DIFFUSE INTERFACE MODELS FOR INCOMPRESSIBLE BINARY FLUIDS
AND THE MASS-CONSERVING ALLEN-CAHN APPROXIMATION

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ABSTRACT. This paper is devoted to the mathematical analysis of some Diffuse Interface systems which model the motion of a two-phase incompressible fluid mixture in presence of capillarity effects in a bounded smooth domain. First, we consider a two-fluids parabolic-hyperbolic model that accounts for unmatched densities and viscosities without diffusive dynamics at the interface. We prove the existence and uniqueness of local solutions. Next, we introduce dissipative mixing effects by means of the mass-conserving Allen-Cahn approximation. In particular, we consider the resulting nonhomogeneous Navier-Stokes-Allen-Cahn and Euler-Allen-Cahn systems with the physically relevant Flory-Huggins potential. We study the existence and uniqueness of global weak and strong solutions and their separation property. In our analysis we combine energy and entropy estimates, a novel end-point estimate of the product of two functions, and a logarithmic type Gronwall argument.

1. INTRODUCTION

The flow of a two-phase or multicomponent incompressible mixture is nowadays one of the most attractive theoretical and numerical problems in Fluid Mechanics (see, for instance, [8, 32, 36, 50, 64] and the references therein). This is mainly due to the interplay between the motion of the interface
separating the two fluids (or phases) and the surrounding fluids. A natural description of this phenomenon is based on a free-boundary formulation. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with \( d = 2, 3 \), and \( T > 0 \). We assume that \( \Omega \) is filled by two incompressible fluids (e.g. two liquids or a liquid and a gas), and we denote by \( \Omega_1(\Omega_2) = \Omega_1(t) \) and \( \Omega_2(\Omega_2) = \Omega_2(t) \) the subsets of \( \Omega \) containing, respectively, the first and the second fluid portions for any time \( t \geq 0 \). The equations of motion are

\[
\begin{cases}
\rho_1(\partial_t u_1 + u_1 \cdot \nabla u_1) - \nu_1 \text{div} D u_1 + \nabla p_1 = 0, & \text{div} u_1 = 0, \quad \text{in } \Omega_1 \times (0, T), \\
\rho_2(\partial_t u_2 + u_2 \cdot \nabla u_2) - \nu_2 \text{div} D u_2 + \nabla p_2 = 0, & \text{div} u_2 = 0, \quad \text{in } \Omega_2 \times (0, T).
\end{cases}
\tag{1.1}
\]

Here, \( u_1, u_2 \) and \( p_1 \) and \( p_2 \) are, respectively, the velocities and pressures of the two fluids, while \( \nu_1, \nu_2 \) and \( \rho_1, \rho_2 \) are the (constant) densities and viscosities of the two fluids, respectively. The symmetric gradient is \( D = \frac{1}{2}(\nabla + \nabla^t) \). The effect of the gravity are neglected for simplicity. Denoting by \( \Gamma = \Gamma(t) \) the (moving) interface between \( \Omega_1 \) and \( \Omega_2 \), system (1.1) can be equipped with the classical free boundary conditions

\[
u_1 D u_1 - \nu_2 D u_2 \cdot n_\Gamma = (p_1 - p_2 + \sigma H) n_\Gamma \quad \text{on } \Gamma \times (0, T),
\tag{1.2}
\]

together with the no-slip boundary condition

\[
u_1 = 0, \quad u_2 = 0 \quad \text{on } \partial \Omega \times (0, T).
\tag{1.3}
\]

The vector \( n_\Gamma \) in (1.2) is the unit normal vector of the interface from \( \partial \Omega_1(t) \), \( H \) is the mean curvature of the interface \( (H = -\text{div} n_\Gamma) \). In this setting, \( \Gamma(t) \) is assumed to move with the velocity given by

\[ V_{\Gamma(t)} = u \cdot n_{\Gamma(t)}. \tag{1.4} \]

The coefficient \( \sigma > 0 \) is the surface tension, which introduces a discontinuity in the normal stress proportional to the mean curvature of the surface. Since \( \frac{d}{dt} \mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} HV_{\Gamma} d\mathcal{H}^{d-1} \), where \( \mathcal{H}^{d-1} \) is the \( d-1 \)-dimensional Hausdorff measure, the (formal) energy identity for system (1.1)-(1.2) is

\[
\frac{d}{dt}\left\{ \sum_{i=1,2} \int_{\Omega_i(t)} \frac{\rho_i}{2} |u_i|^2 \, dx + \sigma \mathcal{H}^{d-1}(\Gamma(t)) \right\} + \sum_{i=1,2} \int_{\Omega_i(t)} \nu_i |D u_i|^2 \, dx = 0. \tag{1.5}
\]

We refer the reader to [1, 19, 62–64, 71, 72] for the analysis of classical and varifold solutions to the system (1.1)-(1.3).

The twofold Lagrangian and Eulerian nature of system (1.1)-(1.3) has led to the breakthrough idea (mainly from numerical analysts, see the review [67]) to reformulate the above system in the Eulerian description by interpreting the effect of the surface tension as a singular force term localized at the interface. Let us introduce the so-called level set function \( \phi : \Omega \times (0, T) \to \mathbb{R} \) such that

\[
\phi > 0 \quad \text{in } \Omega_1 \times (0, T), \quad \phi < 0 \quad \text{in } \Omega_2 \times (0, T), \quad \phi = 0 \quad \text{on } \Gamma \times (0, T),
\]

namely the interface is the zero level set of \( \phi \). We consider the Heaviside type function

\[
K(\phi) = \begin{cases} 
1 & \phi > 0, \\
0 & \phi = 0, \\
-1 & \phi < 0,
\end{cases} \tag{1.6}
\]
and we denote by $u$ the velocity such that $u = u_1$ in $\Omega_1 \times (0, T)$ and $u = u_2$ in $\Omega_2 \times (0, T)$. It was shown in [17, Section 2] that the system (1.1)-(1.3) is formally equivalent to

$$
\begin{aligned}
\rho(\phi) \left( \partial_t u + u \cdot \nabla u \right) - \text{div} \left( \nu(\phi) D u \right) + \nabla P &= \sigma H(\phi) \nabla \phi \delta(\phi), \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= 0,
\end{aligned}
$$

in $\Omega \times (0, T)$, (1.7)

together with the boundary condition (1.3). Here

$$
\rho(\phi) = \rho_1 \frac{1 + K(\phi)}{2} + \rho_2 \frac{1 - K(\phi)}{2}, \quad \nu(\phi) = \nu_1 \frac{1 + K(\phi)}{2} + \nu_2 \frac{1 - K(\phi)}{2}, \quad H(\phi) = \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right).
$$

Here, $\delta$ is the Dirac distribution, and $\nabla \phi$ is oriented as $n_T$. The equation (1.7)$_3$ represents the motion of the interface $\Gamma$ that is simply transported by the flow. This follows from the immiscibility condition, which translates into $(u, 1) \in \text{Tan} \{ (x, t) \in \Omega \times (0, T) : x \in \Gamma(t) \}$. Although (1.7) seems to be more amenable than (1.1)-(1.2), the presence of the Dirac mass still makes the analysis challenging. In the literature, two different approaches have been used to overcome the singular nature of the right-hand side of (1.7)$_1$, which both rely on the idea of continuous transition at the interface. The first approach is the Level Set method developed in the seminal works [17, 60, 61, 70] (see also the review [67]). This approach consists in approximating the Heaviside function $K(\phi)$ by a smoothing regularization $K_\varepsilon(\phi)$. More precisely, for a given $\varepsilon > 0$, we introduce the function

$$
K_\varepsilon(\phi) = \begin{cases} 
1 & \phi > \varepsilon, \\
\frac{1}{2} \left[ \phi + \frac{1}{\pi} \sin \left( \frac{\pi \phi}{\varepsilon} \right) \right] & |\phi| \leq \varepsilon, \\
-1 & \phi < -\varepsilon.
\end{cases}
$$

The resulting approximating system reads as follows

$$
\begin{aligned}
\rho_\varepsilon(\phi) \left( \partial_t u + u \cdot \nabla u \right) - \text{div} \left( \nu_\varepsilon(\phi) D u \right) + \nabla P &= \sigma H(\phi) \nabla \phi \delta_\varepsilon(\phi), \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= 0,
\end{aligned}
$$

in $\Omega \times (0, T)$, (1.9)

where

$$
\rho_\varepsilon(\phi) = \rho_1 \frac{1 + K_\varepsilon(\phi)}{2} + \rho_2 \frac{1 - K_\varepsilon(\phi)}{2}, \quad \nu_\varepsilon(\phi) = \nu_1 \frac{1 + K_\varepsilon(\phi)}{2} + \nu_2 \frac{1 - K_\varepsilon(\phi)}{2}, \quad \delta_\varepsilon = \frac{dK_\varepsilon(\phi)}{d\phi}.
$$

As a consequence of the approximation (1.8), the thickness of the interface is approximately $\frac{2\varepsilon}{|\nabla \phi|}$. This necessarily requires that $|\nabla \phi| = 1$ when $|\phi| \leq \varepsilon$, namely $\phi$ is a signed-distance function near the interface. However, even though the initial condition is suitably chosen, the evolution under the transport equation (1.9)$_3$ does not guarantee that this property remains true for all time. This fact had led to different numerical algorithms aiming to avoid the expansion of the interface (see [67] and the references therein). In addition, as pointed out in [54], another drawback of this approach is that the dynamics is sensitive to the particular choice of the approximation for the surface stress tensor.

The second approach is the so-called Diffuse Interface method (see [8, 24, 32]). This is based on the postulate that the interface is a layer with positive volume, whose thickness is determined by the interactions of particles occurring at small scales. In this context, the auxiliary function $\phi$ represents the difference between the fluids concentrations (or rescaled density/volume fraction). This function
may exhibit a smooth transition at the interface, which is identified as intermediate level sets between the two values 1 and $-1$. The evolution equations for the state variables (density, velocity, concentration) are derived by combining the theory of binary mixtures and the energy-based formalism from thermodynamics and statistical mechanics. In this framework, the surface stress tensor is replaced by a diffuse stress tensor whose action is essentially localized in the regions of high gradients, namely, $-\sigma \text{div}(\nabla \phi \otimes \nabla \phi)$. This tensor is known as (Korteweg) capillary tensor (cf., e.g., [8]). The resulting Diffuse Interface system, also called “complex fluid” model (see, e.g., [50, Sec.5]), is the following

$$
\begin{align*}
\rho(\phi)(\partial_tu + u \cdot \nabla u) - \text{div}(\nu(\phi) Du) + \nabla P &= -\sigma \text{div}(\nabla \phi \otimes \nabla \phi), \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= 0,
\end{align*}
$$

in $\Omega \times (0, T)$, (1.10)

equipped with the no-slip boundary condition

$$
u = 0 \quad \text{on} \quad \partial \Omega \times (0, T).$$

Here

$$
\rho(\phi) = \rho_1 \frac{1+\phi}{2} + \rho_2 \frac{1-\phi}{2}, \quad \nu(\phi) = \nu_1 \frac{1+\phi}{2} + \nu_2 \frac{1-\phi}{2}.
$$

(1.12)

The energy associated to system (1.10) is defined as

$$
E(u, \phi) = \int_{\Omega} \frac{1}{2} \rho(\phi) |u|^2 + \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx,
$$

where $\Psi$ is a double-well potential from $[-1, 1] \to \mathbb{R}$, and the corresponding energy identity is

$$
\frac{d}{dt} E(u, \phi) + \int_{\Omega} \nu(\phi) |Du|^2 \, dx = 0.
$$

(1.13)

This model dissipates energy due to viscosity, but there are no regularization effects for $\phi$. It is worth noting that (1.10) is also related to the models for viscoelastic fluids (see, for instance, [51]) or to the two-dimensional incompressible MHD system without magnetic diffusion (cf., e.g., [65] and the references therein). Notice that, after rescaling the capillary tensor and the free energy by a parameter $\varepsilon$, it is possible to recognize the connection between (1.1)-(1.2) and (1.10). Indeed, we have formally the convergences of the stress tensor (see, for instance, [5, 50, 64] for further details on the sharp interface limit)

$$
-\int_{\Omega} \varepsilon \text{div}(\nabla \phi \otimes \nabla \phi) \cdot v \, dx \xrightarrow{\varepsilon \to 0} \int_{\Gamma} \sigma H n \cdot v \, dH^{d-1},
$$

where the limit integral corresponds to the weak formulation of (1.2), and of the (Helmholtz) free energy $\int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} \Psi(\phi) \right) \, dx$ to the area functional $H^{d-1}(\Gamma)$ (see [57]). Before proceeding with the introduction of diffusive relaxations of the transport equation and their physical motivations, it is important to point out two main properties of (1.10)$_2$-(1.10)$_3$:

1. Conservation of mass:

$$
\int_{\Omega} \phi(t) \, dx = \int_{\Omega} \phi_0 \, dx, \quad \forall t \in [0, T].
$$

(1.14)

2. Conservation of $L^\infty(\Omega)$-norm:

$$
\|\phi(t)\|_{L^\infty(\Omega)} = \|\phi_0\|_{L^\infty(\Omega)}, \quad \forall t \in [0, T],
$$

(1.15)
which implies that
\[-1 \leq \phi_0(x) \leq 1 \quad \text{a.e. in } \Omega \quad \Rightarrow \quad -1 \leq \phi(x,t) \leq 1 \quad \text{a.e. in } \Omega \times (0,T). \tag{1.16}\]

The theory of binary mixtures takes into accounts dissipative mechanisms occurring at the interfaces. The molecules of two fluids interact at a microscopic scale, and their disposition is the result of a competition between the diffusion of molecules and the attraction of molecules of the same fluid (mixing vs demixing or “philic” vs “phobic” effects). This liquid-liquid phase separation phenomenon, though already well-known in Materials Science, has recently become a sort of paradigm in Cell Biology (see, for instance, [6, 11, 42, 68]). This competition is described in the Helmholtz free energy of the system \( \mathcal{E}(\phi) \) defined by
\[
\mathcal{E}(\phi) = \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx.
\]

The first term describes weakly non-local interactions (see [14], cf. also [21]). The potential \( \Psi \) is the Flory-Huggins free energy density\(^1\)
\[
\Psi(s) = \frac{\theta}{2} \left[ (1 + s) \log(1 + s) + (1 - s) \log(1 - s) \right] - \frac{\theta_0}{2} s^2, \quad s \in [-1, 1]. \tag{1.19}\]

We consider hereafter the case \( 0 < \theta < \theta_0 \), which implies, in particular, that \( \Psi \) is a non-convex potential\(^2\). It is worth mentioning that the Landau theory that leads to the well-known Ginzburg-Landau free energy is just an approximation of the above \( \mathcal{E}(\phi) \) obtained through a Taylor expansion of the logarithmic potential \( \Psi \). This choice is very common in the related literature (see, for instance, [13].

\(^1\)For a system of finite number of molecules \( A \) and \( B \) occupying a lattice with \( M \) sites, the thermodynamic properties of the system of molecules are derived from the partition function
\[
Z = \sum_{\Omega} \exp \left( \frac{H(\sigma_1, \ldots, \sigma_M)}{k_B T} \right), \tag{1.17}
\]
where the Hamiltonian \( H(\sigma_1, \ldots, \sigma_M) \) denotes the energy of the arrangement \( \sigma_1, \ldots, \sigma_M \) \((\sigma_n = 1 \text{ if the lattice is occupied by molecule } A, \sigma_n = 0 \text{ otherwise})\), and \( \Omega \) is the set of all possible arrangements. Here \( k_B \) is the Boltzmann constant and \( T \) is the temperature. It is common to describe only nearest neighbor interactions between particles, which lead to the particular Hamiltonian
\[
H(\{\sigma\}) = \frac{1}{2} \sum_{m,n} \left[ e_{AA} \sigma_m \sigma_n + e_{BB}(1 - \sigma_m)(1 - \sigma_n) + e_{AB}(\sigma_m(1 - \sigma_n) + \sigma_n(1 - \sigma_m)) \right], \tag{1.18}
\]
where \( e_{AA}, e_{BB}, \text{ and } e_{AB} \) are coefficients. In the Mean Field approximation the arrangements \( \sigma_n \) and \( 1 - \sigma_n \) are approximated by the probability (average) that a site is occupied by a molecule \( A \) and \( B \), namely \( \phi_A = \frac{N_A}{M} \) and \( \phi_B = \frac{N_B}{M} \) \((N_A \text{ and } N_B \text{ are the number of molecules of type } A \text{ and } B, \text{ and } M = N_A + N_B)\). Then, the partition function is given by
\[
Z = \frac{M!}{N_A! N_B!} \exp \left( \frac{H(\phi_A, \phi_B)}{k_B T} \right), \quad H(\phi_A, \phi_B) = \frac{z M}{2} \left( e_{AA} \phi_A^2 + 2 e_{AB} \phi_A \phi_B + e_{BB} \phi_B^2 \right),
\]
where \( z \) is the number of neighbors in a lattice. By using the Stirling approximation, the free energy density reads as
\[
f(\phi_A, \phi_B) = -\frac{k_B T \ln Z}{M} \approx \frac{k_B T}{\nu} \left[ \phi_A \ln \phi_A + \phi_B \ln \phi_B \right] + \frac{z}{2\nu} \left[ e_{AA} \phi_A^2 + 2 e_{AB} \phi_A \phi_B + e_{BB} \phi_B^2 \right],
\]
where \( \nu \) is the volume of molecules. By defining \( \phi = \phi_A - \phi_B \) (with range \([-1, 1])\), and setting appropriately the constants \( \theta \) and \( \theta_0 \), the Flory-Huggins potential (1.19) immediately follows. As usual, the function \( \Psi \) is meant as the continuous extension at the values \( s = \pm 1 \). Roughly speaking, the logarithmic term accounts for the entropy after mixing and the quadratic perturbation represents the internal energy after mixing. For more details, we refer the reader to [46].

\(^2\)In the case, \( \theta \geq \theta_0 \), mixing prevails over demixing, and no separation takes place.
and [22]). However, it has the main drawback that the solution does not belong in general to the physical interval \([-1,1]\) (cf. (1.16)).

In order to include dissipative mechanisms in the dynamics of the concentration, we define the first variation of the Helmholtz free energy. This is called chemical potential and it is given by

\[ \mu = \frac{\delta \mathcal{E}(\phi)}{\delta \phi} = -\Delta \phi + \Psi'(\phi). \]

Two fundamental relaxation models proposed in the Diffuse Interface theory for binary mixture are the following modifications of the transport equation (1.10)3:

1. **Mass-conserving Allen-Cahn dynamics** ([66, 79])

   \[ \partial_t \phi + \mathbf{u} \cdot \nabla \phi + \gamma (\mu - \overline{\mu}) = 0 \quad \text{in } \Omega \times (0, T), \quad \partial_n \phi = 0 \quad \text{on } \partial \Omega \times (0, T); \]

2. **Cahn-Hilliard dynamics** ([14, 15])

   \[ \partial_t \phi + \mathbf{u} \cdot \nabla \phi - \gamma \Delta \mu = 0 \quad \text{in } \Omega \times (0, T), \quad \partial_n \phi = \partial_n \mu = 0 \quad \text{on } \partial \Omega \times (0, T). \]

Here \(\overline{\mu}\) is the spatial average defined by

\[ \overline{\mu} = \frac{1}{|\Omega|} \int_{\Omega} \mu \, dx, \]

and \(\gamma\) is the elastic relaxation time. We point out that from the thermodynamic viewpoint the relaxation terms describe dissipative diffusional flux at the interface (cf. [38, 54]). As for the transport equation, both the mass-conserving Allen-Cahn and Cahn-Hilliard equations satisfy the conservation properties (1.14) and (1.16). In addition, their dynamics maintain the integrity of the interface: the mixing-demixing mechanism (which also translates into \(\mu\)) allows a balance which avoids uncontrolled expansion or shrinkage of the interface layer (cf. [24]).

In this work, we study a Diffuse Interface model that has been recently derived in [32, Part I, Chap.2, 4.2.1]. It accounts for unmatched densities and viscosities of the fluids, as well as dissipation due to interface mixing. The dynamics of \(\phi\) is described through the following modification of the transport equation

\[ \partial_t \phi + \mathbf{u} \cdot \nabla \phi + \gamma \left( \mu + \rho'(\phi) \frac{|\mathbf{u}|^2}{2} - \xi \right) = 0, \quad \text{in } \Omega \times (0, T), \]

where \(\mathbf{u}\) denotes the volume averaged fluid velocity and

\[ \xi(t) = \frac{1}{|\Omega|} \int_{\Omega} \mu + \rho'(\phi) \frac{|\mathbf{u}|^2}{2} \, dx, \quad \text{in } (0, T). \]

Here the dissipation mechanism is similar to that of the mass-conserving Allen-Cahn dynamics, but it also includes an extra term due to the difference of densities. We thus have the nonhomogeneous Navier-Stokes-Allen-Cahn system

\[
\begin{cases}
\rho(\phi)(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \text{div} \left( \nu(\phi) D \mathbf{u} \right) + \nabla P = -\sigma \text{div} \left( \nabla \phi \otimes \nabla \phi \right), \\
\text{div} \mathbf{u} = 0, \\
\partial_t \phi + \mathbf{u} \cdot \nabla \phi + \gamma \left( \mu + \rho'(\phi) \frac{|\mathbf{u}|^2}{2} - \xi \right) = 0,
\end{cases}
\]

\[
\text{in } \Omega \times (0, T). \quad (1.20)
\]

3This equation differs from the classical Allen-Cahn equation due to the presence of term \(\overline{\mu}\) (see [66, 79], cf. also [7, 12, 18, 35, 58, 74]).
This system is usually subject to a no-slip boundary condition for $u$ and a homogeneous Neumann boundary condition for $\phi$, namely

$$u = 0, \quad \partial_n \phi = 0 \quad \text{on } \partial \Omega \times (0, T). \quad (1.21)$$

In the last part of this work, we will consider the mass-conserving Euler-Allen-Cahn system

$$\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = -\sigma \text{div} (\nabla \phi \otimes \nabla \phi), \\
\text{div } u = 0, \\
\partial_t \phi + u \cdot \nabla \phi + \gamma (\mu - \overline{\mu}) = 0,
\end{cases} \quad \text{in } \Omega \times (0, T), \quad (1.22)$$

endowed with the boundary conditions

$$u \cdot n = 0, \quad \partial_n \phi = 0 \quad \text{on } \partial \Omega \times (0, T). \quad (1.23)$$

The above model is obtained from (1.20) in the case of inviscid flow and matched densities. We observe that other boundary conditions can be considered, for instance, periodic (cf. [32, Part I, Chap.2, 4.2.3]) and also [55] for moving contact lines.

The mathematical literature concerning systems similar to the Navier-Stokes-Allen-Cahn system (1.20)-(1.21) has been widely developed in last years, in terms of both physical modeling and well-posedness analysis. First, we report that there are different ways of accounting for the unmatched densities for incompressible binary mixtures. Among the existing literature, we mention [4, 10, 27, 38, 39, 54]. The model herein studied is derived via an energetic variational approach in [37] (see also [32] and, for the Navier-Stokes-Cahn-Hilliard system [53]). The system (1.20)-(1.21) has been investigated in [37] in the case of constant viscosity and standard Allen-Cahn equation with regular Landau potential $\Psi_0(s) = \frac{1}{4} (s^2 - 1)^2$ and no mass conservation. The authors prove the existence of a global weak solution in three dimensions and the existence as well as uniqueness of the global strong solution in two dimensions. In the latter case, they also show the convergence of a weak solution to a single stationary state and they establish the existence of a global attractor. Thanks to their choice of potential and the absence of mass constraint, the authors can easily ensure that $\phi$ takes values in the physical range $[-1, 1]$. This fact is crucial for their proofs. However, the mass constraint would not allow to establish a comparison principle even if the double-well potential is smooth. We also mention the previous contributions [29, 30, 40, 76–78] for the case with constant density, and [5] for the sharp interface limit in the Stokes case. Additionally, there are works devoted to Navier-Stokes-Allen-Cahn models in which density is regarded as an independent variable (see, for instance, [23, 47, 48]). In these works the potential is the classical Landau double-well and there is no mass conservation. The (non-conserved) compressible case (see [9, 27] for modeling issues) has been analyzed, for instance, in [20, 25, 44, 80] (see also [75] for sharp interface limits). On the other hand, in comparison with the viscous case above-mentioned, only few works have been addressed with the Euler-Allen-Cahn system (1.22)-(1.23). In this respect, we mention [81] (see also [28] for a nonlocal model), where the authors prove local existence of smooth solutions for the Euler-Allen-Cahn in the case of no-mass conservation and Landau potential.

The aim of this paper is to address the existence, uniqueness and (possibly) regularity of the solutions to the aforementioned Diffuse Interface systems\footnote{Without loss of generality, we consider the values of the parameters $\sigma = \gamma = 1$ in our analysis.}: the complex fluid model (1.10)-(1.11), the Navier-Stokes-Allen-Cahn system (1.20)-(1.21), and the Euler-Allen-Cahn system (1.22)-(1.23). On one hand, the purpose of our analysis is to stay as close as possible to a thermodynamically grounded framework...
by keeping densities and viscosities to be dependent on \( \phi \), and the physically relevant Flory-Huggins potential (1.19). Although this choice requires some technical efforts, it provides results which are physically more reasonable. On the other hand, by working in this general setting, we demonstrate that the dynamics originating from a general initial condition become global (in time) when the mass-conserving Allen-Cahn relaxation is taken into account. The latter is achieved in three dimensions for finite energy (weak) solutions, and in two dimensions even for more regular solutions in the case of non-constant density and viscosity and of constant density and zero viscosity.

Before concluding this introduction, we make some more precise comments on the analysis and on the main novelties of our techniques. First, we recall that the existence and uniqueness of local (in time) regular solutions to the complex fluid system (1.10)-(1.11) has been proven in [51, 52] for constant density and viscosity. Here we generalize this result by allowing \( \rho \) and \( \nu \) to depend on \( \phi \) and taking a more general initial datum \((u_0, \phi_0) \in (V_\sigma \cap H^2(\Omega)) \times W^{2,p}(\Omega)\), with \( p > 2 \) in two dimensions and \( p > 3 \) in three dimensions (see Theorem 3.1). Next, we study the Navier-Stokes-Allen-Cahn system (1.20)-(1.21). We prove the existence of a global weak solution with \((u_0, \phi_0) \in H_\sigma \times H^{1}(\Omega)\) (see Theorems 4.1 and 4.2), and the existence of a global strong solution with \((u_0, \phi_0) \in V_\sigma \times H^2(\Omega)\) such that \(\Psi'(\phi_0) \in L^2(\Omega)\) (see Theorem 5.1). For the latter, we combine a classical energy approach, a new end-point estimate of the product of two functions in \(L^2(\Omega)\) (see Lemma 2.1 below), and a new estimate for the Stokes system with non-constant viscosity. The proof is concluded with a logarithmic Gronwall argument that leads to double-exponential control. However, in light of the singularity of the Flory-Huggins potential, the uniqueness of these strong solutions seems to be a hard task. To overcome this issue, we then establish global estimates on the derivatives of the entropy\(^5\) (entropy estimates) provided that \(\|\rho'\|_{L^\infty((-1,1))}\) is small enough and \(F''(\phi_0) \in L^1(\Omega)\). These estimates allow us to prove that \(F''(\phi)^2 \log(1 + F''(\phi)) \in L^1(\Omega \times (0,T))\), and, in turn, the uniqueness of such strong solutions. As a consequence of these entropy estimates, we achieve the so-called uniform separation property. The latter says that \(\phi\) stays uniformly away from the pure states in finite time\(^6\). This fact, besides being physically relevant, entails further regularity properties of the solution. Note that in the case of a smooth potential and no mass conservation (cf. [37]) this issue is trivial since the potential is smooth and a comparison principle holds. Finally, we consider the inviscid case, namely the Euler-Allen-Cahn system (1.22)-(1.23). Although this system turns out to be similar to the MHD equations with magnetic diffusion and without viscosity, the classical argument in the literature (see, e.g., [16]) does not apply because of the logarithmic potential. In our proof, it is crucial to make use of the structure of the incompressible Euler equations (1.22)-(1.23) and the end-point estimate of the product (Lemma 2.1). This gives the existence of global solutions with \((u_0, \phi_0) \in (H_\sigma \cap H^1(\Omega)) \times H^2(\Omega)\) in two dimensions. Next, in light of the entropy estimates, we also prove the existence of smoother global solutions originating from \((u_0, \phi_0) \in (H_\sigma \cap W^{1,p}(\Omega)) \times H^2(\Omega)\) provided that \(p > 2\) and \(\nabla \mu_0 := \nabla(-\Delta \phi_0 + \Psi'(\phi_0)) \in L^2(\Omega)\).

**Plan of the paper.** In Section 2 we introduce the notation, some functional inequalities and then prove an estimate for the product of two functions. In Section 3 we show the local well-posedness of

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\(^5\)We define the (mixing) entropy as \(F(s) = \frac{\phi}{2} [(1+s) \log(1+s) + (1-s) \log(1-s)]\), for \(s \in [-1,1]\). This corresponds to the convex part of (1.19).

\(^6\)It is worth pointing out that the initial concentration \(\phi_0\) for strong solutions is not separated from the pure phases. Indeed, the imposed conditions \(F'(\phi_0) \in L^2(\Omega)\) or \(F''(\phi_0) \in L^1(\Omega)\) allow \(\phi_0\) being arbitrarily close to +1 and −1.
system (1.10)-(1.11). Section 4 is devoted to the existence of global weak solutions for the Navier-Stokes-Allen-Cahn system (1.20)-(1.21). In Section 5 we study the existence and uniqueness of strong solutions to the Navier-Stokes-Allen-Cahn system (1.20)-(1.21). Section 6 is devoted to the global existence of solutions to the Euler-Allen-Cahn system (1.22)-(1.23). In Appendix A we prove a result on the Stokes problem with variable viscosity, and in Appendix B we recall the Osgood lemma and two logarithmic versions of the Gronwall lemma.

