FAST EVALUATION OF B-SPLINE FUNCTIONS AND RENDERING OF MULTIPLE B-SPLINE CURVES USING LINEAR-TIME ALGORITHM FOR COMPUTING THE BERNSTEIN-BÉZIER COEFFICIENTS OF B-SPLINE FUNCTIONS

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Abstract. A new differential-recurrence relation for the B-spline functions of the same degree is proved. From this relation, a recursive method of computing the coefficients of B-spline functions of degree \( m \) in the Bernstein-Bézier form is derived. Its complexity is proportional to the number of coefficients in the case of coincident boundary knots. This means that, asymptotically, the algorithm is optimal. In other cases, the complexity is increased by at most \( O(m^3) \). When the Bernstein-Bézier coefficients of B-spline basis functions are known, it is possible to compute any B-spline function in linear time with respect to its degree by performing the geometric algorithm proposed recently by the authors. Using a similar approach, one can also convert a \( d \)-dimensional B-spline curve of degree \( m \) over one knot span to a Bézier curve in \( O(m^2) \) time and then evaluate it in \( O(m^2d) \) time. Since one only needs to convert each knot span once, this algorithm scales well when evaluating the B-spline curve at multiple points, e.g., in order to render it. When evaluating many B-spline curves at multiple points, such approach has lower computational complexity than using the de Boor-Cox algorithm. The problem of finding the coefficients of the B-spline functions in the power basis can be solved similarly.

Key word. B-spline functions, Bernstein-Bézier form, power form, recurrence relations, Bézier curves, B-spline curves, de Boor-Cox algorithm.

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1. Introduction. The family of Bernstein (basis) polynomials was used by S. N. Bernstein in 1912 in his constructive proof of the Weierstrass approximation theorem, which states that any continuous function can be approximated over a closed and bounded interval with arbitrary precision using polynomials. For more details, see [8, §10.3]. They came into prominence, however, half a century later when they became a basis for a particular family of parametric curves — Bézier curves.

For a fixed \( n \in \mathbb{N} \) and \( i \in \{0, 1, \ldots, n\} \), \( B_n^i \) is the \( i \)th Bernstein (basis) polynomial of degree \( n \) given by the formula

\[
B_n^i(t) := \binom{n}{i} t^i (1-t)^{n-i}.
\]

Remark 1.1. In the sequel, the convention is applied that \( B_n^i \equiv 0 \) if \( i < 0 \) or \( i > n \).

The Bernstein polynomial \( B_n^i \) \((0 \leq i \leq n)\) is non-negative and has exactly one maximum value in the interval \([0, 1]\) (except for the case \( n = 0 \)). It is also important that all Bernstein polynomials of the same degree give the partition of unity, i.e., \( \sum_{i=0}^{n} B_n^i(t) \equiv 1 \). These properties make the Bernstein basis a very powerful tool in the approximation theory, numerical analysis or in CAGD where one can use it, for example, to define a family of parametric curves called Bézier curves.

For any \( t \in \mathbb{R} \), Bernstein polynomials satisfy the recurrence relations connecting
the polynomials of two subsequent degrees:

\[ B^0_k(t) = tB^0_{k-1}(t) + (1-t)B^0_{k-1}(t), \]
\[ B^n_k(t) = \frac{n-k+1}{n+1} B^{n+1}_k(t) + \frac{k+1}{n+1} B^{n+1}_{k+1}(t) \quad (0 \leq k \leq n). \]

It is also well-known that

\[ \left( B^0_k(t) \right)' = n \left( B^0_{k-1}(t) - B^0_{k-1}(t) \right) \quad (0 \leq k \leq n). \]

Using definition (1.1) and Eqs. (1.4), (1.3), one can easily obtain the following identities:

\[ t \left( B^0_k(t) \right)' = kB^0_k(t) - (k+1)B^0_{k+1}(t), \]
\[ \left( B^0_k(t) \right)' = (n-k+1)B^0_{k-1}(t) + (2k-n)B^0_k(t) - (k+1)B^0_{k+1}(t), \]

where \( k = 0, 1, \ldots, n. \)

Polynomial Bézier curves are a particular family of parametric curves which is defined as a convex combination of control points. The points are weighted using Bernstein polynomials (1.1). A Bézier curve \( P_n : [0, 1] \rightarrow \mathbb{R}^d \) of degree \( n \) with control points \( W_0, W_1, \ldots, W_n \in \mathbb{R}^d \) is defined by the formula

\[ P_n(t) := \sum_{k=0}^n B^n_k(t)W_k \quad (0 \leq t \leq 1). \]

Certainly, the curve \( P_n \) is in the convex hull of the control points, i.e., \( P_n([0, 1]) \subseteq \text{conv}\{W_0, W_1, \ldots, W_n\}. \)

One can evaluate a point on a Bézier curve using the famous de Casteljau algorithm which is based on the relation (1.2) and has \( O(dn^2) \) complexity. It is a classic result, covered extensively in literature (see, e.g., [4] or [14]).

A new method for computing a point on a polynomial or rational Bézier curve in optimal \( O(dn) \) time has been recently proposed by the authors in [23]. The new algorithm combines the qualities of previously known methods for solving this problem, i.e., the linear complexity of the Horner’s scheme and the geometric interpretation, the convex hull property, and operating only on convex combinations which are the advantages of the de Casteljau algorithm. Notice that the new method can be used not only for polynomial and rational Bézier curves but also for other rational parametric objects.

The intention behind inventing Bézier curves was to make computer-aided techniques for automobile design possible and intuitive. Pierre Bézier and Paul de Casteljau’s work resulted in settling on a polynomial curve with control points and used Bernstein polynomials as a basis. Such approach gives a family of curves which have very neat properties, allowing the designers to easily control their shape and behavior. For more information about the history of Bézier curves, see, e.g., [4, 1, 2, 3, 11, 10, 12], as well as [14, §1] and [15, §4].

Despite their elegance and some desirable properties, Bernstein polynomials have a significant drawback. For any \( n, i \in \mathbb{N} \) such that \( 0 \leq i \leq n \), the value of a Bernstein polynomial \( B^n_i(t) \) is non-zero for all \( t \in (0, 1) \). In practice, when operating on a Bézier curve (1.7), any change in one control point’s position changes the curve over its whole length.
In particular, \( \left[ x_i, x_{i+1}, \ldots, x_{i+\ell} \right] f \) is the truncated power function (which may be coincident), denoted by \([x_i, x_{i+1}, \ldots, x_{i+\ell}] f\), is defined in the following recursive way:

\[
[x_i, x_{i+1}, \ldots, x_{i+\ell}] f := \begin{cases} 
\frac{[x_{i+1}, \ldots, x_{i+\ell}] f - [x_i, \ldots, x_{i+\ell-1}] f}{x_{i+\ell} - x_i} & (x_i \neq x_{i+\ell}), \\
\frac{f^{(\ell)}(x_i)}{\ell!} & (x_i = \ldots = x_{i+\ell}).
\end{cases}
\]

In particular, \([x_i] f = \frac{f^{(0)}(x_i)}{0!} = f(x_i)\).

**Definition 1.3** ([20, §5.11]). The B-spline function \( N_{mi} \) of degree \( m \in \mathbb{N} \) with knots \( t_i \leq t_{i+1} \leq \cdots \leq t_{m+i+1} \) is defined as

\[
N_{mi}(u) := (t_{i+m+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+m+1}](t - u)^m_+,
\]

where the generalized divided difference acts on the variable \( t \), and

\[
(x - c)^m_+ := \begin{cases} 
(x - c)^m & (x \geq c), \\
0 & (x < c)
\end{cases}
\]

is the truncated power function.

The B-spline function \( N_{mi} \) with knots \( t_i \leq t_{i+1} \leq \cdots \leq t_{m+i+1} \) has support \([t_i, t_{m+i+1}]\), i.e., \( N_{mi}(u) \) can be non-zero only for \( u \in [t_i, t_{m+i+1}] \) (see, e.g., [19, Property 2.2]).

Let \( m, n \in \mathbb{N} \). The knots

\[
\begin{array}{ccc} 
\underbrace{t_{-m} \leq \cdots \leq t_{=0}} & \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n \leq \underbrace{t_{n+1} \leq \cdots \leq t_{n+m}} \end{array}
\]

where \( t_0 < t_n \) (i.e., the knots \( \Omega_n := \{t_0, t_1, \ldots, t_n\} \) provide a partition of the interval \([t_0, t_n]\)) serve as a support for a B-spline basis of degree \( m \) over \([t_0, t_n]\). The B-spline functions \( N_{m,-m}, N_{m,-m+1}, \ldots, N_{m,-1} \) (cf. Definition 1.3) form a basis for the set \( S_m(\Omega_n) \) of all splines of degree \( m \) over \([t_0, t_n]\). Splines are commonly used in a wide variety of applications, e.g., in computer-aided geometric design, approximation theory and numerical analysis. See, e.g., [14, 20, 19, 13, 16].

**Remark 1.4.** In the sequel, a convention is adopted that for any quantity \( Q \), if \( t_k = t_{m+k+1} \) then \( \frac{Q}{t_{m+k+1} - t_k} := 0 \), as well as that \( N_{mi} = 0 \) for \( i < -m \) or \( i \geq n \).

Computing the B-spline functions or their derivatives using Definition 1.3, while possible, is costly. Instead, one can use the recurrence and differential-recurrence relations.

The B-spline functions satisfy the following de Boor-Mansfield-Cox recursion formula (see, e.g., [16, Eq. (7.8)], [9, §2], [6, Eq. (6.1)])

\[
N_{mi}(u) = (u - t_i)\frac{N_{m-1,i+1}(u)}{t_{m+i+1} - t_i} + (t_{m+i+1} - u)\frac{N_{m-1,i+1}(u)}{t_{m+i+1} - t_{i+1}} \quad (-m \leq i < n)
\]
Additionally, it can be checked that

\[ N_{0i}(u) = \begin{cases} 
1 & (u \in [t_i, t_{i+1})), \\
0 & \text{otherwise},
\end{cases} \]

where \( i = 0, 1, \ldots, n - 1 \).

The derivative of a B-spline function can be expressed as ([19, Eq. (2.7)])

\[ N'_m(u) = m \left( \frac{N_{m-1,i}(u)}{t_{m+i} - t_i} - \frac{N_{m-1,i+1}(u)}{t_{m+i+1} - t_{i+1}} \right) \quad (-m \leq i < n). \]

**Theorem 1.5** ([19, Property 2.5]). All derivatives of \( N_{mi} \) exist in the interior of a knot span (where it is a polynomial). At a knot \( N_{mi} \) is \( m-k \) times continuously differentiable, where \( k \) is the multiplicity of the knot. Hence, increasing \( m \) increases continuity, and increasing knot multiplicity decreases continuity.

