Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees

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1 Introduction

In [8], Dirk Kreimer discovered the striking fact that the process of renormalization in quantum field theory may be described, in a conceptual manner, by means of certain Hopf algebras (which depend on the chosen renormalization scheme). A toy model was studied in detail by Alain Connes and Dirk Kreimer in [3]; the Hopf algebra which occurs, denoted by $H_R$, is the polynomial algebra in an infinity of indeterminates, one for each rooted tree, but with a non-cocommutative comultiplication. Some operators, denoted by $N$ and $L$, have been defined on $H_R$. The first one is the natural growth operator, which acts as a derivation; it defines some elements $\delta_k$, for $k \geq 1$, which provide the link between $H_R$ and another Hopf algebra, introduced in [3] by Alain Connes and Henri Moscovici in a completely different context, namely in noncommutative geometry. The operator $L$ is a solution of the “Hochschild equation”, and the pair $(H_R, L)$ is characterized as the solution of a universal problem in Hochschild cohomology. It was proved also in [3] that $H_R$ is in duality with the universal enveloping algebra of a certain Lie algebra $L^\infty$, which has a linear basis indexed by all (non-empty) rooted trees. Let us note that this Lie algebra $L^\infty$ appeared also, very recently, in [2], in the context of pre-Lie algebras and the operad of rooted trees.

In this note we would like to draw the attention to another Hopf algebra built on rooted trees, introduced ten years ago by Robert Grossman and Richard Larson in [6] (see also their survey [7]). This Hopf algebra (denoted by $A$ in what follows) has a linear basis consisting of all (non-empty) rooted trees, a noncommutative product, and is a cocommutative graded connected Hopf algebra, hence, by the Milnor-Moore theorem, it is the universal enveloping algebra of the Lie algebra of its primitive elements, which may also be described explicitly: $P(A)$ has a linear basis consisting of all rooted trees whose root has exactly one child. Using these properties of $A$, Grossman and Larson gave a Hopf algebraic proof of the classical result of Cayley on the number of rooted trees. The construction of the Hopf algebra $A$ is motivated
also by some ideas concerning differential operators and differential equations, in particular the Runge-Kutta method in numerical analysis (see [7]). Let us note also that actually the construction of Grossman and Larson is slightly more general: they associate a Hopf algebra to any family of trees which satisfy a certain list of axioms. Among these families are: the family of all rooted trees (this is the one who gives the Hopf algebra $A$) and the families of all ordered, heap-ordered and respectively labelled rooted trees. The construction of the Hopf algebra associated to such a family is similar to that of $A$.

As noted in [1], [4], the Hopf algebra $H_R$ may also be related to the Runge-Kutta method and the Butcher group, so it is very likely that there is a relation between $H_R$ and $A$. As we shall see, this relation is best expressed by the fact that the Hopf algebras $A$ and $U(L^\infty)$ are isomorphic, which in turn is a consequence of a Lie algebra isomorphism between $P(A)$ and $L^\infty$. This isomorphism is given by sending a rooted tree $t \in P(A)$ to the rooted tree in $L^\infty$ obtained by deleting the root of $t$.

We believe that this relation between $A$ and $H_R$ may be useful for a better understanding of both these Hopf algebras. On one hand, the advantage of the isomorphism between $A$ and $U(L^\infty)$, which allows one to work with $A$ instead of $U(L^\infty)$, is clear, since we know on $A$ a very explicit linear basis, more manageable than the PBW basis of $U(L^\infty)$. On the other hand, some known results for one of the Hopf algebras $A$ and $H_R$ may serve as a motivation and inspiration for obtaining similar results for the other. We shall make here a first step in this direction, by studying two natural operators on $A$. The first is the natural growth operator $N$ (defined exactly as the one introduced by Connes and Kreimer for $H_R$), which will turn out to be a coderivation on $A$; the sequence $\{x_k\}_{k \geq 0}$ defined by $N$ will turn out to generate a commutative cocommutative Hopf subalgebra of $A$, isomorphic to the polynomial algebra in one indeterminate, with its usual Hopf algebra structure. A nice feature of $N$ (considered on $A$) is that, for any rooted tree $t$, $N(t)$ may be described as the product (in $A$) between the rooted tree with two vertices and $t$. The second one, denoted by $M$, is in some sense dual to the operator $L$ on $H_R$: we shall prove that $M$ is a derivation (the right $A$-module structure on $A$ being the one induced by $\varepsilon$) and that the transpose of $M$ gives a solution of the Hochschild equation on the finite dual Hopf algebra $A^0$.

