Chiral magnetic effect for chiral fermion system

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We concisely derive chiral magnetic effect through Wigner function approach for chiral fermion system. Then we derive chiral magnetic effect through solving the Landau levels of chiral fermions in detail. The procedures of second quantization and ensemble average lead to the equation of chiral magnetic effect for righthand and lefthand fermion systems. Chiral magnetic effect only comes from the contribution of the lowest Landau level. We carefully analyze the lowest Landau level, and find that all righthand (chirality is +1) fermions move along positive z-direction and all lefthand (chirality is −1) fermions move along negative z-direction. From this picture chiral magnetic effect can be explained clearly in a microscopic way.

I. INTRODUCTION

Quark gluon plasma (QGP) can be created in high energy heavy ion collisions, which is a extremely hot and dense matter. Very huge magnetic field can be produced for high energy peripheral collisions. One of the predictions in QGP is that positively charged particles and negatively charged particles will separately along the direction of magnetic field, which is related to chiral magnetic effect (CME). Many efforts have been made to find the signal of CME in experiments. But due to the background noise, no definite CME singal has been found. There are also many theoretical methods to study CME, such as AdS/CFT, hydrodynamics, finite temperature field theory, quantum kinetic theory, et al.

In this article we will carefully study CME through solving Landau levels. For massive Dirac fermion system, there were some works on CME related to Landau levels. In their paper, Fukushima et al. proposed four methods to derive CME, in one of which they took use of Landau energy levels for massive Dirac equation with chemical potential \( \mu \) and chiral chemical potential \( \mu_5 \). When \( \mu \) is the zeroth harmonic oscillator wavefunction and \( \mu_5 \) is the zeroth harmonic oscillator wavefunction along x-direction, we can set \( k_y = 0 \). The \( \lambda \)-component of spin operator for single particle is \( S^\lambda = \frac{1}{2} \text{diag}(\sigma_3, \sigma_3) \), which implies \( S^\lambda \psi_{0\lambda} = (+1)^\lambda \psi_{0\lambda} \). When \( \lambda = +1 \), \( E = \sqrt{m^2 + k_z^2} > 0 \), then \( \psi_{0+} \) in Eq.
describes a particle with momentum \( k_z \) and spin projection \( S^z = +\frac{1}{2} \). When \( \lambda = -1, E = -\sqrt{m^2 + k_z^2} < 0 \), then \( \psi_0^- \) in Eq. (1) describes an antiparticle with momentum \(-k_z\) and spin projection \( S^z = -\frac{1}{2} \). So in the homogeneous magnetic background \( B = B_0 \), we obtain a picture for the lowest Landau level (with \( k_y = 0 \)): all particles spin along \((+z)\)-axis while all antiparticles spin along \((-z)\)-axis, but the \( z \)-component momentum of particles and anti-particles can be along \((+z)\)-axis or \((-z)\)-axis. In fact it is very difficult to obtain a net electric current along the magnetic field direction from the picture of the lowest Landau level for massive fermion case.

In this article, we focus on massless fermion (also called “chiral fermion”) system, in which case we will show that it is very easy to obtain a net electric current along the magnetic field direction from the picture of the lowest Landau level. Chiral fermion field can be divided into two independent parts — righthand part and lefthand part. Firstly let us set up notation. The electric charge of a fermion/antifermion is \( \pm e \). The chemical potential for righthand/lefthand fermions is \( \mu_{R/L} \), from which chiral chemical potential and the ordinary chemical potential can be expressed as \( \mu_5 = (\mu_R - \mu_L)/2, \mu = (\mu_R + \mu_L)/2 \). The chemical potential \( \mu \) describes the imbalance of fermions and anti-fermions, while the chiral chemical potential \( \mu_5 \) describes the imbalance of righthand and lefthand chirality. It is worth noting that an introduction of a chemical potential generally correspond to a conserved quantity. The conserved quantity corresponding to the ordinary chemical potential \( \mu \) is total electric charge of the system. But due to chiral anomaly \( \cite{21, 22} \), there is no conserved quantity corresponding to the chiral chemical potential \( \mu_5 \), which is crucial for the existence of CME \( \cite{23} \).

To study CME for chiral fermion system, firstly we show a succinct derivation of CME through Wigner function approach, from which we can obtain CME as a quantum effect of the first order in \( \hbar \) expansion. Then we turn to solve the Landau levels for the chiral fermion system. Since chiral fermions are massless, the equations of righthand and lefthand parts of the chiral fermion field decouple with each other, which allow us to deal with righthand and lefthand fermion fields independently. Taking righthand fermion field as an example, we firstly solve the energy eigenvalue equation of righthand fermion field in an external uniform magnetic field, and obtain a series of Landau levels. Then we perform the second quantization for righthand fermion field which can be expanded by the complete wavefunctions of Landau levels. Finally chiral magnetic effect can be derived through ensemble average, from which we see explicitly that CME only comes from the lowest Landau level. By analyzing the physical picture for the lowest Landau level, we conclude that all righthand (chirality is \(+1\)) fermions move along positive \( z \)-direction and all lefthand (chirality is \(-1\)) fermions move along negative \( z \)-direction. This is the main result of this article. This result can qualitatively explain why there is a macroscopic electric current along the direction of the magnetic field in a chiral fermion system, which is called CME. We emphasize that CME equation is derived from solving Landau levels without the approximation of weak magnetic field.

The rest of this article is organized as follows. In Sec. \( \text{III} \) we give a succinct derivation for CME through Wigner function approach. In Sec. \( \text{IV} \) we solve the Landau levels for righthand fermion field. In Sec. \( \text{V} \) and \( \text{VI} \) we perform second quantization of the righthand fermion system and obtain CME through ensemble average. In Sec. \( \text{VI} \) we discuss the physical picture of the lowest Landau level. At last, we summarize this article in Sec. \( \text{VII} \). We present some of derivation details in the appendixes.

Throughout this article we adopt natural units where \( \hbar = c = k_B = 1 \). The convention for the metric tensor is \( g^{\mu\nu} = \text{diag} (+1, -1, -1, -1) \). The totally antisymmetric Levi-Civita tensor is \( \epsilon^{\mu\nu\rho\sigma} \) with \( \epsilon^{0123} = +1 \) which agrees with Peskin \( \cite{24} \) but not with Bjorken and Drell \( \cite{25} \). Greek indices, such as \( \mu, \nu, \rho, \sigma \), run over \( 0, 1, 2, 3 \), or \( t, x, y, z \), while Roman indices, such as \( i, j, k \), run over \( 1, 2, 3 \) or \( x, y, z \). We use the Heaviside-Lorentz convention for electromagnetism.

