A \textit{q}-ANALOGUE OF DE FINETTI’S THEOREM

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Abstract. A \textit{q}-analogue of de Finetti’s theorem is obtained in terms of a boundary problem for the \textit{q}-Pascal graph. For \textit{q} a power of prime this leads to a characterisation of random spaces over the Galois field \(\mathbb{F}_q\) that are invariant under the natural action of the infinite group of invertible matrices with coefficients from \(\mathbb{F}_q\).

1. Introduction

The infinite symmetric group \(\mathfrak{S}_\infty\) consists of bijections \(\{1, 2, \ldots\} \rightarrow \{1, 2, \ldots\}\) which move only finitely many integers. The group \(\mathfrak{S}_\infty\) acts on the product space \(\{0, 1\}^\infty\) by permutations of the coordinates. A random element of this space, that is a random infinite binary sequence, is called exchangeable if its probability law is invariant under the action of \(\mathfrak{S}_\infty\). De Finetti’s theorem asserts that every exchangeable sequence can be generated in a unique way by the following two-step procedure: first choose at random the value of parameter \(p\) from some probability distribution on the unit interval \([0, 1]\), then run an infinite Bernoulli process with probability \(p\) for 1’s.

One approach to this classical result, as presented in Feller [3, Ch. VII, §4], is based on the following exciting connection with the Hausdorff moment problem. By exchangeability, the law of a random infinite binary sequence is determined by the array \((v_{n,k})\), where \(v_{n,k}\) equals the probability of every initial sequence of length \(n\) with \(k\) 1’s. The rule of addition of probabilities yields the backward recursion

\[ v_{n,k} = v_{n+1,k} + v_{n+1,k+1}, \quad 0 \leq k \leq n, \quad n = 0, 1, \ldots, \]

which readily implies that the array can be derived by iterated differencing of the sequence \((v_{n,0})_{n=0,1,\ldots}\). Specifically, setting

\[ u^{(k)}_l = v_{l+k,k}, \quad l = 0, 1, \ldots, \quad k = 0, 1, \ldots, \]

and denoting by \(\delta\) the difference operator acting on sequences \(u = (u_l)_{l=0,1,\ldots}\) as

\[ (\delta u)_l = u_l - u_{l+1}, \]

the recursion (1) can be written as

\[ u^{(k)} = \delta u^{(k-1)}, \quad k = 1, 2, \ldots. \]
Since \( v_{n,k} \geq 0 \), the sequence \( u^{(0)} \) must be completely monotone, that is, componentwise
\[
\delta \circ \cdots \circ \delta u^{(0)} \geq 0, \quad k = 0, 1, \ldots,
\]
but then Hausdorff’s theorem implies that there exists a representation
\[
v_{n,k} = u_{n-k}^{(k)} = \int_{[0,1]} p^k (1-p)^{n-k} \mu(d\rho)
\]
with uniquely determined probability measure \( \mu \). De Finetti’s theorem follows since \( v_{n,k} = p^k (1-p)^{n-k} \) for the Bernoulli process with parameter \( p \). See [1] for other proofs and extensive survey of generalisations of this result.

The present note is devoted to variations on the \( q \)-analogue of de Finetti’s theorem, which was briefly outlined in Kerov [9] within the framework of the boundary problem for generalised Stirling triangles. The boundary problem for other weighted versions of the Pascal triangle was studied in [4], [6], and for more general graded graphs in [5], [9], [10].

**Definition 1.1.** Given \( q > 0 \), let us say that a random binary sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in \{0, 1\}^\infty \) is \( q \)-exchangeable if its probability law \( P \) is \( S_\infty \)-quasiconvex with a specific cocycle, which is uniquely determined by the following condition: Denoting by \( P(\varepsilon_1, \ldots, \varepsilon_n) \) the probability of an initial sequence \( (\varepsilon_1, \ldots, \varepsilon_n) \), we have for any \( i = 1, \ldots, n-1 \)
\[
P(\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i, \varepsilon_{i+2}, \ldots, \varepsilon_n) = q^{\varepsilon_i - \varepsilon_{i+1}} P(\varepsilon_1, \ldots, \varepsilon_n).
\]
In words: under an elementary transposition of the form \((\ldots, 1, 0, \ldots) \rightarrow (\ldots, 0, 1, \ldots)\), probability is multiplied by \( q \).

**Theorem 1.2.** Assume \( 0 < q < 1 \). There is a bijective correspondence \( P \leftrightarrow \mu \) between the probability laws \( P \) of infinite \( q \)-exchangeable binary sequences and the probability measures \( \mu \) on the closed countable set
\[
\Delta_q := \{1, q, q^2, \ldots\} \cup \{0\} \subset [0, 1].
\]

More precisely, a \( q \)-exchangeable sequence can be generated in a unique way by first choosing at random a point \( x \in \Delta_q \) distributed according to \( \mu \) and then running a certain \( q \)-analogue of the Bernoulli process indexed by \( x \). Each law \( P \) is uniquely determined by the infinite triangular array
\[
v_{n,k} := P(1, \ldots, 1, 0, \ldots, 0), \quad 0 \leq k \leq n < \infty,
\]
which in turn is given by a \( q \)-version of formula (4), with \( \Delta \) being replaced by \( \Delta_q \) (Theorem 3.2). A similar result with switching the roles of 0’s and 1’s and replacing \( q \) by \( q^{-1} \) also holds for \( q > 1 \).

The rest of the paper is organized as follows. In Section 2 we introduce the \( q \)-Pascal graph and formulate the \( q \)-exchangeability in terms of certain Markov chains on this graph. In Section 3 we find a characteristic recursion for the numbers (5), which is a \( q \)-deformation of (1), and we prove the main result, equivalent to Theorem 1.2, using the
method of [10]. In Section 4 we discuss three examples: two $q$-analogues of the Bernoulli process and a $q$-analogue of Pólya’s urn process. Finally, in Section 5 for $q$ a power of a prime number, we provide an interpretation of the theorem in terms of random subspaces in an infinite-dimensional vector space over $\mathbb{F}_q$.

