A NOTE ON RESOLUTION OF RATIONAL AND HYPERSURFACE SINGULARITIES

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Abstract. It is well known that the exceptional set in a resolution of a rational surface singularity is a tree of rational curves. We generalize the combinatoric part of this statement to higher dimensions and show that the highest cohomologies of the dual complex associated to a resolution of an isolated rational singularity vanish. We also prove that the dual complex associated to a resolution of an isolated hypersurface singularity is simply connected. As a consequence, we show that the dual complex associated to a resolution of a 3-dimensional Gorenstein terminal singularity has the homotopy type of a point.

1. Introduction

Let \( o \in X \) be an isolated singularity of an algebraic variety (or an analytic space) \( X \) defined over a field of characteristic 0, \( \dim X \geq 2 \). Consider a good resolution \( f: Y \to X \) (this means that the exceptional locus \( Z \subset Y \) of \( f \) is a divisor with simple normal crossings). Let \( Z = \sum Z_i \), where \( Z_i \) are irreducible. To the divisor \( Z \) we can associate the dual complex \( \Gamma(Z) \). It is a CW-complex whose cells are standard simplexes \( \Delta^j_{i_0...i_k} \) corresponding to the irreducible components \( Z^j_{i_0...i_k} \) of the intersections \( Z_{i_0} \cap \cdots \cap Z_{i_k} = \bigcup_j Z^j_{i_0...i_k} \). The \( k-1 \)-simplex \( \Delta^j_{i_0...i_k} \) is a face of the \( k \)-simplex \( \Delta^j_{i_0...i_k} \) iff \( Z^j_{i_0...i_k} \cap Z^{j'}_{i_0...i_k} \neq \emptyset \). If \( X \) and \( Y \) are surfaces, then \( \Gamma(Z) \) is the usual resolution graph of \( f \). Note that \( \Gamma(Z) \) is a simplicial complex iff all the intersections \( Z_{i_0} \cap \cdots \cap Z_{i_k} \) are irreducible. This can be obtained for a suitable resolution. Also note that if \( \dim X = n \), then \( \dim(\Gamma(Z)) \leq n - 1 \).

The complex \( \Gamma(Z) \) was first studied by G. L. Gordon in connection to the monodromy in families (see [7]). We say that \( \Gamma(Z) \) is the dual complex associated to the resolution \( f \). The main reason motivating the study of the dual complex is the fact that the homotopy type of \( \Gamma(Z) \) depends only on the singularity \( o \in X \)

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but not on the choice of a resolution \( f \). This is a consequence of the Abramovich-Karu-Matsuki-Włodarczyk Weak Factorization Theorem in the Logarithmic Category (see [1]). Indeed, this theorem reduces the problem to the case of a blowup \( \sigma: (X', Z') \to (X, Z) \), where \( X \) and \( X' \) are smooth varieties with divisors \( Z \) and \( Z' \) with simple normal crossings, and the center of the blowup is admissible in some sense. It can be explicitly verified that \( \Gamma(Z) \) is homotopy equivalent to \( \Gamma(Z') \) (see [10]).

For example (it is taken from [7]), consider the singularity
\[
\{x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} + x_1 x_2 \cdots x_n = 0\} \subset \mathbb{C}^n.
\]
A good resolution can be obtained just by blowing up the origin. The reader can easily prove that the exceptional divisor \( Z \) consists of \( n \) hyperplanes in a general position in \( \mathbb{P}^{n-1} \). We see that the complex \( \Gamma(Z) \) is the border of a standard \( n-1 \)-dimensional simplex, and thus it has the homotopy type of the sphere \( S^{n-2} \).

If \( F: (Y, Z) \to (X, o) \) is a resolution of an isolated toric singularity \( (X, o) \), then the complex \( \Gamma(Z) \) has the homotopy type of a point ([16]).

In this paper, we study the dual complex associated to a resolution in the case when \( X \) is a rational or a hypersurface singularity defined over the field \( \mathbb{C} \) of complex numbers. We show that if \( f: Y \to X \) is a good resolution of an isolated rational singularity \( o \in X \), \( \dim X = n \), then \( H^{n-1}(\Gamma(Z), \mathbb{C}) = 0 \) (see Theorem 2.2). The proof is a generalization of M. Artin’s argument from [8] to the \( n \)-dimensional case. The main new ingredient is the lemma on the degeneracy of a spectral sequence associated to a divisor with simple normal crossings on a Kähler manifold (Lemma 2.4).

