On symplectic transformations

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Abstract

This is an English translation of the PhD thesis ‘Over symplectische transformaties’ that Tonny Albert Springer, ‘born in ’s-Gravenhage in 1926’, submitted as thesis for—as is stated on the original frontispiece—the degree of doctor in mathematics and physics at Leiden University on the authority of the rector magnificus Dr. J. H. Boeke, professor in the faculty of law, to be defended against the objections of the Faculty of Mathematics and Physics on Wednesday October 17 1951 at 4 p.m., with promotor Prof. dr. H. D. Kloosterman.

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Introduction

The classification of conjugacy classes of elements of the general linear group $\text{GL}_n(K)$, the group of invertible linear transformations in $n$ variables over a commutative coefficient field $K$, is easily found with the help of the results from the theory of canonical forms of linear transformations.\footnote{For this theory, see, e.g., [vdW, p. 121].} In this thesis we investigate the classification of conjugacy classes in the symplectic group $\text{Sp}_n(K)$, which is the subgroup of $\text{GL}_n(K)$ consisting of all linear transformations that leave invariant a given skew-symmetric form. We assume that the characteristic of $K$ differs from 2.

It is found that, as with the group $\text{GL}_n(K)$, every conjugacy class of $\text{Sp}_n(K)$ can be characterized by certain invariants. These are irreducible polynomials and systems of integers (such invariants also occur for $\text{GL}_n(K)$), and, additionally, equivalence classes of certain Hermitian or quadratic forms (see section 9).
For the sake of completeness, the classification of conjugacy classes in $GL_n(K)$ is treated first. Also, the structure of the normalizer of an element of $Sp_n(K)$ is investigated.

It seems that the classification of conjugacy classes in $Sp_n(K)$ for an arbitrary field $K$ has not been treated before. Special cases have been discussed by L. E. Dickson [LED1], [LED2] ($Sp_4(K)$ and $Sp_6(K)$ if $K$ is a finite field) and by J. Williamson [W] (if $K$ is the field of real numbers).

The classification of conjugacy classes in the other classical linear groups (the orthogonal and unitary groups) can be examined by an analogous method. We hope to return to that.

1. The classification of conjugacy classes in the general linear group.

We assume $u$ to be a linear transformation of $E$. To each element $f = \sum_{\alpha=0}^s Y_{\alpha}X^{\alpha}$ ($Y_{\alpha} \in K$) of the ring $K[X]$ of polynomials in one variable $X$ over $K$ we assign the linear transformation $f(u) = \sum_{\alpha=0}^s Y_{\alpha}u^{\alpha}$ of $E$. This assignment is a homomorphism from $K[X]$ onto the ring of linear transformations $f(u)$. If $d$ is the unique polynomial of lowest possible degree and with leading coefficient 1 for which $d(u) = 0$ holds ($d$ is called the minimal polynomial of $u$), then the ring consisting of the $f(u)$ is isomorphic to the residue class ring $R = K[X]/(d)$.

Now in $K[X]$, $d$ can be written as product of mutually distinct irreducible polynomials with highest coefficient 1. Suppose this decomposition is $d = \prod_{i=1}^s p_i^{k_i}$. Denote by $E_i$ the subspace of $E$ which consists of all $x \in E$ for which $p_i^{k_i}(u)(x) = 0$ holds. Every $E_i$ is transformed into itself by $u$ (because $up_i^{k_i}(u) = p_i^{k_i}(u)u$). Each element of $E$ can be written as a sum of elements from $E_i$ (there are polynomials $g_i$ such that $\sum_{i=1}^s g_i \frac{d}{p_i^{k_i}} = 1$ and for an $x \in E$ one thus has $x = \sum_{i=1}^s x_i$ with $x_i = g_i \frac{d}{p_i^{k_i}}(u)(x) \in E_i$). This can be done in one way only: if $\sum_{i=1}^s x_i = 0$ with $x_i \in E_i$, then $\frac{d}{p_i^{k_i}}(u)(x_i) = 0$ and, as $\frac{d}{p_i^{k_i}}$ and $p_i^{k_i}$ are relatively prime, there exist polynomials $a$ and $b$ such that $a \frac{d}{p_i^{k_i}} + b p_i^{k_i} = 1$, from which it follows that $x_i = (a \frac{d}{p_i^{k_i}} + b p_i^{k_i})(u)(x_i) = 0$. We see that $E$ is the direct sum of the $E_i$ ($1 \leq i \leq s$). The restriction of $u$ to $E_i$ is a linear transformation $u_i$ of $E_i$ with minimal polynomial $p_i^{k_i}$. For $x = \sum_{i=1}^s x_i$, with $x_i \in E_i$ one has $u(x) = \sum_{i=1}^s u_i(x_i)$. We investigate the individual $u_i$ ($1 \leq i \leq s$).

We can assume that $u$ itself is a linear transformation with minimal polynomial $p^{k}$, where $p$ is an irreducible polynomial of degree $\frac{k}{\ell}$. Since each element of $R = K[X]/(p^{k})$ yields a transformation $f(u)$, we can give $E$ the structure of an $R$-module. We call this module $M$. The multiplication with elements of $R$ is thus defined as follows: if $\kappa$ is the canonical homomorphism from $K[X]$ onto $R$, then for $\rho = \kappa(f) \in R$ one has: $\rho x = f(u)(x)$.

The ring $R$ is commutative with unit element and has a simple structure: when $\pi = \kappa(p)$, every nonzero element of $R$ can be written as $\pi^a \epsilon$ ($0 \leq a \leq k$), where $\epsilon$ is a unit (invertible element) in $R$. The radical of $R$ is $(\pi)$. Also, every element $\rho$ of $R$ can be unambiguously written as $\rho = \sum_{i=0}^{\ell-1} a_i \xi^{i}$ ($a_i \in K$), where $\xi = \kappa(X)$. We denote by $R_i$ the residue class ring $R/(\pi^i)$ ($0 \leq i \leq k$). We can then consider $R_i$ as a ring with operators from $R$. Also, as a ring, $R_i$ is isomorphic to $K[X]/(p^i)$, in particular, $R_1$ (the residue class ring with respect to the radical) is isomorphic to the field $L = K[X]/(p)$.

\footnote{1}{i.e., before 1951.}

\footnote{2}{When we speak about an irreducible polynomial, we shall here and in the sequel assume that the leading coefficient of this polynomial is 1.}
We now investigate the $R$-module $M$ in more detail. We will prove:

There are elements $e_i^j$ ($1 \leq i \leq k, 1 \leq j \leq a_i$) of $M$ such that $M$ is the direct sum of the modules $Re_i^j$. 

(As usual, by $Re_i^j$ is meant the submodule of $M$ consisting of the elements $\rho e_i^j$ with $\rho \in R$).

We define the $e_i^j$ as follows. Take as $e_i^j$ ($1 \leq j \leq a_i$) a maximal system of elements of $M$ such that $\pi^{k-1} e_i^j \neq 0$ and such that every module $Re_i^j$ has only the zero element in common with the sum of the others. Suppose that the $e_i^j$ ($n \leq i \leq k$) are already known. If $n > 0$, take as $e_n^j$ ($1 \leq j \leq a_n$) a maximal system of elements of $M$ such that $\pi^n e_n^j = 0$, $\pi^{n-1} e_n^j \neq 0$, and such that each module $Re_n^j$ ($n \leq i \leq k$) has only the zero element in common with the sum of the other $Re_i^j$ ($n \leq i \leq k$). It is of course possible that such $e_n^j$ do not exist. We then set $a_n = 0$ (however, $a_0 \neq 0$). We now prove that every $x \in M$ is a sum of elements from the $Re_i^j$. Take an $x \in M$. Suppose $\pi^h x = 0$ and $\pi^{h-1} x \neq 0$ if $h > 0$. The assertion is proved by complete induction on $h$. For $h = 0$ it is correct. Assume, that the correctness is already proved for all $n < h$. Due to the definition of the $e_i^j$, there are elements $\rho$ and $\rho_i^j$ of $R$ such that $\rho x - \sum_{i=1}^k \sum_{j=1}^{a_i} \rho_i^j e_i^j = 0$ and such that not all $\rho_i^j e_i^j$ are zero. Multiplying by a unit of $R$, one can arrange that $\rho = \pi^n$ ($0 \leq n < h$). We assume this now. Since $\pi^n x = 0$, we have $\sum_{i=1}^k \sum_{j=1}^{a_i} \pi^{n-h} \rho_i^j e_i^j = 0$, which is possible only if $\pi^{n-h} \rho_i^j$ is a multiple of $\pi^h$. We can therefore state that $\rho_i^j = \pi^h \rho_i^j$. It follows that $\pi^n x - \sum_{i=1}^k \sum_{j=1}^{a_i} \rho_i^j e_i^j = 0$. However, according to the inductive assumption, the element in parentheses is a sum of elements from the $Re_i^j$. The same is therefore true for $x$. Thus $M$ is the sum of the $Re_i^j$. That it is a direct sum of the $Re_i^j$ follows from the definition of the $e_i^j$.

We further note that the annihilator of $e_i^j$ is the ideal $(\pi^i)$ ($1 \leq i \leq k$), which shows that $Re_i^j$ is isomorphic to the $R$-module $R_i$. Denoting $R_i^k$ the direct sum of $a_i$ modules that are isomorphic to $R_i$, we see that the result can also be pronounced like this:

$M$ is the direct sum of modules isomorphic to $R_i^k$ ($1 \leq i \leq k$).

It follows from the above that the vector space $E$ over $K$ has a basis consisting of elements $u^k(e_i^j)$ ($0 \leq h < i$, $g$), and that with respect to this basis $u$ is represented by a matrix which is completely determined by $p$, $K$ and the $a_i$. Here $\sum_{i=1}^a a_i = n$. One thus arrives at the well-known canonical forms of matrices.

If $E_1$ is a second $n$-dimensional vector space over $K$, $u_1$ is a linear transformation of $E_1$ and $t$ a linear mapping from $E$ onto $E_1$ such that $tu(x) = u_1 t(x)$ ($x \in E$), then it is easy to see that to $u_1$ belong the same $p$, $K$ and $a_i$ as to $u$. In the special case that $E_1 = E$ and that $t$ is a linear mapping from $E$ onto itself, it follows that the same $p$, $K$ and $a_i$ belong to $u_1 = tu^{-1}$ as belong to $u$. The other way around: if the linear transformations $u$ and $u_1$ of $E$ with minimal polynomial $p^k$ have the same $a_i$, then there are two systems of basis vectors of $E$ with respect to which $u$ and $u_1$ are represented by the same matrix. The transition from one basis to the other then gives a $t$ for which $u_1 = tu^{-1}$.

Finally, let there be given an irreducible polynomial $p$ of degree $g$, integers $k > 0$ and $a_i \geq 0$ ($1 \leq i \leq k$, $a_i > 0$) such that $g \sum_{i=1}^k a_i = n$. Then multiplication by the element $\xi$ of $R$ gives a

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1. edited to make the case $h = 0$ meaningful.
2. The derivation given here is essentially the one of [SvdW, p. 307].

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Each conjugacy class of $\text{GL}_n(K)$ is unambiguously determined by a number of mutually distinct irreducible polynomials $p_i$ different from $X$, with degrees $g_i$, by integers $k_i > 0$ and integers $a_j^i \geq 0$ (1 $\leq i \leq s$, 1 $\leq j \leq k_i$, $a_j^i > 0$) such that
\[
\sum_{i=1}^s g_i \sum_{j=1}^{k_i} j a_j^i = n.
\]

2. The linear transformations that commute with a given one.

We assume to be given a linear transformation $u$ of the vector space $E$ with minimal polynomial $\rho$ and we investigate the linear transformations of $E$ that commute with $u$. The vector space $E$ can again be decomposed into the subspaces $E_i$ (1 $\leq i \leq s$). If $v$ is a linear transformation of $E$ commuting with $u$, then $\rho^i (u)(v(x)) = v(\rho^i (u)(x))$ ($x \in E$). It follows that $v$ transforms every $E_i$ into itself. We can therefore again restrict ourselves to the case where $u$ has a minimal polynomial $\rho$. To $v$ commuting with $u$ belongs a self-map of the $R$-module $M$, which we shall also call $v$. When $\rho = \kappa(f) \in R$, we have $v(\rho x) = v(\kappa(f)x) = v(f(u)(x)) = f(u)(v(x)) = \rho v(x)$. Since $v(x + y) = v(x) + v(y)$, the self map $v$ of $M$ is an $R$-linear map. Conversely, if for all $\rho \in R$, we have $v(\rho x) = \rho v(x)$, then also $v u(x) = v(\xi x) = \xi v(x) = u v(x)$ for all $x \in E$. We see:

To every linear transformation of the vector space $E$ that commutes with $u$ belongs an $R$-linear mapping of the $R$-module $M$ into itself, and vice versa.

