HIGHER ABEL-JACOBI MAPS

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INTRODUCTION

We work over a subfield $k$ of $\mathbb{C}$, the field of complex numbers. For a smooth variety $V$ over $k$, the Chow group of cycles of codimension $p$ is defined (see [5]) as

$$CH^p(V) = \frac{Z^p(V)}{R^p(V)}$$

where the group of cycles $Z^p(V)$ is the free abelian group on scheme-theoretic points of $V$ of codimension $p$ and rational equivalence $R^p(V)$ is the subgroup generated by cycles of the form $\text{div}_W(f)$ where $W$ is a subvariety of $V$ of codimension $(p - 1)$ and $f$ is a non-zero rational function on it. There is a natural cycle class map

$$\text{cl}_p : CH^p(V) \to H^{2p}(V)$$

where the latter denotes the singular cohomology group $H^{2p}(V(\mathbb{C}), \mathbb{Z})$ with the (mixed) Hodge structure given by Deligne (see [4]). The kernel of $\text{cl}_p$ is denoted by $F_1 CH^p(V)$. There is an Abel-Jacobi map (see [8]),

$$\Phi_p : F_1 CH^p(V) \to \text{IJ}^{p}(H^{2p-1}(V))$$

where the latter is the intermediate Jacobian of a Hodge structure, defined as follows

$$\text{IJ}^p(H) = \frac{H \otimes \mathbb{C}}{F^p(H \otimes \mathbb{C}) + H^*}.$$ 

We note for future reference that for a pure Hodge structure of weight $2p-1$ (such as the cohomology of a smooth projective variety) we have the natural isomorphism,

$$\frac{H \otimes \mathbb{R}}{H} \cong \text{IJ}^p(H)$$

The kernel of $\Phi_p$ is denoted by $F_2 CH^p(V)$.

[10] The conjecture of S. Bloch says (see [10]) that there is a filtration $F^*$ on $CH^p(V)$ which extends the $F^1$ and $F^2$ defined above. Moreover, the associated graded group $\text{gr}_F^k CH^p(V)$ is governed by the cohomology groups $H^{2p-k}(V)$ for each integer $k$ (upto torsion). More precisely, if $N^i H^m(V)$ denotes the filtration by co-niveau (see [9]) which is generated by cohomology
classes supported on subvarieties of codimension \( \leq l \), then \( \text{gr}_k^F \text{CH}^p(V) \) is actually governed by the quotient groups,
\[
\frac{H^{2p-k}(V)}{N^{p-k+1} H^{2p-k}(V)}.
\]
Specifically, in the case when \( V \) is a smooth projective surface with geometric genus 0 (so that \( H^2(V) = N^1 H^2(V) \)) this conjecture implies that \( F^2 \text{CH}^2(V) \) is torsion (and thus 0 by a theorem of Roitman [14]).

The traditional Hodge-theoretic approach to study this problem is based on the fact that the intermediate Jacobian \( \otimes \mathbb{Q} \) can be interpreted as the extension group \( \text{Ext}^1(\mathbb{Q}(-p), H) \) in the category of Hodge structures. One can then propose that the associated graded groups \( \text{gr}_k^F \text{CH}^p(V) \otimes \mathbb{Q} \) should be interpreted as the higher extension groups \( \text{Ext}^k(\mathbb{Q}(-p), H^{2p-k}(V)) \) for \( k \geq 2 \). Unfortunately, there are no such extension groups in the category of Hodge structures. Thus it was proposed that all these extension groups be computed in a suitable category of mixed motives.\(^1\)

Even if such a category is constructed a Hodge-theoretic interpretation of these extension groups would be useful. In section 2 we discuss M. Green’s approach (see [5]) called the Higher Abel-Jacobi map. In section 3 we provide a counter-example to show that Green’s approach does not work; a somewhat more complicated example was earlier obtained by C. Voisin (see [19]). In section 4 we introduce an alternative approach based on Deligne-Beilinson cohomology and its interpretation in terms of Morihiko Saito’s theory of Hodge modules; such an approach has also been suggested earlier by M. Asakura and independently by M. Saito (see [1] and [15]). Following this approach it becomes possible to deduce Bloch’s conjecture from some conjectures of Bloch and Beilinson for cycles and varieties defined over a number field (see [10]).

