One-loop matching for gluon lattice TMDs

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ABSTRACT: Transverse-momentum-dependent parton distributions (TMDs) can be calculated from first principles by computing a related set of Euclidean lattice observables and connecting them via a factorization formula. This work focuses on the leading-power factorization formula connecting the lattice quasi-TMD and continuum Collins TMD for gluons. We calculate the one-loop gluon matching coefficient, which is known to be independent of spin and exhibits no mixing with quarks. We demonstrate that this coefficient satisfies Casimir scaling with respect to the quark matching coefficient at one-loop order. Our result facilitates reliable lattice QCD calculations of gluon TMDs.

KEYWORDS: Effective Field Theories of QCD, Factorization, Renormalization Group, Lattice QCD, Parton Distributions

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1 Introduction

Protons are a fundamental building block of almost all the visible matter in the universe, yet over a century after their discovery, their internal dynamics still confound us. The behavior of quarks and gluons inside the proton are described by universal functions appearing in cross sections, such as parton distribution functions (PDFs), transverse-momentum-dependent PDFs (TMDs), and generalized parton distributions (GPDs). Various collider facilities, such as Jefferson Lab, CERN, DESY, and Fermilab, have provided initial data on these distributions, and many have dedicated plans to increase their precision. These efforts will be further bolstered by the planned Electron-Ion Collider (EIC) [1, 2] at Brookhaven National Lab. The imminence of high-precision data about parton distributions underscores the need for developing a corresponding first-principles understanding.

Parton distributions can be rigorously defined in quantum field theory. Unfortunately, they always contain important nonperturbative components that cannot be computed directly using lattice QCD, as they are typically described by matrix elements involving light-like Wilson line paths. Light-like observables depend on a real-valued time variable, which induces a lattice sign problem: a numerical difficulty suspected to be NP-hard in general. Absent direct analytical or numerical approaches, physicists have traditionally constructed models for such observables, which are then compared to experimental data using global fits. Alternatively, one can carry out an indirect first-principles calculation: one constructs a numerically-tractable lattice observable that shares the same IR physics as the target distribution, but may differ in the ultraviolet (UV). This lattice observable must then be perturbatively matched back onto the desired physical continuum distribution via a factorization formula.

Here, we focus on the physical-to-lattice factorization program for TMDs, an important class of observables describing the three-dimensional momentum distribution of quarks and gluons inside a proton. TMDs appear in many cross sections, including semi-inclusive deep-inelastic scattering (SIDIS), the Drell-Yan process, weak boson production, and Higgs
production. Nonperturbative effects for TMDs dominate at small transverse momenta \( k_T \sim \Lambda_{\text{QCD}} \). For \( k_T \gg \Lambda_{\text{QCD}} \), the dependence of the TMD on \( k_T \) can be computed perturbatively [3–13]; but there are still important nonperturbative contributions [14–17]. On the modeling front, high precision global fits with next-to-next-to-next-to-leading logarithmic accuracy on perturbative ingredients have been performed for the existing experimental data [18, 19]. On the lattice front, the calculation of TMD-like correlators was pioneered and developed over the course of a decade using the Musch-Hägler-Engelhardt-Negele-Schäfer (MHENS) scheme [20–25]. Later on, quasi-TMDs were developed for the lattice [26–35] using the framework of large momentum effective theory (LaMET) [36–38]. The leading-power factorization formula connecting quasi-TMDs to the continuum has been proposed in refs. [28, 29, 31, 32], and recently derived to all orders in \( \alpha_s \) in [39].

First lattice results have been presented for various components of quasi-TMDs: the beam function and Collins-Soper kernel [40–45], and soft function [43, 44].

At leading power, the factorization formula connecting the continuum limit of quasi-TMDs (lattice) and the physical Collins TMD reads [39]:

\[
\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x \tilde{P}_z) = C_{\kappa_i}(x \tilde{P}_z, \mu) \exp \left[ \frac{1}{2} \gamma_{\kappa_i}^{\kappa_i}(\mu, b_T) \ln \frac{\tilde{\zeta}}{\zeta} \right] f_{i/h}(x, \vec{b}_T, \mu, \zeta),
\]

where \( f \) is a continuum TMD, \( \tilde{f} \) is the continuum limit of a lattice TMD, \( C_{\kappa_i} \) is a perturbative matching coefficient, and the exponential term evolves the so-called Collins-Soper (CS) scale \( \zeta \to \tilde{\zeta} \) using the CS kernel \( \gamma_{\kappa_i}^{\kappa_i} \). The other parameters include \( x \), which is the fraction of the hadron \( h \)'s longitudinal momentum \( P \) that the parton of type \( i \) carries; the Fourier-conjugate to the parton’s transverse momentum \( b_T \); and a renormalization scale \( \mu \).

Here \( i \) refers to either a gluon (\( i = g \)) or specific quark flavor (\( i = u, d, s, \ldots \)), where \( \kappa_i = q \) is universal for all quarks, but differs from \( \kappa_g = g \) for gluons. No quark-gluon or flavor mixing occurs in the factorization relation, which simplifies calculation of gluon TMDs from the lattice [39]. The definition of each of the TMDs in eq. (1.1) will be discussed in detail in section 1.1.

The quasi-to-Collins matching coefficient \( C_q \) is known for quarks at one loop [26, 28, 29], for logarithmic terms at two loops [32], and for all next-to-leading-logarithmic (NLL) terms [39]. A key missing ingredient is the matching coefficient for gluons. Gluons are known to be responsible for a large part of internal proton dynamics and would be useful to understand from first principles. In this work, we calculate the one-loop gluon matching coefficient \( C_g \).

