Independent joins of tolerance factorable varieties

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Dedicated to Béla Csákány on his eightieth birthday

Abstract. Let $\text{Lat}$ denote the variety of lattices. In 1982, the second author proved that $\text{Lat}$ is strongly tolerance factorable, that is, the members of $\text{Lat}$ have quotients in $\text{Lat}$ modulo tolerances, although $\text{Lat}$ has proper tolerances. We did not know any other nontrivial example of a strongly tolerance factorable variety. Now we prove that this property is preserved by forming independent joins (also called products) of varieties. This enables us to present infinitely many strongly tolerance factorable varieties with proper tolerances. Extending a recent result of G. Czédli and G. Grätzer, we show that if $\mathcal{V}$ is a strongly tolerance factorable variety, then the tolerances of $\mathcal{V}$ are exactly the homomorphic images of congruences of algebras in $\mathcal{V}$. Our observation that (strong) tolerance factorability is not necessarily preserved when passing from a variety to an equivalent one leads to an open problem.

Basic concepts. Given an algebra $A = (A, F)$, a binary reflexive, symmetric, and compatible relation $T \subseteq A \times A = A^2$ is called a tolerance on $A$. The set of tolerances of $A$ is denoted by $\text{Tol}(A)$. A tolerance which is not a congruence is called proper. By a block of a tolerance $T$ we mean a maximal subset $B$ of $A$ such that $B^2 \subseteq T$. Let $\text{Block}(T)$ denote the set of all blocks of $T$. It follows from Zorn’s lemma that, for $X \subseteq A$, we have that

$$X^2 \subseteq T \iff X \subseteq U \text{ for some } U \in \text{Block}(T).$$

(1)

Applying this observation to $X = \{a, b\}$, we obtain that $\text{Block}(T)$ determines $T$. Furthermore, we also conclude that, for each $n$, each $n$-ary $f \in F$, and all $B_1, \ldots, B_n \in \text{Block}(T)$, there exists a $B \in \text{Block}(T)$ such that

$$\{f(b_1, \ldots, b_n) : b_1 \in B_1, \ldots, b_n \in B_n\} \subseteq B.$$

(2)

We say that $A$ is $T$-factorable if, for each $n$, each $n$-ary $f \in F$ and all $B_1, \ldots, B_n \in \text{Block}(T)$, the block $B$ in (2) is uniquely determined. In this case, we define $f(B_1, \ldots, B_n) := B$, and we call the algebra $(\text{Block}(T), F)$ the quotient algebra $A/T$ of $A$ modulo the tolerance $T$. If $A$ is $T$-factorable for all $T \in \text{Tol}(A)$, then we say that $A$ is tolerance factorable. In what follows, we...
focus on the following properties of varieties; $\mathcal{V}$ denotes a variety of algebras. The tolerances of $\mathcal{V}$ are understood as the tolerances of algebras of $\mathcal{V}$.

(P1) $\mathcal{V}$ is tolerance factorable if all of its members are tolerance factorable.

(P2) $\mathcal{V}$ is strongly tolerance factorable if it is tolerance factorable and, for all $A \in \mathcal{V}$ and all $T \in \text{Tol}(A)$, $A/T \in \mathcal{V}$.

(P3) The tolerances of $\mathcal{V}$ are the images of its congruences if for each $A \in \mathcal{V}$ and every $T \in \text{Tol}(A)$, there exist an algebra $B \in \mathcal{V}$, a congruence $\theta$ of $B$ and a surjective homomorphism $\varphi: B \to A$ such that $T = \{(\varphi(a), \varphi(b)) : (a, b) \in \theta\}$.

(P4) $\mathcal{V}$ has proper tolerances if at least one of its members has a proper tolerance.

