Existence and stability of steady-state solutions with finite energy for the Navier–Stokes equation in the whole space

Clayton Bjorland and Maria E Schonbek

Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
E-mail: cbjorland@math.ucsc.edu and schonbek@math.ucsc.edu

Received 28 October 2008, in final form 30 April 2009
Published 4 June 2009
Online at stacks.iop.org/Non/22/1615

Recommended by K Ohkitani

Abstract

We consider the steady-state Navier–Stokes equation in the whole space $\mathbb{R}^3$ driven by a forcing function $f$. We show that there exists a constant $M_0 > 0$ such that for any $M \geq M_0$, provided the source terms $f$ are sufficiently small in a natural norm (the smallness depending only on $M$), and the low frequencies of $f$ are sufficiently controlled then there exists a solution $U$ with bounds $\|U\|_{L^2(\mathbb{R}^3)} \leq M$. These solutions are unique among all solutions with finite energy and finite Dirichlet integral. Using Fourier splitting tools, the constructed solutions will be shown to be stable in the following sense: if $U$ is such a solution then any viscous, incompressible flow in the whole space, driven by $f$ and starting with finite energy, will return to $U$.

Mathematics Subject Classification: 35B35, 35Q30, 76D05

1. Introduction

The classical theory of viscous, incompressible fluid flow is governed by the famous Navier–Stokes equations:

$$
\begin{align*}
  u_t + u \cdot \nabla u + \nabla p &= \nu \Delta u + f, \\
  u(0) &= u_0, \\
  \nabla \cdot u &= 0.
\end{align*}
$$

(1.1)

A large area of modern research is devoted to deducing qualitative properties of solutions for these equations when they are complemented with initial and boundary conditions and certain restraints are placed on $f$ and $\nu$. The investigations in this subject are too numerous to attempt to list here so we will limit ourselves to discussion directly related to the topic of this paper: the steady-state Navier–Stokes equation in the whole space $\mathbb{R}^3$. 

A steady-state (sometimes called stationary in the literature) solution $U$ of the Navier–Stokes equation is one for which $\partial_t U = 0$, that is the solution is constant in time. Such solutions solve the following PDE, which will be our main point of investigation.

$$\begin{align*}
U \cdot \nabla U + \nabla p &= \Delta U + f, \\
\nabla \cdot U &= 0.
\end{align*}$$

(1.2)

This PDE is supplemented with the idea that $U$ tends to a fixed vector $v_\infty \in \mathbb{R}^3$ as $|x|$ becomes large. Depending on whether $v_\infty$ is zero or not the solutions will behave quite differently: when $v_\infty \neq 0$ the underlying linear system is the Oseen system while if $v_\infty = 0$ the underlying system is the Stokes system. Generally speaking the Oseen system is better behaved than the Stokes system for large $|x|$. Here we consider only the case $v_\infty = 0$ made precise by working in functions spaces which are completions of smooth functions with compact support. For convenience we have set $\nu = 1$. This is not a restriction since $U(\nu x)$ is a solution with viscosity $\nu$ and forcing function $\nu f$ whenever $U$ satisfies the above PDE.

Roughly speaking, the investigation of steady-state solutions can be broken into two regimes: bounded and unbounded domains. In the former situation much progress has been made using a Poincaré type inequality ($\|U\|_2 \leq C \|\nabla U\|_2$) to deduce quickly that solutions have finite energy. In the case where there is no Poincaré inequality it is desirable to find conditions on $f$ which will guarantee, a priori, finite energy of a solution.

The main goal of this paper is to investigate the finite energy situation in the whole space and to develop a new technique which will allow, with easily verified conditions on $f$, the construction of solutions for the steady-state Navier–Stokes equation in the whole space with finite energy. The assumptions we impose on $f$ limit the amount of low frequency information and require $f$ small in a natural norm. Once we have established the finite energy of solutions we deduce uniqueness in the class of solutions with finite energy and prove these solutions are stable in a strong sense referred to in the literature as nonlinearly stable. We show that the finite energy (but possibly large) perturbations of the steady-state solution return to the steady state. The proof of stability relies on the Fourier splitting method, a technique used to overcome problems due to working in the whole $\mathbb{R}^3$.

1.1. Statement of results and background

Modern analysis of the steady-state Navier–Stokes equation in unbounded domains can be traced back to [31, 39]. Since that time much has been accomplished in terms of existence, uniqueness and stability of solutions in unbounded domains. The results are too numerous to list here so we will recall a few relevant cases and point to [17] and the references therein for further reading. Also relevant to the topics in this paper are: [2, 3, 5–10, 12, 15, 19, 20–22, 27–29, 34, 35, 46].

When $\Omega \subset \mathbb{R}^3$ is an exterior domain it is known that if $f$ is taken small in a natural function space and small data on $\partial \Omega$ there is a unique solution in the class of solutions which have a finite Dirichlet integral ($\|\nabla U\|_2 < \infty$) and satisfy $\|(1 + |x|)U\|_\infty < C$ (see for example [17]). It is well known that such solutions

‘have finite energy (i.e. $U \in L^2$, and more generally $U \in L^q, q \in (1, 3]$), if and only if a certain non-local compatibility condition is satisfied on $\partial \Omega$.’

For details we refer the reader to the fundamental work of Galdi [17]. In view of these classical results the work in this paper cannot in general, be extended to exterior domains. In the specific case $\Omega = \mathbb{R}^3$ ‘the compatibility condition’ mentioned above is trivially satisfied (see [17]). Hence, the fact that our space of reference is $\Omega = \mathbb{R}^3$ is crucial to our work and, it also allows us to use techniques different from those found in the literature.
In the arguments presented below we introduce a parabolic problem whose solution can be integrated in time formally to find the stationary solution. By computing the decay rate of the associated parabolic solution we can rigorously integrate the parabolic solution in time to recover a finite $L^2$ bound on the stationary solution. This approach is different from those presented in the literature (for example [16, 18]), it does not rely on uniformly bounding the non-steady problem but uses the decay of a related parabolic problem. We link the $L^2$ bound of the stationary solution with forcing function $f$ to the energy decay of an associated homogeneous parabolic equation with initial data taken to be $f$. Since there is a well-known link between low frequencies in initial data and the energy decay rate, this provides a direct connection between the low frequencies of the forcing function $f$ and the existence of a finite $L^2$ norm for the stationary solution, provided $f$ is small. The statement of the theorems below reflects this dependence on the low frequency information in $f$.

Our construction of solutions with finite energy is based on a well-known formal observation: if $\Phi$ is the fundamental solution for the heat equation then $\int_0^{\infty} \Phi(t, \cdot) \, dt$ is the fundamental solution for Poisson’s equation. Using this idea it is possible to make a time dependent PDE similar to the Navier–Stokes equation with $f$ as initial data whose solution can be formally integrated in time to find a solution of (1.2). At this point our analysis turns to the theory of energy decay for fluid equations. If the decay of the new time dependent PDE is fast enough (an integral over all time converges) we can deduce a finite energy bound for (1.2). It is known that the decay rate of solutions for parabolic PDEs in the whole space is intimately related to the shape of the initial data near the origin in Fourier space; the low frequency assumption we make on $f$ is enough to guarantee the convergence of the required time integrals. This idea is further outlined in section 2 and made precise in section 3.

A particularly useful technique for estimating energy decay is the Fourier splitting method which was used in [44] to establish energy decay for initial data $u_0 \in L^2 \cap L^1$ and later for initial data $u_0 \in L^2$ in [38]. Other works in this area include [2, 25, 26, 34, 36, 37, 40–43, 49, 50]. In essence we are trading bounds on Green’s function for energy decay theorems which we base on the Fourier splitting method. Many previous results consider the more complicated cases of external domains which are not handled within but we hope that with decay theorems for external domains one can use the technique presented here to obtain similar results.

We now state precisely the main existence theorem proved. In the following statement $\dot{H}_0^1$ is the completion of smooth divergence free functions of compact support under the norm $\| \nabla \cdot \|_2$. The requirement $f \in \dot{H}_0^{-1}$ implies the classical assumption $\| \nabla U \|_2 < \infty$ known as a finite Dirichlet integral and is sometimes implied by the restriction $|x| f \in L^2$ in the literature.

Our forcing terms besides being in appropriate Sobolev spaces will belong to the following spaces.

**Definition 1.1.** We define the space $A_q$ as follows

$$A_q = \left\{ f \in L^2_0 : \lim_{\rho \to 0} \rho^{-2q} \int_{B(\rho)} |\hat{f}(\xi)| \, d\xi < \infty \right\}$$

where $B(\rho)$ is the ball of radius $\rho$.

These sets consists of solenoidal functions for which $\hat{f}$ will essentially be bounded by $|\xi|^{q-\frac{3}{2}}$ near zero. See [1] for more details.

The following theorem will be established in section 3.
Theorem 1.2. Let \( f \in L^2_\sigma \cap \dot{H}^{-1}_\sigma \cap A_q \) for some \( q > 2 \). There exists a constant \( \mu_q \) so that for any \( M > \mu_q \) there exists a constant \( C_M \) which depends only on \( q \) and \( M \) so that if
\[
\max \{ \| f \|_2, \| f \|_{\dot{H}^{-1}_\sigma} \} \leq C_M
\]
then
(i) The PDE (1.2) has a weak solution \( U \in \dot{H}^1_\sigma \cap L^2_\sigma \). It is a weak solution in the sense that for any divergence free function of compact support \( \phi \),
\[
\langle U \cdot \nabla U, \phi \rangle + \langle \nabla U, \nabla \phi \rangle = \langle f, \phi \rangle \tag{1.4}
\]
(ii) This solution satisfies \( \| U \|_2 \leq M \) and \( \| \nabla U \|_2 \leq \| f \|_{\dot{H}^{-1}_\sigma} \).
(iii) This solution is unique among all solutions satisfying (ii) above.

The assumption \( f \in A_q \) for \( q > 2 \) implies the integral \( \int_0^\infty \| e^{\Delta t} f \|_2 \, dt < \infty \). Here \( e^{\Delta t} f \) is the solution of the heat equation with initial data \( f \). As our \( L^2 \) bounds are attained essentially by integrating the solution of a parabolic equation in time, the minimum value of the \( L^2 \) norm attained cannot be less than the solution of the heat equation with initial data equal to the forcing function, integrated in time. The condition that \( M > \mu_q \) to ensure the specific bound \( \int_0^\infty \| e^{\Delta t} f \|_2 \, dt < M \), that will be needed in the proof. (See lemma 2.1.)

Remark 1.3. The constant \( \mu_q \) can be chosen to be \( \mu_q = 3 \left( \frac{q+1}{q} \right)^q \).

