Non-uniform Observability for Fast Moving Horizon Estimation with application to the SLAM problem

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Abstract

This paper first formalises a new observability concept, called weak regular observability, that is adapted to Fast Moving Horizon Estimation where one aims to estimate the state of a nonlinear system efficiently on rolling time windows in the case of small initial error. Additionally, sufficient conditions of weak regular observability are provided in a problem of Simultaneous Localisation and Mapping (SLAM) for different measurement models. In particular it is shown that following circular trajectories leads to weak regular observability in a second order 2D SLAM problem with several possible types of sensors.

Key words: nonlinear observability; fast moving horizon estimation, circumnavigation, SLAM.

1 Introduction

In tackling nonlinear estimation problems using the machinery of mathematical optimisation, two ideas prevail. The most straightforward one is to define a cost on the complete sequence of past inputs and outputs and to estimate the associated state trajectory by minimising it over state trajectories. The estimator is then built from the resulting optimal state trajectory. This leads to Full Information Estimation (FIE). To reduce the computational cost and memory usage, another idea is to use a truncated version of the input/output sequence on a time window of fixed length and to keep the optimal state trajectories on this moving horizon. This leads to Moving Horizon Estimation (MHE). See [37] for a general survey on these techniques.

In the classical literature on FIE and MHE, robust stability of the estimation error is usually proved under observability assumptions. For example, in [3,30,36], the stability of MHE schemes has been shown by assuming the so-called \textit{N-step observability} property. This assumption means that on a moving time window in a discrete-time framework, small errors between output trajectories must imply small errors in the initial states, for any pair of initial states and uniformly with respect to the control input. In [20] and [32], the FIE and MHE estimators are proved to be Robustly Globally Asymptotically Stable under an assumption of \textit{incremental input/output-to-state stability} (i-IOSS). It can be interpreted as a detectability condition of any initial conditions in the presence of process noise. Note that the quantitative measures of i-IOSS are again independent of any control input. Global stability of classical FIE and MHE schemes require global solutions of the optimal estimation problem which may not be achievable in a general nonlinear case. This claim has notably been made in [1,2,4,13,23,46] where one only searches for state trajectories that are locally optimal. One focuses on trajectories that are ‘close’ to the true one by considering that the initial estimation error is small. However, with online computational constraints, computing local solution to the MHE problem can also be intractable. This has led to the concept of fast or approximate MHE. The main idea is to solve the underlying optimisation problem in MHE schemes partially by only computing a few iterations of a numerical optimisation routine at each time step. A direct consequence of this approximation is that one does not need to be able to distinguish all the states from each other but only the current state from the ones in a small neighbourhood around it. This means that the required observability condition can be weakened accordingly. For instance, in [46], a version of the \textit{N-step observability} property localised around the
actual state of the system is used to show the convergence of an approximate MHE scheme. These weaker assumptions are again made uniformly with respect to the control input. It suggests that the impact of the input trajectory on the performance of the MHE scheme is overlooked. Nevertheless, it is known that general nonlinear observability properties of nonlinear controlled systems cannot be stated independently of the input, see [6]. In particular, some input trajectories might prevent the system from satisfying the N-step observability property. In this regard, the notion of regularly persistent input trajectories happens to be very useful, particularly, in the design of global observers for state-affine systems, see [6]. It defines a class of input trajectories in a continuous time framework that forces the system to satisfy the equivalent of the N-step observability property on the whole statespace. However, this property is so strong that such input trajectories might not exist. It is also unnecessary in fast MHE, as mentioned before. That is why, the first contribution of this paper is to formulate a general concept of weak regularly persistent input trajectories that is adapted to the design of fast MHE schemes. It is made to capture the impact of the input on observability issues in MHE and is consistent with the existing plethora of nonlinear observability notions.

A typical example of nonlinear estimation problem where the input on the system has a direct impact on the estimation performance is SLAM (Simultaneous Localization and Mapping), see [10]. The problem of SLAM consists of reconstructing the state of a robot and a map of its environment at the same time from partial measurements. Note that SLAM is more and more treated using optimization-based methods, see [8] and [21] and that MHE schemes have been recently tried in this context see [14,24,25,26,28,29,31,42]. However, the suitable observability conditions for fast MHE applied to SLAM do not seem to have been derived. In order to fill this gap, our second contribution is to provide sufficient conditions for weak regular observability in the general case of landmark-based SLAM both with a world-centric and a sensor-centric point of view. We show notably that in the sensor-centric point of view, the dynamics of the robot is not important for our concept of observability and only its state and input trajectories matter. Besides, the notion of circumnavigation appears to be useful in SLAM and related problems of localisation of autonomous systems, see [11,17,16,38,40,41]. Inspired by this, we finally show that 2D SLAM systems with different types of sensors can be made simultaneously weakly-regularly observable if the robot tracks any circular trajectory in the position/velocity space.

The rest of the paper is organized as follows. In Section 2, the standard nonlinear observability concepts are recalled, the notion of weakly regularly persistent input and weakly regularly observable systems are introduced and interpreted in terms of optimization concepts and of the Observability Grammian. In Section 3, sufficient conditions for the weak regular observability of a SLAM problem are presented. Finally, a sufficient condition for input trajectories to be jointly weakly regularly persistent for second-order SLAM problems with different sensors is presented in Section 4.

2 Observability properties of general nonlinear controlled systems

This section is dedicated to the presentation of a new observability concept. We introduce the notion of weakly regularly persistent input trajectories that ensures quantitative distinguishability between the true reference trajectory and nearby states while having only access to the past observations on a moving time-window. This property appears to be a consistent extension of classical nonlinear observability and plays an important role in proving the convergence and stability of nonlinear observers.

2.1 Setup and classical nonlinear observability notions

To begin with, several well-known observability concepts are recalled from [6]. We consider the following general nonlinear system:

\[ \dot{x} = f(x,u), \]
\[ y = h(x,u), \]

where

- \( u : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_u} \) is a piece-wise continuous input trajectory, \( x \) is the corresponding state trajectory valued in \( \mathbb{R}^{n_x} \) and \( y \) the corresponding measurement (or output) trajectory valued in \( \mathbb{R}^{n_y} \);
- \( f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x} \) is the controlled vector field of the system and \( h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_y} \) is the observation function, also called output function. Mappings \( f \) and \( h \) are both assumed to be twice continuously differentiable.

For simplicity, the solutions of system (1) are supposed to be uniquely defined at all times. For \( s_2 \geq s_1 \geq 0, \) and \( \xi \in \mathbb{R}^{n_x}, \) we denote by \( \phi_f(s_2; s_1, \xi, u) \) the solution flow of system (1) at time \( s_2 \) with initial condition \( \xi \), initial time \( s_1 \) and input trajectory \( u \). Let \( x_0 \in \mathbb{R}^{n_x} \) be a fixed initial condition and \( t_0 = 0 \) be the reference initial time. In the following, the reference trajectory is defined, for some input trajectory \( u \), by:

\[ x(t) := \phi_f(t; 0, x_0, u). \]

The property of observability of a system is defined as one's ability to distinguish between two initial conditions using only an input trajectory and the corresponding
output trajectories. The definitions of distinguishable and indistinguishable pairs are recalled in Definition 1.

**Definition 1 (Distinguishability)** Let \( u \) be an input trajectory. A pair \((\xi_1, \xi_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}\) is said to be distinguishable using the input trajectory \( u \) if there exists \( t \geq 0 \) such that:

\[
h(\phi(t, 0, \xi_1), u(t)) \neq h(\phi(t, 0, \xi_2), u(t)).
\]

A pair \((\xi_1, \xi_2)\) is said to be distinguishable if there exists an input trajectory \( u \) such that \((\xi_1, \xi_2)\) is distinguishable using the input trajectory \( u \). If \((\xi_1, \xi_2)\) is distinguishable (resp. using input trajectory \( u \)) then it is also said that \( \xi_1 \) is distinguishable from \( \xi_2 \) (resp. using input trajectory \( u \)). If \((\xi_1, \xi_2)\) is not distinguishable, then it is said to be indistinguishable.

Therefore, observable systems are such that every initial state can be distinguished from the other states.

**Definition 2 (Observability)** System \((1)\) is said to be observable at \( x_0 \in \mathbb{R}^{n_x} \) if every state is distinguishable from \( x_0 \). System \((1)\) is said to be observable if every pair of states \((\xi_1, \xi_2)\) is distinguishable.

Note that, contrary to linear systems, observability of nonlinear systems depends highly on input trajectories. In fact, observability as defined in Definition 2 requires the existence of an input trajectory, for any pair of states in the statespace, that enables one to discriminate them. This makes observability a strong property that might not be satisfied by a large class of systems. This justifies the introduction of the concept of weak observability where one focuses on a neighbourhood of some state.

**Definition 3 (Weak observability)** The system \((1)\) is said to be weakly observable at \( x_0 \) if there exists an input trajectory \( u \) and a neighbourhood of \( x_0, U, \) such that for any \( \xi \in U \setminus \{x_0\}, \) there exists \( t \geq 0 \) such that:

\[
h(\phi_f(t; 0, x_0, u), u(t)) \neq h(\phi_f(t; 0, \xi, u), u(t)).
\]

The system \((1)\) is said to be weakly observable if it is weakly observable at \( x_0 \) for any \( x_0 \in \mathbb{R}^{n_x} \).

A slightly stronger concept of observability is used when one also needs to distinguish a pair of states instantly that is to say by staying close to the initial condition. For this reason, the notion of local weak observability has been introduced in [19]. Its definition is recalled in Definition 4.

**Definition 4 (Local weak observability)** The system \((1)\) is said to be locally weakly observable at \( x_0 \) if there exists an input trajectory \( u \) and a neighbourhood of \( x_0, U, \) such that for any neighbourhood of \( x_0, V \subset U \) and any \( \xi \in V \setminus \{x_0\}, \) there exists \( t \geq 0 \) such that:

\[
h(\phi_f(t; 0, x_0, u), u(t)) \neq h(\phi_f(t; 0, \xi, u), u(t)),
\]

\[
\phi_f(t; 0, \xi, u) \in V.
\]

The system \((1)\) is said to be locally weakly observable if it is locally weakly observable at \( x_0 \) for any \( x_0 \in \mathbb{R}^{n_x} \).

