TROPICAL CONVEX HULL COMPUTATIONS

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Abstract. This is a survey on tropical polytopes from the combinatorial point of view and with a focus on algorithms. Tropical convexity is interesting because it relates a number of combinatorial concepts including ordinary convexity, monomial ideals, subdivisions of products of simplices, matroid theory, finite metric spaces, and the tropical Grassmannians. The relationship between these topics is explained via one running example throughout the whole paper. The final section explains how the new version 2.9.4 of the software system polymake can be used to compute with tropical polytopes.

1. Introduction

The study of tropical convexity, also known as “max-plus convexity”, has a long tradition going back at least to Vorobyev [38] and Zimmermann [41]; see also [9, 26, 8] and the references there. Develin and Sturmfels contributed an inherently combinatorial view on the subject, and they established the link to tropical geometry [12]. The subsequent development includes research on how tropical convexity parallels classical convexity [20, 1, 21, 14] and linear algebra [34] as well as investigations on the relationship to commutative algebra [5, 13]. Most recently, topics in algebraic and arithmetic geometry came into focus [25, 36, 22, 17]. The purpose of this paper is to list known algorithms in this area and to explain how the connections between the various topics work. We emphasize the aspects of geometric combinatorics.

Our version of the tropical semi-ring is \((\mathbb{R}, \min, +)\), and we usually write \(\oplus\) instead of “\(\min\)” and \(\odot\) instead of “\(+\)”. Of course, it is just a matter of taste if one prefers “\(\max\)” over “\(\min\)”.

Whenever convenient we will augment the semi-ring with the additively neutral element \(\infty\) which is absorbing with respect to \(\odot\). The tropical \(d\)-torus is the set \(T^{d-1} := \mathbb{R}^d/\mathbb{R}\mathbb{1}_d\), where \(\mathbb{1}_d := (1, 1, \ldots, 1)\) is the all-ones-vectors of length \(d\). Via the maximum-norm on \(\mathbb{R}^d\) and the quotient topology the tropical torus \(T^{d-1}\) carries a natural topology which is homeomorphic to \(\mathbb{R}^{d-1}\). Componentwise tropical addition and tropical scalar multiplication turn \(\mathbb{R}^d\) into a semi-module, and since tropical scalar multiplication with \(\lambda\) is the same as the ordinary addition of the vector \(\lambda \cdot \mathbb{1}_d\) tropical linear combinations of elements in \(T^{d-1}\) are well-defined. For a set \(V \subset T^{d-1}\) of generators the tropical convex hull is defined as

\[
\text{tconv} V := \{ \lambda \odot v \oplus \mu \odot w \mid \lambda, \mu \in \mathbb{R}, v, w \in V \}.
\]

If \(V\) is finite then \(\text{tconv} V\) is a tropical polytope, which can also be called a “min-plus cone”. There are several natural ways to represent a tropical polytope, for instance, as the tropical convex hull of points (as in the definition), as the union of ordinary polytopes [12], or as the intersection of tropical halfspaces [20]. We will discuss several “tropical convex hull algorithms”, that is, algorithms which translate one representation into the other. This turns out to be related to algorithms in ordinary convexity as well as to algorithms known from combinatorial optimization. Here we focus on the key geometric aspects.

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1
Combinatorics enters the stage through the observation [12, Theorem 1] that configurations of \(n\) points in \(\mathbb{T}^{d-1}\) or, equivalently, tropical polytopes with a fixed set of generators, are dual to regular subdivisions of the product of simplices \(\Delta_{n-1} \times \Delta_{d-1}\). Products of simplices in turn occur as the vertex figures of hypersimplices, and these hypersimplices are known to serve as adequate combinatorial models for the Grassmannians. In fact, it turns out that the tropical Grassmannians can be approximated in terms of decompositions of hypersimplices into matroid polytopes [24, 37, 36, 17]. We will show how a configuration of \(n\) points in \(\mathbb{T}^{d-1}\) can be lifted to a matroid decomposition of the hypersimplex \(\Delta(d,n + d)\).

The structure of this paper is as follows. The first section briefly collects some information about the tropical determinant and tropical hyperplanes before the next one introduces the basic combinatorial concepts of tropical convexity. This section also explains how ordinary convex hull algorithms can be used to compute with tropical polytopes. Then we discuss various versions of the tropical convex hull problem. In a section on matroid subdivisions we explore the role of tropical polytopes for the tropical Grassmannians and finite metric spaces. Finally, we show how the new version 2.9.4 of the software system polymake can be used for computing with tropical polytopes.

2. Tropical Determinants and Tropical Hyperplanes

Evaluating ordinary determinants is the key primitive operation of many algorithms in ordinary convexity. Its tropical counterpart is equally fundamental. The \textit{tropical determinant} of a matrix \(M = (m_{ij}) \in \mathbb{R}^{d \times d}\), also called the “min-plus permanent”, is defined by the tropicalized Leibniz formula

\[
\text{tdet} \ M := \bigoplus_{\sigma \in \text{Sym}_d} m_{1,\sigma(1)} \odot m_{2,\sigma(2)} \odot \cdots \odot m_{d,\sigma(d)},
\]

where \(\text{Sym}_d\) is the symmetric group acting on the set \([d] := \{1, 2, \ldots, d\}\). Evaluating \(\text{tdet}\) is the same as solving the linear assignment problem, or “weighted bipartite matching problem”, from combinatorial optimization for the weight matrix \(M\). This can be done in \(O(d^3)\) time [33, Corollary 17.4b].

By definition \(\text{det} \ M\) vanishes if the minimum in the defining Equation (1) is attained at least twice. In this case \(M\) is \textit{tropically singular}, and it is \textit{tropically regular} otherwise. Checking if the tropical determinant vanishes can be translated into solving \(d + 1\) assignment problems as follows: First we evaluate \(\text{tdet} \ M\) by solving one assignment problem. This way we find some optimal permutation \(\sigma\) with \(\text{tdet} \ M = m_{1,\sigma(1)} \odot m_{2,\sigma(2)} \odot \cdots \odot m_{d,\sigma(d)}\). The permutation \(\sigma\) is called a \textit{realizer} of \(\text{tdet} \ M\). We have to check if there are other realizers or not. To this end define \(d\) matrices \(M_1, M_2, \ldots, M_d\) all of which differ from \(M\) in only one coefficient, namely the coefficient in the \(i\)-th row and the \(\sigma(i)\)-th column of \(M_i\) is increased to \(m_{i,\sigma(i)} + 1\). We subsequently compute \(\text{tdet} \ M_1, \text{tdet} \ M_2, \ldots, \text{up to tdet} \ M_d\). Clearly \(\text{tdet} \ M \leq \text{tdet} \ M_i\) for all \(i\). Now \(M\) is tropically singular if there is another permutation \(\tau\) for which the minimum is also attained. As \(\sigma\) and \(\tau\) must differ in at least one place it follows that such a \(\tau\) exists if and only if \(\text{tdet} \ M = \text{tdet} \ M_i\) for some \(i\). We conclude that \(M\) is tropically regular if and only if \(\text{tdet} \ M < \text{tdet} \ M_i\) for all \(i\). Hence deciding if \(M\) is tropically singular or not requires \(O(d^4)\) time.

\textbf{Remark 1.} With the tropical determinant we can express that a square matrix in the tropical world should be considered as having full rank or not. Defining the rank of a general matrix in the tropical setting is much more subtle [11].

