Limit Behaviour of a Singular Perturbation Problem for the Biharmonic Operator

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Abstract
We study here a singular perturbation problem of biLaplacian type, which can be seen as the biharmonic counterpart of classical combustion models. We provide different results, that include the convergence to a free boundary problem driven by a biharmonic operator, as introduced in Dipierro et al. (arXiv:1808.07696, 2018), and a monotonicity formula in the plane. For the latter result, an important tool is provided by an integral identity that is satisfied by solutions of the singular perturbation problem. We also investigate the quadratic behaviour of solutions near the zero level set, at least for small values of the perturbation parameter. Some counterexamples to the uniform regularity are also provided if one does not impose some structural assumptions on the forcing term.

Keywords
Biharmonic operator · Singular perturbation problems · Monotonicity formula

Mathematics Subject Classification 31A30 · 31B30 · 35R35

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1 Introduction

In this article we study bounded solutions \( \{u^\varepsilon\}_{\varepsilon > 0} \) of the singularly perturbed biLaplacian equation

\[
2\Delta^2 u^\varepsilon = -\beta^\varepsilon (u^\varepsilon) \quad \text{in} \quad \Omega, \tag{1.1}
\]

where \( \varepsilon \in (0, 1] \) is a small parameter, \( \Omega \) is a smooth and bounded domain of \( \mathbb{R}^n \),

\[
\beta^\varepsilon(t) := \frac{1}{\varepsilon} \beta \left( \frac{t}{\varepsilon} \right), \tag{1.2}
\]

and \( \beta \) is a smooth, nonnegative function, with support contained in \([0, 1]\) and such that

\[
\int_{\mathbb{R}} \beta(t) \, dt = \int_0^1 \beta(t) \, dt = 1. \tag{1.3}
\]

Equation (1.1) can be seen as the biharmonic counterpart of classical combustion models, see e.g. [18]. We observe that the problem in (1.1) is variational, and indeed solutions of (1.1) are critical points of the functional

\[
J^\varepsilon[v] := \int_{\Omega} |\Delta v(x)|^2 + B^\varepsilon(v(x)) \, dx, \tag{1.4}
\]

where

\[
B^\varepsilon(v) := \int_0^v \beta^\varepsilon(t) \, dt. \tag{1.5}
\]

The factor 2 in Eq. (1.1) has been placed exactly to avoid additional factors \( 1/2 \) in the energy functional (1.4) (and thus to make the comparison with the existing literature more transparent). As a special example, one can consider minimizers of \( J^\varepsilon \) with respect to Navier boundary conditions, that is, given \( u_0 \in W^{2,2}(\Omega) \), one can minimize \( J^\varepsilon \) among the set of competitors given by

\[
\mathcal{A} := \left\{ u \in W^{2,2}(\Omega) \text{ s.t. } u - u_0 \in W^{1,2}_0(\Omega) \right\}.
\]

Then, minimizers of (1.4) are taken in the class \( \mathcal{A} \) and they are solutions of (1.1) with boundary data \( u = u_0 \) and \( \Delta u = 0 \) along \( \partial\Omega \). See for instance the “hinged problem” on the right hand side of Fig. 1a and on page 84 of [20], or Figure 1.5 on page 6 of [9], or the monograph [10] for further information of this type of boundary conditions.

The existence of minimizers of the functional in (1.4) in the class \( \mathcal{A} \) is obtained by the direct methods in the calculus of variations, see Lemma 2.1 in [8].

Some motivations for investigating equations involving the biharmonic operator come from classical models for rigidity problems, which have concrete applications,
for example, in the construction of suspension bridges, see e.g. [15] and the references therein. See also formula (1) in [13] and the references therein for other classical applications of the biharmonic operator in the study of steady state incompressible fluid flows at small Reynolds numbers under the Stokes flow approximation assumption. In our framework, we will present a simple game-theoretical model for the problem in (1.1) in Sect. 2.

The minimizers of $J_\varepsilon$ enjoy suitable regularity and compactness properties, and they are related to a free boundary problem of biharmonic type which has been recently investigated in [8]. To formalize this, we consider the functional

$$J[v] := \int_{\Omega} |\Delta v(x)|^2 + \chi_{(0,+\infty)}(v(x)) \, dx.$$  \hspace{1cm} (1.6)

Though free boundary problems are by now a classical topic of investigation (see [1]), the setting of higher order operators provides only few results available, and the analysis of the free boundary problem in (1.6) has been only recently initiated in [8] (see also [14] where other types of free boundary problems for higher order operators have been considered). Furthermore, obstacle problems involving biharmonic operators have been studied in [2–4,16,17,19].

In this setting, one can relate minimizers of the functional $J_\varepsilon$ in (1.4) with minimizers of the free boundary problem in (1.6), according to the following convergence result:

**Theorem 1.1** Let $\{u_\varepsilon\}$ be a family of minimizers of the functional $J_\varepsilon$, as defined in (1.4), with

$$\sup_{\varepsilon \in (0,1]} \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty.$$  

Then, as $\varepsilon \to 0^+$, up to a subsequence,

- $u_\varepsilon \to u$ locally uniformly in $C^{1,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0,1)$,
- $u_\varepsilon \rightharpoonup u$ in $W^{2,p}_{\text{loc}}(\Omega)$, for every $p > 2$,
- $\Delta u_\varepsilon \to \Delta u$ in $BMO$,
- $u$ is a minimizer of the functional $J$, as defined in (1.6).

We observe that solutions of (1.1), and in particular minimizers of $J_\varepsilon$, naturally develop a notion of limit free boundary. Indeed, if $u_\varepsilon$ is a minimizer of $J_\varepsilon$ which approaches $u$ as $\varepsilon \to 0^+$, one is interested in the geometric properties of the set $\partial \{u_\varepsilon > 0\}$. To analyze and classify this type of sets, it would be extremely desirable to have suitable monotonicity formulas. Differently from the classical case in which the equation is of second order (i.e. the energy functional is induced by the classical Dirichlet form, see [1]), in our setting no general result of this type is available in the literature.

In our framework, we will obtain a monotonicity formula, relying on the following integral equation for solutions of (1.1):
Lemma 1.2 Let $u^\varepsilon$ be a solution of (1.1). Then, for any $\phi = (\phi^1, \ldots, \phi^n) \in C^\infty_0(\Omega, \mathbb{R}^n)$,

\[
2 \int_\Omega \left( 2 \text{tr}(D^2u^\varepsilon(y) \, D\phi(y)) + \nabla u^\varepsilon(y) \cdot \Delta \phi(y) \right) \Delta u^\varepsilon(y) \, dy \\
= \int_\Omega \text{div}(\phi(y)) \left( |\Delta u^\varepsilon(y)|^2 + B_\varepsilon(u^\varepsilon(y)) \right) \, dy.
\] (1.7)

With this, the argument leading to the monotonicity formula is based on the choice of a test function $\phi$ in (1.7) with a particular form, see [22]. More precisely, we focus on the two-dimensional case and we prove the following

**Theorem 1.3** Let $n = 2$ and $\tau > 0$ such that $B_\tau \Subset \Omega$. Let $u^\varepsilon$ be a solution of (1.1), with

\[
u^\varepsilon(0) = 0 \quad \text{and} \quad \nabla u^\varepsilon(0) = 0.
\] (1.8)

Then, there exists a function $E^\varepsilon : (0, \tau) \to \mathbb{R}$, which is bounded in $(0, \tau)$, nondecreasing and such that, for any $\tau_2 > \tau_1 > 0$,

\[
E^\varepsilon(\tau_2) - E^\varepsilon(\tau_1) = \int_{\tau_1}^{\tau_2} \left\{ \frac{1}{r^2} \int_{\partial B_r} \left[ \left( \frac{u^\varepsilon_{\theta r}}{r^2} - \frac{2u^\varepsilon}{r^2} \right)^2 + \left( u^\varepsilon_{rr} - \frac{3u^\varepsilon}{r} + \frac{4u^\varepsilon}{r^2} \right)^2 \right] \right\}.
\] (1.9)

Theorem 1.3 can be also made more precise, since the function $E^\varepsilon$ is given explicitly by

\[
E^\varepsilon(r) = \int_{\partial B_r} \left( \frac{\Delta u^\varepsilon}{2r^2} - \frac{5(u^\varepsilon)^2}{2r^3} - \frac{\Delta u^\varepsilon u^\varepsilon}{r^3} + \frac{6u^\varepsilon u^\varepsilon}{r^4} + \frac{u^\varepsilon_{\theta r} u^\varepsilon_{\theta r}}{r^4} - \frac{4(u^\varepsilon)^2}{r^5} - \frac{3(u^\varepsilon)^2}{2r^5} \right) \\
+ \frac{1}{4r^2} \int_{B_r} (|\Delta u^\varepsilon|^2 + B_\varepsilon(u^\varepsilon)) + \int_0^r \frac{1}{\rho^3} \int_{B_\rho} \beta_\varepsilon(u^\varepsilon) \, u^\varepsilon.
\] (1.10)

The proof of Theorem 1.3 is based on a series of careful integration by parts aimed at spotting suitable integral cancellations, which are possible in dimension 2. In addition, some “high order of differentiability” terms naturally appear in the computations, which need to be suitably removed in order to rigorously make sense of the formal manipulations.

In light of Theorem 1.3, one can pass to the limit and obtain a monotonicity formula for weak solutions of the limit free boundary problem in (1.6). This result extends the monotonicity formula found in [8] for the case of minimizers to the more general setting of weak solutions. To this end, we introduce the following setting.

**Definition 1.4** A function $u \in W^{2,2}(\Omega)$ is said to be a weak solution of the free boundary problem in (1.6) if
$$\Delta^2 u = 0 \text{ in } \{u > 0\} \cup \{u < 0\},$$

(1.7) holds,

and \(\partial \{u > 0\}\) is locally rectifiable, i.e. \(\partial \{u > 0\} = M_0 \cup \left( \bigcup_{k=1}^{\infty} M_k \right)\),

where \(M_k, k \geq 1\) are \(C^1\) hypersurfaces and \(\mathcal{H}^{n-1}(M_0) = 0\).

To formulate next result, we also let

\(u^{\varepsilon_j}\) be a sequence of solutions of (1.1) satisfying (1.8), with \(\varepsilon_j \downarrow 0\) as \(j \to +\infty\),

and \(u\) be a limit of \(u^{\varepsilon_j}\) such that \(u^{\varepsilon_j} \to u \geq 0\) uniformly, (1.11)

and we define

$$E(r) := \int_{\partial B_r} \left( \frac{\Delta u u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6u u_r}{r^4} + \frac{u_u u_{rr}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_r^2}{2r^5} \right)$$

$$+ \frac{1}{4r^2} \int_{B_r} (|\Delta u|^2 + \hat{B}),$$

(1.12)

where \(\hat{B}\) is the weak star limit of \(B_\varepsilon(u^\varepsilon)\). In this setting, we have the following monotonicity formula:

**Theorem 1.5** Let \(n = 2\) and \(u\) be a weak solution satisfying (1.11). Suppose that

$$\lim_{\varepsilon_j \to 0} |\{0 < u^{\varepsilon_j} \leq \varepsilon_j\} \cap B| = 0$$ \hspace{1cm} (1.13)

for every ball \(B \in \Omega\) and

$$\|\Delta u^{\varepsilon_j}\|_{L^\infty(B_r)} \leq C,$$ \hspace{1cm} (1.14)

for some \(C > 0\) independent of \(j\).

