DISCRETIZATION ON HIGH-DIMENSIONAL DOMAINS

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Abstract. Let \( \mu \) be a Borel probability measure on a compact path-connected metric space \( (X, \rho) \) for which there exist constants \( c, \beta > 1 \) such that \( \mu(B) \geq cr^\beta \) for every open ball \( B \subset X \) of radius \( r > 0 \). For a class of Lipschitz functions \( \Phi : [0, \infty) \to \mathbb{R} \) that piecewisely lie in a finite-dimensional subspace of continuous functions, we prove under certain mild conditions on the metric \( \rho \) and the measure \( \mu \) that for each positive integer \( N \geq 2 \), and each \( g \in L^\infty(X, d\mu) \) with \( \|g\|_\infty = 1 \), there exist points \( y_1, \ldots, y_N \in X \) and real numbers \( \lambda_1, \ldots, \lambda_N \) such that for any \( x \in X \),

\[
\left| \int_X \Phi(\rho(x, y)) g(y) \, d\mu(y) - \sum_{j=1}^N \lambda_j \Phi(\rho(x, y_j)) \right| \leq C N^{-\frac{\beta}{2}} \sqrt{\log N},
\]

where the constant \( C > 0 \) is independent of \( N \) and \( g \). In the case when \( X \) is the unit sphere \( S^d \) of \( \mathbb{R}^{d+1} \) with the usual geodesic distance, we also prove that the constant \( C \) here is independent of the dimension \( d \). Our estimates are better than those obtained from the standard Monte Carlo methods, which typically yield a weaker upper bound \( N^{-\frac{1}{2}} \sqrt{\log N} \).

1. Introduction

The theme of this article is the discretization in high-dimensional spaces and, using these discretizations, finding bounds for errors of numerical quadrature formulae. We mention the idea of the numerical approximation of integrals by finite sums here at the beginning, because it is a subject of interest generally in numerical analysis how to approximate the integral of a function by a finite sum. The most basic approach to this is Gauss quadrature, and the starting point of this in turn is using univariate numerical integration employing zeros of orthogonal polynomials as knots \[15\]. The purpose of taking these zeros is to tailor the quadrature formula to provide the optimal order of accuracy in approximating the integral by a finite sum.

The concept of Gauss quadrature (usually called cubature in higher dimensions) can be generalized in many respects, see for instance \[14\] for work on multivariate quadrature, and our goal in this paper is to take a very general approach. To begin with, we shall work in many (arbitrarily high) dimensions, and both in the literature and here, multivariate spheres are of course our prime examples \[15\], \[20\], \[24\].

Secondly, we shall admit a general metric space as the set on which our integrands are defined or over which the integral shall be taken. We shall, third, find dimension-independent upper bounds on the error of cubature that are uniform in...
$x$, where the integrals and sums take the forms
\[
\int_X \Phi(\rho(x, y)) g(y) \, d\mu(y)
\]
and
\[
\sum_{j=1}^N \lambda_j \Phi(\rho(x, y_j)),
\]
respectively. Since these expressions depend on $x \in X$, where $(X, \rho)$ is a compact metric space, they can be considered as a discretization of probability measures and $X$. It is attractive that the upper bounds on the error are dimension-independent, because it allows us to use these methods in high-dimensions without possibly large constants depending on dimensions marring our results.

With a constant $\beta$ depending on the Borel measure $\mu$, our goal is to derive the estimate
\[
\left| \int_X \Phi(\rho(x, y)) g(y) \, d\mu(y) - \sum_{j=1}^N \lambda_j \Phi(\rho(x, y_j)) \right| \leq C \|g\|_{\infty} N^{-\frac{1}{4} - \frac{3}{4} \sqrt{\log N}},
\]
where $C > 0$ is a constant depending on $\Phi$ and certain properties of the measure $\mu$. The degrees of freedom to obtain the order of the estimate in $1/N$ on the right-hand side come from our judicious placement of the $y_j$s and the coefficients $\lambda_j$. It is worthwhile to point out here that such an estimate is better than most typical estimates that can be deduced from the Monte Carlo methods and standard probability techniques (based on various large deviation inequalities), which normally yield the weaker upper bound $N^{-\frac{1}{2}} \sqrt{\log N}$.

Much of our work depends on the concepts of compactness, weak $(\ast)$ topologies and integrability with respect to a measure $\mu$, and therefore – and for the purpose of fixing notation – we shall review some of these points in the next section.

In Section 3, we prove a preliminary result on the discretization of probability measures, which will play a vital role in this paper. To be more precise, let $Q$ be a compact Hausdorff space equipped with a Borel probability measure $\mu$, and let $X_m$ be an $m$-dimensional subspace of $C(Q)$. Using the method of Bourgain and Lindenstrauss [4], we prove that for every $f \in C(Q)$, the integral $\int_Q f(x) \, d\mu(x)$ can be discretized via weighted sums
\[
\sum_{j=1}^{m+2} \lambda_j f(y_j), \quad \lambda_j \geq 0, \quad y_j \in Q,
\]
where the weights $\lambda_j \geq 0$ and the points $y_j \in Q$ are selected randomly according to a probability distribution in such a way that $\sum_{j=1}^{m+2} \lambda_j = 1$ and
\[
\sum_{j=1}^{m+2} \lambda_j f(y_j) = \int_Q f(x) \, d\mu(x), \quad \forall f \in X_m.
\]

Section 4, then, considers regular partitions of compact metric spaces. Our main result in this section, Theorem 4.1, states that for a non-atomic Borel probability measure $\mu$ on a compact path-connected metric space $(\Omega, \rho)$ with diameter $\pi$, there
exists a partition \( \{R_1, \ldots, R_N\} \) of \( \Omega \) such that for each \( 1 \leq j \leq N \), \( \mu(R_j) = \frac{1}{N} \) and \( \text{diam}(R_j) \leq 4\delta \), where \( \delta > 0 \) is a constant satisfying that

\[
\inf_{x \in \Omega} \mu\left(B_{\delta/2}(x)\right) \geq \frac{1}{N}.
\]

The crucial point here lies in the fact that the constant 4 in the estimate of \( \text{diam}(R_j) \) is absolute.

Section 5 provides one of the main results in Theorem 5.2.

If the discretizations are to take place on finite dimensional compact domains, see, e.g., [5], we have Theorem 6.2 as a suitable result.

Sections 7 and 8 give some examples of interest, the example of the unit sphere probably giving the more important case, and Section 8 suggesting some generalisations of the approach which is using piecewise polynomials \( \Phi \) in our expressions above, to piecewise exponential functions instead.

2. Preliminaries

In this section, we list several basic results from functional analysis and probability that will be needed in later sections. Most of the materials in this section can be found in the book [22].

**Theorem 2.1.** Let \( X \) be a real linear topological space with dual space \( X^* \). Then the following statements hold:

(i) Let \( A \) and \( B \) be two nonempty disjoint convex sets in \( X \). If \( A \) is open, then there exists \( \Lambda \in X^* \) such that

\[
\Lambda x < \inf_{y \in B} \Lambda y, \quad \forall \ x \in A.
\]

If \( A \) is compact, \( B \) is closed and \( X \) is locally convex, then there exist \( \Lambda \in X^* \) such that

\[
\sup_{x \in A} \Lambda x < \inf_{y \in B} \Lambda y.
\]

(ii) If \( X \) is an F-space (i.e., a complete vector space with metric that is translation invariant whose multiplications and additions are continuous), then for every compact subset \( K \subset X \), the closure of the convex hull of \( K \) is compact in \( X \).

Next, we recall some basic facts on weak and weak*-topologies. A topology \( \tau_1 \) on a nonempty set \( X \) is said to be weaker than another topology \( \tau_2 \) on \( X \) if \( \tau_1 \subset \tau_2 \).

**Theorem 2.2.** Let \( X \) be a real vector space, and \( X' \) a vector space of linear functionals on \( X \) which separates points in \( X \) (i.e., given any two distinct points \( x_1, x_2 \in X \) there exists \( \Lambda \in X' \) such that \( \Lambda x_1 \neq \Lambda x_2 \)). If \( \tau \) denotes the weakest topology on \( X \) with respect to which every element in \( X' \) is a continuous linear functional on \( X \), then \( (X, \tau) \) is a locally convex space whose dual is \( X' \).

Let \( X \) be a real, locally convex linear topological space with topology \( \tau \) and the dual space \( X^* \). Let \( \tau_w \) denote the weak topology of \( X \), i.e., the weakest topology of \( X \) with respect to which every linear functional in \( X^* \) is continuous. Then \( \tau_w \subset \tau \), and \( X_w = (X, \tau_w) \) is a locally convex space whose dual is also \( X^* \). We denote by \( \tau_{w^*} \) the weak* -topology of \( X^* \); that is, \( \tau_{w^*} \) is the weakest topology of \( X^* \) with respect to which for every \( x \in X \), the linear functional \( f \in X^* \rightarrow f(x) \) is continuous. Then \( (X^*, \tau_{w^*}) \) is a locally convex linear topological space whose dual
is $X$. If $X$ is separable, then every weak* compact set $K$ in $X^*$ is metrizable in the weak* topology.

**Theorem 2.3** (Banach-Alaoglu theorem). For every neighborhood $V$ of 0 in $X$, its polar

$$K := \{ \Lambda \in X^* : |\Lambda x| \leq 1, \forall x \in V \}$$

is weak* compact in $X^*$. If, in addition, $X$ is separable, then $K$ is sequentially compact in the weak* topology.

Third, we review some basic results on vector-valued integration. We start with the following definition:

**Definition 2.4.** Let $X$ be a real locally convex topological vector space, and let $(Q, \mu)$ be a measure space. A vector-valued function $f : Q \to X$ is said to be integrable with respect to $\mu$ if

$$\Lambda(f(\cdot)) = \langle \Lambda, f(\cdot) \rangle \in L^1(Q, \mu), \quad \forall \Lambda \in X^*$$

and there exists $y \in X$ such that

$$\langle \Lambda, y \rangle = \int_Q \langle \Lambda, f(x) \rangle \, d\mu(x), \quad \forall \Lambda \in X^*.$$

If such a vector $y \in X$ exists, it must be unique, and is denoted by $\int_Q f(x) \, d\mu(x)$.

Recall that a positive Borel measure $\mu$ on a topological space $Q$ is regular if

$$\mu(E) = \sup\{ \mu(K) : K \subset E \text{ is compact} \} = \inf\{ \mu(G) : E \subset G, G \text{ is open in } X \}$$

for every Borel set $E \subset Q$. Each Borel probability measure on a locally compact Hausdorff space with a countable base for its topology, or on a compact metric space is regular. If $Q$ is a compact Hausdorff space, and $C(Q)$ is the space of all continuous functions on $Q$ (with the uniform norm), then the dual of $C(Q)$ is the space of all finite regular Borel measures (i.e., Radon measures) on $Q$ (with the norm of total variation).

