Fourier Transforms of Irregular Holonomic D-modules, Singularities at Infinity of Meromorphic Functions and Irregular Characteristic Cycles *

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Abstract

Based on the recent developments in the irregular Riemann-Hilbert correspondence for holonomic D-modules and the Fourier-Sato transforms for enhanced ind-sheaves, we study the Fourier transforms of some irregular holonomic D-modules. For this purpose, the singularities of rational and meromorphic functions on complex affine varieties will be studied precisely, with the help of some new methods and tools such as meromorphic vanishing cycle functors. As a consequence, we show that the exponential factors and the irregularities of the Fourier transform of a holonomic D-module are described geometrically by the stationary phase method, as in the classical case of dimension one. A new feature in the higher-dimensional case is that we have some extra rank jump of the Fourier transform produced by the singularities of the linear perturbations of the exponential factors at their points of indeterminacy. In the course of our study, not necessarily homogeneous Lagrangian cycles that we call irregular characteristic cycles will play a crucial role.

1 Introduction

The theory of Fourier transforms of D-modules is one of the most active areas in algebraic analysis. They interchange algebraic holonomic D-modules on the complex vector spaces $\mathbb{C}^N$ with those on their duals. Until now, the case $N = 1$ has been studied precisely by many mathematicians such as Bloch-Esnault [4], Malgrange [42], Mochizuki [47], Sabbah [63] etc. On the other hand, after a groundbreaking development in the theory of irregular meromorphic connections by Kedlaya [35, 36] and Mochizuki [48], in [9] D’Agnolo and Kashiwara established the Riemann-Hilbert correspondence for irregular holonomic D-modules (for another Riemann-Hilbert correspondence via filtered local systems, see also Sabbah [64]). For this purpose, they introduced enhanced ind-sheaves extending the classical notion of ind-sheaves introduced by Kashiwara-Schapira [29]. Moreover in [32], Kashiwara and Schapira adapted this new notion to the Fourier-Sato transforms of Tamarkin [67] and developed a new theory of Fourier-Sato transforms for enhanced ind-sheaves which correspond to those for algebraic holonomic D-modules. Subsequently

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in [10], by using these results, D’Agnolo and Kashiwara studied Fourier transforms of holonomic D-modules on the affine line \( \mathbb{C} \) very precisely. In this case \( N = 1 \), later Fourier transforms of regular and irregular holonomic D-modules were studied from various points of view by many authors such as D’Agnolo-Hien-Morando-Sabbah [8], Hohl [20], Mochizuki [49] and Barco-Hien-Hohl-Sevennheck [2] etc. However, in contrast to these achievements in \( N = 1 \), we know only very little in the higher-dimensional case \( N \geq 2 \). The aim of this paper is to clarify this situation by extending our previous results for Fourier transforms of regular holonomic D-modules in [22] and [23] to more general holonomic D-modules. For this purpose, we will study the singularities of rational and irregular holonomic D-modules. For Fourier transforms of regular holonomic D-modules we re studied from var-
ious points of view by many authors such as D’Agnolo-Hien-Morando-Sabbah [8], Hohl [66]. (see Verdier [70]). Then the following beautiful theorem is due to Brylinski [6].

Theorem 1.1 ([Brylinski [6]]) Let \( \mathcal{M} \) be an algebraic regular holonomic D-module on \( X = \mathbb{C}^N \). Assume that its solution complex \( \text{Sol}_X(\mathcal{M}) \) is monodromic. Then its Fourier transform \( \mathcal{M}^\wedge \) is regular and \( \text{Sol}_Y(\mathcal{M}^\wedge) \) is monodromic.
In [22] and [23], removing the monodromicity assumption in this theorem, the author and Ito studied the Fourier transforms of general regular holonomic D-modules $\mathcal{M}$ on $X = \mathbb{C}^N$ and described their smooth loci, exponential factors, irregularities and characteristic cycles etc. in terms of the geometry of $\mathcal{M}$. Moreover, as was clarified in [23], if $\mathcal{M}$ is regular holonomic then $\text{Sol}_X(\mathcal{M}^\wedge)$ is monodromic. For the other important contributions in the regular case, see also Daia [11]. In this paper, removing also the regularity assumption in [11], [22] and [23], we study the Fourier transforms of more general holonomic D-modules $\mathcal{M}$. Namely, we aim at finding a way to get a unified generalization of the results in [10] and [22].

Now let us explain our main results more precisely. Mostly in this paper, we consider the following special but basic holonomic D-modules. For a rational function $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \to \mathbb{C}$ ($P, Q \in \Gamma(X; \mathcal{O}_X) \simeq \mathbb{C}[z_1, z_2, \ldots, z_N]$, $Q \neq 0$) on $X = \mathbb{C}^N_\eta$ we set $U := X \setminus Q^{-1}(0)$ and define an exponential $\mathcal{D}_X$-module $\mathcal{E}_{U|X}^f \in \text{Mod}_{h\mathcal{D}}(\mathcal{D}_X)$ as in the analytic case (see Subsection 1.3). In what follows, we always assume that $P$ and $Q$ are coprime. Then $I(f) := P^{-1}(0) \cap Q^{-1}(0) \subset X = \mathbb{C}^N$ is noting but the set of the points of indeterminacy of the rational function $f = \frac{P}{Q}$.

**Definition 1.2.** We say that a holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \text{Mod}_{h\mathcal{D}}(\mathcal{D}_X)$ is an exponentially twisted holonomic D-module if there exist a regular holonomic $\mathcal{D}_X$-module $\mathcal{N} \in \text{Mod}_{r\mathcal{D}}(\mathcal{D}_X)$ and a rational function $f = \frac{P}{Q} : U = X \setminus Q^{-1}(0) \to \mathbb{C}$ ($P, Q \in \Gamma(X; \mathcal{O}_X), Q \neq 0$) on $X = \mathbb{C}^N_\eta$ such that we have an isomorphism

$$\mathcal{M} \simeq \mathcal{N} \otimes_{\mathcal{D}_U|X} \mathcal{E}_{U|X}^f,$$

where for the hypersurface $D := Q^{-1}(0) \subset X$ we have $(\mathcal{N} \otimes_{\mathcal{D}_U|X} \mathcal{E}_{U|X}^f)(*D) \simeq \mathcal{N} \otimes_{\mathcal{D}_U|X} \mathcal{E}_{U|X}^f$ and hence the right hand side is concentrated in degree 0.

Exponentially twisted holonomic D-modules can be considered as natural prototypes or building blocks for general holonomic D-modules, in view of the recent progress in [35, 36, 48] and [9]. Since for the moment there is no efficient way to describe the enhanced solution complexes of general holonomic D-modules in higher dimensions $N \geq 2$, exponentially twisted holonomic D-modules are the most typical holonomic D-modules to which the general theory of [32] is applicable. From now on, we fix an exponentially twisted holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \text{Mod}_{h\mathcal{D}}(\mathcal{D}_X)$ such that $\mathcal{M} \simeq \mathcal{N} \otimes_{\mathcal{D}_U|X} \mathcal{E}_{U|X}^f$ for a regular holonomic $\mathcal{D}_X$-module $\mathcal{N} \in \text{Mod}_{r\mathcal{D}}(\mathcal{D}_X)$ and a rational function $f = \frac{P}{Q} : U = X \setminus Q^{-1}(0) \to \mathbb{C}$ on $X$ and explain our results on its Fourier transform $\mathcal{M}^\wedge$. Let us call $f$ the exponential factor of $\mathcal{M}$. We set

$$K := \text{Sol}_X(\mathcal{N}) \in \mathcal{D}_b^b(X^\text{an}).$$

Here we use the convention that its shift $K[\mathcal{N}] \in \mathcal{D}_c^b(X^\text{an})$ is a perverse sheaf on $X^\text{an}$ and denote the support of $K \in \mathcal{D}_c^b(X^\text{an})$ by $Z \subset X$. Let $\pi : X^\text{an} \times \mathbb{R} \to X^\text{an}$ be the projection. Then for the enhanced sheaf

$$F := \pi^{-1}K \otimes \mathcal{E}_{U|X}^f \simeq \pi^{-1}K \otimes \left(\mathbb{C}_{\{(z,t) \in X \times \mathbb{R} \mid z \in U, t + \text{Re} f(z) \geq \alpha}\} \right) \in \mathcal{E}_b^b(\mathbb{C}_X^\text{an})$$

on the underlying real analytic manifold $X^\text{an} = X^\text{an}$ of $X$, we have an isomorphism

$$\text{Sol}_X^E(\mathcal{M}) \simeq \mathbb{C}_X^\text{an} \otimes F \simeq \left(\lim_{a \to +\infty} \pi^{-1}K \otimes \left(\mathbb{C}_{\{(z,t) \in X \times \mathbb{R} \mid z \in U, t + \text{Re} f(z) \geq a}\} \right) \right).$$
of enhanced ind-sheaves on $X_\mathbb{R}$ (see Section 4). We define its enhanced micro-support $SS^E(F) \subset (T^*X_\mathbb{R}) \times \mathbb{R}$ and the reduced one $SS_{\text{irr}}(F) \subset T^*X_\mathbb{R}$ as in D’Agnolo-Kashiwara [10] and Tamarkin [67]. In this paper, we call $SS_{\text{irr}}(F)$ the irregular micro-support of $F$. Note that for the (not necessarily homogeneous) complex Lagrangian submanifold $\Lambda^f := \{(z, df(z)) \mid z \in U = X \setminus Q^{-1}(0)\} \subset T^*U$ of $T^*U$ via the natural identification $(T^*U)_\mathbb{R} \simeq T^*U_\mathbb{R}$ we have

$$SS_{\text{irr}}(F) \cap T^*U_\mathbb{R} = (SS(K) \cap T^*U_\mathbb{R}) + \Lambda^f.$$  

Moreover we can show that $SS_{\text{irr}}(F)$ is contained in a complex isotropic analytic subset of $(T^*X)_\mathbb{R} \simeq T^*X_\mathbb{R}$ (see Lemma 5.4). However, for the moment it is not clear for us if $SS_{\text{irr}}(F)$ itself is a complex Lagrangian analytic subset or not. This prevents us from applying the theory of Fourier-Sato transforms for enhanced ind-sheaves developed by Kashiwara-Schapira [32] especially when applying the theory of Fourier-Sato transforms for enhanced ind-sheaves developed by Kashiwara-Schapira [32].

First, as a complex analogue of the $\mathbb{R}$-constructible sheaf $F = \pi^{-1}K \otimes E_{\text{fin}}^{\text{Re}f} |_{T^*X_\mathbb{R}}$ on $X_\mathbb{R} \times \mathbb{R}$, by the (not necessarily) closed embedding

$$i_{-f} : U = X \setminus Q^{-1}(0) \hookrightarrow X \times \mathbb{C}, \quad (z \mapsto (z, -f(z)))$$

associated to the rational function $-f : U \rightarrow \mathbb{C}$ we set

$$\mathcal{F} := (i_{-f})(K|_U) \in D^b(X \times \mathbb{C}).$$

Then we can easily show that it is a constructible sheaf on $X \times \mathbb{C}$ and hence its micro-support $SS(\mathcal{F})$ is a homogeneous complex Lagrangian analytic subset of $T^*(X \times \mathbb{C})$. Then as in the definitions of $SS^E(F) \subset (T^*X_\mathbb{R}) \times \mathbb{R}$ and $SS_{\text{irr}}(F) \subset T^*X_\mathbb{R}$, by forgetting the homogeneity of $SS(\mathcal{F})$ we define the following subsets:

$$SS^{E,C}(\mathcal{F}) \subset T^*X \times \mathbb{C}, \quad SS^C_{\text{irr}}(\mathcal{F}) \subset T^*X.$$  

See Section 5 for the details. It is clear that $SS^{E,C}(\mathcal{F})$ is a complex analytic subset of $(T^*X) \times \mathbb{C}$. Let us call it the enhanced micro-support of $\mathcal{F} \in D^b(X \times \mathbb{C})$. We can also show that $SS^C_{\text{irr}}(\mathcal{F})$ is a (not necessarily homogeneous) complex Lagrangian analytic subset of $(T^*X)_\mathbb{R} \simeq T^*X_\mathbb{R}$ and call it the irregular micro-support of $\mathcal{F}$. As a byproduct of the proof, we obtain a (not necessarily homogeneous) Lagrangian cycle in $T^*X \simeq X \times Y$ supported by $SS^C_{\text{irr}}(\mathcal{F}) \subset T^*X$ (see the proof of Lemma 5.6). As we can show that it depends only on $\mathcal{M}$, we call it the irregular characteristic cycle of $\mathcal{M}$ and denote it by $CC_{\text{irr}}(\mathcal{M})$. For a point $w \in Y = \mathbb{C}^N$ we define a rational function $f^w$ on $X = \mathbb{C}^N$ by

$$f^w : U = X \setminus Q^{-1}(0) \hookrightarrow \mathbb{C} \quad (z \mapsto \langle z, w \rangle - f(z)),$$

where for $z = (z_1, \ldots, z_N) \in X$ and $w = (w_1, \ldots, w_N) \in Y$ we set

$$\langle z, w \rangle := \sum_{i=1}^N z_i w_i \in \mathbb{C}. \quad (1.15)$$
Let
\[ X \leftarrow^p X \times Y \xrightarrow{q} Y \tag{1.16} \]
be the projections. Then by using our meromorphic vanishing cycle functors \( \phi^{mero,c}_{f \circ -c}(\cdot) : D_c^b(X^{an}) \rightarrow D_c^b(X^{an}) \) \((c \in \mathbb{C})\) with compact support (see Section 2 for their definition and basic properties), we define the following subset of \((T^*X)_\mathbb{R} \simeq T^*X_\mathbb{R}\).

**Definition 1.3.** For the enhanced sheaf \( F = \pi^{-1}K \otimes E_{f_{\text{can}} X^{an}}^\text{Ref} \in \mathcal{E}_c^b(\mathcal{C}_X^{an})\) we define a subset \( SS_{\text{eva}}(F) \subset T^*X_\mathbb{R} \) of \( T^*X_\mathbb{R} \simeq (T^*X)_\mathbb{R} \) by
\[ SS_{\text{eva}}(F) := \{(z, w) \in (T^*X)_\mathbb{R} \mid \phi^{mero,c}_{f \circ -c}(K)_z \neq 0 \text{ for some } c \in \mathbb{C}\}. \tag{1.17} \]

Note that we can calculate the stalks of the meromorphic vanishing cycles and hence \( SS_{\text{eva}}(F) \) by using resolutions of singularities of \( f \) and the results in [33] (see the final part of Section 2). We then will show that \( SS_{\text{eva}}(F) \) is contained in \( \Lambda := SS^c_{\text{irr}}(\mathcal{F}) \) and there exists a non-empty Zariski open subset \( \Omega \subset Y = \mathbb{C}^N \) such that the restriction of the projection \( q : T^*X \simeq X \times Y \simeq T^*Y \rightarrow \Omega = \Lambda \subset SS^c_{\text{irr}}(\mathcal{F}) \subset T^*X \) is an unramified finite covering over \( \Omega \subset Y \) and satisfies the following properties. For the precise construction of \( \Omega \) see Lemma 6.3. Let \( V \subset \Omega^{an} \) be a contractible open subset of \( \Omega^{an} \) so that for the decomposition
\[ q^{-1}(V) \cap \Lambda = \Lambda_{V,1} \sqcup \Lambda_{V,2} \sqcup \cdots \sqcup \Lambda_{V,d} \tag{1.18} \]
of \( q^{-1}(V) \cap \Lambda \) into its connected components \( \Lambda_{V,i} \) \((1 \leq i \leq d)\) we have \( \Lambda_{V,i} \rightarrow V \) for any \( 1 \leq i \leq d \).

Then by our choice of \( \Omega \subset Y = \mathbb{C}^N \), \( Z_i := p(\Lambda_{V,i}) \subset X = \mathbb{C}_z^N \) \((1 \leq i \leq d)\) are complex submanifolds of \( X = \mathbb{C}^N \) and we can renumber them so that for some \( 1 \leq r \leq d \) we have \( Z_i \subset U = X \setminus Q^{-1}(0) \) (resp. \( Z_i \subset \text{int}(f) = P^{-1}(0) \cap Q^{-1}(0) \)) if \( 1 \leq i \leq r \) (resp. if \( r + 1 \leq i \leq d \)). Let \( g_i : V \rightarrow \mathbb{C} \) \((1 \leq i \leq d)\) be holomorphic functions on \( V \subset \Omega \subset Y = \mathbb{C}^N \) such that
\[ \chi(\Lambda_{V,i}) = \Lambda^{g_i} := \{(w, dg_i(w)) \mid w \in V\} \subset T^*Y, \tag{1.19} \]
where we used the symplectic transformation of [10]:
\[ \chi : T^*X \rightarrow T^*Y \quad ((z, w) \mapsto (w, -z)). \tag{1.20} \]

Note that such \( g_i : V \rightarrow \mathbb{C} \) are uniquely defined modulo constant functions on \( V \). For \( 1 \leq i \leq d \) and \( w \in V \) let \( \zeta^{(i)}(w) \in X = \mathbb{C}_z^N \) be the unique point of \( Z_i \subset X \) such that \( (\zeta^{(i)}(w), w) \in \Lambda_{V,i} \subset \Lambda \) so that we have
\[ \Lambda_{V,i} = \{\{(\zeta^{(i)}(w), w) \mid w \in V\} \subset T^*Y \simeq T^*X. \tag{1.21} \]

Then by \[ (1.10) \] for any \( 1 \leq i \leq r \) and \( w \in V \) the point
\[ (\zeta^{(i)}(w), df^{w}(\zeta^{(i)}(w))) = (\zeta^{(i)}(w), w - df(\zeta^{(i)}(w))) \in (T^*X)_\mathbb{R} \tag{1.22} \]
is contained in the smooth part of \( SS(K|U) \subset (T^*U)_\mathbb{R} \). We denote by \( m(i) \geq 1 \) the multiplicity of the regular holonomic \( D_X \)-module \( N \in \text{Mod}_{h}(D_X) \) (or of the perverse sheaf \( K[N] \)) there. In fact, for any such \( i \) we have an equality
\[ g_i(w) \equiv f(\zeta^{(i)}(w)) - (\zeta^{(i)}(w), w) = -f^{w}(\zeta^{(i)}(w)) \quad (w \in V) \tag{1.23} \]
modulo constant functions on $V$ and can define $g_i : V \to \mathbb{C}$ by

$$g_i(w) := -f^w(\zeta(i)(w)) \quad (w \in V).$$

(1.24)

Also for $r + 1 \leq i \leq d$ and $w \in V$, we can show that there exist only finitely many $c \in \mathbb{C}$ such that

$$\phi_{f^w-c}^\text{mero,c}(K)_{\zeta(i)(w)} \neq 0$$

(1.25)

and for any such $c \in \mathbb{C}$ we have a concentration

$$H^j_{\phi_{f^w-c}^\text{mero,c}(K)_{\zeta(i)(w)}} \simeq 0 \quad (j \neq N - 1).$$

(1.26)

Moreover, by our choice of $\Omega \subset Y = \mathbb{C}^N$ the dimension of the only non-trivial cohomology group $H^{N-1}_{\phi_{f^w-c}^\text{mero,c}(K)_{\zeta(i)(w)}} \neq 0$ is constant with respect to $w \in V \subset \Omega$. For $r + 1 \leq i \leq d$ we thus can set

$$m(i) := \sum_{c \in \mathbb{C}} \dim H^{N-1}_{\phi_{f^w-c}^\text{mero,c}(K)_{\zeta(i)(w)}} \geq 1.$$  

(1.27)

Namely for $r + 1 \leq i \leq d$ the multiplicity $m(i) \geq 1$ is defined by the singularities of the rational functions $f^w \ (w \in V)$ at their points of indeterminacy $\zeta(i)(w) \in I(f^w) = I(f) = P^{-1} \cap Q^{-1}(0)$. In fact, for any such $i$ we can also show that there exist (distinct) constants $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that we have

$$\{c \in \mathbb{C} \mid \phi_{f^w-c}^\text{mero,c}(K)_{\zeta(i)(w)} \neq 0\} = \{a_1 - g_i(w), a_2 - g_i(w), \ldots, a_n - g_i(w)\}$$

(1.28)

for any $w \in V$.

Let $i_Y : Y = \mathbb{C}^N \hookrightarrow \overline{Y} = \mathbb{P}^N$ be the projective compactification of $Y$. We extend the Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_Y)$ to the holonomic $D$-module $\check{\mathcal{M}}^\wedge := i_{Y*}(\mathcal{M}^\wedge) \simeq \mathcal{D}i_{Y*}(\mathcal{M}^\wedge)$ on $\overline{Y}$. Let $\overline{\mathcal{M}}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_{\overline{Y}^\text{an}})$ be the underlying complex manifold of $\overline{Y}$ and define the analytification $\check{\mathcal{M}}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_{\overline{Y}^\text{an}})$ of $\mathcal{M}^\wedge$ by $\check{\mathcal{M}}^\wedge := \mathcal{O}_{\overline{Y}^\text{an}} \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{M}^\wedge$. Then we have the following formula for the enhanced solution complex

$$\text{Sol}_{\overline{Y}}^\wedge(\check{\mathcal{M}}^\wedge) := \text{Sol}_{\overline{Y}^\text{an}}^\wedge(\check{\mathcal{M}}^\wedge) \in \mathcal{E}^b(\text{IC}_{\overline{Y}^\text{an}})$$

(1.29)

of $\check{\mathcal{M}}^\wedge$.

Theorem 1.4. In the situation as above, we have an isomorphism

$$\pi^{-1}C_V \otimes \left(\text{Sol}_{\overline{Y}}^\wedge(\check{\mathcal{M}}^\wedge)\right) \simeq \bigoplus_{i=1}^d \left(\mathcal{E}^b_{\text{Reg},i}(\overline{Y}^\text{an})\right)^{\oplus m(i)}$$

$$\simeq \bigoplus_{i=1}^d \left(\lim_{\substack{\text{ind} \to \text{in} \to \text{in}}} \mathbb{C}\left\{ (w, \theta) \in \overline{Y}^\text{an} \times \mathbb{R} \mid w \in V, \theta \in \text{Reg}, (w, \theta) \geq a \right\}\right)^{\oplus m(i)}$$

of enhanced ind-sheaves on $\overline{Y}^\text{an}$. In particular, the restriction $\mathcal{M}^\wedge|_\Omega$ of the Fourier transform $\mathcal{M}^\wedge$ to $\Omega \subset Y = \mathbb{C}^N$ is an algebraic integrable connection of rank $\sum_{i=1}^d m(i)$.

By Theorem 1.4 we can describe the exponential factors and the irregularities of the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ along various submanifolds of $Y = \mathbb{C}^N$. See Section 6 for the details. By patching the (not necessarily homogeneous) Lagrangian cycles

$$\sum_{i=1}^d m(i) \cdot [A_{V,i}]$$

(1.30)
in the open subsets $q^{-1}(V) = X \times V \subset X \times Y \simeq T^* X$ for various $V \subset \Omega$ we obtain a Lagrangian cycle globally defined in $q^{-1}(\Omega) = X \times \Omega \subset T^* X$. By our definitions of the multiplicities $m(i) \geq 1$, it turns out that it coincides with the restriction of the irregular characteristic cycle $\text{CC}_{\text{irr}}(\mathcal{M})$ of $\mathcal{M}$ to $q^{-1}(\Omega) = X \times \Omega \subset T^* X$. Then the last assertion of Theorem 1.4 means that the generic rank of the holonomic D-module $\mathcal{M}^\wedge$ is equal to the covering degree of $\text{CC}_{\text{irr}}(\mathcal{M})$ over $\Omega \subset Y = \mathbb{C}^N$. We thus find that if $I(f) = P^{-1} \cap Q^{-1}(0) \neq \emptyset$ there may exist some extra rank jump of $\mathcal{M}^\wedge$ produced by the singularities of the rational functions $f^w$ ($w \in \Omega$) i.e. the linear perturbations of the exponential factor $f$ of $\mathcal{M}$. This shows that the structures of the Fourier transforms of irregular holonomic D-modules are much more involved than those of the regular ones studied in [6], [11], [22] and [23] etc.

Our proof of Theorem 1.4 is similar to that of [22, Theorem 4.4] and relies on a Morse theory for the Morse functions $\text{Re} f^w : U = X \setminus Q^{-1}(0) \rightarrow \mathbb{R}$ $(w \in \Omega)$ and $K|_U \in \mathcal{D}^b_c(U^\text{an})$ as well as the theory of Fourier-Sato transforms for enhanced ind-sheaves of Kashiwara-Schapira [32]. However, it requires also more careful analysis on the singularities of the rational functions $f^w : U = X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ $(w \in \Omega)$ at infinity and at their points of indeterminacy. To treat their singularities at infinity, we extend the classical notion of tameness at infinity introduced by Broughton [5] for polynomial functions on $\mathbb{C}^N$ to that for rational and meromorphic functions on smooth subvarieties of $\mathbb{C}^N$ and obtain a transversality theorem (see Proposition 4.4) similar to the ones proved by Némethi and Zaharia [52, 53] (see also [50, Sections 2 and 3] for related results). Let $\mathcal{S}$ be an algebraic Whitney stratification of $Z \cap U \subset U = X \setminus Q^{-1}(0)$ adapted to $K|_U \in \mathcal{D}^b_c(U)$ such that

$$\text{SS}(K|_U) \subset \bigcup_{S \in \mathcal{S}} T^*_S X$$

(1.31)

Then in Lemma 6.16 we will show that for any $w \in \Omega$ and stratum $S \in \mathcal{S}$ such that $T^*_S X \subset \text{SS}(K|_U)$ the restriction $f^w|_S : S \rightarrow \mathbb{C}$ of $f^w$ to $S \subset U$ is tame at infinity in our (generalized) sense. This implies that the morphisms $f^w : U \longrightarrow \mathbb{C}$ $(w \in \Omega)$ and $K|_U \in \mathcal{D}^b_c(U^\text{an})$ satisfy one of the conditions of Kashiwara and Schapira in their non-proper direct image theorem [26, Theorem 4.4.1] for the family of open subsets $U_r \subset U$ ($r > 0$) of $U$ defined by

$$U_r := \{ z \in U \mid ||z|| < r \} \subset U \quad (r > 0)$$

(1.32)

(see Proposition 3.5). We then show that for the given rational function $f : U \rightarrow \mathbb{C}$ we can apply [26, Theorem 4.4.1] to its generic linear perturbations $\tilde{f} = -f^w : U \rightarrow \mathbb{C}$ $(w \in \Omega)$ and $K|_U \in \mathcal{D}^b_c(U^\text{an})$ to obtain the following byproduct of our study of Fourier transforms (see Proposition 3.10 and Lemma 6.17).

**Proposition 1.5.** Let $f : U \rightarrow \mathbb{C}$ and $K \in \mathcal{D}^b_c(X^\text{an})$ be as above. Then for generic linear perturbations $\tilde{f} : U \rightarrow \mathbb{C}$ of $f$ we have: For any $\tau_0 \in \mathbb{C}$ there exist $R \gg 0$ and $0 < \varepsilon \ll 1$ such that for the inclusion maps $i_r : U_r \hookrightarrow U$ ($r > R$) we have isomorphisms

$$\text{R} \tilde{f}(i_r)|_r(K|_U) \simeq \text{R} \tilde{f}(K|_U)$$

(1.33)

on the open subset $\{ \tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C}$ of $\mathbb{C}$. Moreover for such $\tilde{f}$ we have

$$\text{SS}(\text{R} \tilde{f}(K|_U)) \subset \varpi \rho^{-1} \text{SS}(K|_U),$$

(1.34)
where we used the natural morphisms

\[ T^* \mathbb{C} \xleftarrow{\varphi} U \times_{\mathbb{C}} T^* \mathbb{C} \xrightarrow{\rho} T^* U \]  

(1.35)

associated to \( \tilde{f} : U \to \mathbb{C} \).

This in particular implies that for any algebraic constructible sheaf \( K \in D^b_c(X^{an}) \) the geometric condition of Kashiwara and Schapira in [26, Theorem 4.4.1] is satisfied by generic polynomial functions on \( X = \mathbb{C}^N \). To treat the singularities of \( f^w : U \to \mathbb{C} \) \((w \in \Omega)\) at their points of indeterminacy, we develop a general theory of meromorphic vanishing cycle functors (with compact support) by modifying the similar ones studied previously by Raibaut [60] and Nguyen-Takeuchi [57]. See Section 2 for the details.

It is clear that we can readily extend Theorem 1.4 to all holonomic D-modules on \( X = \mathbb{C}^N \) having only exponentially twisted composition factors. Recall that the Fourier transform is an exact functor. Moreover, if for a holonomic D-module \( \mathcal{N} \) on \( X = \mathbb{C}^N \) there exists an enhanced sheaf \( G \) globally defined on \( X_{\mathbb{R}} \) and having a local expression similar to that of the above one \( F \) such that

\[ \text{Sol}^E_{X}(\mathcal{N}) \simeq \mathbb{C}^{E}_{X^{an}} \otimes G, \]  

(1.36)

then we can obtain a formula for the enhanced ind-sheaf

\[ \pi^{-1}\mathbb{C} \otimes \left( \text{Sol}^E_{X}(\mathcal{N}) \right) \in E^b(\text{IC}_{Y^{an}}) \]  

(1.37)

\((V \) is an open subset of \( Y = \mathbb{C}^N \)) on \( Y^{an} \) along the same line as in the proof of Theorem 1.4. Recall that in [9] D’Agnolo and Kashiwara proved that for a complex manifold \( X \) and an exponential D-module \( \mathcal{E}_{f\mid X} \) on it associated to a meromorphic function \( f \in \mathcal{O}_X(\ast D) \) along a closed hypersurface \( D \subset X \) and \( U = X \setminus D \) we have an isomorphism

\[ \text{Sol}^E_{X}(\mathcal{E}_{f\mid X}) \simeq \mathbb{E}^\text{Ref}_{U\mid X} \simeq \lim_{\longrightarrow a \to +\infty} \left( \mathbb{C}\left\{ (z,t) \in X \times \mathbb{R} \mid z \in U, t + \text{Re}f(z) \geq a \right\} \right) \]  

(1.38)

of enhanced ind-sheaves on \( X^{an} = X_{\mathbb{R}} \). This formula for the enhanced solution complex \( \text{Sol}^E_{X}(\mathcal{E}_{f\mid X}) \) played a central role in the proof of the main results in [9]. In Section 7 we shall try to extend it to arbitrary meromorphic connections. Our argument is a higher-dimensional analogue of those of Kashiwara-Schapira [30, Section 7] and Morando [50, Section 2.1]. We believe that this approach would lead us to a full generalization of Theorem 1.4 to arbitrary holonomic D-modules on \( X = \mathbb{C}^N \).

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2 Meromorphic Nearby and Vanishing Cycle Functors

In this section, we recall the definitions of the meromorphic nearby cycle functors introduced in Nguyen-Takeuchi [57] and Raibaut [60] and prove their basic properties. In
this paper we essentially follow the terminology of \[12,21\] and \[27\]. For a topological space \(X\) denote by \(\mathbf{D}^b(X)\) the derived category whose objects are bounded complexes of sheaves of \(\mathbb{C}_X\)-modules on \(X\). If \(X\) is a complex manifold, we denote by \(\mathbf{D}^b_c(X)\) the full subcategory of \(\mathbf{D}^b(X)\) consisting of constructible objects and adopt the convention that \(\mathbb{C}_X[\dim X] \in \mathbf{D}^b_c(X)\) is a perverse sheaf on \(X\).

Let \(X\) be a complex manifold and \(P(z), Q(z)\) holomorphic functions on it. Assume that \(Q(z)\) is not identically zero on each connected component of \(X\). Then we define a meromorphic function \(f(z)\) on \(X\) by
\[
f(z) = \frac{P(z)}{Q(z)} \quad (z \in X \setminus Q^{-1}(0)),
\]
(2.1)

Let us set \(I(f) = P^{-1}(0) \cap Q^{-1}(0) \subset X\). If \(P\) and \(Q\) are coprime in the local ring \(\mathcal{O}_{X,z}\) at a point \(z \in X\), then \(I(f)\) is nothing but the set of the indeterminacy points of \(f\) on a neighborhood of \(z\). Note that the set \(I(f)\) depends on the pair \((P(z), Q(z))\) of holomorphic functions representing \(f(z)\). For example, if we take a holomorphic function \(R(z)\) on \(X\) (which is not identically zero on each connected component of \(X\)) and set
\[
g(z) = \frac{P(z)R(z)}{Q(z)R(z)} \quad (z \in X \setminus (Q^{-1}(0) \cup R^{-1}(0))),
\]
(2.2)
then the set \(I(g) = I(f) \cup R^{-1}(0)\) might be bigger than \(I(f)\). In this way, we distinguish \(f(z) = \frac{P(z)}{Q(z)}\) from \(g(z) = \frac{P(z)R(z)}{Q(z)R(z)}\) even if their values coincide over an open dense subset of \(X\). This is the convention due to Gusein-Zade, Luengo and Melle-Hernández [16] etc. Now we recall the following fundamental theorem due to [16] and [17]. In what follows, we assume that \(X\) is connected and \(P\) is not identically zero on it.

**Theorem 2.1.** ([Gusein-Zade, Luengo and Melle-Hernández [16], [17]]) For any point \(z_0 \in P^{-1}(0)\) there exists \(\varepsilon > 0\) such that for any \(0 < \varepsilon < \varepsilon_0\) and the open ball \(B_\varepsilon(z_0) \subset X\) of radius \(\varepsilon > 0\) with center at \(z_0\) (in a local chart of \(X\)) the restriction
\[
B_\varepsilon(z_0) \setminus Q^{-1}(0) \longrightarrow \mathbb{C}
\]
(2.3)
of \(f : X \setminus Q^{-1}(0) \longrightarrow \mathbb{C}\) is a locally trivial fibration over a sufficiently small punctured disk in \(\mathbb{C}\) with center at the origin \(0 \in \mathbb{C}\).

We call the fiber in this theorem the Milnor fiber of the meromorphic function \(f(z) = \frac{P(z)}{Q(z)}\) at \(z_0 \in P^{-1}(0)\) and denote it by \(F_{z_0}\). For the meromorphic function \(f(z) = \frac{P(z)}{Q(z)}\) let
\[
i_f : X \setminus Q^{-1}(0) \hookrightarrow X \times \mathbb{C}_\tau
\]
(2.4)
be the (not necessarily) closed embedding defined by \(z \longmapsto (z, f(z))\). Let \(\tau : X \times \mathbb{C} \rightarrow \mathbb{C}\) be the second projection. Then for \(\mathcal{F} \in \mathbf{D}^b(X)\) we set
\[
\psi_f^{\text{mero}}(\mathcal{F}) := \psi_{\tau}((\text{Ri}_{i_f})\mathcal{F}|_{X \setminus Q^{-1}(0)}) \in \mathbf{D}^b(X)
\]
(2.5)
(see Nguyen-Takeuchi [57]). We call \(\psi_f^{\text{mero}}(\mathcal{F})\) the meromorphic nearby cycle sheaf of \(\mathcal{F}\) along \(f\). Moreover we set
\[
\psi_f^{\text{mero},c}(\mathcal{F}) := \psi_{\tau}(i_f^!(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \in \mathbf{D}^b(X)
\]
(2.6)
(see Raibaut [60]) and call it the meromorphic nearby cycle sheaf with compact support of \(\mathcal{F}\) along \(f\). Then the proof of the following lemma is similar to those of [57] Lemma 2.1 and Remark 2.3].
Lemma 2.2. (i) The support of $\psi_f^{\text{mero}}(F)$ is contained in $P^{-1}(0)$.

(ii) There exists an isomorphism

$$\psi_f^{\text{mero}}(F) \sim \psi_f^{\text{mero}}(F_{X \setminus (P^{-1}(0) \cup Q^{-1}(0))}).$$

(iii) Assume that the meromorphic function $f(z) = \frac{P(z)}{Q(z)}$ is holomorphic on a neighborhood of a point $z_0 \in X$ i.e. there exists a holomorphic function $g(z)$ defined on a neighborhood of $z_0 \in X$ such that $P(z) = Q(z) \cdot g(z)$ on it. Then we have an isomorphism

$$\psi_f^{\text{mero}}(F)_{z_0} \sim \psi_g(F_{X \setminus Q^{-1}(0)}),$$

for the classical (holomorphic) nearby cycle functor $\psi_g(\cdot)$.

(iv) For any point $z_0 \in P^{-1}(0)$ and $j \in \mathbb{Z}$ we have an isomorphism

$$H^j\psi_f^{\text{mero}}(F)_{z_0} \sim H^j_z(F_{z_0} \setminus Q^{-1}(0); F),$$

where $H^j_z(\cdot)$ stands for the hypercohomology group with compact support.

Example 2.3. Consider the case where $X$ is the complex plane $\mathbb{C}^2$ endowed with the standard coordinate $z = (z_1, z_2) = (x, y)$ and let $P, Q \in \mathbb{C}[x, y]$ be polynomials on $X = \mathbb{C}^2$ coprime each other such that $P(0) = Q(0) = 0$. Assume that the complex curve $P^{-1}(0) \subset X$ (resp. $Q^{-1}(0) \subset X$) has an isolated singular point (resp. smooth) at the origin $0 = (0, 0) \in X = \mathbb{C}^2$. Then for $c \in \mathbb{C}$ the fiber $f^{-1}(c) \subset X \setminus Q^{-1}(0)$ of the rational function $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ is explicitly described as follows:

$$f^{-1}(c) = \{(x, y) \in X \setminus Q^{-1}(0) \mid P(x, y) - c \cdot Q(x, y) = 0\} \subset X \setminus Q^{-1}(0).$$

On the other hand, by [46, Corollary 2.8] the set $\Sigma_f \subset \mathbb{C}$ of the critical values of $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ is finite and hence generic fibers of $f$ are smooth. This implies that for $c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that $0 < |c| < 1$ the complex curve

$$\overline{f^{-1}(c)} = \{(x, y) \in X = \mathbb{C}^2 \mid P(x, y) - c \cdot Q(x, y) = 0\} \subset X$$

is smooth outside the origin $0 = (0, 0) \in X = \mathbb{C}^2$. Moreover by the conditions

$$\left(\frac{\partial P}{\partial x}(0, 0), \frac{\partial P}{\partial y}(0, 0)\right) = (0, 0), \quad \left(\frac{\partial Q}{\partial x}(0, 0), \frac{\partial Q}{\partial y}(0, 0)\right) \neq (0, 0)$$

we see that for such $c \in \mathbb{C}^*$ the complex curve $\overline{f^{-1}(c)} \subset X$ is smooth also at the origin. For a complex number $c \in \mathbb{C}^*$ and $\varepsilon > 0$ such that $0 < |c| < \varepsilon < 1$ let us explain the topology of the Milnor fiber

$$F_0 = (B_\varepsilon(0) \setminus Q^{-1}(0)) \cap f^{-1}(c) = (B_\varepsilon(0) \cap \overline{f^{-1}(c)}) \setminus \{0\}$$

of $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ at the origin $0 = (0, 0) \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$. We set

$L := P^{-1}(0) \cap \partial B_\varepsilon(0) \subset S_\varepsilon(0) := \partial B_\varepsilon(0) \simeq S^3$ and call it the link of the complex curve $P^{-1}(0) \subset X = \mathbb{C}^2$ at the origin. The small sphere $S_\varepsilon(0)$ being compact, the boundary $\partial F_0$ of the closure $\overline{F_0}$ of $F_0$ is homeomorphic to $L$. We know also that the link $L$ is a disjoint
union of some circles $S^1$ embedded in $S_c(0) \simeq S^3$. We denote the number of circles $S^1$ in $L$ by $m$ and shall explain how to calculate it from now. For this purpose, let $F(P)_0$ be the Milnor fiber of $P^{-1}(0) \subset X$ at the origin $0 = (0, 0) \in X$ and consider the monodromy automorphism

$$\Phi(P)_1 : H^1(F(P)_0; \mathbb{C}) \xrightarrow{\sim} H^1(F(P)_0; \mathbb{C})$$

(2.14)

of its first cohomology group $H^1(F(P)_0; \mathbb{C})$. Denote by $N_1$ the number of the Jordan blocks for the eigenvalue $1$ in $\Phi(P)_1$. Then by the results in [46, Section 8] (see also the proof of [51, Lemma 4.3]) we have $m = N_1 + 1$. Since by the classical theory of the mixed Hodge structure of $H^1(F(P)_0; \mathbb{C})$ we know in this case of $\dim X = 2$ that the maximal possible size of such Jordan blocks is one, we can calculate the number $N_1$ by the theory of monodromy zeta functions (see [46, Section 2] for a review of this subject). We thus have seen that the interior of $F_0$ is a smooth Riemann surface with the boundary $\partial F_0$ which is isomorphic to the disjoint union $S^1 \sqcup \cdots \sqcup S^1$ of $N_1 + 1$ copies of $S^1$. Let $M$ be a compact oriented surface obtained by attaching $N_1 + 1$ (2-dimensional) disks to $F_0$ along its boundary $\partial F_0$ and denote its genus by $g(M)$. By taking a triangulation of $F_0$ we see that $H^2_c(F_0; \mathbb{C}_X) \simeq H^2(F_0; \mathbb{C}_X) \simeq 0$. Then by the short exact sequence

$$0 \rightarrow \mathbb{C}_M \setminus F_0 \rightarrow \mathbb{C}_M \rightarrow \mathbb{C}_{F_0} \rightarrow 0$$

(2.15)

we obtain isomorphisms

$$H^2(F_0; \mathbb{C}_X) \simeq \begin{cases} \mathbb{C} & (j = 0) \\ \mathbb{C}^{2g(M)+N_1} & (j = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

(2.16)

Moreover, by $F_0 \setminus Q^{-1}(0) = F_0 \setminus \{0\}$ we can show that there exist isomorphisms

$$H^j_{\phi^{\text{mero},c}_f}(\mathbb{C}_X)_0 \simeq H^j_c(F_0 \setminus Q^{-1}(0); \mathbb{C}_X) \simeq \begin{cases} \mathbb{C}^{2g(M)+N_1} & (j = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

(2.17)

Compare this result with the perversity of $\phi^{\text{mero},c}_f(\mathbb{C}_X[1])$ which will be proved soon below in this section. For the above $c \in \mathbb{C}^*$ and $\varepsilon' > 0$ such that $0 < \varepsilon' \ll |c| \ll \varepsilon$ we set

$$G_0 := (B_{\varepsilon'}(0) \setminus Q^{-1}(0)) \cap f^{-1}(c) = (B_{\varepsilon'}(0) \cap \overline{f^{-1}(c)}) \setminus \{0\}.$$ 

(2.18)

Then the closure $\overline{G_0}$ of $G_0$ is homeomorphic to the $(2$-dimensional) disk and $\overline{G_0} \setminus Q^{-1}(0) = \overline{G_0} \setminus \{0\}$. We thus obtain isomorphisms

$$H^j_c(\overline{G_0} \setminus Q^{-1}(0); \mathbb{C}_X) \simeq 0 \quad (j \in \mathbb{Z}).$$

(2.19)

The difference of the topology of $G_0$ from that of $F_0$ comes from the fact that there exists $r > 0$ such that $\varepsilon' < r < \varepsilon$ and the sphere $S_c(0) = \partial B_{\varepsilon}(0)$ is tangent to the smooth curve $\overline{f^{-1}(c)}$. This explains the reason why one should distinguish $F_0$ from $G_0$ carefully in the study of the Milnor fiber $F_0$ of $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$. 

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The following result is an analogue of [57, Theorem 2.2] for the meromorphic nearby cycle sheaf with compact support of $F$ along $f$.

**Theorem 2.4.** (i) If $F \in D^b(X)$ is constructible, then $\psi^\text{mero, c}_f(F) \in D^b(X)$ is also constructible.

(ii) If $F \in D^b(X)$ is perverse, then $\psi^\text{mero, c}_f(F)[-1] \in D^b(X)$ is also perverse.

*Proof.* Assume that $F \in D^b(X)$ is constructible. Define a hypersurface $W$ of $X \times \mathbb{C}_\tau$ by

$$W = \{(z, \tau) \in X \times \mathbb{C} \mid P(z) - \tau Q(z) = 0\} \quad (2.20)$$

and let $\kappa : W \rightarrow X$ be the restriction of the first projection $X \times \mathbb{C}_\tau \rightarrow X$ to it. Then $\kappa$ induces an isomorphism

$$\kappa^{-1}(X \setminus Q^{-1}(0)) \xrightarrow{\sim} X \setminus Q^{-1}(0) \quad (2.21)$$

and $\kappa^{-1}(X \setminus Q^{-1}(0))$ is nothing but the graph

$$\{(z, f(z)) \in (X \setminus Q^{-1}(0)) \times \mathbb{C} \mid z \in X \setminus Q^{-1}(0)\} \quad (2.22)$$

of $f : (X \setminus Q^{-1}(0)) \rightarrow \mathbb{C}$. In this way, we identify $X \setminus Q^{-1}(0)$ and the open subset $\kappa^{-1}(X \setminus Q^{-1}(0))$ of $W$. Let

$$j_f : (X \setminus Q^{-1}(0)) \times \mathbb{C}_\tau \hookrightarrow X \times \mathbb{C}_\tau \quad (2.23)$$

and $i_W : W \hookrightarrow X \times \mathbb{C}_\tau$ be the inclusion maps. Then as in the proof of [57, Theorem 2.2] we obtain an isomorphism

$$\mathcal{G} := i_W^!(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \simeq j_f^*j_f^{-1}(i_W^!\kappa^{-1}\mathcal{F}). \quad (2.24)$$

From this we see that $\mathcal{G} = i_W^!(\mathcal{F}|_{X \setminus Q^{-1}(0)})$ is constructible. Now the assertion (i) is clear. Assume that $F \in D^b(X)$ is perverse. Then, although $\mathcal{H} := i_W^!\kappa^{-1}\mathcal{F} \in D^b(X \times \mathbb{C})$ is not necessarily perverse (up to some shift) in general, its restriction

$$j_f^{-1}\mathcal{H} \simeq \mathcal{H}|_{(X \setminus Q^{-1}(0)) \times \mathbb{C}_\tau} \quad (2.25)$$

to $(X \setminus Q^{-1}(0)) \times \mathbb{C}_\tau$ is perverse (up to some shift). Moreover, its Verdier dual $\mathbb{D}_{X \times \mathbb{C}}(\mathcal{H}) \in D^b(X \times \mathbb{C})$ satisfies the same property. It follows from the proof of [57, Theorem 2.2 (ii)] that

$$Rj_{f*}j_f^{-1}\mathbb{D}_{X \times \mathbb{C}}(\mathcal{H}) \in D^b(X \times \mathbb{C}) \quad (2.26)$$

is perverse (up to some shift). By the isomorphism

$$\mathbb{D}_{X \times \mathbb{C}}(\mathcal{G}) = \mathbb{D}_{X \times \mathbb{C}}(j_f^*j_f^{-1}(\mathcal{H})) \simeq Rj_{f*}j_f^{-1}\mathbb{D}_{X \times \mathbb{C}}(\mathcal{H}), \quad (2.27)$$

we find that $\mathcal{G} = j_f^*j_f^{-1}(\mathcal{H}) \simeq i_W^!(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \in D^b(X \times \mathbb{C})$ is also perverse (up to some shift). Then the assertion of (ii) immediately follows from the $t$-exactness of the nearby cycle functor

$$\psi_* : D^b(X \times \mathbb{C}) \rightarrow D^b(X). \quad (2.28)$$

$\square$
By this theorem we obtain a functor
\[ \psi_f^{\text{mero},c}(\cdot) : \mathbf{D}^b(X) \to \mathbf{D}^b_c(X). \]  
(2.29)

Let \( i_0 : X \hookrightarrow X \times \mathbb{C}_r \) \( (x \mapsto (x,0)) \) be the inclusion map and for \( \mathcal{F} \in \mathbf{D}^b(X) \) set
\[ \phi_f^{\text{mero},c}(\mathcal{F}) := \phi_r(i_0(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \in \mathbf{D}^b(X). \]  
(2.30)

We call it the meromorphic vanishing cycle sheaf with compact support of \( \mathcal{F} \) along \( f \).