2 The relation between $H_R$ and $A$

Throughout, $k$ will be a fixed field of characteristic zero and all algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$.

We start by recalling some facts from [3], [6], [7], to which we refer for the terminology and more details (the reader will find also in [3] some nice pictures of rooted trees and the operations which may be performed with them).

A rooted tree $t$ is a connected and simply-connected set of oriented edges and vertices such that there is exactly one distinguished vertex with no incoming edges, called the root of $t$. Every edge connects two vertices. The fertility $f(v)$ of a vertex $v$ is the number of edges outgoing from $v$, that is the number of children of $v$. A
**forest** is a finite set of rooted trees.

We denote by $B_-$ the operator which assigns to a rooted tree $t$ a forest, by removing the root of $t$, and by $B_+$ the operator which maps a forest consisting of $n$ rooted trees $t_1, \ldots, t_n$ to a new rooted tree $t$ which has a root $r$ with fertility $f(r) = n$ which connects to the $n$ roots of $t_1, \ldots, t_n$. Obviously, $B_+(B_-(t)) = B_-(B_+(t)) = t$ for any rooted tree $t$. We also set $B_-(e) = \emptyset$, $B_+(\emptyset) = e$, where $e$ is the rooted tree with only one vertex and $\emptyset$ is the empty tree.

If $t$ is a rooted tree, an elementary cut is a cut of $t$ at a single chosen edge, and an admissible cut is a set of elementary cuts such that any path from any vertex of $t$ to the root of $t$ contains at most one elementary cut. If $c$ is an admissible cut, we denote by $|c|$ the number of elementary cuts of $c$. If we perform an admissible cut $c$ in a rooted tree $t$, we obtain a forest, denoted by $P^c(t)$, consisting of the cut branches of $t$, and a trunk, denoted by $R^c(t)$, which is the branch which remains (it is the one which contains the root of $t$).

We can now define the Connes-Kreimer Hopf algebra over $k$, denoted by $H_R$. As an algebra, it is the polynomial algebra in an infinity of indeterminates, one for each (non-empty) rooted tree (we denote also by $t$ the indeterminate corresponding to the rooted tree $t$). The unit is denoted by $1$ (it corresponds to the empty tree). The comultiplication $\Delta$ is defined by:

$$
\Delta(1) = 1 \otimes 1
$$

$$
\Delta(t) = 1 \otimes t + t \otimes 1 + \sum_c P^c(t) \otimes R^c(t)
$$

for any rooted tree $t$, where the sum is over all admissible cuts $c$ of $t$ (with $|c| \geq 1$) and $P^c(t)$ is the monomial corresponding to the forest $P^c(t)$ (as a general rule, we identify any forest with its monomial). An alternative recursive description of $\Delta(t)$ is

$$
\Delta(t) = t \otimes 1 + (id \otimes B_+)(\Delta(B_-(t))
$$

The counit is given by

$$
\varepsilon(1) = 1
$$

$$
\varepsilon(t) = 0
$$

for any rooted tree $t$. The antipode is given iteratively by

$$
S(1) = 1
$$

$$
S(t) = -t - \sum_c S(P^c(t))R^c(t)
$$

for any rooted tree $t$, where the sum is over all admissible cuts $c$ of $t$ (with $|c| \geq 1$).

