Closed expression for the pair vibrational correlation energy of a uniform distribution of single-nucleon levels

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A closed expression is derived for the pair vibrational correlation energy generated in the random phase approximation by the isovector pairing force in the case when Kramers and charge degenerate single-nucleon levels are uniformly distributed in an interval. The expression is used to analyze the spectral density of pair vibrational frequencies relative to that of two-quasinucleon energies. Applications to the analysis of the symmetry energy of the isovector pairing model and to a Strutinskij renormalization of this model are discussed.

I. INTRODUCTION

The separable pairing force is a schematic representation of a part of the interaction of nucleons in the nuclear medium. It was introduced by Belyaev [1] in the wake of the adaption to nuclei by Bohr, Mottelson, and Pines [2], Bogolyubov [3], and Solov’yov [4] of the theory of superconductivity of Bardeen, Cooper, and Schrieffer [5]. Its most common application is in the framework of the nuclear Bardeen-Cooper-Schrieffer (BCS) theory, where it is supposed to act on otherwise independent nucleons in a valence space. The Hartree-Bogolyubov approximation is applied to this Hamiltonian, the pairing Hamiltonian, which amounts in this case to neglecting, in terms of the general Hartree-Fock-Bogolyubov scheme [6], the contribution of the pairing force to the self-consistent single-nucleon potential. The nuclear BCS theory explains many observations, including the even-odd mass staggering [1], the gap in the spectra of doubly even nuclei [2], moments of inertia [1, 7], and enhanced cross sections for two-nucleon transfer [8].

There was from the outset an interest in exploring the correlations induced by the pairing force beyond the Hartree-Bogolyubov approximation. In an early study, Feldman thus diagonalized numerically the pairing Hamiltonian in a simple case [9], Richardson found a reduction of the exact diagonalization of the Hamiltonian to the solution of a system of non-linear equations [10], and Bès and Broglia [11] used the random phase approximation (RPA) [12]. This latter approach was inspired by Bohr’s suggestion [13] that the pair field might vibrate in a way that is analogous to the vibrations of the single-nucleon potential accompanying surface vibrations. In all this and much later work, separate pairing forces were assumed to act on neutrons and protons. This interaction is not charge invariant. The minimal charge invariant extension includes a separable interaction of isovector pairs of a neutron and a proton. The coupling constants of the three components must be equal. The RPA was applied to the resulting, so-called isovector pairing model in the Sixties by Ginocchio and Wesener [14] and recently by me [15–17]. (My model includes a schematic interaction of isospins. As this only contributes an energy proportional to $T(T + 1) - \frac{3}{4}A$, where $A$ is the mass number and $T$ the total isospin [17], the calculation is equivalent to one employing the bare isovector pairing force.) Later the Richardson scheme was extended to the isovector pairing model by Dukelsky et al. [18] in a calculation of three levels in $^{64}$Ge employing the valence space between the magic numbers 20 and 50. Numeric diagonalization of the isovector pairing Hamiltonian in valence spaces including six or seven Kramers and charge degenerate single-nucleon levels was done by Bentley and Frauendorf [19]. (In both these works the Hamiltonian includes an interaction of isospins of the same form as that of Refs. [15–17].)

Various single-nucleon spectra are employed in my calculations in Refs. [15–17]. In Ref. [17] the levels are generated by a Woods-Saxon potential, while in Refs. [15, 16] they are equidistant, forming a so-called picket-fence spectrum. Exploring an equidistant spectrum aims at displaying average effects of the RPA correlations. To eliminate in this context the dependence of the results on the valence space dimension I consider in Ref. [17], besides the Woods-Saxon spectra, a practically infinite picket-fence spectrum. This approach has the disadvantage that when the BCS gap parameter is fixed, the RPA energy goes to minus infinity as the valence space dimension goes to infinity; only the symmetry energy $E_{\text{sym}} = E(A,T) - E(A,0)$, where $E(A,T)$ is the total energy, stays finite. I here approach the aim of displaying average effects of the RPA correlations in a different manner: The finite picket-fence spectrum is replaced by a continuous spectrum in an interval. Strutinskij previously derived in this way a closed expression for the average BCS energy [20]. Similarly I here obtain a closed expression for the average RPA energy.

One application of these closed expressions is in calculations such as those of Bentley, Frauendorf, and me in Ref. [21], to provide smooth counterterms for a Strutinskij renormalization of the isovector pairing model. Preliminary versions of the present expressions, communicated in Ref. [21] without their derivations, were used in this way in Ref. [21]. Another application is demonstrated in Sec. [15]. As shown there, the expression for the RPA energy provides information on the spectral density of pair vibrational frequencies. It also allows analysis of the contribution to the symmetry energy of the non-collective pair vibrational modes in a general way. As discussed in Refs. [15–17] this contribution influences
the shape of the so-called Wigner cusp in the plot of masses along a chain of isobaric nuclei. This is the topic of Sec. VI.

It may be noted finally that several studies [13, 21–24] show the Hartree-Bogolyubov plus RPA to reproduce very accurately the exact ground state energies of the pairing and isovector pairing Hamiltonians. For the latter, this approximation is shown, in particular, in Refs. [13, 21] to be asymptotically exact in the limit of the coupling constant going to infinity. The largest deviations occur for values of the coupling constant near criticality for the onset of a BCS solution with a non-vanishing pair gap parameter in the case that the critical value is not zero, which occurs when the Fermi level lies in an interval between consecutive single-nucleon levels. As the critical value is of the order of the length of this interval, it vanishes for a continuous spectrum. For such a spectrum the BCS gap parameter is thus nonzero down to vanishing of the coupling constant.

As the isovector pairing Hamiltonian is the special case of the Hamiltonian studied in Ref. [17] without the so-called symmetry force, I refer throughout in the following to that article for details of the formalism. Omitting the symmetry force amounts to setting there \( \kappa = 0 \).

The plan of the present article is the following. In Sec. II I review the derivation of Strutinskij’s expression [24] for the BCS energy of a uniform distribution of single-nucleon levels in an interval. This serves to set some notation and give some background for the main discussion in Sec. III of the RPA energy generated by this spectrum. In Sec. IV I use the closed expression obtained in Sec. III to analyze the distribution of RPA frequencies relative to that of the two-quasineutron energies. I then turn to the application of the isovector pairing model to the estimate of nuclear masses. After a discussion in Sec. V of numeric parameters I analyze in Sec. VI the contributions to the symmetry energy of each of the independent-nucleon, BCS, and RPA energies. Finally, before summarizing the article in Sec. VIII I discuss the application of the closed expression derived in Sec. III to a Strutinskij renormalization of the RPA energy of the isovector pairing model.

II. BCS

For a general single-nucleon spectrum the BCS energy \( E_{BCS} \) is the difference of the Hartree-Bogolyubov energy \( E_{HB} \) given by Eq. (19) of Ref. [17] (with \( \kappa = 0 \)) and the sum of occupied single-nucleon levels. For a doubly even nucleus it consists of a neutron part \( E_{BCS,n} \) and a proton part \( E_{BCS,p} \) each given by

\[
E_{BCS,\tau} = 2 \sum_k v_{k\tau}^2 \epsilon_k - \frac{\Delta^2}{G} - 2 \sum_{k \leq N_{\tau}/2} \epsilon_k. \quad (1)
\]

Here \( \epsilon_k \) are the Kramers and charge degenerate single-nucleon levels and

\[
\frac{v_{k\tau}}{v_{k\tau}} = \sqrt{\frac{1}{2} \left( 1 \pm \frac{\epsilon_k - \lambda_{\tau}}{E_{k\tau}} \right)} 
\]

with

\[
E_{k\tau} = \sqrt{(\epsilon_k - \lambda_{\tau})^2 + \Delta_{\tau}^2}. \quad (2)
\]

The chemical potential \( \lambda_{\tau} \) and gap parameter \( \Delta_{\tau} \) are determined uniquely by the equations

\[
2 \sum_k v_{k\tau}^2 = N_{\tau}, \quad \sum_k \frac{1}{E_{k\tau}} = \frac{2}{G}, \quad (3)
\]

if these equations have a solution. Here \( G \) is the pair coupling constant and \( N_n = N \) and \( N_p = Z \) are the numbers of neutrons and protons. These are understood as the numbers of such nucleons occupying states within the valence space, so they may differ from the true numbers if a limited valence space is employed. It may happen that Eqs. (1) have no solution; then \( E_{BCS,\tau} = 0 \).

