Holography in three-dimensional Kerr–de Sitter space with a gravitational Chern–Simons term

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Received 7 January 2008, in final form 14 May 2008
Published 11 June 2008
Online at stacks.iop.org/CQG/25/135003

Abstract
The holographic description of the three-dimensional Kerr–de Sitter space with a gravitational Chern–Simons term is studied, in the context of dS/CFT correspondence. The space has only one (cosmological) event horizon and its mass and angular momentum are identified from the holographic energy–momentum tensor at the asymptotic infinity. The thermodynamic entropy of the cosmological horizon is computed directly from the first law of thermodynamics, with the conventional Hawking temperature, and it is found that the usual Gibbons–Hawking entropy is modified. It is remarked that, due to the gravitational Chern–Simons term, (a) the results go beyond the analytic continuation from AdS, (b) the maximum-mass/N-bound conjecture may be violated and (c) the three-dimensional cosmology is chiral. A statistical mechanical computation of the entropy, from a Cardy-like formula for a dual CFT at the asymptotic boundary, is discussed. Some remarks on the technical differences in the Chern–Simons energy–momentum tensor, from the literature, are also made.

PACS numbers: 04.60.-m, 11.25.Hf, 04.60.Kz

1. Introduction
In recent years, the holographic description of asymptotically anti-de Sitter (AdS) spaces has been extensively studied in the context of AdS/CFT correspondence [1]. On the other hand, recently, some higher derivative corrections to the description have also been studied [2–4], and it is found that, when there are black holes in the bulk spacetimes, there are good agreements even with the higher derivative corrections to the black-hole entropies [5–8].

More recently, the corrections due to a gravitational Chern–Simons term [9] have also been studied and it is found that, when considering the three-dimensional AdS space with a
negative cosmological constant $\Lambda = -1/l^2$, having a black hole, known as the BTZ black hole \cite{10}, the black hole’s mass and angular momentum are modified as

$$M = m + \hat{\beta} j/l, \quad J = j + \hat{\beta} lm,$$

(1.1)

which shows some mixings between the original BTZ black hole’s mass $m$ and angular momentum $j$, with the coupling constant $\hat{\beta}$ of the gravitational Chern–Simons term\textsuperscript{1}. This has been computed in some standard canonical methods \cite{15–19} and in the holographic methods \cite{20, 21}. It is found also that the black-hole entropy which satisfies the first law of thermodynamics with the mass and angular momentum of (1.1) has a term proportional to the inner-horizon’s area, as well as the usual outer-horizon’s \cite{21–23}. Moreover, the entropy agrees with the statistical entropy, computed from the Cardy’s formula for a CFT at the asymptotic infinity \cite{20–22, 24, 25}.

In this paper, I consider the three-dimensional Kerr–de Sitter space (KdS\(_3\)) with a gravitational Chern–Simons term, in the context of dS/CFT correspondence, as a de Sitter counterpart of the above mentioned analysis \cite{26–36}. As far as the higher ‘curvature’ corrections, there are several works already in the literature in the dS/CFT \cite{2, 4}. But, the analysis with the Chern–Simons term, in particular, is interesting for the following reasons. First, the usual analytic continuation \cite{33, 37} is questionable from the different behavior of the Einstein action and the gravitational Chern–Simons term under the continuation \cite{38}: one gets a real (-valued) action for the former but an imaginary for the latter so that one can not get a real total action but a complex action. This implies that the AD mass, angular momentum \cite{18, 19} and the entropy be complex by the formal analytic continuation. Second, it is known recently that the entropic N-bound \cite{39, 40}, stating that the upper bound of the entropy in asymptotically de Sitter space is given by the entropy of pure de Sitter space, and the maximal mass conjecture \cite{33, 35, 36, 41}, stating that any asymptotically de Sitter space cannot have a mass larger than the pure de Sitter case without inducing a cosmological singularity, may be violated with NUT charge, where the entropy is no longer proportional to the area \cite{42–44}. However, in our case with the gravitational Chern–Simons term also, one would have an entropy which is not proportional to the (outer-horizon) area \cite{21–23}, in the real action approach \cite{19} without recourse to the analytic continuation. So, it would be interesting to study whether the two conjectures are violated similarly in our case also. Third, the Chern–Simons term introduces chirality into cosmology in four dimensions which could produce some interesting effects in our real world \cite{45–47}. It would be interesting to study the corresponding effects in the three-dimensional cosmology model. Finally, it is known \cite{26, 33} that the Gibbons–Hawking entropy for the cosmological horizon agrees with the statistical entropy, computed from the Cardy-like formula at the infinite boundary, even though the spacetime gives rise to a non-unitary CFT, due to complex eigenvalues for the Virasoro generators $L_0, \bar{L}_0$, and so the formula does not generally apply. It would be interesting to explore how much this Cardy formula approach can be generalized further with the quite non-trivial modification of the Einstein gravity theory.

The organization of this paper is as follows. In section 2, I review the holographic energy–momentum tensor, at the asymptotic infinity, for Einstein gravity in dS space. In section 3, I describe the gravitational Chern–Simons term, in the real action approach, and compute its contribution to the holographic energy–momentum tensor. In section 4, I consider the Fefferman–Graham expansion and identify the boundary energy–momentum tensor. It is noted also that the Chern–Simons contributions go beyond the analytic continuation. In section 5, I consider KdS\(_3\) space, compute its holographic energy–momentum tensor, and

\textsuperscript{1} Similar phenomena have been known, for sometime, in several other contexts where the masses and angular momenta are completely interchanged \cite{11–14}.

\textsuperscript{2}
identify the conserved mass and angular momentum of the space. I consider the entropy
of the cosmological horizon in the KdS3 space from the first law of thermodynamics, with
the conventional Hawking temperature. It is remarked also that, due to the Chern–Simons
term, the maximum-mass/N-bound conjecture may be violated and the three-dimensional
cosmology is chiral. In section 6, I consider a statistical computation of the entropy from
a Cardy-like formula for a dual CFT at the asymptotic boundary. In appendix A, I present
some details about the Gauss–Codazzi equations and their Fefferman–Graham expansions, by
comparing the cases of the dS and AdS spaces. In appendix B, I present some details about
computing the gravitational Chern–Simons contributions to the energy–momentum, to clarify
some differences in the details with the literature. In appendix C, I present the comparative
computations of the conserved charges for dS and AdS cases to clarify some differences in
the details from the literature. It is remarked also that, due to the gravitational Chern–Simons
term, the results go beyond the analytic continuation from AdS.

I shall omit the speed of light c and the Boltzman’s constant kB in this paper for
convenience, by adopting the units of c ≡ 1, kB ≡ 1. But, I shall keep the Newton’s
constant G and the Planck’s constant ℏ in order to clearly distinguish the quantum (gravity)
effects with the classical ones.

2. Holographic energy–momentum tensor in dS space: Einstein gravity case

Brown and York [48] have given a general description for defining an energy–momentum
tensor, associated with a boundary ∂M of a spacetime M. In order to study the
asymptotic infinite boundary in three-dimensional, asymptotically dS spacetime with a positive
cosmological constant Λ = 1/l2, let me slice the spacetime2

\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -d\eta^2 + \gamma_{ij}(\eta, x^i) \, dx^i \, dx^j \] (2.1)

with two-dimensional hypersurfaces labeled by η. Then, the Einstein–Hilbert action,
accompanied by the extrinsic curvature term on the boundary ∂M [49], is

\[ I_g = \frac{1}{16\pi G} \int_M d^3x \sqrt{|g|} \left( (3)R - 2\Lambda \right) - \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{|\gamma|} \, Tr \, K \]
\[ = \frac{1}{16\pi G} \int_M d^3x \, d\eta \sqrt{|\gamma|} \left( R + (Tr \, K)^2 + Tr(K^2) - 2\Lambda \right). \] (2.2)

Here, I introduced the extrinsic curvature of a fixed-η surface, \( K_{ij} = \partial_\eta \gamma_{ij}/2 \) and used the
decomposition of the three-dimensional Ricci scalar \((3)R\), in terms of the two-dimensional
Ricci scalar of the metric \( \gamma_{ij} \),

\[ (3)R = R + (Tr \, K)^2 + Tr(K^2) + 2\partial_\eta \, Tr \, K, \] (2.3)

where \( Tr \, K = \gamma^{ij} K_{ij}, \, Tr(K^2) = K^{ij} K_{ij} \).

