Optimal Regional Tracking Control of Time-Fractional Diffusion Systems

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Abstract—In this paper, we aim to explore optimal regional trajectory tracking control problems of the anomalous subdiffusion processes governed by time-fractional diffusion systems under the Neumann boundary conditions. Using eigenvalue theory of the system operator and the semigroup theory, we explore the existence and some estimates of the mild solution to the considered system. An approach on finding solution to the optimal problem that minimizes the regional trajectory tracking error and the corresponding control cost over a finite space and time domain is then explored via the Hilbert uniqueness method (HUM). The obtained results not only can be directly used to investigate the systems that are not controllable on the whole domain, but also yield an explicit expression of the control signal in terms of the desired trajectory. Most importantly, it is worth noting that our results in this paper are still novel even for the special case when the order of fractional derivative is equal to one. Finally, we provide a numerical example to illustrate our theoretical results.

Index Terms—Regional tracking control; Optimal Control; Time-fractional diffusion systems; Hilbert uniqueness method.

I. INTRODUCTION

During the past two decades, there is an increasing activity in the discussion of tracking problems for conventional reaction-diffusion dynamic systems, which can be divided into two steps: trajectory planning and tracking control (see e.g., the monographs [1], [2]). In these studies, the trajectory planning step attempts to generate a reference trajectory for the given desired function, while the tracking control focuses on system dynamics and hopes to design a sequence of inputs to track the pre-planned reference trajectory [3]. For the trajectory planning problems, we refer the reader to [4], [5], [6] where the flatness-based feedforward control strategies were presented or to [7] where the hardware and numerical illustrations were carried out. Moreover, to improve the accuracy in tracking control, various controller design techniques such as sliding-mode control [8], robust control [9], iterative learning control [10] and active disturbance rejection control (ADRC) [11] have been developed.

On the other hand, after the pioneering work given by Einstein in [12], it is confirmed that conventional diffusion system can well model the Brownian motion, whose mean-square displacement (MSD) is a linear function of time $t$. However, there exist a great deal of extremely complex transport processes that is characterized by a power-law MSD relation (i.e., $\text{MSD}=t^\alpha, \alpha>0$) including the anomalous subdiffusion case with $\alpha \in (0,1)$ and the anomalous superdiffusion case with $\alpha > 1$. In these anomalous situations, the usual physical laws would never be followed and the mathematical models will diver from the traditional integer-order systems to the fractional-order cases [13], [14], [15], [16]. Besides, we see that for anomalous subdiffusion process, time-fractional diffusion system has been confirmed as a powerful tool to model it [17], [18], [19]. Here the time-fractional diffusion system is a new extension model of conventional diffusion system by replacing the first order time derivative with a fractional-order derivative of order $\alpha \in (0,1]$. This is due to the fact that fractional-order derivative is defined as a kind of convolution hence representing well the dynamics inheriting subdiffusive properties and moreover, the fractional-order derivative would recover the first-order derivative if it approaches to one [20]. Then, based on our previous work on trajectory planning problem of time-fractional reaction-diffusion systems [21], in this paper, we go on investigating the optimal trajectory tracking control problems of linear time-fractional diffusion systems with the Neumann boundary conditions. Further results on optimal tracking control of coupled nonlinear time-fractional diffusion systems under more general boundary conditions will be discussed in our forthcoming works.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial \Omega$ and denote $Q = \Omega \times [0,T]$, $\Sigma = \partial \Omega \times [0,T]$ with $T > 0$. Herein, we consider the following time-fractional diffusion system with a Caputo fractional derivative $\frac{CD^\alpha_t}{0} y(x,t)$ of order $\alpha \in (0,1]$

$$
\begin{cases}
\frac{CD^\alpha_t}{0} y(x,t) = Ay(x,t) + u(x,t) & \text{in } Q, \\
\frac{\partial y}{\partial \nu}(x,t) = 0 & \text{in } \Sigma, \\
y(x,0) = y_0(x) & \text{in } \Omega,
\end{cases}
$$

(1)

where $A$ generates a strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ on the Hilbert space $L^2(\Omega)$. $-A$ is a uniformly elliptic operator (see e.g., the Definition 9.2 of [22]), $u \in L^2(Q)$ denotes the control inputs and $v$ represents the unit outside normal vector of the boundary $\partial \Omega$. Here $L^2(\Omega)$ represents the usual Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and the norm $\| \cdot \|_{L^2(\Omega)}$. As cited in [23], system (1) covers a great deal of real-world applications in a spatially inhomogeneous environment. Typical examples include the reheating processes of heterogeneous metal slabs [16] or the flow through porous media with varying sources or sinks [24] and so on.

