SPECIAL VALUES OF ANTICYCLOMOTIC RANKIN-SELBERG $L$-FUNCTIONS

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Abstract. In this article, we prove an explicit Waldspurger formula for the toric Hilbert modular forms. As an application, we construct a class of anticyclotomic $p$-adic Rankin-Selberg $L$-functions for Hilbert modular forms, generalizing the construction of Bertolini, Darmon, and Prasanna in the elliptic case. Moreover, building on works of Hida, we give a necessary and sufficient condition when the Iwasawa $\mu$-invariant of this $p$-adic $L$-function vanishes and prove a result on the non-vanishing modulo $p$ of central Rankin-Selberg $L$-values with anticyclotomic twists.

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Introduction

The purpose of this article is to (i) construct a class of anticyclotomic Rankin-Selberg $p$-adic $L$-functions and study the vanishing/non-vanishing of the associated Iwasawa $\mu$-invariant, (ii) prove a result on the non-vanishing modulo $p$ of central Rankin-Selberg $L$-values with anticyclotomic twists. Let $F$ be a totally real algebraic number field of degree $d$ over $\mathbb{Q}$ and $E$ be a totally imaginary quadratic extension of $F$. Denote by $z \mapsto \mathfrak{p}$ the nontrivial element in $\text{Gal}(E/F)$. Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ with unitary central character $\omega$. Let $\pi_E$ be a lifting of $\pi$ to $E$ constructed in [Jac72, Thm. 20.6]. Then $\pi_E$ is an irreducible automorphic representation of $\text{GL}_2(\mathbb{A}_E)$, which is cuspidal if $\pi$ is not obtained from the automorphic induction from $E/F$. Let $\lambda : \mathbb{A}_E^\times \rightarrow C^\times$ be a unitary Hecke character of $E^\times$ such that $L(1/2, \pi_E \otimes \lambda) = \omega^{-1}$.

The automorphic representation $\pi_E \otimes \lambda$ is therefore conjugate self-dual, i.e. $\pi_E^\vee \otimes \lambda^{-1} = \pi_E \otimes \lambda$. For each place $v$ of $F$, we can associate a local $L$-function $L(s, \pi_{E_v} \otimes \lambda_v)$ and a local epsilon factor $\varepsilon(s, \pi_{E_v} \otimes \lambda_v, \psi_v)$ (which depends on a choice of non-trivial character $\psi_v : F_v \rightarrow \mathbb{C}^\times$) to the local constituent $\pi_{E_v} \otimes \lambda_v$ of $\pi_E \otimes \lambda$ ([JL70, Thm. 2.18 (iv)]). Let $L(s, \pi_E \otimes \lambda)$ be the global $L$-function obtained by the meromorphic continuation of the Euler product of local $L$-functions at all finite places. In this paper, we study the $p$-adic variation of the central value $L(1/2, \pi_E \otimes \lambda)$ with anticyclotomic twists under certain hypotheses.

To introduce our hypotheses precisely, we need some notation. Fix a CM-type $\Sigma$ of $E$. Namely, $\Sigma$ is a subset of $\text{Hom}(E, \mathbb{C})$ such that $\Sigma \sqcup \Sigma = \text{Hom}(E, \mathbb{C})$; $\Sigma \cap \Sigma = \emptyset$.

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Then $\Sigma$ induces an identification $E \otimes \Q \bigotimes_{\Q} R \simeq C^\Sigma$. We shall identify $\Sigma$ with the set of archimedean places of $F$ via the restriction. For each $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in Z[\Sigma]$, we write $\Gamma_\Sigma(k) = \prod_{\sigma \in \Sigma} \Gamma(k_\sigma)$, and if $x = (x_\sigma) \in (A^\times)^\Sigma$ for an algebra $A$, we let $x^k := \prod_{\sigma \in \Sigma} x_\sigma^k$. A Hecke character $\chi$ of $E^{\times}$ is of infinity type $(k_1, k_2)$ for $k_1, k_2 \in 2^{-1}Z[\Sigma]$ such that $k_1 - k_2 \in Z[\Sigma]$ if

$$\chi(z) = z^{k_1 - k_2} (z\bar{z})^{k_2} \text{ for all } z \in (E \otimes \Q R)^{\times} \simeq (C^\times)^\Sigma.$$

For each ideal $\mathfrak{a}$ of $F$ (resp. ideal $\mathfrak{A}$ of $E$), we have a unique factorization $a = a^+ a^-$ (resp. $\mathfrak{A} = \mathfrak{A}^+ \mathfrak{A}^-$), where $a^+ \mathfrak{A}^+$ is only divisible by primes split in $E$ and $a^- \mathfrak{A}^-$ is divisible by primes inert or ramified in $E$. Let $n = n^+ n^-$ be the conductor of $\pi$. We define the normalized local root number attached to $\pi_{E, v}$ and $\lambda_v$ for each place $v$ by

$$\epsilon^*(\pi_{E, v}, \lambda_v) := \epsilon(\frac{1}{2}, \pi_{E, v} \otimes \lambda_v, \psi_v) \cdot \omega_v(-1).$$

We remark that the value $\epsilon(\frac{1}{2}, \pi_{E, v} \otimes \lambda_v, \psi_v)$ does not depend on the choice of $\psi_v$.

We assume that $\pi$ has infinity type $k = \sum_{\sigma} k_\sigma \sigma \in Z_{>0}[\Sigma]$ and $\lambda$ has infinity type $(\frac{k_1}{2}, -\frac{k_2}{2})$. In other words, $\pi_\sigma$ is a discrete series or limit of discrete series of weight $k_\sigma$ with unitary central character at every archimedean place $\sigma$. In particular, this implies $\{k_\sigma\}_{\sigma \in \Sigma}$ have the same parity and the local root number $\epsilon^*(\pi_{E, v}, \lambda_v) = +1$ at every archimedean place. We further assume the following local root number hypothesis for $(\pi, \lambda)$:

**Hypothesis A.** The local root number $\epsilon^*(\pi_{E, v}, \lambda_v) = +1$ for each $v | n^-.$

Let $p$ be an odd rational prime. Fix an embedding $\iota\colon Q \hookrightarrow C$ and isomorphism $\iota: C \simeq \C_p$ once and for all. Each $\xi \in Z[\Sigma \cup \Sigma]$ will be regarded as an algebraic character $\xi: (E \otimes Q_p)^{\times} \to C_p^{\times}$ via $\iota$. Throughout this article, we make the following assumption

**(ord)** $\Sigma$ is $p$-ordinary.

Let $\Sigma_p$ be the set of $p$-adic places of $E$ induced by $\Sigma$. Then the ordinary assumption $(\text{ord})$ means that $\Sigma_p$ and its complex conjugation $\Sigma_p^\sigma$ gives full partition of the set of $p$-adic places of $E$. Let $E_{F_{\infty}}$ be the maximal anticyclotomic $Z_p[F = \Q]$-extension of $E$. Let $\Gamma^- = \text{Gal}(E_{F_{\infty}} \otimes E)$ and let $\lambda = \Z_p[\Gamma^-]$ be the Iwasawa algebra of $[F : Q]$-variables. If $L$ is a number field, we write $G_L = \text{Gal}(Q/L)$ for the absolute Galois group. Denote by $\text{rec}_E : A^\times E/F \to G_L^{ab}$ the geometrically normalized reciprocity law. To each locally algebraic $p$-adic character $\phi^- : \Gamma^- \to C_p^{\times}$ of weight $(m, -m)$, we can associate a Hecke character $\phi : A^\times E/F \to C^{\times}$ of infinity type $(m, -m)$ defined by

$$\phi(a) := \epsilon^{-1}(\overline{\phi}(\text{rec}_E(a) a_p^{-m} a_{\infty}^{-m} a_{\infty}^{-m} m),$$

where $a_p \in (E \otimes Q_p)^{\times}$ and $a_{\infty} \in (E \otimes Q \R)^{\times}$ are the $p$-component and $\infty$-component of $a$. We say $\overline{\phi}$ is the $p$-adic avatar of $\phi$. Denote by $X_p^{\text{crit}}$ the set of critical specializations, consisting of locally algebraic $p$-adic characters on $\Gamma^-$ of weight $(m, -m)$ with $m \in Z_{\geq 0}[\Sigma]$ (See [5.3]).

Our first result is the construction of the anticyclotomic primitive $p$-adic $L$-function attached to $\pi_{E, \lambda}$ and $\lambda$. We need more notation. Let $D_F$ be the different of $F$ and $D_{E/F}$ be the relative different of $E/F$. Let $\mathfrak{m}$ be the prime-to-$p$ conductor of $\pi_{E, \lambda}$ and $\lambda$. We have a unique decomposition $\mathfrak{m} = \mathfrak{m}^+ \mathfrak{m}^-$ and fix a decomposition $\mathfrak{m}^+ = \mathfrak{m}^+ \mathfrak{m}^-$. We choose $\delta \in E$ such that

- $\delta = -\delta$,
- $\Im \sigma(\delta) > 0$ for all $\sigma \in \Sigma$,
- The polarization ideal $\epsilon(O_E) := D_F^{-1}(2\delta D_{E/F})$ is prime to $p\mathfrak{m}$.\mathfrak{m}$.

Let $(\Omega_{\pi}, \Omega_p)$ be the complex and $p$-adic periods attached to $(E, \Sigma)$ defined in [HT93] (4.4a), (4.4b)]. For each Hecke character $\chi$ of $E^{\times}$, we define the $p$-adic multiplier $E_{\Sigma_p}(\pi, \chi)$ by

$$E_{\Sigma_p}(\pi, \chi) := \prod_{\mathfrak{p} \subseteq \Sigma_p, \mathfrak{p} = \mathfrak{p} \mathfrak{m}^{-1}} \epsilon(\frac{1}{2}, \pi_p \otimes \chi_{\mathfrak{p}}, \psi_p)L(\frac{1}{2}, \pi_p \otimes \chi_{\mathfrak{p}})^{-2}\omega_p^{-1}\lambda_p^{-2}(-2\delta).$$

The shape of $E_{\Sigma_p}(\pi, \chi)$ has been suggested by Coates [Coa91].
Theorem A. In addition to \( \text{ord} \) and Hypothesis \( [A] \) we further assume that
\[
\text{ord} \text{ is square-free}.
\]
Then there exists an element \( \mathcal{P}_\Sigma(\pi, \lambda) \in \Lambda \) such that for every \( \hat{\phi} \in \mathcal{X}_p^{\text{crit}} \) of weight \( (m, -m) \), we have
\[
\frac{\hat{\omega}(\mathcal{P}_\Sigma(\pi, \lambda)^2)}{\Omega_p^{2(k+2m)}} = [\mathcal{O}_E^\times : \mathcal{O}_E^\times]^2 \cdot \frac{\Gamma(k+m+1)}{(4\pi)^{k+2m+1}} \cdot \mathcal{E}_\Sigma(\pi, \lambda, \phi) \cdot \frac{L(\frac{1}{2}, \pi_E \otimes \lambda \phi)}{\Omega_E^{2(k+2m)}} \cdot \phi(\hat{\phi}) \mathcal{C}(\pi, \lambda),
\]
where \( \Omega_E = (2\pi i)^{-1}\Omega_\infty \) and \( \mathcal{C}(\pi, \lambda) \in \mathbb{Z}_p^\times \) is an explicit \( p \)-adic unit independent of \( \phi \), consisting of a product of local epsilon factors outside \( p \).

In the above expression, \( \text{Im} \delta = (\text{Im} \sigma(\delta))_{\sigma \in \Sigma} + 4\pi \) is considered to be the diagonal element \((4\pi)_{\sigma \in \Sigma} \) in \((\mathbb{C}^\times)^2\).

In virtue of the above specialization formula, \( \mathcal{P}_\Sigma(\pi, \lambda) \) is the \( p \)-adic \( L \)-function that interpolates the square root of central \( L \)-values. We shall call \( L(\pi, \lambda) := \mathcal{P}_\Sigma(\pi, \lambda)^2 \) the anticyclotomic \( p \)-adic \( L \)-function for \( \pi_E \otimes \lambda \) with respect to the \( p \)-adic CM type \( \Sigma \). When \( F = \mathbb{Q} \), \( \pi \) is unramified at \( p \) and \( n^\perp \) is only divisible by primes ramified in \( E \), \( \mathcal{L}_\Sigma(\pi, \lambda) \) is constructed in \([BDP]\), and a non-primitive version of this \( p \)-adic \( L \)-function is also considered in \([Bra11a]\) under different hypotheses. We remark that when \( \pi \) is obtained from the automorphic induction of a Hecke character of \( E^\times \), one can show that, up to an explicit unit in \( \Lambda \), \( \mathcal{L}_\Sigma(\pi, \lambda) \) is a product of two \( p \)-adic Hecke \( L \)-functions for CM fields constructed by Katz \([Kat78]\) and Hida-Tilouine \([HT93]\) by comparing the interpolation formulas on both sides.

Our second theorem is to prove the vanishing of the Iwasawa \( \mu \)-invariant \( \mu_{\pi, \lambda, \Sigma} \) of \( \mathcal{P}_\Sigma(\pi, \lambda) \) under certain hypothesis. This in particular implies that the \( p \)-adic \( L \)-function \( \mathcal{L}_\Sigma(\pi, \lambda) \) is non-trivial. Recall that the \( \mu \)-invariant \( \mu_{\pi, \lambda, \Sigma} \) is defined by
\[
\mu_{\pi, \lambda, \Sigma} = \inf \{ r \in \mathbb{Q}_{\geq 0} \mid p^{-r} \mathcal{P}_\Sigma(\pi, \lambda) \neq 0 \pmod{m_p \Lambda} \},
\]
where \( m_p \) is the maximal ideal of \( \mathcal{O}_p^\times \). Let \( \iota_\lambda \) be the conductor of \( \lambda \). For each \( \nu \mid \iota_\lambda \), let \( \Delta_{\lambda, \nu} \) be the finite group \( \lambda(\mathcal{O}_E^\times) \).

Theorem B. With the assumptions in Theorem \([A]\) suppose further that
1. \( p \) is unramified in \( F \),
2. the residual Galois representation \( \overline{\mathcal{P}}_p(\pi_E) := p^r(\pi) \mid_{G_E} \pmod{m_p} \) is absolutely irreducible,
\[
(3) \ p \nmid \prod_{\nu | \iota_\lambda} \zeta_{\Delta_{\lambda, \nu}}^r(\Delta_{\lambda, \nu}).
\]
Then \( \mu_{\pi, \lambda, \Sigma} = 0 \).

The global assumption on the irreducibility of residual Galois representation assures that the new form associated to \( \pi \) is not congruent to theta series from \( E \) or Eisenstein series, and the local assumption \( (3) \) is equivalent to saying that the local residual character \( \lambda_\nu \pmod{m_p} \) is ramified for all \( \nu \mid \iota_\lambda \). In this situation, our Theorem \([B]\) roughly suggests that the \( \mu \)-invariant is essentially contributed by the congruences among primitive but residually non-primitive \( p \)-adic \( L \)-functions. In particular, suppose that \( (n, \mathcal{D}_{E/F}) = 1 \) and \( \rho_p(\pi) \) is residually irreducible. Then we always have \( \mu_{\pi, \lambda} = 0 \) whenever \( \lambda \) has split conductor over \( F \) (i.e. \( \iota_\lambda = (1) \)). In addition, Theorem \([B]\) shares the same flavor with Vatsal's \( \mu \)-invariant formula. In \([Vat03]\), Vatsal obtains the precise \( \mu \)-invariant formula of a different class of anticyclotomic \( p \)-adic Rankin-Selberg \( L \)-functions associated to elliptic new forms of weight two twisted by finite order anticyclotomic characters of \( p \)-power conductors (so the local root number at the archimedean place is \(-1\)). In virtue of his formula, the \( \mu \)-invariant can be positive when we have either Eisenstein congruence or the congruence between forms with opposite signs in the functional equations. Note that the latter congruence does not happen under the assumption \( (3) \).

The \( p \)-adic \( L \)-function \( \mathcal{L}_\Sigma(\pi, \lambda) \) is expected to encode the arithmetic information on certain Selmer groups through the main conjecture à la R. Greenberg \([Gre94]\). To introduce the \( \Lambda \)-adic Selmer groups connected with \( \mathcal{L}_\Sigma(\pi, \lambda) \), we recall that thanks to the works of Deligne, Carayol, Blasius-Rogawski and Taylor et.al.
(BR93, Tay89, Jar97), there exists a finite extension $L_\pi$ of $Q_p$ and a continuous $p$-adic Galois representation
\[ \rho_p(\pi) : G_F \to \GL_2(\OL_{L_\pi}) \]
such that $\rho_p(\pi)$ is unramified outside $p\mathfrak{n}$, and for each finite place $v \mid p$,
\[ L^{-1}(L(s, \rho_p(\pi)(W_{F_v}))) = L(s + \frac{1 - k_{mx}}{2}, \pi_v^\vee) \quad (k_{mx} = \max_{\sigma \in \Sigma} k_\sigma), \]
where $W_{F_v}$ is the Weil group of $F_v$. Let $\varepsilon_{\Lambda} : G_E \to \Gamma \to \Lambda^\times$, $g \mapsto g|_{E_{p^{-\infty}}}$ be the universal $\Lambda$-adic Galois character. We consider the rank two $\Lambda$-adic Galois representation:
\[ \rho_\Lambda := \rho_p(\pi)|_{G_E} \otimes \lambda \varepsilon_{\Lambda} : G_E \to \GL_2(\Lambda), \]
and define the local condition for each $w \mid p$ by
\[ F^+_w\rho_\Lambda = \begin{cases} \rho_\Lambda & \text{if } w \in \Sigma_p, \\ \{0\} & \text{if } w \notin \Sigma_p. \end{cases} \]
The triple $(\rho_\Lambda, (F^+_w\rho_\Lambda)_{w\mid p}, X_{\text{crit}})$ satisfies the Panchishkin condition in [Gre91] §4, p.217. Let $\Lambda^*$ be the Pontryagin dual of $\Lambda$. The associated Greenberg’s Selmer group is defined by
\[ \Sel_\Sigma(\pi, \lambda) := \ker \left\{ H^1(E, \rho_\Lambda \otimes \Lambda^*) \to \prod_{w \notin \Sigma_p} H^1(I_w, \rho_\Lambda \otimes \Lambda^*) \right\}, \]
where $w$ runs over places of $E$ outside $\Sigma_p$ and $I_w$ is the inertia group of $w$ in $G_E$. It is known that the Pontryagin dual $\Sel_\Sigma(\pi, \lambda)^*$ is a finitely generated $\Lambda$-module. Denote by $\char_\Lambda \Sel_\Sigma(\pi, \lambda)^*$ the characteristic ideal of $\Sel_\Sigma(\pi, \lambda)^*$. We formulate the anticyclotomic main conjecture for $\pi_E \otimes \lambda$ (under the hypotheses in Theorem A).

Conjecture. We have the following equality between ideals in $\Lambda$
\[ \char_\Lambda \Sel_\Sigma(\pi, \lambda)^* = (L_\Sigma(\pi, \lambda)). \]

Let $\ell \neq p$ be a rational prime. We next consider the problem of the non-vanishing modulo $p$ of $L$-values twisted by characters of $\ell$-power conductor. This problem has been studied extensively in the literature in various settings (cf. Was78, Var03, Hid04a, Fin06, Hsi12b) and has many arithmetic applications in Iwasawa theory. To state our result along this direction, we introduce some notation. Let $l$ be a prime of $F$ above $\ell$ and let $E_{l^{-\infty}}$ be the anticyclotomic pro-$\ell$ extension in the ray class field over $E$ of conductor $l^\infty$. Let $\Gamma_l^{-} := \Gal(E_{l^{-\infty}}/E)$ and let $X^0_l$ be the set consisting of finite order characters $\phi : \Gamma_l^{-} \to \mu_{l^{\infty}}$. Let $\chi$ be a Hecke character of infinity type $(\frac{1}{2} + m, -\frac{1}{2} - m)$ and of conductor $\ell \chi$. We obtain the following theorem.

Theorem C. With the same assumptions in Theorem A we assume that
\begin{enumerate}
\item \((l, n, D_{E/F}) = 1,\)
\item \((n, D_{E/F}) = 1\) and $\rho_p(\pi)$ is residually irreducible,
\item $p \nmid \prod_{\ell \mid \varnothing} \delta(\Delta_{\chi, v}).$
\end{enumerate}
Then for almost all characters $\phi \in X^0_l$, we have
\[ \left[ \frac{\mathcal{O}_E : \mathcal{O}_E^{\Sigma^2}}{\mathcal{O}_E^{\Sigma^2}} \right]^2 \cdot \frac{\Gamma_{\Sigma}(k + m)\Gamma_{\Sigma}(m + 1)}{(1 + k + 2m + 4\pi k + 2m + 1 + \Sigma)} \cdot \frac{L\left(\frac{1}{2}, \pi_E \otimes \chi(\phi)\right)}{\Omega_{E}^{2(k + 2m)}} \equiv 0 \pmod{m_p}. \]

Here almost all means "except for finitely many $\phi \in X^0_l$" if $\dim_{Q_l} F_l = 1$ and "for $\phi$ in a Zariski dense subset of $X^0_l$ if $\dim_{Q_l} F_l > 1$" [Hid04a, p.737].

In our forthcoming work [HN], we will apply Theorem C to study the non-vanishing (modulo $p$) of certain theta lifts constructed in MS07 and MN12.

The proof of Theorem A is based on an explicit Waldspurger formula, which connects toric period integrals of automorphic forms in $\pi$ and central values of $\pi_E \otimes \lambda$ [Was85]. To obtain the optimal $p$-integrality of central $L$-values, we will consider holomorphic toric cusp forms and calculate explicitly their period integrals. To be precise, let $\chi$ be a Hecke character of $E^\times$ such that $\chi|_{A^\times} = \omega^{-1}$ and let $T \subset A^\times_E$ be the subgroup consisting of
ideles \( z = (z_v) \in \prod_v E_v^\times \) with \( z_v/\pi \in O_v^\times \) for all primes \( v \) split in \( E \). Fixing an embedding \( \iota : E^\times \hookrightarrow \GL_2(F) \), we say an automorphic form \( \varphi : \GL_2(F) \backslash \GL_2(A_F) \to \mathbb{C} \) in \( \pi \) is a toric form of character \( \chi \) if
\[
\varphi(gu(t)) = \chi^{-1}(t)\varphi(g) \quad \text{for all } t \in \mathcal{T}.
\]

In other words, \( \varphi \) belongs to the space \( \Hom_{\mathcal{T}}(L, \pi \otimes \chi) \).

The construction of \( P_{\Sigma}(\pi, \chi) \) is outlined as follows.

1. Construct a toric Hilbert modular form \( \varphi_{\lambda\phi} \) of character \( \lambda\phi \) for each \( \phi \in \chi_{\mathrm{crit}} \) as above by a careful choice of toric local Whittaker functions in local Whittaker models of \( \pi \) (See Def. 3.1 Lemma 3.17).

2. Make an explicit calculation of the Fourier expansion of \( \varphi_{\lambda\phi} \).

3. Via the \( p \)-adic interpolation of the Fourier expansion, construct a \( p \)-adic family of toric forms \( \{\varphi_{\lambda\phi}\}_{\phi \in \chi_{\mathrm{crit}}} \).

The \( p \)-adic \( L \)-function \( P_{\Sigma}(\pi, \chi) \) is obtained by a weighted sum of the evaluation of this family at a finite set of CM points.

The evaluation of \( \varphi_{\chi} \) with \( \chi = \lambda\phi \) at CM points in the step (3) is actually the toric period integral \( P_{\chi}(\varphi_{\chi}) \) given by
\[
P_{\chi}(\varphi_{\chi}) = \int_{A^\times E^\times \backslash A_{E}^\times} \varphi_{\chi}(u(t))\chi(t)dt.
\]

To prove the formula in Theorem A we have to express the square \( P_{\chi}(\varphi_{\chi})^2 \) in terms of the central \( L \)-value \( L(\frac{1}{2}, \pi_E \otimes \chi) \). This is usually referred to as an explicit Waldspurger formula. Such a formula has been exploited widely in the literature based on either explicit theta lifts ([Mur10], [Mur08], [Xue07], [Hid10a] and [BDP]) or the technique of relative trace formula ([MW09]). In this paper we adopt a different approach, making use of a formula of Waldspurger which is indeed proved but not stated explicitly in [Wald85]. This formula decomposes the square \( P_{\chi}(\varphi) \) of the global period toric integral into a product of local period integrals involving local Whittaker functions of \( \varphi \). Explicit computation of these local integrals shows that \( P_{\chi}(\varphi)^2 \) is essentially equal to the central value of the \( L \)-function \( L(s, \pi_E \otimes \lambda) \). We emphasize that this explicit formula does not depend on the choices of explicit Bruhat-Schwartz functions in the classical approach of theta lifting but on choices of local Whittaker functions which reflect the arithmetic of modular forms directly via the Fourier expansion.

We make a few remarks on our assumptions. The restriction (3) is due to the computational difficulty on the local period integrals and the local Fourier coefficients, and it is expected to be unnecessary. We hope to come back to the removal of this in the future. However, Hypothesis A is fundamental, the failure of which makes the period integral \( P_{\chi}(\varphi_{\chi}) \) vanish by a well-known theorem of Saito-Tunnell ([Sai93], [Tun83]).

The proofs of Theorem B and Theorem C are based on the ideas of Hida in [Hid04a] and [Hid10b]. Thanks to Hida’s theorems on the linear independence of modular forms applied by transcendental automorphisms of the local moduli of CM points in Hilbert modular varieties modulo \( p \) loc. cit., the vanishing/non-vanishing modulo \( p \) properties of the algebraic part of \( L(\frac{1}{2}, \pi_E \otimes \chi) \) with anticyclotomic twists can be deduced from the vanishing/non-vanishing of the Fourier expansions of the toric cusp form \( \varphi_{\lambda} \) at cusps \( (a, b) \) such that \( ab^{-1} \) is the polarization of abelian varieties with CM by \( \mathcal{O}_E \). A new ingredient of this paper is the explicit computation of Fourier coefficients of toric new forms \( \varphi_{\lambda} \) and a study on their non-vanishing modulo \( p \) property in [Sai03].

Exploiting the connection between the Fourier coefficients of Hilbert modular forms and the trace of Frobenius of the associated Galois representations, we deduce from the vanishing modulo \( p \) of Fourier coefficients of \( \varphi_{\lambda} \) at these cusps that the trace of residual Galois representation \( \overline{\rho}_{\mathfrak{p}}(\pi) \) is vanishing on the coset \( G_F - G_E \). A simple lemma (Lemma 3.3) on modular representations shows that \( \overline{\rho}_{\mathfrak{p}}(\pi)|_{G_E} \) is reducible.

