Online Submodular Coordination With Bounded Tracking Regret: Theory, Algorithm, and Applications to Multi-Robot Coordination

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Abstract—We enable efficient and effective coordination in unpredictable environments, i.e., in environments whose future evolution is unknown a priori and even adversarial. We are motivated by the future of autonomy that involves multiple robots coordinating in dynamic, unstructured, and adversarial environments to complete complex tasks such as target tracking, environmental mapping, and area monitoring. Such tasks are often modeled as submodular maximization coordination problems. We introduce the first submodular coordination algorithm with bounded tracking regret, i.e., with bounded suboptimality with respect to optimal time-varying actions that know the future a priori. The bound gracefully degrades with the environments’ capacity to change adversarially. It also quantifies how often the robots must re-select actions to “learn” to coordinate as if they knew the future a priori. The algorithm requires the robots to select actions sequentially based on the actions selected by the previous robots in the sequence. Particularly, the algorithm generalizes the seminal Sequential Greedy algorithm by Fisher et al. to unpredictable environments, leveraging submodularity and algorithms for the problem of tracking the best expert. We validate our algorithm in simulated scenarios of target tracking.

Index Terms—Multi-robot systems, unknown environments, online learning, regret optimization, submodular optimization.

I. INTRODUCTION

In the future, robots will be jointly planning actions to complete complex tasks such as:

- **Target Tracking:** How mobile robot networks can collaboratively track multiple evading targets [1]?
- **Environmental Mapping:** How mobile robots can collaboratively map an unknown environment [2]?
- **Area Monitoring:** How robot swarms can collaboratively monitor an area of interest [3].

All the aforementioned coordination tasks have been modeled by researchers in robotics, control, and machine learning via optimization problems of the form

\[ \max_{a_i, t \in V_i, \forall i \in N} f_t( \{ a_i, t \}_i \in N ), \quad t = 1, 2, \ldots, \tag{1} \]

where \( N \) is the robot set, \( a_i, t \) is robot \( i \)'s action at time \( t \), \( V_i \) is robot \( i \)'s set of available actions, and \( f_t : 2^{\Pi(N \times V_i)} \rightarrow \mathbb{R} \) is the objective function that captures the task utility. Particularly, \( f_t \) is considered computable prior to each time step \( t \) given a model about the future evolution of the environment [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]: e.g., in target tracking, a stochastic model for the targets’ future motion is often considered available, and then \( f_t \) can be chosen for example as the mutual information between the position of the robots and that of the targets [2].

Although (1) is generally NP-hard [12], near-optimal polynomial-time approximation algorithms have been proposed when \( f_t \) is submodular [13], a diminishing returns property. For example, the Sequential Greedy (SG) algorithm [13] achieves the near-optimal 1/2 approximation bound when \( f_t \) is submodular.

All aforementioned complex tasks can be modeled as submodular coordination problems, and thus SG and its variants are commonly used in the literature [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

But complex tasks often evolve in environments that change unpredictably, i.e., in environments whose future evolution is unknown a priori. For example, during adversarial target tracking the targets’ actions can be unpredictable since their intentions and maneuvering capability may be unknown [14]. In such challenging environments, the robots that are tasked to track the targets cannot simulate the future to compute \( f_t \) prior to time step \( t \). Hence, the robots have to coordinate their actions by relying on past information only, e.g., by relying only on the retrospective utility of their actions once the evolution of the environment has been observed. In this paper, we aim to solve (1) in unpredictable environments where \( f_t \) is unknown to the robots prior to time step \( t \) and thus the robots need to coordinate actions by relying only on the retrospective utility of their actions. Our goal is to provide polynomial-time algorithms with bounded suboptimality with respect to optimal time-varying multi-robot actions that know the future a priori, i.e., with bounded tracking regret [15] — the optimal actions ought to be time-varying to be effective against a changing environment such as an evading target. To this end, we will leverage tools from the literature of online learning [15].

Related Work: The current algorithms for (1) either (i) assume that \( f_t \) is known prior to each time step \( t \), instead of unknown, or (ii) apply to static environments where the optimal actions are time-invariant, instead of time-varying, or (iii) run in exponential time, instead of polynomial. No polynomial-time coordination algorithm exists that addresses (1) when \( f_t, t = 1, 2, \ldots, \) are unknown a priori and where the optimal actions in hindsight are time-varying.
a) Related Work in Submodular Optimization for Multi-Robot Coordination: The seminal algorithm Sequential Greedy (SG) [13] is the first polynomial-time algorithm for (1) with near-optimal approximation guarantees. Algorithms based on SG have enabled multi-robot coordination for a spectrum of tasks from target tracking [1], [6], [11] and environmental exploration [10], [16] to collaborative mapping [2], [3] and area monitoring [7], [9], [17], [18], [19], [20]. However, these algorithms assume a priori known \( f_i \).

b) Related Work in Online Learning for Submodular Optimization with Static Regret: Online learning algorithms have been proposed for (1) to account for the case where \( f_i \) is unknown a priori [21], [22], [23], [24], [25], [26]. But these algorithms apply only to tasks where the optimal solution is static: they guarantee bounded suboptimality with respect to optimal time-invariant robot actions, instead of optimal time-varying ones.

