RESTRICTIONS TO THE USE OF TIME-DELAYED FEEDBACK CONTROL IN SYMMETRIC SETTINGS

EDWARD HOOTON, PAVEL KRAVETC AND DMITRII RACHINSKII

Department of Mathematical Sciences
The University of Texas at Dallas
Richardson, TX 75080, USA

(Communicated by Georgi S. Medvedev)

Abstract. We consider the problem of stabilization of unstable periodic solutions to autonomous systems by the non-invasive delayed feedback control known as Pyragas control method. The Odd Number Theorem imposes an important restriction upon the choice of the gain matrix by stating a necessary condition for stabilization. In this paper, the Odd Number Theorem is extended to equivariant systems. We assume that both the uncontrolled and controlled systems respect a group of symmetries. Two types of results are discussed. First, we consider rotationally symmetric systems for which the control stabilizes the whole orbit of relative periodic solutions that form an invariant two-dimensional torus in the phase space. Second, we consider a modification of the Pyragas control method that has been recently proposed for systems with a finite symmetry group. This control acts non-invasively on one selected periodic solution from the orbit and targets to stabilize this particular solution. Variants of the Odd Number Limitation Theorem are proposed for both above types of systems. The results are illustrated with examples that have been previously studied in the literature on Pyragas control including a system of two symmetrically coupled Stewart-Landau oscillators and a system of two coupled lasers.

1. Introduction. Stabilization of unstable periodic solutions is an important problem in applied nonlinear sciences. An elegant method suggested by Pyragas [10] is to introduce delayed feedback with the delay equal, or close, to the period $T$ of the target unstable periodic solution $x^*(t)$ to the uncontrolled system $\dot{x}(t) = f(t, x(t))$. This feedback control is typically linear, and the controlled system has the form

$$\dot{x}(t) = f(t, x(t)) + K(x(t - T) - x(t)), \quad x \in \mathbb{R}^n,$$

where $K$ is an $n \times n$ gain matrix. Since the explicit form of the target cycle is not required this method is easy to implement and widely applicable [11,17,18]. Pyragas control is often referred to as non-invasive, since $x^*(t)$ is an exact solution of both the uncontrolled and controlled systems if the delay exactly equals the period of $x^*$. The question is how to choose the gain matrix $K$ to ensure that $x^*$ is a stable solution of (1).

2010 Mathematics Subject Classification. 34H15, 93B52, 37C80.

Key words and phrases. Stabilization of periodic orbits, Pyragas control, delayed feedback, $S^1$-equivariance, finite symmetry group.

Authors are supported by NSF grant DMS-1413223.
Certain limitations to the method of Pyragas are known. It was proved in [8] that if $f$ depends explicitly on $t$ and the target periodic solution $x^*$ of the uncontrolled non-autonomous system is hyperbolic with an odd number of real Floquet multipliers greater than one, then for any choice of $K$, the function $x^*$ is an unstable solution of (1). In [7], this theorem was modified to deal with the case of autonomous systems

$$\dot{x}(t) = f(x(t)) + K(x(t) - T - x(t)).$$  (2)

In this case, the theorem provides necessary conditions on the control matrix $K$ to allow stabilization of an unstable hyperbolic cycle $x^*$ of the autonomous system

$$\dot{x}(t) = f(x(t)).$$  (3)

Under generic conditions it was proven in [16] that stabilization is possible for some gain matrix $K$, construction of which can be guided by these necessary conditions.

In this paper, we consider system (3), which respects some symmetry, that is $f$ commutes with a given group of matrices $A_g \in \text{GL}(n)$, $g \in G$, where $G$ is an abstract finite group and $A_gA_{g'} = A_{gg'}$. Given a periodic solution $x(t)$ of such a system, $A_gx(t)$ will also be a solution. In the language of group theory we say that solutions come in group theoretic orbits, hence there are typically multiple cycles with the same period. This complicates the applicability of Pyragas control because the control acts non-invasively on all those cycles. In particular, for systems with a continuous group of symmetries, a connected continuum of cycles, which all have the same period, is generic. The cycles that form the continuum are not hyperbolic and hence do not satisfy conditions of theorems from [7,8].

On the other hand, a modification of Pyragas control was proposed in [4] for systems with a finite group of symmetries in order to make the control non-invasively only on one selected target cycle, which has been chosen for stabilization. The symmetry of a cycle $x^*$ of period $T$ is described by a collection of pairs $(A_g, T_g)$ where $g$ belongs to a subgroup $H$ of $G$ and $T_g$ satisfy $T_{g_1} + T_{g_2} = T_{g_1g_2}$ (mod $T$) (hence each $T_g$ is a rational fraction of the period of $x^*$). The symmetry is expressed by the property that

$$A_gx^*(t) = x^*(t + T_g)$$  (4)

for all the pairs $(A_g, T_g)$ with $g \in H$. To stabilize $x^*$, it was suggested in [13] to modify (2) by selecting one particular $g$ and introducing control as follows:

$$\dot{x}(t) = f(x(t)) + K(A_gx(t - T_g) - x(t)).$$  (5)

In [1, 4, 9] this type of control (which can be called equivariant Pyragas control) was applied to stabilize small amplitude cycles born via a Hopf bifurcation, while in [15] analysis of the stability of large amplitude cycles was done by exploiting the additional $S^1$ symmetry of Stewart-Landau oscillators.