I now assume that the single-nucleon levels \( \epsilon_k \) are equidistant with a spacing \( 1/g \), and that for each \( \tau \) a number \( \Omega_{\tau\tau} \) of these levels are selected for the action of the isovector pairing force on pairs of nucleons of the kind \( \tau \). The selection is assumed symmetric about a level \( \lambda_{\tau\tau} \), which turns out equal to \( \lambda_{\tau} \). The interaction of pairs of a neutron and a proton is passive in the BCS approximation. In the RPA this is no longer the case. I therefore, in order to prepare the discussion in Sec. III consider also a selection of a number \( \Omega_{np} \) of single-nucleon levels for the action of the neutron-proton pairing force. This selection is supposed symmetric about a level \( \lambda_{np} \), which turns out equal to \( (\lambda_n + \lambda_p)/2 \). Assuming each of the three components of the isovector pairing force to act on selections of single-nucleon levels that are symmetric about the respective chemical potentials is the single simplification made in this article, which allows me to obtain closed expressions for both the BCS and the RPA energy in the continuous limit. For \( N \neq Z \) it implies a deviation from the isobaric invariance of the original Hamiltonian. The simplification might be justified by the expectation that details of the single-nucleon spectrum far from \( \lambda_{\tau\tau} \) should have little influence on these correlation energies. The three cases \( \tau \tau' = nn, pp \) and \( np \) are discussed in a unified manner in the rest of this section, and I drop the index \( \tau \tau' \) when it can be done unambiguously. In the following thus \( \Omega = \Omega_{\tau\tau} \) and \( \lambda = \lambda_{\tau\tau} \). Other quantities introduced in the course of the discussion should also be understood as specific for the case of \( \tau \tau' \). For convenience in the subsequent analysis, \( \Omega \) is supposed to be always even. The modifications required if \( \Omega \) is odd will be evident.

The continuous approximation results from replacing the sums in Eqs. (1) and (2) by integrals. With

\[
\epsilon_{\approx} = \lambda \pm \frac{\Omega}{2g}, \quad (5)
\]
the second Eq. (4) then becomes

$$\int_{\varepsilon^\text{r}}^{\varepsilon^\text{l}} \frac{gde}{\sqrt{(\varepsilon - \lambda^\text{r})^2 + \Delta^2}} = g(a_{\varepsilon^\text{r}} - a_{\varepsilon^\text{l}}) = \frac{2}{G}, \quad (6)$$

where

$$a_{\varepsilon^\text{r}} = \sinh^{-1} \frac{\varepsilon^\text{r} - \lambda^\text{r}}{\Delta^\text{r}}. \quad (7)$$

The first Eq. (4) takes the form

$$\int_{\varepsilon^\text{r}}^{\varepsilon^\text{l}} \left( 1 - \frac{\varepsilon - \lambda^\text{r}}{\sqrt{(\varepsilon - \lambda^\text{r})^2 + \Delta^2}} \right) de = 2(\lambda^0_{\varepsilon^\text{r}} - \varepsilon^\text{l}) \quad (8)$$

with

$$\lambda^0_{\varepsilon^\text{r}} = \lambda + \frac{N_{\varepsilon^\text{r}} - N_{\varepsilon^\text{l}}}{4g}, \quad (9)$$

which can be written

$$\lambda^\text{r} - \lambda^0_{\varepsilon^\text{r}} \approx \frac{1}{2} \left( \frac{2\Delta}{\Omega} \right)^2 (\lambda^0_{\varepsilon^\text{r}} - \lambda), \quad (10)$$

so that \( \lambda^\text{r} = \lambda^0_{\varepsilon^\text{r}} \) is then a good approximation for \( \tau \tau' = np \). For \( \tau = \tau' \) the equation \( \lambda^\text{r} = \lambda^0_{\varepsilon^\text{r}} \) holds exactly by \( \lambda^\text{r} = \lambda \) and Eq. (4).

It is convenient to express other quantities by the parameter

$$a = \frac{1}{2}(a_{\varepsilon^\text{r}} - a_{\varepsilon^\text{l}}), \quad \frac{1}{2}(a_{\varepsilon^\text{r}} - a_{\varepsilon^\text{l}}) = \frac{1}{2} \frac{\lambda^\text{r} + \lambda}{G}. \quad (12)$$

The last expression, which follows from Eq. (6), shows \( a \) to be a dimensionless reciprocal measure of the coupling constant \( G \). Other convenient relations follow from Eqs. (5) and (7):

$$\frac{\Omega}{g} = \Delta(\sinh a_{\varepsilon^\text{r}} - \sinh a_{\varepsilon^\text{l}}) = 2\Delta \sinh a \cosh \left( \frac{a_{\varepsilon^\text{r}} + a_{\varepsilon^\text{l}}}{2} \right),$$

$$\delta \lambda = \lambda^\text{r} - \lambda = \Delta(\sinh a_{\varepsilon^\text{r}} + \sinh a_{\varepsilon^\text{l}}) = -2\Delta \cosh a \left( \frac{a_{\varepsilon^\text{r}} + a_{\varepsilon^\text{l}}}{2} \right). \quad (13)$$

The difference \( \delta \lambda \) of chemical potentials is a second parameter whereby I shall express other quantities. Equations (13) give in particular

$$\delta \lambda = \frac{\Omega}{2g \sinh a} \sqrt{1 - \frac{(g\delta \lambda \tanh a)^2}{\Omega^2}}, \quad (14)$$

which for \( \tau = \tau' \) becomes

$$\Delta = \frac{\Omega}{2g \sinh a}. \quad (15)$$

I can now derive Strutinskij’s expression for \( E_{\text{BCS}, \tau} \).

In this case \( \tau = \tau' \), so \( \lambda = \lambda \) and \( \Delta = \Delta \). By the first Eq. (2) one can replace \( \epsilon_k \) by \( \epsilon_k - \lambda \) simultaneously in the first and last terms on the right of Eq. (1). With the sums replaced by integrals and \( G \) eliminated by Eq. (6), the expression Eq. (1) then becomes

$$E_{\text{BCS}, \tau} = \int_{\varepsilon^\text{r}}^{\varepsilon^\text{l}} \left[ \left( 1 - \frac{\varepsilon - \lambda}{\sqrt{(\varepsilon - \lambda)^2 + \Delta^2}} \right)(\varepsilon - \lambda) - \frac{\Delta^2}{2\sqrt{(\varepsilon - \lambda)^2 + \Delta^2}} \right] gde$$

$$\quad - 2\int_{\varepsilon^\text{r}}^{\varepsilon^\text{l}} (\varepsilon - \lambda) gde$$

$$\quad = -\frac{1}{2}(1 - e^{-2a})g\Delta^2 = -\frac{1}{2}\Omega \Delta e^{-a}. \quad (16)$$

This is equivalent to Eq. (11) of Ref. [20] except that a factor \( 1/2 \) must be missing there by mistake. Belyayev derives in the continuous limit a related expression for the total Hartree-Bogolyubov energy including the sum of single-nucleon levels occupied for \( \Delta = 0 \) [1]. The expression (16) is used in Ref. [21].

