Accidental crossings of eigenvalues in one-dimensional complex PT-symmetric Scarf-II potential

Zafar Ahmed\textsuperscript{1}, Dona Ghosh\textsuperscript{2}, Joseph Amal Nathan\textsuperscript{3}

\textsuperscript{1}Nuclear Physics Division, Bhabha Atomic Research Centre, Mumbai 400 085, India
\textsuperscript{2}Astivinayak Society, Vashi, Navi-Mumbai, 400703, India
\textsuperscript{3}Reactor Physics Design Division, Bhabha Atomic Research Centre, Mumbai 400085, India

(Dated: March 10, 2015)

Abstract

So far, the two branches of real discrete spectrum of complex PT-symmetric Scarf II potential are kept isolated. Here, we suggest that these two need to be brought together as doublets: $E_{\pm}^n(\lambda)$ with $n = 0, 1, 2,...$. Then if strength ($\lambda$) of the imaginary part of the potential is varied smoothly some pairs of eigenvalue curves can intersect and cross each other. This is unlike one dimensional Hermitian potentials. However, we show that the corresponding eigenstates are linearly dependent denying degeneracy in one dimension, once again. Other pairs of eigenvalue curves coalesce to complex-conjugate pairs completing the scenario of spontaneous breaking of PT-symmetry.

\textsuperscript{*}Electronic address: 1:zahmed@barc.gov.in, 2: rimidonaghosh@gmail.com, 3:josephan@barc.gov.in
In PT-symmetric quantum mechanics [1, 2], one considers non-real, non-Hermitian Hamiltonians which are invariant under the joint action of Parity \((P:x \rightarrow -x)\) and Time-reversal \((T:i \rightarrow -i)\). Even the most simple Hamiltonian \(H = p^2 + V(x)\) corresponding to the Schrödinger equation for these potentials has given amazing results. Based on numerical computations Bender and Boettcher [1] conjectured that the spectrum of \(V_{BB}(x, \epsilon) = x^2(ix)^\epsilon\) was entirely real when \(\epsilon \geq 0\). Next, for \(-1 < \epsilon < 0\) the spectrum consisted of a few real and the rest as complex-conjugate pairs of discrete eigenvalues. In the former case the energy eigenstates were also the eigenstates [1] of PT and the PT-symmetry was exact or unbroken. Interestingly, \(V_{BB}(x, 2) = -x^4\) is a real Hermitian barrier (not a well), the real positive discrete spectrum has been aptly interpreted [3] as the reflectivity zeros in scattering from such potentials as \(V(x) = -x^{2n+2}, n = 1, 2, 3,\ldots\).

Several exactly solvable potentials were complexified to produce [4] exactly solvable complex PT-symmetric potentials having real discrete spectrum. Existence of two branches of real discrete spectrum in complex PT-symmetric Scarf II was revealed [5] and interpreted in terms of (unknown) quasi-parity [6]. Complex PT-symmetric Scarf II

\[
V(x) = -V_1 \text{sech}^2 x + iV_2 \text{sech} x \tanh x, \quad V_1 > 0,
\]

was shown to be a exactly solvable model displaying the spontaneous breaking of PT-symmetry when the the strength of the imaginary part, \(|V_2|\), exceeded a critical value of \(V_1 + 1/4\) [7], where \(-V_1(V_1 > 0)\) was the strength of the real part. Such a phase transition of eigenvalue from real to complex conjugate pairs in complex Scarf II has inspired several theoretical investigations [7-11]. On the other hand very interesting experiments in wave propagation and optics [12] have been performed where they realize PT-symmetry as equal gain and loss medium.

