Integrable Hierarchies and Contact Terms in u-plane Integrals of Topologically Twisted Supersymmetric Gauge Theories

Kanehisa Takasaki
Department of Fundamental Sciences, Kyoto University
Yoshida, Sakyo-ku, Kyoto 606, Japan
E-mail: takasaki@yukawa.kyoto-u.ac.jp

Abstract
The $u$-plane integrals of topologically twisted $N = 2$ supersymmetric gauge theories generally contain contact terms of nonlocal topological observables. This paper proposes an interpretation of these contact terms from the point of view of integrable hierarchies and their Whitham deformations. This is inspired by Mariño and Moore’s remark that the blowup formula of the $u$-plane integral contains a piece that can be interpreted as a single-time tau function of an integrable hierarchy. This single-time tau function can be extended to a multi-time version without spoiling the modular invariance of the blowup formula. The multi-time tau function is comprised of a Gaussian factor $e^{Q(t_1,t_2,...)}$ and a theta function. The time variables $t_n$ play the role of physical coupling constants of 2-observables $I_n(B)$ carried by the exceptional divisor $B$. The coefficients $q_{mn}$ of the Gaussian part are identified to be the contact terms of these 2-observables. This identification is further examined in the language of Whitham equations. All relevant quantities are written in the form of derivatives of the prepotential.
1 Introduction

The “u-plane integral” of Moore and Witten [1] gives an exact answer to the contribution of the Coulomb branch in correlation functions of topologically twisted four-dimensional $N = 2$ supersymmetric gauge theories. Moore and Witten considered the case of $SU(2)$ and $SO(3)$ only. Mariño and Moore [2] extended the $u$-plane integral to more general, higher rank gauge groups. Losev et al. studied the $u$-plane [3] integral from a somewhat different point of view (“Gromov-Witten paradigm”).

The correlation functions of those topologically twisted $N = 2$ supersymmetric gauge theories are the Donaldson-Witten invariants of the four-manifold $X$ [4]. Their generating function is given by a path integral of the form

$$Z_{DW}(xS + yP) = \left\langle \exp(xI(S) + yO(P)) \right\rangle$$

(1)

with book-keeping variables $x$ and $y$ coupled to the 2-cycle $S \in H_2(X, \mathbb{Z})$ and the 0-cycle $O \in H_0(X, \mathbb{Z})$. $I(S)$ and $O(P)$ are associated observables (“2-observable” and “0-observable”). The Coulomb branch has a nonvanishing contribution only if $b_2^+ \leq 1$, and this contribution is given by an integral of the following form (“u-plane integral”) over the Coulomb moduli space:

$$Z_u = \int_{\mathcal{M}_{\text{Coulomb}}} d\alpha A(u)^\chi B(u)^\sigma \exp(U + S^2 T(u)) \Psi.$$  

(2)

$\chi$ and $\sigma$ are respectively the Euler characteristic and the signature of $X$; $U$ is a contribution of the 0-observable; $T$ is the “contact term” induced by the 2-observable; $\Psi$ is the photon partition function, which is a lattice sum of the Siegel-Narain type over the tensor product of $H_2(X, \mathbb{Z})$ and the weight lattice of the gauge group $G$. Since the geometry of the Coulomb moduli space is determined by the low energy effective action of Seiberg and Witten (“Seiberg-Witten solution”) [5], this $u$-plane integral gives a complete answer to the physics of the Coulomb branch at least in the topologically twisted theories.

The low energy effective action of the Coulomb branch is generally related to an integrable Hamiltonian system [6]. Our subsequent consideration is mostly focussed on the $SU(N)$ theory without matter hypermultiplet ($N_f = 0$) [7]. In this case, the integrable Hamiltonian system is the $N$-periodic Toda chain. Its spectral curve can be written

$$z^2 - \Lambda^{-N} P(x)z + 1 = 0,$$

(3)
where \( P(x) \) is a polynomial of the form
\[
P(x) = x^N - \sum_{j=2}^{N} u_j x^{N-j}.
\] (4)

This hyperelliptic curve of genus \( g = N - 1 \) now plays the role of the Seiberg-Witten elliptic curves in the \( SU(2) \) theories [3]. The coefficients \( u_j \), which give a Poisson-commuting set of Hamiltonians of the Toda chain, are thereby identified with the Coulomb moduli. \( \Lambda \) is the energy scale of the low energy theory. The low energy effective action is written in terms of a prepotential \( \mathcal{F} = \mathcal{F}(a) \), \( a = (a_1, \cdots, a_{N-1}) \). This prepotential is determined (though as an implicit function) by the functional relation
\[
a_j^D = \frac{\partial \mathcal{F}}{\partial a_j}
\] (5)
of the period integrals
\[
a_j = \oint_{\alpha_j} dS_{SW}, \quad a_j^D = \oint_{\beta_j} dS_{SW}
\] (6)
of the meromorphic differential
\[
dS_{SW} = x d \log z
\] (7)
along a symplectic basis \( \alpha_j, \beta_j \) \((j = 1, \cdots, N - 1)\) of cycles on the Riemann surface of the spectral curve. The differential \( dS_{SW} \) obeys the fundamental relation
\[
\partial_{\omega_j} dS_{SW}|_{z=\text{const}} = d\omega_j.
\] (8)