In this paper, we extend the odd-number limitation type results considered in [7] to treat the case when control of the form (5) is applied to a system with a finite group of symmetries (Section 2); and, the case when the standard Pyragas control such as in (2) is applied to a target cycle, which is not hyperbolic, because the system is $S^1$-symmetric (Section 3).
2. Modified Pyragas control of systems with finite symmetry group.

2.1. Necessary condition for stabilization. Suppose that system (3) with a smooth \( f \) has a periodic solution \( x^* \) with period \( T \). Assume that this system respects some group of symmetries, that is for each \( g \in G \)

\[ A_g f(x) = f(A_g x) \]

and for one particular \( g \) relation (4) holds. We denote by \( \Phi(t) \) the fundamental matrix of the linearization

\[ \dot{y} = B(t)y, \quad B(t) := Df_x(x^*(t)), \]

of system (3) near \( x^*(t) \), where \( Df_x \) denotes the matrix of partial derivatives of \( f \).

Condition (4) implies that

\[ A^{-1}g \Phi(Tg) \psi(0) = \psi(0), \quad \psi(t) := \dot{x}^*(t), \]

i.e. the matrix \( A^{-1}g \Phi(Tg) \) has an eigenvalue 1. We assume that

\((H_1) \ 1 \ is \ a \ simple \ eigenvalue \ for \ the \ matrix \ A^{-1}g \Phi(Tg).\)

Following [12], we introduce a modified Pyragas control as in (5), where we assume that

\[ KA_g = A_g K. \]

This commutativity property can be a natural restriction on feasible controls. For example, it is typical of laser systems. On the other hand, gain matrices, which are simple enough to allow for efficient analysis of stability of the controlled equation (5), also usually satisfy condition (8) (cf. [9,14]).

Using \((H_1)\), denote by \( \psi_0^\dagger \) the normalized left row eigenvector of the matrix \( A^{-1}g \Phi(Tg) \) with the eigenvalue 1:

\[ \psi_0^\dagger A^{-1}g \Phi(Tg) = \psi_0^\dagger; \quad \psi_0^\dagger \psi(0) = 1. \]

Furthermore, denote

\[ \psi^\dagger(t) = \psi_0^\dagger \Phi^{-1}(t). \]

Finally denote by \( N \) the number of real eigenvalues \( \mu \) of the matrix \( A^{-1}g \Phi(Tg) \), which satisfy \( \mu > 1 \).

**Theorem 2.1.** Assume that conditions \((H_0), (H_1)\) and \((8)\) hold. Let

\[ (-1)^N \left( 1 + \int_0^{T_g} \psi^\dagger(t)K\psi(t) \, dt \right) < 0. \]

Then, \( x^*(t) \) is an unstable periodic solution of the controlled system (5).

Hence, the inequality opposite to (11) is a necessary condition for stabilization of the periodic solution \( x^* \). This necessary condition restricts the choice of the gain matrix \( K \).

Linearizing system (5) near \( x^* \) gives

\[ \dot{y}(t) = B(t)y(t) + K(A_g y(t - T_g) - y(t)). \]

To prove that \( x^* \) is an unstable periodic solution of (5) we will show that system (12) has a solution

\[ y_\mu^*(t) = \mu^{t/T_g} p(t), \quad A_g p(t - T_g) = p(t) \]

with \( \mu > 1 \), where the relation \( A_g p(t - T_g) = p(t) \) ensures that \( p \) is periodic. It is easy to see that if the ordinary differential system

\[ \dot{y} = (B(t) + (\mu^{-1} - 1)K) y \]
has a solution \( y_\mu \) of type (13), then \( y_\mu \) is also a solution of (12). Denote by \( \Psi_\mu(t) \) the fundamental matrix of (14). The proof of Theorem 2.1 is based on the following statement.

**Lemma 2.2.** If for some \( \mu > 1 \) the matrix \( A_g^{-1}\Psi_\mu(T_g) \) has the eigenvalue \( \mu \), then system (14) has a solution of type (13) and hence the periodic solution \( x^* \) of (5) is unstable.

**Proof.** Let us denote

\[
C(t) := B(t) + (\mu^{-1} - 1)K.
\]

Differentiating \((H_0)\) and applying the chain rule gives that \( D_x f(A_g x) = A_g D f_s(x) A_g^{-1} \) for all \( x \in \mathbb{R}^n \). This fact and (4) shows that \( B(t + T_g) = A_g B(t) A_g^{-1} \).

It then follows from \((H_1)\) that

\[
C(t + T_g) = A_g C(t) A_g^{-1}.
\]

To complete the proof denote by \( \nu \) the eigenvector of \( A_g^{-1}\Psi_\mu(T_g) \) with the eigenvalue \( \mu \) and consider the solution \( y_0(t) \) of (14) with \( y_0(0) = \nu \). It is clear that \( y_0(t + T_g) \) satisfies the initial value problem

\[
\begin{aligned}
\dot{u} &= C(t + T_g)u = A_g C(t) A_g^{-1}u, \\
u(0) &= \Psi_\mu(T_g)\nu.
\end{aligned}
\]

(15)

By the change of variables \( v = A_g^{-1}u \), we can see that the solution of (15) is given by

\[
y_0(t + T_g) = A_g \Psi_\mu(t) A_g^{-1} \Psi_\mu(T_g) \nu.
\]

However by assumption \( A_g^{-1}\Psi_\mu(T_g) \nu = \mu \nu \), hence

\[
y_0(t + T_g) = \mu A_g \Psi_\mu(t) \nu = \mu A_g y_0(t),
\]

which proves the lemma.

