The study of rigid-body dynamics is an important topic in classical mechanics. As in the case of point particle dynamics, a good understanding of the kinematic description of rigid body motion is essential for understanding its dynamics. One of the crucial results related to the kinematic description of motion of a rigid body is Chasles's theorem [5, 9, 10], which states that a general displacement of a rigid body can be described by a translation and a rotation about an axis. Further, even though the translation vector is not unique, the orientation of the axis of rotation and the angle of rotation will be the same in passing from one configuration to another [2, 14, 16].

This natural splitting of a general motion into translational and rotational parts makes it possible to study the dynamics of translation and rotation separately. The former reduces to a point-particle-like dynamics, and the latter is described using Euler’s equations in rigid-body dynamics.

It turns out that one can use the freedom in choosing the translation vector to make its direction coincide with that of the axis of rotation. This is the content of the Mozzi–Chasles theorem and is central to the study of dynamics of rigid bodies using screw theory [1], which is used extensively in present-day robotics [6, 11]. The Mozzi–Chasles theorem was first proved by the Italian mathematician and astronomer Giulio Mozzi in the year 1763 [4, 10]. It was later analyzed by the French mathematician Michel Chasles in his 1830 work [5].

Chasles’s theorem can be thought of as an extension of a theorem due to Euler, known as Euler’s rotation theorem, which states that every reconfiguration of a rigid body with one of its points fixed is equivalent to a single rotation about an axis passing through the fixed point. In other words, however a sphere might be rotated around its center, a diameter can always be chosen whose direction in the rotated configuration will coincide with that in the original configuration.

The original proof given by Euler is a geometric one [7, 13]. The proof looks at the initial and final configurations of a great circle on the sphere and gives a recipe to construct a point that is subsequently shown to be the point through which the axis of rotation passes. It is interesting to note that Euler’s proof of the rotation theorem that bears his name was published in 1776, three years after Mozzi’s work [7].

There are various other proofs available for the rotation theorem, both geometric [2, 14] and algebraic [12, 13]. The algebraic proofs typically require a familiarity with rotation matrices and their properties or with ideas from group theory. The most commonly seen analytic proof [3, 8, 15] uses the orthogonality property of three-dimensional rotation matrices to show that they always have an eigenvector with eigenvalue equal to one. This proof makes use of the result that eigenvalues of an orthogonal matrix have modulus one.

Another of the geometric proofs [2] involves looking at the displacement of a segment under rotations and constructs planes of symmetry using the endpoints of the original and displaced segments. The intersection of these symmetry planes is then shown to be the axis of rotation. This is by far the most transparent of the existing proofs.

The proof of Euler’s rotation theorem given by Pars [14] involves going from the initial configuration to the final configuration using two rotations: the first a rotation through the angle π about the center of the great circle connecting one of the original points and its final location,
and the second a rotation about an axis passing through this final location. The invariant point is then established by a construction.

The proof presented here came about as a result of the author’s effort to teach Euler’s rotation theorem and the Mozoom-Chasles theorem to an undergraduate class. The proof is geometric in nature and is different from the existing ones. It makes use of two successive rotations about two mutually perpendicular axes to go from one configuration of the rigid body to the other with one of its points fixed. We shall first develop the key ideas involved in the description of a rigid body and use this knowledge in setting up the proof of the theorem. After proving Euler’s rotation theorem, which is the crux of the paper, Chasles’s theorem is derived. The plan of the paper is as follows: We first derive the number of degrees of freedom of a rigid body. We then derive Euler’s rotation theorem, which we follow with a proof of Chasles’s theorem.

To broaden the scope of the discussion and make it more complete, we shall then look at a few consequences of the theorems we have proved involving the idea of screw axis and rigid-body motion in two dimensions.

**Rigid-Body Displacement**

A rigid body is defined as a collection of particles whose mutual distances remain invariant. In three dimensions, $N$ independent particles have $3N$ degrees of freedom. But if the particles constitute a rigid body, the number of degrees of freedom is reduced to 6 (for the case $N \geq 3$). This is so because the constraint equations that come from the invariant interparticle separations makes $3N - 6$ of the original $3N$ variables dependent on the remaining 6. Let us prove this result rigorously.

We first show that a rigid body configuration is completely defined once coordinates of any three noncollinear particles in the rigid body are specified. Assume that the positions of three particles ($A$, $B$, $C$) are given. Consider now a fourth particle, $D$. Since the body is rigid, the distances between particles $A$ and $D$ (say $d_1$), $B$ and $D$ ($d_2$), and $C$ and $D$ ($d_3$) are fixed. Construct a sphere of radius $d_1$ centered at particle $A$, as shown in Figure 1. It is clear that particle $D$ has to reside on the surface of this sphere.