The B-spline functions, like the family of Bernstein polynomials of an arbitrary degree, have properties which make them a good choice for a parameterization of a family of curves.

**Theorem 1.6** ([19, Properties 2.3, 2.4, 2.6]). \( N_{mi}(u) \geq 0 \) for all \( m, i, u \) (non-negativity). For an arbitrary knot span, \( [t_j, t_{j+1}] \), \( \sum_{i=j-m}^{i} N_{mi}(u) = 1 \) for all \( u \in [t_j, t_{j+1}] \) (partition of unity). Except for the case \( m = 0 \), \( N_{mi} \) attains exactly one maximum value.

Due to their properties such as non-negativity and partition of unity, B-spline functions are well-suited to be used as a basis of a parametric curve family, i.e., B-spline curves. From the graphical perspective, B-spline curves are of use and interest due to their properties. While they have good numerical properties, their advantage over Bézier curves lies in their locality — a change to one of the curve’s control points changes only a fragment of the curve which is influenced by the corresponding B-spline function’s support.

A B-spline curve of degree \( m \) over the non-empty interval \([a, b]\) with knots

\[ t_{-m} \leq \ldots \leq t_0 = a \leq t_1 \leq \ldots \leq b = t_n \leq \ldots \leq t_{n+m} \]

and control points \( W_{-m}, W_{-m+1}, \ldots, W_{n-1} \in \mathbb{R}^d \) is defined as

\[ S(t) := \sum_{i=-m}^{n-1} N_{mi}(t)W_i \quad (t \in [a, b]). \]

One can check that \( S([a, b]) \subseteq \text{conv}\{W_{-m}, W_{-m+1}, \ldots, W_{n-1}\} \), and \( S([t_i, t_{i+1}]) \subseteq \text{conv}\{W_{i-m}, W_{i-m+1}, \ldots, W_i\} \) for \( 0 \leq i \leq n - 1 \) (see, e.g., [19, Property 3.5]).

The recurrence relation (1.8) and Eq. (1.9) can be used to evaluate a point on a B-spline curve. This approach, applied to explicitly compute the values of B-spline functions, has been proposed by de Boor in [9, p. 55–57]. The algorithm given in [9, p. 57–59] (see also, e.g., [14, Eq. (8.3)]), which directly computes a point on a B-spline curve is known as the de Boor-Cox algorithm and has \( O(dm^2) \) computational complexity.

A popular choice for the boundary knots is to make them coincident with \( t_0 \) and \( t_n \), i.e.,

\[ t_{-m} = t_{-m+1} = \ldots = t_{-1} = t_0 = a, \quad b = t_n = t_{n+1} = \ldots = t_{n+m}. \]
In this case, $S(a) = W_{-m}$ and $S(b) = W_{n-1}$.

Notice that if $n = 1$ and the boundary knots are coincident (see (1.11)), it can be proved that the B-spline basis reduces to the Bernstein-Bézier basis, i.e.,

$$N_{mi}(u) = B^m_{i+m}(\frac{u-a}{b-a}) \quad (i = -m, -m+1, \ldots, 0)$$

(cf. (1.1)). This means that Bézier curves are a particular subtype of B-spline curves, i.e., when $n = 1$, $t_{-m} = t_{-m+1} = \ldots = t_0 = 0$ and $t_1 = t_2 = \ldots = t_{m+1} = 1$,

$$S(t) = \sum_{i=-m}^{0} N_{mi}(t)W_i = \sum_{i=0}^{m} B^m_i(t)W_{i-m} \quad (t \in [0,1]).$$

See, e.g., [19, Property 3.1].

2. The Problem: Bernstein-Bézier and power coefficients of B-spline functions. In this paper, the following problem is considered.

Let the adjusted Bernstein-Bézier basis form of the B-spline function $N_{mi}$ over a single non-empty knot span $[t_j, t_{j+1}) \subset [t_0, t_n]$ ($j = i, i+1, \ldots, i+m$) be

$$N_{mi}(u) = \sum_{k=0}^{m} b^{(i,j)}_k B^m_k(\frac{u-t_j}{t_{j+1}-t_j}) \quad (t_j \leq u < t_{j+1}),$$

with $b^{(i,j)}_k \equiv b^{(i,j)}_{k,m}$.

**Problem 2.1.** Find the adjusted Bernstein-Bézier basis coefficients $b^{(i,j)}_k$ ($0 \leq k \leq m$) (cf. (2.1)) of all functions $N_{mi}$ over all non-trivial knot spans $[t_j, t_{j+1}) \subset [t_0, t_n]$, i.e., for $j = 0, 1, \ldots, n-1$ and $i = j-m, j-m+1, \ldots, j$.

Notice that if the coefficients $b^{(i,j)}_k$ are already known, it is possible to compute any B-spline function in a linear time with respect to its degree using the geometric algorithm proposed in [23]. One can also simplify, e.g., the evaluation of a point on a B-spline curve or easily perform some operations analytically.

We also study a similar issue related to the adjusted power basis form of the B-spline functions.

**Problem 2.2.** Let the adjusted power basis form of the B-spline function $N_{mi}$ over a single non-empty knot span $[t_j, t_{j+1}) \subset [t_0, t_n]$ ($j = i, i+1, \ldots, i+m$) be

$$N_{mi}(u) = \sum_{k=0}^{m} a^{(i,j)}_k (u-t_j)^k \quad (t_j \leq u < t_{j+1}),$$

with $a^{(i,j)}_k \equiv a^{(i,j)}_{k,m}$: Find all the coefficients $a^{(i,j)}_k$ for $j = 0, 1, \ldots, n-1$, $i = j-m, j-m+1, \ldots, j$ and $k = 0, 1, \ldots, m$.

Explicit expressions for the adjusted power basis coefficients of $N_{mi}$ have been given in [18], and the result can be adapted for the adjusted Bernstein-Bézier form. The serious drawback of this approach, however, is high complexity, which greatly limits the use of this result in computational practice.

Let

$$s(t) := \sum_{i=-m}^{n-1} c_i N_{mi}(t).$$
An algorithm for finding the adjusted power basis coefficients of a spline \( s \) over a knot span \([t_j, t_{j+1})\) can be found in [13]. It uses Taylor series expansion to express the spline as

\[
s(t) = \sum_{r=0}^{m} \frac{s^{(r)}(t_j)}{r!} (t - t_j)^r \quad (t_j \leq u < t_{j+1})
\]

(cf. [13, Eq. (1.41)]). The derivatives \( s^{(r)}(t_j) \),

\[
s^{(r)}(t_j) = \frac{m!}{(m-r)!} \sum_{i=j-m+r}^{j} c_i^r N_{m-r,i}(t_j) \quad (t_j \leq u < t_{j+1}),
\]

can be computed recursively as follows. Set

\[
c_i^0 := c_i \quad (i = j - m, \ldots, j)
\]

(cf. (2.3)). Then

\[
c_i^r = \begin{cases} 
\frac{c_i^{r-1} - c_{i+1}^{r-1}}{t_{m+i+1-r} - t_i} & (t_i < t_{m+i+1-r}), \\
0 & \text{otherwise}
\end{cases}
\]

for \( r \geq 1 \) and \( j - m + r \leq i \leq j \) (cf. [13, Eq. (1.39) and (1.40)]). This allows to compute all the coefficients \( c_i^r \) in \( O(m^2) \) time. All necessary values \( N_{m-r,i}(t_j) \) can be computed using recurrence (1.8) in \( O(m^2) \) time. The adjusted power basis coefficients of a B-spline over one knot span can thus be found in \( O(m^2) \) time.

This approach can be used to find the adjusted power basis coefficients of one B-spline function (cf. Problem 2.2). To find the coefficients of \( N_{mi} \) over \([t_j, t_{j+1})\), it is enough to set

\[
c_k := \begin{cases} 
1 & (k = i), \\
0 & \text{otherwise}
\end{cases}
\]

(see (2.3)). The cost of finding the coefficients \( a_k^{(i,j)} \) of \( N_{mi} \) is \( O(m^2) \). In total, to find the adjusted power basis coefficients over \([t_j, t_{j+1})\) for all B-spline functions \( N_{mi} \) such that \( j - m \leq i \leq j \), one has to do \( O(m^3) \) operations. Let us assume that there are \( n_e \) non-empty knot spans \([t_j, t_{j+1})\) such that \( j = 0, 1, \ldots, n - 1 \). To find the coefficients of all B-spline functions over all non-empty knot spans \([t_j, t_{j+1})\) for \( j = 0, 1, \ldots, n - 1 \), one would need to perform \( O(n_e m^3) \) operations.

With a similar approach, one can find the Bernstein-Bézier coefficients of \( N_{mi} \) over the knot span \([t_j, t_{j+1})\). One can check that

\[
b_k^{(i,j)} = \frac{(m-k)!}{m!} N_{mi}(t_j) - \sum_{\ell=0}^{k-1} (-1)^{k-\ell} \binom{k}{\ell} b_{\ell}^{(i,j)} \quad (k = 0, 1, \ldots, m)
\]

(cf. [14, Eq. (5.25)] and [17, Theorem 4.1]). Just as in the case of the power basis, the Bernstein-Bézier coefficients \( b_k^{(i,j)} \) of \( N_{mi} \) over \([t_j, t_{j+1})\) can be found in \( O(m^2) \) time. In total, to find these coefficients for all B-spline functions over all non-empty knot spans \([t_j, t_{j+1})\) for \( j = 0, 1, \ldots, n - 1 \), it is required to perform \( O(n_e m^3) \) operations.

The approach given in [21] and [5] serves to convert a B-spline curve segment into a Bézier curve. It can be adapted to give an algorithm with \( O(m^3) \) complexity for finding the adjusted Bernstein-Bézier coefficients \( b_k^{(i,j)} \) of a single basis function.
\[ N_{mi}. \] Doing so for each B-spline function in each non-empty knot span takes \( O(n_e m^4) \) operations.

If there are recurrence relations for the coefficients of the B-spline functions over multiple knot spans, one can instead use them to efficiently find each of the coefficients. Over the course of this paper, such computationally simple recurrence relations for the coefficients of the adjusted Bernstein-Bézier and power forms will be derived from a new differential-recurrence relation for the B-spline functions.

**Remark 2.3.** In the sequel, we assume that no inner knot \( t_1, t_2, \ldots, t_{n-1} \) has multiplicity greater than \( m \). This guarantees the B-spline functions’ continuity in \( (t_1, t_n) \).

