EXPONENTIAL DECAY FOR THE COUPLED KLEIN-GORDON-SCHRÖDINGER EQUATIONS WITH LOCALLY DISTRIBUTED DAMPING

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Abstract. The following coupled damped Klein-Gordon-Schrödinger equations are considered

$$
\begin{aligned}
&i\psi_t + \Delta \psi + i\alpha b(x)(-\Delta)^{\frac{1}{2}}b(x)\psi = \phi \psi \chi_\omega \quad \text{in } \Omega \times (0, \infty), \quad (\alpha > 0) \\
&\phi_{tt} - \Delta \phi + a(x)\phi_t = |\psi|^2 \chi_\omega \quad \text{in } \Omega \times (0, \infty),
\end{aligned}
$$

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), \( n = 2 \), with smooth boundary \( \Gamma \) and \( \omega \) is a neighbourhood of \( \partial \Omega \) satisfying the geometric control condition. Here \( \chi_\omega \) represents the characteristic function of \( \omega \). Assuming that \( a, b \in W^{1,\infty}(\Omega) \cap C^\infty(\Omega) \) are nonnegative functions such that \( a(x) \geq a_0 > 0 \) in \( \omega \) and \( b(x) \geq b_0 > 0 \) in \( \omega \), the exponential decay rate is proved for every regular solution of the above system. Our result generalizes substantially the previous results given by Cavalcanti et. al in the reference [7].

1. Introduction. We consider the following model of Klein-Gordon-Schrödinger equations with locally distributed damping

$$
\begin{aligned}
&i\psi_t + \Delta \psi + i\alpha b(x)(-\Delta)^{\frac{1}{2}}b(x)\psi = \phi \psi \chi_\omega \quad \text{in } \Omega \times (0, \infty) \\
&\phi_{tt} - \Delta \phi + a(x)\phi_t = |\psi|^2 \chi_\omega \quad \text{in } \Omega \times (0, \infty) \\
&\psi = \phi = 0 \quad \text{on } \Gamma \times (0, \infty) \\
&\psi(0) = \psi_0 \in H^1_0(\Omega) \cap H^2(\Omega), \\
&\phi(0) = \phi_0 \in H^1_0(\Omega) \cap H^2(\Omega), \\
&\psi_1(0) = \phi_1 \in H^1_0(\Omega),
\end{aligned}
$$

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), \( n = 2 \), with smooth boundary \( \Gamma \) and \( \omega \) is an open subset of \( \Omega \) such that \( \text{meas}(\omega) > 0 \) and satisfying the geometric control condition. In what follows, \( \alpha \) is a positive constant and \( \chi_\omega \) represents the characteristic function, that is, \( \chi = 1 \) in \( \omega \) and \( \chi = 0 \) in \( \Omega \setminus \omega \). We consider \( a, b \in W^{1,\infty}(\Omega) \cap C^\infty(\Omega) \) nonnegative functions such that

$$
a(x) \geq a_0 > 0 \quad \text{in } \omega, \quad \text{and} \quad b(x) \geq b_0 > 0 \quad \text{in } \omega,
$$

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so that the nonlinearity $\psi$ exists where the damping terms

$$a(x)\phi_t \text{ and } i\alpha b(x)(-\Delta)^{\frac{1}{2}} b(x)\psi$$

(2)

are, in fact, effective and reciprocally. If the damping is effective in the whole domain, i.e., $a(x) \geq a_0 > 0$ in $\Omega$ and $b(x) \geq b_0 > 0$ in $\Omega$ we can consider $\chi_\omega \equiv 1$ in $\Omega$. This is required in order to turn the system dissipative. Indeed, the presence of the damping terms given in (2) is not necessary by itself to guarantee that the energy $E(t)$ associated to problem (1) (see the definition of $E(t)$ in (7)) is a non-increasing function of the parameter $t$. This will clarified in section 4. Uniform decay rate estimates to problem (1) has been considered in the previous results due to Cavalcanti et. al [11, 7]. While in [11] a full damping was in place in both equations, in contrast, in [7] a full damping has been considered in the Schrödinger equation but just a localized damping has been considered for the wave equation. The main purpose of the present article is to generalize substantially both previous results just considering two localized dampings in both equations. A much more natural damping for the Schrödinger equation the should be $ib(x)\psi$ instead of $i\alpha b(x)(-\Delta)^{\frac{1}{2}} b(x)\psi$: however, the second stronger one allows us to take advantage of a smoothing effect introduced by Aloui [1] for bounded domains, which will play an important role in the proof, as we will shown in section 4. As far as we are concerned this problem remains open for the weaker localized damping $ib(x)\psi$.

Problem (1) has its origin in the canonical model of the Yukawa interaction of conserved complex nucleon field $\psi$ with neutral real meson field $\phi$ given by

$$\begin{cases}
  iv_t + \Delta \psi = \phi \psi & \text{in } \Omega \times (0, \infty) \\
  \phi_{tt} - \Delta \phi + \mu^2 \phi = |\psi|^2 & \text{in } \Omega \times (0, \infty) \\
  \psi = \phi = 0 & \text{on } \Gamma \times (0, \infty) \\
  \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1.
\end{cases}$$

(3)

Here, $\psi$ is a complex scalar nucleon field while $\phi$ is a real scalar meson one and the positive constant $\mu$ represents the mass of a meson. Since we are considering a bounded domain, the term $\mu^2 \phi$ does not affect our arguments in the proof of the asymptotic stability. So, for simplicity this term will be omitted.

It is important to note that problem (3) is not naturally dissipative. So, the introduction of the dissipative mechanisms given by the terms in (2) are necessary to force the energy to decay to zero when $t$ goes to infinity. In fact, the dissipative K-G-S equation has been widely studied, see for example the following references: [20, 22, 26, 27, 33, 15] and references therein. The majority of works in the literature deal with linear dissipative terms acting in both equations, except for the works [21] and [11]. Very few is known regarding a localized damping acting in the wave equation for this system and, as far as we are concerned, there is no result in the literature dealing with a localized dissipation in both equations. A natural question arises from this context: it would be possible to consider a localized feedback $ib(x)\psi$ acting in the Schrödinger equation (instead of the stronger present mechanism of damping given by the term $i\alpha b(x)(-\Delta)^{\frac{1}{2}} b(x)\psi$) in order to obtain some decay rate? This is a hard open problem to be solved yet since the so called smoothing effect, which is crucial in the proof is non longer valid. In fact, the smoothing effect is a natural property for the whole $\mathbb{R}^n$ or non-trapping exterior domains as considered in the works of Constantin and Saut [14] or Burq, Gérard and Tzvetkov [10]. It has been considered previously in unbounded domains as powerful tool in order to achieve the exponential stability, see, for instance, the works [8] and [12]. On the other hand,
for compact manifolds the smoothing effect has been introduced firstly by Aloui
[1, 2], by forcing it precisely by means of the stronger damping \(iab(x)(-\Delta)\frac{1}{2}b(x)\)
above mentioned where the the region \(\omega\) such that \(b(x) > 0\) satisfies the well-
known geometric control condition (GCC, in short). It is worth mentioning that
the pioneers in using such property in order to stabilize the purely Schrödinger
equation subject to the same kind of dissipation were Bortot and Corrêa [9]. In
the present paper, we shall adopt similar ideas to our context and since we are
considering a linear feedback acting in the Schrödinger equation and also a linear
localized one acting in the wave equation, it is expected that the energy of the
system decays to zero exponentially. To prove this fact is the main goal of this
paper.

We would like to mention other papers in connection with problem (3), namely:
Fukuda and Tsutsumi [16, 17, 18, 19], Bachelot and Chadam [4] and Hayashi and
W. Von Wahl [25]. In the above articles the unique global existence to problem (3)
is established and some conservation laws are verified. We also would like to quote
some nice important papers in connection with Klein-Gordon Schrödinger (KGS)
system from various points of view as: [3, 5, 6, 9, 13, 15, 23, 25, 24, 26, 27, 32, 33,
34, 36, 28, 39, 40, 38, 37, 41, 42] and references therein.

Our paper is organized as follows. In section 2 we give the precise assumptions
and state our main result, in section 3 we give an idea of the proof of existence and
in section 4 we give the proof of the main theorem.

2. Main result. In what follows let us consider the Hilbert space \(L^2(\Omega)\) of complex
valued functions on \(\Omega\) endowed with the inner product
\[
(u, v) = \int_{\Omega} u(x) \overline{v(x)} dx,
\]
and the corresponding norm
\[
||u||^2 = (u, u).
\]
We also consider the Sobolev space \(H^1(\Omega)\) endowed with the scalar product
\[
(u, v)_{H^1(\Omega)} = (u, v) + (\nabla u, \nabla v).
\]
We define the subspace of \(H^1(\Omega)\), denoted by \(H^1_0(\Omega)\), as the closure of \(C_0^\infty(\Omega)\)
in the strong topology of \(H^1(\Omega)\). This space endowed with the norm induced by the
scalar product
\[
(u, v)_{H^1_0(\Omega)} = (\nabla u, \nabla v)
\]
is, thanks to the Poincaré’s inequality
\[
||u||_2 \leq \lambda ||\nabla u||_2, \quad \text{for all } u \in H^1_0(\Omega),
\] (4)
a Hilbert space. We set the norms
\[
||u||_p^p = \int_{\Omega} |u(x)|^p dx, \quad ||u||_{L^p_\Gamma} = \int_{\Gamma} |u(x)|^p d\Gamma, \quad ||u||_{\infty} = \text{ess sup}_{x \in \Omega} |u(x)|.
\]
In the particular case when \(n = 2\) we have the Gagliardo-Nirenberg inequality,
\[
||u||_4 \leq c||u||_2^{\frac{2}{3}}||\nabla u||_2^{\frac{1}{3}}, \quad \text{for all } u \in H^1_0(\Omega).
\] (5)
The following assumptions are made:
Conjecture 1. We assume that $a, b \in W^{1,\infty}(\Omega) \cap C^\infty(\Omega)$ are nonnegative functions such that
\[ a(x) \geq a_0 > 0, \quad \text{in } \omega \]
\[ b(x) \geq b_0 > 0, \quad \text{in } \omega. \]
In addition,

If $a(x) \geq a_0 > 0$ in $\Omega$, then we consider $\chi_\omega \equiv 1$ in $\Omega$
If $b(x) \geq b_0 > 0$ in $\Omega$, then we consider $\chi_\omega \equiv 1$ in $\Omega$.