Then,

$$\Delta u^{\varepsilon_j} \to \Delta u \text{ strongly in } L^2_{\text{loc}}(B_r).$$ \hspace{1cm} (1.15)

In particular,

the energy identity in (1.7) holds for \(u\),

(1.16)

with \(B_\varepsilon\) replaced by \(\hat{B}\), the weak star limit of \(B_\varepsilon(u^\varepsilon)\).

Furthermore, if

$$|D^2 u(x)| \leq C \quad \text{for any } x \in B_1,$$ \hspace{1cm} (1.17)

then
for almost every $t \in (0, \tau)$, the function $E$ is well defined and nondecreasing.

(1.18)

Moreover, if $\tau_2 > \tau_1 > 0$ and $B_{\tau_2} \subseteq \Omega$, then

$$E(\tau_2) - E(\tau_1) = \int_{\tau_1}^{\tau_2} \frac{1}{r^2} \int_{\partial B_r} \left[ \left( \frac{u_{\theta r}}{r} - \frac{2u_r}{r^2} \right)^2 + \left( \frac{u_{rr} - 3u_r}{r} + 4 \frac{u}{r^2} \right)^2 \right] \, \, \, (1.19)$$

In addition,

if $E$ is constant in $(0, \tau)$, then the function $-\frac{u_r}{r} + 2\frac{u}{r^2}$ is constant in $B_\tau$,

and moreover $u$ is a homogeneous function of degree two in $B_\tau$.

(1.20)

Finally, for every sequence $r_k \searrow 0$ there exists a subsequence $r_{k,j}$ such that

the scaled functions $\frac{u(r_{k,j}, x)}{r_{k,j}^2}$ either converge
to zero or to a homogeneous function of degree two.

(1.21)

We point out that condition (1.17) ensures that $E$ remains bounded as $r \to 0^+$.

It is interesting to detect the quadratic behaviour of solutions of (1.1) near the zero level set, at least for small values of $\varepsilon$. To this end, we provide this limit result, valid in any dimension:

**Theorem 1.6** Let $u^\varepsilon$ be a sequence of solutions to (1.1) in $\Omega$. Let $0 \in \Omega$, and $\alpha, \gamma \in \mathbb{R}$. Suppose that

$$u^\varepsilon \text{ converge to } u := \frac{\alpha}{2} (x_1)_+^2 + \frac{\gamma}{2} (x_1)_-^2, \text{ as } \varepsilon \to 0^+, \text{ up to a subsequence, in } W^{2,2}_{\text{loc}}(\Omega).$$

(1.22)

Then:

- If $\alpha, \gamma > 0$, we have that

$$\alpha = \gamma. \quad (1.23)$$

- If $\alpha, \gamma < 0$, we have that

$$\alpha = \gamma. \quad (1.24)$$

- If $\alpha > 0, \gamma \leq 0$, we have that

$$\alpha^2 - \gamma^2 = 1. \quad (1.25)$$
Moreover,

\[
\text{the case } \alpha < 0, \gamma = 0 \text{ cannot hold.} \tag{1.26}
\]

We also observe that, in general, one cannot expect uniform second derivative bounds on solutions of (1.1) without any additional structure (not even in low dimension). For this, we provide the following one-dimensional counterexample, where the forcing term \( \beta_\varepsilon \) satisfies (1.3), but does not fulfill the structural assumption in (1.2):

**Theorem 1.7** There exists \( \delta > 0 \) such that for all \( \varepsilon > 0 \) sufficiently small there exist \( \beta_\varepsilon \in C_0^\infty([0, \varepsilon], [0, +\infty)) \), such that

\[
\int_{\mathbb{R}} \beta_\varepsilon(t) \, dt = 1,
\]

and a solution \( u_\varepsilon \) of (1.1) in \((-\delta, \delta)\), with

\[
\lim_{\varepsilon \to 0^+} \|u_\varepsilon''\|_{L^\infty((-\delta, \delta))} = +\infty.
\]

We also point out the following example of smooth solutions of equations like (1.1), which are uniformly small but do not possess uniform first derivative bounds. In this example, the forcing term \( \beta_\varepsilon \) satisfies the scaling properties in (1.2), but \( \beta \) does not satisfy the structural assumptions.

**Theorem 1.8** There exists \( \beta \in C^\infty(\mathbb{R}, [0, +\infty)) \) such that \( \beta = 0 \) in \((-\infty, 0]\) for which the following statement holds true.

For every \( \varepsilon_0 \in (0, 1] \) there exist \( \varepsilon \in (0, \varepsilon_0] \) and \( u_\varepsilon \in C^\infty(\mathbb{R}) \) such that

\[
2(u_\varepsilon''')' = -\beta_\varepsilon(u) \text{ in } \mathbb{R}, \tag{1.27}
\]
\[
u_\varepsilon = 0 \text{ in } (-\infty, 0], \tag{1.28}
\]
\[
\sup_{x \in \mathbb{R}} |u_\varepsilon'(x)| \leq \varepsilon, \tag{1.29}
\]
\[
\sup_{\varepsilon \in (0, 1]} |(u_\varepsilon)'(1)| = +\infty. \tag{1.30}
\]

Here above, \( \beta_\varepsilon \) is as in (1.2).

The paper is organized as follows. In Sect. 2 we provide a simple motivation for (1.1) based on a game-theoretic model. Section 3 contains the proof of the convergence result in Theorem 1.1.

Section 4 is devoted to the proof of the integral identity stated in Lemma 1.2. Suitable choices of the test function in (1.7) provide the cornerstone to prove Theorem 1.3 in Sect. 5. Section 6 is devoted to the proof of Theorem 1.5. Then, Theorem 1.6 is proved in Sect. 7.

Section 8 contains the counterexamples to the uniform \( C^{1,1} \) bounds stated in Theorems 1.7 and 1.8.

The paper ends with an appendix which provides some decay estimates for the gradient and the Hessian of solutions of (1.1).
2 Motivations: A Simple Game-Theoretic Model for (1.1)

We point out that there is a simple interpretation of (1.1) which comes from game theory and which can somehow favor the intuition of the problem.

Let us run a Gaussian stochastic process in a Cartesian lattice (say, a random walk) of small step scale $h$. The process starts at some point in a given domain $\Omega$ and there is a prize $u_0$ assigned at the boundary. Let us also suppose that there is a penalization function $v = v(x, t)$ which makes the player pay something till it exits the domain $\Omega$ (of course, the “prize” $u_0$ can also attain negative values, and the penalization $v$ can also attain positive values, hence the game can also penalize exits and compensate for remaining in the domain). More precisely, if the process exits the domain at a point $x \in \partial \Omega$, then the player obtains an award $u_0(x)$; in addition, if the player exits at time $T$ by following a trajectory $x : [0, T) \to \Omega$, it has to pay a fee quantified by

$$
\int_0^T v(x(\theta), \theta) d\theta.
$$

A natural question in this model is: assuming that the time step $\tau$ in which the random walk takes place is quadratic with respect to the spacial scale, i.e.

$$
\tau = h^2, \quad (2.1)
$$

and $u = u(x, t)$ denotes the expected value to win for a player situated at a point $x \in (h \mathbb{Z}^n) \cap \Omega$ at time $t \in \tau \mathbb{N}$, how does one describe $u$ with a good approximation?

For this, we give a heuristic, but hopefully convincing argument, not indulging in rigorous convergence details (see e.g. [21] for related discussions). First of all, one can consider that the expected winning value for a player situated at point $x$ at time $t + \tau$ is equal to the expected winning values for a player at time $t$ who is situated at points reachable by the random walk in one iteration (that is, $x \pm h e$, with $e$ being an element of the Euclidean basis $\{e_1, \ldots, e_n\}$), weighted by the probability that such jumping occurred (that is $1/2n$, since the process can go in each coordinate direction), plus the running cost prescribed by the penalization $v$, that is,

$$
\int_t^{t+\tau} v(x(\theta), \theta) d\theta = v(x, t) \tau + o(\tau),
$$

assuming $\tau$ small enough. To write this concept in a formula, assuming also $u$ sufficiently smooth, we have that

$$
u(x, t + \tau) \quad 1
$$
$$
= \frac{1}{2n} \sum_{i=1}^n \left( u(x + he_i, t) + u(x - he_i, t) \right) - v(x, t) \tau + o(\tau)
$$
$$
= \frac{1}{2n} \sum_{i=1}^n \left( u(x, t) + h \nabla u(x, t) \cdot e_i + \frac{h^2}{2} D^2 u(x, t) e_i \cdot e_i + u(x, t) - h \nabla u(x, t) \cdot e_i u(x, t) \cdot e_i
$$

$\square$ Springer
\[ + \frac{h^2}{2} D^2 u(x, t) e_i \cdot e_i - v(x, t) \tau + o(\tau) + o(h^2) \]
\[ = \frac{1}{2n} \sum_{i=1}^{n} \left( 2u(x, t) + h^2 D^2 u(x, t) e_i \cdot e_i \right) - v(x, t) \tau + o(\tau) + o(h^2) \]
\[ = u(x, t) + \frac{1}{2n} h^2 \sum_{i=1}^{n} D^2 u(x, t) e_i \cdot e_i - v(x, t) \tau + o(\tau) + o(h^2). \]

Hence, recalling (2.1),
\[ \frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{1}{2n} \sum_{i=1}^{n} D^2 u(x, t) e_i \cdot e_i - v(x, t) + o(1) \]
\[ = \frac{1}{2n} \sum_{i=1}^{n} \Delta u(x, t) - v(x, t) + o(1), \]
that is, in the limit as \( \tau \to 0^+ \),
\[ \begin{cases} 
\partial_t u(x, t) = \Delta u(x, t) - v(x, t) & \text{if } x \in \Omega \text{ and } t > 0, \\
u(x, t) = u_0(x) & \text{if } x \in \partial\Omega. 
\end{cases} \tag{2.2} \]

Then, one can also consider the case in which the domain penalization fee \( v \) is not deterministic but it also depends on a stochastic process. For instance, one can prescribe \( v \) to vanish on the boundary of \( \Omega \) and to evolve with a random walk in \( \Omega \), which in addition receives an additional increment of size \( c \) if it travels in a region of the domain on which \( u \) changes its sign (like an “interface prize”). This would lead to an equation of the type
\[ \partial_t v = \Delta v + c \mathcal{H}^{n-1} \bigg|_{\partial\{u > 0\}}, \tag{2.3} \]
where the latter can be seen as a \( (n-1) \)-dimensional measure sitting on the interface.

To avoid such a singular measure, one can replace it with a mollified version induced by the function \( \beta_\varepsilon \) in (1.2), since this function charges \( O(\varepsilon^{-1}) \) the regions in which the values of \( u \) range in \((0, \varepsilon)\). In this way, and taking \( c := 1/2 \) for simplicity, one replaces the singular equation in (2.3) by a regularized version, thus obtaining
\[ \begin{cases} 
\partial_t v(x, t) = \Delta v(x, t) + \beta_\varepsilon \left( \frac{u(x, t)}{2} \right) & \text{if } x \in \Omega \text{ and } t > 0, \\
v(x, t) = 0 & \text{if } x \in \partial\Omega. \quad \tag{2.4} 
\end{cases} \]

Of course, the stationary solutions of (2.2) and (2.4) are of particular interest and they lead to the system of equations.
\[
\begin{cases}
\Delta u(x) = v(x) & \text{if } x \in \Omega, \\
\Delta v(x) = -\frac{\beta_\varepsilon(u(x))}{2} & \text{if } x \in \Omega, \\
u(x) = u_0(x) & \text{if } x \in \partial \Omega, \\
v(x) = 0 & \text{if } x \in \partial \Omega.
\end{cases}
\] 

(2.5)

Substituting \( v \) inside the equations in (2.5), one obtains for \( u = u^\varepsilon \) the equation in (1.1), which is the main object of investigation of our paper, with Navier boundary conditions.

