(G, µ)-WINDOWS AND DEFORMATIONS OF (G, µ)-DISPLAYS

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ABSTRACT. Let \( k_0 \) be a finite field of characteristic \( p \), let \( G \) be a smooth affine group scheme over \( \mathbb{Z}_p \), and let \( µ \) be a cocharacter of \( G(W(k_0)) \) such that the set of \( µ \)-weights of \( \text{Lie} \ G \) is a subset of \( \{-1, 0, 1, 2, \ldots\} \). We prove that the groupoid of adjoint nilpotent \( (G, µ) \)-displays as in [6, Definition 1.0.1], [38] and [5, Definition 3.16] is equivalent to the groupoid of \( (G, µ) \)-windows, which are the generalizations of windows as in [36, Definition 5.1] and [5, Definition 3.3.1].

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1. Introduction

Let $p$ be a prime number. The study of $p$-divisible groups by means of semi-linear algebra over the ring of $p$-typical Witt vectors has a long history. Its undisputed point of departure is a perfect field $k \supseteq \mathbb{F}_p$, over which one has a classification in terms of the category of Dieudonné modules of which an object consists of a finitely generated free $W(k)$-module together with a Frobenius-linear endomorphism whose cokernel is annihilated by $p$. The theory owes its ease and popularity to $W(k)$ being a complete discrete valuation ring with uniformizing element $p$ and residue field $k$. Now let $A$ be a $p$-adically separated and complete commutative ring without $p$-torsion and suppose that some $\sigma \in \text{End}(A)$ lifts the Frobenius endomorphism:

$$A/pA \rightarrow A/pA$$

$$x \mapsto x^p$$

Furthermore, let $a \subseteq A$ be a $p$-adically open pd-ideal (in the case at hand this means that $pA \subseteq \sqrt{a}$ and $x^p \in pa$ for every $x \in a$). One calls $(A, a, \sigma)$ a frame over the ring $R = A/a$, in which $p$ is nilpotent. One knows that the following three categories are equivalent:

- **windows** for the frame $(A, a, \sigma)$ in the sense of [51]
- nilpotent displays over $R$ in the sense of [52].
- $p$-divisible formal groups over $R$

Please see loco citato and in particular [41], [37] for the proofs. A display over $R$ is a quadruple $(P, Q, F, V^{-1})$ with the following properties:

- $P$ is a finitely generated projective $W(R)$-module and $Q \subseteq P$ is the inverse image of a direct summand in $R \otimes_{w_0,W(R)} P$, where $w_0 : W(R) \rightarrow R$ is the 0-th ghost coordinate,
- $V^{-1} : Q \rightarrow P$ is an $F$-linear homomorphism whose image generates $P$ as a $W(R)$-module, and $F : P \rightarrow P$ is an $F$-linear homomorphism satisfying

$$(1) \quad V^{-1}(V a \cdot x) = aF(x)$$

for all $a \in W(R)$ and $x \in P$, where $V : W(R) \rightarrow W(R); (a_0, a_1, \ldots) \mapsto (0, a_0, \ldots)$ is the Verschiebung, so that $V a \cdot x \in Q$.

A window is a similar semi-linear structure in which the projection $A \rightarrow R$ replaces the 0-th ghost coordinate $w_0 : W(R) \rightarrow R$, while the Frobenius lift $\sigma$ is a substitute for the absolute Frobenius on $W(R)$, please see [51] for more information e.g. on the nilpotence conditions in these categories. The axiom (1) should be regarded as a sharpening of the classical formula $F|_Q = pV^{-1}$, inspired by older theories on “strongly divisible lattices” and the like, please see [15] and [16] for more information. It is known that $W(R)$ is $p$-adically separated and complete, but it is hard to unravel its structure as a commutative ring, unless $R$ is a perfect field (or a product of these). The equivalence between windows and nilpotent displays can be viewed as a remedy for this, as it gives some freedom to work over rings which are nicer than $W(R)$.

So given that each of these three categories has its own distinct features, we gain a powerful combination of their advantages from equivalences between them: Within the category of $p$-divisible groups, for instance, it is in general not possible to form
a tensor product, but the categories of displays (resp. windows) can be regarded as full subcategories of the natural rigid \(\otimes\)-categories of Langer-Zink displays (resp. windows), which showed up for the first time in [36, Definition 2.4-2.6/Definition 5.1]. This phenomenon has proved to be a fertile inspiration for the recent works [29], [30] and [4]. Spurred by this, along with [6] and [38], this paper addresses the comparison of various concepts of additional structure in the aforementioned three categories, perhaps with an eye towards applications to the numerous moduli spaces that are indigenous to the Langlands program, i.e. Rapoport-Zink spaces, period domains, Shimura varieties and their integral models.

More specifically, our approach to displays with additional structure runs as follows: We fix a smooth affine \(\mathbb{Z}_p\)-group \(G\) together with a cocharacter \(\mu\) which is defined over the ring of integers in an unramified \(\mathbb{Q}_p\)-extension of finite degree, i.e. over \(W(k_0)\), where \(k_0\) is a finite field of characteristic \(p\). Let \(P_\mu \subset G_{W(k_0)}\) be the associated parabolic subgroup, so that \(\text{Lie } P_\mu\) is the subspace of \(\text{Lie } G\) whose \(\mu\)-weights are non-negative. Assuming that \(-1\) is the only negative \(\mu\)-weight of \(\text{Lie } G\), we define a key homomorphism

\[
L^+ G_{W(k_0)} \hookrightarrow H_\mu \xrightarrow{\Phi_\mu} L^+ G_{W(k_0)}
\]

where \(L^+ G\) stands for the affine group scheme with \(L^+ G(R) = G(W(R))\), \(H_\mu\) for the inverse image of \(P_\mu\) via the 0-th ghost map \(w^0_G : L^+ G \to G\) and \(\Phi_\mu\) for the integral model of the group homomorphism \(q \mapsto \hat{L}(\mu(p) g\mu(\frac{1}{p}))\). The existence of \(\Phi_\mu\) is not a triviality and was proved in [6] and [5] only under additional assumptions, most notably that \(G\) be reductive. In this paper we give an improved exposition, using nothing but \(-1\) being a lower bound for the \(\mu\)-weights of \(\text{Lie } G\). At last we let \(B(G, \mu)\) be the fpqc quotient stack of \(L^+ G_{W(k_0)}\) by a certain \(H_\mu\)-action defined thereon, namely the so-called \(\Phi_\mu\)-conjugation:

\[
L^+ G_{W(k_0)} \times_{W(k_0)} H_\mu \to L^+ G_{W(k_0)}; (U, h) \mapsto h^{-1} U \Phi_\mu(h)
\]

According to [6] and [5] a \((G, \mu)\)-display \(D\) over some \(W(k_0)\)-scheme \(X\) is simply defined to be a 1-morphism from \(X\) to \(B(G, \mu)\). Besides \(D\) is called banal if the said 1-morphism factors through the obvious presentation \(L^+ G_{W(k_0)} \to B(G, \mu)\). Under the same requirement on \(G\) and the \(\mu\)-weights of \(\text{Lie } G\) we suggest a natural definition of the groupoid of \((G, \mu)\)-windows for the frame \((A, a, \sigma)\) and we prove the following theorem:

**Theorem (6.4).** The groupoid of \((G, \mu)\)-windows over the frame \((A, a, \sigma)\) is equivalent to the groupoid of adjoint nilpotent \((G, \mu)\)-displays over \(A/a\).

The adjoint nilpotence is an analog of Zink’s nilpotence condition. Unsurprisingly our proof of theorem 6.4 has close ties to the study of lifts of \((G, \mu)\)-displays from \(A/a\) to \(A\). At this point we invoke the deformation theory established by [6, Theorem 3.5.4]. In fact we have to reconsider it slightly, because the proof of loc. cit. (and of similar applicable lemmas in [38] and [5]) seems to require the \(p\)-adic topological nilpotency of the \(p\)-ideal \(a\), which we do not want to impose in theorem 6.4. Apart from this technicality the proof of theorem 6.4 runs smoothly in an
assembly line, which roughly speaking works as follows:

\[(G, \mu)\text{-windows over } (A, a, \sigma) \rightsquigarrow \]
\[(G, \mu)\text{-triple over } (A, a) \rightsquigarrow \]
\[(G, \mu)\text{-displays over } A/a \]

For the definition of the above notion of \((G, \mu)\text{-triple over } (A, a)\) we have to refer to the body of the text; It is modeled on Zink’s notion for GL(h). In fact the stack of displays of dimension \(d\) and height \(h\) is just our \(B(\text{GL}(h)_{\mathbb{Z}_p}, \mu_{h,d})\) for the cocharacter:

\[\mu_{h,d} : \mathbb{G}_m \to \text{GL}(h) ; z \mapsto \text{diag}(1, \ldots, 1, z, \ldots, z)\]

This normalization follows [6], which is somewhat opposite to the ones of [38], but do notice that these stacks neglect non-invertible homomorphisms between displays. The usage of \((W(A), W(a) + I(A), F)\) is suggested by [51], of which the main result can be reobtained from theorem 6.4 by specializing to the case \((G, \mu) = (\text{GL}(h)_{\mathbb{Z}_p}, \mu_{h,d})\).

Our second result concerns the rigidity of quasi-isogenies between \((G, \mu)\text{-displays, for which it is necessary to work over the localization of the Witt ring at the prime } p\). This leads to an interesting 1-morphism from \(B(G, \mu)\) to its rational analog \(B(G)\), which one can think of as the sheafification of the prestack whose fiber over \(R\) is given by the equivalence relation of \(F\text{-}conjugacy\) on the set \(G(\mathbb{Q} \otimes W(R))\). Thus a quasi-isogeny between \((G, \mu)\text{-displays } \mathcal{D}_1 \text{ and } \mathcal{D}_2\) can be regarded as an isomorphism \(\mathcal{D}_1^0 \cong \mathcal{D}_2^0\) in \(B(G)\), where \(\mathcal{D}^0\) stands for the image of some \((G, \mu)\text{-displays over } \mathcal{D}\) under the canonical map \(B(G, \mu) \to B(G)\). We have the following:

**Theorem (4.25).** Let \(\mathcal{D}\) be a \((G, \mu)\text{-display over a connected and locally noetherian } W(k_0)\text{-scheme } X\) whose structural morphism to \(\text{Spec } W(k_0)\) is assumed to be closed in the sense of [19, (2,2.6)]. Assume that the restriction of \(\mathcal{D}\) to the special fiber \(X \times_{W(k_0)} k_0\) is adjoint nilpotent, and let \(\alpha\) be an automorphism of \(\mathcal{D}^0\) (i.e. a self \(\text{ quasi-isogeny of } \mathcal{D}\)). For any \(x \in X\) we write \(\kappa(x)\) for the residue field of the stalk \(\mathcal{O}_{X,x}\) and \(\mathcal{D}_x\) for the pull-back of \(\mathcal{D}\) along the morphism \(\text{Spec } \kappa(x) \to X\). Then \(\alpha\) is the identity if and only if this holds for any of its specializations \(\alpha_x \in \text{Aut}(\mathcal{D}_x^0)\).

The proof of this result is quite technical and we therefore omit an overview here. We should only mention that we make a careful study of the loop group functor \(LG(R) = G(\mathbb{Q} \otimes W(R))\), which turns out to be a sheaf for the fpqc topology, in fact it is an ind-scheme. We expect that theorem 4.25 has consequences for the description of the period map of Rapoport-Zink spaces in the setting of [6].

We note in passing that [45, Definition 3.3] defines an \(F\text{-isocrystal with } G\text{-structure over a connected } \mathbb{F}_p\text{-scheme } S\) to be a faithful \(\mathbb{Q}_p\text{-linear exact } \otimes\text{-functor}\)

\[M : \text{Rep}_{\mathbb{Q}_p}(G) \to F - \text{Isoc}(S),\]

where \(\text{Rep}_{\mathbb{Q}_p}(G)\) is the \(\otimes\text{-category of finite-dimensional representations of } G_{\mathbb{Q}_p}\) while \(F - \text{Isoc}(S)\) is the category of \(F\text{-isocrystals on } S\) in the sense of [31, section 2.1] and [45, section 3.1]. Loc. cit. attributes the underlying tannakian formalism to [48], and it might suggest to call a faithful \(\mathbb{Z}_p\text{-linear exact } \otimes\text{-functor from the}
category of $G$-representations to the category of Langer-Zink displays a “Langer-Zink display with $G$-structure”. Please see [8] for more precise versions and more information on the relationship between the tannakian approach to ours. Please see [30, §2.1] for more information on the relationship between multi-linear maps of displays and multilinear maps of formal groups.

Let us give further evidence for the relevance of $B(G, \mu)$:

(i) Every $(G, \mu)$-display over a perfect field $k \supseteq k_0$ gives rise to an $F$-isocrystal with $G$-structure over $k$ in the sense of [45, Definition 3.3]. In particular one obtains a function from the set of “points” of $B(G, \mu)_{k_0}$ (say in the sense of [39, Définition (5.2)]) to the important set $B(G)$ introduced and studied by Kottwitz and others (cf. [32], [33] and references therein).

(ii) Every adjoint nilpotent $(G, \mu)$-display $D$ over a perfect field $k \supseteq k_0$ possesses a universal formal deformation $\tilde{D}$ over the $W(k)$-algebra $W(k)[[t_1, \ldots, t_r]]$, where $r$ is the codimension of $P_\mu$ in $G_{W(k)}$.

We round off our results with a few further applications of the deformation theory of adjoint nilpotent $(G, \mu)$-displays; Here is a sample:

**Theorem (4.34).** Let $A$ be a $W(k_0)$-algebra and $D$ a $(G, \mu)$-display over $A/\mathfrak{a}$, where the ideal $\mathfrak{a} \subset A$ is assumed to satisfy one of the following:

(i) $\mathfrak{a}$ is $(pA + \mathfrak{a})$-adically separated and complete.

(ii) $\mathfrak{a}$ allows a divided power structure and is $p$-adically separated and complete.

Assume that $D \times_{A/\mathfrak{a}} A/(pA + \mathfrak{a})$ is adjoint nilpotent. Then, for each lift $\tilde{D}$ of $D$ the reduction of automorphisms gives an injection $\text{Aut}(\tilde{D}) \hookrightarrow \text{Aut}(D)$. Moreover there exists at least one such lift.

Under the assumptions of this theorem it is meaningful to ask for the set of deformations of some $(G, \mu)$-display $D$ over $A/\mathfrak{a}$, and our fourth result describes this set rather explicitly, provided that we are in the scenario (ii) of theorem 4.34: Fix a lift $\tilde{D}$ of $D$ and consider the pointed $A$-scheme

$$X_{\tilde{D}} := q_0(\tilde{D}) * P_\mu G_{W(k_0)} / P_\mu,$$

where $q_0(\tilde{D})$ is a certain $P_\mu$-torsor which we define in the body of the text, our $X_{\tilde{D}}$ is a twist of the homogeneous space $G_{W(k_0)} / P_\mu$ over $A$. Our result states that lifts of $D$ are parametrized by the pointed set

$$X_{\tilde{D}}(\mathfrak{a}) = \{x \in X_{\tilde{D}}(A) \mid x \equiv 1 \mod \mathfrak{a}\},$$

i.e. the kernel of $X_{\tilde{D}}(A) \to X_{\tilde{D}}(A/\mathfrak{a})$. We prove this using the aforementioned notion of $(G, \mu)$-triple over $(A, \mathfrak{a})$.

Our results require a significant amount of background material on torsors, Greenberg transforms, loop groups, arithmetic jet schemes, cocharacters of smooth group schemes, and fpqc-sheaves, which we explain in section 2 and in the appendix. We also give as many references as possible and attempt a rather broad generality, which manifests itself in:

(i) Inclusion of non-reductive smooth affine group schemes, whenever this is possible.

(ii) Preference of arbitrary base schemes over merely affine ones.
(iii) Avoidance of noetherian (or similar finiteness) assumptions, whenever this is possible.
(iv) Occurrences of completions of topological ground rings only with respect to fine as possible a linear topology.

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2. Preliminaries

Recall that a topology on a commutative and unital ring $A$ is called linear if it allows a subbase of the form $\{x + a_i \mid x \in A, i \in I\}$ where $I$ is an index set and
\begin{equation}
\{a_i\}_{i \in I}
\end{equation}
a family of ideals. The index set $I$ can be taken to be $\mathbb{N}_0$ if and only if the resulting topology satisfies the first axiom of countability, in which case the generating family (2) can be taken to be a descending chain of ideals:
\begin{equation}
A \supseteq a_0 \supseteq a_1 \supseteq a_2 \supseteq \ldots
\end{equation}
An ideal $J$ in a linearly topologized ring $A$ is called topologically nilpotent if $a + J/a$ is nilpotent for any open ideal $a$. A linearly topologized ring $A$ is called preadmissible if it possesses a topologically nilpotent open ideal (cf. [19, Chapitre 0, Definition(7.1.2)]). By the completion of $A$ one means the filtered inverse limit $\hat{A} := \lim_{\longrightarrow} A/a$, where $a$ runs through the partially ordered set of open ideals. For each open ideal $a$ one also calls
\begin{equation}
\hat{a} := \ker(\hat{A} \to A/a)
\end{equation}
the completion of $a$. Notice that replacing (2) by $\{\hat{a}_i\}_{i \in I}$ turns $\hat{A}$ into a linearly topologized ring and that the natural homomorphism
\begin{equation}
A \to \hat{A}
\end{equation}
is continuous, because the inverse image of (4) is just $a$. The following is straightforward:

Lemma 2.1. The map (5) is injective (resp. surjective) if and only if this holds for the map $a \to \hat{a}$, where $a$ is any open ideal of $A$.

In fact $A$ is called separated (resp. complete) if the map (5) is injective (resp. surjective) and $A$ is called admissible if it is separated, complete and preadmissible, again according to [19, Chapitre 0, Definition(7.1.2)]. A certain sort of preadmissible topology plays a key role in this paper, the following lemma illustrates that:

Lemma 2.2. Let $A$ be a ring and let $b$ and $a$ be ideals.

(i) The descending chain of ideals $\{b^i a\}_{i \in \mathbb{N}_0}$ endows $A$ with the structure of a preadmissible topological ring, satisfying the first axiom of countability.
(ii) The ideal $a$ is $b$-adically separated (resp. complete) as an $A$-module, if and only if $A$ is separated (resp. complete) with respect to the topology described in (i).
(iii) Suppose that \( b = pA + a \), then \( \{ p^i a + a^{i+1} \}_{i \in \mathbb{N}_0} \) is also a neighborhood basis for \( 0_A \) with respect to the topology described in (i).

Proof. The first statement follows from the inclusion \( (ba)^i \subset b^i a \), the second one is a consequence of lemma 2.1 and the last one follows from the inclusion \( (pA + a)^{2i-1} a \subset p^i a + a^{i+1} \subset (pA + a)^i a \).

\[ \square \]

Remark 2.3. All topologies on all rings \( A \) that occur in this paper may be assumed to be linear and first-countable, so that there does exist a neighborhood basis for \( 0_A \) consisting of a descending chain \( \{ a_i \}_{i \in \mathbb{N}_0} \) as in (3). Observe that such a topology is preadmissible if and only if:

\[ \exists k \in \mathbb{N}_0 \forall m \in \mathbb{N}_0 \exists l \in \mathbb{N}_0 : a^l_k \subset a_{k+m} \]

In fact, some readers might like to think of first-countably topologized preadmissible rings as being topologized by a descending chain of ideals \( \{ b_i \}_{i \in \mathbb{N}_0} \) with \( b_0 = A \) and \( b_i \supset b_{i+1} \). Indeed let us choose \( k_0 \in \mathbb{N}_0 \) as indicated in (6) and let us write \( b_n \) for the ideal generated by all products \( \prod_{i=1}^l a_{n_i-1+k_0} \) where \( n \) is the sum of the \( n_i \in \mathbb{N} \). Then we do have \( b_{(l-1)n+1} \subset a_{m+k_0} \subset b_{m+1} \) whenever \( m \) and \( l \) are as indicated in (6).

Lemma 2.4. Let \( Y \) be quasi-projective \( A \)-scheme, where \( A \) is a ring.

(i) Assume that \( A \) is noetherian and \( \alpha \)-adically separated and complete, where \( \alpha \) is an ideal. Then, the canonical map

\[ Y(A) \rightarrow \lim_{\rightarrow n} Y(A/a^n) \]

is bijective.

(ii) Assume that \( A \) is a linearly topologized complete ring. If \( Y \) is smooth, then the canonical map \( Y(A) \rightarrow Y(A/J) \) is a surjection for all closed ideals \( J \subset A \) consisting of topologically nilpotent elements.

Proof. (i) We begin with the injectivity of (7), so let \( x, y \in Y(A) \) satisfy \( x \equiv y \mod a^n \) for every \( n \). Thinking of \((x, y)\) as a morphism \( \text{Spec } A \rightarrow Y \times_A Y \) we deduce the existence of an ideal \( I \subset A \) such that the diagram

\[
\begin{array}{ccc}
\text{Spec } A/I & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec } A & \longrightarrow & Y \times_A Y
\end{array}
\]

is Cartesian, of course the implication “quasi-projective \( \Rightarrow \) separated” is relevant in this context. Now \( x \equiv y \mod a^n \) for every \( n \), implies \( I \subset \bigcap_{n \in \mathbb{N}^+} a^n = 0 \), so that \( x = y \). Towards the surjectivity of (7) we note that \( Y \) can be regarded as a locally closed subscheme of some \( \mathbb{P}^k_A \), by [20, Corollaire (5.3.3)] and [25, Lemme (18.3.2.1)]. However \( \mathbb{P}^k_A \rightarrow \lim_{\leftarrow n} \mathbb{P}^k(A/a^n) \) does hold, due to the moduli interpretation of \( \mathbb{P}^k \) given in [20, Proposition (4.2.3)] together with [25, Lemme (18.3.2.1)], which establishes the projectivity of inverse limits of finitely generated projective modules. It is elementary to deduce our assertion for locally closed subschemes of \( \mathbb{P}^k_A \).

(ii) Fix some \( y \in Y(A/J) \). According to Grothendieck’s “Élimination des hypothèses noethériennes” the finitely presented \( A \)-scheme \( Y \) allows a form \( Y_0 \) over a finitely generated \( \mathbb{Z} \)-algebra \( A_0 \subset A \) together with some \( y_0 \in Y_0(A_0/A_0 \cap J) \) inducing
separated and quasi-compact. After enlarging \( A_0 \) we may assume that \( Y_0 \) is smooth and quasi-projective over \( \text{Spec} \ A_0 \), by [24, Théorème (8.10.5.xiv)]. Let us endow \( A_0 \) with the \( J_0 \)-adic topology, where \( J_0 := A_0 \cap J \). Observe that the natural inclusion \( A_0 \hookrightarrow A \) is a continuous homomorphism, in view of the finite generation of \( J_0 \) and the assumption on \( J \). Passage to completions yields a homomorphism \( A_0 := \lim_{\to} A_0/J_0^n \to A \). According to the smooth quasi-projectivity of \( Y_0 \) and part (i) of our lemma there exists \( \hat{y}_0 \in Y_0(\hat{A}_0) \) which lifts \( y_0 \). The image of \( \hat{y}_0 \) in \( Y_0(A) = Y(A) \) is the requested lift of \( y \), because \( J \) is closed. \( \square \)

2.1. Lifts and inverse limits of torsors. Occasionally we look at inverse limits of smooth group schemes, for the behavior of whose torsors we offer the following easy lemma:

**Lemma 2.5.** Let \( A \) be a ring and let \( G \to \text{Spec} \ A \) be the inverse limit of a projective system of smooth affine group schemes over \( A \)

\[
G_1 \leftarrow G_2 \leftarrow \cdots
\]

such that all transition homomorphisms are surjective and smooth.

(i) An inverse limit of a projective system of \( G_n \)-torsors is a \( G \)-torsor and all \( G \)-torsors arise in this way.

(ii) If \( P \) is a \( G \)-torsor, there exists a suitable direct limit of rings

\[
A \to R_1 \to R_2 \to \cdots
\]

with étale faithfully flat transition maps, such that \( P(\lim_{\to} R_n) \neq \emptyset \).

**Proof.** (i): Let \( \text{Spec} \ A \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \) be a projective system of \( G_n \)-torsors. First and foremost notice that all torsors in question are affine schemes, by part (i) of corollary B.11. In view of this and according to [24, Proposition (8.2.3)] we may consider the inverse limit \( P = \lim_{\to} P_n \), which is again an affine scheme. In fact \( P \) is formally smooth and faithfully flat over \( A \), as all transition maps \( P_{n+1} \to P_n \) are smooth and surjective, which may be deduced from [25, Proposition (17.1.6.ii)] and [23, Proposition (2.1.3.ii)], as \( G_{n+1} \to G_n \) is smooth and surjective.

In the other direction, consider the projective system of \( G_n \)-torsors \( P_n \), arising from some \( G \)-torsor \( P \) by an extension of structure group via \( G \to G_n \). In order to check that \( P \to \lim_{\to} P_n \) is an isomorphism we can use \( G \to \lim_{\to} G_n \) and pass to a faithfully flat extension, which is legitimate according to [24, Proposition (8.2.5)] and [23, Proposition (2.1.3.ii)].

(ii): We build up the direct limit together with a compatible system of sections \( x_n \in P_n(R_n) \) by induction on \( n \). Indeed we may apply [25, Corollaire (17.16.3.ii)] to find a lift \( x_{n+1} \in P_{n+1}(R_{n+1}) \) of \( x_n \in P_n(R_n) \) for some étale faithfully flat \( R_n \)-algebra \( R_{n+1} \). Passing to the limit we get \( P(\lim_{\to} R_n) \neq \emptyset \). \( \square \)

**Lemma 2.6.** Fix an affine group scheme \( G \to \text{Spec} \Lambda \), where \( \Lambda \) is a discrete valuation ring and \( G \) is the inverse limit of a projective system of smooth affine group schemes over \( \Lambda \)

\[
G_1 \leftarrow G_2 \leftarrow \cdots
\]
such that all transition homomorphisms are surjective and smooth. Assume that some \( \Lambda \)-algebra \( A \) is complete with respect to some linear topology and that \( J \subset A \) is a closed ideal consisting of topologically nilpotent elements.

(i) The natural map \( \text{Hom}_G(P, Q) \to \text{Hom}_G(P \times_A A/J, Q \times_A A/J) \) is surjective for any \( G \)-torsors \( P \) and \( Q \) over \( A \).

(ii) Every \( G \)-torsor over \( A/J \) allows a lift to \( A \).

**Proof.** If \( P \) and \( Q \) are as in (i), we let \( P_n \) (resp. \( Q_n \)) be the \( G_n \)-torsor that arises from \( P \) (resp. \( Q \)) by extension of the structure group via \( G \to G_n \). Let \( Y_n = \text{Hom}_{G_n}(P_n, Q_n) \) be the smooth and affine \( A \)-scheme representing

\[
R \mapsto \text{Hom}_{G_n}(P_n \times_A R, Q_n \times_A R)
\]

according to part (ii) of corollary B.11. It is easy to see that all transition maps \( Y_{n+1} \to Y_n \) are smooth while the \( A \)-scheme \( Y := \lim_n Y_n \) represents the functor \( R \mapsto \text{Hom}_G(P \times_A R, Q \times_A R) \). Fix a section \( y \in Y(A/J) \) and let \( y_n \) be its image in \( Y_n(A/J) \). By induction we build up a sequence \( x_n \in Y_n(A) \) such that \( x_{n+1} \) maps to \( x_n \) while \( y_n \) is the \( \text{mod } J \) reduction of \( x_n \). Indeed consider the commutative square:

\[
\begin{array}{ccc}
\text{Spec } A/J & \xrightarrow{y_{n+1}} & Y_{n+1} \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{x_n} & Y_n
\end{array}
\]

Since \( Y_{n+1} \to Y_n \) is smooth we may apply part (ii) of lemma 2.4 to the smooth affine \( A \)-scheme \( X := Y_{n+1} \times_{Y_n \times_A A} A \) and thus obtain an oblique morphism in the above diagram, i.e. a \( x_{n+1} \in Y_{n+1}(A) \) mapping to \( x_n \) and having \( \text{mod } J \) reduction equal to \( y_{n+1} \).

(ii): In order to construct lifts of torsors we proceed in three steps. Let us start with verifying the assertion for the special case \( G = GL(h) \). According to Hilbert 90 it suffices to prove that any finitely generated projective \( A/J \)-module \( M \), say of constant rank \( h \), can be lifted to a projective \( A \)-module of the same rank. Now, it is folklore that the functor associating to a scheme \( X \) above diagram, i.e. a \( GL(h) \)-module of the same rank. Now, it is folklore that the functor associating to a scheme \( T \) the set of all projective quotients \( \mathcal{O}_T^n \to \mathcal{F} \) of rank \( h \) is representable by a smooth projective \( \mathbb{Z} \)-scheme, namely the Grassmanian \( \text{Grass}(h, n) \). So upon a choice of generators \( x_1, \ldots, x_n \) of our \( A/J \)-module \( M \) we obtain a surjection \( (A/J)^n \to M \), and hence an \( A/J \)-valued point of \( \text{Grass}(h, n) \), to which we can apply part (ii) of lemma 2.4.

