A short exposition of the Patak-Tancer theorem on non-embeddability of $k$-complexes in $2k$-manifolds

E. Kogan and A. Skopenkov

Abstract

In 2019 P. Patak and M. Tancer obtained the following higher-dimensional generalization of the Heawood inequality on embeddings of graphs into surfaces. We expose this result in a short well-structured way accessible to non-specialists in the field.

Let $\Delta_k^n$ be the union of $k$-dimensional faces of the $n$-dimensional simplex.

**Theorem.** (a) If $\Delta_k^n$ PL embeds into the connected sum of $g$ copies of the Cartesian product $S^k \times S^k$ of two $k$-dimensional spheres, then $g \geq \frac{n - 2k - 1}{k + 2}$.

(b) If $\Delta_k^n$ PL embeds into a closed $(k - 1)$-connected PL $2k$-manifold $M$, then $(-1)^k(\chi(M) - 2) \geq \frac{n - 2k - 1}{k + 1}$.

1 Introduction

The classical Heawood inequality states that *if the complete graph $K_n$ on $n$ vertices is embeddable (i.e. realizable without self-intersections) in the sphere with $g$ handles, then*

$$g \geq \frac{(n - 3)(n - 4)}{12}.$$ 

Denote by $\Delta_n^k$ the union of $k$-dimensional faces of the $n$-dimensional simplex (i.e. the complete $(k + 1)$-regular hypergraph on $n + 1$ vertices). A higher-dimensional analogue of the Heawood inequality is the Kühnel conjecture whose simplified version states that *if $\Delta_k^n$ embeds into the connected sum of $g$ copies of the Cartesian product $S^k \times S^k$ of two $k$-dimensional spheres, then*

$$g \geq \frac{(n - k - 2)(n - k - 3)\ldots(n - 2k - 2)}{2(k + 1)(k + 2)\ldots(2k + 1)}.$$ 

We present a simplified exposition of the Paták-Tancer [PT20, PT22] estimate

$$g \geq \frac{n - 2k - 1}{k + 2}.$$ 

For this estimate one needs the following definitions and a more general result.

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We are grateful for useful discussions to S. Dzhenzher, R. Karasev and M. Tancer.

Supported in part by Russian Foundation for Basic Research Grant No. 19-01-00169.

E. Kogan: Higher School of Economics.

A. Skopenkov: Moscow Institute of Physics and Technology, Independent University of Moscow. Email: skopenko@mccme.ru. https://users.mccme.ru/skopenko/.
The Euler characteristics of a PL 2k-manifold M is $\chi(M) = a_0 - a_1 + a_2 - \ldots + a_{2k}$, where $a_j$ is the number of j-dimensional faces in some (or, equivalently, in any) triangulation (or cell decomposition) of $M$. A PL manifold $M$ is called (k−1)-connected if for any $j = 0, 1, \ldots, k−1$ any continuous map $\Delta_{j+1}^n \to M$ extends to a continuous map $\Delta_{j+1}^n \to M$. A square matrix with $\mathbb{Z}_2$-entries is called even if all the numbers on the main diagonal are even.

**Theorem 1.1** ([PT22, Theorem 1]). If $\Delta_n^k$ PL embeds into a closed (k−1)-connected PL 2k-manifold $M$, then $(-1)^k(\chi(M) - 2) \geq \frac{n - 2k - 1}{k + 1}$.

If, moreover, the intersection form of $M$ is even, then $(-1)^k(\chi(M) - 2) \geq \frac{2(n - 2k - 1)}{k + 2}$.

For a definition of the intersection form accessible to non-specialists in topology (in particular, to computer scientists) see [KS21e, §1.2, §1.3], [IF], [Sk20, §6, §10].

The above estimate $g \geq \frac{n - 2k - 1}{k + 2}$ is a particular case of Theorem 1.1 because the intersection form of the connected sum $M$ of $g$ copies of $S^k \times S^k$ is even and $\chi(M) = 2 + (-1)^k 2g$.

**Remark 1.2.** (a) The Kühnel conjecture asserts that if $\Delta_n^k$ PL embeds into a closed (k−1)-connected PL 2k-manifold $M$, then

$$(-1)^k(\chi(M) - 2) \geq \frac{(n - k - 2)(n - k - 3) \ldots (n - 2k - 2)}{(k + 1)(k + 2) \ldots (2k + 1)}.$$  

Giving a complete list of earlier results on the Kühnel conjecture is beyond the scope of the present paper. See e.g. references in [PT22, §1.1], [No98, Sp97]. Our exposition could hopefully be useful to obtain stronger results on the Kühnel conjecture.

For corollaries of Theorem 1.1 see [PT22, Theorem 2 and Corollary 3].

(b) For a closed (k−1)-connected PL 2k-manifold $M$ we have $(-1)^k(\chi(M) - 2) = \text{rk } H_k(M; \mathbb{Z}_2)$. For a definition of the homology group $H_k(M; \mathbb{Z}_2)$ accessible to non-specialists in topology (in particular, to computer scientists) see [IF], [Sk20, §6, §10]. If in Theorem 1.2 we replace $(-1)^k(\chi(M) - 2)$ by $\text{rk } H_k(M; \mathbb{Z}_2)$, then the (k−1)-connectedness assumption could be omitted.

(c) The intersection form of any smooth (k−1)-connected 2k-manifold is even when $k \neq 1, 2, 4$.¹ We conjecture that the same result holds for PL manifolds.

(d) A general position PL map $f : \Delta_n^k \to M$ to a 2k-manifold $M$ is called a $\mathbb{Z}_2$-embedding if $|f \sigma \cap f \tau|$ is even for any pair $\sigma, \tau$ of non-adjacent faces. Theorem 1.1 holds (with the same proof) if one replaces ‘embeds into’ by ‘has a $\mathbb{Z}_2$-embedding to’. For $k = 1, 2$ this non-$\mathbb{Z}_2$-embeddability result is stronger than Theorem 1.1 [FK17], [KS21e, Theorem 1.3.1.bc]. For $k = 1$ this result asserts that if $K_n$ is $\mathbb{Z}_2$-embeddable to a surface $M$, then $3\chi(M) \leq 2 - 2s$. This is covered (at least for large $s$) by [FK19, Theorem 1], because $K, n$ is a subgraph of $K_{2n}$.

(e) In spite of being shorter, our proof of Theorem 1.1 is not essentially different from [PT22], but is a different well-structured exposition. Cf. [Sk21d, Remark 1.1.b]. The proof

¹The following proof written by D. Crowley and A. Skopenkov is presumably folklore. Assume that $k \geq 3$ and $k \neq 4$. Let $M$ be given manifold. Take any $x \in H_k(M; \mathbb{Z}_2)$. Since $M$ is (k−1)-connected and $k \geq 3$, we can realize $x$ as a smooth embedding $S^k \to M$. Let $\tau$ be the modulo 2 Euler number of the normal bundle of this embedding. Then $x^2 = -1 = 0 \in \mathbb{Z}_2$. Here the first equality is an exercise and the second equality is a version of the celebrated result on non-parallelizability of spheres [BM58].
of Theorem 1.1 is split into independent parts: the linear algebraic part (Proposition 1.4, cf. [PT22, Proposition 17]), and the topological part (Lemma 1.5, cf. Remark 1.3 and [PT22, Proposition 16.C2] referred later in [PT22] as Proposition 16.(ii)). In particular, we present simplified elementary statement and proof of Propositions 1.4 and 2.2 (without using cycles and homology as in [PT22, Proposition 17 and §4.2], and topological terminology as in [PT22, Lemma 20]). We do not use technical ‘obstruction machinery’ from [PT22, §1.2, §3], as opposed to [PT22, §4.1] (for a simplified exposition of this machinery see [KS21e, §2.5]). Our exposition clarifies the relation of the Paták-Tancer proof to earlier known results (see footnote 2 and Remark 2.3), and to the low-rank matrix completion problem [NKS].

