THE SKRYPNIK DEGREE THEORY AND BOUNDARY VALUE PROBLEMS

A.P. KOVALENOK AND P.P. ZABREIKO

Dedicated to the Memory of Prof. Igor V. Skrypnik

Abstract. The paper presents theorems on the calculation of the index of a singular point and at the infinity of monotone type mappings. These theorems cover basic cases when the principal linear part of a mapping is degenerate. Applications of these theorems to proving solvability and nontrivial solvability of the Dirichlet problem for ordinary and partial differential equations are considered.

Introduction

Boundary value problems (BVP) for ordinary and partial differential equations constitute one of the most natural branches where the homotopic theory of monotone type mappings is successfully applied. It seems to be F. Browder [2] who suggested a natural construction of reducing a boundary value problem to some operator equation with a monotone type mapping. The homotopic theory of monotone type mappings was suggested by I.V. Skrypnik [11] (see also [12]) and developed by many authors (see for ex. [3, 4]). It appeared to be a natural generalization of the classical Brouwer and Hopf theory of finite dimensional mappings (see for ex. [9]) and found its numerous application in qualitative studies of BVP (see for ex. [12]).

The Krasnosel’skii technique of investigating BVP based on calculation of topological characteristics of Leray and Schauder type mappings at zero points and infinity is well known (see for ex. [8, 9]). But the class of problems where monotone type mappings are applied is much more broader than that leading to Leray and Schauder mappings. Moreover, the scheme of reducing BVP to an operator equation with a monotone type mapping is absolutely natural and does not require any extra constructions as such a scheme, concerning Leray and Schauder maps, does.

It is the aim of the paper to develop a technique based on calculating topological characteristics at zero and infinity of monotone type mappings for proving the solvability and existence of nontrivial solution of BVP. In this connection, in part 1 of the paper, we present theorems for calculation of the index of zero and infinity of quasimonotone mappings in Hilbert spaces. In part 2 we consider applications of theorems stated in part 1 to obtaining conditions for solvability and existence of nontrivial solutions of the Dirichlet problem for the 2nd order differential equations.

1. Topological characteristics of quasimonotone mappings

1.1. Preliminaries. Let $X$ be a Hilbert space, $X^*$ be its dual and let $\langle l, x \rangle$ denote the pairing between the elements $x \in X$ and $l \in X^*$.

A mapping $\Phi$ defined on a set $D \subseteq X$ with values in the space $X^*$ is said to be demicontinuous if it maps each sequence $x_n \in D$ converging to $x_0$ into a sequence $\Phi x_n$ weakly converging to $\Phi x_0$ ($\Phi x_n \rightharpoonup \Phi x_0$).

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quasimonotone if for each sequence $x_n \in D$ with $x_n \xrightarrow{w} x_0$ the inequality
\[
\limsup_{n \to \infty} \langle \Phi x_n, x_n - x_0 \rangle \geq 0
\]
holds;

of class $(S_+)$ or to have $(S_+)$ property if for each sequence $x_n \in D$ with $x_n \xrightarrow{w} x_0$ the inequality
\[
\limsup_{n \to \infty} \langle \Phi x_n, x_n - x_0 \rangle \leq 0
\]
implies that $x_n \to x_0$.

The class of mappings with $(S_+)$ property defined on a weakly closed set $D$ involves strictly monotone mappings, i.e. such that
\[
\langle \Phi x - \Phi y, x - y \rangle \geq m||x - y||_X \quad (x, y \in D)
\]
for some $m > 0$. Quasimonotone perturbations of $(S_+)$ mappings have $(S_+)$ property as well. Furthermore, any compact mapping is quasimonotone. So the class of mappings having $(S_+)$ property and defined on a weakly closed set $D$ contains compact perturbations of strictly monotone mapping.

For any triple $(\Phi, D, y)$, where $y \in X^*$ and $\Phi$ is a demicontinuous quasimonotone mapping defined on the closure $\overline{D}$ of the open bounded set $D \subseteq X$ and satisfying the condition
\[
\inf_{x \in \partial D} ||\Phi x - y|| > 0,
\]
one associates an integer topological characteristic — the Skrypnik degree $\text{Deg} (\Phi, D, y)$ of the mapping $\Phi$ of the set $D$ with respect to the point $y$. As well as the classical Brouwer and Hopf and Leray and Schauder degrees the Skrypnik degree is determined by its properties: normality with respect to $J^{-1}$ (we let $J$ denote a Riesz isomorphism between $X$ and $X^*$), additivity under domain and homotopy invariance (see for ex. [12]). Moreover, in Hilbert spaces the Skrypnik degree theory generalizes the classical Leray and Schauder degree theory (see for ex. [4]).