Define now the operator $N$ on $H_R$ (the natural growth operator) which maps a rooted tree $t$ with $n$ vertices to a sum $N(t)$ of $n$ rooted trees $t_i$, each having $n+1$ vertices, by attaching one more outgoing edge and vertex to each vertex of $t$ (the root remains the same under this operation). On products of rooted trees, $N$ acts, by definition, as a derivation. For any $k \geq 1$, define elements $\delta_k \in H_R$ by $\delta_1 = e$, $\delta_{k+1} = N(\delta_k)$, that is $\delta_{k+1} = N^k(e)$ for all $k \geq 1$. We recall that we have denoted by $e$ the tree with one vertex (in order to be consistent with [6]; in [3] by $e$ is denoted
the unit of \( H_R \)). These elements are very important in [3], because they provide the link between \( H_R \) and the Connes-Moscovici Hopf algebra introduced in [3]. They generate a (non-cocommutative) Hopf subalgebra of \( H_R \).

Define now the operator \( L : H_R \to H_R \) as the unique linear map satisfying the condition \( L(t_1 \ldots t_m) = t \) for any rooted trees \( t_1, \ldots, t_m \), where \( t \) is the rooted tree obtained by connecting a new root to the roots of \( t_1, \ldots, t_m \). Obviously it agrees with the map \( B_\chi \) introduced above. It was shown in [3] that the operator \( L \) satisfies the so-called “Hochschild equation”

\[
\Delta \circ L = L \otimes 1 + (id \otimes L) \circ \Delta
\]

(this is a 1-cocycle condition) and the pair \((H_R, L)\) has the following universal property: if \((H_1, L_1)\) is a pair with \( H_1 \) a commutative Hopf algebra and \( L_1 : H_1 \to H_1 \) a linear map satisfying the Hochschild equation on \( H_1 \), then there exists a unique Hopf algebra map \( \rho : H_R \to H_1 \) such that \( L_1 \circ \rho = \rho \circ L \).

Let \( L^\infty \) be the linear span of the elements \( Z_t \), indexed by all (non-empty) rooted trees. Define an operation on \( L^\infty \) by

\[
Z_{t_1} \ast Z_{t_2} = \sum_t n(t_1, t_2; t) Z_t
\]

where the integer \( n(t_1, t_2; t) \) is the number of admissible cuts \( c \) of \( t \) with \( |c| = 1 \) such that the cut branch is \( t_1 \) and the trunk is \( t_2 \) (note that this operation is not associative). Define then a bracket on \( L^\infty \) by

\[
[Z_{t_1}, Z_{t_2}] = Z_{t_1} \ast Z_{t_2} - Z_{t_2} \ast Z_{t_1}
\]

Then it was proved in [3] that \((L^\infty, [,])\) is a Lie algebra and moreover there is a Hopf duality between \( H_R \) and the universal enveloping algebra of \( L^\infty \).

We recall now from [3], [4] the structure of the Grossman-Larson Hopf algebra, which will be denoted in what follows by \( A \). It has a linear basis consisting of all (non-empty) rooted trees. The unit is the tree \( e \) with only one vertex. The multiplication on the basis is given as follows: let \( t_1 \) and \( t_2 \) be two rooted trees, let \( s_1, \ldots, s_r \) be the children of the root of \( t_1 \), let \( n \) be the number of vertices of \( t_2 \); then there are \( n^r \) ways to attach the \( r \) subtrees of \( t_1 \) which have \( s_1, \ldots, s_r \) as roots to the tree \( t_2 \) by making each \( s_i \) the child of some vertex of \( t_2 \). The product \( t_1 t_2 \) is defined to be the sum of these \( n^r \) rooted trees (note that this product is not commutative).

