A $q$-Dwork-type generalization of Rodriguez-Villegas’ supercongruences

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Abstract. Guo and Zudilin [Adv. Math. 346 (2019), 329–358] developed an analytical method, called ‘creative microscoping’, to prove many supercongruences by establishing their $q$-analogues. In this paper, we apply this method to give a $q$-Dwork-type generalization of Rodriguez-Villegas’ supercongruences, which was recently conjectured by Guo and Zudilin.

1. Introduction

Let $p > 3$ be a prime and $(\cdot p)$ be the Legendre symbol modulo $p$. In 2003, E. Mortenson [13,14] proved the following supercongruences involving hypergeometric functions and Calabi-Yau manifolds,

$$\sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

(1.1)

$$\sum_{k=0}^{p-1} \frac{(3k)(2k)}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

(1.2)

$$\sum_{k=0}^{p-1} \frac{(4k)(2k)}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

(1.3)

$$\sum_{k=0}^{p-1} \frac{(6k)(3k)}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

(1.4)
which were first conjectured by F. Rodriguez-Villegas [17]. In 2014, Z.-H. Sun [18] gave an elementary proof of (1.1)–(1.4) by showing that
\[ \sum_{k=0}^{n-1} \binom{-x}{k} \binom{x-1}{k} \equiv (-1)^{\langle -x \rangle_p} \pmod{p^2} \]
for any \( p \)-adic integer \( x \), where \( \langle x \rangle_p \) denotes the least nonnegative residue of \( x \) modulo \( p \). In 2017, J.-C. Liu [11] stated that for any \( x \in \{1/2, 1/3, 1/4, 1/6\} \) and any positive integer \( n \),
\[ \sum_{k=0}^{n-1} \binom{-x}{k} \binom{x-1}{k} \equiv (-1)^{\langle -x \rangle_p} \sum_{k=0}^{n-1} \binom{-x}{k} \binom{x-1}{k} \pmod{p^2}. \]

In the recent years, \( q \)-analogues of supercongruences were widely investigated, and a variety of techniques, such as asymptotic estimate, basic hypergeometric transformation, creative microscoping, \( q \)-WZ pair and \( q \)-Zeilberger algorithm etc., were involved. For example, in [8], Guo and Zudilin introduced a new method called creative microscoping, and used this method to proved several new Ramanujan-type \( q \)-congruences in a unified way. For more related results and the latest progress, the reader is referred to [3–10, 15, 20–22].

In 1969, Dwork [1] studied a question of continuing analytical solutions \( f(z) = \sum_{k=0}^{\infty} A_k z^k \) of linear differential equations via \( p \)-adic analysis. A general strategy was to prove that the truncated sums \( f_r(z) = \sum_{k=0}^{r-1} A_k z^k \), where \( r = 0, 1, 2, \ldots \), satisfy the so-called Dwork congruences [12]
\[ \frac{f_{r+1}(z)}{f_r(z^p)} \equiv \frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^r \mathbb{Z}_p[[z]]} \text{ for } r = 1, 2, \ldots. \]
Moreover, for some \( m \geq 2 \), provided the congruences (1.5) hold modulo a higher power of \( p \), such as,
\[ \frac{f_{r+1}(z)}{f_r(z^p)} \equiv \frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^{mr} \mathbb{Z}_p[[z]]} \text{ for } r = 1, 2, \ldots, \]
we refer to this type of congruences as Dwork-type supercongruences. Recently, Guo and Zudilin [9] proved some Dwork-type supercongruencies by establishing their \( q \)-analogues. For example, they proved that, for any odd positive integer \( n > 1 \) and integer \( r \geq 1 \), modulo \( \prod_{j=1}^{n-r+1} \Phi_{p^j}(q)^2 \),
\[ \sum_{k=0}^{(n-r-1)/d} \frac{2(q; q^2)_{2k} q^{2k}}{(q^2; q^2)_{k}^2 (1 + q^{2k})} = \left( \frac{-1}{n} \right) \sum_{k=0}^{(n-r-1)/d} \frac{2(q^n; q^{2n})_{2k} q^{2nk}}{(q^{2n}; q^{2n})_k^2 (1 + q^{2nk})}. \]
where \( d = 1, 2 \). Here and in what follows, the \( q \)-shifted factorial [2] is defined by
\[ (x; q)_n = \begin{cases} (1-x)(1-xq)\cdots(1-xq^{n-1}) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases} \]
and the \textit{n-th cyclotomic polynomial} is defined as

$$\Phi_n(q) := \prod_{1 \leq k \leq n \atop (n,k)=1} (q - e^{2\pi i \frac{k}{n}}).$$

