Classification of translation invariant topological Pauli stabilizer codes for prime dimensional qudits on two-dimensional lattices

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We prove that on any two-dimensional lattice of qudits of a prime dimension, every translation invariant Pauli stabilizer group with local generators and with code distance being the linear system size, is decomposed by a Clifford circuit of constant depth into \( T^{\oplus n} \oplus Z \) for some integer \( n \geq 0 \), where \( T \) is the stabilizer group of the toric code (abelian discrete gauge theory) on the square lattice and \( Z \) is a stabilizer group whose code space encodes zero logical qudit in any finite periodic lattice. The direct summand \( Z \) is mapped to the trivial stabilizer group for a product state under a locality-preserving automorphism of the complex operator algebra on the lattice which maps every Pauli matrix to a product of Pauli matrices (Clifford QCA). In other words, up to Clifford QCA the integer \( n \) is the complete invariant of such a stabilizer group. Previously, the same conclusion was obtained by assuming nonchirality for qubit codes or the Calderbank-Shor-Steane structure for prime qudit codes; we do not assume any of these.

I. INTRODUCTION AND RESULT

Topological Pauli stabilizer codes are a class of Pauli stabilizer codes [1–4] whose stabilizer group generators define a local Hamiltonian that exhibits topological order [5, 6]. By construction they give exactly solvable lattice models, which demonstrate robust ground state degeneracy and Aharanov-Bohm interaction between quasi-particles. A defining characteristic of this class is that all operators from which the topological data of the quasi-particles are derived are tensor products of Pauli matrices. Hence in two-dimensional lattices the topological excitations are always abelian, i.e., the fusion rules are deterministic and self and mutual statistics are given by phase factors.

Believing in the effective description by unitary modular tensor categories (UMTC) [7, 8], we can tabulate possible topological phases of matter realized by two-dimensional topological Pauli stabilizer codes. Indeed, if we consider generalized Pauli operators over a system of qudits of dimension \( p \) (to be defined below), the topological spins for a Pauli stabilizer code model must be valued in \( p \)-th roots of unity and the UMTC is determined by a quadratic form \( \theta \) over \( \mathbb{Z}/p\mathbb{Z} \); see [9, §5] and references therein. When \( p \) is a prime (so that the topological spins are valued
in a finite field) nondegenerate quadratic forms are particularly simple \[10, 11\]. If we ignore direct summands of hyperbolic planes, which correspond to the toric code phase \[6\], then the only nontrivial possibilities are

1. \((p = 2)\) a two-dimensional form \(\theta_3F(v) = v_1^2 + v_1v_2 + v_2^2\) which corresponds to the three fermion theory,

2. \((p \equiv 3 \mod 4\) so the Witt group is \(\mathbb{Z}/4\mathbb{Z}\) a one-dimensional form \(\theta_1(v) = v^2\), its time-
   reversal conjugate \(-\theta_1\) or a direct sum \(\theta_1 \oplus \theta_1 \cong (-\theta_1) \oplus (-\theta_1)\) and

3. \((p \equiv 1 \mod 4\) so the Witt group is \(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) a one-dimensional form \(\theta_1(v) = v^2\), another one-dimensional form \(\theta_\alpha(v) = \alpha v^2\) where \(\alpha\) is any nonsquare element of \(\mathbb{F}_p\) or their
direct sum \(\theta_1 \oplus \theta_\alpha\).

This motivates us to ask whether these candidates are exhaustive and whether every candidate can be realized in a lattice model. For Pauli stabilizer code models the problem becomes a question on locally generated abelian multiplicative groups of Pauli operators.

Previously, Bombón has studied topological stabilizer groups in two dimensions that are translation invariant on systems of qubits \((p = 2)\) and concluded that, up to Clifford QCA, any such group is a direct sum of copies of the toric code stabilizer group and a trivial stabilizer group for a product state \textit{if} the topological charge content is not chiral, i.e., its decomposition does not contain the three fermion theory \[12\]. We have studied a similar translation invariant case with qudits of a prime dimension under the assumption that the group be generated by tensor products of \(X\) and tensor products of \(Z\) (the Calderbank-Shor-Steane structure \[1, 2\]) and concluded that every such stabilizer group is a direct sum of those for the toric code after a circuit of control-Not gates \[13\].

Recently a new ingredient was obtained \[14\] proving that every translation invariant topological Pauli stabilizer code on prime qudits must have a nontrivial “boson” which we will define below. The existence of a boson rules out all the nontrivial possibilities listed above (because a boson corresponds to a null vector of the quadratic form \(\theta\)), but leaves a question of whether the decomposition of the quadratic forms can be implemented physically. In this paper we answer this question in the affirmative, almost closing the classification problem of two-dimensional topological Pauli stabilizer codes over prime dimensional qudits up to Clifford circuits in the translation invariant case. Simply put, \textit{there is only the toric code.} Let us state our result precisely.

