Majorana corner and hinge modes in second-order topological insulator-superconductor heterostructures

Zhongbo Yan1,*

1School of Physics, Sun Yat-Sen University, Guangzhou 510275, China

As platforms of Majorana modes, topological insulator (quantum anomalous Hall insulator)/superconductor (SC) heterostructures have attracted tremendous attention over the past decade. Here we substitute the topological insulator by its higher-order counterparts. Concretely, we consider second-order topological insulators (SOTIs) without time-reversal symmetry and investigate SOTI/SC heterostructures in both two and three dimensions. Remarkably, we find that such novel heterostructures provide natural realizations of second-order topological superconductors (SOTSCs) which host Majorana corner modes in two dimensions and chiral Majorana hinge modes in three dimensions. As here the realization of SOTSCs requires neither special pairings nor magnetic fields, such SOTI/SC heterostructures are outstanding platforms of Majorana modes and may have wide applications in future.

Over the past decade, topological superconductors (TSCs) have attracted continuous and tremendous attention[1–9]. Among various TSCs, one-dimensional (1d) and two-dimensional (2d) TSCs without time-reversal symmetry (TRS) have attracted particular interest as they harbor Majorana zero modes (MZMs) at their boundaries[10–12] and in the cores of vortices[13–16], respectively. Owing to their fractional nature, MZMs are considered to be nonlocal qubits immune to local decoherence[10]. Moreover, owing to their non-Abelian statistics[17], their braiding operations are found to realize elementary quantum gates. Thus, MZMs are believed to be building blocks of topological quantum computation[18] and have been actively sought in experiments[19–29].

As is known, odd-parity superconductors (SCs) provide natural realizations of TSCs, however, they are unfortunately rare in nature. In a seminal paper[14], Fu and Kane pointed out that topological insulator (TI)/SC heterostructures provide an effective realization of odd-parity superconductivity. Accordingly, in the presence of magnetic field, vortices emerging in such heterostructures are found to carry MZMs. In a later influential paper[30], Qi et al pointed out that quantum anomalous Hall insulator (QAH)[1/2]SC heterostructures provide a simple realization of 2d chiral TSCs which harbor not only vortex-core MZMs, but also chiral Majorana edge modes. These two theoretical works have triggered a lot of experimental works on TI(QAH)/SC heterostructures[28, 29, 31–41], and remarkable progress in detecting vortex-core MZMs has been witnessed in recent years[28, 29, 40, 41].

Very recently, TIs and TSCs have been generalized to include their higher-order counterparts[42–56]. Importantly, higher-order TIs (HOTIs) and TSCs (HOTSCs) have extended the conventional bulk-boundary correspondence. Accordingly, an n-th order TI or TSC in d dimensions host (d−n)-dimensional boundary modes. For instance, a second-order TI (SOTI) in 2d and 3d host zero-dimensional (0d) corner modes and 1d hinge modes, respectively. The existence of HOTIs and the lessons from the study of TI(QAH)/SC heterostructures lead us to ask the natural question that whether

Majorana corner modes (MCMs, i.e., MZMs bound at the corners) or chiral Majorana hinge modes (CMHMs) can also be achieved in a HOTI/SC heterostructure. It is worth noting that such a question is quite timely as recently the electronic material candidates for SOTIs, both in two dimensions (2D) and three dimensions (3D), are growing[57–63]. Moreover, signature of MZM has also been observed in a heterostructure which consists of a bismuth thin film (a SOTI with TRS[57]), a conventional s-wave SC, and magnetic iron clusters[64].

In this work, we consider SOTIs without TRS and investigate SOTI/SC heterostructures in both 2D and 3D. Remarkably, we find that such heterostructures provide natural realizations of second-order topological superconductors (SOTSCs) which host MCMs in 2D and CMHMs in 3D. Furthermore, here the realization of SOTSCs does not require the pairing of SCs to take any specific form. It can be achieved for both unconventional SCs and conventional s-wave SCs. In addition, it does not need magnetic fields or the deposition of magnetic atoms. In comparison to previous proposals[65–90], these merits make SOTI/SC heterostructures stand out, and potentially allow them to have wide applications in topological quantum computation[91–93].

