SUBELLIPTIC WAVE EQUATIONS WITH LOG-LIPSCHITZ
PROPAGATION SPEEDS

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Abstract. In this paper we study the Cauchy problem for the wave equations for
sums of squares of left invariant vector fields on compact Lie groups and also for
hypoelliptic homogeneous left-invariant differential operators on graded Lie groups
(the positive Rockland operators), when the time-dependent propagation speed
satisfies a Log-Lipschitz condition. We prove the well-posedness in the associated
Sobolev spaces exhibiting a finite loss of regularity with respect to the initial data,
which is not true when the propagation speed is a Hölder function. We also indicate
an extension to general Hilbert spaces. In the special case of the Laplacian on \( \mathbb{R}^n \),
the results boil down to the celebrated result of Colombini-De Giorgi and Spagnolo.

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1. Introduction

In this paper we study the well-posedness of a Cauchy problem in two settings, on
compact Lie groups and on graded Lie groups. In Section 2 we deal with the problem

\[
\begin{cases}
\frac{\partial^2 u(t, x)}{\partial t^2} - a(t) \mathcal{L} u(t, x) = 0, & (t, x) \in [0, T] \times G, \\
\frac{\partial u(0, x)}{\partial t} = u_0(x), & x \in G, \\
u(0, x) = u_1(x), & x \in G;
\end{cases}
\]

where \( \mathcal{L} = X_1^2 + X_2^2 + \ldots + X_k^2 \), \( 1 \leq k \leq \text{dim}(G) = n \),
is a second order operator which is the sum of squares of elements of the Lie algebra of \( G \), namely \( X_1, \ldots, X_k \),

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satisfying Hörmander condition of order $l \in \mathbb{N}$. The coefficient function $a : [0, T] \to \mathbb{R}$ is a Log-Lipschitz function, i.e. a function that satisfies
\begin{equation}
|a(t) - a(s)| \leq C|t - s| \log(t - s),
\end{equation}
for some constant $C > 0$ and for all $t, s \in [0, T]$. We also assume that $a(t) > a_0 > 0$.

In [10, Theorem 2.3], it was shown that if $a(t) \geq a_0 > 0$ and also $a \in C^\alpha([0, T])$ is Hölder with index $0 < \alpha < 1$, then the Problem (1) has a unique solution $u \in C^2([0, T], \gamma^\alpha_{L^2}(G))$ provided that
\begin{equation}
u_0, u_1 \in \gamma^\alpha_{L^2}(G) \quad \text{and} \quad 1 \leq s < 1 + \frac{\alpha}{1 - \alpha},
\end{equation}
where $\gamma^\alpha_{L^2}(G)$ are the Gevrey spaces of $G$, based on the sub-Laplacian $L$.

In our work we assume the Log-Lipschitz condition on the coefficient $a(t)$ which is stronger than the Hölder condition $C^\alpha([0, T])$ with $0 < \alpha < 1$, since any Log-Lipschitz function satisfying (2) belongs also to any $C^\alpha([0, T])$ with $0 < \alpha < 1$. Nevertheless, there is a physical motivation to study Log-Lipschitz-type functions since they appear in relation to the well-posedness of the Navier-Stokes equations. For instance, consider the solution $u = u(t, x)$ for the problem analysed by Hantaek Bae and Marco Cannone in [12], to the problem
\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \\
\nabla \cdot u &= 0, \\
\n\{ u(0, x) \} &= u_0(x),
\end{align*}
for $x \in \mathbb{R}^3$, where $u(t, x)$ is the velocity vector field, and $p(t, x)$ is the scalar pressure function.

The authors, in [12, Theorem 1.1], establish the existence of some $\epsilon > 0$ such that for all $u_0 \in H^\frac{1}{2}$ with $\|u_0\|_{H^\frac{1}{2}} < \epsilon$, there exists a global in time solution $u$ satisfying the Log-Lipschitz regularity estimate
\begin{equation}
\|u\|_{LL^\beta} \leq C_\beta \left( \|u_0\|_{L^1} + \|u_0\|_{H^\frac{1}{2}} \right),
\end{equation}
where $\|u\|_{LL^\beta} := \int_0^\infty \sup_{|x - y| < \frac{1}{t}} \frac{|f(t, x) - f(t, y)|}{|x - y|^\beta \left( -\log |x - y| \right)^\beta} dt$, for $\beta > 0$.

If we define $H^\nu_{L^2}(G) := \{ u : \|(I - L)^{\nu} u\|_{L^2(G)} < \infty \}$, we prove the following

**Theorem 1.1.** Let $a : [0, T] \to \mathbb{R}$ be a Log-Lipschitz function such that $a(t) \geq a_0 > 0$. Suppose $(u_0, u_1) \in H^a_{L^2}(G) \times H^{a-1}_{L^2}(G)$ for some $\nu \in \mathbb{R}$. Then the Cauchy problem (1) has a unique solution satisfying
\begin{equation}
\|u(t, \cdot)\|_{H^{a-\frac{4}{T}}_{L^2}}^2 + \|\partial_t u(t, \cdot)\|_{H^{a-\frac{4}{T} - 1}_{L^2}}^2 \leq C(\|u_0\|_{H^a_{L^2}}^2 + \|u_1\|_{H^{a-1}_{L^2}}^2),
\end{equation}
for some $C, \delta > 0$ independent of $u_0, u_1$, and $t \in [0, T]$.