It follows that the group invertible linear transformations of $E$ that commute with $u$ is isomorphic with the group $\text{GL}(M, R)$ of automorphisms of $M$ ($R$-linear maps of $M$ onto itself). We will examine this group in greater detail.

First we note that an endomorphism $v$ of $M$ (i.e., an $R$-linear map of $M$ into itself) is completely determined by the $v(e^i_j)$. If
\[
v(e^i_j) = \sum_{p=1}^k \sum_{q=1}^{a_j^i} \rho_{ip}^{jq} e^q_p \quad (\rho_{ip}^{jq} \in R),
\]
then $v(x e^i_j) = 0$ must hold, from which it follows that $\pi \rho_{ip}^{jq} \equiv 0 \pmod{\pi^q}$ in $R$. Conversely, if one has elements $\rho_{ip}^{jq}$ of $R$, for which these congruences hold, then (1) determines an endomorphism of $M$. Now $v$ is an automorphism of $M$ if and only if $v(x) = 0$ implies $x = 0$. This follows directly from the corresponding assertion for a vector space.

Denote by $M_i$ (0 $\leq i \leq k$) the subspace of $M$ which consists of the $x \in M$ for which $\pi^i x = 0$. Since $\pi^{k-i}x \in M_i$ for every $x \in M$, one may view $M/M_i$ as a module over $R_{k-i}$. Denote by $\pi M_
(0 ≤ i ≤ k) the submodule of M consisting of the elements πi x (x ∈ M). Then M/πi M can be understood as a module over R. Further M0 is the direct sum of the modules R(ei 0) (1 ≤ p ≤ i), R(π(p−i)e 0 i) (i+1 ≤ p ≤ k). When υ is an endomorphism of M one has υ(πi x) = πi υ(x) for all x ∈ M, showing that M0 and πi M are transformed by υ into themselves. When υ is an automorphism, M0 and πi M are mapped onto themselves. An endomorphism υ of M induces an endomorphism of the Rk−i-module M/Mi. Thus one gets a homomorphism from GL(M, R) to GL(M/Mi, Rk−i). We will show that this is a homomorphism onto GL(M/M0, Rk−i).

Since M/Mi is isomorphic to πi M (isomorphism theorem) and since πi M is the direct sum of the modules R(πi e 0 p) (i + 1 ≤ p ≤ k), the Rk−i-module M/Mi is the direct sum of modules Rk−i e 0 p (i + 1 ≤ p ≤ k, 1 ≤ q ≤ a0); if ψi is the canonical homomorphism from M onto M/Mi, then e 0 p = ψ(e 0 p). Let ϕk−i denote the canonical homomorphism from R onto Rk−i. Suppose that for an automorphism ω of M/Mi it holds that w(ϕk−i) = ϕk−i(ω(e 0 p)). Then v(e 0 p) = e 0 p (1 ≤ p ≤ i), v(e 0 p) = ϕk−i(ω(e 0 p)) (i + 1 ≤ p ≤ k) determines an endomorphism υ of M which on M/Mi induces the automorphism w. Suppose that υ(x) = 0 for an x ∈ M. Since w is an automorphism, x lies in M0. Write x = ∑i=1k ∑q=1a 0 ω(e 0 q) + ω(πi y) (ω ∈ R, y ∈ ∑i=1k ∑q=1a 0 e 0 p). Say y ∈ M0, y ∉ M−i/n > i). From v(x) = 0 it follows that ω(e 0 q) = 0, ω(πi y) = 0. Since w is an automorphism of M/Mi, one also has v(y) ∈ M0, v(y) ∉ M−i/n−1. From υ(πi y) = 0 it then follows that α ≥ n, so that υ(πi y) = 0. Then x = 0, so υ is an automorphism of M. We have thus proved:

The assignment that to an automorphism of M associates the automorphism it induces on M/M0 gives a homomorphism from GL(M, R) onto GL(M/M0, Rk−i) (0 ≤ i ≤ k).

Denote by Gi (0 ≤ i ≤ k) the normal subgroup of G = GL(M, R) which consists of all automorphisms υ of M that induce the identity automorphism on M/Mi. Thus Gi consists of all υ ∈ G such that υ(x) − x lies in Mi for all x ∈ M. We examine these normal subgroups, starting with G1.

First we note that for υ ∈ G1 and for any x = πy ∈ πM it holds that υ(x) = x (because υ(x) − x = υ(πy) − y = 0). Denote by G1′ the normal subgroup of G1 which consists of all υ ∈ G1 for which also υ(x) = x for x ∈ M1. For υ ∈ G1′ we therefore have υ(x) = x + w(x), where w is an R-linear mapping from M to M1 which is zero on M1 + πM. One can easily see, that G1′ is an Abelian group, which is isomorphic to the additive group of maps w. Now both M/πM and M1 can be viewed as vector spaces over the field R1. These vector spaces have the same dimension, viz. a1 + a2 + · · · + ak. An R-linear map from M to M1 which is zero on πM determines a linear mapping from the vector space M/πM to the vector space M1. Conversely, one sees without difficulty, that to a linear mapping from the vector space M/πM to the vector space M1 belongs an R-linear mapping from M to M1, which is zero on πM. This shows that G1′ is isomorphic to the additive group of the linear maps from the vector space M/πM to the vector space M1 which are zero on the subspace M1 + πM/πM of dimension a1 over R1. So G1′ is isomorphic to the additive group of matrices with a1 + · · · + ak rows and with a1 + a2 + · · · + ak columns and with entries from R1. Since R1, viewed as a vector space over K, has dimension g, it follows from the above that G1′ is the direct sum of (a1 + · · · + ak)(a1 + a2 + · · · + ak)g groups isomorphic to the additive group Kg of K.

We now consider G1/G1′. To each υ ∈ G1 belongs an automorphism υ1 of M1 such that υ1(x) = x for x ∈ M1 ∩ πM. Conversely, given such an automorphism υ1 of M1, then one may associate a υ to it by υ(e 0 p) = υ1(e 0 p), υ(v(e 0 p)) = e 0 p (2 ≤ p ≤ k). If υ ∈ G1′, then υ1 is the identity automorphism of M1. It follows from the isomorphism theorem that G1/G1′ is isomorphic to the
group of automorphisms \( \nu_1 \) of \( M_1 \) which leave each element of \( M_1 \cap \pi M \) fixed. Now \( M_1 \) is the direct sum of \( M_1 \cap \pi M \) and a module \( V'_1 \) (namely the direct sum of the \( \text{Re} \)). One may view \( M_1, M_1 \cap \pi M \) and \( V'_1 \) as vector spaces over \( R_1 \). To each \( \nu' \in G_1/G'_1 \) belongs an automorphism of \( V'_1 \), since \( V'_1 \) is isomorphic to \( M_1/M_1 \cap \pi M \). Conversely, one can extend an automorphism of \( V'_1 \) directly to an automorphism of \( M_1 \) that fixes every element of \( M_1 \cap \pi M \). For an automorphism \( \nu'_1 \) of \( M_1 \) that fixes every element of \( M_1 \cap \pi M \) and induces the identity automorphism on \( V'_1 \) one has \( \nu'_1(x) = x + w'_1(x) \), where \( w'_1 \) is a linear map from \( M_1 \) to \( M_1 \cap \pi M \) which is zero on \( M_1 \cap \pi M \).

So if we denote by \( G''_1 \) the normal subgroup of \( G_1 \), consisting of the \( \nu ' \in G_1 \), for which also \( \nu(x) - x \in M_1 \cap \pi M \) for \( x \in M_1 \), it follows from the foregoing that \( G_1/G''_1 \) is isomorphic to the group of automorphisms of the vector space \( V'_1 \) and that \( G''_1/G'_1 \) is isomorphic to the additive group of matrices with \( a_1 \) rows and \( a_2 + \cdots + a_k \) columns and entries from \( R_1 \). Therefore \( G_1/G''_1 \) is isomorphic to \( \text{GL}_{a_1}(R_1) \), and \( G''_1/G'_1 \) is direct sum of \( a_1(a_2 + \cdots + a_k)g \) groups isomorphic to \( K^+ \).

Now \( G/G_i \ (1 \leq i \leq k-1) \) is isomorphic to \( \text{GL}(M/M_i, R_{k-i}) \). The normal subgroup of \( \text{GL}(M/M_i, R_{k-i}) \) which plays the role of \( G_1 \) in \( G \), is isomorphic to \( G_{i+1}/G_i \). One thus arrives at the following result:

\[
G = \text{GL}(M, R) \text{ has a series of normal subgroups } G = G_k \supset G_{k-1} \supset G''_{k-1} \supset G'_{k-1} \supset \cdots \supset G_1 \supset G''_1 \supset G'_1 \supset G_0 = [1], \text{ such that }
\]

- \( G_i/G_{i-1} \) is isomorphic to \( \text{GL}_{a_i}(L) \);
- \( G_i/G''_{i-1} \) is isomorphic to \( \text{GL}_{a_i}(L) \);
- \( G''_{i}/G'_i \) is the direct sum of \( a_i(a_{i+1} + \cdots + a_k)g \) groups isomorphic to \( K^+ \);
- \( G'_i/G_{i-1} \) is the direct sum of \( (a_i + \cdots + a_k)(a_{i+1} + \cdots + a_k)g \) groups isomorphic to \( K^+ \) \((1 \leq i \leq k-1)\).

Here

- \( G_i \) is the normal subgroup consisting of the \( \nu \in G \) for which \( \nu(x) - x \in M_i \ (x \in M) \);
- \( G''_{i} \) is the normal subgroup consisting of the \( \nu \in G \) for which \( \nu(x) - x \in M_{i-1} \cap \pi M \ (x \in M) \) and \( \nu(x) - x \in M_{i-1} + M_i \cap \pi M \ (x \in M) \);
- \( G'_i \) is the normal subgroup consisting of the \( \nu \in G \) for which \( \nu(x) - x \in M_i \ (x \in M) \) and \( \nu(x) - x \in M_{i-1} \ (x \in M) \).

As in the case \( i = 1 \), \( M_i/M_{i-1} \) can be understood as a vector space over \( R_1 \). This vector space is the direct sum of \( (M_{i-1} + M_i \cap \pi M)/M_{i-1} \) and a vector space \( V'_i \) of dimension \( a_i \) over \( R_1 \), which is thus isomorphic to \( V_i = (M_i/M_{i-1})/(M_{i-1} + M_i \cap \pi M/M_{i-1}) \ (1 \leq i \leq k) \).

3. **Investigation of the classification of conjugacy classes in the symplectic group, reduction to two cases.**

In the sequel we suppose that the characteristic of \( K \) is not equal to 2.

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\[ \text{See also } \text{(D1)} \text{ for the case of an algebraically closed field } K. \]
We assume to be given a skew-symmetric form \((x, y)\) on the \(n\)-dimensional vector space \(E\) over \(K\) i.e., a map from \(E \times E\) to \(K\) for which \((x_1 + x_2, y) = (x_1, y) + (x_2, y), \quad (\lambda x, y) = \lambda (x, y) \quad (\lambda \in K)\), and \((x, y) = -(y, x)\). It follows that also \((x, y_1 + y_2) = (x, y_1) + (x, y_2), \quad (x, y) = \lambda (x, y)\). We furthermore suppose that this form is non-degenerate, i.e., that from \((x, y) = 0\) for all \(y \in E\) it follows that \(x = 0\). The dimension of \(E\) must then be even.

A linear transformation \(u\) of \(E\) for which it holds that

\[(u(x), u(y)) = (x, y) \quad (x, y \in E)\]  \hspace{1cm} (2)

is called a symplectic transformation. One sees immediately that \(u\) is an invertible transformation (from \(u(x) = 0\) it follows that \((x, y) = 0\) for all \(y \in E\), so that \(x = 0\)). The symplectic transformations form a subgroup \(\text{Sp}_n(K)\) of \(\text{GL}_n(K)\). We want to investigate the classification of conjugacy classes in \(\text{Sp}_n(K)\).

