The Existence and Uniqueness of Positive Solutions of an Ordinary Differential Equation with a Nonlocal Conditions

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Abstract. Many researchers have studied problems with non-local conditions of the second-order differential equations. In this work we study the ordinary differential equation \( v''(t) + g(t, v(t)) = 0, \ t \in (0,1), \) with the nonlocal conditions \( v'(1) = 0, v(0) = D^\alpha v(t)|_{t=1}, \alpha \in (0,1). \) First, we study the existence of at least one positive continuous solution under some assumptions on the function \( g. \) Then we discuss the uniqueness of solution by assume that there exist a constant \( k > 0 \) such that \( |g(t,v) - g(t,\varphi)| \leq k |v - \varphi|, \forall \ t \in [0,1], \forall v, \varphi \in \mathbb{C}[0,1] \) for this ordinary differential equation, a clarifying example was given as an application. The main idea in this paper is to study ordinary differential equations with a fractional order condition.

Keywords: ordinary differential equation, Green's function, fractional condition, positive solution.

1. Introduction

The differential equations is an important field in mathematical it is used in the modeling of many different phenomena in sciences. The nonlocal boundary value problems for differential equations or difference equations arise in a variety of different areas of applied mathematics, physics, chemistry, control of dynamical systems etc. Recently, many researchers paid attention to existence result of solution of the nonlocal value problem for ordinary differential equations and they gave a lot of time and effort to discuss it, such as [1-16].

On the other hand, the study of boundary value problems of integral and fractional conditions is also important part of nonlocal boundary value problems.

In [8], the authors considered that a second-order differential equation had many positive solutions \( x''(t) + q(t)g(x(t)) = 0, \ t \in [0,1] \) with the nonlocal conditions \( x(0) = 0, x(1) = \int_0^1 x(s)dg(s) \) where \( 0 < \alpha < \beta < 1 \) and \( g: [\alpha, \beta] \to R \) is an increasing function.

Besides, the existence of non-negative solutions of a second-order ordinary differential equation has also been investigated by the authors [9]. \( x''(t) + q(t)g(x(t), x'(t)) = 0, \ a.a. t \in [0,1], \) with the nonlocal conditions \( x(0) = 0, x(1) = \int_0^1 x'(s)dg(s). \)

Where \( g: I \times \mathbb{R}^2 \to \mathbb{R} \) and \( q, g: I \to [0,\infty) \) are given functions.

In this paper, the existence and uniqueness of a positive solution to the non-local boundary value problem of the ordinary differential equation are studied.
\[ v''(t) + g(t, v(t)) = 0, \quad t \in (0,1), \quad (1) \]
with the nonlocal conditions
\[ v'(1) = 0, \quad v(0) = D^\alpha v|_{t=1}. \quad (2) \]

Where \(D^\alpha\) is the Riemann-Liouville fractional-order derivative of order \(\alpha \in (0,1)\).

2. Preliminaries

In this part, we bring back some fundamental notes and meanings that will be included in this article.

Let \(C(I)\) indicates the class of continuous functions and \(L^1(I)\) indicates to the class of Lebesgue integrable functions on the interval \(I = [a, b]\), where \(0 \leq a < b < \infty\) and let \(I(.)\) indicates to the gamma function.

**Definition 1.** [17] The fractional-order integral of the function \(g \in L^1[a, b]\) of order \(\beta \in R^+\) is defined by

\[
I^\beta_a g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) \, ds.
\]

**Definition 2.** [17] The Riemann-Liouville fractional - order derivative of \(g\) of order \(\beta \in (0,1)\) is defined by

\[
D^\beta_a g(t) = \frac{d}{dt} \int_a^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} g(s) \, ds.
\]

3. The existence of solution

This section discusses the integral representation of the solution to problems (1) and (2). The presence of a positive solution is investigated.