For the proof of Theorem 1.1 we use the techniques developed in [10] and [9]. As in those papers the global Fourier analysis on compact Lie groups introduced in [18] plays a key role in our work. This and classical results of well-posedness of ordinary first order differential equations will allow us to proof our result.
In Section 3 we study the problem
\begin{equation}
\begin{cases}
\partial_t^2 u(t, x) + a(t)\mathcal{R}u(t, x) = 0, & (t, x) \in [0, T] \times G, \\
u(0, x) = u_0(x), & x \in G, \\
\partial_t u(0, x) = u_1(x), & x \in G,
\end{cases}
\end{equation}
where $G$ is a graded Lie group and $\mathcal{R}$ is a positive self-adjoint Rockland operator. To analyse the well-posedness of this problem we follow the lines in [17], and also in [14]. The reader should note that in the case of $G = \mathbb{R}$ and $\mathcal{R} = -\Delta$, we are dealing with the classical wave equation with the time-dependent propagation speed $a(t)$. In [2] the authors study the Cauchy problem for strictly hyperbolic operators with low regularity coefficients in any space dimension $n \geq 1$. In particular the coefficients of the differential operator are supposed to be Log-Zygmund continuous in time and Log-Lipschitz continuous in space.

The well-posedness results for Hölder regular functions $a(t)$ have been obtained by Colombini, de Giorgi and Spagnolo in [3]. Moreover, it has been shown by Colombini and Spagnolo in [5] that already in the case of $G = \mathbb{R}$, the Cauchy problem (3) does not have to be well-posed in $C^\infty(\mathbb{R})$.

The Fourier analysis in the case of graded Lie groups can be found in [7] and references therein. Also a treatment of $L^p$ estimates for pseudo-differential operators on graded Lie groups can be found in [1]. The technique used is quite similar to the case of compact Lie groups, but with some differences.

2. Compact Lie groups

For a compact Lie group $G$, we denote by $\widehat{G}$ the unitary dual of $G$, consisting of equivalence classes $[\xi]$ of continuous irreducible unitary representations $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$. Let $f \in C^\infty(G)$ be a smooth function, we define its Fourier coefficient at $[\xi] \in \widehat{G}$ by
\[ \widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx. \]
Then we have that
\[ f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) \widehat{f}(\xi)), \]
and
\[ \|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{HS}^2 \right)^{1/2}, \]
where $\|\widehat{f}(\xi)\|_{HS} := \text{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^* \frac{1}{2})$ is the Hilbert-Schmidt norm. For a linear operator
\[ T : C^\infty(G) \to C^\infty(G), \]
define its global symbol by
\[ \sigma_T(x, \xi) := \xi^*(x)(T\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}, \]
where
\[
[(T_\xi)(x)]_{ij} := T(\xi(x))_{ij}.
\]

Using this symbol the following global quantization holds:
\[
T f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x)\sigma_T(x, \xi)\hat{f}(\xi)).
\]

The corresponding symbolic calculus was introduced in [18]. Since \(L\) is formally self-adjoint, the symbol of the operator can be diagonalised by a choice of a suitable basis in representation spaces, and its symbol has constant entries with respect to \(x\)-variable,
\[
\sigma_{-L}(\xi) = \text{Diag}(\nu_1^2(\xi), ..., \nu_{d_\xi}^2(\xi)).
\]

Furthermore, the global Fourier analysis permits to characterise the spaces of smooth functions \(C^\infty(G)\), the Gevrey spaces associated to the operator \(L\) denoted by \(\gamma^s_L(G)\), and the Sobolev spaces \(H^s_L(G)\) by:

\[(4)\quad f \in C^\infty(G) \iff \forall N \exists C_N \text{ such that } \|\hat{f}(\xi)\|_{HS} \leq C_N \langle \xi \rangle^{-N} \forall [\xi] \in \hat{G},\]

\[(5)\quad f \in \gamma^s_L(G) \iff \exists A > 0 : \sum_{[\xi] \in \hat{G}} d_\xi \sum_{j=1}^{d_\xi} e^{A\nu_j(\xi)} \left( \sum_{m=1}^{d_\xi} |\hat{f}(\xi)_{jm}|^2 \right) < \infty,\]

\[(6)\quad f \in H^s_L(G) \iff \sum_{[\xi] \in \hat{G}} d_\xi \sum_{j=1}^{d_\xi} (1 + \nu_j^2(\xi))^s \left( \sum_{m=1}^{d_\xi} |\hat{f}(\xi)_{jm}|^2 \right) < \infty.\]

Now we proceed to study the well-posedness of the initial value Problem (1). The main idea is to apply the Fourier transform to both sides of the differential equation and then to reduce to a first system which can be analysed by the energy method.

