Mirror Symmetry of Minimal Calabi-Yau Manifolds

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Abstract

We perform the mirror transformations of Calabi-Yau manifolds with one moduli whose Hodge numbers $(h^{11}, h^{21})$ are minimally small. Since the difference of Hodge numbers is the generation of matter fields in superstring theories made of compactifications, minimal Hodge numbers of the model of phenomenological interest are $(1,4)$. Genuine minimal Calabi-Yau manifold which has least degrees of freedom for Kähler and complex deformation is $(1,1)$ model. With help of Mathematica and Maple, we derive Picard-Fuchs equations for periods, and determine their monodromy behaviors completely such that all monodromy matrices are consistent in the mirror prescription of the model $(1,4)$, $(1,3)$ and $(1,1)$. We also discuss to find the description for each mirror of $(1,3)$ and $(1,1)$ by combining invariant polynomials of variety on which $(1,5)$ model is defined. The genus 0 instanton numbers coming from mirror transformations in above models look reasonable. We propose the weighted discriminant for genus 1 instanton calculus which makes all instanton numbers integral, except $(1,1)$ case.
1 Introduction

A Calabi-Yau manifold is partially characterized by the Hodge numbers \((h^{11}, h^{21})\). These are topological numbers which count the number of parameters that deform the Kähler class and the complex structure of the manifold. Recently, Calabi-Yau manifolds are paid attention where both Hodge numbers \((h^{11}, h^{21})\) are small \([1, 2, 3]\). Many Calabi-Yau manifolds with various Hodge numbers are provided by construction with hypersurfaces in weighted projective spaces or in toric varieties. However manifolds admitting freely-acting discrete symmetry seem to be rare \([4]\). Models with small Hodge numbers have been found to classify all the freely acting symmetries for the manifolds \([2, 5, 6, 7, 8, 9]\). Besides the way of constructions, the Hodge numbers for the models with symmetry of order four have been calculated recently \([10]\).
Phenomenologically, it is interesting to search string theories with three generations compactified on Calabi-Yau manifolds with small Hodge numbers. Especially model (1,4) with $\chi = -6$ is the minimal theory which have been discussed in [3, 11]. Theoretically, it is worth investigating the case with minimal Hodge numbers (1,1) found in [7], whose enumerative property is not clear so far. These models are made by taking quotient of freely acting symmetry groups, so that Hodge numbers become both small [2, 3, 4, 7], however their defining equations turn out to be complicated. In these cases, it is not obvious to carry out ordinary systematic calculation to derive Picard-Fuchs equation, and to perform the mirror transformation to calculate the instanton corrections.

In this paper, we attend to investigate the mirror transformations for one moduli models with small Euler numbers $|\chi| \leq 8$ with aid of computer algebra systems Mathematica and Maple. Using Mathematica package “Generationgfunctions”, we derive the Picard-Fuchs equation in such models. Monodromy behaviors are determined by numerical integration on Maple [12]. In order to determine the symplectic basis of periods, as well as, topological indices of such models, we evaluate bi-linear form on periods numerically [13, 14]. To check the consistency of the results, we calculate the genus 0 and 1 instanton numbers to be integral values. Also it is interesting to investigate the relation between the models with $(1, h^{21})$ where $h^{21} < 6$, and their mirrors.

Picard-Fuchs equations for periods of Calabi-Yau three-folds have been studied extensively, and many kinds of equations have been found already in physical or mathematical contexts. Restricted to one moduli case, the automated search for 4th order Picard-Fuchs equations of Calabi-Yau type with maximally unipotent monodromy have been carried out [15], and vast results including previously found operators have been summarized in “Calabi-Yau Operators Database” on the web site [15]. Picard-Fuchs equations we derive in this paper are found in the database.

In section 2, we review mirror model of (1,4) which is known example. From the evaluation of periods, we find Picard-Fuchs equation. Bi-linear form on periods gives the characteristic numbers, such as Euler number, Yukawa coupling $K$, $c_2$, to suggest the way to determine the basis of monodromy. Instanton calculations of genus 1 as well as genus 0 are
performed by using mirror map so that all instanton numbers becomes integral, where we propose weighted discriminant for 1-loop level determined from behaviors around singular points. In section 3, we investigate the sequence of manifolds with small Hodge numbers such as mirror models of (1,5), (1,3), (1,1). Starting from the invariant polynomials for (1,5) model, and choosing suitable combinations of them, we propose the definition of mirror models of (1,5), (1,3), (1,1) respectively. The mirror transformations in these models can be carried out in similar ways in section 2. The results about monodromy behaviors and instanton numbers are all consistent, except that instanton numbers for minimal model (1,1) look strange.

2 Minimal model for three generations

As an examples of the Calabi-Yau manifold with small Hodge numbers \((h^{11}, h^{21})\), we investigate the model with (1,4), and its mirror, which were found in \([2, 3]\). This Calabi-Yau manifold is constructed from \(X^{8,44}\), and was found in the course of the project to classify all the freely acting symmetries for the manifolds of the CICY list. Original space \(X^{8,44}\) has Euler number \(-72\), and is invariant under freely acting group \(G\) whose order is 12. As is explained in \([3]\), quotient variety \(X^{8,44}/G\) is smooth and has Euler number \(\chi = -72/12 = -6\). The definition of this model consists of following three curves on six manifolds

\[
p = 1 + s_0s_1 + s_1s_2 + s_2s_0, \quad q = 1 + t_0t_1 + t_1t_2 + t_2t_0, \\
r = s_0s_1s_2t_0t_1t_2 + c_1(s_0t_0 + s_1t_1 + s_2t_2) + c_2(s_0t_1 + s_1t_2 + s_2t_0) \\
+ c_3(s_0t_2 + s_1t_0 + s_2t_1) + c_4(s_1s_2(t_0 + t_1 + t_2) + (s_0 + s_1 + s_2)t_0t_1t_2),
\]

where \(c_i\) are four kinds of moduli parameters.