2. Preliminaries

2.1. Notation. For a real Banach space $X$, its norm is denoted by $\| \cdot \|_X$. The symbol $\langle \cdot, \cdot \rangle_{X',X}$ stands for the duality pairing between $X$ and its dual space $X'$. The boldface letter $X$ denotes the vectorial space endowed with the product structure. We assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with smooth boundary $\partial \Omega$, $n$ is the unit outward normal vector on $\partial \Omega$, and $\partial_n$ denotes the outer normal derivative on $\partial \Omega$. We denote the Lebesgue spaces by $L^p(\Omega)$ ($p \geq 1$) with norms $\| \cdot \|_{L^p(\Omega)}$. When $p = 2$, the inner product in the Hilbert space $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. For $s \in \mathbb{R}$, $p \geq 1$, $W^{s,p}(\Omega)$ is the Sobolev space with corresponding norm $\| \cdot \|_{W^{s,p}(\Omega)}$. If $p = 2$, we use the notation $W^{s,2}(\Omega) = H^s(\Omega)$. For every $f \in (H^1(\Omega))'$, we denote by $\overline{f}$ the generalized mean value over $\Omega$ defined by $\overline{f} = |\Omega|^{-1} \int_{\Omega} f \, dx$. Thanks to the generalized Poincaré inequality, there exists a positive constant $C = C(\Omega)$ such that

$$\| f \|_{H^1(\Omega)} \leq C \left( \| \nabla f \|_{L^2(\Omega)} + \left| \int_{\Omega} f \, dx \right|^2 \right)^{\frac{1}{2}}, \quad \forall f \in H^1(\Omega).$$

We introduce the Hilbert space of solenoidal vector-valued functions

$$H_\sigma = \{ u \in L^2(\Omega) : \text{div} \, u = 0, \ u \cdot n = 0 \quad \text{on} \ \partial \Omega \} = C^\infty_{0,\sigma}(\Omega)^{L^2(\Omega)},$$

$$V_\sigma = \{ u \in H^1(\Omega) : \text{div} \, u = 0, \ u = 0 \quad \text{on} \ \partial \Omega \} = C^\infty_{0,\sigma}(\Omega)^{H^1(\Omega)},$$

where $C^\infty_{0,\sigma}(\Omega)$ is the space of divergence free vector fields in $C^\infty_{0}(\Omega)$. We also use $(\cdot, \cdot)$ and $\| \cdot \|_{L^2(\Omega)}$ for the inner product and the norm in $H_\sigma$. The space $V_\sigma$ is endowed with the inner product and norm $(u, v)_{V_\sigma} = (\nabla u, \nabla v)$ and $\| u \|_{V_\sigma} = \| \nabla u \|_{L^2(\Omega)}$, respectively. We denote by $V_\sigma'$ its dual space. We recall the Korn inequality

$$\| \nabla u \|_{L^2(\Omega)} \leq \sqrt{2} \| Du \|_{L^2(\Omega)} \leq \sqrt{2} \| \nabla u \|_{L^2(\Omega)}, \quad \forall u \in V_\sigma,$$

where $Du = \frac{1}{2} (\nabla u + (\nabla u)^T)$. We define the Hilbert space $W_\sigma = H^2(\Omega) \cap V_\sigma$ with inner product and norm $(u, v)_{W_\sigma} = (Au, Av)$ and $\| u \|_{W_\sigma} = \| Au \|$, where $A$ is the Stokes operator. We recall that there exists $C > 0$ such that

$$\| u \|_{H^2(\Omega)} \leq C \| u \|_{W_\sigma}, \quad \forall u \in W_\sigma.$$

2.2. Analytic tools. We recall the Ladyzhenskaya, Agmon, Gagliardo-Nirenberg, Brezis-Gallouet-Wainger and trace interpolation inequalities:

$$\| f \|_{L^4(\Omega)} \leq C \| f \|_{L^2(\Omega)} \| f \|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall f \in H^1(\Omega), \ d = 2,$$

$$\| f \|_{L^p(\Omega)} \leq C p^{\frac{1}{2}} \| f \|_{L^2(\Omega)} \| f \|_{H^1(\Omega)}^{\frac{1}{2} - \frac{1}{p}}, \quad \forall f \in H^1(\Omega), \ 2 \leq p < \infty, \ d = 2,$$

$$\| f \|_{L^p(\Omega)} \leq C(p) \| f \|_{L^2(\Omega)} \| f \|_{H^1(\Omega)}^{\frac{3(p-2)}{2p}}, \quad \forall f \in H^1(\Omega), \ 2 \leq p \leq 6, \ d = 3,$$
\[ \|f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{1/2} \|f\|_{H^2(\Omega)}^{1/2}, \quad \forall f \in H^2(\Omega), \; d = 2, \tag{2.7} \]
\[ \|\nabla f\|_{W^{1,4}(\Omega)} \leq C \|f\|_{H^2(\Omega)}^{1/2} \|f\|_{L^\infty(\Omega)}^{1/2}, \quad \forall f \in H^2(\Omega), \; d = 3, \tag{2.8} \]
\[ \|f\|_{L^\infty(\Omega)} \leq C \|f\|_{H^1(\Omega)} \log^{1/2} \left( \|f\|_{H^2(\Omega)} \right), \quad \forall f \in H^2(\Omega), \; d = 2, \tag{2.9} \]
\[ \|f\|_{L^\infty(\Omega)} \leq C(p) \|f\|_{H^1(\Omega)} \log^{1/2} \left( C(p) \frac{\|f\|_{W^{1,p}(\Omega)}}{\|f\|_{H^1(\Omega)}} \right), \quad \forall f \in W^{1,p}(\Omega), \; p > 2, \; d = 2, \tag{2.10} \]
\[ \|f\|_{L^2(\partial \Omega)} \leq C \|f\|_{L^2(\Omega)}^{1/2} \|f\|_{H^1(\Omega)}^{1/2}, \quad \forall f \in H^1(\Omega), \; d = 2. \tag{2.11} \]

Here, the constant \( C \) depends only on \( \Omega \), whereas the constant \( C(p) \) depends on \( \Omega \) and \( p \).

We now prove the following end-point estimate for the product of two functions, which will play an important role in the subsequent analysis. This is a generalization of [34, Proposition C.1].

**Lemma 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary. Assume that \( f \in H^1(\Omega) \) and \( g \in L^p(\Omega) \) for some \( p > 2 \), \( g \) is not identical to 0. Then, we have
\[ \|fg\|_{L^2(\Omega)} \leq C \left( \frac{p}{p-2} \right)^{1/2} \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)} \log^{1/2} \left( e|\Omega|^{\frac{p-2}{2}} \|g\|_{L^p(\Omega)} \right), \tag{2.12} \]
for some positive constant \( C \) depending only on \( \Omega \).

**Proof.** Let us consider the Neumann operator \( A = -\Delta + I \) on \( L^2(\Omega) \) with domain \( D(A) = \{u \in H^2(\Omega) : \partial_n u = 0 \text{ on } \partial\Omega \} \). By the classical spectral theory, there exists a sequence of positive eigenvalues \( \lambda_k \) \((k \in \mathbb{N})\) associated with \( A \) such that \( \lambda_1 = 1, \lambda_k \leq \lambda_{k+1} \) and \( \lambda_k \to +\infty \) as \( k \) goes to \( +\infty \). The sequence of eigenfunctions \( w_k \) in \( D(A) \) satisfying \( Aw_k = \lambda_k w_k \) forms an orthonormal basis in \( L^2(\Omega) \) and an orthogonal basis in \( H^1(\Omega) \). Let us fix \( N \in \mathbb{N}_0 \) whose value will be chosen later. We write \( f \) as follows
\[ f = \sum_{n=0}^{N} f_n + f_{N}^\perp, \tag{2.13} \]
where
\[ f_n = \sum_{k: e^n \leq \lambda_k < e^{n+1}} (f, w_k)w_k, \quad f_{N}^\perp = \sum_{k: e^N \leq \lambda_k < e^{N+1}} (f, w_k)w_k. \]

By using the above decomposition and Hölder’s inequality, we have
\[ \|fg\|_{L^2(\Omega)} \leq \sum_{n=0}^{N} \|f_n g\|_{L^2(\Omega)} + \|f_{N}^\perp g\|_{L^2(\Omega)} \leq \sum_{n=0}^{N} \|f_n\|_{L^\infty(\Omega)} \|g\|_{L^2(\Omega)} + \|f_{N}^\perp\|_{L^{p'}(\Omega)} \|g\|_{L^p(\Omega)}, \]
where \( p > 2 \) and \( \frac{1}{p} + \frac{1}{p'} = \frac{1}{2} \). By using (2.5) and (2.7), we obtain
\[ \|fg\|_{L^2(\Omega)} \leq C \sum_{n=0}^{N} \|f_n\|_{L^2(\Omega)}^{1/2} \|f_n\|_{H^2(\Omega)}^{1/2} \|g\|_{L^2(\Omega)} + C \left( \frac{2p}{p-2} \right)^{1/2} \|f_{N}^\perp\|_{L^2(\Omega)} \|f_{N}^\perp\|_{H^1(\Omega)} \|g\|_{L^p(\Omega)}, \]
for some \( C \) independent of \( p \). We recall that
\[ \|f_n\|_{L^2(\Omega)}^2 = \sum_{k: e^n \leq \lambda_k < e^{n+1}} |(f, w_k)|^2 \leq \frac{1}{e^{2n}} \sum_{k: e^n \leq \lambda_k < e^{n+1}} \lambda_k |(f, w_k)|^2 = \frac{1}{e^{2n}} \|f_n\|_{H^1(\Omega)}^2, \]
where we have used the fact $D(\frac{1}{2}) = H^1(\Omega)$. Observing that $\partial_n f_n = 0$ on $\partial \Omega$ ($f_n$ is a finite sum of $w_k$'s), by the regularity theory of the Neumann problem, we have

$$\|f_n\|_{H^2(\Omega)}^2 \leq C\|Af_n\|_{L^2(\Omega)}^2 = C\sum_{k: e^n \leq N_k \leq e^{n+1}} \lambda_k^2 |(f, w_k)|^2 \leq C\sum_{k: e^n \leq N_k \leq e^{n+1}} e^{2(n+1)} \lambda_k^2 |(f, w_k)|^2 \leq Ce^{2(n+1)}\|f_n\|_{H^1(\Omega)}^2.$$ 

Thus, we infer that

$$\|f_n\|_{H^2(\Omega)} \|f_n\|_{H^2(\Omega)} \leq Ce^{\frac{1}{2}}\|f_n\|_{H^1(\Omega)},$$

where the constant is independent of $n$. On the other hand, reasoning as above, we deduce that

$$\|f_n\|_{L^2(\Omega)} \leq \frac{1}{e^{2(N+1)}}\|f_N\|_{H^1(\Omega)}.$$

Combining the above inequalities and applying the Cauchy-Schwarz inequality, we get

$$\|fg\|_{L^2(\Omega)} \leq C\sum_{n=0}^N e^{\frac{1}{2}}\|f_n\|_{H^1(\Omega)}\|g\|_{L^2(\Omega)} + Ce^{-\frac{2p}{p-2}}\|f_n\|_{H^1(\Omega)}\|g\|_{L^p(\Omega)} \leq C\|g\|_{L^2(\Omega)} \left(eN + 1 + \frac{2p}{p-2}\frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right) \left(\sum_{n=0}^N \|f_n\|_{H^1(\Omega)}^2 + \|f_N\|_{H^1(\Omega)}^2\right)^{\frac{1}{2}} \leq C\|g\|_{L^2(\Omega)} \left(eN + 1 + \frac{2p}{p-2}\frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right) \|f\|_{H^1(\Omega)},$$

(2.14)

where we have used the fact $p' = \frac{2p}{p-2}$ and the constant $C$ is independent of $N$. Now, we choose the non-negative integer $N \in \mathbb{N}_0$ such that

$$\frac{p}{2(p-2)} \log \left(e\|\Omega\|^{\frac{p-2}{p}} \frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right) \leq N + 1 < \frac{p}{2(p-2)} \log \left(e\|\Omega\|^{\frac{p-2}{p}} \frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right).$$

We observe that the logarithm term in the above relations is greater than 1 for any function $g \in L^p(\Omega)$ with $p > 2, g \neq 0$. Then by using the choice of $N$ in (2.14), we infer that

$$\|fg\|_{L^2(\Omega)} \leq C\|f\|_{H^1(\Omega)}\|g\|_{L^2(\Omega)} \left(e \left[1 + \frac{p}{2(p-2)} \log \left(e\|\Omega\|^{\frac{p-2}{p}} \frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right)\right] + \frac{2p}{e(p-2)|\Omega|^{\frac{p-2}{p}}} \right)^{\frac{1}{2}} \leq C\|f\|_{H^1(\Omega)}\|g\|_{L^2(\Omega)} \left(3e \frac{p}{2(p-2)} \log \left(e^2\|\Omega\|^{\frac{p-2}{p}} \frac{\|g\|_{L^p(\Omega)}}{\|g\|_{L^2(\Omega)}}\right) + \frac{2p}{e(p-2)|\Omega|^{\frac{p-2}{p}}} \right)^{\frac{1}{2}},$$
which implies the desired conclusion. □

**Remark 2.2.** The conclusion of Lemma 2.1 holds as well in $\mathbb{T}^2$.

**Remark 2.3.** It is well-known that $H^1(\Omega)$ is not an algebra in two dimensions. An interesting application of Lemma 2.1 together with the Brezis-Gallouet-Wainger inequality (2.10) is that

$$\|fg\|_{H^1(\Omega)} \leq C_1 \|f\|_{H^1(\Omega)} \|g\|_{H^1(\Omega)} \log^\frac{1}{2} \left( C_2 \frac{\|g\|_{W^{1,p}(\Omega)}}{\|g\|_{H^1(\Omega)}} \right),$$

for any $f \in H^1(\Omega)$, $g \in W^{1,p}(\Omega)$ for some $p > 2$, where $C_1$ and $C_2$ are two positive constants depending only on $\Omega$ and $p$.

**Remark 2.4.** Lemma 2.1 can be regarded as a generalization of Hölder and Young inequalities. This inequality is sharp since the product between $f$ and $g$ is not defined in $L^2(\Omega)$ if $f \in H^1(\Omega)$ and $g \in L^2(\Omega)$. Indeed, we have the following counterexample in $\mathbb{R}^2$:

$$g(x) = \frac{1}{|x| \log^\frac{3}{4} \left( \frac{1}{|x|} \right)}, \quad f(x) = \log^{\frac{3}{4} - \frac{1}{100}} \left( \frac{1}{|x|} \right), \quad 0 < x \leq 1.$$

We notice that $g \in L^2(B_{\mathbb{R}^2}(0, 1))$ since

$$\int_{B_{\mathbb{R}^2}(0,1)} |g(x)|^2 \, dx = 2\pi \int_0^1 \frac{1}{r \log^3 \left( \frac{1}{r} \right)} \, dr = 2\pi \int_1^{+\infty} \frac{1}{s \log^\frac{3}{4} (s)} \, ds < +\infty.$$

However, $g \notin L^p(B_{\mathbb{R}^2}(0, 1))$ for any $p > 2$ because

$$\int_{B_{\mathbb{R}^2}(0,1)} |g(x)|^p \, dx = 2\pi \int_0^1 \frac{1}{r^{p-1} \log^p \left( \frac{1}{r} \right)} \, dr = 2\pi \int_1^{+\infty} \frac{1}{s^{3-p} \log^p (s)} \, ds = +\infty.$$

We easily observe that $f \in L^2(B_{\mathbb{R}^2}(0, 1))$, but $f \notin L^\infty(B_{\mathbb{R}^2}(0, 1))$ since $\lim_{|x| \to 0} f(x) = +\infty$. This, in turn, implies that $f \notin W^{1,p}(B_{\mathbb{R}^2}(0, 1))$ for any $p > 2$, due to the Sobolev embedding theorem. Nonetheless, we have

$$\partial_{x_i} f(x) = \left( \frac{1}{2} - \frac{1}{100} \right) \frac{x_i}{|x|^2 \log \left( \frac{1}{|x|} \right)} \frac{1}{\log^{\frac{3}{4} + \frac{1}{100}} \left( \frac{1}{|x|} \right)}, \quad i = 1, 2,$$

such that

$$\int_{B_{\mathbb{R}^2}(0,1)} |\partial_{x_i} f(x)|^2 \, dx \leq 2\pi \left( \frac{1}{2} - \frac{1}{100} \right)^2 \int_0^1 \frac{1}{r \log^{2(\frac{3}{4} + \frac{1}{100})^2} \left( \frac{1}{r} \right)} \, dr \leq C \int_0^1 \frac{1}{r \log^{\frac{1}{100}} \left( \frac{1}{r} \right)} < +\infty.$$

Thus, we have $f \in W^{1,2}(B_{\mathbb{R}^2}(0, 1))$. Finally, we observe that

$$\int_{B_{\mathbb{R}^2}(0,1)} |g(x)f(x)|^2 \, dx = \int_{B_{\mathbb{R}^2}(0,1)} \frac{\log^{1 - \frac{1}{100}} \left( \frac{1}{|x|} \right)}{|x|^2 \log^\frac{3}{4} \left( \frac{1}{|x|} \right)} \, dx$$

$$= 2\pi \int_0^1 \frac{1}{r \log^\frac{3}{4} \left( \frac{1}{r} \right)} \, dr = +\infty.$$
namely, the product \( f g \notin L^2(B_{\mathbb{R}^2}(0,1)) \). The above counterexample can be generalized to any pair of functions
\[
g(x) = \frac{1}{|x| \log^\alpha \left(\frac{1}{|x|}\right)}, \quad f(x) = \log^\beta \left(\frac{1}{|x|}\right), \quad x \in B_{\mathbb{R}^2}(0,1),
\]
where \( \frac{1}{2} < \alpha < 1 \) and \( \beta < \frac{1}{2} \) such that \( \alpha - \beta < \frac{1}{2} \).

3. Complex Fluids Model: Local Well-posedness

In this section we consider the complex fluids system
\[
\begin{cases}
\rho(\phi)(\partial_t u + u \cdot \nabla u) - \text{div} (\nu(\phi) D u) + \nabla P = -\text{div}(\nabla \phi \otimes \nabla \phi), \\
\text{div} u = 0, \\
\partial_t \phi + u \cdot \nabla \phi = 0,
\end{cases}
\]
subject to the boundary condition
\[
u = 0 \quad \text{on } \partial \Omega \times (0, T),
\]
and to the initial conditions
\[
u(\cdot, 0) = u_0, \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega.
\]

We recall that \( u \) is the (volume) averaged velocity of the binary mixture, \( P \) is the pressure, and \( \phi \) denotes the difference of the concentrations (volume fraction) of the two fluids. The coefficients \( \rho(\cdot) \) and \( \nu(\cdot) \) represent the density and the viscosity of the mixture depending on \( \phi \). Throughout this work, motivated by the linear interpolation density and viscosity functions in (1.12), we assume that
\[
\rho, \nu \in C^2([-1, 1]): \quad \rho(s) \in [\rho_1, \rho_2], \quad \nu(s) \in [\nu_1, \nu_2] \text{ for } s \in [-1, 1],
\]
where \( \rho_1, \rho_2 \) and \( \nu_1, \nu_2 \) are, respectively, the (positive) densities and viscosities of two homogeneous (different) fluids. In addition, we will use the notation
\[
\rho_* = \min\{\rho_1, \rho_2\} > 0, \quad \nu_* = \min\{\nu_1, \nu_2\} > 0.
\]

The aim of this section is to prove the local existence and uniqueness of regular solutions to problem (3.1)-(3.3) with general initial data. This generalizes [52, Theorem 1.1] to the case with unmatched densities and viscosities depending on the concentration, and to initial data \( \phi_0 \) belonging to \( W^{2,p}(\Omega) \), instead of \( \phi_0 \in H^3(\Omega) \) (see also [51, Theorem 2.2] for the Cauchy problem in \( \mathbb{R}^2 \)).

**Theorem 3.1 (Local well-posedness in 2D).** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). For any initial datum \((u_0, \phi_0)\) such that \( u_0 \in V_\sigma \cap H^2(\Omega), \phi_0 \in W^{2,p}(\Omega) \) for some \( p > 2 \), with \( |\phi_0(x)| \leq 1 \), for all \( x \in \Omega \), there exists a positive time \( T_0 \), which depends only on the norms of the initial data, and a unique solution \((u, \phi)\) to problem (3.1)-(3.3) on \([0, T_0]\) such that
\[
u \in L^\infty(0, T_0; W^{2,p}(\Omega)) \cap W^{1,\infty}(0, T_0; H\sigma(\Omega)), \quad |\phi(x, t)| \leq 1 \text{ in } \Omega \times [0, T_0].
\]
Proof. We perform some a priori estimates for the solutions to problem (3.1)-(3.3), and then we prove the uniqueness. With these arguments, the existence of local solutions to (3.1)-(3.3) follows from the method of successive approximations (Picard’s method). This relies on the definition of a suitable sequence \((u_k, \phi_k)\) via an iteration scheme, a priori bounds on \((u_k, \phi_k)\) in terms of \((u_{k-1}, \phi_{k-1})\), and uniform estimates of \((u_k - u_{k-1}, \phi_k - \phi_{k-1})\) (by arguing as in the uniqueness proof reported below). We refer to [45] for the details of this type of argument in the case of the nonhomogeneous Navier-Stokes equations (see also, e.g., [48] for the Navier-Stokes-Allen-Cahn system).

**First estimate.** Multiplying (3.1) by \(u\) and integrating over \(\Omega\), we find

\[
\frac{1}{2} \int_{\Omega} \rho(\phi) \partial_t |u|^2 \, dx + \frac{1}{2} \int_{\Omega} \rho(\phi) u \cdot \nabla(|u|^2) \, dx + \int_{\Omega} \nu(\phi) |Du|^2 \, dx = - \int_{\Omega} \Delta \phi \nabla \phi \cdot u \, dx.
\]

Taking the gradient of (3.1), we have

\[
\nabla \partial_t \phi + \nabla (u \cdot \nabla \phi) = 0.
\]

Multiplying the above identity by \(\nabla \phi\), integrating over \(\Omega\) and using the no-slip boundary condition of \(u\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi|^2 \, dx - \int_{\Omega} (u \cdot \nabla \phi) \Delta \phi \, dx = 0.
\]

By adding the two obtained equations, and using the identity

\[
\partial_t \rho(\phi) + \text{div} (\rho(\phi)u) = \rho'(\phi) \left( \partial_t \phi + u \cdot \nabla \phi \right) = 0,
\]

we find the basic energy law

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \rho(\phi)|u|^2 + |\nabla \phi|^2 \right) \, dx + \int_{\Omega} \nu(\phi) |Du|^2 \, dx = 0.
\]

Integrating over \([0, t]\), we obtain

\[
E_0(u(t), \phi(t)) + \int_0^t \int_{\Omega} \nu(\phi) |Du|^2 \, dx = E_0(u_0, \phi_0), \quad \forall t \geq 0.
\]

where

\[
E_0(u, \phi) = \frac{1}{2} \int_{\Omega} \rho(\phi)|u|^2 + |\nabla \phi|^2 \, dx.
\]

In addition, the transport equation yields that, for all \(p \in [2, \infty]\), it holds

\[
\|\phi(t)\|_{L^p(\Omega)} = \|\phi_0\|_{L^p(\Omega)}, \quad \forall t \geq 0. \tag{3.5}
\]

Thus, we infer that

\[
u \in L^\infty(0, T; H_\sigma) \cap L^2(0, T; V_\sigma), \quad \phi \in L^\infty(0, T; H^1(\Omega) \cap L^\infty(\Omega)). \tag{3.6}
\]

**Second estimate.** We multiply (3.1) by \(\partial_t u\) and integrate over \(\Omega\). After integrating by parts and using the fact that \(\partial_t u = 0\) on \(\partial\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu(\phi)|Du|^2 \, dx + \int_{\Omega} \rho(\phi) |\partial_t u|^2 \, dx
\]

\[
= \frac{1}{2} \int_{\Omega} \nu'(\phi) \partial_t \phi |Du|^2 \, dx - \int_{\Omega} \rho(\phi) (u \cdot \nabla)u \cdot \partial_t u \, dx + \int_{\Omega} (\nabla \phi \otimes \nabla \phi) : \nabla \partial_t u \, dx.
\]
Combining (2.4), (2.7) and (3.6), together with the relation $\partial_t \phi = -u \cdot \nabla \phi$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu(\phi) |Du|^2 \, dx + \int_{\Omega} \rho(\phi) |\partial_t u|^2 \, dx$$

$$\leq C \|\partial_t \phi\|_{L^2(\Omega)} \|Du\|_{L^4(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)} + \|\nabla \phi\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)}$$

$$\leq C \|\nabla \phi\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)} \|u\|_{H^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)}$$

$$\leq \frac{\rho_1}{4} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{\nu}{4} \|Du\|_{L^2(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)}$$

$$\leq \frac{\rho_1}{4} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{\nu}{4} \|Du\|_{L^2(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)}$$

(3.7)

Here we have also used that $\|\nabla u\|_{L^2(\Omega)}$ is equivalent to $\|Du\|_{L^2(\Omega)}$ thanks to (2.2). Next, we rewrite (3.1)1-(3.1)2 as a Stokes problem with non-constant viscosity

$$\begin{cases}
-\text{div} (\nu(\phi) Du) + \nabla P = f, & \text{in } \Omega \times (0,T), \\
\text{div } u = 0, & \text{in } \Omega \times (0,T), \\
\partial_t u = 0, & \text{on } \partial \Omega \times (0,T),
\end{cases}$$

where $f = -\rho(\phi)(\partial_t u + u \cdot \nabla u) - \text{div}(\nabla \phi \otimes \nabla \phi)$. By exploiting Theorem A.1 with $p = 2$, $s = 2$, $r = \infty$, we infer that

$$\|u\|_{H^2(\Omega)} \leq C \|\rho(\phi) \partial_t u\|_{L^2(\Omega)} + C \|\rho(\phi) (u \cdot \nabla u)\|_{L^2(\Omega)}$$

$$+ C \|\text{div}(\nabla \phi \otimes \nabla \phi)\|_{L^2(\Omega)} + C \|\nabla \phi\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)}$$

$$\leq C \|\partial_t u\|_{L^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} + C \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)} + \|Du\|_{L^2(\Omega)}$$

$$\leq C \|\partial_t u\|_{L^2(\Omega)} + C \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)} + \|Du\|_{L^2(\Omega)}.$$

Here we have used (2.7) and (3.6). Thus, by Young’s inequality we find

$$\|u\|_{H^2(\Omega)} \leq C \|\partial_t u\|_{L^2(\Omega)} + C \|Du\|_{L^2(\Omega)} + C \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)} + \|Du\|_{L^2(\Omega)}.$$

(3.8)

Inserting (3.8) into (3.7), and using again Young’s inequality, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu(\phi) |Du|^2 \, dx + \frac{\rho_1}{2} \int_{\Omega} |\partial_t u|^2 \, dx$$

$$\leq \frac{\nu}{4} \|\partial_t u\|_{L^2(\Omega)}^2 + C \|Du\|_{L^2(\Omega)}^4 + C \|\nabla \phi\|_{L^\infty(\Omega)}^2 \|\phi\|_{H^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 + C \|\phi\|_{H^2(\Omega)}^2.$$

(3.9)

**Third estimate.** We differentiate (3.1) with respect to the time to obtain

$$\rho(\phi) \partial_t \phi + \rho(\phi) \partial_t (u \cdot \nabla u + u \cdot \nabla \phi) + \rho(\phi) \partial_t \phi (\partial_t u + u \cdot \nabla u) - \text{div} (\nu(\phi) \partial_t u)$$

$$- \text{div} (\nu'(\phi) \partial_t \phi Du) + \nabla \partial_t P = -\text{div} (\nabla \phi \otimes \nabla \partial_t \phi + \nabla \partial_t \phi \otimes \nabla \phi).$$
Multiplying the above equation by $\partial_t \mathbf{u}$ and integrating over $\Omega$, we are led to

$$\frac{1}{2} \int_{\Omega} \rho(\phi) \partial_t |\partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \rho(\phi) \left( (\partial_t \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \partial_t \mathbf{u}) \cdot \partial_t \mathbf{u} \right) \, dx + \int_{\Omega} \rho'(\phi) \partial_t \phi (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx + \int_{\Omega} \nu(\phi)|D\partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \nu'(\phi)\partial_t \phi D\partial_t \mathbf{u} : D\partial_t \mathbf{u} \, dx = \int_{\Omega} \left( \nabla \phi \otimes \nabla \partial_t \phi + \nabla \partial_t \phi \otimes \nabla \phi \right) : \nabla \partial_t \mathbf{u} \, dx.$$

Since

$$\frac{1}{2} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx + \frac{1}{2} \int_{\Omega} \rho(\phi) |\mathbf{u} \cdot \nabla |\partial_t \mathbf{u}|^2 \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx - \frac{1}{2} \int_{\Omega} \partial_t \rho(\phi) + \text{div} (\rho(\phi) \mathbf{u}) |\partial_t \mathbf{u}|^2 \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx,$$

we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \nu(\phi)|D\partial_t \mathbf{u}|^2 \, dx$$

$$= - \int_{\Omega} \rho(\phi)(\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx - \int_{\Omega} \rho'(\phi)\partial_t \phi (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx$$

$$- \int_{\Omega} \nu'(\phi)\partial_t \phi D\partial_t \mathbf{u} : D\partial_t \mathbf{u} \, dx + \int_{\Omega} \left( \nabla \phi \otimes \nabla \partial_t \phi + \nabla \partial_t \phi \otimes \nabla \phi \right) : \nabla \partial_t \mathbf{u} \, dx. \quad (3.10)$$

We now estimate the terms on the right-hand side of the above equality. By using $(2.2), (2.4)$, and the equation $(3.1)_a$, we find

$$- \int_{\Omega} \rho(\phi)(\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx \leq C \|\partial_t \mathbf{u}\|_{L^4(\Omega)}^2 \|\nabla \mathbf{u}\|_{L^2(\Omega)}$$

$$\leq \frac{\nu}{16} \|D\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 \|D\mathbf{u}\|_{L^2(\Omega)}^2, \quad (3.11)$$

and

$$- \int_{\Omega} \rho'(\phi)\partial_t \phi (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx$$

$$\leq C \|\partial_t \phi\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^4(\Omega)}^2 + C \|\partial_t \phi\|_{L^2(\Omega)} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^4(\Omega)} \|\partial_t \mathbf{u}\|_{L^4(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}$$

$$\leq C \|\mathbf{u} \cdot \nabla \phi\|_{L^2(\Omega)} \|\mathbf{u}\|_{L^\infty(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)} \|\nabla \partial_t \mathbf{u}\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}$$

$$\leq \frac{\nu}{16} \|D\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\mathbf{u}\|_{L^\infty(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{L^4(\Omega)} \|D\mathbf{u}\|_{L^2(\Omega)}^2 \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2$$

$$\leq \frac{\nu}{16} \|D\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{H^2(\Omega)}^2 \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|D\mathbf{u}\|_{L^2(\Omega)}^2 \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2$$

$$\leq \frac{\nu}{16} \|D\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^3 + C \|D\mathbf{u}\|_{L^2(\Omega)}^2 \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2$$

$$+ C \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)} + \|D\mathbf{u}\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|D\mathbf{u}\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2$$

$$+ C \|D\mathbf{u}\|_{L^2(\Omega)} \|D\mathbf{u}\|_{L^2(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)} + \|D\mathbf{u}\|_{L^2(\Omega)} \|D\mathbf{u}\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2. \quad (3.12)$$
where we have also used (3.6) and (3.8). Moreover, we obtain

\[
- \int_{\Omega} \nu'(\phi) \partial_t \phi \, D \mathbf{u} : D \partial_t \mathbf{u} \, dx \\
\leq C \| \partial_t \phi \|_{L^4(\Omega)} \| D \mathbf{u} \|_{L^4(\Omega)} \| D \partial_t \mathbf{u} \|_{L^2(\Omega)} \\
\leq \frac{\nu_s}{16} \| D \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 + C \| \mathbf{u} \cdot \nabla \phi \|_{L^2(\Omega)} \| \nabla \partial_t \phi \|_{L^2(\Omega)} \| D \mathbf{u} \|_{L^2(\Omega)} \| \mathbf{u} \|_{H^2(\Omega)} \\
\leq \frac{\nu_s}{16} \| D \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)} \| D \mathbf{u} \|_{L^2(\Omega)} \| \mathbf{u} \|_{H^2(\Omega)}^3,
\]

(3.13)

and

\[
\int_{\Omega} \left( \nabla \phi \otimes \nabla \partial_t \phi + \nabla \partial_t \phi \otimes \nabla \phi \right) : \nabla \partial_t \mathbf{u} \, dx \\
\leq C \| \nabla \phi \|_{L^\infty(\Omega)} \| \nabla \partial_t \phi \|_{L^2(\Omega)} \| D \partial_t \mathbf{u} \|_{L^2(\Omega)} \\
\leq \frac{\nu_s}{16} \| D \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 + C \| \nabla \phi \|_{L^\infty(\Omega)}^2 \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2.
\]

(3.14)

It is clear that an estimate of \( \nabla \partial_t \phi \) is needed in order to control the last two terms in (3.13) and (3.14). For this purpose, we have

\[
\nabla \partial_t \phi = (\nabla \mathbf{u})^T \nabla \phi + \nabla^2 \phi \mathbf{u}.
\]

