Quick cut-elimination for strictly positive cuts

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Abstract
In this paper we show that the intuitionistic theory $\hat{ID}^i_{<\omega}(SP)$ for finitely many iterations of strictly positive operators is a conservative extension of the Heyting arithmetic. The proof is inspired by the quick cut-elimination due to G. Mints. This technique is also applied to fragments of Heyting arithmetic.

1 Introduction
Let us consider in this paper the fixed point predicate $I(x)$ for positive formula $\Phi(X,x)$:

$$(FP)^\Phi \forall x[I(x) \leftrightarrow \Phi(I, x)] \tag{1}$$

W. Buchholz showed that an intuitionistic fixed point theory $\hat{ID}^i(\mathcal{M})$ is conservative over the Heyting arithmetic HA with respect to almost negative formulas, in which $\vee$ does not occur and $\exists$ occurs in front of atomic formulas only). The theory $\hat{ID}^i(\mathcal{M})$ has the axioms $(FP)^\Phi$ for fixed points for monotone formula $\Phi(X,x)$, which is generated from arithmetic atomic formulas and $X(t)$ by means of (first order) monotonic connectives $\vee, \wedge, \exists, \forall$. Namely $\rightarrow$ nor $\neg$ does occur in monotone formula. The proof is based on a recursive realizability interpretation.

After seeing the result of Buchholz, we showed that an intuitionistic fixed point (second order) theory is conservative over HA for any arithmetic formulas. In the theory the operator $\Phi$ for fixed points is generated from $X(t)$ and any second order formulas by means of first order monotonic connectives and second order existential quantifiers $\exists f(\in \omega \rightarrow \omega)$. Moreover the same holds for the finite iterations of these operations. The proof is based on N. Goodman’s theorem.

*The paper has been finished when I visited München. I would like to thank to Prof. W. Buchholz for his interests, the valuable comments and the hospitality in my visit.
Next C. Rüede and T. Strahm extends significantly the results in [3] and [1]. They showed that the intuitionistic fixed point theory $\hat{ID}_{\omega}(SP)$ for finitely many iterations of strictly positive operators is conservative over HA with respect to negative and $\Pi^0_2$-formulas.

In this paper we show a full result.

**Theorem 1** For each $n < \omega$, $\hat{ID}_n(SP)$ is conservative over HA with respect to any arithmetic formulas. In other words $\hat{ID}_{\omega}(SP)$ is a conservative extension of HA.

Our proof is based on a quick cut-elimination of strictly positive cuts with arbitrary antecedents, cf. Theorem. The proof is inspired by G. Mints’ quick cut-elimination of monotone cuts in [7], and was found in an attempt to clarify ideas in [2].

We will give a proof of the non-iterated case, $n = 1$, and indicate necessary modifications for the general cases in the subsection 4.1. Let us explain an idea of our proof more closely. The story is essentially the same as in [2]. First the finitary derivations in $\hat{ID}_1(SP)$ are embedded to infinitary derivations, and eliminate cuts partially. This results in an infinitary derivation of depth less than $\varepsilon_0$, and in which there occurs cut inferences with cut formulas $I_\Phi(t)$ for fixed points only. Now the constraint on operator $\Phi$ admits us to eliminate strictly positive cut formulas quickly. In this way we will get an infinitary derivation of depth less than $\varepsilon_0$, and in which there occurs no fixed point formulas.

By formalizing the arguments we see that the end formula is true in HA.

In the section 5 we show that monotone cuts with negative antecedents can be eliminated more quickly. In the final section 6 these techniques are applied to fragments of Heyting arithmetic.

### 2 An intuitionistic theory $\hat{ID}(SP)$

$L_{HA}$ denote the language of the Heyting arithmetic. Logical connectives are $\lor, \land, \rightarrow, \exists, \forall$. $\neg A := (A \rightarrow \bot)$. Let $I$ be a fresh unary predicate symbol not in $L_{HA}$, and $L_{HA}(I)$ denotes $L_{HA} \cup \{I\}$.

Let $SP$ be the class of $L_{HA}(I)$-formulas such that $A \in SP$ iff $I$ occurs only strictly positive in $A$. The class $SP$ is defined inductively.

**Definition 2** Define inductively a class of formulas $SP$ in $L_{HA}(I)$ as follows.

1. Any atomic formula in $L_{HA}$ belongs to $SP$.
2. Any atomic formula $I(t)$ belongs to the class $SP$.
3. If $R, S \in SP$, then $R \lor S, R \land S, \exists x R, \forall x R \in SP$.
4. If $L \in L_{HA}$ and $R \in SP$, then $L \rightarrow R \in SP$. 


Let \( \hat{D}^{i}(SP) \) denote the following extension of HA. Its language is obtained from \( L_{HA} \) by adding a unary set constant \( I \) for a \( \Phi \equiv \Phi(I, x) \in SP \), in which only a fixed variable \( x \) occurs freely. Its axioms are those of HA in the expanded language, i.e., the induction axioms are available for any formulas in the expanded language) plus the axiom \((FP)^{\Phi}, \{I\} \) for fixed points.

### 3 Infinitary derivations

Given an \( \hat{D}^{i}(SP) \)-derivation \( D_{0} \) of an \( L_{HA} \)-sentence \( C_{0} \), let us first embed it to an infinitary derivation in an infinitary calculus \( \hat{D}^{i\infty}(SP) \).

Let \( N \) denote a number which is big enough so that any formula occurring in \( D_{0} \) has logical complexity, which is defined by the number of occurrences of logical connectives) smaller than \( N \). In what follows any formula occurring in infinitary derivations which we are concerned, has logical complexity less than \( N \).

