SUPERPOSITION AND PROPAGATION OF SINGULARITIES FOR EXTENDED GEVREY REGULARITY

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Abstract. We use sequences which depend on two parameters to define families of ultradifferentiable functions which contain Gevrey classes. It is shown that such families are closed under superposition, and therefore inverse closed as well. Furthermore, we study partial differential operators whose coefficients satisfy the extended Gevrey regularity. To that aim we introduce appropriate wave front sets and derive a theorem on propagation of singularities. This extends related known results in the sense that weaker assumptions on the regularity of the coefficients are imposed.

1. Introduction

Gevrey classes serve as an important reservoir of functions in the context of different aspects of general theory of linear partial differential operators such as hypoellipticity, local solvability and propagation of singularities, since they describe regularities stronger than smoothness and weaker than analyticity [1,7,14]. For example, the Cauchy problem for weakly hyperbolic linear partial differential equations (PDEs) is well-posed for certain values of the Gevrey index $t$, while it is ill-posed in the class of analytic functions, cf. [3,19] and the references given there.

Since the union of Gevrey classes is strictly contained in the class of smooth functions, it is of interest to study intermediate spaces of smooth functions by introducing appropriate regularity conditions. This is done in [16] by observing two-parameter dependent sequences of the form $\{p^{\tau p^\sigma}\}_{p \in \mathbb{N}}, \tau > 0, \sigma > 1$, instead of the Gevrey sequence $\{p^t\}_{p \in \mathbb{N}}, t > 1$. The corresponding families of ultradifferentiable functions, denoted by $\mathcal{E}_{\tau,\sigma}(U)$, extend Gevrey regularity, see Section 2 for the precise definition. We refer to [16,22] for the main properties of such spaces, and note that they can be used e.g. in situations when hypoellipticity of a PDE is better than $C^\infty$ but worse than Gevrey hypoellipticity. In particular, the space $\mathcal{E}_{\{1,2\}}(U)$ is recently explicitly used in the study of strictly hyperbolic equations to capture the regularity of the coefficients in the space variable (with low regularity in time), which ensures that

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the corresponding Cauchy problem is well posed in appropriate solution spaces. We refer to [5] for details.

In this paper we give a further insight to the extended Gevrey regularity by proving the superposition theorem for $E_{\tau,\sigma}(U)$, Theorem 2.2, which immediately implies the inverse closedness property. In the proof we employ a generalized version of Faà di Bruno formula and modified version of Faà di Bruno property of the sequences $\left\{\frac{M_p^{\tau,\sigma}}{p!}\right\}_{p\in\mathbb{N}}$ (Lemma 2.3), following the ideas presented in [17].

Another goal of this paper is to derive propagation of singularities when the coefficients $a_\alpha(x)$ of the partial differential operator $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ belong to $E_{\tau,\sigma}(U)$, see Theorem 4.1. Note that analytic coefficients were treated in [6, Theorem 8.6.1], while [16, Theorem 1.1] treats constant coefficients.

It turns out that an additional information is needed in the study of operators with variable coefficients, since it is not possible to use commutativity properties which hold true when the coefficients are constants. The main tools to overcome these difficulties are the inverse closedness property and careful study of summands in generalized Faà di Bruno’s formula, which gives rise to an explicit construction of approximate solution in Subsections 4.1. Apart from this we use a new result in microlocal analysis, Theorem 3.1 which shows that instead of admissible sequences of cut-off functions used in [16], a single cut-off function can be used in the definition of wave-front set $WF_{\tau,\sigma}(u)$, $u \in \mathcal{D}'(U)$. We refer to [16] for a discussion on different types of wave-front sets in the context of ultradifferentiable functions.

We summarize the paper as follows. In Section 2 we discuss regularity conditions related to the sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, $p \in \mathbb{N}$ (cf. [15,16,22]), and introduce the spaces of ultradifferentiable functions $E_{\tau,\sigma}(U)$. In Section 3 we introduce wave front sets $WF_{\tau,\sigma}(u)$, $u \in \mathcal{D}'(U)$, in the context of extended Gevrey regularity and explain enumeration, an important technical tool in our analysis. The main result there is Theorem 3.1 which offers an equivalent definition of $WF_{\tau,\sigma}(u)$ to be used further on. Finally, in Section 4 we prove the propagation of singularities, Theorem 4.1. The proof is given in details since it contains new nontrivial observations and facts in comparison with the proof of [16, Theorem 1.1].

1.1. Notation. Throughout the paper we use the standard notation for sets of numbers and spaces of distributions, e.g. $\mathbb{N}, \mathbb{Z}_+$, $\mathbb{R}_+$ denote the sets of nonnegative integers, positive integers, and positive real numbers, respectively, and Lebesgue spaces over an open set $\Omega \subset \mathbb{R}^d$ are denoted by $L^p(\Omega)$, $1 \leq p < \infty$. For $x \in \mathbb{R}^d$ we put $\langle x \rangle = (1+|x|^2)^{1/2}$. The integer parts (the floor and the ceiling functions) of $x \in \mathbb{R}_+$ are denoted by $\lfloor x \rfloor := \max\{m \in \mathbb{N} : m \leq x\}$ and $\lceil x \rceil := \min\{m \in \mathbb{N} : m \geq x\}$.
\[ N : m \geq x \}. \text{ For a multi-index } \alpha = (\alpha_1, \ldots, \alpha_d) \in N^d \text{ we write } \partial^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_d}, D^\alpha = (-i)^{|\alpha|} \partial^\alpha, \text{ and } |\alpha| = |\alpha_1| + \cdots |\alpha_d|. \] Open ball of radius \( r > 0 \) centered at \( x_0 \in R^d \) is denoted by \( B_r(x_0) \), and \( \text{card } A \) denotes the cardinal number of \( A \). The Fourier transform of \( u \in L^1(R^d) \) is normalized as
\[
\mathcal{F}_{x \to \xi} u(x) = \hat{u}(\xi) = \int_{R^d} u(x) e^{-2\pi i(x, \xi)} \, dx = \int_{R^d} u(x) e^{-2\pi i x \xi} \, dx, \quad \xi \in R^d,
\]
and the convolution of \( f, g \in L^1(R^d) \) is given by \( f \ast g(x) = \int_{R^d} f(x - y) g(y) \, dy \). Both transforms can be extended in different ways.

By \( C^\infty(K) \) we denote the set of smooth functions on a regular compact set \( K \), and \( \mathcal{D}(U) \) and \( \mathcal{E}(U) \) denote test function spaces for the space of Schwartz distributions \( \mathcal{D}'(U) \), and for the space of compactly supported distributions \( \mathcal{E}'(U) \), respectively.

We will use the Stirling formula: \( N! = N^N e^{-N} \sqrt{2\pi N} e^{\frac{\theta_N}{12N}} \), for some \( 0 < \theta_N < 1, N \in Z_+ \), and formulas for multinomial coefficients:
\[
\binom{a}{a_1, a_2, \ldots, a_m} := \binom{|a|}{a_1} \binom{|a| - a_1}{a_2} \cdots \binom{|a| - a_1 - \cdots - a_{m-2}}{a_{m-1}} = \frac{|a|!}{a_1! a_2! \cdots a_m!} = \sum_{k=1}^{m} \binom{|a|}{a_k} \binom{|a| - 1}{a_1, \ldots, a_k - 1, \ldots, a_m},
\]
where \( |a| = a_1 + a_2 + \cdots + a_m, a_k \in N, k \leq m \).

2. Classes \( \mathcal{E}_{\tau, \sigma}(U) \) AND SUPERPOSITION PROPERTY

In this section we introduce test function spaces denoted by \( \mathcal{E}_{\tau, \sigma}(U) \) via defining sequences of the form \( M^\tau_{p, \sigma} = p^{\tau p^\sigma}, p \in N \), depending on parameters \( \tau > 0 \) and \( \sigma > 1 \). The flexibility obtained by introducing the two-parameter dependence enables the study of smooth functions which are less regular than the Gevrey functions. When \( \tau > 1 \) and \( \sigma = 1 \) we recapture the Gevrey classes.

The spaces \( \mathcal{E}_{\tau, \sigma}(U) \) are already studied in [15,16]. Here we recall their basic properties which are used in the rest of the paper, and collect new results in Subsection 2.1. We employ Komatsu’s approach [10] to spaces of ultradifferentiable functions. Another widely used approach is that of Braun, Meise, Taylor, Vogt and their collaborators, see e.g. [2] and the recent contribution [17]. These two approaches are equivalent in many interesting situations, cf. [12] for more details.

Essential properties of the defining sequences are given in the following lemma. We refer to [15] for the proof.

Lemma 2.1. Let \( \tau > 0, \sigma > 1 \) and \( M^\tau_{p, \sigma} = p^{\tau p^\sigma}, p \in Z_+, M^\tau_{0, \sigma} = 1 \). Then there exists an increasing sequence of positive numbers \( C_q, q \in N \), and a constant \( C > 0 \) such that:
\[
(M.1) \ (M^\tau_{p, \sigma})^2 \leq M^\tau_{p-1, \sigma} M^\tau_{p+1, \sigma}, p \in Z_+.
\]
such that $R \parallel E$ are equivalent in $K$ limit topologies, families of spaces by introducing the following projective and inductive limit topologies,

$$\lim_{h \to 0} \lim_{K \subset U} E_{(\tau, \sigma)}(K) = E_{(\tau, \sigma)}(U) = \lim_{h \to 0} \lim_{K \subset U} D_{(\tau, \sigma)}(K) = D_{(\tau, \sigma)}(U).$$

Moreover, there exist constants $A, B, C > 0$ such that

$$M_{p, \sigma}^{\tau, \sigma} \leq A C^{\sigma} |p^\sigma| !^\tau \quad \text{and} \quad |p^\sigma| !^\tau \leq B M_{p, \sigma}^{\tau, \sigma}. \quad (2.1)$$

Note that $M_{p, 1} \leq (2p)^{0.5} C^{0.5} M_{p, 1}^{\tau, \sigma}, \quad \text{and} \quad |p^\sigma| !^\tau \sim (2\pi)^{\tau} p^{\tau} e^{-\frac{1}{2} p^{\tau}} M_{p, 1}^{\tau, \sigma}, \quad \text{as} \quad p \to \infty.$

For any given values $\tau, h > 0, \sigma > 1$ and a regular compact set $K \subset \mathbb{R}^d$, we denote by $E_{(\tau, \sigma, h)}(K)$ the Banach space of functions $\phi \in C^\infty(K)$ such that

$$\|\phi\|_{E_{(\tau, \sigma, h)}(K)} = \sup_{x \in \mathbb{R}^d} \sup_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(x)|}{h^{||\alpha|| \sigma} M_{p, 1}^{\tau, \sigma}} < \infty. \quad (2.1)$$

Obviously, $E_{(\tau_1, \sigma_1, h_1)}(K) \hookrightarrow E_{(\tau_2, \sigma_2, h_2)}(K), \quad 0 < h_1 \leq h_2, \quad 0 < \tau_1 \leq \tau_2, \quad 1 < \sigma_1 \leq \sigma_2,$

where $\hookrightarrow$ denotes the strict and dense inclusion, and from Lemma 2.1 it follows that the norms given by (2.1) and

$$\|\phi\|_{E_{(\tau, \sigma, h)}(K)} = \sup_{x \in \mathbb{R}^d} \sup_{\alpha \in \mathbb{N}^d} \frac{|\partial^\alpha \phi(x)|}{h^{||\alpha|| \sigma} !^\tau \| \alpha \|^\sigma} < \infty, \quad (2.2)$$

are equivalent in $E_{(\tau, \sigma, h)}(K)$. Moreover, instead of $\sup_{x \in \mathbb{R}^d} |\partial^\alpha \phi(x)|$ we may put $\| \partial^\alpha \phi(x) \|_{L^p(K)}, \quad 1 \leq p < \infty$ in (2.1) and (2.2).

