Abstract. The category $\text{Pre}$ of premetrics is the one whose objects are pairs $(X, d)$ - where the only restriction is for $d(x, x) = 0$ - and $\epsilon$-$\delta$ continuous functions as morphisms. This category is only closed under coproducts. The absence of the triangle inequality can only guarantee a faithful functor $\text{Pre} \rightarrow \text{Top}$, where a premetric is sent to the topology it generates. Moreover, the sequential nature of topologies generated from premetrics indicates that this functor is not surjective on objects either. Developed from recent work by Weiss, we illustrate an extension $\text{Pre} \hookrightarrow \text{P}$, topological over $\text{Set}$, together with a faithful and surjective on objects left adjoint functor $\text{P} \rightarrow \text{Top}$ as an extension of $\text{Pre} \rightarrow \text{Top}$. We show this represents an optimal scenario given that $\text{Pre} \rightarrow \text{Top}$ preserves coproducts only. The objects in $\text{P}$ are metric-like objects valued on value distributive lattices whose limits and colimits we show to be generated by free locales on discrete sets.

1. Introduction

In [1] the author generates a collection of metric-like objects - continuity spaces - with the property that any topological space can be naturally generated by one such object. In a nutshell, a continuity space is a triplet $(X, V, d)$ where $X$ is any set, $V = (V, \leq, +)$ is a value quantale, and $d : X^2 \rightarrow V$ so that $d(x, x) = 0$ and $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. The author of [7] further develops the ideas from Flagg by constructing an equivalence of categories $M : M \cong \text{Top} : O$ where the objects of $M$ are Flagg’s continuity spaces. The morphisms in $M$ are extensions of $\epsilon$-$\delta$ continuous functions between metric spaces and are shown to be equivalent to continuous functions. This category $M$ is shown to be a natural extension of $\text{Met}$ - of all metric spaces - and the following diagram is established,

\[
\begin{tikzcd}
\text{Top} \ar{r}{C} \ar{d} & M \ar{d} \\
\text{Top}_M \ar{r}{O} & \text{Met}
\end{tikzcd}
\]

where $\text{Top}_M$ is the category of metrisable spaces and arrows going up are inclusions. The category $\text{Pre}$ is the one whose objects $(X, d)$ are premetrics; the only requirement is that $d(x, x) = 0$ and nothing more. Morphisms in this category are $\epsilon$-$\delta$ continuous functions. There exists the obvious functor $O : \text{Pre} \rightarrow \text{Top}_P$ that extends $O : \text{Met} \rightarrow \text{Top}_M$, where a premetric is sent...
to the topology it generates and $\textbf{Top}_p$ is the category of premetrisable spaces; a subset $O$ of $X$ is $\tau_d$-open if, and only if, for any $x \in O$ we can find $\epsilon > 0$ so that $B_\epsilon(x) \subseteq O$. Here two important issues arise: (a) by the sequential nature of objects in $\textbf{Pre}$ the functor is not surjective on objects$^{1}$ (b) since premetrics are not required to satisfy the triangle inequality, epsilon balls are not necessarily open in the generated topology - actually, the centre of an epsilon ball might not belong to its interior, even if it’s not empty! For (b) it also follows that $\epsilon$-$\delta$ continuous functions are topologically continuous but the converse is certainly not true (we investigate this further in Section 2). In other words, (b) says that $\mathcal{O} : \textbf{Pre} \rightarrow \textbf{Top}$ is not full and it is, thus, not possible to replicate the above equivalences with an extension of $\textbf{Pre}$. However, in light of $M : \textbf{M} \cong \textbf{Top} : \mathcal{O}$ it is natural to ask: how much is lost by dropping the triangle inequality from $\textbf{M}$? Let $\textbf{P}$ be the category of generalised premetrics whose objects are triplets $(X, V, d)$ - where $X$ is any set, $V = (V, \leq, +)$ is a value quantale and $d : X^2 \rightarrow V$ so that $d(x, x) = 0$. Morphisms in $\textbf{P}$ are $\epsilon$-$\delta$ continuous functions like the ones in $\textbf{M}$. In fact, we latter show that $\textbf{M}$ is a reflective subcategory of $\textbf{P}$, thus highlighting a natural procedure for adding the triangle inequality to any generalised premetric. In spite of (a) and (b), we show that $\textbf{P}$ is remarkably similar to $\textbf{Top}$. More precisely, as an obvious extension of $\mathcal{O} : \textbf{Pre} \rightarrow \textbf{Top}$ we show that the functor $\mathcal{O} : \textbf{P} \rightarrow \textbf{Top}$ is left adjoint and that $\textbf{P}$ is topological over $\textbf{Set}$. In other words, for a large collection of categorical constructions in $\textbf{Top}$ it is only necessary to take into account $\mathcal{O}$-images of $\epsilon$-$\delta$ continuous functions. This is most unexpected given the large discrepancy between $\epsilon$-$\delta$ continuity and topological continuity; a fact we highlight in more detail in Section 2 where we investigate several scenarios in which both types of continuity coincide.

The outline of the paper is the following. In Section 2, we begin by exploring premetrics, topological continuity vs. $\epsilon$-$\delta$ continuity and briefly remark some interesting facts regarding various categories thereof. This section naturally leads to Section 3 where we illustrate $\textbf{P}$ as an extension of $\textbf{Pre}$ and explore topological continuity vs. $\epsilon$-$\delta$ continuity in its full generality. This section closes by proving that $\textbf{P}$ is topological and by highlighting some interesting facts regarding limits in $\textbf{P}$. Lastly, Section 4 is concerned with concrete constructions of colimits in $\textbf{P}$ (an existence result from the topologicity of $\textbf{P}$) and their preservation via $\mathcal{O}$.

