Generalization of a going-down theorem in the category of Chow-Grothendieck motives due to N. Karpenko

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Abstract

Let \( M := (M(X), p) \) be a direct summand of the motive associated with a geometrically split, geometrically variety over a field \( F \) satisfying the nilpotence principle. We show that under some conditions on an extension \( E/F \), if \( M \) is a direct summand of another motive \( M \) over an extension \( E \), then \( M \) is a direct summand of \( M \) over \( F \).

I Introduction

Let \( \Lambda \) be a finite commutative ring. Our main reference on the category \( CM(F; \Lambda) \) of Chow-Grothendieck motives with coefficients in \( \Lambda \) is [1].

The purpose of this note is to generalize the following theorem due to N. Karpenko ([2], proposition 4.5). Throughout this paper we understand a \( F \)-variety over a field \( F \) as a separated scheme of finite type over \( F \).

**Theorem I.1.** Let \( \Lambda \) be a finite commutative ring. Let \( X \) be a geometrically split, geometrically irreducible \( F \)-variety satisfying the nilpotence principle. Let \( M \in CM(F; \Lambda) \) be another motive. Suppose that an extension \( E/F \) satisfies

1. the \( E \)-motive \( M(X)_E \in CM(E; \Lambda) \) of the \( E \)-variety \( X_E \) is indecomposable;
2. the extension \( E(X)/F(X) \) is purely transcendental;
3. the motive \( M(X)_E \) is a direct summand of the motive \( M \).

Then the motive \( M(X) \) is a direct summand of the motive \( M \).

We generalize this theorem when the motive \( M(X) \in CM(F; \Lambda) \) is replaced by a direct summand \( (M(X), p) \) associated with a projector \( p \in End_{CM(F; \Lambda)}(M(X)) \). The proof given by N. Karpenko in [2] cannot be used in the case where \( M(X) \) is replaced by a direct summand because of the use on the *multiplicity* ([1], §75) as the multiplicity of a projector in the category \( CM(F; \Lambda) \) is not always equal to 1 (and it can even be 0). The proof given here for its generalization gives also another proof of theorem I.1.

II Suitable basis of the dual module of a geometrically split \( F \)-variety

Let \( X \) be a geometrically split, geometrically irreducible \( F \)-variety satisfying the nilpotence principle. We note \( CH(X; \Lambda) \) as the colimit of the \( CH(X_K; \Lambda) \) over all extensions \( K \) of \( F \). By assumption there is an integer \( n = rk(X) \) such that

\[
CH(X; \Lambda) \cong \bigoplus_{i=0}^{n} \Lambda.
\]

Let \((x_i)_{i=0}^{n}\) be a base of the \( \Lambda \)-module \( CH(X; \Lambda) \). Each element \( x_i \) of the basis is associated with a subvariety of \( X_E \), where \( E \) is a splitting field of \( X \). We note \( \varphi(i) \) for the dimension of the \( E \)-variety associated to \( x_i \).
**Proposition II.1.** Let $X$ be a geometrically split $F$-variety. Then the pairing
\[ \Psi : CH(\overline{X}; \Lambda) \times CH(\overline{X}; \Lambda) \to \Lambda \]
\[ (\alpha, \beta) \mapsto \deg(\alpha \cdot \beta) \]
is bilinear, symmetric and non-degenerate.

The pairing $\Psi$ induces an isomorphism between $CH(\overline{X}; \Lambda)$ and its dual module $\text{Hom}_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda)$. This isomorphism is given by
\[ CH(\overline{X}; \Lambda) \times Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda) \to Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda) \]
\[ x \mapsto \Psi(x, \cdot) \]

Considering the inverse images of the dual basis of $Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda)$ associated with the basis $x_i$, we get another basis $(x_i^*)_{i=0}^n$ of $CH(\overline{X}; \Lambda)$ such that
\[ \Psi(x_i, x_j^*) = \delta_{ij} \]
where $\delta_{ij}$ is the usual Kronecker symbol.

**Proposition II.2.** Let $M$ and $N$ be two motives in $CM(F; \Lambda)$ such that $M$ is split. Then there is an isomorphism
\[ CH^*(M; \Lambda) \otimes CH^*(N; \Lambda) \to CH^*(M \otimes N; \Lambda) \]

**Proof.** c.f. [1] proposition 64.3. \[ \square \]

Let $Y$ be a smooth complete irreducible $F$-variety. We note $M$ for the motive $(M(Y), q)$ associated with a projector $q \in End(M(Y))$. Then we have the following computations.

