Constitutive relations of the endochronic theory of thermoplasticity for high-temperature composites under plane stress state

B S Sarbayev
Bauman Moscow State Technical University, 2-nd Baumanskaya, 5, b.1, 105005, Moscow, Russia
E-mail: bssarbayev@mail.ru

Abstract. A variant of the constitutive relations of the endochronic theory of thermoplasticity for high-temperature composites is proposed. The relations for the plane stress state are considered. Examples of the use of the proposed relations in which the material behavior under complex loading is analyzed are given. The dependence of the elastic modulus of the material on the temperature of pre-heating is shown.

1. Introduction
The strength analysis of structural elements made with the use of high-temperature composite materials is one of the stages in the development of rocket-space and aviation equipment, engine-building, chemical industry and metallurgy. Problems such as the development of methods for calculating and design uncooled rocket motor nozzles, exit cone of jet engines, wing leading edges and nose cone, turbine blades, etc. can be solved with the use of adequate mathematical models of deformation of anisotropic structural materials, operating under intensive thermo-force loading.

A feature of modern high-temperature composite materials, for example, carbon-ceramic-matrix composites, is their irreversible deformation at both room and elevated temperatures. The deformation diagrams are non-linear, during unloading residual strains are formed [1,2]. To describe this effect, theories of thermoplasticity based on the traditional approach using the concepts of yield surface can be used. Such models have been developed in detail for ordinary isotropic structural materials [3,4]. In this models separation of the total strain tensor into elastic, thermal and plastic components are assumed. In this case, the elastic moduli, the coefficient of thermal expansion, and the yield strength depend on temperature. The deformation history of an isotropic body is described, as a rule, by the Odqvist parameter. In such theories, a refined concept of active and passive loading should be used. Algorithms for calculation of structural elements, founded on the concept of yield surface, should include a procedure for determination of the ultimate load at which the material is transferred from an elastic to a plastic state.

Here, for high-temperature composites, we propose a version of the relations of the endochronic theory of thermoplasticity, which describes the effects of nonlinear deformation. In this theory, the key role is played by the parameter of internal time, which phenomenologically describes the irreversible nature of the deformation [5]. Constitutive relations, including this time-like parameter, are written either as relations of hereditary type, or as differential non-linear relations. At the same time, it is not explicitly the concept of yield surface is used, and there is no separation of strains into elastic and plastic
components. The theory leads to satisfactory results for materials having a heterogeneous structure at the microlevel, for example, for composites. As a rule, nonlinear deformation of such materials begins at relatively low stresses, which makes it difficult to experimentally study the yield surface in the deformation process.

It is well known that the endochronic theory of plasticity is effective technique for analysis of inelastic behavior of isotropic structural materials under various kind of loading conditions [6-8]. Excellent results in the description of the deformation of laminated composites, concrete under force loading have been obtained by use of this approach [9-11]. It is obvious that the development of an endochronic theory for the case of thermal and force loading of structural materials is a natural next step.

It should be noted that causes of irreversible strains may vary for structure materials. They may differ from the causes of plastic deformations for traditional isotropic structural materials, such as steels and alloys. In certain cases irreversible strains may be caused by damages of the microstructures (microcracks, fiber ruptures, delamination, etc.), and in other cases they may be caused by plastic microstrains of components of the composite materials. In this article, the deformation is studied by means of a phenomenological method. Processes, which take place at the microlevel and cause an irreversible deformation at the macrolevel, is not considered. Therefore, used term “plastic strain” is understood in the generalized sense as irreversible rate-independent strain.

