TOPOLOGICAL GRAVITY IN MINKOWSKI SPACE

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Abstract. The two-category with three-manifolds as objects, $h$-cobordisms as morphisms, and diffeomorphisms of these as two-morphisms, is extremely rich; from the point of view of classical physics it defines a nontrivial topological model for general relativity.

A striking amount of work on pseudoisotopy theory [Hatcher, Waldhausen, Cohen-Carlsson-Goodwillie-Hsiang-Madsen . . . ] can be formulated as a TQFT in this framework. The resulting theory is far from trivial even in the case of Minkowski space, when the relevant three-manifold is the standard sphere.

Topological gravity [18] extends Graeme Segal’s ideas about conformal field theory to higher dimensions. It seems to be very interesting, even in extremely restricted geometric contexts:

§1 basic definitions

1.1 A cobordism $W : V_0 \to V_1$ between $d$-manifolds is a $D = d + 1$-dimensional manifold $W$ together with a distinguished diffeomorphism

$$\partial W \cong V_0^{op} \coprod V_1 ;$$

a diffeomorphism $\Phi : W \to W'$ of cobordisms will be assumed consistent with this boundary data.

$\text{Cob}(V_0, V_1)$ is the category whose objects are such cobordisms, and whose morphisms are such diffeomorphisms. Gluing along the boundary defines a composition functor

$$\#: \text{Cob}(V', V) \times \text{Cob}(V, V'') \to \text{Cob}(V, V'') .$$

The two-category with manifolds as objects and the categories $\text{Cob}$ as morphisms is symmetric monoidal under disjoint union.

The categories $\text{Cob}$ are topological groupoids (all morphisms are invertible), with classifying spaces

$$|\text{Cob}(V_0, V_1)| = \coprod_{[W : V_0 \to V_1]} B\text{Diff}(W \text{ rel } \partial) .$$

The topological gravity category has these objects as hom-spaces: it is a (symmetric monoidal) topological category.

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1.2 A theory of topological gravity is a representation of such a category in some simpler monoidal category, e.g. Hilbert spaces, or spectra.

The homotopy-to-geometric quotient map

\[ B\text{Diff} = \text{Met} \times \text{Diff} E\text{Diff} \to \text{Met} \times \text{Diff pt} = \text{Met}/\text{Diff} \]

defines a functor from the topological gravity category to a category with the spaces \( \text{Met}/\text{Diff} \) as morphism objects; these are the spaces of states in general relativity (and

\[ g \mapsto \int R(g) \, d\text{vol}_g \]

is a kind of Morse function upon them).

In Segal’s conformal field theory, the corresponding objects are moduli spaces of (complex structures on) Riemann surfaces. Indeed if \( W = \Sigma \) is a Riemann surface of genus \( g > 1 \), its group of diffeomorphisms is homotopically discrete: the map

\[ \text{Diff}(\Sigma) \to \pi_0 \text{Diff}(\Sigma) \]

is a homotopy equivalence. The mapping class group acts with finite isotropy on Teichmüller space, so when \( d = 1 \) the homotopy-to-geometric quotient is close to a rational homology equivalence.

§2 examples

2.1 In recent work Galatius, Madsen, Tillmann and Weiss have identified the classifying space of the cobordism category of oriented \( d \)-manifolds in terms of a twisted desuspension \( MT\text{SO}(D) \) of the classifying space of the special orthogonal group. Their techniques extend more generally, to cobordism categories of manifolds with extra structure on their tangent bundle.

Three-manifolds under Spin cobordism have very interesting connections with the theory of even unimodular lattices \([8,16]\), and the methods of [6] identify the classifying spectrum of this category with the desuspension of \( B\text{Spin}(4) \) by the vector bundle associated to the standard four-dimensional representation of the spin group. Because of well-known coincidences in low-dimensional geometry, \( \text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2) \), so we can identify its classifying space with the product of two copies of infinite-dimensional quaternionic projective space, and the vector bundle defined by the standard representation with the tensor product (over \( \mathbb{H} \)) of the resulting two canonical quaternionic line bundles \( L_\pm \); thus

\[ MT\text{Spin}(4) \sim (\mathbb{H}P_\infty \times \mathbb{H}P_\infty)^{-L_+ \otimes_{\mathbb{H}} L_-}. \]

The generators of \( \pi_0 \Omega^\infty MT\text{Spin}(4) \cong \mathbb{Z}^2 \) can be identified with the signature and Euler characteristic, or alternately with the number of hyperbolic and \( E_8 \) factors in the middle-dimensional intersection form [2] of a spin cobordism.

2.2 There are other extremely interesting variant constructions in dimension four: contact three-manifolds under \( \text{Spin}^- \) cobordism define a natural context for Seiberg-Witten theory, while Lorentz cobordism [20] incorporates an arrow of time; but this note is concerned with 3-manifolds up to \( h \)-cobordism:
Recall that $W : V_0 \to V_1$ is an $h$-cobordism if the two inclusions $V_0 \subset W, V_1 \subset W$ are homotopy equivalences [17].