Term equivalence, in short, equivalence, of varieties was introduced by W. D. Neumann [9]. (He called it rational equivalence.) Instead of recalling the technical definition, we mention that the variety of Boolean algebras is equivalent to that of Boolean rings. The variety of sets (with no operations) is denoted by $\text{Set}$. Although the present paper is self-contained, for more information on tolerances the reader is referred to the monograph I. Chajda [1].

Motivation and the target. Besides $\text{Lat}$ and $\text{Set}$, no other strongly tolerance factorable variety with proper tolerances has been known since 1982. Our initial goal was to find some other ones. We prove that independent joins, see later, preserve each of the properties (P1)–(P4). This enables us to construct infinitely many, pairwise non-equivalent, strongly tolerance factorable varieties with proper tolerances. Also, we show that if a variety is strongly tolerance factorable, then its tolerances are the images of its congruences, but the converse implication fails. Finally, we show that (strong) tolerance factorability is not always preserved when passing from a variety to an equivalent one, and we raise an open problem based on this fact.

Independent joins. Let $n \in \mathbb{N} = \{1, 2, \ldots\}$, and let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be varieties of the same type. These varieties are called independent if there exists an $n$-ary term $t$ in their common type such that, for $i = 1, \ldots, n$, $\mathcal{V}_i$ satisfies the identity $t(x_1, \ldots, x_n) = x_i$. In this case, the join $\mathcal{V}$ of the varieties $\mathcal{V}_1, \ldots, \mathcal{V}_n$ is called an independent join (in the lattice of all varieties of a given type). This concept was introduced by G. Grätzer, H. Lakser, and J. Plonka [6]. Independent joins of varieties are also called (direct) products.

Proposition 1 (W. Taylor [11], G. Grätzer, H. Lakser, and J. Plonka [6]). Assume that a variety $\mathcal{V}$ is the independent join of its subvarieties $\mathcal{V}_1, \ldots, \mathcal{V}_n$.

1. Every algebra $A \in \mathcal{V}$ is (isomorphic to) a product $A_1 \times \cdots \times A_n$ with $A_i \in \mathcal{V}_i$, $\ldots$, $A_n \in \mathcal{V}_n$. These $A_i$ are uniquely determined up to isomorphism.
(2) If \( B \) is a subalgebra of \( A = A_1 \times \cdots \times A_n \) considered above, then there exist subalgebras \( B_i \) of \( A_i \) (\( i = 1, \ldots , n \)) such that \( B = B_1 \times \cdots \times B_n \).

(3) Every tolerance \( T \) of \( A \) is of the form \( T_1 \times \cdots \times T_n \) such that \( T_i \) is a tolerance of \( A_i \) for \( i = 1 \ldots , n \). If \( T \) is a congruence, then so are the \( T_i \).

Although part (3) above is stated only for congruences in [11], the one-line argument “regard \( T \) as a subalgebra of \( A_2^1 \times \cdots \times A_2^n \) and apply part (2)” of [11] also works if \( T \) is a tolerance rather than a congruence.

Results and examples. The properties (P1)–(P4) are not independent from each other and from congruence permutability. We know from H. Werner [12], see also J. D. H. Smith [10], that a variety is congruence permutable iff it has no proper tolerances. Obviously, a variety without proper tolerances is strongly tolerance factorable and its tolerances are the images of its congruences. Also, we present the following statement, which generalizes the result of G. Czédli and G. Grätzer [5]. (The statements of this section will be proved in the next one.)

Proposition 2.

(1) Assume that \( A \) is a tolerance factorable algebra and \( T \in \text{Tol}(A) \). Then there exist an algebra \( B \) (of the same type as \( A \)), a congruence \( \theta \) of \( B \), and a surjective homomorphism \( \phi: B \to A \) such that \( T = \phi(\theta) \), where \( \phi(\theta) = \{ (\phi(x), \phi(x)) : (x, y) \in \theta \} \).

(2) If a variety is strongly tolerance factorable, then its tolerances are the images of its congruences.