### 3 Convergence Properties: Proof of Theorem 1.1

In this section we will study the minimizers \( u^\varepsilon \) of the functional in (1.4). Recalling (1.2) and (1.5), we also define

\[
B(v) := \int_0^v \beta(t) \, dt,
\]

and we observe that

\[
B_\varepsilon(v) = B\left(\frac{v}{\varepsilon}\right).
\]

In particular, recalling (1.3), we have that, for any \( x \in \Omega \),

\[
B_\varepsilon(v(x)) = \begin{cases} 
1 & \text{if } v(x) > \varepsilon, \\
\int_0^{v(x)/\varepsilon} \beta(\tau) \, d\tau & \text{if } v(x) \leq \varepsilon.
\end{cases}
\]

(3.1)

Hence

\[
0 \leq B_\varepsilon(v) \leq 1,
\]

(3.2)

which says that the functions \( B_\varepsilon \) are uniformly bounded in \( \varepsilon \). From this, one can repeat the proof of Theorem 1.1 in [8] (see also [6,7]) and obtain that

\[
\Delta u^\varepsilon \in BMO_{\text{loc}}, \text{ uniformly in } \varepsilon.
\]

In particular, we find that

\[
u^\varepsilon \in W^{2,p}_{\text{loc}}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega) \text{ for every } \alpha \in (0, 1) \text{ and } p \in [1, +\infty), \text{ uniformly in } \varepsilon.
\]

(3.3)
Moreover, from (1.1), it follows that \( u^\varepsilon \) is locally \( C^\infty \) in \( \Omega \), with bounds which in general depend on \( \varepsilon \).

We stress that estimates that are uniform in \( \varepsilon \), as the ones in (3.3), are special, they depend on the structure of the problem taken into account, and they cannot follow from standard elliptic regularity theory (see [11]), as pointed out in Theorem 1.8.

Now, we want to study the behaviour of the minimizer \( u^\varepsilon \) as \( \varepsilon \to 0 \). We start with the following preliminary convergence result:

**Lemma 3.1** For every \( v \in \mathcal{A} \) we have that

\[
\lim_{\varepsilon \to 0} J_\varepsilon[v] = J[v].
\]

**Proof** Recalling the definition of \( B_\varepsilon \) in (1.5), we see that

\[
J[v] - J_\varepsilon[v] = \int_\Omega \left( |\Delta v|^2 + \chi_{\{ v > 0 \}} \right) - \int_\Omega \left( |\Delta v|^2 + B_\varepsilon(v) \right)
= \int_\Omega \left( \chi_{\{ v > 0 \}} - B_\varepsilon(v) \right)
= \int_{\{ 0 < v < \varepsilon \} \cap \Omega} \left( \chi_{\{ v > 0 \}} - B_\varepsilon(v) \right).
\]

(3.4)

Observe that

\[
0 \leq \chi_{\{ v > 0 \}} - B_\varepsilon(v) \leq 1.
\]

Using this observation together with (3.4), we conclude that

\[
0 \leq J[v] - J_\varepsilon[v] \leq \left| \{ 0 < v < \varepsilon \} \cap \Omega \right|.
\]

Hence, to complete the proof, it remains to show that

\[
\left| \{ 0 < v < \varepsilon \} \cap \Omega \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

(3.5)

For this, let \( v \in \mathcal{A} \). Then \( v \) is quasicontinuous, i.e. for every \( \sigma > 0 \) small there exists a compact set \( E_0 \) such that \( v \) is continuous on \( \Omega \setminus E_0 \) and \( \text{cap}_2(E_0) < \sigma \) (see e.g. [12]).

Let \( E := \{ x \in \Omega \setminus E_0 : v(x) > 0 \} \). Then \( E \) is bounded and open. Moreover, we have that

\[
\left| \{ 0 < v < \varepsilon \} \cap \Omega \right| = \int_E \left( \chi_{\{ v > 0 \}} - \chi_{\{ v \geq \varepsilon \}} \right) + \int_{E_0} \chi_{\{ 0 < v < \varepsilon \}}.
\]

(3.6)

Note also that

\[
\int_E \chi_{\{ v > 0 \}} \geq \int_E \chi_{\{ v \geq \varepsilon \}}.
\]
Thus, taking a sequence \( \varepsilon_k \to 0 \), we get from Fatou Lemma
\[
\int_E X_{\{v > 0\}} \geq \liminf_{\varepsilon_k \to 0} \int_E X_{\{v \geq \varepsilon_k\}} \geq \int_E \liminf_{\varepsilon_k \to 0} X_{\{v \geq \varepsilon_k\}} \geq \int_E X_{\{v > 0\}}.
\]
Since \( \int_E X_{\{v \geq \varepsilon\}} \) is nonincreasing in \( \varepsilon \) it follows that
\[
\lim_{\varepsilon \to 0} \int_E (X_{\{v > 0\}} - X_{\{v \geq \varepsilon\}}) = 0.
\]
From this and (3.6) it follows that, for any \( \sigma > 0 \),
\[
|\{0 < v < \varepsilon\} \cap \Omega| \leq C \sigma
\]
for some \( C > 0 \). Now the claim in (3.5) follows if we let \( \sigma \to 0 \). This completes the proof of Lemma 3.1.

With this, we can now prove the following “convergence to minimizers” result:

**Lemma 3.2** Suppose that, for any \( k \in \mathbb{N} \),
\[
J_{\varepsilon_k}[u_{\varepsilon_k}] = \inf_{v \in A} J_{\varepsilon_k}[v],
\]
and that \( u_{\varepsilon_k} \to u \) locally uniformly on the compact subsets of \( \overline{\Omega} \) as \( k \to +\infty \). Then
\[
J[u] = \inf_{v \in A} J(v).
\]

**Proof** Suppose by contradiction that the claim fails. Then, there exists \( \tilde{u} \in A \) such that
\[
J[u] - J[\tilde{u}] = \delta > 0.
\]
Also, by Lemma 3.1, we have that \( J_{\varepsilon_k}[u] \to J[u] \) and \( J_{\varepsilon_k}[\tilde{u}] \to J[\tilde{u}] \), as \( \varepsilon_k \to 0 \). Hence, for sufficiently small \( \varepsilon_k \), we have that
\[
|J_{\varepsilon_k}[u] - J[u]| < \frac{\delta}{4} \quad \text{and} \quad |J_{\varepsilon_k}[\tilde{u}] - J[\tilde{u}]| < \frac{\delta}{4}.
\]
and also
\[
J_{\varepsilon_k}[u] - J_{\varepsilon_k}[\tilde{u}] > \frac{\delta}{2}.
\]
From the proof of Lemma 2.1 in [8] (see in particular the formula in display before (2.5) in [8]), one can see that \( \|u_{\varepsilon_k}\|_{W^{2,2}(\Omega)} \leq C \) uniformly in \( \varepsilon \), for some \( C > 0 \). Moreover, by (3.2), the functions \( B_{\varepsilon} \) are uniformly bounded in \( L^\infty(\mathbb{R}) \). Therefore we can extract a subsequence, still denoted \( u_{\varepsilon_k} \), so that

\[\square\]
\[ u^{\varepsilon_k} \to u \text{ locally uniformly in } \Omega, \]
\[ u^{\varepsilon_k} \to u \text{ weakly in } W^{2,2}(\Omega), \]
\[ B_{\varepsilon_k}(u^{\varepsilon_k}) \to \ell \text{ weak-star in } L^\infty(\Omega). \]

Note that if \( u(x) > 0 \) at some \( x \in \Omega \), then \( u^{\varepsilon_k}(x) > 0 \) for sufficiently large \( k \), possibly depending on \( x \). Hence \( \ell(x) = 1 \) if \( u(x) > 0 \), and so \( \ell(x) \geq \chi_{\{u > 0\}}(x) \). Hence, from Fatou Lemma we have that

\[
\liminf_{\varepsilon_k \to 0} \int_{\Omega} B_{\varepsilon_k}(u^{\varepsilon_k}) \geq \int_{\Omega} \ell(x) \geq \int_{\Omega} \chi_{\{u > 0\}}. \tag{3.10}
\]

Moreover, by (3.8) and (3.9), we have that

\[ J[u] - \delta = J[\tilde{u}] \geq J_{\varepsilon_k}[\tilde{u}] - \frac{\delta}{4} \geq J_{\varepsilon_k}[u^{\varepsilon_k}] - \frac{\delta}{4}, \]

where we also used the minimizing property in (3.7).

As a consequence, using the lower semicontinuity of the \( L^2 \) norm of \( \Delta u^{\varepsilon_k} \) and recalling (3.10),

\[ J[u] - \delta \geq \liminf_{k \to +\infty} J_{\varepsilon_k}[u^{\varepsilon_k}] - \frac{\delta}{4} \geq J[u] - \frac{\delta}{4}, \]

which is a contradiction, and so the proof of Lemma 3.2 is completed. \( \square \)

The statement in Theorem 1.1 is now the summary of the results obtained in this section, since it follows plainly from (3.3) and Lemma 3.2.

### 4 An Integral Identity for Solutions: Proof of Lemma 1.2

We provide here the integral relation satisfied by the solutions of (1.1) stated in Lemma 1.2.

**Proof of Lemma 1.2** We write \( u := u^{\varepsilon} \) for short and we use (1.5) to get that

\[
\nabla(B_{\varepsilon}(u(x))) = \nabla \int_0^{u(x)} \beta_{\varepsilon}(t) \, dt = \beta_{\varepsilon}(u(x)) \nabla u(x).
\]

Hence, by the Divergence Theorem,

\[
\int_{\Omega} \left( |\Delta u|^2 + B_{\varepsilon}(u) \right) \div \phi = \int_{\Omega} \div \left( \left( |\Delta u|^2 + B_{\varepsilon}(u) \right) \phi \right)
- \int_{\Omega} (2\Delta u \nabla \Delta u + \beta_{\varepsilon}(u) \nabla u) \cdot \phi
\tag{4.1}
= - \int_{\Omega} (2\Delta u \nabla \Delta u + \beta_{\varepsilon}(u) \nabla u) \cdot \phi.
\]
On the other hand, in light of (1.1),

\[-\beta\epsilon(u)\nabla u \cdot \phi = 2\int_{\Omega} \Delta^2 u \nabla u \cdot \phi = 2\int_{\Omega} \Delta u \Delta (\nabla u \cdot \phi) = 2\sum_{i=1}^{n} \int_{\Omega} \Delta u \Delta (\partial_i u \phi^i)
\]

\[= 2\sum_{i=1}^{n} \int_{\Omega} \Delta u (\partial_i \Delta u \phi^i + \partial_i u \Delta \phi^i + 2\partial_i \nabla u \cdot \nabla \phi^i)
\]

\[= 2\int_{\Omega} \Delta u (\nabla \Delta u \cdot \phi + \nabla u \cdot \Delta \phi + 2\text{tr}(D^2 u D\phi)),
\]

which, combined with (4.1), leads to (1.7) after a simplification. \(\Box\)

We observe that another proof of (1.7) can be performed by a domain perturbation, looking at

\[u_\eta(x) := u(x + \eta \phi(x)),\]

which can be set into a “vertical perturbation” setting

\[\psi_\eta(x) := \frac{u(x + \eta \phi(x)) - u_\eta(x)}{\eta},\]

finding that \(u_\eta = u + \eta \psi_\eta\) and thus computing the first order perturbation in \(\eta\) of the energy functional in (1.4) gives another proof of (1.7).

We also point out that, as \(\epsilon \to 0^+\), formula (1.7) also recovers formula (4.4) in [8].

