In four-dimensional $\mathcal{N} = 1$ Minkowski superspace, general nonlinear $\sigma$-models with four-dimensional target spaces may be realised in terms of CCL (chiral & complex linear) dynamical variables which consist of a chiral scalar, a complex linear scalar and their conjugate superfields. Here we introduce CCL $\sigma$-models that are invariant under U(1) “duality rotations” exchanging the dynamical variables and their equations of motion. The Lagrangians of such $\sigma$-models prove to obey a partial differential equation that is analogous to the self-duality equation obeyed by U(1) duality invariant models for nonlinear electrodynamics. These $\sigma$-models are self-dual under a Legendre transformation that simultaneously dualises (i) the chiral multiplet into a complex linear one; and (ii) the complex linear multiplet into a chiral one. Any CCL $\sigma$-model possesses a dual formulation given in terms of two chiral multiplets. The U(1) duality invariance of the CCL $\sigma$-model proves to be equivalent, in the dual chiral formulation, to a manifest U(1) invariance rotating the two chiral scalars. Since the target space has a holomorphic Killing vector, the $\sigma$-model possesses a third formulation realised in terms of a chiral multiplet and a tensor multiplet.

The family of U(1) duality invariant CCL $\sigma$-models includes a subset of $\mathcal{N} = 2$ supersymmetric theories. Their target spaces are hyper Kähler manifolds with a non-zero Killing vector field. In the case that the Killing vector field is triholomorphic, the $\sigma$-model admits a dual formulation in terms of a self-interacting off-shell $\mathcal{N} = 2$ tensor multiplet.

We also identify a subset of CCL $\sigma$-models which are in a one-to-one correspondence with the U(1) duality invariant models for nonlinear electrodynamics. The target space isometry group for these sigma models contains a subgroup U(1) $\times$ U(1).
1 Introduction

Within the framework of four-dimensional $\mathcal{N} = 2$ Poincaré supersymmetry, the most general nonlinear $\sigma$-model can be formulated using off-shell polar supermultiplets \cite{1} (see \cite{2, 3} for reviews). From the point of view of $\mathcal{N} = 1$ supersymmetry, the $\mathcal{N} = 2$ polar supermultiplet is equivalent to an infinite set of $\mathcal{N} = 1$ superfields that naturally split into the categories physical and auxiliary. The physical superfields consist of a chiral scalar $\Phi$ and a complex linear scalar $\Sigma$ constrained by

$$\bar{D}_\alpha \Phi = 0, \quad \bar{D}^2 \Sigma = 0,$$

as well as their conjugates $\bar{\Phi}$ and $\bar{\Sigma}$. The auxiliary superfields consist of unconstrained complex scalars $\Upsilon_n$ and their conjugates $\bar{\Upsilon}_n$, where $n = 2, 3, \ldots$ Formulated in $\mathcal{N} = 1$ superspace, the $\sigma$-model action has the form\cite{4}

$$S = \int d^4x d^4\theta \mathcal{L}(\Phi^I, \bar{\Phi}^J, \Sigma^I, \bar{\Sigma}^J, \Upsilon^I_n, \bar{\Upsilon}^J_n),$$

\footnotetext[2]{Capital Latin letters from the middle of the alphabet are used to denote the target-space indices.}
for some Lagrangian $\mathcal{L}$. Since the auxiliary superfields $\Upsilon^I_n$ and $\bar{\Upsilon}^J_n$ are unconstrained and appear in the action without derivatives, they may be in principle integrated out (using the techniques developed over the last 15 years [4, 5, 6, 7, 8, 9, 10, 11, 12]). Then the $\sigma$-model action turns into

$$S = \int d^4x d^4\theta L(\Phi^I, \Phi^J, \Sigma^I, \Sigma^J).$$

(1.3)

Any $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-model can be recast in the $\mathcal{N} = 1$ superfield form (1.3), and for this reason such $\mathcal{N} = 1$ nonlinear $\sigma$-models have been the focus of much attention. The Lagrangian $L$ in (1.3) must obey some nontrivial conditions in order for the action to be invariant under on-shell $\mathcal{N} = 2$ supersymmetry transformations, see [11, 12] for more details. In this note we provide an alternative motivation for study of $\mathcal{N} = 1$ supersymmetric nonlinear $\sigma$-models involving chiral and complex linear (CCL) superfields – i.e. of the form (1.3). In what follows, the supersymmetric theories (1.3) will be often referred to as CCL $\sigma$-models. It should be mentioned that such nonlinear $\sigma$-models were discussed for the first time long ago by Deo and Gates [13].

Chiral and complex linear superfields are known to provide dual off-shell descriptions of the massless scalar supermultiplet [14, 15, 16]. By performing a special superfield Legendre transformation, which is reviewed in section 4, any supersymmetric theory that is realised in terms of a complex linear scalar $\Sigma$ and its conjugate $\bar{\Sigma}$ has a dual formulation in terms of a chiral scalar $\Psi$, $\bar{D}_\alpha \Psi = 0$, and its conjugate $\bar{\Psi}$. When applied to the $\sigma$-model (1.3), this leads to a purely chiral formulation

$$S = \int d^4x d^4\theta K(\Phi^I, \Phi^J, \Psi^I, \bar{\Psi}^J),$$

(1.4)

where $K$ is the Kähler potential for a target space. On the other hand, the inverse Legendre transformation allows us to dualise chiral multiplets into complex linear ones. Both the Legendre transformation and its inverse may be applied simultaneously to the variables $\Phi$ and $\Sigma$ in (1.3). One can construct $\sigma$-models (1.3) that are self-dual under this simultaneous Legendre transformation, in that the dual Lagrangian is equivalent to the original [19].

Self-duality under a Legendre transformation is an example of a discrete duality symmetry. In this note we will be interested in nonlinear $\sigma$-models (1.3) which possess

---

3It is well-known that any $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetric $\sigma$-model can be formulated in terms of $\mathcal{N} = 1$ chiral scalars and their conjugates [17, 18].

4One may also define self-dual $\sigma$-models with manifest $\mathcal{N} = 2$ supersymmetry [19] using the polar-polar duality [4] (see also [2]).
a continuous U(1) duality invariance. It turns out that such σ-models display a remarkable similarity with the U(1) duality invariant models for nonlinear electrodynamics \[20, 21, 22, 23, 24\] (see \[25, 26\] for reviews).

This paper is organised as follows. In section 2 we present the general theory of U(1) duality invariant CCL σ-models. In section 3 we identify a subset of CCL σ-models which are in a one-to-one correspondence with the U(1) duality invariant models for nonlinear electrodynamics. Section 4 is devoted to the dual chiral formulation. Dual Born-Infeld type solutions are considered in section 5. Concluding comments are given in section 6. Finally, the Appendix is devoted to the derivation of the duality equation.

## 2 Duality invariant sigma models

The object of our study is a supersymmetric nonlinear σ-model realised in terms of a chiral scalar Φ, a complex linear scalar Σ and their conjugates. The action is

\[
S_{\text{CCL}} = \int d^4x d^4\theta L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}).
\]

The off-shell constraints can always be solved in terms of unconstrained complex gauge superfields as \(\Phi = \bar{D}_2 \bar{U}\) and \(\Sigma = \bar{D}_\alpha \bar{\rho}^\alpha\), and therefore the equations of motion are

\[
\bar{D}_\alpha \frac{\partial L}{\partial \Sigma} = 0, \quad \bar{D}_2 \frac{\partial L}{\partial \Phi} = 0.
\]

We see that the equations of motion have the same functional form as the off-shell constraints but with Φ and Σ interchanged. As a result, “duality” rotations that mix \(\Phi\) and \(\frac{\partial L}{\partial \Sigma}\), and Σ and \(\frac{\partial L}{\partial \Phi}\), leave the constraints and the equations of motion invariant.

