Classification of nonproduct states with maximum stabilizer dimension

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Nonproduct $n$-qubit pure states with maximum dimensional stabilizer subgroups of the group of local unitary transformations are precisely the generalized $n$-qubit Greenberger-Horne-Zeilinger states and their local unitary equivalents, for $n \geq 3, n \neq 4$. We characterize the Lie algebra of the stabilizer subgroup for these states. For $n = 4$, there is an additional maximal stabilizer subalgebra, not local unitary equivalent to the former. We give a canonical form for states with this stabilizer as well.

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I. INTRODUCTION

The desire to measure and classify entanglement for multiparty states of $n$-qubit systems has been motivated by potential applications in quantum computation and communication that utilize entanglement as a resource [1, 2]. Entanglement classification is also an interesting question in the fundamental theory of quantum information.

It is natural to consider two states to have the same entanglement type if one is transformable to the other by a local unitary transformation [3, 4]. Thus any entanglement measure, as a function on state space, must be invariant under the action of the local unitary group. One such invariant is the isomorphism class of the stabilizer subgroup (the set of local unitary transformations that do not alter a given state) and its Lie algebra of infinitesimal transformations. Inspired by ideas originally laid out for 3-qubit systems [3, 5], the authors of the present paper have achieved a number of results for systems of arbitrary numbers of qubits on the structure of stabilizer Lie subalgebras and in particular, those stabilizers that have maximum possible dimension. Evidence that maximum stabilizer dimension is an interesting property is the fact that states that have such stabilizers also turn out to maximize other known entanglement measures and play key roles in quantum computational algorithms [6, 7]. In [8] we describe a connection between stabilizer structure and the question of when a pure state is determined by its reduced density matrices.

States with maximum stabilizer dimension are products of singlet pairs (with an unentangled qubit when the number of qubits is odd) and their local unitary equivalents [3, 10]. In the present paper, we show an analogous result for nonproduct states with maximum stabilizer dimension. These are the generalized $n$-qubit Greenberger-Horne-Zeilinger (GHZ) states

$$\alpha |00\cdots0\rangle + \beta |11\cdots1\rangle, \quad \alpha, \beta \neq 0$$

and their local unitary equivalents for 3 or more qubits, together with an additional class of states for the special case of 4 qubits. In order to prove these results, we extend the stabilizer analysis of our previous work [11] from stabilizers of pure states to stabilizers of arbitrary density matrices.

II. THE MAIN RESULT

The $n$-qubit local unitary group is

$$G = G_0 \times G_1 \times \cdots \times G_n$$

where $G_0 = U(1)$ is the group of phases and $G_j = SU(2)$ is the 3-dimensional group of single qubit rotations in qubit $j$ for $1 \leq j \leq n$. Its Lie algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

is the set of infinitesimal local unitary transformations acting on the tangent level, where $\mathfrak{g}_0$ is 1-dimensional and each $\mathfrak{g}_j = su(2)$, the set of skew-Hermitian matrices with trace zero. Given an $n$-qubit state vector $|\psi\rangle$, let

$$\text{Stab}_\psi = \{ g \in G : g |\psi\rangle = |\psi\rangle \}$$

denote the subgroup of elements in $G$ that stabilize $\psi$, and let

$$K_\psi = \{ X \in \mathfrak{g} : X |\psi\rangle = 0 \}$$

be the corresponding Lie algebra.

In [11] we show that the maximum possible dimension for $K_\psi$ is $n - 1$ when $\psi$ is not a product state, for $n \geq 3$. Further, we show that for $n \neq 4$, we must have $\dim P_j K_\psi = 1$ for all $j$, where $P_j : \mathfrak{g} \to \mathfrak{g}_j$ is the natural projection. In fact this condition holds for the generalized $n$-qubit GHZ state

$$\alpha |00\cdots0\rangle + \beta |11\cdots1\rangle$$

and its LU equivalents. For $n = 4$, there is an additional possible condition for $K_\psi$ to attain maximum dimension, and that is $\dim P_j K_\psi = 3$ for all $j$ and $K_\psi \cong su(2)$. The main result of the present paper is the converse of these
previous results. We show that the generalized n-qubit GHZ states (and LU equivalents) are the only nonproduct states that have maximum stabilizer dimension, except for the 4-qubit case. For those 4-qubit states with stabilizer isomorphic to $su(2)$, we give a canonical representative for each LU equivalence class. Here is the formal statement.

