THE PLANCHEREL FORMULA, THE PLANCHEREL THEOREM, AND THE FOURIER TRANSFORM OF ORBITAL INTEGRALS

REBECCA A. HERB (UNIVERSITY OF MARYLAND) AND PAUL J. SALLY, JR. (UNIVERSITY OF CHICAGO)

Abstract. We discuss various forms of the Plancherel Formula and the Plancherel Theorem on reductive groups over local fields.

Dedicated to Gregg Zuckerman on his 60th birthday

1. Introduction

The classical Plancherel Theorem proved in 1910 by Michel Plancherel can be stated as follows:

**Theorem 1.1.** Let $f \in L^2(\mathbb{R})$ and define $\phi_n : \mathbb{R} \to \mathbb{C}$ for $n \in \mathbb{N}$ by

$$\phi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x)e^{iyx} dx.$$ 

The sequence $\phi_n$ is Cauchy in $L^2(\mathbb{R})$ and we write $\phi = \lim_{n \to \infty} \phi_n$ (in $L^2$). Define $\psi_n : \mathbb{R} \to \mathbb{C}$ for $n \in \mathbb{N}$ by

$$\psi_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \phi(y)e^{-iyx} dy.$$ 

The sequence $\psi_n$ is Cauchy in $L^2(\mathbb{R})$ and we write $\psi = \lim_{n \to \infty} \psi_n$ (in $L^2$). Then, $\psi = f$ almost everywhere, and

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\phi(y)|^2 dy.$$ 

This theorem is true in various forms for any locally compact abelian group. It is often proved by starting with $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, but it is really a theorem about square integrable functions.

There is also a “smooth” version of Fourier analysis on $\mathbb{R}$, motivated by the work of Laurent Schwartz, that leads to the Plancherel Theorem.

**Definition 1.2 (The Schwartz Space).** The Schwartz space, $\mathcal{S}(\mathbb{R})$, is the collection of complex-valued functions $f$ on $\mathbb{R}$ satisfying:

1. $f \in C^\infty(\mathbb{R})$.
2. $f$ and all its derivatives vanish at infinity faster than any polynomial. That is, $\lim_{|x| \to \infty} |x|^k f^{(m)}(x) = 0$ for all $k, m \in \mathbb{N}$.

**Fact 1.3.** The Schwartz space has the following properties:
(1) The space $S(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

(2) The space $S(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$.

(3) The space $S(\mathbb{R})$ is a vector space over $\mathbb{C}$.

(4) The space $S(\mathbb{R})$ is an algebra under both pointwise multiplication and convolution.

(5) The space $S(\mathbb{R})$ is invariant under translation.

For $f \in S(\mathbb{R})$, we define the Fourier transform as usual by

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{iyx} \, dx.$$  

Of course, there are no convergence problems here, and we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(y)e^{-iyx} \, dy.$$  

This leads to the Plancherel Theorem for functions in $S(\mathbb{R})$ by setting $\hat{f}(x) = \overline{f(-x)}$ and considering $f \ast \hat{f}$ at 0. Using the fact that the Fourier transform carries convolution product to function product, we have

$$\|f\|^2 = \left[ f \ast \hat{f} \right](0) = \frac{1}{2\pi} \int_{\mathbb{R}} f \ast \hat{f}(y)dy = \|\hat{f}\|^2.$$  

It is often simpler to work on the space $C^\infty_c(\mathbb{R})$ of complex-valued, compactly supported, infinitely differentiable functions on $\mathbb{R}$. However, nonzero functions in $C^\infty_c(\mathbb{R})$ do not have Fourier transforms in $C^\infty_c(\mathbb{R})$. On the other hand, the Fourier transform is an isometric isomorphism from $S(\mathbb{R})$ to $S(\mathbb{R})$.

The spaces $C^\infty_c(\mathbb{R})$ and $S(\mathbb{R})$ can be turned into topological vector spaces so that the embedding from $C^\infty_c(\mathbb{R})$ into $S(\mathbb{R})$ is continuous. However, the topology on $C^\infty_c(\mathbb{R})$ is not the relative topology from $S(\mathbb{R})$. A continuous linear functional on $C^\infty_c(\mathbb{R})$ is a distribution on $\mathbb{R}$, and this distribution is tempered if it can be extended to a continuous linear functional on $S(\mathbb{R})$ with the appropriate topology. This situation will arise again in our discussion of the Plancherel Formula on reductive groups.

Work on the Plancherel Formula for non-abelian groups began in earnest in the late 1940s. There were two distinct approaches. The first, for separable, locally compact, unimodular groups, was pursued by Mautner [63], Segal [82], and others. The second, for semisimple Lie groups, was followed by Gelfand–Naimark [23], and Harish–Chandra [24], along with others. Segal’s paper [82] and Mautner’s paper [63] led eventually to the following statement (see [21], Theorem 7.44).

**Theorem 1.4.** Let $G$ be a separable, unimodular, type I group, and let $dx$ be a fixed Haar measure on $G$. There exists a positive measure $\mu$ on $\hat{G}$ (determined uniquely up to a constant that depends only on $dx$) such that, for $f \in L^1(G) \cap L^2(G)$, $\pi(f)$ is a Hilbert–Schmidt operator for $\mu$-almost all $\pi \in \hat{G}$, and

$$\int_G |f(x)|^2 \, dx = \int_{\hat{G}} \|\pi(f)\|^2_{HS} \, d\mu(\pi).$$

Here, of course, $\hat{G}$ denotes the set of equivalence classes of irreducible unitary representations of $G$.

At about the same time, Harish-Chandra stated the following theorem in his paper *Plancherel Formula for Complex Semisimple Lie Groups.*
Theorem 1.5. Let $G$ be a connected, complex, semisimple Lie group. Then, for $f \in C^\infty_c(G)$,

$$f(1) = \lim_{H \to 0} \prod_{\alpha \in P} D_\alpha D_{\overline{\alpha}} \left[ e^{\rho(H) + \overline{\rho}(H)} \int_{K \times N} f \left( u \exp(H) nu^{-1} \right) du dn \right].$$

An explanation of the notation here can be found in [24]. We do note two things. First of all, $f$ is taken to be in $C^\infty_c(G)$, and the formula for $f(1)$ is the limit of a differential operator applied to what may be regarded as a Fourier inversion formula for the orbital integral over a conjugacy class of $\exp(H)$ in $G$. It should also be mentioned that not all irreducible unitary representations are contained in the support of the Plancherel measure for complex semisimple Lie groups. In particular, the complementary series are omitted.

In this note, we will trace the evolution of the Plancherel Formula over the past sixty years. For real groups, we observe that the original Plancherel Formula and the Fourier inversion formula ultimately became a decomposition of the Schwartz space into orthogonal components indexed by conjugacy classes of Cartan subgroups. While this distinction might not have been clear for real semisimple Lie groups, it certainly appeared in the development of the Plancherel Theorem for reductive $p$-adic groups by Harish-Chandra in his paper The Plancherel Formula for Reductive $p$-adic Groups in [40]. See also the papers of Waldspurger [93] and Silberger [89], [90]. For $p$-adic groups, the lack of information about irreducible characters and suitable techniques for Fourier inversion has made the derivation of an explicit Plancherel Formula very difficult.

In this paper, the authors have drawn extensively on the perceptive description of Harish-Chandra’s work by R. Howe, V. S. Varadarajan, and N. Wallach (see [39]). The authors would like to thank Jonathan Gleason and Nick Ramsey for their assistance in preparing this paper. We also thank David Vogan for his valuable comments on the first draft.

2. ORBITAL INTEGRALS AND THE PLANCHEREL FORMULA

Let $G$ be a reductive group over a local field. For $\gamma \in G$, let $G_\gamma$ be the centralizer of $\gamma$ in $G$. Assume $G_\gamma$ is unimodular. For $f$ “smooth” on $G$, define

$$\Lambda_\gamma(f) = \int_{G/G_\gamma} f(x \gamma x^{-1}) \, d\dot{x},$$

with $d\dot{x}$ a $G$-invariant measure on $G/G_\gamma$.