The assumption regarding the multiplicity of the inner knots is very common and intuitive, as it guarantees the continuity of a B-spline curve. It was used, e.g., in [13, 18] and [22, §3].

Let us suppose that there are \( n_e \) non-empty knot spans \([t_j, t_{j+1})\) such that \( 0 \leq j \leq n - 1 \). One of the main goals of the paper is to give a recursive way of computing all \( O(n_e m^2) \) coefficients \( b^{(i,j)}_k \) (cf. (2.1)) or \( a^{(i,j)}_k \) (cf. (2.2)) of the B-spline functions in \( O(n_e m^2) \) time, assuming that all the boundary knots are coincident.

The possible applications of this result can be as follows. Once the adjusted Bernstein-Bézier coefficients \( b^{(i,j)}_k \) are known, each point on a B-spline curve \( S \),

\[
S(u) := \sum_{i=-m}^{n-1} N_{mi}(u)W_i \quad (t_0 \leq u \leq t_n; \ W_i \in E^d),
\]

can be computed in \( O(m^2 + md) \) time using the geometric algorithm proposed recently by the authors in [23]. If there are \( N \) such points on \( M \) curves (each with the same knots), the total complexity is \( O(n_e m^2 + Nm^2 + MNmd) \), compared to \( O(MN m^2 d) \) when using the de Boor-Cox algorithm. Performed experiments confirm that the new method is faster than the de Boor-Cox algorithm even for low \( M \approx 2, 3 \). Using a similar approach, one can also compute the value of any \( N_{mi} \) in \( O(m) \) time.

The paper is organized as follows. In Section 3 we prove the new differential-recurrence relation between the B-spline functions of the same degree. It will be the foundation for new recurrence relations which can be used to formulate an algorithm which computes the coefficients \( a^{(i,j)}_k \) or \( b^{(i,j)}_k \) of B-spline functions over each knot span. In Section 4, the algorithm for finding the coefficients \( b^{(i,j)}_k \) in the adjusted Bernstein-Bézier form if \( t_{-m} = t_0, t_n = t_{n+m} \) and all inner knots \( t_1, t_2, \ldots, t_{n-1} \) have multiplicity 1 is given. The computational complexity of the method is \( O(nm^2) \). This means that, asymptotically, the algorithm is optimal. Section 5 expands upon the new algorithm to compute multiple points on multiple B-spline curves. The assumptions about knot multiplicity which were made in Section 4 are then relaxed in Section 6 to cover all cases (cf. Remark 2.3). In Section 7, the results given in Section 4 are adapted to the adjusted power basis.

**3. New differential-recurrence relation for B-spline functions.** Using the recurrence relation (1.8) which connects B-spline functions of consecutive degrees, one can find a recurrence relation which is satisfied by their coefficients in the chosen basis.

**Lemma 3.1.** The adjusted Bernstein-Bézier coefficients \( b^{(i,j)}_{k,m} \) of the B-spline functions \( N_{mi} \) over each knot span \([t_j, t_{j+1})\) (cf. (2.1)) satisfy the following recurrence
which, after some additional algebra, gives

\[ b_{k,m}^{(i,j)} = \frac{k}{m} \left( \frac{t_{j+1} - t_i}{t_{m+i} - t_i} b_{k-1,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_{j+1}}{t_{m+i+1} - t_{i+1}} b_{k,m-1}^{(i+1,j)} \right) \\
+ \frac{m-k}{m} \left( \frac{t_j - t_i}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_j}{t_{m+i+1} - t_{i+1}} b_{k-1,m-1}^{(i+1,j)} \right), \]

where \( k = 0, 1, \ldots, m \), and \( b_{1,m-1}^{(i,j)} = b_{1,m-1}^{(i+1,j)} = b_{m,m-1}^{(i,j)} = b_{m,m-1}^{(i+1,j)} := 0 \).

Proof. After applying the adjusted Bernstein-Bézier representations of the B-spline functions \( N_{m-1,i}(u) \) and \( N_{m-1,i+1}(u) \) (cf. (2.1)) in the recurrence relation (1.8), one gets

\[
N_{m,i}(u) = \frac{\sum_{k=0}^{m-1} b_{k,m-1}^{(i,j)} (u - t_i) B_{k}^{m-1}(t)}{t_{m+i} - t_i} + \frac{\sum_{k=0}^{m-1} b_{k,m-1}^{(i+1,j)} (t_{m+i} - u) B_{k}^{m-1}(t)}{t_{m+i} - t_i},
\]

where \( t := \frac{u - t_j}{t_{j+1} - t_j} \).

Note that \( (u - t_i) = (t_{j+1} - t_j) t + (t_j - t_i) \) and \( (t_{m+i} - u) = (t_{m+i+1} - t_{j+1}) + (t_{j+1} - t_1) (1 - t) \), which gives

\[
N_{m,i}(u) = \sum_{k=0}^{m-1} \frac{(t_{j+1} - t_j)}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} \cdot t B_{k}^{m-1}(t) \\
+ \sum_{k=0}^{m-1} \left( \frac{t_j - t_i}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_j}{t_{m+i+1} - t_{i+1}} b_{k,m-1}^{(i+1,j)} \right) B_{k}^{m-1}(t) \\
+ \sum_{k=0}^{m-1} \frac{t_{j+1} - t_j}{t_{m+i+1} - t_{i+1}} b_{k,m-1}^{(i+1,j)} \cdot (1 - t) B_{k}^{m-1}(t).
\]

Now, from Eq. (1.3) and Definition 1.1, one can raise the degree of Bernstein polynomials to get

\[
N_{m,i}(u) = \sum_{k=0}^{m-1} \frac{m-k}{m} \left( \frac{t_j - t_i}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_j}{t_{m+i+1} - t_{i+1}} b_{k,m-1}^{(i+1,j)} \right) B_{k}^{m}(t) \\
+ \sum_{k=0}^{m-1} \frac{k+1}{m} \left( \frac{(t_{j+1} - t_i)}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_{j+1}}{t_{m+i+1} - t_{i+1}} b_{k,m-1}^{(i+1,j)} \right) B_{k+1}^{m}(t),
\]

which, after some additional algebra, gives

\[
N_{m,i}(u) = \sum_{k=0}^{m} \left[ \frac{k}{m} \left( \frac{(t_{j+1} - t_i)}{t_{m+i} - t_i} b_{k,m-1}^{(i,j)} + \frac{t_{m+i+1} - t_{j+1}}{t_{m+i+1} - t_{i+1}} b_{k-1,m-1}^{(i+1,j)} \right) \right] B_{k}^{m}(t).
\]

Lemma 3.1 gives a recurrence relation for the adjusted Bernstein-Bézier coefficients of B-spline basis functions of different degrees, defined using the same knot sequence. While this relation can be used to find the values of the coefficients, it is not
optimal in terms of computational complexity as the recurrence scheme is analogous to the one used in the de Boor-Cox algorithm.

At the end of this section, a new differential-recurrence relation for the B-spline functions of the same degree \( m \) will be derived. We show that by using this result, it is possible to find all the Bernstein-Bézier coefficients faster (see Section 4).

However, Lemma 3.1 can be used to prove some properties which indicate that the adjusted Bernstein-Bézier basis is numerically sound for B-spline functions. The following two theorems can be easily proved by induction on the degree \( m \).

**Theorem 3.2.** For \( u \in [t_j, t_{j+1}) \) (\( j = 0, 1, \ldots, n-1 \)), the coefficients \( b_{k,m}^{(i,j)} \) of the adjusted Bernstein-Bézier representation of the B-spline function \( N_{m,i} \) (cf. Eq. (2.1)) are non-negative.

**Theorem 3.3.** For \( u \in [t_j, t_{j+1}) \) (\( j = 0, 1, \ldots, n-1 \)), the following relation holds:

\[
\sum_{i=-m}^{j} b_{k,m}^{(i,j)} = 1 \quad (k = 0, 1, \ldots, m),
\]

where \( b_{k,m}^{(i,j)} \) are the adjusted Bernstein-Bézier coefficients of \( N_{m,i} \) (cf. (2.1)).

Using equations (1.8) and (1.10), one can derive new differential-recurrence relations for the B-spline functions of the same degree. For example, this result can be used to efficiently compute the coefficients of the \( N_{m,i} \) functions (which are polynomial in each of the knot spans) in an adjusted Bernstein-Bézier or power basis.

**Theorem 3.4.** Let \( t_{-m} = t_{-m+1} = \ldots = t_0 < t_1 < \ldots < t_{n-1} < t_n = t_{n+1} = \ldots = t_{n+m} \) (cf. (1.11)). The following relations hold:

\[
\begin{align*}
(3.1) & \quad m N_{m,-m}(u) + (t_1 - u) N'_{m,-m}(u) = 0, \\
(3.2) & \quad N_{m,i}(u) + \frac{t_i - u}{m} N'_{m,i}(u) = \frac{t_{m+i+1} - t_i}{t_{m+i+2} - t_{i+1}} \left( N_{m,i+1}(u) + \frac{t_{m+i+2} - u}{m} N'_{m,i+1}(u) \right) \\
(3.3) & \quad m N_{m,n-1}(u) + (t_n - u) N'_{m,n-1}(u) = 0.
\end{align*}
\]

**Proof.** Equations (3.1) and (3.3) follow easily from equations (1.8) and (1.10), respectively. The relation (3.2) follows directly from taking the expression for \( N'_{m+1,i} \) from Eq. (1.8) and differentiating it, then equating it with the expression for \( N'_{m+1,i} \) given in Eq. (1.10).

Theorem 3.4 can be used to find a recurrence relation satisfied by the adjusted Bernstein-Bézier coefficients of B-spline functions of the same degree, as will be shown in the next section.

4. **Recurrence relations for B-spline functions’ coefficients in adjusted Bernstein-Bézier basis.** Assume that

\[
t_{-m} = t_{-m+1} = \ldots = t_0 < t_1 < \ldots < t_n = t_{n+1} = \ldots = t_{n+m}
\]

(cf. (1.11)).
For each knot span \([t_j, t_{j+1})\) \((j = 0, 1, \ldots, n-1)\), one needs to find the coefficients of \(N_{m,i}\) \((i = j - m, j - m + 1, \ldots, j)\) in the following adjusted Bernstein-Bézier basis form:

\[
N_{m,i}(u) = \sum_{k=0}^{m} b_k^{(i,j)} B_k^{m}(t) \quad (t_j \leq u < t_{j+1}),
\]

where \(b_k^{(i,j)} \equiv b_{k,m}^{(i,j)}\) and

\[
t = t^{(j)}(u) := \frac{u - t_j}{t_{j+1} - t_j}
\]

(cf. Eq. (2.1) and Problem 2.1). Additionally, then, \(u = (t_{j+1} - t_j)t + t_j\).