A point $x_0$ is a zero of a mapping $\Phi$ if $\Phi x_0 = 0$. A zero $x_0$ is called isolated if there exists a ball $B_{r_0}(x_0)$ which does not contain any other zeroes of the mapping $\Phi$. A zero $x_0$ is called strictly isolated if there exists $r_0 > 0$ such that $\Phi x_n \to 0$ and $||x_n - x_0|| \leq r_0$ imply $x_n \to x_0$.

Let $x_0$ be a strictly isolated zero of a demicontinuous quasimonotone mapping $\Phi$ defined in some neighbourhood of $x_0$. Then the degree $\text{Deg} (\Phi, B_r(x_0), 0)$ is the same for all balls $B_r(x_0)$ with the center $x_0$ and a sufficiently small radius $r$, this common degree being referred as the index of the zero $x_0$ and denoted by $\text{ind} (x_0, \Phi)$. Let us note that to define the index $\text{ind} (x_0, \Phi)$ for a demicontinuous mapping with $(S_+)$ property it suffices for $x_0$ to be an isolated zero.

If for a mapping $\Phi$ defined on all elements of the space $X$ with large norms there exists $r_0 > 0$ such that $\Phi x_n \to 0$ and $||x_n|| \geq r_0$ imply $x_n \to \infty$ then $\Phi$ is said to be strictly nondegenerate at infinity. For any demicontinuous and quasimonotone mapping $\Phi$ which is strictly nondegenerate at infinity the degree $\text{Deg} (\Phi, B_r(x_0), 0)$ is the same for all balls $B_r(x_0)$ with an arbitrary center $x_0$ and a sufficiently large radius $r \geq r_0$, this common degree being referred as the index at infinity or asymptotical index of the mapping $\Phi$ and denoted by $\text{ind} (\infty, \Phi)$. It is clear that to define the index at infinity for a demicontinuous mapping with $(S_+)$ property it suffices for the set of zeroes of this mapping to be bounded.

Formulas for calculation of the index play the very important part when studying operator equations. For mappings of monotone type the Kronecker theorem holds (see for ex. [12]). The corollary of this theorem is the fact that (under corresponding assumptions) the asymptotical index of a mapping equals the sum of zeroes indexes of this mapping. So index formulas make one able to prove the solvability and existence of nontrivial solutions, bifurcation points etc.
1.2. Index of a zero. Let $\Phi$ be a demicontinuous and quasimonotone mapping defined in a neighbourhood of its zero point $x_0$. We assume $\Phi$ to be Fréchet differentiable at $x_0$, i.e.

$$\Phi(x_0 + h) = \Phi'(x_0)h + \omega(h), \quad \lim_{h \to 0} \frac{\omega(h)}{||h||_{X^*}} = 0,$$

with the Fréchet derivative $\Phi'(x_0)$ being quasimonotone and such that the mapping $J\Phi'(x_0)$ proves to be a Fredholm operator.

The space $X$ is the direct sum of the subspaces $X_1$ and $X^1$ invariant for $J\Phi'(x_0)$ where $X_1$ is the finite dimensional root subspace of the operator $J\Phi'(x_0)$ and $X^1$ is complement to $X_1$. Let $X_0 = \ker \Phi'(x_0)$ and $X^0$ be complement to $X^0$. It is evident that $X_0 \subseteq X_1$ and without loss of generality we can suppose $X^1 \subseteq X^0$. Let $P_0, P^0 = I - P_0$ and $P_1, P^1 = I - P_1$ denote projectors onto $X_0, X^0$ and $X_1, X^1$ respectively. At last, note that there exists a nondegenerate linear operator $T$ which acts in $X_1$ and such that $TJ\Phi'(x_0)P_1 = P^0P_1$ (see for ex. [13]).