Then, $\Lambda_\gamma$ is an invariant distribution on $G$, that is, $\Lambda_\gamma(yf) = \Lambda_\gamma(yf)$ where $yf(x) = f(xy^{-1})$ for $y \in G$. A major problem in harmonic analysis on reductive groups is to find the Fourier transform of the invariant distribution $\Lambda_\gamma$. That is, find a linear functional $\widehat{\Lambda}_\gamma$ such that

$$\Lambda_\gamma(f) = \widehat{\Lambda}_\gamma(\hat{f}),$$

where $\hat{f}$ is a function defined on the space of tempered invariant “eigendistributions” on $G$. This space should include the tempered irreducible characters of $G$ along with other invariant distributions. For example, if $\Pi$ is an admissible representation of $G$ with character $\Theta_\Pi$, then

$$\hat{f}(\Pi) = \text{tr}(\Pi(f)) = \int_G f(x) \Theta_\Pi(x) dx.$$
The nature of the other distributions is an intriguing problem. The hope is that the Plancherel Formula for $G$ can be obtained through some limiting process for $\Lambda_{\gamma}$. For example, if $G = SU(1, 1) \cong SL(2, \mathbb{R})$, we let

$$\gamma = \begin{bmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{bmatrix}, \theta_0 \neq 0, \pi.$$ 

Then, $\gamma$ is a regular element in $G$, and

$$G_\gamma = T = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \bigg| 0 \leq \theta < 2\pi \right\}.$$ 

After a simple computation, we get

$$F_T f(\gamma) = \int_{\mathbb{R}} \pi^{(+,+)}(f) \sinh(\nu(\theta_0 - \pi/2)) d\nu - \int_{\mathbb{R}} \pi^{(-,+)}(f) \cosh(\nu(\theta_0 - \pi/2)) d\nu.$$ 

The parameter $n \neq 0$ indexes the discrete series and the parameter $\nu$ indexes the principal series representations of $G$. The terms $\pi^{(+,+)}(f)$ and $\pi^{(+,-)}(f)$ represent the characters of the irreducible components of the reducible principal series, and we obtain a “singular invariant eigendistribution” on $G$ by subtracting one from the other and dividing by 2. This is exactly the invariant distribution that makes harmonic analysis work. It is called a supertempered distribution by Harish-Chandra.

This leads directly to the Plancherel Formula. By a theorem of Harish-Chandra, it follows that

$$\lim_{\theta \to 0} \left[ \frac{1}{i} \frac{d}{d\theta} \left[ F_T f(\gamma) \right] \right] = 8\pi f(1)$$

$$= \sum_{n \in \mathbb{Z}} \left| n \right| \chi_{\omega(n)}(f) + 1/2 \int_0^\infty \pi^{(+,\nu)}(f) \nu \coth(\pi/2\nu) d\nu$$

$$+ 1/2 \int_0^\infty \pi^{(-,\nu)}(f) \nu \tanh(\pi/2\nu) d\nu.$$ 

The representations of $SL(2, \mathbb{R})$ were first determined by Bargmann [8]. In his 1952 paper [25], Harish-Chandra gave hints to the entire picture for Fourier analysis on real groups. He constructed the unitary representations, computed their characters, found the Fourier transform of orbital integrals, and deduced the Plancherel Formula. This was done in about four and one-half pages.

We mention again that the support of the Fourier transform of the tempered invariant distribution $\Lambda_{\gamma}$ contains not only the characters of the principal series and the discrete series, but also the tempered invariant distribution

$$\frac{1}{2} \left( \pi^{(+,+)} - \pi^{(+,-)} \right).$$ 

This singular invariant eigendistribution (appropriately normalized) is equal to 1 on the elliptic set and 0 off the elliptic set, thereby having no effect on harmonic analysis of the principal series.
Through the 1950s, along with an intensive study of harmonic analysis on semisimple Lie groups, Harish-Chandra analyzed invariant distributions, their Fourier transforms, and limit formulas related to these. This was mainly with reference to distributions on $C_c^\infty(G)$. He showed that $G$ has discrete series if and only if $G$ has a compact Cartan subgroup. For the rest of this section, we will assume that $G$ has discrete series. He also suspected quite early that the irreducible unitary representations that occurred in the Plancherel Formula would be indexed by a series of representations parameterized by characters of conjugacy classes of Cartan subgroups.

In the 1960s, Harish-Chandra proved deep results about the character theory of semisimple Lie groups, in particular, the discrete series characters. In developing the Fourier analysis on a semisimple Lie group, he had to work with the smooth matrix coefficients of the discrete series. These matrix coefficients vanish rapidly at infinity, but are not compactly supported. This led to the definition of the Schwartz space $C(G)$ [27]. The Schwartz space was designed to include matrix coefficients of the discrete series and slightly more. The Schwartz space is dense in $L^2(G)$, but is not contained in $L^1(G)$. Moreover, the Schwartz space $C(G)$ does not contain the smooth matrix coefficients of parabolically induced representations. Nonetheless, the matrix coefficients of these parabolically induced representations are tempered distributions, that is, if $m$ is such a matrix coefficient and $f \in C(G)$, then $\int_G fm$ converges. Hence, one can consider the orthogonal complement of these matrix coefficients in $C(G)$.

The collection of parabolically induced representations is indexed by non-compact Cartan subgroups of $G$. If $H$ is a Cartan subgroup of $G$ with split component $A$, then the centralizer $L$ of $A$ is a Levi subgroup of $G$. Now the representations corresponding to $H$ are induced from parabolic subgroups with Levi component $L$, and the subspace $C_H(G)$ is generated by so called wave packets associated to these induced representations. Thus, we have an orthogonal decomposition

$$C(G) = \bigoplus C_H(G),$$

where $H$ runs over conjugacy classes of Cartan subgroups. When $H$ is the compact Cartan subgroup of $G$, $C_H(G)$ is the space of cusp forms in $C(G)$. This decomposition of the Schwartz space is a version of the Plancherel Theorem for $G$, and it is in this form that the Plancherel Theorem appears for reductive $p$-adic groups.

As he approached his final version of the Plancherel Theorem and Formula for real semisimple Lie groups, Harish-Chandra presented a development of the Plancherel Formula for functions in $C_c^\infty(G)$ in his paper “Two Theorems on Semisimple Lie Groups” [28]. Here, he shows exactly how irreducible tempered characters decompose the $\delta$ distribution. In particular, for $G$ of real rank 1, he gives an explicit formula for the Fourier transform of an elliptic orbital integral, and derives the Plancherel Formula from this. To understand the Plancherel Theorem for real groups in complete detail, one should consult the three papers [34], [35], [36], and the expository renditions of this material [30], [31], [32].

3. The Fourier Transform of Orbital Integrals, the Plancherel Formula, and Supertempered Distributions

In a paper in Acta Mathematica in 1973 [79], Sally and Warner re-derived, by somewhat different methods, the inversion formula that Harish-Chandra proved in his “Two Theorems” paper [28]. The purpose of the Sally–Warner paper was to
explore the support of the Fourier transform of an elliptic orbital integral. To quote: “In this paper, we give explicit formulas for the Fourier transform of \( \Lambda_y \), that is, we determine a linear functional \( \widehat{\Lambda}_y \) such that

\[
\Lambda_y(f) = \widehat{\Lambda}_y(\hat{f}), \quad f \in C_c^\infty(G).
\]

Here, we regard \( \hat{f} \) as being defined on the space of tempered invariant eigendistributions on \( G \). This space contains the characters of the principal series and the discrete series for \( G \) along with some ‘singular’ invariant eigendistributions whose character-theoretic nature has not yet been completely determined.”

In fact, the character theoretic nature of these singular invariant eigendistributions was determined in a paper \cite{42} by Herb and Sally in 1977. In this paper, the present authors used results of Hirai \cite{53}, Knapp–Zuckerman \cite{58}, Schmid \cite{81}, and Zuckerman \cite{96} to show that, as in the case of \( SU(1,1) \), these distributions are alternating sums of characters of limits of discrete series representations which can be embedded as the irreducible components of certain reducible principal series. In his final published paper \cite{38}, Harish-Chandra developed a comprehensive version of these singular invariant eigendistributions, and he called them “supertempered distributions.” These supertempered distributions include the characters of discrete series along with some finite linear combinations of irreducible tempered elliptic characters that arise from components of reducible generalized principal series. This situation has already been illustrated for \( SL(2, \mathbb{R}) \) in Section 2 of this paper. One notable fact about supertempered distributions is that they appear discretely in the Fourier transforms of elliptic orbital integrals; hence they play an essential role in the study of invariant harmonic analysis. For the remainder of this section, we present a collection of results of the first author related to Fourier inversion and the Plancherel Theorem for real groups.

