An Optimal Block Diagonal Preconditioner for Heterogeneous Saddle Point Problems in Phase Separation

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Abstract

The phase separation processes are typically modeled by Cahn-Hilliard equations. This equation was originally introduced to model phase separation in binary alloys; where phase stands for concentration of different components in alloy. When the binary alloy under preparation is subjected to a rapid reduction in temperature below a critical temperature, it has been experimentally observed that the concentration changes from a mixed state to a visibly distinct spatially separated two phase for binary alloy. This rapid reduction in the temperature, the so-called “deep quench limit”, is modeled effectively by obstacle potential.

The discretization of Cahn-Hilliard equation with obstacle potential leads to a block 2 × 2 non-linear system, where the (1, 1) block has a non-linear and non-smooth term. Recently a globally convergent Newton Schur method was proposed for the non-linear Schur complement corresponding to this non-linear system. The proposed method is similar to an inexact active set method in the sense that the active sets are first approximately identified by solving a quadratic obstacle problem corresponding to the (1, 1) block of the block 2 × 2 system, and later solving a reduced linear system by annihilating the rows and columns corresponding to identified active sets. For solving the quadratic obstacle problem, various optimal multigrid like methods have been proposed. In this paper, we study a non-standard norm that is equivalent to applying a block diagonal preconditioner to the reduced linear systems. Numerical experiments confirm the optimality of the solver and convergence independent of problem parameters on sufficiently fine mesh.

1 Introduction

The Cahn-Hilliard equation was first proposed in 1958 by Cahn and Hilliard [5] to study the phase separation process in a binary alloy. Here the term phase stands for the concentration of different components in the alloy. It has been empirically observed that the concentration changes from the mixed state to a visibly distinct spatially separated two phase state when

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the alloy under preparation is subjected to a rapid cooling below a critical temperature. This
rapid reduction in the temperature the so-called deep quench limit has been found to be
modeled efficiently by obstacle potential proposed by Oono and Puri [17] in 1987 and by
Blowey and Elliot [2, p. 237, (1.14)]. The phase separation has been noted to be highly non-
linear (point nonlinearity to be precise), and the obstacle potential emulates the nonlinearity
and non-smoothness that is empirically observed. However, handling the non-smoothness as
well as designing robust iterative procedure has been the subject of much active research
in last decades. Assuming semi-implicit time discretizations [3] to alleviate the time step
restrictions, most of the proposed methods essentially differ in the way the nonlinearity and
non-smoothness are handled. There are two main approaches to handle the non-smoothness:
regularization around the non-smooth region [4] or an active set approach [9] i.e., identify the
active sets and solve a reduced problem which is linear, in addition to ensuring the global
convergence of the Newton method by proper damping parameter. The non-linear problem
corresponding to Cahn-Hilliard with obstacle potential could be written as a non-linear system
in block $2 \times 2$ matrix form as follows:

$$
\begin{pmatrix}
F & B^T \\
B & -C
\end{pmatrix}
\begin{pmatrix}
u^* \\
w^*
\end{pmatrix}
\ni
\begin{pmatrix}
f \\
g
\end{pmatrix}, \quad u^*, w^* \in \mathbb{R}^n
$$ (1)

where $u^*, w^*$ are unknowns, $F = A + \partial I_K$, where $I_K$ denotes the indicator functional of the
admissible set $K$. The matrices $A, C$ are essentially Laplacian with $A$ augmented by a non-
local term reflecting mass conservation, a necessary condition in Cahn-Hilliard model. Both
nonlinearity and non-smoothness are due to the presence of term $\partial I_K$ in $F$. By nonlinear
Gaussian elimination of the $u$ variables, the system above could be reduced to a nonlinear
Schur complement system in $w$ variables [9], where the nonlinear Schur complement is given
by $C - BF^{-1}B^T$. In [9], a globally convergent Newton method is proposed for this nonlinear
Schur complement system which is interpreted as a preconditioned Uzawa iteration. Note that
$F(x)$ is a set valued mapping due to the presence of set-valued operator $\partial I_K$; to solve the
inclusion $F(x) \ni y$ corresponding to the quadratic obstacle problem, many methods have been
proposed such as projected block Gauss-Seidel [1], monotone multigrid method [12] [13] [15],
truncated monotone multigrid [10], and truncated Newton multigrid [10]. See the excellent
review article [10] that compares these methods. By annihilating the corresponding rows and
columns that belongs to the active sets identified by solving the obstacle problem, we obtain
a reduced linear system as follows

$$
\begin{pmatrix}
\hat{A} & \hat{B}^T \\
\hat{B} & -C
\end{pmatrix}
\begin{pmatrix}
\hat{u} \\
\hat{w}
\end{pmatrix}
= 
\begin{pmatrix}
\hat{f} \\
\hat{g}
\end{pmatrix}, \quad \hat{u}, \hat{w} \in \mathbb{R}^n
$$ (2)

that correspond to new descent direction in the Uzawa iteration. The overall nonlinear iter-
ation is performed in the sense of inexact Uzawa, and the preconditioners are updated with
next available active sets.

In this paper our goal is to design effective preconditioner and hence an iterative solver
for (1) such that the convergence rate is independent of problem parameters. In particular,
we consider a block diagonal preconditioner proposed in [3]; we adapt it to our linear system,
we prove properties relevant for iterative solvers, derive spectral radius of the preconditioned
operator, and show the effectiveness of the preconditioner numerically compared to a Schur
complement preconditioner proposed recently for same model.

The rest of this paper is organized as follows. In Section 3, we describe the Cahn-Hilliard
model with obstacle potential, we discuss the time and space discretizations and variational
formulations. In Section 4, we discuss briefly the solver for Cahn-Hilliard with obstacle problem. In particular, we briefly discuss Nonsmooth Newton Schur method seen as an Uzawa iteration, and the truncated Newton multigrid for the obstacle problem. The preconditioners for the reduced linear systems are discussed in Section 4.5.3. Finally in Section 5, we shown numerical experiments with the proposed preconditioner.

2 Notations

Let SPD and SPSD denote symmetric positive definite and symmetric positive semi definite. Let \( \kappa(M) \) denote the condition number of SPD matrix \( M \). For \( x \in \mathbb{R}, |x| \) denotes the absolute value of \( x \), whereas for any set \( K, |K| \) denotes the number of elements in \( K \). Let \( Id \in \mathbb{R}^{n \times n} \) denote the identity matrix. Let 1 denote \([1,1,1,...,1]\). For a symmetric matrix \( Z \in \mathbb{R}^{n \times n} \), the eigenvalues are denoted and ordered as follows \( \lambda_1(Z) \leq \lambda_2(Z) \leq \cdots \leq \lambda_n(Z) \).