2. The $q$-Pascal Graph

For $q > 0$, the $q$-Pascal graph is a weighted directed graph $\Gamma(q)$ on the infinite vertex set

$$\Gamma = \{(l,k) : l,k = 0,1,\ldots\}.$$  

Each vertex $(l,k)$ has two weighted outgoing edges $(l,k) \to (l+1,k)$ and $(l,k) \to (l,k+1)$ with weights $1$ and $q^l$, respectively. The vertex set is divided into levels $\Gamma_n = \{(l,k) : l+k = n\}$, so $\Gamma = \bigcup_{n\geq0} \Gamma_n$ with $\Gamma_0$ consisting of the sole root vertex $(0,0)$. For a path in $\Gamma$ connecting two vertices $(l,k) \in \Gamma_{l+k}$ and $(\lambda,\kappa) \in \Gamma_{\lambda+\kappa}$ we define the weight to be the product of weights of edges along the path. For instance, the weight of $(2,3) \to (2,4) \to (3,4) \to (3,5)$ is $q^5 = q^2 \cdot 1 \cdot q^3$. Clearly, such a path exists if and only if $\lambda \geq l$, $\kappa \geq k$.

We shall consider certain transient Markov chains $S = (S_n)$, with state-space $\Gamma$, which start at the root $(0,0)$ and move along the directed edges, so that $S_n \in \Gamma_n$ for every $n = 0,1,\ldots$. Thus, a trajectory of $S$ is an infinite directed path in $\Gamma$ started at the root.

**Definition 2.1.** Adopting the terminology introduced by Vershik and Kerov (see [9]), we say that a Markov chain $S$ on $\Gamma(q)$ is central if the following condition is satisfied for each vertex $(n-k,k) \in \Gamma_n$ visited by $S$ with positive probability: given $S_n = (n-k,k)$, the conditional probability that $S$ follows each particular path connecting $(0,0)$ and $(n-k,k)$ is proportional to the weight of the path.

**Remark 2.2.** If we only require the centrality condition to hold for all $(l,k) \in \Gamma_\nu$ for fixed $\nu$, then we have it satisfied also for all $(l,k)$ with $l+k \leq \nu$. From this it is easy to see that the centrality condition implies the Markov property of $S$ in reversed time $n = \ldots, 1, 0$, hence also implies the Markov property in forward time $n = 0,1,\ldots$.

In the special case $q = 1$ Definition 2.1 means that in the Pascal graph $\Gamma(1)$ all paths with common endpoints are equally likely.

Recall a bijection between the infinite binary sequences $(\varepsilon_1,\varepsilon_2,\ldots)$ and infinite directed paths in $\Gamma$ started at the root $(0,0)$. Specifically, given a path, the $n$th digit $\varepsilon_n$ is given the value $0$ or $1$ depending on whether $l$ or $k$ coordinate is increased by $1$. Identifying a path with a sequence $(n-K_n,K_n)$ (where $0 \leq K_n \leq n$), the correspondence can be written as

$$K_n = \sum_{j=1}^{n} \varepsilon_j, \varepsilon_n = K_n - K_{n-1}, \ n = 1,2,\ldots$$

**Proposition 2.3.** By virtue of the bijection between $\{0,1\}^\infty$ and the paths in $\Gamma$, each $q$-exchangeable sequence corresponds to a central Markov chain on $\Gamma(q)$, and vice versa.
Proof. This follows readily from Remark 2.2, Definitions 1.1 and 2.1 and the structure of $\Gamma(q)$. □

We shall use the standard notation

$$[n] := 1 + q + \ldots + q^{n-1}, \ [n]! := [1] \cdot [2] \cdots [n], \ \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

for $q$-integers, $q$-factorials and $q$-binomial coefficients, respectively, with the usual convention that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < 0$ or $k < 0$. Furthermore, we set

$$(x, q)_k := \prod_{i=0}^{k-1} (1 - xq^i), \ 1 \leq k \leq \infty,$$

with the infinite product $(k = \infty)$ considered for $0 < q < 1$.

The following lemma justifies the name of the graph by relating it to the $q$-Pascal triangle of $q$-binomial coefficients.

**Lemma 2.4.** The sum of weights of all directed paths from the root $(0, 0)$ to a vertex $(n-k, k)$, denoted $d_{n,k}$, is given by

$$d_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}.$$  \hspace{1cm} (6)

More generally, $d_{n,k}^{\nu, \kappa}$, the sum of weights of all paths connecting two vertices $(n-k, k)$ and $(\nu - \kappa, \kappa)$ in $\Gamma$ is given by

$$d_{n,k}^{\nu, \kappa} = q^{(\kappa - k)(n-k)} \begin{bmatrix} \nu - n \\ \kappa - k \end{bmatrix}.$$  \hspace{1cm} (8)

Proof. Note that any path from $(0, 0)$ to $(n-k, k)$ has the second component incrementing by 1 on some $k$ edges $(l_i, i-1) \to (l_i, i)$, where $i = 1, 2, \ldots, k$ and $0 \leq l_1 \leq \cdots \leq l_k \leq n-k$, thus the sum of weights is equal to

$$d_{n,k} = \sum_{0 \leq l_1 \leq \cdots \leq l_k \leq n-k} q^{l_1 + \cdots + l_k}. \hspace{1cm} (7)$$

This array satisfies the recursion

$$d_{n,k} = q^{n-k}d_{n-1,k-1} + d_{n-1,k}, \ 0 < k < n$$

with the boundary conditions $d_{n,0} = d_{n,n} = 1$. On the other hand, it is well known that the array of $q$-binomial coefficients also satisfies this recursion [8], hence by the uniqueness $d_{n,k}$ is the $q$-binomial coefficient. In the like way the sum of weights of paths from $(n-k, k)$ to $(\nu - \kappa, \kappa)$ is

$$d_{n,k}^{\nu, \kappa} = \sum_{n-k \leq l_1 \leq \cdots \leq l_k \leq \nu - \kappa} q^{l_1 + \cdots + l_k}, \ k' := \kappa - k.$$
Comparing with (7) we see that this is equal to \( q^{(n-k)k} \left\lfloor \frac{\nu - \kappa}{k'} \right\rfloor \).

