 Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points

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Abstract: In the present paper, we investigate upper bounds on the third Hankel determinants for the starlike and convex functions with respect to symmetric points in the open unit disk.

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1. Introduction

1.1. Hankel determinant

Let \( \mathcal{A} \) denote the family of analytic functions in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) of the form

\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.
\] (1)

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PUBLIC INTEREST STATEMENT

The interplay between geometry and analysis of function of complex variables is the most attracting part of complex analysis. From the beginning of the twentieth century, the work on the coefficients of the Taylor series expansion of analytic univalent function is of great importance in complex analysis. The bounds on Hankel determinants and Fekete–Szegö inequalities of coefficients of Taylor's series expansion of analytic univalent functions have been studied by many peoples. In this paper, we have studied bounds on third Hankel determinants for the functions which are starlike and convex with respect to symmetric points.
A function \( f \) is said to be univalent in a domain \( D \), if it is one-to-one in \( D \). Let \( S \) denote the subclass of \( \mathcal{A} \) consisting of functions which are univalent in \( D \).

The Hankel determinant \( H_{q,n}(f) \) of Taylor’s coefficients of function \( f \in \mathcal{A} \) of the form (1), is defined by

\[
H_{q,n}(f) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-1}
\end{vmatrix}
\quad (a_1 = 1; n, q \in \mathbb{N} = \{1, 2, \cdots\}).
\]

(2)

The Hankel determinant is useful in showing that a function of bounded characteristic in \( D \), i.e. a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational (see Cantor, 1963). Pommerenke (1967) proved that the Hankel determinant of univalent functions satisfy

\[
|H_{q,n}(f)| < Kn^{-1/2} + \beta n/2,
\]

where \( \beta > 1/4000 \) and \( K \) depends only on \( q \). Later, Hayman (1968) proved that \( |H_{q,n}(f)| < An^{1/2} \) (\( A \) is an absolute constant) for a really mean univalent functions. The study of \( |H_{q,n}(f)| \) for various subfamilies of \( \mathcal{A} \) are of interest for many researchers (see Ehrenborg, 2000; Noonan & Thomas, 1976; Noor, 1992; Pommerenke, 1966).

Note that, the \( H_{2,1}(f) = a_3 - a_2^2 \) is the classical Fekete–Szegö functional. Fekete and Szegö (1933) found the maximum value of \( |H_{2,1}(f)| \) over the function \( f \in S \). The problem of calculating \( \max_{f \in S} |H_{2,1}(f)| \) for various compact subfamilies \( \mathcal{P} \) of \( \mathcal{A} \) was considered by many authors (see Bhownik, Ponnusamy, & Wirths, 2011; Keogh & Merkes, 1969; Koepp, 1987; Mishra & Gochhayat, 2008a, 2010, 2011; Srivastava & Mishra, 2000; Srivastava, Mishra, & Das, 2001). Further, for the second Hankel determinant \( H_{2,2}(f) \), the \( \max_{f \in \mathcal{P}} |H_{2,2}(f)| \) has been studied by many researchers (see Janteng, Halim, & Darus, 2006; Lee, Ravichandran, & Supramaniam, 2013; Mishra & Gochhayat, 2008b; Mishra & Kund, 2013; Patil & Sahoo, 2014) and upper bound on the third Hankel determinant \( H_{3,1}(f) \) studied recently by Babalola (2010), Bansal, Maharana, and Prajapat (2015), Prajapat, Bansal, Singh, and Mishra (2015), Raza and Malik (2013), Vamshee Krishna, Venkateswarlu, and RamReddy (2015).

1.2. Toeplitz determinant

Let \( \mathcal{P} \) denote the class of analytic functions \( p \) in \( \mathbb{D} \) with \( \Re(p(z)) > 0 \) and \( p(0) = 1 \). If \( p \in \mathcal{P} \) is of the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},
\]

(3)

then \( |c_n| \leq 2, \ n \in \mathbb{N} = \{1, 2, \cdots\} \). This inequality is sharp and the equality holds for the function \( q(z) = (1 + z)/(1 - z) \) (see Duren, 1983).

