$\mathcal{N} = 3$ Superparticle Model

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Abstract

We consider the formulation and quantization of the $\mathcal{N} = 3$ superparticle model, both with and without central charge. Without the central charge the action possesses $U(3)$ invariance and therefore is naturally quantized in the $\mathcal{N} = 3$ harmonic superspace. The quantization reproduces the $\mathcal{N} = 3$ supergauge strength multiplets, described by analytic $\mathcal{N} = 3$ superfields and a gravitino multiplet as a constrained $\mathcal{N} = 3$ chiral superfield. When the central charge is present, it breaks the $U(3)$ R-symmetry of $\mathcal{N} = 3$ superalgebra down to $SU(2) \times U(1)$, and the corresponding superparticle model is formulated in the $\mathcal{N} = 2$ harmonic superspace extended by a pair of extra Grassmann variables. The quantization of such a model leads to the massive BPS $\mathcal{N} = 3$ vector multiplet. It is shown that upon additional superfield constraints such multiplet reduces to the massive $\mathcal{N} = 2$ vector multiplet.

\footnote{Alexander von Humboldt fellow at Leibniz Universität Hannover.}
1 Introduction

The models of relativistic particles and superparticles have deep relations to string and field theories. They can be considered not only as toy models which hint how to quantize superstring theories, but also describe the dynamics of D0-branes, point-like objects which form a part of the physical content of the type IIA string theory. From the quantum field theory point of view, the quantization of superparticles results in superfield realizations of supersymmetry multiplets with corresponding equations of motion and constraints \(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\). This is especially important for the models with extended supersymmetry since their superfield equations of motion are usually entangled with superfield constraints \(^7\). In particular, the unconstrained superfield formulation of \(\mathcal{N} = 2\) supersymmetric models of hypermultiplets and gauge multiplet are given within the so-called harmonic superspace approach \(^8\)\(^9\). This approach is crucial for the superfield quantization of these models which usually requires the use of unconstrained superfields \(^10\) (see also \(^11\) for applications to problem of effective action).

We point out that free equations of motion for the \(\mathcal{N} = 2\) super Yang-Mills (SYM) and hypermultiplet models can be naturally derived by quantizing the \(\mathcal{N} = 2\) superparticle in harmonic superspace \(^12\)\(^13\)\(^3\). However, the unconstrained superfield formulation for \(\mathcal{N} = 4\) SYM theory has not been achieved yet despite the many attempts made in this direction. The \(\mathcal{N} = 3\) SYM theory in harmonic superspace \(^17\), which is known as a maximally supersymmetric field theory with the unconstrained superfield formulation, deserves special attention. Since the \(\mathcal{N} = 3\) SYM model is equivalent to the \(\mathcal{N} = 4\) one on-shell, it can be considered as an \(\mathcal{N} = 3\) superfield formulation for the \(\mathcal{N} = 4\) SYM model. Some quantum aspects of the \(\mathcal{N} = 3\) SYM model in harmonic superspace were studied in \(^18\)\(^19\).

Inspired by the success of harmonic superspace formulation for the \(\mathcal{N} = 3\) SYM model we pose the question: which \(\mathcal{N} = 3\) supermultiplets, apart from supergauge one, admit the description in terms of the \(\mathcal{N} = 3\) superfields? For this purpose, we study the relativistic superparticle model in the \(\mathcal{N} = 3\) \(d = 4\) harmonic superspace and quantize it. A generic discussion of \(d = 4\) superparticles in harmonic superspaces with \(\mathcal{N} \geq 2\) and the existence in their quantum spectra of supergauge and supergravity multiplets was given in \(^12\)\(^13\)\(^14\).

Note that different models of superparticles can be considered, depending on whether their actions contain mass and central charge terms. In this paper we consider in detail all such \(\mathcal{N} = 3\) superparticles and find the superfield realizations of corresponding supermultiplets. In particular, the Gupta-Bleuler quantization of the massless \(\mathcal{N} = 3\) superparticle without central charge reproduces the \(\mathcal{N} = 3\) SYM supermultiplet realized on superfield strengths. These \(\mathcal{N} = 3\) superfields satisfy the Grassmann and harmonic shortness conditions \(^12\)\(^20\). Another interesting multiplet appearing in this case is the \(\mathcal{N} = 3\) gravitino multiplet (with the highest helicity 3/2) which is described by a chiral

\(^2\)The harmonic superspace approach is also very effective for studying 1d supersymmetric models with extended supersymmetry (see e.g., the recent works \(^15\) and reviews \(^16\) on the supersymmetric mechanics).
\( \mathcal{N} = 3 \) superfield.

The quantization of the \( \mathcal{N} = 3 \) superparticle with the central charge term is not so straightforward. One of the features in this case is that the central charge in the \( \mathcal{N} = 3 \) superalgebra breaks the group of internal automorphisms \( U(3) \) down to \( SU(2) \times U(1) \), and therefore the \( SU(3) \) harmonic superspace approach is not applicable here. Regarding the preserved R-symmetry group one can use \( SU(2) \) harmonics instead. Therefore the appropriate formulation of such a superparticle is given in the \( \mathcal{N} = 2 \) harmonic superspace, which is extended by a pair of additional Grassmann coordinates. We show that the central charge term in this model coincides with the central charge of \( \mathcal{N} = 2 \) harmonic superparticle studied in [12, 13]. Hence, the quantization proceeds in the same way as in the \( \mathcal{N} = 2 \) superparticle model and leads to the supermultiplets of the \( \mathcal{N} = 3 \) supersymmetry with central charge, realized on superfields in the \( \mathcal{N} = 3 \) superspace with \( SU(2) \) harmonic variables.

One of the simplest multiplets appearing in the quantization of the \( \mathcal{N} = 2 \) superparticle with a central charge is the massive \( q^+ \)-hypermultiplet described by an unconstrained \( \mathcal{N} = 2 \) analytic superfield in harmonic superspace [12, 13]. In our case, the quantization results in a similar \( q^+ \) superfield, which depends on the extra Grassmann spinor coordinate in a chiral way. Such a superfield describes the massive \( \mathcal{N} = 3 \) vector multiplet where the mass is related to the central charge of the superalgebra by the BPS condition. The on-shell field content of this multiplet is given by 5 complex scalars, 4 Dirac spinors and 1 complex vector. Thus it has twice as much components compared to the \( \mathcal{N} = 2 \) non-BPS massive vector multiplet [21].

Naturally, one is led to question, whether it is possible to impose such extra constraints on the \( \mathcal{N} = 3 \) massive vector superfield, which eliminate half of the states and reduce the above multiplet to the \( \mathcal{N} = 2 \) massive vector multiplet. We give the positive answer to this question and show that in \( \mathcal{N} = 3 \) superspace these constraints look very similar to the equations of motion in the massive \( \mathcal{N} = 1 \) Wess-Zumino model. These equations preserve \( \mathcal{N} = 3 \) supersymmetry, but violate CPT invariance of the multiplet. The resulting \( \mathcal{N} = 3 \) superfield describes exactly the \( \mathcal{N} = 2 \) massive vector multiplet on-shell. In other words, we show that the massive non-BPS \( \mathcal{N} = 2 \) vector supermultiplet can be described by the \( \mathcal{N} = 3 \) superfield under specific constraints, which violate the CPT invariance of the \( \mathcal{N} = 3 \) superalgebra with central charge.

The paper is organized as follows. In Section 2 we introduce the actions for the \( \mathcal{N} = 3 \) superparticle both with and without the central charge term. The quantization of the massless \( \mathcal{N} = 3 \) superparticle without central charge term is considered in Section 3, where the \( \mathcal{N} = 3 \) gravitino and SYM multiplets are derived. In Section 4 we quantize the \( \mathcal{N} = 3 \) superparticle with the central charge and, in the simplest case, obtain the \( \mathcal{N} = 3 \) massive vector supermultiplet. We also show that this multiplet can be reduced to the massive \( \mathcal{N} = 2 \) vector supermultiplet by imposing extra superfield constraints. In the Conclusion we summarize the results and discuss some unresolved problems.
2 \( \mathcal{N} = 3 \) superparticle action in harmonic superspace

2.1 \( \mathcal{N} = 3 \) superparticle model without central charge term

The \( \mathcal{N} = 3 \) superspace is parameterized by coordinates \( Z^M = \{x^m, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}i}\} \), where \( \alpha, \dot{\alpha}, \ldots \) are the indices of \( SL(2, C) \) and \( i, j, \ldots \) denote the indices of \( SU(3) \). In this coordinate system the superinvariant Cartan forms \( \omega^M = \hat{\omega}^M d\tau \) are

\[
\begin{align*}
\omega^m &= dx^m - id\theta^\alpha_i \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} + i\theta^\alpha_i \sigma^m_{\alpha\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}i} \\
\omega^\alpha_i &= d\theta^\alpha_i \\
\omega^{\dot{\alpha}i} &= d\bar{\theta}^{\dot{\alpha}i}.
\end{align*}
\]

In terms of the Cartan forms (2.1), the massive superparticle action is given by

\[
S_{sp} = -\frac{1}{2} \int d\tau (e^{-1} \omega^m \dot{\omega}_m + \epsilon m^2),
\]

where \( e(\tau) \) is the einbein field, \( m \) is the mass of the superparticle and \( \tau \) is the worldline parameter.

Action (2.2) is invariant under supertranslations

\[
\begin{align*}
\delta_i \theta^\alpha_i &= \xi^\alpha_i, \\
\delta_i \bar{\theta}^{\dot{\alpha}i} &= \bar{\xi}^{\dot{\alpha}i}, \\
\delta_i x^m &= -i\epsilon_i \sigma^m \theta^i + i\theta^i \sigma^m \bar{\epsilon}^i,
\end{align*}
\]

which correspond to conserved charges (supercharges)

\[
Q^i_\alpha = 2 \epsilon^{-1} \omega_m (\sigma^m \theta_i^\alpha), \quad \bar{Q}^{\dot{i}}_{\dot{\alpha}i} = -2 \epsilon^{-1} \bar{\omega}_m (\theta_i \sigma^m)_{\dot{\alpha}}.
\]

Together with the particle momenta, they generate the \( \mathcal{N} = 3 \) superalgebra (3.37) after quantization. The model (2.2) also respects the \( U(3) \) R-symmetry of the \( \mathcal{N} = 3 \) superalgebra.

In general, the arbitrary superfields on the full \( \mathcal{N} = 3 \) superspace with coordinates \( Z^M \) (as well as the arbitrary superfields on any extended superspace) have a large number of components and do not correspond to irreducible representations of the \( \mathcal{N} = 3 \) superalgebra. The construction of the irreducible superfields with fewer number of components is realized within the harmonic superspace formalism [8, 9]. Following this approach, in the case of the \( \mathcal{N} = 3 \) supersymmetry [17], one extends the superspace \( Z^M \) with the harmonic variables \( u^I_i \) which are \( SU(3) \) matrices,

\[
u^I_i u = 1_{3\times 3}, \quad \det u = 1.
\]

Conditions (2.5) can be written in terms of the matrix elements \( u^I_i \) as

\[
\begin{align*}
u^I_i u^J_j &= \delta^I_j, \\
\bar{u}^I_i \bar{u}^J_j &= \delta^I_j, \\
\varepsilon^{ijk} u^1_i u^2_j u^3_k &= 1, \\
\varepsilon_{ijk} \bar{u}^1_i \bar{u}^2_j \bar{u}^3_k &= 1.
\end{align*}
\]
Here the capital Latin indices $I, J, \ldots$ are $SU(3)$ ones with the values 1, 2, 3. The invariant Cartan forms on the $SU(3)$ group \cite{12}

$$\omega_I^J = du_I^j \bar{u}_J^j = -u_I^j d\bar{u}_J^j$$ (2.8)

satisfy the following identity

$$\omega_1^1 + \omega_2^2 + \omega_3^3 = 0,$$ (2.9)

which can be derived from (2.7).

The action for a particle on the coset space $SU(3)/(U(1) \times U(1))$ can be constructed with the use of the Cartan forms (2.8),

$$S_{SU(3)} = S_\omega + S_{WZ} + S_\lambda,$$ (2.10)

$$S_\omega = \frac{R^2}{2} \int d\tau e^{-1}(\dot{\omega}_2^1 \dot{\omega}_1^2 + \dot{\omega}_3^2 \dot{\omega}_2^3 + \dot{\omega}_1^3 \dot{\omega}_3^1),$$ (2.11)

$$S_{WZ} = -\frac{is_1}{2} \int d\tau (\dot{u}_I^i \bar{u}_J^i - u_I^i \bar{\dot{u}}_J^i) + \frac{is_2}{2} \int d\tau (\dot{\bar{u}}_I^i \bar{u}_J^i - \bar{u}_I^i \dot{\bar{u}}_J^i),$$ (2.12)

$$S_\lambda = \int d\tau \left[ \sum_{i,j=1}^3 \lambda_I^j (u_I^i \bar{u}_J^i - \delta_I^j) + \Lambda (\varepsilon^{ijk} u_I^i u_J^j u_K^k + \varepsilon_{ijk} \bar{u}_I^i \bar{u}_J^j \bar{u}_K^k - 2) \right].$$ (2.13)

Action $S_\omega$ is a kinetic term which appears here with the constant $R^2/2$, $S_{WZ}$ is the Wess-Zumino term for harmonic variables with the constants $s_1$, $s_2$, and action $S_\lambda$ takes into account the constraints (2.6,2.7) with the help of Lagrange multipliers $(\lambda_I^j) = (\lambda_J^j)^\dagger$ and $\Lambda = \Lambda^\dagger$. The Lagrange multipliers have ten independent real degrees of freedom and the corresponding constraints thus single out eight independent components from eighteen components of the arbitrary complex $3 \times 3$ matrix $u_I^j$. In addition, there is the local $U(1) \times U(1)$ symmetry which further reduces this number to six, i.e., the particle moves effectively in the coset space $SU(3)/(U(1) \times U(1))$.