**Theorem 1. (Classification of states with maximal stabilizer)** Let $|\psi\rangle$ be a nonproduct n-qubit state with stabilizer subalgebra $K_\psi$ of maximum possible dimension $n - 1$, for some $n \geq 3$. One of two conditions must hold:

(i) $\dim P_j K_\psi = 1$ for $1 \leq j \leq n$, or

(ii) $n = 4$, $\dim P_j K_\psi = 3$ for $1 \leq j \leq n$ and $K_\psi \cong su(2)$.

If (i) holds, then $|\psi\rangle$ is LU equivalent to a generalized n-qubit GHZ state. If (ii) holds, then $|\psi\rangle$ is LU equivalent to a state of the form

$$|\psi\rangle = a(|0011\rangle + |1100\rangle) + b(|0010\rangle + |1001\rangle) + c(|1010\rangle + |0101\rangle)$$

for some complex coefficients $a, b, c$ satisfying $a > 0$, $abc \neq 0$ and $a + b + c = 0$. Furthermore, this is a unique representative of the LU equivalence class for $|\psi\rangle$ of this form.

The proof divides naturally into two sections, one for each of the two conditions on stabilizer in the statement of the theorem. We consider conditions (i) and (ii) in sections [IV] and [V] respectively. It is convenient to work in terms of density matrices and their local unitary stabilizers; the following preliminary section establishes the necessary extension of our previous results to local unitary action on density matrices.

### III. PRELIMINARY PROPOSITIONS ON STABILIZER STRUCTURE

For the set of $n$-qubit pure and mixed density matrices, we may omit the phase factor in local unitary operations, and take the local unitary group to be

$$SU(2)^n = G_1 \times \cdots \times G_n \subset G$$

and its Lie algebra to be

$$su(2)^n = g_1 \oplus \cdots \oplus g_n \subset g.$$  

Given an $n$-qubit density matrix $\rho$, an element $g \in SU(2)^n$ acts on $\rho$ by

$$g \cdot \rho = g \rho g^\dagger.$$  

An element $X \in su(2)^n$ acts on the infinitesimal level by

$$X \cdot \rho = [X, \rho] = X \rho - \rho X.$$  

Let

$$\text{Stab}_\rho = \{g \in SU(2)^n : g \rho g^\dagger = \rho\}$$

denote the subgroup of elements that stabilize $\rho$, and let

$$K_\rho = \{X \in su(2)^n : [X, \rho] = 0\}$$

be the corresponding Lie algebra. We use multi-index notation $I = (i_1, i_2, \ldots, i_n)$, where each $i_k$ is a binary digit, to denote labels for the standard computational basis for state space. We write $I^c$ to denote the bitwise complement of $I$.

One expects a natural correspondence between $K_\rho$ and $K_\psi$, in the case where $\rho = |\psi\rangle \langle \psi|$; and indeed there is.

**Proposition 1.** Let $\rho = |\psi\rangle \langle \psi|$ be a pure n-qubit density matrix. Then $K_\rho = PK_\psi$, where $P : g_1 \oplus \cdots \oplus g_n \rightarrow g_1$ is the natural projection that drops the phase factor.

**Proof.** We begin with a simple observation. Let $X \in g$ and let $|\phi\rangle = X |\psi\rangle$. Skew-Hermiticity of $X$ gives us

$$\langle \psi | X = - \langle \phi | .$$

(1)

To see that $PK_\psi \subset K_\rho$, let $\psi \in K_\psi$, say $\psi = (\langle -it, X_1, X_2, \ldots, X_n \rangle)$. Let $Y = PX$, so $Y |\psi\rangle = it |\psi\rangle$. Then

$$[Y, \rho] = Y |\psi\rangle \langle \psi| - |\psi\rangle \langle \psi| Y = it \rho - it \rho = 0.$$  

To get the second equality, apply (1) to $|\phi\rangle = it |\psi\rangle = Y |\psi\rangle$.