This is a minimal model of string theory with three generations compactified on Calabi-Yau manifold with \(\chi = -6\). Phenomenological aspects about of this model were discussed in detail in \([3, 11]\).
2.1 Mirror prescription

Toric description of this model is also given in [3], and alternative defining curve consists of
four parameter family of invariant Laurent polynomials in terms of homogeneous coordinates
made of polyhedron ∆ as

\[ f = 1 + \sum_{i=1}^{4} \gamma_i Q_i \]  

(2.2)

where

\[ Q_1 = t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4}, \]

\[ Q_2 = (t_1 + \frac{1}{t_1})(t_3 + \frac{1}{t_3}) + (t_4 + \frac{1}{t_4})(t_1 + \frac{1}{t_1})(t_2 + \frac{1}{t_2})(t_3 + \frac{1}{t_3}), \]

(2.3)

\[ Q_3 = (t_1 + \frac{1}{t_1})(t_4 + \frac{1}{t_4}) + (t_2 + \frac{1}{t_2})(t_3 + \frac{1}{t_3}) + (t_1 + \frac{1}{t_1})(t_2 + \frac{1}{t_2})(t_3 + \frac{1}{t_3}), \]

\[ Q_4 = (t_2 + \frac{1}{t_2})(t_4 + \frac{1}{t_4}) + (t_3 + \frac{1}{t_3})(t_1 + \frac{1}{t_1})(t_2 + \frac{1}{t_2}) + (t_1 + \frac{1}{t_1})(t_3 + \frac{1}{t_3}). \]

Obtaining the dual \( \nabla \) is to delete the vertices of ∆, which corresponds to setting pa-
rameters \( \gamma_i \) equal to zero except one. One of defining curve for the mirror of (1,4) model is

\[ P = 1 + \gamma_1(t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4}). \]  

(2.4)

In ordinary cases of toric varieties, you may find the Picard-Fuchs equation for the period
integral \( \int \frac{dz}{P} \) following the Griffiths-Dwork method. Differently it seems difficult to do
in this case. So we first turn to find an exact form of the fundamental period \( \omega_0 \) by picking
up simple poles of period integral. Residues at \( t_1 = t_2 = t_3 = t_4 = 0 \) are calculated by
expanding \( 1/P \) as

\[ \frac{1}{P} = \sum \binom{n}{i} \left( t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + t_3 + \frac{1}{t_3} + t_4 + \frac{1}{t_4} \right)^{n-i} z^n. \]  

(2.5)
Fundamental period is

$$\omega_0 = \sum \binom{n}{i} \binom{l_1}{i-l_1} \binom{m_1}{p_1} \binom{l_2}{i-l_2} \binom{m_2}{p_2} \binom{k_1}{l_1} \binom{k_2}{l_2} \binom{k_3}{l_1-l_2} \binom{m_3}{p_1-p_2} z^n$$  \hspace{1cm} (2.6)

where

$$l_1 = i - 2k_3 - 2k_2, \quad l_2 = i - 2k_1 - 2k_2,$$

$$p_1 = n - i - 2m_3 - 2m_2, \quad p_2 = n - i - 2m_1 - 2m_2.$$  \hspace{1cm} (2.7)

The multiple summations look still hard to derive Picard-Fuchs equation. Then we have recourse to the power of computer. By using Mathematica package "Generationgfunctions", we can find a differential equation for series expanded function. First we expand $w_0$ in Mathematica up to high enough orders, such as $O(z^{70})$. Next we apply the command “GuessRE” which derive the recursion equation among the coefficients of series expansion. After deriving the recursion equation, the command “RE2DE” tells us the differential equation for this period. The result is

$$a_4(z) \frac{\partial^4 f}{\partial z^4} + a_3(z) \frac{\partial^3 f}{\partial z^3} + a_2(z) \frac{\partial^2 f}{\partial z^2} + a_1(z) \frac{\partial f}{\partial z} + a_0(z) f = 0$$  \hspace{1cm} (2.8)

where

$$a_4(z) = - z^3 (2z - 3)^2 (3z - 1) (4z - 1) (4z + 1) (5z + 1) (6z + 1) (12z - 1),$$

$$a_3(z) = - 2 z^2 (2z - 3) (276480z^7 - 478656z^6 - 11232z^5 + 55844z^4 + 1100z^3$$

$$- 1701z^2 - 52z + 9),$$

$$a_2(z) = - z (4976640z^8 - 16982784z^7 + 14544576z^6 + 880992z^5 - 1286856z^4$$

$$- 29468z^3 + 26098z^2 + 555z - 63),$$

$$a_1(z) = - (6635520z^8 - 23846400z^7 + 22194432z^6 + 610656z^5 - 1445856z^4$$

$$- 12968z^3 + 17532z^2 + 156z - 9),$$

$$a_0(z) = - 48 z \left(34560z^6 - 130464z^5 + 132120z^4 + 284z^3 - 6182z^2 + 9z + 36 \right).$$
This Picard-Fuchs equation is found in the database on the web site [15], though corresponding record number is not manifest there. For later convenience, we define the ratio of coefficients as

\[ r_3(z) = \frac{a_3(z)}{a_4(z)}, \quad r_2(z) = \frac{a_2(z)}{a_4(z)}, \quad r_1(z) = \frac{a_1(z)}{a_4(z)}. \] (2.10)

Classical Yukawa coupling

\[ K^c[z] = \frac{6(3 - 2z)}{(1 - 3z)(1 - 4z)(1 + 4z)(1 + 5z)(1 + 6z)(1 - 12z)} \] (2.11)

is basically a quantity \( \exp\left(-\frac{1}{2} \int r_3(z) \, dz\right) \). To check the consistency for Calabi-Yau, we see that

\[ \frac{1}{2} r_2(z) r_3(z) - \frac{1}{8} r_3(z)^3 + r'_2(z) - \frac{3}{4} r_3(z) r'_3(z) - \frac{1}{2} r''_3(z) - r_1(z) = 0 \] (2.12)

The local property of the solutions of Picard-Fuchs equation is summarized the \( P \) symbol as follows.

\[
\begin{array}{cccccccc}
-1/4 & -1/5 & -1/6 & 0 & 1/12 & 1/4 & 1/3 & 3/2 & \infty \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 4 \\
\end{array}
\] (2.13)

From Picard-Fuchs equation, we can have other three independent solutions besides \( \omega_0 \)

\[ \omega_1 = \log z \cdot \omega_0 + \Omega_1(z), \]
\[ \omega_2 = (\log z)^2 \cdot \omega_0 + 2 \log z \cdot \Omega_1(z) + \Omega_2(z), \] (2.14)
\[ \omega_3 = (\log z)^3 \cdot \omega_0 + 3(\log z)^2 \cdot \Omega_1(z) + 3 \log z \cdot \Omega_2(z) + \Omega_3(z), \]

where polynomial part of above solutions \( \Omega_1, \Omega_2, \Omega_3 \) are obtained order by order as

\[ \Omega_1(z) = z + \frac{31}{2} z^2 + \cdots \]
\[ \Omega_2(z) = \frac{2}{3} z + \frac{31}{6} z^2 + \cdots \] (2.15)
\[ \Omega_3(z) = -4z - \frac{25}{2} z^2 + \cdots. \]
Using *Mathematica* you can get these $\Omega_i$ up to orders you need. It is also possible to derive these four periods directly from Picard-Fuchs equation with aid of the software *Maple*. The command “dsolve” with options “series” and “$z = 0$” gives you four independent series solutions up to orders you define, for example “Order := 30”.

### 2.2 Monodromy

In this model, there are seven singular points $z = \frac{1}{3}, \frac{1}{4}, \frac{1}{12}, 0, -\frac{1}{6}, -\frac{1}{5}, -\frac{1}{4}$. It seems complicated to find complete monodromy behavior around every singular point. Since periods we have here are obtained by series expansion around the origin up to finite orders, we can’t anticipate the analytic property enough to determine the monodromy matrices, by continuation to other singular points. Then, following the literature [12], we have to determine the monodromy by numerical calculation with suitable approximations.