Remark 1.3. In this remark we illustrate the idea of the proof of the topological part (Lemma 1.5) by a low-dimensional example. This remark is not formally used in the proof.

Consider the following statements:

1.3.A) For any distinct points $A_1, A_2, A_3, A_4$ in the line there is exactly one ‘intertwined’ coloring of $A_1, A_2, A_3, A_4$ into two colors.

1.3.B) For any distinct points $A_1, A_2, A_3, A_4$ on the circle

$$|A_1A_2 \cap A_3A_4| + |A_1A_3 \cap A_2A_4| + |A_1A_4 \cap A_2A_3| = 1.$$ 

1.3.B’) For any general position PL map $f : K_5 \to \mathbb{R}^2$ the number of intersection points in $\mathbb{R}^2$ formed by images of disjoint edges is odd.

1.3.C) For different numbers $i, j, k \in [5]$ denote by $[ijk]$ the cycle of length 3 in $K_5$ passing through $i, j, k$. Then for any embedding $f : K_5 \to S$ into a 2-surface $S$

$$f[125] \cap f[345] + f[135] \cap f[245] + f[145] \cap f[235] = 1 \in \mathbb{Z}_2.$$ 

Here $\cap$ is the algebraic intersection modulo 2 of closed curves on $S$ [IF].

Here (A) and (A) $\Rightarrow$ (B) $\Rightarrow$ (C) are easy. A simple deduction of (A) $\Rightarrow$ (B’) is presented in [Sk14] (for the linear case, for the PL case the deduction is analogous).

Theorem 1.1 is reduced to the following elementary algebraic result.

We shorten $\{\}$ to $i$. An $\binom{m}{l}$-matrix is a symmetric square matrix with $\mathbb{Z}_2$-entries whose rows and whose columns correspond to all $l$-element subsets of $[m]$.

Proposition 1.4 (cf. [PT22, Proposition 17]). Suppose $A$ is an $\binom{m}{l}$-matrix such that

1; triviality) $A_{P,Q} = 0$ if $P \cap Q = \emptyset$;

2; linear dependence) for each $(l + 1)$-element and $l$-element subsets $F, P \subset [m]$

$$\sum_{i \in F} A_{F-i,P} = 0.$$

(3; non-triviality) for each $i \in [m]$ and $(2l - 2)$-element subset $F \subset [m] - i$ we have

$A_{F,i} = 1$, where

$$A_{F,i} := \sum_{\{\sigma, \tau\} : F = \sigma \cup \tau, |\sigma| = l-1} A_{i \cup \sigma, i \cup \tau}.$$ 

Then $\text{rk} A \geq \frac{m - 2l + 2}{l - 1}$.

If, moreover, $A$ is even, then $\text{rk} A \geq \frac{2(m - 2l + 2)}{l}$.
The topological part of the proof is the following higher-dimensional analogue of (1.3.C).

We identify \((s+1)\)-element subsets of \([n+1]\) with \(s\)-dimensional faces of \(\Delta_n\). For a \((k+2)\)-element subset \(P \subset [n+1]\) denote by \(\partial P\) the boundary sphere of the \((k+1)\)-dimensional face \(P\). Denote by \(\cap \) the intersection modulo 2 of \(k\)-dimensional spheres (or modulo 2 homology cycles) in a \(2k\)-manifold [IF, §2], [Sk20, §6, §10].

**Lemma 1.5** ([PT22, Proposition 16.C2]). For a \((k+1)\)-element subset \(\sigma \subset [2k+2]\) let 
\[
\hat{\sigma} := \partial((2k+3) \cup \sigma); \text{ this is a } k\text{-dimensional sphere in } \Delta_{2k+2}^k.
\]
Then for any embedding \(f : \Delta_{2k+2}^k \to M\) into a \(2k\)-manifold \(M\)
\[
\sum_{\{\sigma, \tau\} : [2k+2]=\sigma \cup \tau, \ |\sigma|=k+1} f\hat{\sigma} \cap f\hat{\tau} = 1.
\]

**Proof.** Just as (1.3.C) is easily implied by (1.3.A), Lemma 1.5 is implied by a higher-dimensional analogue of (1.3.A), i.e., by Theorem 1.6.(odd) below². Namely, the lemma follows because
\[
\sum f\hat{\sigma} \cap f\hat{\tau} \overset{(1)}{=} \sum |(D \cap f\hat{\sigma}) \cap (D \cap f'\hat{\tau})| \mod 2 \overset{(2)}{=} \sum L_{\sigma, \tau} \overset{(3)}{=} 1.
\]

Here
- the sums are over \(\{\{\sigma, \tau\} : [2k+2] = \sigma \cup \tau, \ |\sigma|=k+1\};\)
- \(f' : \Delta_{2k+2}^k \to M\) is an embedding close to \(f\) and in general position to \(f\);
- \(D \subset M\) is a small \(2k\)-disk in general position to \(f\Delta_{2k+2}^k\) and to \(f'\Delta_{2k+2}^k\), containing \(f(2k+3)\) and \(f'(2k+3)\), and such that \(D \cap f\Delta_{2k+2}^k = D \cap f'\Delta_{2k+2}^k = \emptyset\);
- the equality (1) holds because \(f\) is an embedding and \(f'\) is close to \(f\);
- \(L_{\sigma, \tau}\) is the modulo 2 linking number of \(\partial D \cap f\hat{\sigma}\) and \(\partial D \cap f'\hat{\tau}\); by the symmetry of the linking number \(L_{\sigma, \tau} = L_{\tau, \sigma}\), hence the sum over \(\{\sigma, \tau\}\) is well-defined;
- the equality (2) holds by the following well-known statement (cf. [Sk18o, Lemma 2]):
  - for any proper general position PL maps \(g, g' : D^k \to B^{2k}\) such that \(g\partial D^k \cap g'\partial D^k = \emptyset\)
  - the number \(|gD^k \cap g'D^k|\) modulo 2 equals to the modulo 2 linking coefficient of \(g\partial D^k\) and \(g'\partial D^k\) in \(\partial B^{2k}\);
- the equality (3) is Theorem 1.6.(odd).

**Theorem 1.6.** Let \(f : \Delta_{d+2}^d \to \mathbb{R}^d\) be a general position PL map.

(odd) If \(d\) is odd, then the number of unordered pairs of disjoint \((d+1)/2\)-faces the images of whose boundaries are linked modulo 2, is odd.

(even) If \(d\) is even, then the number of intersection points in \(\mathbb{R}^d\) formed by images of disjoint \((d/2)\)-faces is odd.