Conversely, we show that $K_\rho \subset PK_\psi$. Let $\rho \in K_\rho$, and let $|\phi\rangle = Y |\psi\rangle$. Using (1) again, we have

$$0 = [Y, \rho] = Y |\psi\rangle \langle \psi - |\psi\rangle \langle \psi| Y$$

$$= \langle \phi | \psi + \langle \psi | \phi = \langle \phi | \psi + (|\phi\rangle \langle \psi|)$$

so $|\phi\rangle \langle \psi|$ is skew-Hermitian. Thus there is some unitary $U$ such that $U |\phi\rangle \langle \psi| U^\dagger$ is diagonal, with pure imaginary eigenvalues. In fact there is only one nonzero diagonal entry, say $is$ for some real $s$, since $|\phi\rangle \langle \psi|$ has rank one. Therefore $U |\phi\rangle = isU |\psi\rangle$, so $|\phi\rangle = is |\psi\rangle$. Thus $Y$ is the projection of $(-is, Y_1, Y_2, \ldots, Y_n)$ in $K_\psi$. This concludes the proof.

Let us use the symbol $P_j$ to denote both the projection $P_j : g \rightarrow g_j$ and also $P_j : su(2)^n \rightarrow g_j$. Applying $P_j$ to both sides of $K_\rho = PK_\psi$ yields the following corollary.

**Corollary 1.** Let $\rho = |\psi\rangle \langle \psi|$ be a pure n-qubit density matrix. Then $P_j K_\rho = P_j K_\psi$ for $1 \leq j \leq n$.

The next proposition establishes that we may work with either $K_\rho$ or $K_\psi$ to calculate stabilizer dimension.

**Proposition 2.** Let $\rho = |\psi\rangle \langle \psi|$ be a pure n-qubit density matrix. Then $\dim K_\rho = \dim K_\psi$.
Proposition 3. Let \( X_{k} = PX_{k} \) for \( 1 \leq k \leq r \) and suppose that
\[
0 = \sum a_{k}Y_{k}
\]
for some scalars \( a_{k} \). Since \( Y_{k} = X_{k} - it_{k} \) for some \( t_{k} \), we have
\[
\sum a_{k}X_{k} = i\sum a_{k}t_{k}.
\]
Since \( \sum a_{k}X_{k} \) is in \( K_{\psi} \), so is \( i\sum a_{k}t_{k} \), and so both of these sums must be zero. Since the \( X_{k} \) are independent, all the \( a_{k} \) must be zero, and hence the \( Y_{k} \) are also linearly independent.

We choose the basis
\[
A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]
for \( su(2) \). We denote by \( A_{j} \) the element in \( g \) that has \( A \) in the \( j \)th qubit slot and zeros elsewhere. By slight abuse of notation, we use the same symbol for the corresponding element in \( su(2)^{n} \). Analogous notation applies for \( B_{j} \) and \( C_{j} \), for \( 1 \leq j \leq n \).

We will make repeated use of the following basic calculation. Let \( X = \sum t_{k}A_{k} \) be an element of the local unitary Lie algebra \( su(2)^{n} \). It is straightforward to check that
\[
X \cdot \rho = [X, \rho] = \sum_{I,J} \zeta(I, J)\rho_{I,J} |I\rangle \langle J|,
\]
where
\[
\zeta(I, J) = i\sum_{\ell=1}^{n} t_{\ell}[(-1)^{\ell} + (-1)^{\ell+1}] = 2i \sum_{\ell: i \neq j} (-1)^{it_{\ell}}.
\]
As a consequence, we have the following.

Proposition 3. Let \( \rho \) be an \( n \)-qubit density matrix, let \( X = \sum t_{k}A_{k} \), and suppose that \( X \in K_{\rho} \). Then either \( \zeta(I, J) = 0 \) or \( \rho_{I,J} = 0 \) for all \( I, J \).