Moreover, for polynomials $A_1(q), A_2(q), P(q) \in \mathbb{Z}[q]$, we say that $A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}$ if $P(q)$ divides $A_1(q)$ and is relatively prime to $A_2(q)$. More generally, if the numerator of the reduced form of the difference between two rational functions $B(q)$ and $C(q)$ is divisible by $P(q)$, we say that $B(q) \equiv C(q) \pmod{P(q)}$.

Guo and Zudilin [9, Conjecture 3.13] proposed the following conjecture: for any odd integer $n > 1$ and integers $s \geq 1, m \geq 1, r \geq 2$ with $s < m$ and $n \equiv \pm 1 \pmod{m}$,

$$\sum_{k=0}^{n-1} \frac{2(q^s; q^m)_k(q^{m-s}; q^m)_k q^{mk}}{(q^m; q^m)_k^2(1 + q^{mk})} \equiv (-1)^{(-s/m)n} \sum_{k=0}^{n-1} \frac{2(q^{sn}; q^{mn})_k(q^{mn-sn}; q^{mn})_k q^{mnk}}{(q^{mn}; q^{mn})_k^2(1 + q^{mnk})} \pmod{\prod_{j=1}^{r} \Phi_{n^j}(q)^2}. \quad (1.7)$$

Clearly, the $q$-supercongruence (1.6) is just the $(m,s) = (2,1)$ case of (1.7), and there are parametric generalizations of (1.7) (see [3]) for $r = 1$.

Motivated by Guo and Zudilin’s work [9], we shall confirm the $n \equiv 1 \pmod{m}$ case of (1.7).

**Theorem 1.1.** Let $m$ and $s$ be positive integers with $s < m$. Let $n > 1$ be an odd integer with $n \equiv 1 \pmod{m}$. Then, for $r \geq 2$, modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)^2$,

$$\sum_{k=0}^{n-1} \frac{2(q^s; q^m)_k(q^{m-s}; q^m)_k q^{mk}}{(q^m; q^m)_k^2(1 + q^{mk})} \equiv (-1)^{(-s/m)n} \sum_{k=0}^{n-1} \frac{2(q^{sn}; q^{mn})_k(q^{mn-sn}; q^{mn})_k q^{mnk}}{(q^{mn}; q^{mn})_k^2(1 + q^{mnk})}. \quad (1.8)$$

Note that

$$\Phi_d(1) = \begin{cases} p & \text{if } d = p^k \text{ for some prime } p, \\ 1 & \text{otherwise}. \end{cases}$$

Letting $n = p$ be a prime and $q \to 1$ in (1.8), we obtain

$$\left(\sum_{k=0}^{p^r-1} \binom{-s/m}{k} \binom{-(m-s)/m}{k} - (-1)^{(-s/m)p} \sum_{k=0}^{p^r-1} \binom{-s/m}{k} \binom{-(m-s)/m}{k}\right) / p^{2r}$$
is a $p$-adic integer. Moreover, since $\gcd(m, p) = 1$ and $1/\left((-s/m)_{p^{-1}} \cdot (-m-s)/p^{-1}\right) \in \mathbb{Z}_p$, the number

$$W_{p,p^{-1}} = \frac{\sum_{k=0}^{p^{r-1}} (-s/m)_k \cdot (-m-s)/m_k - (-1)^{(s/m)} p \sum_{k=0}^{p^{r-1}-1} (-s/m)_k \cdot (-m-s)/m_k}{p^{2r} \cdot (-s/m)_{p^{-1}} \cdot (-m-s)/p^{-1}}$$

is a $p$-adic integer. This partially confirms the $n = p^{r-1}$ and $p \equiv 1 \pmod{m}$ case of a conjecture of Z.-W. Sun [19, Conjecture 10]. On the other hand, the $n = p$ and $q \to 1$ case of (1.8) with $m = 3, 4, 6$ also confirms some predictions of Roberts and Rodriguez-Villegas from [16].