Conjecture 2. We assume that $\omega$ is a neighborhood of $\Gamma(x_0)$ where
\[ \Gamma(x_0) := \{ x \in \Gamma; (x - x_0) \cdot \nu(x) > 0 \} \]
and $\nu(x)$ is the unit outward normal at $x \in \Gamma$.

As an example of a domain $\Omega$ satisfying the above assumption let us consider the figure 1:

The energy associated to problem (1) is defined by
\[ E(t) := \frac{1}{2} \int_\Omega (|\psi(x,t)|^2 + |\phi(x,t)|^2 + |\phi_t(x,t)|^2) \, dx. \]

Now, we are in position to state our main results.

Theorem 2.1. Given $\{\psi_0, \phi_0, \phi_1\} \in \{H^1_0(\Omega) \cap H^2(\Omega)\}^2 \times H^1_0(\Omega)$ and assuming that assumption 1 holds, then, there exists a unique regular solution to problem (1) such that
\[ \psi \in L^\infty(0, \infty; H^1_0(\Omega) \cap H^2(\Omega)), \quad \psi' \in L^\infty(0, \infty; L^2(\Omega)), \]
\[ \phi \in L^\infty(0, \infty; H^1_0(\Omega) \cap H^2(\Omega)), \quad \phi' \in L^\infty(0, \infty; H^1_0(\Omega)), \]
and $\phi'' \in L^\infty(0, \infty; L^2(\Omega))$.

Setting $\mathcal{H} := \{H^1_0(\Omega) \cap H^2(\Omega)\}^2 \times H^1_0(\Omega)$, in the next theorem, below, we provide a local uniform decay of the energy. Indeed, we shall consider the initial data taken in bounded sets of $\mathcal{H}$, namely, $||\{\psi_0, \phi_0, \phi_1\}||_{\mathcal{H}} \leq L$, where $L$ is a positive constant. This is strongly necessary due to the non linear character of system (1) and since the energy $E(t)$ is not naturally a non increasing function of the parameter $t$. Thus, the constants, $C$ and $\gamma$ which appear in (8) will depend on $L > 0$. We shall denote $d := c||b||_{\infty}L$, where $c$ comes from the embedding $H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$. Now, we are in position to state our stabilization result.
Theorem 2.2. Assume that assumptions (1) and (2) hold and moreover that \( \alpha > \frac{\omega^{-1} \omega_m^{1/2}}{2} \) or \( d \) is sufficiently small. Then, there exist \( C, \gamma \) positive constants such that following decay rate holds

\[
E(t) \leq Ce^{-\gamma t}E(0), \quad \text{for all } t \geq 0.
\]

for every regular solution of problem (1) in the class given in previous theorem, provided the initial data are taken in bounded sets of \( \mathcal{H} \).

The next proved theorem in [1] will be used in the proof of lemma 4.1.

Theorem 2.3. Let us consider \( b \in C^\infty \) such that \( \omega = \{b(x) \neq 0\} \) controls geometrically \( \Omega \) and \( v \) is a solution of the following problem:

\[
\begin{align*}
iv_t + \Delta v + ib(x)(-\Delta)^\frac{1}{2}b(x)v &= 0 \quad \text{in } \Omega \times (0, \infty) \\
v &= 0 \quad \text{on } \Gamma \times (0, \infty) \\
v(0) &= v_0 \quad \text{in } \Omega.
\end{align*}
\]

Then,

(i) There exists \( c > 0 \) such that \( u \), defined by \( u(t) = \int_0^T S(t - \tau)f(\tau)d\tau \) satisfies

\[
\|u\|_{L^2(0, \infty; H^1(\Omega))} \leq C\|\|f\|_{L^2(0,T; L^2(\Omega))}.
\]

(ii) For every \( \phi \in C^\infty((0, \infty)) \), there exists \( c > 0 \) such that \( v \) is a solution of (9) with initial data \( v_0 \in L^2(\Omega) \) satisfying

\[
\|\phi v\|_{L^2(0, T; H^1(\Omega))} \leq c\|v_0\|_{L^2(\Omega)}.
\]

3. Existence and uniqueness. In this section we derive a priori estimates for the solutions of the Klein-Gordon-Schrödinger system (1). In what follows, for simplicity, we will denote \( u = u' \). Let us represent by \( \{\omega_i\} \) a basis in \( H_0^1(\Omega) \cap H^2(\Omega) \) formed by the eigenfunctions of \( -\Delta \), by \( V_m \) the subspace of \( H_0^1(\Omega) \cap H^2(\Omega) \) generated by the first \( m \) vectors and by

\[
\psi_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i, \quad \phi_m(t) = \sum_{i=1}^m h_{im}(t)\omega_i,
\]

where \( \{\psi_m(t), \phi_m(t)\} \) is the solution to the following Cauchy problem

\[
\begin{aligned}
&\psi''_m(t), u - i\Delta \psi_m(t), \nabla u + \alpha b(x)(-\Delta)^\frac{1}{2}b(x)\psi_m(t), u) \\
&= -i\phi_m(t)\chi_{\omega_i}u, \quad \forall u \in V_m, \\
&\phi''_m(t), v + \nabla \phi_m(t), \nabla v + \alpha(x)\phi_m(t), v) \\
&= (\psi_m(t))^{\Delta}x, v, \quad \forall v \in V_m,
\end{aligned}
\]

(12)

The approximate system (12) is a finite system of ordinary differential equations which has a solution in \([0, t_m]\). The extension of the solution to the whole interval \([0, T]\), for all \( T > 0 \), is a consequence of the first estimate we are going to obtain below.

A priori estimates

The First Estimate: Considering \( u = \overline{\psi}_m \) in the first equation of (12) and taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi_m(t)\|_2^2 + \alpha \|b(\cdot)\psi_m(t)\|_{D((-\Delta)^{1/2})}^2 = 0.
\]

(13)
Note that for $n = 2$, $D[(-\Delta)^{\frac{1}{2}}] \equiv H^{\frac{1}{2}}(\Omega) \hookrightarrow L^4(\Omega)$ (see Lions and Magenes [31] (2.7) and (9.1) and standard Sobolev’s embedding). So
\[
\|\psi_m(t)\|_{L^2}^2 + c \int_0^t \|\psi_m(s)\|_{L^4(\omega)}^2 \, ds \leq \|\psi_0m\|_{L^2}^2,
\]
where $c$ is a positive constant which comes the embedding $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^4(\Omega)$.

Then, from convergence $\psi_m(0) = \psi_0m \to \psi_0$ in $H^1_0(\Omega) \cap H^2(\Omega)$, we have
\[
(\psi_m) \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)),
\]
and there exists $C > 0$, such that
\[
\int_0^\infty \|\psi_m(s)\|_{L^4(\omega)}^2 \, ds \leq C = C(\|\psi_0\|_2).
\]

Now, considering $v = \phi_m'$ in the second equation of (12) and making use of H"older and Young inequalities, we deduce
\[
\frac{1}{2} \frac{d}{dt}\{\|\phi_m'(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2\} + \int_\Omega a(x)|\phi_m'(x,t)|^2 \, dx 
\leq \frac{1}{2} \|\phi_m'(t)\|_{L^4(\omega)}^2 + \frac{1}{2} \|\phi_m(t)\|_{L^4(\omega)}^2 \|\phi_m'(t)\|_2^2.
\]

Multiplying (16) by 2, integrating over $(0, t)$, $t \in [0, t_m]$, observing that $\phi_m(0) = \phi_0m \to \phi_0$ in $H^1_0(\Omega) \cap H^2(\Omega)$, $\phi_m(0) = \phi_{1m} \to \phi_1$ in $H^1_0(\Omega)$ and taking (15) into account, we obtain
\[
\|\phi_m'(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 \leq C + \int_0^t \|\phi_m(s)\|_{L^4(\omega)}^2 \|\phi_m'(s)\|_2^2 + \|\nabla \phi_m(s)\|_2^2 \, ds,
\]
where $C := C(\|\psi_0\|_2, \|\phi_1\|_2, \|\nabla \phi_0\|_2)$.

