On Routh-Steiner Theorem and Generalizations*

Elias Abboud

March 25, 2014

Abstract

Following Coxeter we use barycentric coordinates in affine geometry to prove theorems on ratios of areas.

In particular, we prove a version of Routh-Steiner theorem for parallelograms.

1 Introduction

Coxeter in his book [2, p. 211], considered the following theorem of affine geometry;

Theorem 1 If the sides $BC, CA, AB$ of a triangle $ABC$ are divided at $L, M, N$ in the respective ratios $\lambda : 1, \mu : 1, \nu : 1$, the cevians $AL, BM, CN$ form a triangle whose area is

$$\frac{(\lambda \mu \nu - 1)^2}{(\lambda \mu + \lambda + 1)(\mu \nu + \mu + 1)(\nu \lambda + \nu + 1)}$$

times that of $ABC$.

He emphasized that this result was discovered by Steiner, but simultaneously cited two references the first was Steiner’s work [5, p. 163-168] and the second was Routh’s work: [4, p. 82]. Later in his book he referred to the result as ”Routh’s theorem” [2, p. 219] admitting to the contribution of both scientists in revealing the theorem.

Coxeter gave a general proof of this result using barycentric coordinates attributed to Möbius. These are homogeneous coordinates $(t_1, t_2, t_3)$, where $t_1, t_2, t_3$ are masses at the vertices of a triangle of reference $A_1A_2A_3$. In particular $(1, 0, 0)$ is $A_1$, $(0, 1, 0)$ is $A_2$, $(0, 0, 1)$ is $A_3$ and $(t_1, t_2, t_3)$ corresponds to a point $P$ such that the areas of the triangles $PA_2A_3, PA_3A_1, PA_1A_2$ are proportional to the barycentric coordinates $t_1, t_2, t_3$ of $P$, respectively (see Fig. 1). If $t_1 + t_2 + t_3 = 1$ then the normalized barycentric coordinates $(t_1, t_2, t_3)$ are called

*MSC: 51N10
areal coordinates. In this case the areas of the triangles $PA_2A_3, PA_3A_1, PA_1A_2$ are $t_1, t_2, t_3$ times the area of the whole triangle $A_1A_2A_3$, respectively.

Recently, A. Bényi and B. Curgus [1], proved a version of Theorem 1 and got a unification of the theorems of Ceva and Menelaus. In fact they unified two expressions of Routh on area ratios into one from which they derived both theorems of Ceva and Menelaus as special cases.

Putting aside the search after “nice expressions”, Theorem 1 implies the following general property:

**Theorem 2** If the sides of a triangle $A_2A_3A_1$ are divided at $A_{i,1}, A_{i,2}, ..., A_{i,n-1}, 1 \leq i \leq 3$ in the respective ratios $\lambda_{i,1} : \lambda_{i,2} : \ldots : \lambda_{i,n-1}, 1 \leq i \leq 3$, then the ratio of the area of any sub polygon to the area of the whole triangle depends only on $\{\lambda_{i,j}\}$, where $1 \leq i \leq 3$ and $1 \leq j \leq n-1$ (see Fig. 2 for the case $n = 5$).

A sub polygon of the triangle is defined as a polygon whose vertices are points of intersections of the cevians $\{A_iA_{i,j}\}, 1 \leq i \leq 3, 1 \leq j \leq n-1$.

To prove the theorem we compute first the barycentric coordinates for each vertex of the sub polygon; these are points of intersections of the barycentric equations of the cevians $\{A_iA_{i,j}\}, 1 \leq i \leq 3, 1 \leq j \leq n-1$. Next we divide the sub polygon into non-overlapping triangles (in Fig. 2 the polygon $MNOPQR$ is divided into 4 triangles). Then we use the method which Coxeter gave in his book [2, p. 219] for proving Theorem 1 (which will be illustrated in the next section) to conclude that the ratio of the area of each triangle to the area of the whole triangle depends only on $\{\lambda_{i,j}\}$ and the result follows.