**Remark 2.5.** Changing \((l, k)\) to \((k, l)\) yields the dual \(q\)-Pascal graph \(\Gamma^*(q)\), which has the same set of vertices and edges as \(\Gamma(q)\), but different weights: the edge \((l, k) \rightarrow (l, k + 1)\) has now weight 1, and the edge \((l, k) \rightarrow (l + 1, k)\) has weight \(q^k\). The sum of weights of paths in \(\Gamma^*\) from \((0, 0)\) to \((l, k)\) is again (6), which is related to another recursion for \(q\)-binomial coefficients, \(d_{n,k} = d_{n-1,k-1} + q^k d_{n-1,k}\).

Consider the recursion

\[ v_{n,k} = v_{n+1,k} + q^{n-k} v_{n+1,k+1}, \quad \text{with } v_{0,0} = 1, \tag{9} \]

which is dual to (8), and denote by \(V\) the set of nonnegative solutions to (9).

**Proposition 2.6.** Formula

\[ P\{S_n = (n-k, k)\} = d_{n,k} v_{n,k}, \quad (n-k, k) \in \Gamma \]

establishes a bijective correspondence \(P \leftrightarrow v\) between the probability laws of central Markov chains \(S = (S_n)\) on \(\Gamma(q)\) and solutions \(v \in V\) to recursion (9).

**Proof.** Let \(S\) be a central Markov chain on \(\Gamma\) with probability law \(P\). Observe that the property in Definition 2.1 means precisely that the one-step backward transition probabilities (that is, transition probabilities in the inverse time) are of the standard form

\[ P\{S_{n-1} = (n-1, k) \mid S_n = (n, k)\} = \frac{d_{n-1,k}}{d_{n,k}} = \frac{[n-k]}{[n]} \]

(10)

\[ P\{S_{n-1} = (n-1, k-1) \mid S_n = (n, k)\} = \frac{d_{n-1,k-1}q^{n-k}}{d_{n,k}} = q^{n-k} \frac{[k]}{[n]} \]

(11)

for every such \(S\).

Introduce the notation

\[ \tilde{v}_{n,k} := P\{S_n = (n-k, k)\}, \quad (n-k, k) \in \Gamma. \tag{12} \]

Consistency of the distributions of \(S_n\)'s amounts to the rule of total probability

\[ \tilde{v}_{n,k} = P\{S_n = (n, k) \mid S_{n+1} = (n+1, k)\} \tilde{v}_{n+1,k} \]

\[ + P\{S_n = (n, k) \mid S_{n+1} = (n+1, k+1)\} \tilde{v}_{n+1,k+1}. \tag{13} \]

Rewriting (13), using (10) and (11), and setting

\[ v_{n,k} = d_{n,k}^{-1} \tilde{v}_{n,k} \tag{14} \]

we get (9), which means that \(v \in V\). Thus, we have constructed the correspondence \(P \mapsto v\).

Conversely, start with a solution \(v \in V\) and pass to \(\tilde{v} = (\tilde{v}_{n,k})\) according to (14). For each \(n\) consider the measure on \(\Gamma_n\) with weights \(\tilde{v}_{n,0}, \ldots, \tilde{v}_{n,n}\). Since the weight of the root is 1, it follows from (9) by induction in \(n\) that these are probability measures. Again
by \(9\), the marginal measures are consistent with the backward transition probabilities, hence determine the probability law of a central Markov chain on \(\Gamma(q)\). Thus, we get the inverse correspondence \(v \mapsto P\). □

By virtue of Propositions 2.3 and 2.6, the law of \(q\)-exchangeable infinite binary sequence is determined by some \(v \in \mathcal{V}\), with the entries \(v_{n,k}\) having the same meaning as in \(5\). In the sequel this law will be sometimes denoted \(P_v\).

3. THE BOUNDARY PROBLEM

The set \(\mathcal{V}\) is a Choquet simplex, meaning a convex set which is compact in the product topology of the space of functions on \(\Gamma\) and has the property of uniqueness of the barycentric decomposition of each \(v \in \mathcal{V}\) over the set of extreme elements of \(\mathcal{V}\) (see, e.g., [7, Proposition 10.21]).

The boundary problem for the \(q\)-Pascal graph amounts to describing extreme nonnegative solutions to the recursion \(9\). Each extreme solution \(v \in \mathcal{V}\) corresponds to ergodic process \((S_n)\) for which the tail sigma-algebra is trivial. In this context, the set of extremes is also known as the minimal boundary.

With each array \(v \in \mathcal{V}\), \(v = (v_{n,k})\), it is convenient to associate another array \(\tilde{v} = (\tilde{v}_{n,k})\) related to \(v\) via \(14\). Clearly, the mapping \(v \mapsto \tilde{v}\) is an isomorphism of two Choquet simplexes \(\mathcal{V}\) and \(\tilde{\mathcal{V}} = \{\tilde{v}\}\). Recall that the meaning of the quantities \(\tilde{v}_{n,k}\) is explained in \(12\).

