QUANTUM GRAPH HOMOMORPHISMS VIA OPERATOR SYSTEMS

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Abstract. We explore the concept of a graph homomorphism through the lens of $C^*$-algebras and operator systems. We start by studying the various notions of a quantum graph homomorphism and examine how they are related to each other. We then define and study a $C^*$-algebra that encodes all the information about these homomorphisms and establish a connection between computational complexity and the representation of these algebras. We use this $C^*$-algebra to define a new quantum chromatic number and establish some basic properties of this number. We then suggest a way of studying these quantum graph homomorphisms using certain completely positive maps and describe their structure. Finally, we use these completely positive maps to define the notion of a “quantum” core of a graph.

1. Introduction

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs on vertices $V(G) = \{1, ..., n\}$ and $V(H) = \{1, ..., m\}$. The theory of graph homomorphisms is one of the central tools of graph theory and is used in the development of the concept of the core of a graph. More recently, work in quantum information theory has lead to quantum versions of many concepts in graph theory and there is an extensive literature ([1], [3], [15]). In particular, D. Roberson[17] and L. Mancinska [11] developed an extensive theory of quantum homomorphisms of graphs. D. Stahlke[18] interpreted graph homomorphisms in terms of “completely positive (CP) maps on the traceless operator space of a graph”.

These papers motivate us to consider quantum and classical graph homomorphisms as special families of completely positive maps between the operator systems of the graphs.

There is not just a single quantum theory of graphs, but there are really possibly several different quantum theories depending on the validity of certain conjectures of Connes and Tsirelson. In earlier work on quantum chromatic numbers[16, 15], we studied the differences and similarities between the properties of the quantum chromatic numbers defined by the

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possibly different quantum theories. We wish to parallel those ideas for quantum graph homomorphisms. One technique of [15] and [6] was to show that the existence of quantum colorings was equivalent to the existence of certain types of traces on a C*-algebra affiliated with the graph and we wish to expand upon that topic here. This leads us to introduce the C*-algebra of a graph homomorphism and we will show that the existence or non-existence of various types of quantum graph homomorphisms are related to properties of this C*-algebra, e.g., whether or not it has any finite dimensional representations or has any traces.

Finally, we wish to use our correspondence between quantum graph homomorphisms and CP maps to introduce a quantum analogue of the core of a graph.

2. The Homomorphism Game

Given graphs $G$ and $H$ a graph homomorphism from $G$ to $H$ is a mapping $f : V(G) \to V(H)$ such that

\[(v, w) \in E(G) \implies (f(v), f(w)) \in E(H).\]

When a graph homomorphism from $G$ to $H$ exists we write $G \rightarrow H$.

Paralleling the work on quantum chromatic numbers [16], we study a graph homomorphism game, played by Alice, Bob, and a Referee. Given graphs $G$ and $H$, the Referee gives Alice and Bob a vertex of $G$, say $v$ and $w$, respectively, and they each respond with a vertex from $H$, say $x$ and $y$, respectively. Alice and Bob win provided that:

\[v = w \implies x = y,\]
\[v \sim_G w \implies x \sim_H y.\]

If they have some random strategy and we let $p(x, y|v, w)$ denote the probability that we get outcomes $x$ and $y$ given inputs $v$ and $w$, then these equations translate as:

\[p(x \neq y|v = w) = 0\]
\[p(x \sim_H y|v \sim_G w) = 0.\]

Now say $G$ has $n$ vertices and $H$ has $m$ vertices. We consider the sets of correlations studied in [15] and [16]:

\[Q_l(n, m) \subseteq Q_q(n, m) \subseteq Q_{qa}(n, m) \subseteq Q_{qc}(n, m) \subseteq Q_{vect}(n, m).\]

In the appendix, we review the definition and some known facts about these sets.

For $t \in \{l, q, qa, qc, vect\}$ we define:

\[G \xrightarrow{t} H,\]

provided that there exists

\[p(x, y|v, w) \in Q_t(n, m)\]
satisfying (1) and (2). Notice that when we write \( p(x, y|v, w) \in Q_I(n, m) \) we really mean \( (p(x, y|v, w))_{v,w,x,y} \in Q_I(n, m) \). Any \( p(x, y|v, w) \in Q_I(n, m) \) satisfying these conditions we call a winning t-strategy and say that there exists a \textbf{quantum t-homomorphism} from \( G \) to \( H \).

The condition (1) is easily seen to be the \textbf{synchronous condition} defined in [15] and the subset of correlations satisfying this condition was denoted \( Q^t_I(n, m) \). Thus, \( p(x, y|v, w) \) is a winning t-strategy if and only if \( p(x, y|v, w) \in Q^t_I(n, m) \) and satisfies (2).

The following result is known, but we provide a proof since we are using a slightly different (but equivalent) characterization of \( Q_I(n, m) \).

**Theorem 2.1.** Let \( G \) and \( H \) be graphs. Then \( G \to H \) if and only if \( G \xrightarrow{l} H \).

**Proof.** First assume that \( G \to H \). Let \( f : V(G) \to V(H) \) be a graph homomorphism. Let \( \Omega = \{t\} \) be the singleton probability space. For each \( v \in V(G) \) let Alice have the “random variable”, \( f_v(t) = f(v) \) and for each \( w \in V(G) \) let Bob have the random variable \( g_w(t) = f(w) \). Then

\[
    p(x, y|v, w) := \text{Prob}(x = f_v(t), y = g_w(t)) = \begin{cases} 1, & \text{when } x = f(v), y = f(w) \\ 0, & \text{else} \end{cases}
\]

From this it easily follows that \( p(x, y|v, w) \) satisfies (1) and (2).

Conversely, assume that we have a probability space \((\Omega, P)\) and random variables \( f_v, g_w : \Omega \to V(H) = \{1, \ldots, m\} \) so that \( p(x, y|v, w) = P(x = f_v(\omega), y = g_w(\omega)) \) satisfies (1) and (2). By (1), for each \( v \) the set \( B_v = \{\omega : f_v(\omega) = f_v(\omega)\} \) has probability 1. Similarly, for each \( (v, w) \in E(G) \) the set \( Q_{v,w} = \{\omega : (f_v(\omega), g_w(\omega)) \in E(H)\} \) has probability 1. Thus,

\[
    D = \bigcap_{v \in V(G)} B_v \cap \bigcap_{(v, w) \in E(G)} Q_{v,w}
\]

has measure 1, and so in particular is non-empty. Fix any \( \omega \in D \) and define \( f : V(G) \to V(H) \) by \( f(v) := f_v(\omega) = g_w(\omega) \). Then whenever \( (v, w) \in E(G) \) we have that \( (f(v), f(w)) = (f_v(\omega), g_w(\omega)) \in E(H) \). Thus, \( f \) is a graph homomorphism.