III. RPA

The additional energy arising from the RPA extension of the Hartree-Bogolyubov approximation is composed of the terms in Eqs. (35) and (38) of Ref. [17]. The term \( c \) given by Eq. (35), which stems from reordering of nucleon fields, vanishes when the valence space is half filled, so only Eq. (38) needs to be considered. The resulting energy \( E_{\text{RPA}, \tau} \) splits into a neutron part \( E_{\text{RPA}, \text{nn}} \), a proton part \( E_{\text{RPA}, \text{pp}} \), and a neutron-proton part \( E_{\text{RPA}, \text{np}} \), each given by

$$E_{\text{RPA}, \tau \tau'} = -\frac{i}{4\pi} \int_{-\infty}^{\infty} f(\omega)d\omega \quad (17)$$

with

$$f(\omega) = -\sum_{n=1}^{\infty} \frac{1}{n} \text{tr}(\text{VG}_0(\omega))^n. \quad (18)$$
As a reminder I mostly omit an index \( \tau \tau' \), so all of \( f(\omega) \), \( \lambda, \Delta \), etc. are specific to the case of \( \tau \tau' \). In Eq. (17) the matrices \( V \) and \( G_0(\omega) \) have dimensions \( 2 \Omega \times 2 \Omega \) and are composed of \( 2 \times 2 \) blocks (cf. in Ref. [17] Eqs. (40) and (43) and the equations before Eq. (28))

\[
\begin{align*}
V_{kk'} &= -G \begin{pmatrix} -v_{k\tau} v_{k'\tau'} & u_{k\tau} v_{k'\tau'} \\ u_{k\tau} u_{k'\tau'} & -v_{k\tau} v_{k'\tau'} \end{pmatrix} \\
G_0_{kk'}(\omega) &= \begin{pmatrix} 0 & 1 \\ \omega - E_{k\tau} - E_{k'\tau'} + i\eta & 0 \end{pmatrix},
\end{align*}
\]

where \( \eta \) is a positive infinitesimal. Because, as shown in Ref. [17], \( f(\omega) \) is proportional to \( \omega^{-2} \) for large \( \omega \), one can move the integration path in Eq. (17) to the imaginary axis. So far I only assume that \( \omega \) is not real so that the infinitesimal terms in the denominators in Eq. (20) can be dropped. From Eqs. (19) and (20) one gets

\[
\text{tr} (V G_0(\omega))^n = (-G)^n \text{tr} M^n
\]

in terms of the \( 2 \times 2 \) matrix

\[
M = \sum_k X_k \begin{pmatrix} \frac{1}{\omega - E_{k\tau} - E_{k'\tau'}} & 0 \\ 0 & \frac{1}{\omega - E_{k\tau} - E_{k'\tau'}} \end{pmatrix} X_k
\]

with

\[
X_k = \begin{pmatrix} u_{k\tau} u_{k'\tau'} & -v_{k\tau} v_{k'\tau'} \\ -v_{k\tau} v_{k'\tau'} & u_{k\tau} u_{k'\tau'} \end{pmatrix}.
\]

Hence

\[
f(\omega) = -\sum_{n=1}^{\infty} \frac{(-G)^n}{n} \text{tr} M^n = \text{tr} \log (1 + GM) = \log \det (1 + GM),
\]

where 1 is the \( 2 \times 2 \) unit matrix.

By the symmetry of the single-nucleon spectrum about \( \lambda \), and because \( \Omega \) is even, the indices \( k \) form pairs \( (k, k') \) such that \( \epsilon_k + \epsilon_{k'} = 2\lambda \). As then

\[
E_{k'\tau} + E_{k'\tau'} = E_{k\tau} + E_{k\tau'},
\]

\[
X_{k'} = X_k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_k,
\]

the matrix between the two occurrences of \( X_k \) in Eq. (22) can be replaced by the number

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

by the \( k \)-independent orthogonal transformation

\[
\begin{pmatrix} E_{k\tau} + E_{k\tau'} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} E_{k\tau} + E_{k\tau'} & \delta \lambda^2 - 4\Delta^2 \\ \delta \lambda^2 - 4\Delta^2 & (E_{k\tau} + E_{k\tau'})^2 - \delta \lambda^2 \end{pmatrix}.
\]

The reduction in Eq. (31) follows from Eqs. (48), (58), and (59) of Ref. [17], where \( \tau \) and \( \tau' \) may be substi-

uted for \( n \) and \( p \) in the last two equations. (The reader is reminded that \( \lambda = \lambda_{\tau\tau'} = (\lambda_\tau + \lambda_{\tau'})/2 \) and \( \delta \lambda = \lambda_\tau - \lambda_{\tau'} \).) Putting everything together one gets

\[
1 = \sum_k \frac{E_{k\tau} + E_{k\tau'}}{E_{k\tau} E_{k\tau'}}
\]

derived from the second Eq. (43). one gets

\[
1 = \frac{G}{4} \sum_k \left( \frac{1}{E_{k\tau}} + \frac{1}{E_{k\tau'}} \right) = \frac{G}{4} \sum_k \frac{E_{k\tau} + E_{k\tau'}}{E_{k\tau} E_{k\tau'}}.
\]
\[ f(\omega) = \log \left[ \left( \delta \lambda^2 + 4\Delta^2 - \omega^2 \right) \left( \delta \lambda^2 - \omega^2 \right) \right] \]

\[
\left( \frac{G}{4} \sum_k \frac{E_{kr} + E_{kr'}}{(E_{kr} + E_{kr'})^2 - \omega^2} \right)^2. \tag{34}
\]

Now

\[
\frac{1}{(E_{kr} + E_{kr'})^2 - \omega^2} = \frac{(E_{kr} - E_{kr'})^2 - \omega^2}{(E_{kr} + E_{kr'})^2 - \omega^2} - \frac{4E_{kr}E_{kr'}}{(E_{kr} + E_{kr'})^2 - \omega^2}, \tag{35}
\]

and

\[
\frac{(E_{kr} + E_{kr'})^2 - \omega^2}{E_{kr}E_{kr'}} = \frac{E_{kr}^2 - E_{kr'}^2 - \omega^2}{E_{kr}} + \frac{E_{kr'}^2 - E_{kr}^2 - \omega^2}{E_{kr'}}, \tag{36}
\]

where, by the the symmetry in \( \tau \) and \( \tau' \), the two terms contribute equally to the sum in Eq. (34). Equation (3) gives

\[
E_{kr}^2 - E_{kr'}^2 = 2(\epsilon_k - \lambda)\delta \lambda,
\]

\[
(E_{kr}^2 + E_{kr'}^2 - \omega^2)^2 - 4E_{kr}E_{kr'} = 4(\delta \lambda^2 - \omega^2)(\epsilon_k - \lambda)^2 - (\delta \lambda^2 + 4\Delta^2 - \omega^2)\omega^2 = 4(\delta \lambda^2 - \omega^2)(\epsilon_k - \lambda - q^2). \tag{37}
\]

with

\[
q = \frac{r\omega}{2}, \quad r = \sqrt{\frac{\delta \lambda^2 + 4\Delta^2 - \omega^2}{\delta \lambda^2 - \omega^2}}. \tag{38}
\]

Moreover

\[
\frac{2(\epsilon_k - \lambda)\delta \lambda - \omega^2}{(\epsilon_k - \lambda)^2 - q^2} = \frac{1}{r} \left( r\delta \lambda - \omega + \frac{r\delta \lambda + \omega}{\epsilon_k - \lambda - q} \right). \tag{39}
\]

The branch of the square root in Eq. (38) may be chosen such that \( r \) is positive when \( \omega \) is imaginary.