3 Cahn-Hilliard Problem with Obstacle Potential

3.1 The Model

The Ginzburg-Landau (GL) energy functional which is given as follows

\[
E(u) = \int_{\Omega} \frac{\epsilon}{2}|\nabla u|^2 + \psi(u) \, dx, \quad \Omega = (0, 1) \times (0, 1)
\]  

leads to Cahn-Hilliard equation under \( H^{-1} \) gradient flow. Here the constant \( \epsilon \) relates to interfacial thickness and the obstacle potential \( \psi \) is given as follows:

\[
\psi(u) = \psi_0(u) + I_{[-1,1]}(u), \quad \text{where } \psi_0(u) = \frac{1}{2}(1 - u^2).
\]

Here the subscript \([-1,1]\) of indicator function \( I \) above denotes the range of values of \( u \). Let \( u_1 \) and \( u_2 \) be the concentration of the two components in the binary alloy, then \( u = u_1 - u_2 \), where \( u_1, u_2 \in [0, 1] \). Here \( I_{[-1,1]}(u) \) is defined as follows:

\[
I_{[-1,1]} = \begin{cases} 
0, & \text{if } u(i) \in [-1,1], \\
\infty, & \text{otherwise}.
\end{cases}
\]

Moreover, \( u_1 + u_2 \) is assumed to be conserved. We consider weak form of \( H^{-1} \) gradient flow of \( \epsilon E \) as follows

\[
(\bar{\partial} u, v)_{H^{-1}} = \epsilon (-\nabla E(u), v) \iff ((-\Delta)^{-1} \partial u, v)_{L^2} = \epsilon \langle -\nabla E(u), v \rangle.
\]

And strong form reads

\[
(-\nabla)^{-1} \partial u = -\epsilon \nabla E(u) \iff \partial u = \epsilon ( -\nabla)(-\nabla E)(u)) = \Delta \epsilon \partial E(u).
\]

Now setting \( w = \epsilon \partial E(u) \) above. From (3.1) with \( \gamma = 1 \), we have

\[
\partial E(u) = -\epsilon \Delta u + \frac{1}{\epsilon}(\psi_0'(u) + \mu) \implies w = \epsilon \partial E(u) = -\epsilon^2 \Delta u + \psi_0'(u) + \mu.
\]
Putting everything together, the Cahn-Hilliard equation in PDE form with inequality constraints obtained from GL energy (3.1) reads:

$$
\partial_t u = \Delta w, \\
\dot{w} = -\epsilon \Delta u + \psi_0'(u) + \mu, \\
\mu \in \partial I_{[-1,1]}(u), \\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega.
$$

The unknowns $u$ and $w$ are called order parameter and chemical potential respectively. For a given $\epsilon > 0$, final time $T > 0$, and initial condition $u_0 \in K$ where

$$
K = \{ v \in H^1(\Omega) : |v| \leq 1 \},
$$

the equivalent initial value problem for Cahn-Hilliard equation with obstacle potential interpreted as variational inequality reads

$$
\langle \frac{du}{dt}, v \rangle + (\nabla w, \nabla v) = 0, \forall v \in H^1(\Omega),
$$

(4)

$$
\epsilon (\nabla u, \nabla (v - u)) - (u, v - u) \geq (w, v - u), \forall v \in K,
$$

(5)

where we use the notation $\langle \cdot, \cdot \rangle$ to denote the duality pairing of $H^1(\Omega)$ and $H^1(\Omega)'$. Note that we used the fact that $\psi_0'(u) = -u$ in the second term on the left of inequality (3.1) above. The inequalities (3.1) and (3.1) are defined on constrained set $K$, the variational inequality of first kind is also equivalently represented on unconstrained set using indicator functional [6, p. 2]. The existence and uniqueness of the solution of (3.1), (3.1) above has been established in Blowey and Elliot [2]. We next consider an appropriate discretization in time and space for the model.

### 3.2 Time and space discretizations

We consider a fixed non-adaptive grid in time interval $(0,T)$ and in space $\Omega$ defined in (3.1). The time step $\tau = T/N$ is kept uniform. We consider the semi-implicit Euler discretization in time and finite element discretization in space as in Barrett et. al. [11] with triangulation $T_h$ with the following spaces (as in [9]):

$$
S_h = \{ v \in C(\bar{\Omega}) : v|_T \text{ is linear } \forall T \in T_h \}, \\
P_h = \{ v \in L^2(\Omega) : v|_T \text{ is constant } \forall T \in T_h \}, \\
K_h = \{ v \in P_h : |v|_T \leq 1 \text{ } \forall T \in T_h \} = K \cap S_h \subset K,
$$

which leads to the following discrete Cahn-Hilliard problem with obstacle potential:

Find $u^k_h \in K_h, w^k_h \in S_h$ s.t.

$$
\langle u^k_h, v_h \rangle + \tau (\nabla w^k_h, \nabla v_h) = \langle u^{k-1}_h, v_h \rangle, \forall v_h \in S_h, \\
\epsilon (\nabla u^k_h, \nabla (v_h - u^k_h)) - \langle u^k_h, v_h - u^k_h \rangle \geq \langle u^{k-1}_h, v_h - u^k_h \rangle, \forall v_h \in K_h
$$

holds for each $k = 1, \ldots, N$. The initial solution $u^0_h \in K_h$ is taken to be the discrete $L^2$ projection $\langle u^0_h, v_h \rangle = (u_0, v_h), \forall v_h \in S_h$. 

4
Existence and uniqueness of the discrete Cahn-Hilliard equations has been established in [3]. The discrete Cahn-Hilliard equation is equivalent to the set valued saddle point block 2×2 nonlinear system (1) with 

\[ A = \epsilon(\langle \lambda_p, 1 \rangle \langle \lambda_p, 1 \rangle + \langle \nabla \lambda_p, \nabla \lambda_q \rangle)_{p,q \in \mathcal{N}_h}, \]

\[ B = (\langle \lambda_p, \lambda_q \rangle)_{p,q \in \mathcal{N}_h}, \quad C = \tau((\nabla \lambda_p, \nabla \lambda_q))_{p,q \in \mathcal{N}_h}. \]

We write the above in more compact notations as follows

\[ A = \epsilon(K + mm^T), \quad B = M, \quad C = \tau K, \quad (6) \]

where \( m = \langle \lambda_p, 1 \rangle \), \( M \) and \( K \) are mass and stiffness matrices respectively.

### 4 Iterative solver for Cahn-Hilliard with obstacle potential

In [9], a nonsmooth Newton Schur method is proposed which is also interpreted as a preconditioned Uzawa iteration. For a given time step \( k \), the Uzawa iteration reads:

\[ u^{i,k} = F^{-1}(f^k - B^T w^{i,k}), \quad (7) \]

\[ w^{i+1,k} = w^{i,k} + \rho^{i,k} \tilde{S}^{-1}_{i,k}(Bu^{i,k} - Cw^{i,k} - g^k) \quad (8) \]

for the saddle point problem (1). Here \( i \) denotes the \( i^{th} \) Uzawa step, and \( k \) denotes the \( k^{th} \) time step. Here \( f^k \) and \( g^k \) are defined as follows

\[ \langle f, v_h \rangle = \langle u^{k-1}_h, v_h \rangle, \quad \langle g, v_h \rangle = -(u^{k-1}_h, v_h). \]

The time loop starts with an initial value for \( w^{0,0} \) which can be taken arbitrary as the method is globally convergent, and with the initial value \( u^{0,0} \). The Uzawa iteration requires three main computations that we describe below.

#### 4.1 Computing \( u^{i,k} \)

The first step (4) corresponds to solving a quadratic obstacle problem interpreted as a minimization problem as follows

\[ u^{i,k} = \arg \min_{v \in K} \left( \frac{1}{2} \langle Av, v \rangle - \langle f^k - B^T w^{i,k}, v \rangle \right). \]

As mentioned in the introduction, this problem has been extensively studied during last decades [1, 10, 12, 13].