In order to explain the steps needed to derive the Fourier transform for orbital integrals in general, we first look in more detail at the case that \( G \) has real rank one. In this case \( G \) has at most two non-conjugate Cartan subgroups: a non-compact Cartan subgroup \( H \) with vector part of dimension one, and possibly a compact Cartan subgroup \( T \). We assume for simplicity that \( G \) is acceptable, that is, the half-sum of positive roots (denoted \( \rho \)) exponentiates to give a well defined character on \( T \). The characters \( \Theta^T_\chi \) of the discrete series representations are indexed by \( \tau \in \hat{T}' \), the set of regular characters of \( T \), and the characters \( \Theta^H_\chi \) of the principal series are indexed by characters \( \chi \in \hat{H} \). In addition, for \( f \in C_c^\infty(G) \) we have invariant integrals \( F^T_f(t), t \in T \), and \( F^H_f(a), a \in H \). These are normalized versions of the orbital integrals \( \Lambda_\gamma(f), \gamma \in G \), which have better properties as functions on the Cartan subgroups.

The analysis on the non-compact Cartan subgroup is elementary. First, as functions on \( G' \), the set of regular elements of \( G \), the principal series characters are supported on conjugates of \( H \). In addition, for \( \chi \in \hat{H}, a \in H' = H \cap G' \), \( \Theta^H_\chi(a) \) is given by a simple formula in terms of \( \chi(a) \). As a result it is easy to show that the abelian Fourier transform \( F^H_f(\chi), \chi \in \hat{H} \), is equal up to a constant to \( \Theta^H_\chi(f) \), the principal series character evaluated at \( f \). Finally, \( F^H_f \in C_c^\infty(H) \), and so the abelian Fourier inversion formula on \( A \) yields an expansion

\[
F^H_f(a) = c_H \int_H e(\chi(\overline{\chi(a)}) \Theta^H_\chi(f)) d\chi, a \in H,
\]
where \( c_H \) is a constant depending on normalizations of measures and \( \epsilon(\chi) = \pm 1 \).

The situation on the compact Cartan subgroup is more complicated. There are three main differences. First, for \( \tau \in \hat{T}' \), \( t \in T' = T \cap G' \), \( \Theta_T^\tau(t) \) is given by a simple formula in terms of the character \( \tau(t) \). However, \( \Theta_T^\tau \) is also non-zero on \( H' \). Thus for \( \tau \in \hat{T}' \), \( f \in C_c^\infty(G) \), the abelian Fourier coefficient \( \hat{F}_f^\tau(\tau) \) is equal up to a constant to \( \Theta_T^\tau(f) \) plus an error term which is an integral over \( H \) of \( F_f^\tau \) times the numerator of \( \Theta_T^\tau \). Second, the singular characters \( \tau_0 \in \hat{T} \) do not correspond to discrete series characters. They do however parameterize singular invariant eigendistributions \( \Theta_T^{\tau_0} \), and \( \hat{F}_f^{\tau_0}(\tau_0) \) can be given in terms of \( \Theta_T^{\tau_0}(f) \). Finally, \( F_f^{\tau_0} \) is smooth on \( T' \), but has jump discontinuities at singular elements. Because of this there are convergence issues when the abelian Fourier inversion formula is used to expand \( F_f^{\tau_0} \) in terms of its Fourier coefficients.

Sally and Warner were able to compute the explicit Fourier transform of \( F_f^{\tau_0} \) in the rank one situation where discrete series character formulas on the non-compact Cartan subgroup were known. The resulting formula is very similar to the one for the special case of \( SU(1,1) \) given in the previous section. The discrete series characters and singular invariant eigendistributions occur discretely in a sum over \( \hat{T} \) and the principal series characters occur in an integral over \( \hat{A} \) with hyperbolic sine and cosine factors. They were also able to differentiate the resulting formula to obtain the Plancherel Formula.

The key to computing an explicit Fourier transform for orbital integrals in the general case is an understanding of discrete series character formulas on non-compact Cartan subgroups. Thus we briefly review some of these formulas. The results are valid for any connected reductive Lie group, but we assume for simplicity of notation that \( G \) is acceptable. A detailed expository account of all results about discrete series characters presented in this section is given in [51].

Assume that \( G \) has discrete series representations, and hence a compact Cartan subgroup \( T \), and identify the character group of \( T \) with a lattice \( L \subset E = i\mathbb{R} \). For each \( \lambda \in E \), let \( W(\lambda) = \{ w \in W : w\lambda = \lambda \} \) where \( W \) is the full complex Weyl group, and let \( E' = \{ \lambda \in E : W(\lambda) = \{1\} \} \). Then \( \lambda \in E' = L \cap E' \) is regular, and corresponds to a discrete series character \( \Theta_\lambda^T \). For \( t \in T' \), we have the simple character formula

\[
\Theta_\lambda^T(t) = \epsilon(E^+)\Delta(t)^{-1} \sum_{w \in W_K} \det(w)e^{w\lambda}(t),
\]

where \( \Delta \) is the Weyl denominator, \( W_K \) is the subgroup of \( W \) generated by reflections in the compact roots, and \( \epsilon(E^+) = \pm 1 \) depends only on the connected component (Weyl chamber) \( E^+ \) of \( E' \) containing \( \lambda \).

Now assume that \( H \) is a non-compact Cartan subgroup of \( G \), and let \( H^+ \) be a connected component of \( H' \). Then for \( h \in H^+ \),

\[
\Theta_\lambda^H(h) = c(H^+)\epsilon(E^+)\Delta(h)^{-1} \sum_{w \in W} \det(w)c(w : E^+ : H^+)(w \lambda)\xi_{w,\lambda}(h),
\]

where \( c(H^+) \) is an explicit constant given as a quotient of certain Weyl groups and the \( c(w : E^+ : H^+) \) are integer constants depending only on the data shown in the notation. The sum is over the full complex Weyl group \( W \), and for \( w \) such that \( c(w : E^+ : H^+) \) is potentially non-zero, \( \xi_{w,\lambda} \) is a character of \( H \) obtained from \( w \) and \( \lambda \) using a Cayley transform. This formula is a restatement of results of Harish-Chandra in [26]. In that paper, Harish-Chandra gave properties of the constants...
c(w: E⁺: H⁺) which characterize them completely. These properties can in theory be used to determine the constants by induction on the dimension of the vector component of H. This easily yields formulas when this dimension is one or two, but quickly becomes cumbersome for higher dimensions.

With the above notation, it is easy to describe the singular invariant eigendistributions corresponding to λ ∈ L⁺ = L \ L⁺. Let λ₀ ∈ L⁺, and let E⁺ be a chamber with λ₀ ∈ Cl(E⁺). The exponential terms ξ_{w,λ₀}(h), h ∈ H⁺, still make sense, and the “limit of discrete series” Θ^T_{λ₀,E⁺} = lim_{λ→λ₀,λ∈L⁺∩E⁺} Θ^T_λ is given by (3.3) using the constants from E⁺. Zuckerman [96] showed that the limits of discrete series are the characters of tempered unitary representations of G. The singular invariant eigendistribution corresponding to λ₀ is the alternating sum of the limits of discrete series taken over all chambers with closures containing λ₀.

\begin{equation}
\Theta^T_{λ₀} = [W(λ₀)]^{-1} \sum_{w∈W(λ₀)} \det w \ Θ^T_{λ₀,w,E⁺}.
\end{equation}

The main results of [33] are as follows. Let Φ(λ₀) denote the roots of T which are orthogonal to λ₀. Then Θ^T_{λ₀} vanishes if Φ(λ₀) contains any compact roots. Thus we may as well assume that all roots in Φ(λ₀) are non-compact. By using Cayley transforms with respect to the roots of Φ(λ₀) we obtain a Cartan subgroup H and corresponding cuspidal Levi subgroup M. Because the Cayley transform of λ₀ is regular with respect to the roots of H in M, it determines a discrete series character of M, which can then be parabolically induced to obtain a unitary principal series character Θ^H_{λ₀} of G.