By $D_{(\tau, \sigma, h)}^K$ we denote the set of functions from $E_{(\tau, \sigma, h)}(K)$ with support contained in $K$. If $U$ is an open set $\mathbb{R}^d$ and $K \subset U$ then we define families of spaces by introducing the following projective and inductive limit topologies,

$$E_{(\tau, \sigma)}(U) = \lim_{K \subset U} \lim_{h \to 0} E_{(\tau, \sigma, h)}(K), \quad D_{(\tau, \sigma)}(U) = \lim_{K \subset U} \lim_{h \to 0} D_{(\tau, \sigma, h)}^K.$$
In particular, $D_{\tau,1}(U) = \{0\}$ when $0 < \tau \leq 1$, and $E\{1,1\}(U) = E\{1\}(U)$ is the space of analytic functions on $U$.

The space $E\{1,2\}(U)$ appears in the study of strictly hyperbolic equations where it describes the regularity of the coefficients in the space variable (with low regularity in time), which is sufficient to ensure that the corresponding Cauchy problem is well posed in appropriate solution spaces, we refer to [5] for details.

In the following Proposition we capture the main embedding properties between the above introduced family of spaces.

**Proposition 2.1.** [16] Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$ and $\tau > 0$
\[
\lim_{\tau \to \infty} E_{\tau,\sigma_1}(U) \hookrightarrow \lim_{\tau \to 0^+} E_{\tau,\sigma_2}(U).
\]
Moreover, if $0 < \tau_1 < \tau_2$, then
\[
E\{\tau_1,\sigma\}(U) \hookrightarrow E\{\tau_2,\sigma\}(U) \hookrightarrow E\{\tau_2,\sigma\}(U), \quad \sigma \geq 1,
\]
and
\[
\lim_{\tau \to \infty} E\{\tau,\sigma\}(U) = \lim_{\tau \to 0^+} E\{\tau,\sigma\}(U),
\]
\[
\lim_{\tau \to 0^+} E\{\tau,\sigma\}(U) = \lim_{\tau \to 0^+} E\{\tau,\sigma\}(U), \quad \sigma \geq 1.
\]

We conclude that
\[
E_{\tau_0,\sigma_1}(U) \hookrightarrow \bigcap_{\tau > \tau_0} E_{\tau,\sigma_1}(U) \hookrightarrow E_{\tau_0,\sigma_2}(U),
\]
for any $\tau_0 > 0$ whenever $\sigma_2 > \sigma_1 \geq 1$, and in particular,
\[
\lim_{t \to \infty} E\{t\}(U) \hookrightarrow E_{\tau,\sigma}(U) \hookrightarrow C^\infty(U), \quad \tau > 0, \quad \sigma > 1,
\]
so that the regularity in $E_{\tau,\sigma}(U)$ can be thought of as an extended Gevrey regularity.

Non-quasianalyticity condition (M.3)’ provides the existence of partitions of unity in $E\{\tau,\sigma\}(U)$ which we formulate in the next Lemma.

**Lemma 2.2.** Let $\tau > 0$ and $\sigma > 1$. Then there exists a compactly supported function $\phi \in E\{\tau,\sigma\}(U)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$.

Compactly supported Gevrey function from $E\{t\}(U)$ belong to $D\{t,1\}(U)$. However, in the proof of Lemma 2.2 given in [15] we constructed a compactly supported function in $D\{\tau,\sigma\}(U)$ which does not belong to $D\{t\}(U)$, for any $t > 1$.

**Remark 2.1.** Note that the exponent $\sigma$ which appears in the power of term $h$ in (2.1) makes the above definition different from the definition of Carleman class $C^L$, cf. [6]. This difference is essential for many calculations. For example, Carleman classes perform “stability under differential operators” since their defining sequences satisfy Komatsu’s condition (M.2)’. However, if $\tau > 0$ and $\sigma > 1$ then the sequence $M_p^\tau$ does not satisfy (M.2)’.
If \( P = \sum_{|\alpha| \leq m} a_{\alpha}(x) \partial^{\alpha} \) is a partial differential operator of order \( m \) with \( a_{\alpha} \in \mathcal{E}_{\tau,\sigma}(U) \), then \( P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{E}_{\tau,\sigma}(U) \) is a continuous linear map with respect to the topology of \( \mathcal{E}_{\tau,\sigma}(U) \). In particular, \( \mathcal{E}_{\tau,\sigma}(U) \) is closed under pointwise multiplications and finite order differentiation, see [22, Theorem 2.1].

Let \( \tau > 0, \sigma > 1 \), and let \( a_{\alpha} \in \mathcal{E}_{(\tau,\sigma)}(U) \) (resp. \( a_{\alpha} \in \mathcal{E}_{\{\tau,\sigma\}}(U) \)) where \( U \) is an open set in \( \mathbb{R}^d \). Then

\[
P(x, \partial) = \sum_{|\alpha|=0}^{\infty} a_{\alpha}(x) \partial^{\alpha}
\]

is of class \((\tau,\sigma)\) (resp. \(\{\tau,\sigma\}\)) on \( U \) if for every \( K \Subset U \) there exists constant \( L > 0 \) such that for any \( h > 0 \) there exists \( A > 0 \) (resp. for every \( K \Subset U \) there exists \( h > 0 \) such that for any \( L > 0 \) there exists \( A > 0 \) ) such that,

\[
\sup_{x \in K} |\partial^{\beta} a_{\alpha}(x)| \leq Ah^{\beta|\beta| |\beta| |\beta| L^{|\alpha| |\alpha|}} \frac{|\alpha|^{\sigma}}{|\alpha|^{2\sigma-1} |\alpha|^{\sigma}}, \quad \alpha, \beta \in \mathbb{N}^d.
\]

If \( \tau > 1 \) and \( \sigma = 1 \), then \( P(x, \partial) \) of class \((\tau, 1)\) (resp. \(\{\tau, 1\}\)) is Komatsu’s ultradifferentiable operator of class \((p!^\tau)\) (resp. \(\{p!^\tau\}\)), see [11].

The following theorem gives the continuity properties of such differential operators on \( \mathcal{E}_{\tau,\sigma}(U) \), cf. [16, Theorem 2.1] for the proof.

**Theorem 2.1.** Let \( P(x, \partial) \) be a differential operator of class \((\tau, \sigma)\) (resp. \(\{\tau, \sigma\}\)). Then

\[
P(x, \partial) : \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau,2\sigma-1,\sigma}(U)
\]

is a continuous linear mapping, and the same holds for

\[
P(x, \partial) : \lim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \lim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma}(U).
\]

**2.1. Superposition in \( \mathcal{E}_{\tau,\sigma}(U) \).** In this subsection we prove that the classes \( \mathcal{E}_{\tau,\sigma}(U), \tau > 0, \sigma > 1 \), are stable under **superposition**, and conclude that they are inverse closed. We refer to [4,8,17] for related results. We emphasize here that the inverse-closedness of \( \mathcal{E}_{\tau,\sigma}(U) \) plays an essential role in the proof our main result, Theorem 4.1.

Recall, an algebra \( \mathcal{A} \) is inverse-closed in \( C^\infty(U) \) if for any \( \varphi \in \mathcal{A} \) for which \( \varphi(x) \neq 0 \) on \( U \) it follows that \( \varphi^{-1} \in \mathcal{A} \). It is proved in [21] that a Carleman class defined by a sequence \( M_p \) is inverse closed in \( C^\infty(U) \) if there exists \( C > 0 \) such that

\[
\left( \frac{M_p}{p!} \right)^{1/p} \leq C \left( \frac{M_q}{q!} \right)^{1/q}, \quad p \leq q, \quad \text{and} \quad \lim_{p \to \infty} M_p^{1/p} = \infty, \quad (2.3)
\]

where the condition on the left hand side of (2.3) is equivalent to the statement that \((M_p/p!)^{1/p}\) is an almost increasing sequence.
The Stirling formula implies that the sequence \( (M_p/p!)^{1/p} \) is almost increasing if and only if
\[
\frac{M_p^{1/p}}{p} \leq C \frac{M_q^{1/q}}{q}, \quad p \leq q.
\]

For example, \( \mathcal{E}_{(\tau)}(U) \), \( \tau \geq 1 \) are inverse-closed algebras.

Since \( \left( \frac{M_p^{\tau,\sigma}}{p^p} \right)^{1/p} = p^{\tau p^{-1}} \) when \( M_p^{\tau,\sigma} = p^{\tau p^\sigma} \), \( \tau > 0 \), \( \sigma > 1 \), and
\[
p^{\tau p^{-1}} < q^{\rho^{-1}}, \quad [(1/\tau)^{1/(\sigma-1)}] < p < q,
\]
we conclude that \( \left( \frac{M_p^{\tau,\sigma}}{p^p} \right)^{1/p} \) is an almost increasing sequence and for any choice of indices \( k_i, i = 1, \ldots, j \), and \( k = \sum_{i=1}^j k_i \), we have
\[
\frac{M_k^{\tau,\sigma}}{k!} \leq C^{k_i} \left( \frac{M_k^{\tau,\sigma}}{k!} \right)^{k_i/k}, \quad \text{so that} \quad \prod_{i=1}^j \frac{M_k^{\tau,\sigma}}{k_i!} \leq C^k \frac{M_k^{\tau,\sigma}}{k!}. \tag{2.4}
\]

In other words
\[
\prod_{i=1}^j k_i^{\tau_k} \leq C^k \frac{k_i! \cdots k_j!}{k!} k^{\tau_k}, \quad k = \sum_{i=1}^j k_i.
\]

The almost increasing property of defining sequences is used in the proofs of inverse closedness in Carleman classes, see [9, 20, 21].

Instead, we prove more general result on superposition. We will use Faá di Bruno formula as presented in [13]. Let us first fix the notation. A multiindex \( \alpha \in \mathbb{N}^d \) is said to be decomposed into parts \( p_1, \ldots, p_s \in \mathbb{N}^d \) with multiplicities \( m_1, \ldots, m_s \in \mathbb{N} \), respectively, if
\[
\alpha = m_1 p_1 + m_2 p_2 + \cdots + m_s p_s, \tag{2.5}
\]
where \( m_i \in \{0, 1, \ldots, \lfloor \alpha \rfloor \} \), \( |p_i| \in \{1, \ldots, \lfloor \alpha \rfloor \} \), \( i = 1, \ldots, s \).

