Solutions of nonlinear oscillator differential equations using the variational iteration method

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Abstract. In this paper, He's variational iteration method (VIM) is applied to solve nonlinear oscillators. We illustrate that the variational iteration method is very effective and convenient and does not require linearization or small perturbation. Contrary to the conventional methods, the used method admits only single iteration with very high accuracy of the solution. Comparisons are made between the exact solutions and the results of the variational iteration method.

1. Introduction
This paper considers the following general nonlinear oscillator differential equations:

\[ u'' + f(t, u, u') = 0, \]

subject to \( u(0) = a \) and \( u'(0) = b \), where \( t \) is time, \( u \) is the displacement. And the prime denotes differentiation with respect to \( t \).

In this paper we will apply the variational iteration method to the nonlinear oscillator differential equations. The VIM was first proposed by He [1] and used to give approximate solutions of the problem of seepage flow in porous media with fractional derivatives. We will show how to solve nonlinear oscillators by the VIM [2-9], which leads to a very rapid convergence of the solution series, in the most cases only one iteration leads to high accuracy of the solution, providing an effective and convenient mathematical tool for nonlinear equations. The VIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. There is no need of linearization or discretization, and large computational work. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation.

2. Basic idea of He's variational iteration method
To clarify the basic ideas of He's VIM we consider the following differential equation:

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where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( g(t) \) an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

\[
u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(Lu_n(\tau) + Nu_n(\tau) - g(\tau))d\tau,
\]

Where \( \lambda \) is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript \( n \) indicates the \( n \)th approximation and \( \tilde{u}_n \) is considered as a restricted variation, i.e. \( \delta \tilde{u}_n = 0 \).

3. Applications

In this Section, two examples are considered for the comparison and usefulness of the method developed.

3.1. Example 1

Consider the Van Der Pol Oscillator problem [10]:

\[
u'' + \nu' + \nu + \nu^2 \nu' = 2\cos t - \cos^3 t
\]

with the boundary conditions:

\[
u(0) = 0,
\]

\[
u'(0) = 1.
\]

The exact solution of the above differential system is:

\[
u(t) = \sin(t).
\]

To solve Eq. (4) via VIM, one has to find the Lagrangian multiplier, which can be identified by substituting Equation (4) into Equation (3), upon making it stationary leads to the following:

\[
\lambda''(\tau) - \lambda'(\tau) + \lambda(\tau) = 0,
\]

\[
1 - \lambda'(\tau) + \lambda(\tau)\bigg|_{\tau=t} = 0,
\]

\[
\lambda(\tau)\bigg|_{\tau=t} = 0,
\]

Solving the system of Equations (7), yields:

\[
\lambda(\tau) = \frac{2\sqrt{3}}{3} e^{\frac{1}{2\sqrt{2}}(\tau-t)} \sin\left(\frac{\sqrt{3}}{2}(\tau-t)\right),
\]

And the variational iteration formula is obtained in the form:
\[
\begin{align*}
    u_{n+1}(t) &= u_n(t) + \int_0^t \left[ \frac{2\sqrt{3}}{3} e^{\frac{\sqrt{3}}{2}(t-t')} \sin(\frac{\sqrt{3}}{2}(\tau-t)) \right. \\
    &\quad \left. - u''(\tau) + u'(\tau) + u(\tau) + u^3(\tau) u'(\tau) - 2 \cos \tau + \cos^3 \tau \right] dt,
\end{align*}
\]

(9)

Now, we assume that the initial approximation has the form:

\[
u_0 = at + b,
\]

(10)

where \(a\) and \(b\) are unknown constants to be further determined.

By the iteration formula (9), we can directly obtain other components as:

\[
\begin{align*}
    u_1(x) &= -a^3x^2 + (-2a^2b + 2a^3)x - \frac{2}{73} e^{\frac{\sqrt{3}}{2}x} \cos(\frac{\sqrt{3}}{2}x) + \frac{2\sqrt{3}}{3} a^2 b e^{\frac{\sqrt{3}}{2}x} \sin(\frac{\sqrt{3}}{2}x) \\
    &\quad - 2a^2b e^{\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) + \frac{\sqrt{3}}{3} a b^2 e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) + a b^2 e^{\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) \\
    &\quad - 60 \sqrt{3} e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) + \frac{\sqrt{3}}{3} b e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) + \frac{2\sqrt{3}}{3} a e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) \\
    &\quad - \frac{4\sqrt{3}}{3} a e^{\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) - b e^{\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) + \frac{92}{73} \sin(x) + \frac{8}{73} \cos^3(x) \\
    &\quad - \frac{6}{73} \cos(x) - \frac{3}{73} \sin(x) \cos^2(x) + 2a^2b - ab^2.
\end{align*}
\]

(11)

Incorporating the boundary conditions, Equation (5), into \(u_1(x)\), we obtain:

\[
a = 1, \quad b = 0.
\]

(12)

Therefore, we obtain the following first–order approximate solution:

\[
\begin{align*}
    u_1(x) &= -x^2 + 2x - \frac{2}{73} e^{\frac{\sqrt{3}}{2}x} \cos(\frac{\sqrt{3}}{2}x) + \frac{326 \sqrt{3}}{219} e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) \\
    &\quad + \frac{92}{73} \sin(x) + \frac{8}{73} \cos^3(x) - \frac{6}{73} \cos(x) - \frac{3}{73} \sin(x) \cos^2(x).
\end{align*}
\]

(13)

The obtained solution is of remarkable accuracy, as shown in Table 1.