#### 4.1.1 Algebraic Monotone Multigrid for Obstacle Problem

To solve the quadratic obstacle problem (4), we use the monotone multigrid method proposed in [12]. In Algorithm 1 we describe an algebraic variant of the method. The algorithm performs one V-cycle of multigrid; it takes \( u^{i} \) from the previous iteration, and outputs the improved solution \( u^{i+1} \). The initial set of interpolation operators are constructed using aggregation based coarsening [14].
4.2 Computing $\hat{S}^{-1}_{i,k}(Bu_{i,k} - Cw_{i,k} - g^k)$

The quantity $d_{i,k} = \hat{S}^{-1}_{i,k}(Bu_{i,k} - Cw_{i,k} - g^k)$ in (4) is obtained as a solution of the following reduced linear block $2 \times 2$ system:

$$
\begin{pmatrix}
\hat{A} & \hat{B}^T \\
\hat{B} & -C
\end{pmatrix}
\begin{pmatrix}
\tilde{u}^i_{i,k} \\
d^i_{i,k}
\end{pmatrix} =
\begin{pmatrix}
0 \\
g + Cw_{i,k} - Bu_{i,k}
\end{pmatrix},
$$

(9)

where

$$
\hat{A} = TAT + \hat{T}, \quad \hat{B} = BT.
$$

(10)

Here truncation matrices $T$ and $\hat{T}$ are defined as follows:

$$
T = \text{diag}(0, \text{if } u_{i,k}(j) \in \{-1, 1\}, 1, \text{otherwise}), \quad \hat{T} = \text{diag}(1, \text{if } T_{jj} = 0, 0, \text{otherwise}), \quad j = 1, \ldots, |N_h|.
$$

(11)

where $u_{i,k}(j)$ is the $j$th component of $u_{i,k}$, and $T_{jj}$ is the $j$th diagonal entry of $T$. In words, $\hat{A}$ is the matrix obtained from $A$ by replacing the $i$th row and $i$th column by the unit vector $e_i$ corresponding to the active sets identified by diagonal entries of $T$. Similarly, $B$ is the matrix obtained from $B$ by annihilating rows, and $\hat{B}^T$ is the matrix obtained from $B$ by annihilating columns. Rewriting untruncated version of (4.2) in simpler notation as follows:

$$
\begin{pmatrix}
\epsilon K & M \\
M & -\tau K
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
0 \\
b
\end{pmatrix},
$$

where $K = K + mm^T$. By a change of variable $y' = y/\epsilon$, we obtain

$$
\begin{pmatrix}
K & M \\
M & -\eta K
\end{pmatrix}
\begin{pmatrix}
x \\
y'
\end{pmatrix} =
\begin{pmatrix}
0 \\
b
\end{pmatrix}. 
$$

Furthermore, we modify the (2, 2) term of the system matrix above as follows:

$$
-\eta K = -\eta K - \eta mm^T + \eta mm^T = -\eta K + (\eta^{1/2}m)(\eta^{1/2}m^T) = -\eta K + \tilde{m}\tilde{m}^T.
$$

Now the untruncated system may be rewritten as

$$
\tilde{A} = \begin{pmatrix}
K & M \\
M & -\eta K
\end{pmatrix} + \tilde{m}\tilde{m}^T = A + \tilde{m}\tilde{m}^T
$$

(12)

where $\tilde{m} = [0, \tilde{m}^T]$ is a rank one term with proper extension by zero. Now we are in a position to use Sherman-Woodbury inversion for matrix plus rank-one term. In the following, we shall concern ourselves in developing efficient preconditioners to solve with $\tilde{A}$ in (12).

4.3 Computing step length $\rho^i_{i,k}$

The step length $\rho^i_{i,k}$ is computed using a bisection method. We refer the interested reader to [3][p. 88].
Algorithm 1: Monotone Multigrid (MMG) V cycle

Require: Let $V_1 \subset V_2 \subset \cdots \subset V_m$ and let $r_m, b_m \in V_m$.

Require: $u^i, i > 0$ solution from previous cycle or $u^0$ the initial solution

1: Compute residual: $r_m = b_m - A_m u^i$
2: Compute defect obstacles:

\[
\begin{aligned}
\tilde{\delta}_m &= \psi - u^i \\
\tilde{\delta}_m &= \psi - u^i
\end{aligned}
\]

3: for $\ell = m, \cdots, 2$ do
4: Projected Gauss-Seidel Solve

\[
(D_\ell + L_\ell + \partial I_{K^\ell}) v_\ell = r_\ell
\]

where,

\[
K^\ell = \{ v \in \mathbb{R}^{n_\ell} \mid \tilde{\delta}_\ell \leq v \leq \tilde{\delta}_\ell \}
\]

using Algorithm 2
5: Update

\[
\begin{aligned}
r_\ell &:= r_\ell - A_\ell v_\ell \\
\tilde{\delta}_{\ell - 1} &:= \tilde{\delta}_\ell - v_\ell \\
\tilde{\delta}_{\ell - 1} &:= \tilde{\delta}_\ell - v_\ell
\end{aligned}
\]

6: Restrict and compute new obstacle

\[
\begin{aligned}
r_{\ell - 1} &= P_{\ell - 1}^T r_\ell \\
(\tilde{\delta}_{\ell - 1})_i &:= \max \{ (\tilde{\delta}_{\ell - 1})_j \mid (P_{\ell - 1})_{ji} \neq 0 \}, i = 1, \ldots, n_{\ell - 1} \\
(\tilde{\delta}_{\ell - 1})_i &:= \min \{ (\tilde{\delta}_{\ell - 1})_j \mid (P_{\ell - 1})_{ji} \neq 0 \}, i = 1, \ldots, n_{\ell - 1}
\end{aligned}
\]

7: end for
8: Solve

\[
(D_1 + L_1 + \partial I_{D_1}) v_1 = r_1
\]

9: for $\ell = 2, \cdots, m$ do
10: Add corrections

\[
v_\ell := v_\ell + P_{\ell - 1} v_{\ell - 1}
\]

11: end for
12: Compute

\[
u^{i+1} = u^i + v_m
\]

Ensure: improved solution $u^{i+1}$
Algorithm 2: $x^{i+1} \leftarrow \text{PGS}(x^i, A, \psi, \bar{\psi}, b)$

**Require:** $A \in \mathbb{R}^{n\ell \times n\ell}$, $b, \psi, \bar{\psi} \in \mathbb{R}^{n\ell}$, current iterate $x^i \in \mathbb{R}^{n\ell}$

**Ensure:** new iterate $x^{i+1} \in \mathbb{R}^{n\ell}$

1: Compute residual:

\[ r := b - Ax^i \]

2: Compute defect obstacles:

\[ \psi := \psi - x^i \]

\[ \bar{\psi} := \bar{\psi} - x^i \]

3: for $i = 1 : n\ell$ do

4: for $j = 1 : i$ do

5: Compute $y_i$

\[ y_i = \begin{cases} \max \left( \min \left( \frac{(r_i - A_{ij}y_j) / A_{ii}}{\bar{\psi}_i}, \psi_i \right), \psi_i \right), & \text{if } A_{ii} \neq 0, \\
0, & \text{otherwise} \end{cases} \]

6: end for

7: end for

8: $x^{i+1} = x^i + y$

4.4 Mixed Finite Element Formulation of Reduced Linear System

In section 3.2, we already discussed the finite element discretization of the reduced linear system. To fit our problem into the Zulehner’s approach, we rewrite the PDE corresponding to the reduced linear system, the corresponding continuous weak form in product space using mixed bilinear forms. For corresponding weak formulations of (4.2). We first write the corresponding partial differential equations as follows:

\[ \begin{align*}
\epsilon \Delta u + \lambda &= f \quad \text{in } \Omega_I \subset \Omega, \\
u - \tau \Delta \lambda &= g \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \frac{\partial \lambda}{\partial n} = 0 \quad \text{on } \partial \Omega, \\
u &= 0 \quad \text{on } \partial \Omega_I \setminus \partial \Omega. 
\end{align*} \]  

To this end, we choose suitable Hilbert spaces for trial (i.e. weak solution) and test spaces as follows

\[ \hat{V} = \{ v \in H^1(\Omega) : v|_{\Omega_A} = 0 \}, \quad Q = H^1_0(\Omega), \]

where $\Omega_A = \Omega \setminus \Omega_I$. The weak form of the partial differential equations (4.4) corresponding to (4.2) is written as follows:
Find \((u, \lambda) \in \hat{V} \times H^1(\Omega)\):

\[
\hat{a}(u, v) + \hat{b}(v, \lambda) = f(v) \quad \text{for all } v \in \hat{V},
\]

\[
\hat{b}(u, q) - c(\lambda, q) = g(q) \quad \text{for all } q \in Q,
\]

where

\[
\hat{a}(u, v) = \epsilon \left( \langle \nabla u, \nabla v \rangle + \int_{\Omega} u \int_{\Omega} v \right) = \epsilon \left( \langle \nabla u, \nabla v \rangle + \left\langle u, 1 \right\rangle \left\langle v, 1 \right\rangle \right),
\]

\[
\hat{b}(v, \lambda) = \left\langle v, \lambda \right\rangle,
\]

\[
c(\lambda, q) = \tau(\nabla \lambda, \nabla q).
\]

We immediately observe the following trivial properties for the system (4.1)-(4.4).

