Privacy Under Hard Distortion Constraints

Jiachun Liao, Oliver Kosut, Lalitha Sankar
School of Electrical, Computer and Energy Engineering, Arizona State University
Email: {jiachun.liao,lalithasankar,okosut}@asu.edu

Flavio P. Calmon
School of Engineering and Applied Sciences
Harvard University
Email: fcalmon@g.harvard.edu

Abstract—We study the problem of data disclosure with privacy guarantees, wherein the utility of the disclosed data is ensured via a hard distortion constraint. Unlike average distortion, hard distortion provides a deterministic guarantee of fidelity. For the privacy measure, we use a tunable information leakage measure, namely maximal $\alpha$-leakage ($\alpha \in [1, \infty)$), and formulate the privacy-utility tradeoff problem. The resulting solution highlights that under a hard distortion constraint, the nature of the solution remains unchanged for both local and non-local privacy requirements. More precisely, we show that both the optimal mechanism and the optimal tradeoff are invariant for any $\alpha > 1$; i.e., the tunable leakage measure only behaves as either of the two extrema, i.e., mutual information for $\alpha = 1$ and maximal leakage for $\alpha = \infty$.

Index Terms—Privacy-utility tradeoff, maximal $\alpha$-leakage, hard distortion, $f$-divergence.

I. INTRODUCTION

From social networks to medical databases, useful cloud-based services require some form of user data disclosure to a third party. Data disclosure, however, often incurs a privacy risk. In most non-trivial settings, there is a fundamental tradeoff between privacy and utility: on the one hand, disclosing data “as is” can lead to unwanted inferences of private information. On the other hand, perturbing or limiting the disclosed data can result in a reduced quality of service.

The exact nature of the privacy-utility tradeoff (PUT) will depend to varying degrees on the distribution of the underlying data, as well as the chosen metrics (e.g., differential privacy [1], mutual information (MI) [2], [3], $f$-divergence-based leakage measures [4], maximal leakage (MaxL) [5]). Furthermore, most information-theoretic PUs capture utility as a statistical average of desired measures of fidelity [6]–[9]. This, in turn, simplifies the PUT to a single-letter optimization for independent and identically distributed (i.i.d.) datasets [10].

We measure utility in terms of a new hard distortion metric, which constrains the privacy mechanism so that the distortion function between original and released datasets is bounded with probability 1. This distortion metric is quite stringent, particularly when compared to average-case distortion constraints [10], but it has the advantage that it allows the data curator to make any guarantee that the realization of the disclosed dataset has any relationship to the original one.

We adopt maximal $\alpha$-leakage, which we introduced in [11], as an information leakage measure. Maximal $\alpha$-leakage is a tunable privacy metric defined via an $\alpha$-loss function with parameter $\alpha \in [1, \infty)$. For $\alpha = 1$, this metric captures the inference gain by a (soft decision) belief-refining adversary after observing the disclosed data. As $\alpha \to \infty$, this metric captures the reduction in $0 - 1$ loss or, equivalently, the gain of a (hard decision) adversary’s guessing ability after data disclosure. These extreme points correspond to MI and MaxL, respectively. The tunable parameter $\alpha$ allows continuous interpolation between the two extremal adversarial actions by determining how much weight an adversary gives to its posterior belief.

Using the aforementioned utility and privacy measures, we precisely quantify the PUT and show that: (i) the same privacy mechanism achieves the same optimal PUT for all $\alpha > 1$, and both the optimal mechanism and the optimal PUT are independent of the distribution of original data; (ii) For $\alpha = 1$, the optimal privacy mechanism depends on the distribution of original data. More generally, for the sake of completeness, we also consider a larger class of $f$-divergence-based information leakages and derive the optimal PUTs for this class.

The paper is organized as follows: in Sec. II, we review maximal $\alpha$-leakage. In Sec. III, we formulate and solve the PUT problems with maximal $\alpha$-leakage as well as its $f$-divergence-based variants as privacy measures, and using hard distortion as the utility measure. In Sec. IV, we illustrate our results via an example with binary data wherein the distortion function is the distance between types (empirical distributions) of the original and disclosed datasets.