Then there exists a distinguished triangle
\[ i_0^{-1}i_f! (\mathcal{F}|_{X \setminus Q^{-1}(0)}) \to \psi_f^{\text{mero},c}(\mathcal{F}) \to \phi_f^{\text{mero},c}(\mathcal{F}) +1 \to \]  
(2.31)
in \( \mathbf{D}^b(X) \). Moreover, since for the inclusion map \( j : P^{-1}(0) \setminus Q^{-1}(0) \hookrightarrow X \) we have a Cartesian diagram
\[ \begin{array}{ccc}
P^{-1}(0) \setminus Q^{-1}(0) & \xrightarrow{j} & X \\
\downarrow & & \downarrow \phi_0 \\
X \setminus Q^{-1}(0) & \xrightarrow{i_f} & X \times \mathbb{C}_t \end{array} \]  
(2.32)
of inclusion maps, we obtain an isomorphism
\[ i_0^{-1}i_f! (\mathcal{F}|_{X \setminus Q^{-1}(0)}) \simeq j_!(\mathcal{F}|_{P^{-1}(0) \setminus Q^{-1}(0)}) \simeq \mathcal{F}_{P^{-1}(0) \setminus Q^{-1}(0)}. \]  
(2.33)

We thus see that also the functor
\[ \phi_f^{\text{mero},c}(\cdot) : \mathbf{D}^b(X) \to \mathbf{D}^b(X) \]  
(2.34)
preserves the constructibility and the perversity (up to some shift). Moreover by (2.31) and (2.33), for any \( \mathcal{F} \in \mathbf{D}^b(X) \) we have isomorphisms
\[ \phi_f^{\text{mero},c}(\mathcal{F})_{z_0} \simeq \psi_f^{\text{mero},c}(\mathcal{F})_{z_0} \quad (z_0 \in Q^{-1}(0)). \]  
(2.35)

The proof of the following lemma is similar to those of [57, Lemma 2.1 and Remark 2.3].

**Lemma 2.5.**

(i) The support of \( \phi_f^{\text{mero},c}(\mathcal{F}) \) is contained in \( P^{-1}(0) \).

(ii) There exists an isomorphism
\[ \phi_f^{\text{mero},c}(\mathcal{F}) \sim \phi_f^{\text{mero},c}(\mathcal{F}|_{X \setminus Q^{-1}(0)}). \]  
(2.36)

(iii) Assume that the meromorphic function \( f(z) = \frac{P(z)}{Q(z)} \) is holomorphic on a neighborhood of a point \( z_0 \in X \) i.e. there exists a holomorphic function \( g(z) \) defined on a neighborhood of \( z_0 \in X \) such that \( P(z) = Q(z) \cdot g(z) \) on it. Then we have an isomorphism
\[ \phi_f^{\text{mero},c}(\mathcal{F})_{z_0} \simeq \phi_g(\mathcal{F}|_{X \setminus Q^{-1}(0)})_{z_0} \]  
(2.37)
for the classical (holomorphic) vanishing cycle functor \( \phi_g(\cdot) \).

The following result is a cohomological generalization of Theorem 2.1.
Proposition 2.6. Let $\mathcal{F} \in \mathbf{D}^b_c(X)$ be a constructible sheaf on $X$. Then for any $z_0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$ and $c \in \mathbb{C}$ there exist small $0 < \varepsilon < 1$ such that the cohomology sheaves $H^j Rf_!(\mathcal{F}_{\overline{B_\varepsilon(z_0)}\setminus\{\{Q^{-1}(0)\}}) (j \in \mathbb{Z})$ are local systems over a sufficiently small punctured disk centered at $c \in \mathbb{C}$. Moreover we have

$$Rf_!(\mathcal{F}_{\overline{B_\varepsilon(z_0)}\setminus\{\{Q^{-1}(0)\}}) \simeq R(f - c)_!(\mathcal{F}_{\overline{B_\varepsilon(z_0)}\setminus\{\{Q^{-1}(0)\}}) \simeq 0 \quad (2.38)$$

and there exist isomorphisms

$$\phi_{\tau - c}(Rf_!(\mathcal{F}_{\overline{B_\varepsilon(z_0)}\setminus\{\{Q^{-1}(0)\}})) \simeq \psi_{\tau - c}(Rf_!(\mathcal{F}_{\overline{B_\varepsilon(z_0)}\setminus\{\{Q^{-1}(0)\}})) \quad (2.39)$$

$$\simeq \psi_{\text{mero},c}(\mathcal{F})_{z_0} \simeq \phi_{\text{mero},c}(\mathcal{F})_{z_0}. \quad (2.40)$$

Proof. Note that for $P_c := P - c \cdot Q$ we have $f - c = \frac{P_c}{Q}$. Then we obtain the first assertion by (the proof of) Lemma 2.2 (iv) (see also the proof of [66 Theorem 2.6]). We can also show the vanishing (2.38) by the cone theorem proved by [46, Theorem 2.10] and [7, Lemma 3.2]. Indeed, let $X = \sqcup_{\alpha \in A} X_\alpha$ be a stratification of $X$ such that $H^j(\mathcal{F}|_{X_\alpha})$ is locally constant for any $j \in \mathbb{Z}$ and $\alpha \in A$. Then the cone theorem implies that there exist small $0 < \varepsilon < 1$ such that for any $\alpha \in A$ the subsets

$$\overline{X_\alpha \cap P^{-1}(0) \cap \overline{B_\varepsilon(z_0)}}, \quad \overline{X_\alpha \cap P^{-1}(0) \cap Q^{-1}(0) \cap \overline{B_\varepsilon(z_0)}}, \quad (2.41)$$

of $\overline{B_\varepsilon(z_0)}$ are contractible and hence

$$R(f - c)_!(\mathcal{F}_{\overline{X_\alpha \cap P^{-1}(0) \cap \overline{B_\varepsilon(z_0)}}}) \simeq 0. \quad (2.42)$$

By decomposing the support of $\mathcal{F}$ with respect to the stratification $X = \sqcup_{\alpha \in A} X_\alpha$ of $X$ we thus can prove the vanishing (2.38). Together with Lemma 2.2 (iv), we obtain also the remaining assertions. \hfill \square

Proposition 2.7. For a constructible sheaf $\mathcal{F} \in \mathbf{D}^b_c(X)$ on $X$ and a point $z_0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$, there exists a finite subset $\Sigma \subset \mathbb{C} \setminus \mathbb{C}$ such that we have

$$\phi_{\text{mero},c}(\mathcal{F})_{z_0} \simeq 0. \quad (2.43)$$

for any $c \in \mathbb{C} \setminus \Sigma$.

Proof. Note that for the point $z_0 \in I(f)$ (under the natural identification of $X \times \{0\}$ with $X \times \{c\}$) we have an isomorphism

$$\phi_{\text{mero},c}(\mathcal{F})_{z_0} \simeq \phi_{\tau - c}(i_{f!}(\mathcal{F}|_{X \setminus \{Q^{-1}(0)\}}))_{(z_0,c)}. \quad (2.44)$$

Let $\mathbb{P} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the projective compactification of $\mathbb{C}$ and $i : X \times \mathbb{C} \hookrightarrow X \times \mathbb{P}$ the inclusion map. Then by the proof of Theorem 2.3 (i) we see also that $\mathcal{\tilde{G}} := i_{f!}(i_{f!}(\mathcal{F}|_{X \setminus \{Q^{-1}(0)\}})) \in \mathbf{D}^b(X \times \mathbb{P})$ is constructible. Let $\mathcal{S}$ be a Whitney stratification of $X \times \mathbb{P}$ adapted to $\mathcal{\tilde{G}}$ such that

$$\text{SS}(\mathcal{\tilde{G}}) \subset \bigcup_{\tau \in \mathcal{S}} T_\tau(X \times \mathbb{P}). \quad (2.45)$$

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Then by the theorem in [15, page 43], after refining $S$ if necessary, we may assume that for $S$ and a Whitney stratification $S_0$ of $X$ the projection $X \times \mathbb{P} \rightarrow X$ is a stratified fiber bundle as in its assertion. We may assume also that the one point set $\{z_0\} \subset X$ is a stratum of $S_0$. Then there exists a finite subset $\{c_1, c_2, \ldots, c_k\} \subset \mathbb{P}$ such that the strata in $S$ projecting to the one $\{z_0\} \subset X$ in $S_0$ are $\{(z_0, c_i)\} \subset X \times \mathbb{P}$ ($1 \leq i \leq k$) and $\{z_0\} \times (\mathbb{P} \setminus \{c_1, c_2, \ldots, c_k\}) \subset X \times \mathbb{P}$. Let us set $\Sigma := \mathbb{C} \cap \{c_1, c_2, \ldots, c_k\} \subset \mathbb{C}$. Then by (2.44), (2.45) and [27, Proposition 8.6.3] we obtain the desired vanishing
\[
\phi_f^{\text{mero}, c}(\mathcal{F})_{z_0} \simeq 0.
\] (2.46)
for any $c \in \mathbb{C} \setminus \Sigma$.

The following result is an analogue for $\psi_f^{\text{mero}, c}(\cdot)$ and $\phi_f^{\text{mero}, c}(\cdot)$ of the classical one for $\psi_f(\cdot)$ and $\phi_f(\cdot)$ (see e.g. [12, Proposition 4.2.11], [27, Exercise VIII.15] and [57, Proposition 2.4]).

**Proposition 2.8.** Let $\nu : Y \rightarrow X$ be a proper surjective morphism of complex manifolds and $f \circ \nu$ the meromorphic function on $Y$ defined by
\[
f \circ \nu = \frac{P \circ \nu}{Q \circ \nu}.
\] (2.47)
Then we have:

(i) For $\mathcal{G} \in D^b(Y)$ there exists isomorphisms
\[
\psi_f^{\text{mero}, c}(R \nu_* \mathcal{G}) \simeq R \nu_* \psi_{f \circ \nu}^{\text{mero}, c}(\mathcal{G}), \quad \phi_f^{\text{mero}, c}(R \nu_* \mathcal{G}) \simeq R \nu_* \phi_{f \circ \nu}^{\text{mero}, c}(\mathcal{G}).
\] (2.48)

(ii) If moreover $\nu$ induces an isomorphism
\[
Y \setminus \nu^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\sim} X \setminus (P^{-1}(0) \cup Q^{-1}(0)),
\] (2.49)
then for $\mathcal{F} \in D^b(X)$ there exists an isomorphism
\[
\psi_f^{\text{mero}, c}(\mathcal{F}) \simeq R \nu_* \psi_{f \circ \nu}^{\text{mero}, c}(\nu^{-1}\mathcal{F}).
\] (2.50)

(iii) If moreover $\nu$ induces an isomorphism
\[
Y \setminus \nu^{-1}Q^{-1}(0) \xrightarrow{\sim} X \setminus Q^{-1}(0),
\] (2.51)
then for $\mathcal{F} \in D^b(X)$ there exists an isomorphism
\[
\phi_f^{\text{mero}, c}(\mathcal{F}) \simeq R \nu_* \phi_{f \circ \nu}^{\text{mero}, c}(\nu^{-1}\mathcal{F}).
\] (2.52)

**Lemma 2.9.** Assume that the hypersurfaces $P^{-1}(0), Q^{-1}(0) \subset X$ are smooth and intersect transversally in a neighborhood $U \subset X$ of a point $z_0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$. Then we have
\[
\phi_f^{\text{mero}, c}(\mathbb{C}X)_{z_0} \simeq \psi_f^{\text{mero}, c}(\mathbb{C}X)_{z_0} \simeq 0.
\] (2.53)
Proof. Let $\nu : Y \to U$ be the blow-up of $U$ along the complex submanifold $I(f) \cap U \subset U$ of codimension two. Then $\nu$ induces an isomorphism

$$Y \setminus \nu^{-1}(Q|_U)^{-1}(0) \sim \to U \setminus (Q|_U)^{-1}(0)$$

and we can apply Proposition 2.8 (iii) to the meromorphic function $f|_U : U \to \mathbb{C}$ and $\nu : Y \to U$ to obtain an isomorphism

$$\phi^\text{mero,c}_f(C_X)_{z_0} = \phi^\text{mero,c}_{f|U}(C_U)_{z_0} \simeq R\Gamma(\nu^{-1}(z_0); \phi^\text{mero,c}_{(f|U)\circ\nu}(C_Y)).$$

By our construction of the functor $\phi^\text{mero,c}_{(f|U)\circ\nu}(\cdot)$, the support of $\phi^\text{mero,c}_{(f|U)\circ\nu}(C_Y)$ is contained in the proper transform of $P^{-1}(0) \cap U$ in $Y$. Moreover by Lemma 2.5 (iii), we can easily show that the stalk of $\phi^\text{mero,c}_{(f|U)\circ\nu}(C_Y)$ at each point of $\nu^{-1}(z_0)$ is isomorphic to zero (see also e.g. [26] Lemma 2.9).

**Corollary 2.10.** For a point $z_0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$ assume that there exists its neighborhood $U \subset X$ such that the hypersurfaces $P^{-1}(0), Q^{-1}(0) \subset X$ are smooth and intersect transversally in $U \setminus \{z_0\}$. Then we have the concentration

$$H^j\phi^\text{mero,c}_f(C_X)_{z_0} \simeq H^j\phi^\text{mero,c}_f(C_X)_{z_0} \simeq 0 \quad (j \neq \dim X - 1).$$

**Proof.** By our assumption and Lemma 2.9 the support of

$$\phi^\text{mero,c}_f(C_X)|_U \simeq \phi^\text{mero,c}_{f|U}(C_U)$$

is contained in the one point set $\{z_0\} \subset U$. Then the assertion immediately follows from the perversity of $\phi^\text{mero,c}_f(C_X|_{\dim X})[-1]$.

By the proof of Corollary 2.10, we obtain also the following result.

**Proposition 2.11.** Let $\mathcal{K} \in D^b_c(X)$ be a perverse sheaf on $X$ and for a point $z_0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$ assume that there exists its neighborhood $U \subset X$ such that the support of

$$\phi^\text{mero,c}_f(\mathcal{K})|_U \simeq \phi^\text{mero,c}_{f|U}(\mathcal{K}|_U)$$

is contained in the one point set $\{z_0\} \subset U$. Then we have the concentration

$$H^j\phi^\text{mero,c}_f(\mathcal{K})_{z_0} \simeq H^j\phi^\text{mero,c}_f(\mathcal{K})_{z_0} \simeq 0 \quad (j \neq -1).$$

In the situation of Proposition 2.11 we can calculate the dimension of the only non-trivial cohomology group

$$H^{-1}\phi^\text{mero,c}_f(\mathcal{K})_{z_0} \simeq H^{-1}\phi^\text{mero,c}_f(\mathcal{K})_{z_0}$$

as follows. For a constructible sheaf $\mathcal{F} \in D^b_c(X)$ we set

$$\chi(\mathcal{F}, z_0) := \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j\phi^\text{mero,c}_f(\mathcal{F})_{z_0} \in \mathbb{Z}$$

so that for any distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \xrightarrow{+1} \mathcal{F}''$ in $D^b_c(X)$ we have $\chi(\mathcal{F}, z_0) = \chi(\mathcal{F}', z_0) + \chi(\mathcal{F}'', z_0)$. Let us set $D := P^{-1}(0) \cup Q^{-1}(0) \subset X$. Then by Lemma 2.2

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(ii) we may assume that $\mathcal{K}_{X \setminus D} \simeq \mathcal{K}$. By decomposing the the support of the perverse sheaf $\mathcal{K}_{X \setminus D} \simeq \mathcal{K} \in D^b_c(X)$ with respect to a stratification of $X$ adapted to $\mathcal{K}$ and the divisor $D \subset X$ in $X$, it suffices to calculate $\chi(F, z_0) \in \mathbb{Z}$ for some $F \in D^b_c(X)$ such that for a local system $L$ on a stratum $S \subset X$ contained in $X \setminus D$ and the inclusion map $j : S \hookrightarrow X$ we have $F = j_! L$. On the other hand, as in the proof of [15, Theorem 3.6], we can construct a proper morphism $\nu_1 : \tilde{X}_1 \to X$ of a complex manifold $\tilde{X}_1$ which induces an isomorphism over the open subset $X \setminus D \subset X$ such that the divisor $D' := \nu_1^{-1}(D) = (\nu \circ \nu_1)^{-1}(0) \cup (Q \circ \nu_1)^{-1}(0)$ in $\tilde{X}_1$ is normal crossing and the rational function $f \circ \nu_1 := (\nu \circ \nu_1)/(Q \circ \nu_1)$ on $\tilde{X}_1$ has no point of indeterminacy on the whole $\tilde{X}_1$. Namely the pole and zero divisors of the rational function $f \circ \nu_1$ are disjoint in $D'$. Let $g : \tilde{X}_1 \to \mathbb{P}$ be the holomorphic map defined by $f \circ \nu_1$. In this situation, there exists a (not necessarily closed) embedding $i_S : S \hookrightarrow \tilde{X}_1$ such that $\nu_1 \circ i_S = j$. Then we have $\nu_1^{-1} F = \nu_1^{-1} j_! L \simeq i_S^! L$ and hence by Proposition 2.8 (ii) and Lemma 2.2 (iii) we obtain an isomorphism

$$
\psi^\text{mero,c}_j(F)_{z_0} \simeq R\Gamma(\nu_1^{-1}(z_0) \cap g^{-1}(0); \psi^j(i_S L)). \quad (2.62)
$$

We set $S_1 := i_S(S) \subset \tilde{X}_1$. Denote by $\overline{S}_1$ its closure in $\tilde{X}_1$ and let $i_{\overline{S}_1} : \overline{S}_1 \subset \tilde{X}_1$ be the inclusion map. Then there exists a proper morphism $\nu_2 : T \to \overline{S}_1$ of a complex manifold $T$ which induces an isomorphism over the open subset $S_1$ of $\overline{S}_1$ such that $E := \nu_2^{-1}(\overline{S}_1 \setminus S_1) \subset T$ is a normal crossing divisor in $T$. Set $T^0 := T \setminus E \simeq S_1 \simeq S$ and let $\iota : T^0 \hookrightarrow T$ be the inclusion map. Then for the proper morphism $\nu := \nu_1 \circ \iota \circ \nu_2 : T \to X$ and $\tilde{g} := g \circ i_{\overline{S}_1} \circ \nu_2 = (g|_{\overline{S}_1}) \circ \nu_2 : T \to \mathbb{P}$ we obtain an isomorphism

$$
\psi^\text{mero,c}_j(F)_{z_0} \simeq R\Gamma(\nu^{-1}(z_0) \cap \tilde{g}^{-1}(0); \psi^j(\iota_* L)). \quad (2.63)
$$

Since the divisor $E \subset T$ is normal crossing, one can calculate

$$
\sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(\nu^{-1}(z_0) \cap \tilde{g}^{-1}(0); \psi^j(\iota_* L)) \in \mathbb{Z} \quad (2.64)
$$

by the results in [15, Section 5] (see also e.g. [69, Lemma 2.18]).

**Example 2.12.** As in Example 2.3 consider the case where $X$ is the complex plane $\mathbb{C}^2$ endowed with the standard coordinate $z = (z_1, z_2) = (x, y)$ and for two non-zero complex numbers $a, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ set $P(x, y) = ax^2 + bxy + y^3, Q(x, y) = x$ and

$$
f(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{ax^2 + bxy + y^3}{x} \quad ((x, y) \in X \setminus Q^{-1}(0)). \quad (2.65)
$$

Then the complex curve $P^{-1}(0) \subset X$ (resp. $Q^{-1}(0) \subset X$) has an isolated singular point (resp. is smooth) at the origin $0 = (0, 0) \in X = \mathbb{C}^2$. Namely the rational function $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \to \mathbb{C}$ satisfies the conditions in Example 2.3. Since moreover in this case $P^{-1}(0) \subset X$ is Newton non-degenerate at the origin (see [69, Definition 3.5] for the definition), by [69, Theorem 8.9 (i)] the number $N_1$ of the Jordan blocks for the eigenvalue 1 in its first Milnor monodromy is equal to 1. As we explained in Example 2.3 we can show it also by Varchenko’s formula for monodromy zeta functions in [69]. Hence in this case, by the arguments in Example 2.3 the boundary $\partial \overline{F}_0$ of the closure $\overline{F}_0$ of the Milnor
f-fiber \( F_0 \) of \( f \) at the origin \( 0 = (0, 0) \in I(f) \) is homeomorphic to the disjoint union \( S^1 \sqcup S^1 \) of two copies of \( S^1 \) and there exist isomorphisms

\[
H^j_{\nu f}^{\text{mero.c}}(\mathbb{C}X)_0 \simeq \begin{cases} \mathbb{C}^{2g(M)+1} & (j = 1) \\ 0 & (\text{otherwise}), \end{cases}
\]  

(2.66)

where \( g(M) \) is the genus of the compact oriented surface \( M \) obtained by attaching two (2-dimensional) disks to \( F_0 \) along its boundary \( \partial F_0 \). From now, we shall show that \( g(M) \) is equal to 0. For this purpose, we set \( D_1 := P_1^{-1}(0) \), \( D_2 := Q_2^{-1}(0) \) and \( D := D_1 \cup D_2 \) and construct a proper morphism \( \nu : Y \to X \) of a 2-dimensional complex manifold \( Y \) which induces an isomorphism over the open subset \( X \setminus D \subset X \) such that the divisor \( \nu^{-1}(D) \) in \( Y \) is normal crossing as follows. First, let \( \nu_1 : X_1 \to X \) be the blow-up of \( X = \mathbb{C}^2 \) along the origin \( \{0\} \subset X = \mathbb{C}^2 \) and \( E_1 := \nu_1^{-1}(0)(\simeq \mathbb{P}^1) \subset X_1 \) its exceptional divisor. We denote the proper transforms of \( D_1 \) and \( D_2 \) in \( X_1 \) by \( \widetilde{D}_1 \subset X_1 \) and \( \widetilde{D}_2 \subset X_1 \) respectively. Then we can easily see that the set \( E_1 \cap \widetilde{D}_1 \) consists of two points and one of them is the unique point in the one point set \( \widetilde{D}_1 \cap \widetilde{D}_2 \) and the other one is in \( E_1 \setminus \widetilde{D}_2 \). Let \( \nu_2 : X_2 \to X_1 \) be the blow-up of \( X_1 \) along the point \( \widetilde{D}_1 \cap \widetilde{D}_2 \subset X_1 \) and \( E_2(\simeq \mathbb{P}^1) \subset X_2 \) its exceptional divisor. By abuse of notations, we denote the proper transform of \( \widetilde{D}_2 \subset X_1 \) in \( X_2 \) also by \( \widetilde{D}_2 \subset X_2 \). Then the set \( E_2 \cap \widetilde{D}_2 \) consists of one point and the meromorphic function \( f \circ \nu_1 \circ \nu_2 \) on \( X_2 \) has a zero (resp. pole) of order 1 along the divisor \( E_2 \subset X_2 \) (resp. \( \widetilde{D}_2 \subset X_2 \)). This implies that \( f \circ \nu_1 \circ \nu_2 \) still has indeterminacy at the point \( E_2 \cap \widetilde{D}_2 \). In order to eliminate it, let us consider the blow-up \( \nu_3 : X_3 \to X_2 \) of \( X_2 \) along the point \( E_2 \cap \widetilde{D}_2 \subset X_2 \). Set \( Y := X_3 \) and \( \nu := \nu_1 \circ \nu_2 \circ \nu_3 : Y \to X \) and let \( E_3(\simeq \mathbb{P}^1) \subset Y \) be the exceptional divisor of the last blow-up \( \nu_3 : Y = X_3 \to X_2 \). Again by abuse of notations, we denote the proper transforms of \( D_1 \), \( D_2 \), \( E_1 \) and \( E_2 \) in \( Y = X_3 \) by \( \widetilde{D}_1 \), \( \widetilde{D}_2 \), \( \widetilde{E}_1 \) and \( \widetilde{E}_2 \) respectively. Then the meromorphic function \( f \circ \nu \) on \( Y \) has a zero (resp. pole) of order 1 along the divisors \( \widetilde{D}_1, \widetilde{E}_1, \widetilde{E}_2 \subset Y \) (resp. along the divisor \( \widetilde{D}_2 \subset Y \)). Moreover \( f \circ \nu \) has neither zero nor pole along the divisor \( E_3 \subset Y \) and we have \( (\widetilde{D}_1 \cup \widetilde{E}_1 \cup \widetilde{E}_2) \cap \widetilde{D}_2 = \emptyset \). This implies that the meromorphic function \( f \circ \nu \) has no point of indeterminacy on the whole \( Y \). Moreover the divisor

\[
D' := \nu^{-1}(D) = (\widetilde{D}_1 \cup \widetilde{E}_1 \cup \widetilde{E}_2) \cup E_3 \cup \widetilde{D}_2 \subset Y
\]  

(2.67)

in \( Y \) is normal crossing. Note that among the smooth divisors \( \widetilde{D}_1, \widetilde{E}_1 \simeq \mathbb{P}^1, \widetilde{E}_2 \simeq \mathbb{P}^1, E_3 \simeq \mathbb{P}^1 \) in \( Y \) we have only the following four intersection points:

\[
A := \widetilde{D}_1 \cap \widetilde{E}_1, \quad B := \widetilde{D}_1 \cap \widetilde{E}_2, \quad A' := \widetilde{E}_1 \cap \widetilde{E}_2, \quad B' := \widetilde{E}_2 \cap E_3
\]  

(2.68)

and we set

\[
\widetilde{E}_1' := \widetilde{E}_1 \setminus \{A, A'\}, \quad \widetilde{E}_2' := \widetilde{E}_2 \setminus \{B, A', B'\}.
\]  

(2.69)

Let \( g : Y \to \mathbb{P}^1 \) be the holomorphic map defined by the meromorphic function \( f \circ \nu \). Then we have

\[
g^{-1}(0) = \widetilde{D}_1 \cup \widetilde{E}_1 \cup \widetilde{E}_2, \quad \nu^{-1}(0) = \widetilde{E}_1 \cup \widetilde{E}_2 \cup E_3
\]  

(2.70)
and hence by Lemma 2.2 (i), (ii), (iii) and Proposition 2.8 (i) we obtain isomorphisms
\[
\psi_f^{\text{mero, c}}(\mathbb{C}X)_0 \simeq \psi_f^{\text{mero, c}}(\mathbb{C}X \setminus D)_0 \simeq \psi_f^{\text{mero, c}}(\mathbb{R}^n \mathbb{C}Y \setminus D')_0 \\
\simeq \Gamma(\nu^{-1}(0); \psi_f^{\text{mero, c}}(\mathbb{C}Y \setminus D')) \\
\simeq \Gamma(\nu^{-1}(0); \psi_g(\mathbb{C}Y \setminus D')) \\
\simeq \Gamma(\nu^{-1}(0) \cap g^{-1}(0); \psi_g(\mathbb{C}Y \setminus D')) \\
\simeq \Gamma(\tilde{E}_1 \cup \tilde{E}_2; \psi_g(\mathbb{C}Y \setminus D')).
\] (2.71)

Since we know that the complex $\psi_f^{\text{mero, c}}(\mathbb{C}X)_0$ is concentrated in degree 1, we obtain also
\[
\dim H^1 \psi_f^{\text{mero, c}}(\mathbb{C}X)_0 = - \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(\tilde{E}_1 \cup \tilde{E}_2; \psi_g(\mathbb{C}Y \setminus D')).
\] (2.76)

Now let $\chi(\psi_g(\mathbb{C}Y \setminus D'))$ be the $\mathbb{Z}$-valued constructible function on $g^{-1}(0)$ obtained by taking the local Euler-Poincaré index of $\psi_g(\mathbb{C}Y \setminus D')$ at each point of $g^{-1}(0)$. Then we have an equality
\[
\sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(\tilde{E}_1 \cup \tilde{E}_2; \psi_g(\mathbb{C}Y \setminus D')) = \int_{\tilde{E}_1 \cup \tilde{E}_2} \chi(\psi_g(\mathbb{C}Y \setminus D')),
\] (2.77)

where $\int_{\tilde{E}_1 \cup \tilde{E}_2} (\cdot)$ stands for the topological (Euler) integral over the analytic subset $\tilde{E}_1 \cup \tilde{E}_2 \subset g^{-1}(0)$. Moreover by the results in [43, Section 5] we can easily show that
\[
\chi(\psi_g(\mathbb{C}Y \setminus D'))|_{\tilde{E}_1 \cup \tilde{E}_2} = 1_{\tilde{E}_1} + 1_{\tilde{E}_2}.
\] (2.78)

Together with (2.76) we thus obtain the desired equality
\[
\dim H^1 \psi_f^{\text{mero, c}}(\mathbb{C}X)_0 = - \chi(\tilde{E}_1) - \chi(\tilde{E}_2) = 1.
\] (2.79)

3 Singularities at Infinity of Meromorphic Functions

In this section, we study the singularities at infinity of rational and meromorphic functions on complex affine subvarieties of $X = \mathbb{C}^N$. Although we treat only rational functions here, as is clear from the proofs our results hold true also for (possibly multi-valued) meromorphic functions with moderate growth at infinity. We leave their precise formulations to the readers. First, let us consider a complex submanifold $S \subset X = \mathbb{C}^N$ of $X = \mathbb{C}^N$. Denote by $X_R$ (resp. $S_R$) the underlying real analytic manifold of $X$ (resp. $S$). Then for any point $z \in S_R$ the tangent space $T_z S_R$ is naturally identified with a linear subspace of $T_z X_R \simeq \mathbb{C}^N$. In fact, we can easily see that it has also a structure of $\mathbb{C}$-linear subspace of dimension $\dim S$ i.e. $T_z S_R \simeq \mathbb{C}^{\dim S}$. For $u = (u_1, \ldots, u_N), v = (v_1, \ldots, v_N) \in \mathbb{C}^N$ we set
\[
\langle u, v \rangle := \sum_{i=1}^N u_i v_i \in \mathbb{C}.
\] (3.1)

Let
\[
(\cdot, \cdot) : \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C} \quad ((u, v) \mapsto \langle u, v \rangle)
\] (3.2)
be the standard Hermitian inner product of $\mathbb{C}^N$ and for $z \in \mathbb{C}^N$ define its norm $||z|| \geq 0$ by $||z|| = \sqrt{\langle z, z \rangle}$. For a point $z \in \mathbb{C}^N$ set $V := T_zX_\mathbb{R} \cong \mathbb{C}^N$ and denote by $V^*$ its dual space $\text{Hom}_\mathbb{R}(V, \mathbb{R}) \cong T_z^*X_\mathbb{R}$. Then by the perfect pairing $\text{Re}(\cdot, \cdot) : \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{R}$ we obtain an isomorphism

$$\phi : V = T_zX_\mathbb{R} \xrightarrow{\sim} V^* = T_z^*X_\mathbb{R} \quad (v \mapsto \{u \mapsto \text{Re}(u, v)\}).$$

(3.3)

Define a real-valued function $\delta : X_\mathbb{R} \rightarrow \mathbb{R}$ by

$$\delta(z) := \frac{1}{2}||z||^2 = \frac{1}{2}(z, z).$$

(3.4)

Then for any point $z \in X_\mathbb{R}$, the cotangent vector $d\delta(z) \in T^*_zX_\mathbb{R}$ corresponds to the point $z \in T_zX_\mathbb{R} \cong \mathbb{C}^N$ itself via the isomorphism $\phi : V = T_zX_\mathbb{R} \xrightarrow{\sim} V^* = T_z^*X_\mathbb{R}$. Fix a point $z \in S_\mathbb{R} \subset X_\mathbb{R}$ and set $W := T_zS_\mathbb{R} \cong \mathbb{C}^{\dim S} \subset V := T_zX_\mathbb{R} \cong \mathbb{C}^N$. Denote by $V^*$ (resp. $W^*$) the dual space $\text{Hom}_\mathbb{R}(V, \mathbb{R}) \cong T^*_zX_\mathbb{R}$ (resp. $\text{Hom}_\mathbb{R}(W, \mathbb{R}) \cong T^*_zS_\mathbb{R}$). Then by the perfect pairing $\text{Re}(\cdot, \cdot) : W \times W \rightarrow \mathbb{R}$ we obtain also an isomorphism $\psi : W = T_zS_\mathbb{R} \xrightarrow{\sim} W^* = T^*_zS_\mathbb{R}$.

**Lemma 3.1.** Let $\Psi_z : V^* = T^*_zX_\mathbb{R} \rightarrow W^* = T^*_zS_\mathbb{R}$ be the ($\mathbb{R}$-linear) morphism induced by the inclusion map $S_\mathbb{R} \hookrightarrow X_\mathbb{R}$. Then there exists a $\mathbb{C}$-linear morphism $\Phi_z : V = T_zX_\mathbb{R} \rightarrow W = T_zS_\mathbb{R}$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\phi} & V^* \\
\Psi_z & \downarrow & \Psi_z \\
W & \xrightarrow{\psi} & W^*
\end{array}
$$

commutes.

**Proof.** Assume first that $S$ is a complex hypersurface in $X = \mathbb{C}^N$. For a holomorphic function $h$ defined on a neighborhood of the point $z \in S$ in $X = \mathbb{C}^N$ such that $S = h^{-1}(0)$ we set

$$\text{grad}h(z) := \left(\frac{\partial h}{\partial z_1}, \frac{\partial h}{\partial z_2}, \ldots, \frac{\partial h}{\partial z_N}\right) \in \mathbb{C}^N.$$

(3.6)

Then for any smooth curve $p(t) : (-\varepsilon, \varepsilon) \rightarrow S_\mathbb{R} \ (\varepsilon > 0)$ such that $p(0) = z$ we have

$$\left.\frac{d}{dt} h(p(t))\right|_{t=0} = \left\langle \frac{dp}{dt}(0), \text{grad}h(z) \right\rangle = 0.$$ 

(3.7)

This implies that the $\mathbb{C}$-linear subspace $W = T_zS_\mathbb{R} \subset V = T_zX_\mathbb{R} = \mathbb{C}^N$ is explicitly given by

$$W = \{v \in V \mid \langle v, \text{grad}h(z) \rangle = \langle v, \overline{\text{grad}h(z)} \rangle = 0\},$$

(3.8)

where we set

$$\overline{\text{grad}h(z)} := \left(\frac{\overline{\partial h}}{\partial z_1}, \frac{\overline{\partial h}}{\partial z_2}, \ldots, \frac{\overline{\partial h}}{\partial z_N}\right) \in \mathbb{C}^N.$$

(3.9)

Hence we can define a $\mathbb{C}$-linear morphism $\Phi_z : V \rightarrow W$ by the formula

$$\Phi_z(v) = v - \frac{\langle v, \text{grad}h(z) \rangle}{\langle \text{grad}h(z), \text{grad}h(z) \rangle} \cdot \text{grad}h(z) \in W.$$ 

(3.10)

Then it is straightforward to check that the diagram (3.5) commutes. If $\dim S < \dim X - 1 = N - 1$, we repeat this argument. \qed
In what follows, we assume that $S \subseteq X = \mathbb{C}^N$ is a smooth Zariski locally closed (complex) affine subvariety i.e. a smooth quasi-affine subvariety of $X = \mathbb{C}^N$.

**Lemma 3.2.** Let $p(t) : (0, \varepsilon) \rightarrow S \subseteq X = \mathbb{C}^N$ be an analytic curve on $S$ such that
\[
\lim_{t \to 0^+} ||p(t)|| = +\infty
\]
and its expansion at infinity is of the form
\[
p(t) = at^\alpha + \text{(higher order terms)} \quad (a \in \mathbb{C}^N \setminus \{0\}, \alpha < 0).
\]
Define an analytic curve $q(t) : (0, \varepsilon) \rightarrow X = \mathbb{C}^N$ by
\[
q(t) = \Phi_p(t)(p(t)) \quad (0 < t < \varepsilon).
\]
Recall that the element $d\delta(p(t)) \in T^*_p(X) \cong T^*_p(X_R)$ corresponds to $p(t) \in T^*_p(X) = \mathbb{C}^N$ via the isomorphism $T^*_p(X) \cong T^*_p(X_R)$ and the point $\Phi_p(t)(p(t)) \in T^*_p(S_R)$ is considered as an element of $T^*_p(S) = \mathbb{C}^N$. Then the expansion at infinity of $q(t)$ has the same top term as that of $p(t)$:
\[
q(t) = at^\alpha + \text{(higher order terms)} \quad (a \in \mathbb{C}^N \setminus \{0\}, \alpha < 0).
\]

**Proof.** As in the proof of Lemma 3.1 assume first that $S$ is a complex hypersurface in $X = \mathbb{C}^N$. Consider a holomorphic function $h$ defined on a neighborhood of a point $z \in S$ in $X = \mathbb{C}^N$ such that $S = h^{-1}(0)$. First of all, note that we have
\[
p(t) = \frac{t}{\alpha} \cdot \frac{dp}{dt}(t) + \text{(higher order terms)} \quad (3.15)
\]
and set
\[
r(t) := \text{(higher order terms)} = p(t) - \frac{t}{\alpha} \cdot \frac{dp}{dt}(t). \quad (3.16)
\]
Since $\frac{dp}{dt}(t)$ is a tangent vector of the manifold $S_R$ at the point $p(t) \in S_R$, by the proof of Lemma 3.1 we have
\[
\left( \frac{dp}{dt}(t), \text{grad}(p(t)) \right) = \left( \frac{dp}{dt}(t), \text{grad}(p(t)) \right) = 0. \quad (3.17)
\]
Then by (3.10) we obtain
\[
q(t) = \Phi_p(t)(p(t)) = p(t) - \frac{(r(t), \text{grad}(p(t)))}{(\text{grad}(p(t)), \text{grad}(p(t)))} \cdot \text{grad}(p(t)) \quad (3.18)
\]
from which the assertion immediately follows. If $\dim S < \dim X - 1 = N - 1$, we repeat this argument. \(\square\)

From now on, let us consider also a rational function $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ ($P, Q \in \Gamma(X; \mathcal{O}_X) \simeq \mathbb{C}[z_1, z_2, \ldots, z_N], Q \neq 0$) on the smooth algebraic variety $X = \mathbb{C}^N$ and set $U := X \setminus Q^{-1}(0)$. Let $S \subseteq U$ be a smooth quasi-affine subvariety of $U$ and denote the restriction $f|_S : S \rightarrow \mathbb{C}$ of $f$ to it by $g$. Then for the cotangent vector $d\text{Reg}(z) \in T^*_z X_R$ at a point $z \in S_R$ we define its norm $||d\text{Reg}(z)|| \geq 0$ to be that of the element $\text{grad} \text{Reg}(z) \in T_z S_R \subset T_z X_R \simeq \mathbb{C}^N$ which corresponds to it by the isomorphism $\psi : T_z S_R \cong T^*_z X_R$. For polynomial functions $f : \mathbb{C}^N \rightarrow \mathbb{C}$ on $\mathbb{C}^N$ the following condition was first introduced by Broughton [5].
Definition 3.3. Let $S \subset U$ and $g : S \to \mathbb{C}$ be as above. Then we say that $g$ is tame at infinity if there exist $R \gg 0$ and $0 < \varepsilon \ll 1$ such that

$$\{ z \in S \mid ||\text{d}\text{Reg}(z)|| < \varepsilon, ||z|| > R \} = \emptyset. \quad (3.19)$$

For polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$ on $\mathbb{C}^n$ the following result was essentially obtained in Némethi-Zaharia [52] and [53]. Here we modify their arguments with the help of Lemma 3.2. See also the proof of Nguyen-Pham-Pham [55, Theorem 3.1] for a similar result on complete intersection subvarieties $S \subset X = \mathbb{C}^N$ and in a different formulation in terms of Rabier’s norm defined in [59].

Proposition 3.4. Let $S \subset U$ and $g : S \to \mathbb{C}$ be as above and assume that $g$ is tame at infinity. We define a subset $M(g)$ of $S$ by

$$M(g) = \{ z \in S \mid d(\delta|_S)(z) \in \mathbb{R}d\text{Reg}(z) + \mathbb{R}d\text{Im}(g) \}. \quad (3.20)$$

Then for any $\tau_0 \in \mathbb{C}$ there exist $R \gg 0$ and $0 < \varepsilon \ll 1$ such that

$$M(g) \cap \{ z \in S \mid |g(z) - \tau_0| < \varepsilon, ||z|| > R \} = \emptyset. \quad (3.21)$$

Proof. Fix a point $z \in S$ and recall that we set $g = f|_S$. We denote by $\text{gradRe}f(z) \in T_zX_{\mathbb{R}} \simeq \mathbb{C}^n$ the element which corresponds to $d\text{Re}f(z) \in T^*_zX_{\mathbb{R}}$ by the isomorphism $\phi : T_zX_{\mathbb{R}} \to T^*_zX_{\mathbb{R}}$. Then by the Cauchy-Riemann equations we have

$$\text{gradRe}f(z) = \left( \frac{\partial\text{Re}f}{\partial x_1} + i\frac{\partial\text{Re}f}{\partial y_1}, \ldots, \frac{\partial\text{Re}f}{\partial x_N} + i\frac{\partial\text{Re}f}{\partial y_N} \right) = \text{grad}f(z) = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_N} \right), \quad (3.22)$$

where we set $z_i = x_i + iy_i$ (1 $\leq i \leq N$). Moreover we have

$$\text{gradIm}f(z) = \left( \frac{\partial\text{Im}f}{\partial x_1} + i\frac{\partial\text{Im}f}{\partial y_1}, \ldots, \frac{\partial\text{Im}f}{\partial x_N} + i\frac{\partial\text{Im}f}{\partial y_N} \right) = i \cdot \text{gradRe}f(z). \quad (3.24)$$

By Lemma 3.1 this implies that for a point $z \in S_{\mathbb{R}}$ such that $||z|| \gg 0$ the condition

$$d(\delta|_S)(z) \in \mathbb{R}d\text{Reg}(z) + \mathbb{R}d\text{Im}(g) \quad (3.25)$$

is equivalent to the one that there exists $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that

$$\text{grad}(\delta|_S)(z) = \Phi_2(\lambda \cdot \text{gradRe}f(z)). \quad (3.26)$$

Here we used the fact that if $||z|| \gg 0$ we have $d(\delta|_S)(z) \neq 0$ (see e.g. [46, Corollary 2.8]). Since we have $\text{grad}(\delta|_S)(z) = \Phi_2(\text{grad}(\delta)(z)) = \Phi_2(z)$ and $\Phi_2$ is $\mathbb{C}$-linear, it is also equivalent to the one that there exists $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that

$$\lambda \cdot \Phi_2(z) = \Phi_2(\text{gradRe}f(z)) = \text{gradReg}(z). \quad (3.27)$$

Now let us prove the assertion by showing a contradiction. Assume to the contrary that there exists a sequence $z_n \in M(g) \subset S \subset X = \mathbb{C}^N$ $(n = 1, 2, \ldots)$ such that

$$\lim_{n \to +\infty} |g(z_n) - \tau_0| = 0, \quad \lim_{n \to +\infty} ||z_n|| = +\infty. \quad (3.28)$$
Then by the curve selection lemma [53, Lemma 4] of Némethi and Zaharia, there exists an analytic curve $p(t) : (0, \varepsilon) \rightarrow M(g) \subset S \subset X = \mathbb{C}^N$ such that
\[
\lim_{t \to +0} |g(p(t)) - \tau_0| = 0, \quad \lim_{t \to +0} \|p(t)\| = +\infty.
\tag{3.29}
\]
Assume that its expansion at infinity is of the form
\[
p(t) = at^\alpha + \text{(higher order terms)} \quad (a \in \mathbb{C}^N \setminus \{0\}, \alpha < 0).
\tag{3.30}
\]
and define $q(t) : (0, \varepsilon) \rightarrow X = \mathbb{C}^N$ by
\[
q(t) = \Phi_{p(t)}(p(t)) \quad (0 < t < \varepsilon).
\tag{3.31}
\]
Then by Lemma 3.2 the expansion at infinity of $q(t)$ has the same top term as that of $p(t)$:
\[
q(t) = at^\alpha + \text{(higher order terms)} \quad (a \in \mathbb{C}^N \setminus \{0\}, \alpha < 0).
\tag{3.32}
\]
Moreover, by the condition
\[
\lim_{t \to +0} g(p(t)) = \tau_0,
\tag{3.33}
\]
the expansion at infinity of $g(p(t))$ is of the form
\[
g(p(t)) = bt^\beta + \text{(higher order terms)} \quad (b \in \mathbb{C}^*, \beta \geq 0).
\tag{3.34}
\]
Since $g$ is tame at infinity, the expansion at infinity of $\text{gradReg}(p(t)) : (0, \varepsilon) \rightarrow \mathbb{C}^N$ is of the form
\[
\text{gradReg}(p(t)) = ct^\gamma + \text{(higher order terms)} \quad (c \in \mathbb{C}^N \setminus \{0\}, \gamma \leq 0).
\tag{3.35}
\]
Then by the condition $p(t) \in M(g) \subset S (0 < t < \varepsilon)$ and the above argument, there exists an analytic curve $\lambda(t) : (0, \varepsilon) \rightarrow \mathbb{C}^*$ such that
\[
\lambda(t) \cdot q(t) = \lambda(t) \cdot \Phi_{p(t)}(p(t)) = \text{gradReg}(p(t)) \quad (0 < t < \varepsilon).
\tag{3.36}
\]
If the expansion at infinity of $\lambda(t)$ is of the form
\[
\lambda(t) = \lambda_0 t^{\gamma - \alpha} + \text{(higher order terms)} \quad (\lambda_0 \in \mathbb{C}^*),
\tag{3.37}
\]
we have $c = \lambda_0 a \in \mathbb{C}^N \setminus \{0\}$ and
\[
\text{gradReg}(p(t)) = \lambda_0 at^\gamma + \text{(higher order terms)} \quad (\gamma \leq 0).
\tag{3.38}
\]
On the other hand, for the Hermitian inner product $(\cdot, \cdot)$ we have
\[
\frac{d}{dt} g(p(t)) = \frac{d}{dt} f(p(t)) = \left(\frac{dp}{dt}(t), \text{gradRe} f(p(t))\right)
= \left(\frac{dp}{dt}(t), \text{gradRe} g(p(t))\right)
= \left(\frac{dp}{dt}(t), \Phi_{p(t)}(\text{gradRe} g(p(t)))\right)
= \left(\frac{dp}{dt}(t), \text{gradReg}(p(t))\right)
= \left(\frac{dp}{dt}(t), \lambda(t) \cdot q(t)\right)
= (\alpha a, \lambda_0 a) \cdot t^{\alpha - 1 + \gamma} + \text{(higher order terms)},
\]
where we used (3.10) in the fourth equality (see also the proof of Lemma 3.11). Since 
\((aa, \lambda_0a) = \alpha \lambda_0 ||a||^2 \neq 0, \alpha - 1 + \gamma < -1\) and the left hand side \(\frac{d}{dt}g(p(t))\) is of order \(-1\) in \(t\) by (3.31), we obtain the desired contradiction. This completes the proof. \(\square\)

For \(r > 0\) let \(B_r(0) = \{ z \in X = \mathbb{C}^N \mid ||z|| < r \} \subset X = \mathbb{C}^N\) be the open ball of radius \(r\) centered at the origin \(0 \in X = \mathbb{C}^N\). Then Proposition 3.3 means that if \(g : S \to \mathbb{C}\) is tame at infinity the boundary \(\partial B_r(0) \cap S\) of \(B_r(0) \cap S\) for \(r \to 0\) intersects the fibers \(g^{-1}(\tau) \subset S (|\tau - \tau_0| \ll 1)\) of \(g\) transversally. Now let \(K \in D_c^b(X^{an})\) be an algebraic constructible sheaf on \(X^{an}\). Namely we assume that \(K\) is adapted to an algebraic stratification of \(X^{an} = \mathbb{C}^N\). Then the micro-support \(SS(K) \subset T^*X^{an}\) of \(K\) is a homogeneous Lagrangian subvariety of \((T^*X)^{an} \simeq T^*X^{an}\) and there exists an algebraic Whitney stratification \(S\) of \(X = \mathbb{C}^N\) such that

\[
SS(K) \subset \bigcup_{S \in S} T^*_S X. \tag{3.39}
\]

Recall that by the Whitney condition the right hand side is a closed subset in \((T^*X)^{an} \simeq T^*X^{an}\). For \(r > 0\) we define an open subset \(U_r \subset U\) of \(U = X \setminus Q^{-1}(0)\) by

\[
U_r := U \cap B_r(0) = \{ z \in U \mid ||z|| < r \} \subset U \subset X = \mathbb{C}^N. \tag{3.40}
\]

Then in order to apply the direct image theorem \([26\text{, Theorem 4.4.1}]\) for non-proper maps of Kashiwara and Schapira to \(K|_U \in D_c^b(U^{an})\) and \(f : U = X \setminus Q^{-1}(0) \to \mathbb{C}\), we shall prove the following result.