**Theorem 3.5** (Herb–Sally). Θ^H_{λ₀} = \sum_{w∈W(λ₀)} Θ^T_{λ₀,w,E⁺}.

It follows from Knapp [57] that Θ^H_{λ₀} has at most \([W(λ₀)]\) irreducible components. Thus each limit of discrete series character is irreducible, and Θ^T_{λ₀} is the alternating sum of the characters of the irreducible constituents of Θ^H_{λ₀}.

In [44], Herb used the methods of Sally and Warner, and the discrete series character formulas of Harish-Chandra, to obtain a Fourier inversion formula for orbital integrals for groups of arbitrary real rank. As in the rank one case, for any Cartan subgroup H of G we have normalized orbital integrals F^H_f(h), h ∈ H, f ∈ C^∞_c(G). We also have characters Θ^H_λ, λ ∈ H. If H is compact, these are discrete series characters for regular λ and singular invariant eigendistributions for singular λ. If H is non-compact, corresponding to the Levi subgroup M, then they are parabolically induced from discrete series or singular invariant eigendistributions on M. Using standard character formulas for parabolic induction, these characters can also be written using Harish-Chandra’s discrete series formulas for M.

Fix a Cartan subgroup H₀. The goal is to find a formula

\begin{equation}
F^H_{h₀}(h₀) = \int_{H} \Theta^H_λ(f) K^H(h₀,λ) dλ, \ h₀ ∈ H₀’,
\end{equation}

where H runs over a set of representatives of conjugacy classes of Cartan subgroups of G, dλ is Haar measure on H, and K^H(h₀,λ) is a function depending on h₀ and λ. The problem is to compute the functions K^H(h₀,λ), or at least show they exist.

As in the rank one case, for λ₀ ∈ H₀, f ∈ C^∞_c(G), the abelian Fourier coefficient \(\hat{F}^H_{h₀}(λ₀)\) is equal up to a constant to \(Θ^H_{λ₀}(f)\) plus an error term for each of the other Cartan subgroups. The error term corresponding to H is an integral over H of the
numerator of $\Theta^{H_0}_{\lambda_0}$ times $F^H$. Because $\Theta^{H_0}_{\lambda_0}$ is parabolically induced, its character is non-zero only on Cartan subgroups of $G$ which are conjugate to Cartan subgroups of $M_0$, the corresponding Levi subgroup. Thus the error term will be identically zero unless $H$ can be conjugated into $M_0$, but is not conjugate to $H_0$. This implies in particular that the vector dimension of $H$ is strictly greater than that of $H_0$. Thus if $H_0$ is maximally split in $G$ there are no error terms. However, if $H_0 = T$ is compact, then $M_0 = G$ and all non-compact Cartan subgroups contribute error terms.

Let $H$ be a Cartan subgroup of $M_0$ which is not conjugate to $H_0$ and let $M$ be the corresponding Levi subgroup. In analyzing the error term corresponding to $H$, we obtain a primary term involving the characters $\Theta^H_\chi(f), \chi \in \hat{H}$, plus secondary error terms, one for each Cartan subgroup of $M$ not conjugate to $H$. This leads to messy bookkeeping, but the process eventually terminates since the vector dimension of the Cartan subgroups with non-zero error terms increases strictly at each step.

In particular, if $H$ is a Cartan subgroup of $G$ not conjugate to a Cartan subgroup of $M_0$, then it never occurs in a non-zero error term and $K^H$ is identically zero. Our original Cartan subgroup $H_0$ also is not involved in any error term, and we have

\[(3.7) \quad K^{H_0}(h_0, \chi_0) = c_{H_0} \epsilon(\chi_0) \chi_0(h_0), h_0 \in H_0', \chi_0 \in \hat{H}_0.\]

The formulas for $K^H$ become progressively more complicated as the vector dimension of $H$ increases. In particular, if $H$ is maximally split in $G$, then $K^H$ has contributions from error terms at many different steps.

Aside from the proliferation of error terms, the analysis which will lead to the functions $K^H(h_0, \chi)$ involves two main problems that do not occur in real rank one. The main problem is that the final formulas contain the unknown integer constants $c(w; E^+; H^+)$ appearing in discrete series character formulas. These occur in complicated expressions which can be interpreted as Fourier series in several variables. These series are not absolutely convergent and have no obvious closed form. Thus although [43] showed the existence of the functions $K^H(h_0, \chi)$, it does not result in a formula which is suitable for applications. In particular, it cannot be differentiated to obtain the Plancherel Formula for $G$. Second, in the rank one case the analysis can be carried out for any $h \in H'$. However there are cases in higher rank, for example the real symplectic group of real rank three, in which certain integrals diverge for some elements $h \in H'$. However, the analysis is valid on a dense open subset of $H'$.

In order to improve these results and obtain a satisfactory Fourier inversion formula similar to that of Sally and Warner for rank one groups, it was necessary to have more information about the discrete series constants. The first of these improvements came from a consideration of stable discrete series characters and stable orbital integrals.

Assume that $G$ has a compact Cartan subgroup $T$, and use the notation from the earlier discussion of discrete series characters. For $\lambda \in L$ we define

\[(3.8) \quad \Theta^{T, st}_\lambda = [W_K]^{-1} \sum_{w \in W} \Theta^{T}_w.\]
If $\lambda \in L'$, then $\Theta_{\lambda}^{T,\text{st}}$ is called a stable discrete series character. For $\lambda \in L^s$, we have $\Theta_{\lambda}^{T,\text{st}} = 0$. Similarly we define the stable orbital integral

$$
\Lambda^{\text{st}}_t(f) = \sum_{w \in W} \Lambda_{wt}(f), f \in C_c^\infty(G), t \in T'.
$$

If we normalize the orbital integral as usual, we have

$$
F_{T,\text{st}}^T(f)(t) = \Delta(t) \Lambda^{\text{st}}_t(f) = \sum_{w \in W} \det(w) F_{T}^T(wt).
$$

Similarly, for any Cartan subgroup $H$ with corresponding Levi subgroup $M$ there is a series of stable characters $\Theta_{\chi}^{H,\text{st}}, \chi \in \hat{H}$, induced from stable discrete series characters of $M$. We also obtain stable orbital integrals by averaging over the complex Weyl group of $H$ in $M$.

Recall that there is a differential operator $\Pi$ such that

$$
f(1) = \lim_{t \to 1, t \in T'} \Pi F_{T}^T(f)(t).
$$

Since the differential operator $\Pi$ transforms by the sign character of $W$, it follows immediately that we also have

$$
f(1) = [W]^{-1} \lim_{t \to 1, t \in T'} \Pi F_{T,\text{st}}^T(f)(t).
$$

The advantage of stabilizing is that the formulas for the stable discrete series characters on the non-compact Cartan subgroups are simpler than those of the individual discrete series characters. The Fourier inversion formula for stable orbital integrals involves only these stable characters and has the general form

$$
F_{T,\text{st}}^T(f)(t) = \sum_H \int_H \Theta_{\chi}^{H,\text{st}}(f) K_{\chi}(t, \chi) d\chi, \ t \in T'.
$$

When $G$ has real rank one the Fourier inversion formulas for the stable orbital integrals are no simpler than those obtained by Sally and Warner. However when $G$ has real rank two there is already significant simplification, and Sally’s student Chao [13] was able to obtain expressions for the functions $K_{\chi}^{H,\text{st}}(t, \chi)$ in closed form and differentiate them to obtain the Plancherel Formula.

Herb [45], [46] then developed the theory of two-structures and showed that the constants occurring in stable discrete series character formulas for any group can be expressed in terms of stable discrete constants for the group $SL(2, \mathbb{R})$ and the rank two symplectic group $Sp(4, \mathbb{R})$. As a consequence she was able to write each function $K_{\chi}^{H,\text{st}}(t, \chi)$ occurring in (3.13) as a product of factors which occur in the corresponding formulas for $SL(2, \mathbb{R})$ and $Sp(4, \mathbb{R})$.