If \(u\) is a symplectic transformation of \(E\), then it follows from (2) that \((u(x), y) = (x, u^{-1}(y))\) for all \(x\) and \(y\) from \(E\). One finds that

\[(f(u)(x), y) = (x, f(u^{-1})(y)) \quad (x, y \in E, \quad f \in K[X]).\]  \hspace{1cm} (3)

In the sequel, for \(f\) a polynomial of degree \(g\), we denote by \(\tilde{f}\) the polynomial which is determined by \(f(X) = X^g f(X^{-1})\). One then has \(\tilde{f} = f\). Let \(d\) denote the minimal polynomial of the symplectic transformation \(u\). According to (3) one then has \((d(u)(x), y) = (x, d(u^{-1})(y)) = 0\). From this it follows that for all \(y \in E\), \(d(u^{-1})(y) = 0\), so also \(d(u)(y) = 0\). Therefore, \(d\) must be a multiple of \(\tilde{d}\) and since the degree of \(\tilde{d}\) is not greater than that of \(d\), we see that \(d\) is of the form \(ad\) (\(a \in K\)). Because \(d = (\tilde{d}) = a\tilde{d} = a^2 d\) we have \(a = \pm 1\). Write \(d\) as a product of powers of mutually distinct irreducible polynomials in \(K[X]\), say \(d = \prod_{i=1}^r p_i^{k_i}\). Then \(\prod_{i=1}^r p_i^{\tilde{k}_i} = \pm \prod_{i=1}^r p_i^{k_i}\). Thus, the polynomial \(\tilde{p}_i\) occurs amongst the scalar multiples of the polynomials \(p_j\) \((1 \leq j \leq r)\).

It follows that the decomposition of \(d\) can be written as \(d = \gamma \prod_{i=1}^s (p_i\tilde{p}_i)^k \prod_{i=1}^m q_i^m\), where the \(p_i\) \((1 \leq i \leq s)\) are mutually distinct irreducible polynomials for which \(p_i \neq \pm \tilde{p}_i\) and where the \(q_i\) \((1 \leq i \leq t)\) are mutually distinct irreducible polynomials for which \(q_i = \pm q_i\), and where \(\gamma \in K\).

Denote by \(E_i\) the subspace of \(E\) generated by the \(x \in E\) for which either \(p_i(u)(x) = 0\) or \(\tilde{p}_i(u)(x) = 0\) \((1 \leq i \leq s)\). Denote by \(F_j\) the subspace of \(E\) consisting of the \(x \in E\) for which it holds that \(q_i^m(u)(x) = 0\) \((1 \leq j \leq t)\). The subspaces \(E_i\) and \(F_j\) are invariant under \(u\), and \(E\) is the direct sum of the \(E_i\) and \(F_j\) (see section 1). Furthermore, \(E_i\) is the direct sum of two subspaces \(E_i^1\) and \(E_i^2\), consisting of the \(x \in E\) for which \(p_i(u)(x) = 0\), respectively \(\tilde{p}_i(u)(x) = 0\). The subspaces \(E_i^1\) and \(E_i^2\) are also transformed into themselves by \(u\). Suppose that the vectors \(x\) and \(y\) of \(E\) lie in two different subspaces amongst the \(E_i\) and \(F_j\). Then there are polynomials \(f\) and \(g\) such that \(f(u)(x) = 0, g(u)(y) = 0\) and such that \(f\) and \(g\) are relatively prime \((f\) and \(g\) are certain polynomials \(p_i\tilde{p}_i\) or \(q_i^m\)). Then there are two polynomials \(h\) and \(n\) such that \(h\tilde{f} + ng = 1\). According to (3), we have \((x, y) = (x, (h\tilde{f} + ng)(u)(y)) = (x, h\tilde{f}(u)(y)) = (f(u)(x), u^n h(u)(y)) = 0\) \((a\) is the degree of \(f\)). We recall that a subspace \(G\) of \(E\) is called isotropic if there is an \(x \neq 0\) in \(G\), such that for all \(y \in G\) it holds that \((x, y) = 0\). The subspaces \(E_i\) and \(F_j\) that we found here are not isotropic. For instance, if for any \(x \in E_1\) we have \((x, y) = 0\) for all \(y \in E_1\), then, since \(E\)

---

\[\text{See for the notions that follow here p. 3–5 in the book [D] by J. Dieudonné.}\]

\[d=\text{scalar multiple}^{*}\ \text{added: the leading coefficient of } p_i \text{ is } p(0).\]

\[\text{We removed a comma between } p_i^{k_i} \text{ and } \tilde{p}_i^{l_i}.\]

\[[D \ p. 5]\]
is direct sum of the $E_i$ and $F_j$ and since $(x, z) = 0$ for for $z \in E_i$ ($i \neq 1$) and for $z \in F_j$, we get $(x, y) = 0$ for all $y \in E$, from which it follows that $x = 0$. We have thus found

\[ E \text{ is the direct sum of the non-isotropic subspaces } E_i \text{ and } F_j \ (1 \leq i \leq s, \ 1 \leq j \leq t). \]

We want to investigate when two symplectic transformations $u$ and $u'$ lie in the same conjugacy class of $\text{Sp}_n(K)$, i.e., when there is a symplectic transformation $w$ such that $u' = wuw^{-1}$. According to section 1, this requires that $u$ and $u'$ have the same minimal polynomial and that the subspaces $E_i, F_j$ belonging to $u$ have the same dimension as the subspaces $E_i', F_j'$ belonging to $u'$ ($1 \leq i \leq s, \ 1 \leq j \leq t$). From a theorem of Dieudonné it follows that there is then a symplectic transformation $v$ which maps the subspace $E_i'$ onto $E_i$ and the subspace $F_j'$ onto $F_j$ ($1 \leq i \leq s, \ 1 \leq j \leq t$). Then $vu'v^{-1}$ is a symplectic transformation which lies in the same conjugacy class of $\text{Sp}_n(K)$ as $u'$ and involves the same subspaces $E_i$ and $F_j$ as $u$. We assume in the sequel—which thus may be assumed without objection in the investigation of the classification of conjugacy classes in $\text{Sp}_n(K)$—that $u'$ already has this property.

Now assume that there is a symplectic transformation $w$ so that $u' = wuw^{-1}$. Then for an $x \in F_j$ we have $q_j^{m_i}(u)(x) = 0$, so that $0 = q_j^{m_i}(w^{-1}u'w)(x) = w^{-1}q_i^{m_i}(u')(w(x))$. It follows that $w(x) \in F_j$. So $w$ transforms $F_j$ ($1 \leq j \leq t$) into itself, and in an analogous way we prove that $w$ also transforms $E_i$ ($1 \leq i \leq s$) into itself. As the restriction of $w$ to a subspace $E_i$, $F_j$ is a symplectic transformation belonging to the restriction of the given form $(x,y)$ to that subspace, we see that if $u$ and $u'$ are conjugate in the symplectic group of $E$, then the restrictions of $u$ and $u'$ to the subspaces $E_i, F_j$ ($1 \leq i \leq s, \ 1 \leq j \leq t$) are conjugate in the symplectic groups of those subspaces. The converse is immediately apparent.

The investigation of whether or not two symplectic transformations $u$ and $u'$ in the symplectic group are conjugate is thus reduced to the case where $u$ and $u'$ both have minimal polynomials $(p, \bar{p})^k$ or $(p, \bar{p})^k$. We shall therefore in the sequel assume that $E$ itself is one of the subspaces $E_i$ or $F_j$.

4. First case, reduction to the one dealt with in section 1.

We start with the simplest case, namely that $u$ and $u'$ are symplectic transformations of $E$ with minimal polynomial $(p, \bar{p})^k$, where $p$ is an irreducible polynomial of degree $g$ such that $p \neq \pm \bar{p}$. Recall that $E$ is direct sum of subspaces $E^1$ and $E^2$, consisting of the $x \in E$ for which $p^k(u)(x) = 0$, respectively $\bar{p}^k(u)(x) = 0$. The subspaces are both transformed by $u$ into themselves. One can find two polynomials $h$ and $f$ such that $h\bar{p}^k + f^k = 1$. It follows that for $x, y \in E^1$, we have $(x, y) = (x, (h\bar{p}^k + f^k)(u)(y)) = (x, h\bar{p}^k(u), y) = (p^k(u)(x), u^k\bar{h}(u)(y)) = 0$. Similarly, $(x, y) = 0$ for $x, y \in E^2$. Thus $E^1$ and $E^2$ are totally isotropic subspaces of $E$ i.e., subspaces of $E$ such that the restriction of $(x, y)$ to those subspaces is zero. Suppose that $n = 2t$ is the dimension of $E$. A totally isotropic subspace has a dimension $\leq t$. Since $E$ is the direct sum of $E^1$ and $E^2$, one has $\dim(E^1) + \dim(E^2) = n$. Thus $\dim(E^1) = \dim(E^2) = t$, so that $E^1$ and $E^2$ are both maximal totally isotropic subspaces. There is then a basis $(e_i)$ of $E$ such that for $x = \sum_{i=1}^n \xi_i e_i$, $y = \sum_{i=1}^n \eta_i e_i$ one has $(x, y) = \sum_{i=1}^t (\xi_i \eta_{i+1} - \eta_i \xi_{i+1})$ (a basis with this property

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[1] proposition 2, p. 6
[2] One no longer insists that a minimal polynomial must have leading coefficient 1.
[3] [D] p. 6
[4] [D] p. 7
is called a symplectic basis) and further such that $e_1,\ldots,e_t$ forms a basis of $E^1$ and $e_{t+1},\ldots,e_{2t}$ forms a basis of $E^2$. With respect to the basis $(e_i)$ the map $u$ is represented by a matrix of the form

$$
\begin{pmatrix}
A & 0 \\
0 & \bar{A}
\end{pmatrix}
$$

(4)

where $A$ is invertible and $\bar{A}$ is the contragredient matrix (the inverse of the transpose matrix). From this one finds that the restriction of $u$ to $E^2$ is determined by the restriction to $E^1$ (and vice versa).

Similarly, to $u'$ belong two subspaces $E^1_1$ and $E^2_1$ for which the same holds. So there is a symplectic basis $(e_i')$ of $E$ such that $e_1',\ldots,e_t'$ forms a basis of $E^1_1$ and $e_{t+1}',\ldots,e_{2t}'$ forms a basis of $E^2_1$. The map $e_i \mapsto e'_i$ defines a symplectic transformation $w$. Then $w^{-1}u'w$ is a symplectic transformation that lies in the same conjugacy class of the symplectic group as $u'$ and that uses the same $E^1$, $E^2$ subspaces as belong to $u$. We assume henceforth that $u'$ itself already had this property: we suppose that $E^1_1 = E^1$, $E^2_1 = E^2$, and that $e_i = e'_i$.

With respect to the basis $(e_i)$, the map $u'$ is represented by a matrix of the form \(\begin{pmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{pmatrix}\). If $u$ and $u'$ are conjugate in $Sp_n(K)$, then the restrictions of $u$ and $u'$ to $E^1$ ($E^2$) are conjugate in the group $GL(E^1, K)$ ($GL(E^2, K)$): from $u' = tut^{-1}$ it follows that $p^k(u')(t(x)) = tp^k(u)(x) = 0$ for $x \in E^1$, so that also $t(x) \in E^1$. Conversely, if the restrictions of $u$ and $u'$ to $E^1$ are conjugate in $GL(E^1, K)$, then there is an invertible matrix $B$ such that $A_1 = BAB^{-1}$. The matrix \(\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}\) then gives a symplectic transformation $t$, for which $u' = tut^{-1}$. It follows that the conjugacy class of the symplectic transformation $u$ of $E$ is completely determined by the conjugacy class in $GL(E^1, K)$ of the restriction of $u$ to $E^1$.

Finally, given a symplectic basis $(e_i)$ of $E$, let us call $E^1$ ($E^2$) the maximal totally isotropic subspaces of $E$ with basis vectors $e_1,\ldots,e_t$ ($e_{t+1},\ldots,e_{2t}$). Then starting from a linear transformation of $E^1$ with minimal polynomial $p^k$, one can find a symplectic transformation of $E$ with minimal polynomial $(pp)^k$ which on $E^1$ induces this linear transformation. Thus, given a conjugacy class of linear transformations of $E^1$ with minimal polynomial $p^k$, there are symplectic transformations $u$ of $E$ such that the restriction from $u$ to $E^1$ lies in the class. From section 1 it follows that:

To a symplectic transformation $u$ of $E$ with minimal polynomial $(pp)^k$, where $p$ is an irreducible polynomial of degree $g$ such that $p \neq \pm p$, belong a number of integers $a_i \geq 0$ ($1 \leq i \leq k$, $a_k > 0$), such that $g \sum_{i=1}^k ia_i = n/2$; the conjugacy class of $u$ in the symplectic group is unambiguously determined by the numbers $a_i$.

If $u$ is a symplectic transformation of $E$ with minimal polynomial $(pp)^k$ and if $v$ is a symplectic transformation of $E$ commuting with $u$, then, since the minimal polynomials of the restrictions of $u$ to the subspaces $E^1$, $E^2$ are relatively prime, $v$ must transform these subspaces into themselves according to section 2. Then $v$ is represented by a matrix of the form \(\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}\) relative to the basis $(e_i)$. It follows that the group of symplectic transformations commuting with $u$ is isomorphic to the group of invertible linear transformations of $E^1$ commuting with the restriction of $u$ to $E^1$. 10
5. Second case, preparation.