Consider the first term in the parentheses in Eq. (39). When this term is expanded through Eqs. (35)–(37), the denominator in the general term in the sum in Eq. (34) receives a factor

\[
(\delta \lambda^2 - \omega^2)r = \sqrt{\delta \lambda^2 + 4\Delta^2 - \omega^2}(\delta \lambda^2 - \omega^2), \tag{40}
\]

which is canceled by the factors in front of the squared expression in parentheses. The sum of the remaining factors becomes

\[
\frac{1}{2} r\delta \lambda - \omega \sum_k \frac{1}{\epsilon_k - \lambda - q}E_{kr}. \tag{41}
\]

To arrive at the continuous approximation I replace the sum in this expression by the integral

\[
\int_\epsilon^\delta g\,d\epsilon \left( \frac{\sinh a_{\tau>} - \phi}{\cosh a_{\tau>} + \phi} \right) \tag{42}
\]

where \( \phi \) is any root in

\[
2\Delta \sinh \phi = 2q - \delta \lambda = r\omega - \delta \lambda \tag{43}
\]

and the branch of the logarithm is defined by \( \log 1 = 0 \). The root \( \phi \) can be chosen by the second Eq. (13) such that

\[
2\Delta \cosh \phi = r\delta \lambda - \omega, \tag{44}
\]

which brings the expression (41) with the sum replaced by the integral (42) on the form

\[
g \log \frac{\sinh a_{\tau>} - \phi}{\cosh a_{\tau>} + \phi} \frac{a_{\tau<} + \phi}{a_{\tau<} - \phi}. \tag{45}
\]

Including the contribution from the second term in parentheses in Eq. (39) amounts to multiplying the argument of the logarithm by the expression which results from changing the sign of \( \omega \).

By the second Eq. (13) one has

\[
4\Delta \sinh \frac{a_{\tau>} - \phi}{2} \frac{a_{\tau<} + \phi}{2} = 2\Delta \left( \sinh(a - \phi) + \sinh \frac{a_{\tau>} + a_{\tau<}}{2} \right)
\]

\[
= 2\Delta \left[ \sinh a \cos \phi - \cosh a \sin \phi \right] - \frac{\delta \lambda}{\cosh a}
\]

\[
= \delta \lambda \left( r \sinh a + \cosh a - \frac{1}{\cosh a} \right) - \omega(r \cosh a + \sinh a)
\]

\[
= (\delta \lambda \tanh a - \omega)(r \cosh a + \sinh a). \tag{46}
\]

This gives the numerator of the fraction in Eq. (15), expanded by \( 4\Delta \). The denominator results from interchanging \( a_{\tau>} \) and \( a_{\tau<} \), which amounts to changing the sign of \( a \), and the factors from the second term in parentheses in Eq. (39) result from changing the sign of \( \omega \). Totally, the factors \( \pm \delta \lambda \tanh a \mp \omega \) cancel out so that the sum of the expression (45) and its counterpart for the opposite sign of \( \omega \) becomes

\[
2g \log \frac{r \cosh a + \sinh a}{r \cosh a - \sinh a} = 4g \tanh^{-1} \left( \frac{1}{r} \tan a \right) \tag{47}
\]

with the branch of the inverse hyperbolic tangent given by \( \tanh^{-1} 0 = 0 \).
By substituting the expression (47) for the sum in Eq. (34), remembering that the factors in front of the squared expression in parentheses were eliminated, and using Eq. (12), one gets

\[ f(\omega) = 2 \log \left[ \frac{1}{a} \tanh^{-1} \left( \frac{1}{r} \tanh a \right) \right]. \]  

(48)

As \( r \) is by the second Eq. (38) a function of \( \omega^2 \), it is sufficient to do the integral in Eq. (17) along the positive imaginary axis, so

\[ E_{\text{RPA}, \tau' \tau} = -\frac{i}{2\pi} \int_0^{i\infty} f(\omega)d\omega. \]  

(49)

One can bring this relation on a dimensionless form by setting

\[ \delta \lambda = 2\Delta l, \quad \omega = 2i\Delta y. \]  

(50)

This gives

\[ E_{\text{RPA}, \tau' \tau} = \Delta I(a, l) \]  

(51)

with

\[ I(a, l) = \frac{2}{\pi} \int_0^\infty \log \left[ \frac{1}{a} \tanh^{-1} \left( \frac{1}{r} \tanh a \right) \right] dy, \]  

(52)

where, by the second Eq. (38),

\[ \frac{1}{r} = \sqrt{\frac{l^2 + y^2}{1 + l^2 + y^2}}. \]  

(53)

As the integrand in Eq. (52) is evidently negative, \( I(a, l) \) is negative. Some special cases of the general result (51) are discussed in Secs. III A–III B.

A. \( l = 0 \)

For \( \tau = \tau' \) and \( \tau' \tau = np \) for \( N = Z \) one has \( l = 0 \). The substitution

\[ y = \sinh x \]  

(54)

then gives

\[ \frac{1}{r} = \tanh y, \]  

(55)

so that Eq. (52) becomes

\[ I(a, 0) = \frac{2}{\pi} \int_0^\infty \log \left( \frac{1}{2a} \log \frac{\cosh(x + a)}{\cosh(x - a)} \right) \cosh x dx. \]  

(56)

When inserted in Eq. (51), this gives Eq. (12) of Ref. [21].

Figure 1 displays the function \( I(a, 0) \). It is seen to decrease rapidly. However, as seen from Fig. 2, \( I(a, 0)/\sinh a \), which by Eqs. (51) and (14) gives the dependence of \( E_{\text{RPA}, \tau' \tau} \) on \( a \) for constant \( g \) and \( \Omega \), increases and goes to zero for \( a \to \infty \) as required by Eqs. (17)–(19) because this limit corresponds by Eq. (12) to \( G \to 0 \).
Figure I illustrates

\[ I(0, 0) = \frac{2}{\pi} \int_{0}^{\infty} \log \left[ \frac{d}{da} \tanh^{-1} \left( \frac{1}{r_{1=0} \tanh a} \right) \right]_{a=0} dy \]

\[ = \frac{2}{\pi} \int_{0}^{\infty} \left( \log \frac{1}{r_{1=0}} \right) dy \]

\[ = \frac{2}{\pi} \int_{0}^{\infty} \log \left( \frac{y}{\sqrt{1 + y^2}} \right) dy = -1. \quad (57) \]

This limit is not much physically relevant, though, as \( a \to 0 \) corresponds to \( G \to \infty \).

As \( \tanh a \to 1 \) for \( a \to \infty \), the argument of the logarithm in Eq. (52) goes to zero in this limit, so \( I(a, 0) \to -\infty \) as illustrated in Fig. 1. In particular, if \( \Delta \) is fixed and \( a \) determined by Eq. (15) then \( \Omega \to \infty \) implies \( a \to \infty \) and therefore \( \delta \epsilon_{RPA, np} \to -\infty \) by Eq. (51).

This shows that the exact ground state energy of the pairing Hamiltonian, which is well approximated by the Hartree-Bogolyubov plus RPA (see Sec. III C), cannot be renormalized to a given \( \Delta \) in a way approximately independent of \( \Omega \). This is only possible in the BCS approximation, where the term \( e^{-2\delta I} \) in the penultimate expression in Eq. (16) vanishes for \( a \to \infty \).