When a solution is known to have finite energy in bounded domains well developed energy stability arguments using Poincaré’s inequality can be applied when \( f \) is small in appropriate norms, see [7, 8, 45, 47].

In this case, where we do not have Poincaré’s inequality, we are able to establish stability using modified Fourier splitting techniques. The stability is obtained by showing that solutions of a nonlinear parabolic PDE, similar to the Navier–Stokes equation and found by subtracting the steady state, tends to zero as time becomes large. Specifically, we turn to energy decay methods, specifically a method applied to the Navier–Stokes equation in [38], to show the decay. The method consists of estimating the high and low frequencies of the solution separately and the estimates of the high frequency rely on the Fourier splitting method. Section 4 is dedicated to establishing the following theorem.

Theorem 1.4. Let \( f \) satisfy the assumptions of theorem 1.2 then the solution \( U \) of (1.2) is stable in the following sense: if \( w_0 \in L^2_\sigma \) is a perturbation and \( u \) is a solution of the Navier–Stokes equation (1.1) with initial data \( w_0 + U \) which satisfies, for any \( T > 0 \),
\[
u \in L^\infty(0, T; L^2) \cap L^2(0, T; \dot{H}^1_\sigma)
\]
then
(i) For every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\| w_0 \|_2 \leq \delta \quad \text{implies} \quad \sup_{t \in \mathbb{R}^+} \| u(t) - U \|_2 \leq \epsilon
\]
(ii) \( u(t) \) tends to \( U \) as time becomes large, that is
\[
\lim_{t \to \infty} \| u(t) - U \|_2 = 0
\]

We would also like to recall the classical results on nonlinear stability obtained by Maremonti in [33] whose results work both in exterior domains and in the whole space. When considered in exterior domains, this hypothesis is that the limit \( \lim_{|x| \to \infty} U = U_\infty \neq 0 \), where \( U \) is the solution of the stationary Navier–Soques equations (1.2). In the case that \( \Omega = \mathbb{R}^3 \) we note that our results complement the ones obtained in the work of [33] in the following sense. The results in [33] work for perturbations of solutions to the stationary solutions \( U \) that satisfy...
Steady-state Navier–Stokes in $\mathbb{R}^3$ 1619

the following two conditions.

- There exist $M > 0$ and $\eta > 0$ so that
  \[ \|u - U_\infty\|_{2+\eta} + \|u - U_\infty\|_6 + \|U\|_\infty + \|\nabla U\|_3 \leq M. \]
- Require appropriate smallness condition on the Reynolds number corresponding to the stationary solution $U$.

Although we do not need to require the first condition above, we require conditions on the forcing term that are not required for the results in [33].

1.2. Notation

Unless otherwise noted all integrals in this paper are taken over the whole space $\mathbb{R}^3$, $C_0^\infty$ denotes the space of smooth functions with compact support.

\[ \langle f, g \rangle = \int f \cdot g \quad \mathcal{V} = \{ \phi \in C_0^\infty | \nabla \cdot \phi = 0 \}, \]

\[ \| \cdot \|_p = \left( \int | \cdot |^p \right)^{1/p} \quad L^p_\sigma = \{ \text{completion of } \mathcal{V} \text{ under the norm } \| \cdot \|_p \}, \]

\[ \| \cdot \|_{\dot{H}^1_\sigma} = \| \nabla \cdot \|_2 \quad \dot{H}^1_\sigma = \{ \text{completion of } \mathcal{V} \text{ under the norm } \| \cdot \|_{\dot{H}^1_\sigma} \}, \]

\[ H^1_\sigma = L^2_\sigma \cap \dot{H}^1_\sigma \quad \dot{H}^{-1}_\sigma = \{ \text{dual of } \dot{H}^1_\sigma \}, \]

\[ \hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx \quad \tilde{f}(x) = \int f(\xi) e^{2\pi i x \cdot \xi} \, d\xi. \]

We will typically write an element $f \in \dot{H}^{-1}_\sigma$ as $\langle \cdot, \cdot \rangle$ when we really mean the map $\phi \rightarrow \langle f, \phi \rangle$.

To denote general constants we use $C$ which may change from line to line. In certain cases we will write $C(\alpha)$ to emphasize the constants dependence on $\alpha$. In a similar way we write $\rho$ to denote general potentials (used to describe the pressure, one instance of $\rho$ may not be the same as another even on neighbouring lines). The variable $\xi$ is reserved for working in Fourier space.

2. Preliminaries

Existence of weak solutions for (1.2) is well known, see for example [6, 11, 14, 24, 30, 31, 48]. A typical approach to constructing weak solutions for this PDE is to construct approximations with the Galerkin method and use a priori bounds with the Banach–Alaoglu theorem to find a subsequence of approximations converging weakly to a possible solution. Some stronger compactness property is then used to pass the sequence through the nonlinear term and establish the limit is indeed a solution. A good a priori bound for this approach, and a bound we will rely on throughout is

\[ \|\nabla U\|_2^2 \leq \|f\|_{\dot{H}^{-1}_\sigma}^2. \] (2.1)

This is essentially the classical assumption that $U$ has a finite Dirichlet integral but we derive it from our assumption $f \in \dot{H}^{-1}_\sigma$ using the estimate

\[ |\langle f, U \rangle| \leq \|f\|_{\dot{H}^{-1}_\sigma} \|\nabla U\|_2. \]

The bound (2.1) is proved formally by multiplying (1.2) by $U$ and then integrating by parts. Noting the specific form of the nonlinearity,

\[ \langle w \cdot \nabla U, U \rangle = 0. \] (2.2)
This relation holds when \( \nabla \cdot w = 0 \) and the integral is absolutely summable, it can be proved for functions of compact support using integration by parts then extended to other classes of functions with a density argument. It holds in three dimensions when \( w \in L^3_\sigma \) and \( U \in \tilde{H}^1_\sigma \), or when \( U \in L^3_\sigma \cap \tilde{H}^1_\sigma \) and \( w \in \tilde{H}^1_\sigma \) since either assumption implies summability.

Fix \( f \) and \( U \) as a solution to (1.2) (\( U \) does not depend on time), we would like to find conditions on \( f \) which guarantee \( \|U\|_2 < \infty \). One of the key steps of our approach is to establish 'fast' decay of solutions to the system

\[
v_t + U \cdot \nabla v + \nabla p = \Delta v, \quad v(0) = f \quad \nabla \cdot v = 0.
\]

Formally, if \( v \) is a solution of (2.3) then \( \tilde{U} = \int_0^\infty v(t) \, dt \) solves

\[
U \cdot \nabla \tilde{U} + \nabla p = \Delta \tilde{U} + f, \quad \nabla \cdot \tilde{U} = 0.
\]

Recall we have fixed \( U \) earlier and it is also a solution for this PDE since it satisfies (1.2). As this PDE is linear and \( \nabla \cdot U = 0 \), solutions are unique and we may conclude \( \tilde{U} = U \).

Using Minkowski’s inequality for integrals we can see how the \( L^2 \) decay of \( v \) relates to the \( L^2 \) norm of \( U \):

\[
\|U\|_2 = \left\| \int_0^\infty v(t) \, dt \right\|_2 \leq \int_0^\infty \|v(t)\|_2 \, dt.
\]

In summary, if \( \|v(t)\|_2 \leq C(1+t)^{-\beta} \) with \( \beta > 1 \) we can expect \( U \in L^2 \).

Through a standard Fourier splitting argument we can only hope

\[
\|v\|_2 \leq C(1+t)^{-3/4},
\]

where the hold up for faster decay is the initial data. To get around this problem we will measure the difference

\[
w = v - \Phi \quad \text{where} \quad \Phi = e^{\Delta t} f.
\]

Here \( \Phi \) is the solution to the heat equation with initial data \( f \). The function \( w \) satisfies a parabolic equation with zero initial data and a forcing term which we can control by restricting \( f \):

\[
w_t + U \cdot \nabla w + \nabla p = \Delta w - U \cdot \nabla \Phi, \quad \nabla \cdot w = 0 \quad w(0) = 0.
\]

We can expect \( \|w(t)\|_2 \) to decay as \( (1+t)^{-5/4} \) and if the heat flow corresponding to \( f \) decays at least as fast we can say the same about \( v \), to make other parts of the argument work we need \( \Phi \) to decay faster. It is well known that the energy decay of the heat flow corresponding to \( f \) is intimately related to the behaviour of \( \tilde{f} \) near the origin, therefore an assumption made on the decay of \( \Phi \) is really an assumption on \( \tilde{f} \) near the origin. With this in mind make the following assumption on \( f \).

**Lemma 2.1.** If \( f \in A_q \) with \( q > 2 \), where \( A_q \) was defined in (1.3), and \( \Phi = e^{\Delta t} f \), then let \( q = 2 + \alpha \)

\[
\int_0^\infty \|\Phi(t)\|_2^2 \, dt \leq \mu_q (1 + \|f\|_2^2) = M_1,
\]

where \( \mu_q \) depends only on \( q \) and is defined in (2.7), and for any \( \epsilon < 1 \)

\[
\int_0^\infty \|\nabla \Phi(t)\|_2^2 (1+t)^{1+\epsilon} \, dt \leq \gamma_q (1 + \|f\|_2^2) = M_2,
\]

where \( \gamma_q \) depends only on \( q \) and is defined in (2.12).
Proof. First we focus on proving (2.5). Let $q > 2$ and take $\rho_0 > 0$ sufficiently small so that for all $\rho \leq \rho_0$:

$$\rho^{-2q} \int_{B(\rho)} |\hat{\Phi}(0)|^2 \, d\xi \leq 1.$$ 

Since $f \in A_q$ such a $\rho_0$ exists.

