Abstract

The three-dimensional potential equation, motivated by representations of quantum mechanics, is investigated in four different scenarios: (i) In the usual Euclidean space $\mathbb{E}^3$ where the potential is singular but invariant under the continuous inhomogeneous orthogonal group $IO(3)$. The invariance under the translation subgroup is compared to the corresponding unitary transformation in the Schrödinger representation of quantum mechanics. This scenario is well known but serves as a reference point for the other scenarios. (ii) Next, the discrete potential equation as a partial difference equation in a three-dimensional lattice space is studied. In this arena the potential is non-singular but invariance under $IO(3)$ is broken. This is the usual picture of lattice theories and numerical approximations. (iii) Next we study the six-dimensional continuous phase space. Here a phase space representation of quantum mechanics is utilized. The resulting potential is singular but possesses invariance under $IO(3)$. (iv) Finally, the potential is derived from the discrete phase space representation of quantum mechanics, which is shown to be an exact representation of quantum mechanics. The potential function here is both non-singular and possesses invariance under $IO(3)$, and this is proved via the unitary transformations of quantum mechanics in this representation.

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1 Introduction

It is well known that quantum mechanics may be represented in terms of phase space variables [1]. This formulation of quantum mechanics has a history dating back to Wigner [2] and

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Weyl [3]. The analog of the “wave function” in this formulation is a probability distribution function in phase space known as the Wigner function. It is also known that one may describe quantum mechanics in terms of discrete quantities [4], [5]. Finally, the discrete phase space representation of quantum mechanics has been established in references [6] and [7]. The generalization of this representation to relativistic quantum mechanics and field theory has been presented in [8], [9], and [10]. In [10] quantum electrodynamics was formulated in the arena of three-dimensional discrete phase space and continuous time. (Such mixing of discrete and continuous operations is mathematically rigorous and has a long history. For example, see [11]-[20].) The great advantage of this representation is that the S-matrix elements turn out to be divergence free. The mathematical reason for the elimination of the divergences is the fact that partial difference-differential equations representing free fields have non-singular Green’s functions. This discrete phase space representation is relatively new, and has not been studied in any detail in the important physical arenas of non-relativistic quantum mechanics and potential theory, save for specifically the harmonic oscillator [6]. This is the primary motivation for the current study.

In this paper we concentrate on the partial difference equations representing the potential equation in the three-dimensional discrete phase space (which is a subset of the six-dimensional continuous phase space), and study the resulting non-singular potential, which replaces the usual singular Coulomb potential. Before deriving the result in discrete phase space we discuss briefly in section 2 the usual elliptic potential equation in the differentiable manifold $E_3$. The Green’s functions and the potential function $V_0(x^1, x^2, x^3)$ exhibit well known singularities. Outside the singularity, the potential function is invariant under the continuous group $IO(3)$. Although this fact is well known, we will utilize a proof of this invariance from the unitary transformation of position and momentum operators, $Q_j$ and $P_k$, in quantum mechanics according to the Schrödinger representation. This is done as a reference for the study of quantum mechanics and potential theory in the other representations in subsequent sections.

In section 3 we touch upon discrete potential theory [21] in a three-dimensional lattice space. Physical applications of such a discrete potential occur in random walks on a lattice [22] and in finite electrical networks [23]. The representation here is also often utilized in numerical approximations. This discrete potential is non-singular. However, it does not admit invariance under the continuous group $IO(3)$. Also, this difference Laplacian is not capable of providing an exact representation of quantum mechanics.

In section 4 we deal with the representation of quantum mechanics in the six-dimensional continuous phase space manifold [1]. (Originally, Weyl and Wigner suggested such representations [3] [2].) In this section we also formulate the potential equation in continuous phase space, again by using the corresponding quantum mechanics operators $Q^j$ and $P_k$. The potential here is singular. However, it does admit the invariance under the group $IO(3)$ as a subgroup of canonical transformations.

In section 5 we investigate the potential theory in the new discrete phase space scenario. Firstly, quantum mechanical operators $Q^j$ and $P_k$ are formulated in this representation. These are provided by new finite difference operators which we denote as $\delta^i \overset{\circ}{\Delta}$ and $-i\Delta_k \overset{\#}{}$. These are capable of providing an exact representation of quantum mechanics, not just a discrete approximation of the continuum theory. The corresponding Green’s functions are non-singular and the discrete phase space potential, $W(n^1, n^2, n^3)$ is analogous to the

\footnote{The particular representation in [8]-[10] also respects Lorentz symmetry, as was shown in [9]. One confusion comes from the fact that it is tempting to think that the spacetime is discrete in these theories, but it is not.}
potential $V(x^1, x^2, x^3)$ in $\mathbb{E}_3$. The potential equation governing the function $W(n^1, n^2, n^3)$ is proved to be invariant under the continuous group $IO(3)$. The proof here is also based on the unitary transformation involving the corresponding quantum mechanical operators $Q^i$ and $P_k$. We also study the particular solution $W_0(n^1, n^2, n^3)$, analogous to the particular potential $V_0(x^1, x^2, x^3)$ of section 2 (corresponding to the “source” point at the origin). It is shown that $W_0(n^1, n^2, n^3)$ is non-singular and, when analysis is restricted to an axis, $W_0(0, 0, 2n^3)$ can be expressed in terms of gamma functions.

We present the variational derivation of second order partial difference equations and the equations for boundary variations in an appendix for completion [8].

We choose units such that $\hbar = 1$. A new constant, $\ell > 0$, will also appear in subsequent calculations, which is a characteristic length. Except in certain instances where it is required for clarity, we set $\ell = 1$.

2 The potential function in the Euclidean space $\mathbb{E}_3$

In this section we briefly review the usual potential function in $\mathbb{E}_3$. Most of the analysis in this section is well known but this section serves to set the stage for subsequent results for comparisons. Before proceeding we establish the notation as follows. Roman indices take values from \{1, 2, 3\} and summation over repeated indices is implied. Hatted quantities refer to transformed versions of the corresponding unhatted quantities. The ordered triple $(x^1, x^2, x^3) \in \mathbb{R}^3$ denotes a Cartesian coordinate chart for $\mathbb{E}_3$.

