Generalizing Nyquist criteria via conformal contours for internal stability analysis

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By contriving the regularized return difference relationship in linear time-invariant (LTI) feedback systems, we attempt to generalize and validate the Nyquist approach for such internal stability as Lyapunov stability/instability, asymptotic stability, exponential stability and district stability (or $D$-stability), respectively, even when there exist decoupling zeros, by means of what we call the regularized Nyquist loci that are plotted with respect to a Nyquist contour and its conformal one(s). More precisely, miscellaneous open-loop/closed-loop pole cancellations in the return difference relationship that may complicately tangle our stability interpretation but usually neglected in most existing Nyquist criteria are scrutinized. And then, Nyquist-like criteria for internal stability are claimed with the regularized Nyquist loci. These criteria get rid of pole cancellations testing and can be implemented completely independent of open-loop pole distribution knowledge; moreover, the Nyquist criteria for asymptotic/exponential stability are necessary and sufficient, while those for $D$-stability are sufficient. Internal stability of a cart system with an inverted pendulum is examined to illustrate the results.

Keywords: internal stability; open-loop/closed-loop pole cancellation; meromorphism; holomorphism; conformal; regularized; return difference relationship; Nyquist contour

1. Introduction

Nyquist criteria (or Nyquist approach) are a graphical method for stability analysis and stabilization in both linear and nonlinear feedback configurations, whose standard statements can be found in almost all textbooks, see Hespanha (2009). Indeed, Nyquist approach has an important role in Lure’s problem of robust stability analysis (Khalil, 2000; Vidyasagar, 2002). Due to its simple conception and intuitively geometric expression, Nyquist approach is widely and frequently employed and has brought fruitful results in theory and engineering (Brockett & Byrnes, 1981; Fardad & Bammieh, 2008; Stojie & Siljak, 1965; Vidyasagar, Bertschmann, & Sallaberger, 1988). Probably from their classical impression and explicit link to frequency-domain characteristics of systems, one might think that Nyquist criteria are well-developed only for external stability since the frequency-domain characteristics are essentially input/output relationships. In view of these facts, there seems to be no room for applying Nyquist approach for internal stability such as Lyapunov stability/instability, asymptotic stability, exponential stability and district stability (or $D$-stability). In contrast, the linear matrix inequalities involved in Nyquist approach are noticed again (Ferreira & Bhattacharyya, 1977; MacFarlane & Karcanias, 1976; Rosenbrock, 1970; Schrader & Sain, 1989; Watkins, 1969; Wonham, 1979; Zhou, 1994, 1995a). For example, due to non-controllability and/or non-observability, miscellaneous pole cancellations may happen in the conventional return difference equations of multivariable feedback configurations, which draw little attention in most existing Nyquist criteria up to now. However, how to deal with poles that are uncontrollable and/or unobservable in Nyquist-like stability criteria becomes necessary when pole assignment in a sub-system level of large-scale multivariable systems is considered (Zhou, 1995b, 1997). As an answer to such problems, based on what we call the regularized return difference equations under LTI feedback configurations, this paper establishes Nyquist-like criteria via the regularized Nyquist loci that are plotted with respect to a standard Nyquist contour and its conformal contour(s). Our Nyquist criteria are claimed for Lyapunov stability/instability, asymptotical stability, exponential stability and $D$-stability, respectively, by specifying the Nyquist contour and its conformal counterpart(s) as appropriately. Significance of this study includes: (i) Nyquist-like criteria can actually be created for internal stability; (ii) Nyquist loci also graphically reflect intrinsic structural features of multivariable feedback configurations

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systems, and thus possess implication deeper than we have already expected.

Historically, other efforts were done for deriving various extensions of the Nyquist criterion from single-input/single-output LTI systems to multivariable ones, among which the characteristic locus (or generalized Nyquist diagrams) method (Edmunds, 1979; Edmunds & Kouvaritakis, 1979; MacFarlane & Bellutri, 1973; MacFarlane & Postlethwaite, 1977; Postlethwaite, 1977) is worth reflecting, together with fruitful design techniques and application results with the idea (Foss, Edmunds, & Kouvaritakis, 1980; Kouvaritakis, Murray, & MacFarlane, 1979; Mees & Allwright, 1979; Postlethwaite, 1978). Simply speaking, the characteristic loci are also defined on the conventional return difference equation of feedback systems but with eigenvalues of complex function matrices. Hence, eigenvalue analysis and numerical computation play a key role therein, whose advantages and disadvantages are attributed to how one can do the eigenvalue analysis and computation rigorously and conveniently. In contrast, the regularized Nyquist loci has no direct connection with eigenvalue analysis and computation, which is graphically oriented in the sense that we can draw the regularized Nyquist loci without any complex functional analysis and computation rigorously and conveniently. In what follows, working on the open- and closed-loop systems, together with the contrived regulators. Numeric examples of a cart system with inverted pendulum are worked out to explicate the results in Section 4. Finally, Section 5 summarizes our observation.

2. Preliminaries

2.1. Notations and terminologies

Consider the LTI feedback system illustrated in Figure 1, in which we denote by \( \Sigma_G \) and \( \Sigma_H \), respectively, an LTI multi-input/multi-output (MIMO) plant and an MIMO feedback component that possess the following state-space equations.

\[
\Sigma_G : \begin{cases}
\dot{x} = Ax + Be \\
y = Cx + De
\end{cases}, \quad \Sigma_H : \begin{cases}
\dot{\xi} = A\xi + \Gamma \mu, \\
\eta = \Theta \xi + \Pi \mu,
\end{cases}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{l \times n} \) and \( D \in \mathbb{R}^{l \times m} \), respectively, are constant matrices, while \( \Lambda \in \mathbb{R}^{p \times p}, \Gamma \in \mathbb{R}^{p \times l}, \Theta \in \mathbb{R}^{m \times p} \) and \( \Pi \in \mathbb{R}^{m \times l} \) are also constant ones. In the paper, \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of all real and complex numbers, respectively.

Transfer functions for \( \Sigma_G \) and \( \Sigma_H \) are given as follows.

\[
G(s) = C(sI_n - A)^{-1}B + D =: \hat{G}(s) + D, \\
H(s) = \Theta(sI_p - \Lambda)^{-1}\Gamma + \Pi =: \hat{H}(s) + \Pi,
\]

where \( \hat{G}(s) \) and \( \hat{H}(s) \) are the strictly proper portions of \( G(s) \) and \( H(s) \).

Now we turn to construct the state-space equations for the open-loop and closed-loop systems of Figure 1. In the open-loop system, we can write the open-loop state-space equation as

\[
\Sigma_O : \begin{cases}
\dot{\xi} = \frac{A}{\Gamma}C \xi + \frac{B}{\Gamma}D u, \\
\eta = [\Pi C \Theta] \xi + \Pi D u.
\end{cases}
\]

In the closed-loop system, we can write the closed-loop state-space equation as

\[
\Sigma_C : \begin{cases}
\dot{\xi} = \frac{A - B \Xi \Pi C}{\Gamma C - \Gamma D \Xi \Pi C} \xi + \frac{-B \Xi \Theta}{\Gamma C - \Gamma D \Xi \Pi C} \xi, \\
\eta = [\Xi \Pi C \Xi \Theta] \xi + \Pi D \Xi u,
\end{cases}
\]

where \( \Xi = (I_m + \Pi D)^{-1}. \)

2.2. Internal stability proposition

In what follows, working on the open- and closed-loop characteristic polynomials, we will establish Nyquist criteria for internal stability. As preparation, Proposition 1 summarizes internal stability conditions in terms of eigenvalue distribution of the state matrix of an LTI system in light of Bernstein (2009 Proposition 11.8.2), Vidyasagar (2002, Theorem 5.4.29) and those of An and Liu (2004), Leite and Peres (2003), Geromel et al. (1998), Oliveira and Peres (2005) and de Oliveira et al. (2002). The definitions of internal stabilities are omitted for the sake of simplicity. Dynamical significance and mathematical implication of the definitions can be found in Bernstein (2009), Khalil (2000) and Vidyasagar (2002).