This formula can be differentiated to yield the Plancherel Formula. However, the Fourier inversion formulas for stable orbital integrals are of independent interest, and much of the complexity of these distributions is lost when they are differentiated and evaluated at $t = 1$. In particular the functions occurring in the Plancherel Formula, which had already been obtained by different methods by Harish-Chandra [36], reduce to a product of rank one factors which occur in the Plancherel Formula for $SL(2, \mathbb{R})$. The discrete series character formulas and Fourier inversion formula for $F_{T}^T(f)(t)$ require both $SL(2, \mathbb{R})$ and $Sp(4, \mathbb{R})$ type factors coming from the theory of two-structures.

In [47] Herb was able to use Shelstad’s ideas on endoscopy to obtain explicit Fourier inversion formulas for the individual (not stabilized) orbital integrals. The
idea is that certain weighted sums of orbital integrals, $\Lambda_\kappa(f)$, correspond to stable orbital integrals on endoscopic groups. Thus their Fourier inversion formulas can be computed as in [46]. This is done for sufficiently many weights $\kappa$ that the original orbital integrals $\Lambda_\gamma(f)$ can be recovered. Again, the theory of two-structures was important, and the functions $K^H(h_0, \chi)$ occurring in (3.6) can be given in closed form using products of terms coming from the groups $SL(2, \mathbb{R})$ and $Sp(4, \mathbb{R})$.

Although this gave a satisfactory Fourier inversion formula, the derivation is complicated by the use of stability and endoscopy. Stability and endoscopy also combined to yield explicit, but cumbersome, formulas for the discrete series constants $c(w; E^+: H^+)$ occurring in (3.3). In [50], Herb found simpler formulas for these constants that bypass the theories of stability and endoscopy, and are easier to prove independently of these results. Using special two-structures called two-structures of non-compact type, she obtained a formula for the constants $c(w; E^+: H^+)$ directly in terms of constants occurring in discrete series character formulas for $SL(2, \mathbb{R})$ and $Sp(4, \mathbb{R})$. These formulas could be used to give a direct and simpler proof of the Fourier inversion formulas for orbital integrals given in [47].

4. The $p$-adic Case

We now focus on the representation theory and harmonic analysis of reductive $p$-adic groups. Since the 1960s, there has been a flurry of activity related to these groups. Some of this has been generated by the so-called “Langlands Program” (see Jacquet–Langlands [55] and Langlands [61]). However, a number of results in representation theory and harmonic analysis were completed well before this activity related to the Langlands Program by Bruhat [9], Satake [80], Gelfand–Graev [22], and Macdonald [62]. Of particular interest were the results of Mautner [64] that gave the first construction of supercuspidal representations. Here, a supercuspidal representation is an infinite-dimensional, irreducible, unitary representation with compactly supported matrix coefficients (mod the center). In the mid-1960s, for a $p$-adic field $F$ with odd residual characteristic, all supercuspidal representations for $SL(2, F)$ were constructed by Shalika [86], and for $PGL(2, F)$ by Silberger [88]. These two were Mautner’s Ph.D. students. At roughly the same time, Shintani [87] constructed some supercuspidal representations for the group of $n \times n$ matrices over $F$ whose determinant is a unit in the ring of integers of $F$. Shintani also proved the existence of a Frobenius-type formula for computing supercuspidal characters as induced characters. Incidentally, in 1967–1968, the name “supercuspidal” had not emerged, and these representations were called “absolutely cuspidal,” “compactly supported discrete series,” and other illustrative titles.

We also note that, in this same period, Sally and Shalika computed the characters of the discrete series of $SL(2, F)$ as induced characters [75] (see also [2]), derived the Plancherel Formula for $SL(2, F)$ [76], and developed an explicit Fourier transform for elliptic orbital integrals in $SL(2, F)$ [78]. This Fourier transform led directly to the Plancherel Formula through the use of the Shalika germ expansion [55]. The guide for this progression of results was the 1952 paper of Harish-Chandra on $SL(2, \mathbb{R})$ [25].

In the autumn of 1969, Harish-Chandra presented his first complete set of notes on reductive $p$-adic groups [29]. These are known as the “van Dijk Notes”. These notes appear to be the origin of the terms “supercusp form” and “supercuspidal representation”. They present a wealth of information about supercuspidal forms,
discrete series characters, and other related topics. At the end of the introduction, Harish-Chandra states the following: “Of course the main goal here is the Plancherel Formula. However, I hope that a correct understanding of this question would lead us in a natural way to the discrete series for \( G \). (This is exactly what happens in the real case. But the \( p \)-adic case seems to be much more difficult here.)” It seems that that Harish-Chandra favored the prefix “super” as in “supercusp form,” “supertempered distribution,” etc.

We now proceed to the description of Harish-Chandra’s Plancherel Theorem (see [40]) and Waldspurger’s exposition of Harish-Chandra’s ideas [93]. We then give an outline of the current state of the discrete series of reductive \( p \)-adic groups and their characters. Finally, we give details (as currently known) of the Plancherel Formula and the Fourier transform of orbital integrals.

The background for Harish-Chandra’s Plancherel Theorem was developed in his Williamstown lectures [33]. He showed that, using the philosophy of cusp forms, one could prove a formula similar to that for real groups that we outlined in Section 2. He was able to do this despite the lack of information about the discrete series and their characters.

Following the model of real groups, for each special torus \( A \), Harish-Chandra constructed a subspace \( C_A(G) \) from the matrix coefficients of representations corresponding to \( A \). These representations are parabolically induced from relative discrete series representations of \( M \), the centralizer of \( A \). There are two notable differences between the real case and the \( p \)-adic case. First of all, because, in the \( p \)-adic case, there are discrete series that are not supercuspidal (for example, the Steinberg representation of \( SL(2,F) \)), the theory of the constant term must be modified. Second, because of a compactness condition on the dual of \( A \), it is not necessary to consider the asymptotics of the Plancherel measure that are required in the real case because of non-compactness.

Thus, even though the understanding of the discrete series and their characters for \( p \)-adic groups is quite rudimentary, Harish-Chandra succeeded in proving a version of the Plancherel Theorem. This version, as stated by Howe [39], is: “The (Schwartz) space \( C(G) \) is the orthogonal direct sum of wave packets formed from series of representations induced unitarily from discrete series of (the Levi components of) parabolic subgroups \( P \). Moreover if two such series of induced representations yield the same subspace of \( C(G) \), then the parabolics from which they are induced are associate, and the representations of the Levi components are conjugate.” Equivalently, as stated by Harish-Chandra (Lemma 5 of The Plancherel Formula for Reductive \( p \)-adic Groups in [40]), if \( G \) is a connected reductive \( p \)-adic group and \( C(G) \) is the Schwartz space of \( G \), then

\[
C(G) = \sum_{A \in S} C_A(G)
\]

where \( S \) is the set of conjugacy classes of special tori in \( G \) and the sum is orthogonal.

In 2002, Waldspurger produced a carefully designed version of Harish-Chandra’s Plancherel Theorem. This work is executed with remarkable precision, and we quote here from Waldspurger’s introduction (the translation here is that of the authors of the present article).
"The Plancherel formula is an essential tool of invariant harmonic analysis on real or p-adic reductive groups. Harish-Chandra dedicated several articles to it. He first treated the case of real groups, his last article on this subject being [36]. A little later, he proved the formula in the p-adic case. But he published only a summary of these results [40]. The complete proof was to be found in a handwritten manuscript that was hardly publishable in that state. Several years ago, L. Clozel and the present author conceived of a project to publish these notes. This project was not realized, but the preparatory work done on that occasion has now become the text that follows. It is a redaction of Harish-Chandra’s proof, based on the unpublished manuscript.

As this article is appearing more than fifteen years after Harish-Chandra’s manuscript, we had the choice between scrupulously respecting the original or introducing several modifications taking account of the evolution of the subject in the meantime. We have chosen the latter option. As this choice is debatable and the fashion in which we observe the subject to have evolved is rather subjective, let us attempt to explain the modifications that we have wrought.

