Sum rules and asymptotic behaviors of neutrino mixing and oscillations in matter

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Abstract. Similar to the case in vacuum, it is straightforward to describe neutrino oscillations in matter with the effective lepton flavor mixing matrix $\tilde{U}$ and neutrino mass-squared differences $\tilde{\Delta}_{ji} \equiv \tilde{m}_{j}^{2} - \tilde{m}_{i}^{2}$ ($i,j = 1,2,3$). By calculating two sets of sum rules of $\tilde{U}$ and $\tilde{\Delta}_{ji}$, we have derived exact expressions of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{U}_{\alpha i} \tilde{U}_{\beta i}^*$(for $\alpha, \beta = e, \mu, \tau$ and $i = 1,2,3$) and discussed the asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{\Delta}_{ji}$ in very dense matter (i.e., in the limit of the matter parameter $A = 2\sqrt{2}G_{F}N_{e}E$ approaching infinity).

In 1978, L. Wolfenstein pointed out that the coherent forward scattering of neutrinos with electrons and nucleons through charged- and neutral-current interactions must be taken into account when considering the neutrino oscillations in matter [1]. This can cause big (resonance) amplification of the effective flavor mixing angles in matter even when their counterparts in vacuum are small [2, 3]. Such matter effects were subsequently demonstrated to play an important role in solving the long-standing solar neutrino problem and in understanding the data of accelerator neutrino oscillation experiments [4]. With the advent of the era of accurate measurements of neutrino oscillation parameters, a lot of efforts have been made to formulate matter effects on lepton flavor mixing and neutrino oscillations recently [5, 6, 7, 8, 9, 10, 11, 12, 13].

With the help of two sets of sum rules of the effective lepton flavor mixing matrix $\tilde{U}$ and effective neutrino mass-squared differences $\tilde{\Delta}_{ji} \equiv \tilde{m}_{j}^{2} - \tilde{m}_{i}^{2}$ in matter, we are going to derive the exact expressions for the nine moduli of $\tilde{U}$ and nine sides of the effective Dirac unitarity triangles in the complex plane. Compared with the previous formulas of this kind [14], our results are independent of the less intuitive terms $\tilde{m}_{j}^{2} - m_{i}^{2}$ with $m_{i}$ and $\tilde{m}_{i}$ denoting respectively for neutrino masses in vacuum and their effective counterparts in matter (for $i,j = 1,2,3$). Furthermore, a comprehensive and analytical analysis of the asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{\Delta}_{ji}$ (for $\alpha = e, \mu, \tau$ and $i,j = 1,2,3$) in very dense matter (i.e., in the limit of the matter parameter $A = 2\sqrt{2}G_{F}N_{e}E$ approaching infinity) is performed, which is at least conceptually interesting [13].

In the standard three-flavor scenario, the Hamiltonian responsible for the propagation of
neutrinos in matter can be expressed as

\[ \mathcal{H}_m = \frac{1}{2E} \left[ U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \]

where \( I \) stands for a \( 3 \times 3 \) identity matrix, \( A = 2EV_{cc} \) and \( B = \tilde{m}_1^2 - m_1^2 - 2E_{\text{nc}} \) with \( V_{cc} = \sqrt{2}G_F N_e \) and \( V_{\text{nc}} = -G_F N_n/\sqrt{2} \) being matter potential terms arising respectively form weak charged- and neutral-current interactions of neutrinos with electrons and neutrons in matter. By taking the trace of \( \mathcal{H}_m \), we get \( B = (\Delta_{21} + \Delta_{31} + A - \Delta_{21} - \Delta_{31})/3 \). According to the analytical expressions of \( \tilde{m}_i^2 \) (for \( i = 1, 2, 3 \)) given in the literature [15, 16], we have

\[
\begin{align*}
\tilde{\Delta}_{21} &= \frac{2}{3} \sqrt{x^2 - 3y} \sqrt{3(1 - z^2)}, \\
\tilde{\Delta}_{31} &= \frac{1}{3} \sqrt{x^2 - 3y} \left[ 3z + \sqrt{3(1 - z^2)} \right], \\
B &= \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z + \sqrt{3(1 - z^2)} \right]
\end{align*}
\]