We first recall standard definitions. A \textbf{generalized Pauli matrix} for a \(p\)-dimensional qudit
$\mathbb{C}^p$ is any product of
\[
\exp\left(\frac{2\pi i}{p}\right)I, \quad X = \sum_{j \in \mathbb{Z}_p} |j + 1\rangle \langle j| \quad \text{and} \quad Z = \sum_{j \in \mathbb{Z}_p} \exp\left(\frac{2\pi i}{p}j\right) |j\rangle \langle j|.
\]
(1)
These are defined for any integer $p \geq 1$, but in this paper $p$ is always a prime. A **generalized Pauli operator** is any finite tensor product of generalized Pauli matrices. A **Clifford gate** is a finitely supported unitary that maps every generalized Pauli operator to a generalized Pauli operator, and a **Clifford circuit** is a finite composition of layers of nonoverlapping Clifford gates that are supported on balls of a uniformly bounded radius. The number of layers in a circuit is the **depth** of the circuit. A **Clifford QCA** is an automorphism of the operator algebra that maps every generalized Pauli matrix at a site $s$ to a generalized Pauli operator supported on a ball centered at $s$ of a uniformly bounded radius. A **trivial stabilizer group** is the group of generalized Pauli operators generated by $Z$ on every qudit. The **toric code stabilizer group** \cite{6} is an abelian group of generalized Pauli operators on a two-dimensional lattice $\mathbb{Z}^2$ with two $p$-dimensional qudits per lattice point, generated by $X_{s+\hat{x},1} X_{s,1} X_{s+\hat{x},2} X_{s,2}$ and $Z_{s-\hat{x},1} Z_{s,1} Z_{s-\hat{y},2} Z_{s,2}$ for all sites $s$.

**Theorem I.1.** Let $S$ be an abelian group of generalized Pauli operators acting on a two-dimensional square lattice $\mathbb{Z}^2$ with $q \geq 1$ qudits of a prime dimension $p$ per lattice point. Suppose that

1. (No frustration) if $\omega I \in S$ for $\omega \in \mathbb{C}$, then $\omega = 1$,

2. (Translation invariance) if $P \in S$, then for every translate $P'$ of $P$, we have $\omega P' \in S$ for some $p$-th root of unity $\omega \in \mathbb{C}$ and

3. (Topological order) if any generalized Pauli operator $P$ commutes with every operator of $S$, then $\omega P \in S$ for some $\omega \in \mathbb{C}$.

Then, there exists a Clifford circuit of constant depth that maps $S$ into a direct sum $T^\otimes n \oplus Z$ for some $n \geq 0$ where $T$ is the toric code stabilizer group, and the group $Z$ is the image of the trivial stabilizer group under a Clifford QCA. Here the circuit and QCA are translation invariant with respect to a smaller translation group than the original one.

Physically, the stabilizer group as a whole is more important than a generating set since any local generating set can be used to define a gapped Hamiltonian but the quantum phase of matter only depends on the group \cite[§2]{15}. This is why we have stated our theorem in terms of groups. The theorem should be understood as the scope of the topological phases that can be realized by unfrustrated commuting Pauli Hamiltonians.
One might wonder why there is no reference to the length scale of the topological order. This is because the definition of topological order in [16] is for a family of finite lattices, whereas we work with infinite lattices. Our assumption implies the topological order condition of [16] with length scale being the linear system size under periodic boundary conditions [15]. Whenever we refer to some finiteness on the support of operators, it can usually be transcribed into a uniform bound in a family of finite systems.

Note that our theorem assumes a finite dimensional degrees of freedom per site. This is not just a technical convenience but rather a fundamentally important assumption. Indeed, in a limit $p \to \infty$ that would produce a rotor $U(1)$ model, the basic result that any excitation is attached to a string operator and hence is mobile, cease to be true in general; consider 2+1D analogues of [17]. Moreover, we do not have any stability result against perturbations such as [16] when it comes to rotor models but only an instability result [18].

A Clifford QCA is not a Clifford circuit of constant depth in general. In one dimension, the group of all Clifford QCA that respect translation invariance is well understood — every translation invariant Clifford QCA is a Clifford circuit up to shifts [19]. (This result assumes the finest possible translation invariance, but one can easily relax it to a coarser translation group using polynomial methods.) The question in two dimensions is not fully understood. In three dimensions we know one Clifford QCA that is not a Clifford circuit of constant depth [14].

It also remains an open problem to relax the translation invariance. One might be able to promote an arbitrary system to a periodic system [20], but it appears difficult to adapt the present classification proof directly to nonperiodic situations; the argument for the existence of a boson in [14] relies on bilinear forms over a field of fractions in one variable, for which the translation invariance is used.

Even assuming translation invariance, there is a quantitative question left. In our mapping from a given stabilizer group to a direct sum of toric code stabilizer groups, we had to break the translation invariance down to a smaller group. To some extent this is necessary; a translation group may act nontrivially on the fusion group of anyons [21]. However, our choice of smaller translation group is likely not optimal. One can then ask what the precise order (exponent) of the translation group action on the fusion group of anyons is. Generally this question is to be answered as a function of interaction range and unit cell size. A lower bound that is exponential in the interaction range is known [15, §7 Rem. 3], but the upper bound is largely open. In fact, we have not kept track of the index of this translation subgroup.

