The constant astigmatism equation.  
New exact solution

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Abstract
In this paper we present a new solution for the constant astigmatism equation. This solution is parameterized by an arbitrary function of a single variable.

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1. Introduction

The theory of integrable systems is usually associated with the method of the inverse scattering transform, which allows one to construct multi-soliton, multi-gap and similarity solutions, i.e. those parameterized by an arbitrary number of constants. Nevertheless, some integrable nonlinear systems, in partial derivatives, possess solutions parameterized by arbitrary functions of a single variable. The best known example is the three-wave interaction, which was thoroughly investigated in [21].

In this paper we deal with the constant astigmatism equation

\[ u_{tt} = 2 - \left( \frac{1}{u} \right)_{xx} \] (1)
which was considered in a set of papers (see detail in [1, 2, 9–11, 20]). The main aim of the present paper is to obtain the following four first order reductions

\[ u_t = \pm \frac{1}{u} u_x \pm 2\sqrt{u} \]

of the constant astigmatism equation (1). Thus, we determine four particular solutions of the equation (1) parameterized by an arbitrary function of a single variable.

The equation (1) is connected to the remarkable Bonnet (also known as the Sine–Gordon) equation by a reciprocal transformation. However, we show that only the constant astigmatism equation has the particular solution parameterized by an arbitrary function of a single variable.

To illustrate this phenomenon, we draw attention to another remarkable integrable nonlinear system known as the Kaup–Boussinesq system (see [14])

\[
\begin{align*}
    u_t + \left( \frac{u^2}{2} - \eta \right)_x &= 0, \\
    \eta_t + \left( \eta \frac{1}{4} u_{xx} \right)_x &= 0.
\end{align*}
\]

Under the potential substitution \( u = z_x \) and \( \eta = \frac{z^2}{2x} + z_t \), the second equation becomes

\[ z_{tt} + 2z_z z_{xt} + \left( z_t + \frac{3}{2} z_x \right) z_{xx} - \frac{1}{2} z_{xxxx} = 0. \]

This equation possesses the reduction

\[ z_t + \frac{1}{2} \left( \frac{z^2}{2} - z_{xx} \right) = 0, \]

which is nothing but the well-known Burgers equation

\[ u_t + uu_x - \frac{1}{2} u_{xx} = 0, \]

which is linearizable to the heat equation

\[ v_t = \frac{1}{2} v_{xx}, \]

by the Cole–Hopf substitution \( u = -v_x/v \) (indeed, one can see that the first equation in (2) reduces to (3) if \( \eta = u_x/2 \); then the second equation in (2) automatically satisfies). Since, the heat equation has a solution which is parameterized by an arbitrary function of a single variable \( v_0(x) \), then the Kaup–Boussinesq system also admits a particular solution \( u = z_x \) and \( \eta = \frac{z^2}{2x} + z_t \), where:

\[
\begin{align*}
    u &= -\partial_t \ln \left[ \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{2t} - \int_0^y u(\xi, 0) \, d\xi \right) \right], \\
    \eta &= -\frac{1}{2} \partial_x^2 \ln \left[ \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{2t} - \int_0^y u(\xi, 0) \, d\xi \right) \right],
\end{align*}
\]

which depends on an arbitrary function \( u_0(x) = u(x, t)|_{t=0} \).

2. Reduction procedure

The aim of this section is to develop a reduction procedure for (1) within the framework of the theory of differential constraints. The method is based upon appending a set of PDEs to a given governing system of field equations. The method was first applied by Janenko [12] to the gas dynamics model. The auxiliary equations represent the differential constraints because they select the classes of solutions of the system under interest. The method is very general and includes many of the known reduction approaches. Unfortunately often the generality of

Equation (1) This differs from the original equation, which was investigated by Michal Marvan and his colleagues up to the change of the sign of the unknown function \( u \).
method limits the according diversity of the applications. In each particular case, the method of differential constraints utilizes specific features of the corresponding nonlinear systems (see [4–8, 13, 15, 18]).