**Theorem 2.5.** Suppose that

(i) $X$ is a real, locally convex topological vector space;
(ii) $Q$ is a compact Hausdorff space;
(iii) $f : Q \to X$ is continuous;
(iv) $\conv(f(Q))$ is compact in $X$ (this is automatically true if $X$ is an F-space).

Then given any Borel probability measure $\mu$ on $Q$, the function $f : Q \to X$ is integrable with respect to $\mu$ and moreover,

$$y = \int_Q f \, d\mu = \int_{f(Q)} z \, d\mu_f(z) \in \overline{\conv(f(Q))},$$

where $\mu_f$ is a Borel probability measure on $f(Q)$ given by

$$\mu_f(E) = \mu(f^{-1}(E)), \quad E \subset f(Q).$$

Conversely, if $y \in \overline{\conv(f(Q))}$, then there exists a regular Borel probability measure $\mu_f$ on $f(Q)$ such that

$$y = \int_{f(Q)} z \, d\mu_f(z).$$
Theorem 2.6. Suppose that $Q$ is a compact Hausdorff space, $X$ is a Banach space, $f : Q \to X$ is continuous, and $\mu$ is a positive Borel measure on $Q$. Then
\[
\left\| \int_Q f \, d\mu \right\| \leq \int_Q \|f\| \, d\mu.
\]

3. A preliminary result

Let $Q$ be a compact metric space equipped with a Borel probability measure $\mu$. Let $M(Q)$ denote the space of all finite signed Borel measures on $Q$. Then $M(Q)$ is a Banach space with respect to the norm
\[
\|\nu\| := |\nu|(Q) = \sup \left\{ \left| \int_Q f \, d\nu \right| : f \in C(Q), \|f\|_{C(Q)} \leq 1 \right\}.
\]

Such a Banach space is the dual space of $C(Q)$. Note that $C(Q)$ is a separable Banach space. Let $M(Q)^{w^*}$ denote the space $M(Q)$ endowed with the weak*-topology $\tau_{w^*}$. Then $M(Q)^{w^*}$ is a locally convex topological space with dual space $C(Q)$.

Next, let $X_m$ denote an $m$-dimensional linear subspace of $C(Q)$. Let $\Sigma_0 \subset M(Q)$ denote the set of all probability measures $\rho \in M(Q)$ of the form
\[
\rho = \sum_{j=1}^{m+2} \lambda_j(\rho) \delta_{y_j(\rho)},
\]
where $\lambda_j(\rho) \geq 0$, $y_j(\rho) \in Q$ for $j = 1, 2, \ldots, m + 2$ and $\sum_{j=1}^{m+2} \lambda_j(\rho) = 1$.

Let $\Sigma \subseteq \Sigma_0$ denote the set of all probability measures $\rho \in \Sigma_0$ such that
\[
\int_Q f(x) \, d\mu(x) = \int_Q f(x) \, d\rho(x), \quad \forall f \in X_m.
\]

Theorem 3.1. There exists a Borel probability measure $\nu$ on the space $M(Q)^{w^*}$ which is supported in the set $\Sigma \subset M(Q)$ and satisfies
\[
\mu = \int_{\Sigma} \rho \, d\nu(\rho),
\]
where the equality holds in the sense that for any $f \in C(Q)$,
\[
\int_Q f(x) \, d\mu(x) = \int_{\Sigma} \sum_{j=1}^{m+2} \lambda_j(\rho) f(y_j(\rho)) \, d\nu(\rho)
\]
and where $\mu$ is the probability measure we wish to discretise.

Lemma 3.2. The set $\Sigma$ is $w^*$-compact in $M(Q)$.

Proof. Define
\[
S := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{m+2}) \in \mathbb{R}^{m+2} : \lambda_1, \ldots, \lambda_{m+2} \geq 0, \sum_{j=1}^{m+2} \lambda_j = 1 \right\}.
\]
Then $S \times Q^{m+2}$ is a compact topological space with respect to the product topology. Next, consider the mapping $T : S \times Q^{m+2} \to M(Q)^{w^*}$ that takes $(\lambda, x) \in S \times Q^{m+2}$ to the measure $\sum_{j=1}^{m+2} \lambda_j \delta_{e_j} \in M(Q)$. Note that for any $f \in C(Q)$, and any $(\lambda, x), (\alpha, y) \in S \times Q^{m+2}$, we have
Since each \( X \) lies in the \( w \)-closure of the convex hull of \( \Sigma \); that is, \( \text{conv}(\Sigma) \subset X \). This implies that \( \rho \) is \( w \)-compact.

Finally, for each \( f \in C(Q) \), set \( \mu_f := \int_Q f \, d\mu. \) Then

\[
\Sigma := \{ \rho \in \Sigma \cap \text{conv}(\Sigma) : \langle f, \rho \rangle = \mu_f, \forall f \in X_m \}.
\]

Since each \( X_m \subset C(Q) \) and \( C(Q) \) is the dual space of \( M(Q)^{w^*} \), it follows that \( \Sigma \) is a \( w^* \)-closed subset of the \( w^* \)-compact set \( \Sigma_0 \). Thus, \( \Sigma \) is a weak*-compact subset of \( M(Q) \).

**Lemma 3.3.** The probability measure \( \mu \in M(Q) \) is in the weak*-closure of the convex hull \( K \) of \( \Sigma \subset M(Q)^{w^*} \).

**Proof.** Assume to the contrary that \( \mu \notin K = \overline{\text{conv}(\Sigma)}^{w^*} \). Then by the convex separation theorem, there exists \( g \in C(Q) \) such that

\[
\int_Q g \, d\mu > \sup_{\rho \in \Sigma} \int_Q g \, d\rho.
\]

Let \( X_{m+1} = \text{span}\{X_m, g\} \). By Corollary 4.1 of [13], there exist \( x_1, x_2, \ldots, x_{m+2} \in Q \) and \( \lambda_1, \ldots, \lambda_{m+2} \geq 0 \) such that \( \sum_{j=1}^{m+2} \lambda_j = 1 \) and

\[
\int_Q f \, d\mu = \sum_{j=1}^{m+2} \lambda_j f(x_j), \quad \forall f \in X_{m+1}.
\]

This implies that \( \rho = \sum_{j=1}^{m+2} \lambda_j \delta_{x_j} \in \Sigma \) and \( \int_Q g \, d\mu = \int_Q g \, d\rho \), which contradicts (3.1).

**Proof of Theorem 3.1** Let \( X = C(Q) \). Then \( M(Q) = X^* \). By Lemma 3.3, \( \mu \) lies in the \( w^* \)-closure of the convex hull of \( \Sigma \); that is, \( \mu \in K = \overline{\text{conv}(\Sigma)}^{w^*} \). By Lemma 3.2, \( \Sigma \) is compact in the space \( (X^*, w^*) \). Thus, by Theorem 2.5 it is enough to show that \( K \) is also compact in the space \( (X^*, w^*) \). Note that

\[
\Sigma \subset \Sigma_0 \subset B_{X^*} := \{ \nu \in X^* : \|\nu\| \leq 1 \},
\]

which also implies that \( \text{conv}(\Sigma) \subset B_{X^*} \). Since \( B_{X^*} \) is compact in the space \( (X^*, w^*) \), it follows that \( K := \overline{\text{conv}(\Sigma)}^{w^*} \) is a closed subset of \( B_{X^*} \), which also implies that \( K \) is compact in the space \( (X^*, w^*) \). The theorem is proved.

4. Regular partitions on compact metric space

Let \( (\Omega, \rho) \) be a compact metric space. Open balls and closed balls in \( \Omega \) will be denoted by \( B_\zeta(x) := \{ y \in \Omega : \rho(x, y) < \zeta \} \), and \( B_\zeta[x] := \{ y \in \Omega : \rho(x, y) \leq \zeta \} \), respectively. A path connecting two points \( x, y \in \Omega \) is a continuous map \( \gamma : [0, 1] \to \Omega \) with \( \gamma(0) = x \) and \( \gamma(1) = y \). A metric space \( (\Omega, \rho) \) is called path-connected if every two distinct points in \( \Omega \) can be connected with a path. As is well known,
every open connected subset of \( \mathbb{R}^n \) is path-connected. Given a set \( A \subset \Omega \) and a point \( x \in \Omega \), define
\[
\text{dist}(x, A) := \inf_{y \in A} \rho(x, y).
\]

**Theorem 4.1.** Let \( (\Omega, \rho) \) be a compact path-connected metric space with diameter \( \text{diam}(\Omega) := \max_{x,y \in \Omega} \rho(x, y) = \pi \). Let \( \mu \) be a non-atomic Borel probability measure on \( \Omega \), and \( N \geq 2 \) a positive integer. Assume that the inequality
\[
\inf_{x \in \Omega} \mu\left(B_{\delta/2}(x)\right) \geq \frac{1}{N}
\]
holds for some \( \delta > 0 \). Then there exists a partition \( \{R_1, \ldots, R_N\} \) of \( \Omega \) such that
(i) the \( R_j \) are pairwise disjoint subsets of \( \Omega \),
(ii) for each \( 1 \leq j \leq N \), \( \mu(R_j) = \frac{1}{N} \) and \( \text{diam}(R_j) \leq 4\delta \).

Theorem 4.1 with constants depending on certain geometric parameters of the underlying space \( (\Omega, \rho, \mu) \) (e.g. dimension, doubling constants) is probably known in a more general setting. The crucial point here lies in the fact that the constant 4 in the estimates of \( \text{diam}(R_j) \) is absolute.

**Lemma 4.2.** Let \( (\Omega, \rho) \) be a compact path-connected metric space with diameter \( \pi \). Then for each \( \delta \in (0, \pi) \), there exist a finite set \( \Lambda = \{a_1, \ldots, a_M\} \subset \Omega \) with \( M > 1 \) such that \( \Omega = \bigcup_{j=1}^{M} B_\delta(a_j) \) and

\[
\text{dist}(a_j, \Lambda_{j-1}) = \delta, \quad j = 2, 3, \ldots, M,
\]
where \( \Lambda_k := \{a_1, a_2, \ldots, a_k\}, k = 1, \ldots, M \).

**Proof.** Since the metric space \( \Omega \) is path-connected and has diameter \( \pi \geq \delta \), there exist two points \( a_1, a_2 \in \Omega \) such that \( \rho(a_1, a_2) = \delta \). Assume that \( \Lambda_n = \{a_1, \ldots, a_n\} \) is a finite subset of \( \Omega \) such that
\[
\text{dist}(a_j, \Lambda_{j-1}) = \delta, j = 2, \ldots, n,
\]
where \( \Lambda_j = \{a_1, a_2, \ldots, a_j\} \). If \( \Omega = \bigcup_{j=1}^{n} B_\delta(a_j) \), then it is sufficient to use \( M = n \).