The power series (3) converges in \( \mathbb{D} \) to a function in \( \mathcal{P} \), if and only if the Toeplitz determinants

\[
T_n(p) = \begin{vmatrix}
    2 & c_1 & c_2 & \cdots & c_n \\
    c_1 & 2 & c_1 & \cdots & c_{n-1} \\
    c_2 & c_1 & 2 & \cdots & c_{n-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_n & c_{n-1} & c_{n-2} & \cdots & 2
\end{vmatrix}, \quad n \in \mathbb{N}
\]

are positive, where \( c_{-n} = \overline{c_n} \). The only exception is when \( f(z) \) has the form
\[ f(z) = \sum_{n=1}^{m} \rho_n \frac{1 + \epsilon_n z}{1 - \epsilon_n z}, \quad m \geq 1, \]

where \( \rho_n > 0, |\epsilon_n| = 1 \), and \( \epsilon_k \neq \epsilon_l \) if \( k \neq l, k, l = 1, 2, \ldots, m \); we have then \( T_n(p) > 0 \) for \( n \leq (m - 1) \) and \( T_n(p) = 0 \) for \( n \geq m \) (see Grenander & Szegö, 1984).

Recently, in an article (Janteng et al., 2006) the Toeplitz determinant found to be useful to estimate upper bound on the coefficients functional for various subfamilies of analytic functions. Note that for \( n = 2 \)

\[ T_2(p) = \left| \begin{array}{ccc} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{array} \right| = 8 + 2 \Re \{c_1^2 \bar{c}_2\} - 2|c_2|^2 - 4|c_1|^2 \geq 0, \]

is equivalent to

\[ 2c_2 = c_1^2 + x(4 - c_1^2) \quad (4) \]

for some \( x \) with \( |x| \leq 1 \). Similarly, if

\[ T_3(p) = \left| \begin{array}{ccc} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_2 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{array} \right|, \]

then \( T_3(p) \geq 0 \) is equivalent to

\[ |(4c_3 - 4c_1c_2 + c_1^2)(4 - c_1^2) + c_1(2c_2 - c_1^2)x^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \quad (5) \]

Solving (5) with the help of (4), we get

\[ 4c_3 = c_1^2 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2), \quad (6) \]

for some \( x \) and \( z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \). Conditions (4) and (6) are due to Libera and Złotkiewicz (1982, 1983).

1.3. Starlike and convex functions with respect to symmetric points

A function \( f \in A \) is called starlike, if \( f \) is univalent in \( D \) and \( f(D) \) is a starlike domain with respect to the origin. Analytically, \( f \in S \) is called starlike, denoted by \( f \in S^* \), if \( \Re(f(z)/f(z)) > 0 \), \( z \in \mathbb{D} \). A function \( f \in S \) is called convex, denoted by \( f \in C \), if and only if \( zf'(z) \) is starlike.

A function \( f \in A \) is said to be starlike with respect to symmetric points (see Sakaguchi, 1959) if for every \( r \) less than and sufficiently close to one and every \( \eta \) on the circle \( |z| = r \), the angular velocity of \( f(z) \) about the point \( f(-\eta) \) is positive at \( z = \eta \) as \( z \) traverses the circle \( |z| = r \) in the positive direction, i.e.

\[ \Re \left( \frac{zf'(z)}{f(z) - f(-\eta)} \right) > 0 \quad \text{for} \quad z = \eta, \ |\eta| = r. \quad (7) \]

We denote by \( S^*_\alpha \), the class of all functions in \( S \) which are starlike with respect to symmetric points. A function \( f \) in the class \( S^*_\alpha \) is characterized by

\[ \Re \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \mathbb{D}. \quad (8) \]
This can be easily seen that the function \( f(z) - f(-z)/2 \) is a starlike function in \( \mathbb{D} \), therefore functions satisfying (8) are close-to-convex (and hence univalent) in \( \mathbb{D} \).

Further, the class of all functions in \( S \), which are convex with respect to symmetric points is denoted by \( C_s \). The necessary and sufficient condition for the function \( f \in S \) to be univalent and convex with respect symmetric points in \( \mathbb{D} \) is characterized by (see Das & Singh, 1977, Theorem 1)

\[
g_1\left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \mathbb{D}.
\]

In the present paper, we aim to investigate the upper bounds on the third Hankel determinant \( |H_{1,1}(f)| \) for the functions belonging to the classes \( S_s \) and \( C_s \) defined above in (8) and (10). For this purpose, we shall use Equations (4 and 6) and the following known results.