The tropical evaluation of a linear form \(a \in \mathbb{R}^d\) at a vector \(x \in \mathbb{T}^{d-1}\) reads \((a, x)_{\text{trop}} := a_1 \odot x_1 \odot a_2 \odot x_2 \odot \cdots \odot a_d \odot x_d\). Again this expression \textit{vanishes} if the minimum is attained
at least twice. A vector \( a \in \mathbb{R}^d \) defines a tropical hyperplane
\[
H(a) := \{ x \in \mathbb{T}^d \mid \langle a, x \rangle_{\text{trop}} \text{ vanishes} \},
\]
and the point \(-a \in \mathbb{T}^{d-1}\) is called the apex of \( H(a) \). Any two tropical hyperplanes just differ by a translation. We have the following tropical analog to the situation in ordinary linear algebra; see also \([11, 18]\).

**Proposition 2** (\([30, \text{Lemma 5.1}]\)). The matrix \( M \) is tropically singular if and only if its rows (or, equivalently, its columns) considered as points in \( \mathbb{T}^{d-1} \) are contained in a tropical hyperplane.

The ordering of the real numbers allows to refine the linear algebra in \( \mathbb{R}^{d-1} \) to the theory of ordinary convexity. For algorithmic purposes sidedness queries of a point versus an affine hyperplane (spanned by \( d - 1 \) other points) are crucial, and this is computed by evaluating the sign of the ordinary determinant of a \( d \times d \)-matrix with homogeneous coordinate vectors as its rows. This does not directly translate to the tropical situation since the sign of the tropical determinant does not have a geometric meaning. More severely, the complement of a tropical hyperplane in \( \mathbb{T}^{d-1} \) has exactly \( d \) connected components, its open sectors. The remedy is the following. Let us suppose that all the realizers of the tropical determinant of \( M \) share the same parity. In \([20]\) this parity is called the tropical sign of \( M \), denoted as \( \text{tsgn} M \). If, however, \( M \) has realizers of both signs then \( \text{tsgn} M \) is set to zero. The tropical sign is essentially the same as the sign of the determinant in the “symmetrized min-plus algebra” \([3]\). The computation of the tropical sign of a matrix is equivalent to deciding if a directed graph has a directed cycle of even length \([6, \S 3]\). This latter problem is solvable in \( O(d^3) \) time \([31, 27]\).

For fixed \( v_2, v_3, \ldots, v_d \in \mathbb{T}^{d-1} \) the tropical sign gives rise to a map
\[
\tau : \mathbb{T}^{d-1} \to \{0, \pm 1\} : x \mapsto \text{tsgn}(x, v_2, v_3, \ldots, v_d),
\]
where \((x, v_2, v_3, \ldots, v_d)\) is the \( d \times d \)-matrix formed of the given vectors as its rows. The following is a signed version of Proposition 2

**Theorem 3** (\([20, \text{Theorem 4.7 and Corollary 4.8}]\)). Either \( \tau \) is constantly zero, or for \( \epsilon = \pm 1 \) the preimage \( \tau^{-1}(\{0, \epsilon\}) \) is a closed tropical halfspace. Each closed tropical halfspace arises in this way.

Here a closed tropical halfspace in \( \mathbb{T}^{d-1} \) is the union of at least one and at most \( d - 1 \) closed sectors of a fixed tropical hyperplane. A closed sector is the topological closure of an open sector. For \( x \in \mathbb{T}^{d-1} \) let
\[
x^0 := x - \min(x_1, x_2, \ldots, x_d) \cdot \mathbb{1}_d
\]
be the unique representative in the coset \( x + \mathbb{R}\mathbb{1}_d \) with non-negative coefficients and at least one zero. Then the open sectors of the tropical hyperplane \( H(0) \) are the sets \( S_1, S_2, \ldots, S_d \) where
\[
S_i := \{ x \in \mathbb{T}^{d-1} \mid x^0_i = 0 \text{ and } x^0_j > 0 \text{ for } j \neq i \}.
\]
For the closed sectors we thus have \( \bar{S}_i := \{ x \in \mathbb{T}^{d-1} \mid x^0_i = 0 \} \). Clearly, \(-a + \bar{S}_1, -a + \bar{S}_2, \ldots, -a + \bar{S}_d \) are the closed sector of the tropical hyperplane \( H(a) \) and similarly for the closed sectors. It was mentioned in the introduction that \( \mathbb{T}^{d-1} \) is homeomorphic to \( \mathbb{R} \).

One specific homeomorphism is given by the map
\[
(x_1, x_2, \ldots, x_d) + \mathbb{R}\mathbb{1}_d \mapsto (x_2 - x_1, x_3 - x_1, \ldots, x_d - x_1).
\]
In this paper we will often identify \( \mathbb{T}^{d-1} \) with \( \mathbb{R}^{d-1} \) via this particular map. For instance, its inverse translates our pictures below to \( \mathbb{T}^2 \). Moreover, it allows us to discuss matters of ordinary convexity in \( \mathbb{T}^{d-1} \); see \([21]\). Notice that the closed and open sectors of any
halfspace are both tropically and ordinarily convex. For references to ordinary convexity and, particular, to the theory of convex polytopes see [16, 40]. In order to avoid confusion which type of convexity is relevant in a particular statement we will explicitly say either “tropical” or “ordinary” throughout.

3. Vertices, Pseudo-vertices, and Types

The combinatorics sneaks into the picture through the choice of a system of generators of a tropical polytope. Let \( V = (v_1, v_2, \ldots, v_n) \) be a finite ordered sequence of points in \( \mathbb{T}^{d-1} \). This induces a cell decomposition of \( \mathbb{T}^{d-1} \) into types as follows

\[
\text{type}_V(x) := (T_1, T_2, \ldots, T_d),
\]

where \( T_k = \{i \in [n] \mid v_i \in x + S_k\} \). The individual sets \( T_k \) are called type entries. A type vector \( (T_1, T_2, \ldots, T_d) \) with respect to \( V \) satisfies the condition \( T_1 \cup T_2 \cup \cdots \cup T_d = [n] \) (but the converse does not hold). Sometimes it will be convenient to identify the point sequence \( V \) in \( \mathbb{R}^d \) with the matrix \((v_{ij})\) whose \( i \)-th row is the point \( v_i \).

**Proposition 4** ([12, Lemma 10]). The points of a fixed type \( T \) with respect to the generating system \( V \) form an ordinary polyhedron \( X_T \) which is tropically convex. More precisely,

\[
X_T = \{x \in \mathbb{T}^{d-1} \mid x_k - x_j \leq v_{ik} - v_{ij} \text{ for all } j, k \in [d] \text{ and } i \in T_j\}.
\]

A set in \( \mathbb{T}^{d-1} \) is tropically convex if it coincides with its own tropical convex hull. The ordinary polyhedron \( X_T \) is bounded if and only if all entries in the type \( T \) are non-empty. The bounded ordinary polyhedra \( X_T \) are precisely the polytropes studied in [21].

**Theorem 5** ([12, Theorem 15]). The collection of ordinary polyhedra \( X_T \), where \( T \) ranges over all types, is a polyhedral decomposition of \( \mathbb{T}^{d-1} \). The tropical polytope \( \text{tconv} V \) is precisely the union of the bounded types.