Now assume that, in contrast, \( \Omega \neq \bigcup_{j=1}^{n} B_\delta(a_j) \). Then there exists a point \( y \in \Omega \setminus \Lambda_n \) such that
\[
\text{dist}(y, \Lambda_n) \geq \delta.
\]
Without loss of generality, we may assume that \( \text{dist}(y, \Lambda_n) = \rho(y, a_1) \). Let \( \gamma : [0, 1] \to \Omega \) be a path such that \( \gamma(0) = y \) and \( \gamma(1) = a_1 \). Define \( f(t) := \text{dist}(\gamma(t), \Lambda_n) \) for \( t \in [0, 1] \). Clearly, \( f \) is a continuous function on \( [0, 1] \) with
\[
f(0) = \text{dist}(y, \Lambda_n) \geq \delta \quad \text{and} \quad f(1) = \text{dist}(a_1, \Lambda_n) = 0.
\]
Thus, there exists a point \( a_{n+1} = \gamma(t_n) \in \Omega \) for some \( t_n \in [0, 1] \) such that
\[
\text{dist}(a_{n+1}, \Lambda_n) = f(t_n) = \delta.
\]
We may continue this selection procedure with \( \Lambda_{n+1} = \{a_1, \ldots, a_{n+1}\} \). Since \( \Omega \) is compact, this procedure must terminate after a finite number of steps. \( \square \)

**Proof of Theorem 4.1.** Let \( \{a_1, \ldots, a_M\} \) be a finite subset of \( \Omega \) as given in Lemma 4.2.

For \( 1 < j \leq M \), let \( 1 \leq k_j < j \) be an integer such that
\[
\text{dist}(a_j, \Lambda_{j-1}) = \rho(a_j, a_{k_j}) = \delta.
\]
For each $1 \leq j \leq M$, define
\[ V_j := \left\{ x \in \Omega : \rho(x, a_j) = \text{dist}(x, \Lambda) \text{ and } \text{dist}(x, \Lambda) < \min_{1 \leq i < j} \rho(x, a_i) \right\}. \]
That is, $x \in V_j$ if and only if $j$ is the smallest positive integer such that $\text{dist}(x, \Lambda) = \rho(x, a_j)$. Clearly, the sets $V_j$ are pairwise disjoint,
\[
B_{\frac{\ell}{2}}(a_j) \subset V_j \subset B_\ell[a_j], \quad j = 1, 2, \ldots, M,
\]
and $\Omega = \bigcup_{j=1}^M V_j$. Moreover, using (4.1), we have
\[
\rho(V_j) \geq \frac{1}{N}, \quad \forall 1 \leq j \leq M.
\]
Now we construct the desired partition of $\Omega$ as follows via a finite number of steps. In the first step, we write $V^0_j = V_j$ for $j = 1, \ldots, M$, and modify the cells $V_M$ and $V_M$ slightly so that $N\mu(V_M)$ is an integer. Let $E_M \subset V^0_M$ be such that $\mu(E_M) < \frac{1}{N}$ and $N\mu(V^0_M \setminus E_M)$ is a positive integer. We then update the cells as follows:
\[
V^1_j := \begin{cases} 
V^0_j, & \text{if } j \neq M \text{ and } j \neq k_M, \\
V^0_j \setminus E_M, & \text{if } j = M, \\
V^0_j \cup E_M, & \text{if } j = k_M.
\end{cases}
\]
Note that the sets $V^1_j$ are pairwise disjoint, $\Omega = \bigcup_{j=1}^M V^1_j$, $V^0_j \subset V^1_j$ for $1 \leq j \leq M - 1$ and $V^1_M \subset V^0_M$.

In the second step, we continue the process with the collection of the first $M - 1$ updated cells: $V^1_j$, $1 \leq j \leq M - 1$. More precisely, we choose a subset $E_{M-1}$ of $V^0_{M-1}$ such that $\mu(E_{M-1}) < \frac{1}{N}$ and $N\mu(V^0_{M-1} \setminus E_{M-1})$ is a positive integer, and then update the cells as follows:
\[
V^2_j := \begin{cases} 
V^1_j, & \text{if } j \neq M - 1 \text{ and } j \neq k_{M-1}, \\
V^1_j \setminus E_{M-1}, & \text{if } j = M - 1, \\
V^1_j \cup E_{M-1}, & \text{if } j = k_{M-1}.
\end{cases}
\]
It is very important here that the set $E_{M-1}$ is selected as a subset of $V^0_{M-1}$ (rather than a general subset $V^1_{M-1}$) because this way of selection yields a better control of the diameter of the updated cell $V^1_{M-1} := E_{M-1} \cup V^1_{M-1}$.

In general, at the $\ell$-th step with $1 \leq \ell < M$, we modify the cells $V^\ell_{M-\ell+1}$ and $V^\ell_{M-\ell+1}$ in a similar manner. Indeed, let $E_{M-\ell+1} \subset V^\ell_{M-\ell+1} \subset V^\ell_{M-\ell+1}$ be such that $\mu(E_{M-\ell+1}) < \frac{1}{N}$ and $N\mu(V^\ell_{M-\ell+1} \setminus E_{M-\ell+1})$ is a positive integer. We then define
\[
V^\ell_j := \begin{cases} 
V^{\ell-1}_j, & \text{if } j \neq M - \ell + 1 \text{ and } j \neq k_{M-\ell+1}, \\
V^\ell_{M-\ell+1} \setminus E_{M-\ell+1}, & \text{if } j = M - \ell + 1, \\
V^\ell_{M-\ell+1} \cup E_{M-\ell+1}, & \text{if } j = k_{M-\ell+1}.
\end{cases}
\]
Clearly, the sets $V^\ell_j$ are pairwise disjoint, $\Omega = \bigcup_{j=1}^M V^\ell_j$, $V^0_j \subset V^{\ell-1}_j \subset V^\ell_j$ for $j = 1, 2, \ldots, M - \ell$, and for $j = M - \ell + 1, \ldots, M$,
\[
V^\ell_j \subset V^{\ell-1}_j \quad \text{and} \quad N\mu(V^\ell_j) \text{ is a positive integer.}
\]
Furthermore, by the above construction, it is easily seen that for each \(1 \leq j \leq M - \ell\),

\[
V_j^\ell \subset \bigcup_{M - \ell + 1 \leq k \leq M, \rho(a_k, a_j) = d} (V_j^0 \cup V_k^0),
\]

which, using (4.2), implies that \(V_j^\ell \subset B_{2\delta}[a_j]\) and \(\text{diam}(V_j^\ell) \leq 4\delta\) for all \(1 \leq j \leq M\).

The above process will be terminated after the \((M - 1)\)-st step, where we obtain pairwise disjoint subsets \(V_j^{M-1}\), \(j = 1, 2, \ldots, M\), of \(\Omega\) with diameter \(\leq 4\delta\) such that \(\Omega = \bigcup_{j=1}^{M-1} V_j^{M-1}\) and \(N\mu(V_j^{M-1})\) is a positive integer for \(2 \leq j \leq M\). Since \(\mu\) is a probability measure, we have

\[
N = N\mu(\Omega) = \sum_{j=1}^{M} N\mu(V_j^{M-1}).
\]

This implies that \(N\mu(V_1^{M-1})\) is a positive integer as well. Since \(\mu\) is non-atomic, for each \(1 \leq j \leq M\), we may write \(V_j^{M-1}\) as a disjoint union

\[
V_j^{M-1} = \bigcup_{k=1}^{\ell_j} S_{j,k}
\]

such that \(\mu(S_{j,k}) = \frac{1}{\ell_j}\) and \(\text{diam}(S_{j,k}) \leq 4\delta\) for \(1 \leq k \leq \ell_j\). This leads to a partition of \(\Omega\) with the desired properties:

\[
\Omega = \bigcup_{j=1}^{M} \bigcup_{k=1}^{\ell_j} S_{j,k}.
\]

\[\square\]

5. Discretization on compact metric spaces

Let \((X, \rho)\) be a compact metric space with metric \(\rho\) and diameter \(\pi\). For \(x \in X\) and \(0 \leq a < b \leq \pi\), set

\[
E(x; a, b) := \{y \in X : a \leq \rho(x, y) \leq b\}.
\]

A partition of \(X\) consists of finitely many pairwise disjoint subsets of \(X\) whose union is \(X\).

**Definition 5.1.** Let \(0 = t_0 < t_1 < \cdots < t_\ell = \pi\) be a partition of the interval \([0, \pi]\), and let \(r \in \mathbb{N}\). We say \(\Phi \in C[0, \pi]\) belongs to the class \(S_r \equiv S_r(t_1, \ldots, t_\ell)\) if there exists an \(r\)-dimensional linear subspace \(V_r\) of \(C(X)\) such that for any \(x \in X\) and each \(1 \leq j \leq \ell\),

\[
\Phi(\rho(x, \cdot)) \big|_{E(x; t_{j-1}, t_j)} \in \left\{ f \big|_{E(x; t_{j-1}, t_j)} : f \in V_r \right\}.
\]

Next, let \(\mu\) be a Borel probability measure on \(X\) satisfying the following condition for a parameter \(\beta \geq 1\) and some constant \(c_1 > 1\):

(a) for each positive integer \(N\), there exists a partition \(\{X_1, \ldots, X_N\}\) of \(X\) such that \(\mu(X_j) = \frac{1}{N}\) and \(\text{diam}(X_j) \leq \delta_N := c_1 N^{-\frac{1}{\beta}}\) for \(1 \leq j \leq N\).
According to Theorem 4.1, Condition (a) holds automatically with \( c_1 = 20\pi \) if the metric space \( X \) is path-connected, and \( \mu \) is a non-atomic Borel probability measure on \( X \) satisfying that for any \( 0 < t \leq 1 \),

\[
\inf_{x \in X} \mu(B_t(x)) \geq \left( \frac{8}{c_1} \right)^{\beta t^\beta}.
\]

In this section, we shall prove

**Theorem 5.2.** Let \( \Phi \in C[0, \pi] \) satisfy

\[
|\Phi(s) - \Phi(s')| \leq |s - s'|, \quad \forall s, s' \in [0, \pi],
\]

and belong to a class \( S_r(t_1, \ldots, t_\ell) \) for some compact metric space \( (X, \rho) \), where \( r \in \mathbb{N} \) and \( 0 = t_0 < t_1 < \cdots < t_\ell = \pi \). Let \( \mu \) be a Borel probability measure on \( X \) satisfying the condition (a) and the following condition:

(b) for each \( x \in X \) and \( \delta \in (0, \pi) \),

\[
\mu \left( E(x; t_j - \delta, t_j + \delta) \right) \leq c_2 \delta, \quad 1 \leq j \leq \ell,
\]

where \( c_2 > 1 \) is a constant independent of \( \delta \) and \( x \).

Then for each positive integer \( N \geq 4 \), there exist points \( y_1, \ldots, y_{(r+2)N} \in X \) and nonnegative numbers \( \lambda_1, \ldots, \lambda_{(r+2)N} \geq 0 \) such that \( \sum_{j=1}^{(r+2)N} \lambda_j = 1 \) and

\[
\max_{x \in X} \left| \int_X \Phi(\rho(x, y)) \, d\mu(y) - \sum_{j=1}^{(r+2)N} \lambda_j \Phi(\rho(x, y_j)) \right| \leq c_3 N^{-\frac{1}{2} - \frac{3}{2r}} \sqrt{\log N},
\]

where \( c_3 := 8c_2^2 \sqrt{c_2} \sqrt{\beta} \).