The variation of action (2.13) over the Lagrange multipliers gives the following constraints for harmonic variables

$$\chi_J^j \equiv \frac{\delta S_\lambda}{\delta \lambda_J^j} = u_I^j \bar{u}_J^i - \delta_I^j = 0,$$ (2.14)

$$\chi_1 \equiv \frac{\delta S_\lambda}{\delta \Lambda} = \varepsilon^{ijk} u_I^i u_J^j u_K^k + \varepsilon_{ijk} \bar{u}_I^i \bar{u}_J^j \bar{u}_K^k - 2 = 0.$$ (2.15)

Equation (2.14) shows that the matrix $u$ is unitary while (2.15) means Re $\det u = 1$. These constraints uniquely imply $\det u = 1$. Therefore both conditions (2.6,2.7) are satisfied and the matrices $u_I^j$, $\bar{u}_J^i$ belong to the $SU(3)$ group. In particular, for any $SU(3)$ matrix we have $\Im \det u = 0$, or,

$$\chi_2 = \varepsilon^{ijk} u_I^i u_J^j u_K^k - \varepsilon_{ijk} \bar{u}_I^i \bar{u}_J^j \bar{u}_K^k = 0.$$ (2.16)

Hence, (2.16) appears as a consequence of (2.14,2.15). This constraint will be used in the next section for constructing the Dirac bracket.
The unitary matrices $u_i^I$ rotate the Grassmann variables and supersymmetric Cartan forms (2.1),
\[
\begin{align*}
\theta_i^\alpha & \rightarrow \theta_I^\alpha = \bar{u}_I^i \theta_i^\alpha, \quad \bar{\theta}^{\dot{\alpha}} & \rightarrow \bar{\theta}^{I\dot{\alpha}} = u_I^i \bar{\theta}^{\dot{\alpha}}, \\
\omega_i^{\alpha} & \rightarrow \omega_I^{\alpha} = \bar{u}_I^i \omega_i^{\alpha}, \quad \bar{\omega}^{\dot{\alpha}} & \rightarrow \bar{\omega}^{I\dot{\alpha}} = u_I^i \bar{\omega}^{\dot{\alpha}}.
\end{align*}
\] (2.17)
The $\mathcal{N} = 3$ harmonic superspace is parameterized by a set of coordinates $Z_H = \{x^m, \theta_i^\alpha, \bar{\theta}_I^{\dot{\alpha}}, u\}$, where $\theta_i^\alpha, \bar{\theta}_I^{\dot{\alpha}}$ are given by (2.17). Apart from the usual complex conjugation, there is also $\bar{}$ conjugation which acts on the Grassmann variables and harmonics as (cf. [9, 20])
\[
\begin{align*}
u_1^i & \sim \bar{u}_3^i, \quad u_1^i & \sim -\bar{u}_2^i, \quad u_3^i & \sim \bar{u}_1^i, \\
\theta_1^\alpha & \sim \bar{\theta}^{3\dot{\alpha}}, \quad \theta_2^\alpha & \sim -\bar{\theta}^{2\dot{\alpha}}, \quad \theta_3^\alpha & \sim \bar{\theta}^{1\dot{\alpha}}.
\end{align*}
\] (2.18)

This conjugation is natural in harmonic superspace $Z_H$, since the $\mathcal{N} = 3$ SYM action is known to be real under (2.18). Applying the conjugation $\bar{}$ to action (2.10) swaps the constants $s_1, s_2$ in the Wess-Zumino term. Therefore, action (2.10) is real under (2.18) if $s_1 = s_2$.

The action of the $\mathcal{N} = 3$ superparticle without central charges moving in the harmonic superspace $Z_H$ is a sum of (2.2) and (2.10),
\[
S_{sp} + S_{SU(3)} = S_{sp} + S_\omega + S_{WZ} + S_\lambda = \int d\tau L_1,
\] (2.19)
where $L_1$ denotes the Lagrangian of the superparticle.

## 2.2 $\mathcal{N} = 3$ superparticle model with the central charge term

The superparticle action (2.2) admits the extension by the Wess-Zumino term [4],
\[
S_c = -\int d\tau (\dot{Z}^{ij}\theta_i^\alpha \dot{\theta}_j^\alpha + Z_{ij} \dot{\bar{\theta}}^{i\dot{\alpha}} \dot{\bar{\theta}}^{j\dot{\alpha}}).
\] (2.20)

Here, $Z_{ij}$ and its conjugate $\bar{Z}^{ij}$ are the constant antisymmetric matrices,
\[
\dot{Z}^{ij} = -\bar{Z}^{ji}, \quad Z_{ij} = -Z_{ji}, \quad (Z_{ij})^* = \bar{Z}^{ij}.
\] (2.21)

The Wess–Zumino term (2.20) is also invariant under supersymmetry (2.3), but up to a total derivative. Added to action (2.2), it leads (upon taking into account boundary contributions) to the conserved Noether supercharges with central charge terms,
\[
Q_\alpha = 2ie^{-1}\omega_m (\sigma^m \dot{\theta})^\alpha + 2\bar{Z}^{ij} \theta_j^\alpha, \quad \bar{Q}_{\dot{\alpha}} = -2ie^{-1}\bar{\omega}_m (\theta^m \sigma^\dot{\alpha}) + 2\bar{Z}_{ij} \bar{\theta}^j_{\dot{\alpha}}.
\] (2.22)

which generate the $\mathcal{N} = 3$ superalgebra with central charge (4.32) after quantization. Therefore we refer to action (2.20) as a central charge term in the superparticle model.

Since $Z_{ij}$ are constants, they break the $U(3)$ R-symmetry of the $\mathcal{N} = 3$ superalgebra. To understand which symmetry survives, we notice that any $3 \times 3$ antisymmetric matrix is degenerate,
\[
det Z = 0, \quad \det \bar{Z} = 0.
\] (2.23)
Moreover, performing some rotation with the $SU(3)$ matrix $v_j^i$, the matrices $Z_{ij}$, $Z^{ij}$ can be brought to the normal form \[ Z_{ij} = v_k^i v_j^k Z_{kl} = \begin{pmatrix} 0 & -z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.24) \]

\[ Z^{ij} = \bar{v}_k^i \bar{v}_j^k Z_{kl} = \begin{pmatrix} 0 & \bar{z} & 0 \\ \bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.25) \]

Correspondingly, (2.20) takes the following form

\[ S_c = \int d\tau [z(\theta_1^a \dot{\theta}_2^\alpha - \theta_2^a \dot{\theta}_1^\alpha) + \bar{z}(\bar{\theta}_1^\dot{a} \dot{\bar{\theta}}^{2\dot{a}} - \bar{\theta}_2^\dot{a} \dot{\bar{\theta}}^{1\dot{a}})]. \quad (2.26) \]

Note that action (2.26) is nothing but the Wess-Zumino term of the $\mathcal{N} = 2$ superparticle, which respects the $SU(2)$ symmetry realized on the coordinates $\theta_1^a$, $\theta_2^a$ and $\bar{\theta}_1^\dot{a}$, $\bar{\theta}_2^\dot{a}$. There is also the $U(1)$ symmetry which transforms the $\theta_3^a$ and $\bar{\theta}_3^\dot{a}$ variables with a phase factor,

\[ \theta_3^a \rightarrow e^{i\phi} \theta_3^a, \quad \bar{\theta}_3^\dot{a} \rightarrow e^{-i\phi} \bar{\theta}_3^\dot{a}. \quad (2.27) \]

As a result, we conclude that the Wess-Zumino term (2.20) breaks the internal automorphisms symmetry $U(3)$ down to the $SU(2) \times U(1)$.

The Grassmann variables of the $\mathcal{N} = 3$ superspace can be rearranged as

\[ \{\theta_1^a, \theta_2^a, \theta_3^a\} \rightarrow \{\theta_1^a, \theta_2^a, \theta_3^a\}, \quad \{\bar{\theta}_1^\dot{a}, \bar{\theta}_2^\dot{a}, \bar{\theta}_3^\dot{a}\} \rightarrow \{\bar{\theta}_1^\dot{a}, \bar{\theta}_2^\dot{a}, \bar{\theta}_3^\dot{a}\}, \quad (2.28) \]

where the underlined indices $i$, $j$ are the $SU(2)$ ones with the values 1, 2. Action (2.26) can now be written as

\[ S_c = -\int d\tau (z\varepsilon_{ij}^\dot{a} \dot{\theta}_1^a \dot{\theta}_2^\alpha - \bar{z}\varepsilon_{ij}^\dot{a} \dot{\bar{\theta}}_1^\dot{a} \dot{\bar{\theta}}_2^{\dot{a}}), \quad (2.29) \]

where $\varepsilon_{ij}$ is the antisymmetric two-dimensional tensor, $\varepsilon_{12} = -\varepsilon_{21} = 1$. Action (2.29) has manifest $SU(2)$ invariance realized on the indices $i$, $j$. Therefore a natural harmonic extension of such superparticle model is given by the $SU(2)$ harmonic variables $u_\pm^i$,

\[ u_+^i \in SU(2), \quad u_-^i = \varepsilon_{ij}^\dot{a} u_+^j, \quad u_+^i u_-^i = 1. \quad (2.30) \]

The Cartan forms on the $SU(2)$ group

\[ i\omega^{++} = u_+^i du_-^i, \quad i\omega^{--} = du_-^i u_-^i, \quad i\Theta = u_-^i du_+^i \quad (2.31) \]

are used to write down the particle action on a sphere $S^2 \sim SU(2)/U(1)$ \[ S_{SU(2)} = 2R^2 \int d\tau e^{-\frac{i}{2} \omega^{++} \omega^{--}} - \int d\tau \lambda (u_-^i \dot{u}_+^i - 1) - \frac{i}{2} n \int d\tau (u_-^i \dot{u}_+^i - \dot{u}_+^i u_-^i). \quad (2.32) \]
Here $R$ is the radius of the sphere, $n$ is the electric charge of the particle which couples to a magnetic field produced by a monopole situated in the center of the sphere (see [12] for details) and $\lambda$ is the Lagrange multiplier.

The harmonics $u^\pm_\alpha$ convert $SU(2)$ indices $i, j, \ldots$ into $U(1)$ ones $\pm$, e.g.,
\[
\theta^+_\alpha \rightarrow \theta^+_{\alpha} = u^+_\alpha \theta^+_\alpha, \quad \bar{\theta}^+_{\alpha} \rightarrow \bar{\theta}^+_{\alpha} = u^+_\alpha \bar{\theta}^+_{\alpha}.
\] (2.33)

Apart from usual complex conjugation, the harmonic superspace $\{x^m, \theta^\pm_\alpha, \bar{\theta}^\mp_\dot{\alpha}, \theta^3_\alpha, \bar{\theta}^3_{\dot{\alpha}}, u^\pm_i\}$ has also $\tilde{\ }$ conjugation defined as\[3\]
\[
\tilde{u}^\pm_i = u^\mp_i, \quad \tilde{u}^\mp_i = -u^\pm_i, \quad \tilde{\theta}^\pm_{\alpha} = \bar{\theta}^\mp_{\dot{\alpha}}, \quad \tilde{\theta}^\mp_{\dot{\alpha}} = \bar{\theta}^\pm_{\alpha}, \quad \tilde{\theta}^3_\alpha = \bar{\theta}^3_{\dot{\alpha}} = \theta^3_\alpha, \quad \tilde{\theta}^3_{\dot{\alpha}} = \bar{\theta}^3_\alpha,
\] (2.34)
which leaves the action (2.32) invariant.

The action of a superparticle in harmonic superspace $\{x^m, \theta^\pm_\alpha, \bar{\theta}^\mp_\dot{\alpha}, \theta^3_\alpha, \bar{\theta}^3_{\dot{\alpha}}, u^\pm_i\}$ is given by the sum of (2.2), (2.29) and (2.32)
\[
S_{sp} + S_c + S_{SU(2)} = \int d\tau L_2,
\] (2.35)
where $L_2$ denotes the Lagrangian of the $\mathcal{N} = 3$ harmonic superparticle with central charges.

3 Quantization of the $\mathcal{N} = 3$ superparticle without the central charge term

3.1 Hamiltonian formulation and constraints

We start with the action of the $\mathcal{N} = 3$ harmonic superparticle given by (2.19). The corresponding Lagrangian reads
\[
L_1 = -\frac{1}{2e} \dot{\omega}^m \ddot{\omega}_m + \frac{R^2}{2e} (\dot{\omega}_1^2 \dot{\omega}_1^3 + \dot{\omega}_2^2 \dot{\omega}_2^3 + \dot{\omega}_3^2 \dot{\omega}_3^3) - \frac{1}{2} \epsilon m^2 \\
- \frac{i s_1}{2} (u_i^1 \bar{u}_i^1 - u_i^1 \bar{u}_i^1) + \frac{i s_2}{2} (u_i^3 \bar{u}_i^3 - u_i^3 \bar{u}_i^3) \\
+ \sum_{I,J=1}^3 \lambda_I^J (u_I^1 \bar{u}_J^1 - \delta^I_J) + \Lambda (\epsilon^{ijk} u_i^1 u_j^2 u_k^3 + \epsilon_{ijk} \bar{u}_i^1 \bar{u}_j^2 \bar{u}_k^3 - 2).
\] (3.1)

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\[3\]This conjugation is natural in the $\mathcal{N} = 2$ harmonic superspace with coordinates $\{x^m, \theta^\pm_\alpha, \bar{\theta}^\mp_\dot{\alpha}, u^\pm_i\}$ [9] and acts on extra Grassmann variables $\theta^3_\alpha, \bar{\theta}^3_{\dot{\alpha}}$ as the usual complex conjugation.
The canonical momenta are defined by the Lagrangian (3.1) in a standard way,

\begin{align}

p_m &= -\frac{\partial L_1}{\partial \dot{x}^m} = e^{-1}\omega^m, \\
\pi^i_\alpha &= \frac{\partial L_1}{\partial \dot{\theta}^i_\alpha} = ip_m(\sigma^m \dot{\theta}^i_\alpha), \\
\pi_{i\dot{\alpha}} &= \frac{\partial L_1}{\partial \dot{\theta}^{i\dot{\alpha}}} = ip_m(\theta_\alpha \sigma^m)_{\dot{\alpha}} = -\left(\pi^i_\alpha\right)^*, \\
v^1_1 &= -\frac{\partial L_1}{\partial \dot{u}^1_1} = \frac{R^2}{2e} (u^1_2 \dot{\omega}_2^1 + u^1_3 \dot{\omega}_3^1) - \frac{is_1}{2} u^1_1, \\
v^2_i &= -\frac{\partial L_1}{\partial \dot{u}^2_i} = \frac{R^2}{2e} (u^2_1 \dot{\omega}_1^2 + u^3_1 \dot{\omega}_3^2), \\
v^3_i &= -\frac{\partial L_1}{\partial \dot{u}^3_i} = \frac{R^2}{2e} (u^2_1 \dot{\omega}_2^3 + u^3_1 \dot{\omega}_3^3) + \frac{is_2}{2} \bar{u}^3_i, \\
v^1_i &= \frac{\partial L_1}{\partial \dot{\bar{u}}^1_i} = \frac{R^2}{2e} (\bar{u}^1_2 \dot{\bar{\omega}}_2^1 + \bar{u}^1_3 \dot{\bar{\omega}}_3^1) + \frac{is_1}{2} \bar{u}^1_i, \\
v^2_i &= \frac{\partial L_1}{\partial \dot{\bar{u}}^2_i} = -\frac{R^2}{2e} (\bar{u}^2_1 \dot{\bar{\omega}}_1^2 + \bar{u}^3_1 \dot{\bar{\omega}}_3^2), \\
v^3_i &= \frac{\partial L_1}{\partial \dot{\bar{u}}^3_i} = \frac{R^2}{2e} (\bar{u}^2_1 \dot{\bar{\omega}}_2^3 + \bar{u}^3_1 \dot{\bar{\omega}}_3^3) - \frac{is_2}{2} \bar{u}^3_i.
\end{align}

Note that equations (3.3,3.4) are the constraints since they do not allow to express the Grassmann velocities through the corresponding momenta.

