Decoherence of two qubits coupled with one-mode cavity without rotating-wave approximation

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The decoherence of two qubits, coupled with one-mode cavity separately, has been investigated exactly. The results show that, for the resonant case, the decoherence behavior of system is similar to Markovian case when the coupling strength is weak, while the concurrence vanishes in finite time and might recover fractional initial entanglement before it permanently vanishes when the coupling strength is strong. And for detuning case, the entanglement could periodically recover after a period of time from its disappearance. These results are quite different from that of system subjected to Jaynes-Cummings model.

I. INTRODUCTION

Recently, many works have been devoted to investigating the decoherence behavior of entangled qubits, invoked by the phenomenon, termed as “entanglement sudden death” (ESD) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. It is shown that the interaction, between qubits with environment, could lead to finite-time disentanglement, which is quite different from the case of continuous variable two-atom model discussed by Dodd and Halliwell [1, 16].

The decoherence of two atoms, coupled with one-mode cavity separately, has been investigated in Ref. [3] with Jaynes-Cummings model, which neglects counter rotating terms corresponding the emission and absorption of virtual photon without energy conservation, and is widely used in quantum optics [17, 18, 19]. Generally, the rotating-wave approximation (RWA), which neglecting counter rotating, is justified for small detunings and small ratio of the atom-field coupling divided by the atomic transition frequency. In atom-field cavity systems, this ratio is typically of the order $10^{-7} \sim 10^{-6}$. Recently, cavity systems with very strong couplings have been discussed [20]. In solid state systems, the ratio may become so large as to consider the effect of counter-rotating wave terms, which are neglected in Ref. [3], on the decoherence behavior of two atoms coupled with one-mode cavity separately.

In section II the reduced non-perturbative quantum master equation of atom has been derived and its exact solution is obtained. In section III the decoherence of two initially entangled qubits has been discussed. The conclusion will be given in section IV.

II. MODEL AND EXACT SOLUTION

A. non-perturbative master equation and its Exact solution

Now we restrict our attention to two noninteracting two-level atoms A and B, which are in a perfect one-mode cavity, respectively. First, we consider the subsystem Hamiltonian of one atom coupled to one cavity field mode as

$$H = H_a + H_f + H_{af}$$

where

$$H_a = \omega_0 \frac{\sigma_z}{2}$$

$$H_f = \omega a^\dagger a$$

$$H_{af} = g(\sigma_+ + \sigma_-)(a^\dagger + a)$$

where $\omega_0$ is the atomic transition frequency between the ground state $|0\rangle$ and excited state $|1\rangle$. $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are pseudo-spin operators of atom. $a^\dagger$ and $a$ are creation and annihilation operators of

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the cavity field mode corresponding frequency $\omega$. And $g$ is the coupling constant between the transition $|1\rangle - |0\rangle$ and the field mode.

The reduced non-perturbative quantum master equation of atom could be obtained by path integrals

$$\frac{\partial}{\partial t} \rho_a = -iL_0 \rho_a - \int_0^t ds \langle L_{af} e^{-iL_0(t-s)} L_{af} e^{-iL_{af}(s-t)} \rangle f \rho_a$$

(5)

where $L_0$, $L_a$ and $L_{af}$ are Liouvillian operators defined as

$$L_0 \rho = [H_a + H_f, \rho]$$
$$L_a \rho = [H_a, \rho]$$
$$L_{af} \rho = [H_{af}, \rho]$$

and $\langle \cdots \rangle_f$ stands for partial trace of cavity mode.