We are now going to investigate the conjugacy of two symplectic transformations in the second case mentioned in section 3 that of a minimal polynomial $q^n$ with $q = \pm \bar{q}$. For this some preparations are required.

We assume that in the $n$-dimensional vector space $E$ over the field $K$ with characteristic $\neq 2$ the symplectic transformation $\sigma$ has an irreducible polynomial of degree $h$ such that $q = a\bar{q}$ with $a^2 = 1$. Suppose $h \geq 2$. Then $q(1) \neq 0$, $q(-1) \neq 0$. From $q = a\bar{q}$ it follows that $q(1) = a\bar{q}(1)$, $q(-1) = a(-1)\bar{q}(-1)$, which implies (as char($K$) $\neq 2$) that $a = 1$ and $h$ is even. So either $h$ is even and $q = \bar{q}$, or $h = 1$. In the latter case, $q(X) = X \pm 1$.

For the sequel it is convenient to know that the residue class ring $K[X]/(q^n)$ is isomorphic to the residue class ring with respect to the principal ideal $(q^n)$ in the ring $K[X]_q$ of rational fractions $\frac{u}{v}$ ($u, v \in K[X]$ with $v \neq 0$ mod $(q)$). (This is a well-known fact; incidentally, it is also easy to verify.) Henceforth, by $R$ we mean the residue class ring $K[X]/(q^n)$.

We take a closer look at $R$. The assignment $\frac{u(X)}{r(X)} \mapsto \frac{u(X^{-1})}{r(X^{-1})}$ gives an automorphism of $K[X]_q$ because from $v \neq 0$ mod $(q)$ it follows, using $q = \pm \bar{q}$, that $v(\frac{1}{X}) \neq 0$ mod $(q)$. The ideal $(q^n)$ is mapped onto itself. So this induces an automorphism $\rho \mapsto \tilde{\rho}$ ($\rho \in R$) of $R$. If $\kappa$ is the canonical homomorphism from $K[X]_q$ onto $R$, then for $\rho = \kappa(\frac{u(X)}{r(X)})$ one gets $\tilde{\rho} = \kappa(\frac{u(X^{-1})}{r(X^{-1})})$. It is clear that $\rho \mapsto \tilde{\rho}$ is an involutory automorphism, i.e., that $\tilde{\tilde{\rho}} = \rho$. Let us again put $\tilde{\xi} = \kappa(X)$. Then $\tilde{\xi} = \xi^{-1}$.

Since $R$ is generated by $\tilde{\xi}$, it follows that $\rho \mapsto \tilde{\rho}$ is the identity automorphism only if $\tilde{\xi} = \xi$, so if $X^2 - 1 \equiv 0$ mod $(q^n)$. This can only be the case if $q^n = X \pm 1$ (as the characteristic of $K$ is different from 2).

Denote by $S$ the subring of $R$ consisting of the $\sigma \in R$ for which $\sigma \bar{=} \sigma$. Except in the cases $q^n = X \pm 1$, we have $S \neq R$. Every $\sigma \in S$ can be written in the form $\rho + \tilde{\rho}$ with $\rho \in R$ (e.g., with $\rho = \sigma/2$). Conversely, an element $\rho + \tilde{\rho}$ of $R$ lies in $S$. In particular, $\eta = \tilde{\xi} + \frac{1}{\tilde{\xi}}$ lies in $S$ and $\xi$ satisfies the equation $\xi^2 - \eta \xi + 1 = 0$ with coefficients from $S$. When $R$ is a field, $R$ has rank 2 over $S$.

Furthermore, we will need linear forms $l$ on the vector space $R$ over $K$, i.e. $K$-linear maps from $R$ to $K$. These $l$ form a vector space $R'$ over $K$ (the dual vector space).\footnote{not to be confused with $\tilde{f}$ defined for $f \in K[X]$.} To any linear form $l$ and any $\lambda \in R$ one can associate a linear form $l_\lambda$, which is determined by $l_\lambda(\rho) = l(\lambda \rho)$ ($\rho \in R$). The map $\lambda \mapsto l_\lambda$ is a linear map from $R$ into $R'$. The $\lambda$ for which $l_\lambda = 0$, form an ideal $\mathfrak{f}(l)$ in $R$.

We call $l$ degenerate if $\mathfrak{f}(l)$ does not consist of the zero element of $R$ only. If $l$ is non-degenerate, then $\lambda \mapsto l_\lambda$ defines an injective linear mapping from $R$ onto $R'$. Furthermore, we note that for degenerate $l$ the ideal $\mathfrak{f}(l)$ contains the minimal ideal $(\pi^{n-1})$ of $R$, where $\pi = \kappa(q)$. It follows immediately that $l$ is a degenerate linear form if and only if $l(\rho) = 0$ for $\rho \equiv 0$ mod $(\pi^{n-1})$.

If $l(\lambda) = 0$. Then $l(\lambda) = l(\lambda \epsilon)$ follows that $l(\lambda \epsilon) = l(\lambda \bar{\epsilon})$. So $\epsilon \bar{\epsilon} = 1$. We will show that a non-degenerate form $l$ can be found for which $\epsilon = \bar{\epsilon} = 1$.

If there is an element $\rho \neq 0$ from the ideal $(\pi^{n-1})$ which does not lie in $S$, then $S$ is a proper subspace of the vector space $R$. One can thus find a linear form $l$ such that $l(\rho) \neq 0, l(\sigma) = 0$ for $\sigma \in S$. Then $l(\tau + \bar{\tau}) = 0$ for all $\tau \in R$. This $l$ is non-degenerate and the corresponding $\epsilon$ is $-1$.\footnote{For the notions of linear form and dual space, see [BI 4].}
Now suppose, that \( \rho = \bar{\rho} \) for all \( \rho \) from \((\pi^{m-1})\). Then for all polynomials \( f \in K[X] \)

\[
f(X)(q(X))^{m-1} \equiv f\left(\frac{1}{X}\right)(q\left(\frac{1}{X}\right))^{m-1} \pmod{q^m},
\]

from which follows\(^\text{4}\)

\[
a^{m-1}X^{h(m-1)}f(X) \equiv f\left(\frac{1}{X}\right) \pmod{q}.
\]

So

\[
a^{m-1}X^{h(m-1)} \equiv 1 \pmod{q}, \quad [\text{take } f = 1],
\]

and

\[
a^{m-1}X^{h(m-1)}X \equiv \frac{1}{X} \pmod{q}, \quad [\text{take } f = X].
\]

However, from these congruences it follows that

\[
a^{m-1}X^{h(m-1)}(X^2 - 1) \equiv 0 \pmod{q},
\]

which is only possible if \( q = X \pm 1 \). Thus, if \( h \geq 2 \) there is a non-degenerate \( l \), such that \( l(\rho + \bar{\rho}) = 0 \) \((\rho \in R)\).

If \( h = 1 \), then \( q = X \pm 1 \). The ideal \((\pi^{m-1})\) now consists of all multiples \( \delta \pi^{m-1} \) with \( \delta \in K \). Furthermore, \( \pi + \bar{\pi} = \xi \pm 1 + \frac{1}{2} \pm 1 \equiv 0 \mod{\pi^2} \). So for even \( m > 0 \) we have \( \bar{\pi}^{m-1} = -\pi^{m-1} \). Since now \( \pi^{m-1} \) does not lie in \( S \), one can again find an \( l \) such that \( l(\rho + \bar{\rho}) = 0 \) \((\rho \in R)\), \( l(\pi^{m-1}) \neq 0 \).

If, on the other hand, \( m \) is odd, then this is not possible, because now \( \bar{\pi}^{m-1} = \pi^{m-1} \), so that from \( l(\rho + \bar{\rho}) = 0 \) for all \( \rho \in R \) it follows that \( l(\pi^{m-1}) = 0 \). However, now the set of \( \rho - \bar{\rho} \) \((\rho \in R)\) is a proper subspace of the vector space \( R \), and it does not contain \( \pi^{m-1} \). One can therefore find a linear form \( l \) for which it holds true that \( l(\rho - \bar{\rho}) = 0 \), \( l(\pi^{m-1}) \neq 0 \). This \( l \) is non-degenerate and the corresponding \( \epsilon \) equals \( +1 \). Thus, we have proved

**There exist non-degenerate linear forms \( l \) on \( R \) for which it holds true that \( l(\bar{\rho}) = l(\epsilon \rho) \) \((\rho \in R)\), with \( \epsilon = -1 \) if \( h \geq 2 \) and with \( \epsilon = -(-1)^m \) if \( h = 1 \).**

6. **Second case, reduction to the investigation of the equivalence of Hermitian forms on a module over a ring.**

We use the notations of section 5. As in section 1 one can give \( E \) the structure of an \( R \)-module. Call this module \( N \). The skew-symmetric form \((x, y)\) given on \( E \) defines a map from \( N \times N \) to \( K \), which we also denote by \((x, y)\). This map is \( K \)-linear in \( x \) (resp. \( y \)) for fixed \( y \) (resp. \( x \)), and \((x, y) = -(y, x)\). Furthermore, according to formula 3 of section 3

\[
(\rho x, y) = (x, \bar{\rho} y) \quad (x, y \in N, \rho \in R).
\]

Suppose \( M \) is an \( R \)-module such that there exists a bijective \( R \)-linear map \( \phi \) from \( M \) onto \( N \). Like \( N \) (see section 1), \( M \) is a direct sum of modules \( Re_i^j \) \((1 \leq i \leq m, 1 \leq j \leq b)\). If \( \phi_i \) is a second \( R \)-linear mapping from \( M \) onto \( N \), then we have \( \phi_i = \phi t \), where \( t \) is an automorphism of \( M \).

\(^4\)Recall that \( q(X) = \alpha X^n q\left(\frac{1}{X}\right) \), with \( \alpha = \pm 1 \). In the subsequent computation we have inserted \( \alpha^{m-1} \) at appropriate places.
For $x, y \in M$ we have the $K$-linear form $(\rho \phi(x), \phi(y))$ on $R$. If $l$ is a fixed non-degenerate form on $R$ such that $l = l_o$, then one can write

$$(\rho \phi(x), \phi(y)) = l(\rho f(x, y)) \quad \text{with} \quad f(x, y) \in R.$$  \hspace{1cm} (5)

Then $l(\rho f(x_1 + x_2, y)) = (\rho \phi(x_1 + x_2), \phi(y)) = (\rho \phi(x_1), \phi(y)) + (\rho \phi(x_2), \phi(y)) = l(\rho f(x_1, y)) + l(\rho f(x_2, y))$, from which it follows that $l(\rho f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)) = 0$ for all $\rho \in R$. As $l$ is not degenerate it follows that $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ ($x_1, x_2 \in M$).

Also, $l(\rho \sigma f(x, y)) = (\rho \sigma \phi(x), \phi(y)) = (\rho \phi(\sigma x), \phi(y)) = l(\rho f(x, y))$, so that $f(\sigma x, y) = \sigma f(x, y)$. Finally, $l(\rho f(x, y)) = (\rho \phi(x), \phi(y)) = -\epsilon f(x, y)$, thus one can give $\tau$ for which $f(x, y) = -\epsilon f(x, y)$. We see that $f$ satisfies

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$$

$$f(\rho x, y) = \rho f(x, y),$$

$$f(x, y) = -\epsilon f(x, y).$$ \hspace{1cm} (6)

From these relations it follows easily that $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$ and $f(x, \rho y) = \bar{\rho} f(x, y)$. In section \footnote{Here is a footnote.} we saw that we can take $l$ such that $\epsilon = \pm 1$. By analogy with the case where $R$ is a field, we will call an $f$ with the properties \footnote{Another footnote.} a symmetric or skew-symmetric Hermitian form on the $R$-module $M$, depending on whether $-\epsilon = -1$ or $-\epsilon = -1$. We call such a form degenerate, if there is an $x \neq 0$ for which $f(x, y) = 0$ for all $y \in M$. The form $f$ found here is not degenerate; this follows directly from (5) since the given skew-symmetric form on $E$ is non-degenerate.

We call two forms $f$ and $g$ satisfying \footnote{Yet another footnote.} (with the same $\epsilon$) equivalent if there is an automorphism $t$ of $M$ for which $f(t(x), t(y)) = g(x, y)$ holds. Since the map $\phi$ from $M$ onto $N$ is unambiguously determined by $u$ except for an automorphism of $M$, the found form $f$ on $M$ is determined up to equivalence.