Theorem 1.6
- is (1.3.A) for \(d = 1\),
- is the well-known Conway-Gordon-Sachs theorem for \(d = 3\), i.e. for graph \(K_6\) in 3-space,

² In [PT22, §4.1] Lemma 1.5 was essentially reduced to Theorem 1.6.(even), which is a higher-dimensional analogue of (1.3.B'). For a direct proof of the equivalence of the odd and the even cases of Theorem 1.6 see [Sk16, Remark 4.1.aeg]

Lemma 1.5 is a ‘conical version’ of Theorem 1.6.(odd). The ‘conical version’ of the analogue of Theorem 1.6.(odd) for the join of \((d+1)/2\) copies of the four-point set (in place of the \((d+1)/2\)-skeleton of \(\Delta_{d+2}\)) is [Sk03, Lemma 1']. This analogue coincides with Theorem 1.6.(odd) for \(d = 1\) but is different for \(d > 1\). This analogue is closer to (1.3.B) than to (1.3.C).
• is its higher-dimensional version \([SS92]^3\), \([LS98, Ta00]\) for \(d > 3\) odd, and
• is the van Kampen-Flores theorem for \(d\) even, see e.g. survey \([Sk16, §4]\).

This is because any general position PL map \(\Delta^{\lfloor \frac{d}{2} \rfloor}_{d+2} \to \mathbb{R}^d\) extends to a general position PL map \(\Delta^{\lfloor \frac{d}{2} \rfloor}_{d+2} \to \mathbb{R}^d\), and for \(d\) odd any general position PL map \(\Delta^{\lfloor \frac{d}{2} \rfloor}_{d+2} \to \mathbb{R}^d\) is an embedding.

**Proof of Theorem 1.1 assuming Proposition 1.4 and Lemma 1.5.** Let \(f : \Delta^k \to M\) be a PL embedding. Define an \(\binom{[n+1]}{k+2}\)-matrix \(A\) by \(A_{P,Q} = f \partial P \cap f \partial Q\). Then \(A\) is the Gramian matrix (with respect to \(\cap\)) of the homology classes of the images \(f \partial P\). Hence \(\text{rk} H_k(M; \mathbb{Z}_2) \geq \text{rk} A\) by the following well-known result (see a proof e.g. in [Bi21, Lemma 2.1]).

Let \(v_1, v_2, \ldots, v_s\) be vectors in some \(d\)-dimensional linear space over \(\mathbb{Z}_2\) with a bilinear symmetric product. Let \(A\) be the Gramian matrix of \(v_1, v_2, \ldots, v_s\). Then \(\text{rk} A \leq d\).

Let us prove that \(A\) also satisfies the assumptions of Proposition 1.4. Indeed, the triviality and linear dependence are clear. The non-triviality follows by Lemma 1.5.

Hence by Proposition 1.4 we obtain \(\text{rk} A \geq \frac{n-2k-1}{k+1}\) and, for \(A\) even, \(\text{rk} A \geq \frac{2(n-2k-1)}{k+2}\). So we are done by Remark 1.2.b.

## 2 Proof of Proposition 1.4

Denote by \(m_r\) (respectively \(\tilde{m}_r\)) the largest integer \(m\) such that there exists an (respectively an even) \(\binom{[m]}{l}\)-matrix \(A\) of rank at most \(r\) satisfying (1,2,3) of Proposition 1.4. Obviously, \(m_0 = \tilde{m}_0 = 2l - 2\). We also have \(\tilde{m}_1 = m_0\) because there is no even matrix having rank 1. Proposition 1.4 follows by Proposition 2.1.a’,b’. Proposition 2.1.a,b is only required to illustrate the idea of the proof of Proposition 2.1.a’,b’ by proving slightly weaker results giving estimates \(\text{rk} A \geq \frac{m - 2l + 2}{l}\) and, for \(A\) even, \(\text{rk} A \geq \frac{2(m - 2l + 2)}{2l - 1}\).

**Proposition 2.1.** (a) \(\tilde{m}_r \leq \tilde{m}_{r-2} + 2l - 1\);
(b) \(m_r \leq \max\{m_{r-1} + l, \tilde{m}_r\}\);
(a’) \(\tilde{m}_r \leq \tilde{m}_{r-2} + l\);
(b’) \(m_r \leq \max\{m_{r-1} + l - 1, \tilde{m}_r\}\).

**Proof of (a).** Take an even matrix \(A\) as in Proposition 1.4 for \(m = \tilde{m}_r\) and such that \(\text{rk} A = r\). By the non-triviality \(A \neq 0\). Hence there are \(l\)-element subsets \(X, Y \subset [\tilde{m}_r]\) such that \(A_{X,Y} = 1\). Let \(B\) be the ‘restriction’ of \(A\) to \(l\)-element subsets of \([\tilde{m}_r] - X - Y\). Then

\[
\tilde{m}_{r-2} \geq \tilde{m}_{rkB} \geq |[\tilde{m}_r] - X - Y| \geq \tilde{m}_r - 2l + 1,
\]

where

• the inequality (1) holds because \(B\) is even and \(\text{rk} B = \text{rk} B' - 2 \leq r - 2\) for the ‘restriction’ \(B'\) of \(A\) to \(X, Y\) and \(l\)-element subsets of \([\tilde{m}_r] - X - Y\);
• the inequality (2) holds because the properties (1,2,3) of Proposition 1.4 are preserved under restriction, so the matrix \(B\) satisfies these properties for \(m = |[\tilde{m}_r] - X - Y|\);
• the inequality (3) holds because \(A_{X,Y} = 1\), so by the triviality \(X \cap Y \neq \emptyset\).

\(^3\text{Theorem 1.6.(odd) is not explicitly stated in \([SS92]\) but can be proved analogously to \([SS92, Lemma 1.4]\). See details in \([KS20, Theorem 1.7 and footnote 4]\).}
Proof of (b). Take a matrix \( A \) as in Proposition 1.4 for \( m = m_r \) and such that \( \text{rk} A = r \). If \( A \) is even, then \( \text{rk} A \leq \tilde{m}_r \), so we are done. Otherwise there is an \( l \)-element subset \( X \subset [m_r] \) such that \( A_{X,X} = 1 \). Let \( B \) be the ‘restriction’ of \( A \) to \( l \)-element subsets of \( [m_r] - X \). Then

\[
\begin{align*}
(1) \quad m_r - 1 & \geq m_{\text{rk} B} \geq m_r - l, \\
(2) \quad m_r - 2 & \geq m_{\text{rk} B} \geq \tilde{m}_r - l,
\end{align*}
\]

- the inequality (1) holds because \( \text{rk} B = \text{rk} B' - 1 \leq r - 1 \) for the ‘restriction’ \( B' \) of \( A \) to \( X \) and \( l \)-element subsets of \( [m_r] - X \);
- the inequality (2) holds because the properties (1,2,3) of Proposition 1.4 are preserved under the ‘restriction’, so the matrix \( B \) satisfies these properties for \( m = |[\tilde{m}_r] - X| = m_r - l \).