## II. A SUCCINCT DERIVATION OF CME FROM WIGNER FUNCTION APPROACH

In this section we will concisely derive CME from Wigner function approach for chiral fermion system. Our starting point is the following covariant and gauge invariant Wigner function,

\[
\mathcal{W}_{\alpha\beta}(x, p) = \left\langle \frac{1}{(2\pi)^4} \int d^4 y e^{-i p \cdot y} \Psi_\beta(x + \frac{y}{2}) U(x + \frac{y}{2}, x - \frac{y}{2}) \Psi_\alpha(x - \frac{y}{2}) : \right\rangle,
\]

where \( \langle \cdots : \cdots \rangle \) represents ensemble average, \( \Psi(x) \) is the Dirac filed operator for chiral fermions, \( \alpha, \beta \) are Dirac spinor indices, and \( U(x + y/2, x - y/2) \) is the gauge link of a straight line from \((x - y/2)\) to \((x + y/2)\). This specific choice for the path in gauge link in the definition of Wigner function is firstly proposed in \( \cite{26} \), where the authors argued that this type of gauge link can make the variable \( p \) in Wigner function \( \mathcal{W}(x, p) \) becoming a kinetic momentum, although in principle the path in the gauge link is arbitrary. The specific choice of the two points \((x \pm y/2)\) in the integrand in Eq. (2) is based on the consideration of symmetry. In fact we can also replace \((x \pm y/2)\) by \((x + sy)\) and \((x - (1 - s)y)\) where \( s \) is a real parameter \( \cite{27} \).
Suppose that the electromagnetic field $F_{\mu\nu}$ is homogeneous in space and time, then from the dynamical equation satisfied by $\Psi(x)$, one can obtain the dynamical equation for $\mathcal{W}(x,p)$ as follows,

$$\gamma \cdot K \mathcal{W}(x,p) = 0, \quad (3)$$

where $K_\mu = \frac{i}{2} \nabla_\mu + p_\mu$ and $\nabla_\mu = \partial_\mu - eF_{\mu\nu}\partial_\nu$. Since $\mathcal{W}(x,p)$ is a $4 \times 4$ matrix, we can decompose it by the 16 independent $\Gamma$-matrices,

$$\mathcal{W} = \frac{1}{4}(\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{Y}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{J}_{\mu\nu}). \quad (4)$$

The 16 coefficient functions $\mathcal{F}$, $\mathcal{P}$, $\mathcal{Y}_\mu$, $\mathcal{A}_\mu$, $\mathcal{J}_{\mu\nu}$ are scalar, pseudoscalar, vector, pseudovector and tensor, respectively, and they are all real functions due to the fact that $\mathcal{W}^\dagger = \gamma^0 \mathcal{W} \gamma^0$. Vector current and axial vector current can be expressed as the 4-momentum integration of $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$,

$$J_5^\mu(x) = \int d^4 p \mathcal{V}^\mu, \quad (5)$$

$$J_A^\mu(x) = \int d^4 p \mathcal{A}^\mu. \quad (6)$$

If Eq. (3) is multiplied by $\gamma \cdot K$ from the lefthand side, we can obtain the quadratic form of Eq. (3) as follows,

$$\left(K^2 - \frac{i}{2} \sigma^{\mu\nu}[K_\mu, K_\nu]\right)\mathcal{W} = 0. \quad (7)$$

From Eq. (7) we can obtain two off mass-shell equations for $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ (See Appendix A for details),

$$(p^2 - \frac{1}{4} \hbar^2 \nabla^2)\mathcal{V}_\mu = -e\hbar \tilde{F}_{\mu\nu} \mathcal{A}^\nu, \quad (8)$$

$$(p^2 - \frac{1}{4} \hbar^2 \nabla^2)\mathcal{A}_\mu = -e\hbar \tilde{F}_{\mu\nu} \mathcal{V}^\nu, \quad (9)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. Note that we have explicitly shown $\hbar$ factor in Eqs. (8 9). If we expand $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ order by order in $\hbar$ as

$$\mathcal{V}^\mu = \mathcal{V}_{(0)}^\mu + \hbar \mathcal{V}_{(1)}^\mu + \hbar^2 \mathcal{V}_{(2)}^\mu + \cdots, \quad (10)$$

$$\mathcal{A}^\mu = \mathcal{A}_{(0)}^\mu + \hbar \mathcal{A}_{(1)}^\mu + \hbar^2 \mathcal{A}_{(2)}^\mu + \cdots, \quad (11)$$

then at order $o(1)$ and $o(\hbar)$, Eqs. (8 9) become

$$p^2 \mathcal{V}_{(0)}^\mu = 0, \quad (12)$$

$$p^2 \mathcal{A}_{(0)}^\mu = 0. \quad (13)$$

$$p^2 \mathcal{V}_{(1)}^\mu = -e\hbar \tilde{F}_{\mu\nu} \mathcal{A}_{(0)}^\nu, \quad (14)$$

$$p^2 \mathcal{A}_{(1)}^\mu = -e\hbar \tilde{F}_{\mu\nu} \mathcal{V}_{(0)}^\nu. \quad (15)$$

The zeroth order solutions $\mathcal{V}_{(0)}^\mu$ and $\mathcal{A}_{(0)}^\mu$ can be derived by directly calculating Wigner function without gauge link through ensemble average in Eq. (2), which has already been obtained by one of the authors and his collaborators 28. The results for $\mathcal{V}_{(0)}^\mu$ and $\mathcal{A}_{(0)}^\mu$ are

$$\mathcal{V}_{(0)}^\mu = \frac{2}{(2\pi)^3} \rho^\mu \delta(p^2) \sum_s \left[ \theta(p^0) \frac{1}{e^\beta(p^0 - \mu_s) + 1} + \theta(-p^0) \frac{1}{e^\beta(-p^0 + \mu_s) + 1} \right], \quad (16)$$

$$\mathcal{A}_{(0)}^\mu = \frac{2}{(2\pi)^3} \rho^\mu \delta(p^2) \sum_s \left[ \theta(p^0) \frac{1}{e^\beta(p^0 - \mu_s) + 1} + \theta(-p^0) \frac{1}{e^\beta(-p^0 + \mu_s) + 1} \right], \quad (17)$$
where $\beta = 1/T$ is the inverse temperature of the system, $\mu_{R/L}$ is the chemical potential for righthand/lefthand fermions as mentioned in the introduction, and $s = \pm 1$ corresponds to the chirality of righthand/lefthand fermions respectively. Obviously the zeroth order solutions $\gamma^{(0)}_{\mu}$ and $\delta^{(0)}_{\mu}$ satisfy Eqs. (12) (13), which means they are both on shell. From Eqs. (13) (16) we directly obtain the first order solutions,

$$
\gamma^{(1)}_{\mu} = \frac{2}{(2\pi)^3} e\hbar \tilde{F}^{\mu\nu} p_\nu \delta'(p^2) \sum_s \left[ \theta(p^0) \frac{1}{e^{\beta(p^0) - s\mu} + 1} + \theta(-p^0) \frac{1}{e^{\beta(-p^0) + s\mu} + 1} \right], \tag{18}
$$

$$
\delta^{(1)}_{\mu} = \frac{2}{(2\pi)^3} e\hbar \tilde{F}^{\mu\nu} p_\nu \delta'(p^2) \sum_s \left[ \theta(p^0) \frac{1}{e^{\beta(p^0) - s\mu} + 1} + \theta(-p^0) \frac{1}{e^{\beta(-p^0) + s\mu} + 1} \right], \tag{19}
$$

where we have used $\delta'(p^2) = -\delta(p^2)/p^2$. Eqs. (18) (19) are the same as the second term in Eq. (3) in [29].

