ON A POWERED BOHR INEQUALITY

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Abstract. The object of this paper is to study the powered Bohr radius $\rho_p$, $p \in (1, 2)$, of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and such that $|f(z)| < 1$ defined on the unit disk $|z| < 1$. More precisely, if $M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k$, then we show that $M_p^f(r) \leq 1$ for $r \leq r_p$ where $r_p$ is the powered Bohr radius for conformal automorphisms of the unit disk. This answers the open problem posed by Djakov and Ramanujan in 2000. A couple of other consequences of our approach is also stated, including an asymptotically sharp form of one of the results of Djakov and Ramanujan. In addition, we consider a similar problem for sense-preserving harmonic mappings in $|z| < 1$. Finally, we conclude by stating the Bohr radius for the class of Bieberbach-Eilenberg functions.

1. Preliminaries and Main Results

Let $B$ denote the class of analytic functions $f$ defined on the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$, with the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and such that $|f(z)| < 1$ for $z \in D$. Then the classical Bohr’s inequality states that there is a constant $\rho$ such that

$$M^f(r) := \sum_{k=0}^{\infty} |a_k| r^k \leq 1 \quad \text{for all } r = |z| \leq \rho$$

and the value $\rho = 1/3$ is optimal. The number $\rho = 1/3$, known as Bohr’s radius, was originally obtained in 1914 by H. Bohr [6] with $\rho = 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $r = |z| \leq 1/3$. This little article of Bohr generates intensive research activities even after a century of its appearance. We refer to the recent survey article on this topic [4] and the references therein. Multidimensional generalizations of this result were obtained by Boas and Khavinson [5] by establishing upper and lower bounds for the Bohr radius of the unit polydisk $D^n$. Aizenberg [2,3] extended the concept of Bohr radius in several different directions for further studies in this topic. In 2000, Djakov and Ramanujan [10] investigated the same phenomenon from different point of view. For $f \in B$ and a fixed $p > 0$, we consider the powered Bohr sum $M_p^f(r)$ defined by

$$M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k.$$
Observe that for $p = 1$, $M^f_p(r)$ reduces to the classical Bohr sum defined as above by $M^f(r)$. The best possible constant $\rho_p$ for which

$$M^f_p(r) \leq 1$$

is called the (powered) Bohr radius for the family $\mathcal{B}$.

We now introduce

$$M_p(r) := \sup_{f \in \mathcal{B}} M^f_p(r)$$

and

$$r_p := \sup \left\{ r : a^p + \frac{r(1 - a^2)^p}{1 - ra^p} \leq 1, \ 0 \leq a < 1 \right\} = \inf_{a \in [0,1)} \frac{1 - a^p}{a^p(1 - a^p) + (1 - a^2)^p}.$$

Let us first proceed to recall the following results.

**Theorem A.** ([10, Theorem 3]) For each $p \in (1,2)$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belongs to $\mathcal{B}$, we have $M^f_p(r) \leq 1$ for $r \leq T_p$, where

$$m_p \leq T_p \leq r_p.$$

Here $r_p$ is as above and

$$m_p := \frac{p}{(2^{1/(2-p)} + p^{1/(2-p)})^{2-p}}.$$

**Theorem B.** ([10, Theorem 2]) For each $p \in (0,2)$

$$M_p(r) \asymp \left( \frac{1}{1 - r} \right)^{1-p/2}.$$

Our first aim is to investigate the problem posed by Djakov and Ramanujan [10] about the Bohr radius for $M^f_p(r)$. Their question is the following.

**Problem 1.** [10, Question 1, p. 71] What is the exact value of the (powered) Bohr radius $\rho_p$, $p \in (1,2)$? Is it true that $\rho_p = r_p$?

Using the method of proofs of our recent approach from [12][13], we solve this problem affirmatively in the following form.

**Theorem 1.** If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belongs to $\mathcal{B}$ and $0 < p \leq 2$, then

$$M_p(r) = \max_{a \in [0,1]} \left[ a^p + \frac{r(1 - a^2)^p}{1 - ra^p} \right], \quad 0 \leq r \leq 2^{p/2-1},$$

and

$$M_p(r) < \left( \frac{1}{1 - r^{2/(2-p)}} \right)^{1-p/2}, \quad 2^{p/2-1} < r < 1.$$
Proofs of Theorem 1 and a couple of its corollaries will be given in Section 2. Let us remark that $M_p(r) = 1$ for $p \geq 2$ and $r \leq 1$. So, the interesting case is to consider the problem only for $p \in (1, 2)$.

