INARIANT AND HYPERINVARIANT SUBSPACES FOR AMENABLE OPERATORS

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ABSTRACT. There has been a long-standing conjecture in Banach algebra that every amenable operator is similar to a normal operator. In this paper, we study the structure of amenable operators on Hilbert spaces. At first, we show that the conjecture is equivalent to every non-scalar amenable operator has a non-trivial hyperinvariant subspace and equivalent to every amenable operator is similar to a reducible operator and has a non-trivial invariant subspace; and then, we give two decompositions for amenable operators, which supporting the conjecture.

1. INTRODUCTION

Throughout this paper, ℋ denotes a complex separable infinite-dimension Hilbert space and ℬ(ℋ) denotes the bounded linear operators on ℋ. For an algebra 𝔅 in ℬ(ℋ), we write 𝔅 for the commutant of 𝔅 (i.e., 𝔅 = {B ∈ ℬ(ℋ), BA = AB for all A ∈ 𝔅}) and 𝔅 for the double commutant of 𝔅 (i.e., 𝔅 = (𝔖)^’). We also write Lat 𝔅 for the collection of those subspaces which are invariant for every operator in 𝔅. We say 𝔅 is completely reducible if for every subspace M in Lat 𝔅 there exists N in Lat 𝔅 such that ℋ = M ⊕ N (i.e., M ∩ N = {0}) and ℋ is the algebraic direct sum of M and N); 𝔅 is reducible if for every subspace M in Lat 𝔅 we have M+ (the orthogonal complement of M) in Lat 𝔅; 𝔅 is transitive if Lat 𝔅 = \{\{0\}, ℋ\}. If T ∈ ℬ(ℋ), we say that a subspace M of ℋ is a hyperinvariant subspace for T if M is invariant under each operator in 𝔅_T; M is a reducible subspace for T if M, M+ ∈ Lat T.

The concept of amenable Banach algebras was first introduced by B. E. Johnson in [10]. Suppose that 𝔅 is a Banach algebra. A Banach 𝔅-bimodule is a Banach space X that is also an algebra 𝔅-bimodule for which there exists a constant K > 0 such that \|a ∙ x\| ≤ K\|a\|\|x\| and \|x ∙ a\| ≤ K\|a\|\|x\| for all a ∈ 𝔅 and x ∈ X. We note that X∗, the dual of X, is a Banach 𝔅-bimodule with respect to the dual actions

\[ (a ∙ f)(x) = f(x ∙ a), (f ∙ a)(x) = f(a ∙ x), a ∈ 𝔅, x ∈ X, f ∈ X^*. \]

Such a Banach 𝔅-bimodule is called a dual 𝔅-bimodule.

A derivation D : 𝔅 → X is a continuous linear map such that D(ab) = a ∙ D(b) + D(a) ∙ b, for all a, b ∈ 𝔅. Given x ∈ X, the inner derivation δ_x : 𝔅 → X, is defined by δ_x(a) = a ∙ x − x ∙ a.

According to Johnsons original definition, a Banach algebra 𝔅 is amenable if every derivation from 𝔅 into the dual 𝔅-bimodule X* is inner for all Banach 𝔅-bimodules X. If T ∈ ℬ(ℋ), denote the norm-closure of span\{T^k : k ∈ \{0\} ∪ N\} by 𝔅_T, where N is the

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set of natural numbers, $T$ is said to be an amenable operator, if $A_T$ is an amenable Banach algebra. Ever since its introduction, the concept of amenability has played an important role in research in Banach algebras, operator algebras and harmonic analysis. There has been a long-standing conjecture in the Banach algebra community, stated as follows:

**Conjecture 1.1.** A commutative Banach subalgebra of $\mathfrak{B}(\mathcal{H})$ is amenable if and only if it is similar to a $C^*$-algebra.

One of the first results in this direction is due to Willis [16]. Willis showed that if $T$ is an amenable compact operator, then $T$ is similar to a normal operator. In [7] Gifford studied the reduction property for operator algebras consisting of compact operators and showed that if such an algebra is amenable then it is similar to a $C^*$-algebras. In the recent papers [5], [6] Farenick, Forrest and Marcoux showed that if $T$ is similar to a normal operator, then $A_T$ is amenable if and only if $A_T$ is similar to a $C^*$-algebra and the spectrum of $T$ has connected complement and empty interior; If $T$ is a triangular operator with respect to an orthonormal basis of $\mathcal{H}$, then $A_T$ is amenable if and only if $T$ is similar to a normal operator whose spectrum has connected complement and empty interior. For further details see [5] and [6].

In this paper, we give the characterization of the structure of amenable operators. At first, we use the reduction theory of von Neumann to give two equivalent descriptions for Conjecture 1.1; and then, we give two decompositions for amenable operators, which supporting the Conjecture 1.1.

### 2. AN EQUIVALENT FORMULATON OF THE CONJECTURE 1.1

In this section we use the reduction theory of von Neumann to give two equivalent descriptions for Conjecture 1.1. We obtain that every amenable operator is similar to a normal operator if and only if every non-scalar amenable operator has a non-trivial hyperinvariant subspace if and only if every amenable operator is similar to a reducible operator and has a non-trivial invariant subspace.