We now use the Fourier splitting method ([44]), starting with the well-known energy inequality for the heat equation with $0 < \rho(t) \leq \rho_0$.

$$\frac{1}{2} \frac{d}{dt} \| \Phi(t) \|_2^2 \leq -\| \nabla \Phi(t) \|_2^2 \leq -\rho^2 \int_{B(\rho)} |\hat{\Phi}(t)|^2 \, d\xi.$$ 

The pointwise bound $|\hat{\Phi}(\xi, t)| \leq |\hat{f}(\xi)|$ yields

$$\frac{d}{dt} \| \Phi(t) \|_2^2 + 2 \rho^2 \| \Phi(t) \|_2^2 \leq 2 \rho^2 \int_{B(\rho)} |\hat{\Phi}(t)|^2 \, d\xi \leq 2 \rho^{2+2q}.$$ 

Take $m = q + 1$ and choose $\rho(t) = (m/(2(C_0 + t)))^{1/2}$, where $C_0 > 0$ is large enough to guarantee $\rho(0) \leq \rho_0$. Solve the differential inequality with the integrating factor $(C_0 + t)^m$ to find

$$\| \Phi(t) \|_2^2 \leq \beta_q (m - q)^{-1}(C_0 + t)^{-q} + (C_0 + t)^{-m} \| \Phi(0) \|_2^2,$$

where $\beta_q = (q + 1)^{1+q/2}$. Since we can suppose $C_0 > 1$ and $\| \Phi(0) \|_2^2 = \| f \|_2^2$, by the choice of $m$ this implies

$$\| \Phi(t) \|_2^2 \leq \beta_q (1 + t)^{-q} + (1 + t)^{-q-1} \| f \|_2^2$$

$$\int_0^\infty \| \Phi(t) \|_2^2 \leq \beta_q (q - 1)^{-1} + q^{-1} \| f \|_2^2 \leq \mu_q (1 + \| f \|_2^2).$$

Since $\beta_q (q - 1)^{-1} \geq q^{-1}$ we can choose $\mu_q$ as

$$\mu_q = \beta_q (q - 1)^{-1}, \quad (2.7)$$

and (2.5) follows. To prove (2.6) first examine the energy inequality associated with the heat equation:

$$\| \Phi(t) \|_2^2 + 2 \int_0^\infty \| \nabla \Phi(t) \|_2^2 \, dt \leq \| f \|_2^2. \quad (2.8)$$

This implies the following two estimates:

Estimate 1.

$$\int_0^1 \| \nabla \Phi(t) \|_2^2 (1 + t)^{1+q} \, dt \leq 4 \| f \|_2^2. \quad (2.9)$$

Estimate 2. From (2.8) it follows that for some time $T_o \in [0, 1]$, we have $\| \nabla \Phi(T_0) \|_2^2 < \| f \|_2^2$. We again turn to the Fourier splitting method starting at this time but applying it to $\nabla \Phi$ and considering the initial time to be $T_o$. Arguing as before, for $t \geq T_o$

$$\frac{1}{2} \frac{d}{dt} \| \nabla \Phi(t) \|_2^2 \leq -\rho^2 \int_{B(\rho)} |\nabla \Phi(t)|^2 \, d\xi.$$
Hence using that \( |\hat{\Phi}(\xi, t)| \leq |\hat{f}(\xi)| \)
\[
\frac{d}{dt} \|\nabla \Phi(t)\|_{L_{\infty}}^2 + 2\rho^2 \int_{\mathbb{R}^3} |\nabla \Phi(t)|^2 d\xi \leq 2\rho^4 \int_{\mathbb{R}^3} |f|^2 d\xi \leq 2\rho^{4+2q}.
\]
Proceeding as in the first case, for \( t \geq T_0 \), letting \( m = q + 2, \) and let \( C_0 > 1, \) using the same \( \rho \) as above
\[
\|\nabla \Phi(t)\|_{L_{\infty}}^2 \leq \beta_q (m - q - 1)^{-1} (C_0 + t)^{-q-1} + (C_0 + t)^{-m} \|\nabla \Phi(T_0)\|_{L_{\infty}}^2 \leq \beta_q (1 + t)^{-q-1} + (1 + t)^{-m} \|f\|_{L_{\infty}}^2 \quad (2.10)
\]
since \( \max\{\beta_q (\epsilon + \alpha + 1)^{-1}, (q + 1)^{-1}\} \leq \beta_q. \) Thus it follows that
\[
\int_{T_0}^{\infty} \|\Phi(t)\|_{L_{\infty}}^2 (1 + t)^{1+\epsilon} \leq \beta_q (1 + \|f\|_{L_{\infty}}^2).
\]
Thus combining estimate 1 and the last inequality yields
\[
\int_{0}^{\infty} \|\Phi(t)\|_{L_{\infty}}^2 (1 + t)^{1+\epsilon} \leq (1 + \|f\|_{L_{\infty}}^2). \quad (2.11)
\]
Define
\[
\gamma_q = 4 + \beta_q \quad (2.12)
\]
and (2.6) follows. \( \square \)

**Remark 2.2.** This proof is based on theorem 5.7 in [1].

**Remark 2.3.** We note that \( \mu_q \leq 3((q + 1)/2)^q. \) This will be needed in what follows.

### 3. \( L^2 \) bounds for stationary solutions of the NSE

Throughout this section we will assume \( f \in A_q \) and therefore \( \Phi = e^{\Delta t} f \) satisfies (2.5)–(2.6).

We are focused on the study of solutions for the two auxiliary PDEs:

\[
U^i \cdot \nabla U^{i+1} + \nabla p = \Delta U^{i+1} + f, \\
\nabla \cdot U^{i+1} = 0.
\]  \( (3.1) \)

and

\[
U^i \cdot \nabla U^{i+1} + \nabla p = \Delta U^{i+1} - U^i \cdot \nabla \Phi, \\
\nabla \cdot U^{i+1} = 0, \quad w^{i+1}(0) = 0.
\]  \( (3.2) \)

When dealing with either PDE we take the function \( U^i \in H^1_0 \) fixed before hand. These PDEs will be used recursively to find approximate solutions for (1.2) and (2.4) respectively. In section 3.1 we recall existence theorems for these equations and section 3.2 contains the decay rate calculations for \( w^{i+1}. \) In section 3.3 we make precise the notion \( U^i = \int_0^\infty v^i(t) \, dt \) which is then combined with decay calculations in section 3.4 to find uniform bounds on \( U^i \) and show it is a Cauchy sequence in \( H^1_0 \) whose limit is a solution of (1.2).
3.1. Existence theorems

**Theorem 3.1.** Let $U^i \in H^1_0$ and $f \in \hat{H}^{-1}_0$. There exists a unique weak solution $U^{i+1}$ to the PDE (3.1) in the sense that for any $\phi \in \mathcal{V}$,
\[
(U^i \cdot \nabla U^{i+1}, \phi) + (\nabla U^{i+1}, \nabla \phi) = (f, \phi).
\]  
Moreover, this solution satisfies
\[
\|\nabla U^{i+1}\|_2^2 \leq \|f\|_{\hat{H}^{-1}_0}^2. \tag{3.4}
\]

**Proof.** We only outline the proof as similar PDEs are solved in the literature, see [6, 11, 14, 24, 30, 31, 48]. A typical approach is to construct Galerkin approximations by projecting the PDE onto finite dimensional subspaces of $H^1_0$. A uniform bound similar to (3.4) can be proved for each Galerkin approximation using an argument similar to that following (2.4). Once this bound is established it is possible to use the Banach–Alaoglu theorem to find a subsequence $\{U^{i+1}_n\}_{n \in \mathbb{N}}$ that converges weakly in $H^1_0$. The weak convergence is enough to pass to a limit in the linear terms. To pass through the nonlinear term one uses a stronger compactness theorem in the support of the test function $\phi$. \hfill $\square$

**Theorem 3.2.** Let $U^i \in H^1_0$ and $f \in A_\Phi$ with $\Phi = e^{\lambda t} f$. There exists a unique weak solution $w^{i+1} \in L^\infty(\mathbb{R}^+, L^2) \cap L^2(\mathbb{R}^+, \hat{H}^1_0)$ to the PDE (3.2) in the sense that for any $\phi \in C^1(\mathbb{R}^+; \mathcal{V})$,
\[
(w^{i+1}, \phi) + (U^i \cdot \nabla w^{i+1}, \phi) = -(\nabla w^{i+1}, \nabla \phi) - (U^i \cdot \nabla \Phi, \phi),
\]  
\[
\nabla \cdot w^{i+1} = 0, \quad w^{i+1}(0) = 0. \tag{3.5}
\]

Moreover, this solution satisfies
\[
\sup_t \|w^{i+1}(t)\|_2^2 + \int_0^\infty \|\nabla w^{i+1}(s)\|_2^2 \, ds \leq 4\|U^i\|_3^2 \|f\|_2^2. \tag{3.6}
\]

**Proof.** The PDE in question is closely related to the Navier–Stokes equation and we refer to the literature for similar arguments, see [4, 6, 23, 24, 32, 48]. It is typical to construct a sequence of Galerkin approximations which satisfies a uniform estimate similar to (3.6) and then use compactness arguments to pass through the limit. We now give a formal proof of (3.6) which can be used as an a priori estimate in this approach.

Multiply (2.4) by $w^{i+1}$ and integrate by parts, then use the bilinear relation (2.2) to find
\[
\frac{1}{2} \frac{d}{dt} \|w^{i+1}\|_2^2 + \|\nabla w^{i+1}\|_2^2 = (U^i \cdot \nabla w^{i+1}, \Phi),
\]  
\[
= \|U^i\|_3 \|\nabla w^{i+1}\|_2 \|\Phi\|_6,
\]  
\[
\leq \frac{1}{2} \|U^i\|_3^2 \|\Phi\|_6^2 + \frac{1}{2} \|\nabla w^{i+1}\|_2^2.
\]

The last line was obtained using H"older’s inequality and then Cauchy’s inequality. Putting this together with the Sobolev inequality yields
\[
\frac{d}{dt} \|w^{i+1}\|_2^2 + \|\nabla w^{i+1}\|_2^2 \leq 4\|U^i\|_3^2 \|\nabla \Phi\|_2^2,
\]
where the constant 4 comes from the Sobolev inequality
\[
\|u\|_{L^p(\mathbb{R}^3)} \leq C(n, p) \|Du\|_{L^p(\mathbb{R}^3)}, \tag{3.7}
\]
where in our case for $p^* = 6$, $p = 2$, $n = 3$ the constant satisfies $C = 4$, [9].

Integrating (3.7) in time combined with the heat energy estimate $\int_0^\infty \|\nabla \Phi\|_2^2 \, dt \leq \|f\|_2^2$ finishes the proof. \hfill $\square$
Remark 3.3. In the theorems above the assumption $U^i \in H_1^\sigma$ is enough to ensure $U^i \cdot \nabla U^{i+1} \in \dot{H}_\sigma^{-1}$ and $U^i \cdot \nabla w^{i+1} \in \dot{H}_\sigma^{-1}$ a.e. That is

$\langle U^i \cdot \nabla U^{i+1}, \phi \rangle \leq C(U^i) \|\nabla \phi\|_2^2$, and $\langle U^i \cdot \nabla w^{i+1}, \phi \rangle \leq C(U^i) \|\nabla \phi\|_2^2$.

Therefore, we are justified in multiplying the PDEs by $U^{i+1}$ and $w^{i+1}$, respectively, and integrating in space. Indeed, one just chooses a sequence of test functions approximating either $U^{i+1}$ or $w^{i+1}$ and passes the limit through the weak formulation (3.4) or (3.5). This will be a common technique in the rest of the work. In both cases a stronger existence theorem is true but outside the scope of this paper.