### 1.1 Definitions of lattice and physical continuum TMDs

A TMD \( f_{i/h} \) for gluon or quark flavor \( i \) generally is a product of three components: a beam function (hadronic matrix element), a soft function (vacuum matrix element), and their renormalization. The beam and soft matrix elements are defined in terms of Wilson lines:

\[
W^R[\gamma] = P \exp \left[ ig \int d x^\mu A^a_\mu(x) T^R_a \right],
\]

where
where $\gamma$ is the Wilson line’s path and $R$ is its color representation. Here we focus on gluons, taking $R = g$ for the adjoint representation. Beam functions involve staple-shaped Wilson line paths as shown in figure 1,

$$W^g_{\gamma}(b, \eta v, \delta) = W^g_\gamma \left[ \frac{b}{2} \rightarrow \frac{b}{2} + \eta v - \frac{\delta}{2} \rightarrow - \frac{b}{2} + \eta v + \frac{\delta}{2} \rightarrow - \frac{b}{2} \right]. \quad (1.3)$$

A generic notation for gluon beam functions is given by

$$\Omega^{\mu\nu\rho\sigma}_{g/h} (b, P, \epsilon, \eta v, \delta) = \left\langle h(P) \bar{G}_{\mu\nu}(\frac{b}{2}) W^g_{\gamma}(b, \eta v, \delta) G_{\rho\sigma}(-\frac{b}{2}) \right| h(P) \right\rangle. \quad (1.4)$$

Here, $G^{\mu\nu}(b)$ is the gluon field strength tensor. Gluon fields are spatially separated by $b$, which is Fourier-conjugate to the struck parton’s momentum, $h$ denotes a parent hadron with momentum $P$, $\epsilon$ is the UV regulator, and $\eta v$ and $\delta$ characterize the Wilson line path. See refs. [46, 47] for the decomposition of gluon TMDs into independent spin structures.

Note that the correlator $\Omega$ characterizes the TMDs in $b$-space. When we define a full TMD in terms of beam and soft functions, we generally make a Fourier transform $P \cdot b \rightarrow x$ and write

$$\Omega(b, \ldots) \rightarrow B(x, \vec{b}_T, \ldots), \quad (1.5)$$

where $B$ indicates the correlator in these new coordinates.

The gluon soft function is defined as

$$S^g [b, \epsilon, \eta v, \bar{\eta} \bar{v}] = \frac{1}{N_c^2 - 1} \left\langle 0 \right| \text{Tr} \left[ S^g_{\gamma}(b, \eta v, \bar{\eta} \bar{v}) \right] \left| 0 \right\rangle, \quad (1.6)$$

where the trace is over color, and the soft Wilson line is

$$S^g_{\gamma}(b, \eta v, \bar{\eta} \bar{v}) = W^g_\gamma \left[ \frac{b}{2} \rightarrow \frac{b}{2} + \bar{\eta} \bar{v} \rightarrow - \frac{b}{2} + \bar{\eta} \bar{v} \rightarrow - \frac{b}{2} \rightarrow - \frac{b}{2} + \eta v \rightarrow \frac{b}{2} + \eta v \rightarrow \frac{b}{2} \right], \quad (1.7)$$

as shown in figure 1. The Wilson line $S_{\gamma}$ consists of two beam function staples glued together at the points $\pm b/2$; the long sides of the staples run along the $\bar{\eta} \bar{v}$ and $\eta v$ directions.
Note that the figure shows the case where \( b = \vec{b}_T \), and the points are translated to 0 and \(-\vec{b}_T\).

We can now define all off-lightcone gluon TMD schemes in terms of the exact same \( B \) and \( S \):

\[
f^{[s]}_{gh}(x, \vec{b}_T, \mu, P, v, \bar{v}, \delta, \ldots) = \lim_{\text{lightcone} \text{\& UV limits}} Z^g_{UV}(\epsilon, \mu, v, \bar{v}, \delta, \ldots) \lim_{\eta \to \infty} \frac{B^{\rho\sigma}_{gh}(x, \vec{b}_T, P, \epsilon, \eta v, \delta)}{\sqrt{S^g[b, \epsilon, \eta v, \eta \bar{v}]}} ,
\]

where the ellipses and \( Z^g_{UV} \) account for a scheme-dependent renormalization.\(^1\) Each scheme is characterized by a specific choice of the parameters \( b^\mu, \eta v^\mu, P^\mu, \) and \( \delta^\mu \) appearing in the beam and soft functions, as well as the precise nature and order of the lightcone and UV limits in eq. (1.8). The longitudinal staple length \( \eta \) distinguishes physical TMDs (\( \eta = \pm \infty \)) from lattice TMDs (finite \( \eta \)).

The remainder of section 1.1 specifies the order of limits and beam/soft function arguments that define three major TMD schemes: the continuum Collins TMD, the lattice quasi-TMD, and the continuum Large Rapidity (LR) TMD. We summarize this information in table 1, which also includes values taken on by the four-vectors \( b^\mu, v^\mu, \delta^\mu, P^\mu \).

**Collins scheme** [6]. The continuum Collins scheme uses spacelike Wilson lines with directions

\[
n_{\alpha}^A(y_A) \equiv n_{\alpha}^A - e^{-2y_A} n_{\alpha}^B = (1, -e^{-2y_A}, 0 , 0) ,
\]

\[
n_{\alpha}^B(y_B) \equiv n_{\alpha}^B - e^{-2y_B} n_{\alpha}^A = (-e^{-2y_B}, 1, 0 , 0) ,
\]

parametrized by the rapidities \( y_A \) and \( y_B \). (See appendix A for our lightcone coordinate conventions.) The Collins gluon TMD for a hadron \( h \) moving along \( n_{\alpha} \) with rapidity \( y_P \) is

\[
f_{gh}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z^g_{UV}(\epsilon, \mu, \zeta) \lim_{y_B \to -\infty} \frac{B^{\rho\sigma}_{gh}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S^g[b, \epsilon, \zeta, \infty n_{\alpha}(2y_B), -\infty n_{\alpha}(2y_B)]}} ,
\]

where \( |b_\perp| = b_T \) and the beam function is defined as

\[
B^{\rho\sigma}_{gh}(x, \vec{b}_T, \epsilon, y_P - y_B) = \int \frac{db^- e^{-ik^- (xP^+)} }{2\pi} xP^+ \Omega_{g/h}^{\rho+\sigma} [b, P, \epsilon, -\infty n_{\alpha}(y_B), b^- n_{\beta}] ,
\]

with \( \Omega_{g/h}^{\rho+\sigma} = (n_{\beta})_{\mu} \Omega_{g/h}^{\rho+\sigma} (n_{\beta})_{\nu} \). The CS scale \( \zeta \) is defined as

\[
\zeta = 2(xP^+ e^{-y_B})^2 = x^2 m_h^2 e^{2(y_B - y_P)} .
\]