The second step consists of checking the result whenever \( G \) is smooth, so that it allows a closed immersion

\[
i : G \to GL(h)_\lambda
\]

for some \( h \in \mathbb{N} \), according to [11, Exposé VIB, Proposition 11.11(v)] and [11, Remarque 11.11.1]. This allows to mimic the method of [6, Lemma B.0.17b]), so let us start with a \( G \)-torsor \( P_0 \) over \( A/J \): By an extension of structure group we get a \( GL(h)_\lambda \)-torsor \( P_0 \otimes_{GL(h)} GL(h)_\lambda =: Q_0 \) over \( A/J \). According to the previous step there exists a \( GL(h)_\lambda \)-torsor \( Q \) over \( A \) which lifts \( Q_0 \). By lemma B.13 there exists a smooth and quasi-projective \( A \)-scheme \( Y \) representing the fpqc-quotient \( Q/G \). Its functor may be described as \( G \)-torsors of which the extension of structure group via (8) is isomorphic to \( Q \). So our \( G \)-torsor \( P_0 \) together with the isomorphism
$Q_0 = P_0 * G_i: GL(h) \Lambda \cong Q \times_A A/J$ constitutes an $A/J$-valued point of $Y$. We are done, because we may apply part (ii) of lemma 2.4 to this point.

The general case can be reduced to this in a third and last step: Choose $G$-torsors $\tilde{P}_i$ that lift $P * G_i$ from $A/J$ to $A$. Observe that $\tilde{P}_i$ and $\tilde{P}_{i+1} * G_i$ have like restrictions to $\text{Spec} A/J$, and use part (i) of our lemma to choose lifts $\psi_i : \tilde{P}_i \cong \tilde{P}_{i+1} * G_i$ of the natural isomorphisms between their mod $J$-reductions. The projective system thus defined possesses a limit $P$, by part (i) of lemma 2.5, and this is the desired lift of $P$. \hfill \Box

The following is one of our prime tools:

**Corollary 2.7.** Let $\Lambda$ and $G \to \text{Spec} \Lambda$ be as in lemma 2.6. Let $A$ be a $\Lambda$-algebra and assume that a descending chain of ideals $\{a_i\}_{i \in \mathbb{N}_0}$ defines an admissible topology on $A$. Then the natural functor from the category of $G$-torsors over $A$ to the category of projective systems of $G$-torsors over $A/a_i$ is an equivalence of categories.

**Proof.** Let $\{P_i\}_{i \in \mathbb{N}_0}$ be a projective system of $G$-torsors, where $P_i$ is the specimen defined over $A/a_i$. Choose $k_0 \in \mathbb{N}_0$ as indicated in (6), so that $a_{k_0}$ is a topologically nilpotent open ideal in $A$. Thanks to part (ii) of lemma 2.6 we deduce the existence of a $G$-torsor $P$ over $A$ which lifts $P_{k_0}$. We claim that $P$ serves as an inverse limit of the projective system $\{P_i\}_{i \in \mathbb{N}_0}$: Indeed we have an isomorphism $P_{k_0} \cong P \times_A A/a_{k_0}$. Thanks to part (i) of lemma 2.6 we can lift it to obtain an isomorphism $P_{k_0+1} \cong P \times_A A/a_{k_0+1}$. We keep lifting that and thus obtain a preimage of $\{P_i\}_{i \in \mathbb{N}_0}$. \hfill \Box

**Remark 2.8.** In view of the remark 2.3, the above is more general than [6, Lemma B.0.17.b)], which treats reductive group schemes only.

### 2.2. Greenberg transforms.

By means of the usual addition and multiplication one can regard $\mathbb{A}^1$ as a ring scheme over $\mathbb{Z}$. Now pick $n \in \mathbb{N} \cup \{\infty\}$ and a prime number $p$ and recall the ring scheme $W_n$ of Witt vectors of length $n$ with respect to $p$: Its underlying scheme is $\text{Spec} \mathbb{Z}[x_0, \ldots, x_{n-1}]$, on which there exists a unique ring schemes structure, which makes the so-called ghost coordinates

$$w_k : \text{Spec} \mathbb{Z}[x_0, \ldots, x_{n-1}] \to \mathbb{A}^1; (x_0, \ldots, x_{n-1}) \mapsto \sum_{i=0}^{k} p^i x_i^{p^i-1}$$

into homomorphisms from the ring scheme $W_n$ to the ring scheme $\mathbb{A}^1$, for every $k \in \{0, \ldots, n-1\}$. If $n = \infty$ we also write

$$W := W_\infty \cong \varprojlim_n W_n,$$

in which case there is also an additive map $V : W \to W; (x_0, x_1, \ldots) \mapsto (0, x_0, \ldots)$, called the Verschiebung, and the so-called absolute Frobenius $F : W \to W$, which is the unique ring endomorphism that satisfies

$$\forall k \in \mathbb{N}_0 : \quad w_k \circ F = w_{k+1}$$

Notice that $F(V(x)) = px$ and $V(xF(y)) = yV(x)$ hold for any two Witt-vectors $x$ and $y$. We denote Cartier’s diagonal by

$$\Delta : W \to W \circ W,$$

this is the unique transformation satisfying $w_k \circ \Delta = F^k$, and for its truncation of length $n$ we write $\Delta_n : W \to W_n \circ W$. At last we let $I \subset W$ be the closed ideal
scheme defined by the equation \( x_0 = 0 \). One also writes \([x]\) for the Teichmüller lift, which is the Witt-vector \((x, 0, \ldots)\).

We write \(B\) for a fixed commutative unital ring, \(B - \text{alg}\) for the category of commutative \(B\)-algebras with unit and let \(\text{Ens}\) be the category of sets. A covariant functor from \(B - \text{alg}\) to \(\text{Ens}\) will simply be called a \(B\)-functor. Notice that every (not necessarily affine) \(B\)-scheme naturally gives rise to a \(B\)-functor, and that the set of \(B\)-morphisms between any two \(B\)-schemes can be recovered from the functorial transformations between their \(B\)-functors. In this optic every \(B\)-scheme “is” (i.e. represents) a \(B\)-functor. In this subsection we are particularly interested in arithmetic jet space of \(X\) of length \(n\), which is the \(B\)-functor:

\[
M_{n,B}X : B - \text{alg} \rightarrow \text{Ens} : R \mapsto X(W_n(R)),
\]

where \(n \in \mathbb{N} \cup \{\infty\}\) is fixed and \(X\) is a \(W_n(B)\)-scheme. Notice that \(W_n(R)\) can be regarded as a \(W_n(B)\)-algebra, so that the notation \(X(W_n(R))\) is meaningful. Also, note that the construction of \(M_{n,B}\) is compatible with base change, more precisely, for any \(B\)-algebra \(R\), and any \(W_n(B)\)-scheme \(X\) there is a canonical isomorphism

\[
(M_{n,B}X)_R \cong M_{n,R}(X_{W_n(R)}).
\]

Observe that for every \(k \in \{0, \ldots, n - 1\}\) there is a transformation of \(B\)-functors

\[
M_{n,B}X \xrightarrow{w_k} X \times_{W_n(B),w_k} B
\]

that is canonically induced by the ghost maps \(w_k : W_n(R) \rightarrow R\). It seems to be well-known (e.g. see [34, Proposition 29] and references therein), that the functor

\[
B - \text{alg} \rightarrow W_n(B) - \text{alg}
\]

\[
R \mapsto W_n(R)
\]

allows a left-adjoint, say

\[
\Lambda_{n,B} : W_n(B) - \text{alg} \rightarrow B - \text{alg},
\]

so that

\[
M_{n,B} \text{Spec } A \cong \text{Spec } \Lambda_{n,B}(A)
\]

where \(A\) is an arbitrary \(W_n(B)\)-algebra. Please see [2] for much more on this issue, e.g. for generalizations to algebraic spaces. In particular, the affine case of (11) reads:

\[
m_k : A \otimes_{w_k,W_n(B)} B \rightarrow \Lambda_{n,B}(A)
\]

We refer to these as the co-ghost maps (and may occasionally confuse them with the \(W_n(B)\)-linear pendants \(m_k : A \rightarrow \Lambda_{n,B}(A)[w_k]\) for any \(k < n\)). Before we move on, let us list some properties of the functor \(\Lambda_{n,B}\)

**Lemma 2.9.** Let \(B\) be a ring. We have the following facts about the functor \(\Lambda_{n,B}\):

(i) it preserves arbitrary direct limits,
(ii) it turns surjections into surjections
(iii) there is a canonical isomorphism

\[
\Lambda_{n,B}(W_n(B)[t_1, \ldots, t_r]) \cong B[t_{ij}; 1 \leq i \leq r, 1 \leq j \leq n]
\]

where \(t_i\) and \(t_{ij}\) are independent variables.
(iv) it turns finitely generated (resp. finitely presented) $W_n(B)$-algebras into finitely generated (resp. finitely presented) $B$-algebras, provided that $n$ is finite.

(v) it turns formally smooth $W_n(B)$-algebras into formally smooth $B$-algebras.

(vi) more generally, there is a canonical and functorial isomorphism

$$
\Omega^1_{\Lambda_n,B(A)/B} \cong \prod_{k=0}^{n-1} (\Lambda_n,B(A) \otimes m_k A \Omega^1_{A/W_n(B)})
$$

where $A$ is a $W_n(B)$-algebra.

Proof. The first two assertions follow from the fact that $\Lambda_n,B$ has a right adjoint. Statement (3) follow from the adjunction formula (12), and the canonical set-theoretical bijection $W_n(R) \cong \mathbf{R}^\times n$. Statement (4) follows from (2) and (3). Assertion (5) is a consequence of the formula

$$
W_n(C \oplus M) \cong W_n(C) \oplus \prod_{k=0}^{n-1} M\omega_k
$$

where the multiplication on $C \oplus M$ is $(c' + m')(c + m) = c'c + c'm + cm'$ and $M$ is any $C$-module, the dual number case of this appeared for the first time in [44]. Finally, the last assertion would follow by a similar argument. \qed

Now suppose that $X$ is an affine $W(B)$-scheme, so that the $B$-scheme $M_B X := M_{x,B} X$ is well-defined. It will not cause confusion to write $M_{n,B} X$ for $M_{n,B}(X \times_{W(B)} W_n(B))$ (for some $n \in \mathbb{N}$). Moreover, when using this convention one has $M_B X \cong \lim_{n \rightarrow \infty} M_{n,B} X$. We need the following easy variant of [6, Proposition 2.2.1c),d]):

**Lemma 2.10.** Fix $0 \leq n' < n \leq \infty$ and a smooth affine $W_n(B)$-scheme $X$.

(i) The truncation map

$$
M_{n,B} X \rightarrow M_{n',B} X
$$

is smooth unless $n = \infty$, in which case it is still flat and formally smooth.

(ii) The morphism (15) is faithfully flat if and only if $X \times_{W_n(B),\omega_0} B \rightarrow \text{Spec } B$ is surjective or $n' > 0$.

In particular the zeroth ghost map $M_{n,B} X \rightarrow X \times_{W_n(B),\omega_0} B$ (cf. (11)) is faithfully flat and formally smooth.

Proof. Without loss of generality we assume that $n$ is finite, as the case $n = \infty$ follows by a limit process, regardless of whether we are discussing the assertion (i) or (ii). Observe also that [28] or [2, Table 3] settle the smoothness of each $M_{n,B} X \rightarrow \text{Spec } B$, which is the case $n' = 0$. To deduce the smoothness of (15). We observe that $O_{M_{n,B} X} \otimes O_{M_{n',B} X} \rightarrow \Omega^1_{M_{n,B} X/B}$ is a direct factor of $\Omega^1_{M_{n,B} X/B}$, which one can read off from the formula (13). We deduce the smoothness of (15) according to [25, Theoreme 17.11.1], more specifically the implication “c) => b)” therein.

Towards the surjectivity of (15) we let $x$ be a point of $M_{n',B}$ over an algebraically closed field $k$, i.e. $x \in X(W_{n'}(k))$. If the characteristic of $k$ is $p$ we know that $W_n(k) \rightarrow W_n(k)$ is a nilpotent thickening, provided of course, that $n' > 0$! So in this case the existence of a lift of $x$ to $M_{n,B} X$ follows easily from formal smoothness of $X$. If the characteristic of $k$ is different from $p$, we have $W_{n}(k) \cong W_{n'}(k) \times k^{n-n'}$. 


It follows that there exists a $W_n(k)$-linear section $W_{n'}(k) \to W_n(k)$, provided of course, that $n' > 0$. Again we are able to lift $x$ to $M_{n,B}X$. At last note that the scheme $M_{1,B}X$ is the base change of $X$ along $W_n(B) \to B$ and $M_{0,B}X \cong \text{Spec} B$, thus settling the fringe case $n' = 0$. □

**Lemma 2.11.** Let $B$ be a ring and $n \in \mathbb{N} \cup \{\infty\}$. Let $G$ be a smooth affine group scheme over $W_n(B)$.

(i) If $P$ is a $G$-torsor, then $M_{n,B}P$ is a $M_{n,B}G$-torsor.

(ii) If $Q$ is another $G$-torsor, then the natural map from $\text{Hom}_{M_{n,B}G}(M_{n,B}P, M_{n,B}Q)$ to $\text{Hom}_{M_{n,B}P}(M_{n,B}P, M_{n,B}Q)$ is an isomorphism of $B$-schemes.

**Proof.** (i): Observe that $M_{n,B}P$ is a well-defined affine $B$-scheme because $P$ is affine as a consequence of part (i) of corollary B.11. It is clear that the natural action of $M_{n,B}G$ on $M_{n,B}P$ from the right gives the latter the structure of a pseudo-torsor, as $M_{n,B}$ preserves fiber products (cf. [6, Proposition 2.2.1a)]). Since $P \to \text{Spec} W_n(B)$ is surjective we are able to deduce the faithful flatness of $M_{n,B}P \to M_{0,B}X \cong \text{Spec} B$, in view of part (ii) of lemma 2.10. So that we are done according to lemma B.12. □

(ii): According to [23, Proposition (2.7.1.viii)] we can pass to a faithfully flat extension, and use (10).

Now fix a finite field $k_0$ of characteristic $p$ and let $G$ be an affine group scheme over $W(k_0)$. Let $R$ be a $W(k_0)$-algebra. Using the Cartier map $W(k_0) \to W(W(k_0))$, we can endow the ring $W_n(R)$ with the structure of a $W(k_0)$-algebra. We write $L^+G = M_{n,W(k_0)}(G_{W_n(W(k_0))})$, resp. $L^+G = M_{W(k_0)}(G_{W(W(k_0))})$, for the affine group schemes over $W(k_0)$ whose group of $R$-valued points is given by $G(W_n(R))$, resp. $G(W(R))$. Notice that there are homomorphisms of group schemes

$$L^+G \to \varphi L^+G = L^+\varphi G$$

$$\omega^G_k : L^+G \to \varphi^k G,$$

which are canonically induced by the absolute Frobenius $\varphi : W(R) \to W(R)$ and the ghost maps:

$$\omega_k : W(R) \to R_{\{\varphi^k\}}: (x_0, x_1, \ldots) \mapsto \sum_{i=0}^{k} \varphi^i x_i^{p^k-i}$$

Furthermore, we write $L^{+0}G \subset L^+G$ for the kernel of $\omega^G_0 : L^+G \to G$. Also notice that $M_{n,k_0}(G_{W_n(k_0)})$, resp. $M_{k_0}(G)$ are the special fibers of $L^+_nG$, resp. $L^+G$ (see (10)). One more piece of notation will prove useful in the sequel: If $a$ is a commutative but not necessarily unital $W(k_0)$-algebra, so that $W(k_0) \otimes a$ is a unital $W(k_0)$-algebra with augmentation ideal $a$, we write $G(a)$ for the kernel of the map from $G(W(k_0) \otimes a)$ to $G(W(k_0))$. Observe that we have an exact sequence

$$1 \to G(b) \to G(a) \to G(a/b),$$

whenever $b \subset a$ is a $W(k_0)$-submodule with $ba \subset b$, which are the so-called ideals of $a$. In the same vein we have an exact sequence

$$1 \to G(a) \to G(R) \to G(R/a),$$

whenever $a$ happens to be an ideal in a $W(k_0)$-algebra $R$. In this optic one can think of $L^{+0}G(R)$ as the group of $I(R)$-valued points of $G$, where $I(R)$ is regarded as a
non-unital $W(k_0)$-algebra. Recall that $G$ is called a vector group scheme provided that it can be written in the form

\begin{equation}
\text{Spec} \left( \bigoplus_{l=0}^{\infty} \text{Sym}^l W(k_0) \mathfrak{v} \right) =: \mathfrak{v}
\end{equation}

for some free $W(k_0)$-module $\mathfrak{v}$ of finite rank. In this case the map (16) has an important variant, observe that $\mathfrak{v}$, $L^+ \mathfrak{v}$ and $L^{>0} \mathfrak{v}$ represent respectively the group functors on $W(k_0)$-algebras:

$$R \mapsto \begin{cases} 
\mathfrak{v}(R) = R \otimes_{W(k_0)} \mathfrak{v} \\
L^+ \mathfrak{v}(R) = W(R) \otimes_{W(k_0)} \mathfrak{v} \\
L^{>0} \mathfrak{v}(R) = I(R) \otimes_{W(k_0)} \mathfrak{v},
\end{cases}$$

allowing us to define the mutually inverse maps:

\begin{align}
V_{\mathfrak{v}}^{-1} : V_{L^{>0} \mathfrak{v}} &\to L^+ \mathfrak{v} & a \otimes X &\mapsto V_{L^{>0} \mathfrak{v}}^{-1}(a) \otimes X \\
V_{\mathfrak{v}} : F L^+ \mathfrak{v} &\to V_{L^{>0} \mathfrak{v}} & a \otimes X &\mapsto V(a) \otimes X,
\end{align}

where

$$F L^+ \mathfrak{v} = L^+ \mathfrak{v} \times_{W(k_0), F} W(k_0) \cong L^+ W(k_0) \otimes_{F, W(k_0)} \mathfrak{v}.$$ 

\textbf{Lemma 2.12.} Let $B$ be a $p$-adically separated and complete $W(k_0)$-algebra and $n \in \mathbb{N} \cup \{ \infty \}$. Let $G$ be a smooth affine group scheme over $W(k_0)$. The functor:

$$P \mapsto L^+_{n,B} P$$

provides an equivalence from the category of $G$-torsors over $W_n(B)$ to the category of $L^+_{n,B} G$-torsors over $B$.

\textit{Proof.} Corollary 2.7 and part (i) of lemma 2.5 are reducing the problem to the discrete finite-length case, i.e. we are allowed to assume $pB \subset \sqrt{\mathcal{O}_B}$ and $n \neq \infty$.

The full faithfulness of our functor is clear form part (ii) of lemma 2.11. It remains to show the essential surjectivity. Let $Q$ be a $L^+_{n,B} G$-torsor over $B$ and let $\tilde{B}$ be an étale faithfully flat extension that trivializes $Q$. Observe that $W_n(B) \to W_n(\tilde{B})$ is étale faithfully flat too and that we have canonical isomorphisms

$$G(W_n(B)^{\otimes_{W_n(B)} m}) \cong L^+_{n,B} G(B^{\otimes_m})$$

by [35, Proposition A.8] and [35, Proposition A.12]. Hence the result follows from descent theory. \hfill \Box

\textbf{Remark 2.13.} The case $G = \text{GL}(h)_{W(k_0)}$ is nothing but [52, Corollary 34] in disguise. Again, please see [6, Proposition B.0.15b)] for the reductive case.

\textbf{2.3. Basics on algebraic groups and their cocharacters.} Let $k_0$ be a finite field. For any smooth affine group scheme $G$ over $W(k_0)$ we let $\text{Rep}_{W(k_0)}(G)$ be the category of representations of $G$ on free $W(k_0)$-modules of finite rank. Furthermore we write $\omega^G$ for the natural fiber functor from $\text{Rep}_{W(k_0)}(G)$ to the category $W(k_0) - \text{Vec}$ of free $W(k_0)$-modules of finite rank. Let us fix a cocharacter $\mu : \mathbb{G}_{m,W(k_0)} \to G$ and an integer $m$. For a representation $\rho \in \text{Ob}_{\text{Rep}_{W(k_0)}(G)}$ we write

$$\omega^G_\mu(m, \rho) \subset \omega^G(\rho)$$
for the subspace of $\mu$-weight $m$ and 

$$\text{Fil}_m^\mu \rho = \bigoplus_{l=m}^{\infty} \omega^G_\mu(l, \rho)$$

for the filtration which is induced by $\mu$. For every non-negative $m \in \mathbb{N}$ we consider the associated subgroup functor of $G$:

$$R \mapsto \{ g \in G(R) \mid \forall \rho, l_0 : (\rho(g) - 1)(\omega^G_\mu(l_0, \rho)) \subset R \otimes_{W(k_0)} \text{Fil}_m^h \rho \}$$

We note the following facts:

(i) According to [47, Chapitre IV, 2.1.3] there exists a closed subgroup scheme $U_m^\mu \to G$ which represents the subgroup functor (21), moreover $U_m^\mu$ is connected, unipotent and smooth over $W(k_0)$, and its Lie algebra can be computed from

$$\text{Lie} U_m^\mu \cong \text{Fil}_m^\mu \text{Ad}^G,$$

by [47, Chapitre IV, Proposition 2.1.4.1]. Please see [9, 2.1] or [7, 4th part of Theorem 4.1.7] for alternative treatments.

(ii) If $g \in U_m^\mu(R)$ and $h \in U_m^\mu(R)$ then $ghg^{-1}h^{-1} \in U_{m+n}^\mu(R)$, which can be deduced from:

$$1 - \rho(ghg^{-1}h^{-1}) = \rho(g)[1 - \rho(g)^{-1}, 1 - \rho(h)]\rho(h)^{-1}$$

where $\rho \in \text{Ob}_{\text{Rep}_{W(k_0)}(G)}$ is a representations of $G$ on some free $W(k_0)$-module of finite rank.

We will use the notations:

$$P_\mu := U_\mu^0$$

$$U_\mu := U_\mu^1$$

$$p_\mu := \text{Lie} P_\mu \cong \text{Fil}_\mu^0 \text{Ad}^G$$

$$u_\mu := \text{Lie} U_\mu \cong \text{Fil}_\mu^1 \text{Ad}^G$$

$$\mathfrak{g} := \text{Lie} G = \omega^G(\text{Ad}^G)$$

The following lemma can easily be checked by means of an embedding into a suitable general linear group (cf. [6, A.0.9]):

**Lemma 2.14.** There exists an action

$$\text{int}_\mu : \mathbb{A}_1^{W(k_0)} \times_{W(k_0)} P_\mu \to P_\mu$$

of the monoid $\mathbb{A}_1^{W(k_0)}$ on the group scheme $P_\mu$ by group scheme endomorphisms, whose restriction to $\mathbb{G}_{m,W(k_0)}$ is given by the formula:

$$\text{int}_\mu(z, g) = \mu(z)g\mu\left(\frac{1}{z}\right)$$

Proofs of the following result can be found in [7, 4th part of Theorem 4.1.1] and [9, 3rd part of Proposition 2.1.8]:

**Lemma 2.15.** Let $k_0$ and $G$ be as above. Let $\mu : \mathbb{G}_{m,W(k_0)} \to G$ be a cocharacter. The group multiplication $m_G : P_\mu \times_{W(k_0)} U_{\mu^{-1}} \to G$ identifies the product $P_\mu \times_{W(k_0)} U_{\mu^{-1}}$ with an open subscheme $G^\mu \subset G$. 
Notation 2.16. Let $k_0$, $G$ and $\mu$ be as above. Let $A$ be a $W(k_0)$-algebra and $I$ an ideal in $A$. We denote by $\Gamma^\mu_{A,I}$ the preimage of $P_\mu(A/I)$ in $G(A)$ under the projection $G(A) \to G(A/I)$.

We have the following remarkable corollary:

Corollary 2.17. Let $k_0$ and $G$ be as above. Let $\mu : \mathbb{G}_m,W(k_0) \to G$ be a cocharacter and let $A$ be a $W(k_0)$-algebra. Consider an ideal $I$ which is contained in the Jacobson radical $\text{rad}(A)$ of $A$. Then, the multiplication in $G(A)$ gives identifications

$$P_\mu(A) \times U_{\mu^{-1}}(I) \cong \Gamma^\mu_{A,I}$$

$$P_\mu(I) \times U_{\mu^{-1}}(I) \cong G(I)$$

Proof. The second identification follows from the first one. The idea of the proof is as in [6, proposition 3.1.3] and for convenience of the reader we spell out the details: Consider $h \in \Gamma^\mu_{A,I} \subseteq G(A)$. We would like to show that the corresponding $h : \text{Spec}(A) \to G$ factors through $G^*$. Since $G^*$ is an open subscheme of $G$ by the previous lemma, it suffices to show that the set theoretic image of $h$ is contained in $G^*$, which is equivalent to the emptiness of $h^{-1}(G \setminus G^*)$. Notice that every non-empty closed subset of $\text{Spec}(A)$ meets $\text{Spec}A/I$ in at least one point, given that every maximal ideal of $A$ contains $I$. Whence it suffices to check the disjointness of $h^{-1}(G \setminus G^*)$ and $\text{Spec}A/I$. However, from $h \in \Gamma^\mu_{A,I}$ we deduce $h(\text{Spec}A/I) \subseteq P_\mu \subseteq G^*$.

Recall the Tannakian formalism involving the forgetful fiber functor $\omega^G$ on the representation category $\text{Rep}_{W(k_0)}(G)$ of our smooth affine group $G$, which is flat over the Dedekind ring $W(k_0)$: For every $W(k_0)$-algebra $R$ one recovers $G(R)$ as the group of $\otimes$-preserving functorial transformations from the $R$–$\text{Vec}$-valued $\otimes$-functor $\rho \mapsto R \otimes_{W(k_0)} \omega^G(\rho)$ to itself, i.e.:

$$G(R) = \text{Aut}_R^\otimes(R \otimes_{W(k_0)} \omega^G) = \text{End}_R^\otimes(R \otimes_{W(k_0)} \omega^G) \subseteq \text{End}_R(R \otimes_{W(k_0)} \omega^G)$$

We note a useful Lie-theoretic analog: One recovers $R \otimes_{W(k_0)} \text{Lie} G$ as the additive group of $R$-linear transformations $D : R \otimes_{W(k_0)} \omega^G \to R \otimes_{W(k_0)} \omega^G$ such that

$$D(p_1 \otimes p_2) = D(p_1) \otimes_{W(k_0)} \text{id}_{\omega^G(p_2)} + \text{id}_{\omega^G(p_1)} \otimes_{W(k_0)} D(p_2)$$

holds for any two objects $p_1$ and $p_2$ of $\text{Rep}_{W(k_0)}(G)$. In the sequel we refer to this as the group $\text{Der}_R^\otimes(R \otimes_{W(k_0)} \omega^G)$ of $R$-linear $\otimes$-derivations of $R \otimes_{W(k_0)} \omega^G$. Finally, observe that both $G(R)$ and $R \otimes_{W(k_0)} \text{Lie} G$ can be viewed as subfunctors of:

$$R \mapsto \text{End}_R(R \otimes_{W(k_0)} \omega^G) = R \otimes_{W(k_0)} \text{End}_{W(k_0)}(\omega^G)$$

For an arbitrary connected unipotent algebraic group $U$ over a field $K$ of characteristic $0$ there exists a canonical isomorphism of $K$-varieties

$$\exp_U : \text{Lie} U \cong U,$$

which can be described as follows: Let $R$ be a $K$-algebra and consider a $R$-valued point of $\text{Lie} U$, i.e. some $D \in R \otimes_K \text{Lie} U$, which can be viewed as a $\otimes$-derivation of $R \otimes_K \omega^U$ in the above sense (note that since the characteristic of $K$ is zero, $U$ is smooth). Then $\exp_U(D)$ is the $\otimes$-automorphism of $R \otimes_K \omega^U$ given by the classical series

$$\exp_U(D)(\rho) = \sum_{n=0}^\infty \frac{D(\rho)^n}{n!}$$
Note that \( D(\rho)^n \) vanishes on an \( n \)-dimensional representation \( \rho \) of \( U \), and that all terms on the right are elements in the ring \( R \otimes_K \omega(U(\hat{\rho} \otimes K \rho)) \), see [10, Chapter IV.2, Proposition 4.1] for more details.

**Lemma 2.18.** As before we fix a \( W(k_0) \)-rational cocharacter \( \mu \) of some smooth affine \( W(k_0) \)-group \( G \), so that the generic fiber of \( U_\mu \) is a connected unipotent algebraic group over the fraction field \( K := W(k_0)[\frac{1}{p}] \). If the \( \mu \)-weights of \( g \) are strictly less than \( p \), then \( \exp U_\mu \) extends uniquely to an integral morphism \( e : u_\mu \to U_\mu \).

