Necessary and sufficient condition for longitudinal magnetoresistance

H. K. Pal and D. L. Maslov
Department of Physics, University of Florida, Gainesville, FL 32611-8440, USA
(Dated: March 16, 2010)

Since the Lorentz force is perpendicular to the magnetic field, it should not affect the motion of a charge along the field. This argument seems to imply absence of longitudinal magnetoresistance (LMR) which is, however, observed in many materials and reproduced by standard semiclassical transport theory applied to particular metals. We derive a necessary and sufficient condition on the shape of the Fermi surface for non-zero LMR. Although an anisotropic spectrum is a pre-requisite for LMR, not all types of anisotropy can give rise to the effect: a spectrum should not be separable in any sense. More precisely, the combination \( k_\rho v_\rho / v_\phi \), where \( k_\rho \) is the radial component of the momentum in a cylindrical system with the \( z \) axis along the magnetic field and \( v_\rho (v_\phi) \) is the radial (tangential) component of the velocity, should depend on the momentum along the field. For some lattice types, this condition is satisfied already at the level of nearest-neighbor hopping; for others, the required non-separability occurs only if next-to-nearest-neighbor hopping is taken into account.

PACS numbers: 72.15.Gd

I. INTRODUCTION

Magnetoresistance, i.e., a change in the resistance due to a magnetic field, can be distinguished into two types depending on the mutual orientation of the current and the magnetic field: transverse (TMR) and longitudinal (LMR). Although a change in the transverse resistance due to a magnetic field is natural because electrons experience Lorentz force in that direction, the very existence of LMR is somewhat surprising, at least at first glance. Indeed, since Lorentz force is perpendicular to the field, one does not expect the motion of electrons along the field to be affected. A weak point of this argument is that it applies, strictly speaking, only to free electrons but not to electrons in metals. Moreover, LMR is absent in a more realistic (yet still incomplete) “damped Bloch electrons model” (DBEM), in which a phenomenological damping term is introduced into the semiclassical equations of motion for an arbitrary electron spectrum. However, we will argue in this paper that the “damped Bloch electrons model” is not equivalent to the Boltzmann equation, which provides the only complete semiclassical description of semiclassical dynamics of electrons in solids in the presence of scattering. Therefore, absence of LMR in DBEM does not imply its absence in reality.

Experimentally, LMR has been observed in many materials. Theoretically, a general solution of the Boltzmann equation in the magnetic field does not exclude LMR calculations performed for particular metals, e.g., copper, do yield finite LMR. However, it is not clear from this general solution which symmetries must be broken, i.e., how anisotropic the electron spectrum should be for LMR to occur. It is probably why LMR is sometimes viewed as a kind of surprise. In addition to anisotropic spectrum, several more special models have been invoked to explain LMR. It was shown, for example, that LMR can arise due to anisotropic scattering including barrier inhomogeneities in superlattices as well as due to a modification of the density of states by the magnetic field in the ultra-quantum regime, when all but the lowest Landau levels are depopulated. Whereas observed LMR in many cases is likely to be caused by these more evolved mechanisms, it is still necessary to explore whether LMR can arise simply due to anisotropy of the Fermi surface (FS) and to formulate a minimal condition for LMR to occur.

Magnetotransport in non-quantizing fields is described by the Boltzmann equation which gives the conductivity tensor. To find magnetoresistance, one inverts this tensor. It is well known that for any isotropic spectrum the magnetic field dependences of the diagonal and off-diagonal conductivities cancel out, so that both TMR and LMR are absent. While TMR can be made finite by either invoking any kind of anisotropy of the Fermi surface or introducing a multiband picture while keeping the spectrum isotropic, the story with LMR is not so simple. As is shown in this paper, not all types of anisotropy give rise to LMR, e.g., deforming a spherical Fermi surface into an ellipsoidal one is not enough. We derive the necessary and sufficient condition the spectrum must satisfy for LMR to occur and discuss the implications of this condition for several types of bandstructure. For example, metals with face-centered cubic (FCC) and body-centered cubic (BCC) lattices satisfy the necessary and sufficient condition even if only nearest-neighbor hopping is taken into account, whereas a simple cubic (SC) lattice has LMR only due to hopping between next-to-nearest neighbors. The same is true for layered structures, such as hexagonal planes stacked on top of each other, where one has to include out-of-plane next-nearest-neighbor interactions to see the effect.