3.2. Decay of $w$

This section contains energy decay calculations for $w$, the estimates are an application of the Fourier splitting method with bootstrapping. The first step in the procedure is to apply the Fourier splitting method using the bound (3.6) to find a preliminary decay rate. Once established, this preliminary rate is used to deduce a faster decay rate. This procedure is repeated until the recursion does not lower the rate again; in this case the hold up will be from estimates on the nonlinear term. The sequence of lemmas leading to theorem 3.6 set up the bootstrap situation which is the main part of the proof for the theorem, establishing (3.12) is the main goal of this section. The calculations are formal but can be made rigorous by applying them to a sequence of approximating solutions to (2.4) (see [44]) or working directly with the weak formulation (see remark 3.3). We begin with an estimate for $\hat{w}$.

Lemma 3.4. Let $\hat{w}^{i+1}$ be the solution of (3.2) given by theorem 3.2 with $U^i$ and $f$ satisfying the assumptions of the theorem. Again let $\Phi = e^{\Delta t} f$. Then,

$$|\hat{w}^{i+1}| \leq 10 |\xi| \left( \|U^i\|_2 \int_0^t \|w^{i+1}(s)\|_2 \, ds + \|U^i\|_3 \int_0^t \|\nabla \Phi(s)\|_2 \, ds \right).$$

(3.8)

Proof. Through the Fourier transform of (3.2), noting the initial data are zero, we write

$$\hat{w}^{i+1} = -\int_0^t e^{-|\xi|(t-s)} (\xi \cdot \hat{U}^{i} \hat{w}^{i+1} + \xi \cdot \hat{\phi}^{i+1} + \xi \cdot \hat{U}^{i} \Phi(s)) \, ds.$$

Combining Plancherel’s theorem, with the Hölder and Sobolev inequality yields

$$|\hat{U}^{i} \hat{w}^{i+1}| + |\hat{U}^{i} \Phi| \leq \|U^i\|_2 \|w^{i+1}\|_2 + 4\|U^i\|_3 \|\nabla \Phi\|_2.$$

Taking the divergence of (2.4), then the Fourier transform, one has the bound

$$|\hat{\phi}| \leq 9(\|U^i \hat{w}^{i+1}\| + \|U^i \Phi\|).$$

Putting these together completes the proof. □

Lemma 3.5. Let $\hat{w}^{i+1}$ be the solution of (3.2) given by theorem 3.2 with $U^i$ and $f$ satisfying the assumptions of the theorem and $\Phi = e^{\Delta t} f$. Then, for any $m \geq 3$, $\hat{w}^{i+1}$ satisfies the differential inequality

$$\frac{d}{dt} ((1+t)^m \|w^{i+1}\|_2^2) \leq \mathcal{H}_m (1+t)^{m-5} \left( \|U^i\|_2 \int_0^t \|w^{i+1}(s)\|_2 \, ds + \|U^i\|_3 \int_0^t \|\nabla \Phi(s)\|_2 \, ds \right)^2 + K_m \|U^i\|_2^2 \|\nabla \Phi\|_2^2 (1+t)^m,$$

(3.9)

where $K_m = \left( \frac{4}{3} \right)^2 m^7$, and $\mathcal{H}_m = \frac{2m}{2m - 5} K_m$. (3.10)
Proof. Multiplying (3.2) by $w^{i+1}$, after integration by parts then applying the bilinear relation (2.2), we write
\[
\frac{1}{2} \frac{d}{dt} \|w^{i+1}\|^2_2 + \|\nabla w^{i+1}\|^2_2 = \langle U^i \cdot \nabla w^{i+1}, \Phi \rangle \\
\leq \|U^i\|_3 \|\nabla w^{i+1}\|_2 \|\Phi\|_6 \\
\leq 4\|U^i\|^2_3 \|\nabla \Phi\|^2_2 + \frac{1}{2} \|\nabla w^{i+1}\|^2_2.
\] (3.11)

Now we split the viscous term in Fourier space around the ball $B(R)$ using the Plancherel theorem:
\[
-\|\nabla w^{i+1}\|^2_2 \leq - \int_{B(R)^c} |\xi|^2 |\hat{w}^{i+1}|^2 d\xi \\
\leq -R^2 \int_{B(R)^c} |\hat{\nu}^{i+1}|^2 d\xi \\
\leq -R^2 \|\hat{\nu}^{i+1}\|^2_2 + R^2 \int_{B(R)} |\hat{\nu}^{i+1}|^2 d\xi.
\]
Combining this with (3.11)
\[
\frac{d}{dt} \|w^{i+1}\|^2_2 + R^2 \|w^{i+1}\|^2_2 \leq R^2 \int_{B(R)} |\hat{\nu}^{i+1}|^2 d\xi + 8\|U^i\|^2_3 \|\nabla \Phi\|^2_2.
\]
Then using (3.8) we bound
\[
\int_{B(R)} |\hat{\nu}^{i+1}|^2 d\xi \\
\leq \left(\frac{10}{3}\pi\right)^2 \left(\|U^i\|_2 \int_0^t \|w^{i+1}(s)\|_2 ds + \|U^i\|_3 \int_0^t \|\nabla \Phi(s)\|_2 ds\right)^2 R^5.
\]
Hence,
\[
\frac{d}{dt} \|w^{i+1}\|^2_2 + R^2 \|w^{i+1}\|^2_2 \\
\leq \left(\frac{10}{3}\pi\right)^2 R^2 \left(\|U^i\|_2 \int_0^t \|w^{i+1}(s)\|_2 ds + \|U^i\|_3 \int_0^t \|\nabla \Phi(s)\|_2 ds\right)^2 \\
+ \left(\frac{10}{3}\pi\right)^2 \|U^i\|^2_3 \|\nabla \Phi\|^2_2.
\]

In the preceding inequality we choose $R^2 = m(1+t)^{-1}$ and then use $(1+t)^m$ as an integrating factor to establish the lemma. □

Theorem 3.6. Let $w^{i+1}$ be the solution of (2.4) given by theorem 3.2 with $U^i$ and $f$ satisfying the assumptions of the theorem. Suppose $m \geq 4$. Then, $w^{i+1}$ satisfies the decay bound
\[
\|w^{i+1}(T)\|^2_2 \leq C(m, \epsilon)(1 + M_2)\|U^i\|^2_3(1 + \|U^i\|^2_2)^6(1 + \|f\|^2_2)(1 + T)^{-\beta} \tag{3.12}
\]
where $\beta = \min\{1 + \epsilon, \frac{5}{2}\}$.

Remark 3.7. The exponent of $(1 + T)$ is such that $\|w^{i+1}\|_2$ is integrable over all time.
Proof. We recall here, to make it easier to read, the parts of the bounds from (3.6) and (2.6) necessary in the following.

\[
\int_0^\infty \|\nabla \Phi(t)\|^2_t \, dt < \int_0^\infty \|\nabla \Phi(t)\|^2_t (1 + t)^{1+\epsilon} \, dt \leq M_2 < \infty, \quad (3.13)
\]

\[
\|w^{i+1}(s)\|^2_t \, ds \leq 4\|U^i\|^2_t \|f\|^2_t. \quad (3.14)
\]

Combining these two with (3.9) yields

\[
\frac{d}{dt}((1 + t)^m \|w^{i+1}\|^2_t) \leq \frac{d}{dt}(1 + t)^m \|U^i\|^2_t \|f\|^2_t + \|U^i\|_2 M_2^{1/2}^2 \quad (3.15)
\]

where the constants are given by (3.10).

The next step is to integrate in time; the first term on the RHS can be integrated directly

\[
\int_0^T \|\nabla \Phi(t)\|^2_t (1 + t)^m \, dt \leq K_m M_2 \|U^i\|^2_t (1 + T)^{m-1-\epsilon}. \quad (3.16)
\]

Let

\[
\tilde{D}_m = \max \left\{ \frac{4H_m}{2m - 1}, K_m \right\}. \quad (3.17)
\]

Then we get an initial decay bound

\[
\|w^{i+1}(T)\|^2_t \leq \tilde{D}_m (1 + M_2) \|U^i\|^2_t (1 + \|U^i\|_2) (1 + \|f\|^2_t) (1 + T)^{-1/2}. \quad (3.17)
\]

Now we begin the bootstrapping procedure. Proceeding in a nearly identical way to the argument immediately above, use (3.9) with (3.17) instead of (3.6), then integrate in time:

\[
\|w^{i+1}(T)\|^2_t \leq 2\tilde{D} (1 + M_2) \|U^i\|^2_t (1 + \|U^i\|_2)^2 (1 + \|f\|^2_t) (1 + T)^{-1}. \quad (3.18)
\]

This process can be repeated indefinitely but the ‘best’ decay rate will be obtained after (at most) 6 iterations; here ‘best’ is meant in the sense of best decay rate obtainable from (3.9). This is in fact the best decay rate that can be seen by examining the term \(\int_0^t \|w^{i+1}(s)\|^2_t \, ds\) in (3.9).

Once we have established \(\|w^{i+1}\|_2 \leq C(1 + t)^{-\mu}\) for \(\mu > 1\) this term integrates to a constant and we obtain the ‘best decay rate.’ This rate will then be the minimum of \(\frac{2}{\epsilon}\) (from the first term) and \(1 + \epsilon\) (from the second). As the bootstrapping steps are nearly identical to the above arguments and tedious to write out, we skip to the final step:

\[
\|w^{i+1}(T)\|^2_t \leq \tilde{D} (1 + M_2) \|U^i\|^2_t (1 + \|U^i\|_2)^2 (1 + \|f\|^2_t) (1 + T)^{-1/2}, \quad (3.19)
\]

where \(\tilde{D}_m = D\). Note that the constant \(D_m\) will work since when we integrate the new coefficients will all be \(\leq 2D_m\). \(\square\)

3.3. Relation between \(U^i\) and \(w^i\)

In this section we make precise, for our approximate solutions, the formal notion \(U^i = \int_0^\infty v^i(t) \, dt\). We show approximations of the integral \(\int_0^\infty v^i(t) \, dt\) are bounded uniformly in \(L^2\) and are Cauchy with a limit which is a solution of (3.1). Once this is established we apply the decay results from the previous section to find a uniform bound in \(L^2\) for \(U^i\). Throughout this section we use \(\Phi = e^{\Delta t} f\).
Lemma 3.8. Let $w^{i+1}$ be the solution of (2.4) given by theorem 3.2 with $U^i$ and $f$ satisfying the assumptions of the theorem. The function $v^{i+1} = w^{i+1} + \Phi$ satisfies $\int_0^\infty v^{i+1}(t) \, dt \in L_\infty^2$.