If \( X \) is an isolated hypersurface singularity, \( \dim X \geq 3 \), then \( \pi(\Gamma(Z)) = 0 \) (see Theorem 3.1). This result is based on the well known fact that the link of an isolated hypersurface singularity of dimension \( n \geq 3 \) is simply connected ([14]). These results allow us to prove that the homotopy type of the dual complex associated to a resolution of an isolated rational hypersurface 3-dimensional singularity is trivial (Corollary 3.3). As an application, we show that the dual complex associated to a resolution of a 3-dimensional Gorenstein terminal singularity has the homotopy type of a point (Corollary 3.4).

We prove our theorems for algebraic varieties, but everything also holds for analytic spaces (with obvious changes).

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2. Rational singularities

**Definition 2.1.** An algebraic variety (or an analytic space) \( X \) has rational singularities if \( X \) is normal and for any resolution \( f: Y \to X \) all the sheaves \( R^i f_* \mathcal{O}_Y \) vanish, \( i > 0 \).

In the sequel, when we say that \( f \) is a good resolution we additionally assume that all the intersections \( Z_{i_0} \cap \cdots \cap Z_{i_k} \) of prime components of the exceptional divisor \( Z = \bigcup Z_i \) of \( f \) are irreducible, thus \( \Gamma(Z) \) is a simplicial complex.
The following theorem can be considered as a generalization of the classical fact that the exceptional locus in a resolution of a rational surface singularity is a tree of rational curves ([3]).

**Theorem 2.2.** Let \( o \in X \) be an isolated rational singularity of a variety (or an analytic space) \( X \) of dimension \( n \geq 2 \), and let \( f: Y \to X \) be a good resolution with the exceptional divisor \( Z \). Then the highest (complex) cohomologies of the complex \( \Gamma(Z) \) vanish:

\[
H^{n-1}(\Gamma(Z), \mathbb{C}) = 0.
\]

**Proof.** Let \( Z = \sum_{i=1}^{N} Z_i \) be the decomposition of the divisor \( Z \) to its prime components \( Z_i \). We can assume that \( X \) is projective (since the given singularity is isolated) and \( f \) is obtained by a sequence of smooth blowups (Hironaka’s resolution [12]). Thus all \( Z_i \) and \( Y \) are Kähler manifolds.

The sheaves \( R^if_*\mathcal{O}_Y \) are concentrated at the point \( o \). Via Grothendieck’s theorem on formal functions (see [11], (4.2.1), and [9], Ch. 4, Theorem 4.5 for the analytic case) the completion of the stalk of the sheaf \( R^if_*\mathcal{O}_Y \) at the point \( o \) is

\[
\lim_{(r)\to(\infty)} H^i(Z, \mathcal{O}_{Z(\infty)}),
\]

where \( (r) = (r_1, \ldots, r_N) \) and \( Z(\infty) = \sum_{i=1}^{N} r_i Z_i \). If \( (r) \geq (s) \), i.e., \( r_i \geq s_i \forall i \), there is a natural surjective map \( g \) of sheaves on \( Z \):

\[
g: \mathcal{O}_{Z(r)} \to \mathcal{O}_{Z(s)}.\]

Since dimension of \( Z \) is \( n - 1 \), the map \( g \) induces a surjective map of cohomologies

\[
H^{n-1}(Z, \mathcal{O}_{Z(s)}) \to H^{n-1}(Z, \mathcal{O}_{Z(s)}).
\]

Recall that the given singularity \( o \in X \) is rational, and thus the projective limit (1) is 0. Therefore the cohomology group \( H^{n-1}(Z, \mathcal{O}_Z) \) vanishes, too (because the projective system in (1) is surjective). Now it follows from Lemma 2.3 below that \( H^{n-1}(\Gamma(Z), \mathbb{C}) = 0 \).

**Lemma 2.3.** Let \( Z = \sum_{i=1}^{N} Z_i \) be a reduced divisor with simple normal crossings on a compact Kähler manifold \( Y \), \( \dim Y = n \), and assume that \( H^k(Z, \mathcal{O}_Z) = 0 \) for some \( k, 1 \leq k \leq n - 1 \). Then the \( k \)-th cohomologies with coefficients in \( \mathbb{C} \) of the complex \( \Gamma(Z) \) vanish, too:

\[
H^k(\Gamma(Z), \mathbb{C}) = 0.
\]