One may ask about the second inequality of Theorem 1: how close it to be sharp? To get an answer to this question we will use a Bombieri-Bourgain estimate [8] which reads as follows: for a given $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$, such that

$$M_1(\rho) \geq \frac{1}{\sqrt{1 - \rho^2}} - C(\varepsilon) \left( \log \frac{1}{1 - \rho} \right)^{(3/2) + \varepsilon}, \quad \rho \geq 1/\sqrt{2}. $$

The Hölder inequality implies that

$$M_1(f, r^{1/(2-p)}) = \sum_{k=0}^{\infty} |a_k| |r^{k/p} r^{(2k(p-1))/(p(2-p))}|
\leq \left( \sum_{k=0}^{\infty} |a_k|^p r^k \right)^{1/p} \left( \sum_{k=0}^{\infty} r^{2k/(2-p)} \right)^{1-1/p}
= \left( M_p^1(r) \right)^{1/p} \frac{1}{(1 - r^{2/(2-p)})(p-1)/p}
$$

so that

$$M_p^1(r) \geq \left( \frac{1}{\sqrt{1 - r^{2/(2-p)}}} - C(\varepsilon) \left( \log \frac{1}{1 - r^{1/(2-p)}} \right)^{3/2 + \varepsilon} \right)^p (1 - r^{2/(2-p)})^{p-1}, \quad 2^{p-1} < r < 1,$$

or equivalently

$$M_p^1(r) \geq \left( \frac{1}{1 - r^{2/(2-p)}} \right)^{1-p/2} - C_1(\varepsilon)(1 - r^{2/(2-p)})^{(p-1)/2} \left( \log \frac{1}{1 - r^{1/(2-p)}} \right)^{3/2 + \varepsilon}.$$

This estimate together with the second estimate of Theorem 1 implies that

$$M_p(r) - \left( \frac{1}{1 - r^{2/(2-p)}} \right)^{1-p/2} \rightarrow 0 \quad \text{as} \quad r \rightarrow 1^-$$

for $1 < p < 2$ while we do not know whether this fact is true for $p = 1$. Also the last estimate can be considered as an asymptotically sharp form of Theorem B in the case $p > 1$.

**Corollary 1.** Let $p \in (1, 2)$. Then $M_p(r) = 1$ for $r \leq r_p$.

In [16] Corollary 2.8, Paulsen et al. showed that if $f \in \mathcal{B}$, then for $r \in [0, 1)$,

$$M_1^f(r) \leq m(r) = \inf \{ M(r), 1/\sqrt{1 - r^2} \}
$$

where

$$M(r) = \sup \left\{ t + (1 - t^2) \frac{r}{1 - r} : 0 \leq t \leq 1 \right\} = \begin{cases} 
\frac{1}{4r^2 + (1 - r)^2} & \text{for } 0 \leq r \leq 1/3 \\
\frac{4r^2 + (1 - r)^2}{4r(1 - r)} & \text{for } 1/3 < r < 1.
\end{cases}$$

In 2002, Paulsen et al. [16] raised a question whether the inequality (11) is sharp for any $r$ with $1/3 < r < 1$. However, in 1962 this has been answered by Bombieri [7] who
determined the exact value of this constant for $r$ in the range $1/3 \leq r \leq 1/\sqrt{2}$. This constant is

$$m(r) = \frac{3 - \sqrt{8(1 - r^2)}}{r}.$$  

Further results on this and related topics can be found in [13]. On the other hand, it is worth mentioning that the answer to the above question is indeed a consequence of Theorem 1 and so, we state it as a corollary.

**Corollary 2.** We have the following sharp estimate:

$$M_1(r) = \frac{1}{r}(3 - \sqrt{8(1 - r^2)}) \text{ for } r \in \left[\frac{1}{3}, \frac{1}{\sqrt{2}}\right].$$

Finally, we recall the following corollary which was proved in [13] and so we omit the proof.

