APOLLONIUS CIRCLES AND IRREDUCIBILITY CRITERIA FOR POLYNOMIALS

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Dedicated to our friend, Professor Maurice Mignotte

Abstract. We prove the irreducibility of integer polynomials $f(X)$ whose roots lie inside an Apollonius circle associated to two points on the real axis with integer abscissae $a$ and $b$, with ratio of the distances to these points depending on the canonical decomposition of $f(a)$ and $f(b)$. In particular, we obtain irreducibility criteria for the case where $f(a)$ and $f(b)$ have few prime factors, and $f$ is either an Eneström-Kakeya polynomial, or has a large leading coefficient. Analogous results are also provided for multivariate polynomials over arbitrary fields, in a non-Archimedean setting.

1. Introduction

One of the methods to study the irreducibility of polynomials is to use information on the values that they take at some specified integer arguments. A famous result of Pólya [39] considers only the magnitude of the absolute values that a polynomial takes, with disregard to their canonical decomposition:

**Theorem 1.** If for $n$ integral values of $x$, the integral polynomial $f(x)$ of degree $n$ has values which are different from zero, and in absolute value less than

$$ \frac{\lceil n/2 \rceil!}{2^{n/2}}, $$

then $f(x)$ is irreducible over $\mathbb{Q}$.

Since 1919 this result was generalized in many different ways, of which we only mention here two recent ones, corresponding to the setting where the coefficients belong to the ring of integers of an arbitrary imaginary quadratic number field [32], and to the multivariate case over an arbitrary field [12].

Other irreducibility criteria in the literature rely heavily on the canonical decomposition of the value that a given polynomial takes at a single, specified integral argument. The most interesting results of this kind take benefit of the existence in this canonical decomposition of a suitable prime divisor, or prime power divisor. For instance, in [40] Pólya and Szegö give the following nice irreducibility criterion of A. Cohn:

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Theorem 2. If a prime $p$ is expressed in the decimal system as
\[ p = \sum_{i=0}^{n} a_i 10^i, \quad 0 \leq a_i \leq 9, \]
then the polynomial $\sum_{i=0}^{n} a_i X^i$ is irreducible in $\mathbb{Z}[X]$.

Brillhart, Filaseta and Odlyzko [16] extended this result to an arbitrary base $b$:

Theorem 3. If a prime $p$ is expressed in the number system with base $b \geq 2$ as
\[ p = \sum_{i=0}^{n} a_i b^i, \quad 0 \leq a_i \leq b - 1, \]
then the polynomial $\sum_{i=0}^{n} a_i X^i$ is irreducible in $\mathbb{Z}[X]$.

Filaseta [25] obtained another generalization of Cohn’s theorem by replacing the prime $p$ by a composite number $pq$ with $q < b$:

Theorem 4. Let $p$ be a prime number, $q$ and $b$ positive integers, $b \geq 2$, $q < b$, and suppose that $pq$ is expressed in the number system with base $b$ as
\[ pq = \sum_{i=0}^{n} a_i b^i, \quad 0 \leq a_i \leq b - 1. \]
Then the polynomial $\sum_{i=0}^{n} a_i X^i$ is irreducible over the rationals.

Cohn’s irreducibility criterion was also generalized in [16] and [26] by permitting the coefficients of $f$ to be different from digits. For instance, the following irreducibility criterion for polynomials with non-negative coefficients was proved in [26].

Theorem 5. Let $f(X) = \sum_{i=0}^{n} a_i X^i$ be such that $f(10)$ is a prime. If the $a_i$’s satisfy $0 \leq a_i \leq a_n 10^{30}$ for each $i = 0, 1, \ldots, n - 1$, then $f(X)$ is irreducible.

Cole, Dunn, and Filaseta produced in [20] sharp bounds $M(b)$ depending on an integer $b \in [3, 20]$ such that if each coefficient of a polynomial $f$ with non-negative integer coefficients is at most $M(b)$ and $f(b)$ is prime, then $f$ is irreducible.

Some classical related results relying on the canonical decomposition of the value that a polynomial takes at some integral argument may be also found in the works of Stäckel [42], Ore [37], Weisner [43] and Dorwart [24]. For an unifying approach that uses the concept of admissible triples to study irreducibility of polynomials, we refer the reader to [29]. Along with a simultaneous generalization of some classical irreducibility criteria, one may also find in [29] upper bounds for the total number of irreducible factors (counting multiplicities) for some classes of integer polynomials (see also [31] for problems related to the study of roots multiplicities and square free factorization). For further related results and some elegant connections between prime numbers and irreducible polynomials, the reader is referred to [41], [28] and [7], for instance.
Another method to obtain irreducible polynomials is to write prime numbers or prime powers as a sum of integers of arbitrary sign, of which one has sufficiently large modulus, and to use these integers as coefficients of our polynomials. In this respect we refer the reader to \[\text{[8]}, \text{[9]}\], where several irreducibility criteria for polynomials that take a prime value or a prime power value and have a coefficient of sufficiently large modulus have been obtained. Two such irreducibility criteria are given by the following results:

**Theorem 6.** If we write a prime number as a sum of integers \(a_0, \ldots, a_n\), with \(a_0a_n \neq 0\) and \(|a_0| > \sum_{i=1}^n |a_i|2^i\), then the polynomial \(\sum_{i=0}^n a_iX^i\) is irreducible over \(\mathbb{Q}\).

**Theorem 7.** If we write a prime power \(p^s\), \(s \geq 2\), as a sum of integers \(a_0, \ldots, a_n\) with \(a_0a_n \neq 0\), \(|a_0| > \sum_{i=1}^n |a_i|2^i\), and \(a_1 + 2a_2 + \cdots + na_n\) not divisible by \(p\), then the polynomial \(\sum_{i=0}^n a_iX^i\) is irreducible over \(\mathbb{Q}\).

Other recent results where prime numbers play a central role in testing irreducibility refer to linear combinations of relatively prime polynomials \[\text{[17]}, \text{[18]}, \text{[13]}\], and to compositions of polynomials \[\text{[30]}\] and \[\text{[15]}\]. Counterparts of such results for the multivariate case may be found in \[\text{[19]}, \text{[14]}, \text{[10]}, \text{[11]}\]. For some recent fundamental results on reduction, specialization and composition of polynomials \[\text{[30]}\] and \[\text{[15]}\].

The aim of this paper is to provide irreducibility criteria that depend on the information on the canonical decomposition of the values that a polynomial \(f\) takes at two integer arguments, by also using information on the location of their roots, and then to obtain similar results in the multivariate case over an arbitrary field. As we shall see, to obtain sharper irreducibility conditions we will also make use of information on the derivative of \(f\), or on the partial derivatives of \(f\) in the multivariate case. First of all, let us note that if a polynomial \(f(X) \in \mathbb{Z}[X]\) factors as \(f(X) = g(X)h(X)\) with \(g(X), h(X) \in \mathbb{Z}[X]\) and \(\deg g \geq 1, \deg h \geq 1\), then if we fix an arbitrarily chosen integer \(a\) with \(f(a) \neq 0\), the integers \(g(a)\) and \(h(a)\) are not some arbitrary divisors of \(f(a)\), as they must also satisfy the equality \(f'(a) = g'(a)h(a) + g(a)h'(a)\).

It implies that the greatest common divisor of \(g(a)\) and \(h(a)\) divides \(f(a)\) and \(f'(a)\). This suggests the use of the following definition.

**Definition 1.1.** Let \(f\) be a non-constant polynomial with integer coefficients, and let \(a\) be an integer with \(f(a) \neq 0\). We say that an integer \(d\) is an admissible divisor of \(f(a)\) if \(d \mid f(a)\) and

\[
\gcd\left(d, \frac{f(a)}{d}\right) \mid \gcd(f(a), f'(a)),
\]

and we shall denote by \(\mathcal{D}_{ad}(f(a))\) the set of all admissible divisors of \(f(a)\). We say that an integer \(d\) is a unitary divisor of \(f(a)\) if \(d\) is coprime with \(f(a)/d\). We denote by \(\mathcal{D}_u(f(a))\) the set of unitary divisors of \(f(a)\).

We note that condition (1) is symmetric in \(d\) and its complementary divisor \(f(a)/d\), and that if \(\gcd(f(a), f'(a)) = 1\), then \(\mathcal{D}_{ad}(f(a))\) reduces to the set \(\mathcal{D}_u(f(a))\).
The first result that we will prove relies on information on the admissible divisors of \(f(a)\) and \(f(b)\) for two integers \(a, b\). Rather surprisingly, the study of the irreducibility of \(f\) can be connected with the location of the roots of \(f\) inside an Apollonius circle associated to the points on the real axis with integer abscissae \(a\) and \(b\), and ratio of the distances to these two points expressed only in terms of the admissible divisors of \(f(a)\) and \(f(b)\). We recall the famous result of Apollonius, stating that the set of points \(P\) in the plane such that the ratio of distances from \(P\) to two fixed points \(A\) and \(B\) equals some specified \(k\) is a circle (see Figure 1), which may degenerate to a point (for \(k \to 0\) or \(k \to \infty\)) or to a line (for \(k \to 1\)).

![Figure 1. The Apollonius circles with respect to a pair of points in the plane](image)

More precisely, given two points \(A = (a, 0)\) and \(B = (b, 0)\) and \(k > 0\), the set of points \(P = (x, y)\) with \(d(P, B) = k \cdot d(P, A)\) is the Apollonius circle \(Ap(a, b, k)\) given by the equation

\[
\left(x - a + \frac{b - a}{k^2 - 1}\right)^2 + y^2 = k^2 \left(\frac{b - a}{k^2 - 1}\right)^2.
\]

(2)

Our first result that establishes a connection between Apollonius circles and irreducibility testing is the following.