The paradigm model \(V_{BB}(x, \epsilon)\) [1-2] of complex PT-symmetric potential was solved numerically for energy eigenvalues and the obtained eigenvalues can seen as \(E_0, (E_1, E_2), (E_3, E_4), (E_5, E_6), \ldots\) in the increasing order. Only \(E_0\) was unpaired, other levels were paired as doublets. These pair coalesce in the parametric domain \(\epsilon \in (-1, 0)\). For \(\epsilon \geq 0\), they open up to diverge from each other. In another model \(V_A(x) = x^4 + iAx\) [13] when \(|A| < 3.169\), the whole discrete spectrum is real and the eigenvalues can be arranged as \((E_0, E_1), (E_2, E_3), (E_4, E_5),\ldots\), here the pairing of doublets unlike the case of
\( V_{BB}(x) \) starts from the ground states itself. In Hermitian case, doublets mean even(odd) parities or even(odd) numbers of nodes of eigenstates. In complex PT-symmetric quantum mechanics the energy eigenfunctions are complex and they loose definite parity [14] hence the number of nodes are no more meaningful. The interesting question arising here is whether one can assign a quasi-parity to the levels/states of these doublets.

On the contrary for the exactly solvable versatile complex PT-symmetric Scarf II, two separate branches of real discrete spectrum occur by virtue of the two linearly independent analytic solutions of the Schrödinger equation. Here, one can have \( E^n_+ \) and \( E^n_- \) with ± denoting some (unknown) quasi-parity [5,6]. Consequently, due to the isolation of these two branches several interesting features have gone un-noticed. The question arising here is as to what happens if these two branches are brought together.

In Hermitian quantum mechanics in two or more dimensions if two eigenvalues are equal but their corresponding eigenfunctions are different (not linearly dependent) this is called degeneracy. In other words if a parameter in the Hamiltonian is varied slowly and smoothly two energy eigenvalue curves may come very close and avoid crossing each other or they can intersect and cross each other. A degeneracy (crossing of energy eigenvalue curves) may be accidental or may arise due to some symmetry of Hamiltonian. In one dimension it can be shown (see Appendix 1) that irrespective of whether the Hamiltonian is real or complex a degeneracy can not occur.

In this Letter, we study both the branches \([E^n_\pm(V_2)]\) together by fixing \( V_1 \) and varying \( V_2 \) slowly and smoothly to reveal the accidental crossing of two energy eigenvalues curves unlike the Hermitian case in one dimension. However, we will show the corresponding eigenstates become linearly dependent to dislodge degeneracy in one dimension, once again. We demonstrate some more features of eigenvalues including the coalescing of these two branches of real discrete eigenvalues to complex conjugate pairs when PT-symmetry breaks down spontaneously (when \(|V_2| > V_1 + 1/4 \) [7]). In Schrödinger equation, we generally take \( 2\mu = 1 = \hbar^2 \).

Let us define

\[
\begin{align*}
  s &= \sqrt{1/4 + V_1 + |V_2|} \quad \text{and} \quad t = \sqrt{1/4 + V_1 - |V_2|},
\end{align*}
\]

(2)
then the real discrete spectrum of Scarf II is given as

\[ E^+_n(t, s) = -[n - (s \pm t - 1)/2]^2, \, n < 0, 1, 2, \ldots (s \pm t - 1)/2. \tag{3} \]

Here, for a fixed \( n \), \( E^+_n < E^-_n \) so we shall be calling \( E^+_n \) as lower eigenvalue i.e. \( E^+_n \) is the ground state. Unlike the case of real Hermitian Hamiltonians in the case of complex PT-symmetric Hamiltonians the oscillation theorem connecting the nodes of the eigenfunction with the quantum number \( n \) does not follow. The respective eigenstates are given as [7]

\[ \psi^n_\pm(x, t, s) = A_\pm \left( \text{sech}x \right)^{(s \pm t - 1)/2} \exp\left[-i\frac{1}{2}(s \mp t) \tan^{-1}(\sinh x)\right] P^{s+t,-s}_n(i \sinh x). \tag{4} \]

The Jacobi polynomials \( P^{a,b}_n(i \sinh x) \) [15] are polynomials of degree \( n \) in \( \sinh x \) at large values of \( |x| \) their divergence is damped by \( (\text{sech}x)^\nu \) as \( n < \nu = (s \pm t - 1)/2 \). The exponential terms is always finite. Consequently, we get \( L^2 \)-integrable eigenfunctions satisfying Neumann boundary condition.