**Proof.** We have to show that for all \( n \geq 1 \), all \( W(k_0) \)-algebras \( R \), all representations \( \rho \) and all \( D \in R \otimes W(k_0) \) \( \text{Lie} U_\mu \), \( n! \) divides \( D(\rho)^n \). Thanks to the formula

\[
\sum_{j=1}^{n} \frac{a_j \otimes D_j(\rho)^n}{n!} = \sum_{j=1}^{n} \frac{a_j \otimes D_j(\rho)^{i_j}}{i_j!}
\]

it is enough to check that \( n! \) is a divisor of \( D(\rho)^n \) for every \( D \in \text{Lie} U_\mu \). So we prove \( n! \mid D(\rho)^n \). Assume that the statement is true when \( n \) is a power of \( p \). If \( n \) is not a power of \( p \), then we can write \( n = \sum m_i p^l_i \) with \( 0 \leq m_i \leq p-1 \). Since the \( p \)-adic valuations of \( (\sum m_i p^l)! \) and \( \prod (p^l)!^{m_i} \) are equal, \( n! \) divides \( D(\rho)^n \) if and only if \( \prod (p^l)!^{m_i} \) does. We have

\[
\frac{D(\rho)^n}{\prod (p^l)!^{m_i}} = \prod \frac{D(\rho)^{m_i} p^l}{(p^l)!^{m_i}} = \prod \left( \frac{D(\rho)^{p^l}}{(p^l)!} \right)^{m_i}
\]

By the assumption, the quotient \( \frac{D(\rho)^{p^l}}{(p^l)!} \) is integral and thus so is \( \frac{D(\rho)^n}{n!} \). Therefore, it remains to show the statement when \( n \) is a power of \( p \), say \( p^l \). We prove by induction on \( l \). When \( l = 0 \) there is nothing to prove. We have

\[
\frac{D^{p^l}}{(p^l)!} = \frac{(p^l-1)\ldots p}{(p^l)!} \frac{1}{p^l} \left( \frac{D^{p^l-1}}{p^{l-1}} \right)^p
\]

Note that \( \frac{(p^l-1)\ldots p}{(p^l)!} \) is a unit of \( \mathbb{Z}_p \), and so, since by assumption \( \frac{D^{p^l-1}}{p^{l-1}} \) is integral, we have to show that \( \left( \frac{D^{p^l-1}}{p^{l-1}} \right)^p \) is divisible by \( p \). Consider the elements

\[
\sigma = \sum_{j=1}^{p^l} \epsilon_j
\]

\[
\pi = \frac{(p^l)!}{p^l} \prod_{j=1}^{p^l} \epsilon_j
\]

in the ring \( R = W(k_0)[\epsilon_1, \ldots, \epsilon_{p^l}]/(\epsilon_1^2, \ldots, \epsilon_{p^l}^2) \). Let \( R' \) be the sub-\( W(k_0) \)-algebra generated by these two elements. In \( R' \) the ideal \( (p\pi - \sigma \pi, \pi^2, \sigma \pi) \) is trivial. Notice that \( \sigma \pi \) vanishes and that \( \{ \epsilon_k^k \mid k = 1, \ldots, p^l \} \) are given by the elementary symmetric polynomials in \( \epsilon_1, \ldots, \epsilon_{p^l} \). It follows \( \mathbb{Q} \otimes W(k_0)[\sigma]/(\sigma^{p^l+1}) \cong \mathbb{Q} \otimes R' \), so that the \( W(k_0) \)-rank of the free \( W(k_0) \)-module \( R' \) is equal to \( p^l + 1 \). We deduce that the canonical projection \( W(k_0)[\sigma, \pi]/(p\pi - \sigma \pi, \pi^2, \sigma \pi) \to R' \) must be isomorphism,
as both sides have the same $W(k_0)$-rank. Next consider the element

$$g := \prod_{i=1}^{p^l}(1 + \epsilon_i D) \in U_\mu(R),$$

and observe that the formula

$$\frac{(p^{l-1})^p}{p^l!} \cdot \pi \cdot \left( \frac{D^p}{p^l!} \right)^p + \sum_{k=0}^{p^l-1} \sigma^k D^{k \mod p^l} = g$$

holds in $\text{End}_R(R \otimes_{W(k_0)} \omega^{J_{\mu}})$. Consider the ring homomorphism $R' \to k_0[\epsilon]/\epsilon^2$, defined by $\sigma \mapsto 0$ and $\pi \mapsto \epsilon$. We deduce that the $(\text{mod } p)$-reduction of $\frac{D^p}{p^l!}$ is a $k_0$-linear $\otimes$-derivation of $k_0 \otimes_{W(k_0)} \omega^{J_{\mu}}$, but due to $\mu$-weight considerations this can only be the trivial derivation, so that we have $p \mid (\frac{D^p}{p^l!})^p$.

**Corollary 2.19.** As before we fix a $W(k_0)$-rational cocharacter $\mu$ of some smooth affine $W(k_0)$-group $G$, so that the generic fiber of $U_\mu$ is a connected unipotent algebraic group over the fraction field $K := W(k_0)[\frac{1}{p}]$. If the $\mu$-weights of $g$ are bounded by 1, then $\exp_{U_{\mu, K}}$ extends uniquely to an integral isomorphism of group schemes $e : U_\mu \cong U_{\mu}$.

**Proof.** In view of $[u_\mu, u_\mu] \subset \text{Fil}_{2} \text{Ad}^G = 0$, the Baker-Campbell-Hausdorff formula implies that $\exp_{U_{\mu, K}}$ is a homomorphisms of group schemes, hence $e : U_\mu \to U_{\mu}$ is one too. Since $\exp_{U_{\mu, K}}$ induces the identity on the tangent spaces at the neutral section, the same holds for $e$. In view [25, Théorème (17.11.1)] we may deduce that $e$ is étale, so that its kernel is an étale group scheme. In fact, we obtain the triviality of $\ker(e)$ from looking at the generic fiber. So $e$ is an étale monomorphism, i.e. an open immersion, due to [25, Théorème (17.9.1)]. The connectedness of $U_{\mu}$ finishes the proof. \hfill $\Box$

**Remark 2.20.** The above is decisively more general than [38, Lemma 6.3.2] and [6, A.0.13], which treat reductive group schemes only.

**Remark 2.21.** In the more general situation of lemma 2.18 the morphism $e$ fails to respect the group law, but it remains to be an isomorphism of schemes, as can be seen as follows: With a little bit of extra work, along the same lines of the proof of corollary 2.19, one can show that $e$ induces isomorphisms

$$\omega_{\mu}^G(m, \text{Ad}^G) \cong U_{\mu}^m / U_{\mu}^{m+1}$$

(for all $m$). Moreover, observe that $U_{\mu} / U_{\mu}^{m+1}$ is a $U_{\mu}^m / U_{\mu}^{m+1}$-torsor over $U_{\mu} / U_{\mu}^m$. Arguing inductively, one sees that the induced maps of quotients

$$\bigoplus_{l=1}^{m} \omega_{\mu}^G(l, \text{Ad}^G) \to U_{\mu} / U_{\mu}^{m+1}$$

are isomorphisms, because they can be viewed as homomorphisms between $\omega_{\mu}^G(m, \text{Ad}^G)$-torsors over $U_{\mu} / U_{\mu}^m$.

**Corollary 2.22.** Let the assumptions on $k_0$, $G/W(k_0)$ and $\mu : \mathbb{G}_{m, W(k_0)} \to G$ be as in corollary 2.19. Then

$$\Gamma_{\mu^{-1}} \subset \mu(p)G(A)\mu(\frac{1}{p})$$
holds for any torsion-free $W(k_0)$-algebra $A$.

**Proof.** Without loss of generality we can assume $pA \subset \text{rad} \ A$, as $(1 + pA)^{-1} A \times A_p$ is faithfully flat over $A$. By corollary 2.17 we can consider the groups $P_{\mu^{-1}}(A)$ and $U_\mu(pA)$ separately. However,

$$\mu(\frac{1}{p}) P_{\mu^{-1}}(A) \mu(p) \subset P_{\mu^{-1}}(A) \subset G(A)$$

follows immediately from lemma 2.14, and

$$\mu(\frac{1}{p}) U_\mu(pA) \mu(p) \subset U_\mu(A) \subset G(A)$$

follows from corollary 2.19. Observe that for $X \in A \otimes_{W(k_0)} \text{Lie} U_\mu$ we have:

$$\mu(\frac{1}{p}) \exp_{U_\mu}(X) \mu(p) = \exp_{U_\mu}(\text{Ad}^G(\mu(\frac{1}{p}))X)$$

□

2.4. Definition of $H_\mu$ and $\Phi_\mu$. Consider $H_\mu = L^+G \times_{w_0^G, G} P_\mu$, where $w_0^G$ is the zeroth ghost component (cf. (17)). This is a flat and formally smooth $W(k_0)$-group scheme characterized by $H_\mu(R) = \Gamma_{\mu^{-1}(R)}^W(R)^I(R)$, please note that we have the closed immersions:

$$L^{>0}G \hookrightarrow H_\mu \hookrightarrow L^+G$$

**Proposition 2.23.** Assume that $pR$ is contained in the Jacobson radical $\text{rad}(R)$ of $R$. Then multiplication in $L^+G(R)$ gives an identification

$$L^+P_\mu(R) \times L^{>0}U_{\mu^{-1}}(R) \cong H_\mu(R).$$

**Proof.** By part (i) of [6, Lemma 2.1.1] it follows that $I(R) \subset \text{rad}(W(R))$. The result now follows from the definition of $H_\mu$ and corollary 2.17. □

**Proposition 2.24.** There is a group scheme homomorphism

$$\Phi_\mu : H_\mu \to F L^+G = L^{+F}G$$

characterized by the following property: If $g$ is an $R$-valued point of $H_\mu$, where $R$ is a $W(k_0)[\frac{1}{p}]$-algebra, then

$$\Phi_\mu(g) = F(\mu(p) g \mu(\frac{1}{p})) \in F G(W(R)).$$

**Proof.** We begin with the introduction of maps

$$\Phi_0 : L^+P_\mu \to F L^+P_\mu$$

$$\Phi_1 : L^{>0}U_{\mu^{-1}} \to F L^+U_{\mu^{-1}}$$

satisfying (25). We define $\Phi_0$ to be the composition of the relative Frobenius $F : L^+P_\mu \to F L^+P_\mu$ with $L^+(\text{int}_\mu(p))$, i.e. the endomorphism on $L^+P_\mu$ that arises from applying the functor $L^+$ to the extension of $g \to \mu(p) g \mu(\frac{1}{p})$, in the sense of (22), as given by lemma 2.14. In order to define $\Phi_1$ we can use the map $V_{\mu^{-1}}^{-1} : L^{>0}U_{\mu^{-1}} \to F L^+U_{\mu^{-1}}$, as in (20), and we appeal to corollary 2.19 for our transport of structure, namely by the isomorphism $u_{\mu^{-1}} \cong U_{\mu^{-1}}$ which is induced by $\exp_{U_{\mu^{-1}}}$ where $K = W(k_0)[\frac{1}{p}]$. 

Now let \( h \in \mathcal{H}_\mu(A_0) \) be the universal point of \( H_\mu \) with values in its coordinate ring \( A_0 \), which is flat over \( W(k_0) \). It is enough to show \( F(\mu(p)h_\mu(\frac{1}{p})) \in F G(W(A_0)) \). We observe that we can apply proposition 2.23 over the localization \((1+pA_0)^{-1}A_0 = R)\). So there exists a factorization \( h = h_0h_1 \) into elements \( h_0 \in L^+P_\mu(R) \) and \( h_1 \in L^{>0}U_{\mu^{-1}}(R) \). It follows that \( F(\mu(p)h_\mu(\frac{1}{p})) = \Phi_0(h_0)\Phi_1(h_1) \in F G(W(R)) \), which suffices, as we have \( F(\mu(p)h_\mu(\frac{1}{p})) \in F G(W(A_0[\frac{1}{p}])) \) too. Here note that localizations are flat and that \( \{\text{Spec } R, \text{Spec } A_0[\frac{1}{p}]\} \) is an fpqc covering of \( \text{Spec } A_0 \), as any prime ideal \( p \subset A_0 \) with \( p \cap (1+pA_0) \neq \emptyset \) cannot have \( pA_0 \subset p \).

**Remark 2.25.** In the sequel (in particular in the proof of proposition 4.6) we are using Lie-theoretic analogs of \( H_\mu \) and \( \Phi_\mu : H_\mu \to L^+ FG \), namely

\[
\begin{align*}
\mathfrak{h}_\mu(R) &:= L^{>0}u_{\mu^{-1}} \times_{W(k_0)} L^+ G_m \\
\phi_\mu : \mathfrak{h}_\mu &\to L^+ \mathfrak{g}
\end{align*}
\]

which we define by \( V_{\mu^{-1}}^{>0} \) on \( L^{>0}u_{\mu^{-1}} \) and simply by \( p^m F \) on the summand \( L^+ G_m \) of \( \mu \)-weight \( m \geq 0 \). Actually, one recovers the map

\[
\mathfrak{h}_\mu \times H_\mu \to F L^+ \mathfrak{g} \times F L^+ G; \quad (X,k) \mapsto (\phi_\mu(X), \Phi_\mu(k))
\]

when working with \( \mathfrak{g} \times G \) in the proposition 2.24. Please see [6, subsubsection 3.5.3] for a similar construction.

3. \((G, \mu)\)-Displays

For any affine group scheme \( G \) over a ring \( B \) we denote by \( \text{Tor}_S(G) \) the groupoid of \( G \)-torsors over a \( B \)-scheme \( S \), and we write \( \text{Tor}_B^G \) for the fpqc-stack over \( \text{Spec } B \) whose fibers over \( S \) are given by \( \text{Tor}_S(G) := \text{Tor}_S(G) \). By pull-back we obtain \( \text{Tor}_B^G := S \times_{\text{Spec } B} \text{Tor}_B^G \), which is an fpqc-stack over \( S \). We need to introduce a couple of definitions:

**Definition 3.1.** Fix a finite field \( k_0 \). By a display datum over \( W(k_0) \) we mean a pair \((G, \mu)\), where \( G \) is a smooth affine group scheme over \( \mathbb{Z}_p \) and

\[
\mu : G_m, W(k_0) \to G_{W(k_0)}
\]

is a cocharacter all of whose weights on the adjoint representation \( \mathfrak{g} = \omega^G(\text{Ad}^G) \) are contained in the set \( \{-1, 0, 1, 2, \ldots\} \), i.e.

\[
(28) \quad u_{\mu^{-1}} = \omega^G(-1, \text{Ad}^G)
\]

**Definition 3.2.** Fix a finite field \( k_0 \) and let \((G, \mu)\) be a display datum, as above.

1. By the conjugated Frobenius map of \((G, \mu)\) we mean the morphism

\[
H_\mu \xrightarrow{\Phi_\mu} L^+ G_{W(k_0)},
\]

as exhibited in proposition 2.24.
2. The stack of \((G, \mu)\)-displays, denoted by \( \mathcal{B}(G, \mu) \), is the \( W(k_0) \)-stack rendering the diagram
$B(G, \mu) \rightarrow Tor^{L^+ G}_{W(k_0)}$

(29)

$Tor^{H_\mu} \rightarrow Tor^{L^+ G}_{W(k_0)} \rightarrow Tor^{L^+ G}_{W(k_0)} \times_{W(k_0)} Tor^{L^+ G}_{W(k_0)}$

2-Cartesian.

(3) A $(G, \mu)$-display $\mathcal{D}$ over a $W(k_0)$-scheme $S$, is an $S$-point of the stack $B(G, \mu)$, its image under the vertical projection is called the underlying $H_\mu$-torsor and will be denoted by $q(\mathcal{D})$.

(4) The precomposition of the natural 1-morphism $Tor^{H_\mu} \rightarrow Tor^{P_\mu}$, induced by the projection $H_\mu \rightarrow P_\mu$ (cf. subsection 2.4), with the vertical projection is called the lowest truncation and will be denoted by

(30)

$q_0 : B(G, \mu) \rightarrow Tor^{P_\mu}$

Suppose that $S$ is a $W(k_0)$-scheme and that $\mathcal{D}$ is an $S$-valued section of $B(G, \mu)$. We write $P_D \in Ob_{Tor^{L^+ G}(S)}$ (resp. $Q_D \in Ob_{Tor^{H_\mu}(S)}$) for the $L^+ G$-torsor (resp. $H_\mu$-torsor) over $S$, that arises from $\mathcal{D}$ by means of the horizontal (resp. vertical) projection. Moreover, we denote by

$u_D : Q_D \rightarrow P_D$

the morphism making the diagram commute. Sometimes, when there is little risk of confusion, we drop the subscript $\mathcal{D}$ from the notation. Upon identifying $P_D$ with the structure group extension of $Q_D$ along the inclusion $H_\mu \rightarrow L^+ G$, we see that the triple $\mathcal{D} = (P_D, Q_D, u_D)$ is determined by $(Q_D, u_D)$.

The generic fiber of $B(G, \mu)$ has a rather “pathological” behavior:

Lemma 3.3. The generic fiber of the lowest truncation (30) is an isomorphism, and thus yields an equivalence:

$B(G, \mu)_{W(k_0)[\frac{1}{\mu}]} \simeq Tor^{P_\mu}_{W(k_0)[\frac{1}{\mu}]}$

Proof. The diagram (29) being 2-Cartesian suggests immediately to consider the natural 1-morphism

(31)

$Tor^{H_\mu}_{\mu} \rightarrow B(G, \mu)$

where $H_\mu^{\Phi_\mu} \hookrightarrow H_\mu$ is defined to be the equalizer of $\Phi_\mu : H_\mu \rightarrow L^+ G$ and the inclusion of $H_\mu$ into $L^+ G$. In general the 1-morphism (31) fails to be an isomorphism, as the criterion (i) of lemma B.9 is not satisfied. Nevertheless our proof is finished by the following remarks:

- Upon a change of base to $W(k_0)[\frac{1}{\mu}]$ the criterion (i) of lemma B.9 is satisfied.
- Over $W(k_0)[\frac{1}{\mu}]$ the composition

$H_\mu^{\Phi_\mu} \hookrightarrow H_\mu \rightarrow P_\mu$

becomes an isomorphism.
One easily checks the above by using the ghost coordinates of the Witt vectors functor. Indeed, if $R$ is a $\mathbb{Z}[\frac{1}{p}]$-algebra, then the zeroth ghost component induces an isomorphism

$$w_0 : W(R)^{F=\text{id}} \cong R$$

\[ \text{Definition 3.4.} \quad \text{A } (G, \mu)\text{-display } (\mathcal{P}, Q, u) \text{ is called banal if its underlying } H_\mu\text{-torsor } Q \text{ is trivial.} \]

Banal $(G, \mu)$-displays are important for explicit computations, here are some facts:

\[ \text{Lemma 3.5. Let } (G, \mu) \text{ and } k_0 \text{ be as before, and let } D \text{ be a } (G, \mu)\text{-display over an affine scheme } \text{Spec } A, \text{ where } A \text{ is a } W(k_0)\text{-algebra.} \]

(i) Assume that $A$ is a linearly topologized complete ring and that $J$ is a closed ideal consisting of topologically nilpotent elements. Then $D$ is banal if and only if this holds for its pull-back to $\text{Spec } A/J$.  

(ii) Suppose that $pA \subseteq \sqrt{\mathcal{O}_A}$ holds. Then $D$ is banal if and only if $q_0(D)$ has a global section. 

(iii) Suppose that $pA = A$ holds. Then $D$ is banal if and only if $q_0(D)$ has a global section. 

(iv) The groupoid of banal $(G, \mu)$-displays over a $W(k_0)$-scheme $S$ is equivalent to the quotient groupoid $[L^+ G(S)/q_0, H_\mu(S)]$ (cf. appendix A for the notation). In particular the groupoids of banal $(G, \mu)$-displays over $S$ and over $\text{Spec } \Gamma(S, \mathcal{O}_S)$ agree.

\[ \text{Proof. The first statement follows from part (i) of lemma 2.6 while the second one can be handled using the method of [6, Proposition B.0.15a]) (see also [6, Corollary B.0.16]). The third statement is a consequence of lemma 3.3. For the fourth statement consult [6, Subsubsection 3.2.7].} \]

\[ \text{Remark 3.6. Observe that lemma 3.5 implies that a } (G, \mu)\text{-display } D \text{ over a locally noetherian } W(k_0)\text{-scheme } S \text{ allows a banalization over an fpqc covering } \{U_i\}_{i \in I} \text{ where each } U_i \text{ is noetherian. More specifically, if } \{U_i\}_{i \in I} \text{ is an étale covering trivializing } q_0(D) \text{ and such that each } U_i \text{ is affine, say equal to } \text{Spec } A_i, \text{ then } D \text{ is banal over the localization } A_i[\frac{1}{p}] \text{ and the } p\text{-adic completion } \hat{A}_i, \text{ and the collection } \{\text{Spec } A_i[\frac{1}{p}]\}_{i \in I} \cup \{\text{Spec } \hat{A}_i\}_{i \in I} \text{ is indeed an fpqc covering of } S \text{ banalizing } D \text{ and consisting of noetherian schemes.} \]

\[ \text{Remark 3.7. Let } k_0 \subset \Omega \text{ be a field extension with } \Omega \text{ algebraically closed, and let } D \text{ be a } (G, \mu)\text{-display over } \Omega. \text{ Then } D \text{ is banal and so, by lemma 3.5 (iv), it is represented by an element of } G(W(\Omega)). \]

4. Deformation of Banal $(G, \mu)$-Displays

Throughout this section, $k_0$ is a finite field and $(G, \mu)$ is a display datum over $W(k_0)$ (cf. definition 3.1).

The following definition is adopted from [6, Definition 3.4.2] in conjunction with [6, Lemma 3.4.4]. We denote the absolute Frobenius endomorphism of a $\mathbb{F}_p$-algebra $R_0$ by $\text{Frob}_{R_0}$. 
Definition 4.1. Let $R$ be a $W(k_0)$-algebra, and let $\pi$ be the projector on $W(k_0) \otimes_{\mathbb{Z}_p} g$ with $\im(\pi) = u_{p^{-1}}$ and $\ker(\pi) = p_{\mu}$. An element $U \in G(R)$ is said to satisfy the adjoint nilpotence condition if and only if the $p$-linear endomorphism on $R/pR \otimes_{\mathbb{Z}_p} g$ defined by

$$\text{Ad}^G(U) \circ (\text{Frob}_{R/pR} \otimes \text{id}_g) \circ (\text{id}_{R/pR} \otimes \pi)$$

is nilpotent. We will denote by $C_{nil}^R(G, \mu)$ the subset of $G(R)$ consisting of elements satisfying the adjoint nilpotence condition. Note that the construction of $C_{nil}^R(G, \mu)$ is functorial in $R$.

Lemma 4.2. Let $R$ be a $W(k_0)$-algebra, and let $R_0$ be the perfection of $R/\sqrt{pR}$.

(i) The subset $C_{nil}^R(G, \mu) \subset G(R)$ is the preimage of $C_{nil}^{R_0}(G, \mu) \subset G(R_0)$ under the canonical map $G(R) \to G(R_0)$.

(ii) The subset $C_{nil}^W(G, \mu) \subset L^+ G = G(W)$ is the preimage of $C_{nil}^R(G, \mu)$ in $G(R)$ via the zeroth ghost map $G(W) \to G(R)$, and it is invariant under $\Phi_{\mu}$-conjugation, i.e., for all $h \in H_{\mu}(R)$, we have

$$h^{-1}C_{nil}^W(G, \mu) \Phi_{\mu}(h) = C_{nil}^W(G, \mu)$$

Proof: That $C_{nil}^R(G, \mu)$ depends only on $R/pR$ was built into the very definition while passage to $(R/pR)_{\text{red}}$ and $R_0$ does not alter the nilpotence of an $F$-linear endomorphism either. The next claim follows from the formula $I(R)^2 = p I(R)$, which implies the following (note that $p W(R) + I(R)$ is the kernel of the projection $W(R) \to R/pR$):

$$(p W(R) + I(R))^2 = p(p W(R) + I(R)) \subset p W(R) \subset p W(R) + I(R)$$

For the last assertion we may assume that $R$ is perfect (of characteristic $p$) and so consider the $F$-linear endomorphism

$$\phi_{\mu}^{Ad} : W(R)[\frac{1}{p}] \otimes_{\mathbb{Z}_p} g \to W(R)[\frac{1}{p}] \otimes_{\mathbb{Z}_p} g$$

$$X \mapsto \text{Ad}^G(U \Phi_{\mu}(p)^F X).$$

Observe that $p \phi_{\mu}^{Ad}$ preserves the integral structure $\mathfrak{g} = W(R) \otimes_{\mathbb{Z}_p} g$ and that its reduction to $R \otimes_{\mathbb{Z}_p} g$ agrees with the endomorphism defined by the formula (32). It follows that $(\phi_{\mu}^{Ad})^n(\mathfrak{g}) \subset p^{1-n} \mathfrak{g}$ holds for $n >> 0$ if and only if $w_0(U) \in C_{nil}^R(G, \mu)$, which in turn is equivalent to $U \in C_{nil}^W(G, \mu)$. The proof is completed by $\text{Ad}^G(h)^{-1} \circ \phi_{\mu}^{Ad} \circ \text{Ad}^G(h) = \phi_{h^{-1}U \Phi_{\mu}(h)}^{Ad}$ for any $h \in H_{\mu}(R)$. \hfill \Box

Definition 4.3. Let $\mathcal{D}$ be a $(G, \mu)$-display over a $W(k_0)$-scheme $X$, on which $p$ is Zariski-locally nilpotent. Then $\mathcal{D}$ is called adjoint nilpotent if each geometric fiber $\mathcal{D} \times_X \text{Spec} \Omega$ is represented by an element in $C_{nil}^W(G, \mu)$, where $\Omega$ stands for an algebraically closed field. We denote by $\mathcal{B}_{nil}(G, \mu)$ the (full) substack of $\mathcal{B}(G, \mu)$ given by adjoint nilpotent $(G, \mu)$-displays.

Recall that a pd-structure on an ideal $a \subset R$ consists of a family of maps $\{\gamma_k : a \to a\}_{k \in \mathbb{N}}$ such that the following holds:

(i) $\forall n \in \mathbb{N}$, $x, y \in a : \gamma_n(x + y) - \gamma_n(x) - \gamma_n(y) = \sum_{k=1}^{n-1} \gamma_k(x) \gamma_{n-k}(y)$

(ii) $\forall n \in \mathbb{N}$, $x \in R$, $y \in a : \gamma_n(xy) = x^n \gamma_n(y)$

(iii) $\forall n \in \mathbb{N}$, $x \in a$, $n! \gamma_n(x) = x^n$

(iv) $\forall n, m \in \mathbb{N}$, $x \in a : \gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!n!} \gamma_{mn}(x)$
Notice that a pd-structure \( \{ \gamma_k : a \to a \}_{k \in \mathbb{N}} \) on an ideal \( a \subset R \) induces a \( W(R) \)-linear isomorphism

\[
(33) \quad w' : W(a) \overset{\cong}{\to} \prod_{n=0}^{\infty} a_{[w_n]}; \ (x_0, x_1, \ldots) \mapsto (w'_0, w'_1, \ldots),
\]

which is given by Zink's logarithmic ghost coordinates

\[
w'_n(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} (p^i - 1)! \gamma_{p^i}(x_{n-i})
\]

(the usual ones being \( w_n(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} p^{n-i} x_{n-i} \)). Note that the isomorphism above induces a canonical decomposition

\[
W(a) \cong a_{[w_0]} \oplus I(a),
\]

since \( I(a) \cong \prod_{n=1}^{\infty} a_{[w_n]} \) (as \( w'_0 = w_0 \)). We write \( W(a) \to I(a); x \mapsto x^{>0} \) for the associated \( W(R) \)-linear projector whose kernel is \( \bigcap_{n=1}^{\infty} \ker w'_n = a_{[w_0]} \), so

\[
w'_n(x^{>0}) = \begin{cases} w'_n(x) & n > 0 \\ 0 & n = 0 \end{cases}.
\]

Notice that the assignment \( x \mapsto x^{>0} \) is additive and multiplicative, so that we obtain an induced group homomorphism from \( L^+G(a) \) to \( L^{>0}G(a) \).

Before we prove a proposition that shows the significance of adjoint-nilpotence, we need two auxiliary lemmas:

**Lemma 4.4.** Let \( R \) be a \( \mathbb{Z}_p \)-algebra, let \( a \subset R \) be a pd-ideal, and let \( U \) be a \( W(R) \)-valued point of \( G \). Then

(i) for all \( h \in G(W(a)) \), we have \( (UhU^{-1})^{>0} = Uh^{>0}U^{-1} \)

(ii) for all \( X \in W(a) \otimes \mathfrak{g} \), we have \( (\text{Ad}^G(U)X)^{>0} = \text{Ad}^G(U)X^{>0} \).

**Proof.** Observe that conjugation with \( U \in G(W(R)) \) (resp. \( \text{Ad}^G(U) \)) defines an automorphism of \( G \times_{\mathbb{Z}_p} W(R) \) (resp. \( \mathfrak{g} \times_{\mathbb{Z}_p} W(R) \)). The assertions follow as \( W(a) \to I(a); x \mapsto x^{>0} \) is a homomorphism of non-unital \( W(R) \)-algebras (please see subsection 2.2 for our conventions on groups of points over non-unital algebras).

**Lemma 4.5.** Let \( R \) be a \( W(k_0) \)-algebra and let \( a \subset R \) be a pd-ideal with \( pa = 0 \). If \( X \in I(a) \otimes \mathfrak{g} \) corresponds to \( h \in G(I(a)) \) (via \( I(a)^2 = 0 \)), then \( \phi_{\mu}(X) \) corresponds to \( \Phi_{\mu}(h) \), where \( \phi_{\mu} : h_{\mu} \to L^+ \mathfrak{g} \times_{\mathbb{Z}_p} W(k_0) \) is the transformation constructed in remark 2.25.

