Quantum memory-assisted entropic uncertainty and entanglement dynamics: two qubits coupled with local fields and Ornstein Uhlenbeck noise

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Abstract
Entropic uncertainty and entanglement are two distinct aspects of quantum mechanical procedures. The estimation of the entropic uncertainty is required for the increased accuracy and, hence, for the enhanced efficacy of quantum information processing tasks. In this regard, we analyze the entropic uncertainty, entropic uncertainty lower bound, and concurrence dynamics in two non-interacting qubits. The exposure of two qubits is studied in two different qubit-noise configurations, namely, common qubit-noise and independent qubit-noise interactions. To include the noisy effects of the local external fields, a Gaussian Ornstein–Uhlenbeck process is considered. We show that the increase in entropic uncertainty gives rise to the disentanglement in the two-qubit Werner state. Depending on the parameters adjustment and the number of environments coupled, different classical environments have varying capacities to induce entropic uncertainty and disentanglement in quantum systems. The entanglement remains vulnerable to the external classical fields; however, by employing the ideal parameter ranges we provided, prolonged entanglement retention while preventing entropic uncertainty growth can be achieved. Besides, we have also analyzed the intrinsic behavior of the classical fields towards two-qubit entanglement with no imperfection with respect to different parameters.
1 Introduction

In quantum physics, Heisenberg’s uncertainty principle is a fundamental concept. Uncertainty relations in terms of entropies were constructed to solve conceptual inadequacies in the uncertainty principle’s original formulation, and they now play a crucial role in quantum foundations [1]. In the security analysis of some quantum systems, entropic uncertainty relations have lately emerged as a significant component [2]. The uncertainty principle in quantum mechanics is both a fundamental feature and a significant departure from classical physics. Any pair of incompatible observables obeys a specific type of uncertainty relationship, imposing final constraints on measurement accuracy while also laying the theoretical groundwork for future technologies, such as quantum encryption and QIP tasks [3–6]. The newly empirically validated entropic uncertainty principle has piqued interest in its potential applications from various perspectives. A recently revised entropic uncertainty known as the quantum-memory-assisted entropic uncertainty relation has been constructed, according to Renes and Boileau’s concept [7, 8]. The entropic uncertainty relation is used in cryptographic security [9], quantum randomness [10], quantum key distribution [11], probing quantum correlations [12], entanglement witnessing [13], and quantum metrology [14].

Quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects, notwithstanding their spatial separation, must be explained in terms of one another [15]. Therefore, there are correlations between the systems’ observable physical characteristics. Even though quantum physics makes it difficult to forecast which set of measurements will be observed, it is feasible to combine two particles into a single quantum state so that when one is detected as spin-up, the other is always detected as spin-down, and vice versa. Quantum entanglement has been utilized in experiments to establish quantum teleportation [16], and it has potential applications in quantum computing [17], quantum cryptography [18], communications [19], quantum radar [20] and entanglement swapping [21].

The preservation of entanglement and the level of uncertainty in open quantum systems are inextricably linked, and a hotly debated topic. The entropic uncertainty may give rise to different phenomena, such as a rise in entropy, mixedness, and dephasing of the systems. These effects can further cause the state to be disentangled. We are motivated by these likely reasons to link entropic uncertainty and entanglement in two entangled qubits. In this case, we are interested in analyzing the dynamical map of a Werner entangled state, the relationship between entanglement and entropic uncertainty, and the related optimal control. In addition, this is significant because the dynamics of open quantum systems are crucial for the development of quantum protocols and the inter-transmission of information between two locations, such as in QIP [21]. The principal source of uncertainty is that quantum systems cannot be completely isolated from their external mediums, which can accommodate a variety of disorders [22–25]. These disorders generate a variety of noises, which, when superimposed on the phase factors of the systems, reduce the efficiency of quantum
processes and phenomena [26–28]. As a result, research into such topics can help reduce the actual causes of quantum mechanical application failure while improving relative precision and measurement accuracy by reducing and optimally controlling the entropic uncertainty and, hence, entanglement in quantum systems.

In the present work, we discuss the dynamics of entropic uncertainty, entropic uncertainty lower bound, their relative difference, and entanglement dynamics in a two-qubit Werner state under the influence of classical fields. To limit our problem, we consider the generation of OU noise in the classical fields and the main reason for the disentanglement of the two qubits and the relative degree of entropic uncertainty. In the microscopic view, the OU noise is caused by the Brownian motion of the particles, which can be found nearly in every quantum mechanical process [28]. This makes our noise model more significant because of its widespread presence and noisy actions in such operations. We prefer the classical context of environments rather than the non-classical ones because the local former provides more degrees of freedom to examine the dynamical maps of quantum systems [22–24]. Two types of two-qubit spin squeezing models were used to investigate thermal quantum correlations and the entropic uncertainty relation in the presence of quantum memory [29]. For two atoms and the relative dynamics of the entanglement and uncertainty were found to be greatly dependent upon the temperature parameters [30]. For different three-level systems, the dynamics of entropic uncertainty reveal that the corresponding entanglement losses and rise in entropy and uncertainty are regulated by the coupling strengths of the random telegraphs noise [31]. The demonstration of the evolution of entropic uncertainty in the multi-measurement process has shown that the Markovianity and non-Markovianity of the fields can be traced back to the degree of uncertainty and noise [32]. A new type of long-range interaction was used to achieve long-distance entanglement in the spin system [33, 34]. In a similar case, the authors in [35] investigated the dynamics of entropic uncertainty in three qubits and they found that the classical depolarizing noise and environments enhance the entropic uncertainty. Besides, they found that designing the system-environment coupling between the three qubits and environments can also lead to enhanced entanglement preservation and lower entropic uncertainty. Thus, the above literature concludes that entropic uncertainty and entanglement are controlled by the different variables and fields, which must be thoroughly investigated for the practical implementation of the QIP protocols.