**Lemma 1.1** (Sakaguchi, 1959) If \( f \in S_s \) of the form (1), then \( |a_n| \leq 1, \quad n \geq 2 \). Equality holds for the function \( f(z) = z(1 + cz)^{-1} \), \( |c| = 1 \).

**Lemma 1.2** (Das & Singh, 1977) If \( f \in C_s \) of the form (1), then \( |a_n| \leq 1/n, \quad n \geq 2 \). Equality holds for the function \( f(z) = (1/\nu)(1 + cz)^{-1} \), \( |\nu| = 1 \).

**Lemma 1.3** (Shanmugam, Ramachandran, & Ravichandran, 2006, Example 2.3) If \( f \in S_s \) of the form (1), then \( |a_n - a_2| \leq 1 \). This inequality is sharp and the equality is attained for the function \( f(z) = z(1 + cz)^{-1} \), \( |c| = 1 \).

**Lemma 1.4** (Shanmugam et al., 2006, Example 2.5) If \( f \in C_s \) of the form (1), then \( |a_n - a_2| \leq 1/3 \). This inequality is sharp.

2. Main results

**Theorem 2.1** Let the function \( f \) given by (1) be in the class \( S_s \). Then

\[
|a_3 a_2 - a_4| \leq \frac{1}{2} \quad \text{and} \quad |a_3 a_4 - a_2^2| \leq 1.
\]

The inequalities in (10) are sharp.

**Proof** Let \( f \in S_s \), then by (8), we have

\[
\frac{2zf'(z)}{f(z) - f(-z)} = p(z),
\]

where \( p \in P \) is of the form (3). Substituting the series expansion of \( f(z) \) and \( p(z) \) and equating the coefficients, we get

\[
a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{2}c_2 \quad \text{and} \quad a_4 = \frac{1}{8}(2c_3 + c_1 c_2).
\]

Hence

\[
\begin{cases}
|a_3 a_2 - a_4| = \frac{1}{8} |c_1 c_2 - 2c_3| \\
\text{and} \quad |a_3 a_4 - a_2^2| = \frac{1}{16} |2c_3 c_1 + c_1^2 c_2 - 4c_2^2|.
\end{cases}
\]

Using (4) and (6) in (12) for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we get
\[
\begin{align*}
|a_1a_3 - a_4| &= \frac{1}{16} \left| -c_1x(4 - c_1^2) + c_1x^2(4 - c_1^2) - 2(4 - c_1^2)(1 - |x|^2)z \right| \\
\text{and} \\
|a_2a_5 - a_6^2| &= \frac{1}{32} \left| -c_2^2x(4 - c_2^2) - 2x^2(4 - c_2^2)^2 - c_1^2x^2(4 - c_1^2) + 2c_1(4 - c_1^2)(1 - |x|^2)z \right|
\end{align*}
\]

As \(|c_1| \leq 2\), therefore, letting \(c_1 = c\), we may assume without restriction that \(c \in [0, 2]\). Thus applying the triangle inequality with \(\mu = |x|\), we obtain
\[
|a_2a_3 - a_4| \leq \frac{1}{16} \left[ c\mu(4 - c^2) + c\mu^2(4 - c^2) + 2(4 - c^2)(1 - \mu^2) \right] \\
\quad = F(c, \mu)
\]
and
\[
|a_2a_5 - a_6^2| \leq \frac{1}{32} \left[ c^2\mu^2(4 - c^2) + 8\mu^2(4 - c^2) - c^2\mu^2(4 - c^2) + 2c(4 - c^2)(1 - \mu^2) \right] \\
\quad = G(c, \mu).
\]

Now we need to find the maximum value of \(F\) and \(G\) over the region \(\Omega = \{(c, \mu) : 0 \leq c \leq 2, 0 \leq \mu \leq 1\}\). For this, first differentiating \(F\) with respect to \(\mu\) and \(c\), we get
\[
\frac{\partial F}{\partial \mu} = \frac{1}{16} \left[ (4 - c^2)(c + 2c\mu - 4\mu) \right] \\
\frac{\partial F}{\partial c} = \frac{1}{16} \left[ 4\mu + 4\mu^2 - 3c^2\mu - 3c^2\mu^2 - 4c(1 - \mu^2) \right]
\]
(13)