First, we reduce the governing equation (1) to the pair of equations

\[
\begin{align*}
\frac{a_t}{a_x} - b_x &= 0, \\
\frac{b_t}{a_x} - \frac{1}{(a + t^2)} a_t &= 0,
\end{align*}
\]

(4)

where

\[
a(x, t) = u(x, t) - t^2,
\]

(5)

and \(b(x, t)\) is an auxiliary function. The system (4) is strictly hyperbolic and the eigenvalues of the matrix coefficients, as well as the corresponding left eigenvectors, are

\[
\lambda = \mp f(a, t); \quad L^{(\pm)} = (-\lambda; 1),
\]

(6)

where

\[
f(a, t) = \frac{1}{a + t^2}.
\]

(7)

In [16] (see also [17] and references quoted therein) it was proved that the most general first order differential constraint, compatible with a strictly hyperbolic first order system, must adopt a quasilinear form with coefficients proportional to the components of the left eigenvectors of the matrix coefficients involved in the original system. Therefore the most general first order differential constraint admitted by (4) takes the form

\[
L^{(\pm)} \cdot U_x = p(x, t, a, b), \quad \text{with} \quad U = \begin{pmatrix} a \\ b \end{pmatrix},
\]

which in our case specializes to

\[
b_x - \lambda a_x = p(x, t, a, b),
\]

(8)

where the function \(p(x, t, a, b)\) must be determined during the process. In the following, without loss of generality, we consider the case \(\lambda = + f(a, t)\). The consistency requirement between (4) and (8) leads to a linear expression with respect to \(a_x\), whose coefficients depend only on \((x, t, a, b)\). Thus both the coefficients must vanish independently:

\[
\begin{align*}
p_a + fp_b &= -f_x + pf_a, \\
p_t + fp_x &= \frac{f_t + pf_a}{2f}.
\end{align*}
\]

(9)

Both these equations can be integrated by the method of characteristics, i.e. we have

\[
\frac{db}{da} = f, \quad \frac{dp}{da} = -\frac{f_x + pf_a}{2f}, \quad \frac{dx}{dt} = f, \quad \frac{dp}{dt} = \frac{f_t + pf_a}{2f}.
\]

(10)

Taking (7) into account, we can independently solve the first two ordinary differential equations (with respect to \(a\)) and the two other ordinary differential equations (with respect to \(t\)). However, the solution of (9)

\[
p = -2t \pm 2\sqrt{a + t^2}
\]

(11)

can be obtained directly from the compatibility condition (see (10))

\[
\frac{d}{dt} \left( \frac{dp}{da} \right) = \frac{d}{da} \left( \frac{dp}{dt} \right).
\]
Therefore, taking (8) and (11) into account, equations (4) can be written in the form

\[ \begin{align*}
    at - fa x &= p \\
    bt - fb x &= -fp
\end{align*} \]  

(12)

Then, the first equation (see (7) and (11))

\[ a t - \frac{1}{a + t^2} a x = -2t + 2\sqrt{a + t^2} \]  

(13)

can be integrated by the method of characteristics, while the function \( b(x, t) \) satisfies the second equation that can be found in quadratures (see (4))

\[ db = a t \, dx + \frac{ax}{(a + t^2)^2} \, dt = (u t - 2t) \, dx + \frac{u x}{u^2} \, dt, \]

because the compatibility condition \( (b_1)_x = (b_x)_t \) is fulfilled by virtue of (1). Moreover, since \( b_t = f^2 a_t \) and \( b_x = a_t \), one can see that the second equation in (12) is equivalent to the first one in (12).

Of course a similar analysis holds in the case \( \lambda = -f(a, t) \).

In order to obtain the required reduction of the governing equation (1) by writing equations (8) and (12) in terms of the original variable \( u \), we get (see (5) and, for instance, (13) for the case \( \lambda = +f(a, t) \))

\[ u_t = \pm \frac{1}{u} u_x \pm 2\sqrt{u}. \]  

(14)

So that exact solutions of (1) are determined by integrating the quasilinear differential equation in partial derivatives of a first order (14) using the characteristic method. Therefore we proved the following:

**Theorem 1.** Constant astigmatism equation (1) possesses four natural reductions (14). Thus, constant astigmatism equation (1) has four particular solutions parameterized by an arbitrary function of a single variable.

### 3. New particular solution

Let us first prove the following

**Theorem 2.** The first order equation

\[ u_t = F(x, t, u, u_x) \]  

(15)

is a reduction of constant astigmatism equation (1) if it specializes to (14).