**Proof.** Observe that we have a decomposition

\[
G(I(a)) \cong P_{\mu}(I(a)) \times U_{\mu^{-1}}(I(a))
\]

and its Lie analog

\[
I(a) \otimes \mathfrak{g} \cong I(a) \otimes_{W(k_0)} p_{\mu} \oplus I(a) \otimes_{W(k_0)} U_{\mu^{-1}}.
\]

It follows that we may verify the assertions separately for \( h \in P_{\mu}(I(a)) \) and \( h \in U_{\mu^{-1}}(I(a)) \). In either case we deduce the lemma from the explicit descriptions of \( \Phi_{\mu} \) on \( L^+ P_{\mu} \) and \( L^{>0}U_{\mu^{-1}} \) ((26) and (27)).

□
Proposition 4.6. Fix a $p$-adically separated and complete pd-ideal $a$ in a $W(k_0)$-algebra $R$ (i.e. such that $R$ is separated and complete with respect to the topology defined by the descending chain of pd-ideals $\{p^n a\}_{n\in\mathbb{N}}$ by part (ii) of lemma 2.2). Let $O$ and $U$ be elements of $C_{W(R)}^n(G, \mu) \subset G(W(R))$, satisfying $UO^{-1} \in G(W(a))$. Then there exists a unique $h \in G(W(a))$ with $O = h^{-1}U\Phi_\mu(h^{>0})$.

Proof. By a limit process followed by a straightforward induction we may assume $pa = 0$. According to the isomorphism (33) we deduce $I(R)W(a) = 0$, given that

$$w_n(Vx) = \begin{cases} pw_{n-1}(x) & n > 0 \\ 0 & n = 0 \end{cases}$$

implies $w_n(I(R)) \subset pR$. We deduce $pW(a) = 0$ by the same token, so that:

$$(pW(R) + I(R))W(a) = 0 = (pW(R) + W(a) + I(R))I(a)$$

In particular we may and will regard $I(a)$ (resp. $W(a)$) as a module over $R/(pR+a)$ (resp. over $R/pR$). Also, notice that the solution sets

$$A = \{h \in G(W(a)) \mid h = U\Phi_\mu(h^{>0})O^{-1}\}$$
$$B = \{h \in G(I(a)) \mid h = (U\Phi_\mu(h^{-1})O^{-1})^{>0}\}$$

correspond to each other under the mutually inverse bijections:

$$B \to A; \quad h \mapsto U\Phi_\mu(h)O^{-1}$$
$$A \to B; \quad h \mapsto h^{>0}$$

Although the set $A$ seems to be more natural to work with, we nevertheless prefer to work with the set $B$, because of the isomorphism

$$G(I(a)) \cong I(a) \otimes_{\mathbb{Z}_p} g,$$

as $I(a)^2 = 0$. In order to show that the set $B$ has exactly one element, we note that

$$(U\Phi_\mu(h)^{>0}O^{-1})^{>0} = (U\Phi_\mu(h)^{>0}U^{-1})(UO^{-1})^{>0}$$

holds for all $h \in G(I(a))$, according to lemma 4.4. Let $C \in I(a) \otimes g$ correspond to the element $(UO^{-1})^{>0} \in G(I(a))$. Consider the endomorphism of the $R/(pR+a)$-module $I(a) \otimes_{\mathbb{Z}_p} g$, sending an element $X$ to

$$\text{Ad}^G(U)(\phi_\mu(X)^{>0}) + C$$

In view of lemma 4.5, we ought to show that this endomorphism has a unique fixed point. Our claim follows, once we prove that the $F$-linear endomorphism

$$X \mapsto M(X) := \text{Ad}^G(U)(\phi_\mu(X)^{>0}) \quad \text{is nilpotent on } I(a) \otimes_{\mathbb{Z}_p} g$$

in order to accomplish this it suffices to prove the nilpotence of

$$X \mapsto L(X) := \text{Ad}^G(U)\phi_\mu(X^{>0})$$
on $W(a) \otimes_{\mathbb{Z}_p} g$ (if $L^m = 0$, then $M^{m+1} = 0$). We can rewrite $W(a) \otimes_{R/pR} g$ as $W(a) \otimes_{R/pR} (R/pR \otimes_{\mathbb{Z}_p} g)$ and the endomorphism $L$ thereon can be written as the tensor product of the endomorphisms $x \mapsto V^{-1}(x^{>0})$ on $W(a)$ and the endomorphism (32) on $R/pR \otimes_{\mathbb{Z}_p} g$, where the $R$-valued element of $G$ which appears in it is just the zeroth ghost component $w_0(U)$. "
Remark 4.7. Over artinian local rings with perfect residue field \( k \) of characteristic \( p \neq 2 \) one obtains an analog of the above if one replaces \( W(R) \) by the Zink ring, which is the product of \( W(k) \) with the points of the group of formal Witt vectors over the maximal ideal of \( R \). In this theory one does not have to impose a nilpotence condition. Please see [38, Proposition 7.1.5] for details, and [53] for further background material on such “Dieudonné displays”.

Remark 4.8. The essence of the above proposition was already proved in [6, Theorem 3.5.4] under the slightly more restrictive assumption that \( a \) is a \( p \)-adically closed and topologically nilpotent \( \text{pd} \)-ideal in a \( p \)-adically separated and complete \( W(k_0) \)-algebra \( R \). Our peculiar detour from \( G(W(a)) \) via \( G(I(a)) \cong I(a) \otimes g \) to \( W(a) \otimes g \) allows to work with a weaker assumption on \( a \).

The classical method of [52, proof of Theorem 44] also works for all \( p \)-torsion \( \text{pd} \)-ideals \( a \) without any further restrictions. So up to our limit process the above lemma seems to be the “natural” generalization to \((G, \mu)\)-displays!

Remark 4.9. Please see [5, part (i) of lemma 3.24] for some variants.

4.1. Definition of \( \Psi^a_\mu \). Recall that Zink’s logarithmic ghost components induce a decomposition \( W(a) = a_{[w_0]} \oplus I(a) \) whenever \( a \subset R \) is an ideal with \( \text{pd} \)-structure \( \{\gamma_k : a \to a\}_{k \in \mathbb{N}} \). If the ambient ring \( R \) has the structure of a \( W(k_0) \)-algebra we may deduce the decomposition \( L^+G(a) \cong G(a) \times L^0G(a) \), so that \( G(a) \) can be regarded as a (normal) subgroup of both \( G(R) \) and \( L^+G(R) \). Also, we note that the absolute Frobenius \( \hat{F} : W(R) \to W(R) \) induces a shift of the logarithmic ghost components on \( W(a) \), so that \( a_{[w_0]} \) lies in the kernel of \( F \). We deduce that the morphism \( \hat{\Phi}_\mu : H_\mu \to L^+L^+G \) (cf. proposition 2.24) annihilates \( P_\mu(a) = H_\mu(R) \cap G(a) \subset H_\mu(R) \), regarding \( G(a) \) as a subgroup of \( G(W(R)) \) (cf. [5, corollary 3.11]). We need the following generalization, in which a \( \text{pd} \)-morphism from \((R, a, \gamma)\) to \((S, b, \delta)\) is defined to be a ring homomorphism \( f : R \to S \) satisfying \( f(a) \subset b \) and \( f(\gamma_k(x)) = \delta_k(f(x)) \) for all \( k \in \mathbb{N} \) and \( x \in a \), as in [40, Chapter III, Definition (1.5)].

Lemma 4.10. Consider the functors \( \Gamma_\mu \) and \( L^+G \) from the category of \( \text{pd} \)-structures \((R, a, \gamma)\) over \((W(k_0), 0, \mu)\) to the category of groups, sending a triple \((R, a, \gamma)\) to \( \Gamma_\mu^{W(R), W(a) + I(R)} \) and respectively \( L^+G(R) \). There exists a natural transformation \( \Psi_\mu : \Gamma_\mu \to L^+G \) denoted on a triple \((R, a, \gamma)\) by \( \Psi_\mu^a \) satisfying the following two properties:

- the restriction of \( \Psi_\mu^a \) to \( H_\mu(R) = \Gamma_\mu^{W(R), I(R)} \) agrees with \( \hat{\Phi}_\mu \).
- \( \Psi_\mu^a \) vanishes on \( G(a) \) (note that this is a subgroup of \( G(W(a)) \) which is in turn contained in \( \Gamma_\mu^{W(R), W(a) + I(R)} \)).

Proof. Consider the three localisations \( R_\mu, \tilde{R} = (1 + pR)^{-1}R, \tilde{R}_p \) and observe that \( R = \tilde{R} \times \tilde{R}_p R_p \), as \( R \to \tilde{R} \times \tilde{R}_p \) is faithfully flat. Consequently we have:

\[
L^+G(R) \cong L^+G(\tilde{R}) \times_{L^+G(\tilde{R}_p)} L^+G(R_p)
\]

Observe that \( \tilde{a} = \tilde{R} \otimes R a \) is still a \( \text{pd} \)-ideal of \( \tilde{R} \) and so are \( a_p \subset R_p \) and \( \tilde{a}_p \subset \tilde{R}_p \). Furthermore, we have

\[
P_\mu(R/a) \cong P_\mu(\tilde{R}/\tilde{a}) \times_{P_\mu(\tilde{R}_p/\tilde{a}_p)} P_\mu(R_p/a_p),
\]
as $R/a \to \tilde{R}/\tilde{a} \times R_p/a_p$ is still faithfully flat. We deduce
\[ \Gamma_\mu^{W(R), W(a) + I(R)} \cong \Gamma_\mu^{W(\tilde{R}), W(\tilde{a}) + I(\tilde{R})} \times \Gamma_\mu^{W(R_p), W(a_p) + I(R_p)}, \]
as $\Gamma_\mu^{W(R), W(a) + I(R)}$ is the inverse image of $P_\mu(R/a)$ in $L^+ G(R)$, and likewise for $\Gamma_\mu^{W(R_p), W(a_p) + I(R_p)}, \Gamma_\mu^{W(\tilde{R}), W(\tilde{a}) + I(\tilde{R})}$ and $\Gamma_\mu^{W(R_p), W(a_p) + I(R_p)}$. Therefore it is enough to restrict our attention to triples $(R, a, \gamma)$ satisfying either $pR = R$ or $pR \subset \text{rad}(R)$. Recall that $pR \subset \text{rad}(R)$ implies the direct generation of $\Gamma_\mu^{W(R), W(a) + I(R)}$ by $L^+ P_\mu(R)$ and $U_{\mu^{-1}}(W(a) + I(R))$, as $pR \subset \text{rad}(R)$ implies $a \subset \text{rad}(R)$, given that $a \subset \sqrt{pa} \subset \sqrt{pr}$ along with corollary 2.17, i.e.:

\begin{align*}
(34) & \quad L^+ P_\mu(R) \times U_{\mu^{-1}}(W(a) + I(R)) \cong \\
(35) & \quad \Gamma_\mu^{W(R), W(a) + I(R)} \cong H_\mu(R) \times U_{\mu^{-1}}(a)
\end{align*}

Whence it follows $\Gamma_\mu^{W(R), W(a) + I(R)} = G(a)\Gamma_\mu^{W(R), I(R)}$, and the construction of $\Psi_\mu^a$ is evident. In the other case $p$ is invertible in $R$ and we appeal to the formula $\Psi_\mu^a(g) = F(\mu(p)g\mu(\frac{1}{p})) \in L^+ G(R)$. It remains to check that the assignment $(R, a, \gamma) \mapsto \Psi_\mu^a$ is functorial with respect to $W(k_0)$-linear pd-morphisms $f : (R, a, \gamma) \to (S, b, \delta)$, where we may assume that $pS = S$ and $pR \subset \text{rad}(R)$ (the other three subcases are trivial). To this end it is enough to check that $g \mapsto F(\mu(p)g\mu(\frac{1}{p}))$ annihilates $G(b)$, since we already know that $\Phi_\mu$ is a transformation from $H_\mu$ to $F L^+ G = L^+ F G = L^+ G$ (cf. proposition 2.24). We are done as $F$ annihilates $G(b) = \mu(p)G(b)\mu(\frac{1}{p})$.

Observe that $\Phi_\mu(h^{-0}) = \Psi_\mu^a(h)$ for any $h \in L^+ G(a)$. When using this language, proposition 4.6 turns into the following more elegant version:

**Corollary 4.11.** Fix a $p$-adically separated and complete pd-ideal $a$ in a $W(k_0)$-algebra $R$ (i.e. such that $R$ is separated and complete with respect to the topology defined by the descending chain of pd-ideals $\{p^i a\}_{i \in \mathbb{N}}$ by part (ii) of lemma 2.2). Let $O$ and $U$ be elements of $G(W(R))$, satisfying
\[ O, U \in C_{W(R)}(G, \mu) \]
and
\[ h^{-1} U \Psi_\mu^a(h) O^{-1} \in G(W(a)), \]
for some $h \in \Gamma_\mu^{W(R), W(a) + I(R)}$. Then there exists a unique $\tilde{h} \in \Gamma_\mu^{W(R), W(a) + I(R)}$ satisfying $\tilde{h} h^{-1} \in G(W(a))$ and $O = \tilde{h}^{-1} U \Psi_\mu^a(\tilde{h})$.

**4.2. Rigidity of isomorphisms.** This result for automorphisms can be regarded as an analog of a strengthening of [52, Proposition 40]:

**Theorem 4.12.** Let $k_0$ be a finite field and $(G, \mu)$ a display datum over $W(k_0)$. Let $\phi$ be an automorphism of a $(G, \mu)$-display $\mathcal{D}$ over a $W(k_0)$-algebra $R$, such that $\mathcal{D} \times_R R/R_pR$ is adjoint nilpotent. Consider an ideal $a \subset R$ such that one of the following two assertions hold:

(i) $a$ is $(pR + a)$-adically separated.

(ii) $a$ allows a pd-structure and is $p$-adically separated i.e. $\bigcap_{i \in \mathbb{N}_0} p^i a$ vanishes.

Then $\phi = \text{id}_\mathcal{D}$ if and only if $\phi \equiv \text{id}_\mathcal{D} \mod a$. 
Proof. Recall that by lemma 2.2, $a$ is $(pR + a)$-adically separated if and only if \( \bigcap_{i \in \mathbb{N}} (p^i a + a^{i+1}) \) vanishes. Also, observe that the diagonal of our stack of displays with \((G, \mu)\)-structure is separated according to [6, Lemma 3.2.9b)]. Towards checking \( \phi = id_{\mathcal{D}} \) it therefore suffices to prove \( \phi \equiv id_{\mathcal{D}} \mod (a + pR)^i a \) for all \( i \) or \( \phi \equiv id_{\mathcal{D}} \mod p^i a \) for all \( i \), provided that \( a \) is a pd-ideal. By an easy induction it suffices to check this for \( i = 1 \). This means we must check \( \phi = id_{\mathcal{D}} \) under the assumption \( p^a = 0 \) where \( a \) satisfies \( a^2 = 0 \) or has a pd-structure. Note that the former case follows from the latter, as all square-zero ideals can be endowed with a pd-structure. So, let us consider the latter case. After passage to an affine fpqc covering, we can assume that \( \mathcal{D} \) is banal, hence represented by an element \( O \in G(W(R)) \). By assumption, the definition of adjoint-nilpotence and lemma 4.2, we have \( O \in C_{\text{nil}}^{W(R)}(G, \mu) \). The automorphism \( \phi \) is represented by an element \( h \in G(W(R)) \) satisfying
\[
O = h^{-1} O \Phi_{\mu}(h > 0)
\]
The assumption \( \phi \equiv id_{\mathcal{D}} \mod a \) means that \( h \in G(W(a)) \). The uniqueness part of proposition 4.6 then implies that \( h = 1 \), i.e. \( \phi = id_{\mathcal{D}} \). \qed

4.3. Rigidity of quasi-isogenies. In this subsection we give a proof of theorem 4.25. In order to handle concepts of quasi-isogenies we have to introduce some non-representable fpqc sheaves.

Definition 4.13. Let \( a \) be an ideal in a ring \( R \) and let \( X \) be a scheme over \( R \). We denote by \( \bar{X}^a \) (resp. \( X^a \)) the fpqc sheaf on \( \text{Spec} \, R \) which is obtained as the direct image of \( X \times_R \mathbb{Z}[\frac{1}{p}] \) along the closed (resp. open) immersion from \( \text{Spec} \, R/a \) (resp. \( \mathbb{Z}[\frac{1}{p}] \otimes R =: R[\frac{1}{p}] \)) to \( \text{Spec} \, R \), i.e.
\[
\bar{X}^a(T) := X(T \times_{\text{Spec} \, R} \text{Spec} \, R/a)
\]
\[
\bar{X}(T) := X(T \times_{\text{Spec} \, R} \text{Spec} \, R[\frac{1}{p}]),
\]
where \( T \) is any scheme over \( R \).

Remark 4.14. Let \( f : T \to S \) be a morphism of schemes. Any fpqc sheaf \( \mathcal{F} \) on \( S \) (resp. \( \mathcal{G} \) on \( T \)) allows an inverse image \( f^* \mathcal{F} : \text{Et}_{T} \to \text{Ens} ; V \mapsto \mathcal{F}(V(f)) \) (resp. direct image \( f_* \mathcal{G} : \text{Et}_S \to \text{Ens} ; U \mapsto \mathcal{G}(T \times_f S U) \)). Moreover we have the adjunction isomorphism
\[
\text{Hom}_T(f^* \mathcal{F}, \mathcal{G}) \cong \text{Hom}_S(\mathcal{F}, f_* \mathcal{G})
\]
which induces canonical adjunction morphisms \( \mathcal{F} \to f_* f^* \mathcal{F} \) and \( f^* f_* \mathcal{G} \to \mathcal{G} \). In the setting of the above definition 4.13, the \( R \)-scheme \( X \) gives rise to a representable fpqc sheaf \( h_X \) on \( \text{Spec} \, R \), and one obtains \( \bar{X}^a = i_a^* h_X \) and \( \bar{X} = j_a j^* h_X \) from the immersions \( i : \text{Spec} \, R/a \to \text{Spec} \, R \) and \( j : \text{Spec} \, R[\frac{1}{p}] \to \text{Spec} \, R \).

Lemma 4.15. Let \( R \) be a ring and let \( X \) be a smooth and quasi-projective \( R \)-scheme. For any ideal \( a \subset R \), the adjunction homomorphism
\[
X \to \bar{X}^a
\]
is an epimorphism of fpqc-sheaves on \( \text{Spec} \, R \).

Proof. The lemma boils down to the following claim: For every \( R \)-algebra \( S \) and any \( x_0 \in X(S/\alpha S) \) there exists a section \( y \) of \( X \) over a faithfully flat \( S \)-algebra \( T \) of which the \( \mod a \)-reduction in \( X(T/\alpha T) \) agrees with the image of \( x_0 \) therein.
Since smooth quasi-projectivity is stable under base extensions it clearly suffices to assume $S = R$, moreover we can restrict to the noetherian case, according to Grothendieck’s “Élimination des hypothèses noethériennes”, just as in the proof of lemma 2.4. Appealing to part (ii) of lemma 2.4 we may start out out from a lift $\hat{x} \in X(\hat{R})$ of $x_0$, where $\hat{R}$ denotes the $\mathfrak{a}$-adic completion of $R$. Observe that $\hat{R}$ is flat over $R$, but it might not be faithfully flat. So let $f_1, \ldots, f_n$ generate the ideal $\mathfrak{a}$ and choose an étale faithfully flat $R$-algebra $Q$ with $X(Q) \neq \emptyset$, according to [25, Corollaire (17.16.3.ii)]. The construction is completed by taking $T := \hat{R} \times \prod_{i=1}^n Q_{f_i}$, and by taking $y$ to be any element of $X(\hat{R}) \times \prod_{i=1}^n X(Q_{f_i}) = X(T)$ whose projection to $X(\hat{R})$ agrees with $\hat{x}$.

Now let $X$ be a scheme. The assignment $U \mapsto W(\mathcal{O}_X(U))$ defines a Zariski sheaf on $X$ that we denote by $W(\mathcal{O}_X)$. Also, $\mathbb{Q} \otimes W(\mathcal{O}_X)$ denotes the Zariski sheaf obtained by sheafifying the presheaf $U \mapsto \mathbb{Q} \otimes W(\mathcal{O}_X(U))$ on $X$. There is a canonical homomorphism from $\mathbb{Q} \otimes W(\mathcal{O}_X(X))$ to $\Gamma(X, \mathbb{Q} \otimes W(\mathcal{O}_X))$, which is bijective (resp. injective) for quasi-compact and quasi-separated (resp. quasi-compact) $X$. We need the following:

**Lemma 4.16.** For every smooth affine group scheme $H$ over $W(k_0)$ the contravariant group functor

$$LH : \mathfrak{S}ch_{W(k_0)} \to \mathfrak{G}p : X \mapsto H(\Gamma(X, \mathbb{Q} \otimes W(\mathcal{O}_X)))$$

called the loop group of $H$, is a sheaf for the fpqc topology.

**Proof.** Observe that for a covering of a $W(k_0)$-scheme $X$ by a family of open subsets $\{U_i\}_{i \in I}$ one obtains the exactness of

$$H(\Gamma(X, \mathbb{Q} \otimes W(\mathcal{O}_X))) \leftrightarrow \prod_{i \in I} H(\Gamma(U_i, \mathbb{Q} \otimes W(\mathcal{O}_X))) \Rightarrow \prod_{i,j \in I} H(\Gamma(U_i \cap U_j, \mathbb{Q} \otimes W(\mathcal{O}_X)))$$

from the mere sheaf property of $\mathbb{Q} \otimes W(\mathcal{O}_X)$. It remains to check that the same holds for

$$LH(X) \leftrightarrow LH(Y) \Rightarrow LH(Y \times_X Y)$$

for any faithfully flat and quasi-compact morphism $\phi : Y \to X$. By what we just proved, this is local with respect to the Zariski topology on $X$. Without loss of generality, we may then assume that $X$ is of the form $\text{Spec} A$, in which case $Y$ is covered by finitely many open affines, say $\text{Spec} B_1, \ldots, \text{Spec} B_n$. Since it is allowed to refine our covering we may replace $Y$ by the disjoint union of these, so that it becomes affine too, say of the form $\text{Spec} B$. We should therefore show the exactness of the following:

$$H(\Gamma(X, \mathbb{Q} \otimes W(\mathcal{O}_X))) \leftrightarrow H(\Gamma(Y, \mathbb{Q} \otimes W(\mathcal{O}_Y))) \Rightarrow H(\Gamma(Y \times_X Y, \mathbb{Q} \otimes W(\mathcal{O}_{Y \times_X Y})))$$

It is enough to show that the following is exact:

$$\Gamma(X, \mathbb{Q} \otimes W(\mathcal{O}_X)) \leftrightarrow \Gamma(Y, \mathbb{Q} \otimes W(\mathcal{O}_Y)) \Rightarrow \Gamma(Y \times_X Y, \mathbb{Q} \otimes W(\mathcal{O}_{Y \times_X Y}))$$

Since $X$ and $Y$ are affine, and so are quasi-compact and quasi-separated, it follows from what we said just before the lemma, this sequence is identified with the following sequence:

(37) \[ \mathbb{Q} \otimes W(A) \leftrightarrow \mathbb{Q} \otimes W(B) \Rightarrow \mathbb{Q} \otimes W(B \otimes_A B) \]

Since $A \to B$ is faithfully flat, the sequence

$$A \to B \Rightarrow B \otimes_A B$$
is exact (cf. [43, Chapter I, Proposition 2.18]). By [52, Lemma 30] the sequence
\[ W(A) \hookrightarrow W(B) \xrightarrow{\sim} W(B \otimes_A B) \]
is exact. This implies the exactness of (37), since localization is an exact functor.

**Remark 4.17.** One can think of \( LH \) as the fpqc sheafification of the functor \( X \mapsto H(\mathbb{Q} \otimes \Gamma(X, W(\mathcal{O}_X))) \), but since sheafification is not always a well-defined operation in the fpqc topology, our detour via the Zariski topology seems to be inevitable. Please see [50] for some presheaves which do not possess fpqc sheafifications.

The fpqc sheaf \( LH \) is a prime example of a non-representable group functor. The following two lemmas investigate basic properties of the “neutral locus” of an automorphism of a \( LH \)-torsor:

**Lemma 4.18.** Let \( R \) be a \( W(k_0) \)-algebra. The set
\[
\mathfrak{a}_n = \{ x \in R \mid p^n x = p^{n-1} x^p = \cdots = px^{p^{n-1}} = x = 0 \}
\]
is an ideal for each \( n \in \mathbb{N}_0 \). Moreover, one has \( pW(\mathfrak{a}_n) \subset F(W(\mathfrak{a}_n)) \subset W(\mathfrak{a}_{n-1}) \) for all \( n > 0 \).

**Proof.** Note that we have
\[
\mathfrak{a}_n = \{ x \in R \mid px, x^p \in \mathfrak{a}_{n-1} \}
\]
We show by induction that \( \mathfrak{a}_n \) is an ideal. Observe that \( \mathfrak{a}_0 = 0 \) and thanks to the identity
\[
(p + y)^p = px^p + y^p + pxy\left( \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^{p-i} y^{p-i-1} \right)
\]
if \( \mathfrak{a}_{n-1} \) is an ideal, \( \mathfrak{a}_n \) is an ideal too.

The formula \( p = FV \) on \( W(R) \) implies \( pW(\mathfrak{a}_n) \subset F(W(\mathfrak{a}_n)) \). In order to show \( F(W(\mathfrak{a}_n)) \subset W(\mathfrak{a}_{n-1}) \) it suffices to restrict to \( n = 1 \) because of (39). Consider the \( W(k_0) \)-algebra \( R_1 := W(k_0) \oplus \mathfrak{a}_1 \), and notice that \( \mathfrak{a}_1 \) can be recovered as \( \{ x \in R_1 \mid x^p = px = 0 \} \). Observe that \( R'_1 := R_1/pR_1 \cong k_0 \oplus \mathfrak{a}_1 \), and notice that \( \mathfrak{a}'_1 := \{ x \in R'_1 \mid x^p = px = 0 \} \) is still canonically isomorphic to \( \mathfrak{a}_1 \). Since the absolute Frobenius on \( W(R'_1) \) is given by the familiar formula \( (x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots) \) we deduce \( F(W(\mathfrak{a}'_1)) = 0 \), so that \( F(W(\mathfrak{a}_1)) \subset W(pR_1) \). However, we have \( F(W(\mathfrak{a}_1)) \subset W(\mathfrak{a}_1) \) too, and \( pR_1 \cap \mathfrak{a}_1 = 0 \). It follows that \( F(W(\mathfrak{a}_1)) \) vanishes in the ring \( W(R_1) \). At last we easily deduce the result for \( R \), since the canonical map \( R_1 \to R \) induces an isomorphism of the respective \( \mathfrak{a}_1 \)'s.

**Lemma 4.19.** Let \( H \) be a smooth affine group scheme over \( W(k_0) \). Let \( \alpha : P \xrightarrow{\cong} P \) be an automorphism of an \( LH \)-torsor \( P \) over a \( W(k_0) \)-scheme \( X \) and consider the subset \( X_0 = \{ x \in X \mid \alpha_x = \text{id}_{P_x} \} \), where \( \alpha_x : P_x \xrightarrow{\cong} P_x \) denotes the fiber of \( \alpha \) at \( x \).

Let \( \alpha_{X_{\text{red}}} : P_{X_{\text{red}}} \xrightarrow{\cong} P_{X_{\text{red}}} \) stand for the pull-back of \( \alpha \) to the reduction \( X_{\text{red}} \). Then the following hold:

(i) \( X_0 \) is closed with respect to the Zariski topology of \( X \).

(ii) \( \alpha_{X_{\text{red}}} = \text{id}_{P_{X_{\text{red}}}} \) if and only if \( X_0 = X \).

(iii) Assume that \( X \) is locally noetherian and that its generic fiber \( X \times_{W(k_0)} W(k_0)[\frac{1}{p}] \) is reduced. If \( P \) is trivial, then \( \alpha_{X_{\text{red}}} = \text{id}_{P_{X_{\text{red}}}} \) if and only if \( \alpha = \text{id}_P \).
Proof. We can assume that $X$ is affine, say it is $\text{Spec} R$. Statements (i) and (ii) are not changed if we replace $X$ by $X_{\text{red}}$, and so, in those two statements, we assume further that $R$ is reduced. Since by [23, Corollaire (2.3.12)] (see also by [18, Lemme 1.2]) fpqc morphisms are submersive, and $P$ trivializes after base change of $X$ by a faithfully flat morphism, we can assume that $P$ is already trivial over $X$. So, in all three statements, $X = \text{Spec} R$ and $P$ is trivial over $X$ and so, the automorphism $\alpha$ is represented by an element $A \in \text{GL}_n(W(R))$ and we want to investigate when it is equal to the identity element. In (i) and (ii) $R$ is reduced, in (iii) $R$ is noetherian.