It should be pointed out that the $n \equiv -1 \pmod{m}$ case of (1.7) is rather difficult to prove. The same argument for Theorem 1.1 does not work for this case. Nevertheless, we shall establish the following result to support Guo and Zudilin’s conjecture for $n \equiv -1 \pmod{m}$.

**Theorem 1.2.** Let $m, n$ and $s$ be positive integers with $s < m$, $n \equiv -1 \pmod{m}$ and $n$ odd. Then, for $r \geq 2$, modulo $\prod_{j=1}^{\lfloor r/2 \rfloor} \Phi_n c_i(q)^2$,

$$\sum_{k=0}^{n^{r-1}} \frac{2(q^s; q^m)_k (q^{m-s}; q^m)_k q^{mk}}{(q^m; q^m)^2 (1 + q^{mk})} \equiv \sum_{k=0}^{n^{r-2-1}} \frac{2(q^{sn^2}; q^{mn^2})_k (q^{m^2-n^2}; q^{mn^2})_k q^{mn^2 k}}{(q^{mn^2}; q^{mn^2})^2 (1 + q^{mn^2 k})} \pmod{\prod_{j=1}^{r-1} \Phi_n c_j(q)^2}. \quad (1.9)$$

Suppose that (1.7) is true for all odd integers with $n \equiv -1 \pmod{m}$. Using the $q$-congruence (1.7) twice we have

$$\sum_{k=0}^{n^{r-1}} \frac{2(q^s; q^m)_k (q^{m-s}; q^m)_k q^{mk}}{(q^m; q^m)^2 (1 + q^{mk})} \equiv \sum_{k=0}^{n^{r-2-1}} \frac{2(q^{sn^2}; q^{mn^2})_k (q^{m^2-n^2}; q^{mn^2})_k q^{mn^2 k}}{(q^{mn^2}; q^{mn^2})^2 (1 + q^{mn^2 k})} \pmod{\prod_{j=1}^{r-1} \Phi_n c_j(q)^2}. \quad (1.10)$$

Clearly, (1.9) confirms in part the $n \equiv -1 \pmod{m}$ case of (1.10).

The rest of the paper is arranged as follows. The proof of Theorems 1.1 will be given in Sections 2 using the creative microscoping method developed by Guo and Zudilin [8]. Generally speaking, to prove a $q$-supercongruence modulo $\prod_{j=1}^{r} \Phi_n c_j(q)^2$, we prove its generalization with an extra parameter $a$ so that the corresponding congruence holds modulo $\prod_{j=0}^{\lfloor n^{r-1}-1 \rfloor} (1 - a q^{sn(j+1)}) \prod_{j=0}^{\lfloor n^{r-1}-1 \rfloor} (a - q^{(m-s)n(j+1)})$. Since the polynomials $1 - a q^{sn(j+1)}$ and $a - q^{(m-s)n(j+1)}$ are pairwise relatively prime for any $j$ with $0 \leq j \leq \max\{\lfloor n^{r-1}-1 \rfloor, \lfloor n^{r-1}-1 \rfloor\}$, this generalized $q$-congruence can be established modulo these polynomials individually. Finally,
by taking the limit $a \to 1$, we obtain the original $q$-supercongruence of interest. Similarly, we shall prove Theorem 1.2 in Section 3.

2. PROOF OF THEOREM 1.1

We need the following lemma, which was proved by Guo [3, Corollary 1.4].

**Lemma 2.1.** Let $m, n$ and $s$ be positive integers with $\gcd(m, n) = 1$ and $n$ odd. Then, modulo $(1 - aq^{s+m(-s/m)^n})(a - q^{m-s+m((s-m)/m)^n})$,

$$
\sum_{k=0}^{n-1} \frac{2(aq^s; q^m)_k(q^{m-s}/a; q^m)_k q^{mk}}{(q^m; q^m)_k^2 (1 + q^{mk})} \equiv (-1)^{(-s/m)n}.
$$

(2.1)

In order to prove Theorem 1.1, we need to establish the following parametric generalization.