Let \( U \) be the \((n + d - 1)\)-dimensional real vector space \( \mathbb{R}^{n+d}/\mathbb{R}(1_n, -1_d) \); its elements are the equivalence classes of pairs \((y, z) + \mathbb{R}(1_n, -1_d)\) where \( y \in \mathbb{R}^n \) and \( z \in \mathbb{R}^d \). For an \( n \times d \)-matrix \( V = (v_{ij}) \) the polyhedron

\[
\mathcal{P}_V := \{(y, z) \in U \mid y_i + z_j \leq v_{ij} \text{ for all } i \in [n] \text{ and } j \in [d]\}
\]

is unbounded. We call the polyhedron \( \mathcal{P}_V \) the envelope of \( \text{tconv} V \) with respect to \( V \). By [12, Lemma 22] the map \( U \to \mathbb{T}^{d-1} : (y, z) \mapsto z \) sends the bounded faces of \( \mathcal{P}_V \) to the bounded types induced by the generating set \( V \). In particular, \( \text{tconv} V \) is the image of the union of all bounded faces of its envelope. Similarly the unbounded types correspond to the unbounded faces of \( \mathcal{P}_V \). Of particular importance to us are the vertices of \( \mathcal{P}_V \) which we call the pseudo-vertices of \( \text{tconv} V \) with respect to the generating system \( V \). What we just discussed is explicitly stated as Algorithm A below.

**Algorithm A**: Computing the pseudo-vertices via ordinary convex hull.

```
input : V ⊂ \mathbb{T}^{d-1} finite
output: pseudo-vertices of tconv V
compute the vertices W of the envelope \mathcal{P}_V via an arbitrary (dual) ordinary convex hull algorithm
return image of W under projection (y, z) ↦ z
```

The ordinary convex hull problem asks to compute the facets of the ordinary convex hull of a given finite set of points in Euclidean space. Via cone polarity this is equivalent to the dual problem of enumerating the ordinary vertices from an ordinary halfspace description.
The latter also extends without changes to ordinary unbounded polyhedra which do not contain any affine subspace, and this is why a dual convex hull algorithm can be applied to \( P \) as given in [2] in Algorithm A. For an overview of ordinary convex hull algorithms both from the theoretical and the practical point of view see [2, 19]. The complexity of the ordinary convex hull problem in variable dimension is not entirely settled.

**Remark 6.** It is obvious to compute the types of the pseudo-vertices with respect to a given set of generators by checking the definition. More interesting is that it also works the other way around. Once the type of a pseudo-vertex is given one can determine its coordinates from the coordinates of the generators. To see this observe that if \( w \) is a pseudo-vertex with respect to \( V \) then after fixing an arbitrary coordinate, say \( w_1 \), the inequalities in Proposition 4 degenerate to a system of equations which determine \( w_2 - w_1, w_3 - w_1, \ldots, w_d - w_1 \).

For \( \text{tconv} \, V \), just considered as a subset of \( \mathbb{T}^{d-1} \), there is a unique minimal system of generators with respect to inclusion; see [9, Theorem 15.6] and [12, Proposition 21]. These are the *tropical vertices* of \( \text{tconv} \, V \). Among the generators a vertex \( w \) is recognized by the property that at least one of its type entries contains only the index of \( w \) itself; see Algorithm B. This process takes \( O(nd^2) \) time.

```
input : V ⊂ \mathbb{T}^{d-1} finite
output: set of tropical vertices of tconv V
W ← ∅
foreach w ∈ V do
    (T_1, T_2, \ldots, T_d) ← type_{V \setminus \{w\}}(w)
    if T_k = ∅ for some k then
        add w to W
return W

Algorithm B: Computing the tropical vertices.
```

**Example 7.** Consider the point sequence \( V \) with

\[
V = \{(0, 3, 6), (0, 5, 2), (0, 0, 1), (1, 5, 0)\}
\]

![Figure 1. Tropical convex hull of four points in \( \mathbb{T}^2 \).](image-url)
in $\mathbb{T}^2$. This is the same example as the one considered in [5, Figure 1]. It is shown in Figure 1. The tropical polygon $\text{tconv} V$ has precisely ten pseudo-vertices. They have their coordinates and types as listed in Table 1. The first four pseudo-vertices are the given generators, and these are also the tropical vertices. The meaning of the last column will be explained in Section 4 below. Altogether the type decomposition of $\text{tconv} V$ with respect to $V$ has ten vertices, 12 edges, and three two-dimensional faces. This information is collected in the so-called $f$-vector $(10, 12, 3)$.

**Table 1. Pseudo-vertices of the tropical polygon from Example 7 and Figure 1**

| label | $w$       | type$_V(w)$ | facet sectors |
|-------|-----------|-------------|---------------|
| $v_1$ | $(0, 3, 6)$ | 1, 1, 1234  | 3             |
| $v_2$ | $(0, 5, 2)$ | 2, 123, 24  |               |
| $v_3$ | $(0, 0, 1)$ | 123, 3, 34  |               |
| $v_4$ | $(1, 5, 0)$ | 24, 134, 4  |               |
| $w_1$ | $(1, 1, 0)$ | 1234, 3, 4  | 1             |
| $w_2$ | $(0, 5, 0)$ | 2, 1234, 4  | 2             |
| $w_3$ | $(0, 3, 2)$ | 12, 13, 24  | 23            |
| $w_4$ | $(0, 3, 4)$ | 1, 13, 234  | 13            |
| $w_5$ | $(1, 4, 0)$ | 124, 13, 4  |               |
| $w_6$ | $(0, 1, 2)$ | 12, 3, 234  |               |

Let $\Delta_k := \text{conv}\{e_1, e_2, \ldots, e_{k+1}\}$ be the (ordinary) regular $k$-dimensional simplex. Here and in the sequel the standard basis vectors of $\mathbb{R}^d$ are denoted as $e_1, e_2, \ldots, e_d$. The product of two simplices $\Delta_{n-1} \times \Delta_{d-1}$ is an $(n + d - 2)$-dimensional ordinary polytope with $n + d$ facets and $nd$ vertices. We can read our matrix $V$ as a way of assigning the height $v_{ij}$ to the vertex $(e_i, e_j)$ of $\Delta_{n-1} \times \Delta_{d-1}$. Projecting back the lower convex hull of the lifted polytope yields a regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$; see [10] for a comprehensive treatment of the subject. The following is the main structural result about tropical polytopes.

**Theorem 8 ([12, Theorem 1]).** The tropical polytope $\text{tconv} V$ with the induced structure as a polytopal complex by its bounded types is dual to the regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ induced by $V$.

In view of this result the most natural version of the tropical convex hull problem perhaps is the one that asks to compute the pseudo-vertices together with all the bounded types. This can be achieved via applying a suitable modification of an algorithm of Kaibel and Pfetsch [23] for computing the face lattice of an ordinary polytope from its vertex facet incidences to the output of Algorithm A.

