Adaptive resonance and pumping a swing

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Abstract

Resonance, and particularly parametric resonance, is often given as an explanation for the increase in amplitude of a child on a swing, or more generally the actuation of pendulums. However the analysis is based on linearised models of the pendulum. We revisit the definition of parametric resonance and show how the pumping phenomenon is a much more complicated process than standard accounts might suggest. We show that if the frequency is not adapted as a function of the period of the oscillations then parametric forcing leads to a modulated oscillation which grows and then decreases periodically. A weakly nonlinear approximation confirms this observation. This shows that the parametric forcing required to maintain or grow oscillations in an actuated pendulum need to adapt to the period of the current oscillations rather than some abstract ‘natural frequency’. These results are extended to other resonant models of pumping a swing.

Keywords: pumping a swing, parametric resonance, averaging methods

(Some figures may appear in colour only in the online journal)

1. Introduction

Broadly speaking there are two mathematically tractable models of the pumping of a child’s swing. In the first, classic, approach, the child is modelled as a rigid body which oscillates periodically. If the frequency of this periodic oscillation is chosen appropriately then it is possible to show that the linearised equations lead to growing oscillations [1, 2]. These ‘appropriate’ frequencies are close to resonant frequencies of the small amplitude oscillations...
and may use parametric resonance [3], in which the periodic forcing is in the coefficients of the inertial or linear terms in the equations, or more standard resonance in which the forcing is external to the phase variables. In the second approach the child assumes different fixed postures relative to the ropes during different phases of the swing, and changes quickly at crucial moments such as the top of the swing, where the angular velocity vanishes. This re-positioning can lead to jumps in the phase variables [4, 5]. There is a further class of models in which the child and swing are modelled as multi-component articulated objects, which can actuate and influence each other during the motion [6]. Such models will not be considered in this paper since they are generally amenable to computer simulation rather than mathematical analysis.

The two classes considered here involve different underlying assumptions about the mechanisms of pumping. In the classic approach the frequency of the child’s oscillations determines the frequency of the swinging motion, whilst in the adaptive approach the child modifies the frequency of her movement to the frequency of the swing, so it is the amplitude-dependent frequency of the swing which determines the frequency at which the child changes position rather than the other way round.

A standard example of parametric resonance is

\[ \dot{\theta} + \omega^2 (1 + \varepsilon \kappa + \varepsilon \cos 2\omega t) \theta = 0. \]  

If \( \varepsilon \) is a small parameter, \( 0 < \varepsilon \ll 1 \) then perturbation techniques show that the amplitude of oscillations increases provided \( \kappa \) is sufficiently small, \( |\kappa| < \frac{1}{2} \) [7, 8]. Note that \( \kappa \) models a small mismatch in the relationship between the frequency of the pendulum and the forcing frequency.

In (1) the forcing frequency \( 2\omega \) is twice the natural frequency of the unperturbed pendulum, and this is precisely the condition suggested in [9] to model the movement of a child on a swing which causes the swing to oscillate. As the child moves his body he lengthens or shortens the effective length of the swing and if he is able to match twice the frequency of his oscillation to the natural frequency of the swing then parametric resonance should provide the mechanism for an increase in amplitude of the swing. Indeed it does, but this is not enough to maintain the larger amplitude oscillation as the example of figure 1 shows. In this example a slight generalisation of (1) is viewed as the linearization of a nonlinear model in which a \( \sin \theta \) term is replaced by \( \theta \). Solutions to the nonlinear equations have modulated oscillations,
i.e. the oscillations do not grow unboundedly as in the linear model, nor do they saturate as might be expected from some forms of nonlinearity.

In the remainder of this paper a number of issues suggested by this observation are parsed. In section 3 a weakly nonlinear theory is given. Section 4 describes an adaptive model of parametric resonance due to Wirkus et al [5] using a standing and squatting strategy on the swing. This strategy is analysed in [4], so the emphasis here is on the fact that pumping a swing requires adaptive modification of the strategy and is not simply a question of finding the ‘correct’ natural frequency of the system. In section 5 the basic features of the classic approach to the sitting strategy of [2] are described, which involves a mixture of standard periodic forcing and parametric forcing. Then in section 6 the adaptive version of this strategy is analysed.