In order to proof the main theorem, we need to introduce von Neumann’s reduction theory [15] and some lemmas.

Let $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_\infty$ be a sequence of Hilbert spaces chosen once and for all, $\mathcal{H}_n$ having the dimension $n$. Let $\mu$ be a finite positive regular measure defined on the Borel sets of a separable metric space $\Lambda$, and let $\{E_n\}_{n=1}^\infty$ be a collection of disjoint Borel sets of $\Lambda$ with union $\Lambda$. Then the symbol

$$\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) \mu(d\lambda)$$

denotes the set of all functions $f$ defined on $\Lambda$ such that

1. $f(\lambda) \in \mathcal{H}_n \subseteq \mathcal{H}_\infty$ if $\lambda \in E_n$;
2. $f(\lambda)$ is a $\mu$-measurable function with values in $\mathcal{H}_\infty$;
3. $\int_{\Lambda}^{\oplus} |f(\lambda)|^2 \mu(d\lambda) < \infty$.

We put

$$(f,g) = \int_{\Lambda}^{\oplus} (f(\lambda),g(\lambda)) \mu(d\lambda).$$

The set of functions thus defined is called the direct integral Hilbert space with measure $\mu$ and dimension sets $\{E_n\}$ and denoted by $\mathcal{H} = \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) \mu(d\lambda)$. 

An operator on $\mathcal{H}$ is said to be decomposable if there exists a strongly $\mu$-measurable operator-value function $A(\cdot)$ defined on $\wedge$ such that $A(\lambda)$ is a bounded operator on the space $\mathcal{H}(\lambda) = \mathcal{H}_n$ when $\lambda \in E_n$, and for all $f \in \mathcal{H}$, $(Af)(\lambda) = A(\lambda)f(\lambda)$. We write $A = \int_\wedge A(\lambda)\mu(d\lambda)$ for the equivalence class corresponding to $A(\cdot)$. If $A(\lambda)$ is a scalar multiple of the identity on $\mathcal{H}(\lambda)$ for almost all $\lambda$, then $A$ is called diagonal. The collection of all diagonal operator is called the diagonal algebra of $\wedge$. In [15] 1.3, Schwartz showed that an operator $A$ on Hilbert space $\mathcal{H} = \int_\wedge \mathcal{H}(\lambda)\mu(d\lambda)$ is decomposable if and only if $A$ belong to the commutant of the diagonal algebra of $\wedge$. And $||A|| = \mu - \text{ess.sup}_{\lambda \in \wedge} ||A(\lambda)||$.

In [1], Azoff, Fong and Gilfeather used von Neumann’s reduction theory to define the reduction theory for non-selfadjoint operator algebras: Fix a partitioned measure space $\wedge$ and let $\mathcal{D}$ be the corresponding diagonal algebra. Given an algebra $\mathcal{A}$ of decomposable operators. Each operator $A \in \mathcal{A}$ has a decomposition $A = \int_\wedge A(\lambda)\mu(d\lambda)$. Choose a countable generating set $\{A_n\}$ for $\mathcal{A}$, let $\mathcal{A}(\lambda)$ be the strongly closed algebra generated by the $\{A_n(\lambda)\}$. $\mathcal{A} \sim \int_\wedge \mathcal{A}(\lambda)\mu(d\lambda)$ is called the decomposition of $\mathcal{A}$ respect to $\mathcal{D}$. A decomposition $\mathcal{A} \sim \int_\wedge \mathcal{A}(\lambda)\mu(d\lambda)$ of an algebra is said to be maximal if the corresponding diagonal algebra is maximal among the abelian von Neumann subalgebras of $\mathcal{A}'$. The following lemma is a basic result in [1] which will be used in this paper.

**Lemma 2.1.** ([1], Theorem 4.1) Let $\mathcal{A} \sim \int_\wedge \mathcal{A}(\lambda)\mu(d\lambda)$ be a decomposition of a reductive algebra. Then almost all of $\{\mathcal{A}(\lambda)\}$ are reducible. In particular, if the decomposition is maximal, then almost all of the algebras $\{\mathcal{A}(\lambda)\}$ are transitive.

In [7] Gifford studied the reduction property for operator algebras and obtained the following result:

**Lemma 2.2.** ([7] Lemma 4.4, Lemma 4.12) If $\mathcal{A}$ is a commutative amenable operator algebra, then $\mathcal{A}', \mathcal{A}''$ are complete reducible and there exists $M \geq 1$ so that for any idempotent $p \in \mathcal{A}''$ $||p|| \leq M$.

Assume $\mathcal{A}$ is a operator algebra, let $P(\mathcal{A})$ denote the idempotents in $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$ denote the strongly closed algebra generated by $P(\mathcal{A})$. We get the following lemma:

**Lemma 2.3.** If $\mathcal{A}$ is a commutative amenable operator algebra, then $\mathcal{P}(\mathcal{A}'')$ is similar to an abelian von Neumann algebra.

**Proof.** By Lemma 2.2 and [2] Corollary 17.3, it follows that there exists $X \in \mathcal{B}(\mathcal{H})$ such that $XpX^{-1}$ is selfadjoint for each $p \in P(\mathcal{A}'')$. Hence $\mathcal{P}(\mathcal{A}'')$ is similar to a abelian von Neumann algebra. \hfill $\square$

**Lemma 2.4.** ([5]) Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and suppose that $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous homomorphism with $\varphi(\mathcal{A})$ dense in $\mathcal{B}$. If $\mathcal{A}$ is amenable, then $\mathcal{B}$ is amenable.