The rest of this paper constitutes the proof of Theorem I.1.
II. TRANSCRIPTION TO POLYNOMIALS

Following [15] (see also [13] and a summary section [14, §IV.A]) we transcribe the problem into a polynomial framework by regarding translation invariant groups of generalized Pauli operators modulo phase factors, which are abelian, as modules over the translation group algebra $R = \mathbb{F}_p[x^\pm, y^\pm]$. In particular, the abelianized group of generalized Pauli operators is a free module $R^{2q}$ where $q$ is the number of qudits per lattice site (unit cell), equipped with a nondegenerate symplectic form that captures commutation relations. Below we will not distinguish generalized Pauli operators from an element of $R^{2q}$ as the phase factors will not be important.

We use a $\mathbb{F}_p$-linear ring homomorphism $\phi^{(m)} : \mathbb{F}_p[x'^\pm, y'^\pm] \to R$ such that $x' \mapsto x^m$ and $y' \mapsto y^m$ to denote coarse-graining, which induces formally a covariant functor $\phi^{(m)}_\#$ on the category of modules. The domain of this morphism is interpreted as the group algebra for a smaller translation group, enlarging the unit cell of the qudit system $m \times m$ times as large as the original one. We use $\overline{\cdot}$ to denote the $\mathbb{F}_p$-linear involution of $R$ such that $x \mapsto \bar{x} = x^{-1}$ and $y \mapsto \bar{y} = y^{-1}$, and $\dagger$ the involution followed by transpose for matrices over $R$. Let $I_q$ denote the $q \times q$ identity matrix. We fix specific matrices over $R$:

\[ \lambda_q = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}, \quad \epsilon_0 = \begin{pmatrix} x - 1 & y - 1 & 0 & 0 \\ 0 & 0 & \bar{y} - 1 & -\bar{x} + 1 \end{pmatrix} \quad \text{and} \quad \sigma_0 = (\epsilon_0 \lambda_q^{-1})^\dagger \]

where $\epsilon_0$ describes the $\mathbb{Z}_p$ toric code [6] on the square lattice [15, §5 Ex. 2]. For the clarity in notation we define a matrix $E_{i,j}(a)$ for any $a \in R$ as

\[ [E_{i,j}(a)]_{\mu\nu} = \delta_{\mu\nu} + \delta_{\mu i} \delta_{\nu j} a \quad \text{where } \delta \text{ is the Kronecker delta.} \]

**Definition II.1.** For a given positive integer $q$, the following $2q \times 2q$ matrices generate the elementary symplectic group denoted by $\text{ESp}^\times(q; R)$:

- **Hadamard:** $E_{i,i+q}(-1)E_{i+q,i}(1)E_{i,i+q}(-1)$ where $1 \leq i \leq q$,
- **control-Phase:** $E_{i+q,j}(a)E_{j+q,i}(\bar{a})$ where $1 \leq i, j \leq q$,
- **control-Not:** $E_{i,j}(a)E_{j+q,i}(-\bar{a})$ where $1 \leq i \neq j \leq q$,
- **extra gate:** $E_{i,i}(a - 1)E_{i+q,i+q}(a^{-1} - 1)$ where $a \in \mathbb{F}_p^\times, 1 \leq i \leq q$.

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1 Already this perspective implies the following: Note that in Theorem I.1 we do not assume that the stabilizer group is generated by operators whose supports are contained in disks of a uniformly bounded radius; however, since $R$ is Noetherian and the group of all generalized Pauli operators up to phase factors is a finitely generated module over $R$, we see that the stabilizer module is finitely generated, which means that the stabilizer group has local generators.
The symplectic group denoted by $\text{Sp}^+(q; R)$ consists of all $U \in \text{Mat}(2q; R)$ such that $U^\dagger \lambda_q U = \lambda_q$.2

The elementary symplectic group is a subgroup of the symplectic group as one can verify. Note that the control-Phase3 gate with $i = j$ is equivalent to $E_{i+q,i}(f)$ for some $f = \bar{f}$, and conversely any $E_{i+q,i}(f)$ with $f = \bar{f}$ can be written as the control-Phase with $i = j$ since any such $f$ is of form $a + \bar{a}$ for some nonunique $a$. The extra gate is the identity if $p = 2$.

**Theorem II.2** (Implying Theorem I.1). Let $\sigma$ be a $2q \times t$ matrix over $R$, interpreted as a map acting on the left of a column vector of length $t$, such that

$$\ker \sigma^\dagger \lambda_q = \text{im} \sigma.$$  

(4)

Then, there exist an integer $m \geq 1$ and matrices $E \in \text{ESp}^+(mq; R)$ and $U \in \text{Sp}^+(mq; R)$ such that

$$E \text{im} \phi^{(m)}_\#(\sigma) = \left(U \text{im} \begin{pmatrix} I_{mq} \\ 0 \end{pmatrix}\right) \oplus \bigoplus^n \text{im} \sigma_0$$

(5)

where the images are over $R' = F_p[x^{\pm m}, y^{\pm m}] \subseteq R$ and $2n$ is the $F_p$-dimension of the torsion submodule of $\text{coker} \sigma^\dagger$.