**Proof.** Substitution of (15) into (1) leads to

\[ F_t + FF_u + F_x(F_x + F_u u_x + F_y u_{xx}) - \frac{1}{u^2} u_{xx} + \frac{2}{u^4} u_x^2 = 2, \]  

(16)

where we set \( y = u_x \). By requiring (16) to be satisfied \( \forall u_{xx} \) we get

\[ F = \pm \frac{y}{u} + G(x, t, u), \]  

(17)

where the function \( G(x, t, u) \) needs to be determined. Substitution of (17) in (16) yields

\[ G_t + \left( G \pm \frac{y}{u} \right) \left( G_u \mp \frac{y}{u^2} \right) \pm \frac{1}{u} \left( G_t + \left( G_u \mp \frac{y}{u^2} \right) y \right) + \frac{2}{u^3} y^2 = 2. \]  

(18)

Since the relation (18) must be satisfied \( \forall y \), we obtain \( G = \pm 2\sqrt{u} \). Thus, by virtue of (17), the theorem is proved. \( \square \)
Next we solve equation (14) by a slightly modified version of the characteristic method. Under the point transformation $u = v^2$ (14) reduces to the form

$$v_t = \pm \frac{1}{v^2} v_x \pm 1.$$  

This equation can be written in the conservative form

$$w_t \pm \left( \frac{1}{w \pm t} \right)_x = 0,$$

where $v = w \pm t$. Then the potential function $z$ can be introduced such that

$$dz = w \, dx \pm \frac{dt}{w \pm t}.$$ 

In the first case

$$d[z + \ln(w + t)] = w \, dx + \frac{dw}{w + t},$$

The compatibility condition implies $\left( \frac{1}{w + t} \right)_x = 1$. Thus $\frac{1}{w + t} = x + h(w)$, where $h(w)$ is an arbitrary function. So, the first particular solution of the constant astigmatism equation is

$$u_{(1)} = (w + t)^2,$$

where $w(x, t)$ is a solution of the implicit equation

$$(x + h_1(w))(t + w) = 1.$$ 

Analogously, in the second case:

$$u_{(2)} = (w - t)^2,$$

where $w(x, t)$ is a solution of the implicit equation

$$(x + h_2(w))(t - w) = 1.$$ 

In the third case:

$$u_{(3)} = (w + t)^2,$$

where $w(x, t)$ is a solution of the implicit equation

$$(h_3(w) - x)(t + w) = 1.$$ 

In the fourth case:

$$u_{(4)} = (w - t)^2,$$

where $w(x, t)$ is a solution of the implicit equation

$$(h_4(w) + x)(w - t) = 1.$$ 

Remark 1. The constant astigmatism equation preserves itself under the transformation $x \leftrightarrow t$ and $u \to 1/u$. Also its reduction (14) preserves itself under the same transformation.

Remark 2. A relationship between the Sine–Gordon and the constant astigmatism equations was presented in [2]. The corresponding reciprocal transformation (see formulas (29), (30) in this cited paper) contains two distinct expressions

$$\left( \frac{u_1 \pm u_2}{u} \right)^2 - 4u,$$

which vanish if the function $u(x, t)$ satisfies (14). Thus, the four particular solutions above are parameterized by an arbitrary function of a single variable and cannot be transformed into corresponding solutions of the Sine–Gordon equation. However, the geometrical meaning of these new solutions is an open and very interesting problem.
Remark 3. A Cauchy problem for a quasilinear equation in partial derivatives of a second order is based on a choice of two functions $u_0(x) = u|_{t=0}$ and $u_1(x) = u_t|_{t=0}$. Since instead of the second order, we can consider its reduction, which is a quasilinear equation in partial derivatives of a first order, therefore we can investigate a Cauchy problem restricted to a narrow class of solutions. Indeed, in the first case, a Cauchy problem has a solution
\[ u = (W(x, t) + t)^2, \] (21)
where $W(x, t)$ is a solution of an algebraic equation
\[ \left( x + \frac{1}{W} - X(W) \right) (t + W) = 1, \] and the function $X(W)$ is determined by the equation $X(W_0(x)) = x$, where $W_0(x) = \sqrt{u_0(x)}$ (see equation (21) for $t = 0$). Then $u_1(x)$ cannot be an arbitrary function:
\[ u_1(x) = 2 \frac{1 + X'(W)W^2}{X'(W)W} \bigg|_{W=\sqrt{u_0(x)}}. \]

4. Replication of solutions

According to [11], particular solutions of the constant astigmatism equation can be replicated infinitely starting from any initial solution by the formulas
\[ x^{(1)} = \frac{xu}{x^2u - 1}, \quad t^{(1)} = \eta, \quad u^{(1)} = \frac{(x^2u - 1)^2}{u}; \] (22)
\[ x^{(-1)} = \xi, \quad t^{(-1)} = \frac{t}{t^2 - u}, \quad u^{(-1)} = \frac{u}{(t^2 - u)^2}, \] (23)
where
\[ -d\eta = xu \, dx + \left( \frac{xu}{u^2} + \frac{1}{u} + x^2 \right) dt, \quad d\xi = (tu - u - t^2) \, dx + t \frac{u^2}{u^2} \, dt. \] (24)