Proof. For each fixed $i$ define the sequence $\{V_n^{i+1}\}_{n \in \mathbb{N}} \subset L_2^2$ by

$$V_n^{i+1} = \int_0^n v^{i+1}(t) \, dt.$$  

Since $v^{i+1}(t) \in L_2^2$ a.e. the sequence $\{V_n^{i+1}\}$ is well defined. Combining Minkowski’s inequality for integrals with (2.5) and (3.12) shows how the sequence $\{V_n^{i+1}\}$ is bounded uniformly (for $n$) in $L_2^2$. Indeed,

$$\|V_n^{i+1}\|_2 \leq \int_0^n \|w^{i+1}(t)\| \, dt + \int_0^n \|\Phi(t)\| \, dt.$$  

Also,

$$\|V_n^{i+1} - V_m^{i+1}\|_2 \leq \int_n^m \|w^{i+1}\|_2 \, dt.$$  

(3.18)

Observing (3.12) and the decay of $\Phi$ follows since $f \in A_\phi$ and we know the integral $\int_0^\infty \|v^{i+1}\|_2 \, dt$ is finite so the RHS of (3.18) tends to zero as $n \to \infty$. Following well-known arguments to prove a contraction lemma we can quickly deduce $\{V_n^{i+1}\}_{n \in \mathbb{N}}$ is Cauchy in $L_2^2$ and has a limit which we label $\int_0^\infty v^{i+1}(t) \, dt$.

Remark 3.9. The above lemma also implies $\int_0^\infty v^{i+1}(t) \, dt$ is finite a.e. in $\mathbb{R}^3$.

Lemma 3.10. Let $w^{i+1}$ be the solution of (2.4) given by theorem 3.2 with $U^i$ and $f$ satisfying the assumptions of the theorem. The function $v^{i+1} = w^{i+1} + \Phi$ satisfies $\int_0^\infty v^{i+1}(t) \, dt = U^{i+1}$.

Proof. To prove this lemma we show $\int_0^\infty v^{i+1}(t) \, dt$ is a weak solution of (3.1) and then use the uniqueness implied by theorem 3.1 to conclude the desired result. Let $\{V_n^{i+1}\}_{n \in \mathbb{N}}$ be as in the previous proof.

In (3.5) choose $\phi$ to be any member of $\mathcal{V}$ (so that it is constant in time). Use the relation $v^{i+1} = w^{i+1} + \Phi$ and then integrate in time:

$$\int_0^n \left( \frac{d}{dt} v^{i+1}(t), \phi \right) + \left( U^i \cdot \nabla v^{i+1}(t), \phi \right) \, dt$$

$$= -\int_0^n \left( \nabla v^{i+1}(t), \nabla \phi \right) \, dt.$$  

(3.19)

After changing the order of integration and evaluating the first integral this becomes

$$\left( v^{i+1}(n), \phi \right) + \left( U^i \cdot \nabla V_n^{i+1}, \phi \right) = -\left( \nabla V_n^{i+1}, \nabla \phi \right) + \left( f, \phi \right).$$

Observe the first term on the LHS tends to zero as $n \to \infty$. This follows from the decay bound (3.12) which implies $w^{i+1}$ tends to zero on compact sets and a similar well-known property for the heat equation. The strong convergence of $\{V_n^{i+1}\}$ in $L_2$ is enough to pass the limit through the remaining terms. Indeed, if $V^{i+1} = \int_0^\infty v^{i+1}(t) \, dt$ is this limit,

$$\left| \left( U^i \cdot \nabla (V_n^{i+1} - V^{i+1}), \phi \right) \right| \leq \|\nabla U^i\|_2 \| V_n^{i+1} - V^{i+1} \|_2 \|\nabla \phi\|_3.$$  

As $n \to \infty$ this tends to zero for each test function $\phi \in \mathcal{V}$, hence $V^{i+1}$ is a weak solution of (3.1). The uniqueness implied by theorem 3.1 finishes the proof of the lemma. □
Lemma 3.11. Let \( U^{i+1} \) be the solution of (1.2) given by theorem 3.1 with \( U^i \) and \( f \) satisfying the assumptions of the theorem. Then the function \( U^{i+1} \) satisfies
\[
\|U^{i+1}\|_2^2 \leq C(m, \epsilon)(1 + \mathcal{M}_2)\|U^i\|_3^2(1 + \|U^i\|_2^3)^\beta(1 + \|f\|_2^3) + \mathcal{M}_1^3.
\] (3.20)

Proof. Define \( u^{i+1} = u^{i+1} + \Phi \). Just as in the proof of lemma 3.8 combine Minkowski’s inequality for integral with (2.5) and (3.12), but this time use the relation from lemma 3.10.

\[
\|U^{i+1}\|_2 \leq \int_0^\infty \|u^{i+1}(t)\|_2 \, dt + \int_0^\infty \|\Phi(t)\|_2 \, dt
\]
\[
\leq C(m, \epsilon)^{1/2}(1 + \mathcal{M}_2)\|U^i\|_3(1 + \|U^i\|_2^3)^\beta(1 + \|f\|_2^3) + \mathcal{M}_1.
\]

where \( C = C(m, \epsilon) = \mathcal{D}m\epsilon^{-1} \).

3.4. Convergence of \( U^i \)

The goal of this subsection is to find the limit of the approximating sequence \( U^i \) and show this is a solution of the steady state Navier–Stokes equation. Later in the section we make two assumptions on \( f \): they are smallness assumptions and allow a contraction argument to show \( U^i \) is Cauchy. The assumptions will depend on how big we will allow the \( L^2 \) norm of \( U \) and throttle \( \|\nabla U\|_1 \) so that a product of the \( L^2 \) and \( \dot{H}^{1}_q \) norms of \( U \) is small. We will label this maximum value of the \( L^2 \) norm \( M \) (our choice) and keep it fixed throughout the rest of this section.

In what follows we suppose that the constant \( M \) satisfies \( M > \mu_q \), where \( \mu_q \) was defined by (2.7)

Lemma 3.12. Let \( U^{i+1} \) be the solution of (1.2) given by theorem 3.1 with \( U^i \) and \( f \) satisfying the assumptions of the theorem. Suppose additionally that \( f \in L^2 \cap A_q \). Then there exists a constant \( \tilde{B}_M \) so that if
\[
\max\{\|f\|_2, \|f\|_\dot{H}^{-1}\} < \max\left\{ \tilde{B}_M, \left( \frac{M - \mu_q}{\mu_q} \right)^{1/2} \right\}
\]
and \( \|U^i\|_2 \leq M \),

then \( \|U^{i+1}\|_2 \leq M \).

Proof. We note first that for our purposes it suffices to work with \( D_4 \). Define
\[
\mathcal{C}_4 = \mathcal{D}_4 \epsilon^{-1} \gamma_q,
\] (3.21)
where \( \gamma_q \) is defined in (2.12). Going back to (3.20) and using the estimate
\[
\|U^i\|_3^2 \leq \|U^i\|_2 \|U^i\|_6
\]
\[
\leq M \|f\|_\dot{H}^{-1},
\]
the right hand side can be bounded by
\[
\text{RHS} = 2\mathcal{C}_4(1 + \|f\|_2^3)\|f\|_\dot{H}^{-1}M(1 + M^2)^\beta + \mathcal{M}_1^3.
\]

Let \( Y = L^2 \cap \dot{H}^{-1} \) and let \( \mathcal{C}_0 = 4\mathcal{C}_4 \). Then the last inequality yields combined with (3.20)
\[
\|U^{i+1}\|_2^2 \leq \mathcal{C}_0 M(1 + M^2)^\beta \|f\|_Y(1 + \|f\|_2^3) + \mathcal{M}_1.
\]
Since we want to bound \( \|U^{i+1}\|_2^2 \) by \( M^2 \), we make the RHS of the last inequality equal to \( M^2 \).

Let \( Z = \|f\|_Y \). Hence we need to find the positive root \( \tilde{B}_m \) of the following polynomial.
\[
\frac{M^2 - \mathcal{M}_1^3}{\mathcal{C}_0 M(1 + M^2)^\beta} = Z(1 + Z^2).
\] (3.22)
This solution is unique among all solutions which satisfy (2.1) and have a finite positive root $\tilde{B}_M$. It follows easily that $\tilde{B}_M$ is the constant stated in the lemma. □

Define now

$$B_M = \max \left\{ \tilde{B}_M, \left( \frac{M - \mu_q}{\mu_q} \right)^{-1/2} \right\}.$$ (3.23)

**Theorem 3.13.** Let $f$ and $M$ be as in lemma 3.12. There exists a constant $C_M$, that depends only on $M$, $q$ such that if $\max\{\|f\|_2, \|f\|{H^1}\}\leq C_M$ then

(i) The PDE (1.2) has a weak solution $U \in H^1_0$ (in the sense of (1.4)).

(ii) This solution satisfies $\|U\|_2 \leq M$ in addition to (2.1).

(iii) This solution is unique among all solutions which satisfy (2.1) and have a finite $L^2$ norm.

**Proof.**

The constant $C_M$ in the above statement will be shown to be the minimum of the constant in lemma 3.12 and $1/16M$. That is

$$C_M = \min\{B_M, (16M)^{-1}\},$$ (3.24)

where $B_M$ was defined above. Let $U^0 \in H^1_0$ be such that $\|U^0\|_2 \leq M$ and $\|\nabla U^0\|_2^2 \leq \|f\|{H^1}$. Starting with $U^0$, solve (3.1) recursively using theorem 3.1 to find a sequence $\{U^i\}_{i=0}^\infty$ which satisfies $\|\nabla U^i\|_2 \leq \|f\|{H^1}$.