A critical point of \(F(c, \mu)\) must satisfy \(\frac{\partial F}{\partial \mu} = 0\) and \(\frac{\partial F}{\partial c} = 0\). The condition \(\frac{\partial F}{\partial \mu} = 0\) gives \(c = 2\) or \(\mu = \frac{c}{2c - 4}\). Points \((c, \mu)\) satisfying such conditions are not interior points of \(\Omega\). So the function \(F(c, \mu)\) cannot have a maximum in the interior of \(\Omega\). Since \(\Omega\) is closed and bounded and \(F\) is continuous on \(\Omega\), the maximum shall be attained on the boundary of \(\Omega\). It is easy to see that on the boundary line \(c = 0, 0 \leq \mu \leq 1\), we have \(F(0, \mu) = (1 - \mu^2)/2\) and its maximum on this line is equal to 1/2. On the boundary line \(c = 2, 0 \leq \mu \leq 1\), we have \(F(2, \mu) = 0\). Similarly, on the boundary line \(\mu = 0, 0 \leq c \leq 2\), we have \(F(c, 0) = (4 - c^2)/8\) and the maximum on this line is 1/2. Lastly, on the boundary line \(\mu = 1, 0 \leq c \leq 2\), we have \(F(c, 1) = c(4 - c^2)/8\) and the maximum on this line is 2/\sqrt{27}. Comparing the four maxima we get that the maximum value of \(F(c, \mu)\) on \(\Omega\) is 1/2.

To show the sharpness of first inequality in (10), by setting \(c_1 = x = 0, z = 1\) in (4) and (6), we get \(c_2 = 0\) and \(c_3 = 2\). Using these values in (12), we find that the first inequality in (10) is sharp.

Further, to find the maximum value of \(G\) over \(\Omega\), differentiating \(G\) with respect to \(\mu\), we get
\[
\frac{\partial G}{\partial \mu} = \frac{1}{32} \left[ (4 - c^2)(c^2 + 2\mu(8 - c^2 - 2c)) \right] > 0 \text{ if } 0 < c < 2 \text{ and } 0 < \mu < 1.
\]

Note that, \(G\) is a non-decreasing function of \(\mu\) on \([0, 1]\), hence
\[
\max_{0 \leq \mu \leq 1} G(c, \mu) = G(c, 1) = \frac{1}{4}(4 - c^2) = G(c).
\]

Further, it is clear that \(G(c)\) is a decreasing function on \([0, 2]\), hence it attain maximum value at \(c = 0\). Therefore the maximum of \(G(c, \mu)\) is at the point \((0, 1)\). Further, \(\Omega\) is closed and bounded and \(G\) is continuous on \(\Omega\), the maximum shall be attained on the boundary of \(\Omega\). Hence, we look on the boundary of \(\Omega\) as we have done with the function \(F\), it is easy to see that on the boundary line \(c = 0, 0 \leq \mu \leq 1\), we have \(G(0, \mu) = \mu^2\) and its maximum on this line is equal to 1. On the boundary line \(c = 2, 0 \leq \mu \leq 1\), we have \(G(2, \mu) = 0\). Similarly, on the boundary line \(\mu = 0, 0 \leq c \leq 2\), we have \(G(c, 0) = c(4 - c^2)/16\) and the maximum on this line is 1/\sqrt{27}. Lastly, on the boundary line \(\mu = 1, 0 \leq c \leq 2\), we have \(G(c, 1) = (4 - c^2)/4\) and the maximum on this line is 1. Comparing the four maxima we get that the maximum value of
\( G(c, \mu) \) on \( \Omega \) is 1.

