Lattice Approximation of Quantum Statistical Traces at a Complex Temperature

J. Lukkarinen

Helsinki Institute of Physics, P.O.Box 9, 00014 University of Helsinki, Finland

We prove that the simple condition on the potential $V$, $\int \exp(-tV) < \infty$ for all $t > 0$, is sufficient for the lattice approximation of $\text{Tr}(\hat{A}e^{-\beta\hat{H}})$ with $\text{Re}\beta > 0$ to work for all bounded functions $A$ and a large class of potentials. As a by-product we obtain an explicit bound for the real-temperature lattice kernels.

I. INTRODUCTION

Ever since the introduction of the path-integrals by R. Feynman, the path-integral formulas with a complex “temperature” have presented an elusive mathematical problem—for instance, the Wiener measure, which is so useful for positive temperature statistics, cannot be extended to the complex case. In fact, since real time processes and real temperature statistics can be identified with each other via analytic continuation, the present numerical non-perturbative methods in quantum field theories rely mostly on computations with lattice approximations to the imaginary time, i.e. real temperature, path-space measures.

There have already been proposals for numerical evaluation of the complex path-integrals, but only now can we establish rigorously the assumed relations between the complex lattice integrals and the continuum traces. Also, the result of Theorem III.5 has been used as an assumption in the derivation of a lattice approximation to the quantum microcanonical ensemble and the theorem now yields a wide range of potentials for which this approximation is valid.

For an introduction to the mathematics of operators on a Hilbert space see e.g. the second part of the comprehensive series by Reed and Simon. The path-space measures and their applications are discussed in detail in another book by Simon and the proofs of the statements made in the following section can be found from Section 6 of that book.

II. NOTATIONS AND DEFINITIONS

We will now consider quantum mechanics on the separable Hilbert space $L^2(\mathbb{R}^n)$; in the following, $n$ will always denote the number of dimensions of the parameter space and we will use boldface letters for vectors of $\mathbb{R}^n$. For simplicity, we have also adopted the following non-standard notations: $D\mathbf{x}$ denotes the $n$-dimensional Lebesgue measure with respect to the variable $\mathbf{x}$ and $D^N\mathbf{x}$ stands for the product measure $\prod_{k=1}^N D\mathbf{x}_k$. Operators will be

\begin{flushleft}
\text{E-mail address: jani.lukkarinen@helsinki.fi}
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distinguished by the “hat”—especially the multiplication operator defined by a measurable function $F$ will be denoted by $\hat{F}$.

The free Hamiltonian is defined in the standard way,

$$\hat{H}_0 := \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2,$$

with the usual domain for this operator. To simplify the following discussion we will from now on use the natural units $\hbar = 1$ and, in addition, consider only the case $m = 1$. The results for the situation $m \neq 1$ are easily reproduced from the following formulas by simply replacing $t$ by $t/m$ and the potential $V$ by $mV$.

It is well-known that $\hat{H}_0$ generates an analytic semigroup on the right half-plane, where $e^{-\beta \hat{H}_0}$ are bounded integral operators for $\text{Re} \beta > 0$ and they are given by the integral kernels

$$P(a, b; \beta) := \frac{1}{(2\pi \beta)^{n/2}} \exp\left(-\frac{1}{2\beta} |a - b|^2\right).$$

It is also evident that these integral kernels satisfy the semigroup property exactly, i.e. that for all $a, b \in \mathbb{R}^n$ and $\text{Re} \beta_1, \text{Re} \beta_2 > 0$,

$$\int Dc \ P(a, c; \beta_1)P(c, b; \beta_2) = P(a, b; \beta_1 + \beta_2).$$

The potential $V$ is assumed to be locally in $L^1(\mathbb{R}^n)$ and positive, but the results easily generalize for all real $L^1_{\text{loc}}$ potentials bounded from below (just replace $V$ by $V + \text{ess sup}(-V)$). The Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ can now be defined as a sum of quadratic forms, when it is self-adjoint and bounded from below and, therefore, $e^{-\beta \hat{H}}$ (as well as $e^{-\beta \hat{V}}$) is an analytic semigroup on the right half-plane. In addition, we shall also require that

$$\int Da e^{-tV(a)} < \infty, \text{ for all } t > 0, \quad (1)$$

so that $e^{-t\hat{H}}$ is trace-class for $t > 0$ by the Golden, Thompson, Symanzik -inequality.