5 Monotonicity Formula: Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. As already mentioned, the strategy here is obtain suitable integral cancellations by a series of careful integration by parts. We start with some general computations valid in \(\mathbb{R}^n\). In this part of the paper, for the sake of shortness, we suppose that the assumptions of Theorem 1.3 are always satisfied without further mentioning them. We write \(u := u^\epsilon\) for the sake of shortness and, without loss of generality, we also suppose that \(B_2 \Subset \Omega\). Then, we have the following identity:

**Lemma 5.1** For every \(r_1, r_2 \in (0, 3/2),\)

\[4\int_{r_1}^{r_2} R(r) \,dr + 2T(r_2) - 2T(r_1) + D(r_2) - D(r_1) = 0, \quad (5.1)\]
where

\[ R(r) := \frac{1}{r^{n+1}} \sum_{m=1}^{n} \int_{B_r} \Delta u \nabla u_m \cdot e_m - \sum_{m=1}^{n} \int_{\partial B_r} \Delta u \nabla u_m \cdot \frac{x_m x}{r^{n+2}} \]

\[ = \frac{1}{r^{n+1}} \int_{B_r} |\Delta u|^2 - \frac{1}{r^n} \int_{\partial B_r} \Delta u \partial_r u, \]

\[ T(r) := \sum_{m=1}^{n} \int_{\partial B_r} \Delta u \ n_m \frac{x^m}{r^{n+1}} \]

\[ = \frac{1}{r^n} \int_{\partial B_r} \Delta u \partial_r u, \]

and

\[ D(r) := \frac{1}{r^n} \int_{B_r} (|\Delta u|^2 + B_\varepsilon(u)), \]

(5.2)

and the notations \( x := (x^1, \ldots, x^n) \) and \( \partial_r := \frac{x}{|x|} \cdot \nabla \) have been used.

**Proof** Fix \( r \in (0, 3/2) \). We let \( \delta > 0 \) (to be taken as small as we wish in what follows), and consider a smooth function \( \eta = \eta_\delta \) supported in \( B_{r+\delta} \). We also define \( \phi = (\phi^1, \ldots, \phi^n) : \mathbb{R}^n \to \mathbb{R}^n \) as

\[ \mathbb{R}^n \ni x = (x^1, \ldots, x^n) \mapsto \phi^m(x) := x^m \eta(x). \]

We observe that \( \phi^m \) is supported in \( B_2 \), as long as \( \delta \) is sufficiently small. Consequently, for any \( m \in \{1, \ldots, n\} \),

\[ \int_{\Omega} \Delta u \ n_m \Delta \phi^m = \int_{\mathbb{R}^n} \Delta u \ n_m \Delta \phi = - \int_{\mathbb{R}^n} \nabla (\Delta u \ n_m) \cdot \nabla \phi = - \int_{\Omega} \nabla (\Delta u \ n_m) \cdot \nabla \phi, \]

or, in compact notation,

\[ \int_{\Omega} \Delta u \ n \cdot \Delta \phi = - \sum_{m=1}^{n} \int_{\Omega} \nabla (\Delta u \ n_m) \cdot \nabla \phi^m. \]

From this and (1.7), we find that

\[ 0 = 2 \int_{\Omega} \left( 2 \text{tr} (D^2 u \ D \phi) + \nabla u \cdot \Delta \phi \right) \Delta u - \int_{\Omega} \text{div} \phi \left( |\Delta u|^2 + B_\varepsilon(u) \right) \]

\[ = \int_{\Omega} \left( 2 \sum_{m=1}^{n} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u \ n_m) \right) \cdot \nabla \phi^m - \text{div} \phi \left( |\Delta u|^2 + B_\varepsilon(u) \right) \right). \]

(5.3)
Now, we take $\eta \in C^\infty_0(B_{r+\delta})$ such as
\[
\eta(x) := \begin{cases} 
1 & \text{if } x \in B_r, \\
\frac{1}{\delta + r - |x|} & \text{if } x \in B_{r+\delta} \setminus B_{r+\delta^2}, 
\end{cases}
\]
and $|\nabla \eta| \leq 2/\delta$. In this way, we have that
\[
\nabla \eta(x) = -\frac{x}{\delta |x|} \quad \text{for all } x \in B_{r+\delta} \setminus B_{r+\delta^2},
\]
and
\[
\nabla \phi^m(x) = e_m \eta(x) - \frac{x^m x}{\delta |x|} \quad \text{for all } x \in B_{r+\delta} \setminus B_{r+\delta^2},
\]
which also gives that
\[
div \phi(x) = n \eta(x) - \frac{|x|}{\delta} \quad \text{for all } x \in B_{r+\delta} \setminus B_{r+\delta^2}.
\]
Moreover, we see that $\nabla \phi^m = e_m$ in $B_r$ and $|\nabla \phi^m| \leq 4/\delta$ in $(B_{r+\delta} \setminus B_{r+\delta^2}) \cup (B_{r+\delta^2} \setminus B_r)$. As a consequence, we obtain that
\[
\int_{\Omega} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot \nabla \phi^m = \int_{B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot e_m \\
+ \int_{B_{r+\delta} \setminus B_{r+\delta^2}} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot \left( e_m \eta(x) - \frac{x^m x}{\delta |x|} \right) + O(\delta)
= \int_{B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot e_m - \int_{\partial B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot \frac{x^m x}{r} + O(\delta)
\]
and
\[
\int_{\Omega} \text{div} \phi \left( |\Delta u|^2 + B_\varepsilon(u) \right) \\
= n \int_{B_r} \left( |\Delta u|^2 + B_\varepsilon(u) \right) + \int_{B_{r+\delta} \setminus B_{r+\delta^2}} \left( |\Delta u|^2 + B_\varepsilon(u) \right) \left( n \eta(x) - \frac{|x|}{\delta} \right) + O(\delta)
= n \int_{B_r} \left( |\Delta u|^2 + B_\varepsilon(u) \right) - r \int_{\partial B_r} \left( |\Delta u|^2 + B_\varepsilon(u) \right) + O(\delta).
\]
We insert these two pieces of information into (5.3), and we send $\delta \to 0^+$. In this way, we see that
\[
0 = 2 \sum_{m=1}^n \left[ \int_{B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot e_m - \int_{\partial B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot \frac{x^m x}{r} \right]
\]
\begin{align*}
\text{Springer}
\end{align*}
\[- n \int_{B_r} (|\Delta u|^2 + B_\varepsilon(u)) + r \int_{\partial B_r} (|\Delta u|^2 + B_\varepsilon(u)) \]  

(5.4)

Now, recalling (5.2), we see that

\[ D'(r) = \frac{1}{r^n} \int_{\partial B_r} (|\Delta u|^2 + B_\varepsilon(u)) - \frac{n}{r^{n+1}} \int_{B_r} (|\Delta u|^2 + B_\varepsilon(u)), \]

and hence we can write (5.4) as

\[ 0 = \frac{2}{r^{n+1}} \sum_{m=1}^{n} \left[ \int_{B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot e_m ight. \]
\[ - \int_{\partial B_r} \left( 2 \Delta u \nabla u_m - \nabla (\Delta u u_m) \right) \cdot \frac{x^m}{r} + D'(r). \]

(5.5)

We also point out that

\[ \hat{\partial} \nabla (\Delta u u_m) \cdot e_m = \hat{\partial} \nabla (\Delta u u_m) \cdot \frac{x^m}{r} \]

and, changing variable,

\[ \int_{\partial B_r} \nabla (\Delta u(x) u_m(x)) \cdot \frac{x^m}{r^{n+2}} \, d\mathcal{H}^{n-1}(x) = \int_{\partial B_1} \nabla (\Delta u(ry) u_m(ry)) \cdot \frac{y^m}{r} \, d\mathcal{H}^{n-1}(y) \]
\[ = \int_{\partial B_1} \frac{d}{dr} (\Delta u(ry) u_m(ry)) \frac{y^m}{r} \, d\mathcal{H}^{n-1}(y) \]
\[ = \frac{d}{dr} \int_{\partial B_1} (\Delta u(ry) u_m(ry)) \frac{y^m}{r} \, d\mathcal{H}^{n-1}(y) + \int_{\partial B_1} (\Delta u(ry) u_m(ry)) \frac{y^m}{r^2} \, d\mathcal{H}^{n-1}(y) \]
\[ = \frac{d}{dr} \int_{\partial B_r} (\Delta u(x) u_m(x)) \frac{x^m}{r^{n+2}} \, d\mathcal{H}^{n-1}(x) + \int_{\partial B_r} (\Delta u(x) u_m(x)) \frac{x^m}{r^{n+1}} \, d\mathcal{H}^{n-1}(x). \]

These observations and (5.2) give that

\[ \sum_{m=1}^{n} \left[ \int_{B_r} \nabla (\Delta u u_m) \cdot e_m + \int_{\partial B_r} \nabla (\Delta u u_m) \cdot \frac{x^m}{r^{n+2}} \right] \]
\[ = \sum_{m=1}^{n} \frac{d}{dr} \int_{\partial B_r} (\Delta u u_m) \frac{x^m}{r^{n+1}} = T'(r). \]

Thus, inserting this information into (5.5), we find that

\[ 0 = \frac{4}{r^{n+1}} \sum_{m=1}^{n} \left[ \int_{B_r} \Delta u \nabla u_m \cdot e_m - \int_{\partial B_r} \Delta u \nabla u_m \cdot \frac{x^m}{r} \right] + 2T'(r) + D'(r). \]

(5.6)
From this and (5.2), we can write
\[ 0 = 4R(r) + 2T'(r) + D'(r), \]
which, after an integration, gives (5.1), as desired. \(\square\)

The proof of Theorem 1.3 will also rely on the following auxiliary result:

**Lemma 5.2** In the notation stated by (5.2), we have that, for any \( r_1, r_2 \in (0, 3/2) \),
\[
2T(r_1) - 2T(r_2) + D(r_1) - D(r_2)
= 4 \int_{r_1}^{r_2} \left( \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta u \left( \frac{2u}{r} - \partial_r^2 u - 2\frac{u}{r^2} \right) - \frac{1}{r^{n+1}} \int_{B_r} \Delta^2 u \right) \, dr \quad (5.7)
- 4V(r_2) + 4V(r_1),
\]
where
\[
V(r) := \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta uu. \quad (5.8)
\]

**Proof** We observe that
\[
\int_{B_r} |\Delta u|^2 = \int_{B_r} (\text{div}(\Delta u \nabla u) - \nabla \Delta u \cdot \nabla u) = \int_{\partial B_r} \Delta u u_r - \int_{B_r} \nabla \Delta u \cdot \nabla u
= \int_{\partial B_r} \Delta u u_r - \int_{B_r} \text{div}(u \Delta u) + \int_{B_r} \Delta^2 u \quad (5.9)
= \int_{\partial B_r} \Delta u u_r - \int_{\partial B_r} u \Delta u_r + \int_{B_r} \Delta^2 u.
\]
Furthermore, we see that
\[
\frac{d}{dr} \left( \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta uu \right)
= \frac{d}{dr} \left( \frac{1}{r^2} \int_{\partial B_1} \Delta u(r\theta)u(r\theta) \right)
= -\frac{2}{r^3} \int_{\partial B_1} \Delta u(r\theta)u(r\theta) + \frac{1}{r^2} \int_{\partial B_1} \Delta u_r(r\theta)u(r\theta) + \frac{1}{r^2} \int_{\partial B_1} \Delta u(r\theta)u_r(r\theta)
= -\frac{2}{r^{n+2}} \int_{\partial B_r} \Delta u + \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta u_r u + \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta u u_r.
\]
This and (5.9) give that
\[
\frac{1}{r^{n+1}} \int_{B_r} |\Delta u|^2 - \frac{1}{r^n} \int_{\partial B_r} \Delta u \partial_r^2 u
\]
\(\square\) Springer
\[
\begin{align*}
\frac{1}{r^{n+1}} \int_{\partial B_r} \Delta u \frac{u_r}{r} - \frac{1}{r^{n+1}} \int_{\partial B_r} u \Delta u_r + \frac{1}{r^n} \int_{B_r} \Delta^2 u u - \frac{1}{r^n} \int_{\partial B_r} \Delta u \frac{\partial^2 u}{\partial r} = \\
= \frac{1}{r^n} \int_{\partial B_r} \Delta u \left( \frac{2u_r}{r} - \partial_r^2 u - \frac{2u}{r^2} \right) + \frac{1}{r^{n+1}} \int_{B_r} \Delta^2 u u - \frac{d}{dr} \left( \frac{1}{r^{n+1}} \int_{\partial B_r} \Delta u u \right).
\end{align*}
\]