Proof of (a’). Take an even matrix \( A \) as in Proposition 1.4 for \( m = \tilde{m}_r \) and such that \( \text{rk} A = r \). By the non-triviality \( A \neq 0 \). Hence there are \( l \)-element subsets \( X, Y \subset [\tilde{m}_r] \) such that \( A_{X,Y} = 1 \).\] Let \( B \) be the ‘restriction’ of \( A \) to \( l \)-element subsets of \( [\tilde{m}_r] - X \). Then

\[
\begin{align*}
(1) \quad \tilde{m}_r - 1 & \geq m_{\text{rk} B} \geq \tilde{m}_r - l, \\
(2) \quad \tilde{m}_r - 2 & \geq m_{\text{rk} B} \geq \tilde{m}_r - l,
\end{align*}
\]

- the inequality (2) holds because above the matrix \( B \) is even and satisfies the properties (1,2,3) of Proposition 1.4 for \( m = |[\tilde{m}_r] - X| = \tilde{m}_r - l \);
- the inequality (1) holds because \( \text{rk} B \leq r - 2 \) which is proved as follows (using a geometric interpretation of \( B \)). Take a basis of \( \mathbb{Z}_2^{(\tilde{m}_r)} \) corresponding to \( l \)-element subsets of \( [\tilde{m}_r] \). Define a bilinear form \( A \) on \( \mathbb{Z}_2^{(\tilde{m}_r)} \) by setting \( A(P, Q) := A_{P,Q} \) for basic vectors \( P, Q \). For any \( l \)-element subset \( P \) of \( [\tilde{m}_r] \) let \( P_{X,Y} := P + A_{X,P}Y + A_{Y,P}X \). Since \( A_{X,Y} = 1 \) and \( A_{X,X} = A_{Y,Y} = 0 \), we have \( A(P_{X,Y}, X) = A(P_{X,Y}, Y) = 0 \) (i.e. \( P_{X,Y} \) is the orthogonal projection of \( P \) to the orthogonal complement of \( \langle X, Y \rangle \) with respect to \( A \)). By the triviality, for every \( l \)-element set \( P \subset [\tilde{m}_r] - X \) we have \( P_{X,Y} = P + A_{Y,P}X \). Hence for every \( l \)-element sets \( P, Q \subset [\tilde{m}_r] - X \) we have

\[
A(P_{X,Y}, Q_{X,Y}) = A_{P,Q} + 0 + 0 + 0 = B_{P,Q}.
\]

Then \( B \) is the Gramian matrix (with respect to \( A \)) of the projections of subsets of \( [\tilde{m}_r] - X \). Let \( B' \) be the Gramian matrix (with respect to \( A \)) of \( X, Y \) and the projections of subsets of \( [\tilde{m}_r] - X \). We have \( B_{P,Q} = B'_{P,Q} \) for all subsets \( P, Q \) of \( [\tilde{m}_r] - X \). Furthermore, \( B'_{X,P} = B'_{P,X} = B'_{P,Y} = B'_{Y,Y} = 0 \) for any basic vector \( P \neq X, Y \) and \( B'_{X,Y} = B'_{Y,X} = 1, B'_{X,X} = B'_{Y,Y} = 0 \). Thus \( \text{rk} B = \text{rk} B' - 2 \leq r - 2 \). \( \blacktriangleleft \)

Recall that \( A_{F,i} \) is defined in the non-triviality property (3).

**Proposition 2.2** (cf. [PT22, Lemma 20]). If \( A \) is an \( \binom{m}{l} \)-matrix such that the linear dependence (2) holds, then

(heredity) \( A_{G,i,j} = A_{G,j,i} \) for each \( i, j \in [m] \) and \( (2l - 1) \)-element subset \( G \subset [m] - i - j \).

**Remark 2.3** (on Proposition 2.2). Heredity is a higher-dimensional generalization of the following known fact (see e.g. survey [KS21e, Lemma 3.8]):

For any vertex \( i \) of graph \( K_5 \) denote by \( A_i \) the sum modulo 2 of subsets

\[
T_{\{\sigma, \tau\}} := \{\{\alpha, \beta\} : \alpha, \beta \in E(K_5), \alpha \subset \sigma \cup i, \beta \subset \tau \cup i\} \subset \binom{E(K_5)}{2}
\]

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over non-ordered pairs \( \{\sigma, \tau\} \) of disjoint edges formed by the vertices \([5] - i\). Then

\[
A_i = \{\{\alpha, \beta\} : \alpha, \beta \in E(K_5), \alpha \cap \beta = \emptyset\}
\]

and so is independent of \(i\).

(This \(A_i\) is the \textit{quotient deleted product} of \(K_5\). In the notation of Proposition 1.4 for \(m = 5, l = 3, F = F_i = [5] - i\), a general position map \(f : K_5 \to \mathbb{R}^2\) and \(A_{P,Q} = f\partial P \cap f\partial Q\) we have that \(A_{F_i,i}\) is the scalar product with \(A_i\) of the intersection cocycle of \(f\) and so is the \textit{van Kampen number} of \(f\).

\textbf{Proof of Proposition 2.2.} Denote \(\sigma := \{i, j\} \cup \sigma\). Then

\[
A_{G_{ij,i}} + A_{G_{ij,j}} = \sum_{\{(\sigma, \tau) : G = \sigma \cup \tau, |\sigma| = l-2\}} (A_{\sigma,ij\tau} + A_{\sigma,jij\tau}) = \sum_{\{(\sigma, \tau) : G = \sigma \cup \tau, |\sigma| = l-2\}} \sum_{\tau \in \tau} A_{\sigma,\tau\tau\tau} = \sum_{\tau \in \tau} \sum_{\sigma, \tau} (A_{\sigma,\tau\tau\tau} + A_{\tau,\tau\tau\tau}) = 0,
\]

where

\* the first equality holds because \(A_{G_{ij,i}}\) is equal to the sum of the first summands \(A_{\sigma,ij\tau}\), and \(A_{G_{ij,j}}\) is equal to the sum of the second summands \(A_{\sigma,jij\tau}\);

\* the second equality holds because of the linear dependence (2) for \(F = \tau, P = \sigma\);

\* the other equalities are straightforward.

\textbf{Proof of (b').} Take a matrix \(A\) as in Proposition 1.4 for \(m = m_r\) and such that \(rk A = r\). If \(A\) is even, then \(rk A \leq m_r\), so we are done. Otherwise there is a \(l\)-element subset \(X \subset [m_r]\) such that \(A_{X,X} = 1\). Without loss of generality \(X = \{m_r - l + 1, m_r - l + 2, \ldots, m_r\}\). For \(l\)-element subsets \(P, Q \subset [m_r - l + 1]\) define

\[
B_{P,Q} := A_{P,Q} + A_{P,X}A_{Q,X}.
\]

Below we check that \(rk B < r\) and that \(B\) satisfies the properties (1,2,3) of Proposition 1.4 for \(m = m_r - l + 1\). Therefore

\[
m_{r-1} \geq m_{rkB} \geq m_r - l + 1.
\]

In this paragraph we prove that \(rk B < r\). (Analogously to the proof that \(rk B \leq r - 2\) in the proof of (a') we use a geometric interpretation of the formula defining \(B_{P,Q}\).) Take a basis of \(\mathbb{Z}_2^{m_r}\) corresponding to \(l\)-element subsets of \([m_r]\). Define a bilinear form \(A\) on \(\mathbb{Z}_2^{m_r}\) by setting \(A(P, Q) := A_{P,Q}\) for basic vectors \(P, Q\). Let \(P_X\) be the orthogonal projection of \(P\) to the orthogonal complement of \(X\) (with respect to \(A\)), i.e. \(P_X := P + A_{P,X}X\). We have

\[
A(P_X, Q_X) = A(P, Q) + A(A_{P,X}X, Q) + A(P, A_{Q,X}X) + A(A_{P,X}X, A_{Q,X}X) =
\]

\[
= A_{P,Q} + A_{P,X}A_{Q,X} + A_{P,X}A_{Q,X} + A_{P,X}A_{Q,X}A_{X,X} = A_{P,Q} + A_{P,X}A_{Q,X} = B_{P,Q}.
\]