An other interesting form of the eigenstates of (1) can be written as [7]

\[ \psi^n_\pm(x, t, s) = B_\pm (1 - z)^{\pm t/2 + 1/4}(1 + z)^{-s/2 + 1/4} P^{s+t,-s}_n(z), \quad z = i \sinh x, \tag{5} \]

which helps in the sequel. These two branches (\( \pm \)) are kept in isolation. Let us see when and whether the condition

\[ E^+_m(t, s) = E^-_n(t, s), \quad m > n \tag{6} \]

is met. From Eq. (6) the interesting condition is

\[ t = m - n, \quad m > n, \tag{7} \]

Thus, when \( t(V_1, V_2) \) is an integer the two eigenvalue curves can intersect. This amounts to occurrence of degeneracy, we are however in one dimension (see Appendix 1). Here we claim an interesting (un-noticed) property of Jacobi polynomials, namely

\[ P^{-j,s}_n(z) = C (1 - z)^j P^{j,s}_{n-j}(z), \quad j, n \in I^+ + \{0\}, \quad j < n, \quad s \in \mathbb{R}, \tag{8} \]

we prove this in the Appendix 2. Therefore when \( t = j \) then

\[ E^{n-j}_- = E^+_n, \quad \psi^{n-j}_-(x, j, s) = C \psi^n_+(x, j, s), \tag{9} \]
where $C$ is independent of $x$ or $z$. Hence, two intersecting energy eigenvalues will have corresponding eigenstates as linearly dependent, confirming no degeneracy in one dimension even in non-Hermitian PT-symmetric quantum mechanics.

When $|V_2| > V_1 + 1/4$, PT-symmetry is spontaneously broken $t$ becomes purely imaginary i.e., $t = ir$ and all $E_n^\pm$ pairs become complex-conjugate.

$$E_n^\pm = -[n - (s \pm ir - 1)/2]^2, r = \sqrt{|V_2| - V_1 - 1/4} \in \mathbb{R}, n < 0, 1, 2, ... (s - 1)/2. \quad (10)$$

The corresponding eigenstates are [7]

$$\psi_n^\pm(x) = D_{\pm} \text{sech}x^{(s \pm ir - 1)/2} \exp[\frac{1}{2}(-is \mp r) \tan^{-1}(\sinh x)] P_{n+\mp ir-s}(i \sinh x). \quad (11)$$

These eigenstates are $L^2$-integrable as the divergence of the Jacobi polynomial $P_{n+\mp ir-s}(i \sinh x)$ for large values of $|x|$ is suppressed by $(\text{sech}x)^{(s-1)/2}$ as $n < (s-1)/2$. Further, the argument of the exponential term is always finite irrespective of the values of $s, r$. Consequently the eigen states vanish asymptotically satisfying the Neumann boundary condition that $\psi(\pm \infty) = 0$. Now we can clearly see that the eigenstates flip under the action of PT as [16,17]

$$(PT)\psi_n^\pm(x) = \psi_n^\mp(x). \quad (12)$$

For $V_1 = 20$ and $V_2 = 0$ Scarf II is real Hermitian and it has 4 real discrete eigenvalues with states of definite parity: even/odd. Next we vary $V_2$ from 0 to $V_2 (> V_1)$. In Fig. 1(a) see the variation of $E_0^0(V_2)$, $E_+^0$ starts from the eigen value of the real potential (1) i.e., $E_+^0 = 16$, when it is complex with $V_2 \sim 5$ onwards $E_+^0$ starts existing then at $V_2 = 21.25$ they both coalesce in to one pair of complex conjugate eigenvalues (see the straight line ) demonstrating the spontaneous break down of PT symmetry. The rectangle from $V_2 = 0$ to $\sim 5$ is not a part of the curve it only denotes non-existence of any eigenvalue ($E_0^0$). The linear part from $V_2 = V_1 + 1/4$ onwards is the variation the discrete real part of the complex conjugate pair of eigenvalues. Similarly, coalescing of $E_+^1$ and $E_+^2$ to complex conjugate pairs is shown in Figs. 1(b) and 1(c), respectively. Only $E_+^3$ exists and $E_+^3$ does not exist (see Fig. 1(c)).

