Thresholds for contagious sets in random graphs

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Abstract

For fixed \( r \geq 2 \), we consider bootstrap percolation with threshold \( r \) on the Erdős-Rényi graph \( G_{n,p} \). We identify a threshold for \( p \) above which there is with high probability a set of size \( r \) which can infect the entire graph. This improves a result of Feige, Krivelevich and Reichman, which gives bounds for this threshold, up to multiplicative constants.

As an application of our results, we also obtain an upper bound for the threshold for \( K_4 \)-bootstrap percolation on \( G_{n,p} \), as studied by Balogh, Bollobás and Morris. We conjecture that our bound is asymptotically sharp.

These thresholds are closely related to the survival probabilities of certain time-varying branching processes, and we derive asymptotic formulae for these survival probabilities which are of interest in their own right.

1 Introduction

1.1 Bootstrap percolation

The \( r \)-bootstrap percolation process on a graph \( G = (V,E) \) evolves as follows. Initially, some set \( V_0 \subset V \) is infected. Subsequently, any vertex that has at least \( r \) infected neighbours becomes infected, and remains infected. Formally the process is defined by

\[
V_{t+1} = V_t \cup \{ v : |N(v) \cap V_t| \geq r \},
\]

where \( N(v) \) is the set of neighbours of a vertex \( v \). The sets \( V_t \) are increasing, and so converge to some set \( V_\infty \) of eventually infected vertices. We denote the infected set by \( \langle V_0, G \rangle_r = V_\infty \). A contagious set for \( G \) is a set \( I \subset V \) such that if we put \( V_0 = I \) then we have that \( \langle I, G \rangle_r = V \), that is, the infection of \( I \) results in the infection of all vertices of \( G \).

Bootstrap percolation was introduced by Chalupa, Leath and Reich [20], in the context of statistical physics, for the study of disordered magnetic systems. Since then it has been applied diversely in physics, and in other areas, including computer science, neural networks, and sociology, see [1, 2, 4, 21, 22, 25, 26, 27, 28, 30, 31, 37, 40, 43, 46, 47, 48] and further references therein.

Special cases of \( r \)-bootstrap percolation have been analyzed extensively on finite grids and infinite lattices, see for instance [3, 4, 10, 12, 15, 18, 32, 33].
(and references therein). Other special graphs of interest have also been studied, including hypercubes and trees, see [8, 11, 14, 28]. Recent work has focused on the case of random graphs, see for example [4, 5, 16, 35], and in particular, on the Erdős-Rényi random graph $G_{n,p}$. See [36, 45] (and [6, 7, 41] for related results).

The main questions of interest in this field revolve around the size of the eventual infected set $V_\infty$. In most works, the object of study is the probability that a random initial set is contagious, and its dependence on the size of $V_0$. For example, in [36, Theorem 3.1], the critical size for a random contagious set in $G_{n,p}$ is identified for all $r \geq 2$ and $p$ in a range depending on $r$.

More recently, and in contrast with the above results, Feige, Krivelevich and Reichman [24] study the minimal size of a contagious set in $G_{n,p}$. We define a minimal contagious set to be a contagious set of size $r$. This is the minimal possible size of a contagious set. We call a graph susceptible (or say that it $r$-percolates) if it contains such a minimal contagious set. A main result, [24, Theorem 1.2], gives the approximate threshold for $p$ above which $G_{n,p}$ is likely to be susceptible. Our main result identifies sharp thresholds for the susceptibility of $G_{n,p}$, for all $r \geq 2$.

Let $p_c(n,r)$ denote the infimum over $p > 0$ so that $G_{n,p}$ is susceptible with probability at least $1/2$.

**Theorem 1.1.** Fix $r \geq 2$ and $\alpha > 0$. Let

$$p = p(n) = \left( \frac{\alpha}{n \log^{r-1} n} \right)^{1/r}$$

and denote

$$\alpha_r = (r-1)! \left( \frac{r-1}{r} \right)^{2(r-1)}.$$

If $\alpha > \alpha_r$, then with high probability $G_{n,p}$ is susceptible. If $\alpha < \alpha_r$, then there exists $\beta = \beta(\alpha,r)$ so that for $G = G_{n,p}$, with high probability for every $I$ of size $r$ we have $|\langle I, G \rangle_r| \leq \beta \log n$. In particular, as $n \to \infty$,

$$p_c(n,r) = \left( \frac{\alpha_r}{n \log^{r-1} n} \right)^{1/r} (1 + o(1)).$$

Thus $r$-bootstrap percolation undergoes a sharp transition. For small $p$ sets of size $r$ infect at most $O(\log n)$ vertices, whereas for larger $p$ there are minimal contagious sets.

We remark that for $\alpha < \alpha_r$, with high probability $G_{n,p}$ has susceptible subgraphs of size $\Theta(\log n)$. Moreover, our methods identify the largest $\beta$ so that there are susceptible subgraphs of size $\beta \log n$ (see Proposition 2.1 below).

### 1.2 Graph bootstrap percolation and seeds

Let $H$ be some finite graph. Following Bollobás [17], $H$-bootstrap percolation is a rule for adding edges to a graph $G$. Eventually no further edges can be added,
and the process terminates. An edge is added whenever its addition creates a copy of \( H \) within \( G \). Informally, the process completes all copies of \( H \) that are missing a single edge. Formally, we let \( G_0 = G \), and \( G_{i+1} \) is \( G_i \) together with every edge whose addition creates a subgraph which is isomorphic to \( H \). Note that these are not necessarily induced subgraphs, so having more edges in \( G \) can only increase the final result. The vertex set is fixed, and no vertices play any special role.

For a finite graph \( G \), this procedure terminates once \( G_{\tau+1} = G_{\tau} \), for some \( \tau = \tau(G) \). We denote the resulting graph \( G_{\tau} \) by \( \langle G \rangle_H \). If \( \langle G \rangle_H \) is the complete graph on the vertex set \( V \), the graph \( G \) is said to \( H \)-percolate (or that it is \( H \)-percolating). The case \( H = K_4 \) is the minimal case of interest. Indeed, all graphs \( K_2 \)-percolate, and a graph \( K_3 \)-percolates if and only if it is connected. Hence by a classical result of Erdős and Rényi \[23\], \( G_{n,p} \) will \( K_3 \)-percolate precisely for \( p > n^{-1} \log n + \Theta(n^{-1}) \).

The main focus of \[13\] is \( H \)-bootstrap percolation in the case that \( G = G_{n,p} \) and \( H = K_k \), for some \( k \geq 4 \). The critical thresholds are defined as

\[
p_c(n,H) = \inf \{ p > 0 : P(\langle G_{n,p} \rangle_H = K_n) \geq 1/2 \}.
\]

It is expected that this property has a sharp threshold for \( H = K_k \) for any \( k \), in the sense that for some \( p_c = p_c(k) \) we have that \( G_{n,p} \) is \( K_k \)-percolating with high probability for \( p > (1+\delta)p_c \) and is \( K_k \)-percolating with probability tending to 0 for \( p = (1-\delta)p_c \).

Some bounds on \( p_c(n,K_k) \), \( k \geq 4 \), are obtained in \[13\]. One of the main results of \[13\] is that \( p_c(n,K_4) = \Theta(1/\sqrt{n \log n}) \). We improve the upper bound on \( p_c(n,K_4) \) given in \[13\]. We believe that the bound below is asymptotically sharp.

**Theorem 1.2.** Let \( p = \sqrt{\alpha/(n \log n)} \). If \( \alpha > 1/3 \) then \( G_{n,p} \) is \( K_4 \)-percolating with high probability. In particular as \( n \to \infty \), we have that

\[
p_c(n,K_4) \leq \frac{1 + o(1)}{\sqrt{3n \log n}}.
\]

One way for a graph \( G \) to \( K_r+2 \)-percolate is if there is some ordering of the vertices so that vertices 1, \ldots, \( r \) form a clique, and every later vertex is connected to at least \( r \) of the previous vertices according to the order. In this case we call the clique formed by the first \( r \) vertices a seed for \( G \). When \( r = 2 \), the seed is a clique of size 2, so we call it a seed edge.

**Lemma 1.3.** Fix \( r \geq 2 \). If \( G \) has a seed for \( K_{r+2} \)-bootstrap percolation, then \( \langle G \rangle_{K_{r+2}} = K_n \).

**Proof.** We prove by induction that for \( k \geq r \) the subgraph induced by the first \( k \) vertices percolates. For \( k = r \), the definition of a seed implies that the subgraph is complete. Given that the first \( k-1 \) vertices span a percolating graph, some number of steps will add all edges among them. Finally, vertex \( k \) has \( r \) neighbours among these, and so every edge between vertex \( k \) and a previous vertex can also be added by \( K_{r+2} \)-bootstrap percolation. ■
In light of this, Theorem 1.2 above is a direct corollary of the following result.

**Theorem 1.4.** Let \( p = \sqrt{\alpha/(n \log n)} \). As \( n \to \infty \), the probability that \( G_{n,p} \) has a seed edge tends to 1 if \( \alpha > 1/3 \) and tends to 0 if \( \alpha < 1/3 \).

The case of \( K_4 \)-bootstrap percolation, corresponding to \( r = 2 \), appears to be special: We conjecture that existence of a seed edge is the easiest way for a graph to \( K_4 \)-percolate, and consequently that the inequality in Theorem 1.2 can be made an equality, identifying the asymptotic threshold for \( K_4 \)-percolation. This is similar to other situations where a threshold of interest on \( G_{n,p} \) coincides with that of a more fundamental event. For instance, with high probability, \( G_{n,p} \) is connected if and only if it has no isolated vertices (see [23]); \( G_{n,p} \) contains a Hamiltonian cycle if and only if the minimal degree is at least 2 (Komlós and Szemerédi [38]).

Essentially, if \( G \) \( K_4 \)-percolates, then either there is a seed edge, or some other small structure that serves as a seed (i.e., \( K_4 \)-percolates and exhausts \( G \) by adding doubly connected vertices), or else, there are at least two large structures within \( G \) that \( K_4 \)-percolate independently. Since \( p_c \to 0 \), having multiple large percolating structures within \( G \) is less likely.

For \( r > 2 \), having a seed is no longer the easiest way for a graph to \( K_4 \)-percolate. Indeed, by [13], the critical probability for \( K_{r+2} \)-bootstrap percolation is \( n^{-(2r)/(r^2 + 3r - 2)} \) up to poly-logarithmic factors (note that \( r \) in [13] is \( r + 2 \) here). The threshold for having a seed is of order \( n^{-1/r} (\log n)^{1/r - 1} \), which is much larger (see Theorem 5.1).

1.3 A non-homogeneous branching process

Given an edge \( e = (x_0, x_1) \), we can explore the graph to determine if it is a seed edge. The number of vertices that are connected to both of its endpoints is roughly Poisson with mean \( np^2 \). In our context, the interesting \( p \) are \( o(n^{-1/2}) \), and therefore the number of such vertices has small mean, which we denote by \( \varepsilon = np^2 \). If there are any such vertices, denote them \( x_2, \ldots \). We then seek vertices connected to \( x_2 \) and at least one of \( x_0, x_1 \). The number of such vertices is roughly Pois(\( \varepsilon \)). Indeed, the number of vertices connected to the \( k \)th vertex and at least one of the previous vertices is (approximately) Pois(\( k\varepsilon \)).

This leads us to the case \( r = 2 \) of the following non-homogeneous branching process defined by parameters \( r \in \mathbb{N} \) and \( \varepsilon > 0 \). The process starts with a single individual. The first \( r - 2 \) individuals have precisely one child each. For \( n \geq r - 1 \), the \( n \)th individual has a Poisson number of children with mean \( \left( \begin{array}{c} n \\ r-1 \end{array} \right) \varepsilon \), where here \( \varepsilon = np^r \). Thus for \( r = 2 \) the \( n \)th individual has a mean of \( n \varepsilon \) children. The process may die out (e.g., if individual \( r - 1 \) has no children). However, if the process survives long enough the mean number of children exceeds one and the process becomes super-critical. Thus the probability of survival is strictly between 0 and 1. Formally, this may be defined in terms of independent random variables \( Z_n = \text{Poi} \left( \left( \begin{array}{c} n \\ r-1 \end{array} \right) \varepsilon \right) \) by \( X_t = \sum_{n=r-1}^t Z_n - 1 \). Survival is the event \( \{ X_t \geq 0, \forall t \} \).
Theorem 1.5. As $\varepsilon \to 0$, we have that
\[
\mathbb{P}(X_t > 0, \forall t) = \exp\left[ -\frac{(r-1)^2}{r} k_r(1 + o(1)) \right]
\]
where
\[
k_r = k_r(\varepsilon) = \left(\frac{(r-1)!}{\varepsilon}\right)^{1/(r-1)}.
\]

Note that $\varepsilon(\frac{k_r}{r-1}) \approx 1$. Hence $k_r$ is roughly the time at which the process becomes super-critical.

1.4 Outline of the proof

In Section 2, we obtain a recurrence (2.1) for the number of graphs which $r$-percolate with the minimal number of edges. Using this, we estimate the asymptotics of such graphs, and thereby identify a quantity $\beta_r(\alpha)$, so that for $\alpha < \alpha_r$ (and $p$ as in Theorem 1.1), with high probability no $r$-percolation on $\mathcal{G}_{n,p}$ grows to size $\beta \log n$, for some $\beta \geq \beta_r(\alpha) + \delta$. We put $\beta_r(\alpha) = k_r(np^*)$, where $k_r$ is as in Section 1.3. Moreover, we find that $\beta_r(\alpha) = \beta_r(\alpha)$ if and only if $\alpha = \alpha_r$, suggesting that $\alpha_r$ is indeed the critical value of $\alpha$.

In Section 3, we show by the second moment method that, if $\alpha > \alpha_r$, then $\mathcal{G}_{n,p}$ $r$-percolates with high probability. The main difficulty is showing that contagious sets are sufficiently independent. Since vertices in a contagious set need not be connected, it seems that perhaps a straightforward argument is not available. We instead study contagious sets which infect triangle-free subgraphs of $\mathcal{G}_{n,p}$. Modifying the recurrence (2.1), we obtain a recursive lower bound for graphs which $r$-percolate without using triangles, and find that this restriction does not significantly affect the asymptotics. Using Mantel’s theorem, we establish the approximate independence of correspondingly restricted $r$-percolations, which we call $\hat{r}$-percolations, with comparative ease.

A secondary obstacle is the need for a lower bound on the asymptotics of graphs which $\hat{r}$-percolate, with a significant proportion of vertices in the top level (i.e., vertices $v$ of a graph $G = (V,E)$ such that $v \in V_i \setminus V_{i-1}$ where $V_i = V$). Such bounds are required to estimate the growth of super-critical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$, which have grown larger than the critical size $\beta_r(\alpha) \log n$. Using a lower bound for the overall number of graphs which $\hat{r}$-percolate, we obtain a lower bound on the number of such graphs with $i = \Omega(k)$ vertices in the top level. This estimate, together with the approximate independence result, is sufficient to show that with high probability $\mathcal{G}_{n,p}$ has subgraphs of size $\beta \log n$ which $r$-percolate, for some $\beta \geq \beta_r(\alpha) + \delta$ (where for $\alpha > \alpha_r$, $\beta_r(\alpha) < \beta_r(\alpha)$).

Finally, to conclude, we show by the first moment method that for any given $A > 0$, with high probability an $r$-percolation which survives to size $(\beta_r(\alpha) + \delta) \log n$ survives to size $A \log n$. Having established the existence of a subgraph of $\mathcal{G}_{n,p}$ of size $A \log n$, for a sufficiently large value of $A$ (depending on the difference $\alpha - \alpha_r$), it is straightforward to show that with high probability $\mathcal{G}_{n,p}$ $r$-percolates.
2 Lower bound for $p_c(n, r)$

In this section, we prove the sub-critical case of Theorem 1.1 by the first moment method. Throughout this section we fix some $r \geq 2$. More precisely, we prove the following

Proposition 2.1. Let

$$\alpha_r = (r - 1)! \left( \frac{r - 1}{r} \right)^{2(r-1)}, \quad p = \theta_r(\alpha, n) = \left( \frac{\alpha}{n \log^{r-1} n} \right)^{1/r}.$$ 

Define $\beta_*(\alpha)$ to be the unique positive root of

$$r + \beta \log \left( \frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{r!} - \beta(r-2).$$

For any $\alpha < \alpha_r$ and $\delta > 0$, with high probability, for every $I \subset [n]$ of size $r$, we have that

$$|\langle I, G_{n,p} \rangle_r| \leq (\beta_*(\alpha) + \delta) \log n.$$ 

The methods of Section 3 can be used to show that with high probability there are sets $I$ of size $r$ which infect $(\beta_* - \delta) \log n$ vertices. For $\alpha < \alpha_r$, we have the following upper bound

$$\beta_*(\alpha) \leq \left( \frac{(r - 1)!}{\alpha} \right)^{1/(r-1)}.$$ 

This is asymptotically optimal for $\alpha \sim \alpha_r$.

2.1 Small susceptible graphs

As noted in the introduction, a key idea is to study the number of subgraphs of size $k = \Theta(\log n)$ which are susceptible with the minimal number of edges. If none exist, then there can be no contagious set in $G$. Thus an important step is developing estimates for the number of such susceptible graphs of size $k$.