Thus, quantum l-homomorphisms correspond to classical graph homomorphisms.

**Remark 2.2.** In [1] several notions of graph homomorphisms were also introduced, including \( G \xrightarrow{B} H \), \( G \xrightarrow{V} H \) and \( G \xrightarrow{+} H \). A look at their definition shows that

\[
    G \xrightarrow{\text{vect}} H \text{ if and only if } G \xrightarrow{\text{V}} H
\]

**Corollary 2.3.** Let \( G \) and \( H \) be graphs. Then

\[
    G \to H \implies G \xrightarrow{q} H \implies G \xrightarrow{qa} H \implies G \xrightarrow{qc} H \implies G \xrightarrow{\text{vect}} H
\]

**Proof.** This is a direct consequence of the above definitions, Theorem 2.1, and the corresponding set containments. \(\square\)
3. Quantum Homomorphisms and CP Maps

Recall [12] that the operator system of a graph $G$ on $n$ vertices is the subspace of the $n \times n$ complex matrices $M_n$ given by

$$S_G = \text{span}\{E_{v,w} : v = w \text{ or } (v, w) \in E(G)\},$$

where $E_{v,w}$ denotes the $n \times n$ matrix that is 1 in the $(v, w)$-entry and 0 elsewhere.

We now wish to use a winning x-strategy for the homomorphism game to define a CP map from $S_G$ to $S_H$. It will suffice to do this in the case of winning vect-strategies since every other strategy is a subset.

**Proposition 3.1.** Let $p(x, y|v, w) \in Q^s_{\text{vect}}(n, m)$, let $E_{v,w} \in M_n$ and $E_{x,y} \in M_m$ denote the canonical matrix unit bases. Then the linear map $\phi_p : M_n \rightarrow M_m$ defined on the basis by

$$\phi_p(E_{v,w}) = \sum_{x,y} p(x, y|v, w)E_{x,y},$$

is completely positive.

**Proof.** By Choi’s theorem [14], to prove that $\phi_p$ is CP it is enough to prove that the Choi matrix,

$$P := \sum_{v,w} E_{v,w} \otimes \phi_p(E_{v,w}) = \sum_{v,w,x,y} p(x, y|v, w)E_{v,w} \otimes E_{x,y} \in M_n \otimes M_m = M_{nm},$$

is positive semidefinite.

Recall that by the definition and characterization of vector correlations satisfying the synchronous condition in [16] there exists a Hilbert space and vectors $\{h_{v,x}\}$ satisfying:

- $h_{v,x} \perp h_{v,y}$ for all $x \neq y$,
- $\sum_x h_{v,x} = \sum_x h_{w,x}$ for all $v, w$,
- $\|\sum_x h_{v,x}\| = 1$,

such that $p(x, y|v, w) = \langle h_{v,x}, h_{w,y} \rangle$.

Now let $\{e_v\}$ and $\{f_x\}$ denote the canonical orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, let $a_{v,x} \in \mathbb{C}$ be arbitrary complex numbers, so that $k = \sum_{v,x} a_{v,x}e_v \otimes f_x$ is an arbitrary vector in $\mathbb{C}^n \otimes \mathbb{C}^m$. We have that

$$\langle Pk, k \rangle = \sum_{v,w,x,y} \overline{a_{v,x}}a_{w,y}p(x, y|v, w) = \sum_{v,w,x,y} \overline{a_{v,x}}a_{w,y}\langle h_{v,x}, h_{w,y} \rangle = \langle h, h \rangle,$$

where $h = \sum_{v,x} \overline{a_{v,x}}h_{v,x}$.

Thus, $P$ is positive semidefinite and $\phi_p$ is CP. \hfill \Box

**Theorem 3.2.** Let $G$ and $H$ be graphs, let $p(x, y|v, w) \in Q^s_{\text{vect}}(n, m)$ be a winning vect-strategy for a quantum vect-homomorphism from $G$ to $H$ and let $\phi_p : M_n \rightarrow M_m$ be the CP map defined in Proposition 3.1. Then $\phi_p(S_G) \subseteq S_H$ and $\phi_p$ is trace-preserving on $S_G$. 


Moreover, if $p$ is synchronous, then $E_{v,w} \in S_G$ and define Alice and Bob’s random variables $f$, $g$ as in the proof of Theorem 2.1, then we obtain $p(x, y|v, w) = \frac{1}{d}$. The corresponding CP map satisfies $\phi_p(E_{v,w}) = E_{f(v),f(w)}$.

We now wish to turn our attention to the composition of quantum graph homomorphisms. First we need a preliminary result.

**Proposition 3.5.** Let $x \in \{l,q,qa,qc, vect\}$, let $p(x, y|v, w) \in Q_x(n, m)$ and let $q(a, b|x, y) \in Q_x(m, l)$. Then

$$r(a, b|v, w) := \sum_{x,y} q(a, b|x, y)p(x, y|v, w) \in Q_x(n, l).$$

Moreover, if $p$ and $q$ are synchronous, then $r$ is synchronous.
Definition 4.1. Let \( G \) be a graph. A set of projections \( \{ E_{v,x} : v \in V(G), \, x \in V(H) \} \) on a Hilbert space \( \mathcal{H} \) satisfying the following relations:

1. for each \( v \in V(G) \), \( \sum_x E_{v,x} = I_{\mathcal{H}} \),
2. if \( (v, w) \in E(G) \) and \( (x, y) \notin E(H) \) then \( E_{v,x}E_{w,y} = 0 \).

4. \( C^* \)-algebras and Graph Homomorphisms

We wish to define a \( C^* \)-algebra \( \mathcal{A}(G, H) \) generated by the relations arising from a winning strategy for the graph homomorphism game.

**Theorem 3.7.** Let \( x \in \{ l, q, qa, qc, vect \} \), let \( G, H \) and \( K \) be graphs on \( n, m \) and \( m, l \) vertices, respectively, and assume that \( G \twoheadrightarrow H \), \( H \twoheadrightarrow K \). If \( p(x, y)[v, w) \in Q_x(n, m) \) and \( q(a, b|x, y) \in Q_x(m, l) \) are winning quantum \( x \)-strategies for homomorphisms from \( G \) to \( H \) and \( H \) to \( K \), respectively, then \( r(a, b|x, y) \in Q_x(n, l) \) is a winning \( x \)-strategy for a homomorphism from \( G \) and \( K \), so that \( G \twoheadrightarrow K \). In summary,

\[ \text{if } G \twoheadrightarrow H \quad \text{and} \quad H \twoheadrightarrow K, \quad \text{then } G \twoheadrightarrow K. \]
is called a representation of the graph homomorphism game from \( G \) to \( H \). If no set of projections on any Hilbert space exists satisfying these relations, then we say that the graph homomorphism game from \( G \) to \( H \) is not representable.