Now we are in position to formulate the following

**Theorem 1.1.** [7,14,15] Let the operator $\omega(h)$ have a positively homogeneous of order $l > 1$ principal term at infinity, i.e.

$$\omega(h) = C_m(h) + \omega_m(h),$$

with $C_m$ being a positively homogeneous of order $l > 1$ continuous mapping and $||\omega_m(h)||_{X^*} = o(||h||_{X^*})$ when $||h||_X \to 0$.

If $0$ is the unique zero point of the finite dimensional mapping $\Theta_m u = P_0TP_1JC_m u$ ($u \in X_0$) then $x_0$ is a strictly isolated zero of the mapping $\Phi$ and

$$\text{ind} (x_0, \Phi) = (-1)^{n_0 - l} \cdot \text{ind} (\Phi'(x_0); X^1) \cdot \text{ind} (0, \Theta_m; X_0),$$

where $n_0 = \dim X_1$ and $l$ is the number of Jordan blocks in the Jordan form of $J\Phi'(x_0)$ in the space $X_1$.

If, besides, $J\Phi'(x_0)$ has only isolated Fredholm points of the spectrum on $(-\infty, 0)$ then

$$\text{ind} (x_0, \Phi) = (-1)^{\nu(J\Phi'(x_0)) + n_0 - l} \cdot \text{ind} (0, \Theta_m; X_0),$$

where $\nu(J\Phi'(x_0))$ is the sum of multiplicities of negative eigenvalues of $J\Phi'(x_0)$.

It should be noted that in papers [7,14,15] cited above more general cases were considered.

1.3. Asymptotical index. Let $\Phi$ be a demicontinuous and quasimonotone mapping defined on all elements of $X$ with sufficiently large norms. We assume $\Phi$ to be asymptotically differentiable (differentiable at infinity), i.e.

$$\Phi h = \Phi'(\infty)h + \omega(h), \quad \lim_{h \to +\infty} \frac{\omega(h)}{||h||_{X^*}} = 0,$$

with the asymptotical derivative $\Phi'(\infty)$ being quasimonotone and such that the mapping $J\Phi'(\infty)$ proves to be a Fredholm operator.

Similarly to the case of zero point the space $X$ is the direct sum of the subspaces $X_1$ and $X^1$ where invariant for $J\Phi'(\infty)$ the $X_1$ is the finite dimensional root subspace of the operator $J\Phi'(\infty)$ and $X^1$ is complement to $X_1$. Furthermore, let $X_0 = \ker \Phi'(x_0)$ and $X^0$ be complement to $X^0$ such that $X^1 \subseteq X^0$, $P_0, P^0 = I - P_0$ and $P_1, P^1 = I - P_1$ denote projectors onto $X_0, X^0$ and $X_1, X^1$ respectively, $T$ be nondegenerate operator acting in $X_1$ be such that $TJ\Phi'(\infty)P_1 = P^0P_1$.

**Theorem 1.2.** [7,16] Let the operator $\omega(x)$ have a positively homogeneous of order $l > 1$ principal term at infinity, i.e.

$$\omega(x) = C_k(x) + \omega_k(x),$$

with $C_k$ being a positively homogeneous of order $l > 1$ continuous mapping and $\omega_k$ being a positively homogeneous of order $l > 1$ continuous mapping.
with $C_k$ being a positively homogeneous of order $0 \leq k < 1$ continuous mapping and $\|\omega_k(x)\|_{X^*} = o(||x||_X^*)$ when $||x||_X \to +\infty$.

If $0$ is the unique zero point of the finite dimensional mapping $\Theta_k u = P_0 T P_1 J C_k u \ (u \in X_0)$ then the mapping $\Phi$ is strictly nondegenerate at infinity and

$$\text{ind} (\infty, \Phi) = (-1)^{n_0 - l} \cdot \text{ind} (\Phi'(\infty); X^1) \cdot \text{ind} (0, \Theta_k; X_0),$$

when $n_0 = \dim X_1$ and $l$ is the number of Jordan blocks in the Jordan form of $J \Phi'(\infty)$ in the space $X_1$.