**Notation 2.5.** From Lemma 2.3 and 2.1 we always assume that $\mathcal{P}(\mathcal{A}'')$ is a abelian von Neumann algebra, and $\mathcal{A}_{T}$ is a reducible operator algebra in this section.

Now we will prove the main result of this section:

**Theorem 2.6.** The following are equivalent:

1. Every amenable operator is similar to a normal operator;
2. Every non-scalar amenable operator has a non-trivial hyperinvariant subspace;
(3) Every amenable Banach algebra which is generated by an operator is similar to a C*-algebra.

Proof. (1) ⇔ (3) and (1) ⇒ (2) is clear by [5]. Therefore, in order to establish the theorem it suffices to show the implications (2) ⇒ (1).

Assume (2), by Lemma 2.3 choose a maximal decomposition for $A_T = \int_{\mathbb{R}} A_T^r(\lambda)\mu(d\lambda)$ respect to the diagonal algebra $P(A_T^r)$.

Assume $T \sim \int_{\mathbb{R}} T(\lambda)\mu(d\lambda)$ is the decomposition for $T$. Let $\{p_n\}_{n=1}^{\infty}$ denote the all rational polynomials. Then $p_n(T) \sim \int_{\mathbb{R}} p_n(T)(\lambda)\mu(d\lambda)$ is decomposable for all $n$ and there exists a measurable $E \subseteq \mathbb{R}$ such that $\mu(\mathbb{R}-E)=0$ and for any $\lambda \in E$ we have $p_n(T)(\lambda) = p_n(T(\lambda))$ and $||p_n(T(\lambda))|| \leq ||p_n(T)||$ by [15] Lemma I.3.1, I.3.2. Define a mapping $\varphi_\lambda : A_T \rightarrow A_{T(\lambda)}$ by $\varphi_\lambda(p_n(T)) = p_n(T(\lambda))$ for each rational polynomial $p_n$ and $\lambda \in E$. Note that $||p_n(T(\lambda))|| \leq ||p_n(T)||$ for each rational polynomial $p_n$ and furthermore, $\{p_n(T)\}$ is dense in $A_T$. Hence, $\varphi_\lambda$ is well-defined and $\varphi_\lambda$ a continuous homomorphism with $\varphi(A_T)$ dense in $A_{T(\lambda)}$. By Lemma 2.4 $T(\lambda)$ is amenable for almost all $\lambda$.

Now for almost all $\lambda$, $T(\lambda)$ is amenable and $A_T^r(\lambda) \subseteq A_{T(\lambda)}^r$ and $A_T^r(\lambda)$ is transitive by Lemma 2.1. Thus almost all of $T(\lambda)$ are scalar operators, i.e. $T$ is a normal operator. □

Corollary 2.7. Every amenable operator is similar to a normal operator if and only if there exists a non-trivial idempotent in the double-commutant of every non-scalar amenable operator.

Remark 2.8. In [3], Farenick, Forrest and Marcoux showed that if $T \in \mathcal{B}(H)$ is amenable and similar to a normal operator $N$, then the spectrum of $N$ has connected complement and empty interior. According to [14, Theorem 1.23], $N$ is a reducible operator. Hence, there exists an invertible operator $X \in \mathcal{B}(H)$ such that $A_{TX^{-1}}^r$ is a reducible algebra. The following theorem give the equivalent description for Conjecture 1.1 from the existence of invariant subspace for amenable operators.

Theorem 2.9. The following are equivalent:

1. Every amenable operator is similar to a normal operator;
2. For every amenable operator $T \in \mathcal{B}(H)$, there exists an invertible operator $X \in \mathcal{B}(H)$ such that $A_{TX^{-1}}^r$ is a reducible algebra and $T$ has a non-trivial invariant subspace.

Proof. (1) ⇒ (2) is clear by Remark 2.8
(2) ⇒ (1) is trivial modifications adapt the proof of theorem 2.6. □

Remark 2.10. According to theorem 2.6, 2.9 we obtain that the Conjecture 1.1 for operator algebra which is generated by an operator is equivalent to the following statements:

1. Every amenable operator $T$ has a non-trivial invariant subspace and renorm $H$ with an equivalent Hilbert space norm so that under this norm $\text{Lat}A_T$ becomes orthogonally complemented;
2. Every non-scalar amenable operator has a non-trivial hyperinvariant subspace.