2. Theoretical background

To describe the deformation effects of high-temperature composites under a thermal load, we propose a variant of the endochronic theory of thermoplasticity, which takes into account the history of loading. Constitutive relations shall be written in the following form

\[
\sigma_{ij} = \int_0^\xi E_{ijkl} [z(\xi) - z(\xi')] \frac{d \xi'}{d \xi} d \xi' - \int_0^\xi B_{ij} [z(\xi) - z(\xi')] \frac{dT}{d \xi} d \xi'.
\]

Here, \(\sigma_{ij}\) and \(\varepsilon_{ij}\) are the stress and small strain tensors, \(T\) is the temperature change, \(E_{ijkl}[z(\xi)], B_{ij}[z(\xi)]\) \([z(\xi)]\) - material functions, \(z(\xi)\) - is the internal time for which equations \(dz = d\xi/f(\xi)\) are valid, \(f(\xi)\) is a material function, and \(f(\xi) > 0, f(0) = 1\). The expression for the internal time measure \(\xi\) is represented as

\[
(d \xi')^2 = r_{ijkl} d \varepsilon_{ij} d \varepsilon_{ij} + m^2 (dT)^2,
\]

where \(r_{ijkl}\) is a positive definite symmetric tensor of rank IV of material parameters, \(m > 0\) is a material parameter that describes irreversible deformation during temperature heating. In formula (2), the temperature change along with the strain tensor is considered as a loading parameter.

![Figure 1. Plane stress state of an orthotropic body.](image)

We assume that the composite at the macrolevel is an orthotropic body in a plane stress state (Figure 1). Subject to the use of exponential kernel, the constitutive relations (1) in the matrix form can be written as follows
\[ \{\sigma\} = \int_0^z \exp\{-[\alpha_y z(\xi) - z(\xi')]\} [D] \frac{d\{\varepsilon\}}{d\xi'} d\xi' - \int_0^z \exp\{-[\alpha_y z(\xi) - z(\xi')]\} \{B\} \frac{dT}{d\xi'} d\xi'. \quad (3) \]

Here, \( \{\sigma\} = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \) and \( \{\varepsilon\} = (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})^T \) are the stress and strain vectors, respectively, \( [\alpha] = \text{diag}(\alpha_{s1}, \alpha_{s2}, \alpha_{s3}) \), \( [\alpha_T] = \text{diag}(\alpha_{t1}, \alpha_{t2}, 0) \) - the matrix of material parameters.

Stiffness matrix \([D]\) is of the form

\[
[D] = \begin{bmatrix}
E'_1 & E'_2 v_{12} & 0 \\
E''_1 v_{21} & E'_2 & 0 \\
0 & 0 & G_{12}
\end{bmatrix}
\]

where \( E_1, E_2, G_{12}, v_{12}, v_{21} \) - elastic characteristics of an orthotropic material, \( E'_{1,2} = E_{1,2} / (1 - v_{12} v_{21}) \).

\[ \{B\} = [D]\{\beta\}, \quad \{\beta\} = (\beta_{11}, \beta_{22}, 0)^T \] - vector of linear thermal expansion coefficients. The increment of the parameter \( \xi \) in accordance with the relation (2) is calculated by the formula

\[ (d\xi)^2 = \frac{d\{\varepsilon\}^T}{d\xi'} \{R\} \frac{d\{\varepsilon\}}{d\xi'} + m^2 (dT)^2. \quad (4) \]

where \( [R] = \begin{bmatrix} r_{11} & r_{12} & 0 \\
r_{12} & r_{22} & 0 \\
0 & 0 & r_{33} \end{bmatrix} \)

Material parameters are determined from the condition of the best agreement between experimental and theoretical results. For this, numerical procedures can be used, with the help of which the results of basic experiments are processed. These include force uniaxial loading along the OX1 and OX2 axes, under a pure shear in the OX1X2 plane, and also the heating of material specimen.

Further, for convenience of computation, we introduce the following form of the representation for constitutive relations (3)

\[ \{\sigma\} = \{\sigma_s\} - \{\sigma_t\}, \quad (5) \]

where

\[
\{\sigma_s\} = \int_0^z \exp\{-[\alpha_y z(\xi) - z(\xi')]\} [D] \frac{d\{\varepsilon\}}{d\xi'} d\xi', \quad (6)
\]

Based on (5) and (6), the constitutive relations can be represented in the following differential form

\[
\frac{d((\sigma_s + \{\sigma_t\})}{d\xi} + [\alpha_y \frac{d\{\sigma_t\}}{d\xi} + \{\sigma_s\}] = [D] \frac{d\{\varepsilon\}}{d\xi},
\]

\[
\frac{d\{\sigma_t\}}{d\xi} + [\alpha_y \frac{d\{\sigma_t\}}{d\xi} + \{\sigma_s\}] = \{B\} \frac{dT}{d\xi}. \quad (7) \]