The left hand side means differentiating \( dS_{SW} \) while keeping \( z \) constant. \( d\omega_j \) \((j = 1, \cdots, N - 1)\) are a basis of holomorphic differentials normalized as
\[
\oint_{\alpha_k} d\omega_j = \delta_{jk}.
\] (9)

This equation connecting \( dS_{SW} \) and \( d\omega_j \) shows a link with the notion of “Whitham equations” for adiabatic deformations of algebro-geometric solutions of integrable systems.

It is accordingly natural to expect a similar relation of the \( u \)-plane integrals to some integrable systems and associated Whitham equations. Mariño and Moore [2] and Losev et al. [3], independently, presented several interesting remarks towards that direction. The purpose of this paper is to examine their remarks in more detail.
2 Blowup formula and contact terms

One of remarks of Mariño and Moore [2] is that the “blowup formula” of the $u$-plane integrals contains a factor that can be interpreted as a special tau function of the Toda chain (or, more precisely, the Toda lattice hierarchy [8]).

The blowup formula is concerned with the blowup $\tilde{X}$ of the four-manifold $X$ at a point $P$. Let $B$ denote the homology class of the exceptional divisor (i.e., the inverse image of the blowup point), and take the homology class $\tilde{S} = S + tB$ with a parameter $t$ on $\tilde{X}$. Furthermore, let the metric of $\tilde{X}$ be such that the Poincaré dual of $B$ is anti-self-dual, i.e., $B_+ = 0$. In the case of $G = SU(N)$ and $N_f = 0$, the blowup formula shows that the effect of blowup is just to replace the $0$-observable factor $e^U$ as

$$e^U \rightarrow e^U \frac{\alpha}{\beta} \det \left( \frac{\partial u_k}{\partial a_j} \right)^{1/2} \Delta^{-1/8} e^{-i^{2T} \Theta_{\gamma,\delta}} \left( \frac{1}{2\pi} tV \mid T \right). \quad (10)$$

Here $\alpha$ and $\beta$ are some numerical constants, $\Delta$ the discriminant of the above Toda spectral curve, and the last factor is the ordinary theta function ($g = N - 1$)

$$\Theta_{\gamma,\delta}(w \mid T) = \sum_{\ell \in \mathbb{Z}^g} \exp \left[ \pi i < \ell + \gamma, T(\ell + \gamma) > +2\pi i < \ell + \gamma, w + \delta > \right] \quad (11)$$

with the half-characteristic $(\gamma, \delta)$ determined by the setup of the $u$-plane integral. The period matrix $T = (T_{jk})_{j,k=1,\cdots,g}$ is defined by

$$\oint_{\beta_k} d\omega_j = T_{jk}. \quad (12)$$

The vector $V = (V_j)_{j=1,\cdots,g}$ is a gradient vector, with respect to $a = (a_1, \cdots, a_g)$, of a function $P = P(a)$:

$$V_j = \frac{\partial P}{\partial a_j}. \quad (13)$$

This function $P$ appears in the definition of the $2$-observable $I(S)$ as

$$I(S) = \text{const.} \int_S G^2 P, \quad (14)$$

where $G$ is an operator that generates a standard solution of the descent equations for observables [1], [2]. The contact term $T$ also depends on this potential function $P$, therefore should be written $T_P$ more precisely.

It is the last two factors in (10) that Mariño and Moore identified to be a tau function of the Toda lattice hierarchy. This interpretation is very suggestive in the following sense.
• The parameter $t$ is interpreted as a time variable in the hierarchy. This is in sharp contrast with the status of the Toda chain in the aforementioned description of the Coulomb moduli space. The role of the Toda chain is simply to supply a $g$-dimensional family of curves along with a special-geometric structure; the dynamics of the Toda chain, as an integrable Hamiltonian system, plays no role. In the $u$-plane integral, meanwhile, the time variable becomes a physical coupling constant of the observable $I(B)$.

• The contact term $T$ and the directional vector $V$ are determined by the potential $\mathcal{P}$. In this sense, $\mathcal{P}$ is a Hamiltonian of the Toda lattice hierarchy. In fact, any polynomial (or holomorphic function) of the Coulomb moduli can be used as $\mathcal{P}$. One can consider a set of commuting flows with a set of Hamiltonians $\mathcal{P}_1, \mathcal{P}_2, \cdots$ and associated time variables $t_1, t_2, \cdots$. These commuting flows generally form a subhierarchy of the standard full Toda lattice hierarchy with two infinite series of time variables $t_\pm (n = 1, 2, \cdots)$ \[^1\]. Of course, the notion of tau function is also meaningful for such a subhierarchy. This strongly suggests that the blowup formula, too, can be extended in that way.