**Theorem 1.2.** Let \(f(X) = a_0 + a_1X + \cdots + a_nX^n\) be a polynomial with integer coefficients, and assume that for two integers \(a, b\) we have \(0 < |f(a)| < |f(b)|\). Let

\[
q = \max \left\{ \frac{d_2}{d_1} \leq \sqrt{\frac{|f(b)|}{|f(a)|}} : d_1 \in D_{ad}(f(a)), \ d_2 \in D_{ad}(f(b)) \right\}.
\]

(3)

i) If \(q > 1\) and all the roots of \(f\) lie inside the Apollonius circle \(Ap(a, b, q)\), then \(f\) is irreducible over \(\mathbb{Q}\).

ii) If \(q > 1\), all the roots of \(f\) lie inside the Apollonius circle \(Ap(a, b, \sqrt{q})\) and \(f\) has no rational roots, then \(f\) is irreducible over \(\mathbb{Q}\).

iii) Assume that \(q = 1\). If \(b > a\) and all the roots of \(f\) lie in the half-plane \(x < \frac{a+b}{2}\), or if \(a > b\) and all the roots of \(f\) lie in the half-plane \(x > \frac{a+b}{2}\), then \(f\) is irreducible over \(\mathbb{Q}\).

As we shall see in the sequel, in general it is desirable to work with values of \(q\) in the statement of Theorem 1.2 as small as possible, in order to relax the constraints on the two integers \(a\) and \(b\) that we use. For instance, if \(b > a\) and we can prove that \(q = 1\) (which is the minimum possible value of \(q\)), by imposing the condition that \(f(X + \frac{a+b}{2})\) is a Hurwitz
stable polynomial, so that all the roots of $f$ lie in the half-plane $x < \frac{a+b}{2}$, then by Theorem 1.2 iii) we may conclude that $f$ is irreducible over $\mathbb{Q}$. As known, a necessary and sufficient condition for a polynomial to be Hurwitz stable is that it passes the Routh–Hurwitz test. In some applications, instead of testing the conditions in Theorem 1.2, it might be more convenient to consider the maximum of the absolute values of the roots of $f$, as follows.

**Theorem 1.3.** Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n$ be a polynomial with integer coefficients, $M$ the maximum of the absolute values of its roots, and assume that for two integers $a, b$ we have $0 < |f(a)| < |f(b)|$. Let $q$ be given by (3).

i) If $|b| > q|a| + (1 + q)M$, then $f$ is irreducible over $\mathbb{Q}$.

ii) If $|b| > \sqrt{q}|a| + (1 + \sqrt{q})M$ and $f$ has no rational roots, then $f$ is irreducible over $\mathbb{Q}$.

iii) If $q = 1$, $a^2 < b^2$ and $M < \frac{|a+b|}{2}$, then $f$ is irreducible over $\mathbb{Q}$.

We note that one can obtain slightly weaker results by allowing $d_1$ and $d_2$ in the definition of $q$ in the statement of Theorem 1.2 to be arbitrary divisors of $f(a)$ and $f(b)$, respectively. Doing so will potentially increase $q$, which will consequently lead to stronger restrictions on $|b|$. Even in some particular cases when $f(a)$ and $f(b)$ have few prime factors, to derive an effective, explicit formula for $q$ in the statement of Theorem 1.2 is a difficult problem involving inequalities between products of prime powers. However, one may obtain many corollaries of this result on the one hand by using some classical estimates for polynomial roots that provide explicit upper bounds for the absolute values of the roots of $f$, and on the other hand by considering some special cases for the canonical decomposition of the two integers $f(a)$ and $f(b)$.

The problem of finding a sharp estimate for the maximum of the absolute values of the roots of a given polynomial has a long history that goes back centuries ago. Among the earliest such attempts we mention here the bounds due to Cauchy and Lagrange. A generalization for Cauchy’s bound on the largest root of a polynomial was obtained by Mignotte in [36]:

*If a monic polynomial of height $H$ has $k$ roots of maximal modulus $\rho$ then $\rho < 1 + H^{1/k}$.*

For a recent improvement of the bound of Lagrange for the maximum modulus of the roots we refer the reader to Batra, Mignotte, and Ţîţănescu [1]. Further classical refinements rely on the use of some families of parameters, that brings considerably more flexibility, and here we only mention the classical methods of Fujiwara [27], Ballieu [1], [35], Cowling and Thron [21], [22], Kojima [34], or methods using estimates for the characteristic roots for complex matrices [38].

We will only present in this paper some simple corollaries of Theorem 1.3 for some cases when the canonical decompositions of $f(a)$ and $f(b)$ allow one to conclude that $q = 1$.

**Corollary 1.4.** Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n$ be a polynomial with integer coefficients, and $a, b$ two integers such that $a^2 < b^2$ and $|a_n| > \sum_{i=0}^{n-1} |a_i| \left( \frac{|a+b|}{2} \right)^{i-n}$. Then $f$ is irreducible over $\mathbb{Q}$ in each of the following cases:

- If $|b| > q|a| + (1 + q)M$, then $f$ is irreducible over $\mathbb{Q}$.
- If $|b| > \sqrt{q}|a| + (1 + \sqrt{q})M$ and $f$ has no rational roots, then $f$ is irreducible over $\mathbb{Q}$.
- If $q = 1$, $a^2 < b^2$ and $M < \frac{|a+b|}{2}$, then $f$ is irreducible over $\mathbb{Q}$.
i) \(|f(a)| = p^kr, \ |f(b)| = p^{k+1}\) with \(p\) prime and integers \(k, r\) with \(k \geq 0\) and \(0 \leq r < p\);

ii) \(|f(a)| = p^k, \ |f(b)| = p^r\) for some primes \(p, r\) with \(r < p\) and some integer \(k \geq 1\).

Note that the irreducibility of \(f\) will be guaranteed solely by the condition that \(|f(b)|\) is a prime number \(p\) for some integer \(b\) with sufficiently large absolute value, without using any information on \(a\) or on the magnitude of \(p\). Indeed, to conclude that \(f\) is irreducible it suffices to ask \(|f(b)|\) to be prime for some integer \(b\) with \(|b| > M + 1\), where \(M\) is the maximum of the absolute values of the roots of \(f\). For a proof of this elementary fact and for some of its generalisations we refer the reader to [41] or [28], for instance. Thus, if we ask

\[|f(a)| = p^kr, \ |f(b)| = p^{k+1}|\] with \(b\) prime, with \(\tilde{f}\) the reciprocal of \(f\), and asking \(|a_0| > 2|a_{n-1}| + 2^2|a_{n-2}| + \cdots + 2^n|a_0|\), and \(a_0 \neq 0\). If \(f(\mathbb{Z}\setminus\{0\})\) or \(-f(\mathbb{Z}\setminus\{0\})\) contains a prime number, then \(f\) must be irreducible over \(\mathbb{Q}\).

We mention that Theorem 6 is a special case of Corollary 1.5 obtained by asking \(\tilde{f}(1)\) to be prime, with \(\tilde{f}\) the reciprocal of \(f\), and asking \(|a_0| > 2|a_{n-1}| + 2^2|a_{n-2}| + \cdots + 2^n|a_0|\) instead of \(|a_n| > 2|a_{n-1}| + 2^2|a_{n-2}| + \cdots + 2^n|a_0|\).

Using the well-known Eneström–Kakeya Theorem [33], saying that all the roots of a polynomial \(f(X) = a_0 + a_1X + \cdots + a_nX^n\) with real coefficients satisfying \(0 \leq a_0 \leq a_1 \leq \cdots \leq a_n\) must have absolute values at most 1, one can also prove the following two results.

**Corollary 1.6.** Let \(f(X) = a_0 + a_1X + \cdots + a_nX^n\) be an Eneström–Kakeya polynomial of degree \(n\) with integer coefficients, \(a_0 \neq 0\), and \(a, b\) two integers with \(a^2 < b^2\) and \(|a + b| > 2\). Then \(f\) is irreducible over \(\mathbb{Q}\) in each of the following cases:

i) \(|f(a)| = p^kr, \ |f(b)| = p^{k+1}\) with \(p\) prime and integers \(k, r\) with \(k \geq 0\) and \(0 \leq r < p\);

ii) \(|f(a)| = p^k, \ |f(b)| = p^r\) for some primes \(p, r\) with \(r < p\) and some integer \(k \geq 1\).

**Corollary 1.7.** Let \(f(X) = a_0 + a_1X + \cdots + a_nX^n\) be an Eneström–Kakeya polynomial of degree \(n\) with integer coefficients, \(a_0 \neq 0\). If \(f(-1) \neq 0\) and \(|f(b)|\) is a prime number for some integer \(b\) with \(|b| \geq 2\), then \(f\) is irreducible over \(\mathbb{Q}\).

We note here that condition \(f(-1) \neq 0\) cannot be removed, since the reducible Eneström–Kakeya polynomial \(f(X) = X^3 + X^2 + X + 1\) satisfies \(|f(-2)| = 5\) while \(f(-1) = 0\).

The proofs of the results stated so far will be given in Section 2.

