SOME REMARKS ON INFINITESIMALS IN
MV-ALGEBRAS

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Abstract. Replacing \{0\} by the whole ideal of infinitesimals yields a weaker notion of archimedean element that we call quasiarchimedean.

It is known that semisimple MV-algebras with compact maximal spectrum (in the co-Zarisky topology) are exactly the hyperarchimedean algebras. We characterise all the algebras with compact maximal spectrum as being quasihyperarchimedean MV-algebras, which in a sense are non semisimple hyperarchimedean algebras. We develop some basic facts in the theory of MV-algebras along the lines of algebraic geometry, where infinitesimals play the role of nilpotent elements, and prove a MV-algebra version of Hilbert’s Nullstellensatz. Finally we consider the relations (some inedited) between several elementary classes of MV-algebras in terms of the ideals that characterise them, and present elementary (first order with denumerable disjunctions) proofs in place of the set-theoretical usually found in the literature.

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1. Background on MV-algebras of continuos functions

Given an MV-algebra \( A \), \( X_A \subset [0,1]^A \) will denote the set of morphisms of \( A \) into the MV-algebra \([0,1]\). \( X_A \) becomes a compact Hausdorff space with the topology inherited from the product space. It is immediate to see that a base for the product topology is given by the subsets \( W_a = \{ \chi \mid \chi(a) > 0 \} \subset X_A \), that is, the Zariski topology. On the other hand, this set of morphisms also inherited a topology as a subspace of the prime spectrum \( Z_A \) via the map \( \chi \mapsto \text{Kernel}(\chi) \in M_A \subset Z_A \), where \( M_A \) denotes the maximal spectrum (see [4]). We will denote this space by \( X^c_A \cong M_A \), its topology is the coZariski topology with a base of open sets given by the complements of the subsets \( W_a \), that we denote \( W_a^c = \{ \chi \mid \chi(a) = 0 \} \), The \( W_a^c \) are also closed in \( X^c_A \) ([4, 4.2]), which shows that the coZariski topology is finer than the Zariski topology. Given any MV-algebra \( A \),

\[ 1.1 \quad [4, 4.16]: \text{Equality is a continuous bijection } X^c_A \xrightarrow{\cong} X_A. \]
Each element \( a \in A \) determines a continuous function \( X_A \xrightarrow{\hat{a}} [0, 1] \), \( \hat{a}(\chi) = \chi(a) \), for \( \chi \in X_A \). This determines a morphism \( A \rightarrow \text{Cont}(X_A) \), with image denoted \( \hat{A} \subset \text{Cont}(X_A) \). Note that \( W_a^c = \hat{a}^{-1}(0) \).

Consider the MV-algebra \( \text{Cont}(X) \) of \([0, 1]\)-valued continuous functions on a topological space \( X \), and let \( A \subset \text{Cont}(X) \) be a subalgebra. Recall that \( A \) is said to be separating iff for any two distinct point \( x \) and \( y \), there is \( f \in A \) such that \( f(x) = 0 \) and \( f(y) > 0 \). Each \( x \in X \) determines a morphism \( A \rightarrow [0, 1] \) defined by \( \hat{x}(a) = a(x) \). This determines a continuous function \( X \rightarrow X_A \). If \( X \) is compact Hausdorff and \( A \) is separating, we have:

(1.2) \([3, 4.1]\): The map \( \varepsilon : X \rightarrow X_A \) is a homeomorphism.

Given an ideal \( I \subset A \), we denote by \( V(I) \) the locus of roots of the functions \( f \in I \), \( V(I) = \{ x \in X \mid f(x) = 0 \forall f \in I \} \), \( V(I) \subset X \) is a closed subset).

Given a closed subset \( S \subset X \), we denote by \( J(S) \) the set of all functions null on \( S \), \( J(S) = \{ f \in A \mid f(x) = 0 \forall x \in S \} \), \( J(S) \subset A \) is an ideal).

It is immediate to check that the maps \( S \rightarrow J(S) \) and \( I \rightarrow V(I) \) are order reversing and that \( I \subset J(V(I)) \) and \( S \subset V(J(S)) \).

If \( X \) is compact Hausdorff and \( A \) is separating, we have:

(1.3) \([2, 3.4.2]\): \( V(J) \neq \emptyset \) for each proper ideal \( J \).