In the case when the metric space \( X \) is path-connected, we will prove

**Theorem 5.3.** Let \( (X, \rho) \) be a compact path-connected metric space. Let \( \Phi \in C[0, \pi] \) satisfy 5.2 and belong to a class \( S_r(t_1, \ldots, t_\ell) \) for some \( r \in \mathbb{N} \) and \( 0 = t_0 < t_1 < \cdots < t_\ell = \pi \). Let \( \mu \) be a non-atomic Borel probability measure on \( X \) satisfying \( 5.1 \). Assume in addition that the condition (b) in Theorem 5.2 is satisfied. Then for any \( g \in L^\infty(X, d\mu) \) with \( \|g\|_{L^\infty(\mu)} \leq 1 \), and each positive integer \( N \geq 20 \), there exist points \( y_1, \ldots, y_{2(r+2)N} \in X \) and real numbers \( \lambda_1, \ldots, \lambda_{2(r+2)N} \) such that

\[
\max_{x \in X} \left| \int_X \Phi(\rho(x, y)) g(y) \, d\mu(y) - \sum_{j=1}^{2(r+2)N} \lambda_j \Phi(\rho(x, y_j)) \right| \leq 45c_3 N^{-\frac{1}{2} - \frac{3}{2r}} \sqrt{\log N}.
\]

Let us give some examples of the metric spaces \((X, \rho)\) and the associated classes \( S_r \) which satisfy the conditions of Theorem 5.3.

**Example 5.4.** (i) Let \( X = \mathbb{S}^d \) be the unit sphere of \( \mathbb{R}^{d+1} \) equipped with the usual geodesic distance \( \rho(x, y) = \arccos x \cdot y \) for \( x, y \in \mathbb{S}^d \). If \( \varphi \in C[-1, 1] \) is a piecewise algebraic polynomial of degree at most \( n_0 \) on \([-1, 1] \), then the function \( \Phi(\theta) := \varphi(\cos \theta), \theta \in [0, \pi] \) belongs to a class \( S_r \) with \( r \) being the dimension of the space of all spherical polynomials of degree at most \( n_0 \) on the sphere \( \mathbb{S}^d \). In this case, \( \Phi(\rho(x, y)) = \varphi(x \cdot y) \), and the condition \( 5.1 \) implies both the condition (a) and the condition (b).
(ii) Let \( X = B_{\frac{1}{2}}(0) \subset \mathbb{R}^d \) be the Euclidean ball with centre 0 and radius \( \frac{1}{2} \). If \( \varphi \in C[0, \infty) \) is a piecewise algebraic polynomial of degree at most \( n_0 \) on \( [0, \infty) \), then the function \( \Phi(t) := \varphi(t^2), \ t \geq 0 \) belongs to a class \( \mathcal{S}_r \) with \( r \) being the dimension of the space of all algebraic polynomials of degree at most \( 2n_0 \) in \( d \) variables. In this case, \( \Phi(\rho(x, y)) = \varphi(\|x - y\|^2) \), and the condition (5.1) implies both the condition (a) and the condition (b).

We will discuss these examples in detail in Sections 7 and 8.

5.1. Proof of Theorem 5.2 The proof of Theorem 5.2 follows along the same idea as that of [4].

Let \( \{X_1, \ldots, X_N\} \) be a partition of \( X \) satisfying the condition (a). By the inner regularity of the measure \( \mu \), for each \( 1 \leq j \leq N \), there exists a compact subset \( Q_j \subset X_j \) such that

\[
\frac{1}{N} - \mu(Q_j) \leq \frac{1}{2}(1 + \|\Phi\|_\infty)^{-1} N^{-\frac{1}{4} - \frac{1}{16d}}.
\]

Let \( \mu_j \) denote the probability measure on \( Q_j \) given by \( \mu_j(E) = \frac{\mu(E)}{\mu(Q_j)} \) for each Borel subset \( E \subset Q_j \). Then it is easily seen that

\[
\sup_{x \in X} \left| \int_X \Phi(\rho(x, y)) \, d\mu(y) - \frac{1}{N} \sum_{j=1}^N \int_{Q_j} \Phi(\rho(x, y)) \, d\mu_j(y) \right| \leq N^{-\frac{1}{4} - \frac{1}{16d}}.
\]

Let \( \Sigma_j \) denote the set of all Borel probability measures \( \sigma_j \) on \( Q_j \) that take the form

\[
\sigma_j = \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \delta_{y_i(\sigma_j)}, \quad \lambda_i(\sigma_j) \geq 0, \quad y_i(\sigma_j) \in Q_j, \quad 1 \leq j \leq r + 2,
\]

such that \( \sum_{i=1}^{r+2} \lambda_i(\sigma_j) = 1 \) and

\[
\int_{Q_j} f(y) \, d\mu_j(y) = \sum_{i=1}^{r+2} \lambda_i(\sigma_j) f(y_i(\sigma_j)), \quad \forall f \in V_r.
\]

According to Theorem 6.1, there exists a Borel probability measure \( \nu_j \) on \( \Sigma_j \) such that

\[
\int_{Q_j} f \, d\mu_j = \int_{\Sigma_j} \sum_{i=1}^{r+2} \lambda_i(\sigma_j) f(y_i(\sigma_j)) \, d\nu_j(\sigma_j), \quad \forall f \in C(Q_j).
\]

Now we consider the following product probability space:

\[
(\tilde{\Sigma}, \nu) = \prod_{j=1}^N (\Sigma_j, \nu_j).
\]

We first claim that for each fixed \( x \in X \) and parameter \( t > \sqrt{\log 2} \), there exists a subset \( G(x) \subset \tilde{\Sigma} \) with \( \nu(G(x)) \leq 2e^{-t^2} < 1 \) such that for each \( \sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \tilde{\Sigma} \setminus G(x) \),

\[
\left| \frac{1}{N} \sum_{j=1}^N \lambda_i(\sigma_j) \Phi(\rho((x, y_i(\sigma_j))) - \frac{1}{N} \int_{Q_j} \Phi(\rho(x, y)) \, d\mu_j(y) \right| \leq \frac{4}{\sqrt{3}} c_1 c_2 t^2 N^{-\frac{1}{4} - \frac{1}{16d}}.
\]
To show this claim, we consider the following independent random variables on the probability space \((\Sigma, \nu)\):

\[
h_j(\sigma) = h_j(\sigma_j) := \sum_{i=1}^{n+2} \lambda_i(\sigma_j) \Phi \left( \rho(x, y_i(\sigma_j)) \right) - \int_{Q_j} \Phi(\rho(x, y)) \, d\mu_j(y),
\]

where \(\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma\) and \(j = 1, \ldots, N\). By (5.2) and (5.6), we have

\[
\mathbb{E}h_j = 0, \quad h_j \leq \operatorname{diam}(X_j) \leq \delta_N, \quad 1 \leq j \leq N.
\]

For each \(1 \leq j \leq N\), pick a point \(y_j \in Q_j\) and set \(R_j := B_{\delta_N}(y_j)\) so that \(Q_j \subset X_j \subset R_j\). Set

\[
S_i(x) := E(x; t_{i-1}, t_i) = \left\{ y \in X : t_{i-1} \leq \rho(x, y) \leq t_i \right\}, \quad i = 1, \ldots, \ell.
\]

Note that if \(R_j \subset S_k(x)\) for some \(1 \leq k \leq \ell\) and \(1 \leq j \leq N\), then there exists a function \(f_{k,x} \in V_r\) such that

\[
\Phi(\rho(x, \cdot)) \big|_{Q_j} = f_{k,x} \big|_{Q_j},
\]

which, using (5.5), implies that

\[
h_j(\sigma_j) = \sum_{i=1}^{n+2} \lambda_i(\sigma_j) f_{k,x}(\sigma_j) \rho(x, y_i(\sigma_j)) - \int_{Q_j} f_{k,x}(y) \, d\mu_j(y) = 0.
\]

For \(1 \leq k \leq \ell - 1\) and if \(\ell > 1\), let

\[
E_k(x) := \left\{ y \in X : t_k - 2\delta_N \leq \rho(x, y) \leq t_k + 2\delta_N \right\}.
\]

Denote by \(I\) the set of all positive integers \(1 \leq j \leq N\) such that

\[
y_j \in \bigcup_{k=1}^{\ell-1} E_k(x).
\]

Let \(I^c = \{1, 2, \ldots, N\} \setminus I\). Note that if \(j \in I^c\), then there exists \(1 \leq k \leq \ell\) such that \(R_j \subset S_k(x)\), which implies \(h_j = 0\). Furthermore, since

\[
\bigcup_{j \in I} X_j \subseteq \bigcup_{j \in I} R_j \subseteq \bigcup_{k=1}^{\ell-1} \left\{ y \in X : t_k - 2\delta_N \leq \rho(x, y) \leq t_k + 2\delta_N \right\},
\]

it follows by Condition (b) that

\[
\#I \leq 2c_2\ell N \delta_N = 2c_2c_1 \ell N^{1-\beta^{-1}}.
\]

We shall use this in our next estimate. Now setting

\[
\xi_j = \frac{1}{\delta_N} h_j, \quad j = 1, 2, \ldots, N,
\]

and using the Bernstein inequality in probability, we obtain that for any \(\varepsilon > 0\),

\[
\operatorname{Prob} \left\{ \frac{1}{N} \sum_{j=1}^{N} \xi_j > \varepsilon \right\} = \operatorname{Prob} \left\{ \frac{\#I}{1} \sum_{j \in I} \xi_j > \varepsilon N \right\}
\leq 2 \exp \left( -\frac{3}{8} \frac{(\#I)^2 \varepsilon^2 N^2}{(\#I)^2} \right) \leq 2 \exp \left( -\frac{3\varepsilon^2 N^{1+\beta^{-1}}}{16c_1c_2\ell} \right).
\]
It follows that for any $\delta > 0$,
\[
\text{Prob}\left\{ \frac{1}{N} \left| \sum_{j=1}^{N} h_j \right| > \delta \right\} \leq 2 \exp \left( -\frac{3\delta^2 N^{1+3\beta^{-1}}}{16c_1^2c_2\ell} \right).
\]
Given a parameter $t > 0$, setting
\[
\delta := \frac{4}{\sqrt{3}} c_1 \sqrt{c_1 c_2 \ell N^{-\frac{1}{2} - \frac{3\beta}{2}}} t,
\]
we conclude that the inequality
\[
\frac{1}{N} \left| \sum_{j=1}^{N} h_j \right| \leq \frac{4}{\sqrt{3}} c_1 \sqrt{c_1 c_2 \ell \cdot t \cdot N^{-\frac{1}{2} - \frac{3\beta}{2}}}
\]
holds with probability at least $1 - 2e^{-t^2}$ on the probability space $(\tilde{\Sigma}, \nu)$. This proves the claim (5.7).