When we study the admissible divisors of \(f(a)\) and \(f(b)\), we distinguish the particular cases where \(f(n)\) and \(f'(n)\) are coprime for at least one integer \(n \in \{a, b\}\). Consequently, the set
of admissible divisors of \( f(n) \) reduces in these cases to the set \( \mathcal{D}_u(f(n)) \) of unitary divisors of \( f(n) \). In Section 3 we will state and prove the results corresponding to the case that both relations \( \gcd(f(a), f'(a)) = 1 \) and \( \gcd(f(b), f'(b)) = 1 \) hold, where \( q \) will be denoted by \( q_u \), to emphasize the role of unitary divisors of \( f(a) \) and \( f(b) \). One can easily state the results corresponding to the remaining two cases when only one of these relations holds. Thus we will present two results analogous to Theorem 1.2 and Theorem 1.3, namely Theorem 3.1 corresponding to the remaining two cases when only one of these relations holds. We will also prove in Section 4 similar results for multivariate polynomials \( f(X_1, \ldots, X_r) \) over an arbitrary field \( K \). The results for polynomials in \( r \geq 3 \) variables will be deduced from the results in the bivariate case, by writing \( Y \) for \( X_r \), \( X \) for \( X_{r-1} \), and \( K \) for \( K(X_1, \ldots, X_{r-2}) \).

First, we will need the following definition, analogous to Definition 1.1 for the bivariate case.

**Definition 1.8.** Let \( K \) be a field, \( f(X, Y) \in K[X, Y] \) and \( a(X) \in K[X] \) such that \( f(X, a(X)) \neq 0 \). We say that a polynomial \( d(X) \in K[X] \) is an admissible divisor of \( f(X, a(X)) \) if \( d(X) \mid f(X, a(X)) \) and

\[
\gcd \left( d(X), \frac{f(X, a(X))}{d(X)} \right) \mid \gcd \left( f(X, a(X)), \frac{\partial f}{\partial Y}(X, a(X)) \right).
\]

We will denote by \( D_{ad}(f(X, a(X))) \) the set of admissible divisors of \( f(X, a(X)) \). Also, for \( f(X, Y) \) and \( a(X) \) as above we will denote

\[
\mathcal{D}_u(f(X, a(X))) = \{ d \in K[X] : d(X) \mid f(X, a(X)), \ gcd \left( d(X), \frac{f(X, a(X))}{d(X)} \right) = 1 \},
\]

and call it the set of unitary divisors of \( f(X, a(X)) \). We note that in the particular case that \( \gcd(f(X, a(X)), \frac{\partial f}{\partial Y}(X, a(X))) = 1 \), \( D_{ad}(f(X, a(X))) \) reduces to \( \mathcal{D}_u(f(X, a(X))) \).

With this definition, we have the following results.

**Theorem 1.9.** Let \( K \) be a field, \( f(X, Y) = a_0(X) + a_1(X)Y + \cdots + a_n(X)Y^n \in K[X, Y] \), with \( a_0, \ldots, a_n \in K[X], a_0a_n \neq 0 \). Assume that for two polynomials \( a(X), b(X) \in K[X] \) we have \( f(X, a(X))f(X, b(X)) \neq 0 \) and \( \Delta := \frac{1}{2} \cdot (\deg f(X, b(X)) - \deg f(X, a(X))) \geq 0 \), and let

\[
q = \max \{ \deg d_2 - \deg d_1 \leq \Delta : d_1 \in D_{ad}(f(X, a(X))), d_2 \in D_{ad}(f(X, b(X))) \}.
\]

If \( \deg b(X) > \max \{ \deg a(X), \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_{n-i}}{n-i} \} + q \), then \( f(X, Y) \) is irreducible over \( K \).

In particular, for \( a(X) = 0 \) and \( b(X) \) denoted by \( g(X) \), we obtain:

**Corollary 1.10.** Let \( K \) be a field, \( f(X, Y) = a_0(X) + a_1(X)Y + \cdots + a_n(X)Y^n \in K[X, Y] \), with \( a_0, a_1, \ldots, a_n \in K[X], a_0a_n \neq 0 \) and

\[
\deg a_n \geq \max \{ \deg a_0, \deg a_1, \ldots, \deg a_{n-1} \}.
\]
If for a non-constant polynomial \( g(X) \in K[X] \), the polynomial \( f(X, g(X)) \) is irreducible over \( K \), then \( f(X, Y) \) is irreducible over \( K(X) \).

Two additional irreducibility criteria that rely on the unitary divisors of \( f(X, a(X)) \) and \( f(X, b(X)) \) will be also proved in Section 4. Our results are quite flexible, and provide irreducibility conditions for many cases where other irreducibility criteria fail. We will give in the last section of the paper a series of examples of infinite families of polynomials that are proved to be irreducible by using irreducibility criteria proved in previous sections.

2. The case of admissible divisors

Proof of Theorem 1.2 Assume that \( f \) factors as \( f(X) = a_n(X - \theta_1) \cdots (X - \theta_n) \) for some complex numbers \( \theta_1, \ldots, \theta_n \). Now let us assume to the contrary that \( f \) is reducible, so there exist two polynomials \( g, h \in \mathbb{Z}[X] \) with \( \deg g = m \geq 1 \), \( \deg h = n - m \geq 1 \) such that \( f = g \cdot h \). Without loss of generality we may further assume that

\[
g(X) = b_m(X - \theta_1) \cdots (X - \theta_m) \quad \text{and} \quad h(X) = \frac{a_n}{b_m}(X - \theta_{m+1}) \cdots (X - \theta_n),
\]

for some divisor \( b_m \) of \( a_n \). Now, since \( f(a) = g(a)h(a) \neq 0 \) and \( f'(a) = g'(a)h(a) + g(a)h'(a) \), and similarly \( f(b) = g(b)h(b) \) and \( f'(b) = g'(b)h(b) + g(b)h'(b) \), we see that \( g(a) \) is a divisor \( d_1 \) of \( f(a) \), and \( g(b) \) is a divisor \( d_2 \) of \( f(b) \) that must also satisfy the following divisibility conditions

\[
\gcd \left( d_1, \frac{f(a)}{d_1} \right) \mid \gcd(f(a), f'(a)) \quad \text{and} \quad \gcd \left( d_2, \frac{f(b)}{d_2} \right) \mid \gcd(f(b), f'(b)).
\]

Therefore \( d_1 \) and \( d_2 \) are admissible divisors of \( f(a) \) and \( f(b) \), respectively. Similarly, if we denote \( h(a) \) by \( d'_1 \) and \( h(b) \) by \( d'_2 \), we see that \( d'_1 \) and \( d'_2 \) are also admissible divisors of \( f(a) \) and \( f(b) \), respectively. Next, since

\[
\frac{d_2}{d_1} \cdot \frac{d'_2}{d'_1} = \frac{f(b)}{f(a)},
\]

one of the quotients \( \frac{|d_2|}{|d_1|} \) and \( \frac{|d'_2|}{|d'_1|} \), say \( \frac{|d_2|}{|d_1|} \), must be less than or equal to \( \sqrt{\frac{|f(b)|}{|f(a)|}} \). In particular, we have

\[
\frac{|g(b)|}{|g(a)|} \leq q.
\]

We notice here that since \( |f(b)| > |f(a)| \) and 1 is obviously a divisor of \( f(a) \) and \( f(b) \), a possible candidate for \( q \) is 1, so \( q \geq 1 \). Next, we observe that we may write

\[
\frac{g(b)}{g(a)} = \frac{b - \theta_1}{a - \theta_1} \cdots \frac{b - \theta_m}{a - \theta_m},
\]

so in view of (3) for at least one index \( i \in \{1, \ldots, m\} \) we must have

\[
\frac{|b - \theta_i|}{|a - \theta_i|} \leq q^{\frac{1}{m}}.
\]
Now, let us first assume that \( q > 1 \) and all the roots of \( f \) lie inside the Apollonius circle \( \text{Ap}(a, b, q) \). In particular, since \( \theta_i \) lies inside the Apollonius circle \( \text{Ap}(a, b, q) \), it must satisfy the inequality \(|b - \theta_i| > q|a - \theta_i|\). Since \( q > 1 \) and \( m \geq 1 \), we have \( q \geq q^{\frac{1}{m}} \), so we deduce that we actually have

\[
\frac{|b - \theta_i|}{|a - \theta_i|} > q^{\frac{1}{m}},
\]

which contradicts (6). Therefore \( f \) must be irreducible.

Next, assume that \( q > 1 \) and that all the roots of \( f \) lie inside the Apollonius circle \( \text{Ap}(a, b, \sqrt{q}) \). In particular, we have \(|b - \theta_i| > \sqrt{q}|a - \theta_i|\). Since \( f \) has no rational roots, we must have \( m \geq 2 \), so \( \sqrt{q} \geq q^{\frac{1}{m}} \), which also leads us to the desired contradiction

\[
\frac{|b - \theta_i|}{|a - \theta_i|} > q^{\frac{1}{m}},
\]

thus proving the irreducibility of \( f \).

Finally, let us assume that \( q = 1 \), so in this case (6) reads

\[
\frac{|b - \theta_i|}{|a - \theta_i|} \leq 1,
\]

which is equivalent to

\[
(b - \text{Re}(\theta_i))^2 \leq (a - \text{Re}(\theta_i))^2. \quad (7)
\]

It is easy to see that in order to contradict (7), it is sufficient to ask all the roots of \( f \) to lie in the half-plane \( x < \frac{a+b}{2} \) if \( a < b \), or in the half-plane \( x > \frac{a+b}{2} \) if \( a > b \). This completes the proof of the theorem.