Tolerance factorability does not imply strong tolerance factorability. For example, let \( \mathcal{V} \) be a nontrivial proper subvariety of the variety \( \text{Lat} \) of all lattices. We know from G. Czédli [4] that \( \text{Lat} \) is strongly tolerance factorable; see also G. Grätzer and G. H. Wenzel [7] for an alternative proof. Consequently, \( \mathcal{V} \) is tolerance factorable. However, it is not strongly tolerance factorable by G. Czédli [4] Theorem 3.

Our main achievement is the following statement.

Theorem 3. Assume that a variety \( \mathcal{V} \) is the independent join of its subvarieties \( \mathcal{V}_1, \ldots , \mathcal{V}_n \). Consider one of the properties

1. strong tolerance factorability,
2. tolerance factorability,
3. the tolerances of the variety are the images of its congruences.

If this property holds for all the \( \mathcal{V}_i \), then it also holds for \( \mathcal{V} \).

Now we are ready to give several examples for strongly tolerance factorable varieties with proper tolerances. It would be easy to give such examples by taking varieties equivalent to \( \text{Lat} \). (For example, we could replace the binary join by the \( n \)-ary operation \( f(x_1, \ldots , x_n) := x_1 \lor x_2 \).) Hence we will give pairwise non-equivalent varieties even if Example 4 implies the surprising fact
that strong tolerance factorability is not necessarily preserved when passing from a variety to an equivalent one.

For \( 2 \leq n \in \mathbb{N} \) and \( 1 \leq i \leq n \), let \( S_i^{(n)} \) be the variety consisting of all algebras \( (X, f_n) \) such that \( X \) is a nonempty set and \( f_n \) is an \( n \)-ary operation symbol inducing the \( i \)-th projection on \( X \). That is, \( S_i^{(n)} \) is of type \( \{ f_n \} \), and it is defined by the identity \( f_n(x_1, \ldots, x_n) = x_i \). Let \( S^{(n)} = S_1^{(n)} \vee \cdots \vee S_n^{(n)} \) and \( S^{(1)} = \text{Set} \).

**Example 4.** The varieties \( S^{(n)}, n \in \mathbb{N} \), are strongly tolerance factorable and pairwise non-equivalent, and they have proper tolerances.

Notice that \( S^{(2)} \) is the variety of rectangular bands, which are idempotent semigroups satisfying the identity \( xyx = x \). See A. H. Clifford \[3\], who introduced this concept, and B. Jónsson and C. Tsinakis \[8\].

Next, consider lattices with an additional unary operation \( g_n \) that induces an automorphism of the lattice structure such that the identity \( g_n^m(x) = x \) (where \( g_n^m(x) \) denotes the \( m \)-fold iteration \( g_n(\ldots g_n(x) \ldots) \) of \( g_n \) holds. We can call them rotational lattices of order \( n \). The variety of these lattices is denoted by \( RLat_n \). Note that \( RLat_1 \) is equivalent to \( Lat \) while \( RLat_2 \) consists of lattices with involution, which were studied, for example, in I. Chajda and G. Czédli \[2\]. Note also that \( RLat_n \subseteq RLat_m \) iff \( n \mid m \).

**Example 5.** The varieties \( RLat_n, n \in \mathbb{N} \), are strongly tolerance factorable and pairwise non-equivalent, and they have proper tolerances. Moreover, none of them is equivalent to a variety given in Example 4.

Armed with Theorem 3 one can give some more sophisticated examples. For example, we present the following. Let \( h \) be a binary operation symbol, and let \( m, n \in \mathbb{N} \). We consider the type \( \tau_{mn} = \{ \vee, \wedge, g_m, f_n, h \} \). Define the action of \( f_n \) and \( h \) on the algebras of \( RLat_m \) as first projections. This way these algebras become \( \tau_{mn} \)-algebras and they form a variety \( \tau_{(RLat_m)} \). Similarly, on the members of \( S^{(n)} \), we define \( \vee, \wedge \), and \( g_m \) as first projections and \( h \) as the binary second projection. The algebras we obtain constitute a variety \( (S^{(n)})^m \) of type \( \tau_{mn} \). Let \( C_{mn} = \tau_{(RLat_m)} \vee (S^{(n)})^m \).