For ordinary convex hull computations it is known that genericity assumptions allow for additional and sometimes faster algorithms. This is also the case in the tropical situation. The point set $V \subset \mathbb{T}^{d-1}$ is sufficiently generic if no $k \times k$-submatrix of $V$, viewed as a matrix, is tropically singular. This condition is equivalent to the property that the envelope $\mathcal{P}_V$ is a simple polyhedron. This is the case if and only if the induced subdivision of the product of simplices is a triangulation. A $k$-dimensional polyhedron is simple if each vertex is contained in exactly $k$ facets; for details about polytopes and polyhedra see [40]. The maximal cells of the polytopal complex dual to the type decomposition of $\text{tconv} V$ are in bijection with the pseudo-vertices. If $(T_1, T_2, \ldots, T_d)$ is the type of the pseudo-vertex $w$ then the corresponding maximal cell contains precisely those vertices $(e_i, e_j)$ of $\Delta_{n-1} \times \Delta_{d-1}$ which satisfy $i \in T_j$. 
Example 9. In fact, the point set $V$ in Example 7 is sufficiently generic, and hence the subdivision of $\Delta_3 \times \Delta_2$ induced by $V$ is a triangulation. For instance, the tropical vertex $v_1$ has the type $(1, 1, 1, 2, 3, 4)$, and it corresponds to the maximal cell with vertices
\[
\{(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_3), (e_3, e_3), (e_4, e_3)\}
\]
a 5-simplex. See the Table 2 for the complete list of pseudo-vertices versus maximal cells in the triangulation; the vertex $(e_i, e_j)$ is abbreviated as $ij$.

Table 2. Triangulation of $\Delta_3 \times \Delta_2$ dual to $\text{tconv} V$ decomposed into types.

| label | maximal cell in triangulation | min. generator of $I^*$ |
|-------|-------------------------------|-------------------------|
| $v_1$ | 11 12 13 23 33 43 | $x_{21}x_{22}x_{31}x_{32}x_{41}x_{42}$ |
| $v_2$ | 12 21 22 23 32 43 | $x_{11}x_{13}x_{31}x_{33}x_{41}x_{42}$ |
| $v_3$ | 11 21 31 32 33 43 | $x_{12}x_{13}x_{22}x_{23}x_{41}x_{42}$ |
| $v_4$ | 12 21 32 41 42 43 | $x_{11}x_{13}x_{22}x_{23}x_{31}x_{33}$ |
| $w_1$ | 11 21 31 32 41 43 | $x_{12}x_{13}x_{22}x_{23}x_{33}x_{42}$ |
| $w_2$ | 12 21 22 32 42 43 | $x_{11}x_{13}x_{23}x_{31}x_{33}x_{41}$ |
| $w_3$ | 11 12 21 23 32 43 | $x_{13}x_{22}x_{31}x_{33}x_{41}x_{42}$ |
| $w_4$ | 11 12 23 32 33 43 | $x_{13}x_{21}x_{22}x_{31}x_{41}x_{42}$ |
| $w_5$ | 11 12 21 32 41 43 | $x_{13}x_{22}x_{23}x_{31}x_{33}x_{42}$ |
| $w_6$ | 11 21 23 32 33 43 | $x_{12}x_{13}x_{22}x_{31}x_{41}x_{42}$ |

Block and Yu [5] investigated the relationship of tropical convexity to commutative algebra. We will review some of their ideas in the following. The situation is particularly clear if the points considered are sufficiently generic.

Let $\mathbb{K}[x_{11}, x_{12}, \ldots, x_{nd}]$ be the polynomial ring in $nd$ indeterminates over the field $\mathbb{K}$. The indeterminate $x_{ij}$ will be assigned the weight $v_{ij}$, the $j$-th coordinate of the point $v_i$ in our sequence $V$. Weights of monomials are extended additively, that is, the weight of $x^\alpha = \prod x_{ij}^{\alpha_{ij}}$ is $\sum \alpha_{ij}v_{ij}$. The weight $\text{in}_V(f)$ of polynomial $f$ is the sum of its terms of maximal weight. This defines a partial term ordering on $\mathbb{K}[x_{11}, x_{12}, \ldots, x_{nd}]$. Let $J$ be the determinantal ideal generated by all $2 \times 2$-minors. Then
\[
\text{in}_V(J) := \langle \text{in}_V(f) \mid f \in J \rangle
\]
is the initial ideal of $J$ with respect to $V$.

Proposition 10 ([5] Proposition 4]). The points in $V$ are sufficiently generic if and only if $\text{in}_V(J)$ is a square-free monomial ideal.

A square-free monomial in $\mathbb{K}[x_{11}, x_{12}, \ldots, x_{nd}]$ corresponds to a subset of the indeterminates $x_{11}, x_{12}, \ldots, x_{nd}$ and conversely. The unique minimal set of generators of a square-free monomial ideal is the set of minimal non-faces of a unique finite simplicial complex on the vertices $x_{11}, x_{12}, \ldots, x_{nd}$. Starting with our ideal $I := \text{in}_V(J)$ this simplicial complex is the initial complex $\Delta_V(J)$. The reverse construction can be applied to any finite simplicial complex: Its minimal non-faces generate the Stanley-Reisner ideal of the complex. In particular, the initial ideal $I$ is the Stanley-Reisner ideal of the initial complex $\Delta_V(J)$. The complements of the maximal faces of $\Delta_V(J)$ generate another square-free monomial ideal, the Alexander dual $I^*$. For details see [29, Chapter 1]. The whole point of this discussion is that $\Delta_V(J)$ coincides with the triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ induced by $V$ [5, Lemma 5]. We arrive at Algorithm [C] below as an alternative to Algorithm [A].
input: \( V \subset \mathbb{T}^{d-1} \) finite, in general position
output: types of the pseudo-vertices of \( \text{tconv} \, V \)

\[
J \leftarrow \langle 2 \times 2 \text{-minors of the } n \times d \text{-matrix } (x_{ij}) \rangle
\]

\[
I \leftarrow \text{in}_V(J)
\]

\[
I^* \leftarrow \text{Alexander dual of } I
\]

return \( I^* \)

**Algorithm C:** Computing the pseudo-vertices via Alexander duality.

**Example 11.** We continue our Example 7. In this case the initial ideal \( I := \text{in}_V(J) \) reads

\[
I = \langle x_{11}x_{22}, x_{11}x_{42}, x_{12}x_{21}x_{33}, x_{12}x_{31}, x_{13}x_{21}, x_{13}x_{31}, x_{13}x_{32}, x_{13}x_{41}, x_{13}x_{42},
\]

\[
x_{22}x_{31}, x_{22}x_{33}, x_{22}x_{41}, x_{23}x_{31}, x_{23}x_{41}, x_{23}x_{42}, x_{31}x_{42}, x_{33}x_{41}, x_{33}x_{42} \rangle.
\]

Notice that the generator \( x_{12}x_{21}x_{33} \) is of degree three, whence \( I \) is not homogeneous. The Alexander dual \( I^* \) is generated by the monomials listed in the third column of Table 1.

We would like to point out that there are typos in the description of \( I \) in [5, page 109], but the description of \( I^* \) is correct. The initial complex \( \Delta_V(J) \), or equivalently the triangulation dual to \( \text{tconv} \, V \) subdivided into its bounded types, is the pure 5-dimensional simplicial complex whose facets are the complements of the sets corresponding to the ten minimal generators of \( I^* \); the maximal cells of \( \Delta_V(J) \) are listed in the second column of Table 1. It is known that the \( f \)-vector of any two triangulations of a product of simplices (without new vertices) is the same; in this case it reads \( (12, 48, 92, 93, 48, 10) \).

The following is the main algebraic result on this topic.

**Theorem 12 ([5, Theorem 1]).** If the points in \( V \) are sufficiently generic then \( \text{tconv} \, V \) supports a minimal free resolution of the ideal \( I^* \), as a polytopal complex.