Now we can calculate $J_{\mu/V}$ based on Eqs. (5) (6). Since $\gamma^{(1)}_{\mu}, \delta^{(1)}_{\mu}$ are odd functions of 3-momentum $p$, the nonzero contribution to $J^{(1)}_{\mu/V}$ only comes from $\gamma^{(1)}_{\mu}$ and $\delta^{(1)}_{\mu}$. We assume there only exists a uniform magnetic field $B = Be_z$, i.e. $F^{12} = -F^{21} = -B$ and $\tilde{F}^{03} = -\tilde{F}^{30} = -B$ (other components of $F_{\mu\nu}, \tilde{F}_{\mu\nu}$ are zero), which implies $J^{(1)}_{\mu/V} = J^{(1)}_{V/A} = 0$. After integration over the $z$-components of Eqs. (18) (19) we have

$$
J_{\mu/V} = \int d^4p \gamma^{(1)}_{\mu} = \frac{e\hbar \mu_5}{2\pi^2} B, \tag{20}
$$

$$
J^{(1)}_{\mu/V} = \int d^4p \delta^{(1)}_{\mu} = \frac{e\hbar \mu}{2\pi^2} B, \tag{21}
$$

where $\mu_5 = (\mu_R - \mu_L)/2$ and $\mu = (\mu_R + \mu_L)/2$. Eq. (20) means that if $\mu_5 \neq 0$, then there will be a current along the magnetic direction. Since $\hbar$ appears in the coefficient of the magnetic field $B$, we need a very huge magnetic field to produce a macroscopic current, which may be realised in high energy heavy ion collisions. So far we have derived CME for chiral fermion system through Wigner function approach, and we can see that CME is the first order quantum effect in $\hbar$. In fact, Wigner function approach is a type of quantum kinetic theory, which can imply the quantum effect of a multi-particle system, such as CME.

III. LANDAU LEVELS FOR RIGHTEHAND FERMIONS

In this section and following sections we will derive CME for a chiral fermion system through solving Landau levels. The Lagrangian for a chiral fermion field is

$$
\mathcal{L} = \overline{\Psi(x)} i\gamma \cdot D\Psi(x), \tag{22}
$$

with the covariant derivative $D^\mu = \partial^\mu + ieA^\mu$, and the electric charge $\pm e$ for particles/antiparticles. For a uniform magnetic field $B = Be_z$ along $z$-axis, we can choose the gauge potential as $A^\mu = (0, 0, Bx, 0)$. The equation of motion for the field $\Psi(x)$ is

$$
i\gamma \cdot D\Psi(x) = 0, \tag{23}
$$

which can be written as a form of Schrödinger equation,

$$
i\frac{\partial}{\partial t}\Psi(t, x) = i\alpha \cdot D\Psi(t, x), \tag{24}
$$

with $D = -\nabla + ieA$, $A = (0, Bx, 0)$. In the chiral representation of Dirac $\gamma$-matrices where $\gamma^5 = \text{diag} (-1, 1)$, $\alpha = \text{diag} (-\sigma, \sigma)$, we can write $\Psi$ in this form: $\Psi = (\Psi_L^T, \Psi_R^T)^T$. Then Eq. (24) becomes

$$
\left( \begin{array}{c}
\Psi_L(t, x) \\
\Psi_R(t, x)
\end{array} \right) = e^{-i\sigma \cdot D\Psi(t, x)} \left( \begin{array}{c}
-\sigma \cdot D\Psi_L(t, x) \\
\sigma \cdot D\Psi_R(t, x)
\end{array} \right), \tag{25}
$$

which indicates that the two fields $\Psi_{L/R}$, which correspond to eigenvalues $\pm 1$ of the matrix $\gamma^5$, decouple with each other. The two fields $\Psi_{L/R}$ are often called lefthand/righthand fermion fields respectively. Lefthand and righthand fermions are also called chiral fermions.
In the following we will focus on solving the eigenvalue equation for righthand fermion field $\Psi_R$ (Similar results can be obtained for lefthand fermion field $\Psi_L$).

In order to obtain Landau levels, we must solve the eigenvalue equation for righthand fermion field as follows,

$$i\sigma \cdot D\psi_R = E\psi_R,$$ (26)

with $D = (-\partial_x, -\partial_y + ieBx, -\partial_z)$. The details for solving Eq. (26) are put in Appendix B. We list the eigenfunctions and eigenvalues in the following: For $n = 0$ Landau level, the wavefunction with energy $E = k_z$ is

$$\psi_{R0}(k_y, k_z; x) = \begin{pmatrix} \varphi_0(\xi) \\ 0 \end{pmatrix} \frac{1}{L} e^{i(yk_y + zk_z)},$$ (27)

For $n > 0$ Landau level, the wavefunction with energy $E = \lambda E_n(k_z)$ is

$$\psi_{RN\lambda}(k_y, k_z; x) = c_{n\lambda} \begin{pmatrix} \varphi_n(\xi) \\ iF_{n\lambda}\varphi_{n-1}(\xi) \end{pmatrix} \frac{1}{L} e^{i(yk_y + zk_z)},$$ (28)

where $\lambda = \pm 1$, $E_n(k_z) = \sqrt{2neB + k_z^2}$, $F_{n\lambda}(k_z) = [k_z - \lambda E_n(k_z)]/\sqrt{2neB}$, normalised coefficient $|c_{n\lambda}|^2 = 1/(1 + F_{n\lambda}^2)$, and $\varphi_n(\xi) = \varphi_n(\sqrt{eBx} - k_z/\sqrt{eB})$ is the $n$-th order wavefunction of a harmonic oscillator.

For $n > 0$ Landau levels, the wavefunctions with energies $E = \pm E_n(k_z)$ corresponds to fermions and antifermions respectively. For the lowest Landau level, the wavefunction with energy $E = k_z > 0$ corresponds to fermions and with energy $E = k_z < 0$ corresponds to antifermions respectively. Wavefunctions of all Landau levels are orthonormal and complete. For lefthand fermion field, the eigenfunctions of Landau levels are the same as the righthand case but with the sign of the eigenvalues changed.