**Example 6.** The varieties \( C_{mn}, m, n \in \mathbb{N} \), are strongly tolerance factorable and they have proper tolerances. Furthermore, \( C_{mn} \) is equivalent to \( C_{ij} \) iff \( (i, j) = (m, n) \).

Note that the varieties in Example 5 are congruence distributive while those in Examples 4 and 6 satisfy no nontrivial congruence lattice identity.

Next, in the language of lattices, we consider the ternary lattice terms \( t_\vee(x, y, z) = x \vee (y \wedge z) \) and \( t_\wedge(x, y, z) = x \wedge (y \vee z) \). Clearly, the identities \( x \vee y = t_\vee(x, y, y) \) and \( x \wedge y = t_\wedge(x, y, y) \) hold in all lattices. This motivates the following definition of another variety in the language of \( \{ t_\vee, t_\wedge \} \) as follows. In each of the six usual laws defining \( Lat \), we replace \( \vee \) and \( \wedge \) by \( t_\vee(x, y, y) \) and \( t_\wedge(x, y, y) \). For example, the absorption law \( x = x \vee (x \wedge y) \) turns into
the identity \( x = t_v(x, t_\land(x, y, y), t_\land(x, y, y)) \). The six identities we obtain this way together with the identities \( t_v(x, y, z) = t_v(x, t_\land(y, z, z), t_\land(y, z, z)) \) and \( t_\land(x, y, z) = t_\land(x, t_v(y, z, z), t_v(y, z, z)) \) define a variety, which will be denoted by \( \mathbf{TLat} \).

**Example 7.** \( \mathbf{TLat} \) is equivalent to \( \mathbf{Lat} \). Hence the tolerances of \( \mathbf{TLat} \) are the images of its congruences. However, \( \mathbf{TLat} \) is not tolerance factorable.

Let \( A \in \mathbf{TLat} \) and \( T \in \text{ToI}(A) \). Although \( \mathbf{TLat} \) is not tolerance factorable, the fact that it is equivalent to a tolerance factorable variety (which is \( \mathbf{Lat} \)) yields a natural way of defining \( A/T \). Namely, \( A \in \mathbf{TLat} \) has an alter ego \( A' \in \mathbf{Lat} \) with the same tolerances, so we can take the quotient \( B' := A'/T \) defined in \( \mathbf{Lat} \), and we can let \( A/T \) be the alter ego of \( B' \) in \( \mathbf{TLat} \). Clearly, the strong tolerance factorability of \( \mathbf{Lat} \) implies that \( A/T \in \mathbf{TLat} \).

Since \( \mathbf{TLat} \) is only an “artificial” variety, we raise the following problem.

**Problem 8.** Is there a well-known variety \( \mathbf{V} \) such that although \( \mathbf{V} \) is not tolerance factorable, it is equivalent to some tolerance factorable (possibly "artificial") variety?

**Proofs.**

**Proof of Proposition 3.** We generalize the idea of G. Czédli and G. Grätzer [5]. Assume that \( A = (A, F) \) is a tolerance factorable algebra and \( T \in \text{ToI}(A) \). If \( A \) belongs to a strongly tolerance factorable variety \( \mathbf{V} \), then all the algebras we construct in the proof will clearly belong to \( \mathbf{V} \).

The quotient algebra \( A/T = (\text{Block}(T), F) \), defined according to formula (2), makes sense. So does the direct product \( C = A \times (A/T) \). Denoting \( \{(x, Y) \in A \times \text{Block}(T) : x \in Y \} \) by \( D \), the construction implies that \( D = (D, F) \) is a subalgebra of \( C \). This \( D \) will play the role of \( B \).