**MCMS in a 2d SOTI-SC heterostructure.**— A 2d SOTI/SC heterostructure (Fig.1) could be described by a Bogoliubov-de Gennes (BdG) Hamiltonian, $H = \sum_k \Psi^\dagger_k H(k) \Psi_k$, with $\Psi_k = (c_{a,k\uparrow}, c_{b,k\uparrow}, c_{a,k\downarrow}, c_{b,k\downarrow}, c_{1-a,-k\uparrow}, c_{1-b,-k\uparrow}, c_{1-a,-k\downarrow}, c_{1-b,-k\downarrow})^T$ and

$$H(k) = \epsilon(k)\sigma_z \tau_z + \Lambda_s \sin k_x \sigma_z \tau_y + \Lambda_y \sin k_y \sigma_y \tau_z + \Lambda(k) \sigma_z s_z \tau_z + \mu \tau_z + \Delta(k) s_y \tau_y,$$

where $\sigma_z$, $s_z$, and $\tau_z$ are Pauli matrices in orbit $(a, b)$, spin $(\uparrow, \downarrow)$
and particle-hole spaces, respectively; $\epsilon(\mathbf{k}) = m_0 - t_x \cos k_x - t_y \cos k_y$ is the kinetic energy; $\Lambda(\mathbf{k}) = \Lambda_x \cos k_x - \Lambda_y \cos k_y$ is a TRS breaking term crucial for the realization of SOTI; $\mu$ is the chemical potential, and $\Delta(\mathbf{k}) = \Delta_0 + \Delta_x \cos k_x + \Delta_y \cos k_y$ represents the pairing. Such a form is general enough to model $s$-wave, $s_\pm$-wave and $d$-wave pairings[68]. For convenience, the lattice constants have been set to unit, and $t_{x,y}$, $\Lambda_{x,y}$ and $\Delta_{x,y}$ are set to be positive throughout this work.

Let us focus on the normal state first. Without the terms in the second line of Eq. (1), the Hamiltonian describes a 2d first-order TI when $\prod_{\alpha,\beta=\pm}[m_0 + \alpha(t_x \cos k_x + t_y \cos k_y)] < 0$[94]. Accordingly, when open boundary condition is taken, gapless helical modes will appear on the boundary. Adding the $\Lambda(\mathbf{k})$ term breaks TRS and consequently gaps out the helical modes, resulting in a transition from a first-order TI to a SOTI. When open boundary conditions are taken in both the $x$ and $y$ directions, one can find that in the SOTI phase, each corner of the system will harbor one zero-energy bound state with a fractional charge $e/2[43]$. The pinning of the corner modes’ energy to zero is due to the existence of a chiral symmetry (the operator is $\sigma_x \partial_x$). When superconductivity enters, the operator is accordingly modified as $\sigma_y \tau_y \partial_x$. However, this chiral symmetry is just an accidental symmetry, adding an arbitrary term proportional to the identity matrix (e.g., the chemical potential) immediately breaks this symmetry and accordingly shifts the energy away from zero. Nevertheless, whether the chiral symmetry is preserved or not does not affect our following discussions since the particle-hole symmetry of a SC is sufficient to guarantee the topological robustness of MCMs.