### IV. SECOND QUANTIZATION FOR RIGHTHEAND FERMION FIELD

In this section, we secondly quantize the righthand fermion field $\Psi_R(x)$, then the righthand fermion field $\Psi_R(x)$ becomes an operator and satisfies following anticommutative relations,

$$\{\Psi_R(x), \Psi_R^\dagger(x')\} = \delta^{(3)}(x - x'),$$

$$\{\Psi_R(x), \Psi_R(x')\} = 0.$$ (29)

Since all eigenfunctions for the Hamiltonian of the righthand fermion field are orthonormal and complete, we can decompose the righthand fermion field operator $\Psi_R(x)$ by these eigenfunctions as

$$\Psi_R(x) = \sum_{k_y, k_z} [\theta(k_z) a_0(k_y, k_z) \psi_{R0}(k_y, k_z; x) + \theta(-k_z) b_0^\dagger(k_y, k_z) \psi_{R0}(k_y, k_z; x)]$$

$$+ \sum_{n, k_y, k_z} [a_n(k_y, k_z) \psi_{RN}(k_y, k_z; x) + b_n^\dagger(k_y, k_z) \psi_{RN-1}(k_y, k_z; x)].$$ (30)

Different from the general Fourier decomposition for second quantization, we have put two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y, k_z)$ and $b_0^\dagger(k_y, k_z)$ in the decomposition, which is very important for the subsequent procedure of second quantization. From formula (29), we can obtain following anticommutative relations,

$$\{\theta(k_z) a_0(k_y, k_z), \theta(k_z') a_0^\dagger(k_y', k_z')\} = \theta(k_z) \delta_{k_z k'_z} \delta_{k_y k'_y}$$

$$\{\theta(-k_z) b_0(k_y, k_z), \theta(-k_z') b_0^\dagger(k_y', k_z')\} = \theta(-k_z) \delta_{k_z k'_z} \delta_{k_y k'_y}$$

$$\{a_n(k_y, k_z), a_m^\dagger(k_y', k_z')\} = \delta_{nn'} \delta_{k_y k'_y} \delta_{k_z k'_z}$$

$$\{b_n(k_y, k_z), b_m^\dagger(k_y', k_z')\} = \delta_{nn'} \delta_{k_y k'_y} \delta_{k_z k'_z}.$$ (31)
Note that the two theta functions $\theta(\pm k_z)$ are always attached to the lowest Landau level operators such as $a_0, a_0^+, b_0, b_0^+$. The Hamiltonian and total particle number of the righthand fermion system are

$$H = \int d^3x \Psi_R^\dagger(x)i\sigma \cdot D \Psi_R(x)
= \sum_{k_y,k_z} [k_z\theta(k_z)a_0^+(k_y,k_z)a_0(k_y,k_z) + (-k_z)\theta(-k_z)b_0^+(k_y,k_z)b_0(k_y,k_z)]
+ \sum_{n,k_y,k_z} E_n(k_z)[a_n^+(k_y,k_z)a_n(k_y,k_z) + b_n^+(k_y,k_z)b_n(k_y,k_z)],
$$

(32)

$$N = \int d^3x \Psi_R^\dagger(x)\Psi_R(x)
= \sum_{k_y,k_z} [\theta(k_z)a_0^+(k_y,k_z)a_0(k_y,k_z) - \theta(-k_z)b_0^+(k_y,k_z)b_0(k_y,k_z)]
+ \sum_{n,k_y,k_z} [a_n^+(k_y,k_z)a_n(k_y,k_z) - b_n^+(k_y,k_z)b_n(k_y,k_z)],
$$

(33)

where we have dropped the infinite vacuum term. This can be renormalized in physics calculation and does not affect our result on CME coefficient. It is clear that $\theta(k_z)a_0^+(k_y,k_z)a_0(k_y,k_z)$ and $a_n^+(k_y,k_z)a_n(k_y,k_z)$ are occupied number operators of particles for different Landau levels, and $\theta(-k_z)b_0^+(k_y,k_z)b_0(k_y,k_z)$ and $b_n^+(k_y,k_z)b_n(k_y,k_z)$ are occupied number operators of antiparticles for different Landau levels. Note that if we had not introduced the two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y,k_z)$ and $b_0^+(k_y,k_z)$ in the decomposition of $\Psi_R(x)$, we would not do the second quantization procedure successfully.

V. CHIRAL MAGNETIC EFFECT

Suppose that the system of righthand fermions within an external uniform magnetic field $\mathbf{B} = Be_z$ is in equilibrium with a reservoir with temperature $T$ and chemical potential $\mu_R$. Then the density operator $\hat{\rho}$ for this righthand fermion system is

$$\hat{\rho} = \frac{1}{Z}e^{-\beta(H-\mu_R N)},
$$

(34)

where $\beta = 1/T$ is the inverse temperature, and $Z$ is the grand canonical partition function,

$$Z = \text{Tr} e^{-\beta(H-\mu_R N)}.
$$

(35)

The expectation value of an operator $\hat{F}$ in the equilibrium state can be calculated as

$$\langle \hat{F} \rangle = \text{Tr} (\hat{\rho} \hat{F}).
$$

(36)

In Appendix C we have calculated the expectation values of occupied number operators as

$$\langle \hat{\theta}(k_z)a_0^+(k_y,k_z)a_0(k_y,k_z) \rangle = \frac{\theta(k_z)}{e^{\beta(k_z-\mu_R)} + 1},
$$

$$\langle \hat{\theta}(-k_z)b_0^+(k_y,k_z)b_0(k_y,k_z) \rangle = \frac{\theta(-k_z)}{e^{\beta(-k_z+\mu_R)} + 1},
$$

$$\langle \hat{a}_n^+(k_y,k_z)a_n(k_y,k_z) \rangle = \frac{1}{e^{\beta[E_n(k_z)-\mu_R]} + 1},
$$

$$\langle \hat{b}_n^+(k_y,k_z)b_n(k_y,k_z) \rangle = \frac{1}{e^{\beta[E_n(k_z)+\mu_R]} + 1}.
$$

(37)

The macroscopic electric current for righthand fermion system is

$$\mathbf{J}_R = \left\langle :\Psi_R^\dagger(x)\mathbf{\sigma}\Psi_R(x) : \right\rangle.
$$

(38)
According to the rotational invariance of this system along z-axis, \( J_R^R = J_R^L = 0 \). In the following, we will calculate \( J_R^L \). Taking use of Eq. (39), we can see

\[
J_R^L = \langle \Psi_R^L(x) \sigma^3 \Psi_R(x) \rangle
\]

\[
= \sum_{k_y,k_z} \left( \langle \theta(k_z) a_0^\dagger(k_y,k_z) a_0(k_y,k_z) \rangle + \langle \theta(-k_z) b_0(k_y,k_z) b_0^\dagger(k_y,k_z) \rangle \right) \times \psi_{R0}^\dagger(k_y,k_z; x) \sigma^3 \psi_R(k_y,k_z; x)
\]

\[
= \sum_{n,k_y,k_z} \psi_{Rn+}^\dagger(k_y,k_z; x) \sigma^3 \psi_{Rn+}(k_y,k_z; x)
\]

\[
= \sum_{k_y,k_z} \left( \begin{array}{c} \theta(k_z) \\ \frac{1}{e^{\beta(k_z-\mu_R)} + 1} \end{array} \right) \psi_{R0}^\dagger(k_y,k_z; x) \sigma^3 \psi_R(k_y,k_z; x)
\]

\[
= \sum_{n,k_y,k_z} \frac{1}{e^{\beta E_n(k_z)+\mu_R} + 1} \psi_{Rn+}^\dagger(k_y,k_z; x) \sigma^3 \psi_{Rn+}(k_y,k_z; x)
\]