**Proof of the transcription.** Any translation invariant abelian group of generalized Pauli operators corresponds to a submodule $S$ of the $R$-module $P$ of all generalized Pauli operators (forgetting phase factors) with the property that $v^\dagger \lambda_q v = 0$ for any $v \in S$ [15, Prop. 1.2]. Picking a generating set for the module $S$, which amounts to writing $S = \text{im} \sigma$ for some matrix $\sigma$ over $R$, we have a matrix equation $\sigma^\dagger \lambda_q \sigma = 0$. The topological order condition [16] is equivalent to $\ker \sigma^\dagger \lambda_q = \text{im} \sigma$ [15, Lem. 3.1]. An elementary symplectic transformation is induced by a Clifford circuit of finite depth and a symplectic transformation is induced by a Clifford QCA [15, §2.1]. These (elementary) symplectic transformations are forgetful only of a conjugation by a (possibly infinitely supported) generalized Pauli operator [15, Prop. 2.2]. The module $\text{im} \sigma_0$ is the same as the toric code stabilizer group.

III. PROOF

The proof of our main theorem is by induction in the vector space dimension $k$ of the torsion part of $\text{coker} \epsilon$ where $\epsilon = \sigma^\dagger \lambda_q$ is the excitation map of the given stabilizer group. It has been proved

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2 The $^\dagger$ in the notation $(E)\text{Sp}^+$ refers to the fact that we collect matrices $U$ that obeys $U^\dagger \lambda_q U = \lambda_q$. Sometimes a different group consisting of $U$ such that $U^T \lambda_q U = \lambda_q$ is considered in literature, and our notation differentiate them.

3 This includes the induced action by the “phase” gate diag(1,i) but not diag(1,1,1,i) for qubits ($p = 2$). Hence, this terminology may be inconsistent with that elsewhere.
that the excitation map \( \epsilon \) can be chosen (as it depends on a generating set for the stabilizer group) such that \( \text{coker} \, \epsilon \) is a torsion module, and furthermore after a suitable choice of a smaller translation group (coarse-graining) the annihilator of \( \text{coker} \, \epsilon \) becomes a maximal ideal \( m = (x-1, y-1) \subset R \) \cite[§7 Thm. 4]{ref1}. Such \( \epsilon \) must be \( q \times 2q \) to satisfy the topological order condition \( \ker \epsilon = \text{im} \lambda_q \epsilon^\dagger \). Hence, without loss of generality we begin with an extra assumption, whenever \( \text{coker} \, \epsilon \neq 0 \), that \( \text{ann} \, \text{coker} \, \epsilon = m = (x-1, y-1) \). \hfill (6)

This implies that \( k \) is finite. We will extract a copy of the toric code whenever \( k \geq 2 \), decreasing this dimension \( k \) by 2. This is done by trivializing string operators that transports bosons whose existence is proved in \cite[Cor. III.20]{ref2}. The possibility \( k = 1 \) will be ruled out in the course of the proof. The case \( k = 0 \) is treated in \cite[§IV.B]{ref2} where a Clifford QCA that maps the stabilizer group to the trivial stabilizer group for a product is constructed. These will complete the proof.

### A. Topological spin

We consider the topological spins of the excitations of a given topological stabilizer group. We will define them using hopping operators for the excitations in a way that is tailored to our setting. Recall that two generalized Pauli operators represented as two column vectors \( u, v \) over \( R \) commute if and only if the coefficient of \( x^0 y^0 = 1 \in R \) in \( u^\dagger \lambda_q v \), which we denote by \( [u, v] = -[v, u] \in \mathbb{F}_p \), vanishes.

**Definition III.1.** Assuming Eq. (6), an excitation is any element in the codomain of \( \epsilon \) or equivalently any element in the domain of \( \sigma \). An excitation \( e \) is nontrivial if \( e \notin \text{im} \epsilon \). A (topological) charge is then an equivalence class of excitations modulo trivial ones. An \( x \)-mover \( p_x(e) \) for an excitation \( e \) is any generalized Pauli operator \( p \in R^{2q} \) such that \( \epsilon(p) = (x-1)e \). Likewise, a \( y \)-mover \( p_y(e) \) is any \( p \in R^{2q} \) such that \( \epsilon(p) = (y-1)e \). The topological spin \( \theta(e) \in \mathbb{F}_p \) of an excitation is

\[
\theta(e) = \lim_{n \to +\infty} [a_n, b_n] + [b_n, c_n] + [c_n, a_n] \quad (7)
\]

where \( a_n = (x^{-n} + x^{-n+1} + \cdots + x^{-1})p_x(e) \),
\( b_n = (y^{-n} + y^{-n+1} + \cdots + y^{-1})p_y(e) \) and \( c_n = -(1 + x + x^2 + \cdots + x^n)p_x(e) \).

The topological spin can and should be defined much more generally, but we use this narrow definition which is sufficient for this paper. The existence of movers for any excitation is precisely
the content of Eq. (6). Note that movers are not unique since any element of \( \ker \epsilon = \text{im} \sigma \) can be added to them. The definition of topological spin is due to \cite{22} using commutation relations among three hopping operators (movers) attached to an excitation; we have transcribed it into our additive notation. The limit exists because \( a_n, b_n, c_n \) are long “string” operators and string segments far away from the origin of the lattice always commute. Pictorially, \( a_n \) is inserting \( e \) from the left infinity to the origin, \( b_n \) from the bottom, and \( c_n \) from the right.