To compute the energy–momentum tensor let me consider the variation of the action with
respect to metric. In general, the variation produces bulk terms proportional to the equations
of motion plus some boundary terms

\[ \delta I_g = - \frac{1}{16\pi G} \int_M d^3x \sqrt{|g|} \left( (3)R - \frac{1}{2} g^{\mu\nu} \, (3)R + \frac{1}{i} \bar{s}^{\mu\nu} \right) \delta g_{\mu\nu} + \text{(boundary terms)}. \] (2.4)

2 Greek letters (\( \mu, \nu, \ldots \)) denote the three-dimensional indices, whereas Roman letters (i, j, \ldots) denote the two-
dimensional boundary indices.

3 I follow the conventions of Wald [50], i.e., \( (3)R_{\rho\sigma\mu\nu} = \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} - (\nu \leftrightarrow \mu), \Gamma^\rho_{\sigma\nu} = (1/2) \eta^{\mu\nu} (\partial_\rho \Gamma^\mu_{\sigma\nu} + \partial_\sigma \Gamma^\mu_{\rho\nu} - \partial_\nu \Gamma^\mu_{\rho\sigma}), \) \( (3)R_{\rho\sigma} = \Gamma^\lambda_{\rho\sigma,\lambda}, \) \( (3)R = (3)R_{\rho\sigma} + \Lambda, \) \( \bar{s}^{\mu\nu} = (3)R_{\rho\sigma} \eta^{\rho\sigma}. \) These conventions agree with those in [2, 28, 31, 36, 48, 49, 51], but differ from those in [21, 33, 52], in the sign of the Riemann tensor.
But, if one considers the solutions to the bulk equations of motion,
\[(3) R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (3) R + \frac{1}{12} g^{\mu\nu} = 0,\]  
(2.5)
only the boundary terms remain as follows [48]:
\[\delta I_{\text{g(on-shell)}} = \frac{1}{2} \int_{\partial M} d^2 x \sqrt{|\gamma|} T^{ij} \delta \gamma_{ij}.\]  
(2.6)
Here, the Brown–York’s boundary energy–momentum tensor is given by
\[T^{ij} = \frac{1}{8\pi G} (K^{ij} - \text{Tr} K \gamma^{ij}).\]  
(2.7)
Since I want to think of the boundary at \(\eta = \infty\) with a finite energy–momentum tensor [28, 29, 51, 52], I also need to consider additional counter terms which can be fixed by the locality and general covariance,
\[I_{\text{ct}} = \frac{1}{8\pi G l} \int_{\partial M} d^2 x \sqrt{|\gamma|}.\]  
(2.8)
Then, the regulated energy–momentum tensor becomes
\[T^{ij}_{\text{reg}} = T^{ij} + \frac{1}{8\pi G l} \gamma^{ij}.\]  
(2.9)

3. Gravitational Chern–Simons term and its contribution to the energy–momentum tensor

The gravitational Chern–Simons term, in the form notation [20, 53], is given by
\[I_{\text{GCS}} = \beta_{\text{KL}} \int_{M} \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right),\]  
(3.1)
where I have defined the connection 1-form as \(\Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} dx^\mu\), with the usual Christoffel symbols \(\Gamma^\alpha_{\beta\mu}\), and \(\beta_{\text{KL}}\) is the real-valued coupling which agrees with that of [20]. Note that \(I_{\text{GCS}}\) is of the third-derivative order, rather than the second as in the Einstein–Hilbert term in (2.2), and so it is the first higher-derivative correction in three-dimensional spacetimes. Moreover, it has a peculiar property which differs from that of the higher ‘curvature’ corrections: the action is not manifestly invariant under the diffeomorphism but its bulk equations of motion are covariant [53]; this implies that the non-invariance of diffeomorphism propagates only along the boundary and this introduces a gravitational anomaly on the boundary [20].

Now, in order to compute the contribution to the energy–momentum tensor, let me consider the variation of the action \(I_{\text{GCS}}\) with respect to metric. After some computation (see appendix B for some details), one can find that
\[\beta_{\text{KL}}^{-1} \delta I_{\text{GCS}} = 2 \int_{M} \text{Tr} (\delta \Gamma \wedge R) - \int_{\partial M} \text{Tr} (\Gamma \wedge \delta \Gamma)
= - \int_{M} d^3 x \nabla_{\beta} \left( (3) R^{\mu\beta} \epsilon_{\rho\nu}^{\beta} \delta g_{\rho\nu} \right)
+ \int_{\partial M} d^2 x \left[ (3) R^{\epsilon_{ij}}_{\nu\rho} \epsilon_{\rho\nu}^{\epsilon_{ij}} \delta \gamma_{ij} + (-2 K^{ij} \delta K_{ij} - \Gamma^l_{ij} \delta \Gamma^l_{ij}) \epsilon_{ij} \right],\]  
(3.2)
where the curvature 2-form is given by \((3) R^{\mu\nu} = \text{d}\Gamma^{\mu\nu} + \Gamma^{\nu}_{\rho} \wedge \Gamma^{\rho}_{\mu} = (1/2) (3) R_{\nu\rho} \wedge \text{d} x^{\rho} + \text{d} x^{\nu}\) with the usual Riemann tensor \((3) R_{\nu\rho}^{\alpha}\), and \(\nabla_{\beta}\) denotes the covariant derivative with respect to \(g_{\alpha\beta}\). And also, I have used \(\text{d} x^{\gamma} \wedge \text{d} x^{\mu} \wedge \text{d} x^{\nu} = \epsilon^{\gamma\mu\nu} \text{d}^3 x\), \(\text{d} x^{i} \wedge \text{d} x^{j} = \epsilon_{ij} \text{d}^2 x\) with
$\epsilon^{012} \equiv 1$, $\epsilon^{01} \equiv 1$.\(^4\) Note that the bulk term is a manifestly covariant form and this gives a covariant contribution to the equations of motion (2.5) as follows:

\[
(\text{3}) \quad R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (\text{3}) R + \frac{1}{2} \epsilon^{\mu
u\rho} \nabla_\rho (\text{3}) R_{\rho\sigma} \epsilon^{\sigma\beta} / \sqrt{|g|}. \tag{3.3}
\]

Now, by considering the solution to the full equations of motion for the action, in the presence of the gravitational Chern–Simons term $I_{\text{GCS}}$ as well as the Einstein–Hilbert term $I_\gamma$ of (2.2), the boundary term in (3.2) would contribute to the boundary energy–momentum tensor $t^{ij}$ as follows:

\[
\delta I_{\text{GCS}(on-shell)} = \frac{1}{2} \int_{\partial M} d^3 x \sqrt{|\gamma|} t^{ij} \delta \gamma_{ij}
\]

(3.4)

with

\[
t^{ij} = \frac{2}{\sqrt{|\gamma|}} \beta_{KL} (3) R_{ij} \epsilon^{\mu\nu} + \ldots , \tag{3.5}
\]

where ‘\ldots’ denotes the contributions, if there are, from the second term in the boundary terms in (3.2). By just a naive look at the second boundary term in (3.2), it is not clear whether one can define the local quantity through (3.4) since $\delta K_{ij}$ and $\delta F_{ij}$ involve some derivatives of $\delta \gamma_{ij}$. Generally, we need to introduce some appropriate boundary terms in order to compensate these unwanted derivative terms, but these appropriate boundary terms are known only for some limited cases, like as the Einstein–Hilbert action or its modification by the Gauss–Bonnet term [54]. But, fortunately, in our case of asymptotically dS (or AdS) spacetimes these are not needed for an infinite boundary at $\gamma = \infty$, as one can see explicitly in the following section.