Define \( \theta = \{((x_1, Y_1), (x_2, Y_2)) \in D^2 : Y_1 = Y_2 \} \). As the kernel of the second projection from \( D \) to \( A/T \), it is a congruence on \( D \). The first projection \( \varphi : D \to A \), \( (x, Y) \mapsto x \), is a surjective homomorphism since, for every \( x \in A \), (1) allows us to extend \( \{x\} \) to a block of \( T \).

Clearly, if \( ((x_1, Y_1), (x_2, Y_2)) \in \theta \), then \( \{x_1, x_2\} \subseteq Y_1 = Y_2 \in \text{Block}(T) \) implies that \( (\varphi(x_1, Y_1), \varphi(x_2, Y_2)) = (x_1, x_2) \in T \). Conversely, assume that \( (x_1, x_2) \in T \). Then, by (1), there is a \( Y \in \text{Block}(T) \) with \( \{x_1, x_2\} \subseteq Y \). Hence \( (x_1, Y), (x_2, Y) \in D \), \( ((x_1, Y), (x_2, Y)) \in \theta \), and \( x_i = \varphi(x_i, Y) \) yield the desired equality \( T = \{(\varphi(x_1, Y_1), \varphi(x_2, Y_2)) : ((x_1, Y_1), (x_2, Y_2)) \in \theta \} \). \( \square \)

**Lemma 9.** Assume that \( T \) is as in Proposition 3 and \( B \in \text{Block}(T) \). Then there exist \( B_i \in \text{Block}(T_i), i \in \{1, \ldots, n\} \), such that \( B = B_1 \times \cdots \times B_n \), and they are uniquely determined. Furthermore, \( \text{Block}(T) = \text{Block}(T_1) \times \cdots \times \text{Block}(T_n) \).

**Proof.** Let \( \pi_i \) denote the projection map \( A \to A_i \), \( (x_1, \ldots, x_n) \mapsto x_i \). Define \( B_i := \pi_i(B) \). First we show that \( B_1 \in \text{Block}(T_1) \). If \( a_1, b_1 \in B_1 \), then
$(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in B$ for some $a_j, b_j \in A_j$, $2 \leq j \leq n$. Hence $B^2 \subseteq T$ implies that $(a_1, b_1) \in T_1$. This gives that $B_i^2 \subseteq T_i$ for all $i \in \{1, \ldots, n\}$ by symmetric arguments. Thus
\[(B_1 \times \cdots \times B_n)^2 \subseteq T_1 \times \cdots \times T_n = T,\]
which together with $B \subseteq B_1 \times \cdots \times B_n$ implies that
\[B = B_1 \times \cdots \times B_n.\] (3)
The uniqueness of the $B_i$ is trivial. If $B_1 \subseteq C_1 \subseteq A_1$ such that $C_1^2 \subseteq T_1$, then
\[B^2 = (B_1 \times \cdots \times B_n)^2 \subseteq (C_1 \times \cdots \times C_n)^2 \subseteq T_1 \times \cdots \times T_n = T.\]
Hence $B \in \text{Block}(T)$ yields that the first inclusion above is an equality, which implies that $B_1 = C_1$. Thus $B_1 \in \text{Block}(T_1)$ and $B_i \in \text{Block}(T_i)$ for all $i$. This together with $B \subseteq B_1$ proves that $\text{Block}(T) \subseteq \text{Block}(T_1) \times \cdots \times \text{Block}(T_n)$.