**Proposition 3.5.** Let \(S\) be the Whitney stratification of \(X = \mathbb{C}^N\) as above and assume that for any stratum \(S \in S\) in it such that \(S \cap U \neq \emptyset\) and \(T^*_S X \subset SS(K)\) the restriction \(f|_{SS(K)} : S \cap U \to \mathbb{C}\) of \(f\) to \(S \cap U\) is tame at infinity. Then for any \(\tau_0 \in \mathbb{C}\) there exist \(R > 0\) and \(0 < \varepsilon \ll 1\) such that we have

\[
N^*(U_r) \cap \left\{ SS(K|_U \cup \rho(U \times c T^*\mathbb{C}) \right\} \subset T^*_0 U \tag{3.41}
\]

(for the definition of \(N^*(U_r) \subset T^*U\) see \([27\text{, Definition 5.3.6}]\)) over the open subset \(\{ z \in U \mid |f(z) - \tau_0| < \varepsilon \} \subset U\) of \(U\) for any \(r \geq R\), where \(\rho : U \times c T^*\mathbb{C} \to T^*U\) is the natural morphism associated to \(f : U \to \mathbb{C}\). Moreover the same is true even after replacing \(N^*(U_r)\) by its antipodal set \(N^*(U_r)^\mathbb{A}\).

**Proof.** Since the subset \(SS(K|_U \cup \rho(U \times c T^*\mathbb{C}) \subset T^*U\) of \(T^*U\) is stable by the antipodal map of \(T^*U\), it suffices to consider only \(N^*(U_r)\). We prove the assertion by showing a contradiction. Assume to the contrary that there exists a sequence \(z_n \in U \subset X = \mathbb{C}^N (n = 1, 2, \ldots)\) such that

\[
\lim_{n \to +\infty} |f(z_n) - \tau_0| = 0, \quad \lim_{n \to +\infty} ||z_n|| = +\infty \tag{3.42}
\]

and for the non-zero inner conormal vector \(-d\delta(z_n) \in \mathbb{C}^N\) of the real hypersurface \(\partial U_{||z_n||} \subset U\) at the point \(z_n \in U\) we have

\[
(z_n, w_n) := (z_n, -d\delta(z_n)) \in SS(K|_U \cup \rho(U \times c T^*\mathbb{C}) \subset T^*U \tag{3.43}
\]

for any \(n = 1, 2, \ldots\). By taking a subsequence, we may assume that there exists a stratum \(S \in S\) such that \(S \cap U \neq \emptyset\) and \(z_n \in S \cap U\) for any \(n = 1, 2, \ldots\). First, let us consider
the case where $T^*_S X \subset SS(K)$. Then for each $n = 1, 2, \ldots$ there exist a stratum $S' \in S$ such that $S \subset S'$ and $T^*_S X \subset SS(K)$ and sequences $(z_{nm}, w_{nm}) \in T^*_S \cap U X$, $\lambda_{nm} \in \mathbb{C}$ (m = 1, 2, \ldots) such that

$$ \lim_{m \to +\infty} z_{nm} = z_n, \quad \lim_{m \to +\infty} (w_{nm} + \lambda_{nm} \cdot df(z_{nm})) = w_n. \quad (3.44) $$

Since $f|_{S \cap U} : S \cap U \to \mathbb{C}$ is tame at infinity by our assumption, there exists $0 < \varepsilon_0 \ll 1$ such that for large enough $n \gg 0$ we have the condition $||d(f|_{S \cap U})(z_n)|| \geq \varepsilon_0 > 0$. On the other hand, by the Whitney condition of $S$ a subsequence of $w_{nm}/||w_{nm}|| \in \mathbb{C}^N$ (m = 1, 2, \ldots, $||w_{nm}|| \neq 0$) converges to a point in $(T^*_S X)_{z_n}$. As $df(z) \in \mathbb{C}^N$ is holomorphic with respect to $z$ and bounded on a neighborhood of the point $z_n \in S \cap U$ in $U$, these conditions in together imply that all the sequences $w_{nm} \in \mathbb{C}^N$, $\lambda_{nm} \cdot df(z_{nm}) \in \mathbb{C}^N$, $\lambda_{nm} \in \mathbb{C}$ (m = 1, 2, \ldots) are bounded. Indeed, if the sequence $w_{nm} \in \mathbb{C}^N$ is not bounded, by (3.44) a subsequence of $\lambda_{nm} \cdot df(z_{nm}) \in \mathbb{C}^N$ (m = 1, 2, \ldots) goes to at infinity of a direction in $(T^*_S X)_{z_n}$. This contradicts with the condition $||d(f|_{S \cap U})(z_n)|| \geq \varepsilon_0 > 0$. Hence, by taking their subsequences, we may assume that all of them converge. In particular, by the Whitney condition of $S$ we have

$$ \lim_{m \to +\infty} w_{nm} \in (T^*_S X)_{z_n}. \quad (3.45) $$

Hence for any $n = 1, 2, \ldots$ there exists $\lambda_n \in \mathbb{C}$ such that

$$ d(\delta|_{S \cap U})(z_n) = \lambda_n \cdot d(f|_{S \cap U})(z_n) \quad (3.46) $$

and we get a contradiction by Proposition 3.4. Next, let us consider the case where $T^*_S X$ is not contained in $SS(K)$. Also in this case, for each $n = 1, 2, \ldots$ there exist a stratum $S' \in S$ such that $S \subset S'$ and $T^*_S X \subset SS(K)$ and sequences $(z_{nm}, w_{nm}) \in T^*_S \cap U X$, $\lambda_{nm} \in \mathbb{C}$ (m = 1, 2, \ldots) such that

$$ \lim_{m \to +\infty} z_{nm} = z_n, \quad \lim_{m \to +\infty} (w_{nm} + \lambda_{nm} \cdot df(z_{nm})) = w_n. \quad (3.47) $$

By taking their subsequences and using the Whitney condition of $S$, we may assume that the tangent planes $T_{z_n} S' \subset \mathbb{C}^N$ of $S'$ at $z_{nm} \in S'$ converge to a linear subspace $\mathcal{T}_n \subset T_{z_n} X \simeq \mathbb{C}^N$ such that $T_{z_n} S \subset \mathcal{T}_n$ as $m$ tends to $+\infty$. Let

$$ \Phi_n : T^*_n \sim \mathbb{C}^N \to \mathcal{T}^*_n := \text{Hom}_\mathbb{C}(\mathcal{T}_n, \mathbb{C}) \quad (3.48) $$

be the surjective linear map associated to the inclusion map $\mathcal{T}_n \hookrightarrow T_{z_n} X \simeq \mathbb{C}^N$. Then by our assumption for the stratum $S'$, there exists $0 < \varepsilon_0 \ll 1$ such that for large enough $n \gg 0$ we have the condition $||\Phi_n(df(z_n))|| \geq \varepsilon_0 > 0$. Similarly to the previous case, we can thus show that all the sequences $w_{nm} \in \mathbb{C}^N$, $\lambda_{nm} \cdot df(z_{nm}) \in \mathbb{C}^N$, $\lambda_{nm} \in \mathbb{C}$ (m = 1, 2, \ldots) are bounded. By taking subsequences, we can assume also that all of them converge. Hence for any $n = 1, 2, \ldots$ there exists $\lambda_n \in \mathbb{C}$ such that

$$ \Phi_n(df(z_n)) = \lambda_n \cdot \Phi_n(df(z_n)). \quad (3.49) $$

By taking a subsequence of $z_n \in S \cap U$ (n = 1, 2, \ldots), we may assume that there exists a stratum $S' \in S$ such that $S \subset S'$, $T^*_S X \subset SS(K)$, and for any $n = 1, 2, \ldots$ the linear
subspace \( T_n \subset T_{z_n} X \simeq \mathbb{C}^N \) at \( z_n \in S \cap U \) is a limit of some tangent spaces of \( S' \). We fix such \( S' \) once and for all. Set \( l := \dim S' = \dim T_n \) \((n = 1, 2, \ldots)\) and let \( \text{Gr} \) be the complex Grassmann manifold consisting of \( l \)-dimensional linear subspaces of \( \mathbb{C}^N \). Let \( A \) be a subset of \((S \cap U) \times \text{Gr}\) consisting of pairs \((z, T) \in (S \cap U) \times \text{Gr}\) such that there exists a sequence \( z_m \in S' \cap U \) \((m = 1, 2, \ldots)\) in the stratum \( S' \) such that

\[
\lim_{m \to +\infty} z_m = z, \quad \lim_{m \to +\infty} T_{z_m} S' = T
\]

and for the surjective linear map

\[
\Phi_{(z, T)} : T_z^* X \simeq \mathbb{C}^N \to T^* := \text{Hom}_\mathbb{C}(\mathcal{T}, \mathbb{C})
\]

associated to the inclusion map \( T \hookrightarrow T_z X \simeq \mathbb{C}^N \) we have

\[
\Phi_{(z, T)}(d\delta(z)) = \lambda \cdot \Phi_{(z, T)}(df(z))
\]

for some \( \lambda \in \mathbb{C} \). Then we can easily show that \( A \subset (S \cap U) \times \text{Gr} \subset X \times \text{Gr} \) is a semi-analytic subset of \( X \times \text{Gr} \). In the above argument, we obtained a sequence \((z_n, T_n) \in A \subset (S \cap U) \times \text{Gr} \) in \( A \) such that

\[
\lim_{n \to +\infty} |f(z_n) - \tau_0| = 0, \quad \lim_{n \to +\infty} ||z_n|| = +\infty.
\]

Since the complex Grassmann manifold \( \text{Gr} \) is covered by finitely many open subsets isomorphic to \( \mathbb{C}^{(N-l)} \), we can apply the curve selection lemma [53, Lemma 4] of Némethi and Zaharia to find an analytic curve

\[
q(t) = (p(t), \mathcal{T}(t)) : (0, \varepsilon) \to A \subset (S \cap U) \times \text{Gr}
\]

in \( A \) satisfying the conditions

\[
\lim_{t \to +0} |f(p(t)) - \tau_0| = 0, \quad \lim_{t \to +0} ||p(t)|| = +\infty.
\]

Then we obtain a contradiction as in the proof of Proposition 3.4. More precisely, by taking a family of conormal vectors of the planes \( \mathcal{T}(t) \subset \mathbb{C}^N \) which depend analytically on \( t \in (0, \varepsilon) \), we obtain a result similar to Lemma 3.2 and can apply the proof of Proposition 3.4 to our situation. This completes the proof.

By the proof of Proposition 3.5, we obtain also the following simple consequence.

**Lemma 3.6.** In the situation of Proposition 3.5, for any \( \tau_0 \in \mathbb{C} \) there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that we have

\[
df(z) \notin \text{SS}(K|_U)
\]

for any \( z \in U \) such that \( ||z|| \geq R \) and \( |f(z) - \tau_0| < \varepsilon \).

First, let us consider the special case where \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \). Note that this condition is satisfied for example if \( f = \frac{P}{Q} \) is a polynomial i.e. \( Q = 1 \) or \( P = 1 \).

**Theorem 3.7.** In the situation of Proposition 3.5, assume also that \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \). Then we have the following results.
(i) For any $\tau_0 \in \mathbb{C}$ there exist $R \gg 0$ and $0 < \varepsilon \ll 1$ such that for the inclusion maps $i_r : B_r(0) \hookrightarrow X = \mathbb{C}^N$ $(r > R)$ we have isomorphisms

$$\text{R}f_i(i_r)^{-1}K \simeq \text{R}f_iK_{B_r(0)} \sim \text{R}f_iK$$ (3.57)

on the open subset $\{ \tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C}$ of $\mathbb{C}$.

(ii) For any $\tau_0 \in \mathbb{C}$, if we assume also that

$$\{ z \in f^{-1}(\tau_0) \mid df(z) \in \text{SS}(K) \} \subset f^{-1}(\tau_0) \cap \text{supp}(K)$$ (3.58)

is a finite subset of $f^{-1}(\tau_0) \cap \text{supp}(K)$ and let $p_1, p_2, \ldots, p_k$ be the points in it, then for the vanishing cycle $\phi_{\tau-\tau_0}(\text{R}f_iK) \in \text{D}^b(\{\tau_0\})$ of $\text{R}f_iK$ along the function $\tau - \tau_0 : \mathbb{C} \longrightarrow \mathbb{C}$ we have an isomorphism

$$\phi_{\tau-\tau_0}(\text{R}f_iK) \simeq \bigoplus_{i=1}^k \phi_{f-\tau_0}(K)_{p_i}.$$ (3.59)

If we assume moreover that $K[N] \in \text{D}^b_c(X^{an})$ is a perverse sheaf, then we have also a concentration

$$\text{H}^j\phi_{\tau-\tau_0}(\text{R}f_iK) \simeq 0 \quad (j \neq N - 1).$$ (3.60)

**Proof.** By combining [26, Theorem 4.4.1] with Proposition [3.5] we obtain the isomorphism in (i). Let us prove (ii). For $R \gg 0$ such that $p_1, p_2, \ldots, p_k \in B_R(0)$ and $0 < \varepsilon \ll 1$ in (i) let us fix $r > R$. Then the morphism $f$ being proper on the support of

$$(i_r)^{-1}K \simeq K_{B_r(0)},$$ (3.61)

by (i) we obtain an isomorphism

$$\phi_{\tau-\tau_0}(\text{R}f_iK) \simeq \phi_{\tau-\tau_0}(\text{R}f_iK_{B_r(0)}) \simeq \text{R}f(i_r)^{-1}(\tau_0); \phi_{f-\tau_0}(K_{B_r(0)})).$$ (3.62)

Moreover by our assumption, the complex hypersurface $f^{-1}(\tau_0) \subset X = \mathbb{C}^N$ is smooth on a neighborhood of

$$(f^{-1}(\tau_0) \setminus \{p_1, p_2, \ldots, p_k\}) \cap \text{supp}(K)$$ (3.63)

in $X = \mathbb{C}^N$. Then by [27, Proposition 8.6.3] we see that the support of $\phi_{f-\tau_0}(K_{B_r(0)}) \in \text{D}^b(f^{-1}(\tau_0))$ is contained in the set

$$\{p_1, p_2, \ldots, p_k\} \sqcup (f^{-1}(\tau_0) \cap \partial B_r(0)).$$ (3.64)

Now let us take $r' > R$ such that $r > r' > R$. Then by the proof of [26, Theorem 4.4.1] there exists an isomorphism

$$\text{R}f_iK_{B_{r'}(0)} \sim \text{R}f_iK_{B_r(0)}$$ (3.65)

on the open subset $\{ \tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C}$ of $\mathbb{C}$. In other words, we have a vanishing

$$\text{R}f_i(K_{B_{r}(0)\setminus B_{r'}(0)}) \simeq 0.$$ (3.66)
there. This implies that we have
\[ R\Gamma(f^{-1}(\tau_0); \phi_{f-\tau_0}(K_{B_r(0)\setminus B_r^\varepsilon(0)})) \simeq 0. \] (3.67)

Since the support of \( \phi_{f-\tau_0}(K_{B_r(0)\setminus B_r^\varepsilon(0)}) \) is contained in the set
\[ (f^{-1}(\tau_0) \cap \partial B_r(0)) \cup (f^{-1}(\tau_0) \cap \partial B_r^\varepsilon(0)) \] (3.68)
and hence the object \( R\Gamma(f^{-1}(\tau_0) \cap \partial B_r(0); \phi_{f-\tau_0}(K_{B_r(0)})) \) is a direct summand of the one \( R\Gamma(f^{-1}(\tau_0); \phi_{f-\tau_0}(K_{B_r(0)\setminus B_r^\varepsilon(0)})) \simeq 0 \), we obtain also a vanishing
\[ R\Gamma(f^{-1}(\tau_0) \cap \partial B_r(0); \phi_{f-\tau_0}(K_{B_r(0)})) \simeq 0. \] (3.69)

From this, the first assertion of (ii) immediately follows. To obtain the last one, it suffices to use the t-exactness of the functor \( \phi_{f-\tau_0}(\cdot)[-1] \).

Similarly, by Proposition 3.5 and [26, Theorem 4.4.1] we obtain the following result.

**Theorem 3.8.** In the situation of Proposition 3.5 assume also that \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \). Then we have the following results.

(i) For any \( \tau_0 \in \mathbb{C} \) there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that for the inclusion maps
\[ i_r : B_r(0) \hookrightarrow X = \mathbb{C}^N \quad (r > R) \]
we have isomorphisms
\[ Rf_*R(i_r)_*(i_r^{-1}K) \simeq Rf_*K_{B_r(0)} \simto Rf_*K \] (3.70)
on the open subset \( \{ \tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C} \) of \( \mathbb{C} \).

(ii) For any \( \tau_0 \in \mathbb{C} \), if we assume also that
\[ \{ z \in f^{-1}(\tau_0) \mid df(z) \in SS(K) \} \subset f^{-1}(\tau_0) \cap \text{supp}(K) \] (3.71)
is a finite subset of \( f^{-1}(\tau_0) \cap \text{supp}(K) \) and let \( p_1, p_2, \ldots, p_k \) be the points in it, then for the vanishing cycle \( \phi_{\tau-\tau_0}(Rf_*K) \subset D^b(\{ \tau_0 \}) \) of \( Rf_*K \) along the function \( \tau - \tau_0 : \mathbb{C} \to \mathbb{C} \) we have an isomorphism
\[ \phi_{\tau-\tau_0}(Rf_*K) \simeq \bigoplus_{i=1}^k \phi_{f-\tau_0}(K)_{p_i}. \] (3.72)

If we assume moreover that \( K[N] \in D^b_c(X^{an}) \) is a perverse sheaf, then we have also a concentration
\[ H^j\phi_{\tau-\tau_0}(Rf_*K) \simeq 0 \quad (j \neq N - 1). \] (3.73)

Next, let us consider the problem in the general case i.e. in the absence of the condition \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \). For this purpose, we regard the projection \( \tau : X \times \mathbb{C} \to \mathbb{C} \) as a holomorphic function on \( X \times \mathbb{C} \) and denote it by \( h \).

**Definition 3.9.** Let \( S \subset X \times \mathbb{C} \) be a smooth quasi-affine subvariety of \( X \times \mathbb{C} \). Then we say that the restriction \( h|_S \) of the function \( h \) to \( S \) is relatively tame at infinity for the projection \( X \times \mathbb{C} \to X \) if there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that
\[ \{(z, \tau) \in S \mid ||d\text{Re}(h|_S)(z, \tau)|| < \varepsilon, ||z|| > R\} \subset \emptyset. \] (3.74)
For the rational function \( f = \frac{P}{Q} : U = X\setminus Q^{-1}(0) \to \mathbb{C} \) and the algebraic constructible sheaf \( K \in D_c(X^{an}) \) by using the (not necessarily) closed embedding \( i_f : U \to X \times \mathbb{C} \) \((z \mapsto (z, f(z)))\) we set
\[
L := i_f(K|_U) \in D_c(X^{an} \times \mathbb{C}).
\]
(3.75)

Then there exists an algebraic Whitney stratification \( S \) of \( X \times \mathbb{C} \) such that
\[
SS(L) \subset \bigcup_{S \in S} T^*_S(X \times \mathbb{C}).
\]
(3.76)

Moreover we may assume also that for \( S \) and a Whitney stratification \( S_0 \) of \( X = \mathbb{C}^N \) the projection \( X \times \mathbb{C} \to X \) is a stratified fiber bundle as in the assertion of the theorem in [15] page 43. Indeed, after extending \( L \) to a constructible sheaf on \( X \times \mathbb{P} \) we can apply this theorem to the proper morphism \( X \times \mathbb{P} \to X ((z, \tau) \mapsto z) \). Let us fix such Whitney stratifications \( S \) and \( S_0 \). Then we shall say that a stratum \( S \in S \) in \( S \) is horizontal if for its projection \( S_0 \in S_0 \) to \( X = \mathbb{C}^N \) we have \( \dim S_0 = \dim S \) i.e. the surjective submersion \( S \to S_0 \) induced by the projection \( X \times \mathbb{C} \to X \) is a finite covering. Obviously, if \( S \in S \) is not horizontal then for its projection \( S_0 \in S_0 \) to \( X = \mathbb{C}^N \) we have \( \dim S_0 = \dim S - 1 \) and hence the restriction \( h|_S \) of the function \( h \) to \( S \) is relatively tame at infinity for the projection \( X \times \mathbb{C} \to X \). Namely, our relative tameness at infinity of \( h|_S \) is a constraint only for horizontal strata \( S \in S \). Moreover it is easy to see that if for a horizontal stratum \( S \in S \) there exists a holomorphic function \( g \) on its projection \( S_0 \in S_0 \) to \( X \) such that
\[
S = \{(z, g(z)) \mid z \in S_0 \} \subset X \times \mathbb{C}
\]
(3.77)
then the relative tameness at infinity of \( h|_S \) is equivalent to the tameness at infinity of \( g : S_0 \to \mathbb{C} \) in Definition 3.5. The proof of the following proposition is very similar to that of Proposition 3.5 and we omit it.

**Proposition 3.10.** Let \( S \) be the Whitney stratification of \( X \times \mathbb{C} \) as above and assume that for any stratum \( S \in S \) in it such that \( T^*_S(X \times \mathbb{C}) \subset \text{SS}(L) \) the restriction \( h|_S : S \to \mathbb{C} \) of the function \( h \) to \( S \subset X \times \mathbb{C} \) is relatively tame at infinity for the projection \( X \times \mathbb{C} \to X \). Then for any \( \tau_0 \in \mathbb{C} \) there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that we have
\[
N^*(B_r(0) \times \mathbb{C}) \cap \left\{ \text{SS}(L) + \rho((X \times \mathbb{C}) \times \mathbb{C} T^*\mathbb{C}) \right\} \subset T^*_S(X \times \mathbb{C})
\]
(3.78)
over the open subset \( \{(z, \tau) \in X \times \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset X \times \mathbb{C} \) for any \( r \geq R \), where \( \rho : (X \times \mathbb{C}) \times \mathbb{C} T^*\mathbb{C} \to T^*(X \times \mathbb{C}) \) is the closed embedding associated to the projection \( h : X \times \mathbb{C} \to \mathbb{C} \). Moreover the same is true even after replacing \( N^*(B_r(0) \times \mathbb{C}) \) by its antipodal set \( N^*(B_r(0) \times \mathbb{C})^a \).

**Theorem 3.11.** In the situation of Proposition 3.10 we have the following results.

(i) For any \( \tau_0 \in \mathbb{C} \) there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that for the inclusion maps \( i_r : U_r = U \cap B_r(0) \to U \) \((r > R)\) we have isomorphisms
\[
Rf_i(i_r)_!(i_r)^{-1}(K|_U) \simeq Rf_i(K|_{U_r}) \simto Rf_i(K|_U)
\]
(3.79)
on the open subset \( \{\tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C} \) of \( \mathbb{C} \).
(ii) For $\tau_0 \in \mathbb{C}$ we assume also that
\[
\{ z \in f^{-1}(\tau_0) \mid df(z) \in SS(K) \} \subset f^{-1}(\tau_0) \cap \text{supp}(K|_U) \quad (3.80)
\]
and
\[
\{ z \in I(f) \mid \phi_{f^{-\tau_0}}^\text{mero,c}(K)_z \neq 0 \} \subset I(f) \cap \text{supp}(K) \quad (3.81)
\]
are finite subsets of $Z = \text{supp}(K)$ and denote them by $\{ p_1, p_2, \ldots, p_k \}$ and $\{ q_1, q_2, \ldots, q_l \}$ respectively. Then for the vanishing cycle $\phi_{\tau-\tau_0}(Rf_i(K|_U)) \in D^b(\{ \tau_0 \})$ of $Rf_i(K|_U)$ along the function $\tau - \tau_0 : \mathbb{C} \to \mathbb{C}$ we have an isomorphism
\[
\phi_{\tau-\tau_0}(Rf_i(K|_U)) \simeq \left\{ \bigoplus_{i=1}^k \phi_{f^{-\tau_0}}(K)_{p_i} \right\} \oplus \left\{ \bigoplus_{i=1}^l \phi_{f^{-\tau_0}}^\text{mero,c}(K)_{q_i} \right\}. \quad (3.82)
\]
If we assume moreover that $K[N] \in D^b_c(X^\text{an})$ is a perverse sheaf, then we have also a concentration
\[
H^j\phi_{\tau-\tau_0}(Rf_i(K|_U)) \simeq 0 \quad (j \neq N - 1). \quad (3.83)
\]

Proof. By [26, Theorem 4.4.1] and Proposition 3.10 there exist $R \gg 0$ and $0 < \varepsilon \ll 1$ such that for the inclusion maps $j_r : B_r(0) \times \mathbb{C} \hookrightarrow X \times \mathbb{C}$ ($r > R$) we have isomorphisms
\[
R\mathcal{H}(j_r)_!(j_r)^{-1}L \simeq R\mathcal{H}(L_{B_r(0) \times \mathbb{C}}) \sim R\mathcal{H}_L \quad (3.84)
\]
on the open subset $\{ \tau \in \mathbb{C} \mid |\tau - \tau_0| < \varepsilon \} \subset \mathbb{C}$ of $\mathbb{C}$. Moreover we have $R\mathcal{H}_L \simeq Rf_i(K|_U)$ and for any $r > 0$ there exist an isomorphism
\[
R\mathcal{H}_L \simeq R\mathcal{H}_r \left( i_f(K|_U) \otimes \mathbb{C}_B(0) \times \mathbb{C} \right) \simeq Rf_i(K|_U)|_{U_r}. \quad (3.85)
\]
We thus obtain the isomorphism in (i). Let us prove (ii). Let $R \gg 0$ and $0 < \varepsilon \ll 1$ be as in (i). Here we take $R > 0$ so that also the condition $p_1, p_2, \ldots, p_k$, $q_1, q_2, \ldots, q_l \in B_r(0)$ is satisfied. Then by the proof of (i), for any $r > R$ we obtain isomorphisms
\[
\phi_{\tau-\tau_0}(Rf_i(K|_U)) \simeq \phi_{\tau-\tau_0}(R\mathcal{H}_r L_{B_r(0) \times \mathbb{C}})
\simeq \mathcal{R}\Gamma(X \times \{ \tau_0 \}; \phi_{\tau-\tau_0}(L_{B_r(0) \times \mathbb{C}})),
\]
where in the second isomorphism we used the fact that the morphism $h : X \times \mathbb{C} \to \mathbb{C}$ is proper on the support of $L_{B_r(0) \times \mathbb{C}}$. On the other hand, there exist also isomorphisms
\[
\phi_{h^{-\tau_0}}(L) \simeq \phi_{h^{-\tau_0}}(i_f(K|_U)) \simeq \phi_{f^{-\tau_0}}^\text{mero,c}(K) \quad (3.86)
\]
under the natural identification $h^{-1}(\tau_0) = X \times \{ \tau_0 \} \simeq X$. Similarly, we have isomorphisms
\[
\phi_{h^{-\tau_0}}(L_{B_r(0) \times \mathbb{C}}) \simeq \phi_{h^{-\tau_0}}(i_f(K_{B_r(0)}|_U)) \simeq \phi_{f^{-\tau_0}}^\text{mero,c}(K_{B_r(0)}). \quad (3.87)
\]
Then together with our assumptions, this implies that for any $r > R$ the support of the vanishing cycle $\phi_{h^{-\tau_0}}(L_{B_r(0) \times \mathbb{C}})$ is contained in the set
\[
\{ p_1, p_2, \ldots, p_k \} \cup \{ q_1, q_2, \ldots, q_l \} \cup \partial B_r(0). \quad (3.88)
\]
Now let us take \( r, r' > R \) such that \( r > r' \). Then by the proof of \( [26, \text{Theorem 4.4.1}] \) there exists an isomorphism
\[
Rh_{l}L_{B,(0)\times \mathbb{C}} \cong Rh_{l}L_{B,(0)} \times \mathbb{C}
\]
(3.89)
on the open subset \( \{ r \in \mathbb{C} \mid |r - \tau| < \varepsilon \} \subset \mathbb{C} \) of \( \mathbb{C} \). In other words, we have a vanishing
\[
Rh_{l}L_{(B,(0)\setminus B,(0))}\times \mathbb{C} \cong 0
\]
(3.90)there. This implies that we have
\[
R\Gamma(X \times \{ \tau_0 \}; \phi_{h-\tau_0}(L_{(B,(0)\setminus B,(0))}\times \mathbb{C})) \cong 0.
\]
(3.91)Since the support of \( \phi_{h-\tau_0}(L_{(B,(0)\setminus B,(0))}\times \mathbb{C}) \) is contained in the set
\[
\partial B,(0) \sqcup \partial B,(0)
\]
(3.92)and hence the object \( R\Gamma(\partial B,(0); \phi_{h-\tau_0}(L_{B,(0)})) \) is a direct summand of the one \( R\Gamma(X \times \{ \tau_0 \}; \phi_{h-\tau_0}(L_{(B,(0)\setminus B,(0))}\times \mathbb{C})) \cong 0 \), we obtain also a vanishing
\[
R\Gamma(\partial B,(0); \phi_{h-\tau_0}(L_{B,(0)})) \cong 0.
\]
(3.93)From this, the first assertion of (ii) immediately follows. The second one follows from the fact that the functors \( \phi_{f-\tau_0}(\cdot)[-1] \) and \( \phi_{\text{mero}, \tau_0}(\cdot)[-1] \) preserve the perversity. This completes the proof. \( \square \)

Finally, to end this section, we shall introduce some geometric consequences of Proposition 3.4 which generalize the main results of Broughton [5]. Let \( f \in \mathbb{C}[z_1, z_2, \ldots, z_N] \) be a polynomial on \( X = \mathbb{C}^N \) and \( S \subset X \) a smooth subvariety of \( X = \mathbb{C}^N \). Let \( g : S \longrightarrow \mathbb{C} \) be the restriction of \( f : X = \mathbb{C}^N \longrightarrow \mathbb{C} \) to \( S \subset X \). Then it is well-known that there exists a finite subset \( B \subset \mathbb{C} \) of \( \mathbb{C} \) such that the restriction \( g^{-1}(\mathbb{C} \setminus B) \longrightarrow \mathbb{C} \setminus B \) of \( g \) is a \( \mathbb{C}^\infty \)-locally trivial fibration. We denote by \( B_g \) the smallest finite subset of \( \mathbb{C} \) satisfying this property and call it the bifurcation set of \( g \). By this definition, it is clear that the set \( \Sigma_g := g(\text{Sing}_g) \subset \mathbb{C} \) of the critical values of \( g \) is contained in \( B_g \). Note that as was observed in [5] there are a lot of polynomial maps \( g : S \longrightarrow \mathbb{C} \) such that \( B_g \neq \Sigma_g \). Nevertheless, by Proposition 3.4 we can easily show the following result as in the proof of [52, Theorem 1].

**Theorem 3.12.** (cf. Nguyen-Pham-Pham [55, Theorem 3.1]) Assume that \( g : S \longrightarrow \mathbb{C} \) is tame at infinity. Then we have \( B_g = \Sigma_g \).

By Theorem 3.8 we obtain also the analogues of the results in Sabbah [62, Section 8]. Here, instead of Sabbah’s cohomological tameness, we assume our topological one in Definition 3.3. In particular, we obtain the following results.

**Lemma 3.13.** (cf. Sabbah [62, Lemma 8.5]) Assume that \( n := \dim S - 1 \geq 1 \) and \( g : S \longrightarrow \mathbb{C} \) is tame at infinity. Then we have
\[
H^jRg_*\mathbb{C}_S \cong 0 \quad (j \notin [0, n]).
\]
(3.94)Moreover for any \( 1 \leq j \leq n - 1 \) the direct image sheaf \( H^jRg_*\mathbb{C}_S \) is a constant sheaf on \( \mathbb{C} \) of rank \( \dim H^j(S; \mathbb{C}) \).
As is the proof of [62, Lemma 8.5], in the situation of Lemma 3.13 we see that for any point \( c \in \mathbb{C} \) we have isomorphisms
\[
H^j(\text{R}g_! \mathbb{C}_S)_c \simeq H^j(g^{-1}(c); \mathbb{C}) \quad (j \in \mathbb{Z}).
\] (3.95)

We thus obtain the following very simple consequence of Lemma 3.13.

**Corollary 3.14.** In the situation of Lemma 3.13, for the finite subset \( B_g = \Sigma_g \subset \mathbb{C} \) of \( \mathbb{C} \) and a base point \( c \in \mathbb{C} \setminus B_g \) the monodromy representations
\[
\rho_{g,c,j} : \pi_1(\mathbb{C} \setminus B_g; c) \to \text{Aut}\left(H^j(g^{-1}(c); \mathbb{C})\right) \quad (0 \leq j \leq n - 1)
\] (3.96)

are trivial.

By Corollary 3.14 the monodromy representation
\[
\rho_{g,c,n} : \pi_1(\mathbb{C} \setminus B_g; c) \to \text{Aut}\left(H^n(g^{-1}(c); \mathbb{C})\right)
\] (3.97)
in the top degree \( n = \dim g^{-1}(c) \) is the only non-trivial one and we can calculate its eigenvalues by the theory of monodromy zeta functions (see e.g. [66, Section 2]). For a polynomial map \( h : S \to \mathbb{C} \) on \( S \) we denote by \( \mu(h) \) the sum of the Milnor numbers of \( h \). If the set \( \text{Sing} h \subset S \) is not finite, we set \( \mu(h) := +\infty \).

**Lemma 3.15.** (cf. Sabbah [62, Remark just after Lemma 8.5]) In the situation of Lemma 3.13 the generic rank of the constructible sheaf \( \text{H}^n \text{R}g_! \mathbb{C}_S \) on \( \mathbb{C} \) is equal to the number
\[
\mu(g) + \dim H^n(S; \mathbb{C}) - \dim H^{n+1}(S; \mathbb{C}).
\] (3.98)

Moreover, by the proof of [62, Lemma 8.5] and Lemma 3.15 we obtain the following result.

**Corollary 3.16.** In the situation of Lemma 3.13, for a point \( c_0 \in \mathbb{C} \) we denote by \( \mu_{c_0} \) the sum of the Milnor numbers of the hypersurface \( g^{-1}(c_0) \subset S \) and set \( \mu(g)' := \mu(g) + \dim H^n(S; \mathbb{C}) - \dim H^{n+1}(S; \mathbb{C}) \). Then for any point \( c_0 \in \mathbb{C} \) we have
\[
\dim H^n(g^{-1}(c_0); \mathbb{C}) = \begin{cases} 
\mu(g)' & (c_0 \notin B_g = \Sigma_g) \\
\mu(g)' - \mu_{c_0} & (c_0 \in B_g = \Sigma_g).
\end{cases}
\] (3.99)

Now let \( Y = \mathbb{C}_w^N \) be the dual vector space of \( X = \mathbb{C}_z^N \) and consider the Lagrangian subvariety
\[
\Lambda^g := \{(z, dg(z)) \mid z \in S\} \subset T^*S
\] (3.100)
of \( T^*S \) and the natural morphisms
\[
T^*X \xleftarrow{\pi} S \times_X T^*X \xrightarrow{\rho} T^*S
\] (3.101)
associated to the inclusion map \( S \hookrightarrow X \). Then \( \pi \rho^{-1} \Lambda^g \subset T^*X \) is a Lagrangian subvariety of \( T^*X \). For a point \( w = (w_1, w_2, \ldots, w_N) \in Y = \mathbb{C}^N \) we define a linear perturbation \( g^{(w)} : S \to \mathbb{C} \) of \( g : S \to \mathbb{C} \) by
\[
g^{(w)}(z) := g(z) - \sum_{j=1}^{N} w_j z_j \quad (z \in S \subset X).
\] (3.102)
Lemma 3.17. The polynomial map \( g : S \rightarrow \mathbb{C} \) is tame at infinity if and only if the restriction \( \varpi \rho^{-1}\Lambda^g \rightarrow Y \) of the projection \( T^*X \simeq X \times Y \rightarrow Y \) to \( \varpi \rho^{-1}\Lambda^g \subset T^*X \) is proper over a sufficiently small open ball \( B_\varepsilon(0) \subset Y = \mathbb{C}^N \) \((0 < \varepsilon \ll 1)\) centered at the origin \( 0 \in Y = \mathbb{C}^N \).

Proof. First note that the fiber of the morphism \( \varpi \rho^{-1}\Lambda^g \rightarrow Y \) at a point \( w \in Y = \mathbb{C}^N \) is naturally identified with the set

\[
\text{Sing}^g(w) = \{ z \in S \mid dg(w)(z) = 0 \} \subset S. \tag{3.103}
\]

Then the last condition in the lemma is equivalent to the one that there exist \( R \gg 0 \) and \( 0 < \varepsilon \ll 1 \) such that

\[
\{ z \in S \mid dg(w)(z) = 0, ||z|| > R \} = \emptyset \tag{3.104}
\]

for any \( w \in B_\varepsilon(0) \subset Y = \mathbb{C}^N \). By the Cauchy-Riemann equation, for any \( z \in S \) and \( w \in Y = \mathbb{C}^N \) we have also an equivalence

\[
dg(w)(z) = 0 \iff \text{gradReg}^g(w)(z) = 0. \tag{3.105}
\]

Moreover, for the surjective \( \mathbb{C} \)-linear map \( \Phi_z : T_zX_R \simeq \mathbb{C}^N \rightarrow T_zS_R \simeq \mathbb{C}^\dim S \) in Lemma 3.1, by (3.22) we obtain an equality

\[
\text{gradReg}^g(w)(z) = \text{gradReg}(z) - \Phi_z \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}. \tag{3.106}
\]

Since by the proof of Lemma 3.1 the \( \mathbb{C} \)-linear map \( \Phi_z \) is the orthogonal projection to \( T_zS_R \simeq \mathbb{C}^\dim S \) with respect to the Hermitian metric of \( T_zX_R \simeq \mathbb{C}^N \), we see that the open subset

\[
\left\{ -\Phi_z \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} \mid w \in B_\varepsilon(0) \right\} \subset T_zS_R \tag{3.107}
\]

is the \( \varepsilon \)-ball in \( T_zS_R \simeq \mathbb{C}^\dim S \) centered at the origin. From this we immediately obtain the assertion.

By the proof of Lemma 3.17 and the algebraicity of \( \varpi \rho^{-1}\Lambda^g \subset T^*X \) we can easily see that generic linear perturbations \( g(w) \) \((w \in Y = \mathbb{C}^N)\) of \( g \) are tame at infinity. More precisely, we have the following result.

Lemma 3.18. Let \( \Omega \subset Y = \mathbb{C}^N \) be the maximal Zariski open subset of \( Y \) such that the base change of the morphism \( \varpi \rho^{-1}\Lambda^g \rightarrow Y \) by the inclusion map \( \Omega \hookrightarrow Y \) is a (possibly ramified) finite covering. Then for \( w \in \mathbb{C}^N \) the linear perturbation \( g(w) \) of \( g \) is tame at infinity if and only if \( w \in \Omega \).

As in [5, Proposition 3.1] we obtain also the following characterization of the tameness at infinity of \( g : S \rightarrow \mathbb{C} \).

Proposition 3.19. The polynomial map \( g : S \rightarrow \mathbb{C} \) is tame at infinity if and only if \( \mu(g) < +\infty \) and \( \mu(g) = \mu(g(w)) \) for all sufficiently small \( w \in Y = \mathbb{C}^N \).
From now on, we shall introduce our results on the bouquet decompositions of the fibers \( h^{-1}(c) \subset S \) \((c \in \mathbb{C})\) of some tame polynomial maps \( h : S \to \mathbb{C} \). For this purpose, first we recall the following fundamental theorem due to Broughton [5, Theorem 1.2]. For a polynomial map \( h : S \to \mathbb{C} \) having only isolated singular points and a point \( c_0 \in \mathbb{C} \) we denote by \( \mu_{c_0} \) the sum of the Milnor numbers of the hypersurface \( h^{-1}(c_0) \subset S \) and set \( \mu(h,c_0) := \mu(h) - \mu_{c_0} \).

**Theorem 3.20.** (Broughton [5, Theorem 1.2]) Assume that a polynomial map \( h : X = \mathbb{C}^N \to \mathbb{C} \) is tame at infinity. Then for any point \( c_0 \in \mathbb{C} \) its fiber \( h^{-1}(c_0) \subset X = \mathbb{C}^N \) of \( h \) has the homotopy type of a bouquet \( S^{N-1} \vee \cdots \vee S^{N-1} \) of some \((N-1)\)-dimensional spheres \( S^{N-1} \). Moreover the number of the spheres \( S^{N-1} \) in the bouquet decomposition is equal to \( \mu(h,c_0) \).

Recall that this beautiful result was the starting point of the intensive activities in the study of the monodromies at infinity of polynomial maps \( h : X = \mathbb{C}^N \to \mathbb{C} \) (see e.g. [66, Sections 4 and 7] for the details). Motivated by it, we introduce the following class of subvarieties of \( X = \mathbb{C}^N \).

**Definition 3.21.** We say that a smooth complete intersection subvariety \( Z \subset X = \mathbb{C}^N \) is a bouquet variety if it has the homotopy type of a bouquet of some spheres of dimension \( \dim Z \).

Note that if a polynomial map \( h : X = \mathbb{C}^N \to \mathbb{C} \) is tame at infinity then by Theorem 3.20 generic fibers of \( h \) are bouquet varieties.

**Theorem 3.22.** Assume that \( n := \dim S - 1 \geq 1 \), the smooth subvariety \( S \subset X = \mathbb{C}^N \) is a bouquet variety and the polynomial map \( g : S \to \mathbb{C} \) is tame at infinity. Denote by \( \mu_S \) the number of the spheres \( S^{n+1} \) in the bouquet decomposition of \( S \). Then for any \( c_0 \in \mathbb{C} \) its fiber \( g^{-1}(c_0) \subset S \) of \( g \) has the homotopy type of a bouquet \( S^n \vee \cdots \vee S^n \) of some \( n \)-dimensional spheres \( S^n \). Moreover the number of the spheres \( S^n \) in the bouquet decomposition is equal to \( \mu(g,c_0) - \mu_S \).

**Proof.** With Proposition 3.4 and its consequences above at hands, the proof is very similar to that of [3] Theorem 1.2 and for it we use a Morse theory for the Morse function \( \varphi(z) := |g(z) - c_0|^2 \) \((z \in S_R)\) on \( S_R \). First of all, by Proposition 3.4 for a sufficiently small \( 0 < \varepsilon \ll 1 \) the level set \( \{z \in S_R \mid \varphi(z) < \varepsilon\} \subset S_R \) of \( \varphi \) is homotopic to the fiber \( g^{-1}(c_0) \). Moreover by the Cauchy-Riemann equation we can easily see that a point \( z_0 \in S \setminus g^{-1}(c_0) \) is a critical point of the Morse function

\[
\varphi(z) = |g(z) - c_0|^2 = \exp \left( 2 \text{Re} \log(g(z) - c_0) \right)
\]

(3.108)

if and only if it is a critical point of \( g \). As in the proof of [5] Theorem 1.2, by slightly perturbing \( g : S \to \mathbb{C} \) we may assume that any critical point \( z_0 \in S \setminus g^{-1}(c_0) \) of \( g \) is non-degenerate i.e. of complex Morse type. Then we can easily show that any critical point \( z_0 \in S \setminus g^{-1}(c_0) \) of \( \varphi \) is non-degenerate and has the Morse index \( \dim S = n + 1 \). This would then imply that for a sufficiently large \( t \gg 0 \) the level set \( \{z \in S_R \mid \varphi(z) < t\} \subset S_R \) of \( \varphi \) has the homotopy type of the CW complex obtained by attaching some \((n + 1)\)-dimensional cells to \( g^{-1}(c_0) \) and the number of such cells is equal to \( \mu(g,c_0) = \mu(g) - \mu_{c_0} \). However, to justify such a Morse-theoretical argument on the “non-compact” manifold
we have to show that each integral curve of the gradient vector field \( \text{grad} \varphi / ||\text{grad} \varphi ||^2 \) does not go to infinity in a finite time (see the last part of the proof of [5, Theorem 1.2] and that of Milnor [46, Theorem 2.10]). For this purpose, it suffices to show that for some \( R \gg 0 \) the function \( ||\text{grad} \varphi ||^{-1} \) is bounded on the set \( \{ z \in S_R \mid ||z|| \geq R, \varphi(z) \geq \varepsilon \} \).