We record here a proposition proved in [10] (Lemma 3.8) with an alternative proof using Proposition 3.

Proposition 4. (Stabilizer criterion for an unentangled qubit) Let \( \rho = |\psi\rangle \langle \psi| \) be a pure \( n \)-qubit density matrix where \( |\psi\rangle = \sum c_{i} |I\rangle \). If \( A_{k} \in K_{\rho} \), then the \( t \)th qubit is unentangled.

Proof. Choose any nonzero state coefficient \( c_{j'} \). Apply Proposition 3 to \( X = A_{I}, I = I' \) and \( J \) for which \( j \neq j' \). Since \( \zeta(I', J) = \pm 2i \neq 0 \), we conclude that \( c_{j} \) must be zero.

Proposition 5. (Stabilizer criterion for generalized \( n \)-qubit GHZ states) Let \( |\psi\rangle = \sum c_{i} |I\rangle \) be a nonproduct \( n \)-qubit state vector for some \( n \geq 3 \), and let
\[
V = \left\{ \sum_{k=1}^{n} t_{k}A_{k} : \sum t_{k} = 0 \right\}.
\]
We have \( V = K_{\psi} \) if and only if \( \psi \) is a generalized \( n \)-qubit GHZ state.

Proof. In Section V of [11] we show that if \( \psi \) is a generalized \( n \)-qubit GHZ state, then \( K_{\psi} = V \). Now we prove the converse.

Suppose that \( K_{\psi} = V \). Choose a multi-index \( I \) such that \( c_{I} \neq 0 \). If \( J \) is another multi-index different from \( I \) in indices \( K = \{k_{1}, \ldots, k_{p}\} \) and equal to \( I \) in some index \( r \), let
\[
X = \sum_{j \in K} (-1)^{i_{j}}A_{j} - \left( \sum_{j \in K} (-1)^{i_{j}} \right) A_{r},
\]
so that the sum of the coefficients of the \( A_{j} \) is zero, so \( X \) is in \( K_{\psi} \). Viewing \( X \) as an element of \( K_{\rho} \) and applying (2), we have
\[
\zeta(I, J) = 2i \sum_{\ell: i \neq j} (-1)^{\ell}t_{\ell} = 2i \sum_{j=r}^{p} (-1)^{2i_{j}} = 2ip \neq 0
\]
so \( \rho_{I,J} = c_{I}c_{J}^{*} = 0 \) by Proposition 3 so \( c_{j} = 0 \). It follows that \( |\psi\rangle \) has the form \( |\psi\rangle = \alpha |I| + \beta |I'\rangle \) for some nonzero \( \alpha, \beta \). Finally, if \( I \) does not consist of all zeros or all ones, say \( t_{k} = 0, t_{k'} = 1 \), then let \( X = A_{k} - A_{k'} \), then \( \zeta(I, I') = 4i \neq 0 \), so \( \rho_{I,I'} \) would have to be zero, but this is impossible. Thus we conclude that \( |\psi\rangle = \alpha |00\cdots 0\rangle + \beta |11\cdots 1\rangle \).

IV. MAXIMAL STABILIZERS WITH 1-DIMENSIONAL PROJECTIONS IN EACH QUBIT

In this section we consider the consequences of condition (i) in the statement of Theorem [11]. By virtue of Proposition 3 and 2, we may work with \( K_{\rho} \) in place of \( K_{\psi} \). If \( \dim P_{j}K_{\rho} = 1 \) for all \( j \), we can apply an LU transformation so that \( P_{j}X = t(X)A_{j} \) for all \( X \in K_{\rho} \), where \( t(X) \) is a scalar that depends on \( X \). After this adjustment, the stabilizer \( K_{\rho} \) is a codimension 1 subspace of the subspace of \( su(2)^{n} \) spanned by \( A_{1}, A_{2}, \ldots, A_{n} \). Thus there is some nonzero real vector \( (m_{1}, m_{2}, \ldots, m_{n}) \) such that
\[
K_{\rho} = \left\{ \sum_{k=1}^{n} t_{k}A_{k} : \sum m_{k}t_{k} = 0 \right\}.
\]
In the following Proposition, we show that the only way for a nonproduct state to have this stabilizer is to be (LU equivalent to) a generalized \( n \)-qubit GHZ state.