Taking into account that not all the states of time-fractional diffusion systems are reachable in the whole domain of

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interest. To address this issue, regional control ideas such as regional controllability [25, 26], regional observability [27] and regional stability [28, 29] have been employed and well studied due to their advantages of offering potential to reduce computational requirements and being possible to study the systems that are not controllable on the whole domain. With these in mind, the novelty in this paper is promoting to study regional trajectory tracking control problem of the system (1). For this purpose, we focus on employing optimal control strategy to determine control signals by minimizing the proposed tracking cost functional. However, optimal control design for system (1) is very challenging or even impossible due to the infinite dimensionality property of the problem. To overcome this limitation, the HUM provides an alternative approach [25, 30, 31]. In this method, dual system is selected to determine the explicit expression of the corresponding eigenvalue pairing \((\lambda_k, \xi_k)_{k \in \mathbb{N}}\). With these, any \(\varphi \in L^2(\Omega)\) can be expressed as

\[
\varphi(x) = \sum_{k \in \mathbb{N}} (\varphi, \xi_k)_{L^2(\Omega)} \xi_k(x).
\]

Then, the strongly continuous semigroup \(\{\Phi(t)\}_{t \geq 0}\) on \(L^2(\Omega)\) generated by \(A\) satisfies

\[
\Phi(t)\varphi = (\varphi, \xi_0)_{L^2(\Omega)} \xi_0 + \sum_{k \in \mathbb{N}} e^{\lambda_k t} (\varphi, \xi_k)_{L^2(\Omega)} \xi_k
\]

Denote by \(H^2(\Omega)\) and \(H^1_0(\Omega)\) the usual Sobolev spaces (see e.g., [33]), now we are ready to give the following lemma.

**Lemma 1:** [29, 34] If \(y_0 \in L^2(\Omega)\), then there exists a unique mild solution \(y \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega) \cap H^1_0(\Omega))\) to system (1) such that \(D^\alpha y \in C((0,T];L^2(\Omega))\) and the estimate

\[
||y||_{C([0,T];L^2(\Omega))} \leq \gamma_1 ||y_0||_{L^2(\Omega)} + \gamma_2 ||u||_{L^2(\Omega)}
\]

holds true for some \(\gamma_1, \gamma_2 > 0\). Moreover, \(y(x,t)\) satisfies

\[
y(x,t) = \mathcal{M}_\alpha(t)y_0(x) + \int_0^t \mathcal{M}_\alpha(t-\tau) \mu(x,\tau) d\tau,
\]

where

\[
\mathcal{M}_\alpha(t)\varphi(x) = \sum_{k \in \mathbb{N}} E_A(\lambda_k t^\alpha) (\varphi, \xi_k)_{L^2(\Omega)} \xi_k(x),
\]

and

\[
E_{\alpha,\beta}(t) = \sum_{k=0}^{m} \frac{t^\alpha}{\Gamma(\alpha+1)} \alpha, \beta > 0, \ t \geq 0,
\]

denotes the Mittag-Leffler function in two parameters. In particular, we write \(E_{\alpha,1}(t) = E_{\alpha,1}(t)\) for short when \(\beta = 1\).

Let \(y(x,t,\omega)\) denote the solution of system (1) for any given control \(u \in L^2(\Omega)\). Choose \(\omega \subset \Omega\) a positive Lebesgue measure sub-region, we define \(L^2(\omega), \|\cdot\|_{L^2(\omega)}\) as the corresponding Hilbert space on \(\omega\) and set \(Q_\omega = \omega \times [0,T]\). Moreover, let \(\chi_\omega\) be the characteristic function of \(\omega\) given by

\[
\chi_\omega(x) = \begin{cases} 
1, & \text{if } x \in \omega, \\
0, & \text{if } x \in \Omega \setminus \omega.
\end{cases}
\]

We have the following definition.