The standard mass-shell constraint appears from the equation of motion for the ein-bein,

\[ 0 = \frac{\partial L_1}{\partial e} = \frac{1}{2e^2} [\omega^m \omega_m - R^2(\dot{\omega}_2^1 \dot{\omega}_1^2 + \dot{\omega}_3^2 \dot{\omega}_2^3 + \dot{\omega}_1^3 \dot{\omega}_3^1) - e^2 m^2]. \quad (3.11) \]

It is convenient to pass from the canonical harmonic momenta \( v^l_i, \bar{v}^l_i \) to the covariant ones \( D^l_J, C^l_J \),

\[ D^l_J = u^l_i \bar{v}^j_i - \bar{u}^j_i v^l_i, \quad C^l_J = u^l_i \bar{v}^j_i + \bar{u}^j_i v^l_i, \quad (3.12) \]

which can be written manifestly as

\begin{align}

D^l_J &= -R^2 e^{-1} \dot{\omega}_J^l \quad (I \neq J), \\
D^l_1 &= is_1, \quad D^2_2 = 0, \quad D^3_3 = -is_2, \quad (3.13) \\
C^l_J &= 0 \quad \forall I, J = 1, 2, 3. \quad (3.14) \\
\end{align}

Equations (3.14,3.15) are nothing but the constraints for the harmonic variables. In terms of covariant momenta \( D^l_J \), the mass-shell constraint (3.11) reads

\[ p_m = \frac{1}{R^2} (D^3_1 D^1_1 + D^2_2 D^1_1 + D^3_3 D^3_2) - m^2 = 0. \quad (3.16) \]
Let us now define the Poisson brackets in a standard way:

\[
\{x^n, p_m\}_P = -\delta^n_m, \\
\{\theta^\alpha_i, \pi_{\beta j}\}_P = -\delta^i_j \delta^\alpha_\beta, \\
\{\bar{\theta}^{i\dot{\alpha}}, \bar{\pi}_{j\dot{\beta}}\}_P = -\delta^{i\dot{\alpha}}_j \delta^{\dot{\beta}}_\beta, \\
[u^i_{\dot{\alpha}}, \bar{v}^j_{\dot{\beta}}]_P = -\delta^{i\dot{\alpha}}_j \delta^j_i. \tag{3.17}
\]

The harmonic covariant momenta \([3.13]\) form \(su(3)\) algebra with the Cartan generators \(S_1 = D_1^1 - D_2^2, \ S_2 = D_2^2 - D_3^3\) under the Poisson brackets \([3.17]\), e.g.,

\[
[D_2^1, D_3^3]_P = D_3^1, \quad [D_2^1, D_3^1]_P = 0, \quad [D_2^2, D_3^1]_P = 0, \\
[D_1^1, D_2^1]_P = S_1, \quad [D_2^2, D_3^2]_P = S_1, \quad [S_1, D_2^1]_P = 2D_2^1, \quad \text{etc.} \tag{3.18}
\]

There are the following non-trivial Poisson brackets between the functions \(C^I_J, D^I_J\) and the constraints \([2.14, 2.16]\)

\[
[C^I_J, \chi^K_L]_P = 2\delta^K_J \delta^I_L, \quad [D_2^2, \chi_2]_P = 2, \quad [C^I_J, C^K_L]_P = \delta^K_J D^I_L - \delta^I_K D^J_L \equiv A^{IK}_{JL}, \tag{3.19}
\]

which mean that they are second-class. Hence, they can be taken into account by introducing the Dirac bracket

\[
\{f, g\}_D = \{f, g\}_P + \frac{1}{2}\{f, C^I_J\}_P \{\chi^I_J, g\}_P - \frac{1}{2}\{f, \chi^I_J\}_P \{C^I_J, g\}_P \\
+ \frac{1}{2}\{f, D^2_J\}_P \{\chi_2, g\}_P - \frac{1}{2}\{f, \chi^I_J\}_P \{D^2_J, g\}_P - \frac{1}{4}\{f, \chi^I_J\}_P \{A^{IK}_{JL}, \chi^K_L, g\}_P, \tag{3.20}
\]

where \(f\) and \(g\) are arbitrary phase space functions. The applications of the Dirac and Poisson brackets resemble only the harmonic variables and momenta, while for the other superspace coordinates one can freely use the Poisson bracket instead of \([3.20]\).

Equations \([3.14]\) contain the following first-class constraints

\[
S_1 - is_1 \approx 0, \quad S_2 - is_2 \approx 0. \tag{3.21}
\]

There are also the spinor constraints

\[
\bar{D}^i_{\dot{\alpha}} = -\bar{\pi}^i_{\dot{\alpha}} + ip_m(\sigma^m \bar{\theta}^i)_\alpha \approx 0, \quad \bar{\bar{D}}_{i\dot{\alpha}} = \bar{\pi}_{i\dot{\alpha}} - ip_m(\theta_i \sigma^m)_{\dot{\alpha}} \approx 0, \tag{3.22}
\]

which anticommute non-trivially under the Poisson brackets \([3.17]\),

\[
\{\bar{D}^i_{\dot{\alpha}}, \bar{\bar{D}}_{j\dot{\beta}}\}_P = -2i\delta^i_j \sigma^{m}_{\alpha\dot{\beta}} p_m. \tag{3.23}
\]

In general, the constraints \([3.22]\) are second-class. However, if the dynamics of the superparticle is constrained to the surface \(p_m p^m \approx 0\), the matrix in the rhs of \([3.23]\) is degenerate and the superparticle action is invariant under \(\kappa\)-symmetry.\(^4\) In this case both

\(^4\)The \(\kappa\)-symmetry was first observed in the model of a massive \(N = 2, d = 4\) superparticle with the central charge \([4]\) and in the case of a massless \(N = 1\) superparticle in \([23]\). The group-theoretical and geometrical origin of \(\kappa\)-symmetry as a manifestation of local extended supersymmetry of the superparticle worldline was found in \([24]\). This observation leads to the development of the superembedding approach to the description of superbranes (see \([25]\) for a review and references).
first-class and second-class constraints are entangled in (3.22). As follows from (3.16),
the condition $p_m p^m \approx 0$ is satisfied only for the massless superparticle, $m = 0$, with the
following additional constraint for harmonic variables

$$D_2^1 D_1^2 + D_3^2 D_2^3 + D_3^3 D_1^3 \approx 0.$$  (3.24)

As we will show in the next section, this case corresponds exactly to physical supermultiplets upon quantization.

Apart from the constraints considered above, it is meaningful to introduce the following
extra harmonic constraints \[12, 13\]

$$D_2^1 \approx 0, \quad D_3^2 \approx 0, \quad D_3^3 \approx 0.$$  (3.25)

These constraints are first-class and reduce the mass-shell condition (3.16) to the physical
one,

$$p_m p^m - m^2 \approx 0.$$  (3.26)

Equations (3.25) “freeze” the dynamics of harmonic variables leaving only the motion of
the particle in $\{x^m, \theta_{i\alpha}, \bar{\theta}^i_{\dot{\alpha}}\}$ superspace. This fact emphasizes the unphysical meaning of
the harmonic variables in field theory. Indeed, upon quantization, these constrains will
yield the equations which eliminate an infinite number of auxiliary fields with arbitrary
number of $SU(3)$ indices and leave only physical components.

The canonical Hamiltonian is defined using the momenta (3.2)–(3.10) via the Legendre
transform,

$$H_1 = \frac{e}{2} (p_m^m - m^2) + \frac{e}{2R^2} (D_1^3 D_3^1 + D_2^2 D_2^1 + D_3^3 D_3^2) + \hat{\omega}_1^1 (S_1 + i s_1) + \hat{\omega}_3^3 (S_2 - i s_2) - L_\lambda,$$  (3.27)

where $L_\lambda$ corresponds to the last line in the Lagrangian (3.1) with Lagrange multipliers.
This Lagrangian $L_\lambda$ is not essential when the harmonic constraints (2.14,2.15) are taken
into account by the Dirac bracket (3.20). Therefore we omit $L_\lambda$ further, assuming the
use of the Dirac bracket (3.20) in what follows. Note also that the velocities $\hat{\omega}_1^1, \hat{\omega}_3^3$
cannot be eliminated from the Hamiltonian and remain arbitrary functions. They play the
role of Lagrange multipliers (as well as the einbein $e$) and we denote them as $\mu(\tau), \nu(\tau)$, respectively. As a result we obtain the total Hamiltonian which is a linear combination
of first-class constraints,

$$H_1 = -\frac{e}{2} (p_m^m - m^2) + \frac{1}{R^2} (D_1^3 D_3^1 + D_2^2 D_2^1 + D_3^3 D_3^2) - m^2 \right] + \mu (-S_1 + i s_1) + \nu (S_2 - i s_2).$$  (3.28)

The Hamiltonian equation of motion for any phase space coordinate $f$ has the standard
form

$$\dot{f} = [f, H_1]_D.$$  (3.30)

Here we do not write down these equations in detail, however they can be easily figured
out.
3.2 Gupta-Bleuler quantization

According to the postulates of canonical quantization, one replaces the canonical momenta (3.2)–(3.10) by the corresponding differential operators,

\[ p_m \rightarrow i \frac{\partial}{\partial x^m}, \quad \pi^i \rightarrow -i \frac{\partial}{\partial \theta_i}, \quad \bar{\pi}^\dot{i} \rightarrow i \frac{\partial}{\partial \bar{\theta}^\dot{i}}, \quad v^I_i \rightarrow \frac{\partial}{\partial \bar{u}^I_i}, \quad \bar{v}^I_i \rightarrow \frac{\partial}{\partial u^I_i}. \]  

(3.31)

The spinor constraints (3.22) turn into the covariant spinor derivatives,

\[ D^i_\alpha = \frac{\partial}{\partial \theta^i_\alpha} + i (\sigma^m \bar{\theta})^\alpha_m \partial_m, \quad \bar{D}^\dot{i}_\dot{\alpha} = -\frac{\partial}{\partial \bar{\theta}^\dot{i}_\dot{\alpha}} - i (\theta^m \sigma^m)^\dot{\alpha}_\dot{m} \partial_m, \]

(3.32)

and the covariant harmonic momenta (3.12) lead to the covariant harmonic derivatives,

\[ D^I_J = u^I_i \frac{\partial}{\partial u^I_i} - \bar{u}^I_\dot{i} \frac{\partial}{\partial \bar{u}^I_\dot{i}}. \]

(3.33)

Further we will use covariant spinor derivatives contracted with harmonics,

\[ D^i_\alpha = u^I_i D^I_i, \quad \bar{D}^\dot{i}_\dot{\alpha} = \bar{u}^I_\dot{i} \bar{D}^I_\dot{i}, \]

(3.34)

which satisfy the following algebra

\[ \{ D^I_i, \bar{D}^J_\dot{j} \} = -2i \delta^I_J \sigma^m_{\alpha\dot{\alpha}} \partial_m, \quad \{ D^I_i, D^J_\alpha \} = 0, \quad \{ \bar{D}^I_\dot{i}, \bar{D}^J_\dot{j} \} = 0. \]

(3.35)

The supercharges (2.4) are promoted to the operators

\[ Q^i_\alpha = -\frac{\partial}{\partial \theta^i_\alpha} + i (\sigma^m \bar{\theta})^\alpha_m \partial_m, \quad \bar{Q}^\dot{i}_\dot{\alpha} = \frac{\partial}{\partial \bar{\theta}^\dot{i}_\dot{\alpha}} - i (\theta^m \sigma^m)^\dot{\alpha}_\dot{m} \partial_m, \]

(3.36)

which form the \( N = 3 \) superalgebra,

\[ \{ Q^i_\alpha, \bar{Q}^\dot{j}_\dot{\alpha} \} = 2i \delta^i_\dot{j} \sigma^m_{\alpha\dot{\alpha}} \partial_m, \quad \{ Q^i_\alpha, Q^j_\beta \} = 0, \quad \{ \bar{Q}^\dot{i}_\dot{\alpha}, \bar{Q}^\dot{j}_\dot{\beta} \} = 0. \]

(3.37)

The operators (3.31) should be realized in some Hilbert space formed by the functions \(| \Phi \rangle \),

\[ | \Phi \rangle = \Phi(\theta^m, \bar{\theta}^\dot{m}, u). \]

(3.38)

Superfield (3.38) should satisfy some equations of motion and constraints which originate from the superparticle constraints. The superparticle has both first- and second-class constraints. The first-class constraints form closed algebra under the Poisson or Dirac bracket. Therefore, they all should be imposed on states (3.38),

\[ S_1 \Phi^{(s_1, s_2)} = s_1 \Phi^{(s_1, s_2)}, \quad S_2 \Phi^{(s_1, s_2)} = s_2 \Phi^{(s_1, s_2)}, \]

\[ [\partial^m \partial_m + \frac{1}{R^2} X + m^2] \Phi^{(s_1, s_2)} = 0, \]