Finally, to prove the converse inclusion, assume that $U_i \in \text{Block}(T_i)$ for $i = 1, \ldots, n$, and let $U = U_1 \times \cdots \times U_n$. Clearly, $U^2 \subseteq T_1 \times \cdots \times T_n = T$. By Zorn’s lemma, there is a $B \in \text{Block}(T)$ such that $U \subseteq B$. We already know that $B_i \in \text{Block}(T_i)$ and $B \subseteq B_1$. This together with $U \subseteq B$ yields that $U_i \subseteq B_i$. Comparable blocks of $T_i$ are equal, whence $U_i = B_i$, for all $i$. Hence $U = B \in \text{Block}(T)$, proving that $\text{Block}(T_1) \times \cdots \times \text{Block}(T_n) \subseteq \text{Block}(T)$. □

Proof of Theorem 3 Assume first that the $V_i$ are tolerance factorable. Let $T$ be as in Proposition 1. Assume that $s$ is a $k$-ary term in the language of $V$ and $B_1, \ldots, B_k \in \text{Block}(T)$. By Lemma 4 there are uniquely determined $B_{ij} \in \text{Block}(T_j)$ such that
\[B_i = B_{1i} \times \cdots \times B_{ni} \quad \text{for} \quad i = 1, \ldots, k.\] (4)
Assume that $C$ is in $\text{Block}(T)$ such that
\[\{s(b_1, \ldots, b_k) : b_1 \in B_1, \ldots, b_k \in B_k\} \subseteq C.\] (5)
According to $A = A_1 \times \cdots \times A_n$, we can write $b_i = (b_{i1}, \ldots, b_{in})$. Since $s$ acts componentwise,
\[\{s(b_1, \ldots, b_k) : b_1 \in B_1, \ldots, b_k \in B_k\} = \{s(b_{i1}, \ldots, b_{kn}) : b_{ij} \in B_{ij}\}\] (6)
and $s(b_{i1}, \ldots, b_{kn}) : b_{ij} \in B_{ij} \times \cdots \times \{s(b_{i1}, \ldots, b_{kn}) : b_{in} \in B_{in}\}$. By Lemma 4 $C = C_1 \times \cdots \times C_n$ with $C_j \in \text{Block}(T_j)$. Combining this with $B \subseteq B_1$ and $B \subseteq B_1$, we obtain that, for $j \in \{1, \ldots, n\}$,
\[\{s(b_{ij}, \ldots, b_{kj}) : b_{ij} \in B_{ij} \text{ for } i = 1, \ldots, k\} \subseteq C_j.\] (7)
This implies the uniqueness of $C_j$ since $V_j$ is tolerance factorable. Therefore, $C$ in $B \subseteq B_1$ is uniquely determined, and we obtain that $V$ is tolerance factorable.

Next, assume that the $V_i$ are strongly tolerance factorable. Observe that $B_1$ also yields that $C_j = s(B_{1j}, \ldots, B_{kj})$ in the quotient algebra $A_j/T_j$. This, together with $B \subseteq B_1$ and $C = C_1 \times \cdots \times C_n$, implies that $A/T$ is (isomorphic to)
\[ A_j / T_j \times \cdots \times A_n / T_n. \] Since \( V_j \) is strongly tolerance factorable, we conclude that \( A_j / T_j \in V_j \subseteq V. \) Therefore \( A / T \in V, \) proving that \( V \) is strongly tolerance factorable.

Finally, if the tolerances of \( V_i \) are the images of its congruences, for \( i = 1, \ldots, n, \) then Proposition 1 easily implies the same property of \( V. \)

**Proof of Example 4** Each of the \( S_i^{(n)} \) is equivalent to \( \text{Set}, \) whence it is easy to see that the \( S_i^{(n)} \) are strongly tolerance factorable. The operation \( f_n \) witnesses that \( S(n) = S_1^{(n)} \lor \cdots \lor S_n^{(n)} \) is an independent join. Hence \( S(n) \) is strongly tolerance factorable by Theorem 3. The three-element algebra \( A = \{a, b, c\}, f_n, \) where \( f_n \) acts as the first projection, belongs to \( S_1^{(n)} \subseteq S(n). \) Consider \( T \in \text{Tol}(A) \) determined by \( \text{Block}(T) = \{\{ab\}, \{bc\}\}. \) This \( T \) witnesses that \( S(n) \) has proper tolerances.

