Classical Einstein-Langevin Equation and Proposed Applications

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Abstract
We propose to formulate a theory for Classical Stochastic Gravity for certain applications in Astrophysics and Cosmology. This involves the Langevin approach in curved spacetime, which is introduced here, in the form of a classical Einstein-Langevin equation. The domain of applications of such an approach and possible outcomes of this formulation which are quite different than its semiclassical counterpart (which is an active area of research), are discussed. This field of study can be seen to emerge out of well established ideas and results in Brownian motion theory as well as the Stochastic Semiclassical Gravity and related issues in Thermodynamics. A brief calculation, to demonstrate the contribution of stochasticity and induced fluctuations to the background spacetime via heuristic solution of the Einstein Langevin equation is given. The applicability of the proposed formalism can have a wider expanse than is mentioned in this article.

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1 Introduction
In recent years, the semiclassical theory of stochastic gravity has been taking shape [1] [2] and is finding interesting applications as proposed for black hole physics [3] [4] and cosmology [5]. This relies on including the quantum stress tensor fluctuations in the semiclassical Einstein equation, and looking upon its induced effects therein.

In another attempt, [6] for developing a stochastic theory for gravity, a different approach of incorporating the stochastic effects has been proposed. This raises a question about smearing out singularities in spacetimes of interest. However the basic difference there, lies in the way stochasticity is introduced. The approach that we take up here, is that of introducing stochasticity in a more physically relevant fashion using the Langevin formulation. This is in terms of randomness of the stress tensor itself, however the applications for our formalism are quite specific and have to be addressed clearly in terms of the physical picture of randomness.
The theory of classical Brownian motion \cite{7} \cite{8} and elaborately formulated Semiclassical Stochastic Gravity \cite{1} \cite{2} as mentioned above, naturally seem to direct one towards raising a query about possible formulation of a meaningful theory of Classical Stochastic Gravity.

In what we propose, the modified Einstein equation includes the first order fluctuations of the classical stress tensor, as shown in the subsequent sections. This has very different coverage in terms of developments and applications, than the semiclassical Einstein-Langevin equation. Physical insights and direct applicability to relativistic Astrophysics and Cosmology make such a development quite desirable at this stage.

We put forward the idea of analysing a relativistic thermal cloud of collisional gas, using the Classical Einstein-Langevin equation that we propose in this article.

2 Domain of valid applications: The randomness of the classical stress tensor

For astrophysical objects which can be described by giving different models of the stress-energy tensor (namely perfect and imperfect fluids, classical fields etc), the statistical averages of the stress tensor, are of interest.

Specific example is that of a relativistic star modeled by perfect fluid. The microscopic particles of the fluid collide frequently, such that their mean free path is short compared to the scale on which the density changes. A mean stress tensor in this case can be defined. An observer moving with an average velocity \( u^\alpha \) (four velocity), giving the mean velocity field of the fluid will see the collisions randomly distribute the nearby particle velocities, so that the particle distribution appears locally isotropic \cite{9}. The stress tensor can then be treated as a random variable and the source of stochasticity in such cases, are the collisions of microscopic particles. This has been suggested in \cite{10}.

Neutron stars dynamics, stability issues etc, require a detailed knowledge of the stars' microphysics. While working on perturbations of such system which is an area of active interest, one can further think of including these fluctuations since they give contributions which are otherwise ignored.

The source of these fluctuations may necessarily be the microphysics encompassing the quantum phenomena of the interior of the stars, which we may be able to partially capture in fluctuations of the classical matter variables. Here we work with the simplest possible model of the perfect fluid of star, hence we restrict to fluctuations of pressure and energy density of the matter. It is important to mention here that this analysis is very different from that of the semiclassical case, where the stochasticity is based on specific quantum fields coupled to the spacetime and what follows does not have a implications to classical counterpart. In what follows, the classical fluctuations that we mention here of the stress tensor, have no correspondence to the quantum fluctuations as treated in semiclassical stochastic gravity regarding physical phenomena of...
the system under investigation.