(3.39)

(3.40)

\[ 5\text{The operators } D^i_\alpha, \bar{D}^\dot{i}_\dot{\alpha} \text{ are multiplied here also by } -i \text{ for convenience, so that the operators } (3.32) \text{ are related to each other by complex conjugation rather than Hermitian one. The same concerns the derivatives } (4.20)\text{–}(4.28) \text{ as well as the supercharges } (3.36, 4.31) \text{ and } U(1) \text{ charges } S_1, S_2 \text{ in (3.39).} \]
where
\[
X = D_2^2 D_1^1 + D_3^3 D_2^2 + D_3^1 D_3^1.
\] (3.41)

Equations (3.39) mean that the superfield \( \Phi^{(s_1,s_2)} \) is a function of harmonic variables with definite \( U(1) \) charges. Note that the equations (3.39) covariantly constrain the \( SU(3) \) harmonic dynamics to the one on a coset \( SU(3)/(U(1) \times U(1)) \). Equation (3.40) is the mass-shell constraint which gives Klein-Gordon-like equations for all physical fields. Note that the zero modes of the operator (3.41) (states which are annihilated by this operator, \( X \Phi_0^{(s_1,s_2)} = 0 \)) satisfy the standard Klein-Gordon equation without a harmonic term,
\[
(\partial^m \partial_m + m^2) \Phi_0^{(s_1,s_2)} = 0.
\] (3.42)

The second-class constraints are accounted either by constructing the corresponding Dirac bracket or by applying the Gupta-Bleuler method. In our case, the second-class harmonic constraints (3.19) are taken into account by the Dirac bracket (3.20), while the spinorial ones (3.22) should be considered à la Gupta-Bleuler. It means that they have to be divided into two complex conjugate subsets with weakly commutative constraints in each subset. There are four different ways of separation of derivatives (3.32) or (3.34) into such subsets:
\[
\{D_1^1, D_2^2, D_3^3\} \cup \{\bar{D}_{1\dot{a}}, \bar{D}_{2\dot{a}}, \bar{D}_{3\dot{a}}\},
\] (3.43a)
\[
\{D_1^1, \bar{D}_{2\dot{a}}, \bar{D}_{3\dot{a}}\} \cup \{\bar{D}_{1\dot{a}}, D_2^2, D_3^3\},
\] (3.43b)
\[
\{D_1^1, D_2^2, \bar{D}_{3\dot{a}}\} \cup \{\bar{D}_{1\dot{a}}, \bar{D}_{2\dot{a}}, D_3^3\},
\] (3.43c)
\[
\{\bar{D}_{1\dot{a}}, D_2^2, \bar{D}_{3\dot{a}}\} \cup \{\bar{D}_{1\dot{a}}, \bar{D}_{2\dot{a}}, \bar{D}_{3\dot{a}}\}.
\] (3.43d)

Different choices of subsets (3.43a)–(3.43d) lead to different types of quantization of the superparticle. In the following subsections we consider them separately.

### 3.2.1 \( \mathcal{N} = 3 \) gravitino multiplet

In this subsection we will show that the separation of fermionic constraints (3.43a) leads to the \( \mathcal{N} = 3 \) gravitino multiplet with the highest helicity 3/2. First, we consider the massive case, \( m \neq 0 \), where the spinor constraints \( D_i^i \approx 0, \bar{D}_{i\dot{a}} \approx 0 \) are second-class since the matrix in the rhs of (3.23) is invertible. There is no \( \kappa \)-symmetry in the model and, hence, no extra constraints. According to (3.43a), the physical state is annihilated only by the derivative \( \bar{D}_{i\dot{a}} \), while \( D_i^i \) kills the conjugate superfield,
\[
\bar{D}_{i\dot{a}} \Phi^{(s_1,s_2)} = 0, \quad D_i^i \bar{\Phi}^{(s_1,s_2)} = 0.
\] (3.44)

The dynamics of such a field is described by the set of equations (3.39, 3.40, 3.44) which take into account all the superparticle constraints. Note that equations (3.44) are nothing but the chirality conditions for the field \( \Phi^{(s_1,s_2)} \). Therefore we refer to such a quantization as a chiral quantization.

---

6Here we use a particular ordering of the operators \( D_I^J \) although other orderings are also possible.
The non-zero modes of the operator $X$ propagate analogously to zero ones. It means that the superfield $\Phi(s_1, s_2)$ describes an unphysical multiplet with infinite number of component fields. To make it physical, we impose the additional harmonic constraints (3.25),

$$
D^1_s \Phi(s_1, s_2) = 0, \quad D^2_s \Phi(s_1, s_2) = 0, \quad D^3_s \Phi(s_1, s_2) = 0.
$$

(3.45)

In general, function $\Phi(s_1, s_2)$ is given by a series in harmonic variables. Equations (3.46) reduce this series to a monomial

$$
\Phi(s_1, s_2) = \bar{u}^{j_1} \cdots \bar{u}^{j_{s_2}} u^{i_1} \cdots u^{i_{s_1}} \Phi_{j_1 \ldots j_{s_1-1} s_2}^{i_1 \ldots i_{s_2-1} s_1},
$$

(3.46)

where $\Phi_{j_1 \ldots j_{s_1-1} s_2}^{i_1 \ldots i_{s_2-1} s_1}$ is a totally symmetric traceless tensor. Indeed, $D^1_s$, $D^2_s$, $D^3_s$ are raising operators in the $su(3)$ algebra which define the highest weight vector (3.46) [18, 20]. As a result, we obtain the chiral superfield with fixed number of $SU(3)$ indices (symmetric and traceless) on mass-shell,

$$
D_{\alpha} \Phi_{i_1 \ldots i_{s_2-1} s_2}^{j_1 \ldots j_{s_1-1} s_1} = 0, \quad (\partial^m \partial_m + m^2) \Phi_{i_1 \ldots i_{s_2-1} s_2}^{j_1 \ldots j_{s_1-1} s_1} = 0.
$$

(3.47)

Let us consider, e.g., the simplest representation $\Phi$ without $SU(3)$ indices that corresponds to the choice of $U(1)$ charges $s_1 = s_2 = 0$. The solution of the chirality condition $D_{\alpha} \Phi = 0$ is most naturally given in chiral coordinates $y^m = x^m + i\theta_1 \sigma^m \tilde{\theta}^i$,

$$
\Phi(y, \theta) = \phi + \theta_1^i \bar{\psi}_i^\alpha + \theta_\alpha^i \bar{\psi}_j^\beta \varepsilon^{ijk} F^k_{(\alpha \beta)} + \theta_\alpha^i \theta_j \bar{\psi}_k^\gamma \varepsilon^{ijk} T_{(\alpha \beta \gamma)} + \theta_\alpha^i \theta_j \bar{\psi}_k \varepsilon^{ijk} S_{\gamma mn} + \theta_\alpha^i \theta_j \bar{\psi}_k \varepsilon^{ijk} G^l_{\gamma mn} + \theta_\alpha^i \theta_j \bar{\psi}_k \bar{\psi}_{\lambda m} \varepsilon^{ijk} \rho^m_{\lambda mn} + \theta_\alpha^i \theta_j \bar{\psi}_k \bar{\psi}_{\lambda m} \bar{\psi}_{\nu r} \varepsilon^{ijk} \varepsilon^{lmn} U,
$$

where all components depend on $y^m$. Note that both bosonic and fermionic components in (3.48) satisfy the Klein-Gordon equation owing to (3.47), but there are no Dirac equations for spinors. Therefore, such a multiplet is unphysical. We assume that the mass should be introduced not directly but through a central charge, as is shown in the next section. Therefore for the rest of this section we consider only the massless case, $m = 0$.

In the massless case the superparticle action (2.2) is well known to respect the $\kappa$-symmetry since in the matrix in the r.h.s of (3.23) is degenerate. Half of the constraints $D^i_\alpha \approx 0, \bar{D}_{\dot{\alpha}} \approx 0$ turn into first-class ones. But if we deal with the harmonic superparticle with the action (2.10), the standard mass-shell constraint (3.42) is replaced by the equation (3.43) with the harmonic contribution due to operator $X$. Therefore, for the states out from the kernel of operator $X$, the matrix $\sigma^m_{\alpha \dot{\alpha}} \partial_m$ is invertible and the constraints $D^i_\alpha \approx 0, \bar{D}_{\dot{\alpha}} \approx 0$ still belong to the second class. As explained before, the $\kappa$-symmetry of harmonic superparticle is restored and one half of these second-class constraints turn into first-class ones, if the dynamics is constrained by (3.24). Upon quantization, constraint (3.24) is imposed on the states implying the condition $X \Phi = 0$. Namely such states are interesting from the physical point of view.
In particular, on the surface of constraints (3.25), the condition (3.24) is satisfied and the kinetic part of the action for harmonics (2.11) can be omitted. Therefore the dynamics is described effectively by the action

\[ S = -\frac{1}{2} \int d\tau \frac{\dot{\omega}^m \dot{\omega}_m}{e} + S_{WZ} + S_\lambda, \]  

(3.49)

which is invariant under the following transformations of \( \kappa \)-symmetry

\[ \begin{align*}
\delta_\kappa \theta_{i\alpha} &= -i p_m (\sigma^m \kappa_i)_\alpha, \\
\delta_\kappa \bar{\theta}^i_\dot{\alpha} &= i p_m (\kappa^i \sigma^m)_{\dot{\alpha}}, \\
\delta_\kappa x^m &= i \delta_\bar{\kappa} \theta_{i\alpha} \sigma^m \dot{\alpha} - i \theta_{i\alpha} \sigma^m \delta_\bar{\kappa} \dot{\alpha}, \\
\delta_\kappa \epsilon &= -4 (\bar{\kappa} i \bar{\theta}^i + \dot{\theta}_i \kappa^i),
\end{align*} \]

(3.50)

where \( \kappa^i_\alpha(\tau), \bar{\kappa}_i\dot{\alpha}(\tau) \) are anticommuting local parameters. Note that the harmonic terms \( S_{WZ}, S_\lambda \), given by (2.12) and (2.13) respectively, do not violate the \( \kappa \)-symmetry as the harmonics do not transform, \( \delta_\kappa u^i_I = 0, \delta_\kappa \bar{u}^i_I = 0 \). Therefore the fields \( \Phi(s_1, s_2) \) with all values of \( U(1) \) charges should obey the constraints originating from the \( \kappa \)-symmetry.

The transformations (3.50) are generated by the Poisson brackets of coordinates with the following first-class constraints,

\[ \begin{align*}
\psi_{i\alpha} &= i p_m \sigma^m_{a\dot{a}} \bar{D}^\dot{a}_i \approx 0, \\
\bar{\psi}^i_\dot{\alpha} &= -i p_m \sigma^m_{a\dot{a}} D^i_\alpha \approx 0.
\end{align*} \]

(3.51)

Upon quantization, (3.51) turn into the differential operators,

\[ \begin{align*}
\psi_{i\alpha} &= -\partial_m \sigma^m_{a\dot{a}} \bar{D}^\dot{a}_i, \\
\bar{\psi}^i_\dot{\alpha} &= \partial_m \sigma^m_{a\dot{a}} D^i_\alpha,
\end{align*} \]

(3.52)

where \( D^i_\alpha, \bar{D}^\dot{a}_i \) are the covariant spinor derivatives (3.32). These differential operators should annihilate the physical states,

\[ \partial_m \sigma^m_{a\dot{a}} D^i_\alpha \Phi(s_1, s_2) = 0. \]

(3.53)

Recall that the superfield \( \Phi(s_1, s_2) \) is a chiral \( \mathcal{N} = 3 \) superfield (3.44) constrained by (3.39, 3.42, 3.45). First-class constraints (3.42) and (3.53) arise from the following one as the integrability conditions (cf. [12, 13] in \( \mathcal{N} = 2 \) case)

\[ D^i_\alpha D^j_\dot{\alpha} \Phi(s_1, s_2) = 0. \]

(3.54)

It is the constraint (3.54) which leads to the correct component structure of the multiplet and eliminates all auxiliary fields in the decomposition (3.48), despite the fact that it is stronger than (3.42) and (3.53). Therefore, we use further (3.54) rather than (3.42, 3.53). 