In the case of QIP, classical channels have been widely studied for the two-qubit and multi-qubit quantum correlations, and important results have been achieved. For example, in Ref. [36], the authors studied entanglement and entropy of the two-qubit mixed state and found that the two phenomena inversely affect each other. Depending on the nature of the environmental noise assumed, the authors of Ref. [37] found that entanglement and quantum discord decay either with a monotonic behavior or with sudden death and birth revival phenomenon. They also stated that the microscopic structure of the environments to the behavior of noise toward quantum correlations decay can be useful in designing classical channels of interest. Chandra et al. have investigated the decoherence probabilities associated with the sudden changes and preservation of quantum correlations when a two-qubit \( X \) state is exposed locally to independent quantum channels multiple times [38]. Reference [39] explores the dynamics of complementary correlations to reveal the genuine quantum correlations for the bipartite
composite state coupled with decoherent channels. An ultimate threshold to identify entanglement under the influence of decoherence channels, and the phenomenon of sudden changes has also been demonstrated in Ref. [39]. The local noisy channels have been found to freeze bipartite and multipartite quantum correlations subjected to local noisy channels [40]. In particular, they found that the classical noisy channels play a key role in quantum correlations freezing and associated inhomogeneity of the magnetizations of a quantum composite state. Other examples, where quantum correlations are influenced by local noisy channels, can be found in Refs. [15, 25, 26]. Note that in each case, quantum correlations are influenced differently, suggesting that each type of classical decoherent environment has non-matching effects on quantum correlations’ preservation. Therefore, in the current case, we are motivated to encounter the impact of entropic uncertainty in a two-qubit Werner state when coupled with common and independent classical decoherent channels on the preservation of entanglement.

We assume two kinds of system-environment coupling: common qubit-noise (CQN) and independent qubit-noise (IQN) configurations. Both qubits will be coupled to a common environment characterized by an OU noise source in the CQN configuration. The two qubits are coupled with two independent local environments in the IQN configuration case. This will help to conclude the entropy and disentanglement level for the increasing number of environments. Entanglement has already been shown to degrade differently in different types and the number of environments [22–28]. We concentrate on the preservation of quantum information in a two-qubit Werner state in classical environments driven by OU noise. Because the efficient deployment of QIP tasks necessitates the preservation of information. This is accomplished by investigating entropic uncertainty and concurrence dynamics, the former of which will look at the growth of uncertainty and the latter at the degree of entanglement between the two qubits. Such investigation will help us understand the relationship between entropic uncertainty with entanglement and information losses. We also consider how the current study’s various aspects, such as state purity, noise parameters, and system-channel coupling, affect the growth of uncertainty and entanglement losses. The optimal parameter values we provide will improve the QIP protocols and increase their efficiency. One of the primary motivations for studying entropic uncertainty and entanglement concurrently is that classical channels driven by Gaussian processes are rarely studied. Non-local channels are difficult to design because they are affected by a variety of factors, such as temperature, magnetic effects, and the consequent changes in the original quantum states. For the time being, classical channels continue to outperform quantum channels, specifically in the mentioned regimes. Therefore, the current study is more practical than the one for quantum channels. Finally, this study will offer solutions to avoid quantum information losses in QIP.

This paper is organized as: In Sect. 2, we give details of the physical model, estimators of entropic uncertainty and entanglement along with OU noise application. The explicit results and discussion are written in Sect. 3. In Sect. 5, we summarize our results in a few remarks.
2 Suggested model and dynamics

Our model comprises two non-interacting qubits initially prepared in the Werner state coupled with a classical environment. OU noise is considered as being the primary cause of dephasing and entropic uncertainty increase in classical environments. We examine CQN and IQN configurations, which are two different designs of system-environment coupling approaches. In CQN configuration, the dynamical map of the two qubits is studied under the influence of a single OU noise and classical environment. In IQN configuration, the system is considered evolving when coupled to two independent OU noise sources in two independent classical environments. The Hamiltonian, which characterizes the current model, is written as [15, 25]

\[ H(t) = H_A(t) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B(t), \]  

(1)

where \( H_n(t) = \kappa \mathbb{1}_n + \lambda \chi_n(t) \sigma^z_n \) with \( n \in \{A, B\} \), \( \kappa \) represents the energy of the relative qubit, \( \mathbb{1}_n \) and \( \sigma^z_n \) are the identity and Pauli matrices of dephasing classical channels corresponding to the vector space \( n \), while \( \lambda \) is the coupling constant, regulating the strength of coupling between the qubits and classical environments. The terms \( \chi_n(t) \) represent the stochastic parameter of the fields and control the flipping of the qubits between \( \pm 1 \). For the time-evolution of the two qubits in classical fields, we employ the time unitary operation as [15, 25]

\[ \mathbb{X}(t) = \exp \left[ -i \int_0^t H(f) df \right]. \]  

(2)

The time evolved state of the two qubits. When prepared in the initial state \( \rho_0 \) is obtained using the equation [15, 25]

\[ \rho(t) = \mathbb{X}(t) \rho_0 \mathbb{X}^\dagger(t). \]  

(3)

where \( \rho_0 \) is the two-qubit Werner state and can be written as

\[ \rho_0 = \frac{1 - p}{4} \mathbb{1}_A + p |\psi\rangle \langle \psi|, \]  

(4)

where \( |\psi\rangle = \frac{1}{\sqrt{2}} (|0_A 0_B\rangle + |1_A 1_B\rangle) \) is the two-qubit maximally entangled Bell’s state, while \( p \) denotes the purity factor, controlling the initial purity in the system and ranges between \( 0 \leq p \leq 1 \) (Fig. 1).