**Quasi-TMDs** [26–35, 39]. Collins TMDs cannot be directly discretized and computed on the lattice, as their Wilson lines’ explicit dependence on time induces a sign problem. Instead, one defines numerically-tractable quasi-TMD beam functions using finite-length

\(^1\)Note that for the MHENS scheme, renormalization should be performed before the Fourier transform as \( Z^g_{UV} \) depends on the longitudinal component of \( b^\mu \).
Wilson lines lying on equal-time paths. The lattice gluon quasi-TMD takes the form
\[
\tilde{f}_{i/h}^{[\bar{T}]}(x, \vec{b}_T, \mu, \zeta, x \hat{P}^z) = \lim_{\eta \to \infty} Z_{uv}^\eta (\mu, \tilde{\mu}) Z_{uv}^\eta (a, \tilde{\mu}, y_n - y_B) \tilde{B}_{g/h}^{\alpha \beta \sigma} (x, \vec{b}_T, a, \eta, x \hat{P}^z) \sqrt{S^\eta} \left[ b_{\perp}, a, -\eta \frac{n_A(2y_n)}{|n_A(2y_n)|}, -\eta \frac{n_B(2y_B)}{|n_B(2y_B)|} \right], \tag{1.13}
\]
where the bare quasi-function reads as
\[
\tilde{B}_{g/h}^{\alpha \beta \sigma} (x, \vec{b}_T, a, \eta, x \hat{P}^z) = N^{\alpha \beta \sigma} \int \frac{d\hat{P}^z}{2\pi} e^{i\hat{P}^z(x \hat{P}^z)} \Omega_{g/h}^{\alpha \beta \sigma} [\tilde{b}, \tilde{P}, a, \tilde{\eta} \vec{z}, \tilde{\eta} \vec{z}]. \tag{1.14}
\]
Here, \(\alpha, \beta, \rho\) and \(\sigma\) are generic Lorentz indices, and \(N^{\alpha \beta \sigma}\) is a trivial normalization factor. The factor \(Z_{uv}^\eta (a, \tilde{\mu}, y_n - y_B)\) renormalizes UV divergences from lattice regularization, and \(Z_{uv}^\eta (\mu, \tilde{\mu})\) matches the result to the MS scheme. The second equality in eq. (1.13) holds for large \(\hat{P}^z\), and the variable \(\tilde{\zeta}\) is
\[
\tilde{\zeta} = (x m_h e^{y_P + y_n - y_B})^2 \approx (2 x \hat{P}^z e^{y_n - y_B})^2. \tag{1.15}
\]

**Large Rapidity (LR) scheme.** To facilitate deriving the factorization relation between the quasi- and Collins TMDs, ref. [39] introduced the Large Rapidity (LR) scheme,
\[
f_{g/h}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) = \lim_{y_B \to 1} \lim_{\epsilon \to 0} Z_{uv}^{LR}(\epsilon, \mu, y_n - y_B) \sqrt{S_{g/h}^{\epsilon \eta}} [b_{\perp}, \epsilon, -\infty n_A(2y_n), -\infty n_B(2y_B)], \tag{1.16}
\]
which differs from the Collins scheme only by its order of \(\epsilon \to 0\) and \(y_B \to -\infty\) limits.

| TMD | Quasi-TMD (continuum) | Collins TMD (continuum) |
|-----|------------------------|-------------------------|
| \(l\) | \(\lim_{\eta \to \infty} Z_{uv}^\eta (\mu, \tilde{\mu}) Z_{uv}^\eta (a, \tilde{\mu}, y_n - y_B) \tilde{B}_{g/h}^{\alpha \beta \sigma} (x, \vec{b}_T, a, \eta, x \hat{P}^z) \) | \(\lim_{\eta \to \infty} Z_{uv}^\eta (\mu, \tilde{\mu}) Z_{uv}^\eta (a, \tilde{\mu}, y_n - y_B) \tilde{B}_{g/h}^{\alpha \beta \sigma} (x, \vec{b}_T, a, \eta, x \hat{P}^z) \) |
| \(\epsilon\) | \((0, b^-, b)\) | \((0, b^+, b^0, b^\pm)\) |
| \(\delta\) | \((-e^{2y_n}, 1, 0)\) | \((0, 0, 0, -1)\) |
| \(P\) | \(\frac{m_h}{\sqrt{2}} (e^{y_P}, e^{-y_P}, 0, \perp)\) | \(m_h (\cosh y_P, 0, 0, \sinh y_P)\) |

**Table 1.** Definition of the Collins and quasi-TMDs for quarks and gluons. The Large Rapidity and Collins TMDs are identical, except for the order of the limits and expression for \(Z_{uv}^\eta\) in the first row.
Using Lorentz invariance and the large $\hat{P}^z$ expansion ($-y_B \gg 1$), one can demonstrate that when $\tilde{\zeta} = \zeta$ or $y_P = y_P - y_B$, the quasi- and LR TMDs are equivalent at leading order under a large rapidity expansion [39], i.e.,

$$f_{g/h}(x, \vec{b_T}, \mu, \tilde{\zeta}, x \hat{P}^z) = f_{LR}^{g/h}(x, \vec{b_T}, \mu, \tilde{\zeta}, y_P - y_B) + O(y_B e^{-y_B}), \quad (1.17)$$

where $O(y_B e^{-y_B})$ are exponentially suppressed contributions.

2 Calculation of one-loop gluon matching coefficient

We now compute the one-loop quasi-to-Collins matching coefficient $C_g$ for gluons. To do so, we actually carry out a simpler equivalent matching calculation: between the LR and Collins schemes. In the course of proving the quasi-to-Collins factorization formula in eq. (1.1), ref. [39] showed that the quasi-to-Collins and LR-to-Collins matching coefficients are identical. Note that this matching calculation focuses on beam functions: the quasi-soft function has been chosen to reproduce the Collins soft function as $|\tilde{\eta}| \to \infty$.

Let us compute the matrix element of the Collins beam function for a free external gluon state with momentum $p^\mu = (p^+, 0, 0, -)$, transverse polarization vectors $\epsilon^{\alpha}_\perp$ and $\epsilon^{\beta}_\perp$, as well as color indices $a$ and $b$. We work in Feynman gauge and dimensional regularization with space-time dimension $d = 4 - 2\epsilon$. As we are interested in the leading power gluon TMDs, the Lorentz indices $\rho, \sigma$ are both transverse, and for both the LR and Collins schemes $\mu = \nu = +$. In our calculation we leave the choice of $\rho$ and $\sigma$ unspecified, to make clear that our result applies to all spin-dependent gluon TMDs.