Throughout this paper further student projects are indicated. We consider only the case of rigid ropes, and a more adventurous project would be to determine when the tension in the ropes goes to zero and some element of free fall needs to be introduced into the models. Some comments on this case for the standing and squatting strategy of section 4 can be found in [4].

2. Parametric resonance and swings

Parametric forcing refers to forcing through the parameters of an equation rather than an external additive forcing. The standard description of parametric resonance it is an instability which can occur if there is parametric forcing at a resonant frequency of an oscillator [10, 11]. Of course, this defers the definition to what is meant by a resonant frequency, and this seems to be essentially linear in nature (a ‘natural’ frequency of the system). Thus it is unclear how to interpret this definition for a nonlinear oscillator with no obvious resonant frequencies. In most accounts of parametric resonance the problem is quickly reduced to a linear problem. Whilst the small amplitude approximation is clearly of interest, it does not establish whether the mechanism can explain the fully nonlinear phenomenon, and figure 1 shows that for the simple model (4) the oscillations are modulated. Thus the problem is not just a question of definition, but one of modelling strategy.

One way to pump a swing is to stand and squat. The net effect of this (modelling the swing as a massless string and the child as a point mass) is that the distance from the top of the swing to the child’s centre of mass is a function of time, \( L(t) \). By taking moments about the pivot of the swing (or using the Euler–Lagrange equation) the equation of motion for the angle \( \theta \) of the swing to the vertical is

\[
\frac{d}{dt} (L(t) \dot{\theta}) + gL(t) \sin \theta = 0,
\]

(2)

where \( g \) is the acceleration due to gravity. If the average length of the swing is \( \ell \) and the child stands and squats periodically with frequency \( \Omega \) then a simple model is

\[
L(t) = \ell (1 - \varepsilon \kappa - \varepsilon \cos \Omega t).
\]

(3)

It is worth noting that although this model is used to describe the swing [12], earlier work [13] had already pointed out the drawbacks of this model which we address through the revised adaptive strategies of section 4. For the classic parametric resonance we assume that the child oscillates with approximately twice the natural frequency of the linearised frequency so

\[
\Omega = 2\omega > 0, \quad \omega^2 = g/\ell.
\]

(The inclusion of \( \kappa \) in (3) means that the frequency is not \( \varepsilon \) dependent in the final equation (4) below thus allowing a small mismatch between the fundamental frequencies.) If \( \varepsilon \) is small then substituting (3) into (2) and retaining only the terms of order \( \varepsilon \) gives

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Approximating $\sin \theta$ by $\theta$ gives a slight generalisation of (1), and it is (4) which has the modulated solutions illustrated in figure 1.

### 3. Weakly nonlinear theory

Suppose $|\theta|$ is small, $\theta = \sqrt{\varepsilon}u$ say with $0 < \varepsilon \ll 1$. Then by expanding $\sin \theta \approx \theta - \frac{1}{6}\theta^3 + O(\theta^5)$, (4) becomes

$$\ddot{u} + 4\varepsilon\omega \dot{u} \sin 2\omega t + \omega^2 (1 + \varepsilon \kappa + \varepsilon \cos 2\omega t)\sin \theta = 0. \quad (4)$$

There are a number of different ways this equation could be approached via perturbation theory, and we will choose to use the method of averaging. Details of the method can be found in [7, 14, 15]. We begin by posing the ansatz

$$u = A \cos \omega t + B \sin \omega t, \quad \dot{u} = -\omega A \sin \omega t + \omega B \cos \omega t, \quad (6)$$

where $A$ and $B$ are functions of a slow time $T = \varepsilon t$. Substituting back into (5), ignoring terms of order $\varepsilon^2$ and higher, and integrating over one fast period from 0 to $\frac{2\pi}{\omega}$.