(39)

Firstly we sum over \( k_y \) for \( \psi_{R0}^\dagger(k_y,k_z; x) \sigma^3 \psi_R(k_y,k_z; x) \) and \( \psi_{Rn\lambda}^\dagger(k_y,k_z; x) \sigma^3 \psi_{Rn\lambda}(k_y,k_z; x) \) in Eq. (39), the results are

\[
\sum_{k_y} \psi_{R0}^\dagger(k_y,k_z; x) \sigma^3 \psi_{R0}(k_y,k_z; x)
\]

\[
= \frac{1}{L^2} \sum_{k_y} \begin{pmatrix} \varphi_0(\xi) \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_0(\xi) \\ 0 \end{pmatrix}
\]

\[
= \frac{1}{2\pi L} \int_{-\infty}^{\infty} dk_y |\varphi_0(\sqrt{eB} x - k_y/\sqrt{eB})|^2 = \frac{eB}{2\pi L}.
\]

(40)

and

\[
\sum_{k_y} \psi_{Rn\lambda}^\dagger(k_y,k_z; x) \sigma^3 \psi_{Rn\lambda}(k_y,k_z; x)
\]

\[
= \frac{1}{2\pi L} \int_{-\infty}^{\infty} dk_y c_{n\lambda}^2(k_z) \left( [\varphi_n(\xi)]^2 - \frac{2neB[\varphi_{n-1}(\xi)]^2}{[k_z + \lambda E_n(k_z)]^2} \right)
\]

\[
= \frac{eB}{2\pi L} c_{n\lambda}^2(k_z) \left( 1 - \frac{2neB}{[k_z + \lambda E_n(k_z)]^2} \right)
\]

\[
= \frac{eB}{2\pi L} \frac{c_{n\lambda}^2(k_z)[2 - c_{n\lambda}^2(k_z)]}{E_n(k_z)}
\]

(41)
Secondly, we sum over \( k_z \) in the third equal sign of Eq. (39),

\[
J^R = \sum_{k_z} \left( \frac{\theta(k_z)}{e^{\beta(k_z - \mu_R)} + 1} - \frac{\theta(-k_z)}{e^{\beta(-k_z + \mu_R)} + 1} \right) \frac{eB}{2\pi L} k_z
\]

\[
+ \sum_{n,k_z} \left( \frac{1}{e^{\beta|E_n(k_z) - \mu_R|} + 1} + \frac{1}{e^{\beta|E_n(k_z) + \mu_R|} + 1} \right) \frac{eB}{2\pi L} E_n(k_z) k_z
\]

\[
= \frac{eB}{4\pi^2} \int_{-\infty}^{\infty} dk_z \left( \frac{\theta(k_z)}{e^{\beta(k_z - \mu_R)} + 1} - \frac{\theta(-k_z)}{e^{\beta(-k_z + \mu_R)} + 1} \right) + 0
\]

\[
= \frac{eB}{4\pi^2} \mu_R.
\]

Combining Eq. (42) and \( J^R = J^y_R = 0 \) gives

\[
J^y_R = \frac{e\mu_R}{4\pi^2} B.
\] (43)

From the calculation above, we can see that only the lowest Landau level contributes to Eq. (43). Similar calculation for left-hand fermion system shows that

\[
J^y_L = -\frac{e\mu_L}{4\pi^2} B.
\] (44)

Actually, we can also obtain equation (44) from (43) under space inversion: \( J^R \rightarrow -J^L, \mu_R \rightarrow \mu_L, B \rightarrow B \). If the system composes of right-hand and left-hand fermions, then the vector current \( J^y \) and axial current \( J^A \) are

\[
J^y = J^R + J^L = \frac{e\mu_5}{2\pi^2} B,
\] (45)

\[
J^A = J^R - J^L = \frac{e\mu_5}{2\pi^2} B,
\] (46)

where \( \mu_5 = (\mu_R - \mu_L)/2 \) is called chiral chemical potential and \( \mu = (\mu_R + \mu_L)/2 \). So far we have derived CME for chiral fermion system through solving Landau levels. We emphasize that Eqs. (43), (44) are valid for any strength of magnetic field, which is different from the weak magnetic field approximation through Wigner function approach in Sec. 11.

VI. PHYSICAL PICTURE OF THE LOWEST LANDAU LEVEL

In this section we discuss the physical picture of the lowest Landau level. The wavefunction and energy of the lowest Landau level (\( n = 0 \)) for righthand fermion field is

\[
\psi_{R0}(k_y, k_z; x) = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \frac{1}{L} e^{i(yk_y + zk_z)}, \quad E = k_z.
\] (47)

Setting \( k_y = 0 \) in Eq. (47), we calculate the Hamiltonian, particle number, \( z \)-component of momentum, and \( z \)-component of spin angular momentum of the righthand fermion system for the lowest Landau level as follows,

\[
H = \sum_{k_z} [k_z \theta(k_z)a^\dagger_0(0, k_z)a_0(0, k_z) + (-k_z)\theta(-k_z)b^\dagger_0(0, k_z)b_0(0, k_z)],
\]

\[
N = \sum_{k_z} [\theta(k_z)a^\dagger_0(0, k_z)a_0(0, k_z) + (-1)\theta(-k_z)b^\dagger_0(0, k_z)b_0(0, k_z)],
\]

\[
P_z = \sum_{k_z} [k_z \theta(k_z)a^\dagger_0(0, k_z)a_0(0, k_z) + (-k_z)\theta(-k_z)b^\dagger_0(0, k_z)b_0(0, k_z)],
\]

\[
S_z = \sum_{k_z} \left[ \frac{1}{2} \theta(k_z)a^\dagger_0(0, k_z)a_0(0, k_z) + (-\frac{1}{2})\theta(-k_z)b^\dagger_0(0, k_z)b_0(0, k_z) \right].
\] (48)
Then we have a picture for the lowest Landau level: The operator $\theta(k_z) a_0^\dagger(0, k_z)$ produces a particle with charge $e$, energy $k_z > 0$, $z$-component of momentum $k_z > 0$, and $z$-component of spin angular momentum $+\frac{1}{2}$ (helicity $h = +1$); The operator $\theta(-k_z) b_0^\dagger(0, k_z)$ produces a particle with charge $-e$, energy $-k_z > 0$, $z$-component of momentum $-k_z > 0$, and $z$-component of spin angular momentum $-\frac{1}{2}$ (helicity $h = -1$). This picture means that all righthand fermions/antifermions move along $(+z)$-axis, with righthand fermions spinning along $(+z)$-axis and righthand antifermions spinning along $-z$-axis. If $\mu_R > 0$, which means righthand fermions are more than righthand anti-fermions, then there will be net electric current moving along $(+z)$-axis, which is called chiral magnetic effect for righthand fermion system.