The rest of the paper is organized as follows. In Sec. [II] we show that LMR is absent in DBEM and analyze the differences between this and Boltzmann-equation approach. In Sec. [III] we derive the necessary and sufficient condition for LMR in the Boltzmann-equation formalism and discuss the implications of this condition.
As a particular example, we consider the case of Bernal-stacked graphite in Sec. [V]. In graphite, the necessary and sufficient condition is satisfied due to trigonal warping of the Fermi surface. We find, however, that strong non-parabolic LMR observed in highly oriented pyrolytic graphite (HOPG) samples cannot be accounted for LMR arising simply from anisotropy of the Fermi surface. Our conclusions are given in Sec. [V].

II. SEMICLASSICAL EQUATIONS OF MOTION

The effect of weak electric and magnetic fields on electrons in solids can be described by the semiclassical equations of motion:

\[ \mathbf{v} = \frac{\partial \varepsilon_{\mathbf{k}}}{\partial \mathbf{k}}, \quad (1) \]

\[ \frac{d\mathbf{k}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2) \]

where \( e \) is the electron charge and we set \( \hbar = 1 \). We neglect here the anomalous terms in the velocity which, even if present, are small in weak magnetic fields. In the absence of scattering, Eqs. (1) and (2) are valid for an arbitrary spectrum \( \varepsilon(\mathbf{k}) \) and provide an invaluable tool for analyzing collisionless dynamics of electrons in solids. To account for scattering of electrons by impurities, phonons, etc., it is customary to replace Eqs. (1) and (2) by a phenomenological "damped Bloch electron model" (DBEM) with a damping term \(-\mathbf{k}/\tau\) inserted into the right-hand side of Eq. (2). In steady-state, DBEM reduces to

\[ \frac{\mathbf{k}}{\tau} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (3) \]

We are now going to show that this approach eliminates LMR not only for an isotropic but also for an arbitrary spectrum. To find LMR, we assume that the current \( j = ne\mathbf{v} \), where \( n_e \) is the number density of conduction electrons, is along \( \mathbf{B} \) chosen as the z-axis. Then,

\[ v_z(k_x, k_y, k_z) = \frac{j_z}{n_e e}, \quad (4) \]

\[ v_x(k_x, k_y, k_z) = 0, \quad (5) \]

\[ v_y(k_x, k_y, k_z) = 0. \quad (6) \]

Furthermore, the equation of motion for the \( k_z \) component gives

\[ k_z' = e\tau E_z. \quad (7) \]

The set of four equations (4-7) defines an inhomogeneous system for four unknowns: \( k_x, k_y, k_z, \) and \( E_z \). In general, such a system has a unique solution. Therefore, \( E_z \) can be found as a function of \( j_z \) using only Eqs. (4-7). Since none of these equations involve the magnetic field, the longitudinal resistivity \( \rho_{zz} = j_z/E_z \) does not depend on \( B \) either, which implies that LMR is absent for an arbitrary spectrum. On the other hand, components \( E_x \) and \( E_y \) have to be found from the equations of motion for \( k_x \) and \( k_y \) which do involve \( B \), and hence TMR is not zero for an arbitrary spectrum.

If the above conclusion were correct, it would be in variance with experimental observations. As we will show shortly, non-zero LMR can be understood only by using the Boltzmann equation

\[ \frac{df_{\mathbf{k}}}{dt} = \frac{\partial f_{\mathbf{k}}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{r}} + \mathbf{\dot{k}} \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = I_{c}[f_{\mathbf{k}}], \quad (8) \]

where \( I_{c} \) denotes the collision integral. Although the Boltzmann equation is a semi-classical description just like the previous method, there is some conceptual difference between the two approaches. The problem is that while the equations of motions in the absence of scattering can be derived from the Schroedinger equation, the DBEM does not follow from any microscopic approach. Indeed, the momentum \( \mathbf{k} \) in the absence of scattering still has the meaning of the quantum number parameterizing the Bloch state \( \psi_{\mathbf{k}}(\mathbf{r}) \). Hence a (slow) evolution of \( \mathbf{k} \) with time in the presence of electric and magnetic fields describes the evolution of \( \psi_{\mathbf{k}}(\mathbf{r}) \). In the presence of scattering, e.g., by disorder, \( \psi_{\mathbf{k}}(\mathbf{r}) \) becomes a random quantity whose average over disorder realizations does not have a particular meaning.