c) Related Work in Online Learning of Time-Varying Optimal Actions: The problem of learning online a sequence of time-varying actions that are optimal in hindsight constitutes the problem of tracking the best expert [27], [28], [29], [30], [31], [32], [33]. The problem involves an agent that selects actions online to maximize an accumulated utility across a number of time steps. The challenge is that the utility associated with each action is unknown a priori. Although algorithms for the tracking the best expert problem can be applied "as is" to (1), they then require exponential time to run, instead of polynomial. Similarly, although [33] recently leveraged such algorithms to provide polynomial-time online learning algorithms for the problem of cardinality-constrained submodular maximization, since this problem takes the form of max_{\mathcal{S} \subseteq \mathcal{V}, |\mathcal{S}| \leq k} f(\mathcal{S}) given an integer \( k \) and a function \( f : 2^\mathcal{V} \rightarrow \mathbb{R} \), those algorithms in [33] cannot be applied to (1), which has the form max_{a_{i, t} \in \mathcal{V}_i, \forall i \in \mathcal{N}} f(\{a_{i, t}\}_{i \in \mathcal{N}}). \) Instead, inspired by [33], our algorithm addresses the latter problem.

Contribution: We provide the first polynomial-time online learning algorithm with bounded tracking regret for multi-robot submodular coordination in unpredictable environments (Section III). We name the algorithm Online Sequential Greedy (OSG). The algorithm generalizes the Sequential Greedy algorithm [13] from the setting where each \( f_i \) is known a priori to the online setting where \( f_i \) is unknown a priori. As such, the algorithm requires the robots to select actions sequentially based on the actions selected by the previous robots in the sequence. OSG enjoys the properties:

- **Efficiency:** For each agent \( i \), OSG has a running time linear in the number of available actions per time step (Section IV-A).
- **Effectiveness:** OSG guarantees bounded tracking regret (Section IV-B). The bound gracefully degrades with the environments’ capacity to change adversarially. It quantifies the intuition that the agents should be able to effectively adapt to an unpredictable environment when they can re-select actions frequently enough with respect to the environment’s rate of change. Specifically, the bound guarantees asymptotically and in expectation that the agents select actions near-optimally as if they knew the future a priori, matching the performance of the Sequential Greedy algorithm [13].

Inspired by [33], our technical approach innovates by leveraging algorithms for the tracking the best expert problem [33], and the submodularity of the objective functions. Although the direct application of the tracking the best expert framework to (1) results in an exponential-time algorithm, by leveraging submodularity, we obtain the linear-time OSG.

Numerical Evaluations: We evaluate OSG in simulated scenarios of two mobile robots pursuing two mobile targets (Section V). We consider non-adversarial and adversarial targets: the non-adversarial targets traverse predefined trajectories, independently of the robots’ motion; whereas, the adversarial targets maneuver, in response to the robots’ motion. In both cases, the targets’ future motion and maneuvering capacity are unknown to the robots. Across the simulated scenarios, OSG enables the robots to closely track the targets despite being oblivious to the targets’ future motion.

II. ONLINE SUBMODULAR COORDINATION WITH BOUNDED TRACKING-REGRET

We define the problem Online Submodular Coordination with Bounded Tracking-Regret. To this end, we set:

- \( \mathcal{N}_i \triangleq \prod_{i \in \mathcal{N}} \mathcal{V}_i \) is the set of possible action combinations for all the agents \( \mathcal{N} \), given the set of available actions \( \mathcal{V}_i \) for each agent \( i \in \mathcal{N} \);
- \( f(a | \mathcal{A}) \triangleq f(\mathcal{A} \cup \{a\}) - f(\mathcal{A}) \) is the marginal gain of adding \( a \) to \( \mathcal{A} \), given an objective set function \( f : 2^\mathcal{V} \rightarrow \mathbb{R} \), \( a \in \mathcal{V} \), and \( \mathcal{A} \subseteq \mathcal{V} \).
- \( |\mathcal{A}| \) is the cardinality of \( \mathcal{A} \), given a discrete set \( \mathcal{A} \).

The following framework is also considered.

**Agents:** \( \mathcal{N} \) is the set of all agents. The terms “agent” and “robot” are used interchangeably in this paper. The agents coordinate actions via a coordinate descent scheme commonly used in the literature [1], [2], [3], [5], [6], [7], [8], [9], [10], [11] where the agents sequentially choose actions based on the actions selected by all previous agents in the sequence.

**Actions:** \( \mathcal{V}_i \) is a discrete and finite set of actions available to robot \( i \). For example, \( \mathcal{V}_i \) may be a set of (i) motion primitives that robot \( i \) can execute to move in the environment [6] or (ii) robot \( i \)’s discretized control inputs [2].

**Objective Function:** The robots coordinate their actions to maximize an objective function. In information-gathering tasks such as target tracking, environmental mapping, and area monitoring, typical objective functions are the covering functions [9], [17], [34]. Intuitively, these functions capture how much area/information is covered given the actions of all robots. They satisfy the properties defined below (Definition 1).

**Definition 1 (Normalized and Non-Decreasing Submodular Set Function [13]):** A set function \( f : 2^\mathcal{V} \rightarrow \mathbb{R} \) is normalized and non-decreasing submodular if and only if

\[
\begin{align*}
& f(\emptyset) = 0, \\
& f(\mathcal{A}) \leq f(\mathcal{B}), \text{ for any } \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V}; \\
& f(s | \mathcal{A}) \geq f(s | \mathcal{B}), \text{ for any } \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V} \text{ and } s \in \mathcal{V}. \\
\end{align*}
\]

Normalization \( f(\emptyset) = 0 \) holds without loss of generality. In contrast, monotonicity and submodularity are intrinsic to the function. Intuitively, if \( f(\mathcal{A}) \) captures the area covered by a set \( \mathcal{A} \) of activated cameras, then the more sensors are activated, the more area is covered; this is the non-decreasing property. Also, the marginal gain of the covered area caused by activating a camera \( s \) drops when more cameras are already activated; this is the submodularity property.