Now construct a second sphere of radius $d_2$ centered at particle $B$. The rigidity constraint will imply that particle $D$ has to lie on the circle (call it $S$) formed by the intersection of these two spheres. Note that if the spheres do not intersect, the constraints will not be consistent with the assumption that the distances between particle $D$ and particles $A$ and $B$ are $d_1$ and $d_2$ respectively. A third sphere of radius $d_3$ centered at $C$ will intersect the circle $S$ at two points ($D$ and $D'$ in Figure 1), implying that once the three points are fixed, a fourth particle can be placed at one of only two possible points, consistent with the rigidity constraints given by the fixed distance between particle $D$ and the other three.

The two possible points are related to each other by a reflection about the plane containing particles $A$, $B$, and $C$, and they correspond to the mirror images of each other. But rigid-body displacement excludes reflections, and hence only one of the two points will correspond to the possible position of particle $D$. Thus there is a unique position where particle $D$ can be placed, and hence no further coordinates need to be specified.

But particle $D$ is completely arbitrary and could be any of the particles in the rigid body other than the original three particles whose positions were specified.

Thus we see that the number of degrees of freedom of a rigid body in three dimensions is the same as that of a rigid body containing three noncollinear particles. Three independent particles have nine degrees of freedom. Since the body is rigid, there are three constraint equations specifying the mutual separations between the three particles. The difference between these two numbers gives the number of degrees of freedom of a rigid body, namely $9 - 3 = 6$.

The next question we address is how to describe the displacement of a rigid body from one configuration to another. There are multiple ways to do this. We shall adopt a scheme that will be convenient in formulating our proof of Chasles’s theorem. Consider two configurations $I$ and $II$ of the rigid body. Consider three noncollinear points located at $P_1$, $P_2$, $P_3$ in the rigid body in configuration $I$. In the final configuration, let these points move to locations $P'_1$, $P'_2$, $P'_3$ respectively. To go from configuration $I$ to configuration $II$, we will carry out the following three steps:

1. Translate the body by the vector $P_1P'_1$. This ensures that the point at $P_1$ moves to its final position at $P'_1$.

![Figure 1](image-url)

**Figure 1.** The construction to determine the number of degrees of freedom of a rigid body in three dimensions. If three particles $A$, $B$, and $C$ are fixed, then there are only two possible locations for a fourth particle whose distances from the other three are fixed by rigid-body constraints. The possible locations for the fourth particle $D$ are shown as $D$ (red dot) and $D'$ (blue dot) in the figure. They are related by a reflection about the plane containing particles $A$, $B$, and $C$. The dashed circle is the intersection of the spheres centered at $A$ and $B$ (referred to as circle $S$ in the text). Points $D$ and $D'$ are the intersections of this circle with the sphere centered at $C$. 

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2. Let \( P_3' \) be the location of the point at \( P_2 \) after this translation. Consider the plane formed by the vectors \( P_1'P_2' \) and \( P_2'P_3' \). This is the equatorial plane shown in Figure 2. Rotate the rigid body about an axis perpendicular to this plane and passing through the point at \( P_1' \) such that the point at \( P_2'' \) is in its final position at \( P_2' \). Note that if \( P_2'' \) is the same as \( P_2' \), then this step need not be carried out.

3. Let \( P_3'' \) be the location of the original point \( P_3 \) after the above two operations. The rigid body can now be rotated about the axis passing through \( P_3' \) and \( P_3'' \) (see Figure 2) such that the point at \( P_3'' \) is in its final position at \( P_3' \).

These steps will ensure that the rigid body has been displaced to its final configuration.

**Proof of Euler's Rotation Theorem**

We are now in a position to prove Euler's rotation theorem. In order to do so, let us look at steps 2 and 3 in the scheme described above to go from one configuration of a rigid body to another. Note that after step 1, the point at \( P_1 \) is in its final position, and the point at \( P_3 \) has moved over to \( P_3'' \). Steps 2 and 3 involve rotations to be carried out with \( P_1' \) fixed. These operations are shown in Figure 2. We represent these rotations by \( R_{AB} \) and \( R_{P_1P_2} \). Here \( R_{AB} \) is rotation through an angle \( \phi \) about the axis \( AB \) (the axis perpendicular to the vectors \( P_1'P_2' \) and \( P_2'P_3' \) and passing through the point \( P_1' \)), which is the axis involved in step 2 above. The value of \( \phi \) can vary between 0 and \( 2\pi \). Then \( R_{P_1P_2} \) is rotation through an angle \( \theta \) about the axis connecting points \( P_1' \) and \( P_2' \), which corresponds to step 3 above. Here \( \theta \) can take values from \( -\pi \) to \( \pi \). The sphere shown in the figure has \( P_1' \) at its center, and its radius is equal to the distance between points \( P_1' \) and \( P_2' \). For convenience, we have oriented the figure such that the \( AB \)-axis is vertical.