Proposition 6. Let \( \rho = |\psi\rangle \langle \psi| \) be a pure nonproduct \( n \)-qubit density matrix with stabilizer subalgebra
\[
K_{\rho} = \left\{ \sum_{k=1}^{n} t_{k}A_{k} : \sum m_{k}t_{k} = 0 \right\}
\]
where \( 0 \neq (m_{1}, m_{2}, \ldots, m_{n}) \in \mathbb{R}^{n} \). Then all the \( m_{j} \) have the same absolute value and \( |\psi\rangle \) is LU equivalent to a generalized \( n \)-qubit GHZ state.
Proof. We may suppose, after a suitable LU transformation, that \( c_{00\ldots0} \neq 0 \) (the LU transformation \( \exp(\pi/2C_j) = I_d \otimes \cdots \otimes I_d \otimes C_j \otimes I_d \otimes \cdots \otimes I_d \) sends \( |I_j\rangle \) to \( |I_j\rangle \), where \( I_d \) is the \( 2 \times 2 \) identity matrix and \( I_j \) denotes the multi-index obtained from \( I \) by complementing the \( j \)th index, and changes the stabilizer element \( \sum_k t_k A_k \) to \( \sum (-1)^{\delta_{jk}} t_k A_k \), where \( \delta_{jk} \) denotes the Kronecker delta). This operation may change the sign of some of the \( m_j \), but does not change their absolute values.

If any \( m_\ell \) = 0, then \( A_\ell \in K_\rho \). By Proposition 3, it follows that the \( \ell \)th qubit is unentangled. But \( |\psi\rangle \) is not a product, so we can rule out the possibility that any \( m_\ell \) is zero.

Suppose now that there exist two coordinates of \( (m_1, \ldots, m_n) \) with different absolute values. Then \( (m_1, \ldots, m_n) \) and \( J = (j_1, \ldots, j_n) \) are linearly independent, where \( J \) is any nonzero vector whose entries are all 0’s and 1’s. Let \( J \) be a multi-index not equal to 00\cdots0 with 1’s in positions \( k_1, \ldots, k_p \). We may choose a vector \( (t_1, \ldots, t_n) \) that is perpendicular to \( (m_1, \ldots, m_n) \) but not perpendicular to \( J \). It follows that \( X = \sum t_k A_k \) lies in \( K_\rho \), but \( \sum t_k = 0 \), so we conclude that \( c_J = 0 \) by Proposition 3. This means that \( |\psi\rangle \) is the completely unentangled state \( |00\cdots0\rangle \). But this is a contradiction, since \( |\psi\rangle \) is not a product. This establishes that all the \( m_j \) have the same absolute value.

Thus \( |\psi\rangle \) has (possibly after an LU transformation) the stabilizer of a generalized \( n \)-qubit GHZ state, and therefore \( |\psi\rangle \) must be a generalized \( n \)-qubit GHZ state by Proposition 5. \( \square \)

V. THE 4-QUBIT CASE

In this section we consider condition (iii) of Theorem 1. It follows from Lemma 1 and Proposition 1 of 11 that after an LU transformation, if necessary, we may take \( K_\rho \) to be

\[
K_\rho = \left( \sum_{k=1}^{4} A_k, \sum_{k=1}^{4} B_k, \sum_{k=1}^{4} C_k \right).
\]

From this assumption, we derive a canonical form for unique representatives of each LU equivalence class.

Proposition 7. (Classification of 4-qubit states with \( su(2) \) stabilizer) Let \( \rho = |\psi\rangle \langle \psi| \) be a pure 4-qubit density matrix, where \( |\psi\rangle = \sum_I c_I |I\rangle \), and let

\[
V = \left( \sum_{k=1}^{4} A_k, \sum_{k=1}^{4} B_k, \sum_{k=1}^{4} C_k \right).
\]

Then \( K_\rho = V \) if and only if

\[
|\psi\rangle = a(|0011\rangle + |1100\rangle) + b(|1001\rangle + |0110\rangle) + c(|1010\rangle + |0101\rangle)
\]

for some complex coefficients \( a, b, c \) satisfying \( a > 0 \), \( abc \neq 0 \) and \( a + b + c = 0 \). Furthermore, this is a unique representative of the LU equivalence class for \( |\psi\rangle \) of this form.