The analogous analysis can be applied to lefthand fermions. The picture of the lowest Landau level for a lefthand fermion is: all left fermions/antifermions move along $(-z)$-axis, with left fermions spinning along $(+z)$-axis and lefthand antifermions spinning along $(-z)$-axis. If $\mu_L > 0$, which means lefthand fermions are more than lefthand anti-fermions, then there will be net electric current moving along $(-z)$-axis, which is called chiral magnetic effect for lefthand fermion system.

Since the total electric current $J_V$ of the chiral fermion system is the summation of the electric current $J_R$ of the righthand fermion system and the electric current $J_L$ of the lefthand fermion system, whether $J_V$ is along $(+z)$-axis or not will only depend on the sign of $(\mu_R - \mu_L)$. So we have explained CME for chiral fermion system microscopically.

VII. SUMMARY

Chiral magnetic effect (CME) arises from the lowest Landau level both for massive Dirac fermion system and chiral fermion system. For massive case, the physical picture of how the lowest Landau level contributes to CME is not very clear. When we solve the Landau levels for chiral fermion system in the background of a uniform magnetic field, by performing the second quantization for the chiral fermion field, expanding field operator by eigenfunction of Landau levels, and calculating the ensemble average of vector current operator, we naturally obtain the equation for CME. We point out that we had not made any approximation for the strength of magnetic field in the calculation. It is worth mentioning that we had introduced two theta functions $\theta(\pm k_z)$ in front of $a_0(k_y, k_z)$ and $b_0^\dagger(k_y, k_z)$ in the decomposition of $\Psi_R(x)$, which is crucial for performing the subsequent procedure of second quantization successfully. When we carefully analyze the lowest Landau level, we find that all righthand (chirality is $+1$) fermions move along positive $z$-direction and all lefthand (chirality is $-1$) fermions move along negative $z$-direction, and CME can also be explained microscopically within this picture of the lowest Landau level.

VIII. ACKNOWLEDGMENTS

We are grateful to Hai-Cang Ren and Xin-Li Sheng for valuable discussions. R.-H. F. is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11847220. D.-F. H. is in part supported by the National Natural Science Foundation of China (NSFC) under Grant Nos. 11735007, 11890711.

Appendix A: Vlasov equation and off mass-shell equation

The quadratic form for the equation of motion of Wigner function $\mathcal{W}(x, p)$ is

$$\left(K^2 - \frac{i}{2} \sigma^{\mu\nu} [K_\mu, K_\nu]\right) \mathcal{W} = 0. \quad (A1)$$

Taking use of $K^2 = p^2 - \frac{1}{4} \nabla^2 + ip \cdot \nabla$ and $[K_\mu, K_\nu] = -ieF_{\mu\nu}$, Eq. (A1) becomes

$$(p^2 - \frac{1}{4} \nabla^2 + ip \cdot \nabla \pm \frac{1}{2} eF_{\mu\nu}\sigma^{\mu\nu}) \mathcal{W} = 0. \quad (A2)$$

Note that $\mathcal{W}$ and $\sigma^{\mu\nu}$ satisfies $\mathcal{W} = \gamma^0 \mathcal{W} \gamma^0$ and $\sigma^{\mu\nu} = \gamma^0 \sigma^{\mu\nu} \gamma^0$. Taking Hermitian conjugation and then multiplying $\gamma^0$ from both sides of Eq. (A2) yield

$$(p^2 - \frac{1}{4} \nabla^2 - ip \cdot \nabla) \mathcal{W} - \frac{1}{2} eF_{\mu\nu} \mathcal{W} \sigma^{\mu\nu} = 0. \quad (A3)$$
Eq. (A2) minus Eq. (A3) yields the Vlasov equation for $W$,

$$\mathbf{i} p \cdot \nabla W - \frac{1}{4} e F_{\mu \nu} [\sigma^{\mu \nu}, W] = 0. \quad (A4)$$

Eq. (A2) plus Eq. (A3) yields the off mass-shell equation for $W$,

$$\left(p^2 - \frac{1}{4} \nabla^2\right) W - \frac{1}{4} e F_{\mu \nu} \{\sigma^{\mu \nu}, W\} = 0. \quad (A5)$$

To calculate $[\sigma^{\mu \nu}, W]$ and $\{\sigma^{\mu \nu}, W\}$ in Eqs. (A4) (A5), we list following useful identities,

$$[\sigma^{\mu \nu}, 1] = 0,$n
$$[\sigma^{\mu \nu}, i \gamma^5] = 0,$n
$$[\sigma^{\mu \nu}, \gamma^\rho] = -2i g^{[\mu \nu} \gamma^\rho],$$n
$$[\sigma^{\mu \nu}, \gamma^5 \gamma^\rho] = -2i g^{[\mu \nu} \gamma^5 \gamma^\rho],$$n
$$[\sigma^{\mu \nu}, \sigma^{\rho \sigma}] = 2 i g^{[\mu \rho} g^{\nu] \sigma - 2i g^{[\mu \rho} \sigma^\sigma],$$

$$(A6)$$n

$$\{\sigma^{\mu \nu}, 1\} = 2 \sigma^{\mu \nu},$$n
$$\{\sigma^{\mu \nu}, i \gamma^5\} = -e^{\mu \rho \sigma} \sigma_{\rho \sigma},$$n
$$\{\sigma^{\mu \nu}, \gamma^\rho\} = 2 e^{\mu \rho \sigma} \gamma^5 \gamma^\rho \sigma,$n
$$\{\sigma^{\mu \nu}, \gamma^5 \gamma^\rho\} = 2 e^{\mu \rho \sigma} \gamma^5 \sigma,$n
$$\{\sigma^{\mu \nu}, \sigma^{\rho \sigma}\} = 2 g^{[\mu \rho} g^{\sigma^\rho]} + 2 i e^{\mu \rho \sigma} \gamma^5. \quad (A7)$$

Then all matrices appearing in Eqs. (A4) (A5) are the 16 independent $\Gamma$-matrices, whose coefficients must be zero. These coefficient equations are the Vlasov equations and the off mass-shell equations for $F, P, V_{\mu}, A_{\mu}, S_{\mu \nu}$. The Vlasov equations are

$$p \cdot \nabla F = 0,$n
$$p \cdot \nabla P = 0,$n
$$p \cdot \nabla V_{\mu} = e F_{\mu \nu} V^\nu,$n
$$p \cdot \nabla A_{\mu} = e F_{\mu \nu} A^\nu,$n
$$p \cdot \nabla Q_{\mu \nu} = e F_{\mu \nu} Q_{\mu \nu}, \quad (A8)$$

and the off mass-shell equations are

$$\left(p^2 - \frac{1}{4} \nabla^2\right) F = \frac{1}{2} e F_{\mu \nu} Q_{\mu \nu},$$n
$$\left(p^2 - \frac{1}{4} \nabla^2\right) P = \frac{1}{2} e \tilde{F}_{\mu \nu} Q_{\mu \nu},$$n
$$\left(p^2 - \frac{1}{4} \nabla^2\right) V_{\mu} = -e \tilde{F}_{\mu \nu} A^\nu,$n
$$\left(p^2 - \frac{1}{4} \nabla^2\right) A_{\mu} = -e \tilde{F}_{\mu \nu} V^\nu,$n
$$\left(p^2 - \frac{1}{4} \nabla^2\right) Q_{\mu \nu} = e (F_{\mu \nu} F - \tilde{F}_{\mu \nu} P), \quad (A9)$$

where $\tilde{F}_{\mu \nu} = \frac{1}{2} e_{\mu \nu \rho \sigma} F^{\rho \sigma}$.