The first step is to choose a reference point $p$ in items of each singular point. Next for each of the singular points $z_i$, we choose a piecewise linear loop starting and ending at the reference point $p$ and enclosing only one singular point. Using the *Maple* function “dsolve” with options “numeric, method = gear, relerr= $10^{-15}$, abserr = $10^{-15}$” and “Digits := 100”, we can numerically integrate the differential equation along these paths. Comparing integrated solutions to original ones at $p$ yields the monodromy matrices with respect to an arbitrary basis and produces fully filled $4 \times 4$ matrices. The result will be recognized as the integer matrices with the precision of this calculation.

![Figure 1: a loop from $p$ to $p$](image-url)
Symplectic basis we adopt here is

\[
\begin{align*}
\omega_P^0 &= \omega_0, \\
\omega_P^1 &= \frac{1}{2\pi i} \omega_1, \\
\omega_P^2 &= -\frac{c_2}{24} \omega_0 + \alpha \frac{1}{2\pi i} \omega_1 + \frac{K}{2} \frac{1}{(2\pi i)^2} \omega_2, \\
\omega_P^3 &= -\frac{\zeta(3)c_3}{(2\pi i)^3} \omega_0 - \frac{c_2}{24} \frac{1}{2\pi i} \omega_1 - \frac{K}{6} \frac{1}{(2\pi i)^2} \omega_3,
\end{align*}
\]

(2.16)

where \(K\) is Yukawa coupling, \(c_2\) is second Chern class, \(c_3 = \chi\) is Euler number of Calabi-Yau manifold we consider here, and \(\alpha\) is a constant which will be determined. So far this solution is nothing but of the model on \(X^{8,44}\) with Yukawa coupling 216, second Chern class 144, and Euler number \(-72\). As is well known, there is an undetermined overall factor for \(\omega_P^3\) here, and this would be fixed from the topological informations of Calabi-Yau manifold.

In the models which admit freely acting group, these topological numbers will be reduced simultaneously by moding out symmetries while their ratios will be kept. So we would like to use this degree of freedom so that Euler number in this case will be reduced \(\chi = -6\), and all monodromy matrices will be kept integral.

With this basis, monodromy around the origin is given by

\[
M_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\frac{K}{2} + \alpha & K & 1 & 0 \\
-\frac{c_2}{12} - \frac{K}{6} - \frac{K}{2} + \alpha & -1 & 1
\end{pmatrix}
\]

(2.17)

A key combination of topological numbers to notice is \(\frac{c_2}{12} + \frac{K}{6}\) when we reduce \(\chi\). Also we choose \(\alpha\) suitable so that elements of monodromy matrices will be integers, otherwise we set \(\alpha = 0\).

As the result, we can choose indices \(K, c_2, c_3\) with \(\alpha = 0\) as

\[
K = 18, \; c_2 = 12, \; c_3 = -6,
\]

(2.18)

and monodromy matrices \(M_z\) around \(z = \frac{1}{3}, \frac{1}{4}, \frac{1}{12}, 0, -\frac{1}{6}, -\frac{1}{5}, -\frac{1}{4}\) are, respectively,
\[
\begin{pmatrix}
-11 & -12 & -12 & 48 \\
3 & 4 & 3 & -12 \\
-3 & -3 & -2 & -12 \\
-3 & -3 & -3 & 13
\end{pmatrix}
\begin{pmatrix}
-11 & 0 & -12 & 72 \\
2 & 1 & 2 & -12 \\
0 & 0 & 1 & 0 \\
-2 & 0 & -2 & 13
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 12 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(2.19)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
9 & 18 & 1 & 0 \\
-4 & -9 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
-35 & -96 & -24 & 48 \\
18 & 49 & 12 & -24 \\
-72 & -192 & -47 & 96 \\
-27 & -72 & -18 & 37
\end{pmatrix}
\begin{pmatrix}
-71 & -180 & -60 & 144 \\
30 & 76 & 25 & -60 \\
-90 & -225 & -74 & 180 \\
-36 & -90 & -30 & 73
\end{pmatrix}
\]

(2.20)

Is is easy to check the consistency as

\[
M_{\frac{1}{2}} M_{\frac{1}{2}} M_{\frac{1}{2}} M_{0} M_{-\frac{1}{2}} M_{-\frac{1}{2}} M_{-\frac{1}{2}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Above values of indices could be read from the analysis done by [3], however the conditions that matrix elements for monodromy have to be integral, appear to be able to determine these quantity.

### 2.3 Bi-linear form

There is another way to find above indices by explicit evaluation of periods. Bi-linear form on periods \( Bi(f, g) \) was invented as a tool to enumerate the symplectic relations among period integrals [13].

Let us consider some anti-symmetric differential operators \( \partial^k \wedge \partial^{k'} \) acing to the solutions
\( f, \, g \) of Picard-Fuchs equation as
\[
\partial^k \wedge \partial^{k'} (f, g) = \frac{1}{2}(\partial^k f \cdot \partial^{k'} g - \partial^{k'} f \cdot \partial^k g) \tag{2.21}
\]
where \( \partial^k \) are the \( k \)-th order differential operator with respect to moduli parameter. For periods \( \{ f_{\alpha_i}, \, g_{\beta_j} \} \) obtained by integration along the symplectic homology basis \( \{ \alpha_i, \, \beta_j \} \), we can make bi-liner form acting on these periods to have the same symplectic structure as homology cycles
\[
Bi(f_{\alpha_i}, g_{\beta_j}) = -Bi(g_{\beta_j}, f_{\alpha_i}) = \delta_{i,j}, \, Bi(f_{\alpha_i}, f_{\alpha_j}) = Bi(g_{\beta_i}, g_{\beta_j}) = 0, \tag{2.22}
\]
up to normalization. This can be carried out by setting \( Bi(f, g) \) to be some linear combination of \( \partial^k \wedge \partial^{k'} \), and imposing \( \partial Bi(f, g) = 0 \) associated to Picard-Fuchs equation for periods. Using the ratios of coefficients of Picard-Fuchs equation, we take \( Bi(f, g) \) as
\[
Bi(f, g) = \exp \left( \frac{1}{2} \int r_3(z) dx \right) \left\{ \partial \wedge \partial^2 (f, g) - 1 \wedge \partial^3 (f, g) - \frac{1}{2} r_3(z) 1 \wedge \partial^2 (f, g) + \left( \frac{1}{2} \partial r_3(z) + \frac{1}{4} r_3(z)^2 - r_2(z) \right) 1 \wedge \partial (f, g) \right\} \tag{2.23}
\]
Especially in the model, explicit evaluations around the origin show that
\[
Bi(\omega_0, \omega_3) = -2, \, Bi(\omega_1, \omega_2) = \frac{2}{3}, \tag{2.24}
\]
and all other combinations vanish.