Applying Gronwall’s lemma and considering (15) yields
\[
\|\phi_m'(t)\|_2^2 + \|\nabla \phi_m(t)\|_2^2 \leq C,
\]
where $C := C(\|\psi_0\|_2, \|\phi_1\|_2, \|\nabla \phi_0\|_2)$, from which we deduce that
\[
(\phi_m) \text{ is bounded in } L^\infty(0, \infty; H^1_0(\Omega)),
\]
and
\[
(\phi_m') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)).
\]

The Second Estimate: Taking the derivative on time of the first equation in (12), considering $u = \psi_m$ and taking the real part of it, we have
\[
\frac{1}{2} \frac{d}{dt}\|\psi_m(t)\|_2^2 + \rho \|b(\cdot)\psi(t)\|_{D((-\Delta)^{\frac{1}{2}})}^2 \leq \int_\Omega |\phi_m'(x,t)||\psi_m(x,t)||\psi_m'(x,t)| \, dx.
\]
Again, observing that $D[(-\Delta)^{\frac{1}{2}}] \equiv H^{\frac{1}{2}}(\Omega) \hookrightarrow L^4(\Omega)$ for $n = 2$ and using Hölder’s generalized inequality and also considering the inequality $ab \leq \frac{1}{4} a^2 + \varepsilon b^2$ and (14) to estimate the term on the right hand side of (21), we obtain
\[
\frac{1}{2} \frac{d}{dt}\|\psi_m(t)\|_2^2 + (c - \varepsilon)\|\phi_m'(t)\|_{L^4(\omega)}^2 \leq C\varepsilon \|\nabla \phi_m(t)\|_2^2 \|\phi_m(t)\|_{L^4(\omega)}^2.
\]

Taking the derivative in $t$ of the second equation in (12), considering $v = \phi_m''$ and taking the real part, we have
\[
\frac{1}{2} \frac{d}{dt}\|\phi_m''(t)\|_2^2 + \frac{1}{2} \frac{d}{dt}\|\nabla \phi_m'(t)\|_2^2 + \int_\Omega a(x)|\phi_m''(x,t)|^2 \, dx 
\leq 2 \int_\omega |\psi_m(x,t)||\psi_m'(x,t)||\phi_m''(x,t)| \, dx.
\]
So, making use of the Hölder’s generalized inequality combined with the inequality \( ab \leq \frac{1}{4b^2}a^2 + \varepsilon b^2 \) to estimate the term on the right hand side of (23), we get
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \phi_m''(t) \|_2^2 + \| \nabla \phi_m'(t) \|_2^2 \right\} \leq \varepsilon \| \phi_m(t) \|_{L^4(\Omega)}^2 + C \varepsilon \| \psi_m(t) \|_{L^4(\Omega)}^2 \| \phi_m''(t) \|_2^2.
\] (24)

Adding (22) and (24), integrating the result obtained and taking \( \varepsilon \) small enough, we infer
\[
\frac{1}{2} \left( \| \psi_m'(0) \|_2^2 + \| \phi_m''(0) \|_2^2 + \| \nabla \phi_m'(0) \|_2^2 \right) + c \int_0^t \| \psi_m'(s) \|_{L^4(\Omega)}^2 \, ds
\leq C \int_0^t \| \psi_m(s) \|_{L^4(\Omega)}^2 \left( \frac{1}{2} \| \nabla \phi_m'(s) \|_2^2 + \frac{1}{2} \| \phi_m''(s) \|_2^2 + \frac{1}{2} \| \psi_m(s) \|_2^2 \right) + c \int_0^t \| \psi_m'(\tau) \|_{L^4(\Omega)}^2 \, d\tau \, ds
\]
\[
+ \frac{1}{2} \left( \| \psi_m(0) \|_2^2 + \| \phi_m'(0) \|_2^2 + \| \nabla \phi_m(0) \|_2^2 \right),
\]
where \( c \) and \( C \) are positive constants.

Making \( u = \psi_m'(0) \) in the first equation of (12) and \( v = \phi_m'(0) \) in the second equation and observing the convergences in (12), it is verified that there is \( C > 0 \) such that \( \| \psi_m'(0) \|_2^2 + \| \phi_m'(0) \|_2^2 < C \). Then, applying Gronwall’s Lemma in (25), we obtain
\[
\frac{1}{2} \left( \| \psi_m'(t) \|_2^2 + \| \phi_m''(t) \|_2^2 + \| \nabla \phi_m'(t) \|_2^2 \right) + c \int_0^t \| \psi_m'(s) \|_{L^4(\Omega)}^2 \, ds \leq L_1,
\] (26)
where \( L_1 = L_1(||\nabla \psi_0||_2, ||\Delta \phi_0||_2, ||\nabla \phi_0||_2) \) is a positive constant independent of \( T \) and \( m \in \mathbb{N} \). Thus,
\[
(\psi_m') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)),
\] (27)
\[
(\phi_m') \text{ is bounded in } L^\infty(0, \infty; H^1_0(\Omega)),
\] (28)
\[
(\phi_m'') \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)),
\] (29)
and, furthermore,
\[
\int_0^t \| \psi_m'(s) \|_{L^4(\Omega)}^2 \, ds \leq L_1, \quad \text{for all } t > 0.
\] (30)

The Third Estimate: In (12) considering \( u = \overline{\psi}_m \) and taking the imaginary part, we obtain
\[
||\nabla \psi_m(t)||_2^2 = -Im(\psi_m'(t), \psi_m(t)) - \int_\omega \phi_m(x, t)|\psi_m(x, t)|^2 \, dx.
\] (31)

Noting that \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) and making use of the Hölder’s inequality and taking inequality (5) into consideration, we deduce
\[
|Im(\psi_m', \psi_m)| \leq C \|\psi_m''(t)\|_2 \|\nabla \psi_m(t)\|_2,
\] (32)
\[
\left| \int_\omega \phi_m(x, t)|\psi_m(x, t)|^2 \, dx \right| \leq C \|\nabla \phi_m(t)\|_2 \|\psi_m(t)\|_2 \|\nabla \psi_m(t)\|_2.
\] (33)

So, from (14), (19), (26), (32) and (33) we conclude
\[
||\psi_m(t)||_2 \leq C, \quad \text{for all } t > 0,
\]
where \( C = C(||\nabla \psi_0||_2, ||\Delta \phi_0||_2, ||\nabla \phi_0||_2) \).
Thus, \( \psi_m \) is bounded in \( L^\infty(0, \infty; H_0^1(\Omega)) \). \hfill (34)

Taking \( v = \Delta \phi_m \) in the second equation of (12), we have
\[
\| \Delta \phi_m(t) \|^2 \leq \int_\omega |\psi_m(x, t)|^2 |\Delta \phi_m(x, t)| \, dx
\] \hfill (35)
\[
+ \| \phi_m'(t) \|_2 \| \Delta \phi_m(t) \|_2 + \| a \|_\infty \| \phi_m'(t) \|_2 \| \Delta \phi_m(t) \|_2
\]

We estimate the first term on the right hand side of (35) using the Hölder inequality and (5). Then,
\[
\| \Delta \phi_m(t) \|_2 \leq C, \text{ for all } t > 0,
\] \hfill (36)

where \( C = C(\| \nabla \psi_0 \|_2, \| \Delta \phi_0 \|_2, \| \nabla \phi_0 \|_2) \).

Therefore, \( (\phi_m) \) is bounded in \( L^\infty(0, \infty; H_0^1(\Omega) \cap H_0^2(\Omega)) \). \hfill (37)

The rest of the proof follows the same basic steps as the one of [11], Theorem 2.1.

Uniqueness: Let \( \{ \psi_1, \phi_1 \} \) and \( \{ \psi_2, \phi_2 \} \) solutions do problem (1). Then, the uniqueness follow defining \( z = \psi_1 - \psi_2 \) and \( w = \phi_1 - \phi_2 \) and repeating verbatim the same arguments already used in the first estimate.

4. Uniform decay rates. In this section we work it regular solutions \( \{ \psi(t), \phi(t), \phi, \phi_i \} \) to problem (1), that is, those ones that lie in \( H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \) and taking \( \{ \psi_0, \phi_0, \phi_i \} \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) := \mathcal{H} \), such that \( \| \psi_0, \phi_0, \phi_i \|_H \leq L \), where \( L > 0 \). So, multiplying the first equation of (1) by \( \bar{\psi} \), the second equation by \( \phi' \), integrating over \( \Omega \) and making use of Green formula, we deduce that
\[
\int_\Omega \psi \bar{\psi} \, dx + i \int_\Omega |\nabla \psi|^2 \, dx + \alpha \int_\Omega |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx = -i \int_\omega \phi \psi^2 \, dx
\] \hfill (38)
\[
\frac{1}{2} \int_\Omega \left( |\phi'|^2 + |\nabla \phi|^2 \right) \, dx + \int_\Omega a(x) |\phi'|^2 \, dx = \int_\omega |\psi|^2 \phi' \, dx.
\] \hfill (39)

Taking the real part in (38) and adding the obtained result with (39) we obtain
\[
E'(t) + \alpha \int_\Omega |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx + \int_\Omega a(x) |\phi'|^2 \, dx = \int_\omega |\psi|^2 \phi' \, dx.
\] \hfill (40)

Next, we will analyze the last term on the RHS of (40). We have, from Assumption 1, (5), (34), observing that for \( n = 2 \), \( D[(-\Delta)^{\frac{1}{2}}] \equiv H^2(\Omega) \hookrightarrow L^4(\Omega) \) and making use of the Cauchy-Schwarz inequality that
\[
\left| \int_\omega |\psi|^2 \phi' \, dx \right| \leq \frac{a_0^{-1}b_0^{-2} \Delta}{2} \int_\Omega |b(x)\psi|^4 \, dx + \frac{1}{2} \int_\Omega a(x) |\phi'|^2 \, dx
\] \hfill (41)
\[
\leq \frac{a_0^{-1}b_0^{-2} \Delta}{2} \int_\Omega |b(x)\psi|^2 \, dx \|(-\Delta)^{\frac{1}{2}} b(x)\psi\|^2 + \frac{1}{2} \int_\Omega a(x) |\phi'|^2 \, dx
\]
\[
\leq \frac{a_0^{-1}b_0^{-2} \Delta}{2} \int_\Omega |\nabla \psi|^2 \, dx \|(-\Delta)^{\frac{1}{2}} b(x)\psi\|^2 + \frac{1}{2} \int_\Omega a(x) |\phi'|^2 \, dx
\]
\[
\leq \frac{a_0^{-1}b_0^{-2} \Delta}{2} \int_\Omega |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx + \frac{1}{2} \int_\Omega a(x) |\phi'|^2 \, dx
\]
where \( d := c\|b\|_\infty L \).
Combining (40) and (41) and considering \( \alpha \) large enough such that \( \beta := \alpha - a_0^{-1}b_0^{-1}d > 0 \) it holds that

\[
E'(t) \leq -\frac{1}{2} \int_{\Omega} a(x)|\phi'|^2 \, dx - \beta \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx
\]

\[
\leq -k \left[ \int_{\Omega} a(x)|\phi'|^2 \, dx + \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \right].
\]

where \( k = \min\{\frac{1}{2}, \beta\} \).