We consider the continuous U(1) duality rotations\(^5\)

\[
\left( \frac{\Phi'}{\frac{\partial L'(X')}{\partial \Sigma'}} \right) = \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \left( \frac{\Phi}{\frac{\partial L(X)}{\partial \Sigma}} \right)
\]

\(\Phi' = \bar{D}_2 \bar{U}'\) and \(\Sigma' = \bar{D}_\alpha \bar{\rho}'^\alpha\) as gauge invariant field strengths.

\(^5\)These are motivated by the duality rotations in vacuum electrodynamics, which mix the Bianchi identities \(\vec{\nabla} \times \vec{E} = 0\) and \(\vec{\nabla} \cdot \vec{B} = 0\) with equations of motion \(\vec{\nabla} \times \frac{\partial L}{\partial \vec{B}} = 0\) and \(\vec{\nabla} \cdot \frac{\partial L}{\partial \vec{E}} = 0\). In this case, the rotations are termed duality rotations because they mix derivatives of the field strength, \(\partial^a F_{ab}\), with derivatives of the Hodge dual of the field strength, \(\partial^a \tilde{F}_{ab}\). We may think of the superfields \(\Phi = \bar{D}_2 \bar{U}\) and \(\Sigma = \bar{D}_\alpha \bar{\rho}^\alpha\) as gauge invariant field strengths.

\(^6\)The compact U(1) duality transformations may be promoted to non-compact ones in the presence of additional matter fields, in complete analogy with nonlinear electrodynamics.
and
\[
\left( \frac{\Sigma'}{\partial L'(X')} \right) = \begin{pmatrix} \cos \kappa & \sin \kappa \\ -\sin \kappa & \cos \kappa \end{pmatrix} \left( \frac{\Sigma}{\partial L(X)} \right), \quad (2.3b)
\]
where, for notational convenience, the symbol \( X \) has been introduced for the argument \( \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma} \). The left hand sides of these equations define the chiral scalar \( \Phi' \), the complex linear scalar \( \Sigma' \) and the Lagrangian \( L' \). As shown in the Appendix, there is an integrability condition associated with the existence of the Lagrangian \( L'(X') \) defined in this way that forces the condition
\[
\kappa = \lambda . \quad (2.3c)
\]

Duality invariance of the theory is the requirement that the Lagrangians \( L' \) and \( L \) have the same functional form,
\[
L'(X') = L(X') . \quad (2.4)
\]
The implications of this condition are derived in the Appendix, mimicking standard constructions in nonlinear electrodynamics (see, for example, the Appendix in [25]). They are that the Lagrangian \( L(X) = L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \) must obey the differential equation
\[
0 = \Phi \Sigma + \bar{\Phi} \bar{\Sigma} + \frac{\partial L}{\partial \Phi} \frac{\partial L}{\partial \Sigma} + \frac{\partial L}{\partial \bar{\Phi}} \frac{\partial L}{\partial \bar{\Sigma}} . \quad (2.5)
\]

Note that the condition (2.4) is not equivalent to the requirement that the Lagrangian itself is invariant under the duality rotations. Given a duality invariant theory, its Lagrangian varies as
\[
\delta L(X) \equiv L(X') - L(X) = -2\lambda \left( \Phi \Sigma + \bar{\Phi} \bar{\Sigma} \right) , \quad (2.6)
\]
as a consequence of (2.5). However, there exist two general prescriptions to construct duality invariant observables starting from \( L \).

Firstly, making use of eq. (2.5), one finds that
\[
\delta \left( L - \frac{1}{2} \Phi \frac{\partial L}{\partial \Phi} - \frac{1}{2} \bar{\Phi} \frac{\partial L}{\partial \bar{\Phi}} - \frac{1}{2} \Sigma \frac{\partial L}{\partial \Sigma} - \frac{1}{2} \bar{\Sigma} \frac{\partial L}{\partial \bar{\Sigma}} \right) = 0 , \quad (2.7)
\]
and so the expression in parentheses is invariant under arbitrary duality rotations. For a finite duality rotation (2.3), this means that
\[
L(X) - \frac{1}{2} \Phi \frac{\partial L(X)}{\partial \Phi} - \frac{1}{2} \bar{\Phi} \frac{\partial L(X)}{\partial \bar{\Phi}} - \frac{1}{2} \Sigma \frac{\partial L(X)}{\partial \Sigma} - \frac{1}{2} \bar{\Sigma} \frac{\partial L(X)}{\partial \bar{\Sigma}} = L(X') - \frac{1}{2} \Phi' \frac{\partial L(X')}{\partial \Phi'} - \frac{1}{2} \bar{\Phi}' \frac{\partial L(X')}{\partial \bar{\Phi}'} - \frac{1}{2} \Sigma' \frac{\partial L(X')}{\partial \Sigma'} - \frac{1}{2} \bar{\Sigma}' \frac{\partial L(X')}{\partial \bar{\Sigma}'} . \quad (2.8)
\]
Secondly, if the Lagrangian depends on a duality invariant parameter $g$, that is $L = L(X; g)$, then the observable $\partial L(X; g)/\partial g$ is duality invariant. This follows from the fact that the right-hand side of (2.6) is independent of $g$. A nontrivial application of this property is that the supercurrent multiplet of any duality invariant $\sigma$-model is duality invariant.

Even though the $U(1)$ duality invariance is not a symmetry of the Lagrangian, it leads to the existence of a conserved current $j^a$ such that $\partial_a j^a = 0$. It is proportional to the component $(\sigma^a)^{\dot{\gamma}\gamma}[D_\gamma, \bar{D}_{\dot{\gamma}}]J$ of the superfield

$$J := i\left(\Phi \Sigma + \frac{\partial L}{\partial \Phi} \frac{\partial L}{\partial \Sigma}\right) = -i\left(\bar{\Phi} \bar{\Sigma} + \frac{\partial L}{\partial \bar{\Phi}} \frac{\partial L}{\partial \bar{\Sigma}}\right),$$

which is real, as a consequence of (2.5), and obeys the conservation equation

$$D^2 J = D^2 \bar{J} = 0$$

on the mass shell. This is similar to the Gaillard-Zumino conserved current in duality invariant electrodynamics [20].

An important property of any solution of (2.5) is self-duality under a Legendre transformation which dualises the (anti)chiral variables $\Phi$ and $\bar{\Phi}$ into a complex linear superfield $\Gamma$ and its conjugate $\bar{\Gamma}$, and also dualises the complex linear scalar $\Sigma$ and its conjugate $\bar{\Sigma}$ into a chiral scalar $\Psi$ and its conjugate $\bar{\Psi}$. Before discussing self-duality, let us first recall the definition of a dual formulation for the most general $\sigma$-model (2.1) following [19].