**Lemma III.2.** \( \theta(e) \) is independent of the choices of the movers. \( \theta(e') = \theta(e) \) if \( e' - e \) is trivial. Moreover, the expression Eq. (7) gives the same value if \( a_n, b_n, c_n \) are replaced respectively by

\[
\begin{align*}
a'_n &= (x^{-n} + x^{-n+1} + \cdots + x^{-1})p_x(e), \\
b'_n &= -(1 + x + x^2 + \cdots + x^n)p_x(e) \quad \text{and} \\
c'_n &= -(1 + y + y^2 + \cdots + y^n)p_y(e).
\end{align*}
\]

That is, we could have that \( a_n \) is inserting \( e \) from the left infinity to the origin, \( b_n \) from the right, and \( c_n \) from the top. Actually, they can be any string operators as long as they circle around \( e \) counterclockwise.

**Proof.** The movers are unique up to \( \ker \epsilon = \text{im} \sigma \). It suffices to consider a modification by \( s \in \text{im} \sigma \) to \( p_x(e) \) in \( a_n \); other modifications are similarly treated. The change is \( \Delta \theta(e) = [(x^{-n} + x^{-n+1} + \cdots + x^{-1})s, b_n - c_n] \) for some sufficiently large \( n \). Here, \( b_n - c_n \) creates excitations near \((0, -n)\) and \((n, 0)\) where \((x^{-n} + x^{-n+1} + \cdots + x^{-1})s\) does not have any support, and hence the commutator \( \Delta \theta(e) \) vanishes.

The second claim amounts to modifying the \( x \)-mover by \( f - g \) for \( f, g \in \mathbb{R}^{2q} \) such that \( \epsilon(f) = xe - xe' \) and \( \epsilon(g) = e - e' \), and the \( y \)-mover by \( h - \ell \) for \( h, \ell \in \mathbb{R}^{2q} \) such that \( \epsilon(h) = ye - ye' \) and \( \epsilon(\ell) = e - e' \). By the first claim, we may assume \( f = xg, h = yg \) and \( \ell = g \) since \( f - xg, h - yg, \ell - g \in \text{im} \sigma \). Then, the change in \( \theta(e) \) is \([g - x^{-n}g, g - y^{-n}g] + [g - y^{-n}g, g - x^{n+1}g] + [g - x^{n+1}g, g - x^{-n}g]\), which eventually becomes zero for large \( n \).

For the third claim, note that \( a' = a \) and \( b' = c \). The difference between \( \theta(e) \) of Definition III.1 and \( \theta'(e) \) using the primed string operators, is \([b - c, a - c'] + [c - c', b - a]\), which vanishes for a similar reason as in the first claim. \( \square \)

Therefore, \( \theta \) maps from the set of all topological charges to \( \mathbb{F}_p \), which is a quadratic form on the \( \mathbb{F}_p \)-vector space \( \text{coker} \epsilon \).\(^4\) If \( \theta(e) = 0 \), we call \( e \) a **boson**.

\(^4\) The definition of a quadratic form \cite{11} requires that its polar form \( S(e, e') = \theta(e + e') - \theta(e) - \theta(e') \) be bilinear.
Lemma III.3 (Cor. III.20 of [14]). If \( \text{coker} \, \epsilon \neq 0 \), then there exists a nontrivial excitation \( e \) such that \( \theta(e) = 0 \), i.e., a nontrivial boson exists.

B. Simplification of movers and excitation maps

The rest of the proof of Theorem II.2 consists of elementary computation. In Lemma III.4 below we will choose movers such that they commute with any of its translates. In Lemma III.6 we will further simplify the movers to determine two columns of \( \epsilon \). Then, using the exactness condition Eq. (4), we will find an accompanying row of \( \epsilon \); this will rule out the possibility \( k := \dim_{F_p} \text{coker} \, \epsilon = 1 \). The determined columns and a row will be further simplified in Lemma III.7 and in turn will single out a direct summand \( \epsilon_0 \) from \( \epsilon \).

Lemma III.4. Under a choice of a sufficiently large unit cell (reducing the translation group to a subgroup of a finite index via \( \phi^{(m)} \) for some \( m \)), the movers \( p_x(e), p_y(e) \) for a boson \( e \) can be chosen such that

\[
p_x(e) \dagger \lambda_q p_y(e) = p_x(e) \dagger \lambda_q p_x(e) = p_y(e) \dagger \lambda_q p_y(e) = 0.
\]

That is, the movers and all their translates can be made commuting.