**Proof of Theorem 1.3** The proof goes as in the case of Theorem 1.2, and we deduce again that for at least one index \( i \in \{1, \ldots, m\} \) we must have

\[
\frac{|b - \theta_i|}{|a - \theta_i|} \leq q^{\frac{1}{m}}. \quad (8)
\]

On the other hand, if \(|b| > q|a| + (1 + q)M\) we observe that

\[
\frac{|b - \theta_i|}{|a - \theta_i|} \geq \frac{|b| - |\theta_i|}{|a| + |\theta_i|} > \frac{|b| - M}{|a| + M} > q \geq q^{\frac{1}{m}},
\]

since \( q \geq 1 \). This contradicts (8), so \( f \) must be irreducible over \( \mathbb{Q} \).

In our second case, if we assume that \(|b| > \sqrt{q}|a| + (1 + \sqrt{q})M\) and \( f \) has no rational roots, then \( m \geq 2 \), and consequently

\[
\frac{|b - \theta_i|}{|a - \theta_i|} \geq \frac{|b| - M}{|a| + M} > \sqrt{q} \geq q^{\frac{1}{m}},
\]

again a contradiction.

Finally, let us assume that \( q = 1 \), \( a^2 < b^2 \) and \( M < \frac{|a+b|}{2} \). If \( b > a \), then \( a + b > 0 \) and the conclusion follows by Theorem 1.2 iii) since the disk \(|z| \leq M\) containing all the roots of \( f \) lies in the left half-plane \( x < \frac{a+b}{2} \). Finally, if \( a > b \), then \( a + b < 0 \) and the disk \(|z| \leq M\) lies in the right half-plane \( x > \frac{a+b}{2} \), since \(-M > \frac{a+b}{2}\). \( \square \)
Proof of Corollary 1.4. An immediate consequence of Rouché’s Theorem is that the condition \(|a_n| > \sum_{i=0}^{n-1} |a_i| (\frac{|a+b|}{2})^{i-n}\) forces all the roots of \(f\) to have absolute values less than \(\frac{|a+b|}{2}\). Therefore \(M < \frac{|a+b|}{2}\). In the first case we observe that if \(|f(a)| = p^k r\) and \(|f(b)| = p^{k+1}\), with \(p\) prime and \(0 < r < p\), then any positive quotient \(\frac{d_2}{d_1}\) with \(d_1 | f(a)\) and \(d_2 | f(b)\) has the form \(\frac{p^i}{s}\) with \(i\) an integer satisfying \(-k \leq i \leq k + 1\), and \(s\) a divisor of \(r\). For \(i \leq 0\) these quotients will be at most \(1\), while for \(i > 0\) all the corresponding quotients will exceed \(\sqrt{p}\), as \(\frac{p}{s} > \sqrt{\frac{p}{r}}\). Thus \(q = 1\) in this first case.

Next, if \(|f(a)| = p^k\) and \(|f(b)| = p^k r\), any positive quotient \(\frac{d_2}{d_1}\) with \(d_1 | f(a)\) and \(d_2 | f(b)\) has the form \(p^i r^\varepsilon\) with \(i\) an integer satisfying \(-k \leq i \leq k\) and \(\varepsilon \in \{0,1\}\). For \(\varepsilon = 0\), no such quotient other than \(1\) belongs to the interval \([1, \sqrt{p}]\), since \(p > \sqrt{p}\). Finally, we observe that for \(\varepsilon = 1\) no integer \(i\) can satisfy the condition \(1 < p^i r < \sqrt{p}\) since \(p > r\).

So in both cases \(q\) must be equal to \(1\). The conclusion now follows from Theorem 1.3. \(\square\)

Proof of Corollary 1.5. Our assumption on the magnitude of \(|a_n|\) forces all the roots of \(f\) to have absolute values less than \(\frac{1}{2}\), so \(M < \frac{1}{2}\). We may now apply Theorem 1.3 with \(a = 0\) or Corollary 1.4 i) with \(a = k = 0\) to deduce that \(f\) is irreducible over \(\mathbb{Q}\) if \(b \neq 0\). All that remains now is to prove that our condition \(|a_n| > 2|a_{n-1}| + 2^2|a_{n-2}| + \cdots + 2^n|a_0|\) together with the fact that \(|b| \geq 1\) also force the prime number \(|f(b)|\) to exceed \(|a_0|\). Indeed, we successively deduce that

\[
|f(b)| = |a_0 + a_1 b + \cdots + a_n b^n| \geq |b^n| |a_n| - |b^{n-1}a_{n-1}| - \cdots - |b| \cdot |a_1| - |a_0|
\]

\[
> |b^n (2|a_{n-1}| + 2^2|a_{n-2}| + \cdots + 2^n|a_0|) - |b^{n-1}a_{n-1}| - \cdots - |b| \cdot |a_1| - |a_0|
\]

\[
= |b^{n-1}(2|b| - 1)|a_{n-1}| + |b^{n-2}(2^2|b|^2 - 1)|a_{n-2}| + \cdots + (2^n|b^n - 1)|a_0|
\]

\[
\geq (2^n|b^n - 1)|a_0| \geq (2^n - 1)|a_0| \geq |a_0|,
\]

and this completes the proof. \(\square\)

Proof of Corollary 1.6. Here, by the Eneström–Kakeya Theorem all the roots of \(f\) must have modulus at most \(1\), so \(M \leq 1\). Arguing as in the proof of Corollary 1.4 one may prove that \(q = 1\) in both cases, and the proof finishes by applying Theorem 1.3. \(\square\)

Proof of Corollary 1.7. We note here that since an Eneström–Kakeya polynomial \(f\) has all the roots of modulus at most \(1\), it will be irreducible over \(\mathbb{Q}\) if \(|f(b)|\) is a prime for some integer \(b\) with \(|b| \geq 3\). If we consider now an additional integer argument \(a\) and ask \(f(a) \neq 0\) and \(|f(b)| = p\) for some prime number \(p > |f(a)|\), this will force \(q\) to be equal to \(1\), and will guarantee the irreducibility of \(f\) via Theorem 1.3 if \(a^2 < b^2\) and \(|a+b| > 2\). This will also allow us to use the pairs \((a,b) = (1,2)\) and \((a,b) = (-1,-2)\). In the first case, if \(f(2)\) is a prime number, then it will obviously exceed \(f(1)\), since \(f\) has positive coefficients. Let us consider the remaining case \((a,b) = (-1,-2)\). If \(n\) is even, one can easily check that \(f(-2) > f(-1) \geq 0\), so if we ask \(f(-1) \neq 0\), then condition \(|f(-2)| > |f(-1)| > 0\) will be obviously satisfied. On the other hand, if \(n\) is odd, one can check that \(f(-2) < f(-1) \leq 0\),
so if \( f(-1) \neq 0 \), the condition \(|f(-2)| > |f(-1)| > 0\) will be again satisfied. By Theorem 1.3, \( f \) will be irreducible in both cases. \( \square \)

### 3. The case of unitary divisors

The aim of this section is to find irreducibility conditions by studying the unitary divisors of \( f(a) \) and \( f(b) \). Here instead of \( q \) given by (3), we will use a potentially smaller rational number, defined by

\[
q_u = \max \left\{ \frac{d_2}{d_1} \leq \sqrt{\frac{|f(b)|}{|f(a)|}} : d_1 \in \mathcal{D}_u(f(a)), \ d_2 \in \mathcal{D}_u(f(b)) \right\} . \tag{9}
\]

With this notation we have the following irreducibility criterion.

**Theorem 3.1.** Let \( f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{Z}[X] \), and assume that for two integers \( a, b \) we have \( 0 < |f(a)| < |f(b)| \) and \( \gcd(f(a), f'(a)) = \gcd(f(b), f'(b)) = 1 \). Let also \( q_u \) be given by (9).

i) If \( q_u > 1 \) and all the roots of \( f \) lie inside the Apollonius circle \( \text{Ap}(a, b, q_u) \), then \( f \) is irreducible over \( \mathbb{Q} \).

ii) If \( q_u > 1 \), all the roots of \( f \) lie inside the Apollonius circle \( \text{Ap}(a, b, \sqrt{q_u}) \) and \( f \) has no rational roots, then \( f \) is irreducible over \( \mathbb{Q} \).

iii) Assume that \( q_u = 1 \). If \( b > a \) and all the roots of \( f \) lie in the half-plane \( x < \frac{a+b}{2} \), or if \( a > b \) and all the roots of \( f \) lie in the half-plane \( x > \frac{a+b}{2} \), then \( f \) is irreducible over \( \mathbb{Q} \).