This says that the combinatorics of the type decomposition of \( \text{tconv} \, V \) completely controls the relations among the generators of the ideal \( I^* \). For details on cellular resolutions the reader is referred to [29, Chapter 4].

**Example 13.** We explain in slightly more detail where this leads to for our running example. Abbreviating \( S := \mathbb{K}[x_{11}, x_{12}, \ldots, x_{43}] \), the cellular resolution of Theorem 12 is the complex (in the homological sense)

\[
0 \leftarrow S^1 \leftarrow S^{10} \leftarrow S^{12} \leftarrow S^3 \leftarrow 0
\]

of free \( S \)-modules. The degrees 10, 12, and 3 of the modules correspond to the \( f \)-vector of the type decomposition of \( \text{tconv} \, V \); see Example 7. The arrows are homomorphisms of \( S \)-modules (to be read off the combinatorics of \( \text{tconv} \, V \)) with the property that concatenating two consecutive arrows gives the zero map; the precise maps for this specific example are given in [5, Example 10]. Knowing the complex (3) (with its maps) allows to reconstruct the combinatorics of the types induced by \( V \).

Even if one is not interested in algebraic applications, Theorem 12 opens up an additional line of attack for the algorithmic problem to compute all the types together with the pseudo-vertices. This is due to the fact that a simple polyhedron, such as the envelope \( \mathcal{P}_V \) in the sufficiently generic case, supports a unique minimal free resolution on the polytopal subcomplex of its bounded faces [5, Remark 7]. That is to say, in order to compute the types it suffices to compute any free resolution and to simplify it to a minimal one afterwards [5, Algorithm 2].

**Remark 14.** In ordinary convexity there are at least two algorithms to compute convex hulls essentially via sidedness queries: the beneath-and-beyond method (which iteratively
produces a triangulation) and gift wrapping \[2, 19\]. In general, it is not clear to what extent these also exist in tropical versions. However, in the planar case, that is, \(d = 3\), the cyclic ordering of the tropical vertices can be computed by an output-sensitive algorithm in \(O(n \log h)\) time \[20\] Theorem 5.3, where \(n\) is the number of generators and \(h\) is the number of tropical vertices, by a suitable translation from the situation in the ordinary case \[7\].

4. Minimal Tropical Halfspaces

A key theorem about ordinary polytopes states that an ordinary convex polytope, that is, the ordinary convex hull of finitely many points in \(\mathbb{R}^d\) is the same as the intersection of finitely many affine halfspaces \[40\] Theorem 2.15. For tropical polytopes there is a similar statement.

Theorem 15 (\[20\] Theorem 3.6]). The tropical polytopes are precisely the bounded intersections of tropical halfspaces.

For \(a \in \mathbb{R}^{d-1}\) and \(H \subset [d]\) we use the notation \((a, H)\) for the closed tropical halfspace \(a + \bigcup_{i \in H} S_i\). A tropical halfspace is minimal with respect to a tropical polytope \(t\text{conv}\, V\) if it is minimal with respect to inclusion among all tropical halfspaces containing \(t\text{conv}\, V\). This raises the question how these can be computed. We begin with a straightforward procedure to check if one tropical halfspace is contained in another.

Lemma 16. Let \((a, H)\) and \((b, K)\) be tropical halfspaces. Then \((a, H) \subseteq (b, K)\) if and only if \(H \subseteq K\) and \(a - b \in S_k\).

Proof. The property \((a, H) \subseteq (b, K)\) is equivalent to the following: for all \(x \in (0, H)\) we have \(a + x - b \in (0, K)\). So assume the latter. Then, as \(x \in (0, H)\), this implies \(a - b \in (0, K)\). Now we can assume that \(a = b\) without loss of generality and \(H \subseteq K\) is immediate. The converse follows in a similar way. \(\square\)

It is a feature of tropical convexity that \(d\) of the minimal tropical halfspaces of a tropical polytope can be read off the generator matrix \(V\) right away. To this end we define the \(k\text{-th corner}\) of \(t\text{conv}\, V\) as

\[
c_k(V) := (-v_{1,k}) \odot v_1 \odot (-v_{2,k}) \odot v_2 \odot \cdots \odot (-v_{n,k}) \odot v_n.
\]

By construction it is clear that each corner is a point in \(t\text{conv}\, V\).

Lemma 17. The closed tropical halfspace \(c_k(V) + \bar{S}_k\) is minimal with respect to \(t\text{conv}\, V\).

Proof. By symmetry we can assume that \(k = 1\). Then the vectors \((-v_{1,1}) \odot v_1, (-v_{2,1}) \odot v_2, \ldots , (-v_{n,1}) \odot v_n\) are the Euclidean coordinate vectors of the points \(v_1, v_2, \ldots , v_n\), and their pointwise minimum \(c := c_1(V)\) is the “lower left” corner of their tropical convex hull. By construction the tropical halfspace \(c + \bar{S}_1\) contains \(t\text{conv}\, V\). Suppose that \(c + \bar{S}_1\) is not minimal. Then there must be some other tropical halfspace \(w + \bar{S}_K\) contained in \(c + \bar{S}_1\) which still contains \(t\text{conv}\, V\). By Lemma 16 we have \(K = \{1\}\). Moreover, since \(c \in t\text{conv}\, V\), we have \(c - w \in \bar{S}_1\) and so \(c + \bar{S}_1 \subseteq w + \bar{S}_1\), again by Lemma 16. We conclude that \(w = c\), and this completes our proof. \(\square\)

It is now a consequence of the following proposition that the corners are pseudo-vertices.

Proposition 18 (\[20\] Proposition 3.3]). The apex of a minimal tropical halfspace with respect to \(t\text{conv}\, V\) is a pseudo-vertex with respect to \(t\text{conv}\, V\).
The tropical halfspace $c_k(V) + \bar{S}_k$ is called the $k$-th cornered tropical halfspace with respect to $\text{tconv} V$. The cornered hull of $\text{tconv} V$ is the intersection of all $d$ cornered tropical halfspaces. Each tropical halfspace is tropically convex. But a tropical halfspace is convex in the ordinary sense if and only if it consists of a single sector. This implies that the cornered hull of a tropical polytope is tropically convex and also convex in the ordinary sense. Since it is also bounded it is a polytrope. In fact, the cornered hull of $V$ is the smallest polytrope containing $V$.

**Example 19.** For the matrix $V$ introduced in Example 7 the three corners are the pseudo-vertices

$$c_1(V) = w_1 = (1, 1, 0), \quad c_2(V) = w_6 = (0, 5, 0), \quad \text{and} \quad c_3(V) = v_1 = (0, 3, 6).$$

This notation fits with Table 1. The cornered hull is shaded lightly in Figure 1.

**Lemma 20.** Let $(a, H)$ be any tropical halfspace containing $V$, and let $(T_1, T_2, \ldots, T_d) = \text{type}_V(a)$. Then $\bigcup_{j \in H} T_j = [n]$.