**Proposition 1** In the LTI system \( \Sigma : \dot{x} = Ax \) with \( A \) being constant, we say

- \( \Sigma \) is Lyapunov stable if and only if \( \text{Re}(\lambda_i(A)) \leq 0 \) for each \( i \), and in addition, every eigenvalue of \( A \)
having a zero real part is a simple zero of the minimal polynomial of $A$;
- $\Sigma$ is asymptotically stable if and only if $\text{Re}(\lambda_i(A)) < 0$ for each $i$;
- $\Sigma$ is exponentially stable if and only if $\text{Re}(\lambda_i(A)) < -\epsilon$ for each $i$ with some $\epsilon > 0$;
- $\Sigma$ is $D$-stable, if and only if $\text{Re}(\lambda_i(A)) \in D$ for each $i$;
- $\Sigma$ is unstable if it is not Lyapunov stable.

In the above, $\lambda_i(\cdot)$ denotes the $i$th eigenvalue of the matrix $\cdot$, and $D$ is a prescribed district set in the left-hand half of the complex plane; namely, $D \subset \mathbb{C}^-$.

As revealed by Proposition 1, as far as LTI continuous-time systems are concerned, asymptotic stability is essentially equivalent to exponential stability.

In Proposition 1, the internal stability is defined based on state-space representations. However, the internal stability can be determined even state-space representations themselves are not available in some cases. For instance, in an SISO system with the configuration of Figure 1, it is internally stable if the following conditions are satisfied:
1. there is not unstable pole/zero cancellation in $G(s)$;
2. there is not unstable pole/zero cancellation in $H(s)$;
3. $1/(1 + G(s)H(s))$, $G(s)/(I + G(s)H(s))$ and $H(s)/(1 + G(s)H(s))$ are stable. Acturally, under these assumptions, no decoupling zeros will be created by the feedback configuration (Zhou, 1995a) so that the internal stability of the closed-loop system can be reflected by the transfer function.

3. Nyquist criteria with single contour
In this section, we state and prove Nyquist criteria for internal stability analysis that are claimed by plotting the Nyquist loci with respect to a single (standard) Nyquist contour (Vidyasagar, 2002; Watkins, 1969), which in special cases bring us back to the standard Nyquist criteria of textbooks.

3.1. Conventional return difference relationships of $\Sigma_C$
To clearly explicate possible pole cancellations and/or algebraic ones in characteristic polynomials that are usually neglected in the conventional Nyquist approach, we begin with deriving the conventional return difference equation in the feedback configuration $\Sigma_C$.

By definition, the characteristic polynomial for the closed-loop system $\Sigma_C$ is

$$
\beta(s) := \det \left( \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} - \begin{bmatrix} A - B\Sigma & -B\Sigma \\ C\Sigma - D\Sigma & -D\Sigma \end{bmatrix} \right)
= \alpha(s) \det(\Sigma) \det(I_m + H(s)G(s)),
$$

where $\det(\cdot)$ means the determinant of $(\cdot)$ and $s \in \mathbb{C}$, and

$$
\alpha(s) := \det \left( \begin{bmatrix} sI_n - A \\ -\Gamma C \\ sI_p - \Lambda \end{bmatrix} \right) = \det(sI_n - A) \det(sI_p - \Lambda) = \alpha_G(s)\alpha_H(s)
$$

(5)

with $\alpha_G(s) = \det(sI_n - A)$ and $\alpha_H(s) = \det(sI_p - \Lambda)$. Clearly, $\alpha(s)$ is the characteristic polynomial for $\Sigma_G$, while $\alpha_G(s)$ and $\alpha_H(s)$ are the characteristic polynomials for the sub-systems $\Sigma_C$ and $\Sigma_H$, respectively, in the feedback configuration of Figure 1.

By Equation (4), we have the well-known return difference relationship for $\Sigma_C$.

$$
\frac{\beta(s)}{\alpha(s)} = \frac{\beta(s)}{\alpha_G(s)\alpha_H(s)} = \frac{\det(I_m + H(s)G(s))}{\det(I_m + \Gamma\Pi D)}.
$$

(6)

It is the base for the conventional Nyquist approach to work. To distinguish Equation (6) from what we will term as the regularized return difference relationship, Equation (6) is termed the conventional return difference relationship.

Before leaving, we mention that one must be cautious in combining Equation (6) with the argument principle (Stein & Shakarchi, 2003) so as to claim Nyquist-like stability criteria.

- $\beta(s)$ and $\alpha(s)$ are polynomials. The argument of $\beta(s)/\alpha(s)$ at $s = s'$, i.e. $\angle(\beta(s)/\alpha(s'))$, is the sum of all arguments of prime factors in $\beta(s)$ and $\alpha(s)$ at $s = s'$, denoted by $\angle(s' - \lambda_i)$, where $\lambda_i$ means a zero of $\beta(s)$ or $\alpha(s)$. Also since $\beta(s)$ and $\alpha(s)$ are monic by definition, $\det(I_m + H(s)G(s))/\det(I_m + \Gamma\Pi D)$ in Equation (6) contains all complex functional information about the irreducible portion of $\beta(s)/\alpha(s)$, while the left-hand side of Equation (6) provides nothing more than coprimeness/non-coprimeness between $\beta(s)$ and $\alpha(s)$.
- $\det(I_m + H(s)G(s))/\det(I_m + \Gamma\Pi D)$ is meromorphic in $s$. Thus, the argument principle (Stein & Shakarchi, 2003) applies to $\det(I_m + H(s)G(s))/\det(I_m + \Gamma\Pi D)$, whenever it vanishes nowhere on a prescribed Nyquist contour. This non-vanishing requirement is indispensable to rigorous proof arguments for employing the argument principle.
- Invertibility of $I_m + \Gamma\Pi D = (\Sigma^{-1})$ is substantial; indeed, when $I_m + \Gamma\Pi D$ is non-singular, it is said (Khalil, 2000) that for any initial conditions, the state-space equation (3) for $\Sigma_C$ is solvable and the state solution is unique. Clearly, when $\Pi = 0$ and/or $D = 0$, invertibility of $I_m + \Gamma\Pi D$ is automatically true. This is the case when either $H(s)$ or $G(s)$ is strictly proper in the sense of $\lim_{s \to \infty} H(s) = 0$ or $\lim_{s \to \infty} G(s) = 0$.
- Reducible factor cancellations may exist between $\beta(s)$ and $\alpha(s)$, which reflect whole-structural cancellations (namely, those of (1) and (2) below),
sub-structural ones (namely, those of (3) below) and algebraic ones (namely, those of (4) below):

1. Input and/or output decoupling zeros in $\Sigma_G$ and/or $\Sigma_H$, respectively, due to uncontrollable and/or unobservable modes in each of themselves (Schrader & Sain, 1989; Zhou, 1994);

2. Input and/or output decoupling zeros in $\Sigma_C$ caused by zero/pole cancellations between $\Sigma_G$ and $\Sigma_H$, when they are cascaded in $\Sigma_G \Rightarrow \Sigma_H$ and $\Sigma_H \Rightarrow \Sigma_G$ when forming $\Sigma_C$. We carefully alert the reader to the facts that $\Sigma_G \Rightarrow \Sigma_H$ is simply $\Sigma_G$, but $\Sigma_H \Rightarrow \Sigma_G$ may have structural features different from those of $\Sigma_G$ (Zhou, 1995a);

3. Poles of $\Sigma_O$ that are controllable and observable, and moved by the feedback configuration to locations of some zeros of $\Sigma_O$ but remain controllable and observable in $\Sigma_C$, which are related to pole/zero cancellations in sub-systems. This is the case in pole assignment when sub-structural pole distribution of $\Sigma_C$ are concerned (Zhou, 1994, 1995b, 1997).