If \( p_i = (p_{i1}, \ldots, p_{id}) \), \( i \in \{1, \ldots, s\} \), we put \( p_i < p_j \) when \( i < j \), that is when there exists \( k \in \{1, \ldots, d\} \) such that \( p_{i1} = p_{j1}, \ldots, p_{ik-1} = p_{jk-1} \) and \( p_{ik} < p_{jk} \).

Note that \( s \leq \lfloor \alpha \rfloor \) and the same holds for the total multiplicity \( m = m_1 + \cdots + m_s \leq |\alpha| \).

Any decomposition of \( \alpha \) can be therefore identified with the triple \( (s, p, m) \), and the set of all decompositions of the form (2.5) is denoted by \( \pi \). The total number \( \text{card} \pi \) of decompositions given by (2.5) is bounded by \( (1 + |\alpha|)^{d+2} \).

For smooth functions \( f : U \to \mathbb{C} \) and \( g : V \to U \), where \( U, V \) are open in \( \mathbb{R} \) and \( \mathbb{R}^d \), respectively, the generalized Faá di Bruno formula is given by
\[
\partial^\alpha (f(g)) = \alpha! \sum_{(s, p, m) \in \pi} f^{(m)}(g) \prod_{k=1}^s \frac{1}{m_k!} \left( \frac{1}{p_k!} \partial^{p_k} g \right)^{m_k}. \tag{2.6}
\]
We say that the sequence $M_p, p \in \mathbb{N}$ of positive numbers satisfies Faà di Bruno property if there exist a constant $C > 0$ such that for every $j \in \mathbb{Z}_+$ and $k_i \in \mathbb{Z}_+$ we have
\[
M_j \prod_{i=1}^{j} M_{k_i} \leq C \sum_{i=1}^{j} k_i M_{\sum_{i=1}^{j} k_i}.
\] (2.7)

By [17, Lemma 2.2] it follows that if $M_p, p \in \mathbb{N}$ satisfies $(M.2)'$ and if $M_p^{1/p}$ is almost increasing, then $M_p$ satisfies Faà di Bruno property. Since $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, does not satisfy $(M.2)'$ we first prove a modified version of Faà di Bruno property for the sequence $M_p^{\tau,\sigma} / p!$, $p \in \mathbb{N}$.

**Lemma 2.3.** Let there be given $\tau > 0$, $\sigma > 1$ and let $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbb{N}$. Then there exist a constant $C > 0$ such that for every $j \in \mathbb{Z}_+$ and $k_i \in \mathbb{Z}_+$, $i = 1, \ldots, j$, we have
\[
\frac{M_j^{\tau,\sigma}}{j!} \prod_{i=1}^{j} \frac{M_{k_i}^{\tau,\sigma}}{k_i!} \leq C^{k_1} \frac{M_{\sum_{i=1}^{j} k_i}^{\tau,\sigma}}{k_1!},
\] (2.8)
where $\sum_{i=1}^{j} k_i = k$.

**Proof.** We follow the ideas from the proof of [17, Theorem 4.11].

First we note that the assertion is trivial if $j = k$ since then $k_i = 1$ for all $1 \leq i \leq j$ and therefore
\[
\frac{M_j^{\tau,\sigma}}{j!} \prod_{i=1}^{j} \frac{M_{k_i}^{\tau,\sigma}}{k_i!} = \frac{M_k^{\tau,\sigma}}{k!} \left(\frac{M_1^{\tau,\sigma}}{1!}\right)^k = \frac{M_k^{\tau,\sigma}}{k!}.
\]

For $j < k$, set $I = \{i \mid 1 \leq i \leq j, k_i \geq 2\}$ and $\tilde{k}_i = k_i - 1$, $i \in I$. Note that
\[
k = \sum_{i=1}^{j} k_i = \sum_{i \in I} k_i + \sum_{i \notin I, 1 \leq i \leq j} k_i = \sum_{i \in I} k_i + j - \text{card } I = \sum_{i \in I} \tilde{k}_i + j, \quad (2.9)
\]
and since $\left(\frac{M_p}{p!}\right)^{1/p}$ is almost increasing, then the inequality (2.4) implies that
\[
\frac{M_j^{\tau,\sigma}}{j!} \prod_{i \in I} \frac{M_{k_i}^{\tau,\sigma}}{k_i!} \leq C^k \frac{M_k^{\tau,\sigma}}{k!}.
\] (2.10)

Moreover, from $(M.2)'$ and $k_i = \tilde{k}_i + 1$, $i \in I$, we obtain
\[
\frac{M_{k_i}^{\tau,\sigma}}{k_i!} \leq C_1^{\tilde{k}_i} \frac{M_{\tilde{k}_i}^{\tau,\sigma}}{\tilde{k}_i!},
\] (2.11)
for some constant $C_1 > 0$. 

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By combining (2.9), (2.10) and (2.11) we obtain

\[
\frac{M_j^\tau,\sigma}{j!} \prod_{i=1}^j \frac{M_{k_i}^\tau,\sigma}{k_i!} \leq \left( \frac{M_1^\tau,\sigma}{1!} \right)^{j-\text{card } L} \frac{M_j^\tau,\sigma}{j!} \prod_{i \in I} \frac{M_{k_i}^\tau,\sigma}{k_i!} \leq C_1^{(k-j)} \frac{M_j^\tau,\sigma}{j!} \prod_{i \in I} \frac{M_{k_i}^\tau,\sigma}{k_i!} \leq C_2^k \frac{M_j^\tau,\sigma}{k!},
\]

for some constant \(C_2 > 0\) and the Lemma is proved.

The main result of this section reads as follows.

**Theorem 2.2.** Let there be given \(\tau > 0\), \(\sigma > 1\), and let \(U\) and \(V\) be open sets in \(\mathbb{R}\) and \(\mathbb{R}^d\), respectively. If \(f \in \mathcal{E}_{\tau,\sigma}(U)\) and \(g \in \mathcal{E}_{\tau,\sigma}(V)\) is such that \(g : V \to U\), then \(f \circ g \in \mathcal{E}_{\tau,\sigma}(V)\).

**Proof.** For simplicity we show that if \(f \in \mathcal{E}_{\tau,\sigma}(U)\) and \(g \in \mathcal{E}_{\tau,\sigma}(V)\) is such that \(g : V \to U\), then \(f \circ g \in \mathcal{E}_{\tau,\sigma}(V)\), and leave the (so-called Beurling) case \(f \in \mathcal{E}_{\tau,\sigma}(U)\) and \(g \in \mathcal{E}_{\tau,\sigma}(V)\) to the reader.

Let \(K \subset C \subset V\) and \(h > 0\) be fixed so that \(g \in \mathcal{E}_{\tau,\sigma,h}(K)\). Put \(I = \{g(x), x \in K\}\) and note that \(I\) is a compact set, \(I \subset \subset U\). Therefore \(f \in \mathcal{E}_{\tau,\sigma,h}(I)\) for some \(h' > 0\). By the Faà di Bruno formula (2.6), for any \(x \in K\) we have the following estimate

\[
|\partial^\alpha (f \circ g)(x)| \leq |\alpha|! \sum_{(s,p,m) \in \pi} |f^{(m)}(g(x))| \prod_{k=1}^s \frac{1}{m_k! \left( \frac{1}{p_k!} |\partial^{p_k} g(x)| \right)^{m_k}} \leq A^{|\alpha|+1} |\alpha|! \sum_{(s,p,m) \in \pi} \left( h^{m^\tau} \prod_{k=1}^s m_k^{p_k |p_k|^\tau} \right) \frac{m!}{m_1! \ldots m_s!} \frac{m^{r^m}}{m_1! \ldots m_s!} \prod_{k=1}^s \left( \frac{|p_k|^\tau |p_k|^\tau}{|p_k|!} \right)^{m_k}
\]

(2.12)

for some \(A > 0\), and the second sum being taken over all decompositions \(|\alpha| = \sum_{k=1}^s m_k |p_k|\) where \(m = \sum_{k=1}^s m_k, m_k \in \{0,1,\ldots,|\alpha|\}\), \(|p_k| \in \{1,\ldots,|\alpha|\}\), \(k = 1,\ldots,s\) and \(s \leq |\alpha|\).

By Lemma 2.3 we have

\[
\frac{m^{r^m}}{m!} \prod_{k=1}^s \left( \frac{|p_k|^\tau |p_k|^\tau}{|p_k|!} \right)^{m_k} \leq C^{|\alpha|} \frac{|\alpha|^{|\alpha|}}{|\alpha|!}. \tag{2.13}
\]

Moreover,

\[
m^\tau + \sum_{k=1}^s m_k |p_k|^{\tau} \leq |\alpha|^\tau + |\alpha|^{|\alpha|-1} \sum_{k=1}^s m_k |p_k| = 2 |\alpha|^\tau
\]

wherefrom

\[
h^{m^\tau} \prod_{k=1}^s m_k^{p_k |p_k|^\tau} \leq C^{m^\tau + \sum_{k=1}^s m_k |p_k|^\tau} \leq C_1^{2|\alpha|^\tau}, \tag{2.14}
\]

for some constant \(C_1 > 0\).
where $C_1 = \max\{h, h'\}$. From (2.13), (2.14) and (2.12) we conclude that there is a constant $C_2 > 0$ such that

$$|\partial^\alpha (f \circ g)(x)| \leq C_2 |\alpha|^{|\tau|\sigma} \sum_{(s,p,m) \in \pi} \frac{m!}{m_1! \ldots m_s!}, \quad x \in K. \quad (2.15)$$

It remains to estimate $\sum \frac{m!}{m_1! \ldots m_s!}$. Note that without loss of generality we may assume that $s = |\alpha|$ (for $s < |\alpha|$ we may put $m_k = 0$ for $s \leq k < |\alpha|$). Since $|p_k| \in \{1, \ldots, |\alpha|\}$ note that we can write $|\alpha| = \sum_{k=1}^{|\alpha|} m_k p_k$, where $m = \sum_{k=1}^{|\alpha|} m_k$. Hence we conclude that the summation in (2.15) can be taken over all $(m_1, \ldots, m_s) \in \mathbb{N}^s$, $s = |\alpha|$, such that $|\alpha| = \sum_{k=1}^{|\alpha|} k m_k$ and $m = \sum_{k=1}^{|\alpha|} m_k$. Therefore,

$$\sum \frac{m!}{m_1! \ldots m_s!} = 2^{m_1+2m_2+\cdots+|\alpha|m_{|\alpha|-1}} = 2^{|\alpha|-1},$$

and the proof is completed. \qed

As an immediate consequence of Theorem 2.2 we conclude the following:

**Corollary 2.1.** Let $U \subseteq \mathbb{R}^d$ be open. Classes $E_{\tau,\sigma}(U)$, $\tau > 0$, $\sigma > 1$, are inverse-closed in $C^\infty(U)$.