Then \(B\) is the Gramian matrix (with respect to \(A\)) of the projections of subsets of \([m_r - l + 1]\). Let \(B'\) be the Gramian matrix (with respect to \(A\)) of \(X\) and the projections of subsets of
We have $B_{P,Q} = B'_{P,Q}$ for all subsets $P, Q \subset [m_r - l + 1]$. Furthermore, $B'_{X,P} = B'_{P,X} = 0$ for any basic vector $P \neq X$ and $B'_{X,X} = A_{X,X} = 1$. Thus $\text{rk} B = \text{rk} B' - 1 < r$.

In this paragraph we prove that $B$ satisfies the triviality (1). If $P \cap Q = \emptyset$, then either $P \cap X = \emptyset$ or $Q \cap X = \emptyset$. Hence $B_{P,Q} = A_{P,Q} + A_{P,X}A_{Q,X} = 0 + 0 = 0$.

In this paragraph we prove that $B$ satisfies the linear dependence (2). For each $(l + 1)$-element and $l$-element subsets $F, P \subset [m_r - l + 1]$ we have

$$\sum_{i \in F} B_{F-i,P} = \sum_{i \in F} A_{F-i,P} + A_{P,X} \sum_{i \in F} A_{F-i,X} = 0.$$ 

In this paragraph we prove that $B$ satisfies the non-triviality (3). Using heredity of Proposition 2.2 for $B$, we may assume that $i \neq m_r - l + 1$. Then for each summand $B_{i \cup \sigma, i \cup \tau}$ of $B_{F,i}$ at least one of the sets $i \cup \sigma, i \cup \tau$ does not contain $m_r - l + 1$ and hence does not intersect $X$. Hence $B_{i \cup \sigma, i \cup \tau} = A_{i \cup \sigma, i \cup \tau} + A_{i \cup \sigma, X}A_{i \cup \tau, X} = A_{i \cup \sigma, i \cup \tau}$. Thus $B_{F,i} = A_{F,i} = 1$. \hfill \qed

3 Appendix

Remark 3.1. This remark is formed by public letters discussing the questions

- if there is a conflict of interest in publication of the expository part of this paper;
- should we update an arxiv version of our paper when the current arxiv version is under review;
- should a paper in a refereed journal conceal or reveal its methodology.

(The first question was raised by M. Tancer, see letter of May 8 below, and the other two turned out to be relevant.)

Discussion of our actions in such practical situations reveals our understanding ‘for whom research is done’ [Sk21d]. Such an understanding is sometimes concealed, so we are grateful to M. Tancer for stating his opinion, in spite of it being partly different from ours.

These letters might be interesting as an example of an open discussion of a controversial question, carried in full mutual respect of participants of the discussion, and leaving the final decision to the reader. (See the motivation for publicity in the letter of May 13, 2021 below.)

No reply to the letter of June 5 was received by the time of arxiv submission of this paper. So the discussion seems to have its final form ready for a reader’s judgement. (If available, an update of this discussion will be presented here.)

(A. Skopenkov to P. Patak and M. Tancer, May 8, 2021)

Dear Martin, Dear Pavel,

Martin raised the ‘conflict of interests’ question in his February letter. So in our paper with Eugene we need to publish our opinion on that, and to publicly invite you to publish your opinion. It would be nice if we could find a phrase that satisfies all of us. However, there is nothing wrong to present our different opinions so that a reader could make his/her own judgement. Below please find some suggestions for Remark 2f to be added to our paper. Could you please either choose any of them, or suggest your own phrase we could agree with, or make your own public statement?

(1) We are grateful to M. Tancer and P. Patak for confirming that there is no conflict of interest in publication of this paper, in spite of A. Skopenkov has been a referee of [PT20] since March to August of 2020.

(2) We are grateful to M. Tancer and P. Patak for confirming that there is no conflict of interest in publication of this paper, in spite of A. Skopenkov has been a referee of [PT20] since March to August of 2020. This is so because
• the paper \[PT20\] is openly published on arxiv, and so is available for praise, for criticism, as well as for building upon, properly mentioning the authors’ contribution;
• the current paper properly mentions the contribution of \[PT20\], stating that it only presents an exposition of the Partak-Tancer results;
• I am not a referee of \[PT20\] since August of 2020;
• the current paper is based upon suggestions I made in frame of my being referee of \[PT20\], and the authors have chosen not to realize those suggestions, at least in their full extent leading to a short exposition presented here. (Even the improved version of \[PT20\] partly realizing these suggestions from summer of 2020 is not available to the math community in May of 2021 and the authors kindly informed me in Fall 2020 and again in May, 2021 that they have no estimation for the date when they will make that improved version available to the math community.)

(3) In our opinion, there is no conflict of interest in publication of this paper, in spite of A. Skopenkov has been a referee for \[PT20\] since March to August of 2020. This is because \[bullet points from (2)\]. We are grateful to M. Tancer and P. Patak for learning our opinion presented above and stating that they do find a conflict of interest in publication of this paper because \[here a text from MT and PP is to be presented\].

(4) In our opinion, there is no conflict of interest in publication of this paper, in spite of A. Skopenkov has been a referee for \[PT20\] since March to August of 2020. This is because \[bullet points from (2)\]. We asked M. Tancer and P. Patak to read our opinion presented above and publicly state if they find a conflict of interest in publication of this paper. We are sorry they did not make any public statement on that issue.

Best Regards, Arkadiy.

(M. Tancer to A. Skopenkov, May 12, 2021) [A private letter.]

(A. Skopenkov to P. Patak and M. Tancer, May 13, 2021)
Dear Martin and Pavel,
Dear Martin, thank you for your letter. Recall that we need a public not private statement from you. So could you please
— confirm that the statement you sent us May 12, 2021 is public.
— add either ‘M. Tancer’ or ‘M. Tancer and P. Patak’ at the end of the statement.

Then we would be able to publish that statement, together with my explanation why it misrepresents facts. There is nothing wrong if you would modify your statement before confirming that the statement is public.

We strongly need this discussion to be responsible. We do not have enough time to discuss premature ideas, whose invalidity becomes clear when their publication (or a mental experiment of publication) is suggested. So if the statement you sent us May 12, 2021 is not public, the best way is to treat it as non-existent.

Therefore I inform you that each of us can possibly publish any letter on this subject starting from this letter. After presenting the letters we can possibly write whether we agree or disagree, and/or give explanations. If a part of such a public discussion would become obsolete, we could delete that part (only) by our mutual consent.