At tree-level, we must compute the diagrams in figure 2a, which give

$$\Omega_{g/g}^{+\rho+\sigma}(0) (p^+) = \delta^{ab} (p^+) \left[ g^{\rho\beta} g^{\sigma\alpha} \delta (1 - x) + (\alpha \leftrightarrow \beta, x \to -x) \right], \quad (2.1)$$

where the second term in the square bracket comes from the exchange symmetry of bosonic particles. Taking a Fourier transform $b^z \to xp^z$, the tree-level gluon beam function becomes

$$x B_{g/g}^{C,\rho\sigma}(0) (x, \vec{b_T}) = \delta^{ab} \left[ g^{\rho\beta} g^{\sigma\alpha} \delta (1 - x) + (\alpha \leftrightarrow \beta, x \to -x) \right], \quad (2.2)$$

where we moved the normalization factor $1/x$ to the left hand side for clarity. We emphasize that to find the matching coefficient, we need only examine terms proportional to $\delta (1 - x)$ since the factorization relation does not involve a convolution in $x$.

The one-loop Feynman diagrams that contribute to the beam function are shown in figure 2. According to ref. [39], only diagrams containing a rapidity divergence contribute to the matching, significantly simplifying our task. Only figures 2c and 2e could exhibit rapidity divergences, as the other diagrams have Lorentz-covariant loop integrals, which remain unaffected by the large rapidity limit $y_B \to -\infty$ (except for the transformation of
The tree-level (a) and one-loop level (b–h) diagrams contributing to gluon beam function in a free gluon state. Since the external gluons are transversely polarized, they cannot be attached to the Wilson lines in the $n_B$ direction. While depicted in these diagrams as a straight line, the beam function Wilson line actually traverses an infinitely long staple-shaped path, as shown in figure 1a.
tensor structures). Using the Feynman rules in appendix B, we express figures 2c and 2e as

\[
\Omega_{g/g,c}^{\rho+\sigma}(b^\mu) = (n_b)_\mu(n_b)_\nu \int d^4 k \frac{d^d k}{(2\pi)^d}(-ik^\sigma g^{\rho\delta} + ik^\nu g^{\sigma\delta})\delta^{\mu\nu} e^{i k^\mu - i k^\nu} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

\[
\times \left( g^f e_a \epsilon^{a,\alpha,\beta,\delta} \delta - g^{\alpha\beta} (p - 2k)^\alpha + g^{\alpha\beta} (k - 2p)^\beta + g^{\alpha\beta} (p + k)^\gamma \right)
\]

\[
\times g f^{edc} \int ds \gamma(s)e^{-i(p-k)\cdot\gamma(s)} \frac{i}{k^2 + i\delta} \frac{i}{k^2 - i\delta} \tau\nu \sigma\nu \gamma
\]

\[
\times \left( g^f e_a \epsilon^{a,\alpha,\beta,\delta} \delta - g^{\alpha\beta} (p - 2k)^\alpha + g^{\alpha\beta} (k - 2p)^\beta + g^{\alpha\beta} (p + k)^\gamma \right)
\]

\[
\times g f^{edc} \int ds \gamma(s)e^{-i(p-k)\cdot\gamma(s)} \frac{i}{k^2 + i\delta} \frac{i}{k^2 - i\delta} \tau\nu \sigma\nu \gamma
\]

and

\[
\Omega_{g/g,e}^{\rho+\sigma}(b^\mu) = (n_b)_\mu(n_b)_\nu \int d^4 k \frac{d^d k}{(2\pi)^d}(-ik^\sigma g^{\rho\delta} + ik^\nu g^{\sigma\delta})\delta^{\mu\nu} e^{i k^\mu - i k^\nu} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

\[
\times \left( g^f e_a \epsilon^{a,\alpha,\beta,\delta} \delta - g^{\alpha\beta} (p - 2k)^\alpha + g^{\alpha\beta} (k - 2p)^\beta + g^{\alpha\beta} (p + k)^\gamma \right)
\]

\[
\times g f^{edc} \int ds \gamma(s)e^{-i(p-k)\cdot\gamma(s)} \frac{i}{k^2 + i\delta} \frac{i}{k^2 - i\delta} \tau\nu \sigma\nu \gamma
\]

where the Wilson line takes a staple-shaped path:

\[
\gamma(s) : \frac{b}{2} \rightarrow \frac{b}{2} - \frac{b-n_b}{2} - \infty n_B \rightarrow - \frac{b}{2} + \frac{b-n_b}{2} - \infty n_B \rightarrow - \frac{b}{2},
\]

and we have suppressed the contributions from diagrams with external gluon lines exchanged (\(\alpha \leftrightarrow \beta, p \rightarrow -p\)) for conciseness. We also use a coupling in the \(\overline{\text{MS}}\) scheme, which introduced the factors \(\epsilon' = (e^{\gamma E}/(4\pi))\epsilon\).

Simplifying the above,

\[
\Omega_{g/g,c}^{\rho+\sigma}(b^\mu) = -2ig^2 C_A p^+ g^{\sigma \alpha} \sigma^{\beta \rho} e^{i k^\mu - e^{i p^\mu}} \int d^4 k \frac{d^4 k}{(2\pi)^d} \frac{1}{k^2(p-k)^2} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

\[
+ 2ig^2 C_A e^{2g_B p^+} e^{i k^\mu - e^{i p^\mu}} \int d^4 k \frac{d^4 k}{(2\pi)^d} \frac{1}{k^2(p-k)^2} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

\[
- 2ig^2 C_A p^+ g^{\sigma \alpha} \sigma^{\beta \rho} e^{i k^\mu - e^{i p^\mu}} \int d^4 k \frac{d^4 k}{(2\pi)^d} \frac{1}{k^2(p-k)^2} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

\[
\Omega_{g/g,e}^{\rho+\sigma}(b^\mu) = ig^2 C_A e^{2g_B p^+} e^{i k^\mu - e^{i p^\mu}} \int d^4 k \frac{d^4 k}{(2\pi)^d} \frac{1}{k^2(p-k)^2} \left(\frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta}\right)
\]

where we obtain the Wilson line propagators \([n_B \cdot (p-k) \pm i\delta]^{-1}\) by integrating over the path parameter \(s\). Note that in the LR and Collins scheme that the staple Wilson line has infinite extent. We introduce the infinitesimal imaginary part \(\pm i\delta\) to ensure that the path integral contribution from \(s = -\infty\) properly vanishes in Feynman gauge.
In the $y_B \to -\infty$ limit, the rapidity divergence is in the first line of eq. (2.6). The second line is suppressed by a factor of $e^{\gamma \eta} k_T^2/(p^+)^2$ compared to the first line, as $k_T = |k_\perp| \ll k_+, p^+$ in the power counting, and $\alpha, \beta, \rho, \sigma = \perp$. The last line in eq. (2.6) is Lorentz covariant and so is not affected by $y_B \to -\infty$; it thus does not contribute to the matching.