### 3 Insertion of more than one 2-observables

Let us specify the implication of the second point above. The potentials $\mathcal{P}_1, \mathcal{P}_2, \cdots$, determine the 2-observables

$$ I_n(B) = \text{const.} \int_B G^2 \mathcal{P}_n. \quad (15) $$

These 2-observables can be used to deform the correlation functions as

$$ \left\langle \exp\left(I(S) + O(P)\right)\right\rangle \rightarrow \left\langle \exp\left(I(S) + \sum_n t_n I_n(B) + O(P)\right)\right\rangle. \quad (16) $$

(The book-keeping parameters are set to $x = y = 1$.) This will modify the last two factors of (10) as

$$ \exp(-t^2 T) \Theta_{\gamma, \delta}(\frac{1}{2\pi} t V \mid \mathcal{T}) $$

\[^1\]Actually, the Toda lattice hierarchy has another variable $t_0$ — the lattice coordinates. In order to avoid unnecessary complication, we shall not consider it, or just put $t_0 = 0$. 

5
\[ \exp\left( - \sum_{m,n} C(P_m, P_n) t_m t_n \right) \Theta_{\gamma, \delta} \left( \sum_n t_n V_n \mid T \right), \]  

where \( C(P_m, P_n) \) are the two-body contact terms induced by the new 2-observables. Note that since these 2-observables are carried by the same 2-cycle \( B \), higher contact terms do not appear. The negative sign in front of the contact terms originates in the self-intersection number \( B^2 = -1 \) of the exceptional divisor.

Some of these contact terms have been calculated explicitly. The simplest case of \( G = SU(2) \) is due to Moore and Witten [1]; the result is written in terms of elliptic theta functions and Eisenstein series. This result is generalized to \( G = SU(N) \) by Mariño and Moore [2]. Their method is based on the so called RG (renormalization group) equation [9]

\[ \frac{\partial F}{\partial \log \Lambda} = \text{const.} \ u_2 \]  

and the modular transformations

\[ \frac{\partial^2 F}{(\partial \log \Lambda)^2} \rightarrow \frac{\partial^2 F}{(\partial \log \Lambda)^2} - \frac{\partial^2 F}{\partial \log \Lambda \partial a_j} [(C \tau + D)^{-1} C]_{jk} \frac{\partial^2 F}{\partial \log \Lambda \partial a_k}, \]  

\[ \frac{\partial^2 F}{\partial \log \Lambda \partial a_j} \rightarrow [(C \tau + D)^{-1}]_{jk} \frac{\partial^2 F}{\partial \log \Lambda \partial a_k} \]  

under the symplectic transformations of cycles

\[ \beta_j \rightarrow A_{jk} \beta_k + B_{jk} \alpha_k; \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{Z}). \]  

The second quantity above are the components of the directional vector \( V \):

\[ V_j = \text{const.} \ \frac{\partial^2 F}{\partial \log \Lambda \partial a_j}. \]  

Mariño and Moore thus obtained the following formula:

\[ C(u_2, u_2) = \text{const.} \ \frac{\partial^2 F}{(\partial \log \Lambda)^2}. \]  

By the results from the RG equation, this formula can be rewritten

\[ C(u_2, u_2) = \text{const.} \ \frac{2N - N_f}{2N - N_f} \left( 2u_2 - a_j \frac{\partial u_2}{\partial a_j} \right). \]  

(Here the result is presented in a generalized form with \( N_f \) massless hypermultiplets).

Losev et al. [3] derived a more general result:

\[ C(u_2, u_k) = \text{const.} \ \frac{2N - N_f}{2N - N_f} \left( k u_k - a_j \frac{\partial u_k}{\partial a_j} \right). \]
(We have omitted an explicit form of various constants above, which are irrelevant in our subsequent analysis.)

4 Multi-time tau function and contact terms

Let us now proceed to our interpretation of the contact terms $C(\mathcal{P}_m, \mathcal{P}_n)$. As we have noted above, the product of the last two factor in (10) persists to be a tau function of the Toda lattice hierarchy for any choice of the potential $\mathcal{P}$ of the 2-observable $I(B)$. Meanwhile, the notion of the tau function can be readily generalized to the multi-time setting. Therefore, a natural conjecture is that right hand side of (17) will be a multi-time tau function of the Toda lattice hierarchy:

$$
\tau_{\gamma,\delta}(t_1, t_2, \cdots) = \exp \left( - \sum_{m,n} C(\mathcal{P}_m, \mathcal{P}_n) t_m t_n \right) \Theta_{\gamma,\delta}(\sum_n t_n V_n \mid \mathcal{T}).
$$

(25)