1. Green’s Higher Abel-Jacobi Map

The fundamental idea behind M. Green’s construction can be interpreted as follows (see [19]). One expects that the extension groups are effaceable in the abelian category of mixed motives. Thus the elements of \( \text{Ext}^k \) can be written in terms of \( k \) different elements in various \( \text{Ext}^1 \)'s. The latter groups can be understood in terms of Hodge theory, via the Intermediate Jacobians. So we can try to write the \( \text{Ext}^k \) as a sub of a quotient of a (sum of) tensor products of Intermediate Jacobians.

Specifically, consider the case of a surface \( S \). Let \( C \) be a curve, then we have a product map (see [3]),
\[
\text{CH}^1(C) \times \text{CH}^2(C \times S) \to \text{CH}^2(S)
\]
which in fact respects the filtration \( F \) (see [10]), so that we have
\[
F^1 \text{CH}^1(C) \times F^1 \text{CH}^2(C \times S) \to F^2 \text{CH}^2(S).
\]
Conversely, we can use an argument of Murre (see [11]) to show,

\(^1\)Such a category has recently been constructed by M. V. Nori (unpublished).
**Lemma 1.** Given any cycle class $\xi$ in $F^2 CH^2(S)$ there is a curve $C$ so that $\xi$ is in the image of the map,

$$F^1 CH^1(C) \times F^1 CH^2(C \times S) \to F^2 CH^2(S).$$

**Proof.** Let $z$ be a cycle representing the class $\xi$. There is a smooth (see [12]) curve $C$ on $S$ that contains the support of $z$. Hence it is enough to show that there is a homologically trivial cycle $Y$ on $C \times S$ so that $(z, Y) \mapsto z$ for every cycle $z$ on $C$ such that the image under $CH^1(C) \to CH^2(S)$ lies in $F^2 CH^2(S)$. Let $\Gamma$ denote the graph of the inclusion $\iota : C \hookrightarrow S$. Then clearly $(z, \Gamma) \mapsto z$ but $\Gamma$ is not homologically trivial.

Choose a point $p$ on $C$. Now, by a result of Murre (see [11]), for some positive integer $m$ we have an expression in $CH^2(S \times S)$

$$m \Delta_S = m(p \times S + S \times p) + X_{2,2} + X_{1,3} + X_{3,1}$$

where $\Delta_S$ is the diagonal and $X_{i,j}$ is a cycle so that its cohomology class has non-zero Künneth component only in $H^i(S) \otimes H^j(S)$. In particular, $X_{1,3}$ gives a map $F^1 CH^2(S) \to F^1 CH^2(S)$ which induces multiplication by $m$ on $IJ^2(H^3(S))$. Since $p \times S$ and $S \times p$ induce 0 on $F^1 CH^2(S)$ it follows that the correspondence $X_{2,2} + X_{3,1}$ induces multiplication by $m$ on $F^2 CH^2(S)$.

Now, $\Gamma = (\iota \times 1_S)^*(\Delta_S)$ so we have an expression

$$m \Gamma = mC \times p + (\iota \times 1_S)^*X_{2,2} + (\iota \times 1_S)^*X_{1,3}$$

Let $D = p_{2*}(\iota \times 1_S)^*X_{2,2}$. Then the cohomology class of $Y = (\iota \times 1_S)^*X_{2,2} - p \times D$ is 0. Moreover, the map $F^1 CH^1(C) \to F^1 CH^2(S)$ induced by $p \times D$ is zero. Thus, by the above propery of $X_{2,2} + X_{3,1}$ we see that $mz = (z, m\Gamma) = (z, Y)$ for any $z$ in $F^1 CH^1(C)$ whose image lies in $F^2 CH^2(S) \otimes \mathbb{Q}$. By Roitman’s theorem (see [14]) the group $F^2 CH^2(S)$ is divisible. Hence, we conclude the result.

We now use the Abel-Jacobi maps to interpret the two terms on the left-hand side in terms of Hodge theory.

Firstly, we have the classical Abel-Jacobi isomorphisms $F^1 CH^1(C) = J(C) = IJ^1(H^1(C))$. Let $H^2(S)_{tr} = H^2(S)/N^1 H^2(S)$ denote the lattice of transcendental cycles on $S$. Consider the factor $IJ^2(H^1(C) \otimes H^2(S)_{tr})$ of the intermediate Jacobian $IJ^2(H^3(C \times S))$. We can compose the Abel-Jacobi map with the projection to this factor to obtain

$$F^1 CH^2(C \times S) \to IJ^2(H^1(C) \otimes H^2(S)_{tr}).$$

Using the identification $IJ^p(H) = H \otimes (\mathbb{R}/\mathbb{Z})$ for a pure Hodge structure $H$ of weight $2p - 1$ we have