Proof. Let us drop the reference to \( e \) since we fix \( e \). If \( R' = F_p[x^\pm, y^\pm] \) injects into \( R \) by \( \phi^{(m)} \), then a mover \( p_{x'} \) with respect to \( R' \) \( ((x' - 1)e = \epsilon(p_{x'})) \) can be chose as \( s_{\text{head}} + s_{\text{tail}} + (1 + x + \cdots + x^{m-1})p_x \), that is the \( m \) movers aligned along the moving direction with its “head” near \((m,0)\) and “tail” near the origin modified by \( s_{\text{head, tail}} \in \text{im} \sigma \). Similarly, \( p_{y'} = (1 + y + \cdots + y^{m-1})p_y + t_{\text{tail}} \) with \( t_{\text{tail}} \in \text{im} \sigma \) near the origin. For a large enough \( m \), we know

\[
0 = \theta = \begin{bmatrix} x'p_{x'}, y'p_{y'} \end{bmatrix} + \begin{bmatrix} y'p_{y'}, -p_{x'} \end{bmatrix} + \begin{bmatrix} -p_{x'}, x'p_{x'} \end{bmatrix}
\]

(8)

according to Eq. (7). We can choose \( s_{\text{head}} \) such that the first commutator vanishes; by definition \( e \) is some nontrivial commutator with an element of \( \text{im} \sigma \) near the origin. Also, we can choose \( s_{\text{tail}} \) such that the second commutator vanishes. The chosen \( s_{\text{head, tail}} \) are independent of all sufficiently large \( m \). After these choices, the third commutator must vanish. Inspecting the alternative formula for \( \theta \) in Lemma III.2, we see

\[
0 = \begin{bmatrix} x'p_{x'}, -p_{x'} \end{bmatrix} + \begin{bmatrix} -p_{x'}, y'p_{y'} \end{bmatrix} + \begin{bmatrix} -p_{y'}, x'p_{x'} \end{bmatrix}
\]

(9)

over \( F_p \). This is true, and in fact one can show that this modular \( S \) is precisely the commutation relation of all logical operators of the finite dimensional stabilizer code on any sufficiently large but finite 2-torus. In particular, \( S \) is nondegenerate. One can classify the quadratic forms \( \theta \) based only on the fact that it is \( F_p \)-valued and \( S \) is nondegenerate, and the result is in Introduction.
where the first term is equal to the third term of Eq. (8) that vanishes. The second term of Eq. (9) can be made zero by choosing \( t_{tail} \), and then the third term vanishes. Thus, all the heads and tails of \( p_x' \) and \( p_y' \) are commuting.

For a large enough \( m \), and after a possible redefinition of the unit cell, this new mover will be supported only on the unit cell at the origin and the unit cell at \((1, 0)\) or at \((0, 1)\). Thus, the only potentially nontrivial commutators are when the head or tail of a mover meets the head or tail of another mover, but we have made these commutators to vanish.

Lemma III.5. For any nontrivial topological charge \( e \), there exists a free basis for the stabilizer module (the columns of \( \sigma \)) such that \( e \) is represented by one basis element whose all components belong to the ideal \( \mathfrak{m} = (x - 1, y - 1) \subset R \).

Proof. Let \( \sigma \) be chosen such that \( \epsilon = \sigma^\dagger \lambda_q \) satisfies Eq. (6). Apply row operations \( GL(q; \mathbb{F}_p) \) to \( \epsilon \) so that \( \epsilon|_{x=1,y=1} \) is in the reduced row echelon form. The set of all topological charges is \( \text{coker} \, \epsilon \) which is in fact a vector space over \( \mathbb{F}_p \) on which the translation group has trivial action. The dimension \( k = \dim_{\mathbb{F}_p} \text{coker} \epsilon = \dim_{\mathbb{F}_p} \text{coker} \epsilon|_{x=1,y=1} \) is precisely the number of all zero rows of \( \epsilon|_{x=1,y=1} \). Hence, any nonzero element of the codomain of \( \epsilon \) that is supported on these last \( k \) components represents a nontrivial topological charge. Therefore, for any nontrivial charge \( e \), a representative vector in \( \text{coker} \, \epsilon \) can be mapped to a unit vector by some \( GL(k; \mathbb{F}_p) \) acting on the last \( k \) components.

Lemma III.6. Under a choice of a sufficiently large unit cell, the movers \( p_x, p_y \) for a nontrivial boson \( e \) can be mapped to

\[
\begin{align*}
p_x &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}^T, \\
p_y &= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^T
\end{align*}
\]

by some elementary symplectic transformation \( ESp^\dagger(q; R) \) (Clifford circuit).

Proof. As in the proof of Lemma III.4, we may assume that \( p_x \) and \( p_y \) are supported on at most two adjacent unit cells. In particular, \( p_x \) is a Laurent polynomial vector over \( \mathbb{F}_p[x^\pm] \), satisfying \( p_x^\dagger \lambda_q p_x = 0 \). Since \( \mathbb{F}_p[x^\pm] \) is a principal ideal domain, we can find an elementary symplectic transformation that turns \( p_x \) into a vector with a single nonzero component; see the computation in [15, §6]. But the single component \( g \) must be a monomial; otherwise, under the choice of \( \epsilon \) of Lemma III.5 we would have \( fg = (x - 1) \) for some \( f \in \mathfrak{m} = (x - 1, y - 1) \) but \( x - 1 \) is an irreducible
polynomial. By redefining a basis element of the stabilizer module by monomial multiplication, we can bring the monomial $g$ to 1. Thus, $p_x$ has been turned into the promised form.