If, besides, $J \Phi'(\infty)$ has only isolated Fredholm points of the spectrum on $(-\infty, 0)$ then

$$\text{ind} (\infty, \Phi) = (-1)^{\nu(J \Phi'(\infty)) + n_0 - l} \cdot \text{ind} (0, \Theta_k; X_0),$$

where $\nu(J \Phi'(\infty))$ is the sum of multiplicities of negative eigenvalues of $J \Phi'(\infty)$.

## 2. Applications to Boundary Value Problems

Here we consider the Dirichlet problem (in the weak sense) of the form

\begin{equation}
- \text{div} \ (f(x, \text{grad} u) + q(x, u)) + g(x, u) = 0, \quad u\bigg|_{x \in \partial \Omega} = 0,
\end{equation}

where $f(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \ q(x, t) : \Omega \times \mathbb{R} \to \mathbb{R}^n, \ g(x, t) : \Omega \times \mathbb{R} \to \mathbb{R}, \ \Omega \subset \mathbb{R}^n$.

Let us describe in short the scheme (which is quite standard) of our further study. Under natural conditions including restrictions on the growth of vector functions $f(x, \xi)$, $q(x, t)$ and the function $g(x, t)$ the problem (2.1) is equivalent to the operator equation

\begin{equation}
\Phi u = 0
\end{equation}

with the monotone type mapping $\Phi = F + Q + G$ between the Sobolev space $\dot{W}^1_2$ and its dual space $[\dot{W}^1_2]^*$ where operators $F$, $Q$ and $G$ are given by

$$\langle Fu, v \rangle = \int_\Omega \langle f(x, \text{grad} u), \text{grad} v \rangle \, dx,$$

\begin{equation}
\langle Qu, v \rangle = \int_\Omega \langle q(x, u), \text{grad} v \rangle \, dx, \quad \langle Gu, v \rangle = \int_\Omega g(x, u) v \, dx
\end{equation}

when $u, \ v \in \dot{W}^1_2$. For the problem (2.1) to have a solution (nontrivial solution) it is sufficient then, due to the Kronecker theorem (see [12]), to prove that the asymptotical index of $\Phi$ does not equal zero (the index of trivial solution).

We present two types of conditions under which the problem (2.1) is solvable. They differ by the way of calculating the asymptotical index of $\Phi$. The first condition is based on the calculation of the index by the theorem [1.2] and leads to the equation with the sublinear with respect to the variable $t$ function $g(x, t)$. The second condition is the consequence of a priori estimates on $q(x, t)$ and $g(x, t)$ under which $\text{ind} (\infty, \Phi) = 1$ and allows to consider equations with the vector function $q(x, t)$ being asymptotically zero and the function $g(x, t)$ having a one-side superlinear growth with respect to $t$.

Also we use both these ways of calculating the asymptotical index to obtain conditions under which the problem (2.1) has a nontrivial solution. Besides that here we apply the theorem [1.1] and the Skrypnik theorem [12] to calculate the zero index of $\Phi$. The Skrypnik theorem makes one possible to calculate the index when $\Phi$ has just the Gâteaux derivative (nondegenerate in some neighbourhood of the zero and satisfying some other conditions). The theorem [1.1] can be applied
provided \( \Phi \) is the Fréchet differentiable. It is well known that there is no nonlinear superposition operator acting in \( L_2 \) which is Fréchet differentiable. That is why possible applications of the theorem 1.1 are restricted by the class of problems with the linear principal part \( F \). However, corresponding results based on this theorem remain valid for quasilinear equations. So we distinguish for our further discussions the quasilinear problem

\[
- \text{div}(p(x) \text{grad} u + q(x, u)) + g(x, u) = 0, \quad u \bigg|_{x \in \partial \Omega} = 0
\]

where \( p(x) : \mathbb{R}^n \to \mathbb{R}^n \) is a linear with respect to \( \xi \) for any \( x \in \Omega \) operator (matrix), which is the special case of the problem (2.1).