Then, we easily deduce that

\[
\| \nabla \partial_t \phi \|_{L^2(\Omega)} \leq \| \nabla \mathbf{u} \|_{L^2(\Omega)} \| \nabla \phi \|_{L^\infty(\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} \| \mathbf{u} \|_{L^\frac{2p}{p-2}(\Omega)} \\
\leq C \| D \mathbf{u} \|_{L^2(\Omega)} \left( \| \nabla \phi \|_{L^\infty(\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} \right),
\]

(3.15)

for \( p > 2 \). Combining (3.11)-(3.14) and (3.15) with (3.8)-(3.10), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx + \frac{3\nu_s}{4} \int_{\Omega} |D \partial_t \mathbf{u}|^2 \, dx \\
\leq C \| \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 \| D \mathbf{u} \|_{L^2(\Omega)}^2 + \| \partial_t \mathbf{u} \|_{L^2(\Omega)}^3 \\
+ C \| \nabla \phi \|_{L^\infty(\Omega)} (\| \phi \|_{H^2(\Omega)} + \| D \mathbf{u} \|_{L^2(\Omega)}) \| \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 + C \| D \mathbf{u} \|_{L^2(\Omega)} \| \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 \\
+ C \| D \mathbf{u} \|_{L^2(\Omega)} \left( \| \phi \|_{L^\infty(\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} \right) \left( \| \phi \|_{H^2(\Omega)} + \| D \mathbf{u} \|_{L^2(\Omega)} \right) \| \partial_t \mathbf{u} \|_{L^2(\Omega)}^2 \\
+ C \| D \mathbf{u} \|_{L^2(\Omega)} \| \nabla \phi \|_{L^\infty(\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} || \nabla \phi \|_{L^\infty(\Omega)} \frac{2}{3} \| D \mathbf{u} \|_{L^2(\Omega)}^2 \\
+ C \| D \mathbf{u} \|_{L^2(\Omega)} \left( \| \nabla \phi \|_{L^\infty(\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} \right) \| \nabla \phi \|_{L^\infty(\Omega)} \frac{2}{3} \| D \mathbf{u} \|_{L^2(\Omega)}^2
\]

(3.16)

**Fourth estimate.** In light of (3.9) and (3.16), we are left to control the \( W^{2,p}(\Omega) \)-norm of \( \phi \). To this aim, we make use of the following equivalent norm

\[
\| f \|_{W^{2,p}(\Omega)} = \left( \| f \|_{L^p(\Omega)}^p + \sum_{|\alpha|=2} \| \partial^\alpha f \|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},
\]
where $\alpha$ is a multi-index. Next, we apply $\partial^\alpha$ to the transport equation (3.1)3

$$\partial_t \partial^\alpha \phi + \partial^\alpha (u \cdot \nabla \phi) = 0.$$ 

Multiplying the above equation by $|\partial^\alpha \phi|^{p-2} \partial^\alpha \phi$ and integrating over $\Omega$, we get

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\partial^\alpha \phi|^p \, dx + \int_{\Omega} \partial^\alpha (u \cdot \nabla \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx = 0. \quad (3.17)$$

Since $u$ is divergence free, the above can be rewritten as

$$\frac{1}{p} \frac{d}{dt} \|\phi\|_{W^{2,p}(\Omega)}^p = - \sum_{|\alpha|=2} \int_{\Omega} (\partial^\alpha (u \cdot \nabla \phi) - u \cdot \nabla \partial^\alpha \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx. \quad (3.18)$$

By summing over all multi-index of order 2, and using (3.5), we find

$$\frac{1}{p} \frac{d}{dt} \|\phi\|_{W^{2,p}(\Omega)}^p = - \sum_{|\alpha|=2} \int_{\Omega} (\partial^\alpha (u \cdot \nabla \phi) - u \cdot \nabla \partial^\alpha \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx. \quad (3.19)$$

It is easily seen that the right-hand side can be written as

$$\sum_{|\alpha|=2} \int_{\Omega} (\partial^\alpha (u \cdot \nabla \phi) - u \cdot \nabla \partial^\alpha \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx$$

$$= \sum_{|\alpha|=2} \int_{\Omega} (\partial^\alpha u \cdot \nabla \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx + \sum_{|\beta|=1, |\gamma|=1, |\beta+\gamma|=\alpha} \int_{\Omega} (\partial^\beta u \cdot \nabla \partial^\gamma \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx. \quad (3.20)$$

Observe that

$$\sum_{|\alpha|=2} \int_{\Omega} (\partial^\alpha u \cdot \nabla \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx \leq C \|u\|_{W^{2,p}(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{W^{2,p}(\Omega)}^{p-1}, \quad (3.21)$$

and

$$\sum_{|\beta|=1, |\gamma|=1, |\beta+\gamma|=\alpha} \int_{\Omega} (\partial^\beta u \cdot \nabla \partial^\gamma \phi) |\partial^\alpha \phi|^{p-2} \partial^\alpha \phi \, dx \leq C \|u\|_{W^{1,\infty}(\Omega)} \|\phi\|_{W^{2,p}(\Omega)}^p. \quad (3.22)$$

Collecting (3.19)-(3.22) together, and using the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ (with $p > 2$), we obtain

$$\frac{1}{p} \frac{d}{dt} \|\phi\|_{W^{2,p}(\Omega)}^p \leq C \|u\|_{W^{2,p}(\Omega)} \|\phi\|_{W^{2,p}(\Omega)}^p.$$ 

Notice that the above inequality is equivalent to

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_{W^{2,p}(\Omega)}^2 \leq C \|u\|_{W^{2,p}(\Omega)} \|\phi\|_{W^{2,p}(\Omega)}^2. \quad (3.23)$$

Next, by exploiting Theorem A.1 with $s = p > 2$ and $r = \infty$, we deduce that

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|\rho(\phi)\|_{L^p(\Omega)} + \|\partial u \cdot \nabla \phi\|_{L^p(\Omega)} + \|\nabla (\nabla \phi \otimes \nabla \phi)\|_{L^p(\Omega)} + C \|\nabla \phi\|_{L^\infty(\Omega)} \|\phi\|_{L^p(\Omega)} + C \|\partial \phi\|_{L^p(\Omega)} \|\phi\|_{L^p(\Omega)}$$

$$\leq C \|\partial u\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}^2 + C \|u\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} + C \|u\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}^2 + C \|\phi\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}^2.$$
\[ + C \| \nabla \phi \|_{L^\infty(\Omega)} \left( \| \phi \|_{W^{2,p}(\Omega)} + \| u \|_{W^{2,p}(\Omega)}^{\frac{1}{2}} \| u \|_{L^p(\Omega)}^{\frac{1}{2}} \right). \]

Thus, by Young’s inequality we find
\[ \| u \|_{W^{2,p}(\Omega)} \leq C \| \partial_t u \|_{L^2(\Omega)}^2 \| D \partial_t u \|_{L^2(\Omega)}^{\frac{p-2}{p}} + C \| \phi \|_{W^{2,p}(\Omega)} \]
\[ + C \left( \| u \|_{H^2(\Omega)} + \| \nabla \phi \|_{L^\infty(\Omega)} \right) \| D u \|_{L^2(\Omega)} \| \phi \|_{W^{2,p}(\Omega)}^2. \] (3.24)

Inserting (3.8) and (3.24) into (3.23), we are led to
\[ \frac{1}{2} \frac{d}{dt} \| \phi \|_{W^{2,p}(\Omega)}^2 \leq C \left( \| \partial_t u \|_{L^2(\Omega)}^2 \| D \partial_t u \|_{L^2(\Omega)}^{\frac{p-2}{p}} + \| \phi \|_{W^{2,p}(\Omega)} \right) \| \phi \|_{W^{2,p}(\Omega)}^2 \]
\[ + C \left( \| u \|_{H^2(\Omega)} + \| \nabla \phi \|_{L^\infty(\Omega)} \right) \| D u \|_{L^2(\Omega)} \| \phi \|_{W^{2,p}(\Omega)}^2. \] (3.25)

**Final estimate.** By adding (3.9), (3.16) and (3.25) together, and using the embeddings \( W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega), W^{2,p}(\Omega) \hookrightarrow H^2(\Omega) \) for \( p > 2 \), we deduce that
\[ \frac{d}{dt} Y(t) + \rho_\ast \int_\Omega | \partial_t u |^2 \, dx + \nu_\ast \int_\Omega | D \partial_t u |^2 \, dx \leq C (1 + Y^3(t)), \] (3.26)
where
\[ Y(t) = \int_\Omega \nu(\phi(t)) | D u(t) |^2 \, dx + \int_\Omega \rho(\phi(t)) | \partial_t u(t) |^2 \, dx + \| \phi(t) \|_{W^{2,p}(\Omega)}^2. \]

Concerning the initial data, we observe from (3.1) that
\[ \int_\Omega \rho(\phi(0)) | \partial_t u(0) |^2 \, dx \leq C \left( \| u_0 \|_{L^\infty(\Omega)}^2 + \| \phi_0 \|_{W^{2,p}(\Omega)}^2 \right) \| \nabla u_0 \|_{L^2(\Omega)}^2 + C \| u_0 \|_{H^2(\Omega)}^2 + C \| \phi_0 \|_{W^{2,p}(\Omega)}^4, \]
which, in turn, implies
\[ Y(0) \leq Q \left( \| u_0 \|_{H^2(\Omega)}, \| \phi_0 \|_{W^{2,p}(\Omega)} \right), \]
where \( Q \) is a positive continuous and increasing function of its arguments.

Finally, we deduce from (3.26) that there exists a positive time \( T_0 < \frac{1}{2C(1+Y(0)^2)} \), which depends on the parameters of the system and on the norms of the initial data \( \| u_0 \|_{H^2(\Omega)} \) and \( \| \phi_0 \|_{W^{2,p}(\Omega)} \), such that
\[ \int_\Omega | D u(t) |^2 \, dx + \int_\Omega | \partial_t u(t) |^2 \, dx + \| \phi(t) \|_{W^{2,p}(\Omega)}^2 + \int_0^t \| \partial_t u(\tau) \|_{H^1(\Omega)}^2 \, d\tau \leq C_0, \] (3.27)
for all \( t \in [0, T_0] \), where \( C_0 \) is a positive constant depending on \( T_0, \| u_0 \|_{H^2(\Omega)}, \| \phi_0 \|_{W^{2,p}(\Omega)} \). In addition, we learn from (3.24) and (3.27) that
\[ \int_0^t \| u(\tau) \|_{W^{2,p}(\Omega)}^2 \, d\tau \leq C, \quad \forall \, t \in [0, T_0]. \]

Similarly, we also deduce from (3.8), (3.15), and (3.27) that
\[ \| u(t) \|_{H^2(\Omega)} + \| \partial_t \phi(t) \|_{H^1(\Omega)} + \| \partial_t \phi(t) \|_{L^\infty(\Omega)} \leq C, \quad \forall \, t \in [0, T_0]. \]
We have obtained all the necessary \textit{a priori} estimates. Then the existence result follows as outlined at the beginning of the proof.

\textbf{Uniqueness.} Let \((u_1, \phi_1)\) and \((u_2, \phi_2)\) be two solutions to problem \((3.1)-(3.3)\) originating from the same initial datum. The difference of solutions \((u, \phi, P) := (u_1 - u_2, \phi_1 - \phi_2, P_1 - P_2)\) solves the system

\[
\begin{aligned}
\rho(\phi_1) \left( \partial_t u + u_1 \cdot \nabla u + u \cdot \nabla u_2 \right) - \text{div} \left( \nu(\phi_1) D u \right) + \nabla P \\
= - \text{div} \left( \nabla \phi_1 \otimes \nabla \phi \right) - \text{div} \left( \nabla \phi \otimes \nabla \phi_2 \right) - (\rho(\phi_1) - \rho(\phi_2)) \left( \partial_t u_2 + u_2 \cdot \nabla u_2 \right) \\
+ \text{div} \left( (\nu(\phi_1) - \nu(\phi_2)) D u_2 \right), \\
\partial_t \phi + u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2 = 0,
\end{aligned}
\]

for almost every \((x, t) \in \Omega \times (0, T)\), together with the incompressibility constraint \(\text{div} u = 0\). Multiplying \((3.28)\) by \(u\) and integrating over \(\Omega\), we find

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi_1) |u|^2 \, dx + \int_{\Omega} \rho(\phi_1) (u_1 \cdot \nabla) u \cdot u \, dx + \int_{\Omega} \rho(\phi_1) (u \cdot \nabla) u_2 \cdot u \, dx + \int_{\Omega} \nu(\phi_1) |D u|^2 \, dx \\
= \int_{\Omega} (\nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2) : \nabla u \, dx - \int_{\Omega} (\rho(\phi_1) - \rho(\phi_2)) (\partial_t u_2 + u_2 \cdot \nabla u_2) \cdot u \, dx \\
- \int_{\Omega} (\nu(\phi_1) - \nu(\phi_2)) D u_2 : D u \, dx + \frac{1}{2} \int_{\Omega} |u|^2 \rho'(\phi_1) \partial_t \phi_1 \, dx.
\]

Noticing the identity

\[
\int_{\Omega} \rho(\phi_1) (u_1 \cdot \nabla) u \cdot u \, dx = \int_{\Omega} \rho(\phi_1) u_1 \cdot \nabla \left( \frac{1}{2} |u|^2 \right) \, dx = - \frac{1}{2} \int_{\Omega} \rho'(\phi_1) (\nabla \phi_1 \cdot u_1) |u|^2 \, dx,
\]

we can rewrite \((3.30)\) as follows

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi_1) |u|^2 \, dx + \int_{\Omega} \nu(\phi_1) |D u|^2 \, dx \\
= \int_{\Omega} (\nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2) : \nabla u \, dx - \int_{\Omega} (\rho(\phi_1) - \rho(\phi_2)) (\partial_t u_2 + u_2 \cdot \nabla u_2) \cdot u \, dx \\
- \int_{\Omega} (\nu(\phi_1) - \nu(\phi_2)) D u_2 : D u \, dx - \int_{\Omega} \rho(\phi_1) (u \cdot \nabla) u_2 \cdot u \, dx.
\]

By using the embedding \(W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)\) for \(p > 2\), we find that

\[
\int_{\Omega} (\nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2) : \nabla u \, dx \leq \left( \| \nabla \phi_1 \|_{L^\infty(\Omega)} + \| \nabla \phi_2 \|_{L^\infty(\Omega)} \right) \| \nabla \phi \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\Omega)} \leq \frac{\nu_*}{8} \| D u \|^2_{L^2(\Omega)} + C \| \nabla \phi \|^2_{L^2(\Omega)}.
\]

Next, by H"older’s inequality, we have

\[
- \int_{\Omega} (\rho(\phi_1) - \rho(\phi_2)) (\partial_t u_2 + u_2 \cdot \nabla u_2) \cdot u \, dx \\
\leq C\| \phi \|_{L^6(\Omega)} \| \partial_t u_2 + u_2 \cdot \nabla u_2 \|_{L^3(\Omega)} \| u \|_{L^2(\Omega)} \\
\leq C \left( \| \partial_t u_2 \|_{L^3(\Omega)} + \| u_2 \|_{L^\infty(\Omega)} \| \nabla u_2 \|_{L^3(\Omega)} \right) \| \phi \|_{H^1(\Omega)} \| u \|_{L^2(\Omega)},
\]
Collecting the above estimates together, we deduce from (3.31) that

\[
\frac{d}{dt} \int_{\Omega} \rho(\phi_1) |u|^2 \, dx + \frac{3\nu_*}{2} \int_{\Omega} |Du|^2 \, dx \leq C(1 + \|\partial_t u_2\|_{L^2(\Omega)} + \|\nabla u_2\|_{L^2(\Omega)} + \|\nabla u_2\|_{L^4(\Omega)} + \|\nabla u_2\|_{L^\infty(\Omega)}) \times \left(\|u\|_{L^2(\Omega)}^2 + \|\phi\|_{H^1(\Omega)}^2\right).
\]

(3.32)

Next, we multiply (3.29) by \(\phi\) and get

\[
\frac{1}{2} \frac{d}{dt} \|\phi\|_{L^2(\Omega)}^2 + \int_{\Omega} (u \cdot \nabla \phi_2) \phi \, dx = 0.
\]

Then taking the gradient of (3.29) and multiplying the resulting equation by \(\nabla \phi\), we find

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_{L^2(\Omega)}^2 + \int_{\Omega} \nabla (u_1 \cdot \nabla \phi) \cdot \nabla \phi \, dx + \int_{\Omega} \nabla (u \cdot \nabla \phi_2) \cdot \nabla \phi \, dx = 0.
\]

By adding the last two equations, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\phi\|_{H^1(\Omega)}^2 + \int_{\Omega} (\nabla u_1 \nabla \phi) \cdot \nabla \phi \, dx + \int_{\Omega} (\nabla u \nabla \phi_2) \cdot \nabla \phi \, dx + \int_{\Omega} (\nabla^2 \phi_2 u) \cdot \nabla \phi \, dx + \int_{\Omega} (u \cdot \nabla \phi_2) \phi \, dx = 0.
\]

We have

\[
- \int_{\Omega} (\nabla u_1 \nabla \phi) \cdot \nabla \phi \, dx \leq \|\nabla u_1\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}^2,
\]

and by \(W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega), \ p > 2\), we get

\[
- \int_{\Omega} (\nabla u \nabla \phi_2) \cdot \nabla \phi \, dx \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \phi_2\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \leq \frac{\nu_*}{8} \|Du\|_{L^2(\Omega)}^2 + C \|\phi\|_{L^2(\Omega)}^2,
\]

\[
- \int_{\Omega} (u \cdot \nabla \phi_2) \phi \, dx \leq \|u\|_{L^2(\Omega)} \|\nabla \phi_2\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}^2 + C \|\phi\|_{L^2(\Omega)}^2.
\]

Using (2.5), we obtain

\[
- \int_{\Omega} (\nabla^2 \phi_2 u) \cdot \nabla \phi \, dx \leq \|\nabla^2 \phi_2\|_{L^p(\Omega)} \|u\|_{L^{\frac{2p}{p-2}}(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \leq C \|\phi_2\|_{W^{2,p}(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}^2 \leq C \|\phi_2\|_{W^{2,p}(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}^2 \|\nabla \phi\|_{L^2(\Omega)}
\]
By adding (3.32) and (3.33), we end up with the differential inequality
\[
\frac{d}{dt} \left( \|u\|_{L^2(\Omega)}^2 + \|\phi\|_{H^1(\Omega)}^2 \right) + \nu \|\nabla u\|_{L^2(\Omega)}^2 \leq CR(t) \left( \|u\|_{L^2(\Omega)}^2 + \|\phi\|_{H^1(\Omega)}^2 \right),
\]
where
\[
R = 1 + \|\partial_t u_2\|_{L^3(\Omega)} + \|u_2\|_{L^\infty(\Omega)} \|\nabla u_2\|_{L^3(\Omega)} + \|Du_2\|_{L^2(\Omega)}^2 + \|\nabla u_2\|_{L^\infty(\Omega)} + \|\nabla u_1\|_{L^\infty(\Omega)}.
\]
Since \(R \in L^1(0, T_0)\), then the uniqueness of strong solutions follows from Gronwall’s lemma.

**Remark 3.2.** The local well-posedness result stated in Theorem 3.1 is also valid in three dimensional case, provided that the initial condition \(\phi \in W^{2,p}(\Omega)\) for some \(p > 3\). The strategy used in the above proof can be adapted to this case by using the corresponding Sobolev inequalities in three dimensions.

## 4. Mass-Conserving Navier-Stokes-Allen-Cahn System: Weak Solutions

In this section, we consider the Navier-Stokes-Allen-Cahn system for a binary mixture of two incompressible fluids with different densities. This model was proposed in [32, Section 4.2.2] and derived through an energetic variational approach (see also [37] for the case with no mass constraint). The system reads as follows
\[
\begin{cases}
\rho(\phi)(\partial_t u + u \cdot \nabla u) - \text{div} \left( \nu(\phi) Du \right) + \nabla P = -\text{div} \left( \nabla \phi \otimes \nabla \phi \right), \\
\text{div} u = 0,
\end{cases}
\]
\[
\begin{cases}
\partial_t \phi + u \cdot \nabla \phi + \mu + \rho'(\phi) \frac{|u|^2}{2} = \xi, \\
\mu = -\Delta \phi + \Psi'(\phi), \quad \xi = \mu + \rho'(\phi) \frac{|u|^2}{2},
\end{cases}
\]
in \(\Omega \times (0, T)\),
\[
\begin{align*}
u(\phi) & \geq 0, \\
\phi & \geq 0 \text{ on } \partial\Omega \times (0, T),
\end{align*}
\]
subject to the boundary conditions
\[
u(\phi) \geq 0, \quad \phi \geq 0 \quad \text{on } \partial\Omega \times (0, T),
\]
and to the initial conditions
\[
u(\phi) \geq 0, \quad \phi \geq 0 \quad \text{in } \Omega.
\]
Here, \(\rho(\phi)\) and \(\nu(\phi)\) are, respectively, the density and the viscosity of the mixture, which satisfy the assumptions (3.4). The nonlinear function \(\Psi\) is the Flory-Huggins potential (1.19). The total energy of system (4.1)-(4.2) is given by
\[
E(u, \phi) = \int_{\Omega} \frac{1}{2} \rho(\phi) |u|^2 + \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx.
\]

The main results of this section concern with the existence of global weak solutions.
**Theorem 4.1** (Global weak solution). Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary, $d = 2, 3$. Assume that the initial datum $(u_0, \phi_0)$ satisfies $u_0 \in H_\sigma, \phi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)} \leq 1$ and $|\phi_0| < 1$. Then, there exists a global weak solution $(u, \phi)$ to system (4.1)-(4.3) in the following sense:

(i) For all $T > 0$, the pair $(u, \phi)$ satisfies

$$u \in L^\infty(0, T; H_\sigma) \cap L^2(0, T; V_\sigma),$$

$$\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^q(0, T; H^2(\Omega)), \quad \partial_t \phi \in L^q(0, T; L^2(\Omega)),$$

$$\phi \in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T),$$

$$\mu \in L^q(0, T; L^2(\Omega)),$$

with $q = 2$ if $d = 2$, $q = \frac{4}{3}$ if $d = 3$.

(ii) For all $T > 0$, the system (4.1) is solved as follows

$$- \int_0^T \int_\Omega (\rho'(\phi) \partial_t \phi \eta(t) + \rho(\phi) \eta'(t)) u \cdot v \, dx \, dt + \int_0^T \int_\Omega (\rho(\phi) u \cdot \nabla u) \cdot v \eta(t) \, dx \, dt$$

$$+ \int_0^T \int_\Omega \nu(\phi) (Du : Dv) \eta(t) \, dx \, dt = \int_\Omega (\rho(\phi) u_0 v \eta(0) \, dx + \int_0^T \int_\Omega (\nabla \phi \otimes \nabla \phi) : \nabla v) \eta(t) \, dx \, dt,$$

for $v \in V_\sigma, \eta \in C^1([0, T])$ with $\eta(T) = 0$, and

$$\partial_t \phi + u \cdot \nabla \phi - \Delta \phi + \Psi'(\phi) + \rho'(\phi) \frac{|u|^2}{2} = \Psi'(\phi) + \rho'(\phi) \frac{|u|^2}{2}, \text{ a.e. in } \Omega \times (0, T).$$

(iii) The pair $(u, \phi)$ fulfills the regularity $u \in C([0, T]; H_\sigma)$ and $\phi \in C([0, T]; H^1(\Omega))$, for all $T > 0$, and $u|_{t=0} = u_0, \phi|_{t=0} = \phi_0$ in $\Omega$. In addition, $\partial_{\nu} \phi = 0$ on $\partial \Omega \times (0, T)$ for all $T > 0$.

(iv) The energy inequality

$$E(u(t), \phi(t)) + \int_0^t \int_\Omega \nu(\phi)|Du|^2 \, dx \, d\tau + \int_0^t \|\partial_t \phi + u \cdot \nabla \phi\|_{L^2(\Omega)}^2 \, d\tau \leq E(u_0, \phi_0)$$

holds for all $t \geq 0$.

Next, we investigate the special case with matched densities (i.e. $\rho_1 = \rho_2$, so that $\rho \equiv 1$). The resulting model is the homogeneous mass-conserving Navier-Stokes-Allen-Cahn system

$$\begin{cases}
\partial_t u + u \cdot \nabla u - \text{div} (\nu(\phi) Du) + \nabla p = -\text{div} (\nabla \phi \otimes \nabla \phi), \\
\text{div} u = 0,
\end{cases} \quad \text{in } \Omega \times (0, T). \quad (4.5)$$

This system is associated with the boundary and the initial conditions

$$u = 0, \quad \partial_{\nu} \phi = 0 \quad \text{on } \partial \Omega \times (0, T), \quad u(\cdot, 0) = u_0, \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega. \quad (4.6)$$

We first state the existence of global weak solutions, whose proof follows from similar a priori estimates as the ones obtained for the nonhomogeneous case in the proof of Theorem 4.1 below.
**Theorem 4.2** (Global weak solution). Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary. Assume that the initial datum $(u_0, \phi_0)$ satisfies $u_0 \in H^1_\sigma, \phi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)} \leq 1$ and $|\phi_0| < 1$. Then there exists a global weak solution $(u, \phi)$ to problem (4.5)-(4.6). This is, the solution $(u, \phi)$ satisfies, for all $T > 0$,

$$u \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; \mathbb{V}_\sigma),$$

$$\partial_t u \in L^2(0, T; V'_\sigma) \text{ if } d = 2, \quad \partial_t u \in L^3(0, T; V'_\sigma) \text{ if } d = 3,$$

$$\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

$$\phi \in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T),$$

$$\partial_t \phi \in L^2(0, T; L^2(\Omega)) \text{ if } d = 2, \quad \partial_t \phi \in L^3(0, T; L^2(\Omega)) \text{ if } d = 3,$$

and

$$\langle \partial_t u, v \rangle + (u \cdot \nabla u, v) + (\nu(\phi)Du, \nabla v) = (\nabla \phi \otimes \nabla \phi, \nabla v), \quad \forall v \in V_\sigma, \text{ a.e. } t \in (0, T),$$

$$\partial_t \phi + u \cdot \nabla \phi - \Delta \phi + \Psi'(\phi) = \overline{\Psi'(\phi)}, \quad \text{a.e. } (x, t) \in \Omega \times (0, T).$$

Moreover, the initial and boundary conditions and the energy inequality hold as in Theorem 4.1.

Furthermore, due to the particular form of the density function, we are able to prove a uniqueness result in dimension two.

**Theorem 4.3** (Uniqueness of weak solutions in 2D). Assume $d = 2$. Let $(u_1, \phi_1)$ and $(u_2, \phi_2)$ be two weak solutions to problem (4.5)-(4.6) on $[0, T]$ subject to the same initial condition $(u_0, \phi_0)$ which satisfies the assumptions of Theorem 4.2. Moreover, we assume that $\phi_1$ satisfies the additional regularity $L^3(0, T; H^2(\Omega))$ with $\gamma > \frac{12}{5}$. Then $(u_1, \phi_1) = (u_2, \phi_2)$ on $[0, T]$.

**Remark 4.4.** The existence of strong solutions obtained in Section 5 (cf. Remark 5.4), which yields the regularity $\phi \in L^\infty(0, T; H^2(\Omega))$, where $\gamma > \frac{12}{5}$, entails that the Theorem 4.3 can be regarded as a weak-strong uniqueness result for problem (4.5)-(4.6) in two dimensions. That is, the weak solution originating from an initial condition $(u_0, \phi_0)$ such that $u_0 \in V_\sigma$ and $\phi_0 \in H^2(\Omega)$ with $\Psi'(\phi_0) \in L^2(\Omega)$ coincides with the (unique) strong solution departing from the same initial datum.

4.1. **Proof of Theorem 4.1.** First, we derive a priori estimates of problem (4.1)-(4.3) that will be crucial to prove the existence of global weak solutions.

**Mass conservation and energy dissipation.** First, integrating the equation (4.1), over $\Omega$ and using the definition of $\xi$, we observe that

$$\frac{1}{|\Omega|} \int_{\Omega} \phi(t) \text{d}x = \frac{1}{|\Omega|} \int_{\Omega} \phi_0 \text{d}x, \quad \forall t \geq 0.$$

Next, we derive the energy equation associated with (4.1). Multiplying (4.1) by $u$ and integrating over $\Omega$, we obtain

$$\int_{\Omega} \frac{1}{2} \rho(\phi) \partial_t |u|^2 \text{d}x + \int_{\Omega} \rho(\phi) (u \cdot \nabla) u \cdot u \text{d}x + \int_{\Omega} \nu(\phi) |Du|^2 \text{d}x = -\int_{\Omega} \Delta \phi \nabla \phi \cdot u \text{d}x. \quad (4.7)$$
Here we have used the relation $-\Delta \phi \nabla \phi = \frac{1}{2} \nabla |\nabla \phi|^2 - \text{div}(\nabla \phi \otimes \nabla \phi)$ and the incompressibility condition (4.1)\textsubscript{2}. Thanks to the no-slip boundary condition for $u$, we observe that
\[
\int_{\Omega} \rho(\phi)(u \cdot \nabla)u \cdot u \, dx = \int_{\Omega} \rho(\phi)u \cdot \nabla (\frac{1}{2}|u|^2) \, dx
\]
\[
= -\frac{1}{2} \int_{\Omega} \text{div} (\rho(\phi)u)|u|^2 \, dx = -\int_{\Omega} \rho'(\phi)\nabla \phi \cdot u \frac{|u|^2}{2} \, dx.
\]
Next, we multiply (4.1)\textsubscript{3} by $\partial_t \phi + u \cdot \nabla \phi$ and integrate over $\Omega$. Noticing that $\partial_t \phi + u \cdot \nabla \phi = 0$, we get
\[
\|\partial_t \phi + u \cdot \nabla \phi\|^2_{L^2(\Omega)} + \int_{\Omega} \mu \left(\partial_t \phi + u \cdot \nabla \phi\right) \, dx + \int_{\Omega} \rho'(\phi)\frac{|u|^2}{2} \left(\partial_t \phi + u \cdot \nabla \phi\right) \, dx = 0. \tag{4.8}
\]
On the other hand, the following equalities hold
\[
\int_{\Omega} \mu \partial_t \phi \, dx = \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx,
\]
\[
\int_{\Omega} \mu u \cdot \nabla \phi \, dx = \int_{\Omega} -\Delta \phi \nabla \phi \cdot u \, dx + \int_{\Omega} u \cdot \nabla \Psi(\phi) \, dx = \int_{\Omega} -\Delta \phi \nabla \phi \cdot u \, dx,
\]
\[
\int_{\Omega} \rho'(\phi)\frac{|u|^2}{2} \partial_t \phi \, dx = \int_{\Omega} \partial_t (\rho(\phi)) \frac{|u|^2}{2} \, dx.
\]
Thus, by adding (4.7) and (4.8), and using the above identities, we obtain the energy equation
\[
\frac{d}{dt} E(u, \phi) + \int_{\Omega} \nu(\phi)|Du|^2 \, dx + \|\partial_t \phi + u \cdot \nabla \phi\|^2_{L^2(\Omega)} = 0. \tag{4.9}
\]

**Lower-order estimates.** We assume that $\phi \in L^\infty(\Omega \times (0, T))$ such that $|\phi(x, t)| < 1$ almost everywhere in $\Omega \times (0, T)$ (cf. Existence of weak solutions below). Since $\rho$ is strictly positive, it is immediately seen from (4.9) that
\[
E(u(t), \phi(t)) + \int_0^t \int_{\Omega} \nu(\phi)|Du|^2 \, dx \, d\tau + \int_0^t \|\partial_t \phi + u \cdot \nabla \phi\|^2_{L^2(\Omega)} \, d\tau \leq E(u_0, \phi_0), \quad \forall t \geq 0. \tag{4.10}
\]
Therefore, we deduce
\[
u \in L^\infty(0, T; H_s) \cap L^2(0, T; V_s), \quad \phi \in L^\infty(0, T; H^1(\Omega)) \tag{4.11}
\]
and
\[
\partial_t \phi + u \cdot \nabla \phi \in L^2(0, T; L^2(\Omega)). \tag{4.12}
\]
In light of (4.11) and (4.12), when $d = 2$, we have
\[
\|\rho(\phi)|u|^2\|^2_{L^2(\Omega)} \leq C\|u\|^2_{L^4(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},
\]
which entails that $\rho(\phi)|u|^2 \in L^2(0, T; L^2(\Omega))$. Instead, when $d = 3$, we have
\[
\|\rho(\phi)|u|^2\|^2_{L^2(\Omega)} \leq C\|u\|^2_{L^4(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}.
\]
thus $\rho(\phi)|u|^2 \in L^\frac{4}{3}(0, T; L^2(\Omega))$. Since $\rho(\phi)|u|^2 \in L^\infty(0, T)$, we also learn that
\[
\mu - \overline{\mu} \in L^q(0, T; L^2(\Omega)), \tag{4.13}
\]
for $q = 2$ if $d = 2$, $q = \frac{4}{3}$ if $d = 3$. Thanks to the boundary condition for $\phi$, we see that $\Delta \phi = 0$. Then, multiplying (4.14) by $-\Delta \phi$ and integrating by parts, we have

$$\int_{\Omega} |\Delta \phi|^2 + F''(\phi) |\nabla \phi|^2 \, dx = \theta_0 \|\nabla \phi\|^2_{L^2(\Omega)} - \int_{\Omega} (\mu - \overline{\mu}) \Delta \phi \, dx,$$

where $F$ is the convex part of the potential $\Psi$, i.e. $F(s) = \frac{\theta}{2} [(1 + s) \log(1 + s) + (1 - s) \log(1 - s)]$.