Now let $t := \sqrt{A \log N} \geq \sqrt{\log 2}$ with $A > 1$ being a parameter to be specified later. By (5.4) and (5.7), for each $x \in X$, there exists a set $G(x) \subset \tilde{\Sigma}$ with $\nu(G(x)) \leq 2N^{-A}$ such that for each
\[
\sigma = (\sigma_1, \ldots, \sigma_N) \in \tilde{\Sigma} \setminus G(x),
\]
\[
\frac{1}{N} \left| \sum_{j=1}^{N} \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(\rho(x, y_i(\sigma_j))) - \Phi_0(x) \right| \leq \frac{7}{2} c_1 \sqrt{c_1 c_2 \ell \sqrt{\log N} N^{-\frac{1}{2} - \frac{3\beta}{2}}},
\]
where
\[
\Phi_0(x) := \int_X \Phi(\rho(x, y)) \, d\mu(y).
\]
Let $M$ be a positive integer such that
\[
M - 1 < c_1^\beta N^{\frac{\beta}{2} + \frac{3}{2}} \leq M.
\]
Then, using Condition (a) with $M$ in place of $N$, we obtain a partition
\[
\{X'_1, X'_2, \ldots, X'_M\}
\]
of $X$ such that $\mu(X'_j) = \frac{1}{M}$ and
\[
\text{diam}(X'_j) \leq \delta_M = c_1 M^{-\beta^{-1}} \leq N^{-\frac{3\beta}{2} - \frac{3}{2}}
\]
for each $1 \leq j \leq M$. Choose $z_j \in X'_j$ for each $1 \leq j \leq M$, and let $G = \bigcup_{k=1}^{M} G(z_k)$. Then
\[
\nu(G) \leq \sum_{j=1}^{M} \nu(G(z_j)) \leq 2MN^{-A} \leq 3c_1^\beta N^{\frac{\beta}{2} + \frac{3}{2} - A}.
\]
Thus, setting $A = \frac{1 + 2c_1^\beta}{2} + \frac{3}{2}$, we obtain that for $N \geq 4$, $\nu(G)$ is at most
\[
3c_1^\beta N^{-c_1^\beta} = \left( \frac{3c_1}{4c_1} \right)^\beta < 1.
\]
Finally, using (5.2), we have that for each \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \tilde{\Sigma} \setminus G \)
\[
\sup_{x \in X} \left| \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(\rho(x, y_i(\sigma_j))) - \Phi_0(x) \right| 
\leq \max_{1 \leq k \leq M} \left| \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(\rho(z_i, y_i(\sigma_j))) - \Phi_0(x) \right| + \delta_M,
\]
which, using (5.8), is estimated from above by
\[
\left( \frac{7}{2} c_1^4 (c_2 \ell)^{2} \sqrt{\frac{2c_1+1}{2}} \frac{3}{2} + 1 \right) N^{-\frac{1}{2} - \frac{3}{2} \delta} \sqrt{\log N} \leq 8 c_1^2 (c_2 \ell)^{2} \sqrt{\beta N^{-\frac{1}{2} - \frac{3}{2} \delta} \sqrt{\log N}}.
\]
This completes the proof.

5.2. Proof of Theorem 5.3
Let \( h(x) = b(2 + g(x)) \), where \( b \) is a normalizing constant so that \( \|h\|_{L^1(d\mu)} = 1 \). Clearly,
\[
\frac{1}{3} \leq b \leq h(x) \leq 3b \leq 3, \quad \forall x \in X,
\]
because \( \|g\|_{L^\infty} \leq 1 \). Let \( \tau \) denote the Borel probability measure given by \( d\tau = h d\mu \).
By (5.1), we have that for \( N \geq 15 \),
\[
\tau(B_{\tilde{\delta}/8}(x)) \geq b \mu(B_{\tilde{\delta}/8}(x)) \geq \frac{1}{N}, \quad x \in X,
\]
where
\[
\tilde{\delta}_N = c_1([Nb])^{-\beta^{-1}} \leq \left( \frac{5}{4b} \right)^{1/\beta} c_1 N^{-\beta^{-1}} \leq \frac{5}{4b} c_1 N^{-\beta^{-1}},
\]
because \( \beta \geq 1 \). Furthermore, by (5.3), we have that for each \( x \in X \) and \( \delta \in (0, \pi) \),
\[
\tau \left( \bigcup_{j=1}^{\ell-1} \{ y \in X : t_j - \delta \leq \rho(x, y) \leq t_j + \delta \} \right) \leq 3bc_2 \ell \delta.
\]
Since \( X \) is a compact path-connected metric space, using Theorem 5.2 with \( \tau \) in place of \( \mu \), we may find points \( y_1, \ldots, y_{(r+2)N} \in X \) and nonnegative real numbers \( a_1, \ldots, a_{(r+2)N}, \) such that
\[
\max_{x \in X} \left| \int_X \Phi(\rho(x, y)) h(y) d\mu(y) - \sum_{j=1}^{(r+2)N} a_j \Phi(\rho(x, y_j)) \right| \leq \frac{25 \sqrt{3}}{16 b^2} c_3 N^{-\frac{1}{2} - \frac{3}{2} \delta} \sqrt{\log N}.
\]
On the other hand, using Theorem 5.2 we can also find points \( z_1, \ldots, z_{(r+2)N} \in X \) and nonnegative real numbers \( b_1, \ldots, b_{(r+2)N}, \) such that
\[
\max_{x \in X} \left| \int_X \Phi(\rho(x, y)) d\mu(y) - \sum_{j=1}^{(r+2)N} b_j \Phi(\rho(x, z_j)) \right| \leq c_3 N^{-\frac{1}{2} - \frac{3}{2} \delta} \sqrt{\log N}.
\]
Since
\[
\int_X \Phi(\rho(x, y)) g(y) d\mu(y) = \frac{1}{b} \int_X \Phi(\rho(x, y)) h(y) d\mu(y) - 2 \int_X \Phi(\rho(x, y)) d\mu(y)
\]
and $\frac{1}{3} \leq b \leq 1$, it follows that
\[
\sup_{x \in X} \left| \int_X \Phi(\rho(x, y)) g(y) \, d\mu(y) - \frac{1}{b} \sum_{j=1}^{(r+2)N} a_j \Phi(\rho(x, y_j)) + 2 \sum_{j=1}^{(r+2)N} b_j \Phi(\rho(x, z_j)) \right| \\
\leq \left( \frac{25\sqrt{3}}{16b^2} + 2 \right) c_3 N^{-\frac{1}{2}} \frac{1}{\sqrt{2b}} \sqrt{\log N} \leq 45c_3 N^{-\frac{1}{2}} \frac{1}{\sqrt{2b}} \sqrt{\log N}.
\]

The theorem is proved.

6. Discretization on finite-dimensional compact domains

In this section, we shall prove an analogue of Theorem 5.3 for all $g \in L^1(d\mu)$ (instead of $g \in L^\infty(d\mu)$) on finite-dimensional domains. The implied constant in this section will depend on the dimension and the underlying domain.

Let $(X, \| \cdot \|)$ be a finite-dimensional real normed linear space. Let $B_\zeta(x)$ (resp. $B_\zeta[x]$) denote the open balls (resp. closed balls) with centre $x \in X$ and radius $\zeta > 0$ defined with respect to the metric $\rho(x, y) = \|x - y\|$. Here $\| \cdot \|$ is not necessarily the Euclidean norm. Let $\Omega \subset B_1[0]$ be a compact subset of $X$ (not necessarily connected). Let $\mu$ be a Borel probability measure supported on $\Omega$. The main purpose in this section is to discretize integrals of the form
\[
\int_\Omega \Phi(\|x - y\|) g(y) \, d\mu(y)
\]
for a class of piecewisely defined functions $\Phi : [0, \infty) \to \mathbb{R}$.

We assume that the probability measure $\mu$ satisfies the following two conditions:

(i) there exist a positive constant $c_4 > 1$ and a parameter $\beta \geq 1$ such that for any $x \in \Omega$ and $\delta \in (0, 2]$
\[
(6.1) \quad c_4^{-1}\delta^\beta \leq \mu(B_\delta(x)) \leq c_4 \delta^\beta;
\]

(ii) there exists a constant $c_5 > 0$ such that for any $x \in \Omega$ and $t, s \in (0, 2]$,
\[
(6.2) \quad \mu\left( \{y \in \Omega : t \leq \|y - x\| \leq t + s\} \right) \leq c_5 s.
\]

Under these two conditions, we shall prove

**Theorem 6.1.** Let $\Phi : [0, \infty) \to \mathbb{R}$ be a function such that

\[
(6.3) \quad |\Phi(s) - \Phi(s')| \leq |s - s'|, \quad \forall s, s' \in [0, 2].
\]

Assume that there exist a partition $0 = t_0 < t_1 < \cdots < t_\ell = 2$ of $[0, 2]$ and a translation-invariant linear subspace $X_r$ of $C(\Omega)$ with $\dim X_r = r$ such that with $E_j := \{x \in \mathbb{R}^d : t_{j-1} \leq \|x\| \leq t_j\}, j = 1, 2, \ldots, \ell$,
\[
\Phi(\| \cdot \|)|_{E_j} \in \left\{ f \big|_{E_j} : f \in X_r \right\}.
\]
Let \( g \in L^1(\Omega, \mu) \) be such that \( \|g\|_{L^1(\mu)} = 1 \). Then for each positive integer \( n \geq 2 \), there exist points \( y_1, \ldots, y_n \in \Omega \) and real numbers \( \lambda_1, \ldots, \lambda_n \), such that

\[
\sup_{x \in \Omega} \left| \int_{\Omega} \Phi(\|x - y\|)g(y) \, d\mu(y) - \sum_{k=1}^{n} \lambda_k \Phi(\|x - y_k\|) \right| \leq C(X) \left\{ \begin{array}{ll}
\frac{n^{\beta - \frac{3\beta}{2}}(\log n)^{\frac{1}{2}}}{4}, & \text{if } 1 \leq \beta < 3, \\
\frac{n^{-1}(\log n)^{\frac{1}{2}}}{\beta}, & \text{if } \beta = 3, \\
\frac{n^{-\frac{3\beta}{2}(\beta+1)}(\log n)^{\frac{1}{2}}}{4}, & \text{if } \beta > 3,
\end{array} \right.
\]

(6.4)

where the constant \( C(X) \) depends only on \( \dim X, c_4, c_5, r, \ell \) and \( \beta \).