Certainly, \(N_{m,i}(u) \equiv 0\) if \(u < t_i\) or \(u > t_{m+i+1}\), which means that for a given knot span \([t_j, t_{j+1})\), one only needs to find the coefficients of \(N_{m,j-m}, N_{m,j-m+1}, \ldots, N_{m,j}\), as all coefficients of other B-spline functions over this knot span are identical to zero. Thus, in each of \(n\) knot spans, there are \(m + 1\) non-zero B-spline functions, each with \(m + 1\) coefficients.

Solving Problem 2.1 requires computing \(n(m + 1)^2\) coefficients \(b_k^{(i,j)}\). In this section, it will be shown how to do it in \(O(nm^2)\) time — proportionally to the number of coefficients. Theorem 3.4 serves as a foundation of the presented approach. More precisely, the theorem will be used to construct recurrence relations for the coefficients \(b_k^{(i,j)}\) which allow solving Problem 2.1 efficiently.

The results for particular cases will be presented in stages. In §4.1, an explicit expression for the coefficients of \(N_{m,j}\) and \(N_{m,j-m}\) over \([t_j, t_{j+1})\) \((j = 0, 1, \ldots, n-1)\) will be found. This will, in particular, cover the only non-trivial knot span for \(N_{m,-1}\). In §4.2, Eq. (3.2) will be applied to find the coefficients of \(N_{m,i}\) for \(j = -1, 2, \ldots, 0\) and \(i = j - 1, j - 2, \ldots, j - m + 1\).

### 4.1. Stage 1.

For \(j = 0, 1, \ldots, n-1\), one can use Eq. (1.8) for \(i = j\), along with the fact that \(N_{\ell,j+1} \equiv 0\) over \([t_j, t_{j+1})\) \((\ell = m - 1, m - 2, \ldots, 0)\), to find that

\[
N_{m,j}(u) = \frac{(u - t_j)^m}{\prod_{k=1}^{m}(t_{j+k} - t_j)} N_{0,j}(u) = \frac{(t_{j+1} - t_j)^{m-1}}{\prod_{k=2}^{m}(t_{j+k} - t_j)} B_m^{m}(t).
\]

It means that

\[
\begin{align*}
b_k^{(j,j)} &= 0 \quad (k = 0, 1, \ldots, m-1), \\
b_m^{(j,j)} &= \frac{(t_{j+1} - t_j)^{m-1}}{\prod_{k=2}^{m}(t_{j+k} - t_j)},
\end{align*}
\]

where \(0 \leq j \leq n - 1\).

Using the same approach for \(N_{m,j-m}\) over \([t_j, t_{j+1})\) gives

\[
N_{m,j-m}(u) = \frac{(t_{j+1} - t_j)^{m-1}}{\prod_{k=2}^{m}(t_{j+1+k} - t_{j+1-k})} B_m^{m}(t).
\]

The coefficients \(b_k^{(j-m,j)}\) \((k = 0, 1, \ldots, m)\) are thus given by the following formula:

\[
\begin{align*}
b_0^{(j-m,j)} &= \frac{(t_{j+1} - t_j)^{m-1}}{\prod_{k=2}^{m}(t_{j+1} - t_{j+1-k})}, \\
b_k^{(j-m,j)} &= 0 \quad (k = 1, 2, \ldots, m),
\end{align*}
\]
where \(0 \leq j \leq n - 1\). The adjusted Bernstein-Bézier coefficients of \(N_{mj}\) and \(N_{m,j-m}\) over the knot span \([t_j, t_{j+1}]\) (cf. Eq. (2.1)) have been found for \(j = 0, 1, \ldots, n - 1\).

In the sequel, the following observation will be of use.

**Remark 4.1.** Note that
\[
N_{m,n-1}(t_n) = \frac{(t_n - t_{n-1})^{m-1}}{\prod_{k=1}^{m-1} (t_{n+k} - t_{n-1})} B_m^m(1) = 1,
\]
since \(t_n = t_{n+1} = \ldots = t_{n+m}\). The B-spline functions have the partition of unity property and are non-negative (cf. Theorem 1.6), it is thus clear that
\[
N_{mi}(t_n) = 0 \quad (i = -m, -m + 1, \ldots, n - 2).
\]

Similarly,
\[
N_{m,-m}(t_0) = \frac{(t_1 - t_0)^m}{\prod_{k=1}^{m} (t_1 - t_{1-k})} B_m^0(0) = 1,
\]
since \(t_m = t_{m+1} = \ldots = t_0\). It follows that
\[
N_{mi}(t_0) = 0 \quad (i = -m + 1, -m + 2, \ldots, 0).
\]

**4.2. Stage 2.** To compute the coefficients of all functions \(N_{mi}\) over knot spans \([t_j, t_{j+1}]\) such that \(j = n-1, n-2, \ldots, 0\) and \(i = j-1, j-2, \ldots, j-m+1\), Eq. (3.2) will be used. The following identity will be useful when operating on Eq. (3.2):
\[
\left(N_{mi}(u)\right)' = \frac{dN_{mi}(u)}{du} = \sum_{k=0}^{m} b^{(i,j)}_k \frac{dB^m_k(t)}{dt} \cdot \frac{dt}{du} = (t_{j+1} - t_j)^{-1} \sum_{k=0}^{m} b^{(i,j)}_k \left(B^m_k(t)\right)'.
\]
(cf. (4.1)).

Let
\[
v_i \equiv v_{mi} := \frac{t_{m+i+1} - t_i}{t_{m+i+2} - t_{i+1}}.
\]

Substituting the adjusted Bernstein-Bézier forms of \(N_{mi}\) and \(N_{m,i+1}\) in the knot span \([t_j, t_{j+1}]\) and applying Eq. (4.4) into Eq. (3.2) gives
\[
\sum_{k=0}^{m} b_k^{(i,j)} B_k^m(t) + \left(\frac{t_i - t_j}{m(t_{j+1} - t_j)} - \frac{t}{m}\right) \left(\sum_{k=0}^{m} b_k^{(i,j)} B_k^m(t)\right)'
\]
\[
= v_i \left(\sum_{k=0}^{m} b_k^{(i+1,j)} B_k^m(t) + \left(\frac{t_{m+i+2} - t_j}{m(t_{j+1} - t_j)} - \frac{t}{m}\right) \left(\sum_{k=0}^{m} b_k^{(i+1,j)} B_k^m(t)\right)\right)'.
\]

After using identities (1.6) and (1.5) and doing some algebra, one gets
\[
\sum_{k=0}^{m} \left(l_k b_k^{(i,j)} + d_k b_k^{(i,j)} + u_k b_k^{(i,j)}\right) B_k^m(t) =
\]
\[
= v_i \sum_{k=0}^{m} \left(l_{k,m+i+2} b_k^{(i+1,j)} + d_{k,m+i+2} b_k^{(i+1,j)} + u_{k,m+i+2} b_k^{(i+1,j)}\right) B_k^m(t),
\]
where
\[ l_{kr} := k(t_{j+1} - t_r), \quad d_{kr} := (m - k)(t_{j+1} - t_r) + k(t_r - t_j), \quad u_{kr} := (m - k)(t_r - t_j). \]

Matching the coefficients of Bernstein polynomials on both sides gives a set of \( m + 1 \) equations of the form:
\[
\begin{cases}
(t_{j+1} - t_i) b_0^{(i,j)} + (t_i - t_j) b_1^{(i,j)} = v_i \left( (t_{j+1} - t_{m+i+2}) b_0^{(i+1,j)} + (t_{m+i+2} - t_j) b_1^{(i+1,j)} \right), \\
l_{ki} b_{k-1}^{(i,j)} + d_{ki} b_k^{(i,j)} + u_{ki} b_{k+1}^{(i,j)} = v_i \left( l_{k,m+i+2} b_k^{(i+1,j)} + d_{k,m+i+2} b_{k+1}^{(i+1,j)} + u_{k,m+i+2} b_{k+1}^{(i+1,j)} \right) \\
(t_{j+1} - t_i) b_m^{(i,j)} + (t_i - t_j) b_{m+1}^{(i,j)} = v_i \left( (t_{j+1} - t_{m+i+2}) b_m^{(i+1,j)} + (t_{m+i+2} - t_j) b_{m+1}^{(i+1,j)} \right) \\
\end{cases}
\tag{4.6}
\]

**Theorem 4.2.** For \( j = 0, 1, \ldots, n - 1 \) and \( i = j - 1, j - 2, \ldots, j - m + 1 \), assuming that the coefficients \( b_k^{(i+1,j)} \) (\( 0 \leq k \leq m \)) are known, the values \( b_0^{(i,j)}, b_1^{(i,j)}, \ldots, b_m^{(i,j)} \) satisfy a first-order non-homogeneous recurrence relation
\[
(t_{j+1} - t_i) b_k^{(i,j)} + (t_i - t_j) b_{k+1}^{(i,j)} = A(m, i, j, k) \quad (k = 0, 1, \ldots, m - 1),
\]

where
\[ A(m, i, j, k) := v_i \left( (t_{j+1} - t_{m+i+2}) b_k^{(i+1,j)} + (t_{m+i+2} - t_j) b_{k+1}^{(i+1,j)} \right). \]
\(\text{(cf. (4.5)).}\)

**Proof.** Base case \((k = 0 \text{ and } k = m)\): the relation holds and is presented in the first and the last equations of the system (4.6).

Induction step \((k \to k + 1)\): the \((k + 2)\)th equation in the system (4.6) is
\[
l_{k+1,i} b_k^{(i,j)} + d_{k+1,i} b_{k+1}^{(i,j)} + u_{k+1,i} b_{k+2}^{(i,j)} = v_i \left( l_{k+1,m+i+2} b_k^{(i+1,j)} + d_{k+1,m+i+2} b_{k+1}^{(i+1,j)} + u_{k+1,m+i+2} b_{k+2}^{(i+1,j)} \right).
\]

Subtracting sidewise the induction assumption scaled by \( \frac{l_{k+1,i}}{(t_{j+1} - t_i)} = k + 1 \) gives, after some algebra,
\[
(t_{j+1} - t_i) b_{k+1}^{(i,j)} + (t_i - t_j) b_{k+2}^{(i,j)} = v_i \left( (t_{j+1} - t_{m+i+2}) b_{k+1}^{(i+1,j)} + (t_{m+i+2} - t_j) b_{k+2}^{(i+1,j)} \right),
\]
which concludes the proof.\[ \square \]

From Theorem 4.2, it follows that there are \( m \) independent equations in system (4.6), as one of them is redundant. One thus needs an initial value to find the values of all \( b_0^{(i,j)}, b_1^{(i,j)}, \ldots, b_m^{(i,j)} \) using the recurrence relation (4.7).