3. Decomposition of Amenable operators

In this section, we get two decompositions for amenable operators and prove that the two decompositions are the same which supporting Conjecture 1.1.
At first, we summarize some of the details of multiplicity theory for abelian von Neumann algebras. For the most part, we will follow [3]. If \( A \) is an operator on a Hilbert space \( \mathcal{H} \) and \( n \) is a cardinal number, let \( \mathcal{H}^n \) denote the orthogonal direct sum of \( n \) copies of \( \mathcal{H} \), and \( A^{(n)} \) be the operator on \( \mathcal{H}^n \) which is the direct sum of \( n \) copies of \( A \). Whenever \( \mathfrak{A} \) is an operator algebra on \( \mathcal{H} \), \( \mathfrak{A}^{(n)} \) denotes the algebra \( \{ A^{(n)} | A \in \mathfrak{A} \} \). An abelian von Neumann algebra \( \mathfrak{B} \) is of uniform multiplicity \( n \) if it is (unitary equivalent to) \( \mathfrak{A}^{(n)} \) for some maximal abelian von Neumann algebra \( \mathfrak{A} \). By [3], for any abelian von Neumann algebra \( \mathfrak{A} \), there exists a sequens of regular Borel measures \( \{ \mu_n \} \) on a sequens of separable metric space \( \{ X_n \} \) such that \( \mathfrak{A} \) is unitarily equivalent to \( \sum_{n=1}^{\infty} \oplus \mathfrak{B}_n \oplus \mathfrak{B}_\infty \), where \( \mathfrak{B}_n \) is a von Neumann algebra which has uniform multiplicity \( n \) for all \( 1 \leq n \leq \infty \). For further details see [3, II.3].

**Proposition 3.1.** Suppose that \( T \) is amenable operator and \( \mathfrak{A}_T \) contains a subalgebra which is similar to an abelian von Neumann algebra with no direct summand of uniform multiplicity infinite, then \( T \) is similar to a normal operator.

*Proof.* For the sake of simplicity, we assume \( \mathfrak{A}_T \) contains a subalgebra \( \mathfrak{B} \) which is an abelian von Neumann subalgebra with no direct summand of uniform multiplicity infinite. Trivial modifications adapt the proof to the more general case.

By [3, II.3], there exists a sequens of regular Borel measures \( \{ \mu_n \} \) on a sequens of separable metric space \( \{ X_n \} \) such that \( \mathfrak{B} \) is unitarily equivalent to \( \sum_{n=1}^{\infty} \oplus \mathfrak{B}_n \), where \( \mathfrak{B}_n \) is a von Neumann algebra which has uniform multiplicity \( n \) for all \( n \). Hence, \( T = \sum_{n=1}^{\infty} \oplus T_n \), where \( T_n \in \mathfrak{B}_n \). It suffices to show that \( T_n \) is similar to a normal operator for all \( n \), then by [4, Corollary 26], it follows that \( T \) is similar to a normal operator.

Since \( T \in \mathfrak{B}_n \), according to [14, Theorem 7.20], for any \( 1 \leq n < \infty \) there exists a unitary operator \( U_n \in \mathfrak{B}_n \) such that

\[
U_n T_n (U_n)^{-1} = \begin{bmatrix}
N_{11} & N_{12} & \cdots & \cdots & N_{1n} \\
0 & N_{22} & \cdots & \cdots & N_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & N_{nn}
\end{bmatrix}
\]

where \( N_{ij} \) is a normal operator for all \( 1 \leq i, j \leq n \). By [17, Proposition 3.1], it follows that \( T_n \) is similar to \( \oplus_{i=1}^{n} N_{ii} \), i.e. \( T_n \) is similar to a normal operator for all \( n \).

**Corollary 3.2.** Assume \( T \) is amenable operator, then there exists hyperinvariant subspaces \( M, N \) of \( T \) such that \( T \) has the form \( T = T_1 + T_2 \) respect to the space decomposition \( \mathfrak{H} = M + N \), where \( T_1, T_2 \) are amenable operators, \( T_1 \) is similar to a normal operator and \( \mathcal{P}(\mathfrak{A}_T) \) is similar to an abelian von Neumann algebra with uniform multiplicity infinite.

The proof of the following lemma is straightforward and we omit it.

**Lemma 3.3.** Suppose that \( \mathfrak{A} \) is a completely reductive operator algebra and \( p \in \mathcal{P}(\mathfrak{A}) \). Then \( p\mathfrak{A} \) is a completely reductive operator algebra on Ranp.
Proposition 3.4. Assume that $T$ is a amenable operator and there exists a space decomposition $\mathcal{H} = M + N$ such that $T$ has the matrix form $T = \begin{bmatrix} T_1 & T_2 \\ T_2^* & T'\end{bmatrix}$. Then $T$ is similar to a normal operator if and only if $T_1$ and $T_2$ are similar to normal operators.

Proof. Assume that $T$ has the matrix form

$$T = \begin{bmatrix} T_1 & T_2 \\ T_2^* & T'\end{bmatrix}$$

respect to the space decomposition $\mathcal{H} = M \oplus M'$. By [17, Lemma 2.8], there exists an invertible operator $S = \begin{bmatrix} I & S_12 \\ M & M'\end{bmatrix}$ such that $S^{-1}TS = \begin{bmatrix} T_1 & T_2 \\ T_2^* & T'\end{bmatrix}$. Assume that $S$ has the matrix form $S = \begin{bmatrix} I & S_1 \\ M & M'\end{bmatrix}$, we obtain that $T_2 = S_1T_2S_1^{-1}$. By [9, proposition 6.5], we get that $T$ is similar to a normal operator if and only if $T_1$ and $T_2$ are similar to normal operators. \qed

Proposition 3.5. Suppose that $T$ is an amenable operator, $M_1 \in \text{Lat}\mathfrak{A}_T'$ and $M_2 \in \text{Lat}\mathfrak{A}_T'$. Then $M_1 + M_2$ is closed.