For a graph $G$ and initial infected set $V_0$, recall that $V_t = V_t(V_0, G)$ is the set of vertices infected up to and including step $t$. We let $\tau = \inf \{ t : V_t = V_{t+1} \}$. We put $I_0 = V_0$ and $I_t = V_t \setminus V_{t-1}$, for $t \geq 1$. We refer to $I_t$ as the set of vertices infected in level $i$. In particular, the top level of $G$ is $I_\tau$.

For a graph $G$, we let $V(G)$ and $E(G)$ denote its vertex and edge sets, and put $|G| = |V(G)|$.

We call a graph minimally susceptible if it is susceptible and has exactly $r(|G| - r)$ edges. If a graph $G$ is susceptible, it has at least $r(|G| - r)$ edges, since each vertex in $I_t$, $t \geq 1$, is connected to $r$ vertices in $V_{t-1}$.

Definition 2.2. Let $m_r(k)$ denote the number of minimally susceptible graphs $G$ with vertex set $[k]$ such that $[r]$ is a contagious set for $G$. Let $m_r(k, i)$ denote the number of such graphs with $i$ vertices infected in the top level (so that $m_r(k) = \sum_{i=1}^{k-r} m_r(k, i)$).
We note that \( m_r(k, k - r) = 1 \), and claim that for \( i < k - r \),
\[
m_r(k, i) = \binom{k-r-i}{i} \sum_{j=1}^{k-r-i} a_r(k-i,j)^i m_r(k-i,j), \tag{2.1}
\]
where
\[
a_r(x, y) = \binom{x}{r} - \binom{x-y}{r}. \tag{2.2}
\]

To see this, note that removing the top level from a minimally susceptible graph \( G \) of size \( k \) leaves a minimally susceptible graph \( G' \) of size \( k - i \). If the top level of \( G' \) has size \( j \), then all vertices in the top level of \( G \) are connected to \( r \) vertices of \( G' \), with at least one in the top level of \( G' \). Thus each vertex has \( a_r(k-i,j) \) options for the connections. The \( \binom{k-r-i}{i} \) term accounts for the set of possible labels of the top level of \( G \).

To study asymptotics of \( m \) it is convenient to define
\[
\sigma_r(k, i) = \frac{m_r(k, i)}{(k-r)!} \left( \frac{(r-1)!}{k^{r-1}} \right)^k. \tag{2.3}
\]
Substituting this in (2.1) gives
\[
\sigma_r(k, i) = \sum_{j=1}^{k-r-i} A_r(k, i, j) \sigma_r(k-i, j) \quad \text{for } i < k - r, \tag{2.4}
\]
where
\[
A_r(k, i, j) = \frac{j^i}{i!} \left( \frac{k-i}{k} \right)^{(r-1)k} \left( \frac{(r-1)!}{(k-i)^{r-1}} \right)^j a_r(k-i,j). \tag{2.5}
\]

We make the following observation.

**Lemma 2.3.** Let \( A_r(k, i, j) \) be as in (2.5) and put \( A_r(i, j) = \frac{j^i}{i!} e^{-(r-1)i/j!} \). For any \( i < k - r \) and \( j \leq k - r - i \), we have that \( A_r(k, i, j) \) is increasing in \( k \) and converges to \( A_r(i, j) \).

**Proof.** It is well known that for \( m > 0 \) we have \((1 - m/k)^k\) is increasing and tends to \( e^{-m} \). Thus
\[
\frac{j^i}{i!} \left( \frac{k-i}{k} \right)^{(r-1)k} \rightarrow A_r(i, j).
\]
The lemma follows by (2.5) and the following claim, a formula which will also be of later use.

**Claim 2.4.** For all integers \( x \geq r \) and \( 1 \leq y \leq x - r \), we have that
\[
\frac{(r-1)!}{x^{r-1}} a_r(x, y) = \frac{1}{y} \sum_{\ell=1}^{y} \left( \frac{x-\ell}{x} \right)^{r-1}.
\]
Proof. For an integer \( m \geq r \), let \( (m)_r = m!/(m-r)! \) denote the \( r \)th falling factorial of the integer \( m \). Since

\[
(m)_r - (m-1)_r = r(m-1)^{r-1}.
\]

it follows that

\[
\frac{(r-1)! a_r(x,y)}{x^{r-1}} = \frac{(x)_r - (x-y)_r}{ryx^{r-1}} = \frac{1}{y} \sum_{\ell=1}^{y} \left( \frac{x-\ell}{x} \right)^{r-1}
\]

as required.

Since each term on the right of Claim 2.4 is increasing to 1, the same holds for their average. The proof is complete.

\[\Box\]

2.2 Upper bounds for susceptible graphs

Our first task is to derive bounds on the number of minimally susceptible graphs of size \( k \) with \( i \) vertices in the top level. This relies on the recurrence (2.1).

Lemma 2.5. Fix \( r \geq 2 \). For all \( k > r \) and \( i \leq k-r \), we have that

\[
m_r(k,i) \leq e^{-i-(r-2)k} \frac{k^{r-1}}{(r-1)!} \frac{1}{\sqrt{i}}.
\]

Equivalently, \( \sigma_r(k,i) \leq i^{-1/2}e^{-i-(r-2)k} \).

Proof. Since \( m_r(k,k-r) = 1 \), it is straightforward to verify that the claim holds in the case that \( i = k-r \).

For the remaining cases \( i < k-r \), we prove the claim by induction on \( k \). Applying the inductive hypothesis to the right hand sum of (2.4), bounding \( A_r(k,i,j) \) therein by \( A_r(i,j) \) using Lemma 2.3 and extending the sum to all \( j \) we have

\[
\sigma_r(k,i) \leq \sum_{j=1}^{\infty} A_r(i,j) j^{-1/2} e^{-j-(r-2)(k-i)}.
\]

Thus it suffices to prove that this sum is at most \( i^{-1/2}e^{-i-(r-2)k} \). Using the definition of \( A_r(i,j) \) and cancelling the \( e^{-(2-r)k} \) factors, we need the following

Claim 2.6. For any \( i \geq 1 \) we have

\[
\sum_{j=1}^{\infty} \frac{j^i e^{-i}}{i!} j^{-1/2} e^{-j} \leq i^{-1/2} e^{-i}.
\]

This is proved in Appendix A.1.

We remark that Claim 2.6 is fundamentally a pointwise bound on the Perron eigenvector of the infinite operator \( A_2 \). (Other values of \( r \) follow since the influence of \( r \) cancels out.) This eigenvector decays roughly as \( e^{-i} \), but with some lower order fluctuations. It appears that the \( \sqrt{j} \) correction can be replaced by various other slowly growing functions of \( i \). However, Claim 2.6 fails for certain \( i \) without the \( \sqrt{j} \) term.

\[\Box\]
2.3 Susceptible subgraphs of $\mathcal{G}_{n,p}$

With Lemma 2.5 at hand, we obtain upper bounds on the growth probabilities of $r$-percolations on $\mathcal{G}_{n,p}$.

A set $I$ of size $r$ is called $k$-contagious in the graph $\mathcal{G}_{n,p}$, if there is some $t$ so that $|V_t(I, \mathcal{G}_{n,p})| = k$, i.e., there is some time at which there are exactly $k$ infected vertices. The set $I$ is called $(k,i)$-contagious if in addition the number of vertices infected at step $t$ is $i$, i.e., $|V_t(I, \mathcal{G}_{n,p})| = i$. Let $P_r(k,i) = P_r(p,k,i)$ denote the probability that a given $I \subset [n]$, with $|I| = r$ is $(k,i)$-contagious.

Let $P_r(k) = \sum_i P_r(k,i)$ denote the probability that such an $I$ is $k$-contagious. Finally, let $E_r(k,i)$ and $E_r(k)$ denote the expected number of such subsets $I$.

We remark that $P_r(k)$ is not the same as the probability of survival to size $k$, which is given by $\sum_{\ell \geq k} \sum_{i > \ell - k} P_r(\ell,i)$.

**Lemma 2.7.** Let $\alpha > 0$, and let $p = \varrho_r(\alpha,n)$ (as defined in Proposition 2.7) and $\varepsilon = np^r = \alpha / \log^{r-1} n$. For $i \leq k-r$ and $k \leq n^{1/(r(r+1))}$, we have that

$$P_r(k,i) \leq (1 + o(1)) \frac{e^{-\varepsilon(k-\varepsilon)} \varepsilon^{k-r}}{(k-r)!} m_r(k,i)$$

where $o(1)$ depends on $n$, but not on $i,k$.

**Proof.** Let $I \subset [n]$, with $|I| = r$, be given, and put

$$\ell_r(k,i) = \frac{e^{-\varepsilon(k-\varepsilon)} \varepsilon^{k-r}}{(k-r)!} m_r(k,i)$$

so that the lemma states $P_r(k,i) \leq (1 + o(1)) \ell_r(k,i)$. This follows by a union bound: If $I$ is $(k,i)$-contagious, then $I$ is a contagious set for a minimally susceptible subgraph $G \subset \mathcal{G}_{n,p}$ (perhaps not induced) of size $k$ with $i$ vertices infected in the top level, and all vertices in $v \in V(G)^c$ are connected to at most $r - 1$ vertices below the top level of $G$ (so that $V(G) = V_t(I, \mathcal{G}_{n,p})$, for some $t$). There are $\binom{n-r}{k-i}$ choices for the vertices of $G$ and $m_r(k,i)$ choices for its edges. For any such $v$ and $G$, the probability that $v$ is connected to $r$ vertices below the top level of $G$ is bounded from below by

$$\binom{k-i}{r} p^r (1-p)^{k-i-r} > \binom{k-i}{r} p^r (1-p)^k.$$  

Hence

$$P_r(k,i) < \binom{n}{k-r} m_r(k,i) p^r(k-r) \left(1 - \binom{k-i}{r} p^r(1-p)^k \right)^{n-k}.$$  

By the inequalities $\binom{n}{k} \leq n^k / k!$ and $1 - x < e^{-x}$, it follows that

$$\log \frac{P_r(k,i)}{\ell_r(k,i)} < \varepsilon \binom{k-i}{r} \left(1 - (1-p)^k \left(1 - \frac{k}{n} \right) \right).$$
By the inequality \((1 - x)^y \geq 1 - xy\), and since \(k \leq n^{1/(r(r+1))}\), the right hand side is bounded by
\[
\varepsilon k^{r+1}(p + (1 - pk)/n) \leq \varepsilon n^{1/r}(p + 1/n) \ll 1
\]
as \(n \to \infty\). Hence \(P_r(k, i) \leq (1 + o(1))\ell_r(k, i)\), as claimed. ■

As a corollary we get a bound for \(E_r(k, i)\).

**Lemma 2.8.** Let \(\alpha, \beta > 0\). Put \(p = \vartheta_r(\alpha, n)\). For all \(k = \beta \log n\) and \(i = \gamma k\), such that \(\beta \leq \beta_0\), we have that
\[
E_r(k, i) \lesssim n^{\mu} \log^{r-1} n
\]
where
\[
\mu = \mu_r(\alpha, \beta, \gamma) = r + \beta \log \left( \frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{r} (1 - \gamma)^r - \beta (r - 2 + \gamma). \tag{2.6}
\]
Here \(\lesssim\) denotes inequality up to a constant depending on \(\alpha, \beta_0\), but not on \(\beta, \gamma\).

**Proof.** Let \(r \geq 2\) and \(\alpha, \beta_0 > 0\) be given. Put \(\varepsilon = np^r\). By Lemmas 2.5, 2.7 for all \(k = \beta \log n\) and \(i = \gamma k\), with \(\beta \leq \beta_0\), we have that
\[
E_r(k, i) \leq (1 + o(1)) \left( \frac{n^{r-1}}{r} \right)^k \varepsilon^{r-1} e^{-i-(r-2)k-i^{(r-1)}} \lesssim n^{\mu} \log^{r-1} n.
\]
The \(\sqrt{i}\) term from Lemma 2.5 is safely dropped for this upper bound. ■

### 2.4 Sub-critical bounds

In this section, we prove Proposition 2.4

The case of \(\gamma = 0\) in Lemma 2.8 (corresponding to values of \(i\) such that \(i/k \ll 1\)) is of particular importance for the growth of sub-critical \(r\)-percolations. For this reason, we introduce the notation \(\mu^*(\alpha, \beta) = \mu(\alpha, \beta, 0)\). The next result in particular shows that \(\beta^*(\alpha)\), as in Proposition 2.4, is well-defined.

**Lemma 2.9.** Let \(\alpha > 0\). Let \(\alpha_\varepsilon\) be as in Proposition 2.4. Put
\[
\beta_\varepsilon(\alpha) = \left( \frac{(r-1)!}{\alpha} \right)^{1/(r-1)}.
\]

(i) The function \(\mu^*_\varepsilon(\alpha, \beta)\) is decreasing in \(\beta\), with a unique zero at \(\beta^*_\varepsilon(\alpha)\).

(ii) We have that
\[
\mu^*_\varepsilon(\alpha, \beta_\varepsilon(\alpha)) = r - \beta_\varepsilon(\alpha) \frac{(r-1)^2}{r}
\]
and hence \(\beta^*_\varepsilon(\alpha) = \beta_\varepsilon(\alpha)\) (resp. > or <) if \(\alpha = \alpha_\varepsilon\) (resp. > or <).

The quantity \(\beta^*_\varepsilon(\alpha)\) also plays a crucial role in analyzing the growth of super-critical \(r\)-percolations on \(G_{n,p}\), see Section 3.5 below.
Proof. For the first claim, we note that by setting $\gamma = 0$ in (2.6) we obtain

$$\mu^*_r(\alpha, \beta) = r + \beta \log \left( \frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{r!} - \beta(r-2).$$

(2.7)

Therefore

$$\frac{\partial}{\partial \beta} \mu^*_r(\alpha, \beta) = 1 + \log \left( \frac{\alpha \beta^{r-1}}{(r-1)!} \right) - \frac{\alpha \beta^r}{(r-1)!}.$$

Since $\alpha \beta_r(\alpha)^{r-1}/(r-1)! = 1$, the above expression is equal to 0 at $\beta = \beta_r(\alpha)$ and negative for all other $\beta > 0$. Hence $\mu_*(\alpha, \beta)$ is decreasing in $\beta$, as claimed. Moreover, since $\lim_{\beta \to 0^+} \mu^*_r(\alpha, \beta) = r$ and $\lim_{\beta \to \infty} \mu^*_r(\alpha, \beta) = -\infty$, $\beta^*(\alpha)$ is well-defined.

We obtain the expression for $\mu^*_r(\alpha, \beta_r(\alpha))$ in the second claim by (2.7) and the equality $\alpha \beta_r(\alpha)^{r-1}/(r-1)! = 1$. The conclusion of the claim thus follows by the first claim, noting that $\beta^r(\alpha)$ is decreasing in $\alpha$ and $\mu^*_r(\alpha_r, \beta^r(\alpha_r)) = 0$ since $\beta^r(\alpha_r) = (r/(r-1))^2$.

We are ready to prove the main result of this section.

Proof of Proposition (2.4). Let $\alpha < \alpha_r$ and $\delta > 0$ be given. First, we show that with high probability, $G_{n, p}$ contains no $m$-contagious set, for $m = \beta \log n$ with $\beta \in [\beta^*(\alpha) + \delta, \beta_r(\alpha)]$.

Claim 2.10. For all $\beta \leq \beta_r(\alpha)$, we have that $\mu_r(\alpha, \beta, \gamma) \leq \mu^*_r(\alpha, \beta)$.

This is proved in Appendix A.3.

By Lemmas 2.8, 2.9 and Claim 2.10, we find by summing over all $O(\log n)$ relevant $k$ that the probability that such a set exists is bounded (up to a constant) by

$$n^{\mu_*(\alpha, \beta_r(\alpha) + \delta)} \log^{(r-1)+1} n \ll 1.$$

It thus remains to show that with high probability, $G_{n, p}$ has no $m$-contagious set $I$, for some $m \geq \beta_r \log n$. To this end, note that if such a set $I$ exists, then there is some $t$ so that

$$|V_t(I, G_{n, p})| < \beta_r \log n \leq |V_{t+1}(I, G_{n, p})|$$

Letting $k = |V_t(I, G_{n, p})|$, we find that for some $k < \beta_r \log n$ there is a $k$-contagious set $I$, with $m - k$ further vertices with $r$ neighbours in $V_t(I, G_{n, p})$.

The expected number of $k$-contagious sets with $i$ vertices infected in the top level is $E_r(k, i)$. Let $p_r(k, i)$ be the probability that for a given set of size $k$ with $i$ vertices identified as the top level, there are at least $\beta_r \log n - k$ vertices $r$-connected to the set with at least one neighbour in the top level. Hence the probability that $G_{n, p}$ has a $m$-contagious set $I$ for some $m \geq \beta_r \log n$ is at most

$$\sum_{i<k<\beta_r(\alpha)\log n} E_r(k, i)p_r(k, i).$$

The proposition now follows from the following claim, proved in Appendix A.3.
Claim 2.11. For all $k < \beta_r(\alpha) \log n$ and $i \leq k - r$, we have that

$$E_r(k, i)p_r(k, i) \leq n^{\mu^*_r(\alpha, \beta_r(\alpha))} \log^{r(r-1)} n$$

where $\lesssim$ denotes inequality up to constant, independent of $i, k$.