The potential function is a solution to the well known equation

$$\nabla^2 V(x^1, x^2, x^3) := \delta^{jk} \frac{\partial^2}{\partial x_j \partial x_k} V(x^1, x^2, x^3) \equiv \delta^{jk} \partial_j \partial_k V(x^1, x^2, x^3) = 0. \quad (2.1)$$

A particular Green’s function for the elliptic partial differential equation (2.1) is provided by

$$G(x^1, x^2, x^3; \hat{x}^1, \hat{x}^2, \hat{x}^3) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp[i k_j (x^j - \hat{x}^j)]}{[(k_1)^2 + (k_2)^2 + (k_3)^2]} \, dk_1 dk_2 dk_3 = \frac{1}{4\pi} \left[ (x^1 - \hat{x}^1)^2 + (x^2 - \hat{x}^2)^2 + (x^3 - \hat{x}^3)^2 \right]^{1/2}. \quad (2.2)$$

In electrostatic phenomena, the physical interpretation of (2.2) is that it represents the Coulomb potential at field point $(x^1, x^2, x^3)$ due to a positive unit electric charge located at source point $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ (in Heaviside-Lorentz units).

By taking the “source point” at the origin, we can write a particular solution of the potential equation as

$$V_0(x^1, x^2, x^3) := G(x^1, x^2, x^3; 0, 0, 0) = \frac{1}{4\pi} \frac{1}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}$$

for $(x^1)^2 + (x^2)^2 + (x^3)^2 > 0$, \quad (2.3i)

$$\lim_{|x^1| \to \infty} |V_0(x^1, x^2, x^3)| = 0, \quad (2.3ii)$$

$$\lim_{(x^1, x^2, x^3) \to (0, 0, 0)} |V_0(x^1, x^2, x^3)| \to \infty. \quad (2.3iii)$$

The equation (2.3iii) demonstrates the well known fact that the above potential diverges in the limit $(x^1, x^2, x^3) \to (0, 0, 0)$. 


The potential equation (2.1) may be variationally derived from an appropriate action (the Dirichlet integral [24], [25]):

\[ J[V] := \int_{D_3} dx_1 dx_2 dx_3 \ L(y; y_1, y_2, y_3) |_{y = \mathbf{v}(\ldots)} \ , \quad (2.4i) \]

\[ L(y; y_1, y_2, y_3) := \frac{1}{2} \left( \delta^{jk} y_j y_k \right)_{|_0} . \quad (2.4ii) \]

Here, \( D_3 \) is a closed, simply-connected, proper subset of \( \mathbb{E}_3 \). The varied function is chosen to be

\[ \tilde{y} := V(x^1, x^2, x^3) + \varepsilon h(x^1, x^2, x^3) \] with \( \varepsilon \neq 0 \) and \( h \) being an arbitrary bounded function of class \( C^2(D_3; \mathbb{R}) \). The variational principle applied to (2.4i) using (2.4ii) implies that

\[ \delta^{jk} \partial_j \partial_k V(x^1, x^2, x^3) = 0 , \quad (2.6i) \]

and

\[ \int_{\partial D_3} \left[ h(x^1, x^2, x^3) \delta^{jk} \partial_j \partial_k V(x^1, x^2, x^3) \right]_{|_0} d^2 s = 0 . \quad (2.6ii) \]

The equation (2.6i) is the Euler-Lagrange equation in \( D_3 \subset \mathbb{R}^3 \), and the equation (2.6ii) is the boundary term due to the variation on \( \partial D_3 \). In the case of Dirichlet boundary conditions \( h(x^1, x^2, x^3)_{|_{\partial D_3}} = 0 \), and (2.6ii) is identically satisfied. Moreover, the only prescribed Neumann condition \( n^j \partial_j V(x^1, x^2, x^3)_{|_{\partial D_3}} = 0 \) is variationally admissible.

Now we shall briefly deal with the symmetry group admitted by Laplace’s equation (2.1). It is the six-parameter inhomogeneous orthogonal group \( IO(3) \). This Lie group is explicitly provided by the transformation

\[ \tilde{x}^j = c^j + r^j_k x^k , \quad (2.7i) \]

\[ \delta_{jk} r^j_a r^k_b = \delta_{ab} , \quad (2.7ii) \]

\[ a^j_b r^k_c = r^j_a r^k_b = \delta^j_k , \quad (2.7iii) \]

\[ x^k = a^j_k (\tilde{x}^j - c^j) , \quad (2.7iv) \]

with the six independent parameters \( c^j \) and \( r^j_k \) comprising the linear transformation.

The potential function \( V(x^1, x^2, x^3) \), and the potential equation (2.1), treated tensorially under (2.7iv), transform as

\[ \tilde{V}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = V(x^1, x^2, x^3) , \quad (2.8i) \]

\[ \delta^{ij} \partial_i \partial_j \tilde{V}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \delta^{ij} a^j_a r^k_c \delta_{bc} \partial_k \partial_c V(x^1, x^2, x^3) = \delta^{bc} \partial_b \partial_c V(x^1, x^2, x^3) = 0 . \quad (2.8ii) \]

The equation (2.8ii) illustrates the invariance of the potential equation (2.1) under the group \( IO(3) \) expressed in (2.7iv). If one sets \( a^j_k = \delta^j_k \) and \( r^j_k = \delta^j_k \), the transformations comprise the three-parameter Abelian subgroup of translations:

\[ \tilde{V}(x^1 + c^1, x^2 + c^2, x^3 + c^3) = V(x^1, x^2, x^3) , \quad (2.9i) \]

or

\[ \tilde{V}(x^1, x^2, x^3) = V(x^1 - c^1, x^2 - c^2, x^3 - c^3) . \quad (2.9ii) \]
It is known that the potential function $V(x^1, x^2, x^3)$ (also called a harmonic function), locally admits the Taylor series expansion in a star-shaped domain of $\mathbb{R}^3$ [25]. Therefore, we obtain from (2.9ii),

$$
\hat{V}(x^1, x^2, x^3) = V(x^1, x^2, x^3) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \sum_{i_1=1}^{3} \cdots \sum_{i_j=1}^{3} \frac{\partial^j}{\partial x^{i_1} \cdots \partial x^{i_j}} V(x^1, x^2, x^3),
$$