**Definition 4.2.** Let \( G \) and \( H \) be graphs. If a representation of the graph homomorphism game exists, then we let \( A(G,H) \) denote the “universal” \( C^* \)-algebra generated by such sets of projections. If the graph homomorphism game from \( G \) to \( H \) is not representable, then we say that \( A(G,H) \) does not exist. We write \( G \xrightarrow{C^*} H \) if and only if \( A(G,H) \) exists.

By “universal” we mean that \( A(G,H) \) is a unital \( C^* \)-algebra generated by projections \( \{ e_{v,x} : v \in V(G), x \in V(H) \} \) satisfying

(1) for each \( v \in V(G) \), \( \sum_x e_{v,x} = 1 \),

(2) if \((v,w) \in E(G)\) and \((x,y) \notin E(H)\), then \( e_{v,x} e_{w,y} = 0 \),

with the property that for any representation of the graph homomorphism game on a Hilbert space \( \mathcal{H} \) by projections \( \{ E_{v,x} \} \) satisfying the above relations, there exists a *-homomorphism \( \pi : A(G,H) \to B(\mathcal{H}) \) with \( \pi(e_{v,x}) = E_{v,x} \).

Here is one result that relates to existence. Let \( E_m \) be the “empty” graph on \( m \) vertices, i.e., the graph with no edges.

**Proposition 4.3.** Let \( G \) be a graph with at least one edge, \((v,w) \in E(G)\). Then \( A(G,E_m) \) does not exist.

**Proof.** By definition we have that \( e_{v,x} e_{w,y} = 0 \) for all \( x,y \). Thus,

\[
0 = \sum_{x,y} e_{v,x} e_{w,y} = \left( \sum_x e_{v,x} \right) \left( \sum_y e_{w,y} \right) = 1,
\]

contradiction. \( \square \)

In [1, Definition 2] another type of graph homomorphism was defined, denoted by \( G \xrightarrow{B} H \). Briefly, if in our definition of \( Q_{\text{vect}}(n,m) \) we had dropped the requirement that all the inner products be non-negative, then we would obtain a larger set of tuples and \( G \xrightarrow{B} H \) if and only if there exists a \( p(x,y|v,w) \) in this larger set satisfying the conditions (1) and (2) of a winning strategy for the graph homomorphism game. Note that in this case, since these numbers need not be non-negative, we can not interpret them as probabilities.

**Proposition 4.4.** If \( G \xrightarrow{C^*} H \) or \( G \xrightarrow{\text{vect}} H \), then \( G \xrightarrow{B} H \), as defined in [1].

**Proof.** The vect case is obvious from the remarks above. Let \( \{ E_{v,x} : v \in V(G), x \in V(H) \} \) be a set of projections that yields a representation of the graph homomorphism game on a Hilbert space \( \mathcal{H} \) and let \( h \in \mathcal{H} \) be any unit vector.

If we set \( h^v_x = E_{v,x} h \), then set of vectors \( \{ h^v_x \} \) satisfies all the properties of the definition of \( G \xrightarrow{B} H \) in [1, Definition 2]. \( \square \)
Remark 4.5. We do not know necessary and sufficient conditions for $\mathcal{A}(G, H)$ to exist. In particular, we do not know if $G \overset{B}{\rightarrow} H$ implies $G \overset{C^*}{\rightarrow} H$.

Proposition 4.6. If $G \overset{C^*}{\rightarrow} H$ and $H \overset{C^*}{\rightarrow} K$, then $G \overset{C^*}{\rightarrow} K$.

Proof. Since $G \overset{C^*}{\rightarrow} H$ and $H \overset{C^*}{\rightarrow} K$, then we know that there exist projections $\{E_{v,x}\}$ and $\{F_{y,a}\}$ with $v \in V(G)$, $x, y \in V(H)$ and $a \in V(K)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, satisfying (1) and (2). Consider the set of self-adjoint operators on $\mathcal{H} \otimes \mathcal{K}$ defined by $G_{v,a} = \sum_{x \in V(H)} E_{v,x} \otimes F_{x,a}$ for $v \in V(G)$ and $a \in V(K)$. Notice that,

$$G_{v,a}G_{v,a} = (\sum_x E_{v,x} \otimes F_{x,a})(\sum_y E_{v,y} \otimes F_{y,a}) =$$

$$\sum_{x,y} E_{v,x} E_{v,y} \otimes F_{x,a} F_{y,a} = \sum_x E_{v,x} \otimes F_{x,a} = G_{v,a}$$

by (2) and the fact that $E_{v,x}$ and $F_{x,a}$ are projections. Thus, each $G_{v,a}$ is a projection. Furthermore, for each $v \in V(G)$,

$$\sum_a G_{v,a} = \sum_a \sum_x E_{v,x} \otimes F_{x,a} = \sum_x E_{v,x} \otimes (\sum_a F_{x,a}) = (\sum_x E_{v,x}) \otimes I_\mathcal{K} = I_\mathcal{H} \otimes I_\mathcal{K}$$

by (1). Moreover, for each $(v, w) \in E(G)$ and $(a, b) \notin E(K)$,

$$G_{v,a}G_{w,b} = (\sum_x E_{v,x} \otimes F_{x,a})(\sum_y E_{w,y} \otimes F_{y,b}) = \sum_x \sum_y (E_{v,x} \otimes F_{x,a})(E_{w,y} \otimes F_{y,b})$$

$$= \sum_x \sum_{y \sim q} E_{v,x} E_{w,y} \otimes F_{x,a} F_{y,b} = \sum_{x \sim q} E_{v,x} E_{w,y} \otimes F_{x,a} F_{y,b} = 0$$

by (2). Hence, $\{G_{v,a} : v \in V(G), a \in V(K)\}$ is a representation of a graph homomorphism game from $G$ to $K$. \hfill \square

Recall that a trace on a unital C*-algebra $\mathcal{B}$ is any state $\tau$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{B}$.

Theorem 4.7. Let $G$ be a graph and let $x \in \{l, q, qa, qc, vect\}$.

1. $G \overset{qc}{\rightarrow} H$ if and only if there exists a tracial state on $\mathcal{A}(G, H)$,
2. if $G \overset{qc}{\rightarrow} H$, then $G \overset{C^*}{\rightarrow} H$,
3. $G \overset{q}{\rightarrow} H$ if and only if $\mathcal{A}(G, H)$ has a finite dimensional representation,
4. $G \overset{r}{\rightarrow} H$ if and only if $\mathcal{A}(G, H)$ has an abelian representation.