Starting from (2.1), we introduce a first-order action

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ L(X) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma} - \Gamma \Phi - \bar{\Gamma} \bar{\Phi}\right\},$$

where again, for notational convenience, the symbol $X$ denotes the set of scalars $\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}$. Unlike the original action (2.1), here $\Phi$ and $\Sigma$ and taken to be unconstrained complex superfields. The newly introduced superfields $\Psi$ and $\Gamma$ are chosen to be chiral and complex linear respectively,

$$\bar{D}_a \Psi = 0, \quad \bar{D}^2 \Gamma = 0.$$
into (2.1). On the other hand, we may again start from \( S_{\text{first-order}} \) and integrate out the auxiliary superfields \( X \) using the equations of motion for \( \Phi \) and \( \Sigma \),

\[
\Gamma = \frac{\partial L(X)}{\partial \Phi}, \quad \Psi = -\frac{\partial L(X)}{\partial \Sigma},
\]

as well as the conjugate equations. Under quite general assumptions, these equations may be uniquely solved by expressing the original variables \( X := \{\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}\} \) in terms of the dual ones \( X_D := \{\Psi, \bar{\Psi}, \Gamma, \bar{\Gamma}\} \). As a result, we end up with a dual formulation for the \( \sigma \)-model (2.1),

\[
S_D = \int d^4x d^4\theta L_D(\Psi, \bar{\Psi}, \Gamma, \bar{\Gamma}), \quad (2.14)
\]

where

\[
L_D(X_D) = L(X) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma} - \Gamma \Phi - \bar{\Gamma} \bar{\Phi}. \quad (2.15)
\]

In the derivation of (2.14), the original \( \sigma \)-model (2.1) was completely arbitrary. Now, we assume that the Lagrangian \( L \) in (2.1) is a solution of the duality equation (2.5). It is useful to consider a special finite duality rotation (2.3) with \( \lambda = \kappa = \pi/2 \),

\[
\Phi' = \frac{\partial L(X)}{\partial \Sigma}, \quad \Sigma' = \frac{\partial L(X)}{\partial \Phi}, \quad \frac{\partial L(X')}{\partial \Sigma'} = -\Phi, \quad \frac{\partial L(X')}{\partial \Phi'} = -\Sigma. \quad (2.16)
\]

The relation (2.8) becomes

\[
L(X) - \Phi \frac{\partial L(X)}{\partial \Phi} - \Sigma \frac{\partial L(X)}{\partial \Sigma} - \bar{\Phi} \frac{\partial L(X)}{\partial \bar{\Phi}} - \bar{\Sigma} \frac{\partial L(X)}{\partial \bar{\Sigma}} = L \left( \frac{\partial L(X)}{\partial \Sigma}, \frac{\partial L(X)}{\partial \Phi}, \frac{\partial L(X)}{\partial \bar{\Phi}}, \frac{\partial L(X)}{\partial \bar{\Sigma}} \right). \quad (2.17)
\]

Using the equations of motion (2.13) and the definition (2.15), eq. (2.17) yields

\[
L_D(X_D) = L(X_D), \quad (2.18)
\]

establishing the self-duality of the \( \sigma \)-model under the Legendre transformation.

### 3 Sigma model cousins of nonlinear electrodynamics

In this section, we consider a subclass of CCL \( \sigma \)-models (2.1) of the form

\[
S = \int d^4x d^4\theta L(\omega, \bar{\omega}), \quad (3.1)
\]
where the complex variable $\omega$ is defined by

$$\omega = \Phi\Phi - \Sigma\Sigma + i(\Phi\Sigma + \Phi\Sigma) = (\Phi + i\Sigma)(\Phi + i\Sigma).$$  \hfill (3.2)

For such $\sigma$-models, the condition for duality invariance, eq. (2.5), can be recast in the form

$$0 = \omega - \bar{\omega} - 4\omega \left(\frac{\partial L}{\partial \omega}\right)^2 + 4\bar{\omega} \left(\frac{\partial L}{\partial \bar{\omega}}\right)^2.$$

(3.3)

This is equivalent to the equation for duality invariance in nonlinear electrodynamics (see, e.g., [25] for a review) formulated in terms of the self-dual combination

$$\omega = -F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2} \left( \vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B} + 2i\vec{E} \cdot \vec{B} \right) = \frac{1}{2} (\vec{E} + i\vec{B}) \cdot (\vec{E} + i\vec{B})$$

(3.4)

and its conjugate $\bar{\omega}$. As a result, to any duality invariant solution $L(\omega, \bar{\omega})$ of the equation (3.3) in electrodynamics with $\omega = -F_{\alpha\beta}F^{\alpha\beta}$, there corresponds a duality invariant supersymmetric nonlinear $\sigma$-model with Lagrangian $L(\omega, \bar{\omega})$ in which $\omega$ is the superfield combination (3.2).

The trivial solution of the duality invariance condition (3.3) is

$$L = \frac{1}{2} (\omega + \bar{\omega}),$$

(3.5)

which in electrodynamics is Maxwell’s Lagrangian

$$L = \frac{1}{2} (\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}),$$

(3.6)

and in the $\sigma$-model case is

$$L = \Phi\Phi - \Sigma\Sigma.$$

(3.7)

In electrodynamics, a famous nonlinear solution to the duality invariance condition (3.3) is the Born-Infeld Lagrangian

$$L_{BI} = \frac{1}{g} \left( 1 - \frac{1}{2} \Delta \right)$$

(3.8)

with

$$\Delta = 1 - g(\omega + \bar{\omega}) + g^2 \frac{1}{4}(\omega - \bar{\omega})^2,$$

(3.9)

where $g$ is a coupling constant. The corresponding duality invariant supersymmetric nonlinear $\sigma$-model is defined by the Lagrangian (3.8) with

$$\Delta = 1 - 2g(\Phi\Phi - \Sigma\Sigma) - g^2(\Phi\Sigma + \Phi\Sigma)^2.$$

(3.10)

\footnote{Our definition of $\omega$, (3.4), differs in sign from that used in [25].}
Of course, not all supersymmetric Lagrangians $L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ satisfying the condition (2.5) for duality invariance can be expressed in the form $L(\omega, \tilde{\omega})$. An example is a Born-Infeld variant with $\Delta$ in (3.9) replaced by
\[
\tilde{\Delta} = 1 - 2g(\Phi\Phi - \bar{\Sigma}\bar{\Sigma}) - 4g^2\bar{\Phi}\Phi\bar{\Sigma}\Sigma .
\] (3.11)

For those duality invariant Lagrangians that can be expressed in the form $L(\omega, \tilde{\omega})$, the Lagrangian proves to obey two additional differential equations:
\[
0 = \frac{\partial L}{\partial \Phi} \Phi - \frac{\partial L}{\partial \bar{\Phi}} \bar{\Phi} - \frac{\partial L}{\partial \Sigma} \Sigma + \frac{\partial L}{\partial \bar{\Sigma}} \bar{\Sigma} ,
\] (3.12a)
\[
0 = \frac{\partial L}{\partial \bar{\Phi}} \bar{\Sigma} - \frac{\partial L}{\partial \Sigma} \Phi - \frac{\partial L}{\partial \bar{\Phi}} \bar{\Sigma} + \frac{\partial L}{\partial \Sigma} \bar{\Phi} .
\] (3.12b)

The former equation is the condition for invariance under $U(1)$ transformations
\[
\delta \Phi = i\lambda \Phi , \quad \delta \Sigma = -i\lambda \Sigma , \quad \lambda \in \mathbb{R}
\] (3.13)
which leave $\omega$ invariant. Equation (3.12b) it also expresses the invariance of $\omega$ under certain linear transformations: $\delta \Phi = i\lambda \bar{\Sigma}$ and $\delta \bar{\Sigma} = -i\lambda \Phi$. However such a transformation is not a symmetry of the theory under consideration, since it mixes a chiral scalar superfield with a complex linear superfield, and therefore does not respect the off-shell constraints.