Now we integrate the identity above and recall (5.8), to conclude that
\[
\int_{r_1}^{r_2} \left( \frac{1}{r^n} \int_{B_r} |\Delta u|^2 - \frac{1}{r^n} \int_{\partial B_r} \Delta u \frac{\partial^2 u}{\partial r} \right) dr = \\
= \int_{r_1}^{r_2} \left( \frac{1}{r^n} \int_{\partial B_r} \Delta u \left( \frac{2u_r}{r} - \partial_r^2 u - \frac{2u}{r^2} \right) + \frac{1}{r^{n+1}} \int_{B_r} \Delta^2 u u \right) dr - V(r_2) + V(r_1). \tag{5.10}
\]

Hence, recalling (5.2), we can write (5.10) as
\[
\int_{r_1}^{r_2} R(r) dr = \int_{r_1}^{r_2} \left( \frac{1}{r^n} \int_{\partial B_r} \Delta u \left( \frac{2u_r}{r} - \partial_r^2 u - \frac{2u}{r^2} \right) - \frac{1}{r^{n+1}} \int_{B_r} \Delta^2 u u \right) dr \\
= -V(r_2) + V(r_1).
\]

From this and (5.1) we obtain the desired claim in (5.7). \(\square\)

The previous calculations were valid in any dimension \(n\), and we now restrict to the case \(n = 2\).

**Proof of (1.9)** Using using polar coordinates \((r, \theta)\), we compute
\[
- \frac{1}{r^n} \int_{\partial B_r} \Delta u \left( \frac{2u_r}{r} - \partial_r^2 u - \frac{2u}{r^2} \right) = \int_{\partial B_1} \frac{1}{r} \Delta u \left( u_{rr} - 2 \frac{u_r}{r} + 2 \frac{u}{r^2} \right) \\
= \int_{\partial B_1} \frac{1}{r} \left( u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) \left( u_{rr} - 2 \frac{u_r}{r} + 2 \frac{u}{r^2} \right)^{-1} \\
= A(r) + B(r), \tag{5.11}
\]

where
\[
A(r) := \int_{\partial B_1} \frac{1}{r^\frac{3}{2}} u_{\theta\theta} \left( u_{rr} - 2 \frac{u_r}{r} + 2 \frac{u}{r^2} \right), \tag{5.12}
\]
\[
and \quad B(r) := \int_{\partial B_1} \frac{1}{r} \left( u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) \left( u_{rr} - 2 \frac{u_r}{r} + 2 \frac{u}{r^2} \right).
\]

Now we deal with the terms \(A(r)\) and \(B(r)\) separately. To start with, we perform several integrations by parts that involve the terms related to \(A(r)\). We see that
\[
\frac{1}{r^3} \int_{\partial B_1} u_{\theta\theta} u_{rr} = -\frac{1}{r^3} \int_{\partial B_1} u_{\theta} u_{\theta rr} = \\
= -\frac{d}{dr} \int_{\partial B_1} \frac{u_{\theta} u_{r\theta}}{r^3} + \int_{\partial B_1} \frac{u_{\theta\theta}^2}{r^3} - 3 \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^4}. \tag{5.13}
\]

\(\square\ Springer\)
Similarly, we have that

$$-2 \int_{\partial B_1} \frac{1}{r^4} u_{\theta\theta} u_r = 2 \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^4}$$  \hspace{1cm} (5.14)$$

and

$$2 \int_{\partial B_1} \frac{1}{r^5} u_{\theta\theta} u = -2 \int_{\partial B_1} \frac{u_{\theta}^2}{r^5}.$$  \hspace{1cm} (5.15)

Combining (5.12) (5.13), (5.14) and (5.15), we find that

$$A(r) = -\frac{d}{dr} \left( \int_{\partial B_1} \frac{u_{\theta} u_{\theta r} u_r}{r^3} + \int_{\partial B_1} \frac{u_{r} u_{\theta r}}{r^3} - 3 \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^4} + 2 \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^4} - 2 \int_{\partial B_1} \frac{u_{\theta}^2}{r^5} \right)$$

$$= -\frac{d}{dr} \left( \int_{\partial B_1} \frac{u_{\theta} u_{\theta r} u_r}{r^3} + \int_{\partial B_1} \frac{u_{r} u_{\theta r}}{r^3} - \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^4} - 2 \int_{\partial B_1} \frac{u_{\theta}^2}{r^5} \right)$$

$$= -\frac{d}{dr} \left( \int_{\partial B_1} \frac{u_{\theta} u_{\theta r}}{r^3} + \frac{3}{2} \int_{\partial B_1} \frac{u_{\theta}^2}{r^4} \right) + \int_{\partial B_1} \frac{1}{r^3} \left( u_{\theta r} - \frac{2u_r}{r} \right)^2 + \int_{\partial B_1} \frac{1}{r^4} \left( u_{\theta r} - \frac{2u_r}{r} \right)^2.$$  \hspace{1cm} (5.16)

Now we take into account the term $B(r)$. To this end, from (5.12), we see that

$$B(r) = \int_{\partial B_1} \frac{1}{r} \left( u_{rr} - \frac{2u_{rr} u_r}{r} + \frac{2uu_{rr}}{r^2} + u_r u_{rr} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3} \right)$$

$$= \int_{\partial B_1} \frac{1}{r} \left( u_{rr} - \frac{u_{rr} u_r}{r^2} + \frac{2uu_{rr}}{r^2} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3} \right)$$

$$= \int_{\partial B_1} \frac{1}{r} \left( u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 + \int_{\partial B_1} \frac{5u_{rr}}{r} - \frac{6uu_{rr}}{r^2} - \frac{11u_r^2}{r^2} + \frac{26uu_r}{r^3} - \frac{16u^2}{r^4}$$

$$= \int_{\partial B_1} \frac{1}{r^2} \left( u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 + \frac{d}{dr} \left( \int_{\partial B_1} \frac{5u_{rr}}{r^2} - \int_{\partial B_1} \frac{6uu_{rr}}{r^3} + \int_{\partial B_1} \frac{4u^2}{r^4} \right)$$

$$= \int_{\partial B_1} \frac{1}{r^2} \left( u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 + \frac{d}{dr} \left( \int_{\partial B_r} \frac{5u_{rr}}{r^2} - \int_{\partial B_r} \frac{6uu_{rr}}{r^3} + \int_{\partial B_r} \frac{4u^2}{r^5} \right).$$  \hspace{1cm} (5.17)

Using (5.16) and (5.17), we conclude that

$$A(r) + B(r) = \frac{1}{r^2} \int_{\partial B_r} \left[ \left( \frac{u_{\theta r}}{r} - \frac{2u_r}{r^2} \right)^2 + \left( \frac{u_{rr}}{r} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 \right] + W'(r).$$  \hspace{1cm} (5.18)
where

\[ W(r) := \int_{\partial B_r} \left( \frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_{\theta}u_{r\theta}}{r^4} - \frac{3u_{\theta}^2}{2r^5} \right). \]  

(5.19)

On the other hand, in view of (5.7) and (5.11),

\[-4V(r_2) + 4V(r_1) + 2T(r_2) - 2T(r_1) + D(r_2) - D(r_1)\]

\[= -4 \int_{r_1}^{r_2} \left( \int_{\partial B_r} \Delta u \left( \frac{2u_r}{r} - \mathcal{A}^2 u - 2 \frac{u}{r^2} \right) - \frac{1}{r^{n+1}} \int_{B_r} \Delta^2 uu \right) \, dr \]

\[= 4 \int_{r_1}^{r_2} (A(r) + B(r)) \, dr + \int_{r_1}^{r_2} \int_{B_r} \Delta^2 uu.\]

Consequently, by (5.18),

\[-V(r_2) + V(r_1) + \frac{T(r_2) - T(r_1)}{2} + \frac{D(r_2) - D(r_1)}{4} - W(r_2) + W(r_1)\]

\[= \int_{r_1}^{r_2} \left\{ \frac{1}{r^2} \int_{\partial B_r} \left[ \left( \frac{u_{\theta r}}{r} - \frac{2u_r}{r^2} \right)^2 + \left( u_{rr} - \frac{3u_r}{r} + 4 \frac{u}{r^2} \right)^2 \right] \right\} + \int_{r_1}^{r_2} \int_{B_r} \Delta^2 uu.\]

(5.20)

Recalling (1.1), (1.10), (5.2), (5.8) and (5.19), we see that

\[-V(r) + \frac{T(r)}{2} + \frac{D(r)}{4} - \int_{\partial B_r} \left( \frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_{\theta}u_{r\theta}}{r^4} - \frac{3u_{\theta}^2}{2r^5} \right) - \int_0^r \frac{1}{\rho^3} \int_{B_\rho} \Delta^2 uu\]

\[= -\frac{1}{r^3} \int_{\partial B_r} \Delta uu + \frac{1}{2r^2} \int_{\partial B_r} \Delta u \partial_r u + \frac{1}{4r^2} \int_{B_r} (|\Delta u|^2 + B_\varepsilon(u))\]

\[-\int_{\partial B_r} \left( \frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_{\theta}u_{r\theta}}{r^4} - \frac{3u_{\theta}^2}{2r^5} \right) + \int_0^r \frac{1}{\rho^3} \int_{B_\rho} \beta_\varepsilon(u)\]

\[= E(r).\]

This and (5.20) establish the desired claim in (1.9). \qed

6 Strong Convergence of \( \Delta u^{\alpha_j} \) and Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. To this end, we start by proving the strong convergence claimed in (1.15).

**Proof of (1.15)** Our aim is to show that

\[ \lim_{j \to +\infty} \int_{\Omega} (\Delta u^{\alpha_j})^2 \leq \int_{\Omega} (\Delta u)^2. \]  

(6.1)
To prove this, we take \( \eta \in C_0^\infty(\Omega, [0, 1]) \), and we see that

\[
\int_\Omega \eta u^{\varepsilon_j} \Delta^2 u^{\varepsilon_j} = \int_\Omega \Delta u^{\varepsilon_j} \Delta (\eta u^{\varepsilon_j}) = \int_\Omega \eta (\Delta u^{\varepsilon_j})^2 + \Delta u^{\varepsilon_j} (2 \nabla \eta \nabla u^{\varepsilon_j} + u^{\varepsilon_j} \Delta \eta). \tag{6.2}
\]

Moreover, supposing that \( \eta \) is supported in some \( B \supset \Omega \), we have that

\[
\left| \int_\Omega \eta u^{\varepsilon_j} \Delta^2 u^{\varepsilon_j} \right| \leq \frac{\| \eta \|_{L^\infty(B)}}{2} \int_B |u^{\varepsilon_j}| \beta_{\varepsilon_j} (u^{\varepsilon_j}) = \frac{\| \eta \|_{L^\infty(B)}}{2 \varepsilon_j} \int_{B \cap \{0 < u^{\varepsilon_j} \leq \varepsilon_j\}} |u^{\varepsilon_j}| \beta \left( \frac{u^{\varepsilon_j}}{\varepsilon_j} \right) \leq \frac{1}{2} \| \eta \|_{L^\infty(B)} \sup_{\{0 < u^{\varepsilon_j} \leq \varepsilon_j\}} \beta |(0 < u^{\varepsilon_j} \leq \varepsilon_j) \cap B|
\]

which is infinitesimal as \( j \to +\infty \), thanks to (1.13).