$$IJ^1(H^1(C)) \otimes IJ^2(H^1(C) \otimes H^2(S)_{tr}) = H^1(C)^{\otimes 2} \otimes H^2(S) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$$
The pairing $H^1(C)^{\otimes 2} \to H^2(C) = \mathbb{Z}$, given by the cup product, can be used to further collapse the latter term. Thus, we obtain a diagram,

$$
\begin{array}{ccc}
F^1 CH^1(C) \times F^1 CH^2(C \times S) & \to & F^2 CH^2(S) \\
\downarrow & & \downarrow \\
IJ^1(H^1(C)) \times IJ^2(H^1(C) \otimes H^2(S)_{tr}) & \to & H^2(S)_{tr} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}
\end{array}
$$

**Definition 1.** Green’s second intermediate Jacobian $J^2_2(S)$ is defined as the universal push-out of all the above diagrams as $C$ is allowed to vary. The Higher Abel-Jacobi map is defined as the natural homomorphism

$$
\Psi_2^2 : F^2 CH^2(S) \to J^2_2(S).
$$

By the above lemma it follows that $J^2_2(S)$ is a quotient of $H^2(S)_{tr} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$. The question is whether this constructs the required $\text{Ext}^2$.

**Problem 1 (Green).** Is $\Psi_2^2$ injective?

2. **Non-injectivity of Green’s Map**

We now compute Green’s Higher Abel-Jacobi map for the case of a surface of the form $\text{Sym}^2(C)$, where $C$ is a smooth projective curve. Using this we show that this map is not injective when $C$ is a curve of genus at least two whose Jacobian is a simple abelian variety.

**Lemma 2.** Let $Z \in CH^2(D \times C \times S)$ be a cycle, where $D$, $C$ are smooth curves and $S$ a smooth surface. Then we have a commutative diagram

$$
\begin{array}{ccc}
F^1 CH^1(D) \otimes F^1 CH^1(C) & \overset{p_3^*(p_1^!(\omega) \cup Z)}{\longrightarrow} & F^2 CH^2(S) \\
\downarrow & & \downarrow \\
IJ^1(H^1(D)) \otimes IJ^1(H^1(C)) & \overset{z}{\longrightarrow} & H^2(S)_{tr} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}
\end{array}
$$

Here the map $z$ is the composite as follows. The cohomology class of $Z$ gives a map $H^1(D) \otimes H^1(C) \to H^2(S)$; we further project to $H^2(S)_{tr}$. Now tensor with $(\mathbb{R}/\mathbb{Z})^{\otimes 2}$ and identify the resulting left-hand term with the product of the Intermediate Jacobians.

We note that the vertical arrow on the left is an isomorphism.

**Proof.** By the functoriality of the Abel-Jacobi map we have a commutative diagram

$$
\begin{array}{ccc}
F^1 CH^1(D) & \overset{p_3^*(p_1^!(\omega) \cup Z)}{\longrightarrow} & F^1 CH^2(C \times S) \\
\downarrow & & \downarrow \\
IJ^1(H^1(D)) & \overset{1_{(\mathbb{R}/\mathbb{Z})^{\otimes 2}} \otimes p_3^*(p_1^!(\omega) \cup [Z])}{\longrightarrow} & IJ^2(H^1(C) \otimes H^2(S))
\end{array}
$$

By projection we can replace the bottom right corner with $IJ^2(H^1(C) \otimes H^2(S)_{tr})$. Now we tensor this with the Abel-Jacobi map for $C$ to obtain,

$$
\begin{array}{ccc}
F^1 CH^1(D) \otimes F^1 CH^1(C) & \to & F^1 CH^1(C) \otimes F^1 CH^2(C \times S) \\
\downarrow & & \downarrow \\
IJ^1(H^1(D)) \otimes IJ^1(H^1(C)) & \to & IJ^1(H^1(C)) \otimes IJ^2(H^1(C) \otimes H^2(S)_{tr})
\end{array}
$$
The required commutative diagram now follows from the definition of $J_2^2(S)$.

We now apply this lemma to the case $C = D$ and $S = \text{Sym}^2(C)$. In this case we take $Z$ to be the graph of the quotient morphism $q : C \times C \to \text{Sym}^2(C)$. We then compute that the cohomological correspondence given by $[Z]$ factors as

$$H^1(C) \otimes H^1(C) \to H^2(\text{Sym}^2(C))$$

By the above lemma we obtain a factoring,

$$F^1 CH^1(C) \otimes F^1 CH^1(C) \xrightarrow{\text{pr}_1(\omega) \cdot Z} F^2 CH^2(\text{Sym}^2(C))$$

The image of the tensor product of a pair of elements of $\text{IJ}^1(H^1(C))$ of the form $v \otimes a$ and $v \otimes \beta$ must therefore be 0 in $J_2^2(\text{Sym}^2(C))$.