**Theorem 4.1** (Properties of bilinear forms). There holds

1. \(\hat{a}(. , . )\) is symmetric and coercive
2. \(c(., . )\) is symmetric and semi-coercive
3. \(\hat{a}(., . ), \hat{b}(., . ), \text{ and } c(., . )\) are bounded

**Proof.** 1 and 2 follows from Poincaré inequality. Boundedness of \(\hat{b}(., . )\) follows from Cauchy-Schwarz inequality, and boundedness of \(\hat{a}(., . )\) and \(c(., . )\) follows from Cauchy-Schwarz inequality followed by inverse inequality.

The mixed variational problem above can also be written as a variational form on product spaces:

Find \(x \in \hat{V} \times Q\):

\[
\mathcal{B}(x, y) = \mathcal{F}(y) \quad \text{for all } y \in V \times H^1(\Omega)
\]

where \(\mathcal{B}\) and \(\mathcal{F}\) are defined as follows

\[
\mathcal{B}(z, y) = \hat{a}(w, v) + \hat{b}(v, r) + \hat{b}(w, q) - c(r, q), \quad \mathcal{F}(y) = f(v) + g(q)
\]

for \(y = (v, q) \in \hat{V} \times Q\) and \(z = (w, r) \in \hat{V} \times Q\). The corresponding bilinear form for the untruncated system is given as follows

\[
\mathcal{B}(z, y) = a(w, v) + b(v, r) + b(w, q) - c(r, q), \quad \mathcal{F}(y) = f(v) + g(q)
\]

for \(y = (v, q) \in V \times Q\) and \(z = (w, r) \in V \times Q\), where \(V = H^1(\Omega)\). In the rest of this paper, we shall consider norms also proposed in [20] as follows

\[
((v, q), (w, r))_X = (v, w)_V + (q, r)_Q,
\]

where \((., .)_V\) and \((., .)_Q\) are inner products of Hilbert spaces \(V\) and \(Q\), respectively. We will see shortly that such norms lead to block diagonal preconditioners. The boundedness condition for the mixed problem for truncated and untruncated problem is given as follows

\[
\sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{\mathcal{B}(z, y)}{\|z\|_X \|y\|_X} \leq \sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|z\|_X \|y\|_X} \leq \bar{c}_x < \infty.
\]
However, for well-posedness, following well known Babuska-Brezzi condition needs to be satisfied

\[
\inf_{0 \neq z \in \hat{X}} \sup_{0 \neq y \in X} \frac{\hat{B}(z, y)}{\|y\|_{\hat{X}}\|y\|_{X}} \geq \inf_{0 \neq z \in \hat{X}} \sup_{0 \neq y \in X} \frac{B(z, y)}{\|y\|_{\hat{X}}\|y\|_{X}} \geq c_x > 0. \tag{19}
\]

Here (4.4) and (4.4) are consequence of the fact that $\hat{X} \subset X$ as $\hat{V} \subset V$. We shall provide equivalent conditions for (4.4) and (4.4) that are easier to check, but more importantly it leads to optimal norms. But first we need to introduce some notations for operators corresponding to bilinear forms. It is easy to see that $\hat{V} \times H^1(\Omega)$ is a Hilbert space itself as $\hat{V}$ and $H^1(\Omega)$ are themselves Hilbert spaces. It is convenient to associate linear operators for the bilinear forms $a, b,$ and $c$ as follows

\[
\langle \hat{A}w, v \rangle = \hat{a}(w, v), \quad \hat{A} \in L(\hat{V}, \hat{V}^*),
\]

\[
\langle \hat{B}w, q \rangle = \hat{b}(w, q), \quad \hat{B} \in L(\hat{V}, Q^*),
\]

\[
\langle Cr, q \rangle = c(w, v), \quad C \in L(Q, Q^*),
\]

\[
\langle \hat{B}^*r, v \rangle = \langle \hat{B}v, r \rangle, \quad \hat{B}^* \in L(Q, \hat{V}^*).
\]

Consequently $B$ and $F$ in operator notation are given as follows

\[
\hat{A} = \begin{pmatrix} \hat{A} & \hat{B}^* \\ \hat{B} & -C \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad x = \begin{pmatrix} u \\ p \end{pmatrix}.
\]

The problem (4.4) is now given in operator notation as follows

\[
\hat{A}x = \hat{F}. \tag{20}
\]

The corresponding untruncated problem is given as follows

\[
Ax = F,
\]

where $A$ consists of untruncated matrix $A$ in place of $\hat{A}$ and $B$ instead of $\hat{B}$.

In [20], starting from the abstract theory on Hilbert spaces that lead to representation of isometries, a preconditioner is proposed; it is based on non-standard norms or isometries that correspond to block diagonal preconditioner of the following form

\[
B = \begin{pmatrix} I_V & 0 \\ 0 & I_Q \end{pmatrix}.
\]

### 4.5 Choice of norm: a brief introduction to Zulehner’s idea

In the discrete case, we unavoidably introduce an additional parameter, i.e, the mesh size $h$, in addition to the problem parameters $\tau$ and $\epsilon$. Our goal is look for norms that are independent of all these parameters. The content of this section and the notations are inspired from [20].

Before we move further, we introduce some notations. The duality pairing $\langle \cdot, \cdot \rangle_H$ on $H^* \times H$ is defined as follows

\[
\langle \ell, x \rangle_H = \ell(x) \quad \text{for all } \ell \in H^*, \ x \in H.
\]
Let $I_H : H \to H^*$ be an isometric isomorphism defined as follows

$$\langle I_H x, y \rangle = (x, y)_H.$$ 

The inverse $R_H = I^{-1}$ is Riesz-isomorphism, by which functionals in $H^*$ can be identified with elements in $H$ and we have

$$\langle \ell, x \rangle = (R_H \ell, x)_H.$$ 

We already chose the norm (4.4), we now look for explicit representation of isometries or norms in finite dimension. For this norm, we briefly describe how the norms are derived. The main ingredient is the following theorem.

**Theorem 4.2** (Zulehner 2011 [20]). If there are constants $\gamma_v, \overline{\gamma}_v, \gamma_q, \overline{\gamma}_q > 0$ such that

$$\gamma_v \|I_V w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \overline{\gamma}_v \|w\|_V^2, \quad \forall w \in V$$

and

$$\gamma_q \|I_Q r\|_{Q^*}^2 \leq c(r, r) + \|B^* r\|_{V^*}^2 \leq \overline{\gamma}_q \|r\|_{Q^*}^2, \quad \forall r \in Q$$

then

$$c_x \|z\|_X \leq \|Az\|_{X^*} \leq \overline{c}_x \|z\|_X, \quad \forall z \in X$$

is satisfied with constants $c_x, \overline{c}_x > 0$ that depend only on $\gamma_v, \overline{\gamma}_v, \gamma_q, \overline{\gamma}_q$. And, vice versa, if the estimates (4.2) are satisfied with constants $c_x, \overline{c}_x > 0$, then the estimates (4.2) and (4.2) are satisfied.