### 4. Fefferman–Graham expansion and boundary energy–momentum tensor

In order to study the physics associated with the asymptotic boundary at $\gamma = \infty$, it is convenient to consider the large $\gamma$ expansion, known as the Fefferman–Graham expansion [55], of the boundary metric $\gamma_{ij}(\eta, x^i)$ as follows:\(^5\):

\[
\gamma_{ij}(\eta, x^i) = \epsilon^{-1} \gamma_{ij}^{(0)} + \epsilon \gamma_{ij}^{(2)} + O(\epsilon^3) \tag{4.1}
\]

with $\epsilon \equiv \epsilon^{-2n/\eta}$. Then, it follows that

\[
\gamma^{ij} = \gamma^{(0)ij} - \epsilon^2 \gamma^{(2)ij} + \epsilon^3 ([\gamma^{(2)}]^2 - \gamma^{(4)})_{ij} + O(\epsilon^4),
\]

\[
K_{ij} = \frac{1}{l} [\epsilon^{-1} \gamma^{(0)ij} - \epsilon \gamma^{(4)ij}],
\]

\[
K^{ij} = \frac{1}{l} [\gamma^{(0)ij} - \epsilon \gamma^{(2)ij} + \epsilon^2 ([\gamma^{(2)}]^2 - \gamma^{(4)})_{ij}] + O(\epsilon^3),
\]

\[
K^{ij} = \frac{1}{l} [\gamma^{(0)ij} - \epsilon \gamma^{(2)ij} + \epsilon^2 ([\gamma^{(2)}]^2 - \gamma^{(4)})_{ij}] + O(\epsilon^3),
\]

\[
\text{Tr} K = \frac{1}{l} \text{Tr} [\gamma^{(0)} - \epsilon \gamma^{(2)} + \epsilon^2 ([\gamma^{(2)}]^2 - \gamma^{(4)})] + O(\epsilon^3),
\]

and one can obtain the energy–momentum tensor for the Einstein–Hilbert action as follows:

\[
T_{ij}^{\text{reg}} = -\frac{1}{8\pi G l} \epsilon^2 \gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)} + O(\epsilon^3). \tag{4.3}
\]

\(^4\) The result (3.2) differs from that of [20] by the absence of $\sqrt{|\gamma|}$ factor, but agrees with that of [9]. From this difference, there follows the difference in the energy–momentum tensor $t^{ij}$ of (3.5) by the factor of $1/\sqrt{|\gamma|}$, from that of papers [20, 21].

\(^5\) For higher-dimensional (AdS space, (A)dS$_{p+1}$ with $n \geq 2$, there is also the log($\epsilon$) term generally [37, 56]. But, this is not needed in the three-dimensional case.
But, this naively defined quantity vanishes as $\epsilon \to 0$, i.e., $\eta \to \infty$ even though the variation in the action (2.6) is finite

$$\delta I_{\text{GCS(on-shell)}} = \frac{1}{2} \int_{\mathcal{M}} d^2 x \sqrt{|g^{(0)}|} T^{(2)ij}_{\text{reg}} \delta \gamma_{ij}^{(0)} + O(\epsilon)$$

(4.4)

with

$$T^{(2)ij}_{\text{reg}} = \epsilon^2 T^{(2)ij}_{\text{reg}} + O(\epsilon^3).$$

(4.5)

So, the correct definition of the energy–momentum tensor would be the conformally redefined energy–momentum tensor $T^{(2)ij}_{\text{reg}}$

$$T^{(2)ij}_{\text{reg}} = -\frac{1}{8\pi G} (\gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)}) + O(\epsilon),$$

(4.6)

which is finite for a finite boundary metric $\gamma_{ij}^{(0)}$. Here, it is understood that the boundary indices $(i, j, \ldots)$ in $\gamma^{(n)ij}$ and $\gamma^{(0)ij}$ are lowered and raised by $\gamma_{ij}^{(0)}$ and its inverse $\gamma^{(0)ij}$. But, note that $T^{(2)ij}_{\text{reg}}$ depends on $\gamma^{(2)ij}$, i.e., the bulk spacetime, as well as the boundary metric $\gamma^{(0)ij}$.

On the other hand, from the expansion of the Riemann tensor

$$(3) \, R_{\eta \eta}^{(k)} = \frac{1}{2} \left[ \gamma_{ij}^{(0)k} + \epsilon^2 (-\gamma^{(2)} + 3 \gamma^{(4)} \gamma^{(0)}) - \epsilon \delta_{kj} \gamma_{ij}^{(0)} + O(\epsilon^3) \right] + O(\epsilon^3)$$

(4.7)

one also finds that the first term in (3.5) vanishes, $t^{ij} \sim -2\beta_{KL} \epsilon \delta_i^k \epsilon^{lj}/\sqrt{|g^{(0)}|} = 0$, up to the term of the order of $O(\epsilon^3)$. Then, from the second term in the bracket ‘[]’ in the boundary terms of (3.2), one finds the boundary energy–momentum tensor

$$t^{(2)ij} = \beta_{KL} \frac{2}{\sqrt{|g^{(0)}|}} \left[ \epsilon^{kj} \gamma^{(2)kl} + (i \leftrightarrow j) \right] + O(\epsilon)$$

(4.8)

with

$$t^{ij} = \epsilon^2 t^{(2)ij} + O(\epsilon^3),$$

(4.9)

up to some ‘non-differentiable’ boundary terms

$$\delta I_{\text{GCS(on-shell)}} = \frac{1}{2} \int_{\mathcal{M}} d^2 x \sqrt{|g^{(0)}|} T^{(2)ij}_{\text{reg}} \delta \gamma_{ij}^{(0)} - \beta_{KL} \int_{\mathcal{M}} d^2 x \Gamma^l_{ji} \delta \Gamma^l_{kj} \epsilon^{lj} + O(\epsilon).$$

(4.10)

The unwanted boundary terms can be shown to be vanishing in the explicit computations for the cases that I am interested in this paper, i.e., $\Gamma^l_{ji}^{(0)} = 0$ for the expansion of $\Gamma^l_{ji} = \Gamma^l_{ji}^{(0)} + \epsilon \Gamma^l_{ji}^{(2)} + \cdots$, and so I hereafter do not consider these terms further [20].

Then, by collecting the results, one finds the total energy–momentum tensor for the boundary metric $\gamma_{ij}^{(0)}$ as follows:

$$\tau^{(2)ij} = T^{(2)ij}_{\text{reg}} + t^{(2)ij}$$

$$= -\frac{1}{8\pi G} (\gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)}) + \frac{2\beta_{KL}}{\sqrt{|g^{(0)}|}} \frac{1}{T^2} \left[ \epsilon^{kj} \gamma^{(2)kl} + (i \leftrightarrow j) \right].$$

(4.11)

Here, I would like to note several important points in the above result. First, there is no divergence in the action variation (4.4) for the Einstein–Hilbert part, implying that the counter term (2.8) or its contribution in (2.9) correctly cancels the divergent terms in the un-regularized one. Second, there is no additional divergences from the higher derivative term of the gravitational Chern–Simons due to the cancellation of $\delta \gamma_{ij} \epsilon^{ij} = 0$, though this is not clear in a naive manipulation in (3.2) (see appendix B for details); in other words, I do not need to
consider additional counter terms\(^6\) for this special type of higher derivative term. Third, there is no contribution from the first term of (3.5) in the boundary energy–momentum tensor \(\tau^{ij}\) due to some exact cancellation in the order of \(O(\epsilon)\) when considering \((3^\text{rd})R_{\eta k\eta} = -\gamma_{\eta j}^{(0)} + O(1)\). Finally, the Chern–Simons part \(\tau^{(2)ij}\) of (4.11) is not the analytic continuation from the AdS result: under the analytic continuation \(l \rightarrow i l\), accompanied by an additional continuation of the coordinate \(\eta \rightarrow i \eta\), one has the usual continuation with the real-valued energy–momentum tensor \(T^{ij}\) [33, 37], by considering \(T^{ij} \rightarrow -iT^{ij}\) and \(\tau^{ij} \rightarrow -i\tau^{ij}\), whereas \(\tau^{(2)ij}\) becomes imaginary in this procedure, in contrast to the result (4.11). These resulted from the difference in the transformations, \(I_g(\epsilon) \rightarrow i I_g(\epsilon), I^\text{GCS} \rightarrow i I^\text{GCS}\) under the continuation; the total action \(I = I_g + I_{ct} + I^\text{GCS}\) becomes complex, \(\hat{I} = I_g + I_{ct} - iI^\text{GCS}\) under the continuation \(I \rightarrow i\hat{I}\), and so the Chern–Simons contributions go beyond the analytic continuation (see appendix C for more details).