**Appendix B: Landau levels for righthand fermion field**

Now we will solve following eigenvalue equation in detail,

$$i \sigma \cdot \mathbf{D} \psi_R(\mathbf{x}) = E \psi_R(\mathbf{x}), \quad (B1)$$
with $D = (-\partial_x, -\partial_y + ieBx, -\partial_z)$. Since the operator $i\sigma \cdot D$ is commutative with $\hat{p}_y = -i\partial_y, \hat{p}_z = -i\partial_z$, then we can choose $\psi_R$ as the commomn eigenstate of $i\sigma \cdot D, \hat{p}_y$ and $\hat{p}_z$ as follows

$$\psi_R(x, y, z) = \left( \phi_1(x) \frac{1}{L} e^{i(k_y y + k_z z)}, \phi_2(x) \right), \quad (B2)$$

where $L$ is the length of the system in $y$- and $z$- directions. The explicit form of $\sigma \cdot D$ is

$$\sigma \cdot D = \begin{pmatrix} -\partial_z & -\partial_x + i\partial_y + eBx \\ -\partial_x - i\partial_y - eBx & -\partial_z \end{pmatrix}. \quad (B3)$$

Putting Eq. (B2), (B3) into Eq. (B1), we obtain the group of differential equations for $\phi_1(x)$ and $\phi_2(x)$ as

$$\begin{align*}
i(k_z - E)\phi_1 + (\partial_x + k_y - eBx)\phi_2 &= 0, \\
(\partial_x - k_y + eBx)\phi_1 - i(k_z + E)\phi_2 &= 0. \quad (B4, B5)
\end{align*}$$

From Eq. (B4) we can express $\phi_2$ by $\phi_1$, then Eq. (B4) becomes

$$\partial_x^2 \phi_1 + \left( E^2 + eB - k_z^2 - e^2 B^2 x - \frac{k_y}{eB} \right) \phi_1 = 0, \quad (B6)$$

which is a typical harmonic oscillator equation. Define a dimensionless variable $\xi = \sqrt{eB}(x - k_y/eB)$, and $\phi_1(x) = \varphi(\xi)$, then (B6) becomes

$$\frac{d^2 \varphi}{d\xi^2} + \left( \frac{E^2 - k_z^2}{eB} + 1 - \xi^2 \right) \varphi = 0. \quad (B7)$$

With the boundary condition $\varphi \to 0$ as $\xi \to \pm\infty$, we must set

$$\frac{E^2 - k_z^2}{eB} + 1 = 2n + 1, \quad (B8)$$

with $n = 0, 1, 2, \cdots$. So energy $E$ can only take following discrete values,

$$E = \pm E_n(k_z) \equiv \pm \sqrt{2neB + k_z^2}, \quad (B9)$$

where we have defined $E_n(k_z) = \sqrt{2neB + k_z^2}$. The corresponding normalised solution for equation (B6) is

$$\phi_1(x) = \varphi_n(\xi) = N_n e^{-\xi^2/2} H_n(\xi), \quad (B10)$$

where $N_n = (eB)^{1/4} \pi^{-1/4} (2^n n!)^{-1/2}$, and $H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}$. For energy $E = \lambda E_n(k_z)$ ($\lambda = \pm 1$), we can obtain $\phi_2$ as

$$\phi_2(x) = \frac{\sqrt{eB}(\partial_\xi + \xi)\varphi_n(\xi)}{i(k_z + E)} = \frac{i[k_z - \lambda E_n(k_z)]}{\sqrt{2neB}} \varphi_{n-1}(\xi), \quad (B11)$$

where we have used $(\partial_\xi + \xi)\varphi_n(\xi) = \sqrt{2n}\varphi_{n-1}(\xi)$. Define $F_{n\lambda}(k_z) = [k_z - \lambda E_n(k_z)]/\sqrt{2neB}$, then the eigenfunction with eigenvalue $E = \lambda E_n(k_z)$ is

$$\psi_{Rn\lambda}(k_y, k_z; x) = \left( \frac{\varphi_n(\xi)}{iF_{n\lambda}(k_z)\varphi_{n-1}(\xi)} \right) \frac{1}{L} e^{i(yk_y + zk_z)}, \quad (B12)$$

It is very subtle when $n = 0$ in Eq. (B11). When $n = 0, E = k_z$, the first equal sign of Eq. (B11) indicates $\phi_2 = 0$ due to $(\partial_\xi + \xi)\varphi_0(\xi) = 0$. Then the corresponding eigenfunction becomes

$$\psi_{R0}(k_y, k_z; x) = \left( \frac{\varphi_0(\xi)}{0} \right) \frac{1}{L} e^{i(yk_y + zk_z)}. \quad (B13)$$
When \( n = 0, E = -k_z \), the denominator of the first equal sign of Eq. (B11) becomes zero, in which case we must directly deal with Eqs. (B4) (B5). In this case Eqs. (B4) (B5) become

\[
2ik_z \phi_1 + (\partial_x + k_y - eBx) \phi_2 = 0, \tag{B14}
\]
\[
(\partial_x - k_y + eBx) \phi_1 = 0. \tag{B15}
\]

Eq. (B15) gives \( \phi_1(x) \sim \exp[-\frac{1}{2}eBx^2 + xk_y] \), then Eq. (B14) becomes

\[
2ik_z \exp \left( -\frac{1}{2}eBx^2 + xk_y \right) + (\partial_x + k_y - eBx) \phi_2 = 0. \tag{B16}
\]

When \( x \to \pm \infty \), Eq. (B16) tends to

\[
(\partial_x - eBx) \phi_2 = 0, \tag{B17}
\]

whose solution is \( \phi_2 \sim \exp(\frac{1}{2}eBx^2) \) which is divergent as \( x \to \pm \infty \). So there exists no physical solution when \( n = 0, E = -k_z \).