In Fig. 2(a) all the curves of Fig. 1(a,b,c,d) are put together to present the hitherto un-noticed feature of level crossings when $V_2$ takes values special values 11.25, 16.25 and 19.25 (see the condition (7)). Fig. 2(b) shows the variation of $t(V_1 = 20, V_2)$ and at special values
of $V_2$ it becomes 3, 2 and 1. Since discrete levels exist only for $n \leq 3$ we do not consider the case of $t = 4$. Also see the special pairs of eigenvalues becoming identical in Table 1 at these special values of $V_2$. See the claimed level crossings in Table 1. For $V_2 = 11.25$, $t = 2$, we get just one level crossing as we have $E_0^0 = E_3^3$. For $V_2 = 16.25$, $t = 2$ giving two level crossings: $E_0^0 = E_2^2$ and $E_1^1 = E_3^3$. When $V_2 = 19.25$, $t = 1$, notice three level crossings in Table 1 as we have $E_0^0 = E_1^1$; $E_1^1 = E_2^2$; $E_2^2 = E_3^3$. The eigenstates corresponding to these identical levels are linearly dependent as per Eq. (9).

\[ E_0^0 - E_3^3 = E_3^3 + E_0^0 \]

\[ E_1^1 - E_2^2 = E_2^2 + E_1^1 \]

\[ E_2^2 - E_3^3 = E_3^3 + E_2^2 \]

The eigenstates corresponding to these identical levels are linearly dependent as per Eq. (9).

**FIG. 1:** Variation of real part of discrete eigenvalues, $E_{n}^m(V_2)$, of Scarf II (1) for $V_1 = 20$. The rectangles are not the part of the curves they just indicate absence of eigenvalues inside them. The curved parts display real eigenvalues and the linear parts the complex conjugate pairs starting from the critical value of $V_2 = 20.25$. Solid (dashed) lines indicate +(-).
FIG. 2: (a): all the eigenvalue curves of Fig. 1 have been brought together, see several level crossings. These crossings are accidental which occur when (b): \( t(20, V_2) \) takes integer values \( 0, 1, 2, 3, \ldots \); \( t = 0 \) indicating multiplicity of eigenvalues before real discrete eigenvalues coalesces to complex conjugate pairs.

FIG. 3: The variation of transmission probability, \( T(E) \), for negative energies for scarf II (1) when \( V_1 = 20 \) and \( V_2 = 17 \). All six negative energy poles correspond to six real discrete eigenvalues listed in Table 1.

In Table 1, the discrete eigenvalues of scarf II (1) are given for a fixed value of \( V_1 = 20 \). When \( V_2 = 0 \), the Hermitian potential (1) has four real discrete eigenvalues. Even when strength of the imaginary part i.e., \( V_2 = 5 \) we have four real eigenvalues only in the + branch. When \( V_2 > 5 \) the other branch starts picking up for instance when \( V_2 = 9 \) we have \( E^0 \) appearing. Then on, the negative branch start developing.

Had the Scarf II not been solved analytically, then by numerical integration of
TABLE I: Discrete energy eigenvalues of Scarf II (1) for $V_1 = 20(2\mu = 1 = \hbar^2)$ $V_2$ being varied.

Dash signs mean absence of a discrete eigenvalue.