By (4.11) and (4.13), we obtain

$$\|\Delta \phi\|_{L^2(\Omega)} \leq C(1 + \|\mu - \overline{\mu}\|_{L^2(\Omega)}).$$

(4.14)

Then, from the regularity theory of the Neumann problem, we infer that

$$\phi \in L^q(0, T; H^2(\Omega)).$$

(4.15)

From (2.4), (2.6) and the above bounds, we have

$$\|u \cdot \nabla \phi\|_{L^2(\Omega)} \leq C\|u\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4(\Omega)}$$

$$\leq C\|u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)}$$

$$\leq C\|\nabla u\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)}, \quad \text{if } d = 2,$$

and

$$\|u \cdot \nabla \phi\|_{L^2(\Omega)} \leq C\|u\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4(\Omega)}$$

$$\leq C\|u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)}$$

$$\leq C\|\nabla u\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)}, \quad \text{if } d = 3,$$

which implies $u \cdot \nabla \phi \in L^q(0, T; L^2(\Omega))$. Thus

$$\partial_t \phi \in L^q(0, T; L^2(\Omega)).$$

(4.16)

Moreover, we observe that

$$\|\mu - \overline{\mu}\|_{L^2(\Omega)} \leq \|\partial_t \phi\|_{L^2(\Omega)} + \|u \cdot \nabla \phi\|_{L^2(\Omega)} + \|\rho'(\phi) \frac{|u|^2}{2}\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}} - \rho'(\phi) \frac{|u|^2}{2}\|_{L^1(\Omega)}$$

$$\leq \|\partial_t \phi\|_{L^2(\Omega)} + C\|u\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4(\Omega)} + C\|u\|_{L^2(\Omega)}$$

$$\leq \|\partial_t \phi\|_{L^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)} + C\|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}^2$$

$$\leq \|\partial_t \phi\|_{L^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)}^2 + C, \quad \text{if } d = 2,$$

(4.17)

and

$$\|\mu - \overline{\mu}\|_{L^2(\Omega)} \leq \|\partial_t \phi\|_{L^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \|\phi\|_{H^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)}^2 + C, \quad \text{if } d = 3.$$ 

(4.18)

Recalling (4.11) and (4.14), and using Young’s inequality, we find that

$$\|\phi\|_{H^2(\Omega)} \leq C(1 + \|\partial_t \phi\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}), \quad \text{if } d = 2,$$

(4.19)

and

$$\|\phi\|_{H^2(\Omega)} \leq C(1 + \|\partial_t \phi\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}^2), \quad \text{if } d = 3.$$ 

(4.20)
In order to recover the full $L^2$-norm of $\mu$, we observe that

$$\overline{\mu} = \overline{F'(\phi)} - \theta_0 \overline{\phi}. $$

Since $|\overline{\phi(t)}| = |\overline{\phi_0}| < 1$, it is well-known that

$$\int_\Omega |F'(\phi)| \, dx \leq C \int_\Omega F'(\phi)(\phi - \overline{\phi}) \, dx + C$$

for some positive constant $C$ depending on $F$ and $\overline{\phi}$. Multiplying (4.1) by $\phi - \overline{\phi}$ and using the boundary condition on $\phi$, we obtain

$$\|\nabla \phi\|_{L^2(\Omega)}^2 + \int_\Omega F'(\phi)(\phi - \overline{\phi}) = \int_\Omega \mu(\phi - \overline{\phi}) \, dx + \int_\Omega \theta_0 \phi(\phi - \overline{\phi}) \, dx.$$  

Combining the above two relations and exploiting the energy bounds (4.11), we reach

$$\|F'(\phi)\|_{L^1(\Omega)} \leq C(1 + \|\mu - \overline{\mu}\|_{L^2(\Omega)}).$$

This actually implies that

$$\mu \in L^q(0, T; L^2(\Omega))$$

and, in view of (4.15), we also have

$$F'(\phi) \in L^q(0, T; L^2(\Omega)),$$

where $q = 2$ if $d = 2$, $q = \frac{4}{3}$ if $d = 3$.

Besides, we have the following estimate for the time translations of $u$:

**Lemma 4.5.** For any $\delta \in (0, T)$, the following bound holds

$$\int_0^{T-\delta} \|u(t + \delta) - u(t)\|_{L^2(\Omega)}^2 \, dt \leq C\delta^{\frac{1}{4}}.$$  

**Proof.** We only present the proof for the case $d = 3$. The case $d = 2$ follows along the same lines. It follows from (4.11) and the interpolation (2.6) with $p = 3$ that $u \in L^4(0, T; L^3(\Omega))$. Similar to [49] (see also [37, Lemma 3.5]), we have

$$\|\sqrt{\rho(\phi(t + \delta))}(u(t + \delta) - u(t))\|_{L^2(\Omega)}^2 \leq -\int_\Omega (\rho(\phi(t + \delta)) - \rho(\phi(t))) u(t) \cdot (u(t + \delta) - u(t)) \, dx$$

$$- \int_t^{t+\delta} \int_\Omega \nu(\phi(\tau)) (u(\tau) \cdot \nabla) u(\tau) \cdot (u(t + \delta) - u(t)) \, dx \, d\tau$$

$$- \int_t^{t+\delta} \int_\Omega \nu(\phi(\tau)) Du(\tau) : Du(t + \delta) - u(t) \, dx \, d\tau$$

$$+ \int_t^{t+\delta} \int_\Omega \nu(\phi(\tau) \otimes \nabla \phi(\tau)) : \nabla(u(t + \delta) - u(t)) \, dx \, d\tau$$

$$+ \int_t^{t+\delta} \int_\Omega \rho'(\phi) \partial \phi(\tau) u(\tau) \cdot (u(t + \delta) - u(t)) \, dx \, d\tau := \sum_{i=1}^5 J_i.$$
Observe now
\[
\int_0^{T-\delta} J_1(t) \, dt \leq \int_0^{T-\delta} \int_t^T \int_\Omega |\rho(\phi)| |\partial_\tau \phi(\tau)| |u(t)| |(u(t + \delta) + u(t))| \, dx \, d\tau \, dt \\
\leq \int_0^{T-\delta} (|u(t + \delta)|_{L^3(\Omega)} + |u(t)|_{L^3(\Omega)}) |u(t)|_{L^6(\Omega)} \int_t^T \|\partial_\tau \phi(\tau)\|_{L^2(\Omega)} \, d\tau \, dt \\
\leq C\delta^{\frac{1}{2}} \left( \int_0^T \|\nabla u(t)\|_{L^2(\Omega)} \, dt \right) \left( \int_0^T \|\partial_\tau \phi(t)\|_{L^2(\Omega)}^\frac{1}{2} \, dt \right)^{\frac{3}{4}} \leq C\delta^{\frac{1}{2}},
\]
and, in a similar manner,
\[
\int_0^{T-\delta} J_5(t) \, dt \leq \int_0^{T-\delta} (|u(t + \delta)|_{L^3(\Omega)} + |u(t)|_{L^3(\Omega)}) \int_t^T \|u(\tau)\|_{L^6(\Omega)} \|\partial_\tau \phi(\tau)\|_{L^2(\Omega)} \, d\tau \, dt \\
\leq C\delta^{\frac{1}{2}} \left( \int_0^T \|\nabla u(t)\|_{L^2(\Omega)} \, dt \right) \left( \int_0^T \|\partial_\tau \phi(t)\|_{L^2(\Omega)}^\frac{1}{2} \, dt \right)^{\frac{3}{4}} \leq C\delta^{\frac{1}{2}}.
\]
Next, we have
\[
\int_0^{T-\delta} J_2(t) \, dt \\
\leq \int_0^{T-\delta} \int_t^T \|\rho(\phi(\tau))\|_{L^\infty(\Omega)} \|u(\tau)\|_{L^6(\Omega)} \|\nabla u(\tau)\|_{L^2(\Omega)} \, d\tau \|u(t + \delta)\|_{L^3(\Omega)} + |u(t)|_{L^3(\Omega)} \, dt \\
\leq C\delta^{\frac{1}{2}} \int_0^{T-\delta} \left( \int_t^T \|\nabla u(\tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{\frac{1}{2}} \|u(t + \delta)\|_{L^3(\Omega)} + |u(t)|_{L^3(\Omega)} \, dt \\
\leq C\delta^{\frac{1}{2}} \left( \int_0^T \|\nabla u(t)\|_{L^2(\Omega)}^\frac{1}{2} \, dt \right)^{\frac{3}{4}} \int_0^T \|u(t)\|_{L^3(\Omega)} \, dt \leq C\delta^{\frac{1}{2}},
\]
and
\[
\int_0^{T-\delta} J_3(t) \, dt \\
\leq \int_0^{T-\delta} \int_t^T \|\nu(\phi(\tau))\|_{L^\infty(\Omega)} \|Du(\tau)\|_{L^2(\Omega)} \, d\tau \|Du(t + \delta)\|_{L^2(\Omega)} + |Du(t)|_{L^2(\Omega)} \, dt \\
\leq C\delta^{\frac{1}{2}} \int_0^{T-\delta} \left( \int_t^T \|\nabla u(\tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{\frac{1}{2}} \|Du(t + \delta)\|_{L^2(\Omega)} + |Du(t)|_{L^2(\Omega)} \, dt \\
\leq C\delta^{\frac{1}{2}} \left( \int_0^T \|\nabla u(t)\|_{L^2(\Omega)}^\frac{1}{2} \, dt \right)^{\frac{3}{4}} \int_0^T \|\nabla u(t)\|_{L^2(\Omega)} \, dt \leq C\delta^{\frac{1}{2}},
\]
Finally, by using (2.8) we get
\[
\int_0^{T-\delta} J_4(t) \, dt \leq \int_0^{T-\delta} \int_t^T \|\nabla \phi(\tau)\|_{L^6(\Omega)}^2 \, d\tau \|\nabla u(t + \delta)\|_{L^2(\Omega)} + |\nabla u(t)|_{L^2(\Omega)} \, dt 
\]
\[ \leq C\delta^{\frac{1}{4}} \int_{0}^{T-\delta} \left( \int_{t}^{t+\delta} \|\phi(\tau)\|_{H^{2}(\Omega)}^{\frac{4}{3}} \, d\tau \right)^{\frac{3}{4}} \left( \|\nabla u(t+\delta)\|_{L^{2}(\Omega)} + \|\nabla u(t)\|_{L^{2}(\Omega)} \right) \, dt \]

\[ \leq C\delta^{\frac{1}{4}} \left( \int_{0}^{T} \|\phi(t)\|_{H^{2}(\Omega)}^{\frac{4}{3}} \, dt \right)^{\frac{3}{4}} \int_{0}^{T} \|\nabla u(t)\|_{L^{2}(\Omega)} \, dt \leq C\delta^{\frac{1}{4}}. \]

From the above estimate and the fact that \( \rho \) is strictly bounded from below, we obtain the conclusion (4.24). The proof is complete. \( \square \)

**Existence of weak solutions.** With the above \textit{a priori} estimates, we are able to prove the existence of a global weak solution by using a semi-Galerkin scheme similar to [37]. More precisely, for any \( n \in \mathbb{N} \), we find a local-in-time approximating solution \((u_n, \phi_n)\) where \( u_n \) solves (4.1) as in the classical Galerkin approximation and \( \phi_n \) is the (non-discrete) solution to the Allen-Cahn equations (4.1)$_3$–(4.1)$_4$ with the velocity \( u_n \), the singular potential and the nonlocal term. This is achieved via a Schauder fixed point argument. For this approach, it is needed to solve separately a convective nonlocal Allen-Cahn equation. This can be done by introducing a family of regular potentials \( \{\Psi_\varepsilon\} \) that approximates the original singular potential \( \Psi \) by setting (see, e.g., [33])

\[ \Psi_\varepsilon(s) = F_\varepsilon(s) - \frac{\theta_0}{2} s^2, \quad \forall s \in \mathbb{R}, \]

where

\[ F_\varepsilon(s) = \begin{cases} \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(1-\varepsilon) [s-(1-\varepsilon)]^j, & \forall s \geq 1-\varepsilon, \\ F(s), & \forall s \in [-1+\varepsilon, 1-\varepsilon], \\ \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(-1+\varepsilon) [s-(-1+\varepsilon)]^j, & \forall s \leq -1+\varepsilon. \end{cases} \]

Substituting the regular potential \( \Psi_\varepsilon \) into the original Allen-Cahn equation, we are able to prove the existence of an approximating solution \( \phi_\varepsilon \) to the resulting regularized equation using the semigroup approach like in [37, Lemma 3.2] or simply by the Galerkin method. For the approximating solution \( \phi_\varepsilon \), we can derive estimates that are uniform in \( \varepsilon \) and then pass to the limit as \( \varepsilon \to 0 \) to recover the case with singular potential. Here, we would like to remark that, thanks to the singular potential, we can show that the phase field takes values in \([-1, 1]\) (using a similar argument like in [33]), without the additional assumption \( s\rho'(s) \geq 0 \) for \( |s| > 1 \) that was required in [37]. Next, thanks to the \textit{a priori} estimates showed above, it follows that the existence time interval of any solution \((u_n, \phi_n)\) is independent to \( n \). From the same argument, we deduce uniform estimates that allows compactness for the phase field \( \phi \). Then, the key issue is to obtain uniform estimates of translations \( \int_{0}^{T-\delta} \|u(t+\delta) - u(t)\|_{L^{2}(\Omega)}^2 \, dt \) (see Lemma 4.5) that yields compactness of the velocity field in the case of unmatched densities (cf. [49]).

The above two-level approximating procedure is standard and we omit the details here.

**Time continuity and initial condition.** We first observe that the regularity properties (4.11) and (4.16), together with the global bound \( \|\phi\|_{L^\infty(0,T; L^{\infty}(\Omega))} \leq 1 \), entail that \( \phi \in C([0,T]; L^p(\Omega)) \), for any \( 2 \leq p < \infty \) if \( d = 2, 3 \). In addition, since \( \phi \in L^\infty(0,T; H^1(\Omega)) \), we also infer from [69, Theorem 2.1]
that $\phi \in C([0, T]; (H^1(\Omega))_w)$. If $d = 2$, since $\phi \in L^2(0, T; H^2(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$, we deduce that $\phi \in C([0, T]; H^1(\Omega))$. Next, the weak formulation of (4.1)$_1$(4.1)$_2$ can be written as

$$\frac{d}{dt} \langle \mathbb{P}(\rho(\phi)u), v \rangle = \langle \tilde{f}, v \rangle,$$

for all $v \in V_\sigma$, in the sense of distribution on $(0, T)$, where $\mathbb{P}$ is the Leray projection onto $H_\sigma$ and

$$\langle \tilde{f}, v \rangle = (\rho'(\phi)\partial_t \phi u, v) - (\rho(\phi)(u \cdot \nabla)u, v) - (\nu(\phi)Du, \nabla v) + (\nabla \phi \otimes \nabla \phi, \nabla v).$$

Arguing similarly to the proof of Lemma 4.5, we observe that

$$\|f\|_{V_\sigma'} \leq C\|\partial_t \phi\|_{L^2(\Omega)}\|u\|_{L^4(\Omega)} + C\|u\|_{L^4(\Omega)}\|Du\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)} + C\|\phi\|^2_{L^4(\Omega)}$$

$$\leq C\|\partial_t \phi\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)} + C\|\phi\|_{H^2(\Omega)}$$

$$\leq C(1 + \|\partial_t \phi\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)} + C\|\phi\|^2_{H^2(\Omega)}).$$

In light of the regularity of the weak solution, we infer that $\tilde{f} \in L^1(0, T; \mathcal{V}_\sigma')$. By definition of the weak time derivative, this implies that $\partial_t \mathbb{P}(\rho(\phi)u) \in L^1(0, T; \mathcal{V}_\sigma')$. Observing that $\mathbb{P}(\rho(\phi)u) \in L^\infty(0, T; H_\sigma)$, we have $\mathbb{P}(\rho(\phi)u) \in C([0, T]; \mathcal{V}_\sigma')$. As a consequence, we deduce from [69, Theorem 2.1] that $\mathbb{P}(\rho(\phi)u) \in C([0, T]; (H_\sigma)_w)$. It easily follows from the properties of the Leray operator $\mathbb{P}$ that $\mathbb{P}(\rho(\phi)u) \in C((0, T); (L^2(\Omega))_w)$. Now, repeating the argument in [3, Section 5.2], we deduce that $\rho(\phi)u \in C([0, T]; (L^2(\Omega))_w)$. Therefore, since $\rho(\phi) \in C([0, T]; L^2(\Omega))$ and $\rho(\phi) \geq \rho_\ast > 0$, we conclude that $u \in C([0, T]; (L^2(\Omega))_w)$. Finally, thanks to the time continuity of $u$ and $\phi$, a standard argument ensures that $u_{|t=0} = u_0$, $\phi_{|t=0} = \phi_0$ in $\Omega$. \hfill \Box

4.2. Proof of Theorem 4.3. Let us consider two global weak solutions $(u_1, \phi_1)$ and $(u_2, \phi_2)$ to problem (4.5)-(4.6) given by Theorem 4.2. Denote the differences of solutions by $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$. Then we have

$$\langle \partial_t u, v \rangle + (u \cdot \nabla u, v) + (u \cdot \nabla u_2, v) + (\nu(\phi_1)Du, \nabla v) + ((\nu(\phi_1) - \nu(\phi_2))Du_2, \nabla v)$$

$$= (\nabla \phi \otimes \nabla \phi, \nabla v) + (\nabla \phi \otimes \nabla \phi_2, \nabla v)$$

(4.25)

for all $v \in V_\sigma$, almost every $t \in (0, T)$, and

$$\partial_t \phi + u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2 - \Delta \phi + \Psi'(\phi_1) - \Psi'(\phi_2) = \Psi'(\phi_1) - \Psi'(\phi_2)$$

(4.26)

almost every $(x, t) \in \Omega \times (0, T)$. Following the same strategy as in [34], we take $v = A^{-1}u$, where $A$ is the Stokes operator, and we find

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + (\nu(\phi_1)Du, \nabla A^{-1}u) = (u \otimes u_1, \nabla A^{-1}u) + (u_2 \otimes u_1, \nabla A^{-1}u)$$

$$- ((\nu(\phi_1) - \nu(\phi_2))Du_2, \nabla A^{-1}u) + (\nabla \phi_1 \otimes \nabla \phi, \nabla A^{-1}u) + (\nabla \phi \otimes \nabla \phi_2, \nabla A^{-1}u),$$

where $\|u\| = \|\nabla^{-1}u\|_{L^2(\Omega)}$, which is a norm on $V_\sigma'$ equivalent to the usual dual norm. Here, we have used the equality $u_i \cdot \nabla u = \text{div} (u \otimes u_i)$, $i = 1, 2$, due to the incompressibility condition. Multiplying (4.26) by $\phi$, integrating over $\Omega$ and observing that

$$\int_{\Omega} (u_1 \cdot \nabla \phi) \phi \, dx = \int_{\Omega} u_1 \cdot \nabla \phi^2 \, dx = 0, \quad \int_{\Omega} (\Psi'(\phi_1) - \Psi'(\phi_2))\phi \, dx = (\Psi'(\phi_1) - \Psi'(\phi_2))\phi = 0,$$
we obtain
\[ \frac{1}{2} \frac{d}{dt} \| \phi \|^2_{L^2(\Omega)} + \| \nabla \phi \|^2_{L^2(\Omega)} + \int_\Omega (u \cdot \nabla \phi) \phi \, dx + \int_\Omega (F'(\phi_1) - F'(\phi_2)) \phi \, dx = \theta_0 \| \phi \|^2_{L^2(\Omega)}. \]

By adding the above two equations and using the convexity of $F$, we deduce that
\[
\frac{d}{dt} G(t) + (\nu(\phi_1) Du, \nabla A^{-1} u) + \| \nabla \phi \|^2_{L^2(\Omega)} \leq (u \otimes u_1, \nabla A^{-1} u) + (u_2 \otimes u, \nabla A^{-1} u) - ((\nu(\phi_1) - \nu(\phi_2)) Du_2, \nabla A^{-1} u) + (\nabla \phi_1 \otimes \nabla \phi, \nabla A^{-1} u) + (\nabla \phi \otimes \nabla \phi_2, \nabla A^{-1} u) - (u \cdot \nabla \phi_2, \phi) + \theta_0 \| \phi \|^2_{L^2(\Omega)}, \tag{4.27}
\]
where
\[
G(t) = \frac{1}{2} \| u(t) \|^2 + \frac{1}{2} \| \phi(t) \|^2_{L^2(\Omega)}. \]

In order to recover a $L^2(\Omega)$-norm of $u$, which is a key term to control the nonlinear terms on the right-hand side, we obtain by integration by parts (see [34, (3.9)])
\[
(\nu(\phi_1) Du, \nabla A^{-1} u) = (\nabla u, \nu(\phi_1) D A^{-1} u) = -(u, \div(\nu(\phi_1) D A^{-1} u)) = -(u, \nu'(\phi_1) D A^{-1} u \nabla \phi_1) - \frac{1}{2} (u, \nu(\phi_1) \Delta A^{-1} u).
\]

Here we have used that $\div \nabla^i v = \nabla \div v$. Notice that, by definition of Stokes operator, there exists a scalar function $p \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ (unique up to a constant) such that $-\Delta A^{-1} u + \nabla p = u$ for almost any $(x, t) \in \Omega \times (0, T)$. Moreover, we have the following estimates from [31] and [34, Appendix B]
\[
\| p \|_{L^2(\Omega)} \leq C \| \nabla A^{-1} u \|^2_{L^2(\Omega)} \| u \|^2_{L^2(\Omega)}, \| p \|_{H^1(\Omega)} \leq C \| u \|_{L^2(\Omega)}, \| p \|_{H^2(\Omega)} \leq C \| \nabla u \|_{L^2(\Omega)}. \tag{4.28}
\]

Then, we can write
\[
-\frac{1}{2} (u, \nu(\phi_1) \Delta A^{-1} u) = \frac{1}{2} (u, \nu(\phi_1) u) - \frac{1}{2} (u, \nu(\phi_1) \nabla p) = \frac{1}{2} (u, \nu(\phi_1) u) + \frac{1}{2} (\div(\nu(\phi_1) u), p) = \frac{1}{2} (u, \nu(\phi_1) u) + \frac{1}{2} (\nu'(\phi_1) \nabla \phi_1 \cdot u, p).
\]

Hence, recalling that $\nu(\cdot) \geq \nu_* > 0$, we find
\[
(\nu(\phi_1) Du, \nabla A^{-1} u) \geq \frac{\nu_*}{2} \| u \|^2_{L^2(\Omega)} + \frac{1}{2} (\nu'(\phi_1) \nabla \phi_1 \cdot u, p) - (u, \nu'(\phi_1) D A^{-1} u \nabla \phi_1).
\]

Owing to the above estimate, we rewrite (4.27) as follows
\[
\frac{d}{dt} G(t) + \frac{\nu_*}{2} \| u \|^2_{L^2(\Omega)} + \| \nabla \phi \|^2_{L^2(\Omega)} = (u \otimes u_1, \nabla A^{-1} u) + (u_2 \otimes u, \nabla A^{-1} u) + ((\nu(\phi_1) - \nu(\phi_2)) Du_2, \nabla A^{-1} u) + (\nabla \phi_1 \otimes \nabla \phi, \nabla A^{-1} u) + (\nabla \phi \otimes \nabla \phi_2, \nabla A^{-1} u) - (u \cdot \nabla \phi_2, \phi).
\]
Similarly, we obtain

\begin{align*}
(u \otimes u_1, \nabla A^{-1} u) + (u_2 \otimes u, \nabla A^{-1} u) & \\
\leq ||u||_{L^2(\Omega)} (||u_1||_{L^4(\Omega)} + ||u_2||_{L^4(\Omega)}) ||\nabla A^{-1} u||_{L^4(\Omega)} \\
& \leq C(||u_1||_{H^1(\Omega)} + ||u_2||_{H^1(\Omega)})||u||^2_{L^2(\Omega)} ||u||^\frac{1}{2} \\
& \leq \nu^* \frac{1}{20} ||u||^2_{L^2(\Omega)} + C(||u_1||^2_{H^2(\Omega)} + ||u_2||^2_{H^2(\Omega)})||u||^2_{L^2(\Omega)}.
\end{align*}

and

\begin{align*}
(u \cdot \nabla \phi_2, \phi) & \leq ||u||_{L^2(\Omega)} ||\nabla \phi_2||_{L^4(\Omega)} ||\phi||_{L^4(\Omega)} \\
& \leq C||u||_{L^2(\Omega)} ||\nabla \phi_2||_{H^1(\Omega)} ||\phi||_{L^2(\Omega)} ||\nabla \phi||_{L^2(\Omega)} \\
& \leq \nu^* \frac{1}{20} ||u||^2_{L^2(\Omega)} + \frac{1}{12} ||\nabla \phi||^2_{L^2(\Omega)} + C||\phi_2||^2_{H^2(\Omega)} ||\phi||^2_{L^2(\Omega)},
\end{align*}

where we have also used the inequality (2.1) and the conservation of mass \( \bar{\phi} = 0 \). Next, since \( \nu' \) is bounded, by exploiting (2.4) we have

\begin{align*}
(u, \nu'(\phi_1)DA^{-1}u\nabla \phi_1) & \leq C||u||_{L^2(\Omega)} ||DA^{-1}u||_{L^4(\Omega)} ||\nabla \phi||_{L^4(\Omega)} \\
& \leq C||u||^\frac{3}{2}_{L^2(\Omega)} ||u||^\frac{1}{2} ||\nabla \phi_1||^\frac{1}{2}_{H^1(\Omega)} \\
& \leq \nu^* \frac{1}{20} ||u||^2_{L^2(\Omega)} + C||\phi_1||^2_{H^2(\Omega)} ||u||^2_{L^2(\Omega)}.
\end{align*}

By using the Stokes operator (i.e. \( A = \mathbb{P}(-\Delta) \)) and the integration by parts, we infer that

\begin{align*}
-\frac{1}{2}(\nu'(\phi_1)\nabla \phi_1 \cdot u, p) = \frac{1}{2}(\Delta A^{-1}u, \mathbb{P}((\nu'(\phi_1)\nabla \phi_1)p)) \\
& = -\frac{1}{2} \int_{\Omega} (\nabla A^{-1}u)^t \cdot \nabla \mathbb{P}(\nu'(\phi_1)\nabla \phi_1 p) \, dx \\
& \quad + \frac{1}{2} \int_{\partial\Omega} ((\nabla A^{-1}u)^t \mathbb{P}(\nu'(\phi_1)\nabla \phi_1 p)) \cdot n \, d\sigma.
\end{align*}

Thanks to (2.3), (2.11), and the properties of the Leray projection, we find

\begin{align*}
-\frac{1}{2}(\nu'(\phi_1)\nabla \phi_1 \cdot u, p) & \leq C||\nabla A^{-1}u||_{L^2(\Omega)} ||\nabla \mathbb{P}(\nu'(\phi_1)\nabla \phi_1 p)||_{L^2(\Omega)} \\
& \quad + C||\nabla A^{-1}u||_{L^2(\partial \Omega)} ||\mathbb{P}(\nu'(\phi_1)\nabla \phi_1 p)||_{L^2(\partial \Omega)}
\end{align*}
\[ \begin{align*}
\leq C\|u\| + &\|\nu'(\phi_1)\nabla \phi_1 p\|_{H^1(\Omega)} + C\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|L^2(\Omega)\| \|p\|_{L^4(\Omega)} \\
\leq C\|u\| + &\|\nu'(\phi_1)\nabla \phi_1 p\|_{H^1(\Omega)} + C\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|L^2(\Omega)\| \|\nabla (\phi_1 + \nu')\|_{H^1(\Omega)}.
\end{align*} \tag{4.30} \]

Owing to (2.4), (2.9), Lemma 2.1 and (4.28), we observe that
\[ \|\nu'(\phi_1)\nabla \phi_1 p\|_{L^2(\Omega)} \leq C\|\nabla \phi_1\|_{L^2(\Omega)} \|p\|_{L^4(\Omega)} \]
\[ \leq C\|\phi_1\|_{H^2(\Omega)} \|p\|_{L^2(\Omega)} \]
\[ \leq C\|\phi_1\|_{H^2(\Omega)} \|\nabla \phi_1\|_{L^2(\Omega)} \]
\[ \leq C\|\phi_1\|_{H^2(\Omega)} \|p\|_{L^2(\Omega)} \log^2 \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right). \]

Combining the above estimates with (4.30), we are led to
\[ -\frac{1}{2} \nu'(\phi_1)\nabla \phi_1 \cdot u, p) \leq C \left( 1 + \|\phi_1\|_{H^2(\Omega)} \|u\|_{L^2(\Omega)} \log^2 \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \right) \]
\[ + C \left( 1 + \|\phi_1\|_{H^2(\Omega)} \|u\|_{L^2(\Omega)} \log^2 \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \right) \]
\[ \leq \frac{\nu_0}{20} \|u\|_{L^2(\Omega)} + C \left( 1 + \|\phi_1\|_{H^2(\Omega)} \|u\|_{L^2(\Omega)} \log \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \right) \]
\[ + C \left( 1 + \|\phi_1\|_{H^2(\Omega)} \|u\|_{L^2(\Omega)} \log \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \right). \]

In order to handle the logarithmic terms, we recall that \( C\|\nabla \phi_1\|_{L^2(\Omega)} > 1 \). Since \( C\|\nabla \phi_1\|_{L^2(\Omega)} > 1 \), for some \( C' > 0 \) depending on \( \Omega \), we have
\[ \log^\frac{4}{5} \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \leq 1 + \log \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \]
\[ \leq 1 + \log \left( C\|\nabla \phi_1\|_{L^2(\Omega)} \right) \]
\[ \leq 1 + \log \left( 1 + \|\nabla \phi_1\|_{L^2(\Omega)} \right) + \log \left( \frac{C}{\|\nabla \phi_1\|_{L^2(\Omega)}} \right), \]
where \( \tilde{C} > 0 \) is a sufficiently large constant such that \( \log \left( \frac{\tilde{C}}{\| \nu \|_\infty} \right) > 1 \), which holds true in light of (4.11). Thus, we obtain

\[
-\frac{1}{2} (\nu'(\phi_1) \nabla \phi_1 \cdot u, p) \leq \frac{\nu_s}{20} \| u \|_{L^2(\Omega)}^2 + C(1 + \| \phi_1 \|_{H^2(\Omega)}^{14}) \log (1 + \| \nabla u \|_{L^2(\Omega)}) \| u \|_*^2 \\
+ C(1 + \| \phi_1 \|_{H^2(\Omega)}^{14}) \| u \|_*^2 \log \left( \frac{\tilde{C}}{\| u \|_*} \right).
\]

Next, by using Lemma 2.1, we infer that

\[
-((\nu(\phi_1) - \nu(\phi_2)) Du_2, \nabla A^{-1} u) \\
= -\int_\Omega \int_0^1 \nu'(\tau \phi_1 + (1 - \tau) \phi_2) d\tau \phi Du_2 : \nabla A^{-1} u \, dx \\
\leq C \| Du_2 \|_{L^2(\Omega)} \| \phi \nabla A^{-1} u \|_{L^2(\Omega)} \\
\leq C \| u_2 \|_{H^1(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \| u \|_* \log^{\frac{1}{2}} \left( C \frac{\| u \|_{L^2(\Omega)}}{\| u \|_*} \right) \\
\leq \frac{1}{12} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| u_2 \|_{H^1(\Omega)}^2 \| u \|_*^2 \log \left( \frac{\tilde{C}}{\| u \|_*} \right),
\]

where \( \tilde{C} \) is chosen sufficiently large as above.