We assume the OU process, a stochastic mathematical process with applications in both physical sciences and finance, impacts the classical fields to account for noise. This term in physics describes the velocity of a massive Brownian particle under the influence of friction. In many quantum mechanical protocols, the OU method is a static Gaussian-Markov operation with OU noise, and it has been identified as one of the numerous and primary causes of information, coherence, and quantum correlation losses [28]. Besides, the OU process has been used to investigate the spatial and temporal regularity in linear stochastic dynamics under the influence of Lévy white
Fig. 1 Shows the schematic diagram of two qubits $Q_1$ and $Q_2$ connected with square-like boxes means classical environments of two types: common qubit-noise configuration (a), and independent qubit-noise configuration (right) with a linked quantum memory $Q$ for utilizing the concept of quantum-memory-assisted entropic relations. OU denotes the Ornstein Uhlenbeck noisy sources and the yellowish-blue lines represent the connection between the classical channels and noisy sources characterized by the stochastic parameter $\chi_n(t)$. The glow around the qubits shows the entropic action of the environments, while the wavy lines above the qubits represent the dynamics and the relative coupling strengths $\lambda$ and its diminishing size shows the resultant dephasing effects (Color figure online)

noise utilizing the subordination of a Gaussian white noise [41]. A connection between coherent quantum feedback OU process has been established by the authors of Ref. [42] under the influence of phase noise. In Ref. [43], the authors studied entanglement and coherence dynamics influenced by OU noise while providing exact dynamical maps for the two-qubit optical states. Besides, they also provided the details of the influence of a non-Markovian noise on the time evolution of two-qubit entanglement and coherence. The dynamics of entanglement and quantum correlations for a hybrid one-parameter qubit-qutrit class states when exposed to different independent or common classical noises, including OU noise, have been studied [44]. Here, the authors found that the hybrid qubit-qutrit state exhibits complete freezing of entanglement and quantum correlations because of the OU noise. Besides, OU noise has been extensively studied in the case of single qutrit, two-qubit, three qubits, and hybrid qubit-qutrit states and we find that in each case, the degree and behavior of losses are different [26, 45, 46]. The above literature suggests that the OU process and hence, noise has an influence on a wide range of quantum phenomena, and operations and strategies for avoiding such noise can boost the efficacy of such QIP tasks.

Currently, the OU noise is applied to the dynamical map of the two-qubit Werner state, which is initially prepared in the state $\rho_0$. To determine the negative consequences of OU noise, we use a zero-mean Gaussian process to describe the classical field $\mathcal{L}(t)$ affecting the system. In this regard, the OU Gaussian process in terms of second order statistics, i.e., in terms of its mean $\Theta$ and its autocorrelation function $A$ can be written as [47]

$$\Theta(t) = \mathcal{E}(\mathcal{B}(t)) = 0,$$  \hspace{1cm} (5)

$$A(t, t') = \mathcal{E}(\mathcal{B}(t)\mathcal{B}(t')),$$  \hspace{1cm} (6)

where $\mathcal{E}(\ldots)$ is the average of all possible realizations of $\mathcal{B}(t)$. Next, the Gaussian processes are defined by a characteristic function written as
where $g(t)$ represents an arbitrary function of time and if $g = n$ denotes a constant with respect to time, then Eq. 7 takes the following form as

$$E \left[ \exp \left( \pm i \int_0^t B(s) ds \right) \right] = \exp \left( -\frac{1}{2} n^2 \beta(t) \right),$$

(8)

where $\beta(t)$ is called the $\beta$-function and is used to connect the classical noise with the environments in the dynamical map of the two-qubit state, and can be written as [28]

$$\beta_{OU}(t) = \int_0^t \int_0^t A(s - s') ds ds'.$$

(9)

The OU Gaussian process is a diffusion process with friction, and it is characterized by the auto-correlation function written as

$$A(g, t - t') = \frac{g}{2} \exp \left( -g|t - t'| \right),$$

(10)

where $g = 1/\tau$ is called the memory parameter, while $\tau$ denotes the correlation time of the process. Note that, for the increasing memory parameter, the width of the noise spectrum increases, and for $g \gg 1$, the white noise limit is achieved.

Moreover, the final $\beta$-function for the OU noise can be obtained by plugging the auto-correlation function from Eqs. (10) into (9) as

$$\beta_{OU}(t) = \frac{1}{g} [g \tau + \exp[-g \tau] - 1].$$

(11)

### 2.1 Concurrence

To assess entanglement, we use concurrence for the bipartite state, which ranges from $0 \leq C(t) \leq 1$. The state is entangled at $C(t) = 1$, but at the lowest bound $C(t) = 0$, the state becomes completely separable. For the two-qubit state, the concurrence measurement can be carried out using the following expression [48]

$$C = \max \{0, \sqrt{v_1} - \sqrt{v_2} - \sqrt{v_3} - \sqrt{v_4} \},$$

(12)

where $v_i$ with $i \in \{1, 2, 3, 4\}$ are the eigenvalues in decreasing order of the Hermitian matrix $\varrho = \sqrt{\hat{\rho} \hat{\rho} \hat{\rho} \hat{\rho}}$ with $\hat{\rho} = \hat{\rho}_{AB}(\sigma_y \otimes \sigma_y)\hat{\rho}^*(\sigma_y \otimes \sigma_y)$ and $\hat{\rho}^*$ is the complex conjugate of the density matrix state $\hat{\rho}$ while $\sigma_y$ represent the Pauli spin matrix.

