Cepstral Analysis of Random Variables: Muculants

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Abstract—An alternative parametric description for discrete random variables, called muculants, is proposed. In contrast to cumulants, muculants are based on the Fourier series expansion, rather than on the Taylor series expansion, of the logarithm of the characteristic function. We utilize results from cepstral theory to derive elementary properties of muculants, some of which demonstrate behavior superior to those of cumulants. For example, muculants and cumulants are both additive. While the existence of cumulants is linked to how often the characteristic function is differentiable, all muculants exist if the characteristic function satisfies a Paley-Wiener condition. Moreover, the muculant sequence and, if the random variable has finite expectation, the reconstruction of the characteristic function from its muculants converge. We furthermore develop a connection between muculants and cumulants and present the muculants of selected discrete random variables. Specifically, it is shown that the Poisson distribution is the only distribution where only the first two muculants are nonzero.

Index Terms—Higher-order statistics, cepstrum, cumulants, discrete random variables, Fourier series expansion

INTRODUCTION

CUMULANTS are parametric descriptors of random variables (RVs) and are commonly used to analyze non-Gaussian processes [1] or the effects of nonlinear systems [2]. Unlike moments, cumulants are orthogonal descriptors and satisfy a homomorphism property [3].

As attractive as these properties are, there are several drawbacks: First, since cumulants are the Taylor series coefficients of the logarithm of the distribution’s characteristic function [1], they constitute a complete description only for infinitely differentiable characteristic functions. Second, there are no general results regarding the behavior of the sequence of cumulants; the sequence may even diverge (this problem, though, can be mitigated by the definition of q-moments and q-cumulants [4], [5]). Hence, a reconstruction of a distribution function in terms of its cumulants [6], [7], [8] may not converge. Third, the Marcinkiewicz theorem [9] states that any RV either has infinitely many cumulants or up to second order only; hence, every cumulant-based approximation is either Gaussian or does not correspond to a RV at all. It follows that cumulants are not well suited for hypothesis testing, except in the important case of testing Gaussianity. Fourth, if the RV takes values in the set of integers, then the characteristic function is periodic, and the Taylor series expansion fails to remain the most natural approach.

In this paper, motivated by the latter shortcoming, we replace the Taylor series expansion by a Fourier transform. While in general the resulting description is a functional, for integer RVs, the Fourier transform degenerates to a Fourier series expansion. The resulting Fourier coefficients – henceforth called muculants – are a parametric (i.e., finite or countable) description, retaining several properties of cumulants while removing several of their shortcomings. For example, muculants are orthogonal and additive, but truncating the series leads to a bounded approximation error that converges to zero with increasing order. Also the existence of muculants is less problematic than the one of cumulants, as the former is ensured if the characteristic function satisfies a Paley-Wiener condition. Finally, even though the Poisson distribution is the only distribution for which only the first two muculants are nonzero (see Section IV for the muculants of selected discrete distributions), there exist distributions with more than two (but still only finitely many) nonzero muculants.

The sequence of operations – Fourier transform, (complex) logarithm, and inverse Fourier transform – is also essential in cepstral analysis, originally introduced to investigate the influence of echo [10] and to represent nonlinear systems [11]. Today the cepstrum is widely used in speech processing [12]. Our analysis of RVs is thus deeply rooted in signal processing. Existence and properties of muculants, stated in Section II, are based on properties of the cepstrum, or more generally, properties of the Fourier transform. Moreover, a connection between muculants and cumulants, presented in Section III also finds counterparts in cepstral analysis [13], [14].

About terminology: The name cepstrum is derived from reversing the first syllable of the word spectrum: While the cepstrum is situated in the original (e.g., time) domain, this terminology was introduced to emphasize that this sequence of operations can provide fundamentally different insights [10]. Following this approach, we call our parametric descriptors muculants, a reversal of the first syllable of cumulants.