Consider the following assumptions for problems (1) and (2):

(i) \(g: [0,1] \times R \rightarrow R\) is measurable in \(t \in [0,1]\) for every \(v \in R\),

(ii) \(g: [0,1] \times R \rightarrow R\) is continuous in \(v \in R\) for every \(t \in [0,1]\),

(iii) there exists a function \(m \in L^1[0,1]\) such that \(|g| \leq m.\)

We now give the integral representation of the problem's solution. (1) - (2) if it exists. We have the next lemma.

**Lemma 1.** Solving the problem(1)-(2) can be expressed by an integral equation.

\[
v(t) = \frac{L}{\Gamma(2-\alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta - \frac{L}{\Gamma(2-\alpha)} \int_0^{(1-\theta)^{1-\alpha}} g(\theta, v(\theta)) \, d\theta
\]

\[+ t \int_0^1 g(\theta, v(\theta)) \, d\theta - \int_0^t (t-\theta) g(\theta, v(\theta)) \, d\theta. \quad (3)\]

where \(L = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)-1} \)

**proof.** Consider the problem (1) - (2) with boundary value. if we Integrate both sides of equation (1) twice, we obtain

\[ v(t) = C_2 + t C_1 - \int_0^t (t-\theta) g(\theta, v(\theta)) \, d\theta. \]

And from the relation \(v'(1) = 0\), we have

\[ C_1 = \int_0^1 g(\theta, v(\theta)) \, d\theta \]
Then we get
\[ v(t) = C_2 + t \int_0^t g(\theta, v(\theta)) d\theta - \int_0^t (t - \theta)g(\theta, v(\theta)) d\theta \]

Now, if we operate the above equation by \( I^{1-\alpha} \) on both sides, we could get
\[ I^{1-\alpha}v(t) = C_2 \frac{t^{1-\alpha}}{\Gamma(2 - \alpha)} + \frac{t^{2-\alpha}}{\Gamma(3 - \alpha)} \int_0^t g(\theta, v(\theta)) d\theta - I^{2-\alpha}g(t, v(t)) \]

differentiating the last relation, we have
\[ D^{1-\alpha}v(t) = \frac{d}{dt} I^{1-\alpha}v(t) \]
\[ = C_2 \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + \frac{t^{1-\alpha}}{\Gamma(2 - \alpha)} \int_0^t g(\theta, v(\theta)) d\theta - I^{2-\alpha}g(t, v(t)) \]
And from the relation \( v(0) = D^\alpha v(t)|_{t=1} \), we have
\[ C_2 = \frac{C_2}{\Gamma(1 - \alpha)} + \frac{1}{\Gamma(2 - \alpha)} \int_0^1 g(\theta, v(\theta)) d\theta \]
\[ - \int_0^1 (1 - \theta)^{1-\alpha} \frac{\Gamma(2 - \alpha)}{\Gamma(2 - \alpha)} g(\theta, v(\theta)) d\theta \]
\[ = \frac{C_2}{\Gamma(1 - \alpha)} \int_0^1 g(\theta, v(\theta)) d\theta \]
then we get
\[ v(t) = \frac{\Lambda}{\Gamma(2 - \alpha)} \int_0^1 g(\theta, v(\theta)) d\theta - \Lambda \int_0^1 \frac{(1 - \theta)^{1-\alpha}}{\Gamma(2 - \alpha)} g(\theta, v(\theta)) d\theta + t \int_0^1 g(\theta, v(\theta)) d\theta - \int_0^t (t - \theta) g(\theta, v(\theta)) d\theta \]
where \( \Lambda = \left( \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)-1} \right) \)
Now, in the formula below, we can write equation (3)
\[ v(t) = \int_0^1 G(t, \theta) g(t, v(\theta)) d\theta. \quad (4) \]
Where
\[ G(t, \theta) = \begin{cases} \frac{\Lambda - \Lambda (1 - \theta)^{1-\alpha} + t \Gamma(2 - \alpha) - \Gamma(2 - \alpha)(t - 1)}{\Gamma(2 - \alpha)}, & 0 \leq \theta \leq t \leq 1, \\ \frac{\Lambda - \Lambda (1 - \theta)^{1-\alpha} + t \Gamma(2 - \alpha)}{\Gamma(2 - \alpha)}, & 0 \leq t \leq \theta \leq 1. \end{cases} \]