4. Poles of $\Sigma_O$ that are controllable and observable, and altered by the feedback configuration to the same positions where other poles of $\Sigma_O$ are situated but remain controllable and observable in $\Sigma_C$, while neither whole-structural cancellations nor sub-structural ones are associated in the sense of (1), (2) and (3). In this sense, the corresponding cancellations are merely algebraic ones.

- Removing all reducible factors of $\beta(s)$ and $\alpha(s)$, if any, we are led to

$$\frac{\beta(s)}{\alpha(s)} = \frac{\beta'(s)}{\alpha'(s)} = \frac{\det(I_m + H(s)G(s))}{\det(I_m + PD)},$$

where $\deg(\beta'(s)) < \deg(\beta(s))$ and $\deg(\alpha'(s)) < \deg(\alpha(s))$. In other words, when reducible factors between $\beta(s)$ and $\alpha(s)$ exist, not all the closed-loop poles can be reflected by Equation (6). Consequently, one cannot judge internal stability of $\Sigma_C$ according to a Nyquist criterion claimed on Equation (6). In view of this, it is necessary to pre-requisite coprimeness between $\beta(s)$ and $\alpha(s)$ before applying Nyquist approach that will be claimed via Equation (6).

Although whole-structural and sub-structural zero/pole cancellations can be revealed via controllability and/or observability tests, algebraic cancellations between the characteristic polynomials of $\Sigma_C$ and $\Sigma_O$ can only be understood by numerical calculating or coprimeness testing for polynomials. This is undoubtedly a big problem for creating Nyquist criteria in internal stability analysis and stabilization.

### 3.2. Nyquist contours

Now we describe Nyquist contours for internal stability in accordance with Proposition 1 and their geometric implication (Watkins, 1969) in this section.

For brevity, the Nyquist contours are represented with dashed curves, and the arrows paralleling the contours annotate the clockwise direction of taking $s$ along the contours for plotting Nyquist loci. Our standing assumptions about the Nyquist contours defined on the complex plane are: (1) no open-loop poles, i.e. zeros of $\alpha(s) = \alpha_G(s)\alpha_H(s)$, are on them; (2) their interiors are simply connected; and (3) they are simply closed curves.

(a) $N_{ls}(r)$: Nyquist contour for Lyapunov stability/instability and asymptotical stability, see Figure 2. In Figure 2, the crosses indicate possible poles of the open-loop system $\Sigma_O$ on the imaginary axis, where $\det(I_m + H(s)G(s))$ is not well-defined, and thus which should be detoured or bypassed via half-circles from right when computing $\det(I_m + H(s)G(s))$. By $N_{ls}(r)$, we mean the radius of the half-circles is $r$; $N_{ls}(0)$ has no detouring half-circles on the Nyquist contour. Detouring from the right-hand side is due to Proposition 1; namely, closed-loop poles on the imaginary axis, if any, that are simple in the minimal polynomial are stable.

(b) $N_{es}(r)$: Nyquist contour for exponential stability, see Figure 3.

(c) $N_{ds}$: Nyquist contour for $D$-stability, see Figure 4. In Figure 4, the district $D$ is the sectoral area encircled with the solid lines in the left-half plane.

### 3.3. Single-contour nyquist criteria and related issues

To facilitate the statements, we denote by $\text{Int}(\mathcal{N})$ the interior circumscribed by a Nyquist contour $\mathcal{N}$. By definition, $\text{Int}(\mathcal{N})$ is an open set on the complex plane.

**Theorem 1** In the system $\Sigma_C$ of Figure 1, assume that $I_m + PD$ is non-singular, and that the open-loop characteristic polynomial $\alpha(s)$ and the closed-loop counterpart $\beta(s)$ are coprime. Let the numbers of zeros of $\alpha(s)$ in $\text{Int}(N_{ls}(r))$,
In (i) and (ii), the open-loop poles on the imaginary axis mean the zeros of the open-loop characteristic polynomial \( \alpha(s) = \alpha_c(s)\alpha_1(s) \) with zero real parts, if any.

Proof of Theorem 1

In the following arguments, we use \( N \) to denote \( N_{1,\epsilon} \), \( N_{c,\epsilon} \) and \( N_{d,\epsilon} \) as appropriately. As preparation for our ensuing discussion, let us write

\[
 f(s) = \frac{\det(I_m + H(s)G(s))}{\det(I_m + \Pi D)} \beta(s) = \prod_{i=1}^{n+p}(s - \lambda_{c,i}), \quad \alpha(s) = \prod_{i=1}^{n+p}(s - \lambda_{o,i})
\]

and note by Equation (6) and the coprimeness assumption between \( \beta(s) \) and \( \alpha(s) \) that

\[
 \sum_{i=1}^{n+p} (\angle(s - \lambda_{c,i}) - \angle(s - \lambda_{o,i})) = \angle f(s).
\]

Now let \( s \in N \) move from \( s_0 \in N \) counterclockwisely; or we start with \( s = s_0^+ \). When we return to \( s_0 \) from the other side along \( N \), we write \( s = s_0^- \). We assert readily that for each \( \lambda_{c,i} \),

\[
 \angle(s_0^+ - \lambda_{c,i}) - \angle(s_0^- - \lambda_{c,i}) = -2\pi, \quad \lambda_{c,i} \in \text{Int}(N),
\]

\[
 \angle(s_0^+ - \lambda_{c,i}) - \angle(s_0^- - \lambda_{c,i}) = 0, \quad \lambda_{c,i} \notin \text{Int}(N).
\]

We can claim the same for each \( \lambda_{o,i} \). Let \( n_C \) be the number of poles of \( \Sigma_C \) in \( \text{Int}(N) \), and that of \( \Sigma_0 \) in \( \text{Int}(N) \) by \( n_0 \). The above arguments yield that

\[
 -2\pi n_C - (-2\pi n_0) = \angle f(s)|_{s=s_0^+} - \angle f(s)|_{s=s_0^-} =: -2\pi n_f, \quad (7)
\]

where \( n_f \) means how many times the Nyquist locus \( f(s) \) with respect to \( s \in N \) encircles the origin \((0,0)\) counterclockwisely. Equation (7) leads to

\[
 n_C = n_f + n_0. \quad (8)
\]

Since \( n_C \) and \( n_0 \) are uniquely meant, to rigorously validate the relationship of Equation (8), \( n_f \) must exist and be unique. In the sequel, we see that unique existence of \( n_f \) is guaranteed if the Nyquist locus, namely \( f(s)|_{s \in N} \), does not pass through the origin \((0,0)\).\(^3\)

To this end, let us write the complex logarithm of \( f(s) \) by

\[
 \log f(s) = \log | f(s) | + j f(s),
\]

where \( \log | f(s) | \) is the usual logarithm of the positive real number \( | f(s) | \),\(^4\) while \( f(s) \) is some branch evaluation of the argument (different up to an additive integral multiple of \( 2\pi \), or simply \( 2\pi k \)). Since \( \log | f(s) | |_{s=s_0^+} = \log | f(s) | |_{s=s_0^-} \), we see that

\[
 j \angle f(s)|_{s=s_0^+} - j \angle f(s)|_{s=s_0^-} = \log f(s)|_{s=s_0^+} - \log f(s)|_{s=s_0^-}.
\]