**Proof of Theorem 2.1.** In order to use Lemma 2.2, we consider the characteristic polynomial

\[
F(\mu) := \det (\mu \text{Id} - A_g^{-1}\Psi_\mu(T_g))
\]

of the matrix \( A_g^{-1}\Psi_\mu(T_g) \). Observe that equation (14) with \( \mu = 1 \) coincides with (6), hence \( \Psi_1 = \Phi \) and therefore condition \((H_1)\) implies \( F(1) = 0 \). We are going to show that relation (11) implies

\[
F(1 + \varepsilon) < 0, \quad 0 < \varepsilon \ll 1.
\]

(16)

Since \( F(\mu) \to +\infty \) as \( \mu \to +\infty \), relation (16) implies that \( F \) has a root \( \mu > 1 \) and therefore the conclusion of Theorem 2.1 follows from (16) by Lemma 2.2.

By applying the variation of parameters formula to the equation \( \dot{y} = B(t)y + h(t) \) with \( h = (\mu^{-1} - 1)K y \) (cf. (14)), one obtains

\[
\Psi_\mu(t) = \Phi(t) \left( \text{Id} + (\mu^{-1} - 1) \int_0^t \Phi^{-1}(s) K \Psi_\mu(s) \, ds \right)
\]

(this identity can also be verified by differentiation). Setting \( \mu = 1 + \varepsilon \) and \( t = T_g \) in this identity and using the fact that \( \Psi_{1+\varepsilon}(T_g) = \Phi(T_g) + O(\varepsilon) \), we obtain the expansion

\[
\Psi_{1+\varepsilon}(T_g) = \Phi(T_g) \left( \text{Id} - \varepsilon Q \right) + O(\varepsilon^2), \quad Q := \int_0^{T_g} \Phi^{-1}(t) K \Phi(t) \, dt.
\]
Therefore,
\[ F(1 + \varepsilon) = \det U_\varepsilon + O(\varepsilon^2), \quad U_\varepsilon = \text{Id} - A_g^{-1}\Phi(T_\varepsilon) + \varepsilon \left( \text{Id} + A_g^{-1}\Phi(T_\varepsilon)Q \right). \quad (17) \]

Let us define an invertable matrix \( L \) with the properties that the matrix \( L^{-1}A_g^{-1}\Phi(T_g)L \) is in the Jordan normal form. \( \psi(0) \) forms the first column of \( L \) and \( \psi_t \) forms the first row of \( L^{-1} \) (cf. (7) and (9)):
\[ Le_1 = \psi(0), \quad e_1^tL^{-1} = \psi_0^t, \quad e_1 := (1, 0, \ldots, 0)^t \in \mathbb{R}^n, \quad (18) \]
where prime denotes the transposition. The matrix \( \text{Id} - L^{-1}A_g^{-1}\Phi(T_g)L \) has the Jordan structure with the diagonal entries 0, 1 - \( \mu_2 \), 1 - \( \mu_3 \), \ldots, 1 - \( \mu_n \), where \( \mu_k \) are the eigenvalues of \( A_g^{-1}\Phi(T_g) \) different from the simple eigenvalue 1. With this notation, conjugating the matrix \( U_\varepsilon \) in (17) with \( L \) gives the matrix \( L^{-1}U_\varepsilon L \) of the following Jordan form:
\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 - \mu_2 & * & \cdots & 0 \\
0 & 0 & 1 - \mu_3 & * & \cdots \\
\vdots & \vdots & \vdots & \ddots & * \\
0 & 0 & 0 & \cdots & 1 - \mu_n
\end{bmatrix}
+ \varepsilon
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \mu_2 & * & \cdots & 0 \\
0 & 0 & \mu_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & * \\
0 & 0 & 0 & \cdots & \mu_n
\end{bmatrix}
\tilde{Q},
\]
where stars stand for 0 or 1 and
\[ \tilde{Q} = \int_0^{T_g} L^{-1}\Phi^{-1}(t)K\Phi(t)L \ dt. \]

It is simple to see from this form that the determinant of (19) satisfies
\[ F(1 + \varepsilon) = \varepsilon(1 + \tilde{Q}_{11}) \prod_{k=2}^{n} (1 - \mu_k) + O(\varepsilon^2) \]
with \( \tilde{Q}_{11} = e_1^t\tilde{Q}e_1 \). It follows from (7), (10) and (18) that
\[ \tilde{Q}_{11} = \int_0^{T_g} \psi^1(t)K\psi(t) \ dt. \]

The fact that \( \prod_{k=2}^{n} (1 - \mu_k) = (-1)^N \), where \( N \) is the number of eigenvalues \( \mu_k \) satisfying \( \mu_k > 1 \), implies that formula (11) indeed implies (16). \( \square \)

2.2. Example. As an illustrative example of Theorem 2.1 we consider the system of two identical diffusely coupled Landau oscillators described in complex form by
\[
\begin{align*}
\dot{z}_1 &= (\alpha + i + \gamma|z_1|^2)z_1 + a(z_2 - z_1), \\
\dot{z}_2 &= (\alpha + i + \gamma|z_2|^2)z_2 + a(z_1 - z_2)
\end{align*}
\quad (20)
\]
with \( z_1, z_2 \in \mathbb{C} \). Here \( \alpha \) and \( a > 0 \) are real parameters while \( \gamma \) is a complex parameter with \( \text{Re} \, \gamma > 0 \). When \( \alpha \) is treated as a bifurcation parameter, this system undergoes two sub-critical Hopf bifurcations, the first at \( \alpha = 0 \) giving rise to a fully synchronized branch of solutions (with \( z_1 = z_2 \)) and the second at \( \alpha = 2a \) giving rise to an anti-phase branch. The fully synchronized branch has been stabilized using the standard non-equivariant Pyragas method (2) in [4]. The anti-phase branch is