## 2 Patterns

The search after nice expressions of area ratios succeeds in ”symmetric” divisions. In particular we have the following theorem:

**Theorem 3** Suppose the sides of a triangle $A_2A_3A_1$ are divided at $A_{i,1}, A_{i,2}, ..., A_{i,n-1}, 1 \leq i \leq 3$ in the respective ratios $1 : \lambda : 1$ (see Fig. 3). Let $I, J, K, L, M, N$ be the points of intersection of the corresponding cevians as shown in the table:

| point | cevians intersection |
|-------|----------------------|
| $I$   | $A_1A_{1,1} \cap A_3A_{3,2}$ |
| $J$   | $A_2A_{2,1} \cap A_3A_{3,2}$ |
| $K$   | $A_2A_{2,1} \cap A_1A_{1,2}$ |
| $L$   | $A_3A_{3,1} \cap A_1A_{1,2}$ |
| $M$   | $A_2A_{2,2} \cap A_3A_{3,1}$ |
| $N$   | $A_1A_{1,1} \cap A_2A_{2,2}$ |

Then the ratio of the area of the hexagon $IJKLMN$ to the area of the triangle $A_2A_3A_1$ is

$$\frac{2\lambda^2}{(3 + \lambda)(2\lambda + 3)}.$$
Proof. At first note that such a hexagon exists in any triangle since it exists in an equilateral triangle, by symmetry of the division ratios, and every other triangle is affine equivalent to an equilateral triangle. Let $G$ be the centre of gravity of $A_2A_3A_1$. Since the barycentric coordinates of $A_1, A_2, A_3$ are $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively, the barycentric coordinates of $G$ are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. By symmetry, the area of the hexagon $IJKLMN$ is 6 times the area of the triangle $GMN$. Therefore, it is sufficient to compute the barycentric coordinates of $M$ and $N$.

In order to find the barycentric equations of the corresponding cevians, we exhibit the barycentric coordinates of some of the division points. Now, $A_{1,1}$ divides $A_2A_3$ in the ratio $1 : \lambda + 1$, $A_{2,2}$ divides $A_3A_1$ in the ratio $\lambda + 1 : 1$ and $A_{3,1}$ divides $A_1A_2$ in the ratio $1 : \lambda + 1$. Thus the barycentric coordinates are given in the table:

| point | barycentric coordinates |
|-------|-------------------------|
| $A_{1,1}$ | $(1, 0, \lambda + 1)$ |
| $A_{2,2}$ | $(1, \lambda + 1, 0)$ |
| $A_{3,1}$ | $(0, \lambda + 1, 1)$ |

We proceed by computing the equations of the cevians whose points of intersection are $M$ and $N$. The cevian $A_2A_{2,2}$ has the equation

$$
\begin{vmatrix}
0 & 0 & 1 \\
1 & \lambda + 1 & 0 \\
t_1 & t_2 & t_3
\end{vmatrix} = 0
$$

which is equivalent to $-(\lambda + 1)t_1 + t_2 = 0$. The cevian $A_1A_{1,1}$ has the equation

$$
\begin{vmatrix}
0 & 1 & 0 \\
1 & 0 & \lambda + 1 \\
t_1 & t_2 & t_3
\end{vmatrix} = 0
$$

which is equivalent to $-(\lambda + 1)t_1 + t_3 = 0$, and the cevian $A_3A_{3,1}$ has the equation

$$
\begin{vmatrix}
1 & 0 & 0 \\
0 & \lambda + 1 & 1 \\
t_1 & t_2 & t_3
\end{vmatrix} = 0
$$

which is equivalent to $-t_2 + (\lambda + 1)t_3 = 0$. Hence, the point $M$ can be computed by the following system of equations:

$$
\begin{cases}
-(\lambda + 1)t_1 + t_2 = 0 \\
t_2 + (\lambda + 1)t_3 = 0
\end{cases}
$$