5. KdS\(_3\) space and holographic energy–momentum tensor

In the absence of the gravitational Chern–Simons term, a general two-parameter family of the vacuum solution, satisfying (2.5) with a positive cosmological constant, is known as KdS\(_3\) (three-dimensional Kerr–de Sitter) solution [26]. It would be a non-trivial task to find the general solutions for the modified equations (3.3) with the third-order derivatives. However, there is a trivial solution, e.g., the KdS\(_3\) solution, since it satisfies equation (2.5) trivially from \(\nabla_\beta (3^\text{rd})R^{\gamma \rho \mu \beta} = 0\), due to \((3^\text{rd})R = (3^\text{rd})R/g_{\gamma \rho}g_{\mu \beta} - g_{\gamma \mu}g_{\rho \beta}\) and \((3^\text{rd})R = +6/l^2\).

This looks like a too-trivial situation which does not seem to have any higher derivative effect of the gravitational Chern–Simons term. But, actually this is not the case, as one can see, since there are some non-trivial ‘global’ effects via some shifts in the conserved charges. So, let me introduce, first, the KdS\(_3\) solution, which is given by the metric [26]

\[
\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + N^{-2} \mathrm{d}r^2 + r^2 (\mathrm{d}\phi + N^\phi \mathrm{d}t)^2
\]

with

\[
N^2 = 8Gm - \left(\frac{r}{l}\right)^2 + \frac{(8Gj)^2}{4r^2}, \quad N^\phi = +\frac{8Gj}{2r^2}.
\]

Here, two constants of integration \(m\) and \(j\), which parametrize the KdS\(_3\) solution, are identified as the mass and angular momentum of the spacetime, respectively [31–33]. Note the sign convention of \(j\) differs from that of [26, 31–33] but agrees with that of [57]; the reason of this choice will be clear below.

The KdS\(_3\) solution (5.1) has one cosmological event horizon at

\[
r_+ = \frac{l}{\sqrt{2}} \sqrt{8Gm + \sqrt{(8Gm)^2 + \frac{(8Gj)^2}{l^2}}}
\]

and there is no black-hole event horizon. Here, there is no additional constraint on \(m\) and \(j\) in order that the horizon exists, unless \(j\) vanishes: even the negative values of \(m\) are allowed. So, in the case of \(j \neq 0\), the whole mass spectrum is continuous, ranging form \(-\infty\) to \(\infty\), and

\(^6\) A systematic study of the appropriate boundary terms for some arbitrary boundary metric would be still a challenging problem.

\(^7\) These definitions of the actions are different from that of [38], but consistent with the usual conventions in the holographic renormalization [33, 37].
there is no mass gap. For the case of \( j = 0 \), there is no horizon when \( m < 0 \) and one is left with a region which is filled with negative masses. Moreover, if I introduce a real parameter

\[
\mathcal{r}_{(\pm)} \equiv \frac{l}{\sqrt{2}} \sqrt{-8Gm + \sqrt{(8Gm)^2 + (8Gj)^2}}.
\]

the mass and angular momentum can be conveniently written as

\[
m = \frac{r_{(\pm)}^2 - r_{(\pm)}^2}{8Gl^2}, \quad j = \frac{2rr_{(\pm)}}{8Gl}.
\]

The dS\(_3\) space can be identified as the case of \( m = 1/8G, j = 0 \) in the general KdS\(_3\) solution.

The two regions separated by the cosmological horizon \( \mathcal{r}_+ \) are casually disconnected and so the cosmological event horizon acts like as a black-hole horizon. Then, from the Gibbons–Hawking analysis \([58]\) this cosmological horizon produces an isotropic background of thermal radiation with a temperature and chemical potential

\[
T_C = \frac{\hbar}{2\pi \mathcal{r}_+}, \quad \Omega_C = -N^\bullet |_{\mathcal{r}_+} = -\frac{r_{(\pm)}}{l} \mathcal{r}_+.
\]

with the (positive) surface gravity function \( \kappa = |\partial N^2/(2\partial r)| \).\(^8\)

Now, by considering the first law of thermodynamics for an arbitrary variation \( \delta \) as \(^9\)

\[
\delta m = T_C \delta S_C + \Omega_C \delta j
\]

with \( T_C \) and \( \Omega_C \) as the characteristic temperature and angular velocity of the system\(^10\), one can determine the entropy of the cosmological horizon as

\[
S_C = \frac{2\pi \mathcal{r}_+}{4G\hbar}.
\]

which is the same form as the Bekenstein–Hawking entropy for black holes. Here, the mass \( m \) can be regarded as the (positive) mass within the cosmological horizon, i.e., \( \mathcal{r} \geq \mathcal{r}_+ \), which differs from the Gibbons–Hawking definition \([28, 31, 32, 57, 58]\) but agrees with that of \([4, 33–35]\).

On the other hand, in order to study the holographic definition of mass, angular momentum, and its associated entropy, it is convenient to consider the metric in the following proper radial coordinates \([21, 26, 61]\)

\[
dr^2 = -d\eta^2 + \frac{1}{l^2} \left( r_{(\pm)}^2 \sinh^2(\eta/l) + r_{(\pm)}^2 \cosh^2(\eta/l) \right) dr^2
\]

\[
+ \frac{1}{l^2} \left( r_{(\pm)}^2 \cosh^2(\eta/l) + r_{(\pm)}^2 \sinh^2(\eta/l) \right) d\phi^2 + \frac{2rr_{(\pm)}}{l} dt d\phi
\]

\[
= -d\eta^2 + \epsilon^{-1} \left( \frac{r_{(\pm)}^2 + r_{(\pm)}^2}{4l^2} (dr^2 + l^2 d\phi^2) + \left( \frac{-r_{(\pm)}^2 + r_{(\pm)}^2}{2l^2} (dr^2 - l^2 d\phi^2) \right)
\]

\[
+ \frac{2rr_{(\pm)}}{l} dt d\phi + \epsilon \frac{r_{(\pm)}^2 + r_{(\pm)}^2}{4l^2} (dr^2 + l^2 d\phi^2). \quad (5.9)
\]

\(^8\) In dS space, there is a subtlety in defining the temperature, which is associated with the definition of the mass within the cosmological horizon \([59]\). But, here I take the usual convention with the positive surface gravity and temperature \([58]\).

\(^9\) In an integrated form, one can obtain the Smarr-type formula \([60]\): \( m = T_C S_C/2 + \Omega_C j \).

\(^10\) If I used the sign convention \( N^\bullet = -8Gj/2r^2 \) in \((5.2)\), I would obtain a wrong sign for the \( \Omega_C \delta j \) term. This is the reason why I have chosen the sign convention of \((5.2)\), instead of \([26, 31–33]\).
Then, by comparing with (2.1), one can easily determine the non-vanishing coefficients in the Fefferman–Graham expansion of (4.1) as follows:

\[ \gamma_{tt}^{(0)} = l^{-2} \gamma_{\phi \phi}^{(0)} = \frac{r_+^2 + r_-^2}{4l^2}, \]

\[ \gamma_{tt}^{(2)} = -l^{-2} \gamma_{\phi \phi}^{(2)} = -\frac{r_+^2 - r_-^2}{2l^2} = -4Gm, \]

\[ \gamma_{\phi \phi}^{(2)} = \frac{r_+ r_-}{l} = 4Gj. \]