**Proof.** Let us introduce some notation. Let \( Z^0 = \bigcup_i Z_i \) be the disjoint union of the irreducible components \( Z_i \) and \( Z^p = \bigcup_{0 \leq i_1 < \cdots < i_p} Z_{i_1 \cdots i_p} \), where \( Z_{i_1 \cdots i_p} = Z_{i_1} \cap Z_{i_2} \cap \cdots \cap Z_{i_p} \). By \( \varphi_p: Z^p \to Z \) denote the natural map. Consider the structure sheaves \( \mathcal{O}_{Z^p} = \bigoplus_{0 \leq i_1 < \cdots < i_p} \mathcal{O}_{Z_{i_1 \cdots i_p}} \), the sheaves \( \mathcal{A}^{p,q} = \bigoplus_{0 \leq i_1 < \cdots < i_p} \mathcal{A}^{p,q}_{Z_{i_1 \cdots i_p}} \) of differential forms of bidegree \((0, q)\) on \( Z^p \) and their direct images \( \mathcal{K}^p = \varphi_* \mathcal{O}_{Z^p} \) and \( \mathcal{K}^{p,q} = \varphi_* \mathcal{A}^{p,q} \) on \( Z \). The sequence of sheaves \( \{\mathcal{K}^p\}_{p \geq 1} \) forms a complex via the combinatoric differentials \( \delta^p: \mathcal{K}^p \to \mathcal{K}^{p+1} \), where if

\[
a = \bigoplus_{i_0 < \cdots < i_p} a_{i_0 \cdots i_p} \in \mathcal{K}^p(U) \equiv \bigoplus_{i_0 < \cdots < i_p} \varphi_* \mathcal{O}_{Z_{i_0 \cdots i_p}}(U)
\]

is a section of the sheaf \( \mathcal{K}^p \) over an open set \( U \subseteq Z \), then

\[
(\delta(a))_{i_0 \cdots i_{p+1}}(U) = \sum_{j=0}^{p+1} (-1)^j (a_{i_0 \cdots \hat{i}_j \cdots i_p}) |_{Z_{i_0 \cdots i_{p+1}} \cap U}.
\]
Note that there is also a natural injection of $\mathcal{O}_Z$ into $\mathcal{K}^0$. The sequence of sheaves

$$0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{K}^0 \xrightarrow{\delta^0} \mathcal{K}^1 \xrightarrow{\delta^1} \ldots$$

is exact. This is easy to check by considering the stalks; in particular, the exactness at $\mathcal{K}^0$ is a consequence of the following fact which holds locally in a sufficiently small neighborhood of every point of $Z$: if $\{f_i\}$ is a collection of regular functions on $Z_i$ such that their restrictions onto intersections $Z_i \cap Z_j$ coincide, then there exists a regular function $f$ on $Z$ such that $f|_{Z_i} = f_i$ for all $i$ (it is important here that the divisor $Z$ has normal crossings). Therefore the complexes

$$\mathcal{O}^* : \quad \mathcal{O}_Z \rightarrow 0 \rightarrow 0 \rightarrow \ldots$$

and

$$\mathcal{K}^* : \quad \mathcal{K}^0 \xrightarrow{\delta^0} \mathcal{K}^1 \xrightarrow{\delta^1} \mathcal{K}^2 \xrightarrow{\delta^2} \ldots$$

are quasi-isomorphic. It is clear that the hypercohomologies of the first complex are isomorphic to the cohomologies of $Z$ with coefficients in the structure sheaf: $H^p(\mathcal{O}^*) \simeq H^p(Z, \mathcal{O}_Z)$. Now let us calculate the hypercohomologies of the complex $\mathcal{K}^*$ by using the acyclic resolutions $\mathcal{K}^p \rightarrow \mathcal{K}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{K}^{p,1} \xrightarrow{\bar{\partial}} \ldots$, where $\bar{\partial}$ is the Dolbeaux differential.

Consider the bigraded sequence of groups

$$K^{p,q} = H^0(\mathcal{K}^{p,q}, Z) \simeq \bigoplus_{i_0 < \cdots < i_p} H^0(\varphi_{p*} A^{p,q})$$

endowed with the differentials $\bar{\partial}: K^{p,q} \rightarrow K^{p,q+1}$ and $\delta: K^{p,q} \rightarrow K^{p+1,q}$, where $\bar{\partial}$ is the Dolbeaux differential and $\delta$ is the combinatoric differential defined as follows: if $\alpha = \bigoplus \alpha_{i_0 \ldots i_p} \in K^{p,q}$, then

$$(\delta(\alpha))_{i_0 \ldots i_p} i_{p+1} = \sum_{j=0}^{p+1} (-1)^{q+j} (\alpha_{i_0 \ldots \hat{i}_j \ldots i_p+i_{p+1}})|_{Z_{i_0 \ldots i_p}}.$$ 

These differentials satisfy the equality $\bar{\partial} \delta + \delta \bar{\partial} = 0$, thus $(\mathcal{K}^*, \delta, \bar{\partial})$ is a bicomplex.