We let \( f \), \( q \) and \( g \) respectively denote superposition operators generated by the vector functions \( f(x, \xi) \), \( q(x, t) \) and the function \( g(x, t) \). Further, such properties of superposition operators as the action between appropriate spaces, differentiability, asymptotical differentiability, and so on are needed. All these properties can be expressed in terms of growth restrictions on \( f(x, \xi) \), \( q(x, t) \), \( g(x, t) \) and their derivatives (see for ex. [1]). These growth restrictions are determined by spaces where the superposition operator acts and, in our case, depend significantly on embeddings of the Sobolev space \( W_2^1 \) into spaces of integrable functions. To provide the maximal growth, i.e. to cover as wide range of problems involved as possible, we assume for the set \( \Omega \) to have such a smooth boundary (see for ex. [10]) that embeddings

\[
W_2^1 \subset L^2_n,
\]

hold. Let us note that embeddings (2.5) are limit and except the case \( n = 1 \) noncompact.

In what follows we use notation \( \mathcal{H}_{r,p} \) \((r \in \mathbb{R}, \ p \in \mathbb{N})\) for the family of functions \( h(x, \eta) \) which are positively homogeneous of order \( r \) with respect to \( \eta \in \mathbb{R}^p \). Furthermore, we let \( \mathcal{H}_{r,p}^{\text{even}}, \mathcal{H}_{r,p}^{\text{odd}} \) denote respectively the set of even and odd with respect to \( \eta \) functions \( h(x, \eta) \) belonging to \( \mathcal{H}_{r,p} \).

2.1. **Reduction.** We consider the problem (2.1) provided that

- \( A_1 \) there exists \( m > 0 \) such that
  \[
  \langle f(x, \xi) - f(x, \eta), \xi - \eta \rangle_{\mathbb{R}^n} \geq m ||\xi - \eta||_{\mathbb{R}^n}^2 \quad (\xi, \ \eta \in \mathbb{R}^n);
  \]

- \( A_2 \) the superposition operator \( f \) acts from \( L^2_n := L^2 \times \ldots \times L^2 \) into \( L^2_n \);

- \( A_3 \) the superposition operator \( q \) acts as the improving one from \( L^1_n \) into \( L^2_n \);

- \( A_4 \) the superposition operator \( g \) acts as the improving one from \( L^1_n \) into \( L^2_n \).

Due to \( A_1 \) — \( A_4 \) the problem (2.1) is equivalent to the operator equation (2.2) with the continuous operator \( \Phi \) having \((S_+)\) property as the sum of the continuous and strongly monotone operator \( F \) and completely continuous operators \( Q \) and \( G \).

In the case of the linear principal part \( F = P \) where

\[
\langle Pu, v \rangle = \int_{\Omega} \langle p(x) \text{grad} u, \text{grad} v \rangle_{\mathbb{R}^n} dx
\]

the assumptions \( A_1 \) and \( A_2 \) can be written in the form
all the elements of the matrix \( p(x) \) belong to \( L_\infty \) and there exists \( m > 0 \) such that
\[
(p(x)\xi, \xi)_{\mathbb{R}^n} \geq m||\xi||_{\mathbb{R}^n}^2 \quad (x \in \Omega, \xi \in \mathbb{R}^n).
\]

So conditions \( A_1' \), \( A_3 \), \( A_4 \) are supposed to be fulfilled when studying the problem \( (2.4) \).

2.2. Solvability.

2.2.1. We first consider the case when the theorem 1.2 is applicable. For the mapping \( \Phi \) to be asymptotically differentiable we suppose that for any \( x \in \Omega \) the vector functions \( f(x, \xi), q(x, t) \) and the function \( g(x, t) \) have asymptotical derivatives \( f'(x, \infty), q'(x, \infty) \) and \( g'(x, \infty) \) which generate asymptotical derivatives \( f'_\infty, q'_\infty \) and \( g'_\infty \) of the superposition operators \( f, q \) and \( g \).

The simplest situation when the problem \( (2.1) \) has a solution is rather evident and takes place if the linear problem
\[
- \text{div} (f'(x, \infty) \text{grad} u + q'(x, \infty) u) + g'(x, \infty) u = 0, \quad u \bigg|_{x \in \partial \Omega} = 0
\]
has no nontrivial solution, which corresponds to the case of nondegenerate derivative \( \Phi'(\infty) \).

If the resonance phenomenon happens one has to consider higher order terms at infinity. The following result is the consequence of the theorem 1.2.