More precisely, this tau function should be an algebro-geometric tau function [10] determined by a suitable set of data (the Krichever data) on the Toda spectral curve [11]. (We here consider the case of $N_f = 0$ only. Other cases can be treated in a similar way.) Such a tau function, just like the algebro-geometric tau functions of other integrable hierarchies [12], takes the form

$$
\tau(t_1, t_2, \cdots) = e^{Q(t_1,t_2,\cdots)} \Theta\left(\sum_n t_n V_n + w_0 \mid \mathcal{T}\right),
$$

(26)

where $e^{Q(t_1,t_2,\cdots)}$ is a Gaussian factor (including a linear part),

$$
Q(t_1,t_2,\cdots) = \frac{1}{2} \sum_{m,n} q_{mn} t_m t_n + \sum_n r_n t_n + r_0,
$$

(27)

$\Theta(w \mid \mathcal{T})$ is the ordinary theta function without characteristic, and $w_0$ is a constant vector. As we show later, $q_{mn}$ and $V_n$ are determined by the the algebro-geometric data. $r_n$, $r_0$ and $w_0$ are arbitrary constants. Our conjectural tau function (25) can be rewritten into the above form using the relation

$$
\Theta_{\gamma,\delta}(w \mid \mathcal{T}) = e^{2\pi i \langle \gamma, w + \delta \rangle} \Theta(w + \mathcal{T} \gamma + \delta \mid \mathcal{T})
$$

(28)

between the two theta functions. In particular, the arbitrary constants $r_n$, $r_0$ and $w_0$ are determined by the half-characteristic and the period matrix, and $q_{mn}$ turn out to be
essentially the contact terms:

\[ C(P_m, P_n) = -\frac{1}{2} q_{mn}. \]  

(29)

The above conjecture is supported by a modular transformation property of the tau function \( \tau_{\gamma,\delta}(t_1, t_2, \cdots) \) under symplectic transformations of cycles \((20)\). Namely, as we shall show below, the tau function turns out to possess the same \( t \)-INDEPENDENT modular property as the original \( t \)-dependent factors in \((10)\); this implies that the corrected 0-observable factor \((11)\) persists to be modular invariant after modifying the last two factors as in \((17)\). This is strong evidence (though not a proof) of the correctness of the conjecture.

Let us examine modular transformations of the tau function \( \tau_{\gamma,\delta}(t_1, t_2, \cdots) \). Fortunately, this kind of problems have been already studied in the context of free fermion systems on Riemann surfaces \([13]\). (This also suggests a possible link of four-dimensional supersymmetric gauge theories with two-dimensional conformal field theories.) Remarkably, the modular transformation property of the tau function is quite universal, i.e., does not depend on the detail of the Riemann surface or the algebro-geometric data (even nor the integrable system itself!). In the following, therefore, we consider the tau function \( \tau_{\gamma,\delta}(t_1, t_2, \cdots) \) for an arbitrary compact Riemann surface \( \Sigma \) of genus \( g \).

The fundamental algebro-geometric data in the case of the Toda lattice hierarchy \([10]\) are comprised of two marked points \( P_{\infty}^{\pm} \), local complex coordinates \( z_{\pm} \) at each of those points normalized as \( z_{\pm}(P_{\infty}^{\pm}) = 0 \), and the symplectic basis of cycles \( \alpha_j, \beta_j \) \((j = 1, \cdots, g)\). Furthermore, a set of polynomials \( f_{n}^{\pm}(z_{\pm}^{-1}) \) \((n = 1, 2, \cdots)\) have to be given in order to select the “directions” of the \( t_n \)’s in the total space of the standard time variables \( t_{n}^{\pm} \) \((n = 1, 2, \cdots)\) of the Toda lattice hierarchy \([8]\). Each pair \( f_{n}^{+}(z_{+}^{-1}) \) and \( f_{n}^{-}(z_{-}^{-1}) \) determines the direction of the \( n \)-th time (or, equivalently, a Hamiltonian \( P_n \) in the sense already mentioned).

Given these data, the following meromorphic differentials \( d\Omega_n \) \((n = 1, 2, \cdots)\) are uniquely determined:

- \( d\Omega_n \) is holomorphic everywhere except at \( P_{\infty}^{\pm} \). In a neighborhood of \( P_{\infty}^{\pm} \), respectively,

\[ d\Omega_n = df_{n}^{\pm}(z_{\pm}^{-1}) + \text{holomorphic}. \]  

(30)
• The \( \alpha \)-periods all vanish,

\[
\oint_{\alpha_j} d\Omega_n = 0 \quad (j = 1, \cdots, g).
\] (31)

These meromorphic differentials determine \( q_{mn} \) and \( V_n \) as follows. The components of the vector \( V_n \) are given by

\[
V_{jn} = \frac{1}{2\pi i} \oint_{\beta_j} d\Omega_n.
\] (32)