Moreover, if $T|_{M_1}$ and $T|_{M_2}$ are similar to normal operators, then $T|_{M_1 + M_2}$ is similar to a normal operator.

Proof. Let $N_0 = M_1 \cap M_2$, according to Lemma 3.3, there exists $N \in \text{Lat}\mathfrak{A}_T'$ such that $M_2 = N_0 + N$. Choose $q \in P(\mathfrak{A}_T'$), such that $\text{Ran}q = N$. By the assumption, $M_1 \in \text{Lat}\mathfrak{A}_T'$. Hence $qM_1 \subset M_1 \cap N = \{0\}$. Therefore $M_1 \subset (I - q)\mathcal{H}$. We see that $M_1 + M_2 = M_1 + N$ is closed. This establishes the first statement of the proposition.

Since $T|_{M_2}$ is similar to a normal operator, by proposition 3.4, we get that $T|_N$ is similar to a normal operator. By the assumption $T|_{M_1}$ is similar to a normal operator, using proposition 3.4 again, we obtain that $T|_{M_1 + M_2} = T|_{M_1 + N}$ is similar to a normal operator. \qed

Now we will obtain the main theorem of this section.

Theorem 3.6. Assume $T$ is an amenable operator, then there exists hyperinvariant subspaces $M_1, M_2$ of $T$ such that $T$ has the form $T = T_1 + T_2$ respect to the space decomposition $\mathcal{H} = M_1 + M_2$ and satisfies that:

1. $T_1, T_2$ are amenable operators;
2. If $M$ is a hyperinvariant subspace of $T$ and $T|_{M}$ is similar to a normal operator, then $M \subseteq M_1$, i.e. $M_1$ is the largest hyperinvariant subspace on which $T$ is similar to a normal operator;
3. For any $q \in P(\mathfrak{A}_T')$, $T_2|_{\text{Ran}q}$ is not similar to a normal operator;
4. $P(\mathfrak{A}_T')$ is similar to an abelian von Neumann algebra with uniform multiplicity infinite;
5. $\mathfrak{A}_T' = \mathfrak{A}_{T_1}' + \mathfrak{A}_{T_2}'$, $\mathfrak{A}_T'' = \mathfrak{A}_{T_1}'' + \mathfrak{A}_{T_2}''$;
6. There exists no nonzero compact operator in $\mathfrak{A}_T$.

Proof. Case1. For any $p \in P(\mathfrak{A}_T')$, $T|_{\text{Ran}p}$ is not similar to normal operator. According to the proof of Proposition 3.1, we obtain that $P(\mathfrak{A}_T')$ is similar to an abelian von Neumann algebra with uniform multiplicity infinite. Let $M_1 = 0$. 

Case 2. There exists \( p \in P(\mathfrak{A}_{L}^\prime) \) such that \( T|_{\text{Ran}p} \) is similar to a normal operator. Then, by Zorn’s Lemma and the same method in the proof of [14, Corollary 26], we can show that there exists an element \( p_0 \in P(\mathfrak{A}_{L}^\prime) \) which is maximal with respect to the property that \( T|_{\text{Ran}p_0} \) is similar to a normal operator. Using Proposition \[3.3\] \( \text{Ran}p_0 \) is the largest hyperinvariant subspace of \( T \) on which \( T \) is similar to a normal operator. Hence, \( T \) has the form \( T = T_1 + T_2 \) with respect to the space decomposition \( \mathfrak{A} = \text{Ran}p_0 + \text{Ker}p_0 \) where \( T_1 \) is similar to a normal operator, \( T_1, T_2 \) are amenable operators. Let \( M_1 = \text{Ran}p_0, M_2 = \text{Ker}p_0 \).

Next we will prove that for any \( q \in P(\mathfrak{A}_{L}^\prime), T_2|_{\text{Ran}q} \) is not similar to normal operator. Then according to Proposition \[3.1\] \( P(\mathfrak{A}_{L}^\prime) \) is similar to an abelian von Neumman algebra with uniform multiplicity infinite.

Indeed, if there exists \( q \in P(\mathfrak{A}_{L}^\prime) \) such that \( T_2|_{\text{Ran}q} \) is similar to a normal operator and \( q \) has the form \( q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{Ran}q \). Then for any \( A \in \mathfrak{A}_{L}^\prime \), \( A \) has the form

\[
A = \begin{bmatrix} A_{11} & A_{22} \\ A_{22} & A_{33} \end{bmatrix}
\]

Let

\[
R = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \text{Ran}p_0 \text{Ran}q.
\]

Then \( R \in P(\mathfrak{A}_{L}^\prime) \). By the assumption \( T|_{\text{Ran}R} \) is similar to a normal operator which contradicts to the maximal property of \( p_0 \).

At last we will prove that there exists no nonzero compact operator in \( \mathfrak{A}_{L}^\prime \).