### 4 Chiral formulation

As discussed in section 1, any nonlinear $\sigma$-model of the form (2.1) has a purely chiral formulation which is obtained by performing a superfield Legendre transformation that dualises the complex linear superfield $\Sigma$ and its conjugate $\bar{\Sigma}$ into a chiral scalar $\Psi$ and its conjugate $\bar{\Psi}$. It is worth recalling its derivation. Starting from the $\sigma$-model (2.1), we introduce a first-order action
\[
S_{\text{first-order}} = \int d^4xd^4\theta \left\{ L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma} \right\} ,
\] (4.1)
where $\Sigma$ is taken to be an unconstrained complex superfield, while the Lagrange multiplier $\Psi$ is chosen to be chiral,
\[
\bar{D}_{\alpha} \Psi = 0 .
\] (4.2)
The original CCL $\sigma$-model (2.1) is obtained from (4.1) by integrating out the Lagrange multipliers $\Psi$ and $\bar{\Psi}$. Instead, we can integrate out the auxiliary superfield $\Sigma$ and its conjugate $\bar{\Sigma}$ using the corresponding equation of motion

$$\frac{\partial L}{\partial \Sigma} + \Psi = 0$$

and the conjugate equation. This leads to the chiral formulation

$$S_{\text{chiral}} = \int d^4x d^4\theta K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) ,$$

where the corresponding Lagrangian $K$ is the Legendre transform of $L$,

$$K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma} .$$

The dual Lagrangian $K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ is the Kähler potential for a Kähler target space.

If the original Lagrangian $L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ obeys the duality equation (2.5), then the dual chiral action (4.4) possesses a manifest U(1) symmetry. This follows from (2.5), in conjunction with standard properties of Legendre transformations, including the equation of motion (4.3) and the following relation:

$$\frac{\partial L}{\partial \Phi} = \frac{\partial K}{\partial \Phi} .$$

The resulting dual version of eq. (2.5) is

$$0 = \frac{\partial K}{\partial \Psi} \Phi + \frac{\partial K}{\partial \bar{\Psi}} \bar{\Phi} - \frac{\partial K}{\partial \Phi} \Psi - \frac{\partial K}{\partial \bar{\Phi}} \bar{\Psi} .$$

This is the condition for the invariance of $K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ under the holomorphic U(1) transformations

$$\delta \Phi = -\lambda \Psi , \quad \delta \Psi = \lambda \Phi , \quad \lambda \in \mathbb{R} .$$

We conclude that the isometry group of the target Kähler space is nontrivial, since it contains the U(1) subgroup of holomorphic transformations (4.8). This result has several significant implications.

---

8 A direct analogy exists with the case of duality invariant systems of $(p - 1)$ forms and $(d - p - 1)$ forms in $d$ space-time dimensions, as discussed in section 8 of [25].

9 The chiral superfield $\Phi$ plays the role of a parameter in the context of the Legendre transformation described.
First of all, the holomorphic U(1) symmetry \[18\] leads to the existence of a conserved U(1) current contained in the real superfield

\[ J := i \left( \frac{\partial K}{\partial \Psi} \Phi - \frac{\partial K}{\partial \Phi} \Psi \right) = \bar{J} , \] (4.9)

which obeys eq. (2.10) on-shell. One can read off the expression (4.9) from (2.9) by using the standard properties of the Legendre transformation.

Secondly, consider constructing the most general U(1) invariant \(\sigma\)-model. It is described by a Kähler potential

\[ K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \mathcal{F}(\Phi^2 + \Psi^2, \bar{\Phi}^2 + \bar{\Psi}^2, \bar{\Phi}\Phi + \bar{\Psi}\Psi) , \] (4.10)

where \(\mathcal{F}(z, \bar{z}, w)\) is an arbitrary real function of three real variables. Performing the inverse Legendre transformation, we end up with the most general duality invariant \(\sigma\)-model (2.1). Therefore, the chiral formulation provides a generating formalism for U(1) duality invariant CCL \(\sigma\)-models.

Thirdly, it is well known \[27, 28\] that, given a Kähler manifold with a holomorphic Killing vector field, the \(\mathcal{N} = 1\) supersymmetric \(\sigma\)-model associated with this target space may be formulated in terms of a single real linear superfield, \(G = \bar{G}\), constrained by \(\bar{D}^2 G = 0\) (which describes the \(\mathcal{N} = 1\) tensor multiplet \[30\]) and a set of chiral scalars. In our case, such a formulation may be obtained as follows. We note that the chiral superfields

\[ \phi_\pm = \Phi \pm i\Psi \] (4.11)

transform under (4.8) as

\[ \delta \phi_\pm = \pm i \lambda \phi_\pm . \] (4.12)

We may introduce new local complex coordinates for the target space, \(\varphi\) and \(\chi\), that are defined as

\[ \varphi = \phi_+ \phi_- , \quad \phi_+ = e^{\chi} . \] (4.13a)

Their U(1) transformation laws are respectively

\[ \delta \varphi = 0 , \quad \delta \chi = i\lambda . \] (4.13b)

It follows from these transformation laws that, in terms of the new coordinates introduced, the Kähler potential takes the form

\[ K = K(\varphi, \bar{\varphi}, \chi + \bar{\chi}) . \] (4.14)
Next, the (anti) chiral variables $\chi$ and $\bar{\chi}$ can be dualised into a real linear superfield $G$ using the standard procedure \cite{30,27}. The resulting theory is described by a superfield Lagrangian $L(\varphi, \bar{\varphi}, G)$.

It should be remarked that there is a considerable freedom in the choice of chiral superfields $\varphi$ and $\chi$ with the U(1) transformation laws \cite[(4.13b)]{4.13}. Instead of the variables \cite{(4.13a)}, one equally well may deal with chiral superfields $\varphi'$ and $\chi'$ defined by

$$
\varphi \rightarrow \varphi' = f(\varphi), \quad \chi \rightarrow \chi' = \chi + g(\varphi),
$$

with $f(\varphi)$ and $g(\varphi)$ holomorphic functions. Implementing such a holomorphic field redefinition changes the Kähler potential \cite[(4.14)]{14}, $K \rightarrow K' \equiv K'(\varphi', \bar{\varphi'}, \chi' + \bar{\chi}') = K(\varphi, \bar{\varphi}, \chi + \bar{\chi})$, as well as leads to a modified chiral-tensor Lagrangian $L'(\varphi', \bar{\varphi'}, G)$.

A large family of the U(1) invariant chiral $\sigma$-models discussed in this section are in fact $\mathcal{N} = 2$ supersymmetric. For this to hold, the target space must be hyper Kähler \cite{31}, and thus the corresponding Ricci tensor must vanish.\footnote{Any Ricci-flat Kähler manifold of real dimension four is hyper Kähler and vice versa, see e.g. \cite{32}.} Using the condensed notation $\phi^i = (\Phi, \Psi)$ and $\bar{\phi}^i = (\bar{\Phi}, \bar{\Psi})$, the condition for Ricci-flatness is the Monge-Ampère equation (see, e.g., \cite{32})

$$
\partial_k \partial_l \ln \det \left( \partial_i \partial_j K(\phi, \bar{\phi}) \right) = 0 .
$$

The Killing vector

$$
\kappa = i \left( \frac{\partial}{\partial \chi} - \frac{\partial}{\partial \bar{\chi}} \right)
$$

is holomorphic with respect to the diagonal complex structure, but in general it is not triholomorphic.\footnote{It may be a Killing vector that rotates the two-sphere of complex structures, and thus necessarily leaves one of the complex structures invariant \cite{29}. Such a Killing vector naturally originates in the case of $\mathcal{N} = 2$ supersymmetric $\sigma$-models on cotangent bundles of Kähler manifolds \cite{31,35}. Moreover, the hyper Kähler target spaces of $\mathcal{N} = 2$ supersymmetric sigma models in four-dimensional anti-de Sitter space must possess such a Killing vector \cite{33,34}.} However, as demonstrated in \cite{28}, if $\kappa$ is triholomorphic, one can associate with this hyper Kähler manifold an off-shell $\mathcal{N} = 2$ supersymmetric theory describing a self-interacting $\mathcal{N} = 2$ tensor multiplet \cite{27,35}. The $\mathcal{N} = 2$ tensor multiplet theory is dynamically equivalent to the chiral $\sigma$-model constructed.