I. PRELIMINARIES

Let \( X \) be a real-valued RV and let \( \Phi_X(\mu) \) denote its characteristic function

\[
\Phi_X(\mu) := \mathbb{E} \left( e^{i\mu X} \right),
\]

where \( \mu \in \mathbb{R} \). The characteristic function always exists since it is an expectation of a bounded function (see [9], [13], [16] for the theory, properties, and applications of characteristic functions). Two RVs are identically distributed iff (and only if) their characteristic functions are equal. Moreover, every characteristic function satisfies the following properties [17], p. 13:\( i \): \( \Phi_X(\mu) \) is uniformly continuous everywhere, \( ii \): \( \Phi_X(\mu)|_{\mu=0} = 1 \), \( iii \): \(|\Phi_X(\mu)| \leq 1 \), and \( iv \): \( \Phi_X(\mu) \) is Hermitian. Finally, if \( X \) and \( Y \) are independent, then the characteristic function of \( Z = X + Y \) is \( \Phi_Z(\mu) = \Phi_X(\mu) \cdot \Phi_Y(\mu) \).
Let \( X \) be a discrete RV taking values from the set of integers \( \mathbb{Z} \). It can be described by its probability mass function (PMF) \( f_X: \mathbb{Z} \to [0, 1] \), where

\[
\forall \xi \in \mathbb{Z} : \quad f_X(\xi) := \mathbb{P}(X = \xi).
\]

(2)

In this case, \( 1 \) equates to \( \Phi_X(\mu) = \sum_{\xi=-\infty}^{\infty} f_X(\xi) e^{j \mu \xi} \), i.e., \( \Phi_X(\mu) \) is the inverse Fourier transform of \( f_X \) and periodic.

We call a PMF causal if \( f_X(\xi) = 0 \) for \( \xi < 0 \). We call a PMF minimum-phase if it is causal and its \( \hat{PMF} \) has all its zeros inside the unit circle.

The characteristic function is related to this \( \hat{PMF} \)-transform via \( \Phi_X(\mu) = \Psi_X(e^{j \mu}) \).

For a complex function \( f \), the complex logarithm \( \log \) is defined as a single-valued analytic function on a Riemann surface [19]. The computation of such a continuous phase function is essential for the estimation of the complex cepstrum [20, Ch. 13], with the main difference that the cepstrum [20, Ch. 13], with the main difference that the complex cepstrum \( \hat{PMF} \) is continuous for \( \mu \in \mathbb{C} \).

The definition of the \( \mu \)culants follows the definitions of Fourier transform and inverse Fourier transform are transferred from cepstral analysis, several results are based on the fact that we operate on a probability space.

Definition II.1 (Complex Muculants). The complex muculants \( \{\hat{\mu}_X[n]\}_{n \in \mathbb{Z}} \) are the coefficients of the Fourier series expansion (if it exists) of \( \log \Phi_X(\mu) \), i.e.,

\[
\hat{\mu}_X[n] = \frac{d^n \log \Phi_X(\mu)}{d\mu^n} \bigg|_{\mu=0}.
\]

(6)

If \( X \) and \( Y \) are independent, then \( \hat{\mu}_{X+Y}[n] = \hat{\mu}_X[n] + \hat{\mu}_Y[n] \), i.e., cumulants are additive. For elementary results on cumulants the reader is referred to [23] and [24].

II. MUCULANTS: DEFINITION AND PROPERTIES

The definition of the muculants follows the definitions of the cepstrum [20, Ch. 13], with the main difference that the roles of Fourier transform and inverse Fourier transform are reversed. While some properties of the muculants are directly transferred from cepstral analysis, several results are based on the fact that we operate on a probability space.

Theorem 1 (Properties of Complex Muculants). Let \( X \) be an RV with PMF \( f_X \) supported on \( \mathbb{Z} \) and let \( \{\hat{\mu}_X[n]\}_{n \in \mathbb{Z}} \) be the complex muculants defined in Definition II.1. Then, the following properties hold:

1) \( \hat{\mu}_X[n] \in \mathbb{R} \).
2) \( \hat{\mu}_X[0] \leq 0 \).
3) If \( f_X(\xi) = f_X(-\xi) \), then \( \hat{\mu}_X[n] = \hat{\mu}_X[-n] \).
4) If \( \mathbb{E}(X) < \infty \), then \( \sum_n \hat{\mu}_X[n] = 0 \) and the series in (5) converges pointwise.
5) \( \lim_{n \to \infty} \hat{\mu}_X[n] = 0 \). If \( \mathbb{E}(X) < \infty \), then \( \hat{\mu}_X[n] = \mathcal{O}(1/n) \).
6) If \( X \) and \( Y \) are independent, then \( \hat{\mu}_{X+Y}[n] = \hat{\mu}_X[n] + \hat{\mu}_Y[n] \).