(1.4) \([2, 3.4.3]\): \( S = V(J(S)) \) for each closed subset \( S \).

(1.5) It follows that for \( f \in A \), \( f \in J(S) \) \( \iff \) \( f|_S = 0 \), thus:

\[ A/(J(S) \cap A) \cong A|_S \]

Recall:

(1.6) \([3, 4.5]\): Given any compact space \( X \) and any \( f \in \text{Cont}(X) \), we have

\( f \) is archimedean \( \iff \) \( V(\langle f \rangle) = f^{-1}(0) \subset X \) is open.

where \( \langle f \rangle \subset \text{Cont}(X) \) is the ideal generated by \( f \). Note that under the homeomorphism (1.2) \( V(\langle f \rangle) \cong W_f^c \subset X_{\text{Cont}(X)} \).

Given any families of ideals \( \{I_\ell\}_{\ell \in L} \) and of closed subsets \( \{S_\ell\}_{\ell \in L} \), from the universal property which defines supremum and infinum it immediately follows:

(1.7) \( \bigvee_{\ell \in L} V(I_\ell) \subset V(\bigwedge_{\ell \in L} I_\ell) \), \( \bigvee_{\ell \in L} J(S_\ell) \subset J(\bigwedge_{\ell \in L} S_\ell) \).

(1.8) \( \bigwedge_{\ell \in L} V(I_\ell) = V(\bigvee_{\ell \in L} I_\ell) \), \( \bigwedge_{\ell \in L} J(S_\ell) = J(\bigvee_{\ell \in L} S_\ell) \).

(the infima here are the set theoretical intersection, but the suprema not).

**Free MV-algebras**

For each set \( N \), we denote by \( F[N] \) the free MV-algebra on \( N \)-generators. \( F[N] \) is the MV-algebra of terms \( f \) in variables \( \{x_i\}_{i \in N} \).

Note that with the hindsight of category theory free algebras should be considered up to isomorphisms. In this way we associate free algebras to sets, not to cardinals. Any two bijective sets determine isomorphic algebras.

By Chang’s completeness theorem \( F[N] \) can be considered to be the MV-algebra of \([0, 1]\)-valued term functions on the compact space \([0, 1]^N\).
Term functions are continuous and it is not difficult to prove they are separating.

(1.9) [2, 3.4.6]: $F[N]$ can be considered to be the separating subalgebra of term functions $F[N] \subset \text{Cont}(X)$, for the compact space $X = [0, 1]^N$.

Given a MV-algebra with a presentation $A = F[N]/I$, $a_i = [x_i]$, the universal properties of the free algebra and the quotient algebra say (in turn) that the restriction along $N \hookrightarrow F[N]$ determines a continuous bijection $i^* : X_{F[N]} \xrightarrow{\cong} [0, 1]^N$ that restricts to a bijection $i^* : X_A \xrightarrow{\cong} \text{V}(I)$. Since the spaces are compact Hausdorff they are homeomorphisms.

(1.10) If $A = F[N]/I$, then the restriction along $N \hookrightarrow F[N]$ determines a homeomorphism $i^* : X_A \xrightarrow{\cong} \text{V}(I) \subset [0, 1]^N$, $i^*(\chi) = (\chi(a_i))_{i \in N}$.

2. QUASIHYPARCHIMEDEAN ALGEBRAS

Recall that an element $a$ in an MV-algebra $A$ is said to be infinitesimal if for each integer $n \geq 0$, $na \leq \neg a$, equivalently, iff $na \ominus \neg a = na \odot a = 0$.

2.1. Remark. ([2, 3.6.3]) For any infinitesimal element $a > 0$, the sequence $(0 \leq a \leq 2a \leq 3a \leq \ldots \leq na \leq \ldots)$ is strictly increasing. □

Recall that an element $a$ in an MV-algebra $A$ is said to be archimedean if there is an integer $n \geq 0$, such that $(n+1)a \ominus na = 0$, equivalently, iff the sequence $(a \leq 2a \leq 3a \leq \ldots \leq na \leq \ldots)$ is stationary.

Note that it follows that the only archimedean infinitesimal is $0$.