Next, consider an arbitrary \( A \in S(n). \) It is of the form \( A = A_1 \times \cdots \times A_n, \) where \( A_i \in S_i^{(n)} \) for \( i = 1, \ldots, n. \) Let \( s \) be an arbitrary term in the language of \( S(n). \) Since \( S_i^{(n)} \) is equivalent to \( \text{Set}, \) \( s \) induces a projection on \( A_i, \) for \( i = 1, \ldots, n. \) It follows that \( s \) induces an operation on \( A \) that depends on at most \( n \) variables. On the other hand, if none of the \( A_i \) is one-element, then \( f_n \) defines a term function on \( A \) that depends exactly on \( n \) variables. Thus \( n \) is the largest integer \( k \) such that all term functions on algebras in \( S_i^{(n)} \) depend on at most \( k \) variables and there exists an algebra in \( S(n) \) with a term function depending exactly on \( k \) variables. This proves that \( S(n) \) and \( S(m) \) are non-equivalent if \( n \neq m. \)

**Proof of Example 5** Let \( A = (A, \lor, \land, g_n) \in \text{RLat}_n \) and \( T \in \text{Tol}(A). \) Then \( T \) is also a tolerance of the lattice reduct \( (A, \lor, \land), \) and \( \text{Block}(T) \) for the lattice reduct is the same as it is for \( A. \) We claim that, for every \( B \in \text{Block}(T), \)

\[
g_n(B) := \{g_n(b) : b \in B\} \in \text{Block}(T). \tag{8}
\]

By Zorn’s lemma, there is a \( C \in \text{Block}(T) \) such that \( \{g_n(b) : b \in B\} \subseteq C. \) Since \( g_n^{-1} = g_n^{-1} \) preserves \( T, \) \( \{g_n^{-1}(c) : c \in C\} \subseteq T. \) This together with \( B \subseteq \{g_n^{-1}(c) : c \in C\} \) and \( B \in \text{Block}(T) \) yields that \( B = \{g_n^{-1}(c) : c \in C\}. \) Therefore, \( g_n(B) = C \in \text{Block}(T), \) proving (8).

For the lattice operations, \( B \) in (2) is uniquely determined since \( \text{Lat} \) is (strongly) tolerance factorable by G. Czédli [4]. By (8), the same holds for \( g_n. \) Thus \( A / T \) makes sense. \( (A / T, \lor, \land) \) is a lattice since \( \text{Lat} \) is strongly tolerance factorable. We conclude from (8) that \( g_n \) is a permutation on \( A / T, \) whose \( n \)-th power is the identity map. Finally, assume that \( B \lor C = D \in A / T; \) the case of the meet is similar. Then, by (8) and \( \{b \lor c : b \in B, c \in C\} \subseteq D, \)

\[
\{x \lor y : x \in g_n(B), y \in g_n(C)\} = \{g_n(b) \lor g_n(c) : b \in B, c \in C\}
\]

\[
= \{g_n(b \lor c) : b \in B, c \in C\} \subseteq \{g_n(d) : d \in D\} = g_n(D).
\]
Hence \( g_n(B) \lor g_n(C) = g_n(D) \), that is, \( g_n \) is an automorphism of \((A/T, \lor, \land)\).

Therefore, \( RLat_n \) is strongly tolerance factorable. It has proper tolerances since so has \( Lat \), which is equivalent to the subvariety \( RLat_1 \) of \( RLat_n \).

The boolean lattice with \( n \) atoms allows an automorphism \( \varphi \) of order \( n \) such that the subgroup generated by \( \varphi \) acts transitively on the set of atoms, but no such automorphism of smaller order is possible. This implies easily that \( RLat_m \) is not equivalent to \( RLat_k \) if \( m \neq k \). Since \( RLat_n \) is congruence distributive, it is not equivalent to \( S^{(m)} \).