**Theorem 2.1.** [5, 6, 18] Let for some \( 0 < k < 1 \) there exist vector functions \( f^k(x, \xi) \in (H^{odd}_{k,n})_n := \underbrace{H^{odd}_{k,n} \times \ldots \times H^{odd}_{k,n}}_{n \text{ times}}, q^k(x, t) \in (H^{odd}_{k,1})^n \) and a function \( g^k(x, t) \in H^{odd}_{k,1} \) which generate respectively principal terms of order \( k \) of the superposition operators \( f - f'_\infty, q - q'_\infty \) and \( g - g'_\infty \) at infinity.

Then the problem \( (2.1) \) has a solution provided that none of nontrivial solutions of the problem \( (2.6) \) satisfies the problem
\[
- \text{div} (f^k(x, \text{grad} u) + q^k(x, u)) + g^k(x, u) = 0, \quad u \bigg|_{x \in \partial \Omega} = 0.
\]

2.2.2. In our next assertion we drop the demand of the asymptotical differentiability of superposition operators which automatically implies the sublinearity of their generating functions (see for ex. [1]). But the price of this is ”the smallness” of \( q(x, t) \) and ”the hard” estimate for \( g(x, t) \).

**Theorem 2.2.** [17] Let the superposition operator \( q \) be asymptotically zero and for some \( 0 < \delta < \frac{m}{K^2} \), where \( K \) denotes the norm of the operator embedding \( W^1_2 \) into \( L_2 \), the estimate
\[
g(x, t) t \geq -\delta t^2 \quad (x \in \Omega, \ t \in \mathbb{R})
\]
hold.

Then the problem \( (2.1) \) has a solution.

2.3. Existence of nontrivial solutions. In what follows we assume that the zero of the space \( W^1_2 \) is the solution of the problem \( (2.1) \) and \( (2.4) \).

2.3.1. Assume that conditions of the subsection 2.2.1 on the asymptotical differentiability of \( \Phi \) are fulfilled. We consider the case of resonance, when the problem \( (2.6) \) has nontrivial solutions.

The following result is the consequence of the theorem 1.2 and the Skrypnik theorem on the zero index of the Gâteaux differentiable mapping (see [12]).
Theorem 2.3. [5,6,18] Let the following conditions hold

i) the superposition operators $f$, $q$ and $g$ are Gâteaux differentiable on $L^b_0$, $L(n)$ and $L(n)$ respectively, with $f'_k(x,0)$, $q'_k(x,0)$ and $g'_k(x,0)$ being derivatives of the vector functions $f(x,\xi)$, $q(x,t)$ and the function $g(x,t)$ with respect to the second variable at zero.

ii) for some $0 < k < 1$ there exist vector functions $f^k(x,\xi) \in (H^{even}_{k,1})^n$, $q^k(x,t) \in (H^{even}_{k,1})^n$ and a function $g^k(x,t) \in H^{even}_{k,1}$ which generate respectively principal terms of order $k$ of the superposition operators $f - f'_{\infty}$, $q - q'_{\infty}$ and $g - g'_{\infty}$ at infinity.

Then the problem (2.1) has a nontrivial solution provided that none of nontrivial solutions of the problem (2.6) satisfies the problem (2.7).

2.3.2. Assume that conditions of the subsection 2.2.1 concerning the asymptotical differentiability of $q$ and $g$ hold. We also suppose the superposition operators $q$ and $g$ to have respectively derivatives $q'(0)$ and $g'(0)$ at zero, with $q'(x,0)$ and $g'(x,0)$ being derivatives of the vector function $q(x,t)$ and the function $g(x,t)$ with respect to $t$ at zero.

We consider here the most sophisticated "resonance" situation when both the problem

\[
-\text{div}(p(x)\text{grad} \ u + q'(x,\infty)u) + g'(x,\infty)u = 0, \quad u \bigg|_{x \in \partial \Omega} = 0
\]

(2.8)

and the problem

\[
-\text{div}(p(x)\text{grad} \ u + q'(x,0)u) + g'(x,0)u = 0, \quad u \bigg|_{x \in \partial \Omega} = 0
\]

(2.9)

has nontrivial solutions, i.e. derivatives of $\Phi$ at zero and infinity are degenerate.

The following assertion is the consequence of the theorem 1.1 and the theorem 1.2. It combines two cases, when higher order terms of $\Phi$ are odd at infinity and even at zero and vice versa.