**Problem Definition:** In this paper, we focus on:

**Problem 1 (Online Submodular Coordination):** Assume a time horizon \( H \) of operation discretized to \( T \) time steps. The agents select actions \( \{a_{i, t}\}_{i \in \mathcal{N}} \) online such that at each time step \( t = 1, \ldots, T \) they solve the optimization problem

\[
\max_{a_{i, t} \in \mathcal{V}_i, \forall i \in \mathcal{N}} f(t(\{a_{i, t}\}_{i \in \mathcal{N}}), (2)
\]
where \( f_t : 2^{\mathcal{V}} \to \mathbb{R} \) is a normalized and non-decreasing submodular set function, becoming known to the agents only once they have executed their actions \( \{a_{i,t}\}_{i \in \mathcal{N}} \).

Problem 1 assumes that the agents know the full objective function \( f_t : 2^{\mathcal{V}} \to \mathbb{R} \) once they have executed their actions for the time step \( t \). This assumption is known as the full information setting in the literature of online learning [33].

Remark 1 (Feasibility of the Full Information Setting): The full information setting is feasible in practice when the agents can simulate the past upon observation of the environments’ new state by the end of any time step. For example, during target tracking with multiple robots, once the robots have executed their actions and observed the targets’ new positions by the end of the time step, then they can evaluate in hindsight the effect of all possible actions they could have selected instead. That is, \( f_t \) becomes fully known after the robots have acted at time step \( t \).

Remark 2 (Adversarial Environment): The objective function \( f_t \) can be adversarial, i.e., the environment may choose \( f_t \) once it has observed the agents’ actions \( \{a_{i,t}\}_{i \in \mathcal{N}} \). When \( f_t \) changes arbitrarily bad between time steps, then inevitably no algorithm can guarantee near-optimal performance. In this paper, we provide a randomized algorithm (OSG) which guarantees in expectation a suboptimality bound that deteriorates gracefully as the environment becomes more adversarial.

III. ONLINE SEQUENTIAL GREEDY (OSG) ALGORITHM

We present Online Sequential Greedy (OSG). OSG leverages as subroutine an algorithm for the problem of tracking the best expert [33]. Thus, before presenting OSG in Section III-B, we first present the tracking the best expert problem in Section III-A, along with its solution algorithm.

A. The Problem of tracking the best Expert

Tracking the best expert involves an agent selecting actions to maximize a total reward (utility) across a given number of time steps. The challenge is that the reward associated with each action is time-varying and unknown to the agent before the action has been executed. Therefore, to solve the problem, the agent needs to somehow guess the “best expert” actions, i.e., the actions achieving the highest reward at each time step.

To formally state the problem, we use the notation:

- \( \mathcal{V} \) denotes the set of actions available to the agent;
- \( r_{i,t} \) denotes the reward that the agent receives by selecting action \( i \in \mathcal{V} \) at the time step \( t \);
- \( r_i \triangleq \left[ r_{1,i}, \ldots, r_{|\mathcal{V}|,i} \right]^\top \) is the vector of all rewards at \( t \);
- \( i^*_t \in \arg \max_{i \in \mathcal{V}} r_{i,t} \); i.e., \( i^*_t \) is the index of the “best expert” action at time \( t \), that is, of the action that achieves the highest reward at time \( t \);
- \( 1(\cdot) \) is the indicator function, i.e., \( 1(x) = 1 \) if the event \( x \) is true, otherwise \( 1(x) = 0 \).

\[ P(T) \triangleq \sum_{t=1}^{T-1} 1(i^*_t \neq i^*_{t+1}); \] i.e., \( P(T) \) counts how many times the best action changes over \( T \) time steps.

Problem 2 (Tracking the Best Expert [15]): Assume a time horizon \( H \) of operation discretized to \( T \) time steps. The agent selects an action \( a_t \) online at each time step \( t = 1, \ldots, T \) to solve the optimization problem

\[
\max_{a_t \in \mathcal{V}, t=1,\ldots,T} \sum_{t=1}^{T} r_{a_t,t},
\]

where the rewards \( \{r_{i,t}\}_{i \in \mathcal{V}} \) take any value and become known to the agent only once the agent has executed action \( a_t \).

Algorithm 1: Fixed Share Forecaster* (FSF*) [33].