A common approach to the boundary problem calls for identifying a possibly larger Martin boundary (see [10], [6], [4] for applications of the method). To this end, we need to consider multistep backward transition probabilities, which by Lemma 6 are given by a \(q\)-analogue of the hypergeometric distribution

\[
\tilde{v}_{n,k}(\nu, \kappa) := \mathbb{P}\{S_n = (n - k, k) \mid S_\nu = (\nu - \kappa, \kappa)\}
= q^{(\kappa - k)(n-k)} \begin{bmatrix} \nu - n \\ \nu - \kappa - k \\ k \end{bmatrix} / \begin{bmatrix} \nu - \kappa \\ \nu \end{bmatrix}, \quad k = 0, \ldots, n,
\]

(15)

and to examine the limiting regimes for \(\kappa = \kappa(\nu)\) as \(\nu \to \infty\), under which the probabilities \(15\) converge for all fixed \((n - k, k) \in \Gamma\). If the limits exist, the limiting array \(\tilde{v}_{n,k} := \lim_{(\nu, \kappa)} \tilde{v}_{n,k}(\nu, \kappa)\) belongs necessarily to \(\tilde{\mathcal{V}}\).

Suppose \(0 < q < 1\) and introduce polynomials

\[
\Phi_{n,k}(x) := q^{-k(n-k)} x^{n-k}(x, q^{-1})_k, \quad \bar{\Phi}_{n,k} = d_{n,k} \Phi_{n,k}, \quad 0 \leq k \leq n.
\]

(16)

Obviously, the degree of \(\Phi_{n,k}\) is \(n\); we will consider the polynomial as a function on \(\Delta_q\). Observe also that \(\Phi_{n,k}(x)\) vanishes at points \(x = q^{\kappa}\) with \(\kappa < k\), because of vanishing of \((x, q^{-1})_k\).
Lemma 3.1. Suppose $0 < q < 1$, and let in (15) the indices $n$ and $k$ remain fixed, while $\nu \to \infty$ and $\kappa = \kappa(\nu)$ varies in some way with $\nu$. Then the limit of (15) is $\tilde{\Phi}_{n,k}(q^\kappa)$ if $\kappa$ is constant for large enough $\nu$. If $\kappa \to \infty$ then the limit is $\tilde{\Phi}_{n,k}(0) = \delta_{nk}$.

Proof. Assume first $\kappa \to \infty$ and show that the limit of (15) is $\delta_{nk}$. Since the quantities $\tilde{v}_{n,k}(\nu, \kappa)$, where $k = 0, \ldots, n$, form a probability distribution, it suffices to check that the limit exists and is equal to 1 for $k = n$. In this case the right-hand side of (15) becomes

$$\prod_{i=1}^{n} \frac{[\kappa - n + i]}{[\nu - n + i]}.$$ 

Because $\lim_{m \to \infty} [m] = 1/(1 - q)$ for $q < 1$, this indeed converges to 1 provided that $\kappa \to \infty$.

Now suppose $\kappa$ is fixed for all large enough $\nu$. The right-hand side of (15) is 0 for $k > \kappa$. For $k \leq \kappa$ using $\lim_{m \to \infty} [(m - j)!]/[m]! = (1 - q)^j$ we obtain

$$\frac{[\nu - n]}{[\kappa - k]} = \frac{[\nu - n]}{[\kappa - k]} \frac{[\nu - \kappa]}{[\kappa - \kappa]} \to \frac{(1 - q)^k}{[\kappa - k]} = \tilde{\Phi}_{n,n}(q^\kappa). \tag{17}$$

Part (i) of the next theorem appeared in [9, Chapter 1, Section 4, Corollary 6]. Kerov pointed out that the proof could be concluded from the Kerov-Vershik `ring theorem' (see [5, Section 8.7]), but did not give details.

For $\mu$ a measure, we shall write $\mu(x)$ instead of $\mu(\{x\})$, meaning atomic mass at $x$.

Theorem 3.2. Assume $0 < q < 1$.

(i) The formulas

$$\tilde{v}_{n,k} = \sum_{x \in \Delta_q} \tilde{\Phi}_{n,k}(x)\mu(x), \quad v_{n,k} = \sum_{x \in \Delta_q} \Phi_{n,k}(x)\mu(x)$$

establish a linear homeomorphism between the set $\tilde{V}$ (respectively, $V$) and the set of all probability measures $\mu$ on $\Delta_q$.

(ii) Given $\bar{\nu} \in \tilde{V}$, the corresponding measure $\mu$ is determined by

$$\mu(q^\kappa) = \lim_{\nu \to \infty} \tilde{v}_{\nu,\kappa}, \quad \kappa = 0, 1, \ldots; \quad \mu(0) = 1 - \sum_{\kappa \in \{0, 1, \ldots\}} \mu(q^\kappa).$$

Proof. As in [10], the assertions (i) and (ii) are consequences of the following claims (a), (b) and (c).

(a) For each $\nu = 0, 1, 2, \ldots$, the vertex set $\Gamma_\nu$ is embedded into $\Delta_q$ via the map $(\nu, \kappa) \mapsto q^\kappa$. Observe that, as $\nu \to \infty$, the image of $\Gamma_\nu$ in $\Delta_q$ expands and in the limit exhausts the
whole set $\Delta_q$, except point 0, which is a limit point. In this sense, $\Delta_q$ is approximated by the sets $\Gamma_\nu$ as $\nu \to \infty$.

(b) The multistep backward transition probabilities (15) converge to $\tilde{\Phi}_{n,k}(q^\nu)$, for $0 \leq \nu \leq \infty$, in the regimes described by Lemma 3.1.

(c) The linear span of the functions $\tilde{\Phi}_{n,k}(x)$, $(n-k,k) \in \Gamma$, is the space of all polynomials, so that it is dense in the Banach space $C(\Delta_q)$. \hfill $\Box$

Note that part (ii) of the theorem can be rephrased as follows: given $\tilde{v} \in \tilde{V}$, consider the probability distribution on $\Gamma_0$ determined by $\tilde{v}_0$, and take its pushforward under the embedding $\Gamma_\nu \hookrightarrow \Delta_q$. The resulting probability measure on $\Delta_q$ weakly converges to $\mu$ as $n \to \infty$.