**Theorem 2.1.** Let $m, n$ and $s$ be positive integers with $s < m$, $n \equiv 1 \pmod{m}$ and $n$ odd. Let $r \geq 2$ be an integer and $a$ an indeterminate. Then, modulo

$$
\prod_{j=0}^{\left\lfloor \frac{n^r-1}{m} \right\rfloor} (1 - aq^{sn(mj+1)}) \prod_{j=0}^{\left\lfloor \frac{n^r-1}{m} \right\rfloor} (a - q^{(m-s)n(mj+1)}),
$$

(2.2)

we have

$$
\sum_{k=0}^{n^r-1} \frac{2(aq^s; q^m)_k(q^{m-s}/a; q^m)_k q^{mk}}{(q^m; q^m)_k^2 (1 + q^{mk})} \equiv (-1)^{(-s/m)n} \sum_{k=0}^{n^r-1} \frac{2(aq^{sn}; q^{mn})_k(q^{mn-sn}/a; q^{mn})_k q^{mnk}}{(q^{mn}; q^{mn})_k^2 (1 + q^{mnk})}.
$$

(2.3)

**Proof.** It suffices to show that both sides of (2.3) are identical when we take $a = q^{-sn(mj+1)}$ for any $j$ with $0 \leq j \leq \left\lfloor \frac{n^r-1}{m} \right\rfloor$, i.e.,

$$
\sum_{k=0}^{n^r-1} \frac{2(q^{-sn(mj+1)}; q^m)_k(q^{m-s+sn(mj+1)}; q^m)_k q^{mk}}{(q^m; q^m)_k^2 (1 + q^{mk})} \equiv (-1)^{(-s/m)n} \sum_{k=0}^{n^r-1} \frac{2(q^{-snmj}; q^{mn})_k(q^{mn+snmj}; q^{mn})_k q^{mnk}}{(q^{mn}; q^{mn})_k^2 (1 + q^{mnk})},
$$

(2.4)
or \( a = q^{(m-s)n(mj+1)} \) for any \( j \) with \( 0 \leq j \leq \left\lfloor \frac{n^{r-1}-1}{m-s} \right\rfloor \), i.e.,

\[
\sum_{k=0}^{n^{r-1}-1} \frac{2(q^{s(m-s)n(mj+1)}; q^m)_k(q^{m-s-(m-s)n(mj+1)}; q^m)_k q^{mnk}}{(q^m; q^m)_k(1 + q^{mnk})} = (-1)^{\lfloor s/m \rfloor} \sum_{k=0}^{n^{r-1}-1} \frac{2(q^{mj(m-s)n+mn}; q^m)_k(q^{-mj(m-s)n}; q^m)_k q^{mnk}}{(q^m; q^m)_k(1 + q^{mnk})}.
\]

(2.5)

It is easy to see that \( n^r - 1 \geq snj + s(n-1)/m \) for \( 0 \leq j \leq \left\lfloor \frac{n^{r-1}-1}{s} \right\rfloor \), and \( n^r - 1 \geq (m-s)nj + (m-s)(n-1)/m \) for \( 0 \leq j \leq \left\lfloor \frac{n^{r-1}-1}{m-s} \right\rfloor \). Since \( n \equiv 1 \pmod{m} \), we know \( \langle -s/m \rangle_n = s(n-1)/m \) and \( \langle (s - m)/m \rangle_n = (m-s)(n-1)/m \). By Lemma 2.1 the left side of (2.3) is equal to

\[
(-1)^{sj+s(n-1)/m}.
\]

Likewise, the right-side of (2.3) is equal to

\[
(-1)^{s(n-1)/m}(-1)^{s(mj+1)-1)/m} = (-1)^{sj+s(n-1)/m}.
\]

This proves (2.4). Via a similar discussion as above, we can also prove the identity (2.5) is true. Namely, the \( q \)-congruence (2.3) is true modulo (2.2).

\[ \square \]

**Proof of Theorem 1.1.** This time the limit of (2.2) as \( a \to 1 \) has the factor

\[
\prod_{j=1}^{r} \Phi_{n/j}(q)^{\left\lfloor \frac{n^{r-j-1}}{s} \right\rfloor + \left\lfloor \frac{n^{r-j-1}}{m-s} \right\rfloor + 2},
\]

where we use the fact that the sets

\[
\left\{ \{sn(mj + 1) : j = 0, \ldots, \left\lfloor \frac{n^{r-1}-1}{s} \right\rfloor \}, \right. \\
\left. \{(m-s)n(mj + 1) : j = 0, \ldots, \left\lfloor \frac{n^{r-1}-1}{m-s} \right\rfloor \right\}
\]