**Lemma II.3.** For any integers $i, j, k$ and $s$ less than $rk(X) = n$, and for any cycles $y$ and $y'$ in $CH(\overline{Y}; \Lambda)$, with $1$ being the identity class in either $CH(\overline{X}; \Lambda)$ or $CH(\overline{Y}; \Lambda)$ we have

1. \( (x_i \times x_j^*) \circ (x_k \times x_s^*) = \delta_{is}(x_k \times x_j^*) \)
2. \( (x_i \times y \times 1) \circ (x_k \times x_s^*) = \delta_{is}(x_k \times y \times 1) \)
3. \( (y' \times x_j^*) \circ (x_i \times y) = \deg(y' \cdot y)(x_i \times x_j^*) \)

**Proof.** We only compute (2) (other cases are similar).

\[ (x_i \times y \times 1) \circ (x_k \times x_s^*) = \left( (\sum_{i=0}^{n} x_i \times Y) \right)_*(\sum_{i=0}^{n} x_i \times X)^*(x_k \times x_s^*) \cdot (p_X \times Y)^*(x_i \times y \times 1) \] \[ = \left( (\sum_{i=0}^{n} x_i \times Y) \right)_*((x_k \times x_s^* \times 1 \times 1) \cdot (1 \times x_i \times y \times 1)) \] \[ = \left( (\sum_{i=0}^{n} x_i \times Y) \right)_*(x_k \times (x_s^* \cdot x_i) \times y \times 1) \] \[ = \delta_{is}(x_k \times y \times 1) \]
\[ \square \]

## III Rational cycles of a geometrically split $F$-variety

Let $X$ be a geometrically split $F$-variety. We note $(M(X), p)$ the direct summand of $M(X)$ associated with a projector $p \in CH_{dim(X)}(X \times X; \Lambda)$. Considering the motive $M$ defined in the previous section, if $(M(X_E), p_E)$ is a direct summand of $M_E$ for some extension $E/F$, then there exists cycles $f \in CH(X_E \times Y_E; \Lambda)$ and $g \in CH(Y_E \times X_E; \Lambda)$ such that $f \circ g = p_E$. We can write these cycles in suitable basis of $CH(\overline{X} \times \overline{Y}; \Lambda)$, $CH(\overline{Y} \times \overline{X}; \Lambda)$ and $CH(\overline{X} \times \overline{X}; \Lambda)$ by proposition II.2. Thus there are two subsets $F$ and $G$ of $\{0, \ldots, n\}$, scalars $f_i$, $g_j$, $p_{ij}$ and cycles $y_i$, $y_j$ in $CH(\overline{Y}; \Lambda)$ such that

1. \( \mathcal{F} = \sum_{i \in F} f_i(x_i \times y_i) \)
2. \( \mathcal{P} = \sum_{j \in G} g_j(y_j' \times x_j^*) \)

3. \( \mathcal{P} = \sum_{i \in F} \sum_{j \in G} p_{ij}(x_i \times x_j^*) \)

With \( p_{ij} = f_i g_j \deg(y_j' \cdot y_i) \) by lemma II.3 as \( g \circ f = p_E \).

**Notation III.1.** Let \( p \in CH_{\dim(X)}(X \times X) \) be a non-zero projector. Considering \( \mathcal{P} \), the image of \( p \) in a splitting field of the \( F \)-variety \( X \), we can write \( \mathcal{P} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*) \). We define the least codimension of \( p \) (denoted \( \cd \)) by

\[
\cd(p) := \min_{(i,j)} \left( \dim(X) - \varphi(i) \right)
\]

**Proposition III.2.** Let \( p \in CH_{\dim(X)}(X \times X) \) be a non-zero projector. We consider its decomposition \( \mathcal{P} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*) \) in a splitting field of \( X \). Then for any \( i \in P_1 \) and \( j \in P_2 \) we have

\[
p_{ij} = \sum_{k \in P_1 \cap P_2} p_{kj} p_{ik}
\]