This paper is organized as follows. After fixing notation and definitions in §1 we derive a key formula of Waldspurger on the decomposition of global toric period integrals into local toric period integrals (Prop. 2.11) in §2. The bulk of this article is §3 where we give the choices of local toric Whittaker functions \( W_{\chi, v} \) and calculate explicitly these local period integrals attached to \( W_{\chi, v} \). The explicit Waldspurger formula is proved in Theorem 3.18 and a non-vanishing modulo \( p \) of these toric Whittaker functions is proved in Prop. 3.15. After reviewing briefly theory of complex and geometric Hilbert modular forms in §4 we prove Theorem A in §5. The key ingredient is Prop. 5.5, the construction of a p-adic measure \( \mathcal{P}_{\lambda, \varphi} \) on \( \Gamma_{\mathcal{E}} \) with values in the space of p-adic modular forms, and the p-adic \( L \)-function \( P_{\Sigma}(\pi, \lambda) \) is thus obtained by evaluating \( \mathcal{P}_{\lambda, \varphi} \) at suitable CM points. The precise evaluation formula of \( P_{\Sigma}(\pi, \lambda)^2 \) is established in Theorem 5.6. In §6 we study the \( \mu \)-invariant of \( P_{\Sigma}(\pi, \lambda) \) and prove Theorem B in Theorem 6.2. Finally, the non-vanishing of central \( L \)-values modulo \( p \) is considered in §7 and Theorem C is proved in Theorem 7.1.
1. Notation and definitions

1.1. Measures on local fields. We fix some general notation and conventions on local fields through this article. Let $\psi_Q : A_Q/Q \to C^\times$ be the additive character such that $\psi_Q(x_\infty) = \exp(2\pi i x_\infty)$ with $x_\infty \in R$. Let $q$ be a place of $Q$ and let $F$ be a finite extension of $Q_q$. Let $\psi_q$ be the local component of $\psi$ at $q$ and let $\psi_F := \psi_q \circ T_{F/Q_q}$, where $T_{F/Q_q}$ is the trace from $F$ to $Q_q$. Let $dx$ be the Haar measure on $F$ self-dual with respect to the pairing $(x, x') \mapsto \psi_F(xx')$. Let $|\cdot|_F$ be the absolute value of $F$ normalized by $d(ax) = |a|_F dx$ for $a \in F^\times$. We often simply write $|\cdot| = |\cdot|_F$ if it is clear from the context without possible confusion. We recall the definition of the local zeta function $\zeta_F(s)$. If $F$ is non-archimedean, let $\varpi_F$ be a uniformizer of $F$ and let

$$\zeta_F(s) = \frac{1}{1 - |\varpi_F|^s}.$$  

If $F$ is archimedean, then

$$\zeta_R(s) = \pi^{-s/2}\Gamma(s/2); \quad \zeta_C(s) = 2(2\pi)^{-s}\Gamma(s).$$

The Haar measures $d^s x$ on $F^\times$ is normalized by

$$d^s x = \zeta_F(1) |x|_F^{-s} dx.$$  

In particular, if $F = R$, then $dx$ is the Lebesgue measure and $d^s x = |x|_R^{-s} dx$, and if $F = C$, then $dx$ is twice the Lebesgue measure on $C$ and $d^s x = 2\pi^{-1} r^{-1} dr d\theta (x = re^{i\theta})$.

Suppose that $F$ is non-archimedean. Let $O_F$ be the ring of integers of $F$ and let $D_F$ be the absolute different of $F$. Then $D_F^{-1}$ is the Pontryagin dual of $O_F$ with respect to $\psi_F$, and $\text{vol}(O_F, dx) = |D_F|_F^{1/2}$. If $\mu : F^\times \to C^\times$ is a character of $F^\times$, define the local conductor $a(\mu)$ by

$$a(\mu) = \inf \{ n \in Z_{\geq 0} | \mu(x) = 1 \text{ for all } x \in (1 + \varpi_F^n O_F) \cap O_F^0 \}.$$  

1.2. If $L$ is a number field, the ring of integers of $L$ is denoted by $O_L$, $A_L$ is the adele of $L$ and $A_{L,f}$ is the finite part of $A_L$. For $a \in A_L^\times$, we put

$$\mathfrak{u}_L(a) := a(O_L \otimes \hat{Z}) \cap L.$$  

Denote by $G_L$ the absolute Galois group and by $\text{rec}_L : A_L^\times \to G_L^{ab}$ the geometrically normalized reciprocity law. We define $\psi_L : A_L^\times / L \to C^\times$ by $\psi_L(x) = \psi_Q \circ T_{L/Q}(x)$.

Let $v_p$ be the $p$-adic valuation on $C_p$ normalized so that $v_p(p) = 1$. We regard $L$ as a subfield in $C$ (resp. $C_p$) via $i_\infty : Q \subset C$ (resp. $i_p = i^{-1} \circ i_\infty : Q \subset C_p$) and Hom$(L, Q) = \text{Hom}(L, C_p)$.

Let $\mathbf{Z}$ be the ring of algebraic integers of $Q$ and let $\mathbf{Z}_p$ be the $p$-adic completion of $\mathbf{Z}$ in $C_p$. Let $\mathbf{Z}$ be the ring of algebraic integers of $Q$ and let $\mathbf{Z}_p$ be the $p$-adic completion of $\mathbf{Z}$ in $C_p$, with the maximal ideal $m_p$. Let $m = i^{-1}_p(m_p)$.

1.3. Local $L$-functions. Let $F$ be a non-archimedean local field. Let $\mu, \nu : F^\times \to C^\times$ be two characters of $F^\times$. Denote by $I(\mu, \nu)$ the space consisting of smooth and $GL_2(O_F)$-finite functions $f : GL_2(F) \to C$ such that

$$f \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) g = \mu(a) \nu(d) \frac{|a|_F^{|\gamma|_F}}{|d|_F} f(g).$$

Then $I(\mu, \nu)$ is an admissible representation of $GL_2(F)$. Denote by $\pi(\mu, \nu)$ the unique infinite dimensional subquotient of $I(\mu, \nu)$. We call $\pi(\mu, \nu)$ a principal series if $\mu^{-1} \neq |\cdot|_F^\pm$ and a special representation if $\mu^{-1} = |\cdot|_F^\pm$.

Let $E$ be a quadratic extension of $F$ and let $\chi : E^\times \to C^\times$ be a character. We recall the definition of local $L$-functions $L(s, \pi_E \otimes \chi)$ (Jac72 §20)) when $\pi = \pi(\mu, \nu)$ is a subrepresentation of $I(\mu, \nu)$. If $E = F \oplus F$, then we write $\chi = (\chi_1, \chi_2) : F^\times \oplus F^\times \to C^\times$ and put

$$L(s, \pi_E \otimes \chi) = \begin{cases} L(s, \pi \otimes \chi_1)L(s, \pi \otimes \chi_2) & \text{if } \mu^{-1} \neq |\cdot|_F^\pm, \\ L(s, \mu \chi_1)L(s, \mu \chi_2) & \text{if } \mu^{-1} = |\cdot|. \end{cases}$$
If $E$ is a field, then

$$L(s, \pi_E \otimes \chi) = \begin{cases} L(s, \mu' \chi)L(s, \nu' \chi) & \text{if } \mu \nu^{-1} \neq |.|^\pm, \\ L(s, \mu' \chi) & \text{if } \mu \nu^{-1} = |.|. \end{cases}$$

Here $\mu' = \mu \circ N_{E/F}$, $\nu' = \nu \circ N_{E/F}$ are characters of $E^\times$.

1.4. **Whittaker and Kirillov models.** Let $F$ be a local field. Let $\pi$ be an irreducible admissible representation of $\GL_2(F)$ and let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character. We let $W(\pi, \psi)$ be the Whittaker model of $\pi$. Recall that $W(\pi, \psi)$ is a subspace of smooth functions $W : \GL_2(F) \to \mathbb{C}$ such that

1. $W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) g = \psi(x)W(g)$ for all $x \in F$.

2. If $v$ is archimedean, $W\left(\begin{pmatrix} a & \nu \psi(v) \\ 0 & 1 \end{pmatrix} \right) = O(|a|^N)$ for some positive number $N$.

(cf. [JL70] Thm. 6.3). Let $K(\pi, \psi)$ be the Kirillov model of $\pi$. If $F$ is non-archimedean, then $K(\pi, \psi)$ is a subspace of smooth $\mathbb{C}$-valued functions on $F^\times$, containing all Bruhat-Schwartz functions on $F^\times$. A function in $K(\pi, \psi)$ shall be called a *local Fourier coefficient* of $\pi$. In addition, it is well known that we have the following $\GL_2(F)$-equivariant isomorphism

$$W(\pi, \psi) \sim \sim K(\pi, \psi)$$

(1.1)

$$W \mapsto \xi_W(a) := W\left(\begin{pmatrix} a & \nu \psi(1) \\ 0 & 1 \end{pmatrix} \right).$$

2. **Waldspurger formula.**

Let $F$ be a number field and $E$ be a quadratic field extension of $F$. Let $A = A_F$. Let $G = \GL_2/F$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(A)$ with unitary central character $\omega$. Denote by $A(\pi)$ the realization of $\pi$ in the space $A_0(G)$ of cusp forms on $G(A)$. Let $\chi$ be a unitary Hecke character of $E^\times$ such that $|\chi|_{A^\times} = \omega^{-1}$. Let $\pi_E$ be the quadratic base change of $\pi$ to the quadratic extension $E/F$. The existence of $\pi_E$ is established in [Jac72]. The goal of this section is to deduce from results in [Wal85] a formula (Prop. 2.1) which expresses the central value $L\left(\frac{1}{2}, \pi_E \otimes \chi\right)$ in terms of a product of local toric period integrals of Whittaker functions.

Let $\psi := \psi_F : A/F \to \mathbb{C}^\times$ be the standard non-trivial additive character. For a place $v$ of $F$, we let $G_v = G(F_v)$ and let $\chi_v : E_v^\times \to \mathbb{C}^\times$ (resp. $\psi_v : F_v \to \mathbb{C}^\times$) denote the local constituent of $\chi$ (resp. $\psi$).

2.1. For $x \in E$, let $T(x) := x + \mathfrak{o}$ and $N(x) = x\mathfrak{o}$. Let $\{1, \vartheta\}$ be a basis of $E$ over $F$. We let $\iota : E \to M_2(F)$ be the embedding attached to $\vartheta$ given by

$$\iota(a\vartheta + b) = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \quad (a, b \in F).$$

Put

$$J := \begin{pmatrix} -1 & T(\vartheta) \\ 0 & 1 \end{pmatrix}.$$ 

Then $M_2(F) = \iota(E) \oplus \iota(E)J$. It is clear that $J^2 = 1$ and $\iota(t)J = J\iota(t)$ for all $t \in E$.

2.2. **The local bilinear form and toric integral.** For each place $v$ of $F$, denote by $\pi_v$ (resp. $\psi_v$) the local constituent of $\pi$ (resp. $\psi$) at $v$. Define a $\mathbb{C}$-bilinear form $b_v : W(\pi_v, \psi_v) \times W(\pi_v, \psi_v) \to \mathbb{C}$ by

$$b_v(W_1, W_2) := \sum_{n = -\infty}^{\infty} \int_{Z_n \mathfrak{o}_F^\times} W_1\left(\begin{pmatrix} a & \nu \psi(v) \\ 0 & 1 \end{pmatrix}\right)W_2\left(\begin{pmatrix} -a & \nu \psi(v) \\ 0 & 1 \end{pmatrix}\right)\omega^{-1}(a)d^x a \quad (W_1, W_2 \in W(\pi_v, \psi_v)).$$

It is known that this series converges absolutely as $\pi_v$ is a local constituent of a unitary cuspidal automorphic representation. Moreover, the pairing $b_v$ enjoys the property:

$$b_v(\pi(g)W_1, \pi(g)W_2) = \omega(\det g)b_v(W_1, W_2).$$
functions. We suppose that \( \text{Proposition 2.1 (Waldspurger)} \)
\( \chi \)
Here we have used the self-duality condition \( s \). It is well known that \( \Lambda(\varepsilon) \) Waldspurger.
2.3. A formula of Waldspurger. Let \( \Lambda(s, \pi_E \otimes \chi) \) be the completed \( L \)-function of \( \pi_E \otimes \chi \) given by
\( \Lambda(s, \pi_E \otimes \chi) = \prod_v L(s, \pi_E_\varepsilon \otimes \chi_\varepsilon) \cdot \prod_{\varepsilon_\infty} L(s, \pi_{E_\varepsilon} \otimes \chi_\varepsilon). \)
It is well known that \( \Lambda(s, \pi_E \otimes \chi) \) converges absolutely for \( \text{Re } s > 0 \) and has meromorphic continuation to all \( s \in \mathbb{C} \). Moreover, it satisfies the functional equation
\( \Lambda(s, \pi_E \otimes \chi) = \varepsilon(s, \pi_E \otimes \chi) \Lambda(s, \pi_E \otimes \chi). \)
Here we have used the self-duality condition \( \chi|_{\mathbb{A}^s} = \omega^{-1} \). The global toric period integral for \( \varphi \in \mathcal{A}(\pi) \) is defined by
\[ P_\chi(\varphi) := \int_{E \times \mathbb{A}^s \times \mathbb{A}^d_\varepsilon} \varphi(i(t)) \chi(t) dt. \]
The following proposition connects the global toric periods and central \( L \)-values of \( \pi_E \otimes \chi \).

**Proposition 2.1** (Waldspurger). Let \( \varphi_1, \varphi_2 \in \mathcal{A}(\pi) \) and let \( W_{\varphi_1}, W_{\varphi_2} \) be the associated global Whittaker functions. We suppose that \( W_{\varphi_i} = \prod_v W_{\varphi_i, v} \) where \( W_{\varphi_i, v} \in \mathcal{W}(\pi_v, \psi_v) \) such that \( W_{\varphi_i, v}(1) = 1 \) for almost \( v \) \((i = 1, 2)\). Then there exists a finite set \( S_0 \) of places of \( F \) including all archimedean places such that for every finite set \( S \supset S_0 \), we have
\[ P_\chi(\varphi_1)P_\chi(\varphi_2) = \frac{1}{2} \prod_{\varepsilon_\in \mathbb{C}} \prod_{\varepsilon_\infty} \frac{1}{L(\frac{1}{2}, \pi_{E_\varepsilon} \otimes \chi_\varepsilon)} \cdot P(W_{1, v}, W_{2, v}, \chi_v). \]

**PROOF.** The proof is the combination of various formulae established in [Wals]. We first recall some important local integrals. Let \( D = G \times G \). For each place \( v \) of \( F \), let \( S_v = S(M_2(F_v)) \otimes S(F_v^\times) \) and let \( D_v = G_v \times G_v \). Let \( r = r^r \times r^\sigma : G_v \times D_v \to \text{End} \mathcal{S}_v \) be the Weil representation of \( G_v \times D_v \) defined in [Wals] §1.3 p.178.

Let \( \varphi \in \mathcal{A}(\pi) \) be an automorphic form in the automorphic realization of \( \pi \). Recall that the global Whittaker function of \( \varphi \) is defined by
\[ W_{\varphi}(g) = \int_{F \times \mathcal{A}_\varphi} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x) dx. \]
Write \( W_v = \mathcal{W}(\pi_v, \psi_v) \). We further assume that \( W_{\varphi} \) has the factorization \( W_{\varphi} = \prod_v W_{\varphi, v} \in \mathcal{W}_v \) such that \( W_{\varphi, v}(1) = 1 \) for almost \( v \). For each \( v \), let \( U : S_v \to W_v \times W_v, f_v \to U_{f_v} \) be the \( G_v \times G_v \)-equivariant surjective morphism associated to \( W_v \) introduced in [Wals] COROLLAIRED, p.187]. Define the following local integrals:
\[ C(f_v) := \int_{F_{\varepsilon}^\times} U_{f_v} \left( \begin{pmatrix} a & \rho \sigma \\ 1 & 1 \end{pmatrix} \right) \omega^{-1}(a) x d \sigma, \]
\[ B(f_v, 1) := \int_{Z_v \times G_v} \int_{F_{\varepsilon}^\times} W_{\varphi, v}(\sigma) r^r(\sigma) f_v(x, x^{-2}) x d x d \sigma, \]
\[ P(f_v, \chi_v) := \int_{F_{\varepsilon}^\times} U_{f_v} \left( \begin{pmatrix} 1 & \sigma(t) \\ 0 & 1 \end{pmatrix} \right) \chi_v(t) dt. \]

The convergence and analytic properties of these local integrals are studied in [Wals] LEMME 2, LEMME 3, LEMME 5. Moreover, we have
\[ B(f_v, 1) = C(f_v) : \frac{1}{\zeta_{F_v}(1)}. \]
For each \( v \), take a special test function \( f_v \in S_v \) such that
\[ (2.3) \quad U_{f_v} = W_{1, v} \otimes \pi(J) W_{2, v}. \]
Note that \( f_v \) can be chosen to be the spherical test function \( f_v^0 := \mathbb{1}_{M_2(O_{F_v})} \otimes I_{\mathcal{O}_{F_v}^\times} \) for all but finitely many \( v \). With this particular choice of \( f_v \), we have
\[
\begin{align*}
P(f_v, \chi_v, \frac{1}{2}) &= \int_{F_v^\times \backslash E_v^\times} C(r''(i(t), 1)f_v)dt \cdot \frac{1}{\zeta_{F_v}(1)} \\
&= \int_{F_v^\times \backslash E_v^\times} b_v(\pi(i(t))W_1, \pi(J)W_2)\chi_v(t)dt \cdot \frac{1}{\zeta_{F_v}(1)} \\
&= P(W_{1,v}, W_{2,v}, \chi_v) \cdot \frac{1}{L(1, \tau_{E_v}/F_v)}.
\end{align*}
\]

Let \( \mathcal{S} = \otimes \mathcal{S}_v \) be the restricted product with respect to spherical test functions \( \{f_v^0\}_v \). Define the theta kernel for \( f := \otimes f_v \in \mathcal{S} \) by
\[
\theta_f(\sigma, g) := \sum_{(x, u) \in M_2(F) \times F^\times} r(\sigma, g)f(x, u), \quad \sigma \in G(A), \quad (g \in D(A) = G(A) \times G(A)),
\]
and define the automorphic form \( \theta(f, \varphi, g) \) on \( G(A) \times G(A) \) by
\[
\theta(f, \varphi, g) = \int_{G(F) \backslash G(A)} \varphi(\sigma)\theta_f(\sigma, g)ds.
\]

Note that according to (2.3), we have
\[
\theta(f, \varphi, g_1, g_2) = \varphi_1(g_1)\varphi_2(g_2J).
\]

We define the toric period integral \( P(f, \chi) \) by
\[
P(f, \chi) := \int_{[E^\times A^\times \backslash A^\times F]} \theta(f, \varphi, i(t_1), i(t_2))\chi(t_1)\chi(t_2)dt_1dt_2.
\]

By the relation \( J_i(t_2)J = i(t_2) \) and the automorphy of \( \varphi_2 \), we find that
\[
P(f, \chi) = P_\chi(\varphi_1)P_\chi(\varphi_2).
\]

Let \( S_0 \) be a finite set of places of \( F \) such that \( W_{\varphi, v}, W_{\varphi, v} \) and \( f_v \) are spherical for all \( v \notin S_0 \). From [Wal5 Prop. 4, p.196 and LEMME 7, p.219], we deduce the following formula for every finite set \( S \supset S_0 \):
\[
P_\chi(\varphi_1)P_\chi(\varphi_2) = \Lambda\left(\frac{1}{2}, \pi_E \otimes \chi\right) \cdot \prod_{v \in S} P(f_v, \chi_v, \frac{1}{2}) \cdot \frac{L(1, \tau_{E_v}/F_v)}{L(\frac{1}{2}, \pi_{E_v} \otimes \chi)}.
\]

We thus establish the desired formula in virtue of (2.4). \( \square \)

3. Toric period integrals

3.1. Notation. Throughout we suppose that \( F \) is a totally real number field and \( E \) is a totally imaginary quadratic extension of \( F \). We retain the notation in the introduction and (2.1). Let \( \Sigma \) be a fixed CM type of \( E \). Let \( \pi \) be an irreducible automorphic cuspidal representation of \( GL_2(A) \). Let \( n \) be the conductor of \( \pi \). Suppose that \( \pi \) has infinity type \( k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}_{\geq 1}[\Sigma] \). Let \( m = \sum_{\sigma} m_\sigma \sigma \in \mathbb{Z}_{\geq 0}[\Sigma] \) and let \( \chi \) be a Hecke character of infinity type \( (k/2 + m, -k/2 - m) \) such that \( \chi|_{A^\times} = \omega^{-1} \). Let \( \mathfrak{h} \) be the set of finite places of \( F \). Recall that the set of infinite places of \( F \) is identified with the CM-type \( \Sigma \).

In this section, we will choose a special local Whittaker function at each place \( v \) of \( F \) in (3.0) and calculate their associated local toric period integrals in (3.1) and (3.2). Finally, we prove in (3.3) a non-vanishing modulo \( p \) result of these local Whittaker functions. This result plays an important role in the later application to the calculation of the \( \mu \)-invariant.

Let \( \mathfrak{c}_\chi \) (resp. \( \mathfrak{c}_\omega \)) be the conductor of \( \chi \) (resp. \( \omega \)). Let \( \mathfrak{c}_\omega = \mathfrak{c}_\chi \cap F \). We further decompose \( \mathfrak{n}^- = \mathfrak{n}_-^\omega \mathfrak{n}_-^\pi \), where \( \mathfrak{n}_-^\omega \) is prime to \( \mathfrak{c}_\omega \) and \( \mathfrak{n}_-^\pi \) is only divisible by prime factors of \( \mathfrak{c}_\omega \). Put
\[
c_v(\chi) = \inf \{ n \in \mathbb{Z}_{\geq 0} \mid \chi = 1 \text{ on } (1 + \mathfrak{c}_\omega O_E)^\times \},
\]
\[
m_v(\chi, \pi) = c_v(\chi) - v(\mathfrak{n}^-).
\]

It is clear that \( c_v(\chi) = v(\mathfrak{c}_\chi) \). We put
\[
A(\chi) = \{ v \in \mathfrak{h} \mid E_v \text{ is a field, } \pi_v \text{ is special and } c_v(\chi) = 0 \}.
\]
Let \( p > 2 \) be a rational prime satisfying \([\text{ord}]\). The assumption \([\text{ord}]\) in particular implies that every prime factor of \( p \) in \( F \) splits in \( E \). Let \( \Sigma_p \) be the \( p \)-adic places induced by \( \Sigma \) via \( \iota_p \). Thus \( \Sigma_p \) and its complex conjugation \( \overline{\Sigma_p} \) give a partition of the places of \( E \) above \( p \). Let \( \mathfrak{N} \) be the prime-to-\( p \) conductor of \( \pi_E \otimes \chi \). We fix a decomposition \( \mathfrak{N}^+ = \mathfrak{N} \mathfrak{N}' \) such that \( (\mathfrak{N}, \mathfrak{N}') = 1 \).

3.2. **Galois representation attached to \( \pi \).** Let \( \rho_p(\pi) : G_F \to \text{GL}_2(\mathcal{O}_{L_v}) \) be the \( p \)-adic Galois representation associated to \( \pi \) as in the introduction. Let \( v \mid p \) and let \( W_{F_v} \) be the local Weil group at \( v \). Suppose that \( \pi_v = \pi(\mu_v, \nu_v) \) is a subquotient of the induced representations. By the local-global compatibility ([Car86], [Tay89] and [Jar97]), we have

\[
(3.2) \quad \rho_p(\pi)|_{W_{F_v}} \cong \left( \begin{array}{cc}
\mu_v^{-1} & 0 \\
\nu_v^{-1} & 1 \end{array} \right) (k_{mx} = \max k_\sigma).
\]

In particular, this implies that \( \mu_v(\varpi_{F_v}) \) and \( \nu_v(\varpi_{F_v}) \) are \( p \)-adic units in \( \mathcal{O}_E^\times \).

3.3. **Open-compact subgroups.** For each finite place \( v \), we put

\[
K_v^0 = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v \mid a, d \in \mathcal{O}_{F_v}, \ b \in D_{F_v}^{-1} \right\},
\]

and for an integral ideal \( a \) of \( F \), we put

\[
K_v^0(a) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v^0 \mid c \in aD_{F_v}, \ a - 1 \in a \right\},
\]

\[
U_v(a) = \left\{ g \in \text{GL}_2(\mathcal{O}_{F_v}) \mid g \equiv 1 \pmod{a} \right\}.
\]

Let \( K^0 = \prod_{v \in \mathfrak{N}} K_v^0 \) and \( U(a) = \prod_{v \in \mathfrak{N}} U_v(a) \) be open-compact subgroups of \( \text{GL}_2(\mathcal{A}_F) \).

3.4. **The choices of \( \vartheta \) and \( \varsigma_v \).** Fix an integral ideal \( \mathfrak{c} \subset \mathfrak{n}_F \mathfrak{D}_F^\times \) of \( F \). For each finite place, let \( d_{F_v} \) be a generator of the absolute different \( D_{F_v} \). We choose \( \vartheta \in E \) such that

(d1) \( \text{Im} \sigma(\vartheta) > 0 \) for all \( \sigma \in \Sigma \),

(d2) \( \{1, d_{F_v}^{-1} \vartheta \} \) is an \( \mathcal{O}_{F_v} \)-basis of \( \mathcal{O}_{E_v} \) for all \( v \mid pr \),

(d3) \( d_{F_v}^{-1} \vartheta \) is a uniformizer of \( E_v \) for every \( v \) ramified in \( E \).

Then \( \vartheta \) is a generator of \( E \) over \( F \) which determines an embedding \( E \hookrightarrow M_2(F) \) in \([2.1]\). Let

\[
\delta = 2^{-1}(\vartheta - \overline{\vartheta}) \in E^\times.
\]

For each \( v \) split in \( E \), we shall fix a place \( w \) of \( E \) above \( v \) throughout, and decompose \( E_v := E \otimes_F F_v = F_v e_w \oplus F_v e_{\overline{w}} \), where \( e_w \) and \( e_{\overline{w}} \) are the idempotents attached to \( w \) and \( \overline{w} \). If \( v \mid p\mathfrak{N}^+ \), we further require that \( \mathfrak{N} \mathfrak{N}' \mathfrak{N}_p, \text{i.e.} \ w \mid \mathfrak{N} \) or \( w \in \mathfrak{N}_p \). We identify \( \delta \in E_w = F_w \) and write \( \vartheta_v = \vartheta_w e_w + \vartheta_{\overline{w}} e_{\overline{w}} \) for split \( v \).