**Proof.** The type entry $T_j$ contains (the indices of) the generators in $V$ which are contained in the closed sector $a + \bar{S}_j$. Since all the generators are contained in each facet defining tropical halfspace we have $\bigcup_{j \in H} T_j = [n]$. \hfill \Box

From this lemma it is clear that for a minimal tropical halfspace containing $\text{tconv} V$ the type entries of $a$ with respect to $V$ form a set covering of $[n]$ which is minimal with respect to inclusion. It may happen that one pseudo-vertex occurs as the apex of two distinct minimal tropical halfspaces. A tropical halfspace is *locally minimal* if it is minimal with respect to inclusion among all tropical halfspaces with a fixed apex and containing $V$.

```
input : $x \in \mathbb{T}^d$ point in the boundary of a tropical polytope $\text{tconv} V$
output: set of tropical halfspaces with apex $x$ locally minimal with respect to $P$
compute $T = (T_0, \ldots, T_d) = \text{type}_V(x)$
compute all (with respect to inclusion) minimal set coverings of $[n]$ by $T_0, \ldots, T_d$
$H \leftarrow$ pairs of $x$ with all these minimal set covers
return $H$
```

**Algorithm D:** Computing the minimal tropical halfspaces (locally).

The input to Algorithm D below can be the set of pseudo-vertices with respect to any generating system, in particular, with respect to the tropical vertices.

Notice that, as stated, Algorithm D contains the NP-complete problem of finding a set covering of $[n]$ of minimal cardinality as a subproblem. The following example raises the question if, in order to compute all (globally) minimal tropical halfspaces it can be avoided to compute all locally minimal tropical halfspaces for all pseudo-vertices. This is not clear to the author.

**Example 21.** Consider the tropical convex hull $\text{tconv}(v_1, v_2, v_3, v_4)$ shown in Figure 1. The points $w_4 = (0, 3, 2)$ and $w_5 = (0, 3, 4)$ both are pseudo-vertices with respect to the given system of generators, see Table 1. At $w_5$ there are precisely two locally minimal tropical halfspaces, namely $(w_5, \{1, 3\})$ and $(w_5, \{2, 3\})$. The only locally minimal tropical halfspace at $w_4$ is $(w_4, \{2, 3\})$. As $(w_4, \{2, 3\})$ is contained in $(w_5, \{2, 3\})$, the latter cannot be (globally) minimal.

Both $(w_4, \{2, 3\})$ and $(w_5, \{1, 3\})$ are (globally) minimal. Altogether there are five minimal tropical halfspaces: the three remaining ones are $(v_1, \{3\})$, $(w_1, \{1\})$, and $w_6, \{2\})$. For each pseudo-vertex which is the apex of a minimal tropical halfspace the corresponding facet sectors are listed in the final column of Table 1.
input : $W \subset \mathbb{T}^d$ pseudo-vertices of $\text{tconv} V$

output: set of minimal tropical halfspaces with respect to $\text{tconv} V$

$\mathcal{H} \leftarrow \emptyset$

foreach $x \in W$ do

$\mathcal{H}' \leftarrow$ locally minimal tropical halfspaces at $x$ according to Algorithm D

foreach $(x, H') \in \mathcal{H}'$ do

foreach $(y, H) \in \mathcal{H}$ do

if $(x, H')$ is contained in $(y, H)$ then

remove $(y, H)$ from $\mathcal{H}$

else

if $(y, H)$ is contained in $(x, H')$ then

remove $(x, H')$ from $\mathcal{H}'$

end if

end if

end foreach

$\mathcal{H} \leftarrow \mathcal{H} \cup \mathcal{H}'$

return $\mathcal{H}$

Algorithm E: Computing the minimal tropical halfspaces (globally).

5. Matroid Subdivisions and Tree Arrangements

It turns out that tropical polytopes or, dually, regular subdivisions of products of ordinary simplices carry information which can be, in a way, extended to matroid subdivisions of hypersimplices. These are known to be relevant for studying questions on the Grassmannians of $d$-planes in $n$-space over some field $\mathbb{K}$ [24, 37, 17].

An ordinary polytope in $\mathbb{R}^m$ whose vertices are $0/1$-vectors is a matroid polytope if each edge is parallel to a difference of basis vectors $e_i - e_j$ for some distinct $i$ and $j$ in $[m]$. A matroid of rank $r$ on the set $[m]$ is a set of $r$-element subsets of $[m]$ whose characteristic vectors are the vertices of a matroid polytope. The elements of a matroid are its bases. It is a result of Gel’fand, Goresky, MacPherson, and Serganova [15] that this definition describes the same kind of objects as more standard ones [39]. Each $m \times n$-matrix $M$ (with coefficients in an arbitrary field $\mathbb{K}$) gives rise to a matroid $\mathcal{M}$ of rank $r$ on the set $[m]$ as follows: the columns of $\mathcal{M}$ are indexed by $[m]$, and $r$ is the rank of $\mathcal{M}$. Those $0/1$-vectors which correspond to bases of the column space of $\mathcal{M}$ are the bases of $\mathcal{M}$. We denote the matroid polytope of a matroid $\mathcal{M}$ as $P_M$. The $(r, m)$-hypersimplex $\Delta(r, m)$ is the matroid polytope of the uniform matroid of rank $r$ on the set $[m]$, that is, the matroid whose set of bases is $\left(\binom{[m]}{r}\right)$, by which we denote the set of all $r$-element subsets of $[m]$.

The relationship to tropical convexity works as follows. By the Separation Theorem from ordinary convexity [16, §2.2] each vertex $v$ of an ordinary polytope $P$ can be separated from the other vertices by an affine hyperplane $H$. The intersection of $P$ with $H$ is again an ordinary polytope, and its combinatorial type does not depend on $H$; this is the vertex figure $P/v$ of $P$ with respect to $v$. Moreover, whenever we have a lifting function $\lambda$ on $P$ inducing a regular subdivision, then $\lambda$ induces a lifting function of each vertex figure $P/v$, and this way we obtain a regular subdivision of $P/v$. In the case of the $(d, n + d)$-hypersimplex each vertex figure is isomorphic to the product of simplices $\Delta_{d-1} \times \Delta_{n-1}$. Hence regular subdivisions of $\Delta(d, n + d)$ yield configurations of $n$ points in $\mathbb{T}^{d-1}$ at each vertex of $\Delta(d, n + d)$. The situation for the hypersimplices is special in that the converse also holds, that is, each regular subdivision of a vertex figure can be lifted to a (particularly interesting kind of) regular subdivision of $\Delta(d, n + d)$.

Proposition 22 (24 Corollary 1.4.14]). Each configuration of $n$ points in $\mathbb{T}^{d-1}$ can be lifted to a regular matroid decomposition of $\Delta(d, n + d)$.
A matroid subdivision of a matroid polytope is a polytopal subdivision with the property that each cell is a matroid polytope. Kapranov’s original proof [24] uses non-trivial methods from algebraic geometry. Here we give an elementary proof, which makes use of the techniques developed in [17].

Proof. Let $V$ be the $d \times n$-matrix which has the $n$ given points as its columns, and let $\bar{V}$ be the $d \times n$-matrix produced by concatenating $V$ with the tropical identity matrix of size $d \times d$. The tropical identity matrix has zeros on the diagonal and infinity off the diagonal.

Each $d$-subset $S$ of $[n+d]$ defines $d$ columns of $\bar{V}$ and this way a $d \times d$-submatrix $\bar{V}_S$. It is easy to see that the map

$$\pi : S \mapsto \text{tdet } \bar{V}_S$$

is a finite tropical Plücker vector. A finite tropical Plücker vector is a map from the set $\binom{[n+d]}{d}$ to the reals satisfying the following property: For each subset $T$ of $[n+d]$ with $d-2$ elements the minimum

$$\min \{ \pi(Tij) + \pi(Tkl), \pi(Tik) + \pi(Tjl), \pi(Til) + \pi(Tjk) \}$$

is attained at least twice, where $i, j, k, l$ are the pairwise distinct elements of $[n+d]$ \( \setminus T \) and $Tij$ is short for $T \cup \{i, j\}$. Clearly, the map $\pi$ is a lifting function on the hypersimplex $\Delta(d, n+d)$, and thus it induces a regular subdivision. That the tropical Plücker vectors, in fact, induce matroid decompositions is a known fact [24, §1.2], [35, Proposition 2.2].