Under these assumptions $e^{-t\hat{H}}$ is an integral operator for $t > 0$ and it has an integral kernel $K_\infty$, which can be obtained with aid of the conditional Wiener measure $\mu$: for all $a, b \in \mathbb{R}^n$ and for all $t > 0$ define

$$K_\infty(a, b; t) = \int d\mu_{a,b;\omega}(\omega) e^{-\int_0^t ds V(\omega(s))}.$$

In addition to (1) we shall now require the potential $V$ to be such that the Wiener integrals can almost everywhere be obtained from the lattice kernels

$$K_N(a, b; t) := \int D^{N-1} x \left(\frac{N}{2\pi t}\right)^{N/2} \exp \left[-\frac{N}{2t} \sum_{k=1}^N |x_{k-1} - x_k|^2 - \frac{t}{N} \sum_{k=1}^N V(x_k)\right]_{x_0=a, x_N=b},$$

i.e. that

$$K_\infty(a, b; t) = \lim_{N \to \infty} K_N(a, b; t), \text{ for all } t > 0 \text{ and for almost all } a, b \in \mathbb{R}^n. \quad (2)$$
This requirement is clearly satisfied for all $a$ and $b$ if $V$ is continuous, since then $\int ds V(\omega(s))$ can be approximated by suitable Riemann sums leading to (2) by using the definition of the Wiener measure.

We shall also need the operator

$$\hat{T}(\beta) := e^{-\beta H_0}e^{-\beta \tilde{V}},$$

which is clearly a bounded integral operator for $\Re \beta > 0$, since then $\|\hat{T}(\beta)\| \leq \|e^{-\beta H_0}\|\|e^{-\beta \tilde{V}}\| \leq 1$. Notice that $K_N(\cdot, \cdot; \beta)$ is nothing but the integral kernel of the operator $\hat{T}(\frac{\beta}{N})^N$.

### III. TWO LATTICE THEOREMS

In this section we will prove two theorems concerning the convergence of traces of lattice operators. The first theorem will show that also certain unbounded multiplication operators are measurable using the (real-temperature) Gibbs statistics, if the potential increases sufficiently fast at infinity. This will be proven by using the explicit bounds given by the following lemma and its two corollaries. The rather technical proofs of the lemma and the corollaries are included in Section IV—the lemma itself is an extension of a result derived by Symanzik.

**Lemma III.1** For all $a$, $b$ in $\mathbb{R}^n$, for all $t > 0$ and for all positive integers $N$,

$$K_N(a, b; t) \leq C_n t^\frac{n-1}{2} \int Dc P(a, c; t)e^{-tV(c)}P(c, b; t) \left( \frac{1}{|a-c|^{n-1}} + \frac{1}{|c-b|^{n-1}} \right) + \frac{1}{N} e^{-tV(b)}P(a, b; t),$$

where $C_n$ depends only on the dimension $n$.

Taking the limit $N \to \infty$ in the previous lemma shows that (3) is valid for almost all $a$, $b$ and for all $t > 0$ also when $N = \infty$.

**Corollary III.2** Assume that in addition to (3) the potential satisfies

$$\int Da |a|^r e^{-tV(a)} < \infty,$$

for some $r \geq 0$ and for all $t > 0$. Then there exists functions $F_r$ and $G_r$, continuous on $(0, \infty)$, such that for all $t > 0$ and for all $N < \infty$,

$$\int Da |a|^r K_N(a, a; t) \leq F_r(t)$$

and for all $N$, including $\infty$,

$$\int DaDb |a|^r K_N(a, b; t) \leq G_r(t).$$
If the above assumption on the potential is true for 0 and \( r \) then it is clearly true for all values \( s \) in \((0, r)\) by the inequality \(|a|^s \leq 1 + |a|^r\). If the potential has no singularity in a neighborhood of the origin, it is enough to find one \( r \geq 0 \) for the statement to be true for all \( s \) with \( 0 \leq s \leq r \).

**Corollary III.3** For all \( N \), including \( \infty \), and for all \( t > 0 \), \( \text{ess sup}_{a, b \in \mathbb{R}^n} K_N(a, b; t) \leq C'_n t^{-\frac{n}{2}} \).

The convergence properties of the lattice traces are easily derived from these results; we have only to be careful that the limit of the traces of the integral kernels does give the trace of the continuum integral operator. This leads to the appearance of the factor of two in Eq. (4), which will be discussed more after the proof of the theorem.