Indeed, if there exists a nonzero compact operator \( k_0 \in \mathfrak{A}_{L}^\prime \), let \( L_1 \) denote the subspace spanned by the ranges of all compact operators in \( \mathfrak{A}_{L}^\prime \), and \( L_2 \) the intersection of their kernel, by [13, Lemma 3.1], both \( L_1, L_2 \) lie in \( \text{Lat}\mathfrak{A}_{L}^\prime \) and \( L_1 + L_2 = \text{Ker}p_0 \). Considering the restricting \( T_2|_{L_1} \), assume \( T_2_1 = T_2|_{L_1} \), then \( T_2_1 \) is an amenable operator and \( \mathfrak{A}_{L}^\prime \) contain a sufficient set of compact operators. By Lemma \[2.2\] and [12, Theorem 9], \( T_2_1 \) is similar to a normal operator which contradicts to the above discussion.

Trivial modifications adapt the proof of Theorem \[3.6\] we obtain the following theorem which decomposes amenable operators by the invariant subspaces of them. The proof is similar to Theorem \[3.6\] and we omit it.

**Theorem 3.7.** Assume \( T \) is an amenable operator, then there exists invariant subspaces \( N_1, N_2 \) of \( T \) such that \( T \) has the form \( T = A_1 + A_2 \) respect to the space decomposition \( \mathfrak{A} = N_1 + N_2 \) and satisfies that:

1. \( A_1, A_2 \) are amenable operators;
2. If \( N \) is an invariant subspace of \( T \) such that \( N_1 \subseteq N \) and \( T|_{N} \) is similar to a normal operator, then \( N = N_1 \), i.e. \( N_1 \) is the maximal invariant subspace on which \( T \) is similar to a normal operator;
3. For any \( q \in P(\mathfrak{A}_{L}^\prime), T_2|_{\text{Ran}q} \) is not similar to a normal operator;
4. If \( P(\mathfrak{A}_{L}^\prime) \) contains a subalgebra which is similar to an abelian von Neumman algebra then the von Neumann algebra has the uniform multiplicity infinite.
Remark 3.8. If the answer to Conjecture 1.1 is positive, by Theorem 2.6, every amenable is similar to a normal operator. Then, for the above theorem $M_1 = N_1 = \mathfrak{N}$. That is to say, the two decompositions of theorem 3.6 and 3.7 are the same. The remainder of this section, we will prove that the two decompositions are the same which supporting Conjecture 1.1.

Lemma 3.9. [3] If $T \in \mathfrak{B}(\mathfrak{H})$ is an amenable operator and there exist a one-to-one bounded linear map $W : \mathfrak{H} \to \mathfrak{H}_2$, a bounded linear map $V : \mathfrak{H}_1 \to \mathfrak{H}$ with dense range and operators $S_1 \in \mathfrak{B}(\mathfrak{H}_1), S_2 \in \mathfrak{B}(\mathfrak{H}_2)$ which are similar to normal operators such that $TV = VS_1$ and $WT = S_2W$, then $T$ is similar to a normal operator.

Corollary 3.10. Assume $T = B_1B_2$ is an amenable operator, where $B_1, B_2$ are positive operators, then $T$ is similar to a normal operator.

Proof. Assume $B_1, B_2$ have the forms

$$B_2 = \begin{bmatrix} 0 & \tilde{B}_2 \\ \tilde{B}_2 \end{bmatrix}, B_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix},$$

respect to the space decomposition $\mathfrak{H} = \text{Ker}B_2 \oplus (\text{Ker}B_2)^\perp$ where $\tilde{B}_2$ is one-to-one and $B_{11}, B_{22}$ are positive operators. Thus $T$ has the form $T = \begin{bmatrix} 0 & B_{12}\tilde{B}_2 \\ 0 & B_{22}\tilde{B}_2 \end{bmatrix}$ respect to the space decomposition. Since $T$ is an amenable operator, by [17, Lemma 2.8], $T$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & B_{22}\tilde{B}_2 \end{bmatrix}$. Thus without loss of generality, we may assume that $B_2$ is one-to-one.

Assume that $B_1, B_2$ have the form

$$B_1 = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix},$$

respect to the space decomposition $\mathfrak{H} = (\text{Ker}B_1)^\perp \oplus \text{Ker}B_1$ where $\tilde{B}_1$ is one-to-one and has dense range and $B_{11}, B_{22}$ are positive operators. Thus $T$ has the form $T = \begin{bmatrix} \tilde{B}_1B_{11} & \tilde{B}_1B_{12} \\ 0 & 0 \end{bmatrix}$ respect to the space decomposition. Since $T$ is an amenable operator, by [17, Lemma 2.8], $T$ is similar to $\begin{bmatrix} \tilde{B}_1B_{11} & 0 \\ 0 & 0 \end{bmatrix}$ and there exists an operator $S$ such that $\tilde{B}_1B_{12} = \tilde{B}_1B_{11}S$. Note that $\tilde{B}_1, B_2$ are one-to-one, hence $B_{12} = B_{11}$, and $B_{11}$ is one-to-one. Thus without loss of generality, we may assume that $B_1$ has dense range and $B_2$ is one-to-one.

Note that $B_{12}^\frac{1}{2}B_2B_1^\frac{1}{2}, B_{22}^\frac{1}{2}B_1B_{22}^\frac{1}{2}$ are positive operators and $TB_1^\frac{1}{2} = B_1^\frac{1}{2}B_2B_1^\frac{1}{2}B_1^\frac{1}{2}$ and $B_2^\frac{1}{2}T = B_2^\frac{1}{2}B_2B_1B_{22}^\frac{1}{2}$, by Lemma 3.9, $T$ is similar to a normal operator. $\square$

Theorem 3.11. The two decompositions for an amenable operator in Theorem 3.6, 3.7 are the same.