Such a public discussion, although very useful, would require much effort. So let us find a way to avoid it. E.g. if you would set up a reasonable deadline for arxiv publication of update of arXiv:1904.02404v3, then we would be willing to postpone the arxiv publication of our paper so that it would appear after your update. (Actually, Eugene and I worked on our paper very slowly in the hope that such an update would be available before our paper will be ready for arxiv submission.) The deadline being reasonable means that this postponing would not obstruct too much the progress of science. Publication of our paper after an update of yours would make the

\[\footnote{The improved version of \[PT20\] is now available as \[PT22\].} \]
question (of conflict of interest) void. If you want, we can discuss by skype / zoom this or other propositions.

Best, Arkadiy.

(M. Tancer to A. Skopenkov, May 18, 2021)

(A. Skopenkov to P. Patak and M. Tancer, May 27, 2021)

Dear Martin and Pavel,

Attached please find the update of our paper. We would be grateful for any remarks.

We deleted Remark 2e of the previous version. We would be glad to restore that remark if you allow us to do so. That remark praised your paper [PT20] for presenting an argument which can easily be turned into a proof of a certain result stated in my paper in a weaker form and only with a hint to a proof. We deleted that remark because it used information that I received from you while I was a referee of your paper.

Recall that Martin stated that there is a ‘conflict of interests’ in his letters of February and May, 2021. The reasons why we need the discussion to be public are explained in my letter of May 13, 2021.

In my opinion, updating the arxiv version when the current arxiv version is under review, is a friendly action towards the referees, the Editors and math community (in case the update is essential enough). This action allows the referees’ and the Editors’ decision on the paper to be more informed, if they are inclined to read or browse the update. This action does not force the referees and the Editors to read the update if they are not inclined to do so.

If you update your paper on arXiv, then we update our paper by replacing in Remark 1.2 references to your paper with references to the update of your paper. This would make the question of conflict of interest void.

Best, Arkadiy.

(M. Tancer to A. Skopenkov, May 28, 2021)

[A letter starting with ‘this letter is private’ after ‘Therefore I inform you that each of us can possibly publish any letter on this subject starting from this letter’ of May 13 letter.]

(M. Tancer to A. Skopenkov, May 31, 2021)

A. Skopenkov was a referee when we submitted [PT20] to a journal. We know this because he contacted us directly and he requested regular online meetings while we were explaining the contents of our paper. In short, he required some modifications that we could not accept as the authors which yielded his recommendation to reject the paper. This may of course be a legitimate approach. However, we believe that the fact that he immediately started to work on the same topic puts him into the conflict of interests. It therefore raises the question whether his intentions to reject the paper were honest, or whether he wanted to promote his own work.

In addition, the further reason why A. Skopenkov is in conflict of interests is that he had access to an extra information about paper and about our methodology beyond the publicly available version of our paper [PT20]. He requested such an additional information as a referee. In particular he

5In July 2021 M. Tancer requested removal of the current paper from arxiv (arXiv:2106.14010) claiming that publication of the May 18 letter forms a copyright infringement. This claim is unjust because
• the May 18 letter was sent in a reply to ‘Therefore I inform you that each of us can possibly publish any letter on this subject starting from this letter’ of the May 13 letter;
• the May 18 letter did not mention either that ‘this letter is private’ or that ‘this letter is copyrighted’. I am sorry that M. Tancer did the above instead of
• consenting to my suggestion of removing most of the May 18 letter by mutual consent (see the June 5 letter above);
• asking to remove the entire May 18 letter in reply to versions of the current paper containing this letter, which were sent to M. Tancer and P. Patak on May 27 and on June 5, each time asking for remarks. However, most of the May 18 letter is not relevant to the discussion. So we were glad to incorporate here all requests made by M. Tancer on deleting this letter.
had access to numerous intermediate revisions of our text when we tried to rewrite several proofs, definitions, etc. in order to try to satisfy the demands of the referee. The contents of this paper builds on this extra information, at least partially.

(A. Skopenkov to P. Patak and M. Tancer, June 5, 2021)

Dear Martin and Pavel,

Attached please find the update of our paper.

(1) Let me start with a suggestion (approved by Eugene). If you feel that some phrases (before Remark 3.1) describe your contribution in a misleading way, please list these phrases, explain what is wrong and/or suggest alternative phrases. Our paper from its first version gave all the credit for main results to [PT20]. If you feel that besides that, some credit for exposition should be given to you, please name particular places (statements, elements of proofs etc.) for doing that. If there would be many such places, we would be glad to invite you to be coauthors of this paper, on the condition that this would not lead to a significant delay and to making the text less accessible.

Our intention is to facilitate progress in science by presenting without delay a short exposition of your beautiful proof, acknowledging your priority. (In May 2020 I hoped that this intention could be implemented within your own paper.) I would be grateful if you could describe your intention.

(2) I am sorry that your letter misrepresents facts, and so necessarily arrives at a wrong conclusion. Namely, the following passages are wrong: ‘requested’, ‘we were explaining the contents of our paper’ ‘he required some modifications that we could not accept as the authors which yielded his recommendation to reject the paper’⁶, ‘he immediately started to work on the same topic’⁷, ‘He requested such an additional information as a referee’. Indeed,

• The discussions (meetings) were suggested not requested. In these discussions I presented critical remarks to the paper and suggestions on how to work on them. In particular, I did not request any extra information about [PT20] and about the methodology beyond [PT20]. See also my letter to the Editor in Remark 3.2.

• My rejection recommendation was based not on the authors’ refusal to accept my suggestions, but on specific critical remarks (1)-(8) of Remark 3.3 and on the poor work of the authors on these remarks.⁸

• Eugene and I started to work on the current paper in November 2020. This is long since you informed me in July 2020 that you are not going to incorporate my suggestions (in their full extent leading to a short well-structured exposition presented here [added in 2022: and in [KS21e]]). Eugene and I worked on an expository paper giving all the credit to [PT20].

I am sorry that your letter presents no specific examples of ‘extra information’ and ‘the contents of this paper’ mentioned in the phrases ‘he had access to an extra information about paper and about our methodology beyond the publicly available version of our paper’ and ‘The contents of this paper builds on this extra information’. So an outside observer can only conclude that there are no such examples, and your conclusions are wrong. If you would like to present such examples, see (1).

(3) In my opinion, if some version of a paper requires explanations of its contents (even to a colleague working in a close area) and conceals some methodology, then the version should not be published in a refereed journal. If the authors discover such drawbacks in the frame of a refereeing

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⁶ [added in 2022] Formally, this is correct and (as the authors write) legitimate. However, since the May, 31 letter questions my ‘intentions to reject the paper’, the report is presented in Remark 3.3, so that a reader can see that the reliability standards of this report agree with my reliability standards for other papers as exposed in [Sk21d].

⁷ [added in 2022] This (wrong and unjustified) guess is erroneously called ‘the fact’ in the May, 31 letter. The authors misrepresent my wish to help them as my starting to work on the same topic. E.g. suggestions on the proof of Theorem 1.1 were realized in this paper a year after the authors refused to fully incorporate them (see also Remark 1.2.e); suggestions from Remark 3.3.8 were realized as [KS21e, Proposition 2.5.1.RI Remark 2.5.2.b] more than a year after the authors refused to incorporate them (see also [PT22]).

⁸ [added in 2022] See footnote 6.
process, then the best way is to withdraw the paper (and possibly resubmit a revision when it is ready). Please let me know if you have a different opinion.