contain exactly \( \left\lfloor \frac{n^{r-j-1}}{s} \right\rfloor + \left\lfloor \frac{n^{r-j-1}}{m-s} \right\rfloor + 2 \) multiples of \( n^j \) for \( j = 1, \ldots, r \). Here we need to show that \( n^j-1([\frac{n^{r-j-1}}{s}] + [\frac{n^{r-j-1}}{m-s}] + 2) \leq m^j + 1 \) for all integers \( 1 \leq j \leq r \). Hence, letting \( a \to 1 \) in (2.3) we conclude that (1.8) is true modulo \( \prod_{j=0}^{r} \Phi_{n/j}(q)^{2} \).

\[ \square \]

3. **Proof of Theorem 1.2**

We have the following parametric generalization of Theorem 1.2.

**Theorem 3.1.** Let \( m, n \) and \( s \) be positive integers with \( s < m \), \( n \equiv -1 \pmod{m} \) and \( n \) odd. Then, for \( r \geq 2 \), modulo

\[
\prod_{j=0}^{\left\lfloor \frac{n^{r-2}-1}{s} \right\rfloor} (1 - a q^{n^2(mj+1)}) \prod_{j=0}^{\left\lfloor \frac{n^{r-2}-1}{m-s} \right\rfloor} (a - q^{(m-s)n^2(mj+1)}),
\]

(3.1)
we have
\[
\sum_{k=0}^{n_r-1} 2(aq^s; q^m)_k(q^{m-s}/a; q^m)_kq^{mk} = \sum_{k=0}^{n_r-1} 2(aq^{mn^2}; q^{mn^2})_k(q^{mn^2-sn^2}/a; q^{mn^2})_kq^{mn^2k} \frac{(q^{mn^2}; q^{mn^2})_k^2(1 + q^{mn^2k})}{(q^m; q^m)_k^2(1 + q^{mk})}.
\]

Proof. Taking \( a = -q^{m^2(mj+1)} \) for \( 0 \leq j \leq \lfloor \frac{n_r-1}{s} \rfloor \) or \( a = q^{(m-s)n^2(mj+1)} \) for \( 0 \leq j \leq \lfloor \frac{n_r-s}{m-s} \rfloor \), by Lemma 2.1 the left-hand side of (3.2) can be written as
\[
\sum_{k=0}^{n_r-1} 2(q^{sn^2(mj+1)}; q^m)_k(q^{m-s+sn^2(mj+1)}; q^m)_kq^{mk} = (-1)^{(-s/m)(mj+1)s^2} = (-1)^{sn^2j-s(n^2-1)/m} = (-1)^{sj},
\]
and
\[
\sum_{k=0}^{n_r-1} 2(q^{s+(m-s)n^2(mj+1)}; q^m)_k(q^{(m-s)(m-s)n^2(mj+1)}; q^m)_kq^{mk} = (-1)^{(-s/m)(mj+1)s^2} = (-1)^{sn^2j-s(n^2-1)/m} = (-1)^{sj}.
\]

Similarly, the right-hand side of (3.2) is equal to
\[-1)^{(-s/m)(mj+1)} = (-1)^{sj},
\]
and so the \( q \)-congruence (3.2) is true modulo (3.1).

Proof of Theorem 1.2. As in the previous considerations, the limit of (3.1) as \( a \to 1 \) has the factor \( \prod_{j=1}^{[r/2]} \Phi_{n^2j}(q)^{\lfloor \frac{n-r^2j}{s} \rfloor + \lfloor \frac{n-r^2j}{m-s} \rfloor + 2} \), where we use the fact that the sets
\[
\{sn^2(mj+1) : j = 0, \ldots, \lfloor \frac{n-r^2j}{s} \rfloor \},
\[
\{(m-s)n^2(mj+1) : j = 0, \ldots, \lfloor \frac{n-r^2j}{m-s} \rfloor \},
\]
contain exactly \( \lfloor \frac{n-r^2j}{s} \rfloor + \lfloor \frac{n-r^2j}{m-s} \rfloor + 2 \) multiples of \( n^2j \) for \( j = 1, \ldots, [r/2] \). The proof of (1.9) modulo \( \prod_{j=1}^{[r/2]} \Phi_{n^2j}(q)^2 \) then follows by taking the limit as \( a \to 1 \) in (3.1).

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