**Theorem III.4** Let \( A \) be measurable function bounded by a degree \( r \geq 0 \) polynomial and assume that the potential additionally satisfies

\[ \int Da |a|^{2r} e^{-tV(a)} < \infty, \text{ for all } t > 0. \]

Then for all \( t > 0 \) and \( N \geq 2 \) the operators \( \hat{A} e^{-t\hat{H}} \) and \( \hat{A}^{t/N}(t/N)^N \) are both Hilbert-Schmidt and trace-class and

\[ \text{Tr}(\hat{A} e^{-t\hat{H}}) = \lim_{k \to \infty} \text{Tr}(\hat{A}^{t/(2k)} (2k)^2), \]  
\[ \text{Tr}(\hat{A}^{t/N}(t/N)^N) = \int Da A(a) K_N(a, a; t). \]  

**Proof:** The requirement for \( A \) is equivalent to finding a \( C \geq 0 \) such that \(|A(a)| \leq C(1 + |a|^r)\) for all \( a \). It follows now from Corollaries III.2 and III.3 that for \( 1 \leq N \leq \infty \),

\[ \int Da Db |A(a) K_N(a, b; t)|^2 < \infty. \]

The operators \( \hat{A} e^{-t\hat{H}} \) and \( \hat{A}^{t/N}(t/N)^N \) are given by the integral kernels \( A(a) K_N(a, b; t) \) and, therefore, they are Hilbert-Schmidt.

Since \( \hat{A} e^{-t\hat{H}} \) is a product of the two Hilbert-Schmidt operators \( \hat{A} e^{-t\frac{t}{2}\hat{H}} \) and \( e^{-t\frac{t}{2}\hat{H}} \), it is trace-class. Moreover, the trace can be expressed in terms of the integral kernels of these Hilbert-Schmidt operators:

\[ \text{Tr}(\hat{A} e^{-t\hat{H}}) = \int Da Db A(a) K_\infty(a, b; \frac{t}{2}) K_\infty(b, a; \frac{t}{2}). \]

Using Lemma III.1 and Corollary III.3 to give an upper bound for the Lebesque dominated convergence theorem shows now that

\[ \int Da Db A(a) K_\infty(a, b; \frac{t}{2}) K_\infty(b, a; \frac{t}{2}) = \lim_{N \to \infty} \int Da Db A(a) K_N(a, b; \frac{t}{2}) K_N(b, a; \frac{t}{2}) = \lim_{N \to \infty} \int Da A(a) K_{2N}(a, a; t), \]
where the second equality follows from Fubini’s theorem and a careful inspection of the definition of $K_{2N}$. Therefore,

$$\text{Tr}\left(\hat{A}e^{-t\hat{H}}\right) = \lim_{N \to \infty} \int DaA(a)K_{2N}(a,a;t).$$  (6)

It has already been proven that both $\hat{A}\hat{T}(t)$ and $\hat{T}(t)^{-1}$ are Hilbert-Schmidt for all $t > 0$ and thus $\hat{A}\hat{T}(t/N)^N$ is trace-class and its trace can be expressed in terms of the kernels of these operators. It is then a matter of inspection to conclude

$$\text{Tr}\left(\hat{A}\hat{T}(t/N)^N\right) = \int DaA(a)K_N(a,a;t)$$  (7)

and the theorem follows from equations (6) and (7).

If $\lim_N K_N(a,a;t)$ exists for almost all $a$, then an application of the dominated convergence theorem in (6) shows that $\text{Tr}\left(\hat{A}e^{-t\hat{H}}\right) = \lim_N \int DaA(a)K_N(a,a;t)$. Therefore, also odd lattice sizes can be included in the limit, for instance, if $V$ is continuous (then $K_{\infty}$ is continuous and thus finite on the diagonal).

In the second theorem we will work with temperatures having an imaginary part. The following proof relies on a simple property of the Hilbert-Schmidt operators given as Lemma [V.3] in Section [V]. To make the theorem easier to use, we will now state all the assumptions made on the potential explicitly.

