Hereditarily non-topologizable groups *

Gábor Lukács †

March 15, 2022

Abstract

A group \( G \) is non-topologizable if the only Hausdorff group topology that \( G \) admits is the discrete one. Is there an infinite group \( G \) such that \( H/N \) is non-topologizable for every subgroup \( H \leq G \) and every normal subgroup \( N \triangleleft H \)? We show that a solution of this essentially group theoretic question provides a solution to the problem of \( c \)-compactness.

Following Ol’shanskiǐ, we say that a group \( G \) is non-topologizable if the only Hausdorff group topology that \( G \) admits is the discrete one. In 1944, Markov asked whether infinite non-topologizable groups exist (cf. [8]). Markov’s problem was solved in 1980, independently, by Ol’shanskiǐ and Shelah, who constructed infinite non-topologizable groups ([9] and [12]). Although Ol’shanskiǐ’s example is countable and periodic, Klyachko and Trofimov showed that a non-topologizable group need not satisfy either of these properties.

**Theorem 1.** ([4]) There exists a torsion-free finitely generated non-topologizable group. Thus, there exists a torsion-free non-topologizable group of any cardinality.

(The second statement is obtained from the first one using Löwenheim-Skolem theorem.) In a subsequent paper, Trofimov proved that every group embeds into a non-topologizable group of the same cardinality (cf. [14 Thm. 3]).

Markov himself obtained a criterion of non-topologizability for countable groups with a strong algebraic geometric flavour (Theorem 2 below), whose most elegant proof was given by Zeleynuk and Protasov, more than half a century later (cf. [8] and [11, 3.2.4]). Given a monomial \( f(x) = g_0x^{k_1}g_2x^{k_2}g_3\ldots g_{n-1}x^{k_n}g_n \) in a single variable \( x \), with \( g_i \in G \) and \( k_i \in \mathbb{Z} \), the set \( V(f) = \{ g \in G \mid f(g) = e \} \) is closed in any Hausdorff group topology on \( G \), because multiplication must be continuous. Thus, if \( G \setminus \{ e \} \) can be represented as \( V(f_1) \cup \ldots \cup V(f_n) \), where each \( f_i \) is a monomial, then \( \{ e \} \) is open in any Hausdorff group topology on \( G \), and therefore \( G \) is non-topologizable. In this case, one says that \( e \) is algebraically isolated in \( G \). The reverse implication holds only if \( G \) is countable.

*2000 Mathematics Subject Classification: 20F05 22C05 (22A05 54H11)

†I gratefully acknowledge the generous financial support received from the Killam Trusts and Dalhousie University that enabled me to do this research.
Theorem 2. ([8], [11, 3.2.4]) A countable group \(G\) is non-topologizable if and only if \(e\) is algebraically isolated in \(G\).

Shelah’s solution, on the other hand, is uncountable and simple. Thus, his result can be rephrased as follows:

Theorem 3. ([12]) Under the Continuum Hypothesis, there is a group \(G\) such that \(G/N\) is non-topologizable for every \(N \triangleleft G\).

We say that \(G\) is hereditarily non-topologizable if \(H/N\) is non-topologizable for every subgroup \(H \leq G\) and every normal subgroup \(N \triangleleft H\). Motivated by Shelah’s result, we pose the following problem, and show that it is intimately related to the decade-old problem of \(c\)-compactness of topological groups, outlined below.

**Problem I.** Is there an infinite hereditarily non-topologizable group?

By the well known Kuratowski-Mrówka Theorem, a (Hausdorff) topological space \(X\) is compact if and only if for any (Hausdorff) topological space \(Y\) the projection \(p_Y : X \times Y \to Y\) is closed. Inspired by this theorem, one says that a Hausdorff topological group \(G\) is \(c\)-compact if for any Hausdorff group \(H\), the image of every closed subgroup of \(G \times H\) under the projection \(\pi_H : G \times H \to H\) is closed in \(H\). The problem of whether every \(c\)-compact topological group is compact has been an open question for more than ten years. The most extensive study of \(c\)-compact topological groups was done by Dikranjan and Uspenskij in [2], which was a source of inspiration for part of the author’s PhD dissertation, and his subsequent work (cf. [5], [7], [6]).

A Hausdorff topological group \(G\) is minimal if there is no coarser Hausdorff group topology on \(G\) (cf. [13] and [3]). So, a discrete group \(G\) is non-topologizable if and only if it is minimal. One says that a Hausdorff topological group \(G\) is totally minimal if every quotient of \(G\) by a closed normal subgroup is minimal (cf. [11]), or equivalently, if every continuous surjective homomorphism \(f : G \to H\) is open. The following two results of Dikranjan and Uspenskij link between \(c\)-compactness and total minimality.

Theorem 4. ([2, 3.6]) Every closed separable subgroup of a \(c\)-compact group is totally minimal.

Theorem 5. ([2, 5.5]) A countable discrete group \(G\) is \(c\)-compact if and only if every subgroup of \(G\) is totally minimal.