Note that for any specific \( k \), the derivative of \( \log f(s) \) is \( f'(s)/f(s) \) and single-valued. Thus, \( \log f(s)|_{s=s_0^+} -
log \( f(s)|_{s=s_0} \) can be connected to the contour integral 
\[ \oint_{N} \frac{f'(s)}{f(s)} \, ds, \]
which can be calculated by the argument principle (Stein & Shakarchi, 2003) as follows. More precisely, under the assumption that \( f(s) \) is meromorphic and vanishes nowhere on \( N \) and Equation (7), we observe that

\[
\eta_f = \frac{1}{2\pi} \left( \oint_{N} f(s) |_{s=s_0} - j \oint_{N} f(s) |_{s=s_0} \right) = \frac{1}{2\pi} \int_{N} f'(s) f(s) \, ds = -(\text{the number of zeros of } f(s) \text{ inside } N) - \text{the number of poles of } f(s) \text{ inside } N.
\]

Again, since \( f(s) \) is meromorphic, the numbers of zeros and poles of \( f(s) \) inside \( N \) are well-defined and unique. From this the unique existence of \( \eta_f \) follows.

Equipped with Equation (8), we are ready to show the main assertions.

Under the Nyquist loci encircling condition in (i) with \( r > 0 \) small enough, \( f(s) = \beta(s)/\alpha(s) = 0 \) for some \( s = s_0 \) on the imaginary portion of \( N_\epsilon(r); \) namely at least one closed-loop pole, say \( \lambda_0 = s_0, \) is on the imaginary axis. By Proposition 1, the system is not asymptotically/exponentially stable. Since the Nyquist locus provides us with no knowledge about geometrical multiplicity of \( \lambda_0 = s_0 \) as an eigenvalue of \( A \) (the state matrix of \( \Sigma_C \)), we cannot definitely say whether \( \Sigma_C \) is Lyapunov stable or unstable.

Under the Nyquist loci encircling conditions of (ii), (iii) and (iv), \( f(s) \) vanishes nowhere on \( N \); namely, Equation (8) holds for the sufficiency aspects of (ii), (iii) and (iv). We assert from Equation (8) that there are no closed-loop poles in Int\( (N) \). Moreover, there are no closed-loop poles on \( N \) as well. To see this, suppose that \( \lambda_{c,1} \) is a closed-loop pole on \( N \); that is, \( s = \lambda_{c,1} \) is a factor of \( \beta(s) \). Note that \( \lambda_{c,i} \neq \lambda_{a,i} \) for each \( i \) by coprimeness between \( \beta(s) \) and \( \alpha(s) \). We can assert that \( f(s) |_{s:\lambda_{a,i},e\mathbb{N}} = \beta(s)/\alpha(s) |_{s:\lambda_{a,i},e\mathbb{N}} = 0 \). This says that the Nyquist locus passes through the origin \((0, 0)\). This is contradictory to the assumed non-vanishing condition.

In particular, when \( N = N_{\epsilon,s} \), the above facts further imply that

\[
\text{Re}(\lambda_{c,i}) < -\epsilon, \quad \forall i
\]
or \( \Sigma_C \) is exponentially stable. When \( N = N_{\epsilon,s} \), \( \text{Re}(\lambda_{c,i}) < 0 \) for all \( i \), \( \Sigma_C \) is asymptotically stable. When \( N = N_{\epsilon,s} \), \( \lambda_{c,i} \in \mathbb{D} \) for all \( i \), thus \( \Sigma_C \) is \( \mathbb{D} \)-stable.

Necessity of (iii) is shown as follows. When \( \Sigma_C \) is exponentially stable, all closed-loop poles satisfy \( \text{Re}(\lambda_{c,i}) < -\epsilon \) for all \( i \) with \( \epsilon > 0 \). Again, from the facts that open-loop poles are isolated points, and that \( \beta(s) \) and \( \alpha(s) \) are coprime, \( \epsilon \) can be taken such that \( N_{\epsilon,s} \) is a simply closed curve while all open-loop poles are by-passed and all closed-loop poles are excluded from \( \text{Int}(N_{\epsilon,s}) \). This in turn leads that \( \beta(s)/\alpha(s) \) does not vanish over \( s \in N_{\epsilon,s} \). Bearing these facts in mind and recalling Equation (6), we say that \( f(s) \) vanishes nowhere on \( N_{\epsilon,s} \), and Equation (8) holds true. From Equation (8), the encircling conditions follows readily.

Necessity of the assertion (iv) may not be true whenever a Nyquist contour in form of \( N_{\epsilon,s} \) cannot be defined as a simply closed curve for the prescribed \( \mathbb{D} \).

Now we give remarks about Theorem 1.

- The detouring radius \( r > 0 \) being sufficiently small cannot be removed simply. It plays a key role in both the sufficiency and necessity arguments of Theorem 1.
- Although \( n_{s}(r) \) only denotes unstable poles of the open-loop system \( \Sigma_O \), the numbers \( n_{e,s} \) and \( n_{o,s} \) may count into stable poles of \( \Sigma_O \), depending on how \( \epsilon > 0 \) is taken and how \( \mathbb{D} \) is specified. In other words, Nyquist contours have more general implication (Watkins, 1969).
- Coprimeness between \( \beta(s) \) and \( \alpha(s) \) is indispensable for rigorously interpreting sufficiency of the assertions, which must be checked repetitiously whenever \( G(s) \), \( H(s) \) or something else in \( \Sigma_C \) is altered or modified. This inevitably brings us difficulties in exploiting Theorem 1. A question is: can we develop any Nyquist-like criteria which involve no coprimeness about \( \beta(s) \) and \( \alpha(s) \)? The following discussion reveals that such criteria are attainable.

4. Nyquist criteria with multiple conformal contours

To surmount the tangling coprimeness issues between the open-loop and closed-loop characteristic polynomials, a multiple-contour Nyquist approach is considered in this section.

4.1. Regularized return difference relationship of \( \Sigma_C \)

From Equations (5) and (6), we obtain immediately that

\[
\frac{\beta(s)}{\alpha_C(z_1)\alpha_C(z_2)} = \frac{\det(sI_m - A) \det(sI_p - \Lambda) \det(I_m + H(s)G(s))}{\det(z_1I_m - A) \det(z_2I_p - \Lambda) \det(I_m + \Pi D)} = \frac{\Delta(s, z_1, z_2, A, \Lambda) \det(I_m + H(s)G(s))}{\det(I_m + \Pi D)},
\]

where \( \Delta(s, z_1, z_2, A, \Lambda) \) is obviously defined and termed the regulator. Here, \( z_1 \in \mathbb{C} \) and \( z_2 \in \mathbb{C} \) are newly introduced complex variables to avoid factor cancellations between \( \beta(s) \) and \( \alpha(s) \). In the sequel, Equation (9) will be called the regularized return difference relationship.
Now we do observations about Equation (9). Clearly, what we see about Equation (6), except for factor cancellations, can be claimed for Equation (9). Hence, we only pay attention on distinctive points.

- When \( z_1 = z_2 = s \), Equation (9) reduces to Equation (6). If coprimeness between \( \beta(s) \) and \( \alpha(s) \) is mainly concerned, letting \( z_1 = z_2 = z \) is good enough. However, this may deprive us of possible algebraic and geometric insight on \( H(s) \) and \( G(s) \) disappearing from \( H(s)G(s) \).

- We stress that Equation (9) is of more significance rather than merely avoiding pole cancellations. In Equation (9), avoiding factor cancellations between \( \beta(s) \) and \( \alpha(s) \) is attained by multiplying the return difference equation (6) with the regulator \( \Delta(s, z_1, z_2, A, \Lambda) \). In fact, \( \Delta(s, z_1, z_2, A, \Lambda) \) brings back additional structural and algebraic features of the open-loop system.