**Lemma 2.** The function \( G(t, \theta) \) satisfies \( G(t, \theta) > 0 \), for \( t, \theta \in (0, 1) \).

**Proof.** For \( 0 \leq \theta \leq t \leq 1 \), we have
\[ \Lambda + t \Gamma(2 - \alpha) \geq \Lambda (1 - \theta)^{1-\alpha} + \Gamma(2 - \alpha)(t - \theta) \quad (5) \]
where \( \Lambda = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)-1} \) Thus, \( G_1(t, \theta) > 0 \) for \( t, \theta \in (0, 1) \).
Furthermore, clearly \( G_2(t, \theta) \geq 0 \) for \( t, \theta \in [0, 1] \).
Then we get that \( G(t, \theta) > 0 \) for \( t, \theta \in (0, 1) \).
For the existence of a solution, we have the following theorem:

**Theorem 1.** Let the assumption (i)-(ii)-(iii) are satisfied, then the nonlocal boundary value problem (1)-(2) has at least one positive continuous solution \( v \in C[0,1] \).

**Proof.** Define a subset \( Q^+_r \subset C[0,1] \) by
\[
Q^+_r = \{ v(t) > 0, \text{for each } t \in [0,1], \|v\| \leq r \}, \quad r = 5\|m\|_{L_1}.
\]
The set \( Q^+_r \) is nonempty, closed and convex.

First, we going to prove that \( T \) is a completely continuous operator.

Let \( T: Q^+_r \rightarrow Q^+_r \) be an operator defined by
\[
Tv(t) = \frac{A}{\Gamma(2-\alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta
- A \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} g(\theta, v(\theta)) \, d\theta
+ t \int_0^1 g(\theta, v(\theta)) \, d\theta
- \int_0^t (t-\theta) g(\theta, v(\theta)) \, d\theta.
\]

For \( v \in Q^+_r \), let \( \{ v_n(t) \} \) be a sequence in \( Q^+_r \) converges to \( v(t) \), \( v_n(t) \rightarrow v(t), \forall t \in [0,1] \), then
\[
\lim_{n \to \infty} T v_n(t) = \frac{A}{\Gamma(2-\alpha)} \lim_{n \to \infty} \int_0^1 g(\theta, v_n(\theta)) \, d\theta
- A \lim_{n \to \infty} \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} g(\theta, v_n(\theta)) \, d\theta
+ t \lim_{n \to \infty} \int_0^1 g(\theta, v_n(\theta)) \, d\theta
- \lim_{n \to \infty} \int_0^t (t-\theta) g(\theta, v_n(\theta)) \, d\theta.
\]

by the assumption (i)-(ii)-(iii) and by the Lebesgue dominated convergence Theorem [18] we deduce that
\[
\lim_{n \to \infty} T v_n(t) = (Tv)(t).
\]

Then the operator \( T \) is continuous.

Now, let \( v \in Q^+_r \), then
\[
|Tv(t)| = \left| \frac{A}{\Gamma(2-\alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta
- A \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} g(\theta, v(\theta)) \, d\theta
+ t \int_0^1 g(\theta, v(\theta)) \, d\theta
- \int_0^t (t-\theta) g(\theta, v(\theta)) \, d\theta \right|
\leq \frac{A}{\Gamma(2-\alpha)} \int_0^1 |g(\theta, v(\theta))| \, d\theta.
\]

by the assumption (i)-(ii)-(iii) and by the Lebesgue dominated convergence Theorem [18] we deduce that
\[
\lim_{n \to \infty} T v_n(t) = (Tv)(t).
\]
\[
+ \int_0^1 |g(\theta, v(\theta))| \, d\theta + \int_0^1 (1 - \theta) |g(\theta, v(\theta))| \, d\theta
\leq \frac{2\lambda + 2\Gamma(2 - \alpha)}{\Gamma(2 - \alpha)} \int_0^1 |g(\theta, v(\theta))| \, d\theta
\leq 5\|m\| \leq r
\]

Then \( \{Tv(t)\} \) is uniformly bounded in \( Q_r^+ \).