**Remark 1.** From (42) we deduce two facts: (i) the map \( t \in (0, \infty) \mapsto E(t) \) is non increasing, and, in addition, (ii) we have the following inequality of the energy

\[
E(t_2) - E(t_1) \leq -k \int_{t_1}^{t_2} \left[ \int_{\Omega} a(x)|\phi'|^2 \, dx + \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \right] \, dt,
\]

for \( 0 \leq t_1 \leq t_2 < +\infty \), which will be crucial in the proof. We observe that in order to transform the energy \( E(t) \) in a non increasing function we could have considered \( d \) small enough instead of taking \( \alpha \) sufficiently large, which would imply to take the initial data sufficiently small. Well, in any case, some kind of tribute must be paid in order to obtain uniform decay rates of the energy to the present system.

In order to prove Theorem (2.2) we proceed in several steps.

**Step 1.** Multiplying the first equation of problem (1) by \( \overline{\psi} \) and the second equation by \((q \cdot \nabla \phi)\), where \( q \in (W^{1,\infty}(\Omega))^n \), and following (verbatim) the integration by parts of Lemma 3.7, Chap. I, of Lions [30] we deduce the following identity:

\[
\left[ \int_{\Omega} \left( \frac{|\psi|^2}{2} + \phi'(q \cdot \nabla \phi) \right) \, dx \right]_0^T + \frac{1}{2} \int_0^T (\text{div} \, q)||\phi'||^2 - |\nabla \phi|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} \frac{\partial \phi}{\partial x_j} \frac{\partial \eta_k}{\partial x_k} \partial \phi \, dx \, dt - \int_0^T \int_{\Omega} |\psi|^2 (q \cdot \nabla \phi) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} a(x)\phi'(q \cdot \nabla \phi) \, dx \, dt + \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \, dt
\]

\[
= \frac{1}{2} \int_0^T \int_{\Gamma} (q \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 d\Gamma \, dt.
\]

In (43), for simplicity, we have omitted the variables of the functions under the integral signs and, in addition, we have used the convention of summation of repeated indexes.

Employing (43) with \( q(x) = m(x) = x - x_0 \), for some \( x_0 \in \mathbb{R}^n \), and taking (6) into account, we arrive at

\[
\left[ \int_{\Omega} \left( \frac{|\psi|^2}{2} + \phi'(m \cdot \nabla \phi) \right) \, dx \right]_0^T + \frac{n}{2} \int_0^T \int_{\Omega} ||\phi'||^2 - |\nabla \phi|^2 \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} |\nabla \phi|^2 \, dx \, dt - \int_0^T \int_{\Omega} |\psi|^2 (m \cdot \nabla \phi) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega} a(x)\phi'(m \cdot \nabla \phi) \, dx \, dt + \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 d\Gamma \, dt.
\]
Now, multiplying the second equation of problem (1) by $\xi \phi$, with $\xi \in W^{1,\infty}(\Omega)$ and integrating by parts we obtain the following identity:

$$
\left[ \int_{\Omega} \phi \left( \phi' + \frac{\phi a}{2} \right) dx \right]^T_0 = \int_0^T \int_{\omega} |\psi|^2 \xi \phi \, dx \, dt
$$

(45)

$$
+ \int_0^T \int_{\Omega} \xi |\phi'|^2 - |\nabla \phi|^2 \, dx \, dt - \int_0^T \int_{\Omega} \phi (\nabla \phi \cdot \nabla \xi) \, dx \, dt.
$$

Taking $\xi = \delta \in \mathbb{R}$ in (45) and combining the obtained result with (44) we have

$$
\left[ \int_{\Omega} \left( \frac{|\psi|^2}{2} + \phi' (m \cdot \nabla \phi) + \delta \phi \left( \phi' + \frac{\phi a}{2} \right) \right) dx \right]^T_0
$$

(46)

$$
+ \left( \frac{n}{2} - \delta \right) \int_0^T \int_{\Omega} |\phi'|^2 \, dx \, dt + \left( 1 + \frac{n}{2} - \frac{n}{2} \right) \int_0^T \int_{\Omega} |\nabla \phi|^2 \, dx \, dt
$$

$$
- \int_0^T \int_{\omega} |\psi|^2 (m \cdot \nabla \phi) \, dx \, dt + \int_0^T \int_{\Omega} a(x) \phi' (m \cdot \nabla \phi) \, dx \, dt
$$

$$
+ \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{4}} b(x) \psi|^2 \, dx \, dt - \delta \int_0^T \int_{\Omega} |\psi|^2 \phi \, dx \, dt
$$

$$
\leq \frac{1}{2} \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 \, d\Gamma \, dt.
$$

Denoting

$$
\chi := \left[ \int_{\Omega} \left( \frac{|\psi|^2}{2} + \phi' (m \cdot \nabla \phi) + \delta \phi \left( \phi' + \frac{\phi a}{2} \right) \right) dx \right]^T_0
$$

(47)

and choosing $\delta = \frac{n-1}{2}$ (having in mind that $n = 2$ in the present case) we deduce

$$
\int_0^T E(t) \, dt + \chi - \frac{1}{2} \int_0^T \int_{\Omega} |\psi|^2 \, dx \, dt
$$

(48)

$$
- \int_0^T \int_{\omega} |\psi|^2 (m \cdot \nabla \phi) \, dx \, dt + \int_0^T \int_{\Omega} a(x) \phi' (m \cdot \nabla \phi) \, dx \, dt
$$

$$
+ \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{4}} b(x) \psi|^2 \, dx \, dt - \frac{1}{2} \int_0^T \int_{\omega} |\psi|^2 \phi \, dx \, dt
$$

$$
\leq \frac{1}{2} \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 \, d\Gamma \, dt.
$$

Next, we are going to estimate some terms in (48).

Estimate for $I_1 := \int_0^T \int_{\omega} |\psi|^2 (m \cdot \nabla \phi) \, dx \, dt$.

From now on we will denote

$$
R := \sup_{x \in \overline{\Omega}} m(x) = \sup_{x \in \overline{\Omega}} |x - x^0|.
$$

(49)

So, making use of the integral Cauchy-Schwarz inequality, the numerical Hölder inequality, taking (7), (42) and (49) into account and also considering the inequality
\( ab \leq \frac{1}{4} a^2 + \varepsilon b^2 \) we arrive at
\[
|I_1| \leq \frac{R^2}{4\varepsilon} \int_0^T \int_\Omega |\psi|^4 \, dx \, dt + 2\varepsilon \int_0^T E(t) \, dt \tag{50}
\]
\[
\leq \frac{R^2 d}{4\varepsilon b_0 k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) \, dt.
\]

**Estimate for \( I_2 := \frac{1}{2} \int_0^T \int_\Omega |\psi|^2 \phi \, dx \, dt \)**

Using Cauchy-Schwarz inequality, the numerical Hölder inequality, the inequality \( ab \leq \frac{1}{4} a^2 + \varepsilon b^2 \) and taking (4), (7) and (42) into consideration, we can write
\[
|I_2| \leq \frac{\lambda d}{16 b_0 k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) \, dt. \tag{51}
\]

**Estimate for \( I_3 := -\alpha \int_0^T \int_\Omega (\nabla b(x)|\psi|^2 \, dx \, dt \)**

Analogously, from (42) we obtain
\[
|I_3| \leq \frac{\alpha}{k} |E(0) - E(T)|. \tag{52}
\]

**Estimate for \( I_4 := -\int_0^T \int_\Omega a(x)|\phi|^2 (m \cdot \nabla \phi) \, dx \, dt \)**

Making use of the Cauchy-Schwarz inequality, employing the inequality \( ab \leq \frac{1}{4} a^2 + \varepsilon b^2 \), and considering (7), (42) and (49) we obtain
\[
|I_4| \leq \frac{||a||_{\infty} R^2}{2\varepsilon k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) \, dt. \tag{53}
\]

Combining (50)-(53) and choosing \( \varepsilon = 1/8 \), the following inequality holds
\[
\frac{1}{4} \int_0^T E(t) \, dt \leq \frac{1}{2} \int_0^T \int_{\Gamma(x^0)} (m \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 d\Gamma \, dt \tag{54}
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega |\psi|^2 \, dx \, dt + \|\chi\| + C_0 [E(0) - E(T)],
\]
where
\[
C_0 = \left[ \frac{4 R^2 d + ||a||_{\infty} R^2 2 b_0 k + 2 \alpha b_0^2 + \lambda d}{2 b_0 k} \right].
\]

**Step 2.** We now estimate the quantity \( \frac{1}{4} \int_0^T \int_{\Gamma(x^0)} (m \cdot \nu) \left( \frac{\partial \phi}{\partial \nu} \right)^2 d\Gamma \, dt \) in terms of the damping term \( \int_0^T \int_\Omega a(x)|\phi|^2 \, dx \, dt \).