By Riemann’s bilinear relation, this can be rewritten

\[
V_{jn} = -\frac{1}{2\pi i} \oint_{P_\infty^+} f_n^+(z_+^{-1}) d\omega_j - \frac{1}{2\pi i} \oint_{P_\infty^-} f_n^-(z_-^{-1}) d\omega_j,
\] (33)

where the integrals on the right hand side are a contour integral turning once around \( P_n^\pm \) respectively. \( q_{mn} \) is given by replacing \( d\omega_j \) by \( d\Omega_n \):

\[
q_{mn} = -\frac{1}{2\pi i} \oint_{P_\infty^+} f_n^+(z_+^{-1}) d\Omega_m - \frac{1}{2\pi i} \oint_{P_\infty^-} f_n^-(z_-^{-1}) d\Omega_m.
\] (34)

These somewhat complicated expressions are a linear combination of more familiar expressions of the \( q \)’s and \( V \)’s for the standard time variables \( t_n^\pm \) [10].

The modular transformation properties of these quantities can be derived by straightforward calculations. Under the symplectic transformation of cycles (20), the holomorphic and meromorphic differentials transform as

\[
d\omega_j \rightarrow [(CT + D)^{-1}]_{kj} d\omega_k,
\]
\[
d\Omega_n \rightarrow d\Omega_n - [(CT + D)^{-1}C]_{k\ell} \oint_{\beta_\ell} d\Omega_n \ d\omega_k.
\] (35)

Accordingly, \( V_n \) and \( q_{mn} \) transform as

\[
V_n \rightarrow t(CT + D)^{-1} V_n,
\]
\[
q_{mn} \rightarrow q_{mn} - 2\pi i t V_n (CT + D)^{-1} C V_n.
\] (36)

The final piece of the ring is the following modular transformation formula of theta functions [14]:

\[
\Theta_{\gamma,\delta} \left( t(CT + D)^{-1} w \mid (AT + B)(CT + D)^{-1} \right)
= \epsilon \det(CT + D)^{1/2} \exp(\pi < w, t(CT + D)^{-1} w >) \Theta_{\gamma',\delta'}(w \mid T).
\] (37)
Here $\epsilon$ is an 8th root of unity, $\epsilon^8 = 1$, and $(\gamma', \delta')$ is a new half-characteristic, both determined by the symplectic matrix. Combining this formula with the above modular transformations of $q_{mn}$ and $V_n$ lead to the conclusion that the tau function $\tau_{\gamma, \delta}(t_1, t_2, \cdots)$ has a $t$-independent modular transformation property:

$$\tau_{\gamma, \delta}(t_1, t_2, \cdots) \rightarrow \epsilon \det(C^T + D)^{1/2} \tau_{\gamma', \delta'}(t_1, t_2, \cdots).$$

(38)

Understanding that the half-characteristic is also transformed because of its physical origin, one can thus confirm that the tau function in our conjecture possess a desirable modular transformation property.

5 Perspectives from Whitham equations and prepotential

Let us reconsider the meaning of $q_{mn}$, $V_{jn}$ and $P_n$ in the language of Whitham deformations of the integrable hierarchy. The most suggestive in this respect is the formula (22) of $C(u_2, u_2)$ as a second derivative of the prepotential. The relation between RG equations and Whitham equations [9] tells us that, very roughly speaking, the energy scale $\Lambda$ can be identified with the first Whitham time variable $T_1$. In view of this fact, a natural conjecture is that the contact terms of higher observables can be written

$$C(P_m, P_n) = \text{const.} \frac{\partial^2 F}{\partial T_m \partial T_n}.$$  

(39)

This conjecture indeed turns out to fit into the general framework of Whitham equations proposed in our previous work [10], as we show below.

The Whitham equations in the present situation take the form

$$\partial_{T_n}dS|_{z=\text{const.}} = d\Omega_n, \quad \partial_{a_j}dS|_{z=\text{const.}} = d\omega_j,$$

(40)

where $dS$ is given by

$$dS = \sum_n T_n d\Omega_n + \sum_{j=1}^{N-1} a_j d\omega_j,$$

(41)

and “$|_{z=\text{const.}}$”, also here, means differentiating while keeping $z$ constant. These equations give deformations of the Coulomb moduli $u_j = u_j(a, T)$ as the Whitham time variables
$T = (T_1, T_2, \cdots)$ vary from a point of departure (e.g., a point $T = T_{SW}$ where $dS$ coincides with the meromorphic differential $dS_{SW}$). One can redefine the prepotential $F = F(a, T)$, now as a function of $a$ and $T$, by the equations

$$\frac{\partial F}{\partial T_n} = -\oint_{P^\pm} f_n^\pm(z^{-1})dS - \oint_{P^-} f_n^-(z^{-1})dS$$

(42)

and

$$\frac{\partial F}{\partial a_j} = \oint_{\beta_j} dS.$$  

(43)

Recall that $f_n^\pm(z^{-1})$ are the polynomials giving the singular part of $d\Omega_n$ at $P^\pm$. The compatibility (integrability) of the above equations for $F$ is again a consequence of Riemann’s bilinear relation.

