Distances from $B^\alpha$ Functions to $F(p, q, s)$ Space

Yutong Liu$^a$, Yi Qi$^a$

$^a$School of Mathematical Science, Beihang University, Beijing, People’s Republic of China

Abstract. In this paper, we consider several equivalent formulas for the distances from $B^\alpha$ functions to $F(p, q, s)$ space.

1. The first section

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $S^1$ be its boundary. Denote by $H(\Delta)$ the space of all analytic functions on $\Delta$.

For $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $F(p, q, s)$ is the space of all functions $f \in H(\Delta)$ satisfying
\[
\|f\|_{F(p, q, s)} = \left( \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dxdy \right)^{1/p} < \infty,
\]
where $\sigma_a(z) = \frac{z-a}{1-\bar{a}z}$ is a Möbius transformation of $\Delta$ mapping $a$ to 0. $F_0(p, q, s)$ is the subspace of $F(p, q, s)$ such that
\[
\lim_{|z| \to 1} \int_{\Delta} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dxdy = 0.
\]

$F(p, q, s)$ was introduced by Zhao[11] and it is trivial if $q + s \leq -1$. As we known, $F(p, q, s)$ is a Banach space under the following norm
\[
\|f\|_{F(p, q, s)} = |f(0)| + \|f\|_{F_0(p, q, s)}.
\]

It is proved that $F(p, q, s)$ and $F_0(p, q, s)$ are respectively contained in the Bloch-type space $B^{\alpha \frac{2}{p}}$ and $B^{\alpha \frac{2}{p}}_0$. Here $B^\alpha$ ($\alpha > 0$) is the space of all functions $f \in H(\Delta)$ with
\[
\|f\|_{B^\alpha} = \sup_{z \in \Delta} (1 - |z|^2)^\alpha |f'(z)| < \infty.
\]

And $B^\alpha_0$ consists of all functions $f \in B^\alpha$ such that
\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.
\]

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Email addresses: by1509107@buaa.edu.cn (Yutong Liu), 07122@buaa.edu.cn (Yi Qi)
It is well known that $B^1$ is classical Bloch space $B$ and $B^s$ is also a Banach space if it is equipped with the following norm
\[
\|f\|_{B^s} = |f(0)| + \|f\|_{B^s}.
\]
Moreover $B^s_0$ is the closure of polynomials in $B^s$.

It is clear that $F(p, q, s)$ contains lots of the special function spaces. For example, $F(2, 0, 1) = BMOA$, $F(2, 0, s) = Q_s$ ($s > 1$) and $F(p, p - 2, s) = B$ if $0 < p < \infty$, $s > 1$ (in detail see [1–3, 11]).

Suppose $X \subset B$ is an analytic function space. The distance from a Bloch function $f$ to $X$ is defined as follows:
\[
\text{dist}_{B}(f, X) = \inf_{g \in X} \|f - g\|_{B^s}.
\]

In Ghatage-Zheng[6], Xu[9] and Zhao[10], they studied the distances from $BMOA$ and $F(p, p - 2, s)$ to the Bloch space, separately in the following theorems.

For $f \in B$ and $\epsilon > 0$, set
\[
\Omega_\epsilon(f) = \{z \in \Delta : |f'(z)|(1 - |z|^2) \geq \epsilon\}.
\]

**Theorem A.** [6] Suppose $f \in B$. The following quantities are equivalent:
1. $\text{dist}_{B}(f, BMOA)$;
2. $\inf \{\epsilon : \frac{Y_{\Omega_\epsilon(f)}}{\epsilon} dxdy \text{ is a Carleson measure}\}$, where $\chi$ is the characteristic function.