In view of (4.4) and (4.4) and recalling $\hat{X} \subset X$, the following bounds hold for truncated system

$$\|\hat{A}z\|_{\hat{X}^*} \geq \|Az\|_{X^*}, \quad \|\hat{A}z\|_{\hat{X}^*} \leq \|Az\|_{X^*}, \quad \forall z \in \hat{X} \subset X, z \in X.$$

In [20], the terms $\|Bw\|_{Q^*}^2$ and $\|B^* r\|_{V^*}^2$ in (4.2) and (4.2) respectively are defined as follows:

$$\|Bw\|_{Q^*}^2 = \langle B^* I_Q^{-1} Bw, w \rangle, \quad \|B^* r\|_{V^*}^2 = \langle B I_V^{-1} B^* r, r \rangle.$$ 

Then (4.2) and (4.2) are equivalently written as follows

$$\gamma_v (I_V w, w) \leq \langle (A + B^* I_Q^{-1} B) w, w \rangle \leq \gamma_v (I_V w, w) \quad \text{for all } w \in V,$$

$$\gamma_q (I_Q r, r) \leq \langle (C + B I_V^{-1} B^*) r, r \rangle \leq \gamma_q (I_Q r, r) \quad \text{for all } r \in Q.$$ 

In short, in new notation $\sim$ meaning “spectrally similar”, we obtain the following equivalent conditions for isometries

$I_V \sim A + B^* I_Q^{-1} B$ and $I_Q \sim C + B I_V^{-1} B^*$

$\iff$ $I_V \sim A + B^* (C + B I_V^{-1} B^*)^{-1} B$ and $I_Q \sim C + B I_V^{-1} B^*$

$\iff$ $I_Q \sim C + B(A + B^* I_Q^{-1} B)^{-1} B$ and $I_V \sim A + B^* I_Q^{-1} B$
Let $M$ and $N$ be any SPD matrices, consequently, they define inner products and a Hilbert space structure in $\mathbb{R}^n$. The intermediate Hilbert spaces between $M$ and $N$ are given as follows

$$[M, N]_\theta = M^{1/2}(M^{-1/2}NM^{-1/2})^\theta M^{1/2}, \quad \theta \in [0, 1].$$

Continuing from above, in the case when $A$ and $C$ are non singular, the more generic form of the norms are given by the following lemma

\textbf{Lemma 4.1.} Let $A, C$ be nonsingular. Then

$$I_V = A + [A, B^T C^{-1} B]_\theta, \quad I_Q = C + [C, B A^{-1} B^T]_\theta, \quad \theta \in [0, 1].$$

\textit{Proof.} See [20][p. 547-548].

Before we propose preconditioners, we shall need some properties of the $(1,1)$ block of $A$, and that for the negative Schur complement $S = C + \hat{B} \hat{A}^{-1} \hat{B}^T$ in (4.4). These properties are used to prove some bounds and to suggest approximation of norms.

\subsection*{4.5.1 Properties of truncated $(1,1)$ block and Schur complement}

An important property that we shall need shortly when analyzing preconditioners is that the eigenvalues of the truncated matrix is bounded from above and below by the eigenvalues of the untruncated matrix. In this subsection, we also assume that the grid is uniform.

A result that we need later is the following.

\textbf{Lemma 4.2.} $A, A + B$ is SPD.

\textit{Proof.} We have $A = \epsilon (K + mm^T), \epsilon > 0$. $A$ is SPSP except on the vector $1 = [1, 1, 1, \ldots, 1]^T$ which is in the kernel of $A$ but $(mm^T1, 1) > 0$. Also, since $B = M$ is SPD the proof follows.

\textbf{Lemma 4.3} (Permutation preserves eigenvalues). Let $P \in \mathbb{Z}^{n \times n}$ be a permutation matrix, then $P^T A P$ and $\hat{A}$ are similar.

\textit{Proof.} $P$ being a permutation matrix, $P^T P = Id$, hence the proof.

\textbf{Lemma 4.4} (Poincare separation theorem for eigenvalues). Let $Z \in \mathbb{R}^{n \times n}$ be any symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and let $P$ be a semi-orthogonal $n \times k$ matrix such that $P^T P = Id \in \mathbb{R}^{k \times k}$. Then the eigenvalues $\mu_1 \leq \mu_2 \cdots \mu_{n-k+i}$ of $P^T Z P$ are separated by the eigenvalues of $Z$ as follows

$$\lambda_i \leq \mu_i \leq \lambda_{n-k+i}.$$ 

\textit{Proof.} The theorem is proved in [18] p. 337.

\textbf{Lemma 4.5} (Eigenvalues of the truncated matrix). Let $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ be the eigenvalues of $A$, and let $\hat{\lambda}_1 \leq \hat{\lambda}_2 \cdots \leq \hat{\lambda}_n$ be the eigenvalues of truncated matrix $\hat{A}$. Let $k = \sum_{i=1}^n T(i,i)$ be the number of untruncated rows in $A$. Let $\hat{\lambda}_{n_1} \leq \hat{\lambda}_{n_2} \cdots \hat{\lambda}_{n_k}$ be the eigenvalues of $\hat{A}$ due to addition of $T$. Then the following holds

$$\lambda_i \leq \hat{\lambda}_{n_i} \leq \lambda_{n-k+i}.$$
Proof. The proof shall follow by application of Poincare separation theorem, to this end, we need to reformulate our problem. Let $P$ be a permutation matrix that renumbers the rows such that the truncated rows are numbered first, then we have

$$P^T \hat{A} P = \begin{pmatrix} I & R^T P^T \hat{A} P R \end{pmatrix},$$

where $R \in \mathbb{R}^{n \times k}$ is the restriction operator defined as follows

$$R = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}_{n-k \times k} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}_{k \times k}.$$

Clearly $R^T R = Id \in \mathbb{R}^{k \times k}$. From Lemma 4.3 $P^T \hat{A} P$ and $\hat{A}$ are similar. From Lemma 4.4 theorem follows.

**Corollary 4.1.** From Theorem 4.5 we have

$$\lambda_{\min}(\hat{A}) \geq \lambda_{\min}(A) > 0,$$

$$\lambda_{\max}(\hat{A}) \leq \lambda_{\max}(A),$$

hence $\hat{A}$ is SPD. Moreover, $\text{cond}(\hat{A}) \leq \text{cond}(A)$.

We know that the matrix $M$ is SPD, and $K$ is a SPSD. In the following we observe the properties of truncated matrices obtained from these.

**Definition 4.1.** Let $G(A) = (V, E)$ be the adjacency graph of a matrix $A \in \mathbb{R}^{N \times N}$. The matrix $A$ is called irreducible if any vertex $i \in V$ is connected to any vertex $j \in V$. Otherwise, $A$ is called reducible.

**Definition 4.2.** A matrix $A \in \mathbb{R}^{N \times N}$ is called an $M$–matrix if it satisfies the following three properties: $a_{ii} > 0$ for $i = 1, \ldots, N$, $a_{ij} \leq 0$ for $i \neq j, i, j = 1, \ldots, N$, and $A$ is non-singular and $A^{-1} \geq 0$.

**Definition 4.3.** A square matrix $A$ is strictly diagonally dominant if the following holds

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, i = 1, \ldots, N, \tag{24}$$

and it is called irreducibly diagonally dominant if $A$ is irreducible and the following holds

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, i = 1, \ldots, N, \tag{25}$$

where strict inequality holds for at least one $i$. 

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A simpler criteria for $M$–matrix property is given by the following theorem.