Proof. Suppose that \( K_\rho = V \).

First we consider the consequences of the element \( \sum A_k \in K_\rho \). By Proposition 3, for each \( I, J \) for which \( \rho_{I,J} = c_I c_J^* \neq 0 \), we have

\[
\zeta(I, J) = \sum_{i: i \neq k} (-1)^{i_\ell} = 0.
\]

From this it follows that if \( c_I, c_J \neq 0 \) and we have to flip \( n_0 \) zeros and \( n_1 \) ones to transform \( I \) into \( J \), then \( n_0 = n_1 \). If \( m \) is the number of indices which are zeros in both \( I \) and \( J \), then the number of zeros in \( I \) is \( n_0 + m \), and the number of zeros in \( J \) is \( n_1 + m \). It follows that the number of zeros is constant for all multi-indices \( I \) for which \( c_I \neq 0 \).

Next we consider \( \sum C_k \in K_\rho \). Since \( \exp(\pi/2C) = C \), we have

\[
\rho = \exp(\pi/2 \sum C_k) \cdot \rho = \sum_{I,J} \rho_{I,J} |I\rangle \langle J|,
\]

so

\[
c_I c_J^* = c_I c_J^* \tag{3}
\]

for all \( I, J \). Applying (3) to the pair \( I, I \), we obtain \( |c_I| = |c_I|^\ast \). Applying (3) to the pair \( I, I^\ast \), we see that \( c_I c_J^* \) is real, so it must be that \( c_I = c_J \) for all \( I \). Since the number of zeros is constant for all multi-indices for nonzero state coefficients, it follows that this number must be two. Thus we have so far that \( |\psi\rangle \) must be of the form

\[
|\psi\rangle = a(|0011\rangle + |1100\rangle) + b(|1001\rangle + |0110\rangle) + c(|1010\rangle + |0101\rangle)
\]

for some \( a, b, c \). If \( a = 0 \), then \( A_1 \sim A_2 \) would be in \( K_\rho \), violating our assumption. Similarly, neither \( b \) nor \( c \) can be zero.

To see that \( a + b + c \) must equal zero, note that \( (\sum C_k) \cdot \rho = 0 \) means that \( (\sum C_k) \cdot |\psi\rangle = it |\psi\rangle \) for some \( t \). A straightforward calculation shows that

\[
(\sum C_k) \cdot |\psi\rangle = i \sum_{I} \left( \sum_k c_k \right) |I\rangle,
\]

where \( I_k \) denotes the multi-index obtained from \( I \) by complementing the \( k \)th index. The coefficient of \( |1000\rangle \) on the right hand side is \( (a + b + c) |1000\rangle \), but zero on the left hand side, so \( a + b + c = 0 \). Finally, we may take \( a \) to be positive by applying a phase adjustment, if necessary.

Conversely, suppose that \( \psi \) has the form

\[
|\psi\rangle = a(|0011\rangle + |1100\rangle) + b(|1001\rangle + |0110\rangle) + c(|1010\rangle + |0101\rangle)
\]
for some $a, b, c$, $abc \neq 0$ and $a + b + c = 0$. It is easy to check that $K_\rho$ contains $V$. By our analysis of stabilizer structure in \cite{11}, we have $\dim K_\rho \leq 6$. We know $\dim K_\rho = 6$ if and only if $|\psi\rangle$ is (LU equivalent to) a product of two singlet pairs \cite{10}, but we can rule this out (it is easy to check that $|\psi\rangle$ is not a product). Lemma 1 in \cite{11} rules out the possibility that $\dim K_\rho$ could be 4 or 5, so $\dim K_\rho$ must equal 3 and we must have $K_\rho = V$.