B. \( l \neq 0 \)

The difference

\[ \delta I(a, l) = I(a, l) - I(a, 0) \]

\[ = \frac{2}{\pi} \int_{0}^{\infty} \log \left( \frac{\tanh^{-1} \frac{1}{r_{1=0} \tanh a}}{\tanh^{-1} \frac{1}{r_{1=0} \tanh a}} \right) dy \quad (58) \]

describes the increase, in units of \( \Delta \), of \( \delta \epsilon_{RPA, np} \) with increasing neutron or proton excess. Indeed, by taking \( \lambda_{c} = \lambda_{n}^{0} \) in the definition of \( \delta \lambda \) in Eq. (13) one gets from Eq. (9) and the first Eq. (50) that

\[ l = \frac{N - Z}{4g\Delta} \quad (59) \]

for \( \tau \tau' = np \). The function \( \delta I(a, l) \), which, evidently from Eqs. (52) and (53), is an even function of \( l \), is plotted for several \( a \) in Fig. 3. As \( I(a, l) \) is negative, \( \delta I(a, l) \) necessarily levels off at \( -I(a, 0) \), which equals, for example, 2.4 for \( a = 2 \). As long as \( \delta I(a, l) \) is sufficiently far from this limit, it is seen to be well approximated by the

\[ \frac{\tanh^{-1} \left( l \tanh a \sqrt{\frac{1 + u^2}{1 + l^2 (1 + u^2)}} \right)}{\tanh^{-1} \left( lu \tanh a \sqrt{\frac{1}{1 + (lu)^2}} \right)} \to \frac{\sqrt{1 + u^2} \sqrt{1 + l^2 (1 + u^2)}}{u}, \quad l \to 0^{+}. \quad (62) \]
the integral becomes in this limit the negative of the one in Eq. (57) so that
\[ \delta I(a,l) \approx l, \quad (63) \]
or,
\[ \left. \frac{\partial I(a,l)}{\partial l} \right|_{l=0^+} = 1. \quad (64) \]
This derivative is illustrated by the dashed line in Fig. 3.

The result is anticipated because it is equivalent to
\[ \frac{\partial E_{\text{RPA},np}}{\partial \delta \lambda} \bigg|_{g,\Omega,G,\lambda_n+\lambda_p=2\lambda,\delta \lambda=0^+} = \frac{1}{2}. \quad (65) \]

For a discrete single-nucleon spectrum the analogous derivative of \( E_{\text{RPA},np} \) with respect to \( \delta \lambda \) indeed equals 1/2 simply because in Eq. (39) of Ref. [17] (also to be found, for example, in Refs. [16, 22]) the frequency \( |\delta \lambda| \) of a vibrational mode arising from the conservation of isospin, cf. Ref. [14] and Sec. III H of Ref. [17], is the only term in the expression in square brackets that is not analytic at \( \delta \lambda = 0 \). This single frequency continues smoothly into its negative when \( \lambda_n \) passes through \( \lambda_p \). I call this mode a quasi-Goldstone mode because it is similar to a Goldstone or Nambu-Goldstone mode [25] by arising from a spontaneously broken symmetry but does not have, in general, zero frequency.

C. Comparison with an exact calculation

A comparison of the results of the Hartree-Bogolyubov plus RPA with calculations of the exact energy is made in Table I in the case of a six-level picket-fence spectrum populated by 12 nucleons. It is confirmed that the Hartree-Bogolyubov plus RPA is very good. The largest deviations occur about the minimal G for nonvanishing gap parameters \( \Delta = \), which is given for this spectrum by \( gG \approx 0.35 \) almost independently of \( T \). This is explained in Sec. V of Ref. [21]. See also Refs. [14, 22, 24].

For \( T = 0 \) I also show the energies in the continuous limit of the Hartree-Bogolyubov plus RPA, that is, \( E_{\text{BCS}} + E_{\text{RPA}} \) with the terms given by Eqs. (16) and (51). This comparison shows that already for six levels in the discrete picket-fence spectrum the continuous limit is quite representative. I do not make this comparison for \( T > 0 \) because the prerequisite of the derivation in Sec. III that for \( \tau = \tau' \) the single-nucleon spectrum be symmetric about the chemical potentials \( \lambda_n \) is obviously badly violated for \( N - Z = 2T = 4 \) and 8 when the valence space includes only six levels.

IV. SPECTRUM OF PAIR VIBRATIONAL FREQUENCIES

It follows from the derivation of Eq. (39) of Ref. [17] that the discontinuity of \( f(\omega) \) across a branch cut at the real axis describes the cumulated spectral density of pair vibrational frequencies \( \omega_k \) relative to that of the two-quasinucleon energies \( E_{k\tau} + E_{k\tau'} \). The closed expression Eq. (48) therefore allows an analysis of this relative spectral density. Because \( f(\omega) \) is an even function of \( \omega \) and real for imaginary \( \omega \), complex conjugation of \( f(\omega) \) maps to complex conjugation of \( f(\omega) \), so the discontinuity equals the value of \( -i\Im f(\omega)/2 \) immediately below the real axis. By the derivation in Ref. [17] this gives
\[ \Im f(\omega - i\eta) = \pi \sum_k \left[ \theta(\omega - \omega_k) - \theta(\omega - (E_{k\tau} + E_{k\tau'})) \right] \quad (66) \]
for \( \omega > 0 \).

It is convenient to introduce again a dimensionless measure of \( \omega \). This time I set \( \omega = 2\Delta z \) and define accordingly
\[ h(z) = \frac{1}{\pi} f(2\Delta z) = \frac{2}{\pi} \log \left[ \frac{1}{a} \tanh^{-1} \left( \frac{1}{r} \tanh a \right) \right] \quad (67) \]
with
\[ \frac{1}{r} = \sqrt{\frac{l^2 - z^2}{1 + l^2 - z^2}}. \quad (68) \]

Then \( \Im h(z - i\eta) \) gives for \( z > 0 \) directly the cumulated relative spectral density. An example of this function is plotted in Fig. 4.

The shape of the resulting curve is easily understood from the discussion in Sec. III J of Ref. [17]. The cumulated relative spectral density \( \Im h(z - i\eta) \) jumps from 0
TABLE I. Energy in units of the single-nucleon level spacing $1/g$ induced by the isovector pairing force in the case of a six-level picket-fence spectrum populated by 12 nucleons. Shown are the energies calculated by numeric diagonalization of the Hamiltonian (exact), in the Hartree-Bogolyubov (HB) plus RPA, and in the continuous limit of the latter (continuous). The parameter $G$ is the pair coupling constant. I am indebted to Ian Bentley for providing the results of numeric diagonalization.

| $gG$ | $T = 0$ | $T = 2$ | $T = 4$ |
|------|---------|---------|---------|
|      | Exact   | HB+RPA  | Exact   | HB+RPA  |
| 0.2  | −2.05   | −2.12   | −2.23   |         |
| 0.4  | −4.65   | −5.11   | −5.05   |         |
| 0.6  | −7.86   | −7.97   | −8.12   |         |
| 0.8  | −11.59  | −11.59  | −11.74  |         |
| 1.0  | −15.74  | −15.69  | −15.82  |         |
| 1.2  | −20.19  | −20.12  | −20.24  |         |
| 1.4  | −24.87  | −24.78  | −24.89  |         |
| 1.6  | −29.69  | −29.60  | −29.71  |         |
| 1.8  | −34.63  | −34.54  | −34.64  |         |
| 2.0  | −39.65  | −39.56  | −39.66  |         |

To 1 at the quasi-Goldstone frequency $z = l$. It may be noticed that this frequency remains separated from the bulk of the spectrum in the continuous limit. A second vibrational frequency occurs at $z = \sqrt{1 + l^2}$ (Eq. 39 of Ref. 17), but immediately thereafter with increasing $z$ the first twofold degenerate two-quasinucleon energy appears. In the discrete spectrum, $\Im h(z - in)$ therefore first rises to 2 and then drops to 0. As the rest of the vibrational frequencies are also twofold degenerate and each of them is embedded between successive two-quasinucleon energies, this alternation of 2 and 0 continues until the highest two-quasinucleon energy, which occurs at $z = z_\ell$, where

$$z_\ell^2 = \left[ \frac{1}{2\Delta} \left( \sqrt{(\epsilon_\ell - \lambda) + \Delta^2} + \sqrt{(\epsilon_\ell - \lambda)^2 + \Delta^2} \right) \right]^2$$

$$= \left( \frac{1}{2} \left[ \cosh a_{\ell+} + \cosh a_{\ell-} \right] \right)^2$$

$$= \left( \cosh a \left[ \frac{a_{\ell+} + a_{\ell-}}{2} \right] \right)^2$$

$$= \cosh a + \frac{\cosh a \sinh a_{\ell+} + a_{\ell-}}{2} \right)^2$$

$$= \cosh a + \frac{\cosh a \sinh a_{\ell+} + a_{\ell-}}{2} \right)^2$$

$$= \cosh a^2 + \frac{\cosh a \sinh a_{\ell+} + a_{\ell-}}{2} \right)^2$$

by Eqs. (5), (7), (13), and (50). In the continuous limit $\Im h(z - in)$ becomes the average of these 2 and 0 weighted by the lengths of the intervals in which each of them prevails. Figure 4 shows that at the beginning of the continuous spectrum the vibrational frequencies are situated midway between consecutive two-quasinucleon energies. With increasing frequency they then rapidly approach the two-quasinucleon energy above.