Input: Time steps \( T \); and action set \( \mathcal{V} \).
Output: Probability distribution \( p_t \in \{0,1\}^{|\mathcal{V}|} : \|p_t\|_1 = 1 \) over the actions in \( \mathcal{V} \) at each time \( t = 1, \ldots, T \).
1. \( J \leftarrow \lfloor \log_2(T) \rfloor; \gamma \leftarrow \sqrt{\frac{\log_2(J)}{T}}, \beta \leftarrow 1 / T; \)
2. \( \gamma(t) = \sqrt{\log_2(|\mathcal{V}|T)/2^t} \) for all \( j = 1, \ldots, J \);
3. Initialize \( z_1 = [z_1,1, \ldots, z_{|\mathcal{V}|,1}] \) with \( z_{j,1} = 1 \) for all \( j = 1, \ldots, J \);
4. Initialize \( w_{1,j} = [w_{1,j,1}, \ldots, w_{1,j,|\mathcal{V}|}] \) with \( w_{1,j,1} = 1 \), for all \( j = 1, \ldots, J \) and \( i = 1, \ldots, |\mathcal{V}| \);
5. for each time step \( t = 1, \ldots, T \) do
6. \( q_t \leftarrow z_t / \|z_t\|_1, p_t(j) \leftarrow w_{t,j} / \|w_t\|_1 \) for all \( j = 1, \ldots, J \);
7. \( p_t \leftarrow \sum_{j=1}^{J} q_{j,t} p_t(j) \);
8. observe the rewards \( \{r_{i,t}\}_{i \in \mathcal{V}} \) that the agent receives by selecting any action \( i \in \mathcal{V} \) at the time step \( t \);
9. \( r_t \leftarrow \{r_{1,t}, \ldots, r_{|\mathcal{V}|,t}\} \);
10. for \( j = 1, \ldots, J \) do
11. \( v_{t,j,t} \leftarrow w_{t,j} \exp(\gamma(t) r_{i,j,t}), i = 1, \ldots, |\mathcal{V}|; \)
12. \( v_{t,j,t+1} = v_{t,j,t} + \cdots + v_{t,j,t} \cdot \exp(\gamma(t) r_{i,j,t}) \)
13. \( u_{t,j,t+1} = \beta \cdot \frac{j}{|\mathcal{V}|} + (1 - \beta) v_{t,j,t}, i = 1, \ldots, |\mathcal{V}|; \)
14. \( z_{j,t+1} \leftarrow z_{j,t} \exp(\gamma(t) r_{i,j,t}) \);
15. end for
16. end for

where \( \hat{O}[\cdot] \) hides log terms. Thus, if \( i^*_t \) does not change many times across consecutive time steps \( t \), particularly, if \( P(T) \) grows slow enough with \( T \), then in expectation the agent is able to track the “best expert” actions \( i^*_1, i^*_2, \ldots \) unknown a priori to the agent, i.e., despite the challenge that the rewards \( \{r_{i,t}\}_{i \in \mathcal{V}} \) become known only once the agent has executed action \( a_t \). To this end, the algorithm provides the agent with a probability distribution \( p_t \) over the action set \( \mathcal{V} \) at each time step \( t \), from which the agent draws an action \( a_t \). Then, in expectation the agent’s total reward is guaranteed to be [33, Corollary 1]:

\[
\sum_{t=1}^{T} r_t \cdot p_t \geq \sum_{t=1}^{T} r_{i^*_t,t} - \hat{O} \left[ \sqrt{T \cdot P(T)} \right].
\]
Algorithm 2: Online Sequential Greedy (OSG).

**Input:** Time steps $T$; and agents’ action sets $\{V_i\}_{i \in N}$.

**Output:** Agent actions $\{a_{i,t}^{\text{OSG}}\}_{i \in N}$ at each $t = 1, \ldots, T$.

1. Each agent $i \in N$ initializes an FSF* with the value of the parameters $T$ and $V_i$.
2. Denote the FSF* onboard agent $i$ by $\text{FSF}^*|i|$.
3. Order the agents in $N$ such that $N = \{1, \ldots, |N|\}$.
4. For each time step $t = 1, \ldots, T$ do:
   5. For $i = 1, \ldots, |N|$ do:
      6. Get the output $p_{i,t}^{(i)}$ from $\text{FSF}^*|i|$;
      7. Draw an action $a_{i,t}^{\text{OSG}}$ from the distribution $p_{i,t}^{(i)}$;
      8. End for
      9. End for
10. Observe the objective function $f_t : 2^{V_i} \mapsto \mathbb{R}$;
11. $A_{i,t}^{\text{OSG}} \leftarrow \emptyset$;
12. For $i = 1, \ldots, |N|$ do:
   13. $A_{i,t}^{\text{OSG}} \leftarrow A_{i,t-1}^{\text{OSG}} \cup \{a_{i,t}^{\text{OSG}}\}$;
   14. For every $a \in V_i$ do:
      15. $r_{a,t}^{(i)} \leftarrow f_t(a | A_{i,t-1}^{\text{OSG}})$;
      16. End for
      17. $r_{a,t}^{(i)} \leftarrow \{r_{a,t}^{(i)}\}_{a \in V_i}$;
      18. Input $r_{a,t}^{(i)}$ to $\text{FSF}^*|i$ (per line 8 of Algorithm 1);
      19. End for
20. End for

Intuition Behind Algorithm 1: Algorithm 1 computes the probability distribution $p_t$ online given the observed rewards, i.e., given $\{r_{i,t}\}_{i \in V}$ up to $t-1$ (lines 5-15). To this end, Algorithm 1 assigns each action $i \in V$ time-dependent weights $\{w_{i,t}^{(j)}\}_{j=1,\ldots,T}$ that increase the higher $r_{i,1}, \ldots, r_{i,t-1}$ are compared with the corresponding rewards of all other actions (lines 11-14). The effect that the older rewards have on the value of $w_{i,t}^{(j)}$ is controlled by the parameter $\gamma^{(j)}$ (lines 11-14). $\gamma^{(j)}$ can be interpreted as a “learning” rate: the higher the $\gamma^{(j)}$ is, the more $w_{i,t}^{(j)}$ depends on the most recent rewards only, causing $w_{i,t}^{(j)}$ to adapt to (“learn”) the recent environment faster. Thus, higher values of $\gamma^{(j)}$ are desirable when the environment is more adversarial, i.e., when $P(T)$ is larger. To account for that $P(T)$ is unknown a priori, Algorithm 1 computes in parallel multiple weights, the $\{w_{i,t}^{(j)}\}_{j=1,\ldots,J}$, corresponding to the multiple learning rates $\{\gamma^{(j)}\}_{j=1,\ldots,J}$ (lines 11-14). $\gamma^{(j)}$ cover the spectrum from small to large sufficiently enough (lines 1-2) since (4) achieves the best known bound up to log factors [35].