Substituting $t_2 = 1$ we get,

$$
M = \left(\frac{1}{\lambda + 1}, 1, \frac{1}{\lambda + 1}\right).
$$
Similarly, the point $N$ is obtained from the following equations;

$$\begin{align*}
-(\lambda + 1)t_1 + t_3 &= 0 \\
-(\lambda + 1)t_1 + t_2 &= 0
\end{align*}$$

Substituting $t_1 = 1$ we get,

$$N = (1, \lambda + 1, \lambda + 1).$$

Consequently, the ratio of the area of the triangle $GMN$ to the area of the triangle $A_2A_3A_1$ is equal to the determinant

$$\begin{vmatrix}
\frac{1}{\lambda+1} & \frac{1}{\lambda+1} & \frac{1}{\lambda+1} \\
1 & \lambda + 1 & \lambda + 1
\end{vmatrix} = \frac{\lambda^2}{3(\lambda + 1)}$$

divided by the product of the sums of the rows;

$$\left(1 + \frac{2}{\lambda + 1}\right)(2\lambda + 3).$$

Therefore, the ratio of the area of the hexagon $IGKLMN$ to the area of the triangle $A_2A_3A_1$ is

$$\frac{\frac{6\lambda^2}{3(\lambda + 1)}}{1 + \frac{2}{\lambda + 1}} \frac{2\lambda^2}{(\lambda + 3)(2\lambda + 3)},$$

in agreement with the statement of the theorem. ■

2.1 Special cases

1. If $\lambda = 1$ then the ratio of the area of the hexagon to the area of the triangle is $\frac{1}{10}$. This special case is referred to as Marion Walter’s theorem [7] which states the following: If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has one-tenth the area of the original triangle.

2. If $n$ is odd, $n = 2k + 1$, then taking $\lambda = \frac{1}{k}$ implies that the ratio of the area of the hexagon to the area of the triangle is

$$\frac{2}{(3k + 1)(3k + 2)} = \frac{8}{9n^2 - 1}.$$ 

This special case is referred to as Morgan’s theorem [3], which was proved by T. Watanabe, R. Hanson and F. D. Nowosielski [8] using Routh-Steiner theorem several times.
3. If \( n \) is even, then taking \( \lambda = n - 2 \) implies a new result which was not mentioned in the previous discussion. In this case, we have the following:

The ratio of the area of the hexagon to the area of the triangle is

\[
\frac{2(n-2)^2}{(n+1)(2n-1)}.
\]

Note that, if each side of the triangle is divided into \( n \) equal parts, then for odd \( n, n = 2k+1 \), the hexagon is the sub polygon in Theorem 2, formed by the cevians \( \{A_iA_{k,i}, A_iA_{k+1,i}\}, 1 \leq i \leq 3 \). While for even \( n \), the hexagon is formed by the cevians \( \{A_iA_1,i, A_iA_{n-1,i}\}, 1 \leq i \leq 3 \).

3 Parallelograms

We can generalize the Routh-Steiner theorem to parallelograms in the following manner:

**Theorem 4** If the sides \( BC, CD, DA, AB \) of a parallelogram \( ABCD \) are divided at \( K, L, M, N \) in the respective ratios \( \kappa : 1, \lambda : 1, \mu : 1, \nu : 1 \), the cevians \( AK, BL, CM, DN \) form a quadrilateral whose area is

\[
\frac{1}{2} \frac{1}{(2 + \frac{1}{\lambda} - \frac{1}{1+\mu})} (1 + \kappa + \frac{\kappa}{1+\mu}) (1 + \lambda + \frac{\lambda}{1+\mu}) (1 + 2\kappa + \frac{2\kappa}{1+\mu}) + \frac{1}{2} \frac{1}{(2 + \frac{1}{\nu} - \frac{1}{1+\mu})} (1 + \mu + \frac{\mu}{1+\nu}) (1 + \nu + \frac{\nu}{1+\mu}) (1 + 2\lambda + \frac{2\lambda}{1+\mu})
\]

times that of \( ABCD \).