Our endeavour is to enhance the study of perturbations, instabilities and stellar oscillations as established in literature [9], by inclusion of the above mentioned fluctuations.

It may also be possible to extend the same consistently, for a very different scenario, namely the large scale structure of the universe. The relevant scales of interest there are expected to decide the fluctuations of the stress tensor and the randomness. For such cases, one can then phenomenologically put in the fluctuations of $T^{ab}$ in the Einstein Equation which specifies noise, thus giving it a form of a Classical Einstein-Langevin equation.

3 The Classical Einstein-Langevin Equation

The simplest form of E-L equation can be written as,

$$G^{ab}[g + h](x) = T^{ab}[g + h](x) + \tau^{ab}[g](x)$$

where the fluctuations $\tau^{ab}[g]$ defined by $\tau^{ab}(x) = (T^{ab}(x) - < T^{ab}(x) >)$ are taken over the unperturbed background and satisfy the condition $\nabla_a \tau^{ab} = 0$. Here the covariant derivative is taken w.r.t the metric $g^{ab}$. This ensures that the Einstein Langevin equation is covariantly conserved. The term $\tau^{ab}$ gives the equation a stochastic form and is thus defined only through its expectation value. For the equation to be meaningful we put $< \tau^{ab}(x) > = 0$, which holds for a Langevin type noise. The perturbations that are induced by these fluctuations form the solution of this equation.

The magnitude of fluctuations thus defined, needs to be small enough to fulfill the criteria for validity of such a treatment. We are interested in seeking the effect of these fluctuations on the spacetime geometry. This can be obtained by solving the above equation formally for the perturbations $h_{ab}$ of the metric $g_{ab}$. We shall aim for our future work to develop methods for solutions, which are quite involved and would depend on specific models.

In other applications where the classical fields associated with the body are of interest, one can take average of the stress tensor over these e.g electromagnetic, magnetic or electric field, or classical scalar fields in general, associated with the body.

It may be worth mentioning the difference of the averages taken here and the expectations in semiclassical theory. A quantum stress tensor expectation $< \psi | T^{\mu\nu} | \psi >$ over certain quantum states, as in semiclassical Einstein Equation may be treated as classical averages [11]. Also the issues of regularization etc, related to the quantum stress tensor make the semiclassical theory very involved in mathematical developments regarding formulations and solutions of the corresponding Einstein Langevin equation. The averages of the classical stress tensor that we consider here are fundamentally different and quite simple. Here the underlying Physics is in a different domain, so this should not be confused with the averages in the semiclassical case. Similarly, the applications of the semiclassical and classical stochastic gravity do not overlap.

We give as
an example of a gravitating system, a collapsing relativistic star, which can be modeled by the classical Einstein-Langevin equation. The stochastic analysis of such a system, in order to explore statistical behavior of spacetime is the final aim of this study.

As we will see later, that the fluctuations of the pressure and density, induce additional contribution to the metric perturbations, in addition to giving these a stochastic nature. It thus gives scope for statistical analysis of the spacetime structure in various contexts, including near critical point behaviour of collapse, stellar oscillations and so on.

Thus equation (1) above can take the following form,

\[ G^{ab}[g](x) + \delta G^{ab}[\mathcal{h}](x) = T^{ab}[g](x) + \delta T^{ab}[\mathcal{h}](x) + \tau^{ab}[g](x) \]  

which reduces to

\[ \delta G^{ab}[\mathcal{h}](x) = \delta T^{ab}[\mathcal{h}](x) + \tau^{ab}[g](x) \]  

We assume the stochastic term to be of the following form, as it describes the Langevin noise.

\[ \langle \tau^{ab}(x) \rangle = 0, \langle \tau^{ab}(x)\tau^{cd}(x') \rangle = N^{abcd}(x,x') \]  

where \( ab \) correspond to \( x \) and \( cd \) to \( x' \). The fluctuations denoted by \( \tau^{ab} \) can be written as

\[ \tau^{ab}(x) = T^{ab}(x) - \langle T^{ab}(x) \rangle \]  

as mentioned earlier. Thus the two point correlation is given by

\[ \langle \tau^{ab}(x)\tau^{cd}(x') \rangle = \langle (T^{ab}(x) - \langle T^{ab}(x) \rangle)(T^{cd}(x') - \langle T^{cd}(x') \rangle) \rangle \]  

Here \( \langle T^{ab} \rangle \) is the statistical average of the classical stress tensor, since this is treated as a random variable itself. The bitensor \( N^{abcd}(x,x') \) describes the two point correlation and has the following properties.