\footnote{In fact, (3.54) is a consequence of \( \kappa \)-symmetry constraints (3.38) since on the surface of constraints (3.22, 3.26) the following relations \( \hat{d}^i_\alpha \equiv \hat{\theta}^i_{\alpha j} \approx -\bar{D}^\dot{a}_i D^\dot{a}_j, \hat{d}^{ij} \equiv \hat{\theta}^{ij}_\alpha \bar{\psi}^\alpha \approx -D^i_\alpha D^j_\dot{\alpha} \) hold. It is easy to see that the constraints \( \hat{d}^i_\alpha \approx 0, \hat{d}^{ij} \approx 0 \) are the generators \( \kappa \)-transformations (3.50) with the parameters \( \kappa^i_\alpha = k^i_j \theta_{j\alpha}, \bar{\kappa}_i\dot{\alpha} = \bar{k}^i_j \dot{\theta}^j_{\dot{\alpha}}, \) where \( k^{ij}, \bar{k}^{ij} \) are new bosonic local parameters. Analogously, the constraint (4.44) follows from \( \kappa \)-symmetry ones (4.41).}
Let us summarize all the equations for the superfield $\Phi^{(s_1, s_2)}$ in a single list,

$$\begin{align*}
S_1 \Phi^{(s_1, s_2)} &= s_1 \Phi^{(s_1, s_2)}, \\
S_2 \Phi^{(s_1, s_2)} &= s_2 \Phi^{(s_1, s_2)}, \\
D_2^1 \Phi^{(s_1, s_2)} &= D_3^2 \Phi^{(s_1, s_2)} = D_3^3 \Phi^{(s_1, s_2)} = 0, \\
\bar{D}_{i\dot{\alpha}} \Phi^{(s_1, s_2)} &= 0, \\
D^{i\alpha} D^\alpha_{\dot{\alpha}} \Phi^{(s_1, s_2)} &= 0.
\end{align*}$$

(3.55)

The solution of the pure harmonic constraints in the first two lines of (3.55) is given by (3.46). The other constraints in (3.55) give the chirality and linearity conditions,

$$\begin{align*}
\bar{D}_{i\dot{\alpha}} \Phi^{(s_1, s_2)} &= 0, \\
D^{i\alpha} D^\alpha_{\dot{\alpha}} \Phi^{(s_1, s_2)} &= 0.
\end{align*}$$

(3.56)

Consider the equations (3.56) in the simplest case of the scalar superfield $\Phi$ without $SU(3)$ indices,

$$\begin{align*}
\bar{D}_{i\dot{\alpha}} \Phi &= 0, \\
D^{i\alpha} D^\alpha_{\dot{\alpha}} \Phi &= 0.
\end{align*}$$

(3.57)

The component structure of a general chiral superfield $\Phi$ is given by (3.48). The linearity condition eliminates all unphysical components,

$$\Phi = \phi + \theta^i \psi^i + \theta^\alpha \theta^\beta \varepsilon^{ijk} F_{k(\alpha\beta)} + \theta^\alpha \theta^\beta \varepsilon^{ijk} T_{(\alpha\beta\gamma)}.$$  

(3.58)

Owing to (3.57), the component fields in (3.58) satisfy the standard d’Alembert, Weyl and Maxwell equations,

$$\begin{align*}
\Box \phi &= 0 \quad 1 \text{ complex scalar}, \\
\sigma^{m\alpha}_{\dot{\alpha}} \partial_m \psi^i_{\alpha} &= 0 \quad 3 \text{ Weyl spinors}, \\
\sigma^{m\alpha}_{\dot{\alpha}} \partial_m F_{k(\alpha\beta)} &= 0 \quad 3 \text{ vectors}, \\
\sigma^{m\alpha}_{\dot{\alpha}} \partial_m T_{(\alpha\beta\gamma)} &= 0 \quad 1 \text{ gravitino}.
\end{align*}$$

(3.59)

Therefore, superfield $\Phi$ describes the $\mathcal{N} = 3$ supersymmetric gravitino multiplet.

### 3.2.2 $\mathcal{N} = 3$ supergauge multiplet

In this subsection we will show that the separations of constraints (3.43b)–(3.43d) lead to the $\mathcal{N} = 3$ supergauge multiplet [12, 20]. First, we will analyze the separation (3.43b) in details and then will comment on the (3.43c) and (3.43d) cases.

Recall that superfield $\Phi^{(s_1, s_2)}$ satisfies the first-class constraints (3.39–3.40). Now we impose also the spinorial constraints from the first subset in (3.43b),

$$D_1^{1\alpha} \Phi^{(s_1, s_2)} = 0, \quad \bar{D}_{2\dot{\alpha}} \Phi^{(s_1, s_2)} = 0, \quad \bar{D}_{3\dot{\alpha}} \Phi^{(s_1, s_2)} = 0.$$  

(3.60)

Constraints (3.60) show that superfield $\Phi^{(s_1, s_2)}$ is analytic, i.e., it is short in the component expansion. Therefore, we refer to such type of quantization as an analytic quantization. Note that different types of analytic subspaces in full $\mathcal{N} = 3$ harmonic superspace introduced in [20] correspond to different subsets of Grassmann derivatives (3.43b)–(3.43d) annihilating the state.
It is easy to observe that the spinor derivatives in (3.60) do not commute with the operator $X$ given by (3.41),

$$[D^1_a, X] = -D_a^2 D^1_2 - D_a^3 D^1_3, \quad [\bar{D}_{2\dot{a}}, X] = \bar{D}_{1\dot{a}} D^2_1 + D^2_3 \bar{D}_{3\dot{a}}, \quad [\bar{D}_{3\dot{a}}, X] = \bar{D}_{2\dot{a}} D^2_2 + \bar{D}_{1\dot{a}} D^1_3. \quad (3.61)$$

Therefore, the analytic quantization is consistent only if the state $\Phi^{(s_1, s_2)}$ satisfies extra harmonic constraints (3.45). Owing to these constraints, the operators in the rhs of (3.61) vanish on the state while the constraint (3.40) has no harmonic part and turns into the usual mass-shell constraint ($m = 0$),

$$\Box \Phi^{(s_1, s_2)} = 0. \quad (3.62)$$

Equations (3.45) leave only zero modes of the operator $X$, which are massless. For such modes, the harmonic variables are not dynamical and the action (3.49) possesses $\kappa$-symmetry (3.50). Let us project the generators of $\kappa$-symmetry (3.52) with harmonics,

$$\psi_{1\alpha} = -\partial_m \sigma^m_{\alpha a} \bar{D}^\dot{a}_1, \quad \bar{\psi}_{\dot{a}1} = \partial_m \sigma^m_{a\dot{a}} D^a_1. \quad (3.63)$$

Operators (3.63) should annihilate the state $\Phi^{(s_1, s_2)}$ since the constraints (3.51) are first-class. Owing to the analyticity (3.60), it is sufficient to impose three of the six operators (3.63) as the constraints,

$$\partial_m \sigma^m_{\alpha a} \bar{D}^\dot{a}_1 \Phi^{(s_1, s_2)} = 0, \quad \partial_m \sigma^m_{a\dot{a}} D^a_1 \Phi^{(s_1, s_2)} = 0, \quad \partial_m \sigma^m_{\alpha a} \bar{D}^a_1 \Phi^{(s_1, s_2)} = 0. \quad (3.64)$$

These constraints (3.64), as well as the mass-shell condition (3.62), follow from the more general ones

$$(\bar{D}_1)^2 \Phi^{(s_1, s_2)} = 0, \quad (D^2)^2 \Phi^{(s_1, s_2)} = 0, \quad (D^3 D^2) \Phi^{(s_1, s_2)} = 0, \quad (D^3)^2 \Phi^{(s_1, s_2)} = 0. \quad (3.65)$$

In spite of constraints (3.65) being stronger than (3.62) and (3.64), they should be also imposed on the state by the same reasons as constraint (3.51), obtained for the $\mathcal{N} = 3$ gravitino multiplet.

We summarize all the constraints for the superfield $\Phi^{(s_1, s_2)}$ in a single list,

$$S_1 \Phi^{(s_1, s_2)} = s_1 \Phi^{(s_1, s_2)}, \quad S_2 \Phi^{(s_1, s_2)} = s_2 \Phi^{(s_1, s_2)}, \quad D^1_1 \Phi^{(s_1, s_2)} = D^1_3 \Phi^{(s_1, s_2)} = D^1_3 \Phi^{(s_1, s_2)} = 0, \quad \bar{D}_{2\dot{a}} \Phi^{(s_1, s_2)} = \bar{D}_{3\dot{a}} \Phi^{(s_1, s_2)} = 0, \quad \bar{D}_1^2 \Phi^{(s_1, s_2)} = (D^2)^2 \Phi^{(s_1, s_2)} = (D^3 D^2) \Phi^{(s_1, s_2)} = (D^3)^2 \Phi^{(s_1, s_2)} = 0. \quad (3.66)$$

Further we consider some examples of solutions of these constraints for the lowest values of $U(1)$ charges.

Let $s_1 = s_2 = 0$. The corresponding state is described by the chargeless superfield $\Phi$. It is easy to show that under constraints (3.66), this superfield is just a constant, $\Phi = \text{const}$. Therefore this case is trivial.

The next case with $s_1 = 1$, $s_2 = 0$ was considered in [12]. We denote the corresponding superfield by $\Phi^{(1,0)} = W^1$. As a consequence of (3.66), it satisfies the following equations of motion and constraints
\[ D_1^1 W^1 = D_2^1 W^1 = D_3^1 W^1 = 0, \]
\[ D_1^i W^1 = D_2^i W^1 = D_3^i W^1 = 0, \]
\[ (D_1)^2 W^1 = (D_2)^2 W^1 = (D_3)^2 W^1 = (D_2 D_3) W^1 = 0. \]  

Equations (3.67) are known to describe the \( \mathcal{N} = 3 \) superfield strength of the gauge multiplet \([12, 20]\). The component structure of \( W^1 \) can be most easily found in the analytic coordinates,

\[ y^m_A = x^m - i(\theta^a_1 \sigma^m \bar{\theta}^{1\dot{a}} - \theta^a_3 \sigma^m \bar{\theta}^{3\dot{a}} - \theta^2_2 \sigma^m \bar{\theta}^{2\dot{a}}), \]  

in which the spinor derivatives \( D_1^a, D_2\dot{a}, D_3\dot{a} \) take the most simple form,

\[ D_1^a = \frac{\partial}{\partial \theta^a_1}, \quad D_2\dot{a} = -\frac{\partial}{\partial \theta^{2\dot{a}}}, \quad D_3\dot{a} = -\frac{\partial}{\partial \theta^{3\dot{a}}}. \]

Then we have,

\[ W^1 = \phi^1 + \bar{\theta}^{1\dot{a}} \bar{\lambda}_\dot{a} + \theta^2_2 \lambda_{3a} - \bar{\lambda}^a_3 \lambda_{2a} - i\theta^a_2 \bar{\theta}^{1\dot{a}} \sigma^m_{aa} \partial_m \phi^2 + \theta^a_2 \bar{\theta}^{2\dot{a}} \sigma^m_{(aa} \partial_m \lambda_{1\beta)}, \]

where \( \phi^1 = u^1_i \phi^i \) is a triplet of complex scalars, \( \lambda_{1a} = \bar{u}^1_i \lambda_{ia} \) is a triplet of Weyl spinors, \( \bar{\lambda}_{\dot{a}} \) is also a Weyl spinor, and \( F^{(a\beta)} \) is a Maxwell field strength. All these components satisfy the corresponding free equations of motion.

Let us consider briefly the separation of constraints (3.43(c)) leading to other superfield realizations. Consideration for the case when the superfield \( \Phi^{(s_1, s_2)} \) satisfies the following Grassmann shortness conditions

\[ D_1^a \Phi^{(s_1, s_2)} = 0, \quad D_2^a \Phi^{(s_1, s_2)} = 0, \quad D_3\dot{a} \Phi^{(s_1, s_2)} = 0 \]

(3.71) can be done similarly as in the previous case. The physically interesting representation appears if the \( U(1) \) charges take the values \( s_1 = 0, s_2 = 1 \). We denote such a superfield by \( \Phi^{(0, 1)} = \bar{W}_3 \). It has the following equations of motion and constraints

\[ D_2^1 \bar{W}_3 = D_3^2 \bar{W}_3 = D_3^3 \bar{W}_3 = 0, \]
\[ D_1^1 \bar{W}_3 = D_2^2 \bar{W}_3 = D_3\dot{a} \bar{W}_3 = 0, \]
\[ (D_1)^2 \bar{W}_3 = (D_2)^2 \bar{W}_3 = (D_1 D_2) \bar{W}_3 = (D_3)^2 \bar{W}_3 = 0, \]

(3.72) which describe the \( \mathcal{N} = 3 \) Maxwell multiplet as well \([12, 20]\). In particular, in the coordinates \( (y^m_A, \bar{\theta}^1, \bar{\theta}^2, \theta^3) \),

\[ y^m_A = x^m - i(\theta^a_1 \sigma^m \bar{\theta}^{1\dot{a}} - \theta^a_3 \sigma^m \bar{\theta}^{3\dot{a}} - \theta^2_2 \sigma^m \bar{\theta}^{2\dot{a}}), \]

(3.73) \( \bar{W}_3 \) has the following component field decomposition

\[ \bar{W}_3 = \bar{\phi}_3 - \theta^a_3 \lambda_a + \theta^a_1 \bar{\lambda}^{2\dot{a}} - \bar{\theta}^{a\dot{a}} \bar{\lambda}^{1\dot{a}} + i\theta^a_2 \bar{\theta}^{1\dot{a}} \sigma^m_{aa} \partial_m \bar{\phi}_1 - i\theta^2_2 \bar{\theta}^{2\dot{a}} \sigma^m_{a\dot{a}} \partial_m \bar{\phi}_2 \]
\[ -\bar{\theta}^{1\dot{a}} \bar{\theta}^{2\dot{b}} F^{(a\beta)} + i\theta^a_1 \bar{\theta}^{a\dot{b}} \sigma^m \partial_m (\dot{a} \bar{\lambda}^{3\dot{b}}). \]

(3.74)
Here, $\phi_3 = \bar{u}_i \phi_i$ is a triplet of complex scalars, $\lambda_\alpha$ is a Weyl spinor, $\bar{\lambda}_{\dot{\alpha}}$ is a triplet of Weil spinors, and $\bar{F}_{(\dot{\alpha} \dot{\beta})}$ is a Maxwell field strength. It is easy to see that superfields (3.70) and (3.74) are related to each other by the conjugation (2.18).

In concluding this subsection, let us briefly comment on the last case of constraints (3.43d) on the example of a superfield $\Phi^{(-1,1)} = W^2$,

$$D_\alpha^2 W^2 = 0, \quad D_{1\dot{\alpha}} W^2 = 0, \quad D_{3\dot{\alpha}} W^2 = 0.$$  \hspace{1cm} (3.75)

It is easy to see that the harmonic derivatives $D_3^1$, $D_1^2$, $D_3^2$ commute with $D_\alpha^2$, $D_{1\dot{\alpha}}$, $D_{3\dot{\alpha}}$ and therefore should also annihilate this superfield,

$$D_3^1 W^2 = 0, \quad D_1^2 W^2 = 0, \quad D_3^2 W^2 = 0.$$  \hspace{1cm} (3.76)

The $\kappa$-symmetry leads to the following linearity constraints

$$(D^1)^2 W^2 = (D^1 D^3) W^2 = (D^3)^2 W^2 = (D^2)^2 W^2 = 0.$$  \hspace{1cm} (3.77)

Such a superfield under constraints (3.75)–(3.77) also describes the $\mathcal{N} = 3$ Maxwell multiplet which is equivalent to (3.70). The component structure of $W^2$ is similar to (3.70) with the change of index from 1 to 2.