**Theorem B.** [10] Suppose $2 \leq p < \infty$, $0 \leq q < \infty$, $0 \leq s \leq 1$, and $f \in B$. The following quantities are equivalent:
1. $\text{dist}_{B}(f, F(p, p - 2, s))$;
2. $\inf \{\epsilon : \frac{Y_{\Omega_\epsilon(f)}}{\epsilon} dxdy \text{ is an } s\text{-Carleson measure}\}$;
3. $\inf \{\epsilon : \sup_{z \in \Delta} \frac{1}{|z|^2} \int_{\Omega_\epsilon(f)} |f'(z)|^2 |1 - |z|^2|^2 (1 - |g(z)|^2)^2 dxdy < \infty\}$;
4. $\inf \{\epsilon : \sup_{z \in \Delta} \frac{1}{|z|^2} \int_{\Omega_\epsilon(f)} |f'(z)|^2 |1 - |z|^2|^2 |g'(z, a)| dxdy < \infty\}$, where $g(z, a) = \log \frac{1}{|z|^2 + a}$ is the Green function of $\Delta$ with the pole at $a$.

In this paper, we aim to extend the results about the distance from $F(p, q, s)$ to $B^s$ in Section 3.

This paper is organized as follows. Some relevant notations and important results are given in section 2. In section 3, we prove some equivalent quantities of the distance from $F(p, q, s)$ to $B^s$.

Through this paper $f \approx g$ always means $f \leq g \leq f$, where $f \leq g$ means that there is a constant $C > 0$, independent of functions $f$ and $g$, such that $f \leq Cg$.

2. Preliminaries

Given an arc $I$ on $S^1$, the Carleson box $S(I)$ on $\Delta$ is defined as
\[
S(I) = \{z \in \Delta : 1 - |I| \leq |z| \leq 1, z/|z| \in I\},
\]
where $|I|$ is the Lebesgue measure of $I$. A positive measure $\lambda$ on $\Delta$ is called an $s$-Carleson measure ($s > 0$) if
\[
\|\lambda\|_{C^s} = \sup_{I \subseteq S^1} \frac{\lambda(S(I))}{|I|^s} < \infty,
\]
and a compact $s$-Carleson measure in addition if
\[
\lim_{|I| \to 0} \frac{\lambda(S(I))}{|I|^s} = 0.
\]

Obviously, 1-Carleson measure is the classical Carleson measure (see [4]). Denoted by $CM_s(\Delta)$ (or $CM_s(0, \Delta)$) the set of all (compact) $s$-Carleson measures on $\Delta$. 
Lemma A. \[5\] Let \( \alpha > 0, \beta > 0 \) and \( s < 1 + \frac{\alpha}{\beta} \). For a positive measure \( \lambda \) on \( \Delta \), set
\[
\overline{\lambda}(z) = \int_{\Delta} \frac{(1 - |z|^2)^\alpha(1 - |w|^2)^\beta}{|1 - zw|^{\alpha + \beta + 2}} \lambda(w) dudv.
\]
If \( \lambda \in \text{CM}_s(\Delta) \), then \( \overline{\lambda} \in \text{CM}_s(\Delta) \) and there exists a constant \( C > 0 \) such that \( ||\overline{\lambda}||_{C_s} \leq C||\lambda||_{C_s} \) while \( \overline{\lambda} \in \text{CM}_{s,0}(\Delta) \) if \( \lambda \in \text{CM}_{s,0}(\Delta) \).

Lemma B. \[8\] Let \( s > 0 \), a positive measure \( \lambda \) on \( \Delta \) is an \( s \)-Carleson measure if and only if
\[
\sup_{\Delta} \int_{\Delta} \left( \frac{1 - |w|^2}{1 - \overline{w}z} \right)^s \lambda(z) dudv < \infty,
\]
and is a compact \( s \)-Carleson measure if and only if
\[
\lim_{|w| \to 1} \int_{\Delta} \left( \frac{1 - |w|^2}{1 - \overline{w}z} \right)^s \lambda(z) dudv = 0.
\]

Lemma C. \[10\] Suppose that \( k > -1, r, t > 0, \) and \( r + t - k > 2 \). If \( t < k + 2 < r \), then there exists a universal constant \( C > 0 \), such that for all \( z, w \in \Delta \),
\[
\int_{\Delta} \frac{(1 - |w|^2)^k}{|1 - \overline{w}z||1 - \overline{w}z|^t} dudv \leq C \frac{(1 - |z|^2)^{2+k-r}}{|1 - \overline{z}z|^t}.
\]