To see the effect of superconductivity intuitively, let us focus on the case with chiral symmetry first. As is known, when a chiral electronic mode is in proximity to a SC, it becomes a chiral electronic mode instead of a normal mode. This is why the amplitude of the superconducting order parameter is zero in the chiral edge state. Now, let us focus on the edge (I) first. To obtain the corresponding Hamiltonian to edge (I), we follow ref.[30]. Accordingly, the matrix elements of $H_0$ under the basis composed by the four zero-energy solutions, which read

$$\psi_{\alpha}(x) = N_s \sin(n \pi x / 2) e^{i k_0 y} \chi_{\alpha},$$

where $N_s = 2 \sqrt{|\eta_1|^2 + |\eta_2|^2}$, $\eta_1 = \sqrt{2} \lambda \tau_x / t_x$ and $\eta_2 = \lambda \tau_z$. The four spinors $\chi_{\alpha}$ are determined by $\sigma_y \tau_y \partial_x \chi_{\alpha} = \lambda \chi_{\alpha}$. For their concrete forms, here we follow ref.[68]. Accordingly, the matrix elements of $H_0$ under the basis composed by the four zero-energy solutions are

$$H_{\text{LRD}}(k_y) = \int_0^{\infty} \psi_{\alpha}^*(x)(-i \partial_x, k_y) \psi_{\alpha}(x) dx.$$ (5)

The corresponding low-energy Hamiltonian for edge (1) is

$$H_1(k_y) = -\lambda_s k_y \tau_z + M_{\text{LA}} s_y + M_{\text{LS}} s_y \tau_y,$$ (6)

where the two Dirac masses $M_{\text{LA}}$ and $M_{\text{LS}}$ are of different origins, and they are given by

$$M_{\text{LA}} = -\int_0^{\infty} dx \psi_{\alpha}^* \Lambda(-i \partial_x) \psi_{\alpha}(x) = -\Lambda \frac{m_{\Lambda}}{t_x},$$

$$M_{\text{LS}} = \int_0^{\infty} dx \psi_{\alpha}^* \Lambda(-i \partial_x) \psi_{\alpha}(x) = \Delta \frac{m_{\Delta}}{t_x}.$$ (7)

Similarly, the low-energy Hamiltonians for the other three edges are

$$H_{\text{II}}(k_y) = \lambda_y k_y \tau_z + M_{\text{IIA}} s_y + M_{\text{IIS}} s_y \tau_y,$$

$$H_{\text{III}}(k_y) = \lambda_y k_y s_z + M_{\text{IIIA}} s_y + M_{\text{IIS}} s_y \tau_y,$$

$$H_{\text{IV}}(k_y) = -\lambda_s k_y s_z + M_{\text{IVA}} s_y + M_{\text{IVS}} s_y \tau_y,$$ (8)

with $M_{\text{IIA}} = M_{\text{IVA}} = -\Lambda - m_{\Lambda} / t_y$, $M_{\text{IIS}} = M_{\text{IVS}} = \Delta - m_{\Delta} / t_y$, and $M_{\text{IIIA}} = M_{\text{IVA}}, M_{\text{IIS}} = M_{\text{IVS}}$. By using the boundary coordinate, the low-energy Hamiltonian can be written compactly as

$$H_{\text{Edge}} = -i (l \partial_x) \sigma_y \tau_z + M_L(l) s_y + M_S(l) s_y \tau_y.$$ (9)
where \( \lambda(l) \), \( M_\lambda(l) \) and \( M_S(l) \) are step functions with their values following the sequences: \( \lambda(l) = \lambda_x, \lambda_y, \lambda_z, \Lambda_x, M_\lambda(l) = -\Lambda + m\Delta_x/t_x, -\Lambda - m\Delta_y/t_y, -\Lambda + m\Delta_z/t_z, -\Lambda - m\Delta_y/t_y \), and \( M_S(l) = \Delta - m\Delta_x/t_x, \Delta - m\Delta_y/t_y, \Delta - m\Delta_y/t_y \) for (I), (II), (III) and (IV), respectively.