Using \( Bi(f, g) \) on a solution around conifold point, we can estimate topological indices \( c_1, \, c_2, \, K \) with Euler number \( \tilde{\chi} \) of mirror manifold. In this model, conifold solutions around \( z = \frac{1}{12} \) constitute of four kinds of functions whose leading behaviors \((z - \frac{1}{12})^s\) are of \( s = 0,1,1,2 \). We denote the polynomial solution with \( s = 1 \) as \( \omega_c \), which is
\[
\omega_c = 24u - 288u^2 + 3264u^3 - 35712u^4 + \frac{1965312}{5}u^5 - \frac{21731328}{5}u^6 + \cdots \tag{2.25}
\]
where \( u = z - \frac{1}{12} \). The topological indices can be obtained by using the ratio of bi-linear
forms on a period $\omega_c$ and periods around the origin as

$$c_1 = \frac{18\tilde{\chi}(3)}{\pi^2} \cdot \frac{Bi(\omega_1, \omega_c)}{Bi(\omega_3, \omega_c)},$$

$$c_2 = -\frac{18\tilde{\chi}(3)}{\pi^2} \cdot \frac{Bi(\omega_2, \omega_c)}{Bi(\omega_3, \omega_c)},$$

$$K = 6\tilde{\chi}(3) \cdot \frac{Bi(\omega_0, \omega_c)}{Bi(\omega_3, \omega_c)},$$

with $\tilde{\chi} = -c_3$. Denominators are needed for correct normalization. Periods $w_i (i = 0, 1, 2, 3)$ behave well around the origin and bad around the conifold point $z = \frac{1}{12}$. Conversely, a period $\omega_c$ behaves bad around the origin and good around the conifold point. So we expect these quantities behave like constants in the intermediate region between the neighborhood of origin and the neighborhood of conifold point. With help of Mathematica or Maple, we can estimate the value of above expression by plotting from $z = 0$ to $\frac{1}{12}$. Results for $c_2$, $K$ and $c_1$ with $\tilde{\chi} = -c_3 = 6$ are shown in fig.1, fig.2, and fig.3, respectively.

The values of plateau parts of above results are same as the ones obtained from the monodromy matrices.

### 2.4 Instanton calculation

Next we demonstrate the mirror symmetry to calculate the instanton numbers in genus 0 [16] and genus 1 [17] in topological string theory. After compactification, we have following
expansion for the partition function in topological string theory on Calabi-Yau manifold

\[ F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g. \]  

For genus 0 case [16], we have the formula for the quantum Yukawa coupling as the triple derivative of free energy

\[ \partial_t^3 F_0 = K + \sum_{i=1}^{13} \frac{a_i^3 q^i}{1 - q^i} \]  

where \( a_i \)'s are instanton numbers of the topological string for genus 0. In order to calculate these instanton numbers we use the mirror map. As usual, we set flat coordinate

\[ t = \frac{\omega_1(z)}{\omega_0(z)} = \log z + \frac{\Omega_1(z)}{\omega_0(z)}. \]

We define the variable \( q = e^t \) and invert this relation to express \( z \) in terms of \( q \) as \( z(q) \). Quantum Yukawa coupling is given by the the transformation of classical Yukawa coupling \( K_c[z] \) from \( z \) coordinate to \( q \) as

\[ \partial_t^3 F_0 = \frac{1}{z(q)^3} \left( \frac{\partial z(q)}{\partial q} \right)^3 K_c[z(q)] \frac{1}{\omega_0(x(q))^2} \]

Equating (2.32) and (2.43), and expanding in terms of \( q \), we find instanton number \( a_i \).

| \( i \) | \( a_i \) | Value                |
|-------|--------|----------------------|
| 1     | 6      | 4287838548           |
| 2     | 15     | 29153498904          |
| 3     | 30     | 201163103922         |
| 4     | 114    | 1406107987374        |
| 5     | 522    | 9941935540692        |
| 6     | 2529   | 71017384630734       |
| 7     | 12636  | 511976000663130      |
| 8     | 69744  | 3721663648494978     |
| 9     | 405168 | 27257992426100979    |
| 10    | 2449773| 201015705767041110   |
| 11    | 15261150| 1491738880927589808  |
| 12    | 97808574| 11134231701698352462  |
For genus 1 case, a derivative of free energy is the quantity we want to know

$$\partial_t F_1 = -\frac{c_2}{24} + \sum_i (b_i + \frac{a_i}{12}) \frac{q^i \cdot i}{1 - q^i}$$

(2.31)

where $b_i$’s are instanton numbers of genus 1. To compute $\partial_t F_1$, we follow the analysis of holomorphic anomaly [17], and use the formula

$$\partial_t F_1 = -\frac{1}{2} \partial_t \log \left( \frac{z_1^{1-\frac{c_2}{12}} \text{Dis}[z]}{\omega_0^{4-\frac{c_2}{12}} \partial z} \right).$$

(2.32)

Here, we will use the ansatz that $\text{Dis}[z]$ will be the weighted discriminant of the model. In well known examples with one moduli, we use a factor in the discriminant part as $(z - z_c)^{-\frac{1}{6}}$ where $z_c$ are conifold point of the model, since the monodromy matrix around conifold point is usually set to be

$$\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

In this case we have different behavior around conifold as (2.19) because of effects by taking quotient of freely acting group. Our proposal for $\text{Dis}[z]$ in this case is the following

$$\text{Dis}[z] = \prod_i (z - z_i)^{-\frac{\lambda_i}{6}}$$

(2.33)

where exponent $-\frac{\lambda_i}{6}$ will be determined for the monodromy matrix around the corresponding singular points by following method. Suppose $I$ is $4 \times 4$ identity matrix, and $M_{zi}$ is the monodromy matrix around $z_i$ where we take standard order of symplectic basis as $(\omega_0^P, \omega_2^P, \omega_1^P, \omega_3^P)^T$, then $M_{zi} - I$ will be expressed by using a certain integral vector $(v_1, v_2, v_3, v_4)$ as

$$M_{zi} - I = -\lambda_i \begin{pmatrix}
-v_4 \\
v_3 \\
-v_2 \\
v_1
\end{pmatrix} (v_1, v_2, v_3, v_4)$$

(2.34)
From this we can read off $\lambda_i$ for each $z_i$. In this model, we have

$$(z_i, \lambda_i) = \left( \frac{1}{12}, 12 \right), \left( \frac{1}{4}, 2 \right), \left( \frac{1}{3}, 3 \right), \left( -\frac{1}{4}, 12 \right), \left( -\frac{1}{5}, 1 \right), \left( -\frac{1}{6}, 3 \right), (0, 0).$$

(2.35)

With these exponents, we can calculate instanton numbers in genus 1. Validity for the choice of $Dis[z]$ will be checked whether all $b_i$'s are integers or not.

