TRAIN TRACK COMPLEX OF ONCE-PUNCTURED TORUS AND 4-PUNCTURED SPHERE

KEITA IBARAKI

ABSTRACT. Consider a compact oriented surface $S$ of genus $g \geq 0$ and $m \geq 0$ punctured. The train track complex of $S$ which is defined by Hamenstädt is a 1-complex whose vertices are isotopy classes of complete train tracks on $S$. Hamenstädt shows that if $3g - 3 + m \geq 2$, the mapping class group acts properly discontinuously and cocompactly on the train track complex. We will prove corresponding results for the excluded case, namely when $S$ is a once-punctured torus or a 4-punctured sphere. To work this out, we redefine two complexes for these surfaces.

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1. INTRODUCTION

Consider a compact oriented surface $S$ of genus $g \geq 0$ from which $m \geq 0$ points, so-called punctures, have been deleted. The mapping class group $\mathcal{M}(S)$ of $S$ is, by definition, the space of isotopy classes of orientation preserving homeomorphisms of $S$.

There are natural metric graphs on which the mapping class group $\mathcal{M}(S)$ acts by isometries. Among them, we will concern with the curve complex $\mathcal{C}(S)$ (or, rather, its one-skeleton) and train track complex $TT(S)$.

In [Har81], Harvey defined the curve complex $\mathcal{C}(S)$ for $S$. The vertex of this complex is a free homotopy class of an essential simple closed

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curve on $S$, i.e. a simple closed curve which is neither contractible nor homotopic into a puncture. A $k$-simplex of $\mathcal{C}(S)$ is spun by a collection of $k + 1$ vertices which are realized by mutually disjoint simple closed curves.

The train track which is an embedded 1-complex was invented by Thurston [Thu78] and provides a powerful tool for the investigation of surfaces and hyperbolic 3-manifolds. A detailed account on train tracks can be found in the book [Pen92] of Penner with Harer.

Hamenstädt defined in [Ham05a] that a train track is called complete if it is a birecurrent and each of complementary regions is either a trigon or a once-punctured monogon. Hamenstädt defined the train track complex $TT(S)$ for $S$. Vertices of the train track complex are isotopy classes of complete train tracks on $S$.

Suppose $3g - 3 + m \geq 2$, i.e. $S$ is a hyperbolic surfaces but neither a once-punctured torus nor 4-punctured sphere. Both the curve complex and the train track complex can be endowed with a path-metric by declaring all edge lengths to be equal to 1. In these cases, there are a map $\Phi : TT(S) \to \mathcal{C}(S)$ and a number $C > 0$ such that $d(\Phi(\tau), \Phi(\tau')) \leq Cd(\tau, \tau')$ for all complete train track $\tau, \tau'$ on $S$.

If $S$ is a once-punctured torus or a 4-punctured sphere, then any essential simple closed curve must intersect, so that $\mathcal{C}(S)$ has no edge by definition. However, Minsky [Min96] adopt a small adjustment in the definition for curve complex in these two particular case so that it becomes a sensible and familiar 1-complex: two vertices are connected by an edge when the curves they represent have minimal intersection (1 in the case of a once-punctured torus, and 2 in the case of a 4-punctured sphere). It turns out that in both cases, the complex is the Farey graph $\mathcal{F}$.

In addition, if $S$ is a once-punctured torus, there is no complete train track under Hamenstädt’s definition and $TT(S)$ is homeomorphic to empty set. We thus adopt here Penner’s definition [Pen92], i.e. the train track $\tau$ is complete iff $\tau$ is birecurrent and is not a proper subtrack of any birecurrent train track. These two definitions are equivalent except for the once-punctured torus.

Our main theorem is:

**Theorem 1.1.** Suppose $S$ is a once-punctured torus or a 4-punctured sphere. Then the train track complex $TT(S)$ of $S$ is quasi-isometric to the dual graph of the Farey graph $\mathcal{F}$ (see Figure 1).

More precisely, if $S$ is a once-punctured torus, we show:

**Theorem 1.2.** The train track complex of a once-punctured torus is isomorphic to the Caley graph of $PSL(2, \mathbb{Z}) = \langle r, l \mid (lr^{-1})^2 = 1, (lr^{-1})^3 = 1 \rangle$.