Indeed, by Claim 2.11 it follows, by summing over all $O(\log^2 n)$ relevant $i, k$, that the probability that $G_{n, p}$ has an $m$-contagious set for some $m \geq \beta_r(\alpha) \log n$ is bounded (up to a constant) by

$$n^{\mu^*_r(\alpha, \beta_r(\alpha))} \log^{r(r-1)+2} n \ll 1$$

where the last inequality follows by Lemma 2.9, since $\alpha < \alpha_r$ and hence $\mu^*_r(\alpha, \beta_r(\alpha)) < 0$.

\[\blacksquare\]

3 Upper bound for $p_c(n, r)$

In this section, we prove Theorem 1.1. In light of Proposition 2.1, it remains to prove that for $\alpha > \alpha_r$, with high probability $G_{n, p}$ is susceptible. Fundamentally this is done using the second moment method. As discussed in the introduction, the main obstacle is showing that contagious sets are sufficiently independent for the second moment method to apply. To this end, we restrict to a special type of contagious sets, which infect $k$ vertices with no triangles.

As in the previous section, we fix $r \geq 2$ throughout.

3.1 Triangle-free susceptible graphs

Recall that a graph is called triangle-free if it contains no subgraph which is isomorphic to $K_3$.

Definition 3.1. Let $\hat{m}_r(k, i)$ denote the number of triangle-free graphs that contribute to $m_r(k, i)$ (see Section 2.1). Put $\hat{m}_r(k) = \sum_{i=1}^{k-r} \hat{m}_r(k, i)$.

Following Section 2.1 we obtain a recursive lower bound for $\hat{m}_r(k, i)$. We note that $\hat{m}_r(k, k-r) = m_r(k, k-r) = 1$. For $i < k-r$ we claim that

$$\hat{m}_r(k, i) \geq \binom{k-r}{i} \sum_{j=1}^{k-r-i} \hat{a}_r(k-i, j)^i \hat{m}_r(k-i, j)$$

(3.1)

where

$$\hat{a}_r(x, y) = \max\{0, a_r(x, y) - 2rxy^{r-2}\}.$$ (3.2)

Note that (in contrast to the recursion for $m(k, i)$), this is only a lower bound. To see (3.1), we argue that of the $a_r(k-i, j)$ ways to connect a vertex in the top level to lower levels, at most $2rj(k-i)^{r-2}$ create a triangle. This is so since
the number of ways of choosing \( r \) vertices from \( k - i \), including at least one of the top \( j \) and including at least one edge is at most

\[
jr\binom{k-i-2}{r-2} + jr(k-i-r)\binom{k-i-3}{r-3} < 2jr(k-i)^{r-2},
\]

where the first term accounts for an edge including the previous top level and the second term to \( r \) vertices including an edge below the previous top level.

Setting

\[
\hat{\sigma}_r(k,i) = \hat{m}_r(k,i) \frac{(r-1)!}{(k-r-1)!^{k-r}}
\]

reduces to

\[
\hat{\sigma}_r(k,i) \geq \sum_{j=1}^{k-r-i} \hat{A}_r(k,i,j) \hat{\sigma}_r(k-i,j) \tag{3.3}
\]

where

\[
\hat{A}_r(k,i,j) = \frac{j^i}{i!} \binom{k-i}{k}^{(r-1)k} \frac{(r-1)!}{(k-i)^{r-1}} \frac{\hat{a}_r(k-i,j)}{a_r(k,i,j)}
\tag{3.4}
\]

The following observation indicates that restricting to susceptible graphs which are triangle-free does not have a significant effect on the asymptotics.

**Lemma 3.2.** Let \( \hat{A}_r(k,i,j) \) be as in (3.4) and let \( A_r(i,j) \) be as defined in Lemma 2.3. For any fixed \( i,j \geq 1 \), we have that \( \hat{A}_r(k,i,j) \to A_r(i,j) \), as \( k \to \infty \).

**Proof.** Fix \( i,j \geq 1 \). From their definitions we have that

\[
\frac{\hat{A}_r(k,i,j)}{A_r(k,i,j)} = \left( \frac{\hat{a}_r(k,i,j)}{a_r(k,i,j)} \right)^i.
\]

Since \( a_r(k,i,j) \) is of order \( k^i \) and \( \hat{a}_r(k,i,j) - a(k,i,j) = O(k^{i-1}) \), we have \( \hat{a}_r(k,i,j)/a_r(k,i,j) \to 1 \). Since \( i \) is fixed, it follows by Lemma 2.3 that

\[
\lim_{k \to \infty} \hat{A}_r(k,i,j) = \lim_{k \to \infty} A_r(k,i,j) = A_r(i,j).
\]

In order to get asymptotic lower bounds on \( \hat{m}_r(k,i) \) it is useful to further restrict to graphs with bounded level sizes.

**Definition 3.3.** For \( \ell \geq r \), let \( \hat{m}_{r,\ell}(k) \leq \hat{m}_r(k) \) be the number of graphs that contribute to \( \hat{m}_r(k) \) which have level sizes bounded by \( \ell \) (i.e., \( |I_i| \leq \ell \) for all \( i \)). Let \( \hat{m}_{r,\ell}(k,i) \) be the number of such graphs with exactly \( i \) vertices in the top level. Hence \( \hat{m}_{r,\ell}(k) = \sum_{i=1}^{\ell} \hat{m}_{r,\ell}(k,i) \).

Observe that for fixed \( k, \hat{m}_{r,\ell}(k) \) is increasing in \( \ell \), and equals \( m_r(k) \) for \( \ell \geq k-r \).

Lemma 3.2 will be used to prove asymptotic lower bounds for \( \hat{m} \). When \( i \) is small, the resulting bounds are not sufficiently strong. Thus we also make use of the following lower bound on \( \hat{m}_{r,\ell}(k,i) \) for values of \( i \) which are small compared with \( k \). This is also used as a base case for an inductive proof of lower bounds using Lemma 3.2.
Lemma 3.4. For all relevant $i, k$ and $\ell \geq r$ such that $k > r(r^2 + 1) + i + 2$, we have that

$$\hat{m}_{r,\ell}(k, i) \geq \binom{k-r}{i} \hat{b}_r(k, i)^i \hat{m}_{r,\ell}(k-i)$$

where

$$\hat{b}_r(k, i) = \binom{k-i-r-1}{r-1} \left(1 - \frac{r^3}{k-i-r-2}\right).$$

In particular $\hat{m}_{r,\ell}(k, i) > 0$ for such $k$.

Proof. Let $i, k, \ell$ as in the lemma be given. We obtain the lemma by considering the subset $\mathcal{H}$ of graphs contributing to $\hat{m}_{r,\ell}(k, i)$, constructed as follows. To obtain a graph $H \in \mathcal{H}$, select a subset $U \subset [k] - [r]$ of size $i$ for the vertices in the top level of $H$, and a minimally susceptible, triangle-free graph $H'$ on $[k] - U$ so that $[r]$ is a contagious set for $H'$ with all level sizes bounded by $\ell$ and $j$ vertices in the top level, for some $1 \leq j \leq \min\{k-r-i, \ell\}$. Let $v$ denote the vertex in the top level of $H'$ of largest index. For each $u \in U$, select a subset $V_u \subset [k] - U$ of size $r$ which contains $v$ and none of the neighbours of $v$ in $H'$ and so that no $v', v'' \in V_u$ are connected in $H'$. Finally, let $H$ be the minimally susceptible graph on $[k]$ with subgraph $H'$ such that each vertex $u \in U$ is connected to all vertices in $V_u$. By the choice of $H'$ and $V_u$, $H$ contributes to $\hat{m}_{r,\ell}(k, i)$. By the choice of $v$, for any choice of $U$, $H'$ and $V_u$, a unique graph $H$ is obtained. Hence $|\mathcal{H}| \leq \hat{m}_{r,\ell}(k, i)$.

To conclude, we claim that, for each $u \in U$, the number of possibilities for $V_u$ is bounded from below by

$$\binom{k-i-r-1}{r-1} - r(k-i-r-1)\binom{k-i-r-3}{r-3} \geq \hat{b}_r(k, i).$$

To see this, note that of the $r(k-i)$ edges in $H'$, there are $r(r+1)$ that are either incident to $v$ or else connect a neighbour of $v$ in $H'$ to another vertex below the top level of $H'$. Therefore

$$\hat{m}_{r,\ell}(k, i) \geq \binom{k-r}{i} \hat{b}_r(k, i) \sum_j \hat{m}_{r,\ell}(k-i, j) = \binom{k-r}{i} \hat{b}_r(k, i) \hat{m}_{r,\ell}(k-i)$$

(where the sum is over $1 \leq j \leq \min\{k-r-i, \ell\}$) as claimed.

By the choice of $i, k$, $\hat{b}_r(k, i) > 0$. Hence $\hat{m}_{r,\ell}(k, i) > 0$ for all relevant $k, \ell$, as is easily seen (e.g., by considering minimally susceptible, triangle-free graphs of size $k = nr + m$, for some $n \geq 1$ and $m \leq r$, which have $m$ vertices in the top level and $r$ vertices in all levels below, and all vertices in level $i \geq 1$ are connected to all $r$ vertices in level $i-1$).

Lemma 3.5. As $k \to \infty$, we have that

$$m_r(k) \geq \hat{m}_r(k) \geq e^{-o(k)} e^{-(r-2)k(k-r)!} \left(\frac{k^{r-1}}{(r-1)!}\right)^k.$$
Comparing this with Lemma 2.5, we see that the number of triangle-free susceptible graphs of size $k$ is not much smaller than the number of susceptible graphs (up to an error of $o(k)$).

**Proof.** The idea is to use spectral analysis of the linear recursion (3.5). However, some work is needed to write the recursion in a usable form.

Put
\[
\hat{\sigma}_{r,\ell}(k,i) = \hat{m}_{r,\ell}(k,i) \left( \frac{(r-1)!}{k^{r-1}} \right)^k
\]

Restricting (3.3) to $j \leq \ell$, it follows that
\[
\hat{\sigma}_{r,\ell}(k,i) \geq \sum_{j=1}^{\ell} \hat{A}_r(k,i,j) \hat{\sigma}_{r,\ell}(k-i,j) \text{ for } i \leq \ell.
\]

(3.5)

In order to express (3.5) in matrix form, we introduce the following notations. For an $\ell \times \ell$ matrix $M$, let $M_j$ be the $\ell \times \ell$ matrix whose $j$th row is that of $M$ and all other entries are 0. Let
\[
\psi(M) = \begin{bmatrix}
M_1 & M_2 & \cdots & M_{\ell-1} & M_{\ell} \\
I_{\ell} & & & & \\
& I_{\ell} & & & \\
& & \ddots & & \\
& & & I_{\ell} & 
\end{bmatrix}
\]

where $I_\ell$ is the $\ell \times \ell$ identity matrix and all empty blocks are filled with 0's. For all relevant $k$, put
\[
\hat{\Sigma}_k = \hat{\Sigma}_k(r,\ell) = \begin{bmatrix}
\hat{\sigma}_k \\
\hat{\sigma}_{k-1} \\
\vdots \\
\hat{\sigma}_{k-\ell+1}
\end{bmatrix}
\]

where $\hat{\sigma}_k = \hat{\sigma}_k(r,\ell)$ is the $1 \times \ell$ vector with entries $(\hat{\sigma}_k)_j = \hat{\sigma}_{r,j}(k,j)$.

Using this notation, (3.5) can be written as
\[
\hat{\Sigma}_k \geq \psi(\hat{A}_k) \hat{\Sigma}_{k-1},
\]

where $\hat{A}_k = \hat{A}_k(r,\ell)$ is the $\ell \times \ell$ matrix with entries $(\hat{A}_k)_{i,j} = \hat{A}_r(k,i,j)$.

By Lemma 3.2, we have that all coordinates of $\hat{\Sigma}_k$ are positive for all $k$ large enough. Let $A = A(r,\ell)$ denote the $\ell \times \ell$ matrix with entries $A_{i,j} = A_r(i,j)$. For $\varepsilon > 0$, let $A_\varepsilon = A_\varepsilon(r,\ell)$, be the $\ell \times \ell$ matrix with entries $(A_\varepsilon)_{i,j} = A_{i,j} - \varepsilon$. By Lemma 3.2 for $k$ large enough each entry of $\hat{A}_\varepsilon$ is greater than the same entry of $A_\varepsilon$. Since $A > 0$, for some $\varepsilon_{r,\ell} > 0$, we have that $A_\varepsilon > 0$ for all $\varepsilon \in (0,\varepsilon_{r,\ell})$. Hence, by Lemma 3.2 and (3.6), for any such $\varepsilon > 0$, there is a $k_\varepsilon$ so that
\[
\hat{\Sigma}_{k_\varepsilon+k} \geq \psi(A_\varepsilon)^k \hat{\Sigma}_{k_\varepsilon} > 0 \text{ for } k \geq 0,
\]

15
with entries of $\Sigma_k$ positive. Therefore, up to a factor of $e^{-o(k)}$, the growth rate of $\hat{\sigma}_{r,\ell}(k) = \sum_{i} \hat{\sigma}_{r,\ell}(k, i)$ is given by the Perron eigenvalue $\lambda = \lambda(r, \ell)$ of $\psi(A)$.

Let $D_\lambda = \text{diag}(\lambda^{-i}: 1 \leq i \leq \ell)$. We claim that the Perron eigenvalue of $\psi(A)$ is characterized by the property that the Perron eigenvalue of $D_\lambda A$ is $1$.

To see this, one simply verifies that if $D_\lambda A v = v$, then

$$v_\lambda = \begin{bmatrix} \lambda^{\ell-1} v \\ \lambda^{\ell-2} v \\ \vdots \\ v \end{bmatrix}$$

satisfies $\psi(A) v_\lambda = \lambda v_\lambda$. If $v$ has non-negative entries, then $1$ is the Perron eigenvalue of $D_\lambda A$ and $\lambda$ the Perron eigenvalue of $\psi(A)$.

Fix $\delta > 0$. If $\lambda < e^{-(r-2)(1 - \delta/e)}$, we claim that for $\ell$ large enough, every row sum of $D_\lambda A$ is greater than $1$. Indeed, the sum of row $i \leq \ell$ is (using the bound $i! < (i/e)^i$)

$$(e^{r-1}\lambda)^{-\ell} \sum_{j=1}^{\ell} \frac{j^i}{i!} \geq (e^{r-1}\lambda)^{-\ell} \frac{\ell^i}{i!} \geq (e - \delta)^{-\ell} \frac{\ell^i}{i!} \geq \left( \frac{e}{e - \delta} \right)^i > 1.$$ 

Since the spectral radius of a matrix is bounded below by its minimum row sum, it follows that for such $\lambda$, the spectral radius of $D_\lambda A$ is greater than $1$. Since the spectral radius of $D_\lambda A$ is decreasing in $\lambda$, the Perron eigenvalue of $\psi(A)$ is at least $e^{-(r-2)(1 - \delta/e)}$ for $\ell$ large enough, and hence $\liminf_{\ell \to \infty} \lambda(r, \ell) \geq e^{-(r-2)}$. Taking $\ell \to \infty$, we find that

$$\hat{m}_r(k) \geq e^{-o(k)} e^{-(r-2)k} (k-r)! \left( \frac{k^{r-1}}{(r-1)!} \right)^k$$

as claimed.

We require a lower bound for the number of minimally susceptible graphs of size $k$ with $i = \Omega(k)$ vertices in the top level in order to estimate the growth of super-critical $r$-percolations on $G_{n,p}$.

**Lemma 3.6.** Let $\varepsilon \in (0, 1/(r+1))$. For all sufficiently large $k$ and $i \leq (\varepsilon/r)^2 k$, we have that

$$\hat{m}_r(k, i) \geq e^{-\varepsilon r - (r-2)k - o(k)} (k-r)! \left( \frac{(k-i) k^{r-2}}{(r-1)!} \right)^k$$

where $o(k)$ depends on $k, \varepsilon$, but not on $i$.

Although the proof is somewhat involved, the general scheme is straightforward. We use Lemmas 3.3, 3.5 to obtain a sufficient bound for $i, k$ in a range for which $i/k \ll 1$. Then, for all other relevant $i, k$ we proceed by induction, using 3.1. The inductive step (Claim 3.7 below) of the proof appears in Appendix A.4.
Proof. Fix some $k_r$ so that

$$k_r > \max \left\{ e^{r/\varepsilon}, \frac{r(r^2 + 1) + 2}{1 - (\varepsilon/r)^2} \right\}.$$ 

Note that, for all $k > k_r$ and $i \leq \left(\varepsilon/r\right)^2 k$, we have that $k/\log^2 k < \left(\varepsilon/r\right)^2 k$ and that Lemma 3.4 applies to $\hat{m}_r(k,i)$ (setting $\ell = k - r$, so that $\hat{m}_{r,\ell}(k,i) = \hat{m}_r(k,i)$).

For all relevant $i,k$, let

$$\hat{\rho}_r(k,i) = \frac{\hat{m}_r(k,i)}{(k-r)!} \left( \frac{(r-1)!}{(k-i)k^{r-2}} \right)^k. \quad (3.7)$$

By Lemma 3.5 there is some $f_r(k) \ll k$ such that

$$\hat{m}_r(k) \geq e^{-(r-2)k - f_r(k)}(k-r)! \left( \frac{k^{r-1}}{(r-1)!} \right)^k.$$ 

Without loss of generality, we assume $f_r$ is non-decreasing.