After taking a Fourier transform $b^- \to xp^+$, the first line of eq. (2.6) becomes

$$xB_{g/g,c}^{\rho\sigma(1)}(x, b_T, \epsilon, y_B - y_B) = -2ig^2 C_A p^+ g^{\sigma\rho} g^{\beta\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{k^+ e^{-ik_Tb_T}}{k^2(p-k)^2} \left[ \frac{1}{n_B \cdot (p-k) + i\delta} + \frac{1}{n_B \cdot (p-k) - i\delta} \right]$$

The first term of eq. (2.8) is a plus function in $x$; it is unaffected by the $y_B \to -\infty$ limit as long as $x \neq 1$. The second term contains the contribution to the matching coefficient.

Likewise, after a Fourier transform $b^- \to xp^+$, the second line of eq. (2.6) becomes

$$xB_{g/g,e}^{\rho\sigma(1)}(x, b_T, \epsilon, y_B - y_B) = ig^2 C_A e^{2\gamma \eta} g^{\sigma\rho} g^{\beta\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{n_B \cdot k + i\delta} + \frac{1}{n_B \cdot k - i\delta} \left( e^{ik_Tb_T} - 1 \right)$$

The first term is again a plus function. The second term vanishes under parity transforms $k^\pm \to -k^\pm$. Therefore, the diagrams in figure 2e do not contribute to the matching.

All that remains to compute is the term proportional to $\delta(1-x)$ in eq. (2.8). After a change of variables $k \to p - k$, the next step is to evaluate the integral

$$I = \epsilon' \int \frac{d^dk}{(2\pi)^d} \frac{p^+ - k^+}{k^2(p-k)^2} \left( e^{ik_Tb_T} - 1 \right) \left[ \frac{1}{n_B \cdot k + i\delta} + \frac{1}{n_B \cdot k - i\delta} \right].$$

To simplify the integral we can make a change of variable $k^- \to e^{-\gamma \eta} k^-$ and $k^+ \to e^{\gamma \eta} k^+$ which corresponds to a boost to equal-time with $k^z = (k^+ - k^-)/\sqrt{2}$, and yields

$$I = \epsilon' \frac{1}{\sqrt{2}} \int \frac{d^dk}{(2\pi)^d} \frac{p^+ - k^+}{(k^2 + i\delta)(p^+ - k^+)^2 + i0} \left( e^{ik_Tb_T} - 1 \right) \left[ \frac{1}{k^z + i\delta} + \frac{1}{k^z - i\delta} \right].$$

where $p^\mu = (p^+, 0, 0, 0, 0)$. Note that the result involves a principal value integral for $k^z$,

$$\frac{1}{2} \left( \frac{1}{k^z + i\delta} + \frac{1}{k^z - i\delta} \right) = \text{PV} \left( \frac{1}{k^z} \right).$$

We must evaluate the integral in eq. (2.11) with two different orders of limits.
Taking $y_B \to -\infty$ before $\epsilon \to 0$. We first perform a contour integration in the complex $k^0$ plane, then we integrate over $k^z$ to obtain:

$$I = \frac{i}{4} \int \frac{d^d-k}{(2\pi)^{d-1}} \left( e^{ik_T \cdot b_T} - 1 \right) \left( \frac{2}{k_T^2} + \frac{1}{p_T^2} \right) \ln \left( \frac{b_T^2 + 2p_T^2 + \sqrt{b_T^2 + p_T^2 + 2p_T^2}}{b_T^2} \right) \sqrt{1 + k_T^2 / p_T^2} - \frac{4}{k_T^2} \right]. \quad (2.13)$$

Here $p_T^2 = p^+e - 2y_B/2$. The integrand diverges in the $p_T^2 \to \infty$ or $y_B \to -\infty$ limit, so by taking the limit first we carrying out the power expansion in $p_T^2$ before integrating over $k_\perp$, and keeping the $\ln(p_T^2)$ singularity, which leads to

$$I_{y_B \to -\infty} = \frac{i}{4\pi} \int \frac{d^d-k}{(2\pi)^{d-2} k_T^2} \left[ -2 + \ln \frac{4\mu^2}{k_T^2} \right] \left( e^{ik_T \cdot b_T} - 1 \right), \quad (2.14)$$

$$= \frac{i}{4\pi^2} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( 2 + \ln \frac{\mu^2}{4\mu^2} \right) - \frac{1}{\epsilon^2} \ln \left( \frac{b_T^2 + \mu^2}{b_0^2} \right) \right] . \quad (2.15)$$

Again the $1/\epsilon$ pole is a UV divergence, and is subtracted for the $\overline{\text{MS}}$ renormalized result.

Finally we note that the one-loop Collins soft function is the same for quarks and gluons, so can be obtained from the quark result [29] with the replacement of $C_F \to C_A$,

$$S_C^{g,(1)}(b_T, \epsilon, y_n, y_B) = \frac{\alpha_s C_A}{2\pi} \left( \frac{2}{b_T^2} \right) \left[ (y_n - y_B) \frac{1 + e^{2(y_n - y_B)}}{1 - e^{2(y_n - y_B)}} - 2 \right] . \quad (2.16)$$

Here we use the shorthand $S_C^{g}(b_T, \epsilon, y_A, y_B) = S^{g}[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$. This result is not affected by the order of $\epsilon \to 0$ and $y_B \to -\infty$ limits, i.e.,

$$\lim_{\epsilon \to 0, y_B \to -\infty} S_C^{g,(1)}(b_T, \epsilon, 2y_n, 2y_B) = \frac{\alpha_s C_A}{2\pi} \left( \frac{1}{\epsilon} + \ln \frac{b_T^2}{b_0^2} \right) [2 - 2(y_n - y_B)] . \quad (2.17)$$

Therefore this result cancels out for the matching between the LR and Collins schemes.