Without loss of generality, let us focus on the case with \( \Lambda_x = \Lambda_y \) so that \( \Lambda = 0 \). In the absence of pairing, i.e., \( M_S(l) = 0 \), \( H_{\text{Edge}} \) reduces to a \( 2 \times 2 \) matrix. At each corner, \( \lambda(l) \) does not change sign, but \( M_\lambda(l) \) does, realizing a domain wall of Dirac mass which harbors one charged zero mode according to the Jackiw-Rebbi theory. When superconductivity enters, one can see that \( H_{\text{Edge}} \) is the direct sum of two independent parts, i.e., \( H_{\text{Edge}} = H_{\tau_x=1} \oplus H_{\tau_y=1} \) with

\[
H_{\tau_x=1} = -i\lambda(l)\hat{s}_x + (M_\lambda(l) + M_S(l))s_y,
H_{\tau_y=1} = -i\lambda(l)\hat{s}_y + (M_\lambda(l) - M_S(l))s_y.
\]

One can see that the Dirac mass induced by superconductivity takes different signs in the two parts. In the weak-pairing limit, \( |M_S(l)|/|M_\lambda(l)| \ll 1 \), each part realizes one zero mode per corner. As the particle component and the hole component of these zero modes’ wave functions are equal (note \( \tau_y\psi_0(l) = \pm \psi_0(l) \), where \( \psi_0(l) \) denotes the wave function of zero mode), they are MZMs, agreeing with our previous argument that weak superconductivity will transform one charged zero mode to two MZMs. As now each corner harbors two MZMs, these MCMs are not stable. Indeed, we find that any finite \( \mu \) or on-site potential will make them couple (the chemical potential term contains \( \tau_z \), so it makes the \( \tau_y = 1 \) part couple with the \( \tau_y = -1 \) part) and consequently destroy their self-conjugate nature. This can also be understood from the perspective that because \( \mu \) shifts the energy of charged corner modes away from zero, the energy of corner modes will keep taking finite values if the superconductivity is very weak. Therefore, for the square geometry presented in Fig.1, robust MCMs are absent in the weak-pairing limit. Noteworthily, as \( M_\lambda(l) \) is in fact sensitive to the orientation of edge, here we have emphasized the particular square geometry shown in Fig.1. As will see shortly, if the sample’s geometry is appropriately designed, the critical value of pairing amplitude for realizing robust MCMs can be very small, so even weak superconductivity is sufficient.

To see how robust MCMs emerge in a square sample, we take \( s \)-wave pairing for illustration (other more exotic cases can similarly be analyzed). Accordingly, \( M_S(l) = \Delta_0 \) is uniform on the boundary. Without loss of generality, we further assume \( m\Delta_y/t_y > m\Delta_x/t_x \). According to Eq.(10), one can find when \( m\Delta_x/t_x < \Delta_0 < m\Delta_y/t_y \), while the domain walls for \( H_{\tau_x=1} \) are preserved since \( m\Delta_x/t_x + \Delta_0 \) and \( -m\Delta_y/t_y + \Delta_0 \) still take opposite signs, the ones for \( H_{\tau_y=1} \) are removed since now \( m\Delta_x/t_x - \Delta_0 \) and \( -m\Delta_y/t_y - \Delta_0 \) take same sign. As a result, there is only one MZM per corner in this regime, as shown in Fig.2(a). We have numerically checked that these MCMs are robust against local perturbations, doping and random disorder as long as the doping level and disorder strength are small than some critical values (note in Fig.2(a), \( \mu = 0.1 \)).