First, we define a function \( \tilde{\varphi} : X_R \rightarrow \mathbb{R} \) on \( X_R \) by

\[
\tilde{\varphi}(z) := |f(z) - c_0|^2 \quad (z \in X_R).
\]  

Then our Morse function \( \varphi : S_R \rightarrow \mathbb{R} \) is the restriction of \( \tilde{\varphi} \) to \( S_R \subset X_R \). Moreover, for any point \( z \in X_R \) we can easily show an equality

\[
\text{grad} \tilde{\varphi}(z) = 2(f(z) - c_0) \cdot \text{grad}f(z)
\]

in \( T_z X_R \simeq \mathbb{C}^N \). Recall that by (3.22) we also have

\[
\text{grad}f(z) = \text{grad} \text{Re}f(z).
\]

Then by Lemma 3.1, for any point \( z \in S_R \) there exists a surjective \( \mathbb{C} \)-linear map \( \Phi_z : T_z X_R \rightarrow T_z S_R \) such that

\[
||\text{grad} \varphi||^2 = \langle \text{grad} \varphi(z), \text{grad} \varphi(z) \rangle = \langle \text{grad} \tilde{\varphi}(z), \text{grad} \tilde{\varphi}(z) \rangle = \langle \Phi_z(\text{grad} \tilde{\varphi}(z)), \Phi_z(\text{grad} \tilde{\varphi}(z)) \rangle = 4|g(z) - c_0|^2 \cdot ||\Phi_z(\text{grad} \text{Re}f(z))||^2 = 4|g(z) - c_0|^2 \cdot ||\text{grad} \text{Re}g(z)||^2.
\]

Since \( g \) is tame at infinity and hence the function \( ||\text{grad} \text{Re}g|| = ||d \text{Re}g|| \geq 0 \) on \( S_R \) is bounded away from 0 at infinity, now the desired boundedness of \( ||\text{grad} \varphi||^{-1} \) immediately follows. Then as in the proof of [5, Theorem 1.2], by the homotopy exact sequence associated to the pair \((S, g^{-1}(c_0))\) of topological spaces we obtain

\[
\pi_j(g^{-1}(c_0)) \simeq 0 \quad (j < n = \dim g^{-1}(c_0)).
\]

Note also that by the Andreotti-Frankel theorem in [1] and Hamm’s one in [19] the complete intersection subvariety \( g^{-1}(c_0) \subset X = \mathbb{C}^N \) of \( X = \mathbb{C}^N \) has the homotopy type of a CW complex of dimension \( \leq n = \dim g^{-1}(c_0) \). Then we can apply the argument in the proof of Milnor [46, Theorem 6.5] to \( g^{-1}(c_0) \) to show that its (top-dimensional) homology group \( H_n(g^{-1}(c_0)) \) is a free \( \mathbb{Z} \)-module and the first assertion immediately follows from Whitehead’s theorem as in the proof of [5, Theorem 1.2]. We can also show the second assertion by Corollary 3.16. This completes the proof.

Note that in [68, Theorem 4.6 and Corollary 4.7] Tibar also obtained a result similar to the first part of Theorem 3.22 under a different condition at infinity in an appropriately chosen compactification of \( S \). For \( 1 \leq k \leq N \) we define a decreasing sequence

\[
S_0 := X = \mathbb{C}^N \supset S_1 \supset S_2 \supset \cdots \supset S_k
\]  

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of smooth complete intersection subvarieties of \( X = \mathbb{C}^N \) such that \( \dim S_i = N - i \) \((0 \leq i \leq k)\) as follows. First we take a tame polynomial \( g_0 : S_0 = X = \mathbb{C}^N \to \mathbb{C} \) and a point \( c_0 \in \mathbb{C} \setminus B_{g_0} \) and set \( S_1 := g_0^{-1}(c_0) \subset S_0 \). Next we repeat this construction and define \( S_i \) for \( i \geq 2 \) recursively. Namely, for each \( 1 \leq i \leq k - 1 \) after defining \( S_i \) we take a tame polynomial \( g_i : S_i \to \mathbb{C} \) and a point \( c_i \in \mathbb{C} \setminus B_{g_i} \) and set \( S_{i+1} := g_i^{-1}(c_i) \subset S_i \). Then by Theorem 3.22 we see that \( S_1, S_2, \ldots, S_k \) are bouquet varieties. If for each \( 0 \leq i \leq k \) we take a polynomial map \( f_i : X = \mathbb{C}^N \to \mathbb{C} \) such that \( f_i|_{S_i} = g_i \) and set \( h_i := f_i - c_i : X = \mathbb{C}^N \to \mathbb{C} \), then we obtain a smooth complete intersection subvariety

\[
S_k = \{ z \in X = \mathbb{C}^N \mid h_1(z) = h_2(z) = \cdots = h_k(z) = 0 \} \subset X = \mathbb{C}^N
\]

of \( X = \mathbb{C}^N \) which has the homotopy type of a bouquet \( S^{N-k} \vee \cdots \vee S^{N-k} \) of some \( (N-k) \)-dimensional spheres \( S^{N-k} \). Moreover, by Lemmas 3.17 and 3.18 we can explicitly choose such polynomials \( h_1, h_2, \ldots, h_k \). This explains the reason why the results on the monodromies at infinity of the polynomial maps on the complete intersection subvarieties of \( X = \mathbb{C}^N \) studied in Esterov-Takeuchi [13, Section 6] and Matsui-Takeuchi [44, Section 5] are very similar to the ones in the local case of Hamm [18], Esterov-Takeuchi [13, Sections 4 and 5], Matsui-Takeuchi [13, Theorem 3.12] and Oka [58]. Indeed, by Lemma 3.17 we can easily show that in the Newton non-degenerate setting of [13, Section 6] and [44, Section 5] the generic fibers admit a bouquet decomposition (under some weak assumptions).

### 4 An Overview on Ind-sheaves and the Irregular Riemann-Hilbert Correspondence

In this section, we briefly recall some basic notions and results which will be used in this paper. We assume here that the reader is familiar with the theory of sheaves and functors in the framework of derived categories. For them we follow the terminologies in [27] etc. For a topological space \( M \) denote by \( \mathbb{D}^b(C_M) \) the derived category consisting of bounded complexes of sheaves of \( \mathbb{C} \)-vector spaces on it.

#### 4.1 Ind-sheaves

We recall some basic notions and results on ind-sheaves. References are made to Kashiwara-Schapira [29] and [31]. Let \( M \) be a good topological space (which is locally compact, Hausdorff, countable at infinity and has finite soft dimension). We denote by \( \text{Mod}(C_M) \) the abelian category of sheaves of \( \mathbb{C} \)-vector spaces on it and by \( \text{IC}_M \) that of ind-sheaves. Then there exists a natural exact embedding \( \iota_M : \text{Mod}(C_M) \to \text{IC}_M \) of categories. We sometimes omit it. It has an exact left adjoint \( \alpha_M \), that has in turn an exact fully faithful left adjoint functor \( \beta_M \):

\[
\text{Mod}(C_M) \xrightarrow{\iota_M} \text{IC}_M \xrightarrow{\alpha_M} \beta_M.
\]

The category \( \text{IC}_M \) does not have enough injectives. Nevertheless, we can construct the derived category \( \mathbb{D}^b(\text{IC}_M) \) for ind-sheaves and the Grothendieck six operations among
them. We denote by $\otimes$ and $R\mathcal{H}om$ the operations of tensor products and internal homs respectively. If $f : M \to N$ be a continuous map, we denote by $f^{-1}, Rf_*, f^!$ and $Rf!!$ the operations of inverse images, direct images, proper inverse images and proper direct images respectively. We set also $R\mathcal{H}om := \alpha_M \circ R\mathcal{H}om$. We thus obtain the functors

$$
\iota_M : D^b(C_M) \to D^b(I_C M),
\alpha_M : D^b(I_C M) \to D^b(C_M),
\beta_M : D^b(C_M) \to D^b(I_C M),
\otimes : D^b(I_C M) \times D^b(I_C M) \to D^b(I_C M),
R\mathcal{H}om : D^b(I_C M)^{op} \times D^b(I_C M) \to D^b(I_C M),
Rf_* : D^b(I_C M) \to D^b(I_C N),
f^{-1} : D^b(I_C N) \to D^b(I_C M),
Rf!! : D^b(I_C M) \to D^b(I_C N),
f^! : D^b(I_C N) \to D^b(I_C M).
$$

Note that $(f^{-1}, Rf_*)$ and $(Rf!!, f^!)$ are pairs of adjoint functors. We may summarize the commutativity of the various functors we have introduced in the table below. Here, “$\circ$” means that the functors commute, and “$\times$” they do not.

|     | $\otimes$ | $f^{-1}$ | $Rf_*$ | $f^!$ | $\lim\to$ | $\lim\leftarrow$ |
|-----|-----------|----------|--------|--------|------------|---------------|
| $\iota$ | $\circ$   | $\circ$  | $\circ$ | $\times$ | $\times$  | $\circ$       |
| $\alpha$ | $\circ$  | $\circ$ | $\circ$ | $\times$ | $\circ$   | $\circ$       |
| $\beta$ | $\circ$  | $\circ$ | $\times$ | $\times$ | $\circ$   | $\times$      |
| $\lim\to$ | $\circ$  | $\circ$ | $\times$ | $\circ$ | $\circ$   | $\circ$       |
| $\lim\leftarrow$ | $\times$ | $\times$ | $\circ$ | $\times$ | $\times$  | $\times$      |

### 4.2 Ind-sheaves on Bordered Spaces

For the results in this subsection, we refer to D’Agnolo-Kashiwara [9]. A bordered space is a pair $M_\infty = (M, \tilde{M})$ of a good topological space $\tilde{M}$ and an open subset $M \subset \tilde{M}$. A morphism $f : (M, \tilde{M}) \to (N, \tilde{N})$ of bordered spaces is a continuous map $f : M \to N$ such that the first projection $\tilde{M} \times \tilde{N} \to \tilde{M}$ is proper on the closure $\Gamma_f$ of the graph $\Gamma_f$ of $f$ in $\tilde{M} \times \tilde{N}$. If also the second projection $\Gamma_f \to \tilde{N}$ is proper, we say that $f$ is semi-proper. The category of good topological spaces embeds into that of bordered spaces by the identification $\tilde{M} = (M, M)$. We define the triangulated category of ind-sheaves on $M_\infty = (M, \tilde{M})$ by

$$
D^b(I_C M_\infty) := D^b(I_C \tilde{M})/D^b(I_C \tilde{M}\setminus M).
$$

(4.2)
The quotient functor
\[ q : \text{D}^b(\operatorname{IC}_M) \to \text{D}^b(\operatorname{IC}_{M\infty}) \] (4.3)
has a left adjoint \( l \) and a right adjoint \( r \), both fully faithful, defined by
\[ l(qF) := C \otimes F, \quad r(qF) := R\operatorname{Hom}(C, F). \] (4.4)
For a morphism \( f : M_{\infty} \to N_{\infty} \) of bordered spaces, we have the Grothendieck’s operations
\[ \otimes : \text{D}^b(\operatorname{IC}_{M\infty}) \times \text{D}^b(\operatorname{IC}_{M\infty}) \to \text{D}^b(\operatorname{IC}_{M\infty}), \]
\[ R\operatorname{Hom} : \text{D}^b(\operatorname{IC}_{M\infty})^{\text{op}} \times \text{D}^b(\operatorname{IC}_{M\infty}) \to \text{D}^b(\operatorname{IC}_{M\infty}), \]
\[ Rf_* : \text{D}^b(\operatorname{IC}_{M\infty}) \to \text{D}^b(\operatorname{IC}_{N\infty}), \]
\[ f^{-1} : \text{D}^b(\operatorname{IC}_{N\infty}) \to \text{D}^b(\operatorname{IC}_{M\infty}), \]
\[ Rf_! : \text{D}^b(\operatorname{IC}_{M\infty}) \to \text{D}^b(\operatorname{IC}_{N\infty}), \]
\[ f^! : \text{D}^b(\operatorname{IC}_{N\infty}) \to \text{D}^b(\operatorname{IC}_{M\infty}) \]
(see [9, Definitions 3.3.1 and 3.3.4] for the details). Moreover, there exists a natural embedding of categories
\[ \text{D}^b(\operatorname{IC}_M) \hookrightarrow \text{D}^b(\operatorname{IC}_{M\infty}). \] (4.5)

### 4.3 Enhanced Sheaves

For the results in this subsection, see Kashiwara-Schapira [32] and D’Agnolo-Kashiwara [10]. Let \( M \) be a good topological space. We consider the maps
\[ M \times \mathbb{R}^2 \xrightarrow{p_1,p_2,\mu} M \times \mathbb{R} \xrightarrow{\pi} M \] (4.6)
where \( p_1, p_2 \) are the first and the second projections and we set \( \pi(x,t) := x \) and \( \mu(x,t_1,t_2) := (x,t_1 + t_2) \). Then the convolution functors for sheaves on \( M \times \mathbb{R} \) are defined by
\[ F_1 \ast F_2 := R\mu(p_1^{-1}F_1 \otimes p_2^{-1}F_2), \]
\[ R\operatorname{Hom}^+(F_1, F_2) := Rp_1_* R\operatorname{Hom}(p_2^{-1}F_1, \mu^! F_2). \]
We define the triangulated category of enhanced sheaves on \( M \) by
\[ \text{E}^b(\operatorname{IC}_M) := \text{D}^b(\operatorname{IC}_{M\times\mathbb{R}})/\pi^{-1} \text{D}^b(\operatorname{IC}_M). \] (4.7)
Then the quotient functor
\[ Q : \text{D}^b(\operatorname{IC}_{M\times\mathbb{R}}) \to \text{E}^b(\operatorname{IC}_M) \] (4.8)
has fully faithful left and right adjoints \( L^E, R^E \) defined by
\[ L^E (QF) := (C_{\{t \geq 0\}} \oplus C_{\{t \leq 0\}}) \ast F, \quad R^E (QG) := R\operatorname{Hom}^+(C_{\{t \geq 0\}} \oplus C_{\{t \leq 0\}}, G), \] (4.9)
where \( \{ t \geq 0 \} \) stands for \( \{(x, t) \in M \times \mathbb{R} \mid t \geq 0 \} \) and \( \{ t \leq 0 \} \) is defined similarly. The convolution functors are defined also for enhanced sheaves. We denote them by the same symbols \( \boxtimes \), \( R \mathcal{H}om^+ \). For a continuous map \( f : M \to N \), we can define naturally the operations \( E f^{-1} \), \( E f_* \), \( E f^! \), \( E f! \) for enhanced sheaves. We have also a natural embedding \( \varepsilon : D^b(\mathbb{C}_M) \to D^b(\mathbb{C}_M) \) defined by

\[
\varepsilon(f) := Q(\mathbb{C}_{\{ t \geq 0 \}} \otimes \pi^{-1} F).
\]

(4.10)

For a continuous function \( \varphi : U \to \mathbb{R} \) defined on an open subset \( U \subset M \) of \( M \) we define the exponential enhanced sheaf by

\[
E^\varphi_{U|M} := Q(\mathbb{C}_{\{ t + \varphi \geq 0 \}}),
\]

(4.11)

where \( \{ t + \varphi \geq 0 \} \) stands for \( \{(x, t) \in M \times \mathbb{R} \mid x \in U, t + \varphi(x) \geq 0 \} \).

### 4.4 Enhanced Ind-sheaves

We recall some basic notions and results on enhanced ind-sheaves. References are made to D’Agnolo-Kashiwara [9] and Kashiwara-Schapira [33]. Let \( M \) be a good topological space. Set \( \mathbb{R}_\infty := (\mathbb{R}, \mathbb{R}) \) for \( \mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\} \), and let \( t \in \mathbb{R} \) be the affine coordinate. We consider the maps

\[
M \times \mathbb{R}^2_\infty \xrightarrow{p_1:p_2:\mu} M \times \mathbb{R}_\infty \xrightarrow{\pi} M
\]

(4.12)

where \( p_1, p_2 \) and \( \pi \) are morphisms of bordered spaces induced by the projections. And \( \mu \) is a morphism of bordered spaces induced by the map \( M \times \mathbb{R}^2 \ni (x, t_1, t_2) \mapsto (x, t_1 + t_2) \in M \times \mathbb{R} \). Then the convolution functors for ind-sheaves on \( M \times \mathbb{R}_\infty \) are defined by

\[
F_1 \boxtimes F_2 := R\mu !(p_1^{-1} F_1 \otimes p_2^{-1} F_2),
\]

\[
R\mathcal{H}om^+(F_1, F_2) := Rp_1 *, R\mathcal{H}om(p_2^{-1} F_1, \mu^* F_2).
\]

Now we define the triangulated category of enhanced ind-sheaves on \( M \) by

\[
D^b(\mathbb{I}C_M) := D^b(\mathbb{I}C_{M \times \mathbb{R}_\infty})/\pi^{-1} D^b(\mathbb{I}C_M).
\]

(4.13)

Note that we have a natural embedding of categories

\[
D^b(\mathbb{C}_M) \hookrightarrow D^b(\mathbb{I}C_M).
\]

(4.14)

The quotient functor

\[
Q : D^b(\mathbb{I}C_{M \times \mathbb{R}_\infty}) \to D^b(\mathbb{I}C_M)
\]

(4.15)

has fully faithful left and right adjoints \( L^E, R^E \) defined by

\[
L^E(QK) := (\mathbb{C}_{\{ t \geq 0 \}} \oplus \mathbb{C}_{\{ t \leq 0 \}})^+ \otimes K, \quad R^E(QK) := R\mathcal{H}om^+(\mathbb{C}_{\{ t \geq 0 \}} \oplus \mathbb{C}_{\{ t \leq 0 \}}, K),
\]

(4.16)

where \( \{ t \geq 0 \} \) stands for \( \{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, t \geq 0 \} \) and \( \{ t \leq 0 \} \) is defined similarly. The convolution functors are defined also for enhanced ind-sheaves. We denote them by the same symbols \( \boxtimes, R\mathcal{H}om^+ \). For a continuous map \( f : M \to N \), we can define also the operations \( E f^{-1} \), \( E f_* \), \( E f^! \), \( E f! \) for enhanced ind-sheaves. For example, by the
natural morphism $\tilde{f} : M \times \mathbb{R}_\infty \rightarrow N \times \mathbb{R}_\infty$ of bordered spaces associated to $f$ we set $Ef_*(QK) = Q(R\tilde{f}_*(K))$. The other operations are defined similarly. We thus obtain the six operations $\otimes, \mathbb{R}Hom^+, Ef_*, Ef^!, Ef_!$ for enhanced ind-sheaves. Moreover we denote by $D_E^M$ the Verdier duality functor for enhanced ind-sheaves. We have outer hom functors

$$\mathbb{R}Hom^E(K_1, K_2) := \mathbb{R}\pi_* \mathbb{R}Hom(L^EK_1, L^EK_2) \simeq \mathbb{R}\pi_* \mathbb{R}Hom(E^F K_1, E^F K_2),$$

$$\mathbb{R}Hom^E(K_1, K_2) := \alpha_M \mathbb{R}Hom^E(K_1, K_2),$$

with values in $D^b(IC_M), D^b(C_M)$ and $D^b(C)$, respectively. Moreover for $F \in D^b(IC_M)$ and $K \in E^b(IC_M)$ the objects

$$\pi^{-1} F \otimes K := Q(\pi^{-1} F \otimes L^EK),$$

$$\mathbb{R}Hom(\pi^{-1} F, K) := Q(\mathbb{R}Hom(\pi^{-1} F, E^F K)).$$

in $E^b(IC_M)$ are well-defined. Set $C^E_M := Q\left(\lim_{a \rightarrow +\infty}^\right. C_{\{t \geq a\}} \right) \in E^b(IC_M)$. Then we have natural embeddings $\varepsilon, e : D^b(IC_M) \rightarrow E^b(IC_M)$ defined by

$$\varepsilon(F) := Q(C_{\{t \geq 0\}} \otimes \pi^{-1} F),$$

$$e(F) := C^E_M \otimes \pi^{-1} F \simeq C^E_M \hat{\otimes} \varepsilon(F).$$

For a continuous function $\varphi : U \rightarrow \mathbb{R}$ defined on an open subset $U \subset M$ of $M$ we define the exponential enhanced ind-sheaf by

$$E_{U|M}^\varphi := C^E_M \hat{\otimes} E_{U|M}^\varphi = C^E_M \hat{\otimes} QC_{\{t + \varphi \geq 0\}} = Q\left(\lim_{a \rightarrow +\infty}^\right. C_{\{t + \varphi \geq a\}} \right) \quad (4.17)$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, x \in U, t + \varphi(x) \geq 0\}$.

### 4.5 $D$-modules

In this subsection we recall some basic notions and results on $D$-modules. References are made to [21, 29 §7], [29 §8, 9], [33 §3, 4, 7] and [25 §4, 5, 6, 7, 8]. For a complex manifold $X$ we denote by $\mathcal{D}_X$ its complex dimension. Denote by $\mathcal{O}_X, \Omega_X$ and $\mathcal{D}_X$ the sheaves of holomorphic functions, holomorphic differential forms of top degree and holomorphic differential operators, respectively. Let $D^b(\mathcal{D}_X)$ be the bounded derived category of left $\mathcal{D}_X$-modules and $D^b(\mathcal{D}_X^0)$ be that of right $\mathcal{D}_X$-modules. Moreover we denote by $D_{coh}^b(\mathcal{D}_X), D_{good}^b(\mathcal{D}_X), D_{hol}^b(\mathcal{D}_X)$ and $D_{inh}^b(\mathcal{D}_X)$ the full triangulated subcategories of $D^b(\mathcal{D}_X)$ consisting of objects with coherent, good, holonomic and regular holonomic cohomologies, respectively. For a morphism $f : X \rightarrow Y$ of complex manifolds, denote by $\otimes, \mathbb{R}Hom_{\mathcal{D}_X}, Ef_*, Ef^!$ the standard operations for $D$-modules. We define also the duality functor $D_X : D_{coh}^b(\mathcal{D}_X) \rightarrow D_{coh}^b(\mathcal{D}_X)$ by

$$D_X(\mathcal{M}) := RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X]. \quad (4.18)$$
Note that there exists an equivalence of categories $(\cdot)^r : \text{Mod}(\mathcal{D}_X) \simto \text{Mod}(\mathcal{D}_X^{\text{op}})$ given by

$$\mathcal{M}^r := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}. \quad (4.19)$$

The classical de Rham and solution functors are defined by

$$\text{DR}_X : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \Omega_X^L \otimes_{\mathcal{D}_X} \mathcal{M},$$

$$\text{Sol}_X : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)^{\text{op}} \to \mathbf{D}^b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Then for $\mathcal{M} \in \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)$ we have an isomorphism $\text{Sol}_X(\mathcal{M})[d_X] \simeq \text{DR}_X(\mathcal{D}_X \mathcal{M})$. For a closed hypersurface $D \subset X$ in $X$ we denote by $\mathcal{O}_X(\ast D)$ the sheaf of meromorphic functions on $X$ with poles in $D$. Then for $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X)$ we set

$$\mathcal{M}(\ast D) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ast D). \quad (4.20)$$

For $f \in \mathcal{O}_X(\ast D)$ and $U := X \setminus D$, set

$$\mathcal{D}_X e^f := \mathcal{D}_X / \{ P \in \mathcal{D}_X \mid P e^f|_U = 0 \},$$

$$\mathcal{E}^f_U := \mathcal{D}_X e^f(\ast D).$$

Note that $\mathcal{E}^f_U$ is holonomic and there exists an isomorphism

$$\mathbb{D}_X(\mathcal{E}^f_U)(\ast D) \simeq \mathcal{E}^{-f}_U. \quad (4.21)$$

Namely $\mathcal{E}^f_U$ is a meromorphic connection associated to $d + df$.

One defines the ind-sheaf $\mathcal{O}^t_X$ of tempered holomorphic functions as the Dolbeault complex with coefficients in the ind-sheaf of tempered distributions. More precisely, denoting by $\overline{X}$ the complex conjugate manifold to $X$ and by $X_\mathbb{R}$ the underlying real analytic manifold of $X$, we set

$$\mathcal{O}^t_X := \mathcal{R}\text{Ihom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}^b_{\mathbb{R}}_{X_\mathbb{R}}), \quad (4.22)$$

where $\mathcal{D}^b_{\mathbb{R}}_{X_\mathbb{R}}$ is the ind-sheaf of tempered distributions on $X_\mathbb{R}$ (for the definition see [29, Definition 7.2.5]). Moreover, we set

$$\Omega^t_X := \beta_X \Omega_X \otimes_{\beta_X \mathcal{O}_X} \mathcal{O}^t_X. \quad (4.23)$$

Then the tempered de Rham and solution functors are defined by

$$\text{DR}^t_X : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X) \to \mathbf{D}^b(\mathcal{I}_X \mathcal{C}_X), \quad \mathcal{M} \mapsto \Omega^t_X \otimes_{\mathcal{D}_X} \mathcal{M},$$

$$\text{Sol}^t_X : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)^{\text{op}} \to \mathbf{D}^b(\mathcal{I}_X \mathcal{C}_X), \quad \mathcal{M} \mapsto \mathcal{R}\text{Ihom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^t_X).$$

Note that we have isomorphisms

$$\text{Sol}_X(\mathcal{M}) \simeq \alpha_X \text{Sol}^t_X(\mathcal{M}),$$

$$\text{DR}_X(\mathcal{M}) \simeq \alpha_X \text{DR}^t_X(\mathcal{M}),$$

$$\text{Sol}^t_X(\mathcal{M})[d_X] \simeq \text{DR}^t_X(\mathbb{D}_X \mathcal{M}).$$
Let $i : X \times \mathbb{R}_\infty \rightarrow X \times \mathbb{P}$ be the natural morphism of bordered spaces and $\tau \in \mathbb{C} \subset \mathbb{P}$ the affine coordinate such that $\tau |_{\mathbb{R}}$ is that of $\mathbb{R}$. We then define objects $\mathcal{O}_X^E \in \mathbf{E}^b(I\mathcal{D}_X)$ and $\Omega_X^E \in \mathbf{E}^b(I\mathcal{D}_X^{op})$ by

$$
\mathcal{O}_X^E := \text{R}\text{Ihom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}b_{X_s}^T)
\simeq i'_! \left( (\mathcal{E}_{C|\mathbb{P}}^\tau)^L \otimes_{\mathcal{D}_X} \mathcal{O}_{X\times \mathbb{P}}^L \right) [1] \simeq i'_! \text{R}\text{Ihom}_{\mathcal{D}_X} (\mathcal{E}_{C|\mathbb{P}}^\tau, \mathcal{O}_{X\times \mathbb{P}}^L) [2],
$$

$$
\Omega_X^E := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^E \simeq i'_! (\Omega_{X\times \mathbb{P}}^L \otimes_{\mathcal{D}_X} \mathcal{E}_{C|\mathbb{P}}^\tau) [1],
$$

where $\mathcal{D}b_{X_s}^T$ stand for the enhanced ind-sheaf of tempered distributions on $X_\mathbb{R}$ (for the definition see [9, Definition 8.1.1]). We call $\mathcal{O}_X^E$ the enhanced ind-sheaf of tempered holomorphic functions. Note that there exists an isomorphism

$$
i_0^! \mathbf{R}\mathcal{O}_X^E \simeq \mathcal{O}_X^E, \quad (4.24)
$$

where $i_0 : X \rightarrow X \times \mathbb{R}_\infty$ is the inclusion map of bordered spaces induced by $x \mapsto (x, 0)$. The enhanced de Rham and solution functors are defined by

$$
\bar{\mathcal{D}}R_X^E : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X) \rightarrow \mathbf{E}^b(I\mathcal{C}_X), \quad \mathcal{M} \longmapsto \Omega_X^E \otimes_{\mathcal{D}_X} \mathcal{M},
$$

$$
\text{Sol}_X^E : \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)^{\text{op}} \rightarrow \mathbf{E}^b(I\mathcal{C}_X), \quad \mathcal{M} \longmapsto \text{R}\text{Ihom}_{\mathcal{D}_X} (\mathcal{M}, \mathcal{O}_X^E).
$$

Then for $\mathcal{M} \in \mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)$ we have isomorphism $\text{Sol}_X^E(\mathcal{M})[d_X] \simeq \bar{\mathcal{D}}R_X^E(\mathcal{D}_X \mathcal{M})$ and $\text{Sol}_X^E(\mathcal{M}) \simeq i_0^! \mathbf{R}\mathcal{O}_X^E \text{Sol}_X(\mathcal{M})$. We recall the following results of [9].

**Theorem 4.1.**

(i) For $\mathcal{M} \in \mathbf{D}^b_{\text{hol}}(\mathcal{D}_X)$ there is an isomorphism in $\mathbf{E}^b(I\mathcal{C}_X)$

$$
\bar{\mathcal{D}}R_X^E(D\mathcal{M}) \simeq \text{Sol}_X^E(\mathcal{M})[d_X]. \quad (4.25)
$$

(ii) Let $f : X \rightarrow Y$ be a morphism of complex manifolds. Then for $\mathcal{N} \in \mathbf{D}^b_{\text{hol}}(\mathcal{D}_Y)$ there is an isomorphism in $\mathbf{E}^b(I\mathcal{C}_X)$

$$
\text{Sol}_X^E(Df^*\mathcal{N}) \simeq Ef^{-1}\text{Sol}_Y^E(\mathcal{N}). \quad (4.26)
$$

(iii) Let $f : X \rightarrow Y$ be a morphism of complex manifolds and $\mathcal{M} \in \mathbf{D}^b_{\text{good}}(\mathcal{D}_X) \cap \mathbf{D}^b_{\text{hol}}(\mathcal{D}_X)$. If $\text{supp}(\mathcal{M})$ is proper over $Y$ then there is an isomorphism in $\mathbf{E}^b(I\mathcal{C}_Y)$

$$
\text{Sol}_Y^E(Df_*\mathcal{M})[d_Y] \simeq Ef_*\text{Sol}_X^E(\mathcal{M})[d_X]. \quad (4.27)
$$

(iv) For $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{D}^b_{\text{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathbf{E}^b(I\mathcal{C}_X)$

$$
\text{Sol}_X^E(\mathcal{M}_1 \otimes \mathcal{M}_2) \simeq \text{Sol}_X^E(\mathcal{M}_1) \otimes \text{Sol}_X^E(\mathcal{M}_2). \quad (4.28)
$$

(v) If $\mathcal{M} \in \mathbf{D}^b_{\text{hol}}(\mathcal{D}_X)$ and $D \subset X$ is a closed hypersurface, then there are isomorphisms in $\mathbf{E}^b(I\mathcal{C}_X)$

$$
\text{Sol}_X^E(\mathcal{M}(\ast D)) \simeq \pi^{-1}\mathcal{C}_{X\setminus D} \otimes \text{Sol}_X^E(\mathcal{M}),
\bar{\mathcal{D}}R_X^E(\mathcal{M}(\ast D)) \simeq \text{R}\text{Ihom}(\pi^{-1}\mathcal{C}_{X\setminus D}, \bar{\mathcal{D}}R_X^E(\mathcal{M})).
$$
(vi) Let \( D \) be a closed hypersurface in \( X \) and \( f \in \mathcal{O}_X(*D) \) a meromorphic function along \( D \). Set \( U := X \setminus D \subset X \). Then there exists an isomorphism in \( \mathcal{E}^b(\mathcal{I}C_X) \)

\[
\text{Sol}^E_X(\mathcal{E}^f_{U|X}) \simeq \mathbb{E}^\text{Ref}_{U|X}. \tag{4.29}
\]

(vii) For \( \mathcal{L} \in \mathcal{D}^b_{\text{rh}}(\mathcal{D}_X) \) and \( \mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) \), there exists an isomorphism in \( \mathcal{E}^b(\mathcal{I}C_X) \)

\[
DR^E_X(\mathcal{L} \otimes \mathcal{M}) \simeq \text{R} \mathcal{I}hom(\pi^{-1} \text{Sol}^E_X(\mathcal{L}), DR^E_X(\mathcal{M})). \tag{4.30}
\]

We also have the following corollary of Theorem 4.1.

**Corollary 4.2.** For \( \mathcal{L} \in \mathcal{D}^b_{\text{rh}}(\mathcal{D}_X) \) and \( \mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) \), there exists an isomorphism in \( \mathcal{E}^b(\mathcal{I}C_X) \)

\[
\text{Sol}^E_X(\mathcal{L} \otimes \mathcal{M}) \simeq \pi^{-1} \text{Sol}_X(\mathcal{L}) \otimes \text{Sol}^E_X(\mathcal{M}). \tag{4.31}
\]

**Proof.** Let

\[
D^E_X(\cdot) : \mathcal{E}^b(\mathcal{I}C_X) \to \mathcal{E}^b(\mathcal{I}C_X), \quad F \mapsto \text{R} \mathcal{I}hom^+(F, \omega^E_X) \tag{4.32}
\]

be the Verdier dual functor defined in [9, Section 4.8], where we set \( \omega^E_X = C^E_X \otimes \pi^{-1} \omega_X \). Moreover we set \( L = \text{Sol}_X(\mathcal{L}) \) and \( K = \text{Sol}^E_X(\mathcal{M}) \). Then we have isomorphisms

\[
D^E_X(\pi^{-1}L \otimes K) \simeq \text{R} \mathcal{I}hom(\pi^{-1}L, D^E_X(K))
\]

\[
\simeq \text{R} \mathcal{I}hom(\pi^{-1}L, DR^E_X(\mathcal{M})[d_X])
\]

\[
\simeq DR^E_X(\mathcal{L} \otimes \mathcal{M})[d_X],
\]

where the first isomorphism follows from the definition of the bifunctor \( \text{R} \mathcal{I}hom^+(\cdot, \cdot) \) and in the second (resp. third) isomorphism we used Theorem 4.1 (i) (resp. Theorem 4.1 (vii)). By taking the Verdier dual \( D^E_X(\cdot) \) of the both sides, we obtain the desired isomorphism. \( \square \)

We recall also the following theorem of [9].

**Theorem 4.3** ([9, Theorem 9.5.3 (Irregular Riemann-Hilbert Correspondence)]). There exists an isomorphism functorial with respect to \( \mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) : \)

\[
\mathcal{M} \overset{\sim}{\longrightarrow} \text{R} \mathcal{H}om^E(\text{Sol}^E_X(\mathcal{M}), \mathcal{O}^E_X) \tag{4.33}
\]

in \( \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) \).

For the proof of the main results in [9], a key role was played by the following proposition. Here we give a very short new proof deduced directly from [30, Proposition 7.3 and Remark 7.4] (see also [22, Proposition 3.14]).

**Proposition 4.4** ([9, Proposition 6.2.2]). Let \( D \) be a closed hypersurface in \( X \) and \( f \in \mathcal{O}_X(*D) \) a meromorphic function along \( D \). Set \( U := X \setminus D \subset X \). Then there exists an isomorphism in \( \mathcal{D}^b(\mathcal{I}C_X) \)

\[
DR^E_X(\mathcal{E}^f_{U|X}) \simeq \text{R} \mathcal{I}hom\left( C_U, \lim_{a \to +\infty} C_{(\text{Ref} < a)} \right)[d_X]. \tag{4.34}
\]
Proof. Let \( z \) be the standard holomorphic coordinate of \( \mathbb{C} \). As in \([9, \text{Lemma 6.2.5}]\), first we consider the exponential \( D \)-module \( \mathcal{E}^z_{\mathbb{C}|\mathbb{P}} \) on the one dimensional projective space \( \mathbb{P} = \mathbb{P}^1 \) associated to the meromorphic function \( f = z \in \mathcal{O}_{\mathbb{P}}(*\{\infty\}) \). Then for the holomorphic coordinate \( \zeta := \frac{1}{z} \) on a neighborhood of the point \( \infty \in \mathbb{P} \) there exist isomorphisms

\[
\mathcal{E}^z_{\mathbb{C}|\mathbb{P}} \simeq \mathcal{D}_{\mathbb{P}} \text{exp}\left(\frac{1}{\zeta}\right) \simeq \mathcal{D}_{\mathbb{P}}/\mathcal{D}_{\mathbb{P}}(\zeta^2 \partial_\zeta + 1). \tag{4.35}
\]

This implies that on a neighborhood of \( \infty \in \mathbb{P} \) we have isomorphisms

\[
\mathcal{D}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \simeq \mathcal{D}_{\mathbb{P}}/\mathcal{D}_{\mathbb{P}}(-\partial_\zeta \zeta^2 + 1) \simeq \mathcal{D}_{\mathbb{P}}\left\{ \frac{1}{\zeta^2} \exp\left(-\frac{1}{\zeta}\right) \right\} \tag{4.36}
\]

\[
\simeq \mathcal{D}_{\mathbb{P}}\exp\left(-\frac{1}{\zeta}\right) \simeq \mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \tag{4.37}
\]

(cf. \([9, \text{Lemma 6.1.2 and Remark 6.1.3}]\)). In this case, we thus obtain an isomorphism

\[
\mathcal{D}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}})[1]. \tag{4.38}
\]

Consider the distinguished triangle

\[
H^0 \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \rightarrow \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \rightarrow \tau^{\geq 1} \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \xrightarrow{+1} \tag{4.39}
\]

in \( \text{D}^b(\mathcal{I}\mathcal{C}_X) \). Then, according to \([34, \text{Remark 7.4}]\) (see also \([22, \text{Proposition 3.14}]\)), for the closed embedding \( i_{\infty} : \{\infty\} \hookrightarrow \mathbb{P} \) there exists an isomorphism

\[
\tau^{\geq 1} \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \simeq (i_{\infty})_* \mathcal{C}_{\{\infty\}}[-1]. \tag{4.40}
\]

From this and \( i_{\infty}^{-1} \mathcal{C}_\mathbb{C} \simeq 0 \) we obtain the vanishing

\[
\mathcal{R}\text{I\text{hom}}\left(\mathcal{C}_\mathbb{C}, \tau^{\geq 1} \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}})\right) \simeq \mathcal{R}\text{I\text{hom}}\left(i_{\infty}^{-1} \mathcal{C}_\mathbb{C}, \mathcal{C}_{\{\infty\}}[-1]\right) \simeq 0. \tag{4.41}
\]

On the other hand, by \([34, \text{Proposition 7.3}]\) (see also \([22, \text{Proposition 3.14}]\)) we have an isomorphism

\[
H^0 \text{Sol}_{\mathbb{P}}(\mathcal{E}^z_{\mathbb{C}|\mathbb{P}}) \simeq \text{"lim}_{a \rightarrow +\infty} \mathcal{C}_{\{z \in \mathcal{I}\mathcal{C}_{\text{Re}z < a}\}}. \tag{4.42}
\]

Then the assertion follows immediately from the isomorphisms

\[
\mathcal{D}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq \mathcal{D}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}(*\{\infty\})) \simeq \mathcal{R}\text{I\text{hom}}\left(\mathcal{C}_\mathbb{C}, \mathcal{D}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})\right) \simeq \mathcal{R}\text{I\text{hom}}\left(\mathcal{C}_\mathbb{C}, \mathcal{D}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}(*\{\infty\}))\right) \tag{4.43}
\]

(see \([29, \text{Theorem 7.4.12}]\)) and the distinguished triangle \((4.39)\). Next, we consider the general exponential \( D \)-module \( \mathcal{E}_{U|X}^f \) for \( f \in \mathcal{O}_X(*D) \). Let \( \nu : Y \rightarrow X \) be a proper morphism of complex manifolds such that \( E := \nu^{-1}D \subset Y \) is a normal crossing divisor in \( Y \), the restriction \( \nu|_{Y \setminus E} : Y \setminus E \rightarrow X \setminus D \) of \( \nu \) is an isomorphism and the meromorphic function \( g := f \circ \nu \in \mathcal{O}_Y(*E) \) on \( Y \) has no point of indeterminacy on the whole \( Y \). Such a resolution of singularities of \( D \subset X \) always exists. See for example the proof of \([45, \text{Theorem 3.6}]\). Let \( E_0 \) (resp. \( E_1 \)) be the union of the irreducible components of \( E \) along which \( g \) has no pole (resp. has a pole) so that we have \( E = E_0 \cup E_1 \). Set \( V := Y \setminus E \) and
Since we have an isomorphism
\[ \mathcal{E}_{U|X}^{-g} \simeq (D\nu_{*}\mathcal{E}_{Y|V}^{-g})(*)D, \]
by [29, Theorems 7.4.6 and 7.4.12] we obtain an isomorphism
\[ DR_{X}(\mathcal{E}_{U|X}^{-f}) \simeq \text{R}\text{I}\text{hom}\left( \mathcal{C}_{V}, DR_{Y}^{1}(\mathcal{E}_{Y|V}^{-g}) \right). \]
(4.46)

Since the meromorphic function \( \tilde{g} \in \mathcal{O}(\ast E_{1}) \) has no point of indeterminacy on the whole \( Y \), we obtain a holomorphic map from \( Y \) to \( \mathbb{P} \). We denote it by the same letter \( \tilde{g} \) for simplicity. Then by [29, Theorem 7.4.1] it follows from the isomorphism
\[ \mathcal{E}_{V|Y}^{-\tilde{g}} \simeq D\tilde{g}^{*}\mathcal{E}_{C|\mathbb{P}}^{-\tilde{z}} \]
that we obtain an isomorphism
\[ DR_{Y}(\mathcal{E}_{V|Y}^{-\tilde{g}}) \simeq \tilde{g}^{1}DR_{Y}^{1}(\mathcal{E}_{C|\mathbb{P}}^{-\tilde{g}})[1 - d_{X}] \simeq \text{R}\text{I}\text{hom}\left( \mathcal{C}_{Y}, \lim_{a \rightarrow +\infty} \tilde{g}^{1}\mathcal{C}_{(\text{Reg} < a)} \right)[2 - d_{X}]. \]
(4.48)

Combining these results together, we finally get the desired isomorphism as follows:
\[ DR_{X}(\mathcal{E}_{U|X}^{-f}) \simeq \text{R}\nu_{*}\text{R}\text{I}\text{hom}\left( \mathcal{C}_{V}, \lim_{a \rightarrow +\infty} \mathcal{C}_{(\text{Reg} < a)} \right)[d_{X}] \]
\[ \simeq \text{R}\nu_{*}\text{R}\text{I}\text{hom}\left( \mathcal{C}_{V}, \lim_{a \rightarrow +\infty} \mathcal{C}_{(\text{Reg} < a)} \right)[d_{X}] \]
\[ \simeq \text{R}\text{I}\text{hom}\left( \mathcal{C}_{U}, \lim_{a \rightarrow +\infty} \mathcal{C}_{(\text{Reg} < a)} \right)[d_{X}] \]
\[ \simeq \text{R}\text{I}\text{hom}\left( \mathcal{C}_{U}, \lim_{a \rightarrow +\infty} \mathcal{C}_{(\text{Ref} < a)} \right)[d_{X}]. \]

This completes the proof, \( \square \)

5 Several Micro-supports Related to Holonomic D-modules

In this section, we introduce several micro-supports related to holonomic D-modules and study their properties especially for exponentially twisted ones. Let \( X \) be a complex
manifold of dimension $N$ and $E^b_+(\mathbb{C}_X)$ the full subcategory of $E^b(\mathbb{C}_X)$ consisting of objects $F \in E^b(\mathbb{C}_X)$ such that $F \simeq C_{\{t \geq 0\}} \otimes F$. If for a holonomic $D_X$-module $\mathcal{M} \in \text{Mod}_{\text{hol}}(D_X)$ there exists an object $F \in E^b_+(\mathbb{C}_X)$ of $E^b(\mathbb{C}_X)$ such that

$$\text{Sol}^E_\mathcal{M}(\mathcal{M}) \simeq \mathbb{C}_X^+ \otimes F,$$

(5.1)

the following micro-supports of $F$ introduced in Tamarkin [67] are useful in the study of $\mathcal{M}$. Let $X_\mathbb{R}$ be the underlying real analytic manifold of $X$ and by the standard coordinate $(t; t^*)$ of $T^*\mathbb{R}$ identify $(T^*X_\mathbb{R}) \times \mathbb{R}$ with the subset $(T^*X_\mathbb{R}) \times \{t^* = 1\} \subset (T^*X_\mathbb{R}) \times (T^*\mathbb{R})$ of $T^*(X_\mathbb{R} \times \mathbb{R}) \simeq (T^*X_\mathbb{R}) \times (T^*\mathbb{R})$. Let

$$t_\mathbb{R} : (T^*X_\mathbb{R}) \times \mathbb{R} \leftrightarrow T^*(X_\mathbb{R} \times \mathbb{R})$$

(5.2)

be the inclusion map. For an object $F \in E^b_+(\mathbb{C}_X)$ taking $\tilde{F} \in D^b(\mathbb{C}_{X \times \mathbb{R}})$ such that $\mathcal{Q}(\tilde{F}) = F$ we set

$$\text{SS}^E(F) := t_\mathbb{R}^{-1}\text{SS}(\tilde{F}) \subset (T^*X_\mathbb{R}) \times \mathbb{R}.$$ 

(5.3)

This definition does not depend on the choice of $\tilde{F}$ (see [10, Section 2.6]). We call $\text{SS}^E(F)$ the enhanced micro-support of $F$. For a local coordinate $z = x + iy = (z_1, z_2, \ldots, z_N)$ of $X$ let $(x, y; t; x^*, y^*, t^*)$ be the corresponding coordinate of $T^*(X_\mathbb{R} \times \mathbb{R}) \simeq (T^*X_\mathbb{R}) \times (T^*\mathbb{R})$. Then on the open subset $\{t^* > 0\} \subset T^*(X_\mathbb{R} \times \mathbb{R})$ locally we define a real analytic map

$$\gamma : \{t^* > 0\} \longrightarrow (T^*X_\mathbb{R}) \times \mathbb{R},$$

(5.4)

By the conicness of $\text{SS}(\tilde{F})$ we can easily see that

$$\text{SS}^E(F) = \gamma(\text{SS}(\tilde{F}) \cap \{t^* > 0\}).$$

(5.5)

Let $\text{pr}_{T^*X_\mathbb{R}} : (T^*X_\mathbb{R}) \times \mathbb{R} \longrightarrow T^*X_\mathbb{R}$ be the projection and set

$$\text{SS}_{\text{irr}}(F) := \text{pr}_{T^*X_\mathbb{R}}(\text{SS}^E(F)) \subset T^*X_\mathbb{R}$$

(5.6)

(see [10, Section 2.6] for a different notation for it). We call it the irregular micro-support of $F$.