**Definition 3:** The considered optimal regional trajectory tracking control problem for system (1) in \(\omega\) at time \(T\) concerns how to design controller \(u \in L^2(\Omega)\) such that \(y(x,t,\omega)\) starting from \(y_0(x)\) could reach the given target function \(y_{dT} = y_d(\cdot,T) \in L^2(\omega)\) in \(\omega\) as close as possible along with the given trajectory \(y_d(x,t) \in L^2(Q_\omega)\) within \(t \in [0,T]\), i.e.,

\[
y_0(x) \underset{\text{optimal } u \in L^2(\Omega)}{\rightarrow} y_d(x,t), \quad x \in \omega.
\]
The following lemma plays a key role for our later technical development.

**Lemma 2:** [35] Let
\[ I^\alpha_T \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s)ds, \alpha \in (0,1] \] (15)
be the right-sided Riemann-Liouville fractional integral and let us denote \( J^\alpha_T \) the right-sided Riemann-Liouville fractional derivative given by (see Section 2.1 of [20])
\[ J^\alpha_T \varphi(t) = \left\{ \begin{array}{ll}
-\frac{d}{dt} I^{1-\alpha}_T \varphi(t), & \alpha \in (0,1), \\
-\frac{d}{dt} \varphi(t), & \alpha = 1.
\end{array} \right. \] (16)

For any \( \alpha \in (0,1] \), if \( \varphi'_1 \in L^p(0,T), \varphi_2 \in L^q(0,T), p, q \geq 1, 1/p + 1/q \leq 1 + \alpha \) and \( p \neq 1, q \neq 1 \) in the case when \( 1/p + 1/q = 1 + \alpha \), then the formula
\[ \int_0^T \varphi_2(t) J^\alpha_T \varphi_1(t)dt = \left[ J^\alpha_T \varphi_1(t) + \frac{d}{dt} \varphi_1(t) \right]_{t=0}^{t=T} \] (17)
holds true.

**III. OPTIMAL REGIONAL TRACKING CONTROL**

The control objective of this article is the guidance of state trajectories of system \( \dot{y}(t) = Ay(t) + Bu(t) + u(t) \) along a given trajectory \( y_d \in L^2(\omega) \) to reach the final target function \( y_d(t) = \varphi(t) \in L^2(\omega) \) in \( \omega \) at time \( T \). The closer the controlled \( y(x,t) \), \( y(x,T) \) follows the desired trajectory \( y_d(x,t) \) and the target \( y_d(x) \), the better the control target is achieved.

Suppose that \( U_{ad} \) is an nonempty closed, convex subset of \( L^2(Q) \), in this section, we aim to find \( u(t) \) such that
\[ J(u) \leq J(u) \text{ for all } u \in U_{ad}, \] (18)
where \( J \) is the squared difference integrated performance functional given by
\[ J(u) = \frac{1}{2} \int_0^T \int_Q |\varphi_d(x,t) - y_d(x,t)|^2 dx dt + \frac{1}{2} \int_0^T |\varphi_d(x,t)|^2 dx + \frac{1}{2} \int_0^T |u(x,t)|^2 dx \] (19)
and \( r_1, r_2 \geq 0, r_3 > 0 \) are three given constants. Here \( y_d(t) \) may be the equilibrium or any given target functions of system \( \dot{y}(t) = Ay(t) + Bu(t) + u(t) \). For this purpose, since \( U_{ad} \subseteq L^2(Q) \) is a nonempty closed convex subset, the following lemma is necessary.

**Lemma 3:** [30], [31] Assume that the quadratic functional \( u \rightarrow J(u) \) is strictly convex and differentiable that satisfies
\[ J(u) \rightarrow \infty \text{ as } ||u|| \rightarrow \infty, \quad u \in U_b, \] (20)
Then the uniqueness element \( u \in U_{ad} \), satisfying \( J(u) = \inf_{u \in U_{ad}} J(u) \) is characterized by
\[ J(u) \cdot (v - u) \geq 0, \quad \forall v \in U_{ad}. \] (21)

Based on Lemma 3, we get that the unique solution of the optimization problem (18) can be characterized by
\[ J'(u_r) \cdot (u - u_r) \geq 0 \text{ holds true for all } u \in U_{ad} \] (22)
and more precisely,
\[ r_1 \int_0^T \left( \varphi_d(x,t) - y_d(x,t) \right) \left( \varphi(x,t) - y(x,t) \right) dx dt + r_2 \int_\Omega \left( \varphi_d(x,t) - y_d(x,t) \right) \varphi(x,t) dx \geq 0 \quad \text{for all } u \in U_{ad} \] (23)

after a simple duality derivation.