According to the proof of Lemma 2.3 in Lions \[30\] we can construct a neighborhood \( \tilde{\omega} \) of \( \Gamma(x^0) \) such that
\[
\overline{\omega} \cap \Omega \subset \omega \tag{55}
\]
and a vector field \( h \in (C^1(\overline{\Omega}))^n \) such that
\[
h = \nu \text{ on } \Gamma(x^0); \quad h \cdot \nu \geq 0 \text{ a.e. in } \Gamma, \text{ and } \tag{56}
\]
\[
h = 0 \text{ on } \Omega \setminus \tilde{\omega}, \tag{57}
\]
according to the figure 2.
Applying the identity in (43) with \( q = h \) it holds that
\[
\frac{1}{2} \int_0^T \int_{\Gamma(x^0)} \left( \frac{\partial \phi}{\partial \nu} \right)^2 \, d\Gamma \, dt
\leq \frac{1}{2} \int_0^T \int_{\Gamma(x^0)} (h \cdot \nu)^2 \, d\Gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma \setminus \Gamma(x^0)} (h \cdot \nu)^2 \, d\Gamma \, dt
\]
\[
= \left[ \int_\Omega \left( \frac{\left| \psi \right|^2}{2} + \phi'(h \cdot \nabla \phi) \right) dx \right]_0^T + \frac{1}{2} \int_0^T \int_{\Omega} (\text{div} \, h)(\psi')^2 - |\nabla \phi|^2 \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} a(x) \phi'(h \cdot \nabla \phi) \, dx \, dt - \int_0^T \int_{\Omega} |\psi|^2 (h \cdot \nabla \phi) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} a(x) \phi'(h \cdot \nabla \phi) \, dx \, dt + \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \, dt.
\]

In what follows we will estimate some terms on the RHS of (58).

**Estimate for** \( J_1 := \alpha \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{2}} b(x)\psi|^2 \, dx \, dt. \)

Analogously, from (42) we have
\[
|J_1| \leq \frac{\alpha}{k} [E(0) - E(T)].
\]

**Estimate for** \( J_2 := - \int_0^T \int_\omega |\psi|^2 (h \cdot \nabla \phi) \, dx \, dt. \)

Using Cauchy-Schwarz and Hölder inequalities, making use the inequality \( ab \leq \frac{1}{4\varepsilon} a^2 + \varepsilon b^2 \) and taking (7) and (42) into consideration, we can write
\[
|J_2| \leq \frac{\|h\|_{\infty}}{4\varepsilon} \beta_0 \frac{d}{k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) \, dt.
\]

**Estimate for** \( J_3 := \int_0^T \int_\omega a(x) \phi'(h \cdot \nabla \phi) \, dx \, dt. \)

From (7), (42) and analogously we have done above we obtain
\[
|J_3| \leq \frac{\|h\|^2_{\infty} \|a\|_{\infty}}{4\varepsilon} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) \, dt.
\]

**Estimate for** \( J_4 := \frac{1}{2} \int_0^T \int_\omega (\text{div} \, h)(\psi')^2 - |\nabla \phi|^2 \, dx \, dt. \)
From assumption 1, (7) and (42) we infer
\[ |J_4| \leq \frac{\|h\|_{W^{1,\infty}}} {2a_0k} [E(0) - E(T)] + \frac{1}{2} \|h\|_{W^{1,\infty}} \int_0^T \int_\omega |\nabla \phi|^2 \, dx \, dt. \] (62)

Estimate for \( J_5 := \int_0^T \int_\omega \partial \phi \cdot \frac{\partial h}{\partial x_j} \partial \phi \cdot \frac{\partial h}{\partial x_k} \, dx \, dt. \)

We have
\[ |J_5| \leq \|h\|_{W^{1,\infty}} \int_0^T \int_\omega |\nabla \phi|^2 \, dx \, dt. \] (63)

Denoting
\[ Y := \left[ \int_\Omega \left( \frac{|\psi|^2} {2} + \phi'(h \cdot \nabla \phi) \right) \, dx \right]_0^T \] (64)
and combining (58)-(64), we obtain
\[ \frac{R}{2} \int_0^T \int_{\Gamma(x_0')} \left( \frac{\partial \phi}{\partial \nu} \right)^2 \, d\Gamma \, dt \leq R|Y| + 4Re \int_0^T E(t) \, dt \]
\[ + RC_1 [E(0) - E(T)] + \frac{3R} {2} \|h\|_{W^{1,\infty}} \int_0^T \int_\omega |\nabla \phi|^2 \, dx \, dt, \]
where
\[ C_1 := \left[ \frac{\|h\|_{W^{1,\infty}}^2} {2a_0k} + \frac{|h|_{\infty}^2} {4\varepsilon b_0k} + \frac{\|a\|_{\infty}^2 \|h\|_{\infty}^2} {4\varepsilon k} + \frac{\alpha} {k} \right]. \]

Combining (54) and (65) and choosing \( \varepsilon = 1/32R \) we deduce
\[ \frac{1}{16} \int_0^T E(t) \, dt \leq |\chi| + R|Y| \]
\[ + (C_0 + RC_1)[E(0) - E(T)] + \frac{3R} {2} \|h\|_{W^{1,\infty}} \int_0^T \int_\omega |\nabla \phi|^2 \, dx \, dt \]
\[ + \int_0^T \int_\Omega |\psi|^2 \, dx \, dt. \] (66)

Then, we construct, as in Lemma 2.4 in Lions [29] a function \( \eta \in W^{1,\infty}(\Omega) \) satisfying
\[ 0 \leq \eta \leq 1 \text{ a.e. in } \Omega; \quad \eta = 1 \text{ a.e. in } \hat{\omega}, \]
\[ \eta = 0 \text{ a.e. in } \Omega \setminus \omega, \] (67)
\[ \frac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega). \] (68)

Taking \( \xi = \eta \) in the identity (45) it results that
\[ \left[ \int_\omega \phi \eta \left( \phi' + \frac{\phi a}{2} \right) \, dx \right]_0^T = \int_0^T \int_\omega |\psi|^2 \eta \phi \, dx \, dt \]
\[ + \int_\omega \int_\omega |\phi'|^2 - |\nabla \phi|^2 \, dx \, dt - \int_0^T \int_\omega \phi \nabla \phi \cdot \nabla \eta \, dx \, dt \] (70)

Next, let us analyze the terms on the RHS of (70).

Estimate for \( L_1 := \int_0^T \int_\omega |\psi|^2 \eta \phi \, dx \, dt. \)
Analogously to the above estimates and now considering the inequality $ab \leq \frac{d}{4\varepsilon b_0^4}k[E(0) - E(T)] + 2\varepsilon^2 \int_0^T E(t) \, dt$. \hfill (71)

Estimate for $L_2 = \int_0^T \int_\Omega \eta|\phi'|^2 \, dx \, dt$. From assumption 1, (7), (42) and (67) we arrive at
$$|L_2| \leq \frac{1}{a_0k}[E(0) - E(T)].$$ \hfill (72)

Estimate for $L_3 := -\int_0^T \int_\omega \phi(\nabla \phi \cdot \nabla \eta) \, dx \, dt$. From (67)-(69), we can write
$$|L_3| \leq \frac{1}{2} \int_0^T \int_\omega |\phi|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\omega \frac{|\nabla \eta|^2}{\eta} \, dx \, dt.$$

Defining
$$Z := \left[ \int_\omega \phi \eta \left( \phi' + \frac{\phi a}{2} \right) \, dx \right]_0^T$$

and combining (70)-(74) we obtain
$$\int_0^T \int_\omega |\phi|^2 \, dx \, dt \leq |Z| + C_2 [E(0) - E(T)] + 2\varepsilon^2 \int_0^T E(t) \, dt$$
$$+ \frac{1}{2} \int_0^T \int_\omega |\phi|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\omega \frac{|\nabla \eta|^2}{\eta} \, dx \, dt,$$ \hfill (75)

where
$$C_2 := \left[ \frac{d}{4\varepsilon b_0^4} + \frac{1}{a_0k} \right].$$

Combining (66) and (75), choosing $\varepsilon = 1/32\lambda^2$ and having in mind that
$$\int_0^T \int_\omega |\phi|^2 \, dx \, dt = \int_0^T \int_\Omega |\nabla \phi|^2 \, dx \, dt$$
we deduce
$$\frac{1}{16} \int_0^T E(t) \, dt \leq |\chi| + R|Y| + 3RC_2\|h\|_{W^{1,\infty}}|Z|$$
$$+ (C_0 + RC_1 + 3RC_2\|h\|_{W^{1,\infty}})[E(0) - E(T)]$$
$$+ \frac{3R}{2} \|h\|_{W^{1,\infty}} \left\| \frac{|\nabla \eta|^2}{\eta} \right\|_{L^{\infty}(\omega)} \int_0^T \int_\omega |\phi|^2 \, dx \, dt$$
$$+ \frac{1}{2} \int_0^T \int_\Omega |\psi|^2 \, dx \, dt.$$

On the other hand, from (47), (64) and (74), the following estimate holds
$$|\chi| + R|Y| + 3R\|h\|_{W^{1,\infty}}|Z| \leq C_3 [E(0) + E(T)]$$ \hfill (78)
where $C_3$ is a positive constant such that $C_3 = C_3(R, ||a||_\infty, \lambda, ||h||_\infty)$. Then, (76) and (78) yield

$$
\int_0^T E(t) \, dt \leq C [E(0) - E(T)] + C [E(0) + E(T)]
$$

(79)

$$
+ C \left[ \int_0^T \int_\Omega |\phi|^2 \, dx \, dt + \int_0^T \int_\Omega |\psi|^2 \, dx \, dt \right],
$$

where $C$ is a positive constant such that

$$
C = C(R, ||a||_\infty, ||h||_\infty, \lambda, ||h||_{W^{1, \infty}}, k, a_0, b_0, d).
$$