The combination of these two transformations, in a general case, gives an infinite series of particular solutions. In this section we consider the replication of the particular solutions starting from the solution constructed in the previous section. Without loss of generality, we start with the first such particular solution determined by (19), (20), i.e.
\[ u = (w + t)^2, \]
where $w(x, t)$ is a solution of the algebraic equation
\[ x = \frac{1}{t + w} - h(w). \] (25)
Then the functions $\eta, \xi$ (see (24)) can be found in quadratures, i.e.
\[ t^{(1)} \equiv \eta = 2h(w) - \frac{wh^2(w)}{w} + \int h^2(w) \, dw - \frac{h^2(w)}{w} t \]
\[ x^{(-1)} \equiv \xi = -\frac{w^2}{t + w} + 2w + \int w^2h'(w) \, dw, \]
while (22) leads to a first iteration
\[ u^{(1)} = (t^{(1)} + w^{(1)})^2, \]
where $w^{(1)}(x^{(1)}, t^{(1)})$ is a solution of the algebraic equation (cf (25))
\[ x^{(1)} = \frac{1}{t^{(1)} + w^{(1)}} - h^{(1)}(w^{(1)}), \] (26)
In this case
\[ t \equiv \eta^{(1)} = 2h^{(1)}(w^{(1)}) - w^{(1)}[h^{(1)}(w^{(1)})] + \int [h^{(1)}(w^{(1)})] \, dw^{(1)} - [h^{(1)}(w^{(1))}]^t^{(1)}, \]
where
\[ w^{(1)} = -\int h^2(w) \, dw, \quad h^{(1)}(w^{(1)}) = \frac{1}{h(w)}, \quad w = -\int [h^{(1)}(w^{(1))}] \, dw^{(1)}. \]

Meanwhile, transformation (23) leads to a first ‘negative’ iteration
\[ u^{(-1)} = (w^{(-1)} + t^{(-1)})^2, \]
where \( w^{(-1)}(x^{(-1)}, t^{(-1)}) \) is a solution of the algebraic equation (cf (25) and (26))
\[ x^{(-1)} = \frac{1}{t^{(-1)} + w^{(-1)}} - h^{(-1)}(w^{(-1)}). \]

In this case
\[ t^{(-1)} = -\frac{t}{w^2 + 2wt}, \quad t = -\frac{t^{(-1)}}{(w^{(-1)})^2 + 2w^{(-1)}t^{(-1)}}, \]

where
\[ h^{(-1)}(w^{(-1)}) = -\int w^2 h'(w) \, dw, \quad w^{(-1)} = w^{-1}. \]

Thus, we see that both transformations \((x, t, u) \rightarrow (x^{(1)}, t^{(1)}, u^{(1)})\) and \((x, t, u) \rightarrow (x^{(-1)}, t^{(-1)}, u^{(-1)})\) preserve a class of solutions (see remark 2 in the previous section) which cannot be associated with any solutions of the Sine–Gordon equation, i.e. all such iterated solutions are solutions of reduced equation (14). The above formulas just connect an infinite set of solutions determined by different expressions \( h(w) \).

5. Conclusion

It is well known that integrable systems can conditionally be split into two wide classes: \( S \) and \( C \) integrable systems [3]. Usually, we understand that \( C \) integrable systems are linearizable systems by appropriate transformations. General solutions of \( C \) integrable systems can be expressed explicitly, and these solutions are parameterized by arbitrary functions. \( S \) integrable systems possess infinitely many particular solutions which are parameterized, in general, by sufficiently many arbitrary constants. Also, we know that some \( S \) integrable systems can be degenerated (for instance, the sinh-Gordon equation \( u_{xt} = c_1 e^u + c_2 e^{-u} \) in a particular case becomes the Liouville equation \( u_{xt} = e^u \), whose general solution is parameterized by two arbitrary functions of a single variable; see in more detail in [19]). In this paper we considered constant astigmatism equation (1), which is integrable by the inverse scattering transform (see again the details in [1, 2, 9–11, 20]). The constant astigmatism equation also admits particular solutions parameterized by an arbitrary function of a single variable, even in a non degenerate case. We hope that particular solutions parameterized the by arbitrary functions of a single variable for other important integrable systems will be found soon.

In the previous section we described transformations for the constant astigmatism equation, which replicates its reduced version only (14). We hope that a more general transformation connecting solutions of (14) and (1) can be found.
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