Optimal \((r, \delta)-\)Locally Repairable Codes From Simplex Code and Cap Code

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ABSTRACT Locally repairable codes (LRCs) are implemented in distributed storage systems (DSSs) due to their low coordinate overhead. A linear code \(C\) is said to have \((r, \delta)-\)locality if for each coordinate \(i\), there exists a punctured subcode of \(C\) with support containing \(i\), whose length is at most \(r + \delta - 1\), and whose minimum distance is at least \(\delta\). An LRC is called optimal if its minimum distance attains Singleton-type bound was proposed. In this letter, optimal LRCs are considered. We first determine \((r, \delta)-\)locality of three dimensional Simplex code, then using anticode strategy, a class of \([3q, 3, 2q - 1]_q\) LRCs with \((2, q)\) locality are derived for general \(q\). Finally, using an ovoid in \(PG(3, q)\), we construct \([q^2 + 1, 4, q(q - 1)]_q\) and \([4q - 4, 4, 3q - 5]_q\) LRCs with \(r = 3\) and \(\delta = q - 1\). All LRCs constructed in this letter attain the Singleton-type bound.

INDEX TERMS Singleton-type bound, simplex code, optimal locally repairable codes, cap code.

I. INTRODUCTION

With the increasing demand for data centers on distributed storage systems (DSSs), coding for DSSs have attracted much attention from researchers. Locally repairable codes (LRCs) are a family of erasure codes and designed for DSSs to recover data from one failed node or failed node by accessing \(r\) other survived nodes.

LRCs are introduced by Gopalan \textit{et al.} in [1], some basic properties of LRCs are discussed and a Singleton-type bound on minimum distance of an LRC is proposed in [1]:

\[
d \leq n - k - \left\lfloor \frac{k}{r} \right\rfloor + 2.
\]

A code that achieves the bound on the distance with equality will be called an optimal \(r\)-local LRC. For optimal \(r\)-local LRCs, Tamo and Barg derived a family of optimal LRCs from “good” polynomials and data recovery relies on polynomial interpolation [3]. For optimal cyclic \(r\)-local LRCs, see Refs. [4]–[9]. Optimal \(r\)-local LRCs over small fields can be seen in [10]–[14].

All the LRCs above can recover data from one failed node at a time, but multiple nodes failures are common in distributed storage system. When some of the \(r\) repairing symbols are also erased, the LRC can not finish the repair process, which leads to the concept of \((r, \delta)-\)locality. Prakash \textit{et al.} [2] generalized the single node failure solution and presented the \((r, \delta)\)-locality to recover data from multiple device failures. A code symbol of a linear code \(C\) is said to have \((r, \delta)-\)locality if it is contained in a punctured subcode of \(C\), which has length at most \(r + \delta - 1\) and minimum distance at least \(\delta\). A generalization of Singleton-type bound in [1] is also proposed in [2]

\[
d \leq n - k - \left(\left\lfloor \frac{k}{r} \right\rfloor - 1\right) (\delta - 1) + 1
\]

LRCs attaining this bound are called optimal with \((r, \delta)-\)locality and are given in Refs. [15]–[20].

In this letter, two classes of new optimal \((r, \delta)\) LRCs are derived from Simplex code with dimension 3 and ovoid code over \(\mathbb{F}_q\), respectively. Their parameters are \([q^2 + q + 1, 3, q^2]_q\) for \(r = 2\) and \(\delta = q\), \([3q, 3, 2q - 1]_q\) for \(r = 2\) and \(\delta = q\), \([q^2 + 1, 4, q(q - 1)]_q\) for \(r = 3\) and \(\delta = q - 1\) and \([4q - 4, 4, 3q - 5]_q\) for \(r = 3\) and \(\delta = q - 1\), respectively. All these LRCs attain the Singleton-type bound.