For any ideal $I$ it follows by an easy induction:

2.2. Remark. Given $x \in A$ and an integer $n \geq 1$, if $(n+1)x \ominus nx \in I$, then $\forall m > k \geq n$, $mx \ominus kx \in I$. □

2.3. Definition. An element $a$ in an MV-algebra $A$ is said to be quasihyparchimedean if there is an integer $n \geq 0$, such that $(n+1)a \ominus na$ is infinitesimal. A MV-algebra is quasihyparchimedean if every element is quasihyparchimedean.

Clearly archimedean elements are quasihyparchimedean, and hyperarchimedean algebras are quasihyparchimedean.

2.4. Proposition. $a \in A$ is quasihyparchimedean $\iff \widehat{a} \in \widehat{A}$ is archimedean.

Proof. One implication is clear since any morphism preserves quasihyparchimedean elements, and the only infinitesimal in $\widehat{A}$ is $0$. For the other implication, take $n \geq 0$ such that $(n+1)a \ominus na = 0$. Then for all $\chi \in X_A$, $0 = (n+1)\widehat{a}(\chi) \ominus n\widehat{a}(\chi) = (n+1)\chi(a) \ominus n\chi(a) = \chi((n+1)a \ominus na)$. Thus $(n+1)a \ominus na \in \text{Rad}(A) = \sqrt{A}$, that is, it is infinitesimal. □

From (1.6) and Proposition 2.4 it immediately follows:

\footnote{caution: Contrary with common usage, we consider $0$ to be infinitesimal, as in algebraic geometry $0$ is considered to be nilpotent.}
2.5. Proposition. $a \in A$ is quasihyperarchimedean $\iff W^c_a \subset X_A$ is open.

(this corrects the asymmetry in propositions 5.4 and 5.6 of [4]).

We establish now a characterisation of quasihyperarchimedean MV-algebras as those algebras with a compact maximal spectrum. The reader should note that the maximal spectrum $M_A \subset Z_A$ is in this case a compact Hausdorff non-closed subspace of the compact prime spectrum.

2.6. Proposition. The following conditions in a MV-algebra are equivalent:

1. $A$ is quasihyperarchimedean.
2. For all $a \in A$, $W^c_a \subset X_A$ is open (thus clopen).
3. The map $X_A \rightarrow X^c_A \cong M_A$ is continuous (thus a homeomorphism).
4. The maximal spectrum $X^c_A \cong M_A$ is compact.

Proof. Clearly (1) $\iff$ (2), and (3) $\implies$ (4). (4) $\implies$ (3) because then (see 1.1) $X^c_A \rightarrow X_A$ is a continuous bijection between compact Hausdorff spaces. Finally, (2) $\iff$ (3) because the sets $W^c_a$ are an open base of $X^c_A$.

3. The MV-Nullstellensatz.

In this section we develop some basic lines of algebraic geometry in the context of MV-algebras (reference is [5]). As nilpotent elements are considered “infinitesimal” in algebraic geometry, here its role is played by the MV-algebra concept of, properly called, infinitesimal elements.

We start by recalling a first-order (with denumerable disjunctions) characterisation of maximal ideals, which is a key result in the theory of MV-algebras ([2, 1.2.2]). For any MV-algebra $A$ and ideal $I \subset A$,

\[(3.1) \quad I \text{ is maximal } \iff \forall x \in A \ (x \notin I \iff \exists n \geq 1 \mid nx \in I).\]

The intersection of all maximal ideals of a MV-algebra $A$ is an ideal called the radical of $A$, and denoted $Rad(A)$. In the light of this, we define:

3.2. Definition. Given an ideal $I \subset A$, the intersection of all maximal ideals $M \supset I$ containing $I$ is an ideal that we call the radical of $I$, denoted $Rad(I)$. $I$ is called a radical ideal if $I = Rad(I)$.

3.3. Remark. Recall that if $I$ is a prime ideal, then it is contained in a unique maximal ideal [2, 1.2.12]. It follows that $Rad(I)$ is a maximal ideal.

3.4. Proposition. Let $A \xrightarrow{\varphi} B$ be a surjective morphism of MV-algebras, and $I \subset B$ any ideal of $B$. Then:

$\varphi^{-1} Rad(I) = Rad(\varphi^{-1} I)$.