Proof. We have that $G \overset{qc}{\rightarrow} H$ if and only if there exists a winning qc-strategy $p(x, y|v, w) \in Q_{qc}^n(n, m)$. By [15] this strategy must be given by a trace on the algebra generated by Alice’s operators with $p(x, y|v, w) = \tau(A_{v,x}A_{w,y})$. Moreover, in the GNS representation, this trace will be faithful.

We now wish to show that these operators satisfy the necessary relations to induce a representation of $\mathcal{A}(G, H)$. 

By the original hypotheses, we will have that $A_{v,x}A_{v,y} = 0$ for $x \neq y$. When $(v,w) \in E(G)$ and $(x,y) \notin E(H)$, we will have that $\tau(A_{v,x}A_{w,y}) = p(x,y|v,w) = 0$ and hence, $A_{v,x}A_{w,y} = 0$.

Thus, Alice’s operators give rise to a representation of $A(G,H)$ and composing this *-homomorphism with the tracial state on the algebra generated by Alice’s operators gives the trace on $A(G,H)$. The converse follows by setting $p(x,y|v,w) = \tau(A_{v,x}A_{w,y})$.

Clearly, (2) follows from (1).

The proof of (3) is similar to the proof of (1). In this case since $p(x,y|v,w) \in Q_{0}^{s}(n,m)$ the operators all live on a finite dimensional space and hence generate a finite dimensional representation.

The proof of (4) first uses the fact that $G \rightarrow H$ if and only if $G \stackrel{1}{\rightarrow} H$ (2.1). If we let $(\Omega, \lambda)$ be the corresponding probability space and let $f_{v}, g_{v} : \Omega \rightarrow V(H)$ be the random variables for Alice and Bob, respectively, then the conditions imply that $f_{v} = g_{v}$ a.e. If we let $E_{v,x}$ denote the characteristic function of the set $f^{-1}([x])$, then it is easily checked that these projections in $L^{\infty}(\Omega, \lambda)$ satisfy all the conditions needed to give an abelian representation of $A(G,H)$.

Note that saying that $A(G,H)$ has an abelian representation is equivalent to requiring that it has a one-dimensional representation.

We now apply these results to coloring numbers. Let $K_{c}$ denote the complete graph on $c$ vertices.

**Proposition 4.8.** Let $x \in \{l,q,qa,qc,vect\}$, then $\chi_{x}(G)$ is the least integer $c$ for which $G \xrightarrow{x} K_{c}$.

**Proof.** Any winning $x$-strategy for a homomorphism from $G$ to $H$ is a winning strategy for a $x$-coloring. □

The above result motivates the following definition.

**Definition 4.9.** Define $\chi_{C^{*}}(G)$ to be the least integer $c$ for which $G \xrightarrow{C^{*}} K_{c}$.

Similarly, define $\omega_{C^{*}}(G)$ to be the biggest integer $c$ for which $K_{c} \xrightarrow{C^{*}} G$.

We let $\vartheta(G)$ denote the Lovasz theta function of a graph $G$ and we let $\overline{G}$ denote the graph with the same vertex set as $G$ and edges defined by $(v,w) \in E(\overline{G}) \iff v \neq w$ and $(v,w) \notin E(G)$.

**Proposition 4.10.** Let $G$ be a graph, then $\omega_{C^{*}}(G) \leq \vartheta(\overline{G}) \leq \chi_{C^{*}}(G)$.

**Proof.** Let $c := \chi_{C^{*}}(G)$. If we combine 4.4 with [1, Theorem 6] we know that,

$$G \xrightarrow{C^{*}} K_{c} \iff G \xrightarrow{B} K_{c} \iff \vartheta(\overline{G}) \leq \vartheta(\overline{K_{n}}) = c.$$  

Similarly, if you apply the above proof to $K_{d} \xrightarrow{C^{*}} G$, where $d := \omega_{C^{*}}(G)$, you get the remaining inequality. □
Remark 4.11. Since $G \xrightarrow{q_c} K_c \implies G \xrightarrow{C^*} K_c$, we have that $\chi_{q_c}(G) \geq \chi_{C^*}(G)$, but we don’t know the relation between $\chi_{C^*}(G)$ and $\chi_{\text{vect}}(G)$.

This leads to the following results:

Theorem 4.12. Let $G$ be a graph.

1. $\chi(G)$ is the least integer $c$ for which there is an abelian representation of $A(G, K_c)$.
2. $\chi_q(G)$ is the least integer $c$ for which $A(G, K_c)$ has a finite dimensional representation.
3. $\chi_{q_c}(G)$ is the least integer $c$ for which $A(G, K_c)$ has a tracial state.
4. $\chi_{C^*}(G)$ is the least integer $c$ for which $A(G, K_c)$ exists.

Theorem 4.13. Let $G$ be a graph.

1. The problem of determining if $A(G, K_3)$ has an abelian representation is NP-complete.
2. The problem of determining if $A(G, K_3)$ has a finite dimensional representation is NP-hard.
3. The problem of determining if $A(G, K_c)$ has a trace is solvable by a semidefinite programming problem.

Proof. We have shown that $A(G, K_3)$ has an abelian representation if and only if $G$ has a 3-coloring and this latter problem is NP-complete [2].

In [8, Theorem 1], it is proven that an NP-complete problem is polynomially reducible to determining if $\chi_q(G) = 3$. Hence, this latter problem is NP-hard.

In [15], it is proven that for each $n$ and $c$ there is a spectrahedron $S_{n,c} \subseteq \mathbb{R}^{n^2c^2}$ such that for each graph $G$ on $n$ vertices there is a linear functional $L_G: \mathbb{R}^{n^2c^2} \rightarrow \mathbb{R}$ with the property that $\chi_{q_c}(G) \leq c$ if and only if there is a point $p \in S_{n,c}$ with $L_G(p) = 0$. Thus, determining if $\chi_{q_c}(G) \leq c$ is solvable by a semidefinite programming problem. But we have seen that $\chi_{q_c}(G) \leq c$ if and only if $A(G, K_c)$ has a trace. \(\Box\)

Remark 4.14. Currently, there are no known algorithms for determining if $\chi_q(G) \leq 3$, i.e., for determining if $A(G, K_3)$ has a finite dimensional representation.

Remark 4.15. We do not know the complexity level of determining if $A(G, H)$ exists. In particular, we do not know the complexity level of determining if $G \xrightarrow{C^*} K_3$, or any algorithm.

Remark 4.16. In [1] it is proven that $\chi_{\text{vect}}(G) = \lceil \vartheta^+(\overline{G}) \rceil$, which is solvable by an SDP.