**Matching coefficient.** The difference between the renormalized LR and Collins scheme results in eqs. (2.14) and (2.15) gives

$$I_{y_B \to -\infty}^{\text{ren}} - I_{\epsilon \to 0}^{\text{ren}} = \frac{i}{4\pi^2} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{4\mu^2} + \ln \frac{\mu^2}{4\mu^2} + 2 - \frac{\pi^2}{12} \right] . \quad (2.18)$$
which is purely of ultraviolet origin, and hence is perturbative at a scale $\mu^2 \sim 4p_z^2$. Thus the one-loop matching coefficient between the LR and Collins TMD is given by

$$C_g(p^z, \mu) = 1 + \frac{\alpha_s(\mu) C_A}{4\pi} \left[ \ln^2 \frac{\mu^2}{4p_z^2} + 2 \ln \frac{\mu^2}{4p_z^2} + 4 - \frac{\pi^2}{6} \right]. \quad (2.19)$$

Note that here $4p_z^2 = 2(p^+)^2 e^{-2y_B}$. For a hadron with momentum $P$, we must replace $p^+ \to xP^+$, so that

$$4p_z^2 \rightarrow 2(xP^+)^2 e^{-2y_B} \approx 4x^2 m_h^2 \sinh^2(y_P - y_B) = \zeta_{LR} \quad (2.20)$$

in the limit of $-y_B \gg 1$. Therefore, identifying $\zeta_{LR} = (2xP^z)^2$, we find that the final gluon matching coefficient for the quasi-TMD is:

$$C_g(xP^z, \mu) = 1 + \frac{\alpha_s C_A}{4\pi} \left[ \ln^2 \frac{\mu^2}{(2xP^z)^2} + 2 \ln \frac{\mu^2}{(2xP^z)^2} + 4 - \frac{\pi^2}{6} \right]. \quad (2.21)$$

Interestingly this result is identical to the quark result $C_q$ [29], up to changing Casimirs from $C_F \rightarrow C_A$. This occurs because the entire contribution to the matching comes from the part of the Wilson line diagrams associated with the rapidity divergence. The well-known (generalized) Casimir scaling of the cusp anomalous dimension and CS kernel have been validated to four-loop order [49–54]. The fact that the full one-loop matching and cusp-anomalous dimension obey Casimir scaling, implies that the full tower of next-to-leading-logarithms for $C_g$ will obey Casimir scaling as well [39]. This hints at a relationship for quasi-TMD matching coefficients at higher orders.

### 3 Conclusion

In this paper, we calculated the one-loop gluon matching coefficient $C_g$ in the factorization formula relating quasi and Collins TMDs, marking an important step towards calculating gluon TMDs from lattice QCD. As the matching coefficient is independent of spin structure and there is no quark-gluon mixing, our result enables lattice calculation of all eight leading-twist gluon TMDs. Given the encouraging first results for quark TMDs, it is reasonable to expect that lattice QCD will provide highly useful nonperturbative input for gluon TMDs and their Collins-Soper evolution, neither of which has yet been extracted from experiments.

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A Lightcone coordinate conventions

We work in a frame where the hadron momentum $P$ is close to the lightlike unit vectors

$$
\begin{align*}
    n_a^\mu &= \frac{1}{\sqrt{2}}(1,0,0,1), \\
    n_b^\mu &= \frac{1}{\sqrt{2}}(1,0,0,-1),
\end{align*}
$$

(A.1)

which obey $n_a^2 = n_b^2 = 0$ and $n_a \cdot n_b = 1$. We can decompose a four-vector $p^\mu$ as

$$
p^\mu = (p^+, p^-, p_\perp) = p^+ n_a^\mu + p^- n_b^\mu + p_\perp^\mu,
$$

(A.2)

where $p^\pm = (p^0 \pm p^z)/\sqrt{2}$ and $p_\perp^\mu = (0, p^x, p^y, 0) = (0, \vec{p}_T, 0)$. Here $p_T^\mu$ is a Minkowski vector, and $\vec{p}_T$ is the corresponding Euclidean vector with magnitude $p_T = (p_T^2)^{1/2} = (-p_\perp^2)^{1/2}$.

B Feynman rules

The gluon field strength tensor is defined as $i g G_\mu^\alpha \tau^a = [D_\mu, D^\alpha]$, where $iD_\mu = i\partial_\mu + gA_\mu^a \tau^a$ and $\tau^a$ is the SU(3) generator in the fundamental representation. The Feynman rules for the local, three-gluon, and gluon-Wilson line vertices are:

\begin{align*}
    &\begin{array}{c}
        k \\
        \alpha, b
    \end{array} \quad \Rightarrow \quad G_\mu^\alpha(b) = (-ik^\mu g^{\nu\alpha} + ik^\nu g^{\mu\alpha}) \delta^{ab} e^{-ik\cdot b}, \\

    &\begin{array}{c}
        k \\
        \alpha, b
    \end{array} \quad \Rightarrow \quad G_\mu^{\alpha\beta}(b) = g f^{abc} \left[ g^{\nu\beta} g^{\mu\alpha} - g^{\mu\beta} g^{\nu\alpha} \right] e^{-i(k+l)\cdot b}, \\

    &\begin{array}{c}
        a \\
        \alpha
    \end{array} \quad \Rightarrow \quad \begin{array}{c}
        k_1 \\
        k_2 \quad k_3
    \end{array} = g f^{abc} \left[ g^{\alpha\beta}(k_1 - k_2)\gamma^\beta \gamma^\gamma(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\alpha \right], \tag{B.1}

    &\begin{array}{c}
        \mu, a \\
        b
    \end{array} \quad \Rightarrow \quad \begin{array}{c}
        k \\
        \beta
    \end{array} \quad \Rightarrow \quad \begin{array}{c}
        \gamma
    \end{array} = g f^{abc} \gamma^\gamma(s) \mu e^{-ik\cdot(s)}.
\end{align*}

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References

[1] A. Accardi et al., Electron ion collider: the next QCD frontier. Understanding the glue that binds us all, *Eur. Phys. J. A* **52** (2016) 268 [arXiv:1212.1701] [SPIRE].