6.1. **Proof of Theorem 6.1.** The main idea of our proof comes from the paper \[17\]. We need the following Besicovitch covering theorem on finite-dimensional normed linear spaces \[19\]:

**Lemma 6.2.** \[17\] Let \( E \subset X \) be an arbitrarily given nonempty subset of a finite dimensional normed linear space \( X \). Assume that for each \( x \in E \) there exists a closed ball \( B_r(x)[x] \) with centre \( x \) and radius \( r(x) > 0 \). Assume in addition that \( \sup_{x \in E} r(x) < \infty \). Then there exists a sub-collection \( \mathcal{R} \) of the closed balls \( B_r(x)[x] \), \( x \in E \), which covers the set \( E \) and can be written in the form

\[
\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_m
\]

with \( m \leq \mathcal{N}(X) \), and each \( \mathcal{R}_j \) being a collection of pairwise disjoint balls, 1 \( \leq j \leq m \). Here \( \mathcal{N}(X) \) is a positive constant depending only on the normed space \( (X, \| \cdot \|) \).

The best constant \( \mathcal{N}(X) \) for the Besicovitch covering theorem has been well studied in literature (see \[17, 19\] and the references thereon). In the case when \( (X, \| \cdot \|) = \mathbb{R}^d \), it was known \[19\] that \( \mathcal{N}(X) \leq 6^d \). The sharp estimate of this constant appears in \[24\]. A much more general version of the Besicovitch covering theorem can be found in \[16\].

The proof runs along the same line as that of Theorem 5.2. We sketch it as follows.

Without loss of generality, we may assume that \( g \geq 0 \) since otherwise we may write \( g = g^+ - g^- \) with \( g^\pm \geq 0 \). For the rest of the proof, the letter \( C \) denotes a general positive constant depending only on \( \mathcal{N}(X), c_4, c_5, r, \ell \) and \( \beta \).

Let \( \tau \) denote the probability measure given by \( d\tau(x) = g(x) \, d\mu(x) \). Let \( n_1 = \left\lfloor \frac{n}{2\mathcal{N}(X)^{1+r(\beta+1)}} \right\rfloor \). For \( x \in \Omega \), let \( 0 < \theta_x \leq \delta_{n_1} := (c_4/n_1)^\frac{1}{\ell} \) be such that

\[
\int_{B_{\theta_x}[x]} (1 + g(y)) \, d\mu(y) = \frac{1}{n_1}.
\]

(6.5)

By the Besicovitch covering theorem, we can find finitely many open balls \( B_j = B_{\theta_{x_j}}(x_j) \), \( j = 1, 2, \ldots, m \), such that \( \Omega \subset \bigcup_{j=1}^{m} B_j \),

\[
\{ B_1, \ldots, B_m \} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_{\mathcal{N}(X)}
\]

(6.6)

with each \( \mathcal{R}_j \) being a subcollection of pairwise disjoint balls. By (6.5) and (6.6), we then have \( m \leq 2\mathcal{N}(X)n_1 \leq \frac{n_1}{\ell^2} \). Note that (6.2) implies that \( \mu(B_r(x)) = \mu(B_r[x]) \) for any \( x \in \Omega \) and \( r > 0 \). Now define \( Q_1 = \overline{B_1} \) and

\[
Q_j = \overline{B_j} \setminus \bigcup_{i=1}^{j-1} B_i, \quad j = 2, \ldots, m.
\]
Then \( \Omega = \bigcup_{j=1}^{m} Q_j \), \( \tau(Q_i \cap Q_j) = 0 \) for \( 1 \leq i \neq j \leq m \), \( Q_j \subset \overline{B_j} \) and \( \tau(Q_j) \leq \frac{1}{n^r} \) for \( 1 \leq j \leq m \). Without loss of generality, we may also assume that \( \tau(Q_j) > 0 \) for each \( 1 \leq j \leq m \), since otherwise we remove \( Q_j \) from the partition.

For each \( 1 \leq j \leq m \), let \( \Sigma_j \) denote the set of all probability measures \( \sigma_j \) on \( Q_j \) of the form

\[
\sigma_j = \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \delta_{y_i(\sigma_j)}, \quad \lambda_i(\sigma_j) \geq 0, \quad y_i(\sigma_j) \in Q_j,
\]

such that

\[
\frac{1}{\tau(Q_j)} \int_{Q_j} P(x) \, d\tau(x) = \sum_{i=1}^{r+2} \lambda_i(\sigma_j) P(y_i(\sigma_j)), \quad \forall P \in X_r.
\]

By Theorem 6.1, there exists a Borel probability measure \( \nu_j \) on \( \Sigma_j \) such that

\[
\int_{Q_j} f(x) \, d\tau(x) = \int_{\Sigma_j} \sum_{i=1}^{r+2} \tau(Q_j) \lambda_i(\sigma_j) f(y_i(\sigma_j)) \, d\nu_j(\sigma_j), \quad \forall f \in C(Q_j).
\]

Now we consider the product probability space \( (\Sigma, \nu) = \prod_{j=1}^{m} (\Sigma_j, \nu_j) \). Fix \( x \in \Omega \) temporarily. For \( 1 \leq j \leq m \), define

\[
h_{j,x}(\sigma_j) = \tau(Q_j) \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(\|x - y_i(\sigma_j)\|) - \int_{Q_j} \Phi(\|x - y\|) \, d\tau(y).
\]

Then \( \mathbb{E}h_{j,x} = 0 \),

(6.7) \(|h_{j,x}(\sigma_j)| \leq \tau(Q_j) \cdot \text{diam}(Q_j) \leq \tau(B_j) \text{diam}(Q_j) \leq C\theta x_j n^{-1}.

For \( 0 < \theta \leq 2\delta_{n_1} \), we denote by \( I_\theta := I_\theta(x) \) the set of all integers \( 1 \leq j \leq m \) such that \( \theta/2 < \theta x_j \leq \theta \) and \( t_k - \theta \leq \|x - x_j\| \leq t_k + \theta \) for some \( 1 \leq k \leq \ell \). Note that if \( \theta/2 < \theta x_j \leq \theta \) and \( j \notin I_\theta(x) \), then there exists \( k \) in the interval \( 1 \leq k \leq \ell \) such that \( t_k - 1 \leq \|x - y\| \leq t_k \) for every \( y \in Q_j \subset B_j := B_{\theta x_j}(x_j) \), which implies that \( h_{j,x} \equiv 0 \). Note also that

\[
\bigcup_{j \in I_\theta} (B_j \cap \Omega) \subset \left\{ y \in \Omega : t - 2\theta \leq \|x - y\| \leq t + 2\theta \right\}.
\]

It then follows by (6.6) and (6.2) that

\[
\#I_\theta c_4 \left( \frac{\theta}{2} \right)^\beta \leq \sum_{j \in I_\theta} \mu(B_j) \leq 4\mathcal{N}(X)c_5 \theta,
\]

which implies that

(6.8) \( \#I_\theta \leq C_1 \theta^{1-\beta} \).

Note that (6.8) holds trivially if

(6.9) \( \theta \leq \left( \frac{(r + 2)C_1}{n} \right)^{1/\beta} = C_2 n^{-\frac{1}{\beta r}} \)

since \( \#I_\theta \leq m \leq \frac{n}{r + 2} \). Thus, we will mainly consider those index sets \( I_\theta \) with

(6.10) \( C_2 n^{-\frac{1}{\beta r}} \leq \theta \leq 2\delta_{n_1} := 2 \left( \frac{c_4}{n} \right)^{\frac{1}{\beta}} \),

the second bound being the bound on \( \theta \) stated at the beginning of the paragraph.
To be more precise, let \( k_0, k_1 \) be integers such that
\[
2^{k_0} < 2^{-1} (n/c_4)^{\frac{1}{\beta}} \leq 2^{k_0+1}
\]
and
\[
2^{k_1-1} < C_2^{-1} n^{\frac{1}{2\alpha}} \leq 2^{k_1}.
\]
Define \( J_k = J_k(x) := I_{2^{-k}}(x) \) for \( k_0 \leq k \leq k_1 \) and
\[
J_{k_i+1} = J_{k_i+1}(x) = \bigcup_{k=k_i+1}^{\infty} I_{2^{-k}}(x).
\]
Then by (6.11) and the remark after (6.13), we have
\[
\# J_k \leq n_k := C_1^{-1} 2^{k(\beta-1)}, \quad k_0 \leq k \leq k_1 + 1.
\]
Moreover, by (6.7), we have
\[
h_{j,x} \leq C \theta_{x_j} n^{-1} \leq C 2^{-k} n^{-1}, \quad j \in J_k, \quad k_0 \leq k \leq k_1 + 1.
\]
Thus, using (6.12), (6.11), and the Bernstein inequality, we conclude that for each \( k_0 \leq k \leq k_1 + 1 \) and each \( \varepsilon_k > 0 \), the inequality
\[
\left| \sum_{j \in J_k} h_{j,x}(\sigma_j) \right| > \varepsilon_k
\]
holds with probability at most
\[
2 \exp \left( -C \varepsilon_k^2 n^2 2^{-k(\beta-3)} \right).
\]
Now we write
\[
\sum_{j=1}^{m} h_{j,x}(\sigma_j) = \sum_{k=k_0}^{\infty} \sum_{j: 2^{-k} \leq \theta_{x_j} \leq 2^{-k+1}} h_{j,x}(\sigma_j) = \sum_{k=k_0}^{k_1+1} \sum_{j \in J_k} h_{j,x}(\sigma_j).
\]
Given \( \varepsilon > 0 \), let \( \{ \varepsilon_k \}_{k=k_0}^{k_1+1} \) be a sequence of positive numbers such that \( \sum_{k=k_0}^{k_1+1} \varepsilon_k \leq \varepsilon \).
Then using (6.13), we have
\[
\text{Prob}\left\{ \left| \sum_{j=1}^{m} h_{j,x} \right| > \varepsilon \right\} \leq \sum_{k=k_0}^{k_1+1} \text{Prob}\left\{ \left| \sum_{j \in J_k} h_{j,x} \right| > \varepsilon_k \right\}
\]
\[
\leq 2 \sum_{k=k_0}^{k_1+1} \exp \left( -C \varepsilon_k^2 n^2 2^{-k(\beta-3)} \right).
\]
Noting that \( k_0 \sim k_1 \sim \log n \), we may choose for \( k_0 \leq k \leq k_1 + 1 \),
\[
\varepsilon_k = \begin{cases} 
\frac{\varepsilon}{\log n^{1}}, & \text{if } \beta > 3, \\
\frac{\varepsilon}{2^{(k-k_0)^{\frac{1}{\beta}}}} n^{\frac{1}{2\alpha}}, & \text{if } \beta = 3, \\
\frac{\varepsilon}{2^{(k-k_0)^{\frac{1}{\beta}}}} n^{\frac{1}{2\alpha}}, & \text{if } \beta < 3.
\end{cases}
\]
We use here that \( n \neq 1 \) so that \( \log n \neq 0 \).
For simplicity, we shall assume that \( \beta > 3 \). The proof below with slight modifications works equally well for the case \( \beta \leq 3 \). We then obtain from (6.14) that
\[
\text{Prob}\left\{ \left| \sum_{j=1}^{m} h_{j,x} \right| > \varepsilon \right\} \leq C(\log n) \exp \left( -C n^{2-\frac{2}{\beta}} \varepsilon^2 \right).
\]
The last inequality holds since \( \beta > 3 \). The main point here lies in the fact that the upper bounds for \( c \) measure \( n \) holds with probability bounded above by a multiple of \((\log n)^{\frac{\beta+1}{2}}\). Let

\[
t := \sqrt{A \log n} \quad \text{with} \quad A = \frac{\beta(\beta + 1)}{2(\beta - 1)} > 1.
\]

The last inequality holds since \( \beta^2 - 2\beta + 2 \) has no real zeros.