Proof. It follows once we observe that for any pair of ideals $M, I$ in $B$, $M \supset I$ iff $\varphi^{-1} M \supset \varphi^{-1} I$, and $M$ is maximal iff $\varphi^{-1} M$ is maximal (the second equivalence follows easily from (3.1) above).

3.5. Proposition. Let $X$ be a compact space, $A \subset Cont(X)$ a separating subalgebra, and $I \subset A$ any ideal. Then, $Rad(I) = J(V(I))$. Thus, $I$ is a radical ideal iff $I = J(V(I))$. 
Proof. Once we observe that for any point \( x \in X \), \( x \in V(I) \) iff \( I \subset J(\{x\}) \), the proof follows immediately from (1.5) above. \( \square \)

From this proposition and (1.4) above it follows:

3.6. Proposition. Given a compact space \( X \) and a separating subalgebra \( A \subset \text{Cont}(X) \), the correspondence given by \( J \) and \( V \) establishes a bijection between the closed subsets of \( X \) and the radical ideals of \( A \). \( \square \)

We call the set of infinitesimals (see section 2) the infradical of \( A \), and denote it by \( \sqrt{A} \). It is well known that \( \sqrt{A} = \text{Rad}(A) \) \([2, 3.6.4]\), but we will not need this here, neither that the set \( \sqrt{A} \) is an ideal. All this will be a particular case of our more general Theorem 3.12. Note that \( \sqrt{[0,1]} = \{0\} \).

The following definition was communicated to us by R. Cignoli \([1]\), compare with \([5, \text{page} \ 48]\).

3.7. Definition. Let \( I \subset A \) be an ideal of a MV-algebra \( A \). An element \( a \) in \( A \) is said to be \( I \)-infinitesimal iff \( na \ominus \neg a \in I \) for each integer \( n \geq 0 \).

Clearly an element \( a \) is \( I \)-infinitesimal iff \( \rho(a) \) is infinitesimal in the quotient algebra \( A \overset{\rho}{\longrightarrow} A/I \).

We call this set the infradical of \( I \), and we denote it by \( \sqrt{I} \). Since \( na \ominus \neg a = na \oplus a \leq a \), it follows \( I \subset \sqrt{I} \). It is immediate to check the following two propositions.

3.8. Proposition. Let \( \{I_\ell\}_{\ell \in \mathcal{L}} \) be any family of ideals. Then

\[
\sqrt{\bigcap_{\ell \in \mathcal{L}} I_\ell} = \bigcap_{\ell \in \mathcal{L}} \sqrt{I_\ell}.
\]

\( \square \)

3.9. Proposition. Let \( A \overset{\varphi}{\longrightarrow} B \) be any morphism of MV-algebras, and \( I \subset B \) any ideal of \( B \). Then:

\[
\varphi^{-1}\sqrt{I} = \sqrt{\varphi^{-1}I}.
\]

\( \square \)

3.10. Proposition. Let \( X \) be any topological space, and \( A \subset \text{Cont}(X) \) any subalgebra (not necessarily separating). Then:

1) \( \sqrt{J} \subset J(V(J)) \).
2) If \( X \) is compact, \( J(V(J)) \subset \sqrt{J} \)

Thus, for compact \( X \), \( \sqrt{J} = J(V(J)) \).

Proof. 1) Let \( f \) be \( J \)-infinitesimal and \( x \in V(J) \). Then for each integer \( n \geq 0 \), \( nf(x) \ominus f(x) = (nf \ominus f)(x) = 0 \). Since \([0,1]\) has no infinitesimals other than \( 0 \), we have \( f(x) = 0 \).

2) Since any ideal is an intersection of prime ideals \([3, 1.2.14]\), it follows, from (1.7), (1.8) and Proposition 3.8, that we can assume \( I \) to be prime. Suppose that \( f \) is not a \( I \)-infinitesimal, and let \( n \geq 0 \) be such that \( nf \ominus f \notin I \). From the equation \((x \oplus y) \land (y \ominus x) = 0\) it follows that \( \neg f \oplus nf \in I \). That is, \( \neg(n+1)f = \neg(f \oplus nf) \in I \). By (1.3) we can take \( x \in V(I) \). Then \((\neg(n+1)f)(x) = 0 \), thus \((n+1)f(x) = 1 \) which implies \( f(x) > 0 \). Thus \( f \notin J(V(J)) \). \( \square \)

Taking into account (1.10) above, a particular case of Proposition 3.10 yields (compare with \([5, \text{theorem} \ 5.1]\)):
3.11. **Theorem** (Nullstellensatz). For any ideal $I \subset F[N]$, the ideal of term functions vanishing on the common zero locus of $I$, $V(I) \subset [0, 1]^N$, is the infradical of $I$, that is $J(V(I)) = \sqrt{I}$. That is, if $f|_{V(I)} = 0$, then $nf \ominus -f \in I$ for each integer $n \geq 0$.