Note that the proof of Theorem (2.2) holds even if $\sigma = 1$ and $\tau \geq 1$, so that we recover the well known results on stability under superposition of Gevrey (analytic) classes of functions (see [4, 8, 9, 17]).

### 3. Wave front sets related to classes $E_{\tau,\sigma}$

Let $\tau > 0$, $\sigma > 1$, $\Omega \subseteq K \subset \subset U \subseteq \mathbb{R}^d$, where $\Omega$ and $U$ are open in $\mathbb{R}^d$, $K$ is compact in $\mathbb{R}^d$, in and the closure of $\Omega$ is contained in $K$, $\overline{\Omega} \subseteq K$.

Let $u \in D'(U)$. In [15] we investigated the nature of regularity related to the condition

$$|\hat{u}_N(\xi)| \leq A \frac{h^N N!^{\tau/\sigma}}{\xi^{N\sigma}}, \quad N \in \mathbb{N}, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (3.1)$$

where $\{u_N\}_{N \in \mathbb{N}}$ is bounded sequence in $\mathcal{E}'(U)$ such that $u_N = u$ in $\Omega$ and $A, h$ are some positive constants.

Note that the conditions (3.1) can be replaced by an equivalent set of conditions if instead of $N$ we use another positive, increasing sequence $a_N$ such that $a_N \to \infty$, $N \to \infty$ (cf. [16]). This change of variables
called enumeration, “speeds up” or “slows down” the decay estimates of single members of the corresponding sequences, without changing the asymptotic behavior of the whole sequence when $N \to \infty$. After applying the enumeration $N \to a_N$ we can write again $u_N$ instead of $u_{a_N}$, since we are only interested in the asymptotic behavior.

For example, Stirling’s formula and enumeration $N \to N^\sigma$ applied to (3.1) give an equivalent estimate of the form

$$|\hat{u}_N(\xi)| \leq A_1 h_1^{N^\sigma} \frac{N^{\tau N^\sigma}}{N^N}, \quad N \in \mathbb{N}, \xi \in \mathbb{R}^d \setminus \{0\},$$

for some constants $A_1, h_1 > 0$.

Wave-front sets $\text{WF}_{\{\tau, \sigma\}}(u)$ (see Remark 3.2 for $\text{WF}_{\{\tau, \sigma\}}(u)$) are introduced in [16] in the study of local regularity in $\mathcal{E}_{\{\tau, \sigma\}}(U)$. Together with enumeration we used sequences of cutoff functions in a similar way as it is done in [6] in the context of analytic wave front set $\text{WF}_A$.

We recall the definition of $\text{WF}_{\{\tau, \sigma\}}(u)$.

**Definition 3.1.** Let there be given $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and $(x_0, \xi_0) \in U \times \mathbb{R}^d \setminus \{0\}$. Then $(x_0, \xi_0) \notin \text{WF}_{\{\tau, \sigma\}}(u)$ if there exists an open neighborhood $\Omega$ of $x_0$, a conic neighborhood $\Gamma$ of $\xi_0$ and a bounded sequence $\{u_N\}_{N \in \mathbb{N}}$ in $\mathcal{E}'(U)$ such that $u = u_N$ on $\Omega$ and (3.1) holds for all $\xi \in \Gamma$ and for some constants $A, h > 0$.

For a given $u \in \mathcal{D}'(U)$ it immediately follows that $\text{WF}_{\{\tau, \sigma\}}(u)$ is closed subset of $U \times \mathbb{R}^d \setminus \{0\}$. Note that for $\tau > 0$ and $\sigma > 1$

$$\text{WF}_{\{\tau, \sigma\}}(u) \subseteq \text{WF}_{\{1,1\}}(u) = \text{WF}_A(u), \quad u \in \mathcal{D}'(U),$$

where $\text{WF}_A(u)$ denoted the analytic wave front set of a distribution $u \in \mathcal{D}'(U)$, cf. [6].

Next we prove that in the definition of $\text{WF}_{\{\tau, \sigma\}}(u)$ a bounded sequence of cut-off functions $\{u_N\}_{N \in \mathbb{N}} \in \mathcal{E}'(U)$ can be replaced by a single function from $\mathcal{D}_{\{\tau, \sigma\}}(U)$. First we give an example of $\phi \in \mathcal{D}_{\{\tau, \sigma\}}(U)$ such that $\phi = 1$ on particular open sets.

**Example 3.1.** Let there be given $x_0 \in \mathbb{R}^d$, $\tau > 0$, $\sigma > 1$, and let $d = \sum_{p=1}^{\infty} \frac{1}{(2(p + 1))^{\tau p^\sigma - 1}}$. By Lemma 2.2 and [6, Theorem 1.4.2], there exists $\psi \in \mathcal{D}^{B_{d/2}(x_0)}_{\{\tau, \sigma\}}$ such that $\int \psi(x) \, dx = 1$. If $\chi$ denotes the characteristic function of

$$\{y \in \mathbb{R}^d : |x - y| \leq d/2, x \in B_{d/2}(x_0)\},$$

then $\phi = \chi \ast \psi = 1$ on an open neighborhood $\Omega$ of $B_{d/2}(x_0)$. In particular, if $U$ is an open set such that

$$\inf\{|x - y| : x \in U^c, y \in B_{d/2}(x_0)\} > d$$

then $\phi \in \mathcal{D}_{\{\tau, \sigma\}}(U)$. 

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Remark 3.1. In the sequel we will use the following Paley-Wiener type estimates. If \( u \in \mathcal{E}'(U) \), then \( |\hat{u}(\xi)| \leq C(\xi)^M, \xi \in \mathbb{R}^d \), for some constant \( C > 0 \), where \( M \) denotes the order of distribution \( u \).

Similarly, if \( \phi \in \mathcal{D}^K_{\{\tau,\sigma\}} \), where \( K \) is a compact set in \( \mathbb{R}^d \), then

\[
|\hat{\phi}(\xi)| \leq A h^{\alpha|\alpha|} |\tau| N^{\alpha|\alpha|} \langle \xi \rangle^{-|\alpha|}, \quad \alpha \in \mathbb{N}^d, \xi \in \mathbb{R}^d,
\]

for some constants \( A, h > 0 \).

**Theorem 3.1.** Let \( u \in \mathcal{D}'(U), \tau > 0, \sigma > 1 \), and let \((x_0, \xi_0) \in U \times \mathbb{R}^d \setminus \{0\}\). Then \((x_0, \xi_0) \notin \text{WF}_{\{\tau,\sigma\}}(u)\) if and only if there exists a conic neighborhood \( \Gamma_0 \) of \( \xi_0 \), a compact set \( K \subset \subset U \) and \( \phi \in \mathcal{D}^K_{\{\tau,\sigma\}} \) such that \( \phi = 1 \) on a neighborhood of \( x_0 \), and such that

\[
|\hat{\phi u}(\xi)| \leq A h^{N^{|\alpha|} N^{\alpha|\alpha|}} \langle \xi \rangle^{-|\alpha|}, \quad N \in \mathbb{N}, \xi \in \Gamma_0.
\]

for some \( A, h > 0 \).

**Proof.** The necessity is trivial, since if there is \( \phi \in \mathcal{D}^K_{\{\tau,\sigma\}}, \ K \subset \subset U, \phi = 1 \) on a neighborhood \( \Omega \) of \( x_0 \) and such that (3.4) holds in a conic neighborhood \( \Gamma_0 \) of \( \xi_0 \), then by putting \( u_N = \phi u \), for every \( N \in \mathbb{N} \) it follows that \((x_0, \xi_0) \notin \text{WF}_{\{\tau,\sigma\}}(u)\).

Now assume that \((x_0, \xi_0) \notin \text{WF}_{\{\tau,\sigma\}}(u)\), i.e. that there exists an open neighborhood \( \Omega \) of \( x_0 \), a conic neighborhood \( \Gamma \) of \( \xi_0 \) and a bounded sequence \( \{u_N\}_{N \in \mathbb{N}} \) in \( \mathcal{E}'(U) \) such that \( u_N = u \) on \( \Omega \) and such that

\[
|\hat{u_N}(\xi)| \leq A h^{N^{|\alpha|} N^{\alpha|\alpha|}} \langle \xi \rangle^{-|\alpha|}, \quad N \in \mathbb{N}, \xi \in \Gamma.
\]

(3.5)

Choose \( \phi \in \mathcal{D}^{K_{x_0}}_{\{\tau,\sigma\}}, \ K_{x_0} \subset \subset \Omega, \phi = 1 \) on some neighborhood of \( x_0 \), and choose a conic neighborhood \( \Gamma_0 \) of \( \xi_0 \) with the closure contained in \( \Gamma \). Let \( \varepsilon > 0 \) be chosen so that \( \xi - \eta \in \Gamma \) when \( \xi \in \Gamma_0 \) and \( |\eta| < \varepsilon |\xi| \).

Since \( \hat{\phi u} = \hat{\phi u_N} \),

\[
\hat{\phi u}(\xi) = \left( \int_{|\eta| < \varepsilon |\xi|} + \int_{|\eta| \geq \varepsilon |\xi|} \right) \hat{\phi}(\eta) \hat{u_N}(\xi - \eta) \, d\eta = I_1 + I_2, \quad \xi \in \Gamma_0.
\]
To estimate $I_1$ we use that $|\eta| < \varepsilon|\xi|$ implies $|\xi - \eta| > (1 - \varepsilon)|\xi|$. By (3.5) and $|\hat{\phi}(\eta)| \leq B(\eta)^{-d-1}$ for some $B > 0$, we have

$$|I_1| = \left| \int_{|\eta| < \varepsilon|\xi|} \hat{\phi}(\eta)\hat{u}_N(\xi - \eta) \, d\eta \right|$$

$$\leq \int_{|\eta| < \varepsilon|\xi|} |\hat{\phi}(\eta)| A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi - \eta|^{N^\sigma}} \, d\eta$$

$$\leq AB \frac{h^{N^\sigma} N^{\tau N^\sigma}}{(1 - \varepsilon)|\xi|^N} \int_{\mathbb{R}^d} \langle \eta \rangle^{-d-1} \, d\eta$$

$$\leq A_1 \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad \xi \in \Gamma_0, N \in \mathbb{N}, \quad \text{(3.6)}$$

for some constants $A_1, h_1 > 0$. For the last estimate we have used $(1 - \varepsilon)^{-N} < (1 - \varepsilon)^{-N^\sigma}$ when $\sigma > 1$.