**Proof.** We can assume that \( \varphi(i) \) is constant on \( P_1 \). Then a straightforward computation gives

\[
\mathcal{P} \circ \mathcal{P} = \left( \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*) \right) \circ \left( \sum_{k \in P_1} \sum_{s \in P_2} p_{ks}(x_k \times x_s^*) \right)
\]

\[
= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks} (x_i \times x_j^*) \circ (x_k \times x_s^*)
\]

\[
= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks} \delta_{is} (x_k \times x_s^*)
\]

\[
= \sum_{k \in P_1} \sum_{s \in P_2} \left( \sum_{i \in P_1 \cap P_2} p_{ij} p_{ki} (x_k \times x_s^*) \right)
\]

Moreover \( p \circ p = p \), thus if \( (k, s) \in P_1 \times P_2 \) we have \( p_{ks} = \sum_{i \in P_1 \cap P_2} p_{is} p_{ki} \).

\[\square\]

## IV General properties of Chow groups

Embedding the Chow group of the \( F \)-variety \( X \) is quite useful for computations, but the generalization of the theorem I.1 needs a direct construction of some \( F \)-rational cycles \( f \) and \( g \). We study in this section some properties of rational elements in Chow groups and how they behave when the extension \( E(X)/F(X) \) is purely transcendental.

**Proposition IV.1.** Let \( Y \) be an \( F \)-varieties. Let \( E/F \) be a purely transcendental extension. Then the morphism

\[
res_{E/F} : CH(Y; \Lambda) \longrightarrow CH(Y_E; \Lambda)
\]

is an epimorphism.

**Proof.** Indeed the morphism \( res_{E/F} \) coincides with the composition

\[
CH(Y; \Lambda) \longrightarrow CH(Y \times \mathbb{A}_F^n; \Lambda) \longrightarrow CH(Y_E; \Lambda).
\]

As the extension \( E/F \) is purely transcendental, there is an isomorphism between \( E \) and the function field of an affine space \( \mathbb{A}_F^n \) for some integer \( n \). The first map is an epimorphism by the homotopy invariance of Chow groups ([1], theorem 57.13) and the second map is an epimorphism as well ([1], corollary 57.11).
V Generalization of the going-down theorem in the category of Chow-Grothendieck motives

We now have all the material needed to prove the generalization of theorem I.1.

**Theorem V.1.** Let $\Lambda$ be a finite commutative ring. Let $X$ be a geometrically split, geometrically irreducible $F$-variety satisfying the nilpotence principle. Let also $M \in CM(F; \Lambda)$ be a motive. Suppose that an extension $E/F$ satisfies

1. the $E$-motive $(M(X)_E, p_E)$ associated with the $E$-variety $X_E$ is indecomposable;
2. the extension $E(X)/F(X)$ is purely transcendental;
3. the motive $(M(X_E), p_E)$ is a direct summand of the $E$-motive $M_E$.

Then the motive $(M(X), p)$ is a direct summand of the motive $M$.

**Proof.** We can consider that $M = (Y, q)$ for some smooth complete $F$-variety $Y$ and a projector $q \in CH_{\dim(Y)}(Y \times Y; \Lambda)$. If $p$ is equal to 0 then the motive $(M(X), p)$ is the 0 motive and $(M(X), p)$ is a direct summand of $M$. Now suppose that $p$ is not equal to 0.

As $(M(X)_E, p_E)$ is a direct summand of $M_E$, there are $E$-rational cycles $f \in CH_{\dim(X_E)}(X_E \times Y_E; \Lambda)$ and $g \in CH_{\dim(Y_E)}(Y_E \times Y; \Lambda)$ such that $g \circ f = p_E$. We can decompose the images of these cycles in a splitting field of $X$ in suitable basis for computations:

1. $\overline{f} = \sum_{i \in E} f_i(x_i \times y_i)$
2. $\overline{g} = \sum_{j \in G} g_j(y'_j \times x'_j)$
3. $\overline{\varphi} = \sum_{i \in F} \sum_{j \in G} p_{ij}(x_i \times x'_j)$

with $p_{ij} = f_i g_j (y'_j \cdot y_i)$.

Splitting terms whose first codimension is minimal in $\overline{f}$ and $\overline{\varphi}$ by introducing

$$F_1 := \{ i \in F, \varphi(i) = \text{cdmin}(p) \}$$

we get

1. $\overline{f} = \sum_{i \in F_1} f_i(x_i \times y_i) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i)$
2. $\overline{\varphi} = \sum_{i \in F_1} \sum_{j \in G} p_{ij}(x_i \times x'_j) + \sum_{i \in F \setminus F_1} \sum_{j \in G} p_{ij}(x_i \times x'_j)$

As $E(X)$ is an extension of $E$, the cycle $\overline{f}$ is $E(X)$-rational. Proposition IV.1 implies that the change of field $res_{E(X)/F(X)}$ is an epimorphism, hence $\overline{f}$ is an $F(X)$-rational cycle.

