Geometry of three manifolds
and existence of Black Hole
due to boundary effect

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1 Introduction

In this paper, we observe that the brane functional studied in [5] can be used to generalize some of the works that Schoen and I [4] did many years ago. The key idea is that if a three dimensional manifold $M$ has a boundary with strongly positive mean curvature, the effect of this mean curvature can influence the internal geometry of $M$. For example, if the scalar curvature of $M$ is greater than certain constant related to this boundary effect, no incompressible surface of higher genus can exist.
A remarkable statement in general relativity is that if the mean curvature of $\partial M$ is strictly greater than the trace of $p_{ij}$ (the second fundamental form of $M$ in space time), the value of this difference can provide the existence of apparent horizon in $M$. In fact, matter density can even be allowed to be negative if this boundary effect is very strong. Theorem 5.2 is the major result of this paper.

2 Existence of stable incompressible surfaces with constant mean curvature

We shall generalize some of the results of Schoen-Yau [1] and Meeks-Simon-Yau [3].

Let $M$ be a compact three dimensional manifold whose boundary $\partial M$ has mean curvature (with respect to the outward normal) not less than $c > 0$. Assume the volume form of $M$ can be written as $d\Lambda$ where $\Lambda$ is a smooth two form.

Let $f: \Sigma \to M$ be a smooth map from a surface $\Sigma$ into $M$ which is one to one on $\Pi_1(\Sigma)$. We are interesting in minimizing the energy

$$E_c(f) = \frac{1}{2} \int_{\Sigma} |\nabla f|^2 - c \int_{\Sigma} f^* \Lambda.$$

There are two different hypothesis we shall make for the existence of surfaces which minimizes $E_c$.

**Theorem 2.1.** Assume the existence of embedded $\Sigma$ with $\Pi_1(\Sigma) \to \Pi_1(M)$ to be one to one. Assume that for any ball $B$ in $M$, the volume of the ball $B$ is not greater than $\frac{1}{c} \text{Area}(\partial B)$. Then we can find a surface isotopic to $\Sigma$ which minimizes the functional $\text{Area}(\Sigma) - c \int_{\Sigma} \Lambda$.

**Proof.** This follows from the argument of Meeks-Simon-Yau [3]. The hypothesis is method to deal with the cut and paste argument. □

**Theorem 2.2.** Assume that the supremum norm of $\Lambda$ is not greater than $c^{-1}$. Then for any $c' < c$, we can find a conformal map from some conformal structure on $\Sigma$ to $M$ which induces the same map is $f_* = \Pi_1(\Sigma) \to \Pi_1(M)$ and minimize the energy $\frac{1}{2} \int_{\Sigma} |\nabla f|^2 - c' \int_{\Sigma} f^* \Lambda$. 
**Proof.** Since $c' < c$, the energy is greater than a positive multiple of the standard energy. Hence the argument of Schoen-Yau [1] works. □

**Remark.** It should be possible to choose $c' = c$ in this last theorem.

### 3 Second variational formula

Let $\Sigma$ be the stable surface established in section 2. Then the variational formula shows that the mean curvature of $\Sigma$ is equal to $c$. The second variational formula shows that for all $\varphi$ defined on $\Sigma$,

$$
\int_\Sigma |\nabla \varphi|^2 - \int_\Sigma (\text{Ric}_M(\nu, \nu) + \Sigma h^2_{ij}) \varphi^2 \geq 0 \quad (3.1)
$$

where $\text{Ric}_M(\nu, \nu)$ is the Ricci curvature of $M$ along the normal of $\Sigma$ and $h_{ij}$ is the second fundamental form.

The Gauss equation shows that

$$
\text{Ric}_M(\nu, \nu) = \frac{1}{2} R_M - K_\Sigma + \frac{1}{2}(H^2 - \Sigma h^2_{ij}) \quad (3.2)
$$

where $R_M$ is the scalar curvature, $K_\Sigma$ is the Gauss curvature of $\Sigma$ and $H = \text{trace of } h_{ij}$ is the mean curvature.