It is possible to act in a reverse order. Let us start from the most general off-shell $\mathcal{N} = 2$ supersymmetric $\sigma$-model describing a self-interacting $\mathcal{N} = 2$ tensor multiplet...
Formulated in $\mathcal{N} = 1$ superspace, the action is

$$S_{\text{tensor}} = \int d^4x d^4\theta L(\varphi, \bar{\varphi}, G) , \quad (4.18a)$$

where $\varphi$ is a chiral scalar, $\bar{D}_a \varphi = 0$, and $G = \bar{G}$ is a real linear scalar, $\bar{D}^2 G = 0$. The fact that the theory is $\mathcal{N} = 2$ supersymmetric means that the Lagrangian $L$ has to obey the three-dimensional Laplace equation [27],

$$\left( \frac{\partial^2}{\partial \varphi \partial \bar{\varphi}} + \frac{\partial^2}{\partial G \partial \bar{G}} \right) L = 0 . \quad (4.18b)$$

The theory (4.18a) possesses a chiral formulation obtained by dualising $G$ into a chiral scalar $\chi$ and its conjugate $\bar{\chi}$ [27]. The dual chiral action is

$$S = \int d^4x d^4\theta K(\varphi, \bar{\varphi}, \chi + \bar{\chi}) , \quad (4.19)$$

and it is manifestly U(1) invariant. What is special about this particular complex parametrisation of the hyper Kähler target space is the unimodularity condition [27]

$$\frac{\partial^2 K}{\partial \chi \partial \bar{\chi}} \frac{\partial^2 K}{\partial \varphi \partial \bar{\varphi}} - \frac{\partial^2 K}{\partial \chi \partial \bar{\varphi}} \frac{\partial^2 K}{\partial \varphi \partial \bar{\chi}} = 1 , \quad (4.20)$$

which is a stronger version of eq. (4.16). Next, this $\sigma$-model (4.19) can be reformulated in terms of the chiral superfields $\Phi$ and $\Psi$ defined according to eqs. (4.11) and (4.12). The resulting nonlinear $\sigma$-model $S_{\text{chiral}}$, eq. (4.4), can equivalently be realised as a U(1) duality invariant CCL $\sigma$-model.

As a result, we have established a correspondence between self-interacting $\mathcal{N} = 2$ supersymmetric tensor multiplet models and certain U(1) duality invariant CCS $\sigma$-models (2.1) with hidden $\mathcal{N} = 2$ supersymmetry. The condition (4.20) should be imposed in order to fix the freedom (4.15) when going from the CCS formulation to the tensor one.

As noted in section 2, if the original Lagrangian $L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ can be expressed in the form $L(\omega, \bar{\omega})$, then $L$ obeys two differential equations (3.12). In the chiral formulation, the first of these equations, (3.12a), turns into

$$\frac{\partial K}{\partial \Phi} \frac{\partial K}{\partial \bar{\Phi}} + \frac{\partial K}{\partial \Psi} = \frac{\partial K}{\partial \bar{\Phi}} \frac{\partial K}{\partial \bar{\Psi}} + \frac{\partial K}{\partial \bar{\Psi}} . \quad (4.21)$$

This is the condition for the invariance of $K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ under additional U(1) transformations

$$\delta \Phi = i \alpha \Phi, \quad \delta \Psi = i \alpha \Psi, \quad \alpha \in \mathbb{R} . \quad (4.22)$$
Thus the isometry group of the $\sigma$-model target space includes the group $U(1) \times U(1)$ of transformations (4.20) and (4.22). These symmetries imply that the Kähler potential can be represented as a function of two real variables 

$$K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\omega_+, \omega_-),$$  \hspace{1cm} (4.23)

where

$$\omega_\pm = \Phi \bar{\Phi} + \Psi \bar{\Psi} \pm i (\Phi \Psi - \bar{\Phi} \bar{\Psi}).$$  \hspace{1cm} (4.24)

The simplest way to show this is to switch from the chiral scalars $\Phi$ and $\Psi$ to the new chiral variables (1.11). The Abelian transformations (4.20) and (4.22) can be realised on these variables as

$$\delta \phi_+ = i \lambda \phi_+ , \quad \delta \phi_- = 0 ,$$

$$\delta \phi_+ = 0 , \quad \delta \phi_- = i \lambda \phi_- ,$$ \hspace{1cm} (4.25a)

with real parameters $\lambda_\pm \in \mathbb{R}$. Since the Kähler potential must be invariant under the transformations (4.25), we conclude that $K = K(\phi_+ \bar{\phi}_+, \phi_- \bar{\phi}_-)$. It remains to note that the real variables (4.24) can be factorized in the form

$$\omega_+ = \phi_+ \bar{\phi}_+ , \quad \omega_- = \phi_- \bar{\phi}_-. $$  \hspace{1cm} (4.26)

In particular, they are invariant under the transformations (4.25).

So far we have only derived the implications of (3.12a) in the dual chiral representation of duality invariant CCL models (2.1). The counterpart of equation (3.12b) in the dual chiral formulation is

$$0 = \Phi \bar{\Psi} - \bar{\Phi} \Psi + \frac{\partial K}{\partial \Phi} \frac{\partial K}{\partial \bar{\Psi}} - \frac{\partial K}{\partial \bar{\Phi}} \frac{\partial K}{\partial \Psi}. $$  \hspace{1cm} (4.27)

By comparison with (2.5), this looks like a condition for duality invariance in the dual $\Phi$-$\Psi$ sector. Its origin is the formal symmetry $\delta \Phi = i \lambda \Sigma$ and $\delta \Sigma = -i \lambda \Phi$, in the $\Phi$-$\Sigma$ sector, just as the duality equation (2.5) in the $\Phi$-$\Sigma$ sector can be considered as being a consequence of U(1) symmetry (4.7) in the chiral $\Phi$-$\Psi$ sector via a Legendre transform.

In terms of the real variables (4.24), the invariance condition (4.27) for the Kähler potential (4.23) can be expressed in the form

$$0 = \omega_+ - \omega_- - 4 \omega_+ \left( \frac{\partial K}{\partial \omega_+} \right)^2 + 4 \omega_- \left( \frac{\partial K}{\partial \omega_-} \right)^2. $$  \hspace{1cm} (4.28)

This is structurally of the same form as the duality invariance condition (3.3) for nonlinear electrodynamics; however, here $\omega_\pm$ are real variables, whereas in (3.3), $\omega$ is a complex.
A general solution of equation (4.28) has the form

\[ K(\omega_+, \omega_-) = \frac{1}{2} \bar{\phi}_+ \phi_+ + \frac{1}{2} \bar{\phi}_- \phi_- + \frac{1}{2} \sum_{m,n=1}^{\infty} C_{m,n} (\bar{\phi}_+ \phi_+)^m (\bar{\phi}_- \phi_-)^n, \]  

with \( C_{m,n} \) real coefficients. Equation (4.28) proves to determine all the coefficients \( C_{m,n} \) with \( m \neq n \) in terms of those with \( m = n \), with the latter being completely arbitrary. As a result, the general solution of (4.28) involves an arbitrary real function of a real argument, \( f(x) = \sum C_{n,n} x^n \). The situation is completely analogous to that in self-dual nonlinear electrodynamics, see [25] for a review.

It is interesting that equation (4.28) does not allow higher-order Kähler-like contributions to the Kähler potential of the form \( (\bar{\phi}_+ \phi_+)^n \), with \( n > 1 \) (and similarly in the \( \phi_- \) sector). This may be interpreted as a non-renormalisation theorem that keeps the tree-level kinetic term \( \omega_+ = \bar{\phi}_+ \phi_+ \) for the field \( \phi_+ \) and \( \bar{\phi}_+ \) protected against “quantum” nonlinear corrections.