1. \[ N^{abcd}(x,x') = N^{cdab}(x',x) \]  

This is clear from eqn. (4).

2. \[ \nabla_a N^{abcd}(x,x') = \nabla_b N^{abcd}(x,x') = \nabla_c N^{abcd}(x,x') = \nabla_d N^{abcd}(x,x') = 0 \]  

This follows from covariant conservation of \( \tau^{ab} \).

The contributions coming out of the noise can be important, in addition to giving spacetime perturbations a stochastic nature; which is otherwise ignored in a deterministic treatment.

In the next section we attempt to obtain induced perturbations of spacetime geometry, which partially accounts for solution of the E-L equation. This may be seen as a heuristic calculation for spherically symmetric perturbations of a relativistic star.
4 Induced metric perturbations for relativistic star

For slowly rotating stars, the formalism starts with spherical stars and their perturbations.

4.1 The basic framework

A spherically symmetric spacetime in Schwarzschild coordinates is of the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2d\Omega^2$$ (9)

We discuss the spherical (radial) perturbations here. A more complete analysis of general perturbations, which includes non-spherical cases is more involved in terms of solutions of the Einstein Langevin equation, and we reserve it for later work.

The formalism for obtaining perturbations of stars around equilibrium configuration, lies at the core of study of oscillations, stability issues and critical points of collapse of massive stars. For motion that maintains spherical symmetry, we intend to work on perturbed potentials $\delta\nu$ and $\delta\lambda$ describing the spacetime geometry and the Lagrangian displacement of the fluid elements denoted by $\xi^a$. Here we restrict ourselves to the $r$-dependence of $\nu, \lambda$ and $\xi^a$ given by only one component $\xi^r$. The system is described by perturbed Einstein-Euler equation [9].

The fluid’s 3-velocity at fixed $r$ is given by $v = e^{\lambda - \nu^r}$ and its four-velocity takes the form

$$u^a = \frac{e^{-\nu}}{\sqrt{1 - v^2}}(1, \dot{r}, 0, 0) = \frac{1}{\sqrt{1 - v^2}}(e^{-\nu}, e^{-\lambda}v, 0, 0)$$ (10)

The stress-energy tensor is given by

$$T^{ab} = (\epsilon + p)u^a u^b + g^{ab}p$$ (11)

The unperturbed components of field equation are given as

$$G_{tt} = 8\pi T_{tt} : -e^{2(\nu - \lambda)}\left(\frac{1}{r^2} - \frac{2}{r}\dot{\lambda}\right) - e^{2\nu}\frac{2}{r^2} = -8\pi e^{2\nu}\epsilon \frac{1}{1 - v^2}$$ (12)

$$G_{rr} = 8\pi T_{rr} : \left(\frac{1}{r^2} + \frac{2}{r}\dot{\nu}\right) - e^{2\lambda}\frac{2}{r^2} = 8\pi e^{2\lambda}(e - v^2 + \epsilon) + \frac{p}{1 - v^2}$$ (13)

$$G_{tr} = 8\pi T_{tr} : -2\frac{\dot{\lambda}}{r} = 8\pi e^{\lambda + \nu}(\epsilon + p)\frac{v}{1 - v^2}$$ (14)

$$e^{2\lambda}G_{\theta\theta} = 8\pi e^{2\lambda}T_{\theta\theta} :$$

$$\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{r}(\dot{\nu}' - \lambda') = 8\pi e^{2\lambda}p$$ (15)
4.2 The perturbed equations