The derived objects in the calculus \( \hat{D}^{i\infty}(SP) \) are sequents \( \Gamma \Rightarrow A \), where \( A \) is a sentence (in the language of \( \hat{D}^{i}(SP) \)) and \( \Gamma \) denotes a finite set of sentences, where each closed term \( t \) is identified with its value \( \bar{n} \), the \( n \)th numeral.

\( \perp \) stands ambiguously for false equations \( t = s \) with closed terms \( t, s \) having different values. \( \top \) stands ambiguously for true equations \( t = s \) with closed terms \( t, s \) having same values.

The initial sequents are

\[
\Gamma, I(t) \Rightarrow I(t); \quad \Gamma, \perp \Rightarrow A; \Gamma \Rightarrow \top
\]

The inference rules are \((L\vee), (R\vee), (L\wedge), (R\wedge), (L\rightarrow), (R\rightarrow), (L\exists), (R\exists), (L\forall), (R\forall), (LI), (RI) \) and \((\text{cut})\). These are standard ones.

1. 
\[
\frac{\Gamma, \Phi(I, t) \Rightarrow C}{\Gamma, I(t) \Rightarrow C} \quad (LI) \quad \frac{\Gamma \Rightarrow \Phi(I, t)}{\Gamma \Rightarrow I(t)} \quad (RI)
\]

2. 
\[
\frac{\Gamma, A_{0} \Rightarrow C \quad \Gamma, A_{1} \Rightarrow C}{\Gamma, A_{0} \vee A_{1} \Rightarrow C} \quad (L\vee) \quad \frac{\Gamma \Rightarrow A_{i}}{\Gamma \Rightarrow A_{0} \vee A_{1}} \quad (R\vee) \quad (i = 0, 1)
\]

3. 
\[
\frac{\Gamma, A_{0} \wedge A_{1} \Rightarrow C}{\Gamma, A_{0} \wedge A_{1} \Rightarrow C} \quad (L\wedge) \quad \frac{\Gamma \Rightarrow A_{0} \quad \Gamma \Rightarrow A_{1}}{\Gamma \Rightarrow A_{0} \wedge A_{1}} \quad (R\wedge) \quad (i = 0, 1)
\]

4. 
\[
\frac{\Gamma, A \Rightarrow B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \Rightarrow B \Rightarrow C} \quad (L\rightarrow) \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \Rightarrow B} \quad (R\rightarrow)
\]
5.  
\[
\begin{array}{c}
\vdots \quad \Gamma, B(\vec{n}) \Rightarrow C \quad \cdots(n \in \omega) \\
\Gamma, \exists x B(x) \Rightarrow C \quad (L\exists) \\
\quad \Gamma \Rightarrow B(\vec{n}) \quad (R\exists)
\end{array}
\]

6.  
\[
\begin{array}{c}
\Gamma, \forall x B(x), B(\vec{n}) \Rightarrow C \quad (L\forall) \\
\Gamma, \forall x B(x) \Rightarrow C \quad (R\forall)
\end{array}
\]

7.  
\[
\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C \\
\Gamma, \Delta \Rightarrow C \quad \text{(cut)}
\]

The depth of an infinitary derivation is defined to be the depth of the well founded tree.

As usual we see the following proposition. Recall that $N$ is an upper bound of logical complexities of formulas occurring in the given finite derivation $D_0$ of $L_{HA}$-sentence $C_0$.

**Proposition 3**  
1. There exists an infinitary derivation $D_1$ of $C_0$ such that its depth is less than $\omega^2$ and the logical complexity of any sentence, in particular cut formulas occurring in $D_1$ is less than $N$.

2. By a partial cut-elimination, there exist an infinitary derivation $D_2$ of $C_0$ and an ordinal $\alpha_0 < \varepsilon_0$ such that the depth of the derivation $D_2$ is less than $\alpha_0$ and any cut formula occurring in $D_2$ is an atomic formula $I(t)$, and the logical complexity of any formula occurring in it is less than $N$.

The rank $rk(A)$ of sentences $A$ is defined.

**Definition 4** The rank $rk(A)$ of a sentence $A$ is defined by

\[
rk(A) := \begin{cases} 
0 & \text{if } A \in L_{HA} \\
1 & \text{if } A \in SP \setminus L_{HA} \\
2 & \text{otherwise}
\end{cases}
\]

Let us call a cut inference $HA$-cut [$I$-cut] if its cut formula is of rank 0 [of rank 1], resp.

Let $\vdash_\alpha \Gamma \Rightarrow C$ mean that there exists an infinitary derivation of $\Gamma \Rightarrow C$ such that its depth is at most $\alpha$, and its rank is less than $r(\alpha)$, and the logical complexity of any formula occurring in it is less than $N$.

The following Lemmas are seen as usual.

**Lemma 5** (Weakening lemma)  
If $\vdash_\alpha \Gamma \Rightarrow A$, then $\vdash_\alpha \Delta, \Gamma \Rightarrow A$.

**Lemma 6** (Inversion Lemma)  
Assume $\vdash_\alpha \Gamma \Rightarrow A$. 

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1. If $A \equiv B_0 \land B_1$, then $\vdash_1 \Gamma \Rightarrow B_i$ for any $i = 0, 1$.
2. If $A \equiv \forall x B(x)$, then $\vdash_1 \Gamma \Rightarrow B(\bar{n})$ for any $n \in \omega$.
3. If $A \equiv B_0 \rightarrow B_1$, then $\vdash_1 \Gamma, B_0 \Rightarrow B_1$.
4. If $A$ is an atomic formula $I(t)$, then $\vdash_1 \Gamma \Rightarrow \Phi(I, t)$.
5. If $A \equiv \bot$, then $\vdash_1 \Gamma \Rightarrow C$ for any $C$.
6. If $\top \in \Gamma$, then $\vdash_1 \Gamma \Rightarrow A$ for $\Gamma_1 \cup \{\top\} = \Gamma$.