**Corollary 3.** Let $p \in \mathbb{N}$ and $0 \leq m \leq p$, $f(z) = \sum_{k=0}^{\infty} a_{pk+m}z^{pk+m}$ be analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then

$$\sum_{k=0}^{\infty} |a_{pk+m}|r^{pk+m} \leq 1 \text{ for } r \leq r_{p,m},$$

where $r_{p,m}$ is the maximal positive root of the equation

$$-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0.$$  

The extremal function has the form $z^m(z^p - a)/(1 - az^p)$, where

$$a = \left(1 - \frac{\sqrt{1 - r_{p,m}^{2p}}}{\sqrt{2}}\right) \frac{1}{r_{p,m}^{p+1}}.$$

Our next result concerns sense-preserving harmonic mappings defined on the unit disk $\mathbb{D}$. Recall that the family $\mathcal{H}$ of complex-valued harmonic functions $f = h + \overline{g}$ defined on $\mathbb{D}$ and its univalent subfamilies are investigated in details. Here $h$ and $g$ are analytic on $\mathbb{D}$ with the form

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k$$

so that the Jacobian of $f$ is given by $J_f = |f_z|^2 - |f_\bar{z}|^2 = |h'|^2 - |g'|^2$. We say that the harmonic mapping $f$ is sense-preserving if $J_f(z) > 0$ in $\mathbb{D}$. We call $\omega(z) = g'(z)/h'(z)$ the complex dilatation of $f = h + \overline{g}$. Lewy’s theorem implies that every harmonic function $f$ on $\mathbb{D}$ is locally one-to-one and sense-preserving on $\mathbb{D}$ if and only if $|\omega(z)| < 1$ for $z \in \mathbb{D}$. See [9,11] for detailed discussion on the class of univalent harmonic mappings and its geometric subclasses.

**Theorem 2.** Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k$ is a harmonic mapping of the disk $\mathbb{D}$, where $h$ is a bounded function in $\mathbb{D}$ and $|g'(z)| \leq |h'(z)|$ for $z \in \mathbb{D}$ (the later condition obviously holds if $f$ is sense-preserving). If $p \in [0,2]$ then the following sharp inequality holds

$$|a_0|^p + \sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)r^k \leq ||h||_{\infty} \max_{a \in [0,1]} \left\{ a^p + \frac{2r(1 - a^2)^p}{1 - ra^p} \right\}$$

where $r = |z| < 1$.  

for \( r \leq (2^{1/(p-2)} + 1)^{p/2-1} \). In the case \( p > 2 \) we have

\[
|a_0|^p + \sum_{k=1}^\infty (|a_k|^p + |b_k|^p)r^k \leq \|h\|_\infty \max\{1, 2r\}.
\]

**Corollary 4.** Suppose that \( f(z) = h(z) + g(z) = \sum_{k=0}^\infty a_k z^k + \sum_{k=1}^\infty b_k z^k \) is a sense-preserving harmonic mapping of the disk \( \mathbb{D} \), where \( h \) is a bounded function in \( \mathbb{D} \). Then the following sharp inequalities holds:

\[
|a_0| + \sum_{k=1}^\infty (|a_k| + |b_k|)r^k \leq \frac{\|h\|_\infty}{r} (5 - 2\sqrt{6\sqrt{1-r^2}}) \quad \text{for} \quad \frac{1}{5} \leq r \leq \sqrt{\frac{2}{3}},
\]

and

\[
|a_0| + \sum_{k=1}^\infty (|a_k| + |b_k|)r^k \leq \|h\|_\infty \quad \text{for} \quad r \leq \frac{1}{5}.
\]

Proofs of Theorem 2 and Corollary 4 will be given in Section 2. In Section 3 we discuss Bohr radius for the class of Bieberbach-Eilenberg functions.

2. **Proofs of Theorems 1 and 2 and their corollaries**

The proofs of the theorems rely on a couple of lemmas established by the present authors in [12] (see also [13]).