We further conclude that for each \( x \in \Omega \), there exists a set \( G(x) \subset \Sigma \) with \( \nu(G(x)) \leq C_2(\log n) n^{-\lambda} \) such that for any \( \sigma = (\sigma_1, \ldots, \sigma_m) \in \Sigma \setminus G(x) \),

\[
\sum_{j=1}^{m} \tau(Q_j) \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(||x - y_i(\sigma_j)||) - \int_{\Omega} \Phi(||x - y||) \, d\tau(y) \geq Ctn^{-\frac{\beta+1}{2}}.
\]

This holds with probability bounded above by a multiple of \((\log n)e^{-t^2}\). Let

\[
t := \sqrt{A \log n} \quad \text{with} \quad A = \frac{\beta(\beta + 1)}{2(\beta - 1)} > 1.
\]

Finally, let \( \{z_1, \ldots, z_L\} \) be a maximal \( \varepsilon_1 \)-separated subset of \( \Omega \) with \( \varepsilon_1 := n^{-\frac{\beta+1}{2(\beta - 1)}}(\log n)\frac{1}{2} \). By \( \{\square\} \), we have

\[
L \leq c_4 \left( \frac{2}{\varepsilon_1} \right)^{\beta} \leq C_3 n^{\frac{\beta(\beta+1)}{2(\beta - 1)}}(\log n)^{-\frac{1}{2}\beta}.
\]

Setting \( A = \frac{\beta(\beta + 1)}{2(\beta - 1)} \) we have that

\[
\sum_{j=1}^{L} \nu(G(z_j)) \leq C_2C_3(\log n)^{1-\frac{1}{2}\beta}.
\]

Since \( \beta > 3 \), it follows that the following inequality holds with positive probability:

\[
\vartheta := \inf_{x \in \Omega} \sum_{j=1}^{m} \tau(Q_j) \sum_{i=1}^{r+2} \lambda_i(\sigma_j) \Phi(||x - y_i(\sigma_j)||) - \int_{\Omega} \Phi(||x - y||) \, d\tau(y) \leq Cn^{-\frac{\beta+1}{2(\beta - 1)}}(\log n)^{\frac{1}{2}}.
\]

The theorem is proved.

### 7. Discretization on the unit sphere \( S^d \)

In this section, we will estimate the constants \( c_1 \) and \( c_2 \) for the unit sphere \( S^d \subset \mathbb{R}^{d+1} \) denote the unit sphere in \( \mathbb{R}^{d+1} \) equipped with the normalized surface Lebesgue measure \( \mu_d \) and the geodesic distance \( \rho(x, y) = \arccos(x \cdot y) \), \( x, y \in S^d \). We will prove on the unit sphere \( S^d \) that

\[
c_1 \leq 40\pi, \quad c_2 \leq \frac{3}{2} \sqrt{d}, \quad \alpha = \frac{1}{d}.
\]

The main point here lies in the fact that the upper bounds for \( c_1 \) and \( c_2/\sqrt{d} \) are independent of the dimension \( d \).
By (7.1), we also have

\begin{equation}
45 c_3 = 45 \cdot 8 c_1^2 \sqrt{d} \leq 7 \times 10^6 d^2.
\end{equation}

As a consequence of Theorem 4.1 and Lemma 7.4, we have that

Theorem 7.1. For each integer \( N \geq 1 \), there exists a partition \( \{ R_1, \ldots, R_N \} \) of \( S^d \) such that

(i) the \( R_j \) are pairwise disjoint subsets of \( S^d \);
(ii) for each \( 1 \leq j \leq N \), \( \mu_d(R_j) = \frac{1}{N} \) and \( \text{diam}(R_j) \leq 40 \pi N^{-\frac{d}{2}} \).

Again, the main point here is that the upper bound for \( N^\frac{d}{2} \max_j \text{diam}(R_j) \) is independent of the dimension \( d \).

Theorem 7.2. Let \( \Phi : [-1,1] \to \mathbb{R} \) be a piecewise polynomial of degree at most \( r \) with knots \(-1 = s_0 < s_1 < \cdots < s_r = 1\) such that \( |\Phi(s) - \Phi(s')| \leq |s - s'| \) for any \( s, s' \in [-1,1] \). Let \( m_r \) denote the dimension of the space of all spherical polynomials of degree at most \( r \) on \( S^d \). Let \( g \in L^\infty(S^d) \) be such that \( \|g\|_\infty \leq 1 \). Then for each positive integer \( N \geq 20 \), there exist points \( \xi_1, \ldots, \xi_{2(m_r+2)N} \in S^d \) and real numbers \( \lambda_1, \ldots, \lambda_{2(m_r+2)N} \) such that

\[
\max_{x \in S^d} \left| \int_{S^d} \Phi(x \cdot y) g(y) \, d\mu_d(y) - \sum_{j=1}^{2(m_r+2)N} \lambda_j \Phi(x \cdot \xi_j) \right| 
\leq 7 \cdot 10^6 \sqrt{d} N^\frac{d}{2} N^{-\frac{d}{2}} \sqrt{\log N}.
\]

In the case when \( \Phi(t) = |t| \), Theorem 7.2, but with constants depending on the dimension of the sphere, was previously obtained in [4].

7.1. Proof of (7.1). For \( \theta \in (0, \pi) \) and \( x \) in the \( d \)-dimensional sphere \( S^d \), set

\[
B_\theta(x) := \{ y \in S^d : \rho(x, y) < \theta \}, \quad \text{and} \quad B_\theta[x] := \{ y \in S^d : \rho(x, y) \leq \theta \}.
\]

Let \( \omega_d := \frac{2 \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \) denote the surface area of \( S^d \). Using the following known estimates on gamma functions [1],

\[
x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \quad x > 0, \quad s \in (0, 1),
\]

we have that

\begin{equation}
\pi^{-\frac{d}{2}} \left( \frac{d-1}{2} \right)^{\frac{d}{2}} \leq \frac{\omega_{d-1}}{\omega_d} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} \leq \pi^{-\frac{d}{2}} \left( \frac{d+1}{2} \right)^{\frac{d}{2}}. \tag{7.3}
\end{equation}

Lemma 7.3. For \( 0 < \theta \leq \frac{\pi}{2} \) and \( x \in S^d \),

\[
\frac{1}{\sqrt{2d}} \leq \mu_d(B_\theta(x)) = \frac{\omega_{d-1}}{\omega_d} \frac{\sin^d \theta}{\sin \theta} \leq \frac{2}{\sqrt{d}}.
\]

Proof. For \( \theta \in (0, \pi] \), we have

\[
\mu_d(B_\theta(x)) = \frac{\omega_{d-1}}{\omega_d} \int_{\cos \theta}^{1} (1 - t^2)^{\frac{d-2}{2}} \, dt = \frac{\omega_{d-1}}{\omega_d} \int_{0}^{\sin^2 \theta} \frac{d\theta}{t^{\frac{d-2}{2}} (1-t)^{\frac{d}{2}}}.
\]

If \( 0 < \theta \leq \frac{\pi}{4} \), then for any \( 0 \leq t \leq \sin^2 \theta \), we have

\[
1 \leq (1-t)^{-\frac{d}{2}} \leq \sqrt{2}.
\]
Thus,

\[ \frac{\omega_{d-1}}{\omega_d} \frac{2}{d} (\sin \theta)^d \leq \mu_d(A(x, \theta)) \leq \frac{\omega_{d-1}}{\omega_d} \frac{2\sqrt{2}}{d} (\sin \theta)^d, \]

which, using (7.3), implies that

\[ \frac{1}{2\sqrt{d}} \leq \sqrt{\frac{1}{\pi} d - \frac{\pi}{2}} \leq \sqrt{2\pi^{-\frac{d}{2}} d^{-1}(d - 1)^{\frac{d}{2}}} \leq \frac{\mu_d(A(x, \theta))}{\sin^d \theta} \leq 2\pi^{-\frac{d}{2}} d^{-1}(d + 1)^{\frac{d}{2}} \leq \frac{2}{\sqrt{d}}. \]

□

The following lemma shows that \( c_1 \leq 40\pi \):

**Lemma 7.4.** For any positive integer \( N \),

\[ \inf_{x \in \mathbb{S}^d} \mu_d(B_{\delta_N}(x)) \geq \frac{1}{N} \quad \text{with} \quad \delta_N := 5\pi N^{-\frac{1}{d}}. \]

**Proof.** We consider the following two cases:

**Case 1.** \( N \geq 2^{\frac{d}{2}} + 1 \sqrt{d} \).

In this case, set

\[ \delta := \min \left\{ \theta : 0 \leq \theta \leq \frac{\pi}{4}, \quad \frac{1}{2\sqrt{d}} \sin^d \theta \geq \frac{1}{N} \right\} \]

and our condition on the \( N \) ensures that \( \delta \) be well-defined. Using Lemma 7.3 we have that

\[ \mu_d(B_{\delta}(x)) \geq \frac{1}{N}, \quad \forall x \in \mathbb{S}^d. \]

It remains to estimate the constant \( \delta \). By definition of \( \delta \), we have that

\[ \frac{1}{2\sqrt{d}} \sin^d \delta \geq \frac{1}{N} \quad \Rightarrow \quad \frac{1}{2\sqrt{d}} \sin^d \delta \geq \frac{1}{2}. \]

This implies that

\[ \delta \leq \pi \sin \left( \frac{\delta}{2} \right) \leq \pi \left( \frac{2\sqrt{d}}{N} \right)^{\frac{1}{d}} \leq 2\pi e^{\frac{d}{2}} N^{-\frac{1}{2}} < \frac{3\pi N^{-\frac{1}{2}}}{2}. \]

Here we have used the fact that the maximum of \( (\log y)/y \) is attained at \( y = e \).