For each finite place \( v \), we fix a uniformizer \( \varpi_v = \varpi_{F_v} \) of \( F_v \). By (d2), we fix a choice of \( d_{F_v} \) as follows.

\[
d_{F_v} = \begin{cases} 2\delta & \text{if } v \mid pr \text{ is split} \\ \varpi_v^{(D_{F_v})} & \text{otherwise}. \end{cases}
\]

We also fix an \( \mathcal{O}_{F_v} \)-basis \( \{1, \theta_v\} \) of \( \mathcal{O}_{E_v} \) such that \( \theta_v = \vartheta \) except for finitely many \( v \) and

\[
\theta_v = d_{F_v}^{-1} \vartheta \quad \text{for } v \mid pr.
\]

Write \( \theta_v = a_v \vartheta + b_v \) with \( a_v, b_v \in F_v \).

For each place \( v \), we define \( \varsigma_v \in \text{GL}_2(F_v) \) as follows:

\[
\varsigma_v = \begin{pmatrix} \text{Im} \sigma(\vartheta) & \text{Re} \sigma(\vartheta) \\ 0 & 1 \end{pmatrix} \text{ for } v = \sigma \in \Sigma,
\]

\[
\varsigma_v = (\vartheta_w - \vartheta_{\overline{w}})^{-1} \begin{pmatrix} d_{F_v} \vartheta_w & \vartheta_{\overline{w}} \\ d_{F_v} & 1 \end{pmatrix} \text{ for split } v = w \overline{w},
\]

\[
\varsigma_v = \begin{pmatrix} d_{F_v} & -b_v \\ 0 & a_v \end{pmatrix} \text{ for non-split finite } v.
\]

For \( t \in E_v \), we put

\[
\iota_{E_v}(t) := \varsigma_v^{-1} \iota(t) \varsigma_v.
\]
It is straightforward to verify that if \( v = \sigma \in \Sigma \) is archimedean and \( t = x + iy \in \mathbb{C}^\times \), then

\[
\iota_{\varphi}(t) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix},
\]

and if \( v = w\varpi \) is split and \( t = t_1e_w + t_2e_{\overline{w}} \), then

\[
\iota_{\varphi}(t) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}.
\]

Moreover, we note that for all finite places \( v \)

\[
\iota_{\varphi}(O_{E_v}^\times) = \iota_{\varphi}(E_v^\times) \cap K_v^0.
\]

3.5. **Running assumptions.** In this section, we will assume Hypothesis \( \Box \) for \( (\pi, \chi) \) and \( \text{(sf)} \)

\( n^- \) is square-free.

The assumption \( \Box \) implies that \( \pi_v \) is an unramified special representation if \( v|n^- \) and \( \pi_v \) is a ramified principal series if \( v|n^- \). In particular, for every place \( v \) inert or ramified in \( E \), \( \pi_v \) is a sub-quotient of induced representations and the local \( L \)-function \( L(s, \pi_v) \neq 1 \). We shall write \( \pi_v = \pi(\mu_v, \nu_v) \) such that \( L(s, \pi_v) = L(s, \mu_v) \) for \( v|n^- \). Moreover, by the local root number formulas [JL70, Prop. 3.5, Thm. 2.18], under the assumption \( \Box \) Hypothesis \( \Box \) on the sign of local root numbers is equivalent to the following condition:

\[
\text{(R1)} \quad \text{For each } v \in A(\chi), \text{ } v \text{ } \text{is ramified in } E \text{ and } \mu_v^\prime \chi_v(\varpi_{E_v}) = -|\varpi|^{\frac{1}{2}} \quad (\mu_v^\prime = \mu_v \circ N_{E_v/F_v}).
\]

In what follows, we fix a place \( v \) of \( F \). Let \( F = F_v \) and \( E = E_v \). Let \( O = O_F \) and \( \varpi = \varpi_v \) if \( v \) is finite. We shall suppress the subscript \( v \) and write \( \pi = \pi_v, \chi = \chi_v, \kappa = \kappa_v \) and \( \psi = \psi_v \).

3.6. **The choice of local toric Whittaker functions.** If \( v \) is finite, we let \( W_v^0 \) denote the new Whittaker function in \( W(\pi, \psi) \). In other words, \( W_v^0 \) is invariant by \( K_v^0(1) \) and \( W_v^0(1) = 1 \). The existence of \( W_v^0 \) is a consequence of the theory of local new vectors [Cas73]. Now we introduce special local Whittaker functions.

3.6.1. **The archimedean case.** Suppose that \( v = \sigma \in \Sigma \) is an archimedean place and \( F = \mathbb{R} \). Then \( \pi_\sigma = \pi([\frac{\not{m}_-}{\not{m}_+}], \frac{\chi_\sigma}{\kappa_\sigma} \text{sgn}^\sigma) \) is the discrete series of minimal \( \text{SO}(2, \mathbb{R}) \)-type \( k_\sigma \). Let \( W_{k_\sigma} \in W(\pi_\sigma, \psi_\sigma) \) be the Whittaker function given by

\[
W_{k_\sigma}(z \begin{pmatrix} a \\ 1 \end{pmatrix} \kappa_\sigma) = e^{-2\pi a} e^{i k_\sigma \theta} \text{sgn}(z)^{k_\sigma} \quad (z \in \mathbb{R}^\times, \kappa_\sigma = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}).
\]

Let \( V_+ \) and \( V_- \) be the weight raising and lowering differential operators in [JL70] p.165 given by

\[
V_\pm = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes i \in \text{Lie}(\text{GL}_2(\mathbb{R})) \otimes \mathbb{R} \mathbb{C}.
\]

Define the normalized weight raising differential operator \( \tilde{V}_+ \) by

\[
\tilde{V}_+ = \frac{1}{(-8\pi)} \cdot V_+.
\]

Then we have

\[
\tilde{V}_+^{m_\sigma} W_{k_\sigma}(g\kappa_\sigma) = \tilde{V}_+^{m_\sigma} W_{k_\sigma}(g) e^{i(k_\sigma+2m_\sigma)\theta}.
\]
3.6.2. The split case. Suppose that $v = u\mathfrak{m}$ is split with $w|\overline{\mathfrak{m}}\mathfrak{p}_v\mathfrak{p}$ if $v|\mathfrak{p}\mathfrak{p}_v^+$. We introduce some smooth functions $a_{\chi,v}$ on $F^\times$ in the Kirillov model $\mathcal{K}(\pi,\psi)$. Write $\chi = (\chi_w,\chi_\mathfrak{m}) : F^\times \oplus F^\times \to C^\times$. If the local $L$-function $L(s,\pi \otimes \chi_w) = 1$, we simply put

$$a_{\chi,v}(a) = \mathbb{I}_{\mathcal{O}}(a)(\chi_w(a^{-1}).$$

Suppose that $L(s,\pi \otimes \chi_w) \neq 1$. Then $\pi = \pi(\mu,\nu)$ is a principal series or $\pi = \pi(\mu,\nu)$ is special with $\mu \nu^{-1} = \cdot$. If $\mu \chi_w$ is unramified, we set

$$a_{\chi,v}(a) = \mathbb{I}_{\mathcal{O}}(a) \cdot \chi_w^{-1}(a) |\cdot|^{\frac{1}{2}}(a) \sum_{i+j=1, i,j \geq 0} \mu \chi_w(\overline{a^i}) \nu \chi_w(\overline{a^j}).$$

If $\mu \chi_w$ is unramified and $\mu \chi_w$ is ramified for $\{\mu_1, \mu_2\} = \{\mu, \nu\}$, we set

$$a_{\chi,v}(a) = \mu_1 |\cdot|^{\frac{1}{2}}(a) \mathbb{I}_{\mathcal{O}}(a).$$

If $\pi$ is special, we set

$$a_{\chi,v}(a) = \mu |\cdot|^{\frac{1}{2}}(a) \mathbb{I}_{\mathcal{O}}(a).$$

These functions $a_{\chi,v}$ indeed belong to the Kirillov model $\mathcal{K}(\pi,\psi)$ in virtue of the description of the Kirillov models [Jac72, Lemma 14.3]. For each $x \in \mathcal{K}(\pi,\psi)$, by the isomorphism $\psi$ we denote by $W_\pi \in \mathcal{W}(\pi,\psi)$ the unique Whittaker function such that $W_\pi(\begin{pmatrix} a & \ast \\ 0 & 1 \end{pmatrix}) = \psi(a)$. We put

$$W_{\chi,v} := W_{a_{\chi,v}}.$$

It follows from the choice of $W_{\chi,v}$ that

$$W_{\chi,\varphi,v} = W_{\chi,v} \text{ if } \varphi : E^\times \to C^\times \text{ is unramified.}$$

Recall that the zeta integral $\Psi(s,W_{\chi,v})$ for $W \in \mathcal{W}(\pi,\psi)$ is defined by

$$\Psi(s,W_{\chi,v}) := \int_{F^\times} W(\begin{pmatrix} a & \ast \\ 0 & 1 \end{pmatrix}) \chi_w(a) |a|^{-s} d^\times a.$$  

Then the zeta integral for $W_{\chi,v}$ satisfies the following equation:

$$(3.9) \quad \Psi(s,W_{\chi,v},\chi_w) = L(s,\pi \otimes \chi_w) |D_F|^{\frac{1}{2}} \left( \text{vol}(\mathcal{O}_E^\times, d^\times a) = |D_F|^{\frac{1}{2}} \right).$$

Suppose that $v = w\mathfrak{m}$ with $w \in \mathfrak{p}_v$. We define some p-modified Whittaker functions as follows. For each $u \in \mathcal{O}_E^\times$, we put

$$a_{u,v}(a) := \mathbb{I}_{\mathcal{O}}(a,\overline{\mathfrak{m}})(\chi_w(a^{-1}) \text{ and } W_{\chi,u,v} = W_{a_{u,v}}.$$ 

Let $a_{\chi,v}(a) := \mathbb{I}_{\mathcal{O}}(a) \chi_w(a^{-1})$ and let $W_{\chi,v}^p$ be the p-modified Whittaker function given by

$$W_{\chi,v}^p := W_{a_{\chi,v}} = \sum_{u \in U_0} W_{\chi,u,v},$$

where $U_0$ is the torsion subgroup of $\mathcal{O}_E^\times$. It is easy to verify that

$$(3.11) \quad \Psi(s,W_{\chi,v},\chi_w) = 1; \pi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) W_{\chi,v}^p = \chi_w(a) \chi_\mathfrak{m}^{-1}(d) W_{\chi,v}^p \text{ for } a, d \in \mathcal{O}_E^\times, b \in D_F^{-1}.$$

3.6.3. The inert and ramified case. Suppose that $v$ is an inert or ramified finite place. Then $E$ is a non-archimedean local field. Define the operators $\mathcal{R}_v$ and $\mathcal{P}_{\chi,v}$ on $W \in \mathcal{W}(\pi,\psi)$ by

$$\mathcal{R}_v W(g) := W(g \begin{pmatrix} 1 \\ \overline{a} \end{pmatrix}),$$

$$\mathcal{P}_{\chi,v} W(g) := \mathcal{V}_E^{-1} \int_{E^\times/F^\times} \pi(t) W(g)(t) dt = \mathcal{V}_E^{-1} \int_{E^\times/F^\times} W(g^{-1} \pi(t)) \chi(t) dt.$$ 

Note that

$$\mathcal{V}_E = \text{vol}(E^\times/F^\times, dt) = e_v \cdot |D_E|^{\frac{1}{2}} |D_F|^{-\frac{1}{2}}, \quad e_v = \begin{cases} 1 & \text{if } v \text{ is inert}, \\ 2 & \text{if } v \text{ is ramified}. \end{cases}$$
Lemma 3.2. The Whittaker functions $\chi$ character $c$

3.6.4. Define the subgroup $T_v$ of $E^\times$ by

$$T_v = \begin{cases} \mathcal{O}_E^\times \mathbb{F}_v^\times & \text{if } v \text{ is split}, \\ \mathbb{F}_v^\times & \text{if } v \text{ is non-split}. \end{cases}$$

Then $T_v = \{ x \in E \mid x/\mathfrak{p} \in \mathcal{O}_E^\times \}$ if $v$ is finite.

Definition 3.1 (Toric Whittaker functions). We say that $W \in \mathcal{W}(\pi, \psi)$ is a toric Whittaker function of character $\chi$ if

$$\pi(i_\chi(t))W = \chi^{-1}(t) \cdot W \text{ for all } t \in T_v.$$

Lemma 3.2. The Whittaker functions $W_{\chi, v}$ chosen as above are toric. To be precise, we have

1. $W_{\chi, v}$ is a toric Whittaker function of the character $\chi_\sigma : \mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^{k_\sigma + m_\sigma} |z|^{-k_\sigma/2}$.
2. If $v$ is finite, then $W_{\chi, v}$ are toric Whittaker functions of character $\chi_v$.
3. If $v | p$, then $W_{\chi, v}$ is toric, and for $u \in \mathcal{O}_F^\times$

$$\pi(i_\chi(t))W_{\chi, u, v} = \chi^{-1}(t)W_{\chi, u, t^{-1}c, v},$$

where $u, t^{1-c} := ut_\mathfrak{m}t_\mathfrak{m}^{-1}$, $t = t_w e_w + u t_\mathfrak{m} \in \mathcal{O}_E^\times$ with $w \in \mathfrak{T}_p$.

Proof. It follows immediately from the definitions of these Whittaker functions together with (3.3), (3.4) and (3.5).

3.7. Local toric period integrals (I).

3.7.1. Define the local toric period integral for $W \in \mathcal{W}(\pi, \psi)$ by

$$P(W, \chi) := P(W, W, \chi) = \int_{E^\times / \mathbb{F}_v^\times} b_v(\pi(i(t))W, \pi(J)W) d\chi(t) \cdot L(1, L_{E/F}) \zeta_F(1).$$

The main task of this section is to evaluate $P(\pi(s)W_{\chi, v}, \chi)$. Put

$$d(a) = \begin{pmatrix} a \\ 1 \end{pmatrix} \quad (a \in \mathbb{F}_v^\times).$$

We first treat the archimedean and split cases.

3.7.2. The archimedean case. Suppose $v = \sigma \in \Sigma \cong \text{Hom}(F, \mathbb{R})$ is an archimedean place.

Proposition 3.3. We have

$$P(\pi(s) V_{\mathfrak{m}_\sigma} W_{\mathfrak{k}_\sigma}, \chi) = 2^3 \cdot \frac{\Gamma(m_{\sigma} + 1) \Gamma(k_{\sigma} + m_{\sigma})}{(4\pi)^{k_\sigma + 1 + 2m_\sigma}}.$$

Proof. Introduce the Hermitian inner product on $\mathcal{W}(\pi, \psi)$ defined by

$$\langle W_1, W_2 \rangle := b_v(W_1, c(W_2)),$$ where $c(W_2)(g) := W(\begin{pmatrix} -1 \\ 1 \end{pmatrix}) g \omega(\det g)$.

Write $k = k_\sigma$ and $m = m_\sigma$. It is clear that

$$\langle W_k, W_k \rangle = (4\pi)^{-k} \Gamma(k).$$

Since $c(V^m_+ W_k)$ and $\pi(\begin{pmatrix} -1 \\ 1 \end{pmatrix}) V^m_+ W_k$ are both nonzero Whittaker functions of weight $-k - 2m$, there exists some constant $c$ such that

$$\pi(i_\chi(t)) V^m_+ W_k = c \cdot V^m_+ W_k \iff V^m_+ W_k(d(a)) = \gamma \cdot V^m_+ W_k(d(a)) \text{ for all } a \in \mathbb{R}_+.$$
Let \( h_m(x) := V^m W_k(d(x)) \). Then \( h_0(x) = W_k(d(x)) \) is a real-valued function in view of the definition (3.6). A simple calculation shows that
\[
h_{m+1} = 2x \frac{dh_m}{dx} + (2\pi x - k - 2m)h_m,
\]
so by induction \( h_m(x) \) takes value in \( \mathbb{R} \) (cf. [JL70, p.189]). This implies that \( \gamma = 1 \). We thus have
\[
b_v(\pi(\varsigma)V^m W_k, \pi(J\varsigma)V^m W_k) = \langle V^m W_k, V^m W_k \rangle \quad (\varsigma^{-1} J\varsigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}).
\]
To evaluate \( \langle V^m W_k, V^m W_k \rangle \), note that by [JL70, p.166] we have
\[
\langle V^m W_k, V^m W_k \rangle = (-1)^m \langle W_k, V^m W_k \rangle
\]
and hence we find that
\[
\langle V^m W_k, V^m W_k \rangle = (-1)^m \langle W_k, V^m W_k \rangle
\]
Recall that \( dt = 2\pi^{-1} d\theta \) with \( t = e^{i\theta} \), so \( \text{vol}(C^x/\mathbb{R}^x, dt) = 2\pi^{-1} \cdot \pi = 2 \). Combining these with Lemma 3.2 (1), we find that
\[
\begin{align*}
P(\pi(\varsigma)V^m W_k, \chi) &= 2 \cdot (-8\pi)^{-2-m} \cdot b_v(\pi(\varsigma)V^m W_k, \pi(J\varsigma)V^m W_k) \\
&= 2^{\frac{3}{2}} (4\pi)^{-2m-1} \cdot \langle V^m W_k, V^m W_k \rangle \\
&= 2^{\frac{3}{2}} (4\pi)^{-2m-1} \cdot \langle W_k, W_k \rangle
\end{align*}
\]
3.7.3. The split case. Suppose that \( v = w\bar{w} \) is a finite place split in \( E \). Recall that we have assumed \( w|\Sigma_p \delta \) if \( v|p \mathfrak{R}^+ \).

**Lemma 3.4.** We have
\[
P(\pi(\varsigma)W, \chi) = \Psi(\frac{1}{2}, W, \chi^{-1}) \cdot \frac{L(\frac{1}{2}, \pi \otimes \chi)}{L(\frac{1}{2}, \pi \otimes \chi_w)} \cdot \epsilon(\frac{1}{2}, \pi \otimes \chi_w, \psi) \cdot \omega^{-1} \chi_w^{-2} (-d_F) \omega(\det \varsigma).
\]
**Proof.** Let \( \tilde{W}(g) := W(g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \omega^{-1} (\det g) \). By [JL70, Thm. 2.18 (iv)], we have the local functional equation:
\[
\begin{align*}
\frac{\Psi(1-s, \frac{1}{2}, W, \chi^{-1})}{L(1-s, \pi \otimes \chi_w^{-1})} &= \epsilon(s, \pi \otimes \chi_w, \psi) \\
&= \frac{\Psi(s, W, \chi_w)}{L(s, \pi \otimes \chi_w)}
\end{align*}
\]
We note that
\[
\varsigma^{-1} J\varsigma = \begin{pmatrix} 0 & d_F^{-1} \\ d_F & 0 \end{pmatrix}.
\]
A straightforward computation shows that
\[
P(\pi(\varsigma)W, \chi) = \omega(\det \varsigma) \int_{F^x} \int_{F^x} W(d(at_1)) W(d(-a \begin{pmatrix} 0 & d_F^{-1} \\ d_F & 0 \end{pmatrix})) \chi_w(t_1) \omega^{-1}(a) d^x \sigma dt_1
\]
\[
= \omega(- \det \varsigma \cdot d_F) \int_{F^x} \int_{F^x} W(d(t_1)) W(d(adF^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \chi_w(t_1) \omega^{-1}\chi_w^{-1}(a) d^x \sigma dt_1
\]
\[
= \omega(- \det \varsigma) \omega^{-1} \chi_w^{-2} (d_F) \cdot \Psi(\frac{1}{2}, W, \chi_w) \Psi(\frac{1}{2}, \tilde{W}, \chi_w^{-1})
\]
\[
= \omega(\det \varsigma) \omega^{-1} \chi_w^{-2} (-d_F) \Psi(\frac{1}{2}, W, \chi_w)^2 \cdot \epsilon(\frac{1}{2}, \pi \otimes \chi_w, \psi) \cdot \frac{L(\frac{1}{2}, \pi \otimes \chi_w^{-1})}{L(\frac{1}{2}, \pi \otimes \chi_w)}.
\]
The lemma thus follows.
Proposition 3.5. We have
\[
\frac{1}{L(\frac{1}{2}, \pi_F \otimes \chi)} \cdot P(\pi(\zeta)W_{X,v}, \chi) = |D_F| \cdot \begin{cases}
\epsilon(\frac{1}{2}, \pi \otimes \chi_w, \psi)\omega^{-1} \chi_w^{-2}(-2\delta) & \text{if } v \mid \mathfrak{N}^+,
\omega(\det \zeta) & \text{if } v \nmid \mathfrak{N}^+.
\end{cases}
\]
If \( v = w \overline{w} \) with \( w \in \mathfrak{F}_p \), then
\[
\frac{1}{L(\frac{1}{2}, \pi_F \otimes \chi)} \cdot P(\pi(\zeta)W_{X,v}, \chi) = \epsilon(\frac{1}{2}, \pi \otimes \chi_w, \psi) \frac{L(\frac{1}{2}, \pi \otimes \chi_w)^2}{L(\frac{1}{2}, \pi \otimes \chi_w)} \cdot \omega^{-1} \chi_w^{-2}(-2\delta) |D_F|.
\]

Proof. The proposition follows immediately from Lemma 3.4, 3.9 and 3.11 combined with our choices of \( d_F \) for \( v \mid \mathfrak{N}^+ \) and the fact that
\[
\epsilon(\frac{1}{2}, \pi \otimes \chi_w, \psi) \cdot \omega^{-1} \chi_w^{-2}(-d_F) = 1 \quad \text{if } v \nmid \mathfrak{N}^+.
\]

3.8. Local toric period integrals (II). In this subsection, we treat the case \( v \) is inert or ramified. A large part of the computation in this subsection is inspired by [Mur08]. Let
\[
w = \begin{pmatrix} 0 & -d_F^{-1} \\ d_F & 0 \end{pmatrix}
\]
and put
\[
K^0(\varpi) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v^0 \mid a - 1 \in \varpi \mathcal{O}, c \in \varpi \mathcal{D}_F \right\}.
\]

Let \( \theta = \theta_v \in \mathcal{O}_E \) be the element chosen in 3.4 and write \( W^0 \) for the new local Whittaker function \( W^0_v \) at \( v \). Recall that \( \{1, \theta\} \) is an \( \mathcal{O} \)-basis of \( \mathcal{O}_E \) and \( \theta \) is a uniformizer if \( E/F \) is ramified.

3.8.1. We prepare some elementary lemmas.

Lemma 3.6. Suppose that \( v \mid \mathfrak{r} \). Let \( m \) be a non-negative integer and let
\[
B^1(\mathcal{O}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & d \end{pmatrix} \mid x \in \mathcal{D}_F^{-1}, d \in \mathcal{O}^\times \right\},
\]
\[
N(\mathcal{D}_F^{-1}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathcal{D}_F^{-1} \right\}.
\]
If \( y \in \varpi^{m+1} \mathcal{O} \), then we have
\[
d(\varpi^m) \zeta_c(x + y \theta) d(\varpi^{-m}) \in K^0(\varpi).
\]
If \( y \in \varpi^r \mathcal{O}^\times \) and \( 0 \leq r \leq m \), then
\[
N(\mathcal{D}_F^{-1}) d(\varpi^m) \zeta(x + y \theta) d(\varpi^{-m}) B^1(\mathcal{O}) = N(\mathcal{D}_F^{-1}) \begin{pmatrix} \varpi^m-r & \varpi^{-m} \\ y \varpi^{-m} & 1 \end{pmatrix} w B^1(\mathcal{O}).
\]
If \( y \in \varpi \mathcal{O} \), then
\[
N(\mathcal{D}_F^{-1}) d(\varpi^m) \zeta(y + \theta) d(\varpi^{-m}) B^1(\mathcal{O}) = N(\mathcal{D}_F^{-1}) \begin{pmatrix} \varpi^{m+c_v-1} \varpi^{-m} \\ \varpi^{-m} \end{pmatrix} w B^1(\mathcal{O}).
\]

Proof. Recall that if \( v \mid \mathfrak{r} \), then \( \theta = d_F^{-1} \varpi \), \( \zeta = \begin{pmatrix} d_F \\ d_F^{-1} \end{pmatrix} \), and hence
\[
\zeta_c(x + y \theta) = \begin{pmatrix} x + y T(\theta) \\ y d_F^{-1} N(\theta) \end{pmatrix} (x, y \in F).
\]
Then the proof is a straightforward calculation, so we omit the details.