Therefore, it is not surprising that an ad hoc insertion of the damping term into the equation of motion does not capture essential physics. The shortcomings of this procedure become obvious even in the absence of the magnetic field. For example, Eq. (8) predicts that \( \mathbf{k} \) is always parallel to \( \mathbf{E} \) if \( \mathbf{B} = 0 \). However, the average momentum per electron \( \langle \mathbf{k} \rangle = \int d^3k f_{\mathbf{k}}/n_e \), where \( n_e \) is the total number density, is not parallel to \( \mathbf{E} \) for a lattice of sufficiently low symmetry. Indeed, solving Eq. (8) in the relaxation-time approximation at zero temperature yields \( \langle \mathbf{k} \rangle = e\tau \int dS_{\mathbf{F}} (\mathbf{v} \cdot \mathbf{E})/k/v_{F} \) (where \( dS_{\mathbf{F}} \) is the element of the Fermi surface and \( v_{F} \) (k) is the magnitude of the electron velocity at a given point on this surface. For a generic Fermi surface, \( \langle \mathbf{k} \rangle \) and \( \mathbf{E} \) are not parallel. Also, the conductivity given by the DBEM as \( \sigma' = e^2 v_{F}^2 \tau \mathbf{v} (\varepsilon_{F})/3 \), where \( \mathbf{v} (\varepsilon) \) is the density of states, coincides with the result of the Boltzmann equation \( \sigma_{\alpha\beta} = e^2 \int dS_{\mathbf{F}} v_{F} v_{\alpha} v_{\beta} \tau / v_{F} \) (k) only for an isotropic spectrum.

III. MINIMAL CONDITIONS FOR LONGITUDINAL MAGNETORESISTANCE

A. Necessary condition

Having dealt with the inconsistencies of the “damped Bloch electrons model”, we now return to the original problem of finding the minimum requirement for non-zero LMR for an arbitrary spectrum \( \varepsilon = \varepsilon(\mathbf{k}) \). In the linear-response regime, one can rewrite Eq. (8) for the non-equilibrium part of the distribution function \( g(\mathbf{k}) =
rewriting $\hat{\Omega}$ in cylindrical coordinates, the condition (13) 

$$\sigma = \int_{\text{Fermi surface}} \text{d}^3k \sum_{n=0}^\infty \left(-\hat{\Omega}_n\right)^n \frac{\partial f}{\partial \varepsilon}$$

where we have also adopted the relaxation-time approximation (which is exact for isotropic impurity scattering). Since we are interested in the minimal condition, we allow $\tau$ to depend only on $\varepsilon$ but not on the direction of $k$ and assume that all components of $k$ relax at the same rate, i.e., that $1/\tau$ is a scalar rather than a tensor. We will come back to this point later in the paper. For $B||\hat{z}$,

$$\hat{\Omega} = \tau (v \times B) \frac{\partial}{\partial \varepsilon} = e\tau B \left( v_y \frac{\partial}{\partial k_x} - v_x \frac{\partial}{\partial k_y} \right).$$

Following the Zener-Jones method, we express $g(k)$ via an infinite series in the operator $\hat{\Omega}$:

$$g(k) = (1 + \hat{\Omega})^{-1} \left(-\tau e B \cdot v \frac{\partial f_0}{\partial \varepsilon} \right) = \sum_{n=0}^\infty (-\hat{\Omega})^n \left(-\tau e B \cdot v \frac{\partial f_0}{\partial \varepsilon} \right).$$

Note that the operator $\hat{\Omega}$ always yields zero when it acts on any function that depends only on $\varepsilon_k$ but not on the direction of $k$. Hence, in Eq. (13), $\hat{\Omega}$ acts only on $v$. Substituting Eq. (13) into the current $j = 2e \int d^3k v g(k) / (2\pi)^3$, we find the conductivity as

$$\sigma_{\alpha\beta} = 2e^2 \tau \int \frac{d^3k}{(2\pi)^3} \left( \frac{\partial f_0}{\partial \varepsilon} \right) v_\alpha \sum_{n=0}^\infty (-\hat{\Omega})^n v_\beta.$$  

(12)

In the LMR geometry, $E||B||\hat{z}$. If $Bv_z = 0$, all but the $n = 0$ term in Eq. (12) are equal to zero. Therefore, a necessary condition for $\sigma_{zz}$ to depend on the magnetic field is

$$\hat{\Omega}v_z \neq 0.$$  

(13)