Here, \( \{\sigma_t\} = (\sigma_{11t}, \sigma_{22t}, 0)^T \) is an auxiliary vector of thermal stresses. In coordinate form, the equations (7) are written as follows
For thermal stresses we have the following differential relations

\[
\begin{align*}
\frac{d(\sigma_{11} + \sigma_{11u})}{d\xi} + \alpha_{s1} \frac{\sigma_{11} + \sigma_{11u}}{f(\xi)} &= E_1 \frac{d\varepsilon_{11}}{d\xi}, \\
\frac{d(\sigma_{22} + \sigma_{22u})}{d\xi} + \alpha_{s2} \frac{\sigma_{22} + \sigma_{22u}}{f(\xi)} &= E_2 \frac{d\varepsilon_{22}}{d\xi} + E_1 \frac{d\varepsilon_{22}}{d\xi}, \\
\frac{d\sigma_{12}}{d\xi} + \alpha_{s3} \frac{\sigma_{12}}{f(\xi)} &= G_{12} \frac{d\gamma_{12}}{d\xi}.
\end{align*}
\]

The increment of the parameter \(\xi\) describing the loading history is calculated by the formula

\[
(d\xi)^2 = r_1(d\varepsilon_{11})^2 + r_2(d\varepsilon_{22})^2 + 2r_{12}d\varepsilon_{11}d\varepsilon_{22} + r_{33}(d\gamma_{12})^2 + m^2(dT)^2.
\]

It should be noted that for some materials, for example, for carbon-carbon composites based on woven filler, due to the smallness of the coefficients \(\nu_{12}\) and \(\nu_{21}\), the Poisson effect can be neglected. Then the constitutive relations can be simplified and written in this form.

\[
\begin{align*}
\frac{d(\sigma_{11} + \sigma_{11u})}{d\xi} + \alpha_{s1} \sigma_{11} + \sigma_{11u} &= E_1 \frac{d\varepsilon_{11}}{d\xi}, \\
\frac{d(\sigma_{22} + \sigma_{22u})}{d\xi} + \alpha_{s2} \sigma_{22} + \sigma_{22u} &= E_2 \frac{d\varepsilon_{22}}{d\xi} + E_1 \frac{d\varepsilon_{22}}{d\xi}, \\
\frac{d\sigma_{12}}{d\xi} + \alpha_{s3} \sigma_{12} &= G_{12} \frac{d\gamma_{12}}{d\xi}.
\end{align*}
\]

Here for the parameter \(\xi\) we use a simpler expression

\[
(d\xi)^2 = r_1(d\varepsilon_{11})^2 + r_2(d\varepsilon_{22})^2 + r_{33}(d\gamma_{12})^2 + m^2(dT)^2.
\]

For some types of loading, the constitutive relations (7) allow us to obtain analytical expressions. In general, they should be integrated numerically.

Assuming \(\sigma_{12}=\sigma_{22}=0\) and \(\nu_{11}=1\), \(\nu_{22}=0\) in equations (8) and (9), we obtain the relations for the case of uniaxial thermal force loading in the direction of the OX1 axis. Let us write them as follows

\[
\begin{align*}
\frac{d(\sigma_{11} + \sigma_{11u})}{d\xi} + \alpha_{s1} (\sigma_{11} + \sigma_{11u}) &= E_1 \frac{d\varepsilon_{11}}{d\xi}, \\
\frac{d(\sigma_{22} + \sigma_{22u})}{d\xi} + \alpha_{s2} (\sigma_{22} + \sigma_{22u}) &= E_2 \frac{d\varepsilon_{22}}{d\xi} + E_1 \frac{d\varepsilon_{22}}{d\xi}, \\
\frac{d\sigma_{12}}{d\xi} + \alpha_{s3} \sigma_{12} &= E_{12} \frac{d\gamma_{12}}{d\xi}.
\end{align*}
\]