The rest of this paper is organized as follows: Section II will give the necessary basic concept of LRCs and notations. In section III, the \((r, \delta)-\)locality of \(q\)-ary Simplex code are determined then with rearrangement of the columns in generator matrix of \(q\)-ary Simplex code and the anticode method, optimal \([3q, 3, 2q - 1]_q\) LRCs with \(r = 2\) and \(\delta = q\) are
constructed. Section IV deals with an ovoid code and deduce optimal \([4g - 4, 4, 3g - 5]_q\) LRCs for \(r = 3\) and \(\delta = q - 1\). The last is the conclusion.

II. PRELIMINARIES

In this section, the mathematical notations and definitions are summarized.

**Notations 1:**
1) If \(A\) is a matrix, denote \(A\) by the set of columns in \(A\).
2) Let \(q > 2\) be a prime power and \(F_q = \{0, x_0 = 1, x_1, x_2, \ldots, x_{q-2}\}\) be a finite field with \(q\) elements, \(F_q^* = F_q \setminus \{0\}\).
3) For any subset \(X \subseteq F_q^k\), we use \((X)\) to denote the subspace of \(F_q^k\) spanned by \(X\), where \(F_q^k\) is the \(k\)-dimensional vector space over \(F_q\).
4) Let \(w\) be an element of \(F_q^*\) such that \(f(y_1, y_2) = y_1^2 + y_2 y_1 y_2 + y_1^2\) is irreducible and denote \(h(x, y) = -f(1, x y)\). Fixed \(x, h(x, y)\) is a permutation of \(F_q^k\).

A \(k\)-dimensional subspace of \(F_q^k\) is called a linear code and denote by \(C = [n, k, q]\). \(G\) is the generator matrix of \(C\) which can be derived from a basis of \(C\). Let \(G = \{g_1, \ldots, g_n\}\) be a generator matrix of \(C\) and let \(G^* = [g_1, \ldots, g_n]\). We call the vector in \(C\) codeword. The Hamming weight of a codeword \(c \in C\) is \(wt(c) = |\{i | c(i) \neq 0\}|\). The minimum distance of \(C\) is denoted as \(d = \min_{x, y \in C, x \neq y} wt(x - y) = \min_{c \in C, c \neq 0} wt(c)\).

**Definition 1:** [2] The \(i\)th code symbol \(c_i, 1 \leq i \leq n\), in an \([n, k, d]\) linear code \(C\), will be said to have \((r, \delta)\) -locality if there exists a subset \(S_i \subseteq [n]\) such that:

1. \(|S_i| \leq r + \delta - 1\);
2. The minimum distance of the punctured code \(C|S_i\) is at least \(\delta\).

A famous distance property is introduced in [21] as follows:

**Lemma 1:** An \([n, k]\) linear code \(C\) has a minimum distance \(d\), if and only if for every \( S \subseteq G\) having \(\text{Rank}(S) \leq k - 1\), \(\text{Rank}(T) = k\) for every \(T \subseteq G\) of size \(n - d + 1\).

From Lemma 1, the condition 2 in Definition 1 is equivalent to the following condition

\[ (2') \text{Rank}(\{g|l| l \in I\}) = \text{Rank}(G_{i}) \text{ for any subset } I \subseteq S_{i} \text{ of size } |I| = |S_{i}| - \delta + 1, \text{ where } G_{i} = \{g|l| l \in S_{i}\}. \]

**Definition 2:** Let \(PG(k, q)\) be the \(k\)-dimensional projective geometry over \(F_q^*\) defined in \(F_q^* \setminus \{0\}\). Two vectors \(x, y\) are equivalent if there exists a non-zero element \(\lambda \in F_q\) that \(x = \lambda y\). This is an equivalence relation. All the equivalence classes of \(F_q^* \setminus \{0\}\) form the points of \(PG(k, q)\). \(PG(k, q)\) is also called \(k\)-dimensional projective space.