- One might suggest regulators that are completely independent of the open-loop system \( \Sigma_0 \) or other elements in \( \Sigma_C \). In principle, such regulators do produce us Nyquist-like tests. However, the structural features of the open-loop system \( \Sigma_0 \) have no role in such tests. Such stability tests may not be useful in synthesis problems, though their mathematical implication is interesting and remains to be tackled.

### 4.2. Triple-contour Nyquist criteria and related issues

**Theorem 2** *In the system \( \Sigma_C \) of Figure 1, assume that \( I_m + \Pi D \) is non-singular. Let \( N \) be a Nyquist contour defined on the s-plane, and the functions \( f_1 : s \mapsto z_1 \) and \( f_2 : s \mapsto z_2 \) denote two meromorphisms that are holomorphic on \( \mathcal{N} \) and \( \text{Int}(\mathcal{N}) \), and satisfy, respectively,*

\[
\begin{align*}
  s - \lambda_{c,i} &\neq f_1(s) - \lambda_{a,k} \\
  s - \lambda_{c,i} &\neq f_2(s) - \lambda_{a,k}
\end{align*}
\]

*over \( s \in \mathcal{N} \) for any \( i, k = 1, 2, \ldots, n + p \). The meromorphisms \( f_1 \) and \( f_2 \) map \( \mathcal{N} \) into two simply closed image contours \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), respectively, on the \( z_1 \)-plane and \( z_2 \)-plane. Also, \( f_1 \) and \( f_2 \) are taken such that \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) by-pass all zeros of \( \alpha_G(z_1) \) and \( \alpha_H(z_2) \); or equivalently,*

\[ f_1(s) - \lambda_{a,k} \neq 0, \quad f_2(s) - \lambda_{a,k} \neq 0, \quad \forall s \in \mathcal{N}. \]

**Finally, let \( n_1 \) and \( n_2 \), respectively, denote the numbers of zeros of \( \alpha_G(z_1) \) and \( \alpha_H(z_2) \) belonging to \( \text{Int}(\mathcal{N}_1) \) and \( \text{Int}(\mathcal{N}_2) \). Then we have**

(i) \( \Sigma_C \) is neither asymptotically nor exponentially stable, if \( f_1 \) and \( f_2 \) exist such that the regularized Nyquist locus \( \Delta(s, z_1, z_2, A, \Lambda) \det(I_m + H(s)G(s))/\det(I_m + \Pi D)_{\epsilon \in \mathcal{N}_1, z_1 = f_1(s), z_2 = f_2(s)} \) encircles the origin \( (0, 0) \) counter-clockwisely \( n_1 + n_2 \) times;

(ii) \( \Sigma_C \) is asymptotically stable, if and only if there exist \( f_1 \) and \( f_2 \) such that the regularized Nyquist locus \( \Delta(s, z_1, z_2, A, \Lambda) \det(I_m + H(s)G(s))/\det(I_m + \Pi D)_{\epsilon \in \mathcal{N}_1, z_1 = f_1(s), z_2 = f_2(s)} \) encircles the origin \( (0, 0) \) counter-clockwisely \( n_1 + n_2 \) times;

(iii) \( \Sigma_C \) is exponentially stable, if and only if there exist a Nyquist contour defined in the form of \( \mathcal{N}_s \) with \( \epsilon > 0 \), \( f_1 \) and \( f_2 \) such that the regularized Nyquist locus \( \Delta(s, z_1, z_2, A, \Lambda) \det(I_m + H(s)G(s))/\det(I_m + \Pi D)_{\epsilon \in \mathcal{N}_s, z_1 = f_1(s), z_2 = f_2(s)} \) encircles the origin \( (0, 0) \) counter-clockwisely \( n_1 + n_2 \) times;

(iv) \( \Sigma_C \) is \( D \)-stable, if a Nyquist contour \( \mathcal{N}_d(s) \) can be defined according to a prescribed district \( D \subset \mathbb{C}^- \), and \( f_1 \) and \( f_2 \) exist such that the regularized Nyquist locus \( \Delta(s, z_1, z_2, A, \Lambda) \det(I_m + H(s)G(s))/\det(I_m + \Pi D)_{\epsilon \in \mathcal{N}_d, z_1 = f_1(s), z_2 = f_2(s)} \) encircles the origin \( (0, 0) \) counter-clockwisely \( n_1 + n_2 \) times.

**Proof of Theorem 2** Firstly, since \( f_1 : s \mapsto z_1 \) and \( f_2 : s \mapsto z_2 \) are meromorphic and holomorphic on \( \mathcal{N} \) and \( \text{Int}(\mathcal{N}) \), where \( \mathcal{N} \) is one of \( \mathcal{N}_d(s) \), \( \mathcal{N}_s \), and \( \mathcal{N}_d(s) \). It follows by Watanabe, Miyazaki, and Endo (1991, Theorem 7.2) that \( f_1 : s \mapsto z_1 \) and \( f_2 : s \mapsto z_2 \) are conformal mappings. This implies that \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are simply closed; and when \( s \) moves along \( \mathcal{N} \) clockwise, \( z_1 = f_1(s) \) and \( z_2 = f_2(s) \) run along \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) also clockwise. Therefore, the numbers \( n_1 \) and \( n_2 \) are well-defined.

Secondly, since Equation (10) is true, no common factor cancellations between \( \beta(s) \) and \( \alpha_G(z_1) \alpha_H(z_2) \) occur in the left-hand side of Equation (9) as long as \( s \in \mathcal{N} \), \( z_1 \in \mathcal{N}_1 \) and \( z_2 \in \mathcal{N}_2 \). Thus, all the open-loop and closed-loop poles remain in \( \beta(s)/(\alpha_G(z_1) \alpha_H(z_2)) \) and satisfy Equation (9). This means that the argument computations involved are meaningful for any \( s \in \mathcal{N}, z_1 \in \mathcal{N}_1 \) and \( z_2 \in \mathcal{N}_2 \).

Thirdly, let us write

\[
\beta(s) = \Pi_{i=1}^{n+p} (s - \lambda_{c,i}), \quad \alpha_G(z_1) \alpha_H(z_2) = \Pi_{i=1}^{n} (z_1 - \lambda_{G,i}) \Pi_{k=1}^{p} (z_2 - \lambda_{H,k}),
\]

where \( \lambda_{G,i} \) and \( \lambda_{H,k} \) have the obvious meanings. We can assert from Equation (9), the above equations and coprimeness between \( \beta(s) \) and \( \alpha_G(z_1) \alpha_H(z_2) \), that

\[
\sum_{i=1}^{n+p} \angle(s - \lambda_{c,i}) = \sum_{i=1}^{n} \angle(z_1 - \lambda_{G,i}) - \sum_{k=1}^{p} \angle(z_2 - \lambda_{H,k}) = \angle(\Delta(s, z_1, z_2, A, \Lambda)f(s))
\]

for each fixed \( s \in \mathcal{N} \). In the above, \( f(s) \) is the same as defined in the proof of Theorem 1. Recalling that \( \Delta(s, z_1, z_2, A, \Lambda) \) is also meromorphic, we can repeat the
arguments in the proof for Theorem 1 and organize them in an obvious fashion, and then we are led to

\[ n_C = n_f + n_1 + n_2, \]  

(11)

where \( n_C \) again is the number of the closed-loop poles in \( \text{Int}(\mathcal{N}) \), while \( n_f \) means the number that the regularized Nyquist locus \( \Delta(s, z_1, z_2, A, \Delta) f(s) \) with respect to \( s \in \mathcal{N}, z_1 = f_1(s) \) and \( z_2 = f_2(s) \) encircles the origin \((0,0)\) counterclockwise.