One can verify directly that the point configuration $V$ in $T^{d-1}$ is (in the sense of Theorem 8) dual to the regular subdivision induced by $\pi$ at the vertex figure of $\Delta(d, n+d)$ at the vertex $e_{n+1} + e_{n+2} + \cdots + e_{n+d}$. □

Notice that our computation in the proof above makes use of “$\infty$” as a coordinate, that is, from now on we work in the tropical projective space

$$\mathbb{T}P^{d-1} := ((\mathbb{R} \cup \{\infty\})^d \setminus \{(\infty, \infty, \ldots, \infty)\})/\mathbb{R}1_d.$$ 

The tropical projective space $\mathbb{T}P^{d-1}$ is a compactification of the tropical torus $\mathbb{T}^{d-1}$ with boundary. In fact, $\mathbb{T}^{d-1}$ should be seen as an open ordinary regular simplex of infinite size which is compactified by $\mathbb{T}P^{d-1}$ in the natural way.

Example 23. Starting out with the point configuration $V$ from Example 7 with $d = 3$ and $n = 4$ we obtain

$$\bar{V} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \infty & \infty \\ 3 & 5 & 0 & 5 & \infty & 0 & \infty \\ 6 & 2 & 1 & 0 & \infty & \infty & 0 \end{pmatrix}.$$ 

The resulting tropical Plücker vector is given in Table 3, and the induced matroid subdivision of $\Delta(d, n+d)$ is listed in Table 4. For each of the ten maximal cells the corresponding set of bases is listed. These maximal cells are in bijection with the pseudo-vertices of $\text{tconv } V$ with respect to $V$ as given in Tables 1 and 2.

Table 3. Tropical Plücker vector $\pi$ of $\bar{V}$ from Equation (4).

|     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|
| 123:2 | 124:3 | 125:5 | 126:2 | 127:3 | 134:0 | 135:4 |
| 156:6 | 157:3 | 167:0 | 234:0 | 235:2 | 236:1 | 237:0 |
| 267:0 | 345:0 | 346:0 | 347:1 | 356:1 | 357:0 | 367:0 |
| 136:1 | 137:0 | 145:3 | 146:0 | 147:4 |
| 245:5 | 246:0 | 247:5 | 256:2 | 257:5 |
| 456:0 | 457:5 | 467:1 | 567:0 |

The hypersimplex $\Delta(d, n+d)$ is the set of points $(x_1, x_2, \ldots, x_{n+d}) \in \mathbb{R}^{n+d}$ which satisfies the $2(n+d)$ linear inequalities $0 \leq x_i \leq 1$ and the linear equation $\sum x_i = d$. The given inequalities are all facet defining. The facet defined by $x_i = 0$ is the $i$-th deletion facet,
Table 4. Matroid subdivision of $\Delta(3, 7)$ induced by the tropical Plücker vector $\pi$ in Table 3.

| label | matroid bases |
|-------|---------------|
| $v_1$ | 125 126 135 136 145 146 156 157 167 256 257 356 456 457 567 |
| $v_2$ | 124 125 127 145 157 234 235 237 245 246 256 257 267 345 346 356 357 367 456 457 567 |
| $v_3$ | 134 136 137 146 147 157 234 236 237 246 256 257 267 345 346 356 357 367 456 457 567 |
| $v_4$ | 124 127 145 147 157 234 237 246 267 345 346 347 356 357 456 457 567 |
| $w_1$ | 134 136 146 157 234 236 237 246 247 256 267 345 346 356 357 456 457 567 |
| $w_2$ | 124 125 127 145 157 234 235 237 245 246 247 257 267 345 347 356 357 456 457 567 |
| $w_3$ | 123 124 125 126 127 134 135 136 137 145 146 157 167 234 235 237 246 256 267 345 347 357 456 467 567 |
| $w_4$ | 123 125 126 134 135 136 137 145 146 157 167 234 235 237 246 256 267 345 347 356 357 456 567 |
| $w_5$ | 124 127 134 135 136 137 145 146 157 167 234 235 237 246 256 267 345 347 356 357 456 467 567 |
| $w_6$ | 123 126 134 136 137 146 157 234 235 236 237 245 246 247 256 267 345 346 356 357 456 467 567 |

and the facet $x_1 = 1$ is the $i$-th contraction facet of $\Delta(d, n + d)$. Each deletion facet is isomorphic to $\Delta(d, n + d - 1)$, and each contraction facet is isomorphic to $\Delta(d - 1, n + d - 1)$. Recursively, all faces of hypersimplices are hypersimplices. Now $P_M$ is a subpolytope of $\Delta(d, n + d)$ whenever $M$ is a matroid of rank $d$ on the set $[n + d]$, and the intersection of $P_M$ with a facet is again a matroid polytope, a deletion or a contraction depending on the type of the facet. This implies that each matroid subdivision of $\Delta(d, n + d)$ induces a matroid subdivision on each facet.

Table 5. Restriction $\pi_1$ of the tropical Plücker vector $\pi$ from Table 3 to the first contraction facet.

| label | matroid bases |
|-------|---------------|
| $w_1$ | 23:2 24:3 25:5 26:2 27:3 34:0 35:4 36:1 37:0 45:3 46:0 47:4 56:6 57:3 67:0 |

We want to look at the matroid subdivisions of $\Delta(d, n + d)$ for low rank $d$. The hypersimplex $\Delta(1, n + 1)$ is an $n$-dimensional simplex without any non-trivial (matroid) subdivisions. For $d = 2$ the situation is more interesting: Each matroid subdivision of $\Delta(2, n + 2)$ is dual to a tree with $n$ leaves [24, §1.3]. For $d = 3$ we have the following result.

**Theorem 24** ([17, Theorem 4.4]). The regular matroid subdivisions of $\Delta(3, n + 3)$ bijectively correspond to the equivalence classes of arrangements of $n + 3$ metric trees with $n + 2$ labeled leaves.

It is easy to see where the $n + 3$ trees come from: They are dual to the matroid subdivisions induced on the $n + 3$ contraction facets of $\Delta(3, n + 3)$. A sequence $(\delta_1, \delta_2, \ldots, \delta_n)$ is an arrangement of metric trees if $\delta_i$ is a tree metric on the set $[n] \setminus \{i\}$ satisfying

$$
\delta_i(j, k) = \delta_j(k, i) = \delta_k(i, j)
$$

for any three distinct $i, j, k \in [n]$. Arrangements of metric trees are equivalent if they induce the same arrangement of abstract trees, which roughly means that their intersection patterns are the same; see [17] for details.