Remark 4.17. There is a family of finite input, finite output games that are called synchronous games [6], of which the graph homomorphism game is a special case. For any synchronous game $G$ we can construct the $C^*$-algebra of the game $A(G)$ and there are analogues of many of the above theorems. For
instance, the game will have a winning qc-strategy, q-strategy or l-strategy if and only if $\mathcal{A}(G)$ has a trace, finite dimensional, or abelian representation, respectively.

5. Factorization of Graph Homomorphisms

In this section, we show that the CP maps that arise from graph homomorphisms have a canonical factorization involving $\mathcal{A}(G, H)$.

**Proposition 5.1.** Let $G$ and $H$ be graphs on $n$ and $m$ vertices, respectively. The map $\Gamma : M_n \to M_m(\mathcal{A}(G, H))$ defined on matrix units by $\Gamma(E_{v,w}) = \sum_{x,y} E_{x,y} \otimes e_{v,x} e_{w,y}$ is CP.

**Proof.** Let $E_{v,x}, v \in V(G), x \in V(H)$ denote the $n \times m$ matrix units. Let $Z = \sum_{w,y} E_{v,w} \otimes e_{w,y} \in M_{n,m}(\mathcal{A}(G, H))$. Then

$$\Gamma(\sum_{v,w} c_{v,w} E_{v,w}) = Z^*(c_{v,w} E_{v,w} \otimes I)Z,$$

where $I$ denotes the identity of $\mathcal{A}(G, H)$ and $(c_{v,w} E_{v,w} \otimes I) \in M_n(\mathcal{A}(G, H))$. □

Let $p(x, y|v, w) \in Q_{qc}(n, m)$ be a winning qc-strategy for a graph homomorphism from $G$ to $H$. Then there is a tracial state $\tau : \mathcal{A}(G, H) \to \mathbb{C}$ such that $p(x, y|v, w) = \tau(e_{v,x} e_{w,y})$ and hence $\phi_p$ factors as $\phi_p = (id_m \otimes \tau) \circ \Gamma$, where $id_m \otimes \tau : M_m(\mathcal{A}(G, H)) \to M_m$. Conversely, if $\tau : \mathcal{A}(G, H) \to \mathbb{C}$ is any tracial state, then $(id_m \otimes \tau) \circ \Gamma = \phi_p$ for some winning qc-strategy $p(x, y|v, w) \in Q_{qc}(n, m)$.

Similarly, this map $\phi_p$ arises from a winning q-strategy if and only if it arises from a $\tau$ that has a finite dimensional GNS representation and from a winning l-strategy if and only if it arises from a $\tau$ with an abelian GNS representation.

This factorization leads to the following result. Recall that $\vartheta(G)$ denotes the Lovasz theta function of a graph and let $\|\phi\|_{cb}$ denotes the completely bounded norm of a map.

**Lemma 5.2.** Let $G$ be a graph on $n$ vertices, let $\mathcal{H}$ be a Hilbert space, let $P_{v,w} \in B(\mathcal{H}), \forall v, w \in V(G)$ and regard $P = (P_{v,w})$ as an operator on $\mathcal{H} \otimes \mathbb{C}^n$. If

1. $P = (P_{v,w}) \geq 0$,
2. $P_{v,v} = I_\mathcal{H}$,
3. $(v, w) \in E(G) \implies P_{v,w} = 0$,

then $\|P\| \leq \vartheta(G)$.

**Proof.** Any vector $k \in \mathcal{H} \otimes \mathbb{C}^n$ has a unique representation as $k = \sum_v k_v \otimes e_v$, where $k_v \in \mathcal{H}$ and $e_v \in \mathbb{C}^n$ denotes the standard orthonormal basis. Set $h_v = k_v/\|k_v\|$, (with $h_v = 0$ when $k_v = 0$) and $\lambda_v = \|k_v\|$. Let $y = \sum_v \lambda_v e_v \in \mathbb{C}^n$ so that $\|y\|_{\mathbb{C}^n} = \|k\|$. Set $B_k = \langle (P_{v,w}h_v, h_v) \rangle \in M_n = B(\mathbb{C}^n)$, so that $\langle B_k, k \rangle_{\mathcal{H} \otimes \mathbb{C}^n} = \langle B_k y, y \rangle_{\mathbb{C}^n}$. 
This observation shows that if for any \( h_v \in \mathcal{H}, \forall v \in V(G) \) with \( \|h_v\| = 1 \) we let \((P_{v,w}h_w, h_v)\) \( \in M_n = B(\mathbb{C}^n) \), then
\[
\|P\| = \sup\{\|\langle (P_{v,w}h_w, h_v)\|_{M_n} : \|h_v\| = 1\}.
\]

Now by the above hypotheses each matrix \((P_{v,w}h_w, h_v)\) \( \geq 0 \), has all diagonal entries equal to 1 and \((v, w) \in E(G) \Rightarrow \langle P_{v,w}h_w, h_v \rangle = 0 \). Thus, by [10], \( \|\langle (P_{v,w}h_w, h_v)\| \leq \vartheta(G) \).

**Proposition 5.3.** Let \( p(x, y|v, w) \in Q^s_{qc}(n, m) \) be a winning \( qc \)-strategy for a graph homomorphism from \( G \) to \( H \). Then \( \|\varphi_p\|_{cb} \leq \|\Gamma\|_{cb} \). Since this map is CP, by [14] we have that
\[
\|\Gamma\|_{cb} = \|\Gamma(I)\| = \|Z^*Z\| = \|ZZ^*\|.
\]
Since \( e^*_{v,w} = e_{w,v} \), we have
\[
ZZ^* = \sum_{v,w,x,y} (E_{v,x} \otimes e_{v,x})(E_{w,y} \otimes e_{w,y})^* = \sum_{v,w} E_{v,w} \otimes (\sum_x e_{v,x}e_{w,x}).
\]

Now if we let \( p_{v,w} \) denote the \((v, w)\)-entry of the above matrix in \( M_n(A(G, H)) \), then
\[
p_{v,w} = \sum_x e_{v,x} = 1. When (v, w) \in E(G), then by Definition 4.1(3), we have that \( p_{v,w} = 0 \).
\]
Hence, by the above lemma, \( \|ZZ^*\| \leq \vartheta(G) \).

6. Quantum Cores of Graphs

A retract of a graph \( G \) is a subgraph \( H \) of \( G \) such that there exists a graph homomorphism \( f : G \rightarrow H \), called a retraction with \( f(x) = x \) for any \( x \in V(H) \). A core is a graph which does not retract to a proper subgraph [7].

Note that if \( f : G \rightarrow G \) is an idempotent graph homomorphism and we define a graph \( H \) by setting \( V(H) = f(V(G)) \) and defining \((x, y) \in E(H) \) if and only if there exists \((v, w) \in E(G)\) with \( f(v) = x, f(w) = y \), then \( H \) is a subgraph of \( G \) and \( f \) is a retraction onto \( H \). We denote \( H \) by \( f(G) \).