(2.10i)

or,

$$
\hat{V}(x^1, x^2, x^3) = \exp \left[ -c^k \partial_k \right] V(x^1, x^2, x^3),
$$

(2.10ii)

$$
\delta^{ab} \partial_a \partial_b \hat{V}(x^1, x^2, x^3) = \exp \left[ -c^k \partial_k \right] \left[ \delta^{ab} \partial_a \partial_b V(x^1, x^2, x^3) \right] = 0.
$$

(2.10iii)

Thus, we have an alternate proof of the invariance of the potential equation (2.1) under the subgroup of translations. This proof has a quantum mechanical aspect to it and will be also useful for subsequent analysis of quantum mechanics in the discrete phase space representation.

In the Schrödinger representation of quantum mechanics (also known as wave mechanics) the self-adjoint coordinate and momentum operators acting on Hilbert space vectors are provided by

$$
P_k \vec{\psi} := -i \partial_k \psi(x^1, x^2, x^3),
$$

(2.11i)

$$
Q^j \vec{\psi} := x^j \psi(x^1, x^2, x^3),
$$

(2.11ii)

$$
[P_k Q^j - Q^j P_k] \vec{\psi} = -i \delta^j_k \psi(x^1, x^2, x^3).
$$

(2.11iii)

Now consider an unusual quantum mechanical equation

$$
-\delta^{jk} P_j P_k \vec{\psi} = 0,
$$

(2.12i)

or,

$$
\delta^{jk} \partial_j \partial_k \psi(x^1, x^2, x^3) = 0.
$$

(2.12ii)

This corresponds to a free particle with no kinetic or potential energy. The equation (2.12ii) implies that both $\text{Re}(\psi)$ and $\text{Im}(\psi)$ satisfy the potential equation (2.1). The above equations can also be understood as the non-relativistic limit of a massless free particle in relativistic quantum mechanics via

$$
\left[ \delta^{jk} P_j P_k - (P_4)^2 \right] \vec{\psi} = 0,
$$

(2.13i)

or,

$$
-\delta^{jk} \partial_j \partial_k \psi(x^1, x^2, x^3, x^4) + \left( \frac{1}{c} \partial_t \right)^2 \psi(x^1, x^2, x^3, x^4) = 0.
$$

(2.13ii)
Here $c$ is the speed of light and the fourth coordinate, $x^4$ is $ct$. In the non-relativistic limit $c \to \infty$, the above relativistic equation reduces to \((2.12\text{ii})\).

In the Schrödinger picture, the transformations of Hilbert space vectors and position and momentum operators, induced by the translation subgroup of $JO(3)$ are specified by

\[
\hat{P}_j = P_j, \quad \hat{Q}^k = Q^k, \quad (2.14\text{i})
\]

\[
\hat{\psi} = \exp \left[-ic^j \hat{P}_j\right] \psi. \quad (2.14\text{ii})
\]

By equations \((2.11\text{i-iii})\), the transformations above yield

\[
\hat{\psi}(x^1, x^2, x^3) = \exp \left[-c\partial_j\right] \psi(x^1, x^2, x^3). \quad (2.15)
\]

The equation \((2.15)\) above is the exact replica of \((2.10\text{ii})\). Hence the equivalence of such transformations on the potential $V(x^1, x^2, x^3)$ and on the zero-energy non-relativistic quantum mechanical particle. We repeat here that most of the analysis in this section is well known, but it serves as a reference for the analysis of the potential equation and quantum mechanics in discrete representations and phase space. We will also comment on advantages and disadvantages of these other representations in the appropriate sections.

### 3 Discrete space and the potential function

There is a characteristic length $\ell > 0$ here. (Recall that we use units such that $\ell = 1$. This is common in numerical analysis.) There are also three discrete variables $n^j \in \{\ldots, -1, 0, 1, 2\ldots\} =: \mathbb{Z}$. We define various partial difference operators as follows:

The right partial difference operator is defined as

\[
\Delta_j f(n^1, n^2, n^3) := f(\ldots, n^j + 1, \ldots) - f(\ldots, n^j, \ldots). \quad (3.1)
\]

The left difference operator is defined as

\[
\Delta'_j f(n^1, n^2, n^3) := f(\ldots, n^j, \ldots) - f(\ldots, n^j - 1, \ldots). \quad (3.2)
\]

The mean partial difference operator is defined as

\[
\overline{\Delta}_j f(n^1, n^2, n^3) := \frac{1}{2} \left[f(\ldots, n^j + 1, \ldots) - f(\ldots, n^j - 1, \ldots)\right]. \quad (3.3)
\]

The discrete potential equation is taken to be \([21]\)

\[
\delta^{jk} \Delta_j \Delta'_k U(n^1, n^2, n^3) = 0. \quad (3.4)
\]

It is usually defined this way as this combination of discrete derivatives yield a self-adjoint potential difference equation. As well, it is variationally derivable in this form \([26]\) and this is the usual Laplacian of discrete numerical analysis.