| $V_2$ | $t(V_2)$ | $E^0$ | $E^0_+$ | $E^1_-$ | $E^1_+$ | $E^2$ | $E^2_+$ | $E^3$ | $E^3_+$ |
|-------|----------|-------|---------|---------|---------|-------|---------|-------|---------|
| 0.0   | 4        | —     | -16.00  | —       | -9.00   | —     | -4.00   | —     | -1.00   |
| 5.0   | 4.03     | —     | -15.82  | —       | -8.86   | —     | -3.91   | —     | -0.95   |
| 9.0   | 3.35     | -0.27 | -15.06  | —       | -8.30   | —     | -3.53   | —     | -0.77   |
| 11.25 | 3        | -0.65 | -14.48  | -0.50   | -7.87   | —     | -3.26   | —     | -0.65   |
| 16.25 | 2        | -2.31 | -12.39  | -0.27   | -6.35   | —     | -2.31   | —     | -0.27   |
| 17    | 1.80     | -2.72 | -11.92  | -0.42   | -6.01   | —     | -2.11   | —     | -0.20   |
| 19.25 | 1        | -4.59 | -9.87   | -1.30   | -4.59   | -0.02 | -1.30   | —     | -0.02   |
| 20.25 | 0        | -7.19 | -7.19   | -2.82   | -2.82   | -0.46 | -0.46   | —     | —       |
| 20.30 | 0.22i    | -7.19 | -7.19   | -2.82   | -2.82   | -0.45 | -0.45   | —     | —       |
|       |          |       |         |         |         |       |         |       |         |
| 21    | 0.86i    | -7.16 | -7.16   | -2.74   | -2.74   | -0.31 | -0.31   | —     | —       |
|       |          |       |         |         |         |       |         |       |         |
| 25    | 2.17i    | -7.01 | -7.01   | -2.28   | -2.28   | +0.44 | +0.44   | —     | —       |
|       |          |       |         |         |         |       |         |       |         |

Schrödinger equation for instance for $V_2 = 17$ one would have got the energy eigenvalues as $-11.92, -6.01, -2.72, -2.11, -0.42, -0.20$ (see Table 1) in the ascending order even without knowing whether they belong to + or - branch. By virtue of exact analytic results (2-6) of Scarf II (1), we are familiar with two spectral branches of it. For example, we confirm these eigenvalues as negative energy poles (Fig. 3) in the transmission probability, $T(E)$ [18] of Scarf II (1). The sequence of six eigenvalues is like: $E^0_+, E^1_+, E^0_-, E^2_+, E^1_-, E^3_+$ which appears arbitrary with regard to the subscripts and superscripts. We emphasize that there needs to be some theoretical basis under which we should know the number of spectral branches for a given complex PT-symmetric potential regardless of whether it is solved analytically or numerically. Next, all the branches need to be mixed for full discrete spectrum. This would further require a new scheme to re-label the eigenvalues and eigenstates.
When \( V_2 = 20.25 = V_1 + .25 \), all the energy doublets coalesce and become equal, see three equal pairs of real discrete energies in Table 1. Not shown here is the plot of \( T(E) \) for this case which has three clear poles at three negative energies: -7.49, -2.82, -0.46. For \( V_2 = 20.10 \) there will be three closely lying doublets.

When \( V_2 > 20.25 \) all the eigenvalues become complex conjugate pairs and their real part is shown to vary linearly in \( (a,b,c,d) \) parts. In Fig. 1(c) the real part of CCEE of eigenvalue also becomes positive but the corresponding eigenstate remain square integrable boundstates. See Table 1, for \( V_2 = 21 \), we find three pairs CCEE. For \( V_2 = 25 \), we find 3 pairs of CCEE, the last pair is interesting as its real part is positive. It turns out that the eigenstates which are controlled by \( n < (s - 1)/2 \) will remain \( L^2 \)-integrable bound states irrespective of whether real prat of CCEE is positive or negative. Apart from these real or CCEE discrete states the Scarf II will have continuous part of spectrum where in various parametric regimes it displays interesting novel scattering properties [19,20] like non-reciprocity [21], spectral singularity [22], coherent perfect absorption with [23] and without [24] lasing.