4 Quantization of the $\mathcal{N} = 3$ superparticle with a central charge term

4.1 Hamiltonian formulation and constraints

Let us consider the $\mathcal{N} = 3$ harmonic superparticle with central charges described by the action (2.35). The Lagrangian $L_2$ reads

$$L_2 = -\frac{1}{2} \epsilon^{ij} \dot{\omega}_m \dot{\omega}_m - \frac{1}{2} \epsilon \dot{e}^2 - (z \epsilon^{ij} \dot{\theta}_i \dot{\theta}_j - \bar{z} \epsilon^{ij} \dot{\bar{\theta}}_i \dot{\bar{\theta}}_j) + 2 R^2 e^{-1} \dot{\bar{\omega}}^+ \dot{\bar{\omega}}^- - \lambda (u^- u^+ - 1) - \frac{i}{2} n (u^- u^+ - \bar{u}^- \bar{u}^+).$$  \hspace{1cm} (4.1)

Recall that the underlined indices $i, j, \ldots$ denote $SU(2)$ ones with the values 1, 2. The consequent quantization of this model is similar to the one for the $\mathcal{N} = 2$ superparticle in harmonic superspace [12, 13]. Therefore we follow the same steps keeping, however, basic details of calculations.

The Lagrangian (4.1) defines the following canonical momenta for superspace coordi-
Following [12, 13], we introduce the covariant harmonic momenta

\[ p_m = -\frac{\partial L_2}{\partial \dot{\alpha}_m} = e^{-1}\dot{\omega}_m, \]  
\[ \pi^i_\alpha = \frac{\partial L_2}{\partial \dot{\theta}^i_\alpha} = ip_m(\sigma^m \bar{\theta}^i_\alpha) + \bar{z} \bar{z} \bar{\theta}^i_\alpha, \]  
\[ \bar{\pi}^{i\dot{\alpha}} = \frac{\partial L_2}{\partial \dot{\theta}^{i\dot{\alpha}}} = ip_m(\theta^i \sigma^m)_{\dot{\alpha}} + z \bar{z} \bar{\theta}^{i\dot{\alpha}}, \]  
\[ \pi^3_\alpha = \frac{\partial L_2}{\partial \dot{\theta}^3_\alpha} = ip_m(\sigma^m \bar{\theta}^3_\alpha), \]  
\[ \bar{\pi}^{3\dot{\alpha}} = \frac{\partial L_2}{\partial \dot{\theta}^{3\dot{\alpha}}} = ip_m(\theta_3 \sigma^m)_{\dot{\alpha}}, \]  
\[ v^+i = -\frac{\partial L_2}{\partial \dot{u}^+i} = 2R^2 e^{-1}u^+i \omega^{++} - \frac{1}{2}iu^+_i, \]  
\[ v^-i = -\frac{\partial L_2}{\partial \dot{u}^-i} = 2R^2 e^{-1}u^-i \omega^{--} + \frac{1}{2}iu^-_i. \]  

Following [12, 13], we introduce the covariant harmonic momenta

\[ D^{++} = u^+_i v^+i = -2iR^2 e^{-1}i \omega^{++}, \]  
\[ D^{--} = v^-i u^-i = -2iR^2 e^{-1}i \omega^{--}, \]  
\[ D^0 = v^-i u^+i - u^-i v^+i = in, \]  
\[ \chi_2 = v^-i u^+i + u^-i v^+i = 0 \]

and define the Poisson brackets,

\[ [x^n, p_m]_P = -\delta^n_m, \]
\[ \{\theta^i_\alpha, \pi^j_\beta\}_P = -\delta^i_\beta \delta^j_\alpha, \]
\[ \{\theta^3_\alpha, \pi^3_\beta\}_P = -\delta^3_\beta \delta^3_\alpha, \]
\[ [u^+i, v^-j]_P = -\delta^j_i, \]
\[ [u^-i, v^+j]_P = -\delta^j_i. \]

The full list of constraints is given by

\[ p^m p_m + R^{-2} D^{--} D^{++} - m^2 \approx 0, \]  
\[ D^{++}_\alpha = -\pi^i_\alpha + ip_m(\sigma^m \bar{\theta}^i_\alpha) + z \bar{z} \bar{\theta}^i_\alpha \approx 0, \]  
\[ D^{--}_{\dot{\alpha}} = \pi^{i\dot{\alpha}} - ip_m(\theta^i \sigma^m)_{\dot{\alpha}} - z \bar{z} \bar{\theta}^{i\dot{\alpha}} \approx 0, \]  
\[ D^0 = -in \approx 0, \]
\[ \chi_1 = u^-i u^+i - 1 \approx 0, \]
\[ \chi_2 = v^-i u^+i + u^-i v^+i \approx 0. \]

There is also one more extra harmonic constraint

\[ D^{++} \approx 0, \]
which is necessary to “freeze” the harmonic dynamics and to keep only the physical degrees of freedom.

Constraints (4.12,4.17,4.20) are first-class and should be imposed on the state upon quantization. Second-class harmonic constraints (4.18,4.19) are accounted by the Dirac bracket,

$$\{f, g\}_D = \{f, g\}_P + \frac{1}{2}\{f, \chi_2\}_P\{\chi_1, g\}_P - \frac{1}{2}\{f, \chi_1\}_P\{\chi_2, g\}_P.$$  

(4.21)

It is easy to see that spinor constraints (4.15,4.16) are second-class, since we consider here the massive case. They can be taken into account by using the Gupta–Bleuler method. In general, if the mass of the superparticle is arbitrary and is not related with the central charges \(z, \bar{z}\), the spinor constraints (4.13,4.14) belong to the second class. The quantization of such a particle model does not lead to physical supermultiplets. Therefore, we consider further only a special case, when the central charges are correlated with the mass by BPS condition,

$$z\bar{z} = m^2.$$  

(4.22)

In this case, the superparticle model possesses the \(\kappa\)-symmetry which is realized on the superspace coordinates as follows,

$$\delta_\kappa \theta_{i\alpha} = -i p_m (\sigma^m \bar{\kappa}_{i\bar{\alpha}})_\alpha - z \bar{\varepsilon}_i \kappa_{i\alpha}, \quad \delta_\kappa \bar{\theta}_{\bar{i} \dot{\alpha}} = i p_m (\bar{\kappa}_{i\bar{\alpha}} \sigma^m)_\dot{\alpha} + z \varepsilon_i \kappa_{i\alpha},$$  

$$\delta_\kappa x^m = i \delta_\kappa \theta_{i\alpha} \sigma^m \bar{\theta}_{\bar{i} \dot{\alpha}} - i \theta_{i\alpha} \sigma^m \delta_\kappa \bar{\theta}_{\bar{i} \dot{\alpha}}, \quad \delta_\kappa e = -A (\bar{\kappa}_{i\bar{\alpha}} \dot{\bar{\theta}}_{\bar{i} \dot{\alpha}} + \dot{\theta}_{i\alpha} \kappa_{i\alpha}),$$

(4.23)

The generators of \(\kappa\)-symmetry (4.23) given by

$$\psi_{i\alpha} = i p_m \sigma^m_{\alpha \bar{\alpha}} \bar{D}_{\bar{\alpha}i} \approx 0, \quad \bar{\psi}_{\bar{i} \dot{\alpha}} = -i p_m \sigma^m_{\bar{\alpha} i\dot{\alpha}} \bar{D}_{\bar{i} \dot{\alpha}} \approx 0,$$  

(4.24)

correspond to the additional first-class constraints.

### 4.2 Gupta-Bleuler quantization

Upon quantization, the canonical momenta are replaced by the differential operators,

$$p_m \rightarrow i \frac{\partial}{\partial x^m}, \quad \pi_{i\alpha} \rightarrow -i \frac{\partial}{\partial \theta_{i\alpha}}, \quad \bar{\pi}_{\bar{i} \dot{\alpha}} \rightarrow -i \frac{\partial}{\partial \bar{\theta}_{\bar{i} \dot{\alpha}}},$$

$$\bar{\pi}_{\bar{i} \dot{\alpha}} \rightarrow -i \frac{\partial}{\partial \bar{\theta}_{\bar{i} \dot{\alpha}}}, \quad \bar{\pi}_{3\alpha} \rightarrow -i \frac{\partial}{\partial \bar{\theta}_{3\alpha}}, \quad \bar{v}^{+i} \rightarrow \frac{\partial}{\partial u^{+i}}, \quad \bar{v}^{-i} \rightarrow \frac{\partial}{\partial u^{-i}}.$$  

(4.25)
The spinor constraints \([4.13]–[4.16]\), as well as the harmonic momenta \([4.9]–[4.10]\), turn into the covariant spinor and harmonic derivatives,

\[
\begin{align*}
D^\pm_\alpha &= u^\pm_\alpha \frac{\partial}{\partial u^\pm_\alpha} + i(\sigma^m \bar{\theta}^\pm)_\alpha \partial_m - iz\theta^\pm_\alpha, \\
\bar{D}^\pm_\alpha &= u^\pm_\alpha \frac{\partial}{\partial u^\pm_\alpha} - i(\theta^\pm \sigma^m)_\alpha \partial_m + iz\bar{\theta}^\pm_\alpha, \\
D^3_\alpha &= \frac{\partial}{\partial \theta^3_\alpha} + i(\sigma^m \bar{\theta}^3)_\alpha \partial_m, \\
\bar{D}_{3\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{3\dot{\alpha}}} - i(\theta_3 \sigma^m)_{\dot{\alpha}} \partial_m, \\
D^{++} &= u^+_i \frac{\partial}{\partial u^+_i}, \\
D^{--} &= u^-_i \frac{\partial}{\partial u^-_i}, \\
D^0 &= u^+_i \frac{\partial}{\partial u^+_i} - u^-_i \frac{\partial}{\partial u^-_i}
\end{align*}
\]

with the following anticommutation relations

\[
\{D^+_\alpha, D^-_{\beta}\} = 2iz\varepsilon_{\alpha\beta}, \\
\{\bar{D}^+_\alpha, \bar{D}^-_{\beta}\} = -2iz\varepsilon_{\dot{\alpha}\dot{\beta}}, \\
\{D^+_\alpha, \bar{D}^-_{\dot{\beta}}\} = \{D^3_\alpha, \bar{D}_{3\dot{\alpha}}\} = \{-D^-_{\alpha}, \bar{D}^+_\dot{\alpha}\} = -2i\sigma^m_{\alpha\dot{\alpha}} \partial_m, \\
\{D^+_{\alpha}, D^+_{\dot{\beta}}\} = \{-D^-_{\dot{\alpha}}, D^-_{\dot{\beta}}\} = \{D^+_\alpha, \bar{D}^-_{\dot{\beta}}\} = \{D^+_\dot{\alpha}, \bar{D}^-_{\dot{\beta}}\} = 0.
\]

Supercharges \([2.22]\) in the harmonic superspace are described now by the operators,

\[
\begin{align*}
Q^\pm_\alpha &= \pm \frac{\partial}{\partial \theta^\pm_\alpha} + i(\sigma^m \bar{\theta}^\pm)_\alpha \partial_m - iz\theta^\pm_\alpha, \\
Q^3_\alpha &= -\frac{\partial}{\partial \theta^3_\alpha} + i(\sigma^m \bar{\theta}^3)_\alpha \partial_m, \\
\bar{Q}^+_\dot{\alpha} &= \frac{\partial}{\partial \bar{\theta}^{3\dot{\alpha}}} - i(\theta_3 \sigma^m)_{\dot{\alpha}} \partial_m, \\
\bar{Q}^-_{3\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{3\dot{\alpha}}} - i(\theta_3 \sigma^m)_{\dot{\alpha}} \partial_m,
\end{align*}
\]

which form the \(N = 3\) superalgebra with a central charge,

\[
\{Q^+_\alpha, Q^-_{\beta}\} = -2iz\varepsilon_{\alpha\beta}, \\
\{\bar{Q}^+_\dot{\alpha}, \bar{Q}^-_{3\dot{\beta}}\} = 2iz\varepsilon_{\dot{\alpha}\dot{\beta}}, \\
\{Q^+_\alpha, \bar{Q}^-_{3\dot{\alpha}}\} = \{Q^3_\alpha, \bar{Q}^-_{3\dot{\alpha}}\} = 2i\sigma^m_{\alpha\dot{\alpha}} \partial_m.
\]

Let us introduce the state \(|\Phi\rangle = \Phi^{(n)}\), which is a superfield on harmonic superspace with the equations of motion and constraints originating from the superparticle constraints \([4.12]–[4.19]\). The first-class constraint \([4.17]\) leads to the following equation

\[
D^0\Phi^{(n)} = n\Phi^{(n)},
\]

which shows that this superfield has a definite \(U(1)\) charge. The other harmonic constraints \([4.18]–[4.19]\) are accounted by the Dirac bracket \([4.21]\). The mass-shell constraint \([4.12]\) is also first-class, therefore we have

\[
(\partial^m \partial_m - R^{-2}D^{--}D^{++} + m^2)\Phi^{(n)} = 0.
\]

Furthermore, we require that the superfield \(\Phi^{(n)}\) obeys constraint \([4.20]\),

\[
D^{++}\Phi^{(n)} = 0,
\]
which removes all unphysical degrees of freedom. Under this additional constraint (4.35) the mass-shell condition (4.34) simplifies to
\[(\partial^m \partial_m + m^2)\Phi^{(n)} = 0.\] (4.36)

Now we have to take into account the spinorial constraints (4.13)–(4.16) using Gupta-Bleuler method. The covariant spinor derivatives should be divided into two subsets with commuting constraints in each subset. Clearly, there are two ways of separating these constraints into such subsets,
\[
\{D^+_\alpha, D^+_\bar{\alpha}, D^-_{3\alpha}\} \cup \{D^-_\alpha, \bar{D}^-_{\bar{\alpha}}, D^3_{\bar{\alpha}}\},
\] (4.37)
\[
\{D^+_{\alpha}, \bar{D}^+_{\bar{\alpha}}, D^3_{\alpha}\} \cup \{D^-_\bar{\alpha}, \bar{D}^-_\alpha, \bar{D}_{3\bar{\alpha}}\}.\] (4.38)

Both these cases lead to equivalent results. Therefore, we consider only (4.37) in detail. The corresponding constraints
\[
D^+_\alpha \Phi^{(n)} = 0, \quad \bar{D}^+_\bar{\alpha} \Phi^{(n)} = 0, \quad \bar{D}_{3\bar{\alpha}} \Phi^{(n)} = 0,
\] (4.39)
show that \(\Phi^{(n)}\) is analytic with respect to \(\theta^+_{\alpha}, \bar{\theta}^+_{\bar{\alpha}}\) and is chiral in \(\theta_{3\alpha}, \bar{\theta}^3_{\bar{\alpha}}\) variables.