2. A PRIMER ON PREMETRICS AND SEQUENTIAL SPACES

We illustrate some basic facts regarding premetrizability and topological continuity vs $\epsilon$-$\delta$ continuity. The topology $\tau_d$ on a set $X$ generated by a premetric $d : X^2 \rightarrow \mathbb{R}$ is the one for which $U \in \tau_d$ iff for all $x \in U$ there exists an $\epsilon > 0$ so that $U \supseteq B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$ (it’s a relatively straightforward task to show that any such topology is sequential$^{[5]}$). It is important to notice

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$^{1}$Actually, any triplet $(X, V, d)$ where $V$ is a complete linear order will yield a radial topology. It is this fact that forces us to go beyond the realm of linearly ordered sets.
that, in general, $B_r(x)$ might not be open; the interior of such a set might not even contain $x$ itself. Hence, sequential convergence is not the same for $d$ as it is for $\tau_d$; it is only possible to claim that $d$-convergence implies $\tau_d$-convergence. The reader can quickly verify that $\epsilon$-$\delta$ continuity always implies topological continuity between any pair of premetrizable spaces; the converse is not true. As a matter of fact, the equivalence between both types of continuity occurs precisely when the same is true for both types of convergence. Recall that any function $f : X \rightarrow Y$ between sequential spaces is continuous iff $((x_n) \rightarrow x \Rightarrow f(x_n) \rightarrow f(x))$. It is not hard to verify that the very same holds for premetric spaces.

**Lemma 2.1.** A function $f : (X, d) \rightarrow (Y, m)$ between premetric spaces is $\epsilon$-$\delta$ continuous if, and only if, $(x_n) \rightarrow_d x \Rightarrow f(x_n) \rightarrow_m f(x)$.

In fact, as the following result highlights, the coincidence of both types of continuity is only due to the codomain of a function.

**Lemma 2.2.** For any premetric space $(Y, m)$ the following are equivalent.

(1) For any function $(X, d) \rightarrow (Y, m)$, topological continuity and $\epsilon$-$\delta$ continuity coincide.

(2) The notions of $\tau_m$-convergence and $m$-convergence coincide.

**Proof.** ($\Leftarrow$) Whether or not $\tau_m$-convergence implies $m$-convergence, $\epsilon$-$\delta$ continuity implies topological continuity. Hence, assume that $\tau_m$-convergence implies $m$-convergence and let $f : (X, d) \rightarrow (Y, m)$ be any function which is not $\epsilon$-$\delta$ continuous. This means that there exists an $\epsilon > 0$ and an $x \in X$ for which given any $\delta > 0$ we can find $y \in X$ with $d(x, y) < \delta$ but $m(f(x), f(y)) \geq \epsilon$. Consequently, there exists a $d$-convergent (and thus $\tau_d$-convergent) sequence $(y_n) \rightarrow x$ for which $f(y_n) \not\rightarrow f(x)$ with respect to $m$. Since we assumed that $\tau_m$-convergence implies $m$-convergence, then $f(y_n) \not\rightarrow f(x)$ with respect to $\tau_m$ either and $f$ is not continuous.

($\Rightarrow$) Assume $\tau_m$-convergence to be strictly weaker than $m$-convergence and let convergent sequence $(x_n) \rightarrow x$ in $Y$ be a witness of this fact. Take the convergent sequence space $(\omega + 1, d)$ with, say, $d(n, \omega) = \frac{1}{n}$ and the map $\omega \rightarrow Y$ for which $n \mapsto x_n$ and $\omega \mapsto x$. This map is continuous but not $\epsilon$-$\delta$ continuous. □

**Corollary 2.3.** For any function $(X, d) \rightarrow (Y, m)$ between premetric spaces the following conditions imply that both types of continuity coincide.

(1) $m$ satisfies the triangle inequality.

(2) $\tau_m$-sequential limits are unique.

(3) $\tau_m$ is $T_2$.

**Proof.** In all cases we show that $\tau_m$-convergence implies $m$-convergence. The proof of (1) is obvious. For (3), notice that $T_2$ implies uniqueness of limits and we thus focus on proving (2). If $(x_n) \not\rightarrow x$ with respect to $m$ then let $\epsilon > 0$ so that $B_{\epsilon}(x) \cap (x_n) = \emptyset$. For each $y \in B_{\epsilon}(x)$ let $U_y$ be any open set containing
and so that \( U_y \cap (x_n) = \emptyset \). It follows that \( \bigcup_{y \in B(x)} U_y \) is open, contains \( x \) and is disjoint from \((x_n)\). Hence, \((x_n)\) does not \( \tau_m \)-converge to \( x \) either. \( \square \)

Obviously, none of the above conditions are necessary. The previous lemma narrows the scope of candidates for a topologically continuous function \( f : (X, d) \to (Y, m) \) which is not \( \epsilon \)-\( \delta \) continuous: \((Y, m)\) must be at most \( T_1 \) and limits mustn’t be unique. The following example illustrates just that.

**Example 2.4.** Take two countably infinite disjoint sets \( A = \{a_i \mid i \geq 2\}, B = \{b_i \mid i \geq 2\} \) and let \( Y = A \cup B \). Define \( m : Y \times Y \to \mathbb{R} \) - where we assume \( m \) to be symmetric (i.e., \( m(x, y) = m(y, x) \)) and separated (i.e., \( m(x, y) > 0 \) for \( x \neq y \)):

\[
m(x, y) = \begin{cases} 
\frac{1}{n} & \text{if } x \in A \text{ (resp. } B) \text{ and for some } n \in \mathbb{N}, y = b_n \text{ (resp. } a_n) \\
1 & \text{otherwise.}
\end{cases}
\]