II. MAXIMAL $\alpha$-LEAKAGE AND RELATED LEAKAGE MEASURES

Let $X$ and $Y$ represent the original and disclosed data, respectively, and let $U$ represent an arbitrary (potentially random) function of $X$ that the adversary (a curious or malicious observer of the disclosed data $Y$) is interested in learning. Maximal $\alpha$-leakage, introduced in [11], measures various aspects of leakage (ranging from the probability of correctly guessing to the posterior distribution) about data $U$ from the disclosed $Y$. We review the formal definition next.
Definition 1 ([11, Def. 5]). Given a joint distribution $P_{XY}$ on finite alphabets $X \times Y$, the maximal $\alpha$-leakage from $X$ to $Y$ is defined as

$$\mathcal{L}^{\text{max}}_{\alpha}(X \rightarrow Y) = \sup \left\{ \inf_{f : X \rightarrow Y} \mathcal{L}_f(X; Y) : \mathbb{P}(X) > 0 \right\}$$

where $\mathcal{L}_f(X; Y)$ is constrained to have the same support as $P_X$. The expression in (1) can be further simplified to obtain the following theorem.

Theorem 1 ([11, Thm. 2]). For $\alpha \in [1, \infty]$, the maximal $\alpha$-leakage defined in (1) simplifies to

$$\mathcal{L}^{\text{max}}_{\alpha}(X \rightarrow Y) = \left\{ \begin{array}{ll}
\sup_{P_X} \inf_{Q_Y} D_{\alpha}(P_X P_{Y|X} \| P_X \times Q_Y), & \alpha \in (1, \infty] \\
I(X; Y), & \alpha = 1
\end{array} \right. \quad (3a)$$

where in the supremum $P_X$ is constrained to have the same support as $X$, and $D_{\alpha}(\cdot; \cdot)$ is the Rényi divergence [12] of order $\alpha$ given by

$$D_{\alpha}(P_X P_{Y|X} \| P_X \times Q_Y) = \frac{1}{\alpha - 1} \log \left( \sum_{x,y} P_X(x) P_{Y|X}(y|x)^\alpha \frac{Q_Y(y)}{P_Y(y)^\alpha} \right)$$

and it is defined by its continuous extension for $\alpha = 1$ or $\infty$.

The infimum over $Q_Y$ in (3a) is exactly Sibson MI of order $\alpha$ [13, Def. 4]. Note that for $\alpha = 1$ and $\alpha = \infty$, the maximal $\alpha$-leakage simplifies to MI and MaxL, respectively. In [11], we show that maximal $\alpha$-leakage ($\alpha \in [1, \infty]$) satisfies data processing inequalities and a composition theorem.

While we are mainly interested in maximal $\alpha$-leakage, our results apply to a broader class of information leakages derived from $f$-divergences. Recall that for a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(1) = 0$, an $f$-divergence $D_f$ is a measure of the similarity between two distributions given by

$$D_f(P \| Q) = \int dQ f \left( \frac{dP}{dQ} \right). \quad (4)$$

Definition 2. Given a joint distribution $P_{XY} = P_{Y|X} P_X$ and a $f$-divergence $D_f$, a distribution-dependent leakage is defined as

$$\mathcal{L}_f(X; Y) = \inf_{Q_Y} D_f(P_{XY} \| P_X \times Q_Y), \quad (5)$$

and a distribution-independent leakage defined as

$$\mathcal{L}^{\text{max}}_{f}(X \rightarrow Y) = \sup_{P_X} \inf_{Q_Y} D_f(P_X P_{Y|X} \| P_X \times Q_Y), \quad (6)$$

where $P_X$ is constrained to have the same support as $P_X$. Recall that for $\alpha = 1$, the maximal $\alpha$-leakage is MI and is a special case of $\mathcal{L}^{\text{max}}_{f}(X \rightarrow Y)$ in (5) with $f(t) = t \log t$. Furthermore, for $\alpha > 1$, maximal $\alpha$-leakage has a one-to-one relationship with a special case of $\mathcal{L}^{\text{max}}_{f}(X \rightarrow Y)$ for $f$ given by

$$f_\alpha(t) = \frac{1}{\alpha - 1} (t^\alpha - 1), \quad (7)$$

such that $D_f$ is the Hellinger divergence of order $\alpha$ [14]. The following lemma makes precise this observation.