We illustrate in what follows that \( T \) is a completely continuous operator.

second, we going to prove that \( T \) is a compact operator.

For \( t_1, t_2 \in (0, 1) \), \( t_1 < t_2 \) such that \( |t_2 - t_1| < \delta \) we have

\[
|Tv(t_2) - Tv(t_1)| = |t_2 \int_0^1 g(\theta, v(\theta)) \, d\theta - t_1 \int_0^1 g(\theta, v(\theta)) \, d\theta|
\]

\[
\leq \int_0^1 |(t_2 - \theta)g(\theta, v(\theta))| \, d\theta + \int_0^1 (t_2 - \theta) |g(\theta, v(\theta))| \, d\theta
\]

\[
\leq \int_0^1 (t_2 - \theta) |g(\theta, v(\theta))| \, d\theta + \int_0^1 (t_2 - \theta) |g(\theta, v(\theta))| \, d\theta
\]

Thus, the functions class \( \{Tv(t)\} \) is equi-continuous. By Arzela - Ascolis Theorem [18] \( \{Tv(t)\} \) is relatively compact. If all the conditions of the Schauder Theorem are hold, the operator \( T \) shall have a fixed point in \( Q_r^+ \).

There is also at least one positive continuous solution in the integrated equation (3). \( v \in C(0, 1) \).

Now,

\[
\lim_{t \to 0} v(t) = \frac{\lambda}{\Gamma(2 - \alpha)} \lim_{t \to 0} \int_0^1 g(\theta, v(\theta)) \, d\theta
\]

\[
- \lambda \int_0^1 \frac{(1 - \theta)^{1-\alpha}}{\Gamma(2 - \alpha)} g(\theta, v(\theta)) \, d\theta + \lim_{t \to 0} t \int_0^1 g(\theta, v(\theta)) \, d\theta
\]

\[
= \frac{\lambda}{\Gamma(2 - \alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta
\]

And
\[\lim_{t \to 1} v(t) = \frac{\Lambda}{\Gamma(2 - \alpha)} \lim_{t \to 1} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[= -\frac{\Lambda}{\Gamma(2 - \alpha)} \int_{0}^{1} \left(1 - \theta\right)^{1 - \alpha} g(\theta, v(\theta)) \, d\theta \]

\[+ \lim_{t \to 1} t \int_{0}^{t} g(\theta, v(\theta)) \, d\theta \]

\[= \frac{\Lambda}{\Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta - \frac{\Lambda}{\Gamma(2 - \alpha)} \int_{0}^{1} \left(1 - \theta\right)^{1 - \alpha} g(\theta, v(\theta)) \, d\theta \]

\[+ \int_{0}^{1} g(\theta, v(\theta)) \, d\theta - \int_{0}^{t} \left(1 - \theta\right) g(\theta, v(\theta)) \, d\theta \]

That implies that at least one positive continuous solution is found in the integral equation (3). \( v \in C[0,1] \).

To finish the proof differentiating equation (3) twice we obtain the differential equation (1).

Operating on both sides of equation (3) by \( \Gamma^{1 - \alpha} \), we obtain

\[I^{1 - \alpha} v(t) = \frac{\Lambda t^{1 - \alpha}}{\Gamma(2 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[= \frac{\Lambda t^{1 - \alpha}}{\Gamma(2 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[+ \frac{\alpha}{\Gamma(3 - \alpha)} \int_{0}^{t} (t - \theta)^{2 - \alpha} g(\theta, v(\theta)) \, d\theta \]

\[- \int_{0}^{1} (t - \theta)^{2 - \alpha} g(\theta, v(\theta)) \, d\theta \]