In [Ham05a], Hamenstädt also shows that if $3g - 3 + m \geq 2$ the mapping class group $\mathcal{M}(S)$ acts p.d.c., i.e. properly discontinuously...
and cocompactly, on the train track complex $TT(S)$ and $\mathcal{M}(S)$ is quasi-isometric to $TT(S)$. We can prove that the same is true for the once-punctured torus and 4-punctured sphere.

**Corollary 1.3.** Suppose $S$ is the once-punctured torus or 4-punctured sphere. Then $\mathcal{M}(S)$ acts p.d.c. on $TT(S)$.

**Corollary 1.4.** Suppose $S$ is the once-punctured torus or 4-punctured sphere. Then $\mathcal{M}(S)$ is quasi-isometric to $TT(S)$.

In Section 2, we give a brief review of quasi-isometries. In Section 3, we describe train tracks and define the train track complex. In Section 4, we describe how to build a Farey graph which is used for curve complex of a once-punctured torus or a 4-punctured sphere. In Section 5 and 6, we prove the Theorem 1.1. Finally, we describe the action of mapping class groups on train track complexes in Section 7.

2. QUASI-ISOMETRY

A quasi-isometry is one of the fundamental notion in geometric group theory. For details, see [Bow06].

Let $(X,d)$ be a proper geodesic space, i.e. a complete and locally compact geodesic space. Given $x \in X$ and $r \geq 0$, write $N(x,r) = \{ y \in X \mid d(x,y) \leq r \}$ for the closed $r$-neighborhood of $x$ in $X$. If $A \subseteq X$, write $N(A,r) = \bigcup_{x \in A} N(x,r)$. We say that $A$ is cobounded if $X = N(A,r)$ for some $r \geq 0$.

Suppose that a group $\Gamma$ acts on $X$ by isometry. Given $x \in X$, we write $\Gamma x = \{ gx \mid g \in \Gamma \}$ for the orbit of $x$ under $\Gamma$, and $\text{stab}(x) = \{ g \in \Gamma \mid gx = x \}$ for its stabilizer.
We say that the action of $\Gamma$ on $X$ is \textit{properly discontinuous} if for all $r \geq 0$ and all $x \in X$, the set $\{g \in \Gamma \mid d(x, gx) \leq r\}$ is finite. A properly discontinuous action is called \textit{cocompact} if $X/\Gamma$ is compact. We will frequently abbreviate “properly discontinuous and cocompact” by p.d.c.

\textbf{Proposition 2.1} \textit{([Bow06])}. The followings are equivalent:

(i) The action is cocompact,
(ii) Some orbit is cobounded, and
(iii) Every orbit is cobounded.

\textbf{Proof.} Write $N'(\Gamma x, r)$ for a closed $r$-neighborhood of $\Gamma x$ in $X/\Gamma$. Let $\pi: X \to X/\Gamma$ be a quotient map. Then for any $x \in X$ and any $r > 0$ $\pi(N(x, r)) = N'(\Gamma x, r)$ and $\pi^{-1}(N'(\Gamma x, r)) = N(\Gamma x, r)$.

- (iii) $\Rightarrow$ (ii) is clear.
- Suppose that some orbit is cobounded. So, $X = N(\Gamma x, r)$ for some $x \in X$ and some $r > 0$. Thus, $X/\Gamma = \pi(X) = \pi(N(\Gamma x, r)) = N'(\Gamma x, r) = \pi(N(x, r))$. By Proposition 3.1 of [Bow06], $N(x, r)$ is compact and hence $X/\Gamma = \pi(N(x, r))$ is also compact. Now we proved (ii) $\Rightarrow$ (i).
- Suppose the action is cocompact. So, $X/\Gamma$ is compact and hence $X/\Gamma$ is bounded. Thus, for any $x \in X$ there is some $r > 0$, such that $X/\Gamma = N'(\Gamma x, r)$. Since $X = \pi^{-1}(X/\Gamma) = \pi^{-1}(N'(\Gamma x, r)) = N(\Gamma x, r)$, $\Gamma x$ is cobounded and (i) $\Rightarrow$ (iii) is shown.

\textbf{Definition 2.2} (quasi-isometry). Let $(X, d)$ and $(Y, d')$ be metric spaces. A map $\varphi: X \to Y$ is called a \textit{quasi-isometry} if there are constants $k_1 > 0, k_2, k_3, k_4 \geq 0$ such that for all $x_1, x_2 \in X$,

$$k_1 d(x_1, x_2) - k_2 \leq d'(\varphi(x_1), \varphi(x_2)) \leq k_3 d(x_1, x_2) + k_4,$$

and the image $\varphi(x)$ is cobounded in $Y$.