**Proof.** Using the same notations as in the proof of Theorem 1.2 we see that conditions \( \gcd(f(a), f'(a)) = \gcd(f(b), f'(b)) = 1 \) together with the divisibility conditions

\[
\gcd \left( d_1, \frac{f(a)}{d'_1} \right) \mid \gcd(f(a), f'(a)) \quad \text{and} \quad \gcd \left( d_2, \frac{f(b)}{d'_2} \right) \mid \gcd(f(b), f'(b))
\]

will force \( d_1 \) to be a unitary divisor of \( f(a) \), and \( d_2 \) to be a unitary divisor of \( f(b) \). Similarly, \( h(a) \) must be a unitary divisor \( d'_1 \) of \( f(a) \) and \( h(b) \) must be a unitary divisor \( d'_2 \) of \( f(b) \). Since

\[
\frac{d_2}{d_1} \cdot \frac{d'_2}{d'_1} = \frac{f(b)}{f(a)},
\]

one of the quotients \( \frac{d_2}{d_1} \) and \( \frac{d'_2}{d'_1} \), say \( \frac{d_2}{d_1} \), must be less than or equal to \( \sqrt{\frac{|f(b)|}{|f(a)|}} \). In particular, instead of (5), we obtain \( \frac{|g(b)|}{|g(a)|} \leq q_u \). We note that we will still have \( q_u \geq 1 \), since 1 belongs to both \( \mathcal{D}_u(f(a)) \) and \( \mathcal{D}_u(f(b)) \). The proof continues as in the case of Theorem 1.2 with \( q_u \) instead of \( q \). \( \square \)

**Theorem 3.2.** Let \( f(X) = a_0 + a_1 X + \cdots + a_n X^n \) be a polynomial with integer coefficients, \( M \) the maximum of the absolute values of its roots, and assume that for two integers \( a, b \) we have \( 0 < |f(a)| < |f(b)| \) and \( \gcd(f(a), f'(a)) = \gcd(f(b), f'(b)) = 1 \). Let also \( q_u \) be given by relation (9).
i) If $|b| > q_u |a| + (1 + q_u)M$, then $f$ is irreducible over $\mathbb{Q}$.

ii) If $|b| > \sqrt{q_u} |a| + (1 + \sqrt{q_u})M$ and $f$ has no rational roots, then $f$ is irreducible over $\mathbb{Q}$.

iii) If $q_u = 1$, $a^2 < b^2$ and $M < \frac{|a+b|}{2}$, then $f$ is irreducible over $\mathbb{Q}$.

Proof. The proof is similar to that of Theorem 1.3 with $q_u$ instead of $q$. \hfill \square

In particular, we obtain the following irreducibility criterion that complements Corollary 1.4 by allowing one to consider only the unitary divisors of $f(a)$ and $f(b)$.

**Corollary 3.3.** Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n$ be a polynomial with integer coefficients, and $a, b$ two integers such that $a^2 < b^2$ and $|a_n| > \sum_{i=0}^{n-1} |a_i| \left(\frac{|a+b|}{2}\right)^i n$. Then $f$ is irreducible over $\mathbb{Q}$ in each of the following four cases:

i) $|f(a)| = p^{k_1} r$, $|f(b)| = p^{k_2}$ for some prime number $p$ and some integers $k_1, k_2, r$ with $0 \leq k_1 < k_2$, $0 < r < p$, $p \nmid f'(a)f'(b)$ and $r \nmid f'(a);

ii) $|f(a)| = p^{k_1}$, $|f(b)| = p^{k_2}$ for two distinct prime numbers $p, r$ and some positive integers $k, j$ with $p^k > r^j$, $p \nmid f'(a)f'(b)$ and $r \nmid f'(b)$.

iii) $|f(a)| = p^{u} q^{v}$, $|f(b)| = r^{s_1}$ for three distinct prime numbers $p, q, r$ and some positive integers $u, v, t$ with $p^u > q^v$, $p^u > r^t$, $p^u < q^vr^t$, $p \nmid f'(a)$, $q \nmid f'(b)$ and $r \nmid f'(b)$.

iv) $|f(a)| = p^{u} q^{v}$, $|f(b)| = r^{s_1}$ for four distinct prime numbers $p, q, r, s$ and some positive integers $u, v, k, l$ with $p^u > r^k > s^l > q^v$, $p^u q^v < r^k s^l$, $p \nmid f'(a)$, $q \nmid f'(a)$, $r \nmid f'(b)$ and $s \nmid f'(b)$.

Proof. Here, as in the proof of Corollary 1.4 we have $M < \frac{|a+b|}{2}$.

Assuming now that $|f(a)| = p^{k_1} r$ and $|f(b)| = p^{k_2}$ with $0 < r < p$ and $k_1 < k_2$, then any $d_1 \in \mathcal{D}_u(f(a))$ is of the form $a, b \in \mathcal{D}_u(r)$, while a unitary divisor $d_2$ of $f(b)$ is either $1$ or $p^{k_2}$. Therefore $\frac{d_2}{d_1}$ is either $\frac{1}{d_1}$, which is at most $1$, or is of the form $\frac{p^{k_2}}{s}, \frac{p^{k_2-k_1}}{s}$ with $s \in \mathcal{D}_u(r)$. It suffices to observe that for each $s \in \mathcal{D}_u(r)$ we have $\frac{p^{k_2-k_1}}{s} > \sqrt{\frac{p^{k_2}}{p^{k_1} r}}$, as $p^{k_2-k_1} > \frac{s^2}{r}$. Thus $q_u = 1$ in this first case.

For our second case let us assume that $|f(a)| = p^{k_1} r$ and $|f(b)| = p^{k_2}$ for two distinct primes $p, r$ and some positive integers $k, j$ with $p^k > r^j$. Then $\mathcal{D}_u(f(a)) = \{1, p^k\}$ and $\mathcal{D}_u(f(b)) = \{1, p^{k_2}, r^j, p^{k_2} r^j\}$. Therefore any quotient $\frac{d_2}{d_1}$ with $d_1 \in \mathcal{D}_u(f(a))$ and $d_2 \in \mathcal{D}_u(f(b))$ belongs to the set $\{1, p^k, r^j, p^{k_2} r^j, \frac{1}{p^k}, \frac{r^j}{p^{k_2}}\}$, so here again it holds $q_u = 1$, since according to our assumption that $p^k > r^j$, the only such quotient in the interval $[1, r^\frac{k}{j}]$ is $1$.

In our third case we have $\mathcal{D}_u(f(a)) = \{1, p^k\}$ and $\mathcal{D}_u(f(b)) = \{1, q^v, r^t, q^v r^t\}$, so any quotient $\frac{d_2}{d_1}$ with $d_1 \in \mathcal{D}_u(f(a))$ and $d_2 = 1$ will be at most $1$, while any quotient $\frac{d_2}{d_1}$ with $d_1 \in \mathcal{D}_u(f(a))$ and $d_2 = q^v r^t$ will exceed $\sqrt{\frac{q^v r^t}{p^u}}$. We are thus left with the case that

$$\frac{d_2}{d_1} \in \left\{ \frac{q^v}{p^u}, \frac{r^t}{p^u}, \frac{1}{p^k}, \frac{r^j}{p^{k_2}} \right\}.$$ 

It is now plain to see that while $\frac{q^v}{p^u}$ and $\frac{r^t}{p^u}$ are less than $1$, both $q^v$ and $r^t$ exceed $\sqrt{\frac{q^v r^t}{p^u}}$, so here we have $q_u = 1$ too.
In our last case we have $D_u(f(a)) = \{1, p^n, q^n, p^nq^n\}$ and $D_u(f(b)) = \{1, r^k, s^l, r^ks^l\}$. Here we first note that any quotient $\frac{d_2}{d_1}$ with $d_1 \in D_u(f(a))$ and $d_2 = 1$ will be at most 1, and any quotient $\frac{d_2}{d_1}$ with $d_1 \in D_u(f(a))$ and $d_2 = r^ks^l$ will exceed $\sqrt{\frac{r^k s^l}{p^n q^n}}$. We are therefore left with the case that

$$\frac{d_2}{d_1} \in \left\{ \frac{r^k}{p^n}, \frac{s^l}{q^n} : r^k, s^l \leq |p^n q^n| \right\}.$$ 

Using now our hypothesis that $p^a > r^k > s^l > q^v$ it is easy to check that each of the quotients $\frac{r^k}{p^n}, \frac{s^l}{q^n}$ is less than 1, while each of the remaining ones $\frac{r^k}{p^n}, \frac{s^l}{q^n}$ exceeds $\sqrt{\frac{r^k s^l}{p^n q^n}}$, so here $q_u = 1$ as well. The irreducibility of $f$ now follows from Theorem 3.2.

The reader may naturally wonder if there exists a result analogous to Corollary 1.5, that uses information on the prime power values of a polynomial, instead of its prime values. The answer is affirmative, and one can prove the following result, which illustrates a situation when there is no need to impose both conditions $\gcd(f(a), f'(a)) = 1$ and $\gcd(f(b), f'(b)) = 1$.

**Corollary 3.4.** Let $f(X) = a_0 + a_1 X + \ldots + a_n X^n \in \mathbb{Z}[X]$ with $a_0 a_n \neq 0$ and $|a_n| > 2|a_{n-1}| + 2^2|a_{n-2}| + \ldots + 2^n|a_0|$. If for a non-zero integer $m$, a prime number $p$ and an integer $k \geq 2$ we have $|f(m)| = p^k$ and $p \nmid f'(m)$, then $f$ is irreducible over $\mathbb{Q}$.