Differentiating the above relation, we get

\[D^\alpha v(t) = \frac{\Lambda}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[= \frac{\Lambda}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[+ \frac{\alpha}{\Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[- \int_{0}^{t} (t - \theta)^{1 - \alpha} g(\theta, v(\theta)) \, d\theta \]

Let \( t = 1 \) in the above equation, we get

\[D^\alpha v(t)|_{t=1} = \frac{\Lambda}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[= \frac{\Lambda}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[+ \frac{1}{\Gamma(2 - \alpha)} \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[- \int_{0}^{1} (1 - \theta)^{1 - \alpha} g(\theta, v(\theta)) \, d\theta \]

\[= \left( \frac{\Lambda}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} + \frac{1}{\Gamma(2 - \alpha)} \right) \int_{0}^{1} g(\theta, v(\theta)) \, d\theta \]

\[\frac{\alpha}{\Gamma(1 - \alpha) \Gamma(2 - \alpha)} \int_{0}^{1} (1 - \theta)^{1 - \alpha} g(\theta, v(\theta)) \, d\theta \]
\[
\frac{\Lambda}{\Gamma(2-\alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta - \frac{\Lambda}{\Gamma(2-\alpha)} \int_0^1 (1-\theta)^{1-\alpha} g(\theta, v(\theta)) \, d\theta = u(0)
\]

The proof is complete.

4. The uniqueness of solution

We have the following theorem to show that the solution is unique.

**Theorem 2.** Suppose that a constant \( k > 0 \) exists,

\[ |g(t, v) - g(t, \tilde{v})| \leq |v - \tilde{v}|, \forall \ t \in [0,1], \quad \forall v, \tilde{v} \in C[0,1] \]

If

\[ k \left( \frac{\Lambda}{\Gamma(2-\alpha)} + \frac{\Lambda}{\Gamma(1-\alpha)} + \frac{3}{2} \right) < 1, \quad (6) \]

then the boundary value problem (1) - (2) has a unique solution \( v \in C[0,1] \).

**Proof.** Define the operator \( H: C[0,1] \to C[0,1] \) by

\[
Hv(t) = \frac{\Lambda}{\Gamma(2-\alpha)} \int_0^1 g(\theta, v(\theta)) \, d\theta - \Lambda \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} g(\theta, v(\theta)) \, d\theta + t \int_0^1 g(\theta, v(\theta)) \, d\theta - \int_0^t (t-\theta) g(\theta, v(\theta)) \, d\theta. \quad (7)
\]

Assume that \( v, \tilde{v} \in C[0,1] \), then

\[
Hv(t) - H\tilde{v}(t) = \frac{\Lambda}{\Gamma(2-\alpha)} \int_0^1 (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta - \Lambda \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta + t \int_0^1 (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta - \int_0^t (t-\theta) (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta
\]

\[
= \frac{\Lambda}{\Gamma(2-\alpha)} \int_0^1 (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta - \Lambda \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta + t \int_0^1 (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta - \int_0^t (t-\theta) (g(\theta, v(\theta)) - g(\theta, \tilde{v}(\theta))) \, d\theta
\]

\[
|Hv(t) - H\tilde{v}(t)| \leq \frac{k \Lambda}{\Gamma(2-\alpha)} \int_0^1 |v(\theta) - \tilde{v}(\theta)| \, d\theta
\]
\[- k \Lambda \int_0^1 \frac{(1 - \theta)^{1-\alpha}}{\Gamma(2 - \alpha)} |v(\theta) - \bar{v}(\theta)| \, d\theta \]

\[+ k t \int_0^1 |v(\theta) - \bar{v}(\theta)| \, d\theta \]

\[- k \int_0^t (t - \theta) |v(\theta) - \bar{v}(\theta)| \, d\theta \]

\[\leq \frac{k \Lambda}{\Gamma(2 - \alpha)} \sup_{\theta \in [0,1]} |v(\theta) - \bar{v}(\theta)| \int_0^1 d\theta \]

