Discretization of four types of Weyl group orbit functions

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Abstract. The discrete Fourier calculus of the four families of special functions, called $C^\cdot$, $S^\cdot$, $S^\cdot\!\!^\cdot$- and $S^\cdot\!\!^\cdot\!\!^\cdot$-functions, is summarized. Functions from each of the four families of special functions are discretely orthogonal over a certain finite set of points. The generalizations of discrete cosine and sine transforms of one variable — the discrete $S^\cdot\!\!^\cdot$- and $S^\cdot\!\!\!\cdot$-transforms of the group $F_4$ — are considered in detail required for their exploitation in discrete Fourier spectral methods. The continuous interpolations, induced by the discrete expansions, are presented.

1. Introduction

The standard discrete Fourier analysis of one real variable is known to be a very valuable tool in mathematics, physics and elsewhere [3, 13, 14]. Various approaches can be followed to formulate this analysis in higher dimensions. Here we summarize the approach based on Weyl groups — the one dimensional case as well as its straightforward Cartesian product generalization are then a special case of this calculus. Four types of the multidimensional generalizations of cosine and sine functions, which are available for the root systems with two different lengths of roots and form the core of this approach, appeared in various stages of development of the Lie theory [2, 4, 9, 10, 11]. Their discrete Fourier calculus, formulated explicitly and in full generality, has been developed only recently [5, 6, 7]. Summarizing this calculus and adding a specific example of the group $F_4$ is the main purpose of this paper.

Consider the Lie algebra of the compact simply connected simple Lie group $G$ of rank $n$. The set of simple roots of $G$ is denoted by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and forms a basis of the Euclidean vector space $\mathbb{R}^n$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. Only such simple algebras which have two different lengths of roots are considered, namely $B_n$, $n \geq 3$, $C_n$, $n \geq 2$, $G_2$ and $F_4$. For these algebras the set of simple roots consists of short simple roots $\Delta_\alpha$ and long simple roots $\Delta_l$. The following quantities can be deduced from the entire root system $\Delta$ by standard methods: the Cartan matrix $C$, the highest root $\xi \equiv \alpha_0 = m_1\alpha_1 + \cdots + m_n\alpha_n$, the root lattice $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$, the $Z$-dual lattice $P^\vee = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee$, the dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$, where $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle$ and the weight lattice $P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. The set of vectors $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ also forms a root system called the dual root system $\Delta^\vee$ of $G$.

This dual root system determines the highest dual root $\eta \equiv -\alpha_0^\vee = m_1\alpha_1^\vee + \cdots + m_n\alpha_n^\vee$.

The reflections $r_\alpha$, $\alpha \in \Delta$ are given as reflections in $(n-1)$-dimensional ‘mirrors’ orthogonal to the simple roots $\alpha$ and intersecting at the origin. Similarly are defined the reflections $r_\xi$, $r_\eta$ of the highest root $\xi$ and the highest dual root $r_\eta$. The Weyl group $W$ is generated by $n$
reflections $r_\alpha$, $\alpha \in \Delta$ and the affine Weyl group $W^{\text{aff}}$ of $G$ is defined as a semidirect product $W^{\text{aff}} = Q \ltimes W$. The affine Weyl group is generated by $n$ reflections $r_\alpha$ and the affine reflection $r_0$, which composes of the reflection $r_\varepsilon$ and the shift by $\frac{2k}{(\xi, \xi)}$. The fundamental region $F \subset \mathbb{R}^n$ of $W^{\text{aff}}$ consists of precisely one point from each $W^{\text{aff}}$-orbit. It can be chosen as the $n$-simplex [8] of the form $F = \left\{0, \frac{\omega_1}{m_1}, \ldots, \frac{\omega_n}{m_n}\right\}_\kappa$.

The formulation of the discrete Fourier calculus requires careful analysis of the boundaries of the fundamental domain $F$ – for this purpose short and long boundaries are introduced. The points of $F$ which are stabilized by some $r_\alpha$, $\alpha \in \Delta_s$ are collected in the short boundary $H^s$ of $F$. The points of $F$ which are stabilized by some $r_\alpha$, $\alpha \in \Delta_l$ or $r_0$ are collected in the long boundary $H^l$ of $F$. The short and the long fundamental domains $F^s$ and $F^l$ are defined in [7] as