Let $3_2(\beta) := 3_3\beta$.

Theorem 7 Suppose that $\vdash_2 \beta \Rightarrow C$. Then $\vdash_1 3_2(\beta) \Rightarrow C$.

Assuming the Theorem 7, we can show the Theorem for $n = 1$ as follows. Suppose an $L_{HA}$-sentence $C_0$ is provable in $\widehat{ID} (SP)$. By Proposition 3 we have $\vdash_0 \Rightarrow C_0$ for a big enough number $N$ and an $\alpha_0 < \varepsilon_0$. Then Theorem 7 yields $\vdash_1 \beta_0 \Rightarrow C_0$ for $\beta_0 = 3_2(\alpha_0) < \varepsilon_0$.

Let $Tr_N(x)$ denote a partial truth definition for formulas of logical complexity less than $N$. By transfinite induction up to $\beta_0$ we see $Tr_N(C_0)$. Note that any sentence occurring in the witnessed derivation for $\vdash_1 \beta_0 \Rightarrow C_0$ has logical complexity less than $N$, and it is an $L_{HA}$-sentence. Specifically there occurs no fixed point formula $I(t)$ in it. Now since everything up to this point is formalizable in HA, we have $Tr_N(C_0)$, and hence $C_0$ in HA. This shows the Theorem 7 for the case $n = 1$.

Additional informations equipped with infinitary derivations together with the repetition rule ($Rep$)

$$\Gamma \Rightarrow C \quad (Rep)$$

are helpful when we formalize our proof as in [6]. In this paper let us suppress these.

A proof of Theorem 7 is given in the next section.

4 Quick cut-elimination of strictly positive cuts with arbitrary antecedents

In this section we show that strictly positive cuts can be eliminated quickly even if antecedents of cut inferences and endsequents are arbitrary formulas. The only constraint is that any cut formula has to be strictly positive.

Let $\alpha \# \beta$ denote the natural sum or commutative sum, $\alpha \# \beta = \beta \# \alpha$, and $\alpha \times \beta$ the natural product.

Theorem 7 follows from the following Lemma 8.
Lemma 8  For arbitrary $\Gamma, \Delta$ and $C$, if $rk(A) = 1$,
\[
\vdash_1^\alpha \Gamma \Rightarrow A \text{ and } \vdash_2^\beta \Delta, A \Rightarrow C
\]
then
\[
\vdash_1^{\alpha \times 3_2(\beta)} \Delta, \Gamma \Rightarrow C.
\]

Proof of Theorem 7 by induction on $\beta$. Suppose that $\vdash_2^\beta \Gamma \Rightarrow C$.
Consider the case when the last rule is an $I$-cut:
\[
\begin{align*}
\vdash_2^\gamma \Gamma & \Rightarrow A \\
\vdash_2^\gamma \Delta, A & \Rightarrow C
\end{align*}
\]
with $rk(A) = 1$ and $\gamma < \beta$.
By IH (= Induction Hypothesis) we have
\[
\vdash_1^{3_2(\gamma)} \Gamma \Rightarrow A
\]
Then Lemma 8 yields $\vdash_1^{3_2(\beta)} \Gamma \Rightarrow C$ since
\[
\gamma < \beta \Rightarrow 3_2(\gamma) \times 3_2(\gamma) < 3_2(\beta).
\]
This shows Theorem 7 assuming Lemma 8.

Next we show Lemma 8.

As in Lemma 3.2, [7] eliminating procedure is fairly standard, leaving the resulted cut inferences of rank 0, but has to performed in parallel.

$A$ denotes a finite list $A_k, \ldots, A_2, A_1$ ($k \geq 0$) of $SP$-formulas, and $\alpha = \alpha_k, \ldots, \alpha_2, \alpha_1$ a list of ordinals. Then $\vdash_1^\alpha \Gamma \Rightarrow A$ designates that $\vdash_1^{\alpha_i} \Gamma \Rightarrow A_i$ for each $i$.

\[
\sum \alpha := \begin{cases} 
\alpha_1 \# \cdots \# \alpha_k & \text{if } k > 0 \\
1 & \text{if } k = 0
\end{cases}
\]

$A_1$ denotes the list $A_k, \ldots, A_2$, in which $A_1$ is deleted. Likewise $\alpha_1$ denotes the list $\alpha_k, \ldots, \alpha_2$.

Lemma 9  Suppose $\vdash_1^\alpha \Gamma \Rightarrow A$ and $\vdash_2^\beta \Delta, A \Rightarrow C$. Then
\[
\vdash_1^{(\sum \alpha) \times 3_2(\beta)} \Delta, \Gamma \Rightarrow C
\]
(2)

Note that the case $k = 0$ in Lemma 9 is nothing but Theorem 7.
We prove Lemma 9 by main induction on $\beta$ with subsidiary induction on $\sum \alpha + k$, where $k$ is the length of the list $A$.

1. The case when one of $\Gamma \Rightarrow A_i, \Delta, A \Rightarrow C$ is an initial sequent.
First consider the case when $\Delta, A \Rightarrow C$ is an initial sequent.
If $\Delta, A \Rightarrow C$ is an initial sequent such that one of the cases $C \equiv \top, \bot \in \Delta$ or $C \in \Delta$ occurs, then $\Delta \Rightarrow C$, and hence $\Delta, \Gamma \Rightarrow C$ is still the same kind of initial sequent.
If $\Delta, A \Rightarrow C$ is an initial sequent with the principal formula $A \supset A_i \equiv C \equiv I(t)$, then $\Delta, \Gamma \Rightarrow A_i(\equiv C)$ is an initial sequent.

If $A_i \equiv \bot$, then Inversion lemma 6.5 with a weakening yields $\vdash \alpha_i \Delta, \Gamma \Rightarrow C$. ($\sum \alpha_i \times 3_2(\beta) \geq \alpha_i$ yields [2].)