The description of $F^2 CH^2(\text{Sym}^2(C))$ is given by the following lemma that is similar to one in [3],

**Lemma 3.** The homomorphism

$$Z_* : F^1 CH^1(C) \otimes F^1 CH^1(C) \to F^2 CH^2(\text{Sym}^2(C))$$

is surjective.

**Proof.** The following composite map is multiplication by 2

$$F^2 CH^2(\text{Sym}^2(C)) \xrightarrow{q} F^2 CH^2(C \times C) \xrightarrow{q} F^2 CH^2(\text{Sym}^2(C))$$

By the divisibility of $F^2 CH^2(S)$ for a surface $S$ we see that the lemma follows from the following result.

**Sublemma 1.** Fix a base point $p$ on $C$. Then the filtration $F$ of $CH^2(C \times C)$ is explicitly described as follows

$$F^2 CH^2(C \times C) = \text{im}(J(C) \otimes J(C)) \subset$$

$$F^1 CH^2(C \times C) = F^2 CH^2(C \times C) + \text{im}(J(C) \times p) + \text{im}(p \times J(C))$$

$$\subset \text{CH}^2(C \times C) = F^1 CH^2(C \times C) + Z \cdot (p, p)$$

**Proof.** Let $a, b$ be points on $C$; we get points $[a - p]$ and $[b - p]$ of $J(C)$. The image of $[a - p] \otimes [b - p]$ in $\text{CH}^2(C \times C)$ is $(a, b) + (p, p) - (a, p) - (p, b)$. Thus, we have an expression

$$(a, b) = \text{im}([a - p] \otimes [b - p]) + \text{im}((a - p) \times p) + \text{im}(p \times [b - p]) + (p, p)$$

Now, any cycle $\xi$ in $F^1 CH^2(C \times C)$ can be written as $\sum_{i=1}^n (a_i, b_i) - n \cdot (p, p)$. The Albanese variety of $C \times C$ is $J(C) \oplus J(C)$ and the image of $\xi$ under the Albanese map is $\sum_{i=1}^n [a_i - p_i], \sum_{i=1}^n [b_i - p_i]$. Thus, if the cycle is in
\( F^2 \text{CH}^2(C \times C) \), then \( \sum_{i=1}^n [a_i - p] = 0 = \sum_{i=1}^n [b_i - p] \). Now we combine this with the above expression to obtain
\[
\xi = \sum_{i=1}^n \text{im}([a_i - p] \otimes [b_i - p])
\]
which proves the result. 

**Lemma 4.** If \( C \) is a curve of genus at least 2 such that its Jacobian variety is a simple abelian variety then \( \Psi_2 \) has a non-trivial kernel.

**Proof.** By Mumford’s result there are non-trivial classes in \( F^2 \text{CH}^2(S) \). The Jacobian variety \( J(C) = \text{Pic}^0(C) = H^1(C) \otimes \mathbb{R}/\mathbb{Z} \) is spanned by decomposable elements. Moreover \( F^1 \text{CH}^1(C) \cong J(C) \). Thus there is a pair of elements of \( F^1 \text{CH}^1(C) \) of the form \( v \otimes \alpha, w \otimes \beta \) such that the image of their tensor product in \( F^2 \text{CH}^2(S) \) is non-zero. By a result of Roitman, for any fixed class \( f \) in \( F^1 \text{CH}^1(C) \), the collection \( K_f = \{ e \in J(C) | e \otimes f \mapsto 0 \text{ in } F^2 \text{CH}^2(S) \} \) forms a countable union of abelian subvarieties of \( J(C) \). Since \( w \otimes \beta \) does not lie in \( K_v \otimes \alpha \), the latter is a proper subgroup of \( J(C) \). Since \( J(C) \) is assumed to be simple this is forced to be a countable set. In particular, there is an element of the form \( v \otimes \gamma \) which is not in \( K_{v \otimes \alpha} \); so that the product of this with \( v \otimes \alpha \) is non-zero in \( F^2 \text{CH}^2(S) \). But we just saw that all such elements are mapped to 0 in \( J^2_0(S) \).