Thus, a quasi-isometry is bi-Lipshitz with bounded error and its image is cobounded. We note that the quasi-isometry introduces an equivalence relation on the set of metric spaces.

Two metric spaces, $X$ and $Y$, are said to be \textit{quasi-isometric} if there is a quasi-isometry between them.

Let $X$ be a geodesic space and $A$ a finite generating set for a group $\Gamma$. Suppose $\Delta(\Gamma, A)$ is the Cayley graph of $\Gamma$ with respect to $A$. If $B$ is another generating set for $\Gamma$, then $\Delta(\Gamma, A)$ is quasi-isometric to $\Delta(\Gamma, B)$. Thus, we simply denote the Cayley graph of $\Gamma$ by $\Delta(\Gamma)$ without specifying a generating set. A group $\Gamma$ acts p.d.c. on its Cayley graph $\Delta(\Gamma)$.

We define that $\Gamma$ is quasi-isometric to $X$ if $\Delta(\Gamma)$ is quasi-isometric to $X$. Also, two groups $\Gamma, \Gamma'$ are quasi-isometric if $\Delta(\Gamma)$ is quasi-isometric to $\Delta(\Gamma')$. 
The proof of the following claims can be found for example in [Bow06]:

**Theorem 2.3** ([Bow06]). If $\Gamma$ acts p.d.c. on a proper geodesic space $X$, then $\Gamma$ is quasi-isometric to $X$.

**Proposition 2.4.** Let $\Gamma$ be a finitely generated group. Suppose that $G$ is a subgroup of $\Gamma$ of finite index. Then $G$ is finitely generated and quasi-isometric to $\Gamma$.

3. **Train track complex**

A *train track* on $S$ (see [Pen92]) is an embedded 1-complex $\tau \subset S$ whose vertices are called *switches* and edges are called *branches*. $\tau$ is $C^1$ away from its switches. At any switch $v$ the incident edges are mutually tangent and there is an embedding $f : (0,1) \to \tau$ with $f(1/2) = v$ which is a $C^1$ map into $S$. The valence of each switch is at least 3, except possibly for one bivalent switch in a closed curve component.

Finally, we require that every component $D$ of $S - \tau$ has negative generalized Euler characteristic in the following sense: define $\chi'(D)$ to be the Euler characteristic $\chi(D)$ minus $1/2$ for every outward-pointing cusp (internal angle 0). For the train track complementary regions all cusps are outward, so that the condition $\chi'(D) < 0$ excludes annuli, once-punctured disks with smooth boundary, or non-punctured disks with 0, 1 or 2 cusps at the boundary. We will usually consider isotopic train-tracks to be the same.

A train track is called *generic* if all switches are at most trivalent. A *train route* is a non-degenerate smooth path in $\tau$; in particular it traverses a switch only by passing from incoming to outgoing edge or vice versa. The train track $\tau$ is called *recurrent* if every branch is contained in a closed train route. The train track $\tau$ is called *transversely recurrent* if every branch intersects transversely with a simple closed curve $c$ so that $S - \tau - c$ does not contain an embedded bigon, i.e. a disc with two corners. A train track which is both recurrent and transversely recurrent is called *birecurrent*.

A curve $c$ is *carried* by a transversely recurrent train track $\tau$ if there is a *carrying map* $F : S \to S$ of class $C^1$ which is homotopic to the identity and maps $c$ to $\tau$ in such a way that the restriction of its differential $dF$ to every tangent line of $c$ is non-singular. A train track $\tau'$ is *carried* by $\tau$ if there is a carrying map $F$ and every train route on $\tau'$ is carried by $\tau$ with $F$.

A generic birecurrent train track is called *complete* if it is not a proper subtrack of any birecurrent train track.

**Theorem 3.1** ([Pen92]).

(i) If $g > 1$ or $m > 1$, then any birecurrent train track on $S$ is a subtrack of a complete train track, each of whose complementary region is either a trigon or a once-punctured monogon.
(ii) Any birecurrent train track on a once-punctured torus is a sub-track of a complete train track whose unique complementary region is a once-punctured bigon.

It follows:

**Corollary 3.2.** Suppose $\tau$ is a complete train track on $S$ of genus $g$ with $m$ punctures. Then the number of switches of $\tau$ depends only on the topological type of $S$.