In Definition 4, the term ‘weak’ specifically refers to the fact that one is trying to distinguish states that are near \( x_0 \) while the term ‘local’ means that one is able to use arbitrarily short state trajectories to do so. Thus, local weak observability at some initial condition \( x_0 \) means that \( x_0 \) can be distinguished from its neighbours using the input and output trajectories corresponding to state trajectories \( x \) that stay close to \( x_0 \). Its main interest is that it can be checked using a rank condition on the Lie derivatives of \( h \). See [6] for more details.

Note that in Definition 2, 3 and 4, an element of the statespace is fixed and one focuses on the existence of an input trajectory that allows one to distinguish this element from others. There exists another take on observability where one fixes a control trajectory and wonders if it can be used to distinguish every pair of states. Such input trajectories are called universal input trajectories.

**Definition 5 (Universal input)** For \( t \geq 0, \) an input trajectory \( u \) is a universal input trajectory on \([0, t]\) if for any \( \xi_1 \neq \xi_2, \) there exists \( s \in [0, t] \) such that \( h(\phi_f(s; 0, \xi_1, u), u(s)) \neq h(\phi_f(s; 0, \xi_2, u), u(s)) \). An input trajectory is said to be a universal input trajectory if there exists \( t \geq 0 \) such that it is a universal input trajectory on \([0, t]\). System \((1)\) is said to be uniformly observable if all input trajectories are universal.

In the following, we focus on integral formulations of observability as they typically provide more quantitative notions. This leads to the definition of the cumulative output error.

**Definition 6 (Cumulative output error)** For \( 0 \leq t_1 \leq t_2, \) an input trajectory \( u \) and a pair of states \((\xi_1, \xi_2)\) we define the cumulative output error of system \((1)\) on \([t_1, t_2] \) at \((\xi_1, \xi_2)\) with input trajectory \( u, \) denoted by \( l(t_1, t_2, \xi_1, \xi_2, u), \) as follows:

\[
l(t_1, t_2, \xi_1, \xi_2, u) =
\int_{t_1}^{t_2} \| h(\phi_f(s; t_1, \xi_1, u), u(s)) - h(\phi_f(s; t_1, \xi_2, u), u(s)) \|^2 ds,
\]

where \( \| \cdot \| \) denotes the Euclidian norm.

Thus, from Definition 5, one can derive an equivalent integral characterization of universal input trajectories.
Proposition 7 An input trajectory \( u \) is universal if and only if for any \( \xi_1 \neq \xi_2 \), there exists \( t \geq 0 \) such that:
\[
I(0, t, \xi_1, \xi_2, u) > 0.
\] (3)

PROOF. Since \( u \) is assumed piece-wise continuous and \( h \) is continuous, one has for any \( \xi_1 \neq \xi_2 \) and \( t \geq 0 \), that
\[
\int_0^t \| h(\phi_f(s; 0, \xi_1, u), u(s)) - h(\phi_f(s; 0, \xi_2, u), u(s)) \|^2 ds = 0
\]
if and only if for any \( s \in [0, t] \), \( h(\phi_f(s; 0, \xi_1, u), u(s)) = h(\phi_f(s; 0, \xi_2, u), u(s)) \). The result follows from this.

In theory, when a universal input trajectory is available, it should be possible to reconstruct the state of the system at anytime if one waits for a sufficiently long time. However, in practice, one would like to know an upper bound on the time required to distinguish states using some input trajectory. This leads to the definition of persistent input trajectories.

Definition 8 (Persistent input) An input trajectory \( u \) is said to be persistent if there exists \( T > 0 \) such that, for any \( t \geq T \) and any \( \xi_1 \neq \xi_2 \):
\[
l(t - T, T, \xi_1, \xi_2, u) > 0.
\] (4)

Persistent input trajectories allows one to distinguish every state during a time window of bounded length. In other words, one is then able to distinguish every pair of states without having to wait for more than a time span of \( T \). However, in some cases, \( l(t - T, T, \xi_1, \xi_2, u) \) might vanish as \( t \to +\infty \) for fixed \( \xi_1 \) and \( \xi_2 \) making the system potentially less and less observable along the state trajectory. The states would then be numerically harder and harder to distinguish as time goes. This can be avoided by considering a more quantitative property namely the regular persistence of the input trajectory.

Definition 9 (Regularly persistent input) An input trajectory \( u \) is said to be regularly persistent if there exist \( T > 0 \) and \( \kappa : \mathbb{R}^+ \to \mathbb{R}^+ \) continuous increasing with \( \kappa(0) = 0 \) such that for any \( t \geq T \) and any \( \xi_1 \neq \xi_2 \):
\[
l(t - T, T, \xi_1, \xi_2, u) \geq \kappa(\| \xi_1 - \xi_2 \|).
\] (5)

Clearly, from the definitions above, a regularly persistent input trajectory is persistent and a persistent one is universal. Note that Definition 8 and 9 can also be presented on time windows of the form \([t, t + T]\) for \( t \geq 0 \). The function \( \kappa \) in (5) ensures that regularly persistent input trajectories excite the system consistently along the state trajectory so that every pair of state is distinguishable enough on rolling intervals of size \( T \). Regular persistence happens to be a useful property to enforce the convergence of some closed form observers, see Chapter 5 of [18]. It is also very common to look for estimators by trying to minimise the cumulative output error, see [37] for a general review and analysis on the topic. As a consequence, the corresponding observability notions can be interpreted in terms of optimization notions. This is the topic of the next section.

2.2 Nonlinear observability and optimization-based estimation

Optimization-based estimation aims at building a state estimator by minimizing a cost depending on the input and output trajectories on some time interval. In this paper, we focus on this cost being the cumulative output error. One of the main theoretical issue in the deterministic setting is to ensure that the potential multiple solutions of the resulting optimization problems coincide locally or globally with the reference trajectory. In this section, we first link the previously stated observability concepts to Full Information and Moving Horizon Estimation. Then, we introduce the notion of weak regularly persistent input trajectories that is relevant in Fast Moving Horizon Estimation and give a sufficient condition related to the Grammian of Observability.

To avoid confusion with the several definitions of observability stated above, we recall the definition of several concepts of solution of an optimization problem.

Definition 10 Let \( F : \mathbb{R}^{n_x} \to \mathbb{R} \). Consider the optimization problem:
\[
\inf_{\xi \in \mathbb{R}^{n_x}} F(\xi).
\] (6)

It is said that \( \xi^* \in \mathbb{R}^{n_x} \) is a global solution of Problem (6) if for any \( \xi \in \mathbb{R}^{n_x} \), \( F(\xi^*) \leq F(\xi) \). It is said that \( \xi^* \in \mathbb{R}^{n_x} \) is a local solution of Problem (6) if there exists a neighbourhood of \( \xi^* \), \( U \), such that for any \( \xi \in U \), \( F(\xi) \leq F(\xi^*) \). It is said that \( \xi^* \in \mathbb{R}^{n_x} \) is a strict local solution of Problem (6) if there exists a neighbourhood of \( \xi^* \), \( U \), such that for any \( \xi \in U \setminus \{ \xi^* \} \), \( F(\xi) < F(\xi^*) \).

2.2.1 Full Information Estimation (FIE)

The most straightforward optimization-based estimator is called the Full Information estimator. The cost function is usually chosen as the cumulative measurement error on the whole interval \([0, t]\) at the reference trajectory. It leads to the following optimization problem for \( t \geq 0 \) and \( x_0 \in \mathbb{R}^{n_x} \):
\[
\inf_{\xi \in \mathbb{R}^{n_x}} l(0, t, \xi, x_0, u) (FIE_{t, u})
\]
Full Information Estimation requires solving Problem (FIE_{t, u}) globally. Proposition 11 ensures that one recovers the initial condition \( x_0 \), if \( u \) is a universal input trajectory.
_Proposition 11_ For \( t \geq 0 \), \( u \) is a universal input trajectory on \([0, t]\) if and only if, for any \( x_0 \in \mathbb{R}^{n_x} \), \( x_0 \) is the unique global solution of Problem (FIE\(_{t,u}\)).

**PROOF.** First, notice that for any \( \xi \in \mathbb{R}^{n_x} \), \( l(0, t, \xi, x_0, u) \geq 0 = l(0, t, x_0, x_0, u) \) so \( x_0 \) is a global solution of Problem (FIE\(_{t,u}\)) independently of \( u \). Then, by Proposition 7, \( u \) is a universal input trajectory if and only if for any \( \xi \neq x_0 \), \( l(0, t, \xi, x_0, u) > 0 \). This means that \( x_0 \) is the unique global solution of Problem (FIE\(_{t,u}\)).

A well-known drawback of FIE is that its practical solution becomes progressively more difficult to obtain as \( t \) grows. A common alternative is to consider the input/output trajectories only on a time window of fixed length which leads to *Moving Horizon Estimation*.

### 2.2.2 Moving Horizon Estimation (MHE)

As an alternative to Problem (FIE\(_{t,u}\)), one can consider a problem on \([t - T, t]\) for some memory time \( T > 0 \) and look for a *Moving Horizon* estimator by minimising \( l(t - T, t, \xi, x(t - T), u) \). This typically leads to the following optimization problem, for \( t \geq T \):

\[
\inf_{\xi \in \mathbb{R}^{n_x}} l(t - T, t, \xi, x(t - T), u). \tag{MHE\(_{t,T,u}\)}
\]

Problem (MHE\(_{t,T,u}\)) is written in the so-called ‘simultaneous form’ where the goal is to recover \( x(t - T) \) by solving Problem MHE\(_{t,T,u}\) at time \( t \) and reconstruct the rest of the trajectory by applying the flow \( \phi_f \) with the input trajectory \( u \). Similar to Problem (FIE\(_{t,u}\)), persistence of the input trajectory implies in particular uniqueness of a global solution of Problem (MHE\(_{t,T,u}\)).

**Proposition 12** An input trajectory \( u \) is persistent if and only if, there exists \( T > 0 \) such that for any \( t \geq T \) and any initial condition \( x_0 \in \mathbb{R}^{n_x} \), \( x(t - T) = \phi_f(t - T; 0, x_0, u) \) is the unique global solution of Problem MHE\(_{t,T,u}\).

**PROOF.** The proof is very similar to that of Proposition 11.

**Remark 13** *Moving Horizon Estimation* Problems usually contain an additional cost called an arrival cost, that we denote by \( l_a(\xi, x_T) \) where \( x(t - T) \) is some estimate of \( x(t - T) \) based \( y(s) \) for \( s \in [0, t - T] \). It is meant to aggregate the information available on the system before time \( t - T \) and improve the approximation made by switching from a FIE problem to an MHE problem. We omit this term in the context of this paper as it is not necessary to ensure convergence of MHE schemes in the presence of small disturbances or no disturbance at all, as proved in [1, 23, 46].