Lemma 3.7. Suppose that \( \chi|_{F^\times} \) is trivial on \( 1 + \varpi \mathcal{O} \). For each non-negative integer \( r \), we set
\[
X_r := \int_{\varpi \mathcal{O}} \chi(1 + y \theta) d'y,
\]
where \( d'y \) is the Haar measure on \( \mathcal{O} \) such that \( \text{vol}(\mathcal{O}, d'y) = L(1, \tau_{E/F}) |D_E|^{\frac{1}{2}} |D_F|^{-\frac{1}{2}}. \) Then \( X_r = 0 \) if \( c_v(\chi) > 1 \) and \( 0 < r < c_v(\chi) \) and \( X_r = |\varpi^r| \cdot L(1, \tau_{E/F}) |D_E|^{\frac{1}{2}} |D_F|^{-\frac{1}{2}} \) if \( r \geq c_v(\chi) \).
Let \( Q_r := 1 + \varpi^r \mathcal{O}_E / 1 + \varpi^r \mathcal{O} \). If \( 0 < r < c_v(\chi) \), then \( \chi \) is a non-trivial character on the group \( Q_r \). Note that we have a bijection \( \varpi^r \mathcal{O} \overset{\sim}{\rightarrow} Q_r, y \mapsto 1 + y \theta \) and the pull-back of the quotient measure \( dt \) on \( Q_r \) is \( d'y \). Therefore, we have

\[
\int_{\varpi^r \mathcal{O}} d'y = \int_{Q_r} \chi(t) dt = \begin{cases} 0 & \text{if } 0 < r < c_v(\chi) \\ \text{vol}(\varpi^r \mathcal{O}, d'y) & \text{if } r \geq c_v(\chi). \end{cases}
\]

This finishes the proof. \( \square \)

Define the matrix coefficient \( m^0 : \text{GL}_2(F) \rightarrow \mathbb{C} \) by

\[
m^0(g) := b_v(\pi(g) W^0, \pi(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}) W^0) = \int_{F^\times} W^0(d(a)g) W^0(d(a)) \omega^{-1}(a) d^\times a.
\]

Since \( W^0 \) is invariant by \( K^0(\varpi) \), \( m^0(g) \) only depends on the double coset \( K^0(\varpi) g K^0(\varpi) \) by [22]. Put

\[
m = m_v(\chi, \pi) = c_v(\chi) - \nu(\varepsilon^\perp) \geq -1.
\]

We set

\[
P^*(\pi(\varsigma) R^m_v W^0, \chi) := P(\pi(\varsigma) R^m_v W^0, \chi) \cdot \frac{\zeta_F(1)}{L(1, \tau_{E/F})} \omega(\varpi^{-m} \det \varsigma^{-1})
\]

(3.14)

\[
= \int_{E^\times / F^\times} b_v(R^m_v \pi(\varsigma(t))) R^m_v W^0, \pi(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}) W^0) \chi(t) dt
\]

\[
= \int_{E^\times / F^\times} m^0(d(\varpi^m) \varsigma(t) d(\varpi^{-m})) \chi(t) dt.
\]

Here we have used the fact that \( m + \nu(T(\theta)) \geq 0 \) in the second equality. It follows immediately from the definition of the projector \( \mathcal{P}_{X,\varsigma} \) that

\[
P(\pi(\varsigma) W_{X,\varepsilon}, \chi) = P(\pi(\varsigma) \mathcal{P}_{X,\varsigma} R^m_v W^0, \chi)
\]

(3.15)

\[
= P^*(\pi(\varsigma) R^m_v W^0, \chi) \cdot \frac{\omega(\varpi^{-m} \det \varsigma) L(1, \tau_{E/F})}{\zeta_F(1)}.
\]

Using the decomposition

\[
\mathbb{E}^\times = \mathbb{E}^\times (1 + \mathcal{O} \theta) \bigcup \mathbb{E}^\times (\varpi \mathcal{O} + \theta)
\]

and Lemma [3.6], we find that

\[
P^*(\pi(\varsigma) R^m_v W^0, \chi) = \int_{\mathcal{O}} \chi(1 + y \theta) m^0(\varpi^m) \varsigma(1 + y \theta) d(\varpi^{-m})) d'y
\]

(3.16)

\[
+ \int_{\varpi \mathcal{O}} \chi(x + \theta) m^0(\varpi^m) \varsigma(y + \theta) d(\varpi^{-m})) |y + \theta|_E^{-1} d'y
\]

\[
= X_{m+1} \cdot m^0(1) + \sum_{r=0}^m \int_{\varpi^r \mathcal{O} \times E^\times} \chi(1 + y \theta) \omega(\varpi^{-m} y) d'y \cdot m^0(\varpi^{2(m-r)}) w
\]

\[
+ Y_0 \cdot \omega(\varpi^{-m}) m^0(\varpi^{2m + \varepsilon - 1}) w,
\]

where

\[
Y_0 := \int_{\varpi \mathcal{O}} \chi(y + \theta) d'y \cdot |\varpi|^{1 - \varepsilon_v}.
\]

In what follows, we use Lemma [3.7] and (3.16) to calculate \( P(\pi(\varsigma) W_{X,\varepsilon}, \chi) \).
The case $v \nmid n_\sigma^-$. Suppose that $v \nmid n_\sigma^-$, i.e. the central character $\omega$ is unramified. Then $3.7$ implies that $\pi$ is either an unramified principal series or an unramified special representation.

**Proposition 3.8.** Suppose that $\pi$ is an unramified principal series. Then

$$\frac{1}{|L(\frac{1}{2}, \pi_E \otimes \chi)|} \cdot P(\pi(\zeta)W_\psi, \chi) = \omega(\varpi^m)\left|\varpi^{\omega_\psi}(\chi)\right| |D_E|^\frac{1}{2} \cdot \frac{\omega(\det \varsigma)}{L(1, \tau_{E/F})^2} \cdot \begin{cases} \left(\frac{|\det \varsigma|}{|\chi|}|\varpi^{\omega_\psi}(\chi)\right) & \text{if } c_0(\chi) = 0, \\ \left(1 - \alpha \beta^{-1} |\varpi|\right)(1 - \alpha^{-1} \beta |\varpi|) & \text{if } c_0(\chi) > 0. \end{cases}$$

**Proof.** Since $\pi$ is unramified, $\omega$ is unramified and $m = c_0(\chi)$. Write $\pi = \pi(\mu, \nu)$ and let $\alpha = \mu(\varpi)$ and $\beta = \nu(\varpi)$. The matrix coefficient $m^0$ is a spherical function on $GL_2(F)$ in the sense of [Cas79] Definition 4.1, p.150, and $m^0(g)$ only depends on the double coset $K_0^+ g K_0^0$. By Macdonald formula,

$$m^0(1) = \frac{\zeta_F(1)L(1, \Ad \chi)}{\zeta_F(2)} \cdot |D_F|^\frac{1}{2} = \frac{(1 + |\varpi|)\zeta_F(1)}{(1 - \alpha \beta^{-1} |\varpi|)(1 - \alpha^{-1} \beta |\varpi|)} \cdot |D_F|^\frac{1}{2};$$

$$m^0(d(\varpi)) = \frac{|\varpi|^{\frac{1}{2}}}{1 + |\varpi|} \cdot (\alpha + \beta) \cdot m^0(1);$$

$$m^0(d(\varpi^2)) = \frac{|\varpi|^{1 + |\varpi|}}{1 + |\varpi|} \cdot (\alpha^2 + \beta^2 + (1 - |\varpi|) \alpha \beta) \cdot m^0(1).$$

If $v$ is inert and $m = 0$, then

$$\omega(\det \varsigma^{-1})P(\pi(\zeta)W_\psi^0, \chi) = m^0(1) \cdot \frac{L(1, \tau_{E/F})}{\zeta_F(1)} \cdot |D_E|^\frac{1}{2} \cdot |D_F|^{-\frac{1}{2}}$$

$$= \frac{1}{(1 - \alpha \beta^{-1} |\varpi|)(1 - \alpha^{-1} \beta |\varpi|)} \cdot |D_E|^\frac{1}{2}$$

$$= L(\frac{1}{2}, \pi_E \otimes \chi) \cdot |D_E|^\frac{1}{2}. $$

Suppose that either $v$ is ramified or $m > 0$ (so $v | r$ and $\det \varsigma = 1$). Then we deduce from (3.16) that

$$P(\pi(\zeta)W_\psi^0, \chi) = X_0 \cdot m^0(1) + \sum_{r=0}^{m-1} (X_r - X_{r+1}) \omega(\varpi^{r-m}) \cdot m^0(d(\varpi^{2(m-r)}))$$

$$+ Y_0 \cdot \omega(\varpi^{-m}) \cdot m^0(d(\varpi^{2m+\varepsilon+1})).$$

If $v$ is ramified and $m = 0$, then $X_0 = |D_E|^\frac{1}{2} \cdot |D_F|^{-\frac{1}{2}}$ and $Y_0 = \chi(\varpi_E) \cdot D_E|^\frac{1}{2} \cdot |D_F|^{-\frac{1}{2}}$. By (3.20), we find that

$$P(\pi(\zeta)W_\psi^0, \chi) = \frac{(m^0(1) + \chi(\varpi_E) \cdot m^0(d(\varpi)) \cdot \frac{L(1, \tau_{E/F})}{\zeta_F(1)} \cdot |D_E|^\frac{1}{2} \cdot |D_F|^{-\frac{1}{2}}}{|1 + \chi(\varpi_E)|} \cdot m^0(1).$$

$$= \frac{1 + \chi(\varpi_E) \alpha |\varpi|^{\frac{1}{2}} + \chi(\varpi_E) \beta |\varpi|^{\frac{1}{2}}}{1 + |\varpi|} \cdot m^0(1).$$

Suppose that $m > 0$. Note that since $\chi|_{\varpi^\psi} = 1$, $Y_0 = -X_0$ if $v$ is inert and $Y_0 = X_0 = 0$ if $v$ is ramified. Combining with Lemma 3.7, (3.19) and 3.20), we find that

$$P(\pi(\zeta)W_\psi^m, \chi) = X_m \cdot m^0(1) \cdot \omega(\varpi^m) \cdot \omega(\varpi^m) \cdot \frac{L(1, \tau_{E/F})}{\zeta_F(1)} \cdot m^0(1)$$

$$= \omega(\varpi^m) |\varpi^m| \cdot \left(1 - \alpha \beta^{-1} |\varpi|\right)(1 - \alpha^{-1} \beta |\varpi|) \cdot m^0(1)$$

$$\times \frac{L(1, \tau_{E/F})^2}{\zeta_F(1)} \cdot |D_E|^\frac{1}{2} \cdot \left(1 + |\varpi|\right)^{-\frac{1}{2}}$$

$$= \omega(\varpi^m) |\varpi^m| \cdot |D_E|^\frac{1}{2} \cdot L(1, \tau_{E/F})^2.$$

The proposition follows immediately. \(\square\)
Proposition 3.9. Suppose that $\pi$ is an unramified special representation. Then
\[
\frac{1}{L(1, \pi_E \otimes \chi)} \cdot P(\pi(\varsigma)W_{\chi, \nu}, \chi) = \omega(\varpi^m)^2 |D_E|_{E}^{\frac{1}{\nu}} \cdot L(1, \tau_{E/F})^2 \cdot \begin{cases} 
1 & \text{if } c_v(\chi) > 0, \\
2 & \text{if } v \text{ is ramified and } c_v(\chi) = 0.
\end{cases}
\]

Proof. Suppose that $v|n_\nu^\times$. Then $m = m_v(\chi, \pi) = c_v(\chi) - 1$. Recall that $\pi = \pi(\mu, \nu)$ is a special representation with a unramified character $\mu$ and $\nu^{-1} = |.|$. We have
\[
W^0(d(a)) = \mu(a) |a|^\frac{1}{2} 1_{\mathcal{O}}(a), \\
W^0(d(a)w) = -\mu(a) |a|^\frac{1}{2} \varpi |D_{\mathbb{A}}^{-1} \mathcal{O}}(a).
\]
A direct computation shows that
\[
m^0(1) = \frac{|D_E|^\frac{1}{2}}{1 - |\varpi|^2} m^0(w) = (-|\varpi|) m^0(1) = m^0(1) = (-\mu(\varpi)|\varpi|^{-\frac{1}{2}}) \cdot m^0(1).
\]

If $c_v(\chi) > 0$, then it follows from [3,16] and Lemma 3.7 that
\[
P(\pi(\varsigma)\mathcal{R}^m W^0, \chi) = X_{m+1} \cdot (m^0(1) - m^0(w)) \cdot \omega(\varpi^m) \cdot L(1, \tau_{E/F}) \cdot \frac{1}{\zeta(1)}.
\]
If $c_v(\chi) = 0$ ($m = -1$), then $v$ is ramified, $X_0 = |D_E|^\frac{1}{2} |D_F|^{-\frac{1}{2}}$, $Y_0 = \chi(\varpi^2) |D_E|^\frac{1}{2} |D_F|^{-\frac{1}{2}}$, and
\[
P(\pi(\varsigma)\mathcal{R}^m W^0, \chi) = X_0 \cdot m^0(1 + Y_0 - m^0(1)) \cdot \omega(\varpi^{-1}) \cdot \frac{L(1, \tau_{E/F})}{\zeta(1)}
= (1 - \mu(\varpi) \chi(\varpi^2) |\varpi|^{-\frac{1}{2}}) |D_E|^\frac{1}{2} |D_F|^{-\frac{1}{2}} \cdot m^0(1 - |\varpi|) \omega(\varpi^{-1})
= \frac{2 |D_E|^\frac{1}{2} \omega(\varpi^{-1})}{1 + |\varpi|} \cdot (\text{by (R1)})
= 2 |D_E|^\frac{1}{2} \omega(\varpi^{-1}) \cdot L(1, \pi_E \otimes \chi).
\]

3.8.3. The case $v|n^\times$. We consider the case $\pi$ is a ramified principal series. Recall that [53] suggests that $\pi = \pi(\mu, \nu)$, where $\mu$ is unramified and $\nu$ is ramified, and the conductor $a(\nu) = a(\omega) = 1$. Since $\chi|_{\mathbb{A}} = \mathbb{A}^{-1}$, we must have $m = c_v(\chi) - 1 \geq 0$. Let $\delta_v := \theta - \theta$. Let $D_{E/F}$ be the discriminant of $E/F$. We begin with a lemma.

Lemma 3.10. Suppose that $\chi|_{\mathcal{O}} \neq 1$ and $\chi|_{\mathcal{O}} \otimes \chi = 1$. Then
\[
\int_{\mathcal{O}} \chi(y + \theta)dy = \chi(\delta_v) \cdot \frac{c(0, \chi^{-1}, \psi)_{E^2}}{c(-1, \omega, \psi)} \cdot L(1, \tau_{E/F}) \cdot |D_{E/F}|^{\frac{1}{\nu}}.
\]

Proof. By [HKS96] Prop. 8.2, we have
\[
\int_F (y + 2^{-1} \delta_v) dy := \int_F \chi(y + 2^{-1} \delta_v) |y + 2^{-1} \delta_v|_{E^{-1}}^{-\frac{1}{2}} dy = \chi(\delta_v) \cdot \frac{c(0, \chi^{-1}, \psi)_{E^2}}{c(-1, \omega, \psi)}.
\]

By the assumption, for all $r \geq m + 1$ we have
\[
\int_{\mathcal{O}} \chi(y) dy = \chi(\mathbb{A}) \cdot \int_{\mathcal{O}} \chi(y) dy = 0.
\]
Thus
\[
\int_{\mathcal{O}} \chi(y + \theta) dy = \lim_{r \to \infty} \int_{\mathcal{O}} \chi(y + \theta) dy = \int_F \chi(y + \theta) dy = \int_F \chi(y + 2^{-1} \delta_v) dy.
\]
The lemma follows from the fact that
\[ d' y = L(1, \tau_{E/F}) \left| D_{E/F} \right|^\frac{1}{2} \cdot dy. \]

**Proposition 3.11.** Then we have
\[ \frac{1}{L\left(\frac{1}{2}, \pi_E \otimes \chi\right)} \cdot P(\pi(\zeta)W_{\chi,w}, \chi) = \left| \sigma^{-}(\chi) \right| \left| D_{E/F} \right|^\frac{1}{2} \chi(\delta_v d_{E/F}^{-1}) \left| \delta_v \right|^\frac{1}{2} \epsilon(0, \chi, \psi_E) \cdot L(1, \tau_{E/F})^2 \cdot n_v^2, \]
where \( n_v \) is given by
\[ n_v := \frac{\mu(w) \left| \sigma \right|^{m/2} \left| D_F \right|^\frac{1}{2}}{\epsilon(0, \omega, \psi)} \in \mathbb{Z}^+. \]

**Proof.** Note that \( L(s, \pi_E \otimes \chi) = 1 \) and the conductor \( a(\nu) = 1 \). A straightforward computation shows that
\[ W^0(d(a)) = \nu \cdot |\frac{1}{2}(a)\|_O(a), \]
\[ W^0(d(a)w) = \mu \cdot |\frac{1}{2}(a)\|_\infty^{-1}(1 - |\sigma|) \cdot |D_F|^\frac{1}{2}. \]

It is not difficult to show that if \( m = c_v(\chi) - 1 > 0 \), then
\[ \int_{\varpi \cdot \mathcal{O}} \chi(y^{-1} + \theta) d'y = 0 \quad \text{for} \ 0 < r < m \quad \text{and} \]
\[ \int_{\mathcal{O}} \chi(y + \theta) d'y = 0, \]
and that if \( v \) is ramified, then
\[ Y_0 = \int_{\varpi \cdot \mathcal{O}} \chi(y + \theta) d'y = 0. \]

From the above equations, we find that
\[ P^*(\pi(\zeta)R_v^m W^0, \chi) = X_{m+1} \cdot m^0(1) + \sum_{r=0}^{m} \int_{\varpi \cdot \mathcal{O}} \chi(y^{-1} + \theta) d'y \cdot \omega(\varpi^{-m}) m^0(d(\varpi^{2m-2r}) w) \]
\[ + Y_0 \cdot \omega(\varpi^{-m}) m^0(d(\varpi^{2m+\varepsilon^{-1}}) w) \]
\[ = \int_{\varpi \cdot \mathcal{O}} \chi(y + \theta) d'y \cdot \omega(\varpi^{-m}) \left| \varpi^{2m} \right| m^0(w). \]

By Lemma 3.10 we obtain
\[ P(\pi(\zeta)R_v^m W^0, \chi) = P^*(\pi(\zeta)R_v^m W^0, \chi) \cdot \omega(\varpi^m) \frac{L(1, \tau_{E/F})}{\zeta_F(1)} \]
\[ = \left| \varpi^{2m} \right| \cdot \chi(\delta_v) \left| D_{E/F} \right|^\frac{1}{2} \left| \delta_v \right|^\frac{1}{2} \cdot \frac{\epsilon(0, \chi^{-1}, \psi_E)}{\epsilon(-1, \omega, \psi)} \cdot \frac{\mu(\varpi^2) \omega(d_F)}{\epsilon(0, \omega, \psi)(1 - |\varpi|)} \cdot \frac{L(1, \tau_{E/F})^2}{\zeta_F(1)} \]
\[ = \frac{L(1, \tau_{E/F})^2 \mu(\varpi^2) \left| D_F \right|^\frac{1}{2} \left| \varpi^{2m+1} \right|}{\epsilon(0, \omega, \psi)^2} \cdot \left| D_{E/F} \right|^\frac{1}{2} \chi(\delta_v d_{E/F}^{-1}) \left| \delta_v \right|^\frac{1}{2} \epsilon(0, \chi^{-1}, \psi_E). \]

The last equality follows from
\[ \epsilon(-1, \omega, \psi) = \left| \varpi D_F \right|^{-1} \epsilon(0, \omega, \psi). \]

The proposition follows. \( \square \)
3.9. Non-vanishing of the local Fourier coefficients. The function $a_{\chi,v} : F^\times \to \mathbb{C}$ defined by
$$a_{\chi,v}(a) = W_{\chi,v}(d(a))$$
is called the local Fourier coefficient associated to $W_{\chi,v}$.

**Definition 3.12** (Normalized local Fourier coefficients). Let
$$B(\chi) = \{ v \in \mathfrak{h} \mid v \text{ is non-split with } c_v(\chi) > 0 \}.$$For $v \in B(\chi)$, let $n_v$ be defined as in (3.21) if $v|n_v$ and $n_v = 1$ if $v \nmid n_v$. Define the normalized local Fourier coefficient $a_{\chi,v}^*$ by
$$a_{\chi,v}^* := a_{\chi,v} \cdot \begin{cases} n_v^{-1}L(1,\tau_{E_v/F_v})^{-1} & \text{if } v \in B(\chi), \\ 1 & \text{if } v \in A(\chi), \\ e_v & \text{otherwise.} \end{cases}$$Recall that $e_v = 1$ if $v$ is unramified and $e_v = 2$ if $v$ is ramified.

Let $v \nmid p$ be a finite place. We are going to show the normalized local Fourier coefficients $a_{\chi,v}^*$, indeed take value in a finite extension of $\mathbb{Z}_p$. When regarded as a $\mathbb{C}_p$-valued function via $i_p : \mathbb{C} \cong \mathbb{C}_p$ and is not identically zero modulo $m$. This is clear if $v$ is split in view of the definition of $a_{\chi,v}^* = a_{\chi,v}$ in [3.6]. We give the formulae of $a_{\chi,v}$.

**Lemma 3.13.** Suppose that $c_v(\chi) = 0$. Then
$$a_{\chi,v}^*(a) = \begin{cases} W_v^0(d(a)) & \text{if } v \nmid n \text{ is unramified,} \\ W_v^0(d(a)) + W_v^0(d(a\varpi_v)\chi(\varpi_{E_v})) & \text{if } v \nmid n \text{ is ramified.} \end{cases}$$If $v | n$, then $v$ is ramified and
$$a_{\chi,v}^*(a) = \mu(1)\frac{1}{2}(\varpi)(a)\varpi^{-1}L(1,\tau_{E_v/F_v}).$$

**Proof.** It is well-known that if $\pi = \pi(\mu,\nu)$ is an unramified principal series, then
$$W_v^0(d(a)) = \varpi^{-1}L(1,\tau_{E_v/F_v})^{-1} \sum_{i+j=v(a), i,j \geq 0} \mu(\varpi^i)\nu(\varpi^j)$$
(cf. [Bum97 Thm. 4.6.5]). It follows from the definition of $W_{\chi,v}$ that $W_{\chi,v} = W_v^0$ if $v \nmid n$ is unramified and
$$W_{\chi,v}(g) = \frac{1}{2} \cdot W_v^0(g) + \frac{1}{2} \cdot W_v^0(gd(\varpi))\chi(\varpi_E)$$if $v \nmid n$ is ramified.

If $v|n$, then $v \in A(\chi)$. By (3.11) $v$ is ramified, and we find that
$$W_{\chi,v}(g) = \frac{1}{2} \cdot W_v^0(gd(\varpi)) + \frac{1}{2} \cdot W_v^0(g\varpi)\omega(\varpi)\chi(\varpi_E).$$
The assertion follows from the formulas of $W_v^0$ in Prop. [3.9].

To treat the case $v$ is non-split with $c_v(\chi) > 0$, i.e. $v \in B(\chi)$, we need to introduce certain partial Gauss sums. For a non-split place $v$, write $\pi = \pi(\mu,\nu)$ with unramified $\mu$ and $\mu\nu^{-1}(\varpi) \neq |\varpi|^{-1}$ if $\pi$ is unramified or special. Define a character $\tilde{\psi}_{\pi,\chi,v} : E^\times \to \mathbb{C}^\times$ by
$$\tilde{\psi}_{\pi,\chi,v}(t) := \mu(N(t)) \cdot |\chi|_{|E^\times/\mathbb{Q}^\times}|^t(t).$$

**Recall that the partial Gauss sum $\tilde{A}_{\beta}(\psi_{\pi,\chi,v})$ in [Hsi12b (4.17)] is defined by**
$$\tilde{A}_{\beta}(\psi_{\pi,\chi,v}) := \lim_{n \to \infty} \int_{E_{\mathfrak{p}}} \psi_{\pi,\chi,v}^{-1}(x + \theta)\psi(-d_F^{-1}\beta x)dx \quad (\beta \in F^\times).$$

**Lemma 3.14.** Let $v \in B(\chi)$ be a non-split place with $c_v(\chi) > 0$. Then we have
$$\frac{e_v}{L(1,\tau_{E/F})} \cdot a_{\chi,v}(a) = \tilde{A}_{\beta}(\psi_{\pi,\chi,v}) \mu(1) \cdot |\chi(a)| \cdot |\varpi(\varpi^m)| D_F^{-\frac{1}{2}}$$
$$\times \begin{cases} 1 & \text{if } v \nmid n, \\ -1 & \text{if } v \mid n_v, \\ |\varpi^m| \chi(\delta_v) |\delta_v|^\frac{1}{2}(0,0,0) |\varpi(\varpi^m)| D_F^{-\frac{1}{2}} & \text{if } v \mid n_r. \end{cases}$$
PROOF. We recall the Whittaker linear functional $\Lambda : I(\mu, \nu) \to \mathbb{C}$ ([Bum97, (6.9), p.498]) is defined by

\[
\Lambda(f) = \int_{F^\times} f\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\right) \psi(-x)dx := \lim_{n \to \infty} \int_{E^\times \cap O} f\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\right) \psi(-x)dx.
\]

Let $\varsigma = \varsigma_v = \left(\frac{d_F}{d_F^*}\right)$ and $m = m_v(\chi, \pi)$. Define $P_{\chi, v} \mathcal{R}_v^m \in \text{End}_C I(\mu, \nu)$ by

\[
P_{\chi, v} \mathcal{R}_v^m f(g) = \text{vol}(E^\times / F^\times, dt)^{-1} \int_{E^\times / F^\times} f\left(\begin{pmatrix} \chi & \varsigma \\ 0 & 1 \end{pmatrix}\right) \iota_\varsigma(t) dt.
\]

We are going to choose a section $f^0$ in $I(\mu, \nu)^{K^0(\pi)}$ such that

\[
W^0_v(g) = \Lambda(\pi(g)f^0).
\]

Let $f^0_{\chi} = P_{\chi, v} \mathcal{R}_v^m f^0$. Then it is not difficult to see that

\[
a_{\chi, v}(a) = W_{\chi, v}(d(a)) = \Lambda(\pi(d(a))f^0_{\chi}).
\]

Put

\[
f^0_{\chi}(\varsigma)^* := \nu^{-1}|\cdot|^2(\mathcal{R}_v^m) v_E \cdot f^0_{\chi}(\varsigma) \quad (v_E = e_v |D_E|^\frac{1}{2} |D_F|^{-\frac{1}{2}}).
\]

Therefore we have

\[
a_{\chi, v}(a) = \nu|\cdot|^2(a) \int_{F^\times} f^0_{\chi}\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\right) \psi(-ax)dx
\]

\[
= f^0_{\chi}\left(\frac{d_F}{d_F^*}\right) |D_F|^{-\frac{1}{2}} \nu|\cdot|^2(a) \int_{F^\times} \Psi^{-1}_{\pi, v}(x + \theta) \psi(-d_F^{-1}ax)dx
\]

\[
= f^0_{\chi}(\varsigma)^* \nu |\cdot|^2(a) \widetilde{A}_v(\Psi_{\pi, v}) \cdot e_v^{-1} |D_E|^\frac{1}{2} |D_F|^{-\frac{1}{2}} \nu |\cdot|^2(\mathcal{R}_v^m).
\]

It remains to determine $f^0$ and $f^0_{\chi}(\varsigma)$. Since $W^0$ is uniquely characterized by the property that $W^0 \in W(\pi, \psi)$ is right $K^0(\pi)$-invariant with $W^0(1) = 1$, it suffices to construct $f^0 \in \pi(\mu, \nu)^{K^0(\pi)}$ with $\Lambda(f^0) = 1$. To calculate $f^0_{\chi}(\varsigma)$, following (3.16) we have

\[
f^0_{\chi}(\varsigma)^* = \int_{\Omega} \chi(1 + y\theta) f^0(\varsigma \cdot \mathcal{R}_v^m \iota_\varsigma(y + \theta) \mathcal{R}_v^m) dy + \int_{\varpi^\times} \chi(y + \theta) f^0(\varsigma \cdot \mathcal{R}_v^m \iota_\varsigma(y + \theta) \mathcal{R}_v^m) |D_E|^{-\frac{1}{2}} dy
\]

\[
= X_{m+1} f^0(\varsigma) + \sum_{r=0}^{m} \int_{\varpi^\times \cap O^x} \chi(1 + y\theta) \omega(\varpi^{-m}y) dy f^0(\varsigma d(\varpi^{2(m-r)}))
\]

\[
+ Y_0 \cdot \omega(\varpi^{-m}) f^0(\varsigma d(\varpi^{2m+e_v-1})).
\]

Suppose that $\pi$ is a unramified principal series $(v \mid n^-)$ or special representation $(v \mid n^-)$. Then

\[
f^0_{\chi}(\varsigma)^* = X_m f^0(\varsigma) + \sum_{r=0}^{m-1} (X_r - X_{r+1}) \cdot \omega(\varpi^{-m}) \cdot f^0(\varsigma d(\varpi^{2(m-r)}))
\]

\[
+ Y_0 \cdot \omega(\varpi^{-m}) f^0(\varsigma d(\varpi^{2m+e_v-1})).
\]

Let $f^{sp\pi}$ be the unique $K^0(\pi)$-invariant function in $I(\mu, \nu)$ with $f^{sp\pi}(\varsigma) = L(1, \mu^{-1}) |D_F|^\frac{1}{2}$. If $\pi$ is an unramified principal series, then we can take $f^0 = f^{sp\pi}$ ([Bum97, Prop. 4.6.8]), and following the computation of the case $c_v(\chi) > 0$ in Prop. (3.8) we find that

\[
f^0_{\chi}(\varsigma)^* = X_m f^0(\varsigma) - \omega(\varpi^{-m}) f^0(\varsigma d(\varpi^2\varsigma)) = X_m \cdot (1 - \mu^{-v^-1}|\cdot|(\varpi)) \cdot f^0(\varsigma)
\]

\[
= |\varpi^m| |D_E|^\frac{1}{2} L(1, r_{E/F}).
\]

If $\pi$ is special, then

\[
f^0 = f^{sp\pi} - \mu^{-1}|\cdot|^2(\varpi)^{-1} \pi\left(1 + \frac{1}{\varpi}\right) f^{sp\pi},
\]
and following the computation of the case \( c_v(\chi) > 0 \) in Prop. 3.9 we find that
\[
\hat{f}_\chi^0(\varsigma) = X_{m+1} \cdot (f_\chi^0(\varsigma) - f_\chi^0(\varsigma \cdot w))
\]
\[
= X_{m+1} \cdot (-|\varsigma w|^{-1} + |\varsigma|) f^{\text{ph}}(\varsigma)
\]
\[
= (-1) \cdot |\varsigma w| |D_E|_{E}^{\frac{1}{2}} L(1, \tau_{E/F}).
\]

Finally, suppose that \( \pi \) is a ramified principal series with ramified \( \nu \) \((\nu \mid n^-)\). Let \( B(F) \) be the group of upper triangular matrices in \( \text{GL}_2(F) \). Let \( f^0 \in I(\mu, \nu) \) be the function supported in \( B(F)wN(D_F^{-1}) \) such that
\[
f^0(\varsigma wn) = |D_F|^\frac{1}{2} \quad \text{for every } n \in N(D_F^{-1}).
\]
Then one checks easily that \( f^0 \) does the job. Following the computation in Prop. 3.11 we find that
\[
s = |w| \cdot L(1, \tau_{E/F}) \cdot |(w)| \cdot \chi(\delta_v) \frac{\zeta(0, \chi^{-1}, \psi_E)}{\zeta(-1, \omega, \psi)} |D_E|_{E}^{\frac{1}{2}} |D_F|^{-\frac{1}{2}}.
\]
This completes the proof in all cases.