$$F^s = F \setminus H^s, \quad F^l = F \setminus H^l.$$ 

The dual affine Weyl group $\tilde{W}^{\text{aff}}$ of $G$ is defined similarly as a semidirect product $\tilde{W}^{\text{aff}} = Q \rtimes W$. The dual affine Weyl group is generated by $n$ reflections $r_\alpha$ and the affine reflection $r_0^\vee$, which composes of the reflection $r_\varepsilon$ and the shift by $\frac{2k}{(\eta, \eta)}$. The simplex $F^\vee \subset \mathbb{R}^n$ of the form $F^\vee = \left\{0, \frac{\omega_1}{m_1}, \ldots, \frac{\omega_n}{m_n}\right\}_\kappa$ forms the fundamental region of $\tilde{W}^{\text{aff}}$. The points of $F^\vee$ which are stabilized by some $r_\alpha$, $\alpha \in \Delta_s$ or $r_0^\vee$ are collected in the short dual boundary $H^s\vee$ of $F^\vee$. The points of $F^\vee$ which are stabilized by some $r_\alpha$, $\alpha \in \Delta_l$ are collected in the long dual boundary $H^l\vee$ of $F^\vee$. The short and the long fundamental domains $F^s\vee$ and $F^l\vee$ are defined [7] as

$$F^s\vee = F^\vee \setminus H^s\vee, \quad F^l\vee = F^\vee \setminus H^l\vee.$$ 

2. Discrete Fourier calculus of $C-$, $S-$, $S^s-$ and $S^l-$functions

Considering a weight $b \in P$, the normalized $C-$functions and the $S-$functions are given by

$$\Phi_b(a) = \sum_{w \in W} 2^{2\pi i(\langle wb, a \rangle / 2)} \quad \varphi_b(a) = \sum_{w \in W} (\det w) e^{2\pi i \langle wb, a \rangle}.$$ 

The family of $C-$functions and the family of $S-$function could be taken as the generalization of cosine and sine functions, respectively. Two other families, which carry mixed properties of sines and cosines, can also be singled out from the continuum of Heckman-Opdam polynomials [4]. Two 'sign' homomorphisms $\sigma^s, \sigma^l : W \rightarrow \{\pm 1\}$ are defined [11] by their values on the generating reflections $r_\alpha$, $\alpha \in \Delta$ of $W$,

$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l \\ -1, & \alpha \in \Delta_s \end{cases}$$

$$\sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s \\ -1, & \alpha \in \Delta_l \end{cases}.$$ 

These sign homomorphisms $\sigma^s$ and $\sigma^l$ determine the $S^s-$functions and the $S^l-$functions

$$\varphi^s_b(x) = \sum_{w \in W} \sigma^s(w) e^{2\pi i \langle wb, x \rangle}, \quad \varphi^l_b(x) = \sum_{w \in W} \sigma^l(w) e^{2\pi i \langle wb, x \rangle}.$$ 

Contours plots of some lowest $S^s-$ and $S^l-$functions of the group $G_2$ are depicted in Figures 1 and 2.

For the detailed review of $C-$functions see [9]: the $S-$functions, which are well-known from the character functions, are reviewed in [10]. The character functions, which appear in the
Weyl character formula, are ratios of the general $S$–functions and the lowest possible one. Similar ratios for the $S^s$– and $S^l$–functions lead to short and long character functions. Note that $C$–functions together with the three character functions are special cases of the Jacobi polynomials generalized to root systems [4].

Next, building the discrete Fourier calculus, we consider the values of $C$, $S$, $S^s$ and $S^l$ functions on certain finite grids inside the regions $F$, $F^o$, $F^s$ and $F^l$. Given an arbitrary $M \in \mathbb{N}$, the quotient group $\frac{1}{M} P^\vee / Q^\vee$ forms a $W$-invariant finite group. We define the finite grids $F_M$, $\tilde{F}_M$, $F^s_M$ and $F^l_M$ as such elements from $\frac{1}{M} P^\vee / Q^\vee$ which have representative points in $F$, $F^o$, $F^s$ and $F^l$, respectively:

$$F_M = \frac{1}{M} P^\vee / Q^\vee \cap F,$$
$$\tilde{F}_M = \frac{1}{M} P^\vee / Q^\vee \cap F^o,$$
$$F^s_M = \frac{1}{M} P^\vee / Q^\vee \cap F^s,$$
$$F^l_M = \frac{1}{M} P^\vee / Q^\vee \cap F^l.$$

The group $\frac{1}{6} P^\vee / Q^\vee$ and the grids $F_6$, $\tilde{F}_6$, $F^s_6$ and $F^l_6$ of $G_2$, together with the roots $\alpha, \alpha^\vee$ and the weights $\omega, \omega^\vee$, are depicted in Figure 3. The counting formulas for the numbers of points in the grids $F_M$, $\tilde{F}_M$, $F^s_M$ and $F^l_M$ are for all cases derived in [5, 7].