**Remark 14** Proposition 12 states that Moving Horizon Estimation is enabled by persistent input trajectories. In the case of regularly persistent input trajectories, the supplementary presence of function \( \kappa \) in Definition 9 typically allows one to build global nonlinear observers. See Chapter 5 of [6] for an example. Besides, by assuming similar observability properties, robust convergence of the Full Information Estimator and the Moving Horizon Estimator in the presence of process and measurement noise is proven in [3,20,32,36].

**Remark 15** Even if there exist regularly persistent inputs, they can be very hard to find because of the strong nature of the property. Moreover, one cannot hope to solve Problem (MHE\(_{t,T,u}\)) globally but only locally as it is generally nonconvex. Indeed, if one is only able to find local solutions of Problem (MHE\(_{t,T,u}\)), then regular persistence seems unnecessary and one needs a less demanding concept of observability.

### 2.3 Weak regular persistence and Moving Horizon Estimation

Let \( T > 0 \) and \( x_0 \in \mathbb{R}^{n_x} \) be a time horizon and an initial condition. For \( t \geq T \) and \( \xi \in \mathbb{R}^{n_x} \), we rewrite the cumulative measurement error between \( \phi_f(t; t - T, \xi, u) \) and the reference trajectory on \([t - T, t]\) as \( c(t, T, \xi, u) \) such that:

\[
c(t, T, \xi, u) = l(t - T, t, \xi, x(t - T), u). \tag{7}
\]

Note that a regularly persistent input trajectory \( u \) is such that every possible state can be distinguished with the output of the system if one awaits for some non negligible but bounded time. Thus, if one keeps the terminology from Definition 4, regular persistence is a strong and non-local property of the input trajectories. Besides, if \( u \) is a regularly persistent input trajectory then \( x(t - T) \) is assured to be the unique global solution of Problem (MHE\(_{t,T,u}\)) from Proposition 12. However, it is extremely hard to find a global solution of Problem (MHE\(_{t,T,u}\)) for a general nonlinear system as \( c(t, T, \xi, u) \) is generally nonconvex in \( \xi \). It is also very complicated to check whether or not an input trajectory is persistent because it requires to check that (4) holds for every pair of states. As a result, the concepts of persistent and regularly persistent inputs are too strong and irrelevant in many practical applications of MHE. One would rather like to ensure that the true state \( x(t - T) \) is a strict solution of Problem (MHE\(_{t,T,u}\)) for an appropriate choice of input trajectories. By continuity of \( c(t, T, \cdot, u) \), this would impose that \( x(t - T) \) is the unique global solution of Problem (4) in a neighbourhood of \( x(t - T) \).
As a consequence, the first contribution of this paper is to state a definition of weakly regularly persistent input trajectories based on Definition 9 which is adapted to the practical resolution of Moving Horizon Estimation problems by trying to generalise existing conditions and to emphasize the role of the input trajectories in these observability properties. We also give a necessary and a sufficient condition for the weakly regular persistence of input trajectories based on the Observability Grammian.

Definition 16 (Weakly regularly persistent input) Fix an initial condition $x_0 \in \mathbb{R}^n_x$. An input trajectory $u$ is said to be weakly regularly persistent at $x_0$, if there exist $T > 0$ and $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous increasing with $\kappa(0) = 0$ such that, for any $t \geq 0$, and $\xi$ in a neighborhood of $x(t-T) = \phi_f(t-T; 0, x_0, u)$:

$$c(t, T, \xi, u) \geq \kappa(||\xi - x(t-T)||). \quad (8)$$

For $X \subset \mathbb{R}^n_x$, an input trajectory $u$ is said to be weakly regularly persistent on $X$ if $u$ is weakly regularly persistent at $x_0$ for any $x_0 \in X$ and if $(T, \mu)$ from (8) depend only on $X$ and $u$ and in particular do not depend on $x_0$. System (1) is said to be weakly regularly observable if for any $x_0 \in \mathbb{R}^n_x$ there exists a weakly regularly persistent input trajectory at $x_0$.

Roughly speaking, weakly regularly observable systems are such that, for trajectories starting close enough to the current state and for some input trajectory, small rolling cumulative error in the output implies small ‘estimation’ error in the current state. It is clear that regularly persistent input trajectories are weakly regularly persistent. Contrary to the previously discussed observability concepts in the previous section, weakly regularly persistent input trajectories ensures that Problem (MHE$_{t,T,u}$) has only a strict local solution at $x(t-T)$ and potentially allows several global solutions.

First, Lemma 17 states straightforward properties of $c(t, T, \cdot, u)$ and its derivatives that are included for completeness sake. In the following, $d_1c$ and $d_2c$ respectively denote the first and second order differential of $c(t, T, \cdot, u)$.

Lemma 17 For $x_0 \in \mathbb{R}^n_x$, $T > 0$, $t \geq T$, and an input trajectory $u$, $c(t, T, u)$ is twice continuously differentiable, $c(t, T, x(t-T), u) = 0$ and $d_1c(t, T, x(t-T), u) = 0$.

PROOF. Note that $c(t, T, \cdot, u)$ is twice continuously differentiable because $f$ and $h$ are. Besides, $c(t, T, x(t-T), u) = 0$ by definition of $x(t-T)$ and $c$. Finally, $d_1c(t, T, x(t-T), u) = 0$ because $x(t-T)$ is a local solution of problem (MHE$_{t,T,u}$).

Proposition 18 states the first result of the section.

Proposition 18 For $x_0 \in \mathbb{R}^n_x$ and $t \geq 0$, if $u$ is a weakly regularly persistent input trajectory at $x_0$ then $x(t-T)$ is a global solution and a strict local solution of Problem (MHE$_{t,T,u}$).

PROOF. Since $c(t, T, x(t-T), u) = 0$, $x(t-T)$ is an always global solution and a local solution of Problem (MHE$_{t,T,u}$). Subsequently, by definition of weakly regularly persistent input trajectories, there exists a neighborhood of $x(t-T)$, $U$, such that, $\forall \xi \in U \setminus \{x(t-T)\}$:

$$c(t, T, \xi, u) > 0 = c(t, T, x(t-T), u),$$

and $x(t-T)$ is a strict local solution of Problem (FIE$_{t,u}$).

The main advantage of the weak regular persistence property is that it can be checked by computing the Observability Grammian of system (1). Its definition is stated in Definition 19.

Definition 19 (Observability Grammian) The Observability Grammian of system (1) at time $t$, denoted by $C(t, T, u)$ is defined as the Hessian of $c(t, T, \cdot, u)$ taken at $x(t-T)$ and reads:

$$C(t, T, x(t-T), u) = d_2^2c(t, T, \cdot, u)|_{\xi=x(t-T)} = \int_{t-T}^t \Phi_f^T H^T(x(s), u(s))H(x(s), u(s))\Phi_f ds, \quad (9)$$

where $H(x(s), u(s)) = d_h(h(\cdot, u(\cdot)), \xi=x(t-T))$ and $\Phi_f(s; t-T, u) = d_2\phi_f(s; t-T, \cdot, u)|_{\xi=x(t-T)}$.

Notably inspired by [3], Proposition 20 gives a sufficient and a necessary condition for weak regular persistence in terms of a lower bound on the Observability Grammian.

Proposition 20 Let $x_0 \in \mathbb{R}^n_x$ and $u$ be an input trajectory. Assume there exist $T > 0$ and $\mu > 0$ such that for any $t \geq T$:

$$C(t, T, x(t-T), u) \geq \mu I_{n_x}, \quad (10)$$

where $\succeq$ denotes the classical partial order on positive semi-definite matrices and $I_{n_x}$ denotes the $n_x \times n_x$ identity matrix. Then, $u$ is a weakly regularly persistent input trajectory at $x_0$.

Conversely, if $u$ is a weakly regularly persistent input trajectory at $x_0$ and $\kappa$ has finite sensitivity meaning that
there exists $r > 0$ such that
\[ \inf_{\|\xi\| \leq r} \frac{\kappa(\|\xi\|)}{\|\xi\|^2} > 0, \]  \tag{11}\]
then there exists $T > 0$ and $\mu > 0$ such that for any $t \geq T$, (10) holds.

**PROOF.** Let $x_0 \in \mathbb{R}^{nx}$ and assume, as in the proposition statement, that $\exists T, \mu > 0$ such that $\forall t \geq T$:
\[ C(t, T, x(t - T), u) \geq \mu I_{nx}. \]

Note that for $t \geq 0$, $d_2^2 c(t, T, \cdot, u)$ is continuous as $c(t, T, \cdot, u)$ is twice continuously differentiable. Besides, by continuity of the smallest eigenvalue of $d_2^2 c(t, T, \cdot, u)$ at $x(t - T)$, one has for $\xi$ in a neighborhood of $x(t - T)$, that:
\[ d_2^2 c(t, T, \xi, u) \geq \frac{\mu}{2} I_{nx}. \]  \tag{12}\]

From Lemma 17, $c(t, T, x(t - T), u) = 0$ and $d_2 c(t, T, x(t - T), u) = 0$. Moreover, from the mean value form of the Taylor expansion of $c(t, T, \cdot, u)$ at $x(t - T)$ see Equation (b) in Proposition A.23 of [5], one gets for $\xi$ in a neighborhood of $x(t - T)$, that:
\[ c(t, T, \xi, u) = \frac{1}{2} (\xi - x(t - T)) d_2^2 c(t, T, \chi, u)(\xi - x(t - T)), \]  \tag{13}\]
with $\chi = (1 - \theta) x(t - T) + \theta \xi$ and $0 < \theta < 1$. Note that $\chi$ is in a neighborhood of $x(t - T)$ when $\xi$ is. By using (12), one finally obtains that, for $\xi$ in a neighborhood of $x(t - T)$:
\[ c(t, T, \xi, u) \geq \frac{\mu}{4} \|\xi - x(t - T)\|^2, \]
and the result follows by choosing $\kappa(r) = \frac{4r^2}{1}$. For the converse, assume that $u$ is a weakly regularly persistent input trajectory at $x_0$ and $\kappa$ has finite sensitivity as stated in (11). Thus, there exists $T > 0$ and $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous increasing such that $\kappa(0) = 0$ and for any $t \geq T$ and $\xi$ in a neighbourhood of $x(t - T)$:
\[ c(t, T, \xi, u) \geq \kappa(\|\xi - x(t - T)\|). \]  \tag{14}\]
Besides, according to (11), there exists $\tilde{\mu} > 0$ and $r > 0$ such that, for any $\xi$ and any $t \geq T$, if $\|\xi - x(t - T)\| \leq r$ then,
\[ \kappa(\|\xi - x(t - T)\|) \geq \tilde{\mu} \|\xi - x(t - T)\|^2. \]  \tag{15}\]
From the Taylor’s expansion of $c(t, T, \cdot, u)$ at $x(t - T)$, see Equation (c) in Proposition A.23 of [5], and from Lemma 17, one gets that, for $\xi$ in a neighborhood of $x(t - T)$:
\[ c(t, T, \xi, u) = \frac{1}{2} w^T C(t, T, x(t - T), u) w + \|w\|^2 \theta(\xi), \]
where $w = \xi - x(t - T)$ and $\lim_{\xi \rightarrow x(t - T)} \theta(\xi) = 0$. By further combining (14) and (15), one gets for $\xi$ in a neighborhood of $x(t - T)$:
\[ \frac{w^T C(t, T, x(t - T), u) w}{\|w\|^2} \geq 2\tilde{\mu} + 2\theta(\xi), \]
where $\mu = \tilde{\mu}$. Finally, one gets that $C(t, T, x(t - T), u) \geq \mu I_{nx}$ and the result is proven.