Now differentiate the right hand side of the defining equation of $\partial F/\partial T_n$ against $a_j$ and $T_m$. By the Whitham equations, the outcome is nothing but the right hand side of (33) and (34). Thus $V_{jn}$ and $q_{mn}$, which are now functions of $a$ and $T$, can be expressed as second derivatives of the prepotential:

$$V_{jn} = \frac{1}{2\pi i} \frac{\partial^2 F}{\partial a_j \partial T_n}, \quad q_{mn} = \frac{1}{2\pi i} \frac{\partial^2 F}{\partial T_m \partial T_n}.$$  

(44)

Similarly, the matrix elements of the period matrix can be written

$$T_{jk} = \frac{\partial^2 F}{\partial a_j \partial a_k}.$$  

(45)

Since $V_n$ should be the gradient vector of the potential $P_n$ in the $a$-space, we conclude that the potential $P_n$ can be written

$$P_n = \frac{1}{2\pi i} \frac{\partial F}{\partial T_n}.$$  

(46)

Thus all relevant quantities turn out to be written as derivatives of the redefined prepotential. It is remarkable that the last equation resembles the hypothetical Hamilton-Jacobi equation of Losev et al. [3] In our case, however, $P_n$ is a function of both the Coulomb moduli AND the Whitham time variables, the latter enter from $dS$ through the integral formula, and we do not know how to convert it into a function of $a_j$ and $a_j^D = \partial F/\partial a_j$ like the Hamiltonians of Losev et al.

The redefined prepotential is a purely theoretical backbone, and not very suited for explicit calculations. Integral formulae, such as (33), (34) and an integral representation
of $\mathcal{P}_n$ derived from (42), are more convenient. These integral formulae remain valid even if the Whitham time variables are returned to the “departure time” $T = T_{SW}$ where $dS$ is equal to $dS_{SW}$; everything can thereby calculated in terms of the original geometric data on the spectral curve. Calculations of contact terms are thus eventually reduced to residue calculus. This is enough for understanding the four-dimensional problem.

Nevertheless, special families of Whitham deformations can possess some significant implications. Gorsky et al. [13] indeed presented such an example in the case of $G = SU(N)$ and $N_f = 0$. As they remarked, their Whitham deformations exhibit a remarkable similarity with two-dimensional topological Landau-Ginzburg models [13]. The construction starts from the meromorphic differentials

$$d\hat{\Omega}_n = \left( P(x)^{n/N} \right)_+ d\log z,$$

where $(\cdots)_+$ means the polynomial part of the Laurent expansion of $P(x)^{n/N}$ at $x = \infty$. $d\hat{\Omega}_1$ is nothing but the differential $dS_{SW}$. Now introduce a set of time variables $\tilde{T}_n$ and consider the differential

$$dS = \sum_{n=1}^{\infty} \tilde{T}_n d\hat{\Omega}_n.$$

Because of a reason (see below), we have to distinguish between these time variables $\tilde{T}_n$ and the previous ones $T_n$. The period integrals

$$a_j = \oint_{\alpha_j} dS,$$

define a function of the Coulomb moduli $u = (u_2, \cdots, u_N)$ and the time variables $\tilde{T}_1, \tilde{T}_2, \cdots$. One can prove, by a standard method in Seiberg-Witten geometry, that the Jacobian matrix $\det(\partial a_j/\partial u_k)$ does not vanish in a neighborhood of, say, the “Seiberg-Witten point” $\tilde{T}_n = \delta_{n,1}$. Therefore the above relation can be solved for the Coulomb moduli as $u_j = u_j(a, \tilde{T}_1, \tilde{T}_2, \cdots)$. This gives a deformation family of the spectral curve $\Sigma$. Now modify the meromorphic differential $d\hat{\Omega}_n$ into

$$d\hat{\Omega}_n = d\hat{\Omega}_n - \sum_{j=1}^{N-1} \left( \oint_{\alpha_j} d\hat{\Omega}_n \right) d\omega_j,$$

and write $dS$ in the form

$$dS = \sum_{n=1}^{\infty} \tilde{T}_n d\hat{\Omega}_n + \sum_{j=1}^{N-1} a_j d\omega_j.$$
One can then derive, by the method of Itoyama and Morozov \[17\], the Whitham equations

\[
\frac{\partial \tilde{T}_n}{\partial t_n} \bigg|_{z=\text{const.}} = d\tilde{\Omega}_n, \quad \frac{\partial a_j}{\partial t_n} \bigg|_{z=\text{const.}} = d\omega_j. \tag{52}
\]