Note that finite expectation is required in properties 4 and 5, because one can construct characteristic functions which are nowhere differentiable. E.g., the Weierstrass function.

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Note that finite expectation is required in properties 4 and 5 because one can construct characteristic functions which are nowhere differentiable. E.g., the Weierstrass function.
\[ \sum_{n=1}^{\infty} (\frac{4}{n})^n \cos(3^n \mu) \] is nowhere differentiable, but continuous everywhere and a valid characteristic function \textbf{[17] p. 47}. That muculants of an RV with finite expectation must sum to zero makes truncating the Fourier series as problematic as truncating the cumulant expansion, i.e., the truncated series need not correspond to a valid PMF. However, by Parseval’s theorem and the fact that \( \hat{c}_X[n] = O(1/n) \), the squared error between the true and the approximated characteristic function stays bounded. Hence, muculants may behave better than cumulants when used in functionals of distributions, such as entropy or informational divergence. Moreover, while there exists no distribution with finitely many (but more than two) nonzero cumulants, distributions with, e.g., only three nonzero complex muculants exist.

Finally, property \textbf{[6]} states that muculants are, just as cumulants, additive descriptors. This retains the benefits of cumulants while eliminating some of their drawbacks particularly problematic with discrete RVs.

### III. Linking Cumulants and Muculants

The \( z \)-transform points at a close connection between cumulants and the cepstrum \textbf{[13], [14]}, thus suggesting a connection between cumulants and muculants. Suppose that \( \log \Phi_X(\mu) \) is continuous, that its first \( k \)-th derivatives are continuous, and that its \( k \)-th and \( (k+1) \)-th derivatives are piecewise continuous. We then obtain with \textbf{[28] Th 3.22} that

\[
\frac{d^k}{d\mu^k} \log \Phi_X(\mu) = \sum_{n=-\infty}^{\infty} (jn)^k \hat{c}_X[n] e^{jn\mu}. \tag{12}
\]

Evaluating the l.h.s. at \( \mu = 0 \) yields the \( k \)-th cumulant \( f^k \kappa_X[n] \) (if it exists, cf. \textbf{[7]}); evaluating the r.h.s. at \( \mu = 0 \) then yields

\[
\kappa_X[n] = \sum_{n=-\infty}^{\infty} n^k \hat{c}_X[n], \tag{13}
\]

where, abusing terminology by ignoring the fact that the muculants need not be non-negative, we call the r.h.s. the \( k \)-th non-central moment of the complex muculants.

In \textbf{[14]}, the moments of the cepstrum are connected to the moments of the original sequence, yielding a recursive formula to compute the former from the latter. An equation similar to \textbf{[13]}, called Good’s formula \textbf{[29]}, expresses the cumulants in terms of the moments of a random vector.

### IV. Muculants of Selected RVs

In this section, we present the complex muculants for a selection of discrete distributions, summarized in Table \textbf{[1]}

#### A. Poisson Distribution

With \( \lambda > 0 \), the characteristic function of the Poisson distribution is \( \Phi_X(\mu) = e^{\lambda(e^{\mu} - 1)} \). All cumulants exist and equal \( \lambda \). A fortiori, \( \mathbb{E}(X) = \kappa_X[1] = \lambda \), and with \textbf{[3]} we can write

\[
\log(e^{\lambda(e^{\mu} - 1)}) = \lambda e^{\mu} - \lambda = \sum_{n=-\infty}^{\infty} \hat{c}_X[n] \cdot e^{jn\mu}. \tag{14}
\]

Equating the coefficients yields \( \hat{c}_X[0] = -\lambda, \hat{c}_X[1] = \lambda, \) and \( \hat{c}_X[n] = 0 \) for \( n \neq 0, 1 \). With property \textbf{[4]} of Theorem \textbf{[1]} the Poisson distribution is thus the only distribution with all but the first two muculants being zero.

#### B. Degenerate Distribution

The PMF for \( X = M \) is the Kronecker delta, i.e., \( f_X[\xi] = \delta[\xi - M] \) and \( \Phi_X(\mu) = e^{\mu M} \). While \( \hat{c}_X[0] = 0 \), for \( n \neq 0 \) the complex muculants are given as

\[
\hat{c}_X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} jM e^{-jn\mu} d\mu = \frac{M}{n} (-1)^{n+1}. \tag{15}
\]

Since in this case \( \log \Phi_X(\mu) = jM \) is not periodic, one cannot expect \textbf{[13]} to hold: Indeed, while \( \kappa_3 = 0 \), the third non-central moment of the complex muculants diverges.