**Remark 21** For state-affine systems, weak regular persistence and regular persistence are equivalent because the Observability Grammian does not depend on $x(t - T)$ in that case. Besides, weak observability has been assessed in terms of the Observability Grammian in [35].

**Remark 22** One can see from (12) that under the assumptions on the Observability Grammian of Proposition 20, $c(t, T, \cdot, u)$ is strongly convex on a ball around $x(t - T)$. Note that the latter property has already recently been used in Fast Moving Horizon Estimation where one wants to find approximate local solutions of Problem (MHE_{t,T,u}) at each time $t$. In [4,13,23,46,47], the time-independent lower bound on the Observability Grammian denoted by $\mu$ in (12) ensures the local convergence and robustness of approximated Moving Horizon Estimators.

**Remark 23** Although several weak observability has already been defined for MHE notably in [13,23,46], weak regular persistence of the input does not seem to have been stated in in this form and put into perspective with other nonlinear observability and optimization concepts.

To summarise, on one hand, regular persistence is a strong property on the input trajectories that allows one to build estimators that converge for arbitrary initial error while requiring global solutions of Problem (MHE_{t,T,u}). On the other hand, a consequence of the previous remarks is that weakly regularly persistent inputs allows one to build computationally efficient estimators that converge for small initial error at the lower price of approximate local solutions. Interests in
optimization-based estimation have recently grown in the Robotics community notably through the application of SLAM. However, observability properties for Moving Horizon Estimation in SLAM have not been discussed in the literature. The topic of next section is then to study weak regular persistence of SLAM systems to fill the existing gap in Fast Moving Horizon Estimation in this area.

3 Weak regular observability in landmark-based SLAM problems

In classical SLAM, ones tries to localise a mobile robot and reconstruct a map of its environment at the same time using partial measurements. Landmark-based SLAM is a particular version of SLAM in which the environment is represented by a set of discrete landmarks associated with continuous measurements, see [43]. This representation has the advantage of being sufficiently general to match many realistic scenarios while having a lot of structure from the system dynamics point of view. In this section, we focus on specifying the results from Proposition 20 in the case where each landmark is observed through observations, denoted by $h_j(z,\ell,\eta,u)$. The global dynamics can be summed up as follows for $1 \leq j \leq J$ and any initial condition $(z_0,\ell)$:

$$\dot{z} = g(z,u), \quad \dot{\ell}_j = 0, \quad h_j(z,\ell,\eta,u)$$

where $g$ represents the robot dynamics and $u$ an input trajectory. We assume that $z$ contains the 2D position of the robot in the inertial frame denoted by $\chi$ such that

$$\chi = Pz,$$

with $P$ being a projection matrix from $\mathbb{R}^{n_z}$ to $\mathbb{R}^2$. The remaining variables are denoted by $\eta = (I-P)z$. We also assume that the robot state and the landmark positions are observed through observations, denoted by $y$, of the following type:

$$y = h(z,\ell,u),$$

where

$$h(z,\ell,u) = \begin{bmatrix} h_0(z,u) \\ h_1(\chi - \ell_1,\eta,u) \\ \vdots \\ h_J(\chi - \ell_J,\eta,u) \end{bmatrix}, \quad (17)$$

$h_0$ denotes a direct measurement of the robot’s state and $(h_j)_{1 \leq j \leq J}$ represent relative measurements between the robot position and the landmark positions that may also depend on the remaining variables $\eta$. The local weak observability of this type of system has been assessed in [27], [34], [44]. Two points of view appear to be of interest: the world-centric view and the sensor-centric view. In the world-centric view, one considers the joint estimation of $z$ and the $\ell_j$. The SLAM system (16) is shown not to be locally weakly observable in [34] if only relative measurements are available. However, in [27], the authors show that if the position of some landmarks is known then the system can become locally weakly observable. In the sensor-centric view, one only estimates the position of the landmarks assuming the initial state of the robot is known. In this setting, the system is usually locally weakly observable see [27] again for more information. Unfortunately, these observability conditions have low practical interest as they are not quantitative.

The estimation problem for these models has been mostly recently tackled in the literature by using optimization based techniques, see [8,21]. One is usually interested in solving the associated Problem (FIE$_{1,u}$) in an offline way for system (16)-(17). In this context, the variables are the initial robot state and the positions of the landmark written generically as $Q_0$ for the initial state of the robot and $m$ for the set of landmarks. This leads to the following optimization problem:

$$\min_{\zeta_0 \in \mathbb{R}^{n_z},m \in \mathbb{R}^{2J}} \int_{0}^{t_f} \|h(\phi_g(s;0,\zeta_0,u),m,u(s)) - y(s)\|^2 ds$$

where $u$ is a fixed input trajectory, $\phi_g$ is the flow of the robot’s dynamics and $t_f$ is a final time. As mentioned previously, Problem (18) is written in the simultaneous form, see [13]. Solving Problem (18) implies high computational cost if $t_f$ is large. For this reason, one could try to transform Problem (18) into a Moving Horizon one leading for $t \geq 0$ and $T > 0$, to

$$\min_{\zeta_{T-t} \in \mathbb{R}^{n_z},m \in \mathbb{R}^{2J}} \int_{t-T}^{t} \|h(\phi_g(s;t,\zeta_{t-T},u),m,u(s)) - y(s)\|^2 ds$$

Problem (19) has recently been numerically solved in several contexts in the literature, see [14,24,25,26,28,29,31,42]. However, the observability theory related to Problem
(19) seems to have been overlooked.

As a consequence, the second contribution of this paper is the analysis of sufficient conditions for weak regular persistence of the landmark-based SLAM system (16)-(17) in the spirit of Proposition 20 both with a world-centric and sensor-centric view.

3.2 Observability conditions in the world-centric case

In the world-centric case, one considers the position of the robot and the landmarks in an fixed frame with no knowledge of the initial state of the robot. In the following, for any \((n, m, p) \in \mathbb{N}^3\) and for any differentiable function \(f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m\), \(d_x f(x, y)\) denotes the differential of \(f(\cdot, y)\) for any \(y \in \mathbb{R}^m\). Note that \(d_y f(x, y)\) is defined similarly. As a first step, we compute the differential of observation functions from (17). It is denoted by \(d_{(z, \ell)} h(z, \ell, u)\) and is presented in (20) with \(H_0(z) = d_z h_0(z, u), H_j(\xi, \eta, u) = d_\xi h_j(\xi, \eta, u)\) and \(H_j'(\xi, \eta, u) = d_\xi h_j(\xi, \eta, u)\) for \(1 \leq j \leq J\).

The differential of the flow associated with joint dynamics of the robot and the landmarks described by (16) is denoted by \(\Phi\) and reads:

\[
\Phi = \begin{bmatrix} \Phi_g & 0 & \cdots & 0 \\ 0 & I_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & I_2 & \cdots \end{bmatrix},
\]

where \(\Phi_g\) is the differential of \(\phi_g\) as defined in Definition 19. For the sake of clarity, we have dropped the dependency on the input and the initial condition of the Observability Grammian \(C\) and related matrices. Thus, it can be written as follows, for any initial condition \((z_0, \ell)\), any \(t \geq 0\) and \(T > 0\):

\[
C(t, T) = \int_{t-T}^t C(s, z(s), \ell, u(s)) ds,
\]

where:

\[
C(s, z; u) = \Phi^T d_{(s, \ell)} h^{T} d_{(s, \ell)} h \Phi(s, z; u),
\]

\[
= \begin{bmatrix} A_0 + \sum_{j=1}^J D_j \begin{bmatrix} B_1 & A_1 & 0 \\ \vdots & \ddots & \ddots \\ B_J & 0 & A_J \end{bmatrix} - B_j'B_j \\ \end{bmatrix}
\]

with for \(1 \leq j \leq J\):

\[
A_0(s, z, u) = \Phi_g(s, z; u)^T H_0^T(z, u) H_0(z, u) \Phi_g(s, z; u),
\]

\[
A_j(\chi - \ell_j, \eta, u) = H_j^T(\chi - \ell_j, \eta, u) H_j(\chi - \ell_j, \eta, u),
\]

\[
B_j(s, z, \ell_j, \eta, u) = -A_1(\chi - \ell_j, \eta, u) P \Phi_g(s, z; u)
\]

\[
- H_j^T(\chi - \ell_j, \eta, u) H_j'(\chi - \ell_j, \eta, u)(I - P) \Phi_g(s, z; u),
\]

\[
D_j(s, \chi - \ell_j, \eta, u) = \Phi_g^T(s, z; u)((I - P) H_j'(\chi - \ell_j, \eta, u) + PH_j(\chi - \ell_j, \eta, u))\Phi_g(s, z; u).
\]

One can further rewrite \(C(t, T)\) as follows:

\[
C(t, T) = \begin{bmatrix} A_0(t, T) + \sum_{j=1}^J D_j(t, T) - B_1'B_1 \cdots - B_J'B_J \\ -B_1(t, T) & A_1(t, T) & 0 \\ \vdots & \ddots & \ddots \\ -B_J(t, T) & 0 & A_J(t, T) \end{bmatrix}.
\]

where

\[
A_0(t, T) = \int_{t-T}^t A_0(s, z(s), u(s)) ds,
\]

\[
A_j(t, T) = \int_{t-T}^t A_j(\chi(s) - \ell_j, \eta(s), u(s)) ds,
\]

\[
B_j(t, T) = \int_{t-T}^t B_j(\chi(s) - \ell_j, \eta(s), u(s)) ds,
\]

\[
D_j(t, T) = \int_{t-T}^t D_j(\chi(s) - \ell_j, \eta(s), u(s)) ds.
\]

The idea of the sequel is to separate the sufficient observability conditions from Proposition 20 into a condition on the observability of the robot state and a set of independent conditions on the observability of each landmarks. This translates into Assumptions 24 and 25.