**Lemma 4.6.** If the coefficient matrix $A$ is strictly or irreducibly diagonally dominant and satisfies the following conditions

1. $a_{ii} > 0$ for $i = 1, \ldots, N$
2. $a_{ij} \leq 0$ for $i \neq j, i, j = 1, \ldots, N$

then $A$ is an $M$–matrix.

**Remark 4.1.** Note that $K$ is not an $M$–matrix because $K \cdot 1 = 0$, hence, (4.3) is not satisfied. Moreover, mass matrix $M$ has positive off-diagonal entries, hence, it is not an $M$–matrix either. Alternatively, item 3. of Definition 4.2 is not satisfied.

Let $N_h^\bullet = \{i : T(i, i) = 0\}$.

**Lemma 4.7.** Let $|N_h^\bullet| \geq 1$, then $\hat{K}$, $P^T \hat{K}P$, and $R^T P^T \hat{K}P R$ are $M$–matrices.

**Proof.** Since we have $|N_h^\bullet| \geq 1$, for all rows in truncated set $N_h^\bullet$, we have strict diagonal dominance

$$\hat{k}_{ii} = 1 = |\hat{k}_{ii}| = 0 > \sum_{j \neq i} \hat{k}_{ij}, \quad \forall i \in N_h^\bullet, \ j = 1, \ldots, |N_h|,$$

(26)

where as, for rows corresponding to untruncated set $N_h \setminus N_h^\bullet$,

$$\hat{k}_{ii} = k_{ii} = |\hat{k}_{ii}| \geq \sum_{j \neq i} k_{ij} \geq \sum_{j \neq i} \hat{k}_{ij}, \quad \forall i \in N_h \setminus N_h^\bullet, \ j = 1, \ldots, |N_h|.$$

(27)

Moreover, we have

$$\hat{k}_{ij} = \begin{cases} 1, & \forall i \in N_h^\bullet, \\ k_{ii} > 0, & \forall i \in N_h \setminus N_h^\bullet, \\ 0, & \forall i \in N_h. \end{cases}$$

(28)

The sufficient conditions of Lemma 4.6 are now satisfied: from (4.5.1) and (4.5.1), we conclude that $\hat{K}$ is irreducibly diagonally dominant, and (4.5.1) satisfies items 1. and 2. of Lemma 4.6. Hence $\hat{K}$ is an $M$–matrix. $P^T \hat{K}P$ being the symmetric permutation of rows and columns of $\hat{K}$ above reasoning holds. Lastly, $R^T P^T \hat{K}P R$ being a principle submatrix of $P^T \hat{K}P$ is also an $M$–matrix, see proof in [11][p. 114].

**Remark 4.2.** From (3.2) and (4.2), the $(1,1)$ block $\hat{A} = \epsilon(\hat{K} + \bar{m}\bar{m}^T)$, where $\hat{K} = TKT + \hat{T}, \bar{m} = Tm$. To solve with $\hat{A}$, we use the Sherman–Woodbury formula

$$\hat{A}^+ = (\hat{K} + \bar{m}\bar{m}^T)^+ = \hat{K}^+ - \frac{\hat{K}^+ \bar{m}\bar{m}^T \hat{K}^+}{1 + \bar{m}^T \hat{K}^+ \bar{m}}.$$

Here $\hat{K}^+$ denotes pseudo-inverse of $\hat{K}$, however, in our case $\hat{K}$ is a non-singular $M$–matrix, see Definition 4.2. Since $\hat{K}^+$ is an $M$–matrix algebraic multigrid or incomplete Cholesky (which is as stable as exact Cholesky factorization, [12][Theorem 3.2] ) may be used as a preconditioner to solve with $\hat{K}$ inexactly.
We provide a slightly different proof then in [9].

**Theorem 4.3.** The negative Schur complement $S = C + \hat{B} \hat{A}^{-1} \hat{B}^T$ is non-singular, in particular, SPD if and only if $|N_h^{\star}| > 0$.

**Proof.** If $|N_h^{\star}| = 0$, then $\hat{B}$ is zero matrix, consequently $S = C$ is singular. For other implication, we recall that $\hat{B}^T = \hat{M}^T = -TM$ where $T$ is defined in (4.2). The $(i,j)^{th}$ entry of element mass matrix is given as follows

$$M^K_{ij} = \int_K \phi_i \phi_j dx = \frac{1}{12} (1 + \delta_{ij}|K|), \quad i, j = 1, 2, 3,$$

(29)

where $\delta_{ij}$ is the Kronecker symbol, that is, 1 if $i = j$, and 0 if $i \neq j$. Here $\phi_1, \phi_2$, and $\phi_3$ are hat functions on triangular element $K$ with local numbering and $|K|$ is the area of triangle element $K$. From (4.5.1), it is easy to see that

$$M^K = \frac{1}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

(30)

Evidently, entries of global mass matrix $M = \sum_K M^K$ are also all positive, hence all entries of truncated mass matrix $\hat{M}$ remain non-negative. In particular, due to our hypothesis $|N_h^{\star}| > 0$, there are at least one untruncated column, hence, at least few positive entries. Consequently, $1$ is neither in kernel of $M$ nor in the kernel of $\hat{M}$, in particular, $1^T \hat{M}^T 1 > 0$. The proof of the theorem then follows since $C$ is SPD except on 1 for which $\hat{B}^T 1$ is non-zero, and the fact that $\hat{A}$ is SPD yields

$$\langle \hat{B} \hat{A}^{-1} \hat{B}^T 1, 1 \rangle = \langle \hat{A}^{-1}(\hat{B}^T 1), (\hat{B}^T 1) \rangle = \langle \hat{A}^{-1}(-\hat{M}^T 1), (-\hat{M}^T 1) \rangle > 0.$$

4.5.2 Preconditioner I:

$$\hat{B}^T = B, \quad C = \eta A, \eta \neq 0.$$

(31)

Moreover $A$ hence $C$ are non-singular. Specifically for $\theta = 1/2$, and using (4.5.2) and Lemma 4.1 above yields

$$I_V = A + \eta^{-1/2} [A, BA^{-1}B]_{1/2}, \quad I_Q = C + \eta^{1/2} [A, BA^{-1}B]_{1/2}.$$

But $[A, BA^{-1}B]_{1/2} = B$, thus further simplification yields

$$I_V = A + \eta^{-1/2} B, \quad I_Q = C + \eta^{1/2} B.$$

(32)

Choice of $\theta = 0, 1$ brings back Schur Complements. For large problems, it won’t be feasible to solve with $I_V$ and $I_Q$ in (4.5.2) exactly, or not even up to double precision using prohibitively expensive direct methods such as QR or LU factorizations [7].
Remark 4.3. For existence and subsequent application of fast inexact solvers for $\mathcal{I}_V$ and $\mathcal{I}_Q$, an important property to look for is $M$-matrix property, but unfortunately this property is lost in \[\text{(4.5.2)}, \text{consequently, the diagonal dominance of } \mathcal{I}_V \text{ or } \mathcal{I}_Q \text{ may be lost for certain values of } \eta. \] To sketch the proof for $I_Q$, we observe that

$$A^K_{ij} = \left( \int_K \nabla \phi_i \cdot \nabla \phi_j dx + \int_K \phi_i dx \int_K \phi_j dx \right), \quad i, j = 1, 2, 3,$$

$$= (b_ib_j + c_ic_j) \int_K dx + mm^T = (b_ib_j + c_ic_j)[K] + mm^T, \quad i, j = 1, 2, 3.$$

Reusing the definition of element mass matrix in \[\text{(4.5.1)}\], we have

$$\eta A^K_{ij} + \eta^{1/2} B^K_{ij} = \eta A^K_{ij} + \eta^{1/2} M^K_{ij} = \eta(b_ib_j + c_ic_j)[K] + \eta^{1/2} \frac{1}{12} (1 + \delta_{ij}|K|) + mm^T.$$

It is not hard to see that for certain values of $\eta$, diagonal dominance property \[\text{(4.3)}\] is lost. Similarly, diagonal property is lost for $I_Q$. To retain the $M$-matrix property, it is advisable to lump the mass matrix. We proved earlier that the truncated matrix $\hat{K}$ is $M$-matrix if there is at least one truncated node, in that case $I_Q$ can be made $M$-matrix.