B. The OSG Algorithm

OSG is presented in Algorithm 2. OSG generalizes the Sequential Greedy (SG) algorithm [13] to the online setting of Problem 1, leveraging at the agent-level FSF* (Algorithm 1). Particularly, when $f_i$ is known a priori, instead of unknown per Problem 1, then SG instructs the agents to sequentially select actions $\{a_{i,t}^{\text{SG}}\}_{i \in N}$ at each $t$ such that

$$ a_{i,t} = \max_{a \in V_i} f_t(a | \{a_{i,t'}^{\text{SG}}\}_{t' = t-1,\ldots,1}) $$

i.e., agent $i$ selects $a_{i,t}^{\text{SG}}$ after agent $i - 1$, given the actions of all previous agents $\{1, \ldots, i - 1\}$, and such that $a_{i,t}^{\text{SG}}$ maximizes the marginal gain given the actions of all previous agents from 1 through $i - 1$. But since $f_i$ is unknown and adversarial per Problem 1, OSG replaces the deterministic action-selection rule of (5) with a tracking the best expert rule (cf. Remark 3). Thus, OSG is also a sequential action-selection algorithm. In more detail, OSG starts by instructing each agent $i \in N$ to initialize an FSF* —we denote the FSF* onboard each agent $i$ by $\text{FSF}^*|i|$. Specifically, agent $i$ initializes $\text{FSF}^*|i|$ with its action set $V_i$ and with the number $T$ of total time steps (line 1). Then, at each time step $t = 1, \ldots, T$, in sequence:

- Each agent $i$ draws an action $a_{i,t}^{\text{OSG}}$ given the probability distribution $p_{i,t}^{(i)}$ output by $\text{FSF}^*|i|$ (lines 5-8).
- All agents execute their actions $\{a_{i,t}^{\text{OSG}}\}_{i \in N}$ and then observe $f_t : 2^{V_i} \mapsto \mathbb{R}$ (lines 9-10).
- Each agent $i$ receives from agent $i - 1$ the actions of all agents with a lower index, $A_{i-1,t}$, and then computes the marginal gain $f_t(a | A_{i-1,t}^{\text{OSG}})$ of each of its actions $a \in V_i$ (lines 11-16). Thus, in this step, each agent $i$ imitates in hindsight SG’s rule in (5).
- Finally, each agent $i$ makes the vector of its computed marginal gains observable to $\text{FSF}^*|i|$, per the line 8 of Algorithm 1 (lines 17-18). With this input, $\text{FSF}^*|i|$ will compute $p_{i+1}^{(i)}$, i.e., the probability distribution over the agent $i$’s actions for the next time step $t + 1$.

IV. PERFORMANCE GUARANTEES OF OSG

We quantify OSG’s computational complexity and approximation performance (Sections IV-A and IV-B respectively).

A. Computational Complexity of OSG

OSG is the first algorithm for Problem 1 with polynomial computational complexity.

**Proposition 1 (Computational Complexity):** OSG requires each agent $i \in N$ to perform $O(T |V_i|)$ function evaluations and $O(T \log (T))$ additions and multiplications.

The proposition holds true since at each $t = 1, \ldots, T$, OSG requires each agent $i$ to perform $O(|V_i|)$ function evaluations to compute the marginal gains in OSG’s line 14 and $O(\log (T))$ additions and multiplications to run $\text{FSF}^*|i|$.

B. Approximation Performance of OSG

We bound OSG’s suboptimality with respect to the optimal actions the agents’ would select if they knew the $\{f_i\}_{i=1,\ldots,T}$ a priori. Particularly, we bound OSG’s tracking regret, proving that it gracefully degrades with the environment’s capacity to select $\{f_i\}_{i=1,\ldots,T}$ adversarially (Theorem 1).

To present Theorem 1: first, we define tracking regret, particularly, $1/2$-approximate tracking regret (Definition 2); then, we quantify the environment’s capacity to select $\{f_i\}_{i=1,\ldots,T}$ adversarially (Definition 3). To these ends, we use the notation:

- $A_{\text{OPT}} \in \arg \max_{A_{\text{OPT}}} \{a_{i,t}^{\text{opt}}| i \in N\}$ i.e., $A_{\text{OPT}}$ is the optimal actions the agents’ would select at the time step $t$ if they knew the $f_i$ a priori;
- $a_{i,t}^{\text{opt}}$ is agent $i$’s action among the actions in $A_{\text{OPT}}$.
The robots can observe the exact location of the agents' would select if they knew the \( f_t(A_t) \) when the robots’ actions at time step \( t \).

Definition 2 \((1/2\text{-Approximate Tracking Regret})\): Consider any sequence of action sets \( \{A_t\}_{t=1,\ldots,T} \). Then, \( \{A_t\}_{t=1,\ldots,T} \)’s 1/2-approximate tracking regret is

\[
\text{Tracking-Regret}^{(1/2)}_T(\{A_t\}_{t=1,\ldots,T}) = \frac{1}{2} \sum_{t=1}^{T} f_t(A_t^{\text{OPT}}) - \sum_{t=1}^{T} f_t(A_t) .
\]  

Eq. (6) evaluates \( \{A_t\}_{t=1,\ldots,T} \)’s suboptimality with respect to the optimal actions \( \{A_t^{\text{OPT}}\}_{t=1,\ldots,T} \) the agents’ actions to the targets in each of the two scenarios are described in Section V-B and Section V-C, respectively.