**Example 25.** Restricting the tropical Plücker vector $\pi$ on $\Delta(3, 7)$ from Example 23 to the first contraction facet gives a tropical Plücker vector $\pi_1$ on the second hypersimplex $\Delta(2, 6)$ (with coordinate directions labeled $2, 3, \ldots, 7$) shown in Table 5. Since $\pi_1$ happens to take values between 0 and 6 we can set $\delta_1(S) := 3 - (\pi_1(S)/6)$ for $S \in \binom{\{2,3,\ldots,7\}}{2}$, which
yields a metric on the set \{2, 3, \ldots, 7\}. Written as one half of a symmetric matrix we have

\[
\delta_1 = \frac{1}{2} \begin{pmatrix}
0 & 8/3 & 5/2 & 13/6 & 8/3 & 5/2 \\
0 & 3 & 7/3 & 17/6 & 3 \\
0 & 5/2 & 3 & 7/3 \\
0 & 2 & 5/2 \\
0 & 3 \\
0 & 0
\end{pmatrix}.
\]

Notice that the matroid subdivision of the first contraction facet induced by \(\pi\) does not change if we add multiples of \((1,1,\ldots,1)\) to \(\pi\) or scale it by any positive real number. The rescaling chosen above makes sure that \(\delta_1\) takes values between two and three only. This way the triangle inequality becomes valid automatically, which is why \(\delta_1\) is indeed a metric. Moreover, the metric \(\delta_1\) is tree-like, and the corresponding metric tree is shown in Figure 2. For instance, one can check that \(\delta_1(2,3) = 8/3 = 13/12 + 1/6 + 17/12\).

The edge lengths on the tree can be computed via the Split Decomposition Theorem of Bandelt and Dress [4].

So far we looked at a single tree only. Now we want to explore how the \(n + 3\) trees, one for each contraction facet, interact. In the plane \(\mathbb{T}^2\) a tropical line is the same as a hyperplane. That is to say, the tropical hyperplane/line

\[ H(a) = -a + (\mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_3) \]

is the union of the infinite rays into the three coordinate directions emanating from the apex \(-a\). In particular, it can be seen as a metric tree with one trivalent internal node and three edges of infinite length. The taxa sit at the endpoints of those rays in the compactification \(\mathbb{T}\mathbb{P}^2\). The mirror image of \(H(a)\) at its apex is its dual line

\[ H^*(a) = -a - (\mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_3). \]

Notice that \(H^*(a)\) is a tropical line if one takes “max” as the tropical addition rather than “min”. Any two tropical lines either share an infinite portion of some ray or they meet in a single point, and the same holds for their duals. The arrangement of dual tropical lines induced by a point sequence \(V\) in \(\mathbb{T}^2\), denoted as \(H^*(-V)\), is the sequence of tropical lines with apices \(v_1, v_2, \ldots, v_n\). In order to make the connection to matroid decompositions of hypersimplices we can instead also look at the compactified version \(\tilde{H}^*(-V)\) which is the sequence of lines in \(\tilde{T}\mathbb{P}^2\) with the \(n+3\) apices \(v_1, v_2, \ldots, v_n, (0, \infty, \infty), (\infty, 0, \infty), (\infty, \infty, 0)\). Now each dual tropical line \(\tilde{H}^*(-v_i)\), for \(1 \leq i \leq n + 3\), gives rise to a labeled tree by recording the intersection pattern with the other trees.

**Example 26.** The tree arrangement \(H^*(-V)\) and its compactification \(\tilde{H}^*(-V)\) arising from our running example matrix \(V\) introduced in Example 7 are shown in Figure 3 to the left. The tree with apex \(v_5 = (0, \infty, \infty)\) occurs as the dashed line on the top and on the
Figure 3. Arrangement of seven trees, three of which cover the boundary of $\mathbb{T}^2$.

right of the rectangular section of $\mathbb{T}^2$, the tree corresponding to $v_6 = (\infty, 0, \infty)$ is to the left, and the tree corresponding to $v_7 = (\infty, \infty, 0)$ is at the bottom. For instance, the tree corresponding to $v_1$ receives the labeling shown in Figure 2. The subsequent Theorem 27 can be visually verified for our example by comparing Figures 1 and 3.

**Theorem 27** ([1, Theorem 6.3]). Suppose that $V$ is a sufficiently generic sequence of points in $\mathbb{T}^2$. Then the polyhedral subdivision of $\mathbb{T}^2$ induced by the tree arrangement $H^*(−V)$ coincides with the type decomposition induced by $V$.

The abstract arrangement of trees induced by the matroid decomposition of $\Delta(3, n + 3)$ which in turn is induced by the point sequence $V$ is precisely the compactified tree arrangement $\overline{H}^*(−V)$.

**Remark 28.** The Cayley trick from polyhedral combinatorics allows to view triangulations of $\Delta_{n−1} \times \Delta_{d−1}$ as lozenge tilings of the dilated simplex $n\Delta_{d−1}$; see [32]. For our Example 26 the corresponding lozenge tiling of $4\Delta_2$ is shown in Figure 3 to the right. The tree arrangement $H^*(−V)$ partitions the dual graph of the tiling.

6. **Computational Experiments with polymake**

Most algorithms mentioned above are implemented in polymake, version 2.9.4, which can be downloaded from [www.polymake.de](http://www.polymake.de). After starting the program on the command line (with the application tropical as default) an interactive shell starts which receives commands in a Perl dialect. For instance, to visualize our running Example 7 it suffices to say:

```plaintext
> $p = new TropicalPolytope<Rational>(POINTS=>[[0,3,6],[0,5,2],[0,0,1],[1,5,0]]);
> $p->VISUAL;
```

We want to show how polymake can be used for the investigation of tropical polytopes. For instance we can compute the type of a cell in the type decomposition as follows. This is not a standard function, which is why it requires a sequence of commands. Again the pseudo-vertices and the types always refer to the fixed sequence of generators that we started out with above.

```plaintext
> print @{$rows_labeled($p->PSEUDOVERTICES)};
0:0 5 0
```
The first command lists the pseudo-vertices (in no particular order), and the function `rows_labeled` is responsible for making the implicit numbering explicit which arises from the order. The next command lists all the bounded cells of the type decomposition. Each row corresponds to one maximal cell, namely the (tropical or ordinary) convex hull of the pseudo-vertices with the given indices. We obtain two pentagons, one triangle, and one edge. Now we want to compute the type of the first cell (numbered ‘0’):

```
> $indices = $p->ENVELOPE->BOUNDED_COMPLEX->[0];
```

This is just the sequence of indices of the pseudo-vertices defining this cell. In the following we produce an ordinary polytope which has these points as its vertices. One minor technical complication arises from the fact that the leading ‘0’ used for coordinate homogenization in the tropical world must be replaced by a ‘1’ which makes the ordinary polytope homogeneous.

```
> $vertices = $p->PSEUDOVERTICES->minor($indices,range(1,$p->AMBIENT_DIM));
> $all_ones = new Vector<Rational>([ (1)x$n_vertices ]);  
> $cell = new Polytope<Rational>(VERTICES => ($all_ones|$vertices));
> print $cell->VERTICES;
```

Now the type to be computed is just the type of any relatively interior point of this ordinary polytope called `$cell`.

```
> $point = $cell->REL_INT_POINT;
> print $point;
1 4 2/5
```

This point must be translated back into the tropical world, and then we can call a function to compute its type, which turns out to be (2,13,4). Notice that polymake starts the numbering from ‘0’, and hence each index in the output is shifted by one.

```
> $cell_point = poly2trop(new Polytope<Rational>(POINTS=>$point))->POINTS;
> print types($cell_point,$p->POINTS);
{1}
{0 2}
{3}
```
It should be mentioned that there is also the **Maxplus toolbox for Scilab** for computations in tropical convexity \[28\].

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