When $C$, $S$, $S^s$ and $S^l$–functions are sampled on the grids $F_M$, $\tilde{F}_M$, $F^s_M$ and $F^l_M$ they can be labeled only by the labels from the finite subsets of $P$. As weights in the subsets $\Lambda_M$, $\tilde{\Lambda}_M$, $\Lambda^s_M$ and $\Lambda^l_M$, only the following elements of the $W$-invariant finite group $P/MQ$ are taken
\[ \omega_1 = \omega_1^\vee = \xi \]

\[ \omega_2 = \omega_2^\vee = \eta \]

\[ \alpha_1 = \alpha_1^\vee \]

\[ \alpha_2 = \alpha_2^\vee \]

\[ r_0, r_1, r_2 \]

\[ H_s, H_l \]

\[ \Lambda_M = MF^\vee \cap P/MQ, \quad \tilde{\Lambda}_M = M(F^\vee)^0 \cap P/MQ, \]

\[ \Lambda_M^s = MF^s \cap P/MQ, \quad \Lambda_M^l = MF^l \cap P/MQ. \]

Moreover, it is shown in [5, 7] that the numbers of weights in the sets \( \Lambda_M, \tilde{\Lambda}_M, \Lambda_M^s \) and \( \Lambda_M^l \) coincide with the numbers of points in the sets \( F_M, \tilde{F}_M, F_M^s \) and \( F_M^l \) for all cases of \( G \), i.e.

\[ |\Lambda_M| = |F_M|, \quad |\tilde{\Lambda}_M| = |\tilde{F}_M|, \]

\[ |\Lambda_M^s| = |F_M^s|, \quad |\Lambda_M^l| = |F_M^l|. \]

The order of an \( W \)-orbit of a point \( x \in \frac{1}{M}P^\vee/Q^\vee \) is denoted by \( \varepsilon(x) \); the order of a \( W \)-stabilizer of a weight \( \lambda \in P/MQ \) is denoted by \( h_\lambda^s \). The discrete orthogonality of \( C \)- and \( S \)-functions is proven in [5] and extended to the discrete orthogonality of \( S^s \)- and \( S^l \)-functions in [7]. These results from [5, 7] can be summarized as follows.

**Figure 3.** The fundamental regions \( F, F^s \) and \( F^l \) of \( G_2 \). The fundamental domain \( F \) is depicted as the red triangle containing its borders \( H^s \) and \( H^l \), which are drawn as the thick dashed line and dot-and-dashed lines, respectively. The coset representatives of \( \frac{1}{M}P^\vee/Q^\vee \) are depicted as 36 black dots. The three representatives belonging to \( F^s \) and \( F^l \) are crossed with ‘+’ and ‘×’, respectively. The dashed lines represent ‘mirrors’ \( r_0, r_1 \) and \( r_2 \).
Theorem 2.1. For \( \lambda, \lambda' \in \Lambda_M \) it holds that
\[
\sum_{x \in F_M} \varepsilon(x) \Phi_\lambda(x) \Phi_\lambda'(x) = c |W| M^n h^\vee_\lambda \delta_{\lambda, \lambda'}
\]
and for \( \lambda, \lambda' \in \tilde{\Lambda}_M \) it holds that
\[
\sum_{x \in F_M} \varphi_\lambda(x) \varphi_\lambda'(x) = c M^n \delta_{\lambda, \lambda'}.
\]
For \( \lambda, \lambda' \in \Lambda^s_M \) it holds that
\[
\sum_{x \in F^s_M} \varepsilon(x) \phi^s_\lambda(x) \phi^s_\lambda'(x) = c |W| M^n h^\vee_\lambda \delta_{\lambda, \lambda'}
\]
(1)
and for \( \lambda, \lambda' \in \Lambda^l_M \) it holds that
\[
\sum_{x \in F^l_M} \varepsilon(x) \psi_\lambda(x) \psi_\lambda'(x) = c |W| M^n h^\vee_\lambda \delta_{\lambda, \lambda'},
\]
(2)
where \( c \) is the determinant of the Cartan matrix, \( |W| \) is the number of elements of the Weyl group \( W \) and \( n \) is the rank of \( G \).

