κ-generalization of Stirling approximation and multinominal coefficients

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Abstract

Stirling approximation of the factorials and multinominal coefficients are generalized based on the one-parameter (κ) deformed functions introduced by Kaniadakis [Phys. Rev. E 66 (2002) 056125]. We have obtained the relation between the κ-generalized multinominal coefficients and the κ-entropy by introducing a new κ-product operation.

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I. INTRODUCTION

As is well-known in the fields of statistical mechanics, Ludwig Boltzmann clarified the concept of entropy. He considered a macroscopic system consists of a large number of particles. Each particle is assumed to be in one of the energy level $E_i$ ($i = 1, \ldots, k$) and the number of particles in the level $E_i$ is $n_i$. The total number of particles is then $n = \sum_{i=1}^k n_i$ and the total energy of a macrostate is $E = \sum_i n_i E_i$. The number of ways of arranging $n$ particles in $k$ energy levels such that each level has $n_1, n_2, \ldots, n_k$ particles are the multinominal coefficients

$$\left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right] \equiv \frac{n!}{n_1! \cdots n_k!}, \quad (1)$$

which is proportional to the probability of the macrostates if all microscopic configurations are assumed to be equally likely. In the thermodynamic limit $n$ increasing to infinity, we consider the relative number $p_i \equiv n_i/n$ as the probability of the particle occupation of a certain energy level $E_i$. In order to find the most probable macrostate, Eq. (1) is maximized under a certain constraint. The Stirling approximation of the factorials leads to

$$\ln \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right] \approx n S_{\text{BGS}} \left( \frac{n_1}{n}, \ldots, \frac{n_k}{n} \right), \quad (2)$$

where

$$S_{\text{BGS}} \left( \frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) \equiv - \sum_{i=1}^k p_i \ln(p_i), \quad (3)$$

is the Boltzmann-Gibbs-Shannon (BGS) entropy.

Quite recently one of the authors (H.S.) has shown one-parameter generalizations of Gauss’ law of error [1], Stirling approximation [2], and multinominal coefficients [3] based on the Tsallis $q$-deformed functions and associated multiplication operation ($q$-product) [4, 5]. These mathematical structures are quite fundamental for the basis of any generalization of statistical physics as in the standard statistical physics. In particular Ref. [3] has shown that the one-to-one correspondence between the $q$-multinominal coefficient and Tsallis $q$-entropy [6, 7, 8], i.e., the one-parameter ($q$) generalization of Eq. (2).

On the other hand, Kaniadakis [9, 10, 11] has proposed the $\kappa$-generalized statistical mechanics based on the $\kappa$-deformed function, which is another type of one-parameter deformations for the exponential and logarithmic functions. Based on the $\kappa$-deformed functions and
the associated product operation ($\kappa$-product) we have already shown the $\kappa$-generalization of Gauss’ law of error [12].

In this work we show the $\kappa$-generalizations of the Stirling approximation and the relation between the $\kappa$-multinomial coefficient and $\kappa$-entropy. In the next section the $\kappa$-factorial is introduced based on the $\kappa$-product. Then the $\kappa$-generalization of the Stirling approximation is obtained. We see that the naive approach fails in order to relate the $\kappa$-multinominal coefficients with the $\kappa$-entropy. In order to overcome this difficulty, we introduce a new kind of $\kappa$-product in section 3 and show the explicit relation between the $\kappa$-multinominal coefficients and the $\kappa$-entropy. The final section is devoted to our conclusions.

II. $\kappa$-STIRLING APPROXIMATION

Let us begin with the brief review of the $\kappa$-factorial and its Stirling approximation. The $\kappa$-generalized statistics [9, 10, 11] is based on the $\kappa$-entropy

$$S_\kappa \equiv -\sum_i p_i \ln_{\{\kappa\}}(p_i),$$

(4)

where $\kappa$ is a real parameter in the range $(-1, 1)$, and the $\kappa$-logarithmic function is defined by

$$\ln_{\{\kappa\}}(x) \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa}.$$  

(5)

The inverse function is the $\kappa$-exponential function defined by

$$\exp_{\{\kappa\}}(x) \equiv \left[\sqrt{1 + \kappa^2 x^2} + \kappa x\right]^{\frac{1}{\kappa}}.$$  

(6)

In the limit of $\kappa \to 0$ the both $\kappa$-logarithmic and $\kappa$-exponential functions reduce to the standard logarithmic and exponential functions, respectively. We thus see that $S_\kappa$ reduces to the BGS entropy of Eq. (3) in the limit of $\kappa \to 0$.