If the cavity field is initially in vacuum state, the non-perturbative reduced master equation of the atom could be derived from Eq.(5)

$$\frac{\partial}{\partial t} \rho_a = -g^2 (\alpha^R + f(t)) \rho_a - 2i (\omega_0 - g^2 \alpha^I + g^2 f^I(t)) J_0 \rho_a + g^2 (\alpha^* + f(t)) J_+ \rho_a + 2g^2 \alpha^R K_+ \rho_a$$

$$+ 2g^2 (\alpha^R - f^R(t)) K_0 \rho_a + 2g^2 f^R(t) K_- \rho_a$$

(6)

Where $J_0$, $J_+$, $J_-$, $K_0$, $K_+$ and $K_-$ are superoperators defined as

$$J_0 \rho_a = \left[ \frac{\sigma_+}{4}, \rho_a \right]$$
$$J_+ \rho_a = \sigma_+ \rho_a \sigma_+$$
$$J_- \rho_a = \sigma_- \rho_a \sigma_-$$
$$K_0 \rho_a = (\sigma_+ \sigma_- \rho_a + \rho_a \sigma_+ \sigma_- - \rho_a) / 2$$
$$K_+ \rho_a = \sigma_+ \rho_a \sigma_-$$
$$K_- \rho_a = \sigma_- \rho_a \sigma_+$$

and

$$\alpha = \frac{1 - \exp(-i\Delta t)}{i\Delta}$$
$$f(t) = \frac{\exp(i\delta t) - 1}{i\delta}$$

(7)

(8)

where $\Delta = \omega + \omega_0$, $\delta = \omega_0 - \omega$, $\alpha^R$, $\alpha^I$, $\alpha^*$ and $f^R$, $f^I$, $f^*$ are real part, image part and conjugate of $\alpha$ and of $f(t)$, respectively.

Using algebraic approach, the formal solution of Eq.(6) is obtained

$$\rho_a(t) = \exp(-\Gamma_k) \hat{T} \exp \left[ \int_0^t dt (\epsilon_0 J_0 + \epsilon_+ J_+ + \epsilon_- J_-) \right]$$
$$\times \hat{T} \exp \left[ \int_0^t dt (\nu_0 K_0 + \nu_+ K_+ + \nu_- K_-) \rho_a(0) \right]$$

(9)

where $\hat{T}$ is time ordering operator, $\epsilon_0 = -2i (\omega_0 - g^2 \alpha^I + g^2 f^I)$, $\epsilon_+ = g^2 (\alpha + f^*)$, $\epsilon_- = g^2 (\alpha^* + f)$, $\nu_0 = 2g^2 (\alpha^R - f^R)$, $\nu_+ = 2g^2 \alpha^R$, $\nu_- = 2g^2 f^R$, $\Gamma_k = g^2 (\alpha^R + f^R)$ and

$$\hat{\alpha} = \int_0^t \alpha dt = \frac{1 - \exp(-i\Delta t) - i\Delta t}{\Delta^2} \equiv \hat{\alpha}^R + i\hat{\alpha}^I$$
$$F = \int_0^t f(t) dt = \frac{1 + i\delta t - \exp(i\delta t)}{\delta^2} \equiv F^R + iF^I$$

(10)
where $\tilde{\alpha}^R$, $\tilde{\alpha}^I$, $\tilde{\alpha}^*$ and $F^R$, $F^I$, $F^*$ are real part, image part and conjugate of $\tilde{\alpha}$ and of $F$, respectively. Using the decomposition of SU(2) operator, the exact solution of master equation Eq. (6) is obtained

$$\rho_0(t) = e^{-\Gamma t} \tilde{\rho}(t)$$

$$\tilde{\rho}(t) = 
\begin{pmatrix}
1\rho_a^{11}(0) + m\rho_a^{00}(0) & x\rho_a^{10}(0) + y\rho_a^{01}(0) \\
q\rho_a^{01}(0) + r\rho_a^{10}(0) & n\rho_a^{20}(0) + p\rho_a^{02}(0)
\end{pmatrix}$$

$$l = e^{j_0/2} + e^{-j_0/2}k_+ k_-, \quad m = e^{-j_0/2}k_+$$

$$n = e^{-j_0/2}, \quad p = e^{-j_0/2}k_-$$

$$q = e^{-j_0/2}, \quad r = e^{-j_0/2}j_-$$

$$x = e^{j_0/2} + e^{-j_0/2}j_+ j_-$$

where $j_+, j_0, j_-$ and $k_+, k_0, k_-$ satisfy the following equation

$$\dot{X}_+ = \mu_+ - \mu_- X_+ + \mu_0 X_+$$

$$\dot{X}_0 = \mu_0 - 2\mu_- X_+$$

$$\dot{X}_- = \mu_- \exp(X_0)$$

$\mu = \varepsilon$ for $X = j$ and $\mu = \nu$ for $X = k$.