Once $p_x$ is put in the promised form, further coarse-graining does not complicate $p^{x'}$; under $\phi^{(m)}_\#$, the operator $p^{x'} = (1 + x + \cdots + x^{m-1})p_x$ is still supported on one new unit cell, and $\text{ESp}^\dagger(mq; \mathbb{F}_p)$ can bring $p^{x'}$ into the promised form. Therefore, we may assume that $p_x$ is in the promised form with the sole nonzero entry 1 at the first component, and $p_y$ is a polynomial in $y$ with all exponents being 0 or 1 not involving $x$. Since $p^\dagger \lambda_q p_y = 0$, the $q + 1$-st component of $p_y$ must be zero. This forces $q \geq 2$. Now, we look at the $2, 3, \ldots, q, q + 2, q + 3, \ldots, 2q$-th components of $p_y$. If they generate the unit ideal ($= \mathbb{F}_p[y^\pm]$), then clearly $p_y$ can be turned into the promised form. If not, then by the exponent restriction we can use some transformation of $\text{ESp}^\dagger(q - 1; \mathbb{F}_p)$ to turn $p_y$ into one that has $y - v$ at the second component with $v \in \mathbb{F}_p$, some $u \in \mathbb{F}_p$ at the first component and zeros in all the other components. Then, under the choice of $\epsilon$ of Lemma III.5 we would have $(x - 1)u + f(y - v) = y - 1$ for some $f \in \mathfrak{m}$, but $ux - u - y + 1 = -f(y - v)$ is an irreducible polynomial, a contradiction. 

**Proof of Theorem II.2.** Let $k$ be the $\mathbb{F}_p$-dimension of the torsion submodule of coker $\epsilon$. If $k = 0$, then by [15, Lem. 7.1] $\sigma$ can be chosen to be kernel free, and coker $\epsilon$ must be pure torsion, which implies coker $\epsilon = 0$, so the first Fitting ideal of $\epsilon$ is unit, and [15, Cor. 4.2] says that the code on any finite periodic lattice encodes zero logical qudit. Then, [14, Thm. IV.4] provides a Clifford QCA that maps the stabilizer group to the trivial one.

If $k \neq 0$, as remarked earlier we can assume Eq. (6). By Lemma III.3 we have a nontrivial boson, and its movers can be chosen as in Lemma III.6 by reducing the translation invariance to a subgroup of a finite index. Since these simplified movers have only one nonzero entry of 1, under the choice of stabilizer generators in Lemma III.5 we must have

$$\epsilon = \begin{pmatrix}
    x - 1 & y - 1 & * & * & \cdots & * \\
    0 & 0 & * & * & \cdots & * \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & * & * & \cdots & * 
\end{pmatrix} \quad (10)$$

where $*$ indicates an unknown entry, but the first row has entries from $\mathfrak{m} = (x - 1, y - 1)$. Since $\begin{pmatrix} y - 1 & -x + 1 & 0 & \cdots & 0 \end{pmatrix}^T$ is in ker $\epsilon$, the condition ker $\epsilon = \text{im} \lambda_q \epsilon^\dagger$ implies that the rows of $\epsilon$ must generate

$$\begin{pmatrix}
    0 & \cdots & 0 & \bar{y} - 1 & -\bar{x} + 1 & 0 & \cdots & 0 
\end{pmatrix} \quad (11)$$
over $R$ where $\bar{y} - 1$ is at the $q + 1$-th position. But the $R$-linear combination that results in this row vector cannot contain a nonzero summand of the first row in Eq. (10). This make it impossible for $k$ to be 1 since $k$ is the number of rows of $\epsilon$ that becomes zero by setting $x = 1 = y$; see the proof of Lemma III.5.

Let us add Eq. (11) to $\epsilon$ to make $\epsilon'$; this amounts to increasing the number of generators for the stabilizer module by 1. (The new $\epsilon'$ does not satisfy Eq. (6) since $\text{coker } \epsilon'$ is not torsion.)

$$
\epsilon' = \begin{pmatrix}
  x - 1 & y - 1 & \cdots & \star & u & \star & \cdots & \star \\
  0 & 0 & 0 & \cdots & 0 & \bar{y} - 1 & -\bar{x} + 1 & 0 & \cdots & 0 \\
  0 & 0 & \star & \cdots & \star & \star & \cdots & \star \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \vdots & \vdots \\
  0 & 0 & \star & \cdots & \star & \star & \cdots & \star
\end{pmatrix}
$$

(12)

where $u, v$ are unknown. Since the first row of $\epsilon'$ has entries in $m$, the two generator of $m$ in the top left may eliminate all $\star$ in the first row by control-Not and control-Phase gates which act on the right of $\epsilon'$. These operations does not affect the columns of $u, v$. The second row also remains intact. Thus we obtain

$$
\epsilon'' = \begin{pmatrix}
  x - 1 & y - 1 & 0 & \cdots & 0 & u & v & 0 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & 0 & \bar{y} - 1 & -\bar{x} + 1 & 0 & \cdots & 0 \\
  0 & 0 & \star & \cdots & \star & \star & \cdots & \star \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \vdots & \vdots \\
  0 & 0 & \star & \cdots & \star & \star & \cdots & \star
\end{pmatrix}
$$

(13)

Here, each pair of entries in the columns of $u, v$, below the second row, must be a $R$-multiple of $\left(\bar{y} - 1 - \bar{x} + 1\right)$ for $\epsilon'^\dagger \lambda_0 \epsilon' = 0$; the symplectic product with the first row enforces this. Hence, they can be eliminated by $\text{GL}(q + 1; R)$ on the left of $\epsilon'$. Now we use the following elementary fact which will be proved shortly.