3. The group \( F_4 \)
The highest root \( \xi \) and the highest dual root \( \eta \) of the exceptional group \( F_4 \) are determined by the formulas
\[
\xi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \quad \eta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.
\]
The Weyl group of \( F_4 \) has 1152 elements \([1, 8]\). Discretization of the \( C^- \) and \( S^- \) functions of \( F_4 \) is detailed in \([6]\). Here we focus on discretization of the \( S^s^- \) and \( S^l^- \) functions of \( F_4 \). Taking the variables \( u^s_0, u^s_1, u^s_2 \in \mathbb{Z} \geq 0 \) and \( u^s_3, u^s_4 \in \mathbb{N} \), the grid \( F^s_M \) is given by
\[
F^s_M = \left\{ \frac{u^s_0}{M} \omega^1 + \cdots + \frac{u^s_4}{M} \omega^4 \bigg| u^s_0 + 2u^s_1 + 3u^s_2 + 4u^s_3 + 2u^s_4 = M \right\}
\]
and for the numbers of its elements it holds \([7]\):
\[
|F^s_{12k}| = 18k^4 - k^2, \quad \nonumber
|F^s_{12k+1}| = 18k^4 + 6k^3 - \frac{5}{2}k^2 - \frac{1}{2}k, \quad \nonumber
|F^s_{12k+2}| = 18k^4 + 12k^3 + 2k^2, \quad \nonumber
|F^s_{12k+3}| = 18k^4 + 18k^3 + \frac{7}{2}k^2 - \frac{1}{2}k, \quad \nonumber
|F^s_{12k+4}| = 18k^4 + 24k^3 + 11k^2 + 2k, \quad \nonumber
|F^s_{12k+5}| = 18k^4 + 30k^3 + \frac{31}{2}k^2 + \frac{5}{2}k.
\]
For a given function sampled on \( I \), these interpolating functions and \((2)\), respectively. The coefficients functions, which hold for any two functions \( \varphi_\lambda^s, \varphi_\lambda^l \) labeled by \( \lambda, \lambda' \in \Lambda_M^s \), and \( \varphi_\lambda^l, \varphi_\lambda' \) labeled by \( \lambda, \lambda' \in \Lambda_M^l \), are of the form \((1)\) and \((2)\), respectively. The coefficients \( \varepsilon(x) \) and \( h_\lambda^s \) are listed in Table 1 in [6].

4. Discrete \( S^s - \) and \( S^l - \) transforms of \( F_4 \)

For a given function sampled on \( F_M^s \) or \( F_M^l \) of \( F_4 \), we define continuous interpolating functions \( I_M^s, I_M^l \)

\[
I_M^s(x) = \sum_{\lambda \in \Lambda_M^s} c_\lambda^s \varphi_\lambda^s(x), \quad x \in \mathbb{R}^4
\]

\[
I_M^l(x) = \sum_{\lambda \in \Lambda_M^l} c_\lambda^l \varphi_\lambda^l(x), \quad x \in \mathbb{R}^4.
\]

These interpolating functions \( I_M^s \) and \( I_M^l \) are defined as linear combinations of basis functions \( \varphi_\lambda^s, \varphi_\lambda^l \) with expansion coefficients \( c_\lambda^s, c_\lambda^l \) whose values need to be determined from the conditions

\[
I_M^s(x) = f(x), \quad x \in F_M^s
\]

\[
I_M^l(x) = f(x), \quad x \in F_M^l.
\]

The formulas for calculation of \( c_\lambda^s \) and \( c_\lambda^l \), which can be also viewed as discrete \( S^s - \) and \( S^l - \) transforms, are of the explicit form:

\[
c_\lambda^s = (1152 M^4 h_\lambda^s)^{-1} \sum_{x \in F_M^s} \varepsilon(x) f(x) \varphi_\lambda^s(x) 
\]

\[
c_\lambda^l = (1152 M^4 h_\lambda^l)^{-1} \sum_{x \in F_M^l} \varepsilon(x) f(x) \varphi_\lambda^l(x).
\]
5. Concluding remarks

- The products of two $C_-$, $S_-$, $S^s-$ and $S^l-$functions with the affine Weyl group of the same type and the same arguments $x \in \mathbb{R}^n$ and any two weights $\lambda, \lambda' \in P$, decompose into the sum of $C-$functions,

$$
\Phi_\lambda(x) \cdot \Phi_{\lambda'}(x) = \sum_{w \in W} \Phi_{\lambda + w\lambda'}(x), \quad \varphi_\lambda(x) \cdot \varphi_{\lambda'}(x) = \sum_{w \in W} \det(w) \Phi_{\lambda + w\lambda'}(x)
$$

$$
\varphi^s_\lambda(x) \cdot \varphi^s_{\lambda'}(x) = \sum_{w \in W} \sigma^s(w) \Phi_{\lambda + w\lambda'}(x), \quad \varphi^l_\lambda(x) \cdot \varphi^l_{\lambda'}(x) = \sum_{w \in W} \sigma^l(w) \Phi_{\lambda + w\lambda'}(x).
$$

- To any function $f : F \rightarrow \mathbb{C}$ one can assign two functional series $\{I^s_M \}_{M=1}^\infty$, $\{I^l_M \}_{M=1}^\infty$ with the coefficients calculated from (3), (4). Similar functional series can be constructed using $C-$ and $S-$functions [6]. The form of suitable conditions on $f$, which would be similar to the conditions on one-variable functions in [14] and would guarantee convergence of these functional series, poses an open problem.

- The standard relation between the sine and cosine functions of one variable and Chebyshev polynomials of the first and second kind is exploited in various applications as polynomial interpolation and cubature formulas. The discrete orthogonality of the sine and cosine polynomials of the first and second kind is exploited in various applications as polynomial functional series, poses an open problem.

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