Let $M$ be a given $R$-module (with the same dimension over $K$ as $E$) and let there be given an equivalence class of non-degenerate forms $f$ on $M$. We will show that there exists a symplectic transformations $u$ of $E$ with minimal polynomial $q^n$ and such that $u$ determines on $M$ the equivalence class of $f$. So suppose $\tau$ is a birational $K$-linear mapping from $M$ onto $E$, and suppose $f(x, y)$ is a non-degenerate form on $M$ satisfying \footnote{Here is a footnote.}. Then we get a skew-symmetric form $k$ on $E$ determined by $k(\tau(x), \tau(y)) = l(\rho f(x, y))$; it is clear that $k$ is a bilinear form, and that $k$ is a skew-symmetric form follows from the relations $l(\rho f(x, y)) = l(\rho f(y, x)) = -l(\rho f(y, x)) = -k(\tau(x), \tau(y))$. Since $f$ is non-degenerate, $k$ is non-degenerate. Now two non-degenerate skew-symmetric forms on $E$ are equivalent. There is thus an invertible $K$-linear transformation $t$ of $E$ such that $k(x, y) = (t(x), t(y))$ ($x, y \in E$). Setting $U'(\tau(x)) = \tau(\xi x)$ determines a linear transformation $U'$ of $E$ for which $k(U'(\tau(x)), U'(\tau(y)) = l(f(\xi x, \xi y)) = l(f(x, y)) = k(\tau(x), \tau(y))$. Thus, $U'$ is a symplectic transformation belonging to the form $k$. Then $u = U't^{-1}$ is a symplectic transformation belonging to the given form $(x, y)$ on $E$. The ring of linear transformations of $E$ generated by $u$ is isomorphic to $R$, and can be considered as a ring of operators on $E$. Thus one can give $E$ the structure of an $R$-module. Call that $R$-module $N$. Then $\phi = \tau$ is a mapping from $M$ onto $N$ for which $(\phi(x), \phi(y)) = l(f(x, y))$. It is an $R$-linear map: for $x \in M$ we

\footnote{For any skew-symmetric form $s$ there is a basis with respect to which $s$ is given by $s(x, y) = \sum \epsilon_{i, n} (\xi_{1, i, n} \xi_{2, i, n} - \xi_{2, i, n} \xi_{1, i, n})$ ($n = 2l$). See [13] p. 5.}
have \( \phi(\xi x) = t\tau(\xi x) = t u' \tau(x) = ut\tau(x) = u\phi(x) = \xi \phi(x) \); it is clear that \( \phi(x + y) = \phi(x) + \phi(y) \).

Thus, \( u \) is a symplectic transformation of \( E \) that determines on \( M \) the equivalence class of \( f \).

We investigate when two symplectic transformations \( u \) and \( u' \) of \( E \) with minimal polynomial \( q^n \) lie in the same conjugacy class of the symplectic group. Starting from \( u \) or from \( u' \) one can define on \( E \) the structure of an \( R \)-module. Call the modules one gets \( N \) and \( N' \). According to section 5, \( u \) and \( u' \) can only lie in the same class of the symplectic group, if \( N \) and \( N' \) are isomorphic. We will therefore assume this. There are then bijective \( R \)-linear maps \( \phi, \phi' \) from the fixed \( R \)-module \( M \) to \( N \), respectively \( N' \), such that \( u(\phi(x)) = \phi(\xi x) \), \( u'(\phi'(x)) = \phi'(\xi x) \) \( (x \in M) \).

Furthermore, there are forms \( f \) and \( f' \) on \( M \) which satisfy (6) and for which \( (\phi(x), \phi(y)) = l(f(x, y)) \) and \( (\phi'(x), \phi'(y)) = l(f'(x, y)) \) \( (x, y \in M) \). Now suppose that there is a symplectic transformation \( w \) of \( E \) such that \( u' = w^{-1}uw \). Then \( t = \phi^{-1}w\phi' \) is an \( R \)-linear mapping from \( M \) onto itself: \( t(x + y) = t(x) + t(y) \) is obvious and furthermore \( t(\xi x) = \phi^{-1}w\phi' (\xi x) = \phi^{-1}w\phi' (\xi x) = \xi \phi^{-1}w\phi'(x) = \xi t(x) \) \( (x \in M) \), from which it follows that \( t(\rho x) = \rho t(x) \) \( (\rho \in R, x \in M) \). That \( w \) is a symplectic transformation is expressed by \( (w\phi'(x), w\phi'(y)) = (\phi'(x), \phi'(y)) \) viz. by \( (\phi(x), \phi(y)) = (\phi'(x), \phi'(y)) \) \( (x, y \in M) \). It follows that \( l(f(t(x), t(y))) = l(f'(x, y)) \). Because \( l(\rho f(t(x), t(y))) = l(\rho f'(x, y)) \) for all \( \rho \in R \), we have \( f(t(x), t(y)) = f'(x, y) \). Thus \( f \) and \( f' \) are equivalent on \( M \) if \( u \) and \( u' \) lie in the same class of the symplectic group. Conversely, if there is an automorphism \( t \) of \( M \) such that \( f(t(x), t(y)) = f'(x, y) \) then it is easy to verify that \( w = \phi(t\phi')^{-1} \) is a symplectic transformation of \( E \) for which \( u' = w^{-1}uw \).

We summarize what we found in this section:

To a symplectic transformation \( u \) of \( E \) with minimal polynomial \( q^n \), where \( q \) is an irreducible polynomial such that \( q = \lambda_q \), belongs a module \( M \) over \( R = K[X]/(q^n) \) with the same dimension over \( K \) as \( E \) and an equivalence class of non-degenerate symmetric or skew-symmetric Hermitian forms on \( M \). For a given module \( M \) and a given class of non-degenerate forms on \( M \) there is at least one \( u \). Two symplectic transformations \( u \) and \( u' \) with the same \( M \) are conjugate in the symplectic group if and only if the same class of Hermitian forms on \( M \) belongs to \( u \) and \( u' \).

7. Investigation of the equivalence of two Hermitian forms on a module over a ring.

The result of section 6 requires to look further into the equivalence of two symmetric or skew-symmetric Hermitian forms on the \( R \)-module \( M \). We use the same notations as in the previous section. The ring \( R \) has an automorphism \( \rho \mapsto \tilde{\rho} \). Since this automorphism maps the ideal \( (\pi) \) onto itself, there is an induced automorphism of \( R_\ell = R/(\pi^\ell) \). We denote this also by \( \rho \mapsto \tilde{\rho} \) \( (\rho \in R_\ell) \).

We begin with the case where \( q \) has degree \( h \geq 2 \). Then \( h \) is even (see section 5). For \( \pi_1 = \xi^{-h/2} \pi \) one has \( \pi_1 = \pi_\ell \), and \( \pi_1 \) also generates the ideal \( (\pi) \). Since in formula (6) of section 6 we can now take \( \epsilon = -1 \), we need only consider the case of a symmetric Hermitian form on \( M \) for our investigation of symplectic transformations. But first we prove:

If \( f \) is a non-degenerate symmetric or skew-symmetric Hermitian form on \( M \), then for every \( R \)-linear mapping \( x \mapsto \lambda(x) \) from \( M \) into \( R \) there is \( y \in M \) such that \( \lambda(x) = f(x, y) \) for all \( x \in M \).

Recall that \( M \) is direct sum of modules \( R e_i^j \) \( (1 \leq i \leq m, 1 \leq j \leq b_i) \). The \( R \)-module \( M' \) of linear forms \( \lambda(x) \) (the dual module) is then the direct sum of the modules \( R(e_i^j)' \) \( (1 \leq i \leq m, 1 \leq j \leq b_i) \).
We now show that a Hermitian form $f$ on $M$ also induces Hermitian forms on the residue class modules that were introduced in section [2]. We use the notations of section [2]. Let $f$ be a symmetric Hermitian form.

(a) On the $R_{m-i}$-module $M/M_i$ ($1 \leq i \leq m - 1$), $f$ determines a symmetric Hermitian form $f_i$, which is non-degenerate if $f$ is non-degenerate.

Denote by $\phi_{m-i}$ and $\psi_i$ the canonical homomorphisms of $R$ onto $R_{m-i}$ and of $M$ onto $M/M_i$, respectively. Then define $f_i$ by $f_i(\psi_i(x), \psi_i(y)) = \phi_{m-i}(f(x, y))$ ($x, y \in M$). One checks without difficulty that $f_i$ is a symmetric Hermitian form on the $R_{m-i}$-module $M/M_i$. If $f(\psi_i(x), \psi_i(y)) = 0$ for an $x \in M$ and for all $y \in M$, then one has $f(x, y) \equiv 0 \pmod{\pi^{m-i}}$ for all $y \in M$, from which it follows that $f(x, y) = 0$ for all $y \in M$, so that $\pi x = 0$ and $\psi_i(x) = 0$ if $f$ is non-degenerate on $M$.

(b) On the vector space $V_i = (M/M_{i-1})/(M_{i-1} + M_i \cap \pi M/M_{i-1})$ ($1 \leq i \leq m$) over $R_1$, $f$ determines a symmetric Hermitian form $f_i'$, which is non-degenerate if $f$ is non-degenerate.

We first note that for $x \in M_i$, one has $f(x, y) \equiv 0 \pmod{\pi^{m-i}}$. Denote by $\chi_{m-i}$ the $R$-linear mapping from the ideal $(\pi^{m-i})$ onto $R_i$, determined by $\chi_{m-i}(\pi^{m-i}) = \phi_i(\rho)$ ($\rho \in R$). Denote by $\psi_i'$ the canonical homomorphism from $M_i$ onto $V_i$ ($\psi_i'$ is the map obtained by first mapping $M_i$ canonically onto $M_i/M_{i-1}$ and then $M/M_{i-1}$ onto $V_i$). For $\rho \in R$ and $x \in M_i$, one then gets $\psi_i'(\rho x) = \psi_i(x)$ and $\chi_{m-i}(\pi^{m-i}) = \chi_{m-i}(\pi^{m-i}) = \psi_i(\pi^{m-i}) = \psi_i(\rho) = \psi_i(\rho)$.

Now define $f_i'(\psi_i'(x), \psi_i'(y)) = \chi_{m-i}(f(x, y))$ for $x, y \in M$. Since $\chi_{m-i}(f(x, y)) = 0$ when $x$ or $y$ lies in $M_{i-1} + M_i \cap \pi M$, this determines a mapping from $V_i \times V_i$ into $R_1$. One effortlessly verifies that $f_i'$ is a symmetric Hermitian form on $V_i$.

We now show that $f_i'$ is non-degenerate if $f$ is non-degenerate. We assume that $f$ is non-degenerate. First we check for which $x \in M$ one has $f(y, x) = 0$ for all $y \in M_i$. As $\pi^{m-i}f(y, x) = 0$, from which it follows that $f(y, x) \equiv 0 \pmod{\pi}$. Say $f(y, x) = \pi^s g(y)$. Then $g(y)$ is determined modulo $\pi^{m-i}$, and for $y \in M_i$ we have $g(y) \equiv 0 \pmod{\pi^{m-i}}$. Put $g_1(\psi_i(y)) = \phi_{m-i}(g(y))$. Then $g_1$ is a linear form on the $R_{m-i}$-module $M/M_i$. Since, according to (a), $f_i$ is non-degenerate on $M/M_i$ one can find $z \in M$ such that $f_i(\psi_i(z), \psi_i(z)) = 1$ ($y \in M$), viz. $\phi_{m-i}(f(y, z)) = \phi_{m-i}(g(y))$, from which it follows that $f(y, \pi^{m-i} z) = f(y, x)$ for all $y \in M$. This is only possible if $x = \pi^{m-i} z$, i.e., if $x \in \pi^m M$. If, conversely, $x \in \pi^m M$, then it is clear that $f(y, x) = 0$ for all $y \in M_i$.

Now suppose that for certain $x \in M_i$ and for all $y \in M_i$ one has $f_i'(\psi_i(y), \psi_i(x)) = 0$. Then $f_i'(\psi_i(y), \psi_i(z)) = 0$ for all $z \in M_i$, viz. $\phi_{m-i}(f(y, z)) = \phi_{m-i}(g(y))$, from which it follows that $f(y, \pi^{m-i} z) = f(y, x)$ for all $y \in M_i$. Since $x \in M_i$, also $x \in M_{i-1} + M_i \cap \pi M$, so that $\psi_i'(x) = 0$. Therefore $f_i'$ is non-degenerate on $V_i$ if $f$ is non-degenerate.

(c) If all the forms $f_i'$ ($1 \leq i \leq m$) on $V_i$ are non-degenerate, then $f$ is non-degenerate on $M$. 