$$\omega \int \dot{A} \sin^2 \omega t \, dt = \varepsilon \omega^2 \int \sin \omega t \left(4 \sin 2\omega t(-A \sin \omega t + B \cos \omega t)\right) dt$$

$$+ \varepsilon \omega^2 \int \sin \omega t (\kappa + \cos 2\omega t)(A \cos \omega t + B \sin \omega t) - \frac{1}{6}(A \cos \omega t + B \sin \omega t)^3 \right) dt,$$

$$-\omega \int \dot{B} \cos^2 \omega t \, dt = \varepsilon \omega^2 \int \cos \omega t \left(4 \sin 2\omega t(-A \sin \omega t + B \cos \omega t)\right) dt$$

$$+ \varepsilon \omega^2 \int \cos \omega t (\kappa + \cos 2\omega t)(A \cos \omega t + B \sin \omega t) - \frac{1}{6}(A \cos \omega t + B \sin \omega t)^3 \right) dt.$$

Noting that $\dot{A} = \varepsilon A_T$ and averaging using table 1 we find the averaged amplitude equations

$$A_T = \omega B \left(\frac{1}{2} + \kappa - \frac{1}{4}(A^2 + B^2)\right)$$

$$B_T = \omega A \left(\frac{1}{2} - \kappa + \frac{1}{4}(A^2 + B^2)\right). \quad (7)$$

Equations (7) have three fixed points: the origin $(0, 0)$ and $(0, \pm B^*)$ where $B^* = 8 \left(\frac{3}{2} + \kappa\right)$
If $|\kappa| < \frac{3}{2}$ then the origin is a saddle, whilst $(0, \pm B^*)$ are centres. Indeed, the system is Hamiltonian: if

$$F(A, B) = -\frac{1}{2}\omega\left(\frac{3}{2} - \kappa\right)A^2 + \frac{1}{2}\omega\left(\frac{3}{2} + \kappa\right)B^2 - \frac{1}{32}\omega(A^2 + B^2)^2$$

then

$$A_T = \frac{\partial F}{\partial B}, \quad B_T = -\frac{\partial F}{\partial A}$$

and so $F(A, B)$ is constant on solutions. Solutions are level sets of $F$ as sketched in figure 2 which shows that solutions with initial conditions close to zero are periodic, but have order one amplitude. Thus we expect a slow modulation of the fast oscillation, as seen in the numerical integration of the full equations (4) in figure 1.

The addition of a small friction term (a potential student project) stabilises the two centres, and so the oscillations saturate. It is tempting to claim that this resolves the question: the addition of friction stabilises a large amplitude oscillation at about twice the natural frequency of the linearised swing, and a similar phenomenon is observed in numerical simulations of (4) with the addition of a dissipative $kq$ term. Unfortunately there are problems with this interpretation. First, the period of the oscillation is fixed at approximately twice the period of the small amplitude oscillations, whilst experience suggests that the period increases with the amplitude of the oscillation. Second, the approach to the equilibrium amplitude is oscillatory for small friction, which is again contrary to observations but which can be explained by assuming that the friction is large. Third, Colin Furze [16] has constructed a swing and pumps using the standing-squatting strategy that is modelled by the regular modulation of the length of the (rigid) swing ropes, and the motion does not saturate; he is able to pump the swing so that it turns over a full 360°. This is possible in (4), but only if $\varepsilon$ is large enough (around $\varepsilon = 0.15$ for the parameters of figure 2, i.e. a 15% variation in the length of the swing due to the standing and squatting strategy). Furze also appears to change the frequency of his actions to match the changing frequency of the oscillations, which again supports the adaptive strategy. However, a

Figure 2. (a) Numerically computed solutions of (7) in the $(A, B)$-plane with the same parameters as in figure 1. There is of course a pair of separatrices (homoclinic orbits) through the saddle point $(0, 0)$. Periodic solutions in $(A, B)$ correspond to slowly varying modulation of the oscillations of the original system. (b) Amplitude $P = \sqrt{A^2 + B^2}$ as a function of the slow time $T$ for initial condition $(0.1, 0)$. This should be compared to the envelope of the oscillations in figure 1.

$$B^* = 2\sqrt{3 + 2\kappa}. \quad (8)$$
more detailed study of the role of friction and the turnover problem for (4), and the dynamics following turnover, would make another interesting student project. Thus either more physics is needed (the models here are admittedly over-simplified), or the strategy employed is different. We will explore the latter possibility in section 4 leading to the idea of adaptive parametric resonance. For the standing-squatting strategy the growth in amplitude with adaptive parametric resonance has been studied in [4], so in sections 5 and 6 the corresponding analysis for a seated swing strategy is presented, extending a model described in [5].