The assertion (i) follows readily if we do arguments similar to what we do for the assertion i) of Theorem 1. Sufficiency of (iii) and (iv) follows directly from Equation (11). To show sufficiency of (ii), we note that \( \mathcal{N}_{1,\alpha}(0) \) can contain the whole imaginary axis (without bypassing half-circles). This, together with Equation (11), leads that there are no closed-loop poles on the imaginary axis and \( \text{Int}(\mathcal{N}_{1,\alpha}(0)) \). These facts imply asymptotical stability of \( \Sigma_C \).

To show necessity of (ii) and (iii), it remains to examine existence of \( f_1 \) and \( f_2 \). We only consider the case that \( \Sigma_C \) is asymptotically stable; namely, all the closed-loop poles have negative real parts. \( \mathcal{N}_{1,\alpha}(0) = \mathcal{N} \) is well-defined in the sense that \( \beta(s) \) vanishes nowhere on \( \mathcal{N} \). Note that the open-loop poles are isolated points. Therefore, it is always possible to find constants \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) such that \( \beta(s) \) and \( \alpha_G(s + \Delta_1) \alpha_H(s + \Delta_2) \) are coprime, and \( s + \Delta_1 \) and \( s + \Delta_2 \) bypass all the zeros of \( \alpha_G(s + \Delta_1) \alpha_H(s + \Delta_2) \). And thus \( \beta(s)/(\alpha_G(s + \Delta_1) \alpha_H(s + \Delta_2)) \) is well-defined and vanishes nowhere over all \( s \in \mathcal{N} \). Now we re-write \( z_1 = s + \Delta_1 = f_1(s) \) and \( z_2 = s + \Delta_2 = f_2(s) \). Clearly, \( f_1 \) and \( f_2 \) are meromorphic and holomorphic on \( \mathcal{N} \) and \( \text{Int}(\mathcal{N}) \). For \( f_1 \) and \( f_2 \) to satisfy Equation (10), we can simply take \( \Delta_1 > \| A_C \| + \| A \| \) and \( \Delta_2 > \| A_C \| + \| A \| \), where \( A_C \) is the state matrix of \( \Sigma_C \), since \( |\lambda_{G_1}| < \| A_C \| \) and \( |\lambda_{G_1}| < \| A \| \) and \( |\lambda_{H_1}| < \| A \| \).

Similar to Theorem 1, necessity for (iv) of Theorem 2 does not generally hold.

In what follows, we collect remarks about (iv) of Theorem 2.

- **Existence of \( f_1 \) and \( f_2 \) is imperative for Theorem 2.** Since \( f_1 \) and \( f_2 \) are conformal mappings, we say that the Nyquist contours \( \mathcal{N}, \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are conformal. It is from this geometric feature about the multiple contours that the paper is entitled.

- **Although \( \mathcal{N} \) is chosen to reflect whether or not a certain stability is attainable in the closed-loop system \( \Sigma_C \), the situation about \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) in the sense of that stability in \( \Sigma_G \) and \( \Sigma_H \) are generally hard to surmise. Indeed, \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) complicatedly depend upon \( f_1 \) and \( f_2 \) as conceptually illustrated in Figure 5 for the case of \( \mathcal{N} = \mathcal{N}_s \). Thus, \( n_1 \) and \( n_2 \) are not simply related to "unstable poles"in the corresponding sub-systems.

- **If we simply take \( f_1 = f = f : s \mapsto z \), then \( n_1 + n_2 \) means nothing but the number of the zeros of \( \alpha(z) = \alpha_G(z) \alpha_H(z) \) belonging to \( \text{Int}(\mathcal{N}) \), where \( \mathcal{N} \) is the image contour of the domain contour \( \mathcal{N} \) under the holomorphism \( f \). Indeed, under \( f_1 = f_2 = f : s \mapsto z \), Theorem 2 turns out to be a dual-contour Nyquist criterion. However, we alert the reader to the fact that Theorem 2 does not reduce to Theorem 1 by taking \( f_1(s) = f_2(s) = s \).

### 4.3. Triple-contour Nyquist criteria independent of open-loop structures

Consider the necessity aspects of Theorem 2. If we let \( z_1 = s + \Delta_1 \) and \( z_2 = s + \Delta_2 \) for constants \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \), then \( \mathcal{N}_1 = \mathcal{N} + \mathcal{N}_1 \) and \( \mathcal{N}_2 = \mathcal{N} + \mathcal{N}_2 \), which are the image contours by shifting \( \mathcal{N} \) horizontally to the right by \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \), respectively. By the necessity proof of Theorem 2, such \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) always exist, and actually can be taken as large as desired. As a matter of fact, if \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) are taken sufficiently large, and define

\[ f_1(s) = s + \Delta_1, \quad f_2(s) = s + \Delta_2, \]  

(12)

which are clearly meromorphic and holomorphic on \( \mathcal{N} \) and \( \text{Int}(\mathcal{N}) \). It is straightforward to see that the conditions (10) of Theorem 2 need not to be examined practically, and all the zeros of \( \alpha_G(z_1) \) and \( \alpha_H(z_2) \) lie outside \( \text{Int}(\mathcal{N}_1) \) and \( \text{Int}(\mathcal{N}_2) \).

Based on the above observation, we have the following corollary of Theorem 2, which simply involves no open-loop structural features (i.e. open-loop pole distribution and etc.), and provides us with a fairly convenient way for stability analysis and stabilization. This is essentially different from what we usually see in the conventional Nyquist approach.

**Corollary 1.** In the system \( \Sigma_C \) of Figure 1, assume that \( I_m + \Pi D \) is non-singular. Let \( \mathcal{N} \) be a Nyquist contour defined on the \( s \)-plane, and the functions \( f_1 : s \mapsto z_1 \) and \( f_2 : s \mapsto z_2 \) are defined in Equation (12) with \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) being sufficiently large. Then we say

(i) \( \Sigma_C \) is neither asymptotically nor exponentially stable, and Lyapunov stability is indefinite,
that is holomorphic on $N$?

self-evident that $H$ is a corollary of Theorem 2.

Let $s$ be the origin.

Equation (9), we obtain

$\beta(s) = \beta_K(s), \quad \alpha(s) = \alpha_G(s)$,

where $\beta_K(s)$ is obviously meant. Substituting them for Equation (9), we obtain

$$\beta_K(s) = \frac{\det(sl_n - A) \det(I_m + KG(s))}{\det(I_m + KG(s))} = \Delta(s, z, A) \det(I_m + KG(s)).$$  \hspace{1cm} (13)

Here $\Delta(s, z, A) = \det(sl_n - A)/\det(zl_n - A)$, which yields a corollary of Theorem 2.

**Corollary 2** In the system $\Sigma_C$ as assumed in the above. Let $N$ be a Nyquist contour. Also, a meromorphism $f : s \mapsto z$ that is holomorphic on $N$ and $\text{Int}(N)$ exists and satisfies

$$s - f(s) \neq \lambda_{i,j} - \lambda_{G,k}$$  \hspace{1cm} (14)

over $s \in N$ and for all $i, k = 1, \ldots, n$. Also, $f$ maps $N$ to an image contour $\tilde{N}$. $\tilde{N}$ is taken such that $\tilde{N}$ bypasses all zeros of $\alpha_G(z)$. Finally, let $\tilde{n}$ be the number of zeros of $\alpha_G(z)$ in $\text{Int}(\tilde{N})$.