The coalgebra structure of \( A \) is given as follows. If \( t \) is a rooted tree whose root has children \( s_1, \ldots, s_r \), the coproduct \( \Delta(t) \) is the sum of the \( 2^r \) terms \( t_1 \otimes t_2 \), where the children of the root of \( t_1 \) and the children of the root of \( t_2 \) range over all \( 2^r \) possible partitions of the children of the root of \( t \) into two subsets. If \( t = e \), then \( \Delta(e) = e \otimes e \). The counit \( \varepsilon \) is given by \( \varepsilon(e) = 1 \), \( \varepsilon(t) = 0 \) if \( t \neq e \). Obviously \( \Delta \) is cocommutative. Moreover, \( A \) is a graded connected bialgebra, the component \( A_n \) of degree \( n \) having as basis all trees with \( n + 1 \) vertices. Being a graded connected cocommutative bialgebra, \( A \) is a Hopf algebra and by the Milnor-Moore theorem \( A \) is the universal enveloping algebra of its primitives, \( A \simeq U(P(A)) \), where \( P(A) = \{ a \in A/\Delta(a) = e \otimes a + a \otimes e \} \). There is an explicit description of \( P(A) \): it has a basis consisting of all rooted trees whose root has exactly one child.
We can state now the result which expresses the relation between the Hopf algebras $H_R$ and $A$.

**Proposition 2.1** The Lie algebras $\mathcal{L}^\infty$ and $P(A)$ are isomorphic, hence $A$ is isomorphic to $U(\mathcal{L}^\infty)$ as Hopf algebras.

**Proof:** Define $\varphi : P(A) \to \mathcal{L}^\infty$ as the unique linear map which on the basis of $P(A)$ acts as follows: if $t \in P(A)$ is a rooted tree, then $\varphi(t) = Z_{B_-(t)}$. Recall that $B_-(t)$ is the rooted tree obtained by deleting the root of $t$ (here it is a tree, since the root of $t$ has exactly one child). Obviously, $\varphi$ is a linear isomorphism, its inverse being the map $\psi : \mathcal{L}^\infty \to P(A)$, $\psi(Z_t) = B_+(t)$ for any rooted tree $t$.

It remains to prove that $\varphi$ is a Lie algebra map. Let $t_1, t_2 \in P(A)$ be two rooted trees. In $P(A)$, we have $[t_1, t_2] = t_1 t_2 - t_2 t_1$. By the definition of the multiplication of $A$, we obtain that

$$t_1 t_2 = B(t_1 t_2) + \sum_i t^i$$

$$t_2 t_1 = B(t_2 t_1) + \sum_j T^j$$

where $B(t_1 t_2)$ is the rooted tree obtained by identifying the roots of $t_1$ and $t_2$, and each $t^i$ is obtained by identifying the root of $t_1$ with a vertex of $t_2$, except for the root of $t_2$ (and similarly for $t_2 t_1$), so all the rooted trees $t^i$ and $T^j$ are in $P(A)$. Obviously $B(t_1 t_2) = B(t_2 t_1)$, hence

$$[t_1, t_2] = \sum_i t^i - \sum_j T^j$$

We obtain that $\varphi([t_1, t_2]) = \sum_i \varphi(t^i) - \sum_j \varphi(T^j)$. From the definition of the operation $\ast$ on $\mathcal{L}^\infty$, it is easy to see that

$$\sum_i \varphi(t^i) = \varphi(t_1) \ast \varphi(t_2)$$

$$\sum_j \varphi(T^j) = \varphi(t_2) \ast \varphi(t_1)$$

hence we obtain

$$\varphi([t_1, t_2]) = [\varphi(t_1), \varphi(t_2)]$$

that is $\varphi$ is a Lie algebra map.

Define now the natural growth operator $N$ on $A$, by the same formula as the one defined by Connes and Kreimer on $H_R$, that is $N$ is the linear map $N : A \to A$ such that, for any rooted tree $t$, $N(t)$ is the sum of the rooted trees obtained from $t$ by attaching one more outgoing edge and vertex to each vertex of $t$. The properties of $N$ are collected in the following

**Proposition 2.2**

1. $N(e)b = N(b)$ for all $b \in A$.
2. $N(ab) = N(a)b$ for all $a, b \in A$. 

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(3) \(N^k(e)b = N^k(b)\) for all \(k \geq 1, b \in A\).

(4) \(N^k(e)N(e) = N^{k+1}(e) = N(e)N^k(e)\) for all \(k \geq 1\).