4. Adaptive parametric resonance and pumping

Both the numerical experiments and the weakly nonlinear theory suggest that simple parametric resonance in which the child forces the swing at twice the low amplitude (‘natural’) frequency of the swing can increase the amplitude of the swing, but that this amplitude is not maintained even when the forcing continues [13]. Hence the interpretation of parametric resonance needs to be changed if the phenomenon is to explain how a child pumps a swing to high amplitude and maintains it at that amplitude.

For the classic unforced nonlinear pendulum

\[ \ddot{\theta} + \frac{g}{l} \sin \theta = 0, \]  

(12)

there is really no natural frequency. The period of oscillations depends on the amplitude and is given by elliptic integrals (the solution to (12) can be written in terms of cnoidal functions), and varies from \( 2\pi \sqrt{\frac{l}{g}} \) to infinity as the amplitude increases.

The natural analogue of the classic definition of parametric resonance is for the child to force the swing at close to twice the current frequency of the oscillations. So how can a child implement this strategy? One possibility is that the child (or rather the child’s brain) monitors the period of the most recent oscillation, and adapts his own oscillation accordingly. A second, and in my view more convincing (in the absence of experimental verification), explanation is that there are key parts of the swing which are easy to perceive. The momentary pause with \( \dot{\theta} = 0 \) at the top of each swing, and the moment at which the swing is vertical, with \( \theta = 0 \), are obvious examples. These provide cues from which the child can choose to change her position to pump the swing.

This is precisely the information required to implement the piecewise smooth strategies of [5]. In the standing-squatting strategy the child squats down at the top of the swing, stands up (thus shortening the effective length of the swing) when \( \theta = 0 \) and squats down again at the top of the next swing when \( \dot{\theta} = 0 \), so the child adapts her motion to a frequency which is twice the frequency of the current swinging oscillation. Wirkus et al [5] model this as an instantaneous change in position which leads to jump conditions on some of the variable, but of course smoother versions which take account of the time taken to change position on the swing could also be implemented. Indeed, a smoothed version is used in [9], but in that case the period is fixed and no adaptive strategy is used. In the full nonlinear theory, the standing-squatting strategy with adaptive parametric resonance does lead to models in which the swinger eventually turns over a full 360° as observed [4, 16]. Indeed, over each oscillation the amplitude increases by a constant amount, leading inevitably to turnover.

5. The resonant case for pumping from a sitting position

The standard strategies used by children to pump a swing involve a phase in which the body is stretched out away from the ropes followed by a phase in which the body is aligned with
Thus, in contrast to the standing-squatting strategy described in section 4 the swing is pumped at approximately the frequency of the swing rather than twice this frequency. In the remainder of this paper we will develop a similar set of ideas for this case, basing the calculations around a minor simplification of the sitting model of [2] which lends itself to the generalisation of the sitting model in [5] described in section 6.

Case and Swanson [2] model the child as three masses on a rigid rod. For simplicity we will use two masses \( m_1 \) and \( m_2 \), the former a length \( a_1 \) from the seat of the swing and the latter a length \( a_2 \) from the seat of the swing as shown in figure 3. Let \( \psi \) denote the angle of the child’s body to the ropes of the swing and \( \ell \) denote the length of the swing ropes and suppose \( a_k < \ell, \ k = 1, 2 \). Using elementary geometry the position \((x_k, z_k)\) of the mass \( m_k \) is

\[
x_k = \ell \sin \theta + (-1)^k a_k \sin(\theta + \psi), \quad z = -\ell \cos \theta - (-1)^k a_k \cos(\theta + \psi),
\]