Lemma 1. For $\alpha > 1$, maximal $\alpha$-leakage can be written as

$$\mathcal{L}^{\text{max}}_{\alpha}(X \rightarrow Y) = \frac{1}{\alpha - 1} \log \left( 1 + (\alpha - 1) \mathcal{L}^{f_{\alpha}}_{\text{max}}(X \rightarrow Y) \right), \quad (8)$$

where $\mathcal{L}^{f_{\alpha}}_{\text{max}}(X \rightarrow Y)$ is the $\mathcal{L}^{f_{\alpha}}_{\text{max}}(X \rightarrow Y)$ in (6) for $f_\alpha$ given by (7) such that $D_f$ is the Hellinger divergence of order $\alpha$.

III. PRIVACY-UTILITY TRADEOFF WITH A HARD DISTORTION CONSTRAINT

We now consider PUT problems minimizing either maximal $\alpha$-leakage or its related $f$-divergence-based variants in Def. 2, subject to a hard distortion constraint. Such a constraint can be written as $d(X, Y) \leq D$ with probability 1, where $d(\cdot, \cdot)$ is a distortion function and $D$ is the maximal permitted distortion. In other words, for any input $x \in X$, the output $y$ of the privacy mechanism must lie in a ball $B_D(x)$ given by

$$B_D(x) = \{ y : d(x, y) \leq D \}. \quad (9)$$

We henceforth denote an optimal PUT as $\text{PUT}_{HD, \mathcal{L}^*}$, where HD and $\mathcal{L}^*$ in the subscript indicate the utility and privacy measures, respectively. The following two theorems characterize $\text{PUT}_{HD, \mathcal{L}^*}$ and $\text{PUT}_{HD, \mathcal{L}^{\text{max}}_{f}}$ with detailed proofs in appendices A and B, respectively.

Theorem 2. For any distribution-dependent leakage $\mathcal{L}_f$ in (5) and a distortion function $d(\cdot, \cdot)$ with $B_D(x)$ in (9), the optimal PUT is given by

$$\text{PUT}_{HD, \mathcal{L}_f}(D) = \inf_{P_Y \mid X \text{ s.t. } d(X, Y) \leq D} \mathcal{L}_f(X; Y)$$

$$= f(0) + \inf_{Q_Y} \mathbb{E} \left( Q_Y(B_D(X)) \left( \frac{1}{Q_Y(B_D(X))} - f(0) \right) \right). \quad (10)$$

If there exists a distribution $Q_Y^*$ achieving the infimum in (11), an optimal mechanism $P_{Y|X}^*$ is given by

$$\frac{dP_{Y|X}^*}{dQ_Y^*}(y) = \frac{1}{Q_Y^* (B_D(x))} 1_{d(x, y) \leq D} \frac{dP_Y^*}{dQ_Y^*}(y). \quad (12)$$

1The independence is with respect to the distribution of $X$. This “distribution-independent” measure depends on the distribution of $X$ only through its support. In contrast, the distribution-dependent measure $\mathcal{L}_f$ depends fully on the distribution of $X$. Both measures depend on the chosen mechanism $P_Y|X$. 


For any distribution-independent leakage $\mathcal{L}_{\max}^\text{PUT}$ in (6), a distortion function $d(\cdot, \cdot)$ and $B_D(x)$ in (9), the optimal PUT is given by

$$\text{PUT}_{\text{HD, } \mathcal{L}_{\max}^\text{PUT}}(D) = \inf_{P_{Y|X} : d(X,Y) \leq D} \mathcal{L}_{\max}^\text{PUT}(X \rightarrow Y)$$

for $q^*$ defined as

$$q^* = \sup_{x \in \mathcal{X}} q_Y(B_D(x)).$$

Moreover, if there exists $Q_Y^*\text{ achieving the supremum in (15), an optimal mechanism } P_{X,Y}^* \text{ is given by (12).}$$

The PUTs in (11) and (14) simplify to finding an output distribution $Q_Y$ that can be viewed as a “target” distribution, i.e., the optimal mechanism aims to produce this distribution as closely as possible, subject to the utility constraint. In particular, the resulting optimal mechanism (derived from (12)), for any input, distributes the outputs according to $Q_Y$ while conditioning the output to be within a ball of radius $D$ about the input. The optimization in (15) ensures that all inputs are uniformly masked while (11) provides average guarantees.