Proof. Assume first that for a polynomial $f$ with integer coefficients and two integers $a$, $b$ we have $0 < |f(a)| < |f(b)| = p^k$ for some prime number $p$ and some positive integer $k$. Let us also assume that $p \nmid f'(b)$. Then $D_{ad}(f(b)) = D_u(f(b))$ and any positive quotient $\frac{d_2}{d_1}$ in the definition of $q$ in Theorem 1.3 either has the form $\frac{1}{d_1}$ with $d_1$ an admissible divisor of $f(a)$, and hence is at most 1, or is equal to $\frac{p^k}{d_1}$, which obviously exceeds $\sqrt{\frac{p^k}{|f(a)|}}$. Therefore in this case $q$ must be equal to 1. In particular, if we take $a = 0$ and $b = m$, and assume that $0 < |a_0| < |f(m)| = p^k$ for some prime number $p$ and some integer $k \geq 2$, and also assume that $p \nmid f'(m)$, then the corresponding $q$ must be equal to 1. Since all the roots of $f$ have absolute values less than $\frac{1}{2}$, we have $M < \frac{1}{2}$. By Theorem 1.3 we conclude that $f$ is irreducible over $\mathbb{Q}$ if $|m| > 2M$, or equivalently, if $|m| \geq 1$, since $M < \frac{1}{2}$. All that remains is to prove that the inequalities $|m| \geq 1$ and $|a_n| > 2|a_{n-1}| + 2^2|a_{n-2}| + \ldots + 2^n|a_0|$ actually force $|f(m)|$ to exceed $|a_0|$. As in the case of Corollary 1.5, we deduce successively that

$$|f(m)| = |a_0 + a_1 m + \ldots + a_n m^n| \geq |m|^n |a_n| - |m|^{n-1} |a_{n-1}| - \ldots - |m| \cdot |a_1| - |a_0| > |m|^n (2|a_{n-1}| + 2^2|a_{n-2}| + \ldots + 2^n|a_0|) - |m|^{n-1} |a_{n-1}| - \ldots - |m| \cdot |a_1| - |a_0| \geq |m|^{n-1} (2|m| - 1)|a_{n-1}| + |m|^{n-2} (2^2 |m|^2 - 1)|a_{n-2}| + \ldots + (2^n |m|^n - 1)|a_0| \geq (2^n |m|^n - 1)|a_0| \geq (2^n - 1)|a_0| \geq |a_0|,$$

completing the proof. We note that using Theorem 3.2 instead of Theorem 1.3 would impose here the unnecessary additional condition $\gcd(f(0), f'(0)) = 1$, that is $\gcd(a_0, a_1) = 1$.

We mention here that Theorem 7 is a special case of Corollary 3.4 obtained by considering the reciprocal of $f$ instead of $f$. 

4. The case of multivariate polynomials

Proof of Theorem 1.9 We will first introduce a nonarchimedean absolute value $| \cdot |$ on $K(X)$, as follows. We first fix an arbitrary real number $\rho > 1$, and for any polynomial $F(X) \in K[X]$ we define $|F(X)|$ by the equality

$$|F(X)| = \rho^{\deg F(X)}.$$ 

We then extend this absolute value $| \cdot |$ to $K(X)$ by multiplicativity, that is, for any polynomials $F(X), G(X) \in K[X], G(X) \neq 0$, we let $\left| \frac{F(X)}{G(X)} \right| = \frac{|F(X)|}{|G(X)|}$. Here we must note that for any non-zero element $F$ of $K[X]$ one has $|F| \geq 1$.

Let now $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$, and let us fix an extension of our absolute value $| \cdot |$ to $\overline{K(X)}$, which we will also denote by $| \cdot |$.

Suppose $f$ as a polynomial in $Y$ with coefficients in $K[X]$ factorizes as

$$f(X, Y) = a_n(X)(Y - \theta_1) \cdots (Y - \theta_n)$$

for some $\theta_1, \ldots, \theta_n \in \overline{K(X)}$.

Next, we will prove that

$$\max\{|\theta_1|, \ldots, |\theta_n|\} \leq \rho \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_n}{n-i}. \quad (10)$$

To prove this claim, let $\lambda := \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_n}{n-i}$, and let us assume to the contrary that $f$ has a root $\theta$ with $|\theta| > \rho^{\lambda}$. Since $\theta \neq 0$ and our absolute value also satisfies the triangle inequality, we successively deduce that

$$0 = \left| \sum_{i=0}^{n} a_i \theta^{i-n} \right| \geq |a_n| - \left| \sum_{i=0}^{n-1} a_i \theta^{i-n} \right| \geq |a_n| - \max_{0 \leq i \leq n-1} |a_i| \cdot |\theta|^{i-n}$$

$$> |a_n| - \max_{0 \leq i \leq n-1} |a_i| \cdot \rho^{(i-n)\lambda},$$

yielding $|a_n| < \max_{0 \leq i \leq n-1} |a_i| \cdot \rho^{(i-n)\lambda}$, or equivalently

$$\deg a_n < \max_{0 \leq i \leq n-1} \{\deg a_i + (i-n)\lambda\}. \quad (11)$$

Let us select now an index $k \in \{0, \ldots, n-1\}$ for which the maximum in the right side of (11) is attained. Then we deduce that

$$\deg a_n < \deg a_k + (k-n)\lambda,$$

which leads us to

$$\frac{\deg a_k - \deg a_n}{n-k} > \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_n}{n-i},$$

a contradiction. Therefore (10) holds, so $|\theta_i| \leq \rho^{\lambda}$ for $i = 1, \ldots, n$. 

Now let us assume to the contrary that \( f \) is reducible, so by the celebrated Gauss’ Lemma there exist two polynomials \( g, h \in K[X,Y] \) with \( \deg_Y g = m \geq 1, \deg_Y h = n - m \geq 1 \) such that \( f = g \cdot h \). Without loss of generality we may further assume that

\[
g(X,Y) = b_m(X)(Y - \theta_1) \cdots (Y - \theta_m) \quad \text{and} \quad h(X,Y) = \frac{a_n(X)}{b_m(X)}(Y - \theta_{m+1}) \cdots (Y - \theta_n),
\]

for some divisor \( b_m(X) \) of \( a_n(X) \). Since we have \( f(X,a(X)) = g(X,a(X))h(X,a(X)) \) and \( f(X,b(X)) = g(X,b(X))h(X,b(X)) \), and also

\[
\frac{\partial f}{\partial Y}(X,a(X)) = \frac{\partial g}{\partial Y}(X,a(X))h(X,a(X)) + g(X,a(X))\frac{\partial h}{\partial Y}(X,a(X)),
\]

\[
\frac{\partial f}{\partial Y}(X,b(X)) = \frac{\partial g}{\partial Y}(X,b(X))h(X,b(X)) + g(X,b(X))\frac{\partial h}{\partial Y}(X,b(X)),
\]

we see that \( g(X,a(X)) \) is a divisor \( d_1 \) of \( f(X,a(X)) \), and \( g(X,b(X)) \) is a divisor \( d_2 \) of \( f(X,b(X)) \) that must also satisfy the following divisibility conditions

\[
\gcd\left(d_1, \frac{f(X,a(X))}{d_1}\right) \mid \gcd\left(f(X,a(X)), \frac{\partial f}{\partial Y}(X,a(X))\right) \quad \text{and} \quad \gcd\left(d_2, \frac{f(X,b(X))}{d_2}\right) \mid \gcd\left(f(X,b(X)), \frac{\partial f}{\partial Y}(X,b(X))\right).
\]

Recalling condition (4), we see that \( g(X,a(X)) \) is actually an admissible divisor \( d_1 \) of \( f(X,a(X)) \), and \( g(X,b(X)) \) is an admissible divisor \( d_2 \) of \( f(X,b(X)) \). Similarly, \( h(X,a(X)) \) is an admissible divisor \( d_1' \) of \( f(X,a(X)) \), and \( h(X,b(X)) \) is an admissible divisor \( d_2' \) of \( f(X,b(X)) \). Therefore we have

\[
g(X,b(X)) = \frac{d_2(X)}{d_1(X)} \quad \text{and} \quad h(X,b(X)) = \frac{d_2'(X)}{d_1'(X)}
\]

with \( d_1, d_1' \in D_{ad}(f(X,a(X))) \) and \( d_2, d_2' \in D_{ad}(f(X,b(X))) \). Now, since

\[
\frac{d_2(X)}{d_1(X)} \cdot \frac{d_2'(X)}{d_1'(X)} = \frac{f(X,b(X))}{f(X,a(X))},
\]

by applying \(| \cdot |\), we see that one of the quotients \( \frac{|d_2(X)|}{|d_1(X)|} \) and \( \frac{|d_2'(X)|}{|d_1'(X)|} \), say \( \frac{|d_2(X)|}{|d_1(X)|} \), must be less than or equal to \( \sqrt{\frac{|f(X,b(X))|}{|f(X,a(X))|}} \). In particular, we have

\[
\frac{|g(X,b(X))|}{|g(X,a(X))|} \leq \rho^q.
\]

(12)

We notice now that \( q \geq 0 \) because \( \Delta \geq 0 \) and \( d_1 = 1, d_2 = 1 \) is obviously a pair of divisors of \( f(X,a(X)) \) and \( f(X,b(X)) \), respectively, so that \( 0 = \deg 1 - \deg 1 \) is a possible candidate for \( q \). Next, we observe that we may write

\[
\frac{g(X,b(X))}{g(X,a(X))} = \frac{b(X) - \theta_1}{a(X) - \theta_1} \cdots \frac{b(X) - \theta_m}{a(X) - \theta_m}.
\]
so in view of (12) for at least one index $i \in \{1, \ldots, m\}$ we must have
\[
\frac{|b(X) - \theta_i|}{|a(X) - \theta_i|} \leq \rho^\frac{a}{m}.
\] (13)

On the other hand, since our absolute value also satisfies the triangle inequality, we see that
\[
\frac{|b(X) - \theta_i|}{|a(X) - \theta_i|} \geq \frac{|b(X)| - |\theta_i|}{|a(X)| + |\theta_i|} \geq \frac{|b(X)| - \rho^\lambda}{|a(X)| + \rho^\lambda} = \frac{\rho^{\deg b(X)} - \rho^\lambda}{\rho^{\deg a(X)} + \rho^\lambda}.
\]