\[ \chi^*_a \varphi(x) = \left\{ \begin{array}{ll}
\varphi(x), & x \in \omega, \\
0, & x \in \Omega \setminus \omega
\end{array} \right. \] (24)
denotes the adjoint operator of \( \chi^*_a \) and \( \varphi = \chi^*_a \varphi \). Further, to simplify above equation (23), let us introduce the following adjoint system
\[ \begin{cases}
J^\alpha_T z(x,t) = A^*z(x,t) + r_1 \left( \varphi_d(x,t) - y_d(x,t) \right) \\
\frac{d}{dt} Rz(x,t) + \left( \varphi_d(x,t) - y_d(x,t) \right) = 0
\end{cases} \quad \text{in } Q,
\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{d}{dt} Rz(x,t) &= r_2 \left( \varphi_d(x,t) - y_d(x,t) \right) \\
\chi^*_a y_d(t) &= 0 \text{ in } \Sigma,
\end{aligned} \] (25)
where \( A^* \) is the adjoint operator of \( A \) and \( R \) is an operator given by \( Rz_1(t) = z_1(T-t) \) as in property 2.7 of [36] satisfying
\[ \begin{align*}
R(I^a_T z_1(t)) &= \frac{1}{1-a} \int_0^T (s-T)^{a-1}z_1(s)ds \\
R(I^a_T z_1(t)) &= \frac{1}{1-a} \int_0^T (T-t)^{a-1}z_1(t)ds
\end{align*} \] (26)
\[ R(I^a_T z_1(t)) = \frac{1}{1-a} \int_0^T (s-T)^{a-1}z_1(s)ds, \] (27)
as a consequence of Eq. (15) and Eq. (16). To establish the existence of a unique solution to system (25), it is supposed that the eigenvalue pairing of operator \( A^* \) under the Neumann boundary conditions is \( (\lambda_k^*, \xi_k^*) \) \( k \in \mathbb{N} \), where \( \{ \xi_k^*(x) \} \) \( k \in \mathbb{N} \) also forms an orthonormal basis of \( L^2(\Omega) \). Recall from Lemma 1 and the Lemma 1 of [25], given any \( y \in L^2(Q) \) and \( y_d \in L^2(\Omega) \) based on the property of operator \( R \), we have the following result.

**Lemma 4:** Given any \( y \in L^2(Q) \) and \( y_d \in L^2(\Omega) \), if conditions of Lemma 3 are satisfied, then system (25) admits a unique mild solution \( z \in C([0,T]; L^2(\Omega)) \) \( \cap \ C((0,T]; H^2(\Omega) \cap H^1(\Omega)) \) as the following lemma is necessary.

\[ \chi^*_a \varphi(x) = \sum_{k \in \mathbb{N}} E_{\alpha,\lambda_k^*} \phi \xi_k^* \] (29)
for any \( \varphi \in L^2(\Omega) \).

**Proof:** Taking \( R \)-transformation defined above on both sides of system (25), using Eq. (27), one has
\[ \begin{cases}
\frac{d}{dt} Rz(x,t) = A^*Rz(x,t) + r_1 \left( \varphi_d(x,t) - y_d(x,t) \right) \\
\lim_{t \rightarrow 0^+} \frac{d}{dt} Rz(x,t) = r_2 \left( \varphi_d(x,t) - y_d(x,t) \right)
\end{cases} \quad \text{in } Q,
\begin{aligned}
\chi^*_a y_d(t) &= 0 \text{ in } \Sigma,
\end{aligned} \] (26)
It follows that [25] and Lemma 1 that a unique mild solution $z \in C \left(\left[0,T\right];L^2(\Omega)\right) \cap C \left(\left(0,T\right];H^2(\Omega) \cap H_0^1(\Omega)\right)$ to system 25 exists and moreover, it satisfies

$$
z(x,t) = R_2 \int_0^T \delta \frac{x(t)}{\omega} A y(t) + (p_o y(x,T,u_r) - \chi_{\omega y_d}(x)) + R_1 \int_0^T \delta \frac{x(t)}{\omega} A y(x,t,u_r) - \chi_{\omega y_d}(x,T) d\tau,
$$

$$
+ r_2 \delta \frac{x(t)}{\omega} A y(x,T,u_r) - \chi_{\omega y_d}(x) + r_1 \delta \frac{x(t)}{\omega} A y(x,t,u_r) - \chi_{\omega y_d}(x,T) d\tau.
$$

This finishes the proof.