The associated boundary energy–momentum tensor (4.11) can be computed as

\[ T_{tt}^{(2)}_{\text{reg}} = -\frac{1}{8\pi Gl} \gamma_{tt}^{(2)} = \frac{m}{2\pi l}, \]

\[ T_{\phi \phi}^{(2)}_{\text{reg}} = -\frac{1}{8\pi Gl} \gamma_{\phi \phi}^{(2)} = \frac{ml}{2\pi r}, \]

and

\[ T_{tt}^{(2)}_{\text{reg}} = -\frac{1}{8\pi Gl} \gamma_{tt}^{(2)} = \frac{j}{2\pi l}. \]

\[ t_{tt}^{(2)} = \frac{4\beta_{KL}}{l^2 \sqrt{|\gamma^{(0)}|}} \gamma_{\phi \phi}^{(2)} \gamma_{tt}^{(0) \phi \phi} \epsilon_{\phi \phi} = \frac{16G\beta_{KL} j}{l^4}, \]

\[ t_{\phi \phi}^{(2)} = \frac{4\beta_{KL}}{l^2 \sqrt{|\gamma^{(0)}|}} \gamma_{\phi \phi}^{(2)} \gamma_{tt}^{(0) \phi \phi} \epsilon_{\phi \phi} = \frac{16G\beta_{KL} j}{l}, \]

\[ t_{tt}^{(2)} = \frac{2\beta_{KL}}{l^2 \sqrt{|\gamma^{(0)}|}} (\gamma_{tt}^{(2)} \gamma_{tt}^{(0) tt} \epsilon_{\phi \phi} + \gamma_{\phi \phi}^{(2)} \gamma_{\phi \phi}^{(0) \phi \phi} \epsilon_{\phi \phi}) = \frac{16G\beta_{KL} m}{l}, \]

where I have used Tr(\gamma^{(2)}) = 0 for the solution (5.10) in (5.11) and \( \epsilon_{tt} = \gamma_{\phi \phi}^{(0)} \gamma_{tt}^{(0) \phi \phi} \epsilon_{\phi \phi} = +\text{det} (\gamma^{(0)}) \epsilon_{\phi \phi} \equiv 1 \) with \( t_{ij} = \gamma_{ik} \gamma_{jl}^{(2)} \) in (5.12).

Now, according to the usual definition of mass and angular momentum [20, 52], one obtains

\[ M = l \oint d\phi t_{tt}^{(2)} = m + \frac{32\pi G\beta_{KL} j}{l^2}, \]

\[ J = -l \oint d\phi t_{\phi \phi}^{(2)} = j - 32\pi G\beta_{KL} m \]

with \( t_{ij}^{(2)} = T_{ij}^{(2)}_{\text{reg}} + t_{ij}^{(2)} \). In the absence of the gravitational Chern–Simons contributions, these agree with [26, 33]. Note that the relative signs in the corrections terms are different. This sign difference may be understood conveniently from the AdS case also, by considering (i) \( l \rightarrow il \), (ii) \( (m, M) \rightarrow (-m, -M) \), (iii) \( (j, J) \rightarrow (-j, -J) \) [26]11 but this cannot be obtained by the conventional analytic continuation \( l \rightarrow il \), \( \eta \rightarrow i\eta \) as I have remarked in the previous section. Then, with these modified mass and angular momentum, one can easily determine their associated entropy of the cosmological horizon as

\[ S = S_C + \frac{16\pi^2 \beta_{KL}}{l\hbar} r_{-}, \]

which satisfies the first law of thermodynamics

\[ \delta M = T_C \delta S + \Omega_C \delta J. \]

11 The last step ‘(iii)’, which is absent in the convention of [26], is due to our definition of (5.2) and (5.5).
with the same $T_C$ and $\Omega_C$ as the characteristic temperature and angular velocity of the system. The entropy correction from the gravitational Chern–Simons term does not satisfy the usual area law [62], similar to the BTZ black hole in the AdS space [8, 21–25]. But here, there is no special meaning of the dependence of $r_{\pm}$, in contrast to the inner horizon $r_-$ in the BTZ black hole. Moreover, the dS$_3$ vacuum solution with $m = 1/(8G)$ and $j = 0$ has a permanent rotation as well, in the new context,

$$M = \frac{1}{8G}, \quad J = -4\pi \beta_{KL}. \quad (5.16)$$

This corresponds to the chirality in the four-dimensional cosmology with a Chern–Simons term.

Before finishing this section, I remark that the maximum-mass conjecture [33, 35, 36, 41] may be violated by the modified mass $M$ since this can be larger than that of the dS$_3$ vacuum, for $\beta_{KL} > 0$, even though the original system satisfies the conjecture [33], i.e., $m < 1/(8G)$. Moreover, the N-bound in the entropy [39, 40] may be violated from the correction term in (5.14) which does not depend on the (outer-horizon) area and can be also arbitrarily larger than that of the dS$_3$ vacuum, for $\beta_{KL} > 0$, even though it is satisfied originally. This situation is quite similar to that of the asymptotically dS solution with NUT charge, where the entropy is no longer proportional to the area [42–44].

### 6. Statistical entropy

In order to compute the statistical entropy we need to know about the holographic anomalies, first. To this end, I start by noting that our three-dimensional gravity system would have the Weyl (or trace) anomaly in the two-dimensional boundary, which is dictated by the non-vanishing trace of the boundary energy–momentum tensor. And also, I note that there is the gravitational anomaly, due to the gravitational Chern–Simons term, which is dictated by the non-conservation of the energy–momentum tensor or the diffeomorphism non-invariance of the action.

As for the Weyl anomaly [1], one usually considers the asymptotic isometry diffeomorphism which produces some anomalous transformations [52, 33]. But, an equivalent and easier way is to consider the relation

$$\tau_{(2)i} = T_{(2)i} + t_{(2)i}$$

$$= +\frac{1}{8\pi G l} \text{Tr}(\gamma_{(2)})$$

$$= +\frac{l R^{(0)}}{16\pi G}, \quad (6.1)$$

where, in the third line, I have used $t_{(2)i} \sim \gamma_{(2)}^{ik} \epsilon^{ik} = 0$ and the bulk Einstein equation $l^2 R^{(0)} = +2 \text{Tr}(\gamma_{(2)})$ at the order of $O(\epsilon)$. (The two-dimensional Ricci tensor is expanded as $R_{ij} = R_{ij}^{(0)} + \epsilon R_{ij}^{(2)} + \cdots$ and $R^{(0)} = R^{(0)i}i$.) Then, by comparing with the Weyl anomaly of the two-dimensional gravity$^{13}$

$$\tau_{(2)i} \equiv -i\hbar \frac{c - \tilde{c}}{48\pi} R^{(0)i}, \quad (6.2)$$

$^{12}$The non-differentiable boundary term of (4.10) might have some corrections in this general context, without considering the explicit metric (5.9). However, even in this case, those are absent [20].

$^{13}$I have introduced an imaginary factor ‘i’ so that it agrees with the result of [26]. I have also introduced $\hbar$ in order to recover the correct $1/\hbar$ factor in the entropy [25].
one may identify the central charges
\[ c - \bar{c} = i \frac{3l}{Gh} \] (6.3)
for the two central charges \( c \) and \( \bar{c} \) for the holomorphic and anti-holomorphic sectors, generally. For the pure imaginary central charges and \( \bar{c} = c^* \), this agrees with the result of [26]. But, even for the complex valued \( c \) and \( \bar{c} \), their imaginary parts are not perturbed by the gravitational Chern–Simons term. Of course, \( R^{(0)} \) vanishes identically, i.e., \( \tau^{(2i)} = 0 \), if one uses the metric \( g^{(0)}_{ij} \) in the explicit solution (5.9) so that the computation of the central charge cannot be justified for this metric14. But, an important point is that the relation (6.1) is generally valid for the asymptotic isometries which produces non-vanishing \( R^{(0)} \), and so the computation of the central charge from (6.2) is quite a robust procedure.