Note that in other words this theorem means:

Given any MV-algebra $A$ with a presentation $A = F[N]/I$, then $A$ is isomorphic to the algebra $F[N]|_{V(I)}$ of term-functions restricted to the zero-set $V(I) \subset [0, 1]^N$, if and only if, $\sqrt{A} = \{0\}$, i.e, $A$ has no infinitesimals other than 0.

Using now that $F[N]$ is a separating subalgebra, (1.10) above, we have the following corollary of theorem 3.11 ([1, Th. 0.1]).

3.12. **Theorem.** For any MV-algebra $A$ and ideal $I \subset A$, $\text{Rad}(I) = \sqrt{I}$, in particular, $\sqrt{A} = \text{Rad}(A)$.

**Proof.** Take $N$ such that $F[N] \xrightarrow{\rho} A$ is a quotient. It suffices to prove $\rho^{-1}\text{Rad}(I) = \rho^{-1}\sqrt{I}$. We have:

$$\rho^{-1}\text{Rad}(I) = \text{Rad}(\rho^{-1}I) = J(V(\rho^{-1}I)) = \sqrt{\rho^{-1}I} = \rho^{-1}\sqrt{I}.$$ 

These equalities follow (in order) by Proposition 3.4, Proposition 3.5, Theorem 3.11, and Proposition 3.9. \qed

3.13. **Corollary.** For any MV-algebra $A$ and ideal $I \subset A$, the set of all $I$-infinitesimals is an ideal.

3.14. **Corollary.** An ideal $I \subset A$ of a MV-algebra $A$ is a radical ideal (Definition 3.2) if and only if $I = \sqrt{I}$.

4. **The relations between some classes of MV-algebras**

In this section we prove (except for Proposition 4.11) in a syntactic elementary way, meaning first order with denumerable disjunctions, several implications (some inedited) between elementary classes of MV-algebras which in the literature are usually proved in a set theoretical semantical way. In the following the variables $x, y, \ldots$ are assumed to range on some MV-algebra $A$.

In view of the characterisation 3.1 of maximal ideals we set:

4.1. **Definition.** An ideal $I \subset A$ of a MV-algebra $A$ is quasimaximal $\iff \forall x \in A (x \notin I \iff \exists n \geq 1 \mid -nx \in \sqrt{I})$.

For a MV-algebra $A$, the ideals $I$ such that the quotient algebra $A/I$ is hyperarchimedean will be called hyperradical. Thus:

4.2. **Definition.** An ideal $I \subset A$ of a MV-algebra $A$ is hyperradical if for any $x \in A$, there exists an integer $n \geq 1$ such that $(n+1)x \ominus nx \in I$.

For a MV-algebra $A$, the ideals $I$ such that the quotient algebra $A/I$ is quasihyperarchimedean will be called quasihyperradical. Thus:

4.3. **Definition.** An ideal $I \subset A$ of a MV-algebra $A$ is quasihyperradical if for any $x \in A$, there exists an integer $n \geq 1$ such that $(n+1)x \ominus nx \in \sqrt{I}$. 


4.4. **Remark.** Clearly an ideal $I$ is quasihyperradical if and only if $\sqrt{I}$ is hyperradical.

This illustrates a correspondence between classes of MV-algebras and notions of ideals. We have the following table:

| Class                  | Correspondence                |
|------------------------|-------------------------------|
| simple                 | maximal                       |
| quasisimple            | quasimaximal                  |
| semisimple             | radical                       |
| chain                  | prime                         |
| hyperarithmetician     | hyperradical                  |
| quasihyperarithmetian  | quasihyperradical             |

The next proposition is clear:

4.5. **Proposition.** An ideal is hyperradical if and only if it is quasihyperradical and radical (that is, an MV-algebra is semisimple quasihyperarchimedean if and only if it is hyperarchimedean).