Based on the above $\kappa$-deformed functions, the $\kappa$-product is defined by

$$x \otimes_\kappa y \equiv \exp_{\{\kappa\}}[\ln_{\{\kappa\}}(x) + \ln_{\{\kappa\}}(y)]$$

$$= \left[\frac{x^\kappa - x^{-\kappa}}{2} + \frac{y^\kappa - y^{-\kappa}}{2} + \sqrt{1 + \left\{\frac{x^\kappa - x^{-\kappa}}{2} + \frac{y^\kappa - y^{-\kappa}}{2}\right\}^2}\right]^{\frac{1}{\kappa}},$$

(7)
which reduces to the standard product \(x \cdot y\) in the limit of \(\kappa \to 0\). Similarly the \(\kappa\)-division is defined by
\[
x \oslash_{\kappa} y \equiv \exp_{\{\kappa\}} \left[ \ln_{\{\kappa\}}(x) - \ln_{\{\kappa\}}(y) \right]
\]
\[
= \left[ \left( \frac{x^\kappa - x^{-\kappa}}{2} \right) - \left( \frac{y^\kappa - y^{-\kappa}}{2} \right) + \sqrt{1 + \left\{ \left( \frac{x^\kappa - x^{-\kappa}}{2} \right) - \left( \frac{y^\kappa - y^{-\kappa}}{2} \right) \right\}^2} \right]^\frac{1}{\kappa},
\]
which reduces to the standard division \(x/y\) in the limit of \(\kappa \to 0\). By utilizing this \(\kappa\)-product, the \(\kappa\)-factorial \(n!_{\kappa}\) with \(n \in \mathbb{N}\) is defined by
\[
n!_{\kappa} \equiv 1 \oslash_{\kappa} 2 \oslash_{\kappa} \cdots \oslash_{\kappa} n
\]
\[
= \left[ \sum_{k=1}^{n} \left( \frac{k^\kappa - k^{-\kappa}}{2} \right) + \sqrt{\left\{ \sum_{k=1}^{n} \left( \frac{k^\kappa - k^{-\kappa}}{2} \right) \right\}^2 + 1} \right]^\frac{1}{\kappa} = \exp_{\{\kappa\}} \left[ \sum_{k=1}^{n} \ln_{\{\kappa\}}(k) \right].
\]
Now we come to the Stirling approximation of the \(\kappa\)-factorials. For sufficient large \(n\), the summation is well approximated with the integral as follows.
\[
\ln_{\{\kappa\}}(n!_{\kappa}) = \sum_{k=1}^{n} \ln_{\{\kappa\}}(k) \approx \int_{0}^{n} dx \ln_{\{\kappa\}}(x) = \frac{n^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{n^{1-\kappa}}{2\kappa(1-\kappa)}.
\]
Clearly this reduces to the standard Stirling approximation as
\[
\lim_{\kappa \to 0} \ln_{\{\kappa\}}(n!_{\kappa}) \approx \lim_{\kappa \to 0} \frac{n}{1 - \kappa^2} \left\{ \ln_{\{\kappa\}}(n) - \frac{n^{\kappa} + n^{-\kappa}}{2} \right\} = n \left( \ln n - 1 \right).
\]
Next the \(\kappa\)-multinominal coefficient is defined by utilizing the \(\kappa\)-product and \(\kappa\)-division as follows
\[
\left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{\kappa} \equiv n!_{\kappa} \oslash_{\kappa} \left( n_1!_{\kappa} \oslash_{\kappa} \cdots \oslash_{\kappa} n_k!_{\kappa} \right),
\]
where we assume
\[
n = \sum_{i=1}^{k} n_i.
\]
In the limit of \(\kappa \to 0\), Eq. (12) reduces to the standard multinominal coefficient of Eq. (1).

Let us try to relate the \(\kappa\)-multinominal coefficients with the \(\kappa\)-entropy. Taking the \(\kappa\)-logarithm of Eq. (12) and applying the \(\kappa\)-Stirling approximation leads to
\[
\ln_{\{\kappa\}} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{\kappa} = \ln_{\{\kappa\}}(n!_{\kappa}) - \sum_{i=1}^{k} \ln_{\{\kappa\}}(n_i!_{\kappa}),
\]
we then obtain
\[
\ln_{\kappa}\left[\begin{array}{c}
\frac{n}{n_1 \cdots n_k}
\end{array}\right]_\kappa \approx \frac{n^{1+\kappa}}{2\kappa(1+\kappa)} \left\{ 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1+\kappa} \right\} + \frac{n^{1-\kappa}}{2\kappa(1-\kappa)} \left\{ \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1-\kappa} - 1 \right\}.
\]
(15)

From this relation, we see that the above naive approach fails. Since the r.h.s. consists of the two terms with different factors (one is proportional to \(n^{1+\kappa}/(1 + \kappa)\) and the other is proportional to \(n^{1-\kappa}/(1 - \kappa)\)), this cannot be proportional to the \(\kappa\)-entropy, which can be written by
\[
S_{\kappa}\left(\frac{n_1}{n}, \ldots, \frac{n_k}{n}\right) \equiv -\sum_{i=1}^{k} \left( \frac{n_i}{n} \right) \ln_{\kappa}\left( \frac{n_i}{n} \right).
\]
(16)