\[ \text{1} \]The fully synchronized branch is always present in systems of coupled identical oscillators with Laplacian coupling. The equivariant Pyragas control (5) for such a branch is simply a sum of the non-equivariant Pyragas control such as in (2) and a non-delayed linear control.
defined for $\alpha < 2a$ and is given explicitly by the formula
\[ z_1^*(t) = -z_2^*(t) = r(\alpha)e^{i\omega(\alpha)t}, \quad (21) \]
where $r = r(\alpha) := \sqrt{(2a - \alpha)/\Re \gamma}$ and $\omega = \omega(\alpha) := 1 + r^2(\alpha) \Im \gamma$. In [4] this branch was stabilized by introducing equivariant Pyragas control to system (20) in the following way:
\[
\begin{align*}
\dot{z}_1 &= (\alpha + i + \gamma|z_1|^2)z_1 + a(z_2 - z_1) + b(z_2(t - \pi/\omega) - z_1), \\
\dot{z}_2 &= (\alpha + i + \gamma|z_2|^2)z_2 + a(z_1 - z_2) + b(z_1(t - \pi/\omega) - z_2)
\end{align*}
\quad (22)
\]
with a complex gain parameter $b$. For this branch we can compute the necessary condition (11) analytically and compare this result with the one obtained in [4]. For larger systems of oscillators with larger groups of symmetries the diversity of symmetric branches grows and analytic formulae for inequality (11) are a challenge.

For larger systems of oscillators with larger groups of symmetries the diversity of symmetric branches grows and analytic formulae for inequality (11) are a challenge. The modified (equivariant) Pyragas control scheme has been applied to $S_4$- and $O_4$-symmetric systems near Hopf bifurcation points in [9] and [6], respectively.

To compute inequality (11) we first transform equations (22) to a system of 4 real-valued equations with real variables
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} =
\begin{pmatrix}
\Re(z_1 + z_2) \\
\Im(z_1 + z_2) \\
\Re(z_1 - z_2) \\
\Im(z_1 - z_2)
\end{pmatrix}
\quad (22)
\]

In these coordinates,
\[
T_g = \frac{\pi}{\omega}, \quad A_g = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad K = \begin{bmatrix}
\Re b & -\Im b & 0 & 0 \\
\Im b & \Re b & 0 & 0 \\
0 & 0 & \Re b & -\Im b \\
0 & 0 & \Im b & \Re b
\end{bmatrix}
\]
and the periodic eigenfunction $\psi$, which is the normalized time derivative of the periodic solution (21), is given by
\[
\psi(t) = (0, 0, -\sin(\omega t), \cos(\omega t))^\prime. \quad (23)
\]

Further, in this coordinate system, the linearization (6) has the block-diagonal matrix
\[
B(t) = \begin{pmatrix}
2a \Id + B(t) & 0 \\
0 & B(t)
\end{pmatrix}
\quad (24)
\]

In particular, (23) defines a solution $u = (-\sin(\omega t), \cos(\omega t))^\prime$ of the two-dimensional system $\dot{u} = B(t)u$. Another linearly independent solution $(u_1(t), u_2(t))^\prime$ is easy to obtain by the reduction of order method applied to $\dot{u} = B(t)u$. It defines a solution $\varphi(t) = (0, 0, u_1(t), u_2(t))^\prime$ of (6). Equation (24) implies that
\[
\dot{\psi}(t) = e^{2at}(-\sin(\omega t), \cos(\omega t), 0, 0)^\prime, \quad \dot{\varphi}(t) = e^{2at}(0, 0, u_1(t), u_2(t))^\prime
\]
are also solutions of (6). With 4 linearly independent solutions $\psi, \varphi, \dot{\psi}, \dot{\varphi}$ of (6), one obtains the fundamental matrix $\Phi(t)$ of the linearization of system (20) near the solution (21) and, using (9) and (10), the neutral eigennode of the adjoint system:
\[
\psi^1(t) = \frac{1}{\Re \gamma}(0, 0, -\Im(\gamma e^{i\omega t}), \Re(\gamma e^{i\omega t})).
\]
This gives us all the ingredients for computing (11) and obtaining the following necessary condition for stabilization of any of the anti-phase cycles (21):

\[ 1 + \pi \frac{\text{Re} \gamma \text{Re} b + \text{Im} \gamma \text{Im} b}{\omega \text{Re} \gamma} < 0 \]  

(25)

(note that \( N = 3 \)).

To study stability of the anti-phase branch close to the bifurcation point in system (22), linear stability analysis of the origin combined with explicit knowledge of the branch made it possible to find sufficient conditions under which for some interval of \( \alpha \) sufficiently close to \( 2a \) the branch is stable [4]. Our formula (25) is one of the simultaneous conditions given in [4] (formula (6.10)), which define the stability domain for small periodic orbits at the point \( \alpha = 2a, \omega = 1 \). Note that (25) is necessary for stabilization not only at this point, but for any value of the parameter \( \alpha \). In particular, for a fixed gain parameter \( b \), condition (25) provides an upper bound for the interval of \( \alpha \) where the anti-phase cycle is stable.