**Theorem III.5** Assume that $A$ is an essentially bounded measurable function and $V \in L^1_{\text{loc}}$ is a potential bounded from below for which (3) holds and

$$\int Da e^{-tV(a)} < \infty, \text{ for all } t > 0.$$  

Then for all complex $\beta$ with $\text{Re } \beta > 0$, $\hat{A}e^{-\beta\hat{H}}$ is trace-class and

$$\text{Tr}\left(\hat{A}e^{-\beta\hat{H}}\right) = \lim_{N \to \infty} \int DaA(a)K_{2N}(a,a;\beta).$$

**Proof:** Let $f_N(\beta) = \int DaA(a)K_{2N}(a,a;\beta)$ and $g(\beta) = \text{Tr}\left(\hat{A}e^{-\beta\hat{H}}\right)$. Both are then well-defined on the right half-plane: $f_N$ is an $2N$-fold integral over an absolutely integrable function and, by the Schwarz inequality, self-adjointness of $\hat{H}$ and Theorem [III.4],

$$|g(\beta)|^2 \leq \text{Tr}\left(|\hat{A}|^2 e^{-\text{Re } \beta\hat{H}}\right) \text{Tr}\left(e^{-\text{Re } \beta\hat{H}}\right) < \infty.$$  

In addition, $g$ and all $f_N$ are analytic on the right half-plane, as can be seen e.g. by a standard application of Morera’s theorem—not that

$$g(\beta) = \sum_n \langle \psi_n | \hat{A} \psi_n \rangle e^{-\beta E_n},$$

where $\{E_n\}$ are the eigenvalues of $\hat{H}$ and $\{\psi_n\}$ is a corresponding orthonormal basis.
Since Theorem [III.4] states that \( \lim_{N \to \infty} f_N(\beta) = g(\beta) \) for positive \( \beta \), the theorem will follow from the Vitali convergence theorem if we can only prove that the sequence \( (f_N) \) is uniformly bounded on every compact subset of the half-plane.

An argument similar to one in the proof of Theorem [III.4] shows that \( f_N(\beta) = \text{Tr} \left( \hat{A} \left( \frac{\beta}{2N} \right)^{2N} \right) \). Therefore, Lemma [IV.1] establishes the bound \( |f_N(\beta)| \leq \| \hat{A} \| \text{Tr} (\hat{T}^* \hat{T})^N \),

where \( \hat{T} = \hat{T} \left( \frac{\beta}{2N} \right) \).

However, as \( \hat{T}^* \hat{T} = e^{-\frac{\partial^2}{2N}} e^{-\frac{\partial^2}{2N}} e^{-\frac{\partial^2}{2N}} \),

\[
\text{Tr} (\hat{T}^* \hat{T})^N = \text{Tr} \left( \hat{T} (\hat{\beta}/N)^N \right) \leq F_0 (\hat{\beta}/N)^N \]

by Corollary [III.2] and Theorem [III.4]. If \( S \) is a compact set in the right half-plane, it follows that \( F_0 (\hat{\beta}/N) \) is bounded since \( F_0 \) was continuous. This proves that \( (f_N) \) is uniformly bounded on every compact subset of the right half-plane and hence completes the proof. □

If \( V \) is continuous, then the limit can be taken by using all values of \( N \): Define \( h_N(\beta) = \int \mathcal{D}a A(a) K_N(a, a; \beta) \). By the remark after Theorem [III.4], \( \lim_N h_N = g \) on the positive real axis. Also,

\[
|h_{2N+1}(\beta)| = \left| \text{Tr} \left( \hat{A} \left( \frac{\beta}{2N+1} \right)^{2N+1} \right) \right| \leq \| \hat{A} \| \| \hat{T} \| \text{Tr} (\hat{T}^* \hat{T})^N \leq \| \hat{A} \| F_0 (\hat{\beta}/N)^{2N+1},
\]

and thus the sequence \( (h_N) \) is uniformly bounded on the right half-plane.

**IV. PROOFS OF THE LEMMAS**

**Proof of Lemma [III.1]**: By the relation between geometric and arithmetic means, \( \exp \left( -\frac{1}{N} \sum_{k=1}^{N} V(x_k) \right) \leq \frac{1}{N} \sum_{k=1}^{N} \exp \left( -tV(x_k) \right) \). Using this and the semigroup property of the integral kernels \( P \), we arrive at the inequality

\[
K_N(a, b; t) \leq \frac{1}{N} e^{-tV(b)} P(a, b; t) + \int \mathcal{D}c e^{-tV(c)} \frac{1}{N} \sum_{k=1}^{N-1} P(a, c; k \frac{t}{N}) P(c, b; (N-k) \frac{t}{N}).
\]