As the rotation \( R_{AB} \) is carried out, the great circle arc \( AP_1'B \) will move over into the great circle arc \( AP_2'B \). And the entire region that lies between these two arcs before rotation will now lie between the great arcs \( AP_1'B \) and \( AP_2'B \) (see Figure 2). In particular, the great circle arc \( ADB \) that bisects the region \( AP_1'BP_2'A \) (see Figure 3) will move over to the great circle arc \( ADB \). Consider now an arc of latitude such as \( LMN \), where \( L \) lies on \( AP_1'B \), \( N \) on \( AP_2'B \), and \( M \) on \( ADB \). Note that \( M \) is the midpoint of the arc. Under rotation \( R_{AB} \), \( LMN \) will move over to the latitude arc \( NM'N' \). Similarly, the latitude arc \( HQQS \) (\( Q \) being the midpoint) will move over to \( SQ'S' \) under rotation (Figure 3).

It is interesting to note what happens to points like \( M' \) and \( Q' \) under the second rotation (step 3 above). They are candidates for points that could fall back to their original positions after the two rotations! This is so because the great circle arc \( PM \) \((P'Q')\) is equal in magnitude to the great circle arc \( P'M' \) \((P'Q')\), and hence under rotation about an axis passing through \( P_3' \) and \( P_1' \), both points will fall on the same latitude circle with \( P_3' \) as the pole.

We will now argue that depending on the value of \( \theta \), there is going to be exactly one such point that will return to its original position (that is, the position before step 2). Figure 4 shows a few of the candidate points that can return to their original locations. It is clear from the figure that the angles shown have the following ordering:

\[
DP_2'D' = \pi > MP_2'M' > QP_2'Q' > AP_2'A = 0. \tag{1}
\]

Even though this ordering is apparent from the figure, one can establish it more rigorously in the following manner. The spherical triangle \( P_2'DQ \) (see Figure 3) contains the spherical triangle \( P_2'DM \). This is because the base \( P_2'D \) is common to both triangles, and the great circle arcs \( DQ \) and \( DM \) are part of the same great circle, with \( DM \) longer than \( DM \). This implies that the angle \( NP_2'M \) is larger than \( SP_2'Q \) (both being defined as angles between the corresponding great circle arcs). But we have \( MP_2'M' = 2MP_2'N \) and \( QP_2'Q' = 2QP_2'S \). The relationship in equation (1) follows.

Thus for every positive value of \( \theta \) in the interval from 0 to \( \pi \), one and only one of the points of type \( M' \) that lie in the hemisphere containing point \( A \) will return to its original position. There will be a similar point in the diametrically opposite side of the sphere. If \( \theta \) were negative and lay
Similarly, point $Q$, being the midpoint of the arc, moves over to back to their original positions under the rotation about $P$. Similarly, point $Q$, which is the midpoint of the arc of latitude $HQS$, goes to point $Q'$. Points like $D'$, $M'$, and $Q'$ can move back to their original positions under the rotation about $P_1P_2'$ (see Figure 4).

between $0$ and $-\pi$, there would be a point in the lower region below the equatorial plane that would return to its original position and a corresponding point in the diametrically opposite side. Thus for given values of $\phi$ and $\theta$, there are diametrically opposite pairs of points that do not change position under steps 2 and 3.

This implies that the effect of both the rotations considered above should be the same as that due to a single rotation about an axis that passes through these invariant points and the center of the sphere ($P'_1$). Since the effect of an arbitrary set of rotations with $P'_1$ fixed can be described using steps 2 and 3 above, we see that the net effect of these rotations can be attained by a single rotation about an axis. This proves Euler’s rotation theorem. If we know the value of $\theta$, we can find the invariant point by construction. To find this, choose the point (say $X$) on the great circle arc $ADB$ such that the angle between the great circle arcs $P_2X$ and $P'_2A$ is $\theta/2$.

**Proof of Chasles’s Theorem**

We have already shown that the last two steps in our procedure to represent a general displacement of a rigid body correspond to a rotation about a single axis. But step 1 involved a pure translation that took point $P_1$ to $P'_1$. Thus we can conclude that a general displacement of the rigid body can be obtained by a translation and a rotation about an axis. To complete the proof of Chasles’s theorem, we also need to show that a different choice of point (instead of $P_1$) for translation will not alter the direction and amount of rotation to be carried out. To prove this, imagine we had chosen a different point $Q_1$ instead of $P_1$ for translation. Choose points $Q_2$ and $Q_3$ such that $P_1P_2 = Q_1Q_2$ and $P_1P_3 = Q_1Q_3$, as shown in Figure 5. One can now repeat the arguments above for proving Euler’s rotation theorem. Since the vectors involved in steps 2 and 3 ($Q_1Q_2$ and $Q_1Q_3$) in this case are identical to the old ones (even though the new displacement vector could be different), we will end up with the same rotation axis and angle. This completes the proof of Chasles’s theorem. It may well be that there is no material point in the rigid body at the location of $Q_2$ or $Q_3$. One can nevertheless think of imaginary points rigidly attached to the body and moving in accordance with the rigidity constraints. In fact, the displacing points (such as $P_1$ and $Q_1$) themselves need not form part of the rigid body.