To estimate $I_2$ we use that $|\eta| > \varepsilon|\xi|$ implies $|\xi - \eta| < |\xi| + |\eta| < (1 + 1/\varepsilon)|\eta|$. For a given $N \in \mathbb{N}$, we put $|\alpha| = N + M + d + 1$, where $M > 0$ is the order of distribution $u$. Then by (3.3) there exist constants $A, h > 0$ such that

$$|I_2| = \left| \int_{|\eta| \geq \varepsilon|\xi|} \hat{\phi}(\eta)\hat{u}_N(\xi - \eta) \, d\eta \right|$$

$$\leq A h_1^{N^\sigma} \left( N + M + d + 1 \right)^{\tau(N + M + d + 1)^\sigma}$$

$$\int_{|\eta| \geq \varepsilon|\xi|} \langle \eta \rangle^{-M-d-1} C |\xi - \eta|^M \, d\eta$$

$$\leq A_1 h_1^{N^\sigma} N^{\tau N^\sigma} \quad \xi \in \Gamma_0, N \in \mathbb{N}, \quad \text{(3.7)}$$

where $h_1 = \max\{h, h^{2\sigma-1}\}, A_1 = A \max\{1, h^{2\sigma-1}(M+d+1)\}$.

In the last inequality we used $|\alpha| + |\beta| \leq |\alpha + \beta| \leq 2^{\sigma-1}(|\alpha|^\sigma + |\beta|^\sigma), \quad \alpha, \beta \in \mathbb{N}^d$.

and (M.2)\textsuperscript{7} property of $M_{p, \sigma} = p^{\tau p^\sigma}$.

Thus, (3.4) follows and the theorem is proved. \hfill \Box

Remark 3.2. In the Beurling case, for $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and $(x_0, \xi_0) \in U \times \mathbb{R}^d \setminus \{0\}$ we have that $(x_0, \xi_0) \notin \text{WF}_{(\tau, \sigma)}(u)$ if there exists open neighborhood $\Omega$ of $x_0$, a conic neighborhood $\Gamma$ of $\xi_0$ and a bounded sequence $\{u_N\}_{N \in \mathbb{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on $\Omega$ and such that for every $h > 0$ there exists $A > 0$ such that

$$|\hat{u}_N(\xi)| \leq A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^{N^\sigma}}, \quad N \in \mathbb{N}, \xi \in \Gamma.$$
Note that Theorem 3.1 can be formulated for the Beurling case as well with \( \phi \in \mathcal{D}_K \) such that (3.4) holds for every \( h > 0 \) and for some \( A = A(h) > 0 \). More precisely, for any \( h > 0 \) we can choose \( \phi \in \mathcal{D}_{(\tau,\sigma, C_h)} \) where \( C_h = \min\{h, h^\frac{\tau}{\tau-1}\} \) and obtain \( \phi \in \mathcal{D}_K \) with the desired properties.

Thus the results concerning \( \text{WF}_{(\tau,\sigma)}(u) \) are analogous to those for \( \text{WF}_{\{\tau,\sigma\}}(u) \), and we will consider only the later wave-front sets in the sequel.

We end this section an auxiliary result which will be used in the proof of Theorem 4.1.

**Lemma 3.1.** Let \( u \in \mathcal{D}'(U) \), \( \tau > 0 \), \( \sigma > 1 \), \( \Omega \subset K \subset U \), where \( U \) and \( \Omega \) are open. If \( F \) is a closed cone such that \( \text{WF}_{\{\tau,\sigma\}}(u) \cap (K \times F) = \emptyset \) and \( \phi \in \mathcal{D}_{\{\tau,\sigma\}} \), \( \phi = 1 \) on \( \Omega \), then for some \( A, h > 0 \) it holds

\[
|\hat{\phi} u(\xi)| \leq A \frac{h^N \sigma^N \sigma^{N^\sigma}}{|\xi|^N}, \quad N \in \mathbb{N}, \xi \in F.
\]  

(3.8)

**Proof.** Let \((x_0, \xi_0) \in K \times F\), and set \( r_0 := r_{x_0, \xi_0} > 0 \). Furthermore, let \( \phi \in \mathcal{D}_{\{\tau,\sigma\}}(B_{r_0}(x_0)) \), \( B_{r_0}(x_0) \subseteq \Omega \subseteq K \).

Since \((x_0, \xi_0) \notin \text{WF}_{\{\tau,\sigma\}}(u)\) by Theorem 3.1 there exists \( \psi \in \mathcal{D}_{\{\tau,\sigma\}}(U) \), \( \psi = 1 \) on \( \Omega \), and a conical neighborhood \( \Gamma \) of \( \xi_0 \), such that

\[
|\hat{\psi} u(\xi)| \leq A \frac{h^N \sigma^N \sigma^{N^}\sigma}{|\xi|^N}, \quad N \in \mathbb{N}, \xi \in \Gamma,
\]  

(3.9)

for some \( A, h > 0 \).

Let \( \Gamma_0 \) be an open conical neighborhood of \( \xi_0 \) with the closure contained in \( \Gamma \). We write

\[
\hat{\phi} u(\xi) = \left( \int_{|\eta| < \varepsilon |\xi|} + \int_{|\eta| \geq \varepsilon |\xi|} \right) \hat{\phi}(\eta) \hat{\psi} u(\xi - \eta) d\eta = I_1 + I_2, \quad \xi \in \Gamma_0,
\]

and arguing in a similar way as in the proof of Theorem 3.1 we obtain (3.8) for \((x, \xi) \in B_{r_0}(x_0) \times \Gamma_0\).

In order to extend the result to \( K \times F \) we use the same idea as in the proof of [6, Lemma 8.4.4]. Since the intersection of \( F \) with the unit sphere is a compact set, there exists a finite number \( n \) of balls \( B_{r_{x_0, \xi_j}}(x_0) \), such that \( F \subset \bigcup_{j=1}^n \Gamma_j \). Note that (3.8) remains true if \( \phi \) is chosen so that \( \text{supp} \phi \subseteq B_{r_{x_0}} := \bigcap_{j=1}^n B_{r_{x_0, \xi_j}}(x_0) \), \( \xi_j \in \Gamma_j \).

Moreover, since \( K \) is compact set, it can be covered by a finite number \( m \) of balls \( B_{r_k} \), \( k \leq m \). Since \( M_{p,\sigma} = \sigma^\sigma \) satisfies (M.1) and (M.3)', then there exist non-negative functions \( \phi_k \in \mathcal{D}_{\{\tau,\sigma\}}(B_{r_k}) \), \( k \leq n \), such that \( \sum_{k=1}^n \phi_k = 1 \) on a neighborhood of \( K \) (cf. [10, Lemma 5.1]).
To conclude the proof we note that if $\phi \in D_{\{\tau,\sigma\}}^K$ then $\phi \phi_k \in D_{\{\tau,\sigma\}}(B_{r_k})$ and consequently (3.8) holds if we replace $\phi$ by $\phi \phi_k$. Since $\sum_{k=1}^{n} \phi \phi_k = \phi$, the proof is finished. □

4. Main result

We first recall the definition of the characteristic set of an operator and the main property of its principal symbol, cf. [18].

If $P(x, D) = \sum_{|\alpha| \leq m} a_{\alpha}(x)D^\alpha$ is a differential operator of order $m$ on $U$ and $a_{\alpha} \in C^\infty(U)$, $|\alpha| \leq m$, then its characteristic variety at $x \in U$ is given by

$$Char_x(P) = \{(x, \xi) \in U \times \mathbb{R}^d \setminus \{0\} \mid P_m(x, \xi) = 0\},$$

and its characteristic set on $U$ is given by

$$Char(P) = \bigcup_{x \in U} Char_x(P).$$

Here $P_m(x, \xi) = \sum_{|\alpha| = m} a_{\alpha}(x)\xi^\alpha \in C^\infty(U \times \mathbb{R}^d \setminus \{0\})$ is the principal symbol of $P(x, D)$.

By the homogeneity of the principal symbol it follows that $Char(P)$ is a closed conical subset of $U \times \mathbb{R}^d \setminus \{0\}$.

If $(x_0, \xi_0) \not\in Char(P)$ then there exists an open neighborhood $\Omega$ of $x_0$ and a conical neighborhood $\Gamma$ of $\xi_0$ such that $P_m(x, \xi) \neq 0$, $x \in \Omega$ and $\xi \in \Gamma$. Moreover, since the principal symbol is homogeneous we have

$$\left| P_m(x, \frac{\xi}{|\xi|}) \right| = \frac{1}{|\xi|^m} |P_m(x, \xi)| \geq C, \quad x \in \Omega, \xi \in \Gamma,$$

so that for any compact set $K \subset \subset \Omega$ there are constants $0 < C_1 < C_2$ such that

$$C_1 |\xi|^m \leq |P_m(x, \xi)| \leq C_2 |\xi|^m, \quad x \in K, \xi \in \Gamma.$$

The main result of this section, Theorem 4.1 extends [16, Theorem 1.1] to operators with variable coefficients. We recall that in [6, Theorem 8.6.1] operators with real analytic coefficients are observed, while in Theorem 4.1 we allow the extended Gevrey regularity of the coefficient. In particular, by the inspection of the proof, we conclude that Theorem 4.1 remains valid even if $\sigma = 1$ and $\tau > 1$, that is, if the coefficients are Gevrey regular. In that sense Theorem 4.1 extends [6, Theorem 8.6.1] as well.

**Theorem 4.1.** Let there be given $\tau > 0$, $\sigma > 1$, $u \in \mathcal{D}'(U)$ and let $P(x, D) = \sum_{|\alpha| \leq m} a_{\alpha}(x)D^\alpha$ be partial differential operator of order $m$
such that $a_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $|\alpha| \leq m$. Then

$$\WF_{2\sigma^{-1},\tau}(f) \subseteq \WF_{2\sigma^{-1},\tau}(u) \subseteq \WF_{\{\tau,\sigma\}}(f) \cup \Char(P(x,D)),$$

where $P(x,D)u = f$ in $\mathcal{D}'(U)$. In particular,

$$\WF_{0,\infty}(f) \subseteq \WF_{0,\infty}(u) \subseteq \WF_{0,\infty}(f) \cup \Char(P(x,D)),$$

where $\WF_{0,\infty}(u) = \bigcup_{\sigma > 1} \bigcap_{\tau > 0} \WF_{\{\tau,\sigma\}}(u)$.

**Proof.** The pseudolocal property $\WF_{2\sigma^{-1},\tau}(f) \subseteq \WF_{2\sigma^{-1},\tau}(u)$ is proved in in [22], see also [16], so it remains to prove the second inclusion in (4.1).

Assume that $(x_0,\xi_0) \notin \WF_{\{\tau,\sigma\}}(f) \cup \Char(P(x,D))$. Then there exists a compact set $K$ containing $x_0$ and a closed cone $\Gamma$ containing $\xi_0$ such that $P_m(x,\xi) \neq 0$ when $(x,\xi) \in K \times \Gamma$ and such that

$$(K \times \Gamma) \cap \left( \WF_{\{\tau,\sigma\}}(f) \cup \Char(P(x,D)) \right) = \emptyset.$$

Since $K$ is fixed, the distributions $u$ and $f$ involved in the proof are of finite order denoted by the same letter $M$ for the sake of simplicity.