An \(n\)-cap in \(PG(k, q)\) is a set of \(n\) points no three of which are collinear. If we write the \(n\) points of an \(n\)-cap in \(PG(k, q)\) as columns of a matrix, we obtain a \((k + 1) \times n\) matrix such that every set of three columns is linearly independent; hence the check matrix of a linear code has minimum distance \(d \geq 4\).

It follows that an \(n\)-cap in \(PG(k, q)\) is equivalent to a \(q\)-ary linear \([n, n-k-1, 4]_q\) code. An \(n\)-cap in \(PG(k, q)\) of maximal size is called a maximal cap in \(PG(k, q)\). For more details about caps, see [23]. A code \(C\) is called a projective code if the minimum distance \(d\) of its dual is equal to or greater than 3.

Next, we introduce the concepts of anti-code and Simplex code.

**Definition 3:** [22] An anticode \(A\) with length \(n\) and maximal distance \(\delta\) is a multiset of codewords in \(F_q^k\) such that the maximum Hamming distance between any pair of codewords is less than or equal to \(\delta\). Denote its generator matrix by \(G\), the anticode by \(A_G\) then we have \(\delta = \max_{x \in A_G} wt(x)\).

**Definition 4:** [23] Denote the matrix \(S_k\) such that the columns are representatives for the one-dimensional subspaces in \(F_q^k\). Then we call the code generated by \(S_k\) \(q\)-ary Simplex code with dimension \(k\), denote it by \(S_k\). Obviously, the dual of Simplex code is Hamming code.

III. OPTIMAL LRCs WITH \(r = 2\) AND \(\delta = q\)

This section will discuss the construction of optimal LRCs with parameters \(r = 2\) and \(\delta = q\). First, we determine the \((r, \delta)\)-locality of \(q\)-ary Simplex code with dimension 3, then deduce a class of optimal LRC from threedimensional Simplex code.

**A. \((r, \delta)\) LOCALITY OF \(q\)-ARY SIMPLEX CODE WITH DIMENSION 3**

In order to determine the \((r, \delta)\)-locality of \(q\)-ary Simplex code with dimension 3, we need to construct a proper generator matrix of Simplex code.

Before constructing LRCs, let us construct the points in \(PG(2, q)\) in the form we wish. The first thing to be determined is the number of points in \(PG(2, q)\):

\[ q^{2-1} = q^2 + q + 1. \]

Let \(0 \leq i \leq q - 2\) and construct

\[ A_i = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & x_i \\ x_i & 1 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_i & x_i & \cdots & x_i \\ 1 & x_i & \cdots & x_{q-2} \end{pmatrix}. \]

Then \(G = [I_3|A_0 \cdots A_{q-2}|B_0 \cdots B_{q-2}]\) is the matrix formed by all the points in \(PG(2, q)\) and \(G\) generates a Simplex \([q^2 + q + 1, 3, q^2]_q\) code \(C\). The number of columns with weight 1, 2, 3 in \(C\) are 3, 3\((q-1)\), \((q-1)^2\), respectively.

**Example 1:** Let \(q = 3\) and \(F_3 = \{0, 1, 2\}\), the Simplex \([13, 3, 9]_3\) code has a generator matrix

\[ G_{3,13} = \begin{pmatrix} 100 & 011 & 011 & 1111 \\ 010 & 101 & 102 & 1122 \\ 001 & 110 & 220 & 1223 \end{pmatrix}, \]

there are 3, 6, 4 columns in \(G_{3,13}\) with weight 1, 2 and 3.

In [24], the authors determined the availability of \(q\)-ary Simplex code and derived more LRCs using anticode strategy.

Next, we will focus on the parameters \(\delta\) of \(q\)-ary Simplex code with dimension 3, then deduce the \((r, \delta)\)-locality of \(q\)-ary Simplex code. Further, we deduce a class of optimal \((r, \delta)\) LRCs, whose length \(n\) is field-size.
Lemma 2: Let $S_3$ be $q$-ary Simplex code with dimension 3, then $S_3$ have $(2, q)$-locality and are optimal.