V. Numerical Evaluation in Multi-Target Tracking Tasks With Multiple Robots

We evaluate OSG in simulated scenarios of target tracking. We first consider non-adversarial targets (Section V-B), i.e., targets whose motion is non-adaptive to the robots’ motion. Then, we consider adversarial targets (Section V-C), i.e., targets whose motion adapts to the robots’ motion.

Common Simulation Setup across Simulated Scenarios: We consider two robots pursuing two targets. Particularly:

a) Robots: The robots can observe the exact location of the targets. The challenge is that the robots are unaware of the targets’ future motion and, as a result, the robots cannot coordinate their actions by projecting where the targets are going to be. Instead, the robots have to coordinate their actions based only on the history of past observations and somehow guess the future. To move in the environment, each robot \( i \in \mathcal{N} \) can perform either of the actions (“upward,” “downward,” “left,” “right”) at a speed of \((1, 2) \) units/s.

b) Targets: The targets are either non-adversarial or adversarial, moving on the same 2D plane as the robots. The available actions to the targets in each of the two scenarios are described in Section V-B and Section V-C.

c) Objective Function: The robots coordinate their actions to maximize at each time step \( t \)

\[
f_t(\{a_{i,t}\}_{i \in \mathcal{N}}) = \sum_{j \in \mathcal{T}} \max_{i \in \mathcal{N}} \frac{1}{d_i(a_{i,t}, j)} ,
\]  

For example, (9) holds true in environments whose evolution in real-time is unknown yet predefined, instead of being adaptive to the agents’ actions. Then, \( \Delta(T) \) is uniformly bounded since increasing the discretization density of time horizon \( H \), i.e., increasing the number of time steps \( T \), does not affect the environment’s evolution. Thus, \( \Delta(T)/T \to 0 \) for \( T \to +\infty \), which implies (9). The result agrees with the intuition that the agents should be able to adapt to an unknown but non-adversarial environment when they re-select actions with high enough frequency.

Remark 4 (“Learning” to be Near-Optimal): When (9) holds true, OSG enables the agents to asymptotically “learn” to coordinate as if they knew \( f_t, f_{t+1}, \ldots \) a priori, matching the performance of the near-optimal Sequential Greedy algorithm [13]: the Sequential Greedy algorithm guarantees \( f_t(A_t) \geq 1/2 f_t(A_t^{\text{OPT}}) \) when \( f_t \) is known a priori, and (9) asymptotically guarantees \( f_t(A_t) \geq 1/2 f_t(A_t^{\text{OPT}}) \) in expectation despite \( f_t \) is unknown a priori.

Definition 3 (Environment’s Total Adversarial Effect): The environment’s total adversarial effect over \( T \) time steps is

\[
\Delta(T) = \sum_{t=1}^{T} \sum_{i \in \mathcal{N}} 1(\text{a}_{i,t}^{\text{OPT}} \neq \text{a}_{i,t+1}) .
\]  

\( \Delta(T) \) captures the environment’s total effect in selecting \( \{f_t\}_{t=1,\ldots,T} \) adversarially by counting how many times the optimal actions of the agents must shift across the \( T \) steps to adapt to the changing \( f_t \). That is, any changes that do not necessitate the agents’ actions to adapt are ignored.

In sum, the larger is the environment’s capacity to select \( \{f_t\}_{t=1,\ldots,T} \) adversarially, the larger \( \Delta(T) \) is, and the more frequently the agents need to change actions to remain optimal.

Theorem 1 (Approximation Performance): OSG instructs the agents to select actions, \( \{a_{i,t}\}_{i \in \mathcal{N}, t=1,\ldots,T} \) guaranteeing

\[
\mathbb{E}\left[ \text{Tracking-Regret}^{(1/2)}_T(\{a_{i,t}\}_{i \in \mathcal{N}, t=1,\ldots,T}) \right] \leq \hat{O}\left( \sqrt{\mathcal{N}T}(\Delta(T) + |\mathcal{N}|) \right) ,
\]  

where \( \mathbb{E}[\cdot] \) denotes expectation with respect to OSG’s randomness, and \( \hat{O}\{\cdot\} \) hides log terms.

Theorem 1 bounds the expected tracking regret of OSG. The bound is a function of the number of robots, the total time steps \( T \), and the environment’s total adversarial effect.

The upper bound in (8) implies that if the environment’s total adversarial effect grows slowly enough with \( T \), then in expectation the agents are able to 1/2-approximately track the unknown optimal actions \( A_t^{\text{OPT}}, \ldots, A_T^{\text{OPT}} \): if

\[
\hat{O}\left( \sqrt{|\mathcal{N}|T}(\Delta(T) + |\mathcal{N}|) \right) / T \to 0 \text{ for } T \to +\infty , \tag{9}
\]  

then (8) implies \( f_t(A_t) \to 1/2 f_t(A_t^{\text{OPT}}) \) in expectation.\(^4\)

\(^2\)Definition 2 generalizes existing notions of tracking regret [27], [33] to the online submodular coordination Problem 1.

\(^3\)Definition 3 generalizes existing notions of an environment’s total adversarial effect [33] to the online submodular coordination Problem 1.

\(^4\)Discretizing time finely, such that \( T \to +\infty \) is feasible, is limited in practice by the robots’ inability to replan and execute actions instantaneously.