**Proof.** If $S$ is the once-punctured torus, then $\tau$ have 2 vertices.

In the other case, let $n_t$ be the number of triangle component of $S-\tau$, $n_s$ be the number of switches of $\tau$ and $n_b$ be the number of branches of $\tau$. Since $\tau$ is generic, $2n_b = 3n_s$. By Theorem 3.1 $n_s = 3n_t + m$. By Euler characteristic, $n_t - n_b + n_s = 2 - 2g - m$. Now we get $n_s = 4(3g - 3 + m)$. □

A half-branch $\tilde{b}$ in a generic train track $\tau$ incident on a switch $v$ is called large if the switch $v$ is trivalent and if every arc $\rho: (\varepsilon, \varepsilon) \rightarrow \tau$ of class $C^1$ which passes through $v$ meets the interior of $\tilde{b}$. A branch $b$ in $\tau$ is called large if each of its two half-branches is large; in this case $b$ is necessarily incident to two distinct switches.

There is a simple way to modify a complete train track $\tau$ to another complete train track. Namely, if $e$ is a large branch of $\tau$ then we can perform a right or left split of $\tau$ at $e$ as shown in Figure 2. A complete train track $\tau$ can always be at least one of the left or right split at any large branch $e$ to a complete train track $\tau'$ (see [Ham05b]). We note that $\tau'$ is carried by $\tau$.

![Figure 2. a split](image-url)

**Definition 3.3** (train track complex). A train track complex $TT(S)$ is defined as follow: The set of vertices of $TT(S)$ consists of all isotopy classes of complete train tracks on $S$. Two Complete train tracks $\tau, \tau'$ is connected with an edge if $\tau'$ can be obtained from $\tau$ by a single split.

For each switch $v$ of $\tau$, fix a direction of the tangent line to $\tau$ at $v$. The branch $b$ which is incident to $v$ is called incoming if the direction at $v$ coincides with the direction from $b$ to $v$, and outgoing if not.
transverse measure on $\tau$ is a non-negative function $\mu$ on the set of branches satisfying the switch condition: For any switch of $\tau$ the sums of $\mu$ over incoming and outgoing branches are equal. A train track $\tau$ is recurrent if and only if it supports a transverse measure which is positive on every branch (see [Pen92]).

For a recurrent train track $\tau$ the set $P(\tau)$ of all transverse measures on $\tau$ is a convex cone in a linear space. A vertex cycle (see [MM99]) on $\tau$ is a transverse measure $\mu$ which spans an extremal ray in $P(\tau)$. Up to scaling, a vertex cycle $\mu$ is a counting measure of a simple closed curve $c$ which is carried by $\tau$. This means that for a carrying map $F : c \rightarrow \tau$ and every open branch $b$ of $\tau$ the $\mu$-weight of $\tau$ equals the number of connected components of $F^{-1}(b)$. We also use the notion, a vertex cycle, for the simple closed curve $c$.

**Proposition 3.4** ([Ham05b]). Let $\tau$ be a complete train track. Suppose $c$ is a vertex cycle on $\tau$ with a carrying map $F$. Then, $F(c)$ passes through every branch of $\tau$ at most twice, and with different orientation if any.

Proposition 3.4 and Corollary 3.2 imply that the number of vertex cycles on a complete train track on $S$ is bounded by a universal constant (see [MM99]). Moreover, there is a number $D > 0$ with the property that for every complete train track $\tau$ on $S$ the distance in $C(S)$ between any two vertex cycles on $\tau$ is at most $D$ (see [Ham05b], [MM04]).

4. **Farey graph**

Let $S$ be the once-punctured torus or the 4-punctured sphere. The essential simple closed curves on $S$ are well known to be in one-to-one correspondence with rational numbers $p/q$ with $1/0 = \infty$. Thus the 0-skeleton of $C(S)$ is identified with $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ in the circle $S^1 = \mathbb{R} \cup \infty$.

There are numerous ways to build a Farey graph $\mathcal{F}$, any of them produces an isomorphic graph. One can start with the rational projective line $\hat{\mathbb{Q}}$, identifying 0 with 0/1 and $\infty$ with 1/0, and take this to be the vertex set of $\mathcal{F}$. Then, two projective rational numbers $p/q, r/s \in \hat{\mathbb{Q}}$, where $p$ and $q$ are coprime and $r$ and $s$ are coprime, are deemed to span an edge, or 1-simplex, if and only if $|ps - rq| = 1$. The result is a connected graph in which every edge separates. The graph $\mathcal{F}$ can be represented on a disc; see Figure 3. We shall say a graph is a Farey graph if it is isomorphic to $\mathcal{F}$.