If \( j = n - 1 \), Remark 4.1 can be used to find that
\[ N_{mi}(t_n) = b_m^{(i,n-1)} = 0 \quad (i = n - 2, n - 3, \ldots, n - m). \]
In this case, the recurrence relation given in Theorem 4.2 simplifies to
\[ (t_n - t_i)b_k^{(i,n-1)} = (t_{n-1} - t_i)b_k^{(i,n-1)} + v_i(t_n - t_{n-1})b_{k+1}^{(i+1,n-1)}. \]
It means that, for \( i = n - 2, n - 3, \ldots, n - m \), the following relation holds:
\[
\begin{align*}
\begin{cases}
    b_m^{(i,n-1)} = 0, \\
    b_k^{(i,n-1)} = & \frac{t_n - t_i}{t_n - t_{i+1}} b_k^{(i,n-1)} + \frac{t_n - t_{n-1}}{t_n - t_{i+1}} b_{k+1}^{(i+1,n-1)} & (k = m - 1, m - 2, \ldots, 0).
\end{cases}
\end{align*}
\]

For \( i = n - 2, n - 3, \ldots, n - m \), assuming that the coefficients \( b_k^{(i+1,n-1)} \) are known \((k = 1, 2, \ldots, m)\), Eq. (4.8) has an explicit solution
\[
\begin{align*}
\begin{cases}
    b_m^{(i,n-1)} = 0, \\
    b_k^{(i,n-1)} = & \frac{t_n - t_i}{t_n - t_{i+1}} b_k^{(i,n-1)} + \sum_{\ell=0}^{m-k-1} \left( \frac{t_n - t_i}{t_n - t_{i+1}} \right)^\ell b_{k+1+\ell}^{(i+1,n-1)} & (k = 0, 1, \ldots, m - 1).
\end{cases}
\end{align*}
\]

To find the initial value, if \( j < n - 1 \) and \( i = j - 1, j - 2, \ldots, j - m + 1 \), the right continuity condition will be used, i.e.,
\[ N_{mi}(t^-_{j+1}) = N_{mi}(t^+_{j+1}). \]

More precisely,
\[
N_{mi}(t^-_{j+1}) = \sum_{k=0}^{m} b_k^{(i,j)} B_k^m(1) = b_m^{(i,j)}
\]
and
\[
N_{mi}(t^+_{j+1}) = \sum_{k=0}^{m} b_k^{(i,j+1)} B_k^m(0) = b_0^{(i,j+1)},
\]
which gives the relation
\[
(4.10) \quad b_m^{(i,j)} = b_0^{(i,j+1)}.
\]

This completes the recurrence scheme for \( j = n - 2, n - 3, \ldots, 0 \) and \( i = j - 1, j - 2, \ldots, j - m + 1 \):
\[
\begin{align*}
\begin{cases}
    b_m^{(i,j)} = b_0^{(i,j+1)}, \\
    b_k^{(i,j)} = & \frac{t_j - t_i}{t_{j+1} - t_i} b_k^{(i,j)} + \frac{v_i}{t_{j+1} - t_i} \left( (t_{j+1} - t_{m+i+2}) b_k^{(i+1,j)} + (t_{m+i+2} - t_j) b_{k+1}^{(i+1,j)} \right) & (k = m - 1, m - 2, \ldots, 0).
\end{cases}
\end{align*}
\]

From Eq. (4.11) follows an explicit formula for the coefficients \( b_k^{(i,j)} \) \((0 \leq k \leq m)\), assuming that the coefficients \( b_0^{(i,j+1)} \) and \( b_k^{(i+1,j+1)} \) \((k = 0, 1, \ldots, m)\) are known:
\[
(4.12) \quad b_k^{(i,j)} = \left( \frac{t_j - t_i}{t_{j+1} - t_i} \right)^{m-k} b_0^{(i,j+1)} + \sum_{\ell=0}^{m-k-1} \left( \frac{t_j - t_i}{t_{j+1} - t_i} \right)^\ell \frac{v_i}{t_{j+1} - t_i} q_{k+\ell},
\]
where
\[ q_\ell := (t_{j+1} - t_{m+i+2}) b_\ell^{(i+1,j)} + (t_{m+i+2} - t_j) b_{\ell+1}^{(i+1,j)}, \]
and \( 0 \leq j \leq n - 2, j - m + 1 \leq i \leq j - 1 \).

The coefficients of \( N_{mi} \) have been found for \( j = 0, 1, \ldots, n - 1 \) and \( i = j - 1, j - 2, \ldots, j - m + 1 \).
4.3. The theorem and the algorithm. The results presented in §4.1 and §4.2 can be combined to prove the following theorem.

**Theorem 4.3.** Let us assume that

\[ t_{-m} = t_{-m+1} = \ldots = t_0 < t_1 < \ldots < t_{n-1} < t_n = t_{n+1} = \ldots = t_{n+m} \]

(cf. (1.11)). The \( n(m+1)^2 \) adjusted Bernstein-Bézier coefficients \( \beta_k^{(i,j)} \) of the B-spline functions \( N_{mi} \) over each knot span \([t_j, t_{j+1}]\) (cf. (2.1)), for \( j = 0, 1, \ldots, n-1, \ i = j-m, j-m+1, \ldots, j \) and \( k = 0, 1, \ldots, m \), can be computed in the computational complexity \( O(nm^2) \) in the following way:

1. For \( j = 0, 1, \ldots, n-1 \) and \( k = 0, 1, \ldots, m \), the coefficients \( \beta_k^{(j,j)} \) and \( \beta_k^{(j-m,j)} \) are given explicitly in equations (4.2) and (4.3), respectively.
2. For \( j = n-1 \), \( i = n-2, n-3, \ldots, n-m \) and \( k = m, m-1, \ldots, 0 \), the coefficients \( \beta_k^{(i,n-1)} \) (\( k = 0, 1, \ldots, m \)) are computed by the recurrence relation (4.8) (for their explicit forms, see (4.9)).
3. For \( j = n-2, n-3, \ldots, 0 \), \( i = j-1, j-2, \ldots, j-m+1 \) and \( k = m, m-1, \ldots, 0 \), the coefficients \( \beta_k^{(i,j)} \) are computed by the recurrence relation (4.11) (for their explicit forms, see (4.12)).

**Example 4.4.** Let us set \( m := 3, n := 5 \). Let the knots be

| \( t \) | \( t \) | \( t \) | \( t \) | \( t \) | \( t \) | \( t \) | \( t \) | \( t \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 5 | 6 | 9 | 10 | 10 |

Figure 1 illustrates the approach to computing all necessary adjusted Bernstein-Bézier coefficients of B-spline functions, given in Theorem 4.3. Arrows denote recursive dependence. Diagonally striped squares are computed using Eq. (4.2). Horizontally striped squares are computed using Eq. (4.3). White squares are computed using either the recurrence (4.8) (for \( u \in [t_4, t_5] \)) or (4.11) (for \( u < t_4 \)).

4.3.1. Implementation. Algorithm 4.1 implements the approach proposed in Theorem 4.3. This algorithm returns a sparse array \( B \equiv B[0..n-1, -m..n-1, 0..m] \), where

\[ B[j, i, k] = \beta_k^{(i,j)} \quad (0 \leq j < n, -m \leq i < n, 0 \leq k \leq m) \]

(cf. (2.1)).

For each of the \( n \) knot spans, one has to compute the coefficients of \( m+1 \) functions \( \beta_k^{(i,j)} \) (\( i \) coefficients in total). Computing all coefficients of one B-spline function in a given knot span requires \( O(m) \) operations. In total, then, the complexity of Algorithm 4.1 is \( O(nm^2) \) — giving the optimal \( O(1) \) time per coefficient.

4. Fast computation of multiple points on multiple B-spline curves. Let \( u \in [t_j, t_{j+1}] \) and \( t := \frac{u - t_j}{t_{j+1} - t_j} \). By solving Problem 2.1, the Bernstein-Bézier coefficients of the B-spline functions are found. A point on a B-spline curve (2.4) can thus be expressed as

\[ S(u) = \sum_{i=j-m}^{j} \left( \sum_{k=0}^{m} b_k^{(i,j)} B_k^m(t) \right) W_i. \]
The inner sums

\[ p_i(u) := \sum_{k=0}^{m} b_k^{(i,j)} B^m_k(t) \equiv N_{mi}(u) \quad (i = j - m, j - m + 1, \ldots, j) \]

can be treated as polynomial Bézier curves with control points \( b_k^{(i,j)} \in \mathbb{E}^1 \) and thus can be computed using the geometric algorithm given in [23] in total time \( O(m^2) \) — more precisely, \( O(m) \) per each of \( m + 1 \) sums. It also means that — when all the coefficients \( b_k^{(i,j)} \) are already known — any B-spline function may be computed in linear time with respect to its degree.

**Example 5.1.** A comparison of the new method of evaluating B-spline functions and using recurrence relation (1.8) has been done. The results have been obtained on a computer with Intel Core i5-6300U CPU at 2.40GHz processor and 4GB RAM, using GNU C Compiler 11.2.0 (single precision).

For each \( n \in \{10, 15, 20, 25, 30, 35, 40, 45, 50\} \), and \( m = 3, 4, \ldots, 15 \), a sequence of knots has been generated 100 times. The knot span lengths \( t_{j+1} - t_j \in [1/50, 1] \) \((j = 0, 1, \ldots, n - 1; t_0 = 0)\) have been generated using the `rand()` C function. The boundary knots are coincident. Then, \( 50 \cdot n + 1 \) points such that \( t_{j+\ell} := t_j + \ell/50 \times (t_{j+1} - t_j) \) for \( j = 0, 1, \ldots, n - 1 \) and \( \ell = 0, 1, \ldots, 49 \), with the remaining point being \( t_{n0} \equiv t_n \), are generated.

At each point \( t_{j+\ell} \in [t_j, t_{j+1}) \), all \( m + 1 \) B-spline functions \( N_{mi} \ (i = j - m, j - m + 1, \ldots, j) \) which do not vanish at \( t_{j+\ell} \) are evaluated using both algorithms. Due to the size of the table, the resulting running times are available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF-Example-5-1.xlsx.