[2] R. Abdul Khalek et al., Science requirements and detector concepts for the electron-ion collider: EIC yellow report, arXiv:2103.05419 [SPIRE].

[3] J.C. Collins and D.E. Soper, Back-to-back jets: Fourier transform from $b \rightarrow k_T$, *Nucl. Phys. B* **197** (1982) 446 [SPIRE].

[4] S.M. Aybat and T.C. Rogers, TMD parton distribution and fragmentation functions with QCD evolution, *Phys. Rev. D* **83** (2011) 114042 [arXiv:1101.5057] [SPIRE].

[5] S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, Vector boson production at hadron colliders: hard-collinear coefficients at the NNLO, *Eur. Phys. J. C* **72** (2012) 2195 [arXiv:1209.0158] [SPIRE].

[6] J. Collins, Foundations of perturbative QCD, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **32** (2011) 1 [SPIRE].

[7] T. Gehrmann, T. Luebbert and L.L. Yang, Calculation of the transverse parton distribution functions at next-to-next-to-leading order, *JHEP* **06** (2014) 155 [arXiv:1403.6451] [SPIRE].

[8] M.G. Echevarria, I. Scimemi and A. Vladimirov, Unpolarized transverse momentum dependent parton distribution and fragmentation functions at next-to-next-to-leading order, *JHEP* **09** (2016) 004 [arXiv:1604.07869] [SPIRE].

[9] T. Lübbert, J. Oredsson and M. Stahlhofen, Rapidity renormalized TMD soft and beam functions at two loops, *JHEP* **03** (2016) 168 [arXiv:1602.01829] [SPIRE].

[10] M.-X. Luo, X. Wang, X. Xu, L.L. Yang, T.-Z. Yang and H.X. Zhu, Transverse parton distribution and fragmentation functions at NNLO: the quark case, *JHEP* **10** (2019) 083 [arXiv:1908.03831] [SPIRE].

[11] M.-X. Luo, T.-Z. Yang, H.X. Zhu and Y.J. Zhu, Quark transverse parton distribution at the next-to-next-to-next-to-leading order, *Phys. Rev. Lett.* **124** (2020) 092001 [arXiv:1912.05778] [SPIRE].

[12] M.-X. Luo, T.-Z. Yang, H.X. Zhu and Y.J. Zhu, Unpolarized quark and gluon TMD PDFs and FFs at $N^3$LO, *JHEP* **06** (2021) 115 [arXiv:2012.03256] [SPIRE].

[13] M.A. Ebert, B. Mistlberger and G. Vita, Transverse momentum dependent PDFs at $N^3$LO, *JHEP* **09** (2020) 146 [arXiv:2006.05329] [SPIRE].

[14] A. Bacchetta, F. Delcarro, C. Pisano, M. Radici and A. Signori, Extraction of partonic transverse momentum distributions from semi-inclusive deep-inelastic scattering, Drell-Yan and Z-boson production, *JHEP* **06** (2017) 081 [Erratum ibid. **06** (2019) 051] [arXiv:1703.10157] [SPIRE].

[15] I. Scimemi and A. Vladimirov, Systematic analysis of double-scale evolution, *JHEP* **08** (2018) 003 [arXiv:1803.11089] [SPIRE].

[16] G. Lustermans, J.K.L. Michel, F.J. Tackmann and W.J. Waalewijn, Joint two-dimensional resummation in $q_T$ and 0-jettiness at NNLL, *JHEP* **03** (2019) 124 [arXiv:1901.03331] [SPIRE].
[17] M.A. Ebert, J.K.L. Michel, I.W. Stewart and Z. Sun, *Disentangling long and short distances in momentum-space TMDs*, JHEP 07 (2022) 129 [arXiv:2201.07237] [INSPIRE].

[18] I. Scimemi and A. Vladimirov, *Non-perturbative structure of semi-inclusive deep-inelastic and Drell-Yan scattering at small transverse momentum*, JHEP 06 (2020) 137 [arXiv:1912.06532] [INSPIRE].

[19] A. Bacchetta et al., *Disentangling long and short distances in momentum-space TMDs, JHEP 07 (2022) 129 [arXiv:2201.07237] [INSPIRE].

[20] P. Hagler, B.U. Musch, J.W. Negele and A. Schafer, *Intrinsic quark transverse momentum in the nucleon from lattice QCD*, EPL 88 (2009) 61001 [arXiv:0908.1283] [INSPIRE].

[21] B.U. Musch, P. Hagler, J.W. Negele and A. Schafer, *Exploring quark transverse momentum distributions with lattice QCD*, Phys. Rev. D 83 (2011) 094507 [arXiv:1011.1213] [INSPIRE].

[22] B.U. Musch, P. Hagler, M. Engelhardt, J.W. Negele and A. Schafer, *Sivers and Boer-Mulders observables from lattice QCD*, Phys. Rev. D 85 (2012) 094510 [arXiv:1111.4249] [INSPIRE].

[23] M. Engelhardt, P. Hägler, B. Musch, J. Negele and A. Schäfer, *Lattice QCD study of the Boer-Mulders effect in a pion*, Phys. Rev. D 93 (2016) 054501 [arXiv:1506.07826] [INSPIRE].

[24] B. Yoon et al., *Lattice QCD calculations of nucleon transverse momentum-dependent parton distributions using clover and domain wall fermions*, in 33rd International Symposium on Lattice Field Theory, SISSA, Trieste, Italy (2015) [arXiv:1601.05717] [INSPIRE].

[25] B. Yoon et al., *Nucleon transverse momentum-dependent parton distributions in lattice QCD: renormalization patterns and discretization effects*, Phys. Rev. D 96 (2017) 094508 [arXiv:1706.03406] [INSPIRE].

[26] X. Ji, P. Sun, X. Xiong and F. Yuan, *Soft factor subtraction and transverse momentum dependent parton distributions on the lattice*, Phys. Rev. D 91 (2015) 074009 [arXiv:1405.7640] [INSPIRE].

[27] X. Ji, Y. Liu and Y.-S. Liu, *Transverse-momentum-dependent parton distribution functions from large-momentum effective theory*, Phys. Lett. B 811 (2020) 135946 [arXiv:1911.03840] [INSPIRE].