To investigate the \( p \)-integrality of \( \text{a}_{\chi,v}^* \), we define the local invariant \( \mu_p(\Psi_{\chi,v}) \) by
\[
\mu_p(\Psi_{\chi,v}) := \inf_{x \in E_v^*} u_p(\Psi_{\chi,v}(x) - 1).
\]
By [Hsi12b (4.17)], \( \tilde{A}_\beta(\Psi_{\chi,v}) \) is indeed an algebraic integer. Moreover, it is proved in [Hsi12b Lemma 6.4] that
\[
\mu_p(\Psi_{\chi,v}) > 0 \iff \tilde{A}_\beta(\Psi_{\chi,v}) \equiv 0 \pmod m \quad \text{for all } \beta \in F^\times.
\]
Therefore, it follows from Lemma 3.14 that if \( v \in B(\chi) \), then \( \text{a}_{\chi,v}^* \) takes values in \( \mathbb{Z}_p \) and
\[
\text{a}_{\chi,v}^* \equiv 0 \pmod m \iff \mu_p(\Psi_{\chi,v}) > 0.
\]
We summarize our discussion in the following proposition.

**Proposition 3.15.** Let \( \mathcal{O} \) be the finite extension of \( \mathcal{O}_{L^n} \) generated by \( \{ \text{a}_{\chi,v}^*(1) \}_{v \in B(\chi)} \) and the values of \( \hat{\chi} \). Then we have

1. the normalized local Fourier coefficient \( \text{a}_{\chi,v}^* \) takes values in \( \mathcal{O} \) for every finite place \( v \nmid p \),
2. if either \( v \not\in B(\chi) \) is unramified or \( v \in A(\chi) \), then \( \text{a}_{\chi,v}^*(1) = 1 \),
3. if \( v \mid n \) is ramified with \( c_v(\chi) = 0 \), then \( \text{a}_{\chi,v}(\omega^{-1}) = 1 \),
4. if \( v \in B(\chi) \), then \( \mu_p(\Psi_{\chi,v}) = 0 \) if and only if there exists \( \eta_v \in F^\times \) such that
\[
\text{a}_{\chi,v}^*(\eta_v) \not\equiv 0 \pmod m.
\]

### 3.10. The global toric period integral

We return to the global situation. Let \( W_{\chi,f}^{(p)} \) be the prime-to-\( p \) Whittaker function given by
\[
W_{\chi,f}^{(p)} = \prod_{v \in \mathbf{h}, v \nmid p} W_{\chi,v} \in \bigotimes_{v \in \mathbf{h}, v \nmid p} W(\pi_v, \psi_v).
\]

**Definition 3.16.** Let \( W_{\chi,\infty} := \prod_{\sigma \in \Sigma} W_{k_{\sigma}} \). Define the \( p \)-modified toric Whittaker function \( W_{\chi} \) by
\[
W_{\chi} = W_{\chi,\infty} \cdot W_{\chi,f}^{(p)} \prod_{v \mid p} W_{\chi,v} \in W(\pi, \psi).
\]

Let \( u = (u_v) \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times = \prod_v \mathcal{O}_{F_v}^\times \). The \( u \)-component \( W_{\chi,u} \) of \( W_{\chi} \) is defined by
\[
W_{\chi,u} = W_{\chi,\infty} \cdot W_{\chi,f}^{(p)} \prod_{v \mid p} W_{\chi,u,v}.
\]

Recall that the automorphic form \( \varphi_W \in \mathcal{A}(\pi) \) associated to \( W \in W(\pi, \psi) \) is defined by
\[
\varphi_W(g) := \sum_{\beta \in F} W_{\chi} \left( \begin{pmatrix} \beta \\ 1 \end{pmatrix} g \right).
\]
Let \( \varphi \) (resp. \( \varphi_{x,u} \)) be the automorphic form associated to \( W \) (resp. \( W_{x,u} \)). Let \( U_p = \prod_{v \mid p} U_v \) be the torsion subgroup of \((\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_p)^{\times}\). It follows immediately from the definition (3.10) that

\[
\varphi = \sum_{u \in \mathcal{U}_p} \varphi_{x,u}.
\]

Choose a sufficiently small prime-to-\( p \) integral ideal \( n_1 \) such that \( W_{x,v} \) is invariant by \( U_v(n_1) \) for all \( v \mid p \). Let \( K = \prod_v K_v \subset \text{GL}_2(\mathbb{A}_f) \) be an open-compact subgroup such that

\[
K_v = K_v^0 \text{ if } v \mid p; \ K_v \subset U_v(n_1) \text{ if } v \nmid p.
\]

For each positive integer \( n \), put

\[ K^n_1 := \left\{ g \in K \mid g_v = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^n} \text{ for all } v \mid p \right\}. \]

It is easy to verify that \( W \) and \( W_{x,u} \) (and hence \( \varphi \) and \( \varphi_{x,u} \)) are invariant by \( K^n_1 \) for sufficiently large \( n \).

The following lemma immediately follows from Lemma 3.2.

**Lemma 3.17.** Let \( T = \prod_v T_v \subset \mathbb{A}_E^{\times} \). Then \( \varphi \) is a toric automorphic form in the sense that for all \( t \in T \), we have

\[
\pi(\iota_v(t)) \tilde{V}^m_+ \varphi = \chi^{-1}(t) \tilde{V}^m_+ \varphi.
\]

In addition, for all \( t \in T_f = \prod_{v \in B} T_v \), we have

\[
\pi(\iota_v(t)) \varphi_{x,u} = \chi^{-1}(t) \varphi_{x,u} t^{1-c},
\]

where \( u \cdot t^{1-c} := ut_{\Sigma_v^{-1} |_{F_v}} \in \left( \mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_p \right)^{\times} \).

Decompose \( c_v^- = c_v^{1,-1} c_v^{2,-2} \) such that \( (c_v^{1,-1}, n_v^-) = 1 \) and \( c_v^{2,-2} \) has the same support with \( n_v^- \). Define a constant \( C'(\pi, \chi) \) by

\[
C'(\pi, \chi) := 2^{\Delta(\chi) + 3|E:F|} N_{E/F} \left( c_v^{1,-1} \right)^{-1} \omega(c_v^{1,-1}) \omega(n_v^-)^{-1} \prod_{v \mid p} \omega(\det c_v)
\]

\[
\times \prod_{w \mid \mathfrak{q}, \nu = \mathfrak{w}\mathfrak{q}} \epsilon\left( \frac{1}{2}, \pi_v \otimes \chi_{\mathfrak{w}}, \psi_v \right) \omega^{-1} \chi_{\mathfrak{w}}^{-2}(-2\delta) \cdot \prod_{e | n_v^-} \chi_v(-\delta_v d_{F_v}^{-1}) |\delta_v|_{E_v}^{-\frac{1}{2}} \epsilon(0, \chi_v^{-1}, \psi_e)
\]

Note that \( C'(\pi, \chi) \) is actually a \( p \)-adic unit as \( p > 2 \) and \( (p, 3n^-) = 1 \). We introduce the normalization factor \( N(\pi, \chi) \) given by

\[
N(\pi, \chi) := \prod_{v \in B(\chi)} L(1, \tau_{E_v/F_v}) n_v.
\]

We have the following central value formula of the toric integral \( P_{\chi}(\pi(\chi)) \tilde{V}^m_+ \varphi) \).

**Theorem 3.18.** We have

\[
P_{\chi}(\pi(\chi)) \tilde{V}^m_+ \varphi)^2 = |D_E|^{-\frac{1}{2}} \frac{\Gamma_{\mathfrak{a}}(k+m)\Gamma_{\mathfrak{a}}(m+1)}{(4\pi)^{k+2m+1.2}} \cdot E_{\Sigma_p}(\pi, \chi) \cdot L\left( \frac{1}{2}, \pi_E \otimes \chi \right) \cdot C'(\pi, \chi) N(\pi, \chi)^2,
\]

where \( E_{\Sigma_p}(\pi, \chi) \) is the Coates’ \( p \)-adic multiplier given by

\[
E_{\Sigma_p}(\pi, \chi) = \prod_{w \in \Sigma_p, \nu = \mathfrak{w}\mathfrak{q}} \epsilon\left( \frac{1}{2}, \pi_v \otimes \chi_{\mathfrak{w}}, \psi_v \right) L\left( \frac{1}{2}, \pi_v \otimes \chi_{\mathfrak{w}} \right)^{-2} \omega^{-1}(\chi_{\mathfrak{w}}^{-2}(-2\delta)).
\]

**Proof.** Note that \( \tilde{V}^m_+ \varphi \) is the automorphic form associated to the Whittaker function

\[ \tilde{W}_\chi^m = \tilde{W}_\chi^m \cdot W_{\chi, \infty}^{(p)} \cdot \prod_{v \mid p} W_{\chi, v}^p, \]
Hence, by Prop. 2.1 we find that
\[
P_{\chi}(\pi(\bar{\chi})) = \prod_{\sigma \in \Sigma} P(\pi(\bar{\chi})) \frac{1}{L\left(\frac{1}{2}, \pi_{E_{\sigma}} \otimes \chi_{\sigma}\right)} \cdot P(\pi(\bar{\chi})) W_{\chi,v}^{g} \chi_{v} \\
\times \prod_{v \in h \cup \mathfrak{p}} \frac{1}{L\left(\frac{1}{2}, \pi_{E_{v}} \otimes \chi_{v}\right)} \cdot P(\pi(\bar{\chi})) W_{\chi,v} \chi_{v} \cdot L\left(\frac{1}{2}, \pi_{E} \otimes \chi\right).
\]

Combining the local calculations of toric integrals of our Whittaker functions (Prop. 3.3 Prop. 3.5 Prop. 3.8 Prop. 3.9 and Prop. 3.11) yields the central value formula.

**Remark 3.19.** Let \( \varphi^2 \) be the automorphic form associated to the toric Whittaker function \( W^2 \) := \( W_{\chi, \infty} \). Then we obtain the following central value formula:
\[
P_{\chi}(\pi(\bar{\chi})) = |D_{E}|^{\frac{1}{2}} \frac{\Gamma_{\Sigma}(k+m+1)}{(4\pi)^{k+2m+1}} \cdot L\left(\frac{1}{2}, \pi_{E} \otimes \chi\right) \cdot C'(\pi, \chi) \cdot N(\pi, \chi)^2.
\]

### 4. Review of Hilbert modular forms

In this section, we review some standard facts about Hilbert modular Shimura varieties and Hilbert modular forms. The main purpose of this section is to recall the notation in

4.1. Let \( V = Fe_1 \oplus Fe_2 \) be a two dimensional \( F \)-vector space and \( \langle , \rangle : V \times V \to F \) be the \( F \)-bilinear alternating pairing defined by \( \langle e_1, e_2 \rangle = 1 \). Let \( \mathcal{Z} = \mathcal{O}_F e_1 \oplus \mathcal{O}_F e_2 \) be the standard \( \mathcal{O}_F \)-lattice in \( V \), which is self-dual with respect to \( \psi(\langle , \rangle) \). For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \), we define an involution \( g \mapsto g' := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

We identify vectors in \( V \) with row vectors according to the basis \( e_1, e_2 \), so \( G(F) = GL_2(F) \) has a natural right action on \( V \). If \( g \in G(F) \), then \( g' = g^{-1} \cdot g \). Define a left action of \( G \) on \( V \) by \( g * x := x \cdot g' \), \( x \in V \).

Hereafter, we let \( K \) be an open-compact subgroup of \( G(A_f) \) satisfying (4.28) and the following conditions:

- \( K \) is neat and \( \det(K) \cap \mathcal{O}_{F, +}^\times \subset (K \cap \mathcal{O}_{F, +})^2 \).
- We also fix a prime-to-\( p \) positive integer \( N \) such that \( U(N) \subset K \).

4.2. **Kottwitz models.** We recall Kottwitz models of Hilbert modular Shimura varieties following the exposition in [Hid04b].

**Definition 4.1 (S-quadruples).** Let \( \square \) be a finite set of rational primes not dividing \( N \) and let \( U \) be an open-compact subgroup of \( K_0 \) such that \( U(N) \subset U \). Let \( \mathcal{W}_U = \mathbf{Z}[[\mathfrak{g}]][\zeta_N] \) with \( \zeta_N = \exp(\frac{2\pi i}{N}) \). Define the fibered category \( \mathcal{A}_U^{(\square)} \) over the category \( SCH_{/W_U} \) of schemes over \( W_U \) as follows. Let \( S \) be a locally noetherian connected \( W_U \)-scheme and let \( \pi \) be a geometric point of \( S \). Objects are abelian varieties with real multiplication (AVRM) over \( S \) of level \( U \), i.e. a \( S \)-quadruple \( (A, \lambda, \iota, \pi^{(\square)})_S \) consisting of the following data:

1. \( A \) is an abelian scheme of dimension \( d \) over \( S \).
2. \( \iota : \mathcal{O}_F \to \text{End}_S A \otimes_{\mathbf{Z}} \mathbf{Z}^{(\square)} \).
3. \( \lambda \) is a prime-to-\( \square \) polarization of \( A \) over \( S \) and \( \lambda \) is the \( \mathcal{O}_{F,(\square), +} \)-orbit of \( \lambda \). Namely
   \[ \lambda = \mathcal{O}_{F,(\square), +} \lambda := \{ \lambda' \in \text{Hom}(A, A^t) \otimes_{\mathbf{Z}} \mathbf{Z}^{(\square)} | \lambda' = \lambda \circ a, a \in \mathcal{O}_{F,(\square), +} \}. \]
4. \( \pi^{(\square)} = \eta^{(\square)} U^{(\square)} \) is a \( \pi_1(S, \mathcal{S}) \)-invariant \( U^{(\square)} \)-orbit of isomorphisms of \( \mathcal{O}_F \)-modules \( \eta^{(\square)} : \mathcal{L} \otimes_{\mathbf{Z}} \mathcal{A}_f^{(\square)} \to V^{(\square)}(\mathcal{A}_\mathfrak{g}) := H_1(A_f, \mathcal{Z}^{(\square)}) \otimes_{\mathbf{Z}} \mathcal{A}_f^{(\square)} \).

Furthermore, \( (A, \lambda, \iota, \pi^{(\square)})_S \) satisfies the following conditions:

- Let \( \iota \) denote the Rosati involution induced by \( \lambda \) on \( \text{End}_S A \otimes \mathbf{Z}^{(\square)} \). Then \( \iota(b) = \iota(b), \forall b \in \mathcal{O}_F \).
- Let \( \iota \) denote the Weil pairing induced by \( \lambda \). Lifting the isomorphism \( \mathbf{Z}/N\mathbf{Z} \simeq \mathbf{Z}/N\mathbf{Z}(1) \) induced by \( \zeta_N \) to an isomorphism \( \zeta : \mathbf{Z} \simeq \mathbf{Z}(1) \), we can regard \( \iota \) as an \( F \)-alternating form \( \iota : V^{(\square)}(A) \times V^{(\square)}(A) \to \mathcal{D}_F \otimes_{\mathbf{Z}} \mathcal{A}_f^{(\square)} \). Let \( \eta \) denote the \( F \)-alternating form on \( V^{(\square)}(A) \) induced by \( \eta(x, x') = \langle x\eta, x'\eta \rangle \).

Then
\[ e^\lambda = u \cdot e^\eta \text{ for some } u \in \mathcal{A}_f^{(\square)}. \]
• As $\mathcal{O}_F \otimes \mathbb{Z} \mathcal{O}_S$-modules, we have an isomorphism $\text{Lie} A \simeq \mathcal{O}_F \otimes \mathbb{Z} \mathcal{O}_S$ locally under Zariski topology of $S$.

For two $S$-quadruples $A = (A, \lambda, \tau, \eta)_{S}$ and $A' = (A', \lambda', \tau', \eta')_{S}$, we define morphisms by

$$\text{Hom}_{A^{(\square)}}(A, A') = \left\{ \phi \in \text{Hom}_{\mathcal{O}_F}(A, A') \mid \phi \circ \eta = \lambda, \phi \circ \eta' = \lambda' \right\} .$$

We say $A \sim A'$ (resp. $A \simeq A'$) if there exists a prime-to-$\square$ isogeny (resp. isomorphism) in $\text{Hom}_{A^{(\square)}}(A, A')$.

We consider the cases when $\square = \emptyset$ and $\{p\}$. When $\square = \emptyset$ is the empty set and $U$ is an open-compact subgroup in $G(A_f^{(\square)}) = G(A_f)$, we define the functor $\mathcal{E}_U : \text{SCH}_{/W_U} \to \text{SETS}$ by

$$\mathcal{E}_U(S) = \left\{ (A, \lambda, \tau, \eta)_{S} \in A_K(S) \right\} / \sim .$$

By the theory of Shimura-Deligne, $\mathcal{E}_U$ is represented by $\text{Sh}_{U}$ which is a quasi-projective scheme over $W_U$. We define the functor $\mathcal{E}_U : \text{SCH}_{/W_U} \to \text{SETS}$ by

$$\mathcal{E}_U(S) = \left\{ (A, \lambda, \tau, \eta)_{S} \in A_U^{(\square)}(S) \mid \eta(\square) \mathcal{L} \otimes \mathbb{Z} \mathcal{Z} = H_1(A_\square, \mathcal{Z}) \right\} / \sim .$$

By the discussion in [Hid04b], we have $\mathcal{E}_U \to \mathcal{E}_K$ under the hypothesis (heat).

When $\square = \{p\}$ and $U = K$, we let $W = W_K = \mathcal{Z}_{\{p\}}$ and define functor $\mathcal{E}_K^{(p)} : \text{SCH}_{/W} \to \text{SETS}$ by

$$\mathcal{E}_K^{(p)}(S) = \left\{ (A, \lambda, \tau, \eta^{(p)})_{S} \in A_K^{(p)}(S) \right\} / \sim .$$

It is shown in [Hid04b] §4.2.1 that $\mathcal{E}_K^{(p)} \simeq \mathcal{E}_K^{(p)}$.

Let $c$ be a prime-to-$pN$ ideal of $\mathcal{O}_F$ and let $c \in (A_f^{(pN)})^\times$ such that $c = \text{if}(c)$. We say $(A, \lambda, \tau, \eta^{(p)})$ is $c$-polarized if $\lambda \in \mathcal{L}$ such that $e^\lambda = uc \eta$, $u \in c \text{det}(K)$. The isomorphism class $[(A, \lambda, \tau, \eta^{(p)})]$ is independent of a choice of $\lambda$ in $\mathcal{L}$ under the assumption (heat) (cf. [Hid04b] p.136). We consider the functor

$$\mathcal{E}_{c,K}^{(p)}(S) = \left\{ \text{c-polarized S-quadruple } [(A, \lambda, \tau, \eta^{(p)})]_S \in \mathcal{E}_K^{(p)}(S) \right\} .$$

Then $\mathcal{E}_{c,K}^{(p)}$ is represented by a geometrically irreducible scheme $\text{Sh}_{K}^{(p)}(c)/W$, and we have

$$\text{Sh}_{K}^{(p)}(c) = \bigcup_{[c] \in \text{Cl}^{(p)}(K)} \text{Sh}_{K}^{(p)}(c)/W,$$

where $\text{Cl}^{(p)}(K)$ is the narrow ray class group of $F$ with level $\text{det}(K)$.

4.3. Igusa schemes. Let $n$ be a positive integer. Define the functor $I_{K,n}^{(p)} : \text{SCH}_{/W} \to \text{SETS}$ by

$$S \mapsto I_{K,n}^{(p)}(S) = \left\{ (A, \lambda, \tau, \eta^{(p)}, j)_S \right\} / \sim ,$$

where $(A, \lambda, \tau, \eta^{(p)})_S$ is a $S$-quadruple, $j$ is a level $p^n$-structure, i.e. an $\mathcal{O}_F$-group scheme morphism

$$j : D_F^{-1} \otimes \mathbb{Z} \mu_{p^n} \to A[p^n],$$

and $\sim$ means modulo prime-to-p $\mathbb{Z}$ isogeny. It is known that $I_{K,n}^{(p)}$ is relatively representable over $\mathcal{E}_K^{(p)}$ (cf. [HLS06] Lemma (2.1.6.4)) and thus is represented by a scheme $I_{K,n}$.

Now we consider $S$-quintuples $(A, \lambda, \tau, \eta^{(p)}, j)_S$ such that $[(A, \lambda, \tau, \eta^{(p)})] \in \mathcal{E}_{c,K}^{(p)}(S)$. Define the functor $I_{K,n}^{(p)}(c) : \text{SCH}_{/W} \to \text{SETS}$ by

$$S \mapsto I_{K,n}^{(p)}(c)(S) = \left\{ (A, \lambda, \tau, \eta^{(p)}, j)_S \text{ as above} \right\} / \sim .$$
Then $\mathcal{I}_{K,n}^{(p)}(c)$ is represented by a scheme $I_{K,n}(c)$ over $Sh_{K}^{(p)}(c)$, and $I_{K,n}(c)$ can be identified with a geometrically irreducible subscheme of $I_{K,n}$ ([DRM] Thm. (4.5)). For $n \geq n' > 0$, the natural morphism $\pi_{n,n'}: I_{K,n}(c) \rightarrow I_{K,n'}(c)$ induced by the inclusion $D_{F}^{-1} \otimes \mu_{p^{n'}} \rightarrow D_{F}^{-1} \otimes \mu_{p^{n}}$ is finite étale. The forgetful morphism $\pi: I_{K,n}(c) \rightarrow Sh_{K}^{(p)}(c)$ defined by $\pi: (\underline{A}, j) \mapsto \underline{A}$ is étale for all $n > 0$. Hence $I_{K,n}(c)$ is smooth over Spec $\mathcal{W}$. We write $I_{K}(c)$ for $\lim_{n} I_{K,n}(c)$.

4.4. **Complex uniformization.** We describe the complex points $Sh_{U}(C)$ for $U \subset G(A)$. Put

$$X^{+} = \{ \tau = (\tau_{\sigma})_{\sigma \in \Sigma} \in C^{\Sigma} \mid \text{Im } \tau_{\sigma} > 0 \text{ for all } \sigma \in \Sigma \}.$$  

The action of $g = (g_{\sigma})_{\sigma \in \Sigma} \in G(F \otimes Q R)$ with $g_{\sigma} = \left(\begin{array}{cc} a_{\sigma} & b_{\sigma} \\ c_{\sigma} & d_{\sigma} \end{array} \right)$ and $\text{det } g_{\sigma} > 0$ on $X^{+}$ is given by $\tau \mapsto g\tau = \left(\frac{a_{\sigma} \tau_{\sigma} + b_{\sigma}}{c_{\sigma} \tau_{\sigma} + d_{\sigma}}\right)$. Let $F_{+}$ be the set of totally positive elements in $F$ and let $G(F)^{+} = \{ g \in G(F) \mid \text{det } g \in F_{+} \}$. Define the complex Hilbert modular Shimura variety by

$$M(X^{+}, U) := G(F)^{+} \backslash X^{+} \times G(A) / U.$$  

It is well known that $M(X^{+}, K) \sim Sh_{U}(C)$ by the theory of abelian varieties over $C$ (cf. [Hid04b] § 4.2). For $\tau = (\tau_{\sigma})_{\sigma \in \Sigma} \in X^{+}$, we shall let $p_{\tau}$ be the isomorphism $V \otimes Q R \sim C^{\Sigma}$ defined by $p_{\tau}(a_{\sigma}e_{1} + b_{\sigma}e_{2}) = a\tau + b$ with $a, b \in F \otimes Q R = R^{\Sigma}$. We can associate a AVRM to $(\tau, g) \in X^{+} \times G(A)$ as follows.