A discrete group \(G\) is hereditarily non-topologizable if and only if the discrete topology is totally minimal on every subgroup of \(G\). Thus, Theorem 5 yields:

Corollary 6. A countable discrete group is \(c\)-compact if and only if it is hereditarily non-topologizable.

Recall that a topological group \(G\) has small invariant neighborhoods (or briefly, \(G\) is SIN), if any neighborhood \(U\) of \(e \in G\) contains an invariant neighborhood \(V\) of \(e\), that is, a neighborhood \(V\) such that \(g^{-1}Vg = V\) for all \(g \in G\). Equivalently, \(G\) is SIN if its left and right uniformities
coincide. In a former paper, the author showed that the problem of \(c\)-compactness for locally compact SIN groups can be reduced to the countable discrete case (cf. [6, 4.5]). Therefore, Problem [1] is equivalent to a special case of the problem of \(c\)-compactness.

**Theorem 7.** The following statements are equivalent:

(i) every locally compact \(c\)-compact group admitting small invariant neighborhoods is compact;

(ii) every countable hereditarily non-topologizable group is finite. \(\square\)

We conclude with an algebraic consequence of hereditary non-topologizability. We denote by \(H^{(k)}\) the \(k\)-th derived group of a group \(H\), that is, \(H^{(1)} = [H, H]\), and \(H^{(k)} = [H^{(k-1)}, H^{(k-1)}]\).

**Theorem 8.** Let \(G\) be a hereditarily non-topologizable group, and let \(H \leq G\) be a subgroup. Then:

(a) \(H\) has a smallest subgroup \(N\) of finite index, and \(N = [N, N]\);

(b) \(H^{(k)}\) has finite index in \(H\) for every \(k \in \mathbb{N}\);

(c) there is \(n \in \mathbb{N}\) such that \(H^{(n)} = H^{(n+1)}\);

(d) if \(H\) is soluble, then \(H\) is finite.

In light of Corollary [6] Dikranjan and Uspenskij’s results imply Theorem [8] (cf. [2, 3.7-3.12]). Nevertheless, for the sake of completeness, we provide here a direct proof that does not rely on the Prodanov-Stoyanov theorem (cf. [10]).

**PROOF.** (a) Let \(\{N_\alpha\}\) be the collection of normal subgroups of finite index in \(H\), and set \(N = \bigcap N_\alpha\). Since \(G\) is hereditarily non-topologizable, the discrete topology is the only Hausdorff group topology on \(H/N\). On the other hand, \(H/N\) embeds into the product \(P = \prod H/N_\alpha\), and \(P\) is compact Hausdorff, because each \(H/N_\alpha\) is finite. Thus, the image of \(H/N\) in \(P\) can be discrete only if it is finite. Therefore, \(N\) has finite index in \(H\). (The closure of the image of \(H/N\) in \(P\) is called the *pro-finite completion* of \(H/N\).)

Let \(H_1 \leq H\) be such that \(|H : H_1| = l\). Then \(H_1\) contains a normal subgroup \(N_1\) of \(H\) such that \(|H : N_1| \leq l!\), and hence \(N \leq N_1 \leq H_1\), as desired. By (a), \([N, N]\) has finite index in \(N\), and so the last statement follows.

(b) Let \(A = H/\langle H, H \rangle\) be the maximal abelian quotient of \(H\). We denote by \(\hat{A}\) the group \(\text{hom}_\mathbb{Z}(A, \mathbb{T})\), where \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). Consider the homomorphism

\[
\rho_A : A \rightarrow \mathbb{T}^{\hat{A}}
\]

\[
a \mapsto (\chi(a))_{\chi \in \hat{A}}.
\]

For every non-zero \(a \in A\), there is a homomorphism \(\chi_a : \langle a \rangle \rightarrow \mathbb{T}\) such that \(\chi_a(a) \neq 0\). Since \(\mathbb{T}\) is injective, \(\chi_a\) can be extended to \(\hat{\chi}_a : A \rightarrow \mathbb{T}\). Thus, \(\rho_A(a) \neq 0\), and so \(\rho_A\) is injective. The *Bohr topology* on \(A\) is the initial topology induced by \(\rho_A\), that is, the subgroup topology when \(A\) is viewed as a subgroup of \(\mathbb{T}^{\hat{A}}\). Since \(G\) is hereditarily non-topologizable, the only Hausdorff group topology on \(A\) is the discrete one. On the other hand, the subgroup \(\rho_A(A)\) of the compact Hausdorff group \(\mathbb{T}^{\hat{A}}\) is discrete if and only if \(A\) is finite. Therefore, \(A\) is finite. Hence, the statement follows by an inductive reiteration of this argument.

(c) and (d) follow from (a) and (b). \(\square\)
ACKNOWLEDGMENTS

My deepest thanks go to Walter Tholen, for introducing me to the problem of $c$-compactness, and for his attention to my work.

I wish to thank Robert Paré for the valuable discussions that were of great assistance in writing this paper.

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Department of Mathematics and Statistics
Dalhousie University
Halifax, B3H 3J5, Nova Scotia
Canada

e-mail: lukacs@mathstat.dal.ca