On the other hand, the gravitational anomaly may be conventionally computed from the variation
\[ \delta_\xi I_{\text{GCS}} = \beta_{\text{KL}} \int_{\partial M} \text{Tr}(v d\Gamma) \equiv \hbar \frac{c + \bar{c}}{96\pi} \int_{\partial M} \text{Tr}(v d\Gamma) \] (6.4)
under a diffeomorphism \( \delta_\xi x^\mu = -\xi^\mu(x) \), \( \delta_\xi g_{\mu\nu} = v_{\mu\nu} + v_{\nu\mu} \) with \( v_{\alpha\beta} = \partial_\xi \alpha / \partial_\xi \beta \), giving
\[ c + \bar{c} = 96\pi \beta_{\text{KL}} / \hbar. \] (6.5)
Note that this is a genuine effect of the gravitational Chern–Simons term, from \( \delta_\xi I_g = 0 \). Now, by combining (6.3) and (6.5), one can get
\[ c = \frac{i 3l}{2Gh} + \frac{48\pi \beta_{\text{KL}}}{h}, \quad \bar{c} = -i \frac{3l}{2Gh} + \frac{48\pi \beta_{\text{KL}}}{h}. \] (6.6)
Note also that these central charges are genuine data of the spacetimes, independently on the local structures, i.e., regardless of the existence of the horizons. In other words, the existence of the central charges does not necessarily mean some non-vanishing entropy. In order to have an entropy, the system needs to have some ‘energies’ and these are represented by the Virasoro generators. So, I introduce, following [26], the zero-mode Virasoro generators as
\[ L_0 = \frac{1}{2\hbar} (ilM + J) + \frac{c}{24}, \quad \bar{L}_0 = \frac{1}{2\hbar} (-ilM + J) + \frac{\bar{c}}{24}. \] (6.7)
since these satisfy the usual hermiticity condition \( L_0 = +\bar{L}_0 \) and \( c^* = \bar{c} \). With a unitary representation of the Virasoro algebras of \( L_m, \bar{L}_m \) in the standard form, which are defined on the plane, one can use the Cardy formula for the entropy of a CFT [26, 63–68]
\[ S_{\text{stat}} = 2\pi \sqrt{\frac{1}{6} (c - 24L_{0\text{min}}) \left( L_0 - \frac{c}{24} \right) + 2\pi \sqrt{\frac{1}{6} (\bar{c} - 24\bar{L}_{0\text{min}}) \left( \bar{L}_0 - \frac{\bar{c}}{24} \right)}}, \] (6.8)
which is real valued and positive semi-definite, by construction. Of course, a priori justification of this formula for the complex valued \( c, \bar{c} \) and \( L_0, \bar{L}_0 \) which make the corresponding CFT
non-unitary, is still missing, but I will just assume this formula and see what happens in our
non-trivial circumstance. Then, one can easily get \((γ \equiv 1 + i \frac{3\pi G β_{KL}}{π G L})\)
\[ S_{\text{stat}} = \frac{π}{4G\hbar} \left[ \sqrt{-γ^2(r_+ - ir_-)^2} + \sqrt{-γ^2(r_+ + ir_-)^2} \right] \]
\[ = \frac{2π}{4G\hbar} \left| r_+ + \frac{32π G β_{KL}}{l} r_- \right|, \quad (6.9) \]
where I have chosen \(L_{0(\min)} = L_{0(\min)} = 0\), as usual [69–71], which corresponds to
the dS_3 vacuum solution with \(m = 1/8G\) and \(j = 0\) in the usual context, but with
\(M = 1/8G, J = -4πβ_{KL}\) in the new context. For a positive coupling \(β_{KL}\) or negative
\(β_{KL}\) satisfying \(β_{KL} ≥ -lr_+/(32π Gr_-)\), this agrees exactly with the thermodynamic entropy
formula (5.14). The disagreement for negative couplings \(β_{KL} < -lr_+/(32π Gr_-)\) would
not be so strange since the thermodynamic entropy (5.14) becomes negative, whereas the
statistical entropy is positive semi-definite, by the construction.

Finally, I note that this agreement is quite non-trivial. Actually, with the real-valued
central charges \(c_L = (3l/2G\hbar) + (48π β_{KL}/\hbar), c_R = (3l/2G\hbar) - (48π β_{KL}/\hbar)\) as in the
literature [28, 32, 33], one cannot construct a simple formula, like (6.8), anymore due to the
Chern–Simons contributions. From this non-trivial agreement, the assumed formula (6.8),
which does not generally apply to non-unitary theories, might have some deep meaning.
However, its physical interpretation would be quite different from that of AdS_3, and remains
to be fully understood.

Acknowledgments

I would like to thank Robert B Mann, Si-Young Nam, Chanyong Park and Chaiho Rim for
useful correspondences. This work was supported by the Korea Research Foundation Grant
funded by the Korean Government(MOEHRD) (KRF-2007-359-C00011).

Appendix A. The Gauss–Codazzi equations and the Fefferman–Graham expansion

In this appendix, I consider the Gauss–Codazzi equations for the slicings as in (2.1)
and their expansions \(a la\) Fefferman–Graham. I consider arbitrary dimensions with a positive or
negative cosmological constant, for the sake of generality.

I first start by considering the following slicing of the spacetime with \((d-1)\)-dimensional
hypersurfaces labeled by \(η\) (\(d\) denotes the total spacetime dimensions)
\[ ds^2 = ±dη^2 + γ_{ij} dx^i dx^j, \quad (A.1) \]
where the upper (lower) sign denotes the dS (AdS) space. The extrinsic curvature of a fixed-\(η\)
surface is defined as \(K_{ij} = ±\frac{1}{2} \partial_η γ_{ij}\) and the non-vanishing Christoffel symbols are given by
\[ Γ^η_{ij} = ±K_{ij}, \quad Γ^i_{ηj} = K^j_i = \frac{1}{2} γ^{ik} γ_{kj}, \]
\[ Γ^k_{ij} = \frac{1}{2} γ^{kl}(\partial_η γ_{jl} + \partial_j γ_{il} - \partial_i γ_{lj}), \quad (A.2) \]
where \(γ_{ik} ≡ \partial_η γ_{ik}\).
The components of the curvature tensors, known as the Gauss–Codazzi equations\textsuperscript{,15} are given by
\[
\begin{align*}
(d) R_{ijk}^a &= \pm(-D_i K_{jk} + D_j K_{ik}) \\
&= \pm \frac{1}{2}(-D_i \gamma_{jk} + D_j \gamma_{ik}), \\
(d) R_{ijk}^a &= \pm(\partial_i K_{jk} - K_i^k K_{j}^l) \\
&= \pm \frac{1}{2} \left( \partial_i \gamma_{jk} - \frac{1}{2} \gamma_{il} \gamma_{jk} \right), \\
(d) R_{ij}^k &= R_{ij}^k = R_{ij}^k = R_{ij}^k \mp (K_{ij} K_{kl} - K_{lj} K_{ik}) \\
&= R_{ij}^k \mp \frac{1}{2} (\partial_j \gamma_{ik} - \frac{1}{2} \gamma_{kl} \gamma_{ij}). \\
(d) R_{ij} = R_{ij} \pm (K_{ij} \text{Tr} K + \partial_h K_{ij} - 2(\text{Tr} K^2)_{ij}) \\
&= R_{ij} \pm \frac{1}{2} \gamma_{ij} \pm \frac{1}{2} \gamma_{ij} \text{Tr}(\gamma^{-1} \gamma) \mp \frac{1}{2} (\gamma^{-1} \gamma^{-1} \gamma^{-1} \gamma)_{ij}, \quad (A.3) \\
(d) R_{ij} = R_{ij} = R_{ij} = R_{ij} \pm (\partial_h K_{ij}) \\
&= R_{ij} \pm \frac{1}{2} \gamma_{ij} \pm \frac{1}{2} \gamma_{ij} \text{Tr}(\gamma^{-1} \gamma) \mp \frac{1}{2} (\gamma^{-1} \gamma^{-1} \gamma^{-1} \gamma)_{ij}, \\
(d) R_{ij} = R_{ij} = R_{ij} = R_{ij} \pm (\partial_h K_{ij}) \\
&= R_{ij} \pm \frac{1}{2} \gamma_{ij} \pm \frac{1}{2} \gamma_{ij} \text{Tr}(\gamma^{-1} \gamma) \mp \frac{1}{2} (\gamma^{-1} \gamma^{-1} \gamma^{-1} \gamma)_{ij}, \\
(d) R = (d) R_{ij}^a + (d) R_{ij}^a = R \pm (\text{Tr} K) \pm (\text{Tr} K^2) \pm 2 \partial_h (\text{Tr} K) \\
&= R \pm \text{Tr}(\gamma^{-1} \gamma) \pm \frac{1}{2} \text{Tr}(\gamma^{-1} \gamma)^2 \pm \frac{1}{2} \text{Tr}(\gamma^{-1} \gamma^{-1} \gamma^{-1} \gamma). \\
\end{align*}
\]

Now, for the Fefferman–Graham expansion (4.1), the expansions of the extrinsic curvatures are given by (4.2), regardless of the sign of the cosmological constant and the spacetime dimensions. The expansion of the Christoffel symbols $\Gamma_{ij}^k$ is given by
\[
\Gamma_{ij}^k = \Gamma_{ij}^{(0)k} + \epsilon \Gamma_{ij}^{(2)k} + \cdots, \quad (A.4)
\]
where $\Gamma_{ij}^{(0)k}$ is formed by $\gamma^{(0)}$.