By Lemma 3.4, we find that for all $k > k_r$ and relevant $i$, $\hat{\rho}_r(k,i)$ is bounded from below by

$$\hat{b}_r(k,i) \geq \frac{(k-i-2r)^{r-1}}{(r-1)!} \left( 1 - \frac{r^3}{k - i - r - 2} \right).$$

By the bound $\binom{n}{k} \geq (n-k)^k/k!$,

$$\hat{b}_r(k,i) \geq \frac{(k-i-2r)^{r-1}}{(r-1)!} \left( 1 - \frac{r^3}{k - i - r - 2} \right).$$

Therefore the lower bound for $\hat{\rho}_r(k,i)$ above is bounded from below by (using the inequality $i! < i^i$)

$$C_r(k,i)g_r(k,i)e^{-(r-2)k - f_r(k-i) - i \log i}$$

where

$$C_r(k,i) = \left( 1 - \frac{2r}{k-i} \right)^{(r-1)i} \left( 1 - \frac{r^3}{k - i - r - 2} \right)^i$$

and

$$g_r(k,i) = e^{(r-2)i} \left( \frac{k-i}{k} \right)^{(r-2)k}.$$ 

If $r = 2$, then $g_r \equiv 1$. We note that, for $r > 2$,

$$\frac{\partial}{\partial t} g_r(k,i) = -\frac{(r-2)i}{k-i} g_r(k,i) < 0$$
and so, for any such $r$, for any relevant $k$, $g_r(k,i)$ is decreasing in $i$. By the inequality $(1 - x)^y > 1 - xy$, for any $k > k_r$ and $i \leq (\varepsilon/r)^2 k,$

$$C_r(k,i) > 1 - \frac{r^2i}{k - i} - \frac{r^3i}{k - i - r - 2}$$

$$> 1 - \frac{2\varepsilon^2}{1 - (\varepsilon/r)^2} - \frac{r\varepsilon^2}{1 - (\varepsilon/r)^2 - (r + 2)/k}$$

$$> 1 - \frac{2/(r + 1)^2}{1 - 1/r^4} - \frac{1/r}{1 - 1/r^4 - (r + 2)/kr}$$

$$> 0$$

since $k_r > e^r/\varepsilon > e^{r(r+1)}$, $r \geq 2$, and $\varepsilon < 1/(r + 1)$ (and noting that the second line is increasing in $r$). Altogether, for some $\xi'(r) > 0$, we have that

$$\hat{\rho}_r(k,i) \geq \xi'(r)e^{-(r-2)k-h_r(k)} \quad \text{for } k > k_r \text{ and } i \leq k/\log^2 n \quad (3.8)$$

where

$$h_r(k) = f_r(k) - \log g_r \left( k, \frac{k}{\log^2 k} \right) + \frac{k}{\log^2 k} \log \left( \frac{k}{\log^2 k} \right). \quad (3.9)$$

We note that $h(k) \ll k$ as $k \to \infty$.

**Claim 3.7.** For some $\xi = \xi(r, \varepsilon) > 0$, for all $k > k_r$ and $i \leq (\varepsilon/r)^2 k$, we have that $\hat{\rho}_r(k,i) \geq \xi e^{-(r-2)k-h_r(k)}$.

Claim 3.7 is proved in Appendix A.4.

Since $h_r(k) \ll k$ and $\xi$ depends only on $r, \varepsilon$, the lemma follows by Claim 3.7 and 3.7.

### 3.2 $\hat{r}$-bootstrap percolation on $G_{n,p}$

We define $\hat{r}$-percolation, a restriction of $r$-percolation, which informally halts upon requiring a triangle. Formally, recall the definitions of $I_r(I,G)$ and $V_r(I,G)$ given in Section 2.1. Let $\hat{I}_t = I_t$ if $G$ contains a triangle-free subgraph $H$ such that $V_r(I,H) = V_r(I,G)$, and otherwise put $\hat{I}_t = \emptyset$. Put $\hat{V}_t = \bigcup_{s \leq t} \hat{I}_s$.

**Definition 3.8.** Let $\hat{P}_r(k,i) = \hat{P}_r(p,k,i)$, for some $p = p(n)$, denote the probability that for a given $I \subset [n]$, with $|I| = r$, we have that $|\hat{V}_t(I,G_{n,p})| = k$ and $|\hat{I}_t(I,G_{n,p})| = i$, for some $t$. Let $\hat{E}_r(k,i)$ denote the expected number of such subsets $I$. We put $\hat{P}_r(k) = \sum_{i=1}^{k-2} \hat{P}_r(k,i)$ and $\hat{E}_r(k) = \sum_{i=1}^{k-2} \hat{E}_r(k,i)$.

Using Lemma 3.6, we obtain lower bounds on the growth probabilities of $\hat{r}$-percolations on $G_{n,p}$.

**Lemma 3.9.** Let $\alpha > 0$. Put $p = \vartheta(\alpha,n)$ and $\varepsilon = np^r = \alpha/\log^{r-1} n$. For $i \leq k - r$ and $k \leq n^{1/(r(r+1))}$, we have that

$$\hat{P}_r(k,i) \geq (1 - o(1))\frac{e^{-\varepsilon(k-r)}}{(k-r)!} \hat{m}_r(k,i)$$

where $o(1)$ depends on $n$, but not on $i,k$. 

18
Proof. Let $I \subset [n]$, with $|I| = r$, be given. Put

$$\hat{\ell}_r(k, i) = \frac{e^{-\varepsilon(r-1)/k} e^{k-r}}{(k-r)!} \hat{m}_r(k, i).$$

If for some $V \subset [n]$ with $|V| = k$ and $I \subset V$ we have that the subgraph $G_V \subset \mathcal{G}_{n,p}$ induced by $V$ is minimally susceptible and triangle-free, $I$ is a contagious set for $V$ with $i$ vertices in the top level, and all vertices in $v \in V^c$ are connected to at most $r$ vertices below the top level of $G_V$, then it follows that $|\hat{V}_t(I, \mathcal{G}_{n,p})| = k$ and $|\hat{V}_t(I, \mathcal{G}_{n,p})| = i$ for some $t$. Hence

$$\hat{P}_r(k, i) \geq \left(\frac{n-r}{k-r}\right)^k \hat{m}_r(k, i) p^{r(k-r)}(1-p)^{k^2} \left(1 - \left(\frac{k-i}{r}\right)p^r\right)^n.$$

By the inequalities $\binom{n}{k} \geq (n-k)^k/k!$ and $(1-x/n)^n \geq e^{-x}(1-x^2/n)$, it follows that

$$\frac{\hat{P}_r(k, i)}{\hat{\ell}_r(k, i)} \geq \left(1 - \frac{k}{n}\right)^k (1-p)^{k^2} \left(1 - \left(\frac{k-i}{r}\right)^2 \frac{2x^2}{n}\right).$$

For all large $n$, the right hand side is bounded from below by

$$\left(1 - \frac{k}{n}\right)^k \left(1 - \frac{1}{n^{1/r}}\right)^{k^2} \left(1 - \frac{k^2 - 2x}{n}\right) \sim 1$$

since $k \leq n^{1/(r(r+1))} \ll n^{1/(2r)}$, as $r \geq 2$. It follows that $\hat{P}_r(k, i) \geq (1 - o(1))\hat{\ell}_r(k, i)$, where $o(1)$ depends on $n$, but not on $i, k$, as required.

\section{3.3 Super-critical bounds}

In this section we show that, for $\alpha > \alpha_*$, the expected number of super-critical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$ which grow larger than a critical size $\beta_*(\alpha) \log n > \beta_*(\alpha) \log n$ is large. The importance of $\beta_*(\alpha)$ is established in Section 3.3 below. Subsequent sections establish the existence of sets $I$ of size $r$ so that $\hat{r}$-percolation initialized at $I$ grows larger than $\beta_*(\alpha) \log n$.

\textbf{Lemma 3.10.} Let $\alpha, \beta_0 > 0$ and $\varepsilon \in (0, 1/(r+1))$. Put $p = \vartheta(\alpha, n)$. For all sufficiently large $k = \beta \log n$ and $i = \gamma k$, with $\beta \leq \beta_0$ and $\gamma \leq (\varepsilon/r)^2$, we have that

$$\hat{E}_r(k, i) \geq n^{\mu_* - o(1)}$$

where

$$\mu_* = \mu_{r, \varepsilon}(\alpha, \beta, \gamma) = r + \beta \log \left(\frac{\alpha \beta^{r-1}(1-\gamma)}{r-1}\right) - \frac{\alpha \beta^r}{r!} (1-\gamma)^r - \beta (r-2 + \varepsilon \gamma)$$

and $o(1)$ depends on $\alpha, \varepsilon, \beta_0$, but not on $\beta, \gamma$. 

19
Proof. Put $\delta = np^2$. By Lemmas 3.6,3.9 for large $k = \beta \log n$ and $i = \gamma k$, with $\beta \leq \beta_0$ and $\gamma \leq (\varepsilon/r)^2$, 

$$\hat{E}_r(k,i) \geq \xi(n) \left( \frac{n}{r} \binom{\delta(k-i)k^{r-2}}{r} \right)^k \left( \frac{\delta}{r-1} \right)^k e^{-\delta k} \delta^{-\delta k} \gamma^{-\delta k} - o(k) = n^{\mu_r - o(1)}$$

where $\xi(n) \sim 1$ depends only on $n$, and $o(k)$ depends only on $r,\varepsilon,\beta_0$. \hfill \blacksquare 

We note that, for any $\alpha, \varepsilon > 0$, 

$$\mu_{r,\varepsilon}(\alpha, \beta, 0) = \mu^*_r(\alpha, \beta). \quad (3.10)$$

We now state the main result of this section.

**Lemma 3.11.** Let $\varepsilon < 1/(r+1)$. Put $\alpha_{r,\varepsilon} = (1+\varepsilon)\alpha_r$ and $p = \vartheta(\alpha_{r,\varepsilon}, n)$. For some $\delta(r, \varepsilon) > 0$ and $\zeta(r, \varepsilon) > 0$ we have that if $k_n/\log n \in [\beta_*(\alpha_{r,\varepsilon}), \beta_*(\alpha_{r,\varepsilon}) + \delta]$ for all large $n$, then $\hat{E}_r(k_n) \gg n^\zeta$ as $n \to \infty$.

The proof appears in Appendix A.5. The argument is technical but straightforward: the basic idea is to show that, for some $\zeta > 0$ and all large $n$, for all relevant $k$ there is some $i$ so that $E_r(k, i) > n^\zeta$. For $k > \beta_* \log n$, values of $i$ with this property are on the order of $k$. We shall thus require Lemma 3.6.

### 3.4 $\hat{r}$-percolations are almost independent

For a set $I \subset [n]$, with $|I| = r$, let $\hat{E}_I(I)$ denote the event that $\hat{r}$-percolation on $G_{n,p}$ initialized by $I$ grows to size $k$, i.e., we have that $|\hat{V}_I(I)| = k$ for some $t$. Hence $\hat{P}_r(k) = \mathbb{P}(\hat{E}_I(I))$. In this section we show that for sets $I \neq I'$ of size $r$ and suitable values of $k, p$, the events $\hat{E}_I(I)$ and $\hat{E}_I(I')$ are approximately independent. Specifically, we establish the following

**Lemma 3.12.** Let $\alpha, \beta > 0$ and put $p = \vartheta_r(\alpha, n)$. Fix sets $I \neq I'$ such that $|I| = |I'| = r$ and $|I \cap I'| = m$. For $\beta \log n \leq k \leq n^{1/(r(r+1))}$, we have that 

$$\mathbb{P}(\hat{E}_k(I')|\hat{E}_k(I)) \leq \left( \frac{\xi}{n} \right)^{-m} + O(k^{2r}(kp)^{-r-m}) + \begin{cases} (1 + o(1))\hat{P}_r(k) & \text{if } m = 0, \\ o\left( \left( \frac{n}{r} \right)^m \right)\hat{P}_r(k) & \text{if } 0 < m < r, \end{cases}$$

where $o(1)$ depends only on $n$.

For sets $I \subset V$ of sizes $r$ and $k$, let $\hat{E}(I, V)$ be the event that for some $t$ we have $\hat{V}_I(\hat{V}_I(V)) = V$. By symmetry these events all have the same probability. Since for a fixed $I$ and different sets $V$ these events are disjoint, we have $\hat{P}_r(k) = \binom{n-r}{k-r}\mathbb{P}(\hat{E}(I, V))$.

**Lemma 3.13.** Fix sets $I \subset V$ with $|I| = r$ and $|V| = k$.

(i) For any set of edges $E \subset [n]^2 - V^2$, the conditional probability that $E \subset E(G_{n,p})$, given $\hat{E}(I, V)$, is at most $p^{|E|}$. 

20
where $\ell$ vertices below the top level of $H$ in the set of edges of $G$.

Proof. Let $G_V$ denote the subgraph of $G_{n,p}$ induced by $V$. The event $\hat{E}(I, V)$ occurs if and only if for some $t$ and triangle-free subgraph $H \subset G_V$, we have that $V_t(I, H) = V_t(I, G_V) = V$ and all vertices in $V'$ are connected to at most $r-1$ vertices below the top level of $H$ (i.e., $V - I_t(I, H)$). This event is increasing in the set of edges of $G_V$, and decreasing in edges outside $V$. By the FKG inequality,

$$\mathbb{P}(E \subset E(G_{n,p}) | \hat{E}(I, V)) \leq \mathbb{P}(E \subset E(G_{n,p})) = p^{|E|}. $$

For claim (ii), let $G$ be a possible value for $G_V$ on $\hat{E}(I, V)$, with a subgraph $H$ as above and $i \leq k-r$ vertices infected in the top level (i.e., $I_t(I, H) = i$). The conditional probability that $u$ is connected to all vertices in $W$, given $\hat{E}(I, V)$ and $G_V = G$, is equal to

$$\frac{p^r \sum_{\ell_0=0}^{r-1} \left( k-i-\ell_0 \right) p^i (1-p)^{k-i-\ell_0-\ell} \sum_{\ell=0}^{r-1} \left( k-\ell \right) p^\ell (1-p)^{k-\ell-\ell}}$$

where $\ell_0 < r$ is the number of vertices in $W$ below the top level of $H$. Bounding the numerator by the $\ell = 0$ term and the denominator by 1, the above expression is at least $p^r (1-p)^{k-i-\ell_0} \geq p^r (1-p)^k$. Hence, summing over the possibilities for $G$ we obtain the second claim.

The following result, a special case of Turán’s Theorem [44], plays an important role in establishing the approximate independence of $\hat{r}$-percolations.

**Lemma 3.14** (Mantel’s Theorem [39]). If a graph $G$ is triangle-free, then we have that $e(G) \leq \lfloor v(G)^2/4 \rfloor$.

In other words, a triangle-free graph has edge-density at most $1/2$. The number $2r - 1$ is key, since $\lfloor (2r-1)^2/4 \rfloor = r(r-1)$, and thus

$$r(2r - 1) - \lfloor (2r-1)^2/4 \rfloor = r^2. \quad (3.11)$$

**Lemma 3.15.** Let $\alpha > 0$ and $k \leq n^{1/(r(r+1))}$. Put $p = \vartheta_\alpha(\alpha, n)$. Fix sets $I \subset V$ and $I'$ such that $|I| = |I'| = r$, $|V| = k$ and $\ell = |V \cap I'| < r$. Let $\hat{E}_{k,q}(I')$ denote the event that for some $t$ we have that $\hat{V}_t(I') = V'$ for some $V'$ such that $|V'| = k$ and $|V \cap V'| = q$. Then

$$\mathbb{P}(\hat{E}_{k,q}(I') | \hat{E}(I, V)) \leq \begin{cases} (1 + o(1))\hat{P}_r(k) & q = 0, \\ o((n/k)^t)\hat{P}_r(k) & 1 \leq q < 2r - 1, \\ k^{2r-1}(kp)^{r(\ell-\ell)} & q \geq 2r - 1, \end{cases}$$

where $o(1)$ depends only on $n$.  

21
Proof. Case i \((q < 2r - 1)\). We claim that

\[
\mathbb{P}(\hat{\mathcal{E}}_{k,q}(I') | \hat{\mathcal{E}}(I, V)) \leq \left( \left( \frac{n}{k} \right)^{\ell} \left( \frac{k^2}{np^{r/4}} \right)^q \right)^{k-r} \sum_{i=1}^{k-r} \hat{Q}_r(k, i) \tag{3.12}
\]

where \(\hat{Q}_r(k, i)\) is equal to

\[
\left( \frac{n}{k-r} \right) \hat{m}_r(k, i) p^{r(k-r)} \left( 1 - \left( \frac{k-i}{r} \right) - \left( \frac{q}{r} \right) \right) p^r (1-p)^{2k} n^{-2k}.
\]

To see this, note that if \(\hat{\mathcal{E}}_{k,q}(I)\) occurs then for some \(V'\) such that \(|V'| = k, I' \subset V'\), and \(|V \cap V'| = q\), we have that \(I'\) is a contagious set for a triangle-free subgraph \(H' \subset \mathcal{G}_{n,p}\) on \(V'\) with \(i\) vertices in the top level, for some \(i \leq k-r\), and all vertices in \((V \cup V')^c\) are connected to at most \(r-1\) vertices below the top level of \(H'\). There are at most

\[
\left( \frac{k}{q-\ell} \right) \left( \frac{n-(q-\ell)}{k-r-(q-\ell)} \right) \leq \left( \frac{n}{k} \right)^{\ell} \left( \frac{k^2}{n} \right)^q \left( \frac{n}{k-r} \right)
\]

such subsets \(V'\). By Lemma 3.14 for any such \(V'\) and \(i\) as above, the conditional probability that such a subgraph \(H'\) exists, given \(\hat{\mathcal{E}}(I, V)\), is bounded by \(\hat{m}(k, i) p^{r(k-r)-q^2/4}\), since at most \(q^2/4\) edges of \(H'\) join vertices in \(V \cap V'\).