Proof: From Ref. [24], we know $q$-ary Simplex code with dimension 3 has locality $r = 2$.

For parameter $\delta$, we use the expression

$$G = [I_3 | A_0 \cdots A_{q-2} | B_0 \cdots B_{q-2}].$$

For convenience, denote $I_3 = (e_1, e_2, e_3)$. From $r = 2$, let $X_i = (\{i\} j \in [3])$, where $1 \leq i \leq 3$. It’s not difficult to determine every $X_i$ is a 2-dimensional subspace of $\mathbb{F}_q^3$. Considering $X_1$, this subspace has $(q + 1)$ vectors as follows

$$T = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & x_{q-2}
\end{pmatrix}.$$

From the discussion above, we know the rank of matrix $T$ is 2 and because any two columns in $T$ are distinct, hence any two columns also have the same rank. From the equivalent condition of definition 1, we have constructed

$$S_j = \{s|\alpha_i = \beta_s, 1 \leq i \leq q + 1, 1 \leq s \leq q^2 + q + 1\},$$

where $\alpha_i$ and $\beta_s$ are the columns of $T$ and $G$, respectively. Obviously, $|S_j| = r + \delta - 1 = q + 1$. Hence $\delta = q$ means any $q - 1$ columns in $T$ can be recovered by left two columns. For the vectors in $T$, they have $(2, q)$-locality. With a similar discussion on other columns with weight less than 3, they still have the same locality.

For the columns in $G$ with weight 3, we choose a vector with weight 3 in the following form

$$\alpha = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

then all column vectors in subspace $(\alpha, e_3)$ are

$$\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
0 & x & x & \cdots & x \\
1 & x_0 & x_1 & \cdots & x_{q-1}
\end{pmatrix}.$$

Hence for the columns with weight 3 above, they have $(2, q)$-locality. With a similar discussion for the other columns with weight 3 in $G$, we obtain the same result. Thus $q$-ary Simplex code with dimension 3 have $(2, q)$-locality.

For the optimality of $q$-ary Simplex code with dimension 3, we need to consider $n - k = (\lceil \frac{q}{2} \rceil - 1)(\delta - 1) + 1 = q^2 + q + 1 - 3 - (\lceil \frac{q}{2} \rceil - 1)(q - 1) + 1 = q^2 = d$, we finish the proof.

B. OPTIMAL LRCs FROM q-ARY SIMPLEX CODE WITH DIMENSION 3

This subsection will discuss optimal LRCs from $q$-ary Simplex code with dimension 3 under the anticode strategy.

Lemma 3: Let $q$ be prime power, $n = 3q$, then there exist an optimal $[3q, 3, 2q - 1]_q$ LRC with $(2, q)$-locality.

Proof: First, we construct $[3q, 3, 2q - 1]_q$ linear code then determine its locality.

Let $S_3$ be $q$-ary Simplex code with dimension 3 with generator matrix $S_3$.

Construct

$$\tilde{G} = (B_0, \cdots, B_{q-2}),$$

there are $(q - 1)^2$ column vectors in $\tilde{G}$. In the anticode generated by $\tilde{G}$, we have

$$\max_{e \in A_{\tilde{G}}} \text{wt}(e) = (q - 1)^2.$$ 

Hence, a linear code generated by $S_3 \tilde{G}$ have parameters

$$[q^2 + q + 1 - (q - 1)^2, 3, q^2 - (q - 1)^2]_q = [3q, 3, 2q - 1]_q.$$

As the discussion of vectors with weight less than 3 in Section III-A, we can prove linear $[3q, 3, 2q - 1]_q$ code have $(2, q)$-locality.

The optimality of the obtained code can be derived from the fact that $n - k = (\lceil \frac{q}{2} \rceil - 1)(\delta - 1) + 1 = 3q - 3 - (\lceil \frac{q}{2} \rceil - 1)(q - 1) + 1 = 2q - 1 = d$.