\( k = 1, 2 \), and so the Lagrangian is

\[
L(\theta, \dot{\theta}, \psi, \dot{\psi}) = \frac{1}{2} m_1 (\ell^2 \dot{\theta}^2 + a_1^2 (\dot{\theta} + \dot{\psi})^2 - 2a_1 \ell \dot{\theta} (\dot{\theta} + \dot{\psi}) \cos \psi)
\]

\[
+ \frac{1}{2} m_2 (\ell^2 \dot{\theta}^2 + a_2^2 (\dot{\theta} + \dot{\psi})^2 + 2a_2 \ell \dot{\theta} (\dot{\theta} + \dot{\psi}) \cos \psi)
\]

\[
+ m_1 g (\ell \cos \theta - a_1 \cos(\theta + \psi))
\]

\[
+ m_2 g (\ell \cos \theta + a_2 \cos(\theta + \psi)).
\]

(13)
The Euler–Lagrange equation of motion in $\theta$ is
\[
\frac{\partial L}{\partial \dot{\theta}} = ML^2\ddot{\theta} + A(\dot{\theta} + \psi) - 2a \ell \ddot{\psi} \cos \psi - a \ell \dot{\psi} \cos \psi,
\]
where
\[
M = m_1 + m_2, \quad A = m_1a_1^2 + m_2a_2^2, \quad a = m_1a_1 - m_2a_2,
\]
and assume further that $a > 0$. The equation of motion for the $\theta$ variable is
\[
(ML^2 + A - 2a \ell \cos \psi)\ddot{\theta} = -(A - a \ell \cos \psi)\dot{\psi} - a \ell (2\dot{\theta} + \dot{\psi}) \sin \psi
\]
\[- M\ell \sin \theta + a g \sin(\theta + \psi).
\]

The corresponding equation of motion for $\psi$ depends on the unknown moment exerted by the child to maintain $\psi$ at the required value or change it as appropriate. It would be interesting to investigate the energy expended by the child in more detail using the $\psi$ equation, but the remainder of our dynamic investigation concentrates on (16).

Case and Swanson [2] assume that the pumping is periodic
\[
\psi = \psi_0 \cos \omega t,
\]
for some constants $\psi_0$ and $\omega$. The linearised natural frequency if $|\psi_0|$ is small is
\[
\omega_0^2 = g \frac{ML - a}{ML^2 + A - 2a \ell} = \frac{m_1(\ell - a_1) + m_2(\ell + a_2)}{m_1(\ell - a_1)^2 + m_2(\ell + a_2)^2}.
\]

Case and Swanson [2] describe oscillations dominated by additive resonant forcing at small angles with parametric resonance dominating at larger angles. A further student project would be to examine this claim and the associated scaling regimes of the variables.

The linearised model has unbounded solutions [2]. The behaviour of the full nonlinear problem (16) with $\psi$ given by (17) and $\omega = \omega_0$ is shown in figure 4. It shows the modulated oscillations we have come to expect, but which does not correspond either to the motion generated by a competent swinger or to the linearised behaviour. Rather than pursue the classical perturbation method approach to this problem (another potential student project: assume $a_k \ll 1$) we shall develop an adaptive resonant strategy following [4, 5].

6. Adaptive resonant approach to the sitting strategy

In the sitting strategy, the child’s body is aligned with the ropes of the swing ($\psi = 0$) on the backwards swing with $\dot{\theta} < 0$ and is held at a constant angle $\psi_c$ during the forwards swing. There is a rapid transition between the two positions at the top of the swing when $\dot{\theta} \approx 0$ which will be treated using the methods of [4, 5].

If $\dot{\psi} = 0$ with $\dot{\psi} = \dot{\psi} = 0$ then the equation of motion from (16) is
\[
\dot{\theta} = -\omega_0^2 \sin \theta,
\]
with $\omega_0 > 0$ given by (18). Hence a solution starting at $(x, 0)$ with $0 < x < \pi$ strikes $\dot{\theta} = 0$ again for the first time at $(-x, 0)$ and the time taken if $x$ is small is approximately $\frac{\pi}{\omega_0}$.