3. Pyragas control of systems with \( S^1 \) spatial symmetry.

3.1. Necessary condition for stabilization. Suppose that equation (3) is \( S^1 \)-equivariant:

\[ f(e^{\theta J}x) = e^{\theta J}f(x) \]  

(26)

for all \( \theta \in \mathbb{R}, x \in \mathbb{R}^n \), where the skew-symmetric non-zero matrix \( J \) satisfies \( e^{2\pi J} = \mathbb{I} \). We assume that equation (3) has a non-stationary periodic solution \( x^*(t) \) of a period \( T \), which is not a relative equilibrium (recall that a relative equilibrium is a solution of the form \( x = e^{wtJ}x_0 \)). Hence, equation (3) has an orbit of \( T \)-periodic non-stationary solutions \( e^{\theta J}x^*(t + \tau) \) with arbitrary \( \theta, \tau \). Therefore, the linearization (6) has two linearly independent zero modes (periodic solutions):

\[ \psi_1(t) = \dot{x}^*(t), \quad \psi_2(t) = Jx^*(t) \]  

(27)

with the Floquet multiplier 1. We additionally assume that

\[ (H_2) \quad \text{The eigenvalue 1 of the monodromy matrix } \Phi(T) \text{ of system (6) has multiplicity exactly 2.} \]

Then, there are two periodic eigenfunctions of the adjoint system (solutions of the equation \( \dot{y} = -yB'(t) \)) that can be normalized as follows:

\[ \psi_1^\dagger(t)\psi_1(t) = \psi_2^\dagger(t)\psi_2(t) = 1, \quad \psi_1^\dagger(t)\psi_2(t) = \psi_2^\dagger(t)\psi_1(t) = 0. \]  

(28)

**Theorem 3.1.** Assume that conditions (26) and (H2) hold. Let

\[ (-1)^N(1 + c_{11})(1 + c_{22}) - c_{12}c_{21} < 0, \]  

(29)

where \( N \) is the number of real eigenvalues \( \mu \) of the monodromy matrix \( \Phi(T) \), which satisfy \( \mu > 1 \), and

\[ c_{ij} = \int_0^T \psi_i^\dagger(t)K\psi_j(t)dt. \]  

(30)

Then, \( x^*(t) \) is an unstable periodic solution of the controlled system (1).

**Proof.** Up to the asymptotic expansion (17) the proof of Theorem 3.1 is a modification of the proof of Theorem 2.1 where \( A_g \) is replaced by the identity matrix \( \mathbb{I} \) and \( T_g \) is replaced by \( T \). In this case, the counterpart of relation (17) is given by
describing this system can be written as coupled lasers, see for example \[19\]. In dimensionless form, the rate equations are

\[\begin{align*}
F(1 + \varepsilon) &= \det V_{\varepsilon} + O(\varepsilon^2), \\
V_{\varepsilon} &= \text{Id} - \Phi(T) + \varepsilon \left( \text{Id} + \Phi(T) \int_0^T \Phi^{-1}(t) K \Phi(t) \, dt \right).
\end{align*}\]

(31)

We again denote by \(L\) the similarity transformation to a basis in which the matrix \(L^{-1}\Phi(T)L\) has the Jordan form and agree that \(\psi_1(0)\) and \(\psi_2(0)\) are the first and second columns of \(L\) (cf. (22)), i.e.

\[\begin{align*}
Le_1 &= \psi_1(0), \quad e_1 := (1, 0, \ldots, 0)' \in \mathbb{R}^n, \\
Le_2 &= \psi_2(0), \quad e_2 := (0, 1, \ldots, 0)' \in \mathbb{R}^n.
\end{align*}\]

Hence, the matrix \(\text{Id} - L^{-1}\Phi(T)L\) has the Jordan structure with the diagonal entries \(0, 0, 1 - \mu_3, \ldots, 1 - \mu_n\), where \(\mu_k\) are the eigenvalues of \(\Phi(T)\), which are different from 1. With this notation, the matrix \(L^{-1}V_{\varepsilon}L\) (see (31)) has the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 - \mu_3 & * & \cdots & 0 \\
0 & 0 & 0 & 1 - \mu_4 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & \cdots & 1 - \mu_n
\end{pmatrix} + \varepsilon \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \mu_3 & * & \cdots & 0 \\
0 & 0 & 0 & \mu_3 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & \cdots & \mu_n
\end{pmatrix}
\]

where

\[
\tilde{Q} = \int_0^T L^{-1} \Phi^{-1}(t) K \Phi(t) L \, dt.
\]

Formula (31) implies

\[
F(1 + \varepsilon) = \varepsilon^2 (\tilde{Q}_{11} \tilde{Q}_{22} - \tilde{Q}_{12} \tilde{Q}_{21}) \prod_{k=3}^n (1 - \mu_k) + O(\varepsilon^3),
\]

(32)

where

\[
\tilde{Q}_{ij} = c_i' \tilde{Q} e_j.
\]

(33)

The same argument as in the proof of Theorem 2.1 shows that

\[
\text{sgn} F(1 + \varepsilon) = (-1)^N ((1 + c_{11})(1 + c_{22}) - c_{12}c_{21}),
\]

where \(c_{ij}\) is defined by (30). Combining this with the case of Lemma 2.2 where \(A_g = \text{Id}\) and \(T_g = T\), and the fact that \(F(\mu) \to +\infty\) as \(\mu \to +\infty\) completes the proof. 

\[\square\]