The following remark relates the eigenvalues of $A$ to the eigenvalue of $A$ by a change of variable

Remark 4.4. To this end we rewrite the system as follows

$$\begin{pmatrix} \eta^{1/2} \hat{A} & \eta^{1/2} B^T \\ \hat{B} & (\eta^{1/2})\hat{A} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{A} \end{pmatrix}. \quad (33)$$

Let $(\lambda, u)$ be an eigenpair of $\hat{A} + \hat{B}$, then $(u^T, u^T)^T$ is an eigenvector of \[\text{(4.4)}\]. Similarly, let $(\mu, v)$ be an eigenpair of $\hat{A} - \hat{B}$, then $(v^T, -v^T)^T$ is also an eigenvector of \[\text{(4.4)}\]. This implies that eigenvalues of \[\text{(4.4)}\] are union of eigenvalues of $\hat{A} + \hat{B}$ and $\hat{A} - \hat{B}$. We notice that the eigenvalues come in pairs with real part of eigenvalues in each pair having opposite sign.

The following theorem estimates the spectral radius of the preconditioned operator.

Lemma 4.8. The spectral radius of the preconditioned operator is given as follows

$$\rho\left( \begin{bmatrix} \hat{K} + \eta^{-1/2} M & 0 \\ 0 & \eta\hat{K} + \eta^{1/2} M \end{bmatrix}^{-1} \begin{bmatrix} \hat{K} & M \\ M & -\eta\hat{K} \end{bmatrix} \right) \leq \frac{1}{(1 + \eta)} + \frac{Ch^4}{(1 + \eta)}.$$ 

Proof. Consider the generalized eigenvalue problem

$$\begin{bmatrix} \hat{K} & M \\ M & -\eta\hat{K} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \lambda \begin{bmatrix} \hat{K} + \eta^{-1/2} M & 0 \\ 0 & \eta\hat{K} + \eta^{1/2} M \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}.$$ 

Taking inner product on both sides with $[v^T, u^T]$, we have

$$[v^T, u^T] \begin{bmatrix} \hat{K} & M \\ M & -\eta\hat{K} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \lambda [v^T, u^T] \begin{bmatrix} \hat{K} + \eta^{-1/2} M & 0 \\ 0 & \eta\hat{K} + \eta^{1/2} M \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}.$$ 

$$v^T \hat{K} v + v^T M u + u^T M v - \eta v^T \hat{K} u = \lambda [v^T, u^T] \begin{bmatrix} \hat{K} + \eta^{-1/2} M v \\ M v - \eta\hat{K} u \end{bmatrix}.$$ 

$$v^T \hat{K} v + 2Re(M u, v) - \eta v^T \hat{K} u = \lambda [v^T, u^T] \begin{bmatrix} \hat{K} + \eta^{-1/2} v^T M v + \eta u^T \hat{K} u + \eta^{1/2} u^T M u \\ \eta K u + \eta^{1/2} u^T M u \end{bmatrix}.$$ 

$$v^T \hat{K} v + \eta u^T \hat{K} u = \lambda [v^T, u^T] \begin{bmatrix} \hat{K} + \eta^{-1/2} v^T M v + \eta u^T \hat{K} u + \eta^{1/2} u^T M u \\ \eta K u + \eta^{1/2} u^T M u \end{bmatrix}.$$
Taking absolute value, we get

\[ |v^T \bar{K} v + 2 \text{Re}(Mu, v) + \eta u^T \bar{K} u| = |\lambda \left[ v^T \bar{K} v + \eta^{-1/2} v^T M v + \eta u^T \bar{K} u + \eta^{1/2} u^T M u \right]| \]

\[ \|v\|^2_K + \eta \|u\|^2_K + 2 |\text{Re}(Mu, v)| = |\lambda| \left[ \|v\|^2_K + \eta^{-1/2} \|v\|^2_M + \eta \|u\|^2_K + \eta^{1/2} \|u\|^2_M \right] \]

\[ |\lambda| = \frac{\|v\|^2_K + \eta \|u\|^2_K + 2 |\text{Re}(Mu, v)|}{\|v\|^2_K + \eta^{-1/2} \|v\|^2_M + \eta \|u\|^2_K + \eta^{1/2} \|u\|^2_M} \]

We note that \( M \) being mass matrix \( \|M\| \leq C h^2 \) for any finite dimensional norm. We now estimate \( |\text{Re}(Mu, v)| \)

\[ |(Mu, v)| \leq C_1 h^2 \|v\|_M \|u\|_M \] (Cauchy-Schwarz ineq.)

\[ \leq C_1 h^2 \|v\|_K \|u\|_M \] (Equivalence of norms in finite dimension)

\[ \leq C_1 C_2 h^2 \left( \frac{\|v\|^2_K + \|u\|^2_M}{2} \right) \] (Young’s inequality)

Thus we have

\[ 2 |(Mu, v)| \leq \frac{C h^4}{2} \left( \|v\|^2_K + \|u\|^2_M \right) \] (set \( C = C_1 C_2 C_3 \))

With this, the estimate for \( |\lambda| \) becomes

\[ |\lambda| \leq \frac{\|v\|^2_K + \eta \|u\|^2_K + Ch^4 (\|v\|^2_K + \|u\|^2_M)}{\|v\|^2_K + \eta^{-1/2} \|v\|^2_M + \eta \|u\|^2_K + \eta^{1/2} \|u\|^2_M} \]

\[ \leq \frac{\|v\|^2_K + \eta \|u\|^2_K + \eta^{-1/2} \|v\|^2_M + \|u\|^2_K + \eta^{1/2} \|u\|^2_M}{\|v\|^2_K + \eta^{-1/2} \|v\|^2_M + \|u\|^2_K + \eta^{1/2} \|u\|^2_M} + \frac{Ch^4}{\|v\|^2_K + \|u\|^2_M} \]

\[ = \frac{1}{\|v\|^2_K + \eta^{-1/2} \|v\|^2_M + \eta \|u\|^2_K + \eta^{1/2} \|u\|^2_M} + \frac{Ch^4}{\|v\|^2_K + \|u\|^2_M} \]

\[ \leq \frac{1}{1 + \eta \left( \frac{\|v\|^2_M + \eta \|u\|^2_M}{\|v\|^2_K + \eta \|u\|^2_K} \right) + \frac{Ch^4}{\|v\|^2_K + \|u\|^2_M}} \]

\[ \leq \frac{1}{1 + \eta \left( \frac{\|v\|^2_M + \eta \|u\|^2_M}{\|v\|^2_K + \eta \|u\|^2_K} \right) + \frac{Ch^4}{1 + \left( \frac{\eta \|u\|^2_K + \eta^{1/2} \|u\|^2_M}{\|v\|^2_K + \|u\|^2_M} \right)}} \] (since \( \eta^{-1/2} > 1 \))

\[ \leq \frac{1}{1 + \eta \left( \frac{\|v\|^2_M + \eta \|u\|^2_M}{\|v\|^2_K + \eta \|u\|^2_K} \right) + \eta \left( \frac{\|u\|^2_K + \|u\|^2_M}{\|v\|^2_K + \|u\|^2_M} \right)} \] (since \( \eta = \min(\eta, \eta^{1/2}) \))

Using equivalence of norms in finite dimension

\[ \|u\|_M \leq C_4 \|u\|_K, \quad \|v\|_M \leq C_5 \|v\|_K \]
we have

$$|\lambda|^2 \leq \frac{1}{(1 + \eta C_6)} + \frac{Ch^4}{(1 + \eta)}$$

The following Lemma shows that condition number is of the order one.