Similarly, if $\dot{\psi} = \psi_c$, $0 \leq \psi_c \leq \frac{\pi}{2}$, with $\dot{\psi} = \dot{\psi} = 0$ then the equilibrium solution of (16) is $\theta = \Theta(\psi_c)$ where
\[
\tan \Theta = \frac{a \sin \psi}{ML - a \cos \psi}.
\]
and setting \( \phi = \theta - \Theta(\psi_c) \) the equation of motion (16) is

\[
\dot{\phi} = \omega_c^2 \sin \phi, \quad \omega_c^2 = g \frac{M \ell - a}{M \ell^2 + A - 2a \ell \cos \psi_c}.
\] (21)

Hence a solution starting at \((\theta, \dot{\theta}) = (-x, 0)\) with \(x > 0\) strikes \(\dot{\theta} = 0\) again for the first time at \((2\Theta(\psi_c) + x, 0)\) and the time taken if \(x\) is small is approximately \(\frac{x}{\omega_c}\).

The function \(\Theta\) satisfies the identity

\[
\Theta(\psi) \equiv -\frac{1}{2} \psi + \tan^{-1} \left( \frac{M \ell + a}{M \ell - a} \tan \frac{\psi}{2} \right),
\] (22)

which can be established by taking the tangent function of both sides and using (20) on the left-hand side and the formula for the tangent of a sum on the right-hand side. This will be useful later in this section. Moreover, \(\Theta(\psi) > 0\) if \(\psi > 0\).

Now assume that the re-positioning of the child’s body is almost instantaneous. If \(\dot{\theta} = 0\) and \(\theta < 0\) the child changes her orientation to the swing from \(\psi = 0\) to \(\psi = \psi_c > 0\), whilst if \(\dot{\theta} = 0\) with \(\theta > 0\) the child moves back in line with the swings. Thus if square brackets are used to denote the change in a variable, \([\psi] = \psi_0\) at the change with \(\theta < 0\), and \([\psi] = -\psi_0\) at the change with \(\theta > 0\). These changes can lead to jumps in the \(\theta\) variable which can be found by integrating the full equations of motion over a small time \(\Delta t\) and then letting \(\Delta t \to 0\). For the examples in [4, 5] this integration is relatively simple, but here the asymmetry leads to terms which require rather more work to determine.

Integrating the Euler–Lagrange equation from \(t = 0\) to \(t < \Delta t\), noting that \(\frac{\partial L}{\partial \dot{\theta}}\) is bounded, we obtain

\[
\left[ \frac{\partial L}{\partial \theta} \right]_{t=0}^{t=\Delta t} = O(\Delta t), \quad \text{i.e. from (14)}
\]

\[
(M \ell^2 + A) \dot{\theta} + A \ddot{\psi} = K + O(\Delta t),
\] (23)

where \(K\) is the initial value of the left-hand side of (23). Integrating from 0 to \(\Delta t\) and letting \([x]\) denote the change in value of a variable over this time, \([x] = x(\Delta t) - x(0)\)
\[(M\ell^2 + A)[\dot{\theta}] + A[\psi] - a\ell[\sin \psi] - 2a\ell \int \dot{\theta} \cos \psi \, dt = O(\Delta t). \quad (24)\]

To simplify the remaining integral note that
\[\int \dot{\theta} \cos \psi \, dt = \int \frac{\dot{\psi}}{\cos \psi} \, d\psi\]
and that from (23)
\[\frac{\dot{\psi}}{\psi} = \frac{A - a\ell \cos \psi}{M\ell^2 + A - 2a\ell \cos \psi} + \frac{K}{\psi(M\ell^2 + A - 2a\ell \cos \psi)} + O(\Delta t).\]

Thus \[\int \frac{K \cos \psi + O(\Delta t)}{M\ell^2 + A - 2a\ell \cos \psi} \, dt = \int \frac{A - a\ell \cos \psi}{M\ell^2 + A - 2a\ell \cos \psi} \cos \psi \, d\psi. \quad (25)\]

The first integral on the right-hand side of (25) is \(O(\Delta t)\) as the integrand is bounded. To evaluate the second note that two judicious additions of zero to match the numerator with factors of the denominator establishes that
\[\frac{(R + S \cos \psi) \cos \psi}{U + V \cos \psi} = \frac{S}{V} \cos \psi + \frac{RV - SU}{V^2} - \frac{RV - SU}{V^2} \left(1 + \frac{1}{V} \cos \psi \right).\]