in the case of normal mass ordering (NMO) with \( m_1 < m_2 < m_3 \), where

\[
\begin{align*}
x &= \Delta_{21} + \Delta_{31} + A, \\
y &= \Delta_{21} \Delta_{31} + A \left( \Delta_{21} \left| U_{e2} \right|^2 + \Delta_{31} \left| U_{e3} \right|^2 \right), \\
z &= \cos \left[ \frac{1}{3} \arccos \left( \frac{2x^3 - 9xy + 27A\Delta_{21}\Delta_{31}|U_{e1}|^2}{2(x^2 - 3y)^3} \right) \right].
\end{align*}
\]

Given Eq. (1), a direct calculation of the nine elements of \( \mathcal{H}_m \) and \( \mathcal{H}_m^2 \) leads to two sets of sum rules. These sum rules, together with the unitarity conditions of \( U \) and \( \tilde{U} \), constitute a full set of linear equations of \( \tilde{U}_{\alpha_1} \tilde{U}_{\beta_1}^\dagger, \tilde{U}_{\alpha_2} \tilde{U}_{\beta_2}^\dagger \) and \( \tilde{U}_{\alpha_3} \tilde{U}_{\beta_3}^\dagger: \)

\[
\begin{align*}
\sum_{i=1}^{3} \tilde{U}_{\alpha_1} \tilde{U}_{\beta_i}^\dagger &= \sum_{i=1}^{3} U_{\alpha_1} U_{\beta_i}^\dagger = \delta_{\alpha\beta}, \\
\sum_{i=1}^{3} \tilde{U}_{\alpha_1} \tilde{U}_{\beta_i}^\dagger \Delta_{i1} &= \sum_{i=1}^{3} U_{\alpha_1} U_{\beta_i}^\dagger \Delta_{i1} + A\delta_{\alpha\beta} \delta_{\epsilon\beta} - B\delta_{\alpha\beta}, \\
\sum_{i=1}^{3} \tilde{U}_{\alpha_1} \tilde{U}_{\beta_i}^\dagger \Delta_{i1} (\Delta_{i1} + 2B) &= \sum_{i=1}^{3} U_{\alpha_1} U_{\beta_i}^\dagger \Delta_{i1} \left[ \Delta_{i1} + A(\delta_{\alpha\beta} + \delta_{\beta\epsilon}) \right] + A^2\delta_{\alpha\beta} \delta_{\epsilon\beta} - B^2\delta_{\alpha\beta},
\end{align*}
\]

where the Greek and Latin subscripts run over \((e, \mu, \tau)\) and \((1, 2, 3)\), respectively. Solving Eq. (4) in the case \( \alpha = \beta \) and \( \alpha \neq \beta \), we can get the exact expressions of the nine moduli \( |\tilde{U}_{\alpha_1}|^2 \) and nine sides of the Dirac unitarity triangles \( \tilde{U}_{\alpha_1} \tilde{U}_{\beta_1}^\dagger \) in terms of \( U_{\alpha_1} U_{\beta_1}^\dagger, \Delta_{ji}, \Delta_{ji} \) and \( A \). Thus we get rid of the less intuitive terms \( \tilde{m}_i^2 - m_i^2 \) in which the effective neutrino masses \( \tilde{m}_i \) do not have a definite physical meaning. Note that the above formulas only apply to the neutrino beam with normal mass ordering. For the inverted mass ordering (IMO) case with \( m_3 < m_1 < m_2 \), we need...
Table 1. The analytical expressions of $\tilde{\Delta}_{ji}$ (for $ji = 21, 31$) and $\tilde{U}$ in the $A \to \infty$ limit.