The Riemann tensors are expanded as
\[
\begin{align*}
(d) R_{ijk}^a &= -\frac{1}{l^2} \left[ \epsilon^{-1} \gamma_{ik}^{(0)} + \gamma_{ik}^{(2)} + \epsilon \left( - \gamma^{(2)^2} + 4 \gamma^{(4)} \right)_{ik} \right] + O(\epsilon^2), \\
(d) R_{ijk}^a &= -\frac{1}{l} \left[ \epsilon^{-1} D_i \gamma_{jk}^{(0)} - \epsilon D_i \gamma_{jk}^{(2)} \right] - (i \leftrightarrow j). \quad (A.5)
\end{align*}
\]

And also, by raising the indices $i, j, \ldots$ by the metric $\gamma^{ij}$, these become
\[
\begin{align*}
(d) R_{ij}^k &= -\frac{1}{l^2} \left[ \gamma_{ij}^{(2)k} + \epsilon^2 \left( - \gamma^{(2)^2} + 3 \gamma^{(4)} \right)_{ij} \right] + O(\epsilon^3), \\
(d) R_{ij}^{(2)k} &= -\frac{1}{l^2} \left[ \gamma_{ij}^{(2)k} + \epsilon^2 \left( - \gamma^{(2)^2} + 3 \gamma^{(4)} \right)_{ij} \right] + O(\epsilon^3), \\
(d) R_{ij}^{(2)k} &= -\frac{1}{l^2} \left[ \gamma_{ij}^{(2)k} + \epsilon^2 \left( - \gamma^{(2)^2} + 3 \gamma^{(4)} \right)_{ij} \right] + O(\epsilon^3), \quad \text{and} \\
&= -\frac{1}{l^2} \left[ \gamma_{ij}^{(2)k} + \epsilon^2 \left( - \gamma^{(2)^2} + 3 \gamma^{(4)} \right)_{ij} \right] + O(\epsilon^3).
\end{align*}
\]

Here, I note that there is an exact cancellation in the order $O(\epsilon)$ of $\epsilon^{-1} D_i \gamma_{jk}^{(0)} = \epsilon D_i \gamma_{jk}^{(2)}$ by the contraction process. This fact is crucial when considering the finite energy–momentum tensor $t^{(2)ij}$ in (4.11).

\textsuperscript{15}I follow the conventions summarized in the footnote 4. The $(d - 1)$-dimensional curvatures are denoted by $R_{ijk}^l, \ldots$, etc without the superscript of $(d - 1)$. $D_i$ denotes the covariant derivatives with respect to the metric $\gamma_{ij}$. 

13
The Ricci tensor and scalar are given by
\[(d) R_{\eta \eta} = - \frac{1}{L^2}(d - 1) + \frac{1}{L^2} \epsilon^2 \text{Tr} \left( \gamma^{(0)}_2 - \gamma^{(4)} \right) + O(\epsilon^3),\]
\[(d) R_{\eta i} = \frac{1}{L} \epsilon \left( \partial_i \text{Tr} \gamma^{(2)} - D_i \gamma^{(2)j} \right) - \frac{1}{L} \epsilon^2 \left[ \partial_i \text{Tr} \left( \gamma^{(0)}_2 - \gamma^{(4)} \right) - D_j \left( \gamma^{(0)}_2 - \gamma^{(4)} \right) \right] + O(\epsilon^3),\]
\[(d) R_{i j} = \pm \frac{1}{L^2} \delta^i_j (d - 1) + \frac{1}{L^2} \epsilon \left[ \frac{1}{\epsilon^2} R^{(0)}_{i j} \pm (3 - d) \gamma^{(2)j} \mp \delta^i_j \text{Tr} \gamma^{(2)} \right] \]
\[\mp \frac{1}{L^2} \epsilon^2 \left[ - \frac{1}{L} \epsilon (R^{(2)j}_{i k} - \gamma^{(2)j} \gamma^{(2)k}) \right] \pm \delta^i_j \text{Tr} \left( \gamma^{(2)} - \gamma^{(4)} \right) \mp \gamma^{(2)j} \text{Tr} \gamma^{(2)} + O(\epsilon^3),\]
\[(d) R = \pm \frac{1}{L^2} (d - 1) + \frac{1}{L^2} \epsilon \left[ \frac{1}{\epsilon^2} R^{(0)} \pm 2(2 - d) \text{Tr} \gamma^{(2)} \right] - \frac{1}{L^2} \epsilon^2 \left[ - \frac{1}{L} \epsilon (R^{(2)} - \text{Tr} \gamma^{(2)}) \pm (-2d + 6 \mp 1) \text{Tr} \left( \gamma^{(2)} - \gamma^{(4)} \right) \right] + O(\epsilon^3).\]
Then, by considering the bulk Einstein equation \[(d) R = \left( \frac{2d}{d-2} \right) \Lambda,\] with the cosmological constant \(\Lambda = \pm (d - 1)(d - 2)/2L^2,\) one obtains the iterative (Einstein) equations
\[R^{(0)} = \mp 2(2 - d) L^{-2} \text{Tr} \gamma^{(2)},\]  
\[R^{(2)} = \text{Tr} \gamma^{(2)} \pm (-2d + 6 \mp 1)L^{-2} \text{Tr} \left( \gamma^{(2)} - \gamma^{(4)} \right),\]  
\[\vdots\]  
\[\text{etc.}\]

The first equation (A.8) has been used to get the trace anomaly of (6.1).

Appendix B. Computing the energy–momentum tensor \(\epsilon^{ij}\) for the gravitational Chern–Simons term

In this appendix, I compute the gravitational Chern–Simons contributions to the energy–momentum tensor, from the term \(\int_{\mathcal{B}M} \text{Tr}(\Gamma^\sqcap \delta \Gamma)\) in (3.2). To this end, I first start by noting
\[
\int_{\mathcal{B}M} \text{Tr}(\Gamma^\sqcap \delta \Gamma) = \int_{\mathcal{B}M} \Gamma^a_{b i} \delta \Gamma^b_{aj} \text{d}x^i \wedge \text{d}x^j
\]
\[= \int_{\mathcal{B}M} \left[ \pm 2 K^{ij}_l \delta K_{lj} \pm K^m_{ij} K^{l} j \delta y_{ml} + \Gamma^m_{ij} \delta \Gamma^l_{mj} \right] \epsilon^{ij} \text{d}^2 x,\]
where the second term is cancelled by \(\epsilon^{ij}\) factor and the third term would not contribute, as I have noted in the section 4. Now, by using the Fefferman–Graham expansion of (4.2) one has
\[K^i_{ji} \delta K_{lj} = \frac{1}{L^2} \epsilon^{-1} \delta y^{(0)}_{ij} - \frac{1}{L^2} \epsilon \gamma^{(2)ij} \delta y^{(0)}_{ij} + \frac{1}{L^2} \epsilon \left[ - \delta y^{(4)}_{ij} + \left( (\gamma^{(2)})^2 - \gamma^{(4)} \right) \delta y^{(0)}_{ij} \right] + O(\epsilon^2),\]  
(B.2)
which has a divergence in the first term. But, by considering the factor $\epsilon^{ij}$ again, this term and the first term in the bracket do not contribute. So, I finally have

$$
\delta I_{\text{GCS(on-shell)}} = \mp 2 \beta_{KL} \int_{\partial M} d^2 x \epsilon^{ij} K^i \delta K^j
$$

$$
= \mp 2 \beta_{KL} \int_{\partial M} d^2 x \left[ -\frac{1}{l^2} \epsilon^{ij} \gamma^{(2)i} \delta \gamma^{(0)j} + \frac{1}{l^2} \epsilon \epsilon^{ij} \gamma^{(2)i} \delta \gamma^{(0)j} \right] + O(\epsilon^2),
$$

(B.3)

which produces the boundary energy–momentum tensor of (4.10)

$$
t^{(2)ij} = \pm \beta_{KL} \frac{2}{\sqrt{|\gamma^{(0)ij}|}} \left[ \epsilon^{ij} \gamma^{(2)i} \delta \gamma^{(0)j} \right] + O(\epsilon)
$$

(B.4)

with $t^{ij} = \epsilon^2 t^{(2)ij} + O(\epsilon^3)$. Here, I note that the overall sign differs from that of (5.6) in [20] and this results a different sign in the gravitational Chern–Simons corrections to the mass and angular momentum. (See the following section for details.)