**Lemma 1.** [12] Let \( |a| < 1 \) and \( 0 < R \leq 1 \). If \( g(z) = \sum_{k=0}^\infty b_k z^k \) belongs to \( \mathcal{B} \), then the following sharp inequality holds:

\[
\sum_{k=1}^\infty |b_k|^2 R^k \leq R \frac{(1 - |b_0|^2)^2}{1 - |b_0|^2 R}.
\]

**Lemma 2.** For all \( p \in (0, 2) \), we have \( r_p < (1/2)^{1-p/2} \).

**Proof.** Let \( r = r_p \) and set \( a = (1/2)^{1-p/2} \). Then we conclude that

\[
a^p + r \frac{(1 - a^2)^p}{1 - ra^p} = 2 \left( \frac{1}{2} \right)^{p/2} > 1
\]

which contradicts to the definition of \( r_p \). \( \square \)
Proof of Theorem 1. Let \( |a_0| = a > 0 \) and \( r \leq 2^{p/2 - 1} \). At first we suppose that \( a > r^{1/(2-p)} \). In this case we have

\[
M_p^f(r) = a^p + \sum_{k=1}^{\infty} \rho^k |a_k|^p \left( \frac{r}{\rho} \right)^k
\]

\[
\leq a^p + \left( \sum_{k=1}^{\infty} (\rho^k |a_k|^p)^{2/p} \right)^{p/2} \left( \sum_{k=1}^{\infty} \left( \frac{r}{\rho} \right)^{2k/(2-p)} \right)^{1-p/2}
\]

\[
= a^p + \left( \sum_{k=1}^{\infty} (\rho^{2/p} |a_k|^2)^{2/p} \right)^{p/2} \left( \sum_{k=1}^{\infty} \left( \frac{r}{\rho} \right)^{2k/(2-p)} \right)^{1-p/2}
\]

\[
\leq a^p + \left( \frac{\rho^{2/p}(1 - a^2)^2}{1 - a^2 \rho^{2/p}} \right)^{p/2} \left( \frac{(r/\rho)^{2/(2-p)}}{1 - (r/\rho)^{2/(2-p)}} \right)^{(2-p)/2}
\]

(by Lemma 1).

\[
= a^p + r \left( \frac{1 - a^2}{1 - a^2 \rho^{2/p}} \right)^{p/2} \left( \frac{1}{1 - (r/\rho)^{2/(2-p)}} \right)^{(2-p)/2}.
\]

Setting \( \rho = r^{p/2}a^{(p-2)/p^2} \) we obtain the inequality

\[
M_p^f(r) \leq a^p + r \frac{(1 - a^2)^p}{1 - ra^p},
\]

which proves the theorem in the case \( a > r^{1/(2-p)} \).

In the case \( a \leq r^{1/(2-p)} \), we set \( \rho = 1 \) and obtain

\[
M_p^f(r) = \sum_{k=0}^{\infty} |a_k|^p r^k \leq a^p + r \frac{(1 - a^2)^{p/2}}{(1 - r^{2/(2-p)})^{1-p/2}}.
\]

Let us remark that the inequality \( M_p^f(r) \leq 1 \) is valid in the cases \( a = 0 \) and \( a = r^{1/(2-p)} \).

This fact can be established as a limiting case of the previous case. Finally, we let \( t = a^2 \).

We have then to maximize the expression

\[
A(t) = t^{p/2} + r \frac{(1 - t)^{p/2}}{(1 - r^{2/(2-p)})^{1-p/2}}, \quad t \leq r^{2/(2-p)}.
\]

Using differentiation we obtain the stationary point

\[
t = 1 - r^{2/(2-p)}
\]

which must satisfy under the restriction \( t \leq r^{2/(2-p)} \) which is impossible because \( r \leq 2^{p/2 - 1} \).

However, in the case \( r > 2^{p/2 - 1} \) the critical point \( t \) is admissible so that

\[
A(t) = t^{p/2} + r \frac{(1 - t)^{p/2}}{(1 - r^{2/(2-p)})^{1-p/2}} = \left( \frac{1}{1 - r^{2/(2-p)}} \right)^{1-p/2}.
\]

This observation shows that

\[
M_p^f(r) \leq \left( \frac{1}{1 - r^{2/(2-p)}} \right)^{1-p/2}, \quad 2^{p/2 - 1} < r < 1.
\]
Now let us show that this inequality cannot be sharp. To do this we will use the method presented by Bombieri and Bourgain [8].