We will now prove that for a sufficiently large $\rho$ one has
\[
\frac{\rho^{\deg b(X)} - \rho^\lambda}{\rho^{\deg a(X)} + \rho^\lambda} > \rho^q \geq \rho^\frac{a}{m},
\]
and this will contradict (13). The inequality $\rho^q \geq \rho^\frac{a}{m}$ obviously holds for an arbitrary $\rho > 1$ since $q \geq 0$ and $m = \deg g \geq 1$. Finally, all that remains to see is that the first inequality is equivalent to
\[
\rho^{\deg b(X)} > \rho^{q + \deg a(X)} + \rho^{q + \lambda} + \rho^\lambda,
\]
which will obviously hold for a sufficiently large $\rho$, since according to our assumption on the magnitude of $\deg b(X)$ we have $\deg b(X) > \max\{q + \deg a(X), q + \lambda, \lambda\}$. Therefore $f$ must be irreducible over $K(X)$, and this completes the proof. \[ \square \]

Proof of Corollary 1.10 We may apply Theorem 1.9 with $a(X) = 0$ and $b(X)$ any nonconstant polynomial (denoted here by $g(X)$) such that $f(X, b(X))$ is irreducible over $K$. To see this, we first observe that $f(X, a(X)) = f(X, 0) = a_0 \neq 0$ and since $f(X, b(X))$ is irreducible, we also have $f(X, b(X)) \neq 0$. Next, the inequality $\deg a_n \geq \max\{\deg a_0, \ldots, \deg a_{n-1}\}$ shows that
\[
\deg f(X, b(X)) = n \deg b + \deg a_n > \deg a_0 = \deg f(X, a(X)),
\] (14)
so the condition $\Delta \geq 0$ is also satisfied. It remains to prove that in this case we have $q = 0$. To prove this equality, we note that any divisor $d_2$ of the irreducible polynomial $f(X, b(X))$ either has degree 0 or has degree $n \deg b + \deg a_n$, so $\deg d_2 - \deg d_1$ in the definition of $q$ is equal either to $-\deg d_1$, which is at most 0, or to $n \deg b + \deg a_n - \deg d_1$, which exceeds $\Delta$, in view of inequality (12). Therefore our condition $\deg b(X) > \max\{\deg a(X), \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_{n-i}}{n-i}\} + q$ reduces in this case to $\deg b(X) > 0$. \[ \square \]

We mention that in analogy to the univariate case, when testing the irreducibility of $f(X, Y)$ in terms of two of its values $f(X, a(X))$ and $f(X, b(X))$, we don’t necessarily need to impose conditions on both partial derivatives $\frac{\partial f}{\partial Y}(X, a(X))$ and $\frac{\partial f}{\partial Y}(X, b(X))$. However, we will only state here a result for the case that $f(X, a(X))$ and $\frac{\partial f}{\partial Y}(X, a(X))$ are relatively prime, and $f(X, b(X))$ and $\frac{\partial f}{\partial Y}(X, b(X))$ are also relatively prime.
Theorem 4.1. Let $K$ be a field, $f(X, Y) = a_0(X) + a_1(X)Y + \cdots + a_n(X)Y^n \in K[X, Y]$, with $a_0, \ldots, a_n \in K[X]$, $a_0a_n \neq 0$. Assume that for two polynomials $a(X), b(X) \in K[X]$ we have $f(a(X))f(X, b(X)) \neq 0$ and let $\Delta := \frac{1}{2} \cdot (\deg f(X, b(X)) - \deg f(a(X))) \geq 0$, and let

$$q_u = \max\{\deg d_2 - \deg d_1 \leq \Delta : d_1 \in D_u(f(X, a(X))), d_2 \in D_u(f(X, b(X)))\}.$$ 

If $\gcd(f(a(X)), \frac{\partial f}{\partial Y}(X, a(X))) = 1$, $\gcd(f(X, b(X)), \frac{\partial f}{\partial Y}(X, b(X))) = 1$ and

$$\deg b(X) > \max \left\{ \deg a(X), \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_n}{n-i} \right\} + q_u,$$

then $f(X, Y)$ is irreducible over $K(X)$.

Proof. Here, with the same notations as in the proof of Theorem 1.9 we see that $d_1$ and $d_1'$ must belong to $D_u(f(X, a(X)))$, while $d_2$ and $d_2'$ must belong to $D_u(f(X, b(X)))$. We notice here that even if the defining set for $q_u$ is in this case smaller than the one for $q$ in Theorem 1.9, we will still have $q_u \geq 0$, since 1 belongs to both $D_u(f(X, a(X)))$ and $D_u(f(X, b(X)))$. The rest of the proof is identical to that of Theorem 1.9 and will be omitted. 

In particular, we obtain as a special case the following irreducibility criterion that complements Corollary 4.2 by allowing $f(X, g(X))$ to be a power of an irreducible polynomial, instead of an irreducible polynomial.

Corollary 4.2. Let $K$ be a field, $f(X, Y) = a_0(X) + a_1(X)Y + \cdots + a_n(X)Y^n \in K[X, Y]$, with $a_0, a_1, \ldots, a_n \in K[X]$, $a_0a_n \neq 0$ and

$$\deg a_n \geq \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-1}\}.$$ 

If for a non-constant polynomial $g(X) \in K[X]$, the polynomial $f(X, g(X))$ is a power of an irreducible polynomial over $K$, and $f(X, g(X))$ and $\frac{\partial f}{\partial Y}(X, g(X))$ are relatively prime, then $f(X, Y)$ is irreducible over $K(X)$.

Proof. Here we may apply Theorem 4.1 with $a(X) = 0$, and $b(X)$ any non-constant polynomial (denoted here by $g(X)$) such that $f(X, b(X)) = h(X)^k$ with $k \geq 1$ and $h \in K[X]$, $h$ irreducible over $K$. Indeed, in this case $f(X, a(X))f(X, b(X)) \neq 0$, and the fact that $\Delta \geq 0$ follows again by (14). It remains to prove that we must have $q_u = 0$. Any unitary divisor $d_1$ of $f(X, a(X)) = a_0$ is of degree at most $\deg a_0$, and any divisor $d_2$ of $h(X)^k$ which is relatively prime to $\frac{b(X)^k}{2^2}$ either has degree 0 or has degree $k \deg h = n \deg b + \deg a_n$, so $\deg d_2 - \deg d_1$ in the definition of $q_u$ is equal either to $-\deg d_1$, which is at most 0, or to $n \deg b + \deg a_n - \deg d_1$, which exceeds $\Delta$, according to (14). Therefore our condition $\deg b(X) > \max\{\deg a(X), \max_{0 \leq i \leq n-1} \frac{\deg a_i - \deg a_n}{n-i} \} + q_u$ reduces here to $\deg b(X) > 0$ too. 

5. Examples

1) For any fixed, arbitrarily chosen integers $a_1, \ldots, a_{n-1}$ and $k \geq 0$, the polynomial

$$f(X) = p^k + a_1 X + \cdots + a_{n-1} X^{n-1} + (p^{k+1} - p^k - a_1 - \cdots - a_{n-1}) X^n$$
is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers $p$. To prove this, we note that $f(0) = p^k$ and $f(1) = p^{k+1}$, so we may apply Corollary [1.4 i] with $a = 0$, $b = 1$, provided that

$$|p^{k+1} - p^k - a_1 - \cdots - a_{n-1}| > 2^n p^k + \sum_{i=1}^{n-1} 2^{n-i} |a_i|,$$

and this will obviously hold for sufficiently large prime numbers $p$.

To see an explicit example where a lower bound for $p$ can be easily derived, one can take $a_1 = \cdots = a_{n-1} = 1$ and $k \geq 1$ to conclude that the polynomial

$$f(X) = (p^{k+1} - p^k - n + 1)X^n + X^{n-1} + \cdots + X + p^k$$

is irreducible over $\mathbb{Q}$ for all primes $p \geq 2^n + 3$. Indeed, the condition

$$|a_n| = p^{k+1} - p^k - n + 1 > 2^n - 2 + 2^n p^k = \sum_{i=0}^{n-1} 2^{n-i} |a_i|$$

will obviously hold for $p \geq 2^n + 3$.

2) For any integers $k \geq 1$, $n \geq 1$, and any prime numbers $p$, $r$ with $p > r \geq 2^n + 1$, the polynomial $f(X) = (p^k + 1)(2^n - 1)X^n - X^{n-1} - \cdots - X - p^k$ is irreducible over $\mathbb{Q}$. Here we observe that $|f(0)| = p^k$, $f(1) = p^k r$, and

$$|a_n| = p^k(r + 1) + n - 1 > 2^n - 2 + 2^n p^k = \sum_{i=0}^{n-1} 2^{n-i} |a_i|$$

for $p > r \geq 2^n + 1$. The conclusion follows by Corollary [1.4 ii] with $a = 0$ and $b = 1$.

3) For any integers $n \geq 1$ and $m > 2^{n+1} - 2$ such that $(m + 1) - 2^n - 1$ is a prime number, the polynomial $f(X) = 1 + X + \cdots + X^{n-1} + mX^n$ is irreducible over $\mathbb{Q}$. To see this, we note that $f(2) = (m + 1) - 2^n - 1$, which is a prime number, and since the inequality $m > 2^{n+1} - 2$ is precisely the condition $|a_n| > 2|a_{n-1}| + 2^2 |a_{n-2}| + \cdots + 2^n |a_0|$ applied to the coefficients of $f$, the conclusion follows by Corollary [1.3].

4) Let $f(X) = 254X^6 - 4X^5 + X^4 - X^3 - X^2 - 3$. We observe that $f(2) = 127^2$, which is a prime number, and $f'(2) = 48464$, which is not divisible by 127. Then, since $a_6$ satisfies $|a_6| = 254 > 228 = \sum_{i=0}^{5} 2^{6-i} |a_i|$, we see from Corollary [3.4] that $f$ too is irreducible over $\mathbb{Q}$.