**Assumption 24** For any \((z_0, \ell) \in \mathbb{R}^{n+m+2J}\), there exists an input trajectory \(u, T > 0\) and \(\mu > 0\) such that for any
Proposition 26

Consequently, the following proposition states that, in order to obtain weak regular observability in the world-centric view, one can consider the persistence with respect to the state of the robot \( z \) and the landmarks \( \ell_j \) in a split way through Schur complements.

Proposition 26 Under Assumptions 24 and 25, for any \((z_0, \ell) \in \mathbb{R}_+^{n_z+2J}\), for \((u, T, \mu)\) satisfying (23)-(24), there exists \( 0 < \mu_0 < \mu \) such that for any \( t \geq T \),

\[
C(t, T) \geq \mu_0 I_{n_z+2J}.
\]

In particular, under Assumption 24 and 25, system (16) is weakly regularly observable.

PROOF.

Fix \((z_0, \ell) \in \mathbb{R}_+^{n_z+2J}\), the associated \( T \) and \( \mu \) from Assumption 24, \( t \geq 0 \) and \( 0 < \mu_0 < \mu \). The goal of the following is to show that \( C(t, T) - \mu_0 I_{n_z+2J} \geq 0 \). To do so, consider the following decomposition of \( C(t, T) - \mu_0 I_{n_z+2J} \):

\[
C(t, T) - \mu_0 I_{n_z+2J} = \begin{bmatrix} C_0(t, T) - \mu_0 I_{n_z} & B^T(t, T) \\ B(t, T) & A(t, T) - \mu_0 I_{2J} \end{bmatrix},
\]

where:

\[
C_0(t, T) = A_0(t, T) + \sum_{j=1}^{J} D_j(t, T),
\]

\[
B(t, T) = \begin{bmatrix} \chi(t, T) \\ \chi(t, T) \end{bmatrix},
\]

\[
A(t, T) = \begin{bmatrix} \chi(t, T) & 0 \\ \chi(t, T) & A(t, T) \end{bmatrix}.
\]

Let \( C_0/A(t, T) \) be the Schur complement of \( A(t, T) - \mu_0 I_{2J} \) in \( C(t, T) - \mu_0 I_{n_z+2J} \). Besides, since \( A_j - \mu_0 I_2 \geq (\mu - \mu_0) I_2 > 0 \), one has for any \( 1 \leq j \leq J \):

\[
C_0/A(t, T) = C_0(t, T) - \mu_0 I_{n_z} - \sum_{j=1}^{J} B_j^T(A_j - \mu_0 I_2)^{-1}B_j, \]

\[
C_0/A(t, T) = A_0 + \sum_{j=1}^{J} D_j - B_j^T(A_j - \mu_0 I_2)^{-1}B_j - \mu_0 I_{2J}. \]

By assumption, \( \mu_0 A_j^{-1} \geq \frac{\mu_0}{\mu} I_2 \). Then, by applying the Woodbury matrix identity on \((A_j - \mu_0 I_2)^{-1}\), one has for any \( 1 \leq j \leq J \):

\[
(A_j - \mu_0 I_2)^{-1} = \sum_{k=0}^{\infty} \mu_0^k A^{-1}(k+1),
\]

\[
(A_j - \mu_0 I_2)^{-1} = A_j^{-1} + \mu_0 \sum_{k=0}^{\infty} \mu_0^k A_j^{-1}(k+2). \]

By combining (26) and (27), one gets:

\[
C_0/A(t, T) = A_0 + \sum_{j=1}^{J} D_j - B_j^T A_j^{-1} B_j. \]
\[-\mu_0 \left( \sum_{j=1}^{J} B_j^T \sum_{k=0}^{+\infty} \mu_0^k A_j^{-(k+2)} B_j + I_{n_x} \right). \]  

From (24) in Assumption 24,

\[ \sum_{j=1}^{J} \sum_{k=0}^{+\infty} \mu_0^k A_j^{-(k+2)} B_j \geq \mu^{-2} \left( \sum_{k=0}^{+\infty} \left( \frac{\mu_0}{\mu} \right)^k \right) \sum_{j=1}^{J} B_j^T B_j. \]

From Assumption 25,

\[ \sum_{j=1}^{J} \sum_{k=0}^{+\infty} \mu_0^k A_j^{-(k+2)} B_j \geq \mu^{-2} \frac{\sigma \mu}{\mu - \mu_0} I_{n_x}. \]  

Finally, by combining (29), (30) and (23), one gets:

\[ C_0 / A(t, T) \geq \left( \mu - \mu_0 - \frac{\mu \sigma}{\mu (\mu - \mu_0)} \right) I_{n_x}, \]  

which proves that, there exists \( \mu_0 > 0 \) such that \( C_0 / A(t, T) > 0 \).

From Assumption 24,

\[ A(t, T) - \mu_0 I_{2J} > 0. \]  

Thus, from (31), (32) and Theorem 1.12 of [48], \( C(t, T) - \mu_0 I_{n_x+2J} \succeq 0 \) and the first result is proved. For the second one, notice that, from Proposition 20, (25) directly implies that the system (16) is weakly regularly observable.

### 3.3 Observability conditions in the sensor-centric case

In the sensor-centric case, one considers that \( z_0 \) is known so that the frame of study is centered at \( \chi_0 \). In our deterministic framework, since the input \( u \) is known, this is equivalent to having \( h_0(z, u) = z \). Therefore, Assumption 24 can be weakened to recover the result of Proposition 26.

**Assumption 27** For any \( (z_0, \ell) \in \mathbb{R}^{n_x+2J} \), there exists an input trajectory \( u, T > 0 \) and \( \mu > 0 \) such that for any \( t \geq T \) and any \( 1 \leq j \leq J \):

\[ A_j(t, T) \geq \mu I_2. \]  

Finally, Proposition 28 states that in the sensor-centric view, only the weak regular persistence with respect to the position of the landmarks is required.

**Proposition 28** Under Assumptions 25 and 27 and if \( h_0(z, u) = z \) then for any \( (z_0, \ell) \in \mathbb{R}^{n_x+2J} \), for \( (u, T, \mu) \) satisfying (24), there exists \( \mu_0 \) such that for any \( t \geq 0 \),

\[ C(t, T) \geq \mu_0 I_{n_x+2J}. \]  

In particular, under Assumption 24, system (16) is weakly regularly observable.

**PROOF.** Under the assumptions of the proposition, for any \( (z_0, \ell) \in \mathbb{R}^{n_x+2J} \), there exists an input trajectory \( u, T > 0 \) and \( \mu > 0 \) such that for any \( t \geq T \) and any \( 1 \leq j \leq J \):

\[ A_0(t, T) = TI_{n_x}. \]  

Besides, for any \( 1 \leq j \leq J \),

\[ D_j - B_j^T A_j^{-1} B_j \succeq 0, \]

as

\[ \begin{bmatrix} A_j & B_j \\ B_j^T & D_j \end{bmatrix} \succeq 0 \]

and \( A_j > 0 \). This leads to:

\[ A_0 + \sum_{j=1}^{J} D_j - B_j^T A_j^{-1} B_j \succeq TI_{n_x}. \]

This means that (23) and (24) from Assumption 24 are satisfied with \( \mu' = \min(T, \mu) \). The rest of the result follows then from the proof of Proposition 26.

**Remark 29** For simplicity of the presentation, it is assumed in this section that the landmarks are observed at all times by the robot. Yet, it is not completely realistic as one needs in practice to match the measurements with the landmarks. This is represented by a family of data association functions \( (a_j(\cdot))_{1 \leq j \leq J} \) such that \( a_j : \mathbb{R}^+ \rightarrow \{0, 1\} \) takes the value 1 when the landmark \( j \) is seen by robot and 0 if not. However, if we consider fixed and known data association functions \( (a_j(\cdot))_{1 \leq j \leq J} \), the result from this section can be recovered by defining \( \tilde{h}_j(t, \chi - \ell_1, \eta, u) = a_j(t) h_1(\chi - \ell_j, \eta, u) \). In particular, \( \tilde{A}_j \) can be written as follows

\[ \tilde{A}_j(t, T) = \int_{t-T}^{t} a_j(s) H_j^T(s) H_j(s), \]

where \( H_j(s) = H_j^T(\chi(s) - \ell_j, u(s)) \). One can then recover Assumption 27 and Proposition 28. In this case, Assumption 27 requires, broadly speaking, that the input \( u \) only be persistent for the landmark \( j \) only when it is seen by the robot.

**Remark 30** Assumption 25 is not restrictive as in most SLAM problems the area to explore is bounded a priori.
as well as the state and input trajectories of the system. This will be illustrated later in Section 4.

Remark 31 One can notice that Assumption 27 does not depend on the dynamics of system (16). This means that, in the sensor-centric case, the study of weak regular observability can be decomposed in two steps. First, for fixed state and input spaces, one can look for state and input trajectories that satisfy Assumption 27. Secondly, for some robot dynamics $g$ one can check if the previous trajectories can be tracked by state and input trajectories (or state trajectories only if the measurement does not depend explicitly on $u$) that are compatible with the corresponding dynamical constraint (16).

Following from Remark 29, the last contribution of this paper is to show that several relevant simple landmark-based SLAM systems with different observation types are made weakly regularly observable by tracking circular paths.

4 Joint weak regular observability by circumnavigation for second order SLAM problems with bearing, range, optical flow or Doppler shift measurements

Circumnavigation to ensure observability in localisation and SLAM problems with bearing measurements or range measurements has notably been studied in [11,40]. Optical flow measurements are also well-known in SLAM, see [9]. An observability analysis of optical flow measurement for inertial navigation can be found in [12,45]. Observability of Doppler-shift measurements in SLAM and localisation problems have been studied in [22,39]. The goal of this section is to shed light on the similarities of these four types of measurements by carrying out a joint observability analysis in the framework developed in the previous sections. This joint study does not seem to be present in the existing literature. In particular, we show that, in the case of a second order SLAM system with one landmark, Assumption 25 and 27 are jointly satisfied for the four types of measurements when the robot tracks circular paths around any point. As discussed below, it is without loss of generality that we can consider only one landmark.