Suppose that the estimate sharp in this case. Then by analyzing Hölder’s inequality we immediately conclude that

$$|a_k| = \sqrt{1 - r^{2/(2-p)} k^{(2-p)}} \quad k \geq 0.$$  

Also it is easy to show that the extremal function must be a Blashke’s product with a finite degree \( d \geq 1 \). Computing the area, one obtains that

$$\pi d = \text{Area } f(\mathbb{D}) = \pi \sum_{k=1}^{\infty} k|a_k|^2 = \pi \frac{\lambda^2}{1 - \lambda^2}, \quad \lambda = r^{1/(2-p)}.$$

From here we easily deduce that \( d = \lambda^2/(1 - \lambda^2) \) and thus, \( \lambda = \sqrt{d/(d + 1)} \), which gives

$$\sqrt{\frac{d}{d + 1}} = r^{1/(2-p)}, \quad \text{i.e.} \quad r = \left( \frac{d}{d + 1} \right)^{1-(p/2)}.$$

Therefore our inequality could be sharp for these values only. Now let us show that this is possible for \( d = 1 \) only. Using the same reasoning as in [8] (in fact we apply their considerations in which \( r \) is replaced by \( r^{1/(2-p)} \)) we arrive at the identity

$$\sqrt{1 - r^{2/(2-p)}} = r^{d/(2-p)}$$

which together with (2) implies that

$$\sqrt{1 - \frac{d}{d + 1}} = \left( \frac{d}{d + 1} \right)^{d/2}.$$

From here we easily deduce that \( d = 1 \) and this completes the proof of Theorem 1. \( \square \)

**Proof of Corollary 1.** Easily follows from Theorem 1 and Lemma 2. \( \square \)

**Proof of Corollary 2.** Theorem 1 for \( p = 1 \) gives that

$$M_p(r) = \max_{a \in [0,1]} \left\{ a + \frac{r(1 - a^2)}{1 - ra} \right\}.$$

By using differentiation it is easy to show that in the case \( 1/3 \leq r \leq 1/\sqrt{2} \) the maximum of the last expression is achieved at the point

$$a = \left( 1 - \frac{\sqrt{1 - r^2}}{\sqrt{2}} \right) \frac{1}{r}$$

and consequently, we obtain that

$$M_1(r) = \frac{1}{r} (3 - 2\sqrt{2}\sqrt{1 - r^2}).$$

The proof is complete. \( \square \)
Proof of Theorem \[\Box\] Without lost of generality we may assume that \(\|h\|_\infty = 1\). As in \[14\], the condition \(|g'(z)| \leq |h'(z)|\) gives that for each \(r \in [0, 1)\),

\[
\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} |a_k|^r.
\]

Let \(|a_0| = a > 0\). Then, by using the same method as in the previous theorem in the case \(a > \frac{1}{2-p}\), we obtain

\[
|a_0|^p + \sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p) r^k \leq a^p + 2r \frac{(1-a^2)^p}{1-ra^p}.
\]

In the case \(a \leq \frac{1}{2-p}\), we let \(\rho = 1\) and obtain

\[
\sum_{k=0}^{\infty} |a_k|^r \leq a^p + 2r - \frac{(1-a^2)^p/2}{(1-r^{2/(2-p)})^{1-p/2}}.
\]

We set \(t = a^2\). We have to maximize the expression

\[
B(t) = t^{p/2} + 2r \frac{(1-t)^{p/2}}{(1-r^{2/(2-p)})^{1-p/2}}, \quad t \leq r^{2/(2-p)}.
\]

Using differentiation we see that the function \(B(t)\) is increasing on the interval

\[
0 \leq t \leq \frac{1 - r^{2/(2-p)}}{1 + (2r)^{2/(2-p)} - r^{2/(2-p)}}.
\]

The upper bound of this interval is greater than or equal to \(2^{p-1}\) in the case \(r \leq (2^{1/(p-2)} + 1)^{p/2-1}\). It means that the function \(B(t)\) has maximum at the point \(t = r^{2/(2-p)}\) which corresponds to the case \(a = \frac{1}{2-p}\) so that we can apply our previous case. This completes the proof Theorem \[\Box\]

Let \(p = 1\) and then we apply the previous theorem. As a result, we obtain the inequality

\[
|a_0| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \leq \max_{a \in [0,1]} \left\{ a + 2r(1-a^2) \frac{1-ra}{1-ra} \right\} \text{ for } r \leq \sqrt{2/3}.
\]

Straightforward calculations confirm the proof of Corollary \[\Box\] In Section \[3\] we present the Bohr radius for the class of Bieberbach-Eilenberg functions.