3. Main Results

Suppose \( X \subset B^\alpha \) is an analytic function space. The distance from a function \( f \in B^\alpha \) to \( X \) is defined as follows:
\[
\text{dist}_{B^\alpha}(f, X) = \inf_{g \in X} \|f - g\|_{B^\alpha}.
\]
For \( f \in B^\alpha \) and \( \epsilon > 0 \), set
\[
\Omega_{\epsilon}(f) = \{ z \in \Delta : |f'(z)(1 - |z|^2)^\alpha \geq \epsilon \}.
\]
The idea of establishing equivalent forms is from Lou-Chen\[7\] and we state it in detail as follows. The distance formula between a function \( f \in B^\alpha \) and the subspace \( X \subset B^\alpha \) is to decompose \( f \) properly into two parts so that one part is in the space \( X \) and the \( \alpha \)-Bloch norm of the other part is equivalent to the distance \( \text{dist}_{B^\alpha}(f, X) \). It is expected that such a decomposition is non-linear and is not unique.

For \( f \in B^\alpha \), it is easy to get the following formula (see Lemma 4.2.8 in \[12\])
\[
f(z) = f(0) + \int_{\Delta} \frac{f'(w)(1 - |w|^2)}{1 - \overline{w}z} dudv, \quad z \in \Delta.
\]
Let
\[
E_{\epsilon}(f)(z) = \int_{\Delta \setminus \Omega_{\epsilon}(f)} \frac{f'(w)(1 - |w|^2)}{1 - \overline{w}z} dudv + C_{\epsilon,f}
\]
and
\[
P_{\epsilon}(f)(z) = f(z) - E_{\epsilon}(f)(z),
\]
where \( C_{\epsilon,f} \) is a constant such that \( E_{\epsilon}(f)(0) = 0 \). It is clear that
\[
P_{\epsilon}(f)(z) = f(0) - C_{\epsilon,f} + \int_{\Omega_{\epsilon}(f)} \frac{f'(w)(1 - |w|^2)}{1 - \overline{w}z} dudv, \quad z \in \Delta.
\]
Lemma 1. If \( f \in B^\alpha \) and \( 0 < \alpha < 2 \). Then \( P_\epsilon(f) \in B^\alpha \).

Proof. From (3) and \( f \in B^\alpha \), we have
\[
|P_\epsilon'(f)| = \left| \iint_{\Omega(f)} \frac{2f'(w)(1 - |w|^2)}{(1 - \overline{w}z)^3} dudv \right| \leq 2\|f\|_{B^\alpha} \iint_{\Delta} \frac{(1 - |w|^2)^{1-\alpha}}{|1 - \overline{w}z|^3} dudv. \tag{4}
\]

By Lemma 4.2.2 in [12], we get
\[
\iint_{\Delta} \frac{(1 - |w|^2)^{1-\alpha}}{|1 - \overline{w}z|^3} dudv \leq \frac{1}{(1 - |z|^2)^s}.
\]

Hence \( |P_\epsilon'(f)(1 - |z|^2)^s| \leq C\|f\|_{B^\alpha} \), which implies \( P_\epsilon(f) \in B^\alpha \). \( \square \)

Theorem 1. Suppose \( 0 < p < \infty \), \( -2 < q < \infty \), \( 0 < s < \infty \) and \( 0 < s < \alpha = \frac{q+2}{p} < 2 \). If \( f \in B^\alpha \) and \( 1 < \alpha + s < 3 \), then the following quantities are equivalent:
1. \( \text{dist}_{B^\alpha}(f, F(p, q, s)) \);
2. \( \inf \{ \epsilon : \frac{1}{\|f\|_{B^\alpha}} \iint_{\Delta} |f(z)|^p(1 - |z|^2)^q \mu \} < \infty \};
3. \( \sup_{\epsilon \in \mathbb{A}} \iint_{\Omega(f)} \frac{|f'(z)|^p(1 - |z|^2)^q \sigma}{|1 - \overline{z}w|^3} dxdy < \infty \);
4. \( \inf \{ \epsilon : \sup_{\epsilon \in \mathbb{A}} \iint_{\Omega(f)} |f'(z)|^p(1 - |z|^2)^q \sigma dxdy < \infty \} \).