This Whitham deformation family is slightly distinct from those that we have considered. A natural choice of local coordinates \(z\) at \(P^\pm\) is the following:

\[
z_+ = z^{1/N}, \quad z_- = z^{1/N}. \tag{53}
\]

The polynomials \(f^\pm_n(z_\pm^{-1})\) giving the singular part of \(d\tilde{\Omega}_n\) at \(P^\pm\) can be written

\[
\begin{align*}
    f_n(z_+^{-1}) &= \frac{N}{n} \Lambda^n z_+^{-n} + \cdots, \\
    f_n(z_-^{-1}) &= -\frac{N}{n} \Lambda^n z_-^{-n} + \cdots. \tag{54}
\end{align*}
\]

The difference lies in the tail part “\(\cdots\)”. This part vanishes for \(n < 2N\), but remains for \(n \geq 2N\), and the coefficients of this part are NOT a numerical constant but a polynomial of the moduli \(u_j\)'s. This is the reason that we changed the notation of the time variables. This somewhat strange situation forces us a careful treatment of the prepotential. For instance, when differentiating contour integrals like those in (42) against \(T_m\) and \(a_k\), we use the fact that the part of \(f^\pm_n(z_\pm^{-1})\) may be considered constant; this is not permitted in the above example.

Extending this example to other gauge groups, such as \(SO(2N)\), is an interesting problem. We shall consider this issue elsewhere.

6 Conclusion

Inspired by the work of Mariño and Moore \[2\], we have proposed an extension of the blowup formula with more than one 2-observables of the form \(I_n(B)\) supported on the exceptional divisor \(B\). Our strategy is simply to replace the single-time tau function of the Toda lattice hierarchy (in the sense of Mariño and Moore) by a multi-time tau function. The time variables \(t_n\) are interpreted as the coupling constants for the insertion of \(I_n(B)\).

The tau function is an algebro-geometric tau function comprised of a Gaussian factor \(e^{Q(t_1, t_2, \ldots)}\) and a theta function \(\Theta_{\gamma, \delta}(\sum_n t_n V_n \mid \mathcal{T})\). The coefficients of the Gaussian part are identified to be the contact terms of the 2-observables. We have partly confirmed the
validity of our proposal by showing that the tau function possesses a desirable modular transformation property.

This proposal has been further examined in the language of the Whitham equations that underlie the integrable hierarchy. We have shown that the contact terms, as well as other relevant quantities, are written in the form of derivatives of the prepotential with respect to the Whitham time variables $T_n$. This also clearly explains why the RG equation takes place in the description of contact terms of the quadratic Casimir $u_2$.

These observations should be further checked in a field-theoretic language. The superfield formalism of Losev et al. [3] will provide a suitable framework for this purpose.

It is remarkable that the integrable hierarchy and the Whitham hierarchy are both linked with two-dimensional field theories. The integrable hierarchy (the Toda lattice hierarchy in the case of $G = SU(N)$ and $N_f = 0$) is related to massless free fermions on the spectral curve [18]. (This is also an implicit message from the work of Gorsky et al. [15]. They used the Szegő kernels, which are correlation functions of free fermions.) The modular transformation property of the tau function is physically a consequence of conformal invariance of the two-dimensional massless free fermion theory. Meanwhile, as recent attempts at an analogue of the WDVV equations [13] suggest, the Whitham hierarchy are related to two-dimensional topological CFT’s, in particular, topological Landau-Ginzburg models.

These two-dimensional structures deserve to be studied in more detail. Of particular interest will be to examine, from our point of view, the mirror-like structure pointed out by Ito and Yang [13].

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References
[1] G. Moore and E. Witten, Integration over the $u$-plane in Donaldson Theory, hep-th/9709193.

[2] M. Mariño and G. Moore, Integrating over the Coulomb branch in N=2 gauge theory, Nucl. Phys. Proc. Suppl. 68 (1998), 336-347; The Donaldson-Witten function for gauge groups of rank greater than one, Commun. Math. Phys. 199 (1998), 25-69.

[3] A. Losev, N. Nekrasov and S. Shatashvili, Issues in topological gauge theory, Nucl. Phys. B534 (1998), 549-611; Testing Seiberg-Witten solution, hep-th/9801061.

[4] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988), 353-386; Supersymmetric Yang-Mills theory on a four-manifold, J. Math. Phys. 35 (1994), 5101-5135.

[5] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in N = 2 supersymmetric Yang-Mills theory, Nucl. Phys. B426 (1994), 19-52; Monopoles, duality and chiral symmetry breaking in N = 2 supersymmetric QCD, Nucl. Phys. B431 (1994), 484-550.