Next assume $\Gamma \Rightarrow A_i$ is an initial sequent for an $i$. This implies $k > 0$. For simplicity assume $i = 1$.

If $A_1 \equiv \top$, then Inversion lemma 6.6 yields $\vdash \beta \Delta, \Gamma \Rightarrow C$, and by SIH we have $\vdash (\sum \alpha_1) \times 3_2(\beta) \Delta, \Gamma \Rightarrow C$ with $A_1 \in \Gamma$ and $\sum \alpha_1 = 1$ by the definition.

Otherwise by SIH we have $\vdash (\sum \alpha_1) \times 3_2(\beta) \Delta, \Gamma \Rightarrow A_{k+1}$ with $\sum \alpha_1 = 1$ by the definition.

MIH (=Main Induction Hypothesis) yields $\vdash (\sum \alpha_1) \times 3_2(\beta) \Delta, \Gamma \Rightarrow C$, and once again by MIH and $\sum \alpha_1 \times 3_2(\beta) \leq (\sum \alpha_1) \times 3_2(\beta)$

we conclude $\vdash (\sum \alpha_1) \times 3_2(\beta) \Delta, \Gamma \Rightarrow C$.

We will depict a ‘derivation’ to illustrate the arguments.

In what follows assume that none of $\Gamma \Rightarrow A_i$, $\Delta, A \Rightarrow C$ is an initial sequent.

2. Consider the case when $\Delta, A \Rightarrow C$ is a lower sequent of an $I$-cut. For a $\gamma < \beta$

\[
\begin{array}{c}
\vdash_1^\alpha \Gamma \Rightarrow A \\
\vdash_2^\beta \Delta, A \Rightarrow C \\
\Delta, \Gamma \Rightarrow C
\end{array}
\]

\[
\Delta, \Gamma \Rightarrow C
\]

with $rk(A_{k+1}) = 1$.

MIH (=Main Induction Hypothesis) yields $\vdash (\sum \alpha_1) \times 3_2(\gamma) \Delta, \Gamma \Rightarrow A_{k+1}$, and once again by MIH and

\[
(\sum \alpha \#(\sum \alpha) \times 3_2(\gamma)) \times 3_2(\gamma) \leq (\sum \alpha) \times 3_2(\beta)
\]

we conclude $\vdash (\sum \alpha_1) \times 3_2(\beta) \Delta, \Gamma \Rightarrow C$.

In what follows assume that $\Delta, A \Rightarrow C$ is a lower sequent of an inference rule $J$ other than $I$-cut.
3. If the principal formula of $J$ if any is not in $A$, then lift up the left upper part: for a $\gamma < \beta$

\[ \cdots \vdash \Delta_i, A \Rightarrow C_i \cdots \]

\[ \Delta, \Gamma \Rightarrow C \]

\[ \vdash \Delta_i, A \Rightarrow C_i \quad MIH \]

\[ \cdots \vdash (\sum \alpha) \times 3_2(\gamma) \Delta_i, \Gamma \Rightarrow C_i \cdots \]

\[ \vdash (\sum \alpha) \times 3_2(\beta) \Delta, \Gamma \Rightarrow C \]

Note that $(\sum \alpha) < (\sum \alpha) \times 3_2(\beta)$, i.e., $(\sum \alpha) > 0$ by the definition.

4. Finally suppose that the principal formula of $J$ is a cut formula $A_i \in A$ of $rk(A_i) = 1$. For simplicity suppose $i = 1$. Use the Inversion Lemma \[ \text{if available. Otherwise examine the left upper part } \vdash \Delta, \Gamma \Rightarrow A. \]

(a) The case when $A_i \equiv \exists x B(x) \in A$.

\[ \cdots \vdash \Delta, A_1, B(\bar{n}) \Rightarrow C \cdots \]

\[ \vdash \Delta, A \Rightarrow C \]

where $A_1 \not\in A_1$.

We will examine the last rule in $\vdash \Delta, \Gamma \Rightarrow A_1(\equiv \exists x B(x))$.

i. If $\exists x B(x)$ is derived by an $(R \exists)$,

\[ \vdash \Delta_i, A \Rightarrow C_i \quad MIH \]

\[ \vdash \Delta, A \Rightarrow C \]

\[ \vdash \Delta_0, B(\bar{n}) \Rightarrow \exists x B(x) \quad (R \exists) \]

then

\[ \vdash \Delta, \Gamma \Rightarrow \exists x B(x) \]

\[ \sum \alpha \not\in A_1 \]

ii. If the last rule is a left rule, then postpone it.

For example $\exists y D(y) \in \Gamma$

\[ \cdots \vdash \Delta, D(\bar{n}) \Rightarrow \exists x B(x) \cdots \]

\[ \vdash \Delta, \Gamma \Rightarrow \exists x B(x) \quad (L \exists) \]

Then $\alpha_0 < \alpha_1$, and hence $\sum \alpha_1 \# \alpha_0 < \sum \alpha_1 \# \alpha_1 = \sum \alpha$. Thus SIH yields

\[ \vdash (\sum \alpha_1 \# \alpha_0) \times 3_2(\beta) \Delta, \Gamma, D(\bar{n}) \Rightarrow C \]
for each $n$.

\[
\frac{
\vdash_{\alpha_1} \Gamma \Rightarrow A_1 \quad \vdash_{\alpha_0} \Gamma, D(n) \Rightarrow \exists x B(x) \quad \vdash_{\beta} \Delta, A_1, \exists x B(x) \Rightarrow C
}{\Delta, \Gamma, D(n), A_1 \Rightarrow C}
\]

\[\vdash_{\alpha_1} \sum_{\alpha_1, \# \alpha_0} \times 3 \exists \beta \Delta, \Gamma \Rightarrow C \]

\[\vdash_{\alpha_1} \Delta, \Gamma, D(n) \Rightarrow C \]