3.2. Example. In order to illustrate Theorem 3.1, we consider a model of two coupled lasers, see for example [19]. In dimensionless form, the rate equations describing this system can be written as

\[
\begin{align*}
\dot{E}_1 &= i \delta E_1 + (1 + i \alpha) N_1 E_1 + \eta e^{-i\varphi} E_2, \\
\dot{N}_1 &= \varepsilon \left[ J - N_1 - (1 + 2N_1) |E_1|^2 \right], \\
\dot{E}_2 &= (1 + i \alpha) N_2 E_2 + \eta e^{-i\varphi} E_1, \\
\dot{N}_2 &= \varepsilon \left[ J - N_2 - (1 + 2N_2) |E_2|^2 \right],
\end{align*}\]

(34)

(35)

(36)

(37)

where the complex-valued variables \(E_1, E_2\) are optical fields and the real-valued variables \(N_1, N_2\) are carrier densities in two laser cavities, respectively. This system is symmetric under the action of the group \(S^1\) of transformations \((E_1, N_1, E_2, N_2) \rightarrow (e^{i\theta} E_1, N_1, e^{i\theta} E_2, N_2)\). Hence, the system admits relative equilibria of the form

\[
(E_1, N_1, E_2, N_2) = (a_1 e^{i\omega t}, n_1, a_2 e^{i\omega t}, n_2)
\]

(38)
with $\omega, n_1, n_2 \in \mathbb{R}$ and $a_1, a_2 \in \mathbb{C}$. The problem of stabilization of unstable relative equilibria for this system was considered in [5]. System (34)–(37) can also have relative periodic orbits, i.e. solutions of the form
\[ (e^{i\omega t} E_1^*(t), N_1^*(t), e^{i\omega t} E_2^*(t), N_2^*(t)), \]
where $E_1^*(t), N_1^*(t), E_2^*(t), N_2^*(t)$ are $T$-periodic. That is in the rotating coordinates they are periodic functions. In the present section we choose a relative periodic solution as a target state for stabilization.

Following the analysis presented in [5], we use the phase $\varphi$ of coupling between the lasers as the bifurcation parameter. Varying $\varphi$ one observes Hopf bifurcations on the branches of relative equilibria. These bifurcations give rise to branches of relative periodic solutions. Figure 1 features the bifurcation diagram for system (34)–(37) with the same parameter set as in [5]. We are interested in the unstable part of the branch of relative periodic solutions born via a subcritical Hopf bifurcation.

In order to stabilize the solution (39), we add the modified Pyragas control term
\[ E_b(t) := b_0 e^{i\beta} \left( e^{-i\omega T} E_1(t - T) - E_1(t) \right) \]
(40) to the right hand side of equation (34). Here the parameters $b_0 > 0$ and $\beta \in \mathbb{R}$ measure the amplitude and phase of the control, respectively; and $T, \omega$ are the parameters of the target relative periodic solution (39). Introducing the rotating coordinates $(\tilde{E}_1, \tilde{N}_1, \tilde{E}_2, \tilde{N}_2) = (e^{-i\omega t} E_1, N_1, e^{-i\omega t} E_2, N_2)$ transforms equations (34)–(37) to an autonomous system that has an orbit of non-stationary $T$-periodic solutions $(\tilde{E}_1, \tilde{N}_1, \tilde{E}_2, \tilde{N}_2) = (e^{i\theta} E_1^*(t + \tau), N_1^*(t + \tau), e^{i\theta} E_2^*(t + \tau), N_2^*(t + \tau))$ with arbitrary $\theta, \tau$. This change of variables transforms relative periodic solutions to just periodic solutions and the control term (40) to the standard Pyragas form
\[ \tilde{E}_b(t) = b_0 e^{i\beta} (\tilde{E}_1(t - T) - \tilde{E}_1(t)). \]

(41)
Hence, system (34)–(37) in the rotating coordinates with the control (41) can be written in the real form (2) with the gain matrix

\[
K = b_0 \begin{bmatrix}
\cos \beta & -\sin \beta & 0 & 0 & 0 & 0 \\
\sin \beta & \cos \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(42)


![Figure 2. Domains of stability of the target relative periodic solution. Parameters correspond to the gray dot in Figure 1. Black region: sufficient condition (29) for instability is satisfied; white region: relative periodic solution is stable; gray region: relative periodic solution is unstable.](image)

Note that relative periodic solutions satisfy condition \((H_2)\), hence we can use Theorem 3.1 to establish the values of \(b_0\) and \(\beta\) for which the control cannot stabilize the solution (39). In order to check condition (29) we compute the monodromy matrix \(\Phi(T)\), the eigenfunctions \(\psi_1(t), \psi_2(t)\) and the eigenfunctions \(\psi_1^\dagger(t), \psi_2^\dagger(t)\) of the adjoint system numerically using routines from DDE-BIFTOOL, a Matlab package for bifurcation analysis. We first find the relative equilibrium state (38) of system (34)–(37) from a system of algebraic equations for a fixed set of parameter values. Then the package uses the parameter continuation technique to trace the relative equilibrium for varied value of the parameter \(\varphi\) (see the ‘eight’-shaped curve in Figure 1) and outputs the Lyapunov exponents of this solution. This allows us to find the Hopf bifurcation point denoted by \(H\) in Figure 1 and use the package to continue the branch of relative periodic solutions starting from \(H\) to the point of interest denoted by the gray dot. The package outputs the numerical relative periodic solution \(x^*\) of the form (39) at this point. This gives us (numerical) eigenfunctions (27). Further, the software solves the linearized system \(\dot{y} = B(t)y\) evaluated on the solution \(x^*\) and outputs the fundamental matrix \(\Phi(t)\) (at all points of the mesh of \(t\)). Then, we find two linearly independent left eigenvectors \(\psi_1^\dagger(0), \psi_2^\dagger(0)\) of \(\Phi(T)\) corresponding to the eigenvalue 1 and normalized to satisfy \(\psi_i^\dagger(0)\psi_j(0) = \delta_{ij}\) (where \(\delta_{ij}\) is Kronecker delta) and obtain the periodic eigenfunctions \(\psi_i(t) = \psi_i^\dagger(0)\Phi^{-1}(t)\) of the adjoint
system. This allows us to evaluate numerically the integrals \( \int_0^T \psi_i(t)^\dagger \psi_j(t) \, dt \) of the components of the eigenfunctions. Finally, plugging these values into (29), (30) allows us to obtain the necessary condition for stabilization in the form of a quadratic inequality with respect to the entries of the control matrix (42).