The partial difference equation in discrete analysis, \((3.4)\), is an analogue of the usual potential equation \((2.1)\) in the continuum. The discrete potential equation \((3.4)\) implies that

\[
U(n^1, n^2, n^3) = \frac{1}{6} \left\{ \begin{array}{l}
[U(n^1 + 1, n^2, n^3) + U(n^1 - 1, n^2, n^3)] \\
+ [U(n^1, n^2 + 1, n^3) + U(n^1, n^2 - 1, n^3)] \\
+ [U(n^1, n^2, n^3 + 1) + U(n^1, n^2, n^3 - 1)]
\end{array} \right\}. \quad (3.5)
\]
Thus, the value $U(n^1, n^2, n^3)$ of the discrete potential (or discrete harmonic function) is the arithmetic mean of its values at the six neighboring points of $(n^1, n^2, n^3)$. This property is analogous to the mean-value theorem for the harmonic function $V(x^1, x^2, x^3)$ of the continuous variables $(x^1, x^2, x^3)$ studied in the preceding section. A Green’s function for the discrete potential equation (3.4) is furnished by

$$G(n^1, n^2, n^3, \hat n^1, \hat n^2, \hat n^3) := \frac{1}{4(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\left[i k_j (n^j - \hat n^j)\right] \sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right) + \sin^2\left(\frac{k_3}{2}\right) \, dk_1 dk_2 dk_3, \quad (3.6i)$$

$$\delta^{ij} \Delta_i \Delta_j \cdot G(\cdot \cdot \cdot) = -\delta_{n^1} \delta_{\hat n^1} \delta_{n^2} \delta_{\hat n^2} \delta_{n^3} \delta_{\hat n^3}. \quad (3.6ii)$$

A discrete potential function $U_{(0)}(n^1, n^2, n^3)$, analogous to $V_{(0)}(x^1, x^2, x^3)$ of (2.3i) is provided by

$$U_{(0)}(n^1, n^2, n^3) := G(n^1, n^2, n^3, 0, 0, 0)$$

$$= \frac{1}{4(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\left[i k_j n^j\right] \sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right) + \sin^2\left(\frac{k_3}{2}\right) \, dk_1 dk_2 dk_3. \quad (3.7)$$

An interesting property of $U_{(0)}(n^1, n^2, n^3)$ is

$$U_{(0)}(0, 0, 0) = \frac{1}{2} \left[ 18 + 12 \sqrt{2} - 10 \sqrt{3} - 7 \sqrt{6} \right] \left\{ \left(\frac{2}{\pi}\right) K \left[ (2 - \sqrt{3})(\sqrt{3} - \sqrt{2}) \right] \right\}^2, \quad (3.8i)$$

$$|U_{(0)}(0, 0, 0)| < \infty. \quad (3.8ii)$$

Here, the function $K[\cdot \cdot \cdot]$ denotes the complete elliptic integral. Thus, the function $U_{(0)}(n^1, n^2, n^3)$ in (3.7) is everywhere non-singular. For large values of $n^j$ this function possesses the following asymptotic behavior

$$U_{(0)}(n^1, n^2, n^3) = \frac{1}{4\pi ||n||} + \frac{1}{32\pi ||n||^3} \left\{ 5[(n^1)^4 + (n^2)^4 + (n^3)^4] \over ||n||^4 \right\} - 3 \right\} + O\left(\frac{1}{||n||^5}\right), \quad (3.9i)$$

$$\lim_{||n|| \to \infty} |U_{(0)}(n)| = 0, \quad (3.9ii)$$

with $||n|| := \sqrt{(n^1)^2 + (n^2)^2 + (n^3)^2}$. From the expression in braces in (3.9i) it can be inferred that the potential violates the symmetry group $O(3)$. Therefore, although this discrete potential remedies the singularity, it is not invariant under the continuous rotation group. This, of course, is due to the fact that the underlying lattice structure of the theory does not possess rotational invariance. The invariance group of the discrete potential equation (3.4) is a crystallographic space group [27].

We finish this section with a brief comment that the difference operators in (3.1)-(3.3) are not capable of providing an exact representation of quantum mechanics, although an approximation in the form of the discrete Schrödinger equation can be provided [28]. The discrete Schrödinger equation has been shown to have applicability in tight-binding models for crystal solids [29] and highly localized phases. The new representation of section 5 yields non-singular Green’s functions and an exact representation of the algebra of quantum mechanics.
Continuous phase space and the potential function

This section relies heavily on the representations of quantum mechanics. The usual Schrödinger representation was briefly introduced in equations (2.11i)-(2.11iii). Although not common, quantum mechanics can also be formulated in phase space [1]. We provide here a continuous phase space representation of quantum mechanics in the following:

\[ P_j \tilde{\psi} := \frac{1}{\sqrt{2}} (p_j - i \partial_q) \psi (q^1, q^2, q^3; p_1, p_2, p_3), \]  

\[ Q_k \tilde{\psi} := \frac{1}{\sqrt{2}} (q^k + i \partial_p) \psi (q^1, q^2, q^3; p_1, p_2, p_3), \]  

\[ [P_j Q^k - Q^k P_j] \tilde{\psi} = -i \delta_{jk} \psi (q^1, q^2, q^3; p_1, p_2, p_3). \]  

The eigenfunctions of the momentum operator are given by:

\[ \psi_{(k)}(q^1, q^2, q^3; p_1, p_2, p_3) = A(p_1, p_2, p_3) \exp \left[ i (\sqrt{2} k_j - p_j) q^j \right], \]  

with

\[ P_j \tilde{\psi}_{(k)} = k_j \tilde{\psi}_{(k)}. \]  

Here, \( A(p_1, p_2, p_3) \neq 0 \) is an arbitrary function. Similarly, the eigenfunctions of the position operator are furnished by:

\[ \psi_{(x)}(q^1, q^2, q^3; p_1, p_2, p_3) = B(q^1, q^2, q^3) \exp \left[ -i (\sqrt{2} x_j - q_j) p^j \right], \]  

with \( B(q^1, q^2, q^3) \neq 0 \) being another arbitrary function.