5) Let $p$ be a prime number and let

$$f(X, Y) = p + (p - 1)XY + (p^2X + p + 1)Y^2 + pXY^3 + X^2Y^4.$$ 

We observe that if we write $f$ as a polynomial in $Y$ with coefficients in $\mathbb{Z}[X]$ as $f = \sum_{i=0}^{4} a_i(X)Y^i$ with $a_i(X) \in \mathbb{Z}[X]$, we have $\deg a_4 > \max_{0 \leq i \leq 3} \{ \deg a_i \}$. On the other hand, we note that

$$f(X, X) = p + 2pX^2 + p^2X^3 + pX^4 + X^6,$$

which is Eisensteinian with respect to the prime $p$, and hence irreducible over $\mathbb{Q}$, so one may apply Corollary [1.10] with $g(X) = X$ to conclude that $f$ is irreducible over $\mathbb{Q}$. 
Let \( p \) be a prime number and let
\[
f(X, Y) = p^2 + (2p^3 + p^4X)Y + (2pX + 2p^2X^2)Y^2 + X^2Y^4.
\]
If we write \( f \) as
\[
f = \sum_{i=0}^{4} a_i(X)Y^i
\]
with \( a_i(X) \in \mathbb{Z}[X] \), we have \( \deg a_4 \geq \max_{0 \leq i \leq 3} \{ \deg a_i \} \). We observe next that \( f(X, X) = (p + p^2X + X^3)^2 \), with \( p + p^2X + X^3 \) irreducible over \( \mathbb{Q} \), being Eisensteinian with respect to \( p \). Since
\[
\frac{\partial f}{\partial Y}(X, X) = 2p^3 + p^4X + 4pX^2 + 4p^2X^3 + 4X^5,
\]
we have
\[
\frac{\partial f}{\partial Y}(X, X) = 4X^2(p + p^2X + X^3) + 2p^3 + p^4X,
\]
so \( \frac{\partial f}{\partial Y}(X, X) \) is not divisible by \( p + p^2X + X^3 \), which shows that \( f(X, X) \) and \( \frac{\partial f}{\partial Y}(X, X) \) are relatively prime. We may therefore apply Corollary 4.2 with \( g(X) = X \) to conclude that \( f \) is irreducible over \( \mathbb{Q} \).

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References

[1] R. Ballieu, Sur les limitations des racines d’une équation algébrique, Acad. Roy. Belg. Bull. Cl. Sci. (5) 33 (1947), 747–750.
[2] L. Bary-Soroker, Irreducible values of polynomials, Adv. Math. 229 (2)(2012), 854–874.
[3] L. Bary-Soroker, A. Entin, Explicit Hilbert’s Irreducibility Theorem in Function Fields. [arXiv:1912.05162]
To appear in Jarden, Shaska (eds.), Abelian varieties and number theory, Contemporary Mathematics, 2020.
[4] P. Batra, M. Mignotte, and D. Ștefănescu, Improvements of Lagrange’s bound for polynomial roots, J. Symbolic Comput. 82 (2017), 19–25.
[5] A. Bodin, P. Débes and S. Najib, The Schinzel hypothesis for polynomials. [arXiv:1902.08155] To appear in Trans. Amer. Math. Soc.
[6] A. Bodin, P. Débes and S. Najib, Families of polynomials and their specializations, J. Number Theory 170 (2017), 390–408.
[7] A. Bodin, P. Débes and S. Najib, Prime and coprime values of polynomials. Enseign. Math. (2) 66 (2020), 169–182.
[8] A.I. Bonciocat and N.C. Bonciocat, The irreducibility of polynomials that have one large coefficient and take a prime value, Canad. Math. Bull. 52 (2009), no. 4, 511–520.
[9] A.I. Bonciocat, N.C. Bonciocat, and A. Zaharescu, On the irreducibility of polynomials that take a prime power value, Bull. Math. Soc. Sci. Math. Roumanie 54 (102) (2011), no. 1, 41–54.
[10] N.C. Bonciocat and A. Zaharescu, Irreducible multivariate polynomials obtained from polynomials in fewer variables, J. Pure Appl. Algebra 212 (2008), 2338–2343.
[11] N.C. Bonciocat and A. Zaharescu, Irreducible multivariate polynomials obtained from polynomials in fewer variables, II, Proc. Indian Acad. Sci. (Math. Sci.) 121 (2011), no. 2, 133–141.
[12] N.C. Bonciocat, Y. Bugeaud, M. Cipu, and M. Mignotte, *Some Pólya-Type Irreducibility Criteria for Multivariate Polynomials*, Comm. Alg. 40 (2012), no. 2, 3733–3744.

[13] N.C. Bonciocat, Y. Bugeaud, M. Cipu, and M. Mignotte, *Irreducibility criteria for sums of two relatively prime polynomials*, Int. J. Number Theory 9 (2013), no. 6, 1529–1539.

[14] N.C. Bonciocat, Y. Bugeaud, M. Cipu, and M. Mignotte, *Irreducibility criteria for sums of two relatively prime multivariate polynomials*, Publ. Math. Debrecen 87 (2015), no. 3–4, 255–267.

[15] N.C. Bonciocat, Y. Bugeaud, M. Cipu, and M. Mignotte, *Irreducibility criteria for compositions of polynomials with integer coefficients*, Monatsh. Math. 182 (2017), no. 3, 499–512.

[16] J. Brillhart, M. Filaseta, and A. Odlyzko, *On an irreducibility theorem of A. Cohn*, Canad. J. Math. 33 (1981), no. 5, 1055–1059.

[17] M. Cavachi, *On a special case of Hilbert's irreducibility theorem*, J. Number Theory 82 (2000), no. 1, 96–99.

[18] M. Cavachi, M. Vâjâitu, and A. Zaharescu, *A class of irreducible polynomials*, J. Ramanujan Math. Soc. 17 (2002), no. 3, 161–172.

[19] M. Cavachi, Vâjâitu, and A. Zaharescu, *An irreducibility criterion for polynomials in several variables*, Acta Math. Univ. Ostrav. 12 (2004), no. 1, 13–18.

[20] M. Cole, S. Dunn, and M. Filaseta, *Further irreducibility criteria for polynomials with non-negative coefficients*, Acta Arith. 175 (2016), no. 2, 137–181.

[21] V.F. Cowling and W.J. Thron, *Zero-free regions of polynomials*, Amer. Math. Monthly 61 (1954), 682–687.

[22] V.F. Cowling and W.J. Thron, *Zero-free regions of polynomials*, J. Indian Math. Soc. (N.S.) 20 (1956), 307–310.

[23] P. Dèbes, *Reduction and specialization of polynomials*, Acta Arith. 172 (2) (2016), 175–197.

[24] H.L. Dorwart *Irreducibility of polynomials*, Amer. Math. Monthly 42 (1935), no. 6, 369–381.

[25] M. Filaseta, *A further generalization of an irreducibility theorem of A. Cohn*, Canad. J. Math. 34 (1982), no. 6, 1390–1395.

[26] M. Filaseta, *Irreducibility criteria for polynomials with non-negative coefficients*, Canad. J. Math. 40 (1988), no. 2, 339–351.

[27] M. Fujiwara, *Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung*, Tôhoku Math. J. 10 (1916), 167–171.

[28] K. Girstmair, *On an Irreducibility Criterion of M. Ram Murty*, Amer. Math. Monthly 112 (2005), no. 3, 269–270.

[29] N.H. Guersenzvaig, *Simple arithmetical criteria for irreducibility of polynomials with integer coefficients*, Integers 13 (2013), 1–21.

[30] N.H. Guersenzvaig, *Elementary criteria for irreducibility of $f(X^r)$*, Israel J. Math. 169 (2009), 109–123.

[31] N.H. Guersenzvaig and F. Szechtman, *Roots multiplicity and square free factorization of polynomials using companion matrices*, Linear Algebra Appl. 436 (9) (2012), 3160-3164.

[32] K. Győry, L. Hajdu, and R. Tijdeman, *Irreducibility criteria of Schur-type and Pólya-type*, Monatsh. Math. 163 (2011), no. 4, 415–443.

[33] S. Kakeya, *On the Limits of the Roots of an Algebraic Equation with Positive Coefficients*, Tôhoku Mathematical Journal (First Series), 2 (1912), 140–142.

[34] T. Kojima, *On a theorem of Hadamard’s and its application*, Tôhoku Math. J. 5 (1914), 54–60.

[35] M. Marden, *Geometry of polynomials*, Mathematical Surveys and Monographs No. 3, American Mathematical Society, Providence, RI, 1966.

[36] M. Mignotte, *An inequality on the greatest roots of a polynomial*, Elem. Math. 46 (1991), no. 3, 85–86.
[37] O. Ore, *Einige Bemerkungen über Irreduzibilität*, Jahresbericht der Deutschen Mathematiker-Vereinigung 44 (1934), 147–151.
[38] O. Perron, *Algebra. II Theorie der algebraischen Gleichungen*, Walter de Gruyter & Co., Berlin, 1951.
[39] G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresber. Deutschen Math. Ver., 28 (1919), 31–40.
[40] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1964.
[41] M. Ram Murty, *Prime numbers and irreducible polynomials*, Amer. Math. Monthly 109 (2002), no. 5, 452–458.
[42] P. Stäckel, *Arithmetischen Eigenschaften ganzer Funktionen*, Journal für Mathematik 148 (1918), 101–112.
[43] L. Weisner, *Criteria for the irreducibility of polynomials*, Bull. Amer. Math. Soc. 40 (1934), 864–870.

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