Table 1. Comparison of the first-order approximate solution with exact solution

| t   | Exact solution | Approximate solution | Error |
|-----|----------------|----------------------|-------|
| 0.0 | 0.000000000000 | 0.000000000000 | 0.000000000000E+0 |
| 0.1 | 9.983341665E-2 | 9.983338930E-2     | 2.73500000E-8     |
| 0.2 | 0.1986693308   | 0.1986676148       | 1.71600000E-6     |
| 0.3 | 0.2955202067   | 0.2955011070       | 1.90997000E-5     |
| 0.4 | 0.3894183423   | 0.3893138342       | 1.04508100E-4     |
| 0.5 | 0.4794255386   | 0.4790386507       | 3.86889000E-4     |
| 0.6 | 0.5646424734   | 0.5635252927       | 1.11718070E-3     |
| 0.7 | 0.6442176872   | 0.6415028932       | 2.71479400E-3     |
3.2. Example 2
Consider the nonlinear oscillator differential equation [11]:

\[ u^{\prime\prime} - u + u^2 + u^{\prime 2} - 1 = 0. \]  \hspace{1cm} (14)

With the boundary conditions:

\[ u(0) = 2, \]
\[ u'(0) = 0. \]  \hspace{1cm} (15)

The exact solution of the above differential system is:

\[ u(t) = 1 + \cos(t). \]  \hspace{1cm} (16)

To solve Equation (14) via VIM, one has to find the Lagrangian multiplier, which can be identified by substituting Equation (14) into Equation (3), upon making it stationary leads to the following:

\[ \lambda^\prime(\tau) - \lambda(\tau) = 0, \]  \hspace{1cm} (17a)
\[ 1 - \lambda'(\tau) \bigg|_{\tau=t} = 0, \]  \hspace{1cm} (17b)
\[ \lambda(\tau) \bigg|_{\tau=t} = 0. \]  \hspace{1cm} (17c)

Solving the system of Equations (17), yields:

\[ \lambda(\tau) = \frac{1}{2} e^{(\tau-x)} + \frac{1}{2} e^{(t-\tau)}, \]  \hspace{1cm} (18)

And the variational iteration formula is obtained in the form:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \left\{ \frac{1}{2} e^{(\tau-x)} + \frac{1}{2} e^{(t-\tau)} \right\}(u^{\prime\prime}(\tau) - u(\tau) + u^2(\tau) + u^{\prime 2}(\tau) - 1) \ dt, \]  \hspace{1cm} (19)

Now, we assume that the initial approximation has the form

\[ u_0 = a t + b, \]  \hspace{1cm} (20)

where \( a \) and \( b \) are unknown constants to be further determined.

By the iteration formula (19), we can directly obtain other components as:

\[ u_1(x) = \frac{1}{2} e^x + \frac{1}{2} e^{-x} + \frac{a}{2} e^x - \frac{a}{2} e^{-x} + \frac{b}{2} e^x + \frac{b}{2} e^{-x} - \frac{3}{2} a^2 e^x - \frac{3}{2} a^2 e^{-x} - a b e^x + a b e^{-x} - \frac{b^2}{2} e^x - \frac{b^2}{2} e^{-x} + a^2 x^2 + 3a^2 + 2b x + b^2 - 1. \]  \hspace{1cm} (21)

Incorporating the boundary conditions, Equation (15), into \( u_1(x) \), we obtain:
Therefore, we obtain the following first–order approximate solution:

\[ u_1(x) = -\frac{1}{2} e^x - \frac{1}{2} e^{-x} + 3. \]  

The obtained solution is of remarkable accuracy, as shown in Table 2.

| t   | Exact solution | Approximate solution | Error          |
|-----|----------------|----------------------|----------------|
| 0.0 | 2.000000000    | 2.000000000          | 0.0000000E+0   |
| 0.1 | 1.995004165    | 1.994995832          | 8.3330000E-6   |
| 0.2 | 1.980066578    | 1.979933244          | 1.3334000E-4   |
| 0.3 | 1.955336489    | 1.954661486          | 6.7500300E-4   |
| 0.4 | 1.921060994    | 1.918927628          | 2.1336600E-3   |
| 0.5 | 1.877582562    | 1.872374035          | 5.2085270E-3   |
| 0.6 | 1.825335615    | 1.814534782          | 1.0800833E-2   |
| 0.7 | 1.764842187    | 1.744830994          | 2.001193E-2    |
| 0.8 | 1.696706709    | 1.662565054          | 3.4141655E-2   |
| 0.9 | 1.621609968    | 1.566913614          | 5.4696354E-2   |
| 1.0 | 1.540302306    | 1.456919365          | 8.3382941E-2   |

4. Conclusions

The variational iteration method is remarkably effective for solving boundary value problems. Comparison between the approximate and exact solutions shows that the one iteration is enough. This method is a very promoting method, which will be certainly found widely applications.

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