5 Dual Born-Infeld type solutions

As discussed in section 3, the Lagrangian

\[ L_{BI}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \frac{1}{g} \left( 1 - \Delta \frac{1}{2} \right) \]  

with

\[ \Delta = 1 - 2g(\bar{\Phi}\Phi - \bar{\Sigma}\Sigma) - g^2(\Phi\Sigma + \bar{\Phi}\bar{\Sigma})^2 \]  

is duality invariant. When expressed in terms of the the complex variables \( \omega \) and \( \bar{\omega} \) defined in (3.2), the functional form of this Lagrangian is the same as that for the famous Born-Infeld action for nonlinear electrodynamics.

Dualisation of the complex linear superfields \( \Sigma \) in favour of chiral scalar superfields \( \Psi \) is via the the Legendre transform

\[ L_D(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{g} \left( 1 - \Delta \frac{1}{2} \right) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma}. \]  

Eliminating \( \Sigma \) and \( \bar{\Sigma} \) by their equations of motion, we obtain the implicit equation

\[ \begin{pmatrix} \Sigma \\ \bar{\Sigma} \end{pmatrix} = \Delta \frac{1}{2} (1 - A)^{-1} \begin{pmatrix} \bar{\Psi} \\ \Psi \end{pmatrix}, \]  

\[ 14 \]
where
\[ A = g \begin{pmatrix} \bar{\Phi}\Phi & \bar{\Psi}\Psi \\ \Phi\Phi & \bar{\Phi}\Phi \end{pmatrix} \].

(5.5)

Substituting into (5.3), the dual Lagrangian is
\[ L_D(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = 1 - (1 - 2g \bar{\Phi}\Phi)^{1/2} (1 - g\alpha + g^2\beta^2)^{-1/2} (1 - g\gamma) \]

(5.6)

with
\[
\alpha = (\Psi^\dagger \Psi) (1 - A)^{-2} \left( \bar{\Psi} \Psi \right), \quad \beta = (\Phi^\dagger \Phi) (1 - A)^{-1} \left( \bar{\Psi} \Psi \right), \]

\[
\gamma = (\Psi^\dagger \bar{\Psi}) (1 - A)^{-1} \left( \bar{\Psi} \Psi \right).
\]

(5.7)

(5.8)

Representing the matrix \( A \) in the form
\[ A = g \begin{pmatrix} \bar{\Phi} \\ \Phi \end{pmatrix} \begin{pmatrix} \Phi & \bar{\Phi} \end{pmatrix} \]

(5.9)

allows \( \alpha, \beta \) and \( \gamma \) to be expressed as
\[
\alpha = 2\bar{\Psi}\Psi + 2(1 - 2g \bar{\Phi}\Phi)^{-2} (\bar{\Psi}\Phi + \bar{\Phi}\Psi)^2,
\]

\[
\beta = (1 - 2g \bar{\Phi}\Phi)^{-1} (\bar{\Psi}\Phi + \bar{\Phi}\Psi),
\]

\[
\gamma = 2\bar{\Psi}\Psi + (1 - 2g \bar{\Phi}\Phi)^{-1} (\bar{\Psi}\Phi + \bar{\Phi}\Psi)^2.
\]

(5.10)

(5.11)

(5.12)

Substituting into (5.6) yields
\[ L_D(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{g} (1 - \Delta_D^{1/2}) \]

(5.13)

where
\[ \Delta_D = 1 - 2g (\bar{\Phi}\Phi + \bar{\Psi}\Psi) - g^2 (\bar{\Phi}\Phi - \bar{\Psi}\Psi)^2. \]

(5.14)

This can be expressed as in terms of the real variables \( \omega_\pm \), defined in (4.24), as
\[ \Delta_D = 1 - g (\omega_+ + \omega_-) - \frac{g^2}{4} (\omega_+ - \omega_-)^2, \]

(5.15)

which exhibits a Born-Infeld type functional form for the dual chiral Lagrangian. By construction, this dual Lagrangian is a solution of (4.28).

Using similar techniques for the variant Born-Infeld type Lagrangian involving \( \bar{\Delta} \) defined in (3.11), the dual Lagrangian is
\[ \bar{L}_D = \frac{1}{g} (1 - \bar{\Delta}_D^{1/2}), \]

(5.16)

with
\[ \bar{\Delta}_D = 1 - 2g (\bar{\Phi}\Phi + \bar{\Psi}\Psi). \]

(5.17)
6 Conclusion

Recently, there has been a revival of interest in the duality invariant dynamical systems of Abelian vector fields, see [36, 37, 38, 39] and references therein. This interest was mainly inspired by the desire to achieve a better understanding of the UV properties of extended supergravity theories. The recent studies have concentrated, in particular, on the following two problems: (i) consistent deformations of duality invariant systems [36, 37, 38]; and (ii) the fate of duality invariance in quantum theory [39]. Duality invariant CCL σ-models may shed some light on both of these issues.

Let us first briefly discuss the problem (i). In the case of nonlinear electrodynamics, the condition of duality invariance (3.3) is a nonlinear differential equation on the Lagrangian. This means that the problem of consistent deformations of duality invariant theories is rather nontrivial. In order to develop a systematic procedure to generate duality invariant theories, the authors of [36, 37] put forward the so-called “twisted self-duality constraint” and provided simple perturbative applications of this scheme. However, it has been demonstrated by Ivanov and Zupnik [40] that the non-supersymmetric constructions of [36, 37] naturally originate within the more general approach developed a decade earlier in [41, 42]. Specifically, the twisted self-duality constraint corresponds to an equation of motion in the approach of [41, 42]. The approach of [41, 42] has also been extended [43, 44] to the cases of duality invariant \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) (locally) supersymmetric vector multiplet models [45, 25, 46, 47]. The main idea of the generating formalism [41, 42] consists in reformulating the U(1) duality invariant models by introducing auxiliary variables in such a way that the self-interaction is manifestly U(1) invariant. The original theory is obtained by integrating out the auxiliary variables. Since any choice of U(1) invariant self-interaction proves to lead to a U(1) duality invariant model, the Ivanov-Zupnik approach [41, 42] is an efficient scheme to generate duality invariant systems.

In the case of U(1) duality invariant CCL σ-models, there is no need to introduce auxiliary variables as a mechanism to generate such dynamical systems. As we demonstrated in section 4, the dual chiral representation plays the role of such a generating formalism.

Regarding the problem (ii) raised e.g. in [39], the main issue is that duality invariance is not a manifest symmetry of the action. As a result, the precise realisation of this symmetry at the S-matrix level, or in the framework of the effective action, requires an additional definition. This issue is nontrivial, for instance, in the case of the duality invariant models for nonlinear electrodynamics (barring the non-renormalizability of such models). In the case of duality invariant CCL σ-models, we have a natural way out. These
Theories possess the dual chiral formulation in which U(1) duality symmetry turns into a manifest U(1) symmetry. Switching to the dual chiral formulation requires us to make use of the superfield Legendre transformation described in section 4, and the latter can naturally be implemented within the path integral.

We see that there is a conceptual difference between the two families of U(1) duality invariant theories: (i) models for nonlinear electrodynamics and (ii) CCL $\sigma$-models. This difference is that only for the latter family do we have the ability to realise continuous duality symmetries as manifest U(1) symmetries in a dual formulation of a theory. Still, there is a way to relate these two types of theories. Specifically, we can start with a four-dimensional duality invariant model for nonlinear electrodynamics and dimensionally reduce it to three dimensions resulting in a duality invariant theory involving a vector field and a scalar field. By Legendre transformation, this theory can be cast purely in terms of scalar fields or purely in terms of vector fields, each involving a manifest U(1) symmetry that has its origin in the duality invariance of the original theory.