Proof. Let \( d_1, d_2, d_3 \) and \( d_4 \) be the quantities of (1), (2), (3) and (4) in Theorem 1, respectively. We would show that \( d_1 \approx d_2, d_2 = d_3 \) and \( d_3 \approx d_4 \) by three parts.

Part 1: (1) To prove \( d_1 \leq Cd_2 \).
We firstly prove that \( P_\epsilon(f) \in F(p, q, s) \) if \( \epsilon > d_2 \).
Note that
\[
1 - |\sigma(z)|^2 = \frac{(1 - |\sigma|^2)(1 - |\sigma|^2)}{|1 - \overline{z}w|^2}. \tag{5}
\]

By Lemma 1, (4) and (5), we have
\[
L := \sup_{\epsilon \in \mathbb{A}} \iint_{\Delta} |P_\epsilon'(f)|^p(1 - |z|^2)^q \sigma dxdy
\leq \|P_\epsilon(f)\|_{B^\alpha}^{p-1} \sup_{\epsilon \in \mathbb{A}} \iint_{\Delta} (\frac{2f'(w)(1 - |w|^2)}{|1 - \overline{w}z|^3}) dudv \times (1 - |z|^2)^{q-2}(1 - |\sigma|^2)^{s-2} dxdy
\leq C\|f\|_{B^\alpha}^{p-1} \sup_{\epsilon \in \mathbb{A}} \iint_{\Omega(f)} (1 - |\sigma|^2)^{s-2}(1 - |\sigma|^2)^{s} dxdy \times (\iint_{\Delta} \frac{(1 - |\sigma|^2)^{s-3}}{|1 - \overline{w}z|^3} dudv).
\]

From Lemma C, we get
\[
\iint_{\Delta} \frac{(1 - |\sigma|^2)^{s-2} - (1 - |\sigma|^2)^s}{|1 - \overline{w}z|^3} dxdy \leq C\frac{(1 - |\sigma|^2)^{s+3}}{|1 - \overline{w}z|^3} \tag{6}
\]

From (6), we know
\[
L \leq C\|f\|_{B^\alpha} \sup_{\epsilon \in \mathbb{A}} \iint_{\Omega(f)} \frac{(1 - |\sigma|^2)^{s-2} - (1 - |\sigma|^2)^s}{|1 - \overline{w}z|^3} dudv.
\]

Since \( \frac{X_{\alpha+s}(\epsilon)}{(1 - |\sigma|)^{\alpha+s}} dxdy \) is an \( s \)-Carleson measure, by Lemma B, we know \( L \) is finite. So \( P_\epsilon(f) \in F(p, q, s) \) if \( \epsilon > d_2 \).

Since \( E_\epsilon(f)(0) = 0 \), we have
\[
d_1 = \text{dist}_{B^\alpha}(f, F(p, q, s)) \leq \|f - P_\epsilon(f)\|_{B^\alpha} = \|E_\epsilon(f)\|_{B^\alpha} = \|E_\epsilon(f)\|_{B^\alpha} \leq Ce \text{ if } \epsilon > d_2.
\]
To conclude \( d_1 \leq Cd_2 \), we left to show that \( \|E_\epsilon(f)\|_{B^\alpha} \leq Ce \) if \( \epsilon > d_2 \).
Hence we have

\[ |E_v(f)(z)| = \int_{\Omega(f)} \frac{2f'(\omega)(1-|\omega|^2)}{(1-\omega \bar{z})^3} \, du \, dv \leq 2 \epsilon \int_{\Omega} \frac{(1-|\omega|^2)^{-\alpha}}{|1-\omega \bar{z}|^3} \, du \, dv = \frac{2\epsilon}{(1-|z|^2)^{\alpha}}. \]

By \( |E_v(f)(z)| \leq C \epsilon d \) and the definition of \( d_2 \), we have \( d_1 \leq d_2 \).