(5) \(N\) is a coderivation, that is, for all \(b \in A\), we have

\[
\Delta(N(b)) = (id \otimes N + N \otimes id)(\Delta(b))
\]

**Proof:** (1) obviously, for any rooted tree \(t\), we have \(N(e)t = N(t)\), from which we obtain \(N(e)b = N(b)\) for all \(b \in A\).

(2) using (1), we have \(N(ab) = N(e)ab = N(a)b\).

(3) follows easily by induction, and (4) follows from (3).

(5) by (1), we obtain that \(\Delta(N(b)) = \Delta(N(e)b) = \Delta(N(e))\Delta(b)\). We shall use the \(\Sigma\)-notation, so we write \(\Delta(b) = \sum b_{(1)} \otimes b_{(2)}\); since \(N(e)\) is a primitive element, we have then:

\[
\Delta(N(b)) = (e \otimes N(e) + N(e) \otimes e)(\sum b_{(1)} \otimes b_{(2)}) = \\
\sum b_{(1)} \otimes N(e)b_{(2)} + \sum N(e)b_{(1)} \otimes b_{(2)} = \\
\sum b_{(1)} \otimes N(b_{(2)}) + \sum N(b_{(1)}) \otimes b_{(2)} = \\
(id \otimes N + N \otimes id)(\Delta(b))
\]

Now, for any \(k \geq 0\), define the element \(x_k = N^k(e) \in A\). These elements are analogous to the elements \(\delta_k\) of [3].

**Proposition 2.3** The elements \(x_k\) have the following properties:

(1) \(x_mx_n = x_nx_m = x_{m+n}\)

(2) \(x_0 = e\)

(3) \(\varepsilon(x_m) = \delta_{0,m}\)

(4) \(\Delta(x_m) = \sum_{i=0}^{m} \binom{m}{i} x_i \otimes x_{m-i}\)

(5) \(S(x_m) = (-1)^m x_m\)

for all \(m, n \geq 0\), where \(S\) is the antipode of \(A\). Moreover, the elements \(\{x_k\}_{k \geq 0}\) are linearly independent. Hence, the subspace of \(A\) generated by the elements \(x_k\) with \(k \geq 0\) is a commutative cocommutative Hopf subalgebra of \(A\), isomorphic (as a Hopf algebra) to the polynomial algebra \(k[X]\) with its usual Hopf algebra structure.

**Proof:** (1) follows easily from the previous proposition; (2) and (3) are obvious; (4) and (5) follow by induction, using the facts that \(x_{m+1} = N(e)x_m\) and \(N(e)\) is primitive. Since \(x_k \in A_k\) for all \(k \geq 0\) (hence \(\deg(x_m) \neq \deg(x_n)\) if \(m \neq n\)) it follows that the elements \(\{x_k\}_{k \geq 0}\) are linearly independent. The isomorphism between the Hopf subalgebra given by the elements \(x_k\) and \(k[X]\) is determined by \(x_k \mapsto X^k\) for all \(k \geq 0\).
Define now the $k$-linear map $M : A \rightarrow A$, by $M(e) = 0$ and $M(t) = tN(e)$ for all rooted trees $t \neq e$. As we shall see, this operator $M$ is in some sense dual to the operator $L$ on $H_R$.

**Proposition 2.4** (1) $M(b) = (b - \varepsilon(b))N(e)$ for all $b \in A$.

(2) $M(ab) = aM(b) + M(a)\varepsilon(b)$ for all $a, b \in A$ (that is, $M$ is a derivation from $A$ into the bimodule $A$, where the left $A$-module structure of $A$ is the one given by multiplication and the right $A$-module structure is given via $\varepsilon$).

(3) $\Delta(M(b)) = (id \otimes M + M \otimes id)(\Delta(b)) + (b - \varepsilon(b)) \otimes N(e) + N(e) \otimes (b - \varepsilon(b))$ for all $b \in A$.