- The complex abelian variety $A_{g}(\tau) = C^{\Sigma \prime} / p_{\tau}(g \cdot \mathcal{L}).$
- The $F_{+}$-orbit of polarization $(\cdot)_{\text{can}}$ on $A_{g}(\tau)$ is given by the Riemann form $(\cdot, \text{can}) := (\cdot) \circ p_{\tau}^{-1}.$
- The $\iota_{C} : O \rightarrow \text{End} A_{g}(\tau) \otimes Z Q$ is induced from the pull back of the natural $F$-action on $V$ via $p_{\tau}$.
- The level structure $\eta_{g} : \mathcal{L} \otimes \mathcal{Z} A_{[g]} \rightarrow (g \cdot \mathcal{L}) \otimes \mathcal{Z} A_{f} = H_{1}(A_{g}(\tau), A_{f})$ is defined by $\eta_{g}(v) = g \ast v$.

Let $A_{g}(\tau)$ denote the $C$-quadruple $(A_{g}(\tau), (\cdot)_{\text{can}}, \iota_{C}, K, \eta_{g})$. Then the map $[(\tau, g)] \mapsto [A_{g}(\tau)]$ gives rise to an isomorphism $M(X^{+}, U) \sim Sh_{U}(C)$.

For a positive integer $n$, the exponential map gives the isomorphism $\exp(2\pi i -) : p^{-n}Z / Z \simeq \mu_{p^{n}}$ and thus induces a level $p^{n}$-structure $j(g_{p})$:

$$j(g_{p}) : D_{F}^{-1} \otimes \mu_{p^{n}} \rightarrow D_{F}^{-1} e_{2} \otimes Z p^{-n}Z / Z \rightarrow \mathcal{L} \otimes Z p^{-n}Z / Z \rightarrow A_{g}(\tau)[p^{n}].$$

Put

$$K^{n}_{\tau} := \left\{ g \in K \mid g_{p} \equiv \left(\begin{array}{cc} 1 & \ast \\ 0 & \ast \end{array} \right) \pmod{p^{n}} \right\}.$$  

We have a non-canonical isomorphism:

$$M(X^{+}, K^{n}_{\tau}) \sim I_{K,n}(C)$$

$$[(\tau, g)] \mapsto ((A_{g}(\tau), (\cdot)_{\text{can}}, \iota_{C}, \eta_{g}^{(p)}, j(g_{p}))).$$

Let $z = \{z_{\sigma}\}_{\sigma \in \Sigma}$ be the standard complex coordinates of $C^{\Sigma}$ and $d_{z} = \{dz_{\sigma}\}_{\sigma \in \Sigma}$. Then $\mathcal{O}_{F}$-action on $d_{z}$ is given by $\iota_{C}(\alpha) \ast dz_{\sigma} = \sigma(\alpha)dz_{\sigma}, \sigma \in \Sigma \simeq \text{Hom}(F, C)$. Let $z = z_{\text{id}}$ be the coordinate corresponding to $\iota_{\text{id}} : F \rightarrow Q \rightarrow C$. Then

$$(\mathcal{O}_{F} \otimes Z C)dz_{\gamma} = H^{0}(A_{g}(\tau), \Omega_{A_{g}(\tau) / C}).$$

4.5. **Hilbert modular forms.** Let $k = \sum_{\sigma} k_{\sigma} \sigma \in Z_{\geq 1}[\Sigma]$ such that

$$k_{\sigma_{1}} \equiv k_{\sigma_{2}} \equiv \cdots \equiv k_{\sigma_{d}} \pmod{2} \text{ for all } \sigma_{1}, \ldots, \sigma_{d} \in \Sigma.$$  

For $\tau = (\tau_{\sigma})_{\sigma \in \Sigma} \in X^{+}$ and $g = \left(\begin{array}{cc} a_{\sigma} & b_{\sigma} \\ c_{\sigma} & d_{\sigma} \end{array} \right)$, we put

$$J^{k}(g, \tau)^{k} = \prod_{\sigma \in \Sigma} (c_{\sigma} \tau_{\sigma} + d_{\sigma})^{k_{\sigma}}.$$
Definition 4.2. Let $k_{max} = \max_{\alpha \in \Sigma} k_{\alpha}$. Denote by $\mathcal{M}_k(K^n_1, \mathbb{C})$ the space of holomorphic Hilbert modular forms of weight $k$ and level $K^n_1$. Each $f \in \mathcal{M}_k(K^n_1, \mathbb{C})$ is a $\mathbb{C}$-valued function $f: X^+ \times G(A_f) \to \mathbb{C}$ such that the function $f(-, g_f) : X^+ \to \mathbb{C}$ is holomorphic for each $g_f \in G(A_f)$, and for $u \in K^n_1$ and $\alpha \in G(F)^+$,

$$f(\alpha(\tau, g_f)u) = (\det \alpha)^{-\frac{(k_{max} - 2)\Sigma + k}{2}} f(\tau, g_f).$$

Here $\det \alpha$ is considered to be the element $(\sigma(\det \alpha))_{\alpha \in \Sigma}$ in $(\mathbb{C}^\times)^{\Sigma}$.

For every $f \in \mathcal{M}_k(K^n_1, \mathbb{C})$, we have the Fourier expansion

$$f(\tau, g_f) = \sum_{\beta \in \mathcal{F} \cup \{0\}} W_\beta(f, g_f) e^{2\pi i \text{Tr}_{F/Q}(\beta \tau)}.$$

For a semi-group $L$ in $F$, let $L_+ = F_+ \cap L$ and $L_{\geq 0} = L_+ \cup \{0\}$. If $B$ is a ring, we denote by $B[L]$ the set of all formal series

$$\sum_{\beta \in L} a_\beta q^\beta, a_\beta \in B.$$

Let $a, b \in (A_f(pN))^\times$ and let $a = iF(a)$ and $b = iF(b)$. The $q$-expansion of $f$ at the cusp $(a, b)$ is given by

$$(4.3) \quad f_{|(a, b)}(q) = \sum_{\beta \in (N^{-1}ab)_{\geq 0}} W_\beta(f, \begin{pmatrix} a^{-1} & 0 \\ 0 & b \end{pmatrix}) q^\beta \in \mathbb{C}[(N^{-1}ab)_{\geq 0}].$$

If $B$ is a $\mathcal{W}$-algebra in $\mathbb{C}$, we put

$$\mathcal{M}_k(\mathbb{C}, K^n_1, B) = \left\{ f \in \mathcal{M}_k(K^n_1, \mathbb{C}) \mid f_{|(a, b)}(q) \in B[(N^{-1}ab)_{\geq 0}] \text{ for all } (a, b) \text{ such that } ab^{-1} = \epsilon \right\}.$$

4.5. Tate objects. Let $\mathcal{F}$ be a set of $d$ linearly $\mathbb{Q}$-independent elements in $\text{Hom}(F, \mathbb{Q})$ such that $l(F_+) > 0$ for $l \in \mathcal{F}$. If $L$ is a lattice in $F$ and $n$ a positive integer, let $L_{\mathcal{F}, n} = \{ x \in L \mid l(x) > -n \text{ for all } l \in \mathcal{F} \}$ and put $B((L; \mathcal{F})) = \lim_{n \to \infty} B[L_{\mathcal{F}, n}]$. To a pair $(a, b)$ of two prime-to-$pN$ fractional ideals, we can attach the Tate $\mathbb{A}$-algebra $\mathcal{T}_{a,b}(q) := \mathfrak{a}^* \otimes \mathbb{Z} \mathbb{G}_m/q^b$ over $\mathcal{Z}(\mathfrak{a}; \mathcal{F}))$ with $\mathfrak{a}^* := a^{-1}D_1^{-1}$. As described in [Kat78], $\mathcal{T}_{a,b}(q)$ has a canonical ab$^{-1}$-polarization $\lambda_{can}$, and also carries $\omega_{can}$ a canonical $\mathcal{O}_{\mathbb{F}} \otimes \mathbb{Z}(\mathfrak{a}; \mathcal{F})$-generator of $\Omega_{\mathcal{T}_{a,b}(q)}$ induced by the isomorphism $\text{Lie}(\mathcal{T}_{a,b}(q), \mathcal{Z}(\mathfrak{a}; \mathcal{F})) = \mathfrak{a}^* \otimes \mathbb{Z} \text{Lie}(\mathbb{G}_m) \simeq \mathfrak{a}^* \otimes \mathbb{Z}(\mathfrak{a}; \mathcal{F})$. Since $a$ is prime to $p$, the natural inclusion $\mathfrak{a}^* \otimes \mathbb{Z} \mathbb{G}_m \hookrightarrow \mathfrak{a}^* \otimes \mathbb{Z} \mathbb{G}_m$ induces a canonical level $p^n$-structure $\mathbb{G}_m$-valuation $D_1^{-1} \otimes \mathbb{Z} \mathbb{G}_m \hookrightarrow \mathfrak{a}^* \otimes \mathbb{Z} \mathbb{G}_m$ on $\mathcal{T}_{a,b}(q)$. Let $\mathcal{L}_{a,b} = \mathcal{F} \cdot \begin{pmatrix} b \\ a^{-1} \end{pmatrix} = b\mathcal{E}_1 \oplus a^*\mathcal{E}_2$. Then we have a level $N$-structure $\mathcal{L}_{a,b} = \mathcal{L}_{a,b} \otimes \mathcal{F}$, and $\mathcal{F}_{a,b}(N) = \mathcal{T}_{a,b}(q)[N]$ over $\mathcal{Z}(\mathfrak{a}; \mathcal{F})(N^{-1}ab; \mathcal{F})$ induced by the fixed primitive $N$-th root of unity $\zeta_N$. We write $\mathcal{T}_{a,b}(q)$ for the Tate $\mathcal{Z}(\mathfrak{a}; \mathcal{F})$-quadruple $(\mathcal{T}_{a,b}(q), \lambda_{can}, \epsilon_{can}, \eta_{can}, \mathcal{H}_{can})$ at $(a, b)$.

4.5.2. Geometric modular forms. We collect here definitions and basic facts of geometric modular forms. The whole theory can be found in [Kat78] and [Hid01b]. Let $T$ be the algebraic torus over $W$ such that $T(R) = (\mathcal{O}_F \otimes \mathbb{Z} R)^{\times}$ for every $\mathbb{W}$-algebra $R$. Let $k \in \text{Hom}(T, \mathbb{G}_m/\mathbb{W})$. Let $B$ be a $\mathbb{W}$-algebra. Consider \([\mathcal{A}, j] = [(A, \lambda, \omega, \eta^{(p)}, j)] \in \mathcal{K}(\mathbb{A}, C)(\mathbb{C})\) (resp. \([\mathcal{A}, j] = [(A, \lambda, \omega, \eta^{(p)}, j)] \in \mathcal{K}(\mathbb{A}, C)\)) for a $B$-algebra $C$ with a differential form $\omega$ generating $\mathcal{H}(A, \Omega_A/C)$ over $\mathcal{O}_F \otimes \mathbb{Z} C$. A geometric modular form over $B$ of weight $k$ on $\mathcal{K}(\mathbb{A}, C)$ (resp. $I_{K,n}(\mathbb{C})$) is a functorial rule of assigning a value $f(\mathcal{A}, j, \omega) \in C$ satisfying the axioms:

(G1) $f(\mathcal{A}, j, \omega) = f(\mathcal{A}', j', \omega') \in C$ if $(\mathcal{A}, j, \omega) \simeq (\mathcal{A}', j', \omega')$ over $C$,

(G2) For a $B$-algebra homomorphism $\varphi : C \to C'$, we have

$$f((\mathcal{A}, j, \omega) \otimes \mathbb{C} C') = \varphi(f((\mathcal{A}, j, \omega)),$$

(G3) $f((\mathcal{A}, j, \omega)) = k(a^{-1})f((\mathcal{A}, j, \omega))$ for all $a \in T(C) = (\mathcal{O}_F \otimes \mathbb{Z} C)^{\times}$,

(G4) $f(T\mathcal{T}_{a,b}, \omega_{can}) \in B[(N^{-1}ab)_{\geq 0}]$ at all cusps $(a, b)$ in $I_{K,n}(\mathbb{C})$ (resp. $I_{K,n}$).

For each $k \in \mathbb{Z}[\Sigma]$, we regard $k \in \text{Hom}(T, \mathbb{G}_m/\mathbb{W})$ as the character $x \mapsto x^k$, $x \in (\mathcal{O}_F \otimes \mathbb{Z} \mathcal{W})^{\times}$. We denote by $\mathcal{M}_k(c, K^n_1, B)$ (resp. $\mathcal{M}_k(K^n_1, B)$) the space of geometric modular forms over $B$ of weight $k$ on $I_{K,n}(\mathbb{C})$ (resp. $I_{K,n}$). For $f \in \mathcal{M}_k(K^n_1, B)$, we write $f|_c \in \mathcal{M}_k(c, K^n_1, B)$ for the restriction $f|_{I_{K,n}(\mathbb{C})}$. 
For each \( f \in \mathcal{M}_k(K_1^n, C) \), we regard \( f \) as a holomorphic Hilbert modular form of weight \( k \) and level \( K_1^n \) by

\[
 f(\tau, g) = f(A_g(\tau), \lambda_{can}, \iota_c, \tau_g, 2\pi i dz),
\]

where \( dz \) is the differential form in \((4.2)\). By GAGA this gives rise to an isomorphism \( \mathcal{M}_k(K_1^n, C) \cong \mathcal{M}_k(K_1^n, C) \) and \( \mathcal{M}_k((\epsilon, K_1^n, C) \cong \mathcal{M}_k((\epsilon, K_1^n, C) \). Moreover, as discussed in [Katz78 §1.7], we have the following important identity which bridges holomorphic modular forms and geometric modular forms

\[
 f|_{(a,b)}(q) = f(\text{Tate}_{(a,b)}, \omega_{can}) \in C[[N^{-1}ab \geq 0]].
\]

By the \( q \)-expansion principle, if \( B \) is \( W \)-algebra in \( C \) and \( f \in \mathcal{M}_k((\epsilon, K_1^n, B) = \mathcal{M}_k((\epsilon, K_1^n, C) \), then \( f|_c \in \mathcal{M}_k((\epsilon, K_1^n, B) \).

### 4.5.3. \( p \)-adic modular forms.

Let \( B \) be a \( p \)-adic \( W \)-algebra in \( C_p \). Let \( V((\epsilon, K, B) \) be the space of Katz \( p \)-adic modular forms over \( B \) defined by

\[
 V((\epsilon, K, B) := \lim_{\rightarrow} H^0(I_{K,n}(\epsilon)/B/p^{m}B, \mathcal{O}_{I_{K,n}}).
\]

In other words, Katz \( p \)-adic modular forms consist of formal functions on the Igusa tower.

Let \( C \) be a \( B/p^m \)-\( B \)-algebra. For each \( C \)-point \( [(\underline{A}, j)] \in \{(A, \lambda, \eta, j) \in I_K((\epsilon)(C) = \lim_{\rightarrow} I_{K,n}(\epsilon)(C) \) the \( p^\infty \)-level structure \( j \) induces an isomorphism \( j_* : D_F^{-1} \otimes_Z C \simeq \text{Lie A} \) which in turn gives rise to a generator \( \omega(j) \) of \( H^0(A, \Omega_A) \) as an \( \mathcal{O}_F \otimes Z C \)-module. Then we have a natural injection

\[
 \mathcal{M}_k((\epsilon, K_1^n, B) \hookrightarrow V((\epsilon, K, B)
\]

which preserves the \( q \)-expansions in the sense that \( \hat{f}|_{(a,b)}(q) := \hat{f}(\text{Tate}_{(a,b)} = f|_{(a,b)}(q) \). We call \( \hat{f} \) the \( p \)-adic avatar of \( f \).

### 4.6. CM points.

Recall that we have fixed \( \vartheta \in E \) in \((3.4)\) satisfying (d1-3) and the associated embedding \( \iota : E \hookrightarrow M_2(F) \) in \((2.1)\). Let \( q_\vartheta : E \rightarrow V = F e_1 \oplus F e_2 \) be the isomorphism given by \( q_\vartheta(a\vartheta + b) = ae_1 + be_2 \). Then

\[
 q_\vartheta(x) = q_\vartheta(x)\iota(\alpha) \text{ for all } \alpha \in E,
\]

and \( p_\vartheta := q_\vartheta^{-1} : V \otimes Q \rightarrow E \otimes Q \simeq C^\Sigma \) is the period map associated to the point \( \vartheta : (\sigma(\vartheta))_{\sigma\in \Sigma} \in X^+ \).

Let \( \zeta = \prod_\sigma \zeta_\sigma \in G(A) \) where \( \zeta_\sigma \in G_\sigma \) for each place \( v \) is defined in \((3.3)\). Let \( \zeta_f \in G(A_f) \) be the finite part of \( \zeta \). According to our choices of \( \zeta_\sigma \), we have

\[
 \zeta_f * (\mathcal{L} \otimes \mathcal{Z}) = (\mathcal{L} \otimes \mathcal{Z}) \cdot \zeta_f = q_\vartheta(O_E \otimes \mathcal{Z}).
\]

Define \( x : A_E^\infty \rightarrow X^+ \times G(A_f) \) by

\[
 a = (a_\infty, a_f) \mapsto x(a) := (\vartheta, \iota(a_f)\zeta_f).
\]

Let \( a \in (A_E^{(p\infty)})^\times \) and let

\[
 (A(a), j(a)) \in C = (A_{\iota_{a}}\zeta_f)(\vartheta, \zeta_f, \iota_{\text{can}}, \epsilon_{\text{can}}, \eta^{(p)}(a), j(a))
\]

be the \( C \)-quintuple associated to \( a(x) \) as in \((4.4)\). The alternating pairing \( \langle \cdot \rangle : E \times E \rightarrow F \) defined by \( \langle x, y \rangle = (\tau y - x \tau)/((\vartheta - \overline{\vartheta}) \) induces an isomorphism \( O_E \wedge O_F \mathcal{O}_E = c(O_E)^{-1}D_F^{-1} \) for the fractional ideal \( c(O_E) = D_F^{-1}((\vartheta - \overline{\vartheta})D_E^{-1}p) \). The hypothesis (d2) on \( \vartheta \) implies that

\[
 c(O_E) \text{ is prime to } \rho_{\zeta_f}nD_{E/F}.
\]

Note that \( c(O_E) \) descends to a fractional ideal of \( O_p \) and that \( c(O_E) \) is the polarization of \( x(1) = (\underline{A}(1), j(1)) \). In addition, \( x(a) = (A(a), j(a)) \in C \) is an abelian variety with CM by \( O_E \) with the polarization ideal of \( x(a) \) given by

\[
 c(a) := c(O_E)N(a)^{-1} \quad (a = i_{E}(a)).
\]

It thus gives rise to a complex point \([x(a)] \) in \( I_K(c(a))(C) \). Let \( W_p \) be the \( p \)-adic completion of the maximal unramified extension of \( Z_p \) in \( C_p \). The general theory of CM abelian varieties shows that \([x(a)] \) indeed descends to a point in \( I_K(c(a))(W_p) \hookrightarrow I_K(W_p) \), which is still denoted by \( x(a) \). The collection \( \{[x(a)]\}_{a \in (A_E^{(p\infty)})^\times} \subset I_K(W_p) \) are called CM points in Hilbert modular Shimura varieties.
5. Anticyclotomic $p$-adic Rankin-Selberg $L$-functions

5.1. Toric forms.

Definition 5.1 (The toric form). We define the complex Hilbert modular form $f_{\chi} : X^+ \times G(A_f) \to \mathbb{C}$ associated to $\varphi_{\chi}$ by

\[
f_{\chi}(\tau, g_f) = \varphi_{\chi}(g) \cdot \mathcal{J}(g_\infty, i)^k (\det g_\infty)^{-\frac{(k_{mx} - 2)\Sigma_1}{2}} |\det g|_{\mathbb{A}_f}^{k_{mx}/2 - 1},
\]

where $i = (\sqrt{-1})_{\sigma \in \Sigma}$, $g = (g_\infty, g_f)$, $g_\infty i = \tau$, $\det g_\infty > 0$.

Let $f^*_\chi$ be the normalization of $f_{\chi}$ given by

\[
f^*_\chi = N(\pi, \chi)^{-1} |\det \varsigma|^{}_{\mathbb{A}_f}^{1-k_{mx}/2} \cdot f_{\chi}.
\]

Let $\delta_k^m$ be the Shimura-Maass differential operator (cf. [HT93] (1.21)). Then the normalized differential operator $\widetilde{\delta}_k^m$ is the representation theoretic avatar of $\delta_k^m$ in the following sense:

\[
\delta_k^m f_{\chi}(\tau, g_f) = (\widetilde{\delta}_k^m, \varphi_{\chi})(g_\infty, g_f)\mathcal{J}(g_\infty, i)^k (\det g_\infty)^{-\frac{k_{mx}E+k}{2} - m} |\det g|_{\mathbb{A}_f}^{k_{mx}/2 - 1}
\]

(cf. [Hsi12a] §4.5). We call $\delta_k^m f_{\chi}$ the normalized toric form of character $\chi$.

Similarly, for each $u \in (O_E \otimes \mathbb{Z} \mathbb{Z}^p)^\times$, we let $f_{\chi,u}^*$ be the normalized modular form associated to the $u$-component $\varphi_{\chi,u}$ (cf. $W_{\chi,u}$ in (5.25)). It is clear from (3.27) that

\[
f_{\chi}^* = \sum_{u \in \mathcal{U}_p} f_{\chi,u}^*.
\]

Let $K^n_1$ be the open-compact subgroup defined in (3.29). Then $f_{\chi}^*$ and $\{f_{\chi,u}^*\}_{u \in \mathcal{U}_p}$ belong to $M_k(K_1^n, \mathbb{C})$ for sufficiently large $n$.

For $a \in (A_{E_f}^\times)^\times \times (O_E \otimes \mathbb{Z} \mathbb{Z}^p)^\times$, consider the Hecke action $[a]$ given by

\[
[a] : M_k(c(a), K^n_1, \mathbb{C}) \to M_k(c(a)K^n_1, \mathbb{C}) \quad (c(a)K^n_1 := c(a)K^n_1 c(a^{-1})),
\]

\[
f \mapsto f([a](\tau, g_f)) := f(\tau, g_f c(a)).
\]

The Hecke action $[a]$ can be extended to the spaces of $p$-integral modular forms (cf. [Hsi12a] §2.6). It follows from Lemma 3.17 immediately that

\[
f_{\chi,u}^*([a]) = \chi^{-1} \cdot |\mathbf{A}_{E_f}^\times|^{k_{mx}/2 - 1} (a) \cdot f_{\chi,u}^* |a|_{\mathbf{A}_{E_f}}^{a_{-1} - c} \quad \text{for all } a \in \mathcal{T}_f \quad (a_{-1} = a|_{\mathbf{A}_{E_f}^\times} a_{-1} \in \mathbb{Z}^p).
\]

5.2. The toric period integral. Next we consider the toric period integral of $f_{\chi}^*$. Let $U_E = (E \otimes R) \times (O_E \otimes \mathbb{Z} \mathbb{Z}^p)^\times$ be a subgroup of $A_{E_f}^\times$ and let $CL_- = E^\times A_{E_f}^\times \setminus A_{E_f}^\times / U_E$. Let $\mathcal{R}$ be the subgroup of $A_{E_f}^\times$ generated by $E_{\mathbb{Z}}^\times$ for all ramified places $v$ and let $CL_-^{alg}$ be the subgroup of $CL_-$ generated by the image of $\mathcal{R}$. By Lemma 3.17 and the fact that $\mathcal{T} = \mathbb{A}^\times U_E \mathbb{R}$, we have

\[
P_\chi(\pi(\varsigma) \varphi_{\chi}) = vol(U_E, dt) \xi(CL_-^{alg}) \cdot \sum_{[t] \in CL_- / CL_-^{alg}} \varphi_{\chi}(t(\varsigma)) \chi(t).
\]

Let $D_1$ be a set of representatives of $CL_- / CL_-^{alg}$ in $(A_{E_f}^\times)^\times$. We define the $\chi$-isotypic toric period by

\[
P_\chi(\delta_k^m f_{\chi}^*) := \sum_{a \in D_1} \delta_k^m f_{\chi}^*(x(a))|\chi|^{}_{\mathbb{A}_{E_f}}^{1-k_{mx}/2}(a).
\]

Proposition 5.2. Let $D_{E/f}$ be the discriminant of $E/F$. We have

\[
P_\chi(\delta_k^m f_{\chi}^*)^2 = \left[ O_{E_f}^\times : O_F^\times \right]^2 \cdot \frac{\Gamma_\Sigma(k+m) \Gamma_\Sigma(m+1)}{(\lim \theta)^{k+2m}(4\pi)^{2m+k+1}} \cdot L(1/2, \pi \otimes \chi) \cdot E_{\Sigma}(\pi, \chi) \cdot C(\pi, \chi),
\]

where

\[
C(\pi, \chi) = C'(\pi, \chi) \cdot A_{E_f}^\times |N_{F/E}(D_{E/f})|^{1/2} R \left( \frac{2(CL_- h_F)}{2(CL_-^{alg} h_E)} \right)^2 \in \mathbb{Z}^\times_{(p)}.
\]
PROOF. By definition, we have
\[ f_\chi(x(a)) = \varphi_\chi(\iota(a_f) c)(\Im \theta)^{-k/2} \cdot |N(a) \det \gamma f^{k/2}|. \]
By (5.4), we find that
\[ \vol(U_E, d^x t^z(C^\infty)) \cdot P_\chi(\delta_k^m f_\chi^*) = \frac{1}{N(\pi, \chi) \cdot (\Im \theta)^{k/2+m}} \cdot P_\chi(\pi(\zeta) \varGamma^m_+ \varphi_\chi). \]
From the well-known formula
\[ 2L(1, \tau_{E/F}) = (2\pi)^{|F:Q|} \cdot \frac{h_E/h_F}{|D_E|^{1/2} |D_F|^{-1/2} \cdot |O_E^2 : O_F^2|}, \]
we see that
\[ \vol(U_E, dt) = \vol(E^\times A^\times, dt) \cdot (C^\infty)^{-1} \]
\[ = 2\pi^{-|F:Q|} L(1, \tau_{E/F}) \cdot (C^\infty)^{-1} = \frac{2|F:Q|}{|D_E|^{1/2} |O_E^2 : O_F^2|} \cdot \frac{h_E}{h_F} \cdot (C^\infty)^{-1}. \]
The proposition follows form Theorem 3.13 immediately. Note that the ratio \( \frac{h_F}{h_E} \) is a power of 2, so the constant \( C(\pi, \chi) \) is a p-adic unit. \( \square \)

5.3. The Fourier expansion of \( f_{\chi,u}^* \). Let \( u = (u_v) \in U_p \). We give an expression of the Fourier expansion of \( f_{\chi,u}^* \). Let \( W_{\chi,u,f} \) be the finite part of \( W_{\chi,u} \). By the definition of \( f_{\chi,u} \), we have
\[ f_{\chi,u}^*(\tau, g_f) = \sum_{\beta \in F_F} W_{\chi,u,f}(\begin{pmatrix} \beta \\ 1 \end{pmatrix} g_f) W_{\chi,\infty}(\begin{pmatrix} \beta \\ 1 \end{pmatrix} \begin{pmatrix} y_\infty \\ 0 \\ x_\infty \\ 1 \end{pmatrix}) \cdot y^{-k/2} \]
(5.5)
\[ = \sum_{\beta \in F_F} W_{\chi,u,f}(\begin{pmatrix} \beta \\ 1 \end{pmatrix} g_f) \beta^{k/2} e^{2\pi i \Tr_{F/Q}(\beta \tau)}. \]
(\( \tau = x_\infty + iy_\infty = (x_\sigma + iy_\sigma)_{\sigma \in \Sigma} \in X^+ \))

The second equality follows from the choice of Whittaker functions at the archimedean places \( \Sigma \).