| $i$ | $b_i$   | 13         | 14         | 15         | 16         | 17         | 18         | 19         | 20         |
|-----|---------|------------|------------|------------|------------|------------|------------|------------|------------|
| 1   | 7       | 107873536349 | 854827410657 |            |            |            |            |            |            |
| 2   | 41      |            |            | 6809292590762 |            |            |            |            |            |
| 3   | 233     |            |            |            | 54489457053320 |            |            |            |            |
| 4   | 1393    |            |            |            |            | 43777878426585 |            |            |            |
| 5   | 10121   |            |            |            |            |            | 3529641546245282 |            |            |
| 6   | 72022   |            |            |            |            |            |            | 28547903108757361 |            |
| 7   | 518960  |            |            |            |            |            |            |            | 231550298613514152 |
| 8   | 3878268 |            |            |            |            |            | 1882881825812617783 |            |            |
| 9   | 29437440 |            |            |            |            |            |            | 15346314478913958426 |            |
| 10  | 225911060 |           |            |            |            |            |            |            | 125342659401860309785 |
| 11  | 1750966967 |           |            |            |            |            |            | 1025721457879954913034 |            |
| 12  | 13694924062 |           |            |            |            |            |            | 8408667562177554413449 |            |

Before closing this section, we mention that we can produce same results by using the original defining curve eq.(2.1) with reduced parameterization, say $c_1 = c_2 = c_3 = 0$, and with the period integral $\int \prod dt_i ds_i \frac{1}{p_{qr}}$.

## 3 Mirror transformations of (1,5), (1,3), (1,1) model

### 3.1 Six manifolds with quaternionic symmetry

In this section, we discuss three models with small Hodge numbers such as (5,1), (3,1), (1,1) by restricting the parameter of model coming from the manifolds $X^{4,68}$. For this manifold it is possible to write a defining polynomial that is transverse, as well as invariant and fixed
point free under the group $\mathbb{H} \times \mathbb{Z}_2$, where $\mathbb{H} = \{1, i, j, k, -1, -i, -j, -k\}$ is the quaternion group [2]. There are $3^4 = 81$ tetraquadric monomials in the $s_\alpha$ where $\alpha \in \mathbb{H}$. One of these is the fundamental monomial, $\Pi_\alpha \in \mathbb{H}s_\alpha$, that is invariant under the full group. Of the other 80 monomials, 40 are even under $(s_\alpha, s_{-\alpha}) \rightarrow (s_\alpha, -s_{-\alpha})$ and 40 odd. The 40 even monomials fall into five parameter family of invariant polynomials. We change the variable as $s_1s_{-1} = t_1$, $s_is_{-i} = t_2$, $s_js_{-j} = t_3$, $s ks_{-k} = t_4$, five invariant polynomials are

$$P_1 = \left(t_1t_2 + \frac{1}{t_1t_2}\right)\left(t_3t_4 + \frac{1}{t_3t_4}\right) + \left(t_2 + \frac{t_1}{t_2}\right)\left(t_4 + \frac{t_3}{t_4}\right),$$

$$P_2 = \left(t_2 + \frac{t_1}{t_2}\right)\left(t_3t_4 + \frac{1}{t_3t_4}\right) + \left(t_1t_2 + \frac{1}{t_1t_2}\right)\left(t_4 + \frac{t_3}{t_4}\right),$$

$$P_3 = \left(t_3 + \frac{1}{t_3}\right)\left(t_4 + \frac{1}{t_4}\right) + \left(t_1 + \frac{1}{t_1}\right)\left(t_2 + \frac{1}{t_2}\right),$$

$$P_4 = \left(t_2 + \frac{1}{t_2}\right)\left(t_4 + \frac{1}{t_4}\right) + \left(t_1 + \frac{1}{t_1}\right)\left(t_3 + \frac{1}{t_3}\right),$$

$$P_5 = \left(t_1 + \frac{1}{t_1}\right)\left(t_4 + \frac{1}{t_4}\right) + \left(t_2 + \frac{1}{t_2}\right)\left(t_3 + \frac{1}{t_3}\right).$$

(3.36)

Manifolds which are defined by these polynomials have been found in [2]. $X^{4,68}$ modulo $\mathbb{H} \times \mathbb{Z}_2$ is a model with Hodge numbers (1,5), because $\chi = -128/(8 \times 2) = -8$, which is smallest combination of Hodge numbers constructed on this manifold. The model we consider here are given by defining polynomial as

$$f = 1 + \sum_{i=1}^{5} c_i P_i. \quad (3.37)$$

In this section, we pursuit the possibility that the mirror model of $(1, h^{2,1})$ with $h^{2,1} \leq 5$ would be obtained by suitable restriction of parameter $c_i$’s of the above curve reducing the number of moduli to 1. It is natural to get the mirror of (1,5) model with above curve because it is just the restriction of defining curve of (1,5). Besides this case, it is interesting to get mirror models of (1,3) and (1,1) with the family of this curves associated with invariant polynomial on $X^{4,68}$. 15
3.2 Mirror transformation of (1,5)

First we consider the model defined by one of $P_i$’s, for example,

$$f = 1 + cP_3.$$  \hspace{1cm} (3.38)

We denote this model as (5,1) because the indices we find below will be $\tilde{\chi} = 8$. Fundamental period $\omega_0$ of this model is calculated in the same way as in previous section. It is easy see that the curve with any $P_i$ will produce the same fundamental period. We have Picard-Fuchs equation for periods, whose coefficients are given by

$$a_4(z) = z^3(64z - 1)(16z - 1),$$
$$a_3(z) = 6z^2 - 640z^3 + 10240z^4,$$
$$a_2(z) = 7z - 1172z^2 + 25344z^3,$$
$$a_1(z) = 1 - 424z + 14592z^2,$$
$$a_0(z) = -8 + 768z.$$  \hspace{1cm} (3.39)

This Picard-Fuchs equation is “AESZ 16” in the database [15]. The local property of the solution is given by

$$\begin{pmatrix}
0 & 1/64 & 1/16 & \infty \\
0 & 0 & 0 & \frac{1}{2} \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 2 & 2 & \frac{3}{2}
\end{pmatrix}$$  \hspace{1cm} (3.40)
Monodromy matrices are found as following form with $\alpha = \frac{1}{2}$

$$M_{\frac{1}{16}} = \begin{pmatrix}
1 & 0 & 0 & 16 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad M_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 \\
-1 & -1 & -1 & 1 \\
\end{pmatrix},$$

$$M_{\frac{1}{64}} = \begin{pmatrix}
-7 & 8 & -16 & 64 \\
2 & -1 & 4 & -16 \\
1 & -1 & 3 & -8 \\
-1 & 1 & -2 & 9 \\
\end{pmatrix}.$$  \hspace{1cm} (3.41)

Monodromy around $\infty$ is not trivial in this case, so including this contribution we can check the consistency as $M_0 M_{\frac{1}{64}} M_{\frac{1}{16}} M_{\infty} = I$. With this result we see that indices of this model are

$$K = 3, \quad c_2 = 6, \quad -\tilde{\chi} = c_3 = -8.$$  \hspace{1cm} (3.42)

These are checked directly by using bi-linear form $Bi(f, g)$.