In this paper, our discussion was restricted to $\mathcal{N} = 1$ SUSY in four dimensions. Plain dimensional reduction leads to analogous results for $\mathcal{N} = 2$ SUSY in three dimensions and $\mathcal{N} = (2,2)$ SUSY in two dimensions. In fact, it is well known that the two-dimensional $\mathcal{N} = (2,2)$ Poincaré supersymmetry allows the existence of new superfield types that are impossible in higher dimensions: twisted chiral and semi-chiral. In terms of such supermultiplets one may define new families of duality invariant $\sigma$-models.

We have also restricted our discussion to the case of U(1) duality invariant CCL $\sigma$-models with a single chiral multiplet and a single complex linear multiplet. Inclusion of $n > 1$ chiral–complex linear doublets is expected to allow non-Abelian duality groups. It is also of interest to consider gauging the isometries in the purely chiral chiral theory and investigate the consequences in the dual chiral–complex linear theory.

Acknowledgement: The work of SK is supported in part by the Australian Research Council.

---

12 This is a specific example of more general duality invariant systems of $(p-1)$ forms and $(d-p-1)$ forms in $d$ space-time dimensions, as discussed in section 8 of [25].

13 CCL $\sigma$-models with non-Abelian duality groups may be obtained by generalising the construction of the general duality invariant systems of $(p-1)$ forms and $(d-p-1)$ forms in $d$ space-time dimensions, see section 8 of [25]. Alternatively, one may start from a chiral $\sigma$-model with a manifest non-Abelian symmetry group and then switch to the dual chiral-complex linear formulation.
A The duality equation

Here, we adapt standard arguments (as reviewed in [25]) from nonlinear electrodynamics to derive an integrability condition associated with the consistency of the definition of $L'(X')$ via equations (2.3a) and (2.3b) that requires $\kappa = \lambda$. We then derive the requirement (2.5) for duality invariance of nonlinear $\sigma$-models (2.1) under the duality rotations (2.3). As in earlier sections, we use the notation $X$ for the argument $\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}$.

In infinitesimal form, the U(1) duality rotations (2.3) preserving the constraints and the equations of motion are

\[
\frac{\partial L'(X')}{\partial \Sigma'} = \frac{\partial L(X)}{\partial \Sigma} - \lambda \Phi', \quad \Phi' = \Phi + \lambda \frac{\partial L(X)}{\partial \Sigma}, \tag{A.1}
\]

and

\[
\frac{\partial L'(X')}{\partial \Phi'} = \frac{\partial L(X)}{\partial \Phi} - \kappa \Sigma', \quad \Sigma' = \Sigma + \kappa \frac{\partial L(X)}{\partial \Phi}. \tag{A.2}
\]

Using the notation

\[
\delta X \frac{\partial L(X)}{\partial X} \equiv \delta \Phi \frac{\partial L(X)}{\partial \Phi} + \delta \bar{\Phi} \frac{\partial L(X)}{\partial \bar{\Phi}} + \delta \Sigma \frac{\partial L(X)}{\partial \Sigma} + \delta \bar{\Sigma} \frac{\partial L(X)}{\partial \bar{\Sigma}}, \tag{A.3}
\]

then to first order in $\delta X$,

\[
L'(X') = L(X) + \Delta L(X) + \delta X \frac{\partial L'(X)}{\partial X}, \tag{A.4}
\]

where

\[
\Delta L(X) = L'(X) - L(X). \tag{A.5}
\]

Using the chain rule to convert derivatives with respect to $\Sigma'$ into derivatives with respect to $\Phi, \Sigma, \bar{\Phi}$ and $\bar{\Sigma}$, and retaining only terms linear in the infinitesimal parameters $\kappa$ and $\lambda$, the first equation in (A.1) becomes

\[
-\lambda \Phi = \frac{\partial}{\partial \Sigma} \left( \Delta L(X) + \delta X \frac{\partial L(X)}{\partial X} \right) - \kappa \frac{\partial^2 L(X)}{\partial \Sigma \partial \Phi} \frac{\partial L(X)}{\partial \Sigma} - \kappa \frac{\partial^2 L(X)}{\partial \Sigma \partial \bar{\Phi}} \frac{\partial L(X)}{\partial \bar{\Sigma}} - \lambda \frac{\partial^2 L(X)}{\partial \Sigma^2} \frac{\partial L(X)}{\partial \Phi} - \lambda \frac{\partial^2 L(X)}{\partial \Sigma \partial \bar{\Sigma}} \frac{\partial L(X)}{\partial \bar{\Phi}}. \tag{A.6}
\]

The left hand side of this relation can be expressed as the derivative $\frac{\partial}{\partial \Sigma}(-\lambda \Phi \Sigma)$, so consistency requires that the right hand side can be expressed as a derivative with respect to $\Sigma$. This is only so if $\kappa = \lambda$, in which case (A.6) is

\[
0 = \frac{\partial}{\partial \Sigma} \left( \Delta L(X) + \lambda \Phi \Sigma + \delta X \frac{\partial L(X)}{\partial X} - \lambda \frac{\partial L(X)}{\partial \Phi} \frac{\partial L(X)}{\partial \Sigma} - \lambda \frac{\partial L(X)}{\partial \Phi} \frac{\partial L(X)}{\partial \bar{\Sigma}} \right). \tag{A.7}
\]
Integrating and requiring reality of $L(X)$ yields
\[
\Delta L(X) = -\lambda \Phi \Sigma - \lambda \bar{\Phi} \bar{\Sigma} - \lambda \frac{\partial L(X)}{\partial \Phi} \frac{\partial L(X)}{\partial \Sigma} - \lambda \frac{\partial L(X)}{\partial \bar{\Phi}} \frac{\partial L(X)}{\partial \bar{\Sigma}}.
\] (A.8)

Imposing the requirement (2.4) for duality invariance means that $\Delta L(X)$ defined in (A.5) vanishes. Equation (A.8) then yields the condition (2.5) for duality invariance of the Lagrangian $L(X)$.

References

[1] U. Lindstrøm and M. Roček, “New hyperkähler metrics and new supermultiplets,” Commun. Math. Phys. 115, 21 (1988).

[2] U. Lindstrøm and M. Roček, “Properties of hyperkähler manifolds and their twistor spaces,” Commun. Math. Phys. 293, 257 (2010) [arXiv:0807.1368 [hep-th]].

[3] S. M. Kuzenko, “Lectures on nonlinear sigma models in projective superspace,” J. Phys. A 43, 443001 (2010) [arXiv:1004.0880 [hep-th]].

[4] S. J. Gates Jr. and S. M. Kuzenko, “The CNM-hypermultiplet nexus,” Nucl. Phys. B 543, 122 (1999) [arXiv:hep-th/9810137].

[5] S. J. Gates Jr. and S. M. Kuzenko, “4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kähler manifolds,” Fortsch. Phys. 48, 115 (2000) [arXiv:hep-th/9903013].

[6] M. Arai and M. Nitta, “Hyper-Kähler sigma models on (co)tangent bundles with SO(n) isometry,” Nucl. Phys. B 745, 208 (2006) [arXiv:hep-th/0602277].

[7] M. Arai, S. M. Kuzenko and U. Lindstrøm, “Hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace,” JHEP 0702, 100 (2007) [arXiv:hep-th/0612174].

[8] M. Arai, S. M. Kuzenko and U. Lindstrøm, “Polar supermultiplets, Hermitian symmetric spaces and hyperkähler metrics,” JHEP 0712, 008 (2007) [arXiv:0709.2633 [hep-th]].