Note that the curve complexes of a once-punctured torus and 4-punctured sphere are Farey graphs (see [Min96], [APS06]).

**Remark.** The Farey graph $\mathcal{F}$ is quasi-isometric to the dual graph.
5. **Train tracks on the once-punctured torus**

Let $S_{1,1}$ be the once-punctured torus and $\tau$ a complete train track on $S_{1,1}$. By Theorem 3.1 and Corollary 3.2, $\tau$ has the unique complementary region that is a once-punctured bigon and the number of switches of $\tau$ equals 2. It follows that every complete train track on $S_{1,1}$ is orientation preserving $C^1$-diffeomorphic to the one illustrated in Figure 4.

![Figure 4. complete train track on Once-punctured torus](image)

$\tau$ has exactly two vertex cycles $c_1, c_2$ whose intersection number $i(c_1, c_2)$ equals 1. Thus, $c_1$ and $c_2$ is connected by an edge in Farey graph. Conversely, if we fix simple closed curve $c_1, c_2$ on $S_{1,1}$ whose intersection number $i(c_1, c_2)$ equals 1, then there is only two complete train tracks whose vertex cycles are $c_1$ and $c_2$.

Write $V(G)$ for the vertex set of a graph $G$ and $E(G)$ for the set of all edges in $G$. We define a map $\varphi : V(TT(S_{1,1})) \to E(\mathcal{F})$ as follow:
Let $\tau \in S_{1,1}$. Suppose $c_1, c_2$ are vertex cycles on $\tau$. We define $\varphi(\tau)$ as the edge of $\mathcal{F}$ which connects $c_1$ and $c_2$.

We construct the graph $G_1$ of $\varphi$ as follows: Let $V(G_1) = E(\mathcal{F})$. We connect vertices $e_1, e_2 \in E(\mathcal{F})$ if some $\tau_1 \in \varphi^{-1}(e_1)$ and some $\tau_2 \in \varphi^{-1}(e_2)$ are connected by an edge in $TT(S_{1,1})$. $\varphi$ can be naturally extended to $\varphi : TT(S_{1,1}) \to G_1$.

**Lemma 5.1.** $G_1$ is quasi-isometric to the dual graph of the Farey graph.

**Proof.** Suppose $\tau_1$ and $\tau_2$ are different complete train tracks on $S_{1,1}$ which have common vertex cycles $c_1, c_2$. $\tau_1$ have a unique large edge $b$ and there are two complete train tracks $\tau_1', \tau_1''$ which are obtained by a left or right split of $\tau_1$ at $b$ (see Figure 5). We can see that one vertex cycle on $\tau_1'$ is the same as $c_1$ on $\tau_1$, and the other vertex cycle $c_3$ on $\tau_1'$ intersects $c_2$ on $\tau_1$ at one point. Thus $\varphi(\tau_1')$ and $\varphi(\tau_1)$ are adjacent edges in $\mathcal{F}$. In this case, vertex cycles on $\tau_1''$ are $c_2$ and $c_3$. Hence $\varphi(\tau_1)$, $\varphi(\tau_1')$ and $\varphi(\tau_1'')$ span a triangle on $\mathcal{F}$. Similarly, we can get complete train tracks $\tau_2', \tau_2''$ by a split of $\tau_2$, and $\varphi(\tau_2)$, $\varphi(\tau_2')$ and $\varphi(\tau_2'')$ span another triangle on $\mathcal{F}$. (see Figure 6)

The mapping class group $\mathcal{M}(S_{1,1})$ acts isometrically on $TT(S_{1,1})$ and $\mathcal{F}$ and acts transitively on $V(G_1) = E(\mathcal{F})$. It follows that every edge in $G_1$ connects an adjacent edge of $\mathcal{F}$ and every adjacent edge of $\mathcal{F}$ is connected with a direct edge in $G_1$. Thus $G_1$ is the line graph of the dual of $\mathcal{F}$, i.e. vertices of $G_1$ represent edges of the dual of $\mathcal{F}$ and two vertices are adjacent iff their corresponding edges share a common endpoint (see Figure 7). It’s now obvious that $G_1$ is quasi-isometric to the dual of the Farey graph. \qed