4.1 Model description

In this section we are interested in a 2D SLAM system where the state of the robot is represented by position and velocity variables, $z = (\chi, v) \in \mathbb{R}^4$. Since Assumption 27 is distributed over the landmarks, we can assume without loss of generality that $J = 1$ and our SLAM system has only one unknown landmark position $\ell \in \mathbb{R}^2$. Our system can be written in the following form:

$$\dot{\chi} = g(\chi, v, u), \quad \dot{v} = 0,$$

where $u$ is some input trajectory and $g$ is twice continuously differentiable. In this application, the remaining variables $\eta$ are exactly the velocity variables so that $\eta = v$. The state of the robot is supposed to be observed through a relative measurement in position that can also depend on the velocity variables. In the sequel, we consider the following types of measurements:

1. Bearing measurements where one measures the direction from the robot to the landmark such that:

$$h^{(1)}(\chi - \ell, v) = p$$

where $p = (\ell - \chi) / \| \ell - \chi \|$.

2. Range measurements where one measures the distance between the robot and the landmark such that:

$$h^{(2)}(\chi - \ell, v) = r$$

where $r = \| \ell - \chi \|$.

3. Optical flow measurements where one measures the angular velocity of the landmark in the referential of the robot such that:

$$h^{(3)}(\chi - \ell, v) = \langle v, Qp \rangle / r$$

where $\langle \cdot \rangle$ denotes the canonical scalar product on $\mathbb{R}^2$ and $Q$ is the rotation matrix of angle $\frac{\pi}{2}$.

4. Doppler shift measurements where one measures a frequency shift between the landmark and the robot such that:

$$h^{(4)}(\chi - \ell, v) = \alpha(v, p),$$

where $\alpha > 0$ is a constant.

We consider this SLAM system in the sensor-centric view so the trajectory of the robot can be seen as fully observed such that:

$$h_0(z) = z.$$
for any \( t \geq T \) and \( i \in \{1, 2, 3, 4\} \):
\[
\mathcal{A}^{(i)}(t, T) = \int_{t-T}^{t} H^{(i)T}(s)H^{(i)}(s)ds \geq \mu I_2,
\]
where \( H^{(i)}(s) = H^{(i)}(\chi(s) - \ell, v(s)) \).

The topic of the next section is precisely to show that if the robot travels in circle around any point then (42) is satisfied simultaneously for the aforementioned four types of relative measurements.

4.2 Main results

To define the circular paths considered in this section, we fix \( \chi_c = (\chi_{c,1}, \chi_{c,2}) \in \mathbb{R}^2 \), \( \bar{x}_0 = (\bar{x}_{0,1}, \bar{x}_{0,2}) \in \mathbb{R}^2 \), \( \omega > 0 \) and \( r_c > 0 \). Then, the circular position and velocity trajectories around \( \chi_c \) of radius \( r_c \), denoted by \( (\bar{x}, \bar{v}) \), read for any \( s \geq 0 \):
\[
\bar{x}(s) = \chi_c + v_c \begin{bmatrix} \cos(\omega s + \bar{\psi}(0)) \\ \sin(\omega s + \bar{\psi}(0)) \end{bmatrix}, \tag{43}
\]
\[
\bar{v}(s) = \omega r_c \begin{bmatrix} -\sin(\omega s + \bar{\psi}(0)) \\ \cos(\omega s + \bar{\psi}(0)) \end{bmatrix}, \tag{44}
\]
where \( \bar{\psi}(0) = \arctan(\bar{x}_{0,1} - \chi_{c,1}, \bar{x}_{0,2} - \chi_{c,2}) \). From this, one can define, \( \mathcal{A}^{(i)}(t, T) \) as the Observability Gramian with respect to the landmark position associated with the circular trajectory for measurement of type \( i \). It reads, for \( \ell \in \mathbb{R}^2 \), \( T > 0 \), \( t \geq T \), and \( i \in \{1, 2, 3, 4\} \):
\[
\mathcal{A}^{(i)}(t, T) = \int_{t-T}^{t} \bar{H}^{(i)T}(s)\bar{H}^{(i)}(s)ds, \tag{45}
\]
where \( \bar{H}^{(i)}(s) = H^{(i)}(\bar{x}(s) - \ell, \bar{v}(s)) \).

One can now state the first result of the section which is contained in Proposition 32.

**Proposition 32** For any \( r_c > 0 \), \( \omega > 0 \), \( \chi_c \in \mathbb{R}^2 \), \( \chi_0 \in \mathbb{R}^2 \), \( \ell \in \mathbb{R}^2 \) and any \( i \in \{1, 2, 4\} \) there exist \( T(i) > 0 \) and \( \mu(i) > 0 \) such that, for any \( t \geq T(i) \):
\[
\mathcal{A}^{(i)}(t, T(i)) \geq \mu(i)I_2. \tag{46}
\]
Additionally, if \( r_c \neq \|\chi_c - \ell\| \), then (46) also holds for \( i = 3 \).

**PROOF.** See Appendix A.

Going back to the notion of weak regular input trajectories, we can now state the main result of the section.

**Theorem 33** If, for any \((\chi_0, v_0, \ell) \in \mathbb{R}^6 \), \( r_c > 0 \), \( \omega > 0 \), \( \chi_c \in \mathbb{R}^2 \), \( \bar{x}_0 \in \mathbb{R}^2 \), \( \ell \in \mathbb{R}^2 \) and bounded input trajectory, \( u \), for the dynamics (36) such that \( r_c \neq \|\chi_c - \ell\| \), there exists \( \mu > 0 \) and \( L > 0 \) such that if
\[
\epsilon < \frac{\mu}{L},
\]
and the corresponding circular path, \((\bar{x}, \bar{v})\), defined by (44) and the corresponding position and velocity solution flow, \((\chi, v)\), defined, for any \( s \geq 0 \) by \((\chi(s), v(s)) = \phi_g(s; 0, (\chi_0, v_0), u)\) satisfy:
\[
\sup_{s \geq 0} \|z(s) - \bar{z}(s)\| < +\infty, \tag{47}
\]
\[
\int_{t}^{T} \|z(s) - \bar{z}(s)\|ds \leq \epsilon, \tag{48}
\]
where \( \|z(s) - \bar{z}(s)\| = (\|\chi(s) - \bar{\chi}(s)\|^2 + \|v(s) - \bar{v}(s)\|^2)^{1/2}, \) then \( u \) is a weak regular input trajectory at \((\chi_0, v_0, \ell)\) for the systems (36)-(41) and the systems (36)-(41) are weakly regularly observable.

**PROOF.** Since systems (36), (41), (37)-(40) evolve in the sensor-centric view, the result can be obtained by applying Proposition 28. To do so, one need to check if Assumption 25 and 27 are satisfied with the settings of the theorem. Therefore, we first fix \((\chi_0, v_0, \ell) \in \mathbb{R}^6 \), \( r_c > 0 \), \( \omega > 0 \), \( \chi_c \in \mathbb{R}^2 \), \( \chi_0 \in \mathbb{R}^2 \), \( \ell \in \mathbb{R}^2 \) and an input trajectory, \( u \), for the dynamics (36) such that the corresponding circular path, \( \bar{z} = (\bar{\chi}, \bar{v}) \), defined by (44) and the corresponding position and velocity solution flow, \( z = (\chi, v) \), defined, for any \( s \geq 0 \) by \((\chi(s), v(s)) = \phi_g(s; 0, (\chi_0, v_0), u)\) satisfies (47) and (48).

Concerning Assumption 25, as \((\bar{x}, \bar{v})\) is bounded by nature and \((\chi, v)\) verifies (47), \((\chi, v)\) is also bounded. We recall that \( u \) is also bounded. Note that the observation functions defined by (37)-(40) are continuously differentiable. Then, keeping the notations from (22) and by boundedness of \( z \) and \( u \), we have that for any \( T > 0 \) there exists \( L > 0 \) such that, for any \( t \geq T \):
\[
\|B(t, T)\| \leq L \int_{t-T}^{t} \|\Phi_f(s; t - T, z(t - T), u)\|ds.
\]
Let us define \( f(\chi, v, u) = \begin{bmatrix} v \\ g(\chi, v, u) \end{bmatrix} \). According to Theorem 2.3.2 of [7], for any \( T > 0 \), \( t \geq T \) and \( s \in [t - T, t] \), \( \Phi_f(s; t - T, z(t - T), u) = M(s, t - T) \) is the solution of the following matrix-valued linear Cauchy problem:
\[
d_s M(s; t, t) = d_s f(\chi(s), v(s), u(s))M(s, t), \quad M(t - T, t - T) = I_6.
\]
By integrating on \([t-T, t]\) and taking the norm, one gets for any \(T > 0, t \geq T\) and \(s \in [t-T, t]:

\[
\|M(s, t-T)\| \leq \|M(t-T, t-T)\|
+ \int_{t-T}^{t} \|d_z f(\chi(s), v(s), u(s))\| \|M(s, t-T)\| ds
\]

(49)

By assumption, \(d_z f\) is continuous and the trajectories \((\chi, v, u)\) are bounded so there exists \(\sigma_1 > 0\) such that for any \(T > 0, t \geq T\) and \(s \in [t-T, t]:
\[
\|d_z f(\chi(s), v(s), u(s))\| \leq \sigma_1,
\]

leading to:
\[
\|M(s, t-T)\| \leq 1 + \sigma_1 \int_{t-T}^{t} \|M(s, t-T)\| ds,
\]

By Gronwall Lemma,
\[
\|M(s, t-T)\| \leq \exp(\sigma_1 T),
\]

Integrating again on \([t-T, t]\) yields
\[
\int_{t-T}^{t} \|M(s, t-T)\| ds \leq T \exp(\sigma_1 T),
\]

\[
\|B(t, T)\| \leq L_1 T \exp(\sigma_1 T).
\]

Thus, Assumption 25 holds. Concerning Assumption 27, from Proposition 32, for \(i \in \{1, 2, 3, 4\}\) there exist \(T(i) > 0\) and \(\mu(i) > 0\) such that, for any \(t \geq T(i):\)
\[
\bar{A}(i)(t, T(i)) \geq \mu(i) I_2.
\]