\[+ k \Lambda \sup_{\theta \in [0,1]} |v(\theta) - \bar{v}(\theta)| \int_0^1 \frac{(1 - \theta)^{1-\alpha}}{\Gamma(2 - \alpha)} \, d\theta \]

\[+ k t \sup_{\theta \in [0,1]} |v(\theta) - \bar{v}(\theta)| \int_0^1 d\theta \]

\[+ k \sup_{\theta \in [0,1]} |v(\theta) - \bar{v}(\theta)| \int_0^t (t - \theta) \, d\theta \]

\[\| Hv - H\bar{v} \| \leq \frac{k \Lambda}{\Gamma(2 - \alpha)} \| v - \bar{v} \| + \frac{k \Lambda}{\Gamma(1 - \alpha)} \| v - \bar{v} \| \]

\[+ k \| v - \bar{v} \| + \frac{k}{2} \| v - \bar{v} \| \]

but since \( k \left( \frac{\Lambda}{\Gamma(2 - \alpha)} + \frac{\Lambda}{\Gamma(1 - \alpha)} + \frac{3}{2} \right) < 1 \) then we get

\[\| Hv - H\bar{v} \| \leq \| v - \bar{v} \| \]

therefore the operator \( H : C[0,1] \to C[0,1] \) is contraction and (3) has a unique fixed point \( v \in C[0,1] \).

5. An example

In this part, we show an example of one of our main findings. Consider the following example of a boundary value problem.

\[v''(t) + \frac{e^{-t}|v(t)|}{(14 + e^t)(1 + |v(t)|)} = 0, \quad t \in (0,1), \quad (8)\]

\[v'(1) = 0, \quad v(0) = D^\alpha v(t)|_{t=1}. \quad (9)\]

Set

\[g(t, v) = \frac{e^{-t}|v(t)|}{(14 + e^t)(1 + |v(t)|)}, \quad (t, v) \in [0,1] \times [0, \infty).\]

Let \( v, \bar{v} \in [0, \infty) \) and \( t \in [0,1] \). Then we have

\[|g(t, v) - g(t, \bar{v})| = \frac{e^{-t} \wedge}{(14 + e^t)(1 + |v(t)|)} \frac{v}{1 + v} - \frac{\bar{v}}{1 + \bar{v}} \]

\[\leq \frac{e^{-t} |v - \bar{v}|}{(14 + e^t)(1 + v)(1 + \bar{v})} \]

\[\leq \frac{1}{15} |v - \bar{v}| \]
Hence the condition of theorem (3.1) holds with $\alpha = \frac{1}{15}$. We shall check that condition (6) is satisfied for appropriate values of $\alpha \in (0,1)$.

$$k \left( \frac{A}{\Gamma(2-\alpha)} + \frac{A}{\Gamma(1-\alpha)} + \frac{3}{2} \right) < 1$$

$$\iff \frac{1}{15} \left( \frac{2 - 0.2}{\Gamma(2 - 0.2) - 1 + 0.29} + \frac{3}{2} \right) < 1$$

- If $\alpha = \frac{1}{5}$ then
  $$\frac{1}{15} \left( \frac{2 - 0.2}{\Gamma(2 - 0.2) - 1 + 0.29} + \frac{3}{2} \right) = 0.8514088917 < 1$$

- If $\alpha = 0.9$ then
  $$\frac{1}{15} \left( \frac{2 - 0.9}{\Gamma(2 - 0.9) - 1 + 0.29} + \frac{3}{2} \right) = 0.1861326 < 1$$

The unique value problem (8) has, therefore, a special solution on $[0, \infty)$ by Theorem 4.1.

6. Conclusion

For an ordinary differential equation with fractional-order derivative condition, we proved the existence and uniqueness of continuous positive solutions. Using fixed point theorem methods such as the Banach contraction principle and the Schauder fixed point theorem. Finally, we give an example to make our results clear.

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7. References

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