Hence

$$
\int_\Sigma |\nabla \varphi|^2 \geq \frac{1}{2} \int_\Sigma (R_M + \Sigma h^2_{ij} + H^2) \varphi^2 - \int_\Sigma K_\Sigma \varphi^2. \quad (3.3)
$$

Since $\Sigma h^2_{ij} \geq \frac{1}{2} H^2$, we conclude that

$$
\int_\Sigma |\nabla \varphi|^2 \geq \frac{1}{2} \int_M \left( R_M + \frac{3}{2} H^2 \right) \varphi^2 - \int_\Sigma K_\Sigma \varphi^2. \quad (3.4)
$$

If $\chi(\Sigma) \leq 0$, we conclude by choosing $\varphi = 1$, that

$$
\int_\Sigma \left( R_M + \frac{3}{2} c^2 \right) \leq 0. \quad (3.5)
$$
Theorem 3.1. If $R_M + \frac{3}{2}c^2 \geq 0$, any stable orientable surface $\Sigma$ with $\chi(\Sigma) \leq 0$ must have $\chi(\Sigma) = 0$ and $R_M + \frac{3}{2}c^2 = 0$ along $\Sigma$. Furthermore $\Sigma$ must be umbilical.

Let us now see whether stable orientable $\Sigma$ with $\chi(\Sigma) = 0$ can exist or not. If $R_M + \frac{3}{2}c^2 > 0$ at some point of $M$ and $R_M + \frac{3}{2}c^2 \geq 0$ everywhere. We can deform the metric conformally so that $R_M + \frac{3}{2}c^2 > 0$ everywhere while keeping mean curvature of $\partial M$ not less than $c$. (This can be done by arguments of Yamabe problem.) In this case, incompressible torus does not exist.

Hence we may assume $R_M + \frac{3}{2}c^2 = 0$ everywhere. In this case, we deform the metric to $g_{ij} - t(R_{ij} - \frac{R}{3}g_{ij})$. By computation, one sees that unless $R_{ij} - \frac{R}{3}g_{ij}$ everywhere, the (new) scalar curvature will be increased.

Let $\Sigma$ be the stable surface with constant mean curvature with respect to the metric $g_{ij}$. We can deform the surface $\Sigma$ along the normal by multiplying the normal with a function $f$. For this surface $\Sigma_f$, we look at the equation $H_t(\Sigma_f) = c$ where $H_t$ is the mean curvature with respect to the new metric at time $t$. As a function of $t$ and $f$, $H_t(\Sigma_f)$ define a mapping into the Hilbert space of functions on $\Sigma$. The linearized operator with respect to the second (function) variable is $-\Delta - (\text{Ric} (\nu, \nu) + \Sigma h^2_{ij})$. This operator is self-adjoint and if there is no kernel, we can solve the equation $H_t(\Sigma_f) = c$ for $t$ small.

We conclude that if $-\Delta - (\text{Ric} (\nu, \nu) + \Sigma h^2_{ij})$ has no kernel and if the metric is not Einstein, we can keep mean curvature constant and scalar curvature greater than $-\frac{3c^2}{2}$. On the other hand, if the metric is Einstein, we can use argument in [5] to prove that $M$ is the warped product of the flat torus with $R$.

If $-\Delta - (\text{Ric} (\nu, \nu) + \Sigma h^2_{ij})$ has kernel, it must be a positive function $f$ defined on $\Sigma$. (This comes from the fact that it must be the first eigenfunction of the operator.) Hence

$$\Delta(\log f) + |\nabla \log f|^2 = -(\text{Ric} (\nu, \nu) + \Sigma h^2_{ij})$$
$$\leq -\frac{1}{2} \left( R_M + \frac{3}{2}H^2 \right) + K_\Sigma.$$
Since 

\[ \int_{\Sigma} \left( R_M + \frac{3}{2} H^2 \right) \geq 0 \]

and

\[ \int_{\Sigma} K_{\Sigma} = 0, \]

\( f \) must be a constant and

\[ K_{\Sigma} \geq -\frac{1}{2} \left( R_M + \frac{3}{2} H^2 \right) = 0. \]

Hence \( K_{\Sigma} = 0 \) and \( R_M = -\frac{3}{2} H^2 \) is constant along \( \Sigma \). Also \( h_{ij} = \frac{H}{2} g_{ij} \) and \( \text{Ric} (\nu, \nu) + \Sigma h_{ij}^2 = 0 \) along \( \Sigma \).