Rewriting $\hat{\Omega}$ in cylindrical coordinates, the condition (13) can be re-expressed as:

$$\left( \frac{\partial \varepsilon}{\partial \phi} \frac{\partial}{\partial k_\rho} - \frac{\partial \varepsilon}{\partial k_\rho} \frac{\partial}{\partial \phi} \right) v_z \neq 0,$$

(14)

or

$$\frac{\partial}{\partial k_z} \left( \frac{\partial \varepsilon}{\partial \phi} \right) \neq 0.$$  

(15)

On the other hand, Eq. (15) is not a sufficient condition because even if $\hat{\Omega}^n v_z \neq 0$ for the $n^{th}$ term in the series, the contribution of this term to $\sigma_{zz}$ may vanish upon integrating over the Fermi surface. For example, since $\sigma_{zz}$ must be an even function of $B$, all odd terms in the series must vanish.

Equation (15) implies that the minimum condition on the spectrum is that the ratio of $\partial \varepsilon / \partial \phi$ and $\partial \varepsilon / \partial k_\rho$ (equal to $k_\rho v_\phi / v_\rho$) must depend on $k_z$. Geometrically, this means that the angle between the component of velocity perpendicular to the field and the radial direction at a given point on the Fermi surface must vary with $k_z$. It can be easily seen that if the spectrum does not depend on $\phi$, condition (15) is trivially violated and there is no LMR. Therefore, angular anisotropy of the FS about the magnetic-field direction is a prerequisite. However, anisotropy must be of a special kind. For example, spectra such as $\varepsilon_k = \varepsilon_1(k_\rho, \phi) + \varepsilon_2(k_\rho)$ and $\varepsilon_k = \varepsilon_1(k_\rho, \phi)\varepsilon_2(k_z)$, which are arbitrarily anisotropic in the $\phi$ direction but separable in $k_z$, violate condition (15) and thus do not lead to LMR. As an example, let us consider a SC lattice with lattice parameter $a$. In the tight-binding model with nearest-neighbor hopping (parameterized by coupling $t_1$), the energy spectrum is given by $\varepsilon_k = -2t_1 [\cos(k_x a) + \cos(k_y a) + \cos(k_z a)]$, which, being separable in all three coordinates, clearly violates the LMR condition. If next-to-nearest-neighbor hopping (parameterized by coupling $t_2$) is taken into account, additional terms $-4t_2 [\cos(k_x a/2)\cos(k_y a/2) + \cos(k_x a/2)\cos(k_z a/2) + \cos(k_y a/2)\cos(k_z a/2)]$ occur in the spectrum, which no more violates the LMR condition. Thus, the effect comes only from next-to-nearest-neighbor hopping for an SC lattice. On the other hand, an FCC lattice satisfies the condition already at the nearest-neighbor level because the spectrum in this case $\varepsilon_k = -4t_1 [\cos(k_x a/2)\cos(k_y a/2) + \cos(k_x a/2)\cos(k_z a/2) + \cos(k_y a/2)\cos(k_z a/2)]$ is non-separable; the same is true for a BCC lattice. On the other hand, layered, e.g., hexagonal, structures, will require coupling between an atom located in one plane and another atom in the adjacent plane but situated obliquely from the former, if the magnetic field is perpendicular to the planes (more on this later for the specific case of graphite).

A quantity measured in a typical experiment is not the conductivity but the resistivity. Generally speak-
ing, the dependence of the conductivity on the magnetic field does not automatically imply a dependence of the resistivity on the field – a well known case is the isotropic spectrum, when the (transverse) diagonal components of the conductivity depend on \( B \) but the diagonal components of the resistivity do not. It is necessary, therefore, to make sure that Eq. (15) is not only a necessary condition for longitudinal magnetocconductance but also for magnetoresistance. It is difficult to prove that non-zero magnetocconductance implies non-zero LMR for an arbitrary spectrum. To proceed further, we relax a condition on the energy spectrum, assuming that \( B \) is perpendicular to the plane of symmetry, i.e., that \( \varepsilon(k_x,k_y,k_z) = \varepsilon(k_x,k_y,-k_z) \). This constraint is stronger than that imposed by time reversal symmetry (in the absence of the spin-orbit interaction and magnetic structure), i.e., \( \varepsilon(k_x,k_y,k_z) = \varepsilon(k_x,k_y,k_z) \). In this case, \( v_x \) is odd while \( v_z \) and \( v_y \) are even in \( k_z \), and the off-diagonal components \( \sigma_{xz} \) (\( \alpha \neq z \)) vanish both in zero and finite magnetic fields. For example, all terms in the expression for \( \sigma_{xz} \) vanish upon integration over \( k_z \):

\[
\sigma_{xz} = 2e^2\tau \sum_{n=0}^{\infty} (-e\tau B)^n \times \int \frac{d^3k}{(2\pi)^3} v_x \left( v_y \frac{\partial}{\partial k_x} - v_z \frac{\partial}{\partial k_y} \right) v_z \frac{\partial f^0}{\partial k} = 0, \tag{16}
\]