**Corollary 3.3.** For $0 < q < 1$ we have:

(i) The extreme elements of $V$ are parameterised by the points $x \in \Delta_q$ and have the form

$$v_{n,k} = \Phi_{n,k}(x), \quad 0 \leq k \leq n. \quad (18)$$

(ii) The Martin boundary of the graph $\Gamma(q)$ coincides with its minimal boundary and can be identified with $\Delta_q \subset [0,1]$ via the function $v \mapsto v_{1,0}$.

**Proof.** All the claims are immediate. We only comment on the fact the parameter $x \in \Delta_q$ is recovered as the value of $v_{1,0}$: this holds because $\Phi_{1,0}(x) = x$. \hfill $\Box$

Letting $q \to 1$ we have a phase transition: the discrete boundary $\Delta_q$ becomes more and more dense and eventually fills the whole of $[0,1]$ at $q = 1$.

As is seen from (16), the polynomial $\Phi_{n,k}(x)$ can be viewed as a $q$-analogue of the polynomial $x^{n-k}(1-x)^k$, so that (18) is a $q$-analogue of (4). Keep in mind that $x = q^\nu$ is a counterpart of $1-p$, the probability of $\varepsilon_1 = 0$. The following $q$-analogue of the Hausdorff problem of moments emerges. Introduce a modified difference operator acting on sequences $u = (u_l)_{l=0,1,\ldots}$ as

$$\delta_q u_l = q^{-l}(u_l - u_{l+1}), \quad l = 0, 1, \ldots.$$

**Corollary 3.4.** Assume $0 < q < 1$. A real sequence $u = (u_l)_{l=0,1,\ldots}$ with $u_0 = 1$ is a moment sequence of a probability measure $\mu$ supported by $\Delta_q \subset [0,1]$ if and only if $u$ is ‘$q$-completely monotone’ in the sense that for every $k = 0, 1, \ldots$ we have componentwise

$$\underbrace{\delta_q \circ \cdots \circ \delta_q}_{k} u \geq 0, \quad k = 0, 1, \ldots.$$

**Proof.** Using the notation $v_{l+k,k} = u^{(k)}_l$ as in (2), we see that the recursion (2) is equivalent to $u^{(k)}_l = \delta_q u^{(k-1)}_l$, cf. (3). Then we use the fact that $\Phi_{n,0}(x) = x^n$ and repeat in the reverse order the argument of Section 3. \hfill $\Box$
The case \( q > 1 \). This case can be readily reduced to the case with parameter \( 0 < \tilde{q} < 1 \), where \( \tilde{q} := q^{-1} \). It is convenient to adopt a more detailed notation \([n]_q\) for the \( q \)-integers.

**Lemma 3.5.** For every \( q > 0 \), \( \tilde{q} = q^{-1} \), the backward transition probabilities (10), (11) for the graph \( \Gamma(q) \) and the dual graph \( \Gamma^*(\tilde{q}) \) are the same.

**Proof.** Indeed, by virtue of (10), (11), this is reduced to the equality
\[
\frac{[n-k]_q}{[n]_q} = \tilde{q}^k \frac{[n-k]_{\tilde{q}}}{[n]_{\tilde{q}}}.
\]

The lemma implies that the boundary problem for \( q > 1 \) can be treated by passing to \( \tilde{q} < 1 \) and changing \((l,k)\) to \((k,l)\). In terms of the binary encoding of the path, this means switching 0’s with 1’s.

Kerov [9, Chapter 1, Section 2.2] gives more examples of ‘similar’ graphs, which have different edge weights but the same backward transition probabilities.

### 4. Examples

**A \( q \)-analogue of the Bernoulli process.** Our first example is a description of the extreme \( q \)-exchangeable infinite binary sequences.

With each infinite binary sequence we associate some \( T \)-sequence \((T_0, T_1, T_2, \ldots)\) of nonnegative integers, where \( T_j \) is the length of \( j \)-th run of 0’s. That is to say, \( T_0 \) is the number of 0’s before the first 1, \( T_1 \) is the number of 0’s between the first and second 1’s, \( T_2 \) is the number of 0’s between the second and third 1’s, and so on. Clearly, this is a bijection, i.e. a binary sequence can be recovered from its \( T \)-sequence as
\[
(0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots).
\]

If \( q = 1 \), then the Bernoulli process with parameter \( p \) has a simple description in terms of the associated random \( T \)-sequence: all \( T_i \) are independent and have the same geometric distribution with parameter \( 1 - p \).

**Proposition 4.1.** Assume \( 0 < q < 1 \). For \( x \in \Delta_q \), let \( v(x) = (v_{n,k}(x)) \) be the extreme element of \( \mathcal{V} \) corresponding to \( x \). Consider \( q \)-exchangeable infinite binary sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \) under the probability law \( \mathbb{P}_{v(x)} \) and let \((T_0, T_1, \ldots)\) be the associated random \( T \)-sequence.

(i) If \( x = q^\kappa \) with \( \kappa = 1, 2, \ldots \) then \( T_0, \ldots, T_{\kappa-1} \) are independent, \( T_\kappa \equiv \infty \), and \( T_i \) has geometric distribution with parameter \( q^{\kappa-i} \) for \( 0 \leq i \leq \kappa - 1 \).

(ii) If \( x = 1 \) then \( T_0 \equiv \infty \), which means that with probability one \( \varepsilon \) is the sequence \((0, 0, \ldots)\) of only 0’s.