Summing up, we arrive at the differential inequality

\[
\frac{d}{dt} G(t) + \frac{\nu_s}{4} \| u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla \phi \|_{L^2(\Omega)}^2 \leq CS(t) G(t) \log \left( \frac{\tilde{C}}{G(t)} \right), 
\]

where

\[
S(t) = \left( 1 + \| u_1 \|_{H^1(\Omega)}^2 + \| u_2 \|_{H^1(\Omega)}^2 + \| \phi_2 \|_{H^2(\Omega)}^2 + \| \phi_1 \|_{H^2(\Omega)}^2 + \| \phi_1 \|_{H^2(\Omega)}^{14} (1 + \log (1 + \| \nabla u \|_{L^2(\Omega)}) \right).
\]

Here we have used that the function \( s \log \left( \frac{\tilde{C}}{\tau} \right) \) is increasing on \( (0, \frac{\tilde{C}}{\tau}) \). We observe that \( S \in L^1(0, T) \) provided that \( \phi_1 \in L^{7}(0, T; H^2(\Omega)) \) with \( \gamma > \frac{12}{5} \). Indeed, we recall that \( \log(1 + s) \leq C(\kappa)(1 + s)^\kappa \), for any \( \kappa > 0 \). Taking \( \kappa = \frac{2(5\gamma - 12)}{5\gamma} \), we have

\[
\int_0^T \| \phi_1(\tau) \|_{H^2(\Omega)}^{12} \log \left( 1 + \| \nabla u(\tau) \|_{L^2(\Omega)} \right) \, d\tau \\
\leq C \int_0^T \| \phi_1(\tau) \|_{H^2(\Omega)}^{12} \left( 1 + \| \nabla u(\tau) \|_{L^2(\Omega)} \right)^{2(5\gamma - 12)} \, d\tau \\
\leq C \int_0^T \| \phi_1(\tau) \|_{H^2(\Omega)}^\gamma + \| \nabla u_1(\tau) \|_{L^2(\Omega)}^2 + \| \nabla u_2(\tau) \|_{L^2(\Omega)}^2 \, d\tau.
\]

Throughout the rest of the proof, we will assume that \( S \in L^1(0, T) \).

Integrating (4.31) on the time interval \([0, t] \), we find

\[
G(t) \leq G(0) + C \int_0^t S(s) G(s) \log \left( \frac{\tilde{C}}{G(s)} \right) \, ds,
\]
for almost every $t \in [0, T]$. We observe that $\int_0^1 \frac{1}{s \log(\frac{1}{s})} \, ds = \infty$. Thus, if $G(0) = 0$, applying the Osgood lemma B.1, we deduce that $G(t) = 0$ for all $t \in [0, T]$, namely $u_1(t) = u_2(t)$ and $\phi_1(t) = \phi_2(t)$. This demonstrates the uniqueness of solutions in the class of weak solutions satisfying the additional regularity $\phi_1 \in L^\gamma(0, T; H^2(\Omega))$ with $\gamma > \frac{12}{5}$. Indeed, we are able to deduce a continuous dependence estimate with respect to the initial datum. To this end, we define $\mathcal{M}(s) = \log(\log(\frac{C}{s}))$. By the Osgood lemma, for $G(0) > 0$, we are led to

$$- \log\left( \log \left( \frac{C}{G(t)} \right) \right) + \log \left( \log \left( \frac{C}{G(0)} \right) \right) \leq C \int_0^t S(s) \, ds$$

(4.32) for almost every $t \in [0, T]$. Taking the double exponential of (4.32), we eventually infer the control

$$G(t) \leq \tilde{C} \left( \frac{C}{G(0)} \right)^{\exp(-C \int_0^t S(s) \, ds)} \quad \forall \ t \in [0, T],$$

(4.33) where $T_0 > 0$ is defined by

$$\log \left( \log \left( \frac{C}{G(0)} \right) \right) \geq C \int_0^{T_0} S(s) \, ds.$$

The proof is complete. \hfill \Box

**Remark 4.6.** We note that the same existence result as in Theorem 4.2 holds for $\Omega = \mathbb{T}^d$, $d = 2, 3$. In the particular case $\Omega = \mathbb{T}^2$, the uniqueness of weak solutions can be achieved, without the additional regularity $\phi \in L^\gamma(0, T; H^2(\Omega))$ as in Theorem 4.3. Indeed, in this case the solutions of the Stokes operator $A^{-1} u$ and $p$ are given by (see [73, Chapter 2.2])

$$A^{-1} u = \sum_{k \in \mathbb{Z}^2} g_k e^{2 \pi i k \cdot \frac{x}{L}}, \quad p = \sum_{k \in \mathbb{Z}^2} p_k e^{2 \pi i k \cdot \frac{x}{L}},$$

where

$$g_k = - \frac{L^2}{4 \pi^2 |k|^2} \left( u_k - \frac{(k \cdot u_k) k}{|k|^2} \right), \quad p_k = \frac{L k \cdot u_k}{2 i \pi |k|^2}, \quad k \in \mathbb{Z}^2, \ k \neq 0,$$

$L > 0$ is the cell size. Here $u_k$ is the $k$-mode of $u$. We observe that we only consider the case $k \neq 0$ since $\overline{u}$ is conserved for (4.5) on $\mathbb{T}^2$, and so we can choose $\overline{u} = 0$. Moreover, since $u \in H_\sigma$, we have that $k \cdot u_k = 0$ for any $k \in \mathbb{Z}^2$, which implies that $p_k = 0$ for any $k \in \mathbb{Z}^2$. Thus, following the above proof, we are led to the differential inequality (4.29) without the last term on the right-hand side, i.e.

$$-\frac{1}{2} \nu' \nabla \phi \cdot \nabla u + p.$$ Hence, we eventually end up with

$$\frac{d}{dt} G(t) + \frac{\nu}{4} \| u \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \phi \|^2_{L^2(\Omega)} \leq C \tilde{S}(t) G(t) \log \left( \frac{C}{G(t)} \right),$$

where

$$\tilde{S}(t) = \left( 1 + \| u_1 \|^2_{H^1(\Omega)} + \| u_2 \|^2_{H^1(\Omega)} + \| \phi_1 \|^2_{H^2(\Omega)} + \| \phi_2 \|^2_{H^2(\Omega)} \right).$$

Since $\tilde{S}(t) \in L^1(0, T)$ for any couple of weak solutions, an application of the Osgood lemma as above entails the uniqueness of weak solutions (without additional regularity) and a continuous dependence estimate with respect to the initial data, i.e. (4.33).
5. Mass-conserving Navier-Stokes-Allen-Cahn System: Strong Solutions

This section is devoted to the analysis of global strong solutions to the nonhomogeneous Navier-Stokes-Allen-Cahn system (4.1)-(4.3) in two dimensions. The main results are as follows.

**Theorem 5.1** (Global strong solution in 2D). Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^2 \). Assume that \( u_0 \in V_\sigma, \phi_0 \in H^2(\Omega) \) such that \( \partial_n \phi_0 = 0 \) on \( \partial \Omega \), \( F'(\phi_0) \in L^2(\Omega) \), \( \|\phi_0\|_{L^\infty(\Omega)} \leq 1 \) and \( |\phi_0| < 1 \).

1. There exists a global strong solution \((u, \phi)\) to problem (4.1)-(4.3) satisfying, for all \( T > 0 \),
   \[
   u \in L^\infty(0, T; V_\sigma) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; H_\sigma),
   \]
   \[
   \phi \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,p}(\Omega)),
   \]
   \[
   \partial_t \phi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
   \]
   \[
   F'(\phi) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^p(\Omega))
   \]
   where \( p \in (2, \infty) \). The strong solution satisfies the system (4.1) almost everywhere in \( \Omega \times (0, \infty) \). Besides, \( |\phi(x, t)| < 1 \) for almost any \((x, t) \in \Omega \times (0, \infty)\) and \( \partial_n \phi = 0 \) on \( \partial \Omega \times (0, \infty) \).

2. There exists \( \eta_1 > 0 \) depending only on the norms of the initial data and on the parameters of the system:
   \[
   \eta_1 = \eta_1(E(u_0, \phi_0), \|u_0\|_{V_\sigma}, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}, \theta, \theta_0).
   \]
   If, in addition, \( \|\rho'\|_{L^\infty(-1,1)} \leq \eta_1 \) and \( F''(\phi_0) \in L^1(\Omega) \), then, for any \( T > 0 \), we have
   \[
   F''(\phi) \in L^\infty(0, T; L^1(\Omega)), \quad F'' \in L^q(0, T; L^p(\Omega)),
   \]
   where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p \in (1, \infty) \), and
   \[
   (F''(\phi))^2 \log(1 + F''(\phi)) \in L^1(\Omega \times (0, T)).
   \]
   In particular, the strong solution satisfying (5.2) is unique.

**Theorem 5.2** (Propagation of regularity in 2D). Let the assumptions in Theorem 5.1-(1) be satisfied. Assume that \( \|\rho'\|_{L^\infty(-1,1)} \leq \eta_1 \). Given a strong solution from Theorem 5.1-(1), for any \( \sigma > 0 \), there holds
   \[
   (F''(\phi))^2 \log(1 + F''(\phi)) \in L^1(\Omega \times (\sigma, T)),
   \]
   and
   \[
   \partial_t u \in L^\infty(\sigma, T; H_\sigma) \cap L^2(\sigma, T; V_\sigma), \quad \partial_t \phi \in L^\infty(\sigma, T; H^1(\Omega)) \cap L^2(\sigma, T; H^2(\Omega)).
   \]
   In addition, for any \( \sigma > 0 \), there exists \( \delta = \delta(\sigma) > 0 \) such that
   \[
   -1 + \delta \leq \phi(x, t) \leq 1 - \delta, \quad \forall x \in \overline{\Omega}, \ t \geq \sigma.
   \]

**Remark 5.3.** The smallness assumption on \( \rho' \) (see (5.38) below for the explicit form) can be reformulated in terms of the difference of the (constant) densities of the two fluids when \( \rho \) is a linear interpolation function. In this case, we have
   \[
   \rho(s) = \rho_1 \frac{1 + s}{2} + \rho_2 \frac{1 - s}{2}, \quad \rho'(s) = \frac{\rho_1 - \rho_2}{2}, \quad \forall s \in [-1, 1].
   \]
Roughly speaking, the results given by Theorem 5.1 and Theorem 5.2 imply that uniqueness and further regularity of strong solutions to the nonhomogeneous system hold provided that the two fluids have similar densities ($\rho_1 \approx \rho_2$).

**Remark 5.4 (Matched densities).** It is worth noticing that Theorem 5.1 and Theorem 5.2 hold true in the case of constant density $\rho \equiv 1$ (i.e. $\rho_1 = \rho_2$) without any smallness assumption.

### 5.1. Proof of Theorem 5.1

We perform higher-order *a priori* estimates that are necessary for the existence of global strong solutions.

**Higher-order estimates.** Multiplying (4.1) by $\partial_t \phi$, integrating over $\Omega$, and observing that

$$
\int_{\Omega} \nu(\phi) Du \cdot D\partial_t u \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu(\phi) |Du|^2 \, dx - \frac{1}{2} \int_{\Omega} \nu'(\phi) \partial_t \phi |Du|^2 \, dx,
$$

we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu(\phi) |Du|^2 \, dx + \int_{\Omega} \rho(\phi) |\partial_t u|^2 \, dx
= -\left(\rho(\phi)u \cdot \nabla u, \partial_t u\right) - \int_{\Omega} \Delta \phi \nabla \phi \cdot \partial_t u \, dx + \frac{1}{2} \int_{\Omega} \nu'(\phi) \partial_t \phi |Du|^2 \, dx.
$$

Next, differentiating (4.1)$_3$ in time, multiplying the resultant by $\partial_t \phi$ and integrating over $\Omega$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\partial_t \phi\|_{L^2(\Omega)}^2 + \int_{\Omega} \partial_t u \cdot \nabla \phi \partial_t \phi \, dx + \|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 + \int_{\Omega} F''(\phi) |\partial_t \phi|^2 \, dx
\leq -\left(\rho(\phi)u \cdot \nabla u, \partial_t u\right) - \int_{\Omega} \Delta \phi \nabla \phi \cdot \partial_t u \, dx + \frac{1}{2} \int_{\Omega} \nu'(\phi) \partial_t \phi |Du|^2 \, dx + \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2
\leq \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2 - \int_{\Omega} \Delta \phi \nabla \phi \cdot \partial_t u \, dx + \frac{1}{2} \int_{\Omega} \nu'(\phi) \partial_t \phi |Du|^2 \, dx + \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2,
$$

where

$$
H(t) = \frac{1}{2} \int_{\Omega} \nu(\phi) |Du|^2 \, dx + \frac{1}{2} \|\partial_t \phi\|_{L^2(\Omega)}^2.
$$

In (5.5), we have used that $\rho$ is strictly positive ($\rho(s) \geq \rho_* > 0$). In addition, we simply infer from (4.1) that

$$
\|\partial_t \phi\|_{L^2(\Omega)} \leq C \left(1 + \|u\|_{H^1(\Omega)}\right) \|\phi\|_{H^2(\Omega)} + C \|F'(\phi)\|_{L^2(\Omega)} + C \|u\|_{H^1(\Omega)}^2.
$$

Therefore, it follows from the assumptions on the initial data that $H(0) < +\infty$.

We proceed to estimate the right-hand side of (5.5). By using (2.2) and (2.10), we have

$$
-(\rho(\phi)u \cdot \nabla u, \partial_t u) \leq \|\rho(\phi)\|_{L^\infty(\Omega)} \|u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\partial_t u\|_{L^2(\Omega)}
\leq C \|Du\|_{L^2(\Omega)}^2 \log^2 \left(C \frac{\|u\|_{W^{1,2}(\Omega)}}{\|Du\|_{L^2(\Omega)}} \right) \|\partial_t u\|_{L^2(\Omega)}^2.
$$
2.1

\[ \frac{\partial}{\partial x} + 2.1 \]

for some \( p > 2 \). Moreover, it holds

\[
- \int_\Omega \Delta \phi \nabla \phi \cdot \partial_t u \, dx \leq \| \Delta \phi \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \| \partial_t u \|_{L^2(\Omega)}
\]

\[
\leq C \| \Delta \phi \|_{L^2(\Omega)} \| \nabla \phi \|_{H^1(\Omega)} \| \partial_t u \|_{L^2(\Omega)}
\]

\[
\leq C \| \Delta \phi \|_{L^2(\Omega)} \| \nabla \phi \|_{H^1(\Omega)} \|
\]

Next, by exploiting Lemma 2.1, together with (2.1) and \( \overline{\partial_t \phi} = 0 \), we obtain

\[
\frac{1}{2} \int_\Omega \nu' \phi \partial_t \phi D u^2 \, dx \leq \| \nu' \phi \|_{L^\infty(\Omega)} \| \partial_t \phi \|_{L^2(\Omega)} \| D u \|_{L^2(\Omega)}
\]

\[
\leq C \| \nabla \partial_t \phi \|_{L^2(\Omega)} \| D u \|_{L^2(\Omega)}
\]

\[
\leq \frac{1}{8} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + C \| D u \|_{L^2(\Omega)}^2 \log \left( C \| D u \|_{L^2(\Omega)} \right). 
\]

It remains to control the last three terms on the right-hand side of (5.5). By using (2.4) and (4.11), we obtain

\[- \int_\Omega \partial_t u \cdot \nabla \phi \partial_t \phi \, dx \leq \| \partial_t u \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \| \partial_t \phi \|_{L^2(\Omega)}
\]

\[
\leq \frac{\rho_u}{8} \| \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{8} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \phi \|_{L^2(\Omega)} \| \partial_t \phi \|_{L^2(\Omega)}^2,
\]

\[- \int_\Omega \rho''(\phi) \| \partial_t \phi \|_2^2 \frac{\| u \|_2^2}{2} \, dx \leq C \| \rho''(\phi) \|_{L^\infty(\Omega)} \| \partial_t \phi \|_{L^2(\Omega)}^2 \| u \|_{L^2(\Omega)}^2
\]

\[
\leq \frac{1}{8} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \partial_t \phi \|_{L^2(\Omega)}^2 \| D u \|_{L^2(\Omega)}^2,
\]

and

\[- \int \rho'(\phi) \partial_t u \cdot \partial_t u \partial_t \phi \, dx \leq \| \rho'(\phi) \|_{L^\infty(\Omega)} \| u \|_{L^2(\Omega)} \| \partial_t u \|_{L^2(\Omega)} \| \partial_t \phi \|_{L^2(\Omega)}
\]

\[
\leq \frac{\rho_u}{8} \| \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{8} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \partial_t \phi \|_{L^2(\Omega)}^2 \| D u \|_{L^2(\Omega)}.
\]

Combining (5.5) and the above inequalities, we deduce that

\[
\frac{dH(t)}{dt} + \frac{\rho_u}{2} \| \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2
\]
\[ \leq C \|
abla \phi \|_{L^2(\Omega)}^2 + C \left( \| Du \|_{L^2(\Omega)}^2 + \| \phi \|_{H^2(\Omega)}^2 \right) \|
abla \phi \|_{L^2(\Omega)}^2 + C \| Du \|_{L^2(\Omega)}^4 \log \left( \frac{C \| u \|_{W^{1,p}(\Omega)}}{\| Du \|_{L^2(\Omega)}} \right) + C \| \nabla \phi \|_{L^2(\Omega)}^2 \| \nabla \phi \|_{H^1(\Omega)}^2 \log \left( \frac{C \| \nabla \phi \|_{W^{1,p}(\Omega)}}{\| \nabla \phi \|_{H^1(\Omega)}} \right). \]

From the inequalities
\[ x^2 \log \left( \frac{Cy}{x} \right) \leq x^2 \log (Cy) + 1, \quad \forall \ x, \ y > 0, \tag{5.7} \]
\[ \frac{\nu_*}{2} \| Du \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla \phi \|_{H^1(\Omega)}^2 \leq H(t) \leq C \left( \| Du \|_{L^2(\Omega)}^2 + \| \nabla \phi \|_{L^2(\Omega)}^2 \right), \tag{5.8} \]
and the estimate (4.19), we can rewrite the above differential inequality as follows
\[ \frac{d}{dt} H(t) + \frac{\nu_*}{2} \| Du \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla \phi \|_{L^2(\Omega)}^2 \leq C \left( 1 + H(t) + H^2(t) \right) + CH^2(t) \log \left( C \| \phi \|_{W^{1,p}(\Omega)} \right) + C \left( 1 + H^2(t) \right) \log \left( C \| \phi \|_{W^{1,p}(\Omega)} \right). \tag{5.9} \]

Let us now estimate the argument of the logarithmic terms on the right-hand side of (5.9). First, we rewrite (4.1) as a Stokes problem with non-constant viscosity
\[
\begin{align*}
\begin{cases}
-\text{div} \left( \nu(\phi) Du \right) + \nabla P = f, & \text{in } \Omega \times (0, T), \\
\text{div} u = 0, & \text{in } \Omega \times (0, T), \\
u_0 = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\end{align*}
\]
where \( f = -\rho(\phi) \left( \partial_t u + u \cdot \nabla u \right) - \Delta \phi \nabla \phi \). We now apply Theorem A.1 with the following choice of parameters \( p = 1 + \varepsilon, \varepsilon \in (0, 1) \), and \( r \in (2, \infty) \) such that \( \frac{1}{p} = \frac{1}{1+\varepsilon} - \frac{1}{2} \). We infer that
\[
\| u \|_{W^{2,1+\varepsilon}(\Omega)} \leq C \left( \| \partial_t u \|_{L^{1+\varepsilon}(\Omega)} + \| u \cdot \nabla u \|_{L^{1+\varepsilon}(\Omega)} + \| \Delta \phi \nabla \phi \|_{L^{1+\varepsilon}(\Omega)} \right) + C \| Du \|_{L^2(\Omega)} \| \nabla \phi \|_{L^r(\Omega)}
\leq C \left( \| \partial_t u \|_{L^2(\Omega)} + \| u \|_{L^{2(1+\varepsilon)}(\Omega)} \| \nabla u \|_{L^2(\Omega)} + \| \nabla \phi \|_{L^{2(1+\varepsilon)}(\Omega)} \| \Delta \phi \|_{L^2(\Omega)} \right)
\]
\[ + C \| Du \|_{L^2(\Omega)} \| \phi \|_{H^2(\Omega)} \leq C \| \partial_t u \|_{L^2(\Omega)} + C \| Du \|_{L^2(\Omega)}^2 + \| \phi \|_{H^2(\Omega)}^2 \]
\[ \leq C \| \partial_t u \|_{L^2(\Omega)} + C \| \nabla \phi \|_{L^2(\Omega)}^2 \| \phi \|_{H^2(\Omega)}.
\]
where the constant \( C \) depends on \( \varepsilon \). We recall the Sobolev embedding \( W^{2,1+\varepsilon}(\Omega) \hookrightarrow W^{1,p}(\Omega) \) where \( \frac{1}{p} = \frac{1}{1+\varepsilon} - \frac{1}{2} \). Therefore, for any \( p \in (2, \infty) \) there exists a constant \( C \) depending on \( p \) such that
\[
\| u \|_{W^{1,p}(\Omega)} \leq C \| \partial_t u \|_{L^2(\Omega)} + C \left( 1 + H(t) \right). \tag{5.10}
\]

Next, by reformulating the equation (4.1) as the elliptic problem
\[
\begin{align*}
\begin{cases}
-\Delta \phi + F'(\phi) = \mu + \theta_0 \phi, & \text{in } \Omega \times (0, T), \\
\partial_n \phi = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\end{align*}
\]
We infer from the elliptic regularity theory (see, e.g., [2, Lemma 2] and [34]) that
\[
\| \phi \|_{W^{2,p}(\Omega)} + \| F'(\phi) \|_{L^p(\Omega)} \leq C \left( 1 + \| \phi \|_{L^2(\Omega)} + \| \mu + \theta_0 \phi \|_{L^p(\Omega)} \right)
\]
\[ \leq C(1 + \|\phi\|_{L^p(\Omega)} + \|\mu\|_{L^p(\Omega)}), \quad (5.12) \]

for \( p \in (2, \infty) \). On the other hand, from the equation (4.1)3, we see that

\[ \mu = -\partial_t \phi - u \cdot \nabla \phi - \rho'(\phi) \frac{|u|^2}{2} + \mu + \rho'(\phi) \frac{|u|^2}{2}. \]

Observe now that

\[ \left\| - \rho'(\phi) \frac{|u|^2}{2} \right\|_{L^p(\Omega)} \leq C\|u\|_{L^{2p}(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \]

Then, owing to Sobolev embedding and (2.1), we have

\[ \|\mu - \bar{\mu}\|_{L^p(\Omega)} \leq \|\partial_t \phi\|_{L^p(\Omega)} + \|u \cdot \nabla \phi\|_{L^p(\Omega)} + \left\| \rho'(\phi) \frac{|u|^2}{2} - \rho'(\phi) \frac{\rho|u|^2}{2} \right\|_{L^p(\Omega)} \]

\[ \leq C\|\nabla \partial_t \phi\|_{L^2(\Omega)} + C\|u\|_{H^s(\Omega)} \|\phi\|_{H^s(\Omega)} + C\|\nabla u\|_{L^2(\Omega)}. \]

In light of (4.17) and (4.21), the above inequality yields

\[ \|\mu\|_{L^p(\Omega)} \leq C\|\mu - \bar{\mu}\|_{L^p(\Omega)} + C|\bar{\mu}| \]

\[ \leq C\|\mu - \bar{\mu}\|_{L^p(\Omega)} + C(1 + \|\mu - \bar{\mu}\|_{L^2(\Omega)}) \]

\[ \leq C(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t)). \quad (5.13) \]

Thus, for any \( p > 2 \), we deduce from the above estimate and (5.12) that

\[ \|\phi\|_{W^{2,p}(\Omega)} \leq C(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t)), \quad (5.14) \]

for some positive constant \( C \) depending on \( p \).

We recall the generalized Young inequality

\[ xy \leq \Phi(x) + \Upsilon(y), \quad \forall \ x, \ y > 0, \quad (5.15) \]

where

\[ \Phi(s) = s \log s - s + 1, \quad \Upsilon(s) = e^s - 1. \]

Then we have

\[ H(t) \log(1 + \|\partial_t u\|_{L^2(\Omega)}) \leq H(t) \log H(t) + 1 + \|\partial_t u\|_{L^2(\Omega)}. \]

Thus, using the above estimate and the elementary inequality

\[ \log(x + y) < \log(1 + x) + \log(1 + y), \quad x, \ y > 0, \]

we can estimate the second term on the right-hand side of (5.9) as follows

\[ \begin{align*}
    CH^2(\log(C\|u\|_{W^{1,p}(\Omega)}) & \leq CH^2(t) \log(C\|\partial_t u\|_{L^2(\Omega)} + C(1 + H(t))) \\
    & \leq CH^2(t)(1 + \log(1 + \|\partial_t u\|_{L^2(\Omega)} + \log(1 + H(t))) \\
    & \leq CH^2(t) + CH(t)(H(t) \log H(t) + 1) + CH(t)\|\partial_t u\|_{L^2(\Omega)} + CH^2(t) \log(1 + H(t)) \\
    & \leq \frac{D_\mu}{4}\|\partial_t u\|_{L^2(\Omega)}^2 + C(1 + H^2(t)) + CH^2(t) \log(e + H(t)).
\end{align*} \]

(5.16)

In a similar manner, we have

\[ H(t) \log(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)}) \leq H(t) \log H(t) + 1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)}. \]
Then, using (5.14), the third term on the right-hand side of (5.9) can be estimated as follows
\[
C(1 + H^2(t)) \log(C\|\phi\|_{W^{2,p}(\Omega)}) \leq C(1 + H^2(t)) \log(C(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t))) \\
\leq C(1 + H^2(t)) \log(C(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t)) + H^2(t) \log(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t))) \\
\leq C(1 + H^2(t)) + C(1 + \|\nabla \partial_t \phi\|_{L^2(\Omega)} + H(t)) + C\|\nabla \partial_t \phi\|_{L^2(\Omega)} H(t) \\
+ H^2(t) \log(1 + H(t)) \\
\leq \frac{1}{8}\|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 + C(1 + H^2(t)) + CH(t)(e + H(t)) \log(e + H(t)). \tag{5.17}
\]
Hence, by (5.16) and (5.17), we easily deduce from (5.9) that
\[
\frac{\partial}{\partial t}(e + H(t)) + \rho_s\|\partial_t u\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 \leq C + CH(t)(e + H(t)) \log(e + H(t)). \tag{5.18}
\]
Thanks to (4.10), (4.19), and (5.8), we obtain
\[
\int_t^{t+1} H(\tau) \ d\tau \leq Q(E_0), \quad \forall t \geq 0, \tag{5.19}
\]
where \(Q\) is independent of \(t\), and \(E_0 = E(u_0, \phi_0)\). We now apply the generalized Gronwall lemma B.2 to (5.18) and find the estimate
\[
\sup_{t \in [0, 1]} H(t) \leq C(e + H(0)) e^{Q(E_0)}. \tag{5.20}
\]
Moreover, by using the generalized uniform Gronwall lemma B.3 together with (5.19), we infer that
\[
\sup_{t \geq 1} H(t) \leq C e^{(e + Q(E_0)) e^{(t + Q(E_0))}}. \tag{5.21}
\]
By combining the above inequalities, we get
\[
\sup_{t \geq 0} H(t) \leq Q(E_0, \|u_0\|_{V_\sigma}, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}). \tag{5.22}
\]
In addition, integrating (5.18) on the time interval \([t, t + 1]\), we have, for all \(t \geq 0\),
\[
\int_t^{t+1} \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 + \|\nabla \partial_t \phi(\tau)\|_{L^2(\Omega)}^2 \ d\tau \leq Q(E_0, \|u_0\|_{V_\sigma}, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}). \tag{5.23}
\]
Then we can deduce that
\[
u \in L^\infty(0, T; V_\sigma) \cap H^1(0, T; H_\sigma) \quad \partial_t \phi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \tag{5.24}
\]
Thanks to (4.19) and (5.14), we also get,
\[
\sup_{t \geq 0} \|\phi(t)\|_{H^2(\Omega)} \leq Q(E_0, \|u_0\|_{V_\sigma}, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}), \tag{5.25}
\]
and, for all \(t \geq 0\),
\[
\int_t^{t+1} \|\phi(\tau)\|_{W^{2,p}(\Omega)}^2 \ d\tau \leq Q(E_0, \|u_0\|_{V_\sigma}, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}), \tag{5.26}
\]
for any \( p \in (2, \infty) \). This entails that \( \phi \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,p}(\Omega)) \). According to (4.17), (4.21) and (5.13), it follows that \( \mu \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^p(\Omega)) \) and, as a consequence, \( F'(\phi) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^p(\Omega)) \).

Finally, by exploiting Theorem A.1 with \( p = 2 \) and \( r = \infty \), together with the regularity of \( \phi \) obtained above, we have, for all \( t \geq 0 \),

\[
\int_t^{t+1} \| u(\tau) \|^2_{H^2(\Omega)} \, d\tau \leq Q(E_0, \| u_0 \|_v, \| \phi_0 \|_{H^2(\Omega)}, \| F'(\phi_0) \|_{L^2(\Omega)}),
\]

which yields that \( u \in L^2(0, T; H^2(\Omega)) \).