The perturbed equations, including the fluctuations of the random stress tensor are given by (corresponding to the first two equations above)

\[
G_{tt}^{(1)} = 8\pi T_{tt}^{(1)}(x) + \tau_{tt}(x) : \quad (16)
\]

\[
e^{-2\lambda}\left\{ (2\delta\lambda)\left( \frac{1}{r^2} - \frac{2}{r} \lambda' \right) - \frac{2}{r} \delta\lambda' \right\} = -8\pi(\delta\epsilon) - \frac{1}{1 - \eta^2} + 8\pi\tau_{tt}
\]

\[
G_{rr}^{(1)}(x) = 8\pi T_{rr}^{(1)}(x) + 8\pi\tau_{rr}(x) : \quad (17)
\]

\[
e^{-2\lambda}\left[ \frac{1}{r}\delta\nu' - \left( \frac{2}{r} \nu' + \frac{1}{r^2} \right)\delta\lambda \right] = 4\pi\delta\rho + 4\pi\tau_{rr}
\]

\[
G_{tr}^{(1)}(x) = 8\pi T_{tr}^{(1)}(x) + 8\pi\tau_{tr}(x) : \quad (18)
\]

\[
\delta\lambda = -4\pi r e^{2\lambda}(\epsilon + p)\xi + 4\pi r \int \tau_{tr} dt \quad (19)
\]

From equation (12) and (13) one can write

\[
\nu' + \lambda' = 4\pi(\epsilon + p)e^{2\lambda}r
\]

thus giving

\[
\delta\lambda = -(\nu' + \lambda')\xi + 4\pi r \int \tau_{tr} dt \quad (20)
\]

\[
\Delta\rho = \frac{1}{2}(\epsilon + p)q_{ab}\Delta g_{ab}
\]

\[
\Delta\rho = -\frac{1}{2}\Gamma_1 p q^{ab}\Delta g_{ab}
\]

where \(\Gamma_1\) is the adiabatic index given by

\[
\Gamma_1 = \frac{\epsilon + p}{\partial \epsilon}
\]
In our case we easily obtain
\[ q^{ab} \Delta g_{ab} = 2\delta - \frac{e^{-\lambda}}{r^2} [e^\lambda r^2 \xi]' = \frac{2}{r^2} e^\nu (e^{-\nu} r^2 \xi)' - 16\pi r \int \tau_{tt} dt \] (27)

thus giving
\[ \delta p = -\Gamma_1 p \frac{1}{r^2} e^\nu (e^{-\nu} r^2 \xi)' + 8\pi \Gamma_1 pr \int \tau_{rr} dt - \xi p' \] (28)
\[ \delta \epsilon = - (\epsilon + p) \frac{e^\nu}{r^2} (e^{-\nu} r^2 \xi)' + 8\pi (\epsilon + p) r \int \tau_{tt} dt - \epsilon' \xi \] (29)

From equation (17) one obtains,
\[ \delta \nu(t,r) = \int f_1(r) dr + 4\pi \int f_2(r)(\int \tau_{tt} dt) dr + 4\pi \int r \tau_{rr} dr \] (30)

where
\[ f_1(r) = e^{2\lambda} \left\{ - (\nu' + \lambda') \xi \left[ \frac{1}{r} + 8\pi p \right] + 4\pi r \left[ -\Gamma_1 p \frac{1}{r^2} e^\nu (e^{-\nu} r^2 \xi)' - \xi p' \right] \right\} \]

and
\[ f_2(r) = e^{2\lambda} r \left\{ 2\Gamma_1 p - \frac{1}{r} - 8\pi rp \right\} \]