**Lemma 5.2.** $TT(S_{1,1})$ is connected.
Proof. Let $e \in E(F)$ and $\varphi^{-1}(e) := \{\tau, \tau'\}$. All we need is to show that $\tau$ and $\tau'$ are connected in $TT(S_{1,1})$, since $G_1$ is connected. $\tau$ can be a right(left) split to a complete train track $\tau_1$. Then, there is a complete train track $\tau_2$ which can be a right(left) split to $\tau'$ and can be a left(right) split to $\tau_1$ (see Figure 8). Thus $\tau$ and $\tau'$ are connected and $d(\tau, \tau') = 3$. □

Lemma 5.3. $\varphi$ is a quasi-isometry.

Proof. Let $\tau, \tau' \in V(TT(S_{1,1}))$. Suppose $\alpha$ is geodesic on $G_1$ from $\varphi(\tau)$ to $\varphi(\tau')$. $\tau, \tau' \in \varphi^{-1}(\alpha)$ and $\text{diam}(\varphi^{-1}(\alpha)) \leq 4d(\varphi(\tau), \varphi(\tau')) + 3$ since
$diam(\varphi^{-1}(e)) = 3$ for any $e \in E(F)$. Thus $d(\tau, \tau') \leq 4d(\varphi(\tau), \varphi(\tau')) + 3$.

It follows that $\varphi$ is a quasi-isometry. $\square$

**Proof of Theorem 1.1 (a once-puncture torus case).** By Lemma 5.1 and 5.3, $TT(S_{1,1})$ is quasi-isometric to the dual graph of the Farey graph. $\square$

$TT(S_{1,1})$ is obtained by extending one vertex of $G_1$ to two vertices. When we think the action of the mapping class group, we see that $TT(S_{1,1})$ is isomorphic to the graph as in Figure 9. We can notice that this graph is isomorphic to the Cayley graph of $PSL(2,\mathbb{Z}) = \langle r, l \mid (lr^{-1}l)^2 = 1, (lr^{-1})^3 = 1 \rangle$, and thus Theorem 1.2 is proved.

![Figure 9.](image)

6. **Train tracks on the 4-punctured sphere**

Let $S_{0,4}$ be the 4-punctured sphere. A train track complex of $S_{0,4}$ is similar to that of the once-punctured torus but more complicated.

Orientation preserving $C^1$–diffeomorphism classes of a complete train tracks $\tau$ depends on combination of switches and branches. By Proposition 3.2, number of switches and branches of $\tau$ are constants. Thus, number of orientation preserving $C^1$–diffeomorphism classes of the complete train tracks is finite. In fact, complete train tracks on $S_{0,4}$ are classified into 13 classes, illustrated in (1) to (8) of Figure 10 and their mirror images, though (1), (4) and (8) can move these mirrors by orientation preserving $C^1$–diffeomorphism.

We can see that all of those train tracks have exactly two vertex cycles $c,c'$ whose intersection number $i(c,c')$ equals 2 or 4.
Figure 10.
First, we confirm connectivity of $\mathcal{T}\mathcal{T}(S_{0,4})$:

**Proposition 6.1.** $\mathcal{T}\mathcal{T}(S_{0,4})$ is connected.

First we look at a $C^1$–diffeomorphism class which has two large edges and whose two vertex cycles intersect at two points (1 of Figure 10). We write $A_1$ for the collection of these train tracks.

Let $\tau \in A_1$. $\tau$ can split at two large edges $b_1, b_2$. We can get another complete train track $\tau_1$ by a right split $\tau$ at $b_1$ ($\tau_1$ is $C^1$–diffeomorphic to (2) of Figure 10). $\tau_1$ can be right split at $b_2$ to a complete train track $\tau'$, and we can find that $\tau' \in A_1$. Incidentally, if we left split $\tau_1$ at $b_2$, we cannot get a complete train track (see Figure 11). In the same way, we can get $\tau'' \in A_1$ by being left splits $\tau$ at both $b_1$ and $b_2$. That is to say, we can get two complete train tracks $\tau', \tau'' \in A_1$ by splits at both $b_1$ and $b_2$ of $\tau$.