Consider next \((L \rightarrow)\). Let \(D \rightarrow E \in \Gamma\).

\[
\frac{
\vdash_{\alpha_0} \Gamma \Rightarrow D \quad \vdash_{\alpha_0} \Gamma, E \Rightarrow \exists x B(x)
}{\vdash_{\alpha_1} \Gamma \Rightarrow \exists x B(x)}
\]

\[\vdash_{\alpha_1} \sum_{\alpha_1, \# \alpha_0} \times 3 \exists \beta \Delta, \Gamma, E \Rightarrow C \]

\[\vdash_{\alpha_1} \Delta, \Gamma \Rightarrow C \]

Finally consider an HA-cut with \(rk(D) = 0\).

\[
\frac{
\vdash_{\alpha_0} \Gamma \Rightarrow D \quad \vdash_{\alpha_0} \Gamma, D \Rightarrow \exists x B(x)
}{\vdash_{\alpha_1} \Gamma \Rightarrow \exists x B(x)}
\]

\[\vdash_{\alpha_1} \sum_{\alpha_1, \# \alpha_0} \times 3 \exists \beta \Delta, \Gamma, D, A_1 \Rightarrow C \]

\[\vdash_{\alpha_1} \Delta, \Gamma, D \Rightarrow C \]

\[\vdash_{\alpha_1} \Delta, \Gamma \Rightarrow C \]

(b) The case when \(A_1 \equiv H \rightarrow A_0 \in A\) with an \(H \in L_{HA}\) and an \(A_0 \in SP\). For a \(\gamma < \beta\)

\[
\frac{
\vdash_{\gamma} \Delta, A \Rightarrow H \quad \vdash_{\gamma} \Delta, A_1, A_0 \Rightarrow C
}{\vdash_{\gamma} \Delta, A \Rightarrow C}
\]

where \(A_1 \not\subset A_1\).

\[
\frac{
\vdash_{\alpha_1} \Gamma \Rightarrow A \quad \vdash_{\gamma} \Delta, A \Rightarrow H
}{\Delta, \Gamma, H \Rightarrow \exists x B(x)}
\]

\[\frac{
\vdash_{\alpha_1} \Gamma \Rightarrow A_1 \quad \vdash_{\alpha_1} \Gamma, H \Rightarrow A_0 \quad \vdash_{\alpha_1} \Delta, A_0 \cup A_1 \Rightarrow C
}{\Delta, A_1, \Gamma, H \Rightarrow C}
\]

\[\vdash_{\alpha_1} \Delta, \Gamma, H \Rightarrow C \]

\[\vdash_{\alpha_1} \Delta, \Gamma \Rightarrow C \]

where \(\vdash_{\alpha_1} \Gamma, H \Rightarrow A_0\) by inversion, and \(rk(H) = 0\).
(c) The case when $A_i \equiv \forall x B(x) \in A$. For a $\gamma < \beta$

$$\vdash \gamma \Delta, A, B(\bar{n}) \Rightarrow C \quad (\forall)$$

By $(\sum \alpha \# \alpha_1) \times 3_2(\gamma) \leq (\sum \alpha) \times 3_2(\beta)$ and inversion $\vdash \gamma^1 \Gamma \Rightarrow B(\bar{n})$ we have

$$\vdash \gamma^1 \Gamma \Rightarrow A \quad \vdash \gamma^1 \Delta \Rightarrow B(\bar{n}) \Rightarrow C \quad \Delta, \Gamma \Rightarrow C$$

$d$ The case when $A_i \equiv B_0 \lor B_1 \in A$ is treated as in the Case (4a) for existential quantifier.

$e$ The case when $A_i \equiv B_0 \land B_1 \in A$ is treated as in the Case (4c) for universal quantifier.

$f$ The case when $A_i$ is a formula $I(t)$. Use Inversion $\vdash \gamma^1 \Gamma \Rightarrow \Phi(I, t)$.

This completes a proof of (2), and hence of Lemma 9.

4.1 Finite iterations

Our proof is easily extended to finite iterations of fixed points for strictly positive operators. The theory $\hat{ID}_n(SP)$ has the following axiom for formulas $\Phi(X, Y, x)$ in which $X$ occurs only strictly positive:

$$\forall i < n \forall x[x \in I^\Phi_i \leftrightarrow \Phi(I^\Phi_i, I^\Phi_{<i}, x)]$$

where $I^\Phi_j = \{(y, j) : y \in I^\Phi_i \text{ & } j < i\}$.

Let us explain how to modify the proof. For simplicity consider the case $n = 2$. Drop the superscript $\Phi$ in $I^\Phi$, and identify $I_{<1}$ with $I_0$. Let $\Phi_i(X, x) :\Leftrightarrow \Phi(X, I_{<i}, x)$. The initial sequents $\Gamma, I_i(t) \Rightarrow I_i(t)$ and the inference rules $(LI), (RI)$ are for each $i = 0, 1$

$$\Gamma, \Phi_i(I_i(t), t) \Rightarrow C \quad (LI)_i ; \quad \Gamma \Rightarrow \Phi_i(I_i(t), t) \quad (RI)_i$$

The rank $rk(A) \leq 3$ of sentences $A$ is defined. Let $SP_i$ denote the set of formulas in which $I_i$ occurs only strictly positive. Let $SP_{-1} := L_{HA}$.

**Definition 10** The rank $rk(A)$ of a sentence $A$ is defined by

$$rk(A) := \begin{cases} 
0 & \text{if } A \in L_{HA} \\
 i + 1 & \text{if } A \in SP_i \setminus SP_{i-1} (i = 0, 1) \\
 3 & \text{otherwise}
\end{cases}$$

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Then Theorem 7 runs as follows.