By the Onsager principle, \( \sigma_{yx} = 0 \). Therefore, the matrix of \( \sigma_{\alpha\beta} \) is block-diagonal and \( \rho_{zz} = 1/\sigma_{zz} \). Thus Eq. (15) is a necessary condition for non-zero LMR as well, provided that the spectrum is symmetric on inversion of \( k_z \).

### B. Sufficient condition

The condition presented in Eq. (15) is only a necessary condition for LMR, as the integral in Eq. (12) may still vanish due to some symmetry even if the integrand satisfies Eq. (16). To formulate a sufficient condition, we approach the problem from the strong-magnetic-field limit. In this limit, it is convenient to use the method of Lifshitz, Azbel’ and Kaganov, in which the \( k \)-space is mapped onto a space defined by the set of variables \( \varepsilon \equiv \varepsilon_k, k_z \) and \( t_1 \), where \( t_1 \), defined by the equation

\[
\frac{dk}{dt_1} = ev \times B, \tag{17}
\]

is the time spent by an electron on the orbit in the \( k \)-space in the presence of the magnetic field only. Accordingly, the integration measure is transformed as

\[
\int \int dk_z dk_x dk_y = eB \int \int dt_1 d\varepsilon dk_z. \tag{18}
\]

The non-equilibrium correction to the distribution function can be written as

\[
g = e \frac{\partial f^0}{\partial \varepsilon} E \cdot s, \tag{19}
\]

where \( s \) satisfies

\[
\frac{ds}{dt_1} = I_c [s] + v. \tag{20}
\]

Adopting the relaxation-time approximation for \( I_c \) and keeping only the leading term in \( 1/B \), it is easy to see that

\[
s_z = \tau(v_z), \tag{21}
\]

where \( \langle v_z \rangle = \frac{1}{T} \int v_z dt_1 \) with \( T \) being either the period of an orbit (for closed orbits) or the time over which an orbit reaches the boundary of the Brillouin zone (for open orbits). The \( \sigma_{zz} \) component of the conductivity tensor in this limit is then equal to

\[
\sigma_{zz}(\infty) = \frac{2e^2\tau}{(2\pi)^3} B \int \int d\varepsilon dk_z dt_1 \langle v_z \rangle \left( -\frac{\partial f^0}{\partial \varepsilon} \right) = \frac{2e^2\tau}{(2\pi)^3} \int d\ell \int dk_z \int \frac{dz}{v_z} \langle v_z \rangle \left( -\frac{\partial f^0}{\partial \varepsilon} \right), \tag{22}
\]

where \( d\ell \) is a line element along the orbit and \( v_\perp = \sqrt{v_x^2 + v_y^2} \). Obviously, \( \sigma_{zz}(\infty) \) does not depend on \( B \). On the other hand, the zero-field value of \( \sigma_{zz} \) is

\[
\sigma_{zz}(0) = \frac{2e^2\tau}{(2\pi)^3} \int d^3k v_z^2 \left( -\frac{\partial f^0}{\partial \varepsilon} \right). \tag{23}
\]

Pippard suggested that the ratio \( \sigma_{zz}(\infty)/\sigma_{zz}(0) \) may be used to get information about the scattering mechanisms on the FS. We, however, use this ratio to construct a sufficient condition for LMR. Keeping the same constraint on the energy spectrum \( \varepsilon(k_x,k_y,k_z) = \varepsilon(k_x,k_y,-k_z) \) so that \( \rho_{zz} = 1/\sigma_{zz} \), the sufficient condition for LMR can now be formulated as follows: if \( \sigma_{zz}(\infty) \neq \sigma_{zz}(0) \), we have non-zero LMR. It is only a sufficient condition because, even if it is violated, LMR can still exist. Indeed, even if asymptotic limits of the function \( \sigma_{zz}(B) \) coincide, it is not necessarily a constant. To formulate the sufficient condition in more transparent terms, we use the following trick. The integration measure in the expression (22) for the zero-field conductivity can formally be re-written in terms of variables \( \varepsilon, k_z \) and \( t_1 \), as specified by transformation (13). Since the result does not depend on the magnetic field, this transformation can be applied for any value of the field but, to compare the zero- and strong-field values, we choose the same \( B \) as in the first line of Eq. (22). Then,