|       | (NMO, $\nu$)                       | (NMO, $\bar{\nu}$) |
|-------|------------------------------------|---------------------|
| $\tilde{\Delta}_{21}$ | $\Delta_{31}(1 - |U_{e3}|^2) - \Delta_{21}|U_{e1}|^2$ | $A$                |
| $\tilde{\Delta}_{31}$ | $A$                                 | $A$                |

\[
\tilde{U} = \begin{pmatrix}
0 & 0 & 1 \\
\sqrt{1 - |U_{\mu3}|^2} & |U_{\mu3}| & 0 \\
-|U_{\mu3}| & \sqrt{1 - |U_{\mu3}|^2} & 0
\end{pmatrix}
\]

\[
\tilde{U} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{1 - |U_{\mu3}|^2} & |U_{\mu3}| \\
0 & -|U_{\mu3}| & \sqrt{1 - |U_{\mu3}|^2}
\end{pmatrix}
\]

to change $\tilde{\Delta}_{21}$, $\tilde{\Delta}_{31}$ and $B$ in Eq. (2) to their counterparts in this case [13]. When it comes to an antineutrino beam, one should make the replacements $U \to U^*$, $V_{cc} \to -V_{cc}$ ($A \to -A$) and $V_{nc} \to -V_{nc}$.

Now let us discuss the the asymptotic behaviors of $|\tilde{U}_{ai}|^2$ in the $A \to \infty$ limit using their exact formulas derived above. In Table 1, we show the analytical expressions of $\tilde{\Delta}_{ji}$ (for $ji = 21, 31$) and $\tilde{U}$ in this extreme case, where a neutrino beam with the normal mass ordering (NMO, $\nu$) and an antineutrino beam with the normal mass ordering (NMO, $\bar{\nu}$) are considered, respectively. We find that only one degree of freedom is needed to describe $\tilde{U}$ in the $A \to \infty$ limit. Considering the standard parametrization of $\tilde{U}$ in terms of three effective mixing angles ($\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}$) and the effective Dirac CP phase ($\tilde{\delta}$), it is always possible to remove $\tilde{\delta}$ from $\tilde{U}$ if the $A \to \infty$ limit is taken, and then we are left with a trivial flavor mixing angle (e.g., $\tilde{\theta}_{13} = \pi/2$ or 0) and a nontrivial flavor mixing angle which is neither $\tilde{\theta}_{12}$ nor $\tilde{\theta}_{23}$. This kind of subtle parameter redundancy was not noticed in the previous papers (see, e.g., Refs. [10, 12]), where specific but misleading values of $\tilde{\delta}, \tilde{\theta}_{12}$ and $\tilde{\theta}_{23}$ have been obtained in the $A \to \infty$ limit. The other two cases — a neutrino beam with the inverted mass ordering (IMO, $\nu$) and an antineutrino beam with the inverted mass ordering (IMO, $\bar{\nu}$) can similarly be discussed [13]. In Fig. 1, we illustrated how each of the nine effective quantities $|\tilde{U}_{ai}|^2$ evolves with the matter parameter $A$ in the two case (NMO, $\nu$) and (NMO, $\bar{\nu}$). One can see that $\tilde{U}$ asymptotically approaches a constant matrix in the $A \to \infty$ limit when taking the value of $A$ larger than $10^{-2}$ eV$^2$. This means that we can use one degree of freedom to approximately describe the flavor mixing effects in this range of $A$.

In summary, we have established some more intuitive formulas for both nine $|\tilde{U}_{ai}|^2$ and nine $\tilde{U}_{ai}\tilde{U}_{betai}^*$, by which we have analytically unravelled the asymptotic behaviors of $|\tilde{U}_{ai}|^2$ and $\tilde{\Delta}_{ji}$ in very dense matter for the first time. We conclude that $\tilde{U}$ contains only a single degree of freedom in the $A \to \infty$ limit. In this extreme case, we get trivial $\tilde{\theta}_{13}$ and no CP violation in neutrino oscillations; and thus it is meaningless to discuss the individual values of $\tilde{\theta}_{12}$ and $\tilde{\theta}_{23}$.

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Figure 1. The evolution of $|\tilde{U}_{\alpha i}|^2$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) with the matter parameter $A$ in the normal neutrino mass ordering case, where the best-fit values of six neutrino oscillation parameters have been input [17].

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