We define the global prime-to-\( p \) Fourier coefficient \( a_\chi^{(p)} : (A_f^{(p)})^\times \rightarrow \mathbb{C} \) by
\[ a_\chi^{(p)}(a) := N(\pi, \chi)^{-1} \cdot W_{\chi,f}(\begin{pmatrix} a \\ 1 \end{pmatrix}) (a = (a_v) \in A_f^\times) \]
(5.6)
\[ = \prod_{v \in B(\chi), v \neq p} a_{\chi,v}(a_v) \]
\[ = \prod_{v \in B(\chi), v \neq p} a_{\chi,v}^*(a_v). \]

Here \( a_{\chi,v}^* \) are the local Fourier coefficients defined in Def. 3.12.

**Proposition 5.3.** Let \( \mathfrak{c} \) be a prime-to-\( p \) ideal of \( F \) and let \( \mathfrak{c} \in (A_f^{(p)})^\times \) such that \( \mathfrak{u}_F(\mathfrak{c}) = \mathfrak{c} \). Then the Fourier expansion of \( f_{\chi,u}^* \) at the cusp \( (O_F, \mathfrak{c}) \) is given by
\[ f_{\chi,u}^*(|O_F, \mathfrak{c}))(q) = \sum_{\beta \in (N^{-1}\mathfrak{c})_+} a_\beta(f_{\chi,u}^*, \mathfrak{c}) q^\beta, \]
where
\[ a_\beta(f_{\chi,u}^*, \mathfrak{c}) = \beta^{k/2} a_\chi^{(p)}(\beta \mathfrak{c}^{-1}) \prod_{w \in \Sigma_{\mathfrak{p},\mathfrak{c}}} \chi_w(\beta^{-1}) \mathfrak{u}_{\mathfrak{c}}(1 + \mathfrak{c}_w O_F)(\beta). \]

In particular, \( f_{\chi,u}^* \in M_k(K_F^\times, 0) \) by Prop. 3.26, and the Fourier expansion of \( f_{\chi,u}^* \) at the cusp \( (O_F, \mathfrak{c}) \) is given by
\[ f_{\chi,u}^*(|O_F, \mathfrak{c}))(q) = \sum_{\beta \in (N^{-1}\mathfrak{c})_+} a_\beta(f_{\chi,u}^*, \mathfrak{c}) q^\beta. \]
where
\[ a_\beta(f^*_\lambda, c) = \beta^{k/2}a^{(p)}(\beta c^{-1}) \prod_{w \in \Sigma_p} \chi_w(\beta^{-1}) \cdot L_{\Gamma^{	imes}}^\infty(\beta) \].

PROOF. It follows from the definition of \(W_{\chi, u}\) that
\[ W_{\chi, u, f}(\beta c^{-1}) = W_{\chi, f}(\beta c^{-1}) \prod_{w \in \Sigma_p} W_{\chi, u, w}(\beta) \]
\[ = W_{\chi, f}(\beta c^{-1}) \prod_{w \in \Sigma_p, v \mid w} \chi_w^{-1}(a_v) L_{\Gamma^{	imes}}^0(1 + \psi_v, \chi_w)(\beta). \]

The proposition follows from (5.5) immediately. The Fourier expansion of \(f^*_\chi\) follows from (5.2).

5.4. \(p\)-adic \textit{L}-functions. We go back to the setting in the introduction. Let \(E_p^{\infty}\) be the maximal anticyclotomic \(\mathbb{Z}_p^{F, Q^{-1}}\)-extension of \(E\) and let \(\Gamma^- = \text{Gal}(E_p^{\infty} / E)\). The reciprocity law \(\text{rec}_E\) at \(\Sigma_p\) induces a morphism
\[ \text{rec}_{\Sigma_p} : (F \otimes Q_{p})^\times \cong \prod_{w \in \Sigma_p} E^\times_{w} \xrightarrow{\text{rec}_E} \Gamma^-. \]

Let \(\chi^\text{crit}\) be the set of critical specializations, consisting of \(p\)-adic characters \(\phi : \Gamma^- \to C_p^\times\) such that for some \(m \in \mathbb{Z}_{\geq 0}[\Sigma] \cong \mathbb{Z}_{\geq 0}[\Sigma_p]\),
\[ \phi(\text{rec}_E(x)) = x^m \text{ for all } x \in (O_F \otimes Z_p)^\times \text{ sufficiently close to } 1. \]

Let \(\phi\) be an anticyclotomic Hecke character of \(p\)-power conductor and of infinity type \((m, -m)\) with \(m \in \mathbb{Z}_{\geq 0}[\Sigma]\). Then \(\phi\) is unramified outside \(p\) and \(\phi|_{\Lambda^\times} = 1\). The \(p\)-adic avatar \(\hat{\phi}\) of \(\phi\) belongs to \(\chi^\text{crit}\). To be precise, let \(\hat{\phi}_{\Sigma_p} := \prod_{w \in \Sigma_p} \phi_w\). Then we have
\[ \hat{\phi}(\text{rec}_E(x)) = \phi_{\Sigma_p}(x)x^{-m} \text{ for every } x \in (F \otimes Q_{p})^\times. \]

Hereafter, we let \(\lambda\) be a Hecke character of \(E^\times\) and assume that Hypothesis \(A\) and \(E_{\lambda}\) hold for \((\pi, \lambda)\). Note that Hypothesis \(A\) and \(E_{\lambda}\) also hold for \((\pi, \lambda \phi)\). We will apply our calculations in \(\text{5.3}\) to the pair \((\pi, \chi) = (\pi, \lambda \phi)\).

Lemma 5.4. Let \(\phi\) be as above. Then
\[ (1) \ a_{\lambda \phi}^{(p)} = a_{\lambda}^{(p)}. \]
\[ (2) \ C'(\pi, \lambda \phi) = C'(\pi, \lambda)(\hat{\phi}(\hat{\theta})). \]

PROOF. If \(v \parallel p\) is split, we have remarked that \(W_{\chi, \phi, -1, v} = W_{\chi, v}\). If \(v\) is inert or ramified, then \(\phi_v = 1\) as \(\phi_v\) is unramified and \(p > 2\). Therefore, we have \(W_{\chi, f}^{(p)} = W_{\chi, f}^{(p)}\). Part (1) follows from the definition of \(a_{\lambda \phi}^{(p)}\) immediately. Next, recall that we have defined \(C'(\pi, \chi)\) for a Hecke character \(\chi\) in \(\text{5.3}\). Since \(\phi\) is anticyclotomic and unramified outside \(p\), part (2) follows from the well-known fact that
\[ \epsilon\left( \frac{1}{2}, \pi_v \otimes \chi_v, \psi_v \right) = \epsilon\left( \frac{1}{2}, \pi_v \otimes \chi_v, \psi \right) = \epsilon\left( \frac{1}{2}, \pi_v \otimes \chi_v, \psi \right) \lambda \phi\left(D_{\Gamma^{	imes}}^0 \hat{\theta}\right) \quad (v = w\Sigma, w | \Sigma). \]

Let \(O_p := O_F \otimes Z_p\) and let \(\Gamma' := \text{rec}_E(1 + pO_p)\) be an open subgroup of \(\Gamma^-\). Let \(\{\theta(\sigma)\}_{\sigma \in \Sigma}\) be the Dwork-Katz \(p\)-adic differential operators (Kat78, Cor. (2.6.25)) and let \(m := \prod_{\sigma \in \Sigma} \theta(\sigma)^{m_{\sigma}}\).

Proposition 5.5. There exists a unique \(V(\chi, K, \mathbb{Z}_p)\)-valued \(p\)-adic measure \(\mathcal{F}_{\lambda, \epsilon}\) on \(\Gamma^-\) such that
\[ \text{(i) } \mathcal{F}_{\lambda, \epsilon}\text{ is supported in } \Gamma', \]
\[ \text{(ii) for every } \phi \in \chi^\text{crit} \text{ of weight } (m, -m), \text{ we have} \]
\[ \int_{\Gamma^-} \hat{\phi} d\mathcal{F}_{\lambda, \epsilon} = \theta^m \hat{f}^{\epsilon\lambda \phi}_-. \]

PROOF. We denote by \(\mathcal{F}_{\lambda, \epsilon}(q)\) the \(p\)-adic measure with values in the space of formal \(q\)-expansions such that for every \(\varphi \in C(\Gamma^-, \mathbb{Z}_p)\),
\[ \int_{\Gamma^-} \varphi d\mathcal{F}_{\lambda, \epsilon}(q) = \sum_{\beta \in (N^{-1} \epsilon)_+} a_\beta(f^*_\lambda, c) \varphi(\text{rec}_E(\beta^{-1})) q^\beta. \]
Note that $a_\beta(f_{x, \omega}) = 0$ unless $\beta \in O_{F,(p)}^X$ and that $\mathcal{F}_\chi$ has support in $\Gamma'$ by definition.

Let $\hat{\phi}$ be the $p$-adic avatar of a Hecke character $\phi$ of infinity type $(m, -m)$. By [Kat78 (2.6.27)] (cf. [HT93 §1.7 p.205]), the $q$-expansion of $\theta^m \hat{f}_{x, \phi}$ is given by

$$
\theta^m \hat{f}_{x, \phi}|(O_E, \omega)(q) = \sum_{\beta \in (N-1)^+} a_\beta(f_{x, \phi}, \omega) \phi_{\overline{\gamma_p}}(\beta^{-1}) \beta^m q^\beta.
$$

Therefore, by Lemma 5.4 and (5.7) we find that

$$
\int_{\Gamma^-} \hat{\phi}\mathcal{F}_\chi(q) = \theta^m \hat{f}_{x, \phi}(q).
$$

By the $q$-expansion principle, this measure descends to the $p$-adic measure $\mathcal{F}_\chi$ with values in the space of $p$-adic modular forms $V(\omega, K, \mathbb{Z}_p)$.

Let $P_{\Sigma}(\pi, \lambda)$ be the $p$-adic measure on $\Gamma^-$ such that for each $\varphi \in C(\Gamma^-, \mathbb{Z}_p)$,

$$
\int_{\Gamma^-} \varphi dP_{\Sigma}(\pi, \lambda) = \sum_{a \in D_1} \lambda(a) \int_{\Gamma^-} \varphi[a] d\mathcal{F}_{\chi, \omega}(a)(x(a)) = (\lambda \cdot \cdot 1_{\Lambda_k}^{1-k_m/2}).
$$

Here $\varphi[a](x) := \varphi(x \text{ rec}_{E_{\infty}/E}(a))$. Let $(\Omega_\infty, \rho_p) \in (C^\times)^2 \times (\mathbb{Z}_p)^\Sigma$ be the complex and $p$-adic CM periods of $(E, \Sigma)$ introduced in [HT93 (4.4 a,b) p.211] (cf. (\Omega, \rho) \in [Kat78 (5.1.46), (5.1.48)] and let $\Omega_E = (2\pi i)^{-1} \Omega_\infty$. We have the following evaluation formula of $P_{\Sigma}(\pi, \lambda)$.

**Theorem 5.6.** Suppose that Hypothesis $A$ and $B$ hold. Then for each $p$-adic character $\hat{\phi} \in \mathbb{X}_p^{\text{crit}}$ of weight $(m, -m)$, we have the evaluation formula

$$
\left(\frac{1}{\Omega_{E}^{k+m+2m}} \int_{\Gamma^-} \hat{\phi} dP_{\Sigma}(\pi, \lambda) \right)^2 = |O_E : O_F|^2 \cdot \frac{\Gamma_{\Sigma}(k+m) \Gamma_{\Sigma}(m+1)}{(\text{Im} \phi)^{k+2m}(4\pi)^{k+2m+1}} \times E_{\Sigma, \omega}(\pi, \lambda) \cdot \frac{L(\Sigma, \pi \otimes \rho_p)}{\Omega_E^{2(k+2m)}} \cdot \phi(\hat{\phi}) C(\pi, \lambda).
$$

**Proof.** From [Kat78 (2.4.6), (2.6.8), (2.6.33)] we can deduce that

$$
\frac{1}{\Omega_{E}^{k+2m}} \hat{f}_{x, \phi}(x(a)) = \frac{1}{\Omega_{E}^{k+2m}} \delta_k \hat{f}_{x, \phi}(x(a)).
$$

Therefore, we have

$$
\int_{\Gamma^-} \hat{\phi} dP_{\Sigma}(\pi, \lambda) = \sum_{a \in D_1} \lambda(a) \varphi(\omega \| \phi) \int_{\Gamma^-} \hat{\phi}^m \hat{f}_{x, \phi}(x(a)) = \frac{1}{\Omega_{E}^{k+2m}} \sum_{a \in D_1} \lambda(a) \varphi(\omega \| \phi) \delta_k \hat{f}_{x, \phi}(x(a)) = \frac{1}{\Omega_{E}^{k+2m}} \cdot P_{\Sigma}(\delta_k \hat{f}_{x, \phi}).
$$

Combined with Prop. 5.2 and Lemma 5.4 (2), the above equation yields the proposition. \qed

6. The $\mu$-invariant of $p$-adic $L$-functions

In this section, we use the explicit computation of Fourier coefficients of $\{f_{x, \chi}^*\}_{\chi \in \mathbb{X}_p}$ to study the $\mu$-invariant of the $p$-adic measure $P_{\Sigma}(\pi, \lambda)$ by the approach of Hida [Hid10b].

6.1. The $t$-expansion of $p$-adic modular forms. We begin with a brief review of the $t$-expansion of $p$-adic modular forms. A functorial point in $A_{\mathbb{Z}}(c)$ can be written as $[([A], j) = [(\mathbb{A}, \lambda, \phi, \eta, j)]$. Enlarging $W_p$ if necessary, we let $W_p$ be the $p$-adic ring generated by the values of $\lambda$ on finite ideles over the Witt ring $W(\mathbb{F}_p)$. Let $mwp$ be the maximal ideal of $W_p$ and fix an isomorphism $W_p/mwp \cong \mathbb{F}_p$. Let $T := \mathcal{O}_p \otimes \mathbb{Z} \mu_\infty$ and let $\hat{T} := \lim_{\rightarrow_{m \rightarrow \infty}} T/W_p/mwp = \mathcal{O}_p \otimes \mathbb{Z} \tilde{G}_m$. Let $\{\xi_1, \ldots, \xi_\ell\}$ be a basis of $\mathcal{O}_p$ over $\mathbb{Z}$ and let $t$ be the character $1 \in \mathcal{O}_F = X^\times(\mathbb{O}_p \otimes \mathbb{Z} \mathcal{G}_m) = \text{Hom}(\mathcal{O}_p \otimes \mathbb{Z} \mathcal{G}_m, \mathbb{G}_m)$. Then we have $\mathcal{O}_F \hat{\to} W_p[t^{\xi_1 - 1}, \ldots, t^{\xi_\ell - 1}]$. For $y = (\mathbb{A}, \eta) \in IK(c)(\mathbb{F}_p) \subset IK(\mathbb{F}_p)$, it is well known that the deformation space $S_y$ of $y$ is isomorphic to the
formal torus $\hat{T}$ by the theory of Serre-Tate coordinate ([Kat81]). The $p^\infty$-level structure $j_y$ of $A_y$ induces a
canonical isomorphism $\varphi_y: \hat{T}/\mathcal{W}_y \to \hat{S}_y = \text{Spf} \, \hat{O}_{F}(e), y$ (cf. [Hai10b] (3.15)).

Now let $x := x(1)/\mathcal{W}_p \in I_K(e)(W_p)$ be a fixed CM point of type $(E, \Sigma)$ and let $x_0 = x \otimes_{W_p} \bar{p} = (\Delta_0, j_0)$. For a deformation $z = (\Delta, j)/\mathcal{R}$ of $x_0$ over an artinian local ring $\mathcal{R}$ with the maximal ideal $\mathfrak{m}_\mathcal{R}$ and the residue field $\bar{p}$, we let $t(\Delta, j) := t(\varphi^{-1}_x((\Delta, j)/\mathcal{R})) \in 1 + \mathfrak{m}_\mathcal{R}$. Then $x$ is the canonical lifting of $x_0$, i.e. $t(x) = 1$. For $f \in V(\varsigma, K, W_p)$, we define

$$f(t) := \varphi^*_x(f) \in O_{\bar{p}} = W_p[T_1, \ldots, T_d] \quad (T_i = t^{k_i} - 1).$$

The formal power series $f(t)$ is called the $t$-expansion around $x_0$ of $f$. For each $u \in O_{p}^\times$, let $u_z := (\Delta, u^z)$ is a deformation of $ux_0$. Then we have $t(uz) = t(z^u)$ and hence $\varphi^*_x(uz)(f)(t^u) = f(t^u)$.

6.2. The vanishing of the $\mu$-invariant. Let $\pi^- : (A_{E,f})^\times \to \Gamma^\ast$ be the natural map induced by the
reciprocity law. Let $Z' = \pi^{-1}(\Gamma')$ be a subgroup of $(A_{E,f})^\times$ and let $CL_{/CL_{/CL}}$ be the image of $Z'$ in $CL_{/CL}$.
Let $D'_1$ (resp. $D''_1$) be a set of representative of $CL_{/CL}_{/CL}$ (resp. $CL_{/CL}$) in $(A_{E,f})^\times$. Let $D_1 := D'_1D''_1$ be a set of representative of $CL_{/CL}$. Recall that $U_p$ is the torsion subgroup of $O_p$. Let $U$ be the torsion subgroup of $E^\times$ and let $U_{alg} = (E^\times)^{1-e-c}$ be a subgroup of $U$. We regard $U_{alg}$ as a subgroup of $O_p$ by the imbedding induced by $\bar{\mathcal{p}}$. Let $D_0$ be a set of representatives of $U_{p}/U_{alg}$ in $U_p$. Fix $\varsigma := \varsigma(O_K)$ to be the polarization ideal of the CM point $x(1)$. The following theorem reduces the calculation of the $\mu$-invariant $\mu^-_{\pi, \lambda, \Sigma}$ to the determination of the $q$-expansion of $f^*_{\lambda, u}$.

**Theorem 6.1.** Suppose that $p$ is unramified in $F$. Then

$$\mu^-_{\pi, \lambda, \Sigma} = \inf_{(a, u) \in U_p \times D_0} v_p(a_b(f^*_{\lambda, u}(\varsigma(a)))),$$

**Proof.** For every pair $(u, a) \in U_p \times D_0$, we let $f^*_{a, u}(\varsigma(a))$ is defined over $O_{L}$ and hence $F_{u, a}[a] \in V(\varsigma, O_L)$. For each $z \in Z'$, let $\langle z \rangle$ be the unique element in $1 + pO_p$ such that $rec_{\bar{\mathcal{p}}}(\langle z \rangle) = \pi^- = (z) \in \Gamma^-$. For $(a, b) \in D_1 \times D'_1$, we define

$$\tilde{T}_a(t) = \sum_{u \in U_p} T_{u, a}(t^{u-1}),$$

$$\tilde{G}^b(t) = \sum_{a \in bD_1} \lambda(ab^{-1})T_{a}[a](t^{ab^{-1}}).$$

Let $\mathcal{P}^b_{\Sigma}(\pi, \lambda)$ be the $p$-adic measure on $1 + pO_p \simeq \Gamma'$ obtained by the restriction of $\mathcal{P}^\ast_{\Sigma}(\pi, \lambda)$ to $\pi^- \ast (b)$. In other words,

$$\int_{\Gamma} \varphi d\mathcal{P}^\ast_{\Sigma}(\pi, \lambda) \colon = \int_{\Gamma} \mathbb{I}_{b, \Gamma'} \cdot \varphi[|b^{-1}|]d\mathcal{P}^\ast_{\Sigma}(\pi, \lambda)$$

$$= \sum_{a \in bD_1} \lambda(a) \int_{\Gamma^-} \varphi[|ab^{-1}|]d\mathcal{P}_{\lambda, t}(x(a)).$$

Here the second equality follows from the fact that $F_{\lambda, t}(a)$ has support in $\Gamma'$ (Prop. 5.3 (i)). The argument of [Hai12a] Prop. 5.2] shows that $\tilde{G}^b(t)$ is the power series expansion of the measure $\mathcal{P}^b_{\Sigma}(\pi, \lambda)$ regarded as a $p$-adic measure on $O_p$, and that

$$\mu^-_{\pi, \lambda, \Sigma} = \inf_{b \in D'_1} \mu(\tilde{G}^b),$$

where

$$\mu(\tilde{G}^b) := \inf \left\{ r \in \mathbb{Q}_{\geq 0} \mid p^{-r}\tilde{G}^b \neq 0 (\text{mod } m_p) \right\}.$$ 

By (3.3) we find that

$$\tilde{T}_a(t) = \tilde{z}(U_{alg}) \cdot \sum_{u \in D_0} T_{u, a}(t^{u-1}),$$
and hence
\[ \mathcal{I}^b(t) = 2(\mathcal{I}^\text{alg}) \cdot \sum_{(a,b) \in D_0 \times BD_1'} \hat{\lambda}(ab^{-1})\mathcal{I}_{u,v}[a](\ell(ab^{-1})u^{-1}). \]

Proceeding along the same lines in [Hsi12a, Thm. 5.5], we can deduce the theorem from the above equation by the linear independence of $p$-adic modular forms modulo $p$ acted by the automorphisms in $D_0 \times D_1'$ ([Hsi10b Thm. 3.20, Cor. 3.21]) and the $q$-expansion principle for $p$-adic modular forms.

**Theorem 6.2.** In addition to Hypothesis A and (5.3), we suppose that $p$ is unramified in $F$ and the residual Galois representation $\overline{\rho}_E(\pi_E) := \rho_p(\pi)|_{E_F}$ is absolutely irreducible.

Then $\mu_{\pi,\lambda,\Sigma} = 0$ if and only if
\[ \sum_{v|\mathfrak{c}_\lambda} \mu_p(\Psi_{\pi,\lambda,\Sigma}) = 0, \]
where $\mu_p(\Psi_{\pi,\lambda,\Sigma})$ are the local invariants defined as in (6.2).

**Proof.** It is not difficult to deduce from the formula of $a_\beta(f_{\lambda,u}^*, c(a))$ in Prop. 5.3 and Prop. 5.13 that
\[ \mu_p(\Psi_{\pi,\lambda,\Sigma}) > 0 \text{ for some } v|\mathfrak{c}_\lambda \Rightarrow a_\beta(f_{\lambda,u}^*, c(a)) \equiv 0 \text{ (mod } m) \text{ for all } a \in A_f^\times, \]
and hence $\mu_{\pi,\lambda,\Sigma} > 0$ by Theorem 6.1.

Conversely, we suppose that $\mu_p(\Psi_{\pi,\lambda,\Sigma}) = 0$ for all $v|\mathfrak{c}_\lambda$. We are going to show $\mu_{\pi,\lambda,\Sigma} = 0$ by contradiction.

Assume that $\mu_{\pi,\lambda,\Sigma} > 0$. By Prop. 5.3 Theorem 6.1 for each $a \in A_{E,F}^{pN}$ we find that
\[ a_\beta(f_{\lambda,u}^*, c(a)) \equiv 0 \text{ (mod } m) \text{ for all } u \in U_p \text{ and } \beta \in F^+_\mathfrak{c}_\lambda. \]

\[ \iff a_\lambda^{(p)}(\beta c^{-1}N(a^{-1})) \equiv 0 \text{ (mod } m) \text{ for all } \beta \in O_{E,F}^\times(p). \]

Therefore, as a function on $(A_f^p)^\times$, we have
\[ a_\lambda^{(p)}(a) \equiv 0 \text{ (mod } m) \text{ for all } a \in O_{E,F}^\times(p)c^{-1}\det(U(N))N((A_{E,F}^{pN})^\times) = F^xc^{-1}N((A_{E,F}^p)^\times). \]

By Prop. 6.13 there exists $\eta = \eta_v \in \prod_{v|\mathfrak{c}_\lambda} F_v^\times$ such that $a^{(p)}_\lambda(\eta_v) \not\equiv 0 \text{ (mod } m)$ for each $v|\mathfrak{c}_\lambda$. We extend $\eta$ to be the idele in $A_f^\times$ such that $\eta_v = 1$ at $v \nmid \mathfrak{c}_\lambda$. Therefore, together with the factorization formula of $a_\lambda^{(p)}$ (6.6) imply that for each uniformizer $\varpi_v$ at $v \mid p\mathfrak{c}_\lambda$, we have
\[ a_\lambda^{(p)}(\eta \varpi_v) \equiv 0 \text{ (mod } m) \iff W_v^{(p)}\left(\begin{array}{c} \varpi_v \\ 1 \end{array}\right) \equiv 0 \text{ (mod } m) \text{ whenever } \varpi_v \in [\eta^{-1}c^{-1}] := F^xc^{-1}N((A_{E,F}^p)^\times). \]

On the other hand, by (6.2), we find that
\[ \text{Tr } \rho_p(\pi)(\text{Frob}_v) = \omega(\varpi_v)^{-1}|\varpi_v|^{-k_{\text{max}}/2}W_v^{(p)}\left(\begin{array}{c} \varpi_v \\ 1 \end{array}\right) \text{ for all } v \mid pm. \]

Let $\text{rec}_{E/F} : A_{E,F}^\times \to \text{Gal}(E/F)$ be the surjection induced by the reciprocity law. Combined with (6.2), the above equation yields that
\[ \text{Tr } \rho_p(\pi)(\text{Frob}_v) \equiv 0 \text{ (mod } m) \text{ whenever } \text{Frob}_v \mid E = \text{rec}_{E/F}(\varpi_v) = \text{rec}_{E/F}(\eta^{-1}c^{-1}). \]

This in particular implies that $\text{rec}_{E/F}(\eta^{-1}c^{-1})$ must be the complex conjugation $c$, and hence we arrive at a contradiction to (6.2) by the following Lemma 6.3.