The trivial $h$-cobordism $W = V \times I$, where $I$ is an interval, is an interesting example. In dimensions $\geq 5$, the $s$-cobordism theorem classifies $h$-cobordisms by elements of the Whitehead group

$$\text{Im} \left[ \pm \pi_1 V \to K_1(\mathbb{Z}[\pi_1(V)]) \right] := \text{Wh}(\pi_1(V)),$$

and there are invariants for parametrized $h$-cobordisms taking values in higher homotopy groups of certain pseudoisotopy spaces, which have been studied by Hatcher, Waldhausen, Igusa, . . .

This category has a monoidal structure, but it is relatively trivial, so that it is natural to assume that the manifolds $V$ are connected.

2.3 Here I will be concerned mostly with the case $V = S^3$: by Minkowski space I really mean the universal cover $S^3 \times \mathbb{R}$ of Penrose’s (and others’) conformal compactification $S^3 \times_{\pm 1} S^1$ of Minkowski space; this contains, in particular, a copy of Einstein’s static universe [11]. Its time-like intervals define trivial $h$-cobordisms of $S^3$.

Note that there are lots of wild $S^3 \times \mathbb{R}$’s: remove a point from a fake $\mathbb{R}^4$. It would be very interesting to construct a semigroup of such things, under some kind of boundary gluing, as Segal did with topological annuli; current work of Gompf [10 §7, cf. also [3]] seems close to this. It is not clear at the moment if nontrivial smooth $h$-cobordisms of the three-sphere exist; the question is closely connected to the smooth four-dimensional Poincaré conjecture.

§3 double categories

3.1 Boundary value problems involve the interplay between diffeomorphisms of a manifold and diffeomorphisms of its boundary. Tillmann [21] suggests that double categories provide a natural framework for such questions. In this context, the primary objects are certain rectangular diagrams

$$W : \quad \begin{array}{c} V_0 \longrightarrow V_1 \\ \phi \downarrow \quad \phi_0 \downarrow \phi_1 \\ W' : \quad V'_0 \longrightarrow V'_1. \end{array}$$

with cobordisms displayed horizontally, and diffeomorphisms (which preserve some boundary framing) presented vertically; these can be patched together in either direction. More recently, Getzler [7] has used manifolds, together with suitable (eg separating) codimension one submanifolds, to define morphisms in such contexts; this seems particularly suited to the millefeuille examples of Gompf, which (if I understand correctly) can be regarded as smooth $h$-cobordisms between topological, but not necessarily smooth, three-spheres.

3.2 In any case, the double category $D$ of trivial $h$-cobordisms between ordinary three-spheres is already extremely interesting. I don’t know how to associate a
topological category to a double category in general, but in this case pseudoisotopy theory defines an equivalence with the two-category
\[ \prod \left[ \left\{ V \right\}/C(V) \right] \]
having manifolds \( V \) as its objects, and Cerf’s group \( C(V) \) \([13 \S 6.2]\) of pseudoisotopies (regarded as a category with one object) as its category of automorphisms:

These pseudoisotopies are diffeomorphisms of the cylinders \( V \times I \), equal to the identity map on \( V \times 0 \). There is a fibration
\[ \text{Diff}(V \times I \text{ rel } \partial) \to C(V) \to \text{Diff}(V) \]
of groups, and concordance
\[ \Phi, \Psi \mapsto \Phi \# (\phi_1 \times 1_I) \circ \Psi \]
of pseudoisotopies defines a homomorphism
\[ C(V) \times C(V) \to C(V) . \]
The classifying space \( BC(V) \) is thus a monoid, and the topological category associated to this rectification of \( D \) defines an ad hoc topologification (with one object for each \( V \), and the topological monoid \( BC(V) \) for its space of endomorphisms). The classifying \( B^2C(V) \) space of that topological category is the totalization of the bisimplicial space defined by the category of trivial \( h \)-cobordisms of \( V \).

3.3 There is a natural stabilization map from \( B^2C(V) \) to Waldhausen’s ring spectrum \( A(V) \). In the language of TQFT’s, this defines a functor from the gravity category of trivial \( h \)-cobordisms of \( V \) to the category with \( \{ V \} \) as its object, and the group ring \( S^0[\Omega \text{Wh}^d(V)] \) as its endomorphism object. [The map from \( \Omega B^2C(V) \) to \( \Omega A(V) \) factors through the space \( H\text{Cobord}^d(V) \sim \Omega \text{Wh}^d(V) \) of stabilized \( h \)-cobordisms of \( V \) [22].] This reveals Whitehead torsion (regarded as an element of \( \mathbb{Z}[\text{Wh}] \)) as perhaps the primordial example of a TQFT!