IV. OPTIMAL LRCs FROM $(q^2 + 1)$-CAP IN PG(3, q)

In this section, a $(q^2 + 1)$-cap in PG(3, q) will be constructed first and then the parameters of responding cap code is determined. Finally, using this cap code, a class of optimal LRCs will be obtained. First, a proper construction of a $(q^2 + 1)$-cap is needed.

A. CONSTRUCTION OF $(q^2 + 1)$-CAP

The points $(y_1, y_2, y_3, y_4)^T$ in PG(3, q) satisfying $f(y_1, y_2) + y_3y_4 = 0$ will form a $(q^2 + 1)$-cap [25]. Obviously, the points $(1, y_2, y_3, h(y_2, y_3))^T$ satisfies the equation $f(1, y_2) + y_3y_4 = 0$. According to the equation $f(y_1, y_2) + y_3y_4 = 0$ and different $y_1, y_2$, we will discuss the construction of these points in four cases.

(I) If $(y_1, y_2) = (0, 0)$, the points in PG(3, q) satisfying $f(y_1, y_2) + y_3y_4 = 0$ are

$$E_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
10 & 1 & \cdots & 1
\end{pmatrix} = (e_3|e_4);$$

(II) If $(y_1, y_2) = (0, 1)$, the points in PG(3, q) satisfying $f(0, 1) + y_3y_4 = 0$ are

$$E_2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
x_0 & x_1 & \cdots & x_{q-2} \\
-x_0 & -x_1 & \cdots & -x_{q-2}
\end{pmatrix};$$

(III) If $(y_1, y_2) = (1, 0)$, the points in PG(3, q) satisfying $f(1, 0) + y_3y_4 = 0$ are

$$E_3 = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
x_0 & x_1 & \cdots & x_{q-2} \\
-x_0 & -x_1 & \cdots & -x_{q-2}
\end{pmatrix};$$
TABLE 1. Comparisons of our results and the previously known ones.

| Comment | n and q | Method                  | \((d, r, \delta)\) |
|---------|--------|-------------------------|-------------------|
| Ref. [3] | \(n \leq q\) | polynomials               | \((-r, r, \delta)\) |
| Ref. [6] | \(n < q\)  | polynomials               | \((-r, n, \delta)\) |
| Ref. [8] | \(n \leq q + 2\sqrt{q}\) | elliptic curves           | \((n - 3r, 2, \delta)\); \((n - (t - 1)(r + 1), r, \delta)\) for \(r = 3, 5, 7, 11, 23\) over different fields |
| Ref. [7] | unbounded length | cyclic code            | \((\delta \in \mathbb{Z}, 2)\) |
| Ref. [18] | unbounded length | cyclic code            | \((\delta \in \mathbb{Z}, 2, r, \delta)\) |
| Ref. [17] | \(n(q + 1)\) or \(n(q - 1)\) | cyclic code            | See Table III in [17] |
| Ref. [20] | \(n(q + 1)\) | constacyclic code        | See Table II in [20] |
| our results | \(q^2 + q + 1\) | Simplex code            | \((q^2, 2, q)\) |
| our results | \(3q\)  | anti-code               | \((2q - 1, 2, q)\) |
| our results | \(q^2 + 1\) | cap code                | \((q(q - 1), 3, q - 1)\) |
| our results | \(4q - 4\) | anti-code               | \((3q - 5, 3, q - 1)\) |

(IV) If \(y_1 = 1\) and \(y_2 \neq 0\).

\[
J_i = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
-1 & -1 & \cdots & -1 \\
x_i & x_i & \cdots & x_i \\
x_0 & x_0 & \cdots & x_0 \\
h(x_i, x_0) & h(x_i, x_1) & \cdots & h(x_i, x_{q-2})
\end{pmatrix}
\]

for \(0 \leq i \leq q - 2\), let \(\tilde{J} = [J_0 \cdots J_{q-2}]\), the number of columns in \(\tilde{F}\) is \((q - 1)^2\).