The following result is central to proofs of the existence of cores of graphs.

**Theorem 6.1** ([7]). Let \( f \) be an endomorphism of a graph \( G \). Then there is an \( n \) such that \( f^n \) is idempotent and a retraction onto \( R = f^n(G) \).

Our goal in this section is to attempt to define a quantum analogue of the core using completely positive maps, in particular we will use the above theorem as a guiding principle.

For \( A = (a_{ij}) \in M_n \), denote \( \|A\|_1 = \sum_{i,j} |a_{ij}| \) and \( \sigma(A) = \sum_{i,j} a_{ij} \). Let \( \varphi_p : M_n \rightarrow M_m \), \( \varphi_p(E_{vw}) = \sum_{x,y} p(x, y|v, w)E_{xy} \), for some \( p(x, y|v, w) \in Q^s_{tec}(n, m) \). Before we continue our discussions on cores we will need the following facts,

**Lemma 6.2.**
\[
\sigma(\varphi_p(A)) = \sigma(A)
\]
Proof. By linearity it is enough to show the claim for matrix units, 

\[ \sigma(\phi_p(E_{vw})) = \sum_{x,y} p(x, y|v, w) = \sum_{x,y} \langle h_{v,x}, h_{w,y} \rangle = \langle \sum_x h_{v,x}, \sum_y h_{w,y} \rangle = \langle \eta, \eta \rangle = 1 = \sigma(E_{vw}) \]

\[ \square \]

Lemma 6.3. Let \( A = (a_{vw}) \) be a matrix, then

\[ ||\phi_p(A)||_1 \leq ||A||_1 \]

If the entries of \( A \) are non-negative, then \( \|\phi_p(A)\|_1 = \|A\|_1 \).

Proof. We have,

\[ ||\phi_p(A)||_1 = \sum_{x,y} |\sum_{v,w} p(x, y|v, w)a_{v,w}| \leq \sum_{v,w} |a_{v,w}|(\sum_{x,y} p(x, y|v, w)) \]

\[ = \sum_{v,w} |a_{vw}| = ||A||_1 \]

When the entries of \( A \) are all non-negative, the first inequality is an equality.

\[ \square \]

For the next step in our construction we need to recall the concept of a Banach generalized limit. A Banach generalized limit, is a positive linear functional \( f \) on \( \ell^\infty(\mathbb{N}) \), such that:

- if \( (a_k) \in \ell^\infty(\mathbb{N}) \) and \( \lim_k a_k \) exists, then \( f((a_k)) = \lim_k a_k \),
- if \( b_k = a_{k+1} \), then \( f((b_k)) = f((a_k)) \).

The existence and construction of these are presented in [4], along with many of their other properties. Often a Banach generalized limit functional is written as \( \text{glim} \).

Now fix a Banach generalized limit \( \text{glim} \), assume that \( n = m \), and that \( \phi_p : M_n \to M_n, \phi_p(E_{vw}) = \sum_{x,y} p(x, y|v, w)E_{xy} \), for some \( p(x, y|v, w) \in Q_{\text{qc}}(n, n) \). Fix a matrix \( A \in M_n \) and set

\[ a_{x,y}(k) = \langle \phi^k_p(A)e_y, e_x \rangle \]

so that \( \phi^k_p(A) = \sum_{x,y} a_{x,y}(k)E_{x,y} \). By Lemma 6.3, for every pair, \( (x, y) \) the sequence \( (a_{x,y}(k)) \in \ell^\infty(\mathbb{N}) \).

We define a map, \( \psi : M_n \to M_n \) by setting

\[ \psi_p(A) = \sum_{x,y} \text{glim}((a_{x,y}(k)))E_{x,y}. \]

Alternatively, we can write this as,

\[ \psi_p(A) = (id_n \otimes \text{glim})\phi^k_p(A). \]

Proposition 6.4. Let \( (p(x, y|v, w)) \in Q_{\text{vec}}(n, n) \) and let \( \psi_p : M_n \to M_n \) be the map obtained as above via some Banach generalized limit, \( \text{glim} \). Then:
(1) \( \psi_p \) is CP,
(2) \( \sigma(\psi_p(A)) = \sigma(A) \) for all \( A \in M_n \),
(3) \( \|\psi_p(A)\|_1 \leq \|A\|_1 \),
(4) \( \psi_p \circ \phi_p = \phi_p \circ \psi_p = \psi_p \),
(5) \( \psi_p \circ \psi_p = \psi_p \).

**Proof.** The first two properties follow from the linearity of the \( \text{glim} \) functional. For example, if \( A = (a_{x,y}) \) and \( h = (h_1, ..., h_n) \) \( \in \mathbb{C}^n \), then

\[
\langle \psi_p(A)h, h \rangle = \sum_{x,y} \text{glim}(a_{x,y}(k))h_yh_x = \text{glim}\left( \sum_{x,y} a_{x,y}(k)h_yh_x \right)
\]

\[
= \text{glim}\left( \langle \phi^k_p(A)h, h \rangle \right)
\]

If \( A \geq 0 \), then \( \phi^k(A) \geq 0 \) for all \( k \), and so is the above function of \( k \). Since \( \text{glim} \) is a positive linear functional, we find \( A \geq 0 \) implies \( \langle \psi_p(A)h, h \rangle \geq 0 \), for all \( h \). This shows that \( \psi_p \) is a positive map. The proof that it is CP is similar, as is the proof that it preserves \( \sigma \).

The proof of the third property is similar to the proof of Lemma 6.3. For the next claim, we have that

\[
\psi_p(\phi_p(A)) = (id \otimes \text{glim})(\phi^{k+1}_p(A)) = (id \otimes \text{glim})(\phi^k_p(A)) = \psi_p(A).
\]

If we set \( \psi_p(A) = \sum_{v,w} b_{v,w}E_{v,w} \), with \( b_{v,w} = \text{glim}(a_{v,w}(k)) \), then

\[
\phi_p(\psi_p(A)) = \sum_{x,y,v,w} p(x, y|v, w)b_{v,w}E_{x,y}
\]

\[
= \sum_{x,y} \text{glim}\left( \sum_{v,w} p(x, y|v, w)a_{v,w}(k) \right)E_{x,y} = \sum_{x,y} \text{glim}(a_{x,y}(k+1))E_{x,y} = \psi_p(A)
\]

Finally, to see the last claim, we have that

\[
\psi_p(\psi_p(A)) = (id \otimes \text{glim})(\phi^k_p(\psi_p(A))) = (id \otimes \text{glim})(\psi_p(A)) = \psi_p(A),
\]

since the \( \text{glim} \) of a constant sequence is equal to the constant. \( \square \)