**Proof.** If the barycentric coordinates of \( A, B, C \) are \((0,1,0), (0,0,1), (1,0,0)\) respectively, then the barycentric coordinates of \( D \) are \((1,1,-1)\). This is true since the diagonals in the parallelogram bisect each other and the barycentric coordinates of the midpoint of \( AC \) are \((\frac{1}{2}, \frac{1}{2}, 0)\) (see Fig. 4). The barycentric coordinates of \( K, L, M, N \) are shown in the following table:

| point | barycentric coordinates |
|-------|-------------------------|
| \( K \) | \((\kappa, 0, 1)\) |
| \( L \) | \((\lambda + 1, \lambda, -\lambda)\) |
| \( M \) | \((1, \mu + 1, -1)\) |
| \( N \) | \((0, 1, \nu)\) |

Proceeding as in the proof of Theorem 2, the cevians \( AK, BL, CM, DN \) have the following equations:

| cevian | barycentric equation |
|--------|----------------------|
| \( AK \) | \(-t_1 + \kappa t_3 = 0\) |
| \( BL \) | \(-\lambda t_1 + (\lambda + 1)t_2 = 0\) |
| \( CM \) | \(t_2 + (\mu + 1)t_3 = 0\) |
| \( DN \) | \((1 + \nu)t_1 - \nu t_2 + t_3 = 0\) |
Now, let \( X, Y, Z, W \) be the points of intersections of pairs of cevians \( BL \) and \( CM \), \( CM \) and \( DN \), \( DN \) and \( AK \), \( AK \) and \( BL \), respectively (see Fig. 4).

Then the barycentric coordinates of these points are given in the following table:

| point | barycentric coordinates |
|-------|-------------------------|
| \( X \) | \( \left( \frac{\lambda+1}{\lambda}, 1, -\frac{1}{\mu+1} \right) \) |
| \( Y \) | \( \left( -\frac{\lambda+\nu+\mu}{\lambda+1}, -\mu - 1, 1 \right) \) |
| \( Z \) | \( \left( \kappa, \frac{1+\kappa+\nu}{\lambda+1}, 1 \right) \) |
| \( W \) | \( \left( \kappa, \frac{\kappa}{\lambda+1}, 1 \right) \) |

The area of the quadrilateral \( XYZW \) equals the sum of the areas of the triangles \( XYZ \) and \( ZWX \). Normalizing the barycentric coordinates of \( X, Y, Z, W \) and dividing by 2 which is the area of the parallelogram \( ABCD \), we get that the ratio of the area of the quadrilateral \( XYZW \) to the area of the parallelogram \( ABCD \) equals \( \frac{1}{2} (r_1 + r_2) \) where,

\[
 r_1 = -\frac{\begin{vmatrix} \frac{\lambda+1}{\lambda} & 1 & -\frac{1}{\mu+1} \\ -\frac{\lambda+\nu+\mu}{\lambda+1} & -\mu - 1 & 1 \\ \kappa & \frac{1+\kappa+\nu}{\lambda+1} & 1 \end{vmatrix}}{(2 + \frac{1}{\lambda} - \frac{1}{\mu+1})(1 + \kappa + \frac{\lambda \kappa}{\lambda+1})(1 + 2\kappa + \frac{\kappa+1}{\mu})} \tag{2}
\]

and

\[
 r_2 = \frac{\begin{vmatrix} \kappa & \frac{1+\kappa+\nu}{\lambda+1} & 1 \\ \frac{\lambda}{\lambda+1} & \frac{\lambda}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} & \frac{\lambda}{\lambda+1} & 1 \end{vmatrix}}{(2 + \frac{1}{\lambda} - \frac{1}{\mu+1})(1 + \kappa + \frac{\lambda \kappa}{\lambda+1})(1 + 2\kappa + \frac{\kappa+1}{\nu})} \tag{3}
\]