Using the relations (10), we consider some deformation effects predicted by the proposed theory. In the calculations we will use the well-known expression \(f(\xi) = 1 + b\xi\) which allows in some cases to obtain analytical solutions [5].
3. Deformation under complex loading

Let us consider the following variant of complex loading. Let at the first stage the material specimen is subjected to free heating, at which the force load is absent, i.e. \( \sigma_{11}=0, dT\neq0 \). In the initial state at \( \xi=0 \) we have \( \sigma_{11t}=0, \varepsilon_{11}=0, T=0 \). Next, we take into account that for heating \( dT>0 \), and when cooling take place - \( dT<0 \). Writing the third equation of system (10) in the form

\[
\frac{d}{d\xi} \left( \frac{d\varepsilon_{11}}{d\xi} \right) + m^2 \left( \frac{dT}{d\xi} \right)^2 = 1
\]

and using the first two equalities of this system, we obtain the following differential equation

\[
\frac{d\sigma_{11u}}{d\xi} = \pm \rho \sqrt{E_1^2 (1+\rho^2) f^2(\xi) - (\alpha_s-\alpha_t)^2 \sigma_{11u}^2 - (\alpha_s \rho^2 + \alpha_t) \sigma_{11u}}
\]

(11)

where \( \rho = \beta_{11}/m \). From the equation obtained, as well as from the first two equalities of system (10), the differential equations follow.

\[
\frac{d\varepsilon_{11k}}{d\xi} = \pm \rho \sqrt{E_1^2 (1+\rho^2) f^2(\xi) - (\alpha_s-\alpha_t)^2 \sigma_{11u}^2 + (\alpha_s - \alpha_t) \sigma_{11u}}
\]

(12)

\[
\frac{dT}{d\xi} = \pm \frac{E_1^2 (1+\rho^2) f^2(\xi) - (\alpha_s - \alpha_t)^2 \sigma_{11u}^2 - \rho (\alpha_s - \alpha_t) \sigma_{11u}}{m E_1 (1+\rho^2) f(\xi)}
\]

(13)

Equations (11) - (13) form a system of nonlinear differential ordinary equations, which can be integrated numerically for given initial conditions. When heating, one should take a plus sign, when cooling - minus.

Let us determine the deformation during the heating of the material specimen to a certain temperature \( T_0 \), with further cooling it to its initial state. Thus, within the interval \( 0 \leq \xi \leq \xi_0 \) there will be heating, and within \( \xi_0 \leq \xi \leq \xi_k \) - cooling. Value \( \xi_0 \) corresponds to the time of the end of heating, and the value of \( \xi_k \) – the moment of return of the material specimen to its initial temperature state.

In the numerical example, let the specimen be heated to a temperature \( T_{max}=2140^\circ \text{C} \), which corresponds to the value \( \xi_0=0.005 \). For material parameters, let us take the following values: \( E_1=20 \, \text{GPa}; \quad \alpha_1=150; \quad \alpha_t=30; \quad b=5; \quad \beta_{11}=1.8-10^{-6} \, /\text{deg}; \quad m=0.24-10^{-6} \, /\text{deg}. \) The system of equations integrates numerically via the Runge-Kutta method. Deformation diagram is shown in Figure 2, wherein \( \xi_{11} = \varepsilon_{11} / \varepsilon_{max}, \bar{T} = (T - 293 \, \text{K})/T_{max} \). \( \varepsilon_{max}=0.005 \) is obtained as a result of the calculation. As seen, at heating nonlinear deformation occurs while increasing tangential coefficient of temperature expansion \( \beta_{11k} = \frac{d\varepsilon_{11}}{dT} \). Note that during cooling, the theory predicts formation of residual strains (see Figure 2). In this example, when cooled to the initial temperature, a residual strain of \( \varepsilon_{11r}=0.0017 \) is formed. This value corresponds to \( \xi_k=0.0084 \).
At the second stage, when $\xi \geq \xi_0$ the specimen is subjected to uniaxial tension at a constant temperature $T = T_{\text{max}}$. In this case, we have $dT/d\xi > 0$. The system of equations (10) takes the following form:

$$
\begin{align*}
\frac{d\sigma_{11s}}{d\xi} + \alpha_{1s} \frac{\sigma_{11s}}{f(\xi)} &= E_1, \\
\frac{d\sigma_{11t}}{d\xi} + \alpha_{1} \frac{\sigma_{11t}}{f(\xi)} &= 0, \\
d\xi &= d\varepsilon_{11s},
\end{align*}
$$

where $\sigma_{11s} = \sigma_{11} + \sigma_{11t}$. This system of equations can be integrated analytically using the following initial conditions $\sigma_{11s}(\xi_0) = \sigma_{11t}(\xi_0) = \sigma_{11t}^*$. The value $\sigma_{11t}^*$ was obtained in the numerical solution presented above. As a result, when $\xi \geq \xi_0$, we have the following expression:

$$
\sigma_{11}(\xi) = \frac{E_1}{\alpha_{1s} + \beta} \left[ 1 - \left( \frac{1 + \beta \frac{\xi}{\xi_0}}{1 + \beta \xi} \right)^{\alpha_{1s} - 1} \right] + \sigma_{11t}^* \left[ 1 + \frac{\beta \frac{\xi}{\xi_0}}{1 + \beta \xi} \right]^{\alpha_{1s} - 1} - \left( 1 + \beta \frac{\xi}{\xi_0} \right)^{\alpha_{1s} - 1},
$$

where $\xi = \xi_0 + \varepsilon_{11s}$. By differentiating this expression with respect to $\xi$, we can obtain a formula for calculating the tangential stiffness of the material. At the initial moment of force loading at $\xi = \xi_0$ we will have

$$
E_{1k}(\xi_0) = \frac{d\sigma_{11}}{d\varepsilon_{11s}} = E_1 - \sigma_{11t}^* \frac{\alpha_{1s} - \alpha_{11t}}{1 + \beta \xi_0}.
$$

Formula (14) shows that the tangent modulus at the initial moment of force loading decreases as a result of preheating. This deformation of the material is confirmed by experimental studies of composites [12]. In addition, formula (14) allows to reasonably choose the values of the material parameters $\alpha_{1s}$ and $\alpha_{11t}$.

4. Conclusions

The proposed constitutive relations of the theory of thermoplasticity are the development of the endochronic approach in the theory of inelastic deformation of structural materials. In applications, it is convenient to write them in the form of differentially non-linear relations. In particular cases of loading, they can lead to relatively simple analytical relations. In the general case, they are integrated numerically. Constitutive relations describe the effects of nonlinear deformation of the material under...
thermal force loading, in particular, non-linear change of thermal deformations with free heating, the formation of residual strains during cooling, the dependence of the elastic modulus on the preheating temperature.

References
[1] Sarbayev B S, Baryshev A N 2017 Vestnik MGTU im. N.E. Baumana. Seria «Mashinostroenie» (in Russian) 4 65.
[2] Bobrov A V, Sarbaev B S, Shirshov Yu Yu 2016 J.of Mach.Manuf. And Reliab. 45 145.
[3] Kachanov L M 1969 Osnovy teorii plastichnosti (in Russian). Moscow: Nauka. 420.
[4] Birger I A, Demynushko I V 1968 Mechanika tverdogo tela. Inzhenerny Zhurnal (in Russian). 6 70.
[5] Valanis K C 1971 Arch. of Mech. 23 517.
[6] Valanis K C, Lee C F 1984 Trans. ASME. J. of Appl. Mech. 51 367.
[7] Wu H C, Yip M C 1981 Trans. ASME. J. of Eng. Mater. And Technol. 103 212.
[8] Kadashevich U I and Pomitkin S P 2010 Izvestia RAN. Mechanika tverdogo tela (in Russian). 6 123.
[9] Sarbayev B S 1995 Comput. Mater. Science. 4 220.
[10] Pindera M -J, Herakovich C T 1983 Mechanics of Composite Materials. Recent Advances, ed. Z Hashin, C T Herakovich. New York: Pergamon Press. pp. 367-381.
[11] Bazant Z P, Bhat P D 1976 J. of the Eng. Mech.Div. 4 701.
[12] Vasiliev V V, Morozov E V 2007 Advanced Mechanics of Composite Materials (Amsterdam: Elsevier). p 491.