Proof. According to Theorem 3.6, 3.7, and Proposition 3.5, it is suffices to proof that $N_1 \in \text{Lat}\mathfrak{H}_T'$. 

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Moreover, we can assume that \( Y \) has dense range. Using \( T \) is amenable again, there exists \( L = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} N_1 \in \mathfrak{U}_T \), where \( X \neq 0 \), by Lemma 2.8. Similar to the decomposition to \( S \) and \( T \), we get that \( S \), \( L \) and \( T \) have the form

\[
S = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{Y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \tilde{X} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 & 0 \\ T_{31} & T_{32} & T_{33} \\ T_{34} & T_{34} & T_{34} \end{bmatrix},
\]

respect to the space decomposition \( \mathfrak{S} = N_1 \oplus \ker X \oplus (N_2 \oplus \ker X) \), where \( \tilde{X} \) is one-to-one, and \( \tilde{Y}_1, \tilde{Y}_2 \) has dense range. Note that \( LT = TL \), we get that \( T_{33} = 0 \). Using \( T \) is amenable again, there exists an operator \( C : N_2 \oplus \ker X \rightarrow \ker X \) such that

\[
\begin{bmatrix} I & 0 & 0 \\ I & C & I \\
I & \tilde{X} & I \end{bmatrix} \begin{bmatrix} T_1 & 0 & 0 \\ T_{31} & T_{32} & T_{33} \\ T_{34} & T_{34} & T_{34} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ I & C & I \\
I & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \tilde{X} \\ 0 & 0 & 0 \end{bmatrix},
\]

and

\[
\begin{bmatrix} I & 0 & 0 \\ I & C & I \\
I & \tilde{Y}_1 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{Y}_1 + C \tilde{Y}_2 & 0 & 0 \end{bmatrix}.
\]

Moreover, \( \tilde{Y}_2 T_1 = T_{34} \tilde{Y}_2, T_1 \tilde{X} = \tilde{X} T_{34} \), and \( T_1 \) is similar to a normal operator, by Lemma 3.9. \( T_{34} \) is similar to a normal operator, which contracts to Theorem 3.7.
Corollary 3.12. Assume \( T \) is an amenable operator, then \( M \) is a maximal invariant subspace such that \( T|_M \) is similar to a normal operator if and only if \( M \) is the largest invariant subspace such that \( T|_M \) is similar to a normal operator.

Corollary 3.13. Assume \( T \) is an amenable operator and which is quasisimilar to a compact operator, then \( T \) is similar to a normal operator.

Proof. Suppose, \( TV = VK, WT = KW \) with \( V, W \) injective operators having dense ranges and \( K \) is a compact operator. Then \( TVKW = VKWT \). Let \( C = VKW, C \in \mathfrak{A}_T \), and \( C \) is a compact operator. According to Theorem 3.6 \( C \) has the form \( \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \) respect to the space decomposition in the Theorem. If \( Cx = 0, VWTx = Cx = 0 \), thus \( T'x = 0 \). It follows that there is no part of \( T_2 \), i.e. \( T \) is similar to a normal operator.

4. (Essential) Operator valued roots of abelian analytic functions

In this section, we will study the structure of an operator which is an (essential) operator valued roots of abelian analytic functions and then we get that if such an operator is also amenable, then it is similar to a normal operator. In [8] Gilfeather introduces the concept of operator valued roots of abelian analytic functions as follows: Let \( \mathfrak{A} \) be an abelian von Neumann algebra and \( \psi(z) \), an \( \mathfrak{A} \) valued analytic function on a domain \( \mathcal{D} \) in the complex plane. We may decompose \( \mathfrak{A} \) into a direct integral of factors such that for \( A \in \mathfrak{A} \), there exists a unique \( g \in L_\infty(\Lambda, \mu) \) such that \( \mathfrak{A} = \int_\Lambda g(\lambda)I(\lambda)\mu(d\lambda) \). If \( T \in \mathfrak{A}' \) and \( \sigma(T) \subseteq \mathcal{D} \), let

\[
\psi(T) = (2\pi i)^{-1} \int_\Lambda (T - zI)^{-1} \psi(z) dz.
\]

An operator \( T \) is called a (essential) roots of the abelian analytic function \( \psi \), if \( \psi(T) = 0 \) (compact, respectively). The structure of roots of a locally nonzero abelian analytic function has been given in [8], in this section we mainly study the structure of essential roots of a locally nonzero abelian analytic function.

Lemma 4.1. Assume \( T \in \mathfrak{B}(\mathfrak{H}) \), \( f \) is a locally nonzero analytic function on the neighborhood of \( \sigma(T) \) and assume \( f(T) \) is a compact operator, then \( T \) is a polynomial compact operator.

Proof. Let \( \widehat{T} \) denote the image of \( T \) in the Calkin algebra, then \( \widehat{f(T)} = 0 \). Since \( f \) is a locally nonzero analytic function on \( \sigma(T) \), there exists a polynomial \( p \) such that \( p(T) = 0 \). i.e. \( T \) is a polynomial compact operator.