![Figure 11.](image)

We construct graph $T_1$ as follow: $V(T_1)$ is $A_1$. We connect $\tau, \tau' \in A$ by an edge if $\tau'$ can be obtained by splits at two large edges of $\tau$. Clearly, $T_1$ is homeomorphic to subgraph of $\mathcal{T}\mathcal{T}(S_{0,4})$.

**Lemma 6.2.** $T_1$ is connected and is quasi-isometric to the dual of $\mathcal{F}$.
Proof. Let $\tau \in V(T_1)$. $\tau$ has two vertex cycles connected by an edge in $\mathcal{F}$. Thus, we can think it just the same as $TT(S_{1,1})$. As a result, we can get this Lemma. \hfill $\Box$

**Lemma 6.3.** Let $\tau$ be any complete train track of $S_{0,4}$. Then there is $\tau' \in V(T_1)$ obtained from $\tau$ by at most 5 splits.

**Proof.** We can easily see that each $C^1$–diffeomorphism class of $V(TT(S_{0,4}))$ has a train track $\tau$ which implements $d(\tau, V(T_1)) \leq 5$ (see Figure 12). Meanwhile, $d(\tau, V(T_1))$ depends only on a $C^1$–diffeomorphism class of $\tau$, because the mapping class group $\mathcal{M}(S_{0,4})$ acts isometrically on $TT(S_{1,1})$. Now this Lemma is proved. \hfill $\Box$

**Proof of Proposition 6.1.** We obtained this theorem by Lemma 6.2 and 6.3. \hfill $\Box$

Similar to the construction of $\varphi$ in Section 5, we construct the map $\psi : V(TT(S_{0,4})) \to E(\mathcal{F})$ as follow : Let $\tau \in V(TT(S_{1,1}))$. Suppose $c_1, c_2$ are vertex cycles on $\tau$. If $i(c_1, c_2) = 2$, there is an edge $e$ which connects $c_1$ and $c_2$ in $\mathcal{F}$. We define $\psi(\tau)$ as $e$. If $i(c_1, c_2) = 4$, there are two simple closed curves $c_3, c_4$ that implement $i(c_3, c_4) = 2$ ($j = 1, 2, k = 3, 4$, $i(c_3, c_4) = 2$. We define $\psi(\tau)$ as the edge which connects $c_3$ and $c_4$. We construct graph $G_2$ as the same as $G_1$ of Section 5 and extends to $\psi : TT(S_{0,4}) \to G_2$.

**Lemma 6.4.** $G_2$ is quasi-isometric to the dual of the Farey graph.

**Proof.** It is possible to think this just as $G_1$.

Let $\tau \in V(TT(S_{0,4}))$ and $\tau'$ obtained by a single split of $\tau$. The relation between vertex cycles on $\tau$ and $\tau'$ can fall into the following 3 types (see also Table I):

(i) $\tau$ and $\tau'$ have the same vertex cycles. Thus $\phi(\tau)$ and $\phi(\tau')$ are the same edge in $\mathcal{F}$.

(ii) One vertex cycle $c_1$ on $\tau$ and $c'_1$ on $\tau'$ are the same. Another vertex cycles $c_2$ on $\tau$ and $c'_2$ on $\tau'$ intersect at 2 points. Thus $\phi(\tau)$ and $\phi(\tau')$ are an adjacent edge in $\mathcal{F}$.

(iii) One vertex cycle $c_1$ on $\tau$ and $c'_1$ on $\tau'$ are the same. Another vertex cycles $c_2$ on $\tau$ and $c'_2$ on $\tau'$ intersect at 4 points. Thus $\phi(\tau)$ and $\phi(\tau')$ are the next but one edge in $\mathcal{F}$.

So, the edges of $G_2$ connect adjacent or next but one edges in $\mathcal{F}$. Thus, $G_2$ is quasi-isometric to $G_1$ and the dual of $\mathcal{F}$. \hfill $\Box$

We note that $G_2$ is isomorphic to Figure 13.

**Lemma 6.5.** $\psi : TT(S_{0,4}) \to G_2$ is a quasi-isometry.

**Proof.** It can be proved in the same way as in Lemma 5.3.
Figure 12.
Let $\tau, \tau' \in V(\mathcal{T}\mathcal{T}(S_{0,4}))$. There are only finitely many complete train tracks if vertex cycles are fixed. Thus, $\psi^{-1}(e)$ is finite. Since $\mathcal{M}(S_{0,4})$ acts isometrically on $\mathcal{T}\mathcal{T}(S_{1,1})$ and acts transitively on $V(G_2)$, $a := \text{diam}(\psi^{-1}(e))$ is constant for all $e$. It follows that $d(\psi(\tau), \psi(\tau')) \leq d(\tau, \tau') \leq (a + 1)d(\psi(\tau), \psi(\tau')) + a$ and $\psi$ is a quasi-isometry.