**Theorem 4.4.** The asymptotic condition number is given as follows

$$\kappa \left( \begin{bmatrix} \bar{K} + \eta^{-1/2}M & 0 \\ 0 & \eta \bar{K} + \eta^{1/2}M \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} \bar{K} & M \\ M & -\eta \bar{K} \end{bmatrix} \right) = O(1).$$

For sake of comparison, we also consider block triangular preconditioners of the form used in Bosch et. al. [4]. In the following, we briefly describe this preconditioner in our notation.

### 4.5.3 Preconditioner II

In Bosch et. al. [4], a preconditioner is proposed in the framework of a semi-smooth Newton method combined with Moreau-Yosida regularization for the same problem. However, the preconditioner was constructed for a linear system which is different from the one we considered here (4.2). The preconditioner proposed in [4] has the following block lower triangular form

$$B = \begin{pmatrix} \bar{K} & 0 \\ M & -S \end{pmatrix}, \quad (34)$$

where $S = C + M \bar{K}^{-1}M^T$ is the Schur complement. Note that such preconditioners are also called inexact or preconditioned Uzawa preconditioners for linear saddle point problems. Both $\bar{K}$ and $S$ are invertible [10]. Hence by block $2 \times 2$ inversion formula we have

$$B^{-1} = \begin{pmatrix} \bar{K} & 0 \\ M & -S \end{pmatrix}^{-1} = \begin{pmatrix} \bar{K}^{-1} & 0 \\ -S^{-1}M^T \bar{K}^{-1} & -S^{-1} \end{pmatrix}.$$  \( (34) \)

Let $\tilde{S}$ be any approximation of Schur complement $S$ in $B$ in (4.5.3), then the new preconditioner $\tilde{B}$, and the corresponding preconditioned operator $\tilde{B}^{-1}A$ is given as follows

$$\tilde{B} = \begin{pmatrix} \bar{K} & 0 \\ M & -\tilde{S} \end{pmatrix}, \quad \tilde{B}^{-1}A = \begin{pmatrix} I & \bar{K}^{-1}M^T \\ 0 & \tilde{S}^{-1} \end{pmatrix}. \quad (35)$$

In this paper we choose $\tilde{S}$ as follows

$$\tilde{S} = S_1 \bar{K}^{-1}S_2 = (M + \sqrt{\eta} \bar{K}) \bar{K}^{-1}(M + \sqrt{\eta} \bar{K})$$

We note the following trivial result.

**Fact 4.1.** Let $B$ be defined as in (4.5.3), then there are $|N_{h_1}|$ eigenvalues of $B^{-1}A$ equal to one, and the rest are the eigenvalues of the preconditioned Schur complement $\tilde{S}^{-1}S$.

**Remark 4.5.** When using GMRES [13], right preconditioning is preferred. Similar result as for the left preconditioner above Theorem 4.1 holds.
Let \( x = [x_1, x_2], b = [b_1, b_2] \). The preconditioned system \( B^{-1}Ax = B^{-1}b \) is given as follows

\[
\begin{pmatrix}
I & \hat{A}^{-1}\hat{B}^T \\
0 & \hat{S}^{-1}S
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
\hat{A}^{-1} & 0 \\
S^{-1}\hat{B}^T\hat{A}^{-1} & -S^{-1}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
\]

from which we obtain the following set of equations

\[
x_1 + \hat{A}^{-1}\hat{B}^T x_2 = \hat{A}^{-1}b_1, \quad \hat{S}^{-1}S x_2 = S^{-1}(\hat{B}^T\hat{A}^{-1}b_1 - b_2).
\]

**Algorithm 4.1.** *Objective:* Solve \( B^{-1}Ax = B^{-1}b \)

1. Solve for \( x_2 \): \( \hat{S}^{-1}S x_2 = \hat{S}^{-1}(\hat{B}^T\hat{A}^{-1}b_1 - b_2) \)
2. Set \( x_1 = \hat{A}^{-1}(b_1 - \hat{B}^T x_2) \)

Here if Krylov subspace method is used to solve for \( x_1 \), then matrix vector product with \( S \) and a solve with \( \hat{S} \) is needed. However, when the problem size, i.e., \( |N_h| \) is large, it won’t be feasible to do exact solve with \( \hat{A} \), and we need to solve it inexact, for example, using algebraic multigrid methods. In the later case, the decoupling of \( x_1 \) and \( x_2 \) as in Algorithm 4.1 is not possible; then the preconditioned Schur complement \( \hat{S}^{-1}S \) is not symmetric, so we use GMRES in Saad [19, p. 269] that allows nonsymmetric preconditioners.

## 5 Numerical Experiments

All the experiments were performed in double precision arithmetic in MATLAB. The Krylov solver used was GMRES with subspace dimension of 200, and maximum number of iterations allowed was 300. The iteration was stopped as soon as the relative residual was below the tolerance of \( 10^{-7} \).

### 5.1 Spectrum Analysis

We consider two samples of active set configurations that occur when a square region evolves as shown in figures (1(a) and (1(b)). The region between the two squares and the circles is the interface between two bulk phases taking values +1 and -1; initially we chose random values between -0.3 and 0.5 in the interface region. The width of the interface is kept to be 10 times the chosen mesh size. The time step \( \tau \) is chosen to be equal to \( \epsilon \). We compare various mesh sizes leading to number grid points up to just above 1 million, and compare various values of epsilon for each mesh sizes. We observe that the number of iterations remain independent of the mesh size, however it depends on \( \epsilon \). But we observe that for a fixed epsilon, with finer mesh, the number of iterations actually decrease significantly. For example the number of iterations for \( h = 2^{-7}, \epsilon = 10^{-6} \) is 84 but the number of iterations for \( h = 2^{-10}, \epsilon = 10^{-6} \) is 38, a reduction of 46 iterations! It seems that finer mesh size makes the preconditioner more efficient. We also observe that the time to solve is proportional to number of iterations; the inexact solve for the (1,1) block remains optimal because the (1,1) block is essentially Laplacian for which AMG remains very efficient.
Figure 1: Active set configurations: Square and Circle

| $h$   | $\epsilon$ | square time | circle time |
|-------|-------------|-------------|-------------|
| $2^{-9}$ | e-2        | 10 49.24    | 10 33.04    |
|       | e-3        | 11 54.25    | 10 36.63    |
|       | e-4        | 36 173.67   | 34 117.32   |
|       | e-5        | 101 418.05  | 89 334.28   |
| $2^{-10}$ | e-2       | 10 162.05   | 9  149.18   |
|        | e-3        | 12 175.00   | 10 167.47   |
|        | e-4        | 22 313.02   | 19 276.98   |
|        | e-5        | 73 1113.2   | 65 954.67   |

Table 1: Iteration count for various $\epsilon$ and $h$; Left: Preconditioner I; Right: Preconditioner II
6 Conclusion

For the solution of large scale optimization problem corresponding to Cahn-Hilliard problem with obstacle problem, we proposed an efficient preconditioning strategy that requires two elliptic solves. In our initial experiments up to over million unknowns, the preconditioner remains mesh independent. Although, for coarser mesh there seems to be strong dependence on the epsilon, but as the mesh becomes finer, we observe a significant reduction in iteration count, thus making the preconditioner effective and useful on finer meshes. It is likely that the the iteration count further decreases on finer meshes.

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