(2) We now prove \( d_2 \leq d_1 \) by contradiction.

Indeed, suppose that \( d_1 < d_2 \), then there exists \( 0 < \epsilon_1 < \epsilon \) and \( f_\epsilon \in F(p,q,s) \) such that \( \frac{X_{\Omega(f)}(z)}{(1-|z|^2)^{p}} \, dx \, dy \) is not an \( s \)-Carleson measure and \( \| f - f_\epsilon \|_b \leq \epsilon_1 \).

For \( z \in \Lambda \), we have

\[ |f'(z)(1-|z|^2)^3| \leq |f'(z)(1-|z|^2)^3| + \| f - f_\epsilon \|_b \leq |f'(z)(1-|z|^2)^3| + \epsilon_1 \]

This means \( \Omega_\epsilon(f) \subset \Omega_{<\epsilon_1}(f_\epsilon) \) and

\[ X_{\Omega_\epsilon(f)}(z) \leq \frac{|f'(z)(1-|z|^2)^3|}{\epsilon - \epsilon_1}, \]

which implies

\[ X_{\Omega_\epsilon(f)}(z) \leq \frac{|f'(z)|^p(1-|z|^2)^{3p}}{(\epsilon - \epsilon_1)^p}. \]

Note that \( |f'(z)|^p(1-|z|^2)^{3p} \, dx \, dy \) is an \( s \)-Carleson measure, by Lemma B and (5), which implies that \( \frac{X_{\Omega_\epsilon(f)}(z)}{(1-|z|^2)^{p}} \, dx \, dy \) is also an \( s \)-Carleson measure. This is a contradiction. Hence \( d_2 \leq d_1 \). Moreover \( d_1 \approx d_2 \).

Part 2: (1) To prove \( d_4 \leq d_2 \), we start with the assumption that \( \frac{X_{\Omega(f)}(z)}{(1-|z|^2)^{p}} \, dx \, dy \) is an \( s \)-Carleson measure. By Lemma B and 5, we know that

\[ \sup_{a \in \Lambda} \int_{\Omega(f)} \frac{(1-|\sigma_a(z)|^2)^p}{(1-|z|^2)^2} < \infty. \]

Hence we have

\[ \sup_{a \in \Lambda} \int_{\Omega(f)} |f'(z)|^p(1-|z|^2)^{q} \, dx \, dy \leq \| f \|_b^p \sup_{a \in \Lambda} \int_{\Omega(f)} \frac{(1-|\sigma_a(z)|^2)^p}{(1-|z|^2)^2} < \infty. \]

(2) \( d_2 \leq d_4 \) is obvious from the following estimate and Lemma B. Because \( |f'(z)|(1-|z|^2)^3 \geq \epsilon \), we have

\[ \sup_{a \in \Lambda} \int_{\Omega(f)} \frac{(1-|\sigma_a(z)|^2)^p}{(1-|z|^2)^2} \, dx \, dy \leq \frac{1}{\epsilon} \sup_{a \in \Lambda} \int_{\Omega(f)} |f'(z)|^p(1-|z|^2)^{q}(1-|\sigma_a(z)|)^2 \, dx \, dy. \]

Part 3: Since \( 1-|z|^2 \leq 2 \log \frac{1}{|z|} \) for \( z \in \Lambda \), we have \( d_4 \leq C d_2 \). We only need to prove \( d_3 \leq C d_4 \). Set

\[ I = \int_{\Omega(f)} |f'(z)|^p(1-|z|^2)^q g(z,a) \, dx \, dy := I_1 + I_2, \]

where \( \Delta(a,r) = \{ z \in \Delta : |z-a| < r \} \) and

\[ I_1 = \int_{\Omega(f) \cap \Delta(a,\frac{1}{4})} |f'(z)|^p(1-|z|^2)^q g(z,a) \, dx \, dy, \]

\[ I_2 = \int_{\Omega(f) \cap \Delta(a,\frac{1}{4})} |f'(z)|^p(1-|z|^2)^q g(z,a) \, dx \, dy. \]