According to the criterion \( m\Delta_x/t_x < \Delta_0 < m\Delta_y/t_y \), one may make the conclusion that if the underlying pairing is \( s \)-wave, anisotropy is necessary for the realization of SOTSC. That is, if \( \Lambda_x = \Lambda_y \), \( t_x \neq t_y \) must be satisfied. However, anisotropy is in fact unnecessary. For the isotropic case with \( t_x = t_y \) and \( \Lambda_x = \Lambda_y \), \( M_\lambda(l) \) follows the angle dependence \( M_\lambda(l) \approx m\Lambda_x \cos \theta/l_x \), where \( \theta \) represents the angle relative to edge (I). This indicates that on the edge whose orientation is pointing to \( \theta = \pi/4, M_\lambda(l) = 0 \). As a result, one can find that for the \( \pi/4 \)-angle corner formed by edge (I) and the \( \theta = \pi/4 \)-orientation edge, it will harbor one MZM as long as \( 0 < |\Delta_0| < m\Delta_x/t_x \). We demonstrate the validity of this analysis numerically, as shown in Figs.2(b)(c). According to the phase diagram in Fig.2(c), one can see that for an isosceles-right-triangle geometry, MCMs can exist for a quite broad range of \( \mu \) and for infinitely weak pairing amplitude.
As for a SOTI, $M_N(l)$ inevitably vanishes along some direction, this implies that a judicious design of the corners is always able to realize MCMs even though the superconductivity is weak. Clearly, this conclusion also holds for other unconventional SCs.

**CMHMs in a 3d SOTI/SC heterostructure.**—The scenario above can straightforwardly be generalized to 3D. For example, if we have a 3d SOTI at hand, we can grow a thin film of s-wave SC on its surface (see Figs.3(a)(b)). Accordingly, the system could be modeled by $H = \sum_{l} \Psi_{l}^{\dagger}H(k)\Psi_{l}$ with

$$H(k) = \xi(k)\sigma_z\tau_z + \sum_{i=x,y,z} \lambda_i \sin k_{i}\sigma_i \xi_i + \lambda_0 \sin k_{D}\sigma_y \tau_z$$

$$+ \Lambda(k)\sigma_y + \mu \tau_z + \Delta_0 \sigma_y \tau_y,$$

where $\xi(k) = m_0 - t_x \cos k_x - t_y \cos k_y - t_z \cos k_z$. Without the terms in the second line, the Hamiltonian describes a strong TI when $\prod_{\alpha,\beta=x,y}(m_0 + \alpha t_{\alpha} + \beta t_{\beta} + \gamma t_{\gamma}) < 0$. Accordingly, when open boundary condition is taken, gapless Dirac surface states will appear on the boundary. The presence of the term $\Lambda(k)$ gaps out the Dirac surface states on the four lateral surfaces (in 3D, we take open boundary condition in both the $x$ and $y$ directions, and periodic boundary condition in the $z$ direction) and leaves one chiral electronic mode per hinge\[46\].

As mentioned before, when superconductivity enters, one chiral electronic mode becomes two chiral Majorana modes in the weak-pairing limit\[30\]. Unlike the MCMs in two dimensions, while here the wave functions of the two chiral Majorana modes also overlap in space, they are stable against perturbations since they are chiral in nature. Therefore, in the weak-pairing regime, there are two robust chiral Majorana modes per hinge (see Figs.3(a)(c)). Interestingly, we find that with the increase of pairing amplitude, a topological phase transition will take place on the boundary and accordingly a new SOTSC which host one robust chiral Majorana mode per hinge will be realized (see Figs.3(b)(d)). It is worth noting that when doing the calculation of the energy spectra presented in Figs.3(c)(d), the superconductivity has been taken to be uniform throughout the whole sample. It is apparent that this assumption is unrealistic for the heterostructure since deep in the bulk the superconductivity induced by proximity effect should vanish, however, the low-energy physics within the gap can be well captured since the in-gap states are located on the surfaces which are well in contact with the SC. In other words, here the derivation from real situation only has strong impact on the bulk states. In fact, if we focus on the in-gap states, we can also adopt the edge theory as in 2D. For the geometry shown in Figs.3(a)(b), one can easily find that the criterion for realizing the SOTSC phase with one robust chiral Majorana mode per hinge is also $m_{\Lambda} l_{x} - \Delta_0 < m_{\Lambda} l_{y}$ ($m = t_{x} + t_{y} + t_{z} - m_0 > 0$, $\mu = 0$, and $\Lambda_x = \Lambda_y$ are also presumed). One can see that the results presented in Figs.3(c)(d) are consistent with this criterion. Similar to the 2d situation, the critical pairing amplitude can also be tuned to take a very small value if the sample’s geometry is appropriately designed.