**Lemma III.7.** For $u, v \in R$, if a matrix

$$
\epsilon = \begin{pmatrix}
x - 1 & y - 1 & u & v \\
0 & 0 & \bar{y} - 1 & -\bar{x} + 1
\end{pmatrix}
$$

(14)

satisfies $\epsilon \lambda_2 \epsilon^\dagger = 0$, then there exist $A \in \text{GL}(2; R)$ and $B \in \text{ESp}^\dagger(2; R)$ such that $A \epsilon B = \epsilon_0$. 

Then we obtain

$$
\epsilon'' = \begin{pmatrix}
  x - 1 & y - 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & 0 & y - 1 & -x + 1 & 0 & \cdots & 0 \\
  0 & 0 & * & \cdots & * & 0 & 0 & \cdots & * \\
  \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
  0 & 0 & * & \cdots & * & 0 & 0 & \cdots & *
\end{pmatrix}.
$$

(15)

It is evident that $\epsilon_0$ is a direct summand. The direct summand of $\epsilon''$ that is complementary to $\epsilon_0$ gives a stabilizer module that satisfies the conditions of our theorem. This proves the induction step decreasing $k$ by 2, and thus completes the proof of the theorem.

\[ \square \]

**Proof of Lemma III.7.** By long division we can write $u = (\bar{y} - 1)u' + u''$ where $u'' \in \mathbb{F}_p[x^\pm]$. By a row operation on the left of $\epsilon$, we can eliminate $(\bar{y} - 1)u'$ and thus we may assume that $u = u'' \in \mathbb{F}_p[x^\pm]$. The equation $\epsilon\lambda_2 \epsilon^2 = 0$ implies that $(x - 1)\bar{u} + (y - 1)\bar{v} = (\bar{x} - 1)u + (\bar{y} - 1)v$, which can be rearranged as

$$(x - 1)(\bar{u} + \bar{x}u) = -(y - 1)(\bar{v} + \bar{y}v).$$

The left hand side is a Laurent polynomial in $x$, and therefore, if nonzero, it is not divisible by $y - 1$. Hence, $\bar{u} + \bar{x}u = 0 = \bar{v} + \bar{y}v$. Let $u = \sum_j u_j x^j$ where $u_j \in \mathbb{F}_p$ be the expansion of $u$. It follows that $u_{-j} + u_{j+1} = 0$ for all $j$. This implies that $u|_{x=1} = 0$ so $u = (x - 1)h$ for some $h \in \mathbb{F}_p[x^\pm]$. Substituting, we have $\bar{u} + \bar{x}u = (\bar{x} - 1)\bar{h} - (\bar{x} - 1)h = 0$ or $h = \bar{h}$. Thus the control-Phase gate on the first qudit can eliminate $u$.

We are left with an equation $\bar{v} + \bar{y}v = 0$. Write $v = (x - 1)s + f$ where $f \in \mathbb{F}_p[y^\pm]$. Then, we have $(x - 1)(-\bar{x}s + \bar{y}s) + \bar{f} + \bar{y}f = 0$. Since $f$ is constant in $x$, we have $f + \bar{y}f = 0$ implying $f = (y - 1)g$ for some $g = \bar{y} \in \mathbb{F}_p[y^\pm]$ as before and hence $f$ can be eliminated by the control-Phase on the second qudit.

The remaining term $(x - 1)s$ of $v$ satisfies $\bar{x}y\bar{s} = s$. Let $z = \bar{x}y$ to avoid typos. Write $s = \sum_j y^j s_j$ where $s_j \in \mathbb{F}_p[\bar{z}^\pm]$; such an expression is unique as seen by considering a ring isomorphism where $z \mapsto \bar{x}y, y \mapsto y$. We have $s_{-j} = z\bar{s}_j$ for all $j$ and hence $s_0 = (z + 1)\ell$ for some $\ell = \bar{\ell} \in \mathbb{F}_p[z^\pm]$. Let $s_+ = \sum_{j>0} y^j s_j$. Then $s = s_+ + z\bar{s}_+ + (z + 1)\ell = y(x\ell + xs_+) + \bar{x}(x\ell + xs_+)$. But this form of $s$ is precisely the form that can be implemented by the control-Phase between the first and second qudits with $a = y(x\ell + xs_+)$ and a simultaneous row addition from the second to first by $x\ell + xs_+$ to keep $u$ intact. Therefore, both $u$ and $v$ can be eliminated by the asserted $A$ and $B$. \[ \square \]
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