We have
\[(7)\quad \partial_t^2 \widehat{u}(t, \xi) + a(t)\sigma_{-L}(\xi)\hat{u}(t, \xi) = 0,\]
for any \([\xi] \in \hat{G}\) fixed. In matrix components, the equation (7) can be written as
\[(8)\quad \partial_t^2 \widehat{u}_{mk}(t, \xi) + a(t)\nu_m^2(\xi)\widehat{u}_{mk}(t, \xi) = 0,\]
for \(1 \leq m, k \leq d_\xi\).

It is then natural to analyse the problem
\[(9)\quad \partial_t^2 \widehat{v}(t, \xi) + a(t)|\xi|_\nu^2 \widehat{v}(t, \xi) = 0,\]
where for simplicity we denote \(|\xi|_\nu := \nu_m(\xi)\). Using the transformation
\[
V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} i|\xi|_\nu \widehat{v} \\ \partial_t \widehat{v} \end{pmatrix}
\]
and taking
\[ A = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}, \]
we obtain the first order linear differential equation
\[ \partial_t V(t, \xi) = i|\xi|_\nu A(t)V(t, \xi) \]
and the initial condition
\[ V(0, \xi) = \begin{pmatrix} i|\xi|_\nu \tilde{c}_0(\xi) \\ \tilde{c}_1(\xi) \end{pmatrix}. \]

Now we look for a solution in the form
\[ V(t, \xi) = \frac{1}{\det(H(t))} e^{-\rho(t) \log|\xi|_\nu} H(t)W(t, \xi), \]
where \( \rho \in C^1([0, T]) \) is a real-valued function to be chosen later. Also \( W = W(t, \xi) \) is to be determined. Take \( \psi \in C^\infty_c(\mathbb{R}), \psi \geq 0, \int_\mathbb{R} \psi = 1, \) and \( \psi_\epsilon(t) := \frac{1}{\epsilon} \psi(\frac{t}{\epsilon}). \) We take \( H = H(t) \) to be
\[ H(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}, \]
with \( \lambda_1(t) =: (-\sqrt{a} \ast \psi_\epsilon) \) and \( \lambda_2(t) =: (\sqrt{a} \ast \psi_\epsilon), \) the mollified approximations for the coefficient \( a(t). \)

By substitution of this in equation (18), we obtain that
\[
e^{-\rho(t) \log|\xi|_\nu}(\det H)^{-1} H \partial_t W + e^{-\rho(t) \log|\xi|_\nu} (-\rho'(t) \log(|\xi|_\nu)(\det H)^{-1} HW
\]
\[ -e^{-\rho(t) \log|\xi|_\nu} \left( \frac{\partial_t \det H}{(\det H)^2} \right) HW + e^{-\rho(t) \log|\xi|_\nu} (\det H)^{-1}(\partial_t H)W \]
\[ = i|\xi|_\nu e^{-\rho(t) \log|\xi|_\nu}(\det H)^{-1} AHW. \]

Multiplying both sides by \( e^{\rho(t) \log|\xi|_\nu}(\det H)H^{-1} \) we obtain
\[ \partial_t W - \rho'(t) \log|\xi|_\nu W - \frac{\partial_t \det H}{\det H} W + H^{-1}(\partial_t H)W = i|\xi|_\nu H^{-1} AHW. \]

Now,
\[ \partial_t|W(t, \xi)|^2 = 2Re(\partial_t W(t, \xi), W(t, \xi)) \]
\[ = 2\rho'(t) \log|\xi|_\nu |W(t, \xi)|^2 + 2\left( \frac{\partial_t \det H}{\det H} \right) |W(t, \xi)|^2 \]
\[ -2H^{-1}\partial_t H|W(t, \xi)|^2 - |\xi|_\nu(H^{-1} AHW - (H^{-1} AH)^* W). \]

Then,
\[ |\partial_t|W(t, \xi)|^2| \leq 2(|\rho'(t)| \log|\xi|_\nu + \left| \frac{\partial_t \det H}{\det H} \right| ||H^{-1}\partial_t H|| \]
\[ + |\xi|_\nu ||H^{-1} AHW - (H^{-1} AH)^* W|| W(t, \xi)|^2. \]
We estimate, taking into account that the coefficient function $a = a(t)$ is Log-Lipschitz and $\sqrt{a(t)} \geq a_0 > 0$ for some $a_0$, the following terms:

(i) $\left| \frac{\partial_t \det H}{\det H} \right|$
(ii) $\left\| H^{-1} \partial_t H \right\|$
(iii) $\left\| H^{-1} AH - (H^{-1} AH)^* \right\|

Observe that $\left| \frac{\partial_t \det H}{\det H} \right| = \frac{\lambda_2(t) - \lambda_1(t)}{\lambda_2(t) - \lambda_1(t)} \leq \frac{1}{2a_0} |\lambda_2(t) - \lambda_1(t)|$ since $|\lambda_2(t) - \lambda_1(t)| = |2 \int_{\mathbb{R}} \sqrt{a(t)} \psi(t) \left( \frac{t - \tau}{\epsilon} \right) \epsilon^{-1} d\tau| \geq 2a_0$.