The results for one particular set of parameters (gray dot in Figure 1) are presented in Figure 2 showing three regions in the parameter space \((\beta, b_0)\) of matrix (42). The black region corresponds to the values of \(b_0\) and \(\beta\) for which condition (29) is satisfied, hence the target state is not stabilizable by control (40). The white and gray regions correspond to stable and unstable target state in the controlled system, respectively. Figure 3 shows the change of the spectrum of the target state after applying Pyragas control (40).

**Figure 3.** Panel (a): Floquet multipliers of the target relative periodic orbit in the uncontrolled system (34)–(37). Panel (b): Floquet multipliers of the same relative periodic orbit in the controlled system with the parameters \(b_0 = 0.3036\) and \(\beta = 6\) of control (40).

4. Discussion and conclusions. We have obtained necessary conditions for stabilization of unstable periodic solutions to symmetric autonomous systems by the Pyragas delayed feedback control. For systems with a finite symmetry group, we used a modification (5) of Pyragas control proposed in [4]. This control is designed to act non-invasively on one particular solution from an orbit of periodic solutions with a specific targeted symmetry. Inequality (11) which makes the stabilization by this control impossible is similar to its counterpart known for the standard Pyragas control of non-symmetric systems such as (2) (see [7]):

\[
(-1)^N \left( 1 + \int_0^T \psi_i(t) K \psi(t) \, dt \right) < 0,
\]

where \(\psi = \dot{x}^*\) is the periodic Floquet mode of the target periodic solution \(x^*\), \(\psi^\dagger\) is the normalized periodic eigenfunction of the adjoint linear system, and \(T\) is the period of \(x^*\). However, the necessary condition for stabilization by the modified (equivariant) Pyragas control (cf. (5)) is typically more restrictive than the necessary condition for the standard Pyragas control (cf. (2)). This can be seen from the example of Section 2.2. The fully synchronized branch of cycles of system (20) \((z_1 = \ldots = z_n = 0)\)
z_2), which bifurcates from zero at \( \alpha = 0 \), can be stabilized by the standard Pyragas control, at least near the Hopf bifurcation point \([4]\). The necessary condition for stabilization (i.e., the inequality opposite to (43)) has the form

\[
1 + 2\pi \frac{\text{Re} \gamma \text{Re} b + \text{Im} \gamma \text{Im} b}{\omega \text{Re} \gamma} < 0.
\]  

(44)

The same control fails to stabilize the anti-phase branch of cycles \( z_1(t) = -z_2(t - T/2) \), which bifurcates from zero at another Hopf point \( \alpha = 2\alpha \). On the other hand, the modified Pyragas control (cf. (22)) with a proper choice of the parameter \( b \) successfully stabilizes the anti-phase branch near the Hopf point \([4]\). Comparing the necessary conditions (25) and (44) for the two controls, we see that (25) is more restrictive because the value of the integral in (43) is twice the value of the integral in (11) since \( T_g = T/2 \) and \( A_g = -\text{Id} \). It is important to note that condition (25) is necessary for stabilizing any cycle of the global anti-phase branch by the modified control. In particular, it is part of the set of sufficient conditions obtained in \([4]\) for stabilizing small cycles.

We have further considered \( S^1 \)-equivariant systems with the usual Pyragas control such as in (2). Due to symmetry, (relative) periodic solutions of such systems come in an \( S^1 \)-orbit and form a two-dimensional torus in the phase space. The control aims to stabilize all the solutions of this torus. The necessary condition for stabilization here, i.e. the inequality opposite to (29), is more subtle than its counterpart for non-symmetric systems because the solutions on the torus have the characteristic multiplier 1 of multiplicity 2 while in the non-symmetric case this multiplier is simple. The control preserves the multiplier 1 with its multiplicity.

It should be noted that any of the inequalities (11), (29) or (43) prevents the stabilization because it implies the existence of a real unstable characteristic multiplier \( \mu > 1 \) (as shown in the above proofs). At the same time, our results do not help to control complex characteristic multipliers. This can be seen from the example of Section 3.2. On the border between the white stability domain and the black instability region (see Figure 2) a real characteristic multiplier passes through the value 1, and its stability is controlled by the sign of the left hand side of inequality (29). On the other hand, on the border between the white domain and the gray instability region, the change of stability is due to a pair of complex characteristic multipliers crossing the unit circle.

Theorems 2.1, 3.1 provide some qualitative information for choosing a class of candidates for the gain matrix \( K \). In particular, relations (11) and (29) show that in the case of an odd \( N \) the Pyragas control with a gain matrix \( K \) of small norm cannot be successful, while for an even \( N \) the periodic solution may be stabilizable by small controls. In this sense, stabilization of a periodic solution is more challenging in the case when the number \( N \) of real characteristic multipliers which are greater than 1 is odd than in the case when \( N \) is even. On the other hand, a control with a too large amplitude is generally not successful in either case because it pushes some characteristic multipliers out of the unit circle. As another example, Theorem 2.1 excludes the “intuitive” control \( K = b\text{Id} \) with a real \( b > 0 \) because it does not satisfy the necessary condition for stabilization (25).