Finally, we have to take into account constraints (4.24) originating from the \(\kappa\)-symmetry (4.23). Upon quantization, the generators of \(\kappa\)-transformations (4.24) turn into the differential operators
\[
\psi^+_\alpha = -\sigma^m_{\alpha\bar{\alpha}} \partial_m \bar{D}^-_{\bar{\alpha}} + \bar{z} D^+_\alpha, \quad \bar{\psi}^+_\bar{\alpha} = \sigma^m_{\alpha\bar{\alpha}} \partial_m D^+_{\alpha} - z \bar{D}^-_{\bar{\alpha}}.
\] (4.40)

Note that owing to the analyticity (4.39), constraints \(\psi^+_\alpha \Phi^{(n)} = 0, \quad \bar{\psi}^+_\bar{\alpha} \Phi^{(n)} = 0\) are satisfied automatically, while the “−” projections in (4.40) lead to the equations
\[
(-\sigma^m_{\alpha\bar{\alpha}} \partial_m \bar{D}^-_{\bar{\alpha}} + \bar{z} D^-_{\alpha}) \Phi^{(n)} = 0, \quad (\sigma^m_{\alpha\bar{\alpha}} \partial_m D^-_{\alpha} - z \bar{D}^-_{\bar{\alpha}}) \Phi^{(n)} = 0.
\] (4.41)

Let us introduce the operator
\[
Y^{--} = \frac{i}{4} (-\bar{z} \bar{D}^-_{\bar{\alpha}} \bar{D}^-_{\bar{\alpha}} + \bar{z} D^-_{\alpha} D^-_{\alpha}),\] (4.42)
which commutes with covariant spinor derivatives as
\[
\frac{1}{\bar{z}} [D^+_\alpha, Y^{--}] = \psi^-_{\alpha}, \quad \frac{1}{\bar{z}} [\bar{D}^+_{\bar{\alpha}}, Y^{--}] = \bar{\psi}^-_{\bar{\alpha}}.
\] (4.43)

As the superfield \(\Phi^{(n)}\) is analytic, both constraints (4.41) follow from
\[
(z \bar{D}^-_{\bar{\alpha}} \bar{D}^-_{\bar{\alpha}} - \bar{z} D^-_{\alpha} D^-_{\alpha}) \Phi^{(n)} = 0.
\] (4.44)

Despite this equation being stronger than the pair (4.41), it should be imposed on the physical states as well, since the first-class constraint (4.42) is a function of spinorial constraints (4.13–4.14) and forms the algebra (4.43) with the generators of \(\kappa\)-symmetry. As a result, all superparticle constraints are accounted by the corresponding equations for the field \(\Phi^{(n)}\).
4.2.1 $\mathcal{N} = 3$ massive vector multiplet

Let us consider a solution of constraints for the superfield $\Phi^{(n)}$ on the physically interesting example of a superfield $\Phi^{(1)} \equiv q^+$ with $U(1)$ charge $+1$. We will show that such a superfield describes the $\mathcal{N} = 3$ massive vector multiplet.

To begin with, we list once again all the constraints for the $q^+$ superfield,

$$D^0 q^+ = q^+, \quad (4.45)$$

$$D^+_\alpha q^+ = \bar{D}^+_\alpha q^+ = 0, \quad (4.46)$$

$$D^{++} q^+ = 0, \quad (4.47)$$

$$\bar{D}^{++} q^+ = 0, \quad (4.48)$$

$$[\bar{z}(D^-)^2 - z(D^-)^2] q^+ = 0, \quad (4.49)$$

$$(\partial^m \partial_m + m^2) q^+ = 0. \quad (4.50)$$

Equation (4.45) is satisfied automatically, while the pair (4.46,4.47) follows from (4.46,4.47). To solve (4.46), we pass from the central coordinates to the analytic ones,

$$x^m_A = x^m - i(\theta^+ \sigma^m \theta^- + \theta^- \sigma^m \theta^+), \quad (4.51)$$

and transform Grassmann and harmonic derivatives as well as the superfield $q^+$ such that $D^+_\alpha, \bar{D}^+_\alpha$ become short,

$$D^+_\alpha \rightarrow D^+_\alpha = e^\Omega D^+_\alpha e^{-\Omega} = \frac{\partial}{\partial \theta^{-\alpha}}, \quad (4.52)$$

$$\bar{D}^+_\alpha \rightarrow \bar{D}^+_\alpha = e^\Omega \bar{D}^+_\alpha e^{-\Omega} = \frac{\partial}{\partial \theta^{-\alpha}}, \quad (4.53)$$

$$D^-_\alpha \rightarrow D^-_\alpha = e^\Omega D^-_\alpha e^{-\Omega} = -\frac{\partial}{\partial \theta^{+\alpha}} + 2i(\sigma^m \theta^-)_\alpha \frac{\partial}{\partial x^m_A} - 2iz\theta^-_\alpha, \quad (4.54)$$

$$\bar{D}^-_\alpha \rightarrow \bar{D}^-_\alpha = e^\Omega \bar{D}^-_\alpha e^{-\Omega} = -\frac{\partial}{\partial \theta^{+\alpha}} - 2i(\theta^- \sigma^m)_\alpha \frac{\partial}{\partial x^m_A} + 2iz\bar{\theta}^-_\alpha, \quad (4.55)$$

$$D^{++} \rightarrow D^{++} = e^\Omega D^{++} e^{-\Omega} = D^{++} + iz(\theta^+)^2 + iz(\bar{\theta}^+)^2, \quad (4.56)$$

$$q^+ \rightarrow q^+ = e^\Omega q^+, \quad (4.57)$$

where

$$\Omega = -iz\theta^+ \theta^- - iz\bar{\theta}^+ \bar{\theta}^- \quad (4.58)$$

In this representation, constraints (4.46) are solved automatically if $q^+$ does not depend on $\theta^-_\alpha, \bar{\theta}^-_\alpha$,

$$q^+ = q^+ (x^m_A, \theta^+_\alpha, \bar{\theta}^+_\alpha, \theta^-_\alpha, \bar{\theta}^-_\alpha, \bar{u}^+_\alpha). \quad (4.59)$$

Next, we expand (4.59) over $\theta^+_\alpha, \bar{\theta}^+_\alpha$ and solve the equation $D^{++} q^+ = 0$ which follows from (4.47). As a result we have

$$q^+ = u^+_\alpha F^\alpha + \bar{\theta}^+ \bar{\Psi} + \bar{\bar{\theta}}^+ \bar{\Xi} - iz(\theta^+)^2 F^\alpha u^-_\alpha - iz(\bar{\theta}^+)^2 F^\alpha u^{-}_\alpha + 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m F^\alpha u^{-}_\alpha, \quad (4.60)$$
where all the components depend on \((x^m, \theta_{3a}, \bar{\theta}^3_{\dot{a}})\). Here \(F^i\) is a doublet of complex scalars satisfying Klein-Gordon equation,

\[
(\Box + zz) F^i = 0, \tag{4.61}
\]

and \(\left(\frac{\Psi_{\dot{a}}}{\bar{\Xi}_{\dot{a}}}\right)\) is a massive Dirac spinor,

\[
\sigma^{m\dot{a}} \partial_m \Psi_{\dot{a}} - i z \bar{\Xi}_{\dot{a}} = 0, \quad i \sigma_{m\dot{a}} \partial_m \bar{\Xi}_{\dot{a}} + i z \Psi_{\dot{a}} = 0. \tag{4.62}
\]

In what follows, the mass is correlated with the central charges as

\[
m = i z = -i \bar{z}. \tag{4.63}
\]

Now we recall that \(q^+\) depends also on \(\theta_{3a}, \bar{\theta}^3_{\dot{a}}\) variables in a chiral way,

\[
\bar{D}_{3a} q^+ = 0. \tag{4.64}
\]

In the chiral coordinates \(y^m = x^m + i \theta_{3a} \sigma^m \bar{\theta}^3\), the derivative \(\bar{D}_{3a}\) is short, \(\bar{D}_{3a} = -\frac{\partial}{\partial \theta_{3a}}\), and all components in (4.60) depend on \(\theta^3\), but not on \(\bar{\theta}^3\),

\[
F^i = F^i(y^m, \theta^3), \quad \Psi_{\dot{a}} = \Psi_{\dot{a}}(y^m, \theta^3), \quad \bar{\Xi}^\dot{a} = \bar{\Xi}^\dot{a}(y^m, \theta^3). \tag{4.65}
\]

Let us consider the decomposition of these components in the series over \(\theta^3\),

\[
F^\dot{a}(y, \theta^3) = f^\dot{a}(y) + \theta^3 \sigma^{\dot{a}}(y) + (\theta^3)^2 g^\dot{a}(y), \\
\Psi_{\dot{a}}(y, \theta^3) = \psi_{\dot{a}}(y) + \theta^3 \bar{\Psi}^{\dot{a}}(y) + (\theta^3)^2 \chi_{\dot{a}}(y), \\
\bar{\Xi}^\dot{a}(y, \theta^3) = \bar{\chi}^\dot{a}(y) + \theta_{3a} \bar{A}^{\dot{a}a}(y) + (\theta^3)^2 \rho^\dot{a}(y). \tag{4.66}
\]

Owing to (4.61), the fields \(f^\dot{a}\), \(g^\dot{a}\), \(\sigma^{\dot{a}}\) obey the Klein-Gordon equation. Unfortunately, there is no Dirac equation for the spinors \(\sigma^{\dot{a}}\). However, the two Weyl spinors \(\sigma^{\dot{a}}\) with Klein-Gordon equation are equivalent to a pair of Dirac spinors satisfying usual Dirac equation.\footnote{Let \(\psi_{\dot{a}}\) be a function with the spinor index satisfying Klein-Gordon equation, \(\partial_m \partial^m \psi_{\dot{a}} + m^2 \psi_{\dot{a}} = 0\). Factorizing the box operator we rewrite this equation as \(i \sigma^{m\dot{a}} \partial_m \frac{1}{m} \sigma^{a\dot{a}} \partial_a \psi_{\dot{a}} - m \psi_{\dot{a}} = 0\). By denoting \(\bar{\chi}_{\dot{a}} = \frac{1}{m} \sigma^{m\dot{a}} \partial_m \psi_{\dot{a}}\) the Klein-Gordon equation can be rewritten as a pair \(i \sigma^{m\dot{a}} \partial_m \bar{\chi}_{\dot{a}} - m \psi_{\dot{a}} = 0, \quad i \sigma^{m\dot{a}} \psi_{\dot{a}} + m \bar{\chi}_{\dot{a}} = 0\) that is nothing but the Dirac equation for the spinor \(\left(\frac{\psi_{\dot{a}}}{\bar{\chi}_{\dot{a}}}\right)\).}

Let us study the consequences of the Dirac equations (4.62). They imply that \(\left(\frac{\psi_{\dot{a}}}{\bar{\chi}_{\dot{a}}}\right)\) are usual Dirac spinors while the components \(\bar{A}_{\dot{a}a}, C, F_{a\beta}\) obey

\[
\sigma_{a\dot{a}} \partial_m C + \sigma^{m\beta} \bar{A}_{a\dot{a}} F_{a\beta} - z \bar{A}_{a\dot{a}} = 0, \tag{4.67}
\]

\[
\sigma^{m \dot{a}} \partial_m A_{a\dot{a}} + \bar{z} \varepsilon_{a\dot{a}} C + \bar{z} F_{a\beta} = 0. \tag{4.68}
\]

Equations (4.67) have the following solutions

\[
C = -\frac{1}{\bar{z}} \partial_m A^m, \quad F_{a\beta} = -\frac{1}{\bar{z}} \sigma^{mn} \bar{F}_{mn}, \tag{4.69}
\]
\[ (\Box + zz)\bar{A}_m = 0, \]  
(4.70)

where \( F_{mn} = \partial_m \bar{A}_n - \partial_n \bar{A}_m, \bar{A}_m = \frac{1}{2} \sigma^a_{\alpha m} \bar{A}_{\alpha n} \) and \( \sigma_{alm}^{mn} = -\frac{i}{4} (\sigma_{\alpha n}^m \sigma_{\beta}^n \bar{A}_{\alpha m} - \sigma_{\alpha m}^n \sigma_{\beta}^m \bar{A}_{\alpha n}) \). As a result, vector \( \bar{A}_n \) corresponds to the complex massive vector field (with Proca equation) plus a complex scalar \( C \) with Klein-Gordon equation.

Summarizing these results we have the following field content in \( q^+ \) subject to (4.45)–(4.50):

- Complex massive vector \( \bar{A}_m \) describes 6 bosonic degrees of freedom;
- \( f^i, g^i, C \) are complex scalars with 10 bosonic degrees of freedom;
- \( (\psi_\alpha, \bar{\chi}_\dot{\alpha}), (\lambda_\alpha, \bar{\rho}_\dot{\alpha}) \) are massive Dirac spinors with 8 fermionic degrees of freedom;
- The doublet of spinors \( \sigma^a_\dot{\alpha} \) does not satisfy Dirac equation since it originates from the scalars \( F^i \) obeying only the Klein-Gordon equation (4.62). However, as noted above, \( \sigma^a_\dot{\alpha} \) correspond to two Dirac spinors with 8 fermionic degrees of freedom.

This is nothing but the field content of \( \mathcal{N} = 3 \) massive vector multiplet with BPS mass \([21]\). Note that it has double the number of components in comparison with the massive (non-BPS) \( \mathcal{N} = 2 \) vector multiplet.

### 4.2.2 Superfield reduction of components in massive vector multiplet

In the previous subsection, we have shown that the quantization of the \( \mathcal{N} = 3 \) harmonic superparticle with central charges correlated with the mass as in (4.63) leads to the \( \mathcal{N} = 3 \) massive vector BPS multiplet. As this multiplet has double the number of states in comparison with the massive (non-BPS) \( \mathcal{N} = 2 \) vector multiplet, it is natural to ask whether it is possible to impose additional constraints on the \( q^+ \) superfield which reduce its component content to the massive \( \mathcal{N} = 2 \) supergauge multiplet. As we will show, this is possible if one relaxes the condition of the CPT invariance of the multiplet.