(i): Pick a closed embedding $\rho : H \hookrightarrow \text{GL}_n$ over $W(k_0)$. The element $\rho(A) - 1 \in \mathcal{M}_n(Q \otimes W(R))$ can be written as $\frac{1}{p^m}B$ where $m \geq 0$ is a natural number and $B \in \mathcal{M}_n(W(R))$. It then follows that $X_0$ is the locus of vanishing of all Witt components of all entries of the matrix $B$, which is closed in $X$.

(ii): Assume that $X_0 = X$ and let $B$ be as in (i). By the construction of $X_0$ in (i) and by assumption, all Witt components of all entries of $B$ belong to all prime ideals of $R$. Since $R$ is reduced, $B$ is the zero matrix.

(iii): We ought to show that $A$ is the identity element of $H(Q \otimes W(R))$ given that its reduction in $H(Q \otimes W(R/\sqrt{\mathcal{O}_R}))$ is the identity. We are going to show that the canonical morphism

$$Q \otimes W(R) \to Q \otimes W(R/\sqrt{\mathcal{O}_R})$$

is in fact an injection, which would then imply that $A$ is the identity. It is enough to show that $W(\sqrt{\mathcal{O}_R})$ is a subset of $W(R)_{\text{tor}}$, the $p$-power torsion elements of $W(R)$. Consider the ascending chain of ideals $a_n$, as in (38). For all $n \geq 0$ we have:

$$W(a_n) \subset W(R)_{\text{tor}}$$

(40)

The union of $a_n$ is equal to $R_{\text{tor}} \cap \sqrt{\mathcal{O}_R}$, which is equal to $\sqrt{\mathcal{O}_R}$, since by assumption $R \otimes_{W(k_0)} W(k_0)[\frac{1}{p}]$ is reduced. On the other hand, since $R$ is noetherian, this union is equal to $a_n$ for $n \gg 0$. To summarize, for $n \gg 0$, we have $a_n = \sqrt{\mathcal{O}_R}$. Thus, by (40), we have $W(\sqrt{\mathcal{O}_R}) = W(a_n) \subset W(R)_{\text{tor}}$ as desired. □

From now on we work again with the display datum $(G, \mu)$ over $W(k_0)$. The relevance of $LG$ in this context stems from $F_{\mu(p)} \in LG(W(k_0))$ and the commutative diagram

$$
\begin{array}{ccc}
L^+ G & \xrightarrow{F} & L^+ G \\
\uparrow & & \uparrow_{\text{int}(F_{\mu(p)})} \\
H_\mu & \xrightarrow{\Phi_\mu} & L^+ G \\
\end{array}
$$

which implies that for any $H_\mu$-torsor $Q$, we have a canonical isomorphism:

$$F(Q * LG) \xrightarrow{\cong} Q * \Phi_\mu LG$$
Also, observe that we have a commutative diagram:

\[
\begin{array}{ccc}
H_\mu & \longrightarrow & H_\mu^G \\
\downarrow & & \downarrow \\
L^+G & \longrightarrow & LG
\end{array}
\]

where \(H_\mu^G := \overline{P}_\mu \times_{\overline{G},w_G^G} LG\).

**Definition 4.20.** Let \(X\) be a scheme over \(\mathbb{Z}_p\) (resp. over \(W(k_0)\)). By a \(G\)-isodisplay (resp. filtered \((G,\mu)\)-isodisplay) we mean a pair \((P, b)\) where \(P\) is a torsor under \(LG\) (resp. \(H_\mu^G\)) over \(X\) and \(b\) is an isomorphism

\[b : F(P \ast LG) \xrightarrow{\cong} P \ast LG\]

of \(LG\)-torsors over \(X\). We call \((P, b)\) banal if \(P\) is the trivial torsor.

A morphism between \(G\)-isodisplays (resp. filtered \((G,\mu)\)-isodisplays) \((P_1, b_1)\) and \((P_2, b_2)\) is a morphism \(\alpha : P_1 \to P_2\) between their underlying torsors, such that \(b_2 \circ F \alpha \circ b_1^{-1} = \alpha\) (these are morphisms between the \(LG\)-torsors \(P_1 \ast LG\) and \(P_2 \ast LG\) over \(X\)).

**Definition 4.21.** Let \(D = (Q, u)\) be a \((G, \mu)\)-display over \(X\). Then, as we discussed above, we have canonical isomorphisms:

\[F(Q \ast LG) \xrightarrow{\cong} Q \ast_{\mathbb{G}_m} LG \xrightarrow{u} Q \ast LG\]

Therefore, by change of group from \(H_\mu\) to \(LG\) (resp. \(H_\mu^G\)), the \((G,\mu)\)-display \(D\) gives rise to a canonical \(G\)-isodisplay (resp. filtered \((G,\mu)\)-isodisplay), which we denote by \(D^0\) (resp. \(D^0\)). Let \(D_1\) and \(D_2\) be \((G,\mu)\)-displays over \(X\). A quasi-isogeny from \(D_1\) to \(D_2\) over \(X\), denoted by \(D_1 \longrightarrow D_2\) is an element of \(\text{Hom}(D_1^0, D_2^0)\).

**Remark 4.22.** The groups \(\text{Hom}(D_1^0, D_2^0), \text{Hom}(D_1^G, D_2^G)\), and \(\text{Hom}(D_1, D_2)\) are closely related to each other, observe that we have:

\[\text{Hom}(D_1^0, D_2^0) = \text{Hom}(H_\mu^G(D_1^0, D_2^0) \times_{\text{Hom}(LG, D_1^0, D_2^0)} \text{Hom}(D_1^0, D_2^0))\]

It follows that we have a natural injection \(\text{Hom}(D_1^G, D_2^G) \hookrightarrow \text{Hom}(D_1^0, D_2^0)\), since \(H_\mu^G\) is a subsheaf of \(LG\). In particular, the content of theorem 4.25 remains unaltered if each occurrence of “\(D^0\)” is replaced by the arguably more natural “\(D^G\)”.

**Proposition 4.23.** Let \(A\) be a \(W(k_0)\)-algebra and fix an element \(U \in C^{nil}_{W(\mathcal{A})}(G,\mu)\). Then the equation

\[g^{-1}(U F^\mu(p))^F g = U F^\mu(p)\]

has no non-trivial solutions in elements \(g \in G(\mathbb{Q} \otimes W(\mathfrak{a}))\), in each of the following two cases:

(i) \(a = \sqrt{p^\mathfrak{a}}\) and \(A\) is noetherian and \(p\)-adically separated and complete.

(ii) \(A\) is a complete discrete valuation ring with maximal ideal \(a\) and its residue field contains \(k_0\).

**Proof.** Let \(b := U F^\mu(p)\), so that \(g^{-1}b^F g = b\), where \(g\) is as in the proposition. Lemma 4.18 shows that if \(b\) is an ideal, and there exists an \(n\) such that for all \(x \in b\), \(x^n = 0 = p^n x\), then \(W(b)\) is \(p\)-power torsion, and so \(\mathbb{Q} \otimes W(b) = 0\). So, in order
to deal with (i) we may replace $a = \sqrt{pA}$ by $pA$, since $\mathbb{Q} \otimes W(\sqrt{pA}/pA) = 0$ and therefore we have:
\[
\mathbb{Q} \otimes W(pA) \cong \mathbb{Q} \otimes W(\sqrt{pA})
\]
Moreover, we can assume that $A$ has no $p$-torsion, which is due to
\[
\mathbb{Q} \otimes W(pA) \cong \mathbb{Q} \otimes W(pA_0),
\]
where $A_0$ is the quotient of $A$ by its $p$-torsion (again, we have $\mathbb{Q} \otimes W(\ker(pA \rightarrow pA_0)) = 0$). Now observe that $F$ maps $W(pA)$ to $pW(pA)$, so there exists some $s \in \mathbb{N}$ with $F^s g = \tilde{g} \in G(W(pA))$. Let us put $F^s U =: \tilde{U}$ and $F^s \mu =: \tilde{\mu}$ and $F^s b = \tilde{U}F\tilde{\mu}(p) =: \tilde{b}$. Observe that $\tilde{U}$ is an element of $C_W(A)(G, \tilde{\mu})$ and that $\tilde{g}^{-1} \tilde{b} F \tilde{g} = \tilde{b}$ holds. It follows that
\[
\tilde{g}^{-1} \tilde{U} F (\tilde{\mu}(p) \tilde{g} \tilde{\mu}(1)) = \tilde{U}
\]
and so, $\tilde{g}$ is trivial by proposition 4.6. By iterating the identity $g = b F gb^{-1}$, one concludes that $g$ and $F^s g = \tilde{g}$ are conjugates and so $g$ is trivial as well.

In case (ii) we must focus on the equal characteristic case, since we could appeal to part (i) otherwise. Let $k$ denote the residue field of $A$. Our proof starts out from the Cohen structure theorem, which implies that $A$ is isomorphic to the ring $k[[t]]$ of power series with coefficients in $k$ in an indeterminate $t$. It does no harm to assume that $k$ is algebraically closed. Observe that the equation $g = b F gb^{-1}$ implies that $g$ is a unipotent element (of say $G(\mathbb{Q} \otimes W(k[[t]]))$, indeed look at some faithful representation $\rho : G_{\mathbb{Q}_p} \rightarrow GL(n, \mathbb{Q}_p)$ and consider the characteristic polynomial $\chi \in (\mathbb{Q} \otimes W(k[[t]]))[x]$ of $\rho(g)$. Notice that since $g$ is conjugated to $F g$, the coefficients of $\chi$ must be fixed by $F$, and so, they are elements of $\mathbb{Q} \otimes W(\{ a \in k((t)) \mid a = a^p \}) = \mathbb{Q}_p$. Also, notice that
\[
\chi \equiv (x - 1)^n \mod \mathbb{Q} \otimes W(tk[[t]]),
\]
as $\rho(g)$ specializes to the identity matrix at $t = 0$. We deduce $\chi = (x - 1)^n$, which is due to $\mathbb{Q}_p \cap \mathbb{Q} \otimes W(tk[[t]]) = 0$. Since $g$ is unipotent it is allowed to consider its logarithm $\log(g) =: X$, which satisfies the identity $X = \text{Ad}(b)F X$ and the rest of the proof is the lemma 4.24.

\begin{lemma}
\label{lemma}
Let $k$ be an algebraically closed field of characteristic $p$ and let $B$ be a $d \times d$-Matrix with entries in $\mathbb{Q} \otimes W(k[[t]])$. If there exists a number $s \in \mathbb{N}$ with
\[
(41) \quad B \cdot F B \cdots F^{s-1} B \in \mathbb{Q} \otimes W(k[[t]]),
\]
then the equation
\[
(42) \quad x = B \cdot F x
\]
has no non-trivial solutions in column vectors $x$ with entries in $\mathbb{Q} \otimes W(tk[[t]])$.
\end{lemma}

\begin{proof}
Observe that [14, Definition 5.15; Proposition 5.16] introduced, for any $r \in \mathbb{R}_{\geq 0}$, valuations:
\[
(43) \quad v_r : \mathbb{Q} \otimes W(k[[t]])^{\text{perf}} \rightarrow \mathbb{R} \cup \{ \infty \}; \quad \sum_{Z \ni a \rightarrow -x} p^i[a_i] \mapsto \inf \{ ri + v(a_i) \mid i \in \mathbb{Z} \}
\]
where $v : k((t))^{\text{perf}} \rightarrow \mathbb{Z}[\frac{1}{p}] \cup \{ \infty \}$ is the usual $t$-adic valuation (see also [42]).
When regarding $\sum_{i \in \mathbb{Z}} p^i[a_i]$ as a bi-Witt vector its $i$th component is given by $u_i^p$,
Theorem 4.25

We are now ready to prove the following:

Now assume that (42) holds for a column vector \( x \) because of the identity

\[
\nu(x) = \min\{\nu(x_1), \ldots, \nu(x_d)\} \geq \frac{r}{\log p} (1 + \log \log p - \log r),
\]

simply because the \( i \)-th Witt component \( a_i^0 \) of \( a \) is an element of \( tk[[t]] \), while

\[
ri + \frac{1}{p^i} \geq \frac{r}{\log p} (1 + \log \log p - \log r)
\]

by a little bit of high school calculus (look at \( i = \frac{\log \log p - \log r}{\log p} \), which is the only zero of \( r - \frac{\log \log p - \log r}{\log p} \) being the derivative with respect to \( i \) of the left hand side).

Now assume that (42) holds for a column vector \( x = (x_1, \ldots, x_d)^T \), and let \( s \) be as in (41), so that we have

(44)

\[
x = C \cdot F^s x
\]

where \( C := B \cdot F B \cdots F^{s-1} B \in \frac{1}{p^s} W(k[[t]]) \). Notice that \( \nu(F x_i) = \nu_T(p x_i) \) holds for \( i \in \{1, \ldots, d\} \), so that looking at the function

\[
\nu(r) := \min\{\nu(x_1), \ldots, \nu(x_d)\} \geq \frac{r}{\log p} (1 + \log \log p - \log r),
\]

allows us to deduce

\[
\nu(r) \geq (1 - s)n r + p^s n \nu\left(\frac{r}{p^s}\right)
\]

because of the identity

\[
x = C \cdot F^s C \cdots F^{s(n-1)} C \cdot F^{sn} x
\]

which follows by iterating (44). It follows that we have

\[
\nu(r) \geq rn + \frac{r}{\log p} (1 + \log \log p - \log r)
\]

for every \( n \), which shows that the valuations of \( x_i \) are not bounded and therefore \( x_i \) are zero as desired. \( \square \)

We are now ready to prove the following:

**Theorem 4.25 (Rigidity of Quasi-Isogenies).** Let \( k_0 \) be a finite field and \( (G, \mu) \) a display datum over \( W(k_0) \). Let \( D \) be a \( (G, \mu) \)-display over a connected and locally noetherian \( W(k_0) \)-scheme \( X \) whose structural morphism to \( \text{Spec} W(k_0) \) is assumed to be closed in the sense of [19, (2.2.6)] or [49, Tag 004E]. Assume that the restriction of \( D \) to the special fiber of \( X \) is adjoint nilpotent, and let \( \alpha \) be an automorphism of \( D^0 \) (i.e. a self quasi-isogeny of \( D \)). For any \( x \in X \) we write \( \kappa(x) \) for the residue field of \( X \) at \( x \) and \( D_x \) for the pull-back of \( D \) along the morphism \( \text{Spec} \kappa(x) \to X \). Then \( \alpha \) is the identity if and only if for some \( x \in X \) the specialization \( \alpha_x \in \text{Aut}(D^0_x) \) is the identity.

**Proof.** One direction is trivial, and so, assume that the specialization of \( \alpha \) to some point of \( X \) is the identity. Let \( X_0 \subset X \) be the set of all \( x \) such that \( \alpha_x \) agrees with the identity element of \( \text{Aut}(D^0_x) \). By assumption \( X_0 \) is not empty and by part (i) of lemma 4.19 \( X_0 \) is closed. In order to prove that \( X_0 \) is open, it suffices to check that it is stable under generation, in view of [22, Théorème (1.10.1)] or [49, Tag 0542]
as closed sets are certainly constructible. So assume that some \( x \in X \) specializes to some \( x_0 \in X_0 \). It does no harm to assume that the characteristic of \( \kappa(x_0) \) is \( p \), because \( X \to \text{Spec} \ W(k_0) \) is a closed map and so the special point of \( \text{Spec} \ W(k_0) \) belongs to the image of the non-empty closed subset \( \{x\} \cap X_0 \). According to a classical specialization result (e.g. [40, Chapter III, Lemma (1.8)]) there exists a discrete valuation ring \( R \) and a morphism \( \text{Spec} \ R \to X \) that sends the generic point to \( x \) and the special point to \( x_0 \), furthermore we may assume that \( R \) is complete and has an algebraically closed residue field (of characteristic \( p \)). It comes in handy that a \((G, \mu)\)-display over such a ring \( R \) is banal. Therefore we are allowed to apply part (ii) of proposition 4.23 and deduce \( x \in X_0 \), as required. The connectedness of \( X \) enforces \( X_0 = X \).

Now that we have established that all specializations of \( \alpha \) are identity, we will use a descent argument to show that \( \alpha \) is the identity (globally). So choose an affine étale covering \( \{U_i\}_{i \in I} \) trivializing \( \mathfrak{q}_0(D) \). Let us write \( A_i \) for the noetherian \( W(k_0) \)-algebra with \( \text{Spec} A_i = U_i \) and pick \( f_i \in 1 + p A_i \) with \( f_i(\cap_{n \in \mathbb{N}} p^n A_i) = 0 \) (use Artin-Rees and Nakayama). Observe that \( U_i \) and \( D(f_i) = \text{Spec} A_i[\frac{1}{f_i}] \) have the same fiber over \( \text{Spec} k_0 \), as \( f_i \equiv 1 \pmod{p} \). It follows that the special fiber of \( X \) is contained in the image of the map

\[
\prod_{i \in I} D(f_i) \to X
\]

which is an open subset of \( X \) (the composition \( D(f_i) \to U_i \to X \) is flat, and so is an open map). Since \( X \to \text{Spec} \ W(k_0) \) is a closed map, the image of the above map must be the whole of \( X \) and therefore, \( \{D(f_i)\}_{i \in I} \) is still an étale covering of \( X \). We are going to show that the pull-back of \( \alpha \) along this covering is the identity and therefore, it must be the identity as well. So, we need to show that the pull-back of \( \alpha \) to each \( A_i[\frac{1}{f_i}] \) is the identity. Using parts (i) and (ii) of lemma 3.5, we conclude that the pull-back of \( \mathcal{D} \) to the \( p \)-adic completion \( \hat{A}_i \) of \( A_i \) is banal. Moreover, the reducedness of \( A_i/\sqrt{pA_i} \) allows to apply part (ii) of lemma 4.19 and we deduce that the pull-backs of \( \alpha \) to each of those \( A_i/\sqrt{pA_i} \)'s are identities. Furthermore, over each \( A_i \) we can take advantage of part (i) of proposition 4.23 and deduce that the pull-backs of \( \alpha \) to each of those \( A_i \)'s are still identities. At last the pull-backs of \( \alpha \) to each \( A_i[\frac{1}{f_i}] \) must agree with the identity too, because the map \( A_i[\frac{1}{f_i}] \to \hat{A}_i[\frac{1}{f_i}] \) is injective. This is what we wanted to show and it achieves the proof. \( \square \)

4.4. \((G, \mu)\)-triples. For any ideal \( a \) in any \( W(k_0) \)-algebra \( R \) we wish to consider the following fpqc sheaves of groups on \( \text{Spec} R \):

\[
\hat{P}_a^\mu := \hat{P}_a^\mu \times G^s G \\
\hat{H}_a^\mu := \hat{P}_a^\mu \times_{G,w_0^s} L^+ G
\]

So, for any \( R \)-algebra \( S \), in the previous notations, we have \( \hat{P}_a^\mu(S) = \Gamma_\mu^S a_S \). We also have a canonical monomorphism

\[
H_\mu \times_{W(k_0)} R \hookrightarrow \hat{H}_a^\mu
\]

of fpqc sheaves of groups over \( \text{Spec} R \). Suppose that \( a \) is equipped with divided powers \( \{\gamma_k : a \to a\}_{k \in \mathbb{N}} \). Observe that for any flat \( R \)-algebra \( S \), a fundamental observation of Berthelot (see e.g. [40, Chapter III, Lemma (1.8)]) grants a unique
extension of \( \gamma \) to a pd-structure \( \gamma_S \) on the ideal \( a \otimes_R S = aS \). Note that in the notations of lemma 4.10, \( H^a_S(S) \) is nothing but the group \( \Gamma_\mu(S,aS,\gamma_S) \), and so, we have a group homomorphism

\[
\hat{H}^a_S(S) = \hat{H}^a_S(S) \xrightarrow{\psi^a_S} L^+G(S)
\]

Whence we obtain a mini-morphism from \( \hat{H}^a_\mu \) to \( L^+G \times_{W/(k_0)} R \), in the sense of appendix B.5 (i.e., a morphism of functors on the full subcategory of flat \( R \)-algebras). Following the arguments therein (cf. (74)) we construct an awkward functor:

\[
\text{Tor}_R(\hat{H}^a_\mu) \to \text{Tor}_R(L^+G) : \hat{Q} \mapsto \hat{Q} \ast \psi^a_\mu L^+G
\]

With these preparations we are now in the position to define the notion of \((G, \mu)\)-triple:

**Definition 4.26.** Let \( R \) be a \( W(k_0) \)-algebra and let \( a \subset R \) be an ideal with divided powers \( \{ \gamma_k : a \to a \}_{k \in \mathbb{N}} \).

1. A \((G, \mu)\)-triple over \((R, a, \gamma)\) is a pair \((\hat{Q}, u)\), where \( \hat{Q} \) is a \( \hat{H}^a_\mu \)-torsor equipped with a \( L^+G \)-equivariant isomorphism

\[
u : \hat{Q} \ast \psi^a_\mu L^+G \xrightarrow{\sim} \hat{Q} \ast L^+G
\]

2. We write \( \text{Tri}_{R,a,\gamma}(G, \mu) \) for the groupoid of \((G, \mu)\)-triples over \((R, a, \gamma)\). Equivalently \( \text{Tri}_{R,a,\gamma}(G, \mu) \) is the groupoid rendering the diagram:

\[
\begin{array}{ccc}
\text{Tri}_{R,a,\gamma}(G, \mu) & \xrightarrow{} & \text{Tor}_R(L^+G) \\
\downarrow & & \downarrow \Delta_{\text{Tor}_R(L^+G)} \\
\text{Tor}_R(\hat{H}^a_\mu) & \xrightarrow{(\psi^a_\mu \ast \text{id}) \circ \Delta_{\text{Tor}_R(\hat{H}^a_\mu)}} & \text{Tor}_R(L^+G)^2
\end{array}
\]

2-Cartesian.

3. The passage from a \((G, \mu)\)-triple \( \mathcal{D} \) over \((R, a, \gamma)\) to its underlying \( \hat{H}^a_\mu \)-torsor (i.e. the vertical projection in (49)) will be denoted by

\[
q : \text{Tri}_{R,a,\gamma}(G, \mu) \to \text{Tor}_R(\hat{H}^a_\mu), \ (\hat{Q}, u) \mapsto \hat{Q}
\]

and \( \mathcal{D} \) is called banal if \( q(\mathcal{D}) \) possesses a global section, i.e. is trivial.

4. The precomposition of the natural 1-morphism \( \text{Tor}_R^H_{\mu} \to \text{Tor}_R^{F_{\mu}} \), induced by \( w_0^G : L^+G \to G \) with \( q \) is called the lowest truncation and will be denoted by \( q_0 : \text{Tri}_{R,a,\gamma}(G, \mu) \to \text{Tor}_R^{F_{\mu}} \).

**Remark 4.27.**

(a) In principle, it would have been possible to define a notion of \((G, \mu)\)-triple over an arbitrary \( W(k_0) \)-scheme which is equipped with a pd-ideal sheaf. We refrained from doing that, because we did not see any applications, very much unlike \((G, \mu)\)-displays over non-affine bases.

(b) Notice that a \((G, \mu)\)-triple over \((R, 0_R, 0)\) is nothing but a \((G, \mu)\)-display over \( R \). Furthermore, a \((G, \mu)\)-triple over \((R, a, \gamma)\) allows a pull-back along any pd-morphism say \((R, a, \gamma) \to (S, b, \delta)\), in particular any \((G, \mu)\)-triple over \((R, a, \gamma)\) determines a \((G, \mu)\)-display over \( R/a \), which we will refer to as its canonical reduction.
Definition 4.28. We call a \((G, \mu)\)-triple over \((R, a, \gamma)\) to be adjoint nilpotent if the \((\text{mod } p)\) reduction of its canonical reduction is an adjoint nilpotent \((G, \mu)\)-display over \(R/(pR + a)\). The class of adjoint nilpotent \((G, \mu)\)-triples over \((R, a, \gamma)\) forms a natural full subcategory, which will be denoted by \(\text{Tri}^{\text{nil}}_{R, a, \gamma}(G, \mu)\).

Remark 4.29. Notice that the full subcategory of banal \((G, \mu)\)-triples over \((R, a, \gamma)\) is equivalent to the groupoid \(\text{L} \times G \times \gamma \times \hat{G} \times \mu\) (cf. appendices A and B). Its full subcategory of adjoint nilpotent banal triples over \((R, a, \gamma)\) is equivalent to \(\text{Tri}^{\text{nil}}_{R, a, \gamma}(G, \mu)\). In the sequel we will quite often assume that \(a\) is \(p\)-adically separated and complete. In this case a \((G, \mu)\)-triple is banal if and only if its canonical reduction is banal, by lemma 2.6 (i).

Unfortunately, we need another preparation for the final result of this section, namely an extension of corollary 2.7 to the categories of torsors under our non-representable sheaves of groups \(\hat{H}^\mu\) :

Lemma 4.30. Let \(a\) be an ideal in a \(W(k_0)\)-algebra \(R\).

(i) The natural 2-commutative diagram

\[
\begin{array}{ccc}
\text{Tor}_R(\hat{H}_\mu^a) & \longrightarrow & \text{Tor}_R(L^+G) \\
\downarrow & & \downarrow \\
\text{Tor}_{R/a}(P_\mu) & \longrightarrow & \text{Tor}_{R/a}(G)
\end{array}
\]

is 2-Cartesian.

(ii) Assume that \(a\) is closed with respect to some admissible topology \(\{a_i\}_{i \in \mathbb{N}_0}\) on \(R\). Then we have:

\[
\text{Tor}_R(\hat{H}_\mu^a) \xrightarrow{\cong} 2 - \lim_{\longrightarrow} \text{Tor}(\hat{H}_\mu^{(a_i + a)/a_i})
\]

Proof. Observe, that there are equivalences of categories:

\[
\begin{align*}
\text{Tor}_R(\hat{G}^a) & \xrightarrow{\cong} \text{Tor}_{R/a}(G) \\
\text{Tor}_R(\hat{P}_\mu^a) & \xrightarrow{\cong} \text{Tor}_{R/a}(P_\mu)
\end{align*}
\]

So the first assertion follows from lemma B.9 and the faithful flatness of \(w^G_{\bar{a}} : L^+G \to \bar{G}^a\) together with \(G \to \bar{G}^a\) being an epimorphism, which in turn follows from the lemmas 2.10 and 4.15.

The second assertion is a straightforward consequence of the first one together with corollary 2.7, here notice that the descending chain of ideals \(\{(a + a_i)/a\}_{i \in \mathbb{N}_0}\) defines an admissible topology on \(R/a\).

Let \(R\) be a \(W(k_0)\)-algebra and \((R, a, \gamma)\) a pd-structure. Let \(S\) be a flat \(R\)-algebra. As we said before, \((S, aS, \gamma_S)\) is then a pd-structure over \((R, a, \gamma)\). The pull-back of triples along the pd-morphism

\[
(R, a, \gamma) \to (S, aS, \gamma_S)
\]

deserves special attention and we will write

\[
\text{Tri}_{R, a, \gamma}(G, \mu) \to \text{Tri}_{S, aS, \gamma_S}(G, \mu) \\
\hat{D} \to \hat{D}_S
\]
where \( \hat{D} \) is a \((G, \mu)\)-triple over \((R, a, \gamma)\). Notice that this is well defined only for flat \(R\)-algebras \(S\!\!\!.\!

**Theorem 4.31.** Let \(k_0\) be a finite field and \((G, \mu)\) a display datum over \(W(k_0)\). Let \(R\) be a \(W(k_0)\)-algebra, and let \(a \subset R\) be a \(p\)-adically separated and complete ideal with pd-structure \(\{\gamma_k : a \to a\}_{k \in \mathbb{N}_0}\) so that \(R\) is separated and complete with respect to the topology defined by the descending chain of pd-ideals \(\{p^i a\}_{i \in \mathbb{N}_0}\). The canonical reduction establishes an equivalence of categories from the category of adjoint nilpotent \((G, \mu)\)-triples over \((R, a, \gamma)\) to the category of \((G, \mu)\)-displays over \(R/a\) satisfying the adjoint nilpotence condition over \(R/(pR + a)\).

**Proof.** Let us first settle the banal case. Observe that \(W(a)\) is a \(p\)-adically separated and complete pd-ideal. Part (ii) of lemma 2.4 yields the surjectivity of \(G(W(R)) \to G(W(R/a))\) (N.B.: The requested topological nilpotence of \(W(a)\)-elements follows from the pd-property as \(x^{p^n} \in p^n a\) for \(x \in W(a)\)). By the same token we obtain the surjectivity of \(\hat{H}^0_p(R) \to H_p(R/a)\). Moreover, even \(C_{W(R/a)}^\text{nil}(G, \mu) \to C_{W(R)}^\text{nil}(G, \mu)\) is surjective, as any lift of an \(C_{W(R/a)}^\text{nil}(G, \mu)\)-element to \(G(W(R))\) lies automatically in \(C_{W(R)}^\text{nil}(G, \mu)\), according to lemma 4.2 together with \(a \subset \sqrt{p}a \subset \sqrt{p}R\). Together with corollary 4.11 this implies the equivalence:

\[
[C_{W(R/a)}^\text{nil}(G, \mu)/\phi_p \hat{H}^0_p(R)] \xrightarrow{\sim} [C_{W(R)}^\text{nil}(G, \mu)/\phi_p H_p(R/a)],
\]

which is nothing but the banal case of the requested result, in view of remark 4.29.