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For \( m = 1 \) this assertion is true (then \( M = V_1 \)). Suppose it has already been proved for modules \( M \) over a ring \( K[X]/(q^n) \) with \( n < m \). Just like for \( M \) one has the vector spaces \( V_1 \) and the forms \( f'_i \) \((1 \leq i \leq m)\). And one has for \( M/M_a (1 \leq i \leq m - a) \) vector spaces \( W_i \) over \( R_i \) and forms \( f''_i \) on these vector spaces \((1 \leq i \leq m - a)\). The vector space \( W_i \) has the same dimension over \( R_1 \) as \( V_{i+a} \) and from the definition of the forms it further follows easily that there is a linear map \( t_i \) from \( W_i \) onto \( V_{i+a} \) such that \( f''_i(x, y) = f'_{i+a}(t_i(x), t_i(y)) \) \((x, y \in W_i)\).

Now suppose that \( f(x, y) = 0 \) for some \( x \in M \) and for all \( y \in M \). Then \( f_1(\psi_1(x), \psi_1(y)) = 0 \) for all \( \psi_1(y) \in M/M_1 \). Since, according to the induction hypothesis, \( f_1 \) is non-degenerate on \( M/M_1 \), one has \( \psi_1(x) = 0 \), i.e., \( x \in M_1 \). From \( f(x, y) = 0 \) for all \( y \in M_1 \) it further follows that \( f'_1(\psi'_1(x), \psi'_1(y)) = 0 \) for all \( \psi'_1(y) \in V_1 \). Since \( f'_1 \) is non-degenerate, one has \( \psi'_1(x) = 0 \), i.e. \( x \in M_1 \cap \pi M \). Say \( x = \pi z \). Then \( f(z, y) \equiv 0 \) mod \((\pi^{m-1})\) for all \( y \in M \). So \( f_1(\psi(z), \psi(y)) = 0 \) for all \( \psi(y) \in M/M_1 \). It follows that \( \psi(z) = 0 \), or \( z \in M_1 \). But then \( x = \pi z = 0 \). Therefore \( f \) is non-degenerate on \( M \).

Now suppose that \( f \) and \( g \) are two non-degenerate symmetric forms on \( M \). To \( f \) and \( g \) belong non-degenerate symmetric forms \( f_i \) and \( g_i \) on \( M/M_i \), as well as \( f'_i \) and \( g'_i \) on \( V_i \). We prove then:

For the forms \( f \) and \( g \) to be equivalent on \( M \) it is necessary and sufficient that the forms \( f'_i \) and \( g'_i \) are equivalent on \( V_i \) \((1 \leq i \leq m)\).

The necessity can be seen immediately by observing that an automorphism of \( M \) which transforms \( f \) into \( g \) induces on \( V_1 \) an automorphism that transforms \( f'_1 \) into \( g'_1 \).

We prove the other part of the assertion by complete induction on \( m \). For \( m = 1 \), we have \( M = V_1 \) and the assertion is true. Assume that for modules \( M \) over the ring \( K[X]/(q^n) \) with \( n < m \) the assertion has already been proved. On the \( R_{m-1} \)-module \( M/M_1 \) the forms \( f_i, g_i \) are then equivalent. Since \( M_1 \) is direct sum of \( M_1 \cap \pi M \) and a vector space \( V'_1 \) isomorphic to \( V_1 \) (see section 2), and \( f(x, y) = 0 \) for \( x \in \pi M, y \in M_1 \), the equivalence of \( f \) and \( g \) on \( M_1 \) follows from the equivalence of \( f'_1 \) and \( g'_1 \). Since, according to the result of section 2, there is an automorphism of \( M \) which induces given automorphisms on \( M/M_1 \) and on \( V_1 \), there exists an automorphism \( u \) of \( M \) such that

\[
\begin{align*}
\begin{cases}
 f(u_1(x), u_1(y)) &\equiv g(x, y) \pmod{\pi^{m-1}} \quad (x, y \in M), \\
f(u_1(x), u_1(y)) &\equiv g(x, y) \quad (x, y \in M_1).
\end{cases}
\end{align*}
\]

In order to derive the claim from (7), we need three auxiliary results.

1) If \( h \) and \( k \) are two non-degenerate symmetric Hermitian forms on \( M \) such that, for certain \( j \) \((1 \leq j \leq m - 1)\), we have

\[
\begin{align*}
\begin{cases}
 h(x, y) &\equiv k(x, y) \pmod{\pi^{m-1}} \quad (x, y \in M), \\
h(x, y) &\equiv k(x, y) \quad (x, y \in M_j \text{ or } x \in M_{j-1}, y \in M),
\end{cases}
\end{align*}
\]

then there is an automorphism \( \nu \) of \( M \) such that

\[
\begin{align*}
\begin{cases}
 h(v(x), v(y)) &\equiv k(x, y) \pmod{\pi^{m-1}} \quad (x, y \in M), \\
h(v(x), v(y)) &\equiv k(x, y) \quad (x \in M_j, y \in M).
\end{cases}
\end{align*}
\]

We will repeatedly use \( \nu \), where the original used and [in Dutch: en].
We try to find \( v \) with \( \nu(x) = x + \pi_i^{j-1}w(x) \), where \( w \) is an \( R \)-linear mapping from \( M \) into \( M_j \) such that \( \pi_i^{j-1}w(x) = 0 \) for \( x \in M_j \). It is easy to see that such \( v \) is an automorphism of \( M \) and that the first relation in (9) is satisfied. It follows from the other relation in (8), that \( w \) must satisfy
\[
\pi_i^{j-1}h(x, w(y)) = k(x, y) - h(x, y) \quad (x \in M_j, y \in M).
\]
Write \( k(x, y) - h(x, y) = \pi_i^{m-1}p(x, y) \). Then it follows that for \( x \in M_j \) it must hold true that
\[
h'_j(\psi'_j(x), \psi'_j(w(y))) = \phi_j(p(x, y)). \tag{9}
\]
For \( x \in M_{j-1} + M_j \cap \pi M \), we have \( k(x, y) - h(x, y) = 0 \), so \( p(x, y) \equiv 0 \mathrm{ mod } (\pi) \). This shows that \( \lambda(\phi'_j(x)) = \phi_j(p(x, y)) \) defines a linear form \( \lambda \) on \( V_j \). This form is zero if \( y \in M_j \). Recall the elements \( e_p^q \) \((1 \leq p \leq m, 1 \leq q \leq b_p)\) of \( M \) such that \( M \) is the direct sum of the modules \( R e_p^q \). Since \( h'_j \) on \( V_j \) is not degenerate, we can define for any element \( e_p^q \) an element \( w(e_p^q) \) so that \( \mathcal{E} \) is satisfied. One can further suppose that \( w(e_p^q) = 0 \) for \( 1 \leq p \leq j \). The \( w(e_p^q) \) determine an endomorphism \( w \) of \( M \). One easily verifies that \( \pi_i^{j-1}w(x) = 0 \) for \( x \in M_j \). Thus, for the \( v \) derived from this \( w \), \( \mathcal{E} \) is satisfied.

II) If \( h \) and \( k \) are two non-degenerate symmetric Hermitian forms on \( M \) such that for certain \( j \) \((1 \leq j \leq m - 1)\) one has
\[
\begin{align*}
\{h(x, y) & \equiv k(x, y) \mod \pi^{m-1} \quad (x, y \in M), \\
h(x, y) & = k(x, y) \quad (x \in M_j, y \in M),
\end{align*}
\]
then there is an automorphism \( \nu' \) of \( M \) such that
\[
\begin{align*}
\{h(\nu'(x), \nu'(y)) & \equiv k(x, y) \mod \pi^{m-1} \quad (x, y \in M), \\
h(\nu'(x), \nu'(y)) & = k(x, y) \quad (x, y \in M_{j+1} \text{ or } x \in M_j, y \in M). \tag{10}
\end{align*}
\]
We try to find \( \nu' \) with \( \nu'(x) = x + \pi_i^jw'(x) \), where \( w' \) is a linear mapping from \( M \) into \( M_{j+1} \) such that \( w'(e_p^q) = 0 \) for \( p \neq j + 1 \). One easily sees that such \( \nu' \) is an automorphism of \( M \) and that then the first relation in (10) is already satisfied. The other relation in (10) gives
\[
\pi_i^j(h(w'(x), y) + h(w(x), y) + \pi_i^j h(w'(x), w'(y)) = k(x, y) - h(x, y) \quad (x, y \in M_{j+1} \text{ or } x \in M_j, y \in M). \tag{11}
\]
Since \( w'(x) \in M_{j+1} \), we have \( h(w'(x), w'(y)) \equiv 0 \mod \pi^{m-j-1} \). It follows that the third term on the left hand side of (11) is zero. Furthermore, it is immediately apparent that (11) is satisfied for \( x \in M_j, y \in M \). Now put \( k(x, y) - h(x, y) = \pi_i^{m-1}p'(x, y) \). Then it follows from (11) that \( w' \) needs to satisfy
\[
\nu'_j(\psi'_j(w'(x)), \psi'_j(x), \psi'_j(w'(y))) = \phi_j(p'(x, y)) \tag{12}
\]
If one puts \( q(\psi'_j(x), \psi'_j(y)) = \phi_j(p'(x, y)) \), then \( q \) is a symmetric Hermitian form on \( V_{j+1} \). Auxiliary result II) then follows from
III) If $V$ is a vector space over the field $R_1$, $h$ is a non-degenerate symmetric Hermitian form on $V$ and $n$ a symmetric Hermitian form on $V$, then there is a linear transformation $t$ of $V$ such that
\[ h(t(x), y) + h(x, t(y)) = n(x, y). \]

This is easy to prove: for any fixed $y \in V$, we see that $\frac{1}{2}n(x, y)$ is a linear form on $V$. One can then find a linear transformation $t$ with the intended property from $\frac{1}{2}n(x, y) = h(t(x), y)$.

The assertion about the equivalence of the forms $f$ and $g$ that we wanted to prove is a direct consequence of I) and II): starting from (7) we can, by application of I) and II), find an automorphism $u_i$ of $M$ for each $i$ ($1 \leq i \leq m$) such that $f(u_i(x), u_i(y)) = g(x, y)$ is satisfied for $x$, $y \in M$, and for $x \in M_{i-1}$, $y \in M$. Then $u_0$ is such that $f(u_0(x), u_0(y)) = g(x, y)$ for all $x$ and $y$ from $M$.

Finally, we prove:

Given forms $f_i$ on the $V_i$ ($1 \leq i \leq m$), there is a symmetric Hermitian form $f$ that induces them.

Namely, suppose that $x = \sum_{p=1}^{m} \sum_{q_1=1}^{b_p} e_p^{\phi} \bar{e}_p^{\phi}$ and $y = \sum_{p=1}^{m} \sum_{q_1=1}^{b_p} \eta_p^{\phi} \bar{e}_p^{\phi}$ are two elements of $M$. Then for a symmetric Hermitian form $f$, one has $f(x, y) = \sum_{p=1}^{m} \sum_{q_1=1}^{b_p} \sum_{q_2=1}^{b_p} \alpha^{\phi}_{pq} e_p^{\phi} \bar{e}_p^{\phi} \eta^{\phi}_{pq} \bar{e}_p^{\phi}$ with $\alpha^{\phi}_{pq} = \alpha q_{pq}$, $\pi^p \alpha^{\phi}_{pq} = \pi^p \alpha^{\phi}_{pq} = 0$. One easily sees that if $\alpha^{\phi}_{pq} = \pi^p_{m-1} f^{\phi}_{pq}$, one has for $x$, $y \in M$ that $f_i(\psi_i'(x), \psi_i'(y)) = \phi_i(\sum_{q=1}^{b_p} f^{\phi}_{pq} e_p^{\phi} \bar{e}_p^{\phi})$. By appropriately choosing the $\alpha^{\phi}_{pq}$, it is possible to achieve that $f_i'$ becomes equal to a given form.

8. Continuation of section 7

In section 7 we examined the equivalence of the forms occurring in the study of symplectic transformations in the case where the polynomial $q$ has degree $h \geq 2$. We now consider the case $h = 1$. In the study of symplectic transformations with a minimal polynomial $q^m$ we now get to deal with a form $f$ satisfying relations (5) of section 6, where $-\epsilon = (-1)^m$. (See the result of section 5.) Furthermore, we now have $\pi + \tilde{\epsilon} \equiv 0 \mod (\pi)$, and the automorphism which is induced on $R_1$ is the identity automorphism (as $\xi - \tilde{\xi} \equiv 0 \mod (\pi)$, one has $\rho + \tilde{\rho} \equiv 0 \mod (\pi)$ for each $\rho \in R$).

Our starting point is a Hermitian form $f$ on $M$ which satisfies (5) of section 6 with $\epsilon = \pm 1$ (we thus do not yet suppose that $-\epsilon = (-1)^m$). As in section 7 one can then define a form $f_i$ on $M/M_i$, which is symmetric or skew-symmetric precisely if $f$ is so, and which is non-degenerate if $f$ is non-degenerate.