\(^5\)The objective function in (10) is a non-decreasing and submodular function. The proof is presented in Appendix B.
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Fig. 1. Non-Adversarial Target Tracking with Multiple Robots. 2 robots pursue 2 targets that traverse trajectories that are non-adaptive to the robots motion: (a)–(c) the targets traverse the dashed straight trajectories; (g)–(i) the targets traverse the dashed noisy-rectangular trajectories that result from nominal rectangular trajectories corrupted with Gaussian noise of mean zero and variance 2. The green and red solid lines are the robots’ trajectories, starting from the solid dots, and the blue and orange dashed lines are the targets’ trajectories, starting from the crosses. Across (a)–(c) and (g)–(i), the number \( T \) of time steps per the given horizon \( H \) varies, resulting in the robots to re-select actions with varying frequency. (d)–(f) and (j)–(l) depict the minimum distance from each target to a robot, averaged over 50 instances of the simulation scenarios shown in (a)–(c) and (g)–(i) respectively.

A. Non-Adversarial Targets: Non-Adaptive Target Trajectories

Simulation Setup: We consider two scenarios of non-adaptive target trajectories: (i) the targets traverse predefined straight lines over a time horizon \( H = 50 \) s with speed 1 unit/s (Fig. 1(a)–(c)), and (ii) the targets traverse noisy rectangular-like trajectories over a time horizon \( H = 100 \) s (Fig. 1(g)–(i)); specifically, the rectangular-like trajectories are generated by targets that follow a nominal rectangular trajectory with speed 1 unit/s while randomizing their lateral speed by sampling from a Gaussian distribution with zero mean and variance 2. For each case, we evaluate OSG when the robots’ action re-selection frequency varies from 10 Hz to 20 Hz to 50 Hz.

Results: The simulation results are presented in Fig. 1. They reflect the theoretical analyses in Section IV-B. At 10 Hz, the robots fail to “learn” the targets’ future motion, failing to reduce their distance to them (Fig. 1(a), (d), (g), (j)). The situation improves at 20 Hz (Fig. 1(b), (e), (h), (k)), and even further at 50 Hz (Fig. 1(c), (f), (i), (l)), in which case the robots closely track the targets. The average minimum distances improve from (10 Hz) to 2 (20 Hz) to 0.3 (50 Hz) for the line case (Fig. 1(d)–(f)), and from 8 (10 Hz) to 4 (20 Hz) to 2 (50 Hz) units for the noisy-rectangular case (Fig. 1(j)–(l)).

B. Adversarial Targets: Adaptive Target Trajectories

Simulation Setup: We consider targets that maneuver when the robots are close enough. As long as the robots are more than 1.5 units away from a target \( j \), the target \( j \) will keep moving “right” on a nominal straight line at a speed 1 unit/s. But once a robot is within 1.5 units away, then target \( j \) will perform a maneuver: target \( j \) will first choose from moving “upward” or “downward” at 2 units/s for 1 s to maximize the distance from the robots, and then will move diagonally for 0.05 s to return back to the nominal path with vertical speed 40 units/s and horizontal speed to the “right” 30 units/s.

To demonstrate the need for randomization in adversarial environments (Remark 3), we compare OSG with a deterministic algorithm that selects actions at each \( t \) with respect to the previously observed \( f_{t-1} \). We denote the algorithm by \( \hat{SG} \) since it is a (heuristic) extension of the Sequential Greedy algorithm [13] to Problem 1’s setting per the rule:

\[
a_{i,t}^{\hat{SG}} \in \max_{a \in V_i} f_{t-1}(a | \{a_{i-1,t}, \ldots, a_{i-1,t-1}\}). \quad (11)
\]

The Sequential Greedy algorithm [13] instead selects actions per the rule in (5), given the a priori knowledge of \( f_t \).

Results: The simulation results are presented in Fig. 2. OSG performs better than \( \hat{SG} \), as expected since \( \hat{SG} \) prescribes actions to the robots “blindly” by deterministically following the target’s previous position, instead of accounting for the whole history of

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Code: Our code is available at: https://gitlab.umich.edu/iral-code/ral23-online-submodular-coordination.
the targets’ past motions as OSG does: OSG triggers 50% more target maneuvers than \(\text{OSG} \) (31 maneuvers in Fig. 2(a) vs. 20 maneuvers in Fig. 2(b)), implying OSG tracks the targets closer than \(\text{OSG} \), particularly, once OSG has “learned” the target’s future motion (after the 30-th second in Fig. 2(c)), OSG results in average minimum distances to each target that are 20% smaller than \(\text{OSG} \)’s.

VI. CONCLUSION

Summary: We introduced the first algorithm for efficient and effective online submodular coordination in unpredictable environments (OSG): OSG is the first polynomial-time algorithm with bounded tracking regret for Problem 1. The bound gracefully degrades with the environments’ capacity to change adversarially, quantifying how frequently the agents must re-select actions to “learn” to coordinate as if they knew the future a priori. OSG generalizes the seminal Sequential Greedy algorithm [13] to Problem 1’s online optimization setting. To this end, we leveraged the FSG* algorithm for the problem of tracking the best expert. We validated OSG in simulated scenarios of target tracking.