When PT-symmetry breaks down spontaneously as a parameter of the potential passes over a critical value, each of these pairs change over to CCEE states. Before and after this transition the eigenstates would be orthogonal in the proposed way \[30\]

\[
\int_{-\infty}^{\infty} \psi_1(x)\psi_2(x) \, dx = 0, \quad E_1 \neq E_2.
\] (13)

We conclude that if one isolates two branches of discrete eigenvalues of complex PT-symmetric Scarf II, the coalescing of real eigenvalues to complex conjugate energy eigenvalue (CCEE) pairs and hence the spontaneous breakdown of PT-symmetry can not be discussed. And if one bring these two branches together, interesting variations of discrete eigenvalues emerge. In this, the most interesting is the accidental crossings of two eigenvalues. Since level crossing mean degeneracy which can not occur in one dimension, we could succeed in showing the linear dependence of such pairs of eigenstates by proposing and proving an un-noticed property of Jacobi polynomials. Given a complex PT-symmetric potential regardless of whether it is solved numerically or analytically there is a need of a criterion to tell the number of spectral branches: one or more. Further, the spectral branches need to be mixed for full discrete spectrum.

9
Appendix 1

Proposition 1:

Let $\psi_m(x), \psi_n(x)$ be two $L^2$-integrable solutions of one dimensional time-independent Schrödinger equation satisfying Neumann boundary condition: $\psi(\pm\infty)$ with an equal eigenvalue $E$, then $\psi_m(x)$ and $\psi_n(x)$ are linearly independent.

**Proof:** Let the potential $V(x)$ (real or complex) in Schrödinger $(2\mu = 1 = \hbar^2)$ equation gives rise to two solutions $\psi_m(x)$ and $\psi_n(x)$ with the same energy eigenvalue $E$, then we write

\[
\frac{d^2\psi_m(x)}{dx^2} + [E - V(x)]\psi_m(x) = 0, \quad (A-1)
\]
\[
\frac{d^2\psi_n(x)}{dx^2} + [E - V(x)]\psi_n(x) = 0. \quad (A-2)
\]

Multiply the first by $\psi_n(x)$ and the second by $\psi_m(x)$ and by subtracting them we get

\[
\psi_m(x)\frac{d^2\psi_n(x)}{dx^2} - \psi_n(x)\frac{d^2\psi_m(x)}{dx^2} = 0 \Rightarrow \frac{d}{dx} \left( \psi_m \frac{d\psi_n}{dx} - \psi_n \frac{d\psi_m}{dx} \right) = 0, \quad (A-3)
\]

leading to

\[
\left( \psi_m(x)\frac{d\psi_n(x)}{dx} - \psi_n(x)\frac{d\psi_m(x)}{dx} \right) = C, \quad (A-4)
\]

where $C$ is constant independent of $x$ which can as well be determined at $x = \pm\infty$. As the eigenstates satisfy $\psi_j(\pm\infty) = 0$, we get $C = 0$. Further we get, $\frac{\psi_m(x)}{\psi_m} = \frac{\psi_n(x)}{\psi_n}$ implying linear dependence: $\psi_m(x) = C'\psi_n(x)$. Thus, like in Hermitian quantum mechanics, here too the degeneracy can not occur.

Appendix 2

One of the representations of Jacobi polynomials [15] is

\[
P_n^{a,b}(z) = \frac{\Gamma(a + n + 1)}{n! \Gamma(a + b + n + 1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(a + b + n + m + 1)}{\Gamma(a + m + 1)} \left( \frac{z - 1}{2} \right)^m. \quad (A-1)
\]

Putting $a = -j$, $b = s$ in the above expression we get

\[
P_n^{-j,s}(z) = \frac{\Gamma(n - j + 1)}{n! \Gamma(s + n - j + 1)} \sum_{m=j}^{n} \binom{n}{m} \frac{\Gamma(s + n + m - j + 1)}{\Gamma(m - j + 1)} \left( \frac{z - 1}{2} \right)^m,
\]
the summation, instead of \( m = 0 \), will effectively start from \( m = j \) as Gamma of zero and negative integers is \( \pm \infty \). Changing the summation index to \( p = m - j \) gives,

\[
P_n^{-j,s}(z) = \frac{\Gamma(n - j + 1)}{n! \Gamma(s + n - j + 1)} \sum_{p=0}^{n-j} \binom{n}{p+j} \frac{\Gamma(s + n + p + 1)}{\Gamma(p + 1)} \left( \frac{z - 1}{2} \right)^{p+j}.
\]

