]

Take OPEs in \(123\), one gets that

\[
(ab)_{-2}(w) = \partial(ab)_{-3}(w).
\]

Thus the coefficient at \((z - w)^{-2}\) equals to the coefficient at \((z - w)^{-3}\).

**Step 3.** Let us calculate a coefficient at \((z - w)^{-3}\) in the expansion of the numerator in \(117\), that is a coefficient at \((z - w)^{-3}\) in the expansion of

\[
\sum_{I} \sum_{I = I_1 \sqcup I_2} (F_{I_1} P f F_{I_2 J})(z) P f F_I(w).
\]

To calculate this coefficient let us write

\[
\sum_{I = I_1 \sqcup I_2} (-1)^{(I_1, I_2)} (F_{I_1} P f F_{I_2 J})(z) P f F_I(w) = \sum_{I = I_1 \sqcup I_2} (-1)^{(I_1, I_2)} P f F_I(w) (F_{I_1} P f F_{I_2 J})(z) =
\]

\[
= \sum_{I = I_1 \sqcup I_2} (-1)^{(I_1, I_2)} \frac{1}{2\pi i} \int \frac{dx}{x - w} P f F_I(w) F_{I_1}(x) P f F_{I_2}(z) + F_{I_1}(x) P f F_I(w) P f F_{I_2}(z) = \]

\[
= \sum_{I_1, I_2} (-1)^{(I_1, I_2)} \oint \frac{dx}{x - w} F_{I_1}(x) P f F_I(w) P f F_{I_2}(z) + F_{I_1}(x) P f F_{I_2}(z) P f F_I(w).
\]

\[(126)\]
Consider two summands in (126) separately. Let us substitute the expression for the contraction $F_{I_1}(x) P f F_{I_2}(w)$ which is given by Proposition 4. After an explicit calculation one gets that the term of order $(z - w)^{-3}$ equals to

$$\sum_{I_1, I_2} (-1)^{|I_1| |I_2|} (k + 2) P f F_{P f F_{I_1} F_{I_2} J} (z - w)^3. \tag{127}$$

Note that $P f F_{I_1} = (-1)^{|I_1| |I_2|} F_{I_2}$. Thus the numerator equals $-P f F_J$, if exactly one of the elements of $I_2$ belongs to $J$ and equals to zero otherwise. There are

$$4(N - 4)(N - 2)(N - 3) \quad \text{pair of such subsets } I_1, I_2 \subset \{1, ..., N\}.$$ 

Indeed the factor 4 corresponds to the choice of the element of $I_2$, that does not belong to $J$, the factor $(N - 4)$ corresponds to the choice of the second element, the denominator $2!$ corresponds to the fact that the set $I_2$ is unordered. The factor $\frac{(N-2)(N-3)}{2!}$ is the number of choices of the set $I_1$.

Thus the first summand in (126) equals

$$- \frac{(N - 2)(N - 3)(N - 4)(k + 2) P f F_J (w)}{(z - w)^3}. \tag{129}$$

Consider the second summand in (126). The contraction $P f F_{I_1} (z) P f F_{I_2} (w)$ is given by Proposition 5. The numerators in this expression depend on $z$. After integration $\oint \frac{dx}{x - w}$ one gets an expression where the term of order $(z - w)^{-3}$ is given by expression

$$\sum_{I_1, I_2} (-1)^{|I_1| |I_2|} (k + 2) P f F_{I_1} P f F_{I_2} ((F_{I_2} J \setminus I_1)) (w) \quad \text{for } I_1 \cup I_2 = \{1, ..., N\} \setminus I_1. \tag{130}$$

Consider firstly the expression

$$\sum_{I_1, I_2} (-1)^{|I_1| |I_2|} (F_{I_1} P f F_{I_2} ((F_{I_2} J \setminus I_1)) (w) \tag{131}$$

Let us do the summation over partitions $I = I_1 \cup I_2$. By Proposition 6

$$\text{Let } F_{I_1} ((F_{I_2} J \setminus I_1) \neq 0 \text{ if and only if } I_1 \subset F_{I_2} J, \text{ and one of the elements } I_2 \text{ belongs to } F_{I_2} J, \text{ and the other does not belong.}$$
For each such partition, doing the same calculation as in the proof of Lemma 2 (using everywhere $J = F_{I_2}J$ instead of $J$), one gets that

$$\sum_{I=I_1 \cup I_2} (F_{I_1}PfF_{I_2}'((F_2J)\setminus I_1))(w) = 3(F_{I_1}F_{I_2}\Delta(F_{I_2}J))(w),$$  \(132\)

where $I \Delta (F_{I_2}J)$ is a symmetric difference of sets. Thus for $I = \{a, b, c, d\}$ and $F_{I_2}J = \{a, b, c, e\}$ one has $I \Delta (F_{I_2}J) = \{d, e\}$.

Find the sum $\sum_I \sum_{I=I_1 \cup I_2}$ of the expressions (132). Let $J = \{a, b, c, e\}$. The set $I_1$ must be a subset in $J$, one of the elements of $I_2$ must be contained in $J$ and the other not. Thus the possibilities for $I_1$ and $I_2$ are the following

$I_1 = \{a, b\}$, $I_2 = \{d\}$, $F_{I_2}J = \{a, b, d, e\}$, $I \Delta (F_{I_2}J) = \{c, e\}$

$I_1 = \{a, c\}$, $I_2 = \{b, d\}$, $F_{I_2}J = \{a, d, c, e\}$, $I \Delta (F_{I_2}J) = \{b, e\}$

$I_1 = \{b, c\}$, $I_2 = \{a, d\}$, $F_{I_2}J = \{d, b, c, e\}$, $I \Delta (F_{I_2}J) = \{a, e\}$

(133)

Thus the sum of expressions (132) equals

$$-3PfF_J(w)$$  \(134\)

Hence the second summand in (126) equals

$$-\frac{6(k + 2)PfF_J(w)}{(z - w)^3}$$  \(135\)

So, the expression (126) equals

$$-\frac{(k + 2)((N - 2)(N - 3)(N - 4) + 6)PfF_J(w)}{(z - w)^3}.$$  \(136\)

**Conclusion.** Thus

the coefficient at $(z - w)^{-4}$ in the expansion of (117) equals

$$-(k + 2)((N - 2)(N - 3)(N - 4) + 6)PfF_J(w),$$

the coefficient at $(z - w)^{-3}$ in the expansion of (117) equals

$$-(k + 2)((N - 2)(N - 3)(N - 4) + 6)\partial PfF_J(w).$$

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5The set $I$ is fixed
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