On the other hand

$|\lambda_2(t) - \lambda_1(t)| = 2|\int_{\mathbb{R}} \sqrt{a(t)} \psi(t) \left( \frac{t - \tau}{\epsilon} \right) \epsilon^{-1} d\tau|$

$= 2\epsilon^{-1} |\int_{\mathbb{R}} \sqrt{a(t - \epsilon s)} \psi(s) ds|$

$= 2\epsilon^{-1} \int_{\mathbb{R}} \left( \sqrt{a(t - \epsilon s)} - \sqrt{a(t)} \right) \psi(s) ds|.$

Notice that

$|\sqrt{a(t - \epsilon s)} - \sqrt{a(t)}| = \frac{|a(t - \epsilon s) - a(t)|}{\sqrt{a(t - \epsilon s)} + \sqrt{a(t)}} \leq \frac{1}{2a_0} \epsilon |s \log(\epsilon s)|,$

and also that $\int_{\mathbb{R}} \sqrt{a(t)} \psi'(s) ds = 0$ since $\psi$ has compact support.

We conclude that

(11) $\left| \frac{\partial_t \det H}{\det H} \right| \leq M_1 |\log(\epsilon)|$

for $M_1 > 0$ constant. For example, if we take the support of the function $\psi(s)$ to be $\text{supp} \psi \subset [1, 2]$, and take $0 < \epsilon < \frac{1}{4}$, then

$M_1 \leq \frac{1}{a_0} \int_1^2 |s \psi'(s)| ds,$

because $|\log(\epsilon s)| < |\log \epsilon|$ for $\epsilon$ and $s$ satisfying the condition.

To estimate $\|H^{-1} \partial_t H\|$, we compute directly:

$H^{-1} \partial_t H = \frac{1}{\lambda_2(t) - \lambda_1(t)} \begin{pmatrix} -\lambda_1(t) & -\lambda_2(t) \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}.$

Now, we see that the matrix $H^{-1} \partial_t H$ is symmetric whose eigenvalues are $\beta_1 = 0$ and $\beta_2 = 2\lambda_2(t)$. Then we have $\|H^{-1} \partial_t H\| = \frac{1}{a_0} \lambda_2(t)$, from which we obtain

(12) $\|H^{-1} \partial_t H\| \leq M_2 |\log(\epsilon)|$

for $M_2 = M_1 = \frac{1}{a_0} \int_1^2 |s \psi'(s)| ds$.

Now we compute $H^{-1} AH - (H^{-1} AH)^*$

$= \frac{1}{\lambda_2(t) - \lambda_1(t)} \begin{pmatrix} 0 & \lambda_2(t) + \lambda_1(t) - 2a(t) \\ \lambda_2(t) - \lambda_1(t) & 2a(t) \end{pmatrix}$.
We will estimate for $i = 1, 2$,

$$|\lambda_i^2(t) - a(t)| = \left| \left( e^{-1} \int_{\mathbb{R}} \sqrt{a(s)} \psi\left( \frac{t-s}{\epsilon} \right) ds \right)^2 - a(t) \right|.$$ 

We observe that

$$\left| \left( e^{-1} \int_{\mathbb{R}} \sqrt{a(s)} \psi\left( \frac{t-s}{\epsilon} \right) ds \right)^2 - a(t) \right| = \left| \left( e^{-1} \int_{\mathbb{R}} \sqrt{a(s)} \psi\left( \frac{t-s}{\epsilon} \right) ds \right)^2 - \left( e^{-1} \int_{\mathbb{R}} \sqrt{a(t)} \psi\left( \frac{t-s}{\epsilon} \right) ds \right)^2 \right| =$$

$$\left| \left( e^{-1} \int_{\mathbb{R}} (\sqrt{a(s)} - \sqrt{a(t)}) \psi\left( \frac{t-s}{\epsilon} \right) ds \right) \left( e^{-1} \int_{\mathbb{R}} (\sqrt{a(s)} + \sqrt{a(t)}) \psi\left( \frac{t-s}{\epsilon} \right) ds \right) \right|.$$ 

It is clear that

$$\left| \left( e^{-1} \int_{\mathbb{R}} (\sqrt{a(s)} + \sqrt{a(t)}) \psi\left( \frac{t-s}{\epsilon} \right) ds \right) \right| \leq 2 \|\sqrt{a}(\cdot)\|_{\infty},$$

and

$$\left| \left( e^{-1} \int_{\mathbb{R}} (\sqrt{a(s)} - \sqrt{a(t)}) \psi\left( \frac{t-s}{\epsilon} \right) ds \right) \right| \leq \frac{1}{2a_0} \left( \int_{1}^{2} \tau \psi(\tau) d\tau \right) \epsilon |\log(\epsilon)|,$$

if $\psi$ is as chosen above. Finally, taking into account the zeros in the anti-diagonal of the matrix we have that, the norm of the matrix as an operator is precisely $2|\lambda_2^2 - a^2(t)|$. For this reason,

$$\|H^{-1}AH - (H^{-1}AH)^*\| \leq M_3 \epsilon |\log(\epsilon)|$$

for $M_3 = \frac{1}{2a_0} \left( \int_{1}^{2} \tau \psi(\tau) d\tau \right)$. Applying the estimates (11), (12), and (13) we obtain

$$\partial_t |W(t, \xi)|^2 \leq$$

$$2 \left( \rho'(t) \log |\xi|_\nu + M_1 |\log(\epsilon)| + M_2 |\log(\epsilon)| + M_3 \epsilon |\log(\epsilon)| \right) |W(t, \xi)|^2.$$ 