**Proof:** (1) Let $b \in A$, written as $b = b_0e + \sum_{t_i \neq e} b_it_i$, with $b_0, b_i \in k$. Then we have:

$$M(b) = b_0M(e) + \sum b_iM(t_i) = \sum b_it_iN(e) =$$

$$\sum b_it_iN(e) + b_0N(e) - b_0N(e) = bN(e) = (b - \varepsilon(b))N(e)$$

(2) using (1), we have $M(ab) = (ab - \varepsilon(a)\varepsilon(b))N(e)$, and

$$aM(b) + M(a)\varepsilon(b) = a(b - \varepsilon(b))N(e) + (a - \varepsilon(a))\varepsilon(b)N(e) =$$

$$(ab - \varepsilon(b)a + \varepsilon(b)a - \varepsilon(a)\varepsilon(b))N(e) = (ab - \varepsilon(a)\varepsilon(b))N(e)$$

(3) write $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$; then we have:

$$\Delta(M(b)) = \Delta((b - \varepsilon(b))N(e)) = \Delta(b)\Delta(N(e)) - \varepsilon(b)\Delta(N(e)) =$$

$$\sum b_{(1)}N(e) \otimes b_{(2)} + \sum b_{(1)} \otimes b_{(2)}N(e) - \varepsilon(b) \otimes N(e) - N(e) \otimes \varepsilon(b) =$$

$$\sum M(b_{(1)}) \otimes b_{(2)} + \sum \varepsilon(b_{(1)})N(e) \otimes b_{(2)} + \sum b_{(1)} \otimes M(b_{(2)}) +$$

$$+ \sum b_{(1)} \otimes \varepsilon(b_{(2)})N(e) - \varepsilon(b) \otimes N(e) - N(e) \otimes \varepsilon(b) =$$

$$(id \otimes M + M \otimes id)(\Delta(b)) + (b - \varepsilon(b)) \otimes N(e) + N(e) \otimes (b - \varepsilon(b))$$

Recall from [10], [4] that we can associate to $A$ the so-called *finite dual*, which is also a Hopf algebra, and which consists of the elements $f \in A^*$ such that $\text{Ker}(f)$ contains a cofinite ideal of $A$. If we denote by $m$ the multiplication of $A$, then an element $f \in A^*$ belongs to $A^0$ if and only if $m^*(f) \in A^* \otimes A^*$, which in turn is equivalent to the fact that there exist some elements $f_i, f'_i \in A^*$, with $i$ in some finite set, such that $f(ab) = \sum f_i(a)f'_i(b)$ for all $a, b \in A$. Moreover, we have that $m^*(A^0) \subseteq A^0 \otimes A^0$ and the comultiplication of $A^0$ is $\Delta = m^*|A^0$.

Now, let $f \in A^0$, $m^*(f) = \sum f_i \otimes f'_i$, with $f_i, f'_i \in A^0$. By using the condition $M(ab) = aM(b) + M(a)\varepsilon(b)$ satisfied by $M$, we can compute:

$$f(M(ab)) = f(M(a))\varepsilon(b) + f(aM(b))$$
for all \(a, b \in A\), which may be rewritten as
\[
f(M(ab)) = f(M(a))\varepsilon(b) + \sum f_i(a)f_i'(M(b))
\]
that is
\[
M^*(f)(ab) = M^*(f)(a)\varepsilon(b) + \sum f_i(a)M^*(f'_i)(b)
\]

hence \(M^*(f) \in A^0\). So, we have \(M^*(A^0) \subseteq A^0\), and we denote by \(M^0\) the restriction of \(M^*\) to \(A^0\). Also, for \(f \in A^0\), since \(M^*(f), \varepsilon, f_i, M^*(f'_i) \in A^0\) for all \(i\), we obtain finally that in \(A^0\) the following relation holds:
\[
\Delta(M^0(f)) = M^0(f) \otimes \varepsilon + (id \otimes M^0)(\Delta(f))
\]

which expresses the fact that \(M^0\) is a solution for the Hochschild equation on \(A^0\). Hence, since \(A^0\) is commutative (because \(A\) is cocommutative), by the universal property of the pair \((H_R, L)\) we obtain the following

**Proposition 2.5** There exists a unique Hopf algebra map \(\rho : H_R \to A^0\) such that \(M^0 \circ \rho = \rho \circ L\).

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