Opening \( \binom{n}{p+j} \) and manipulating the Gamma functions we get

\[
P_n^{-j,s}(z) = \frac{(n - j)!}{(n - j)! \Gamma(s + n - j + 1)} \sum_{p=0}^{n-j} \binom{n - j}{p} \frac{\Gamma(s + n + p + 1)}{\Gamma(p + j + 1)} \left( \frac{z - 1}{2} \right)^{p+j}.
\] (A-2)

Using Eq.(A-1), to substitute for \( P_{n-j}^{j,s}(z) \), in Eq.(A-2) we prove

\[
P_n^{-j,s}(z) = (-2)^{-j} \frac{(n - j)!}{n! \Gamma(s + n - j + 1)} \frac{\Gamma(s + n + 1)}{\Gamma(s + n - j + 1)} (1 - z)^j P_{n-j}^{j,s}(z).
\] (A-3)

References

[1] C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.

[2] C.M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[3] Z. Ahmed, C.M. Bender and M.V. Berry, 38 (2005) L627.

[4] M. Znojil, J. Phys. A: Math. Gen. 21 (2000) L61.

[5] B. Bagchi and C. Quesne, Phys. Lett. A 273 (2000) 285.

[6] B. Bagchi, C. Quesne, and M. Znojil, Mod. Phys, Lett. A 16 (2001) 2047.

[7] Z. Ahmed, Phys. Lett. A 282 (2001) 343; 287 (2001) 295;

[8] G. Levai, Czech. J. Phys. 54 (2004) 77.

[9] C-S. Jia, X-L, Zeng, L-T. Sun, Phys. Lett. A 294 (2002) 185.

[10] A. Sinha and P. Roy, J. Phys. A: Math. & Gen. 39 (2002) L377.

[11] B. Bagchi and C. Qesne, arxiv: 0209031v1 [quant-Ph] 2002; GROUP24 (IOP, Bristol, 2003) 589.
[12] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D.N. Christodoulides, Phys. Rev. Lett. 100 (2008) 030402; A. Guo, G.J. Salamo, D. Duchesne, R. Morondotti, M. Volatier-Ravat, V. Amex, G. A. Siviloglou and D.N. Christodoulides, Phys. Rev. Lett. 103 (2009) 093902; C.E. Rüter, G. E. Makris, R.El-Ganainy, D.N. Christodoulides, M. Segev, D. Kip, 6 (2010) 192.

[13] C.M. Bender, M.V. Berry, O.N. Meisinger, Van M. Savage and Mehmet Simsek, J. Phys. A : Math. Gen. 34 (2001) L31.

[14] Z. Ahmed, Phys. Lett. A 360 (2006) 238.

[15] M. Abramowitz and I. A. Stegun, ‘Handbook of Mathematical Functions’, Dover, N.Y. (1970).
G.Szego, ‘Orthogonal Polynomials’, Am. Math. Soc. Rhode Island (1939) 58-99.

[16] Y.D. Chong, Li Ge, and A.D. Stone, Phys. Rev. Lett. 106 (2011) 093902.

[17] Z. Ahmed, D. Ghosh and J. A. Nathan arXiv:1502.0483[quant-ph] (2015)

[18] A. Kahre and U.P. Sukhatme, J. Phys. A 21 (1988) L501. G. Levai, F. Cannata and A. Ventura, J. Phys. A: Math. Gen. 34 (2001) 839.

[19] Z. Ahmed, J. Phys. A: Math. Theor. 42 (2009) 472005; 45 (2012) 032004.

[20] Z. Ahmed, J. Phys. A: Math. Theor. 47 (2014) 485303.

[21] Z. Ahmed, Phys. Rev. A 64(2001) 042716; Phys. Lett. A 324(2004) 152.

[22] A. Mostafazadeh, Phys. Rev. Lett. 102 (2009) 220402;

[23] Y. D. Chong, Li Ge. Hui Cao and A. D. Stone Phys. Rev. Lett. 105 (2010) 053901.

[24] S. Longhi, Phys. Rev. A 82(2010) 031801 (R).

[25] Z. Ahmed and J. A. Nathan, Phys. Lett. A 379 (2015) 865.