III. INTRODUCING A NEW \(\kappa\)-PRODUCT

In order to overcome the above difficulty, we introduce another \(\kappa\)-generalization of the products based on the following function defined by
\[
u_{\kappa}\left(\kappa\ln(x)\right) \equiv \frac{x^{\kappa} + x^{-\kappa}}{2} = \cosh \left(\kappa \ln(x)\right).
\]
(17)

We here call it the \(\kappa\)-generalized unit function since \(\lim_{\kappa \to 0} u_{\kappa}(x) = 1\). The basic properties of \(u_{\kappa}(x)\) are as follows.

\[
u_{\kappa}(x) = u_{\kappa}\left(\frac{1}{x}\right),
\]
(18)

\[
u_{\kappa}(x) \geq 1, \text{ because } u_{\kappa}(x) - 1 = \frac{(x^{\frac{\kappa}{2}} - x^{-\frac{\kappa}{2}})^2}{2} \geq 0.
\]
(19)

The inverse function of \(u_{\kappa}(x)\) can be defined by
\[
u_{\kappa}^{-1}(x) \equiv \left[ \sqrt{x^2 - 1} + x \right]^{\frac{1}{\kappa}}, \quad (x \geq 1).
\]
(20)

In Ref. [13], the canonical partition function associated with the \(\kappa\)-entropy is obtained in terms of this \(U_{\kappa}\) function. Note that
\[
u_{\kappa}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} = \frac{1}{\kappa} \sinh \left(\kappa \ln(x)\right),
\]
(21)
the two kinds of the $\kappa$-deformed functions ($\ln_{\{\kappa\}}(x)$ and $\exp_{\{\kappa\}}(x)$; $u_{\{\kappa\}}(x)$ and $u_{\{\kappa\}}^{-1}(x)$) are thus associated each other. This can be seen from the following relations

$$\sqrt{1 + \kappa^2 \ln_{\{\kappa\}}^2(x)} = u_{\{\kappa\}}(x),$$  \hspace{1cm} (22)

$$u_{\{\kappa\}}^{-1} \left( \sqrt{1 + \kappa^2 x^2} \right) = \exp_{\{\kappa\}}(x), \quad \text{for} \ x \geq 0.$$ \hspace{1cm} (23)

Now by utilizing these functions, a new $\kappa$-product is defined by

$$x \circ_{\kappa} y \equiv u_{\{\kappa\}}^{-1} \left[ u_{\{\kappa\}}(x) + u_{\{\kappa\}}(y) \right]$$

$$= \left[ \left( \frac{x^\kappa + x^{-\kappa}}{2} \right) + \left( \frac{y^\kappa + y^{-\kappa}}{2} \right) + \sqrt{ \left( \frac{x^\kappa + x^{-\kappa}}{2} + \frac{y^\kappa + y^{-\kappa}}{2} \right)^2 - 1 } \right]^{\frac{1}{\kappa}}. \hspace{1cm} (24)$$

Similarly the corresponding $\kappa$-division is defined by

$$x \oslash_{\kappa} y \equiv u_{\{\kappa\}}^{-1} \left[ u_{\{\kappa\}}(x) - u_{\{\kappa\}}(y) \right]$$

$$= \left[ \left( \frac{x^\kappa + x^{-\kappa}}{2} \right) - \left( \frac{y^\kappa + y^{-\kappa}}{2} \right) + \sqrt{ \left( \frac{x^\kappa + x^{-\kappa}}{2} - \frac{y^\kappa + y^{-\kappa}}{2} \right)^2 - 1 } \right]^{\frac{1}{\kappa}}, \hspace{1cm} (25)$$

where $\left( \frac{x^\kappa + x^{-\kappa}}{2} \right) - \left( \frac{y^\kappa + y^{-\kappa}}{2} \right) \geq 1$.

Note that $\lim_{\kappa \to 0} x \circ_{\kappa} y = \infty$, consequently there is no corresponding operation in the standard case of $\kappa = 0$. However unless $\kappa = 0$ this new product satisfies

$$x \circ_{\kappa} y = y \circ_{\kappa} x, \hspace{1cm} \text{comutativity} \hspace{1cm} (26)$$

$$(x \circ_{\kappa} y) \circ_{\kappa} z = x \circ_{\kappa} (y \circ_{\kappa} z), \hspace{1cm} \text{associativity} \hspace{1cm} (27)$$

There exists no unit element within a real number. Therefore real numbers and this product consist a semigroup!