**Proof of Theorem 1.1 (4-punctured sphere case).** By Lemma 6.4 and 6.5, $\mathcal{T}\mathcal{T}(S_{0,1})$ is quasi-isometric to the dual of the Farey graph.
7. Action of the mapping class group

It is well known that $\mathcal{M}(S_{1,1})$ is isomorphic to $SL(2,\mathbb{Z})$ (see for instance [Tak01]). Also $\mathcal{M}(S_{0,4})$ has a subgroup of finite index which is isomorphic $PSL(2,\mathbb{Z})$.

First, we prove Corollary 1.3 and 1.4.

Proof of Corollary 1.3. Let $\tau$ be a complete train track on the once-punctured torus $S_{1,1}$. The train track complex is locally finite. The stabilizer $\text{stab}(\tau)$ under the action of $\mathcal{M}(S_{1,1})$ is finite. Thus, $\{\sigma \in \mathcal{M}(S_{1,1}) \mid d(\tau, \sigma \tau) \leq r\}$ is finite for all $r \geq 0$. It follows that the action of $\mathcal{M}(S_{1,1})$ on $\mathcal{T}(S_{1,1})$ is properly discontinuous.

$\mathcal{M}(S_{1,1})$ acts transitively on $V(\mathcal{T}(S_{1,1}))$, since all complete train tracks on $S_{1,1}$ are $C^1$–diffeomorphism. It follows that the orbit $\mathcal{M}(S_{1,1})\tau = V(\mathcal{T}(S_{1,1}))$ and thus $N(\mathcal{M}(S_{1,1})\tau, 1) = \mathcal{T}(S_{1,1})$. That is to say, any orbit $\mathcal{M}(S_{1,1})\tau$ is cobounded. By Proposition 2.1, the action of $\mathcal{M}(S_{1,1})$ on $\mathcal{T}(S_{1,1})$ is cocompact.

Let $\tau$ be a complete train track on the 4-punctured sphere $S_{0,4}$. Just as in the case of $\mathcal{M}(S_{1,1})$, the action of $\mathcal{M}(S_{0,4})$ on $\mathcal{T}(S_{0,4})$ is properly discontinuous.

$\mathcal{M}(S_{0,4})$ acts transitively on $V(T_1)$. Thus the orbit $\mathcal{M}(S_{0,4})\tau$ of $\tau \in V(T_1)$ is $V(T_1)$. By Lemma 6.3, $N(\mathcal{M}(V(T_1), 6) = \mathcal{T}(S_{0,4})$. So, some orbit is cobounded. By Proposition 2.1, the action of $\mathcal{M}(S_{0,4})$ on $\mathcal{T}(S_{0,4})$ is cocompact.

Proof of Corollary 1.4. The train track complex $\mathcal{T}(S)$ is a locally finite graph. Thus $\mathcal{T}(S)$ is a proper space. Since the mapping class group $\mathcal{M}(S)$ is finitely generated, by Theorem 2.3 and Corollary 1.3 $\mathcal{T}(S_{1,1})$ is quasi-isometric to $\mathcal{M}(S)$.

As already stated in Section 5, $\mathcal{T}(S_{1,1})$ is isomorphic to the Cayley graph of $PSL(2,\mathbb{Z})$. Similarly, we can notice that $T_1$ in Section 6 is isomorphic to $\mathcal{T}(S_{1,1})$ and hence the Cayley graph of $PSL(2,\mathbb{Z})$. Thus, $PSL(2,\mathbb{Z})$ acts freely and p.d.c. on $T_1$ and on $V(T_1)$ transitively. Meanwhile, we can easily show that $\mathcal{M}(S_{0,4})$ acts p.d.c. on $T_1$ and the stabilizer $\text{stab}(\tau)$ for $\tau \in T_1$ is isomorphic to dihedral group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence index of $PSL(2,\mathbb{Z})$ on $\mathcal{M}(S_{0,4})$ equals 4.

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Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

E-mail address: ibaraki4@is.titech.ac.jp