**Conclusions.**—We have shown that SOTI/SC heterostructures provide promising new platforms of MCMs and CMHMs. As our proposed scheme requires neither special pairings nor magnetic fields, we believe it should be simple to implement experimentally. Consider the fast growth of material candidates for SOTIs\[57–63\], we can foresee that such novel heterostructures will be synthesised and investigated in the near future. Experimentally, MCMs and CMHMs can be probed by STM techniques\[64\] and transport experiments\[97\].

**Acknowledgements.**—We would like to acknowledge the support by a startup grant at Sun Yat-sen University.

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Supplemental Material

In this supplemental material, we provide the details about the determination of the phase diagram. Let us first rewrite down the Hamiltonian, which is

\[ H(\mathbf{k}) = \epsilon(\mathbf{k})\sigma_i x^i + \lambda_4 \sin k_x \sigma_z s_x + \lambda_4 \sin k_y \sigma_y s_z + \Lambda(\mathbf{k})\sigma_i x^i \tau_x + \mu \tau_z + \Delta(\mathbf{k}) s_y \tau_y, \]

where \( \sigma_i, s_i \) and \( \tau_i \) are Pauli matrices in orbit \( (a, b) \), spin \( (\uparrow, \downarrow) \) and particle-hole spaces, respectively; \( \epsilon(\mathbf{k}) = m_0 - t_x \cos k_x - t_y \cos k_y \) is the kinetic energy; \( \Lambda(\mathbf{k}) = \Lambda_4 \cos k_x - \Lambda_4 \cos k_y \) is a time-reversal symmetry breaking term crucial for the realization of SOTI; \( \mu \) is the chemical potential, and \( \Delta(\mathbf{k}) = \Delta_0 + \Delta_4 \cos k_x + \Delta_4 \cos k_y \) represents the pairing. Here we focus on conventional \( s \)-wave superconductor, so we let \( \Delta_4 = 0 \).

The Hamiltonian has an intrinsic particle-hole symmetry, i.e., \( PH(\mathbf{k})P^{-1} = -H(-\mathbf{k}) \) with \( P = \tau_1 K \) \( (K \) denotes the charge conjugate). For the special case with \( \mu = 0 \), the Hamiltonian has an additional chiral symmetry, i.e., \( \{ C, H(\mathbf{k}) \} = -H(\mathbf{k}) \) with \( C = \sigma_3 s_y \tau_z \). The energy spectra of this Hamiltonian are given by

\[ E(\mathbf{k}) = \pm \sqrt{F(\mathbf{k}) \pm 2 \sqrt{G(\mathbf{k})}}, \]

where \( F(\mathbf{k}) = \epsilon^2(\mathbf{k}) + \lambda_4^2 \sin^2 k_x + \lambda_4^2 \sin^2 k_y + \Lambda^2(\mathbf{k}) + \frac{\mu^2}{2} \), and \( G(\mathbf{k}) = \mu^2(\epsilon^2(\mathbf{k}) + \lambda_4^2 \sin^2 k_x + \lambda_4^2 \sin^2 k_y + \Lambda^2(\mathbf{k}) + \Lambda^2(\mathbf{k}) \Lambda_4^2) \). If without the terms in the second line the Hamiltonian in Eq.(12) describes an insulator, then the above energy spectra are always gapped as long as \( \Delta_0 \neq 0 \).

Now we consider that without the terms in the second line in Eq.(12), the Hamiltonian describes a first-order topological insulator with gapless helical edge modes on the boundary. As mentioned in the main text, adding the \( \Lambda(\mathbf{k}) \) term drives the system to a second-order topological insulator with localized corner modes. When superconductivity enters, as bulk energy spectra keep gapped no matter what value the pairing amplitude and the chemical potential take, this implies that the first-order topological property is always trivial.