Considering the morphism $Spec(F(X)) \to X$ associated with the generic point of the geometrically irreducible variety $X$, we get a morphism

$$\epsilon : (X \times Y)_{F(X)} \to X \times Y \times X$$

This morphism induces a pull-back $\epsilon^* : CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda) \to CH_{\dim(X)}(\overline{X} \times \overline{Y}; \Lambda)$ mapping any cycle of the form $\alpha \times \beta \times 1$ on $\alpha \times \beta$ and vanishing on elements $\alpha \times \beta \times \gamma$ if $\text{cdim}(\gamma) > 0$. Moreover $\epsilon^*$ induces an epimorphism of $F$-rational cycles onto $F(X)$-rational cycles ([1], corollary 57.11). We can thus choose a $F$-rational cycle $f_1 \in CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda)$ such that $\epsilon^*(f_1) = \overline{f}$.

By the expression of the pull-back $\epsilon^*$ we can assume

$$\overline{f}_1 = \sum_{i \in F_1} f_i(x_i \times y_i \times 1) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i \times 1) + \sum (\alpha \times \beta \times \gamma)$$

where the codimension of the cycles $\gamma$ is non-zero.
Considering $f_1$ as a correspondence from $\overline{X}$ to $\overline{X} \times \overline{Y}$, we consider $f_2 := f_1 \circ p$ which is also a $F$-rational cycle. We have

\[ \overline{f}_2 = (\sum_{i \in F_1} f_i(x_i \times y_i \times 1)) \circ (\sum_{i \in F_1, j \in G} p_{ij}(x_i \times x_j^*)) + \sum_{i \in F_1, j \in G} \sum_{i \in \lambda_{ij}} (x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \]

\[ = \sum_{i \in F_1, j \in F_1 \cap G} f_jp_{ij}(x_i \times y_j \times 1) + \sum_{i \in F_1 \setminus F_1, j \in G} \sum_{i \in \lambda_{ij}} (x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \]

where the cycles $\tilde{\gamma}$ are of non-zero codimension, the cycles $\tilde{\alpha}$ are such that $\text{codim}(\tilde{\alpha}) \geq \text{cdim}(p)$ and where elements $\lambda_{ij}$ are scalars.

We now consider the diagonal embedding

\[ \Delta : \overline{X} \times \overline{Y} \times \overline{X} \times \overline{Y} \times \overline{X} \]

The morphism $\Delta$ induces a pull-back $\Delta^* : C^H_{\text{dim}(X)}(\overline{X} \times \overline{Y} \times \overline{X} ; \Lambda) \rightarrow C^H_{\text{dim}(X)}(\overline{X} \times \overline{Y} ; \Lambda)$

We note $f_3 := \Delta^*(f_2)$, which is also a $F$-rational cycle and whose expression in a splitting field of $X$ is

\[ f_3 = \sum_{i \in F_1, j \in F_1 \cap G} f_jp_{ij}(x_i \times y_j) + \sum_{i \in F_1 \setminus F_1, j \in G} \sum_{i \in \lambda_{ij}} (x_i \times y_j) + \sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta} \]

where $\text{codim}(\tilde{\alpha} \cdot \tilde{\gamma}) > \text{cdim}(p)$ as $\text{codim}(\tilde{\alpha}) \geq \text{cdim}(p)$ and $\text{codim}(\tilde{\gamma}) > 0$.

We can compute the $g \circ f_3$:

\[ g \circ \overline{f}_3 = g \circ (\sum_{i \in F_1, j \in G} f_jp_{ij}(x_i \times y_j)) + g \circ (\sum_{i \in F_1 \setminus F_1, j \in G} \sum_{i \in \lambda_{ij}} (x_i \times y_j)) + \sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta} \]

(V.3)

\[ = \sum_{i \in F_1, j \in G} \sum_{i \in F_1 \setminus F_1, j \in G} g_{s_i}f_jp_{ij}(y_s' \times x_s^*)(x_i \times y_j) + (\sum \tilde{\alpha} \times \tilde{\beta}) \]

(V.4)