Lemma 3.12 provides the uniform bound $\|\nabla U^i\|_2 \leq M$ and so its limit, if it exists, must also satisfy this bound. We will now show this sequence is Cauchy in $H^1_0$ and a limit does indeed exist. The difference $Y^{i+1} = U^{i+1} - U^i$ solves

$$U^i \cdot \nabla Y^{i+1} + Y^i \cdot \nabla U^i + \nabla p = \Delta Y^{i+1}.$$ 

After multiplying this by $Y^{i+1}$, integrating by parts and using the bilinear relation (2.2) one can deduce

$$\|\nabla Y^{i+1}\|_2^2 = \langle Y^i \cdot \nabla Y^{i+1}, U^i \rangle$$

$$\leq \|Y^i\|_6 \|\nabla Y^{i+1}\|_2 \|U^i\|_3$$

$$\leq 8\|\nabla Y^i\|_2^2 \|U^i\|_2 \|\nabla U^i\|_2^{1/2} + \frac{1}{2} \|\nabla Y^{i+1}\|_2^2.$$ 

The above sequence relies on Hölder’s inequality, Cauchy’s inequality and the Gagliardo–Nirenberg–Sobolev inequality (3.7) with $p^* = 6, p = 2, n = 3$. This implies

$$\|\nabla Y^{i+1}\|_2^2 \leq 16 \|U^i\|_2 \|\nabla U^i\|_2 \|\nabla Y^i\|_2^2$$

$$\leq 16M \|f\|{H^1} \|\nabla Y^i\|_2^2.$$ (3.25)

Note the multiplication by $Y^{i+1}$ is justified since all $U^i$ (and hence all $Y^i$) are bounded in $H^1_0$. Using this bound recursively one finds

$$\|\nabla Y^{i+1}\|_2^2 \leq (16M \|f\|{H^{-1}})^i \|\nabla Y^1\|_2^2$$

$$\leq (16M \|f\|{H^{-1}})^{i+1} 2 \|f\|{H^{-1}}.$$ (3.26)

The last step relies on the uniform bound for $\|\nabla U^i\|_2 \leq \|f\|{H^{-1}}$. By assumption, $\|f\|{H^{-1}} < 1/16M$, hence $Y^i$ tends to zero in $H^1_0$ which implies $U^i$ is Cauchy, call its limit $\tilde{U}$. Through this construction we can also be sure $\|U\|_2 \leq M$. Using standard arguments this strong
convergence is enough to pass the limit through (3.1) and show \( \tilde{U} \) is a solution of (1.2). For completeness we will demonstrate how to pass through the nonlinear term:

\[
\langle \tilde{U} \cdot \nabla \tilde{U}, \phi \rangle - \langle U^i \cdot \nabla U^i, \phi \rangle \leq I + II,
\]

\[
I = |\langle \tilde{U} \cdot \nabla (\tilde{U} - U^i), \phi \rangle|,
\]

\[
II = |\langle (\tilde{U} - U^i) \cdot \nabla U^i, \phi \rangle|.
\]

To show \( I \to 0 \) use Hölder's inequality:

\[
I \leq \| \tilde{U} \|_3 \| \nabla (\tilde{U} - U^i) \|_2 \| \phi \|_6.
\]

Since the \( L^3 \) norm of \( \tilde{U} \) and the \( L^6 \) norm of \( \phi \) are bounded, the strong convergence \( U^i \to \tilde{U} \) in \( \tilde{H}^1_0 \) shows the RHS tends to zero. The term \( II \) is handled in a nearly identical way.

It remains to establish that \( \tilde{U} \) is the unique solution of (1.2) among all solutions which satisfy (2.1) and have finite \( L^2 \) norm. Let \( U \) be any other solution which satisfies (2.1) and has a finite \( L^2 \) norm. The difference \( Y = U - \tilde{U} \) satisfies

\[
U \cdot \nabla Y + Y \cdot \nabla \tilde{U} + \nabla p = \Delta Y.
\]

(3.27)

Since \( U \) and \( \tilde{U} \) are bounded in \( L^2 \) and \( \tilde{H}^1_0 \), we are allowed to multiply this equation by \( Y \). Then, proceeding in the same way as the lines leading to (3.25),

\[
\| \nabla Y \|_2^2 \leq 16M \| f \|_{\tilde{H}^{-1}_0} \| \nabla Y \|_2^2.
\]

The assumption on \( f \) made earlier in this proof is enough to guarantee \( \| f \|_{\tilde{H}^{-1}_0} < 1/16M \) and implies the solution is unique.

\[\square\]

**Remark 3.14.** This is exactly theorem 1.2. Examining (3.27) it seems \( Y \in \tilde{H}^1_0 \) might be enough to obtain uniqueness as the first term on the LHS would formally integrate to zero after multiplying by \( Y \). Unfortunately, following techniques used in this paper, we are not able to multiply (3.27) by \( Y \) unless we also know \( Y \in L^3 \). Indeed, using Hölder’s inequality and the Gagliardo–Nirenberg–Sobolev inequality one can see \( U \cdot \nabla Y \in \tilde{H}^1_0 \cap L^3 \subseteq \tilde{H}^1_0 \), but not \( U \cdot \nabla Y \in \tilde{H}^1_0 \).

The uniqueness in this theorem, with the exact same proof, could instead be stated, ‘This solution is unique among all solutions which satisfy (2.1) and have a finite \( L^3 \) norm’ and using some other technique it may be possible to expand this uniqueness theorem further.

**Remark 3.15.** The last theorem establishes theorem 1.2 stated in the introduction.

### 4. Stability of solutions

An important property of steady state solutions for physical problems is stability: ‘If a steady state solution is perturbed will it return to the same solution?’ In the setting of the Navier–Stokes equation we investigate the stability of a solution \( U \) for (1.2) by considering a perturbation \( w_0 \) and examining the long term behaviour of the solution of the Navier–Stokes equation (1.1) with initial data \( u(0) = U + w_0 \). In particular, one would like to know what conditions on \( w_0 \) will guarantee solutions of (1.1) approach \( U \) as time becomes large. An equivalent problem, found by subtracting (1.2) from the (1.1), is to determine when solutions of the following PDE tend to 0:

\[
w_t + u \cdot \nabla w + w \cdot \nabla U + \nabla p = \Delta w,
\]

\[
\nabla \cdot w = 0 \quad w(0) = w_0.
\]

(4.1)
A thorough examination of stability for the steady state Navier–Stokes equation is currently outside the reach of modern techniques (even in bounded domains) but using energy techniques we can prove strong stability results for perturbations of finite energy under certain restraints on \( f \). Following the literature we introduce the following notion of stability, commonly called nonlinear stability, and find conditions on \( f \) which guarantee this type of stability.

**Definition 4.1.** We say a solution \( U \in H^1_\sigma \) of (1.2) corresponding to \( f \in X \) and given by theorem 1.2 is nonlinearly stable if it satisfies the following: if \( w_0 \in L^2_\sigma \) is a perturbation and \( u \) is the solution of the Navier–Stokes equation (1.1) with initial data \( w_0 + U \) which satisfies, for any \( T > 0 \),

\[
    u \in L^\infty_\sigma (0, T; L^2) \cap L^2(0, T; H^1_\sigma)
\]

then

(i) For every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
    \| w_0 \|_2 \leq \delta \quad \text{implies} \quad \sup_{t \in \mathbb{R}} \| u(t) - U \|_2 \leq \epsilon
\]

(ii) \( u(t) \) tends to \( U \) as time becomes large, that is

\[
    \lim_{t \to \infty} \| u(t) - U \|_2 = 0
\]

To start we can multiply (4.1) by \( w \), then integrate by parts and use the bilinear relation (2.2) to find a formal energy inequality

\[
    \frac{1}{2} \frac{d}{dt} \| w \|_2^2 + \| \nabla w \|_2^2 = -\langle w \cdot \nabla U, w \rangle 
\]

\[
    \leq \| w \|_6 \| U \|_3 \| \nabla w \|_2 
\]

\[
    \leq C \| U \|^{\frac{3}{2}} \| \nabla U \|_2^{\frac{1}{2}} \| \nabla w \|_2^2. 
\]

The last two lines above were obtained with a combination of Hölder’s inequality, Cauchy’s inequality and the Gagliardo–Nirenberg–Sobolev inequality. If \( f \) is chosen so that \( U \) is small in \( H^1_\sigma \) norm \( (C \| U \|_2^{\frac{1}{2}} \| \nabla U \|_2^{1/2} \leq 1/2) \) the above inequality becomes

\[
    \frac{d}{dt} \| w \|_2^2 + \| \nabla w \|_2^2 \leq 0. \tag{4.2}
\]

This differential inequality implies \( \| w \|_2^2 \) is bounded uniformly. As \( U \in L^2_\sigma \) we can say the same about \( u = w + U \). In other words, if one considers a solution \( U \) given by theorem 1.2 then any finite energy perturbation will stay close to \( U \). This is condition (i) in definition 4.1. The rest of this section will be spent proving the stronger statement, that all finite energy perturbations return to \( U \).

### 4.1. Existence theorems

Here we state the existence theorems and properties of the two PDEs examined in this section. Proofs of these theorems can be found in the literature and are omitted.

**Theorem 4.2.** Given \( T > 0 \), initial data \( u_0 \in L^2_\sigma \) and a forcing function \( f \in \dot{H}^{1}_\sigma \), the PDE (1.1) has a weak solution

\[
    u \in L^\infty([0, T]; L^2_\sigma) \cap L^2([0, T]; H^1_\sigma) \tag{4.3}
\]

in the whole space \( \mathbb{R}^3 \) which satisfies, for any \( \phi \in \mathcal{V} \),

\[
    \langle u_\cdot, \phi \rangle + \langle \nabla u, \nabla \phi \rangle + \langle u \cdot \nabla u, \phi \rangle = \langle f, \phi \rangle
\]
Proof. See [4, 6, 23, 24, 32, 48].

**Theorem 4.3.** Let \( u \) satisfy (4.3) and \( U \in H^1 \). There is a constant \( C \) such that if \( \| U \|_{H^1} \leq C \) the PDE (4.1) has a unique weak solution
\[
w \in L^\infty(\mathbb{R}^+, L^2) \cap L^2(\mathbb{R}^+, H^1)
\]
satisfying, for any \( \phi \in \mathcal{V}, \)
\[
\langle w_t, \phi \rangle + \langle \nabla w, \nabla \phi \rangle + \langle u \cdot \nabla w, \phi \rangle + \langle w \cdot \nabla U, \phi \rangle = \langle f, \phi \rangle
\]
as well as the following energy inequalities:
\[
d \frac{d}{dt} \| w \|_2^2 + \| \nabla w \|_2^2 \leq 0 \tag{4.4}
\]
\[
\sup_{t \in \mathbb{R}^+} \| w(t) \|_2^2 + \int_0^\infty \| \nabla w \|_2^2 \leq \| w(0) \|_2^2. \tag{4.5}
\]

**Proof.** The proof of this theorem is similar to the one immediately preceding with the exception of the a priori bounds which are argued formally preceding (4.2). The linearity of the equation implies uniqueness (which is not known for general solutions of (1.1)).

4.2. Decay of \( w \)

In this section we prove a decay property for solutions \( w \) given by theorem 4.3. We show \( \| w \|_2 \to 0 \) using a method developed for the Navier–Stokes equation in [38]. The method relies on generalized energy inequalities for the solutions which allow the energy to be decomposed in high and low frequencies. These are estimated independently and shown to approach zero.