The forms $f_i'$ on $V_i$ ($1 \leq i \leq m$), however, must now be defined differently. Denote by $\chi_{m-i}$ the $R$-linear mapping from the ideal $(\pi^{m-i})$ into $R_1$ determined by $\chi_{m-i}(\pi^{m-i}) = \phi_i(\bar{p})$ ($\rho \in R$). Then, since $\pi + \tilde{\epsilon} \equiv 0 \mod (\pi^2)$, we get $\chi_{m-i}(\pi^{m-i}) = \chi_{m-i}((-\pi)^{m-i} \tilde{p} + \pi^{m-i+1} \sigma)$, with certain $\sigma \in R$ for each $\rho \in R$. It follows that $\chi_{m-i}(\pi^{m-i}) = (-\rho)^{m-i} \phi_i(\bar{p}) = (-1)^{m-i} \phi_i(\rho) = (-1)^{m-i} \chi_{m-i}(\pi^{m-i})$. Now define $f_i'$ by $f_i'(\psi_i'(x), \psi_i'(y)) = \chi_{m-i}(f(x, y))$ ($x$, $y \in M_i$). Then $f_i'(\psi_i'(x), \psi_i'(y)) = \chi_{m-i}((-\bar{f}(y, x)) = -\epsilon(-1)^{m-i} \chi_{m-i}(f(y, x)) = -\epsilon(-1)^{m-i} f_i'(\psi_i'(y), \psi_i'(x))$. Thus $f_i'$ is now a symmetric or skew-symmetric bilinear form on $V_i$, depending on whether $-\epsilon(-1)^{m-i}$
equals +1 or −1. One proves in the same way as in section 7 that \( f_{\alpha} \) is non-degenerate if \( f \) is non-degenerate and that \( f \) is non-degenerate if all \( f_{\alpha} \) are non-degenerate (1 \leq i \leq m).

For the case \(-\epsilon = (-1)^m\), which we have to deal with for symplectic transformations, we note that it follows that the \( f_{\alpha} \) with odd \( i \) are non-degenerate skew-symmetric bilinear forms on the vector space \( V_i \) of dimension \( b_i \) over \( R_1 \). So \( b_i \) is even if \( i \) is odd.

Again, for the equivalence of the non-degenerate symmetric or skew-symmetric Hermitian forms \( f \) and \( g \) on \( M \) it is necessary and sufficient that the forms \( f_{\alpha} \) and \( g_{\alpha} \) be equivalent on \( V_i \). This can be deduced from claims I) and II) of section 7 (which must then be slightly modified in obvious ways). I) is proved in the same way for the present case. The proof of II) however, becomes slightly different. Instead of (11) one gets the relation \( \pi^j h(w'(x),y) + \bar{\pi}^j h(x,w'(y)) = k(x,y) - h(x,y) \) \((x,y \in M_{j+1})\). Put \( k(x,y) - h(x,y) = \pi^{m-1} p'(x,y) \). Then \( \pi^{m-1} p'(x,y) = -\epsilon \pi^{m-1} p'(y,x) - \epsilon (-\pi)^{m-1} p'(y,x) \), from which it follows that \( \phi_1(p'(x,y)) = -\epsilon (-1)^{m-1} \phi_1(p'(y,x)) \). In place of (12) we now need

\[
\begin{align*}
&h'_{j+1}(\psi'_{j+1}(w'(x)),\psi'_{j+1}(y)) + (-1)^j h'_{j+1}(\psi'_{j+1}(x),\psi'_{j+1}(w'(y))) = \\
&\phi_1(p'(x,y)) \quad (x,y \in M_{j+1}).
\end{align*}
\]

Here \( h'_{j+1} \) is such that \( h'_{j+1}(x,y) = -\epsilon (-1)^{m-1} h'_{j+1}(y,x) \) \((x,y \in V_{j+1})\). For III) we must now substitute

III’) If \( V \) is a vector space over the field \( R_1 \), \( h \) is a non-degenerate bilinear form on \( V \) such that \( h(x,y) = \epsilon_1 h(y,x) \) \((\epsilon_1^2 = 1)\), \( n \) is a bilinear form on \( V \) such that \( n(x,y) = \epsilon_2 n(y,x) \) \((\epsilon_2^2 = 1)\), then there is a linear transformation \( t \) of \( V \) such that

\[ h(t(x),y) + \epsilon_1 \epsilon_2 h(x,t(y)) = n(x,y). \]

(Thus in (13) one has \( \epsilon_1 = -\epsilon (-1)^{m-1} \), \( \epsilon_2 = -\epsilon (-1)^{m-1} \).)

The proof is easy: one may determine \( t \) by \( \frac{1}{2} n(x,y) = h(x,t(y)). \)

It is proved in the same way as in section 7 that there are forms \( f \) which induce given forms \( f_{\alpha} \) on the \( V_i \).

9. The classification of conjugacy classes in the symplectic group.

We summarize what we found in sections 3 through 8 about the classification of conjugacy classes in the group \( \text{Sp}_n(K) \) (where the characteristic of \( K \) differs from 2):

Every conjugacy class of \( \text{Sp}_n(K) \) is unambiguously determined by
(a) a number of mutually distinct irreducible polynomials \( p_i \in K[X] \), different from \( X \), of degrees \( g_i \), and such that \( p_i \neq \pm \tilde{p}_i \) \((1 \leq i \leq s)\), integers \( k_i > 0 \) and integers \( a_{ij} \geq 0 \) \((1 \leq i \leq s, 1 \leq j \leq k_i, a_{ij}^2 > 0)\);
(b) a number of mutually distinct irreducible polynomials \( q_i \in K[X] \) of degrees \( h_i \geq 2 \) and such that \( q_i = \tilde{q}_i \) \((1 \leq i \leq t)\), integers \( m_i > 0 \), integers \( b_i^j \geq 0 \) \((1 \leq i \leq t, 1 \leq j \leq m_i, b_i^j_m > 0) \) and equivalence classes \( K_i^j \) of non-degenerate skew-symmetric Hermitian forms on the vector space of dimension \( b_i^j \) over the field \( L_i = K[X]/(q_i) \) \((1 \leq i \leq t, 1 \leq j \leq m_i)\);
(c) powers \((X+1)^m\) and \((X-1)^m\) of the polynomials \( X + 1 \) and \( X - 1 \) respectively,
Every irreducible polynomial then has the degree 1, so 10. The symplectic transformations which commute with a given symplectic transformation

Finally, we investigate the group of the symplectic transformations \( \nu \) which commute with a given symplectic transformation \( u \). With the help of what was proved in section 6, it is easily seen
that we can restrict ourselves to the case where \( u \) has a minimal polynomial \((p\bar{p})^k\) \((p \neq \pm \bar{p})\) or a minimal polynomial \(q^m\) \((q = \pm \bar{q})\). The first case has already been covered at the end of section \([4]\), so here we suppose that \( u \) has minimal polynomial \(q^m\), where \( q \) is an irreducible polynomial of degree \( h \) such that \( q = \pm \bar{q} \). We use the notations of the previous sections.

With a linear transformation \( v \) commuting with \( u \) belongs an automorphism \( t \) of the module \( M \) that was introduced in section \([8]\) (see section \([2]\)). This \( t \) is determined by \( \phi(t(x)) = v\phi(x) \) \((x \in M)\). If \( v \) is a symplectic transformation, then \((\phi(t(x)), \phi(t(y))) = (\phi(x), \phi(y))\), from which it follows that \( f(t(x), t(y)) = f(x, y) \). This shows that the group of symplectic transformations commuting with the symplectic transformation \( u \) is isomorphic to the group \( U = U(M, R, f) \) of automorphisms \( t \) of the \( R \)-module \( M \) for which \( f(t(x), t(y)) = f(x, y) \). We investigate this group in the same way as the group \( \text{GL}(M, R) \) (see section \([2]\)).

As in section \([3]\) an automorphism \( t \) of \( M \) for which \( f(t(x), t(y)) = f(x, y) \) induces an automorphism \( v_t \) on the \( R_{m-i} \)-module \( M/M_i \) for which \( f(v_t(x), v_t(y)) = f(x, y) \) \((x, y \in M/M_i)\). Conversely, assuming a \( v_t \) for which this holds, one can find an automorphism \( t' \) of \( M \) which on \( M/M_i \) induces the automorphism \( v_t \) and which is such that \( t'(e_i^p) = e_i^q \) \((1 \leq p \leq i)\) (see section \([2]\)). For \( i = 1 \), by applying I) and II) from section \([7]\) (resp. section \([8]\)), one can construct from this \( t' \) an automorphism \( t \) of \( M \) which on \( M/M_1 \) induces the automorphism \( v_t \) and which is such that \( f(t(x), t(y)) = f(x, y) \). When \( i > 1 \), one can apply this to the module \( M/M_{i-1} \) and thus find an automorphism \( v_{t-1} \) of \( M/M_{i-1} \) which on \( M/M_i \) induces the automorphism \( v_t \) and which is such that \( f(v_{t-1}(x), v_{t-1}(y)) = f_{t-1}(x, y) \) \((x, y \in M/M_{i-1})\). From the foregoing it then follows without difficulty:

Assigning to an automorphism from the group \( U(M, R, f) \) the automorphism it induces on \( M/M_i \) gives a homomorphism from \( U(M, R, f) \) onto \( U(M/M_i, R_{m-i}, f) \).

By \( G, G' \) and \( G'' \) we shall mean the groups introduced in section \([2]\). Then \( U_i = U \cap G_i \) is the normal subgroup of \( U \) which consists of the \( t \in U \) that induce the identity automorphism on \( M/M_i \). Furthermore, we set \( U'_i = U \cap G'_i \), \( U''_i = U \cap G''_i \) \((1 \leq i \leq m - 1)\). We are going to examine these normal subgroups in more detail.

First, suppose that \( h \geq 2 \). We can then suppose that \( f \) is a symmetric Hermitian form on \( M \) (see section \([9]\)). Now \( U'_i \) consists of all automorphisms \( t \) of \( M \) for which \( t(x) = x + w(x) \) with \( w(x) \in M_1 \), \( w(x) = 0 \) for \( x \in M_1 \), while also \( f(t(x), t(y)) = f(x, y) \), so

\[
 f(x, w(y)) + f(w(x), y) + f(w(x), w(y)) = 0. \quad (x, y \in M).
\]

Since \( w(x) = 0 \) for \( x \in M_1 \), we have \( f(x, w(y)) = 0 \) for \( x \in M_1 \), from which it follows that \( w(y) \in \pi M \) (see section \([7]\)). However, then \( f(w(x), w(y)) = 0 \) \((x, y \in M)\). It follows that \( U'_1 \) is isomorphic to the additive group of linear maps \( w \) from \( M \) into \( M_1 \cap \pi M \) which are zero on \( M_1 \) and for which \( f(x, w(y)) + f(y, w(x)) = 0 \) \((x, y \in M)\). Such a \( w \) induces a linear mapping \( w' \) from the vector space \( V = (M/\pi M)/(M_1 + \pi M/\pi M) \) over \( R_1 \) into the vector space \( W = M_1/\pi M \) over \( R_1 \). If we denote by \( \chi \) the canonical homomorphism from \( M \) onto \( V \), then \( f \) determines a mapping \( h(x, y) \) \((x \in V, y \in W)\) from \( V \times W \) into \( R_1 \) which is linear in \( x \) and anti-linear in \( y \), where \( h(x, y) \) is defined by \( h(x, y) = \chi_{m-1}(f(x, y)) \) \((x \in M, y \in W, \chi_{m-1} \) is the linear mapping from \((\pi M)^{m-1}) \) into \( R_1 \) that was introduced in section \([7]\). From \( h(x, y) = 0 \) for all \( x \in M \) follows \( f(x, y) = 0 \) for all \( x \in M \), and from this follows \( y = 0 \). For the map \( w' \) one has

\[
 h(x, w'(y))) + h(x, w'(y))) = 0. \tag{14}
\]
Conversely, one can easily see that given a linear mapping \( f \) from \( V \) into \( W \) satisfying this relation one gets an unambiguously determined \( w \).