Let $\phi \in \mathcal{D}_K^{\{\tau,\sigma\}}$ such that $\phi = 1$ on some neighborhood of $x_0$. By Theorem 3.1 it is enough to prove that

$$|\widehat{\phi u}(\xi)| \leq A \frac{h^N N^{2\sigma-1} N^\sigma}{|\xi|^N}, \quad \xi \in \Gamma, \ N \in \mathbb{N}.$$

We divide the proof in several steps.

**Step 1.** Since $u$ is of order $M$, Paley-Wiener type estimate (see Remark 3.1) implies

$$|\xi|^N |\widehat{\phi u}(\xi)| \leq A (N^{2\sigma-1} N^{\sigma-1})^N (N^{2\sigma-1} N^{\sigma-1})^M \leq A h^N N^{2\sigma-1} N^\sigma, \quad N \in \mathbb{N},$$

where $A, h > 0$ do not depend on $N$, and the last inequality follows from $M 2^{\sigma-1} N^{\sigma-1} \ln N \leq M 2^{\sigma-1} N^\sigma$ after taking the exponentials.

This gives the desired estimate when $|\xi| \leq N^{2\sigma-1} N^{\sigma-1}$, $\xi \in \Gamma$.

**Step 2.** It remains to estimate $|\widehat{\phi u}(\xi)|$ when $\xi \in \Gamma$, $|\xi| > N^{2\sigma-1} N^{\sigma-1}$ and for $N \in \mathbb{N}$ large enough. We refer to Subsection 4.1 for calculations which lead to

$$\phi(x) = e^{ix} P^\tau(x,D) \left( \frac{e^{-ix}}{P_m(x,\xi)} w_N(x,\xi) \right) + e_N(x,\xi), \quad x \in K, \ \xi \in \Gamma,$$

where

$$w_N(x,\xi) = \sum_{k \in K_1} \sum_{\Theta_k = 0}^{N-m} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi),$$

and

$$e_N(x,\xi) = \sum_{k \in K_2} \sum_{\Theta_k = N-m+1}^{N} (R_{j_1} R_{j_2} \ldots R_{j_k} \phi)(x,\xi),$$

for $x \in K, \ \xi \in \Gamma$. 

\[ S_k = j_1 + j_2 + \cdots + j_k, \quad j_i \in \{1, \ldots, m\}, \quad 1 \leq i \leq k, \text{ and we put} \]
\[ K_1 = \{ k \in \mathbb{N} \mid 0 \leq mk \leq N - m \}, \quad (4.6) \]

and
\[ K_2 = \{ k \in \mathbb{N} \mid N - m < mk \leq N \}. \quad (4.7) \]

The functions \( R_j \) in (4.4) and (4.5) can be written as
\[ R_j(x, \xi) = \sum_{|\alpha| \leq j} c_{\alpha,j}(x, \xi) D^\alpha, \quad (4.8) \]

for suitable functions \( c_{\alpha,j}(x, \xi) \) which are homogeneous of order \(-j\) (with respect to \(\xi\)) and such that
\[ |D^\beta c_{\alpha,j}(x, \xi)| \leq |\xi|^{-j} Ah^{|\beta|} |\beta|^{|\beta|}, \quad \beta \in \mathbb{N}^d, x \in K, \xi \in \Gamma \]
for some \( A, h > 0 \) and for all \(|\alpha| \leq j\), see Subsections 4.1 and 4.2.

From (4.3) it follows that
\[ \hat{\mu}(\xi) = \int u(x)e_N(x, \xi)e^{-ix\cdot\xi}dx + \int u(x)P_T(x, D) \left( \frac{e^{-ix\cdot\xi}w_N(x, \xi)}{P_m(x, \xi)} \right) dx \]
\[ = \int u(x)e_N(x, \xi)e^{-ix\cdot\xi}dx + \int P(x, D)u(x) \left( \frac{e^{-ix\cdot\xi}w_N(x, \xi)}{P_m(x, \xi)} \right) dx, \quad (4.9) \]

\( x \in K, \xi \in \Gamma, \) and in the next steps we estimate terms on the right hand side of (4.9).

**Remark 4.1.** Since the number of summands in \( w_N(x, \xi) \) and \( e_N(x, \xi) \) is the same as in the case when \( R_j \) have constant coefficients we refer to [16, Subsection 4.1] where it is shown that the upper bound for the number of summands is of the form \( A \cdot C \cdot N^{\sigma} \) for suitable constants \( A, C > 0 \). In fact, from [16, Subsection 4.1] it follows that the number of summands in
\[ e_N(x, \xi) = \sum_{k \in K_2} \sum_{\Theta_k = \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} (R_{j_1}R_{j_2} \ldots R_{j_k} \phi)(x, \xi) \]
is bounded by \( A \cdot C \cdot \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor \) and the calculations remain the same after replacing \( \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor \) with \( N \).

**Step 3.** Note that the operators \( R_j, 1 \leq j \leq m \), given in (4.8) do not commute. For that reason we must use different arguments than those given in [16] where the operators with constant coefficients were studied. If \( M \) denotes the order of distribution \( u \), then the estimates of \( D^\beta(R_{j_1} \ldots R_{j_k} \phi) \) from Subsection 4.3 (cf. (4.29)) imply
\[ |\langle u(x), e_N(x, \xi)e^{-ix\cdot\xi} \rangle| \leq A \sum_{|\alpha| \leq M} |D^\alpha(e_N(x, \xi)e^{-ix\cdot\xi})| \quad (4.10) \]
\[
\leq A'|\xi|^M|\xi'|^{m-N}|h^{N^\sigma}(N+M)^{(N+M)^\sigma} = A'\frac{h^{N^\sigma}(N+M)^{(N+M)^\sigma}}{|\xi|^{N-m-M}},
\]
x \in K, \xi \in \Gamma, for suitable constants $A', h > 0$, and $N \in \mathbb{N}$ large enough. After enumeration $N \to N + m + M$ we conclude that (4.10) is equivalent to

\[
|\langle u(x), e_N(x, \xi)e^{-ix\cdot\xi}\rangle| \leq A'\frac{h^{(N+m+N)^\sigma}(N + m + 2M)^{(N + m + 2M)^\sigma}}{|\xi|^N}, \quad x \in K, \xi \in \Gamma,
\]

where for the last inequality we used (M.2) of the sequence $M^{r,s}_p = p^{r,s}$. This is the estimate for the first term on the righthand side of (4.9).

**Step 4.** To estimate the second term on the righthand side of (4.9) for $|\xi| > N^{2-1+N^{-1}}$, note that since $(x_0, \xi_0) \notin \text{WF}_{\{r,s\}}(f)$, by Lemma 3.1, there exists a compact set $\tilde{K} \subset \subset U$ such that $\psi \in \mathcal{D}_{\{r,s\}}(U), \psi = 1$ on a neighborhood of $\tilde{K}$, and a conical neighborhood $V$ of $\xi_0$ such that $\Gamma \subset V$ and

\[
|\mathcal{F}(\psi f)(\eta)| \leq A\frac{h^{N^\sigma}N^{rN^\sigma}}{|\xi|^N}, \quad \eta \in V, N \in \mathbb{N}, \quad (4.11)
\]

for some $A, h > 0$. In the sequel we write $v = \psi f$. Since $w_Nf = w_Nv$ in $\mathcal{D}'(U)$, we have

\[
\langle f(\cdot)e^{-ix\cdot\xi}, w_N(\cdot, \xi)/P_m(\cdot, \xi) \rangle = \mathcal{F}_{x \to \eta}(v(x)\frac{w_N(x, \xi)}{P_m(x, \xi)})(\xi)
\]

\[
= \int_{\mathbb{R}^d} \mathcal{F}(v)(\xi - \eta)\mathcal{F}_{x \to \eta}(\frac{w_N(x, \xi)}{P_m(x, \xi)})(\eta, \xi) \, d\eta = I_1 + I_2,
\]

where

\[
I_1 = \int_{|\eta| < \epsilon|\xi|} \mathcal{F}(v)(\xi - \eta)\mathcal{F}_{x \to \eta}(\frac{w_N(x, \xi)}{P_m(x, \xi)})(\eta, \xi) \, d\eta,
\]

\[
I_2 = \int_{|\eta| \geq \epsilon|\xi|} \mathcal{F}(v)(\xi - \eta)\mathcal{F}_{x \to \eta}(\frac{w_N(x, \xi)}{P_m(x, \xi)})(\eta, \xi) \, d\eta;
\]

and $0 < \epsilon < 1$ is chosen so that $\xi - \eta \in V$ when $\xi \in \Gamma$, $|\xi| > N^{2-1+N^{-1}}$, and $|\eta| < \epsilon|\xi|$.

**Step 5.** Let $j_1, \ldots, j_k \in \{1, \ldots, m\}$ be fixed. Since the coefficients of $P_m(\cdot, \xi)$ are in $C^\infty(U)$, and $P_m(x, \xi) \neq 0$ when $x \in K$ and $\xi \in \Gamma$, it follows that $R_{j_1}R_{j_2}\ldots R_{j_k}\phi(\cdot, \xi)/P_m(\cdot, \xi)$ belongs to $C^\infty(K)$ when $\xi \in \Gamma$, and moreover it is homogeneous of order $-m - \mathcal{G}_k$. Hence, by Paley-Wiener type estimates it follows that there exist a constant $C > 0$, such that
we obtain the desired estimate for $I$ when $\xi > N^{2^{\sigma-1} \tau N^{\sigma-1}}$.

This estimate, and the estimate for number of terms in (4.4) (see remark 4.1) imply that there exist constants $A, C > 0$ such that

$$|F_{x \rightarrow \eta}(\frac{R_{j_1}R_{j_2} \ldots R_{j_k} \phi(x, \xi)}{P_m(x, \xi)})(\eta, \xi)| \leq C|\xi|^{-m-\Theta_k}(\eta)^{-d-1}$$

$$\leq C(\eta)^{-d-1}, \quad \eta \in \mathbb{R}^d,$$

when $\xi \in \Gamma, |\xi| > N^{2^{\sigma-1} \tau N^{\sigma-1}}$.

Step 6. It remains to estimate $I_2$. We note that $|\xi - \eta| \leq (1 + 1/\varepsilon)|\eta|$ when $|\eta| \geq \varepsilon|\xi|$. Since $f$ is a distribution of order $M$, the Paley-Wiener type estimate for $v = \psi f \in \mathcal{S}(U)$ implies that $|\mathcal{F}(v)(\eta)| \leq C(\eta)^M$, for some constant $C > 0$. Therefore

$$|I_2| \leq \int_{|\eta| < |\xi|} |\mathcal{F}(v)(\xi - \eta)||F_{x \rightarrow \eta}(\frac{w_N(x, \xi)}{P_m(x, \xi)})(\eta, \xi)| d\eta$$

$$\leq \int_{|\eta| < |\xi|} A \frac{h^{N^{\sigma} N^{\tau N^{\sigma}}}}{|\xi - \eta|^N}|F_{x \rightarrow \eta}(\frac{w_N(x, \xi)}{P_m(x, \xi)})(\eta, \xi)| d\eta$$

$$\leq A \frac{h^{N^{\sigma} N^{\tau N^{\sigma}}}}{(1 - \varepsilon)|\xi|)^N} \int_{\mathbb{R}^d} C^N(\eta)^{-d-1} d\eta$$

$$\leq A \frac{h^{N^{\sigma} N^{\tau N^{\sigma}}}}{|\xi|^N}, \quad \xi \in \Gamma, |\xi| > N^{2^{\sigma-1} \tau N^{\sigma-1}}.$$
for a sufficiently large $N \in N$, and then we use this estimate to bound $|I_2|$. 