In what follows, we proceed to simplify [23] based on the adjoint system (25).

Indeed, for the first term of above equation (23), using Lemma 2 it yields that

$$
\begin{align*}
& r_1 \left[ \int_0^T (p_o y(x,t,u_r) - \chi_{\omega y_d}(x))(y(x,t,u) - y(x,t,u_r)) dx dt \right] \\
& = \int_0^T \left[ \int_0^T \left( \partial_y (x,u) - \partial_y (x,u_r) \right) dx dt \right].
\end{align*}
$$

Then, using formula (30), we have

$$
\begin{align*}
& r_1 \left[ \int_0^T (p_o y(x,t,u_r) - \chi_{\omega y_d}(x))(y(x,t,u) - y(x,t,u_r)) dx dt \right] \\
& = \int_0^T \left[ \int_0^T \left( \partial_y (x,u) - \partial_y (x,u_r) \right) dx dt \right].
\end{align*}
$$

This, together with the boundary conditions of considered systems, yields that

$$
\begin{align*}
& r_1 \left[ \int_0^T (p_o y(x,t,u_r) - \chi_{\omega y_d}(x))(y(x,t,u) - y(x,t,u_r)) dx dt \right] \\
& + r_2 \left[ \int_0^T (p_o y(x,t,u_r) - \chi_{\omega y_d}(x))(y(x,t,u) - y(x,t,u_r)) dx dt \right] \\
& = \int_0^T \left[ \int_0^T \left( \partial_y (x,u) - \partial_y (x,u_r) \right) dx dt \right].
\end{align*}
$$

Therefore, the optimality condition (23) can be simplified to

$$
\int_0^T (r_3 u_r(x,t) + z(x,t,u_r)) dx dt \geq 0
$$

for all $u \in U_{ad}$.

Now we summarize the following result and omit the detailed proof.

**Theorem 1:** Given any target trajectory $y_d \in L^2(Q)$ and the target function $y_{\omega y_d} = \gamma(\cdot,T) \in L^2(\Omega)$, the optimal problem (18) admits a unique optimal solution $u_r$ that is determined by the system (1) and the adjoint system (25) satisfying the variational inequality (31).

In particular, when $U_{ad} = L^2(Q)$, since $r_3 > 0$, (31) holds true if

$$
u_r(x,t) = -\frac{1}{r_3} \zeta(x,t,u_r) \quad \text{for all } (x,t) \in Q.
$$

Furthermore, in order to obtain the optimal control (18) in feedback form, according to Lemma 1 [28] and (43), we have

$$
u_r(x,t) = -\frac{1}{r_3} \zeta(x,t,u_r)
$$

and

$$
\begin{align*}
& = \frac{1}{r_3} \zeta(x,t,u_r)
\end{align*}
$$

for all $(x,t) \in Q$. This allows us to give the explicit expression of the designed optimal controller and at the same time, to minimize the tracking cost functional (19).

**IV. NUMERICAL EXAMPLE**

This section aims to present a numerical simulation illustrating our obtained results. For the sake of simplicity, we let $\Omega = (0,1) \subseteq \mathbb{R}$ and claim that the higher-dimensional spatial domain case can be considered in a similar way.

Let us consider the following example

$$
\begin{align*}
& \zeta D_{0.5}^5 y(x,t) = 1.5 \frac{d^5 y(x,t)}{d x^5} - y(x,t) + u(x,t)
\end{align*}
$$

in $(0,1) \times [0,0.6,$

$$
\frac{d^4 y(x,t)}{d x^4} = 0 \quad \text{in } [0,0.6],
$$

$$
y(x,0) = 100(x - 0.7)^2 \quad \text{in } (0,1)
$$

Obviously, $\alpha = 0.5$, $T = 0.6$ and $A = 1.5 \frac{d^5}{d x^5} - 1$ is a uniformly elliptic operator. Under the Neumann boundary conditions $\frac{\partial y(x,t)}{\partial x} = \frac{\partial y(x,t)}{\partial x} = 0$ for all $t \in [0,0.6]$, the eigenvalue pairing of operator $A$ satisfies [5]

$$
\lambda_0 = -1, \lambda_k = -1.5k^2\pi^2 - 1
$$

(36)
The evolution of the tracking error in control problem

In what follows, we aim to solve the following optimal trajectory

\[ y_d(x,t) = \begin{cases} 1, & \text{if } k = 0, \\ \sqrt{2} \cos(k \pi x), & \text{if } k \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{37} \]