There are several changes of notation: we have used those which seemed to us to be the most common and which have been used since Arthur’s work on the trace formula. We work on a base field of any characteristic, positive characteristic causing only the slightest disturbance. We have eliminated the notion of the Eisenstein integral in favor of the equivalent and more popular coefficient of the induced representation. We have used the algebraic methods introduced by Bernstein. They allow us to demonstrate more naturally that certain functions are polynomial or rational, where Harish-Chandra proved their holomorphy or meromorphy. At the end of the article, we have slightly modified the method of extending the results obtained for semi-simple groups to reductive groups, in particular, the manner in which one treats the center. In fact, the principal change concerns the ‘constant terms’ and the intertwining operators. Harish-Chandra began with the study of the ‘constant terms’ of the coefficients of the induced representations and deduced from this study the properties of the intertwining operators. These latter having seemed to us more popular than the ‘constant terms,’ we have inverted the order, first studying the intertwining operators, in particular their rational extension, and having deduced from this the properties of the ‘constant terms.’ All of these modifications remain, nevertheless, minor and concern above all the preliminaries. The proof of the Plancherel formula itself (sections VI, VII and VIII below) has not been altered and is exactly that of Harish-Chandra.”

It remains to address the current status of the three central problems of harmonic analysis on reductive p-adic groups. These are the construction of the discrete series, the determination of the characters of the discrete series, and the derivation of the Fourier transform of orbital integrals as linear functionals on the space of supertempered distributions.

There is a long list of authors who have attacked the construction of discrete series of p-adic groups over the past forty years. We limit ourselves to a few of the major stepping stones. The work of Howe [54] on GL(n) in the tame case set the stage for a great deal of the future work. Howe’s supercuspidal representations for
GL(n) were proved to be exhaustive by Moy in [68]. Further work in the direction of tame supercuspidals may be found in the papers [66] and [67] of L. Morris.

In the mid 1980s, Bushnell and Kutzko attacked GL(n) in the wild case. Their main weapon was the theory of types, and the definitive results for GL(n) and SL(n) were published in [10], [11], and [12]. While in the tame case, one gets a reasonable parameterization in terms of characters of tori, it does not seem that such a parameterization can be expected in the wild case. It is difficult to associate certain characters with any particular torus, as well as to tell when representations constructed from different tori are distinct. We also mention the work of Corwin on division algebras in both the tame [14] and the wild [15] case.

A big breakthrough came in J.-K. Yu’s construction of tame supercuspidal representations for a wide class of groups in [95]. In this paper, Yu points to the fact that he was guided by the results of Adler [1] at the beginning of this undertaking. Under certain restrictions on p, Yu’s supercuspidal representations were proved to be exhaustive by Ju-Lee Kim [56] using tools from harmonic analysis in a remarkable way. Throughout this period, the work of Moy–Prasad [69], [70] was quite influential. Also, Stevens [92] succeeded in applying the Bushnell–Kutzko methods to the classical groups to obtain all their supercuspidal representations as induced representations when the underlying field has odd residual characteristic. Finally, major results have been obtained by Mœglin and Tadic for non-supercuspidal discrete series in [65]. There is still much work to be done, but considerable progress has been made.

The theory of characters has been slower in its development. There are two avenues of approach that have been cultivated. The first is the local character expansion of Harish-Chandra. If O is a G-orbit in g, then O carries a G-invariant measure denoted by μO (see, for example, [74]). The Fourier transform of the distribution f ↦→ μO(f) is represented by a function ˆμO on g that is locally summable on the set of regular elements g′ in g. The local character expansion is:

**Theorem 4.1.** Let π be an irreducible smooth representation of G. There are complex numbers cO(π), indexed by nilpotent orbits O, such that

Θπ(exp Y) = ∑ O cO(π) ˆμO(Y)

for Y sufficiently near 0 in g′.

This result is presented in Harish-Chandra’s Queen’s Notes [37] and is fully explicated in [41]. The local character expansion could be a very valuable tool if three problems are overcome. These are: (1) determine the functions ˆμO, (2) find the constants cO(π), and (3) determine the domain of validity of the expansion. For progress in these directions, see Murnaghan [71], [72], Waldspurger [94], DeBacker–Sally [19], and DeBacker [17].

The second approach is the direct use of the Frobenius formula for induced characters to produce full character formulas on the regular elements in G. See Harish-Chandra [29] (p. 94), Sally [77], and Rader–Silberger [73]. This approach has been used by DeBacker for GL(ℓ), ℓ a prime [20], and Spice for SL(ℓ), ℓ a prime [91]. Recent work of Adler and Spice [3] and DeBacker and Reeder [15] shows some promise in this direction, but their results are still quite limited. The paper [3] of Adler and Spice gives an interesting report on the development and current status of character theory on reductive p-adic groups. For additional results on the theory
of characters, consult the papers of Cunningham and Gordon [16] and Kutzko and Pantoja [59]. We finish this paper with an update on the Plancherel Theorem, the Plancherel Formula, and the Fourier transform of orbital integrals in the $p$-adic case. As regards the Plancherel Theorem, it seems that some flesh is beginning to appear on the bones. Thus, for some special cases, an explicit Plancherel measure related to the components in the Schwartz space decomposition has been found (see Shahidi [83], [84], Kutzko–Morris [60], and Aubert–Plymen [6], [7]). The results seem to be applicable mainly to $GL(n)$ and $SL(n)$. In some cases, restrictions on the residual characteristic have been completely avoided. These methods seem to a great extent to be independent of explicit character formulas. It would be interesting to determine how far these techniques can be carried for general reductive $p$-adic groups.

It is one of the purposes of this paper to point out the nature of the Plancherel Formula in the theory of harmonic analysis on reductive $p$-adic groups. As was the case originally with Harish-Chandra, the Plancherel Formula should be considered as the Fourier transform of the $\delta$ distribution regarded as an invariant distribution on a space of smooth functions on the underlying group. This is achieved in the real case by determining the Fourier transform of an elliptic orbital integral and applying a limit formula involving differential operators to deduce an expression for $f(1)$ as a linear functional on the space of tempered invariant distributions. This space is directly connected to the space of tempered irreducible characters of $G$ along with some additional super tempered virtual characters. It appears to be the case that, to accomplish this goal, one has to have a full understanding of the irreducible tempered characters of $G$. This, of course, requires a detailed knowledge of the discrete series. This is exactly the approach that was detailed in Section 3.

As pointed out by Harish-Chandra, a complete knowledge of the discrete series and their characters would yield the Plancherel measure for $p$-adic groups exactly as in the real case. In the $p$-adic case, the role of differential operators in the limit formula to obtain $f(1)$ is assumed by the Shalika germ expansion.

Shalika Germs

For a connected semi-simple $p$-adic group $G$, Shalika defines in [85]

$$I_f(x) = \int_{G(x)} f d\mu,$$

where $x$ is a regular element in $G$, $G(x)$ is its conjugacy class, $\mu$ is a $G$-invariant measure on $G(x)$, and $f \in C_c^{\infty}(G)$. Shalika shows that $I_f(x)$ has an asymptotic expansion in terms of the integrals

$$\Lambda_{\mathcal{O}}(f) = \int_{\mathcal{O}} f d\mu$$

of $f$ over the unipotent conjugacy classes $\mathcal{O}$. Here, for $\mathcal{O} = \{1\}$, we take $\Lambda_{\mathcal{O}}(f) = f(1)$. The coefficients $C_{\mathcal{O}}(x)$ occurring in this expansion are called the Shalika germs.

We start with $G = SL(2,F)$ where $F$ has odd residual characteristic, and then use Shalika germs to produce the Plancherel Formula for $G$. This result of Sally and Shalika was proved in 1969 and is presented in detail in [78]. We repeat it
here to indicate the role that such a formula can play in the harmonic analysis on a reductive \( p \)-adic group.

Let \( T \) be a compact Cartan subgroup of \( G \). For each nontrivial unipotent orbit \( \mathcal{O} \), there is a subset \( T_{\mathcal{O}} \) of the set of regular elements in \( T \) such that the following asymptotic expansion holds.

\[
F_T^f(t) = |D(t)|^{1/2} I_f(t) \sim -A_T |D(t)|^{1/2} f(1) + B_T \sum_{\dim \mathcal{O} > 0} C_\mathcal{O}(t) \Lambda_\mathcal{O}(f)
\]

where the Shalika germ \( C_\mathcal{O}(t) \) is the characteristic function of \( T_{\mathcal{O}} \). The constants \( A_T \) and \( B_T \) depend on normalization of measures and whether \( T \) is ramified or unramified.