**Theorem 11**

1. Suppose that \( \vdash_2 \beta \Gamma_0 \Rightarrow C_0 \). Then \( \vdash_3 ^{3\beta} \Gamma_0 \Rightarrow C_0 \).

2. There exists an \( m < \omega \) for which the following holds.

Suppose that \( \vdash_3 \beta \Gamma_0 \Rightarrow C_0 \). Then \( \vdash_2 ^{2m(3\beta)} \Gamma_0 \Rightarrow C_0 \).

Suppose that \( \vdash_3 \beta \Gamma_0 \Rightarrow C_0 \). To prove Theorem 11.2, first eliminate cut inferences with cut formulas \( \Delta \), in which \( I_1 \) does occur strictly positive. The proof is the same as in one for Theorem 7. Then the depth of the resulting derivation \( D_1 \) is bounded by \( 3^{2\beta} \). Unfortunately this derivation \( D_1 \) might be still of rank 2 since for example if \( H \rightarrow A_0 \in SP_1 \), then \( H \) is an arbitrary formula in \( L_{HA}(I_0) \). In other words \( I_0 \) might occur in \( H \) arbitrarily, and we left the cut inference of cut formula \( H \) in the Case (4b) of the proof of Lemma 9.

Now observe that \( H \) is a subformula of the fixed operator \( \Phi_1(I_1,n) \equiv \Phi(I_1,I_0,n) \). This means that the logical complexity of \( H \) can be bounded in advance. Let \( m \) be the number of occurrences of logical symbols in \( \Phi \). Then in \( D_1 \) eliminate cut inferences of rank 2(, but \( I_1 \) does not occur in their cut formulas), to get a derivation \( D_2 \) of depth \( 2^m(3\beta) \) and of rank 1. In \( D_2 \) any cut formula is either an HA-formula or of the form \( I_0(t) \). Then apply Theorem 11.1 to get an HA-derivation of depth less than \( \varepsilon_0 \).

Obviously this elimination procedure can be iterated finite times as you need, and the depth of the resulting HA-derivation is less than \( \varepsilon_0 \). This proves the general case in Theorem 1.

### 5 Quick cut-elimination of monotone cuts with negative antecedents

We show that monotone cuts with negative antecedents can be eliminated more quickly. In this section we consider the Heyting arithmetic \( HA \) and its infinitary counterpart \( HA^\infty \). First let us introduce a class \( \mathcal{N}M \) of \( L_{HA} \)-formulas. \( \mathcal{N} \) is the class of negative formulas.

**Definition 12** \( \mathcal{N} \) denotes the class of negative formulas, in which no disjunction and existential quantifier occur.

Define inductively a class of formulas \( \mathcal{N}M \) in \( L_{HA} \) as follows.

1. Any atomic formula \( s = t \) belongs to \( \mathcal{N}M \).

2. If \( R, S \in \mathcal{N}M \), then \( R \lor S, R \land S, \exists x R, \forall x R \in \mathcal{N}M \).

3. If \( L \in \mathcal{N} \) and \( R \in \mathcal{N}M \), then \( L \rightarrow R \in \mathcal{N}M \).

It is easy to see \( \mathcal{N} \subset \mathcal{N}M \).

Note that by the equivalence

\[
[\exists x A(x) \rightarrow B] \leftrightarrow \forall x [A(x) \rightarrow B]
\]  

(3)
∃x A(x) → B for A ∈ N, B ∈ NM is equivalent to the NM-formula ∀x[A(x) → B].

The rank rk(A) of sentences A is redefined as follows.

**Definition 13** The rank rk(A) of a sentence A is defined by

\[
 rk(A) := \begin{cases} 
 0 & \text{if } A \in N \\
 1 & \text{if } A \in NM \setminus N \\
 2 & \text{otherwise}
\end{cases}
\]

Let HA∞ denote an infinitary system in the language LHA, whose initial sequents and inference rules are obtained from those of \( \hat{D}^∞(SP) \) by deleting the initial sequents \( \Gamma, I(t) \Rightarrow I(t) \) and inference rules (LI), (RI).

By restricting antecedents to negative (or Harrop) formulas we have a stronger inversion.

**Lemma 14** (Inversion Lemma with negative antecedents)

Assume \( \vdash_1 \Gamma \Rightarrow A \) such that \( \Gamma \subseteq N \).

1. If \( A \equiv B_0 \lor B_1 \), then \( \vdash_1 \Gamma \Rightarrow B_i \) for an \( i = 0, 1 \).
2. If \( A \equiv \exists x B(x) \), then \( \vdash_1 \Gamma \Rightarrow B(\bar{n}) \) for an \( n \in \omega \).

**Theorem 15** Let \( C_0 \) denote an NM-sentence, and \( \Gamma_0 \) a finite set of N-sentences. Suppose that \( \vdash_2 \Gamma_0 \Rightarrow C_0 \). Then \( \vdash_1 \Gamma_0 \Rightarrow C_0 \).

Again Theorem 15 follows from the following Lemma 16 for quick cut-elimination in parallel.

\( A \) denotes a non-empty finite list \( A_0, \ldots, A_2, A_1 \) (\( k > 0 \)) of NM-formulas, and \( \alpha \) an ordinal. Then \( \vdash_1 \Gamma \Rightarrow A \) designates that \( \vdash_1 \Gamma \Rightarrow A_i \) for any \( i \). Note here that the depth \( \alpha \) of the derivations of \( \Gamma \Rightarrow A_i \) is independent from \( i \).