If we compute the first order deformation of the mean curvature of \( \Sigma \) along the normal, it is trivial as \( h_{ij} = \frac{H}{2} g_{ij} \), \( R = -\frac{3}{2} H^2 \) and \( R(\nu, \nu) = -\Sigma h_{ij} \).

In conclusion, the mean curvature is equal to \( H \) up to first order in \( t \) while we can increase the scalar curvature of \( M \) up to first order (unless \( R_{ij} = \frac{H}{2} g_{ij} \) everywhere). We can therefore prove the following

**Theorem 3.2.** Let \( M \) be a three dimensional complete manifold with scalar curvature not less than \( -\frac{3}{2} c^2 \) and one of the component of \( \partial M \) is an orientable incompressible surface with nonpositive Euler number and mean curvature \( \geq c \). Suppose that for any ball \( B \) in \( M \), the area of \( \partial B \) is not less than \( c \text{Vol} (B) \). Then \( M \) is isometric to the warped product of the flat torus with a half line.

4 Geometry of manifolds with lower bound on scalar curvature.

In this section, we generalize the results of Schoen-Yau [4].

Given a region \( \Omega \) and a Jordan curve \( \Gamma \subset \partial \Omega \) which bounds an embedded disk in \( \Omega \) and a subdomain in \( \partial \Omega \) we define \( R_{\Gamma} \) to be the supremum of \( r > 0 \) so that \( \Gamma \) does not bound a disk inside the tube of \( \Gamma \) with radius \( r < R_{\Gamma} \). We define \( \text{Rad} (\Omega) \) to be the supremum of all such
Let $R$ be the scalar curvature of $M$ and $h$ be a function defined on $M$ and $k$ be a function defined on $\partial M$ so that for any smooth function $\varphi$

$$\int_M |\nabla \varphi|^2 + \frac{1}{2} \int_M R \varphi^2 + \int_{\partial M} k \varphi^2 \geq \int_M h \varphi^2. \quad (4.1)$$

Let $f$ be the positive first eigenfunction of the operator $-\Delta + \frac{1}{2} R - h$ so that

$$\begin{cases}
-\Delta f + \frac{1}{2} R f - h f = \lambda f \\
\frac{\partial f}{\partial \nu} + k f = 0 \quad \text{on} \quad \partial M.
\end{cases} \quad (4.2)$$

Let $\Gamma$ be a Jordan curve on $\partial \Omega$ which defines $\text{Rad}(\Omega)$ up to a small constant. Let $\Sigma$ be a disk in $\Omega$ with boundary $\Gamma$ such that $\Sigma$ together with a region on $\partial \Omega$ bounds a region $\Omega_\Sigma$.

Assume that $\partial \Omega$ has mean curvature $H$ so that $f(H - k)$ is greater than $cf$. Then we define a functional

$$L(\Sigma) = \int_\Sigma f - c \int_{\Omega_\Sigma} f. \quad (4.3)$$

Let us now demonstrate that $\partial \Omega$ forms a “barrier” for the existence of minimum of $L(\Sigma)$.

Let $r$ be the distance function to $\partial \Omega$. Let us assume that $\Gamma$ is in the interior of $\Omega$. If $\Sigma$ touches $\partial \Omega$, we look at the domain $\Omega_\Sigma \cap \{0 < r < \varepsilon\} = \Omega_{\Sigma, \varepsilon}$. Then

$$\int_{\partial(\Omega_{\Sigma, \varepsilon})} f \frac{\partial r}{\partial \nu} = \int_{\Omega_{\Sigma, \varepsilon}} f \Delta r + \int_{\Omega_{\Sigma, \varepsilon}} \nabla f \cdot \nabla r. \quad (4.4)$$

When $\varepsilon$ is small, $f \Delta r + \nabla f \cdot \nabla r$ is close to the boundary value $-\frac{\partial f}{\partial \nu} - H f$ on $\partial \Omega$. Hence

$$\int_{\partial(\Omega_{\Sigma, \varepsilon})} f \frac{\partial r}{\partial \nu} < -\int_{\Omega_{\Sigma, \varepsilon}} cf. \quad (4.5)$$
Since $|\partial r/\partial \nu| \leq 1$ and $\partial r/\partial \nu = 1$ along $r = \epsilon$, we conclude that if we replace $\Sigma$ by $(\Sigma \setminus \partial \Omega_{\Sigma, \epsilon}) \cup (\partial \Omega_{\Sigma, \epsilon} \cap \{r = \epsilon\})$, then the new surface will have strictly less energy than $L_f(\Sigma)$. Hence when we minimize $L$, $\partial \Omega$ forms a barrier.