By summing products of characters, we are led to the following expression.

\[
\mu(T) I_f(t) = \sum_{\Pi \in \mathcal{D}} \chi_\Pi(t) \hat{f}(\Pi) + \frac{1}{2} \sum_{\Pi \in \text{RPS}_V} \chi_\Pi(t) \hat{f}(\Pi)
\]

\[
- \frac{q^2}{2q} \mu(A_1) \int_{\xi \in \hat{F}^\times |\Gamma(\xi)|^{-2} \hat{f}(\xi) d\xi}
\]

\[
+ \frac{q}{2} \mu(A_1) \kappa_{\mathcal{O}} |D(t)|^{-1/2} \int_{\xi \in \hat{F}^\times \xi|_{A_{h_0+1}} = 1} \hat{f}(\xi) d\xi
\]

This is the Fourier transform of the elliptic orbital integral corresponding to the regular element \( t \). Note the occurrence of the characters of the reducible principal series, denoted \( \text{RPS}_V \), corresponding to the three \text{sgn} characters on \( F^\times \). As in the case of \( SL(2, \mathbb{R}) \), each represents the difference of two characters divided by 2, and that difference is 0 except on the compact Cartan subgroups corresponding to the \text{sgn} character associated to the quadratic extension \( V \). So again, these singular tempered invariant distributions (see \[52\]) appear in the Fourier transform of an elliptic orbit.

Using Shalika germs, we are led directly to the Plancherel Formula for \( SL(2, F) \).

\[
\mu(K) f(1) = \sum_{\Pi \in \mathcal{D}} \hat{f}(\Pi) d(\Pi) + \frac{1}{2} \left( \frac{q^2 - 1}{q} \right) \mu(A_1) \int_{\xi \in \hat{F}^\times |\Gamma(\xi)|^{-2} \hat{f}(\xi) d\xi
\]

It is clear that a complete theory of the Fourier transform of orbital integrals would lead to direct results about lifting, matching, and transferring orbital integrals. Such a theory would entail a deep understanding of discrete series characters and their properties. A start in this direction may be found in papers of Arthur \[4\], \[5\] and Herb \[48\], \[49\]. We expect to return to this subject in the near future.

REFERENCES

[1] Jeffrey D. Adler. Refined anisotropic \( K \)-types and supercuspidal representations. Pacific J. Math., 185(1):1–32, 1998.

[2] Jeffrey D. Adler, Stephen DeBacker, Paul J. Sally, Jr., and Loren Spice. Supercuspidal characters of \( SL_2 \) over a \( p \)-adic field. To appear in Harmonic Analysis on reductive, \( p \)-adic groups (Contemp. Math.).

[3] Jeffrey D. Adler and Loren Spice. Supercuspidal characters of reductive \( p \)-adic groups. Amer. J. Math., 131(4):1137–1210, 2009.

[4] James Arthur. On elliptic tempered characters. Acta Math., 171(1):73–138, 1993.

[5] James Arthur. On the Fourier transforms of weighted orbital integrals. J. Reine Angew. Math., 452:163–217, 1994.
[6] Anne-Marie Aubert and Roger Plymen. Explicit Plancherel formula for the $p$-adic group $GL(n)$. C. R. Math. Acad. Sci. Paris, 338(11):843–848, 2004.

[7] Anne-Marie Aubert and Roger Plymen. Plancherel measure for $GL(n,F)$ and $GL(m,D)$: explicit formulas and Bernstein decomposition. J. Number Theory, 112(1):26–66, 2005.

[8] V. Bargmann. Irreducible unitary representations of the Lorentz group. Ann. of Math. (2), 48:568–640, 1947.

[9] François Bruhat. Sur les représentations des groupes classiques $P$-adiques. I. II. Amer. J. Math., 83:321–338, 343–368, 1961.

[10] Colin J. Bushnell and Philip C. Kutzko. The admissible dual of $GL(N)$ via compact open subgroups, volume 129 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1993.

[11] Colin J. Bushnell and Philip C. Kutzko. The admissible dual of $SL(N)$. I. Ann. Sci. École Norm. Sup. (4), 26(2):261–280, 1993.

[12] Colin J. Bushnell and Philip C. Kutzko. The admissible dual of $SL(N)$. II. Proc. London Math. Soc. (3), 68(2):317–379, 1994.

[13] Wen-Min Chao. Fourier inversion and Plancherel formula for semisimple Lie groups of real rank two. University of Chicago Thesis, 1977.

[14] Lawrence Corwin. Representations of division algebras over local fields. Advances in Math., 13:259–267, 1974.

[15] Lawrence Corwin. The unitary dual for the multiplicative group of arbitrary division algebras over local fields. J. Amer. Math. Soc., 2(3):565–598, 1989.

[16] Clifton Cunningham and Julia Gordon. Motivic proof of a character formula for $SL(2)$. Experiment. Math., 19(1):1–44, 2009.

[17] Stephen DeBacker. Homogeneity results for invariant distributions of a reductive $p$-adic group. Ann. Sci. École Norm. Sup. (4), 35(3):391–422, 2002.

[18] Stephen DeBacker and Mark Reeder. Depth-zero supercuspidal $L$-packets and their stability. Ann. of Math. (2), 169(3):795–901, 2009.

[19] Stephen DeBacker and Paul J. Sally, Jr. Germs, characters, and the Fourier transforms of nilpotent orbits. In The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), volume 68 of Proc. Sympos. Pure Math., pages 191–221. Amer. Math. Soc., Providence, RI, 2000.

[20] Stephen M. DeBacker. On supercuspidal characters of $GL_\ell$, $\ell$ a prime. University of Chicago Thesis, 1997.

[21] Gerald B. Folland. A course in abstract harmonic analysis. Studies in Advanced Mathematics. CRC Press.

[22] I. M. Gel’fand and M. I. Graev. Representations of the group of second-order matrices with elements in a locally compact field and special functions on locally compact fields. Uspehi Mat. Nauk, 18(4 (112)):29–99, 1963.

[23] I. M. Gel’fand and M. A. Naimark. Unitarnye predstavleniya klassieckikh grupp. Trudy Mat. Inst. Steklov., vol. 36. Izdat. Nauk SSSR, Moscow-Leningrad, 1950.

[24] Harish-Chandra. Plancherel formula for complex semisimple Lie groups. I. Construction of invariant eigendistributions. Acta Math., 113:241–318, 1965.

[25] Harish-Chandra. Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. Acta Math., 113:241–318, 1965.

[26] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. Acta Math., 116:111–118, 1966.

[27] Harish-Chandra. Two theorems on semi-simple Lie groups. Ann. of Math. (2), 83:74–128, 1966.

[28] Harish-Chandra. Harmonic analysis on reductive $p$-adic groups. Lecture Notes in Mathematics, Vol. 162. Springer-Verlag, Berlin, 1970. Notes by G. van Dijk.

[29] Harish-Chandra. Harmonic analysis on semisimple Lie groups. Bull. Amer. Math. Soc., 76:529–551, 1970.

[30] Harish-Chandra. Some applications of the Schwartz space of a semisimple Lie group. In Lectures in Modern Analysis and Applications. II, Lecture Notes in Mathematics, Vol. 140, pages 1–7. Springer, Berlin, 1970.
Harish-Chandra. On the theory of the Eisenstein integral. In Conference on Harmonic Analysis (Univ. Maryland, College Park, Md., 1971), pages 123–149. Lecture Notes in Math., Vol. 266. Springer, Berlin, 1972.

Harish-Chandra. Harmonic analysis on reductive $p$-adic groups. In Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pages 167–192. Amer. Math. Soc., Providence, R.I., 1973.

Harish-Chandra. Harmonic analysis on real reductive groups. I. The theory of the constant term. J. Functional Analysis, 19:104–204, 1975.

Harish-Chandra. Harmonic analysis on real reductive groups. II. Wavepackets in the Schwartz space. Invent. Math., 36:1–55, 1976.

Harish-Chandra. Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula. Ann. of Math. (2), 104(1):117–201, 1976.