Then, the corresponding strongly continuous semigroup \((\Phi(t))_{t \geq 0}\) satisfies

\[ \Phi(t) \varphi = \sum_{k=0}^{\infty} e^{\lambda_k t} \langle \varphi, \xi_k \rangle_{L^2(0,1)} \xi_k, \quad \varphi \in L^2(0,1). \tag{38} \]

By Lemma 1 it yields that

\[ M_\alpha(t) \varphi = \sum_{k=0}^{\infty} E_{\alpha} (\lambda_k t^{2\alpha}) \langle \varphi, \xi_k \rangle_{L^2(0,1)} \xi_k, \quad \varphi \in L^2(0,1). \tag{39} \]

and

\[ X_\alpha(t) \varphi = \sum_{k=0}^{\infty} E_{\alpha} (\lambda_k t^{2\alpha}) \langle \varphi, \xi_k \rangle_{L^2(0,1)} \xi_k, \quad \varphi \in L^2(0,1). \]

Let the subregion \(\omega = [0.3, 0.7] \subseteq (0,1)\) and the desired trajectory \(y_d(x,t), x \in \omega\) be

\[
y_d(x,t) = 100x(x - 0.7)^2 \left( 0.6 - 0.5x^3 \right) + 4.5r(0.6 - t) + \frac{5t}{3} (-0.5x^4 + 2x^3 - 2.8x^2 + 1.38x - 0.05). \tag{40}
\]

One has

\[
y_d(x,0.6) = -0.5x^4 + 2x^3 - 2.8x^2 + 1.38x - 0.05, \quad x \in \omega.
\]

In what follows, we aim to solve the following optimal control problem

\[
\min_{u \in L^2((0,1) \times [0,0.6])} J(u)
\]

with

\[
J(u) = 10^4 \int_0^{0.7} \left( X_{[0.3,0.7]} y(x,t,u) - y_d(x,t) \right)^2 dxdt + 10^7 \int_0^{0.7} \left( X_{[0.3,0.7]} y(x,T,u) - y_d(x,T) \right)^2 dx + \frac{1}{2} \int_0^T |u(x,t)|^2 dxdt.
\]

According to Theorem 1, the optimal control problem admits a unique optimal solution \(u^*_e\) governed by

\[
u^*_e(x,t) = -z(x,t,u^*_e) \quad \text{for all} \quad (x,t) \in (0,1) \times [0,0.6].
\]

Moreover, to illustrate the effectiveness of our results, we set

\[
h(x,t) = \begin{cases} 0.25y_d(0,3,0) & \text{if } x \in (0,0.3), \\ y_d(x,t) & \text{if } x \in \omega = [0.3,0.7], \\ y_d(0,3,0) & \text{if } x \in (0.7,1), \end{cases}
\]

and then plot Figures 1(a) – (d), which show how closely does the state evolution track along the desired trajectory \(y_d(x,t) \in L^2((0.3,0.7) \times [0,0.6])\) and the final state reach the target function \(y_{Tr} = y_d(.,T) \in L^2(0.3,0.7)\) in \(\omega = [0.3,0.7]\) with the error

\[
\|X_{[0.3,0.7]} y(.,T) - y_{Tr}(x)\|_{L^2(0.3,0.7)} \leq 0.005,
\]

\[
\|X_{[0.3,0.7]} y(.,T) - y_d(x,.,T)\|_{L^2(0.3,0.7)} \leq 2.99. \tag{46}
\]

The corresponding optimal solution of the optimal control problem is depicted in Figure 1(e) with the costs \(\|u^*_e\|_{L^2(0.1) \times L^2(0.0,0.6)} = 12.57\).
V. Conclusion

Sufficient and necessary conditions for optimal regional trajectory tracking control problem of linear time-fractional diffusion systems are obtained in this paper by using the HUM. The obtained results not only can be used directly to discuss the systems that are not controllable on the whole domain, but also yield an explicit expression of the control signal in terms of the desired trajectory and minimize the proposed tracking cost functional as well. This is very appealing in practical applications and pose many new theoretically challenges at the same time. Moreover, we claim that the main results in this paper can be extended to more complex fractional-order distributed parameter systems (see those in [37] for example) and various open questions such as optimal actuation configuration problems for regional tracking control of the coupled nonlinear space-time fractional diffusion systems are still open.

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