**Lemma 16** Suppose \( \Gamma \cup \Delta \subseteq N \) and \( A \cup \{C\} \subseteq NM \). If

\[
 \vdash_1 \Gamma \Rightarrow A \text{ and } \vdash_2 \Delta, A \Rightarrow C
\]

then

\[
 \vdash_1^{\alpha+2^\beta} \Delta, \Gamma \Rightarrow C.
\]

We can prove Lemma 16 by induction on \( \beta \) as in Lemma 9. In Case (1) we don’t need to examine the left upper parts \( \vdash_1 \Gamma \Rightarrow A \). In Case (1) the Inversion Lemma on the succedent is always available since the antecedent \( \Gamma \) consists solely of negative formulas. Note that in the Case (4b) the remaining cut formula \( H \in N \) is in the class NM.

This completes a proof of Lemma 16 and of Theorem 15.

Note that the procedure leaves cuts with negative cut formulas \( H \) in Case (4b). If we restrict to eliminate monotone cuts, then cuts are eliminated quickly and completely.
Theorem 17 Let \( C_0 \) denote an \( \mathcal{N}, \mathcal{M} \)-sentence, and \( \Gamma_0 \) a finite set of \( \mathcal{N} \)-sentences. Suppose that there exists a derivation of \( \Gamma_0 \Rightarrow C_0 \) in which any cut formula is a monotone formula, and whose depth is at most \( \beta \). Then there exists a cut-free derivation of \( \Gamma_0 \Rightarrow C_0 \) in depth \( 2^\beta \).

Let us iterate this procedure for monotone cuts.

In what follows \( \Phi \) denotes a class of arithmetic formulas such that any atomic formula is in \( \Phi \), and \( \Phi \) is closed under substitution of terms for variables and renaming of bound variables.

Given such a class \( \Phi \) of formulas, introduce a hierarchy \( \{ \mathcal{M}_n(\Phi) \} \) of arithmetic formulas.

Definition 18 First set \( \mathcal{M}_1(\Phi) = \Phi \).

Define inductively classes of formulas \( \mathcal{M}_{n+1}(\Phi) \) \((n \geq 1)\) in \( L_{HA} \) as follows.

1. \( \mathcal{M}_n(\Phi) \subset \mathcal{M}_{n+1}(\Phi) \).
2. If \( R, S \in \mathcal{M}_{n+1}(\Phi) \), then \( R \lor S, R \land S, \exists x R, \forall x R \in \mathcal{M}_{n+1}(\Phi) \).
3. If \( L \in \mathcal{M}_n(\Phi) \) and \( R \in \mathcal{M}_{n+1}(\Phi) \), then \( L \rightarrow R \in \mathcal{M}_{n+1}(\Phi) \).

We have \( \bigcup_{n<\omega} \mathcal{M}_n(\Phi) = L_{HA} \).

For \( \Phi = \Sigma_1 \), \( \mathcal{M}_n(\Sigma_1) \) coincides with the class \( \Theta_n \) introduced by W. Burr [4]. Note that by (3) for any \( n \geq 2 \), each formula in \( \mathcal{M}_n(\Delta_0) \) is equivalent to a formula in \( \mathcal{M}_n(\Delta_0) \), where \( \Delta_0 \) is the class of all atomic formulas. Also each formula in \( \Theta_2 \) is equivalent to a monotone formula in \( \mathcal{M} \).

The rank \( rk(A; \Phi) \) of sentences \( A \) relative to the class \( \Phi \) is defined.

Definition 19 The rank \( rk(A; \Phi) \) of a sentence \( A \) is defined by

\[
 rk(A; \Phi) := \min\{ n - 1 : A \in \mathcal{M}_n(\Phi) \}.
\]

Let \( \vdash_r^\alpha \Gamma \Rightarrow C \) designate that there exists an infinitary derivation of \( \Gamma \Rightarrow C \) such that the depth of the derivation tree is bounded by \( \alpha \) and any cut formula occurring in it has rank less than \( r \). \( \vdash_r^\alpha \Gamma \Rightarrow C \) means that in the witnessed derivation of depth \( \alpha \) any cut formula is in the class \( \mathcal{M}_2(\Phi) \).

Theorem 20 Suppose that \( \vdash_{r+1}^\beta \Gamma_0 \Rightarrow C_0 \). Then \( \vdash_r^{3\beta} \Gamma_0 \Rightarrow C_0 \) for \( r \geq 2 \).

Proof: This is seen as in the proof of Theorem 11 but leave the cut inference of cut formula \( H \) with \( rk(H; \Phi) < r \) in the Case (4b). \( \square \)

6 Applications to fragments of Heyting arithmetic

Finally let us remark an application of quick cut-eliminations to fragments of Heyting arithmetic.
Definition 21 Let $\Phi$ be a class of arithmetic formulas such that any atomic formula is in $\Phi$, and $\Phi$ is closed under substitution of terms for variables and renaming of bound variables.

$i\Phi$ denotes the fragment of HA in which induction axioms are restricted to formulas in $\Phi$.

$$A(0) \land \forall x [A(x) \rightarrow A(x + 1)] \rightarrow \forall x A(x) \ (A \in \Phi).$$

For a class of formulas $\Psi$, $RFN_{\Psi}(i\Phi)$ denotes the $\Psi$-(uniform) reflection principle for $i\Phi$:

$$RFN_{\Psi}(i\Phi) = \{ \Pr_{i\Phi}([\varphi(\dot{x})]) \rightarrow \varphi(x) : \varphi \in \Psi \}$$

where $Pr_{i\Phi}$ denotes a standard provability predicate for $i\Phi$ and $\dot{x}$ is the $x$-th formalized numeral.

When $\Psi = L_{HA}$ the subscript $\Psi$ in $RFN_{\Psi}(i\Phi)$ is dropped.