**Entropy bound in** \( L^\infty(0, T; L^1(\Omega)) \). First of all, we observe that, for all \( s \in (-1, 1) \),

\[
F'(s) = \frac{\theta}{2} \log \left( \frac{1 + s}{1 - s} \right), \quad F''(s) = \frac{\theta}{1 - s^2}, \quad F'''(s) = \frac{2\theta s}{(1 - s)^2(1 + s)^2}
\]

and

\[
F''(s) = \frac{2\theta(1 + 3s^2)}{(1 - s)^2(1 + s)^3} > 0.
\]

Next, we compute

\[
\frac{d}{dt} \int_\Omega F''(\phi) \, dx = \int_\Omega F'''(\phi) \partial_t \phi \, dx
\]

\[
= \int_\Omega F'''(\phi) \left( \Delta \phi - u \cdot \nabla \phi - F'(\phi) + \theta_0 \phi - \rho'(\phi) \frac{|u|^2}{2} + \xi \right) \, dx.
\]

Since

\[
\int_\Omega F'''(\phi) u \cdot \nabla \phi \, dx = \int_\Omega u \cdot \nabla(F'''(\phi)) \, dx = 0,
\]

and exploiting the integration by parts, we rewrite the above equality as follows

\[
\frac{d}{dt} \left( \int_\Omega F''(\phi) \, dx + \int_\Omega F'''(\phi) \, dx \right) = \int_\Omega F'''(\phi) \left( \theta_0 \phi - \rho'(\phi) \frac{|u|^2}{2} + \xi \right) \, dx.
\]

In particular, by using (5.27), we have

\[
\frac{d}{dt} \int_\Omega F''(\phi) \, dx + \int_\Omega F'''(\phi) F'(\phi) \, dx \leq \int_\Omega F'''(\phi) \left( \theta_0 \phi - \rho'(\phi) \frac{|u|^2}{2} + \xi \right) \, dx.
\]

It follows from (5.15) that

\[
xy \leq \varepsilon x \log x + e^\frac{xy}{\varepsilon}, \quad \forall x > 0, y > 0, \varepsilon \in (0, 1).
\]

which implies

\[
\int_\Omega -F''(\phi) \rho'(\phi) \frac{|u|^2}{2} \, dx \leq \int_\Omega |F'''(\phi)||\rho'(\phi)| \frac{|u|^2}{2} \, dx
\]

\[
\leq \varepsilon \int_\Omega |F'''(\phi)| \log(|F'''(\phi)|) \, dx + \int_\Omega e^{\frac{|u'|}{e}} \frac{|u|^2}{2} \, dx.
\]
We observe that, for all $s \in [0, 1)$, it holds
\[
\log(|F''(s)|) = \log(F''(s)) = \log\left(\frac{2\theta s}{(1-s)^2(1+s)^2}\right)
\]
\[
= 2 \log\left(\frac{1 + s \sqrt{2\theta s}}{1 - s (1+s)^2}\right) \leq 2 \log\left(\frac{\sqrt{2\theta} (1+s)}{1-s}\right) = \log(2\theta) + \frac{4}{\theta} F'(s).
\]
Since both $F'(s)$ and $F''(s)$ are odd, we easily deduce that
\[
\log(|F''(s)|) \leq C_0 + \frac{4}{\theta} |F'(s)|, \quad \forall s \in (-1, 1),
\]
where $C_0 = \log(2\theta)$ (without loss of generality, we assume in the sequel that $C_0 > 0$). Then, using the fact that $F''(s)F'(s) \geq 0$ for all $s \in (-1, 1)$, we obtain
\[
|F''(s)||\log(|F''(s)|)| \leq C_0 |F''(s)| + \frac{4}{\theta} F''(s)F'(s), \quad \forall s \in (-1, 1).
\]
Fix the constant $\alpha \in (0, 1)$ such that $F'(\alpha) = 1$. We infer that
\[
|F''(s)||\log(|F''(s)|)| \leq C_1 + C_2 F''(s)F'(s), \quad \forall s \in (-1, 1).
\]
where
\[
C_1 = C_0 F''(\alpha), \quad C_2 = \frac{4}{\theta} + C_0.
\]
Taking $\varepsilon = \frac{1}{2C_2}$ in (5.31), we arrive at
\[
\int_{\Omega} -F'''(\phi)\rho'(|\phi|)\frac{|u|^2}{2} \, dx \leq \frac{C_1|\Omega|}{2C_2} + \frac{1}{2} \int_{\Omega} F'''(\phi)F'(\phi) \, dx + \int_{\Omega} e^{C_2|\rho(\phi)||u|^2} \, dx.
\]
Arguing in a similar way ($\varepsilon = \frac{1}{4C_2}$), we obtain
\[
\int_{\Omega} F'''(\phi) (\theta_0|\phi| + \xi) \, dx \leq \frac{C_1|\Omega|}{4C_2} + \frac{1}{4} \int_{\Omega} F'''(\phi)F'(\phi) \, dx + \int_{\Omega} e^{4C_2|\theta_0| + \xi} \, dx.
\]
Since $\phi$ is globally bounded ($\|\phi\|_{L^\infty(\Omega \times (0,T))} \leq 1$) and $\|\xi\|_{L^\infty(0,T)} \leq C_2$, we get
\[
\int_{\Omega} F'''(\phi) (\theta_0 + \xi) \phi \, dx \leq \frac{1}{4} \int_{\Omega} F'''(\phi)F'(\phi) \, dx + \frac{C_1|\Omega|}{4C_2} + e^{4C_2(\theta_0 + C_2)}|\Omega|.
\]
Combining (5.29) with (5.33) and (5.34), we deduce that
\[
\frac{1}{d} \int_{\Omega} F''(\phi) \, dx + \frac{1}{4} \int_{\Omega} F'''(\phi)F'(\phi) \, dx
\]
\[
\leq \frac{3C_1|\Omega|}{4C_2} + e^{4C_2(\theta_0 + C_2)}|\Omega| + \int_{\Omega} e^{C_2|\rho(\phi)||\nabla u|^2_{L^2(\Omega)}} \left(\frac{|u|^2}{\nabla u|^2_{L^2(\Omega)}}\right) \, dx.
\]
In order to control the last term on the right-hand side of (5.35), we shall use the Trudinger-Moser inequality (see, e.g., [59]). Namely, let $f \in H_0^1(\Omega)$ ($d = 2$) such that $\int_{\Omega} |\nabla f|^2 \, dx \leq 1$. Then, there exists a constant $C_{TM} = C_{TM}(\Omega)$ (which depends only on the domain $\Omega$) such that
\[
\int_{\Omega} e^{4\pi|f|^2} \, dx \leq C_{TM}(\Omega).
\]
Next, as a consequence of (5.20), we have the following uniform estimate
\[
\sup_{t \geq 0} \left\| \nabla u(t) \right\|_{L^2(\Omega)} \leq Q(E(u_0, \phi_0), H(0)) =: R_0,
\]
(5.37)
where \( R_0 \) is independent of time. The exact value of \( R_0 \) can be estimated in terms of the norm of the initial conditions. Now we make the following assumptions:
\[
|\rho'(s)|_{L^\infty(-1,1)} \leq \frac{4\pi}{C_2 R_0^2}.
\]
(5.38)
Thanks to (5.38), we conclude that
\[
\frac{d}{dt} \int_{\Omega} F''(\phi) \, dx + \frac{1}{4} \int_{\Omega} F'''(\phi) F'(\phi) \, dx \leq \frac{3C_1|\Omega|}{4C_2} + e^{AC_2(\theta_0 + C_2^2)}|\Omega| + C_{TM}(\Omega).
\]
(5.39)
Observe now that, for \( s \in \left[\frac{1}{2}, 1\right)\),
\[
F''(s) = \frac{\theta}{1 - s^2} = \frac{(1 - s)(1 + s)}{2s} F''(s) \leq \frac{3}{4F'(\frac{1}{2})} F'''(s) F'(s).
\]
This gives
\[
F''(s) \leq C_3 + C_4 F'''(s) F'(s), \quad \forall \ s \in (-1, 1),
\]
(5.40)
where
\[
C_3 = F''(\frac{1}{2}), \quad C_4 = \frac{3}{4F'(\frac{1}{2})}.
\]
Hence, we are led to
\[
\frac{d}{dt} \int_{\Omega} F''(\phi) \, dx + \frac{1}{4C_4} \int_{\Omega} F''(\phi) \, dx \leq C_5,
\]
where
\[
C_5 = \frac{3C_1|\Omega|}{4C_2^2} + e^{AC_2(\theta_0 + C_2^2)}|\Omega| + C_{TM}(\Omega) + \frac{C_3|\Omega|}{4C_4}.
\]
We recall that \( F''(\phi_0) \in L^1(\Omega) \). Then, an application of the Gronwall lemma entails that
\[
\int_{\Omega} F''(\phi(t)) \, dx \leq \left\| F''(\phi_0) \right\|_{L^1(\Omega)} e^{-\frac{t}{4C_4}} + 4C_4 C_5, \quad \forall \ t \geq 0.
\]
(5.41)
In addition, integrating (5.39) on the time interval \([t, t+1]\), we find
\[
\int_t^{t+1} \int_{\Omega} F''(\phi) F'(\phi) \, dx \, d\tau \leq 4\left\| F''(\phi_0) \right\|_{L^1(\Omega)} + C_6, \quad \forall \ t \geq 0,
\]
(5.42)
where
\[
C_6 = 4C_5 - \frac{C_3|\Omega|}{C_4}.
\]
This allows us to improve the integrability of \( F''(\phi) \). Indeed, arguing similarly to (5.40), we have for \( s \in \left[\frac{1}{2}, 1\right)\)
\[
(F''(s))^2 \log(1 + F''(s)) = \frac{\theta^2}{(1 - s)^2(1 + s)^2} \log \left( 1 + \frac{\theta}{1 - s^2} \right)
\]
\[
\leq \theta F'''(s) \log \left( 1 + s \frac{1 - s^2 + \theta}{1 - s^2} \left( 1 + s \right)^2 \right)
\]
\[
\leq 2F''(s)F'(s) + \theta F''(s) \log \left( \frac{1}{2} + \frac{2\theta}{3} \right) 
\leq C_7 F''(s)F'(s).
\]

Hence, we infer that
\[
(F''(s))^2 \log(1 + F''(s)) \leq C_7 F''(s)F'(s) + C_8, \quad \forall s \in (-1, 1).
\]

In light of (5.42), we deduce (5.2). Indeed, we have
\[
\int_t^{t+1} \int_{\Omega} (F''(\phi))^2 \log(1 + F''(\phi)) \, dx \, d\tau \leq 4C_7 \|F''(\phi_0)\|_{L^1(\Omega)} + C_6 C_7 + C_8, \quad \forall t \geq 0. \tag{5.43}
\]

We notice that, by keeping the (non-negative) term \(F(\phi) |\nabla \phi|^2\) (cf. (5.28)) on the left-hand side of (5.39) in the above argument, we can also deduce that
\[
\int_t^{t+1} \int_{\Omega} F(\phi)|\nabla \phi|^2 \, dx \, d\tau \leq C_9, \quad \forall t \geq 0,
\]
where \(C_9\) depends on \(\|F''(\phi_0)\|_{L^1(\Omega)}, R_0, \theta, \theta_0 \) and \(\Omega\). Since \(\left(\frac{s}{\sqrt{1-s}}\right)' = (1-s)^{-\frac{3}{2}}\), we infer that
\[
\int_t^{t+1} \int_{\Omega} |\nabla \left( \frac{\phi}{\sqrt{1-\phi^2}} \right)|^2 \, dx \, d\tau \leq \frac{C_9}{2\theta}, \quad \forall t \geq 0.
\]

Setting \(\psi = \frac{\phi}{\sqrt{1-\phi^2}}\), and observing that \(F''(s) = \theta \left[ (\frac{s}{\sqrt{1-s}})^2 + 1 \right]\), we have (cf. (5.39))
\[
\|\psi(t)\|^2_{L^2(\Omega)} + \int_t^{t+1} \|\nabla \psi(\tau)\|^2_{L^2(\Omega)} \, d\tau \leq C_{10}, \quad \forall t \geq 0.
\]

This implies that \(\psi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\). By Sobolev embedding, we also have that \(\psi \in L^q(0, T; L^p(\Omega))\) where \(\frac{1}{2} = \frac{1}{p} + \frac{1}{q}, p \in (2, \infty)\). As a consequence, we conclude that
\[
\int_t^{t+1} \|F''(\phi(\tau))\|^q_{L^p(\Omega)} \, d\tau \leq C_{11}, \quad \forall t \geq 0, \tag{5.44}
\]
where \(1 = \frac{1}{p} + \frac{1}{q}, p \in (1, \infty)\).

**Uniqueness of strong solutions.** Let us consider two strong solutions \((u_1, \phi_1, P_1)\) and \((u_2, \phi_2, P_2)\) to system (4.1)-(4.3) satisfying the entropy bound (5.2) and originating from the same initial datum. The solutions difference \((u, \phi, P) := (u_1 - u_2, \phi_1 - \phi_2, P_1 - P_2)\) solves
\[
\rho(\phi_1)(\partial_t u + u_1 \cdot \nabla u + u \cdot \nabla u_2) - \text{div} (\nu(\phi_1) Du) + \nabla P \\
= -\Delta \phi_1 \nabla \phi - \Delta \phi \nabla \phi_2 - (\rho(\phi_1) - \rho(\phi_2)) (\partial_t u_2 + u_2 \cdot \nabla u_2) + \text{div} ((\nu(\phi_1) - \nu(\phi_2)) Du) \tag{5.45}
\]
and
\[
\partial_t \phi + u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2 - \Delta \phi + \Psi'(\phi_1) - \Psi'(\phi_2) \\
= -\rho'(\phi_1) \frac{|u_1|^2}{2} + \rho'(\phi_2) \frac{|u_2|^2}{2} + \xi_1 - \xi_2, \tag{5.46}
\]
for almost every \((x, t) \in \Omega \times (0, T)\), together with the incompressibility constraint \(\text{div} u = 0\).
It follows that $\overline{\phi}(t) = 0$. Multiplying (5.45) by $u$ and integrating over $\Omega$, we obtain
\[
\frac{d}{dt} \int_{\Omega} \rho(\phi_1) \frac{1}{2} |u|^2 \, dx + \int_{\Omega} \rho(\phi_1) (u_1 \cdot \nabla) u \cdot u \, dx + \int_{\Omega} \rho(\phi_1) (u \cdot \nabla) u_2 \cdot u \, dx + \int_{\Omega} \nu(\phi_1) |Du|^2 \, dx
\]
\[
= - \int_{\Omega} \Delta \phi_1 \nabla \cdot u \, dx - \int_{\Omega} \Delta \phi \nabla u_1 \cdot u \, dx - \int_{\Omega} \left( \rho(\phi_1) - \rho(\phi_2) \right) (\partial_t u_2 + u_2 \cdot \nabla u_2) \cdot u \, dx
\]
\[
- \int_{\Omega} (\nu(\phi_1) - \nu(\phi_2)) Du_2 : Du \, dx + \int_{\Omega} \frac{1}{2} |u|^2 \rho'(\phi_1) \partial_t \phi_1 \, dx. \quad (5.47)
\]

Next, multiplying (5.46) by $-\Delta \phi$ and integrating over $\Omega$, we find
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 \, dx + \|\Delta \phi\|^2_{L^2(\Omega)} = \int_{\Omega} (u_1 \cdot \nabla \phi) \Delta \phi \, dx + \int_{\Omega} (u_2 \cdot \nabla \phi) \Delta \phi \, dx
\]
\[
+ \int_{\Omega} \left( F'(\phi_1) - F'(\phi_2) \right) \Delta \phi \, dx + \theta_0 \|\nabla \phi\|^2_{L^2(\Omega)} + \int_{\Omega} \left( \rho'(\phi_1) \frac{|u_1|^2}{2} - \rho'(\phi_2) \frac{|u_2|^2}{2} \right) \Delta \phi \, dx. \quad (5.48)
\]
Here we have used the fact that $\overline{\Delta \phi} = 0$ which implies that $\int_{\Omega} (\xi_1 - \xi_2) \Delta \phi \, dx = 0$. Adding (5.47) and (5.48), together with the bound from below of the viscosity, we have
\[
\frac{d}{dt} \left( \int_{\Omega} \frac{\rho(\phi_1)}{2} |u|^2 \, dx + \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 \, dx \right) + \nu_s \|Du\|^2_{L^2(\Omega)} + \|\Delta \phi\|^2_{L^2(\Omega)}
\]
\[
\leq - \int_{\Omega} \rho(\phi_1) (u_1 \cdot \nabla) u \cdot u \, dx - \int_{\Omega} \rho(\phi_1) (u \cdot \nabla) u_2 \cdot u \, dx - \int_{\Omega} \Delta \phi_1 \nabla \phi \cdot u \, dx
\]
\[
- \int_{\Omega} \left( \rho(\phi_1) - \rho(\phi_2) \right) (\partial_t u_2 + u_2 \cdot \nabla u_2) \cdot u \, dx - \int_{\Omega} (\nu(\phi_1) - \nu(\phi_2)) Du_2 : Du \, dx
\]
\[
+ \int_{\Omega} \frac{1}{2} |u|^2 \rho'(\phi_1) \partial_t \phi_1 \, dx + \int_{\Omega} (u_1 \cdot \nabla \phi) \Delta \phi \, dx + \int_{\Omega} \left( F'(\phi_1) - F'(\phi_2) \right) \Delta \phi \, dx
\]
\[
+ \theta_0 \|\nabla \phi\|^2_{L^2(\Omega)} + \int_{\Omega} \left( \rho'(\phi_1) \frac{|u_1|^2}{2} - \rho'(\phi_2) \frac{|u_2|^2}{2} \right) \Delta \phi \, dx. \quad (5.49)
\]

We now proceed by estimating the terms on the right hand side of the above differential equality. We would like to mention that most of the bounds obtained below are easy applications of the Sobolev embedding theorem and interpolation inequalities in view of the estimates for global strong solutions that have been obtained before. Nevertheless, less standard is the estimate of the term involving the difference of the nonlinear terms $(F'(\phi_1) - F'(\phi_2))$ which makes use of the entropy bound (5.43).

By using the regularity of strong solutions, (2.2) and (2.4), we have
\[
- \int_{\Omega} \rho(\phi_1) (u_1 \cdot \nabla) u \cdot u \, dx \leq C \|u_1\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}
\]
\[
\leq \frac{\nu_s}{12} \|Du\|^2_{L^2(\Omega)} + C \|u_1\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)},
\]
\[
- \int_{\Omega} \rho(\phi_1) (u \cdot \nabla) u_2 \cdot u \, dx \leq C \|\nabla u_2\|_{L^2(\Omega)} \|u\|^2_{L^2(\Omega)}
\]
\[
\leq \frac{\nu_s}{12} \|Du\|^2_{L^2(\Omega)} + C \|u\|^2_{L^2(\Omega)},
\]
Using the generalized Young inequality (5.15) and the standard Young inequality, for \( x > 0, y > 0, z > 0 \) with \( Cz > y \), we obtain
\[
x^2y^2 \log \left( \frac{Cz}{y} \right) \leq xy^2 \left( x \log x + \frac{Cz}{y} \right) \\
\leq \varepsilon z^2 + x^2y^2 \log x + C^2\varepsilon^{-1}x^2y^2, \quad \forall \varepsilon > 0.
\] (5.50)

By making use of (2.9) and (5.50), we obtain that
\[
\int_\Omega (F'(\phi_1) - F'(\phi_2)) \Delta \phi \, dx
\]
applying Jensen’s inequality, we have
\[ 48 \text{ GIORGINI-GRASSELLI-WU} \]
Using the explicit form of
\[ \text{Here we have used that } \rho \]
Collecting the above bounds, we find the differential inequality
\[ \text{To this aim, we introduce the function } \]
\[ \text{where } \]
\[ \text{and } \]
\[ \text{Here we have used that } \rho(s) \geq \rho_* \text{ for all } s \in (-1, 1). \]
In order to apply the Gronwall lemma, we are left to show that
\[ \int_0^T \| F''(\phi_i) \|^2_{L^2(\Omega)} \log \left( \| F''(\phi_i) \|_{L^2(\Omega)} \right) d\tau \leq C(T), \quad i = 1, 2. \]
To this aim, we introduce the function
\[ g(s) = s \log(C^* s), \quad \forall s \in (0, \infty), \]
where \( C^* \) is a positive constant. It is easily seen that \( g \) is continuous and convex \( (g''(s) = \frac{1}{s} > 0) \). By applying Jensen’s inequality, we have
\[ g\left( \frac{1}{|\Omega|} \int_\Omega |F''(\phi)|^2 \, dx \right) \leq \frac{1}{|\Omega|} \int_\Omega g(|F''(\phi)|^2) \, dx. \]
Using the explicit form of \( g \), this is equivalent to
\[ \frac{1}{|\Omega|} \| F''(\phi) \|^2_{L^2(\Omega)} \log \left( \frac{C^*}{|\Omega|} \| F''(\phi) \|^2_{L^2(\Omega)} \right) \leq \frac{1}{|\Omega|} \int_\Omega |F''(\phi)|^2 \log(C^* |F''(\phi)|^2) \, dx. \]
Taking $C^* = |\Omega|$ and integrating the above inequality over $[0, T]$, we find
\[
\int_0^T \| F''(\phi) \|^2_{L^2(\Omega)} \log \left( \| F''(\phi) \|^2_{L^2(\Omega)} \right) \, d\tau \leq \int_0^T \int_\Omega |F''(\phi)|^2 \log(\|\Omega\|F''(\phi)|^2) \, dx \, d\tau. \tag{5.53}
\]
Then, (5.52) immediately follows from the entropy bounds (5.43) and (5.53). As a consequence, both $W_1$ and $W_2$ belong to $L^1(0, T)$. Finally, an application of the Gronwall lemma entails the uniqueness of strong solutions. \hfill \Box

**Remark 5.5 (Entropy Estimates in $L^p$, $p > 1$).** Notice that the entropy estimate in $L^1(\Omega)$ proved in Theorem 5.1-(2) can be generalized to the $L^p(\Omega)$ case with $p > 1$. More precisely, for any $p \in \mathbb{N}$, there exists $\eta_p > 0$ with the latter depending on the norms of the initial data and on the parameters of the system
\[
\eta_p = \eta_p(E(u_0, \phi_0), \|u_0\|_V, \|\phi_0\|_{H^2(\Omega)}, \|F'(\phi_0)\|_{L^2(\Omega)}, \theta, \theta_0)
\]
such that, if $\|\rho'\|_{L^\infty(-1, 1)} \leq \eta_p$ and $F''(\phi_0) \in L^p(\Omega)$, then, for any $T > \sigma$, we have
\[
F''(\phi) \in L^\infty(0, T; L^p(\Omega)), \quad |F''(\phi)|^{p-1} F'''(\phi) F'(\phi) \in L^1(\Omega \times (0, T)).
\]
Such result follows from the above proof by replacing $\frac{d}{dt} \int_\Omega F''(\phi) \, dx$ by $\frac{d}{dt} \int_\Omega (F''(\phi))^p \, dx$, and the observation that, for any $p > 2$, there exist two positive constants $C^1_p$ and $C^2_p$ such that
\[
|(F''(s))^{p-1} F'''(s)| \log \left( \left| (F''(s))^{p-1} F'''(s) \right| \right) \leq C^1_p + C^2_p (F''(s))^{p-1} F'''(s) F'(s), \quad \forall s \in (-1, 1).
\]

### 5.2. Proof of Theorem 5.2.

We now prove the propagation of entropy bound as stated in Theorem 5.2. For every strong solution given by Theorem 5.1-(1), we have
\[
\|u \cdot \nabla \phi\|_{H^1(\Omega)} \leq \|u\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4} + \|\nabla u\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4(\Omega)} + \|u\|_{L^\infty(\Omega)} \|\phi\|_{H^2(\Omega)}
\]
\[
\leq C + C\|u\|_{H^2(\Omega)}^2 \|\phi\|_{H^2(\Omega)} + C\|\phi\|_{H^2(\Omega)},
\]
and
\[
\|\rho'(\phi)|u|^2\|_{H^1(\Omega)} \leq C\|u\|_{L^4(\Omega)}^2 + C\|\nabla \phi\|_{L^\infty(\Omega)} \|u\|_{H^2(\Omega)}^2 + C\|\nabla u\|_{L^4(\Omega)} \|u\|_{L^4(\Omega)} \leq C + C\|\phi\|_{W^{2,3}(\Omega)} + C\|u\|_{H^2(\Omega)},
\]
which imply that
\[
\int_t^{t+1} \|u(\tau) \cdot \nabla \phi(\tau)\|_{H^1(\Omega)} + \|\rho'(\phi(\tau)) \frac{|u(\tau)|^2}{2}\|_{H^1(\Omega)}^2 \, d\tau \leq C, \quad \forall t \geq 0,
\]
for some $C$ independent of $t$. In light of (5.21), it follows that
\[
\int_t^{t+1} \| - \Delta \phi(\tau) + F'(\phi(\tau))\|_{H^1(\Omega)}^2 \, d\tau \leq C, \quad \forall t \geq 0.
\]

By using [33, Lemma 7.4], we infer that, for any $p \geq 1$, there exists $C = C(p)$ such that
\[
\|F''(\phi)\|_{L^p(\Omega)} \leq C \left( 1 + e^{C\|\Delta \phi + F'(\phi)\|_{H^1(\Omega)}^2} \right) \text{ a.e. in } (0, T). \tag{5.54}
\]
Notice that we are not able to conclude that the right hand side of (5.54) is $L^1(0, T)$. Nevertheless, since integrable function are finite almost everywhere, the above inequality entails that there exists some $\sigma \in (0, 1)$ (actually $\sigma$ can be taken arbitrarily small but positive) such that

$$F''(\phi(\sigma)) \in L^p(\Omega) \quad \text{with} \quad \|F''(\phi(\sigma))\|_{L^p(\Omega)} \leq C(p, \sigma), \quad \forall p \in [1, \infty). \quad (5.55)$$

Then, under the condition (5.38) but without the additional assumption $F''(\phi_0) \in L^1(\Omega)$ on the initial datum, we are able to deduce that the previous estimates (5.41)-(5.43) hold for $t \geq \sigma > 0$. More precisely, we have

$$\int_{t}^{t+1} \int_{\Omega} (F''(\phi))^2 \log(1 + F''(\phi)) \, dx \, d\tau \leq C(\sigma), \quad \forall t \geq 0. \quad (5.56)$$

Differentiating (4.1) with respect to time and testing the resultant by $\partial_t u$, integrating over $\Omega$, we have

$$\frac{1}{2} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \rho(\phi) (\partial_t \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \partial_t \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx + \int_{\Omega} \rho' (\phi) \partial_t \phi (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx$$

$$+ \int_{\Omega} \nu (\phi) |D \partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \nu' (\phi) \partial_t \phi D \mathbf{u} : D \partial_t \mathbf{u} \, dx = \int_{\Omega} \partial_t (\nabla \phi \otimes \nabla \phi) : \nabla \partial_t \mathbf{u} \, dx.$$

Since

$$\frac{1}{2} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx \leq \frac{1}{2} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx - \frac{1}{2} \int_{\Omega} \rho' (\phi) \partial_t \phi |\partial_t \mathbf{u}|^2 \, dx,$$

we find

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\phi) |\partial_t \mathbf{u}|^2 \, dx + \int_{\Omega} \nu(\phi) |D \partial_t \mathbf{u}|^2 \, dx$$

$$= - \int_{\Omega} \rho(\phi) (\partial_t \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \partial_t \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx \leq \frac{1}{2} \int_{\Omega} \rho' (\phi) \partial_t \phi |\partial_t \mathbf{u}|^2 \, dx$$

$$- \int_{\Omega} \rho' (\phi) \partial_t \phi (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx - \int_{\Omega} \nu' (\phi) \partial_t \phi D \mathbf{u} : \nabla \partial_t \mathbf{u} + \int_{\Omega} \partial_t (\nabla \phi \otimes \nabla \phi) : \nabla \partial_t \mathbf{u} \, dx.$$

In view of (5.22), by using (2.4), we have

$$- \int_{\Omega} \rho(\phi) (\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx \leq C \|\partial_t \mathbf{u}\|_{L^4(\Omega)}^2 \|\nabla \mathbf{u}\|_{L^2(\Omega)}$$

$$\leq \frac{\nu_s}{16} \|D \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2,$$

and

$$- \int_{\Omega} \rho(\phi) (\mathbf{u} \cdot \nabla \partial_t \mathbf{u}) \cdot \partial_t \mathbf{u} \, dx \leq C \|\mathbf{u}\|_{L^4(\Omega)} \|\nabla \partial_t \mathbf{u}\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^4(\Omega)}$$

$$\leq \frac{\nu_s}{16} \|D \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + C \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2.$$

Similarly, we obtain

$$- \frac{1}{2} \int_{\Omega} \rho' (\phi) \partial_t \phi |\partial_t \mathbf{u}|^2 \, dx \leq C \|\partial_t \phi\|_{L^2(\Omega)} \|\partial_t \mathbf{u}\|_{L^4(\Omega)}^2.$$
Here we have used that

\[ \int_\Omega \rho'(\phi) \partial_t \phi (u \cdot \nabla u) \cdot \partial_t u \, dx \leq C \| \partial_t \phi \|_{L^4(\Omega)} \| u \|_{L^4(\Omega)} \| \nabla u \|_{L^4(\Omega)} \| \partial_t u \|_{L^4(\Omega)} \]

and

\begin{align*}
&\leq C \| \nabla \partial_t \phi \|_{L^2(\Omega)} \frac{1}{2} \| u \|_{H^2(\Omega)} \| \partial_t u \|_{L^2(\Omega)} \| D \partial_t u \|_{L^2(\Omega)} \\
&\leq \frac{\nu_*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + C \| \partial_t u \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + C \| u \|_{H^2(\Omega)}.
\end{align*}

Besides, by means of (2.7), we deduce that

\begin{align*}
- \int_\Omega \rho'(\phi) \partial_t \phi D u : D \partial_t u \, dx \\
&\leq C \| \partial_t \phi \|_{L^\infty(\Omega)} \| D u \|_{L^2(\Omega)} \| D \partial_t u \|_{L^2(\Omega)} \\
&\leq \frac{\nu_*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + C \| \partial_t \phi \|_{L^2(\Omega)} \| \partial_t u \|_{H^2(\Omega)} \\
&\leq \frac{\nu_*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2,
\end{align*}

and

\begin{align*}
\int_\Omega \partial_t (\nabla \otimes \nabla \phi) : \nabla \partial_t u \, dx \\
&\leq \| \nabla \phi \|_{L^4(\Omega)} \| \nabla \partial_t \phi \|_{L^4(\Omega)} \| D \partial_t u \|_{L^2(\Omega)} \\
&\leq \frac{\nu_*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)} \| \nabla \partial_t \phi \|_{H^1(\Omega)} \\
&\leq \frac{\nu_*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2.
\end{align*}

Next, we differentiate (4.1) with respect to time, multiply the resultant by \(-\Delta \partial_t \phi\), and integrate over \Omega to obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 \\
= \theta_0 \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + \int_\Omega F''(\phi) \partial_t \phi \Delta_\theta \phi \, dx + \int_\Omega (u \cdot \nabla \phi) \Delta_\theta \phi \, dx \\
+ \int_\Omega (u \cdot \nabla \partial_t \phi) \Delta_\theta \phi \, dx + \frac{1}{2} \int_\Omega \rho''(\phi) \partial_t \phi ||u||^2 \Delta_\theta \phi \, dx + \int_\Omega \rho'(\phi) (u \cdot \partial_t u) \Delta_\theta \phi \, dx.
\end{align*}