Hence, \( d \) fails only to satisfy the triangle inequality. Next, generate a topology \( \tau \) on \( Y \) as usual: \( O \in \tau_m \) iff for any \( x \in O \) there exists an \( \epsilon > 0 \) so that \( B_\epsilon(x) \subseteq O \). By design: \( B \) converges to \( A \), \( A \) converges to \( B \). Also, it is simple to observe that any open set is cofinite. The converse is also true. Let \( O \) be cofinite and \( p, q, r \in \mathbb{N} \) so that \( p \) (resp. \( q, r \)) is the least number so that \( \forall i \geq p, a_i \in O \) (resp. \( b_i \in O, c_i \in O \)). For each \( a_i \in O \) let \( \delta = \frac{1}{\max \{p, q, r\}} \) and notice that \( B_\delta(a_i) \subseteq O \); the same is true for all \( b_i \in O \). Hence, \( O \) is open. Next, split the rationals into two mutually dense sets \( C, D \) and let \( f : \mathbb{Q} \to Y \) be a bijection for which \( f(C) = A, f(D) = B \). Assume \( d \) is the usual metric on \( \mathbb{Q} \) and notice that, since \( \tau_m \) is the cofinite topology on \( Y \), then \( f \) is topologically continuous. However, \( f \) fails to be \( \epsilon \)-\( \delta \) continuous about each and every point in its domain. Indeed, WLOG, let \( x \in C \) and notice that for any \( \delta > 0 \), \( B_\delta(x) \) contains infinitely many points from \( D \). Whence, choosing \( \epsilon = \frac{1}{2} \) yields that for all \( \delta > 0 \) \( f(B_\delta(x)) \not\subseteq B_\epsilon(f(x)) \) and that \( f \) is not \( \epsilon \)-\( \delta \) continuous.

Let \( \text{Pre}, \ Top \) and \( \text{Seq} \) denote the categories of premetrics (with \( \epsilon \)-\( \delta \) continuous), premetrizable topologies and sequential topologies (with continuous functions as morphisms in the latter two), respectively. It is known that the complete and cocomplete category \( \text{Seq} \) is a coreflective subcategory of \( \text{Top} \). It is not closed under topological limits; limits in \( \text{Seq} \) are constructed by applying the convergent-open topology to underlying products and subsets (see [2] and [3]). Not all sequential spaces are premetrizable. In fact, more is true: neither Fréchet nor premetrizability imply each other. In [3] Example 5.1 illustrates a premetrizable space that is not Fréchet. For a Fréchet space that is not premetrizable consider the following example.

**Example 2.5.** Let \( X \) be the topological products of countably many copies of \( \omega + 1 \) and quotient all of \( \omega \times \{\omega\} \), and denote this point \( \infty \) (this is the Fréchet Fan). The resulting space is Fréchet but fails to be premetrizable. Indeed, one can easily verify that epsilon balls must be open in \( X \) for any premetric on it.
since $X \setminus \infty$ is discrete. However, the Fréchet Fan is not first countable and, thus, can’t be premetrized.

The Fréchet Fan is an excellent example of a space that cannot be generated by an evaluating map on a linear order. Both conditions are weaker than first-countability (that Fréchet is weaker is a well-known result).

**Lemma 2.6.** Any first countable space is premetrizable.

**Proof.** Take any first countable space $X$, and for each point $x \in X$ select a countable nested collection of neighbourhoods about $x$, $U_{n}(x)$. Let $d : X^2 \to \mathbb{R}$ as

$$d(x, y) = \begin{cases} 
0 & \text{if } y \in \bigcap_{n} U_{n}(x), \\
\frac{1}{n} & \text{if } y \in U_{n}(x) \setminus U_{n+1}(x), \\
1 & \text{otherwise.} 
\end{cases}$$

The category $\textbf{Pre}$ is closed under equalizers and coproducts. The former are simple to construct: for a premetric space $(X, d)$ and $E \subseteq X$, take $m$ on $E$ as the restriction of $d$ on $E$. The functor $\textbf{Pre} \to \textbf{Top}$ does not preserve equalizers. Example 5.1 in [3] illustrates a sequential space with a non-sequential subspace and since $\textbf{Pre}$ is closed under equalizers the claim follows. As for coproducts, let $\phi : \mathbb{R} \to [0, 1)$ be any order isomorphism and for a collection $(X_i, d_i)_{i \in I}$ define $Y = \prod_i X_i$ and $m : Y^2 \to \mathbb{R}$ so that $m(x, y) = \phi \circ d_i(x, y)$ when $x, y \in X_i$ (for some $i$), and 1 otherwise. A moment’s thought verifies that $\textbf{Pre} \to \textbf{Top}$ preserves these colimits and that they also exist in $\textbf{Topp}$. We use Example 2.5 to show that coequalizers do not exist in $\textbf{Pre}$: let $(X, d)$ be the sum in $\textbf{Pre}$ of $\omega$ copies of the convergent sequence $(\frac{1}{n}) \cup \{0\}$ with the usual metric. Let $Y$ be the quotient set where by all limits points (i.e., the 0’s) are glued together, and assume that for some $r : Y^2 \to \mathbb{R}$, $(Y, r)$ represents the coequalizer of the above scenario in $\textbf{Pre}$. Since the quotient map $q : X \to Y$ must be $\epsilon$-$\delta$ continuous it follows that any epsilon ball about $q(0) = 0$ contains all but finitely many elements of each convergent sequence in $X$. Next, notice that for any open set in the Fréchet Fan, one can forge a metric on $Y$, say $s$, that would witness such an open set making the function $q : (X, d) \to (Y, s)$ $\epsilon$-$\delta$ continuous. Since $(Y, r)$ was assumed to be the coequalizer, then $id_Y : (Y, r) \to (Y, s)$ must also be $\epsilon$-$\delta$ continuous and, in turn, $(Y, r)$ would generate the Fréchet Fan on $Y$. A contradiction.

### 3. Constructing $\mathbf{P}$ and $\mathcal{O}$

#### 3.1. The Category $\mathbf{P}$

We begin by illustrating the construction of $\mathbf{P}$ (based on Flagg’s continuity spaces [11]) and later describe its relationship with $\textbf{Top}$. For a lattice $L$ and any pair $x, y \in L$, $y \succ x$ is the *well-above* relation defined by $y \succ x$ if whenever $x \geq \bigwedge S$, with $S \subseteq L$, there exists some $s \in S$ such that $y \geq s$. A well-known characterization of completely distributive lattices is as follows ([6])
Theorem 3.1. A lattice $L$ is completely distributive iff for all $y \in L$
\[ y = \wedge \{ a \in L \mid a \succ y \} . \]

Definition 3.2. A value distributive lattice is a completely distributive lattice $L$ for which $L_{\wedge} = \{ a \in L \mid a \succ 0 \}$ forms a filter.