Because the path of integration in Eq. (17) can be transformed as described in Sec. III C of Ref. 17, the integral of $\Im h(z - in)$ for $z > 0$ is $-I(a, l)$. The shape of the plot of $\Im h(z - in)$ as displayed in Fig. 4 thus provides a deeper understanding of the behavior of the function $I(a, l)$, including, in particular, its deviation from linearity in $l$. Thus, if $\Im h(z - in)$ were replaced by a constant in the interval $l < z < z_c$, as suggested by the dashed line in Fig. 4 and $z_\ell$ were constant, then the integral would be strictly linear in $l$ due to the displacement with $l$ of the lower edge of the curve. The displacement of the upper edge adds a positive term equal to the increase of $\epsilon_\ell$. Finally the subtracted area between the solid and dashed curves shrinks with $z_\ell - \sqrt{1 + l^2}$, which decreases with increasing $l$. This adds another positive contribution. Both these contributions are quadratic in $l$ to the lowest order and give the deviation from linearity. This argument is seen to provide also an alternative derivation of Eq. 64.

The shape of the curve in Fig. 4 is also easily understood from the expressions (67) and (68). First notice that the imaginary part of $h(z)$ is $2/\pi$ times the complex argument of $\phi(z) = \tanh^{-1} \psi(z)$, $\psi(z) = \frac{1}{r} \tanh a$. (70)

I discuss how this develops with increasing $z > 0$. For $z < l$ the square root in Eq. 68 is real and less than one. As also $\tanh a < 1$, the functions $\psi(z)$ and $\phi(z)$ are real. At $z = l$ the square root branches off in opposite imaginary values, the one below the cut being positive. Then $\phi(z)$ is also positive imaginary, so $\arg \phi(z)$ jumps to $\pi/2$. It stays there until $z = \sqrt{1 + l^2}$, when the square root becomes positive real again. At this point, however, $\psi(z)$ is infinite, so $\phi(z)$ has reached $i\pi/2$. As $\psi(z)$ then descends from infinity through positive real values, $\phi(z)$ takes values of an increasing positive real number plus $i\pi/2$ by continuity, and so $\arg \phi(z)$ descends. This continues until $r = \tanh a$, when $\Re \phi(z)$ is infinite so that...
arg $\phi(z)$ vanishes. At this point $\phi(z)$ thus has a pole. Walking around the pole the branches above and below the cut then join in a real value and $\phi(z)$ stays real. It is easily shown by Eqs. (68) and (69) that $r = \tanh a$ is equivalent to $z = z_\succ$.

V. PARAMETERS FOR NUMERIC ESTIMATES

Four parameters enter the expressions for the BCS and RPA energies derived in Secs. II and III:

1. the number $\Omega$ of Kramers and charge degenerate single-nucleon levels supposed to participate in pair correlations,
2. the density $g$ of such levels, understood to pertain to the neighborhood of the Fermi level,
3. the pair coupling constant $G$,
4. in $E_{\text{RPA},np}$ for $N \neq Z$, the difference $\delta \lambda$ of neutron and proton chemical potentials.

I discuss the choice of these parameters for the purpose of numeric estimates.

There is no obvious way to extract $\Omega$ from data. It is desirable, however, to use a recipe that is the simplest possible, involves the least possible structural assumptions, and is consistent with the prerequisite of the derivations in Secs. II and III that the single-nucleon levels be symmetrically distributed about the respective chemical potentials. These criteria are satisfied if in each of the three cases $\tau \tau' = nn$, $pp$, and $np$ one includes all levels from the bottom of the spectrum to the Fermi level and equally many levels upwards from there. This amounts to taking $\Omega = N_\tau$ for $\tau = \tau'$ and $\Omega = A/2$ for $\tau \tau' = np$. Incidentally these are also the approximate numbers of bound levels in the nuclear potential well.

For the single-nucleon level density $g$ I adopt the value extracted by Kataria, Ramamurthy and Kapoor [26] from observed neutron resonances in spherical nuclei,

$$\frac{\pi^2}{6} \times 4g = 0.176 \text{ MeV}^{-1}(A - A^{-2/3}). \tag{71}$$

The pair coupling constant $G$ is generally extracted in some manner from observed odd-even mass differences. Strutinskij [20] thus calculates $G$ from Eqs. (12) and (14) by presumably (cf. Ref. [27]) identifying $\Delta$ with the odd-even mass difference $\Delta_{oe}$ and adopting Bohr’s and Mottelson’s fit [28]

$$\Delta_{oe} = 12A^{-1/2} \text{ MeV} \tag{72}$$

to the observed values. This procedure makes $G$ somewhat dependent on $\Omega$. The dependence is seen, however, to be logarithmic. On the other hand identifying $\Delta$ directly with $\Delta_{oe}$ is a severe simplification. Its rationale is that in the BCS theory, $\Delta$ is the energy of a quasineutron excitation at the Fermi level, cf. Eq. (3). This

excitation blocks, however, the Fermi level from taking part in the pair correlations, thus reducing the effective density of participating levels. The reduction of the effective $g$ increases the parameter $a$ by Eq. (12) and thus reduces by Eqs. (14) and (16) and Fig. 2 the absolute values of both $E_{\text{BCS}}$ and $E_{\text{RPA}}$. For a reliable determination of $G$ one therefore needs to do a full calculation of the binding energies of both the odd-$A$ nuclei and their doubly even neighbours and then fit $G$ to reproduce the observed differences.

Moreover, twice the expression (72) is seen from Fig. 3 of Ref. [19] or Fig. 6 of Ref. [21] to underestimate greatly the observed $T = 0$ doubly odd–doubly even mass differences above $^{56}\text{Ni}$. In Ref. [21], Bentley, Frauendorf and I fit the $T = 0$ doubly odd–doubly even mass differences from $A = 24$ to $A = 100$ in a full, Strutinskij renormalized calculation, cf. Sec. VII, based on the isovector pairing Hamiltonian with the above $\Omega$. We find

$$G = 8.6A^{-4/5} \text{ MeV} \tag{73}$$

to be optimal. I therefore adopt this expression.

Equations (12), (71) and (73) give

$$\frac{4.35}{A^{1/5} - A^{-2/15}}. \tag{74}$$

For example $a = 3.5$ for $A = 24$, $a = 2.6$ for $A = 56$ and $a = 2.2$ for $A = 100$. On can then infer from Eq. (14) that

$$-\frac{1}{2} \frac{(2g\Delta)^2}{\Omega} \leq \frac{1}{2(\sinh a)^2} \leq 0.025, \tag{75}$$

where the final bound corresponds to $a = 2.2$. It then follows from Eq. (11) that $\lambda_\tau = \lambda_0^0$ is a very good approximation, improving with smaller values of $a$. In this approximation

$$\delta \lambda = \frac{N - Z}{2g} \tag{76}$$

follows from Eq. (9).

VI. SYMMETRY ENERGY

I discuss in Refs. [15, 17] the symmetry energy of the isovector pairing model. In this section I examine which new insights the closed expressions derived in Secs. II and III might bring to this discussion. For the modeling of the symmetry energy it is sufficient to consider the isobaric analog with the maximal $N$, so $T$ can be identified with $(N - Z)/2$.