By choosing $\epsilon := |\xi|_\nu^{-1} < \frac{1}{2}$, we get

$$\partial_t |W(t, \xi)|^2 \leq 2 \left( \rho'(t) \log |\xi|_\nu + (M_1 + M_2 + M_3) \log |\xi|_\nu \right) |W(t, \xi)|^2.$$ 

We have, for $|\xi|_\nu \geq 1$, that

$$\rho'(t) \log |\xi|_\nu + (M_1 + M_2 + M_3) \log |\xi|_\nu \leq 0$$

provided that $\delta > M_1 + M_2 + M_3$ where we assume $\rho(t) = \rho(0) - \delta t$.

We deduce that $\partial_t |W(t, \xi)|^2 \leq 0$ under the conditions $|\xi|_\nu > 2$, and

$$\delta > M_1 + M_2 + M_3 = \frac{1}{a_0} \left( \int_{1}^{2} \tau \psi(\tau) d\tau \right) + \frac{1}{2a_0} \left( \int_{1}^{2} \tau \psi(\tau) d\tau \right).$$
This implies that
\[
|V(t, \xi)| = \exp(-\rho(t) \log |\xi|_\nu) \frac{1}{\det(H(t))} \|H(t)\| |W(t, \xi)|
\]

\[
\leq \exp(-\rho(t) \log |\xi|_\nu) \frac{1}{\det(H(t))} \|H(t)\| |W(0, 0) - W(0, \xi)|
\]

\[
= \exp((-\rho(t) + \rho(0)) \log |\xi|_\nu) \frac{|\det(H(0))|}{\det(H(t))} \|H(t)\| \|H^{-1}(0)\| |V(0, \xi)|.
\]

From this, we have that
\[
|V(t, \xi)| \leq M_4 |\xi|_\nu^{|\delta T|} |V(0, 0)|
\]
for \(M_4 > 0\). This implies that
\[
|\xi|^2 |\hat{g}|^2 + |\partial_\xi \hat{\nu}|^2 \leq M_4^2 |\xi|^2 |\hat{\nu}|^2\quad (|\xi|^2 |\hat{\nu}|^2 + |\hat{\nu}|^2)
\]

Coming back to the functions \(u_{mk}\), we obtain
\[
\nu_m^2 |\hat{u}_{mk}|^2 + |\partial_\xi \hat{u}_{mk}|^2 \leq M_4^2 \nu_m^{\delta T} \left( \nu_m^2 |\hat{u}_{0mk}|^2 + |\hat{u}_{1mk}|^2 \right).
\]

Multiplying both sides by \(\nu_m^{2s-2-\delta T}\) we get
\[
\nu_m^{2s-2-\delta T} |\hat{u}_{mk}|^2 + \nu_m^{2s-2-\delta T} |\partial_\xi \hat{u}_{mk}|^2 \leq M_4^2 \left( \nu_m^{2s} |\hat{u}_{0mk}|^2 + \nu_m^{2s-2} |\hat{u}_{1mk}|^2 \right),
\]

which says that
\[
\|u(t, \cdot\|_{H^{s-\frac{\delta T}{2}}} + \|\partial_\xi u(t, \cdot\|_{H^{s-\frac{\delta T}{2}}} - 1 \leq C(\|u_0(t, \cdot\|_{H^{s-\frac{\delta T}{2}}} + \|u_1(t, \cdot\|_{H^{s-\frac{\delta T}{2}}} - 1,
\]

proving Theorem 1.1.

3. Graded Lie groups

In this section we use the Fourier analysis on graded Lie groups, see e.g. [7] and [14], to analyse the Cauchy Problem (3). A Lie group \(G\) is called graded if its Lie algebra \(\mathfrak{g}\) can be decomposed in the form
\[
\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i,
\]
such that \([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\) and \(\mathfrak{g}_{i+j} = \{0\}\) for \(i + j > k\).

The gradation induces a homogeneous structure on \(\mathfrak{g}\) by the dilations \(D_r := \exp(A \log r)\), where \(A : \mathfrak{g} \to \mathfrak{g}\) is a diagonalisable operator acting by \(AX = jX\) for \(X \in \mathfrak{g}_j\). Notice that each \(D_r\) is morphism of the Lie algebra \(\mathfrak{g}\) for all \(r > 0\). With this algebraic structure it is possible to extend the basics of Fourier analysis in \(\mathbb{R}^n\) to graded Lie groups \(G\).