**Lemma 4.4.** Let \( u \) and \( U \) satisfy the assumptions of theorem 4.3 with \( \| U \|_{H^1} \) less than the given constant. Let \( \phi = e^{-|\xi|^2}, \psi = 1 - \phi \) and \( E(t) \in C^1([0, \infty); L^\infty) \). The solution given by theorem 4.3 satisfies the following two generalized energy inequalities:
\[
\| \hat{\phi} * w(t) \|_2^2 \leq \| e^{\Delta(t-s)} \hat{\phi} * w(s) \|_2^2 + 2 \int_s^t |\langle u \cdot \nabla w, e^{2\Delta(t-\tau)} \hat{\phi} * w \rangle| \, d\tau
\]
\[
+ 2 \int_s^t |\langle w \cdot \nabla U, e^{2\Delta(t-\tau)} \hat{\phi} * w \rangle| \, d\tau \tag{4.6}
\]
\[
E(t) \| \hat{\psi} \hat{w}(t) \|_2^2 \leq E(s) \| \hat{\psi} \hat{w}(s) \|_2^2 - 2 \int_s^t E(\tau) \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau + \int_s^t E'(\tau) \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau
\]
\[
+ 2 \int_s^t E(\tau) \| \langle \hat{u} \cdot \nabla w, (1 - \psi^2) \hat{w} \rangle \| \, d\tau
\]
\[
+ 2 \int_s^t E(\tau) \| \langle \hat{u} \cdot \nabla w, \hat{w} \rangle \| \, d\tau \tag{4.7}
\]
Proof. We give a formal proof here which can be made precise by considering an approximating sequence, see [38] for details.

To see the first inequality multiply the PDE (4.1) by $e^{2\Delta(t-s)}\hat{\phi} \ast \hat{\phi} \ast w$ and integrate from $s$ to $t$. The assumptions are enough to ensure all integrals are finite and this multiplication makes sense. After integration by parts

$$
\|\hat{\phi} \ast w(t)\|_2^2 \leq \|e^{\Delta(t-s)}\hat{\phi} \ast w(s)\|_2^2 - \int_s^t \|\nabla(\Delta(t-\tau)\hat{\phi}) \ast w\|_2^2 \, d\tau
$$

$$
+ \int_s^t (\partial_\tau (e^{\Delta(t-\tau)}\hat{\phi} \ast w), e^{2\Delta(t-\tau)}\hat{\phi} \ast w) \, d\tau
$$

$$
+ 2 \int_s^t |(u \cdot \nabla w, e^{2\Delta(t-\tau)}\hat{\phi} \ast w)| \, d\tau
$$

$$
+ 2 \int_s^t |(w \cdot \nabla U, e^{2\Delta(t-\tau)}\hat{\phi} \ast w)| \, d\tau
$$

e^{\Delta(t+s)}\hat{\phi}$ describes a heat flow so the second and third terms on the RHS add to zero, this proves (4.6). For the second inequality, take the Fourier transform of (4.1) then multiply by $\hat{\psi}^2\hat{w}$. After integration by parts one finds

$$
E(t)\|\psi \hat{w}(t)\|_2^2 \leq E(s)\|\psi \hat{w}(s)\|_2^2 - 2 \int_s^t E(\tau)\|\xi \psi \hat{w}(\tau)\|_2^2 \, d\tau
$$

$$
+ \int_s^t E'(\tau)\|\psi \hat{w}(\tau)\|_2^2 \, d\tau + 2 \int_s^t E(\tau)\|\hat{w} \cdot \nabla U, \psi^2 \hat{w}\| \, d\tau
$$

$$
+ 2 \int_s^t E(\tau)\|\hat{u} \cdot \nabla \hat{w}, \psi^2 \hat{w}\| \, d\tau.
$$

The bilinear relation (2.2) and the Plancherel theorem imply $\langle \hat{u} \cdot \nabla \hat{w}, \hat{w} \rangle = 0$ and (4.7) follows immediately.

Theorem 4.5. Let $u$ and $U$ satisfy the assumptions of theorem 4.3 with $\|U\|_{H^1}$ less than the given constant and $\sup_t \|u\|_2 < \infty$. The energy of the solution given by theorem 4.3 decays to zero. That is,

$$
\lim_{t \to 0} \|w(t)\|_2 = 0.
$$

(4.8)

Proof. Following [38] we bound first the low frequencies using (4.6) and then the high frequencies using (4.7) and the Fourier splitting method.

To show the low frequencies tend to zero we start by estimating the integrals on the RHS of (4.6):

$$
|\langle u \cdot \nabla w, e^{2\Delta(t-\tau)}\hat{\phi} \ast w \rangle| = |\langle \hat{\phi} \ast u \cdot w, e^{2\Delta(t-\tau)}\nabla w \rangle|
$$

$$
\leq \|\hat{\phi} \ast u \cdot w\|_2 \|\nabla w\|_2
$$

$$
\leq C \|u\|_2 \|\nabla w\|_2^2.
$$

This estimate was obtained using integration by parts, the Cauchy–Schwartz inequality, Young’s inequality and the Gagliardo–Nirenberg–Sobolev inequality. Similarly,

$$
|\langle w \cdot \nabla U, e^{2\Delta(t-\tau)}\hat{\phi} \ast w \rangle| \leq C \|U\|_2 \|\nabla w\|_2^2.
$$
Combining these two bounds with (4.6) yields
\[ \| \hat{\phi} * w(t) \|_2^2 \leq \| e^{\Delta(t-s)} \hat{\phi} * w(s) \|_2^2 + C \left( \sup_{\tau \in \mathbb{R}^+} \| u(\tau) \|_2^2 + \| U \|_2^2 \right) \int_s^t \| \nabla w \|_2^2 \, d\tau. \]

Heat energy is known to approach zero as time becomes large so
\[ \limsup_{t \to \infty} \| \hat{\phi} * w(t) \|_2^2 \leq C \left( \sup_{\tau \in \mathbb{R}^+} \| u(\tau) \|_2^2 + \| U \|_2^2 \right) \int_0^\infty \| \nabla w \|_2^2 \, d\tau. \]

The LHS is independent of \( s \), noting the energy bound (4.5) we see the RHS tends to zero as \( s \to \infty \). Using the Plancherel theorem we conclude
\[ \lim_{t \to \infty} \| \hat{\phi} \hat{w}(t) \|_2^2 = \lim_{t \to \infty} \| \hat{\phi} * w(t) \|_2^2 = 0. \] (4.9)

We begin work with the high frequencies on a similar path, bounding the integrals on the RHS of (4.7). Note
\[ (\hat{w} \cdot \nabla U, \psi^2 \hat{w}) = |\langle \xi \cdot \hat{w}, \psi^2 \hat{w} \rangle| \]
\[ \leq \| \hat{w} \cdot \nabla U \|_2 \| \xi \|_2 \| \hat{w} \|_2 \]
\[ \leq C \| w \|_6 \| \nabla w \|_2 \]
\[ \leq C \| \nabla w \|_2^2. \]

This chain of inequalities used the Cauchy–Schwartz inequality, the Plancherel theorem, Hölder’s inequality and the Gagliardo–Nirenberg–Sobolev inequality. Similarly, but this time making use of the rapid decay properties of \( 1 - \psi^2 \),
\[ |\langle u \cdot \nabla w, (1 - \psi^2) \hat{w} \rangle| \leq \| (1 - \psi^2) u \cdot \hat{w} \|_2 \| \xi \|_2 \| \hat{w} \|_2 \]
\[ \leq C \| (1 - \psi^2) u \|_6 \| w \|_6 \| \nabla w \|_2 \]
\[ \leq C \| u \|_2 \| \nabla w \|_2^2. \]

Use these two bounds with (4.7) to find
\[ \| \hat{\psi} \hat{w}(t) \|_2^2 \leq \frac{E(s)}{E(t)} \| \hat{\psi} \hat{w}(s) \|_2^2 + \int_s^t \frac{E(\tau)}{E(t)} \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau \]
\[ - 2 \int_s^t \frac{E(\tau)}{E(t)} \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau + \int_s^t \frac{E'(\tau)}{E(t)} \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau \]
\[ + C \left( \sup_{\tau \in \mathbb{R}^+} \| u(\tau) \|_2^2 + \| U \|_2^2 \right) \int_s^t \frac{E(\tau)}{E(t)} \| \nabla w \|_2^2 \, d\tau. \] (4.10)

Now split the viscous term and the term with \( E' \) around the ball with radius \( \rho(\tau) > 0 \), \( B(\rho) \):
\[ -2 \int_s^t \frac{E(\tau)}{E(t)} \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau + \int_s^t \frac{E'(\tau)}{E(t)} \| \hat{\psi} \hat{w}(\tau) \|_2^2 \, d\tau \]
\[ \leq \frac{1}{E(t)} \int_s^t (E'(\tau) - 2E(\tau) \rho(\tau)^2) \int_{B(\rho)^c} |(1 - \phi) \hat{w}|^2 \, d\xi \, d\tau \]
\[ + \int_s^t \frac{E'(\tau)}{E(t)} \int_{B(\rho)} |(1 - \phi) \hat{w}|^2 \, d\xi \, d\tau. \]
Upon choosing $E(\tau) = (1 + \tau)^\alpha$ and $\rho^2 = \alpha/(1 + \tau)$ ($\alpha > 3$), (4.10) becomes
\begin{align*}
\| (1 - \phi) \hat{w}(t) \|_2^2 &\leq (1 + s)^\alpha (1 + \tau)^\alpha \| (1 - \phi) \hat{w}(s) \|_2^2 \\
&\quad + \int_s^t \frac{\alpha (1 + \tau)^{\alpha - 1}}{(1 + \tau)^\alpha} \int_{B(\rho)} |(1 - \phi) \hat{w}|^2 \, d\xi \, d\tau \\
&\quad + C \sup_{\tau \in \mathbb{R}^+} \| u(\tau) \|_2^2 + \| U \|_2^2 \int_s^t \| \nabla w \|_2^2 \, d\tau.
\end{align*}

Note $|1 - \phi| \leq ||\xi||^2$ if $||\xi|| < 1$, so for large values of $s$,
\begin{align*}
\int_{B(\rho)} |(1 - \phi) \hat{w}|^2 \, d\xi &\leq C (1 + \tau)^{-2} \| w(\tau) \|_2^2.
\end{align*}

This implies, again for large $s$,
\begin{align*}
\int_s^t \frac{\alpha (1 + \tau)^{\alpha - 1}}{(1 + \tau)^\alpha} \int_{B(\rho)} |(1 - \phi) \hat{w}|^2 \, d\xi \, d\tau &\leq C \sup_{\tau \in \mathbb{R}^+} \| w(\tau) \|_2^2 \int_s^t \frac{(1 + \tau)^{\alpha - 3}}{(1 + \tau)^\alpha} \, d\tau \\
&\leq C \sup_{\tau \in \mathbb{R}^+} \| w(\tau) \|_2^2 (1 + \tau)^{-2}.
\end{align*}

Taking into account the energy bound (4.5), this tends to zero as $t$ becomes large. For any large $s$ we are now justified in writing
\begin{align*}
\limsup_{t \to \infty} \| (1 - \phi) \hat{w}(t) \|_2^2 &\leq C \left( \sup_{\tau \in \mathbb{R}^+} \| u(\tau) \|_2^2 + \| U \|_2^2 \right) \int_s^\infty \| \nabla w \|_2^2 \, d\tau.
\end{align*}

Again relying on (4.5) and then letting $s \to \infty$ we find
\begin{align*}
\lim_{t \to \infty} \| (1 - \phi) \hat{w}(t) \|_2^2 &= 0.
\end{align*}

Using this limit in the triangle inequality with the low frequency limit (4.9) completes the proof.