The following theorem characterizes the optimal tradeoff

$$\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}^\text{PUT}}$$

for maximal $\alpha$-leakage. Recall that for $\alpha = 1$, $\mathcal{L}_{\alpha}^\text{PUT}$ equals $\mathcal{L}_f$ with $f(t) = t \log t$. For $\alpha > 1$, from the one-to-one relationship between $\mathcal{L}_{\alpha}^\text{PUT}$ and $\mathcal{L}_{\alpha}^\text{max}$ in (8), we know that finding $\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}^\text{PUT}}$ is equivalent to finding the optimal tradeoff $\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}^\text{max}}$ in (13) for $\mathcal{L}_f = \mathcal{L}_{\alpha f}^\text{max}$. Due to space constraints, we omit details.

Theorem 4. For maximal $\alpha$-leakage $\mathcal{L}_{\alpha}^\text{PUT}$, a distortion function $d(\cdot, \cdot)$ and $B_D(x)$ in (9), the optimal PUT is given by

$$\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}^\text{PUT}}(D) = \inf_{P_{Y|X} : d(X,Y) \leq D} \mathcal{L}_{\alpha}^\text{max}(X \rightarrow Y)$$

for $q^*$ defined in (15). Moreover, an optimal mechanism is given by (12), where for $\alpha = 1$, $Q_Y^*$ achieves the infimum in (17a); and for $\alpha > 1$, $Q_Y^*$ achieves the supremum in (15).

Remark 1. Note that subject to a hard distortion constraint, the optimal privacy mechanism is always given by (12). In particular, for maximal $\alpha$-leakage, the optimal mechanism as well as the optimal PUT are identical for all $\alpha > 1$. 

IV. EXAM PLE: HARD DISTOR TION FOR BINARY TYPES

When considering dataset disclosure under privacy constraints, a reasonable goal is to design privacy mechanisms that are uniformly masked while (11) provides average guarantees. Theorem 4.

For maximal $\alpha$-leakage, the optimal mechanism as well as the optimal PUT are identical for all $\alpha > 1$.

Remark 1. Note that subject to a hard distortion constraint, the optimal privacy mechanism is always given by (12). In particular, for maximal $\alpha$-leakage, the optimal mechanism as well as the optimal PUT are identical for all $\alpha > 1$.

IV. EXAMPLE: HARD DISTOR TION FOR BINARY TYPES

When considering dataset disclosure under privacy constraints, a reasonable goal is to design privacy mechanisms that preserve the statistics of the original dataset while preventing inference of each individual record (e.g., a sample or a row of the dataset). Since the type (empirical distribution) of a dataset captures its statistics, we quantify distortion as the distance between the type of the original and disclosed datasets. We use maximal $\alpha$-leakage to capture the gain of an adversary (with access to the disclosed dataset) in inferring any function of the original dataset.

Let $X^n$ be a random dataset with $n$ entries and $Y^n$ be the corresponding disclosed dataset generated by a privacy mechanism $P_{Y^n|X^n}$. Entries of both $X^n$ and $Y^n$ are from the same alphabet $\mathcal{X}$. For a pair of input and output datasets $(x^n, y^n)$ of $P_{Y^n|X^n}$, let $P_{x^n}$ and $P_{y^n}$ indicate the types, respectively. We define the distortion function as

$$d(x^n, y^n) = \max_{x \in \mathcal{X}} |P_{x^n}(x) - P_{y^n}(x)|,$$

and therefore, obtain $\text{PUT}_{\text{HD, } \mathcal{L}_{\max}^\text{PUT}}$ as in (16) but with datasets $X^n, Y^n$ in place of single letters $X, Y$. Let the fraction $\frac{m}{n}$ $(m \in [0, n])$ be the upper bound $D$ in (16), where $\{0, n\}$ indicates the set of integers from $0$ to $n$.