(i) $\Sigma_C$ is neither asymptotically nor exponentially stable, while Lyapunov stability is indefinite, if $f$ exists such that the regularized Nyquist locus $\Delta(s, z, A) \det(I_m + KG(s))|_{s = f(s)}$ passes through the origin $(0,0)$;

(ii) $\Sigma_C$ is asymptotically stable, if and only if $f$ exists such that the regularized Nyquist locus $\Delta(s, z, A) \det(I_m + KG(s))|_{s = f(s)}$ encircles the origin $(0,0)$ counter-clockwise $\tilde{n}$ times;

(iii) $\Sigma_C$ is exponentially stable, if and only if there exist a Nyquist contour defined in the form of $N_{\epsilon}, s > 0$, and $f$ such that the regularized Nyquist locus $\Delta(s, z, A) \det(I_m + KG(s))|_{s = f(s)}$ encircles the origin $(0,0)$ counter-clockwise $\tilde{n}$ times;

(iv) $\Sigma_C$ is $D$-stable, if a Nyquist contour $N_{\epsilon}$ can be defined according to a prescribed district $D \subset \mathbb{C}^-$, and $f$ exists such that the regularized Nyquist locus $\Delta(s, z, A) \det(I_m + KG(s))|_{s = f(s)}$ encircles the origin $(0,0)$ counter-clockwise $\tilde{n}$ times.

Although the first two assertions of Theorem 2 are claimed explicitly by means of $N_{\epsilon}(0)$ instead of a general contour like $N_{\epsilon}(r)$ with $r > 0$, the same results can be derived by using any $N_{\epsilon}(r)$ with $r > 0$ being sufficiently small if additional conditions are imposed on open-loop poles on the imaginary axis, as explicated in Corollary 3. Without such conditions, we may not be able to do so directly; for example, in the case when some open-loop poles on the imaginary axis remain at the same positions as closed-loop poles for non-controllability and/or non-observability.

**Corollary 3** Under the assumptions of Corollary 2, $\Sigma_C$ is Lyapunov stable if

(i) each pole of $\Sigma_G$ on the imaginary axis is simple and reflects an unobservable mode;

(ii) there exist Nyquist contour $N_{\epsilon}(r)$ with $r > 0$ sufficiently small, and a meromorphism $f$ that is holomorphic on $N_{\epsilon}(r)$ and over $s \in \text{Int}(N_{\epsilon}(r))$, and bypasses all the poles of $\Sigma_G$ on the imaginary axis;

(iii) the regularized Nyquist locus $\Delta(s, z, A) \det(I_m + KG(s))|_{s = f(s)}$ encircles the origin $(0,0)$ counter-clockwise $\tilde{n}$ times.

More precisely, the poles of $\Sigma_G$ on the imaginary axis are zeros of the characteristic polynomial $\alpha_G(s) = |sI_n - A|$ with zero real parts.

**Proof of Corollary 2** With regard to the feedback configuration of Figure 1, we see that the state-space expressions for $\Sigma_O$ and $\Sigma_C$ are given by

$$\Sigma_O : \begin{cases} \dot{\xi} = Ax + Bu, \\ \eta = Kc, \end{cases} \quad \Sigma_C : \begin{cases} \dot{\xi} = (A - BK)c + Bu, \\ \eta = Cy. \end{cases}$$

Consequently, the open-loop characteristic polynomial $\alpha(s) = \alpha_G(s) = |sI_n - A|$; namely, the poles of $\Sigma_G$ are equivalently the open-loop poles of $\Sigma_O$. Using this and repeating the sufficiency arguments for the assertion (ii) of Theorem 1 and that of Theorem 2, we see that under the given assumptions, there are no closed-loop poles in $\text{Int}(\tilde{N}_{\epsilon}(r))$ and on $\tilde{N}_{\epsilon}(r)$.
Hence, to complete the proof, it remains to show that the poles of $\Sigma_G$ on the imaginary axis, the corresponding closed-loop pole remains there and is simple. To see this, we note that $r > 0$ is sufficiently small, thus there exist no other closed-loop poles on the imaginary axis except for those remaining at the same positions of the open-loop poles on the imaginary axis, each of which is simple by the assumption.

It is well known by the Kalman’s decomposition theorem that there is a similarity transform $T$ such that $\Sigma_G$ can be transformed into an observability canonical form given by

$$
\tilde{A} = T^{-1}AT = \begin{bmatrix}
\tilde{A}_0 & 0 \\
\tilde{A}_1 & \tilde{A}_0
\end{bmatrix}, \\
\tilde{B} = T^{-1}B = \begin{bmatrix}
\tilde{B}_0 \\
\tilde{B}_1
\end{bmatrix},
$$

and

$$
\tilde{C} = CT = \begin{bmatrix}
\tilde{C}_0 & 0
\end{bmatrix},
$$

where all sub-matrices are obviously meant and have compatible dimensions.

Using these facts, we turn to observe that

$$T^{-1}(A - BK)T = \tilde{A} - \tilde{B}K\tilde{C} = \begin{bmatrix}
\tilde{A}_0 - \tilde{B}_0K\tilde{C}_0 & 0 \\
\tilde{A}_1 - \tilde{B}_1K\tilde{C}_0 & \tilde{A}_0
\end{bmatrix}.
$$

The specific lower triangular form of the above matrix implies that the eigenspace for any eigenvalue of $\tilde{A}_0$ is completely independent of the feedback gain $K$. Together with the fact that similarity transforms do not change eigenspace, it follows that if an eigenvalue of the open-loop state matrix $A$ that is related to $A_0$ is simple and on the imaginary axis, then it will remain at the same position and be simple in the closed-loop system as well.

Several remarks about Corollary 3.

- The Nyquist contour $N_{\Sigma}(s)$ cannot be replaced with $N_{\Sigma}(0)$ simply. Indeed, if there do exist some closed-loop poles on the imaginary axis, we will run into the situation described by the assertion (i) of Corollary 2 when using $N_{\Sigma}(s) = N_{\Sigma}(0)$.
- Corollary 3 renders us a simple case, where the Nyquist loci can be utilized for Lyapunov stability analysis. Actually it reveals that it is generally quite difficult for us to do Lyapunov stability analysis with a Nyquist-like criterion, while other internal stabilities can be simply dealt with via the suggested Nyquist approach.

This is not surprising if we notice that Lyapunov stability involves eigenvalue simplicity, which relates to both geometric and algebraic multiplicity about matrix eigenvalues. In view of this, Nyquist approach is seemingly poor in connecting these concepts.

5. Numeric examples

We consider the cart with an inverted pendulum (Chen, 1999) as shown in Figure 6, where we implement a constant feedback matrix $H(s) = K$ between $u$ and $y$. Here, $u$ is the control input imposed on the cart and the measured output $y$ will be specified case by case.

It is shown by Chen (1999) that when the swing angle $\theta$ of the pendulum arm and the angular speed $\dot{\theta}$ are relatively small, a state-space equation for the cart dynamics is given by

$$
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & (M + m)g/M & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
\dfrac{1}{M} \\
0 \\
\dfrac{1}{Ml}
\end{bmatrix} u
$$

$$= Ax + Bu,
$$

where $x = [x_1, x_2, x_3, x_4]^T$ and $x_1 = \rho, x_2 = \dot{\rho}, x_3 = \theta$ and $x_4 = \dot{\theta}$. And the (open-loop) characteristic polynomial for the state matrix $A$ is

$$
a_{\Sigma}(s) = \det(sI_4 - A) = s^2 - \dfrac{(M + m)g}{(Ml)}. \quad (15)
$$

Throughout the numeric computation, we evaluate the model at $M = 2$ kg, $m = 0.2$ kg and $l = 0.5$ m, while the gravitational acceleration $g = 9.8 \text{ m/s}^2$. Accordingly, the open-loop poles are 0, 0 and $\pm 4.6433$; namely, the open-loop system possesses two poles on the imaginary axis, one pole with positive real part.