Let $(\mathcal{K}^*, d)$, $K^n = \bigoplus_{p+q=n} K^{p,q}$, $d = \delta + \bar{\partial}$ be the associated complex. Now we can write that $H^p(\mathcal{K}^*) = H^p(\mathcal{K}^*, d)$.

There is a filtration

$$F^p K^n = \bigoplus_{p' + q = p} K^{p', q}$$

on the complex $\mathcal{K}^*$. It is known (see [10], Ch. 3, section 5) that the spectral sequence $E_r$ associated to the filtration $F^p K^n$ converges to the cohomologies $H^*(\mathcal{K}^*, d)$ and

$$E_0^{p,q} \simeq K^{p,q};$$
$$E_1^{p,q} \simeq H^q(\bar{\partial}(\mathcal{K}^*));$$
$$E_2^{p,q} \simeq H^q(\delta(\mathcal{K}^*)).$$
In particular,

\[ E_1^{p,0} \simeq H^0_\partial \left( \bigoplus_{i_0 < \cdots < i_p} H^0(p_{\ast}, A^{p,\ast}) \right) \]

\[ \simeq \bigoplus_{i_0 < \cdots < i_p} \mathbb{C}. \]

Therefore the cochain complex

\[ 0 \to E_1^{0,0} \delta \to E_1^{1,0} \delta \to \ldots \]

is isomorphic to the cochain complex that one uses to calculate cohomologies of \( \Gamma(Z) \) (here we denote by the same letter \( \delta \) the map between cohomologies induced by the combinatoric differential). It follows that

\[ E_2^{p,0} \simeq H^p(\Gamma(Z), \mathbb{C}). \]

We shall show that the spectral sequence \( E_r \) degenerates in \( E_2 \). The method of the proof is based on the standard technique of the theory of the mixed Hodge structures. We learned this from [13], Chapter 4, §2. Also compare [8]. Since the result about \( E_r \) can be of a particular interest, we state it as a separate lemma.

**Lemma 2.4.** Let \( Z = \sum Z_i \) be a reduced divisor with simple normal crossings on a compact Kähler manifold \( Y \), and let \( E_r \) be the associated spectral sequence as described above. Then \( d_r = 0 \) for all \( r \geq 2 \), i.e., this spectral sequence degenerates in \( E_2 \).

**Proof.** Consider the diagram

\[ \begin{array}{ccccccccc}
\cdots & \delta & K^{p,q} & \delta & K^{p+1,q} & \delta & K^{p+2,q} & \delta & \cdots \\
\downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \cdots \\
\cdots & \delta & K^{p,q-1} & \delta & K^{p+1,q-1} & \delta & K^{p+2,q-1} & \delta & \cdots \\
\downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \downarrow \bar{\partial} & \uparrow \delta & \cdots \\
\end{array} \]

where \( \bar{\partial} \delta + \delta \bar{\partial} = 0 \). First we take cohomologies in the vertical rows and obtain the sequences

\[ \begin{array}{c}
\cdots \\
\cdots \to H^q(K^{p,\ast}) \xrightarrow{\delta} H^q(K^{p+1,\ast}) \xrightarrow{\delta} H^q(K^{p+2,\ast}) \to \cdots \\
\cdots \to H^{q-1}(K^{p,\ast}) \xrightarrow{\delta} H^{q-1}(K^{p+1,\ast}) \xrightarrow{\delta} H^{q-1}(K^{p+2,\ast}) \to \cdots \\
\end{array} \]