We concentrate on binary datasets and let $\mathcal{X} = \{0, 1\}$. Note that for binary datasets, we can simply write $d(x^n, y^n) = |P_{x^n}(0) - P_{y^n}(0)|$. For a $n$-length binary dataset, the number of types is $n + 1$. Therefore, all input and output datasets can be categorized into $n + 1$ type classes defined as

$$T(i) = \{x^n : n P_{x^n}(0) = i\} = \{y^n : n P_{y^n}(i) = i\}, i \in [0, n].$$

Theorem 5. Given an arbitrary pair of $(n, m) \in [1, \infty) \times [0, n]$, the minimal leakage for $\alpha > 1$ is

$$\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}^\text{PUT}}(\frac{m}{n}) = \log \left[\frac{n+1}{m+1}\right].$$

An optimal privacy mechanism maps all input datasets in a type class to a unique output dataset which is feasible and belongs to a type class in the set $T^*$ given by

$$T^* = \{T(j) : j = l + (2m + 1)k, k \in [0, \left\lceil \frac{n+1}{m+1} \right\rceil - 1]\},$$

where $l = m$ if $m + \left(\left\lceil \frac{n+1}{m+1} \right\rceil - 1\right)(2m + 1) \leq n$, and otherwise, $l = n - \left(\left\lceil \frac{n+1}{m+1} \right\rceil - 1\right)(2m + 1)$.

A detailed proof is in Appendix C. Let $(n, m) = (8, 2)$ such that from Thm. 5, we have $T^* = \{T(2), T(7)\}$. Fig. 1 shows the optimal mechanism, which maps all input datasets in $\{T(i) : i \in [0, 4]\}$ (resp. $\{T(i) : i \in [5, 8]\}$) to a unique output dataset in $T(2)$ (resp. $T(7)$) with probability 1.
V. CONCLUSION

We have explored PUTs in the context of hard distortion utility constraints. This utility constraint has the advantage that it allows the data curator to make specific, deterministic guarantees on the quality of the published dataset. Focusing on maximal $\alpha$-leakage and its $f$-divergence-based variants, under a hard distortion constraint, we have shown that: (i) for all $\alpha > 1$, we obtain the same optimal privacy mechanism and optimal PUT, which are independent of the distribution of the original data (or datasets); (ii) for $\alpha = 1$, the optimal mechanism differs and depends on the distribution of the original data (or data sets). In other words, for this distortion measure, the tunable privacy measure behaves as either MI or MaxL. Possible future directions include verifying whether the observed behavior holds for average distortion constraints and more complicated data models.

APPENDIX

A. Proof of Theorem 2

The feasible ball $B_D(x)$ around $x$ is defined in (9). For the distribution dependent PUT in (10), we have

$$\text{PUT}_{\text{HD}^c}(D) = \inf_{p_Y^{f}|X,d(X,Y)\leq D} \sup_{Q_Y} \inf_Q D_f(P_Y^{f}\|P_X \times Q_Y)$$

$$= \inf_{q_Y^{f}} \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int dP_X D_f(P_Y^{f}|X=x\|Q_Y)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$+ Q_Y(B_D(x)) \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg)$$

$$\geq \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= f(0) + \inf_{q_Y^{f}} \int dP_X Y \bigg( \text{Q}_Y(B_D(x)) f \bigg( \frac{1}{Q_Y(B_D(x))} - f(0) \bigg) \bigg)$$

where

- (22) follows from the fact that $D_f(P_Y^{f}|X,d(X,Y)\leq D)$ is convex in $(P_Y^{f}|X,d(X,Y)\leq D)$ for fixed $P_X$.
- (24) is from the Jensen’s inequality and the equality holds if and only if there is a mechanism $P_Y^{f}|X$ satisfying

$$\frac{dP_Y^{f}|X(y|x)}{dQ_Y(y)} = 1(y \in B_D(x)) \quad \frac{Q_Y(B_D(x))}{Q_Y(B_D(x))} - f(0)$$

B. Proof of Theorem 3

The feasible ball $B_D(x)$ around $x$ is defined in (9). For the distribution independent PUT in (13), we have

$$\text{PUT}_{\text{HD}^c}(D) = \inf_{p_Y^{f}|X,d(X,Y)\leq D} \sup_{Q_Y} \inf_Q D_f(P_Y^{f}\|P_X \times Q_Y)$$

$$= \inf_{q_Y^{f}} \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int dP_X D_f(P_Y^{f}|X=x\|Q_Y)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

$$= \inf_{q_Y^{f}} \int dP_X Y \bigg( \inf_{P_Y^{f}|X,d(X,Y)\leq D} \int_{Y \in B_D(x)} Q_Y f \bigg( \frac{dP_Y^{f}|X=x}{dQ_Y} \bigg) \bigg)$$