3. CONCLUDING REMARKS

Let \(\mathcal{BE}\) denote the class of all functions \(f(z) = \sum_{k=1}^{\infty} a_k z^k\) analytic in \(D\) such that \(f(z_1)f(z_2) \neq 1\) for all pairs of points \(z_1, z_2\) in \(D\). Each \(f \in \mathcal{BE}\) is called a Bieberbach-Eilenberg function. Clearly, \(\mathcal{BE}\) contains the class \(\mathcal{B}_0\), where \(\mathcal{B}_0 = \{ f \in \mathcal{B} : f(0) = 0 \}\). In 1970, Aharonov \[1\] and Nehari \[15\] independently showed that

\[
\sum_{k=1}^{\infty} |a_k|^2 \leq 1 \text{ and } |f(z)| \leq \frac{|z|}{\sqrt{1-|z|^2}}.
\]
hold for every $f \in \mathcal{BE}$. Equality holds only for the functions
\[ f(z) = \frac{\eta z}{R \pm (\sqrt{R^2 - 1})i\eta z}, \quad R > 1, \quad |\eta| = 1. \]
Since $\mathcal{B}_0 \subset \mathcal{BE}$, it is natural to ask for the Bohr radius for the family $\mathcal{BE}$. Indeed, we see below that the Bohr radius for $\mathcal{BE}$ and the class $\mathcal{B}_0$ remains the same.

**Theorem 3.** Assume that $f(z) = \sum_{k=1}^{\infty} a_k z^k$ belongs to $\mathcal{BE}$. Then
\[ \sum_{k=1}^{\infty} |a_k|r^k \leq 1 \quad \text{for} \quad |z| = r \leq 1/\sqrt{2}. \]

The number $1/\sqrt{2}$ is sharp.

**Proof.** Because $f \in \mathcal{BE}$ satisfies the coefficient inequality (4), it follows that
\[
\sum_{k=1}^{\infty} |a_k|r^k \leq \sqrt{\sum_{k=1}^{\infty} |a_k|^2} \sqrt{\sum_{k=1}^{\infty} r^{2k}} \leq \frac{r}{\sqrt{1 - r^2}}
\]
which is less than or equal to 1 if $0 \leq r \leq 1/\sqrt{2}$. The number $1/\sqrt{2}$ is sharp as the function $f(z) = z(a - z)/(1 - az)$ shows, where $a = 1/\sqrt{2}$. The proof is complete. \(\square\)

**Theorem 4.** Suppose that $f(z) = h(z) + g(z) = \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k$ is a harmonic mapping of the disk $\mathbb{D}$, where $h \in \mathcal{BE}$ and $|g'(z)| \leq |h'(z)|$ for $z \in \mathbb{D}$. Then for any $p \geq 1$ and $r < 1$, the following inequality holds:
\[ \sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{1/p} r^k \leq \max\{2^{(1/p)-1/2}, 1\} \frac{\sqrt{2}r}{\sqrt{1 - r^2}}. \]

**Proof.** By hypothesis, (3) holds and thus, letting $r$ approach 1, we get
\[ \sum_{k=1}^{\infty} |b_k|^2 \leq \sum_{k=1}^{\infty} |a_k|^2 \leq 1. \]
Consequently, we obtain
\[
\sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{1/p} r^k \leq \sqrt{\sum_{k=1}^{\infty} (|a_k|^p + |b_k|^p)^{2/p} \sum_{k=1}^{\infty} r^{2k}} \leq \sqrt{\max\{2^{(2/p)-1}, 1\} \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2) \frac{r}{\sqrt{1 - r^2}}} \leq \max\{2^{(1/p)-1/2}, 1\} \frac{\sqrt{2}r}{\sqrt{1 - r^2}}
\]
and the proof is complete. \(\square\)
Theorem 4 for \( p = 1 \) shows that for \( r \leq 1/\sqrt{5} \),
\[
\sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \leq 1.
\]

Similarly, for \( p = 2 \), we see that for \( r \leq 1/\sqrt{3} \),
\[
\sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2)^{1/2} r^k \leq 1.
\]

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