Here \( \delta \) is the induced map between cohomologies. Then we calculate \( \delta \)-cohomologies and obtain the differential

\[ d_2: H_\delta^p(H^q_\partial(K^{\ast,\ast})) \to H_\delta^{p+2}(H^q_\partial(K^{\ast,\ast})) \]

that acts as described below.
An element
\[ \tilde{a} \in H_3^p(H_3^3(K^{*+})) \]
is a class of those \( \tilde{a} \in H_3^3(K^{p,*}) \) modulo \( \delta(H_3^3(K^{p-1,*})) \) that map to 0 under the action of \( \delta: \delta(\tilde{a}) = 0 \) in \( H_3^3(K^{p+1,*}) \). But \( \tilde{a} \) is a class of \( a \in K^{p,q} \) mod \( \partial K^{p,q-1} \) such that \( \partial a \in 0 \in K^{p,q+1} \). Therefore we can choose a representative \( a \in K^{p,q} \) for \( \tilde{a} \) such that \( \partial a = 0 \) and \( \delta a = 0 \) modulo \( \partial K^{p+1,q-1} \) in \( K^{p+1,q} \). It follows that there exists an element \( a' \in K^{p+1,q-1} \) such that \( \partial a' = \delta a \). Map this \( a' \) down to \( K^{p+1,q-1}: \delta(a') \in K^{p+2,q-1} \).

It can be verified by standard methods that \( \delta(a') \) correctly determines a class \( \delta(a') \) in \( H_3^{p+2}(H_3^{q-1}(K^{*+})) \) and the differential \( d_2 \) is defined as follows:
\[ d_2(\tilde{a}) = \delta(a'). \]

Differentials \( d_r, r \geq 3 \), can be obtained by iterating this construction. For example, \( d_3 \) is defined for \( a \in K^{p,q} \) such that \( \delta(a') = 0 \) modulo \( \partial K^{p+2,q-1} \), thus there is an \( a'' \in K^{p+2,q-2} \) such that \( \delta(a'') = \delta(a') \), and \( d_3: E_3^{p,q} \to E_3^{p+3,q-2} \) is induced by correspondence \( a \to \delta(a'') \) (see, e.g., [5], A.3.13.4).

Our aim is to show that \( d_r = 0 \), \( r \geq 2 \). The differential \( d_2 \) is trivial if the representative \( a \in K^{p,q} \) can be chosen in such a way that \( \delta(a) \) is exactly 0 but not only 0 modulo \( \partial K^{p+1,q-1} \). But this is true because there are harmonic differential forms in the class \( a \in H_3^3(K^{p,*}) \) and we can take \( a \) to be a harmonic form. The form \( \delta a \) is defined by means of restrictions onto subvarieties and linear operations. All varieties we consider are Kähler, hence \( \delta a \) is also harmonic. But it is 0 mod \( \partial K^{p+1,q-1} \) and thus is exactly 0. Further, this \( \delta a = 0 \) can be lifted to \( a' = 0 \) in \( K^{p+1,q-1} \), thus \( \delta(a') = 0 \) and so forth. This also shows that \( d_r = 0 \) for all \( r \geq 3 \).

Now let us come back to the proof of Lemma 2.3. We have \( E_2 = E_\infty \); therefore
\[ H^p(\Gamma(Z), \mathbb{C}) \simeq E_2^{p,0} \]
is a subgroup of
\[ H^p(K^*, d) \simeq H^p(K^*) \simeq H^p(O^*) \simeq H^p(Z, \mathcal{O}_Z). \]

If \( H^p(\Gamma(Z)) \) is not trivial, then \( H^p(Z, \mathcal{O}_Z) \) is also not trivial. \( \square \)

3. Hypersurface singularities

**Theorem 3.1.** Let \( o \in X \) be an isolated hypersurface singularity of an algebraic variety (or an analytic space) \( X \) of dimension at least 3 defined over the field \( \mathbb{C} \) of complex numbers. If \( f: Y \to X \) is a good resolution of \( o \in X \) and \( Z \) its exceptional divisor, then the fundamental group of \( \Gamma(Z) \) is trivial:
\[ \pi(\Gamma(Z)) = 0. \]

**Proof.** Let \( n \) be the dimension of \( X \). We can assume that \( X \) is a hypersurface in \( \mathbb{C}^{n+1} \) and the singular point \( o \) coincides with the origin. Consider the link \( M \) of singularity \( o \in X \), i.e., the intersection of \( X \) with a sphere \( S^{2n+1} \) of sufficiently small radius around the origin. The link \( M \) is an \((n-2)\)-connected smooth manifold (12, Corollary 2.9, Theorem 5.2); in particular, \( M \) is simply connected.