We list the numbers of columns of all matrices constructed above as follows

| \(E_1\) | \(E_2\) | \(E_3\) | \(\tilde{F}\) |
|---------|--------|--------|--------|
| 2       | \((q - 1)\) | \((q - 1)\) | \((q - 1)^2\) |

Hence, the matrix

\[
T = [E_1|E_2|E_3|\tilde{J}]
\]

has \((q^2 + 1)\) columns and forms a \((q^2 + 1)\)-cap.

Example 2: Let \(q = 4\), \(\mathbb{F}_4 = \{0, 1, \omega, \sigma\}\) and \(f(x, y) = x^2 + \omega xy + y^2\) be an irreducible polynomial over \(\mathbb{F}_4\). Denote the points of \(\text{PG}(3, 4)\) by vectors \((x, y, z, t)\). The canonical form for an elliptic quadric in \(\text{PG}(3, 4)\) has equation \(x^2 + \omega xy + y^2 + zt = 0\) [27], [28]. There are 17 points in \(\text{PG}(3, 4)\) which satisfy \(f(x, y) + zt = 0\) and form a 17-cap. For convenience, denote \(\omega\) and \(\sigma\) by 2 and 3, respectively. These 17 points in \(\text{PG}(3, 4)\) are as follows:

\[
G_{4,17} = \begin{pmatrix}
1000 & 0111 & 0111 & 11111 \\
0100 & 1031 & 1022 & 12133 \\
0010 & 3202 & 2301 & 32113 \\
0001 & 1130 & 2310 & 22321
\end{pmatrix}
\]

Hence, these 17 points form an elliptic quadric.

B. OPTIMAL LRCs FROM \((q^2 + 1)\)-CAP

Lemma 4 A \((q^2 + 1)\)-cap determines an optimal \([q^2 + 1, 4, q(q - 1)]_{q^2}\) LRC with \((3, q - 1)\)-locality.

Proof: The code length, dimension and minimum distance can be seen in [23].

For any two columns of \(E_2\), their distance is 2, hence, these two columns and \(e_3, e_4\) are linear dependent. For the columns in \(E_3\) or \(J_i\), any two columns in the same block matrix and \(e_3, e_4\) are also linear dependent. Hence the locality \(r\) of \([q^2 + 1, 4, q(q - 1)]_{q}\) is 3.

For parameter \(\delta\), we consider another aspect of this ovoid code. The \([q^2 + 1, 4, q(q - 1)]_{q}\) cap code is a two-weight code with weight \(q(q - 1)\) and \(2\) \([26]\). Hence for a generator matrix of \([q^2 + 1, 4, q(q - 1)]_{q}\) cap code, there are \((q + 1)\) columns beginning with entry 0. Denote subspace \((\beta, e_3, e_4)\) of \(\mathbb{F}_q^4\) by \(\mathcal{S}\), where \(\beta\) is a column of \(E_2\). We can obtain

\[
|\mathcal{S} \cap \{E_2, e_3, e_4\}| = q + 1.
\]

One aspect, any three columns in this \(q + 1\) columns are linear independent from this \((q^2 + 1)\)-cap. Another aspect, these \(q + 1\) columns have \(rank\) 3. Hence for code symbols which are located in these \((q + 1)\) columns have parameter \(\delta = q + 1 + 1 - r = q - 1\).

With a similar discussion on the columns in block matrix \(E_3\) and \(\tilde{J}\), we can deduce \([q^2 + 1, 4, q(q - 1)]_{q}\) code have \((3, \delta = q - 1)\)-locality.

Considering \(q^2 + 1 - 4 - (\lceil \frac{q}{2} \rceil - 1)(q - 1) + 1 = q^2 - q = d\), we prove the optimality and finish the proof.