Consequently, recalling (6.2),

\[
- \lim_{j \to +\infty} \int_\Omega \eta (\Delta u^{\varepsilon_j})^2 = \lim_{j \to +\infty} \int_\Omega \Delta u^{\varepsilon_j} (2 \nabla \eta \cdot \nabla u^{\varepsilon_j} + u^{\varepsilon_j} \Delta \eta). \tag{6.3}
\]

Furthermore,

\[
\lim_{j \to +\infty} \left| \int_\Omega \Delta u^{\varepsilon_j} \nabla \eta \cdot \nabla u^{\varepsilon_j} - \int_\Omega \Delta u \nabla \eta \cdot \nabla u \right| \leq \lim_{j \to +\infty} \left| \int_\Omega (\Delta u^{\varepsilon_j} - \Delta u) \nabla \eta \cdot \nabla u \right| + \int_\Omega |\Delta u^{\varepsilon_j}| \| \nabla \eta \| \| \nabla u^{\varepsilon_j} - \nabla u \|_{L^2(B)} \leq \lim_{j \to +\infty} \| \eta \|_{C^1(B)} \| \Delta u^{\varepsilon_j} \|_{L^2(B)} \| \nabla (u^{\varepsilon_j} - u) \|_{L^2(B)} = 0,
\]

thanks to the weak convergence of \( \Delta u^{\varepsilon_j} \) and the Sobolev embedding, and, similarly,

\[
\lim_{j \to +\infty} \left| \int_\Omega \Delta u^{\varepsilon_j} u^{\varepsilon_j} \Delta \eta - \int_\Omega \Delta uu \Delta \eta \right| = 0.
\]

These observations and (6.3) yield that

\[
- \lim_{j \to +\infty} \int_\Omega \eta (\Delta u^{\varepsilon_j})^2 = \int_\Omega \Delta u (2 \nabla \eta \cdot \nabla u + u \Delta \eta). \tag{6.4}
\]
By Stampacchia’s Theorem, we also know that $\nabla u = 0$ a.e. in $\{u = 0\}$, hence we can write (6.4) in the form

$$-\lim_{j \to +\infty} \int_{\Omega} \eta(\Delta u^{\varepsilon_j})^2 = \int_{\{u > 0\}} \Delta u (2\nabla \eta \cdot \nabla u + u \Delta \eta).$$

(6.5)

Next, we exploit the Sard Theorem in Sobolev spaces (see [5]) to see that $\{u = s_k\}$ has smooth boundary, for an infinitesimal sequence $s_k$. Hence, after some integrations by parts,

$$\int_{\{u > s_k\}} \eta(\Delta u)^2 = \int_{\{u = s_k\}} \eta \partial_v u \Delta u - \int_{\{u > s_k\}} \nabla u \cdot \nabla (\Delta u \eta)$$

$$= \int_{\{u = s_k\}} \eta \partial_v u \Delta u - \int_{\{u = s_k\}} (u - s_k) \partial_v (\eta \Delta u) + \int_{\{u > s_k\}} (u - s_k) \Delta (\Delta u \eta)$$

$$= \int_{\{u = s_k\}} \eta \partial_v u \Delta u + \int_{\{u > s_k\}} (u - s_k) (2\nabla \Delta u \cdot \nabla \eta + \Delta u \Delta \eta)$$

$$= \int_{\{u = s_k\}} \eta \partial_v u \Delta u + 2 \int_{\{u = s_k\}} (u - s_k) \partial_v \eta \Delta u$$

$$- \int_{\{u > s_k\}} 2\nabla u \cdot \nabla \eta \Delta u + (u - s_k) \Delta u \Delta \eta$$

$$= \int_{\{u = s_k\}} \eta \partial_v u \Delta u - \int_{\{u > s_k\}} 2\nabla u \cdot \nabla \eta \Delta u + (u - s_k) \Delta u \Delta \eta,$$

(6.6)

where $v$ is the exterior normal to $\{u > s_k\}$. As a technical detail, we point out that the term $\partial_v \Delta u$ is not really well defined in our setting, hence, to justify (6.6), one should first approximate $u$ with a mollification and then take the limit.

Now, we claim that

$$\lim_{k \to +\infty} \int_{\{u = s_k\}} \eta \partial_v u \Delta u = 0.$$

(6.7)

To see this we recall (1.14) and we find that

$$\int_{\{u = s_k\}} \eta \partial_v u \Delta u = \int_{\{u = s_k\}} \eta \partial_v u (\Delta u + C) - C \int_{\{u = s_k\}} \eta \partial_v u$$

$$= -\int_{\{u = s_k\}} \eta |\nabla u|(\Delta u + C) - C \int_{\{u = s_k\}} \eta \partial_v u.$$

(6.8)

Moreover, we observe that

$$\int_{\{u = s_k\}} \eta \partial_v u = \int_{\{u > s_k\}} \text{div}(\eta \nabla u)$$

$$= \int_{\{u > s_k\}} \nabla \eta \cdot \nabla u + \eta \Delta u,$$
and, thus, taking the limit,

$$\lim_{k \to +\infty} \int_{\{u = s_k\}} \eta \partial_{\nu} u = \int_{\{u > 0\}} \nabla \eta \cdot \nabla u + \eta \Delta u = \int_{\partial \{u > 0\}} \eta \partial_{\nu} u = 0. \quad (6.9)$$

Also, in light of (1.14),

$$0 \leq \int_{\{u = s_k\}} \eta |\nabla u| (\Delta u + C) \leq \left( \sup_B |\Delta u| + C \right) \int_{\{u = s_k\}} \eta |\nabla u| = - \left( \sup_B |\Delta u| + C \right) \int_{\{u = s_k\}} \eta \partial_{\nu} u,$$

and therefore

$$\lim_{k \to +\infty} \int_{\{u = s_k\}} \eta |\nabla u| (\Delta u + C) = 0,$$

thanks to (6.9).

Using this, (6.8) and (6.9), we establish (6.7), as desired.

Then, combining (6.6) with (6.7), we conclude that

$$\int_{\{u > 0\}} \eta (\Delta u)^2 = \lim_{k \to +\infty} \int_{\{u > s_k\}} \eta (\Delta u)^2 = - \lim_{k \to +\infty} \int_{\{u > s_k\}} 2 \nabla u \cdot \nabla \eta \Delta u + (u - s_k) \Delta u \Delta \eta = - \int_{\{u > 0\}} 2 \nabla u \cdot \nabla \eta \Delta u + u \Delta u \Delta \eta.$$

From this and (6.5), we see that

$$\lim_{j \to +\infty} \int_{\Omega} \eta (\Delta u_j^\delta)^2 = \int_{\{u > 0\}} \eta (\Delta u)^2 \leq \int_{\Omega} (\Delta u)^2. \quad (6.10)$$

Furthermore, fixing $\delta > 0$ and taking $\eta$ such that $|\Omega \setminus \{\eta = 1\}| \leq \delta$, recalling (1.14) we see that

$$\left| \int_{\Omega} (1 - \eta) (\Delta u_j^\delta)^2 \right| \leq C^2 \delta.$$

In light of this and (6.10), we thereby obtain that

$$\lim_{j \to +\infty} \int_{\Omega} (\Delta u_j^\delta)^2 \leq C^2 \delta + \int_{\Omega} (\Delta u)^2.$$

Hence, by taking $\delta$ as small as we wish, we complete the proof of (6.1).
The weak convergence of $\Delta u^\varepsilon_j$ also implies that
\[
\lim_{j \to +\infty} \int_\Omega (\Delta u^\varepsilon_j)^2 = \lim_{j \to +\infty} \int_\Omega (\Delta u^\varepsilon_j - \Delta u)^2 + 2\Delta u^\varepsilon_j \Delta u - (\Delta u)^2 \\
\geq \lim_{j \to +\infty} \int_\Omega 2\Delta u^\varepsilon_j \Delta u - (\Delta u)^2 = \int_\Omega (\Delta u)^2.
\]
This and (6.10) give that
\[
\lim_{j \to +\infty} \int_\Omega (\Delta u^\varepsilon_j)^2 = \int_\Omega (\Delta u)^2,
\]
which in turn implies (1.15).

\hfill \Box

**Strong convergence of Hessian and proof of (1.16).** It is easy to check that
\[
\int_\Omega u^\varepsilon_{ij} u^\varepsilon_{ij} \eta = - \int_\Omega u^\varepsilon_{ij} (u^\varepsilon_{ji} \eta + u^\varepsilon_{jj} \eta_i) \\
= - \int_\Omega u^\varepsilon_{ij} u^\varepsilon_{jj} \eta_i + \int_\Omega (u^\varepsilon_{ij})^2 \eta + u^\varepsilon_i \eta_j u^\varepsilon_{ij}.
\]
Hence the strong convergence of the Hessian follows from the strong convergence of the Laplacian in (1.15). Taking the limits, this proves (1.16).

\hfill \Box

**Proof of the boundedness of $E$, of (1.18), and of (1.19).** By (1.17), we know that
\[
|u(x)| \leq \tilde{C} |x|^2, \quad |
abla u(x)| \leq \tilde{C} |x| \quad \text{and} \quad |D^2 u(x)| \leq \tilde{C},
\]
for all $x \in B_{1/2}$, for a suitable $\tilde{C} > 0$. This gives that the function $E$ in (1.12) is well defined and bounded.