From now on, we assume that $X$ is a smooth algebraic variety over $\mathbb{C}$ and denote by $X^{\text{an}}$ its underlying complex manifold that we sometimes denote by $X$ for short. We set $X_\mathbb{R} := (X^{\text{an}})_\mathbb{R}$. We shall consider the micro-supports of the objects $F \in E^b_+(\mathbb{C}_{X^{\text{an}}})$ related to the following special but basic holonomic $D$-modules. The problem being local, we may assume that $X$ is affine. Then for a rational function $f = \frac{P}{Q} : X \setminus Q^{-1}(0) \longrightarrow \mathbb{C}$ ($P, Q \in \Gamma(X; \mathcal{O}_X), \ Q \neq 0$) on $X$ we set $U := X \setminus Q^{-1}(0)$ and define an algebraic exponential $D_X$-module $\mathcal{E}_{U/X}^f \in \text{Mod}_{\text{hol}}(D_X)$ as in the analytic case. Let $i_U : U \hookrightarrow X$ be the inclusion map and $\mathcal{O}_U(f) \in \text{Mod}_{\text{hol}}(D_U)$ the algebraic integrable connection on $U$ associated to the regular function $f : U \longrightarrow \mathbb{C}$. Then there exist isomorphisms

$$\mathcal{E}_{U/X}^f \cong D_{i_U^*}\mathcal{O}_U(f) \cong (i_U)_*\mathcal{O}_U(f).$$

(5.7)
We define the analytification $(E_{U|X}^{f})^{an} \in \text{Mod}_{hol}(D_{X^{an}})$ of $E_{U|X}^{f}$ by $(E_{U|X}^{f})^{an} := O_{X^{an}} \otimes_{X} E_{U|X}^{f}$ and set

$$\text{Sol}_{X}^{E}(E_{U|X}^{f}) := \text{Sol}_{X}^{E}((E_{U|X}^{f})^{an}) \in \mathcal{E}^{\text{b}}(\mathcal{I}C_{X^{an}}). \quad (5.8)$$

Then by Theorem 4.1 (vi) we have an isomorphism

$$\text{Sol}_{X}^{E}(E_{U|X}^{f}) \cong E_{U|X^{an}}^{\text{Re}f}. \quad (5.9)$$

**Definition 5.1.** We say that a holonomic $D_{X}$-module $\mathcal{M} \in \text{Mod}_{hol}(D_{X})$ is an exponentially twisted holonomic $D$-module if there exist a regular holonomic $D_{X}$-module $\mathcal{N} \in \text{Mod}_{rh}(D_{X})$ and a rational function $f = \frac{P}{Q} : U = X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ ($P, Q \in \Gamma(X; O_{X}), Q \neq 0$) on $X$ such that we have an isomorphism

$$\mathcal{M} \cong \mathcal{N} \otimes E_{U|X}^{f}. \quad (5.10)$$

Let $\mathcal{M} \in \text{Mod}_{hol}(D_{X})$ be an exponentially twisted holonomic $D_{X}$-module such that $\mathcal{M} \cong \mathcal{N} \otimes E_{U|X}^{f}$ for a regular holonomic $D_{X}$-module $\mathcal{N} \in \text{Mod}_{rh}(D_{X})$ and a rational function $f = \frac{P}{Q} : U = X \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ on $X$. Let us set

$$K := \text{Sol}_{X}^{E}(\mathcal{N}) \in \mathcal{D}^{b}_{\mathcal{E}}(X^{an}). \quad (5.11)$$

We denote the support of $K \in \mathcal{D}^{b}_{\mathcal{E}}(X^{an})$ by $\mathcal{Z} \subset X$. Note that $\mathcal{Z} = \text{supp}(\mathcal{N})$ and if $\mathcal{Z} \subset Q^{-1}(0)$ then we have $\mathcal{M} \cong 0$. Hence in what follows, we assume that $\mathcal{Z}$ is not contained in $Q^{-1}(0)$. Then in this algebraic case, considering $f$ as a rational function on a (possibly singular) compactification $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ and using the enhanced solution complex of $\mathcal{M}$, we see that the restriction of $f$ to $\mathcal{Z}$ is uniquely determined by $\mathcal{M}$ modulo constant functions on $\mathcal{Z}$. Note also that the shift $K[N] \in \mathcal{D}^{b}_{\mathcal{E}}(X^{an})$ of $K$ is a perverse sheaf on $X^{an}$. Then by Theorem 4.1 (vi) and Corollary 4.2 for the enhanced sheaf $F := \pi^{-1}K \otimes_{\pi_{X|X^{an}}^{\text{Re}f}} \in \mathcal{E}^{b}_{\mathcal{E}}(\mathcal{C}_{X^{an}})$ on $X^{an}$ we have an isomorphism

$$\text{Sol}_{X}^{E}(\mathcal{M}) \cong \mathcal{C}_{X^{an}}^{+} \otimes F. \quad (5.12)$$

From now on, we shall study the enhanced micro-support $SS^{E}(F) \subset (T^{*}X_{\mathbb{R}}) \times \mathbb{R}$ and the irregular one $SS_{\text{irr}}^{E}(F) \subset T^{*}X_{\mathbb{R}}$ of the enhanced sheaf $F = \pi^{-1}K \otimes_{\pi_{X|X^{an}}^{\text{Re}f}} \in \mathcal{E}^{b}_{\mathcal{E}}(\mathcal{C}_{X^{an}})$. It turns out that over the open subset $T^{*}U_{\mathbb{R}} \subset T^{*}X_{\mathbb{R}}$ of the structures of these two micro-supports are very simple. Define a (not necessarily $\mathbb{C}^{*}$-conic) complex Lagrangian submanifold $\Lambda^{f}$ of $T^{*}U$ by

$$\Lambda^{f} := \{(z, df(z)) \mid z \in U = X \setminus Q^{-1}(0)\} \subset T^{*}U. \quad (5.13)$$

Then we can easily show that via the natural identification $(T_{U})_{\mathbb{R}} \cong T^{*}U_{\mathbb{R}}$ we have

$$SS_{\text{irr}}^{E}(F) \cap T^{*}U_{\mathbb{R}} = \left( SS(K) \cap T^{*}U_{\mathbb{R}} \right) + \Lambda^{f} \quad (5.14)$$

and

$$SS^{E}(F) \cap \left( (T^{*}U_{\mathbb{R}}) \times \mathbb{R} \right) = \left\{ (x, y; x^{*}, y^{*}, -\text{Re}f(x + iy) \mid (x, y; x^{*}, y^{*}) \in \left( SS(K) \cap T^{*}U_{\mathbb{R}} \right) + \Lambda^{f} \right\}.$$

Moreover we have the following results.

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Lemma 5.2. We denote the preimage of the locally closed subset \( Q^{-1}(0) \setminus I(f) = Q^{-1}(0) \setminus P^{-1}(0) \subset X \) by the projection \((T^*X_R) \times \mathbb{R} \longrightarrow X_R\) simply by \{\(P \neq 0, \ Q = 0\}\}. Then we have

\[
SS^F(F) \cap \{P \neq 0, \ Q = 0\} = \emptyset.
\]

(5.15)

In particular, if we assume that \(I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset\), then we have

\[
SS_{tor}(F) = (SS(K) \cap T^*U_R) + \Lambda^f.
\]

(5.16)

Proof. The problem being local, after shrinking \(X\) we may assume that \(I(f) = \emptyset\). First, we consider the special case where \(X = \mathbb{C}_z\), \(K = \mathbb{C}_X\) and \(P(z) = 1, \ Q(z) = z, \ f(z) = \frac{1}{z}\). In this case, we have

\[
F = \mathbb{C}_{\{z \neq 0, \ t + \text{Re}(\frac{1}{z}) \geq 0\}}.
\]

(5.17)

Then it suffices to show that for any point \((0, t_0) \in (Q^{-1}(0) \setminus I(f)) \times \mathbb{R} = Q^{-1}(0) \times \mathbb{R}\) there exists no covector of the form \(\xi dx + dt\) in \(T^*_{(0,t_0)}(X_R \times \mathbb{R}) \cap SS(F)\). For such a point \((0, t_0)\) we can easily see that there exists its neighborhoods \(W, W'\) in \(X_R \times \mathbb{R} \cong \mathbb{R}^3\) and a diffeomorphism \(\Phi : W \sim W'\) between them such that \(\Phi((0, t_0)) = (0, t_0)\),

\[
\Phi\{\{z \neq 0, \ t + \text{Re}(\frac{1}{z}) \geq 0\} \cap W\} = \{z \neq 0, \ t_0 + \text{Re}(\frac{1}{z}) \geq 0\} \cap W'
\]

(5.18)

and the tangent map

\[
T_{(0,t_0)}\Phi : T_{(0,t_0)}(X_R \times \mathbb{R}) \longrightarrow T_{(0,t_0)}(X_R \times \mathbb{R})
\]

(5.19)

of \(\Phi\) at it is the identity. Then locally we can replace \(F\) by the sheaf

\[
F' = \mathbb{C}_{\{z \neq 0, \ t_0 + \text{Re}(\frac{1}{z}) \geq 0\}}
\]

(5.20)

to check the condition. The same argument can be applied even if we replace \(f(x)\) by \(\frac{1}{m^r}\) \((m \geq 1)\). Next consider the case where \(X = \mathbb{C}_z^N\), \(K = \mathbb{C}_X\) and \(P(z) = 1, \ Q(z) = z_{m_1}^r \cdot \ldots \cdot z_{m_k}^r\) \((m_i \geq 1)\), \(f(z) = \frac{1}{Q(z)}\) for some \(1 \leq k \leq N = \dim X\). In this case, for the sheaf

\[
F = \mathbb{C}_{\{Q(z) \neq 0, \ t + \text{Re}(f(z)) \geq 0\}}
\]

(5.21)

on \(X_R \times \mathbb{R}\) we can similarly check the condition. Let us now consider the more general case where \(K = \mathbb{C}_X\) and \(P(z) = 1, \ Q(z) \neq 0 \ f(z) = \frac{1}{Q(z)}\). Let \(\nu : \tilde{X} \longrightarrow X\) be a proper morphism of complex manifolds such that the restriction \(\tilde{X} \setminus \nu^{-1}Q^{-1}(0) \longrightarrow X \setminus Q^{-1}(0)\) of \(\nu\) is an isomorphism and \(\nu^{-1}Q^{-1}(0) \subset \tilde{X}\) is a normal crossing divisor in \(\tilde{X}\). Then for the sheaf \(F\) on \(X_R \times \mathbb{R}\) and the morphism \(\tilde{\nu} := \nu \times \text{id}_R : \tilde{X}_R \times \mathbb{R} \longrightarrow X_R \times \mathbb{R}\) we have the condition

\[
SS^E(\tilde{\nu}^{-1}F) \cap \{P \circ \nu \neq 0, \ Q \circ \nu = 0\} = \emptyset
\]

(5.22)

and there exists an isomorphism

\[
F \sim \mathbb{R}\tilde{\nu}_*\tilde{\nu}^{-1}F.
\]

(5.23)

Hence we can check the condition on \(F\) by [27, Proposition 5.4.4]. Finally, we consider the general case. Let \(S\) be a stratification of \(Z\) adapted to \(K\) such that \(Q^{-1}(0) \cap Z\) is a
Then for the sheaf $S \in \mathcal{S}$ in $\mathcal{S}$ such that $S \subset Z \setminus Q^{-1}(0)$ and any local system $L$ on $S$ we have the condition

$$\text{SS}^E \left( \pi^{-1}(i^!L) \otimes \mathcal{E}^{\text{Re}(f_{\nu})}_{U|X} \right) \cap \{ P \neq 0, \ Q = 0 \} = \emptyset,$$

where $i : S \hookrightarrow X$ is the inclusion map. Let $\nu : T \rightarrow X$ be a proper morphism of complex manifolds such that $\nu(T) = \overline{T}$ and the restriction $T^\circ := T \setminus \nu^{-1}(\overline{T} \setminus S) \rightarrow S$ of $\nu$ is an isomorphism and $\nu^{-1}Q^{-1}(0), \nu^{-1}(\overline{T} \setminus S) \subset T$ are normal crossing divisors in $T$. We thus can identify $T^\circ$ with $S$ and regard $L$ as a local system on $T^\circ$. Let us consider the meromorphic function $f \circ \nu := (P \circ \nu)/(Q \circ \nu)$ on $T$ and the inclusion map $i' : T^\circ \hookrightarrow T$.

Then for the sheaf $\pi^{-1}(i'!L) \otimes \mathcal{E}^{\text{Re}(f_{\nu})}_{U|T^\circ}$ on $T^\circ \times \mathbb{R}$ we can similarly show the condition

$$\text{SS}^E \left( \pi^{-1}(i'!L) \otimes \mathcal{E}^{\text{Re}(f_{\nu})}_{U|T^\circ} \right) \cap \{ P \circ \nu \neq 0, \ Q \circ \nu = 0 \} = \emptyset.$$

Moreover, for the morphism $\tilde{\nu} := \nu \times \text{id}_\mathbb{R} : T^\circ \times \mathbb{R} \rightarrow X_{\mathbb{R}} \times \mathbb{R}$ there exists an isomorphism

$$\pi^{-1}(i'!L) \otimes \mathcal{E}^{\text{Re}(f_{\nu})}_{U|X} \xrightarrow{\sim} \tilde{\nu}_* \left( \pi^{-1}(i_!L) \otimes \mathcal{E}^{\text{Re}(f_{\nu})}_{U|T^\circ} \right).$$

Then we obtain (5.24) by [27, Proposition 5.4.4]. This completes the proof.

**Lemma 5.3.** Assume that there exists a stratification $\mathcal{S}$ of $Z$ adapted to $K$ such that any stratum $S \in \mathcal{S}$ in it is not contained in $P^{-1}(0) \cap Z \subset Z$ nor $Q^{-1}(0) \cap Z \subset Z$ and the meromorphic function $f|_{S \setminus Q^{-1}(0)} : S \setminus Q^{-1}(0) \rightarrow \mathbb{C}$ on $S$ satisfies the condition: For any point of $S \cap P^{-1}(0) \cap Q^{-1}(0)$ there exists a local coordinate $z_1, z_2, \ldots, z_l$ ($l := \dim S$) of $S$ around it such that $(P|_S)(z_1, \ldots, z_l) = z_1$, $(Q|_S)(z_1, \ldots, z_l) = z_2^m$ for some $m \geq 1$ and hence

$$(f|_S)(z_1, \ldots, z_l) = \frac{z_1}{z_2^m}.$$  

We denote the preimage of the closed subset $I(f) = P^{-1}(0) \cap Q^{-1}(0) \subset X$ by the projection $(T^*X_{\mathbb{R}}) \times \mathbb{R} \rightarrow X_{\mathbb{R}}$ simply by $\{ P = Q = 0 \} \subset (T^*X_{\mathbb{R}}) \times \mathbb{R}$. Then we have

$$\text{SS}^E(F) \cap \{ P = Q = 0 \} = \emptyset.$$  

**Proof.** The problem being local, we may assume that locally $X = \mathbb{C}^N$. First, we consider the special case where $N = 2$, $Z = X = \mathbb{C}^2$, $K = \mathbb{C}X$ and $P(z_1, z_2) = z_1$, $Q(z_1, z_2) = z_2$, $f(z_1, z_2) = \frac{z_1}{z_2}$. We set $z = (z_1, z_2) = (x, y)$ so that we have $f(x, y) = \frac{x}{y}$. We shall show $\text{SS}^E(F) \cap \{ x = y = 0 \} = \emptyset$. By an explicit calculation, we can easily show that the closure of the set

$$\text{SS}^E(F) \setminus \{ x = y = 0 \} \subset (T^*X_{\mathbb{R}}) \times \mathbb{R} \setminus \{ x = y = 0 \}$$

in $(T^*X_{\mathbb{R}}) \times \mathbb{R}$ does not intersect $\{ x = y = 0 \}$. Let $\nu : \tilde{X} \rightarrow X$ be the blow-up of $X = \mathbb{C}^2$ along the origin $\{(0, 0)\} \subset X = \mathbb{C}^2$ and $M = \mathbb{C}_{u,y}^2$ the affine chart of $\tilde{X}$ such that we have

$$\nu|_M : M = \mathbb{C}_{u,y}^2 \rightarrow X = \mathbb{C}^2 \quad ((u, y) \mapsto (uy, y)).$$

Then for the exceptional divisor $\nu^{-1}\{(0, 0)\} \simeq \mathbb{P} := \mathbb{P}^1$ we have $\nu^{-1}\{(0, 0)\} \cap M = \mathbb{C}_{u} \times \{ 0 \} \subset M = \mathbb{C}_{u,y}^2$ and $(f \circ \nu)(u, y) = \frac{uy}{y} = u$ on $M = \mathbb{C}_{u,y}^2$. Let $M' = \mathbb{C}_{x,v}^2$ be the affine chart of $\tilde{X}$ such that we have

$$\nu|_{M'} : M' = \mathbb{C}_{x,v}^2 \rightarrow X = \mathbb{C}^2 \quad ((x, v) \mapsto (x, vx)).$$
Then we have $M \cup M' = \tilde{X}$, $\nu^{-1}((0,0)) \cap M' = \{0\} \times \mathbb{C}_v = \{x = 0\} \subset M' = \mathbb{C}^2_{x,v}$ and $(f \circ \nu)(x,v) = \frac{u}{v} = \frac{1}{v}$ on $M' = \mathbb{C}^2_{x,v}$. Hence the meromorphic function $f \circ \nu$ on $\tilde{X}$ has no point of indeterminacy on the whole $\tilde{X}$ and its pole is contained in $\tilde{X} \setminus M = \{v = 0\} \subset M' = \mathbb{C}^2_{x,v}$. Now we define an enhanced sheaf $\tilde{F}$ on $\tilde{X}$ by

$$\tilde{F} := E_{\nu}^{\text{Re}(f \circ \nu)}(\mathbb{C}^{\mathbb{R}}_{\tilde{X}}) \in E_b^{\mathbb{R}}(\mathbb{C}^{\mathbb{R}}_{\tilde{X}})$$

and set $\tilde{\nu} : \tilde{X} \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ so that there exists an isomorphism

$$R\tilde{\nu}_*\tilde{F} \simeq F.$$  

Let

$$T^*(X \times \mathbb{R}) \xleftarrow{\tilde{\nu}^*} (\tilde{X} \times \mathbb{R}) \times_{(X \times \mathbb{R})} T^*(X \times \mathbb{R}) \xrightarrow{\rho} T^*(\tilde{X} \times \mathbb{R})$$

be the natural morphisms associated to $\tilde{\nu}$. Then by [27 Proposition 5.4.4] we have

$$SS(F) \subset \varpi \rho^{-1}SS(\tilde{F}).$$

Let

$$(0,0,t; \xi, \eta, 1) \in \{t^* > 0\} \subset T^*(X \times \mathbb{R})$$

be a point which corresponds to the covector $\xi dx + \eta dy + dt \in T_{(0,0,t)}(X \times \mathbb{R})$ at the point $(0,0,t) \in \{P = Q = 0\} \subset X \times \mathbb{R}$. Then its pull-back by the map

$$\tilde{\nu}|_{M \times \mathbb{R}} : M \times \mathbb{R} \longrightarrow X \times \mathbb{R} \quad ((u, y, t) \longmapsto (uy, y, t))$$

is calculated as

$$(\tilde{\nu}|_{M \times \mathbb{R}})^* (\xi dx + \eta dy + dt) = (\xi u + \eta) dy + dt.$$  

Since $\text{Re}(f \circ \nu)(u, y) = \text{Re}u$ on $M = \mathbb{C}^2_{u,y}$ and hence together with Lemma 5.2 there is no covector of the form $(\xi u + \eta) dy + dt$ in $\{t^* = 1\} \cap SS(\tilde{F})$ over the exceptional divisor $\nu^{-1}((0,0))$, by (5.35) we obtain the assertion $SS^E(F) \cap \{x = y = 0\} = \emptyset$. We can treat also the case where $N = 2$, $Z = X = \mathbb{C}^2, K = \mathbb{C}_X$ and $P(z_1, z_2) = z_1, Q(z_1, z_2) = z_2^m, f(z_1, z_2) = \frac{z_1^m}{z_2^m}$ for some $m \geq 2$ by repeating blow-ups (see e.g. the proof of [45 Theorem 3.6]). The general case can be treated similarly by decomposing the support of $K$ with respect to the stratification $\mathcal{S}$. This completes the proof.

By the natural identification $(T^*X)_\mathbb{R} \simeq T^*X_{\mathbb{R}}$ we regard $T^*X_{\mathbb{R}}$ as a complex symplectic manifold endowed with the canonical holomorphic symplectic 2-form $\sigma_\mathbb{R}$.

**Lemma 5.4.** The irregular micro-support $SS_{\text{irr}}(F) \subset T^*X_{\mathbb{R}}$ of $F = \pi^{-1}K \otimes E^{\text{Re}f}_{U|Z} \in E_b^{\mathbb{R}}(\mathbb{C}^{\mathbb{R}}_{\mathbb{C}_X})$ is contained in a complex isotropic analytic subset of $(T^*X)_{\mathbb{R}} \simeq T^*X_{\mathbb{R}}$.

**Proof.** Let us consider the exact sequence

$$0 \longrightarrow F_{\pi^{-1}(X \setminus P^{-1}(0))} \longrightarrow F \longrightarrow F_{\pi^{-1}(P^{-1}(0))} \longrightarrow 0$$

of enhanced sheaves on $X$. Since there exists an isomorphism

$$F_{\pi^{-1}(P^{-1}(0))} \simeq \pi^{-1}K \otimes \mathbb{C}_{(z, t) \in U \times \mathbb{R} | z \in U \cap P^{-1}(0), t \geq 0},$$

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we see that $SS_{\text{irr}}(F_{\pi^{-1}(P^{-1}(0))}) = SS(K_{U \cap P^{-1}(0)})$ is a Lagrangian analytic subset of $(T^*X)_R \simeq T^*X_R$. Let $\nu : \tilde{X} \to X$ be a proper morphism of complex manifolds such that the restriction $\tilde{X} \setminus \nu^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \to X \setminus (P^{-1}(0) \cup Q^{-1}(0))$ of $\nu$ is an isomorphism, $\nu^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \subset \tilde{X}$ is a normal crossing divisor, and the meromorphic function $f \circ \nu := (P \circ \nu)/(Q \circ \nu)$ on $\tilde{X}$ has no point of indeterminacy on the whole $\tilde{X}$ (see e.g. the proof of [15, Theorem 3.6]). We thus obtain a function $f \circ \nu : \nu^{-1}(1) = \tilde{X} \setminus (Q \circ \nu)^{-1}(0) \to C$. Moreover we define a complex Lagrangian submanifold $\Lambda^{f_{\nu}} \subset T^*\tilde{X}$ by

$$\Lambda^{f_{\nu}} := \{(z, d(f \circ \nu)(z)) | z \in \nu^{-1}(U)\} \subset T^*\tilde{X} \quad (5.41)$$

and set $\tilde{\nu} := \nu \times \text{id}_R : \tilde{X} \times R \to X \times R$. Then for the enhanced sheaf $\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))}$ on $\tilde{X}$ we can easily show that

$$SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))}) = SS(\nu^{-1}K_{U \cap P^{-1}(0)}) + \Lambda^{F_{\nu}} \quad (5.42)$$

in $T^*\tilde{X}$. This implies that $SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))})$ is a Lagrangian analytic subset of $T^*\tilde{X}$. Let

$$T^*X \xleftarrow{\tilde{\nu}} \tilde{X} \times_X T^*X \xrightarrow{\text{pr}} T^*\tilde{X}$$

be the natural morphisms associated to $\nu : \tilde{X} \to X$. Since $\nu$ is proper and there exists an isomorphism

$$F_{\pi^{-1}(X \setminus P^{-1}(0))} \sim \text{R}\tilde{\nu}_*\tilde{\nu}^{-1}(F_{\pi^{-1}(X \setminus P^{-1}(0))}) \quad (5.44)$$

by [27, Proposition 5.4.4] we have

$$SS_{\text{irr}}(F_{\pi^{-1}(X \setminus P^{-1}(0))}) \subset \varpi^{-1}SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))}) \quad (5.45)$$

It is clear that the right hand side is an analytic subset of $T^*X$. Moreover, by [13, page 43] there exist some stratifications of it and $\varpi^{-1}SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))})$ such that the morphism

$$\varpi^{-1}SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))}) \to \varpi\rho^{-1}SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))})$$

induced by $\varpi$ is a stratified fiber bundle over each stratum. Then we can slightly modify the proof of [21, Proposition A.54] to show that the symplectic 2-form $\sigma_X$ vanishes on $\varpi\rho^{-1}SS_{\text{irr}}(\tilde{\nu}^{-1}F_{\pi^{-1}(X \setminus P^{-1}(0))})$. We thus obtain the assertion. \(\square\)

In the special case where $X$ is $\mathbb{C}^N_2$, $Y$ is its dual vector space $\mathbb{C}^N_w$ and $T^*X \simeq X \times Y$, we shall introduce a new subset of $(T^*X)_R \simeq T^*X_R$ as follows. For a point $w_0 \in Y = \mathbb{C}^N$ we define a rational function $f_{w_0}^w$ on $X = \mathbb{C}^N$ by

$$f_{w_0}^w : U = X \setminus Q^{-1}(0) \to \mathbb{C} \quad (z \mapsto (z, w_0) - f(z)). \quad (5.47)$$

**Definition 5.5.** In the case where $X$ is $\mathbb{C}^N_2$, $Y$ is its dual vector space $\mathbb{C}^N_w$ and $T^*X \simeq X \times Y$, by using $K$ and $f$ in $F = \pi^{-1}K \otimes \text{Ref}_{U \setminus X_{an}}$ we define a subset $SS_{\text{eva}}(F) \subset (T^*X)_R$ of $(T^*X)_R \simeq T^*X_R$ by

$$SS_{\text{eva}}(F) := \{(z_0, w_0) \in (T^*X)_R | \phi_{f_{w_0}^w}^{\text{mero}, c}(K)_{z_0} \neq 0 \text{ for some } c \in \mathbb{C}\}. \quad (5.48)$$

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Unfortunately, we do not know if this set $SS_{\text{eva}}(F) \subset T^*X_R$ is contained in the irregular micro-support $SS_{\text{irr}}(F)$ of $F$ or not. With Lemmas 5.2, 5.3 and 5.4 at hands, now we have obtained a rough picture of the irregular micro-support $SS_{\text{irr}}(F) \subset T^*X_R$. However, for the moment it is not clear for us if it is a complex Lagrangian analytic subset of $(T^*X)_R \simeq T^*X_R$ or not. For these reasons, we define a (not necessarily homogeneous) complex Lagrangian analytic subset of $(T^*X)_R \simeq T^*X_R$ in a different way as follows and use it instead of $SS_{\text{irr}}(F)$. We return to the case where $X$ is a smooth algebraic variety over $\mathbb{C}$. First, as a complex analogue of the $\mathbb{R}$-constructible sheaf $F = \pi^{-1}K \otimes E^{\text{ref}}_{\text{trans}}$ on $X_R \times \mathbb{R}$, by applying the proof of Theorem 2.4 (i) to the (not necessarily) closed embedding

$$i_{-f} : U = X \setminus Q^{-1}(0) \hookrightarrow X \times \mathbb{C}, \quad (z \mapsto (z, -f(z)))$$

(5.49)

associated to the function $-f : U \longrightarrow \mathbb{C}$, we obtain a complex constructible sheaf

$$\mathcal{F} := (i_{-f})_!(K|_U) \in D^b_c(X \times \mathbb{C})$$

(5.50)

on $X \times \mathbb{C}$. Then for the morphism $id_X \times \text{Re} : X \times \mathbb{C} \longrightarrow X \times \mathbb{R}$ induced by the one $\text{Re} : \mathbb{C} \longrightarrow \mathbb{R} (\tau \longmapsto \text{Re}\tau)$ we have an isomorphism

$$F \simeq C_{\{t \geq 0\}} \otimes \text{Re}(id_X \times \text{Re})_!\mathcal{F}.$$  

(5.51)

Note that $\mathcal{F}[N]$ is a perverse sheaf and its micro-support $SS(\mathcal{F}[N]) = SS(\mathcal{F})$ is a homogeneous complex Lagrangian analytic subset of $T^*(X \times \mathbb{C})$. Then as in the definitions of $SS^E(F) \subset (T^*X)_R \times \mathbb{R}$ and $SS_{\text{irr}}(F) \subset T^*X_R$, by forgetting the homogeneity of $SS(\mathcal{F})$ we define two subsets:

$$SS^E, C(\mathcal{F}) \subset T^*X \times \mathbb{C}, \quad SS_{\text{irr}}^C(\mathcal{F}) \subset T^*X \quad (5.52)$$

in the following way. First, by the closed embedding

$$\iota : (T^*X) \times \mathbb{C} \hookrightarrow (T^*(X \times \mathbb{C}) \quad (((z, w), \tau) \longmapsto (z, \tau; w, 1))$$

(5.53)

we set

$$SS^E, C(\mathcal{F}) := \iota^{-1}SS(\mathcal{F}) \subset (T^*X) \times \mathbb{C}.$$  

(5.54)

It is clear that $SS^E, C(\mathcal{F})$ is a complex analytic subset of $(T^*X) \times \mathbb{C}$. Let us call it the enhanced micro-support of $\mathcal{F} \in D^b_c(X \times \mathbb{C})$. Then we define $SS_{\text{irr}}^C(\mathcal{F}) \subset T^*X$ to be the closure of the image of $SS^E, C(\mathcal{F})$ by the projection $(T^*X) \times \mathbb{C} \longrightarrow T^*X$. We call it the irregular micro-support of $\mathcal{F}$. As the restriction of $f$ to $Z$ is uniquely determined by $\mathcal{M}$ modulo constant functions on $Z$, we see that $SS_{\text{irr}}^C(\mathcal{F})$ does not depend on the expression $\mathcal{N} \otimes \mathcal{E}_{U|X}$ of $\mathcal{M}$. It is easy to see that the results analogous to Lemmas 5.2 and 5.3 holds true also for $SS_{\text{irr}}^C(\mathcal{F}) \subset T^*X$.

**Lemma 5.6.** The irregular micro-support $SS_{\text{irr}}^C(\mathcal{F}) \subset T^*X$ of $\mathcal{F} \in D^b_c(X \times \mathbb{C})$ is a complex Lagrangian analytic subset of $(T^*X)_R \simeq T^*X_R$. If we assume moreover that $I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset$, then we have

$$SS_{\text{irr}}^C(\mathcal{F}) = (SS(K) \cap T^*U_R) + \Lambda^f.$$  

(5.55)

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Proof. By the inclusion map $j : X \times \mathbb{C} \hookrightarrow X \times \mathbb{P}$ we obtain an algebraic constructible sheaf

$$j_*\mathcal{F} = j_!(i_*j^!(K|_U)) \in D^b(X \times \mathbb{P})$$

(5.56)
on $X \times \mathbb{P}$. Then by the theorem in [15, page 43] there exist Whitney stratifications $\mathcal{S}$ and $\mathcal{S}_0$ of $X \times \mathbb{P}$ and $X$ respectively such that

$$\text{SS}(j_*\mathcal{F}) \subseteq \bigcup_{S \in \mathcal{S}} T^*_S(X \times \mathbb{P}),$$

(5.57)

$X \times \mathbb{C}$ is a union of some strata in $\mathcal{S}$ and the projection $X \times \mathbb{P} \rightarrow X$ is a stratified fiber bundle as in its assertion. Note that for each stratum $S_0 \in \mathcal{S}_0$ in $\mathcal{S}_0$ there exist only finitely many strata $S \in \mathcal{S}$ projecting to it such that $S \subseteq X \times \mathbb{C}$, dim $S = \dim S_0$ and the enhanced micro-support $\text{SS}^{E, \mathbb{C}}(\mathcal{F}) \subseteq (T^*X) \times \mathbb{C}$ of $\mathcal{F}$ is determined only by the conormal bundles $T^*_S(X \times \mathbb{P})$ of such “horizontal” strata $S \in \mathcal{S}$ such that $S \subseteq X \times \mathbb{C}$. Then we obtain the first assertion by simple calculations. We obtain also the second one as in the proof of Lemma 5.2.

By the proof of Lemma 5.6, considering also the multiplicities of $\mathcal{F}$ we obtain a Lagrangian cycle in $T^*X$ supported by the irregular micro-support $\text{SS}^{E, \mathbb{C}}(\mathcal{F}) \subseteq (T^*X) \times \mathbb{C}$ of $\mathcal{F}$ as in the proof of Lemma 5.2.

Similarly to the case of $\mathcal{M}$, we call it the irregular characteristic cycle of $\mathcal{M}$ and denote it by $\text{CC}_{\text{irr}}(\mathcal{M})$. Moreover it satisfies also the following functorial property. Let $\phi : X \rightarrow Y$ be a proper morphism of smooth algebraic varieties and $\mathcal{M}' := D\phi_*\mathcal{M} \in D^b_{\text{hol}}(D_Y)$ the direct image of the exponentially twisted holonomic $D_X$-module $\mathcal{M}$ by it. Then by Theorem 4.3 (iii) for the enhanced sheaf $F' := E\phi_*\mathcal{F}[d_X - d_Y] \in E^b_Y(C_{\text{Yan}})$ on $Y^{\text{an}}$ we have an isomorphism

$$\text{Sol}^E_Y(\mathcal{M}') \simeq C^+_Y \otimes F'.$$

(5.58)

Similarly to the case of $\mathcal{M}$, also for $\mathcal{M}' = D\phi_*\mathcal{M}$ we can define a Lagrangian cycle in $T^*Y$ by using the complex constructible sheaf $\mathcal{F}' := R(\phi \times \text{id}_\mathbb{C})_*\mathcal{F}[d_X - d_Y] \in D^b_{\text{hol}}(Y \times \mathbb{C})$. Denote it by $\text{CC}_{\text{irr}}(\mathcal{M}')$ and let

$$T^*Y \leftarrow \mathbb{C} \times Y T^*Y \xrightarrow{\rho} T^*X$$

(5.59)

be the natural morphisms associated to $\phi : X \rightarrow Y$. Then there exists an equality

$$\text{CC}_{\text{irr}}(\mathcal{M}') = \varpi_*\rho^*\text{CC}_{\text{irr}}(\mathcal{M})$$

(5.60)

of Lagrangian cycles in $T^*Y$ (see [27, Proposition 9.4.2]). Note also that by the commutative diagram

$$\begin{array}{ccc}
X \times \mathbb{C} & \xrightarrow{\text{id}_X \times \text{Re}} & X \times \mathbb{R} \\
\phi \times \text{id}_\mathbb{C} & \downarrow & \phi \times \text{id}_\mathbb{R} \\
Y \times \mathbb{C} & \xrightarrow{\text{id}_Y \times \text{Re}} & Y \times \mathbb{R}
\end{array}$$

(5.61)

we obtain an isomorphism

$$F' \simeq C_{\{t \geq 0\}} \otimes R(\text{id}_Y \times \text{Re})_*\mathcal{F}'.$$

(5.62)

Also for a morphism $\psi : Y \rightarrow X$ of smooth algebraic varieties and the inverse image $\mathcal{M}' := D\psi^*\mathcal{M} \in D^b_{\text{hol}}(D_Y)$ of $\mathcal{M}$ by it, we can define a Lagrangian cycle $\text{CC}_{\text{irr}}(\mathcal{M}')$.
in $T^*Y$ and obtain a formula which expresses it in terms of $CC_{irr}(M)$ under some non-characteristic condition. Note that in this case the inverse image $M' = D\psi^*M$ is also of exponentially twisted type.

**Proposition 5.7.** In the case where $X$ is $\mathbb{C}^N$, $Y$ is its dual vector space $\mathbb{C}^N_w$ and $T^*X \simeq X \times Y$, for a point $(z_0, w_0)$ of $T^*X \simeq X \times Y$ assume that there exists a complex number $c \in \mathbb{C}$ such that

$$\phi_{\omega_{0,-c}}^\text{mero}(K)_{z_0} \neq 0$$

and set $\tau_0 := c - (z_0, w_0) \in \mathbb{C}$. Then we have $(z_0, w_0, \tau_0) \in SS_{\mathbb{C}}(F)$, equivalently $(z_0, \tau_0, w_0, 1) \in SS(F)$.

**Proof.** Let us consider the (not necessarily) closed embedding

$$i_{f^w_0} : U = X \setminus Q^{-1}(0) \hookrightarrow X \times \mathbb{C}, \quad (z) \mapsto (z, f^w_0(z))$$

associated to the rational function $f^w_0 : U \rightarrow \mathbb{C}$. Then we obtain a non-vanishing

$$\phi_{\omega_{0,-c}}^\text{mero}(K)(z_0) \simeq \phi_{\omega_{0,-c}}^\text{mero}(K)_{z_0} \neq 0.$$  \hspace{1cm} (5.63)

By [27, Proposition 8.6.3], this implies that the point $(z_0, 0, 1) \in T^*(X \times \mathbb{C})$ is contained in $SS((i_{f^w_0})!(K|_U)) \subset T^*(X \times \mathbb{C})$. On the other hand, for the automorphism $T_{w_0}$ of $X \times \mathbb{C}$ defined by

$$T_{w_0} : X \times \mathbb{C} \rightarrow X \times \mathbb{C}, \quad ((z, \tau)) \mapsto (z, \tau + (z, w_0))$$

we have an isomorphism

$$(T_{w_0})*(F) = (T_{w_0})*(i_!)(K|_U) \simeq (i_{f^w_0})!(K|_U).$$  \hspace{1cm} (5.65)

Then the assertion $(z_0, \tau_0, w_0, 1) \in SS(F)$ immediately follows from it. \hfill $\square$

By Proposition 5.7, the subset $SS_{\text{eva}}(F) \subset (T^*X)_R$ is contained in the irregular micro-support $SS_{irr}^C(F)$ of $F \in \mathcal{D}_{\text{hol}}(X \times \mathbb{C})$. It is also clear that we have

$$SS_{irr}^C(F) \cap T^*U_R = (SS(K) \cap T^*U_R) + \Lambda_f.$$  \hspace{1cm} (5.68)

### 6 Fourier Transforms of Exponentially Twisted Holonomic D-modules

In this section, we study the Fourier transforms of exponentially twisted holonomic D-modules. We inherit the situation and the notations in Section 4. Let

$$X \leftarrow P X \times Y \rightarrow q Y$$

be the projections. Then by Katz-Laumon [33], for an algebraic holonomic $\mathcal{D}_X$-module $M \in \text{Mod}_{\text{hol}}(\mathcal{D}_X)$ we have an isomorphism

$$M^\wedge \simeq Dq_*(\mathcal{D}p^*M \otimes \mathcal{O}_{X \times Y}e^{-(z,w)})$$  \hspace{1cm} (6.2)
where $Dp^*, Dq_*, \otimes^D$ are the operations for algebraic D-modules and $O_{X \times Y} e^{-(z,w)}$ is the integral connection of rank one on $X \times Y$ associated to the canonical paring $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$. In particular the right hand side is concentrated in degree zero. Let $\tilde{X} \simeq \mathbb{P}^N$ (resp. $\tilde{Y} \simeq \mathbb{P}^N$) be the projective compactification of $X$ (resp. $Y$). By the inclusion map $i_X : X = \mathbb{C}^N \hookrightarrow \tilde{X} = \mathbb{P}^N$ we extend a holonomic $D_X$-module $\mathcal{M} \in \text{Mod}_\text{hol}(D_X)$ on $X$ to the one $\tilde{\mathcal{M}} := i_X^* \mathcal{M} \simeq D\tilde{i}_{X,*} \mathcal{M}$ on $\tilde{X}$. Denote by $\tilde{X}^\text{an}$ the underlying complex manifold of $\tilde{X}$ and define the analytification $\tilde{\mathcal{M}}^\text{an} \in \text{Mod}_\text{hol}(D_{\tilde{X}^\text{an}})$ of $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{M}}^\text{an} := O_{\tilde{X}^\text{an}} \otimes_{\tilde{O}_{\tilde{X}}} \tilde{\mathcal{M}}$. Then we set

$$\text{Sol}^E_{\tilde{X}}(\tilde{\mathcal{M}}) := \text{Sol}^E_{\tilde{X}^\text{an}}(\tilde{\mathcal{M}}^\text{an}) \in \text{E}^b(\text{IC}_{\tilde{X}^\text{an}}).$$

Similarly for the Fourier transform $\tilde{\mathcal{M}}^\text{\wedge} \in \text{Mod}_\text{hol}(D_Y)$, by the inclusion map $i_Y : Y = \mathbb{C}^N \hookrightarrow \tilde{Y} = \mathbb{P}^N$ we define $\tilde{\mathcal{M}}^\text{\wedge}$ and $\text{Sol}^E_{\tilde{Y}}(\tilde{\mathcal{M}}^\text{\wedge}) \in \text{E}^b(\text{IC}_{\tilde{Y}^\text{an}})$.

Let

$$\tilde{X}^{\text{an}} \xrightarrow{\tilde{p}} \tilde{Y}^{\text{an}} \xrightarrow{\tilde{f}} \tilde{Y}^\text{an}$$

be the projections. Then the following theorem is essentially due to Kashiwara-Schapira [32] and D’Agnolo-Kashiwara [10]. For $F \in \text{E}^b(\text{IC}_{\tilde{Y}^\text{an}})$ we set

$$L^F := \text{E}T_* \left( \text{E}^b \text{F} \right) \otimes \text{E}^{\text{Re}(z,w)}_{X \times Y|X \times Y}[N]) \in \text{E}^b(\text{IC}_{\tilde{Y}^\text{an}})$$

(here we denote $X^{\text{an}} \times Y^{\text{an}}$ etc. by $X \times Y$ etc. for short) and call it the Fourier-Sato (Fourier-Laplace) transform of $F$.

**Theorem 6.1.** For $\mathcal{M} \in \text{Mod}_\text{hol}(D_X)$ there exists an isomorphism

$$\text{Sol}^E_{\tilde{Y}}(\tilde{\mathcal{M}}^\text{\wedge}) \simeq L \text{Sol}^E_{\tilde{X}}(\tilde{\mathcal{M}}).$$

By [8, Lemma 2.5.1] we can take an enhanced sheaf $F \in \text{E}^b(\mathbb{C}_{X^\text{an}})$ on $X^\text{an}$ such that $F \simeq \mathbb{C}_{(t \geq 0)} \otimes^D F$ and

$$\text{Sol}^E_{\tilde{X}}(\tilde{\mathcal{M}}) \simeq \mathbb{C}^E_{X^\text{an}} \otimes \text{E}i_{X!} F.$$

For an enhanced sheaf $G \in \text{E}^b(\mathbb{C}_{\tilde{Y}^\text{an}})$ on $\tilde{Y}^\text{an}$ we define its Fourier-Sato (Fourier-Laplace) transform $L^G \in \text{E}^b(\mathbb{C}_{\tilde{Y}^\text{an}})$ by

$$L^G := \text{E}T_* \left( \text{E}^b \text{G} \right) \otimes \text{E}^{\text{Re}(z,w)}_{X \times Y|X \times Y}[N]) \in \text{E}^b(\mathbb{C}_{\tilde{Y}^\text{an}}).$$

Since by [9, Proposition 4.7.17] we have

$$L^\left( \mathbb{C}^E_{X^\text{an}} \otimes (\cdot) \right) \simeq \mathbb{C}^E_{\tilde{Y}^\text{an}} \otimes L(\cdot),$$

by Theorem 6.1 and (6.7) we obtain an isomorphism

$$\text{Sol}^E_{\tilde{X}}(\tilde{\mathcal{M}}) \simeq \mathbb{C}^E_{\tilde{Y}^\text{an}} \otimes L(\text{E}i_{X!} F).$$

Hence it suffices to study the enhanced sheaf

$$L(\text{E}i_{X!} F) \simeq \text{E}i_{Y!} \text{E}q_!(\text{E}p^* F \otimes \text{E}^{\text{Re}(z,w)}_{X \times Y|X \times Y}[N]) \in \text{E}^b(\mathbb{C}_{\tilde{Y}^\text{an}}).$$
exists an isomorphism

\[ \exists \text{ Fourier transform} \]

such that \( M \cong N \). Moreover for the enhanced sheaf

\[ \chi : T^*X^{an} \rightarrow T^*Y^{an}, \quad (z, w) \mapsto (w, -z). \]

Theorem 6.2 (Theorem 3.6). For \( F \in E_+^h(C_{X^{an}}) \) and \( L(F) = Eq(Ep^{-1}F \otimes E_{-Re(z,w)}^{X,Y}[N]) \in E_+^h(C_{Y^{an}}) \) we have

\[ SS^E(L(F)) = \tilde{\chi}(SS^E(F)), \quad SS_{irr}(L(F)) = \chi(SS_{irr}(F)). \]

For a rational function \( f = \frac{P}{Q} : X \setminus Q^{-1}(0) \rightarrow C \) \( (P, Q) \in \Gamma(X; O_X) \cong \mathbb{C}[z_1, z_2, \ldots, z_N], \ Q \neq 0 \) on \( X = \mathbb{C}^n \) we set \( U := X \setminus Q^{-1}(0) \) and define an exponential \( D_X \)-module \( E_{U/X}^f \in \text{Mod}_{hol}(D_X) \) as in the analytic case. Then by Theorem 6.1 (vi) for its extension \( (E_{U/X}^f)^{-} := i_{X!}E_{U/X}^f \simeq Di_{X!}E_{U/X}^f \) to \( X = \mathbb{P}^N \) we have an isomorphism

\[ Sol_{E}(E_{U/X}^f)^{-} \simeq E_{U^{an},X^{an}}^{Ref}. \]

From now on, we fix an exponentially twisted holonomic \( D_X \)-module \( M \in \text{Mod}_{hol}(D_X) \) such that \( M \cong N \otimes E_{U/X}^f \) for a regular holonomic \( D_X \)-module \( N \in \text{Mod}_{nh}(D_X) \) and a rational function \( f = \frac{P}{Q} : U = X \setminus Q^{-1}(0) \rightarrow C \) on \( X \) and study the basic properties of its Fourier transform \( M^\wedge \). Let us set

\[ K := Sol_X(N) \in D_c^b(X^{an}). \]

Note that the shift \( K[N] \in D^b_c(X^{an}) \) of \( K \) is a perverse sheaf on \( X^{an} \). We denote the support of \( K \in D^b_c(X^{an}) \) by \( Z \subset X \). Then by Theorem 4.1 (vi) and Corollary 4.2 there exists an isomorphism

\[ Sol^E_{X}(\hat{M}) \simeq \{ \pi^{-1}(i_X)_!K \} \otimes E_{U^{an},X^{an}}^{Ref}. \]

Moreover for the enhanced sheaf

\[ \{ \pi^{-1}(i_X)_!K \} \otimes E_{U^{an},X^{an}}^{Ref} \simeq E_{i_X!}^{\Gamma} \{ \pi^{-1}K \otimes E_{U^{an}|X^{an}}^{Ref} \} \quad \in E_+^h(C_X^{an}) \]

on \( X^{an} \) we have an isomorphism

\[ Sol^E_{X}(\hat{M}) \simeq C_X^{an} \otimes E_{U^{an}|X^{an}}^{Ref}. \]

It follows from Theorem 6.1 that there exists an isomorphism

\[ Sol^E_{X}(\hat{M}^\wedge) \simeq C_X^{an} \otimes E_{U^{an}|X^{an}}^{Ref}. \]
Hence it suffices to study the enhanced sheaf

$$L\left\{ \{ \pi^{-1}(i_X)_! K \} \otimes E^{\Ref}_{U^\an|X^\an} \right\} \in E^b(C^\an_{X^\an}).$$  \hfill (6.21)

From now on, for the enhanced sheaf

$$F := \pi^{-1} K \otimes E^{\Ref}_{U^\an|X^\an} \in E^b(C^\an_{X^\an})$$  \hfill (6.22)
on $X^\R$ we shall study $L(F) = \mathcal{E}_F(\mathcal{E}_F^{-1} F \otimes E^{\Ref}_{X \times Y(x) | X \times Y([N])}) \in E^b(C^\an_{Y^\an})$. For this purpose, as a complex analogue of the $\R$-constructible sheaf $F = \pi^{-1} K \otimes E^{\Ref}_{U^\an|X^\an}$ on $X^\R \times \R$, by applying the proof of Theorem 2.4 (i) to the (not necessarily) closed embedding

$$i^{-1}_f : U = X \setminus Q^{-1}(0) \hookrightarrow X \times \C, \quad (z \mapsto (z, -f(z)))$$  \hfill (6.23)

associated to the rational function $-f : U \longrightarrow \C$, we obtain a complex constructible sheaf

$$\mathcal{I} := (i^{-1}_f)_!(\mathcal{K}|_U) \in D^b_c(X \times \C)$$  \hfill (6.24)
on $X \times \C$. Note that $\mathcal{I}[N]$ is a perverse sheaf and its micro-support $SS(\mathcal{I}[N]) = SS(\mathcal{I})$ is a homogeneous complex Lagrangian analytic subset of $T^*(X \times \C)$. Then by forgetting the homogeneity of $SS(\mathcal{I})$ we define two subsets:

$$SS^E(\mathcal{I}) \subset T^*X \times \C, \quad SS^C_{\text{irr}}(\mathcal{I}) \subset T^*X$$  \hfill (6.25)

(see Section 5 for the details). Recall that the irregular micro-support $SS^C_{\text{irr}}(\mathcal{I})$ of $\mathcal{I} \in D^b_c(X \times \C)$ is a (not necessarily homogeneous) complex Lagrangian analytic subset of $T^*X$ and $SS^C_{\text{evan}}(\mathcal{I}) \subset (T^*X)^\R$ is contained in $SS^C_{\text{irr}}(\mathcal{I})$. In what follows, we set $\Lambda := SS^C_{\text{irr}}(\mathcal{I}) \subset (T^*X)^\R$. Recall that we have

$$\Lambda \cap T^*U^\R = SS^C_{\text{irr}}(\mathcal{I}) \cap T^*U^\R = (SS(K) \cap T^*U^\R) + \Lambda'$$  \hfill (6.26)

Since $\Lambda$ is Lagrangian and $\dim \Lambda = \dim X = N$, by e.g. the theorem in [15, page 43] we can easily prove the following lemma.