Using classical Yukawa coupling

$$K_c[z] = \frac{3}{(1 - 16z)(1 - 64z)}$$  \hspace{1cm} (3.43)

we can calculate genus 0 instanton numbers $a_i$. 
| i  | a_i  | 13 | 17788009827334944 |
|----|------|----|------------------|
| 1  | 12   | 14 | 497375443061477076 |
| 2  | 60   | 15 | 14145255850235272728 |
| 3  | 644  | 16 | 408279490665349434096 |
| 4  | 9216 | 17 | 11938435093860094144356 |
| 5  | 157536 | 18 | 353131094729321849805456 |
| 6  | 3083604 | 19 | 10553174109736271978455644 |
| 7  | 66250884 | 20 | 318296315795274110349024768 |
| 8  | 1522656816 | 21 | 9680349870962148118941442064 |
| 9  | 36850292240 | 22 | 296637049016097525560121350484 |
| 10 | 929119768416 | 23 | 9152575814156431319768582760780 |
| 11 | 24217533456516 | 24 | 284178186604373405325304089538064 |
| 12 | 648807231571968 | 25 | 88745134564110811718556119122168 |

From the monodromy matrices, we also have the exponents of singular points of the weighted discriminant for the genus 1 free energy

$$\log dis[z] = -\frac{1}{6} \{ \log(1 - 16z) + 16 \log(1 - 64z) \}. \quad (3.44)$$

Genus 1 instanton numbers $b_i$ are
Next we consider the mirror model whose defining curve is made out of two kinds of invariant polynomials \( \{ P_i \} \). We anticipate to have some extra symmetry \( \mathbb{Z}_2 \) to reduce Euler number from \( -8 \) to \( -4 \) by combining \( \{ P_i \} \). Recently in [10] the (1,3) model has been discovered as a quotient \( \mathbb{H} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) on \( X^{1,65} \), which may be related to the model we are going to construct here. The criteria to adopt a combination for defining curve as the mirror model of (1,3) are following:

1. derived Picard-Fuchs equation which is satisfied by calculated period integral is of 4th order equation and of Calabi-Yau type.

2. assuming Euler number \( \chi = -4 \), monodromy matrices are all integral and consistent.

For defining the model there would be several possibilities. For example, combinations such as \( \{ P_1, P_2 \} \), \( \{ P_3, P_4 \} \), \( \{ P_3, P_5 \} \), \( \{ P_4, P_5 \} \) would become Calabi-Yau manifolds whose Hodge number could not be small.
A choice we take here is

\[ f = 1 + c(P_1 + 2P_3 + 8). \] (3.45)

Due to the relation

\[ P_1 + 2P_3 + 8 = \left( t_1 t_2 + \frac{1}{t_1 t_2} + 2 \right) \left( t_3 t_4 + \frac{1}{t_3 t_4} + 2 \right) + \left( \frac{t_2}{t_3} + \frac{t_1}{t_3} + 2 \right) \left( \frac{t_4}{t_3} + \frac{t_3}{t_4} + 2 \right), \] (3.46)

and by changes of variables, this curve is expressed as follows

\[ f = 1 + c \left( (\zeta_1 + \frac{1}{\xi_1})^2 (\zeta_2 + \frac{1}{\xi_2})^2 + (\xi_1 + \frac{1}{\xi_1})^2 (\xi_2 + \frac{1}{\xi_1})^2 \right). \] (3.47)

The quadratic form of this curve may enhance the symmetry of the models. We do not investigate here singularities and degrees of freedom of deformations corresponding to this model in detail, we anticipate that three polynomials \( \{P_1 + 2P_i + 8\} \) \( (i = 3, 4, 5) \) would define \((1, 3)\) Calabi-Yau space somehow. We would like to refer the model defined by eq. (3.47) as the mirror of \((1, 3)\) due to Euler number \( \tilde{\chi} = -c_3 \) coming from the monodromy as we will see below.

Picard-Fuchs equation is expressed by following coefficients

\[ a_4(z) = z^3(1 - 32z)(16z - 1)^2(32z - 3)^2, \]
\[ a_3(z) = -2z^2(16z - 1)(32z - 3)(81920z^3 - 14336z^2 + 688z - 9), \]
\[ a_2(z) = -z(-63 + 9132z - 410240z^2 + 7860224z^3 - 66977792z^4 + 209715200z^5), \] (3.48)
\[ a_1(z) = 9 - 3000z + 188928z^2 - 4259840z^3 + 38797312z^4 - 125829120z^5, \]
\[ a_0(z) = -(512z - 24)(16384z^3 - 4096z^2 + 336z - 3). \]

This operator is found as “AESZ 23” in the database [15]. The local properties of the solutions are read as

\[
\begin{pmatrix}
0 & 1/32 & 1/16 & 3/32 & \infty \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1/2 & 1 & 1 \\
0 & 1 & 1/2 & 3 & 1 \\
0 & 2 & 1 & 4 & 2 \\
\end{pmatrix}
\] (3.49)
Using bi-linear form, we first estimate the relation between $K$, $c_2$ and $c_3$ numerically

\[
K = -3c_3, \quad c_2 = -3c_3.
\]  

(3.50)

Next we analyze monodromy behavior around $z = 0, \frac{1}{32}, \frac{1}{16}$ with unknown $c_3$ as

\[
M_{\frac{1}{32}} = \begin{pmatrix}
1 & 0 & 0 & -\frac{32}{c_3} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad M_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-\frac{3}{2}c_3 & -3c_3 & 1 & 0 \\
-\frac{3}{4}c_3 & \frac{3}{2}c_3 & -1 & 1
\end{pmatrix},
\]

(3.51)

\[
M_{\frac{1}{16}} = \begin{pmatrix}
1 & -8 & 0 & \frac{64}{c_3} \\
-1 & 3 & \frac{4}{c_3} & -\frac{32}{c_3} \\
-\frac{c_3}{2} & 0 & 3 & -8 \\
-\frac{c_3}{4} & \frac{c_3}{2} & 1 & -7
\end{pmatrix},
\]

where we set $\alpha = 0$. This result shows that indices for this models must be

\[
c_3 = -4, \quad K = 12, \quad c_2 = 12, \quad \tilde{\chi} = 4.
\]  

(3.52)

This is the reason why we refer this model as the mirror of (1,3).