Next, we present a class of optimal LRCs through modifying \([q^2 + 1, 4, q(q - 1)]_{q}\) code.

Lemma 5: Let \(q\) be a prime power, there exists an optimal \([4q - 4, 4, 3q - 5]_{q}\) LRC with \((3, q - 1)\)-locality.

Proof: Noticing that in the construction of \((q^2 + 1)\)-cap, \(\tilde{J}\) have \((q - 1)^2\) columns with weight 4. Hence an anticode derived from \(\tilde{J}\) have

\[
\max_{c \in \mathcal{A}_q^4} \text{wt}(c) = (q - 1)^2.
\]

A linear code from \(T^\perp \tilde{J}\) has parameters \([4q - 4, 4, 3q - 5]_{q}\). The \((r, \delta) = (3, q - 1)\) can be obtained from the discussion in Lemma IV-B.

Considering \(4q - 4 - 4 - (\lceil \frac{q}{2} \rceil - 1)(q - 1) + 1 = 3q - 5 = d\), we prove the optimality and finish the proof.
the generator matrices of antcodes in [24] consist of columns weight 2 and more than 2 (Theorem 10, 11 in [24]).

V. CONCLUSION

In this letter, constructions of \((r, \delta)\) optimal LRCs are discussed. After analysis of construction of Simplex code, we determined its locality \((r, \delta) = (2, q)\) and derived a class of optimal \([3q, 3, 2q - 1] \) LRCs with \( (2, q) \). Based on \((q^2 + 1)-\)cap, a \([q^2 + 1, 4, q(q - 1)]_q\) LRC with \( (3, q - 1) \) was constructed and a class of \([4g, 4, 3q - 5]_q\) LRC with \( (3, q - 1) \) was obtained. All the codes attain the Singleton-type bound [2]. The advantage is the length of the obtained codes from Simplex code and cap code is field-size. And for clarifying the difference between our codes and the known results, we make a comparison, see Table 1. Although all the LRCs we constructed are optimal, their code rate are not high enough and code parameters are not flexible. This will lead us to study optimal LRCs with better code rate and flexible parameters in the future.

REFERENCES

[1] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, “On the locality of codeword symbols,” IEEE Trans. Inf. Theory, vol. 58, no. 11, pp. 6925–6934, Nov. 2012.

[2] N. Prakash, G. M. Kamath, V. Lalitha, and P. V. Kumar, “Optimal linear codes with a local-error-correction property,” in Proc. IEEE Int. Symp. Inf. Theory, Jul. 2012, pp. 2776–2780.

[3] I. Tamo and A. Barg, “A family of optimal locally recoverable codes,” IEEE Trans. Inf. Theory, vol. 60, no. 8, pp. 4661–4676, Aug. 2014.

[4] S. Goparaju and R. Calderbank, “Binary cyclic codes that are locally repairable,” in Proc. IEEE Int. Symp. Inf. Theory, Jun. 2014, pp. 676–680.

[5] I. Tamo, A. Barg, S. Goparaju, and R. Calderbank, “Cyclic LRC codes and their subfield subcodes,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2015, pp. 1262–1266.

[6] J. Liu, S. Mesnager, and L. Chen, “New constructions of optimal locally recoverable codes via good polynomials,” IEEE Trans. Inf. Theory, vol. 64, no. 2, pp. 889–899, Feb. 2018.

[7] Y. Luo, C. Xing, and C. Yuan, “Optimal locally repairable codes of distance 3 and 4 via cyclic codes,” IEEE Trans. Inf. Theory, vol. 65, no. 2, pp. 1048–1053, Feb. 2019.

[8] X. Li, L. Ma, and C. Xing, “Optimal locally repairable codes via elliptic curves,” IEEE Trans. Inf. Theory, vol. 65, no. 1, pp. 108–117, Jan. 2019.