\[
\sigma_{zz}(0) = \frac{2e^2\tau}{(2\pi)^3} eB \int \int d\varepsilon dk_z dt_1 v_z^2 \left( -\frac{\partial f^0}{\partial \varepsilon} \right). \tag{24}
\]

Comparing this equation with the first line of Eq. (22), we see that the sufficient condition is equivalent to

\[
\int \int \int d\varepsilon dk_z dt_1 \left( -\frac{\partial f^0}{\partial \varepsilon} \right) (v_z \langle v_z \rangle - v_z^2) \neq 0. \tag{25}
\]
Integrating over $t_1$, we rewrite the last equation as
\[
\int \int d\varepsilon dk_z \left( -\frac{\partial f^0}{\partial \varepsilon} \right) (\langle v_z^2 \rangle - \langle v_z \rangle^2) = \int \int d\varepsilon dk_z \left( -\frac{\partial f^0}{\partial \varepsilon} \right) (\langle v_z^2 \rangle - \langle v_z \rangle^2)^{\frac{1}{2}} \neq 0.
\]

Since the integrand is non-negative, the integral can only vanish if $v_z = \langle v_z \rangle$, which is the case if $v_z$ does not depend on $t_1$. Hence, the sufficient condition is equivalent to the requirement that
\[
\frac{\partial v_z}{\partial t_1} - \frac{\partial v_z}{\partial k_x} \frac{\partial k_x}{\partial t_1} + \frac{\partial v_z}{\partial k_y} \frac{\partial k_y}{\partial t_1} \neq 0. \tag{26}
\]
Recalling that $k$ satisfies Eq. (17), we re-write the last equation as
\[
\left( v_y \frac{\partial}{\partial k_x} - v_z \frac{\partial}{\partial k_y} \right) v_z \neq 0 \tag{27}
\]
or, recalling the definition of the operator $\hat{\Omega}$ in Eq. (10), as
\[
\hat{\Omega} v_z \neq 0. \tag{28}
\]

Since the sufficient condition (28) coincides with the necessary condition in Eq. (13), we conclude that Eq. (15) is both a necessary and sufficient condition for LMR. As a corollary, it also follows that the strong-field value $\sigma_{zz} (\infty)$ is always smaller than or equal to $\sigma_{zz} (0)$, implying that if LMR is finite, it is positive.

Before concluding this section, we would like to comment that our aim was to establish a minimal condition for the appearance of LMR in materials. Specifically, we wanted to explore whether, in the simplest model for scattering, anisotropy of the bandstructure alone can give rise to LMR; the answer turns out to be in the affirmative. It should be pointed out that LMR can also occur due to anisotropic scattering. Indeed, as was shown by Jones and Sondheimer\footnote{A. S. Jones and J. Sondheimer, Proc. Phys. Soc. (London) 65, 589 (1952).}, who chose a special form of the scattering probability to solve the Boltzmann equation exactly, non-zero LMR can occur even for an isotropic spectrum, if the scattering probability is appropriately anisotropic. In general, scattering of Bloch electrons is to be described by a tensor of relaxation times, because different components of momentum relax at different rates. In lieu of a fully microscopic description, we adopt here an heuristic model, in which the relaxation time, being still a scalar, depends on the point in the $k$ space, $\tau = \tau (k)$. It is easy to see that the necessary condition for non-zero LMR in this case is modified to:
\[
\hat{\Omega} (\tau v_z) \neq 0. \tag{29}
\]
That means that even if the spectrum alone violates our previous condition (15), i.e., $\hat{\Omega} v_z = 0$, Eq. (29) may still be satisfied because $\hat{\Omega} \tau$ may be non-zero. If this is the case, LMR is finite as well. On the other hand, an attempt to prove that Eq. (29) is also a sufficient condition in this case fails because of the following reason. With $\tau = \tau (k)$, expressions for the high-field and zero-field longitudinal conductivities are still given by Eqs. (24) and (26), except that now $\tau$ is inside the integrals. Following the same reasoning as before, a sufficient condition for non-zero LMR would be $\sigma_{zz} (B = \infty) \neq \sigma_{zz} (B = 0)$, which now implies that $\int \int (\frac{\partial f^0}{\partial \varepsilon}) (\langle v_z^2 \rangle - \langle v_z \rangle^2) d\varepsilon dk_z \neq 0$. Unlike the previous case, however, the integrand cannot be proven to be a positive function; therefore, a non-zero integrand does not guarantee that the integral is also non-zero. Therefore, the sufficient condition can only be formulated in the integral form, as given above.