**Proof of Example 6.** Since \( h \) takes care of independence, Examples 4 and 5 together with Theorem 3 yield that \( C_{mn} \) is strongly tolerance factorable and it has proper tolerances. Suppose for a contradiction that \((m, n) \neq (u, v)\) but \( C_{mn} \) is equivalent to \( C_{uv} \).

Suppose first that \( m = u \) and \( n \neq v \). Let, say, \( v < n \). Take the \( 2^n \)-element \( A \in (S^{(n)})^m \subseteq C_{mn} \) for all which the \( A_i \) in Proposition 1 are 2-element. Let \( s \) be a binary term in the language of \( C_{mn} \). Since all terms induce projections on \( A_i \), the identity \( s(x, s(y, x)) = x \) holds in \( A_i \) for \( i = 1, \ldots, n \). Therefore, \( A \) satisfies the same identity, for every binary term \( s \). Observe that, up to now, we did not use the assumption on the size of \( A_i \), whence

\[
s(x, s(y, x)) = x \quad \text{holds in} \quad S^{(n)} \quad \text{for all binary terms} \quad s.
\]

By the assumption, there is a \( C_{mv} \)-structure \( B \) on the set \( A \) such that \( B \) and \( A \) have the same term functions. By the definition of \( C_{mv} = C_{uv} \), \( B \) is (isomorphic to) \( C \times D \), where \( C \in \gamma(RLat_m) \) and \( D \in (S^{(v)})^m \). Since \( C \) is a homomorphic image of \( B \) and \( B \) has the same term functions as \( A \), the identity \( s(x, s(y, x)) = x \) holds in \( C \) for all binary terms \( s \). Thus \( C \) is one-element since otherwise \( s(x, y) = x \lor y \) would fail this identity. Hence the term functions of \( B \) are the same as those of its \( S^{(v)} \)-reduct. Now, we can obtain a contradiction the same way as in the last paragraph of the proof of Example 4. \( A \) has an \( n \)-ary term function that depends on all of its variables while all term functions of \( B \) depend on at most \( v \) variables. This proves that \( n = v \).

Secondly, we suppose that \( m \neq u \). Let, say, \( m > u \). Consider the algebra \( A \in \gamma(RLat_m) \subseteq C_{mn} \) such that the \( RLat_m \)-reduct of \( A \) is the \( 2^m \)-element boolean lattice and \( g_m \) is a lattice automorphism of order \( m \) that acts transitively on the set of atoms. (That is, the restriction of \( g_m \) to the set of atoms is a cyclic permutation of order \( m \).) Since \( C_{mn} \) is equivalent to \( C_{mn} = C_{uv} \), there exist algebras \( C \in \gamma(RLat_u) \) and \( D \in (S^{(u)})^m \) such that \( B := C \times D \in C_{uv} \) is equivalent to \( A \). Observe that \( D \), which is a homomorphic image of \( B \), has a lattice reduct. Hence, like in the first part of the proof, \( D \) easily implies that \( D \) is a one-element algebra. Therefore, \( A \) is equivalent to \( C \), that is, to a member of \( \gamma(RLat_u) \). Hence the \( RLat_m \)-reduct of \( A \) is equivalent to a member of \( RLat_u \). This leads to a contradiction the same way as in the last paragraph of the proof of Example 5. \( \Box \)
Proof of Example 7. Consider the lattice $L$ in Figure 1 as an algebra of $\mathbf{TLat}$. A tolerance $T \in \text{Tol}(L)$ is given by its blocks $A = [a_0, a_1], \ldots, E = [e_0, e_1]$. (It is easy to check, and it follows even more easily from G. Czédli [4, Theorem 2], that $T$ is a tolerance.) Since

$$\{t_{\vee}(x, y, z) : x \in A, y \in B, z \in C\} = [c_0, a_1],$$

this set is a subset of two distinct blocks, $A$ and $C$. Hence $\mathbf{TLat}$ is not tolerance factorable. The rest is trivial. □

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