**Theorem 4.6.** Let $f$ satisfy the assumptions of theorem 1.2 and be such that $\| f \|_X$ is less than the constant given by the theorem and $\| U \|_{H^1}$ is less than the constant given by theorem 4.3. The solution $U$ of (1.2) nonlinearly stable in the sense of definition 4.1.

**Proof.** Let $u$ be given by theorem 4.2, the difference $v = u - U$ solves (4.1) and $u, U, f$ meet the criteria of theorem 4.5. This proves (ii) in definition 4.1. Integrating (4.4) in time proves (i) in definition 4.1.

**Remark 4.7.** The last theorem establishes theorem 1.4 stated in the introduction.

**Acknowledgments**

The authors would like to thank two anonymous referees, for many very helpful suggestions.

The work of M Schonbek was partially supported by NSF Grant DMS-0600692 and the work of C Bjorland was partially supported by NSF grant OISE-0630623 and UCSC Chancellors Dissertation-Year Fellowship.
References

[1] Bjorland C and Schonbek M E 2009 Poincaré’s inequality and diffusive evolution equations Adv. Differ. Eqns. 14 (March/April)

[2] Borchers W and Miyakawa T 1992 $L^2$-decay for Navier-Stokes flows in unbounded domains, with application to exterior stationary flows Arch. Ration. Mech. Anal. 118 273–95

[3] Borchers W and Miyakawa T 1995 On stability of exterior stationary Navier–Stokes flows Acta Math. 174 311–82

[4] Caffarelli L., Kohn R and Nirenberg L 1982 Partial regularity of suitable weak solutions of the Navier–Stokes equations Commun. Pure Appl. Math. 35 771–831

[5] Chen Z M 1993 Solutions of the stationary and nonstationary Navier–Stokes equations in exterior domains Pacific J. Math. 159 227–40

[6] Constantin P and Foias C 1988 Navier–Stokes equations (Chicago Lectures in Mathematics) (Chicago, IL: University of Chicago Press)

[7] Doering C R and Gibbon J D 1995 Applied analysis of the Navier–Stokes equations (Cambridge Texts in Applied Mathematics) (Cambridge: Cambridge University Press)

[8] Drazin P G and Reid W H 2004 Hydrodynamic stability (Cambridge Mathematical Library) (Cambridge: Cambridge University Press) 2nd edn (With a foreword by John Miles)

[9] Evans C L 1991 Partial Differential Equations Graduate Studies in Mathematics vol 19 (Providence, RI: American Mathematical Sociey)

[10] Farwig R 1998 The stationary Navier–Stokes equations in a 3D-exterior domain Recent topics on mathematical theory of viscous incompressible fluid (Tsukuba, 1996) (Lecture Notes Numer. Appl. Anal. vol 16) pp 53–115 (Tokyo: Kinokunuya)

[11] Finn R 1959 On steady-state solutions of the Navier–Stokes partial differential equations Arch. Ration. Mech. Anal. 3 381–96

[12] Finn R 1961 On the steady-state solutions of the Navier–Stokes equations III. Acta Math. 105 197–244

[13] Finn R 1965 On the exterior stationary problem for the Navier–Stokes equations, and associated perturbation problems Arch. Ration. Mech. Anal. 19 363–406

[14] Fujita H 1961 On the existence and regularity of the steady-state solutions of the Navier–Stokes theorem J. Fac. Sci. Univ. Tokyo Sect. 1 9 59–102

[15] Galdi G P, Heywood J G and Shibata Y 1997 On the global existence and convergence to steady state of Navier-Stokes flow past an obstacle that is started from rest Arch. Ration. Mech. Anal. 138 307–18

[16] Galdi G and Sohr H 2004 Existence and uniqueness of time-periodic physically reasonable Navier–Stokes flow past a body Arch. Ration. Mech. Anal. 172 363–406

[17] Galdi G P 1994 An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Vol. II Springer Tracts in Natural Philosophy (New York: Springer) Nonlinear steady problems

[18] Galdi G P 2003 Steady flow of a Navier–Stokes fluid around a rotating obstacle J. Elasti. 71 1–31 Essays and papers dedicated to the memory of Clifford Ambrose Truesdell III, Vol. II

[19] Galdi G P 2007 Further properties of steady-state solutions to the Navier–Stokes problem past a three-dimensional obstacle J. Math. Phys. 48 065207, 43

[20] Galdi G P and Padula M 1990 A new approach to energy theory in the stability of fluid motion Arch. Ration. Mech. Anal. 110 187–286

[21] Galdi G P and Silvestre A L 2007 The steady motion of a Navier–Stokes liquid around a rigid body Arch. Ration. Mech. Anal. 184 371–400

[22] Heywood J G 1970 On stationary solutions of the Navier–Stokes equations as limits of nonstationary solutions Arch. Ration. Mech. Anal. 37 48–60

[23] Heywood J G 1980 The Navier–Stokes equations: on the existence, regularity and decay of solutions Indiana Univ. Math. J. 29 639–81

[24] Hopf E 1951 Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen Math. Nachr. 4 213–31

[25] Kajikiya R and Miyakawa T 1986 On $L^2$ decay of weak solutions of the Navier–Stokes equations in $R^n$ Math. Z. 192 135–48

[26] Kato T 1984 Strong $L^r$-solutions of the Navier–Stokes equation in $R^n$, with applications to weak solutions Math. Z. 187 471–80

[27] Kozono H and Ogawa T 1994 On stability of Navier–Stokes flows in exterior domains Arch. Ration. Mech. Anal. 128 1–31

[28] Kozono H and Sohr H 1993 On stationary Navier–Stokes equations in unbounded domains Ric. Mat. 42 69–86
Steady-state Navier–Stokes in $\mathbb{R}^3$

[29] Kozono H and Yamazaki M 1998 On a larger class of stable solutions to the Navier–Stokes equations in exterior domains Math. Z. 228 751–85

[30] Ladyzhenskaya O A 1969 The Mathematical Theory of Viscous Incompressible Flow 2nd Engl. edn, revised and enlarged. Translated from the Russian by R A Silverman and J Chu (Mathematics and its Applications vol 2 (New York: Gordon and Breach)

[31] Leray J 1933 Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique J. Math. Pures Appl. 12 1–82

[32] Leray J 1934 Sur le mouvement d’un liquide visqueux emplissant l’espace Acta Math. 63 193–248

[33] Maremonti P 1986 Stabilità assintotica in media per moti fluidi viscosi in domini esterni Ann. Mat. Pura Appl. Ser. Quarta 142 57–75

[34] Miyakawa T 1995 On uniqueness of steady Navier–Stokes flows in an exterior domain Adv. Math. Sci. Appl. 5 411–20

[35] Miyakawa T 1997 On $L^1$-stability of stationary Navier–Stokes flows in $\mathbb{R}^n$ J. Math. Sci. Univ. Tokyo 4 67–119

[36] Miyakawa T 2000 On space-time decay properties of nonstationary incompressible Navier–Stokes flows in $\mathbb{R}^n$ Funkcial. Ekvac. 43 541–57

[37] Miyakawa T and Sohr H 1988 On energy inequality, smoothness and large time behavior in $L^2$ for weak solutions of the Navier–Stokes equations in exterior domains Math. Z. 199 455–78

[38] Ogawa T, Rajopadhye S V and Schonbek M E 1997 Energy decay for a weak solution of the Navier–Stokes equation with slowly varying external forces J. Funct. Anal. 144 325–58

[39] Oseen C W 1927 Neuere Methoden und Ergebnisse in der Hydrodynamik (Leipzig: Akademische Verlagsgesellschaft M.B.H.)

[40] Schonbek M E 1986 Large time behaviour of solutions to the Navier–Stokes equations Commun. Partial Differ. Eqns 11 733–63

[41] Schonbek M E 1995 Large time behaviour of solutions to the Navier–Stokes equations in $H^m$ spaces Commun. Partial Differ. Eqns 20 103–17

[42] Schonbek M E and Schonbek T P 2000 On the boundedness and decay of moments of solutions to the Navier–Stokes equations Adv. Differ. Eqns 5 861–98

[43] Schonbek M E and Wiegner M 1996 On the decay of higher-order norms of the solutions of Navier–Stokes equations Proc. R. Soc. Edinb. Sect. A 126 677–85

[44] Schonbek M E 1985 $L^2$ decay for weak solutions of the Navier–Stokes equations Arch. Ration. Mech. Anal. 88 209–22

[45] Serre J 1959 On the stability of viscous fluid motions Arch. Ration. Mech. Anal. 3 1–13

[46] Sohr H 2001 The Navier–Stokes Equations Birkhäuser Advanced Texts: Basler Lehrbücher (Birkhäuser Advanced Texts: Basel Textbooks) (Basel: Birkhäuser Verlag) An elementary functional analytic approach

[47] Straughan B 1992 The Energy Method, Stability, and Nonlinear Convection (Applied Mathematical Sciences vol 91) (New York: Springer)

[48] Temam R 2001 Navier–Stokes Equations (Providence, RI: AMS Chelsea Publishing) Theory and numerical analysis Reprint of the 1984 edn

[49] Wiegner M 1987 Decay results for weak solutions of the Navier–Stokes equations on \( \mathbb{R}^d \) J. Lond. Math. Soc. 35 303–13

[50] Wiegner M 2000 Decay estimates for strong solutions of the Navier–Stokes equations in exterior domains Conf. on Navier–Stokes Equations and Related Nonlinear Problems (Ferrara, 1999), Ann. Univ. Ferrara Sez. VII (N.S.) 46 61–79