**Step 3.** Let $T_0 > 0$ considered sufficiently large for our purpose. We will prove the following lemma:

**Lemma 4.1.** For all $T > T_0$ there exists a positive constant $C = C(T)$ such that if $(\psi, \phi)$ is the regular solution of (1) with initial data $\{\psi_0, \phi_0, \phi_1\}$ we have

$$
\int_0^T \int_\Omega |\phi|^2 \, dx \, dt + \int_0^T \int_\Omega |\psi|^2 \, dx \, dt
$$

(80)

$$
\leq C(T) \left[ \int_0^T \int_\Omega a(x)|\phi'|^2 \, dx \, dt + \int_0^T \int_\Omega (|(-\Delta)^{1/4} b(x)\psi|^2 \, dx \, dt \right].
$$

**Proof.** We argue by contradiction. Let us suppose that (80) is not verified and let

\[
\psi_k(0), \phi_k(0), \phi'_k(0)
\]

be a sequence of initial data where the corresponding solutions $\{\psi_k, \phi_k\}$ with $E_k(0)$ uniformly bounded in $k$, verifies

$$
\lim_{k \to +\infty} \frac{\int_0^T \int_\Omega |\phi|^2 \, dx \, dt + \int_0^T \int_\Omega |\psi|^2 \, dx \, dt}{\int_0^T \int_\Omega a(x)|\phi'|^2 \, dx \, dt + \int_0^T \int_\Omega (|(-\Delta)^{1/4} b(x)\psi|^2 \, dx \, dt} = +\infty.
$$

(81)

Since $E_k(t)$ is non-increasing and $E_k(0)$ remains bounded then, we obtain a subsequence, still denoted by $\{\psi_k, \phi_k\}$ which verifies

$$
\psi_k \rightharpoonup \psi \text{ weak star in } L^\infty(0, T; L^2(\Omega)),
$$

(82)

$$
\phi_k \rightharpoonup \phi \text{ weak star in } L^\infty(0, T; H^1_0(\Omega)),
$$

(83)

$$
\phi'_k \rightharpoonup \phi' \text{ weak star in } L^\infty(0, T; L^2(\Omega)).
$$

(84)

We also have, employing compactness results (see Theorem 5.1 in Lions [29]) that

$$
\phi_k \to \phi \text{ strongly in } L^2(0, T; L^2(\Omega)).
$$

(85)

Now, from (81), (82) and (83) we deduce that

$$
\lim_{k \to +\infty} \int_0^T \int_\Omega a(x)|\phi'|^2 \, dx \, dt = 0,
$$

(86)

$$
\lim_{k \to +\infty} \int_0^T \int_\Omega (|(-\Delta)^{1/4} b(x)\psi|^2 \, dx \, dt = 0,
$$

(87)

On the other hand, from Assumption 1, namely, $b(x) \geq b_0 > 0$ in $\omega$, taking (87) into account and considering $D([(-\Delta)^{1/4} \equiv H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$ for $n = 2$, we deduce

$$
\lim_{k \to +\infty} \int_0^T \int_\omega |\psi_k|^4 \, dx \, dt = 0.
$$

(88)
From now on let us focus our attention on the coupled wave equation
\[ \phi''_k - \Delta \phi_k + a(x)\phi'_k = |\psi_k|^2 \chi_\omega \text{ in } \Omega \times (0, T) \quad (89) \]

Let us divide our proof in two cases (in what concerns the limit \( \phi \) above):

(a) \( \phi \neq 0 \).

Passing to the limit when \( k \to +\infty \) in (89) taking into account the above convergence, we deduce that
\[
\begin{cases}
\phi'' - \Delta \phi = 0 \text{ in } L^2(0, T; H^{-1}(\Omega)) \\
\phi = 0 \text{ on } L^2(0, T; H^{1/2}(\Gamma)) \\
\phi' = 0 \text{ a. e. in } \omega \times (0, T),
\end{cases}
\]
and for \( \phi' = v \), we obtain, in the distributional sense that
\[
\begin{cases}
v'' - \Delta v = 0 \text{ in } D'(\Omega \times (0, T)) \\
v = 0 \text{ on } \Gamma \times (0, T) \text{ in } H^{-1}(0, T; H^{1/2}(\Gamma)) \\
v = 0 \text{ a. e. in } \omega \times (0, T).
\end{cases}
\]

From Holmgren’s uniqueness theorem for the wave equation we conclude that \( v \equiv 0 \), that is, \( \phi' \equiv 0 \). Returning to (90) we obtain the following elliptic equation for a. e. \( t \in (0, T) \):
\[
\begin{cases}
-\Delta \phi = 0 \text{ in } \Omega \\
\phi = 0 \text{ on } \Gamma, \\
\phi' = 0 \text{ in } \omega,
\end{cases}
\]

Multiplying (92) by \( \phi \) we deduce that \( \int_\Omega |\nabla \phi|^2 \, dx = 0 \), which implies that \( \phi \equiv 0 \), which is a contradiction.

Now, we consider the other case when

(b) \( \phi \equiv 0 \).

Defining
\[
c_k := \left( \int_0^T \int_\Omega |\phi_k|^2 \, dx \, dt + \int_0^T \int_\Omega |\psi_k|^2 \, dx \, dt \right)^{1/2}
\]
we obtain
\[
\int_0^T \int_\Omega |\hat{\phi}_k|^2 \, dx \, dt + \int_0^T \int_\Omega |\hat{\psi}_k|^2 \, dx \, dt = 1.
\]

Besides,
\[
\hat{E}_k(t) = \frac{1}{2} \left( \int_\Omega |\hat{\psi}_k|^2 \, dx + \int_\Omega |\hat{\phi}_k'|^2 \, dx + \int_\Omega |\nabla \hat{\phi}_k|^2 \, dx \right)
= \frac{1}{2c_k^2} \left( \int_\Omega |\psi_k|^2 \, dx + \int_\Omega |\phi_k'|^2 \, dx + \int_\Omega |\nabla \phi_k|^2 \, dx \right),
\]
that is,
\[
\hat{E}_k(t) = \frac{E_k(t)}{c_k^2}.
\]

(96)
On the other hand, integrating (40) over $(0, T)$, we deduce

$$E_k(T) = E_k(0) - \alpha \int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt$$  \hspace{1cm} (97)

\[ - \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt + \int_0^T \int_\omega |\psi_k|^2 \phi_k' dx dt. \]

From the fact that $E_k(t) \geq E_k(T)$ for all $t \in [0, T]$ and taking (97) into account, we obtain

$$\int_0^T E_k(t) \, dt \geq T E_k(T)$$  \hspace{1cm} (98)

$$= T E_k(0) - \alpha T \int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt$$

\[ - T \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt + T \int_0^T \int_\omega |\psi_k|^2 \phi_k' dx dt. \]

Combining (79), (98) and making use of Cauchy-Schwarz inequality and considering the Assumption 1, we infer

$$T E_k(0) \leq [2(\alpha + a \omega^{-1} b_0^4 d + 3)] T \left\{ \int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt \right. \right.$$  

$$\left. + \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt \right\}$$

$$+ C E_k(0) + C \int_0^T \int_\Omega |\phi_k|^2 dx dt + C \int_0^T \int_\Omega |\psi_k|^2 dx dt.$$  

The last inequality yields for a large $T$,

$$E_k(0) \leq C(T, a_0, b_0, \alpha, d) \left\{ \int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt \right.$$  

$$\left. + \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt + \int_0^T \int_\Omega |\phi_k|^2 dx dt + \int_0^T \int_\Omega |\psi_k|^2 dx dt \right\}$$  \hspace{1cm} (99)

Having in mind that $E_k(t) \leq E_k(0)$ for all $t \in [0, T]$, applying inequality (99) and dividing both sides by $\int_0^T \int_\Omega |\phi_k|^2 dx dt + \int_0^T \int_\Omega |\psi_k|^2 dx dt$ it holds that

$$\frac{E_k(t)}{\int_0^T \int_\Omega |\phi_k|^2 dx dt + \int_0^T \int_\Omega |\psi_k|^2 dx dt} \leq C(T, a_0, b_0, \alpha, d) \left\{ \frac{\int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt + \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt}{\int_0^T \int_\Omega |\phi_k|^2 dx dt + \int_0^T \int_\Omega |\psi_k|^2 dx dt} \right\} + 1$$  \hspace{1cm} (100)

Since in view of (81) we have

$$\lim_{k \to +\infty} \frac{\int_0^T \int_\Omega \left| (-\Delta)^{\frac{1}{2}} b(x) \psi_k \right|^2 dx dt + \int_0^T \int_\Omega a(x) |\phi_k'|^2 dx dt}{\int_0^T \int_\Omega |\phi_k|^2 dx dt + \int_0^T \int_\Omega |\psi_k|^2 dx dt} = 0,$$  \hspace{1cm} (101)
then, from (100) there exists $M > 0$ such that
\[ E_k(t) \leq C(T, a_0, b_0, \alpha, d)(M + 1), \] for all $t \in [0, T]$ and for all $k \in \mathbb{N}$.