4.6. **Proposition.** Hyperradical ideals are radical ideals (that is, hyperarchimedean algebras are semisimple).

**Proof.** The reader can easily check that the following holds for any ideal $I$:

$$d(x \lor y, x) \in I \iff y \ominus x \in I.$$  

Assuming $x$ to be $I$-infinitesimal, it follows that for any integer $n \geq 1$, $d(\neg x \lor nx, \neg x) \in I$. Equivalently, $d(\neg(-\neg x \lor nx), x) \in I$. But:

$$\neg(\neg x \lor nx) = x \land \neg nx = \neg nx \ominus (nx \ominus x) = (n+1)x \ominus nx.$$  

Thus, $d((n+1)x \ominus nx, x) \in I$. Take $n \geq 1$ such that $(n+1)x \ominus nx \in I$, it follows that $x \in I$, proving that $I$ is a radical ideal (compare this proof with the remark after [2, definition 3.6.3]).

4.7. **Proposition.** Maximal ideals are hyperradical ideals (that is, simple algebras are hyperarchimedean).

**Proof.** If $x \in I$, clearly $2x \ominus x \leq 2x \in I$. Assume $x \notin I$, and by 3.1 take an integer $n \geq 1$ such that $\neg nx \in I$. $nx \leq (n+1)x$, so also $\neg (n+1)x \in I$. Then, $(n+1)x \ominus nx \leq d(nx, (n+1)x) = d(\neg nx, \neg (n+1)x) \in I$.

4.8. **Proposition.** Quasimaximal ideals are quasihyperradical ideals (that is, quasisimple algebras are quasihyperarchimedean).

**Proof.** The reader can check that the same proof in the previous proposition applies here.

4.9. **Proposition.** Prime hyperradical ideals are maximal ideals (that is, hyperarchimedean chains are simple algebras).

**Proof.** The reader can easily check that the following holds for any ideal $I$:

(a) $(x \ominus y \in I, y \in I \Rightarrow x \in I)$. 

Let $I$ be a prime hyperradical ideal, justified by 3.1, it is enough to prove that if $x \notin I$, then there exist an integer $m \geq 1$ such that $-mx \in I$.

Take $n$ such that $(n+1)x \oplus -nx = (n+1)x \odot nx \in I$. Assume (absurdum hypothesis) that $(n+1)x \odot nx \in I$. By distributivity of $\odot$ over $\lor$ it follows $(n+1)x \ominus (-nx \lor nx) = (n+1)x \odot (-nx \lor nx) \in I$.

But $-(-nx \lor nx) = nx \land -nx = -nx \odot (nx \oplus nx) = 2nx \odot nx$. Then, by (2.2) $2nx \odot nx \in I$. It follows by (a) above that $(n+1)x \in I$, which implies $x \in I$, contrary with our primary assumption. Thus we have $(n+1)x \ominus -nx = (n+1)x \odot nx \notin I$. Since $I$ is prime, it follows that $-nx \ominus (n+1)x \in I$.

Finally:

$-nx \oplus (n+1)x = -nx \ominus -(n+1)x = -(nx \oplus (n+1)x) = -(2n+1)x$.

Thus, $-mx \in I$ for $m = 2n + 1$.

4.10. **Comment.** In order to develop an elementary proof of the next two propositions it would be necessary to prove in the style of propositions 4.5 to 4.9 that if $I$ is a prime ideal, then $\sqrt{I}$ is maximal.

4.11. **Proposition.** Prime ideals are quasihyperradical ideals (that is, chains are quasihyperarquimedean algebras).

*Proof.* By Remark 3.3 and Theorem 3.12 it follows that if $I$ is a prime ideal, $\sqrt{I}$ is maximal, thus by 4.7 it is hyperradical. Then, Remark 4.4 finishes the proof.

4.12. **Proposition.** Prime radical ideals are maximal ideals (that is, semisimple chains are simple algebras).

*Proof.* By proposition 4.11 the ideal is quasihyperradical and radical, thus by 4.5 it is hyperradical. The proof finishes by proposition 4.9.

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