We can also consider \( M \) as the border of a tubular neighborhood of the exceptional divisor \( Z \subset Y \). It is known (see [2]) that there is a surjective map \( \varphi: M \to Z \) whose fibers are tori. It follows that the induced map \( \varphi^*: \pi(M) \to \pi(Z) \) is also surjective and hence \( \pi(Z) = 0. \)
It remains to show that $\pi(Z) = 0$ implies $\pi(\Gamma(Z)) = 0$. It is enough to construct a surjective map $\psi: Z \to \Gamma(Z)$ with connected fibers. The following lemma is, essentially, a partial case of the general construction of a map from a topological space $Z = \bigcup Z_i$ to the nerve $\Gamma(Z)$ corresponding to the covering $\{Z_i\}$ (see [4], p. 355). As in section 2, we assume that the intersections $Z_{i_0\ldots i_p}$ are irreducible so that $\Gamma(Z)$ is a simplicial complex.

**Lemma 3.2.** Let $Z$ be a divisor with simple normal crossings on an algebraic variety or an analytic space $X$, and let $\Gamma(Z)$ be the corresponding dual complex. Then there exists a map $\psi: Z \to \Gamma(Z)$ which is (i) simplicial in some triangulations of $Z$ and $\Gamma(Z)$, (ii) surjective, and (iii) has connected fibers.

**Proof.** First, let us take a triangulation $\Sigma'$ of $Z$ such that all the intersections $Z_{i_0\ldots i_p}$ are subcomplexes. Next we make the barycentric subdivision $\Sigma$ of $\Sigma'$ and the barycentric subdivision of the complex $\Gamma(Z)$. Now let $v$ be a vertex of $\Sigma$ belonging to the subcomplex $Z_{i_0\ldots i_p}$ but not to any smaller subcomplex $Z_{i_0\ldots i_p,i_{p+1}}$:

$$v \in Z_{i_0\ldots i_p}, \quad v \notin Z_{i_0\ldots i_p,i_{p+1}} \forall i_{p+1}.$$ 

Then let

$$\psi(v) = \text{the center of the simplex } \Delta_{i_0\ldots i_p}.$$ 

This determines the map $\psi$ completely as a simplicial map (depending on the triangulation $\Sigma'$). It is clear from our construction that $\psi$ is surjective.

We claim that the map $\psi$ has connected fibers. Indeed, first observe that $\psi$ can be represented as a composition of topological contractions of connected subcomplexes

$$Z_{i_0\ldots i_p} \setminus \text{(the open neighborhoods of intersections of } Z_{i_0\ldots i_p} \text{ with } Z_{i_{p+1}} \forall i_{p+1} \neq i_0, \ldots, i_p).$$

By an open neighborhood of a subcomplex $K$ we mean the union of interior points of all simplicial stars of $\Sigma$ with centers on $K$. All complexes in (2) are connected because the codimension of $Z_{i_0\ldots i_p} \cap Z_{i_{p+1}}$ in $Z_{i_0\ldots i_p}$ is 2.

Further, the contraction of a subcomplex $K$ from (2) can be factored into one-by-one contraction of maximal simplices of $K$. The preimage of every connected set under such a contraction is connected since the preimage of every simplex is a simplex. Therefore the map $\psi$ has all the needed properties.

Some important types of singularities are rational hypersurface. Combining Theorems 2.2 and 3.1 we can obtain some precise results in the 3-dimensional case.

**Corollary 3.3.** Let $o \in X$ be an isolated rational hypersurface singularity of dimension 3. If $f: Y \to X$ is a good resolution with the exceptional divisor $Z$, then the dual complex $\Gamma(Z)$ associated to the resolution $f$ has the homotopy type of a point.

**Proof.** We know from Theorems 2.2 and 3.1 that $\Gamma(Z)$ is simply connected and $H^2(\Gamma(Z), \mathbb{C}) = 0$. Since $\dim X = 3$, we have $\dim(\Gamma(Z)) \leq 2$. Thus $H_2(\Gamma(Z), \mathbb{Z}) = 0$. Now Corollary 3.3 follows from the Inverse Hurevich and Whitehead Theorems.
For instance, 3-dimensional Gorenstein terminal singularities are exactly isolated compound Du Val points (up to an analytic equivalence; for details see [15]). Here it is sufficient to us that compound Du Val points are hypersurface singularities. On the other hand, terminal singularities (and, moreover, canonical) are rational (see [6]). Combining these results with Corollary 3.3, we get

**Corollary 3.4.** Let \( o \in X \) be a 3-dimensional Gorenstein terminal singularity and let \( f : Y \to X \) be a good resolution with the exceptional divisor \( Z \). Then the dual complex \( \Gamma(Z) \) of \( f \) has the homotopy type of a point.

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