By standard geometric measure theory, we can find a surface $\Sigma$ which minimize the functional $L_f$. (We start to minimize the functional $\int_\Sigma f - tc\int_{\Omega_\Sigma} f$ when $t$ is small.)

For this surface, we can compute both the first variational and second variational formula and obtain from the first variational formula

$$\frac{\partial f}{\partial \nu} + Hf = cf.$$  \hspace{1cm} (4.6)

The second variational formula has contributions from two terms. The second term gives rise to

$$\int c \left( \frac{\partial f}{\partial \nu} + Hf \right) = c^2 \int f.$$  \hspace{1cm} (4.7)

Using (4.6) the first term of the second variational formula gives

$$0 \leq \int |\nabla \varphi|^2 f$$

$$- \int \left( \frac{1}{2} R_M - K_\Sigma \right) \varphi^2 f - \int \det(h_{ij}) \varphi^2 f$$

$$+ \int \left( \Delta_M f - \Delta_\Sigma f - H \frac{\partial f}{\partial \nu} \right) \varphi^2 - \int (\Sigma h_{ij}^2 - H^2) \varphi$$

$$+ 2 \int \frac{\partial f}{\partial \nu} H \varphi^2 - \int c^2 f \varphi^2$$

$$\leq \int |\nabla \varphi|^2 + \int \left( \Delta_M f - \frac{1}{2} R_M f \right) \varphi^2$$

$$- \int \left( \Delta_\Sigma f - K_\Sigma f \right) \varphi^2 + \frac{1}{4} \int H^2 f \varphi^2 + \int \frac{\partial f}{\partial \nu} H \varphi^2 - c^2 \int f \varphi^2$$

where $\Delta_M$ and $\Delta_\Sigma$ are the Laplacian of $M$ and $\Sigma$ respectively and $\varphi$ is any function vanishing on $\partial \Sigma$. 
We conclude from $\frac{\partial f}{\partial \nu} + H f = cf$ that

$$
\frac{1}{4} \int_{\Sigma} H^2 f \varphi^2 + \int_{\Sigma} \frac{\partial f}{\partial \nu} H \varphi^2 - c^2 \int_{\Sigma} f \varphi^2 
\leq -\frac{3}{4} \int_{\Sigma} H^2 f \varphi^2 + c \int_{\Sigma} H f \varphi^2 - c^2 \int_{\Sigma} f \varphi^2 
\leq -\frac{2}{3} c^2 \int_{\Sigma} f \varphi^2,
$$

(4.9)

$$
\int_{\Sigma} |\nabla \varphi|^2 f - \int_{\Sigma} (\Delta_{\Sigma} f - K_{\Sigma} f) \varphi^2 - \int_{\Sigma} \left( h + \lambda + \frac{2c^2}{3} \right) f \varphi^2 \geq 0
$$

(4.10)

where $\lambda$ is the first eigenvalue of the operator $-\Delta + \frac{R}{2} - h$ with boundary value given by $\frac{\partial f}{\partial \nu} + k f = 0$.

By the argument of [4], we see that for any point $p \in \Sigma$, there exists a curve $\sigma$ from $p$ to $\partial \Sigma$ with length $l$ such that

$$
\int_{0}^{l} \left( h + \lambda + \frac{2c^2}{3} \right) \varphi^2 \leq \frac{3}{2} \int_{0}^{l} (\varphi')^2
$$

(4.11)

where $l$ is the length of the curve $\sigma$ and $\varphi$ vanishes at 0 and $l$.

**Theorem 4.1.** Let $M$ be a three dimensional manifold so that (4.1) holds. Let $\lambda$ be the first eigenvalue of the operator (4.2). Suppose that the mean curvature of $\partial M$ minus $k$ is greater than a constant $c > 0$. Then for any closed curve $\Gamma \subset M$, there is a surface $\Sigma$ that $\Gamma$ bounds in $M$ so that for any point $p \in \Sigma$, there is a curve $\sigma$ from $p$ to $\partial \Sigma$, inequality (4.11) holds.