To show that $|\psi\rangle$ is the unique representative of its LU class with the form stated in the Proposition, we use local unitary invariants. First, we consider invariants of the form $\text{tr}((\text{tr}_A |\rho\rangle\langle\rho|)^2)$ where $A$ is a subsystem consisting of a subset of the 4 qubits. From these trace invariants, a straightforward derivation produces invariants $I_1, I_2, I_3$ in the table below.

| Qubit pair in $A$ | Invariant |
|------------------|-----------|
| $(1, 2)$         | $I_1(\psi) = |a|b|$ |
| $(1, 3)$         | $I_2(\psi) = |a|c|$ |
| $(1, 4)$         | $I_3(\psi) = |b|c|$ |

It follows that the norms of the state coefficients for $|\psi\rangle$ are LU invariant (for example, we have $|a| = \sqrt{I_1(\psi)I_2(\psi)/I_3(\psi)}$). Since we are taking $a$ to be positive, and since $a + b + c = 0$, a simple geometric or algebraic argument shows that $b$ and $c$ are determined up to conjugate.

To determine the imaginary parts of $b$ and $c$, we use a class of invariants given by

$$P_{\sigma,\tau,\phi}^m(\psi) = \sum_{I_1, I_2, \ldots, I_m} c_{I_1} \cdots c_{I_m} c_{I_1}^* \cdots c_{I_m}^*,$$

where $I_1, \ldots, I_m$ is an $m$-tuple of 4-qubit multi-indices $I^k = (i^k_1, \ldots, i^k_4)$, and

$$J^k = (i^k_{\sigma(1)}, i^k_{\tau(1)}, i^k_{\phi(1)})$$

for $1 \leq k \leq m$. We consider the invariant for $m = 3$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, and $\phi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The imaginary part of this invariant acting on $|\psi\rangle$ is

$$-24a^2 b_1 b_2 (b_2^2 + b_1^2 + ab_1)$$

where $b_1, b_2$ are the real and imaginary parts of $b$, respectively. From this we see that if $b_1 \neq 0$, then $b_2$ is determined. If $b_1 = 0$, then $b_2$ is not completely determined; there remains the ambiguity of conjugation for $b$ and $c$. To settle this, let $\alpha(r, s, t)$ denote the state

$$r(0, 0, 1) + |1, 0, 0\rangle$$

$$s(|0, 1, 0\rangle + |1, 0, 0\rangle)$$

$$t(|0, 1, 0\rangle + |1, 0, 0\rangle)$$

for state coefficients $r, s, t$. The (not local unitary) operation that permutes qubits 3 and 4 takes $\alpha(a, b, c)$ to $\alpha(a, c, b)$. If $b$ is pure imaginary, then $c$ is not, so our invariant $P_{\sigma,\tau,\phi}^3$ tells us that $\alpha(a, c, b)$ is not LU equivalent to $\alpha(a, c^*, b^*)$. It follows that $\alpha(a, b, c)$ is not LU equivalent to $\alpha(a, b^*, c^*)$, for if the LU operation $g_1 \otimes g_2 \otimes g_3 \otimes g_4$ takes $\alpha(a, b, c)$ to $\alpha(a, b^*, c^*)$, then the LU operation $g_1 \otimes g_2 \otimes g_4 \otimes g_3$ takes $\alpha(a, c, b)$ to the LU inequivalent state $\alpha(a, c^*, b^*)$.

Having exhausted all cases, we see that $|\psi\rangle$ is the unique state in its LU equivalence class of the form stated in the Proposition.

\[\square\]

VI. CONCLUSION

We have described the stabilizer subalgebra structure and have a complete description of states with stabilizers of maximum dimension, both for product and nonproduct states. We have shown evidence that warrants further study of stabilizer structure. There are at least three natural directions in which to further pursue this analysis: seek complete descriptions, perhaps a classification, of all possible stabilizer subalgebras; seek complete description and perhaps canonical LU forms in the spirit of Proposition \[\square\] for states that have those stabilizers; and extend these pursuits to mixed states.

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