[9] S. M. Kuzenko and J. Novak, “Chiral formulation for hyperkähler sigma-models on cotangent bundles of symmetric spaces,” JHEP 0812, 072 (2008) [arXiv:0811.0218 [hep-th]].

[10] S. M. Kuzenko, “N = 2 supersymmetric sigma models and duality,” JHEP 1001, 115 (2010) [arXiv:0910.5771 [hep-th]].

[11] S. M. Kuzenko, “Comments on N = 2 supersymmetric sigma models in projective superspace,” J. Phys. A 45, 095401 (2012) [arXiv:1110.4298 [hep-th]].

[12] D. Butter, S. M. Kuzenko, U. Lindstrøm and G. Tartaglino-Mazzucchelli, “Extended supersymmetric sigma models in AdS$_4$ from projective superspace,” JHEP 1205, 138 (2012) [arXiv:1203.5001 [hep-th]].
[13] B. B. Deo and S. J. Gates Jr., “Comments on non-minimal N=1 scalar multiplets,” Nucl. Phys. B 254, 187 (1985).

[14] B. Zumino, “Superspace,” in Unification of the Fundamental Particle Interactions, S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Eds.), Plenum Press, 1980, p. 101.

[15] S. J. Gates Jr. and W. Siegel, “Understanding constraints in superspace formulations of supergravity,” Nucl. Phys. B 163, 519 (1980); “Variant superfield representations,” Nucl. Phys. B 187, 389 (1981).

[16] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, “Superspace, or one thousand and one lessons in supersymmetry,” Front. Phys. 58, 1 (1983) [arXiv:hep-th/0108200].

[17] B. Zumino, “Supersymmetry and Kähler manifolds,” Phys. Lett. B 87, 203 (1979).

[18] C. M. Hull, A. Karlhede, U. Lindström and M. Roček, “Nonlinear sigma models and their gauging in and out of superspace,” Nucl. Phys. B 266, 1 (1986).

[19] S. M. Kuzenko, U. Lindström, R. von Unge, “New supersymmetric sigma-model duality,” JHEP 1010, 072 (2010) [arXiv:1006.2299 [hep-th]].

[20] M. K. Gaillard and B. Zumino, “Duality rotations for interacting fields,” Nucl. Phys. B 193, 221 (1981).

[21] G. W. Gibbons and D. A. Rasheed, “Electric-magnetic duality rotations in nonlinear electrodynamics,” Nucl. Phys. B 454, 185 (1995) [arXiv:hep-th/9506035].

[22] G. W. Gibbons and D. A. Rasheed, “SL(2,R) invariance of non-linear electrodynamics coupled to an axion and a dilaton,” Phys. Lett. B 365, 46 (1996) [hep-th/9509141].

[23] M. K. Gaillard and B. Zumino, “Self-duality in nonlinear electromagnetism,” in Supersymmetry and Quantum Field Theory, J. Wess and V. P. Akulov (Eds.), Springer Verlag, 1998, p. 121 [arXiv:hep-th/9705226].

[24] M. K. Gaillard and B. Zumino, “Nonlinear electromagnetic self-duality and Legendre transformations,” in Duality and Supersymmetric Theories, D.I. Olive and P.C. West eds., Cambridge University Press, 1999, p. 33 [hep-th/9712103].

[25] S. M. Kuzenko and S. Theisen, “Nonlinear self-duality and supersymmetry,” Fortsch. Phys. 49, 273 (2001) [arXiv:hep-th/0007231].

[26] P. Aschieri, S. Ferrara and B. Zumino, “Duality rotations in nonlinear electrodynamics and in extended supergravity,” Riv. Nuovo Cim. 31, 625 (2008) [arXiv:0807.4039 [hep-th]].

[27] U. Lindström and M. Roček, “Scalar tensor duality and N=1, 2 nonlinear sigma models,” Nucl. Phys. B 222, 285 (1983).

[28] P. S. Howe, A. Karlhede, U. Lindström and M. Roček, “The geometry of duality,” Phys. Lett. B 168, 89 (1986).

[29] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkähler metrics and supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[30] W. Siegel, “Gauge spinor superfield as a scalar multiplet,” Phys. Lett. B 85, 333 (1979).
[31] L. Alvarez-Gaumé and D. Z. Freedman, “Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model,” Commun. Math. Phys. 80, 443 (1981).

[32] A. L. Besse, Einstein Manifolds, Springer, Berlin, 1987.

[33] D. Butter and S. M. Kuzenko, “N=2 supersymmetric sigma-models in AdS,” Phys. Lett. B 703, 620 (2011) [arXiv:1105.3111 [hep-th]].

[34] D. Butter and S. M. Kuzenko, “The structure of N=2 supersymmetric nonlinear σ-models in AdS₄,” JHEP 1111, 080 (2011) [arXiv:1108.5290 [hep-th]].

[35] A. Karlhede, U. Lindström and M. Roček, “Self-interacting tensor multiplets in N = 2 superspace,” Phys. Lett. B 147, 297 (1984).

[36] G. Bossard and H. Nicolai, “Counterterms vs. dualities,” JHEP 1108, 074 (2011) [arXiv:1105.1273 [hep-th]].

[37] J. J. M. Carrasco, R. Kallosh and R. Roiban, “Covariant procedures for perturbative non-linear deformation of duality-invariant theories,” Phys. Rev. D 85, 025007 (2012) [arXiv:1108.4390 [hep-th]].

[38] J. Broedel, J. J. M. Carrasco, S. Ferrara, R. Kallosh and R. Roiban, “N=2 supersymmetry and U(1)-duality,” Phys. Rev. D 85, 125036 (2012) [arXiv:1202.0014 [hep-th]].

[39] R. Roiban and A. A. Tseytlin, “On duality symmetry in perturbative quantum theory,” JHEP 1210, 099 (2012) [arXiv:1205.0176 [hep-th]].

[40] E. A. Ivanov and B. M. Zupnik, “Bispinor auxiliary fields in duality-invariant electrodynamics revisited,” Phys. Rev. D 87, 065023 (2013) [arXiv:1212.6637 [hep-th]].

[41] E. A. Ivanov and B. M. Zupnik, “New representation for Lagrangians of self-dual nonlinear electrodynamics,” hep-th/0202203.

[42] E. A. Ivanov and B. M. Zupnik, “New approach to nonlinear electrodynamics: Dualities as symmetries of interaction,” Phys. Atom. Nucl. 67, 2188 (2004) [Yad. Fiz. 67, 2212 (2004)] [hep-th/0303192].

[43] S. M. Kuzenko, “Duality rotations in supersymmetric nonlinear electrodynamics revisited,” JHEP 1303, 153 (2013) [arXiv:1301.5194 [hep-th]].

[44] E. Ivanov, O. Lechtenfeld and B. Zupnik, “Auxiliary superfields in N=1 supersymmetric self-dual electrodynamics,” arXiv:1303.5962 [hep-th].

[45] S. M. Kuzenko and S. Theisen, “Supersymmetric duality rotations,” JHEP 0003, 034 (2000) [arXiv:hep-th/0001068].

[46] S. M. Kuzenko and S. A. McCarthy, “Nonlinear self-duality and supergravity,” JHEP 0302, 038 (2003) [hep-th/0212039].

[47] S. M. Kuzenko, “Nonlinear self-duality in N=2 supergravity,” JHEP 1206, 012 (2012) [arXiv:1202.0120 [hep-th]].

[48] S. J. Gates Jr., C. M. Hull and M. Roček, “Twisted multiplets and new supersymmetric nonlinear sigma models,” Nucl. Phys. B 248, 157 (1984).

[49] T. Buscher, U. Lindström and M. Roček, “New supersymmetric sigma models with Wess-Zumino terms,” Phys. Lett. B 202, 94 (1988).