Note that :

\[
\begin{vmatrix} \frac{\lambda+1}{\lambda} & 1 & -\frac{1}{\mu+1} \\ -\frac{\lambda+\nu+\mu}{\lambda+1} & -\mu - 1 & 1 \\ \kappa & \frac{1+\kappa+\nu}{\lambda+1} & 1 \end{vmatrix} = \begin{vmatrix} \frac{\lambda}{\lambda+1} & 1 & -\frac{1}{\mu+1} \\ -\frac{\lambda+\nu+\mu}{\lambda+1} & -\mu - 1 & 1 \\ \kappa & \frac{1+\kappa+\nu}{\lambda+1} & 1 \end{vmatrix}
\]

and

\[
\begin{vmatrix} \kappa & \frac{1+\kappa+\nu}{\lambda+1} & 1 \\ \frac{\lambda}{\lambda+1} & \frac{\lambda}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} & \frac{\lambda}{\lambda+1} & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & \frac{1+\kappa+\nu}{\lambda+1} \\ 0 & \frac{\lambda}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} & \frac{\lambda}{\lambda+1} & 1 \end{vmatrix}.
\]

After evaluating the determinants and substituting in (2) and (3) the result follows.

In particular, if \( \kappa = \lambda = \mu = \nu \) then the expressions in the denominators of (1) are equal to:

\[
\frac{(2\lambda^2 + 2\lambda + 1)^3}{\lambda^2(\lambda + 1)^2},
\]
while the expressions in the numerators of (1) are equal to:
\[
\frac{(2\lambda^2 + 2\lambda + 1)^2}{\lambda^2(\lambda + 1)^2}.
\]

Hence, we have the following:

**Corollary 5** If the sides $BC, CD, DA, AB$ of a parallelogram $ABCD$ are divided at $K, L, M, N$ in the ratio $\lambda : 1$, then the cevians $AK, BL, CM, DN$ form a quadrilateral whose area is
\[
\frac{1}{2\lambda^2 + 2\lambda + 1}
\] times that of $ABCD$.

Substituting $\lambda = \frac{1}{p-1}$, $p \geq 2$, we get the equivalent expression $\frac{p^2 - 2p + 1}{p^2 + 1}$, which was discovered by M. De Villiers [6]. He did not provide any proof but pointed out the following: "Since a square is affinely equivalent to a parallelogram, the easiest way to derive and prove this formula is to consider the special case of a square".

We leave to the reader to see what happens in another particular case of the theorem: if $\kappa = \mu$ and $\lambda = \nu$.

Finally, the fascinating theorem of Routh-Steiner attracted the interest of mathematics education researchers, probably because of the use of a dynamic geometry software to rediscover the theorem, but it still attracts the interest of mathematics researchers as well.

### References

[1] A. Bényi and B. Ćurgus, A Generalization of Routh’s Triangle Theorem, *The American Mathematical Monthly* 9 (2013) 841-846.

[2] H. S. M. Coxeter, *Introduction to Geometry*, second edition. Wiley, New York, 1989.

[3] R. Morgan, Reader Reflections: No restrictions needed, *Mathematics teacher* 87 (1994) 726, 743.

[4] E. J. Routh, *A Treatise on Analytical Statics with Numerous Examples*, Vol. 1, second edition. Cambridge University Press, 1896.

[5] J. Steiner, *Gesammelte Werke*, Vol. 1, Reimer, Berlin, 1882.
Figure 1: The areas of the triangles $PA_2A_3, PA_3A_1, PA_1A_2$ are proportional to the barycentric coordinates $t_1, t_2, t_3$, respectively.
Figure 2: The ratio of the area of any sub-polygon to the area of the whole triangle depends only on \( \{\lambda_{i,j}\} \)
Figure 3: Dividing each side in the ratios $1 : \lambda : 1$
Figure 4: The sides of a parallelogram are divided in the ratios $\kappa : 1, \lambda : 1, \mu : 1, \nu : 1$