For the proof in the non-banal case, we use descent theory to reduce to the banal case: First and foremost, notice that part (ii) of lemma 4.30 allow us to restrict to the \(p\)-adically discrete case i.e. \(p^i a = 0\) for \(i \gg 0\), here notice that the condition \(\mu = 0\) is local for the fpqc topology, while \(a\) is \(p\)-adically separated and complete” is not. The full faithfulness of the canonical reduction is straightforward: If \(D\) (resp. \(D'\)) are the canonical reductions of adjoint nilpotent \((G, \mu)\)-triples \(D = (\hat{Q}, u)\) (resp, \(D' = (\hat{Q}', u')\), then we choose a faithfully flat \(R\)-algebra \(S\), such that \((S, a \otimes_R S, \gamma_S)\) banalizes \(D\) and \(D'\). The result follows from the full faithfulness of the canonical reduction over \(S\) together with descent theory. The determination of the essential image is a little trickier, because it is not so clear how to choose our trivialisation \(S\): Starting out from a \((G, \mu)\)-display \(D = (Q, u)\) over \(R/a\) such that \(D \times_{R/a} R/pR + a\) is adjoint nilpotent, we need to check the following auxiliary assertion: There exists a faithfully flat extension \(R \to S\) such that \(S/aS\) banalizes \(D\). Indeed, part (ii) of lemma 2.5 yields a section of \(Q\) over, say \(\lim_n A_n\), where \(R/a \to A_1 \to A_2 \ldots\) is a direct system with étale faithfully flat transition maps. By [25, Théorème (18.1.2)] each of the étale \(R/a\)-algebras \(A_n\) has a unique lift, which gives rise to a direct system \(R \to S_1 \to S_2 \ldots\) again with étale faithfully flat transition maps. The limit \(S = \lim_n S_n\) is still faithfully flat and so it solves our auxiliary assertion (although \(S\) may not even be finitely generated let alone étale). Now it is clear that there exists an adjoint nilpotent \((G, \mu)\)-triple \(D\) over \((S, aS, \gamma_S)\) of which the canonical reduction agrees with \(D \times_{R/a} S/aS\). It remains to construct an effective descent datum on \(\hat{D}\), which may be deduced from the full faithfulness of the canonical reduction. \(\square\)
4.5. Lifts of \((G, \mu)\)-displays. Now we can complete our description of the set of lifts of a \((G, \mu)\)-display. As before, let \(R\) be a \(W(k_0)\)-algebra and \((R, a, \gamma)\) a pd-structure. Our point of departure is the canonical monomorphism (47) together with its truncated pendant \(P_{\mu} \times_{W(k_0)} R \hookrightarrow \hat{P}_{\mu}^a\). These lead to a commutative diagram

\[
\begin{array}{ccc}
B(G, \mu)(R) & \longrightarrow & \text{Tor}_R(H_\mu) \\
\downarrow & & \downarrow \\
\text{Tri}_{R, a, \gamma}(G, \mu) & \longrightarrow & \text{Tor}_R(\hat{H}_\mu^a) \\
\end{array}
\]

within which both squares are 2-Cartesian (cf. part (i) of lemma 4.30). Fix a \(\hat{P}_\mu^a\)-torsor \(\hat{Q}_0\). Denote its extension of structure group via \(\hat{P}_\mu^a \rightarrow G \times_{zp} R\) by \(P\). Moreover, let \(Q_0\) be the \(P_{\mu}\)-torsor over \(R/a\) obtained by pulling back \(\hat{Q}_0\) (N.B.: The pull-back of \(\hat{P}_\mu^a\) to \(R/a\) agrees with \(P_{\mu} \times_{W(k_0)} R\)). Lemma B.13 justifies to consider the \(R\)-scheme \(X_{\hat{Q}_0} := \hat{P}_{\mu}/P_{\mu}\), of which the mod \(a\)-reduction is pointed, as \(X_{\hat{Q}_0} \times_R R/a \cong Q_0 \ast G_{W(k_0)}/P_{\mu}\) (N.B.: The \(P_{\mu}\)-action on \(G_{W(k_0)}/P_{\mu}\) preserves the neutral element therein). It follows that the inverse image of the aforementioned base point, say \(x_{\hat{Q}_0} \in X_{\hat{Q}_0}(R/a)\), under the mod \(a\)-reduction:

\[
X_{\hat{Q}_0}(R) \rightarrow X_{\hat{Q}_0}(R/a)
\]

agrees with the set of \(P_{\mu}\)-structures on the \(\hat{P}_\mu^a\)-torsor \(\hat{Q}_0\). Summing up, we have proven the following:

**Corollary 4.32.** Let \((a, \gamma)\) be a \(p\)-adically separated and complete pd-ideal in some \(W(k_0)\)-algebra \(R\). Let \(D\) be a \((G, \mu)\)-display over \(R/a\) of which the \((\text{mod } p)\)-reduction is assumed to be adjoint nilpotent. Then there exists a canonical bijection between the set of isomorphism classes of lifts of \(D\) to a \((G, \mu)\)-display over \(R\) and the set of lifts of \(x_{\hat{Q}_0} \in X_{\hat{Q}_0}(R/a)\) to an \(R\)-valued point of \(X_{\hat{Q}_0}\), where \(\hat{Q}_0\) is the lowest truncation of the \((G, \mu)\)-triple \(D\) over \((R, a, \gamma)\) which is associated to \(D\) according to theorem 4.31.

**Remark 4.33.** Notice that there exists at least one such lift of \(D\) as a consequence of lemma 2.6 while (50) is surjective by part (ii) of lemma 2.4.

Combining theorem 4.12 and corollary 4.32, we obtain the following:

**Theorem 4.34.** Let \(k_0\) be a finite field and \((G, \mu)\) a display datum over \(W(k_0)\). Let \(R\) be a \(W(k_0)\)-algebra and \(a \subset R\) an ideal. Finally, let \(D\) a \((G, \mu)\)-display over \(R/a\), where \(a\) is assumed to satisfy one of the following:

(i) \(a\) is \((pR + a)\)-adically separated and complete.

(ii) \(a\) allows a divided power structure and is \(p\)-adically separated and complete.

Assume that \(D \times_{R/a} R/(pR + a)\) is adjoint nilpotent. Then, for each lift \(\hat{D}\) of \(D\) the reduction of automorphisms gives an injection \(\text{Aut}(\hat{D}) \hookrightarrow \text{Aut}(D)\). Moreover there exists at least one such lift.

5. \((G, \mu)\)-Windows

Fix a finite field \(k_0\), a smooth affine group scheme \(G\) over \(W(k_0)\), and a cocharacter \(\mu : G_m, W(k_0) \rightarrow G\) as in subsection 2.3.

**Definition 5.1.** Let \(P\) be a \(G\)-torsor over a flat \(W(k_0)\)-algebra \(A\).
(1) By a modification of $P$ we mean a pair $(P_1, \eta_1)$, where $P_1$ is a $G$-torsor over $A$ and
\[ \eta_1 : P \times_A A[\frac{1}{p}] \to P_1 \times_A A[\frac{1}{p}] \]
is a $G$-equivariant isomorphism.

(2) Let $\eta_0 : Q_0 \times_G A \to P \times_A A/pA$ be a $G$-equivariant isomorphism, where $Q_0$ is a $P_\mu$-torsor over $A/pA$. We say that a modification $(P_1, \eta_1)$ is adapted to $(Q_0, \eta_0)$ if there exist a faithfully flat $A$-algebra $B$, and elements $x_0 \in Q_0(B/pB), x_1 \in P_1(B)$ and $x \in P(B)$ such that the $(\mod p)$ reduction of $x$ agrees with $\eta_0(x_0) \in P(B/pB)$ while $\eta_1(x) = x_1\mu(p)$ holds in $P_1(B[\frac{1}{p}])$, i.e.:
\[ P_1(B[\frac{1}{p}]) \ni P_1(B)\mu(p) \ni x_1\mu(p) \xrightarrow{\eta_1} x \leadsto \eta_0(x_0) \in P(B/pB) \]

**Definition 5.2.** If $(P_2, \eta_2)$ is another modification of $P$, we say that $(P_1, \eta_1)$ and $(P_2, \eta_2)$ are equivalent and write $(P_1, \eta_1) \sim (P_2, \eta_2)$ if the $G$-equivariant isomorphism
\[ \eta_2 \circ \eta_1^{-1} : P_1 \times_A A[\frac{1}{p}] \to P_2 \times_A A[\frac{1}{p}] \]
is already defined over $A$.

**Lemma 5.3.** Let $G/W(k_0)$ and $\mu : G_{m,k_0} \to G$ be as above, but assume that $-1$ is the only negative $\mu$-weight of Lie $G$. Let $P$ be a $G$-torsor over a flat $W(k_0)$-algebra $A$.

(i) Up to equivalence, there exists at most one modification of $P$ which is adapted to a given $P_\mu$-structure $(Q_0, \eta_0)$ on $P \times_A A/pA$.

(ii) Let $\eta_1 : P \times_A A[\frac{1}{p}] \cong P_1 \times_A A[\frac{1}{p}]$ be a modification adapted to a $P_\mu$-structure $(Q_0, \eta_0)$ on $P \times_A A/pA$. If $A$ is $p$-adically separated and complete and $x_0$ is a global section of $Q_0$, then there exist $x_1 \in P_1(A)$ and $x \in P(A)$ satisfying
\[ P_1(A[\frac{1}{p}]) \ni x_1\mu(p) \xrightarrow{\eta_1} x \leadsto \eta_0(x_0) \in P(A/pA), \]
in particular $P_1$ and $P$ are trivial too.

**Proof.** We start with assertion (i), so let $(P_1, \eta_1)$ and $(P'_1, \eta'_1)$ be modifications of $P$ that are adapted to the same $P_\mu$-structure $(Q_0, \eta_0)$ on the $(\mod p)$ reduction $P \times_A A/pA$. Over a sufficiently large faithfully flat extension $A \to B$ there exist elements $x_1 \in P_1(B), x'_1 \in P'_1(B)$, and $x, x' \in P(B)$ and $x_0, x'_0 \in Q_0(B/pB)$ satisfying
\[ P_1(B[\frac{1}{p}]) \ni x_1\mu(p) \xrightarrow{\eta_1} x \leadsto \eta_0(x_0) \in P(B/pB) \]
\[ P'_1(B[\frac{1}{p}]) \ni x'_1\mu(p) \xrightarrow{\eta'_1} x' \leadsto \eta_0(x'_0) \in P(B/pB) \]
according to the definition. There is a unique element $g \in G(B)$ with $x'g = x$ and likewise one and only one $g_0 \in P_1(B/pB)$ with $x'_0g_0 = x_0$. Observe that $g_0$ agrees with the reduction of $g$ modulo $p$, as $\eta_0$ sends $x_0$ (resp. $x'_0$) to the reduction of $x$ (resp. $x'$) modulo $p$. Also, observe that the isomorphism $\eta := \eta'_1 \circ \eta_1^{-1}$ from the $G$-torsor $P_1 \times_A A[\frac{1}{p}]$ to $P'_1 \times_A A[\frac{1}{p}]$ sends $x_1 \in P_1(B)$ to $x'_1\mu(p)g\mu(\frac{1}{p}) \in P'_1(B[\frac{1}{p}])$. According to corollary 2.22 we have $\mu(p)g\mu(\frac{1}{p}) \in G(B)$, as
\[ G(B) \ni g \leadsto g_0 \in P_\mu(B/pB) \]
Towards the assertion (ii), we observe that part (ii) of lemma 2.4 yields a section $\mu$ deduce by taking $B$ over $p$.

It is clear that Frobenius $\mu$ for the isomorphism induced by $b$ and therefore the integral section 1 = $x_1 \in G(A)$ would yield the desired element $x_1 \in P_1(A)$.

5.1. Frobenius $G$-torsors. Suppose that $\sigma$ is a Frobenius lift on a torsion-free $p$-adically separated and complete ring $A$. For any $n \in \mathbb{N}$, the pull-back along $\sigma^n : A \to A$ will be denoted with a superscripted “$\sigma^n$”, e.g. $\sigma^n M := M \otimes_{\sigma^n A} A$ for any $A$-module $M$ and $\sigma^n X := X \times A$ for any $A$-scheme $X$. It is folklore to call a pair $(M, \phi)$, where $M$ is a finitely generated projective $A$-module and

$$\phi : \sigma^n A \otimes A A[1/p] \equiv M \otimes A A[1/p]$$

is an isomorphism, a Frobenius module. We would like to spell out the analog in the setting of $G$-torsors:

**Definition 5.4.** Let $\sigma : A \to A$ be as above and let $G$ be a smooth affine $\mathbb{Z}_p$-group. By a Frobenius $G$-torsor over $(A, \sigma)$ we mean a pair $(P, b)$, where $P$ is a $G$-torsor over $A$ and $b$ is a $G$-equivariant isomorphism from $\sigma^n P \times A A[1/p]$ to $P \times A A[1/p]$. If $(P, b)$ is a Frobenius $G$-torsor over $(A, \sigma)$ we write $P^{\text{Ad}}$ for the finitely generated projective $A$-module corresponding to the GL($g$)-torsor $P^{\text{Ad}}{\text{GL}(g)}$ and

$$b^{\text{Ad}} : \sigma^n P^{\text{Ad}} \otimes A A[1/p] \to P^{\text{Ad}} \otimes A A[1/p]$$

for the isomorphism induced by $b$. A Frobenius $G$-torsor $(P, b)$ over $(A, \sigma)$ is called adjoint-nilpotent if one (and hence both) of the following equivalent properties is fulfilled:

- There exists $n \in \mathbb{N}$ such that the composition $\sigma^n \circ b^{\text{Ad}} \circ \sigma(b^{\text{Ad}}) \circ \cdots \circ \sigma^n(b^{\text{Ad}})$ is integral (i.e. sends $\sigma^{n+1} P^{\text{Ad}}$ to $P^{\text{Ad}}$).

- The $\sigma$-linear endomorphism $\sigma \circ b^{\text{Ad}} \circ (\text{id}_{P^{\text{Ad}}} \otimes \sigma)$ is $p$-adically topologically nilpotent on $P^{\text{Ad}} \otimes A A[1/p]$.

We call $(P, b)$ banal if $P$ is a trivial $G$-torsor, i.e. $P(A) \neq \emptyset$. A morphism between Frobenius $G$-torsors $(P_1, b_1)$ and $(P_2, b_2)$ is a morphism $\alpha : P_1 \to P_2$ of their underlying torsors, such that $b_2 \circ \sigma(\alpha) \circ b_1^{-1} = \alpha$ holds over $A[1/p]$.

It is clear that Frobenius $G$-torsors over $(A, \sigma)$ form a groupoid which we denote by $B_{A, \sigma}(G)$. Its full subcategory consisting of adjoint nilpotent Frobenius $G$-torsors over $(A, \sigma)$ will be denoted by $B^{\text{nil}}_{A, \sigma}(G)$. In either case the full subcategories of banal objects read:

$$G(A[1/p]/\sigma G(A)) \to B_{A, \sigma}(G)$$

$$C^{\text{nil}}_{A}(G)/\sigma G(A) \to B^{\text{nil}}_{A, \sigma}(G)$$
where
\[ C_A^{nil}(G) := \{ b \in G(A[p]) \mid \exists n \in \mathbb{N} : p^n \text{Ad}^G(b) \cdots \text{Ad}^G(b)(g) \subset A \otimes_{\mathbb{Z}_p} g \}. \]
equivalently, \( C_A^{nil}(G) \) is the subset of \( G(A[\frac{1}{p}]) \) consisting of elements \( b \) such that \( p^n (\text{Ad}^G(b) \circ (\sigma \otimes \text{id}_g))^n \) converges \( p \)-adically towards 0. Observe that we have an equality (cf. definition 4.1):
\[ G(A) \cap C_A^{nil}(G) \mu^F(A\{\frac{1}{p}\}) = C_A^{nil}(G, \mu) \]
In order to see this, consider \( b = U^F \mu(p) \) for some \( U \in G(A) \), and observe that the endomorphism \( p \text{Ad}^G(b) \circ (\sigma \otimes \text{id}_g) \) preserves \( A \otimes_{\mathbb{Z}_p} g \) and that its \((\text{mod} \ p)\)-reduction is given by the formula (32). Hence \( U \) satisfies the adjacent nilpotence condition if and only if \( b \) is contained in \( C_A^{nil}(G) \).

5.2. Frames. From now on we fix a finite field \( k_0 \supseteq F_p \) and a display datum \((G, \mu)\) over \( W(k_0)\) (cf. definition 3.1).

**Definition 5.5.** By a \((W(k_0), \mu)\)-frame we mean a triple \((A, a, \sigma)\) where \( a \) is a \( \mu \)-adically closed pd-ideal in a torsion-free \( \mu \)-adically separated and complete \( W(k_0) \)-algebra \( A \) on which \( \sigma \) is a Frobenius lift.

Recall, that pd-structures on ideals in torsion-free rings are unique. Furthermore, given the existence of a lift \( \sigma \) of the absolute Frobenius, we observe that the existence of a pd-structure on \( A \) implies a crucial inclusion
\[ a \subset \sigma^{-1}(pA) \]
as \( \sigma^{-1}(pA) \) is in fact the largest pd-ideal of \( A \). When working with one \((W(k_0), \mu)\)-frame \((A, a, \sigma)\) at a time, we would write
\[ \sigma_0 : A/a \rightarrow A/pA. \]
for the homomorphism induced by \( \sigma \). Pull-back along this morphism will be indicated by superscripted “\( \sigma_0 \)”, in particular we let \( \sigma_0 X := X \times_{A/a, \sigma_0} A/pA \) for any \( A/a \)-scheme \( X \). In the same vein, observe that a \( \mu \)-structure \((Q_0, \eta_0)\) on the \( \text{mod} \ p \) reduction of some \( G \)-torsor \( P \) over \( A \) gives rise (by pulling-back along \( \sigma_0 \)) to a \( \mu \)-structure on the \( \text{mod} \ p \) reduction of \( \sigma P \), which we denote by \((\sigma_0 Q_0, \sigma_0(\eta_0))\).

In the sequel we write \( \hat{\Phi}_{A, \mu}^{A, \sigma, e-1(pA)} : G(A) \rightarrow G(A) \) for the composition of the following group homomorphisms:
\[ \Gamma_{\mu}^{\sigma^{-1}(pA)} \rightarrow G(A) \]

Notice that the second map is well-defined, because of corollary 2.22. Clearly one has \( \hat{\Phi}_{\mu}^{A, \sigma, e}(g) = \sigma(\mu(p)g\mu(\frac{1}{p})) = \sigma(\mu(p)\sigma(g)\mu(\frac{1}{p})) \). Observe that \([C_A^{nil}(G, \mu) / \hat{\Phi}_{\mu}^{A, \sigma}, \Gamma_{\mu}^{A, \sigma}]\) agrees with \([C_A^{nil}(G, \mu) / \mu, \Gamma_{\mu}^{A, \sigma}]\). We would like to add some general comments on morphisms between frames:

**Definition 5.6.** Fix a finite field \( k_0 \) and let \((A, a, \sigma)\) and \((B, b, \tau)\) be \((W(k_0), \mu)\)-frames. By a morphism \( \varphi \) from the former to the latter we mean a \((W(k_0), \mu)\)-algebra homomorphism \( A \rightarrow B \) such that \( \varphi(a) \subset b \) and \( \varphi \circ \sigma = \tau \circ \varphi \). At last we call \( \varphi \) a \( p \)-adic covering if and only if its special fiber \( A/pA \rightarrow B/pB \) is faithfully flat while \( b \) is topologically generated by \( \varphi(a) \) with respect to the \( p \)-adic topology.
Lemma 5.7. Let $\Lambda$ be a ring, $M \subset \Lambda$ an ideal and $\varpi : A \to B$ a morphism of flat $\Lambda$-algebras. If the induced morphism $A/M A \to B/MB$, is flat then the same holds for $A/M^n A \to B/M^n B$, for any $n \in \mathbb{N}$.

Proof. Let us prove this by induction on $n$, so given an ideal $I \subset A/M^n A$, we need to show the injectivity of $I \otimes_{A/M^n A} B/M^n B \to B/M^n B$. Consider the ideal $I \cap MA/M^n A =: J \subset MA/M^n A$ and the commutative diagram

$$
\begin{array}{c}
J \otimes_{A/M^n A} B/M^n B \\
\downarrow \\
MB/M^n B \\
\downarrow \\
B/M^n B \\
\downarrow \\
B/MB
\end{array}
$$

Notice the identifications

\[
\begin{align*}
MA/M^n A & \otimes_{A/M^n A} B/M^n B \cong MB/M^n B \\
J \otimes_{A/M^n-1 A} B/M^{n-1} B & \cong J \otimes_{A/M^n A} B/M^n B \\
I/J \otimes_{A/M^n A} B/M^n B & \cong I/J \otimes_{A/M A} B/MB
\end{align*}
\]

The map

$$J \otimes_{A/M^n A} B/M^n B \to B/M^{n-1} B$$

is injective by the induction hypothesis. The map

$$I/J \otimes_{A/M A} B/MB \to B/MB$$

is injective by the assumed flatness of $B/MB$ over $A/MA$. At last observe the exactness of the sequence

$$J \otimes_{A/M^n A} B/M^n B \to I \otimes_{A/M^n A} B/M^n B \to I/J \otimes_{A/M^n A} B/M^n B$$

so that a straightforward diagram chase yields the result. \hfill $\Box$

Definition 5.8. Let $(A, a, \sigma)$ be a $W(k_0)$-frame. Consider a Frobenius $G$-torsor $(P, b)$ over $(A, \sigma)$ together with a $P_\mu$-structure $(Q_0, \eta_0)$ on $P_0 := P \times_A A/a$, in other words, we have:

- $P$ is a $G$-torsor over $A$.
- $b$ is a $G$-equivariant isomorphism from $^G P \times_A A[\frac{1}{p}]$ to $P \times_A A[\frac{1}{p}]$.
- $Q_0$ is a $P_\mu$-torsor over $A/a$.
- $\eta_0$ is a $G$-equivariant isomorphism from $Q_0 \ast G$ to $P_0$.

We call the quadruple $(P, Q_0, \eta_0, b)$ a $(G, \mu)$-window over $(A, a, \sigma)$ if and only if the following axioms hold:

(i) The modification of $^G P$ given by $b : ^G P \times_A A[\frac{1}{p}] \cong P \times_A A[\frac{1}{p}]$ is adapted to the $P_\mu$-structure $(^G Q_0, \sigma_0(\eta_0))$ on $^G P \times_A A/pA$, which is the pull-back of $(Q_0, \eta_0)$ along the morphism $\sigma_0 : A/a \to A/pA$.

(ii) The Frobenius $G$-torsor $(P, b)$ over $(A, \sigma)$ is adjoint nilpotent, in the sense of definition 5.4.

A $(G, \mu)$-window $(P, Q_0, \eta_0, b)$ over $(A, a, \sigma)$ is called banal if and only if $Q_0$ is the trivial $P_\mu$-torsor over $A/a$. 

For any $p$-adic covering $\varpi : (A, a, \sigma) \to (B, b, \tau)$ we claim that the maps $A/p^n A \to B/p^n B$ are flat for every $n$, in fact we have the following:
It is clear that \((G, \mu)\)-windows over \((A, a, \sigma)\) form a groupoid, which we denote by \(B^\text{nil}_{A,a,\sigma}(G, \mu)\). Note that this is a full subcategory of the groupoid of quadruples \((P, Q_0, \eta_0, b)\) satisfying (i) only, which shall be denoted by \(B_{A,a,\sigma}(G, \mu)\). Let us elucidate their full subcategories of banal quadruples \(W = (P, Q_0, \eta_0, b)\): Given the triviality of \(Q_0\) we may choose isomorphisms \(P_0 \times W(k_0) A / a \cong Q_0\) and \(G\times_{Z_p} A \cong P\), of whose the latter is a lift of \(G \times_{Z_p} A / a \cong Q_0 \ast G \cong P_0 \times A / P_0\) (cf. part (i) of lemma 2.6). This allows to rewrite \(W\) in the form \(G \times_{Z_p} A, P_0 \times W(k_0) A / P_0\), where \(b \in \text{Hom}_G(P \times A [\frac{1}{p}], P \times A [\frac{1}{p}]) = G(A[\frac{1}{p}])\), as \(P = G \times W(k_0) A = P\).

It remains to analyze which elements of \(G(A[\frac{1}{p}])\) satisfy the adaptedness condition (i) of definition 5.8. So \(P\) is an adapted modification of \(P\) if and only if there exists \(x_0 \in P_0\mu(A/pA)\) and \(x, x_1 \in G(A)\) such that the \(\text{mod } p\) reduction of \(x\) agrees with \(x_0\) while \(bx = x_1\mu(p)\). In view of corollary 2.22 this means that \(b\) can be written as \(U^F\mu(p)\) for some \(U \in G(A)\). At last, it is easy to see that the set of \(B_{A,a,\sigma}(G, \mu)\)-morphisms from another \(B_{A,a,\sigma}(G, \mu)\)-object of the form \(W' = (G \times_{Z_p} A, P_0 \times W(k_0) A / P_0, \text{id}, b')\) to \(W\) is given by

\[
\text{Hom}(W', W) = \{k \in \Gamma_{\mu,\sigma}^A \mid b' = k^{-1}\sigma r(k)\} = \{k \in \Gamma_{\mu,\sigma}^A \mid U' = k^{-1}U\Phi_{\mu,\sigma}^A(k)\},
\]

where \(b' = U^F\mu(p)\). In the language of part A of the appendix this means that the full subcategory of banal \((G, \mu)\)-windows over \((A, a, \sigma)\) (resp. banal \(B_{A,a,\sigma}(G, \mu)\)-objects) is equivalent to \([C^\text{nil}_A(G, \mu)F\mu(p)/\sigma\Gamma_{\mu,\sigma}^A] \cong [C^\text{nil}_A(G, \mu)/\Phi_{\mu,\sigma}^A\Gamma_{\mu,\sigma}^A]\) (resp. \([G(A)F\mu(p)/\sigma\Gamma_{\mu,\sigma}^A] \cong [G(A)/\Phi_{\mu,\sigma}^A\Gamma_{\mu,\sigma}^A]\)). Note also that there is a natural commutative diagram of groupoids:

\[
\begin{array}{ccc}
B^\text{nil}_{A,a,\sigma}(G, \mu) & \longrightarrow & B_{A,a,\sigma}(G, \mu) \\
\downarrow & & \downarrow \\
B^\text{nil}_{A,\sigma}(G) & \longrightarrow & B_{A,\sigma}(G)
\end{array}
\]

of which the banal pendant is:

\[
\begin{array}{ccc}
[C^\text{nil}_A(G, \mu)F\mu(p)/\sigma\Gamma_{\mu,\sigma}^A] & \longrightarrow & [G(A)F\mu(p)/\sigma\Gamma_{\mu,\sigma}^A] \\
\downarrow & & \downarrow \\
[C^\text{nil}_A(G)/\sigma G(A)] & \longrightarrow & [G(A[\frac{1}{p}])\sigma G(A)]
\end{array}
\]

**Remark 5.9.** We note some elementary properties of these diagrams:

(a) All horizontal arrows in these diagrams are fully faithful.
(b) Both of these diagrams are 2-Cartesian.
(c) All vertical arrows in these diagrams are faithful.

### 5.3. A reformulation

In this subsection we give a technical reformulation of the concept of \((G, \mu)\)-window over a \(W(k_0)\)-frame \((A, a, \sigma)\). This is based on the formalism of pull-back along so-called mini-morphism, as developed in part B.5 of the appendix: Let us write \(\text{int}_\mu\) for the mini-morphism from \(\hat{\mu}W(k_0)\) to \(G_W(k_0)\) whose effect on any flat \(W(k_0)\)-algebra \(R\) is given by

\[
\hat{\mu}W(k_0)(R) = \Gamma_{\mu,pr}^A \rightarrow G(R); \quad g \mapsto \mu(p)g\mu(\frac{1}{p})
\]
(see corollary 2.22). Following the formalism (74) this induces a functor

\[(55) \quad \mathrm{mo}_\mu : \text{Tor}_R(\hat{\mathcal{P}}_\mu^W(k_0)) \to \text{Tor}_R(G); \hat{P} \mapsto \hat{P} \ast_{\text{int}_\mu} G\]

for any such \( R \). Observe also that for any \( \hat{\mathcal{P}}_\mu^W(k_0) \)-torsor \( \hat{P} \) over \( R \) there is a canonical isomorphism between the generic fibers of \( \hat{P} * G \) and \( \hat{P} \ast_{\text{int}_\mu} G \), i.e., the latter is a canonical modification of the former in the sense of definition 5.1. We have to start with another result:

**Lemma 5.10.** Let \( A \) be a flat \( W(k_0) \)-algebra.

(i) The natural 2-commutative diagram

\[
\begin{array}{ccc}
\text{Tor}_A(\hat{\mathcal{P}}_\mu^W(k_0)) & \longrightarrow & \text{Tor}_A(G) \\
\downarrow & & \downarrow \\
\text{Tor}_{A/p}(P_\mu) & \longrightarrow & \text{Tor}_{A/p}(G)
\end{array}
\]

is 2-Cartesian, where the top horizontal map is induced by change of group along the natural inclusion \( \hat{\mathcal{P}}_\mu^W(k_0) \to G \).