#### C. Minimum-Phase Distribution

Suppose that \( f_X \) is minimum-phase. Its \( z \)-transform

\[
\Psi_X(z) = A \cdot \prod_{k=1}^{\infty} \frac{1 - o_k z^{-1}}{1 - p_k z^{-1}} \tag{16}
\]

has all poles \( p_k \) and zeros \( o_k \) inside the unit circle, i.e., \( |o_k| < 1 \) and \( |p_k| < 1 \) for all \( k \). This applies, for example, to the geometric distribution, the Bernoulli distribution and the binomial distribution with \( p < 0.5 \), and the negative binomial distribution. Exploiting \( \Phi_X(\mu) = \Psi_X(e^{\mu}) \), \textbf{[5]}, and the Mercator series, which for \( |z| \leq 1 \) and \( z \neq -1 \) reads

\[
\log(1+z) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} z^m, \tag{17}
\]

the complex muculants are obtained by, cf \textbf{[20] p. 1011]

\[
\log \Phi_X(\mu) = \log A + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{p_k^n - o_k^n}{n} e^{-jn\mu}. \tag{18}
\]

Specifically, for minimum-phase PMFs, we obtain \( \hat{c}_X[n] = 0 \) for \( n < 0 \). Note further that \textbf{[17]} can be used to derive a recursive relation among the complex muculants which, at least under special conditions, admits computing the complex muculants from the PMF without requiring a Fourier transform or a complex logarithm \textbf{[20] p. 1022].

### V. Discussion and Future Work

We have argued that truncating the muculant series \textbf{[8]} results in a bounded error that decreases with increasing number of summands, a property that does not hold for the cumulant series \textbf{[6]}. Although this is an advantage of muculants over cumulants, the question under which circumstances a truncated muculant series represents a PMF remains open. One may investigate, for example, the approximation of a distribution by one with a given number of nonzero muculants. A second line of research could investigate expressions for functionals of discrete distributions (such as entropy and informational divergence) based on muculants, thus complementing cumulant-based expressions for continuous distributions, cf. \textbf{[30]}. 


TABLE I
MUCULANTS $\hat{c}_X[n]$ OF SELECTED DISTRIBUTIONS. WE ALSO PRESENT THE CUMULANTS $\kappa_X[n]$ AND THE CHARACTERISTIC FUNCTIONS $\Phi_X(\mu)$.

| Distribution | $f_X[\xi]$ | $\Phi_X(\mu)$ | $\kappa_X[n]$ | $\hat{c}_X[n]$ |
|-------------|------------|---------------|---------------|---------------|
| Poisson     | $e^{-\lambda \xi}$, $\xi \geq 0$; $0$, else | $e^{\lambda(e^{\mu} - 1)}$ | $\kappa_X[n] = \lambda$ | $-\lambda$, if $n = 0$; $\lambda$, if $n = 1$; otherwise, $0$ |
| Degenerate  | $\delta[\xi - M]$ | $e^{\mu M}$ | $\kappa_X[n] = M$, if $k = 1$; $0$, else | $0$, if $n = 0$; $\frac{M}{n}$, if $n > 1$ |
| Bernoulli ($p < 0.5$) | $\{1 - p, \text{ if } \xi = 0; p, \text{ if } \xi = 1\}$ | $1 - p + pe^{\mu}$ | $\kappa_X[n + 1] = p(1 - p) \frac{d\log |\Phi_X[\mu]|}{d\mu}$ | $\log(1 - p)$, if $n = 0$; $\log(\frac{1 - n}{n} p^{-1})$, if $n > 0$ |
| Bernoulli ($p > 0.5$) | $\{1 - p, \text{ if } \xi = 0; p, \text{ if } \xi = 1\}$ | $1 - p + pe^{\mu}$ | $\kappa_X[n + 1] = p(1 - p) \frac{d\log |\Phi_X[\mu]|}{d\mu}$ | $\log p$, if $n = 0$; $\log(\frac{1 - n}{n} p^{-1} (1 + \frac{1 - p}{p})^{-1})$, if $n > 0$ |
| Geometric   | $\{1 - p\} \rho^k$, $\xi \geq 0$; $0$, else | $\frac{1 - p}{1 - pe^{\mu}}$ | $\kappa_X[n + 1] = \rho(1 + \rho) \frac{d\log |\Phi_X[\mu]|}{d\mu}$ | $\log(1 - p)$, if $n = 0$; $\rho^2$, if $n > 0$ |
| Negative    | $\{\xi \leq n\} \rho^k(1 - p)^{n-k}$, $\xi \geq 0$; $0$, else | $\left(\frac{1 - p}{1 - pe^{\mu}}\right)^n$ | N times the cumulant of the geometric distribution | N times the cumulant of the geometric distribution |
| Binomial    | $\{\xi \leq n\} \rho^k(1 - p)^{n-k}$, $\xi \geq 0$; $0$, else | $\left(\frac{1 - p}{1 - pe^{\mu}}\right)^n$ | N times the cumulant of the Bernoulli distribution | N times the cumulant of the Bernoulli distribution |