Consequently, from (96) it results that
\[ \hat{E}_k(t) \leq C(T, a_0, b_0, \alpha, d)(M + 1), \] for all $t \in [0, T]$ and for all $k \in \mathbb{N}$ (102)

Then, in particular, from (101) we deduce
\[
\lim_{k \to +\infty} \int_0^T \int_{\Omega} a(x)|\hat{\phi}'_k|^2 \, dx \, dt = \lim_{k \to +\infty} \int_0^T \int_{\Omega} a(x)|\phi'_k|^2 \, dx \, dt + \int_0^T \int_{\Omega} |\psi_k|^2 \, dx \, dt
\]
\[ = 0, \] (103)

and
\[
\lim_{k \to +\infty} \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{4}} b(x)\hat{\psi}_k|^2 \, dx \, dt
\]
\[ = \lim_{k \to +\infty} \int_0^T \int_{\Omega} |(-\Delta)^{\frac{1}{4}} b(x)\psi_k|^2 \, dx \, dt + \int_0^T \int_{\Omega} |\psi_k|^2 \, dx \, dt
\]
\[ = 0, \] (104)

and from (102), for a subsequence \{\psi_k, \phi_k\}, we obtain
\[ \hat{\psi}_k \rightharpoonup \hat{\psi} \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \] (105)
\[ \hat{\phi}_k \rightharpoonup \hat{\phi} \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \] (106)
\[ \hat{\phi}'_k \rightharpoonup \hat{\phi}' \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \] (107)
\[ \hat{\phi}_k \to \hat{\phi} \text{ strongly in } L^2(0, T; L^2(\Omega)). \] (108)

In addition, \( \hat{\phi}_k \) satisfies the equation
\[
\begin{cases}
\hat{\phi}'' - \Delta \hat{\phi}_k + a(x)\hat{\phi}'_k = \frac{|\psi_k|^2}{c_k} \text{ in } \Omega \times (0, T) \\
\hat{\phi}_k = 0 \text{ on } \Gamma \times (0, T) \\
\hat{\phi}'_k = 0 \text{ a. e. in } \omega \times (0, T).
\end{cases}
\] (109)

Passing to the limit when $k \to +\infty$ taking the above convergences and (104) into account, we get
\[
\begin{cases}
\hat{v}'' - \Delta \hat{v} = 0 \text{ in } \Omega \times (0, T) \\
\hat{v} = 0 \text{ on } \Gamma \times (0, T) \\
\hat{v}' = 0 \text{ a. e. in } \omega \times (0, T).
\end{cases}
\] (110)

Then, \( v = \hat{v}' \) verifies, in the distributional sense
\[
\begin{cases}
\psi'' - \Delta \psi = 0 \text{ in } D'((0, T)) \\
v = 0 \text{ on } \Gamma \times (0, T) \text{ (in a weak sense)} \\
v = 0 \text{ a. e. in } \omega \times (0, T).
\end{cases}
\] (111)

Applying Holmgren’s uniqueness theorem it comes that $v = \hat{v}' = 0$. Returning to (110) we obtain, for a. e. $t \in (0, T)$ that
\[
\begin{cases}
-\Delta \hat{v} = 0 \text{ in } \Omega \\
\hat{v} = 0 \text{ on } \Gamma \\
\hat{v}' = 0 \text{ in } \omega.
\end{cases}
\] (112)
Multiplying the above equation by \( \dot{\phi} \), we deduce

\[
0 = - \int_{\Omega} \Delta \dot{\phi} \dot{\phi} \, dx = \int_{\Omega} |\nabla \dot{\phi}|^2 \, dx,
\]

that is, \( \dot{\phi} = 0 \).

Moreover, \( \dot{\psi}_k \) satisfies the equation

\[
\begin{aligned}
& i \dot{\psi}_k + \Delta \dot{\psi}_k + ib(x)(-\Delta)^{1/2} b(x) \dot{\psi}_k = \dot{\phi}_k \psi_k \chi_\omega \quad \text{in } \Omega \times (0, T) \\
& \dot{\psi}_k = 0 \quad \text{on } \Gamma \times (0, T) \\
& \dot{\psi}_k(0) = \dot{\psi}_k^0 \quad \text{in } \Omega.
\end{aligned}
\]  

(113)

Now, we will use the effect smoothing effect due to Aloui given in Theorem 2.3. Indeed, since \( \dot{\psi}_k \) satisfies (113), we have that \( (\dot{\psi}_k) \) satisfies the integral equation

\[
\dot{\psi}_k = S(t)\dot{\psi}_k(0) + \int_0^T S(T - \tau)F(\dot{\psi}_k)(\tau) \, d\tau,
\]

where \( S(t) \) is the semigroup generated by

\[
A : D(A) = H^1_0(\Omega) \cap H^2(\Omega) \to L^2(\Omega)
\]

\( y \mapsto Ay := i\Delta y - b(x)(-\Delta)^{1/2} b(x)y, \)

and \( F(\dot{\psi}_k) := \dot{\phi}_k \psi_k \chi_\omega \). We observe that

\[
\begin{aligned}
\int_0^T \int_\Omega |F(\dot{\psi}_k)|^2 \, dx \, dt &\leq \int_0^T \left( \int_\Omega |\dot{\phi}_k|^4 \, dx \right)^{1/2} \left( \int_\Omega |\dot{\psi}_k|^4 \, dx \right)^{1/2} \, dt \\
&= \int_0^T \|\dot{\phi}_k(t)\|_{L^4(\omega)}^2 \|\dot{\psi}_k(t)\|_{L^4(\omega)}^2 \, dt \\
&\leq C\|\dot{\phi}_k\|_{L^\infty(0,T;H^1_0(\Omega))^2} \int_0^T \|\dot{\psi}_k(t)\|^2_{L^4(\omega)} \, dt.
\end{aligned}
\]

(115)

Thus, from (88), (106) and (115), we have \( F(\dot{\psi}_k) \to 0 \) strongly in \( L^2(0, T; L^2(\Omega)) \). Employing the smoothing effect given in (10), we infer

\[
\|\omega_k\|_{L^2(0,T;H^1_0(\Omega))} \leq C\|F(\dot{\psi}_k)\|_{L^2(0,T;L^2(\Omega))},
\]

(116)

where

\[
\omega_k = \int_0^t S(t - \tau)F(\dot{\psi}_k)(\tau) \, d\tau.
\]

Moreover, since \( \|\dot{\psi}_k(0)\| \leq C(M + 1) \) and having in mind (116), we obtain the following estimmative:

\[
\|\dot{\psi}_k\|_{L^2(0,T;H^1_0(\Omega))} \leq C(\|\dot{\psi}_k(0)\|_{L^2(\Omega)} + \|F(\dot{\psi}_k)\|_{L^2(0,T;L^2(\Omega))}) \\
\leq C(\|\dot{\psi}_k(0)\|_{L^2(\Omega)} + \|\dot{\phi}_k\psi_k \chi_\omega\|_{L^2(0,T;L^2(\Omega))}) \\
\leq \hat{C}.
\]

This allows us to say that

\[
\{\dot{\psi}_k\} \text{ is bounded in } L^2(0, T; H^1_0(\Omega)).
\]

(117)

Recalling (113), we note that for a.e \( t \in (0, T) \)

\[
\dot{\psi}_k^\prime = i\Delta \dot{\psi}_k + ib(x)(-\Delta)^{1/2} b(x) \dot{\psi}_k - i\dot{\phi}_k \psi_k \chi_\omega \in H^{-1}(\Omega).
\]

(118)
Therefore,\[ \{ \hat{\psi}_k' \} \text{ is bounded in } L^2(0,T; H^{-1}(\Omega)). \quad (119) \]

Combining (117), (119) and the embedding chain \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \), due to Aubin-Lions Theorem (see Theorem 5.1 in Lions [29]), we have,
\[ \hat{\psi}_k \to \hat{\psi} \text{ strongly in } L^2(0,T; L^2(\Omega)). \quad (120) \]

Moreover, from (104) and (120) we get,
\[ \hat{\psi}_k \to \tilde{\psi} \text{ strongly in } L^2(0,T; L^2(\Omega)), \quad (121) \]

where
\[ \tilde{\psi} = \begin{cases} \hat{\psi} & \text{in } \Omega \setminus \omega \\ 0 & \text{in } \omega. \end{cases} \quad (122) \]

Note that if \( \hat{\psi} = 0 \), so from (108), (121), (95) and observing that \( \hat{\phi} = 0 \) we have a contradiction.

On the other hand, if \( \hat{\psi} \neq 0 \), by passing limit on (113), we have, thanks to (115) and (104), that
\[ \begin{cases} i \hat{\psi}' + \Delta \hat{\psi} = 0 \text{ in } \Omega \times (0,T) \\ \hat{\psi} = 0 \text{ on } \Gamma \times (0,T) \\ \hat{\psi} = 0 \text{ a.e. in } \omega \times (0,T). \end{cases} \quad (123) \]

from Holmgren’s uniqueness theorem we conclude that \( \hat{\psi} = 0 \text{ a.e. in } \Omega \). Therefore we have \( \hat{\psi} = 0 \) and \( \hat{\phi} = 0 \), which is a contradiction by (108), (121), and (95). So, Lemma 3.1 is proved.

Combining (79) and (80) it holds that
\[ \int_0^T E(t) \, dt \leq C_1 E(0) \quad (124) \]

where \( C_1 = C(R, ||a||_\infty, ||b||_\infty, ||h||_\infty, \lambda, ||h||_{W^{1,\infty}}, k, n, \alpha, \beta, a_0, b_0, d) \) and \( C_2 = C_2(T) \).

Take \( T_0 > 0 \) large enough. From (124) we have we can write
\[ \int_0^{T_0} E(t) \, dt \leq C_1 E(0) + C_2 \int_0^{T_0} D(t) \, dt, \]
from which we conclude, once \( E(t) \) is non-increasing that
\[ T_0 E(T_0) \leq C_1 E(0) + C_2 \int_0^{T_0} D(t) \, dt, \quad (125) \]

where \( C_1 \) does not depend on \( T_0 \) but \( C_2 = C_2(T_0) \).