The expression for \( \delta \lambda \) as obtained in equation (21) has been used in the above analysis. However in order to be able to see the effect on two point correlations and the effect of noise, another expression for the same as obtained from equation (12) would be useful. This can be shown to be of the form
\[ \delta \lambda(t,r) = \int m_1(r) dr + 8\pi \int m_2(r)(\int \tau_{tt} dt) dr + 8\pi \int \frac{e^\lambda}{r^{3/2}} \tau_{tt} dr \] (31)

where
\[ m_1 = \frac{1}{1 - \nu^2} \frac{1}{r^{1/2}} \left[ 2(\nu' + \lambda') e^\nu (e^{-\nu} r^2 \xi)' + 8\pi e^\lambda \xi \epsilon' \right] \]

and
\[ m_2 = \frac{2\pi e^\lambda}{r^{1/2}} \frac{1}{1 - \nu^2} (\nu' + \lambda') e^{-2\lambda} \]

4.3 Model of Noise

The model of noise that we use in the above, decides the stochastic behaviour of the perturbations of the metric.

For the perfect fluid stress tensor in spherically symmetric spacetime given by (11) we have
\[ \tau_{tt}(x) = g_{tt}(x) \gamma_\varepsilon(x) \]
\[ \tau_{rr}(x) = g_{rr}(x) \gamma_p(x) \]
\[ \tau_{tt}(x) = u_t(x) u_r(x) (\gamma_\varepsilon(x) + \gamma_p(x)) \]
\[ \tau_{\theta\theta}(x) = g_{\theta\theta}(x) \gamma_p(x) \]
\[ \tau_{\phi\phi}(x) = g_{\phi\phi}(x) \gamma_p(x) \] (32)
where we denote the fluctuations in pressure and density by $\gamma_p$ and $\gamma_\epsilon$. We note that for the stress tensor that we have considered in this article, the above are all of the non-vanishing components of the fluctuations. For our noise model we assume $<\gamma_p> = <\gamma_\epsilon> = 0$, which implies the vanishing of $<\xi_{ab}>$ as expected.

The two point correlations of $\xi_{ab}$, namely the Noise Kernel then can be defined as

$$N_{abcd}(x, x') = K_{abcd}(x, x') <\gamma_i(x)\gamma_j(x')> ; \text{ where } \{i, j\} = \{p, \epsilon\} \tag{33}$$

with the following relevant components for the model:

$$
K_{tttt}(x, x') = g_{tt}(x)g_{tt}(x'); \quad K_{rrrr}(x, x') = g_{rr}(x)g_{rr}(x') \\
K_{trtr}(x, x') = u_t(x)u_t(x')u_r(x)u_r(x'); \quad K_{trcd}(x, x') = u_t(x)u_r(x)g_{cd}(x') \\
K_{rr\theta\theta}(x, x') = g_{rr}(x)g_{\theta\theta}(x'); \quad K_{rr\phi\phi}(x, x') = g_{rr}(x)g_{\phi\phi}(x') \\
K_{\theta\theta\phi\phi}(x, x') = g_{\theta\theta}(x)g_{\phi\phi}(x') \tag{34}
$$

We assume no correlation between pressure and density fluctuations for simplicity. A gaussian distribution may be well suited model for this noise, such that all the higher order correlations may be defined in terms of the two point correlations. A coloured noise model may be more favourable than a white noise in the case of curved spacetime.

To be consistent with our simplified case of spherically symmetric spacetime and radial perturbations of fluid variables, one can clearly see that $g_{tt}$ and $g_{rr}$ are functions of $r$, with pressure and density having $t, r$ dependence. It would be meaningful here to choose a $\delta$-correlation for $r$ dependence in the noise model while keeping the $t$ dependence such that $<\gamma_p(t, r)\gamma_p(t, r')> = p_0\mu(t-t')\delta(r-r')$ and $<\gamma_\epsilon(t, r)\gamma_\epsilon(t', r')> = q_0\mu(t-t')\delta(r-r')$. Here $p_0$ and $q_0$ define the strength of the correlations.