Harish-Chandra. Admissible invariant distributions on reductive $p$-adic groups. In Lie theories and their applications (Proc. Ann. Sem. Canad. Math. Congr., Queen’s Univ., Kingston, Ont., 1977), pages 281–347. Queen’s Papers in Pure Appl. Math., No. 48. Queen’s Univ., Kingston, Ont., 1978.

Harish-Chandra. Supertempered distributions on real reductive groups. In Studies in applied mathematics, volume 8 of Adv. Math. Suppl. Stud., pages 139–153. Academic Press, New York, 1983.

Harish-Chandra. Collected papers. Vol. I. Springer-Verlag, New York, 1984. 1944–1954, Edited and with an introduction by V. S. Varadarajan, With introductory essays by Nolan R. Wallach and Roger Howe.

Harish-Chandra. Collected papers. Vol. IV. Springer-Verlag, New York, 1984. 1970–1983, Edited by V. S. Varadarajan.

Harish-Chandra. Admissible invariant distributions on reductive $p$-adic groups, volume 16 of University Lecture Series. American Mathematical Society, Providence, RI, 1999. Preface and notes by Stephen DeBacker and Paul J. Sally, Jr.

R. A. Herb and P. J. Sally, Jr. Singular invariant eigendistributions as characters. Bull. Amer. Math. Soc., 83(2):252–254, 1977.

R. A. Herb and P. J. Sally, Jr. Singular invariant eigendistributions as characters in the Fourier transform of invariant distributions. J. Funct. Anal., 33(2):195–210, 1979.

Rebecca A. Herb. Fourier inversion of invariant integrals on semisimple real Lie groups. Trans. Amer. Math. Soc., 249(2):281–302, 1979.

Rebecca A. Herb. Fourier inversion and the Plancherel theorem. In Noncommutative harmonic analysis and Lie groups (Marseille, 1980), volume 880 of Lecture Notes in Math., pages 197–210. Springer, Berlin, 1981.

Rebecca A. Herb. Fourier inversion and the Plancherel theorem for semisimple real Lie groups. Amer. J. Math., 104(1):9–58, 1982.

Rebecca A. Herb. Discrete series characters and Fourier inversion on semisimple real Lie groups. Trans. Amer. Math. Soc., 277(1):241–262, 1983.

Rebecca A. Herb. Elliptic representations for Sp(2n) and SO(n). Pacific J. Math., 161(2):347–358, 1993.

Rebecca A. Herb. Supertempered virtual characters. Compositio Math., 93(2):139–154, 1994.

Rebecca A. Herb. Discrete series characters and two-structures. Trans. Amer. Math. Soc., 350(8):3341–3369, 1998.

Rebecca A. Herb. Two-structures and discrete series character formulas. In The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), volume 68 of Proc. Sympos. Pure Math., pages 285–319. Amer. Math. Soc., Providence, RI, 2000.

Rebecca A. Herb, Nick Ramsey, and Paul J. Sally, Jr. Some remarks on the representations of $p$-adic SL$_2$. To appear.

Takeshi Hirai. Invariant eigendistributions of Laplace operators on real simple Lie groups. III. Methods of construction for semisimple Lie groups. J. Math. (N.S.), 2(2):269–341, 1976.

Rogier E. Howe. Tamely ramified supercuspidal representations of GL$_n$. Pacific J. Math., 73(2):437–460, 1977.

H. Jacquet and R. P. Langlands. Automorphic forms on GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin, 1970.
[56] Ju-Lee Kim. Supercuspidal representations: an exhaustion theorem. *J. Amer. Math. Soc.*, 20(2):273–320 (electronic), 2007.

[57] A. W. Knapp. Commutativity of intertwining operators. II. *Bull. Amer. Math. Soc.*, 82(2):271–273, 1976.

[58] A. W. Knapp and Gregg Zuckerman. Classification of irreducible tempered representations of semi-simple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 73(7):2178–2180, 1976.

[59] Philip Kutzko and Gregg Zuckerman. Explicit Plancherel theorems for $SL_2(F)$ and $SL_2$. *Pure Appl. Math. Q.*, 5(1):435–467, 2009.

[60] R. P. Langlands. Problems in the theory of automorphic forms. In *Lectures in modern analysis and applications, III*, pages 18–61. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[61] I. G. Macdonald. Spherical functions on a $p$-adic Chevalley group. *Bull. Amer. Math. Soc.*, 74:520–525, 1968.

[62] F. I. Mautner. Unitary representations of locally compact groups. II. *Ann. of Math. (2)*, 52:528–556, 1950.

[63] F. I. Mautner. Spherical functions over $p$-adic fields. II. *Amer. J. Math.*, 86:171–200, 1964.

[64] Colette Moeglin and Marko Tadić. Construction of discrete series for classical $p$-adic groups. *J. Amer. Math. Soc.*, 15(3):715–786 (electronic), 2002.

[65] Lawrence Morris. Tamely ramified supercuspidal representations of classical groups. I. Filtrations. *Ann. Sci. École Norm. Sup. (4)*, 24(6):705–738, 1991.

[66] Lawrence Morris. Tamely ramified supercuspidal representations of classical groups. II. Representation theory. *Ann. Sci. École Norm. Sup. (4)*, 25(3):233–274, 1992.

[67] Allen Moy. Local constants and the tame Langlands correspondence. *Amer. J. Math.*, 108(4):863–930, 1986.

[68] Allen Moy and Gopal Prasad. Unrefined minimal $K$-types for $p$-adic groups. In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 303–340. Springer, Berlin, 1984.

[69] Fiona Murnaghan. Characters of supercuspidal representations of classical groups. *Ann. Sci. École Norm. Sup. (4)*, 29(1):49–105, 1996.

[70] Fiona Murnaghan. Local character expansions and Shalika germs for $GL(n)$. *Math. Ann.*, 304(3):423–455, 1996.

[71] Paul J. Sally, Jr. Some remarks on discrete series characters for reductive $p$-adic groups. In *Representations of Lie groups, Kyoto, Hiroshima, 1986*, volume 14 of *Adv. Stud. Pure Math.*, pages 337–348. Academic Press, Boston, MA, 1988.
[85] J. A. Shalika. A theorem on semi-simple $p$-adic groups. *Ann. of Math.* (2), 95:226–242, 1972.
[86] Joseph A. Shalika. Representation of the two by two unimodular group over local fields. In *Contributions to automorphic forms, geometry, and number theory*, pages 1–38. Johns Hopkins Univ. Press, Baltimore, MD, 2004.
[87] Takuro Shintani. On certain square-integrable irreducible unitary representations of some $p$-adic linear groups. *J. Math. Soc. Japan*, 20:522–565, 1968.
[88] Allan J. Silberger. $\text{PGL}_2$ over the $p$-adics: its representations, spherical functions, and Fourier analysis. *Lecture Notes in Mathematics*, Vol. 166. Springer-Verlag, Berlin, 1970.
[89] Allan J. Silberger. Harish-Chandra’s Plancherel theorem for $p$-adic groups. *Trans. Amer. Math. Soc.*, 348(11):4673–4686, 1996.
[90] Allan J. Silberger. Correction to: “Harish-Chandra’s Plancherel theorem for $p$-adic groups” [Trans. Amer. Math. Soc. 348 (1996), no. 11, 4673–4686; MR1370652 (99c:22026)]. *Trans. Amer. Math. Soc.*, 352(4):1947–1949, 2000.
[91] Loren Spice. Supercuspidal characters of $\text{SL}_l$ over a $p$-adic field, $l$ a prime. *Amer. J. Math.*, 127(1):51–100, 2005.
[92] Shaun Stevens. The supercuspidal representations of $\text{PGL}_2$ over classical groups. *Invent. Math.*, 172(2):289–352, 2008.
[93] J.-L. Waldspurger. La formule de Plancherel pour les groupes $p$-adiques (d’après Harish-Chandra). *J. Inst. Math. Jussieu*, 2(2):235–333, 2003.
[94] Jean-Loup Waldspurger. Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés. *Astérisque*, (269):vi+449, 2001.
[95] Jiu-Kang Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14(3):579–622 (electronic), 2001.
[96] Gregg Zuckerman. Tensor products of finite and infinite dimensional representations of semisimple Lie groups. *Ann. Math.* (2), 106(2):295–308, 1977.