Here we have used that \(\Delta \partial_t \phi = 0\) since \(\partial_n \partial_t \phi = 0\) on the boundary \(\partial \Omega\). Exploiting (2.9), we get

\begin{align*}
\int_\Omega F''(\phi) \partial_t \phi \Delta_\theta \phi \, dx &\leq \| F''(\phi) \|_{L^2(\Omega)} \| \partial_t \phi \|_{L^\infty(\Omega)} \| \Delta_\theta \phi \|_{L^2(\Omega)} \\
&\leq \| F''(\phi) \|_{L^2(\Omega)} \| \nabla \partial_t \phi \|_{L^2(\Omega)} \log^{\frac{1}{2}} \left( \frac{\| \Delta_\theta \phi \|_{L^2(\Omega)}}{\| \nabla \partial_t \phi \|_{L^2(\Omega)}} \right) \| \Delta_\theta \phi \|_{L^2(\Omega)} \\
&\leq \frac{1}{28} \| \Delta_\theta \phi \|_{L^2(\Omega)}^2 + C \| F''(\phi) \|_{L^2(\Omega)} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 \log \left( \frac{\| \Delta_\theta \phi \|_{L^2(\Omega)}}{\| \nabla \partial_t \phi \|_{L^2(\Omega)}} \right).
\end{align*}
Recalling (5.50), we obtain
\[ \int_{\Omega} F''(\phi) \partial_t \phi \Delta \partial_t \phi \, dx \leq \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| F''(\phi) \|_{L^2(\Omega)}^2 \log (\| F''(\phi) \|_{L^2(\Omega)}) \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2. \] (5.57)

Next, using (2.4) and (5.22), we see that
\[ \int_{\Omega} (\partial_t u \cdot \nabla \phi) \Delta \partial_t \phi \, dx \leq \| \partial_t u \|_{L^4(\Omega)} \| \nabla \phi \|_{L^4(\Omega)} \| \Delta \partial_t \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{\nu^*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \partial_t u \|_{L^2(\Omega)}^2, \]
and
\[ \int_{\Omega} (u \cdot \nabla \partial_t \phi) \Delta \partial_t \phi \, dx \leq \| u \|_{L^4(\Omega)} \| \nabla \partial_t \phi \|_{L^4(\Omega)} \| \Delta \partial_t \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2. \] (5.58)

Finally, in a similar manner we find that
\[ \frac{1}{2} \int_{\Omega} \rho''(\phi) \partial_t \phi \| u \|^2 \Delta \partial_t \phi \, dx \leq C \| \partial_t \phi \|_{L^4(\Omega)} \| u \|_{L^4(\Omega)}^2 \| \Delta \partial_t \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2, \]
and
\[ \int_{\Omega} \rho'(\phi) (u \cdot \partial_t u) \Delta \partial_t \phi \, dx \leq C \| u \|_{L^4(\Omega)} \| \partial_t u \|_{L^4(\Omega)} \| \Delta \partial_t \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{\nu^*}{16} \| D \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{14} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 + C \| \partial_t u \|_{L^2(\Omega)}^2. \]

From the above estimates, we deduce that
\[ \frac{d}{dt} L(t) + \frac{\nu^*}{2} \| D \partial_t u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 \leq C K(t) L(t) + C \| u \|_{H^2(\Omega)}^2, \] (5.59)

where
\[ L(t) = \frac{1}{2} \int_{\Omega} \rho(\phi) |\partial_t u(t)|^2 \, dx + \frac{1}{2} \| \nabla \partial_t \phi(t) \|_{L^2(\Omega)}^2, \]
\[ K(t) = 1 + \| F''(\phi) \|_{L^2(\Omega)}^2 \log (\| F''(\phi) \|_{L^2(\Omega)}). \]

Recalling estimates (5.21) and (5.25), we have
\[ \int_{t}^{t+1} L(\tau) + \| u(\tau) \|_{H^2(\Omega)}^2 \, d\tau \leq C, \quad \forall \, t \geq 0, \]
where $C$ is independent of $t$. As a consequence, there exists $\sigma \in (0, 1)$ ($\sigma$ can be chosen arbitrary small but positive) such that
\[ L(\sigma) \leq C(\sigma). \] (5.60)
Notice that, without loss of generality, this value of \( \sigma \) can be chosen equal to the one in (5.55). Then, by exploiting (5.56) and the Jensen inequality (cf. (5.53)), we obtain

\[
\int_t^{t+1} K(\tau) \, d\tau \leq C, \quad \forall \, t \geq \sigma,
\]

where \( C \) depends on \( \sigma \), but is independent of \( t \). Thus, by using the Gronwall lemma on the time interval \([\sigma, 1]\) and the uniform Gronwall lemma for \( t \geq 1 \), we deduce that

\[
L(t) + \int_t^{t+1} \| D\partial_t u \|_{L^2(\Omega)}^2 + \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 \, d\tau \leq C(\sigma), \quad \forall \, t \geq \sigma.
\]

Hence we have

\[
\partial_t u \in L^\infty(\sigma, T; H_\sigma) \cap L^2(\sigma, T; V_\sigma), \quad \partial_t \phi \in L^\infty(\sigma, T; H^1(\Omega)) \cap L^2(\sigma, T; H^2(\Omega)).
\]

In light of (5.10) and (5.20), we infer that

\[
u \in L^\infty(\sigma, T; W^{1,p}(\Omega)), \quad \forall \, p \in (2, \infty).
\]

An immediate consequence of the above regularity results is that

\[
\bar{\mu} = -\Delta \phi + F'(\phi) \in L^2(\sigma, T; L^\infty(\Omega)).
\]

Thanks to [33, Lemma 7.2], we deduce that \( F'(\phi) \in L^2(\sigma, T; L^\infty(\Omega)) \). This property entails that there exists \( \sigma' \in (\sigma, \sigma + 1) \) such that

\[
\| F'(\phi(\sigma')) \|_{L^\infty(\Omega)} \leq C(\sigma).
\]

Note that \( \sigma' \) can also be chosen arbitrarily close to \( \sigma \).

Now, we rewrite (4.1) as follows

\[
\partial_t \phi + u \cdot \nabla \phi - \Delta \phi + F'(\phi) = U(x,t),
\]

where \( U = \theta_0 \phi - \rho'(\phi) \frac{|u|^2}{2} + \xi \). Thanks to the above regularity, it is easily seen that \( U \in L^\infty(0, T; L^\infty(\Omega)) \).

In particular, \( \sup_{t \geq \sigma} \| U(t) \|_{L^\infty(\Omega)} \leq C(\sigma) \). For any \( p \geq 2 \), we compute

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega |F'(\phi)|^p \, dx = \int_\Omega |F'(\phi)|^{p-2} F'(\phi) F''(\phi) \partial_t \phi \, dx
\]

\[
= \int_\Omega |F'(\phi)|^{p-2} F'(\phi) F''(\phi) \left( -u \cdot \nabla \phi + \Delta \phi - F'(\phi) + U \right) \, dx.
\]

Since

\[
\int_\Omega |F'(\phi)|^{p-2} F'(\phi) F''(\phi) u \cdot \nabla \phi \, dx = \int_\Omega u \cdot \nabla \left( \frac{1}{p} |F'(\phi)|^p \right) \, dx = 0,
\]

we deduce that

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega |F'(\phi)|^p \, dx + \int_\Omega \left( (p-1) |F'(\phi)|^{p-2} F''(\phi)^2 + |F'(\phi)|^{p-1} F'(\phi) F'''(\phi) \right) |\nabla \phi|^2 \, dx
\]

\[
+ \int_\Omega |F'(\phi)|^p U \, dx = \int_\Omega |F'(\phi)|^{p-2} F'(\phi) F''(\phi) U \, dx.
\]

We notice that the second term on the left-hand side is non-negative. Next, we observe that

\[
F''(s) \leq \theta_0 \phi \hat{\phi} |F'(s)|, \quad \forall \, s \in (-1, 1).
\]
Owing to the above inequality, and using the fact that \( s \leq e^x \) for \( s \geq 0 \), we deduce that

\[
\log \left( |F'(s)|^{p-1} F''(s) \right) \leq \log(\theta) + \left( 1 + \frac{2}{\theta} \right) (p - 1) |F'(s)|, \quad \forall \, s \in (-1, 1).
\]

Thus, we get

\[
|F'(s)|^{p-1} F''(s) \log \left( |F'(s)|^{p-1} F''(s) \right) \leq C_1 p |F'(s)|^p F''(s) + C_2, \quad \forall \, s \in (-1, 1),
\]

for some \( C_1, C_2 > 0 \) independent of \( p \). Recalling

\[
xy \leq \varepsilon x \log x + e^\varepsilon, \quad \forall \, x > 0, y > 0, \quad \varepsilon \in (0, 1),
\]

and taking \( \varepsilon = \frac{1}{2C_1} \), we arrive at

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega |F'(\phi)|^p \, dx + \frac{1}{2} \int_\Omega |F'(\phi)|^p F''(\phi) \, dx \leq \frac{C_2 |\Omega|}{2C_1} + \int_\Omega e^{2C_1 p |U|} \, dx.
\]

Since \( U \) is globally bounded, we obtain

\[
\frac{C_2 |\Omega|}{2C_1} + \int_\Omega e^{2C_1 p |U|} \, dx \leq \frac{C_2 |\Omega|}{2C_1} + |\Omega| e^{2C_3 p} \leq C_4 e^{C_5 p},
\]

for some \( C_4, C_5 > 0 \) independent of \( p \) and \( t \). Observing that \( F''(s) \geq \theta \) for all \( s \in (-1, 1) \), we rewrite the above differential inequality for \( p \geq 2 \) as follows

\[
\frac{d}{dt} \int_\Omega |F'(\phi)|^p \, dx + \theta \int_\Omega |F'(\phi)|^p \, dx \leq C_4 p e^{C_5 p}.
\]

By applying the Gronwall lemma on the time interval \([\sigma', \infty)\), we infer that

\[
\|F'(\phi(t))\|_{L^p(\Omega)}^p \leq \|F'(\phi(\sigma'))\|_{L^p(\Omega)}^p e^{-\theta(t-\sigma')} + \frac{C_4 p e^{C_5 p}}{\theta}, \quad \forall \, t \geq \sigma'.
\]

We recall the elementary inequality for \( q < 1 \)

\[
(x + y)^q \leq x^q + y^q, \quad \forall \, x > 0, y > 0.
\]

Choosing \( q = \frac{1}{p} \), with \( p \geq 2 \), we find

\[
\|F'(\phi(t))\|_{L^p(\Omega)} \leq \|F'(\phi(\sigma'))\|_{L^p(\Omega)} e^{-\frac{\theta(t-\sigma')}{p}} + \left( \frac{C_4 p}{\theta} \right)^{\frac{1}{p}} e^{C_5}, \quad \forall \, t \geq \sigma'.
\]

Recalling (5.61) and taking the limit as \( p \to +\infty \), we deduce that

\[
\|F'(\phi(t))\|_{L^\infty(\Omega)} \leq \|F'(\phi(\sigma'))\|_{L^\infty(\Omega)} + e^{C_5}, \quad \forall \, t \geq \sigma'.
\]

As a result, there exists \( \delta = \delta(\sigma) > 0 \) such that

\[
-1 + \delta \leq \phi(x, t) \leq 1 - \delta, \quad \forall \, x \in \overline{\Omega}, \, t \geq \sigma'.
\]

The proof is complete. \( \square \)
6. Mass-conserving Euler-Allen-Cahn System in Two Dimensions

In this section, we study the dynamics of ideal two-phase flows in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, which is described by the mass-conserving Euler-Allen-Cahn system:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= - \text{div} (\nabla \phi \otimes \nabla \phi), \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi + \mu &= \overline{\mu}, \\
\mu &= -\Delta \phi + \Psi'(\phi),
\end{aligned}
\]

in $\Omega \times (0, T)$.

(6.1)

The above system corresponds to the inviscid NS-AC system (4.1) (i.e. $\nu \equiv 0$) with matched densities (i.e. $\rho \equiv 1$). The system is subject to the following boundary conditions

\[
u \cdot n = 0, \quad n \phi = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

and initial conditions

\[
u(\cdot, 0) = u_0, \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega.
\]

(6.2)

(6.3)

The main result of this section is as follows:

**Theorem 6.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$.

1. Assume that $u_0 \in H_\sigma \cap H^1(\Omega), \phi_0 \in H^2(\Omega)$ such that $F'(\phi_0) \in L^2(\Omega), \|\phi_0\|_{L^\infty(\Omega)} \leq 1, |\phi_0| < 1$ and $\partial_n \phi = 0$ on $\partial \Omega$. Then, there exists a global solution $(u, \phi)$ which satisfies the problem (6.1)-(6.3) in the sense of distribution on $\Omega \times (0, \infty)$ and, for all $T > 0$,

\[
\begin{aligned}
\phi &\in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T), \\
\phi &\in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T),
\end{aligned}
\]

where $p \in (2, \infty)$. Moreover, $\partial_n \phi = 0$ on $\partial \Omega \times (0, \infty)$.

2. Assume that $u_0 \in H_\sigma \cap W^{1,p}(\Omega), p \in (2, \infty), \phi_0 \in H^2(\Omega)$ such that $F'(\phi_0) \in L^2(\Omega), F''(\phi_0) \in L^1(\Omega), \|\phi_0\|_{L^\infty(\Omega)} \leq 1, |\phi_0| < 1, \partial_n \phi = 0$ on $\partial \Omega$, and in addition $\nabla \mu_0 = \nabla (-\Delta \phi + F'(\phi_0)) \in L^2(\Omega)$. Then, there exists a global solution $(u, \phi)$ which satisfies the problem (6.1)-(6.3) almost everywhere in $\Omega \times (0, \infty)$ and, for all $T > 0$,

\[
\begin{aligned}
\phi &\in L^\infty(\Omega \times (0, T)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T).
\end{aligned}
\]

In addition, for any $\sigma > 0$, there exists $\delta = \delta(\sigma) > 0$ such that

\[-1 + \delta \leq \phi(x, t) \leq 1 - \delta, \quad \forall x \in \overline{\Omega}, t \geq \sigma.
\]

To prove Theorem 6.1, we first derive formal estimates leading to the required estimates of solutions. Then the existence results can be proved by a suitable approximation scheme with fixed point arguments and then passing to the limit, which is standard owing to uniform estimates obtained in the first step. Hence, here below we only focus on the *a priori* estimates and omit further details.
6.1. Case 1. Let us first consider initial datum \((u_0, \phi_0)\) such that
\[
   u_0 \in H_s \cap H^1(\Omega), \quad \phi_0 \in H^2(\Omega), \quad \partial_u \phi_0 = 0 \quad \text{on} \ \partial \Omega,
\]
with
\[
   \|\phi_0\|_{L^\infty(\Omega)} \leq 1, \quad |\bar{\phi}_0| < 1 \quad \text{and} \quad F'(\phi_0) \in L^2(\Omega).
\]

**Lower-order estimate.** As in the previous section, we have the conservation of mass
\[
   \overline{\phi}(t) = \overline{\phi}_0, \quad \forall \ t \geq 0.
\]
By the same argument for (4.9), we deduce the energy balance
\[
   \frac{d}{dt} E(u, \phi) + \|\partial_t \phi + u \cdot \nabla \phi\|_{L^2(\Omega)}^2 = 0.
\]
Integrating the above relation on \([0, t]\), we find
\[
   E(u(t), \phi(t)) + \int_0^t \|\partial_t \phi + u \cdot \nabla \phi\|_{L^2(\Omega)}^2 \, d\tau = E(u_0, \phi_0), \quad \forall \ t \geq 0.
\]
This implies that
\[
   u \in L^\infty(0, T; H_s), \quad \phi \in L^\infty(0, T; H^1(\Omega)), \quad \partial_t \phi + u \cdot \nabla \phi \in L^2(0, T; L^2(\Omega)),
\]
where the last property also implies \(\mu - \overline{\Pi} \in L^2(0, T; L^2(\Omega))\). In addition, it follows from the estimates (4.14) and (4.21) that
\[
   \phi \in L^2(0, T; H^2(\Omega)), \quad \mu \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad F'(\phi) \in L^2(0, T; L^2(\Omega)).
\]
The latter entails that \(\phi \in L^\infty(\Omega \times (0, T))\) such that \(|\phi(x, t)| < 1\) almost everywhere in \(\Omega \times (0, T)\). We remark that in comparison with the viscous case, it is not possible at this stage to prove that \(\partial_t \phi \in L^2(\Omega \times (0, T))\).

**Higher-order estimates.** In the two dimensional case, it is convenient to consider the equation for the vorticity \(\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\) that reads as follows
\[
   \partial_t \omega + u \cdot \nabla \omega = \nabla \mu \cdot (\nabla \phi)^{\perp},
\]
where \(\mathbf{v}^{\perp} = (v_2, -v_1)\) for any \(\mathbf{v} = (v_1, v_2)\). Multiplying (6.6) by \(\omega\) and integrating over \(\Omega\), we obtain
\[
   \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\Omega)}^2 = \int_\Omega \nabla \mu \cdot (\nabla \phi)^{\perp} \omega \, dx.
\]
On the other hand, differentiating (6.1)_3 with respect to time, multiplying by \(\partial_t \phi\) and integrating over \(\Omega\), we find
\[
   \frac{1}{2} \frac{d}{dt} \|\partial_t \phi\|_{L^2(\Omega)}^2 + \int_\Omega \partial_t u \cdot \nabla \phi \partial_t \phi \, dx + \|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 + \int_\Omega F''(\phi) \partial_t \phi \, dx = \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2.
\]
Here we have used the following equalities
\[
   \int_\Omega u \cdot \nabla \partial_t \phi \, dx = \int_\Omega u \cdot \nabla \left(\frac{1}{2} |\partial_t \phi|^2 \right) \, dx = 0 \quad \text{and} \quad \int_\Omega \partial_t \phi \, dx = 0.
\]
We now define
\[
   H(t) = \frac{1}{2} \|\omega\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\partial_t \phi\|_{L^2(\Omega)}^2.
\]
By adding together (6.7) and (6.8), we infer from the convexity of $F$ (i.e. $F'' > 0$) that
\[
\frac{d}{dt} H(t) + \|\nabla \partial_t \phi \|_{L^2(\Omega)}^2 \leq \int_{\Omega} \nabla \mu \cdot (\nabla \phi) \cdot \omega \, dx - \int_{\Omega} \nabla \mu \cdot (\nabla \phi) \cdot \omega \, dx - \int_{\partial \Omega} n \cdot \nabla \phi \partial_t \phi \, d\sigma + \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2.
\] (6.9)

Before proceeding to control the terms on the right-hand side of (6.9), we rewrite the second one using the Euler equation. We first observe that
\[
\partial_t u = P(\mu \nabla \phi - u \cdot \nabla u),
\]
where $P$ is the Leray projection operator. Thus, we write
\[
\int_{\Omega} \partial_t u \cdot \nabla \phi \partial_t \phi \, dx = \int_{\Omega} P(\mu \nabla \phi - u \cdot \nabla u) \cdot \nabla \phi \partial_t \phi \, dx
\]
\[
= \int_{\Omega} \mu \nabla \phi \cdot P(\nabla \phi \partial_t \phi) \, dx - \int_{\Omega} (u \cdot \nabla u) \cdot P(\nabla \phi \partial_t \phi) \, dx
\]
\[
= - \int_{\Omega} \mu \nabla \phi \cdot P(\nabla \phi \partial_t \phi) \, dx + \int_{\Omega} \nabla (u \otimes u) \cdot P(\nabla \phi \partial_t \phi) \, dx
\]
\[
= - \int_{\Omega} \mu \nabla \phi \cdot P(\nabla \phi \partial_t \phi) \, dx + \int_{\Omega} (u \otimes u) : \nabla P(\nabla \phi \partial_t \phi) \, dx
\]
\[
= - \int_{\Omega} \mu \nabla \phi \cdot P(\nabla \phi \partial_t \phi) \, dx + \int_{\Omega} (u \otimes u) : \nabla P(\nabla \phi \partial_t \phi) \, dx.
\]

Here we have used that $P(\nabla v) = 0$ for any $v \in H^1(\Omega)$, the relation $\text{div}(S'v) = S^t \nabla v + \text{div} S \cdot v$ for any $d \times d$ tensor $S$ and vector $v$, and the no-normal flow condition $u \cdot n = 0$ at the boundary. As a consequence, we rewrite (6.9) as follows
\[
\frac{d}{dt} H(t) + \|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \nabla \mu \cdot (\nabla \phi) \cdot \omega \, dx + \int_{\Omega} \mu \nabla \phi \cdot P(\nabla \phi \partial_t \phi) \, dx
\]
\[
- \int_{\Omega} (u \otimes u) : \nabla P(\nabla \phi \partial_t \phi) \, dx + \theta_0 \|\partial_t \phi\|_{L^2(\Omega)}^2.
\] (6.10)

We now turn to estimate the right-hand side of (6.10). By Hölder’s inequality, we have
\[
\int_{\Omega} \nabla \mu \cdot (\nabla \phi) \cdot \omega \, dx \leq \|\nabla \mu\|_{L^2(\Omega)} \|\nabla \phi\|_{L^{\infty}(\Omega)} \|\omega\|_{L^2(\Omega)}.
\] (6.11)

By taking the gradient of (6.1)$_\Omega$, we observe that
\[
\|\nabla \mu\|_{L^2(\Omega)} \leq \|\nabla \partial_t \phi\|_{L^2(\Omega)} + \|\nabla^2 \phi u\|_{L^2(\Omega)} + \|\nabla \phi \cdot \nabla \phi\|_{L^2(\Omega)}.
\]

Recalling the elementary inequality
\[
\|v\|_{H^1(\Omega)} \leq C \left(\|v\|_{L^2(\Omega)} + \|\text{div} v\|_{L^2(\Omega)} + \|\text{curl} v\|_{L^2(\Omega)} + \|v \cdot n\|_{H^\frac{1}{2}(\partial \Omega)} \right), \quad \forall v \in H^1(\Omega),
\]
and exploiting Lemma 2.1 as well as (2.10), we find that
\[
\|\nabla \mu\|_{L^2(\Omega)} \leq \|\nabla \partial_t \phi\|_{L^2(\Omega)} + C\|\mathbf{u}\|_{H^{1}(\Omega)}\|\nabla^2 \phi\|_{L^2(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla^2 \phi\|_{L^p(\Omega)}}{\|\nabla^2 \phi\|_{L^2(\Omega)}} \right) \\
+ \|\nabla \mathbf{u}\|_{L^2(\Omega)}\|\nabla \phi\|_{L^\infty(\Omega)} \\
\leq \|\nabla \partial_t \phi\|_{L^2(\Omega)} + C(1 + \|\omega\|_{L^2(\Omega)})\|\nabla^2 \phi\|_{L^2(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla^2 \phi\|_{L^p(\Omega)}}{\|\nabla^2 \phi\|_{L^2(\Omega)}} \right) \\
+ C(1 + \|\omega\|_{L^2(\Omega)})\|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla \phi\|_{W^{1,p}(\Omega)}}{\|\nabla \phi\|_{H^1(\Omega)}} \right),
\]
for some \( p > 2 \). Using (5.7), we rewrite the above estimate as follows
\[
\|\nabla \mu\|_{L^2(\Omega)} \leq \|\nabla \partial_t \phi\|_{L^2(\Omega)} + C(1 + \|\omega\|_{L^2(\Omega)}) \left( \|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\|\nabla \phi\|_{W^{1,p}(\Omega)} + 1 \right) \right).
\]
Then, using again the inequality (2.10), (6.11) can be controlled as follows
\[
\int_{\Omega} \nabla \mu \cdot (\nabla \phi)^\perp \omega \, dx \leq \|\nabla \partial_t \phi\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega)}\|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla \phi\|_{W^{1,p}(\Omega)}}{\|\nabla \phi\|_{H^1(\Omega)}} \right) \\
+ C\|\omega\|_{L^2(\Omega)} \left( \|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\|\nabla \phi\|_{W^{1,p}(\Omega)} + 1 \right) \right) \\
\times \|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla \phi\|_{W^{1,p}(\Omega)}}{\|\nabla \phi\|_{H^1(\Omega)}} \right) \\
\leq \|\nabla \partial_t \phi\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega)} \left( \|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\|\nabla \phi\|_{W^{1,p}(\Omega)} + 1 \right) \right) \\
+ C(1 + \|\omega\|_{L^2(\Omega)}^2) \left( \|\nabla \phi\|_{H^1(\Omega)} \log \left( C\|\nabla \phi\|_{W^{1,p}(\Omega)} + 1 \right) \right) \\
\leq \frac{1}{6}\|\nabla \partial_t \phi\|_{L^2(\Omega)}^2 + C(1 + \|\omega\|_{L^2(\Omega)}^2) \left( \|\phi\|_{H^2(\Omega)}^2 \log \left( C\|\phi\|_{W^2,p(\Omega)} + 1 \right) \right),
\]
for some \( p > 2 \). Next, since \( \phi \) is globally bounded, we have
\[
\int_{\Omega} \mu\nabla \phi \cdot \mathbf{P}(\phi \nabla \partial_t \phi) \, dx \leq C\|\mu\|_{L^2(\Omega)}\|\nabla \phi\|_{L^\infty(\Omega)}\|\nabla \partial_t \phi\|_{L^2(\Omega)} \\
\leq C\|\mu\|_{L^2(\Omega)}\|\nabla \phi\|_{H^1(\Omega)} \log^{\frac{1}{2}} \left( C\frac{\|\nabla \phi\|_{W^1,p(\Omega)}}{\|\nabla \phi\|_{H^1(\Omega)}} \right) \|\phi\|_{L^\infty(\Omega)}\|\nabla \partial_t \phi\|_{L^2(\Omega)} \\
\leq C\|\mu\|_{L^2(\Omega)} \left( \|\phi\|_{H^2(\Omega)} \log^{\frac{1}{2}} \left( C\|\phi\|_{W^2,p(\Omega)} + 1 \right) \right)\|\nabla \partial_t \phi\|_{L^2(\Omega)},
\]
for some \( p > 2 \). In order to estimate the \( L^2 \)-norm of \( \mu \), we notice that
\[
\|\mu - \overline{\mathbf{m}}\|_{L^2(\Omega)} \leq \|\partial_t \phi\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \nabla \phi\|_{L^2(\Omega)} \\
\leq \|\partial_t \phi\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^4(\Omega)}\|\nabla \phi\|_{L^4(\Omega)} \\
\leq \|\partial_t \phi\|_{L^2(\Omega)} + C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla \phi\|_{L^2(\Omega)}^\frac{1}{2}\|\phi\|_{H^2(\Omega)}^\frac{1}{2} \\
\leq \|\partial_t \phi\|_{L^2(\Omega)} + C(1 + \|\omega\|_{L^2(\Omega)})^\frac{1}{2}(1 + \|\mu - \overline{\mathbf{m}}\|_{L^2(\Omega)})^\frac{1}{2} \\
\leq \|\partial_t \phi\|_{L^2(\Omega)} + C(1 + \|\omega\|_{L^2(\Omega)}) + \frac{1}{2}\|\mu - \overline{\mathbf{m}}\|_{L^2(\Omega)}.
Here we have used the equation (6.13), the Ladyzhenskaya inequality, and the estimates (4.14), (6.5). Since \( \|\mu\|_{L^2(\Omega)} \leq C(1 + \|\mu - \mathbb{P}\|_{L^2(\Omega)}) \) (recalling (4.21)), we then infer that
\[
\|\mu\|_{L^2(\Omega)} \leq C(1 + \|\partial_t \phi\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}).
\]
Thus, we can deduce that
\[
\int_{\Omega} \mu \nabla \phi \cdot \mathbb{P}(\phi \nabla \partial_t \phi) \, dx
\leq \frac{1}{6} \|\nabla \partial_t \phi\|^2_{L^2(\Omega)} + C(1 + \|\partial_t \phi\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}) \left( \|\phi\|^2_{H^2(\Omega)} \log \left( C\|\phi\|_{W^{2,p}(\Omega)} + 1 \right) \right).
\]
Recalling that \( \mathbb{P} \) is a bounded operator from \( H^1(\Omega) \) to \( H^s \cap H^1(\Omega) \), and using the inequalities (2.4), (2.10), Poincaré’s inequality and Lemma 2.1, we have
\[
- \int_{\Omega} (u \otimes u) : \nabla \mathbb{P}(\nabla \phi \partial_t \phi) \, dx
\leq \|u\|^2_{L^4(\Omega)} \|\mathbb{P}(\nabla \phi \partial_t \phi)\|_{H^1(\Omega)}
\leq C\|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \|\nabla \phi \partial_t \phi\|_{H^1(\Omega)}
\leq C(1 + \|\omega\|_{L^2(\Omega)}) \left( \|\nabla \phi \partial_t \phi\|_{L^2(\Omega)} + \|\nabla^2 \phi \partial_t \phi\|_{L^2(\Omega)} + \|\nabla \phi \nabla \partial_t \phi\|_{L^2(\Omega)} \right)
\leq C(1 + \|\omega\|_{L^2(\Omega)}) \left[ \|\nabla \phi\|_{L^{\infty}(\Omega)} \|\nabla \partial_t \phi\|_{L^2(\Omega)} + \|\nabla^2 \phi\|_{L^2(\Omega)} \log^\frac{1}{2} \left( C\|\nabla^2 \phi\|_{L^2(\Omega)} \right) \right]
\leq C(1 + \|\omega\|_{L^2(\Omega)}) \|\nabla \partial_t \phi\|_{L^2(\Omega)} \left( \|\nabla \phi\|_{H^1(\Omega)} \log^\frac{1}{2} \left( C\|\nabla \phi\|_{W^{1,p}(\Omega)} \right) \right)
\leq C(1 + \|\omega\|_{L^2(\Omega)}) \|\nabla \partial_t \phi\|_{L^2(\Omega)} \left( \|\phi\|_{H^2(\Omega)} \log^\frac{1}{2} \left( C\|\phi\|_{W^{2,p}(\Omega)} + 1 \right) \right)
\leq \frac{1}{6} \|\nabla \partial_t \phi\|^2_{L^2(\Omega)} + C(1 + \|\omega\|^2_{L^2(\Omega)}) \left( \|\phi\|^2_{H^2(\Omega)} \log \left( C\|\phi\|_{W^{2,p}(\Omega)} + 1 \right) \right),
\]
forsome \( p > 2 \).
Combining the above estimates together with (6.10), we arrive at the differential inequality
\[
\frac{d}{dt} H(t) + \frac{1}{2} \|\nabla \partial_t \phi\|^2_{L^2(\Omega)} \leq C(1 + H(t)) \left( \|\phi\|^2_{H^2(\Omega)} \log \left( C\|\phi\|_{W^{2,p}(\Omega)} + 1 \right) \right).
\]
In order to close the estimate, we are left to absorb the logarithmic term on the right-hand side of the above differential inequality. To this aim, we first multiply \( \mu = -\Delta \phi + \Psi'(\phi) \) by \( |F'(\phi)|^{p-2}F'(\phi) \), for some \( p > 2 \), and integrate over \( \Omega \). After integrating by parts and using the boundary condition for \( \phi \), we obtain
\[
\int_{\Omega} \left( p - 1 \right) |F'(\phi)|^{p-2}F''(\phi) |\nabla \phi|^2 \, dx + \|F'(\phi)\|^p_{L^p(\Omega)} = \int_{\Omega} \left( \mu + \theta_0 \phi \right) |F'(\phi)|^{p-2}F'(\phi) \, dx.
\]
By Young’s inequality and the fact that \( F'' > 0 \), we deduce
\[
\| F'(\phi) \|_{L^p(\Omega)} \leq C(1 + \| \mu \|_{L^p(\Omega)}).
\]

Using a well-known elliptic regularity result, together with the above inequality and (6.5), we obtain that (cf. (5.12))
\[
\| \phi \|_{W^{2,p}(\Omega)} \leq C(1 + \| \mu \|_{L^p(\Omega)}).
\]