### 4.4 The two point correlations of potentials

Using the perturbed potentials as obtained in section 4.2 and the noise model described above, the two point correlations are

$$<\delta\nu(t, r)\delta\nu(t', r')> = \int \int f_1(r)f_1(r')drdr' + (8\pi)^2p_0\int e^{4\mu(r)}\{\mu(t-t')r^2 - f_2(r)\int \dot{\mu}(t-t')dt\}dr \tag{35}$$

$$<\delta\lambda(t, r)\delta\lambda(t', r')> = \int \int m_1(r)m_1(r')drdr' + (8\pi)^2q_0\int \frac{e^{2\mu(r)}}{r} \left\{\mu(t-t') + m_2(r)r^{1/2}e^{(\lambda(r)+2\nu(r))} \right\} \int \dot{\mu}(t-t')dt \tag{36}$$

In evaluating the above expressions, we have assumed $dr/dt' = 0$, while $\dot{r} = dr/dt$ remains non-vanishing. The second term in equation (35) and (36) can be seen to arise out of the fluctuations of the stress tensor. These are additional
contributions to the perturbations, which arise out of stochasticity taken into account in the model. The strength of the noise $q_0$ and $p_0$ decide the magnitude of these, and we would elaborate more on this and a fluctuation dissipation theorem in an upcoming article. These are deeper issues, and need to be worked upon separately, while considering a solution of the Einstein Langevin equation in terms of metric perturbations $h_{ab}$ rather than just the potentials. It may then, be more appropriate to give physically relevant results in this context.

Here, our purpose has been to establish this formalism and state few basic consequences of including the effects of fluctuations of the classical stress tensor.

It may be of interest to associate other meaningful models of noise for relativistic stars and seek the corresponding behavior.

5 Further Directions

Though the framework here is inspired by semiclassical stochastic gravity, classical Brownian motion developments should be looked upon for the first principles. It may be interesting to see if, general relativity in this domain (statistical), would have at the fundamental level a much richer structure and content, than the theory of Brownian motion in Newtonian Physics. This can be expected, due to curved spacetime here, which gets coupled to matter. There is indication towards this, from certain domains where we see the effect of underlying geometry of space as that in fractal structures, which come up in a very interesting way [18]. One can seek to analyse the Einstein Langevin equation in terms of Markovian or Non-Markovian criteria, in addition to the background spacetime structure affecting the same.

Apart from this general interest in stochastic theory development, the present article directs one, towards a well defined research program which includes the following:

- A formal solution of the classical Einstein Langevin equation in terms of general perturbation of the background spacetime metric to be obtained by devising methods to do so. Thus solvable models have to be identified, for which this could be done analytically, along with cases where one may need numerical solutions.

- Few cases of interest would be the rotating neutron star and binary star systems, which are of interest to Graviational Waves.

- This analysis with additional contributions to the the spacetime structure and its perturbations, also can be used to study instabilities in collapsing bodies. A formal study towards non-nonequilibrium statistical physics can be provided in this context. The instabilities in gravitational collapse of relativistic stars (viz a neutron star or a system of neutron stars), would be amenable to analysis in terms of studying the correlations of the stress tensor fluctuations and their effects near different critical regions during the collapse scenario.
Another question that can be raised, is about an associated radiation of classical waves that a gravitating body such as a black hole may emit, that of superradiance. The contribution of fluctuations to the superradiance and its behaviour may be an interesting direction to probe in order to see if it gives significant results.

For the complete gravitational collapse, where one necessarily gets a singularity as the end state of the collapse, the effect of these stochastic contributions can play interesting role in deciding the stability criteria for the end states. This can modify the present established results, regarding parameter values that decide the occurrence of naked or covered singularity. Also this may enhance the area by giving it a way, to work out issues using statistical physics.

The few possible applications of the proposed classical theory of stochastic gravity in this article are mentioned, to physically motivate such a formulation and bring to notice the directions where this would lead to meaningful study. One can always suggest many more sub-areas and problems in astrophysics or cosmology, where this may find valid applications.

Introducing the idea of such an approach as that of a Langevin equation in classical gravity, is just the very first step. This necessarily needs to be followed by the appropriate solutions of the Einstein-Langevin equation, including mathematical developments for specific cases. This is our endeavour in immediate future.

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