### 2.2 The entropic uncertainty measure

Consider the following two users, Bob and Alice: Bob generates a qubit in the quantum state of his choice and delivers it to Alice, who must choose between the two
measurements and broadcast her decision to Bob. Now, we can reduce the uncertainty in the outcome using the measurement results obtained by Bob. The standard deviation uncertainty relation for two observables $\Omega$ and $D$ can be written as [49, 50]

$$\Delta \Omega \Delta D \geq \frac{1}{2} |\langle [\Omega, D] \rangle|.$$  

(13)

Deutsch proposed the entropic uncertainty relation for any pair of observables by describing uncertainty in terms of Shannon entropy rather than using standard deviation [51]. Maassen and Uffink devised a tighter entropic uncertainty expression based on Deutsch’s approach can be written as [52]

$$S(\Omega) + S(D) \geq \log_2(\frac{1}{c}),$$  

(14)

where $S(\Omega)$ is the Shannon entropy, which represents the probability distribution when $\Omega$ is measured, and $S(D)$ is the Shannon entropy when $D$ is measured. $c$ denotes the complementarity between $\Omega$ and $D$, and $c = \max_{a, b} |\langle \psi | \phi \rangle|^2$ for non-degenerate observables, where $|\psi\rangle$ and $|\phi\rangle$ are the eigenvectors of $\Omega$ and $D$, respectively. The current definition has been updated into a new form known as quantum-memory-assisted entropic uncertainty relation [53], which has also been experimentally tested [54] and can be written as

$$S(\Omega|B) + S(D|B) \geq S(A|B) + \log_2(\frac{1}{c}).$$  

(15)

where $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy of the density matrix $\rho_{AB}$ with $S(\rho_{AB}) = -\text{tr}(\rho_{AB} \log_2 \rho_{AB})$. $\rho(X|B) = S(\rho_{XB}) - S(\rho_B)$ is the post measurement state with $X \in \{\Omega, D\}$, $S(\rho_{XB}) = \sum_{\lambda} (|\psi_\lambda\rangle \langle \psi_\lambda|) \otimes I \rho_{AB} (|\psi_\lambda\rangle \langle \psi_\lambda| \otimes I)$, and $\rho_B = tr_A(\rho_{AB})$. Note that $I$ is the identical operator.

In Eq. (15), $L(\tau)$ and $R(\tau)$ are used to denote the left-hand and right-hand sides of the entropic uncertainty equation. To find the difference between the two sides, we use the equation:

$$U(\tau) = L(\tau) - R(\tau),$$  

(16)

where $U(\tau)$ is the tightness of the uncertainty relation, where the entropic uncertainty and the related lower bound can be written as

$$L(\tau) = S(\rho_{\sigma_x B}) + S(\rho_{\sigma_z B}) - 2S(\rho_B),$$  

(17)

$$R(\tau) = S(\rho_{AB}) - S(\rho_B) + 1.$$  

(18)
3 Main results

In this section, we present the dynamics of entanglement, tightness, entropic uncertainty, and entropic uncertainty lower bound using Eqs. (12), (16), (17), and (18). The system’s time unitary matrix, obtained using Eq. (2), has the following form [55]

\[
X(t) = \begin{bmatrix}
e^{it(2\kappa + \omega_1 \lambda)} & 0 & 0 & 0 \\
0 & e^{-it(2\kappa + \omega_2 \lambda)} & 0 & 0 \\
0 & 0 & e^{-it(2\kappa + \omega_3 \lambda)} & 0 \\
0 & 0 & 0 & e^{-it(2\kappa + \omega_1 \lambda)}
\end{bmatrix},
\]  

(19)

where

\[
\omega_1 = \chi_A(t) + \chi_B(t), \quad \omega_2 = \chi_B(t) - \chi_A(t), \quad \omega_3 = \chi_A(t) - \chi_B(t).
\]

Note that all operations are carried out using the time evolved density matrix given in Eq. (3). For the CQN configuration, the time evolved density matrix takes the form

\[
\rho_{CQN}(t) = \begin{bmatrix}
\frac{1+p}{4} & 0 & 0 & \frac{1}{2}e^{4it\chi_A(t)\lambda}p \\
0 & \frac{1-p}{4} & 0 & 0 \\
0 & 0 & \frac{1-p}{4} & 0 \\
\frac{1}{2}e^{-4it\chi_A(t)\lambda}p & 0 & 0 & \frac{1+p}{4}
\end{bmatrix},
\]

(20)

where \(\chi_A(t) = \chi_B(t)\).

For the IQN configuration where \(\chi_A(t) \neq \chi_B(t)\), the time evolved density matrix of the two-qubit state has the following form

\[
\rho_{IQN}(t) = \begin{bmatrix}
\frac{p+1}{4} & 0 & 0 & \frac{1}{2}e^{2it(\chi_A(t)+\chi_B(t))\lambda}p \\
0 & \frac{1-p}{4} & 0 & 0 \\
0 & 0 & \frac{1-p}{4} & 0 \\
\frac{1}{2}e^{-2it(\chi_A(t)+\chi_B(t))\lambda}p & 0 & 0 & \frac{p+1}{4}
\end{bmatrix}.
\]

(21)