Future Work: We plan for the future work:

a) Partial Information Feedback: OSG selects actions at each \(t\) based on full information feedback, i.e., based on the availability of the functions \(f_k : 2^N \to \mathbb{R}\) for each \(k \leq t - 1\). The availability of these functions relies on the assumption that the agents can simulate the environment from the beginning of each step \(k\) till its end. But the agents may lack the resources for this. We will enable OSG to rely only on partial information, i.e., only on the observed values \(f_k(\{a_{i,k}\}_{i \in N})\) per the executed actions \(\{a_{i,k}\}_{i \in N}\).

b) Best of Both Worlds (BoBW): OSG selects actions assuming the environment may evolve arbitrarily in the future. The assumption is pessimistic when the environment’s evolution is governed by a stochastic (yet unknown) model. For example, OSG’s performance against the non-adversarial targets in Section V-B, although it becomes near-optimal for large \(T\), is pessimistic: the deterministic heuristic \(\text{OSG} \) can be shown to perform better since by construction \(f_{t-1} \simeq f_t\) in Section V-B. We will extend OSG such that it offers BoBW suboptimality guarantees [37]. BoBW guarantees become especially relevant under partial information feedback since then heuristics such as the \(\text{OSG} \) cannot apply in the first place.

APPENDIX

A. Proof of Theorem 1

We use the notation:

- \(A_{t-1,t}\) is the optimal solution set for the first \(i - 1\) agents at time step \(t\);
- \(\Delta_i(T) = \sum_{t=1}^{T} 1(a_{i,t}^{OPT} \neq a_{i,t}^{OPT})\) is the change of environment for agent \(i\), i.e., \(\sum_{t=1}^{T} \Delta_i(T) = \Delta(T)\).

We have:

\[
\sum_{t=1}^{T} f_t(A_t^{OPT}) \leq \sum_{t=1}^{T} f_t(A_t^{OPT} \cup A_t^{OSG}) \tag{12}
\]

\[
= \sum_{t=1}^{T} f_t(A_t^{OSG}) + \sum_{t=1}^{T} \sum_{i \in N} f_t(a_{i,t}^{OPT} | A_{t-1,t}) \tag{13}
\]

\[
\leq \sum_{t=1}^{T} f_t(A_t^{OSG}) + \sum_{t=1}^{T} \sum_{i \in N} f_t(a_{i,t}^{OPT} | A_{t-1,t}) \tag{14}
\]

\[
= 2 \sum_{t=1}^{T} f_t(A_t^{OSG}) + \sum_{t=1}^{T} \sum_{i \in N} f_t(a_{i,t}^{OPT} | A_{t-1,t}) \tag{15}
\]

\[
\geq 2 \sum_{t=1}^{T} f_t(A_t^{OSG}) + \sum_{t=1}^{T} \sum_{i \in N} r^{(i)}_{a_{i,t}^{OPT},t} - r^{(i)}_{a_{i,t}^{OSG},t} \tag{16}
\]

where (12) holds from the monotonicity of \(f_t\); (13) and (15) are proved by telescoping the sums; (14) holds from the submodularity of \(f_t\); and (16) holds from the definition of \(r^{(i)}\) (OSG’s line 14). We now complete the proof:

\[
\mathbb{E}\left[\text{Tracking-Regret}_{T}^{(1/2)}(A_t^{OSG})\right] = \mathbb{E}\left[\frac{1}{2} \sum_{t=1}^{T} f_t(A_t^{OPT}) - f_t(A_t^{OSG})\right] \tag{17}
\]

\[
\leq \frac{1}{2} \sum_{t=1}^{T} \sum_{i \in N} \mathbb{E}\left[r^{(i)}_{a_{i,t}^{OPT},t} - r^{(i)}_{a_{i,t}^{OSG},t}\right] \tag{18}
\]

\[
= \frac{1}{2} \sum_{t=1}^{T} \sum_{i \in N} r^{(i)}_{a_{i,t}^{OPT},t} - r^{(i)}_{a_{i,t}^{OSG},t} P_t \tag{19}
\]

\[
\leq \frac{1}{2} \times 8 \sum_{i \in N} \sqrt{T((\Delta_i(T) + 1) \log (|V_i(T)| + 1))} \tag{20}
\]
TABLE I

| $s \leq \min(b_1, b_2)$ | $\max(s | B_1)$ | $\max(s | B_1 \cup B_2)$ |
|-------------------------|-----------------|------------------------|
| $b_1 < s < b_2$         | $s - b_1$       | $0$                    |
| $b_2 < s < b_1$         | $0$             | $s - \max(b_1, b_2)$  |
| $\max(b_1, b_2) \leq s$| $s - b_1$       | $0$                    |
| $b_1 = b_2$             | $\max(0, s - b_1)$ | $\max(0, s - b_2)$ |

\[
\begin{align*}
\leq 4\sqrt{|W| T \sum_{i \in \mathcal{N}} ((\Delta_i(T) + 1) \log |V_i(T) + 1 |) \\
\leq 4\sqrt{|W| T ((\Delta(T) + |V|) \log (\max_{i \in \mathcal{N}} |V_i(T) + |V_i|)},
\end{align*}
\]

B. Proof of Monotonicity and Submodularity of Function (10)

It suffices to prove that $\max: 2^\mathcal{R} \rightarrow \mathcal{R}$ is non-decreasing and submodular. Indeed, if $A \subseteq B \subseteq 2^\mathcal{R}$, then $\max(A) \leq \max(B)$, i.e., $\max$ is non-decreasing. Next, consider finite and disjoint $B_1 \in 2^\mathcal{R}$ and $B_2 \in 2^\mathcal{R}$, and an arbitrary real number $s$. Then, using Table I we verify that $\max(s | B_1) \geq \max(s | B_1 \cup B_2)$ holds true, i.e., $\max$ is submodular.

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