[6] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Integrability and Seiberg-Witten exact solution, Phys. Lett. 335B (1995), 466-474.
E. Martinec and N.P. Warner, Integrable systems and supersymmetric gauge theory, Nucl. Phys. B459 (1996), 97-112.
T. Nakatsu and K. Takasaki, Whitham-Toda hierarchy and N = 2 supersymmetric Yang-Mills theory, Mod. Phys. Lett. A11 (1996), 157-161.
R. Donagi and E. Witten, Supersymmetric Yang-Mills theory and integrable systems, Nucl. Phys. B460 (1996), 299-344.
A. Gorsky and A. Marshakov, Towards effective topological gauge theories on spectral curves, Phys. Lett. B375 (1996), 127.
H. Itoyama and A. Morozov, Integrability and Seiberg-Witten theory, Nucl. Phys. B477 (1996), 855-877; Prepotential and the Seiberg-Witten theory, Nucl. Phys. B491 (1997), 529-573.
C. Ahn and S. Nam, Integrable structure in supersymmetric gauge theories with massive hypermultiplets, Phys. Lett. B387 (1996), 304-309.
I.M. Krichever and D.H. Phong, On the integrable geometry of soliton equations and N = 2 supersymmetric gauge theories, J. Diff. Geom. 45 (1997), 349-389.

[7] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Simple singularities and N = 2 supersymmetric Yang-Mills theory, Phys. Lett. 344B (1995), 169-175.
P. Argyres and A. Faraggi, The vacuum structure and spectrum of N = 2 supersymmetric SU(n) gauge theory, Phys. Rev. Lett. 73 (1995), 3931-3934.

[8] K. Ueno and K. Takasaki, Toda lattice hierarchy, Adv. Stud. Pure Math. 4 (1984), 1-95.
T. Takebe, Toda lattice hierarchy and conservation laws, Commun. Math. Phys. 129 (1990), 281-318.

[9] G. Bonelli and M. Matone, Nonperturbative renormalization group equation and beta function in N = 2 SUSY Yang-Mills, Phys. Rev. Lett. 76 (1996), 4107-4110.
E. D’Hoker, I.M. Krichever and D.H. Phong, The renormalization group equations in N = 2 supersymmetric gauge theories, Nucl. Phys. B494 (1997), 89-104.
E. D’hoker and D.H. Phong, Calogero-Moser systems in SU(N) Seiberg-Witten theory, Nucl. Phys. B513 (1998), 405-444.
A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, RG equations from Whitham hierarchy, Nucl. Phys. 527 (1998), 690-716.

[10] T. Nakatsu and K. Takasaki, cited above [8].

[11] I.M. Krichever, Algebraic curves and non-linear difference equations, Russ. Math. Surveys 33:4 (1978), 215-216; Theta functions and non-linear equations, Russ. Math. Surveys 36:2 (1981), 11-92 (Appendix).

[12] M. Jimbo and T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, Physica 2D (1981), 407-448; ditto III, Physica 4D (1981), 26-46.
E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation theory for soliton equations III, J. Phys. Soc. Japan 50 (1982), 3806-3812; ditto IV, J. Phys. Soc. Japan 50 (1982), 3813-3818; ditto V, Physica 4D (1982), 343-365; ditto VI, Publ. RIMS., Kyoto Univ., 18 (1982), 1077-1110.
T. Shiota, Characterization of jacobian varieties in terms of soliton equations, Invent. Math. 83 (1986), 333-382.

[13] L. Alvarez-Gaumé, G. Moore and C. Vafa, Theta functions, modular invariance and strings, Commun. Math. Phys. 106 (1986), 1-40.
N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, Geometric realization of conformal field theory on Riemann surfaces, Commun. Math. Phys. 116 (1988), 247-308.

[14] D. Mumford, Tata lectures on theta (Birkhäuser, Boston, 1983).

[15] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, cited above [9].

[16] Dijkgraaf, R., Verlinde, E., and Verlinde, H., Topological strings in $d < 1$, Nucl. Phys. B352 (1991), 59-86.

[17] H. Itoyama and A. Morozov, cited above [9].

[18] L. Alvarez-Gaumé, C. Gomez and C. Raina, Loop groups, grassmannians and string theory, Phys. Lett. B190 (1987), 55.
N. Ishibashi, Y. Matsuo and H. Ooguri, Soliton equations and free fermions on Riemann surfaces, Mod. Phys. Lett. A2 (1987), 119.
C. Vafa, Operator formulation on Riemann surfaces, Phys. Lett. B190 (1987), 47.

[19] G. Bonelli and M. Matone, Nonperturbative relations in $N = 2$ SUSY Yang-Mills and WDVV equation, Phys. Rev. Lett. 77 (1996), 4712-4715.
A. Marshakov, A. Mironov and A. Morozov, WDVV-like equations in $N = 2$ SUSY Yang-Mills Theory, Phys. Lett. B389 (1996), 43-52; WDVV equations from algebra of forms, Mod. Phys. Lett. A12 (1997), 773-788.
K. Ito and S.-K. Yang, The WDVV equations in $N = 2$ supersymmetric Yang-Mills theory, hep-th/9803126.