Figure 2 shows the time evolution of entanglement detection in two qubits prepared in the state \(\rho_0\). Because the values of these parameters are equal, i.e., \(\pm 1\), we set \(\chi_A = \chi_B = \chi\) in this section. Compared to those with defects, understanding the intrinsic behavior of classical environments without defects is critical for promoting and preserving non-local correlations. Therefore, we employ entanglement witness (\(EW(t)\)), a tool for detecting entanglement. Mathematically, \(EW(t) = -Tr[r(t).E_x]\) is an easy to compute measure where \(r(t)\) is the time evolved density ensemble given in Eq. (20) or (21), and \(E_x = \frac{1}{2}I - \rho_0\) is the \(EW(t)\) operator. As shown in Fig. 2, the two qubits preserved entanglement while experiencing revivals that show non-Markovian behavior. Using Fig. 2a, we investigated the effect of increasing the intensity of \(\lambda\) on entanglement revivals. When \(\lambda\) was increased, the robustness of the entanglement revivals improved. The results in Fig. 2b are consistent with those in Fig. 2a, implying...
that as $\lambda$ increases, the revival speed increases. Meanwhile, the parameter $\chi_n = \pm 1$ (with $n \in \{A, B\}$) and $\lambda$, simply toggle the system between relative maximum and minimum. We discovered that the two-qubit state entanglement is only effective in the purity range $0.6 < p \leq 1$, by plotting several values of the parameter $p$ against $t$ in Fig. 2c. The results of Fig. 2d match those of Fig. 2c, revealing the same $p$ region where the system remains entangled. It is worth noting that the two qubits and their surrounding classical environment effectively exchange information periodically, demonstrating that classical channels are vital resources for QIP applications. The behavior of time evolved density ensemble in common and independent coupling cases will remain the same, as when no noise is involved, we get $\chi_A = \chi_B = \chi$.

3.1 Entropic uncertainty relations and entanglement dynamics in CQN configuration

We discuss the dynamics of entropic uncertainty, tightness, and entanglement when two non-interacting qubits are both coupled with a common OU noise source in a single local random field. To include the OU noisy effects in the matrix given in Eq. (20), we take the average of the time evolved density matrix of the system for the CQN configuration as follows:
\[ \rho_{CQN}(\tau) = \langle \rho(\phi_A, t) \rangle_{\theta(\tau)A}, \]  
\tag{22} 

where \( \phi_A \) is the combined factor of system and environments, while \( \theta_A \) is the superimposed noise factor over the system. In Eq. (22), \( \phi_A(t) = i n \chi_A(t) \) and we set \( \chi_A(t) = \chi_B(t) \) where \( \theta(\tau) = -\frac{1}{2} n^2 \beta(\tau) \). The explicit form of Eq. (22) obtained can be put into the following form:

\[ \rho_{CQN}(\tau) = \begin{bmatrix} \frac{1+p}{4} & 0 & 0 & \frac{1}{2} e^{-8\beta(\tau)} p \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ \frac{1}{2} e^{-8\beta(\tau)} p & 0 & 0 & \frac{1+p}{4} \end{bmatrix}, \]  
\tag{23} 

where \( \beta \)-function is given in Eq. (9). The analytical results of Eq. (16) and Eq. (12) take the form as

\[ U(\tau) = \frac{1}{\log(16)} e^{-\theta(\tau)} \left( v_1 + e^{\theta(\tau)} (v_2 + v_3) \right), \]  
\tag{24} 
\[ C(\tau) = -\sqrt{1 - p - \frac{v_4}{2} + \frac{v_5}{2}}, \]  
\tag{25} 

where

\[ v_1 = -4 \arctan \left( e^{-\theta(\tau)} \right) - 2 \log \left( 1 - 2 e^{-\theta(\tau)} + p \right) + 2 \log \left( 1 + 2 e^{-\theta(\tau)} + p \right), \]
\[ v_2 = -\log(16) - 2(1 + p) \log(1 + p) - 2 \log \left( \frac{1}{4} - \frac{1}{4} e^{-\theta(\tau)} \right) - 2 \log \left( 1 + e^{-\theta(\tau)} \right), \]
\[ v_3 = (1 + p) \log \left( 1 - 2 e^{-\theta(\tau)} + p \right) + (1 + p) \log \left( 1 + 2 e^{-\theta(\tau)} + p \right), \]
\[ v_4 = \sqrt{e^{-\theta(\tau)} \left( -2 + e^{\theta(\tau)} + e^{\theta(\tau)} p \right)}, \]
\[ v_5 = \sqrt{e^{-\theta(\tau)} \left( 2 + e^{\theta(\tau)} + e^{\theta(\tau)} p \right)}. \]

When the CQN configuration is considered, Fig. 3 shows the entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence dynamics utilizing Eqs. (15) and (12) in the dynamical map of two qubits. When OU noise dephasing effects are present, the two-qubit entanglement decreases, and entropic uncertainty relations increase in classical environments. In two qubits, the entropic uncertainty functions \( L(\tau) \) and \( R(\tau) \) are increasing, while the tightness and entanglement functions, \( U(\tau) \) and \( C(\tau) \) are found decreasing functions with time. Although the difference between \( L(\tau) \) and \( R(\tau) \) is insignificant, however, the high rate of entropic uncertainty increase cannot be omitted. The results of \( U(\tau) \), which show dynamics in a small restricted elevation, confirm the minimal difference between the \( L(\tau) \) and \( R(\tau) \). Henceforth, \( L(\tau) \) and \( R(\tau) \) results are in good agreement with \( U(\tau) \). From the \( C(\tau) \) results, we can see that classical fields with Brownian motion disorders cause entanglement to degrade. It is simple to deduce that the rate of disentanglement lags the entropic uncertainty growth by comparing the entropic uncertainty and concurrence dynamics. This means that an increase in entropic uncertainty causes the disentanglement of...
the two qubits. Despite this, the noise parameter $g$ regulates entropic uncertainty and entanglement loss, and as $g$ rises, entropic uncertainty rises and entanglement falls. Under the current noise and parameter settings, the system becomes completely separable because of high entropic uncertainty. We find that the current results differ completely from those described in [56], where revivals in $U(\tau)$, $R(\tau)$, and $L(\tau)$ are observed. The maximum values of the two sides of uncertainty at comparable times do not match, which is important. The qualitative monotone behavior and dynamical map of the two non-equivalent sides of the entropic uncertainty as well as that of the concurrence entanglement, on the other hand, were similar and depict that the entanglement and information loss are irreversible in the current example.