Theorems 2.1, 3.1 further provide quantitative information for the choice of the gain matrix. Notice that verifying the necessary conditions provided by these theorems requires to know (at least approximately) the periodic solution to be stabilized.
Sometimes, this solution can be evaluated numerically using the continuation technique as in Example 3.2. If this information is available, then one can compute (numerically) the periodic eigenfunctions $\psi$ and $\psi^\dagger$ of the uncontrolled system and the integrals $d^{k\ell} = \int_0^T \psi^\dagger_k(t) \psi^\ell(t) \, dt$ of their components. Once this has been done, Theorem 2.1 (resp., Theorem 3.1) provides a linear (resp., quadratic) inequality for the entries of the matrix $K$, which helps restrict the choice of the gain matrix.

Acknowledgments. The authors acknowledge the support of NSF through grant DMS-1413223.

REFERENCES

[1] G. Brown, C. Postlethwaite and M. Silber, Time-delayed feedback control of unstable periodic orbits near a subcritical hopf bifurcation, Physica D: Nonlinear Phenomena, 240 (2011), 859–871.

[2] K. Engelborghs, T. Luzyanina and D. Roose, Numerical bifurcation analysis of delay differential equations using dde-biftool, ACM Trans. Math. Softw., 28 (2002), 1–21, URL http://doi.acm.org/10.1145/513001.513002.

[3] K. Engelborghs, T. Luzyanina and G. Samaey, DDE-BIFTOOL v. 2.00: A Matlab Package for Bifurcation Analysis of Delay Differential Equations. Department of Computer Science, Technical report, KU Leuven, Technical Report TW-330, Leuven, Belgium, 2001.

[4] B. Fiedler, V. Flunkert, P. Hövel and E. Schöll, Delay stabilization of periodic orbits in coupled oscillator systems, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 368 (2010), 319–341.

[5] B. Fiedler, S. Yanchuk, V. Flunkert, P. Hövel, H.-J. Wünsche and E. Schöll, Delay stabilization of rotating waves near fold bifurcation and application to all-optical control of a semiconductor laser, Physical Review E, 77 (2008), 066207, 9pp, URL http://link.aps.org/doi/10.1103/PhysRevE.77.066207.

[6] E. Hooton, Z. Balanov, W. Krawcewicz and D. Rachinskii, Noninvasive Stabilization of Periodic Orbits in $O_4$-Symmetrically Coupled Systems Near a Hopf Bifurcation Point, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 27 (2017), 1750087, 18, URL http://dx.doi.org/10.1142/S0218127417500870.

[7] E. Hooton and A. Amann, Analytical limitation for time-delayed feedback control in autonomous systems, Physical Review Letters, 109 (2012), 154101.

[8] H. Nakajima, On analytical properties of delayed feedback control of chaos, Physics Letters A, 232 (1997), 207–210.

[9] C. Postlethwaite, G. Brown and M. Silber, Feedback control of unstable periodic orbits in equivariant Hopf bifurcation problems, Phil. Trans. R. Soc. A, 371 (2013), 20120467, 20pp.

[10] K. Pyragas, Continuous control of chaos by self-controlling feedback, Physics Letters A, 170 (1992), 421–428.

[11] S. Schikora, P. Hövel, H.-J. Wünsche, E. Schöll and F. Hennelberger, All-optical noninvasive control of unstable steady states in a semiconductor laser, Physical Review Letters, 97 (2006), 213902.

[12] I. Schneider, Delayed feedback control of three diffusively coupled Stuart–Landau oscillators: a case study in equivariant Hopf bifurcation, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 371 (2013), 20120472, 10pp.

[13] I. Schneider, Equivariant Pyragas Control, Master’s thesis, Freie Universität Berlin, 2014.

[14] I. Schneider and M. Bosewitz, Eliminating restrictions of time-delayed feedback control using equivariance, Discrete and Continuous Dynamical Systems Series A, 36 (2016), 451–467.

[15] I. Schneider and B. Fiedler, Symmetry-breaking control of rotating waves, in Control of Self-Organizing Nonlinear Systems, Springer, 2016, 105–126.

[16] J. Sieber, Generic stabilizability for time-delayed feedback control, in Proc. R. Soc. A, vol. 472, The Royal Society, 2016, 20150593.

[17] J. Sieber, A. Gonzalez-Buelga, S. Neild, D. Wagg and B. Krauskopf, Experimental continuation of periodic orbits through a fold, Physical Review Letters, 100 (2008), 244101.

[18] M. Tlidi, A. Vladimirov, D. Pierroux and D. Turaev, Spontaneous motion of cavity solitons induced by a delayed feedback, Physical Review Letters, 103 (2009), 103904.
[19] S. Yanchuk, K. R. Schneider and L. Recke, Dynamics of two mutually coupled semiconductor lasers: Instantaneous coupling limit, Physical Review E, 69 (2004), 056221, URL http://link.aps.org/doi/10.1103/PhysRevE.69.056221.

Received February 2017; revised June 2017.

E-mail address: exh121730@utdallas.edu
E-mail address: pxk142530@utdallas.edu
E-mail address: dmitry.rachinskii@utdallas.edu