Due to the convexity of $f$, we have $f(q^{-1}) - f(0) \leq f'(q^{-1})(q^{-1} - f(0))$, from which, the derivative $g'(q) = f(q^{-1}) - q^{-1}f(q^{-1}) - f(0) \leq 0$. Therefore, the function $g$ in (35) is non-increasing, such that (34) is be simplified as $g(q^*)$, where $q^*$ is given by

$$q^* \triangleq \sup_{q_Y^{f}} \inf_{X} \text{Q}_Y(B_D(x))$$

□
C. Proof of Theorem 5

Define the feasible ball around an input dataset \( x^n \) as
\[
B_D(x^n) \triangleq \left\{ y^n : |P_{x^n}(0) - P_{y^n}(0)| \leq \frac{m}{n} \right\}. \tag{37}
\]
From Thm. 4, to find an optimal mechanism \( P_{Y^n | X^n} \), we need to find an output distribution \( Q_{Y^n} \), which optimizes (15) with \( x^n \) and \( y^n \) in place of \( x,y \).

Note that for the hard distortion \( |P_{x^n}(0) - P_{y^n}(0)| \leq \frac{m}{n} \), all datasets in a type class share the same group of feasible output datasets, and this feasible group can be represented by the type classes. Therefore, for any \( x^n \in T(i) \) (\( i \in [0, n] \)), we rewrite \( B_D(x^n) \) as
\[
B_D(x^n) = B_D(T(i)) \triangleq \{ T(j) : |i - j| \leq m, j \in [0, n] \}. \tag{40}
\]
We define an distribution \( Q_T \) of type classes for outputs as
\[
Q_T(j) = \sum_{y^n \in T(j)} Q_{Y^n}(y^n), \text{ for } j \in [1, n], \tag{38}
\]
such that
\[
q^* = \sup_{Q_T} \inf_{i \in [0, n]} Q_T(B_D(T(i))). \tag{39}
\]
The optimal distribution \( Q_T \) is determined by both upper and lower bounding \( q^* \) in (39). The upper bound is determined by restricting the optimization in (39) to a judicious choice of a small set of input types. The lower bound is a constructive scheme. Let \( l = m + (2m + 1) \left\lceil \frac{n+1}{2m+1} \right\rceil - n \). We define an index set \( \mathcal{I}_T \subset [0, n] \) for types as
\[
\mathcal{I}_T \triangleq \begin{cases} m + (2m + 1)k : k \in [0, \left\lceil \frac{n+1}{2m+1} \right\rceil - 1] & l \leq 0 \\ l + (2m + 1)k : k \in [0, \left\lceil \frac{n+1}{2m+1} \right\rceil - 1] & l > 0. \end{cases} \tag{40}
\]
From the expression of \( \mathcal{I}_T \) in (40), we observe that: (i) for \( l \leq 0 \) (resp. \( l > 0 \)), the first (resp. last) element is \( m \) (resp. \( n \)); (ii) for \( l \leq 0 \) (resp. \( l > 0 \)), the last (resp. first) element is no less (resp. less) than \( n - m \) (resp. \( m + 1 \)); (iii) for both cases, the difference between adjacent elements is \( 2m + 1 \).

Therefore, it is not difficult to see that feasible balls of input type classes indexed by \( I_T \) are a partition of the set of all type classes, i.e.,
\[
B_D(T(i_1)) \cap B_D(T(i_2)) = \emptyset \quad i_1, i_2 \in \mathcal{I}_T, \tag{41a}
\]
\[
\{ T(j) : j \in [0, n] \} = \bigcup_{i \in \mathcal{I}_T} B_D(T(i)). \tag{41b}
\]
Therefore, the problem in (39) is upper bounded by
\[
q^* \leq \sup_{Q_T} \inf_{i \in \mathcal{I}_T} Q_T(B_D(T(i))) \tag{42}
\]
\[
\leq \sup_{Q_T} \frac{1}{|\mathcal{I}_T|} \sum_{i \in \mathcal{I}_T} Q_T(B_D(T(i))) \tag{43}
\]
\[
= \sup_{Q_T} \left( \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right) \sum_{j \in [1, n]} Q_T(T(j)) \tag{44}
\]
\[
= \left( \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right). \tag{45}
\]
Construct an distribution \( Q'_{T} \) as
\[
Q'_T(j) = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right)^{-1} \text{ for } j \in I_T, \tag{46}
\]