The change of topological property (or say topological phase transition) is associated with the close of energy gap. For a first-order topological phase, topological phase transition is associated with the close of bulk energy gap. Accordingly, for an \( n \)-th-order topological phase in \( d \) dimensions, the topological phase transition is associated with the close of energy gap of the \((d-n+1)\)-dimensional boundary modes. Guided by this principle, in the following we investigate the phase diagram.

For concreteness, we consider the isosceles-right-triangle geometry (see Fig.2(b) in the main text) and focus on the case with isotropic parameters, i.e., \( t_x = t_y = \lambda_4 = \lambda_4 = \Lambda_4 = 1 \). Let us first focus on the edge whose orientation is in parallel to the \( y \) direction. To obtain the corresponding energy spectra of edge modes, it is more convenient to consider that the system takes open boundary condition in the \( x \) direction and periodic boundary condition in the \( y \) direction. We first consider the case without superconductivity. As shown in Fig.4(a), the in-gap edge modes are gapped, which is consistent with the trivialness of first-order topological property.

For convenience, we label the gap of edge-mode energy spectra as \( E_g \). When \( |\mu| < E_g/2 \), there is no boundary Fermi surface, therefore when superconductivity enters, the topological property of this edge corresponds to the strong pairing regime[13]. For a fixed pairing amplitude, with the increase of \( \mu \), the boundary topological property will undergo a transition from strong pairing regime to weak pairing regime[13]. Accordingly, \( E_g \) will undergo an “open-to-closed-to-open” transition (see Figs.4(b)(c)(d)). At the critical point, it gets closed (see Fig.4(c)). In comparison, as the \( \Lambda(\mathbf{k}) \) term vanishes along the \( k_x = k_y \) and \( k_z = -k_y \) directions, the energy spectra of edge modes on the edge with orientation pointing to \( \theta = \pi/4 \) (\( \theta \) is defined in relative to the positive \( y \) direction) will keep gapless before the superconductivity enters, implying that the topological property on the \( \theta = \pi/4 \)-orientation edge always corresponds to the weak pairing regime. As a result, when the topological property of the \( \theta = 0 \)-orientation edge corresponds to the strong pairing regime, the \( \pi/4 \)-angle corner, which is the intersection of \( \theta = 0 \)-orientation edge and \( \theta = \pi/4 \)-orientation edge, is a domain wall which harbors one robust Majorana zero mode. According to this principle, the phase diagram can be mapped out, as shown in Fig.5(a). We have confirmed that the phase boundary determined by using this principle is consistent with the approach based on the direct diagonalization of the real-space Hamiltonian (see Fig.5(b)(c)(d)). Owing
FIG. 5. (a) Phase diagram for an isosceles-right-triangle geometry. Common parameter are $t_x = t_y = A_x = A_y = \Lambda_x = \Lambda_y = 1$, and $m_0 = 1.5$. To show that the phase diagram can be determined by simply investigating the gap of edge-mode energy spectra, we focus on the two dashed lines shown in (a) for illustration. (b) $E_g$-vs-$\mu$. For a fixed pairing amplitude, with the increase of $\mu$, the gap of edge-mode energy spectra for the edge with orientation in parallel to the $y$ direction will undergo an “open-to-closed-to-open” transition. Accordingly, the topological property of this edge undergoes a transition from the strong pairing regime to weak pairing regime. (c)(d) Energy spectra for an isosceles-right-triangle sample whose length of the two right-angle sides are equal to 40. Here only the part near zero energy has been shown. (c) $\Delta_0 = 0.1$; (d) $\Delta_0 = 0.2$. In (c)(d), the red lines correspond to the energy spectra of the two Majorana corner modes. The blue lines correspond to the energy spectra of two bound states located at the right-angle corner, one can see that once $\mu$ goes away from zero, their energies are split. One can infer from (b)(c)(d) that using the gap close of edge-mode energy spectra can faithfully determine the phase boundary.