## B. Concurrence

Throughout the paper, we use Wootters concurrence. For simplicity, we assume that the two subsystems have the same parameters. The concurrence of the whole system could be obtained

$$C_\xi = \max \{0, c_1, c_2\}, (\xi = \Phi, \Psi)$$

$$c_1 = 2e^{-2\Gamma_\xi} (\sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}})$$

$$c_2 = 2e^{-2\Gamma_\xi} (\sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}})$$

corresponding to the initial states of $|\Phi\rangle = \beta|01\rangle + \eta|10\rangle$ and $|\Psi\rangle = \beta|00\rangle + \eta|11\rangle$, respectively. Where $\beta$ is real and $0 < \beta < 1$, $\eta = |\eta|e^{i\phi}$ and $\beta^2 + |\eta|^2 = 1$. The reduced joint density matrix of the two atoms, in the standard product basis $B = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$, could be written as

$$\rho_{AB} = e^{-2\Gamma_\xi} \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix}$$

here the diagonal elements are

$$\rho_{11} = l^2\rho_{11}(0) + lm\rho_{22}(0) + ml\rho_{33}(0) + m^2\rho_{44}(0)$$

$$\rho_{22} = l^2\rho_{11}(0) + lm\rho_{22}(0) + mp\rho_{33}(0) + mn\rho_{44}(0)$$

$$\rho_{33} = l^2\rho_{11}(0) + pm\rho_{22}(0) + ml\rho_{33}(0) + mn\rho_{44}(0)$$

$$\rho_{44} = p^2\rho_{11}(0) + pm\rho_{22}(0) + np\rho_{33}(0) + n^2\rho_{44}(0)$$

and the nondiagonal elements are

$$\rho_{14} = x^2\rho_{14}(0) + xy\rho_{23}(0) + yx\rho_{32}(0) + y^2\rho_{41}(0)$$

$$\rho_{23} = x\rho_{14}(0) + xq\rho_{23}(0) + yr\rho_{32}(0) + yq\rho_{41}(0)$$

$$\rho_{32} = rx\rho_{14}(0) + ry\rho_{23}(0) + qx\rho_{32}(0) + qy\rho_{41}(0)$$

$$\rho_{41} = r^2\rho_{14}(0) + r\rho_{23}(0) + qr\rho_{32}(0) + q^2\rho_{41}(0)$$
In order to compare the results with that of two Jaynes-Cummings atoms in Ref. [3], we primarily investigate the decoherence for resonant case $\omega = \omega_0$.

First, we focus on the decoherence of two qubits with initial state of $|\Phi\rangle$. For the RWA model in Ref. [3], Fig. 1 shows that the concurrence periodically vanishes and revives. And the change of $C_\Phi$ against $\beta^2$ is symmetrical because of the symmetry of the initial state $|\Phi\rangle$. The decoherence, of the non-RWA model in this paper, has been shown in Fig. 2.

(A) As $\omega_0 = 1.5g$, Fig. 2(a) reveals that the concurrence $C_\Phi$ decreases to zero in a finite time, vanishes for a period of time, revives with small amplitude and then vanishes permanently. This characteristic will hold on for more stronger coupling constant.

(B) From Fig. 2(b), we could find that, when $\omega_0 = 3g$, the concurrence $C_\Phi$ first decreases to a certain value and maintain it for a period of time before it vanishes, like a rumple, then it revives with a smaller amplitude than that in Fig. 2(a) and vanishes permanently. As the coupling constant $g$ decreases, there are more rumples and a smaller amplitude revival.

(C) When $\omega_0 = 30g$, Fig. 2(c) exhibits that the concurrence $C_\Phi$ decreases monotonically and exponentially to zero without revival of entanglement. This characteristic will hold on for more weaker coupling constant.