(50)

Recall the definition of \(A(i)(t, T(i))\) for any \(i \in \{1, 2, 3, 4\}\) and \(t \geq T(i):\)
\[
A(i)(t, T(i)) = \int_{t-T(i)}^{t} H(i)^T(s) H(i)(s) ds,
\]

where \(H(i)(s) = H(i)(\chi(s) - \ell, v(s))\). Moreover, since the observation functions defined by (37)-(40) are twice continuously differentiable then for any \(i \in \{1, 2, 3, 4\}\) \(H(i)^T H(i)\) is locally Lipschitz. We recall that the trajectories \(z\) and \(\bar{z}\) are bounded. Thus, there exists \(L_1 > 0\) such that for any \(i \in \{1, 2, 3, 4\}\), \(t \geq T(i)\) and \(s \in [t-T, t]:\)
\[
\|H(i)^T(s)H(i)(s) - H(i)^T(s)\bar{H}(i)(s)\| \leq L_1 \|z(s) - \bar{z}(s)\|.
\]

Therefore, from (48), one has for any \(i \in \{1, 2, 3, 4\}\) and \(t \geq T(i),\)
\[
\|A(i)(t, T(i)) - A(i)(t, T(i))\| \leq L \int_{0}^{+\infty} \|z(s) - \bar{z}(s)\| ds,
\]

\[
\|A(i)(t, T(i)) - A(i)(t, T(i))\| \leq L e, \quad (51)
\]

where \(L = TL_1\). Finally by substituting (51) in (50), one gets the following matrix inequality, for any \(i \in \{1, 2, 3, 4\}\) and \(t \geq T(i):\)
\[
A(i)(t, T(i)) \geq \lambda I_2,
\]

(52)

where \(\lambda = \mu - L e\) with \(\mu = \min(\mu(i))\). Observe that if \(\epsilon < \frac{\lambda}{2}\) then \(\lambda > 0\) and the input trajectory \(u\) satisfies Assumption 27. As mentioned at the beginning of the proof, the result follows from Proposition 28.

We would like to end this section by adding some remarks.

**Remark 34** Informally, Theorem 33 states that for a fixed initial condition, if a controlled trajectory of system (36) tracks any circular trajectory of the form (44) then the corresponding input trajectory is weakly regularly persistent at this initial state. As an immediate corollary, the result also holds if \(\int_{0}^{+\infty} \|z(s) - \bar{z}(s)\| ds = 0\), which corresponds to the case where the state trajectory \(z\) is exactly a circular path.

**Remark 35** Theorem 33 has a direct application in control design. In fact, as any circular trajectory makes the system weakly regularly observable then one does not need to know the position of \(\ell\) to ensure observability through an adequate choice of \(u\). In other words, only the level of observability, represented by \(\lambda\) in (52), depends on \(\ell\) but not the qualitative property of weak regular observability. This is to be nuanced in the case of optical flow measurement as the result does not hold if \(\|\ell - \chi_c\| = r_\epsilon\), even if this condition seems not to be satisfied generically.

**Conclusion**

In this paper, we have introduced the notion of weakly regularly persistent input along with its connection to the Observability Gramian and the existence and uniqueness of solutions to the problem of Moving Horizon Estimation. We have applied this study to the case of landmark-based SLAM both in the world-centric and sensor-centric case and provided several sufficient conditions for weak regular observability. Finally, we have shown that these conditions are simultaneously satisfied in a SLAM problem with a second order dynamics and various measurements when the robot trajectory tracks a circular path.
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A Proof of Proposition 32

We first present a lemma that allows one to find a lower bound of integrals of the form (42) that is independent of $t$. It is inspired from section 3.3 of [33]. For any $n \in \mathbb{N}^*$, we denote by $S_n$ the set of $n \times n$ symmetric matrices.

**Lemma 36** Let $n \in \mathbb{N}^*$, and $S : \mathbb{R}^+ \times \mathbb{R}^+ \to S_n$. If there exist $\nu > 0$, $S_0 > 0$ and a function $S_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to S_n$ such that for any $t \geq 0$ and $\tau \geq 0$:

\[
S(t, \tau) \geq \tau S_0 + S_1(t, \tau), \quad (A.1)
\]

\[
||S_1(t, \tau)|| \leq \nu, \quad (A.2)
\]

then there exist $T > 0$ and $\mu > 0$ such that for any $t \geq 0$:

\[
S(t, T) \geq \mu I_n. \tag{A.3}
\]

**PROOF.** Assume that there exist $\nu > 0$, $S_0 > 0$ and a function $S_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to S_n$ satisfying (A.1) and (A.2). In particular, by the definition of the operator norm and the Cauchy-Swarz inequality, one has for any $t \geq 0$, $\tau \geq 0$ and $x \in \mathbb{R}^n$:

\[
-x^T S_1(t, \tau) x \leq |x^T S_1(t, \tau) x| \leq \nu ||x||^2,
\]

leading to,

\[
S_1(t, \tau) \geq -\nu I_n.
\]

Therefore, by denoting the smallest eigenvalue of $S_0$ by $\mu_0$, one gets for any $t \geq 0$ and $\tau \geq 0$ that:

\[
S(t, \tau) \geq (\tau \mu_0 - \nu) I_n,
\]

where $\mu_0 > 0$ as $S_0 > 0$. As a consequence, by setting $T = \frac{2\nu}{\mu_0}$ and $\mu = T \mu_0 - \nu$, one obtains $\mu > 0$, $T > 0$ and $\mu I_n \geq (T \mu_0 - \nu) I_n$.
for any \( t \geq 0 \),

\[
S(t,T) \geq \mu I_n,
\]

which proves the result.

**PROOF.** [Proof of Proposition 32] The goal of the proof is to apply Lemma 36 to \( S(t,\tau) = A^{(i)}(t+\tau,\tau) \) for \( i \in \{1,2,3,4\} \). First, we fix \( r_c > 0 \), \( \omega > 0 \), \( \chi_c \in \mathbb{R}^2 \), \( \bar{\chi}_0 \in \mathbb{R}^2 \) and \( \ell \in \mathbb{R}^2 \). By defining \( \psi(s) = \arctan(\bar{\chi}_1(s) - \chi_c,1, \bar{\chi}_2(s) - \chi_c,2) \), for any \( s \geq 0 \), one can rewrite (44) as follows, for any \( \tau > 0 \), \( t \geq 0 \) and \( s \in [t,t+\tau] \):

\[
\bar{\chi}(s) = \chi_c + r_c \begin{bmatrix} \cos(\omega(s-t) + \psi(t)) \\ \sin(\omega(s-t) + \psi(t)) \end{bmatrix}, \tag{A.3}
\]

\[
\bar{v}(s) = \omega r_c \begin{bmatrix} -\sin(\omega(s-t) + \psi(t)) \\ \cos(\omega(s-t) + \psi(t)) \end{bmatrix}. \tag{A.4}
\]

The rest of the proof is decomposed into four parts, one for each type of measurements.

(1) **Bearing measurements**

A consequence of (37), \( H^{(1)} = d_{\chi} h^{(1)} \) reads, for any \( (\chi,v,\ell) \in \mathbb{R}^6 \)

\[
H^{(1)}(\chi - \ell, v) = \frac{1}{r^3} \begin{bmatrix} (-\chi_2 - \ell_2)^2 & (\chi_1 - \ell_1)(\chi_2 - \ell_2) \\ (\chi_1 - \ell_1)(\chi_2 - \ell_2) & -(\chi_1 - \ell_1)^2 \end{bmatrix}, \tag{A.5}
\]

where we recall that \( r = \| \ell - \chi \| \). After straightforward computations one gets that, for any \( \tau > 0 \) and \( t \geq 0 \):

\[
\bar{A}^{(1)}(t+\tau,\tau) = \int_{t}^{t+\tau} \frac{1}{r^3(s)} \begin{bmatrix} \bar{e}_2(s)^2 & -\bar{e}_1(s)\bar{e}_2(s) \\ -\bar{e}_1(s)\bar{e}_2(s) & \bar{e}_1(s)^2 \end{bmatrix} ds, \tag{A.6}
\]

where \( \bar{e}_1(s) = \bar{\chi}_1(s) - \ell_1, \bar{e}_2(s) = \bar{\chi}_2(s) - \ell_2 \) and \( \bar{r}(s) = \| \bar{\chi}(s) - \ell \| \).

Besides, by the triangle inequality, one has for any \( s \geq 0 \):

\[
\bar{r}(s) \leq \| \bar{\chi}(s) - \chi_c \| + \| \chi_c - \ell \|, \tag{A.7}
\]

\[
\bar{r}(s) \leq r_c + \| \chi_c - \ell \|, \tag{A.8}
\]

where \( R_c = r_c + \| \chi_c - \ell \| \).

By substituting (A.8) in (A.6), one obtains the following matrix inequality, for any \( \tau > 0 \) and \( t \geq 0 \):

\[
\bar{A}^{(1)}(t+\tau,\tau) \geq \frac{1}{R_c^3} \int_{t}^{t+\tau} \begin{bmatrix} \bar{e}_2(s)^2 & -\bar{e}_1(s)\bar{e}_2(s) \\ -\bar{e}_1(s)\bar{e}_2(s) & \bar{e}_1(s)^2 \end{bmatrix} ds,
\]

\[
\geq \frac{1}{R_c^3} \begin{bmatrix} I_1 & I_3 \\ I_3 & I_2 \end{bmatrix}, \tag{A.9}
\]

where:

\[
I_1 = \int_{t}^{t+\tau} \bar{e}_2(s)^2 ds, \quad I_2 = \int_{t}^{t+\tau} \bar{e}_1(s)^2 ds, \tag{A.10}
\]

\[
I_3 = \int_{t}^{t+\tau} -\bar{e}_1(s)\bar{e}_2(s) ds.
\]

Observe that \( I_1 \) can be expanded as follows from (A.3) and (A.4):

\[
I_1 = \int_{t}^{t+\tau} (e_{c,2} + r_c \sin(\omega(s-t) + \psi(t)))^2 ds,
\]

\[
= \tau e_{c,2}^2 + 2 e_{c,2} r_c \int_{t}^{t+\tau} \sin(\omega(s-t) + \psi(t)) ds
\]

\[
+ r_c^2 \int_{t}^{t+\tau} \sin^2(\omega(s-t) + \psi(t)) ds,
\]

where \( e_{c,2} = \chi_c,2 - \ell_2 \). It follows that:

\[
I_1 = \tau e_{c,2}^2 + 2 e_{c,2} r_c \int_{t}^{t+\tau} \sin(\omega(s-t) + \psi(t)) ds
\]

\[
+ r_c^2 \frac{1}{2} \left( \tau - \frac{1}{2\omega} (\sin(2\omega\tau + 2\psi(t)) - \sin(2\bar{\psi}(t))) \right). \tag{A.11}
\]