[9] P. Tan, Z. Zhou, H. Yan, and U. Parampalli, “Optimal cyclic locally repairable codes via cyclotomic polynomials,” IEEE Commun. Lett., vol. 23, no. 2, pp. 202–205, Feb. 2019.

[10] P. Huang, E. Yaakobi, H. Uchikawa, and P. H. Siegel, “Binary linear locally repairable codes,” IEEE Trans. Inf. Theory, vol. 62, no. 11, pp. 6268–6283, Nov. 2016.

[11] Q. Fu, R. Li, L. Guo, and L. Lv, “Locality of optimal binary codes,” Finite Fields Their Appl., vol. 48, pp. 371–394, Nov. 2017.

[12] M.-Y. Nam and H.-Y. Song, “Binary locally repairable codes with minimum distance at least six based on partial t-spreads,” IEEE Commun. Lett., vol. 21, no. 8, pp. 1683–1686, Aug. 2017.

[13] J. Hao, S.-T. Xia, and B. Chen, “Some results on optimal locally repairable codes,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jul. 2016, pp. 440–444.

[14] J. Hao, S.-T. Xia, and B. Chen, “On optimal ternary locally repairable codes,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2017, pp. 171–175.

[15] W. Song, S. H. Dau, C. Yuen, and T. J. Li, “Optimal locally repairable linear codes,” IEEE J. Sel. Areas Commun., vol. 32, no. 5, pp. 1019–1036, May 2014.

[16] T. Ernvall, T. Westerback, R. Freij-Hollanti, and C. Hollanti, “Constructions and properties of linear locally repairable codes,” IEEE Trans. Inf. Theory, vol. 62, no. 3, pp. 1129–1143, Mar. 2016.

[17] B. Chen, S.-T. Xia, J. Hao, and F.-W. Fu, “Constructions of optimal cyclic \((r, \delta)\) locally repairable codes,” IEEE Trans. Inf. Theory, vol. 64, no. 4, pp. 2499–2511, Apr. 2018.

[18] W. Fang and F.-W. Fu, “Optimal cyclic \((r, \delta)\) locally repairable codes with unbounded length,” in Proc. IEEE Inf. Theory Workshop (ITW), Nov. 2018, pp. 1–5.

[19] B. Chen and J. Huang, “A construction of optimal \((r, \delta)\) -Locally recoverable codes,” IEEE Access, vol. 7, pp. 180349–180353, 2019.

[20] B. Chen, W. Fang, S.-T. Xia, and F.-W. Fu, “Constructions of optimal \((r, \delta)\) locally recoverable codes via constacyclic codes,” IEEE Trans. Commun., vol. 67, no. 8, pp. 5253–5263, Aug. 2019.

[21] M. Tfasman, S. Vladut, and D. Nogin, Algebraic Geometric Codes: Basic Notions, Providence, RI, USA: Amer. Math. Soc., 2007.

[22] P. G. Farrell, “Linear binary anticodes,” Electron. Lett., vol. 6, no. 13, pp. 419–421, Jun. 1970.

[23] J. Bierbrauer, Introduction to Coding Theory, 2nd ed. Boca Raton, FL, USA: CRC Press, 2017.

[24] N. Silberstein and A. Zeh, “Anticode-based locally repairable codes with high availability,” Des., Codes Cryptogr., vol. 86, pp. 419–454, 2018.

[25] R. C. Bose, “Mathematical theory of the symmetrical factorial design” Sankhāyā, vol. 8, no. 2, pp. 107–166, 1947.

[26] R. Calderbank and W. M. Kantor, “The geometry of two-weight codes,” Bull. London Math. Soc., vol. 18, no. 2, pp. 97–122, Mar. 1986.

[27] J. W. P. Hirschfeld, Projective Geometries Over Finite Fields, 2nd ed. London, U.K.: Oxford Univ. Press, p. 474, 1979.

[28] J. W. P. Hirschfeld, Finite Projective Spaces of Three Dimensions. London, U.K.: Oxford Univ. Press, p. 316, 2000.