On the other hand, we infer from equation (6.1) that
\[
\| \mu - \bar{\mu} \|_{L^p(\Omega)} \leq \| \partial_t \phi \|_{L^p(\Omega)} + \| u \cdot \nabla \phi \|_{L^p(\Omega)}.
\]

Then by Poincaré’s inequality and a Sobolev embedding theorem, we find
\[
\| \mu \|_{L^p(\Omega)} \leq C\| \mu - \bar{\mu} \|_{L^p(\Omega)} + C|\bar{\mu}|
\leq C\| \nabla \partial_t \phi \|_{L^2(\Omega)} + C\| u \|_{H^1(\Omega)} \| \phi \|_{H^2(\Omega)} + C(1 + \| \mu - \bar{\mu} \|_{L^2(\Omega)})
\leq C\| \nabla \partial_t \phi \|_{L^2(\Omega)} + C(1 + \| \omega \|_{L^2(\Omega)})(1 + \| \mu - \bar{\mu} \|_{L^2(\Omega)})
\leq C\| \nabla \partial_t \phi \|_{L^2(\Omega)} + C(1 + \| \omega \|_{L^2(\Omega)})(1 + \| \partial_t \phi \|_{L^2(\Omega)} + \| \omega \|_{L^2(\Omega)}).
\]

Thus, for \( p > 2 \), we reach
\[
\| \phi \|_{W^{2,p}(\Omega)} \leq C(1 + \| \nabla \partial_t \phi \|_{L^2(\Omega)} + H(t)),
\]
which, in turn, allows us to rewrite (6.12) as
\[
\frac{d}{dt} H(t) + \frac{1}{2} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 \leq C(1 + H(t)) \left( \| \phi \|_{H^2(\Omega)}^2 \log \left( C(1 + \| \nabla \partial_t \phi \|_{L^2(\Omega)} + H(t)) \right) + 1 \right). \tag{6.13}
\]

We now observe that, for any \( \varepsilon > 0 \), the following inequality holds
\[
x \log(Cy) \leq \varepsilon y + x \log \left( \frac{Cy}{\varepsilon} \right) \quad \forall \, x, y > 0.
\]

By using the above inequality with \( x = 1 + H(t) \), \( y = 1 + \| \nabla \partial_t \phi \|_{L^2(\Omega)} + H(t) \) and \( \varepsilon = 1 \), we deduce that
\[
\frac{d}{dt} H(t) + \frac{1}{2} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 \leq \| \phi \|_{H^2(\Omega)}^4 + C(1 + \| \phi \|_{H^2(\Omega)}^2)(1 + H(t)) \log \left( C(1 + H(t)) \right).
\]

By Young’s inequality, we obtain
\[
\frac{d}{dt} H(t) + \frac{1}{4} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 \leq \| \phi \|_{H^2(\Omega)}^4 + C(1 + \| \phi \|_{H^2(\Omega)}^2)(1 + H(t)) \log \left( C(1 + H(t)) \right).
\]

Recalling that \( \| \phi \|_{H^2(\Omega)}^2 \leq C(1 + H(t)) \), we are finally led to the differential inequality
\[
\frac{d}{dt} H(t) + \frac{1}{4} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 \leq C(1 + \| \phi \|_{H^2(\Omega)})(1 + H(t)) \log \left( C(1 + H(t)) \right). \tag{6.14}
\]

Since \( \phi \in L^2(0,T;H^2(\Omega)) \), then applying the generalized Gronwall lemma B.2, we find the double exponential bound
\[
\sup_{t \in [0,T]} \left( \| \partial_t \phi(t) \|_{L^2(\Omega)}^2 + \| \omega(t) \|_{L^2(\Omega)}^2 \right)
\leq C \left( 1 + \| u_0 \|_{H^1(\Omega)}^2 \| \phi_0 \|_{H^2(\Omega)}^2 + \| \phi_0 \|_{H^2(\Omega)}^2 + \| \Psi'(\phi_0) \|_{L^2(\Omega)}^2 + \| u_0 \|_{H^1(\Omega)}^2 \right)^{\frac{1}{2} + \| \phi(\cdot) \|_{H^2(\Omega)}^2} d\alpha,
\]
for some constant $C > 0$. Here we have used that
\[ \| \partial_t \phi(0) \|_{L^2(\Omega)} \leq C \| \mathbf{u}_0 \|_{H^1(\Omega)} \| \phi_0 \|_{H^2(\Omega)} + C \| \phi_0 \|_{H^2(\Omega)} + C \| \Psi'(\phi_0) \|_{L^2(\Omega)}. \]
Hence, we get
\[ \partial_t \phi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \omega \in L^\infty(0, T; L^2(\Omega)), \] which, in turn, entail that
\[ \mathbf{u} \in L^\infty(0, T; H^1(\Omega)), \quad \phi \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,p}(\Omega)), \] for any $p \in [2, \infty)$.

6.2. Case 2. We now consider an initial condition $$(\mathbf{u}_0, \phi_0)$$ such that
\[ \mathbf{u}_0 \in \mathbf{H}_0 \cap \mathbf{W}^{1,p}(\Omega), \quad \phi_0 \in H^2(\Omega), \quad \partial_n \phi_0 = 0 \text{ on } \partial \Omega, \] for $p \in (2, \infty)$, with $\| \phi_0 \|_{L^\infty(\Omega)} \leq 1$, $|\phi_0| < 1$ and
\[ F'(\phi_0) \in L^2(\Omega), \quad F''(\phi_0) \in L^1(\Omega), \quad \nabla \mu_0 = \nabla (-\Delta \phi_0 + F'(\phi_0)) \in L^2(\Omega). \]
Thanks to the first part of Theorem 6.1, we have a solution $(\mathbf{u}, \phi)$ satisfying (6.15) and (6.16). Moreover, repeating the same argument performed in Section 5, we have (cf. (5.35))
\[ \frac{d}{dt} \int_\Omega F''(\phi) \, dx + \frac{1}{4} \int_\Omega F'''(\phi) F'(\phi) \, dx \leq C, \] for some positive constant $C$ only depending on $\Omega$ and the parameters of the system. Since $F''(\phi_0) \in L^1(\Omega)$, we learn, in particular, that (cf. (5.43))
\[ \int_t^{t+1} \int_\Omega |F''(\phi)|^2 \log(1 + F''(\phi)) \, dx \, dt \leq C, \quad \forall t \geq 0. \] (6.17)
Multiplying (6.6) by $|\phi|^{p-2} \phi$ ($p > 2$) and integrating over $\Omega$, we obtain
\[ \frac{1}{p} \frac{d}{dt} \| \phi \|_{L^p(\Omega)}^p = \int_\Omega \nabla \mu \cdot (\nabla \phi)^\perp |\phi|^{p-2} \phi \, dx. \]
By Hölder’s inequality, we easily get
\[ \frac{1}{p} \frac{d}{dt} \| \phi \|_{L^p(\Omega)}^p \leq \| \nabla \mu \cdot (\nabla \phi)^\perp \|_{L^p(\Omega)} \| \phi \|_{L^p(\Omega)}^{p-1}, \] which, in turn, implies
\[ \frac{1}{2} \frac{d}{dt} \| \phi \|_{L^p(\Omega)}^2 \leq \| \nabla \mu \cdot (\nabla \phi)^\perp \|_{L^p(\Omega)} \| \phi \|_{L^p(\Omega)}. \]
Next, differentiating (6.13) with respect time, then multiplying the resultant by $-\Delta \partial_t \phi$ and integrating over $\Omega$, we obtain
\[ \frac{d}{dt} \frac{1}{2} \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + \| \Delta \partial_t \phi \|_{L^2(\Omega)}^2 = \theta_0 \| \nabla \partial_t \phi \|_{L^2(\Omega)}^2 + \int_\Omega F''(\phi) \partial_t \phi \Delta \partial_t \phi \, dx + \int_\Omega (\mathbf{u} \cdot \nabla \phi) \Delta \partial_t \phi \, dx + \int_\Omega (\mathbf{u} \cdot \nabla \partial_t \phi) \Delta \partial_t \phi \, dx. \]
Here we have used the fact that $\Delta \partial_t \phi = 0$ since $\partial_n \partial_t \phi = 0$ on $\partial \Omega$. Collecting the above estimates, we find that

$$\frac{d}{dt} \left( \frac{1}{2} \| \omega \|^2_{L^p(\Omega)} + \frac{1}{2} \| \nabla \partial_t \phi \|^2_{L^2(\Omega)} \right) \leq \| \nabla \mu \cdot (\nabla \phi)^\perp \|_{L^p(\Omega)} \| \omega \|_{L^p(\Omega)} + \theta_0 \| \nabla \partial_t \phi \|^2_{L^2(\Omega)} + \int_{\Omega} F''(\phi) \partial_t \phi \Delta \partial_t \phi \, dx$$

$$+ \int_{\Omega} (\partial_t \mathbf{u} \cdot \nabla \phi) \Delta \partial_t \phi \, dx + \int_{\Omega} (\mathbf{u} \cdot \nabla \partial_t \phi) \Delta \partial_t \phi \, dx.$$

Notice that, by (6.1)_3, we have the relation $\nabla \mu = \nabla \partial_t \phi + (\nabla \mathbf{u})^\perp \nabla \phi + (\mathbf{u} \cdot \nabla) \nabla \phi$. By exploiting this identity, we obtain

$$\| \nabla \mu \cdot (\nabla \phi)^\perp \|_{L^p(\Omega)} \| \omega \|_{L^p(\Omega)} \leq \left( \| \nabla \partial_t \phi \|_{L^p(\Omega)} + \| (\nabla \mathbf{u})^\perp \nabla \phi \|_{L^p(\Omega)} + \| (\mathbf{u} \cdot \nabla) \nabla \phi \|_{L^p(\Omega)} \right) \| \nabla \phi \|_{L^\infty(\Omega)} \| \omega \|_{L^p(\Omega)}.$$

Using the Gagliardo-Nirenberg inequality (2.5) and the following inequality for divergence free vector fields satisfying the boundary condition (6.2)_1

$$\| \nabla \mathbf{u} \|_{L^p(\Omega)} \leq C(p) \| \omega \|_{L^p(\Omega)}, \quad p \in [2, \infty), \quad (6.18)$$

we deduce that

$$\| \nabla \mu \cdot (\nabla \phi)^\perp \|_{L^p(\Omega)} \| \omega \|_{L^p(\Omega)} \leq C \| \nabla \partial_t \phi \|^2_{L^2(\Omega)} \| \Delta \partial_t \phi \|^\frac{2}{p} \| \nabla \phi \|_{L^\infty(\Omega)} \| \omega \|_{L^p(\Omega)} + C \| \nabla \mathbf{u} \|_{L^p(\Omega)} \| \nabla \phi \|^2_{L^\infty(\Omega)} \| \omega \|_{L^p(\Omega)} \leq \frac{1}{8} \| \Delta \partial_t \phi \|^2_{L^2(\Omega)} + C \| \nabla \phi \|^2_{L^\infty(\Omega)} \| \nabla \partial_t \phi \|^\frac{4}{p} \| \nabla \phi \|^\frac{4}{p} \| \omega \|^\frac{2p}{p^2} \| \| \mathbf{u} \|_{W^{2,p}(\Omega)} \| \nabla \phi \|_{L^\infty(\Omega)} \right) \left( 1 + \| \omega \|^2_{L^p(\Omega)} \right).$$

Next, using (6.1)_1 together with the bounds (6.15), we have

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \nabla \phi \Delta \partial_t \phi \, dx$$

$$\leq \int_{\Omega} \mathbb{P}(\mathbf{u} \cdot \nabla \phi - \Delta \phi \nabla \phi) \cdot \nabla \phi \Delta \partial_t \phi \, dx$$

$$\leq C \| \mathbb{P}(\mathbf{u} \cdot \nabla \phi) \|_{L^2(\Omega)} \| \nabla \phi \Delta \partial_t \phi \|_{L^2(\Omega)} + C \| \mathbb{P}(\Delta \phi \nabla \phi) \|_{L^2(\Omega)} \| \nabla \phi \Delta \partial_t \phi \|_{L^2(\Omega)} \leq \frac{1}{8} \| \Delta \partial_t \phi \|^2_{L^2(\Omega)} + C \| \mathbf{u} \|^2_{L^\infty(\Omega)} \| \nabla \phi \|^2_{L^2(\Omega)} \| \nabla \phi \|^2_{L^\infty(\Omega)} + C \| \Delta \phi \|^2_{L^2(\Omega)} \| \nabla \phi \|^4_{L^\infty(\Omega)} \leq \frac{1}{8} \| \Delta \partial_t \phi \|^2_{L^2(\Omega)} + C \| \mathbf{u} \|^2_{L^\infty(\Omega)} \| \nabla \phi \|^2_{L^\infty(\Omega)} + C \| \nabla \phi \|^4_{L^\infty(\Omega)}.$$

Arguing as for (5.57) and (5.58), we have

$$\int_{\Omega} F''(\phi) \partial_t \phi \Delta \partial_t \phi \, dx \leq \frac{1}{8} \| \Delta \partial_t \phi \|^2_{L^2(\Omega)} + C \| F''(\phi) \|^2_{L^2(\Omega)} \log \left( C \| F''(\phi) \|_{L^2(\Omega)} \right) \| \nabla \partial_t \phi \|^2_{L^2(\Omega)},$$
\[
\int_{\Omega} (\mathbf{u} \cdot \nabla \partial_t \phi) \Delta \partial_t \phi \, dx \leq \frac{1}{8} \|\Delta \partial_t \phi\|^2_{L^2(\Omega)} + C \|\nabla \partial_t \phi\|^2_{L^2(\Omega)}.
\]

Collecting the above estimates and using Young’s inequality, we arrive at the differential inequality
\[
\frac{d}{dt} \left( \|\omega\|^2_{L^p(\Omega)} + \|\nabla \partial_t \phi\|^2_{L^2(\Omega)} \right) + \|\Delta \partial_t \phi\|^2_{L^2(\Omega)} \leq R_1(t) \left( \|\omega\|^2_{L^p(\Omega)} + \|\nabla \partial_t \phi\|^2_{L^2(\Omega)} \right) + R_2(t),
\]
where
\[
R_1 = C \left( 1 + \|\nabla \phi\|^2_{L^\infty(\Omega)} + \|F''(\phi)\|^2_{L^2(\Omega)} \log \left( C \|F''(\phi)\|_{L^2(\Omega)} \right) \right)
\]
and
\[
R_2 = C \|\phi\|^4_{W^{2,p}(\Omega)} + C \left( \|\nabla \phi\|^4_{L^\infty(\Omega)} + 1 \right).
\]

By using (2.10), and recalling (5.7), we see that
\[
\|\nabla \phi\|^4_{L^\infty(\Omega)} \leq C \|\nabla^2 \phi\|^4_{L^2(\Omega)} \log^2 \left( \|\nabla^2 \phi\|_{L^p(\Omega)} \right) + 1 \leq C \log^2 \left( \|\phi\|_{W^{2,p}(\Omega)} \right) + 1,
\]
for \( p > 2 \). In light of (6.16), we infer that both \( R_1 \) and \( R_2 \) belong to \( L^1(0, T) \). Thanks to Gronwall’s lemma, we obtain
\[
\|\omega(t)\|^2_{L^p(\Omega)} + \|\nabla \partial_t \phi(t)\|^2_{L^2(\Omega)} \leq \left( \|\omega(0)\|^2_{L^p(\Omega)} + \|\nabla \partial_t \phi(0)\|^2_{L^2(\Omega)} + \int_0^T R_2(\tau) \, d\tau \right) e^{\int_0^T R_1(\tau) \, d\tau},
\]
for any \( t \in [0, T] \). Since \( \|\omega(0)\|_{L^p(\Omega)} \leq \|\nabla \mathbf{u}_0\|_{L^p(\Omega)} \) and
\[
\|\nabla \partial_t \phi(0)\|_{L^2(\Omega)} \leq \|\nabla \mathbf{u}_0\|_{L^p(\Omega)} \|\nabla \phi_0\|_{L^2(\Omega)} + \|\nabla \phi_0\|_{L^2(\Omega)} + \|\nabla \mu_0\|_{L^2(\Omega)} \leq C \|\nabla \mathbf{u}_0\|_{L^p(\Omega)} \|\phi_0\|_{H^2(\Omega)} + \|\nabla \phi_0\|_{L^2(\Omega)} + \|\nabla \mu_0\|_{L^2(\Omega)},
\]
we deduce that for any \( p \in (2, \infty) \)
\[
\omega \in L^\infty(0, T; L^p(\Omega)), \quad \partial_t \phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).
\]
This, in turn, implies that
\[
\mathbf{u} \in L^\infty(0, T; W^{1,p}(\Omega)), \quad \phi \in L^\infty(0, T; W^{2,p}(\Omega)).
\]
As a consequence, the above estimates yield that
\[
\tilde{\mu} = -\Delta \phi + F'(\phi) \in L^2(0, T; L^\infty(\Omega)).
\]
The rest part of the proof is the same as the proof of Theorem 5.2 with the choice \( \sigma > 0 \).
The proof of Theorem 6.1 is complete.

7. Conclusions and Future Developments

In this paper we present mathematical analysis of some Diffuse Interface models that describe the evolution of incompressible binary mixture having (possibly) different densities and viscosities. We focus on the mass-conserving Allen-Cahn relaxation of the transport equation with the physically relevant Flory-Huggins potential. We show the existence of global weak solution in three dimension and of global strong solutions in two dimensions. For the latter, we discuss additional properties, such as uniqueness, regularity and the separation property. On the other hand, several still unsolved questions concern the analysis of the complex fluid, Navier-Stokes-Allen-Cahn and Euler-Allen-Cahn systems.
in the three dimensional case, which will be the subject of future investigations. We conclude by mentioning some interesting open problems related to the results proved in this work:

- An important possible development of this work is to show the existence of global solutions to the complex fluids system \( (3.1)-(3.3) \) originating from small perturbation of some particular equilibrium states. We mention that some remarkable results in this direction have been achieved in [51, 52, 65] (see also [50] and the references therein). In addition, it would be interesting to study the global existence of weak solutions as in [41] and to generalize Theorem 3.1 to the case with zero viscosity (cf. [51, Theorem 3.1]).

- Two possible improvements of this work concern the Navier-Stokes-Allen-Cahn system \( (4.1)-(4.3) \). The first question is whether the entropy estimates in Theorem 5.1 can be achieved for strong solutions with small initial data, but without restrictions on the parameters of the system, or even without any condition on the initial data. The second issue is to show the uniqueness of strong solutions given from Theorem 5.1-(1), without relying on the entropy estimates in Theorem 5.1-(2). Also, we mention the possibility of considering moving contact lines for the Navier-Stokes-Allen-Cahn system (see [55] for numerical).

- Interesting open issues regarding the Euler-Allen-Cahn system \( (6.1)-(6.3) \) are the existence and the uniqueness of solutions corresponding to an initial datum \( \omega_0 \in L^\infty(\Omega) \) as well as the study of the inviscid limit on arbitrary time intervals (cf. [81] for short times).

ACKNOWLEDGMENTS

Part of this work was carried out during the first and second authors’ visit to School of Mathematical Sciences of Fudan University whose hospitality is gratefully acknowledged. M. Grasselli is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilit e le loro applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). H. Wu is partially supported by NNSFC grant No. 11631011 and the Shanghai Center for Mathematical Sciences at Fudan University.

COMPLIANCE WITH ETHICAL STANDARDS

The authors declare that they have no conflict of interest. The authors also confirm that the manuscript has not been submitted to more than one journal for simultaneous consideration and the manuscript has not been published previously (partly or in full).

APPENDIX A. STOKES SYSTEM WITH VARIABLE VISCOITY

We prove an elliptic regularity result for the following Stokes problem with concentration depending viscosity

\[
\begin{align*}
-\text{div} (\nu(\phi) Du) + \nabla P &= f, & \text{in } \Omega, \\
\text{div } u &= 0, & \text{in } \Omega, \\
 u &= 0, & \text{on } \partial \Omega.
\end{align*}
\]  

(A.1)

This result is a variant of [2, Lemma 4].

**Theorem A.1.** Let \( \Omega \) be a bounded domain of class \( C^2 \) in \( \mathbb{R}^d \), \( d = 2, 3 \). Assume that \( \nu \in W^{1,\infty}(\mathbb{R}) \) such that \( 0 < \nu_* \leq \nu(\cdot) \leq \nu^* \) in \( \mathbb{R} \), \( \phi \in W^{1,r}(\Omega) \) with \( r > d \), and \( f \in L^p(\Omega) \) with \( 1 < p < \infty \) if
$d = 2$ and $\frac{6}{5} \leq p < \infty$ if $d = 3$. Consider the (unique) weak solution $u \in V_\sigma$ to (A.1) such that $(\nu(\phi) Du, \nabla w) = (f, w)$ for all $w \in V_\sigma$. We have:

1. If $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ then there exists $C = C(p, \Omega) > 0$ such that

\[
\|u\|_{w^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} + C\|\nabla \phi\|_{L^r(\Omega)} \|Du\|_{L^2(\Omega)}.
\]  (A.2)

2. Suppose that $u \in V_\sigma \cap W^{1,s}(\Omega)$ with $s > 2$ such that

\[
\frac{1}{p} = \frac{1}{s} + \frac{1}{r}, \quad r \geq \frac{2s}{s-1}.
\]

Then, there exists $C = C(s, p, \Omega) > 0$ such that

\[
\|u\|_{w^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} + C\|\nabla \phi\|_{L^r(\Omega)} \|Du\|_{L^s(\Omega)}.
\]  (A.3)

**Proof.** We denote by $B$ the Bogovskii operator. We recall that $B : L^q_0(\Omega) \to W^{1,q}_0(\Omega), 1 < q < \infty,$ such that $\text{div} \, B f = f$. It is well-known (see, e.g., [31, Theorem III.3.1]) that, for all $1 < q < \infty,$

\[
\|B f\|_{W^{1,q}(\Omega)} \leq C\|f\|_{L^q(\Omega)}.
\]  (A.4)

In addition, by [31, Theorem III.3.4], if $f = \text{div} \, g$, where $g \in L^q(\Omega)$, $1 < q < \infty$, is such that $\text{div} \, g \in L^q(\Omega)$, and $g \cdot n = 0$ on $\partial \Omega$, we have

\[
\|B f\|_{L^q(\Omega)} \leq C\|g\|_{L^q(\Omega)}.
\]  (A.5)

For the sake of simplicity, we start proving the second part of Theorem A.1, and then we show the first part.

**Case 2.** Let us take $v \in C_{0,\sigma}^\infty(\Omega)$. As in [2, Lemma 4], we define $w = \frac{v}{\nu(\phi)} - B[\text{div} \left( \frac{v}{\nu(\phi)} \right)]$. We observe that $w \in W^{1,r}_0(\Omega)$ with $\text{div} \, w = 0$. In particular, $w \in V_\sigma$. Taking $w$ in the weak formulation, we obtain

\[
(Du, \nabla v) = \left( f, \frac{v}{\nu(\phi)} - B \left[ \text{div} \left( \frac{v}{\nu(\phi)} \right) \right] \right) - \left( \nu(\phi) Du, v \otimes \nabla \left( \frac{1}{\nu(\phi)} \right) \right) + \left( \nu(\phi) Du, \nabla B \left[ \text{div} \left( \frac{v}{\nu(\phi)} \right) \right] \right)
\]

Since $\frac{2s}{s-1} \leq r$, we deduce that $r \geq p'$ ($\frac{1}{p'} = 1 - \frac{1}{p}$). This implies that $\text{div} \left( \frac{v}{\nu(\phi)} \right) \in L^{p'}(\Omega)$. By using the assumptions on $\nu$ and the estimate (A.5) with $q = p'$, we find

\[
\left| \left( f, \frac{v}{\nu(\phi)} - B \left[ \text{div} \left( \frac{v}{\nu(\phi)} \right) \right] \right) \right| \leq \|f\|_{L^p(\Omega)} \left( \frac{1}{\nu_\ast} \|v\|_{L^{p'}(\Omega)} + \left\| B \left[ \text{div} \left( \frac{v}{\nu(\phi)} \right) \right] \right\|_{L^{p'}(\Omega)} \right)
\]

\[
\leq C\|f\|_{L^p(\Omega)} \left( \frac{1}{\nu_\ast} \|v\|_{L^{p'}(\Omega)} + \left\| \frac{v}{\nu(\phi)} \right\|_{L^{p'}(\Omega)} \right)
\]

\[
\leq C\|f\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.
\]

Also, we have

\[
\left| \left( \nu(\phi) Du, v \otimes \nabla \left( \frac{1}{\nu(\phi)} \right) \right) \right| = \left| \left( Du, \frac{\nu'(\phi)}{\nu^2(\phi)} v \otimes \nabla \phi \right) \right|
\]

\[
\leq C\|Du\|_{L^s(\Omega)} \|\nabla \phi\|_{L^r(\Omega)} \|v\|_{L^{p'}(\Omega)}.
\]
Recalling that $\text{div} \, v = 0$ and $r > s'$, by using (A.4) we obtain
\[
\left| \left( \nu(\phi) Du, \nabla B \left[ \text{div} \left( \frac{v}{\nu(\phi)} \right) \right] \right) \right| \leq \| Du \|_{L^s(\Omega)} \left\| \nabla B \left[ \nabla \left( \frac{1}{\nu(\phi)} \right) \cdot v \right] \right\|_{L'(\Omega)}
\leq C \| Du \|_{L^s(\Omega)} \| \nabla \left( \frac{1}{\nu(\phi)} \right) \cdot v \|_{L'(\Omega)}
\leq C \| Du \|_{L^s(\Omega)} \| \nabla \phi \|_{L^r(\Omega)} \| v \|_{L'(\Omega)}.
\]

Therefore, by the Riesz representation theorem and a density argument, we find
\[
(Du, \nabla v) = (\tilde{f}, v) \quad \forall v \in V_\sigma,
\]
where
\[
\| \tilde{f} \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} + C \| Du \|_{L^s(\Omega)} \| \nabla \phi \|_{L^r(\Omega)}.
\]
for some $C$ depending on $s, p$ and $\Omega$. By the regularity of the Stokes operator (see, e.g., [31, Theorem IV.6.1]), the claim easily follows.

**Case 1.** We consider $\phi_n \in C_c(\Omega)$ such that $\phi_n \to \phi$ in $W^{1,r}(\Omega)$ as $n \to \infty$. For any $n \in \mathbb{N}$, we define $u_n$ as the solution to
\[
(\nu(\phi_n) Du_n, \nabla w) = (f, w) \quad \forall w \in V_\sigma.
\]
Since $\nu(\cdot) \geq \nu_* > 0$, by taking $w = u_n$, it is easily seen that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $V_\sigma$ independently of $n$. In addition, recalling that $W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega)$, we have $\nu(\phi_n) \to \nu(\phi)$ in $L^\infty(\Omega)$. By uniqueness of the weak solution $u$ to (A.1), we deduce that $u_n \rightharpoonup u$ weakly in $V_\sigma$.

Let us take $w = \frac{v}{\nu(\phi_n)} - B(\text{div} \left( \frac{v}{\nu(\phi_n)} \right))$ with $v \in C_0(\Omega)$. Then we find
\[
(Du_n, \nabla v) = \left( f, \frac{v}{\nu(\phi_n)} - B \left[ \text{div} \left( \frac{v}{\nu(\phi_n)} \right) \right] \right)
- \left( \nu(\phi_n) Du_n, v \otimes \nabla \left( \frac{1}{\nu(\phi_n)} \right) \right) + \left( \nu(\phi_n) Du_n, \nabla B \left[ \text{div} \left( \frac{v}{\nu(\phi_n)} \right) \right] \right).
\]
Note that, by construction, $\frac{v}{\nu(\phi_n)} \in W^{1,q}(\Omega)$ for all $q \in [1, \infty]$. Therefore, by repeating the same computations carried out above with $s = 2$, we arrive at
\[
(Du_n, \nabla v) = (\tilde{f}, v) \quad \forall v \in V_\sigma,
\]
where
\[
\| \tilde{f} \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} + C \| Du_n \|_{L^2(\Omega)} \| \nabla \phi_n \|_{L^r(\Omega)},
\]
for some $C$ depending on $p$ and $\Omega$. By the regularity theory of the Stokes operator, we infer
\[
\| u_n \|_{W^{2,p}(\Omega)} \leq C \| f \|_{L^p(\Omega)} + C \| Du_n \|_{L^2(\Omega)} \| \nabla \phi_n \|_{L^r(\Omega)}.
\]
Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $V_\sigma$ and $\phi_n \to \phi$ in $W^{1,r}(\Omega)$, $u_n$ is bounded in $W^{2,p}(\Omega)$ independently of $n$. By the choice of the parameters $r > d$ and $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$, $W^{2,p}(\Omega) \cap V_\sigma$ is compactly embedded in $V_\sigma$.

In particular, $\| Du_n \|_{L^2(\Omega)} \to \| Du \|_{L^2(\Omega)}$ as $n \to \infty$. As a consequence, by the lower semi-continuity of the norm with respect to the weak topology, the conclusion follows. The proof is complete. □
APPENDIX B. SOME LEMMAS ON ODE INEQUALITIES

For convenience of the readers, we collect some useful results concerning ODE inequalities that have been used in this paper. First, we report the Osgood lemma.

**Lemma B.1.** Let $f$ be a measurable function from $[0, T]$ to $[0, a]$, $g \in L^1(0, T)$, and $W$ a continuous and nondecreasing function from $[0, a]$ to $\mathbb{R}^+$. Assume that, for some $c \geq 0$, we have

$$f(t) \leq c + \int_0^t g(s)W(f(s)) \, ds, \quad \text{for a.e. } t \in [0, T].$$

- If $c > 0$, then for almost every $t \in [0, T]$
  $$-\mathcal{M}(f(t)) + \mathcal{M}(c) \leq \int_0^T g(s) \, ds, \quad \text{where } \mathcal{M}(s) = \int_s^a \frac{1}{W(s)} \, ds.$$

- If $c = 0$ and $\int_0^a \frac{1}{W(s)} \, ds = \infty$, then $f(t) = 0$ for almost every $t \in [0, T]$.

Next, we report two generalizations of the classical Gronwall lemma and of the uniform Gronwall lemma.

**Lemma B.2.** Let $f$ be a positive absolutely continuous function on $[0, T]$ and $g$, $h$ be two summable functions on $[0, T]$ that satisfy the differential inequality

$$\frac{d}{dt} f(t) \leq g(t) f(t) \ln (e + f(t)) + h(t),$$

for almost every $t \in [0, T]$. Then, we have

$$f(t) \leq (e + f(0))^{\int_0^t g(\tau) \, d\tau} e^{\int_0^t e^{\int_0^\tau g(s) \, ds} h(\tau) \, d\tau}, \quad \forall t \in [0, T].$$

**Lemma B.3.** Let $f$ be an absolutely continuous positive function on $[0, \infty)$ and $g$, $h$ be two positive locally summable functions on $[0, \infty)$ which satisfy the differential inequality

$$\frac{d}{dt} f(t) \leq g(t) f(t) \ln (e + f(t)) + h(t),$$

for almost every $t \geq 0$, and the uniform bounds

$$\int_t^{t+r} f(\tau) \, d\tau \leq a_1, \quad \int_t^{t+r} g(\tau) \, d\tau \leq a_2, \quad \int_t^{t+r} h(\tau) \, d\tau \leq a_3, \quad \forall t \geq 0,$$

for some positive constants $r$, $a_1$, $a_2$, $a_3$. Then, we have

$$f(t) \leq e^{\left(\frac{r+a_1}{r+a_2}\right)a_2}, \quad \forall t \geq r.$$

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