To show the sharpness in the second inequality of (10), by setting \( c_4 = 0, x = 1 \) in (4) and (6), we get \( c_3 = 2 \) and \( c_1 = 0 \). Using these values in (12), we find that the second inequality in (10) is sharp. \( \square \)

**Theorem 2.2** Let the function \( f \) given by (1) be in the class \( S^*_c \). Then

\[
|H_{\delta,1}(f)| \leq \frac{5}{2}.
\]

**Proof** Using Lemma 1.1, Lemma 1.3, Theorem 2.1 and applying the triangle inequality, we get

\[
|H_{\delta,1}(f)| \leq |\alpha_4| |\alpha_2 \alpha_4 - \alpha^2_3| + |\alpha_4| |\alpha_2 \alpha_3 - \alpha_4| + |\alpha_4| |\alpha_3 - \alpha^2_4|
\]

\[
\leq 1 + \frac{1}{2} + 1 = \frac{5}{2}.
\]

**Theorem 2.3** Let the function \( f \) given by (1) be in the class \( C_p \). Then

\[
|a_2 a_3 - a_4| \leq \frac{4}{27} \quad \text{and} \quad |a_2 a_4 - a^2_3| \leq \frac{1}{9}.
\]

The second inequality in (14) is sharp.

**Proof** Let \( f \in C_p \) then by (9), we have

\[
\frac{2(zf'(z))'}{(f(z) - f(-z))'} = p(z),
\]

where \( p \in \mathcal{P} \) is of the form (3). From the definitions of the class \( S^*_c \) and \( C_p \), it follows that the function \( f(z) \in C_p \) if and only if \( zf'(z) \in S^*_c \). Thus replacing \( \alpha_4 \) by \( n \alpha_4 \) in (11), we get

\[
a_2 = \frac{1}{4} c_1, \quad a_3 = \frac{1}{6} c_2 \quad \text{and} \quad a_4 = \frac{1}{32} (2c_3 + c_1 c_2).
\]

Hence

\[
\begin{align*}
|a_2 a_3 - a_4| &= \frac{1}{96} |c_1 c_2 - 6c_3| \\
|a_2 a_4 - a^2_3| &= \frac{1}{1152} |18c_1 c_3 + 9c_2^2 c_2 - 32 c_2^3|.
\end{align*}
\]

Using (4) and (6) in (16) for some \( x \) and \( z \) such that \( |x| \leq 1 \) and \( |z| \leq 1 \), we get

\[
\begin{align*}
|a_2 a_3 - a_4| &= \frac{1}{192} \left| -2c^3_1 + (4 - c^2_1) (-5c_1 x + 3c_2 x^2 - 6(1 - |x|^2) iz) \right| \\
|a_2 a_4 - a^2_3| &= \frac{1}{2304} \left| 2c^3_1 + (4 - c^2_1) \left( -5c_1^2 x^2 - 16x^2 (4 - c^2_1) - 9c_2^2 x^2 + 18c_3 (1 - |x|^2) iz \right) \right|.
\end{align*}
\]

As \( |c_1| \leq 2 \), therefore, letting \( c_4 = c \), we may assume without restriction that \( c \in [0, 2] \). Thus applying the triangle inequality with \( \mu = |x| \), we obtain

\[
|a_2 a_3 - a_4| \leq \frac{1}{192} \left[ 2c^3 + (4 - c^2) (5c_1 \mu + 3c_2 \mu^2 + 6(1 - \mu^2)) \right]
\]

\[
:= X(c, \mu)
\]

and

\[
|a_2 a_4 - a^2_3| \leq \frac{1}{2304} \left[ 2c^3 + (4 - c^2) (5c^2_1 \mu + 64c_2 \mu^2 - 7c_3^2 \mu^2 + 18c(1 - \mu^2)) \right]
\]

\[
:= Y(c, \mu).
\]
Now to find the maximum value of \( X \) and \( Y \) over the region \( \Omega \). First differentiating \( X \) with respect to \( \mu \) and \( c \), we get
\[
\frac{\partial X}{\partial \mu} = \frac{1}{192} \left[ (4 - c^2)(5c + 6c\mu - 12\mu) \right],
\frac{\partial X}{\partial c} = \frac{1}{192} \left[ 6c^2 + (4 - c^2)(5\mu + 3\mu^2) - 2c(5c\mu + 3c\mu^2 + 6 - 6\mu) \right].
\]