That the Poisson distribution has only two nonzero muculants (cf. Section IV-A) makes the presented theory attractive for hypothesis testing. Future work shall thus develop numerical methods to estimate muculants from data, together with estimator variances and confidence bounds. Based on that, a test whether a distribution is Poisson or not shall be formalized and compared to existing hypothesis tests.

We presented a condition for the existence of muculants in (11); however, it is not clear whether there exist distributions violating this condition. The search for a concrete example, or preferably, for a more general statement about existence is within the scope of future work. Similar investigations shall be devoted to the convergence of (8), which for the moment is guaranteed only for distributions with finite expectation (cf. property 1 of Theorem 1).

Finally, the theory of muculants shall be extended to continuous RVs, even though this requires a muculant function rather than a muculant sequence. In this context, the connection between the muculant function and/or cumulants of a continuous RV and the muculants of a discrete RV obtained by uniformly quantizing the former might be of interest. A fundamental step towards these results lies in the fact that quantization periodically repeats the characteristic function [31].

APPENDIX A
PROOF OF THEOREM 1

1) The Fourier transform of a Hermitian function is real-valued [20, p. 83]; $\Phi_X(\mu)$ is Hermitian, and so is $\log \Phi_X(\mu)$.

2) We have:

$$\hat{c}_X[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\Phi_X(\mu)| d\mu \leq 0$$

where (a) follows from (4) and the fact that the phase of $\Phi_X(\mu)$ has odd symmetry; (b) then follows from $|\Phi_X(\mu)| \leq 1$. 3) If the PMF is an even function, then $\Phi_X(\mu)$ and $\log \Phi_X(\mu)$ are real; the muculants have even symmetry by Fourier transform properties.

4) If $\mathbb{E}(X) < \infty$, then $\Phi_X(\mu)$ is uniformly continuous and continuously differentiable; since $\log$ is piecewise continuous, we have that $\log \Phi_X(\mu)$ is piecewise continuous and piecewise differentiable; indeed,

$$\frac{d}{d\mu} \log \Phi_X(\mu) = \frac{\Phi_X(\mu)^{'} \Phi_X(\mu)}{\Phi_X(\mu)}$$

where $\Phi_X(\mu)$ vanishes on an at most countable set (otherwise the muculants would not exist, cf. (11)). Since the Fourier transform of a periodic, piecewise continuous and piecewise differentiable function converges pointwise [28, p. 105], pointwise convergence of (8) follows. That, in this case, the muculants sum to zero, follows from evaluating (8) at $\mu = 0$ and the fact that $\Phi_X(\mu) |_{\mu = 0} = 1$.

5) That $\lim_{n \to \pm \infty} \hat{c}_X[n] = 0$ is a direct consequence of the Lebesgue-Riemann theorem and the fact that $\log \Phi_X(\mu)$ is absolutely integrable [28, p. 94]. If $\mathbb{E}(X) < \infty$, note that pointwise convergence implies bounded variation [32, p.70]; the result about the order of convergence follows from [23].

6) If $X$ and $Y$ are independent RVs, then $\Phi_{X+Y}(\mu) = \Phi_X(\mu) \cdot \Phi_Y(\mu)$. The desired result follows from (5) and the linearity of the Fourier series expansion.

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