(iii) If \( x = 0 \) then \( T_0 \equiv T_1 \equiv \cdots \equiv 0 \), which means that with probability one \( \varepsilon \) is the sequence \((1, 1, \ldots)\) of only 1’s.
Proof. Consider the central Markov chain $S = (S_n)$ corresponding to the extreme element $v(q^\kappa)$. Computing the forward transition probabilities, from (18) and (10), for $0 \leq k \leq \kappa$ we have

$$
P\{S_{n+1} = (n + 1 - k, k) \mid S_n = (n - k, k)\} = \frac{(q^{n+1-k} - 1)}{(q^n - 1)} \frac{d_{n+1,k} \Phi_{n+1,k}(q^\kappa)}{d_{n,k} \Phi_{n,k}(q^\kappa)} = q^{\kappa-k}. \tag{19}
$$

This implies (i) and (ii). In the limit case $x = 0$ corresponding to $\kappa \to +\infty$, the above probability equals 0, which entails (iii). □

The analogy with the Bernoulli process is evident from the above description of the binary sequence $\varepsilon(q^\kappa)$. Moreover, the Bernoulli process appears as a limit. Indeed, fix $p \in (0,1)$ and suppose $\kappa$ varies with $q$, as $q \uparrow 1$, in such a way that $\kappa \sim -\log(1-p)/(1-q)$.

In this limiting regime, $q^{\kappa-k} \to 1-p$ for every $k$, hence $(T_0, T_1, \ldots)$ weakly converges to an infinite sequence of i.i.d. geometric variables with parameter $1-p$, and the random binary sequence $\varepsilon(q^\kappa)$ converges in distribution to the Bernoulli process with the frequency of 0’s equal to $1-p$.

Another $q$-analogue of Bernoulli process. Following [9], another $q$-analogue of Bernoulli process is suggested by the $q$-binomial formula (see [8])

$$
(-\theta, q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k.
$$

For $\theta \in [0, \infty]$ we define a probability law $\mathbb{P}_{w^\theta}$ for $S = (S_n)$ by setting

$$
w^\theta_{n,k} := \frac{\theta^k q^{k(k-1)/2}}{(-\theta, q)_n}, \quad \mathbb{P}_{w^\theta}\{S_n = (n - k, k)\} := d_{n,k} w^\theta_{n,k}, \quad (n, k) \in \Gamma. \tag{20}
$$

Checking (9) is immediate. Computing forward transition probabilities,

$$
\mathbb{P}_{w^\theta}\{S_{n+1} = (n + 1 - k, k) \mid S_n = (n - k, k)\} = 1/(1 + \theta q^n),
$$

shows that under $\mathbb{P}_{w^\theta}$ the process $S_n = (n - K_n, K_n)$ has independent inhomogeneous increments, with probability $\theta q^{n-1}/(1 + \theta q^{n-1})$ for increment $K_n - K_{n-1} = 1$. For $q = 1$ we are back to the ergodic Bernoulli process, but for $0 < q < 1$ the process is not extreme. To obtain the barycentric decomposition of $w^\theta$ over extremes,

$$
w^\theta = \sum_{0 \leq \kappa \leq \infty} v^\kappa \mu(q^\kappa),
$$
we can apply Theorem 3.2(ii) to compute from (20)

\[ \mu(q^\kappa) = \lim_{n \to \infty} \mathbb{P}_{\omega^\theta}\{S_n = (n - \kappa, \kappa)\} = \frac{1}{(-\theta, q)_\infty (1 - q)^\kappa} q^{\kappa(\kappa - 1)/2 \theta \kappa}. \]

This measure \( \mu \) may be viewed as a \( q \)-analogue of the Poisson distribution.

A \( q \)-analogue of Pólya’s urn process. The conventional Pólya’s urn process is described in [3, Section 7.4]. Here we provide its natural deformation.

Fix \( a, b > 0 \) and \( 0 < q < 1 \). Consider the Markov chain \((S_n)\) on \( \Gamma \) with the forward transition probabilities from \((n - k, k)\) to \((n + 1 - k, k)\) and from \((n - k, k)\) to \((n - k, k + 1)\) given by

\[
\frac{[b + n - k]}{[a + b + n]} \quad \text{and} \quad \frac{[a + k]}{[a + b + n]} q^{n-k+b},
\]

respectively. Then the distribution at time \( n \) is

\[
\mathbb{P}\{S_n = (n - k, k)\} = \binom{n}{k} q^{bk} \times \frac{[a][a + 1] \cdots [a + k - 1][b][b + 1] \cdots [b + n - k - 1]}{[a + b][a + b + 1] \cdots [a + b + n - 1]}.
\]

Checking consistency (39) is easy. The conventional Pólya’s urn process appears in the limit \( q \to 1 \). The corresponding probability measure \( \mu \) is computable from Theorem 3.2(ii) as

\[
\lim_{n \to \infty} \mathbb{P}\{S_n = (n - \kappa, \kappa)\}
\]

For \( a = 1 \), the limit distribution of the coordinate \( \kappa \) is geometric with parameter \( 1 - q^b \).

For general \( a, b \) we obtain a measure on \( \Delta_q \)

\[ \mu(q^\kappa) = \frac{(q^a, q)_\infty(q^b, q)_\infty}{(q, q)_\infty(q^{a+b}, q)_\infty} q^{\kappa b}, \quad q^\kappa \in \Delta_q, \]

which may be viewed as a \( q \)-analogue of the beta distribution on \([0, 1]\).

5. Grassmannians over a finite field

For \( q \) a power of a prime number, let \( \mathbb{F}_q \) be the Galois field with \( q \) elements. Define \( V_n \) to be the \( n \)-dimensional space of sequences \((\xi_1, \xi_2, \ldots)\) with entries from \( \mathbb{F}_q \), which satisfy \( \xi_i = 0 \) for \( i > n \). The spaces \( \{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \) comprise a complete flag, and the union \( V_\infty := \bigcup_{n \geq 0} V_n \) is a countable, infinite-dimensional space over \( \mathbb{F}_q \).

By the Grassmannian \( \text{Gr}(V_\infty) \) we mean the set of all vector subspaces \( X \subseteq V_\infty \). Likewise, for \( n \geq 0 \) let \( \text{Gr}(V_n) \) be the set of all vector subspaces in \( V_n \), with \( \text{Gr}(V_0) \) being a singleton. Consider the projection \( \pi_{n+1,n} : \text{Gr}(V_{n+1}) \to \text{Gr}(V_n) \) which sends a subspace of \( V_{n+1} \) to its intersection with \( V_n \).
Lemma 5.1. There is a canonical bijection $X \leftrightarrow (X_n)$ between the Grassmannian $\text{Gr}(V_\infty)$ and the set of sequences $(X_n \in \text{Gr}(V_n), \ n \geq 0)$ satisfying the consistency condition $X_n = \pi_{n+1,n}(X_{n+1})$ for each $n$.