We start with \(\pi\) a representation of \(G\) on the separable Hilbert space \(H_\pi\). A vector \(v \in H_\pi\) is said to be smooth or of type \(C^\infty\) if the function
\[
G \ni x \mapsto \pi(x)v \in H_\pi
\]
is smooth. The vector space of smooth vectors of a representation \( \pi \) is denoted by \( H^\infty_\pi \). For a function \( f \in S(G) = \{ f : f \circ \exp_G \in S(\mathfrak{g}) \} \) in the Schwartz space of \( G \), the Fourier transform evaluated at \( \pi \in \hat{G} \) is the operator acting on \( H^\infty_\pi \) defined by

\[
\mathcal{F}_G(f)(\pi) := \hat{f}(\pi) := \pi(f) = \int_G f(x) \pi(x)^* dx.
\]

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). For every \( X \in \mathfrak{g}, \ v \in H^\infty_\pi \) smooth, and for a given \( \pi \in \hat{G} \), we recall the definition of the infinitesimal representation

\[
d\pi(X)v := \lim_{t \to 0} \frac{1}{t} (\pi(\exp_G(tx))v - v),
\]

which is a representation of \( \mathfrak{g} \) on \( H^\infty_\pi \).

According to the Poincaré-Birkhoff-Witt Theorem, any left-invariant differential operator \( T \) on \( G \), can be written in a unique way as a finite sum

\[
T = \sum_{|\alpha| \leq M} c_\alpha X^\alpha
\]

where all but finitely many of the coefficients \( c_\alpha \in \mathbb{C} \) are zero and \( X^\alpha = X_1 \ldots X_{|\alpha|} \), for \( X_i \in \mathfrak{g} \). This allows one to look at any left-invariant differential operator \( T \) on \( G \) as an element of the universal enveloping algebra \( U(\mathfrak{g}) \) of the Lie algebra of \( G \). In this case the symbol of the operator \( T \) is the family of infinitesimal representations

\[
\{ d\pi(T) := \pi(T) | \pi \in \hat{G} \}.
\]

A linear operator \( T : C^\infty(G) \rightarrow D'(G) \) is homogeneous of degree \( \nu \in \mathbb{C} \) if for every \( r > 0 \) the equality

\[
T(f \circ D_r) = r^\nu(Tf) \circ D_r
\]

holds for every \( f \in D(G) \). A Rockland operator is a left-invariant differential operator \( \mathcal{R} \) which is homogeneous of positive degree \( \nu = \nu_\mathcal{R} \) and such that, for every unitary irreducible non-trivial representation \( \pi \in \hat{G}, \pi(\mathcal{R}) \) is injective on \( H^\infty_\pi \); \( \sigma_\mathcal{R}(\pi) = \pi(\mathcal{R}) \) is the symbol associated to \( \mathcal{R} \).

Hulanicki, Jenkins and Ludwig showed in [13] that the spectrum of \( \pi(\mathcal{R}) \), with \( \pi \in \hat{G} \setminus \{1\} \), is discrete and lies in \( (0, \infty) \). Any Rockland operator is a Fourier multiplier and we have that

\[
\mathcal{F}(\mathcal{R}f)(\pi) = \pi(\mathcal{R})\hat{f}(\pi),
\]

where \( \pi(\mathcal{R}) = \text{Diag}(\pi_1^2, \pi_2^2, \ldots) \) with \( \pi_i \in \mathbb{R}^+ \) for \( i \in \mathbb{N} \).

For every \( \pi \in \hat{G} \), the Kirillov trace character \( \Theta_\pi \) defined by

\[
(\Theta_\pi, f) := \text{Tr}(\hat{f}(\pi)),
\]

is a tempered distribution on \( S(G) \). The identity \( f(e_G) = \int_{\hat{G}} (\Theta_\pi, f) d\pi \), implies the Fourier inversion formula

\[
f = \mathcal{F}_G^{-1}(\hat{f}),
\]

where

\[
(F_G^{-1}(\sigma))(x) := \int_{\hat{G}} \text{Tr}(\pi(x) \sigma(\pi)) d\pi, \ x \in G, \ F_G^{-1} : S(\hat{G}) \rightarrow S(G),
\]
is the inverse Fourier transform. In this context, the Plancherel theorem takes the form \( \|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(\hat{G})} \), where

\[
L^2(\hat{G}) := \int_{\hat{G}} H_\pi \otimes H^*_\pi d\mu(\pi),
\]

is the Hilbert space endowed with the norm

\[
\|\sigma\|_{L^2(\hat{G})} = \left( \int_{\hat{G}} \|\sigma(\pi)\|_{H^s}^2 d\mu(\pi) \right)^{\frac{1}{2}},
\]

with \( d\mu(\pi) \) denoting the Plancherel measure on \( \hat{G} \).