**Lemma 6.3.** There exists a non-empty Zariski open subset $\Omega \subset Y = \C^N_Y$ of $Y$ such that the restriction $q^{-1}(\Omega) \cap \Lambda \longrightarrow \Omega$ of the projection $q : X \times Y \longrightarrow Y$ is an unramified finite covering and any connected component of the open subset $q^{-1}(\Omega) \cap \Lambda \subset \Lambda$ is a fiber bundle over a complex submanifold of $X = \C^N_Y$ contained in $U = X \setminus Q^{-1}(0)$ or $I(f) = P^{-1}(0) \cap Q^{-1}(0)$.

**Remark 6.4.** In the special case where $I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset$, we have $\Lambda = (SS(K) \cap T^*U^\R) + \Lambda'$ by Lemma 5.6 and hence there exists a (unique) maximal Zariski open subset $\Omega \subset Y = \C^N_Y$ satisfying the conditions in Lemma 6.3.

Let $\Omega \subset Y = \C^N_Y$ be as in Lemma 6.3 and $V \subset \Omega$ a contractible open subset of it. Then for the decomposition

$$q^{-1}(V) \cap \Lambda = \Lambda_{V,1} \sqcup \Lambda_{V,2} \sqcup \cdots \sqcup \Lambda_{V,d}$$  \hfill (6.27)
of $q^{-1}(V) \cap \Lambda$ into its connected components $\Lambda_{V,i} \subset \Lambda$ ($1 \leq i \leq d$) the morphism $q|_\Lambda$ induces an isomorphism $\Lambda_{V,i} \simeq V$ for any $1 \leq i \leq d$. Now let us consider the symplectic transformation of [10]:

$$\chi : T^*X \simeq T^*Y \quad ((z, w) \mapsto (w, -z)), \quad (6.28)$$

where we used the natural identification $T^*Y \simeq Y \times X$. Then $\chi(\Lambda_{V,i}) \subset T^*Y$ is a Lagrangian submanifold of $T^*Y$ for which the natural projection $T^*Y \rightarrow Y$ induces an isomorphism $\chi(\Lambda_{V,i}) \simeq V$. It follows that there exists a holomorphic function $g_i$ on $V$ such that

$$\chi(\Lambda_{V,i}) = \Lambda^i := \{(w, dg_i(w)) \mid w \in V\} \subset T^*Y. \quad (6.29)$$

Namely, if for a point $w \in V$ we denote by $\zeta^{(i)}(w) = (\zeta_1^{(i)}(w), \zeta_2^{(i)}(w), \ldots, \zeta_N^{(i)}(w)) \in X = \mathbb{C}_z^N$ the unique point of $X$ such that $(\zeta^{(i)}(w), w) \in \Lambda_{V,i} \subset \Lambda$, then we have

$$\left(\frac{\partial g_i}{\partial w_1}(w), \frac{\partial g_i}{\partial w_2}(w), \ldots, \frac{\partial g_i}{\partial w_N}(w)\right) = -(\zeta_1^{(i)}(w), \zeta_2^{(i)}(w), \ldots, \zeta_N^{(i)}(w)). \quad (6.30)$$

More precisely, we have the following higher-dimensional analogue of [10] Lemma-Definition 7.4.2]. For $1 \leq i \leq d$ by our choice of $\Omega \subset Y = \mathbb{C}^N$ in Lemma 6.3, $Z_i := p(\Lambda_{V,i}) \subset X = \mathbb{C}_z^N$ is a complex submanifold of $X$. We renumber them so that for some $0 \leq r \leq d$ we have $Z_i \subset U = X \setminus Q^{-1}(0)$ (resp. $Z_i \subset I(f) = P^{-1}(0) \cap Q^{-1}(0)$) if $1 \leq i \leq r$ (resp. if $r + 1 \leq i \leq d$).

**Lemma 6.5.** For any $1 \leq i \leq r$ there exists a unique holomorphic function $g_i$ on $V \subset \Omega$ such that $\chi(\Lambda_{V,i}) = \Lambda^i$ and the equality

$$g_i(w) = f(\zeta^{(i)}(w)) - \langle \zeta^{(i)}(w), w \rangle \quad (6.31)$$

holds on $V$.

**Proof.** By the condition $Z_i = p(\Lambda_{V,i}) \subset U = X \setminus Q^{-1}(0)$ and (6.29), the subset $\Lambda_{V,i} - \Lambda^f \subset (T^*U)_R$ of $(T^*U)_R$ is contained in the $C^*$-conic Lagrangian analytic set $\text{SS}(K) \cap (T^*U)_R$. By [21], Lemma A.52] this implies that we have $\Lambda_{V,i} \subset T^*_Z X + \Lambda^f$. Choose a holomorphic function $g_i$ on $V \subset \Omega$ such that $\chi(\Lambda_{V,i}) = \Lambda^i$ and set

$$k_i(w) := g_i(w) - f(\zeta^{(i)}(w)) + \langle \zeta^{(i)}(w), w \rangle \quad (w \in V). \quad (6.32)$$

Then by using the conditions $\zeta^{(i)}(w) \in Z_i$ ($w \in V$) and $(\zeta^{(i)}(w), w) \in \Lambda_{V,i} \subset T^*_Z X + \Lambda^f$ ($w \in V$) we can easily show that the condition (6.30) is equivalent to the one

$$\frac{\partial k_i}{\partial w_j}(w) = 0 \quad (1 \leq j \leq N, w \in V). \quad (6.33)$$

This implies that $k_i$ is constant on $V$. Then by substracting its value from $g_i$ we obtain the assertion. \[\square\]
Example 6.6. Consider the case where $N = 2$ and set $z = (z_1, z_2) = (x, y)$ and $w = (w_1, w_2) = (\xi, \eta)$. Assume that $U = \{(x, y) \in X = \mathbb{C}^2 \mid x \neq 0\} \subset X$, $K = \mathbb{C}X$ and the function $f : U \rightarrow \mathbb{C}$ is defined to be the ratio of $P(x, y) = y$ and $Q(x, y) = x$ as
\[
f(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{y}{x}.
\] (6.34)

Then by (an analogue of) Lemma 5.3 we have
\[
\Lambda = \text{SS}^C_{irr}(\mathcal{F}) = \Lambda^f = \{(x, y, -\frac{y}{x^2}, \frac{1}{x}) \mid (x, y) \in U\} \subset T^*X \simeq X \times Y.
\] (6.35)

By solving the equation $\xi = -\frac{y}{x^2}$, $\eta = \frac{1}{x}$, we find that $d = 1$, $\Omega = \{(\xi, \eta) \in Y = \mathbb{C}^2 \mid \eta \neq 0\} \subset Y$ and for $(\xi, \eta) \in \Omega$ we have
\[
x = \zeta_1(\xi, \eta) := \frac{1}{\eta}, \quad y = \zeta_2(\xi, \eta) := -\frac{\xi}{\eta^2}.
\] (6.36)

We can also verify that the function $g(\xi, \eta) := f\left(\frac{1}{\eta}, -\frac{\xi}{\eta^2}\right) - \left\langle \left(\frac{1}{\eta}, -\frac{\xi}{\eta^2}\right), (\xi, \eta)\right\rangle = -\frac{\xi}{\eta}$ satisfies the condition $\chi(\Lambda) = \Lambda^g$.

Example 6.7. Let $N = 2$ and $z = (z_1, z_2) = (x, y)$, $w = (w_1, w_2) = (\xi, \eta)$ be as in Example 6.6. Assume that $U = \{(x, y) \in X = \mathbb{C}^2 \mid x \neq 0\} \subset X$, $K = \mathbb{C}X$ and the function $f : U \rightarrow \mathbb{C}$ is defined to be the ratio of $P(x, y) = x - y^3$ and $Q(x, y) = x$ as
\[
f(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{x - y^3}{x} = 1 - \frac{y^3}{x}.
\] (6.38)

Then we have
\[
\Lambda_0 := \text{SS}^C_{irr}(\mathcal{F}) \cap (T^*U)_\mathbb{R} = \left\{(x, y, \frac{y^3}{x^2}, \frac{-3y^2}{x}) \mid (x, y) \in U\right\} \subset T^*X \simeq X \times Y.
\] (6.39)

Let us set
\[
\Omega_0 := \{(\xi, \eta) \in Y = \mathbb{C}^2 \mid \xi \neq 0, \eta \neq 0\} \subset Y.
\] (6.40)

Then by solving the equation $\xi = \frac{y^3}{x^2}$, $\eta = -\frac{3y^2}{x}$, we find that for $(\xi, \eta) \in \Omega_0$ we have
\[
x = \zeta_1(\xi, \eta) := -\frac{\eta^3}{27\xi^2}, \quad y = \zeta_2(\xi, \eta) := \frac{\eta^2}{9\xi}.
\] (6.41)

We can also verify that the function $g(\xi, \eta) := f\left(-\frac{\eta^3}{27\xi^2}, \frac{\eta^2}{9\xi}\right) - \left\langle \left(-\frac{\eta^3}{27\xi^2}, \frac{\eta^2}{9\xi}\right), (\xi, \eta)\right\rangle = 1 - \frac{\eta^3}{27\xi}$ (6.42)
on $\Omega_0$ satisfies the condition $\chi(\Lambda_0) \cap q^{-1}(\Omega_0) = \Lambda^g$. 59
Recall that for the closed embedding
\[ \iota : (T^*X) \times \mathbb{C} \hookrightarrow T^*(X \times \mathbb{C}) \quad \left( ((z, w), \tau) \mapsto (z, \tau, w, 1) \right) \] (6.43)
we have
\[ \text{SS}^{E, \mathbb{C}}(\mathcal{F}) = \iota^{-1} \text{SS}(\mathcal{F}). \] (6.44)

Then for \( r + 1 \leq i \leq d \), by the proof of Lemma 5.6, we obtain the following description of \( \text{SS}^{E, \mathbb{C}}(\mathcal{F}) \cap (\Lambda_{V,i} \times \mathbb{C}) \). For \( r + 1 \leq i \leq d \) let \( \rho_i : Z_i \times_X T^*X \rightarrow T^*Z_i \) be the natural morphism associated to the inclusion map \( Z_i \hookrightarrow X = \mathbb{C}^N \).

**Lemma 6.8.** For any \( r + 1 \leq i \leq d \) there exist holomorphic functions \( h_{ik} : Z_i \rightarrow \mathbb{C} \) \((1 \leq k \leq n_i)\) on \( Z_i \) and a neighborhood \( W \) of \( \Lambda_{V,i} \) in \( T^*X \) such that for the complex submanifolds
\[ \Gamma_k := \{(z, -h_{ik}(z)) \in X \times \mathbb{C} \mid z \in Z_i\} \subset X \times \mathbb{C} \quad (1 \leq k \leq n_i) \] (6.45)
of \( X \times \mathbb{C} \) we have
\[ \text{SS}^{E, \mathbb{C}}(\mathcal{F}) = \bigcup_{k=1}^{n_i} \iota^{-1} \left\{ T_{\Gamma_k}^*(X \times \mathbb{C}) \right\} \] (6.46)
in the open subset \( W \times \mathbb{C} \subset (T^*X) \times \mathbb{C} \) and
\[ \rho_i(\Lambda_{V,i}) = \Lambda^{h_{ik}} \quad (1 \leq k \leq n_i), \] (6.47)
where we set
\[ \Lambda^{h_{ik}} := \{(z, dh_{ik}(z)) \mid z \in Z_i\} \subset T^*Z_i. \] (6.48)

**Proof.** Note that as \( \Lambda_{V,i} \simeq V \) the complex manifold \( Z_i = p(\Lambda_{V,i}) \subset I(f) \) is contractible. Let \( \Lambda^0 \subset q^{-1}(\Omega) \cap \Lambda = q^{-1}(\Omega) \cap \text{SS}^{E, \mathbb{C}}(\mathcal{F}) \) be the (unique) connected component of \( \Lambda^0 := q^{-1}(\Omega) \cap \Lambda \) containing the connected set \( \Lambda_{V,i} \simeq V \). Then by our choice of \( \Omega \subset Y = \mathbb{C}^N \) in Lemma 5.3, the smooth variety \( \Lambda^0 \) is a fiber bundle over a smooth (quasi-affine) subvariety \( S_i \) of \( X = \mathbb{C}^N \) contained in \( I(f) \) such that \( Z_i \subset S_i \). Note that even if \( i \neq i' \) we may have \( \Lambda^0_i = \Lambda^0_{i'} \). As the natural morphism \( S_i \times_X T^*X \rightarrow T^*S_i \) associated to the inclusion map \( S_i \hookrightarrow X = \mathbb{C}^N \) is an extension of \( \rho_i \), we denote it also by \( \rho_i \). Note that the closure \( \overline{\Lambda^0_i} \) of \( \Lambda^0_i \) in \( T^*X \) is an irreducible component of \( \Lambda \). Moreover by the proof of Lemma 5.6 there exists a “horizontal” stratum \( M_i \subset X \times \mathbb{C} \) over a Zariski open subset of \( S_i \) such that the closure of the smooth Lagrangian subvariety of \( T^*X \) obtained by forgetting the homogeneity of the conormal bundle \( T^*_{M_i}(X \times \mathbb{C}) \subset T^*(X \times \mathbb{C}) \) is equal to \( \overline{\Lambda^0_i} \). This implies that the subset \( \overline{\Lambda^0_i} \cap (S_i \times_X T^*X) \) of \( S_i \times_X T^*X \) is a union of some fibers of \( \rho_i \). Namely there exists a Lagrangian subvariety \( \Lambda_{S_i} \subset T^*S_i \) such that
\[ \overline{\Lambda^0_i} \cap (S_i \times_X T^*X) = \rho_i^{-1} \Lambda_{S_i}. \] (6.49)

Recall that \( \Lambda^0_i \) is smooth and a fiber bundle over \( S_i \). Since \( \Lambda^0_i \) is an open subset of \( \rho_i^{-1} \Lambda_{S_i} \), for any point \( (z_0, w_0) \in \Lambda^0_i \) there exists its neighborhood \( W(z_0, w_0) \) in \( S_i \times_X T^*X \) such that \( \Lambda^0_i \) is a union of some fibers of \( \rho_i \) in \( W(z_0, w_0) \). This implies that the open subset \( \Lambda^0_{S_i} := \rho_i(\Lambda^0_i) \) of \( \Lambda_{S_i} \) is a complex Lagrangian submanifold of \( T^*S_i \) and the morphism
\( \Lambda_i^o \rightarrow \Lambda^o_{S_i} \) induced by \( \rho_i \) is a submersion. As the structure morphism \( \Lambda_i^o \rightarrow S_i \) of the fiber bundle \( \Lambda_i^o \) over \( S_i \) is the composite of the morphisms

\[
\Lambda_i^o \rightarrow \Lambda^o_{S_i} \rightarrow S_i,
\]

(6.50)

the morphism \( \Lambda^o_{S_i} \rightarrow S_i \) induced by \( T^*S_i \rightarrow S_i \) is also a submersion. Moreover by the dimensional reason it is a finite unramified covering. We denote its degree by \( e_i \geq 1 \).

Restricting the covering \( \Lambda^o_{S_i} \rightarrow S_i \) to the contractible open subset \( Z_i \subset S_i \), we obtain a trivial covering \( \Lambda_{Z_i} := Z_i \times S_i^I \Lambda^o_{S_i} \rightarrow Z_i \) of \( Z_i \) of degree \( e_i \). Since \( \rho_i(\Lambda_{V,i}) \subset \Lambda_{Z_i} \) is connected, it is contained in only one connected component of \( \Lambda_{Z_i} \). Together with the condition \( p(\Lambda_{V,i}) = Z_i \) we obtain \( \rho_i(\Lambda_{V,i}) \simeq Z_i \). Then there exists a holomorphic function \( h : Z_i \rightarrow \mathbb{C} \) on \( Z_i \) such that

\[
\rho_i(\Lambda_{V,i}) = \Lambda^h := \{(z, dh(z)) \mid z \in Z_i \}
\]

(6.51)

in \( T^*Z_i \). Such \( h \) is uniquely determined up to constant functions on \( Z_i \). Since \( \Lambda_{V,i} \) is contained in the smooth part of the Lagrangian subvariety \( \Lambda = SS_{irr}(F) \subset T^*X \), we then immediately obtain the assertion. This completes the proof.

Moreover by the condition (6.47) we see also that for any \( 1 \leq k, k' \leq n_i \) such that \( k \neq k' \) we have \( h_{ik} = h_{ik'} \), and hence \( h_{ik} - h_{ik'} : Z_i \rightarrow \mathbb{C} \) is a non-zero constant function on \( Z_i \). Then as in the proof of Lemma 6.5 we obtain the following result.

**Lemma 6.9.** For any \( r + 1 \leq i \leq d \) and \( 1 \leq k \leq n_i \) there exists a unique holomorphic function \( g_{ik} \) on \( V \subset \Omega \) such that \( \chi(\Lambda_{V,i}) = \Lambda^{g_{ik}} \) and the equality

\[
g_{ik}(w) = h_{ik}(\zeta^{(i)}(w)) - \langle \zeta^{(i)}(w), w \rangle
\]

(6.52)

holds on \( V \).

**Example 6.10.** We consider the situation in Example 6.7 and inherit the notations there. Moreover we assume that \( Z = X = \mathbb{C}^2 \) and \( K = \mathbb{C}_X \). For \( w = (\xi, \eta) \in \Omega_0 \) the meromorphic function \( f^w : X \setminus Q^{-1}(0) = \mathbb{C}^2 \setminus \{x = 0\} \rightarrow \mathbb{C} \) is written as

\[
f^w(x, y) = (\xi x + \eta y) - f(x, y) = \frac{\xi x^2 + \eta xy + y^3 - x}{x}.
\]

(6.53)

For \( t \in \mathbb{C} \) we set also

\[
R^w_t(x, y) := (\xi x^2 + \eta xy + y^3 - x) - tx
\]

(6.54)

so that we have

\[
f^w(x, y) - t = \frac{R^w_t(x, y)}{Q(x, y)} \quad (z = (x, y) \in U = X \setminus Q^{-1}(0))
\]

(6.55)

and

\[
(f^w)^{-1}(t) = (R^w_t)^{-1}(0) \subset X = \mathbb{C}^2.
\]

(6.56)
Then for \( t \neq -1 \) the complex curve \((R^w_t)^{-1}(0)\) in \( X = \mathbb{C}^2\) is smooth on a neighborhood of the origin \( \{(0,0)\} = P^{-1}(0) \cap Q^{-1}(0)\). If \( t = -1 \), for any \( w = (\xi, \eta) \in \Omega_0 \) the defining polynomial

\[
R^w_{-1}(x, y) := \xi x^2 + \eta xy + y^3 \tag{6.57}
\]
of the curve \((R^w_t)^{-1}(0)\) is Newton non-degenerate at the origin \( \{(0,0)\} \) (see [66, Section 3] for the details). Then by applying the classical Kouchnirenko’s theorem (see [38]) to it, we see that for any \( w = (\xi, \eta) \in \Omega \) its Milnor number at the origin is equal to 1. Moreover in Example 2.12 we studied the Milnor fiber of the rational function

\[
f^w(x, y) = -\frac{R^w_{-1}(x, y)}{Q(x, y)} = \frac{\xi x^2 + \eta xy + y^3}{x} \quad (z = (x, y) \in U = X \setminus Q^{-1}(0)) \tag{6.58}
\]
precisely and obtained the isomorphisms

\[
H^j \phi_{f^w(-1)}^{\text{mero,c}}(K[2])(0,0) \simeq \begin{cases} 
\mathbb{C} & (j = -1) \\
0 & \text{(otherwise)}.
\end{cases} \tag{6.59}
\]

By Proposition 5.7 this implies that for the conormal bundle \( T^*_\{(0,0)\}X(\simeq \{(0,0)\} \times Y) \subset T^*X(\simeq X \times Y) \) of the complex submanifold \( \{(0,0)\} \subset X \) the subset \( (T^*_\{(0,0)\}X) \times \{-1\} \subset (T^*X) \times \mathbb{C} \) is contained in \( \text{SS}^\mathbb{C}_{\text{irr}}(\mathcal{F}) \). We thus obtain

\[
\text{SS}^\mathbb{C}_{\text{irr}}(\mathcal{F}) = \left( T^*_\{(0,0)\}X \right) \cup \Lambda_0 \tag{6.60}
\]
(for the definition of \( \Lambda_0 \) see Example 6.7). In particular, we can take \( \Omega \) to be \( \Omega_0 \) in this case.

For a point \( w \in V \subset \Omega \) we define two regular functions \( \ell^w : X = \mathbb{C}^N \to \mathbb{C} \) and \( f^w : U = X \setminus Q^{-1}(0) \to \mathbb{C} \) by

\[
\ell^w : X \to \mathbb{C} \quad (z \mapsto \langle z, w \rangle) \tag{6.61}
\]
and

\[
f^w = \ell^w |_U - f : U = X \setminus Q^{-1}(0) \to \mathbb{C} \quad (z \mapsto \langle z, w \rangle - f(z)) \tag{6.62}
\]
respectively.

**Lemma 6.11.** For any \( 1 \leq i \leq r \) and \( w \in V \subset \Omega \) the holomorphic function \( f^w|_{Z_i} : Z_i \to \mathbb{C} \) on \( Z_i \) has a non-degenerate (complex Morse) critical point at \( \zeta^{(i)}(w) \in Z_i \).

**Proof.** In the proof of Lemma 6.5 for \( 1 \leq i \leq r \) we saw that \( \Lambda_{V,i} \) is an open subset of \( T^*_{Z_i}X + \Lambda^i \subset T^*U \). In other words, for the diffeomorphism

\[
\Phi : T^*U \to T^*U \quad ((z, \omega) \mapsto (z, \omega + df(z))) \tag{6.63}
\]
of \( T^*U \) we have the inclusion

\[
\Lambda_{V,i} \subset \Phi(T^*_{Z_i}X). \tag{6.64}
\]
Moreover, for $w \in V$ the condition $(\zeta^{(i)}(w), w) \in \Lambda_{V,i}$ implies
\[
(\zeta^{(i)}(w), df^w(\zeta^{(i)}(w))) = (\zeta^{(i)}(w), w - df(\zeta^{(i)}(w))) \in T^*_Z X
\] (6.65)
and hence the holomorphic function $f^w|_{Z_i} : Z_i \to \mathbb{C}$ has a critical point at $\zeta^{(i)}(w) \in Z_i$. By the choice of $\Omega$, for any $w \in V \subset \Omega$ the fiber $q^{-1}(w) \simeq X = \mathbb{C}^N$ of $q : X \times Y \to Y$ intersects $\Lambda_{V,i} \subset \Phi(T^*_Z X)$ transversally. Moreover, we have $q^{-1}(w) = \Lambda^w$. This implies that $\Phi^{-1}(\Lambda^w) = \Lambda^{f^w}$ intersects $\Phi^{-1}(\Lambda_{V,i}) \subset T^*_Z X$ transversally. Then by Kashiwara-Schapira [26] Lemma 7.2.2, the holomorphic function $f^w|_{Z_i} = (f^w - f)|_{Z_i} : Z_i \to \mathbb{C}$ on $Z_i$ has a non-degenerate (complex Morse) critical point at $\zeta^{(i)}(w) \in Z_i$. □

As in the proof of Lemma 6.11 by Lemma 6.8 we obtain also the following result. For $r + 1 \leq i \leq d$, $1 \leq k \leq n_i$ and $w \in V \subset \Omega$ we define a holomorphic function $h^w_{ik} : Z_i \to \mathbb{C}$ on $Z_i$ by
\[
h^w_{ik} := \ell^w|_{Z_i} - h_{ik} : Z_i \to \mathbb{C}, \quad (z \mapsto (z, w) - h_{ik}(z))
\] (6.66)
so that we have
\[
g_{ik}(w) = h_{ik}(\zeta^{(i)}(w)) - \langle \zeta^{(i)}(w), w \rangle = -h^w_{ik}(\zeta^{(i)}(w)) \quad (w \in V). \tag{6.67}
\]

**Lemma 6.12.** For any $r + 1 \leq i \leq d$, $1 \leq k \leq n_i$ and $w \in V \subset \Omega$ the function $h^w_{ik} : Z_i \to \mathbb{C}$ on $Z_i$ has a non-degenerate (complex Morse) critical point at $\zeta^{(i)}(w) \in Z_i$.

Note that for $1 \leq i \leq r$ and $w \in V \subset \Omega$ the point
\[
(\zeta^{(i)}(w), df^w(\zeta^{(i)}(w))) = (\zeta^{(i)}(w), w - df(\zeta^{(i)}(w))) \in (T^*X)_\mathbb{R}
\] (6.68)
is contained in the intersection of $T^*_Z X$ and the smooth part of $\text{SS}(k[V]) \subset (T^*U)_\mathbb{R}$. We denote by $m(i) \geq 1$ the multiplicity of the regular holonomic $\mathcal{D}_X$-module $\mathcal{N} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X)$ (or of the perverse sheaf $K[N]$) there.

**Proposition 6.13.** For any $1 \leq i \leq r$ and $w \in V \subset \Omega$ we have isomorphisms
\[
H^j\phi^\text{mero,c}_{f^w-c}(K[N])_{\zeta^{(i)}(w)} \simeq H^j\phi^w_{f^w-c}(K[N])_{\zeta^{(i)}(w)} \simeq \begin{cases}
\mathbb{C}^{m(i)} & (j = -1) \\
0 & \text{(otherwise)},
\end{cases}
\] (6.69)
where we set $c := f^w(\zeta^{(i)}(w)) = (\zeta^{(i)}(w), w) - f(\zeta^{(i)}(w)) \in \mathbb{C}$.

**Proof.** Since for $1 \leq i \leq r$ we have $\zeta^{(i)}(w) \in Z_i \subset U = X \setminus Q^{-1}(0)$ and hence the function $f^w - c$ is holomorphic on a neighborhood of $\zeta^{(i)}(w)$, there exists an isomorphism
\[
\phi^\text{mero,c}_{f^w-c}(K[N])_{\zeta^{(i)}(w)} \simeq \phi_{f^w-c}(K[N])_{\zeta^{(i)}(w)}.
\] (6.70)
Moreover by our definition of $m(i) \geq 1$ the perverse sheaf $K[N] \in \mathcal{D}^b_c(X^{an})$ is isomorphic to $\mathcal{C}^{\text{dim}(i)}_{Z_i}[\text{dim}Z_i] \in \mathcal{D}^b_c(X^{an})$ in the localized category $\mathcal{D}^b_c(X^{an}, \{(\zeta^{(i)}(w), df^w(\zeta^{(i)}(w)))\})$ at the point $(\zeta^{(i)}(w), df^w(\zeta^{(i)}(w))) \in T^*X^{an}$ (see [27] Definition 6.1.1] for the definition). Then by [27] Proposition 8.6.3] we obtain isomorphisms
\[
\phi_{f^w-c}(K[N])_{\zeta^{(i)}(w)} \simeq \phi_{f^w-c}(K[N])(\mathcal{C}^{\text{dim}(i)}_{Z_i}[\text{dim}Z_i])_{\zeta^{(i)}(w)}
\simeq \phi_{f^w|_{Z_i} - f^w(\zeta^{(i)}(w))}(\mathcal{C}^{\text{dim}(i)}_{Z_i}[\text{dim}Z_i])_{\zeta^{(i)}(w)},
\] 63
Recall now that by Lemma 6.11 the holomorphic function $f^w|_{Z_i} : Z_i \to \mathbb{C}$ on $Z_i$ has a non-degenerate (complex Morse) critical point at $\zeta^{(i)}(w) \in Z_i$. Then the Milnor fiber at it is equal to one and hence we obtain the desired isomorphisms

$$H^j\phi^{f^w \rightarrow c}(K[N])_{\zeta^{(i)}(w)} \simeq H^j\phi^{f^w|_{Z_i}}(\mathbb{C}^{|\dim Z_i|}(\zeta^{(i)}(w)))$$

\begin{align*}
&\simeq \\
&\begin{cases}
\mathbb{C}^{|\dim Z_i|} & (j = -1) \\
0 & \text{(otherwise)}
\end{cases}
\end{align*}

(see e.g. [66, Theorem 2.6]).

\[ \square \]

Note that by (6.46) for $r + 1 \leq i \leq d$, $1 \leq k \leq n_i$ and $w \in V \subset \Omega$ the point

$$(\zeta^{(i)}(w), -h_{ik}(\zeta^{(i)}(w)), w, 1) \in T^*(X \times \mathbb{C})$$

is contained in the smooth part of $SS(F[N]) = SS(F) \subset T^*(X \times \mathbb{C})$. We denote by $m(i, k) \geq 1$ the multiplicity of of the perverse sheaf $F[N] = (i_{-f})_!(K[N]|_U) \in D^b_c(X^{an} \times \mathbb{C})$ there.

**Proposition 6.14.** For any $r + 1 \leq i \leq d$, $1 \leq k \leq n_i$ and $w \in V \subset \Omega$ we have isomorphisms

$$H^j\phi^{mero,c}_{f^w \rightarrow c_k}(K[N])_{\zeta^{(i)}(w)} \simeq \begin{cases}
\mathbb{C}^{|\dim Z_i|} & (j = -1) \\
0 & \text{(otherwise)},
\end{cases}$$

(6.72)

where we set $c_k := h_{ik}^w(\zeta^{(i)}(w)) = \langle \zeta^{(i)}(w), w \rangle - h_{ik}(\zeta^{(i)}(w)) \in \mathbb{C}$. Moreover, for any complex number $c \in \mathbb{C}$ satisfying the condition $c \neq c_k (1 \leq k \leq n_i)$ we have a vanishing

$$\phi^{mero,c}_{f^w \rightarrow c}(K[N])_{\zeta^{(i)}(w)} \simeq 0.$$

(6.73)

**Proof.** Note that for the (not necessarily closed) embedding $i_{f^w} : U \hookrightarrow X \times \mathbb{C} (z \mapsto (z, f^w(z)))$ and any $c \in \mathbb{C}$ we have an isomorphism

$$\phi_{f^w \rightarrow c}(K[N])_{\zeta^{(i)}(w)} \simeq \phi_{f^w \rightarrow c}(i_{f^w})_!(K[N]|_U)_{\zeta^{(i)}(w), c}.$$ 

(6.74)

As in the proof of Proposition 5.7, let us consider the automorphism $T_w$ of $X \times \mathbb{C}$ defined by

$$T_w : X \times \mathbb{C} \to X \times \mathbb{C}, \quad ((z, \tau) \mapsto (z, \tau + \langle z, w \rangle)).$$

(6.75)

Then we have $i_{f^w} = T_w \circ i_{-f}$ and hence there exists an isomorphism

$$(T_w)_*(F[N]) = (T_w)_*(i_{-f})_!(K[N]|_U) \simeq (i_{f^w})_!(K[N]|_U).$$

(6.76)

On the other hand, by (6.46) and our definition of $m(i, k) \geq 1$ the perverse sheaf $F[N] \in D^b_c(X^{an} \times \mathbb{C})$ is isomorphic to $\mathbb{C}^{|\dim Z_i|}(\zeta^{(i)}(w), w, 1) \in D^b_c(X^{an} \times \mathbb{C})$ in the localized category $D^b_c(X^{an} \times \mathbb{C}, \zeta^{(i)}(w), w, 1))$ at the point $(\zeta^{(i)}(w), -h_{ik}(\zeta^{(i)}(w)), w, 1) \in T^*(X^{an} \times \mathbb{C})$ (see [27, Definition 6.1.1]). This implies that for the closed embedding

$$i_k : Z_i \hookrightarrow X \times \mathbb{C}, \quad (z \mapsto (z, h_{ik}^w(z)))$$

(6.77)
we have an isomorphism
\[(i_{fw})(K[N]|_U) \simeq (i_k)_*(C^{\oplus m(i,k)}_Z)[\dim Z_i]\] (6.78)
in the localized category $D^b(X^{an} \times \mathbb{C}, \{(\zeta^{(i)}(w), c_k, 0, 1)\})$ at the point $(\zeta^{(i)}(w), c_k, 0, 1) \in T^*(X^{an} \times \mathbb{C})$. Then by Proposition 8.6.3 we obtain isomorphisms
\[
\begin{align*}
\phi_{\tau - c_k}((i_{fw})(K[N]|_U))_{(\zeta^{(i)}(w), c_k)} &\simeq \phi_{\tau - h_{ik}^w(\zeta^{(i)}(w))}((i_k)_*(C^{\oplus m(i,k)}_Z)[\dim Z_i])_{(\zeta^{(i)}(w), h_{ik}^w(\zeta^{(i)}(w)))} \\
&\simeq \phi_{h_{ik}^w - h_{ik}^w(\zeta^{(i)}(w))}((C^{\oplus m(i,k)}_Z)[\dim Z_i])_{\zeta^{(i)}(w)}. 
\end{align*}
\]
Recall that by Lemma 6.12 the holomorphic function $h_{ik}^w|_{Z_i} : Z_i \rightarrow \mathbb{C}$ on $Z_i$ has a non-degenerate (complex Morse) critical point at $\zeta^{(i)}(w) \in Z_i$. Then the first assertion immediately follows from the standard fact that the Milnor number at it is equal to one (see e.g. [60, Theorem 2.6]). Similarly, we can show the second assertion.

By the proof of Proposition 6.14 and (6.46), for any $r + 1 \leq i \leq d$, $w \in V \subset \Omega$ and $c \in \mathbb{C}$ such that $c \neq h_{ik}^w(\zeta^{(i)}(w))$ ($1 \leq k \leq n_i$) we have also a vanishing
\[
\phi_{f^{\text{mero},c}}(K[N])_{\zeta^{(i)}(w)} = 0. 
\]
(6.79)
Together with Proposition 6.14 this implies that for any $r + 1 \leq i \leq d$ the positive integer
\[
m(i) := \sum_{k=1}^{n_i} m(i, k) \geq 1 
\]
satisfies the condition
\[
m(i) = \sum_{c \in \mathbb{C}} \dim H^{N-1}(\phi_{f^{\text{mero},c}}(K))_{\zeta^{(i)}(w)} \quad (w \in V). 
\]
(6.81)

**Corollary 6.15.** For the open subset $q^{-1}(\Omega) = X \times \Omega \subset X \times Y$ of $X \times Y \simeq T^*X$ we have
\[
SS_{ir}^C(\mathcal{F}) \cap q^{-1}(\Omega) = SS_{eva}(F) \cap q^{-1}(\Omega). 
\]
(6.82)

**Proof.** By Proposition 5.7 it suffices to prove only the inclusion
\[
\Lambda \cap q^{-1}(\Omega) = SS_{ir}^C(\mathcal{F}) \cap q^{-1}(\Omega) \subset SS_{eva}(F) \cap q^{-1}(\Omega). 
\]
(6.83)

For a point $(z_0, w_0) \in \Lambda \cap q^{-1}(\Omega)$ we take a contractible open subset $V \subset \Omega$ such that $q((z_0, w_0)) = w_0 \in V$ and use the several notations that we introduced for it above. Then by Propositions 6.13 and 6.14 we obtain the desired condition $(z_0, w_0) \in SS_{eva}(F) \cap q^{-1}(\Omega)$.

**Lemma 6.16.** Let $S$ be a Whitney stratification of $Z \cap U \subset U = X \setminus Q^{-1}(0)$ adapted to $K|_U \in D^b_c(U)$ such that
\[
SS(K|_U) \subset \bigcup_{S \in S} T^*_S X 
\]
(6.84)
and \( S_{\text{red}} \subset S \) its subset consisting of strata \( S \in S \) satisfying the condition \( T^*_S X \subset \text{SS}(K|_U) \) so that we have
\[
\text{SS}(K|_U) = \bigcup_{S \in S_{\text{red}}} T^*_S X. \tag{6.85}
\]

Then for any \( w \in V \subset \Omega \) and \( S \in S_{\text{red}} \) the holomorphic function \( f^w|_S : S \rightarrow \mathbb{C} \) on \( S \) is tame at infinity.

**Proof.** Since by definition the morphisms \( \zeta^{(i)} : V \rightarrow U = X \setminus Q^{-1}(0) \) (1 \( \leq i \leq r \)) are holomorphic, there exist \( 0 < \varepsilon \ll 1 \) and \( R \gg 0 \) such that for the open ball \( B_\varepsilon(w) \subset V \subset \Omega \) (resp. \( B_R(0) \subset X = \mathbb{C}^N \)) with radius \( \varepsilon > 0 \) (resp. \( R > 0 \)) centered at \( w \in V \) (resp. the origin \( 0 \in X = \mathbb{C}^N \)) we have the inclusion
\[
\zeta^{(i)}(B_\varepsilon(w)) \subset U \cap B_R(0) \quad (1 \leq i \leq r). \tag{6.86}
\]

In other words, the subset
\[
\Lambda \cap p^{-1}(U) \cap q^{-1}(B_\varepsilon(w)) = \left\{ \text{SS}(K|_U) + \Lambda \right\} \cap q^{-1}(B_\varepsilon(w)) \tag{6.87}
\]
of \( p^{-1}(U) \cap q^{-1}(B_\varepsilon(w)) \) is contained in \( p^{-1}(U \cap B_R(0)) \cap q^{-1}(B_\varepsilon(w)) \). This implies that for the stratum \( S \in S_{\text{red}} \) and any \( u \in B_\varepsilon(0) \subset Y = \mathbb{C}^N \) the linear perturbation of \( f^w|_S \):
\[
(f^{w+u}|_S)(z) = (f^w|_S)(z) + \langle z, u \rangle \quad (z \in S) \tag{6.88}
\]
has no critical point in the set
\[
S \setminus B_R(0) = \{ z \in S \mid \|z\| \geq R \}. \tag{6.89}
\]

Suppose now that there exists a point \( z \in S \setminus B_R(0) \) such that \( \|d\text{Re}(f^w|_S)(z)\| < \varepsilon \). Then, the image of \( B_\varepsilon(0) \subset \mathbb{C}^N \) by the surjective linear map \( \Psi_z : T^*_z X \simeq \mathbb{C}^N \rightarrow T^*_z S \mathbb{R} \) being also an open ball with radius \( \varepsilon > 0 \), we have the equality \( d\text{Re}(f^w|_S)(z) = \Psi_z(u) \) for some \( u \in B_\varepsilon(0) \). Moreover, for the holomorphic function \( \ell_\mathbb{P} : S \rightarrow \mathbb{C} \) on \( S \) defined by
\[
\ell_\mathbb{P}(z) := \langle z, \overline{u} \rangle = \langle z, u \rangle \quad (z \in S) \tag{6.90}
\]
we have \( d\text{Re}(\ell_\mathbb{P})(z) = \Psi_z(u) \) by (3.22). This implies that the holomorphic function \( (f^{w-\mathbb{P}}|_S)(z) \) on \( S \) satisfies the condition \( d\text{Re}(f^{w-\mathbb{P}}|_S)(z) = 0 \) at the point \( z \in S \setminus B_R(0) \). Then by the Cauchy-Riemann equation, we obtain also \( d(f^{w-\mathbb{P}}|_S)(z) = 0 \). Since we have \( -\overline{u} \in B_\varepsilon(0) \), we get a contradiction. \( \Box \)

Similarly, by the proofs of Lemma 6.8 and Proposition 6.14 we obtain the following result. For a point \( w \in \Omega \) by the (not necessarily closed) embedding \( i_{f^w} : U \hookrightarrow X \times \mathbb{C} \) \((z \mapsto (z, f^w(z))) \) we set
\[
L^w := (i_{f^w})_!(K|_U) \in \text{D}^b_c(X \times \mathbb{C}). \tag{6.91}
\]

**Lemma 6.17.** For a point \( w \in V \subset \Omega \) let \( S \) and \( S_0 \) be Whitney stratifications of \( X \times \mathbb{C} \) and \( X \) respectively such that
\[
\text{SS}(L^w) \subset \bigcup_{S \in S} T^*_S (X \times \mathbb{C}) \tag{6.92}
\]
and the projection \( X \times \mathbb{C} \rightarrow X \) is a stratified fiber bundle as in the assertion of the theorem in [15, page 43]. Then for any stratum \( S \in S \) in \( S \) such that \( T^*_S (X \times \mathbb{C}) \subset \text{SS}(L^w) \) the restriction \( h|_S : S \rightarrow \mathbb{C} \) of the function \( h : X \times \mathbb{C} \rightarrow \mathbb{C} \) \((z, \tau) \mapsto \tau \) to \( S \subset X \times \mathbb{C} \) is relatively tame at infinity for the projection \( X \times \mathbb{C} \rightarrow X \) in the sense of Definition 3.9.
With Theorem 3.7, Lemmas 6.5, 6.11 and 6.16 and Proposition 6.13 at hand, in the special case where \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \) we obtain the following result as in the proof of [22, Theorem 4.4] (see the proof of Theorem 6.19 below for the details).

**Theorem 6.18.** In the situation as above, assume also that \( I(f) = P^{-1}(0) \cap Q^{-1}(0) = \emptyset \) so that we have \( r = d \). Then we have an isomorphism

\[
\pi^{-1}C_V \otimes (\text{Sol}^{E_Y}(\mathcal{M}^\kappa)) \simeq \bigoplus_{i=1}^{d} (E^{\text{Reg}_i}_{V_{\text{an}}}|_{Y_{\text{an}}}) \oplus m(i)
\]

(6.93)

of enhanced ind-sheaves on \( \overline{Y_{\text{an}}} \).

In the general case i.e. if we do not assume the condition \( I(f) = 0 \), we have the following result. Note that by Lemma 6.9 for any \( r+1 \leq i \leq d \) and \( 1 \leq k, k' \leq n_i \) such that \( k \neq k' \) the difference \( g_{ik} - g_{ik'} : Z_i \rightarrow \mathbb{C} \) is a non-zero constant function on \( Z_i \). Then for any \( r+1 \leq i \leq d \) and \( 1 \leq k, k' \leq n_i \) we have an isomorphism

\[
E^{\text{Reg}_i}_{V_{\text{an}}|_{Y_{\text{an}}}} \simeq E^{\text{Reg}_{i,k'}}_{V_{\text{an}}|_{Y_{\text{an}}}}
\]

(6.94)

of enhanced ind-sheaves on \( \overline{Y_{\text{an}}} \). For this reason, for each \( r+1 \leq i \leq d \) by choosing an index \( 1 \leq k \leq n_i \) we set \( g_i(w) := g_{ik}(w) \) \( (w \in V) \) and consider the enhanced ind-sheaf \( E^{\text{Reg}_i}_{V_{\text{an}}|_{Y_{\text{an}}}} \) in what follows.

**Theorem 6.19.** In the situation as above, we have an isomorphism

\[
\pi^{-1}C_V \otimes (\text{Sol}^{E_Y}(\mathcal{M}^\kappa)) \simeq \bigoplus_{i=1}^{d} (E^{\text{Reg}_i}_{V_{\text{an}}|_{Y_{\text{an}}}}) \oplus m(i)
\]

(6.95)

of enhanced ind-sheaves on \( \overline{Y_{\text{an}}} \).