**Case 2.** \( 1 \leq N < 2^{\frac{d}{2} + 1} \sqrt{d} \).

In this case,

\[ N^{-\frac{1}{2}} > 2^{-\frac{d}{2}} d^{-\frac{1}{2}} > 2^{-\frac{d}{2}} e^{-\frac{1}{2}} > 0.2, \]

and

\[ \delta_N = 5\pi N^{-\frac{1}{2}} \geq \pi. \]

Hence, (7.3) holds trivially in this case. □

The following lemma shows that \( c_2 \leq \frac{4}{3} \sqrt{d} \):

**Lemma 7.5.** For any \( \delta > 0 \), \( x \in \mathbb{S}^d \) and \( t \in (0, \pi) \),

\[ \mu_d \left( \left\{ y \in \mathbb{S}^d : t - \delta \leq \rho(x, y) \leq t + \delta \right\} \right) \leq \frac{3}{2} \sqrt{d} \delta. \]
Proof. Without loss of generality, we may assume that $0 < t \leq \frac{\pi}{2}$. Setting

$$S_\delta(x) := \{ y \in \mathbb{S}^d : t - \delta \leq \rho(x, y) \leq t + \delta \},$$

and using (7.3), we have

$$\mu_d(S_\delta(x)) = \omega_d^{-1} \int_{\max\{t-\delta,0\}}^{t+\delta} \sin^{d-1} u \, du \leq \frac{\pi^{-1/2}}{2} \left( \frac{d+1}{2} \right)^{1/2} \delta \leq \frac{2}{\sqrt{\pi}} \sqrt{d} < \frac{3}{2} \sqrt{d}.
$$

\[\square\]

8. Further Examples

Further examples for our results stem from the fact that not only piecewise polynomials are suitable for our spaces $V_r$ of dimension $r$, but also piecewise exponentials [21], [11] and [12], as well as radial basis functions of compact support [9], [10] and [7], [8].

All these function spaces are defined not over piecewise polynomials (splines) with a simple continuity condition, but for instance over piecewise exponentials.

In the most general form, see [21], the exponential splines of compact support are, say, in $d$ dimensions of degree $n - 1$ for equally spaced knots defined as distributions $B$ that satisfy

$$B(\varphi) = \int_{[0,1]^n} \varphi(\Xi t) \exp(\lambda \cdot t) \, dt,$$

where $\varphi$ is a test-function from the Schwartz space $S$, $\Xi$ is a linear map $\mathbb{R}^n \to \mathbb{R}^d$ and $\lambda$ is a vector from $\mathbb{R}^n$ to define the exponentials. Alternatively we can write for $\phi \in L^1_{\text{loc}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} B(x) \phi(x) \, dx = \int_{[0,1]^n} \phi(\Xi t) \exp(\lambda \cdot t) \, dt.$$

In the multivariate setting, these functions are called exponential box-splines, in the univariate case they are exponential B-splines. The piecewise polynomial case corresponds to $\lambda \equiv 0$. They may also be conveniently defined by their Fourier transforms

$$\prod_{j=1}^{n} \frac{\exp(\lambda_j - i\xi_j \cdot x)}{\lambda_j - i\xi_j \cdot x}.$$

Here, $\lambda = (\lambda_j)_{j=1}^n$ and $\Xi = (\xi_j)_{j=1}^n$.

The univariate piecewise polynomial case corresponds to $\Xi = (1, 1, \ldots, 1) \in \mathbb{R}^n$, $d = 1$. In this case the splines are defined over the interval or cube for $d = 1$ and $d > 1$, respectively, $\Xi[0, h]^n$, e.g., $h = 1/\ell$ as in our cases. The $V_r$ space is here the space of univariate exponential splines spanned by the exponential B-splines with $d + 1$ knots.

More generally, we can define space of piecewise exponentials including piecewise polynomials and exponentials as the span of

$$x^{r_i} \exp(\lambda_i x), \quad r_i = 0, 1, \ldots, \tau_i - 1, \quad i = 1, 2, \ldots, n,$$

on each subinterval between two knots, now of no longer necessarily equally spaced knots, of dimension $r = \sum_{i=1}^{n} \tau_i$ when they are required to be continuous. The $\lambda_i$s may be complex and must be pairwise distinct.
Special cases \[11\] are \( \lambda \equiv 0 \) (piecewise polynomials), \( \lambda \in i\mathbb{R} \) (\( V_r \) containing piecewise trigonometric functions \( \sin, \cos \) and constants) and \( \lambda \in \mathbb{R} \) (\( V_r \) containing \( \sinh \) and \( \cosh \) and constants). In fact, it is usual (but not necessary) to restrict the exponents that form the components of \( \lambda \) to \( \mathbb{R} \cup i\mathbb{R} \). Examples for the spaces are the polynomials for some fixed maximal degree (classical spline case) or the spans of, e.g.,

\[
1, \cos(\text{Im} \lambda t), \sin(\text{Im} \lambda t), t \cos(\text{Im} \lambda t), t \sin(\text{Im} \lambda t),
\]
or

\[
1, \cosh(\text{Re} \lambda t), \sinh(\text{Re} \lambda t), t \cosh(\text{Re} \lambda t), t \sinh(\text{Re} \lambda t).
\]

These two examples are the suitable generalisations of the \( \Phi(t) = |t| \) case (piecewise linears) referred to in the paragraph after the statement of Theorem 7.1. For higher powers, larger \( r \) and more exponentials, the other piecewise polynomials used in the first sentence of the statement of Theorem 7.1 are generalised.

Univariate piecewise polynomial B-splines on equally spaced knots can be generated in a computational useful, recursive way by convolutions \[3\] but now, for exponential splines we get a weight function, so that, for the B-spline of degree \( n \), the exponential spline

\[
e^{\lambda \cdot t} H(t) - e^{\lambda \cdot (t-1)}e^{\lambda \cdot (t-1)}
\]
needs to be convolved with itself \( n \)-times, once for the case of piecewise linears multiplied with exponentials. In the display, \( H \) denotes the Heaviside function which is identically zero for negative argument and identically one for positive argument.

This results from the identities which we stated already in \( s \) dimensions

\[
B \ast f = \int_{[0,1]^n} \exp(\lambda \cdot t)f(\cdot - \Xi t) \, dt
\]
or

\[
B = \int_0^1 \exp(\lambda_{1,\gamma} \xi \gamma t) \tilde{B}(\cdot - \xi \gamma t) \, d\xi
\]
Here \( B \) is the exponential box-spline as above, \( \tilde{B} \) is the same with the direction \( \xi \gamma \) removed from \( \Xi \).

As with the piecewise polynomials and the special case of piecewise constants above, we consider the special case of piecewise exponentials only (no polynomials as in our example with \( \sin, \cos, \cosh, \sinh \)).

For this, consider again the vector of exponents \( \lambda \), set \( n = d \) and let \( \tilde{\lambda} = \lambda \Xi^{-1} \).

Then the spline is

\[
B(x) = \frac{1}{|\det \Xi|} \exp(\tilde{\lambda} \cdot x)\chi_{[0,1]^d}(\Xi^{-1} x), \quad x \in \mathbb{R}^d.
\]
Here, \( \chi \) is the characteristic function. Starting from this piecewise “constant” function (i.e., one that contains no polynomials, just one exponential), \textit{other} splines can be generated recursively by

\[
B(x) = e^{\mu \cdot x} \int_0^1 \tilde{B}(x - t \xi) \, dt,
\]
where \( B \) is the exponential spline with one direction \( \xi \) more in the direction set and the \( \mu \)s are chosen arbitrarily from \( \mathbb{R}^n \).
The corresponding radial basis functions of compact support with exponentials are
\[
\left(\frac{1}{e - \exp(-x)}\right)^\nu_+
\]
and
\[
\left(1 - \exp(-(1 - x)^\mu_+)\right)^\mu
\]
which are positive definite for suitable parameters $\mu$ and $\nu$ depending on the dimension because they are logarithmically monotone of order $\mu$ in the first case and of order $\min(\mu, \nu)$ in the second case [7]. The $V_\nu$'s are then defined by the translates
\[
\left(\frac{1}{e - \exp(-|x|)}\right)^\nu_+
\]
and
\[
\left(1 - \exp(-(1 - |x|)^\nu_+)ight)^\mu,
\]
respectively.

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1964.
[2] A. Bondarenko, D. Radchenko, and M. Viazovska, Well-separated spherical designs., Constr. Approx. 41 (2015), no. 1, 93-112.
[3] Carl de Boor, A practical guide to splines, Springer, 1978.
[4] J. Bourgain and J. Lindenstrauss, Distribution of points on spheres and approximation by zonotopes, Israel J. Math. 64 (1988), no. 1, 25–31.
[5] G. Brown and F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Analysis 220 (2005), no. 2, 401–423.
[6] Martin Buhmann, Radial basis functions: theory and implementations, Cambridge University Press, 2003.
[7] Martin Buhmann and Janin Jäger, Multiply and monotone functions for radial basis function interpolation: extensions and new kernels, preprint, JLU Giessen.
[8] __________, Pólya type criteria for conditional strict positive definiteness of functions on spheres, preprint, JLU Giessen.
[9] Martin Buhmann, A new class of radial functions with compact support, Mathematics of Computation 70 (2001), 307–318.
[10] __________, Radial functions on compact support, Proceedings of the Edinburgh Mathematical Society 41 (1998), 33–46.
[11] Constanza Conti, Mariantonia Cotronei, and Lucia Romani, Beyond B-splines: exponential pseudo-splines and subdivision schemes reproducing exponential polynomials, Dolomites Research Notes on Approximation 10 (2017), 31–42.
[12] Constanza Conti, L. Gemignani, and Lucia Romani, Exponential pseudo-splines: looking beyond exponential B-splines, Journal of Mathematical Analysis and Applications 439 (2016), 32–56.
[13] F. Dai, A. Primak, V. N. Temlyakov, and S. Yu. Tikhonov, Integral norm discretization and related problems, Uspekhi Mat. Nauk 74 (2019), no. 4(448), 3–58 (Russian, with Russian summary); English transl., Russian Math. Surveys 74 (2019), no. 4, 579–630.
[14] Feng Dai and Heping Wang, Optimal cubature formulae in weighted Besov spaces with $A_\infty$-weights on multivariate domains, Constructive Approximation 37 (2013), 167–194.
[15] Feng Dai and Yuan Xu, Approximation Theory and Harmonic Analysis on Spheres and Balls, Springer-Verlag, New York, 2013.
[16] Herbert Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR257325
[17] Zoltan Füredi and Peter A. Loeb, On the best constant for the Besicovitch covering theorem, Proc. AmerMath. Soc. 121 (1994), no. 4, 1063–1073, DOI 10.2307/2161215. MR1249875
[18] Walter Gautschi, Orthogonal Polynomials, Numerical Mathematics and Scientific Computation, Oxford University Press, 2004.
[19] Steven G. Krantz, The Besicovitch covering lemma and maximal functions, Rocky Mountain J. Math. 49 (2019), no. 2, 539–555, DOI 10.1216/RMJ-2019-49-2-539. MR3973239

[20] G. Petrova, Cubature formulae for spheres, simplices and balls Journal of Computational and Applied Mathematics, Journal of Computational and Applied Mathematics 162 (2004), 483–496.

[21] Amos Ron, Exponential box splines, Constructive Approximation 4 (1988), 357–378.

[22] Walter Rudin, Functional analysis, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[23] John M. Sullivan, Sphere packings give an explicit bound for the Besicovitch covering theorem, J. Geom. Anal. 4 (1994), no. 2, 219–231, DOI 10.1007/BF02921548, MR1277507

[24] Yuan Xu, Orthogonal polynomials and cubature formulae on spheres and on simplices, SIAM Journal of Mathematical Analysis 29 (2006), no. 3, 779–793.

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