The terms in the last sum are explicitly

\[
\left( \frac{1}{(2\pi t)} \frac{N}{\sqrt{k(N-k)}} \right)^n \exp \left[ -\frac{1}{2t} \left( \frac{N}{k} |a - c|^2 + \frac{N}{N-k} |c - b|^2 \right) \right]
\]

\[
= P(a, c; t) P(c, b; t) \frac{N}{\sqrt{k(N-k)}} \left( \frac{N-k}{k} \right)^{n-1} \exp \left[ -\frac{k}{N-k} \frac{|c-b|^2}{2t} \right]
\]

\[
\times \left( \frac{N}{N-k} \right)^{n-1} \exp \left[ -\frac{k}{N-k} \frac{|c-b|^2}{2t} \right] = P(a, c; t) P(c, b; t) \frac{N}{\sqrt{k(N-k)}} \left( \frac{k}{N-k} \right)^{n-1} \exp \left[ -\frac{k}{N-k} \frac{|c-b|^2}{2t} \right]
\]

\[
\times \left( \frac{N}{k} \right)^{n-1} \exp \left[ -\frac{N-k}{k} \frac{|a-c|^2}{2t} \right].
\]
Assume now that \( n > 1 \) and consider first the case \( 1 \leq k \leq \frac{N}{2} \). Since, \( t^* e^{-tr} \leq \left( \frac{t}{te} \right)^{ \frac{n-1}{2} } \) for all positive \( r, s \) and \( t \), the expression (3) has an upper bound

\[
P(a, c; t)P(c, b; t) \frac{N}{\sqrt{k(N-k)}} 2^{n-1} \left( \frac{(n-1)t}{e|a-c|^2} \right)^{\frac{n-1}{2}}.
\]

On the other hand, if \( \frac{N}{2} \leq k \leq N-1 \), then the same reasoning can be applied to (3) leading to the same bound apart from the change of \(|a-c|\) to \(|c-b|\). Since in both cases these upper bounds are positive, the terms under inspection are for all \( 1 \leq k \leq N-1 \) bounded by

\[
P(a, c; t)P(c, b; t) \left( \frac{4(n-1)}{e} \right)^{\frac{n-1}{2}} t^{n-1} \left( \frac{1}{a-c|^{n-1}} + \frac{1}{|c-b|^{n-1}} \right) \frac{N}{\sqrt{k(N-k)}}.
\]

Now \( \frac{1}{N} \sum_{k=1}^{N-1} [\frac{k}{N}(1-\frac{k}{N})]^{-\frac{1}{2}} \) is a lower Riemann sum for the integral \( \int_0^1 ds\frac{1}{\sqrt{s(1-s)}} = \pi \) and, therefore for all \( N \),

\[
\frac{1}{N} \sum_{k=1}^{N-1} \frac{N}{\sqrt{k(N-k)}} \leq \pi
\]

and the Lemma follows if we choose \( C_n = \pi 2^{n-1}(n-1)^{-\frac{n-1}{2}} e^{-\frac{n-1}{2}} \).

The case \( n = 1 \) is even simpler to prove, as then both of the exponentials in (3) can be majorized by one and the Lemma, with the choice \( C_1 = \pi \), follows from (10).

**Proofs of Corollaries III.2 and III.3.** Define the following two functions on \((0, \infty)\)

\[
I_s(t) := \int D(a)|a|^s e^{-tV(a)}, \quad J_s(t) := \int D(a)|a|^s P(0, a; t),
\]

where \( s \) is a real parameter. \( J_s \) is finite for \( s > -n \) and, in fact, it is then continuous, since a change of variables and relying on \( n \)-dimensional spherical coordinates shows that

\[
J_s(t) = (2t)^{s/2} \frac{\Gamma \left( \frac{n+s}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}.
\]

Under the assumptions of the theorem, also \( I_0 \) and \( I_r \) are everywhere finite. They are then also continuous, since for every converging sequence \((t_k)\) in \((0, \infty)\) dominated convergence can be used to prove \( \lim_{k \to \infty} I_s(t_k) = I_s(\lim t_k) \).

The rest is just a matter of bookkeeping when integrating both sides of the inequality (3). Using Jensen’s inequality for \( r \geq 1 \) and the obvious relation \((1 + |a|)^r - |a|^r \leq 1 \) for \( 0 \leq r < 1 \) it is easy to prove that for \( r \geq 0 \) and \( a, b \in \mathbb{R}^n \)

\[
|a|^r \leq 2^r(|a-b|^r + |b|^r).
\]

Applying this result, equation (11) and a little algebra shows that

\[
\int D(a)|a|^r K_N(a, a; t) \leq C^{(1)} t^{-\frac{r}{2}} I_r(t) + C^{(2)} t^{-\frac{r+1}{2}} I_0(t),
\]

\[
\int D(a)D(b)|a|^r K_N(a, b; t) \leq C^{(3)} I_r(t) + C^{(4)} t^\frac{r}{2} I_0(t),
\]
where the constants \( C^{(k)} \) depend only on \( n \) and \( r \). The functions on the right hand sides are continuous and can therefore be chosen as \( F_r \) and \( G_r \).