(4) In reply to your letter of May 28, recall that this discussion is public, see my letter of May 13. So, if I receive any letter on this subject described to be a private letter, I’ll have to delete it unread (to avoid confusion). Recall that I suggested avoiding a public discussion by having a (private) skype / zoom discussion. Also, I would be glad to discuss any other subject in any form you like (until that subject would require a public discussion for reasons explained in my letter of May 13).

Best regards, Arkadiy.9

(A. Skopenkov to P. Patak and M. Tancer, February 26, 2022)

Dear Martin and Pavel,

Attached please find the project of the update of arXiv:2106.14010. We would be grateful for any remarks. Some annoying flaws in our and your papers survived to arXiv publication because of our lack of exchange on these papers.

We would be glad to remove Remarks 3.2 and 3.3 if you publicly state something like ‘The reliability standards of A. Skopenkov’s report to [PT20] agree with his reliability standards for other papers as exposed in [Sk21d]. So, as opposed to our letter of May, 31, 2021, there is no reason to believe that his reasons for recommending rejection were dishonest.’

We would be glad to remove Remark 3.1 if you publicly withdraw your claim (shown to be wrong by Remark 3.1) that there is a conflict of interest in publication of this paper, see (1) and (2) of my May, 8, 2021 letter.

Best regards, Arkadiy.

Remark 3.2 (A. Skopenkov. A letter to an Editor of June 2020). Dear ..., Hope you are fine and healthy.

The Patak-Tancer paper submitted to ... has high potential but needs a thorough revision. Instead of thoroughly justifying these statements in a formal report I suggested to the authors (and they accepted) a more effective way of discussing specific remarks and suggestions directly with the authors and helping them to prepare a revision. (Cf. arXiv:2003.12285v1, Remark 3a: Journal publications practically rule the mathematical world. So writing a referee report on a paper is a responsible task involving double-checking. In this time-consuming form it is much harder to help the author than via informal discussions.).

The result of these discussions would be my final report (hopefully with acceptance recommendation) together with a list of my suggestions (hopefully minor) which the authors intentionally did not realize. The authors are making good progress on my suggestions. There are points where we disagree, but we are likely to reach a compromise. For this compromise we would need an opinion of yours (or of another referee) on those points, or just a report from another referee. We plan to send a letter describing our question in about a week’s time. Since (I am sorry) I missed the deadline 30 May 2020, I decided to inform you right away on our plans.

I am sorry if this non-standard approach, however more effective than a formal report, will take more of your time than you intend to spend. I am willing to be as consistent with the standard system as not to do ineffective work. E.g. please let me know if I should upload this letter as a ‘letter to the Editor’ to the Editorial system.

Best wishes, Arkadiy.

9Arkadiy suggested to delete this by mutual consent:

PS Could you please approve or disapprove my suggestions to delete by mutual consent some material including this footnote.
PSS There is nothing wrong in delaying your answer. I do not consider a 5-days reply to be a delay. I suggest omitting publication of this PS and of PS from your May 18 letter by our mutual consent (see my letter of May 13), so as not to flood the main topic of the discussion.

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Remark 3.3 (A. Skopenkov. A letter to an Editor of August 2020). Here I present a public report to \[\text{PT20}\] (this report is prepared for some journal). Although this report is public, I only plan to publish it, together with objections to it, if some objections would be raised outside our private discussions with the authors.\(^{10}\) I will not answer to any objections which are not public. Motivations for that are presented in ‘Work on critical remarks’ below.

I invited the authors to add to this public text their objections (if any). They disagreed. Still, the report below is too detailed at some places to rule out potential objections (which appeared in our discussions).

I am so sorry I have to spend time on writing and double-checking this text rather than on informal private skype discussions helping the authors to improve the paper.

Recommendation.

I recommend to reject the paper in its current form (i.e. in the submitted form), and to invite the resubmission of a shorter paper containing bright results (Theorem 5, its corollary Theorem 6 and Corollary 7; the shorter paper can well have a short remark on the van Kampen obstruction).

Presumably problems from my critical remarks below could be fixed. However, in my opinion our work on these remarks in June-July (but not our work in April-May) described below shows that the amount of time and efforts required from the referee and the Editors to bring the part of the paper involving the van Kampen obstruction to publishable form is not worth the corresponding results of the paper. This justifies the rejection recommendation.

The authors’ work on my remarks in April-May (not presented here) was very fruitful and so justifies the invitation recommendation.

Critical remarks.

Unless otherwise indicated, I presented these remarks (some of them in a less explicit form) in our April-May skype discussions with the authors. I use references as listed in the paper and the following reference.

[BKK] M. Bestvina, M. Kapovich and B. Kleiner. Van Kampen’s embedding obstruction for discrete groups, Invent. Math. 150 (2002) 219–235. \texttt{arXiv:math/0010141}.

(1) I suggest to postpone technical results and definitions to later (sub)sections and bring bright results to earlier (sub)sections. Most important example of this is to present the (generalized) van Kampen obstruction after the results not involving the obstruction (or even to extract all considerations of the obstruction to a separate paper to be submitted to a less rated journal). E.g. to move §1.1 after §1.3 (or maybe even later), §2.2 after §2.3 (or maybe even later). See (2)-(7) below.

(2) We agree with the authors that bright results of this paper are Theorem 5, its corollary Theorem 6 and Corollary 7. Statements of these results do not use the van Kampen obstruction. Proofs of these results are non-trivial and interesting. The proofs do not use (or can easily be simplified not to use) the van Kampen obstruction. They need much less technical (generalization of) \(\mathbb{Z}_2\)-valued van Kampen invariant, cf. \texttt{arXiv:1805.10237}, §1.4. Thus is misleading to call these bright results applications of the van Kampen obstruction, i.e of Theorem 1. See [Sk21d, Remark 1.1.b].

(3) Previous research suggests that the van Kampen obstruction is useful for algorithmic applications [MTW11], but only provides unnecessary sophistication for proofs of bright results. Compare the following papers:

B. Ummel. The product of nonplanar complexes does not imbed in 4-space, Trans. Amer. Math. Soc., 242 (1978) 319–328.

\(^{10}\)The report is kept here as a justification of footnote 6, although most of the criticism is not relevant to [PT22]. In this report references are updated and grammar typos are corrected, but no other changes are made.
M. Skopenkov. Embedding products of graphs into Euclidean spaces, Fund. Math. 179 (2003), 191–198, arXiv:0808.1199.

The introduction does not present algorithmic applications except minor Theorem 10 (cf. (2) above). In my opinion, bright results and their proofs are potentially more useful than technical versions of known constructions which so far did not yield any bright results.

(4) There are many versions of the classical van Kampen obstruction. See [Mel09], [BKK] and the following paper.¹¹

[RS] D. Repovš and A. B. Skopenkov. A deleted product criterion for approximability of a map by embeddings, Topol. Appl. 1998. 87 P. 1-19.

Not citing these references is a negligible drawback if the authors present their yet another version of the van Kampen obstruction outside the introduction. Since the authors consider their version as important as to be presented at the beginning of the introduction, not citing [BKK, RS] is a more serious drawback.

Yet another new version (Theorem 1) is not interesting enough, unless it yielded some bright results whose statements do not involve technical description of this version. According to the introduction, Theorem 1 and Corollary 3 did not yield such results.