### 5 Existence of Black Holes

For a general initial data set for the Einstein equation, we have two tensors $g_{ij}$ and $p_{ij}$. The local energy density and linear momentum are given by

$$
\mu = \frac{1}{2} \left[ R - \Sigma p^{ij} p_{ij} + (\Sigma p^k_i) \right]
$$

(5.1)

$$
J^i = \sum_{j} D_j \left[ p^{ij} - (\Sigma p^k_i) g^{ij} \right]
$$
In [2], Schoen and I studied extensively the following equation initiated by Jung

\[
\sum_{i,j} \left( g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2} \right) \left( \frac{D_i D_j f}{(1 + |\nabla f|^2)^{1/2}} - p_{ij} \right) = 0. \tag{5.2}
\]

For the metric

\[
\tilde{g}_{ij} = g_{ij} + \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j},
\]

one has the following inequality

\[
2(\mu - |J|) \leq \bar{R} - \sum_{i,j} (h_{ij} - p_{ij})^2 \tag{5.3}
\]

\[
- 2 \sum (h_{i4} - p_{i4})^2 + 2 \sum D_i (h_{i4} - p_i).
\]

Hence for any function \( \varphi \)

\[
2 \int_M (\mu - |J|) \varphi^2 \leq \int_M \bar{R} \varphi^2 - 2 \int_M \sum (h_{i4} - p_{i4})^2 \tag{5.4}
\]

\[
- \int_M 4 \varphi (\nabla_i \varphi) (h_{i4} - p_{i4}) + 2 \int_{\partial M} (h_{\nu4} - p_{\nu4}) \varphi^2
\]

\[
\leq \int_M \bar{R} \varphi^2 + 2 \int_M |\nabla \varphi|^2 + 2 \int_{\partial M} (h_{\nu4} - p_{\nu4}) \varphi^2.
\]

Hence in (4.1) we can take

\[
h = (\mu - |J|), \quad k = h_{\nu4} - p_{\nu4}. \tag{5.5}
\]

Let \( e_1, e_2, e_3 \) and \( e_4 \) be orthonormal frame of the graph so that \( e_1, e_2 \) is tangential to \( \partial M \) (assuming \( f = 0 \) on \( \partial M \)) and \( e_3 \) is tangential to the graph but normal to \( \partial M \). By assumption,

\[
h_{\nu4} = h_{34} = \langle \nabla_{e_4} e_4, e_3 \rangle. \tag{5.6}
\]

Let \( w \) be the outward normal vector of \( \partial M \) in the horizontal space where \( f = 0 \). Hence

\[
h_{34} = \langle e_4, w \rangle \langle \nabla_w e_4, e_3 \rangle \tag{5.7}
\]

\[
= -\langle e_4, w \rangle \langle e_4, \nabla_w e_3 \rangle
\]

\[
= \frac{-\langle e_4, w \rangle}{\langle e_3, w \rangle} \langle e_4, \nabla_{e_3} e_3 \rangle.
\]
Since $-\sum_{i=1}^{3} \langle e_4, \nabla e_i e_i \rangle$ is the mean curvature of the graph of $f$ which is $\text{tr}p$, we conclude that

$$h_{34} = \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \left( \text{tr}p + \sum_{i=1}^{2} \langle e_4, \nabla e_i e_i \rangle \right)$$

$$= \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \left( \text{tr}p + \langle e_4, w \rangle \sum_{i=1}^{2} \langle w, \nabla e_i e_i \rangle \right).$$

(5.8)

The mean curvature of $\partial \Omega$ with respect to the metric $g_{ij} + \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$ is given by

$$-\sum_{i=1}^{2} \langle \nabla e_i e_i, e_3 \rangle = -\sum_{i=1}^{2} \langle \nabla e_i e_i, w \rangle \langle w, e_3 \rangle.$$

Hence the difference between mean curvature and $k$ is given by

$$-\frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} \text{tr}p + p_{34} + \left( \frac{\langle w, e_3 \rangle + \langle w, e_4 \rangle^2}{\langle w, e_3 \rangle} \right) H_{\partial \Omega}$$

(5.9)

where $H_{\partial \Omega}$ is the mean curvature of $\partial \Omega$ with respect to the metric $g_{ij}$.