Since \( e^{-tV(c)} P(c, b; t) \leq (2\pi t)^{-n/2} \), Corollary [III.3] is an immediate consequence of (11) and Lemma [II.4]. \( \square \)

**Lemma IV.1** If \( \widehat{A} \) is bounded, \( \widehat{T} \) is Hilbert-Schmidt and \( N \geq 1 \), then

\[
\left| \text{Tr} \left( \widehat{A} \widehat{T}^{2N} \right) \right| \leq \| \widehat{A} \| \text{Tr} \left( \widehat{T}^* \widehat{T} \right)^N.
\]

**Proof:** By the Schwarz inequality,

\[
\left| \text{Tr} \left( \widehat{A} \widehat{T}^{2N} \right) \right|^2 \leq \text{Tr} \left[ (\widehat{A} \widehat{T}^N)^* (\widehat{A} \widehat{T}^N) \right] \text{Tr} \left[ (\widehat{T}^N)^* \widehat{T}^N \right]
\]

and since \( \text{Tr} \left[ (\widehat{A} \widehat{T}^N)^* (\widehat{A} \widehat{T}^N) \right] = \text{Tr} \left[ |\widehat{A}|^2 \widehat{T}^N (\widehat{T}^N)^* \right] \leq \| \widehat{A} \|^2 \text{Tr} \left[ (\widehat{T}^N)^* \widehat{T}^N \right], \) we get

\[
\left| \text{Tr} \left( \widehat{A} \widehat{T}^{2N} \right) \right| \leq \| \widehat{A} \| \text{Tr} \left[ (\widehat{T}^N)^* \widehat{T}^N \right].
\]

Using a method similar to the so called min-max principle, K. Fan has shown\(^4\) that for any finite-dimensional operator (i.e. matrix) \( \widehat{B} \), \( \text{Tr} \left[ (\widehat{B}^N)^* \widehat{B}^N \right] \leq \text{Tr} \left( \widehat{B}^* \widehat{B} \right)^N \). The finite-dimensional operators are dense in \( \mathcal{T}_2 \) = the Hilbert space of Hilbert-Schmidt operators with the inner product \( \langle \cdot, \cdot \rangle_{\text{HS}} \). Since the product, the inner product and the norm are continuous on \( \mathcal{T}_2 \) and \( \text{Tr} \left[ (\widehat{B}^N)^* \widehat{B}^N \right] = \| \widehat{B}^N \|^2_{\text{HS}}, \) \( \text{Tr} \left( \widehat{B}^* \widehat{B} \right)^N = (\widehat{B}^* \widehat{B}, (\widehat{B}^* \widehat{B})^{N-1})_{\text{HS}} \), it is obvious that the result of Fan is valid for every Hilbert-Schmidt operator. \( \square \)

**V. DISCUSSION**

The main results of this paper are the two theorems, [III.4] and [III.5]. The first one justifies the faith of the lattice community that also expectation values of polynomials are measurable with the lattice Monte Carlo methods if the potential increases sufficiently fast at infinity. The second one establishes the possibility of using the lattice integrals for computations of the expectation values even when the temperature has an imaginary part.

There are a few shortcomings of the present results. First, the exact class of potentials for which the integral kernels \( K_N \) can be used was not determined. This is not a serious problem since, by accepting the necessity for a double limit, statistics given by any potential in \( L_\text{loc} \) can be approached with the lattice integrals by replacing \( V \) by a sequence \( V_k \in \mathcal{C}^\infty \) which converges to it pointwise almost everywhere. The second shortcoming is the class of observables in Theorem [III.5]: the present proof cannot be easily extended to include also polynomial observables for polynomial potentials, although it is very natural to expect the integrals to converge to the right limit by Theorem [III.4]. Finally, the necessity of using only even number of lattice points has no natural explanation and as such it is expected to be an artifact of the present approach. Indeed, as has been noted earlier, the requirement of even lattice sizes can be dropped for continuous potentials.
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