In the Hartree-Bogolyubov plus RPA total energy $E$ includes the sum $E_{\text{indep}}$ of single-nucleon levels subtracted in Eq. (1). Thus

$$E = E_{\text{indep}} + E_{\text{BCS}} + E_{\text{RPA}}. \tag{77}$$
The term $E_{\text{indep}}$ is composed of a neutron part $E_{\text{indep},n}$ and a proton part $E_{\text{indep},p}$, each given by

$$E_{\text{indep},\tau} = 2 \sum_{k \leq N_{\tau}/2} \epsilon_k.$$  

(78)

In the continuous limit this becomes

$$E_{\text{indep},\tau} = 2g \int_{\epsilon_<}^{\lambda} \epsilon \, d\epsilon = g(\lambda^2 - \epsilon_<^2) = \epsilon_< N_{\tau} + \frac{N^2_\tau}{4g},$$  

(79)

by Eq. (5) and $\Omega = N_{\tau}$. As the filling of the single-nucleon spectrum from the bottom for both $\tau$ implies $\epsilon_{n,<} = \epsilon_{p,<} := \epsilon_<$, adding the neutron and proton contributions results in

$$E_{\text{indep}} = \epsilon_< A + \frac{A^2}{8g} + \frac{T^2}{2g}. \quad (80)$$

The contribution to $E_{\text{sym}}$ is $T^2/2g$.

In the approximation (63) the part $\Delta \delta I(a,l)$ of $E_{\text{RPA}}$ equals $T/2g$ by Eqs. (71) and (59). It is noteworthy that this term depends on neither $\Omega$ nor $G$, which are empirically the least well determined parameters. In combination with the contribution from $E_{\text{indep}}$ it gives a total term in $E_{\text{sym}}$ equal to $T(T + 1)/2g$. It is well known [28] that $1/2g$ is much less than the empirical symmetry energy coefficient. For example, for $A = 56$ Eq. (71) gives $1/2g = 0.45$ MeV, while the coefficient of $T(T + 1)$ in the semiempirical mass formula, Eq. (1) of Ref. [29] by Mendoza-Temis, Hirsch, and Zucker, is 1.29 MeV. The difference must be attributed to isovector-dependent interactions [28]. In Refs. [15,17] I consider a schematic two-nucleon interaction, called the symmetry force in Refs. [16,17],

$$V_{12} = \kappa \mathbf{t}_1 \cdot \mathbf{t}_2, \quad (81)$$

where $\kappa$ is a coupling constant and $\mathbf{t}$ is the nucleonic isospin. The symmetry force contributes an energy

$$\frac{1}{2} \kappa [T(T + 1) - \frac{3}{4} A], \quad (82)$$

so that, totally so far,

$$E_{\text{sym}} = \frac{1}{2} \left( \frac{1}{g} + \kappa \right) T(T + 1). \quad (83)$$

The most important lesson to be learned from this result is that it supports taking in semiempirical mass formulas the symmetry energy proportional to $T(T + 1)$, such as done for example by Mendoza-Temis, Hirsch, and Zucker, rather than proportional to $T^2$ as is more usual. It should be noticed that this conclusion does not rest on the present very schematic model where otherwise independent nucleons in a valence space interact by the isovector pairing and symmetry forces. It applies to any Hamiltonian which produces a self-consistent Hartree, Hartree-Fock, Hartree-Bogolyubov, or Hartree-Fock-Bogolyubov self-consistent state that is not an eigenstate of $N - Z$. If $E_{\text{sc}}(T)$ is the self-consistent energy, a generalization of the discussion in Sec. III H of Ref. [17] implies, in fact, that the quasi-Goldstone RPA mode restoring isobaric invariance has frequency $E_{\text{sc}}^{\Omega}(T)$. The only requirement for this relation to hold is that the RPA stability matrix, Eq. (8.73) of Ring and Schuck [30], is the Hessian matrix of $E_{\text{sc}}$ with respect to variations about self-consistency. Adding to $E_{\text{sc}}(T)$ the term in Eq. (39) of Ref. [17] from the quasi-Goldstone frequency gives $E_{\text{sc}}(T) + \frac{1}{2} E_{\text{sc}}^{\Omega}(T)$. In the neighborhood of $T = 0$ the self-consistent energy $E_{\text{sc}}(T)$ rises proportionally to $T^2$, so $E_{\text{sc}}(T) + \frac{1}{2} E_{\text{sc}}^{\Omega}(T)$ rises proportionally to $T(T + 1)$.

I now discuss the remaining contributions to $E_{\text{sym}}$ in the expression (77). Throughout I understand $\Omega = N_{\tau}$ for $\tau = \tau'$ and $\Omega = A/2$ for $\tau \tau' = np$ to be substituted wherever $\Omega$ occurs in formulas. Equations (15) and (16) then give

$$E_{\text{BCS}} = \frac{N^2 + Z^2}{4g(e^{2\kappa a} - 1)} = -\frac{A^2/4 + T^2}{2g(e^{2\kappa a} - 1)}. \quad (84)$$

The BCS energy thus generates a small term quadratic in $T$. Its coefficient $1/(2g(e^{2\kappa a} - 1))$, is easily estimated to amount to at most a few permille of the coefficient of $T(T + 1)$ in Eq. (1) of Ref. [29]. From Eqs. (51) and (15) one gets

$$E_{\text{RPA},nn} + E_{\text{RPA},pp} = \frac{AI(a,0)}{2g \sinh a} := 2E_{\text{RPA}}^0, \quad (85)$$

which does not depend on $T$. With $g$ and $a$ given by Eqs. (71) and (77), the energy $E_{\text{RPA}}^0$ equals $-5.4$ MeV for $A = 24$, $-6.4$ MeV for $A = 56$, and $-7.1$ MeV for $A = 100$.

It remains to discuss the terms in $E_{\text{RPA},np}$ in excess of $E_{\text{RPA}}$ and the term $T/2g$ from the quasi-Goldstone RPA mode. With the square root factor in Eq. (14) denoted by $s$ one has

$$E_{\text{RPA},np}^{\text{res}} = E_{\text{RPA},np} - E_{\text{RPA}}^0 - \frac{T}{2g}$$

$$= (s - 1) E_{\text{RPA}}^0 + \frac{sA}{4g \sinh a} (\delta I(a,l) - l) \quad (86)$$

with

$$l = \frac{2T \sinh a}{sA} \quad (87)$$

by Eq. (59). Equation (9) gives

$$s = \sqrt{1 - \left( \frac{2T \tanh a}{A} \right)^2}. \quad (88)$$

The two terms in Eq. (86) have opposite signs but the second one dominates. Figure 6 shows $E_{\text{RPA},np}^{\text{res}}$ as a function of $T/A$ for $A = 24$, 56 and 100 and $T < 0.2A$. It is is seen to take in this range negative values of the order of a few MeV with the numerically largest values occurring for the
lowest \( A \). Also shown are least square fits by quadratic functions. Their coefficients make up \(-5.9, -1.2,\) and \(-0.4\%\), respectively, of the coefficient of \( T(T+1) \) in Eq. (1) of Ref. [24]. It follows that in a fit of the symmetry energy of the isovector pairing plus symmetry force model with a \( \kappa \) that reproduces approximately the empirical masses, by a function proportional to \( T(T+X) \), the optimal \( X \) is 1/0.941, 1/0.988 and 1/0.996 respectively. The quadratic approximation is seen to be poor, however, and poorest for the lowest \( A \). Anyway, the deviations do not exceed some hundred MeV, which is within the accuracy of semiempirical mass formulas. The bending down of \( E_{RPA, np}(A, T) - E_{RPA, np}(A, 0) \) from linearity in \( T \) at high \( T \) displayed in Fig. 5 is well known from my previous studies of discrete single-nucleon spectra. It is quantified in a simple form by Eq. [86].