By Lemma 3.13 for any \(u \in (V \cup V')^c\) and set \(V''\) of \(r\) vertices below the top level of \(H'\) with at most \(r-1\) vertices in \(V \cap V'\), the conditional probability that \(u\) is connected to all vertices in \(V''\) is at least \(p^r (1-p)^k\). Hence any such \(u\) is connected to all vertices in such a \(V''\) with conditional probability at least \(\left( \frac{k-i}{r} \right) - \left( \frac{q}{r} \right) p^r (1-p)^{2k}\). The claim follows.

To conclude, let \(\hat{\ell}_r(k, i)\) be as in the proof of Lemma 3.3, which recall shows that \(\hat{P}_r(k, i) \geq (1-o(1)) \hat{\ell}_r(k, i)\) as \(k \to \infty\), where \(o(1)\) depends only on \(n\). We have, by the inequalities \(\binom{k}{i} \leq n^k/k!\) and \(1-x < e^{-x}\), that

\[
\log \frac{\hat{Q}_r(k, i)}{\hat{\ell}_r(k, i)} < x \left( \frac{k-i}{r} \right) (1-(1-p)^k) \left( 1- \left( \frac{k}{n} \right) \right) + x q^r.
\]

By the inequality \((1-x)^y \geq 1 - xy\), and since \(k \leq n^{1/(r(r+1))}\), it follows that the right hand side is at most \(\varepsilon n^{1/r} (p+1/n) + \varepsilon q^r \sim 0\), and so

\[
\hat{Q}_r(k, i) \leq (1+o(1)) \hat{\ell}_r(k, i) \leq (1+o(1)) \hat{P}_r(k, i)
\]

where \(o(1)\) depends only on \(n\). Hence

\[
\sum_{i=1}^{k-r} \hat{Q}_r(k, i) \leq (1+o(1)) \sum_{i=1}^{k-r} \hat{P}_r(k, i) = (1+o(1)) \hat{P}_r(k).
\]

Finally, case (i) follows by 3.12 and noting that

\[
\frac{np^{r/4}}{k^2} > \frac{np^{r/2}}{k^2} \geq n^{1/2-2/(r(r+1))} \left( \frac{\alpha}{\log^2 n} \right)^{1/2} \gg 1
\]
since \( q < 2r, k \leq n^{1/(r+1)} \) and \( r \geq 2 \).

**Case ii** \((q \geq 2r - 1)\). Put \( q_* = 2r - 1 - \ell \). If \( \hat{\mathcal{E}}_{k,q}(I') \) occurs, then for some \( \{v_j\}_{j=1}^{q_*} \subset V - I' \) and non-decreasing sequence \( \{t_j\}_{j=1}^{q_*} \), we have that \( v_j \in \hat{I}_j(I') \) and \( \hat{V}_j = \hat{I}_j-I_j(I') \) satisfy \( |V_q| < k \) and \( \hat{V}_j \cap (V-I') \subset \bigcup_{i<j} \{v_i\} \). Informally, \( t_j \) is the \( j \)th time that \( \hat{v} \)-percolation initialized by \( I' \) affects a vertex in \( V - I' \).

It follows that \( G_{n,p} \) contains a triangle-free subgraph on \( \{v_j\}_{j=1}^{q_*} \cup \hat{V}_q \). Since \( v_j \in \hat{I}_j(I') \), note that \( v_j \) is \( r \)-connected to \( \hat{V}_j \). Hence, by Lemma 3.12 and 3.11, there are at least

\[
q_* - [(2r - 1)^2/4] = r(r - \ell)
\]

edges between \( \{v_j\}_{j=1}^{q_*} \) and \( \hat{V}_q - V \). Thus, by Lemma 3.14, the conditional probability of \( \hat{\mathcal{E}}_{k,q}(I') \), given \( \hat{\mathcal{E}}(I,V) \), is bounded by \( k^{r+\ell}(kp)^{(r-\ell)} \leq k^{2r-1}(kp)^{(r-\ell)} \), as claimed.

Using Lemma 3.15 we establish the main result of this section.

**Proof of Lemma 3.12** Fix a sequence of sets \( \{V_{\ell}\}_{\ell=0}^m \) such that \( I \subset V_\ell \) and \( \ell = |V_\ell \cap I'| \). By symmetry, we have that

\[
P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I)) = \left(\frac{n-r}{k-r}\right)^{-1} \sum_{\ell=m}^r \left(\frac{n-r-(\ell-m)}{k-r-(\ell-m)}\right) P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_\ell)) \leq \sum_{\ell=m}^r \frac{(k/n)^{\ell-m} P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_\ell))}{(k/n)^{\ell-m} P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_\ell))}.
\]

If \( \ell = m \), then by Lemma 3.15 summing over \( q \in [\ell,k] \), we get

\[
P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_m)) \leq \begin{cases} (1 + o(1)) \hat{P}_r(k) + k^{2r}(kp)^2 & m = 0, \\ o((n/k)^m) \hat{P}_r(k) + k^{2r}(kp)^{(r-m)} & 1 \leq m < r. \end{cases}
\]

Likewise, for any \( m < \ell < r \),

\[
(k/n)^{\ell-m} P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_\ell)) \leq (k/n)^{\ell-m} \left( o((n/k)^m) \hat{P}_r(k) + k^{2r}(kp)^{(r-\ell)} \right)
\]

\[
= o((n/k)^m) \hat{P}_r(k) + k^{2r}(kp)^{(r-m)}(np^r k^{r-1})^{m-\ell} 
\]

\[
\leq o((n/k)^m) \hat{P}_r(k) + k^{2r}(kp)^{(r-m)}(\alpha^r)^{m-\ell} 
\]

\[
= o((n/k)^m) \hat{P}_r(k) + O(k^{2r}(kp)^{(r-m)}).
\]

Finally, for \( \ell = r \) we bound \( P(\hat{\mathcal{E}}(I')|\hat{\mathcal{E}}(I,V_r)) \leq 1 \). Summing over \( \ell \in [m,r] \) we obtain the result.

**3.5 Terminal \( r \)-percolations**

In this section, we establish the importance of \( \beta_*(\alpha) \) to the growth of supercritical \( r \)-percolations. Essentially, we find that an \( r \)-percolation on \( G_{n,p} \), having grown larger than \( \beta_*(\alpha) \log n \), with high probability continues to grow.
Definition 3.16. We say that $I \subset [n]$ is a terminal $(k, i)$-contagious set for $G_{n,p}$ if $|V(\tau(I, G_{n,p}, r))| = k$ and $|L(\tau(I, G_{n,p}, r))| = i$.

Lemma 3.17. Let $\alpha > \alpha_r$ and $\beta^*_{r}(\alpha) < \beta_1 < \beta_2$. Put $p = \vartheta(\alpha, n)$. With high probability, $G_{n,p}$ has no terminal $m$-contagious set, with $m = \beta \log n$, for all $\beta \in [\beta_1, \beta_2]$.

Proof. If $r$-percolation initialized by $I \subset [n]$ terminates at size $k$ with $i$ vertices in the top level, then $I$ is a contagious set for some subgraph $H \subset G_{n,p}$ of size $k$ with $i$ vertices in the top level, and all vertices in $V(H)^c$ are connected to at most $r-1$ vertices in $V(H)$. Hence the probability that a given $I$ is as such is bounded by

$$\left(\frac{n}{k-r}\right)m_r(k, i)p^r(k-r)\left(1 - \left(\frac{k}{r}\right)p^r(1 - p)^r\right)^{n-k}.$$  

For $k \leq \beta_2 \log n$ and relevant $i$, we have that

$$1 - \left(\frac{k}{r}\right)p^r(1 - p)^r = 1 - \left(\frac{k}{r}\right)p^r + O(n^{-1})$$

where $O(n^{-1})$ depends on $\alpha, \beta_2$, but not on $k/\log n$ and $i/k$. Put $\varepsilon = np^r$. By Lemma 2.5 (and the inequalities $\left(\frac{n}{k}\right) \leq n^k/k!$ and $1 - x < e^{-x}$), it follows that the expected number of terminal $(k, i)$-contagious sets, with $k = \beta \log n$ and $i = \gamma k$, for some $\beta \leq \beta_2$, is bounded (up to a constant) by

$$\left(\frac{n}{r}\right)\left(\frac{\varepsilon k^{r-1}}{(r-1)!}\right)^k \varepsilon^{-r(i - (r-2)k - \varepsilon)} \lesssim n^{\mu^*_{r}(\alpha, \beta) - \beta \gamma \log^{r}(r-1) n}$$

where $\lesssim$ denotes inequality up to a constant depending on $\alpha, \beta_2$, but not on $\beta, \gamma$.

By Lemma 2.9 we have that $\mu^*_{r}(\alpha, \beta) \leq \mu^*_{r}(\alpha, \beta_1) < 0$ for all $\beta \in [\beta_1, \beta_2]$. Hence, summing over the $O(\log^2 n)$ relevant values of $i, k$, we find that the probability that $G_{n,p}$ contains a terminal $m$-contagious set for some $m = \beta \log n$, with $\beta \in [\beta_1, \beta_2]$, is bounded (up to a constant) by

$$n^{\mu^*_{r}(\alpha, \beta_1) \log^{r}(r-1) + 2} n \ll 1$$

as required. \[\blacksquare\]

3.6 Almost sure susceptibility

Finally, we complete the proof of Theorem 1.1. Using Lemmas 3.11, 3.12, 3.17 we argue that if $\alpha > \alpha_r$, then with high probability $G_{n,p}$ contains a large susceptible subgraph. By adding independent random graphs with small edge probabilities, we deduce that percolation occurs with high probability.
Proof of Theorem 1.1. Proposition 2.1 gives the sub-critical case $\alpha < \alpha_r$. Assume therefore that $\alpha > \alpha_r$. Let $G_i$, for $i \geq 0$, be independent random graphs with edge probabilities $p = \theta_r(\alpha_r + \varepsilon, n)$ and $p_i = 2^{-i(r-1)/r} p$, where $p_r = \theta_r(\varepsilon, n)$. Moreover, let $\varepsilon > 0$ be sufficiently small so that $G = \bigcup_{i \geq 0} G_i$ is a random graph with edge probabilities at most $\theta_r(\alpha, n)$. It thus suffices to show that $G$ is susceptible.

Claim 3.18. Let $A > 0$. With high probability, the graph $G_i$ contains a susceptible subgraph on some set $U_0 \subset [n]$ of size $|U_0| \geq A \log n$.

Proof. Using Lemmas 3.11,3.12 we show by the second moment method that, with high probability, $G_i$ contains a susceptible subgraph of size at least $\beta_\ast(\alpha) + \delta_0 \log n$, for some $\delta_0 > 0$. By Lemma 3.17 this gives the claim.

Recall that Lemma 3.11 provides $\delta, \zeta > 0$ so that if $k_n / \log n \in [\beta_\ast(\alpha) + \delta/2, \beta_\ast(\alpha) + \delta]$, then $E_r(k_n) \gg n^{\zeta}$. Fix such a sequence $k_n$. For each $n$, fix $I_n \subset [n]$ with $|I_n| = r$. By Lemma 3.12 it follows that

$$
\sum_I \frac{P(\hat{E}_{k_r}(I) \bigcap \hat{E}_{k_n}(I_n))}{\hat{E}_{r}(k_n)} \leq 1 + o(1) + \left(\frac{n}{r}\right)^{-1} \sum_{m=1}^{r-1} \left(\frac{n-m}{r-m}\right) o((n/k_n)^m)

+ n^{-\zeta} \sum_{m=0}^{r-1} \left(\frac{n-m}{r-m}\right) \left(O(k_n^{2r}(k_n p)^{r(r-m)}) + (k_n/n)^{r-m}\right)

\leq 1 + o(1) + \sum_{m=1}^{r-1} o((r/k_n)^m)

+ n^{-\zeta} \sum_{m=0}^{r-1} \left(O(k_n^{2r}(k_n p)^{r(r-m)}) + (k_n)^{r-m}\right)

= 1 + o(1) + O(n^{-\zeta} \log^{3r} n)

\sim 1
$$

where the sum is over $I \neq I_n$ with $|I| = r$, and $|I \cap I_n| = m$ for some $0 \leq m < r$. Hence, by the second moment method, with high probability some $\hat{r}$-percolation on $G_{n,p}$ grows to size $k_n$ and thus $G_{n,p}$ contains a suscepible subgraph of size $k_n$, as required. As discussed, the claim follows by the choice of $k_n$ and Lemma 3.17.

Claim 3.19. There is some $A = A(\varepsilon)$ so that if $U_0$ is a set of size $|U_0| \geq A \log n$, then with high probability, $r$-percolation on $\bigcup_{i \geq 1} G_i$ initialized at $U_0$ infects a set of vertices of order $n/\log n$.

Proof. Let $A = 2r(16r/\varepsilon)^{1/(r-1)}$. Moreover assume that $n$ is sufficiently large and $\varepsilon$ is sufficiently small so that $A > 2$ and $A(2^{1-r}\varepsilon/\log n)^{1/r} < 1/2$.

We define a sequence of sets $U_i$ as follows. Given $U_i$, we consider all vertices not in $U_0, \ldots, U_i$, and add to $U_{i+1}$ some $2^{i+1} A \log n$ vertices that are $r$-connected in $G_{i+1}$ to $U_i$ (say, those of lowest index).
We first argue that, as long as at most \( n/2 \) vertices are included in \( \bigcup_{j=1}^{i} U_j \) and \( 2^i < n/\log^2 n \), the probability that we can find \( 2^{i+1} A \log n \) vertices to populate \( U_{i+1} \) is at least \( 1 - n^{-1} \). Indeed, a vertex not in \( \bigcup_{j=1}^{i} U_j \) is at least \( r \)-connected in \( G_{i+1} \) to \( U_i \) with probability bounded from below by

\[
\left( \frac{|U_i|}{r} \right) \left( 1 - \frac{|p_{i+1}|}{r} \right)^{r} \left( 1 - \frac{|p_{i+1}|}{r} \right) \geq \frac{1}{2} \left( \frac{|U_i|}{r} \right)^{r},
\]

since, for all large \( n \),

\[
|U_i|p_{i+1} = 2^{-(r-1)/r}(A \log n) \left( \frac{2^{i}\varepsilon}{n \log^{r-1} n} \right)^{1/r} \leq A \left( \frac{2^{1-r} \varepsilon}{\log n} \right)^{1/r} \leq \frac{1}{2}.
\]

Hence the expected number of such vertices is at least

\[
\frac{n}{2} \left( \frac{|U_i|p_{i+1}}{r} \right)^r = \frac{\varepsilon}{4r} \left( \frac{A}{2r} \right)^{r-1} (2^i A \log n) = 2^{i+2} A \log n
\]

by the choice of \( A \). Therefore by Chernoff’s bound, such a set \( U_{i+1} \) of size \( 2^{i+1} A \log n \) can be selected with probability at least \( 1 - \exp(-2^i A \log n) \leq 1 - n^{-1} \), since \( A > 2 \) and \( i \geq 0 \), as required.

Since the number of levels before reaching \( n/2 \) vertices is at most \( \log n \), the claim follows.

By Claims \( 3.18, 3.19 \) with high probability, \( G_0 \cup \bigcup_{i \geq 1} G_i \) contains an \( r \)-infectious subgraph on some \( U \subseteq [n] \) of order \( n/\log n \). To conclude, we observe that given this, by adding \( G_0 \) we have that \( G = G_0 \cup \bigcup_{i \geq 0} G_i \) is susceptible with high (conditional) probability. Indeed, the expected number of vertices in \( U^c \) which are connected in \( G_0 \) to at most \( r - 1 \) vertices of \( U \) is bounded from above by

\[
n \sum_{j=0}^{r-1} \left( \frac{|U_j|}{r} \right) p_0 (1 - p_0)^{|U| - j} \ll n (|U| p_0)^r e^{-p_0 (|U| - r)} \ll n^r e^{-n^{(1-r)/2}} \ll 1.
\]

Hence \( G \) is susceptible with high probability, as required.

### 4 Time dependent branching processes

In this section, we prove Theorem \( 1.5 \) giving estimates for the survival probabilities for a family of non-homogenous branching process which are closely related to contagious sets in \( G_{n,p} \).

Recall that in our branching process, the \( n \)th individual has a Poisson number of children with mean \( (\binom{n}{r}) \varepsilon \). This does not specify the order of the individuals, i.e. which of these children is next. While the order would affect the resulting tree, the choice of order clearly does not affect the probability of survival. In light of this, we can use the breadth first order: Define generation 0 to
be the first \( r - 1 \) individuals, and let generation \( k \) be all children of individuals from generation \( k - 1 \). All individuals in a generation appear in the order before any individual of a later generation. Let \( Y_t \) be the size of generation \( t \), and \( S_t = \sum_{i < t} Y_t \).