The potential equation, according to (2.12i) and (4.1i,ii) is provided by

\[ -\delta^{jk} P_j P_k \tilde{\psi} = 0, \]  

or,

\[ -\frac{1}{2} \left[ \delta^{jk} (p_j - i \partial_q) (p_k - i \partial_q) \right] \Omega(q^1, q^2, q^3; p_1, p_2, p_3) = 0. \]  

A special solution, which is the analogue in this representation of the potential function in (2.3i-iii) is given by

\[ \Omega_{(0)}(q^1, q^2, q^3; p_1, p_2, p_3) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ i (\sqrt{2} k_j - p_j) q^j \right] d k_1 d k_2 d k_3, \]  

or,

\[ \Omega_{(0)}(q^1, q^2, q^3; p_1, p_2, p_3) = \frac{e^{-ip_j q^j}}{4 \sqrt{2} \pi \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}} \text{ for } ||q|| > 0, \]  

\[ \lim_{||q|| \to \infty} |\Omega_{(0)}(\cdots)| = 0, \]  

\[ \lim_{||q|| \to 0^+} |\Omega_{(0)}(\cdots)| \to \infty. \]
Here, we have a singular potential function (as well as a singular Green’s function).

Now we shall touch upon the invariance of the potential equation (4.5ii). In phase space
the symmetry group is the group of canonical transformations, which preserve the canonical
symplectic form. The group $I/O(3)$ is a subgroup of the canonical transformations. The
required transformation is explicitly furnished by

$$
\hat{q}^i \, d\hat{p}_j + p_j dq^i = dS(\hat{p}, q),
$$

(4.6i)

$$
S(\hat{p}, q) := \left( c^i + r^j_k q^k \right) \hat{p}_j,
$$

(4.6ii)

$$
\hat{q}^i = \frac{\partial S(\cdot)}{\partial \hat{p}_j} = c^i + r^j_k q^k,
$$

(4.6iii)

$$
p_j = \frac{\partial S(\cdot)}{\partial q^i} = r^k_j p_k,
$$

(4.6iv)

$$
\hat{\Omega}(\hat{q}; \hat{p}) = \Omega(q; p),
$$

or,

(4.6v)

$$
\hat{\Omega}(q; p) = \Omega \left[ a^k_j (q^i - c^i); r^k_j p_k \right],
$$

($a^k_j$ being the inverse transformation of $r^k_j$ and the parameters are in accordance with equations (2.7i-iv)). Note that the particular transformation in (4.6iii) exactly corresponds to the
group $I/O(3)$ of equation (2.7i). Therefore, the group $I/O(3)$ is also a symmetry group of the
phase space potential $\Omega$.

5 Discrete phase space: quantum mechanics and the potential function

It is well known that in the one-dimensional idealized oscillator (reinstating the constants $\hbar$
and $\ell > 0$), the energy eigenvalues are provided by

$$
\frac{1}{2} \left[ \frac{\ell^2}{\hbar^2} P^2 + \frac{1}{\ell^2} Q^2 \right] \tilde{\psi}_N = \left( N + \frac{1}{2} \right) \tilde{\psi}_N,
$$

(5.1)

$$
N \in \{0, 1, 2, \cdots \}.
$$

Note that the operator $\frac{\ell^2}{\hbar^2} P^2 + \frac{1}{\ell^2} Q^2$ has discrete eigenvalues in spite of $P$ and $Q$ both pos-
sessing continuous spectra. The equation (5.1), interpreted phenomenologically, yields the
discretization of the phase plane as

$$
\frac{1}{2} \left[ \frac{\ell^2}{\hbar^2} (p)^2 + \frac{(x)^2}{\ell^2} \right] = n + \frac{1}{2},
$$

(5.2)

$$
n \in \{0, 1, 2, \cdots \}.
$$

The equation (5.2) represents denumerably infinite numbers of confocal ellipses in the back-
ground of the continuous $p - x$ phase plane. Discrete phase space in this section comprises
of three such equations
\[ \frac{1}{2} \left[ \ell^2 \left( p_j \right)^2 + \frac{(x_j)^2}{\ell^2} \right] - \frac{1}{2} = n^j. \] (5.3)
\[ n^j \in \{0, 1, 2, \ldots \}. \]

In other words, the particle’s accessible phase space is discrete in this case in such a way that the above combination of \( x_j \) and \( p_j \) is integer valued \([8]\). In much of the remaining section we again choose units so that the characteristic length \( \ell = 1 \) and \( \hbar = 1 \).

Previously we had defined three partial difference operators in (3.1), (3.2), and (3.3). Here we need to introduce two more partial difference operators in the following. The weighted mean difference operator is
\[ \Delta_j^# f(n^1, n^2, n^3) := \frac{1}{\sqrt{2}} \left[ \sqrt{n^j + 1} f(\ldots, n^j + 1, \ldots) - \sqrt{n^j} f(\ldots, n^j - 1, \ldots) \right], \] (5.4)
and the weighted average difference operator is
\[ \Delta_j^\circ f(n^1, n^2, n^3) := \frac{1}{\sqrt{2}} \left[ \sqrt{n^j + 1} f(\ldots, n^j + 1, \ldots) + \sqrt{n^j} f(\ldots, n^j - 1, \ldots) \right], \] (5.5)

Moreover \( f(n^1, n^2, n^3) := 0 \) for any \( n^j < 0 \). The physical significance of the operators (5.4) and (5.5) is that they provide an exact discrete representation of quantum mechanics via
\[ P_j \psi := -i \Delta_j^# \psi(n^1, n^2, n^3), \] (5.6i)
\[ Q^k \psi := \delta^k_l \Delta_l \psi(n^1, n^2, n^3), \] (5.6ii)
\[ \left[ P_j Q^k - Q^k P_j \right] \psi = -i \delta_j^k \psi(n^1, n^2, n^3). \] (5.6iii)

Now, we denote a Hermite polynomial, \( H_n(k) \) and define a related polynomial as follows:
\[ \xi_{n^j}(k_j) := \frac{(i)^{n^j} e^{-k_j^2} H_{n^j}(k_j)}{\pi^{1/4} 2^{n^j/2} \sqrt{n^j}!} ; \] (5.7i)
for \( n^j \in \{0, 1, 2, \ldots \} \),
\[ \int_{-\infty}^{\infty} \xi_{n^j} \xi_{\tilde{n}^j} \, dk_j = \delta_{n^j, \tilde{n}^j} , \] (5.7ii)
\[ -i \Delta_j^# \xi_{n^j}(k_j) = k_j \xi_{n^j}(k_j) . \] (5.7iii)
(Here the index \( j \) is not summed.) We can infer from (5.7iii) that \( \xi_{n^j}(k_j) \) is the normalized eigenfunction of the momentum operator \( P_j \equiv -i \Delta_j^# \).