One can get a similar expression for \( I_2 \):

\[
I_2 = \tau e_{c,1}^2 + 2 e_{c,1} r_c \int_{t}^{t+\tau} \sin(\omega(s-t) + \psi(t)) ds
\]

\[
+ r_c^2 \frac{1}{2} \left( \tau - \frac{1}{2\omega} (\sin(2\omega\tau + 2\psi(t)) - \sin(2\bar{\psi}(t))) \right), \tag{A.12}
\]

where \( e_{c,1} = \chi_c,1 - \ell_1 \). By expanding \( I_3 \),

\[
I_3 = - \tau e_{c,1} e_{c,2} - 2 e_{c,1} r_c \int_{t}^{t+\tau} \sin(\omega(s-t) + \psi(t)) ds
\]

\[
- 2 e_{c,2} r_c \int_{t}^{t+\tau} \cos(\omega(s-t) + \psi(t)) ds
\]

\[
- r_c^2 \int_{t}^{t+\tau} \cos(\omega(s-t) + \psi(t)) \sin(\omega(s-t) + \psi(t)) ds,
\]
where:

$$\mathbf{A}(t+\tau)\geq\tau\mathbf{S}_0(t) + \mathbf{S}_1(t,\tau),$$

(A.14)

By further combining (A.11)-(A.13) with (A.9), one gets for any $t \geq 0$ and $\tau > 0$ that

$$\mathbf{A}(t+\tau)\geq\tau\mathbf{S}_0(t) + \mathbf{S}_1(t,\tau),$$

(A.14)

where:

$$\mathbf{S}_0 = \frac{r_c^2}{2R_c^2}\mathbf{I}_2 + \frac{1}{R_c^2}\begin{bmatrix} e_{c,2}^2 & -e_{c,1}e_{c,2} \\ -e_{c,1}e_{c,2} & e_{c,1}^2 \end{bmatrix} > 0.$$}

and $\mathbf{S}_1(t,\tau) \in S_2$ is a linear combination of sines and cosines. In particular, $\mathbf{S}_1(t,\tau)$ is bounded with respect to $t$ and $\tau$. Finally, from Lemma 36, there exist $\mu(1) > 0$ and $T(1) > 0$ such that for any $t \geq 0$,

$$\mathbf{A}(t+T(1),T(1)) \geq \mu(1)\mathbf{I}_2.$$}

By noticing that for any $\tau > 0$ and any $t \geq \tau$, $\mathbf{A}(t+\tau) = \mathbf{A}(t,\tau)$, one gets that for any $t \geq T(1)$:

$$\mathbf{A}(t,T(1)) \geq \mu(1)\mathbf{I}_2.$$}

(2) **Range measurements**

After standard computations from (38), one obtains that, for any $(\chi,v,\ell) \in \mathbb{R}^6$:

$$H^{(2)}(\chi - \ell,v) = \frac{1}{r}(\ell - \chi)^T.$$}

Therefore, for any $t \geq 0$ and $\tau > 0$:

$$\mathbf{A}^{(2)}(t+\tau) = \int_t^{t+\tau} \frac{1}{r^2(s)}\begin{bmatrix} e_{c,2}^2(s) & e_{c,1}(s)e_{c,2}(s) \\ e_{c,1}(s)e_{c,2}(s) & e_{c,1}^2(s) \end{bmatrix}ds.$$}

(A.16)

From (A.8) and (A.16),

$$\mathbf{A}^{(2)}(t+\tau) \geq \frac{1}{R_c^2}\begin{bmatrix} \mathbf{I}_2 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_1 \end{bmatrix}.$$}

(A.17)

By combining (A.11)-(A.13) with (A.17) as in the bearing case, one gets that, for any $t \geq 0$ and $\tau > 0$:

$$\mathbf{A}^{(2)}(t+\tau) \geq \tau \mathbf{S}_0^{(2)} + \mathbf{S}_1^{(2)}(t,\tau),$$

where:

$$\mathbf{S}_0^{(2)} = \frac{r_c^2}{2R_c^2}\mathbf{I}_2 + \frac{1}{R_c^2}\begin{bmatrix} e_{c,1}^2 & e_{c,1}e_{c,2} \\ e_{c,1}e_{c,2} & e_{c,2}^2 \end{bmatrix} > 0,$$

and $\mathbf{S}_1^{(2)}(t,\tau) \in S_2$ is also bounded with respect to $t$ and $\tau$. Finally, from Lemma 36, there exist $\mu(2) > 0$ and $T(2) > 0$ such that for any $t \geq T(2)$,

$$\mathbf{A}^{(2)}(t,T(2)) \geq \mu(2)\mathbf{I}_2.$$}

(3) **Optical flow measurements**

From (39), one can show that, for any $(\chi,v,\ell) \in \mathbb{R}^6$ and by setting $e = \chi - \ell$:

$$H^{(3)}(e,v) = \frac{1}{r^T} \begin{bmatrix} -v_2e_2 + 2v_1e_1 \langle e_1e_2 \rangle \\ v_1e_1^2 + 2v_2e_1e_2 - v_1e_1e_2 \end{bmatrix}.$$}

(A.20)

By defining the $q$ and $q^\perp$ as follows

$$q = \begin{bmatrix} e_1^2 - e_2^2 \\ 2e_1e_2 \end{bmatrix},$$

$$q^\perp = Qq,$$

one can rewrite (A.20) so that:

$$H^{(3)}(e,v) = \frac{1}{r^T} \begin{bmatrix} \langle v,q^\perp \rangle & \langle v,q \rangle \end{bmatrix}.$$}

(A.21)

Consequently, from (A.21) one gets that for any $t \geq 0$ and $\tau > 0$:

$$\mathbf{A}^{(3)}(t+\tau) = \int_t^{t+\tau} \frac{1}{r(s)^8} \begin{bmatrix} \langle \ddot{v}(s),q^\perp(s) \rangle^2 & \langle \ddot{v}(s),q^\perp(s) \rangle \langle \ddot{v}(s),q(s) \rangle \\ \langle \ddot{v}(s),q^\perp(s) \rangle & \langle \ddot{v}(s),q(s) \rangle \end{bmatrix}ds.$$}

(A.22)

After introducing (A.8), one gets:

$$\mathbf{A}^{(3)}(t+\tau) \geq \cdots$$

(A.23)
\[
\frac{1}{R^6} \int_t^{t+\tau} \left[ \frac{\langle \bar{v}(s), \bar{q}_1(s) \rangle^2}{\langle \bar{v}(s), \bar{q}_1(s) \rangle} \frac{\langle \bar{v}(s), \bar{q}_1(s) \rangle \langle \bar{v}(s), \bar{q}(s) \rangle}{\langle \bar{v}(s), \bar{q}(s) \rangle^2} \right] ds,
\]
where \( \bar{q}(s) = \left[ \bar{e}_1^2(s) - \bar{e}_2^2(s) \right]. \) Further development of (A.24) leads to a large expression that could not fit in this proof. To be able to assess a lower bound of \( \tilde{\mathcal{A}}^{(3)}(t + \tau, \tau) \) in the spirit of previous computations, we used MATLAB’s sym toolbox as presented in [15]. We were able to find an explicit expression of the right-hand side of (A.24). After an eigenvalue computation, one can find the following lower bound, for any \( t \geq 0 \) and \( \tau > 0 \):

\[
\tilde{\mathcal{A}}^{(3)}(t + \tau, \tau) \geq \tau S_0^{(3)} + S_1^{(3)}(t, \tau),
\]
where

\[
S_0^{(3)} = \frac{\omega^2}{2R^6} \min(\|\chi_c - \ell\|^4 + 10r_c^2\|\chi_c - \ell\|^2 + r_c^4, (\|\chi_c - \ell\|^2 - r_c^2)^2)I_2 \quad \text{and} \quad S_1^{(3)}(t, \tau) \text{ is bounded with respect to } t \text{ and } \tau \text{ as a linear combination of sines and cosines. If } r_c \neq \|\chi_c - \ell\| \text{ then } S_0^{(3)} > 0 \text{ and from Lemma 36 there exists } \mu^{(3)} > 0 \text{ and } T^{(3)} > 0 \text{ such that for any } t \geq T^{(3)},
\]

\[
\tilde{\mathcal{A}}^{(3)}(t, T^{(3)}) \geq \mu^{(3)} I_2.
\]

(4) Doppler drift measurements

From (40), one can show that, for any \( (\chi, v, \ell) \in \mathbb{R}^6 \) with \( e = \chi - \ell \):

\[
H^{(4)}(e, v) = \frac{\alpha}{v^3} \left[ -v_1 e_2^2 + v_2 e_1 e_2 \\ -v_2 e_1^2 + v_1 e_1 e_2 \right]^T.
\]

From (A.8) and using MATLAB’s sym toolbox with steps presented in [15], one can find the following lower bound for any \( t \geq 0 \) and \( \tau > 0 \):

\[
\tilde{\mathcal{A}}^{(4)}(t + \tau, \tau) \geq \tau S_0^{(4)} + S_1^{(4)}(t, \tau),
\]
where

\[
S_0^{(4)} = \frac{(\omega r_c)^2}{8R^6} \min(r_c^2 (4r_c^2 + \|\chi_c - \ell\|^2), (4r_c^2 + \|\chi_c - \ell\|^2)^2 + 19r_c^2\|\chi_c - \ell\|^2)I_2 \quad \text{and} \quad S_1^{(4)}(t, \tau) \text{ is bounded with respect to } t \text{ and } \tau. \text{ Note that } S_0^{(4)} > 0 \text{ as } r_c > 0, \alpha > 0 \text{ and } \omega > 0. \text{ Therefore, from Lemma 36, there exists } \mu^{(4)} > 0 \text{ and } T^{(4)} > 0 \text{ such that for any } t \geq T^{(4)},
\]

\[
\tilde{\mathcal{A}}^{(4)}(t, T^{(4)}) \geq \mu^{(4)} I_2.
\]

Finally, the result follows from (A.15), (A.19), (A.25) and (A.26).