A simple example is the extended positive real line $[0, \infty]$. Given any value distributive lattice, $V$, a $V$-space is a pair $(X, d)$ so that $d : X \times X \rightarrow V$ for which $d(x, x) = 0$ for all $x \in X$. We follow Flagg's terminology of the triple $(V, X, d)$ as a continuity space. The category $P$ will be that of all continuity spaces; its objects are the triples $(V, X, d)$ where $V$ is a value distributive lattice and $(X, d)$ is a $V$-space. A morphism $(V, X, d) \rightarrow (W, Y, m)$ is a function $f : X \rightarrow Y$ such that for every $x \in X$ and for every $\epsilon \in W_{\prec}$ there exists $\delta \in V_{\prec}$ such that for all $x' \in X$: if $d(x', x) \prec \delta$ then $m(f(x'), f(x)) \prec \epsilon$; we will refer to these morphisms as $\epsilon$-$\delta$ continuous functions. Every ordinary premetric space $(X, d)$ is a $V$-space for $V = [0, \infty]$. Hence, $Pre$ is a full subcategory of $P$.

Definition 3.3. Let $(X, d)$ be a $V$-space and $\epsilon \in V$ with $\epsilon \succ 0$. $B_\epsilon(x) = \{ y \in X \mid d(y, x) \prec \epsilon \}$ is the $\epsilon$-ball with radius $\epsilon$ about the point $x \in X$.

Lemma 3.4. Let $(X, d)$ be a $V$-space. Declaring a set $U \subseteq X$ to be open if for every $x \in U$ there exists $\epsilon \succ 0$ such that $B_\epsilon(x) \subseteq U$ defines a topology on $X$.

Proof. For a continuity space $(X, V, d)$ let $\tau$ be the collection of all $U \subseteq X$ for which the hypothesis is satisfied. Clearly, $\tau$ is closed under unions. The rest follows from the well-above relation. That is, let $U_1, U_2 \in \tau$ and $x \in U_1 \cap U_2$. By definition, we can find $\epsilon_1, \epsilon_2 \in V_{\prec}$ so that $B_{\epsilon_1}(x) \subseteq U_1$ and $B_{\epsilon_2}(x) \subseteq U_2$. Since $V$ is a value distributive lattice, then $\delta = \epsilon_1 \wedge \epsilon_2 \in V_{\prec}$ and $B_\delta(x) \subseteq B_{\epsilon_1}(x) \cap B_{\epsilon_2}(x) \subseteq U_1 \cap U_2$. \hfill \Box

For a set any collection of sets $X$ and $A \subseteq X$, we say that $A$ is downwards closed provided that $B, C \in X$ and $B \subseteq C$, and $C \in A$ then $B \in A$. Also, we follow standard set-theoretic notation in that for any set $X$, we let $[X]^{\leq \omega}$ denote the collection of all finite subsets of $X$. The following construction is key for limits and colimits in $P$.

Lemma 3.5. For any set $X$ let $\Omega(X) = \{ A \subseteq [X]^{\leq \omega} \mid A$ is downwards closed$\}$. Ordering $\Omega(X)$ by reverse inclusion yields $(\Omega(X), \leq)$ as a value distributive lattice, where $p \succ 0$ if, and only if, $p$ is finite.

Proof. As a ring of sets, complete distributivity is clear for $\Omega(X)$. Next, if $p$ is finite and for some $S = \{ S_k \}_{k \in I} \subseteq \Omega(X)$ we have that $\wedge S_k \leq 0$ (i.e. $\cup S_k = [X]^{\leq \omega}$) then for at least one $k \in I$, $\cup p \subset S_k$. Indeed, each $S_k$ is a downward-closed collection of finite subsets of $X$ whose union must yield $[X]^{\leq \omega}$ and, hence, at least one must contain $\cup p$. Since we are dealing with lower sets, then $\cup p \in S_k \Rightarrow p \subseteq S_k$ and thus $p \geq S_k$. Conversely, notice that 
\[ 0 = \bigcup_{z \in [X]^{\leq \omega}} [X]^{\leq \omega} = \bigwedge_{z \in [X]^{\leq \omega}} [X]^{\leq \omega} \]
and that no infinite set is contained in any of the previous sets.

**Theorem 3.6.** The functor $O : P \to \text{Top}$ which sends a continuity space to the topology it generates is surjective and faithful.

**Proof.** Verifying that $\epsilon$-$\delta$ continuous functions are also continuous is done in very much the same way as with premetric spaces. The following is due to Flagg and can be found in [1] pg. 273: to show surjectivity of $O$ take any topological space $(X, \tau)$ construct an $\Omega(\tau)$-space $(X, d)$ for which

$$d(x, y) = \{ F \in [\tau]^{\omega} \mid \text{for all } U \in F \text{ if } x \in U \text{ then } y \in U \}.$$ 

Let $x \in U \in \tau$ and denote $\epsilon = \{ \emptyset, \{ U \} \}$. Construct $B_\epsilon(x)$ and notice

$$y \in B_\epsilon(x) \Rightarrow d(x, y) < \epsilon \Rightarrow d(x, y) \supseteq \epsilon \Rightarrow y \in U.$$

□

In Section 3.2 we illustrate how $O : P \to \text{Top}$ is left adjoint and $P$ topological over $\text{Set}$. The construction $\Omega(X)$ for a set $X$ is the dual of the free locale on $X$ [4]. We will frequently employ this construction in the sequel when developing limits and colimits in $P$.