As discussed in Refs. [15, 17] the term in Eq. (83) linear in \( T \) as well as the bending down from linearity described by the term \( E_{RPA, np} \) contribute to the average shape of the Wigner cusp in the plot of masses along an isobaric chain. In particular the vanishing of \( E_{RPA, np} \) at large \( T \) is reminiscent of the behavior of the phenomenological “Wigner energy” of exponential form proposed by Myers and Swiatecki [31]. It should be borne in mind, however, that the contribution from \( E_{RPA, np} \) makes up less than half of the total linear term in Eq. (83) and that shell effects contribute very significantly to the Wigner cusp of an individual isobaric chain [17, 19, 21].

Notice finally that for \( T = 0 \) the total RPA energy equals \( 3E_{\text{RPA}}^{0} \). It thus takes values about \((-15)\)–\((-20)\) MeV for \( A = 24–100 \).

VII. STRUTINSKIJ RENORMALIZATION

The idea of the Strutinskij theory [20] is to view in a first approximation the nucleus as a liquid drop whose properties may be derived from semiempirical mass formulas. The deviation of the actual mass from the liquid drop average is viewed as a “shell correction” which must be calculated microscopically. As only this small correction needs to be calculated from a microscopic model, the model need not be very accurate; in the simplest version of the theory the microscopic energy is just the sum of occupied levels in a potential well. To calculate the shell correction one must subtract from the microscopic energy an average that depends smoothly on the parameters of the model. Replacing this average by the liquid drop energy is known as a Strutinskij renormalization. The notation of the present section is such that a symbol without a tilde denotes a quantity calculated from the microscopic model and the same symbol with a tilde its smooth counterterm. If \( x \) is any quantity, \( \delta x = x - \tilde{x} \).

The pairing and isovector pairing models are crude models offering themselves to Strutinskij renormalization. Strutinskij in fact uses his expression (16) to provide the smooth counterterm for a renormalization of the of BCS energy. The expression (51) may be applied analogously to renormalize the RPA energy. It is used in a preliminary form in this way by Bentley, Frauendorf, and me in Ref. [21].

An important role is played in the Strutinskij theory by a smooth single-nucleon level density \( \tilde{g} \), which is a function of the single-nucleon energy \( \epsilon \). It is obtained by spreading each microscopic single-nucleon level over an interval of the order of the distance of the major shells. In terms of this function one can define smooth chemical potentials \( \lambda \) by

\[
\int_{-\infty}^{\lambda} \tilde{g}(\epsilon) d\epsilon = N_{\tau}.
\]

In his calculation [20] of \( \tilde{E}_{\text{BCS}} \), Strutinskij uses formulas equivalent to those of Sec. [11] with \( g = \tilde{g}(\lambda_{\tau}) \). The parameter \( \Omega \) is taken as the number of single-nucleon levels included in the microscopic BCS calculation. The approach to the calculation of \( E_{\text{RPA}} \) taken in Ref. [21] is analogous with \( g = \tilde{g}(\lambda_{np}) \) in \( \tilde{E}_{\text{RPA, np}} \). Here \( \lambda_{np} \) is defined by

\[
\int_{-\infty}^{\lambda_{np}} \tilde{g}(\epsilon) d\epsilon = \frac{A}{2}.
\]

In these calculations the lowest \( A/2 \) single-nucleon levels are included in all parts of the microscopic calculations and \( \Omega \) accordingly set to \( A/2 \) for all \( \tau \tau' \) in the calculations of the counterterms.

At the time of these calculation Eq. (52) had not been derived. Only Eq. (56) was known to us and used for \( N = Z \). For \( N > Z \) we used an approximation which is essentially equivalent to neglecting \( E_{\text{RPA, np}}^{0} \) namely
Eq. (13) of Ref. [21], where, unfortunately, a factor 1/2 is missing in the last term by mistake. As seen from Fig. 5, $E_{\text{RPA, np}}^{\text{res}}$ can take values of minus several MeV. Using Eq. (51) diminishes $\tilde{E}_{\text{RPA}}$ by the absolute value of this amount and thus increases the total energy by the same absolute value. It is noticed that this change is largest for the smallest $A$ and the largest $T$. A full calculation by the scheme of Ref. [21] using Eq. (51) instead of the preliminary $\tilde{E}_{\text{RPA}}$ requires a refit to the data of $G$ and the liquid drop parameters. Such work is in progress.

In Fig. 6, the RPA energies of Ref. [21] are compared to their smooth counterterms calculated by Eq. (51). The RPA correction $\delta E_{\text{RPA}}$ is seen to be almost constant about 0.7 MeV. As such a constant term can largely be absorbed by adjustment of the liquid drop parameters, the RPA correction thus turns out to have little significance for the reproduction of the observed doubly even masses. The shell correction $\Delta E_{\text{indep}} + \Delta E_{\text{BCS}}$ suffices for this purpose within the general accuracy of the model. It is crucial, however, for the reproduction of the masses in the vicinity of $N = Z$ and thus, in particular, of the Wigner cusp that the liquid drop symmetry energy be proportional to $T(T + 1)$ rather than $T^2$. The proportionality to $T(T + 1)$ can be motivated microscopically only by the reasoning in Sec. VII. Moreover an RPA contribution to the $T = 0$ doubly even–doubly odd mass staggering remains.

The fact that $E_{\text{RPA}}$ is consistently greater than $\tilde{E}_{\text{RPA}}$ could be understood from the fact that at equilibrium deformation the effective microscopic single-nucleon density $\rho$ is always at the Fermi level lower than $\bar{\rho}(\lambda)$. Therefore the effective $a$ is higher by Eq. (12) and, consequently, $E_{\text{RPA}}$ is less negative by Eq. (85) and Fig. 2.

**VIII. SUMMARY**

The main result of this article is the closed expression (51) for the pair vibrational correlation energy generated in the random-phase approximation (RPA) by the isovector pairing force in the case when Kramers and charge-degenerate single-nucleon levels are uniformly distributed in an interval. Using this expression I analyzed the distribution of pair vibrational frequencies relative to that of two-quasinucleon energies. This analysis revealed among other results that, like for the previously studied discrete single-nucleon spectra, quasi-Goldstone pair vibrational frequencies produced by the breaking of isobaric invariance by the self-consistent Bogolyubov quasimucleon vacuum are in the continuous limit isolated from the bulk of the spectrum. The total distribution of pair vibrational frequencies was found to account in a simple way for features of the neutron-proton pair vibrational correlation energy observed both in the previous studies and presently: a linear increase with the isospin $T$ near $T = 0$ and a bending down from linearity at higher $T$.

The emergence in the isovector pairing model, possibly amended by a schematic interaction of isospins, of a symmetry energy proportional to $T(T + 1)$ for low $T$ was reviewed in terms of Eq. (51), and the universal character of this result as a consequence of the breaking of isobaric invariance by the self-consistent state was pointed out. The deviation from proportionality to $T(T + 1)$ at higher $T$ was expressed by a simple formula and found to be largest for the lowest mass numbers $A$.

Finally the application of Eq. (51) to a Strutinskij renormalization of the RPA energy of the isovector pairing model was discussed. Significant modifications of the calculated masses were found to result from using Eq. (51) instead of an approximation to this expression applied in recent calculations by Bentley, Frauendorf and me. The difference between the microscopic RPA energy and a smooth counterterm expressed by Eq. (51) turned out to be almost constant about 0.7 MeV. Therefore, upon Strutinskij renormalization, the RPA contribution is insignificant for the reproduction of the observed doubly even masses. It is crucial, however, for such masses near $T = 0$ to be well described that the proportionality of the symmetry part of the total smooth energy to $T(T + 1)$ rather than $T^2$ for low $T$ be preserved in the replacement of the smooth energy by a liquid drop energy.
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