So far we obtain the eigenfunctions and eigenvalues of the Hamiltonian of the righthand fermion field as follows:

For \( n = 0 \) Landau level, the wavefunction with energy \( E = k_z \) is

\[
\psi_{R0}(k_y, k_z; x) = \left( \frac{\varphi_0}{0} \right) \frac{1}{L} e^{i(yk_y + zk_z)}. \tag{B18}
\]

For \( n > 0 \) Landau level, the wavefunction with energy \( E = \lambda E_n(k_z) \) are

\[
\psi_{Rn\lambda}(k_y, k_z; x) = c_{n\lambda} \left( \frac{\varphi_n}{iF_{n\lambda}\varphi_{n-1}} \right) \frac{1}{L} e^{i(yk_y + zk_z)}, \tag{B19}
\]

where \( \lambda = \pm 1, E_n(k_z) = \sqrt{2neB + k_z^2}, F_{n\lambda}(k_z) = [k_z - \lambda E_n(k_z)]/\sqrt{2neB}, |c_{n\lambda}|^2 = 1/(1 + F_n^{2\lambda}) \).

**Appendix C: Expectation value of occupied number operators**

In the following, we will calculate the expectation values of particle number operators. From the expression of Hamiltonian and total particle number operator in Eqs. (22) (23), we can easily get following commutative relations,

\[
[N, \theta(k_z)a_0^\dagger(k_y, k_z)] = \theta(k_z)a_0^\dagger(k_y, k_z),
\]
\[
[N, \theta(-k_z)b_0^\dagger(k_y, k_z)] = -\theta(-k_z)b_0^\dagger(k_y, k_z),
\]
\[
[N, a_n^\dagger(k_y, k_z)] = a_n^\dagger(k_y, k_z),
\]
\[
[N, b_n^\dagger(k_y, k_z)] = -b_n^\dagger(k_y, k_z), \tag{C1}
\]
\[
[H, \theta(k_z)a_0^\dagger(k_y, k_z)] = k_z\theta(k_z)a_0^\dagger(k_y, k_z),
\]
\[
[H, \theta(-k_z)b_0^\dagger(k_y, k_z)] = (-k_z)\theta(-k_z)b_0^\dagger(k_y, k_z),
\]
\[
[H, a_n^\dagger(k_y, k_z)] = E_n(k_z)a_n^\dagger(k_y, k_z),
\]
\[
[H, b_n^\dagger(k_y, k_z)] = E_n(k_z)b_n^\dagger(k_y, k_z), \tag{C2}
\]

where we have used \([AB, C] = A\{B, C\} - \{A, C\}B\). Define

\[
\theta(k_z)a_0^\dagger(k_y, k_z; \beta) = e^{-\beta(H-\mu R N)} \theta(k_z)a_0^\dagger(k_y, k_z) e^{\beta(H-\mu R N)},
\]
\[
\theta(-k_z)b_0^\dagger(k_y, k_z; \beta) = e^{-\beta(H-\mu R N)} \theta(-k_z)b_0^\dagger(k_y, k_z) e^{\beta(H-\mu R N)},
\]
\[
a_n^\dagger(k_y, k_z; \beta) = e^{-\beta(H-\mu R N)} a_n^\dagger(k_y, k_z) e^{\beta(H-\mu R N)},
\]
\[
b_n^\dagger(k_y, k_z; \beta) = e^{-\beta(H-\mu R N)} b_n^\dagger(k_y, k_z) e^{\beta(H-\mu R N)}. \tag{C3}
\]

For \( \theta(k_z)a_0^\dagger(k_y, k_z; \beta) \), we can see
\[
\frac{\partial}{\partial \beta} \left[ \theta(k_z) a_0^1(k_y, k_z; \beta) \right] = -[H - \mu_R N, \theta(k_z) a_0^1(k_y, k_z; \beta)]
\]
\[
= -e^{-\beta(H - \mu_R N)}[H - \mu_R N, \theta(k_z) a_0^1(k_y, k_z)] e^{\beta(H - \mu_R N)}
\]
\[
= -(k_z - \mu_R) \theta(k_z) a_0^1(k_y, k_z; \beta),
\]
(C4)

with the boundary condition \( \theta(k_z) a_0^1(k_y, k_z; 0) = \theta(k_z) a_0^1(k_y, k_z) \), which implies
\[
\theta(k_z) a_0^1(k_y, k_z; \beta) = \theta(k_z) a_0^1(k_y, k_z) e^{-\beta(k_z - \mu_R)}.
\]
(C5)

Similarly we can obtain
\[
\theta(-k_z) b_0^1(k_y, k_z; \beta) = \theta(-k_z) b_0^1(k_y, k_z) e^{-\beta(k_z + \mu_R)}
\]
\[
a_0^1(k_y, k_z; \beta) = a_0^1(k_y, k_z) e^{-\beta[E_n(k_z) - \mu_R]}
\]
\[
b_0^1(k_y, k_z; \beta) = b_0^1(k_y, k_z) e^{-\beta[E_n(k_z) + \mu_R]}
\]
(C6)

Now we calculate the expectation value of \( \langle \theta(k_z) a_0^1(k_y, k_z) a_0(k_y, k_z) \rangle \). We can see
\[
\langle \theta(k_z) a_0^1(k_y, k_z) a_0(k_y, k_z) \rangle = \text{Tr} \left[ \rho \theta(k_z) a_0^1(k_y, k_z) a_0(k_y, k_z) \right]
\]
\[
= \frac{1}{Z} \text{Tr} \left( \theta(k_z) a_0^1(k_y, k_z; \beta) e^{-\beta(H - \mu_R N)} a_0(k_y, k_z) \right)
\]
\[
= \frac{1}{Z} \text{Tr} \left( \theta(k_z) a_0(k_y, k_z) a_0^1(k_y, k_z; \beta) e^{-\beta(H - \mu_R N)} \right)
\]
\[
= \langle \theta(k_z) a_0(k_y, k_z) a_0^1(k_y, k_z; \beta) \rangle
\]
\[
= \langle \theta(k_z) a_0(k_y, k_z) a_0^1(k_y, k_z) \rangle e^{-\beta(k_z - \mu_R)}
\]
\[
= \theta(k_z) e^{-\beta(k_z - \mu_R)} - \langle \theta(k_z) a_0^1(k_y, k_z) a_0(k_y, k_z) \rangle e^{-\beta(k_z - \mu_R)},
\]
(C7)

so we obtain
\[
\langle \theta(k_z) a_0^1(k_y, k_z) a_0(k_y, k_z) \rangle = \frac{\theta(k_z)}{e^{\beta(k_z - \mu_R)} + 1}.
\]
(C8)

Similar calculations obtain
\[
\langle \theta(-k_z) b_0^1(k_y, k_z) b_0(k_y, k_z) \rangle = \frac{\theta(-k_z)}{e^{\beta(-k_z + \mu_R)} + 1}
\]
\[
\langle a_0^1(k_y, k_z) a_0(k_y, k_z) \rangle = \frac{1}{e^{\beta[E_n(k_z) - \mu_R]} + 1}
\]
\[
\langle b_0^1(k_y, k_z) b_0(k_y, k_z) \rangle = \frac{1}{e^{\beta[E_n(k_z) + \mu_R]} + 1},
\]
(C9)

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