IV. EXAMPLE: LONGITUDINAL MAGNETORESISTANCE IN GRAPHITE

As a particular example of a material with significant LMR, we consider the case of graphite, where a huge up-to three orders of magnitude- LMR effect is observed when both the current and magnetic field are along the $c$ axis.\footnote{J. M. Kosterlitz, Phys. Rev. Lett. 20, 1001 (1968).} The crystal crystal structure of graphite consists of Carbon atoms arranged in hexagonal layers stacked on top of each other in the Bernal way (ABABAB...). Each unit cell has 4 C atoms with two inequivalent C atoms in each layer. The resulting Brillouin zone is a hexagonal prism with very thin elongated FSs along the edges of the Brillouin zone extended in the direction perpendicular to the plane of the layers. The energy spectrum of graphite is well described by the Słonczewski Weiss McClure (SWMc) model\footnote{E. Słonczewski, J. Phys. Chem. Solids 1, 24 (1956); J. W. McClure, Phys. Rev. 124, 71 (1961).} which involves 7 parameters $\gamma_0, ..., \gamma_5$, describing different kinds of interactions between lattice points. Here, $\gamma_0$ and $\gamma_1$ denote in- and out-of-plane nearest neighbor interactions, respectively, $\gamma_2, ..., \gamma_5$ describe various next-nearest neighbor interactions, while $\gamma_6$ embodies the difference in the on-site energies of two inequivalent C atoms in each layer. Parameter $\gamma_3$ plays a special role as it breaks rotational symmetry of the FS. Without $\gamma_3$, the FS is cylindrically symmetric about the Brillouin zone edge. Therefore, the LMR condition is clearly violated. However, finite $\gamma_3$ leads to “trigonal warping”, i.e., a three-fold deformation of the FS. An expression for energy spectrum of electrons and holes with $\gamma_3$ included in a perturbative way can be written as\footnote{M. Orgad, Phys. Rev. B 2, 1128 (1970).}
\[
\varepsilon = \varepsilon_3^0 + A \sigma^2 \pm \{B^2 \sigma^4 + 2 B \gamma_3 \Gamma \sigma^3 \cos(3\alpha) + \gamma_3^2 \Gamma^2 \sigma^2 \}^{1/2}, \tag{30}
\]
where $\sigma = \sqrt{\frac{4\pi k_F}{2}}, \Gamma = \cos(k_z a_0)/2$, and $\alpha = \pi/2 + \phi$, with $a_0$ and $c_0$ being the in-plane and out-of-plane lattice constants, respectively. Also in Eq. (30), $\varepsilon_3^0$, $A$ and $B$ are all functions of $k_z$ and contain other interaction parameters. Neglecting all the next-to-nearest neighbor couplings except for $\gamma_3$ in the spectrum, we have $\varepsilon_3^0 = A = 0$ and $B = \gamma_3^0 / \gamma_1 \Gamma$. With this approximation,
find that relative magnetoresistance $\Delta$ nearly match the experiment quantitatively. Namely, we non-zero LMR in the first place, the curve does not although this explains qualitatively why graphite has virtually saturates at large fields. However, we note that the spectrum must satisfy a particular non-separability condition given by Eq. (15). We also show that a phenomenological "damped Bloch electrons" model does not capture essential physics of semiclassical transport in anisotropic materials. In particular, this model predicts that LMR is absent not only for isotropic but also anisotropic transport, which is not consistent either with the predictions of the Boltzmann-equation theory or experiment.

In general, the limiting values of the longitudinal conductivities in the zero- and high-field limits differ only in how the square of the $z$-component of the electron velocity is averaged over the FS. Excluding some pathological situations, these two averages can only differ by a numerical coefficient on the order of unity. Therefore, an LMR effect can, in principle, result from FS anisotropy if its magnitude does not exceed or comparable to 100%. If, on the other hand, the lattice structure is such that LMR is only possible only due nearest-neighbor-hopping, one should expect even smaller values of LMR. In many materials, for example in copper and Sr$_2$RuO$_4$, the observed LMR effect is on the order of 10%, which is well within the anisotropic-MS mechanism. However, gigantic LMR effects, such as the one observed in graphite, require explanations which involve macroscopic inhomogeneities of the sample.