We now prove (1.18). This is somehow a delicate point, since one cannot simply take the limit of the function $E^\varepsilon$ since the last term in (1.10) is not necessarily infinitesimal in $\varepsilon$ (this possible pathology can be understood, for instance, by making a direct computation assuming that $u^\varepsilon$ is quadratic). To cope with this difficulty, it is convenient to define
\[
\tilde{E}^\varepsilon(r) := \int_{\partial B_r} \left( \frac{\Delta u^\varepsilon}{2r^2} u^\varepsilon_{rr} - \frac{5(u^\varepsilon)^2}{2r^3} - \frac{\Delta u^\varepsilon u^\varepsilon}{r^3} + \frac{6u^\varepsilon u^\varepsilon_{rr}}{r^4} - \frac{u^\varepsilon_{rr} u^\varepsilon}{r^4} - \frac{4(u^\varepsilon)^2}{r^5} - \frac{3(u^\varepsilon)^2}{2r^5} \right) \\
+ \frac{1}{4r^2} \int_{B_r} (|\Delta u^\varepsilon|^2 + B_r(u^\varepsilon)).
\]
By (1.10), we have that
\[
E^\varepsilon(r) = \tilde{E}^\varepsilon(r) + \int_0^r \frac{1}{\rho^3} \int_{B_\rho} \beta^\varepsilon(u^\varepsilon) u^\varepsilon.
\]
Moreover, by the strong convergence of the Hessian that we have just proved, we know that

\[
\lim_{j \to +\infty} \tilde{E}^{\epsilon_j}(r) = E(r).
\]  

(6.13)

We now fix \( \tau_2 > \tau_1 > 0 \), with \( B_{\tau_2} \subseteq \Omega \). Then, we have that

\[
\left| \int_{\tau_1}^{\tau_2} \frac{1}{\rho^3} \int_{B_{\rho}} \beta_\epsilon(u^\epsilon) \ u^\epsilon \right| \leq \int_{\tau_1}^{\tau_2} \frac{1}{\rho^3} \int_{B_{\rho} \cap \{0 < u^\epsilon \leq \epsilon\}} \beta \left( \frac{u^\epsilon}{\epsilon} \right) \frac{u^\epsilon}{\epsilon}
\]

\[
\leq \sup_{[0,1]} \beta \frac{1}{\tau_1^3} \int_{\tau_1}^{\tau_2} |B_{\rho} \cap \{0 < u^\epsilon \leq \epsilon\}|
\]

\[
\leq \sup_{[0,1]} \beta \frac{\tau_2 - \tau_1}{\tau_1^3} |B_{\tau_2} \cap \{0 < u^\epsilon \leq \epsilon\}|.
\]

As a consequence, by (1.13),

\[
\lim_{j \to +\infty} \int_{\tau_1}^{\tau_2} \frac{1}{\rho^3} \int_{B_{\rho}} \beta_{\epsilon_j}(u^{\epsilon_j}) \ u^{\epsilon_j} = 0.
\]

Using this, (6.12) and (6.13), we thus conclude that

\[
E(\tau_2) - E(\tau_1) = \lim_{j \to +\infty} \tilde{E}^{\epsilon_j}(\tau_2) - \tilde{E}^{\epsilon_j}(\tau_1)
\]

\[
= \lim_{j \to +\infty} E^{\epsilon_j}(\tau_2) - E^{\epsilon_j}(\tau_1) - \int_{\tau_1}^{\tau_2} \frac{1}{\rho^3} \int_{B_{\rho}} \beta_{\epsilon_j}(u^{\epsilon_j}) \ u^{\epsilon_j}
\]

\[
= \lim_{j \to +\infty} E^{\epsilon_j}(\tau_2) - E^{\epsilon_j}(\tau_1).
\]  

(6.14)

From this and (1.9) we obtain (1.18), as desired.

Also, using (6.14), (1.9) and the strong convergence of the Hessian, we obtain (1.19).

\[\Box\]

**Proof of (1.20)** If \( E \) is constant in \((0, \tau)\), we deduce from (1.19) that

\[-\frac{\partial}{\partial \theta} \left( -\frac{u_r}{r} + \frac{2u}{r^2} \right) = \frac{u_{r\theta}}{r^2} - \frac{2u_{\theta}}{r} = 0\]

and

\[-r \frac{\partial}{\partial r} \left( -\frac{u_r}{r} + \frac{2u}{r^2} \right) = u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} = 0.\]

As a consequence, we have that

\[\nabla \left( -\frac{u_r}{r} + \frac{2u}{r^2} \right) = 0,\]

which implies that the function \(-\frac{u_r}{r} + \frac{2u}{r^2}\) is constant for \(|x| \in (0, \tau)\).
Accordingly, we see that
\[ -\frac{u_r}{r} + \frac{2u}{r^2} = c, \]  
for some \( c \in \mathbb{R} \). Let now
\[ v(r, \theta) := u(r, \theta) + cr^2 \log r. \]  
From (6.15), we have
\[ v_r = u_r + 2cr \log r + cr = \frac{2u}{r} + 2cr \log r = \frac{2v}{r}. \]  
Integrating this equation, fixed \( \bar{r} \in (0, \tau) \), we find that
\[ v(r, \theta) = \frac{r^2 v(\bar{r}, \theta)}{\bar{r}^2}. \]  
This and (6.16) give that
\[ u(r, \theta) = \frac{r^2 v(\bar{r}, \theta)}{\bar{r}^2} - cr^2 \log r. \]  
Hence, recalling (1.17),
\[ C \geq \frac{|u(r, \theta)|}{r^2} \geq |c| \frac{1}{r^2} |\log r| - \frac{|v(\bar{r}, \theta)|}{\bar{r}^2}, \]  
and therefore
\[ |c| \leq \lim_{r \to 0} \frac{|v(\bar{r}, \theta)|}{r^2 |\log r|} + \frac{C}{|\log r|} = 0. \]  
This gives that \( c = 0 \) and in this way we can write (6.15) as \( -\frac{u_r}{r} + \frac{2u}{r^2} = 0 \), or, equivalently, \( \nabla u(x) \cdot x = 2u(x) \) for any \( x \in B_\tau \). The latter is the Euler equation for homogeneous functions of degree two, and accordingly we find that \( u \) is necessarily homogeneous of degree two. \( \square \)

**Proof of (1.21)** The proof of (1.21) is now standard (for instance, one can repeat the argument in the proof of Theorem 1.14 in [8]). The proof of Theorem 1.5 is thereby complete. \( \square \)

### 7 Quadratic Detachment: Proof of Theorem 1.6

The proof of Theorem 1.6 relies on the integral identity in Lemma 1.2, and it goes as follows:
Proof of Theorem 1.6 We let $\psi \in C^\infty_0(\Omega)$ and we exploit (1.7) with $\phi(x) := (\psi(x), 0, \ldots, 0)$. In this way, we obtain that
\[
\int_{\Omega} \left[ 2 \left( 2 \sum_{j=1}^n u_j^\varepsilon \psi_j + u_1^\varepsilon \Delta \psi \right) \Delta u^\varepsilon - \psi_1 \left( |\Delta u^\varepsilon|^2 + B_{\varepsilon}(u^\varepsilon) \right) \right] = 0. \quad (7.1)
\]

We also remark that
\[
\lim_{\varepsilon \to 0^+} B_{\varepsilon}(u^\varepsilon(x)) = \begin{cases} 
1 & \text{if } x \in \{ u > 0 \}, \\
0 & \text{if } x \in \{ u < 0 \}.
\end{cases} \quad (7.2)
\]

Indeed, if $u(x) > 0$, we have that $u^\varepsilon(x) > \frac{u(x)}{\varepsilon}$ if $\varepsilon$ is small enough and hence, in view of (3.1), we know that $B_{\varepsilon}(u^\varepsilon(x)) = 1$. Conversely, if $u(y) < 0$, we have that $u^\varepsilon(y) < 0$ for small $\varepsilon$ and thus
\[
B_{\varepsilon}(u^\varepsilon(y)) = \int_0^{u^\varepsilon(y)/\varepsilon} \beta(t) \, dt = 0,
\]

since $\beta = 0$ in $\left( \frac{u^\varepsilon(y)}{\varepsilon}, 0 \right)$. These observations establish (7.2).

Thanks to (3.2) and the Dominated Convergence Theorem, we can take the limits inside the integral and find that
\[
\lim_{\varepsilon \to 0^+} \int_{\Omega \cap \{ u \neq 0 \}} \psi_1 B_{\varepsilon}(u^\varepsilon) = \int_{\Omega \cap \{ u > 0 \}} \psi_1.
\]

Plugging this identity inside (7.1), and exploiting the convergence in (1.22), we conclude that
\[
0 = \lim_{\varepsilon \to 0^+} \int_{\Omega} \left[ 2 \left( 2 \sum_{j=1}^n u_j^\varepsilon \psi_j + u_1^\varepsilon \Delta \psi \right) \Delta u^\varepsilon - \psi_1 \left( |\Delta u^\varepsilon|^2 + B_{\varepsilon}(u^\varepsilon) \right) \right] \\
= \int_{\Omega} \left[ 2 \left( 2 \sum_{j=1}^n u_j \psi_j + u_1 \Delta \psi \right) \Delta u - \psi_1 |\Delta u|^2 \right] - \lim_{\varepsilon \to 0^+} \int_{\Omega \cap \{ u = 0 \}} \psi_1 B_{\varepsilon}(u^\varepsilon) - \int_{\Omega \cap \{ u > 0 \}} \psi_1. \quad (7.3)
\]

Now, since in all the cases under consideration \{ $u = 0$ \} has zero Lebesgue measure, we can write (7.3) as
\[
0 = \int_{\Omega} \left[ 2 \left( 2 \sum_{j=1}^n u_j \psi_j + u_1 \Delta \psi \right) \Delta u - \psi_1 |\Delta u|^2 \right] - \int_{\Omega \cap \{ u > 0 \}} \psi_1. \quad (7.4)
\]
In addition, from (1.22), we know that
\[
\begin{align*}
    u_1 &= \alpha x_1 \chi_{\{x_1 > 0\}} + \gamma x_1 \chi_{\{x_1 < 0\}} \\
    \Delta u &= u_{11} = \alpha x_1 \chi_{\{x_1 > 0\}} + \gamma x_1 \chi_{\{x_1 < 0\}} \quad \text{a.e. in } \Omega,
\end{align*}
\]
and \(u_j = 0\) if \(j \neq 1\), therefore (7.4) becomes
\[
0 = \int_{\Omega \cap \{x_1 > 0\}} \left[ (4\alpha \psi_1 + 2\alpha x_1 \Delta \psi) \alpha - \psi_1 \alpha^2 \right] + \int_{\Omega \cap \{x_1 < 0\}} \left[ (4\gamma \psi_1 + 2\gamma x_1 \Delta \psi) \gamma - \psi_1 \gamma^2 \right] - \int_{\Omega \cap \{u > 0\}} \psi_1. \tag{7.5}
\]
Since
\[
\int_{\Omega \cap \{x_1 > 0\}} x_1 \Delta \psi = -\int_{\Omega \cap \{x_1 > 0\}} \nabla x_1 \cdot \nabla \psi = -\int_{\Omega \cap \{x_1 > 0\}} \psi_1,
\]
and similarly
\[
\int_{\Omega \cap \{x_1 < 0\}} x_1 \Delta \psi = -\int_{\Omega \cap \{x_1 < 0\}} \psi_1,
\]
we deduce from (7.5) that
\[
0 = \alpha^2 \int_{\Omega \cap \{x_1 > 0\}} \psi_1 + \gamma^2 \int_{\Omega \cap \{x_1 < 0\}} \psi_1 - \int_{\Omega \cap \{u > 0\}} \psi_1. \tag{7.6}
\]
Now, if \(\alpha, \gamma > 0\), it follows that \(\Omega \cap \{u > 0\} = \Omega \cap \{x_1 \neq 0\}\) and consequently
\[
\int_{\Omega \cap \{u > 0\}} \psi_1 = \int_{\Omega \cap \{x_1 \neq 0\}} \psi_1 = \int_{\Omega} \psi_1 = 0. \tag{7.7}
\]
On the other hand, if \(\alpha, \gamma < 0\), it follows that \(\Omega \cap \{u > 0\}\) is void, and consequently
\[
\int_{\Omega \cap \{u > 0\}} \psi_1 = 0. \tag{7.8}
\]
Hence, in light of (7.7) and (7.8), we see that if either \(\alpha, \gamma > 0\) or \(\alpha, \gamma < 0\), we can write (7.6) as
\[
0 = -\alpha^2 \int_{\Omega \cap \{x_1 = 0\}} \psi + \gamma^2 \int_{\Omega \cap \{x_1 = 0\}} \psi,
\]
which leads to (1.23) and (1.24) in these cases.
If instead $\alpha > 0$ and $\gamma \leq 0$, we have that $\Omega \cap \{u > 0\} = \Omega \cap \{x_1 > 0\}$ and consequently, by (7.6),

$$0 = (\alpha^2 - 1) \int_{\Omega \cap \{x_1 > 0\}} \psi_1 + \gamma^2 \int_{\Omega \cap \{x_1 < 0\}} \psi_1$$

$$= -(\alpha^2 - 1) \int_{\Omega \cap \{x_1 = 0\}} \psi + \gamma^2 \int_{\Omega \cap \{x_1 = 0\}} \psi,$$

which leads to (1.25).