The new method consistently performs faster than evaluating B-spline functions.
Algorithm 4.1 Computing the coefficients of the adjusted Bernstein-Bézier form of the B-spline functions

1: procedure BSPLINEBBF($n,m,[t_{-m},t_{-m+1},\ldots,t_{n+m}]$)
2: \hspace{1em} $B \leftarrow$ SparseArray[0..n-1, -m..n-1, 0..m] (fill=0)
3: \hspace{1em} for $j \leftarrow 0, n-1$ do
4: \hspace{2em} $B[j,j,m] \leftarrow \frac{(t_{j+1} - t_{j})^{m-1}}{\prod_{k=2}^{m}(t_{j+k} - t_{j})}$
5: \hspace{2em} $B[j,j,0] \leftarrow \frac{(t_{j+1} - t_{j})^{m-1}}{\prod_{k=2}^{m}(t_{j+1} - t_{j+k})}$
6: \hspace{1em} end for
7: \hspace{1em} for $i \leftarrow n-2, n-m$ do
8: \hspace{2em} for $k \leftarrow m-1, 0$ do
9: \hspace{3em} $B[n-1,i,k] \leftarrow \frac{t_{n-1} - t_{i}}{t_{n} - t_{i}} \cdot B[n-1,i,k+1] + \frac{t_{n} - t_{n-1}}{t_{n} - t_{i+1}} \cdot B[n-1,i+1,k+1]$
10: \hspace{2em} end for
11: \hspace{1em} end for
12: \hspace{1em} for $j \leftarrow n-2, 0$ do
13: \hspace{2em} for $i \leftarrow j-1, j-m+1$ do
14: \hspace{3em} $v \leftarrow \frac{t_{m+i+2} - t_{i+1}}{t_{m+i+1} - t_{i}}$
15: \hspace{3em} $B[j,i,m] \leftarrow B[j+1,i,0]$
16: \hspace{2em} for $k \leftarrow m-1, 0$ do
17: \hspace{3em} $B[j,i,k] \leftarrow \frac{t_{j} - t_{i}}{t_{j+1} - t_{i}} \cdot B[j,i,k+1] + \frac{v}{t_{j+1} - t_{i}} \cdot (t_{j+1} - t_{m+i+2}) \cdot B[j,i+1,k+1]$ \hspace{1em} \hspace{1em}
18: \hspace{2em} end for
19: \hspace{2em} end for
20: \hspace{1em} end for
21: return $B$
22: end procedure

\textbf{using recurrence relation (1.8). The new method reduced the running time for any dataset by 33–47\%, while the total running time was reduced by 45\%. The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF.c.}

Note that the sums $p_i$ (cf. (5.1)) do not depend on the control points. Afterwards, computing a convex combination of $m+1$ points from $\mathbb{R}^d$, i.e.,

$$S(u) = \sum_{i=j-m}^{j} p_i(u) W_i,$$

requires $O(md)$ arithmetic operations. Observe that these values may also be computed using the geometric method proposed in [23, Algorithm 1.1]. In total, then, assuming that the Bernstein-Bézier coefficients of the B-spline functions over each knot span $[t_j, t_{j+1}) \ (j = 0, 1, \ldots, n - 1)$ are known, $O(m(m + d))$ arithmetic operations are required to compute a point $S(u) \ (u \in [t_j, t_{j+1}))$ on a B-spline curve.

When it is required to compute the values of $S$ for many parameters $u_0, u_1, \ldots, u_N$, one would have to perform $O(nm^2)$ arithmetic operations to find the Bernstein-Bézier
coefficients of the B-spline functions over each knot span and then do \(O(m(m + d))\) operations for each of \(N + 1\) points that are to be computed. In total, the computational complexity of this approach is \(O(nm^2 + Nm(m + d))\).

Due to the fact that the sums \(p_i\) (cf. (5.1)) do not depend on the control points, they can be used for computing a point on multiple B-spline curves, all of degree \(m\), with the same knots.

**Problem 5.2.** For \(M\) B-spline curves \(S_0, S_1, \ldots, S_{M-1}\) with the knots

\[
t_{-m} = t_{-m+1} = \ldots = t_0 < t_1 < \ldots < t_n = t_{n+1} = \ldots = t_{n+m},
\]

and the control points of \(S_k\) being

\[
W_{k,-m}, W_{k,-m+1}, \ldots, W_{k,n-1} \in \mathbb{E}^d \quad (k = 0, 1, \ldots, M-1),
\]

compute the value of each of the B-spline curve \(S_k\) at points \(u_0, u_1, \ldots, u_{N-1}\) such that \(t_0 \leq u_k \leq t_n\) for all \(k = 0, 1, \ldots, N - 1\). More precisely, for \(k = 0, 1, \ldots, M - 1\) and \(\ell = 0, 1, \ldots, N - 1\), compute all the points \(S_k(u_\ell)\).

One can efficiently solve Problem 5.2 in the following way. Using Algorithm 4.1 allows to compute all the adjusted Bernstein-Bézier coefficients of B-spline functions (cf. Problem 2.1) in \(O(nm^2)\) time. Now, one needs to compute the values

\[
p_i(u_\ell) \quad (\ell = 0, 1, \ldots, N-1, \ u_\ell \in [t_j, t_{j+1}), \ i = j-m, j-m+1, \ldots, j)
\]

(cf. (5.1)), which takes \(O(Nm^2)\) time. Using these values, computing

\[
S_k(u_\ell) = \sum_{i=j-m}^{j} p_i(u_\ell)W_{ki} \quad (\ell = 0, 1, \ldots, N-1, \ k = 0, 1, \ldots, M-1, \ u_\ell \in [t_j, t_{j+1}])
\]

takes \(O(MNmd)\) time (see [23]). In total, then, the complexity of this approach is \(O(nm^2 + Nm^2 + NMmd)\), compared to the complexity of using the de Boor-Cox algorithm to solve Problem 5.2, i.e., \(O(NMm^2d)\).

A comparison of running times is given in Example 5.3. The new algorithm is compared to executing the de Boor-Cox algorithm and to an alternative way of computing the B-spline functions based on the recurrence relation (1.8) (see [9, p. 55–57]) and then evaluating the point in the same way as in the new method.

**Example 5.3.** Table 1 shows the comparison between the running times of the de Boor-Cox algorithm, an algorithm which computes the values of B-spline function using the recurrence relation (1.8) and then computes the points, and the new method described above and using Algorithm 4.1.

The results have been obtained on a computer with Intel Core i5-6300U CPU at 2.40GHz processor and 4GB RAM, using GNU C Compiler 11.2.0 (single precision).

The following numerical experiments have been conducted. For fixed \(n = 20\) and \(d = 2\), for each \(M \in \{1, 5, 10, 20, 50, 100\}\) and \(m \in \{3, 5, 7, 9, 11\}\), a sequence of knots and control points has been generated 100 times. The control points \(W_{ki} \in [-1, 1]^d\) \((i = -m, -m+1, \ldots, n-1, \ k = 0, 1, \ldots, M-1)\) and the knot span lengths \(t_{j+1} - t_j \in [1/50, 1]\) \((j = 0, 1, \ldots, n-1; \ t_0 = 0)\) have been generated using the \texttt{rand()} C function. The boundary knots are coincident. Each algorithm is then tested using the same knots and control points. Each curve is evaluated at 1001 points which are \(t_j + \ell/50 \times (t_{j+1} - t_j)\) for \(j = 0, 1, \ldots, n - 1\) and \(\ell = 0, 1, \ldots, 49\), with the remaining point being \(t_n\). Table 1 shows the total running time of all \(100 \times 1001 \times M\) curve evaluations for each method.
Table 1

| $M$ | $m$ | de Boor-Cox | eval splines | new method |
|-----|-----|--------------|--------------|------------|
| 1   | 3   | **0.032**    | 0.046        | 0.035      |
| 1   | 5   | 0.036        | 0.050        | **0.033**  |
| 1   | 7   | 0.062        | 0.079        | 0.050      |
| 1   | 9   | 0.100        | 0.122        | **0.075**  |
| 1   | 11  | 0.154        | 0.178        | **0.103**  |
| 5   | 3   | 0.085        | 0.049        | 0.041      |
| 5   | 5   | 0.195        | 0.094        | **0.073**  |
| 5   | 7   | 0.340        | 0.126        | **0.096**  |
| 5   | 9   | 0.533        | 0.188        | 0.132      |
| 5   | 11  | 0.732        | 0.240        | **0.167**  |
| 10  | 3   | 0.170        | 0.081        | **0.072**  |
| 10  | 5   | 0.372        | 0.129        | 0.109      |
| 10  | 7   | 0.651        | 0.190        | **0.152**  |
| 10  | 9   | 1.028        | 0.255        | **0.199**  |
| 10  | 11  | 1.453        | 0.319        | **0.244**  |
| 20  | 3   | 0.339        | 0.139        | **0.127**  |
| 20  | 5   | 0.721        | 0.210        | **0.185**  |
| 20  | 7   | 1.269        | 0.295        | **0.250**  |
| 20  | 9   | 2.022        | 0.388        | **0.323**  |
| 20  | 11  | 2.889        | 0.493        | **0.401**  |
| 50  | 3   | 0.845        | 0.314        | **0.301**  |
| 50  | 5   | 1.786        | 0.469        | **0.427**  |
| 50  | 7   | 3.305        | 0.671        | **0.610**  |
| 50  | 9   | 5.267        | 0.864        | **0.756**  |
| 50  | 11  | 9.436        | 1.328        | **1.196**  |
| 100 | 3   | 1.734        | 0.631        | **0.618**  |
| 100 | 5   | 3.776        | 0.954        | **0.901**  |
| 100 | 7   | 7.000        | 1.323        | **1.225**  |
| 100 | 9   | 10.458       | 1.590        | **1.470**  |
| 100 | 11  | 14.722       | 1.882        | **1.712**  |

Running times comparison (in seconds) for Example 5.3. The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF.c.

Example 5.4. Experiments similar to Example 5.3, with a wider choice of parameters, has been performed. The results have been obtained on the same computer, software, and precision.

More precisely, for each $d \in \{1, 2, 3\}$, $n \in \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$, $M \in \{1, 2, 3, 4, 5, 10, 15, 20, 25, 30, 50, 100\}$ and $m = 3, 4, \ldots, 15$, a sequence of knots and control points has been generated 100 times. The boundary knots are coincident. Each algorithm is then tested using the same C function.
knots and control points. Each curve is evaluated at 50 · n + 1 points which are
\( t_j + \ell/50 \times (t_{j+1} - t_j) \) for \( j = 0, 1, \ldots, n - 1 \) and \( \ell = 0, 1, \ldots, 49 \), with the remaining
point being \( t_n \). Due to the size of the table, the resulting running times are available
at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF-Example-5-4.xlsx.