V. CONCLUDING REMARKS

To conclude, we have derived a necessary and sufficient condition that an electronic spectrum should satisfy in order to show non-zero longitudinal magnetoresistance within the semiclassical regime of electron transport. We find that anisotropy is essential for non-zero LMR although this anisotropy is to be a special kind, namely, the predictions of the Boltzmann-equation theory or experiment.

Animally coefficient on the order of unity. Therefore, an LMR effect can, in principle, result from FS anisotropy if its magnitude does not exceed or comparable to 100%. If, on the other hand, the lattice structure is such that LMR is only possible only due nearest-neighbor-hopping, one should expect even smaller values of LMR. In many materials, for example in copper and Sr$_2$RuO$_4$, the observed LMR effect is on the order of 10%, which is well within the anisotropic-MS mechanism. However, gigantic LMR effects, such as the one observed in graphite, require explanations which involve macroscopic inhomogeneities of the sample.

Acknowledgments

We acknowledge stimulating discussions with D. B. Gutman, A. F. Hebard, S. Hill, P. Kumar, E. I. Rashba, C. Stanton, and S. Tongay. This work was supported in part by NSF-DMR-0908029.

1 C. Kittel, Introduction to Solid State Physics, (Wiley, New York, 2004) 8th edition; p. 152.
2 M. Ali Omar, Elementary Solid State Physics, (Wesley, Reading), revised printing; p. 263.
3 M. P. Marder, Condensed Matter Physics, (Wiley, New York, 2000), corrected printing; p. 437.
4 A. B. Pippard, Magnetoresistance in Metals (Cambridge University Press, Cambridge, 1989).
5 A. B. Pippard, Proc. Roy. Soc. (London) A 282, 1391 (1964).
6 E. M. Lifshitz and L. P. Pitaevskii, Course of Theoretical Physics: Physical Kinetics, Vol. 10, (Butterworth-Heinemann, Oxford, 1981).
7 N. E. Hussey, A. P. Mackenzie and J. R. Cooper, Phys. Rev. B 57, 5505 (1998).
8 D. L. Miller and B. Laihtman, Phys. Rev. B 54, 10669
E. H. Sondheimer, Proc. Roy. Soc. (London) A 268, 100 (1962).

D. Stroud and F. P. Pan, Phys. Rev. B 13, 1434 (1976).

P. N. Argyres and E. N. Adams, Phys. Rev. 104, 900 (1956).

N. B. Brandt, S. M. Chudinov, and Ya. G. Ponomarev, *Semimetals: I. Graphite and its compounds*, (North-Holland, Amsterdam, 1988).

N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Brooks/Cole, Australia, 1976).

See G. Sundaram and Q. Niu, Phys. Rev. B 59, 14915 (1999) and references therein.

M. P. Marder, *Condensed Matter Physics*, (Wiley, New York, 2000), corrected printing; p. 428.

J. M. Ziman, *Electrons and Phonons: The Theory of Transport Phenomena in Solids* (Clarendon Press, Oxford, 1967).

I. M. Lifshitz, M. Ya. Azbel’, and M. I. Kaganov, JETP 4, 41 (1957).

A. A. Abrikosov, *Fundamentals of the Theory of Metals*, (Elsevier, Amsterdam, 1988).

M.C. Jones and E. H. Sondheimer, Phys. Rev. 155, 567 (1967).

J. W. McClure, Phys. Rev. 108, 612 (1957).

S. Ono, J. Phys. Soc. Jpn. 40, 498 (1976).

I. L. Spain and J. L. Woollam, Solid State Commun. 9, 1581 (1971).

K. Matsubara, K. Sugihara and T. Tsuzuku, Phys. Rev. B 41, 969 (1990).

D. B. Gutman and D. L. Maslov, Phys. Rev. Lett. 99, 196602 (2007); D. B. Gutman and D. L. Maslov, Phys. Rev. B 77, 035115 (2008).

D. L. Maslov, V. I. Yudson, A. M. Somoza and M. Ortuño, Phys. Rev. Lett. 102, 216601 (2009).