One of strange things in this model is that monodromy matrix around $z = \frac{1}{16}$ does not have standard form, and its square becomes the one we expected

\[
M^2_{\frac{1}{16}} = \begin{pmatrix}
-7 & 0 & -8 & 32 \\
2 & 1 & 2 & -8 \\
0 & 0 & 1 & 0 \\
-2 & 0 & -2 & 9
\end{pmatrix}.
\]  

(3.53)

Classical Yukawa coupling is

\[
K_c[z] = \frac{4(3 - 32z)}{(1 - 16z)^2(1 - 32z)}.
\]  

(3.54)

therefore we have genus 0 instanton numbers $a_i$ as
For genus 1 case, we have to get the exponents of singular points of $Dis[z]$ from monodromy matrices. As we have mentioned, monodromy around $z = \frac{1}{16}$ are not usual, so we propose that logarithm of $Dis[z]$ would be

$$\log Dis[z] = -\frac{1}{6} \left\{ \lambda_1 \log(1 - 16z)^2 + \lambda_2 \log(1 - 32z) \right\}, \quad (3.55)$$

where $\lambda_1$ is the exponent from $M^2_1$. With exponents $\lambda_1 = 2, \lambda_2 = 8$, the result for genus 1 instanton calculation is
| $i$ | $b_i$   | $b_i$ |
|-----|---------|-------|
| 1   | 8       | 13    | $177583895318624$ |
| 2   | 82      | 14    | $2579924022491086$ |
| 3   | 856     | 15    | $37674030557685648$ |
| 4   | 10321   | 16    | $552612289406933025$ |
| 5   | 128864  | 17    | $813788233521859928$ |
| 6   | 1677110 | 18    | $12025275347028012752$ |
| 7   | 22506040| 19    | $1782584199002075687048$ |
| 8   | 308025697| 20    | $26497544221536391150192$ |
| 9   | 4282495040| 21    | $394868536532868752512$ |
| 10  | 60292530504| 22    | $589779177076681458656334$ |
| 11  | 857470990104| 23    | $88272996476471800976727752$ |
| 12  | 12296761625755| 24    | $1323704678176645827390945547$ |
| 13  | 177583895318624| 25    | $19884364383541833676997550064$ |

Before closing this subsection we mention about results given by similar calculations based on mirror transformations in two different models known as (1,3) in [2, 18]. First model is a quotient $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$ constructed on $X^{5,45}$. Following the literature, defining curve for a mirror of this model would be

$$
p_1 = 1 + a_1(t_4t_5 + t_5t_1 + t_1t_2 + t_2t_3 + t_3t_4) + a_2(t_3t_5 + t_4t_1 + t_5t_2 + t_1t_3 + t_2t_4) + a_3(t_1t_2t_3t_4t_5(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5})),
$$

$$
p_2 = t_1t_2t_3t_4t_5 + a_1(t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_5 + t_4t_5t_1 + t_5t_1t_2) + a_2(t_1t_2t_4 + t_2t_3t_5 + t_3t_4t_1 + t_4t_5t_2 + t_5t_1t_3) + a_3(t_1 + t_2 + t_3 + t_4 + t_5). \quad (3.56)
$$

with restriction $a_1 = a_2 = 0$. We can derive Picard-Fuchs equation (AESZ 34 [15]), and expect consistent result for $\chi = -4$ with $K = 6$, $c_2 = 6$. However, monodromy matrices can not be integral, thus we would conclude this is a mirror of (1,5) model of $\chi = -8$ with $K = 12$, $c_2 = 12$.

Second one is the model on $X^{19,19}$ modulo $\text{Disc}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. In the literature, explicite
curve to define a mirror of this model is not found, so we propose following form

\[ p = 1 + s_0s_1 + s_1s_2 + s_2s_0, \quad q = 1 + t_0t_1 + t_1t_2 + t_2t_0, \]
\[ r = s_0s_1t_0t_1t_2 + c(s_0t_0 + s_1t_1 + s_2t_2 + s_0t_1 + s_1t_2 + s_2t_0 + s_0t_2 + s_1t_0 + s_2t_1). \] (3.57)

Calculations lead us to Picard-Fuchs equation (AESZ 103 [15]), and results about monodromy and instanton calculations done by mirror transformation are all consistent for \( \chi = -4 \) with \( K = 3, \ c_2 = 6. \)

### 3.4 Mirror transformation of minimal model (1,1)

The model which has minimal Hodge numbers is (1,1). This model is originally found by studying 24-cell in [7]. A example of curve to define this model in \( \mathbb{C}^8 \) is

\[ p = 1 + \sum_{i=1}^{8} x_i + x_1x_3 + x_1x_5 + x_2x_6 + x_3x_7 + x_3x_5 + x_6x_8 + x_1x_3x_7 \]
\[ + x_3x_6(x_1 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_3x_6) + \varphi x_3x_6, \] (3.58)

with identifications

\[ x_1x_3 = x_2x_4, x_1x_5 = x_4x_6, x_1x_7 = x_2x_6, x_1x_8 = x_2x_5 = x_3x_6 = x_4x_7, \]
\[ x_2x_8 = x_3x_7, x_3x_5 = x_4x_8, x_5x_7 = x_6x_8. \] (3.59)

Reducing the number of variables by using above identifications from eight to four, and changing variables, effective curve would be

\[ f = 1 + c \left( s_1 + s_2 + s_3 + s_4 + \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s_4} \right. \]
\[ + s_2s_3 + \frac{1}{s_2s_3} + \frac{1}{s_4} + \frac{1}{s_4} + \frac{1}{s_1} + \frac{1}{s_1} + \frac{1}{s_3} + \frac{1}{s_3} \]
\[ + \left. \frac{s_1}{s_2s_3} + \frac{s_2s_3}{s_1} + \frac{s_1}{s_2s_4} + \frac{s_2s_4}{s_1} + \frac{s_4}{s_2s_3} + \frac{s_2s_3}{s_4} + \frac{s_4}{s_1s_3} + \frac{s_1s_3}{s_4} \right), \] (3.60)

where moduli \( c = 1/\varphi \)

As this minimal model (1,1), we propose alternative definition of curve made out of invariant polynomials \( \{ P_i \} \) on \( X^{48} \)

\[ f = 1 + c(P_3 + P_4 + P_5). \] (3.61)
This is unique definition of using three kinds of invariant polynomials with $Z_3$ symmetry. This definition is different from the curve (3.78), however we can show that periods, monodromy matrices, and instanton numbers obtained by both definitions are completely same.