With cycles $\overline{\pi}$ such that $\text{codim}(\overline{\pi}) > \text{cdim}(p)$. Computing the component of $g \circ f_3$ for elements of the form $x_k \times x_s^*$ with $\varphi(k) = \text{cdim}(p)$ we get

\[ g \circ \overline{f}_3 = \sum_{i \in F_1, j \in G} \sum_{i \in F_1 \setminus F_1, j \in G} g_{s_i}f_jp_{ij}(y_s' \times x_s^*)(x_i \times y_j) + (\sum \tilde{\alpha} \times \tilde{\beta}) \]

(V.5)

\[ = \sum_{i \in F_1, j \in G} \sum_{i \in F_1 \setminus F_1, j \in G} g_{s_i}f_jp_{ij} \deg(y_s' \cdot y_j)(x_i \times x_s^*) \]

(V.6)

Now we can see that if $k \in F_1$, then the coefficient of $g \circ f_3$ relatively to an element $x_k \times x_s^*$ is equal to $\sum_{i \in F_1 \cap G} g_{s_i}f_ip_{ki} \deg(y_i' \cdot y_i^*)$. Moreover proposition III.2 says that

\[ \sum_{i \in F_1 \cap G} g_{s_i}f_ip_{ki} \deg(y_i' \cdot y_i^*) = \sum_{i \in F_1 \cap G} p_{is}p_{ki} = p_{ks} \]

Since $p$ is non-zero, there exists $(k, s)$ with $k \in F_1$ and $p_{ks} \neq 0$, thus we have shown that the cycle $g \circ f_3$ as a decomposition

\[ g \circ f_3 = p_{ks}(x_k \times x_s^*) + \sum_{(i,j) \neq (k,s)} p_{ij}(x_i \times x_j^*) + \sum (\overline{\pi} \circ \overline{\beta}) \]

where $\text{codim}(\overline{\pi}) > \text{cdim}(p)$. Since $p$ is a projector, for any integer $n$ the $n$-th power of $g \circ f_3$ as always a non-zero component relatively to $x_k \times x_s^*$ which is equal to $p_{ks}$, that is to say

\[ \forall n \in \mathbb{N}, (g \circ f_3)^\circ n = p_{ks}(x_k \times x_s^*) + \sum_{(i,j) \neq (k,s)} p_{ij}(x_i \times x_j^*) + \sum (\overline{\pi} \circ \overline{\beta}) \]

where $\text{codim}(\overline{\pi}) > \text{cdim}(p)$. 

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As the ring $\Lambda$ is finite, there is a power of $g \circ (f_3)_E$ which is a non-zero idempotent (cf [2] lemma 3.2). Since the $E$-motive $(M(X)_E, p_E)$ is indecomposable this power of $g \circ (f_3)_E$ is equal to $p_E$. Thus we have shown that there exists an integer $n_1$ such that 

$$(g \circ (f_3)_E)^{n_1} = p_E$$

In particular if $g_1 := (g \circ (f_3)_E)^{n_1 - 1} \circ g$ we get $g_1 \circ (f_3)_E = p_E$.

Now we can transpose the last equality and get 

$$t_1(f_3)_E \circ t_1 g_1 = t_1 p_E.$$ 

Repeating the same process as before, we get a $F$-rational cycle $\tilde{g}$ and an integer $n_2$ such that 

$$t_1(f_3)_E \circ (\tilde{g})_E = t_1 p_E.$$ 

Now setting $\hat{g} := (\tilde{g} \circ (f_3))^{n_2 - 1} \circ \tilde{g}$, we have two $F$-rational cycles $\hat{g}$ and $f_3$ such that 

$$\hat{g}_E \circ (f_3)_E = p_E.$$ 

Using the nilpotence principle again, there is an integer $\varpi \in \mathbb{N}$ such that 

$$(\hat{g} \circ f_3)^\varpi = p$$

Hence if $\hat{f} := f_3 \circ (\hat{g} \circ f_3)^{-1}$, $\hat{f}$ is a $F$-rational cycle satisfying 

$$\hat{g} \circ \hat{f} = p$$

Thus we have shown that the motive $(M(X), p)$ is a direct summand of the motive $M$. \hfill \Box
References

[1] R. Elman, N. Karpenko, and A. Merkurjev. *The Algebraic and Geometric Theory of Quadratic Forms*. American Mathematical Society, 2008.

[2] N. Karpenko. *Hyperbolicity of hermitian forms over biquaternion algebras*. 2009.