**Proof.** By (6.20) and (6.94), it suffices to prove that there exists an isomorphism

\[
\pi^{-1}C_V \otimes \bigg\{ \pi^{-1}(i_X)_{!}K \bigg\} \otimes E^{\text{Reg}_{i,k}}_{U_{\text{an}}|X_{\text{an}}}
\]

\[
\simeq \bigg( \bigoplus_{i=1}^{r} (E^{\text{Reg}_i}_{V_{\text{an}}|_{Y_{\text{an}}}}) \oplus m(i) \bigg) \bigg( \bigoplus_{i=r+1}^{d} \bigg( \bigoplus_{k=1}^{n_i} (E^{\text{Reg}_{i,k}}_{V_{\text{an}}|_{Y_{\text{an}}}}) \oplus m(i,k) \bigg) \bigg)
\]

of enhanced sheaves on \( V \subset \Omega \subset Y \). Let

\[
X \times \mathbb{R}_s \leftarrow \left( X \times \mathbb{R}_s \right) \times (Y \times \mathbb{R}_t) \xrightarrow{p_2} Y \times \mathbb{R}_t
\]

(6.96)

be the projections. Then by D’Agnolo-Kashiwara [10, Lemma 7.2.1] on \( Y_{\text{an}} \subset \overline{Y_{\text{an}}} \) we have an isomorphism

\[
L \bigg\{ \pi^{-1}(i_X)_{!}K \bigg\} \otimes E^{\text{Reg}_{i,k}}_{U_{\text{an}}|X_{\text{an}}}
\]

\[
\simeq \mathcal{Q} \left( \text{R}p_2! \left( p_1^{-1} \left( \pi^{-1}K \otimes E^{\text{Reg}_{i,k}}_{U_{\text{an}}|X_{\text{an}}} \right) \otimes \mathbb{C}_{\{ \text{Re}(z,w) \geq 0 \}} [N] \right) \right)
\]

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where \( Q : D^b(C_{\text{Y}^{\text{an}}}) \to E^b(C_{\text{Y}^{\text{an}}}) \) is the quotient functor. For a point \((w, t) \in Y^{\text{an}} \times \mathbb{R}\) we have isomorphisms

\[
\left( R\pi_2 \left( p_1^{-1} \left\{ \left( \pi^{-1} K \right) \otimes E^\text{Ref}_{U^{\text{an}}|X^{\text{an}}} \right\} \otimes C_{\{ t-s-\text{Re}(z,w) \geq 0 \}} \right) \right)_{(w,t)} \simeq R\Gamma_c \left( \left\{ (z,s) \in U^{\text{an}} \times \mathbb{R} \mid t-s-\text{Re}(z,w) \geq 0, s+\text{Re}(f(z)) \geq 0; \pi^{-1} K[N] \right\} \right) 
\]

\[
\simeq R\Gamma_c \left( U^{\text{an}}; \mathcal{M}_c \left[ (z,s) \in U^{\text{an}} \times \mathbb{R} \mid t-s-\text{Re}(z,w) \geq 0, s+\text{Re}(f(z)) \geq 0 \right] \otimes \pi^{-1} K[N] \right) 
\]

\[
\simeq R\Gamma_c \left( \{ z \in U^{\text{an}} \mid \text{Re} f^w(z) \leq t \}; K[N] \right),
\]

where we used

\[
R\pi_! C_{\{ (z,s) \in U^{\text{an}} \times \mathbb{R} \mid t-s-\text{Re}(z,w) \geq 0, s+\text{Re}(f(z)) \geq 0 \}} \simeq C_{\{ z \in U^{\text{an}} \mid \text{Re}(z,w)-\text{Re}(z) \leq t \}} \tag{6.97}
\]

in the last isomorphism. Fix a point \( w \in V \subset \Omega \subset Y = \mathbb{C}^N \). Then as in the proof of [22, Theorem 4.4] we can prove the vanishing

\[
R\Gamma_c \left( \{ z \in U^{\text{an}} \mid \text{Re} f^w(z) \leq t \}; K[N] \right) \simeq 0 \tag{6.98}
\]

for \( t \ll 0 \) as follows. Let us set

\[
L(Sol_X(\mathcal{M})) := \left( R\pi_2 \left( p_1^{-1} \left\{ \left( \pi^{-1} K \right) \otimes E^\text{Ref}_{U^{\text{an}}|X^{\text{an}}} \right\} \otimes C_{\{ t-s-\text{Re}(z,w) \geq 0 \}} \right) \right)_{(w,t)}.
\]

Then for \( t \in \mathbb{R} \) its stalk at \((w,t) \in Y \times \mathbb{R}\):

\[
\left( L(Sol_X(\mathcal{M})) \right)_{(w,t)} \simeq R\Gamma_c \left( \{ z \in U^{\text{an}} \mid \text{Re} f^w \leq t \}; K[N] \right) \tag{6.99}
\]

is calculated as follows:

\[
R\Gamma_c \left( \{ z \in U^{\text{an}} \mid \text{Re} f^w \leq t \}; K[N] \right) 
\]

\[
\simeq R\Gamma_c \left( \{ \tau \in \mathbb{C} \mid \text{Re} \tau \leq t \}; Rf^w(K[N]) \right).
\]

Since the morphism \( f^w : U = X \setminus Q^{-1}(0) \to \mathbb{C} \) is algebraic, the proper direct image \( Rf^w(K[N]) \) of the perverse sheaf \((K[N]) \in D^b_c(U^{an})\) is constructible. Then in particular, for \( t \ll 0 \) the restrictions of its cohomology sheaves to the closed half space \( \{ \tau \in \mathbb{C} \mid \text{Re} \tau \leq t \} \subset \mathbb{C} \) are locally constant. Thus for \( t \ll 0 \) we obtain the vanishing

\[
R\Gamma_c \left( \{ \tau \in \mathbb{C} \mid \text{Re} \tau \leq t \}; Rf^w(K[N]) \right) \simeq 0. \tag{6.100}
\]

For \( 1 \leq i \leq r \) we set

\[
c(i) := f^w(\zeta^{(i)}(w)) = \langle \zeta^{(i)}(w), w \rangle - f(\zeta^{(i)}(w)) \in \mathbb{C}. \tag{6.101}
\]

Moreover for \( r+1 \leq i \leq d \) and \( 1 \leq k \leq n_i \) we set

\[
c(i,k) := h_{ik}^w(\zeta^{(i)}(w)) = \langle \zeta^{(i)}(w), w \rangle - h_{ik}(\zeta^{(i)}(w)) \in \mathbb{C}. \tag{6.102}
\]

Let \( B \subset \mathbb{C} \) be the union of these points in \( \mathbb{C} \) and write it as

\[
B = \{ c_1, c_2, \ldots, c_l \} = \{ c_j \in \mathbb{C} \mid 1 \leq j \leq l \} \subset \mathbb{C} \tag{6.103}
\]

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so that for any \( j \neq j' \) we have \( c_j \neq c_{j'} \). Then for any \( 1 \leq i \leq r \) there exists a unique index \( 1 \leq j \leq l \) such that \( c_j = c(i) \) and we denote it by \( j(i) \). Similarly, for any \( r + 1 \leq i \leq d \) and \( 1 \leq k \leq n_i \) there exists a unique index \( 1 \leq j \leq l \) such that \( c_j = c(i, k) \) and we denote it by \( j(i, k) \). Then by Corollary 6.15 and Lemma 6.17 for any \( 1 \leq j \leq l \) we can apply Theorem 3.11 to obtain an isomorphism

\[
\phi_{\tau - c_j}(\mathcal{R}f^u_!(K[U][N])) \\
\simeq \bigoplus_{i: j(i) = j} \phi_{f^u - c_j}(K[N])_{\zeta(i)(w)} \bigoplus \bigoplus_{(i, k): j(i, k) = j} \phi_{\text{mero}, c}(K[N])_{\zeta(i)(w)},
\]

where in the first (resp. the last) direct sum \( \bigoplus \) the index \( 1 \leq i \leq r \) (resp. the pair \( (i, k) \)) of \( r + 1 \leq i \leq d \) and \( 1 \leq k \leq n_i \) ranges over the ones satisfying the condition \( j(i) = j \) (resp. \( j(i, k) = j \)). Recall that for any \( r + 1 \leq i \leq d \) there exists at most one index \( 1 \leq k \leq n_i \) such that \( j(i, k) = j \). By Proposition 6.13 for any \( 1 \leq i \leq r \) such that \( j(i) = j \) we have isomorphisms

\[
H^n \phi_{f^u - c_j}(K[N])_{\zeta(i)(w)} \simeq \begin{cases} 
\mathbb{C}^{m(i)} & (n = -1) \\
0 & \text{(otherwise)}. 
\end{cases}
\quad (6.104)
\]

Moreover by Proposition 6.14 for any pair \( (i, k) \) of \( r + 1 \leq i \leq d \) and \( 1 \leq k \leq n_i \) such that \( j(i, k) = j \) we have isomorphisms

\[
H^n \phi_{\text{mero}, c}(K[N])_{\zeta(i)(w)} \simeq \begin{cases} 
\mathbb{C}^{m(i, k)} & (n = -1) \\
0 & \text{(otherwise)}. 
\end{cases}
\quad (6.105)
\]

For \( 1 \leq j \leq l \) let us set

\[
d_j := \left( \sum_{i: j(i) = j} m(i) \right) + \left( \sum_{(i, k): j(i, k) = j} m(i, k) \right).
\quad (6.106)
\]

Then for any \( 1 \leq j \leq l \) we thus obtain isomorphisms

\[
H^n \phi_{\tau - c_j}(\mathcal{R}f^u_!(K[U][N])) \simeq \begin{cases} 
\mathbb{C}^{d_j} & (n = -1) \\
0 & \text{(otherwise)}. 
\end{cases}
\quad (6.107)
\]

Again by Theorem 3.11 for any \( c \in \mathbb{C} \) such that \( c \notin B = \{ c_1, c_2, \ldots, c_l \} \) we can apply Proposition 6.14 to obtain a vanishing

\[
\phi_{\tau - c}(\mathcal{R}f^u_!(K[U][N])) \simeq 0.
\quad (6.108)
\]

By the proof of 23 Lemma 2.1] this implies that the constructible sheaf \( \mathcal{R}f^u_!(K[U][N]) \in \mathcal{D}^b_{\mathbb{Q}, c}(\mathbb{C}) \) on \( \mathbb{C} \) is smooth outside the finite subset \( B = \{ c_1, c_2, \ldots, c_l \} \subset \mathbb{C} \). For \( 1 \leq j \leq l \) set \( \gamma_j := \text{Re} c_j \in \mathbb{R} \). For the fixed point \( w \in V \subset \Omega \), after reordering \( \gamma_1, \gamma_2, \ldots, \gamma_l \in \mathbb{R} \) we may assume that

\[
\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_l.
\]
If for some $1 \leq i < j \leq l$ such that $\gamma_i < \gamma_j$ the open interval $(\gamma_i, \gamma_j) \subset \mathbb{R}$ does not intersect the set $\{\gamma_1, \gamma_2, \ldots, \gamma_l\}$, then for any $t_1, t_2 \in \mathbb{R}$ such that $\gamma_i < t_1 < t_2 < \gamma_j$ we can show an isomorphism

$$R\Gamma_c(\{z \in U^m \mid \text{Re}f^w(z) \leq t_2\}; K[N]) \xrightarrow{\cong} R\Gamma_c(\{z \in U^m \mid \text{Re}f^w(z) \leq t_1\}; K[N]).$$

(6.109)

Equivalently, we shall show an isomorphism

$$R\Gamma_c(\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t_2\}; Rf^w_1(K|_U[N])) \xrightarrow{\cong} R\Gamma_c(\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t_1\}; Rf^w_1(K|_U[N])).$$

(6.111)

This follows from Kashiwara’s non-characteristic deformation lemma (see [27 Proposition 2.7.2]) as follows. Let $\iota : \mathbb{C} = \mathbb{R} \times \sqrt{-1}\mathbb{R} \hookrightarrow \mathbb{R} \times \sqrt{-1}\mathbb{R}$ be the inclusion map. Then there exists a continuous map $\ell : \mathbb{R} \times \sqrt{-1}\mathbb{R} \to \mathbb{R}$ which extends the one $\text{Re} : \mathbb{C} \to \mathbb{R}$. Now, by applying Kashiwara’s non-characteristic deformation lemma to the Morse function $\ell : \mathbb{R} \times \sqrt{-1}\mathbb{R} \to \mathbb{R}$, we obtain an isomorphism

$$R\Gamma_c(\{\tau \in \mathbb{R} \times \sqrt{-1}\mathbb{R} \mid \ell(\tau) \leq t_2\}; \iota \ell Rf^w_1(K|_U[N])) \xrightarrow{\cong} R\Gamma_c(\{\tau \in \mathbb{R} \times \sqrt{-1}\mathbb{R} \mid \ell(\tau) \leq t_1\}; \iota \ell Rf^w_1(K|_U[N])),$$

(6.113)

which is equivalent to the desired one. For $1 \leq j \leq l$ we define a closed half space $G_j \subset \mathbb{C}$ of $\mathbb{C}$ by

$$G_j := \{\tau \in \mathbb{C} \mid \text{Re}\tau \geq \gamma_j\} \subset \mathbb{C}.$$  

(6.115)

Then for any $1 \leq j \leq l$ we have isomorphisms

$$H^n R\Gamma_{G_j}(Rf^w_1(K|_U[N]))_{c_j} \simeq H^n\phi_{t-c_j}(Rf^w_1(K|_U[N])[-1]$$

$$\cong \begin{cases} 
\mathbb{C}^{d_j} & (n = 0) \\
0 & \text{(otherwise)}.
\end{cases}$$

(6.116)  

(6.117)

Starting from the situation (6.109), by Morse theory, we can show that for any $t = \gamma_j = \text{Re}c_j \in \mathbb{R}$ $(1 \leq j \leq l)$ there exists $0 < \epsilon \ll 1$ such that

$$R\Gamma_c(\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t + \epsilon\}; Rf^w_1(K|_U[N]))$$

$$\simeq R\Gamma_c(\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t\}; Rf^w_1(K|_U[N]))$$

$$\simeq R\Gamma_c(\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t - \epsilon\}; Rf^w_1(K|_U[N])) \oplus \mathbb{C}^{d_j}.$$  

This implies that the restriction of $L(Sol_X(M))$ to the fiber $\pi^{-1}(w) \simeq \mathbb{R}$ of the point $w \in V \subset \Omega$ is isomorphic to that of the sheaf

$$\left(\bigoplus_{i=1}^{r} (\mathbb{C}_{\{t + \text{Re}\gamma_i(w) \geq 0\}})^{\oplus m(i)}\right) \oplus \left\{\bigoplus_{i=r+1}^{d} \left(\bigoplus_{k=1}^{n_i} (\mathbb{C}_{\{t + \text{Re}\gamma_k(w) \geq 0\}})^{\oplus m(i,k)}\right)\right\}.$$  

$$\cong \left(\bigoplus_{i=1}^{r} (\mathbb{E}_{V^\text{an}(\gamma_i)^{\text{Reg}}})^{\oplus m(i)}\right) \oplus \left\{\bigoplus_{i=r+1}^{d} \left(\bigoplus_{k=1}^{n_i} (\mathbb{E}_{V^\text{an}(\gamma_i)^{\text{Reg}}}^{\text{Reg}})^{\oplus m(i,k)}\right)\right\}.$$  

Since the subsets $(V \times \mathbb{R}) \cap \{t + \text{Re}\gamma_i(w) \geq 0\} \simeq V \times \mathbb{R}_{\geq 0}$ and $(V \times \mathbb{R}) \cap \{t + \text{Re}\gamma_k(w) \geq 0\} \simeq V \times \mathbb{R}_{\geq 0}$ of $V \times \mathbb{R}$ are connected and simply connected, we can extend this isomorphism to the whole $V \times \mathbb{R} \subset Y^\text{an} \times \mathbb{R}$. This completes the proof.  

\[\square\]
Corollary 6.20. In the situation of Theorem 6.19, the restriction of the Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_Y)$ of $\mathcal{M}$ to $\Omega \subset Y = \mathbb{C}^N$ is an integrable connection. Moreover its rank is equal to

$$\sum_{i=1}^d m(i) = \left( \sum_{i=1}^r m(i) \right) + \sum_{i=r+1}^d \left( \sum_{k=1}^{n_i} m(i,k) \right). \quad (6.118)$$

Since the proof of this corollary is completely the same as that of [22, Corollary 4.5], we omit it here. Next fix a point $w \in Y = \mathbb{C}^N$ such that $w \neq 0$ and set

$$\mathbb{L} := \mathbb{C} w = \{ \lambda w \mid \lambda \in \mathbb{C} \} \subset Y = \mathbb{C}^N.$$

Then $\mathbb{L}$ is a complex line isomorphic to $\mathbb{C}_\lambda$. Assume that $\mathbb{L}$ is not contained in $D := Y \setminus \Omega \subset Y = \mathbb{C}^N$. Let $\mathbb{P} := \mathbb{L} \sqcup \{ \infty \} \subset \overline{Y} = \mathbb{P}^N$ be the projective compactification of $\mathbb{L}$ and $i_{\mathbb{P}} : \mathbb{P} \hookrightarrow \overline{Y} = \mathbb{P}^N$ the inclusion map. Then by Corollary 6.20 the holonomic D-module $\mathcal{L} := \mathcal{H}^0(\mathbb{P})^* \mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_\mathbb{P})$ on $\mathbb{P}$ is a meromorphic connection on a neighborhood of the point $\infty \in \mathbb{P}$. The following result is a generalization of [14, Theorem 5.6] and [22, Theorem 4.6]. Let $\varpi_{\mathbb{P}} : \tilde{\mathbb{P}} \to \mathbb{P}$ be the real oriented blow-up of $\mathbb{P}$ along the divisor $\{ \infty \} \subset \mathbb{P}$.

**Theorem 6.21.** In the situation of Theorem 6.19, for any point $\theta \in \varpi^{-1}\{ \infty \} \simeq S^1$ there exists its open neighborhood $W$ in $\tilde{\mathbb{P}}$ such that we have an isomorphism

$$\mathcal{L}^A|_W \simeq \left( \bigoplus_{i=1}^d ((\mathcal{E}_{L|\mathbb{P}}^{g_i(\lambda w)})^\oplus m(i)) \right)|_W$$

of $\mathcal{D}^A$-modules (see Section 7 for the definition) on $W$. In particular, the functions $g_i(\lambda w)$ of $\lambda$ are the exponential factors of the meromorphic connection $\mathcal{L}$ at the point $\infty \in \mathbb{P}$.

Moreover the multiplicity of $g_i(\lambda w)$ is equal to $m(i)$.

**Proof.** By [22, Proposition 3.5], for any point $\theta \in \varpi^{-1}\{ \infty \} \simeq S^1$ there exists its sectorial neighborhood $V_\theta \subset \mathbb{P} \setminus \{ \infty \}$ such that we have isomorphisms

$$\pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}(\mathcal{E}_{L|\mathbb{P}}^{g_i(\lambda w)}) \simeq \mathbb{P}^{\text{Reg}_i(\lambda w)}_{V_\theta} \simeq \pi^{-1}C_{V_\theta} \otimes \left( \lim_{a \to +\infty} \mathbb{C}_{(t \geq -\text{Reg}_i(\lambda w) + a)} \right). \quad (6.119)$$

On the other hand, by Theorem 6.19 there exists an isomorphism

$$\pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}(\mathcal{L}) \simeq \pi^{-1}C_{V_\theta} \otimes \left( \lim_{a \to +\infty} \mathbb{C}^{\oplus m(i)}_{(t \geq -\text{Reg}_i(\lambda w) + a)} \right). \quad (6.120)$$

We thus obtain an isomorphism

$$\pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}(\mathcal{L}) \simeq \pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P} \left( \bigoplus_{i=1}^d ((\mathcal{E}_{L|\mathbb{P}}^{g_i(\lambda w)})^\oplus m(i)) \right).$$

Then the assertions follow from [22, Corollary 3.11 and Theorem 3.18].

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As in Esterov-Takeuchi [14, Remark 5.7], by Theorems 6.19 and 6.21 we easily obtain the Stokes lines of the meromorphic connection $\mathcal{L} \in \text{Mod}_\text{hol}(\mathcal{D}_\mathbb{P})$ at $\infty \in \mathbb{P}$. We leave the precise formulation to the readers.

By Theorem 6.19 at generic points $v \in D = Y \setminus \Omega \subset Y = \mathbb{C}^N$ at which $D$ is a smooth hypersurface in $Y = \mathbb{C}^N$ we obtain also the irregularity and the exponential factors of $\mathcal{M}^\wedge$ along it as follows. Let $D_{\text{reg}} \subset D$ be the smooth part of $D$ and $v \in D_{\text{reg}}$ such a generic point. Take a subvariety $M \subset Y$ of $Y = \mathbb{C}^N$ which intersects $D_{\text{reg}}$ at $v$ transversally. We call it a normal slice of $D$ at $v$. By definition $M$ is smooth and of dimension 1 on a neighborhood of $v$. Let $i_M : M \hookrightarrow Y = \mathbb{C}^N$ be the inclusion map and define (possibly multi-valued) holomorphic functions $\varphi_i : M \setminus \{v\} \rightarrow \mathbb{C}$ ($1 \leq i \leq d$) by

$$\varphi_i(u) = g_i(i_0(u)).$$

Then it is easy to see that $\varphi_i(u)$ are Laurent Puiseux series of $u$ (see Kirwan [37, Section 7.2] etc.). For each Laurent Puiseux series

$$\varphi_i(u) = \sum_{a \in \mathbb{Q}} c_{i,a} u^a \quad (c_{i,a} \in \mathbb{C})$$

set $r_i = \min\{a \in \mathbb{Q} \mid c_{i,a} \neq 0\}$ and define its pole order $\text{ord}_{\{v\}}(\varphi_i) \geq 0$ by

$$\text{ord}_{\{v\}}(\varphi_i) = \begin{cases} -r_i & (r_i < 0) \\ 0 & \text{(otherwise)}. \end{cases}$$

Then we obtain the following theorem.

**Theorem 6.22.** The exponential factors appearing in the Hukuhara-Levelt-Turrittin decomposition of the meromorphic connection $\mathcal{K}(\star \{v\})$ at $v \in M$ are the pole parts of $\varphi_i$ ($1 \leq i \leq d$). Moreover for any $1 \leq i \leq d$ the multiplicity of the pole part of $\varphi_i$ is equal to $m(i)$. In particular, the irregularity of the meromorphic connection $\mathcal{K}(\star \{v\})$ along $v \in M$ is given by

$$\text{irr}(\mathcal{K}(\star \{v\})) = \sum_{i=1}^{d} m(i) \cdot \text{ord}_{\{v\}}(\varphi_i).$$

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7 Toward the Study of Fourier Transforms of General Holonomic D-modules

7.1 Preliminary Results for Holonomic D-modules

In this subsection, we prove some new formulas which might be useful to extend our results in Section 4 to arbitrary holonomic D-modules. First of all, let us recall some notions and results in [9, §7]. Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Denote by $\varpi_X : \tilde{X} \to X$ the real oriented blow-up of $X$ along $D$ (sometimes we denote it simply by $\varpi$). Then we set

$$O^L_X := R \text{Hom}_{\varpi^{-1}\mathcal{D}_{\tilde{X}}}(\varpi^{-1}O_{\tilde{X}}, \mathcal{D}_{\tilde{X}}),$$

$$A^L_{\tilde{X}} := \alpha_{\tilde{X}}O^L_X,$$

$$D^A_{\tilde{X}} := A^L_{\tilde{X}} \otimes_{\varpi^{-1}O_{\tilde{X}}} \varpi^{-1}\mathcal{D}_{\tilde{X}},$$

where $\mathcal{D}_{\tilde{X}}$ stands for the ind-sheaf of tempered distributions on $\tilde{X}$ (for the definition see [9, Notation 7.2.4]). Recall that a section of $\mathcal{A}_{\tilde{X}}$ is a holomorphic function having moderate growth at $\varpi^{-1}(D)$. Note that $\mathcal{A}_{\tilde{X}}$ and $\mathcal{D}^A_{\tilde{X}}$ are sheaves of rings on $\tilde{X}$. For $M \in \text{D}^b(\mathcal{D}_X)$ we define an object $\mathcal{M}^A \in \text{D}^b(\mathcal{D}_{\tilde{X}})$ by

$$\mathcal{M}^A := D^A_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M \cong A^L_{\tilde{X}} \otimes_{\varpi^{-1}O_{\tilde{X}}} \varpi^{-1}M.$$

Note that if $M$ is a holonomic $\mathcal{D}_X$-module such that $M \xrightarrow{\sim} M(\ast D)$ and $\text{sing, supp}(M) \subset D$, then one has $\mathcal{M}^A \cong \mathcal{D}^A_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M$ (see [9, Lemma 7.3.2]). Moreover we have an isomorphism $\mathcal{M}^A \xrightarrow{\sim} \mathcal{M}(\ast D)^A$ for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ (see [9, Lemma 7.2.2]). Let us take local coordinates $(u, v) = (u_1, \ldots, u_l, v_1, \ldots, v_{n-l})$ of $X$ such that $D = \{u_1u_2 \cdots u_l = 0\}$. We define a partial order $\leq$ on the set $\mathbb{Z}^l$ by

$$a = (a_1, \ldots, a_l) \leq a' = (a'_1, \ldots, a'_l) \iff a_i \leq a'_i \ (1 \leq i \leq l).$$

Then for a meromorphic function $\varphi \in \mathcal{O}_X(\ast D)$ on $X$ having a pole along $D$ by using its Laurent expansion

$$\varphi = \sum_{a \in \mathbb{Z}^l} c_a(\varphi)(v) \cdot u^a \in \mathcal{O}_X(\ast D)$$

with respect to $u_1, \ldots, u_l$ we define its order $\text{ord}(\varphi) \in \mathbb{Z}^l$ to be the minimum

$$\min \left\{ \{a \in \mathbb{Z}^l \mid c_a(\varphi) \neq 0\} \cup \{0\} \right\}$$

if it exists. In [93, Chapter 5] Mochizuki defined the notion of good sets of irregular values on $(X, D)$ to be finite subsets $S \subset \mathcal{O}_X(\ast D)/\mathcal{O}_X$ such that

(i): $\text{ord}(\varphi)$ exists for any $\varphi \in S$ and if $\varphi \neq 0$ then its leading term $c_{\text{ord}(\varphi)}(\varphi)(v)$ does not vanish at any point $v \in Y := \{u_1 = \cdots = u_l = 0\} \subset D$.

(ii): $\text{ord}(\varphi - \psi)$ exists for any $\varphi \neq \psi$ in $S$ and then $\text{ord}(\varphi - \psi) \in \mathbb{Z}^l_{\geq 0} \setminus \{0\}$ and the leading term $c_{\text{ord}(\varphi - \psi)}(\varphi - \psi)(v)$ does not vanish at any point $v \in Y = \{u_1 = \cdots = u_l = 0\} \subset D$.

(iii): the subset $\{\text{ord}(\varphi - \psi) \mid \varphi, \psi \in S, \varphi \neq \psi\} \subset \mathbb{Z}^l$ is totally ordered with respect to the order $\leq$ on $\mathbb{Z}^l$. 

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Definition 7.1. Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Then we say that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a normal form along $D$ if

(i) $\mathcal{M} \cong \mathcal{M}(\ast D)$

(ii) sing supp($\mathcal{M}$) $\subset D$

(iii) for any $\theta \in \varpi^{-1}(D) \subset \tilde{X}$, there exist an open neighborhood $U \subset X$ of $\varpi(\theta) \in D$ in $X$, a good set $S = \{[\varphi_1], [\varphi_2], \ldots, [\varphi_k]\} \subset \mathcal{O}_X(\ast D)/\mathcal{O}_X (\varphi_i \in \mathcal{O}_X(\ast D))$ of irregular values on $(U, D \cap U)$, positive integers $m_i > 0$ ($1 \leq i \leq k$) and an open neighborhood $W$ of $\theta$ with $W \subset \varpi^{-1}(U)$ such that

$$\mathcal{M}^A|_W \cong \bigoplus_{i=1}^k \left( (\mathcal{E}^\varphi_i)^A|_W \right)^{\otimes m_i}. \quad (7.1)$$

By [22] Proposition 3.19] the good set $S \subset \mathcal{O}_X(\ast D)/\mathcal{O}_X$ of irregular values for $\mathcal{M}$ in this definition does not depend on the point $\theta \in \varpi^{-1}(D)$. Moreover by [22] Proposition 3.5 for any $\theta \in \varpi^{-1}(D \cap U)$ there exists its sectorial open neighborhood $V \subset U \setminus D$ such that

$$\pi^{-1}C_V \otimes S_{\mathcal{O}_X} \cong \bigoplus_{i=1}^k \left( E_{\mathcal{O}_X}^\varphi_i \right)^{\otimes m_i}. \quad (7.2)$$

Lemma 7.2. In the situation as above, there exists a sectorial open neighborhood $V \subset U \setminus D$ of $\theta \in \varpi^{-1}(D \cap U)$ such that for any $1 \leq i, j \leq k$ the natural morphism

$$\text{Hom}^E(E_{\mathcal{O}_X}^{\varphi_i}, E_{\mathcal{O}_X}^{\varphi_j}) \longrightarrow \text{Hom}^E(E_{\mathcal{O}_X}^{\varphi_i}, E_{\mathcal{O}_X}^{\varphi_j}) \quad (7.3)$$

is an isomorphism.

Proof. The proof is similar to that of [10] Lemma 5.2.1 (ii)]. It suffices to consider only pairs $(i, j)$ such that $i \neq j$. Then by $[\varphi_i] \neq [\varphi_j]$ the function $\varphi_j - \varphi_i \in \mathcal{O}_X(\ast D)$ has a pole along the normal crossing divisor $D \subset X$. For a local coordinate system $(u, v) = (u_1, \ldots, u_l, v_1, \ldots, v_{n-l})$ of $X$ such that $D = \{u_1u_2\cdots u_l = 0\}$ and $\varpi(\theta) \in Y := \{u_1 = u_2 = \cdots = u_l = 0\} \subset D$ let

$$(\varphi_j - \varphi_i)(u, v) = \sum_{a \in \mathbb{Z}^l} c_{\alpha}(\varphi_j - \varphi_i)(v) \cdot u^a \in \mathcal{O}_X(\ast D) \quad (7.4)$$

be the Laurent expansion of $\varphi_j - \varphi_i$ with respect to $u_1, u_2, \ldots, u_l$. Then by the goodness of the set $S$ the order $\alpha := \text{ord}(\varphi_j - \varphi_i) \in \mathbb{Z}^l$ is defined and satisfies the condition

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \in (\mathbb{Z}_{\leq 0})^l \setminus \{0\}. \quad (7.5)$$

If there exists a sectorial open neighborhood $V \subset X \setminus D$ of $\theta \in \varpi^{-1}(D)$ such that $\text{Re}(\varphi_j - \varphi_i) \leq 0$ on $V$, then by the proof of [10] Lemmas 3.1.1 and 3.2.2] we have isomorphisms

$$\text{Hom}^E(E_{\mathcal{O}_X}^{\varphi_i}, E_{\mathcal{O}_X}^{\varphi_j}) \cong \mathbb{C}, \quad \text{Hom}^E(E_{\mathcal{O}_X}^{\varphi_j}, E_{\mathcal{O}_X}^{\varphi_j}) \cong \mathbb{C}. \quad (7.6)$$

Otherwise, the point $\theta \in \varpi^{-1}(D)$ is contained in the closure $\overline{R}$ of the open subset

$$R := \{(u, v) \in X \setminus D \mid \text{Re}(\varphi_j - \varphi_i)(u, v) > 0\} \quad (7.7)$$

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of $\tilde{X}$. As the meromorphic function $\varphi_j - \varphi_i \in \mathcal{O}_X(*D)$ has no point of indeterminacy by the goodness of the set $S$, we can easily show that $R$ is a subanalytic open subset of $\tilde{X}$. Then by the curve selection lemma there exists a real analytic curve $\gamma(t) : [0, \varepsilon) \rightarrow \overline{R}$ ($\varepsilon > 0$) (defined on a neighborhood of $0 \in \mathbb{R}$) such that $\gamma(0) = \theta \in \mathbb{R}^{-1}(D) \cap \overline{R}$ and $\gamma(t) \in R$ for any $t \in (0, \varepsilon)$. By the leading term $c_\alpha(\varphi_j - \varphi_i)(v) \cdot u^\alpha$ of the Laurent expansion of $\varphi_j - \varphi_i$, we define a complex valued real analytic function $\psi : \mathbb{R}^{-1}(Y) \rightarrow \mathbb{C}$ on the real analytic manifold $\mathbb{R}^{-1}(Y) \simeq (S^1)^l \times Y$ (defined on a neighborhood of the point $\theta \in \mathbb{R}^{-1}(Y)$) by

$$\psi(e^{i\delta_1}, e^{i\delta_2}, \ldots, e^{i\delta_l}, v) := c_\alpha(\varphi_j - \varphi_i)(v) \cdot e^{i\alpha_1\delta_1} e^{i\alpha_2\delta_2} \cdots e^{i\alpha_l\delta_l}.$$  

(7.8)

Recall that the holomorphic function $c_\alpha(\varphi_j - \varphi_i)$ on $Y$ does not vanish at any point $v \in Y$. Suppose that $\text{Re}\psi(\theta) < 0$. Then for the real analytic curve $\gamma$ we have

$$\lim_{t \rightarrow +0} \text{Re}(\varphi_j - \varphi_i)(\gamma(t)) = -\infty.$$  

(7.9)

Then contradicts to the condition $\gamma(t) \in R$ ($t \in (0, \varepsilon)$). We thus obtain $\text{Re}\psi(\theta) \geq 0$. Then, by the above definition of $\psi$, for any sectorial open neighborhood $V \subset X \setminus D$ of $\theta \in \mathbb{R}^{-1}(Y)$ there exists a point $\theta' = (e^{i\delta_1}, \ldots, e^{i\delta_l}, v) \in \mathbb{R}^{-1}(Y) \cap \text{Int}(\overline{V})$ such that $\text{Re}\psi(\theta') > 0$. Define a line $\gamma'(t) : (0, \varepsilon') \rightarrow V (\varepsilon' > 0)$ in $V$ by

$$\gamma'(t) = (te^{i\delta_1}, te^{i\delta_2}, \ldots, te^{i\delta_l}, v) \in V \quad (0 < t < \varepsilon')$$  

(7.10)

so that we have

$$\lim_{t \rightarrow +0} \gamma'(t) = \theta' = (e^{i\delta_1}, \ldots, e^{i\delta_l}, v).$$  

(7.11)

Then by $\text{Re}\psi(\theta') > 0$ we obtain

$$\lim_{t \rightarrow +0} \text{Re}(\varphi_j - \varphi_i)(\gamma'(t)) = +\infty,$$  

(7.12)

which implies also that $\theta' \in \overline{R}$. By the proof of [10] Lemma 3.1.1 (i)] there exist isomorphisms

$$\text{Hom}^E\left(\mathcal{E}_{V|M}, \mathcal{E}_{V|M}\right) \simeq \text{Hom}_{\mathcal{C}_X}(\mathcal{C}_{V \cap \{\text{Re}(\varphi_j - \varphi_i) \leq 0\}}, \mathcal{C}_X) \simeq 0.$$  

(7.13)

Moreover by the proof of [10] Lemma 3.2.2 (i) we have

$$\text{Hom}^E\left(\mathcal{E}_{V|X}, \mathcal{E}_{V|X}\right) \simeq \lim_{c \rightarrow +\infty} \text{Hom}_{\mathcal{C}_X}(\mathcal{C}_{V \cap \{\text{Re}(\varphi_j - \varphi_i) \leq c\}}, \mathcal{C}_X) \simeq 0.$$  

(7.14)

We thus obtain the desired isomorphism

$$\text{Hom}^E\left(\mathcal{E}_{V|M}, \mathcal{E}_{V|M}\right) \sim \text{Hom}^E\left(\mathcal{E}_{V|X}, \mathcal{E}_{V|X}\right)$$  

(7.15)

for the pair $(i,j)$. Clearly we can take a sectorial open neighborhood $V \subset X \setminus D$ of $\theta \in \mathbb{R}^{-1}(D)$ so that this isomorphism holds for any pair $(i,j)$. \hfill \Box

A ramification of $X$ along the normal crossing divisor $D \subset X$ on a neighborhood $U$ of $x \in D$ is a finite map $\rho : X' \rightarrow U$ of complex manifolds of the form $w \mapsto z = (z_1, z_2, \ldots, z_n) = \rho(w) = (w_1^{d_1}, \ldots, w_i^{d_i}, w_{i+1}, \ldots, w_n)$ for some $(d_1, \ldots, d_i) \in (\mathbb{Z}_{>0})^l$, where $(w_1, \ldots, w_n)$ is a local coordinate system of $X'$ and $(z_1, \ldots, z_n)$ is that of $U$ such that $D \cap U = \{z_1 \cdots z_l = 0\}$.
**Definition 7.3.** Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Then we say that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a quasi-normal form along $D$ if it satisfies the conditions (i) and (ii) of Definition 7.1, and if for any point $x \in D$ there exists a ramification $\rho : X' \to U$ on a neighborhood $U$ of it such that $\mathcal{D}\rho^*(\mathcal{M}|_U)$ has a normal form along the normal crossing divisor $\rho^{-1}(D \cap U)$.

Note that $\mathcal{D}\rho^*(\mathcal{M}|_U)$ as well as $\mathcal{D}\rho_!\mathcal{D}\rho^*(\mathcal{M}|_U)$ is concentrated in degree zero and $\mathcal{M}|_U$ is a direct summand of $\mathcal{D}\rho_!\mathcal{D}\rho^*(\mathcal{M}|_U)$. Now let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module having a quasi-normal form along the normal crossing divisor $D \subset X$. Then for any point $x \in D$ there exists a ramification $\rho : X' \to U$ on a neighborhood $U$ of it such that $\mathcal{D}\rho^*(\mathcal{M}|_U)$ has a normal form along the normal crossing divisor $D' := \rho^{-1}(D \cap U) \subset X'$. Note that $\rho^{-1}(x) \subset D'$ is a point and denote it by $x'$. Let $\varpi' : \tilde{X}' \to X'$ be the real oriented blow-up of $X'$ along $D'$ and $\tilde{\rho} : \tilde{X}' \to \tilde{X}$ the morphism induced by $\rho$. Then by [22] Propositions 3.5 and 3.19] there exist a unique good set $S = \{[\varphi_1], [\varphi_2], \ldots, [\varphi_k]\} \subset \mathcal{O}_{\tilde{X}'}(\ast D')/\mathcal{O}_{X'}(\varphi_i \in \mathcal{O}_{X'}(\ast D'))$ of irregular values on a neighborhood of $x' \in D'$ in $X'$ and positive integers $m_i > 0$ ($1 \leq i \leq k$) such that for any $\vartheta ' \in (\varpi ')^{-1}(D')$ and its sufficiently small sectorial open neighborhood $V' \subset X' \setminus D'$ we have an isomorphism

$$
\pi^{-1}\mathcal{C}_{V'} \otimes \text{Sol}_{\chi'}^E(\mathcal{D}\rho^*(\mathcal{M}|_U)) \cong \bigoplus_{i=1}^{k} \left( \mathbb{F}^{\text{Ref}}_{V'|X'} \right)^{\oplus m_i}. \quad (7.16)
$$

For a point $\theta \in \varpi^{-1}(D \cap U)$ and its sufficiently small sectorial open neighborhood $V \subset U \setminus D$ we take a point $\theta ' \in (\varpi ')^{-1}(D')$ such that $\tilde{\rho}(\theta ') = \theta$ and its sectorial open neighborhood $V' \subset X' \setminus D'$ such that $\rho|_{V'} : V' \to V$. Define holomorphic functions $f_i : V \to \mathbb{C}$ ($1 \leq i \leq k$) by $f_i := \varphi_i \circ (\rho|_{V'})^{-1}$. Then by [22] Proposition 3.5] we obtain an isomorphism

$$
\pi^{-1}\mathcal{C}_{V} \otimes \text{Sol}_{\chi}^E(\mathcal{M}) \cong \bigoplus_{i=1}^{k} \left( \mathbb{F}^{\text{Ref}_{f_i}}_{V|X} \right)^{\oplus m_i}. \quad (7.17)
$$

As $\tilde{\rho} : \tilde{X}' \to \tilde{X}$ is locally an isomorphism, then it is also clear that on an open neighborhood $W$ of $\theta$ in $\tilde{X}$ we have an isomorphism

$$
\mathcal{M}^A|_W \cong \bigoplus_{i=1}^{k} \left( \mathcal{C}^A_{\text{Ref}_{f_i}^{1, D|U}} |_W \right)^{\oplus m_i}. \quad (7.18)
$$

Moreover, by the proof of Lemma 7.2 we obtain the following result.

**Lemma 7.4.** In the situation as above, there exists a sectorial open neighborhood $V \subset U \setminus D$ of $\theta \in \varpi^{-1}(D \cap U)$ such that for any $1 \leq i, j \leq k$ the natural morphism

$$
\text{Hom}^E(\mathbb{F}^{\text{Ref}_{f_i}}_{V|\mathcal{M}}, \mathbb{F}^{\text{Ref}_{f_j}}_{V|\mathcal{M}}) \to \text{Hom}^E(\mathbb{F}^{\text{Ref}_{f_i}}_{V'|X}, \mathbb{F}^{\text{Ref}_{f_j}}_{V'|X}) \quad (7.19)
$$

is an isomorphism.

In order to improve (7.17) and obtain a higher-dimensional analogue of D’Agnolo-Kashiwara [10] Proposition 5.4.5], let us prepare some notations (see [10] Section 5) for
the details in the one dimensional case). For the real oriented blow-up \( \varpi : \tilde{X} \to X \) of \( X \) along the normal crossing divisor \( D \subset X \) consider the following commutative diagram

\[
\begin{array}{ccc}
\varpi^{-1}(D) & \xrightarrow{j} & \tilde{X} \\
\downarrow{\tilde{\imath}} & & \downarrow{\varpi} \\
X \setminus D & \xrightarrow{j} & X,
\end{array}
\]

(7.20)

where \( \tilde{\imath}, \tilde{j}, j \) are the natural embeddings. For an open subset \( \Omega \subset \tilde{X}, \ f \in \Gamma(\Omega; \tilde{j}^*j^{-1}\mathcal{O}_X) \simeq \Gamma(\tilde{\imath}^{-1}\Omega; \mathcal{O}_{\tilde{X} \setminus D}) \) and \( \theta \in \Omega \cap \varpi^{-1}(D) \) we say that \( f \) admits a Puiseux expansion along \( D \subset X \) at \( \theta \) if there exist a ramification \( \rho : X' \to U \) of a neighborhood \( U \) of \( \varpi(\theta) \in D \) along \( D \cap U \subset U \), a sectorial neighborhood \( V \subset U \setminus D \) of \( \theta \) contained in \( \tilde{\imath}^{-1}\Omega = \varpi(\Omega \setminus \varpi^{-1}(D)) \subset X \setminus D \) and a meromorphic function \( g \in \mathcal{O}_{X'}(D') \) along the normal crossing divisor \( D' := \rho^{-1}(D) \subset X' \) defined on an open neighborhood \( W \subset \rho^{-1}(V \cap D) = \rho^{-1}(V) \cap D' \) in \( X' \) such that the pull-back of \( f|_V \in \mathcal{O}_X(V) \) by \( \rho \) coincides with \( g \) on the open subset \( W \cap \rho^{-1}(V) \subset W \). We denote by \( \mathcal{P}_\tilde{X} \) the subsheaf of \( \tilde{j}^*j^{-1}\mathcal{O}_X \) whose sections are defined by

\[
\Gamma(\Omega; \mathcal{P}_\tilde{X}) := \{ f \in \Gamma(\Omega; \tilde{j}^*j^{-1}\mathcal{O}_X) \mid \text{For any } \theta \in \Omega \cap \varpi^{-1}(D), \ f \text{ admits a Puiseux expansion along } D \subset X \text{ at } \theta. \}
\]

for open subsets \( \Omega \subset \tilde{X} \). Then we define the sheaf of Puiseux germs \( \mathcal{P}_{\varpi^{-1}(D)} \) on \( \varpi^{-1}(D) \) by

\[
\mathcal{P}_{\varpi^{-1}(D)} := \tilde{\imath}^{-1}\mathcal{P}_\tilde{X}.
\]

(7.21)

For a point \( \theta \in \varpi^{-1}(D) \) if we take a local coordinate \( (u, v) = (u_1, \ldots, u_l, v_1, \ldots, v_{n-l}) \) of \( X \) on a neighborhood of \( \varpi(\theta) \in D \) in \( X \) such that \( \varpi(\theta) = (0, 0, 0) = D \) then the stalk of \( \mathcal{P}_{\varpi^{-1}(D)} \) at \( \theta \) is isomorphic to the ring

\[
\bigcup_{p \in \mathbb{Z}_{\geq 1}} \mathbb{C}\{u_1^{\frac{1}{p}}, \ldots, u_l^{\frac{1}{p}}, v_1, \ldots, v_{n-l}\}\left[u_1^{-\frac{1}{p}}, \ldots, u_l^{-\frac{1}{p}}\right].
\]

(7.22)

of Puiseux series along \( D \subset X \). We denote by \( \mathcal{P}_{\varpi^{-1}(D)}^{\leq 0} \) the subsheaf of \( \mathcal{P}_{\varpi^{-1}(D)} \) consisting of sections locally contained in the ring

\[
\bigcup_{p \in \mathbb{Z}_{\geq 1}} \mathbb{C}\{u_1^{\frac{1}{p}}, \ldots, u_l^{\frac{1}{p}}, v_1, \ldots, v_{n-l}\}
\]

(7.23)

for some (hence, any) local coordinate \( (u, v) = (u_1, \ldots, u_l, v_1, \ldots, v_{n-l}) \) of \( X \) as above. By this definition, it is clear that for any point \( x \in D \) there exist its neighborhood \( U \) in \( X \) and a subsheaf \( \mathcal{P}'_{\varpi^{-1}(D \cap U)} \subset \mathcal{P}_{\varpi^{-1}(D \cap U)} \) of \( \mathcal{C}_{\varpi^{-1}(D \cap U)} \)-modules defined on the open subset \( \varpi^{-1}(D \cap U) \subset \varpi^{-1}(D) \) such that the natural morphism

\[
\mathcal{P}'_{\varpi^{-1}(D \cap U)} \to \mathcal{P}_{\varpi^{-1}(D \cap U)}/\mathcal{P}_{\varpi^{-1}(D \cap U)}^{\leq 0}
\]

(7.24)

is an isomorphism. We call such \( \mathcal{P}'_{\varpi^{-1}(D \cap U)} \) a representative subsheaf of \( \mathcal{P}_{\varpi^{-1}(D \cap U)} \). By slightly modifying the definition of the multiplicities in D’Agnolo-Kashiwara [10] Section 5.3], we shall use the following one (cf. [39] Definition 2.4)].
Definition 7.5. (cf. [10 Section 5.3] and [39 Definition 2.4]) In the situation as above, we say that a morphism \( N : \mathcal{P}_{\varpi^{-1}(D \cap U)} \to (\mathbb{Z}_{\geq 0})_{\varpi^{-1}(D \cap U)} \) of sheaves of sets is a multiplicity along \( D \cap U \subset U \) if there exists a ramification \( \rho : X' \to U \) of \( U \) along \( D \cap U \subset U \) such that for any \( \theta \in \varpi^{-1}(D) \), the subset \( N_{\theta}^{> 0} := \{ f \in N_{\theta}^{-1}(\mathbb{Z}_{\geq 0}) \} \subset \mathcal{P}_{\varpi^{-1}(D \cap U), \theta} \) is finite and the pull-backs of its elements \( f \in N_{\theta}^{> 0} \) by \( \rho \) are meromorphic functions on \( X' \) along \( D' := \rho^{-1}(D) \subset X' \) and form a good set \( \{ f \circ p \mid f \in N_{\theta}^{> 0} \} \subset \mathcal{O}_{X'}(\star D')/\mathcal{O}_{X'} \) of irregular values on \( (X', D') \) on a neighborhood of the point \( \rho^{-1}(\varpi(\theta)) \in D' \).

Definition 7.6. (cf. [10 Definition 5.3.1] and [39 Definition 2.5]) In the situation as above, we say that an \( \mathbb{R} \)-constructible enhanced sheaf \( F \in \mathbf{E}^{b}(\mathcal{C}_{X}) \) on \( X \) has a quasi-normal form along the normal crossing divisor \( D \cap U \subset U \) if there exists a multiplicity \( N : \mathcal{P}_{\varpi^{-1}(D \cap U)} \to (\mathbb{Z}_{\geq 0})_{\varpi^{-1}(D \cap U)} \) such that any point \( \theta \in \varpi^{-1}(D \cap U) \) has its sectorial open neighborhood \( V_{\theta} \subset U \setminus D \subset \tilde{X} \) for which we have an isomorphism

\[
\pi^{-1}C_{V_{\theta}} \otimes F \cong \bigoplus_{f \in N_{\theta}^{> 0}} \left( E_{\mathcal{V}_{\theta}/X}^{Ref} \right)^{N(f)} .
\]

(7.25)

Enhanced ind-sheaves having a quasi-normal form along the normal crossing divisor \( D \cap U \subset U \) are defined similarly.

Lemma 7.7. Assume that a holonomic \( \mathcal{D}_{X} \)-module \( \mathcal{M} \) has a quasi-normal form along the normal crossing divisor \( D \subset X \). Then for any point \( x \in D \) there exists a subanalytic open neighborhood \( U \) of \( x \) in \( X \) such that the \( \mathbb{R} \)-constructible enhanced ind-sheaf

\[
\pi^{-1}C_{U} \otimes \text{Sol}^{E}_{X}(\mathcal{M}) \cong \pi^{-1}C_{U \setminus D} \otimes \text{Sol}^{E}_{X}(\mathcal{M})
\]

(7.26)

has a quasi-normal form along the normal crossing divisor \( D \cap U \subset U \).

Proof. The proof is similar to that of [10 Lemma 5.4.4]. With the representative subsheaf \( \mathcal{P}_{\varpi^{-1}(D \cap U)} \) of \( \mathcal{P}_{\varpi^{-1}(D \cap U)} \) at hand, it suffices to use [147, 148] and [22, Propositions 3.10 and 3.19].