**Theorem 6.5.** Let \( G \) be a graph on \( n \) vertices, let \( x \in \{l, qa, qc, vect\} \) and let \( p(x, y|v, w) \in Q^*_x(n, n) \) be a winning \( x \)-strategy implementing a quantum graph \( x \)-homomorphism from \( G \) to \( G \). Set \( p_1(x, y|v, w) = p(x, y|v, w) \) and recursively define,

\[
p_{k+1}(x, y|v, w) = \sum_{a,b} p(x, y|a, b)p_k(a, b|v, w).
\]

If we set \( r(x, y|v, w) = \text{glim}(p_k(x, y|v, w)) \), then \( r(x, y|v, w) \in Q^*_x(n, n) \) is a winning \( x \)-strategy implementing a graph \( x \)-homomorphism from \( G \) to \( G \) such that:

(1) \( \psi_r = \phi_r \),
(2) \( r(x, y|v, w) = \sum_{a,b} r(x, y|a, b)r(a, b|v, w) \).
Proof. By Theorem 3.7, \( \phi^k_p = \phi_{p_k} \), and \( p_k \) is a winning \( x \)-strategy for a graph \( x \)-homomorphism from \( G \) to \( G \). Thus,

\[
\psi_p(E_{v,w}) = (id \otimes \lim)(\phi^k_p(E_{v,w})) = (id \otimes \lim)(\phi_{p_k}(E_{v,w})) = \sum_{x,y} \lim(p_k(x,y|v,w))E_{x,y} = \phi_r(E_{v,w}).
\]

Thus, (1) follows.

Since \( \phi_r \circ \phi_r = \psi_p \circ \psi_p = \psi_p = \phi_r \), (2) follows from Proposition 3.6.

Finally, if a bounded sequence of matrices \( A_k = (a_{v,w}(k)) \) all belong to a closed set, then it is not hard to see that \( A = (\lim(a_{v,w}(k))) \) also belongs to the same closed set. Thus, since \( (p_k(x,y|v,w)) \) is in the closed set \( Q^s_x(n,n) \) for all \( k \), we have that \( (r(x,y|v,w)) \in Q^s_x(n,n) \). Also, since \( p_k \) is a winning \( x \)-strategy for a graph \( x \)-homomorphism of \( G \), for all \( k \), we have that for all \( k \), \( (p_k(x,y|v,w)) \) is zero in certain entries. Since the \( \lim \) of the 0 sequence is again 0, we will have that \( (r(x,y|v,w)) \) is also 0 in these entries. Hence, \( r \) is a winning \( x \)-strategy for a graph \( x \)-homomorphism.

\[ \blacksquare \]

Remark 6.6. In the case that \( p \) is a winning \( q \)-strategy implementing a graph \( q \)-homomorphism, all we can say about \( r \) is that it is a winning \( qa \)-strategy implementing a graph \( qa \)-homomorphism, since we do not know if the set \( Q^s_q(n,n) \) is closed.

There is a natural partial order on idempotent CP maps on \( M_n \). Given two idempotent maps \( \phi, \psi : M_n \rightarrow M_n \) we set \( \psi \leq \phi \) if and only if \( \psi \circ \phi = \phi \circ \psi = \psi \).

Theorem 6.7. Let \( x \in \{l, qa, qc, vect\} \), then there exists \( r(x,y|v,w) \in Q^s_x(n,n) \) implementing a quantum \( x \)-homomorphism, such that \( \phi_r : M_n \rightarrow M_n \) is idempotent and is minimal in the partial order on idempotent maps of the form \( \phi_p \) implemented by a quantum \( x \)-homomorphism of \( G \).

Proof. Quantum \( x \)-homomorphisms always exist, since the identity map on \( G \) belongs to the \( l \)-homomorphisms, which is the smallest set. By the last theorem we see that beginning with any correlation \( p \) implementing a quantum \( x \)-homomorphism, there exists a correlation \( r \) implementing a quantum \( x \)-homomorphism with \( \phi_r \) idempotent.

It remains to show the minimality claim. We will invoke Zorn’s lemma and show that every totally ordered set of such correlations has a lower bound. Let \( \{p_t(x,y|v,w) : t \in T\} \subset Q^s_x(n,n) \) and \( T \) a totally ordered set, where all \( p_t(x,y|v,w) \) implement a quantum \( x \)-homomorphisms, with \( \phi_{p_t} \) idempotent, and \( \phi_{p_t} \leq \phi_{p_s} \), whenever \( s \leq t \).

These define a net in the compact set \( Q^s_x(n,n) \) and so we may choose a convergent subnet. Now it is easily checked that if we define \( p(x,y|v,w) \) to be the limit point of this subnet, then it implements a quantum \( x \)-homomorphism, \( \phi_p \) is idempotent, and \( \phi_p \leq \phi_{p_t} \) for all \( t \in T \). \[ \blacksquare \]
Remark 6.8. It is important to note that we are not claiming that $\phi_r$ can be chosen minimal among all idempotent CP maps, just minimal among all such maps that implement a quantum $x$-homomorphism of $G$.

Definition 6.9. Let $x \in \{l, qa, qc, vect\}$, then a quantum $x$-core for $G$ is any $r(x, y|v, w) \in Q^x(n, n)$ that implements a quantum $x$-homomorphism such that $\phi_r$ is idempotent and minimal among all $\phi_p$ implemented by a quantum $x$-homomorphism of $G$.

Appendix: Background Material

Let $I$ and $O$ be two finite sets called the input set and output set, respectively.

Definition 6.10. A set of real numbers $p(x, y|v, w)$, $v, w \in I$, $x, y \in O$ is called a local or classical correlation if there is a probability space $(\Omega, \mu)$ and random variables, $f_v, g_w : \Omega \to O$ for each $v, w \in I$ such that,

$$p(x, y|v, w) = \mu(\{\omega \mid f_v(\omega) = x, g_w(\omega) = y\})$$

To motivate this definition, imagine that there are two people, Alice and Bob, when Alice receives input $v$, she uses the random variable $f_v$ and when Bob receives input $w$ he uses the random variable $g_w$. In this case $p(x, y|v, w)$ represents the probability of getting outcomes $x$ and $y$ respectively, given that they received inputs $v$ and $w$, respectively.

Definition 6.11. Given a Hilbert space $\mathcal{H}$, a collection $\{E_x : x \in O\}$ of bounded operators on $\mathcal{H}$ is called a projection valued measure (PVM) on $\mathcal{H}$, provided that each $E_x$ is an orthogonal projection and $\sum_{x \in O} E_x = I_\mathcal{H}$. The set is called a positive operator valued measure (POVM) on $\mathcal{H}$, provided that each $E_x$ is a positive semidefinite operator on $\mathcal{H}$ and $\sum_{x \in O} E_x = I_\mathcal{H}$.