From the series expansion of fundamental period, we have Picard-Fuchs equation of the form

\[ a_4(z) = -z^3(8z + 1)(24z - 1)(3z + 1)(4z + 1)(12z + 1)(1 + 18z)^2, \]
\[ a_3(z) = 6z^2 + 204z^3 - 1948z^4 - 184248z^5 - 3322944z^6 - 26476416z^7 - 9551488z^8 - 125411328z^9, \]
\[ a_2(z) = 7z + 164z^2 - 8310z^3 - 455148z^4 - 8595936z^5 - 77054976z^6 - 319997952z^7 - 483729408z^8, \]
\[ a_1(z) = 1 - 14z - 5574z^2 - 274788z^3 - 5818176z^4 - 60943104z^5 - 300589056z^6 - 537477120z^7, \]
\[ a_0(z) = -384z - 22752z^2 - 606528z^3 - 7921152z^4 - 48771072z^5 - 107495424z^6. \]

This is nothing but “AESZ 366” in the database [15]. The local property for periods is summarized as

\[
\begin{pmatrix}
-1/3 & -1/4 & -1/8 & -1/12 & -1/18 & 0 & 1/24 & \infty \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 3 & 0 & 1 & 2 \\
2 & 2 & 2 & 2 & 4 & 0 & 2 & 3
\end{pmatrix}
\]  

(3.63)
Monodromy matrices are found as

\[
M_{\frac{1}{24}} = \begin{pmatrix} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & -1 & 1 \end{pmatrix},
\]

\[
M_{-\frac{1}{12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ -2 & -4 & -1 & 0 \\ -1 & -2 & -1 & 1 \end{pmatrix}, \quad M_{-\frac{1}{8}} = \begin{pmatrix} -23 & -48 & -48 & 72 \\ 16 & 33 & 32 & -48 \\ -16 & -32 & -31 & 48 \\ -8 & -16 & -16 & 25 \end{pmatrix},
\]

\[
M_{-\frac{1}{4}} = \begin{pmatrix} -95 & -144 & -264 & 576 \\ 44 & 67 & 121 & -264 \\ -24 & -36 & -65 & 144 \\ -16 & -24 & -44 & 97 \end{pmatrix}, \quad M_{-\frac{1}{2}} = \begin{pmatrix} -95 & -96 & -288 & 768 \\ 36 & 37 & 108 & -288 \\ -12 & -12 & -35 & 96 \\ -12 & -12 & -36 & 97 \end{pmatrix}.
\]

From these matrices, we would read indices of this models with \(\alpha = 0\) as

\[
K = 4, \quad c_2 = 4, \quad c_3 = -\tilde{\chi} = 0.
\]  

(3.65)

Apart from previous examples, we are not able to check by bi-linear form \(Bi(f, g)\) in this case, because \(\tilde{\chi} = 0\).

Curious results appear about genus 0 instanton numbers. Following same procedure as before, and using classical Yukawa coupling

\[
K_e[z] = \frac{4(1 + 18z)}{(1 + 3z)(1 + 4z)(1 + 8z)(1 + 12z)(1 - 24z)},
\]

(3.66)

we find that instanton numbers in genus 0 level for even order become negative.
For genus 1 instanton numbers, we have to find correct exponents of $\text{Dis}[z]$. Direct calculations about monodromy matrices lead us the form of $\log \text{Dis}[z]$ as

\[
\log \text{Dis}[z] = -\frac{1}{6} \left\{ 12 \log (1 + 3z) + \log (1 + 4z) + 8 \log (1 + 8z) + 24 \log (1 - 24z) + \log (1 + 12z) \right\}
\]  

(3.67)

Differently from the previous models, this discriminant produces wrong genus 1 behaviors whose instanton numbers are half-integers, such as

\[
b_1 = 35, \ b_2 = \frac{753}{2}, \ b_3 = 3175, \ b_4 = 45510, \ b_5 = 501917, \ b_6 = \frac{1583609}{2}, \ \cdots
\]

(3.68)

Surprisingly, if we set the coefficient of $\log (1 + 12z)$ in $\log \text{Dis}[z]$ to be 5 mod 6, every genus 1 instanton number becomes integer up to 50th orders.

Lastly we add some comments about results of a model made of another combination of three $P_i$, whose defining curve is $f = 1 - c(P_1 + P_2 + 4P_3)$, though this is not relevant to the model (1,1). Picard-Fuchs equation of this model looks ordinary Calabi-Yau type (AESZ 107 [15]), however topological indices turn out to be unusual values as $K = 4$, $c_2 = 4$, $\chi = 10$.  

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|}
\hline
$i$ & $a_i$ \\
\hline
1 & 12 \\
2 & -16 \\
3 & 256 \\
4 & -1012 \\
5 & 17168 \\
6 & -102432 \\
7 & 1768032 \\
8 & -12810048 \\
9 & 226260008 \\
10 & -1831410544 \\
11 & 33000429000 \\
12 & -286340050052 \\
13 & 5252822116016 \\
14 & -47718467477584 \\
15 & 890108488876160 \\
16 & -8340130846927456 \\
17 & 158096635640838140 \\
18 & -1512328959263997360 \\
19 & 29129403982340313132 \\
20 & -282368793768124234092 \\
21 & 5527396080871599103212 \\
22 & -53986928091516821971440 \\
23 & 1074486987800843943995916 \\
24 & -10525957761076292523611520 \\
25 & 213137290904593560452816768 \\
\hline
\end{tabular}
\end{center}
\end{table}
Monodromy matrices are consistent, and instanton numbers are all integers in genus 0 and 1 level, however some of them become negative.

4 Conclusion and Discussions

We have presented mirror transformations of Calabi-Yau manifolds whose Hodge numbers \((h^{11}, h^{21})\) are both small. We have determined the monodromy of the models completely, and enumerated genus 0 and 1 instanton numbers of the models by using weighted discriminant for genus 1 level. Results based on the mirror models of \((1,5)\) and \((1,4)\) are consistent. Since the Yukawa coupling as well as instanton numbers in genus 0 in these quotient models are directly related to the quantities on originated manifolds by division of freely acting group, genus 1 calculations are more significant. We have also proposed the description for mirrors of \((1,3)\) and \((1,1)\) models by using invariant polynomials of \((1,5)\) model. Results in \((1,3)\) case look reasonable, however in minimal case \((1,1)\) negative and half integer value of instanton numbers appear against our expectation. Special treatment might be needed when you calculate instanton numbers in the model with Euler number \(\chi \geq 0\).

We attempted as many combinations as possible of invariant polynomials that could be viewed as definitions of mirror models of Calabi-Yau with small Hodge numbers discussed in section 3, however we couldn’t find an appropriate one to a mirror of \((1,2)\) model. The extention to include sets of invariant polynomials of \((1,4)\) model on \(X^{8,44}\), or \((1,3)\) model on \(X^{5,45}\) did not work well so far. It is interesting to recognize how to describe the model \((1,2)\) in a way suggested in [4] and its mirror.

The numerical integration around singular points to fix monodromy behaviors would be applicable to several modulus case. This method may help us to perform mirror transformations for various string compactifications. It is interesting if \((2,2)\) model could be analyzed in a view point of mirror symmetry transformation by applying methods we discussed here, as well as the conifold transition to other models such as \((1,3)\) and \((1,4)\).

Also investigations to apply these methods to the mirror symmetry with small Hodge numbers in open string theories including D-branes would be interesting.
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