(5) No motivations for technical description of the van Kampen obstruction in §1.1 is given. Statements of the main results are started in §1.1 with ‘We need some technical preliminaries. Also, for some notions we will not give a precise definition yet as we would need too many preliminaries in the introduction, but all notions are explained in Section 2.’ After those sentences (and a meaningless formula, see (6) below) a reader is likely to put away the paper and not to reach the bright results (Theorem 5, Theorem 6 and Corollary 7).

(6) The van Kampen obstruction is required for the statements of Theorem 1, Corollary 3, Theorem 4 and for the proofs (not for the statements) of Proposition 8 and Theorem 10. These statements and proofs are not rigorous because they rely on definitions involving meaningless formula

\[ (\ast) \quad \xi(\sigma \times \tau) = (-1)^k \xi(\tau \times \sigma). \]

This formula of §1.1 is meaningless (according to common definition of chains with integer coefficients, see textbooks by Fomenko-Fuchs, Hatcher or https://en.wikipedia.org/wiki/Simplicial_homology#Definition).

Indeed, no orientation on \( \sigma \times \tau \) (or on \( \sigma, \tau \)) is chosen, so the number \( \xi(\sigma \times \tau) \) is not defined (only its absolute value is defined).¹² Analogously, the number \( \xi(\tau \times \sigma) \) is not defined. Observe that the sign is important in the formula (\( \ast \)).

Presumably the authors use some non-specified non-standard orientation convention which lead them to skew-symmetric cochains on \( \tilde{K} \) rather than symmetric cochains on \( \tilde{K} \) isomorphic to ordinary cochains on \( \tilde{K}/\mathbb{Z}_2 \) as in [Sha57, §3] and in the survey [Sko08, §4]. No motivation for this sophistication is presented.

(7) The construction of §1.1 is not invariant, i.e. it uses (co)chains, not (co)cycles and (co)homology classes; this makes it less valuable from a theoretical point of view.

¹¹Possibly I did not mention [BKK, RS] in April-May discussions with the authors. However, if the authors consider their generalization of the van Kampen obstruction as important as to be presented at the beginning of the introduction, then it would be natural to search earlier publications on such generalizations before submission, or at least after our April-May discussions (and before presenting the version justifying the authors’ wish to present van Kampen obstruction at the beginning of the introduction, see ‘Work on critical remarks’ below).

¹²I did not mention this particular formula in our April-May skype discussions with the authors. However, I did mention that fixing orientation is necessary to work with \( \mathbb{Z} \)-coefficients, in relation to other part of the text, and I did refer to Remark 1.6.4 of arXiv:1805.10237. So it would be natural to correct analogous mistake in the version justifying the authors’ wish to present van Kampen obstruction at the beginning of the introduction, see ‘Work on critical remarks’ below).
A possible compromise is to state mod 2 version of Theorem 1 in the following user-friendly way similar to the Sarkaria’s definition of the obstructor complex, see e.g. [BKK, Definition 4 in §2]. (There is perhaps an analogous statement over integers, with analogous proof; hopefully this invariant statement needs neither technical condition (H) nor (H’).)

**Theorem 1.** Let $K$ be a $k$-complex, $M$ a PL 2k-manifold, and $f : [K] \to M$ be a generic map homotopic to an embedding. Assume that

- (H’) the restriction of $f$ the $(k-1)$-skeleton of $K$ is null-homotopic.
- Denote $\widetilde{K} := \cup \{ \sigma \times \tau \in K \times K : \sigma \cap \tau = \emptyset \}$. Let $C$ be any 2k-cycle modulo 2 in $\widetilde{K}/\mathbb{Z}_2$ whose preimage in $\widetilde{K}$ equals to $\sum_j (\alpha_j \times \beta_j + \beta_j \times \alpha_j)$ for some $k$-cycles $\alpha_j, \beta_j$ in $K$. Then

$$\sum_j |f\alpha_j \cap f\beta_j| \equiv \sum_{(\sigma, \tau) \in C} |g\sigma \cap g\tau| \mod 2.$$

(The later sum equals to the value on $C$ of the classical van Kampen obstruction of $K$.)

**Remark 2.** In more theoretical terms the conclusion of Theorem 1 means (at least for $(k-1)$-connected $M$) that the classical van Kampen obstruction of $K$ has to be the image of the mod 2 intersection form of $M$ under the composition

$$\text{Hom}_{\text{sym}}(H^k(M) \otimes H^k(M); \mathbb{Z}_2) \to (H^k(M) \otimes H^k(M))_{\text{sym}} \xrightarrow{\kappa} H^{2k}(M \times M)_{\text{sym}} \xrightarrow{(f \times f)^*}$$

$$\rightarrow H^{2k}(K \times K)_{\text{sym}} \xrightarrow{r} H^{2k}_{\text{sym}}(\widetilde{K}) \xrightarrow{\sim} H^{2k}(\widetilde{K}/\mathbb{Z}_2).$$

Here coefficients $\mathbb{Z}_2$ are omitted, $(H^k(M) \otimes H^k(M))_{\text{sym}}$ is the subgroup of $H^k(M) \otimes H^k(M)$ generated by elements $\alpha \otimes \alpha$ and $\alpha \otimes \beta + \beta \otimes \alpha$, the homomorphism $\kappa$ comes from the Künneth theorem and $r$ is the restriction.

**Work on critical remarks.**

Upon my suggestion and agreement of the authors and the Editor, in April-May the authors and I discussed my critical remarks and suggestions by skype. My identity as a referee was revealed upon my wish. Our discussions were very effective, cf. [Sk21d, Remark 1.4.a]. The authors greatly improved the text.

For the parts of the paper involving the van Kampen obstruction we did not agree and we wanted to ask the Editor (and possibly another referee) to step in. Namely,

- we agreed that Theorem 4, Proposition 8 and Theorem 10 are less important than other results of the introduction;
- we only disagreed on the comparative value of Theorem 1 (+Corollary 3) versus the amount of motivations and technical details required for its statement.

The authors prepared updated version of the paper to illustrate the differences and to justify their point of view. For justification of my rejection recommendation it suffices to know that the above critical remarks on the submitted version are also pertinent for the updated version, although I presented these remarks in our April-May skype discussions (with exceptions mentioned in the footnotes above, which are irrelevant for the rejection recommendation). Since it was not so easy for the authors to prepare an update properly motivating and rigorously defining the van Kampen obstruction, it would be hard for a reader to understand the motivation and the definition.

We also planned a common letter to the editor containing descriptions of our different opinions. In preparation of this letter there appeared less-responsible and/or unjustified passages (i.e. those which would not stand the test of making them public). So I required a public discussion where none of our letters is changed after its writing (the resulting dialogue to be sent to the Editor, but not to be put it on the internet unless we will be forced to). I had to make our letter public in order to

- make discussion of our different opinions more responsible;
- obstruct unjustified criticism of my report (if such a criticism would appear);
• submit to justified criticism of my report (if such a criticism would appear).

The authors disagreed. Instead, they agreed to incorporate my April-May suggestions on the van Kampen obstruction (at least in a compromise form we considered in April-May). Besides my repeating critical remarks in written form and impossibility of further less-responsible discussion, there could be other reasons for the authors’ change of mind. Whatever the authors’ reasons, I spent much more time on convincing them to rigorously define the van Kampen obstruction and to properly motivate it (to justify its place in the paper), than time sufficient to make corresponding changes myself.

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