To begin with, we introduce a conjugation “\( \sim \)” which acts as a standard conjugation \( \sim \) (2.34) in harmonic superspace and changes the signs of the central charges (and mass), \( \tilde{z} = -\bar{z}, \tilde{\bar{z}} = -z \). For instance, the superfield conjugated to (4.60) is

\[
\bar{q}^+ = u_\alpha^+ \bar{F}^\alpha_\dot{\alpha} - \theta^+\alpha \Xi_\alpha + \bar{\theta}^+\dot{\alpha} \bar{\Psi}_\dot{\alpha} - iz(\theta^+)^2 \bar{F}^\alpha_\dot{\alpha} u_\alpha^+ + \bar{\theta}^+\dot{\alpha} \bar{\sigma}^m \partial_m \bar{F}^\alpha_\dot{\alpha} u_\alpha^- + 2i\theta^+ \bar{\sigma}^m \bar{\theta}^\alpha \partial_m \bar{F}^\alpha_\dot{\alpha} u_\alpha^- \quad (4.71)
\]

Note that \( q^+ \) depends on \( \theta_3^\alpha, \bar{\theta}_3^\dot{\alpha} \) in a chiral way while \( \bar{q}^+ \) is antichiral with respect to these variables. It is natural to restrict the dependence on \( \theta_3^\alpha, \bar{\theta}_3^\dot{\alpha} \) by imposing equations which are similar to the ones in the massive Wess-Zumino model,

\[
\frac{1}{4} (D_3)^2 q^+ + iz\bar{q}^+ = 0, \quad \frac{1}{4} (\bar{D}^3)^2 \bar{q}^+ - i\bar{z}q^+ = 0. \quad (4.72)
\]
Equations (4.72) are conjugate to each other with respect to $\sim$ conjugation and have the following consequences:

\[
\begin{align*}
\frac{1}{4}(D^3)^2 F^i + iz F_i &= 0, & \frac{1}{4}(\bar{D}_3)^2 \bar{F}_i - i\bar{z} \bar{F}_i &= 0, \\
\frac{1}{4}(D^3)^2 \Psi_\alpha - iz \Xi_\alpha &= 0, & \frac{1}{4}(\bar{D}_3)^2 \Xi_\alpha + i\bar{z} \Psi_\alpha &= 0, \\
\frac{1}{4}(D^3)^2 \Xi^{\dot{\alpha}} + iz \bar{\Psi}^{\dot{\alpha}} &= 0, & \frac{1}{4}(\bar{D}_3)^2 \bar{\Psi}^{\dot{\alpha}} - i\bar{z} \Xi^{\dot{\alpha}} &= 0.
\end{align*}
\]

(4.73)

Clearly, (4.73) are nothing but the usual Wess-Zumino equations for the $\mathcal{N} = 1$ superfields $F_i, \bar{F}_i$. Therefore for the components of these superfields we have

\[
\begin{align*}
(\Box + z\bar{z}) f^i &= 0, & (\Box + z\bar{z}) \bar{f}_i &= 0, \\
iz\sigma^m_{\alpha\dot{\alpha}} \partial_m \sigma^{\dot{\alpha}} + iz\sigma^{\dot{\alpha}} &= 0, & iz\sigma^m_{\alpha\dot{\alpha}} \partial_m \sigma^{\dot{\alpha}} + iz\sigma^{\dot{\alpha}} &= 0, \\
g^i = iz\bar{f}_i, & \bar{g}^i = -iz f_i.
\end{align*}
\]

(4.76)

(4.77)

One can easily construct a Dirac spinor \( \sigma^{\dot{\alpha}}_\alpha \equiv (\sigma^{\dot{\alpha}}_{\alpha} \bar{m}) \) from the spinors \( \sigma^{\dot{\alpha}}_\alpha \) satisfying (4.77). Note that the conjugated spinor \( (\bar{m})_{\dot{\alpha}} \equiv (\bar{m}^{\dot{\alpha}}_\alpha) \) satisfies the Dirac equation with opposite sign of the mass.

Let us consider the pair of equations (4.74). They lead to the Klein-Gordon equations for spinors \( \chi_\alpha, \psi_\alpha \), while \( \lambda_\alpha, \rho_\alpha \) are expressed from them,

\[
\lambda_\alpha = -iz\chi_\alpha, \quad \rho_\alpha = iz\psi_\alpha.
\]

(4.79)

The other components obey the following equations

\[
\begin{align*}
\sigma^m_{\dot{\alpha}} \partial_m A_{\alpha\dot{\alpha}} + \bar{z}z \epsilon_{\alpha\beta} C + \bar{z} F_{\alpha\beta} &= 0, \\
\sigma^m_{\alpha\dot{\beta}} \partial_m F_{\alpha\beta} + \sigma^m_{\alpha\dot{\beta}} \partial_m C + z A_{\alpha\dot{\beta}} &= 0,
\end{align*}
\]

(4.80)

(4.81)

which are solved by

\[
\begin{align*}
C &= \frac{1}{\bar{z}} \partial_m A^m, & F_{\alpha\beta} &= -\frac{1}{\bar{z}} \sigma^m_{\alpha\dot{\beta}} F_{mn}, \\
(\Box + z\bar{z}) A_m &= 0.
\end{align*}
\]

(4.82)

(4.83)

Considering (4.82) together with (4.69) we conclude that field strength \( F_{mn} = \partial_m A_n - \partial_n A_m \) is real, \( F_{mn} = \bar{F}_{mn} \), while \( \partial_m A^m \) is imaginary and corresponds to a real scalar \( B = i\partial_m A^m \). As a result, the complex vector \( A_m \) splits into a real vector obeying Proca equations, and a real scalar with Klein-Gordon equation. The resulting multiplet exactly corresponds to a massive \( \mathcal{N} = 2 \) vector multiplet:

- Two complex scalars \( f^i \) and the real one \( B \) give 5 real bosonic degrees of freedom;
- Dirac spinors \( (\psi_\alpha, \bar{\chi}^\alpha), (\bar{m}^{\dot{\alpha}}_\alpha) \) describe 8 fermionic degrees of freedom;
Real massive vector field $A_m$ has 3 bosonic components on-shell.

To show that equations (4.72) preserve the $\mathcal{N} = 3$ supersymmetry with a central charge, we note that the supercharges (4.31) are conjugate to each other with respect to $\sim$ conjugation rather than to $\tilde{}$, 

\[
\begin{align*}
\tilde{Q}_+^+ &= -\tilde{Q}_\alpha^+, & \tilde{Q}_-^- &= -\tilde{Q}_{\bar{\alpha}}^-, & \tilde{Q}_+^+ &= Q_\alpha^+, & \tilde{Q}_-^- &= Q_{\bar{\alpha}}^-.
\end{align*}
\] (4.84)

Hence, the supersymmetry variation is real under such a conjugation,

\[
\delta_\epsilon = -\epsilon^+\alpha Q_\alpha^- + \epsilon^-\bar{\alpha} \tilde{Q}_\alpha^+ + \epsilon^-\alpha \tilde{Q}_\alpha^+ + \epsilon^+\bar{\alpha} Q_{\bar{\alpha}}^- + \epsilon^+\alpha Q_{\bar{\alpha}}^- + \epsilon^-\bar{\alpha} \tilde{Q}_{\bar{\alpha}}^+ \equiv \tilde{\delta}_\epsilon .
\] (4.85)

Therefore, both superfields $q^+$ and $\tilde{q}^+$ transform in the same way under supersymmetry, and the conjugation $\sim$ in equations (4.72) does not break the $\mathcal{N} = 3$ supersymmetry. However, the resulting multiplet is not CPT selfconjugated and the CPT symmetry is lost. In terms of superfields it is obvious since the conjugation $\sim$ involves the change of the sign of the mass and the central charge. In components it leads to the fact that the spinors \((\psi_\alpha, \bar{\psi}_{\dot{\alpha}})\), \((\rho_\alpha, \bar{\rho}_{\dot{\alpha}})\) are standard Dirac ones while their conjugates \((\bar{\psi}_{\dot{\alpha}}, \bar{\psi}_{\bar{\alpha}})\), \((\bar{\rho}_{\dot{\alpha}}, \bar{\rho}_{\bar{\alpha}})\) satisfy the Dirac equation with opposite sign of the mass.

As a result, equations (4.72) reduce the number of components in the $\mathcal{N} = 3$ vector multiplet by half resulting in the $\mathcal{N} = 2$ massive (non-BPS) vector multiplet. In other words, the $\mathcal{N} = 2$ massive vector multiplet is equivalent to a half of the $\mathcal{N} = 3$ vector multiplet which respects the $\mathcal{N} = 3$ supersymmetry with central charge, but is not CPT selfconjugated. Although this fact is well known [21], we establish this correspondence by superfield considerations. In particular, this multiplet is realized as a single constrained $\mathcal{N} = 3$ superfield. It would also be very tempting to find a supersymmetric action for superfields $q^+$, $\tilde{q}^+$ reproducing the corresponding equations of motion.

## 5 Conclusion

Let us summarize the results obtained in the quantization of $\mathcal{N} = 3$ superparticle.

1. The models of the $\mathcal{N} = 3$ superparticles both with and without central charge term, are considered in the harmonic superspace. Since the $\mathcal{N} = 3$ superalgebra with central charge possesses $SU(2) \times U(1)$ R-symmetry rather than $U(3)$, the description of the $\mathcal{N} = 3$ superparticle with central charge is achieved in the $\mathcal{N} = 2$ harmonic superspace (with $SU(2)$ harmonic variables) extended by a pair of extra Grassmann variables: \([x^m, \theta^\pm_\alpha, \theta^\pm_{\alpha}, \theta^3_3, \bar{\theta}^3_\dot{\alpha}, u^\pm_\alpha]\).

2. By quantizing the $\mathcal{N} = 3$ superparticle without central charge, we obtain $\mathcal{N} = 3$ superfield realizations of the $\mathcal{N} = 3$ supergauge multiplet and the gravitino multiplet (with highest helicity 3/2). The latter is described by a chiral $\mathcal{N} = 3$ superfield satisfying linearity constraints, while the former is given by the superfield strengths,
which are short superfields in the $\mathcal{N} = 3$ harmonic superspace subject to Grassmann and harmonic shortness conditions. These superfields were originally introduced in [12] and studied in [20].

3. The $\mathcal{N} = 3$ superparticle with central charge is quantized similarly to the $\mathcal{N} = 2$ superparticle in harmonic superspace [12, 13]. The resulting massive $\mathcal{N} = 3$ supergauge multiplet is given by 5 complex scalars, 4 Dirac spinors and 1 complex vector on-shell. It is embedded into a superfield $q^+\bar{q}$, which is analytic in $\theta^{\pm \alpha}, \bar{\theta}^{\pm \dot{\alpha}}$ variables and chiral with respect to $\theta_3, \bar{\theta}_3$. The equation of motion for this superfield is similar to the one for $q$-hypermultiplet, $D^{++}q^+=0$.

4. We notice that the number of states of the massive $\mathcal{N} = 3$ supergauge multiplet is doubled compared to the massive (non-BPS) $\mathcal{N} = 2$ vector multiplet with 5 real scalars 4 spinors and 1 real vector. The doubling of states in the representations of the $\mathcal{N} = 3$ superalgebra with central charge is required for CPT invariance [21]. However, if we abandon the CPT invariance, the numbers of states in these two multiplets coincide. We have shown how this can be achieved at a superfield level: by imposing the extra superfield constraints on the $\mathcal{N} = 3$ superfield $q^+$ we reduce the number of states by one half, arriving at the massive $\mathcal{N} = 2$ supergauge multiplet realized as a constrained $\mathcal{N} = 3$ superfield. Of course, these constraints break the CPT invariance manifestly. In components, the loss of CPT invariance means that the Dirac spinor $\left(\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}\right)$ and its conjugate $\left(\nu_\alpha, \bar{\nu}^{\dot{\alpha}}\right)$ satisfy Dirac equations with different signs of the mass.

The results of the quantization of the massless $\mathcal{N} = 3$ superparticle without a central charge term are rather expected: the superfield realizations of the $\mathcal{N} = 3$ supergauge and gravitino multiplets are achieved. However, the quantization of the $\mathcal{N} = 3$ superparticle with a central charge leads to a superfield description for the massive $\mathcal{N} = 3$ supergauge multiplet which was previously unknown. It would be interesting to develop the classical and quantum field theory of this multiplet in a superfield realization. If a Lagrangian superfield formulation of this model is achievable, the classical and quantum properties would be as interesting as for the massive $\mathcal{N} = 2$ vector superfield [26, 27, 28].

We have shown the relations between the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$ massive vector multiplets to be even deeper. Indeed, the $\mathcal{N} = 2$ massive vector multiplet is described by an $\mathcal{N} = 3$ superfield under specific superfield constraints which manifestly break CPT symmetry of $\mathcal{N} = 3$ superalgebra with central charge. In other words, the $\mathcal{N} = 2$ massive vector multiplet can be viewed as a half of the $\mathcal{N} = 3$ massive vector multiplet, which does not have its CPT conjugate. This means that the free $\mathcal{N} = 2$ massive vector multiplet possesses $\mathcal{N} = 3$ supersymmetry with a central charge, if we neglect CPT invariance. Finally, the $\mathcal{N} = 3$ superfield under the additional constraints can be considered as an alternative formulation for the massive $\mathcal{N} = 2$ vector multiplet. It would be interesting to study whether it is possible to include the non-Abelian selfinteraction of this multiplet in terms of $\mathcal{N} = 3$ superfields, and to build some action directly in the $\mathcal{N} = 3$ superspace.
Another obvious continuation of this research would be the study of the \( \mathcal{N} = 4 \) superparticle in a similar way. The case of the massive superparticle with a central charge would be the most tempting and should lead to a massive \( \mathcal{N} = 4 \) vector multiplet realized as an \( \mathcal{N} = 4 \) superfield. This model would be of high interest both at the classical and at the quantum level.

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