The discrete phase space potential equation, analogous to the equations (2.1), is chosen to be
\[ -\delta_{jk} P_j P_k \hat{W} = \delta_{jk} \Delta_j^# \Delta_k^# W(n^1, n^2, n^3) = 0 . \] (5.8)
A Green’s function for the partial difference equation (5.8), which is analogous to that in (2.2) is provided by:

\[ G(n_1, n_2, n_3; \hat{n_1}, \hat{n}_2, \hat{n}_3) := \int \int \int_{-\infty}^{\infty} \frac{3 \prod_{j=1}^{3} \xi_{n,j}(k_j) \xi_{\hat{n},j}(k_j)}{[(k_1)^2 + (k_2)^2 + (k_3)^2]} dk_1 dk_2 dk_3, \quad (5.9i) \]

\[ \delta^j_\# \Delta^{\#}_k G(n_1, n_2, n_3; \hat{n}_1, \hat{n}_2, \hat{n}_3) = - \delta_{n_1, \hat{n}_1} \delta_{n_2, \hat{n}_2} \delta_{n_3, \hat{n}_3}, \quad (5.9ii) \]

\[ G(0, 0, 0; 0, 0, 0) = 2. \quad (5.9iii) \]

In analogy with (2.3i), we present the following potential from (5.9i) in this representation:

\[ W_{(0)}(n_1, n_2, n_3) = G(n_1, n_2, n_3; 0, 0, 0) \]

\[ = \frac{(8^{1/2})^{n_1 + n_2 + n_3}}{\pi^{3/2} 2^{n_1 + n_2 + n_3} \Gamma(n_1 + n_2 + n_3 + 1)^{1/2}} \int \int \int_{-\infty}^{\infty} e^{-(k_1^2 + k_2^2 + k_3^2)} \frac{3 \prod_{j=1}^{3} H_{n,j}(k_j)}{[(k_1)^2 + (k_2)^2 + (k_3)^2]} dk_1 dk_2 dk_3. \quad (5.10) \]

Restricting our analysis along the third axis for the moment, \( W_{(0)}(0, 0, 2n) \) admits the closed form expression:

\[ W_{(0)}(0, 0, 2n) = \frac{(-1)^n}{\pi^{3/2} 2^n \sqrt{2n}} \int \int \int e^{-(k_1^2 + (k_2^2 + k_3^2) H_{2n}(k_3)} \frac{3 \prod_{j=1}^{3} H_{n,j}(k_j)}{[(k_1)^2 + (k_2)^2 + (k_3)^2]} dk_1 dk_2 dk_3, \quad (5.11i) \]

\[ = \frac{2^{n+1} n!}{(2n + 1) \sqrt{(2n)!}} > 0, \quad (5.11ii) \]

\[ \left[ \frac{W_{(0)}(0, 0, 2n + 2)}{W_{(0)}(0, 0, 2n)} \right] = \left( \frac{2n + 2}{2n + 3} \right) \frac{\sqrt{2n + 1}}{2n + 2} < 1, \quad (5.11iii) \]

\[ W_{(0)}(0, 0, 0) = 2. \quad (5.11iv) \]

Thus, \( \{ W_{(0)}(0, 0, 2n) \}_{n=0}^{n=\infty} \) is a positive-valued, monotone decreasing sequence. Now we shall explore the asymptotic behavior in the limit \( n \to \infty \) \( W_{(0)}(0, 0, 2n) \). For that purpose, we express (5.11iii) as

\[ W_{(0)}(0, 0, 2n) = \frac{2^{n+1} \Gamma(n + 1)}{(2n + 1) \sqrt{\Gamma(2n + 1)}}. \quad (5.12) \]

The asymptotic expression for the gamma function is provided by

\[ \Gamma(n) = \sqrt{2\pi(n)^{n-\frac{1}{2}}} e^{-n} \left[ 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{5184n^3} - \frac{571}{248832n^4} + O \left( \frac{1}{n^5} \right) \right]. \quad (5.13) \]
Therefore, equations (5.11ii), (5.12), and (5.13) imply that
\[
W(0,0,2n) = \frac{\pi^{1/4}}{\sqrt{e}} \left\{ \frac{1}{n^{3/4}} \left[ 1 + \frac{1}{2n} \right]^n \left[ 1 + \frac{1}{2n} \right]^{5/4} \right\} \times \left[ 1 + \frac{1}{12(n+1)} + O \left( \frac{1}{(n+1)^2} \right) \right] + O \left( \frac{1}{(2n+1)^2} \right), \tag{5.14i}
\]
\[
\lim_{n \to \infty} W(0,0,2n) = \frac{\pi^{1/4}}{\sqrt{e}} \left\{ 0 \cdot \sqrt{e} \cdot 1 \cdot 1 \right\} = 0. \tag{5.14ii}
\]
We plot \(W(0,0,2n)\) in figure 1 for values of \(n\) near the origin.

**Figure 1:** The discrete phase space potential \(W(0,0,2n)\) for \(\ell = 0.1\). Note that this potential is finite everywhere and has a value of 2 at the origin.