Since

$$p_{34} = \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} p(e_3, w)$$

$$= \frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} p(e_3, e_3).$$

We conclude that the expression (5.9) is given by

$$-\frac{\langle e_4, w \rangle}{\langle e_3, w \rangle} (\text{tr} \partial \Omega p) + \frac{1}{\langle e_3, w \rangle} H_{\partial \Omega}$$

(5.10)

$$\geq (H_{\partial \Omega} - |\text{tr} \partial \Omega p|) \langle e_3, w \rangle^{-1}.$$}

We shall assume $H_{\partial \Omega} > |\text{tr} \partial \Omega p|$ and we can choose $c$ to be lower bound of $H_{\partial \Omega} - |\text{tr} \partial \Omega p|$.

We need to solve the Dirichlet problem for $f$ with $f = 0$ on $\partial \Omega$. While most of the estimates were made in [2], we need to construct a barrier for the boundary valued problem.
Let $\varphi$ be an increasing function defined on the interval $[0, \varepsilon]$ so that $\varphi'(\varepsilon) = \infty$. Let $d$ be the distance function from $\partial \Omega$ measured with respect to $g_{ij}$.

Then $\varphi(d)$ can be put in (5.2) and when $\varepsilon$ is small, we obtain the expression

$$\frac{\varphi'}{\sqrt{1 + (\varphi')^2}} \left( -H_{\partial \Omega} - \text{tr}_{\partial \Omega} p \right) + \frac{\varphi''}{(1 + \varphi'^2)^{3/2}} - \frac{p_{\nu \nu}}{1 + \varphi'^2}. \quad (5.11)$$

To construct a supersolution, we need this expression to be nonpositive. When $\varepsilon$ is small, and $\varphi'$ is very large, the condition is simply $H_{\partial \Omega} > \text{tr}_{\partial \Omega} p$. Similarly, we can construct a subsolution using $-\varphi(d)$.

The conclusion is that we can solve (5.2) if $H_{\partial \Omega} > |\text{tr}_{\partial \Omega} p|$. We have therefore arrived at the following conclusion

**Theorem 5.1.** Let $M$ be a space like hypersurface in a four dimensional spacetime. Let $g_{ij}$ be the induced metric and $p_{ij}$ be the second fundamental form. Let $\mu$ and $J$ be the energy density and local linear momentum of $M$. Suppose the mean curvature $H$ of $\partial M$ is greater than $\text{tr}_{\partial M}(p)$. Assume that $H - |\text{tr}_{\partial M}(p)| \geq c \geq 0$. Let $\Gamma$ be a Jordan curvature in $\partial M$ that bounds a domain in $\partial M$. If $M$ admits no apparent horizon, then there exists a surface $\Sigma$ in $M$ bounds by $\Gamma$ so that for any point $p \in \Sigma$, there is a curve $\sigma$ with length $l$ from $p$ to $\Gamma$ and

$$\int_0^l \left( (\mu - |J|) + \frac{3}{2} c^2 \right) \varphi^2 ds \leq \int_0^l |\nabla \varphi|^2 ds \quad (5.12)$$

where $\varphi$ is any function vanishing at 0 and $l$.

**Theorem 5.2.** Let $M$ be a space like hypersurface in a spacetime. Let $g_{ij}$ be its induced metric and $p_{ij}$ be its second fundamental form. Assume that the mean curvature $H$ of $\partial M$ is strictly greater than $|\text{tr}_{\partial M}(p)|$. Let $c = \min(H - |\text{tr}_{\partial M}(p)|)$ if $\text{Rad}(M) \geq \frac{\sqrt{3}}{2} \frac{H}{\Lambda}$ where $\Lambda = \frac{2}{3} c^2 + \mu - |J|$, then $M$ must admit apparent horizons in its interior.

An important point here is that the curvature $H - |\text{tr}_{\partial M}(p)|$ of the boundary itself can give rise to Black Hole.

The inequality actually shows that as long as $\mu - |J| \geq 0$ everywhere, $\frac{2c^2}{3} + \mu - |J|$ to be large in a reasonable ringed region and $\text{Rad}(\Omega)$ is large, an apparent horizon will form in $M$. 


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