3.1.1 Explicit dynamics of the two qubits in common qubit-noise configuration

The entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence dynamics are displayed in Fig. 4 when the system is coupled with a single classical environment. Initially, $R(\tau) = L(\tau) = U(\tau) = 0$ and $C(\tau) = 1$ suggesting the state has initially no uncertainty and is maximally entangled. When the interaction between the system and field is switched on, the entropic uncertainty rises and causes the entanglement to degrade. The current results are qualitatively similar to those in Fig. 3, although they differ in quantitative terms. Lower $g$ values allowed the $L(\tau)$ and $R(\tau)$ to achieve final maximum saturation levels, however, taking a much longer
time, which is the main reason for the disparity between Figs. 3, and 4. The difference between the $L(\tau)$ and $R(\tau)$ is minor when compared to the results obtained for $g = 0.4$ in Figs. 3 and 4. This shows that entropic uncertainty is primarily caused and controlled by the noise parameter $g$ and that the two are intrinsically related. The results of $U(\tau)$ show that $L(\tau) > R(\tau)$ and depict that the gap between the two sides of the uncertainty relation becomes narrow and finally vanishes with time. The entanglement decays monotonically under the influence of OU noise, as shown by the $C(\tau)$ measure. As the values of $g$ increase, the entanglement decreases. The current findings suggest that rising entropic uncertainty directly affects the degree of entanglement between qubits and that entropic uncertainty increases faster than entanglement diminishes. This suggests that entropic uncertainty is a critical contributor to quantum correlated systems losing their entanglement. We discovered that for low $g$ values, we could maintain entanglement for a long time, even though the state eventually becomes separable. The current entropic uncertainty results contradict those reported in [56, 57], where the qualitative and quantitative features of the dynamics of entropic uncertainties, tightness, and entanglement dynamics are vastly different.

Figure 5 shows the dynamics of entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence in the two-qubit Werner state when exposed to local fields and OU noise. The qualitative dynamics of the current results against different values of purity factor differ from those seen in Fig. 4. In the two-qubit
Werner entangled state, we find that the purity factor significantly affects the initial entanglement and level of entropic uncertainty. As seen that the relative entropic uncertainty increases proportionally as $p$ decreases, with minimum disorder in the system occurring at the upper bound of $p$ and maximum disorder at $p = 0.10$. When $p > 0.9$ and $p < 0.1$, the $U(\tau)$ predicts minor variations between the entropic uncertainty and entropic uncertainty lower bound. When compared to various purity factor values, the qualitative dynamics of the current results differ from those shown in Fig. 4. After a finite interval of time, the two-qubit Werner state becomes separable for all ranges of $p$. Under the OU noise, bipartite entanglement was preserved for longer intervals than that shown under different noisy situations explored in Refs. [15, 23, 24, 27–29].

### 3.2 Entropic uncertainty relations and entanglement dynamics in IQN configuration

The dynamics of two-qubit entropic uncertainty, entropic uncertainty lower bound, tightness, and entanglement in the presence of the independent classical fields characterized by OU noise are discussed in this section. The final density matrix for the IQN configuration is obtained by averaging the time evolved density matrix given in Eq. (3) as
\[ \rho_{IQN}(t) = \langle \langle \rho(\phi_A, \phi_B, t) \rangle_{\theta(\tau)_A} \rangle_{\theta(\tau)_B}, \]  

where \( \chi_A \neq \chi_B \). The corresponding numerical form of the density matrix can be put into the following form as

\[
\rho_{IQN}(t) = \begin{bmatrix}
\frac{1+p}{4} & 0 & 0 & \frac{1}{2} e^{-4\beta} p \\
0 & \frac{1-p}{4} & 0 & 0 \\
0 & 0 & \frac{1-p}{4} & 0 \\
\frac{1}{2} e^{-4\beta} p & 0 & 0 & \frac{1+p}{4}
\end{bmatrix}
\]  

The presence of diagonal and off-diagonal components in the obvious matrix indicates that the state is entangled and coherent. Therefore, time evolution limitations and noise parameter choices have a role in further restricting entanglement and promoting entropic uncertainty. Next, the analytical expressions obtained for the \( U(\tau) \) and \( C(\tau) \) can be given as

\[
U(\tau) = \frac{1}{\log(16)} e^{-4\beta} \left( -2p \mu_1 + e^{4\beta} (\mu_2 - 2\mu_3 + \mu_4) \right),
\]

\[
C(\tau) = -\sqrt{1-p} - \frac{1}{2} \sqrt{1+p - 2e^{-4\beta} p} + \frac{1}{2} \sqrt{e^{-4\beta} (e^{4\beta} + 2p + e^{4\beta} p)}. 
\]

where,

\[
\mu_1 = 2 \arctan \left( e^{-4\beta} p \right) + \log \left( 1 + p - 2e^{-4\beta} p \right) - \log \left( 1 + p + 2e^{-4\beta} p \right),
\]

\[
\mu_2 = -2(1+p) \log(1+p) + (1+p) \log \left( 1 + p - 2e^{-4\beta} p \right),
\]

\[
\mu_3 = \log \left( 1 - e^{-4\beta} p \right) + \log \left( 1 + e^{-4\beta} p \right),
\]

\[
\mu_4 = (1+p) \log \left( 1 + p + 2e^{-4\beta} p \right).
\]