### 3.2. Topological continuity vs $\epsilon$-$\delta$ continuity in $P$. 

In much the same spirit as with premetrics, we show that when topological net convergence implies $\epsilon$-$\delta$ net convergence then topological continuity is equivalent to $\epsilon$-$\delta$ continuity.

**Definition 3.7.** Let $(X, V, d)$ be any continuity space and $(x_i)_{i \in I}$ be any net in $X$. We say that $(x_i)_{i \in I}$ $d$-converges to a point $x \in X$ whenever for all $\epsilon > 0$ there exists $i_0 \in I$ so that for all $i \geq i_0$, $x_i \in B_\epsilon(x)$.

Obviously, $d$-convergence is stronger than $\tau_d$-convergence. Recall that topological continuity can also be characterized in terms of nets: a function $f : X \to Y$ is continuous iff it preserves net convergence. The following is then straightforward to prove.

**Lemma 3.8.** A function $f : (X, V, d) \to (Y, W, m)$ between continuity spaces is $\epsilon$-$\delta$ continuous if, and only if, $f$ preserves net convergence.

**Lemma 3.9.** For any continuity space $(Y, W, m)$ the following are equivalent.

1. For any function $(X, V, d) \to (Y, W, m)$, topological continuity and $\epsilon$-$\delta$ continuity coincide.

2. The notions of $\tau_m$-convergence and $m$-convergence coincide.

Mimicking the behaviour of premetrics, the following holds for continuity spaces.

**Corollary 3.10.** For any function $(X, V, d) \to (Y, W, m)$ between premetric spaces the following conditions imply that both types of continuity coincide.
(1) $B_\epsilon^n(x)$ is open, for every $x \in X$ and $\epsilon > 0$.
(2) $m$ satisfies the triangle inequality.
(3) $\tau_m$-net limits are unique.
(4) $\tau_m$ is $T_2$.

**Corollary 3.11.** The functor $O : P \to \textbf{Top}$ is left adjoint.

**Proof.** Given any topological space $(X, \tau)$, Theorem 3.6 generates a $P$-object $X_0 = (X, d, \Omega(\tau))$ so that $O(X_0) = (X, \tau)$. Moreover, one can easily verify that epsilon balls in $X_0$ are open sets in $(X, \tau)$. Next, consider a continuity space $Y_0 = (Y, m, W)$ with a topologically continuous function $f : O(Y_0) \to (X, \tau)$. From part (1) of the previous corollary we obtain that $f$ is also $\epsilon$-$\delta$ continuous and thus $O$ is left adjoint. □

In view of the above result and the equivalence $M \leftrightarrows \textbf{Top}$ highlighted in Section 1 (see [7]), it is clear that the category $M$ becomes a reflective subcategory of $P$. This highlights a cohesive way in which to add the triangle inequality to any object in $P$.

**Corollary 3.12.** The category $M$ is a reflective subcategory of $P$.

Next we show that the forgetful functor $P \to \textbf{Set}$ exposes $P$ as topological over $\textbf{Set}$. Because $\textbf{Set}$ is complete and cocomplete then, based on the following result, so must be $P$. In view of this fact, Section 5 might seem redundant. This is certainly not the case: some of the most interesting features of $P$ can be appreciated when constructing its limits and colimits. For instance, given a pair of continuity spaces $(X, V, d), (Y, W, m)$ the value distributive lattice of their product will be shown to be $\Omega((V)_\prec \times (W)_\prec)$ while its coproduct counterpart will be $\Omega((V)_\prec \sqcup (W)_\prec)$. A largely more complex construction will be that of the coequalizer for a continuity space $(X, V, d)$ and an equivalence relation on it: here we will need to consider all $V_\prec$ valued functions from $X$ and capture it as a value quantale. That is, the value distributive lattice in this case will be $\Omega((V_\prec)^X)$. In addition, in Section 4 we show that $O$ preserves these colimits. To facilitate notation, we will suppress the subscript dummy indexing in the product notation. For instance, $\prod_{j \in J} V_j$ will become $\prod V_j$ (where the indexing set will be understood from context).

**Theorem 3.13.** $P$ is topological over $\textbf{Set}$

**Proof.** Fix a set $X$, a collection of continuity spaces $\{(X_j, V_j, d_j) \mid j \in J\}$, and for each $j$ a function $f_j : X \to X_j$. For each $j$ one can construct $(X, V_j, m_j)$ where $m_j(x, y) = d_j(f_j(x), f_j(y))$ and, thus, endow each $f_j$ with $\epsilon$-$\delta$ continuity. Next we construct a value distributive lattice $V$ (based on each $V_j$) and $m : X^2 \to V$ (based on all $d_j$) making all $f_j$ $\epsilon$-$\delta$ continuous (in addition to the usual cohesion properties so as to render $P$ as topological). First notice that letting $V = \prod(V_j)$ and $m(x, y) \in V$ so that $\pi_j \circ m(x, y) = m_j(x, y)$ does turn each function $f_j$ into an $\epsilon$-$\delta$ continuous function. However, $\prod(V_j)$ is not value distributive (the well-above elements do not form a filter). Indeed, take a two
value distributive lattice product $V \times W$. Its well-above zero elements are of the form $(\top_V, a)$ and $(b, \top_W)$ where $a \succ \bot_W$ and $b \succ \bot_V$, respectively. Thus, the meet of any pair $(\top_V, a), (b, \top_W)$ is not well-above zero in $V \times W$. In order to fix this, we order-embed $\prod V_j$ into a suitable value distributive lattice $V$ and define $m : X^2 \rightarrow V$ accordingly. Recall that for any lattice $L$, the set $L_\prec := \{ a \in L \mid a \succ 0 \}$. Let $U := \prod f(V_j)_\prec$ (that is, $a \in U$ implies that for only finitely many $i \in J, a_i \neq \top_i$) and $V := \Omega(U)$. The injection $\phi : \prod V_j \rightarrow V$ is defined as follows: for a given $x \in \prod V_j$ let
\[ \phi(x) = x_\uparrow = \{ A \in [U]^{< \omega} \mid A \subseteq x_\uparrow \} \]
and $x_\uparrow = \{ a \in U \mid \forall j \in J, a_j \succ x_j \}$. Notice that since all $V_j$ are completely distributive lattices then for any $x \in \prod V_j$, $x_\uparrow$ uniquely determines $x$. Consequently, we have $\wedge(\cup x_\uparrow) = \wedge(x_\uparrow) = x$ in $\prod V_j$.