Arguing in the similar way as in the proof of [16, Theorem 1.1], it is sufficient to prove

$$\sup_{x \in K} |D^\gamma \left( \frac{w_N(x, \xi)}{P_m(x, \xi)} \right) | \leq Ah^{N_\sigma} N^\sigma, \quad \beta \in N^d, |\beta| = N + M + d + 1,$$

for some constants $A, h > 0$, when $\xi \in \Gamma, |\xi| > N^\sigma$. Recall (see Subsection 4.2),

$$\sup_{x \in K} \left| D^\gamma \frac{1}{P_m(x, \xi)} \right| \leq |\xi|^{-m} C |\gamma|^{\sigma + 1} |\tau|^{\gamma}, \quad \gamma \in N^d, \xi \in \Gamma,$n

for some constant $C > 0$. Moreover, from (4.29) (see Subsection 4.3) it follows that

$$\sup_{x \in K} |D^\gamma w_N(x, \xi)| \leq Ah^{N_\sigma} \sum_{k \in K_1} \sum_{\xi = 0}^{N-m} \left| \xi \right|^{-\xi_k} (G_k + |\xi|)^{\sigma} (G_k + |\xi|)^{\sigma},$$n

for some constants $A, h > 0$, when $\xi \in \Gamma$. Therefore, for $x \in K$ and $\xi \in \Gamma, |\xi| > N^\sigma$, we obtain

$$\left| D^\gamma \frac{w_N(x, \xi)}{P_m(x, \xi)} \right| \leq \sum_{k \in K_1} \sum_{\xi \in \Gamma} \left( \beta \right) \left( \gamma \right) \left| D^\gamma \frac{1}{P_m(x, \xi)} \right| |D^\gamma w_N(x, \xi)|$$

$$\leq Ah^{N_\sigma} \sum_{\gamma \leq \beta} \sum_{k \in K_1} \sum_{\xi = 0}^{N-m} \left| \xi \right|^{-\xi_k} (G_k + |\xi|)^{\sigma} (G_k + |\xi|)^{\sigma},$$

for $\beta \in N^d, |\beta| = N + M + d + 1$, where we used $(M.1)$ property of the sequence $M^{\sigma, \beta} = p^{\rho^\sigma}$.

Since $G_k \leq k$ it follows that $N > G_k$ and therefore

$$|\xi| > N^\sigma - 1, N^\sigma - 1 > G_k^{2^\sigma - 1} G_k^{\sigma - 1}.$$n

Now $(M.2)$ property of $M^{\sigma, \beta} = p^{\rho^\sigma}$ implies

$$|\xi|^{-\xi_k} (G_k + |\beta|)^{\sigma} \leq \frac{(G_k + |\xi|)^{\sigma}}{G_k^{2^\sigma - 1} G_k^{\sigma - 1}}$$

$$\leq C G_k^{\sigma + |\beta|} \frac{G_k^{2^\sigma - 1} G_k^{\sigma - 1} |\beta|^{2^\sigma - 1} |\beta|^{\sigma}}{G_k^{2^\sigma - 1} G_k^{\sigma - 1}}$$

$$= C G_k^{\sigma + |\beta|} (N + M + d + 1)^{2^\sigma - 1} (N + M + d + 1) \leq C_1 N^\sigma N^{2^{\sigma - 1} N^\sigma},$$

(4.16)
for some constant $C_1 > 0$ where the last inequality follows from $(M.2)'$ property of $M_{f,\sigma}^\tau$. Using the estimate for number of terms in $w_N$, by (4.15) and (4.16), the estimate (4.14) follows.

By the similar arguments as in the proof of [16, Theorem 1.1], (4.13) follows from (4.14) since

$$\pi_1(\text{supp} \frac{w_N(x, \xi)}{P_m(x, \xi)}) \subseteq K.$$ 

Therefore

$$|I_2| \leq A \frac{h^{N^\tau} N^{2^\sigma-1} \tau N^\sigma}{|\xi|^N}$$

(4.17)

for suitable constants $A, h > 0$ and $N$ sufficiently large, and the theorem is proved. \qed

4.1. Representing $\hat{\phi}u(\xi)$ by an approximate solution. In this subsection we derive (4.3), (4.4) and (4.5).

Let $P^T(x, D) = \sum_{|\alpha| \leq m} b_\alpha(x)D^\alpha$, $b_\alpha(x) \in E_{\{\tau, \sigma\}}(U)$ be the transpose of $P(x, D)$. If $v(x, \xi)$ is the solution of the equation

$$e^{ix\xi} P^T(x, D)v(x, \xi) = \phi(x), \quad x \in K, \xi \in \Gamma,$$

(4.18)

then

$$\hat{\phi}u(\xi) = \int u(x)\phi(x)e^{-ix\xi} dx = \int u(x) P^T(x, D)v(x, \xi) dx, \quad \xi \in \Gamma.$$ 

Similarly as in [6] and [15] we may assume that $v(x, \xi) = \frac{e^{-ix\xi} w(x, \xi)}{P_m(x, \xi)}$, for some $w(\cdot, \xi) \in C^\infty(K)$, so that the left hand side of (4.18) becomes

$$e^{ix\xi} P^T(x, D)(\frac{w(x, \xi)e^{-ix\xi}}{P_m(x, \xi)}) = e^{ix\xi} \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) b_\alpha(x)D^{\alpha-\beta}(e^{-ix\xi})D^\beta\left( \frac{w(x, \xi)}{P_m(x, \xi)} \right)$$

$$= \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} \left( \frac{\alpha}{\beta} \frac{\beta}{\gamma} \right) b_\alpha(x)(-\xi)^{\alpha-\beta} D^\gamma\left( \frac{1}{P_m(x, \xi)} \right) D^{\beta-\gamma}w(x, \xi),$$

(4.19)

$$= (I - R(x, \xi))w(x, \xi), \quad x \in K, \xi \in \Gamma,$$

where

$$R(x, \xi) = \sum_{j=1}^m R_j(x, \xi), \quad R_j(x, \xi) = \sum_{|\alpha| \leq j} c_{\alpha,j}(x, \xi)D^\alpha,$$

for suitable functions $c_{\alpha,j}(x, \xi)$ which are homogeneous of order $-j$ and

$$|D^\beta c_{\alpha,j}(x, \xi)| \leq |\xi|^{-j}A_k|\beta|^{|\beta|\sigma}, \quad \beta \in \mathbb{N}^d, x \in K, \xi \in \Gamma$$

(21)
for some $A, h > 0$ and for all $|\alpha| \leq j$. We refer to Subsection 4.2 for the calculus which shows how (4.19) implies (4.20).

Therefore (4.18) can be rewritten in the following convenient form:

$$ (I - R(x, \xi))w(x, \xi) = \phi(x) \quad x \in K, \xi \in \Gamma. \tag{4.21} $$

which gives rise to approximate solutions as follows.

Note that the order of operator $R_k, k \in \mathbb{N}$, is $mk$. We compute

$$ \sum_{k \in \mathcal{K}_1} R^k - R \sum_{k \in \mathcal{K}_1} R^k = \sum_{k \in \mathcal{K}_1} R^k - \sum_{k \in \mathcal{K}_2} R^k = I - \sum_{k \in \mathcal{K}_2} R^k \tag{4.22} $$

where $\mathcal{K}_1$ is given by (4.6) and in the last equality we used

$$ \mathcal{K}_1 \cap \{k \in \mathbb{N} \mid m \leq mk \leq N\} = \{k \in \mathbb{N} \mid m \leq mk \leq N - m\}. $$

Moreover, since the operators $R_j, 1 \leq j \leq m$, do not commute we can write

$$ \sum_{k \in \mathcal{K}_1} R_k = \sum_{k \in \mathcal{K}_1 \backslash \mathcal{G}_k} \sum_{s = 1}^{N-m} R_{j_{1}}R_{j_{2}} \ldots R_{j_{k}}, $$

and

$$ \sum_{k \in \mathcal{K}_2} R_k = \sum_{k \in \mathcal{K}_2 \backslash \mathcal{G}_k} \sum_{s = N-m+1}^{N} R_{j_{1}}R_{j_{2}} \ldots R_{j_{k}} $$

where $\mathcal{G}_k = j_1 + j_2 + \cdots + j_k, j_i \in \{1, \ldots, m\}, 1 \leq i \leq k$.

Now,

$$ (I - R(x, \xi)) \left( \sum_{k \in \mathcal{K}_1} R_k \phi(x) \right) = \left( \sum_{k \in \mathcal{K}_1} R_k - R \sum_{k \in \mathcal{K}_1} R^k \right) \phi(x) $$

$$ = (I - \sum_{k \in \mathcal{K}_2} R_k) \phi(x) = \phi(x) - \sum_{k \in \mathcal{K}_2} R_k \phi(x), $$

and if we put $w_N = \sum_{k \in \mathcal{K}_1} R_k \phi$ and $e_N = \sum_{k \in \mathcal{K}_2} \phi$ we conclude that

$$ (I - R)w_N(x, \xi) = \phi(x) - e_N(x, \xi), \quad N \in \mathbb{N}, x \in K, \xi \in \Gamma, $$

with $w_N$ and $e_N$ given by (4.4) and (4.5) respectively.

### 4.2. Estimates for $c_{\alpha,j}(x, \xi)$.

In this subsection we show that (4.19) implies (4.20). An essential argument in this part of the proof is the inverse-closedness property presented in Theorem 2.1.