Figure 6 investigates the dynamics of the entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence in two qubits coupled with two independent environments characterized by two OU noise sources. The qualitative dynamics of the entropic uncertainty and entropic uncertainty lower bound appear to be identical. However, the inequality remains quantitatively valid, and the two sides are not equal. It can be validated by looking at the \( U(\tau) \) findings, which show that there is a quantitative difference between the two sides, but it is lower than that seen in the CQN configuration. The dynamical mappings of \( L(\tau) \) and \( R(\tau) \) remained the growing functions of entropic uncertainty. On the other hand, \( U(\tau) \) and \( C(\tau) \) remained decreasing functions of tightness and entanglement. The present dynamical map of entanglement under IQN configurations appears to be more suppressed than \( C(\tau) \) dynamics in the CQN configuration. Hence, entanglement is better retained in CQN than in IQN configurations. The rate of entropic uncertainty rise and entanglement decrease remained
Fig. 6 Dynamics of $R(\tau)$ (a), $L(\tau)$ (b), $U(\tau)$ (c) and $C(\tau)$ (d) for a system of two qubits initially prepared in Werner state form $\rho_0$ given in Eq. (4) for independent qubit-noise configuration with parameter settings: $g = 0.4$, $p = 1$ against time parameter $\tau$

3.2.1 Explicit dynamics of the two qubits in IQN configuration

When the system is coupled with two independent classical environments, the entropic uncertainty, entropic uncertainty lower bound, as well as tightness and concurrence dynamics are presented in Fig. 7. The current findings agree with those obtained in Fig. 3, although there are some quantitative differences. As seen from $R(\tau)$ and $L(\tau)$ functions, the present case’s entropic uncertainty growing rate is slower than that observed in the CQN configuration scenario. The dynamical maps of the tripartite states under classical noises, on the other hand, demonstrate that the entropic uncertainty proportionate, and entanglement decay is witnessed to occur later than the entropic uncertainty rise. As a result, it may be deduced that relative entropic uncertainty causes disentanglement in the two qubits. The entropic uncertainty rises monotonically until it reaches its maximum value and finally saturates. $C(\tau)$ has a monotonic qualitative dynamical behavior that matches $L(\tau)$, $R(\tau)$, and $U(\tau)$. The new results show a lower degree of uncertainty than the tripartite entropic uncertainty explored for the three-qubit GHZ state [57]. In contrast, the GHZ state remained entangled depending on the type of system-environment coupling, but the current bipartite Werner state becomes completely separable. Different types of quantum systems remained entangled depending on the type of system-environment interaction given in Refs. [15, 28, 35, 45, 46], but the current bipartite Werner state becomes completely disentangled.
rise is much smaller in the current case [57]. At lower $g$ values, both the $L(\tau)$ and $R(\tau)$ curves reached the ultimate saturation levels, taking a longer time. We found the two sides deviating from each other in the curves with an insignificant narrow gap. Compared to [56], the present difference between the two sides is smaller. Even though no entropic revivals were observed in the current dynamical setup, we noticed that the occurrence of revivals in the dynamical maps is due to the suppression of entropic uncertainty, as shown in [58]. Although the preservation intervals are longer under the OU noise case, they are consistent with the results obtained for the two-qubit Bell’s state dynamics when subjected to classical fields with static noise [25]. Despite that, as seen in Fig. 4, entanglement decreases, and entropic uncertainty increases faster for larger values of $g$. In contrast, we noticed that the values of the entropic uncertainty relation are lower for smaller $g$ values.

In Fig. 8, we illustrate the dynamics of the two sides of entropic uncertainty relations, tightness, and concurrence in the bipartite entangled state coupled to local random fields with OU noise against various purity factor values. The current findings can be traced back to different values of $p$ in Fig. 5, and it can be seen that as $p$ decreases, the initial entropic uncertainty increases. Compared to the CQN setup investigated for different $p$ values, the entropic uncertainty increase in the current case is smaller. The measurements of $R(\tau)$, and $L(\tau)$ agree, demonstrating that the initial uncertainty is lowest for $p = 0.99$ and grows for all other values. Besides, the $U(\tau)$ functions
has shown larger variation between the range $0.9 > p > 0.1$, and at maximum and minimum bounds of $p$, the variation between $R(\tau)$ and $L(\tau)$ functions becomes insignificant. When the difference between $R(\tau)$ and $L(\tau)$ approaches zero, the tightness curves eventually reach a minimal saturation level. The entanglement preservation duration appears to be influenced by the initial purity of the two qubits. Because of the decoherence and entropic uncertainty nature of the classical environments, the entanglement quickly fades as the initially encoded purity lowers. The current dynamical maps in IQN configuration offered less entropic uncertainty and a longer entanglement retention time than in CQN configuration, showing that it is a good resource for practical QIP applications. This contradicts the findings of the tripartite non-local correlations, which remained more robust and preserved in the presence of a common noise source as compared to the current two-qubit Werner entangled state given in [24, 28, 35, 59, 60].

### 3.3 Purity factor, degree of entropic uncertainty relations and entanglement

The dynamics of entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence as functions of the purity factor of the two-qubit mixed entangled state are discussed in this section.
In Fig. 9, the entropic uncertainty, entropic uncertainty lower bound, tightness, and concurrence are displayed as a function of purity parameter $p$. The $R(p)$ and $L(p)$ are both the maximum at $p = 0$ and minimum at $p = 1$, according to the current results. This means that the entropic uncertainty functions reach their maximums when the two-qubit Werner state becomes completely separable. However, in the CQN configuration, both the $L(p)$ and $R(p)$ entropies are higher than in the IQN setup. The difference between the $L(p)$ and $R(p)$ in the range $0.9 \leq p \leq 0.15$ is smaller than the top and lower limits of $p$. The results of $U(p)$ agree with the $L(p)$ and $R(p)$ and show that for the upper and lower bounds, the difference between the two sides is insignificant and vice versa. At $p = 0$, the state of the $C(p)$ becomes completely zero, suggesting separability of the state, and at $p = 1$, it becomes maximally entangled. The variance in entanglement and entropic uncertainty relations becomes completely insignificant for minimum $g$ values against the purity factor of the state, as seen from entropic uncertainty and concurrence functions. Compared to the CQN configuration, the entanglement in the IQN configuration appears to be better preserved. As a result, the IQN configuration may simulate QIP protocols realistically.