**CLAIM:** the function $\phi : \prod V_j \rightarrow V$ so that $x \mapsto x_\uparrow$ is an order-embedding.

**Proof.** Take $x = (x_j)$ and $y = (y_j)$ in $\prod V_j$ so that $x \neq y$. Notice that $\wedge(\cup x_\uparrow) = \wedge\{ a \in \cup x_\uparrow \mid a_i \geq x_i \} = x \neq y = \wedge\{ a \in \cup y_\uparrow \mid a_i \geq y_i \} = \wedge(\cup y_\uparrow)$ and, hence, that $\phi$ is injective. Also, if $x > y$, then clearly $x_\uparrow \subset y_\uparrow$ and $x_\uparrow > y_\uparrow$. \hfill \Box

Next, we define $m : X^2 \rightarrow R$ as follows: for $x, y \in X$ let $d(x, y) \in \prod V_j$ so that $\pi_j(d(x, y)) = d_j(x, y)$ and
\[ m(x, y) = \phi \circ d(x, y). \]

**CLAIM:** for each $j \in J$, $f_j : (X, V, m) \rightarrow (Y, V_j, d_j)$ is $\epsilon$-$\delta$ continuous.

**Proof.** Fix an $i \in J$ and let $\epsilon \succ \bot_i$. Let $\hat{\epsilon} \in \prod f(V_j)_\prec$ so that $\pi_j(\hat{\epsilon}) = \top_j$ when $j \neq i$ and $= \epsilon$ otherwise. Notice that $\hat{\epsilon} = \{ \emptyset, \{ \hat{\epsilon} \} \} \succ \bot_V$ and that $m(x, y) \prec \hat{\epsilon} \Rightarrow d_i(x, y) \prec \epsilon$. Thus, $f_i$ is as claimed. \hfill \Box

Lastly, choose any continuity space $(Z, W, s)$ and function $f : Z \rightarrow X$, and assume that all compositions $h_j := f \circ f$ are $\epsilon$-$\delta$ continuous. Choose any $z \in Z$ and $\epsilon \succ \bot_V$ and recall that the latter is equivalent to $|\epsilon| \in \omega$. In particular, $|\cup \epsilon| \in \omega$ also. By construction, for each $p \in \cup \epsilon$ only finitely many projections onto their respective value distributive lattices are different than the largest element of the given lattice at the given coordinate. Let
\[ k = \{ t \mid \exists p \in \cup \epsilon \text{ and } j \in J, \pi_j(p) = t < \top_j \}. \]
For the chosen $z$ and each $t \in k$ there exists a $\delta_t$ so that $s(x, y) < \delta_t \Rightarrow d_j(h_j(x), h_j(y)) < t$ (since each $f_j$ is $\epsilon$-$\delta$ continuous), where $t = \pi_j(p)$ for some $j \in J$ and $p \in \cup \epsilon$. Since $W$ is a value distributive lattice, then $\delta := \wedge_{t \in k} \delta_t > 0_W$ and one can easily verify that $s(x, y) < \delta \Rightarrow m(f(z), f(y)) < \epsilon$. \hfill \Box

3.3. **Limits in P.** The forging of products and limits in $P$ follows directly from the construction illustrated in Theorem 3.13. In sum, we have the following.
3.3.1. Products. Take an arbitrary collection of continuity spaces \( \{ (X_j, V_j, d_j) \} \), where \( j \in J \), let \( X = \prod X_j \) and \( U = \prod_j (V_j)_{\prec} \), so that if \( a \in U \) then for only finitely many \( i \in J \), \( a_i \neq 1_i \) and let \( V = \Omega(U) \). The injection \( \phi : \prod V_j \to V \) is defined as follows: for a given \( x \in \prod V_j \) let
\[
\phi(x) = x^\uparrow = \{ A \in [U]^{<\omega} \mid A \subseteq x^\uparrow \}
\]
and \( x^\uparrow = \{ a \in U \mid \forall j \in J, a_j \succ x_j \} \). For \( x = (x_j), y = (y_j) \in X \) let \( d(x,y) \in \prod V_j \) so that \( \pi_j(d(x,y)) = d_j(x_j, y_j) \) and
\[
m(x,y) = \phi \circ d(x,y).
\]

**Theorem 3.14.** \( \prod_P (X_j, V_j, d_j) = ((X,V,m), \pi_i : X \to X_i) \).

3.3.2. Equalizers. Equalizers are the simplest of all constructions. Take a pair of continuity spaces \((V,X,d_X)\) and \((Y,W,d_Y)\) with \(\epsilon\)-\(\delta\) continuous \(f,g : (X,V,d_X) \Rightarrow (Y,W,d_Y)\). The equalizer is simply \(Z = \{ x \in X \mid f(x) = g(x) \}\) with \(V\) and \(d_Z : X \times X \to V\) as the restriction of \(d_X\) onto \(Z\). The inclusion function \(Z \to X\) is clearly \(\epsilon\)-\(\delta\) continuous.

**Remark 3.15.** The functor \(O\), in general, does not preserve limits. In fact, it is simple to show that limits in \(P\) are mapped to topologies at least as fine as their corresponding limits in \(Top\) and that equality is rarely guaranteed.

4. Cocompleteness and cocontinuity

In contrast with the previous section, we allocate separate subsections for coproducts and coequalizers; the forging of such in \(P\) requires of a more delicate process than with limits. This is particularly true of coequalizers. Section 3.2 shows that \(O\) is left adjoint. In spite of this, we present more topological proofs for how colimits are preserved by \(O\).