Furthermore, if $\alpha < 0$ and $\gamma = 0$, we have that $\Omega \cap \{u > 0\}$ is void, and hence (7.6) gives that

$$0 = \alpha^2 \int_{\Omega \cap \{x_1 > 0\}} \psi_1 + \gamma^2 \int_{\Omega \cap \{x_1 < 0\}} \psi_1 = -\alpha^2 \int_{\Omega \cap \{x_1 = 0\}} \psi.$$

As a consequence, we find that $\alpha = 0$, against our assumption, and then we obtain (1.26), as desired. \hfill $\Box$

### 8 Counterexamples to Uniform $C^{1,1}$ Bounds: Proofs of Theorems 1.7 and 1.8

Here we construct the one-dimensional counterexamples claimed in Theorem 1.7, using a suitable logarithmic bifurcation from a quadratic function, and in Theorem 1.8.

**Proof of Theorem 1.7** We let

$$u^\varepsilon(x) := \begin{cases} -x^2 \log(\varepsilon + x^4) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

see Fig. 1.
We observe that
\[-\frac{x^4 + \varepsilon}{2x} \frac{du^\varepsilon}{dx}(x) = (x^4 + \varepsilon) \log(x^4 + \varepsilon) + 2x^4 \leq (x^4 + \varepsilon)(\log(x^4 + \varepsilon) + 2) < 0\]
if \(x \in (0, \sqrt[4]{e^{\varepsilon^2} - \varepsilon})\), and so in particular if \(x \in (0, e^{-3/4})\) as long as \(\varepsilon\) is small enough. This says that the function \(u^\varepsilon : (0, e^{-3/4}) \rightarrow \mathbb{R}\) is strictly increasing and we denote by \(\zeta^\varepsilon\) its inverse. We observe that
\[u^\varepsilon(e^{-3/4}) = -e^{-3/2} \log(e^{-3} + \varepsilon) \geq 2e^{-3/2}\]
as long as \(\varepsilon\) is sufficiently small, hence we can define \(\zeta^\varepsilon\) in \((0, 2e^{-3/2})\).

In this way, for any \(t \in (0, 2e^{-3/2})\), we can write that
\[u^\varepsilon(\zeta^\varepsilon(t)) = t.\]

We let \(\iota^\varepsilon \in (0, \varepsilon/2)\). For all \(t \in (0, \iota^\varepsilon)\), we define
\[\beta^\varepsilon(t) := -16(\zeta^\varepsilon(t))^2 \left((\zeta^\varepsilon(t))^4 - 11\varepsilon(\zeta^\varepsilon(t))^8 + 135\varepsilon^2(\zeta^\varepsilon(t))^4 - 45\varepsilon^3\right) \left((\zeta^\varepsilon(t))^4 + \varepsilon\right)^4.\]

We notice that
\[\lim_{t \to 0^+} \left((\zeta^\varepsilon(t))^4 - 11\varepsilon(\zeta^\varepsilon(t))^8 + 135\varepsilon^2(\zeta^\varepsilon(t))^4 - 45\varepsilon^3\right) = -45\varepsilon^3 < 0,\]
and hence we can suppose that \(\beta^\varepsilon \geq 0\) in \((0, \iota^\varepsilon)\), provided that \(\iota^\varepsilon\) is sufficiently small (possibly in dependence of \(\varepsilon\)). We can also extend \(\beta^\varepsilon\) to be smooth, zero outside \((0, \varepsilon)\), and with integral 1.

Then, we see that, when \(x > 0\) is sufficiently small,
\[(u^\varepsilon)^{'''}(x) = -\frac{d^4}{dx^4}(x^2 \log(x + x^4)) = \frac{8x^2(x^{12} - 11\varepsilon x^8 + 135\varepsilon^2 x^4 - 45\varepsilon^3)}{(x^4 + \varepsilon)^4} = -\frac{1}{2}\beta^\varepsilon(u^\varepsilon(x)),\]
and so \(u^\varepsilon\) is a local solution of (1.1). Nevertheless, it does not possess second derivative bounds in \(L^\infty\) that are uniform in \(\varepsilon\) since, for \(x > 0\) sufficiently small,
\[(u^\varepsilon)^{''}(x) = \frac{2(6x^8 + 14\varepsilon x^4 + (x^4 + \varepsilon)^2 \log(x^4 + \varepsilon))}{(x^4 + \varepsilon)^2},\]
which converges to \(12 + 8 \log x\) as \(\varepsilon \to 0^+\), and
\[(u^\varepsilon)^{''}(\sqrt[4]{\varepsilon}) = 10 + 2 \log(2\varepsilon),\]
which becomes unbounded as $\varepsilon \to 0^+$.

**Proof of Theorem 1.8** We take $\phi \in C^\infty(\mathbb{R}, [0, +\infty))$ such that $\phi = 0$ in $(-\infty, 0]$, $\phi > 0$ in $(0, +\infty)$, $\phi(k) = k^2$ for every $k \in \mathbb{N}$ and

$$
\int_{\mathbb{R}} \phi(t) \, dt = 1.
$$

Let also

$$
u(x) := \int_{-\infty}^{x} \phi(t) \, dt = \int_{0}^{x} \phi(t) \, dt.
$$

We point out that

$$u = 0 \text{ in } (-\infty, 0].
$$

Moreover, we see that $u' = \phi > 0$ in $(0, +\infty)$, and

$$
\sup_\mathbb{R} u = \lim_{x \to +\infty} u(x) = \int_{\mathbb{R}} \phi(t) \, dt = 1. 
$$

Hence we can invert $u|_{[0, +\infty)}$ and we denote its inverse by $v$. In this way, $v : [0, 1] \to [0, +\infty)$ and for all $x \in (0, +\infty)$ we have that

$$v(u(x)) = x.
$$

Now, for all $t \in [0, 1]$ we define

$$
\beta(t) := -2u'''(v(t)).
$$

Let also

$$
u^\varepsilon(x) := \varepsilon u\left(\frac{x}{\sqrt{\varepsilon}}\right).
$$

Notice that

if $x \in (0, +\infty)$, then $u^\varepsilon(x) \in (0, \varepsilon)$;

moreover $u^\varepsilon = 0$ in $(-\infty, 0]$,

thanks to (8.3) and (8.4). In particular, the claims in (1.28) and (1.29) follow from (8.7).

It also follows from (8.7) that

$$
\frac{u^\varepsilon(x)}{\varepsilon} \in (0, 1) \quad \text{for all } x > 0,
$$

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and therefore, by (1.2) and (8.6), we have that, for all $x > 0$,

$$-eta \varepsilon \left( u^\varepsilon (x) \right) = -\frac{1}{\varepsilon} \beta \left( \frac{u^\varepsilon (x)}{\varepsilon} \right) = \frac{2}{\varepsilon} u''' \left( \frac{v \left( \frac{u^\varepsilon (x)}{\varepsilon} \right)}{\varepsilon} \right) = 2 \frac{u'''}{\varepsilon} (v) \left( \frac{x}{\sqrt{\varepsilon}} \right) = 2 \frac{u'''}{\varepsilon} (x).$$

(8.8)

In addition, from (8.2), we see that

$$u'''(0) = \phi'''(0) = \lim_{t \to 0^-} \phi'''(t) = 0.$$  

This and (8.6) give that

$$\beta(0) = -2u'''(v(0)) = -2u'''(0) = 0.$$  

Accordingly, from (1.2) and (8.7), for all $x \leq 0$,

$$-2(u^\varepsilon )'''(0) = \frac{2}{\varepsilon} u''' \left( 0 \frac{1}{\sqrt{\varepsilon}} \right) = 0 = \beta(0) = \beta \varepsilon \left( u^\varepsilon (x) \right).$$

This and (8.8) establish (1.27).

Finally,

$$(u^\varepsilon)'(x) = \sqrt{\varepsilon} u' \left( \frac{x}{\sqrt{\varepsilon}} \right) = \sqrt{\varepsilon} \phi \left( \frac{x}{\sqrt{\varepsilon}} \right).$$

Hence, defining $\varepsilon_k := k^{-2}$, we see that $\varepsilon_k$ is as small as we wish for large $k$, and

$$(u^\varepsilon_k)'(1) = \frac{1}{k} \phi(k) = k,$$

which gives (1.30), as desired.

\[\square\]

**Appendix A: Decay Estimates for the Gradient and the Hessian**

Here, we present some decay estimates for the gradient and the Hessian of solutions to (1.1).

**Proposition A.1** Suppose that $u^\varepsilon$ is a solution of (1.1) such that $|u^\varepsilon| \leq 1$ in $\Omega$. Let $D \subseteq \Omega$ and $R_0 \in (0, \text{dist}(D, \partial \Omega))$. Suppose that

$$\hat{C} := \inf_{x \in D \setminus \Delta} \int_{B_{R_0}(x)} \Delta u^\varepsilon > -\infty.$$  

(A.1)
Then, we have
\[
\frac{1}{R^{n+2}} \int_{B_R(x_0)} |\nabla u^\epsilon|^2 + \frac{1}{R^n} \int_{B_R(x_0)} |D^2 u^\epsilon|^2 \\
\leq \frac{C}{R^{n+4}} \int_{B_{4R}(x_0)} (u^\epsilon - m)^2 + \frac{\hat{C}}{R^{n+2}} \int_{B_{4R}(x_0)} (u^\epsilon - m),
\] (A.2)
for any \(x_0 \in D\) and any \(R \in (0, R_0/4)\), where
\[
m = m^\epsilon := \min_{B_{4R}(x_0)} u^\epsilon,
\] (A.3)
and \(C > 0\) depends only on \(n\).

**Proof** The proof follows from the argument used to prove Lemma A.1 in [8]. We briefly sketch the argument here. Up to a translation, we suppose \(x_0 := 0\). From the super biharmonicity of \(u^\epsilon\) we get
\[
0 \geq \int_{\Omega} \Delta u^\epsilon \Delta \phi = \sum_{i,j=1}^n \int_{\Omega} u^\epsilon_{ij} \phi_{ij}
\]
for every \(\phi \in C_0^\infty(\Omega), (0, +\infty))\), where two integration by parts are performed in the latter step. Choosing \(\phi := (u^\epsilon - m^\epsilon)\eta^2\), where \(m^\epsilon\) is as in (A.3), and \(\eta\) is a standard cut-off function supported in \(B_{2R} \subset \Omega\), such that \(\eta = 1\) in \(B_R\) and \(\eta = 0\) outside \(B_{2R}\) we get
\[
\sum_{i,j=1}^n \int_{\Omega} (u^\epsilon_{ij})^2 \eta^2 \leq \frac{C}{R^2} \int_{B_{2R}} |\nabla u^\epsilon|^2 + \frac{C}{R^4} \int_{B_{2R}} (u^\epsilon - m)^2,
\] (A.4)
for some universal constant \(C > 0\) (compare, e.g. with formula (A.6) in [8]).

On the other hand, using the mean value property \(\Delta u^\epsilon(x) \geq \int_{B_r(x)} \Delta u^\epsilon\) and the lower bound in (A.1), we obtain the Caccioppoli-type inequality
\[
\int_{B_{2R}} |\nabla u^\epsilon|^2 \leq \frac{C}{R^2} \int_{B_{4R}} (u^\epsilon - m^\epsilon)^2 + C \int_{B_{4R}} (u^\epsilon - m^\epsilon),
\] (A.5)
see e.g. formula (7.7) in [8]. Combining (A.4) and (A.5), we finish the proof. \(\square\)

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