Recall,

$$ D^\alpha \left( \frac{1}{P_m(x, \xi)} \right) = \alpha! \sum_{(s, p, j) \in \pi} \frac{(-1)^j j!}{(P_m(x, \xi))^{j+1}} \prod_{k=1}^{s} \frac{1}{j_k!} \left( \frac{1}{p_k!} D^{p_k} P_m(x, \xi) \right)^{j_k}, \tag{4.23} $$
for \( \alpha \in \mathbb{N}^d \), where sum is taken over all decompositions \( (s, p, j) \) of the form
\[
\alpha = j_1p_1 + j_2p_2 + \cdots + j_sp_s,
\]
with \( j = \sum_{i=1}^{s} j_i \in \{0, 1, \ldots, |\alpha|\} \), \( p_i \in \mathbb{N}^d \), \( |p_i| \in \{1, \ldots, |\alpha|\} \) for \( i \in \{1, \ldots, s\} \), \( s \leq |\alpha| \). (see Subsection 2.1)

Since the coefficients of \( P_m(x, \xi) \) belong to \( \mathcal{E}_{\{\tau, \sigma\}}(U) \) it follows that
\[
\sup_{x \in K} |D^{p_k} P_m(x, \xi)| \leq Ah|p_k|^\tau|\xi|^m, \quad (4.24)
\]
for some \( A, h > 0 \). Moreover, from \((K \times \Gamma) \cap \text{Char}(P) = \emptyset\) it follows that
\[
\sup_{x \in K} |P_m(x, \xi)| \geq C'|\xi|^m. \quad (4.25)
\]

Hence, by using (4.23), (4.24) and (4.25) we obtain
\[
|D^{\alpha} \left( \frac{1}{P_m(x, \xi)} \right) | \leq |\alpha|! \sum_{(s, p, j) \in \pi} \frac{j_1! \cdots j_s!}{j_1! \cdots j_s!} |P_m(x, \xi)|^{j+1}
\]
\[
\times \prod_{k=1}^{s} \left( \frac{1}{p_k!} |D^{p_k} P_m(x, \xi)| \right)^{j_k}
\]
\[
\leq |\alpha|! \sum_{(s, p, j) \in \pi} \left| \frac{1}{j_1! \cdots j_s!} |p_k|^{j_k} \right| |\xi|^{m(j+1)}
\]
\[
\times \prod_{k=1}^{s} \left( \frac{1}{p_k!} A h|p_k|^\tau|\xi|^m \right)^{j_k}
\]
\[
\leq |\xi|^{-m} Ah' |\alpha|^{\tau|\alpha|},
\]
for some \( A, A', h, h' > 0 \), where the last inequality follows by calculation from the proof of Theorem 2.1.

In particular, we have proved that \( \frac{1}{P_m(\cdot, \xi)} \in \mathcal{E}_{\{\tau, \sigma, h\}}(K) \) for some \( h > 0 \) and for every \( \xi \in \Gamma \). From the algebra property of extended Gevrey classes it follows that \( b_\alpha(\cdot) \partial^\gamma \frac{1}{P_m(\cdot, \xi)} \in \mathcal{E}_{\{\tau, \sigma, h'\}}(K) \) for some \( h' > 0 \), where \( |\gamma| \leq |\alpha| \leq m \) and \( b_\alpha(x) \) are the coefficients of \( P^T(x, D) \).

These estimates, together with (4.19) give (4.20).

4.3. Estimates for \( D^\beta(R_{j_1} \ldots R_{j_k} \phi) \). In this subsection we follow the idea presented in [6, Lemmas 8.6.2 and 8.6.3]. As in Subsection 4.1 we put
\[
\mathcal{G}_k = j_1 + \cdots + j_k, \quad N - m \leq \mathcal{G}_k \leq N,
\]
for \( k \in \mathbb{N} \) such that \( mk \leq N \), and let \( |\beta| \leq M \) where \( M \) is order of distribution \( u \).
Recall, \( R_j(x, \xi) = \sum_{|\alpha| \leq j} c_{\alpha,j}(x, \xi)D^\alpha \), and note that by successive applications of the Leibniz rule \( D^\beta(R_j \ldots R_k \phi) \) can be written as a sum of terms of the form

\[
(D^{\gamma_0}c_{\alpha_1,j_1}(x, \xi))(D^{\gamma_1}c_{\alpha_2,j_2}(x, \xi)) \ldots (D^{\gamma_{k-1}}c_{\alpha_{k-1},j_{k-1}}(x, \xi))(D^{\gamma_k} \phi(x)).
\]

Put \( a_i = |\gamma_i| \) so that

\[
a_0 + \cdots + a_k = \mathfrak{S}_k + |\beta|, \quad (4.26)
\]

\[
a_0 \leq |\beta|, \quad (4.27)
\]

and

\[
a_i \leq \sum_{t=1}^{i} j_t + |\beta|, \quad 1 \leq i \leq k. \quad (4.28)
\]

From (4.20) it follows that

\[
|D^{\gamma_{i-1}}c_{\alpha_{i-1},j_{i-1}}(x, \xi)| \leq |\xi|^{-j_i} A h^p a_{i-1}^\sigma, \quad \gamma_{i-1} \in \mathbb{N}^d, x \in K, \xi \in \Gamma,
\]

for some constants \( A, h > 0 \) and for all \( |\alpha_i| \leq j_i, i = 1, \ldots, k \).

Observe that the number of multiindices \( \gamma_0, \ldots, \gamma_k \) with the property (4.26) is \( \begin{pmatrix} \mathfrak{S}_k + |\beta| \\ a_0, \ldots, a_k \end{pmatrix} \). In the sequel we write \( \sum \) when the sum is taken over all multiindices \( \gamma_0, \ldots, \gamma_k \) which satisfies (4.26)-(4.28).

Since \( \phi \in D^K_{\tau, \sigma}, \) for \( x \in K \) and \( \xi \in \Gamma, \) we estimate

\[
|D^\beta R_j \ldots R_k \phi(x, \xi)| \leq \sum \left( \mathfrak{S}_k + |\beta| \right) \left( \prod_{i=1}^{k} |D^{\gamma_{i-1}}c_{\alpha_{i-1},j_{i-1}}(x, \xi)| \right) \cdot |D^{\gamma_k} \phi(x)|
\]

\[
\leq |\xi|^{-\mathfrak{S}_k} \sum \left( \mathfrak{S}_k + |\beta| \right) \left( \prod_{i=1}^{k} Ah^{p a_{i-1}^\sigma} \right) \cdot \left( Ah^{p a_k^\sigma} \right)
\]

\[
\leq |\xi|^{m-N A^p h^{-p} a_0 \cdots a_k} \sum \left( \mathfrak{S}_k + |\beta| \right) \left( \prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} \right),
\]

for some \( A', h' > 0 \). By the almost increasing property of \( M_p^\tau = p^\tau \), it follows that

\[
\prod_{i=1}^{k+1} a_{i-1}^{\tau a_{i-1}^\sigma} \leq C^{a_0 + \cdots + a_k} a_0! \cdots a_k! (a_0 + \cdots + a_k)^{p a_0 \cdots a_k}
\]

\[
= \frac{a_0! \cdots a_k!}{(\mathfrak{S}_k + |\beta|)!} (\mathfrak{S}_k + |\beta|)^{p (\mathfrak{S}_k + |\beta|)^\sigma},
\]

\[24\]
for some $C > 0$, wherefrom
\[\sum_{k=0}^{\infty} \left( \sum_{\beta} \left( \mathcal{S}_k + |\beta| \right) \left( \prod_{i=1}^{k+1} a_{i-1}! \right) \right) \leq \prod_{i=1}^{k+1} a_{i-1}! \sum_{k=0}^{\infty} \frac{a_0! \cdots a_k!}{(\mathcal{S}_k + |\beta|)!} \cdot C^{\mathcal{S}_k + |\beta|} \left( \mathcal{S}_k + |\beta| \right)!^{(\mathcal{S}_k + |\beta|)\sigma} \leq C^N (N + M)^{(N+M)\sigma} \sum_{a_0 + \cdots + a_k = \mathcal{S}_k + |\beta|} 1.\]

for suitable $C' > 0$.

Hence we conclude that there exist constants $A, h > 0$ such that
\[|D^\beta R_j \cdots R_j \phi(x, \xi)| \leq A|\xi|^{m-N} h^{N\sigma} (N + M)^{(N+M)\sigma}, \quad (4.29)\]
\[x \in K, \xi \in \Gamma, \] which gives the desired estimate.

REFERENCES

[1] P. Albano, A. Bove, M. Mughetti: Analytic Hypoellipticity for sums of squares and the Treves conjecture, Arxiv, 12 May 2016.
[2] R. W. Braun, R. Meise, B. A. Taylor, Ultra-differentiable functions and Fourier analysis, Results Math. 17 (3-4), 206–237, (1990)
[3] H. Chen, L. Rodino, General theory of PDE and Gevrey classes in General theory of partial differential equations and microlocal analysis, Pitman Res. Notes Math. Ser., Longman, Harlow, 349 (1996), 6–81.
[4] C. Fernández; A. Galbis Superposition in classes of ultradifferentiable functions, Publ. Res. Inst. Math. Sci. 42 (2) (2006), 399419.
[5] M. Cicognani; D. Lorentz, Strictly hyperbolic equations with coefficients low-regular win time and smooth in space, arXiv:1611.09548v2, 30Nov2016.
[6] L. Hörmander, The Analysis of Linear Partial Differential Operators. Vol. I: Distribution Theory and Fourier Analysis, Springer-Verlag, 1983.
[7] L. Hörmander: A counterexample of Gevrey class to the uniqueness of the Cauchy problem, Math. Research Letters 7 (2000) 615–624.
[8] M. Ider On the superposition of functions in Carleman classes, Bull. Austral. Math. Soc. 39 (3) (1989), 471476.
[9] A. Klotz, Inverse closed ultradifferential subalgebras, J. Math. Anal. Appl. 409 (2) (2014), 615-629.
[10] H. Komatsu, Ultradistributions, I: Structure theorems and a characterization. J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 20 (1) (1973), 25–105.
[11] H. Komatsu, An introduction to the theory of generalized functions. Lecture notes, Department of Mathematics Science University of Tokyo 1999.
[12] O. Liess, Y. Okada, Ultra-differentiable classes and intersection theorems Math. Nachr. 287, 638–665 (2014)
[13] T. Ma, Higher Chain formula proved by Combinatorics, The Electronic Journal of Combinatorics, 16 (2009), #N21.
[14] G. Menon: Gevrey class regularity for the attractor of the laser equations, Nonlinearity 12 (6) (1999), 1505-1510.
[15] S. Pilipović, N. Teofanov, and F. Tomić, On a class of ultradifferentiable functions. Novi Sad Journal of Mathematics, 45 (1) (2015), 125–142.
[16] S. Pilipović, N. Teofanov, and F. Tomić, Beyond Gevrey regularity. Journal of Pseudo-Differential Operators and Applications, 7, (2016), 113–140.
[17] A. Rainer, G. Schindl, Composition in ultradifferentiable classes, Studia Math., 224 2, 97 – 131 (2014)
[18] J. Rauch, Partial Differential Equations. Springer-Verlag, 1991.
[19] L. Rodino, Linear Partial Differential Operators in Gevrey Spaces. World Scientific, 1993.
[20] W. Rudin, Division in algebras of infinitely differentiable functions, J. Math. Mech., 11 (1962), 797–810.
[21] J. A. Siddiqi, Inverse-closed Carleman algebras of infinitely differentiable functions, Proc. Amer. Math. Soc. 109 (2) (1990), 357-367.
[22] N. Teofanov, F. Tomić: Ultradifferentiable functions of class $M^\tau_{p,\sigma}$ and microlocal regularity, to appear in Generalized functions and Fourier analysis, Birkhäuser, (2017).

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