3. Absolute Deligne-Beilinson Cohomology

The fundamental idea underlying the following constructions and definitions is as follows. A variety \( V \) over \( \mathbb{C} \) can be thought of as a family of varieties over the algebraic closure \( \overline{\mathbb{Q}} \subset \mathbb{C} \) of the field of rational numbers. Even when the variety is defined over \( \mathbb{Q} \) the Chow group of such a variety (when considered over \( \mathbb{C} \)) may contain cycles that are defined over larger fields. In particular, the usual examples of non-trivial elements in \( F^2 \text{CH}^2(S) \) are defined over fields of transcendence degree 2 (see [18]). Thus, in order to detect such cycles we must use the full force of such a “family”-like structure.

For any variety \( V \) over \( \mathbb{C} \) we consider the collection of Cartesian diagrams
\[
\begin{array}{ccc}
V & \rightarrow & \mathcal{V} \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \rightarrow & S
\end{array}
\]
where \( S \) and \( \mathcal{V} \) are varieties defined over \( \overline{\mathbb{Q}} \), and the lower horizontal arrow factors through the generic point of \( S \). Assume for the moment that \( V \) is smooth projective, and that \( S \) and \( \mathcal{V} \) are smooth and \( \mathcal{V} \rightarrow S \) is proper and smooth. Then the relative de Rham cohomology groups \( H^i_{\text{dR}}(\mathcal{V}/S) \) carry the Gauss-Manin connection; moreover, after base change to \( S \otimes \mathbb{C} \) the associated local system is a variation of Hodge structure. This has been generalised by M. Saito (see [18]) for all \( V \) and all choices of \( S \) and \( \mathcal{V} \) as
There is a (mixed) Hodge module $R^i_{dR}(\mathcal{V}/S)$ on $S$ in the above context so that its pull-back via $\text{Spec } \mathbb{C} \to S$ is the (mixed) Hodge structure on the cohomology of $V$. The category $\text{MHM}(S)$ of Hodge modules over $S$ is an abelian category which has non-trivial $\text{Ext}^2$’s when $S$ has dimension at least 1. Moreover, we have a natural spectral sequence

$$E_1^{a,b} = \text{Ext}^b_{\text{MHM}(S)}(\mathbb{Q}(c), R^a_{dR}(\mathcal{V}/S)) \Rightarrow \text{Ext}^{a+b}_{\text{MHM}(\mathcal{V})}(\mathbb{Q}(c), \mathbb{Q})$$

We are interested in the case $a = 2p - k$, $b = k$ and $c = -p$. In this case the latter term can be identified with the Deligne-Beilinson cohomology $H^{2p}\text{Db}_D(V, \mathbb{Q}(p))$ (see [15] and [2]).

**Definition 2.** Let us define the absolute Deligne-Beilinson cohomology of $V$ as the direct limit

$$H^n_{\text{ADb}}(V, \mathbb{Q}(c)) = \lim_{\rightarrow} \text{Ext}^n_{\text{MHM}(\mathcal{V})}(\mathbb{Q}(c), \mathbb{Q})$$

where the limit is taken over all diagrams such as the one above.

Since any algebraic cycle on $V$ (and $V$ itself) is defined over some finitely generated field, we have

$$\text{CH}^p(V) = \lim_{\rightarrow} \text{CH}^p(V)$$

The cycle class map in Deligne-Beilinson cohomology then gives us a cycle class map

$$\text{cl}^p_{\text{ADb}} : \text{CH}^p(V) \to H^{2p}_{\text{ADb}}(V, \mathbb{Q}(p))$$

The filtration on the latter group induced by the above spectral sequence induces a filtration on $\text{CH}^p(V)$. We can then ask whether this is the filtration as required by Bloch’s conjecture.

It is well known (see [3]) that the cycle class map for Deligne-Beilinson cohomology combines the usual cycle class map to singular cohomology with the Abel-Jacobi map. Thus, the following conjecture implies that $\text{cl}^p_{\text{ADb}}$ is injective.

**Conjecture 1 (Bloch-Beilinson).** If $V$ is a variety defined over a number field then $F^2 \text{CH}^p(V) = 0$.

We (of course) offer no proof of this conjecture. However, there are examples due to C. Schoen and M. V. Nori (see [17]), discovered independently by M. Green and the third author, which show that one cannot relax the conditions in this conjecture. A paper [3] by M. Green and and the third author contains these and other examples showing that $F^2 \text{CH}^p(V)$ can be non-zero for $V$ a variety over a field of transcendence degree at least one.

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2 Since we have chosen an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ we can think of Hodge modules as being associated with varieties over $\overline{\mathbb{Q}}$ rather than with varieties over $\mathbb{C}$.
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