The results show that the new method is significantly faster than the de Boor-
Cox algorithm except for the case \( M = 1 \). While the acceleration with respect to the
approach which utilizes Eq. (1.8) is smaller, it is also consistent, getting lower running
time in 99.95% test cases.

Some statistics regarding the experiments are given in Table 2.

| Algorithm       | Total running time [s] | Relative to new method |
|-----------------|-------------------------|------------------------|
| de Boor-Cox     | 14769.25                | 6.81                   |
| eval splines    | 2600.92                 | 1.20                   |
| new method      | 2167.92                 | —                      |

| Algorithm       | New method win % | Max time rel. to new method | Min time rel. to new method |
|-----------------|-----------------|-----------------------------|-----------------------------|
| de Boor-Cox     | 97.01%          | 11.664                      | 0.688                       |
| eval splines    | 99.95%          | 1.808                       | 0.998                       |

Table 2

Statistics for Example 5.4. The source code in C which was used to perform the tests is available
at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF.c.

6. Generalizations. The approach presented in Section 4 (as well as in Section 7)
can be generalized so that the inner knots may have their multiplicity higher
than 1 or the boundary knots are of multiplicity lower than \( m + 1 \).

6.1. Inner knots of any multiplicity. When an inner knot has multiplicity
over 1, some knot spans \([t_j, t_{j+1})\) \((j = 0, 1, \ldots, n - 1)\) are empty. It is only necessary to
find the B-spline functions' coefficients over the non-empty knot spans. If there are \( n_e \)
such knot spans, one only needs to find \( n_e (m + 1)^2 \) coefficients, and the algorithm will have \( O(n_e m^2) \) complexity. To use the continuity condition, the following definition
will be useful.

**Definition 6.1.** The left neighbor of a given knot \( t_k \) is the knot \( t_\ell \) if \( \ell \) is the
largest natural number such that \( t_\ell < t_k \), i.e., \([t_\ell, t_{\ell+1})\) is non-empty and \( t_{\ell+1} = t_k \).

The right neighbor of a given knot \( t_k \) is the knot \( t_r \) if \( r \) is the smallest natural
number such that \( t_k < t_r \), i.e., \([t_{r-1}, t_r)\) is non-empty and \( t_k = t_{r-1} \).

Note that in the case considered in Section 4, the right neighbor of \( t_j \) \((j = 0, 1, \ldots, n - 1)\) is always \( t_{j+1} \).

From Remark 2.3, it follows that each B-spline function is continuous in \((t_0, t_n)\).
The only modification then is in the continuity condition in Eq. (4.11). Let us consider
a non-empty knot span \([t_j, t_{j+1})\) \((j = 0, 1, \ldots, n - 2)\). Let \( t_r \) be the right neighbor of
\( t_{j+1} \), i.e., \( t_{r-1} = t_{j+1} \). In this case, the continuity property at \( t_{j+1} \) is

\[
\sum_{k=0}^{m} b^{(i,j)}_k B^m_k \left( \frac{t_{j+1} - t_j}{t_{j+1} - t_j} \right) = \sum_{k=0}^{m} b^{(i,r-1)}_k B^m_k \left( \frac{t_{j+1} - t_{r-1}}{t_r - t_{r-1}} \right),
\]
which simplifies to
\[ b_m^{(i,j)} = b_0^{(i,r-1)} \]
(cf. Eq. (4.10)). In such case, the recurrence relation (4.11) takes the form
\[
\begin{cases}
  b_m^{(i,j)} = b_0^{(i,r-1)}, \\
  b_k^{(i,j)} = \frac{t_{j+1} - t_i}{t_{j+1} - t_i} b_{k+1}^{(i,j)} + \\
  \frac{1}{t_{j+1} - t_i} \left( (t_{j+1} - t_{m+i+2}) b_k^{(i+1,j)} + (t_{m+i+2} - t_j) b_{k+1}^{(i+1,j)} \right)
\end{cases}
\]
(cf. Eq. (4.5)), where \( t_r \) is the right neighbor of \( t_{j+1} \), and \( j = n - 2, n - 3, \ldots, 0 \), \( i = j - 1, j - 2, \ldots, j - m + 1 \). Example 6.2 presents this approach.

**Example 6.2.** Let us set \( m := 3 \), \( n := 5 \). Let the knots be
\[
\begin{array}{cccccccccccc}
  t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
  0 & 0 & 0 & 3 & 3 & 5 & 9 & 10 & 10 & 10 & 10 & 0
\end{array}
\]
The knot \( t_1 \) is of multiplicity 2. To compute the adjusted Bernstein-Bézier coefficients of the B-spline functions over \([t_0, t_1]\) a continuity condition with the knot span \([t_2, t_3]\) is used, as \( t_1 = t_2 \). Figure 2 illustrates this approach to computing all necessary coefficients, analogous to Example 4.4.

![Figure 2](image-url)

**Fig. 2.** An illustration of Example 6.2.

### 6.2. Boundary knots of multiplicity lower than \( m + 1 \)
First, note that in Section 4, only the assumption that \( t_n = t_{n+m} \) is used, therefore if that condition
holds, Theorem 4.3 and Algorithm 4.1 still apply, regardless of the multiplicity of boundary knots \( t_{-m}, t_{-m+1}, \ldots, t_0 \).

If the boundary knot \( t_n \) has multiplicity lower than \( m+1 \), the problem can be reduced so that it can be solved using Theorem 4.3. Its drawback, however, is higher complexity.

The idea is to inflate the multiplicity of \( t_{n+m} \) up to \( m+1 \). More precisely, let \( t_{n+m-\ell-1} < t_{n+m-\ell} = t_{n+m} = \ldots = t_{n+m+\ell} = t_{n+2m-\ell} \). Let the \( m-\ell \) new knots \( t_{n+m+1} = t_{n+m+2} = \ldots = t_{n+2m-\ell} \) be defined so that \( t_{n+m} = t_{n+m+1} \). This allows to execute Algorithm 4.1 with \( m_1 := n + m - \ell, m_1 := m \) and

\[
\begin{align*}
&\text{boundary knots} \quad t_{-m} \leq \ldots \leq t_{-1} \leq t_0 \leq \ldots \leq t_{n+m-1} \leq t_{n+m} = t_{n+m+1} = \ldots = t_{n+2m-\ell}.
&\text{inner knots}
\end{align*}
\]

It remains then to return the coefficients of \( N_{mi} \) over \( [t_j, t_{j+1}] \) for \( j = 0, 1, \ldots, n-1 \) and \( i = j-m, j-m+1, \ldots, j \). This approach requires the computation of \( O((n+m-\ell)m^2) \) coefficients and is presented in Example 6.3.

**Example 6.3.** Let us set \( m := 3, n := 2 \). Let the knots be

\[
\begin{array}{cccccccc}
t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
\]

After adding the knots \( t_6 = t_7 = t_8 \) such that \( t_5 = t_8 \) (thus increasing \( n \) by 3), the problem takes the form

\[
\begin{array}{cccccccc}
t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5
\end{array}
\]

Figure 3 illustrates the application of Algorithm 4.1 (cf. Example 4.4) computing all the adjusted Bernstein-Bézier coefficients of the inflated problem. The coefficients which are relevant to the solution of the primary problem are in the frame drawn in bold.

### 7. B-spline functions’ coefficients in adjusted power basis.

Let us consider Problem 2.2. The coefficients of the B-spline functions in the adjusted power basis can be computed using a very similar approach to the one presented in Section 4 for the Bernstein-Bézier basis. In this section, the approach will be outlined, with some details omitted when they are analogous to the Bernstein-Bézier case.

As in Section 4, let us assume that the boundary knots are coincident and all inner knots are of multiplicity 1, i.e., \( t_{-m} = t_{-m+1} = \ldots = t_0 < t_1 < \ldots < t_n = t_{n+1} = \ldots = t_{n+m} \) (cf. (1.11)).

Recall that \( N_{mi} \) over \( [t_j, t_{j+1}] \) in the adjusted power basis has the form (2.1).

**Remark 7.1.** In the sequel, it will be assumed that \( u \in [t_j, t_{j+1}] \).

For \( j = 0, 1, \ldots, n-1 \) and \( i = j \), using Eq. (1.8) gives an explicit representation in the adjusted power basis:

\[
N_{mi}(u) = \frac{u-t_i}{t_{m+i}-t_i} N_{m-1,i}(u) = \ldots = \prod_{j=1}^{m} \frac{u-t_i}{t_{j+i}-t_i} N_{0,i}(u) = \frac{(u-t_i)^m}{\prod_{j=1}^{m} (t_{j+i}-t_i)}.
\]

In the same way, an expression for \( N_{mi} \) over \( [t_{m+i}, t_{m+i+1}] \) could be found. However, the recurrence relation (1.8) would give it in the \( (u-t_{m+i+1})^k \) basis and one would
need \(O(m^2)\) operations to convert it to the \((u - t_{m+i})^k\) basis. Due to that, Eq. (3.2) will be used to find the coefficients of \(N_{m,j-m}\) over the knot span \([t_j, t_{j+1})\) \((j = 0, 1, \ldots, n - 1)\).

For \(j = n - 1, n - 2, \ldots, 0\) and \(i = j - 1, j - 2, \ldots, j - m\), Eq. (3.2) is be used to get \(m\) equations:

\[
(m-k)a_k^{(i,j)} + (k+1)(t_i-t_j)a_{k+1}^{(i,j)} = v_i \left( (m-k)a_k^{(i+1,j)} + (k+1)(t_{m+i+2} - t_j)a_{k+1}^{(i+1,j)} \right) \quad (k = 0, 1, \ldots, m - 1)
\]

(cf. Eq. (4.5)). In a particular case of \(j = 0\), the recurrence simplifies to

\[
a_k^{(i,0)} = v_i \left( a_k^{(i+1,0)} + \frac{(k+1)(t_{m+i+2} - t_0)}{m-k}a_{k+1}^{(i+1,0)} \right) \quad (k = 0, 1, \ldots, m - 1),
\]

giving the expressions for all coefficients except \(a_m^{(i,0)}\).