It can be shown that a Lie group \( G \) is graded if and only if there exists a differential Rockland operator on \( G \). If the Rockland operator is formally self-adjoint, then \( R \) and \( \pi(R) \) admit self-adjoint extensions on \( L^2(G) \) and \( H_\pi \), respectively.

**Definition 3.1.** Let \( G \) be a graded Lie group and let \( R \) be a positive Rockland operator of homogeneous degree \( \nu \). The Sobolev space \( H^s_R(G) \) is the subspace of \( S'(G) \) obtained by completion of the Schwartz space \( S(G) \) with respect to the Sobolev norm

\[
\|f\|_{H^s_R(G)} := \| (1 + \pi(R))^{\frac{s}{2}} f \|_{L^2(G)}.
\]

Now we have the necessary tools to study the Problem (3). These spaces have been extensively analysed in [7] and [8].

**Theorem 3.1.** Consider the Problem (3) where \( R \) is a Rockland operator with homogeneous degree \( \nu \). For initial data \((u_0, u_1) \in H^s \times H^{s-\nu} \), the problem is well posed and the solution \( u \) satisfy that

\[
\|u\|^2_{H^s} + \|\partial_t u\|^2_{H^{s-\nu}} \leq M(\|u_0\|^2_{H^s} + \|u_1\|^2_{H^{s-\nu}}),
\]

for some constants \( \delta, M > 0 \).

**Proof.** We apply the group Fourier transform to both sides of the equation to obtain

\[
\partial^2_t \hat{u}(t, \pi) + a(t)\pi(R) \hat{u}(t, \pi) = 0.
\]

Following the lines of Section 2, and taking into account the diagonal form of the symbol of the Rockland operator \( R \), write \( \hat{u}(t, \pi) = [\hat{u}_{mk}] \), also for the initial data \( \hat{u}_0(t, \pi) = [\hat{u}_{0mk}], \hat{u}_1(t, \pi) = [\hat{u}_{1mk}] \). For the eigenvalues of the symbol of the Rockland Operator \( R \), we write \( \pi_m \).

The idea is to analyse the first order linear differential equation (18)

\[
\partial_t V(t, \xi) = i|\xi|_\pi A(t)V(t, \xi)
\]

with initial condition

\[
V(0, \xi) = \left( i|\xi|_\pi \hat{v}_0(\xi) \overline{\hat{v}_1(\xi)} \right).
\]

We look for a solution in the form

\[
V(t, \xi) = \frac{1}{\det(H(t))} e^{-\rho(t)\log |\xi|_\pi} H(t) W(t, \xi),
\]
where \( \rho \in C^1([0, T]) \) is a real valued function to be chosen later. For \( \psi \in C_c^\infty(\mathbb{R}) \), \( \psi \geq 0, \int_\mathbb{R} \psi = 1 \), and \( \psi_{\epsilon}(t) := \frac{1}{\epsilon} \psi(\frac{t}{\epsilon}) \), we take \( H = H(t) \) to be

\[
H(t) = \left( \frac{1}{\lambda_1(t)} \frac{1}{\lambda_2(t)} \right),
\]

with \( \lambda_1(t) := (-\sqrt{a * \psi_{\epsilon}}) \) and \( \lambda_2(t) := (\sqrt{a * \psi_{\epsilon}}) \), the mollified approximations for the coefficient function \( a(t) \). Using the estimates from Section 2 we arrive to, in complete analogy with equation (16):

\[
\pi^2 m |\hat{u}_{mk}|^2 + |\partial_t \hat{u}_{mk}|^2 \leq M \left( \pi_m^{2+\delta T} |\hat{u}_{0mk}|^2 + \pi_m^{\delta T} |\hat{u}_{1mk}|^2 \right),
\]

for some \( \delta > 0 \). In order to deduce the estimate for functions in the Sobolev spaces, multiplying by \( \pi^{\frac{4s}{\nu} - \delta T - 2} \) we obtain

\[
\pi^{\frac{4s}{\nu} - \delta T} |\hat{u}_{mk}|^2 + \pi^{\frac{4s}{\nu} - \delta T - 2} |\partial_t \hat{u}_{mk}|^2 \leq M \left( \pi_m \pi_0 |\hat{u}_{0mk}|^2 + \pi_m^{-2} |\hat{u}_{1mk}|^2 \right).
\]

Recall that

\[
\| F\{ (1 + \pi(\mathcal{R}))^{\frac{s}{\nu}} u \} \|_{HS}^2 = \sum_m (1 + \pi_m^{\frac{2s}{\nu}}) \sum_j |\hat{u}_{mj}|^2.
\]

Then we can see that \( u \in H_{\mathbb{R}}^s \) is characterised by

\[
\int_G \sum_m (\pi_m^{\frac{2s}{\nu}}) \sum_j |\hat{u}_{mj}|^2 d\mu(\pi) < \infty.
\]