On the other hand, from the inequality of the energy (42) yields
\[ E(T_0) - E(0) \leq -k \int_0^{T_0} D(t) \, dt, \]
from which follows that
\[ C_2 \int_0^{T_0} D(t) \, dt \leq \frac{C_2}{k} E(T_0) + \frac{C_2}{k} E(0). \quad (126) \]
Combining (125) and (126) we deduce
\[
\left(T_0 + \frac{C_2}{k}\right) E(T_0) \leq \left(C_1 + \frac{C_2}{k}\right) E(0),
\]
which implies, choosing \(T_0\) sufficiently large that
\[
E(T_0) \leq \alpha E(0), \quad \text{with } 0 < \alpha < 1.
\]
So, taking \(T_0\) large enough for \(T > T_0\) we obtain
\[
E(T) \leq E(T_0) \leq \alpha E(0).
\]
Thus,
\[
E(T) \leq \alpha E(0), \quad \text{for all } T > T_0,
\]
where, \(\alpha < 1\).

Proceeding in a similar way to that done previously from \(T\) to \(2T\) and we deduce as before, \(E(2T) \leq \alpha E(T)\), for all \(T > T_0\), and, consequently,
\[
E(2T) \leq \alpha^2 E(0), \quad \text{for all } T > T_0.
\]
In general,
\[
E(nT) \leq \alpha^n E(0), \quad \text{for all } T > T_0.
\]
Let us consider, now, \(t > T_0\), then \(t = nT_0 + r\) for \(0 \leq r < T_0\). Thus,
\[
E(t) \leq E(t-r) = E(nT_0) \leq \alpha^n E(0) = \alpha^\frac{r}{T_0} E(0) = e^{\frac{r}{T_0} \ln \alpha} E(0),
\]
which implies the exponential stability.

REFERENCES

[1] L. Aloui, Smoothing effect for regularized Schrödinger equation on bounded domains, Asymptot. Anal., 59 (2008), 179–193.
[2] L. Aloui, Smoothing effect for regularized Schrödinger equation on compact manifolds, Collect. Math., 59 (2008), 53–62.
[3] A. Bachelot, Problème de Cauchy pour des systèmes hyperboliques semi-linéaires, Ann. Inst. H. Poincaré Anal. non Linéaire, 1 (1984), 453–478.
[4] J. B. Baillon and J. M. Chadam, The Cauchy problem for the coupled Klein-Gordon-Schrödinger equations, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations, North Holland, Amsterdam, (1978), 37–44.
[5] C. Banquet, L. C. F. Ferreira and E. J. Villamizar-Roa, On existence and scattering theory for the Klein-Gordon-Schrödinger system in an infinite L2-norm setting, Ann. Mat. Pura Appl., 194 (2015), 781–804.
[6] P. Biler, Attractors for the system of Schrödinger and Klein-Gordon equations with Yukawa coupling, SIAM J. Math. Anal., 21 (1990), 1190–1212.
[7] V. Bisognin, M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Soriano, Uniform decay for the Klein-Gordon-Schrödinger equations with locally distributed damping, NoDEA, Nonlinear Differ. Equ. Appl., 15 (2008), 91–113.
[8] C. A. Bortot and M. M. Cavalcanti, Asymptotic stability for the damped Schrödinger equation on noncompact Riemannian manifolds and exterior domains, Comm. Partial Differential Equations, 39 (2014), 1791–1820.
[9] C. A. Bortot and W. J. Corrêa, Exponential stability for the defocusing semilinear Schrödinger equation with locally distributed damping on a bounded domain, Differential and Integral Equations, 31 (2018), 273–300.
[10] N. Burq, P. Gérard and N. Tzvetkov, On nonlinear Schrödinger equations in exterior domains, [Equations de Schrödinger non linéaires dans des domaines extérieurs,] Annales de l’Institut Henri Poincaré. Analyse Non Linéaire, 21 (2004), 295–318.
[11] M. Cavalcanti and V. Domingos Cavalcanti, Global existence and uniform decay for the coupled Klein-Gordon-Schrödinger equations, NoDEA, Nonlinear differ. equ. appl., 7 (2000), 285–307.
[12] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, F. Natali, Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: exponential and polynomial stabilization, *J. Differential Equations*, 248 (2010), 2955–2971.

[13] J. Colliander, J. Holmer, N. Tzirakis, Low regularity global well-posedness for the Zakharov and Klein- Gordon-Schrödinger systems, *Trans. Amer. Soc. Math.*, 360 (2008), 4619–4638.

[14] P. Courtin and J. C. Saut, Local smoothing properties of dispersive equation, *Journal of the American Mathematical Society*, 1, 413–439.

[15] Z. Dai and P. Gao, Exponential attractor for dissipative Klein-Gordon-Schrödinger equations in $\mathbb{R}^3$, *Chim. Ann. Math. Ser. A.*, 21 (2000), 241–250.

[16] I. Fukuda and M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations I, *Bull. Sci. Engrg. Res. Lab. Waseda Univ.*, 69 (1975), 51–62.

[17] I. Fukuda and M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations II, *J. Math. Analysis Applic.*, 66 (1978), 358–378.

[18] I. Fukuda and M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations III - Higher order interaction, decay and blow-up, *Math. Japonica*, 24 (1979), 307–321.

[19] I. Fukuda and M. Tsutsumi, On the Yukawa-coupled Klein-Gordon-Schrödinger equations in three space dimensions, *Proc. Japan Acad.*, 51 (1975), 402–405.

[20] B. Guo and Y. Li, Attractor for dissipative Klein-Gordon-Schrödinger equations, *J. Differential Equations*, 136 (1997), 356–377.

[21] B. Guo and Y. Li, Attractor for the dissipative generalized Klein-Gordon-Schrödinger equations, *J. Partial Differ. Equations*, 11 (1998), 260–272.

[22] B. Guo and Y. Li, Asymptotic smoothing effect of solutions to weakly dissipative Klein-Gordon-Schrödinger equations, *J. Math. Anal. Appl.*, 282 (2003), 256–265.

[23] Y. Han, On the Cauchy problem for the coupled Klein-Gordon-Schrödinger system with rough data, *Discret. Contin. Dyn. Syst.*, 12 (2005), 233–242.

[24] N. Hayashi, Global strong solutions of coupled Klein-Gordon-Schrödinger equations, *Funkcialaj Ekvacioj*, 29 (1986), 299–307.

[25] N. Hayashi and W. Von Wahl, On the global strong solutions of coupled Klein-Gordon-Schrödinger equations, *J. Math. Soc. Japan*, 39 (1987), 489–497.

[26] H. Lange and B. Wang, Regularity of the global attractor for the Klein-Gordon-Schrödinger equation, *Math. Methods Appl. Sci.*, 22 (1999), 1535–1554.

[27] H. Lange and B. Wang, Attractors for the Klein-Gordon-Schrödinger equation, *J. Math. Phys.*, 40 (1999), 2445–2457.

[28] Y. Li, Q. Shi, C. Wang and S. Wang, Well-posedness for the nonlinear Klein-Gordon-Schrödinger equations with heterointeractions, *J. Math. Phys.*, 51 (2010), 032–102.

[29] J. L. Lions, Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires, Dunod, Paris, 1969.

[30] J. L. Lions, *Controlabilité exacte, perturbations et Stabilisation de systèmes distribués*, Tome 1, Masson, Paris, 1988.

[31] J.L. Lions-E.Magenes, *Problèmes Aux Limites Non Homogènes et Applications*, Dunod, Paris, 1968, Vol. 1.

[32] C. Miao and G. Xu, Global solutions of the Klein-Gordon-Schrödinger system with rough data in $\mathbb{R}^{2+1}$, *J. Differ. Equ.*, 227 (2006), 365–405.

[33] O. Goubet, A. Hakim and A. Mostafa, Regularity of the global attractor for a coupled Klein-Gordon-Schrödinger system, *Diff. Integral Equ.*, 16 (2003), 573–581.

[34] M. Ohta, Stability of Stationary States for the Coupled Klein-Gordon-Schrödinger Equations, *Nonlinear Analysis, Theory, Methods and Appl.*, 27 (1996), 455–461.

[35] T. Ozawa and Y. Tsutsumi, Asymptotic behaviour of solutions for the coupled Klein-Gordon-Schrödinger equations, *Adv. Stud. Pure Math.*, 23 (1994), 295–305.

[36] H. Pecher, Global solutions of the Klein-Gordon-Schrödinger system with rough data, *Differ. Int. Equ.*, 17 (2004), 179–214.

[37] M. N. Poulou and N. M. Stavrakakis, Global attractor for a system of Klein-Gordon-Schrödinger type in all $\mathbb{R}$, *Nonlinear Anal.*, 74 (2011), 2548–2562.

[38] M. N. Poulou and N. M. Stavrakakis, Uniform decay for a local dissipative Klein-Gordon-Schrödinger type system, *Electron. J. Differential Equations*, (2012), No. 179. 16pp.

[39] A. Shimomura, Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions, *J. Math. Sci. Univ. Tokyo*, 10 (2003), 661–685.

[40] A. Shimomura, Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions. II, *Hokkaido Math. J.*, 34 (2005), 405–433.
[41] N. Tzirakis, The Cauchy problem for the Klein-Gordon-Schrödinger system in low dimensions below the energy space, *Comm. Partial Differ. Equ.*, **30** (2005), 605–641.

[42] B. Wang, Classical global solutions for non-linear Klein-Gordon-Schrödinger equations, *Math. Methods Appl. Sci.*, **20** (1997), 599–616.

[43] H. Yukawa, On the interaction of elementary particles I, *Proc. Physico-Math. Soc. Japan*, **17** (1935), 48–57.

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