Now we want to compare the continuous phase space potential \(\Omega_0(q^1, q^2, q^3; p_1, p_2, p_3)\) to the discrete phase space potential \(W_{(0)}(0,0,2n)\). Note that \(\Omega_0(q^1, q^2, q^3; p_1, p_2, p_3)\) is a complex potential, and hence we really wish to compare the real, monotone \(W_{(0)}(0,0,2n)\) with the quantity \(\sqrt{|\Omega_0(0,0,2n)|}\). This comparison is also equivalent to comparing \(W(0,0,2n)\) to the potential \(V(0,0,x^3) = (1/4\pi)1/|x^3|\) noting that the \(p_j\) dependence drops out in the magnitude \(|\Omega_0(0,0,q^3; p_1, p_2, p_3)|\). Hence, \(\sqrt{|\Omega_0(0,0,q^3; p_1, p_2, p_3)|^2}\) and \(V(0,0,x^3)\) differ only by a numerical factor of \(1/\sqrt{2}\).

From (5.3) we can use the approximation
\[
n^3 = (1/2\ell^2)(x^3)^2 - 1/2 + O(\ell^2) \tag{5.15}
\]
for extremely small values of \(\ell\). Specifically \(\ell\) is assumed small in comparison to \(h\), perhaps even as small as the Planck length. Some caution must be applied in the comparison as the \(\ell\) and \(h\) are not of the same dimension and hence the approximation (5.15) only really
becomes valid for large values of \( x \) and provided the momentum is not large. In this limit the discrete phase space potential function essentially goes over to a configuration space potential function, making the comparison meaningful. Hence we limit our analysis away from the origin. (We have already shown in (5.11 iv) and figure 1 that the potential \( W_{(0)}(n^1, n^2, n^3) \), without the approximations, is finite at the origin.) This comparison is done in figure 2 for the various potentials.

![Comparison of potentials away from the origin](image)

Figure 2: A comparison of the potentials \( V_0 \) (dashed), \( |U_{(0)}| \) (black) and \( W_{(0)} \) (diamonds, for \( \ell = 0.1 \)) away from the origin. The discrete phase space potential is actually finite at the origin (see main text and previous plot) whereas the other two potentials diverge at the origin.

The most general transformation of the scalar field \( W(n^1, n^2, n^3) \) under the six-parameter (proper) continuous group \( IO^+(3) \) is given by\(^3\)

\[
\hat{W}(n^1, n^2, n^3) = \exp \left\{ -c^j \Delta_j^\# - \frac{\omega^{jk}}{4} \left[ \tilde{\Delta}_j^\# \Delta_k^\# - \tilde{\Delta}_k^\# \Delta_j^\# + \Delta_k^\# \tilde{\Delta}_j^\# - \Delta_j^\# \tilde{\Delta}_k^\# \right] \right\} W(n^1, n^2, n^3),
\]

(5.16i)

\[
\omega^{kj} \equiv -\omega^{jk}.
\]

(5.16ii)

The translation subgroup is parameterized by the continuous parameters \( c^j \) and rotations are encoded in the continuous parameters \( \omega^{jk} \). The discrete phase space potential equation remains invariant since the generators \( \Delta_j^\# \) and

\[
\left[ \tilde{\Delta}_j^\# \Delta_k^\# - \tilde{\Delta}_k^\# \Delta_j^\# + \Delta_k^\# \tilde{\Delta}_j^\# - \Delta_j^\# \tilde{\Delta}_k^\# \right]
\]

of the group \( IO^+(3) \) commute with the Casimir invariant operator \( \delta^{jk} \Delta_j^\# \Delta_k^\# \). Therefore the potential is also invariant under the continuous group \( IO(3) \).

Finally, we shall investigate the variational derivation of the discrete phase space potential

\(^3\)Here, \( O^+(3) \) denotes the proper subgroup (rotation group) of the orthogonal group \( O(3) \).
Consider a class of action sum

\[ J[f] := \sum_{n^1 = N_1^1}^{N_2^1} \sum_{n^2 = N_1^2}^{N_2^2} \sum_{n^3 = N_1^3}^{N_2^3} L(y; y_1, y_2, y_3)_{y = f(n), y_j = \Delta^#_j f(n)} \cdot \quad (5.17) \]

The variational principle states:

\[ \lim_{\varepsilon \to 0} \left\{ \frac{J[f + \varepsilon h] - J[f]}{\varepsilon} \right\} = 0. \quad (5.18) \]

The equation (5.18) implies the following Euler-Lagrange equations:

\[ \frac{\partial L(\cdots)}{\partial y} \bigg|_{\cdots} - \Delta^#_j \left[ \frac{\partial L(\cdots)}{\partial y_j} \right]_{\cdots} = 0 \quad (5.19) \]

and boundary terms. The derivation of (5.18) and the explicit boundary terms will be shown in the Appendix. In the case of the discrete phase space potential equation, the appropriate Lagrangian is furnished by

\[ L(y; y_1, y_2, y_3)_{y = W(n), y_j = \Delta^#_j W(n)} := \frac{1}{2} \delta^{jk} \Delta^#_j W(n) \Delta^#_k W(n). \quad (5.20) \]

(Recall that it is the momentum operator which yields the potential equation Laplacian and hence \( \Delta^#_j \) (but not \( \Delta^\circ_j \)) appears here.)

6 Concluding remarks

In this paper we have studied potential theory in discrete phase space. The Green’s functions are non-singular as are their associated potentials. Unlike the usual discrete theory which arises as an approximation of continuous systems, these Green’s functions have been shown here to be rotationally covariant. As well, the operators giving rise to these Green’s functions provide an exact representation of quantum mechanics. This provides a novel representation for quantum mechanics. It is known that singular Green’s functions, for example, lead to divergences in \( S \)-matrix elements in relativistic quantum field theory. With the relativistic version of the representation presented here, the corresponding \( S \)-matrix elements have been previously shown to be finite [10].