Proof. Indeed, the mapping $X \mapsto (X_n)$ is given by setting $X_n = X \cap V_n$ for each $n$, while the mapping $(X_n) \mapsto X$ is defined by $X = \bigcup X_n$. □

The lemma shows that $\text{Gr}(V_\infty)$ can be identified with a projective limit of the finite sets $\text{Gr}(V_n)$, the projections being the maps $\pi_{n+1,n}$. Using this identification we endow $\text{Gr}(V_\infty)$ with the corresponding topology, in which $\text{Gr}(V_\infty)$ becomes a totally disconnected compact space. For $X \in \text{Gr}(V_\infty)$, a fundamental system of its neighborhoods is comprised of the sets of the form $\{X' \in \text{Gr}(V_\infty) : X_n' = X_n\}$, where $n = 1, 2, \ldots$.

Let $\mathcal{G}_n = GL(n, \mathbb{F}_q)$ be the group of invertible linear transformations of the space $V_n$, realised as the group of transformations of $V_\infty$ which may only change the first $n$ coordinates. We have then $\{e\} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ and we define $\mathcal{G}_\infty := \bigcup \mathcal{G}_n$. The countable group $\mathcal{G}_\infty$ consists of infinite invertible matrices $(g_{ij})$, such that $g_{ij} = \delta_{ij}$ for large enough $i + j$. The group $\mathcal{G}_\infty$ acts on $V_\infty$ hence also acts on $\text{Gr}(V_\infty)$.

A probability distribution on $\text{Gr}(V_\infty)$ defines a random subspace of $V_\infty$. We look at random subspaces of $V_\infty$ whose distribution is invariant under the action of $\mathcal{G}_\infty$. Observe that the action of $\mathcal{G}_n$ splits $\text{Gr}(V_n)$ into orbits

$$G(n, k) = \{X \in \text{Gr}(V_n), \dim X = k\}, \ 0 \leq k \leq n,$$

where $\#G(n, k) = d_{n,k}$ is the number of $k$-dimensional subspaces of $V_n$. Therefore, a probability distribution on $\text{Gr}(V_\infty)$ is $\mathcal{G}_\infty$-invariant if and only if the conditional distribution on each $G(n, k)$ is uniform.

It must be clear that this setting of ‘$q$-exchangeability’ of linear spaces is analogous to the framework of de Finetti’s theorem: exchangeability of a random binary sequence means that the conditional measure is uniform on sequences of length $n$ with $k$ 1’s. See [1], [2] for more on symmetries and sufficiency.

Lemma 5.2. Formula

$$\tilde{v}_{n,k} = P\{X \in \text{Gr}(V_\infty) : X \cap V_n \in G(n, k)\}, \ (n, k) \in \Gamma$$

establishes a linear homeomorphism between $\tilde{V}$ and $\mathcal{G}_\infty$-invariant probability measures on the Grassmannian $\text{Gr}(V_\infty)$.

Proof. We first spell out more carefully the remark before the lemma. Consider projections

$$\pi_{n,n} : \text{Gr}(V_\infty) \to \text{Gr}(V_n), \ \ X \mapsto X \cap V_n, \ X \in \text{Gr}(V_\infty), \ \ n = 1, 2, \ldots.$$ 

If $P$ is a Borel probability measure on the space $\text{Gr}(V_\infty)$, then, for any $n$, the pushforward $P_n := \pi_{n,n}(P)$ is a probability measure on $\text{Gr}(V_n)$, and the measures $P_n$ are consistent with respect to the projections $\pi_{n+1,n}$, that is,

$$P_n = \pi_{n+1,n}(P_{n+1}), \ \ n = 0, 1, 2, \ldots.$$
Conversely, if a sequence \((P_n)\) of probability measures is consistent, then it determines a probability measure \(P\) on \(\text{Gr}(V_\infty)\). Moreover, \(P\) is \(\mathcal{G}_\infty\)-invariant if and only if each \(P_n\) is \(\mathcal{G}_n\)-invariant. Next, observe that if \(P_n\) is a \(\mathcal{G}_n\)-invariant probability measure, then it assigns the same weight to each \(k\)-dimensional space \(X_n \in G(n, k)\); let us denote this weight by \(v_{n,k}\).

Fix \(X_n \in G(n, k)\). We claim that there are precisely \(q^{n-k}+1\) subspaces \(X_{n+1} \in \text{Gr}(V_{n+1})\) such that \(X_{n+1} \cap V_n = X_n\): one subspace from \(G(n+1, k)\) and \(q^{n-k}\) subspaces from \(G(n+1, k+1)\). Indeed, \(\dim X_{n+1}\) equals either \(k\) or \(k+1\). In the former case \(X_{n+1} = X_n\), while in the latter case \(X_{n+1}\) is spanned by \(X_n\) and a nonzero vector from \(V_{n+1} \setminus V_n\). Such a vector is defined uniquely up to a scalar multiple and addition of an arbitrary vector from \(X_n\). Therefore, the number of options is equal to the number of lines in \(V_{n+1}/X_n\) not contained in \(V_n/X_n\), which equals

\[
\frac{q^{n+1-k} - 1}{q - 1} - \frac{q^{n-k} - 1}{q - 1} = q^{n-k}.
\]

Now, let \(P\) be a \(\mathcal{G}_\infty\)-invariant probability measure on \(\text{Gr}(V_\infty)\), with projections \((P_n)\) specified by the corresponding array of weights \(v = (v_{n,k})\). Then the relations \(P_n = \pi_{n+1,n}(P_{n+1})\) together with the dimension computation imply that \(v\) satisfies (\ref{eq:dimension}).