Then, we focus on the decoherence of two qubits with initial state of $|\Psi\rangle$. For the RWA model in Ref. [12], Fig. 3 shows that the entanglement represented by $C_\Psi$ can fall abruptly to zero, and will remain zero for a period of time before it recovers. The length of time interval for the zero entanglement is dependent on the degree of initial entanglement. The time interval for $\beta^2 < 1/2$ is longer than that for $\beta^2 > 1/2$. There are periodical disappearance and revival of entanglement in time scale. For the non-RWA model, the decoherence of the system was also categorized into three cases

(A) When $\omega_0 = 1.5g$, Fig. 4(a) reveals that, for $\beta^2 > 1/2$, the concurrence $C_\Psi$ decreases exponentially to zero, remain zero for a period of time, revives fractional initial entanglement and then vanishes permanently, while $C_\Phi$ vanishes permanently after a finite time for $\beta^2 < 1/2$, similar to the Markovian case [1].

(B) As $\omega_0 = 3g$, Fig. 4(b) shows that the entanglement represented by $C_\Psi$ has a similar behavior to that of $C_\Phi$ for
\( \beta^2 > 1/2 \) in Fig.2(b). In contrast, for \( \beta^2 < 1/2 \), it decreases smoothly to zero at finite time, then vanishes permanently after a small amplitude entanglement revival link that for \( \beta^2 > 1/2 \).

(C) When \( \omega_0 = 30g \), Fig.4(c) exhibits that it first decreases to zero at short time, then vanishes permanently after a small amplitude entanglement revival. Unlike the two cases above, the evolution behavior of concurrence \( C_\Psi \) becomes symmetric, like that of \( C_\Phi \), because of the strong interaction of atom with reservoir through the emission and absorption of virtual photon.

From Fig.2 and Fig.4, we find that the entanglement will decreases to zero finally for no-RWA and there are no periodical disappearance and revival of entanglement, like that in Fig.1 and Fig.3 for RWA case. It is the enhancement of spontaneous emission, as an atom resonantly coupled with a cavity, leads to the disappearance of entanglement [25]. We could conclude that the neglect of counter-rotating wave terms in Hamiltonian of RWA model leads to the different characteristics of \( C_\phi \).

Finally, we investigate whether there are periodical disappearance and revival of entanglement for detuning case. Concurrence \( C_\Phi \) and \( C_\Psi \) for detuning case as a function of \( gt \) and \( \beta^2 \) are shown in Fig.5 and Fig.6 respectively.

From Fig.5 and Fig.6, we could find that the initial entanglement will recover after a period of time from its disappearance because the spontaneous emission could be greatly inhibited, as an atom non-resonantly coupled with cavity mode [25]. The revival time interval is dependent on the detuning \( \delta \). The bigger the detuning is, the shorter the time interval is. We also find that the entanglement will change little when the detuning is large and coupling strength is weak.

**IV. CONCLUSION**

The decoherence of two initially entangled atoms, coupled with two one-mode cavities separately, has been discussed exactly. The results show that the decoherence behavior of two qubits is dependent on the coupling strength and the
FIG. 5: Concurrence $C_\Psi$ and its contour for non-resonant case as a function of $gt$ and $\beta^2$ with $\omega_0 = 10g$, $\delta = 0.1\omega_0$.

FIG. 6: Concurrence $C_\Psi$ and its contour for non-resonant case as a function of $gt$ and $\beta^2$ with $\omega_0 = 10g$, $\delta = 0.1\omega_0$.

detuning between atom transition frequency and the cavity mode.

Firstly, there are no periodical disappearance and revival of entanglement for resonant case like that for Jaynes-Cummings model in Ref. [3]. Secondly, for detuning case, the entanglement could periodically recover after a period of time from its disappearance. Thirdly, for resonant case, the decoherence behavior of system is similar to Markovian case when the coupling strength is weak, while the concurrence will vanishes in finite time and might recover fractional initial entanglement before it permanently vanishes when the coupling strength is strong.

The results also exhibit that the RWA in Hamiltonian leads to the existence of revival of entanglement for resonant case and it might be improper to take RWA when the interaction between atom and external field is correlated and strong.

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