The integrals of the first two terms are easy, whilst if \(U > V\) the third is standard after the substitution \(t = \tan \frac{1}{2} \psi\). Substituting the result back into (24) with \(R = A, S = -a\ell, U = M\ell^2 + A\) and \(V = -2a\ell\) gives the final jump condition \((\Delta t \to 0)\)
\[[\theta] = -\frac{1}{2} [\psi] + \left[ \frac{M\ell^2 - A}{2(M\ell^2 + A)^2 - 4a\ell^2} \tan^{-1} \left( \frac{M\ell^2 + A + 2a\ell}{\sqrt{M\ell^2 + A - 2a\ell}} \tan \frac{\psi}{2} \right) \right]. \quad (26)\]

Equation (22) can be used to rewrite this as
\[[\theta] = [\Theta(\psi)] - [T(\psi)], \quad (27)\]

where
\[T(\psi) = \tan^{-1} \left( \frac{M\ell + a}{M\ell - a} \tan \frac{\psi}{2} \right) - \frac{M\ell^2 - A}{\sqrt{M\ell^2 + A^2 - 4a\ell^2}} \tan^{-1} \left( \frac{M\ell^2 + A + 2a\ell}{\sqrt{M\ell^2 + A - 2a\ell}} \tan \frac{\psi}{2} \right). \quad (28)\]

Note that \(T(0) = 0\) and \(T(\psi) > 0\) if \(0 < \psi < \frac{\pi}{2}\) and \(m_2 = 0\) (recall \(a > 0\)). At the end of the swing to the left during which the child’s body is aligned with the swing and \(\psi = 0\), the child throws her body backwards making an angle \(\psi_s\) to the ropes, so \([\psi] = \psi_s, [\Theta(\psi)] = \Theta(\psi_s)\) and \([T(\psi)] = T(\psi_s)\). Similarly, at the other end of the swing \([\psi] = -\psi\) and the signs of the changes are reversed.

Now suppose that a solution initially crosses \(\dot{\theta} = 0\) at \(\theta = x_0, 0 < x_0 < \pi\). Then evolving under (19) it will strike \(\dot{\theta} = 0\) again at \((-x_0, 0)\) at which point the child moves quickly into the pushed back position making an angle \(\psi_s\) with the ropes, and there is an instantaneous jump given by (27) to \((-x_0 + \Theta(\psi_s) - T(\psi_s), 0)\). If \(-x_0 + \Theta(\psi_s) > \pi - \Theta(\psi_s)\) this will have pushed the swing over a full rotation, otherwise the solution starting at \((-x_0 + \Theta(\psi_s) - T(\psi_s), 0)\) evolves under (21) and, as argued below equation (21), will return to \(\dot{\theta} = 0\) with \(\theta = x_0 - \Theta(\psi_s) + T(\psi_s) + 2\Theta(\psi_s)\). Provided this is less than \(\pi\) it is followed by an immediate re-positioning of the child’s body, re-aligning with the ropes, causing a jump in \(\theta\) of \(-\Theta(\psi_s) + T(\psi_s)\) given by (27) and so the total increase in \(\theta\) is \(2T(\psi_s)\). In other words, an intersection at \((x_n, 0), x_n > 0\), is followed by either by a complete revolution or the next
oscillation (starting in $\theta > 0$ again) will be from $(x_n + 1, 0)$ where

$$x_{n+1} = x_n + 2T(\psi) .$$

(29)

Thus this adaptive resonant forcing strategy inevitably leads to the swing turning over a full revolution unless the strings lose tension, an issue addressed in greater detail in [4].

7. Conclusion

There are many other effects that could improve the modelling, in particular the strings can be kinked when they are pulled on and so they do not remain approximately straight, or the ropes can become slack. However, from the point of view of actuating and maintaining the amplitude of the swing the analysis above suggests that the term parametric resonance needs to be interpreted loosely if it is to be used at all in the context of pumping a swing. In particular resonance is achieved by constantly adapting the strategy to the current frequency of the swing.

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