Let \( \Psi_r(k, i) \) be the probability that for some \( t \) we have \( S_t = k \) and \( Y_t = i \).

**Lemma 4.1.** We have that

\[
\Psi_r(k, i) = \frac{e^{-\varepsilon(k_r)} \varepsilon^{k_r - r}}{(k_r - r)!} m_r(k, i).
\]

**Proof.** We first give an equivalent branching process. Instead of each individual having a number of children, children will have \( r \) parents. We start with \( r \) individuals (indexed \( 0, \ldots, r - 1 \)), and every subset of size \( r \) of the population gives rise to an independent \( \text{Poi}(\varepsilon) \) additional individuals. Thus the initial set of \( r \) individuals produces \( \text{Poi}(\varepsilon) \) further individuals, indexed \( r, \ldots \). Individual \( k \) together with each subset of \( r - 1 \) of the previous individuals has \( \text{Poi}(\varepsilon) \) children, so overall individual \( k \) has \( \text{Poi}\left(\binom{k}{r-1}\varepsilon\right) \) children where \( k \) is the maximal parent.

Let \( X_S \) be the number of children of a set \( S \) of individuals. A graph contributing to \( m_r(k, i) \) requires \( \text{Poi}(\varepsilon) \) variables to equal \( X_S \), so the probability is \( \prod e^{-\varepsilon} X_S / X_S! \). Up to generation \( t \) this considers \( \binom{k}{r-1} \) sets, and \( \sum X_S = k - r \), giving the terms involving \( \varepsilon \) in the claim. The combinatorial terms \( \prod X_S! \) and \( (k-r)! \) come from possible labelings of the graph. \( \square \)

**Proof of Theorem (\texttt{1.5})** Up to the \( o(1) \) term appearing in the statement of the theorem, the survival of \( (X_t) \) is equivalent to the probability \( p_S \) that for some \( t \) we have that \( S_t \geq k_r \), where \( (S_t)_{t \geq 0} \) is as defined above Lemma 4.1 and \( k_r = k_r(\varepsilon) \) is as in the theorem. By Lemma 4.1

\[
p_S \geq \sum_i \Psi_r(k_r, i) \geq \frac{e^{-\varepsilon(k_r)} \varepsilon^{k_r - r}}{(k_r - r)!} \sum_i m_r(k_r, i) \geq \frac{e^{-\varepsilon(k_r)} \varepsilon^{k_r - r}}{(k_r - r)!} m_r(k_r).
\]

By Lemma 3.5 as \( \varepsilon \to 0 \), the right hand side is bounded from below by

\[
e^{-o(k_r)} e^{-(r-2)k_r - \varepsilon(k_r)} \left(\frac{\varepsilon}{k_r - 1}\right)^{k_r} \varepsilon^{-r} = e^{-(r-1)k_r(1+o(1))}.
\]

On the other hand, we note that the formula for \( \Psi_r(k, i) \) in Lemma 4.1 agrees with the upper bound for \( P_r(k, i) \) in Lemma 2.1 (up to the \( 1 + o(1) \) factor). Hence, using the bounds in Lemma 2.1 and slightly modifying of the proof of Proposition 2.1 (since here we have Poisson random variables instead of Binomial random variables), it can be shown that

\[
p_S \leq e^{o(k_r)} e^{-\varepsilon(k_r)} \frac{\varepsilon^{k_r - r}}{(k_r - r)!} m_r(k_r) = e^{-(r-1)k_r(1+o(1))}
\]

completing the proof. \( \square \)
5 Graph bootstrap percolation

Fix \( r \geq 2 \) and a graph \( H \). We say that a graph \( G \) is \((H, r)\)-susceptible if for some \( H' \subset G \) we have that \( H' \) is isomorphic to \( H \) and \( V(H) \) is a contagious set for \( G \). We call such a subgraph \( H' \) a contagious copy of \( H \). Hence a seed, as discussed in Section 1.2, is a contagious clique. Let \( p_c(n, H, r) \) denote the infimum over \( p > 0 \) such that \( G_{n, p} \) is \((H, r)\)-susceptible with probability at least \( \frac{1}{2} \).

By the arguments in Sections 2, 3, with only minor changes, we obtain the following result. We omit the proof.

**Theorem 5.1.** Fix \( r \geq 2 \) and \( H \subset K_r \) with \( e(H) = \ell \). Put

\[
\alpha_{r, \ell} = \left(\frac{(r-1)!}{(r^2 - \ell)^{r-1}}\right).
\]

As \( n \to \infty \),

\[
p_c(n, H, r) = \left(\frac{\alpha_{r, \ell}}{n \log^{r-1} n}\right)^{1/r} (1 + o(1)).
\]

We obtain Theorem 1.4, from which Theorem 1.2 follows, as a special case.

**Proof of Theorem 1.4.** The result follows by Theorem 5.1, taking \( r = 2 \) and \( \ell = 1 \), in which case \( \alpha_{2, 1} = 1/3 \). ■

A Technical lemmas

We collect in this appendix several technical results used above.

A.1 Proof of Claim 2.6

**Proof of Claim 2.6** By the bound \( i! > \sqrt{2\pi i(i/e)^i} \), it suffices to verify that

\[
\frac{(e/i)^i}{\sqrt{2\pi}} \Lambda(i) \leq 1 \quad \text{for } i \geq 1,
\]

where \( \Lambda(i) = \text{Li}\left(-i + 1/2, 1/e\right) \) and \( \text{Li}(s, z) = \sum_{j=1}^{\infty} z^j j^{-s} \) is the polylogarithm function.

Let \( \Gamma \) denote the gamma function. >From the relationship between \( \text{Li} \) and the Herwitz zeta function, it can be shown that \( \Lambda(i)/\Gamma(i + 1/2) \sim 1 \), as \( i \to \infty \), and hence \( (e/i)^i \Lambda(i) \to \sqrt{2\pi} \), as \( i \to \infty \). It appears (numerically) that \( (e/i)^i \Lambda(i) \) increases monotonically to \( \sqrt{2\pi} \), however this is perhaps not simple to verify (or in fact true). Instead, we find a suitable upper bound for \( \Lambda(i) \).

**Claim A.1.** For all \( i \geq 1 \), we have that

\[
\Lambda(i) < \Gamma(i + 1/2)(1 + ab^i)
\]

where \( a = \zeta(3/2) \) and \( b = e/(2\pi) \), and \( \zeta \) is the Riemann zeta function.
Proof. For all $|u| < 2\pi$ and $s \notin \mathbb{N}$, we have the series representation

$$\mathrm{Li}(s, e^u) = \Gamma(1 - s)(-u)^{s-1} + \sum_{\ell=0}^\infty \frac{\zeta(s - \ell)}{\ell!} u^\ell.$$ 

Hence

$$\Lambda(i) = \Gamma(i + 1/2) + \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!} \zeta(1/2 - i - \ell). \quad (A.2)$$

Recall the functional equation for $\zeta$,

$$\zeta(x) = 2^x \pi^{x-1} \sin(\pi x/2) \Gamma(1-x) \zeta(1-x).$$

Therefore, since $\zeta(1/2 + x) > 0$ is decreasing in $x \geq 1$ we have that, for all relevant $i, \ell$,

$$|\zeta(1/2 - i - \ell)| \leq a \sqrt{\frac{2 \pi}{\ell}} \frac{\Gamma(\ell + i + 1/2)}{(2\pi)^{\ell+1}} < a \frac{\Gamma(\ell + i + 1/2)}{(2\pi)^{\ell+1}}. \quad (A.3)$$

Applying (A.2) and (A.3) (and the inequalities $\Gamma(x + \ell) < (x + \ell - 1)^{\ell} \Gamma(x)$, $\ell! > \sqrt{2\pi \ell (\ell/e)\ell}$, and $(1 + x/\ell)^\ell < e^\ell$), we find that, for all $i \geq 1$,

$$\frac{\Lambda(i)}{\Gamma(i + 1/2)} - 1 < \frac{a}{(2\pi)^i} \sum_{\ell=0}^\infty \frac{(\ell + i - 1/2)^\ell}{(2\pi)^\ell \ell!}$$

$$< \frac{a b^i}{e^i} \left(1 + \sum_{\ell=1}^\infty \frac{1}{\sqrt{2\pi \ell}} \left(\frac{e}{2\pi} \left(1 + \frac{i - 1/2}{\ell}\right)^\ell\right)\right)$$

$$< a b^i \left(\frac{1}{e} + \frac{1}{\sqrt{2\pi}} \sum_{\ell=1}^\infty \left(\frac{e}{2\pi}\right)^\ell\right)$$

$$< a b^i$$

establishing the claim. □

By Claim A.1 the formula

$$\Gamma(i + 1/2) = \sqrt{\pi \frac{i!}{4^i} \left(\frac{2i}{i}\right)^i},$$

and the bounds

$$\left(\frac{2i}{i}\right) < \frac{4^i}{\sqrt{\pi i}} \left(1 - \frac{1}{9i}\right)$$

and

$$i! < \sqrt{2\pi i} \left(\frac{i}{e}\right)^i \left(1 - \frac{1}{12i}\right)^{-1}$$

(valid for all $i \geq 1$), we find that

$$\frac{(e/i)^i}{\sqrt{2\pi}} \Lambda(i) < \frac{4}{3} \frac{9i - 1}{12i - 4} (1 + ab^i) \quad \text{for } i \geq 1. \quad (A.4)$$
Differentiating the right hand side of (A.4), and dividing by the positive term $\frac{4}{3(12i^2 - 1)}$, we obtain

$$3 + ab^i \left(3 + \log(b)(108i^2 - 12i + 1)\right)$$

which, for $i \geq 11$, is bounded from below by

$$3 + 108ab^i \log(b)i^2 > 3 - 237b^i i^2 > 0.$$ 

Hence, for $i \geq 11$, the right hand side of (A.4) increases monotonically to 1 as $i \to \infty$. It follows that (A.1) holds for all $i \geq 11$. Inequality (A.1), for $i \leq 10$, can be verified numerically (e.g., by interval arithmetic), completing the proof of Claim 2.6. 

\[\Box\]

### A.2 Proof of Claim 2.10

**Proof of Claim 2.10.** By (2.6), we have that

$$\frac{\partial^2}{\partial \gamma^2} \mu_r(\alpha, \beta, \gamma) = -\frac{\alpha \beta^r}{(r-2)!} (1-\gamma)^{-2} < 0.$$ 

The result thus follows, noting that

$$\frac{\partial}{\partial \gamma} \mu_r(\alpha, \beta, \gamma) = -\beta \left(1 - \frac{\alpha \beta^r}{(r-1)!} (1-\gamma)^{r-1}\right)$$

and hence for any $\xi < 1$ and $\gamma \in (0, 1)$,

$$\frac{\partial}{\partial \gamma} \mu_r(\alpha, \xi \beta, \gamma) = -\xi \beta (1 - (\xi(1-\gamma))^{r-1}) < 0.$$ 

\[\Box\]

### A.3 Proof of Claim 2.11

**Proof of Claim 2.11.** By Lemma 2.8, for all $k = \beta \log n$ and $i = \gamma k$ as in the lemma, we have that

$$E_r(k,i) \lesssim n^{\mu_r(\alpha, \beta, \gamma)} \log^{r(r-1)} n. \quad (A.5)$$

We find a suitable upper bound for $p_r(k,i)$ as follows. For $\beta < \beta_r(\alpha)$, put $\ell_\beta = \xi_\beta \log n$, where $\xi_\beta = \beta_r(\alpha) - \beta$. For a given set $V$ of size $k$ with $i$ vertices identified as the top level, there are $a_r(k, i)$ ways to select $r$ vertices in $V$ with at least one in the top level. Hence, for $k = \beta \log n$ with $\beta < \beta_r(\alpha)$, it follows that

$$p_r(k, i) \leq \binom{n}{\ell_\beta} (a_r(k, i) p^r)^{\ell_\beta}.$$ 

By Claim 2.3, we have that $a_r(k, i) < i^{k^r-1}/(r-1)!$. Hence, applying the bound $(\binom{n}{\ell}) \leq (ne/\ell)^\ell$, we find that

$$p_r(k, i) \leq \left(\frac{e \alpha \beta^r \gamma}{\xi_\beta (r-1)!}\right)^{\ell_\beta}.$$ 

30
Hence, by Lemma 2.8,
\[ E_r(k,i)p_r(k,i) \lesssim n^{\bar{\mu}_r(\alpha,\beta,\gamma)} \log^{r(r-1)} n \]  \( \text{(A.6)} \)
where
\[ \bar{\mu}_r(\alpha, \beta, \gamma) = \mu_r(\alpha, \beta, \gamma) + \xi_{\beta} \log \left( \frac{e^{\alpha \beta r \gamma}}{(r-1)!} \right) \].  \( \text{(A.7)} \)

Therefore, by (A.5), (A.6), we obtain Claim 2.11 by the following fact.

**Claim A.2.** For any \( \gamma \in (0, 1) \), we have that
\[ \min\{\mu_r(\alpha, \beta, \gamma), \bar{\mu}_r(\alpha, \beta, \gamma)\} \leq \mu^*_r(\alpha, \beta_r) \]
for all \( \beta \in (0, \beta_r(\alpha)) \).

**Proof.** For convenience, we simplify notations as follows. Put \( \beta_r = \beta_r(\alpha) \). We parametrize \( \beta \) using a variable \( \delta \): for \( \delta \in (0, 1] \), let \( \beta_\delta = \delta \beta_r \). For \( \gamma \in (0, 1) \), let \( \mu_r(\delta, \gamma) = \mu_r(\alpha, \beta_\delta, \gamma) \), \( \bar{\mu}_r(\delta, \gamma) = \bar{\mu}_r(\alpha, \beta_\delta, \gamma) \), and \( \delta_r = \delta_r(r) = 1 - \sqrt{\gamma/r} \). Finally, put \( \mu^*_r = \mu_r(1, 0) = \mu^*_r(\alpha, \beta_r) \). In this notation, Claim A.2 states that
\[ \min\{\mu_r(\delta, \gamma), \bar{\mu}_r(\delta, \gamma)\} \leq \mu^*_r, \quad \text{for } \delta \in (0, 1]. \]

Since \( \alpha \beta_r^{-1}/(r-1)! = 1 \), it follows that \( \alpha \beta_\delta^{-1}/(r-1)! = \delta^{r-1} \). Therefore, by (2.6), (A.7), we have that
\[ \mu_r(\delta, \gamma) = r - \beta_r \left( \frac{\delta^r}{r} (1 - \gamma)^r + \delta (r - 2 + \gamma) - (r - 1) \delta \log \delta \right) \]  \( \text{(A.8)} \)
and
\[ \bar{\mu}_r(\delta, \gamma) = \mu_r(\delta, \gamma) + \beta_r(1 - \delta) \log \left( \frac{e^{\gamma \delta^r}}{1 - \delta} \right) \].  \( \text{(A.9)} \)

We obtain Claim [A.2] by the following subclaims (as we explain below the statements).

**Sub-claim A.3.** For any fixed \( \gamma \in (0, 1) \), we have that \( \mu_r(\delta, \gamma) \) and \( \bar{\mu}_r(\delta, \gamma) \) are convex and concave in \( \delta \in (0, 1) \), respectively.

**Sub-claim A.4.** For \( \gamma \in (0, 1) \), we have that
\( (i) \) \( \mu_r(1, \gamma) < \mu^*_r \),
\( (ii) \) \( \mu_r(\delta_r, \gamma) < \mu^*_r \), and
\( (iii) \) \( e \gamma \delta_r^r/(1 - \delta_r) < 1 \).

Indeed, by Sub-claim [A.4(ii),(iii)], we have that \( \bar{\mu}_r(\delta_r, \gamma) < \mu_r(\delta_r, \gamma) < \mu^*_r \). Therefore, noting that \( \lim_{\delta \to 1} \bar{\mu}_r(\delta, \gamma) = \mu_r(1, \gamma) \), \( \lim_{\delta \to 0^+} \mu_r(\delta, \gamma) = r \), and \( \lim_{\delta \to 0^+} \bar{\mu}_r(\delta, \gamma) = -\infty \) (see (A.8),(A.9)), we then obtain Claim [A.2] by applying Sub-claimes [A.3],[A.4].
Proof of Sub-claim A.3. By (A.8), for any $\gamma \in (0, 1)$, we have that
\[
\frac{\partial^2}{\partial \delta^2} \mu_r(\delta, \gamma) = \left(\frac{r-1}{\delta} - (1 - \delta r^{-1}(1 - \gamma)^r) > 0
\right.
\]
for all $\delta \in (0, 1)$. Moreover, by (A.8), (A.9), the above expression, and noting that
\[
\frac{\partial^2}{\partial \delta^2} (1 - \delta) \log \left(\frac{e^{\gamma \delta r}}{1 - \delta}\right) = -\frac{r - (r - 1)\delta^2}{\delta^2(1 - \delta)}
\]
\[
\frac{\partial^2}{\partial \delta^2} \bar{\mu}_r(\delta, \gamma) = -\frac{\beta_r}{\delta^2(1 - \delta)} \left(r - (r - 1)\delta^2 - \delta(1 - \delta)(1 - \delta^{-1}(1 - \gamma)^r)\right)
\]
\[
= -\frac{\beta_r}{\delta^2(1 - \delta)} \left(1 + (r - 1)(1 - \delta)(1 + \delta^r(1 - \gamma)^r)\right)
\]
\[
< 0
\]
for all $\delta \in (0, 1)$. The claim follows.