Both \( V \) and \( W \) are vector spaces of dimension \( b_2 + \cdots + b_m \) over \( R_1 \). Choose bases in \( V \) and \( W \). If \( X \) (resp. \( Y \)) represents the matrix of components of the vector \( x \) (resp. \( y \)) from \( V \) (resp. \( W \)) relative to the basis of \( V \) (resp. \( W \)), then we have \( h(x, y) = X^tHY \), where \( H \) is an invertible matrix with entries from \( R_1 \) (here, \( X \) and \( Y \) are matrices with one column, and if \( A \) is a matrix with entries from \( R_1 \), we denote by \( A' \) the transposed matrix and by \( \bar{A} \) the matrix obtained from \( A \) by replacing each entry \( \rho \in R_1 \) of the matrix by \( \bar{\rho} \)). If to the linear transformation \( w' \) belongs a matrix \( L \), then the relation
\[
(14) \quad X^tHY + Y^tHLX = 0.
\]
holds for \( L \). The result is that \( \bar{H}L + L'\bar{H}' = 0 \). Putting \( HLY = T \) we get then \( T + T' = 0 \), and \( L = H^{-1}T \). This shows that the group \( \psi \) is isomorphic to the additive group of the skew-symmetric Hermitian matrices with \( b_2 + \cdots + b_m \) rows and columns and with entries in \( R_1 \). One easily verifies that this group of matrices is the direct sum of \( (b_2 + \cdots + b_m)^2 \) groups isomorphic to \( K^+ \). The same thus holds for \( U_1' \).

To each \( t \in U_1 \) belongs an automorphism \( t_1 \) of \( M_1 \) for which \( t_1(x) = x \) for \( x = 1 \cap \pi M \) and \( f(t_1(x), t_1(y)) = f(x, y) \).\(^3\) Conversely, one can also find a \( t \) for a \( t_1 \) (this again turns out to be easy using \( \mathcal{I} \) and \( \mathcal{I} \)) from section \( \mathcal{I} \). The \( t_1 \)’s for which \( t_1 \) is the identity automorphism are those from \( U_1' \). Thus \( U_1'/U_1' \) is isomorphic to the group of automorphisms \( t_1 \)’s. And \( U_1'/U_1' \) is isomorphic to the group of automorphisms \( t_1 \)’s of \( M_1 \) that satisfy \( t_1(x) = x \) for \( x \in 1 \cap \pi M \), \( t_1(x) = x \) for \( x \in 1 \cap \pi M \). \( f(t_1(x), t_1(y)) = f(x, y) \) (\( x, y \in M_1 \)). However, if the first two relations are satisfied, then the last relation is also satisfied: this follows directly from \( f(x_1, x_1) = 0 \) for \( x_1 \in 1 \cap \pi M \). Thus \( U_1'/U_1' \) is isomorphic to \( G_1'/G_1' \) and in section \( \mathcal{I} \) we have seen that this quotient group is direct sum of \( b_1(b_2 + \cdots + b_m) \) subgroups isomorphic to \( K^+ \). Furthermore, it is easy to see that \( U_1'/U_1' \) is isomorphic to the group \( U_{b_1}(f_1', R_1) \) of the automorphisms of \( V_1 \) which leave invariant the non-degenerate symmetric Hermitian form \( f_1' \).

In the same way as in section \( \mathcal{I} \) the investigation of the groups \( U_i \) with \( i > 1 \) can be reduced to the above case. The result is (with \( L = K[X]/(q) \)):

If \( h \geq 2 \) then the group \( U = U(M, R, f) \) has a series of normal subgroups
\[
U = U_m \supset U_{m-1} \supset U_{m-1} \supset \cdots \supset 1 \supset U_{i+1} \supset U_i \supset U_0 = \{1\}, \text{ such that }
\]
- \( U_i/U_{i+1} \) is isomorphic to \( U_{b_i}(L, f_{b_i}) \);
- \( U_i'/U_{i+1}' \) is isomorphic to \( U_{b_i}(L, f_i') \);
- \( U_i''/U_e' \) is the direct sum of \( b_1(b_2 + \cdots + b_m) \) groups isomorphic to \( K^+ \);
- \( U_i'/U_{i-1} \) is the direct sum of \( (b_{i+1} + \cdots + b_m)^2 \) groups isomorphic to \( K^+ \).

We now consider the case \( h = 1 \). The result then becomes slightly different. Assume that \( f(x, y) = -\bar{e}f(y, x) \) (\( \epsilon = \pm 1 \)).\(^3\) In the same way as above, we find that \( U_1' \) is isomorphic to the additive group of linear maps \( w \) from \( M \) into \( 1 \cap \pi M \) which are zero on \( M_1 \) and for which \( f(x, w(y)) - \bar{e}f(y, w(x)) = 0 \) (\( x, y \in M \)). Define \( V, W \) and \( \chi \) in the same way and now define \( h \) by \( h(y, x) = \chi''_m(f(x, y)) \), where in this case \( \chi''_m \) is the linear map from \( \pi^{m-1} \) into \( R_1 \) introduced in section \( \mathcal{I} \). We get that \( w \) induces a linear mapping \( w' \) from the vector space \( V \) into the vector space \( W \), for which
\[
h(\chi(x), w'(\chi(y))) + (-1)^m \bar{e}h(\chi(y), w'(\chi(x))) = 0.
\]

\(^3\) bar added.
Introducing bases in $V$ and $W$ shows that $U'_1$ is isomorphic to the additive group of symmetric or skew-symmetric matrices with $b_2 + \cdots + b_m$ rows and columns and with entries in $R_1$ according to whether $(-1)^m \epsilon = -1$ or $+1$. It follows without difficulty that $U'_1$ is direct sum of $\frac{1}{2}(b_2 + \cdots + b_m)\bar{\epsilon}$ groups isomorphic to $K'$. Again, $U''_1$ is isomorphic to $G'_1/G'_1$ and $U_1/U''_1$ is isomorphic to $Sp_b(R_1)$ if $\epsilon = (-1)^{m-1}$ and with the group $O_h(R_1, f'_1)$ of orthogonal transformations belonging to the quadratic form $f'_1$ on $V_1$ if $\epsilon = (-1)^m$.

Using this, one can also investigate the other $U_i$. One finds, since $R_1$ is now isomorphic to $K$, for the case $\epsilon = (-1)^{m-1}$, which we have to deal with for the case of symplectic transformations:

If $h = 1$ and $f(x, y) = (-1)^m f(y, x)$, then the group $U = U(M, R, f)$ has a series of normal subgroups $U = U_m \supset U_{m-1} \supset U''_{m-1} \supset \cdots \supset U_1 \supset U''_1 \supset U'_1 \supset U_0 = \{1\}$, such that

- $U_i/U_{i-1}$ is isomorphic to $O_h(K, f'_m)$ if $m$ is even and to $Sp_{b_0}(K)$ if $m$ is odd;
- $U''_i/U'_i$ is isomorphic to $O_h(K, f'_i)$ if $i$ is even and to $Sp_{b_0}(K)$ if $i$ is odd;
- $U''_i/U'_i$ is the direct sum of $b_1(b_1+\cdots+b_m)$ groups isomorphic to $K'$;
- $U'_i/U_{i-1}$ is the direct sum of $\frac{1}{2}(b_1+\cdots+b_m)(b_1+\cdots+b_m+(-1)^{i+1})$ groups isomorphic to $K'$.

Using these results, one can find, e.g., in the case where $K$ is a finite field of characteristic different from 2, the order of the normalizer of an element of the symplectic group. For $n = 4$ these numbers were calculated by L. E. Dickson.

Summary

In this thesis we have studied the classification of conjugacy classes of elements of the symplectic group $Sp_b(K)$, i.e. the group of linear transformations with $n$ variables and coefficients in the commutative field $K$ leaving invariant an alternating bilinear form. We have limited ourselves to the consideration of the case where the characteristic of $K$ is different from 2.

The result we have arrived at can be stated as follows: each conjugacy class of the group $Sp_b(K)$ is characterized by a system of invariants. These invariants are first of all, as in the case of the general linear group (the group of all invertible linear transformations), irreducible polynomials and systems of non-negative integers, but secondly also equivalence classes of certain Hermitian forms and of certain quadratic forms.

Here are some indications of how we have treated the problem. Let us denote, for a one-variable polynomial $f$ of degree $g$ with coefficients in $K$, by $\tilde{f}$ the polynomial defined by $\tilde{f}(X) = X^g f(\frac{1}{X})$. It is easy to see that the minimal polynomial of a symplectic transformation of a vector space $E$ on $K$ satisfies $f = \pm \tilde{f}$. It is shown that we may focus on two cases:

1) $f = (p \tilde{p})^k$, where $p$ is an irreducible polynomial such that $p \neq \pm \tilde{p}$
2) $f = q^n$, where $q$ is an irreducible polynomial such that $q = \pm \tilde{q}$.

In the first case, the class of $u$ in the symplectic group is determined by a conjugacy class of a general linear group. One can thus use the known theory of canonical forms of linear transformations.

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\footnote{The text mistakenly said $p = \pm \tilde{p}$.}
In the second case we proceed as follows. By virtue of the relation \( q = \pm \bar{q} \), the ring \( R = K[X]/(q^m) \) has an involutive automorphism \( \rho \mapsto \bar{\rho} \) (\( \rho \in R \)). One can define on \( E \) a module structure with respect to \( R \). Letting \( M \) be a copy of this \( R \)-module, we show that \( u \) and the alternating bilinear form given on the vector space \( E \), define on \( M \) an equivalence class of antisymmetric Hermitian forms, or symmetric Hermitian forms, where by such a form is meant a map \( f(x, y) \) from \( M \times M \) to \( R \) such that

\[
\begin{align*}
&f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y), \\
&f(\rho x, y) = \rho f(x, y) \quad (\rho \in R), \\
&f(x, y) = \epsilon f(y, x) \quad (\epsilon = \pm 1; \text{one can even assume that } \epsilon = +1, \text{except in the case where } q = X \pm 1, m \text{ odd}).
\end{align*}
\]

If two symplectic transformations \( u \) and \( u' \) with minimal polynomial \( q^m \) are conjugate in the general linear group, then we can take the same module \( M \) for \( u \) and for \( u' \). We show that \( u \) and \( u' \) are conjugate in the symplectic group if and only if \( u \) and \( u' \) give the same equivalence class of Hermitian forms on \( M \). Finally, we prove that two Hermitian forms on \( M \) are equivalent if and only if certain ordinary\(^8\)Hermitian forms (if the degree of \( q \) is \( > 2 \)) or certain quadratic forms (if the degree of \( q \) is \( 1 \)) are equivalent.

We have also studied the structure of the normalizer of an element of the symplectic group \( G \). One finds results analogous to those found by J. Dieudonné in the case of the general linear group \([D1]\). The study of the conjugacy classes of the other classical groups (i.e. the orthogonal and unitary groups) can be done in a similar way. We hope to return to this on another occasion.

**Propositions**

1. The elements of the Lie algebra belonging to a skew-symmetric or quadratic form over a field \( K \), can be divided into classes of elements that can be obtained from each other with a symplectic resp. orthogonal transformation. One can investigate this division into classes by the same method as was used in this thesis in the investigation of the classification of conjugacy classes in the symplectic group.

2. One can define the trace of a linear transformation of a vector space without using a basis of the vector space.

3. It is possible to prove purely algebraically that any ordinary orthogonal transformation is a product of two-dimensional rotations and of reflections.

4. It is probable that an irreducible representation of the finite group \( \text{GL}_n(\mathbb{F}_q) \) (where \( \mathbb{F}_q \) is the finite field with \( q \) elements), is characterized by certain invariants which bear a lot of resemblance with the invariants characterizing the conjugacy classes of this group.

5. The theorem proved by E. Hecke that every irreducible representation by matrices of the group \( \text{PSL}_2(\mathbb{F}_q) \) is equivalent to a representation by matrices all entries of which lie in the field produced by the characters, can also be proved without using arithmetic tools.

E. Hecke, Math. Ann., Bd. 116 (1939), p. 469–510.

6. Some of the irreducible representations of the modular group modulo \( p^4 \) can be calculated in an algebraic way.

H. D. Kloosterman, Ann. of Math., vol. 47 (1946), p. 317–447.

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\(^8\)here “ordinary” means they take values in a field.

\(^*\)As was customary, the PhD thesis ends with a separate list of ‘Propositions’ (‘Stellingen’) without proofs, about which the author could be questioned at the oral defense.
7. It is desirable that in the theory of partial differential equations attention should be given to a strict definition of the concept of integral surface.

8. The inequality derived by H. Weyl for the powers of the two types of eigenvalues of a linear transformation can also be proved using elementary matrix and differential calculus.

   H. Weyl, Proc. Nat. Ac. Sc. vol. 35 (1949), p. 408–411.

9. Tensor calculus is not indispensable for a clear formulation of differential geometric properties.

10. The calculation by K. Husimi and I. Syôzi of the state sum belonging to a planar hexagonal lattice can be brought into a simpler form.

   K. Husimi and I. Syôzi, Progr. Theor. Phys., vol. V (1950) p.177–186, I. Syôzi, ibid., p. 341-351.

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