4 Significance of the current study

The presence of uncertainty, indeed, negatively affects the efficacy of QIP. For example, in the form of less precision and accuracy, leading to inaccurate and faulty implemen-
tation of QIP tasks. Besides, information is needed to be encoded in the state, for which different strategies are devised. But the fruitful and optimal manipulation of the state is again restricted by uncertainty between the targeted observables. Similar types of drawbacks can be interpreted when the systems are set to communicate information, which needs an appropriate coupling medium. In this regard, the compatibility between the system and the coupled medium can be measured in terms of uncertainty. For example, a configuration of system-field coupling that outputs a reduced uncertainty value when subjected to entropic uncertainty measurement will be more optimal and have high compatibility. Besides, the uncertainty between the energy of the system and the time of information processing can provide us with related preservation limits. There are many aspects of the entropic uncertainty utilized in the current study which can be turned into a useful practical implementation of QIP.

Entanglement, on the other hand, is an essential asset for the successful implementation of QIP. However, because of the inevitable decoherence and uncertainty effects, entanglement quickly dissipates, causing QIP failure. Therefore, studying the uncertainty effects, as well as the entanglement simultaneously, provided us with the exact relation between them. One of the primary reasons, we chose to investigate the entanglement and entropic uncertainty was that in the case of classical channels; they were not analyzed. As classical channels are still reliable resources for information processing, therefore, we believe, it highly suits the current physical models of QIP implementations, such as in the case of teleportation. Hence, we studied the exact relation between the two quantum criteria and this is intriguing to know that compared to the magnetic fields, and other related types of mediums, the classical channels offered smaller entropic uncertainty. Therefore, ensuring the higher success of QIP protocols while using classical communication means.

In this study, we provided the exact relationship between the entropic uncertainty and entanglement influenced by the classical channels. We find that the entropic uncertainty relations cause the entanglement to degrade. The statement can be verified by looking into Figs. 3-9 where the entropic uncertainty slope always leads and the entanglement decay lags. This suggests that the increasing uncertainty between the two qubits causes the associated entanglement to degrade. Compared to the external magnetic fields, the entropic uncertainty caused by the classical channels is much lesser, for example, see Refs. [61–65]. Not only this, but the entropic uncertainty causes disentanglement of the quantum systems, however, much later than that observed under the presence of magnetic fields. Moreover, there are certain examples [66–69], where there exists nonzero initial entropic uncertainty, or the associated eventual rise is much higher and took comparably a very short time to cause disentanglement of quantum systems compared to that observed in the current CQN and IQN configuration. In agreement with Ref. [50, 70], we also conclude that the growth rate, as well as the total occurrence of the entropic uncertainty pushing the quantum systems from quantum to classical domains greatly depends upon the initial purity of the state, and the types of channels utilized. Besides, the current results also appraise the possibility that the growth speed of the entropic uncertainty can be resisted by optimization of the parameters related to the configuration in use.
5 Conclusion

We study the dynamics of entropic uncertainty, entropic uncertainty lower bound, tightness, and entanglement in two qubits exposed to classical fields described by the OU process. The two qubits are prepared initially in an entangled Werner state, regulated by a purity factor. In addition, we consider two different system-environment coupling schemes, namely common qubit-noise and independent qubit-noise configurations. Besides, we analyzed the entropic uncertainty, uncertainty bound, tightness and entanglement under different parameters’ setup to obtain the optimal procedure for achieving longer entanglement and correlation time. In addition, we primarily focused on finding the relation between the entropic uncertainty and related entanglement decaying effects in the two-qubit entangled state.

Our findings show that both entropic uncertainty and entanglement in quantum systems are related, in the sense that when entropic uncertainty rises, entanglement falls. Entropic uncertainty and entropic bound functions, both increased due to the noisy action of the classical fields in the dynamical map of two qubits. The discrepancy between the entropic uncertainty and entropic uncertainty lower bound is controlled primarily by the purity parameter. However, for the common qubit-noise configurations, the related uncertainty functions have shown better growth as compared to that in independent qubit-noise configurations. Besides, the difference between the two entropic uncertainty functions develops more when the noise parameter $g$ is raised. In the case of purity factor $p$, some unusual behavior has been observed. In the ranges of $0.9 \geq p \geq 0.15$, the entropic uncertainty gap between the two sides is larger, but it becomes exceptionally small at $p > 0.9$ and $p < 0.15$. Our findings show that the left-hand side of the entropic uncertainty relation is more effective than the right-hand side, which is consistent with the findings given in [56]. In the case of concurrence, entanglement decreases as the entropic uncertainty between the qubits increases, and it remains a crucial cause for the disentanglement of the two qubits.

The entanglement preservation intervals are controlled by the intensity of the noise in the classical channel(s), which decrease proportionately as $g$ rises. In contrast, the initial state entanglement and entropic uncertainty solely depend on the state’s purity and both are directly proportional.

Finally, under any parameter optimization values, the two-qubit state eventually reaches separability, with no ultimate solution to avoid the corresponding disentanglement and decoherence. It’s worth mentioning, however, that the phase factors of the current system can be leveraged to accomplish longer entanglement preservation via the optimal setting of the parameters, in particular, when $g \leq 10^{-3}$ is employed. In contrast to the bipartite and tripartite states reported in [12, 15, 23, 25, 56], entanglement in the current study has been maintained for extended periods. The entropic uncertainty relations can also be depicted similarly. Moreover, the entropic uncertainty relations and concurrence all showed a monotonic Markovian behavior with no revivals, thus predicting the permanent loss of information in the current classical dephasing channels.

Data availability This work is theoretical research and has no associated data.
Declarations

Conflict of Interest  All authors declare that they have no recognized competing financial interests that could have emerged to influence the work reported in this research.

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