By the result of Buchholz\cite{3} we see that HA proves the consistency of the intuitionistic arithmetic $iM$ for the class $M$ of monotone formulas since $\bar{ID}^i(M)$ can define the truth of monotone formulas, and the consistency statement $\text{CON}(iM)$ is an almost negative formula. Observe that any prenex $\Pi^0_1$-formula is a monotone formula, and any monotone formula is equivalent to a prenex formula.

Moreover using truth definition for $\Theta_n$-formulas and a partial truth definition we see that for each $n \geq 2 \ \bar{ID}^i_{n-1}(M)$ proves the soundness $RFN(i\Theta_n)$ of $i\Theta_n$. Hence HA $\vdash RFN(i\Theta_n)$ by the full conservativity of $\bar{ID}^i_n(M)$ over HA in \cite{1}.

However this does not show that $\{iM_n(\Phi)\}_n$ forms a proper hierarchy. Burr\cite{4}, Corollary 2.25 shows that $I\Pi^0_{n+1}$ proves the $2$-consistency $RFN_{I\Pi^0_2}(\Pi^0_n)$ of $I\Pi^0_n$ and hence of $i\Theta_n$, by the result of Burr we see that $i\Theta_{n+1}$ proves the $2$-consistency of $i\Theta_n$. Thus $\{i\Theta_n\}_n$ forms a proper hierarchy.

Let us show that $i\Theta_3$ proves the soundness of $i\Theta_2$ with respect to $\Theta_2$, $RFN_{\Theta_2}(i\Theta_2)$. Recall that $\Theta_2$, monotone formulas and formulas in prenex formulas are equivalent each other.

Let $<$ denote a standard $\varepsilon_0$-well ordering. Let $Prg[A] :\Leftrightarrow \forall x [\forall y < x A(y) \rightarrow A(x)]$ and for a class $\Phi$ of formulas, $TI(< \alpha, \Phi)$ denote the transfinite induction schema $Prg[A] \rightarrow \forall x < \beta A(x)$ for each $\beta < \alpha$ and $A \in \Phi$.

Also let $\omega_1 := \omega$ and $\omega_{m+1} := \omega^{\omega_m}$.

**Proposition 22** If $m + k \leq n + 2$, then

$$iM_n(\Phi) \vdash TI(< \omega_m, M_k(\Phi)).$$
Proof. Let

\[ j[A](α) :⇔ ∀β[∀γ < β A(γ) → ∀γ < β + ω^n A(γ)]. \]

Then for \( A ∈ M_n(Φ) \) we have \( j[A] ∈ M_{n+1}(Φ) \)

\[ HA(M_n(Φ)) ⊢ Prg[A] → Prg[j[A]] \]

and \( HA(M_n(Φ)) ⊢ TI(< ω_1, M_{n+1}(Φ)) \). The proposition follows from these. \( ∎ \)

Corollary 23

1. For \( n ≥ 2 \)

\[ iΘ_{2n−1} ⊢ RFN_{Θ_n}(iΘ_n). \]

For example \( iΘ_3 \) proves the soundness of prenex induction with prenex consequences.

2. For any \( m, k, n ≥ 1 \)

\[ iM_{2m+k}(Π^n_0) ⊢ RFN_{M_k(Π^n_0)}(iM_m(Π^n_0)). \]

Proof. 23.1 follows from Theorems 20, 17 and Proposition 22. Namely embed a finitary derivation of a monotone sentence \( C \) in \( iM_n(∆_0) \) to an infinitary one. Apply first Theorem 20 \((n − 2)\)-times, to get a derivation of \( C \) such that any cut formula occurring in it is a monotone formula and its depth is bounded by \( 3z_{2n−2}(ω^2) = ω_{2n−3} \). Then apply Theorem 17 to get a cut-free derivation of \( C \) in depth \( 2z_{2n−3} = ω_{2n−2} \). By Proposition 22 \( TI(< ω_{2n−1}, Θ_2) \) is provable in \( iΘ_{2n−1} \). Since any formula occurring in the cut-free derivation is a subformula of the monotone \( C ∈ Θ_2 \), by a \( Θ_2 \)-truth definition of subformulas of \( C \) we knows that \( C \) is true in \( iΘ_{2n−1} \).

23.2 follows from Theorem 20, quick cut-elimination of monotone cuts with arbitrary antecedents and Proposition 22. Namely embed a finitary derivation of a sentence \( C_0 ∈ M_k(Π^n_0) \) in \( iM_m(Π^n_0) \) to an infinitary one. Eliminate cuts by applying Theorem 20 \( m \)-times, and get a derivation of \( C_0 \) in depth \( 3z_{2m}(ω^2) = ω_{2m+1} \), and in which any cut formula is in \( Π^n_0 \). Any formula occurring in the derivation is either a subformula of \( C_0 ∈ M_k(Π^n_0) \) or a \( Π^n_0 \)-formula. Therefore using \( M_k(Π^n_0) \)-truth definition of sequents occurring in the derivation and \( TI(< ω_{2m+2}, M_k(Π^n_0)) \) we conclude that \( C_0 \) is true in \( iM_{2m+k}(Π^n_0) \). \( ∎ \)

Next consider conservations.

The following Corollary 24 shows, for example that \( iΘ_2 \) is \( Π^n_0 \)-conservative over \( iΠ^n_k \) for any \( k \), and generalizes a theorem by A. Visser and K. Wehmeier (cf. Theorem 3 in [9] and Corollary 2.28 in [4]) stating that \( iΘ_2 \) is \( Π^n_2 \)-conservative over \( iΠ^n_2 \).

Corollary 24 For any \( Φ ⊂ Θ_2 \), \( iΘ_2 \) is \( Φ \)-conservative over \( iΦ \).

Proof. Embed a finitary derivation of a monotone sentence \( C \) in \( iM_2(∆_0) \) to an infinitary one. Apply Theorem 17 to get a cut-free derivation of \( C \) in depth less than \( ω_2 \). \( ∎ \)
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