Conversely, given \(v \in \mathcal{V}\), we can construct a sequence \((P_n)\) of measures such that \(P_n\) lives on \(\text{Gr}(V_n)\), is invariant under \(\mathcal{G}_n\) and agrees with \(P_{n+1}\) under \(\pi_{n+1,n}\). Since \(P_0\), which lives on a singleton, is obviously a probability measure, we obtain by induction that all \(P_n\) are probability measures. Taking their projective limit we get a \(\mathcal{G}_\infty\)-invariant probability measure \(P\) on \(\text{Gr}(V_\infty)\).

Rephrasing Theorem 3.2 we have from the lemma

**Corollary 5.3.** The ergodic \(\mathcal{G}_\infty\)-invariant probability measures on \(\text{Gr}(V_\infty)\) are parameterised by \(\kappa \in \{0, 1, \ldots, \infty\}\). For \(\kappa = 0\) the measure is the Dirac mass at \(V_\infty\), for \(\kappa = \infty\) it is the Dirac mass at \(V_0\), and for \(0 < \kappa < \infty\) the measure is supported by the set of subspaces of \(V_\infty\) of codimension \(\kappa\).

The following random algorithm describes explicitly the dynamics of the growing space \(X_n \in \text{Gr}(V_n)\) as \(n\) varies, under the ergodic measure with parameter \(\kappa\). Recall the notation \(\bar{q} = q^{-1}\). Start with \(X_0 = V_0\). With probability \(\bar{q}^\kappa\) choose \(X_1 = V_1\), and with probability \(1 - \bar{q}^\kappa\) choose \(X_1 = X_0\). Suppose \(X_n \subseteq V_n\) has been constructed and has dimension \(n-k\) with \(k \leq \kappa\). Then let \(X_{n+1} = X_n\) with probability \(1 - \bar{q}^{\kappa-k}\), and with probability \(\bar{q}^{\kappa-k}\) choose uniformly at random a nonzero vector \(\xi \in V_{n+1} \setminus V_n\) and let \(X_{n+1}\) be the linear span of \(X_n\) and \(\xi\).

**Duality.** We finish with a dual version of our construction. Let \(V_\infty\) denote the set of all sequences \(\eta = (\eta_1, \eta_2, \ldots)\) with entries from \(\mathbb{F}_q\). This is again a vector space over \(\mathbb{F}_q\), strictly larger than \(V_\infty\) since we do not require \(\eta\) to have finitely many nonzero entries. That is to say, \(V_\infty\) is just the infinite product space \((\mathbb{F}_q)^\infty\), which we endow with the product topology. Let \(\text{Gr}(V_\infty)\) denote the set of all closed subspaces \(Y \subseteq V_\infty\). A dual...
version of Lemma 5.1 says that such subspaces $Y$ are in a bijective correspondence with the sequences $(Y_n \in \text{Gr}(V_n), n \geq 0)$ such that $Y_n = \pi'_n V_n$, where $\pi'_n$ is induced by the projection map $V_n \to V_n$ which sets the $(n+1)$th coordinate of a vector $\xi \in V_n$ equal to 0. The branching of $G(n, k)$'s under these projections corresponds to the dual $q$-Pascal graph.

**Lemma 5.4.** The operation of passing to the orthogonal complement with respect to the bilinear form

$$\langle \xi, \eta \rangle := \sum_{i=1}^{\infty} \xi_i \eta_i, \quad \xi \in V_\infty, \quad \eta \in V^\infty,$$

is a bijection $\text{Gr}(V_\infty) \leftrightarrow \text{Gr}(V^\infty)$.

**Proof.** First of all, note that the bilinear form is well defined, because the coordinates $\xi_i$ of $\xi \in V_\infty$ vanish for $i$ large enough. This form determines a bilinear pairing $V_\infty \times V^\infty \to \mathbb{F}_q$. We claim that it brings the spaces $V_\infty$ and $V^\infty$ into duality, where $V^\infty$ is viewed as a vector space with nontrivial topology, and the topology on $V_\infty$ is discrete.

Indeed, it is evident that the pairing is nondegenerate and that any linear functional on $V_\infty$ is given by a vector of $V^\infty$. A minor reflection also shows that, conversely, any continuous linear functional on $V^\infty$ is given by a vector from $V_\infty$. Thus, the spaces $V_\infty$ and $V^\infty$ are indeed dual to one another. They are also dual as commutative locally compact topological groups: one is discrete and the other is compact.

Using the duality, it is readily checked that if $X$ is an arbitrary subspace in $V_\infty$, then its orthogonal complement $X^\perp$ is a closed subspace in $V^\infty$, whose orthogonal complement $(X^\perp)^\perp$ coincides with $X$. Likewise, starting with a closed subspace $Y \subseteq V^\infty$, we have $Y^\perp \subseteq V_\infty$ and $(Y^\perp)^\perp = Y$. Thus, the operation of taking the orthogonal complement is a bijection. \qed

The group $G_\infty$ acts on both $V_\infty$ and $V^\infty$ and preserves the pairing between these vector spaces. Under the identification $\text{Gr}(V^\infty) = \text{Gr}(V_\infty)$, the group $G_\infty$ acts by homeomorphisms on this compact space. In the dual picture, the ergodic measures with $\kappa < \infty$ live on the set of $\kappa$-dimensional subspaces of $V^\infty$. The case $\kappa = \infty$ corresponds then to the zero subspace in $V_\infty$ (or the full space $V^\infty$). There is a simple explanation why we have to fix codimension in the $V_\infty$-picture and dimension in the $V^\infty$-picture, and not vice versa. Namely, the subspaces in $V_\infty$ of fixed nonzero finite dimension form a countable set, which is a single $G_\infty$-orbit, and such a $G_\infty$-space cannot carry a finite invariant measure.

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