(ii) Suppose that \( \hat{P} \) is the \( \hat{\mathcal{P}}_\mu^W(k_0) \)-torsor that corresponds to a \( P_\mu \)-structure \((Q_0, \eta_0)\) on the \((\text{mod } p)\)-reduction of some \( G \)-torsor \( P \) over \( A \), as in the diagram above. Then the modification \( P_1 := \hat{P} \ast_{\text{int}_\mu} G \) is adapted to \((Q_0, \eta_0)\).

In particular, adapted modifications always exist (and they are unique by part (i) of lemma 5.3).

**Proof.** Part (i) is a canonical consequence of the criterion (i) of lemma B.9. In order to deal with part (ii) we choose a section \( \hat{x} \in \hat{P}(B) \), where \( B \) is a faithfully flat \( A \)-algebra. This gives rise to sections \( x \in P(B) \) and \( x_0 \in Q_0(B/pB) \) satisfying \( \hat{x} \sim_{\eta_0} x_0 \in P(B/pB) \). Moreover \( P_1 \) has a section over \( B \) too, namely \( x_1 := \hat{x} \ast 1 \). At last the canonical homomorphism from \( P \times A A[\frac{1}{p}] \) to \( P_1 \times_A A[\frac{1}{p}] \) sends \( x \) to \( x_1 \mu(p) \), as can be seen immediately from the formula (73). □

We are now in the position to give the reformulation of definition 5.8, as promised at the beginning of this subsection. Let \((A, a, \sigma)\) be a \( W(k_0) \)-frame and let us write \( F X \) (resp. \( \hat{X} \)) for the pull-back of an fpqc sheaf \( X \) on \( \text{Spec} W(k_0) \) (resp. \( \text{Spec} A \)) along the morphism \( \text{Spec} W(k_0) \xrightarrow{\sigma} \text{Spec} W(k_0) \) (resp. \( \text{Spec} A \xrightarrow{\sigma} \text{Spec} A \)). In particular, observe that the pull-back \( \sigma \hat{P}_\mu^a \) is naturally isomorphic to \( \hat{P}_{\sigma(a)}^F \), as \( F P_\mu \cong P_{\sigma(a)}^F \). Likewise, pulling back some \( \hat{P}_\mu^a \)-torsor \( \hat{Q} \) on \( \text{Spec} A \) yields a \( \hat{P}_{\sigma(a)}^F \)-torsor \( \sigma \hat{Q} \) on \( \text{Spec} \sigma \). The crucial inclusion (53) allows to precompose with an extension of structure groups \( \hat{P}_{\sigma(a)}^F \subset \hat{P}_\mu^F \), thus arriving at a functor:

\[(56) \quad \text{Tor}_A(\hat{P}_\mu^a) \to \text{Tor}_A(\hat{P}_{\sigma(a)}^F); \hat{Q} \to \sigma \hat{Q} \ast_{\text{int}_{\sigma(a)}^F} \hat{P}_{\sigma(a)}^F \]

Composing the functor \( \mathrm{mo}_{\sigma(a)} \), as defined in (55), with (56) yields:

\[(57) \quad \mathrm{mo}_\sigma : \text{Tor}_A(\hat{P}_\mu^a) \to \text{Tor}_A(G); \hat{Q} \to (\sigma \hat{Q} \ast_{\text{int}_{\sigma(a)}^F} \hat{P}_{\sigma(a)}^F) \ast_{\text{int}_\mu} G \]

**Corollary 5.11.** Let \((A, a, \sigma)\) be a \( W(k_0) \)-frame and \((G, \mu)\) a \( W(k_0) \)-display datum, where \( k_0 \) is a finite field.
(i) The natural category structure on pairs \((\hat{Q}, u)\) consisting of a \(\hat{P}_\mu^u\)-torsor \(\hat{Q}\) together with a \(G\)-equivariant isomorphism \(u : \text{mo}^u_\sigma(\hat{Q}) \xrightarrow{\cong} \hat{Q} \ast G\) is naturally equivalent to the groupoid \(\mathcal{B}_{A,a,\sigma}(G, \mu)\).

(ii) Its full subcategory consisting of those pairs \((\hat{Q}, u)\) such that its scalar extension to \(A[\frac{1}{\mu}]\) satisfies the adjoint nilpotence condition (cf. definition 5.4) is equivalent to the groupoid \(\mathcal{B}^{nil}_{A,a,\sigma}(G, \mu)\) of \((G, \mu)\)-windows over \((A, a, \sigma)\).

**Proof.** Our point of departure is part (i) of lemma 5.10, which says that giving \((Q_0, \eta_0)\) on its \(\text{mod } a\)-reduction, i.e. a \(G\)-isomorphism \(\eta_0 : Q_0 \ast F_{\mu} G \xrightarrow{\cong} P \times_A A/a\).

In order to turn \((P, Q_0, \eta_0)\) into a \(\mathcal{B}_{A,a,\sigma}(G, \mu)\)-object \((P, Q_0, \eta_0, b)\) we need to give a \(G\)-isomorphism \(b : P \times A A[\frac{1}{\mu} \mu] \xrightarrow{\cong} P \times A A[\frac{1}{\mu}]\) with respect to which \(P\) becomes a \((\sigma_0 Q_0, \sigma_0(\eta_0))\)-adapted modification of \(\sigma P\). Applying part (ii) of lemma 5.10 to the \(P_{\mu}^B\)-torsor \(\sigma \hat{Q} \ast \hat{P}_{\mu}^B\) reveals that there is a unique such \(G\)-torsor, namely \(\text{mo}^u_\sigma(\hat{Q})\). We deduce that the datum “\(b\)” is equivalent to giving an isomorphism \(u\) from \(\text{mo}^u_\sigma(\hat{Q})\) to \(P\), which is \(\hat{Q} \ast G\).

**Remark 5.12.** One might want to think of this as yet another 2-Cartesian diagram, namely:

\[
\begin{array}{ccc}
\mathcal{B}_{A,a,\sigma}(G, \mu) & \rightarrow & \text{Tor}_A(G) \\
\downarrow & & \downarrow \Delta_{\text{Tor}_A(G)} \\
\text{Tor}_A(\hat{P}_\mu^u) & \underset{(\text{mo}^u_\sigma, \text{id})}{\rightarrow} & \text{Tor}_A(G)^2
\end{array}
\]

6. **The Main Theorem**

The strategy of the proof of theorem 6.4 consists of passing from a given \(W(k_0)\)-frame \((A, a, \sigma)\) to the so-called Witt-frame, i.e. \((W(A), W(a) + I(A), F)\). In the banal case this is an intricate application of corollary 4.11 and in the non-banal case one also has to worry about “banalization” via passage to other frames, supplemented by a study of descent properties. A last and final ingredient is the equivalence

\[
\mathcal{B}^{nil}_{W(A), W(a) + I(A), F}(G, \mu) \xrightarrow{\cong} \text{Tri}^{nil}_{A,a,\gamma}(G, \mu),
\]

of which the banal pendant is a tautology, namely:

\[
[C^{nil}_{W(A)}(G, \mu)/\phi_{W(A)}(F) \mu W(A), W(a) + I(A)] \xrightarrow{\cong} [C^{nil}_{W(A)}(G, \mu)/\phi_{W(A)}(F) \mu W(A)]
\]

Every morphism of \(W(k_0)\)-frames \(\kappa : (A, a, \sigma) \rightarrow (B, b, \tau)\) induces “compatible” maps

\[
\Gamma^A_{\mu} \rightarrow \Gamma^B_{\mu}
\]

\[
C^A_A(G, \mu) \rightarrow C^B_B(G, \mu)
\]

along with functors

\[
[C^{nil}_A(G, \mu)^F \mu(p) / \sigma \Gamma^A_{\mu}] \rightarrow [C^{nil}_B(G, \mu)^F \mu(p) / \sigma \Gamma^B_{\mu}]
\]

\[
\mathcal{B}^{nil}_{A,a,\sigma}(G, \mu) \rightarrow \mathcal{B}^{nil}_{B,b,\tau}(G, \mu)
\]
as well as \( B_{A,\sigma}^{nil}(G) \to B_{B,A}^{nil}(G) \) and \( B_{A,\sigma}(G) \to B_{B,A}(G) \). If \( \sigma \) is injective, it is easy to see that all of the aforementioned functors are faithful.

6.1. **Descent.** Let \( k_0 \) and \( (G, \mu) \) be as above and let \( \chi : (A, a, \sigma) \to (B, b, \tau) \) be \( W(k_0) \)-linear \( p \)-adic covering. Consider the \( p \)-adically completed \( m \)-fold tensor product of \( B \), i.e. the \( A \)-algebra

\[
B_{[m]} := B^\otimes A_m = B^\otimes A_{m},
\]

and let us write \( b_{[m]} \) for the \( p \)-adically closed \( B_{[m]} \)-ideal which is topologically generated by the set \( \bigcup_{k+l=m-1} B^\otimes A l \otimes A b \otimes A B^\otimes A k \), so that \( B_{[m]}/b_{[m]} \) is isomorphic to the \( p \)-adically completed \( m \)-fold tensor product of the \( A/a \)-algebra \( B/b \). Let us write \( \tau_{[m]} : B_{[m]} \to B_{[m]} \) for the endomorphism induced by \( x_1 \otimes \cdots \otimes x_m \mapsto \tau(x_1) \otimes \cdots \otimes \tau(x_m) \). With the usual notations for \( \chi_{12}, \chi_{23}, \chi_{13} : B_{[2]} \to B_{[3]} \) (resp. \( \chi_1, \chi_2 : B \to B_{[2]} \)) we obtain further morphism \( (B_{[2]}, b_{[2]}, \tau_{[2]}) \to (B_{[3]}, b_{[3]}, \tau_{[3]}) \) (resp. \( (B, b, \tau) \to (B_{[2]}, b_{[2]}, \tau_{[2]}) \)). In order to give a sober formulation of descent we need the following notions:

- Let \( W \) be a \((G, \mu)\)-window over \((B, b, \tau)\). By a \( \chi \)-descent datum on \( W \) we mean a \( B_{B_{[2]}, b_{[2]}, \tau_{[2]}}^{nil}(G, \mu) \)-morphism \( \alpha : \chi_1(W) \to \chi_2(W) \). The descent datum is said to satisfy the cocycle condition if \( \chi_{23}(\alpha) \circ \chi_{12}(\alpha) = \chi_{13}(\alpha) \) holds in \( B_{B_{[3]}, b_{[3]}, \tau_{[3]}}^{nil}(G, \mu) \).

- Let \( \alpha' \) be another \( \chi \)-descent datum on another \( B_{B_{[2]}, b_{[2]}, \tau_{[2]}}^{nil}(G, \mu) \)-object \( W' \). By an isomorphism from \((W, \alpha)\) to \((W', \alpha')\), we mean a \( B_{B_{[2]}, b_{[2]}, \tau_{[2]}}^{nil}(G, \mu) \)-morphism \( \eta : W \to W' \) satisfying \( \alpha' \circ \chi_1(\eta) = \chi_2(\eta) \circ \alpha \).

**Proposition 6.1.** Let \( k_0 \) and \( (G, \mu) \) be as above and let \( \chi : (A, a, \sigma) \to (B, b, \tau) \) be a \( W(k_0) \)-linear \( p \)-adic covering. Then \( \chi \) induces a fully faithful functor from \( B_{A,\sigma}^{nil}(G, \mu) \) to the groupoid of \((G, \mu)\)-windows over \((B, b, \tau)\) with \( \chi \)-descent datum. The essential image of this functor consists of all pairs \((W, \alpha)\), where \( \alpha \) satisfies the cocycle condition.

**Proof.** In order to descend the underlying \( G \)-torsor \( P \) (resp. the \( P_0 \)-structure \((Q_0, \eta_0)\) on its \( \mod \) \( a \)-reduction) one simply combines corollary 2.7 with the usual descent along the maps \( A/p^n A \to B/p^n B \) (resp. \( A/(p^n A + a) \to B/(p^n B + b) \)). Due to lemma 5.7 these maps are faithfully flat, so that one has effective descent. In order to descend the Frobenius of the window i.e. the datum \( "b" \) in the quadruple of definition 5.8 we use the reformulation in terms of the \( "a" \) given in corollary 5.11, together with the aforementioned faithful flatness of \( A/p^n A \to B/p^n B \). Descending the applicable adjoint nilpotence condition is straightforward. \( \square \)

Observe that the triple \((W(A), W(a) + I(A), F)\) is a \( W(k_0) \)-frame whenever \( a \) is a \( p \)-adically closed pd-ideal in a \( p \)-adically separated and complete torsion-free \( W(k_0) \)-algebra \( A \). We write \( \chi_\sigma : (A, a, \sigma) \to (W(A), W(a) + I(A), F) \) for Cartier’s diagonal map, which is associated to our Frobenius lift \( \sigma \). This Frobenius lift induces by functoriality of the Witt vectors, a morphism on \((W(A), F)\), which we will still denote by \( \sigma \). Note that \( \chi_\sigma \) identifies \( A \) with the equalizer of the two endomorphisms \( F \) and \( \sigma \) on \( W(A) \). Concerning a change of frame via \( \chi_\sigma \) we have the following result, which is one of the main ingredients of for the proof of the main theorem:

**Proposition 6.2.** Fix a \( W(k_0) \)-frame \((A, a, \sigma)\). The frame change

\[
B_{A,\sigma}^{nil}(G, \mu) \to B_{W(A), W(a) + I(A), F}^{nil}(G, \mu),
\]

as well as \( B_{A,\sigma}^{nil}(G) \to B_{B,A}^{nil}(G) \) and \( B_{A,\sigma}(G) \to B_{B,A}(G) \). If \( \sigma \) is injective, it is easy to see that all of the aforementioned functors are faithful.
which is induced from Cartier’s diagonal morphism

\[ \kappa_\sigma : (A, a, \sigma) \to (W(A), W(a) + I(A), F) \]

is an equivalence of categories.

**Proof.** Observe that we have \( W(A)/(W(a) + I(A)) \cong A/a \), so that the \( P_\mu \)-torsor \( Q_0 \) which is implicit in giving a window (being the 2nd datum in the quadruple of definition (5.8)) is not altered by the functor (65). In particular a \((G, \mu)\)-window over \((A, a, \sigma)\) is banal if and only if its base change under \( \kappa_\sigma \) is a banal \((G, \mu)\)-window over \((W(A), W(a) + I(A), F)\). We are going to first prove the equivalence of the frame change functor restricted to the banal objects and then prove the general case by descent to this case. So let us consider the banal pendant of the functor (65), which reads:

\[
[C^\text{nil}_A(G, \mu)^F, \mu(p)/\sigma \Gamma^A, a] \to [C^\text{nil}_W(A)(G, \mu)^F, \mu(p)/\sigma \Gamma W(A), W(a)+I(A)]
\]

Recall that for any frame \((B, b, \tau)\) we have a canonical identification

\[
[C^\text{nil}_B(G, \mu)^F, \mu(p)/\sigma \Gamma^B, b] \cong [C^\text{nil}_B(G, \mu)/\hat{\Phi}_\mu^B, \Gamma^B, b]
\]

We are going to work with the description on the right hand side. Our proof starts with the full faithfulness of (66). Its faithfulness is clear, from \( \kappa_\sigma \) inducing an inclusion \( \Gamma^A, a \to \Gamma W(A), W(a)+I(A) \). Let us consider \( O, U \in C^\text{nil}_A(G, \mu) \), so that \( \kappa_\sigma(O), \kappa_\sigma(U) \in C^\text{nil}_W(A)(G, \mu) \). Suppose that \( k \in \Gamma W(A), W(a)+I(A) \) satisfies

\[
\kappa_\sigma(O) = k^{-1} \kappa_\sigma(U) \hat{\Phi}_\mu W(A), F(k)
\]

Applying \( F \) and \( \sigma \) yields

\[
F \kappa_\sigma(O) = F k^{-1} F \kappa_\sigma(U) \hat{\Phi}_\mu W(A), F(F(k))
\]

and

\[
F \kappa_\sigma(O) = \sigma(k)^{-1} F \kappa_\sigma(U) \hat{\Phi}_\mu W(A), F(\sigma(k))
\]

as \( F \kappa_\sigma(O) = \sigma(\kappa_\sigma(O)) \) and \( F \kappa_\sigma(U) = \sigma(\kappa_\sigma(U)) \). In view of proposition 4.6 we are allowed to deduce \( F k = \sigma(k) \) from \( F k \equiv \sigma(k) \mod W(pA) \) together with the fact that \( pA \) is a pd-ideal of \( A \). It follows that \( k \) lies in the image of \( \kappa_\sigma \). At last, notice that

\[
\begin{array}{ccc}
\Gamma^A, a & \xrightarrow{\kappa_\sigma} & \Gamma W(A), W(a)+I(A) \\
\downarrow & & \downarrow \\
G(A) & \xrightarrow{\kappa_\sigma} & G(W(A))
\end{array}
\]

is a Cartesian diagram, so that we have \( k \in \kappa_\sigma(\Gamma^A, a) \), which is what we wanted.

We continue with the essential surjectivity of (66). Let us consider some \( U \in C^\text{nil}_W(A)(G, \mu) \). Observe that we have \( \tau(U) \equiv F(U) \mod W(pA) \), so that by proposition 4.6 and the fact that \( pA \) is a pd-ideal implies there exists some \( k \in G(W(pA)) \) with

\[
\tau(U) = k^{-1} F(U) \hat{\Phi}_\mu W(A), F(k)
\]

Consider, for each \( i \), the ghost coordinate \( k_i := w_i(k) \) and the element \( h_i := k_{i-1} \tau(k_{i-2}) \ldots \tau^{-1}(k_0) \) of \( G(A) \). Since \( h_{i+1} \tau(h_i)^{-1} = k_i \equiv 1 \mod p^{i+1} \), there exists an element \( h \in G(W(A)) \) whose \( i \)th ghost coordinate agrees with \( h_i \) for
any \( i \), in fact \( h \) lies in \( G(I(A)) \subset \Gamma^{W(A)}_\mu,\langle W(A) + I(A) \rangle \), since \( h_0 = 1 \). Now it follows that \( U' := h^{-1}U \hat{\Phi}^{W(A)}_\mu.F(h) \) lies in the image of \( \varphi \), as \( F(U') = \tau(U') \) in view of \( F(h) \tau(h)^{-1} = k \). This achieves the proof in the banal case.

We are now in the position to return to the functor (65), and again we start with the full faithfulness. Consider a \( B^{\text{gr}}_0(W(A),W(A) + I(A),F)(G,\mu) \)-isomorphism \( h \) between \( B^{gr}_{\alpha,n}G,\mu,(G,\mu) \)-objects \((P',Q'_0,\eta'_0,b')\) and \((P,Q_0,\eta_0,b)\). According to [25, Corollaire (17.16.3.ii)] we may choose a \( \acute{\text{e}} \)tale faithfully flat \( A/(pA + a) \)-algebra \( \hat{A}_0 \) satisfying \( Q'_0(\hat{A}_0) \neq \emptyset \neq Q_0(\hat{A}_0) \). Let us write \( \hat{A}_i \) for the \( \acute{\text{e}} \)tale faithfully flat \( A/p^iA \)-algebra satisfying \( \hat{A}_i/(p\hat{A}_i + a\hat{A}_i) \cong \hat{A}_0 \), which exists up to a unique isomorphism (cf. [25, Théorème (18.1.2)]). The uniqueness implies that there are natural morphisms \( \hat{A}_{i+1} \rightarrow \hat{A}_i \) inducing an isomorphism \( \hat{A}_{i+1}/p^{i+1} \cong \hat{A}_i \). By Lemma 6.3 below it is possible to extend \( \varphi \) to a Frobenius lift \( \hat{\varphi} \) on the \( p \)-adically formally \( \acute{\text{e}} \)tale, separated and complete \( A \)-algebra \( \hat{A} := \lim_i \hat{A}_i \). At last writing \( \hat{a} \) for the \( p \)-adic closure of \( a \hat{A} \), yields a \( W(k_0) \)-linear \( p \)-adic covering \( \varphi : (A,a,\sigma) \rightarrow (\hat{A},\hat{a},\hat{\sigma}) \). We write \( (\hat{P}',\hat{Q}'_0,\hat{\eta}'_0,\hat{b}') \) (resp. \((\hat{P},\hat{Q}_0,\hat{\eta}_0,\hat{b})\)) for the \( B^{gr}_{\hat{A},\hat{a},\hat{\sigma}} G,\mu,(G,\mu) \)-objects that arise from \((P',Q'_0,\eta'_0,b')\) (resp. \((P,Q_0,\eta_0,b)\)) by base change along \( \varphi \). From the full faithfulness of (66) we deduce an isomorphism, say \( \hat{h} : (\hat{P}',\hat{Q}'_0,\hat{\eta}'_0,\hat{b}') \rightarrow (\hat{P},\hat{Q}_0,\hat{\eta}_0,\hat{b}) \) whose Cartier diagonal agrees with the base change of \( h \) to the frame \((W(A),W(\hat{a}) + I(\hat{A}),F)\). We have to show that \( \hat{h} \) descends to \((A,a,\sigma)\). We therefore have to check that its images in \((A_{[2]},\hat{a}_{[2]},\hat{\sigma}_{[2]}))\) agree (see notations at the beginning of subsection 6.1), which can be done using the previously established faithfulness of \( \varphi_{[2]} \).

It remains to check the essential surjectivity of (65), so consider a \((G,\mu)\)-window \((P',Q'_0,\eta'_0,b')\) over \((W(A),W(A) + I(A),F)\). Let us choose a \( W(k_0) \)-linear \( p \)-adic covering \( \varphi : (A,a,\sigma) \rightarrow (\hat{A},\hat{a},\hat{\sigma}) \) such that \((P',Q'_0,\eta'_0,b')\) becomes banal over \((W(\hat{A}),W(\hat{a}) + I(\hat{A}),F)\), here notice that \( W(\hat{A})/W(\hat{a}) + I(\hat{A}) \cong \hat{A}/\hat{a} \), so that we can construct \( A \) by means of [25, Corollaire (17.16.3.ii)] and [25, Théorème (18.1.2)]. The self-explanatory descent argument is completed by means of the faithfulness of \( \varphi_{[2]} \) and the full faithfulness of \( \varphi_{[2]} \). \( \square \)

**Lemma 6.3.** Let \( A \) be a ring and \( B \) be a \( p \)-adically formally smooth, separated and complete \( A \)-algebra. Let \( \sigma : A \rightarrow A \) be a Frobenius lift. Then there exists a Frobenius lift \( \tau : B \rightarrow B \) that restricts to \( \sigma \) on \( A \).

**Proof.** In view of \( B \) being \( p \)-adically separated and complete, we should construct a compatible system of Frobenius lifts \( \tau_n : B/p^n \rightarrow B/p^n \), extending \( \sigma_n : A/p^n \rightarrow A/p^n \). We can therefore assume that \( p^nA \) vanishes for some \( n \). The case \( n = 1 \) is trivial. By induction we may assume the result for all formally smooth \( A/p^{n-1}A \)-algebras, so that we have a Frobenius lift extending \( \sigma_{n-1} \)

\[
\tau_{n-1} : B/p^{n-1} \rightarrow B/p^{n-1}
\]

The homomorphism \( \tilde{\tau}_{n-1} : B \rightarrow B/p^{n-1} \) fits into a commutative diagram
\[
A \xrightarrow{\tau_n} B \xrightarrow{\tau_n-1} B/p^{n-1}
\]

in which the formal smoothness yields the desired oblique morphism \( \tau_n : B \to B \)
filling the diagram. \( \Box \)

We are now in a position to prove the following:

**Theorem 6.4.** Let \((A, \alpha, \sigma)\) be a \(W(k_0)\)-frame and \((G, \mu)\) a \(W(k_0)\)-display datum, where \(k_0\) is a finite field. The natural functor from \(B_{A,\alpha,\sigma}(G, \mu)\) to the \(A/\alpha\)-valued objects of \(B(G, \mu)\) restricts to an equivalence from the groupoid of \((G, \mu)\)-windows over \((A, \alpha, \sigma)\) to the groupoid of \((G, \mu)\)-displays over \(A/\alpha\) of which the \((\mod p)\)-reduction is adjoint nilpotent. If \(a\) is \(p\)-adically open, one has in particular:

\[
B_{A,\alpha,\sigma}^{nil}(G, \mu) \xrightarrow{\cong} B^{nil}(G, \mu)(A/\alpha)
\]

**Proof.** In view of theorem 4.31 and \(\chi_{\sigma} : B_{A,\alpha,\sigma}^{nil}(G, \mu) \xrightarrow{\cong} B_{W(A),W(\alpha)+I(A)},F(G, \mu)\) (i.e. proposition 6.2) it only remains to check (59). However, this follows from our reformulation i.e. corollary 5.11. We merely have to observe that

- \(\text{Tor}_{W(A)}(G) \xrightarrow{\cong} \text{Tor}_{A}(L^+G)\) by lemma 2.12.
- \(\text{Tor}_{W(A)}(\hat{H}_{\mu}^{W(\alpha)+I(A)}) \xrightarrow{\cong} \text{Tor}_{A}(\hat{H}_{\mu}^{\alpha})\) again by lemma 2.12 in conjunction with part (i) of lemma 4.30
- The 2-commutativity of the diagram:

\[
\begin{array}{ccc}
\text{Tor}_{W(A)}(\hat{H}_{\mu}^{W(\alpha)+I(A)}) & \xrightarrow{\Psi_{\mu}^{\alpha}} & \text{Tor}_{A}(\hat{H}_{\mu}^{\alpha}) \\
\downarrow \Psi_{\mu} & & \uparrow \equiv \\
\text{Tor}_{W(A)}(\hat{H}_{\mu}^{W(k_0)}) & \xrightarrow{\text{mod}_n} & \text{Tor}_{W(A)}(G)
\end{array}
\]

\( \Box \)

**Appendix A. Groupoids**

Let \(\Gamma\) be an abstract group acting from the right on a set \(X\). We denote by \([X/\Gamma]\) the following category: objects are elements of \(X\) and if \(x, y \in X\), we set \(\text{Hom}_{[X/\Gamma]}(x, y) := \{\gamma \in \Gamma \mid x = y \cdot \gamma\}\). This defines a groupoid, as every morphism has an inverse.

Suppose that \(\Xi, \Phi : \Gamma \to M\) are homomorphisms to another abstract group \(M\). By \((\Xi, \Phi)\)-conjugation we mean the action

\[
(67) \quad M \times \Gamma \to M; \quad (m, \gamma) \mapsto \Xi(\gamma)^{-1}m\Phi(\gamma),
\]

and we let \([M/\Xi, \Phi]\Gamma\) be the resulting quotient groupoid, where \(M\) is regarded as a right \(\Gamma\)-set, according to the \((\Xi, \Phi)\)-conjugation (67). If \(M\) contains \(\Gamma\) as a subgroup and \(\Xi\) is the inclusion, we drop \(\Xi\) from the notation. Note that this construction is functorial in quadruples \((\Xi, \Phi, \Gamma, M)\) in the obvious way. Whenever some subset \(X \subset M\) is invariant under \((\Xi, \Phi)\)-conjugation, we also write \([X/\Xi, \Phi]\Gamma\) for the full subcategory of \([M/\Xi, \Phi]\Gamma\) whose class of objects is \(X\).
Appendix B. Torsors

B.1. Fpqc sheaves. For every scheme $S$ we let $\mathcal{S}ch_S$ be the category of schemes over the base $S$. Unless otherwise said we endow $\mathcal{S}ch_S$ with the so-called fpqc topology, which is the Grothendieck topology generated by all faithfully flat and quasi-compact coverings together with the usual Zariski ones (cf. [11, Exposée IV 6.3]). Recall that a covering $\mathcal{V} = \{V_j\}_{j \in J}$ of some $S$-scheme $U$ is said to refine another $\mathcal{U} = \{U_i\}_{i \in I}$ one, if for each $j \in J$ there exists $i \in I$ such that $V_j \to U$ factors through $U_i \to U$. Recall also that a presheaf on $S$ is nothing but a contravariant set-valued functor on $\mathcal{S}ch_S$. Let us recall the sheaf concepts for this particular site:

Notation B.1. 
- $\mathcal{U} \times \mathcal{V} := \{U_i \times_U V_j\}_{i \in I, j \in J}$ is the “canonical common refinement” of coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ of some $S$-scheme $U$.
- For every presheaf $X: \mathcal{S}ch_S^{op} \to Ens$ and every covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $U$, we let $X(\mathcal{U}) := \prod_{i \in I} X(U_i)$. Moreover,

$\text{pr}_1, \text{pr}_2 : X(\mathcal{U}) \to X(\mathcal{U} \times \mathcal{U})$

(68)

denote the projections to the two factors. Observe that $U_{ij} := U_i \times_U U_j$ are the “intersections” and that $X(\mathcal{U} \times \mathcal{U}) = \prod_{i,j \in I} X(U_{ij})$.
- A presheaf $X: \mathcal{S}ch_S^{op} \to Ens$ is called a sheaf if (69) agrees with the equalizer of (68), whenever $\{U_i\}_{i \in I}$ is a covering of $U$.