In the following, the conventional Nyquist loci will be plotted with solid curves, while the regularized ones are represented with dotted lines. The arrows indicate the locus directions when $s$ (resp., $z$) moves clockwise along the Nyquist contour (resp., the conformal contour).

5.1. When the measured output is the state vector itself, i.e. $y = x$

Now the state-space expression for the cart system is written as follows.

$$
\Sigma_G : \begin{bmatrix}
\dot{x} = Ax + Bu, \\
y = x = Cx,
\end{bmatrix} \quad (16)
$$

where $C = I_4$. By the Popov–Belevitch–Hautus test, $\Sigma_G$ is controllable and observable.
Now we consider a static output feedback in form of $u = -Ky$ with $K = [K_1, K_2, K_3, K_4]$, where $K_1$, $K_2$, $K_3$ and $K_4$ are scalars. Since the output $y = x$, $u = -Ky$ is essentially a state feedback. The closed-loop characteristic polynomial is $\beta_K(s) = \det(sI_4 - A + BK)$. 

Case (a): Let $K = [-0.12, -0.1, -25.18, -0.55]$. The corresponding closed-loop poles are 

$$\lambda_{c,1,2} = -0.1058 \pm j1.7638,$$

$$\lambda_{c,3,4} = -0.1442 \pm j0.5985.$$ 

Clearly, the closed-loop cart system is asymptotically stable by Proposition 1.

Now we examine stability by Theorem 1 and Corollary 2, respectively. The Nyquist contour for Theorem 1 is specified as $\mathcal{N} = \mathcal{N}_d(0.5)$ with a detouring half-circle around the origin by the radius $r = 0.5$ as in Figure 2, or simply denoted as $\mathcal{N}_{d}(0.5)$. The Nyquist contour for Corollary 1 is $\mathcal{N}_d(0)$, whose conformal contour is $\tilde{\mathcal{N}} = \mathcal{N}_d(0) + \Delta$ with a shifting factor $\Delta = 0.35$.

On the one hand, note that $\lambda_{G,3} = 4.6433 \in \text{Int}(\mathcal{N}_d(0.5))$ and that the Nyquist locus in Figure 7 encircles counter-clockwisely the origin $(0, 0)$ only once. This, together with the fact that the open-loop and closed-loop characteristic polynomials are coprime, leads us also to asymptotic stability of the closed-loop system by Theorem 1. On the other hand, by the definition of $\tilde{\mathcal{N}}$, there is one open-loop pole in $\text{Int}(\tilde{\mathcal{N}})$; that is, $\lambda_{G,3} = 4.6433 \in \text{Int}(\tilde{\mathcal{N}})$. In Figure 7, the regularized Nyquist locus encircles counter-clockwisely the origin $(0, 0)$ only once; thus the closed-loop system is asymptotically stable by Corollary 2.

### 5.2. When the measured output is a partial state vector $y = [\theta, \dot{\theta}]^T$

Now the state-space expression for $\Sigma_G$ is as follows.

$$\Sigma_{p,G} : \begin{cases} \dot{x} = Ax + Bu \\ y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = : C_p x, \end{cases} \quad (17)$$

where the subscript ‘p’ means partial. $\Sigma_{p,G}$ is controllable but not observable.

Now we consider a static output feedback $K_p = [K_3, K_4]$, where $K_3$ and $K_4$ are scalars. The closed-loop characteristic polynomial is $\beta_{K_p} = \det(sI_4 - A + BK_p C_p)$.

The Nyquist contour for Corollary 3 is specified once again as $\tilde{N} = \mathcal{N}_d(0.5)$ with a detouring half-circle around the origin by $r = 0.5$, while the conformal contour is $\tilde{N} = \mathcal{N}_d(0.5) + \Delta$ with a shifting factor $\Delta = 0.35$. Let us implement the feedback gain $K_p = [-22.56, -0.50]$. Then, the corresponding closed-loop poles are given by

$$\lambda_{c,1} = \lambda_{c,2} = 0,$$

$$\lambda_{c,3,4} = -0.2500 \pm j0.9682.$$ 

By matrix indice computation (Bernstein, 2009), we see that $\lambda_{c,1} = 0$ and $\lambda_{c,2} = 0$ are two simple zeros in the minimal polynomial of $A - BK_p C_p$. Hence, Proposition 1 says that $\Sigma_C$ is Lyapunov stable.

Noting that $\text{Int}(\tilde{N}) = \text{Int}(\mathcal{N}_d(0.5) + \Delta)$ contains only one open-loop pole, that is $\lambda_{G,3} = 4.6433$, and from Figure 8 that the regularized Nyquist locus encircles
counter-clockwisely around the origin only once, we conclude by Corollary 3 that $\Sigma_C$ is Lyapunov stable. Clearly, we cannot draw Lyapunov stability from Theorem 1 with the Nyquist contour $N_{1,4}(0.5)$ and the corresponding Nyquist locus plotted in Figure 8. We must emphasize that there is a common factor $s^2$ between the open-loop and closed-loop characteristic polynomials.

6. Conclusion

In this study, we begin with examining miscellaneous pole cancellation issues about the conventional return difference relationships, based on which most existing Nyquist criteria are stated for LTI feedback systems; as a matter of fact, pole cancellations are simply neglected so that most conventional Nyquist criteria may not work for internal stability.

To avoid the pole cancellation problems and validate the Nyquist approach for internal stability, coprimeness assumption between the open- and closed-loop characteristic polynomials is added as in Theorem 1, which collects single-contour Nyquist criteria. To completely get rid of the pole cancellation problems, we further develop the regularized Nyquist loci that are plotted along the Nyquist contour and its conformal contour(s), by means of which several multiple-contour Nyquist criteria are stated and proved. The stability criteria are based on the regularized return difference equations, which are suggested for the first time by the best knowledge of the author. The results are claimed for Lyapunov stability/instability, asymptotic stability, exponential stability and $\mathcal{D}$-stability in LTI feedback systems by specifying the Nyquist contour and its conformal one(s). The details are summarized in Theorem 2, and through Corollaries 1–3.

Last but not the least, implication of the study is that structural features of LTI multivariable systems may be lost or neglected to a great extent, if a single variable frequency(complex)-domain framework is adopted in the Nyquist approach. Frequency(complex)-domain analysis and synthesis approaches in multiple variables (Hörmander, 1990) will be considered in our future study.

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Notes

1. That is, these cancellations are pole/zero cancellations that happen only in some sub-systems of $\Sigma_C$, which cannot be classified into (1) and (2) by the definition where the system $\Sigma_C$ is viewed as a whole. One should interpret these pole/zero cancellations from a sub-system point of view.

2. The proof arguments are similar to what is known in standard textbooks about Nyquist criteria. Since necessity and/or sufficiency may vary in each assertion, the details are retained to reveal such distinctions.

3. Indeed, if $f(s) = 0$ at some $s = s' \in \mathcal{N}$, then $\angle f(s')$ has no definite definition; or in some abused words the phase vector

Figure 8. Nyquist locus and regularized Nyquist locus when $\mathcal{N} = N_{1,4}(r)$. 
\[ f(s') = 0 \] can encircle the origin as many times as desired. Hence, \( g(s') \) is not unique.

4. Clearly, \( |f(s)| > 0 \) over \( s \in N \) if \( f(s)|_{s \in N} \) does not pass through the origin (0,0).

5. Clearly, they are equivalent to the numbers of zeros of \( \alpha_N(s) \) and \( \alpha_F(s) \) belonging to \( \text{Int}(N_1) \) and \( \text{Int}(N_2) \).

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