4.1. Coproducts. For an arbitrary set-indexed family \(\{(X_j, V_j, d_j)\} (j \in J)\) of continuity spaces, we seek a construction of an continuity space \((X,V,m)\) that generates the topological sum of the topologies generated by \(\{(X_j, V_j, d_j)\}\).

In particular, we want \(R\) and \(d_Z\) to be defined in terms of the \(V_i\)'s and the \(d_j\)'s respectively. Let \(U = \bigsqcup_{j \in J} (V_j)_{\prec}\) and \(X = \bigsqcup_{j \in J} X_j\). It is not difficult to verify that \(U\) is not a value distributive lattice (in fact, it’s not even a lattice).

That said, all information about the \(V_i\)'s is contained in \(U\) and we construct \(V\) in much the same way as with products: \(V = \Omega(U)\). The distance function \(m : X \times X \to V\) must be defined by parts: for a fixed \(i \in J\) consider the function \(\phi_i : V_i \to V\) so that for \(a \in V_i\),
\[
a \mapsto a^\uparrow := a^\uparrow \cup \left( \bigcup_{i \neq j} (V_j)_{\prec} \right)^{<\omega}
\]
where \(a^\uparrow = \{ \epsilon \succ 0 \mid \epsilon \succ a \}\) and \(0_i\) is the bottom element of \(V_i\). In order to lighten some of the notational burden, for any \(j \in J\) and a point \(x \in i_j(X_j) \subseteq \)}
X, we let \( x_j = i_j^{-1}(x) \) where \( i_j : X_j \to X \) denotes the obvious injection. Lastly, define \( d : X \times X \to V \) for all \( x, y \in Z \) as follows:

\[
d(x, y) = \begin{cases} 
\phi_j \circ d_j(x, y_j) & x, y \in i_j(X_j), \\
\psi_V & \text{otherwise.}
\end{cases}
\]

The following are easy to verify.

**Lemma 4.1.** For \( \{ (X_j, V_j, d_j) \} \) and \( (X, V, d) \) as previously described

1. For any \( j \in J \) the function \( \phi_j \) is an order-embedding.

2. For any \( j \in J \), all \( \epsilon > 0 \) and any pair \( x, y \in i_j(X_j) \) then

\[
d_j(x_j, y_j) < \epsilon \Leftrightarrow d(x, y) < \epsilon = \{ \epsilon \}, \emptyset \).
\]

3. For any \( \forall V > \epsilon > \perp_V \) and \( x, y \in X \): \( d(x, y) < \epsilon \) if, and only if, for some \( j \in J \) we have \( x, y \in i_j(X_j) \) and \( d_j(x_j, y_j) < \delta \), \( \forall \delta \in \cup \epsilon \cap V_j \).

**Theorem 4.2.** \( \bigsqcup \mathcal{P}(X_j, V_j, d_j) = ((X, V, d), i_j : X_j \to X) \) and \( \mathcal{O} \) preserves it.

**Proof.** First we show that the injections \( i_j : (X_j, V_j, d_j) \to (X, V, d) \) are \( \epsilon-\delta \) continuous. Fix a \( k \in J \), an \( x \in i_k(X_k) \) and take any \( p \in V_\epsilon \). If \( \forall V > \epsilon \) then there is nothing to prove so we assume \( \forall V > p \). Since \( p > \perp_V \), there are only finitely many \( p_n \in \cup p \cap (V_j)_\perp \) and thus \( \delta := \wedge p_n > \perp_k \). Moreover, \( y_k \in B_\delta(x_k) \) implies \( d_k(x_k, y_k) < p_n \) (for all \( n \)) and that \( d(x, y) < p \). Hence, the injection \( i_k \) is \( \epsilon-\delta \) continuous. Next, take a continuity space \( (Y, W, m) \) in conjunction with a collection \( \epsilon-\delta \) continuous functions \( f_j : X_j \to Y \) and define \( h : X \to Y \) so that \( x \mapsto f_j(x_j) \). Take \( x \in X \) and \( \epsilon > \perp_W \) where, wlog, \( x \in i_k(X_k) \) for some \( k \in J \). Since \( f_k \) is \( \epsilon-\delta \) continuous there exists \( \delta > \perp_k \) so that \( d_k(x_k, y_k) < \delta \Rightarrow m(x, y) < \epsilon \). Letting \( \mathfrak{d} = \{ \delta \}, \emptyset \) we have that \( y \in B_\delta(x) \Rightarrow y \in B_\delta(x) \). Thus, \( m(h(x), h(y)) = m(f_k(x_k), f_k(y_k)) < \epsilon \).

We are left with showing that \( \mathcal{O} \) preserves this coproduct. In what follows, \( (X, \tau_\cup) = \bigsqcup (X_j, \tau_j) \) from the collection of topological spaces \( \{(X_j, \tau_j) \mid j \in J \} \) generated by \( \{(X_j, V_j, d_j)\} \) and \( (X, \tau_V) = \mathcal{O}(X, V, d) \).

**CLAIM:** \( (X, \tau_V) = (X, \tau_\cup) \).

**Proof.** \( \supseteq \) Fix a \( k \in J \) and let \( x \in O \in \tau_\cup \) with \( x \in i_k(X_k) \), and \( \epsilon > 0 \) so that \( i_k(B_\epsilon(x_k)) \subseteq O \). It follows that if \( y \in B_\epsilon(x) \) then \( y_k \in B_{\epsilon}(x_k) \) and thus \( y \in O \).