**Lemma 6.3.** Let $p > 2$ be a prime. Let $G$ be a finite group and $H \leq G$ be a index two subgroup. Let $\rho : G \to \text{GL}_2(\mathbb{F}_p)$ be a faithful irreducible representation of $G$. Let $T = \text{Tr } \rho : G \to \mathbb{F}_p$ be the trace function. Assume that

1. There exists an order two element $c \in G - H$.
(2) \( T(hc) = 0 \) for all \( h \in H \).
Then \( \rho|_H \) is reducible.

**Proof.** The assumption (2) implies that \( T(c) = 0 \), and hence \( \det \rho(c) = -1 \). We may assume that \( \rho(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Suppose that \( p \nmid \sharp(G) \). By the usual representation theory of finite groups, we have

\[
1 = \langle T, T \rangle := \frac{1}{\sharp(G)} \sum_{g \in G} T(g)T(g^{-1}) = \frac{1}{\sharp(H)} \sum_{h \in H} T(h)T(h^{-1}) = \frac{1}{2} \cdot \langle T|_H, T|_H \rangle.
\]

Since \( \langle T|_H, T|_H \rangle = 2 \), we conclude that \( \rho|_H \) is not irreducible.

Now we assume that \( p \mid \sharp(H) \). For each \( b \in M_{2}(\mathbb{F}_p) \) with \( b^2 = 0 \), define the \( p \)-subgroup \( P_b \) of \( \rho(H) \) by

\[
P_b = \{ h \in \rho(H) \mid h = 1 + xb \text{ for some } x \in \mathbb{F}_p \}.
\]

Let \( h \in H \) be an element of \( p \)-power order. It is well known that \( (\rho(h) - 1)^2 = 0 \), and hence \( T(h) = 2 \) and \( \det \rho(h) = 1 \). Combined with \( T(hc) = 0 \), these equations imply that

\[
\rho(h) \in P_{b_1} \text{ or } P_{b_2} \text{ with } b_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.
\]

Note that either \( P_{b_1} \) or \( P_{b_2} \) is trivial. Indeed, if \( h_1 \neq 1 \in P_{b_1} \) and \( h_2 \neq 1 \in P_{b_2} \). Then \( h_1 h_2 \in H \) and \( \text{Tr}(h_1 h_2 c) \neq 0 \), which is a contradiction. In particular, we conclude that elements of \( p \)-power order in \( H \) are commutative to each other and that there is only one \( p \)-Sylow subgroup of \( H \), which we denote by \( P \). It is clear that \( H \) normalizes \( P \). Since \( P \neq \{1\} \), there is a unique line fixed by \( \rho(P) \), which is an invariant subspace of \( \rho(H) \). We find that \( \rho|_H \) is reducible if \( p \mid \sharp(H) \).

**Remark 6.4.** The assumption (3) in Theorem 6.3 in the introduction implies the vanishing of \( \mu_{p}(\Psi_{\pi,\lambda,v}) \) for all \( v|c_{\lambda} \).

### 7. Non-vanishing of central \( L \)-values with anticyclotomic twists

In this section, we consider the problem of non-vanishing of \( L \)-values modulo \( p \) with anticyclotomic twists. Let \( \ell \neq p \) be a rational prime and let \( \ell \) be a prime of \( F \) above \( \ell \). Let \( \Gamma_{\ell}^{-} := \text{Gal}(E_{\ell}^{-}/E) \) be the Galois group of the maximal anticyclotomic pro-\( \ell \) extension \( E_{\ell}^{-} \) in the ray class field of \( E \) of conductor \( \Gamma_\ell \). Let \( \mathcal{X}_{0}^{0} \) be the set consisting of finite order characters \( \phi : \Gamma_{\ell}^{-} \to \mu_{\ell}^{-} \). Fix a Hecke character \( \chi \) of infinity type \((k/2 + m, -k/2 - m)\). For each \( \phi : \Gamma_{\ell}^{-} \to \mu_{\ell}^{-} \) in \( \mathcal{X}_{0}^{0} \), we put

\[
L^{\text{alg}}(1, \pi_E \otimes \chi) := \left[ O_{\mathcal{F}}^{X} : O_{\mathcal{F}}^{0} \right]^2 \cdot \frac{\Gamma_{\ell}(m)\Gamma_{\ell}(k + m)}{\text{Im}(\vartheta) \cdot 2\pi i^{k + 2m + 1}} \cdot \frac{L(\frac{1}{2}, \pi_E \otimes \chi)}{\Omega_{\mathcal{F}}^{2(k + 2m)}}.
\]

For simplicity, we assume

\[
(p, nD_{E/F}) = 1.
\]

In particular, \( \pi \) is unramified at \( l \) and every place above \( p \). It can be shown that \( L^{\text{alg}}(1, \pi_E \otimes \chi) \in \mathbb{Z}_{(p)} \) at least when \( p \nmid D_{E} \). This section is devoted to proving the following result:

**Theorem 7.1.** With the same assumptions in Theorem 6.3, we further assume that

1. \( (p, n\zeta_{\ell} D_{E/F}) = 1 \).
2. \( \mu_{p}(\Psi_{\pi,\lambda,v}) = 0 \) for all \( v|c_{\lambda}^{-} \).

Then for almost all \( \phi \in \mathcal{X}_{0}^{0} \) we have

\[
L^{\text{alg}}(1, \pi_E \otimes \chi) \neq 0 \text{ (mod } m \text{)}.
\]

Here almost all means "except for finitely many \( \phi \in \mathcal{X}_{0}^{0} \)" if \( \dim_{\mathbb{Q}} F_{1} = 1 \) and "for \( \phi \) in a Zariski dense subset of \( \mathcal{X}_{0}^{0} \)" if \( \dim_{\mathbb{Q}} F_{1} > 1 \) (See [Hi04, p.737]).

When \( F = \mathbb{Q} \), a non-primitive version of the above result under different assumptions is treated in [Bra11b].
7.1. After introducing some notation, we outline the approach of Hida \cite{Hida2004a} to study this problem. We shall take \( r = \zeta \chi n_{D_{EF}} f \) to be the fixed ideal in \( \mathbb{Z} \). For every \( n \in \mathbb{Z} \), let \( R_{n} := \mathcal{O}_{F} + t^{n} \mathcal{O}_{E} \) be the order in \( E \) of conductor \( t^{n} \). Let \( U_{n} := (\mathbb{E} \otimes \mathbb{Q} R)^{\times} (R_{n} \otimes \mathbb{Z} \mathbb{Z})^{\times} \) and let \( \mathcal{C}L_{n}^{-} := E^{\times} A_{n}^{\times} \mathcal{A}^{\times} U_{n} \) be the anticyclotomic ideal class group of conductor \( t^{n} \). Denote by \( \varepsilon_{n} : A_{n}^{\times} \rightarrow \mathcal{C}L_{n}^{-} \) the quotient map. Let \( \mathcal{C}L_{n}^{-} = \lim_{\rightarrow n} \mathcal{C}L_{n}^{-} \). Let \( I_{1} \) be the \( l \)-adic Iwahori subgroup of \( K_{1}^{0} \) given by

\[
I_{1} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{1}^{0} \mid c \in \varpi I_{1} \right\}.
\]

Let \( K_{0}(l) := K^{1} I_{l} = \{ g \in K \mid g_{l} \in I_{l} \} \) be an open-compact subgroup of \( GL_{2}(A_{l}) \). Recall that the \( U_{l} \)-operator on \( M_{k}(K_{0}(l), \mathbb{C}) \) is given by

\[
F | U_{l}(\tau, g_{l}) = \sum_{u \in \mathcal{O}_{F}/l^{n} \mathcal{O}_{F}} F(\tau, g_{l}) \left( \varpi l^{n} \begin{array}{c} ud_{F_{l}}^{-1} \\ 1 \end{array} \right).
\]

We briefly outline the approach of Hida to prove Theorem 7.1 as follows:

1. Construct a suitable \( p \)-adic modular form \( \tilde{f}_{l}^{\dagger} \) which is an eigenfunction of \( U_{l} \)-operator with \( p \)-adic unit eigenvalue.
2. Consider Hida’s measure \( \varphi_{l}^{\dagger} \) on \( \mathcal{C}L_{l}^{-} \) attached to \( \tilde{f}_{l}^{\dagger} \) and show the evaluation formula of this measure is related to central values \( L^{\mathbb{Z}}(1, \pi_{E} \otimes \chi_{\phi}) \) (Prop. 7.3).
3. The Zariski density of CM points in Hilbert modular varieties modulo \( p \) reduces the proof of Theorem 7.1 to the non-vanishing of certain Fourier coefficients of \( \tilde{f}_{l}^{\dagger} \) at some cusp (\cite{Hida2004a} Thm. 3.2 and Thm. 3.3).

We remind the reader that the proof is very close to Theorem 7.2 and Theorem 7.2. The essential new inputs in this section are the choice of \( U_{l} \)-eigenforms and the computation of local period integral at \( l \).

7.2. CM points of conductor \( t^{n} \). Let \( n \in \mathbb{Z} \). We choose \( \zeta_{l}^{(n)} \in G_{l} \) as follows. If \( l = 2 \mathbb{Z} \) splits in \( E \), writing \( \vartheta = \vartheta_{E} e_{E} + \vartheta_{\mathbb{Q} / \mathbb{Z}} \) (so \( \mathcal{D}_{E} = \vartheta_{E} - \vartheta_{\mathbb{Q} / \mathbb{Z}} \) is a generator of \( \mathcal{D}_{E} \)), we put

\[
\zeta_{l}^{(n)} = \begin{pmatrix} \vartheta_{E} & -1 \\ 1 & 0 \end{pmatrix} \left( \varpi_{l}^{n} \begin{array}{c} 1 \\ 1 \end{array} \right).
\]

If \( l \) is inert, then we put

\[
\zeta_{l}^{(n)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \varpi_{l}^{n} \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

Let \( \zeta^{(n)} := \zeta_{l}^{(n)} \prod_{\mathbb{Z} \neq l} \zeta_{0} \). According to this choice of \( \zeta_{l}^{(n)} \), we have

\[
\sigma_{f}^{(n)}( \mathbb{Z} \otimes \mathbb{Z} \mathbb{Z} ) = q_{\phi}(R_{n}).
\]

Define \( x_{n} : A_{n}^{\times} \rightarrow X^{+} \times G(A_{f}) \) by

\[
x_{n}(a) := (\vartheta_{E}, a_{f}^{(n)}). \]

This collection \( \{ x_{n}(a) \in A_{n}^{\times} \} \) of points is called CM points of conductor \( t^{n} \). As discussed in \( \mathbb{Z} \mathbb{Z} \) \( \{ x_{n}(a) \} \in A_{n}^{\times} \mathcal{A}_{n}^{\times} U_{n} \) descend to CM points in \( I_{K}(W_{p}) \).

7.3. The measures associated to \( U_{l} \)-eigenforms. We construct the \( U_{l} \)-eigenform \( f_{l}^{\dagger} \) as follows. Write \( \pi_{l} = \pi(\mu_{l}, \nu_{l}) \). Define the local Whittaker function \( W_{l}^{\dagger} \in \mathcal{W}(\pi_{l}, \psi_{l}) \) by

\[
W_{l}^{\dagger}(g) = W_{l}^{0}(g) - \mu_{l} \cdot \vartheta(1) W_{l}^{0}(g) \left( \varpi_{l}^{1} \begin{array}{c} 1 \\ 1 \end{array} \right).
\]

It is not difficult to verify that

- \( W_{l}^{\dagger} \) is invariant by \( I_{l} \),
- \( W_{l}^{\dagger} \left( \begin{array}{c} a \\ 1 \end{array} \right) = \nu_{l} \cdot \vartheta(1) \mathcal{O}_{F_{l}}(a) \),
- \( W_{l}^{\dagger} \) is an \( U_{l} \)-eigenfunction with the eigenvalue \( \nu_{l}(\varpi_{l}) \varpi_{l}^{-\frac{1}{4}} \).
Define the normalized global Whittaker function $W^\dagger_\chi$ by
\[
W^\dagger_\chi := N(\pi, \chi)^{-1} \left| \det \chi \right|_{A_d} \prod_{\sigma \in \Sigma} W_{k_\sigma} \cdot \prod_{v \in \mathcal{V} \setminus \{1\}} W_{\chi,v} \cdot W^\dagger_1,
\]
where $N(\pi, \chi)$ is the normalization factor in (3.30). Let $\varphi^\dagger_1$ be the automorphic form associated to $W^\dagger_1$ as in (3.26) and let $f^\dagger_1$ be the associated Hilbert modular form as in Def. 5.1.

The following lemma follows from the choices of our Whittaker function $W^\dagger_1$ and the construction of $f^\dagger_1$.

**Lemma 7.2.** Recall that $\mathcal{R}$ is the group generated by $E^\times_v$ for all ramified places $v$ in $A^\times_\mathbb{E}$. We have

1. $f^\dagger_1$ is toric of character $\chi$ outside $I$, and
\[
f^\dagger_1(x_n(ta)) = f^\dagger_1(x_n(t)) \chi^{-1} \cdot \left| k_{mx}/2 \right|^{\chi(\sigma)}(a) \text{ for all } a \in \mathcal{R} \cdot (R^{\mathbb{A}} \otimes \mathbb{Z})^\times.
\]

2. $f^\dagger_{\chi,\phi} = f^\dagger_1$ for every $\phi \in \mathfrak{X}_1^0$.

**Proof.** Part (1) follows immediately from the fact that $W^\dagger_1$ is a toric Whittaker function outside $I$ in view of Lemma 3.2. In addition, for every $\phi \in \mathfrak{X}_1^0$, $\phi$ is anticyclotomic and unramified outside $I$. We thus have $W^\dagger_1 = W^\dagger_{\chi,\phi}$, which verifies part (2) (cf. Lemma 6.4).

Following [Hid04a (3.9)], we define a $p$-adic $\mathfrak{Z}_p$-valued measure $\varphi^\dagger_1$ on $Cl_{\mathbb{Z}}^-$ attached to the $p$-adic avatar $\hat{f}^\dagger_1$ of $f^\dagger_1$ as follows. For a locally constant function $\phi : Cl_{\mathbb{Z}} \to \mathfrak{Z}_p$ factors through $Cl_{\mathbb{Z}}^-$, we define
\[
\int_{Cl_{\mathbb{Z}}^-} \phi d\varphi^\dagger_1 = \alpha^{-n}_1 \sum_{\alpha \in \mathfrak{X}_1^0} \theta_m f^\dagger_1(\sigma_n(a)) \hat{\chi}(a) \phi([\alpha])
\]
where $\alpha_1 = \nu_1([\alpha_1])|\alpha_1|^{\frac{1-k_{mx}}{2}}$ and $\hat{\chi}$ is the $p$-adic avatar of $\chi|_{A^\times_\mathbb{E}}^{1-k_{mx}/2}$. One checks that the right hand side does not depend on the choice of $n$ since $f^\dagger_1$ is an $U_1$-eigenform with the eigenvalue $\alpha_1$.

Let $\phi \in \mathfrak{X}_1^0$ be a character of conductor $l^n$. We view $\phi$ as a character on $Cl_{\mathbb{Z}}^-$ by the reciprocity law. Following the arguments in Prop. 5.2 and Theorem 5.9 we can write the measure as a toric period integral of $\widehat{V}_+^{\dagger} \varphi^{\dagger}_{\chi,\phi}$:
\[
\frac{\Omega_{l^n}}{\Omega_{l^n}^{k_{mx}+2m}} \cdot \int_{Cl_{\mathbb{Z}}^-} \phi d\varphi^\dagger_1 = \alpha^{-n}_1 \text{vol}(U_{l^n}, dt)^{1} \cdot \left( \text{Im } \theta \right)^{k_{mx}/2 + m} \cdot P_{\chi,\phi}(\pi(\sigma^{(n)})) \widehat{V}_+^{\dagger} \varphi^{\dagger}_{\chi,\phi}
\]
(7.2)
\[
= \text{vol}(U_{l^n}, dt)^{-1} \cdot \left( \text{Im } \theta \right)^{k_{mx}/2 + m} \cdot \frac{\alpha^{-n}_1}{L(1, \tau_{E,F_1})|\sigma_1|^{n}} \cdot P_{\chi,\phi}(\pi(\sigma^{(n)})) \widehat{V}_+^{\dagger} \varphi^{\dagger}_{\chi,\phi}.
\]
Here we used the fact that
\[
\text{vol}(U_{l^n}, dt) = \text{vol}(U_{l^n}, dt) \cdot L(1, \tau_{E,F_1}) |\sigma_1|^{n}.
\]
We have the following evaluation formula.

**Proposition 7.3.** Suppose that $(l, \mathfrak{c}, nD_{E/F}) = 1$. For $\phi \in \mathfrak{X}_1^0$ of conductor $l^n$ with $n \geq 1$, we have
\[
\left( \frac{1}{\Omega_{l^n}^{k_{mx}+2m}} \cdot \int_{Cl_{\mathbb{Z}}^-} \phi d\varphi^\dagger_1 \right)^2 = |\sigma_1|^{-n} \cdot L_{\text{alg}}^{k_{mx}/2} \cdot L(1, \tau_{E,F_1}) \cdot C(\pi, \chi) \phi(\mathfrak{f}).
\]

**Proof.** In view of (7.2), it remains to compute $P_{\chi,\phi}(\pi(\sigma^{(n)})) \widehat{V}_+^{\dagger} \varphi^{\dagger}_{\chi,\phi}$, which can be written as a product of local toric period integrals as in the proof of Theorem 3.18. We have computed these local period integrals in 3.7 and 3.8 except for the local integral at $I$, which will be carried out in the following Lemma 7.4. The desired formula is obtained by combining these calculations.

**Lemma 7.4.** Suppose that $\chi_1$ is unramified and $\phi \in \mathfrak{X}_1^0$ has conductor $l^n$, $n \geq 1$. Then
\[
P(\pi(\sigma^{(n)})) W^\dagger_1, \phi) = |D_{E,F_1}|^{k_{mx}/2} \cdot \omega_1(\sigma_1) \cdot \omega_1(n) \cdot L(1, \tau_{E,F_1})^2.
\]
PROOF. Write \( F = F_1 \) (resp. \( E = E_1 \)) and \( \varpi = \varpi_1 \). For \( t \in E \), we put
\[
i_{\xi(n)}(t) := (\xi_{\xi(n)}')^{-1}i(t)\xi_{\xi(n)}.
\]
First we suppose \( I \) is split. A direct computation shows that
\[
i_{\xi(n)}(t) = \begin{pmatrix} 1 & -\varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix};
\]
\[
(\xi_{\xi(n)}')^{-1}J_{\xi(n)} = \begin{pmatrix} 1 & 0 \\ \varpi^n d_F & -1 \end{pmatrix}.
\]
We find that
\[
\omega^{-1}(\det \xi_{\xi(n)}')P(\pi(\xi_{\xi(n)}')W_{\xi_1}^+, \chi\phi)
= \int_{F^\times} \int_{F^\times} W_{\xi_1}^+ \left( \frac{ax}{1} \begin{pmatrix} 1 & -\varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix} \right) W_{\xi_1}^+ \left( \begin{pmatrix} 1 & -\varpi^{-n}d_F^{-1} \\ 0 & 1 \end{pmatrix} \right) \omega^{-1}(\det \xi_{\xi(n)}') \chi\phi \phi_{\xi_1}(x) d^\times x d^\times a
\]
\[
= \int_{O_F} \psi(-d_F^{-1}\varpi^{-n}x)\nu(\chi_{\xi_1})|\tilde{\varpi}(x)|^2 x \cdot \int_{O_F} \psi(d_F^{-1}\varpi^{-n}a)\nu^{-1}(\chi_{\xi_1})|\tilde{\varpi}(a)|^2 a
\]
\[
= \epsilon(1, \phi_{\xi_1}^{-1}, \psi)\phi_{\xi_1}(-1)\epsilon(1, \phi_{\xi_1}, \psi) \cdot \zeta_F(1)^2 = |\varpi^n D_F| L(1, \tau_{E/F})^2.
\]
This proves the formula in the split case.

Now we suppose that \( I \) is inert. We shall retain the notation in \( \S 3.8 \). Define \( m^I : G_1 \to C \) by
\[
m^I(g) := b_i(\pi(g)W_{\xi_1}^+, W_{\xi_1}^+).
\]
Then \( m^I(g) \) only depends on the double coset \( I_1 g I_1 \). Put
\[
P^* := \int_{E^\times / F^\times} m^I(I_{\xi(n)}(t))\chi\phi(t)dt.
\]
It is clear that
\[
(7.3) \quad P(\pi(\xi_{\xi(n)}')W_{\xi_1}^+, \chi\phi) = P^* \cdot \frac{L(1, \tau_{E/F})\omega(\det \xi_{\xi(n)}')}{\zeta_F(1)}.
\]
For \( y \in \varpi^r O_F \), it is easy to verify that \( \iota_{\xi(n)}(1 + y\theta) \in I_1 \) if \( r \geq n \) and
\[
\iota_{\xi(n)}(1 + y\theta) \in I_1 \varpi \begin{pmatrix} \varpi^{-r} & \varpi^{-n} \\ \varpi^{-n} & \varpi^{-r} \end{pmatrix} I_1 \text{ if } 0 \leq r < n
\]
\[
(\varpi = \begin{pmatrix} 0 & -d_F^{-1} \\ d_F & 0 \end{pmatrix}).
\]
If \( x \in \varpi O_F \), then
\[
\iota_{\xi(n)}(x + \theta) \in I_1 \varpi \begin{pmatrix} \varpi^{-n} & \varpi^{-n} \\ \varpi^{-n} & \varpi^{-n} \end{pmatrix} I_1.
\]
Note that \( n = c_1(\phi) = c_1(\chi\phi) \) as in \( \S 3.1 \). Combined with the above observations and Lemma \( 3.7 \) a direct computation shows that
\[
P^* = X_n \cdot m^I(1) + (-X_n) \cdot m^I(\varpi \varpi^{-1})
\]
\[
= b_i(W_{\xi_1}^0 - \pi(\varpi^{-1} \varpi)W_{\xi_1}^0 W_{\xi_1}^+) \cdot X_n
\]
\[
= \left( \frac{1}{\mu(\varpi)} - \frac{\nu(\varpi)}{\mu(\varpi)} \right) \cdot \frac{1}{\mu(\varpi)} \cdot \frac{1}{\nu(\varpi)} \cdot |\varpi|^{1/2} \cdot X_n
\]
\[
= \frac{1}{1 - |\varpi|} \cdot L(1, \tau_{E/F}) |\varpi^n| |D_E|^{1/2}.
\]
The formula in the inert case follows from (7.3) immediately. \( \square \)
7.4. Proof of Theorem 7.1. We prove Theorem 7.1 in this subsection. By the evaluation formula Prop. 7.2 it boils down to proving that

\[
\prod_{\phi \in \mathfrak{X}^0} \phi d\varphi_{\chi} \neq 0 \pmod{m} \quad \text{for almost all } \phi \in \mathfrak{X}^0.
\]

By [Hid04a Thm. 3.2, 3.3] together with the toric property of \( f_{\chi}^+ \) Lemma 7.2 (cf. [Hsi12b] Lemma 6.1 and Remark 6.2)], the validity of \( 7.3 \) is reduced to verifying the following condition:

(H') For every \( u \in \mathcal{O}_{F_1} \) and a positive integer \( r \), there exist \( \beta \in \mathcal{O}_{F,(p)}^{\times} \) and \( a \in (\mathbb{A}_{E,F})^{\times} \) such that \( \beta \equiv u \pmod{\Gamma'} \) and

\[
a_{\beta}(f_{\chi}^+(a)) \neq 0 \pmod{m}.
\]

The verification of (H') under the assumptions \( \left( \frac{a}{\mathcal{O}_F} \right) \) and \( \mu_p(\Phi_{\pi,\chi,v}) = 0 \) for all \( v|\mathfrak{c}^- \) follows from the same argument in Theorem 6.2. Note that for a polarization ideal \( \mathfrak{c}^- \) (\( \mathfrak{c} = \mathcal{O}_E \), \( a \in (\mathbb{A}_{E,F})^{\times} \)) and a totally positive \( \beta \in \mathcal{O}_{F,(p)}^{\times} \cap \mathcal{O}_{F_1} \), we have \( (\mathfrak{c}(a),p) = 1 \) and

\[
a_{\beta}(f_{\chi}^+(a)) = \beta^{k/2} \prod_{v|\mathfrak{c}^-} a_{\chi,v}^{-1}(\beta c_{\mathfrak{c},v}^{-1}(a_{\mathfrak{c},v}^{-1})) \cdot \nu_{|\beta|_{\mathfrak{p}_F}^{-1}} \phi_{\mathfrak{p}_F}(\beta) \quad (\mathfrak{c}(a) = \mathcal{O}_E).
\]

Let \( u \in \mathcal{O}_{F_1} \) and let \( \eta_{\mathfrak{c}}^u = (\eta_{\mathfrak{c}}^u) \) be the idele in \( \mathbb{A}_{\chi} \) such that \( a_{\chi,v}^{-1}(\eta_{\mathfrak{c}}^u) \neq 0 \pmod{m} \) for all \( v|\mathfrak{c}^- \), \( \eta_{\mathfrak{c}}^u = u \) and \( \eta_{\mathfrak{c}}^u = 1 \) for all \( v \not| \mathfrak{c}_{\mathfrak{c}} \). To verify (H'), we simply proceed the Galois argument in Theorem 6.2 replacing \( \eta \) in by \( \eta^u \) therein. This completes the proof of Theorem 7.1.

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