[28] M.A. Ebert, I.W. Stewart and Y. Zhao, *Determining the nonperturbative Collins-Soper kernel from lattice QCD*, Phys. Rev. D 99 (2019) 034505 [arXiv:1811.00026] [INSPIRE].

[29] M.A. Ebert, I.W. Stewart and Y. Zhao, *Towards quasi-transverse momentum dependent PDFs computable on the lattice*, JHEP 09 (2019) 037 [arXiv:1901.03685] [INSPIRE].

[30] M.A. Ebert, I.W. Stewart and Y. Zhao, *Renormalization and matching for the Collins-Soper kernel from lattice QCD*, JHEP 03 (2020) 099 [arXiv:1910.08569] [INSPIRE].

[31] X. Ji, Y. Liu and Y.-S. Liu, *TMD soft function from large-momentum effective theory*, Nucl. Phys. B 955 (2020) 115054 [arXiv:1910.11415] [INSPIRE].

[32] X. Ji, Y. Liu and Y.-S. Liu, *Transverse-momentum-dependent parton distribution functions from large-momentum effective theory*, Phys. Lett. B 811 (2020) 135946 [arXiv:1911.03840] [INSPIRE].

[33] M.A. Ebert, S.T. Schindler, I.W. Stewart and Y. Zhao, *One-loop matching for spin-dependent quasi-TMDs*, JHEP 09 (2020) 099 [arXiv:2004.14831] [INSPIRE].
[34] X. Ji, Y. Liu, A. Schäfer and F. Yuan, Single transverse-spin asymmetry and Sivers function in large momentum effective theory, *Phys. Rev. D* **103** (2021) 074005 [arXiv:2011.13397] [INSPIRE].

[35] X. Ji and Y. Liu, Computing light-front wave functions without light-front quantization: a large-momentum effective theory approach, *Phys. Rev. D* **105** (2022) 076014 [arXiv:2106.05310] [INSPIRE].

[36] X. Ji, Parton physics on a Euclidean lattice, *Phys. Rev. Lett.* **110** (2013) 262002 [arXiv:1305.1539] [INSPIRE].

[37] X. Ji, Parton physics from large-momentum effective field theory, *Sci. China Phys. Mech. Astron.* **57** (2014) 1407 [arXiv:1404.6680] [INSPIRE].

[38] X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and Y. Zhao, Large-momentum effective theory, *Rev. Mod. Phys.* **93** (2021) 035005 [arXiv:2004.03543] [INSPIRE].

[39] M.A. Ebert, S.T. Schindler, I.W. Stewart and Y. Zhao, Factorization connecting continuum & lattice TMDs, *JHEP* **04** (2022) 178 [arXiv:2201.08401] [INSPIRE].

[40] P. Shanahan, M.L. Wagman and Y. Zhao, Nonperturbative renormalization of staple-shaped Wilson line operators in lattice QCD, *Phys. Rev. D* **101** (2020) 074505 [arXiv:1911.00800] [INSPIRE].

[41] P. Shanahan, M. Wagman and Y. Zhao, Collins-Soper kernel for TMD evolution from lattice QCD, *Phys. Rev. D* **102** (2020) 014511 [arXiv:2003.06063] [INSPIRE].

[42] M. Schlemmer, A. Vladimirov, C. Zimmermann, M. Engelhardt and A. Schäfer, Determination of the Collins-Soper kernel from lattice QCD, *JHEP* **08** (2021) 004 [arXiv:2103.16991] [INSPIRE].

[43] LATTICE PARTON collaboration, Lattice-QCD calculations of TMD soft function through large-momentum effective theory, *PoS LATTICE2021* (2022) 477 [Phys. Rev. Lett. **125** (2020) 192001] [arXiv:2005.14572] [INSPIRE].

[44] Y. Li et al., Lattice QCD study of transverse-momentum dependent soft function, *Phys. Rev. Lett.* **128** (2022) 062002 [arXiv:2106.13027] [INSPIRE].

[45] P. Shanahan, M. Wagman and Y. Zhao, Lattice QCD calculation of the Collins-Soper kernel from quasi-TMDPDFs, *Phys. Rev. D* **104** (2021) 114502 [arXiv:2107.11930] [INSPIRE].

[46] P.J. Mulders and J. Rodrigues, Transverse momentum dependence in gluon distribution and fragmentation functions, *Phys. Rev. D* **63** (2001) 094021 [hep-ph/0009343] [INSPIRE].

[47] M.G. Echevarria, T. Kasemets, P.J. Mulders and C. Pisano, QCD evolution of (un)polarized gluon TMDPDFs and the Higgs qT-distribution, *JHEP* **07** (2015) 158 [Erratum ibid. **05** (2017) 073] [arXiv:1502.05354] [INSPIRE].

[48] Y. Li, D. Neill and H.X. Zhu, An exponential regulator for rapidity divergences, *Nucl. Phys. B* **960** (2020) 115193 [arXiv:1604.00392] [INSPIRE].

[49] G.P. Korchemsky and A.V. Radyushkin, Renormalization of the Wilson loops beyond the leading order, *Nucl. Phys. B* **283** (1987) 342 [INSPIRE].

[50] S. Moch, J.A.M. Vermaseren and A. Vogt, The three loop splitting functions in QCD: the nonsinglet case, *Nucl. Phys. B* **688** (2004) 101 [hep-ph/0403192] [INSPIRE].
[51] J.M. Henn, G.P. Korchemsky and B. Mistlberger, The full four-loop cusp anomalous dimension in $\mathcal{N} = 4$ super Yang-Mills and QCD, JHEP 04 (2020) 018 [arXiv:1911.10174] [INSPIRE].

[52] A. von Manteuffel, E. Panzer and R.M. Schabinger, Cusp and collinear anomalous dimensions in four-loop QCD from form factors, Phys. Rev. Lett. 124 (2020) 162001 [arXiv:2002.04617] [INSPIRE].

[53] I. Moult, H.X. Zhu and Y.J. Zhu, The four loop QCD rapidity anomalous dimension, arXiv:2205.02249 [INSPIRE].

[54] C. Duhr, B. Mistlberger and G. Vita, The four-loop rapidity anomalous dimension and event shapes to fourth logarithmic order, arXiv:2205.02242 [INSPIRE].