(69)

is injective for all coverings $\{U_i\}_{i \in I}$ of $U$.
- A separated presheaf $X: \mathcal{S}ch_S^{op} \to Ens$ is called a separated presheaf if (69) agrees with the equalizer of (68), whenever $\{U_i\}_{i \in I}$ is a covering of $U$.

Notice that the sheaf property implies the convenient formula $X(\bigcup_{i \in I} U_i) = X(\mathcal{U})$, since $\mathcal{U} = \{U_i\}_{i \in I}$ is a Zariski covering of $\bigcup_{i \in I} U_i$. A homomorphism between sheaves or (separated) presheaves is defined to be a transformation of contravariant functors.

Remark B.2. Let $\{U_i\}_{i \in I}$ be a covering of $U$ and let $X_i$ be a sheaf on $U_i$. By a glueing datum on the family $\{X_i\}_{i \in I}$ one means a family of isomorphisms $\psi_{ij}$ between sheaves $X_j|_{U_{ij}}$ and $X_i|_{U_{ij}}$ on $U_{ij}$ satisfying $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$. The family of restrictions to $U_i$ of a sheaf $X$ on $U$ possesses a tautological glueing datum. One knows that:

(a) The set of homomorphisms $\alpha$ between sheaves $X'$ and $X$ on $U$ can be recovered as the family of homomorphisms $\alpha_i$ between the respective restrictions $X'_i = X'|_{U_i}$ and $X_i = X|_{U_i}$, which preserve their glueing data $\psi_{ij}'$ and $\psi_{ij}$ respectively in the sense that $\alpha_i|_{U_{ij}} \circ \psi_{ij}' = \psi_{ij} \circ \alpha_j|_{U_{ij}}$ holds.

(b) To every glueing datum $\{\psi_{ij}\}_{i,j \in I}$ on $\{X_i\}_{i \in I}$ there is a sheaf $X$ on $U$ together with a family $e_i : X_i \to X|_{U_i}$ satisfying $e_i|_{U_{ij}} \circ \psi_{ij} = e_j|_{U_{ij}}$. The pair $(X, \{e_i\}_{i \in I})$ is unique up to unique isomorphism.

B.2. Torsors. In this subsection we recall some basic definitions and facts on torsors. The main references are [17, Chapitre III] and [43, Chapter III]. So let $G$ be a fpqc sheaf in groups over $S$ and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an fpqc covering of some $S$-scheme $U$. Let us endow the set $G(\mathcal{U} \times \mathcal{U})$ with a right $G(\mathcal{U})$-action via:

$$(g, \tilde{h}) \mapsto \text{pr}_1(\tilde{h})^{-1} g \text{ pr}_2(\tilde{h})$$
whenever \( g \in G(U \times U) \) and \( h \in G(U) \). In the language of appendix A this is just our \((p_{r_1}, p_{r_2})\)-conjugation. Recall that a family \( g := \{g_{ij}\}_{i, j \in I} \in G(U \times U) \) satisfying the cocycle relation

\[
\forall i, j, k : \quad (g_{ij}|_{U_{ij}})(g_{jk}|_{U_{jk}}) = g_{ik}|_{U_{ik}}
\]

is called a 1-cocycle for \( U \), according to [17, Chapitre III, Definition 3.6.1.(i)]. The set of all 1-cocycles for \( U \) forms a \( G(U) \)-invariant subset, which we denote by \( Z^1(U/S, G) \subset G(U \times U) \). Two 1-cocycles \( g = \{g_{ij}\}_{i, j \in I} \) and \( f = \{f_{ij}\}_{i, j \in I} \) for \( U \) are cohomologous if they lie in the same \( G(U) \)-orbit (i.e. such that there exists some \( \tilde{h} = \{h_i\}_{i \in I} \in G(U) \) satisfying

\[
(\tilde{h}_i|_{U_{ij}})g_{ij} = f_{ij}(\tilde{h}_j|_{U_{ij}})
\]

in accordance with [17, Chapitre III, Definition 3.6.1.(ii)]).

The set of cohomology classes of 1-cocycles modulo this equivalence relation is denoted by \( H^1(U/S, G) \). Notice that \( H^1(U/S, G) \) has a distinguished element, namely, the class of trivial 1-cocycles: Here, a cocycle is called trivial if it lies in the \( G(U) \)-orbit of the neutral element (i.e. if it can be written as \( g = \{(e_i^{-1}|_{U_{ij}})(e_j|_{U_{ij}})\}_{i, j \in I} \), where \( \{e_i\}_{i \in I} = \tilde{e} \in G(U) \).

**Definition B.3.** Let \( G \) be an fpqc sheaf of groups over \( S \). A pseudo \( G \)-torsor on an \( S \)-scheme \( U \) is an fpqc sheaf \( X \) on \( U \) on which \( G \) acts freely and transitively from the right, i.e. such that the morphism

\[
X \times G|_U \rightarrow X \times X, \quad (x, g) \mapsto (x, x \cdot g)
\]

is an isomorphism. For any two pseudo \( G \)-torsors \( X \) and \( Y \) over \( U \), we write \( \text{Hom}_G(X, Y) \) for the \( G \)-equivariant morphism between these two fpqc-sheaves on \( U \).

**Remark B.4.** Recall that for every pseudo \( G \)-torsor \( X \) on \( U \) any global section \( s \in X(U) \) gives rise to a canonical isomorphism:

\[
G|_U \xrightarrow{\cong} X, \quad g \mapsto s \cdot g
\]

Therefore we have canonical identifications:

\[
\text{Isom}_{G|_U}(X, G|_U) \cong \text{Isom}_{G|_U}(G|_U, X) \cong \text{Hom}_{G|_U}(G|_U, X) \cong X(U)
\]

We deduce that a pseudo \( G \)-torsor whose set of global sections is not empty must be isomorphic to \( G|_U \) regarded as a pseudo torsor under itself. If this happens the pseudo \( G \)-torsor \( X \) is called (globally) trivial. Also, notice that we have a canonical identification \( \text{Aut}_{G|_U}(G|_U) = G(U) \), but do notice that the isomorphism \( \text{Aut}_{G|_U}(X) = G(U) \) for an arbitrary trivial \( G \)-torsor \( X \) may depend on the choice of \( G|_U \cong X \), i.e. on the base point \( s \in X(U) \).

**Definition B.5.** Let \( G \) be an fpqc sheaf of groups over \( S \). A pseudo \( G \)-torsor \( X \) over an \( S \)-scheme \( U \) is called a \( G \)-torsor if and only if there exists a covering \( U = \{U_i\}_{i \in I} \) of \( U \) such that each \( X|_{U_i} \) is trivial. We denote by \( \text{Tor}_U(G) \) the category of \( G \)-torsors over \( U \) and by \( \text{Tor}_{G|U}(G) \) the full subcategory of \( G \)-torsors that are trivialized by a given covering \( U \). We let \( \text{PHS}_U(G) \) and \( \text{PHS}_{U|U}(G) \) be the respective classes of isomorphism classes of objects in these categories.

At last, we write \( \text{Tor}_{G|U}(G) \) for the category of \( G \)-torsors over \( U \) that are equipped with a specific trivialization over \( U \).
It is folklore that the set of isomorphism classes of torsors are in bijection with the first cohomology with non-abelian coefficients. We would like to make this statement precise, the bijection explicit and present a variant of it. To this end we fix a covering \( \{ U_i \}_{i \in I} = \mathcal{U} \) of \( U \). An element \( \{ g_{ij} \}_{i, j \in I} = g \in \tilde{Z}^1(\mathcal{U}/U, G) \) may be interpreted as a glueing datum on the family \( X_i := G|_{U_i} \) of trivial \( G \)-torsors on \( U_i \).

Whence we obtain a \( G \)-torsor \( X \) over \( U \) together with a family \( \{ x_i \}_{i \in I} = \bar{x} \in X(\mathcal{U}) \). We denote the assignment thus obtained by:

\[
\epsilon : \tilde{Z}^1(\mathcal{U}/U, G) \to \text{Tor}_{\mathcal{U}/U}(G); \ g \mapsto (X, \bar{x})
\]

Notice that we have \( x_i|_{U_{ij}} g_{ij} = x_j|_{U_{ij}} \) in this setting.

**Proposition B.6.** Let \( G \) be an fpqc sheaf of groups on a scheme \( S \) and fix a covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) of an \( S \)-scheme \( U \):

(i) The assignment \( \epsilon \) induces a bijection between the set \( \check{H}^1(\mathcal{U}/U, G) \) and the set of isomorphism classes of the category \( \text{Tor}_{\mathcal{U}/U}(G) \).

(ii) The assignment \( \epsilon \) induces a bijection between \( \check{Z}^1(\mathcal{U}/U, G) \) and the set of isomorphism classes of the (discrete!) category \( \text{Tor}_{\mathcal{U}/U}(G) \).

(iii) The assignment \( \epsilon \) induces an equivalence

\[
\tilde{Z}^1(\mathcal{U}/U, G)/_{\text{pr}_1, \text{pr}_2} G(\mathcal{U}) \cong \text{Tor}_{\mathcal{U}/U}(G)
\]

in the sense that there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{Z}^1(\mathcal{U}/U, G) & \xrightarrow{\epsilon} & \text{Tor}_{\mathcal{U}/U}(G) \\
\downarrow & & \downarrow \\
[\tilde{Z}^1(\mathcal{U}/U, G)/_{\text{pr}_1, \text{pr}_2} G(\mathcal{U})] & \cong & \text{Tor}_{\mathcal{U}/U}(G)
\end{array}
\]

both of whose horizontal arrows are category equivalences.

**Proof.** We could just refer to [17, Chapitre III, Proposition 3.6.3.(i)], but for the convenience of the reader we explain in our own words what is going on. Part (ii) follows from the remarks B.2 and B.4. We continue with the construction of a fully faithful functor from \( \tilde{Z}^1(\mathcal{U}/U, G)/_{\text{pr}_1, \text{pr}_2} G(\mathcal{U}) \) to \( \text{Tor}_{\mathcal{U}/U}(G) \); All we have to do is exhibit a bijection

\[
(71) \quad \{ \bar{h} \in G(\mathcal{U}) \mid \text{pr}_1(\bar{h}) g = \text{f pr}_2(\bar{h}) \} \cong \text{Hom}_{\text{Tor}_{\mathcal{U}/U}(G)}(X, Y)
\]

for any two instances

\[
\tilde{Z}^1(\mathcal{U}/U, G) \ni g = \{ g_{ij} \}_{i, j \in I} \mapsto (X, \bar{x})
\]

\[
\tilde{Z}^1(\mathcal{U}/U, G) \ni f = \{ f_{ij} \}_{i, j \in I} \mapsto (Y, \bar{y})
\]

of our glueing construction \( \epsilon \). Here, \( \bar{x} \in X(\mathcal{U}) \) and \( \bar{y} \in Y(\mathcal{U}) \) are respective trivialisations of \( G \)-torsors \( X \) and \( Y \) over \( U \), as objects of the category \( \text{Tor}_{\mathcal{U}/U}(G) \) are “pointed” \( G \)-torsors over \( U \). These elements satisfy:

\[
x_i|_{U_{ij}} g_{ij} = x_j|_{U_{ij}} \\
y_i|_{U_{ij}} f_{ij} = y_j|_{U_{ij}}
\]

For any \( i \in I \) and \( h \in G(U_i) \) we will now write \( \alpha_i(h) : X_i \to Y_i \) for the unique \( G \)-morphism which sends \( x_i \) to \( y_i h \). Let us see for which families \( \{ h_i \}_{i \in I} = \bar{h} \in G(\mathcal{U}) \)
one obtains a global morphism $\alpha : X \to Y$ from those $\alpha_i(h_i)$'s: The restriction of $\alpha_i(h_i)$ to $U_{ij}$ sends $x_j|_{U_{ij}}$ to:

$$y_i|_{U_{ij}}, h_i|_{U_{ij}} g_{ij}$$

The restriction of $\alpha_j(h_j)$ to $U_{ij}$ sends $x_j|_{U_{ij}}$ to:

$$y_i|_{U_{ij}}, f_{ij} h_j|_{U_{ij}}$$

It follows that $\alpha_i(h_i)$ and $\alpha_j(h_j)$ agree on the overlap $U_{ij}$ if and only if $h_i|_{U_{ij}} g_{ij} = f_{ij} h_j|_{U_{ij}}$, in which case we do obtain our global homomorphism. This yields the bijection (71), and we leave it to the reader to check that it is compatible with compositions. At last (i) is a mere corollary to (iii). $\square$

**Remark B.7.** Let $\mathcal{U}$ be a covering of an $S$-scheme $U$ and let $\mathcal{V}$ be a refinement of $\mathcal{U}$. There are canonical arrows

$$\text{Tor}_{\mathcal{U}/U}(G) \to \text{Tor}_{\mathcal{V}/U}(G)$$

$$\text{PHS}_{\mathcal{U}/U}(G) \to \text{PHS}_{\mathcal{V}/U}(G),$$

which do not depend on the specific factorisations $V_j \to U_i$, which are implicitly present when saying that $\mathcal{V}$ refines $\mathcal{U}$! Taking the direct limit over all coverings, we obtain

$$\lim_{\mathcal{U}} \text{Tor}_{\mathcal{U}/U}(G) \cong \text{Tor}_U(G)$$

$$\lim_{\mathcal{U}} \text{PHS}_{\mathcal{U}/U}(G) \cong \text{PHS}_U(G),$$

and all of these maps are “pointed” in the sense that they preserve the distinguished element (cf. [17, Remarque 3.6.5]). It is somewhat more fiddly to describe the whole groupoid $\text{Tor}_U(G)$ in terms of “non-abelian Cech cohomology 1-classes”. Rather than taking the direct limit we consider the disjoint union over all coverings

$$\hat{Z}^1(U, G) := \bigcup_{\mathcal{U}} \hat{Z}^1(\mathcal{U}/U, G),$$

which is the proper class consisting of pairs $(\mathcal{U}, g)$ where $\mathcal{U}$ is a covering of $U$ and $g \in \hat{Z}^1(\mathcal{U}/U, G)$. Next we define a groupoid structure on $\hat{Z}^1(U, G)$ by decreeing that a homomorphism from $(\mathcal{U}, \{g_{ij}\}_{i, j \in I})$ to $(\mathcal{V}, \{f_{kl}\}_{k, l \in J})$ shall consist of $\hat{h} \in G(\mathcal{U} \times \mathcal{V})$ satisfying

$$(h_{ik}|_{U_{ij} \times U_{kl}})(g_{ij}|_{U_{ij} \times U_{kl}}) = (f_{kl}|_{U_{ij} \times U_{kl}})(h_{jl}|_{U_{ij} \times U_{kl}})$$

This groupoid is equivalent to $\text{Tor}_U(G)$, but it does not fit into the framework of appendix A.

**B.3. Twists and change of group.** We will say that $X$ is a $G$-sheaf if each $X(U)$ is endowed with the structure of a left $G(U)$-set, functorially in $U \in \text{Ob}_{\text{Sch}_S}$. In several settings we need to twist $G$-sheaves with $G$-torsors. The ultimate reference for this is [17, Chapitre III, Proposition 2.3.1], but again we would like to spell out a version that fits the style of our paper:

**Proposition B.8.** Let $G$ be an fpqc sheaf of groups on $S$, let $X$ be a $G$-sheaf and let $P$ be a $G$-torsor. For any $S$-scheme $U$ consider the equivalence relation

$$(p', x') \sim_G (p, x) \iff \exists g \in G(U) : (p', x') = (pg^{-1}, gx)$$
on the set $P(U) \times X(U)$. Then the presheaf

$$P \times X/ \sim_G : \mathcal{G} \rightarrow \operatorname{Ens} : U \mapsto P(U) \times X(U)/ \sim_G$$

possesses a sheafification denoted by $P \ast^\ast G X$.

**Proof.** Let us choose a trivialization of $P$ over some covering $U = \{U_i\}_{i \in I}$ of $S$, and consider the 1-cocycle $\{(g_{ij})\}_{i, j \in I} = g \in \tilde{Z}^1(U/U, G)$ that goes with it. Let $X_i$ be the pull-back of $X$ to $U_i$. The family of maps

$$\psi_{ij} : X_j|_{U_{ij}} \rightarrow X_i|_{U_{ij}} ; x \mapsto g_{ij}x$$

may be interpreted as a glueing datum on the family $\{X_i\}_{i \in I}$ in the sense of remark B.2. This yields a sheaf $P \ast^\ast G X$, which is equipped with a canonical morphism $P \times X \rightarrow P \ast^\ast G X$, as $P$ is obtained by an analogous glueing construction. 

Let $G$ and $H$ be sheaves of groups over $S$ and let $\phi : G \rightarrow H$ be a homomorphism between them. Since $H$ may be regarded as a $G$-sheaf proposition B.8 can be used to construct a functor

$$(72) \quad \text{Tor}_S(G) \rightarrow \text{Tor}_S(H) ; P \mapsto P \ast^\ast G, \phi H$$

It is noteworthy, that the isomorphism class of $(72)$ depends only on the conjugacy class of $\phi$. Indeed if $Q$ is any $H$-torsor and $h$ any global section of $H$, then there is a canonical isomorphism

$$(73) \quad Q \xrightarrow{\cong} Q \ast^\ast H, \text{int}(h) H ; q \mapsto q \ast h$$

as $\text{int}(h) : H \rightarrow H$ is the inner automorphism $g \mapsto h|_U \cdot g \cdot h|_U^{-1}$ for varying $U \in \mathfrak{G} \times \mathcal{G} S$ and $g \in H(U)$.

**B.4. Cartesianism of Tor$_S$.** It turns out useful to have a criterion for the Cartesianism of Tor$_S$:

**Lemma B.9.** Consider fpqc sheaves of groups $H$, $G$ and $G'$ on $S$ together with homomorphisms $\phi : G \rightarrow H$ and $\phi' : G' \rightarrow H$. The following statements are equivalent:

(i) The composition of the group law $\mu_H : H^2 \rightarrow H$ with the outer product

$$\phi \times \phi' : G \times G' \rightarrow H^2$$

induces an epimorphism of fpqc sheaves $G \times G' \rightarrow H$.

(ii) The natural 2-commutative diagram

$${\begin{array}{ccc}
\text{Tor}_S(G \times_{\phi, H, \phi'} G') & \xrightarrow{\text{pr}_2} & \text{Tor}_S(G') \\
\downarrow{\text{pr}_1} & & \downarrow{\phi'} \\
\text{Tor}_S(G) & \xrightarrow{\phi} & \text{Tor}_S(H)
\end{array}}$$

is 2-Cartesian.

In particular (ii) holds if one of $\phi$ or $\phi'$ is already an epimorphism.

**Proof.** Suppose that (i) holds and that we are given torsors $P$ (resp. $P'$) under $G$ (resp. $G'$) together with $\alpha : P' \ast^\ast G, \phi H \xrightarrow{\cong} P \ast^\ast G, \phi H = Q$. We have to construct a $G'' := G \times_{\phi, H, \phi'} G'$-torsor $P''$ with compatible isomorphisms $P'' \ast^\ast G', \text{pr}_1 G \cong P$ and $P'' \ast^\ast G', \text{pr}_2 G' \cong P'$. The sheaf $P'' := P \times_{\phi Q, \alpha \circ \phi'} P'$ is a pseudo-torsor under $G''$. In order to check that $P''$ is a torsor we may assume that all of $P'$, $P$ and $P \ast^\ast G, \phi H = Q$ are trivial. In view of (i) we may also assume that $\alpha$ is the identity, as it does no harm to replace it by $\phi(\gamma) \circ \alpha \circ \phi'(\gamma')$ for any $\gamma \in \operatorname{Aut}_G(P)$ and $\gamma' \in \operatorname{Aut}_{G'}(P')$. 

However $P = G$, $P' = G'$ and $\alpha = \text{id}_G$ gives rise to $P'' = G''$ and we are done. We leave the implication (ii) $\Rightarrow$ (i) to the reader because it is not used in the text. \qed

B.5. **Functoriality properties of** $\mathbf{Tor}_S$. In fact we need a slight generalization of the obvious functoriality (72): Let us write $G_{\text{small}}$ (resp. $H_{\text{small}}$) for the restrictions of fpqc groups $G$ (resp. $H$) on $S$ to the full subcategory $\text{Flat}_S \subset \text{Sch}_S$ consisting of flat $S$-schemes. By a *mini-morphism* from $G$ to $H$ we understand a transformation of functors $G_{\text{small}} \to H_{\text{small}}$. This is a family of homomorphisms $\phi_U : G(U) \to H(U)$ where $U$ runs through the class of flat $S$-schemes, and which is functorial in the sense that the diagrams

$$
\begin{array}{ccc}
G(V) & \xrightarrow{\phi_V} & H(V) \\
\downarrow & & \downarrow \\
G(U) & \xrightarrow{\phi_U} & H(U)
\end{array}
$$

commute for any (not necessarily flat) $S$-morphism between any flat $S$-schemes $V$ and $U$. Let $\mathcal{U}$ be a covering of $S$. Observe that our mini-morphism $\phi$ induces maps $\check{H}^1(\mathcal{U}/S, G) \to \check{H}^1(\mathcal{U}/S, H)$ and $\check{Z}^1(\mathcal{U}/S, G) \to \check{Z}^1(\mathcal{U}/S, H)$, and a functor:

$$
[\check{Z}^1(\mathcal{U}/S, G)/_{\text{pr}_1, \text{pr}_2} G(\mathcal{U})] \to [\check{Z}^1(\mathcal{U}/S, H)/_{\text{pr}_1, \text{pr}_2} H(\mathcal{U})]
$$

Using transport of structure and proposition B.6 we obtain maps $
\begin{align*}
\text{PHS}_{\mathcal{U}/S}(G) & \to \text{PHS}_{\mathcal{U}/S}(H) \\
\text{Tor}_{\mathcal{U}/S}(G) & \to \text{Tor}_{\mathcal{U}/S}(H)
\end{align*}
$

and a functor:

$$
\text{Tor}_{\mathcal{U}/S}(G) \to \text{Tor}_{\mathcal{U}/S}(H)
$$

At last, passage to the limit (cf. remark B.7) yields another map

$$
\text{PHS}_S(G) \to \text{PHS}_S(H)
$$

and yet another functor:

(74)  
$$
\text{Tor}_S(G) \to \text{Tor}_S(H)
$$

By slight abuse of notation we continue to denote its effect on $G$-torsors $P$ by:

$$
P \mapsto P * H, \phi_G
$$

B.6. **Representability of twists.** In the last part of this appendix we discuss representability issues. Whenever $\mathcal{L}$ is a line bundle on a $S$-scheme $X$, we write $\text{Aut}_S(X, \mathcal{L})$ for the set of pairs $(\alpha, m)$ consisting of an $S$-automorphism $\alpha : X \xrightarrow{\sim} X$ and an isomorphism $m : \alpha^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{L}$. This is a group, the group law being: $(\alpha, m)(\beta, n) := (\alpha \circ \beta, n \circ \beta^*(m))$. Furthermore, we let $\text{Aut}_S(X, \mathcal{L})$ be the fpqc sheaf of groups on $S$ whose sections on an $S$-scheme $U$ are given by

$$
\Gamma(U, \text{Aut}_S(X, \mathcal{L})) = \text{Aut}_U(X_U, \mathcal{L}_U),
$$

where $\mathcal{L}_U$ stands for the pull-back of $\mathcal{L}$ along the projection of $X_U = X \times_S U$ to the first factor. Moreover, $\text{Aut}_S(X)$ shall denote the fpqc sheaf of groups given by $U \mapsto \text{Aut}_U(X_U, \mathcal {L}_U)$. \qed
Aut\(_U(X_L)\). It goes without saying that there is a natural morphism Aut\(_S(X, \mathcal{L}) \rightarrow\) Aut\(_S(X)\).

**Proposition B.10.** Consider an \(S\)-scheme \(X\).

(i) Assume that \(X \rightarrow S\) is affine and consider \(G := \text{Aut}_S(X)\). Then, for any \(G\)-torsor \(P\) on \(S\), the twist \(P \ast^G X\) is representable by a relatively affine \(S\)-scheme.

(ii) Assume that \(X\) is of finite type over \(S\), let \(\mathcal{L}\) be a relatively ample line bundle on the \(S\)-scheme \(X\) and consider \(G := \text{Aut}_S(X, \mathcal{L})\). Then, for any \(G\)-torsor \(P\), the twist \(P \ast^G X\) is representable by a quasi-projective \(S\)-scheme.

**Proof.** Part (i) can be proved using faithfully flat descent à la [18, Théorème 2], which owes its effectivity to [18, Théorème 1] together with the mere fact that relatively affine \(S\)-schemes are relative spectra of Zariski sheaves of \(O_S\)-algebras on \(S\). Part (ii) is a straightforward consequence of [3, Chapter 6.1, Theorem 7], see also [13]. □

**Corollary B.11.** Let \(G\) be a relatively affine group scheme over \(S\).

(i) Every \(G\)-torsor is representable by a relatively affine \(S\)-scheme.

(ii) For any two \(G\)-torsors \(P\) and \(Q\), the sheaf

\[
U \mapsto \text{Hom}_{G(U)}(P|_U, Q|_U)
\]

is representable by a relatively affine \(S\)-scheme.

**Proof.** Observe that \(P \ast^G G \cong P\), so that (i) is a special case of part (i) of proposition B.10. Towards (ii) notice that the sheaf in question is isomorphic to \(Q \ast^G P^{opp}\), where \(P^{opp}\) stands for \(P\) equipped with the left \(G\)-action:

\[
G \times P \rightarrow P; \quad (g, x) \mapsto g^{-1}x.
\]

Again we are done by part (i) of proposition B.10. Alternatively one could consult [43, Chapter III, Theorem 4.3]. □

We write \(\text{Hom}_G(P, Q)\) for the sheaf (75).

**Lemma B.12.** Let \(G\) be a flat group scheme over \(S\) and let \(P\) be a representable pseudo \(G\)-torsor over \(S\). Then \(P\) is a \(G\)-torsor if and only if the structural morphism \(P \rightarrow S\) is faithfully flat.

**Proof.** This is because \(P \rightarrow S\) is an fpqc covering in this case while \(P\) becomes trivial over \(P\), cf. [43, Chapter III, Proposition 4.1]. □

**Lemma B.13.** Let \(i : H \hookrightarrow G\) be a closed immersion of smooth affine group schemes over a discrete valuation ring \(\Lambda\). For every \(G\)-torsor \(P\) over some \(\Lambda\)-scheme \(S\), the fpqc-quotient sheaf \(P/H_S\) is representable by a quasi-projective scheme over \(S\).

**Proof.** By Chevalley's trick \(H\) can be recovered as the subgroup stabilizing some line \(L \subset V\), where

\[
\rho : G \rightarrow \text{GL}(V/\Lambda)
\]

is a suitable representation of \(G\) on a free \(\Lambda\)-module \(V\) of finite rank. Let \(\Gamma\) be the stabilizer of the same line in the larger group \(\text{GL}(V/\Lambda)\), so that \(P_{\Lambda}(V)\) represents the quotient of fppf sheaves \(\text{GL}(V/\Lambda)/\Gamma\). Our representation \(\rho\) induces a natural morphism

\[
G/H \rightarrow \text{GL}(V/\Lambda)/\Gamma,
\]
of which its source and target are the quotients in the category of fppf-sheaves. Here observe that [1, Théorème 4.C] (see also [12, Exposé XVII, Appendice III, Théorème 1.3] or [46, Théorème 1, Example ii]) are granting both of these quotients to be representable by separated schemes of finite type over $\Lambda$. Two more remarks on this scenario are in order: First, it turns out that (76) is a radicial morphism, as one can show by inspecting the injectivity of all geometric fibers. In particular it is separated and quasi-finite too. Second, recall that separated and quasi-finite morphisms are quasi-affine, as a consequence of Grothendieck’s generalization of Zariski’s main theorem, cf. [25, Proposition (18.12.6)] resp. [24, Théorème (8.12.6)] (see also [25, Corollaire (18.12.13)]). Let us write $R$ for the $GL_p(V)\{\Lambda_R^{-}\}$-torsor $P_{\hat{G},\rho}$, observe that

$$R \ast^ {GL(V/\Lambda)} \mathbb{P}_\Lambda(V) \cong R/\Gamma_S$$

is (represented by) a familiar scheme, namely $P_S(\mathcal{F})$, where $\mathcal{F}$ is the locally free $\mathcal{O}_S$-module corresponding to the $GL(V/\Lambda)$-torsor $R$ via Hilbert 90. In fact, the whole of (76) allows a twist with the $G$-torsor $P$ as can be seen by means of part (ii) of proposition B.10. In this context it is relevant that there exists a $G$-equivariant ample line bundle $\mathcal{L}$ on $G/H$, namely the pull-back along (76) of the canonical line bundle $\mathcal{O}_{\mathbb{P}_\Lambda(V)}(1)$, which is a $GL(V/\Lambda)$-equivariant ample line bundle on $\mathbb{P}_\Lambda(V)$ (N.B.: pull-back along quasi-affine morphisms preserves ampleness).

**Remark B.14.** In the setting of the above lemma, one can identify the sections of $P/H_S$ over some $S$-scheme $T$ with the set of “$H$-structures” on the $G$-torsor $P \times_S T$, i.e. $H$-torsors $Q$ over $T$ equipped with an isomorphism $Q \ast^H G \cong P \times_S T$.

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