Proof of Sub-claim A.4. Since, by (A.8),
\[
\mu_r(1, \gamma) = r - \beta_r \left(\frac{1 - \gamma)^r}{r} + (r - 2 + \gamma\right)
\]
claim (i) follows immediately by the inequality $(1 - x)^y \geq 1 - xy$.

Next, we note that, by (A.8) and the bound $\log x < (x^2 - 1)/(2x)$,
\[
\mu_r^* - \mu_r(\delta, \gamma) > \beta_r \left(\frac{\delta^r}{r}(1 - \gamma)^r + \delta(r - 2 + \gamma) + \frac{r - 1}{2}(1 - \delta^2) - \frac{(r - 1)^2}{r}\right).
\]

Hence, to establish claim (ii), if suffices to verify that $f_r(\delta, \gamma) > (r - 1)^2/r$, where
\[
f_r(\delta, \gamma) = \frac{\delta^r}{r}(1 - \gamma)^r + \delta(r - 2 + \gamma) + \frac{r - 1}{2}(1 - \delta^2).
\]
The case $r = 2$ is straightforward, since in this case
\[
f_2(\delta, \gamma) = f_2(1 - \sqrt{\gamma/2}, \gamma) = \frac{1}{2} \left(1 + \sqrt{2\gamma^{3/2}(1 - \gamma) + \gamma^{3/2}/2}\right) > \frac{1}{2}
\]
for all $\gamma \in (0, 1)$. For the remaining cases $r > 2$, we show that $f_r(\delta, \gamma)$ is increasing in $\gamma$. Since $\lim_{\gamma \to 0^+} f_r(\delta, \gamma) = (r - 1)^2/r$ this implies the claim. To this end, we note that
\[
\frac{\partial}{\partial \delta} f_r(\delta, \gamma) = \delta r^{-1}(1 - \gamma)^r + r - 2 + \gamma - (r - 1)\delta,
\]
\[
\frac{\partial}{\partial \gamma} f_r(\delta, \gamma) = -\frac{1}{2\sqrt{\gamma^{3/2}}},
\]
32
(recalling that $\delta_\gamma = 1 - \sqrt{\gamma/r}$) and
\[
\frac{\partial}{\partial \gamma} f_r(\delta, \gamma) = \delta - \delta_\gamma^{-1}(1 - \gamma)^{-1}.
\]
Hence, differentiating $f_r(\delta, \gamma)$ with respect to $\gamma$, we obtain
\[
\delta_\gamma - \delta_\gamma^{-1}(1 - \gamma)^{-1} \left( \frac{1 - \gamma}{2\sqrt{\gamma r}} + \delta_\gamma \right) - \frac{r - 2 + \gamma - (r - 1)\delta_\gamma}{2\sqrt{\gamma r}}.
\]
By the inequality $(1 - x)^y < 1/(1 + xy),
\[
\delta_\gamma^{-1}(1 - \gamma)^{-1} < \frac{1}{(1 + (r - 1)\gamma)(\delta_\gamma + \sqrt{\gamma r})}.
\]
Therefore the expression at (A.10) multiplied by
\[
2\sqrt{\gamma r}(1 + (r - 1)\gamma)(\delta_\gamma + \sqrt{\gamma r}) > 0
\]
is bounded from above by
\[
(1 + (r - 1)\gamma)(\delta_\gamma + \sqrt{\gamma r})(1 - \gamma + 2\sqrt{\gamma r}\delta_\gamma) - (1 - \gamma + 2\sqrt{\gamma r}\delta_\gamma).
\]
which after some straightforward manipulations reduces to
\[
\frac{(r - 1)\gamma}{r} \left( \sqrt{\gamma r}(2r - 3((r - 1)\gamma + 1)) + (r^2 - 3r - 1)\gamma + 2r + 1 \right).
\]
Put
\[
F_r(\gamma) = \sqrt{\gamma r}(3\gamma + 2r - 3 - 3r\gamma) + r^2\gamma - 3r\gamma + 2r + 1 - \gamma.
\]
We claim that $F_r(\gamma) > 0$ for all $r > 2$ and $\gamma \in (0, 1)$. Note that this implies that the expression at (A.10) is positive for all such $r, \gamma$, and hence that $f_r(\delta, \gamma)$ is increasing in $\gamma$ for any such $r$, as desired. To see this, we observe that
\[
\frac{\partial^2}{\partial \gamma^2} F_r(\gamma) = -\frac{1}{4\sqrt{\gamma^3}(r - 3 + 9(r - 1)\gamma)} < 0,
\]
lim$_{\gamma \to 0^+} F_r(\gamma) = 2r + 1 > 0$, and lim$_{\gamma \to 1^-} F_r(\gamma) = r(r - \sqrt{r} - 1) > 0$ for all $r > 2$ and $\gamma \in (0, 1)$. Altogether, we have established claim (ii) for all $r \geq 2$.
Finally, for claim (iii), let $g_r(\delta, \gamma) = e\gamma\delta^r/(1 - \delta)$. In this notation, claim (iii) states that $g_r(\delta, \gamma) < 1$. To verify this inequality, we note that
\[
\frac{\partial}{\partial \delta} g_r(\delta, \gamma) = \frac{e\gamma\delta^{r-1}}{(1 - \delta)^2}(r - (r - 1)\delta)
\]
and hence
\[
\frac{\partial}{\partial \delta} g_r(\delta, \gamma) = e\delta_\gamma^{-1}(r + (r - 1)\sqrt{\gamma r}).
\]
Therefore, noting that
\[ \frac{\partial}{\partial \gamma} g_r(\delta, \gamma) \bigg|_{\delta = \delta_r} = \frac{e^{\delta_r \gamma}}{1 - \delta_r \gamma} = e^{\delta_r \gamma - 1} \left( \frac{r}{\gamma} - 1 \right) \]
and recalling that
\[ \frac{\partial}{\partial \gamma} \delta_r = -\frac{1}{2\sqrt{r}} \]
it follows that
\[ \frac{\partial}{\partial \gamma} g_r(\delta_r, \gamma) = e^{\delta_r \gamma - 1} \left( \frac{r}{\gamma} - (r + 1) \right). \]
Therefore, for any \( r \geq 2 \), \( g_r(\delta_r, \gamma) \) is maximized at \( \gamma = \frac{r}{r + 1} \). By the inequality \((1 - x/n)^n < e^{-x}\), we find that
\[ g_r(r/(r + 1)^2) = e^{r/(r + 1)^2} < \frac{r}{r + 1} \left( 1 - \frac{1}{r + 1} \right)^{-1} = 1 \]
giving the claim. ■

As discussed, Sub-claims A.3, A.4 imply Claim A.2. ■

To conclude, we recall that Claim A.2 implies Claim 2.11. ■

A.4 Proof of Claim 3.7

Proof of Claim 3.7. We recall the relevant quantities defined in the proof of Lemma 3.6 see (3.7), (3.8), (3.9). We have that
\[ \hat{\rho}(k, i) \geq \xi e^{(r - 2)k - h_r(k)} \text{ for } k > k_r \text{ and } i \leq k/\log n \]
where
\[ h_r(k) = f_r(k) - \log g_{r, k} \left( k, \frac{k}{\log^2 k} \right) + \frac{k}{\log^2 k} \log \left( \frac{k}{\log^2 k} \right), \]
\[ f_r(k) \text{ is non-decreasing and } f_r(k) \ll k, \text{ and } g_{r, k}(i, k/i) = e^{(r - 2)i} \left( \frac{k}{r} \right)^{(r - 2)k}. \]
Claim 3.7 states that for some \( \xi > 0 \), for all large \( k \) and \( i \leq (\varepsilon/\xi)^2 k \), we have that
\[ \hat{\rho}(k, i) \geq \xi e^{-(r - 2)k - h_r(k)} \]

Sub-claim A.5. For all \( k > k_r \), we have that \( h_r(k) \) is increasing in \( k \).

Proof. Since \( f_r(k) \) is non-decreasing and \( k/\log^2 k \) is increasing, it suffices by (3.9) to show that \( g_{r, k}(k/\log^2 k) \) is decreasing for \( k > k_r \) (and assuming \( r > 2 \), as else \( g_r \equiv 1 \) and so there is nothing to prove). To this end, we note that
\[ \frac{\partial}{\partial i} g_{r, k}(i) = -\frac{(r - 2)i}{k - i} g_{r, k}(i), \]
\[ \frac{\partial}{\partial k} \frac{k}{\log^2 k} = \frac{\log k - 2}{\log^3 k}, \]
and
\[ \frac{\partial}{\partial k} g_r(k, i) = \frac{r - 2}{k - i} \left( (k - i) \log \left( \frac{k - i}{k} \right) + i \right) g_r(k, i) \]

Hence, differentiating \( g_r(k, k/\log^2 k) \) with respect to \( k \), and dividing by
\[ -\frac{(r - 2)k}{k(1 - \log^2 k) \log^3 k} g_r(k, k/\log^2 k) < 0 \]

we obtain
\[ (\log k)(1 - \log^{-2} k) \log \left( \frac{\log^2 k}{\log^2 k - 1} \right) - \frac{\log^3 k - \log k + 2}{\log^2 k}. \]

By the inequality \( \log x > 2(x - 1)/(x + 1) \) (valid for \( x > 1 \)), the above expression is bounded from below by
\[ \frac{\log^3 k - 4 \log^2 k - \log k + 2}{(\log^2 k)(2 \log^2 k - 1)} > \log k > 5 > 2 \log^2 k - 1 > 0 \]

for all \( k > k_r \), since \( k_r > e^{r/\varepsilon} > e^{r(1 + 2)} \) and \( r > 2 \). The claim follows.

By Sub-claim A.5, fix some \( k^* = k^*(r, \varepsilon) > k_r \) so that \( k/\log k \) is larger than \( 9(r/\varepsilon)^4 \) and \( (r + 2)!/(1 - \varepsilon) \) for all \( k \geq k^* \), and \( h_r(k) \) is increasing for all \( k \geq (1 - (\varepsilon/r)^2)k^* \). By (3.8), select some \( \xi(r, \varepsilon) \leq \xi' \) so that the claim holds for all \( k > k_r \) and relevant \( i \), provided either \( i \leq k/\log k \) or \( k \leq k^* \).

We establish the remaining cases, \( k > k^* \) and \( k/\log k < i \leq (\varepsilon/r)^2 \), by induction. To this end, let \( k > k^* \) be given, and assume that the claim holds for all \( k' < k \) and relevant \( i \). By (3.11) it follows that
\[ \hat{\rho}_r(k, i) \geq \sum_{j=1}^{k-r} \hat{B}_r(k, i, j) \hat{\rho}_r(k - i, j) \quad (i < k - r) \quad (A.11) \]

where
\[ \hat{B}_r(k, i, j) = \frac{j^i}{i!} \left( \frac{k - i}{k} \right)^{(r-2)k} \left( \frac{k - i - j}{k - i} \right)^{k - i} \left( \frac{(r - 1)!}{(k - i)^{r-1}} \frac{\hat{a}_r(k - i, j)}{j} \right)^i. \]

Sub-claim A.6. For all \( (r + 2)! \leq i, j \leq k/r^2 \), we have that
\[ \hat{B}_r(k, i, j) \geq \frac{j^i}{i!} \left( \frac{k - i}{k} \right)^{(r-2)k} \left( \frac{k - i - j}{k - i} \right)^{k + (r-2)i}. \]

Proof. By the formula for \( \hat{B}_r(k, i, j) \) above, it suffices to show that
\[ \frac{(r - 1)!}{(k - i)^{r-1}} \frac{\hat{a}_r(k - i, j)}{j} > \left( \frac{k - i - j}{k - i} \right)^{r-1}. \]
To this end, we note that by (3.2) and Claim 2.4 the left hand side is bounded from below by

\[
\frac{1}{j} \sum_{\ell=1}^{j} \left( \frac{k-i-\ell}{k-i} \right)^{r-1} - \frac{2r!}{k-i}
\]

Since, for any integer \( m \), \((1 - y/x)^m - (1 - (y + 1/2)/x)^m \) is decreasing in \( y \), for \( y < x \), it follows that

\[
\frac{1}{j} \sum_{\ell=1}^{j} \left( \frac{k-i-\ell}{k-i} \right)^{r-1} \geq \left( \frac{k-i-(j+1)/2}{k-i} \right)^{r-1}.
\]

Thus, applying the inequalities \( 1 - xy \leq (1 - x)^y \leq 1/(1 + xy) \), we find that

\[
\frac{(r-1)!}{(k-i)^{r-1}} \frac{\hat{a}_{r}(k-i,j)}{j} - \left( \frac{k-i-j}{k-i} \right)^{r-1}
\]

is bounded from below by

\[
1 - \frac{(j+1)(r-1)}{2(k-i)} - \frac{2r!}{k-i} - \frac{1}{1 + j(r-1)/(k-i)}
\]

which equals

\[
\frac{(r-1)j - (r+4r!-1))(k-i) - ((r-1)j + (r+4r!-1))(r-1)j}{2(k-i)(k-i+(r-1)j)}.
\]

It thus remains to show that the numerator in the above expression is non-negative, for all \( i, j \) as in the claim. To see this, we observe that \( r + 4r! - 1 < (r-1)(r+2)! \) for all \( r \geq 2 \). Hence, for \( (r+2)! \leq i, j \leq k/r^2 \) and \( r \geq 2 \), the numerator divided by \((r-1)k > 0 \) is bounded from below by

\[
(j - (r+2)!)(1 - \frac{1}{r^2}) - (j + (r+2)!)(\frac{1}{r^2}) = \left(1 - \frac{2}{r^2}\right)(j - (r+2)! \geq 0
\]

as required. The claim follows. ■

Applying Sub-claim A.6 to the inductive hypothesis, and the bound \( i! < 3\sqrt{i}(i/e)^i \) to (A.11), it follows that

\[
\hat{p}_{r}(k, i) > \xi \frac{e^{-(r-2)k+(r-1)i-h_{r}(k-i)}}{3\sqrt{i}} \left( \frac{k-i}{k} \right)^{(r-2)k} \sum_{j \in J_{r,e}} \psi_{r,e}(i/k, j/i)^{k} \quad \text{(A.12)}
\]

where \( J_{r,e}(k, i) \) is the set of \( j \) satisfying \((r+2)! \leq j \leq (\varepsilon/r)^2(k-i) \), and

\[
\psi_{r,e}(\gamma, \delta) = \delta^\gamma e^{-\delta\gamma} \left(1 - \frac{\delta\gamma}{1 - \gamma}\right)^{1+\gamma(r-2)}.
\]
Sub-claim A.7. Put $\delta_\varepsilon = 1 - \varepsilon$ and $\delta_{r,\varepsilon} = \delta_\varepsilon + (\varepsilon/r)^2$. For any fixed $\gamma \leq (\varepsilon/r)^2$, we have that $\psi_{r,\varepsilon}(\gamma, \delta)$ is increasing in $\delta$, for $\delta \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$.

Proof. Differentiating $\psi_{r,\varepsilon}(\gamma, \delta)$ with respect to $\delta$, we obtain

$$
\frac{\psi_{r,\varepsilon}(\gamma, \delta)}{\delta (1 - \gamma - \delta)} (\varepsilon \gamma \delta^2 - (1 + \varepsilon + \gamma (r - 1 - \varepsilon)) \delta + 1 - \gamma).
$$

Hence, to establish the claim, it suffices to show that

$$
\varepsilon \gamma \delta^2 - (1 + \varepsilon + \gamma (r - 1 - \varepsilon)) \delta + 1 - \gamma
$$

is positive for relevant $\gamma$. Moreover, since the above expression is decreasing in $\gamma$, we need only verify the case $\gamma = (\varepsilon/r)^2$. Setting $\gamma$ as such in the above expression, and then dividing by $\varepsilon^2/r^6$, we obtain

$$
r^6 - (1 - \varepsilon) r^5 - (1 + 3\varepsilon^2 - \varepsilon^3) r^4 - 3\varepsilon^2 + \varepsilon^2 (1 + 3\varepsilon - 2\varepsilon^2) r^2 + \varepsilon^5.
$$

For $\varepsilon < 1/r$ and $r \geq 2$, this expression is bounded from below by

$$
r(r^5 - r^4 - (1 + 3/r^2) r^3 - 1) \geq r > 0
$$

as required, giving the claim.