### Appendix C. Comparative computation of the conserved charges in dS/AdS and the analytic continuation

In this appendix, I consider the computation of the conserved charges by comparing the dS and AdS cases, for completeness, and the analytic continuation between them. This will clarify a sign difference with the literature.

To this end, I first start by considering the Einstein–Hilbert action, with the Gibbons–Hawking boundary term, in arbitrary $d$-dimensions and with arbitrary cosmological constants $\Lambda = \pm (d - 1)(d - 2)/2l^2$, is

$$
I_g = \frac{1}{16\pi G} \int_{M} d^3 x \sqrt{|g|} \left( R^{(d)} - 2\Lambda \right) = \frac{1}{16\pi G} \int_{\partial M} d^2 x \sqrt{|\gamma|} \text{Tr} K
$$

$$
= \frac{1}{16\pi G} \int_{\partial M} d^{d-1} x \sqrt{|\gamma|} (R \pm \text{Tr} K)^2 \pm \text{Tr} (K^2) - 2\Lambda).
$$

(C.1)

(My conventions are the same as those of [2, 20], but different from those of [21, 28, 32, 33, 52, 37].) Then, the variation produces, when applying the bulk equations of motion,

$$
\delta I_{g(\text{on-shell})} = \frac{1}{2} \int_{\partial M} d^{d-1} x \sqrt{|\gamma|} T^{ij} \delta \gamma_{ij}
$$

(C.2)

with

$$
T^{ij} = \pm \frac{1}{8\pi G} \left( K^{ij} - \text{Tr}(K) \gamma^{ij} \right).
$$

(C.3)

Now, with the counter terms [30, 33, 51, 52], which can be fixed by the locality and general covariance,

$$
I_{ct} = \pm \frac{1}{8\pi G} \int_{\partial M} d^{d-1} x \sqrt{|\gamma|} \left[ \frac{d-2}{l} \pm \frac{l}{2(d-3)} R \right]
$$

(C.4)

one obtains the regulated energy–momentum tensor

$$
T^{ij}_{\text{reg}} = T^{ij} \pm \frac{1}{8\pi G} \left[ \frac{d-2}{l} \gamma^{ij} \pm \frac{l}{d-3} \left( R^{ij} - \frac{1}{2} R \gamma^{ij} \right) \right].
$$

(C.5)

In the Fefferman–Graham expansion, this becomes

$$
T^{ij}_{\text{reg}} = \pm \frac{1}{8\pi G} \frac{\epsilon^2}{l} \left[ (4-d) \gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)} \right] + O(\epsilon^3).
$$

(C.6)
For the three-dimensional case, i.e., \( d = 3 \), this reduces to

\[
T_{ij}^{\text{reg}} = \mp \frac{1}{8 \pi G} \epsilon^2 \frac{1}{l^3} \left[ \gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)} + O(\epsilon^3) \right],
\]

which agrees with (4.3). Here, I note that these results can be also obtained by the analytic continuation \( l \to i l, R_{ij} \to R_{ij}, I_{s} \to i I_{s}, I_{ct} \to i I_{ct} \), and \( T_{ij}^{\text{(reg)}} \to - i T_{ij}^{\text{(reg)}} \) [33, 37].

By combining \( t^{(2)ij} \) of (B.4) from the gravitational Chern–Simons term, the total boundary energy–momentum tensor in \( d = 3 \) is given by

\[
\tau^{ij} = \epsilon^2 t^{(2)ij} + O(\epsilon^3)
\]

with

\[
\tau^{(2)ij} = \mp \frac{1}{8 \pi G l} \left[ \gamma^{(2)ij} - \gamma^{(0)ij} \text{Tr} \gamma^{(2)} \right] \mp \frac{2\beta_{KL}}{l^3 \sqrt{|\gamma^{(0)}|}} \left[ \epsilon^{ijk} \gamma^{(2)kl} + (i \leftrightarrow j) \right].
\]

Here, note that the Chern–Simons contributions can not be analytically continued, in contrast to \( T_{ij}^{\text{reg}} \) parts since the Chern–Simons contributions become imaginary under the continuation \( t^{ij} \to - i t^{ij} \) with \( l \to i l \).

I also note the appearance of \( 1/\sqrt{|\gamma^{(0)}|} \) factor in the second term, in contrast to the literatures [20, 21]. Then, from the Fefferman–Graham expansion of the metric [21, 26, 61]

\[
\gamma^{(0)}_{\alpha\beta} = \pm t^{-2} \gamma^{(0)}_{\alpha\beta} = \pm \left( \frac{r^2 \pm r_{-}^2}{4l^2} \right),
\]

\[
\gamma^{(2)}_{\alpha\beta} = \mp t^{-2} \gamma^{(2)}_{\alpha\beta} = \mp \left( \frac{r^2 \pm r_{-}^2}{2l^2} \right) = \mp 4Gm,
\]

\[
\gamma^{(4)}_{\alpha\beta} = \pm t^{-2} \gamma^{(4)}_{\alpha\beta} = \pm \left( \frac{r^2 \pm r_{-}^2}{4l^2} \right),
\]

one can find the total boundary energy–momentum tensors

\[
\tau^{(2)}_{\alpha\beta} = \frac{m}{2\pi l} \pm \frac{16G\beta_{KL}m}{l^3},
\]

\[
\tau^{(2)}_{\alpha\phi} = \mp \frac{ml}{2\pi l} - \frac{16G\beta_{KL}m}{l},
\]

\[
\tau^{(2)}_{\phi\phi} = - \frac{j}{2\pi l} \pm \frac{16G\beta_{KL}m}{l},
\]

where I have used the usual definition \( \epsilon_{\alpha\beta} = +\text{det} \gamma^{(0)} (\epsilon_{\alpha\beta} \equiv 1) \) and \( \text{det} \gamma^{(0)}/|\text{det} \gamma^{(0)}| = \text{sign}(\gamma^{(0)}) = \pm 1 \) with \( \text{det} \gamma^{(0)} = \pm (r^2 \pm r_{-}^2)^2/(4l^2)^2 \).

Then, the conserved charges becomes

\[
M = l \oint d\phi \tau^{(2)}_{\alpha\beta} = m \pm \frac{32\pi G\beta_{KL}m}{l^2},
\]

\[
J = -l \oint d\phi \tau^{(2)}_{\phi\phi} = j - 32\pi G\beta_{KL}m.
\]

These final results agree with those of [20, 21, 33, 52] and (1) with \( \beta = -32\pi G\beta_{KL}/l \) [25], but the details in the computations are different. The asymptotic metric \( \gamma_{ij}^{(0)} \) has the arbitrary conformal factors \( (r^2 \pm r_{-}^2)/4l^2 \), which are functions of \( m \) and \( j \), in contrast to the literature. There is no difference in the computation of \( T_{ij}^{(2)}_{\text{reg}} \) since there is no contribution of \( \gamma_{ij}^{(0)} \) for the solution (C.10). However, the conformal factors are crucial in the computation of \( t^{(2)}_{ij} \) in
order to get the correct results of (C.11): in [20, 21] (similarly also in [33, 52]), $\gamma_{ij}^{(0)}$ has been considered as $\gamma_{ij}^{(0)} = -l^{-2} \gamma_{ij}^{(0)} = -1$ with det $\gamma^{(0)} = -l^2$, even with the same higher metric $\gamma^{(n)}$ as in (C.10).\textsuperscript{16} and some strange normalization $\epsilon^{\theta\phi} = 1$, in contrast to the usual one as above. In contrast, with the correct factor of $1/\sqrt{|\gamma^{(0)}|}$ in the gravitational Chern–Simons part in (C.9) or (5.12), one can find the correct results, even for the asymptotic metric $\gamma^{(0)}$ with arbitrary masses and angular momenta.

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\textsuperscript{16} This implies that the generic $K_{\mathrm{dS}_3}$/BTZ metric cannot be ‘smoothly’ transformed into the forms in [20, 21, 33, 52]. In the context of the subtraction procedure [48, 72], this corresponds to choosing the reference spacetime as $\gamma_{ij}^{(0)}$ in (C.10), which depends on the mass and angular momentum.
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