In the situation of Lemma 7.7, let \( N : \mathcal{P}_{\varpi^{-1}(D \cap U)} \to (\mathbb{Z}_{\geq 0})_{\varpi^{-1}(D \cap U)} \) be the multiplicity for which the enhanced ind-sheaf \( F \cong \pi^{-1}C_{U} \otimes \text{Sol}^{E}_{X}(\mathcal{M}) \in \mathbf{E}^{b}(\mathcal{C}_{X}) \) has a quasi-normal form along the normal crossing divisor \( D \cap U \subset U \). Then by the proof of Lemma 7.7, the sections of the subsheaf \( N_{\geq 0} = N_{\leq 0}(\mathbb{Z}_{\geq 0})_{\varpi^{-1}(D \cap U)} \subset \mathcal{P}_{\varpi^{-1}(D \cap U)} \) of \( \text{Sol}^{E}_{X}(\mathcal{M}) \) are the exponential factors of \( \mathcal{M} \). Moreover, if the divisor \( D \cap U \subset U \) is smooth and connected, then the non-negative rational number

\[
\sum_{f \in N_{\theta}^{> 0}} N_{\theta}(f) \cdot \text{ord}_{D \cap U}(f) \in \mathbb{Q}_{\geq 0}
\]

(7.27)

associated to a point \( \theta \in \varpi^{-1}(D \cap U) \) is an integer and does not depend on the choice of \( \theta \in \varpi^{-1}(D \cap U) \), where for the exponential factor \( f \in N_{\theta}^{> 0} \) of \( \mathcal{M} \) the rational number \( \text{ord}_{D \cap U}(f) \geq 0 \) stands for the pole order of \( f \) along \( D \cap U \). We call it the irregularity of \( \mathcal{M} \) along \( D \cap U \) and denote it by \( \text{irr}_{D \cap U}(\mathcal{M}) \). If \( D \subset X \) itself is smooth and connected, we define the irregularity \( \text{irr}_{D}(\mathcal{M}) \in \mathbb{Z}_{\geq 0} \) of \( \mathcal{M} \) along \( D \subset X \) similarly. By Lemmas 7.4 and 7.7, we obtain the following higher-dimensional analogue of [10 Proposition 5.4.5]. For a precise explanation of the proof of [10 Proposition 5.4.5], see [39 Remark 2.10].
Proposition 7.8. Assume that a holonomic $D_X$-module $\mathcal{M}$ has a quasi-normal form along the normal crossing divisor $D \subset X$. Then for any point $x \in D$ there exist a subanalytic open neighborhood $U$ of $x$ in $X$ and an $\mathbb{R}$-constructible enhanced sheaf $F \in E^b(\mathbb{C}_X)$ on $X$ having a quasi-normal form along the normal crossing divisor $D \cap U \subset U$ such that

$$\pi^{-1}C_U \otimes \text{Sol}_{E}^{Z}(\mathcal{M}) \simeq \pi^{-1}C_{U \cap D} \otimes \text{Sol}_{E}^{Z}(\mathcal{M}) \simeq \mathbb{C}_X^{\dagger} \otimes F. \quad (7.28)$$

The following fundamental result is due to Kedlaya and Mochizuki.

Theorem 7.9 ([35, 36, 48]). For a holonomic $D_X$-module $\mathcal{M}$ and $x \in X$, there exist an open neighborhood $U$ of $x$, a closed hypersurface $Y \subset U$, a complex manifold $X'$ and a projective morphism $\nu : X' \to U$ such that

(i) sing.$\text{supp}(\mathcal{M}) \cap U \subset Y$,

(ii) $D := \nu^{-1}(Y)$ is a normal crossing divisor in $X'$,

(iii) $\nu$ induces an isomorphism $X \backslash D \simeq U \backslash Y$,

(iv) $(D \nu^* \mathcal{M})(*D)$ has a quasi-normal form along $D$.

This is a generalization of the classical Hukuhara-Levelt-Turrittin theorem to higher dimensions. Now let $X$ be a compact complex manifold, $Y \subset X$ a closed hypersurface and $\mathcal{M}$ a holonomic $D_X$-module such that $\text{sing.} \text{supp}(\mathcal{M}) \subset Y$ and $\mathcal{M}(\ast Y) \simeq \mathcal{M}$. Assume that there exist a projective morphism $\nu : Z \to X$ of a compact complex manifold $Z$ such that $D := \nu^{-1}(Y) \subset Z$ is a normal crossing divisor and $\nu|_{Z \backslash D} : Z \backslash D \to X \backslash Y$ is an isomorphism. Assume also that there exist a ramification $\rho : Z' \to U$ on a neighborhood $U$ of $D$ in $Z$, meromorphic functions

$$\varphi_1, \varphi_2, \ldots, \varphi_k \in \Gamma \left( Z' : \mathcal{O}_{Z'}(\ast \rho^{-1}(D)) \right) \quad (7.29)$$

and positive integers $m_i > 0$ ($1 \leq i \leq k$) such that $D \rho^*(D \nu^* \mathcal{M}|_U)$ has a normal form along the normal crossing divisor $D' := \rho^{-1}(D) \subset Z'$ for the set

$$S = \{ [\varphi_1], [\varphi_2], \ldots, [\varphi_k] \} \subset \mathcal{O}_{Z'}(\ast D')/\mathcal{O}_{Z'} \quad (7.30)$$

which is good at each point of $D'$ and $m_i > 0$ ($1 \leq i \leq k$). Then by the proof of Proposition 7.8 we can also show that there exist a semi-analytic open neighborhood $V \subset U$ of $D$ in $Z$ and an $\mathbb{R}$-constructible enhanced sheaf $F \in E^b(\mathbb{C}_Z)$ on $Z$ having a quasi-normal form along the normal crossing divisor $D$ at each point of $D$ such that

$$\pi^{-1}C_{V \backslash D} \otimes \text{Sol}_{Z}^{E}(D \nu^* \mathcal{M}) \simeq \mathbb{C}_Z^{\dagger} \otimes F. \quad (7.31)$$

Since in this situation there exists an isomorphism

$$\mathcal{M} \sim \rightarrow D \nu_* (D \nu^* \mathcal{M}), \quad (7.32)$$

for the $\mathbb{R}$-constructible enhanced sheaf $G := E \nu_* F \in E^b(\mathbb{C}_X)$ on $X$ and the open neighborhood $W := \nu(V)$ of $Y$ in $X$ we obtain an isomorphism

$$\pi^{-1}C_{W \backslash Y} \otimes \text{Sol}_{X}^{E}(\mathcal{M}) \simeq \mathbb{C}_X^{\dagger} \otimes G. \quad (7.33)$$
As in Kudomi-Takeuchi [39], one can also slightly modify the $\mathbb{R}$-constructible enhanced sheaf $G \in E^b(\mathbb{C}_X)$ so that we have isomorphisms

$$\text{Sol}_{X}^E(M) \simeq \pi^{-1}C_{X|Y} \otimes \text{Sol}_{X}^E(M) \simeq C_{X}^{\perp} \oplus G. \quad (7.34)$$

By this very explicit description of the enhanced solution complex $\text{Sol}_{X}^E(M)$, in the one dimensional case $d_X = \dim X = 1$ we can apply the Morse theoretical argument in [22] to improve the main results in [10]. See [39] for the details.

### 7.2 Fourier Transforms of Standard Holonomic D-modules

In this subsection, let $X$ be the affine space $\mathbb{C}^N$ of dimension $N$ and regard it as an algebraic variety endowed with the Zariski topology. Let $S \subset X$ be a smooth and connected quasi-affine subvariety of $X = \mathbb{C}^N$ of dimension $n$ and $\mathcal{N}$ an algebraic integrable connection on it. For some technical reason, we assume here that there exists an algebraic hypersurface $H \subset X$ of $X$ such that $S \subset X \setminus H$ and $S$ is closed in $X \setminus H$. This in particular implies that the inclusion map $i_S : S \hookrightarrow X$ is affine. Then we set

$$\mathcal{M} := Di_{S,*}\mathcal{N} \simeq i_{S,*}\mathcal{N} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X). \quad (7.35)$$

By the standard operations for algebraic D-modules, for the initial study of Fourier transforms of general holonomic D-modules on $X = \mathbb{C}^N$ it suffices to study those for such holonomic $\mathcal{D}_X$-modules. Recall that the Fourier transform is an exact functor. For this reason, let us call $\mathcal{M}$ a standard holonomic D-module on $X$. Let $\overline{X} = \mathbb{P}^N$ be the projective compactification of $X = \mathbb{C}^N$ and $\overline{S} \subset \overline{X}$ the closure of $S$ in it. Let $\nu : Z \longrightarrow \overline{X}$ be a proper morphism of a smooth variety $Z$ such that $\nu(Z) = \overline{S}$, the restriction $\nu^{-1}S \longrightarrow S$ of $\nu$ is an isomorphism and $D := \nu^{-1}(\overline{S}\setminus S) = Z \setminus \nu^{-1}S \subset Z$ is a normal crossing divisor in $Z$. Let $i_S : S \simeq \nu^{-1}S \hookrightarrow Z$ be the inclusion map and consider the algebraic meromorphic connection

$$\mathcal{M}' := D i_{S,*}\mathcal{N} \simeq i_{S,*}\mathcal{N} \in \text{Mod}_{\text{hol}}(\mathcal{D}_Z) \quad (7.36)$$

on $Z$ for which we have an isomorphism

$$\widetilde{\mathcal{M}} \simeq D\nu_*\mathcal{M}'. \quad (7.37)$$

Assume that the analytification $(\mathcal{M}')^{an}$ of $\mathcal{M}'$ on $Z^{an}$ has a quasi-normal form along the normal crossing divisor $D^{an} \subset Z^{an}$ and satisfies the nice property that we assumed at the end of Subsection 7.1. Then by the proof of Proposition 7.8 there exists a semi-analytic open neighborhood $\mathcal{V}$ of $D^{an}$ in $Z^{an}$ and an $\mathbb{R}$-constructible enhanced sheaf $F \in E^b(\mathbb{C}_{Z^{an}})$ on $Z^{an}$ such that we have an isomorphism

$$\pi^{-1}C_{\mathcal{V}\setminus D^{an}} \otimes \text{Sol}_{Z}^E(\mathcal{M}') \simeq C_{Z^{an}}^{\perp} \oplus F. \quad (7.38)$$

For the $\mathbb{R}$-constructible enhanced sheaf $G := E\nu_*F \in E^b(\mathbb{C}_{X^{an}})$ on $X^{an}$ and the semi-analytic open subset $\mathcal{W} := \nu(\mathcal{V}) \subset S^{an}$ of $S^{an}$ containing $\partial S := (\overline{S} \setminus S)^{an}$ we thus obtain an isomorphism

$$\pi^{-1}C_{\mathcal{W}\setminus \partial S} \otimes \text{Sol}_{X}^E(\widetilde{\mathcal{M}}) \simeq C_{X^{an}}^{\perp} \oplus G. \quad (7.39)$$
Let us explain the structure of $G$ more precisely. First, by the construction of $F$ there exist (possibly multi-valued) holomorphic functions $f_i : \mathcal{V} \backslash D^\text{an} \rightarrow \mathbb{C} (1 \leq i \leq k)$ on $\mathcal{V} \backslash D^\text{an}$ and positive integers $m_i > 0 (1 \leq i \leq k)$ for which the enhanced sheaf $F \in \mathcal{E}^b(\mathbb{C}^\times)$ has a quasi-normal form along the normal crossing divisor $D^\text{an} \subset Z^\text{an}$ in the sense of Definition 7.4. Recall that we constructed $F$ by glueing some enhanced sheaves $F_j$ on open sectors $V_j \subset \mathcal{V} \backslash D^\text{an} \subset Z^\text{an} \backslash D^\text{an}$ along $D^\text{an}$ and for any $1 \leq i \leq k$ the analytic continuation of $f_i$ along any curve $\gamma \subset \mathcal{V} \backslash D^\text{an}$ concides with $f_j$ for some $1 \leq j \leq k$ (see the proof of [10, Proposition 5.4.5]). Then we define a complex hypersurface $\mathcal{V} \backslash D^\text{an} \subset (\mathcal{V} \backslash D^\text{an}) \times \mathbb{C}$ of $(\mathcal{V} \backslash D^\text{an}) \times \mathbb{C}$ by

$$\mathcal{V} \backslash D^\text{an} := \bigcup_{i=1}^k \{ (x, -f_i(x)) \mid x \in \mathcal{V} \backslash D^\text{an} \} \subset (\mathcal{V} \backslash D^\text{an}) \times \mathbb{C},$$

where in the right hand side we take the union considering all the possible branches of the (possibly multi-valued) functions $f_i$. Since the pull-backs of $f_i$ ($1 \leq i \leq k$) by a ramification $\rho : \mathcal{V}' \rightarrow \mathcal{V}$ of $\mathcal{V}$ along the normal crossing divisor $D^\text{an} \subset \mathcal{V}$ are single-valued and form a good set of irregular values in the sense of Mochizuki [18, Chapter 5], after shrinking $\mathcal{V}$ if necessary we may assume that $\mathcal{V} \backslash D^\text{an}$ is a smooth hypersurface in $(\mathcal{V} \backslash D^\text{an}) \times \mathbb{C}$. Then by using the positive integers $m_i > 0 (1 \leq i \leq k)$ and the transition matrices that we used to glue the enhanced sheaves $F_j$ in the construction of $F$ we define a local system $\mathcal{L}$ on the smooth hypersurface $\mathcal{V} \backslash D^\text{an} \subset (\mathcal{V} \backslash D^\text{an}) \times \mathbb{C}$. Let $i: \mathcal{V} \backslash D^\text{an} \rightarrow \mathcal{V} \times \mathbb{C}$ be the (not necessarily closed) embedding of $\mathcal{V} \backslash D^\text{an}$ into $\mathcal{V} \times \mathbb{C}$ and set

$$\mathcal{F} := i_* \mathcal{L} \in \mathcal{D}^b(\mathcal{V} \times \mathbb{C}).$$

Lemma 7.10. The object $\mathcal{F}[n] \in \mathcal{D}^b(\mathcal{V} \times \mathbb{C})$ is a perverse sheaf on $\mathcal{V} \times \mathbb{C}$.

Proof. Let $\rho : \mathcal{V}' \rightarrow \mathcal{V}$ be a ramification of $\mathcal{V}$ along the normal crossing divisor $D^\text{an} \subset \mathcal{V}$ such that the function $f_i \circ \rho$ is single-valued for any $1 \leq i \leq k$ and $\rho \times \text{id}_{\mathbb{C}} : \mathcal{V}' \times \mathbb{C} \rightarrow \mathcal{V} \times \mathbb{C}$ the morphism associated to it. Then by the proof of Theorem 2.4 (i), $(\rho \times \text{id}_{\mathbb{C}})^{-1} \mathcal{F}$ is a constructible sheaf on $\mathcal{V}' \times \mathbb{C}$. Moreover we can easily see that the canonical morphism

$$\mathcal{F} ightarrow (\rho \times \text{id}_{\mathbb{C}})_*(\rho \times \text{id}_{\mathbb{C}})^{-1} \mathcal{F}$$

(7.42)

of sheaves is injective and hence $\mathcal{F}$ is a subsheaf of the constructible sheaf $(\rho \times \text{id}_{\mathbb{C}})_*(\rho \times \text{id}_{\mathbb{C}})^{-1} \mathcal{F}$. As the support of $\mathcal{F}$ coincides with that of $(\rho \times \text{id}_{\mathbb{C}})_*(\rho \times \text{id}_{\mathbb{C}})^{-1} \mathcal{F}$ in $(\mathcal{V} \backslash D^\text{an}) \times \mathbb{C} \subset \mathcal{V} \times \mathbb{C}$ and $\mathcal{F}|_{\mathcal{V} \times \mathbb{C}} \simeq 0$, we then see that $\mathcal{F}$ itself is constructible. It follows that the restriction of $\mathcal{F}[n]$ to $(\mathcal{V} \backslash D^\text{an}) \times \mathbb{C}$ is perverse. Now let $j : (\mathcal{V} \backslash D^\text{an}) \times \mathbb{C} \rightarrow \mathcal{V} \times \mathbb{C}$ be the inclusion map. Then there exists an isomorphism

$$\mathcal{F}[n] \simeq j_* j^{-1} \mathcal{F}[n].$$

(7.43)

From this the perversity of $\mathcal{F}[n]$ follows as in the proof of Theorem 2.4 (ii). $\square$

Let $\widetilde{W} \subset \overline{X}^\text{an}$ be an open subset of $\overline{X}^\text{an}$ such that $\mathcal{W} = \mathcal{S}^\text{an} \cap \widetilde{W}$ and set $W := \widetilde{W} \cap \overline{X}^\text{an} \subset X^\text{an}$. Then for the restriction $\nu|_{\mathcal{V}} : \mathcal{V} \rightarrow \widetilde{W}$ of $\nu$ and the morphism $\nu|_{\mathcal{V}} \times \text{id}_{\mathbb{C}} : \mathcal{V} \times \mathbb{C} \rightarrow \widetilde{W} \times \mathbb{C}$ associated to it, we set

$$\tilde{\mathcal{G}} := R(\nu|_{\mathcal{V}} \times \text{id}_{\mathbb{C}})_* \mathcal{F} \simeq (\nu|_{\mathcal{V}} \times \text{id}_{\mathbb{C}})_* \mathcal{F} \in \mathcal{D}^b(\widetilde{W} \times \mathbb{C})$$

(7.44)
Lemma 7.11. The object \( \mathcal{G}[n] \in D^b(W \times \mathbb{C}) \) is a perverse sheaf on \( W \times \mathbb{C} \).

**Proof.** As \( \nu|_V \times \text{id}_\mathbb{C} \) is a finite map, the constructibility of \( \mathcal{G}[n] \) is clear. By the construction of \( \mathcal{G} \), it is also clear that the restriction of \( \mathcal{G}[n] \) to the open subset \((W \setminus H^\text{an}) \times \mathbb{C}\) of \( W \times \mathbb{C} \) is perverse. Let \( j_0 : (W \setminus H^\text{an}) \times \mathbb{C} \hookrightarrow W \times \mathbb{C} \) be the inclusion map. Then there exists an isomorphism

\[
\mathcal{G}[n] \simeq j_0\circ j_0^{-1}\mathcal{G}[n].
\]

From this the perversity of \( \mathcal{G}[n] \) follows as in the proof of Theorem 2.4 (ii). \( \square \)

By Lemma 7.11 the micro-support \( SS(\mathcal{G}[n]) \) of the perverse sheaf \( \mathcal{G}[n] \) is a homogeneous complex Lagrangian analytic subset of \( T^*(W \times \mathbb{C}) \). Then as in Section 5 by forgetting the homogeneity of \( SS(\mathcal{G}[n]) \) we define the following subsets:

\[
SS^E, C(\mathcal{G}[n]) \subset (T^*W) \times \mathbb{C}, \quad SS^C(\mathcal{G}[n]) \subset T^*W.
\]

(7.47)

Similarly we can show that \( \Lambda := SS^C(\mathcal{G}[n]) \) is a (not necessarily homogeneous) complex Lagrangian analytic subset of \( T^*W \). We call it the irregular micro-support of \( \mathcal{G}[n] \). Moreover, we obtain a Lagrangian cycle \( CC(\mathcal{M}) \) supported on \( \Lambda \subset T^*W \) and call it the irregular characteristic cycle of \( \mathcal{M} \) (for the case \( N = 1 \) see Kudomi-Takeuchi [39]). By the isomorphism \( V \setminus D^\text{an} \simeq W \setminus \partial S \) we regard \( f_i \) \((1 \leq i \leq k)\) as (possibly multi-valued) holomorphic functions on the open subset \( W_0 := W \setminus \partial S \subset S^\text{an} \) of \( S^\text{an} \) and set

\[
\Lambda_0 := \bigcup_{i=1}^k \{(x, df_i(x)) \mid x \in W_0\} \subset T^*W_0 \subset T^*S^\text{an},
\]

(7.48)

where in the right hand side we take the union considering all the possible branches of the (possibly multi-valued) functions \( f_i \). Set \( W_0 := W \setminus H^\text{an} \) and let

\[
T^*W_0 \xleftarrow{\varpi_0} W_0 \times_{W_0} T^*W_0 \xrightarrow{\rho_0} T^*W_0
\]

(7.49)

be the natural morphisms associated to the closed embedding \( W_0 \hookrightarrow W_0 \). Then it is easy to see that for the open subset \( T^*W_0 \subset T^*W \) of \( T^*W \) we have

\[
\Lambda \cap (T^*W_0) = \varpi_0\rho_0^{-1}\Lambda_0.
\]

(7.50)

**Remark 7.12.** Although the pull-backs of \( f_i \) \((1 \leq i \leq k)\) by a ramification \( \rho : Z' \rightarrow V \) of \( V \) along the normal crossing divisor \( D^\text{an} \subset V \) form a good set of irregular values in the sense of [38, Chapter 5], even after shrinking \( V \) the complex Lagrangian analytic subset \( \Lambda_0 \subset T^*W_0 \) may be singular.

From now on, assume also that there exists a non-empty Zariski open subset \( \Omega \subset Y \) of \( Y = \mathbb{C}^N \) and \( R > 0 \) such that for the open subset \( A(R) := \{w \in Y^\text{an} \mid ||w|| > R\} \) of \( Y^\text{an} \) the restriction \( q^{-1}(\Omega \cap A(R)) \cap \Lambda \rightarrow \Omega \cap A(R) \) of the projection \( q : X \times Y \rightarrow Y \) is an unramified finite covering and any connected component of the open subset \( q^{-1}(\Omega \cap A(R)) \cap \Lambda \subset \Lambda \) is a fiber bundle over a complex manifold in \( W \subset S^\text{an} \) contained
in $W \setminus \partial S \subset S^\text{an}$ or $\partial S \cap X^\text{an}$ (see Lemma 6.3). Fix a sufficiently large such $R > 0$ and let $V \subset \Omega \cap A(R)$ be a contractible open subset of $\Omega \cap A(R)$. Then for the decomposition

$$q^{-1}(V) \cap \Lambda = \Lambda_{V,1} \sqcup \Lambda_{V,2} \sqcup \cdots \sqcup \Lambda_{V,d} \quad (7.51)$$

of $q^{-1}(V) \cap \Lambda$ into its connected components $\Lambda_{V,i} \subset \Lambda$ ($1 \leq i \leq d$) the morphism $q|_{\Lambda}$ induces an isomorphism $\Lambda_{V,i} \xrightarrow{\sim} V$ for any $1 \leq i \leq d$. In this situation, as in Section 6 by $\Lambda$ and $G[n] \in \mathcal{D}_\text{hol}(W \times \mathbb{C})$ we define holomorphic functions $g_i : \Lambda \to \mathbb{C}$ ($1 \leq i \leq d$) and positive integers $m(i) \geq 1$ ($1 \leq i \leq d$) to obtain the following result. For the proof, we use also a method similar to the one used in Kudomi-Takeuchi [39].

**Theorem 7.13.** In the situation as above, for the standard holonomic $D$-module $\mathcal{M} = D_{iS^*N} \simeq i_{S^*N} \mathcal{N} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X)$ on $X$ there exists an isomorphism

$$\pi^{-1}C_V \otimes (\text{Sol}_X^E(\mathcal{M}^a)) \simeq \bigoplus_{i=1}^d \left( \mathbb{E}_{V,2}(\mathbb{T}^a) \right)^{\otimes m(i)} \quad (7.52)$$

of enhanced ind-sheaves on $\mathcal{T}^a$.

### 7.3 Some Auxiliary Results

Recall that by Theorem 4.1 (vi) for an exponential $D$-module $\mathcal{E}_U^f\chi$ on $X$ associated to a meromorphic function $f \in \mathcal{O}_X(*D)$ along $D \subset X$ and $U = X \setminus D$ we have an isomorphism in $\mathcal{E}\mathcal{b}(\mathbb{C}X)$

$$\text{Sol}_X^E(\mathcal{E}_U^f\chi) \simeq \mathcal{E}\mathcal{b}^\text{Ref}(\mathcal{E}_U^f\chi). \quad (7.53)$$

This formula for the enhanced solution complex $\text{Sol}_X^E(\mathcal{E}_U^f\chi)$ played a central role in the proof of the main results in [9]. Aiming at the study of the Fourier transformers of general holonomic $D$-modules, we shall extend it to more general meromorphic connections. Our argument below is a higher-dimensional analogue of those of Kashiwara-Schapira [30] Section 7] and Morando [50, Section 2.1]. Assuming that $M$ is a real analytic manifold we recall some basic definitions and results.

**Definition 7.14.** Let $U \subset M$ be an open subset and $f : U \to \mathbb{C}$ a $\mathbb{C}$-valued $C^\infty$-function on it. Then for a point $p \in M$ we say that $f$ has polynomial growth at $p$ if for a local coordinate system $x = (x_1, x_2, \ldots, x_n)$ of $M$ around $p$ there exists a compact neighborhood $K$ of $p$ in $M$ and $N > 0$ such that

$$\sup_{x \in K \setminus U} \text{dist}(x, K \setminus U)^N \cdot |f(x)| < +\infty. \quad (7.54)$$

Obviously $f$ has polynomial growth at any point of $U$. We say that $f$ is tempered at $p \in M$ if all its derivatives have polynomial growth at $p$. Moreover we say that $f$ is tempered on $U$ if it is tempered at any point $p \in M$.

For an open subset $U \subset M$ we denote by $C^\infty_M(U)$ the $\mathbb{C}$-vector space consisting of $\mathbb{C}$-valued $C^\infty$-functions on $U$ that are tempered on $U$. Then for any pair $(U_1, U_2)$ of relatively compact subanalytic open subsets $U_1, U_2 \subset M$ we have an exact sequence

$$0 \to C^\infty_M(U_1 \cup U_2) \to C^\infty_M(U_1) \oplus C^\infty_M(U_2) \to C^\infty_M(U_1 \cap U_2) \to 0 \quad (7.55)$$

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and hence we get a sheaf $C_M^{\infty}$ on the subanalytic site of $M$ in the sense of [29]. Note that this important result is an immediate consequence of the following Lojasiewicz’s inequality (see Bierstone-Milman [33] and Lojasiewicz [41] for the details).

**Proposition 7.15** (Lojasiewicz’s inequality). Let $U \subset \mathbb{R}_x^n$ be an open subset of $\mathbb{R}_x^n$ and $Z_1, Z_2 \subset U$ its closed subanalytic subsets. Then for any point $p \in Z_1 \cap Z_2$ there exist an open neighborhood $V \subset U$ of $p$ in $U$ and positive real numbers $C, r > 0$ such that we have

$$\text{dist}(x, Z_1) + \text{dist}(x, Z_2) \geq C \cdot \text{dist}(x, Z_1 \cap Z_2)^r$$

(7.56)

for any $x \in V$.

Following [50] Definition 1.1.4], for a real analytic curve $\gamma(t) : [0, \varepsilon) \rightarrow M$ ($\varepsilon > 0$) on $M$ (defined on a neighborhood of $0 \in \mathbb{R}$) we call the subset $\Gamma := \gamma([0, \varepsilon)) \subset M$ of its image a semi-analytic arc with an endpoint $\gamma(0) \in \overline{\Gamma}$. In what follows, let $X$ be a complex manifold and consider $C$-valued $C^{\infty}$-functions on its underlying real analytic manifold $X_R$. Then we have the following higher-dimensional analogue of Kashiwara-Schapira [30, Lemma 7.2] and Morando [50, Proposition 2.1.1].

**Proposition 7.16.** Let $X$ be a compact complex manifold, $D \subset X$ a normal crossing divisor on it, and $f \in \mathcal{O}_X(*D)$ a meromorphic function along $D$. Set $U := X \setminus D$. Assume that $f$ does not have any point of indeterminacy on the whole $X$ and has a pole along each irreducible component of $D$. Then for any relatively compact subanalytic open subset $W \subset U = X \setminus D$ the following conditions are equivalent.

(i) The function $\text{Ref}|_W : W \rightarrow \mathbb{R}$ is bounded from above.

(ii) The $C$-valued $C^{\infty}$-function $\exp(f)|_W : W \rightarrow \mathbb{C}$ is is tempered on $W \subset X_R$.

**Proof.** Since $f$ has no point of indeterminacy on the whole $X$, it defines a holomorphic map from $X$ to $\mathbb{P} = \mathbb{P}^1$. We denote it also by $f$. This non-constant map $f : X \rightarrow \mathbb{P}$ being proper and open, we obtain a relatively compact subanalytic open subset $f(W) \subset \mathbb{P}$ of $\mathbb{P}$ such that $f(W) \subset \mathbb{P} \setminus \{\infty\} = \mathbb{C}$. Since the proof of (i) $\implies$ (ii) is trivial, we shall prove only (ii) $\implies$ (i). We prove it by showing a contradiction. Suppose that the function $\text{Ref}|_W : W \rightarrow \mathbb{R}$ is not bounded from above. Then there exists a sequence $p_k \in W$ ($k = 1, 2, 3, \ldots$) such that

$$\lim_{k \rightarrow +\infty} \text{Ref}(p_k) = +\infty.$$  

(7.57)

By taking a subsequence of it, we may assume also that it converges to a point $p \in X$. Then by the condition $f \in \mathcal{O}_X(*D)$, we have $p \in D \setminus \overline{W}$. By the curve selection lemma, there exists a real analytic curve $\gamma(t) : [0, \varepsilon) \rightarrow \overline{W}$ ($\varepsilon > 0$) (defined on a neighborhood of $0 \in \mathbb{R}$) such that $\gamma(0) = p \in D \setminus \overline{W}$, $\gamma(t) \in W$ for any $t \in (0, \varepsilon)$, and

$$\lim_{t \rightarrow 0} \text{Ref}(\gamma(t)) = +\infty.$$  

(7.58)

Consider the semi-analytic arc $\Gamma := \gamma([0, \varepsilon)) \subset W \subset X_R$ with an endpoint $p = \gamma(0) \in D \setminus \overline{W}$. Let $z = (z_1, z_2, \ldots, z_n)$ be a holomorphic coordinate system of $X$ on a neighborhood $V$ of the point $p \in D$ such that we have $\{p\} = \{z = 0\}$ and

$$D = \{z_1z_2 \cdots z_l = 0\}, \quad f(z) = \frac{1}{z_1^{m_1} z_2^{m_2} \cdots z_l^{m_l}}$$  

(7.59)

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for some $1 \leq l \leq n$ and $m_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq l$). Shrinking the semi-analytic arc $\Gamma \subset W$ if necessary, we may assume that $\overline{\Gamma} \subset V \subset \mathbb{C}^n$. Then we can apply Proposition 7.15 the pair $(Z_1, Z_2)$ of the closed subanalytic subsets $Z_1 = \overline{\Gamma}$ and $Z_2 = \partial W$ of $V \subset \mathbb{C}^n$ (satisfying the condition $Z_1 \cap Z_2 = \{p\} = \{z = 0\}$) to show that there exist an open neighborhood $V_1 \subset V$ of $p$ in $V$ and positive real numbers $C_1, r_1 > 0$ such that we have

$$||z|| = \text{dist}(z, \{p\}) \leq C_1 \cdot \text{dist}(z, \partial W)^{r_1}$$

for any $z \in \overline{\Gamma} \cap V_1 \subset V \subset \mathbb{C}^n$. On the other hand, by the condition $\{p\} = \{z = 0\} \subset D$ we have also

$$\text{dist}(z, D) \leq \text{dist}(z, \{p\}) = ||z||$$

for any $z \in \overline{\Gamma} \cap V_1$. Set

$$g(z) := \frac{1}{f(z)} = z_1^{m_1} z_2^{m_2} \cdots z_l^{m_l}.$$  

(7.62)

Then we can easily show that there exist an open neighborhood $V_2 \subset V$ of $p$ in $V$ and positive real numbers $C_2, r_2 > 0$ such that we have

$$|g(z)| = \frac{1}{|f(z)|} \leq C_2 \cdot \text{dist}(z, D)^{r_2}$$

(7.63)

for any $z \in V_2 \setminus D = V_2 \cap U$. Now let us set $V_0 := V_1 \cap V_2 \subset V$. Then there exist positive real numbers $C, r > 0$ such that we have

$$|g(z)| \leq C \cdot \text{dist}(z, \partial W)^r$$

(7.64)

for any $z \in (V_0 \cap \overline{\Gamma}) \setminus D = (V_0 \cap \overline{\Gamma}) \cap U$. Shrinking the semi-analytic arc $\Gamma$ once again, we may assume that $\Gamma \subset V_0 \cap W$. Note that the image $f(\Gamma) \subset \mathbb{P} \setminus \{\infty\} = \mathbb{C}^n$ of $\Gamma$ by $f : X \rightarrow \mathbb{P}$ is a semi-analytic arc in $\mathbb{P}$ with an endpoint $\infty \in \mathbb{P}$. Indeed, for the real analytic curve $(f \circ \gamma)(t) : [0, \varepsilon) \rightarrow \mathbb{P}$ ($\varepsilon > 0$) we have $(f \circ \gamma)(0) = \infty \in \mathbb{P}$. It satisfies also the condition

$$\lim_{t \rightarrow +0} \text{Re}(f \circ \gamma)(t) = +\infty.$$  

(7.65)

Then by the proofs of [30, Lemma 7.2] and [50, Proposition 2.1.2], we see that for any $M > 0$ and $N > 0$ there exists a point $\tau \in f(\Gamma) \subset \mathbb{C}$ such that

$$|\exp(\tau)| > M|\tau|^N.$$  

(7.66)

In particular, if we take increasing sequences $M_k \to > 0$, $N_k \to > 0$ ($k = 1, 2, 3, \ldots$) satisfying the condition

$$\lim_{k \rightarrow +\infty} N_k = +\infty, \quad \lim_{k \rightarrow +\infty} \frac{M_k}{C^{N_k}} = +\infty,$$  

(7.67)

there exist points $\tau_k \in f(\Gamma) \subset \mathbb{C}$ ($k = 1, 2, 3, \ldots$) such that

$$|\exp(\tau_k)| > M_k|\tau_k|^{N_k}.$$  

(7.68)

We may assume also that $\lim_{k \rightarrow +\infty} \tau_k = \infty \in \mathbb{P}$. For each $k \geq 1$ we choose a point $z_k \in \Gamma$ such that $f(z_k) = \tau_k$. Then we obtain

$$|\exp(f(z_k))| > M_k|f(z_k)|^{N_k} = M_k|g(z_k)|^{-N_k}.$$  

(7.69)
for any $k \geq 1$. By the condition
\[
\lim_{k \to +\infty} f(z_k) = \lim_{k \to +\infty} \tau_k = \infty, \tag{7.70}
\]
the sequence $z_k \in \Gamma (k = 1, 2, 3, \ldots)$ satisfies the condition
\[
\lim_{k \to +\infty} |g(z_k)| = 0. \tag{7.71}
\]
Then it follows also from Proposition [7.13] that we have
\[
\lim_{k \to +\infty} \text{dist}(z_k, D) = \lim_{k \to +\infty} ||z_k|| = 0. \tag{7.72}
\]
Hence we get
\[
\lim_{k \to +\infty} z_k = p \in D. \tag{7.73}
\]
On the other hand, by (7.64) and (7.69) we have
\[
|\exp(f(z_k))| > M_k \frac{C^N_k}{N_k} \cdot \text{dist}(z_k, \partial W) - r N_k \tag{7.74}
\]
for any $k \geq 1$. By (7.67) and (7.73) this implies that the $\mathbb{C}$-valued $C^\infty$-function $\exp(f)|_W : W \to \mathbb{C}$ does not have polynomial growth at $p \in D \cap \overline{W}$. This completes the proof.

As in the the proofs of [30, Proposition 7.3] and [50, Lemma 3.1.1], by Proposition [7.16] we obtain the following result.

**Theorem 7.17.** In the situation of Proposition [7.16] for $R > 0$ we set
\[
W_R := \{ z \in U \mid \text{Re}(z) < R \} \subset X. \tag{7.75}
\]
Then we have an isomorphism
\[
H^0 \text{Sol}_X^i (E^f_U|_X) \simeq \lim_{R \to +\infty} (\mathbb{C}_X)^{W_R}. \tag{7.76}
\]

**Proposition 7.18.** Let $D \subset X$ be a normal crossing divisor in $X$ and assume that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a quasi-normal form along $D$. Let $i_D : D \hookrightarrow X$ be the inclusion map. Then for any $j \geq 1$ there exists a complex constructible “sheaf” $F_j$ on $D$ such that we have
\[
H^j \text{Sol}_X^i (\mathcal{M}) \simeq (i_D)_* F_j. \tag{7.77}
\]

**Proof.** For a point $x \in D$, by our assumption there exists a ramification map $\rho : X' \to U$ on a neighborhood $U$ of $x$ such that $D \rho^*(\mathcal{M}|_U)$ has a normal form along the normal crossing divisor $D' := \rho^{-1}(D \cap U) \simeq D \cap U$ in $X'$. Let $i_{D'} : D' \hookrightarrow X'$ be the inclusion map. Then by the proof of [22, Proposition 3.14] for any $j \geq 1$ there exists a complex constructible “sheaf” $G_j$ on $D'$ such that we have
\[
H^j \text{Sol}_X^i (D \rho^*(\mathcal{M}|_U)) \simeq (i_{D'})_* G_j. \tag{7.78}
\]
Moreover, by [29] Theorem 7.4.6] and [24, Proposition 4.39] we have an isomorphism
\[
\text{Sol}_U^j (D \rho_* D \rho^*(\mathcal{M}|_U)) \simeq R \rho_* \text{Sol}_X^i (D \rho^*(\mathcal{M}|_U)). \tag{7.79}
\]
Theorem 7.19. In the situation as above, we have an isomorphism
\[ H^j \text{Sol}^\prime_U (\mathbf{D} \rho_* \mathbf{D} \rho^* (\mathcal{M}|_U)) \simeq (\rho \circ i_D)_* G_j. \]  
(7.80)

Since \( \mathcal{M}|_U \) is a direct summand of \( \mathbf{D} \rho_* \mathbf{D} \rho^* (\mathcal{M}|_U) \), the same is true also for \( \text{Sol}^\prime_U (\mathcal{M}|_U) \) and \( \text{Sol}^\prime_U (\mathcal{M}|_U) \). Hence we verified the assertion locally on \( U \subset X \). We can easily check that it holds also globally on \( X \).

From now on, let \( \mathcal{M} \) be a holonomic D-module on \( X \) such that for a closed hypersurface \( D \subset X \) of \( X \) we have
(i) \( \mathcal{M} \twoheadrightarrow \mathcal{M}(sD) \)
(ii) \( \text{sing, supp}(\mathcal{M}) \subset D \).

Set \( U := X \setminus D \) and let \( j : U \hookrightarrow X \) be the inclusion map. Then \( L := \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{M}|_U, \mathcal{O}_U) \) is a local system on \( U \). We denote its rank by \( r \geq 0 \). Let \( \mathcal{R} \in \text{Mod}_{\text{hol}} (\mathcal{D}_X) \) be the regular meromorphic connection on \( X \) along \( D \) such that we have an isomorphism
\[ \text{Sol}_X (\mathcal{R}) \simeq j_i L \]  
(7.81)
(see e.g. [21] Theorem 5.3.8]). We thus obtain an isomorphism
\[ \Phi : L = \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{M}|_U, \mathcal{O}_U) \simeq \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{R}|_U, \mathcal{O}_U) \]  
(7.82)
of local systems on \( U \). Fix a point \( p \in U \) and let \( \phi_1, \ldots, \phi_r \in \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{R}|_U, \mathcal{O}_U) \) (resp. \( \psi_1, \ldots, \psi_r \in \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{M}|_U, \mathcal{O}_U) \) be the \( \mathbb{C} \)-basis of the local system \( \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{R}|_U, \mathcal{O}_U) \) (resp. \( L = \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{M}|_U, \mathcal{O}_U) \) on a neighborhood of \( p \) such that \( \Phi (\psi_i) = \phi_i \) for any \( 1 \leq i \leq r \). Assume that there exist holomorphic functions \( f_1, \ldots, f_r \) defined on a neighborhood of \( p \) such that
\[ \psi_i = \exp (f_i) \cdot \phi_i \quad (1 \leq i \leq r). \]  
(7.83)
As \( \phi_i \) (resp. \( \psi_i \)) \( 1 \leq i \leq r \) extend to multi-valued global sections of the local system \( \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{R}|_U, \mathcal{O}_U) \) (resp. \( \mathcal{H} \text{om}_{\mathcal{D}_U} (\mathcal{M}|_U, \mathcal{O}_U) \)), the same is true also for the holomorphic functions \( f_i \) \( 1 \leq i \leq r \). Since we have \( \Phi (\psi_i) = \phi_i \) even after the extensions, their real parts \( \text{Re} f_i : U \rightarrow \mathbb{R} \) are single-valued. Moreover, the isomorphism \( \Phi \) of sheaves being compatible with restrictions, if the section \( \phi_i \) is continued to \( \sum_{i=1}^r c_i \phi_i \) \( c_i \in \mathbb{C} \) along a closed continuous curve \( \gamma (t) : [0, 1] \rightarrow U \) in \( U \) such that \( \gamma (0) = \gamma (1) = p \), the section \( \psi_i \) is continued to \( \sum_{i=1}^r c_i \psi_i \) along it. By this observation, we see that the enhanced sheaf
\[ \bigoplus_{i=1}^r \mathbb{C}_{\{(z, t) \in X \times \mathbb{R} \mid z \in U, \ t + \text{Re} f_i (z) \geq 0\}} \]  
(7.84)
defined on a neighborhood of \( p \) can be naturally extended to the one \( F \) on the whole \( X \) such that there exists a surjective morphism \( \pi^{-1} j_i L \rightarrow F \) and we have \( \pi^{-1} \mathbb{C}_U \otimes F \simeq F \).

**Theorem 7.19.** In the situation as above, we have an isomorphism
\[ \text{Sol}^F_X (\mathcal{M}) \simeq \mathbb{C}^F_X \otimes F. \]  
(7.85)

**Proof.** Let \( \nu : Y \rightarrow X \) be a projective morphism of complex manifolds such that \( E := \nu^{-1} D \subset Y \) is a normal crossing divisor in \( Y \), the restriction \( \nu|_{Y \setminus E} : Y \setminus E \rightarrow U = X \setminus D \) of
\( \nu \) is an isomorphism and the holonomic D-module \( \mathcal{N} := \mathbf{D} \nu^* \mathcal{M} \) on \( Y \) has a quasi-normal form along \( E \). Set \( V := Y \setminus E = \nu^{-1}U \). By Theorem 4.11 (ii) and (v), we have

\[
\pi^{-1}C_U \otimes \text{Sol}^E_0(\mathcal{M}) \simeq \text{Sol}^E_X(\mathcal{M})
\]  

(7.86)

and

\[
\pi^{-1}C_V \otimes \text{Sol}^E_0(\mathcal{N}) \simeq \text{Sol}^E_0(\mathcal{N}) \simeq \mathbf{D} \nu^{-1} \text{Sol}^E_X(\mathcal{M}).
\]  

(7.87)

Since \( \nu|_{Y \setminus E} : V \to U \) is an isomorphism, there exist also isomorphisms

\[
\mathbf{D} \nu_* \text{Sol}^E_Y(\mathcal{N}) \simeq \mathbf{D} \nu_* \left( \pi^{-1}C_V \otimes \mathbf{D} \nu^{-1} \text{Sol}^E_X(\mathcal{M}) \right)
\]

\[
\simeq \pi^{-1}C_U \otimes \text{Sol}^E_X(\mathcal{M}) \simeq \text{Sol}^E_X(\mathcal{M}).
\]

Then it suffices to calculate \( \text{Sol}^E_Y(\mathcal{N}) \) for \( \mathcal{N} \) having a quasi-normal form along \( E \subset Y \). Let

\[
i : Y \times \mathbb{R}_\infty \to Y \times \mathbb{P}
\]

(7.88)

be the morphism of bordered spaces obtained by composing the natural ones \( Y \times \mathbb{R}_\infty \to Y \times P \) and \( Y \times P \to Y \times \mathbb{P} \) for the real projective line \( P = \mathbb{R} \cup \{ \infty \} \). Then we have an isomorphism

\[
\text{Sol}^E_Y(\mathcal{N}) \simeq i^! \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right)[2]
\]

(7.89)

for the holonomic D-module \( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \) on \( Y \times \mathbb{P} \) having a quasi-normal form along the normal crossing divisor

\[
E' := (E \times \mathbb{P}) \cup (Y \times \{ \infty \}) \subset Y \times \mathbb{P}
\]

(7.90)

in \( Y \times \mathbb{P} \). Note that by Theorem 4.11 (v) we have also

\[
\text{Sol}^E_Y(\mathcal{N}) \simeq \pi^{-1}C_V \otimes i^! \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right)[2]
\]

(7.91)

Now let us consider the distinguished triangle

\[
\text{Hom}^0 \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right) \to \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right) \to \tau^{\geq 1} \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right) \xrightarrow{\text{+1}} .
\]

(7.92)

Then by Proposition 7.18 we can easily show the vanishing

\[
\pi^{-1}C_V \otimes i^! \left\{ \tau^{\geq 1} \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right) \right\} \simeq 0.
\]

(7.93)

Moreover, by the proofs of Proposition 7.16 and Theorem 7.17 we obtain an isomorphism

\[
\text{Hom}^0 \text{Sol}^E_{Y \times \mathbb{P}} \left( \mathcal{N} \boxtimes \mathcal{E}^{\tau}_{\mathbb{C} \mathbb{P}} \right) \simeq \text{"lim}_{\mathcal{R} \to +\infty} \nu_!(\text{id}_Y \times \text{Re})^{-1} \mathcal{G}_R,
\]

(7.94)

where \( i : Y \times \mathbb{C} \hookrightarrow Y \times \mathbb{P} \) is the inclusion map, the morphism \( \text{id}_Y \times \text{Re} : Y \times \mathbb{C} \to Y \times \mathbb{R} \) is induced by the one \( \text{Re} : \mathbb{C} \to \mathbb{R} (\tau \mapsto \text{Re} \tau) \) and the enhanced sheaf \( \mathcal{G}_R (R \geq 0) \) on \( Y \) is a natural extension of the one

\[
\bigoplus_{i=1}^r \mathbb{C} \{ (w,t) \in Y \times \mathbb{R} \mid w \in V, t + \text{Re}(f_{i,w})(w) < R \}
\]

(7.95)
defined on a neighborhood of the point \( q = \nu^{-1}(p) \in V \) such that there exists an injective morphism \( G_R \hookrightarrow \pi^{-1}\nu^{-1}(j_L) \) and we have \( \pi^{-1}C_V \otimes G_R \simeq G_R \). Indeed, the sheaf \( \nu_!(\text{id}_Y \times \text{Re})^{-1}G_R \) on \( Y \times \mathbb{P} \) is an extension of the one

\[
\bigoplus_{i=1}^{r} \mathbb{C}\{ (w,\tau) \in Y \times \mathbb{P} \mid w \in V, \tau \in \mathbb{C}, \text{Re}(\tau + (f_{i\nu})(w)) < R \}
\]

(7.96)

to the whole \( Y \times \mathbb{P} \). Then, as in the proof of [9, Lemma 9.3.1] we obtain isomorphisms

\[
\text{Sol}_X^E(\mathcal{M}) \simeq \pi^{-1}C_V \otimes \left\{ H^0\text{Sol}_Y^E \left( \mathcal{N} \otimes \mathcal{E}_{C_Y}^r \right) \right\}[2]
\]

\[
\simeq \pi^{-1}C_V \otimes \left( \text{lim} \dfrac{G_R[1]}{R \rightarrow +\infty} \right)
\]

\[
\simeq \pi^{-1}C_V \otimes \left( C_Y^E \otimes \nu^{-1}F \right)
\]

\[
\simeq \pi^{-1}C_V \otimes \text{E}^{-1}(C_Y^E \otimes F),
\]

where we set \( \nu := \nu \times \text{id}_\mathbb{R} : Y \times \mathbb{R} \rightarrow X \times \mathbb{R} \) and in the third isomorphism we used the one \( G_0[1] \simeq \nu^{-1}F \) in the category \( \text{E}^b_\mathbb{C}(C_Y) \). It follows that we get the desired isomorphism

\[
\text{Sol}_X^E(\mathcal{M}) \simeq \text{E}^{-1}(\text{Sol}_Y^E(\mathcal{N}) \simeq C_X^E \otimes F).
\]

(7.97)

This completes the proof. \( \square \)

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