Definition 6.12. A density $p$ is called a quantum correlation if it arises as follows:

Suppose Alice and Bob have finite dimensional Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ and for each input $v \in I$ Alice has PVMs $\{F_{v,x}\}_{x \in O}$ on $\mathcal{H}_A$ and for each input $w \in I$ Bob has PVMs $\{G_{w,y}\}_{y \in O}$ on $\mathcal{H}_B$ and they share a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$, then

$$p(x, y|v, w) = \langle F_{v,x} \otimes G_{w,y} \psi, \psi \rangle$$

This is the probability of getting outcomes $x, y$ given that they conducted experiments $v, w$.

Definition 6.13. A density $p$ is called a quantum commuting correlation if there is a single Hilbert space $\mathcal{H}$, such that for each $v \in I$ Alice has
PVMs $\{F_{v,x}\}_{x \in O}$ on $\mathcal{H}$ and for each $w \in I$ Bob has PVMs $\{G_{w,y}\}_{y \in O}$ on $\mathcal{H}$ satisfying,

$$F_{v,x} G_{w,y} = G_{w,y} F_{v,x}, \forall v, w, x, y$$

and

$$p(x, y|v, w) = \langle F_{v,x} G_{w,y} \psi, \psi \rangle$$

where $\psi \in \mathcal{H}$ is a shared state.

**Remark 6.14.** Suppose we have projection value measures $\{P_{v,i}\}_{i=1}^m$ and $\{Q_{w,j}\}_{j=1}^m$ on $\mathcal{H}$ as in 6.13. Set $X_{v,i} = P_{v,i} k, Y_{w,j} = Q_{w,j} k$. Then

1. $X_{v,i} \perp X_{v,j}$ for every $i \neq j$.
2. $Y_{w,i} \perp Y_{w,j}$ for every $i \neq j$.
3. $\sum_i X_{v,i} = \sum_j Y_{w,j}$ for every $v, w$ and $\|\sum_i X_{v,i}\| = 1$.
4. $\langle X_{v,i}, Y_{w,j} \rangle \geq 0$ since $\langle X_{v,i}, Y_{w,j} \rangle = \langle P^{2}_{v,i}, Q^{2}_{w,j} \rangle = \langle Q_{w,j} P_{v,i} k, Q_{w,j} P_{v,i} k \rangle = \|Q_{w,j} P_{v,i} k\|^2 \geq 0$ where the second equality results from the fact that $Q_{w,j}$ and $P_{v,i}$ are commuting projections.

**Definition 6.15.** A density $p$ is called a vectorial correlation if $p(i, j|v, w) = \langle X_{v,i}, Y_{w,j} \rangle$ for sets of vectors $\{X_{v,i}\}, \{Y_{w,j}\}$ satisfying (1) through (4) in 6.14.

Let $n := |I|$ and $m := |O|$, we let:

- $Q_{loc}(n, m)$ denote the set of all densities that are local correlations
- $Q_{q}(n, m)$ denote the set of all densities that are quantum correlations
- Set $Q_{qa}(n, m) := Q_{q}(n, m)$, the closure of $Q_{q}(n, m)$.
- $Q_{qc}(n, m)$ denote the set of all densities that are quantum commuting correlations
- $Q_{vect}(n, m)$ denote the set of all densities that are vectorial correlations
- For $x \in \{loc, q, qa, qc\}$, we let $Q_{x}^s(n, m)$ denote the set of synchronous correlations in $Q_{x}(n, m)$.

**Remark 6.16.** Results in [16] and [15] show that the possibly larger sets that one obtains by using the larger collection of all POVMs in the definitions of $Q_{q}, Q_{qa}$ and $Q_{qc}$ in place of PVMs, yields the same sets. These equalities essentially follow from Stinespring’s theorem. Also, while earlier versions of [15] use the notation $Q_{t}(n, m)$, which we have adopted here, this notation was changed to $C_{t}(n, m)$ in later versions.

**Remark 6.17.** In addition to $Q_{vect}(n, m)$ being a natural relaxation of the other sets, determining membership in this set reduces to standard problems in linear algebra. Another important reasons for studying $Q_{vect}(n, m)$ is Tsirelson’s 1980 [9] attempted proof that $Q_{q}(n, m) = Q_{vect}(n, m)$. He attempted to show that $Q_{q}(n, m) = Q_{vect}(n, m)$, from which the other equality would follow, by starting with vectors satisfying (1) through (4) and attempting to build projections $\{P_{v,i}\}, \{Q_{w,j}\}$ on finite dimensional Hilbert space,
and a vector $k$ such that $X_{e,i} = P_{e,i}k$ and $Y_{w,j} = Q_{w,j}k$ commuted. In [1] a graph on 15 vertices is constructed for which $\chi_q(G) = 8 \neq \chi_{\text{vect}}(G) = 7$, giving a definitive proof that $Q_q(15, 7) \neq Q_{\text{vect}}(15, 7)$, hence showing that for some such set of vectors, one cannot construct corresponding projections. Later, for this same graph [15] proved that $\chi_{qc}(G) = 8 \neq \chi_{\text{vect}}(G)$ showing that $Q_{qc}(15, 7) \neq Q_{\text{vect}}(15, 7)$.

Here are some further facts and open problems about these sets that show their importance.

- $Q_{\text{loc}}(n,m) \subseteq Q_q(n,m) \subseteq Q_{qa}(n,m) \subseteq Q_{qc}(n,m) \subseteq Q_{\text{vect}}(n,m)$.
- $Q_{\text{loc}}(n,m), Q_{qa}(n,m), Q_{qc}(n,m)$, and $Q_{\text{vect}}(n,m)$ are closed.
- Bounded entanglement conjecture: $Q_q(n,m) = Q_{qa}(n,m) \forall n,m$, i.e., is $Q_q(n,m)$ closed.
- Tsirelson conjecture [9]: $Q_q(n,m) = Q_{qc}(n,m) \forall n,m$.
- Ozawa [13] proved that Connes’ embedding conjecture [5] is true if and only if $Q_{qa}(n,m) = Q_{qc}(n,m) \forall n,m$.
- Paulsen and Dykema [6] proved that Connes’ embedding conjecture is true if and only if $Q_{qa}(n,m) = Q_{qc}(n,m) \forall n,m$.
- The synchronous approximation conjecture: $Q_{qa}^*(n,m) = Q_{qc}^*(n,m) \forall n,m$.
- If Tsirelson’s conjecture is true, then the Connes’ embedding conjecture and the bounded entanglement conjecture are true.
- If Connes’ embedding conjecture is true, then the synchronous approximation conjecture is true.

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