and otherwise, \( Q'_T(j) = 0 \). By (41) for each \( i \in [0, n] \), there is a unique \( k \) satisfying \( |i - I_T(k)| \leq m \), where \( I_T(k) \) is the \( k \)-th element of \( I_T \). Therefore, we lower bound (39) by
\[
q^* \geq \inf_{i \in [0, n]} Q_T(B_D(T(i))) \tag{47}
\]
\[
= \inf_{i \in [0, n]} Q_T \left( \bigcup_{|i-j| \leq m} T(j) \right) \tag{48}
\]
\[
= \inf_{k} Q_T'(T(I_T(k))) = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right)^{-1}. \tag{49}
\]
Thus, \( q^* = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right)^{-1} \) and the \( Q'_T \) in (46) is optimal.

From (38) and the \( Q'_T \), we derive an optimal privacy mechanism, which assigns the same non-zero probability to only one dataset of each type classes indexed by \( I_T \), i.e., \( Q'_Y(y^n) \) for one \( y^n \in T(j) \) for each \( j \in I_T \). Therefore, from (12) we have the corresponding optimal privacy mechanism, which maps all input datasets in one input type class to one feasible output dataset with probability 1.

\[\square\]

References

[1] C. Dwork, “Differential privacy: A survey of results,” in Theory and Applications of Models of Computation: Lecture Notes in Computer Science. New York:Springer, Apr. 2008.

[2] F. du Pin Calmon and N. Fawaz, “Privacy against statistical inference,” in 2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2012.

[3] L. Sankar, S. R. Rajagopalan, and H. V. Poor, “Utility-privacy trade-offs in databases: An information-theoretic approach,” IEEE Trans. on Inform. For. and Sec., vol. 8, no. 6, pp. 838–852, 2013.

[4] B. Rassouli and D. Gündüz, “Optimal utility-privacy trade-off with the total variation distance as the privacy measure,” in arXiv:1801.02505v1 [cs.IT], 2018.

[5] I. Issa, S. Kamath, and A. B. Wagner, “An operational measure of information leakage,” in 2016 Annual Conference on Information Science and Systems (CISS), 2016.

[6] J. C. Duchi, M. I. Jordan, and M. J. Wainwright, “Local privacy and statistical minimax rates,” in 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, 2013.

[7] Q. Geng, P. Kairouz, S. Oh, and F. Viswanath, “The staircase mechanism in differential privacy,” IEEE Journal of Selected Topics in Signal Processing, vol. 9, no. 7, pp. 1176–1184, 2015.

[8] J. Liu, L. Sankar, V. Y. F. Tan, and F. P. Calmon, “Hypothesis testing under mutual information privacy constraints in the high privacy regime,” IEEE Transactions on Information Forensics and Security, vol. 13, no. 4, pp. 1058–1071, 2018.

[9] H. Wang, M. Diaz, F. P. Calmon, and L. Sankar, “The utility cost of robust privacy guarantees,” in arXiv:1801.05926v1, 2018.

[10] A. Rényi, “On measures of entropy and information,” in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. The Regents of the University of California, 1961, pp. 547–561.

[11] A. Rényi, “α–mutual information,” in 2015 Information Theory and Applications Workshop (ITA), 2015.

[12] F. Liese and I. Vajda, “On divergences and informations in statistics and information theory,” IEEE Trans. on Inform. For. and Sec., vol. 8, no. 6, pp. 838–852, 2013.

[13] S. Verdú, “On divergences and informations in statistics and information theory,” IEEE Transactions on Information Theory, vol. 52, no. 10, pp. 4394–4412, Oct 2006.

Footnotes:

2From (40), \( I_T(k) \) is either \( m + (2m + 1)k \) or \( l + (2m + 1)k \).