An important corollary of Chasles’s theorem is the Mozzi–Chasles theorem, which states that the general displacement of a rigid body can be obtained by a rotation about an axis and a translation along the same axis. For completeness, we give here a proof of this theorem.
which are related to \( Q \) of rotation can be seen by considering points constancy in the direction of the axis of rotation and the angle parallel to \( \vec{a} \) a vector pointing along it. One chooses the translating point to be \( Q_1, \) instead of \( P_1, \) the constancy in the direction of the axis of rotation and the angle of rotation can be seen by considering points \( Q_2 \) and \( Q_3, \) which are related to \( Q_1 \) as \( P_2 \) and \( P_3 \) are to \( P_1. \)

Consider a rigid body displacement described by the displacement vector \( \vec{F} \) and the rotation in the direction \( \hat{n} \) by an amount \( \Theta, \) as shown in Figure 6. It may be that \( \vec{F} \) is not parallel to \( \hat{n}. \) We will assume that the rigid displacement affects all the points in space and not just those belonging to the rigid body.

The translation vector \( \vec{F} \) can be expressed as the sum of a vector pointing along \( \hat{n} \) (\( \vec{g} \) in the figure) and a vector lying in the plane perpendicular to \( \hat{n} \) (\( \vec{g} \) in the figure). Under the rotation about \( \hat{n}, \) the different points in space will undergo displacements that lie in a plane perpendicular to \( \hat{n}. \) For every value of \( \Theta, \) the set of displacement vectors will contain all possible vectors in the plane. This is because the magnitude of the rotation vector will vary from zero to infinity as the distance of the points from the axis of rotation increases from zero to infinity, and all points lying on a circle at fixed distance from the axis of rotation will generate displacements in all possible directions in the plane.

Thus one should be able to find points whose displacement is \(-\vec{s}\) under the rotation. If one chooses one of these points as the translating point, it will ensure that the displacement vector is along \( \hat{n} \) itself, proving the corollary. The common axis that is involved in this description, in the direction of \( \hat{n}, \) is referred to as the screw axis or Mozzi axis.

The Mozzi–Chasles theorem leads to another important result concerning the motion of a rigid body in two dimensions. The counterpart of Chasles’s theorem in two dimensions, sometimes referred to as Euler’s first rotation theorem, states that every displacement of a rigid body in two dimensions can be achieved by either a single rotation or a translation. There is a straightforward geometric proof by construction for this theorem [2]. We shall prove the result using the Mozzi–Chasles theorem. For this, note that in two dimensions, the axis of rotation is always perpendicular to the plane. By the Mozzi–Chasles theorem (since two-dimensional displacements are a subset of possible displacements in three dimensions), the rigid displacement can be achieved by a translation along an axis and rotation about that axis. Since the only possible translation along the rotation axis is one with zero magnitude, there must be a point that does not change its position under rigid displacement in two dimensions. Thus the rigid displacement can then be achieved through a pure rotation about an axis through this point. The other possibility is a pure translation in the plane, in which case the screw axis will lie in the plane and the rotation about the screw axis will be zero. It follows that in two dimensions, a rigid displacement is either a pure translation (screw axis lies in the plane) or a pure rotation (screw axis is normal to the plane).

**Concluding Remarks**

We have derived Euler’s rotation theorem using a novel geometric proof. The proof involves using a set of three steps that takes the rigid body from its initial state to its final state. Euler’s rotation theorem was derived using the last of the two steps in this procedure. The proof is presented in a manner that helps in the visualization of how the invariant points arise and is therefore of pedagogical interest. But it should be kept in mind that the sequence
The first part of Chasles’s theorem, which asserts that the general displacement of a rigid body is a combination of translation and a rotation about an axis, follows immediately from Euler’s theorem and the first step in the procedure for carrying out rigid displacement. The fact that the orientation of the axis of rotation and angle of rotation are independent of the translation vector involved was proved by a separate construction. For completeness, we have also presented a proof of the existence of a screw axis for motion in three dimensions and a proof that in two dimensions, every rigid displacement can be achieved by a pure rotation or a translation. The proof of Euler’s rotation theorem presented here should be accessible to a non-specialist with some familiarity with high-school geometry. Also, it offers yet another point of view in looking at these important theorems associated with the mechanics of rigid bodies.

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