A APPENDIX - Variational principle in discrete phase space

We will consider the two-dimensional lattice function \( f(n^1, n^2) \) for the sake of simplicity as generalizations to higher dimensions is straightforward. We use capital Roman indices to take values from \( \{1, 2\} \). The action functional is defined by

\[ J[f] := \sum_{n^1 = N_1^1}^{N_2^1} \sum_{n^2 = N_1^2}^{N_2^2} L(y; y_1, y_2)_{y = f(n), y_A = \Delta^#_A f(n)} \cdot \quad (A.1) \]

Subsequently, we denote \( |(n^1, n^2) \rangle \equiv |y = f(n^1, n^2), \Delta^#_A f(n^1, n^2) \rangle \) for brevity.
The variational principle states that

\[ 0 = \lim_{\varepsilon \to 0} \left\{ \frac{J[f + \varepsilon h] - J[f]}{\varepsilon} \right\} \]

\[ = \sum_{n^1=N_1^A}^{N_1^B} \sum_{n^2=N_1^A}^{N_2^B} \left\{ \frac{\partial L(\cdots)}{\partial y} \left. h(n^1, n^2) + \frac{\partial L(\cdots)}{\partial y_A} \left. \Delta^# h(n^1, n^2) \right|_{(n^1, n^2)} \right\} \right. \]  

(A.2)

We now use the following relation [8]

\[ \phi(n^1, n^2) \Delta^# h(n^1, n^2) + h(n^1, n^2) \Delta^# \phi(n^1, n^2) \]

\[ = \Delta_A \left\{ \sqrt{2} \phi(n^1, n^2) h(., n^A - 1, .) + h(n^1, n^2) \phi(., n^A - 1, .) \right\} \]  

(A.3)

where \( \Delta_A \) has been defined in (3.1). (In the above equation the summation convention has been suspended.)

Next we utilize the discrete Gauss’ theorem [21] for a discrete vector field \( j^A(n^1, n^2) \):

\[ \sum_{n^1=N_1^A}^{N_1^B} \sum_{n^2=N_1^A}^{N_2^B} \Delta_A j^A(n^1, n^2) \]

\[ = \sum_{n^2=N_2^B}^{N_2^B} \left[ j^1(N_2^B + 1, n^2) - j^1(N_1^A, n^2) \right] + \sum_{n^1=N_1^A}^{N_1^B} \left[ j^2(n^1, N_2^B + 1) - j^2(n^1, N_1^A) \right] . \]

(A.4)

Using (A.3) and (A.4) in (A.1), and utilizing the discrete Dubois-Reymond lemma [24], [8] we derive the following:

The Euler-Lagrange equations:

\[ \frac{\partial L(\cdots)}{\partial y} \left. \right|_{(n^1, n^2)} - \Delta^# \left[ \frac{\partial L(\cdots)}{\partial y_j} \right] \right|_{(n^1, n^2)} = 0 \]  

(A.5i)

for \( N_1^A < n^A < N_2^A + 1 \);

The boundary terms:

\[ \sum_{n^2=N_1^B}^{N_2^B} \left\{ \sqrt{2} \frac{N_2^B + 1}{2} \left[ \frac{\partial L(\cdots)}{\partial y_1} \left. \right|_{(N_2^B, n^2)} h(N_2^B + 1, n^2) + \frac{\partial L(\cdots)}{\partial y_1} \left. \right|_{(N_2^B, n^2)} h(N_2^B + 1, n^2) \right] \right. \]

\[ - \sqrt{2} \frac{N_1^B}{2} \left[ \frac{\partial L(\cdots)}{\partial y_1} \left. \right|_{(n^2, n^2)} h(N_1^B - 1, n^2) \right] \]

\[ + \left[ \frac{\partial L(\cdots)}{\partial y} \left. \right|_{(n^1, n^2)} - \Delta_A \frac{\partial L(\cdots)}{\partial y_A} \left. \right|_{(n^1, n^2)} \right] - \sqrt{2} \frac{N_1^B}{2} \left[ \frac{\partial L(\cdots)}{\partial y_1} \left. \right|_{(n^1, n^2)} \right] \]

\[ \sum_{n^1=N_1^A}^{N_2^B} \left\{ \sqrt{2} \frac{N_2^B + 1}{2} \left[ \frac{\partial L(\cdots)}{\partial y_2} \left. \right|_{(n^2, N_2^B + 1)} h(n^1, N_2^B + 1) + \frac{\partial L(\cdots)}{\partial y_2} \left. \right|_{(n^2, N_2^B + 1)} h(n^1, N_2^B + 1) \right] \right. \]
\[-\sqrt{\frac{N_1^2}{2}} \left[ \frac{\partial L(\cdots)}{\partial y_{\{n_1,N_1^1\}}} h(n_1^1, N_1^2 - 1) \right] \]

\[+ \left[ \frac{\partial L(\cdots)}{\partial y} \right]_{(n_1^1, N_1^2)} - \Delta_A \left[ \frac{\partial L(\cdots)}{\partial y_A} \right]_{(n_1^1, N_1^2)} - \sqrt{\frac{N_1^2}{2}} \frac{\partial L(\cdots)}{\partial y_{\{n_1,N_1^2 - 1\}}} \right] h(n_1^1, N_1^2) \right\} = 0. \] (A.5ii)

Note that the discrete domain \( D_2 \) for the validity of the Euler-Lagrange equations is given by
\[D_2 := \{(n_1^1, n_2^2) : N_1^1 \leq n_1^1 < N_1^2 + 1, \ N_1^2 < n_2^2 < N_2^2 + 1\}. \] (A.6)

However, the variationally admissible boundary of the discrete domain is provided by
\[\partial^\# D_2 = \{(n_1^1, n_2^2) : n_1^1 = N_1^2 + 1, N_2^1, N_1^1 - 1, N_1^1; N_1^2 \leq n_2^2 \leq N_2^2 \}
\cup \{(n_1^1, n_2^2) : n_1^1 \leq n_1^1 \leq N_2^1; n_2^2 = N_2^2 + 1, N_2^2, N_1^2 - 1, N_1^2 \} \] (A.7)

The domain \( D_2 \) and the boundary \( \partial^\# D_2 \) are depicted in figure 3.

**Figure 3:** The domain and boundary for the discrete variational principle in a two dimensional lattice space.

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