Taking this into account we have that

\[
\| \pi(\mathcal{R})^{\frac{s}{\nu} - \frac{4s}{\nu} - \delta T} \hat{u} \|_{HS}^2 + \| \pi(\mathcal{R})^{\frac{s}{\nu} - \frac{4s}{\nu} - \delta T} \partial_t \hat{u} \|_{HS}^2
\]

\[
\leq M \left( \| \pi(\mathcal{R})^{\frac{s}{\nu}} \hat{u}_0 \|_{HS}^2 + \| \pi(\mathcal{R})^{\frac{s}{\nu}} \hat{u}_1 \|_{HS}^2 \right).
\]

after integration with respect to the Plancherel measure on \( \hat{G} \) on both sides we obtain the proof of the Theorem. \( \square \)

**Remark 3.1.** In [16] and [9] we can find a result on embedding between the Sobolev spaces: the spaces \( \mathcal{H}^s_L := \{ f | (1 - \mathcal{L})^{\frac{s}{2}} f \in L^2(G) \} \) where the operator \( \mathcal{L} = X_1^2 + \ldots + X_k^2 \) is a sum of squares of left invariant vector fields satisfying Hörmander condition of length \( l \), and the classical ones \( \mathcal{H}^s := \{ f | (1 - \Delta)^{\frac{s}{2}} f \in L^2(G) \} \) associated to the Laplace operator \( \Delta \). Indeed,

\[
\mathcal{H}^s \subset \mathcal{H}^s_L \subset \mathcal{H}^s.
\]

From this we can deduce the well-posedness assuming data in \( \mathcal{H}^s_L \) and obtaining solution in classical Sobolev spaces.
4. An Extension to Hilbert Spaces

We observe that the technique applied in the sections above can be used to study the problem \( (1) \) in the context that \( u(t) \in \mathcal{H} \) where \( \mathcal{H} \) is a separable Hilbert space.

Let \( (e_j)_{j \in \mathbb{N}} \subset \mathcal{H} \) be an orthonormal basis. We define the Fourier transform of the element \( u \in \mathcal{H} \) by \( \hat{u}(j) := \langle u, e_j \rangle_{\mathcal{H}} \). Clearly
\[
u = \sum_{j \in \mathbb{N}} \hat{u}(j)e_j.
\]

Consider an operator \( \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H} \) which acts on \( u \) as
\[
\mathcal{A}u = \sum_{j \in \mathbb{N}} \lambda_j^2 \hat{u}(j)e_j,
\]
for a sequence of real numbers \( (\lambda_j^2)_{j \in \mathbb{N}} \). Notice that \( \hat{\mathcal{A}}u(j) = \lambda_j^2 \hat{u}(j) \). We proceed to define the induced Sobolev spaces \( \mathcal{H}_{\mathcal{A}}^s \) by:
- \( \mathcal{H}_{\mathcal{A}}^0 := \mathcal{H} \).
- \( \mathcal{H}_{\mathcal{A}}^s := \{ u \in \mathcal{H} : \sum_{j \in \mathbb{N}} \lambda_j^{2s} |\hat{u}(j)|^2 < \infty \} \), for \( s \in \mathbb{R} \).

Now we have introduced analogous tools of the Fourier analysis. We establish the version of problem \( (1) \) in this setting. For \( a : [0, T] \to \mathbb{R} \) being a Log-Lipschitz function, consider the problem of finding a function \( u : [0, T] \to \mathcal{H} \) such that
\[
\begin{align*}
\partial_t^2 u(t) - a(t)A u(t) &= 0, \quad t \in [0, T], \\
u(0) &= u_0, \\
\partial_t u(0) &= u_1,
\end{align*}
\]
the initial data is in suitable Sobolev spaces.

**Theorem 4.1.** Let \( a : [0, T] \to \mathbb{R} \) be a Log-Lipschitz function such that \( a(t) > a_0 > 0 \). Suppose \( (u_0, u_1) \in H_{\mathcal{A}}^s(G) \times H_{\mathcal{A}}^{s-1}(G) \) for some \( s \in \mathbb{R} \). Then the Cauchy problem \( (22) \) has a unique solution satisfying
\[
\|u(t, \cdot)\|_{H_{\mathcal{A}}^s}^2 + \|\partial_t u(t, \cdot)\|_{H_{\mathcal{A}}^{s-1}}^2 \leq C(\|u_0\|_{H_{\mathcal{A}}^s}^2 + \|u_1\|_{H_{\mathcal{A}}^{s-1}}^2),
\]
for some \( C, \delta > 0 \) independent of \( u_0, u_1 \), and \( t \in [0, T] \).

**Proof.** Taking Fourier transform on both sides of the equation, we obtain for each \( k \in \mathbb{N} \),
\[
\langle \partial_t^2 u(t), e_k \rangle - a(t) \langle A u(t), e_k \rangle = 0,
\]
or
\[
\partial_t^2 \hat{u}(t)(k) - a(t) \lambda_k^2 \hat{u}(t)(k) = 0.
\]
After denoting \( \hat{u}(t) = \beta(t) \), we deal with the equation
\[
\partial_t^2 \beta(t) - a(t) \lambda_k^2 \beta(t) = 0.
\]
This is clearly the equation (9). The conclusions obtained in Theorem 1.1 are then valid in this case.

\[\square\]

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