Note also that $x \oslash_{\kappa} x \neq 1$ because $u_{\{\kappa\}}^{-1}(0)$ does not exist by the definition Eq. (20). However we see that the following identity holds.

$$x \circ_{\kappa} y \oslash_{\kappa} x = y, \quad (\kappa \neq 0). \hspace{1cm} (28)$$

Using the new $\kappa$-product, the associated $\kappa$-factorial can be introduced as

$$n!_{\kappa} \equiv 1 \circ_{\kappa} 2 \circ_{\kappa} \cdots \circ_{\kappa} n$$

$$= \left[ \sum_{k=1}^{n} \left( \frac{k^\kappa + k^{-\kappa}}{2} \right) + \sqrt{ \left\{ \sum_{k=1}^{n} \left( \frac{k^\kappa + k^{-\kappa}}{2} \right) \right\}^2 - 1 } \right]^{\frac{1}{\kappa}}. \hspace{1cm} (29)$$
Similar to Eq. (10), the $\kappa$-Stirling approximation can be obtained as

$$u_{\{\kappa\}}(n!^\kappa) = \sum_{k=1}^{n} u_{\{\kappa\}}(k) \approx \int_{0}^{n} dx \ u_{\{\kappa\}}(x)$$

$$= \frac{n^{1+\kappa}}{2(1+\kappa)} + \frac{n^{1-\kappa}}{2(1-\kappa)}. \tag{30}$$

Also the corresponding $\kappa$-multinominal coefficient can be defined by

$$\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \equiv n!^\kappa \bigotimes_{\kappa} \left( n_1!^\kappa \otimes \cdots \otimes n_k!^\kappa \right). \tag{31}$$

Applying the above Stirling approximation to Eq. (31) we obtain

$$u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \approx \frac{n^{1+\kappa}}{2(1+\kappa)} \left\{ 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1+\kappa} \right\} - \frac{n^{1-\kappa}}{2(1-\kappa)} \left\{ \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1-\kappa} - 1 \right\}. \tag{32}$$

This is the complemental relation to Eq. (15), and by combining Eqs (15) and (32) we have

$$\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \pm \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \approx \pm \frac{n^{1\pm\kappa}}{\kappa(1 \pm \kappa)} \left\{ 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1\pm\kappa} \right\}, \tag{33}$$

We thus obtain the final result

$$\left( \frac{1 - \kappa}{2n^{1-\kappa}} \right) \left( \ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa - \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \right)$$

$$+ \left( \frac{1 + \kappa}{2n^{1+\kappa}} \right) \left( \ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa + \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa \right)$$

$$= \frac{1}{2\kappa} \left\{ 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1+\kappa} \right\} + \frac{1}{2\kappa} \left\{ \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{1-\kappa} - 1 \right\} = S_{\kappa} \left( \frac{n_1}{n}, \ldots, \frac{n_k}{n} \right). \tag{34}$$

Note that since

$$\lim_{\kappa \to 0} \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^\kappa = 0, \tag{35}$$

as shown in the Appendix, Eq. (34) reduces to the standard case Eq. (2) in the limit of $\kappa \to 0$. 

7
IV. CONCLUSION

We have generalized Stirling approximation of the factorials and multinomial coefficients based on the $\kappa$-deformed function introduced by Kaniadakis, which is different one-parameter generalization from the $q$-deformed functions by Tsallis. In order to relate the $\kappa$-generalized multinominal coefficients to the $\kappa$-entropy, we showed the naive approach, which is similar to that [3] for the $q$-generalization, failed. In order to overcome this difficulty, we have introduced a new kind of the $\kappa$-product operations, which never exist in the standard case of $\kappa = 0$, and have obtained the relation between the $\kappa$-generalized multinominal coefficients and the $\kappa$-entropy.

The final result Eq. (34) clearly states that the maximizing $\kappa$-entropy is equivalent to the maximizing the l.h.s. of Eq. (34) as same as in the standard case of Eq. (2). A next step in a future work is thus to clarify what microscopic states are described by the both $\kappa$-multinominal coefficients Eqs. (12) and (31).

V. APPENDIX

We here show the proof of Eq. (35).

\[
\frac{1}{\kappa} u_{\{\kappa\}} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right] = \frac{1}{\kappa} \left( \sum_{\ell=1}^{n} u_{\{\kappa\}}(\ell) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} u_{\{\kappa\}}(j) \right),
\]

(36)

In the limit of $\kappa \to 0$ the numerator reduces to $n - \sum_{i=1}^{k} n_i$ since $\lim_{\kappa \to 0} u_{\{\kappa\}} = 1$. Consequently both the denominator $\kappa$ and the numerator become null. Then applying l’Hopital’s rule and using the relation

\[
\frac{d}{d\kappa} u_{\{\kappa\}}(x) = \ln x \left( \frac{x^\kappa - x^{-\kappa}}{2} \right),
\]

(37)

we obtain Eq. (35).

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