Theorem 4.2. Let \( \psi \) be a locally nonzero abelian analytic function on \( \mathcal{D} \) taking values in the von Neumann algebra \( \mathfrak{A} \). If \( T \) is an essential roots of \( \psi \) and is amenable, then \( T \) is similar to a normal operator.

Proof. Since \( \mathfrak{A} \) is an abelian von Neumann algebra, \( \mathfrak{A} \) is unitarily equivalent to \( \sum_{n=1}^{\infty} \mathfrak{B}_n \oplus \mathfrak{B}_\infty \), where \( \mathfrak{B}_n \) is a von Neumann algebra which has uniform multiplicity \( n \) for all \( 1 \leq n \leq \infty \). Note \( T \in \mathfrak{A}' \) is an amenable operator, thus \( T = T_1 \oplus T_2 \), where \( T_1 \) is similar to a normal operator, and \( T_1 \in \left( \sum_{n=1}^{\infty} \mathfrak{B}_n \right)' \), \( T_2 \in \mathfrak{B}_\infty' \). Let \( \sigma_1(\sigma_2) \) denote the continuous (atom, respectively) parts of the spectrum of \( \mathfrak{B}_\infty \), then \( \mathfrak{B}_\infty = \mathfrak{C}_\infty \oplus \mathfrak{D}_\infty \), where \( \mathfrak{C}_\infty \) and
\( D_\infty \) are uniform multiplicity \( \infty \) von Neumann algebra and \( \sigma(C_\infty) = \sigma_1, \sigma(D_\infty) = \sigma_2 \) and 

\[ T_2 = T_3 \oplus T_4, \] where \( T_3 \in \sigma(C_\infty)', T_4 \in \sigma(D_\infty)' \).

Assume \( \psi \) is a locally nonzero abelian analytic function on \( D \) and \( \sigma(T) \subseteq D \), then \( \psi(T) = \psi(T_1) \oplus \psi(T_3) \oplus \psi(T_4) \), note that \( \psi(T_3) \) is a compact operator and \( \sigma(C_\infty) = \sigma_1 \), so \( \psi(T_3) = 0 \). Since \( D_\infty \) are uniform multiplicity \( \infty \) and \( \sigma(D_\infty) = \sigma_2 \), by Lemma \[ \text{[4.1]} \] it follows that \( T_4 \) is direct sum of polynomial compact operators. According to \[ \text{[8, Theorem 2.1]} \], there exists a sequence of mutually orthogonal projections \( \{ P_n, Q_m \} \) in \( A \) with \( I = \sum P_n + \sum Q_m \) so that \( T|_{P_n} \) is finite type spectral operator and \( T|_{Q_m} \) is polynomial compact operator. By \[ \text{[17, Theorem 3.5, 4.5]} \], we get that \( T \) is similar to a normal operator. \( \square \)

References

[1] E. A. Azoff; C. K. Fong and F. Gilfeather, A reduction theory for non-self-adjoint operator algebras. Trans. Amer. Math. Soc. 224 (1976) 351–366 (1977).
[2] K. R. Davidson, Nest algebras, Longman group UK limited, Essex, 1988.
[3] K. R. Davidson, (3-WTRL) \( C^* \)-algebras by example. (English summary) Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.
[4] C. K. Fong, Operator algebras with complemented invariant subspace lattices. Indiana Univ. Math. J. 26 (1977) 1045–1056.
[5] D. R. Farenick, B.E. Forrest and L. W. Marcoux, Amenable operators on Hilbert spaces, J. reine angew. Math. 582(2005) 201-228.
[6] D. R. Farenick, B.E. Forrest and L. W. Marcoux, Amenable operators on Hilbert spaces, J. reine angew. Math. 602(2007) 235.
[7] J. A. Gifford, Operator algebras with a reduction propriety, J. Aust. Math. Soc 80 (2006) 279–315.
[8] F. Gilfeather, Operator valued roots of abelian analytic functions, Pacific J. Math. 55 (1974), 127–148.
[9] D. W. Hadwin, An asymptotic double commutant theorem for \( C^* \)-algebras. Trans. Amer. Math. Soc. 244 (1978) 273–297.
[10] B. E. Johnson. Cohomology in Banach Algebras. Mem. Amer. Math. Soc. Vol. 127 (Amer. Math. Soc., 1972).
[11] R. G. Douglas, On operators similar to normal operators, Rev. Roum. Math. Pures Appl. 14(1969) 193-197.
[12] S. Rosenoer, Completely reducible operator algebras and spectral synthesis, Canad. J. Math. 34 (1982), no. 5, 1025–1035.
[13] S. Rosenoer, Completely reducible algebras containing compact operators, J. Operator Theory. 29 (1993), no. 2, 269–285.
[14] H. Radjavi and P. Rosenthal, Invariant subspaces. Second edition. Dover Publications, Inc., Mineola, NY, 2003.
[15] J. T. Schwartz, \( W^* \)-algebras, Gordon and Breach, New York, 1967.
[16] G. A. Willis, When the algebra generated by an operator is amenable, J. Operator Theory. 34 (1995) 239–249.
[17] Y. Q. Ji and L. Y. Shi, Amenable operators on Hilbert spaces, Houston Journal of Mathematics(to appear).

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