By the following inequality

\[ g(z,a) = \log \frac{1}{|\sigma_a(z)|} \left\{ \begin{array}{ll} \geq \log 4 \geq 1, & |\sigma_a(z)| \leq \frac{1}{4}, \\ \leq 4(1-|\sigma_a(z)|^2), & |\sigma_a(z)| \geq \frac{1}{4}. \end{array} \right. \]
Hence
\[ I_2 \leq 4 \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha g'(z, a) \, dx \, dy \]
and
\[ I_1 \leq \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha g'(z, a) \, dx \, dy \leq \|f\|^p_{B^\alpha} \int_{\Omega(f)} (1 - |z|^2)^2 g'(z, a) \, dx \, dy \leq C, \]
where C is a constant number independent of a. Therefore \( d_3 \leq Cd_4. \)

Corollary 1. Suppose \( 0 < p < \infty, -2 < q < \infty, 0 < s < \infty \) and \( 0 < s < \alpha = \frac{q+2}{p} < 2. \) If \( f \in H(\Delta) \) and \( 1 < \alpha + s < 3. \) Then the following conditions are equivalent:

1. \( f \) is in the closure of \( F(p, q, s) \) in \( B^\alpha; \)
2. \( \frac{X_{\alpha}(f)}{(1 - |z|^2)^{2\alpha}} \, dx \, dy \) is an \( s \)-Carleson measure for all \( e > 0; \)
3. \( \sup_{e\in\Lambda} \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha g'(z, a) \, dx \, dy < \infty \) for all \( e > 0; \)
4. \( \sup_{e\in\Lambda} \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha (1 - |a(z)|^2)^2 \, dx \, dy < \infty \) for all \( e > 0. \)

Corollary 2. Suppose \( 0 < p < \infty, -2 < q < \infty, 0 < s < \infty \) and \( 0 < s < \alpha = \frac{q+2}{p} < 2. \) If \( f \in H(\Delta), 1 < \alpha + s < 3 \) and \( s_1 < s_2. \) Then
\[ \text{dist}_{B^\alpha}(f, F(p, q, s_1)) = \text{dist}_{B^\alpha}(f, F(p, q, s_2)). \]

Remark:

1. If \( s > 1, F(p, q, s) = B^\alpha. \)
2. If \( q + s \leq -1, F(p, q, s) \) is trivial.

Similarly, we have the following result.

Theorem 2. Suppose \( 0 < p < \infty, -2 < q < \infty, 0 < s < \infty \) and \( 0 < s < \alpha = \frac{q+2}{p} < 2. \) If \( f \in B^\alpha \) and \( 1 < \alpha + s < 3. \) Then the following quantities are equivalent:

1. \( \text{dist}_{B^\alpha}(f, F_0(p, q, s)); \)
2. \( \text{dist}_{B^\alpha}(f, B^\alpha_\alpha); \)
3. \( \inf e: \frac{X_{\alpha}(f)}{(1 - |z|^2)^{2\alpha}} \, dx \, dy \) is a compact \( s \)-Carleson measure;
4. \( \inf e: \lim_{\mu \to 1} \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha g'(z, a) \, dx \, dy = 0; \)
5. \( \inf e: \lim_{\mu \to 1} \int_{\Omega(f)} |f'(z)|^p (1 - |z|^2)^\alpha (1 - |a(z)|^2)^2 \, dx \, dy = 0. \)

Corollary 3. Suppose \( 0 < p < \infty, -2 < q < \infty, 0 < s < \infty \) and \( 0 < s < \alpha = \frac{q+2}{p} < 2. \) If \( f \in B^\alpha_\alpha \) and \( 1 < \alpha + s < 3. \) Then \( f \in F_0(p, q, s) \) if and only if \( \frac{X_{\alpha}(f)}{(1 - |z|^2)^{2\alpha}} \, dx \, dy \) is a compact \( s \)-Carleson measure for all \( e > 0. \)

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