A critical point of \( X(c, \mu) \) must satisfy \( \frac{\partial X}{\partial \mu} = 0 \) and \( \frac{\partial X}{\partial c} = 0 \). The condition \( \frac{\partial X}{\partial \mu} = 0 \) gives \( c = 2 \) or \( \mu = \frac{6c - 12}{5c} \). Points \((c, \mu)\) satisfying such conditions are not interior points of \( \Omega \). So the function \( X(c, \mu) \) cannot have a maximum in the interior of \( \Omega \). Since \( \Omega \) is closed and bounded and \( X \) is continuous the maximum shall be attained on the boundary of \( \Omega \). It is easy to see that on the boundary line \( c = 0 \), \( 0 \leq \mu \leq 1 \), we have \( X(0, \mu) = (1 - \mu^2)/8 \) and its maximum on this line is equal to 1/8. On the boundary line \( c = 2 \), \( 0 \leq \mu \leq 1 \), we have \( X(2, \mu) = 1/12 \). Similarly, on the boundary line \( \mu = 0 \), \( 0 \leq c \leq 2 \), we have \( X(c, 0) = (c^3 - 3c^2 + 12)/96 \) and the maximum on this line is 1/8. Lastly, on the boundary line \( \mu = 1 \), \( 0 \leq c \leq 2 \), we have \( X(c, 1) = (16 - 3c^2)/96 \) and the maximum on this line is 4/27. Comparing the four maxima we get that the maximum value of \( X(c, \mu) \) on \( \Omega \) is 4/27.

Further, to find the maximum value of \( Y \) over \( \Omega \), differentiating \( Y \) with respect to \( \mu \), we get
\[
\frac{\partial Y}{\partial \mu} = \frac{1}{2304} \left[ (4 - c^2)(5c^2 + 2(64 - 7c^2 - 18c)) \right] > 0 \quad \text{if} \quad 0 < c < 2 \quad \text{and} \quad 0 < \mu < 1.
\]

Note that, \( Y(c, \mu) \) is a non-decreasing function of \( \mu \) on \([0, 1]\), hence
\[
\max_{0 < c < 1} Y(c, \mu) = Y(c, 1) = \frac{1}{1152}(13c^4 - 36c^2 + 128) = Y(c).
\]

It is clear that \( Y(c) \) is a decreasing function on \([0, 2]\) and it attained maximum value at \( c = 0 \). Therefore, the maximum of \( Y(c, \mu) \) is at the point \((0, 1)\). Further, \( \Omega \) is closed and bounded and \( Y \) is continuous, the maximum shall be attained on the boundary of \( \Omega \). Hence, we look on the boundary of \( \Omega \), it is easy to see that on the line \( c = 0 \), \( 0 \leq \mu \leq 1 \), we have \( Y(0, \mu) = \mu^2/9 \) and its maximum on this line is equal to 1/9. On the boundary line \( c = 2 \), \( 0 \leq \mu \leq 1 \), we have \( Y(2, \mu) = 1/72 \). Similarly, on the boundary line \( \mu = 0 \), \( 0 \leq c \leq 2 \), we have \( Y(c, 0) = c(c^3 - 9c^2 + 36)/1152 \) and the maximum on this line is less than 1/9. Lastly, on the boundary line \( \mu = 1 \), \( 0 \leq c \leq 2 \), we have \( Y(c, 1) = (2c^4 - 36c^2 + 128)/1152 \) and the maximum on this line is 1/9. Comparing the four maxima we get that the maximum value of \( Y(c, \mu) \) on \( \Omega \) is 1/9.

To show the sharpness in the second inequality of (14), by setting \( c_1 = 0 \), \( x = 1 \) in (4) and (6), we get \( \psi_1 = 2 \) and \( \psi_2 = 0 \). Using these values in (16), we find that the second inequality in (14) is sharp. This completes the proof of the theorem.

**Theorem 2.4** Let the function \( f \) given by (1) be in the class \( C_\infty \). Then
\[
|H_{3, 1}(f)| \leq \frac{19}{135}.
\]
**Proof** Using Lemma 1.2, Lemma 1.4, Theorem 2.3 and applying the triangle inequality, we get
\[
|H_{3, 1}(f)| \leq |a_3||a_4 - a_3| + |a_4||a_2 - a_1| + |a_3||a_4 - a_3| \leq \frac{1}{3} \times \frac{1}{9} + \frac{1}{4} \times \frac{4}{27} + \frac{1}{5} \times \frac{1}{3} = \frac{19}{135}.
\]
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