Note that Cerf’s maps define a fibration
\[ B\text{Diff}(V) \to B^2\text{Diff}(V \times I \text{ rel } \partial) \to B^2C(V) \]
which looks like a presentation of this ad hoc classifying space for a double category as a fibration
\[ |\text{Vertical}| \to |\text{Horizontal}| \to |\text{Double}| \]
built from classifying spaces for its component (vertical and horizontal) morphisms; but I don’t know enough about double categories to guess if this might be an instance of something more general.

§4 about \( A(S^n) \)

4.1 Through the efforts of many researchers, a great deal is known about the algebraic \( K \)-theory of spaces; in particular, if \( X \) is simply connected (and of finite type) its \( A \)-theory can be calculated (at least \( p \)-locally \([4 \S 1.3]\)) from the topological cyclic homology \([14 \S 7.3.14]\) of \( S^0[\Omega X] \).

Since this pretends to be a paper about physics, however, I will be content with some remarks about \( A_*(X) \otimes \mathbb{Q} \), which is accessible in more elementary terms. [I want to record here my thanks to Bruce Williams and Bjorn Dundas for walking
me through a great deal of literature in this field, without suggesting that they bear any responsibility for the excesses of this paper.]

4.2 In particular, old results [12] of Hsiang and Staffeldt imply that (when $n > 1$) the rationalization of $A(S^n)$ splits as a copy of $A(pt) \otimes \mathbb{Q} \cong K\text{alg}(\mathbb{Z}) \otimes \mathbb{Q}$ and the suspension of what is essentially the (reduced) topological cyclic homology of $S^n$, which can be computed effectively as the abelianization of $\tilde{H}_*(\Omega S^n, \mathbb{Q})$ regarded as a graded Lie algebra; hence

$$\pi_*(\Omega \text{Wh}_d(S^n)) \otimes \mathbb{Q} \cong K\text{alg}_{2+1}(\mathbb{Z}) \otimes \mathbb{Q} \oplus \tilde{H}_*(\Omega S^n, \mathbb{Q})_{ab}.$$ 

The Whitehead product structure on a wedge of spheres is rationally free, so the graded Lie algebra structure has nontrivial commutators only when $n$ is even. When $n = 2m + 1$ is odd, the rational homology is polynomial on a single generator $x_{2m}$; it follows that

$$A_{2+1}(S^3) \otimes \mathbb{Q} = \mathbb{Q}\langle \zeta_k, x_{2k}^l \rangle$$

is spanned as a rational vector space by elements $x_{2k}^l$ of degree $2k$ and elements $\zeta_k$ of degree $4k$ corresponding to the odd zeta-values $\zeta(2k+1)$ which appear as regulators in Borel’s calculations of $K\text{alg}_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q}$.

This can be made more precise; when $X$ is simply-connected then a reduced version $\Omega \tilde{\text{Wh}}(X)$ of loops on the Whitehead space is closely connected to a similarly reduced version $Q(\tilde{L}X_{hT})$ of (the infinite loopspace defined by) the suspension spectrum of the homotopy quotient (by its natural circle action) of the free loopspace of $X$.

4.3 The construction $Q = \Omega^\infty \Sigma^\infty$ sends a space to the infinite loopspace representing its suspension spectrum: this sends the rational homology of a space to its symmetric algebra. The cohomological invariants defined by the space of trivial $h$-cobordisms of the three-sphere thus resemble the ‘big’ phase spaces [9] studied in quantum cohomology: for example, the stable rational homology of the Riemann moduli space is essentially with the symmetric algebra on the homology of $\mathbb{C}P^\infty$, and is thus a polynomial algebra with one generator of each even degree.

The rational cohomology of the infinite loopspace $\Omega^{n+1} A(S^3)$ seems similar in many ways: it is again a polynomial algebra, now with one set of generators indexed by even integers, the other by integers $\equiv 0$ modulo four. Physicists see these symmetric algebras as Fock representations associated to certain polarized symplectic vector spaces. In our context this seems to be related to an ‘almost’ splitting

$$HC_{per} \sim HC \oplus \text{Hom}_{\mathbb{Q}[u]}(HC, \mathbb{Q}[u]),$$

of periodic cyclic homology [5] These representations have symmetries closely related to the Virasoro algebra, which lead [19] to interesting integrable systems.

This connection between 4D topological gravity and the equivariant free loopspace of the three-sphere resembles in many ways a purely mathematical instance of a phenomenon physicists [1] call ‘holography’, in which one physical model on the interior of a manifold is described by some other model on its boundary. Rather than proceed any further with speculations along these lines, I’d like to close by raising a mathematical question:
A trivial $h$-cobordism between three-spheres is an example of a four-dimensional spin cobordism; this defines a monoidal functor, and hence a morphism

$$\Sigma^{-1}A(S^3) \to M\text{Spin}(4) \sim (\mathbb{H}P_{\infty} \times \mathbb{H}P_{\infty})_+^{L^p \hat{\otimes} \mathbb{L}^-}$$

of spectra. Could it possibly be nontrivial?

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