Theorem 2.4. [5,6,18] Let the following conditions hold

i) for some $0 < k < 1$ there exists a vector function $q^k(x,t) \in (H^{odd}_{k,1})^n$ $(q^k(x,t) \in (H^{even}_{k,1})^n)$ and a function $g^k(x,t) \in H^{odd}_{k,1} (g^k(x,t) \in H^{even}_{k,1})$ which generate respectively principal terms of order $k$ of the superposition operators $q - q'_{\infty}$ and $g - g'_{\infty}$ at infinity;

ii) for some $l > 1$ ($l < \frac{n}{n-2}$ when $n > 2$) there exists a vector function $q'(x,t) \in (H^{even}_{l,1})^n$ $(q'(x,t) \in (H^{odd}_{l,1})^n)$ and a function $g'(x,t) \in H^{even}_{l,1} (g'(x,t) \in H^{odd}_{l,1})$ which generate respectively principal terms of order $l$ of the superposition operators $q - q'(0)$ and $g - g'(0)$ at zero.

Then the problem (2.4) has a nontrivial solution provided that none of nontrivial solutions of the problem (2.8) satisfies the problem

\[
-\text{div} \ q^k(x,u) + g^k(x,u) = 0, \quad u \bigg|_{x \in \partial \Omega} = 0
\]

and none of nontrivial solutions of the problem (2.9) satisfies the problem

\[
-\text{div} \ q'(x,u) + g'(x,u) = 0, \quad u \bigg|_{x \in \partial \Omega} = 0.
\]

(2.10)

Let us note that in the case when the problem (2.9) (the problem (2.8)) turns out to have no nontrivial solutions we have to consider even higher order terms at infinity (at zero) only. It seems to be quite clear how to formulate the same results as the theorem above for such cases (see [5,6,18]). That is why we omit the precise formulating here.
2.3.3. Similarly to the subsection 2.2.2 we study here the case of the problem (2.9) whose function \( q(x, t) \) can have one side superlinear growth with respect to \( t \). We assume conditions of the theorem 2.2 to be fulfilled. Furthermore, we suppose assumptions of the previous subsection 2.3.2 concerning the differentiability of \( q \) and \( g \) at zero to hold.

We consider the degenerate case, when the linearization at zero (2.9) of the problem (2.4) has nontrivial solutions.

**Theorem 2.5.** [17] Let for some \( l > 1 \) (\( l < \frac{n}{n-2} \) when \( n > 2 \)) there exists a vector function \( q'(x, t) \in (H_{l,1}^{even})^n \) and a function \( g'(x, t) \in H_{l,1}^{even} \) which generate respectively principal terms of order \( l \) of the superposition operators \( q - q'(0) \) and \( g - g'(0) \) at zero.

Then the problem (2.4) has a nontrivial solution provided that no solution of the problem (2.9) satisfies the problem (2.10).

2.3.4. As the reader could notice in this section we studied only situations when the mapping \( \Phi \) had the degenerate derivative at zero or at infinity (or both). We should say that the technique of proving the existence of nontrivial solutions when both derivatives (in the subsections 2.3.1 and 2.3.2) or the derivative at zero (in the subsection 2.3.3) of \( \Phi \) are nondegenerate is quite different. In such cases we do not have to consider higher order terms but to establish the the parity of the sums of negative eigenvalues multiplicities of corresponding linearizations.

So for the existence of a nontrivial solutions of the problem (2.4) under assumptions of the subsection 2.3.2 when both the problem (2.8) and the problem (2.9) has no nontrivial solutions it is sufficient to prove that the sums of negative eigenvalues multiplicities of problems

\[
-\text{div} (p(x) \text{grad} u + q'(x, 0)u) + g'(x, 0)u = -\lambda \text{div} \text{grad} u, \quad u \bigg|_{x \in \partial \Omega} = 0,
\]

\[
-\text{div} (p(x) \text{grad} u + q'(x, \infty)u) + g'(x, \infty)u = -\lambda \text{div} \text{grad} u, \quad u \bigg|_{x \in \partial \Omega} = 0
\]

have different parities.

In a similar manner one can study analogous situations if they appear in the subsections 2.3.1 and 2.3.3 (see [5, 6, 17, 18]).

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Institute of Mathematics of NAS of Belarus, 11 Surganova Str., 220072 Minsk, Belarus