If \(j = n - 1\), one can use Remark 4.1 to complete the system of equations:

\[
\begin{bmatrix}
0 & s_1 & s_2 & \cdots & s_m \\
l_0 & d_0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
l_k & d_k & \ddots & \ddots & 0 \\
0 & \cdots & 0 & l_{m-1} & d_{m-1}
\end{bmatrix}
\begin{bmatrix}
a_0^{(i,n-1)} \\
a_1^{(i,n-1)} \\
\vdots \\
a_k^{(i,n-1)} \\
a_m^{(i,n-1)}
\end{bmatrix}
= v_i
\begin{bmatrix}
0 \\
l_0a_0^{(i+1)} + r_0a_1^{(i+1)} \\
\vdots \\
l_k a_k^{(i+1)} + r_k a_{k+1}^{(i+1)} \\
l_{m-1} a_{m-1}^{(i+1)} + r_{m-1} a_m^{(i+1)}
\end{bmatrix},
\]
where \( a^{(j)}_k \equiv a^{(j,n-1)}_k \) and
\[
l_k := m-k, \quad d_k := (k+1)(t_i - t_{n-1}), \quad r_k := (k+1)(t_n - t_{n-1}), \quad s_k := (t_n - t_{n-1})^k.
\]

One can use Gaussian elimination so that only the last element in the first row of the matrix is non-zero. For \( k = 0, 1, \ldots, m-1 \), let \( g_k \) be the factor by which the \((k+2)\)th row (i.e., the row containing \( l_k \) and \( d_k \)) is multiplied before being subtracted from the first row. More precisely, the following recurrence relation with an initial value needs to be satisfied:

\[
\begin{cases}
g_0 = m^{-1}, \\
g_k = \frac{(t_n - t_{n-1})^k}{m-k} - g_{k-1} \frac{k(t_i - t_{n-1})}{m-k} \quad (k = 1, 2, \ldots, m-1).
\end{cases}
\]

It is clear that one can compute all \( g_0, g_1, \ldots, g_{m-1} \) in \( O(m) \) time. One can check that
\[
g_k = \frac{(t_n - t_{n-1})^k}{m(m-1) \cdots (m-k)} \sum_{h=0}^k \binom{m}{h} \left( -\frac{t_n - t_{n-1}}{t_i - t_{n-1}} \right)^h \quad (k = 0, 1, \ldots, m-1),
\]

however, it will simplify the expressions and computations if the recursive form is used instead. After performing the elimination, the first row of the matrix gives an expression for \( a^{(i,n-1)}_{m,n} \):
\[
a^{(i,n-1)}_{m,n} = \frac{-v_i}{s_m - g_{m-1}d_{m-1}} \sum_{k=0}^{m-1} g_k (l_k a^{(i+1,n-1)}_k + r_k a^{(i+1,n-1)}_{k+1}).
\]

This can be computed in \( O(m) \) time. The remaining coefficients can be then found using the recurrence relations given in other rows of the matrix:
\[
\begin{cases}
a^{(i)}_m = \frac{-v_i}{s_m - g_{m-1}d_{m-1}} \sum_{k=0}^{m-1} g_k (l_k a^{(i+1)}_k + r_k a^{(i+1)}_{k+1}), \\
a^{(i)}_k = l_k^{-1} \left( -d_k a^{(i)}_{k+1} + v_i (l_k a^{(i+1)}_k + r_k a^{(i+1)}_{k+1}) \right) \quad (k = m-1, m-2, \ldots, 0),
\end{cases}
\]

where \( a^{(j)}_k \equiv a^{(j,n-1)}_k \) and
\[
l_k := m-k, \quad d_k := (k+1)(t_i - t_{n-1}), \quad r_k := (k+1)(t_n - t_{n-1}), \quad s_k := (t_n - t_{n-1})^k,
\]

and the values \( g_k \) are given by (7.1).

For \( j = 0, 1, \ldots, n-2 \), finding the initial value for the recurrence scheme can be done by using the continuity condition at \( t_{j+1} \). The knot \( t_{j+1} \) has multiplicity 1, therefore from Theorem 1.5, it follows that the \((m-1)\)th derivative of \( N^{(i-1)}_{mi} \) is continuous at \( t_{j+1} \):
\[
N^{(m-1)}_{mi}(t_{j+1}^-) = N^{(m-1)}_{mi}(t_{j+1}^+).
\]

It is easy to check that
\[
N^{(m-1)}_{mi}(t_{j+1}^-) = a^{(i,j)}_{m-1}(m-1)! + a^{(i,j)}_m m!(t_{j+1} - t_j)
\]
and
\[
N^{(m-1)}_{mi}(t_{j+1}^+) = a^{(i,j+1)}_{m-1}(m-1)!,
\]
which completes the recurrence scheme.

This, together with the previously found equation, i.e.,

$$a_{m-1}^{(i,j)} + m(t_i - t_j)a_{m-1}^{(i,j)} = v_i \left( a_{m-1}^{(i+1,j)} + m(t_{m+i+2} - t_j)a_{m}^{(i+1,j)} \right),$$

allows to find an expression for $a_{m}^{(i,j)}$ (assuming that $a_{m-1}^{(i+1,j)}$ and $a_{m}^{(i+1,j)}$ are known):

$$m(t_{j+1} - t_j)a_{m}^{(i,j)} = a_{m-1}^{(i,j+1)} - v_i \left( a_{m-1}^{(i+1,j)} + m(t_{m+i+2} - t_j)a_{m}^{(i+1,j)} \right),$$

which completes the recurrence scheme.

This proves the following theorem.

**Theorem 7.2.** Let $t_m = t_{m+1} = \ldots = t_0 < t_1 < \ldots < t_{n-1} < t_n = t_{n+1} = \ldots = t_{n+m}$ (cf. (1.11)). The $n(m + 1)^2$ coefficients $a_{k}^{(i,j)}$ of the B-spline functions $N_{mi}$ $(i = -m, -m + 1, \ldots, n - 1)$ over each knot span $[t_j, t_{j+1})$ $(0 \leq j \leq n-1, i = j-m, j-m+1, \ldots, j)$ are given by the recurrence relation (cf. Eq. (2.2))

1. For $j = 0, 1, \ldots, n-1$ and $i = j$, we have

\[
\begin{align*}
\begin{cases}
    a_{k}^{(j,j)} &= 0 & (k = 0, 1, \ldots, m-1), \\
    a_{m}^{(j,j)} &= \left( \prod_{\ell=1}^{m}(t_{j+\ell} - t_j) \right)^{-1}.
\end{cases}
\end{align*}
\]

2. For $j = n-1$ and $i = n-2, n-3, \ldots, n-1-m$, the coefficients of $N_{mi}$ are given by the recurrence relation

\[
\begin{align*}
\begin{cases}
    a_{m}^{(i,n-1)} &= \frac{-v_i}{s_m - g_{m-1}d_{m-1}} \sum_{k=0}^{m-1} g_k \left( l_k a_{k}^{(i+1,n-1)} + r_k a_{k+1}^{(i+1,n-1)} \right), \\
    a_{k}^{(i,n-1)} &= t_k^{-1} \left( -d_k a_{k+1}^{(i,n-1)} + v_i \left( l_k a_{k}^{(i+1,n-1)} + r_k a_{k+1}^{(i+1,n-1)} \right) \right),
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
l_k := m - k, & \quad d_k := (k+1)(t_i - t_{n-1}), \\
r_k := (k+1)(t_n - t_{n-1}), & \quad s_k := (t_n - t_{n-1})^k,
\end{align*}
\]

and the values $g_k$ are given by (7.1).

3. For $j = n-2, n-3, \ldots, 0$ and $i = j-1, j-2, \ldots, j-m$, the coefficients of $N_{mi}$ over $[t_j, t_{j+1})$ are given by the recurrence relation

\[
\begin{align*}
\begin{cases}
    a_{m}^{(i,j)} &= \frac{a_{m-1}^{(i+1,j)} - v_i \left( a_{m-1}^{(i+1,j)} + m(t_{m+i+2} - t_j)a_{m}^{(i+1,j)} \right)}{m(t_{j+1} - t_i)}, \\
    a_{k}^{(i,j)} &= \frac{(k+1)(t_j - t_i)}{m - k} a_{k+1}^{(i,j)} \quad + \quad v_i \left( a_{k}^{(i+1,j)} + \frac{(k+1)(t_{m+i+2} - t_j)a_{k+1}^{(i+1,j)}}{m - k} \right),
\end{cases}
\end{align*}
\]

$(k = 0, 1, \ldots, m-1).$
Algorithm 7.1 implements the approach given in Theorem 7.2. This algorithm returns a sparse array \( A \equiv A[0..n-1,-m..n-1,0..m] \), where

\[
A[j, i, k] = a_k^{(j)} \quad (0 \leq j < n, -m \leq i < n, 0 \leq k \leq m)
\]

(cf. (2.2)).

Similarly to the case of Bernstein-Bézier basis, the complexity of Algorithm 7.1 is \( O(nm^2) \) — giving the optimal \( O(1) \) time per coefficient.

**Algorithm 7.1** Computing the coefficients of the power form of the B-spline functions

1: procedure BSplinePF\((n, m, [t_{-m}, t_{-m+1}, \ldots, t_{n+m}])\)
2: \( A \leftarrow \text{SparseArray}[0..n-1, -m..n-1, 0..m](\text{fill}=0) \)
3: for \( i \leftarrow 0, n-1 \) do
4: \( A[i, i, m] \leftarrow \left( \prod_{\ell=1}^{m}(t_{i+\ell} - t_i) \right)^{-1} \)
5: end for
6: for \( i \leftarrow n-2, n-1-m \) do
7: \( A[n-1, i, 0..m] \leftarrow \text{Eq. (7.2)} \)
8: end for
9: for \( j \leftarrow n-2, 0 \) do
10: for \( i \leftarrow j-1, j-m \) do
11: \( v \leftarrow \frac{t_{m+i+1} - t_i}{t_{m+i+2} - t_{i+1}} \)
12: \( w \leftarrow A[j, i+1, m-1] + m \cdot (t_{m+i+2} - t_j) \cdot A[j, i+1, m] \)
13: \( A[j, i, m] \leftarrow \frac{A[j+1, i, m-1] - v \cdot w}{m \cdot (t_{j+1} - t_i)} \)
14: end for
15: end for
16: for \( k \leftarrow m-1, 0 \) do
17: \( q \leftarrow \frac{m-k}{k+1} \)
18: \( w \leftarrow A[j, i+1, k] + (t_{m+i+2} - t_j) \cdot q \cdot A[j, i+1, k+1] \)
19: \( A[j, i, k] \leftarrow (t_j - t_i) \cdot q \cdot A[j, i, k+1] + v \cdot w \)
20: end for
21: end for
22: return \( A \)
23: end procedure

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