\( \subseteq \) Let \( x \in O \in \tau_V \) and \( p \in V_\epsilon \) so that \( B_p(x) \subseteq O \). Without loss of generality, assume that \( x \in i_k(X_k) \) for some \( k \in J \). Since the injection is \( i_k \) is \( \epsilon-\delta \) continuous then we can find a \( \delta > \perp_k \) so that \( i_k(B_\delta(x_k)) \subseteq B_p(x) \subseteq O \). In turn, \( i_k(O) \) is \( \tau_k \)-open and the proof is complete. \( \Box \)
4.2. Coequalizers. Begin with a continuity space \((X, V, d)\) and an equivalence relation \(\sim \in Eq(X)\); since \(P\) is complete and the forgetful functor \(P \to \text{Set}\) is continuous we must consider all equivalence relations on \(X\). Denote \(Z = X/\sim\); we seek a value quantale \(Q\) and a distance assignment \(m : Z \times Z \to Q\) that will generate that quotient topology on \(Z\) inherited from the topology generated by \((X, V, d)\) on \(X\). The value quantale \(Q\) will be based on \((V_\sim)^X\) (the collection of all functions from \(X\) to \(V_\sim\)) while \(m\) will be constructed from epsilon balls. As usual, let \(Q = \Omega((V_\sim)^X)\) as described previously. There is the obvious injection \(\phi : V^X \to Q\) where

\[ \phi(x) = \{ A \in ((V_\sim)^X)^\omega | \forall y \in A, p \in X, \text{we have } g(p) > x(p) \}. \]

For any \(f \in (V_\sim)^X\) and \([a] \in Z\) say \([[a], [z]] < f\) precisely when for a finite sequence \(x_1, \ldots, x_n\) we have: (a) \(a \sim x_1\) and \(z \sim x_n\), (b) for \(i\) odd, \(x_{i+1} \in B_{f(x_i)}(x_i)\), and (c) for \(i\) even \(x_i \sim x_{i+1}\). In view of the above balls, the most natural way to assign a distance for a pair \([x], [y] \in Z\) is by first by letting \(\overline{T} = \{ \emptyset, \{ f \} \}\) and defining:

\[ m([x], [y]) = \{ A \in ((V_\sim)^X)^\omega | \forall f \in A, ([x], [y]) < f \}. \]

Clearly, \(m([x], [y]) \in Q\) and one can easily show that

\[ B_{\overline{T}}([a]) = \{ [x] | m([a], [x]) \prec \overline{T} \} \]

and, in general, that for any \(\epsilon \succ \bot_Q\),

\[ B_{\epsilon}([a]) = \{ [x] | m([a], [x]) \prec \overline{T}, \text{ for all } f \in \epsilon \}. \]

That the quotient map is \(\epsilon-\delta\) continuous follows directly from the definition of the balls \(B_{\epsilon}([a])\).

**Lemma 4.3.** The category \(P\) contains all coequalizers and \(O\) preserves them.

**Proof.** To show that \(P\) contains all coequalizers, let \((X, V, d)\) and \((Z, R, m)\) be as before and consider any other continuity space \((Y, W, m)\) with an \(\epsilon-\delta\) continuous function \(g : X \to Y\) for which there exists \(h : Z \to Y\) with \(g = h \circ q\). Pick any \(\epsilon \succ 0_W\) and let \(f \in (V_\sim)^X\) so that for each \(x \in X\), \(g(B_f(x))(x) \subseteq B_\epsilon(g(x))\). It follows that for any \([x] \in Z\), \(h(B_f([x]))\) is also contained in \(B_\epsilon(g(x))\) and since \(g = h \circ q\) it follows that \(h\) is \(\epsilon-\delta\) continuous and \(P\) contains all coequalizers. We complete this proof by showing that \(O\) preserves them. Let \((X, \tau)\) be the topological space generated by the continuity space \((X, V, d)\) and \((Z, \tau_q)\) be the quotient topological space.

**CLAIM:** \(O(Z, V, m) = (Z, \tau_q)\).

**Proof.** \((\subseteq)\) Let \(\epsilon \succ \bot_V\) and notice that it suffices to assume that \(\epsilon = \overline{T}\) for some \(f \in (V_\sim)^X\). Let \([a] \in Z\) and take any \(x \in q^{-1}(B_{\epsilon}([a]))\). By definition of \(B_{\epsilon}([a])\), it must be that there exists a finite \(a \sim x_1, \ldots, x_n \sim x\) with \(x_{i+1} \in B_{f(x_i)}(x_i)\) and \(x_{2k} \sim x_{2k+1}\). Notice that one can extend this chain as \(a \sim \overline{T}\) for some \(f \in (V_\sim)^X\) and take any \(x \in q^{-1}(B_{\epsilon}([a]))\). By definition of \(B_{\epsilon}([a])\), it must be that there exists a finite \(a \sim x_1, \ldots, x_n \sim x\) with \(x_{i+1} \in B_{f(x_i)}(x_i)\) and \(x_{2k} \sim x_{2k+1}\). Notice that one can extend this chain as \(a \sim \overline{T}\) for some \(f \in (V_\sim)^X\) and take any \(x \in q^{-1}(B_{\epsilon}([a]))\). By definition of \(B_{\epsilon}([a])\), it must be that there exists a finite \(a \sim x_1, \ldots, x_n \sim x\) with \(x_{i+1} \in B_{f(x_i)}(x_i)\) and \(x_{2k} \sim x_{2k+1}\). Notice that one can extend this chain as \(a \sim
$x_1, \ldots, x_n, x$ and that for any $y \in B_{f(x)}(x)$ it must be that $[y] \in B_f([a])$. It follows that $B_{f(x)}(x) \subseteq f^{-1}(B_f([a]))$ and $f^{-1}(B_f([a]))$ is a saturated open set in $(X, \tau)$. Lastly, one can easily show that $B([x]) = \bigcap_{f \in \pi_\prec(x)} B_f([x])$.

(2) For each $U \in \tau_q$ and $x \in O = f^{-1}(U)$, with there corresponds an $\epsilon_x \in V_x$ so that $B_{\epsilon_x}(x) \subseteq O$. Also, since $O$ must be saturated then $[x] \subseteq O$. The choice of $f \in (V_\prec)^X$ is the one for which $f(x) = \epsilon_x$ for $x \in O$ and $\top$ otherwise. In particular, for each $x \in O$, $B_f([x]) \subseteq U$. □

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