By the choice of $k_*$ and since $k > k_*$, for all relevant $k/\log^2 k \leq i \leq (\varepsilon/r)^2 k$, we have that $\delta_{r,\varepsilon} i \geq (r + 2)!$ and

$$
\frac{\delta_{r,\varepsilon} i}{k - i} \leq \frac{(\varepsilon/r)^2}{1 - (\varepsilon/r)^2} \leq (\varepsilon/r)^2
$$

where the second inequality follows since

$$
-1 - \varepsilon + (\varepsilon/r)^2
$$

is increasing in $\varepsilon, \gamma$. Setting $\gamma = (\varepsilon/r)^2$, we obtain

$$
\frac{\partial}{\partial \varepsilon} \frac{1 - \varepsilon + (\varepsilon/r)^2}{1 - (\varepsilon/r)^2} = -r^2 \cdot \frac{(r^2 + \varepsilon^2 - 4\varepsilon)}{(r - \varepsilon)^2 (r + \varepsilon)^2} < 0
$$

for all $r \geq 2$. Hence, for all such $i, k$, we have that $j \in J_{r,\varepsilon}(k, i)$ for all $j \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$. Therefore, for any such $i, k$, by (A.12) and Sub-claim A.7, we have that

$$
\rho_r(k, i) > \varepsilon^2 \frac{(r - 2)k + (r - 1)i - h_r(k - i)}{3 \sqrt{i}} (k - i)^{(r-2)k} \sum_{\delta_{i,j} \leq \delta_{r,\varepsilon}} \psi_{r,\varepsilon}(i/k, j/i)^k
$$

$$
> \varepsilon^2 \frac{(\delta_{r,\varepsilon} - \delta_\varepsilon) \sqrt{i}}{3} e^{-(r-2)k + (r-1)i - h_r(k-i)} \left( \frac{k - i}{k} \right)^{(r-2)k} \psi_{r,\varepsilon}(i/k, \delta_\varepsilon)^k
$$

$$
> \varepsilon e^{-(r-2)k + (r-1)i - h_r(k-i)} \left( \frac{k - i}{k} \right)^{(r-2)k} \psi_{r,\varepsilon}(i/k, \delta_\varepsilon)^k
$$

where the last inequality follows since for any such $i, k$, by the choice of $k_*$ and since $k > k_*$, we have that $\delta_{r,\varepsilon} - \delta_\varepsilon = (\varepsilon/r)^2 > 3/\sqrt{r}$. 

37
Sub-claim A.8. Fix $k / \log^2 k \leq i \leq (\varepsilon/r)^2 k$, and define $\zeta_r(k, i)$ such that

$$\hat{\rho}_{r}(k, i) = \xi e^{-\zeta_{r}(k, i)(\varepsilon-\varepsilon)(r-2)k-h_r(k)}.$$  

We have that $\zeta_r(k, i) < 1$.

Proof. Letting $\gamma = i/k$, it follows by the bound for $\hat{\rho}_{r}(k, i)$ above, and since $k > k_*$ and hence $h_r(k) < h_r(k)$ by the choice of $k_*$, that $\zeta_r(k, i)$ is bounded from above by

$$\delta_x = \frac{r-1}{\varepsilon} - \frac{r-2}{\varepsilon\gamma} \log(1-\gamma) - \frac{1}{\varepsilon} \log \delta_x - \frac{1}{\varepsilon\gamma} \log \left( \frac{1 - \delta_x\gamma}{1 - \gamma} \right).$$

Recall that $\delta_x = 1 - \varepsilon$. Applying the bound $-\log(1-x) \leq x/(1-x)$ for $x = \gamma$ and $x = \delta_x\gamma/(1-\gamma)$, and the bound $-\log(1-x) \leq x + (1+x)x^2/2$ for $x = \varepsilon$ (valid for any $x < 1/3$, and so for all relevant $\varepsilon < 1/(r+1)$ with $r \geq 2$), we find that the expression above is bounded from above by

$$\nu(\varepsilon, \gamma) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{1 - (r-1)\gamma}{\varepsilon(1-\gamma)} + \frac{(1-\varepsilon)(1+(r-2)\gamma)}{\varepsilon(1-(2-\varepsilon)\gamma)}.$$

Therefore, noting that

$$\frac{\partial}{\partial \gamma} \nu(\varepsilon, \gamma) = \frac{r-2}{\varepsilon(1-\gamma)^2} + \frac{(1-\varepsilon)(r-\varepsilon)}{\varepsilon(1-(2-\varepsilon)\gamma)^2} > 0,$$

to establish the subclaim, it suffices to verify that $\nu(\varepsilon, (\varepsilon/r)^2) < 1$ for all $r \geq 2$ and $\varepsilon < 1/(r+1)$. Furthermore, since

$$\nu(\varepsilon, (\varepsilon/r)^2) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{r^2 - \varepsilon^2(r-1)}{\varepsilon(r^2 - \varepsilon^2)} + \frac{(1-\varepsilon)(r^2 + \varepsilon^2(r-2))}{\varepsilon(r^2 - 2\varepsilon^2 + \varepsilon^3)}$$

and hence

$$\frac{\partial}{\partial r} \nu(\varepsilon, (\varepsilon/r)^2) = -\frac{\varepsilon(r(r-4) + \varepsilon^2)}{(r^2 - \varepsilon^2)^2} - \frac{\varepsilon(1-\varepsilon)(r(r-2\varepsilon) + \varepsilon^2(2-\varepsilon))}{(r^2 - 2\varepsilon^2 + \varepsilon^3)^2} < 0$$
or all $k \geq 4$ and $\varepsilon < 1$, we need only verify the cases $r \leq 4$.

To this end, let $\eta(r, \varepsilon)$ denote the difference of the numerator and denominator of $\nu(\varepsilon, (\varepsilon/r)^2)$ (in its factorized form), namely

$$-\varepsilon^7 + 3\varepsilon^6 + (r^2 - 4)\varepsilon^5 - 2(2r^2 - 2r + 1)\varepsilon^4 + (5r^2 - 6r + 8)\varepsilon^3 + r^2(r^2 - 2r - 2)\varepsilon^2 - r^2(r - 2)^2\varepsilon.$$  

For all $\varepsilon < 1/3$, we have that

$$\eta(2, \varepsilon) = -\varepsilon^2(1-\varepsilon)(2-\varepsilon)(2+\varepsilon)(2-2\varepsilon + \varepsilon^2) < -\varepsilon^2 < 0.$$  

Similarly,

$$\eta(3, \varepsilon) = -\varepsilon(9 - 9\varepsilon - 35\varepsilon^2 + 26\varepsilon^3 - 5\varepsilon^4 - 3\varepsilon^5 + \varepsilon^6) < -\varepsilon < 0.$$

38
and
\[
\eta(4, \varepsilon) = -\varepsilon(64 - 96\varepsilon - 64\varepsilon^2 + 50\varepsilon^3 - 12\varepsilon^4 - 3\varepsilon^5 + \varepsilon^6) < -\varepsilon < 0.
\]

It follows that \( \nu(\varepsilon, (\varepsilon/r)^2) < 1 \) for all \( \varepsilon < 1/3 \) and \( k \leq 4 \), and hence for all \( k \geq 2 \), giving the subclaim.

By Sub-claim A.8, we find that \( \hat{\rho}_r(k, i) = \xi e^{-\varepsilon i - (r - 2)k - h_r(k)} \) for all \( i, k \) such that \( k/\log^2 k \leq i \leq (\varepsilon/r)^2k \), completing the induction, and thus giving Claim 3.7.

A.5 Proof of Lemma 3.11

Proof of Lemma 3.11. Put \( \alpha_{r, \varepsilon} = (1 + \varepsilon) \alpha_r \). Let \( \beta_r = \beta_r(\alpha_{r, \varepsilon}) \) and \( \beta_* = \beta_*(\alpha_{r, \varepsilon}) \). For \( \beta > 0 \) and \( \gamma \in [0, 1) \), let \( \mu_{r, \varepsilon}(\beta, \gamma) = \mu_{r, \varepsilon}(\alpha_{r, \varepsilon}, \beta, \gamma) \) and \( \mu_*^r(\beta) = \mu_*^r(\alpha_{r, \varepsilon}, \beta) \). Let \( \gamma_{r, \varepsilon}^*(\beta) \) denote the maximizer of \( \mu_{r, \varepsilon}(\beta, \gamma) \) over \( \gamma \in [0, 1) \), which is well-defined, since for all \( \gamma \in (0, 1) \),

\[
\frac{\partial^2}{\partial \gamma^2} \mu_{r, \varepsilon}(\beta, \gamma) - \frac{\beta}{(1 - \gamma)^2} - \frac{\alpha_{r, \varepsilon} \beta}{(r - 2)!} (1 - \gamma)^{r - 2} < 0 \quad \text{(A.13)}
\]

and \( \lim_{\gamma \to 1^-} \mu_{r, \varepsilon}(\beta, \gamma) = -\infty \). Finally, put \( \gamma_{r, \varepsilon}(\beta) = \min\{\gamma_{r, \varepsilon}^*(\beta), (\varepsilon/r)^2\} \).

We show that \( \mu_{r, \varepsilon}(\beta, \gamma_{r, \varepsilon}(\beta)) \) is bounded away from 0 for \( \beta \in [\beta_*^r, \beta_*^r + \delta] \), for some \( \delta > 0 \). By Lemma 3.10, the result follows.

Claim A.9. For \( \gamma \in (0, 1) \), let

\[
\beta_{r, \varepsilon}(\gamma) = \frac{1/(1 - \gamma) + \varepsilon}{1 - \gamma} \beta_r
\]

and put

\[
\beta_{r, \varepsilon} = \lim_{\gamma \to 0^+} \beta_{r, \varepsilon}(\gamma) = (1 + \varepsilon)^{1/(r - 1)} \beta_r.
\]

We have that

(i) \( \gamma_{r, \varepsilon}^*(\beta) = 0 \), for all \( \beta \leq \beta_{r, \varepsilon} \),

(ii) for \( \beta > \beta_{r, \varepsilon} \), \( \gamma = \gamma_{r, \varepsilon}^*(\beta) \) if and only if \( \beta = \beta_{r, \varepsilon}(\gamma) \), and

(iii) \( \gamma_{r, \varepsilon}^*(\beta) \) is increasing in \( \beta \), for \( \beta \geq \beta_{r, \varepsilon} \).

Proof. By (A.13), we have that \( \mu_{r, \varepsilon}(\beta, \gamma) \) is concave in \( \gamma \). Therefore, since

\[
\frac{\partial}{\partial \gamma} \mu_{r, \varepsilon}(\beta, \gamma) - \beta \left( \frac{1}{1 - \gamma} + \varepsilon - \frac{\alpha_{r, \varepsilon} \beta^{r - 1}}{(r - 1)!} (1 - \gamma)^{r - 1} \right)
\]

and hence, for any \( \xi > 0 \),

\[
\frac{\partial}{\partial \gamma} \mu_{r, \varepsilon}(\xi \beta_r, \gamma) = -\xi \beta_r \left( \frac{1}{1 - \gamma} + \varepsilon - \xi^{r - 1} (1 - \gamma)^{r - 1} \right),
\]

the first two claims follow. The third claim is a consequence of the second claim and the fact that \( \beta_{r, \varepsilon}(\gamma) \) is increasing in \( \gamma \).

39
By the following claims, we obtain the lemma (as we discuss below the statements).

**Claim A.10.** For $\beta > 0$ and $\gamma \in [0, 1)$, let

$$\omega_{r,\varepsilon}(\beta, \gamma) = \mu_{r,\varepsilon}(\beta, \gamma) - \mu^*_r(\beta).$$

We have that

(i) $\omega_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta)) = 0$, for all $\beta \leq \beta_{r,\varepsilon}$, and

(ii) $\omega_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ is increasing in $\beta$, for $\beta \geq \beta_{r,\varepsilon}$.

**Claim A.11.** We have that $\beta_{r,\varepsilon} < \beta^*_s$.

Indeed, the claims together imply that $\omega_{r,\varepsilon}(\beta^*_s, \gamma_{r,\varepsilon}(\beta^*_s)) > 0$. Therefore, since $\mu^*_r(\beta^*_s) = 0$, we thus have that $\mu_{r,\varepsilon}(\beta^*_s, \gamma_{r,\varepsilon}(\beta^*_s)) > 0$. Therefore, by the continuity of $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ in $\beta$, it follows that $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta)) > 0$ for all $\beta \in [\beta^*_s, \beta^*_s + \delta]$, for some $\delta > 0$. As discussed the lemma follows, applying Lemma 3.10.

**Proof of Claim A.10.** The first claim follows by (3.10) and Claim A.9(i).

For the second claim, we show that (a) $\omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))$ is increasing in $\beta$, for $\beta \geq \beta_{r,\varepsilon}$ such that $\gamma^*_{r,\varepsilon}(\beta) \leq (\varepsilon/r)^2$, and (b) $\omega_{r,\varepsilon}(\beta, (\varepsilon/r)^2)$ is increasing in $\beta$, for $\beta \geq \beta_{r,\varepsilon}$. By Claim A.9(ii), this implies the claim.

Since $\gamma^*_{r,\varepsilon}(\beta)$ maximizes $\mu_{r,\varepsilon}(\beta, \gamma)$, and so $\partial \omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))/\partial \gamma = 0$, it follows that

$$\frac{\partial}{\partial \beta} \omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta)) = 0.$$  

Hence, by Claim A.9(ii), to establish (a) we show that for all $\gamma \leq (\varepsilon/r)^2$, $\partial \omega_{r,\varepsilon}(\beta, \gamma, \gamma)/\partial \beta > 0$. To this end, we observe that

$$\frac{\partial}{\partial \beta} \omega_{r,\varepsilon}(\beta, \gamma) = \log(1 - \gamma) - \varepsilon \gamma + \frac{\alpha_{r,\varepsilon} \beta^{-1}}{(r - 1)!} (1 - (1 - \gamma)^r).$$  

(A.14)

Setting $\beta = \beta_{r,\varepsilon}(\gamma)$, the above expression simplifies as

$$\log(1 - \gamma) - \varepsilon \gamma + \frac{1/(1 - \gamma) + \varepsilon}{(1 - \gamma)^{r-1}} (1 - (1 - \gamma)^r).$$

By the inequalities $(1 - x)^y \leq 1/(1 + xy)$ and $\log(1 - x) \geq -x/(1 - x)$, this expression is bounded from below by

$$-\frac{\gamma}{1 - \gamma} - \varepsilon \gamma + (1 + (r - 1)\gamma) \left( \frac{1}{1 - \gamma} + \varepsilon \right) \left( 1 - \frac{1}{1 + \gamma r} \right)$$

which factors as

$$\gamma (1 + \varepsilon (1 - \gamma)) (1 - \gamma)(1 + \gamma r) (r - 1 + \gamma r (r - 2)) > 0$$

and (a) follows.
Similarly, we note that by (A.14), for any $\beta \geq \beta_{r,\varepsilon}$ and $\gamma > 0$,

$$\frac{\partial}{\partial \beta} \omega_{r,\varepsilon}(\beta, \gamma) \geq \log(1 - \gamma) - \varepsilon \gamma + \frac{\alpha_{r,\varepsilon} \beta_{r,\varepsilon}^{-1}}{(r - 1)!} (1 - (1 - \gamma)\gamma')$$

$$= \log(1 - \gamma) - \varepsilon \gamma + (1 + \varepsilon)(1 - (1 - \gamma)\gamma').$$

Hence, using the same bounds for $(1 - x)^y$ and $\log(1 - x)$ as above, we find that for all such $\beta \geq \beta_{r,\varepsilon}$, $\partial \omega_{r,\varepsilon}(\beta, (\varepsilon/r)^2)/\partial \beta$ is bounded from below by

$$\frac{\varepsilon^2(r^3(r - 1)(1 + \varepsilon) - 2r^2\varepsilon^2 - r(2r - 1)e^3 + e^5)}{(r - \varepsilon)(r + \varepsilon)(r + \varepsilon^2)r^2}.$$

For $\varepsilon < 1/r$, the numerator is bounded from below by

$$\varepsilon^2 \left( r^3(r - 1) - 2 - \frac{2r - 1}{r^2} \right) = \frac{\varepsilon^2}{r} \left( r^6 - r^5 - 2r^2 - 2r + 1 \right) > 0$$

since $r \geq 2$. Hence $\partial \omega_{r,\varepsilon}(\beta, (\varepsilon/r)^2)/\partial \beta > 0$, giving (b), and thus completing the proof of the second claim.

**Proof of Claim [A.11].** By Lemma 2.9 the claim is equivalent to $\mu^*_r(\beta_{r,\varepsilon}) > 0$. To see this, we note that

$$\beta_r = \left( \frac{(r - 1)!}{\alpha_{r,\varepsilon}} \right)^{1/(r - 1)} = \left( \frac{1}{1 + \varepsilon} \right)^{1/(r - 1)} \left( \frac{r - 1}{r} \right)^2,$$

and hence by (2.7), for any $\xi > 0$, we have that

$$\mu^*_r(\xi^{1/(r - 1)}\beta_r) = r - \xi^{1/(r - 1)}\beta_r \left( r - 2 + \frac{\xi}{r} - \log \xi \right)$$

$$= r - \left( \frac{r}{r - 1} \right)^2 \left( \frac{\xi}{1 + \varepsilon} \right)^{1/(r - 1)} \left( r - 2 + \frac{\xi}{r} - \log \xi \right).$$

In particular,

$$\mu^*_r(\beta_{r,\varepsilon}) = r - \left( \frac{r}{r - 1} \right)^2 \left( r - 2 + \frac{1 + \varepsilon}{r} - \log(1 + \varepsilon) \right).$$

Therefore, by the bound $\log(1 + x) > x/(1 + x)$, we find that

$$\mu^*_r(\beta_{r,\varepsilon}) > \frac{\varepsilon r(r - 1 - \varepsilon)}{(1 + \varepsilon)(r - 1)^2} > 0$$

as required.

As discussed, Lemma 3.11 follows by Claim A.10, A.11.
Acknowledgments

The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, and the organizers of the program Random Geometry, supported by EPSRC Grant Number EP/K032208/1, during which progress on this project was made. OA was further supported by the Simons foundation and NSERC. BK was supported by NSERC and Killam Trusts, and attended the above program with the support of an NSERC Michael Smith Foreign Study Supplement.

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