Filtered formal groups, Cartier duality, and derived algebraic geometry

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Abstract

We develop a notion of formal groups in the filtered setting and describe a duality relating these to a specified class of filtered Hopf algebras. We then study a deformation to the normal cone construction in the setting of derived algebraic geometry. Applied to the unit section of a formal group $\hat{G}$ this provides a $\mathbb{G}_m$-equivariant degeneration of $\hat{G}$ to its tangent Lie algebra. We prove a unicity result on complete filtrations, which, in particular, identifies the resulting filtration on the coordinate algebra of this deformation with the adic filtration on the coordinate algebra of $\hat{G}$. We use this in a special case, together with the aforementioned notion of Cartier duality, to recover the filtration on the filtered circle of [MRT19]. Finally, we investigate some properties of $\hat{G}$-Hochschild homology, set out in loc. cit., and describe “lifts” of these invariants to the setting of spectral algebraic geometry.

1 Introduction

The starting point of this work arises from the construction in [MRT19] of the filtered circle, an object of algebro-geometric nature, capturing the $k$-linear homotopy type of $S^1$, the topological circle. This construction is motivated by the schematization problem due to Grothendieck, stated most generally in finding a purely algebraic description of the $\mathbb{Z}$-linear homotopy type of an arbitrary topological space $X$.

In the process of doing this, the authors realized that there was an inextricable link between this construction, and the theory of formal groups and Cartier duality, as set out in [Car62]. We briefly review the relationship. The filtered circle is obtained as the classifying stack $B\mathbb{H}$ where $\mathbb{H}$ is a $\mathbb{G}_m$-equivariant family of group schemes parametrized by the affine line, $\mathbb{A}^1$. This family of schemes interpolates between two affine group schemes, $\text{Fix}$ and $\text{Ker}$; these can be traced to the work of [SS01] where they are shown to arise via Cartier duality from the formal multiplicative and formal additive groups, $\mathbb{G}_m$ and $\mathbb{G}_a$ respectively. The filtered circle $S^1_{fil}$ is then obtained as $B\mathbb{H}$, the classifying stack over $\mathbb{A}^1/\mathbb{G}_m$ of $\mathbb{H}$. By taking the derived mapping space out of $S^1_{fil}$ in $\mathbb{A}^1/\mathbb{G}_m$-parametrized derived stacks, one recovers precisely Hochshild homology together with a functorial filtration.

There is no reason to stop at $\mathbb{G}_m$ or $\mathbb{G}_a$ however. In loc. cit., the authors proposed, given an arbitrary 1-dimensional formal group $\hat{G}$, the following generalized notion of Hochshild homology of simplicial commutative rings:

$$ \text{HH}^\mathfrak{G}(-) : \text{sCAlg}_k \to \text{sCAlg}_k, \ A \mapsto \text{HH}^\mathfrak{G}(A) := R\Gamma(\text{Map}_{\text{dStk}_k}(B\mathfrak{G}^\vee, \text{Spec}A)). $$

The right hand side is the derived mapping space out of $B\mathfrak{G}^\vee$, the classifying stack of the Cartier dual of $\hat{G}$. For $\hat{G} = \mathbb{G}_m$ one recovers Hochshild homology, via a natural equivalence of derived schemes

$$ \text{Map}(B\text{Fix}, X) \to \text{Map}(S^1, X) $$
and for $\hat{G} = \hat{G}_a$ one recovers the derived de Rham algebra (cf. [TV11]) via an equivalence

$$\text{Map}(B\text{Ker}, X) \simeq T_{X/k}[-1] = \text{Spec}(\text{Sym}(L_{X/k}[1]))$$

with the shifted (negative) tangent bundle. One may now ask the following natural questions: if one replaces $\hat{G}_m$ with an arbitrary formal group $\hat{G}$, does one obtain a similar degeneration? Is there a sense in which such a degeneration is canonical?

The overarching aim of this paper is to address some of these questions by further systematizing some of the above ideas, particularly using further ideas from spectral and derived algebraic geometry.

### 1.1 Filtered formal groups

The first main undertaking of this paper is to introduce a notion of filtered formal group. For now, we give the following rough definition, postponing the full definition to Section 4:

**Definition 1.1** (cf. Definition 4.29). A filtered formal group is an abelian cogroup object $A$ in the category of complete filtered algebras $\text{CAlg}(\hat{\text{Fil}}_R)$ which are discrete at the level of underlying algebras.

Heuristically, these give rise to stacks

$$\hat{G} \to \mathbb{A}^1/G_m,$$

for which the pullback $\pi^*(\hat{G})$ along the the smooth atlas $\pi: \mathbb{A}^1 \to \mathbb{A}^1/G_m$ is a formal group over $\mathbb{A}^1$ in the classical sense.

From the outset we restrict to a full subcategory of complete filtered algebras, for which there exists a well-behaved duality theory. Our setup is inspired by the framework of [Lur18] and the notion of smooth coalgebra therein. Namely, we restrict to complete filtered algebras that arise as the duals of smooth filtered coalgebras (cf. Definition 4.14). The abelian cogroup structure on a complete filtered algebra $A$ then corresponds to the structure of an abelian group object on the corresponding coalgebra. As everything in sight is discrete, hence 1-categorical (cf. Remark 3.3) this is precisely the data of a comonoid in smooth coalgebras, i.e. a filtered Hopf algebra. Inspired by the classical Cartier duality correspondence over a field between formal groups and affine group schemes, we refer to this as as filtered Cartier duality.

**Remark 1.2.** We acknowledge that the phrase “Cartier duality” has a variety of different meanings throughout the literature (e.g. duality between finite group schemes, $p$-divisible groups, etc.) For us, this will always mean a contravariant correspondence between (certain full subcategories of) formal groups and affine group schemes, originally observed by Cartier over a field in [Car62].

**Remark 1.3.** In this paper we are concerned with filtered formal groups $\hat{G} \to \mathbb{A}^1/G_m$ whose “fiber over $\text{Spec}k \to \mathbb{A}^1/G_m$” recovers a classical (discrete) formal group. We conjecture that the duality theory of Section 3 holds true in the filtered, spectral setting. Nevertheless, as this takes us away from our main applications, we have stayed away from this level of generality.

As it turns out, the notion of a complete filtered algebra, and hence ultimately the notion of a filtered formal group is of a rigid nature. To this effect, we demonstrate the following unicity result on complete filtered algebras $A_n$ with a specified associated graded:
**Theorem 1.4.** Let $A$ be an commutative ring which is complete with respect to the $I$-adic topology induced by some ideal $I \subset A$. Let $A_n \in \text{CAlg}(\widehat{\text{Fil}}_k)$ be a (discrete) complete filtered algebra with underlying object $A$. Suppose there is an inclusion

$$A_1 \rightarrow I$$

of $A$-modules inducing an equivalence

$$\text{gr}(A_n) \simeq \text{gr}(F^*_I(A)) = \text{Sym}_{\text{gr}}(I/I^2)$$

of graded objects, where $I/I^2$ is of pure weight 1. Then $A_n = F^*_I A$, namely the filtration in question is the $I$-adic filtration.

Hence, if $A$ is an augmented algebra, there can only be one (multiplicative) filtration on $A$ satisfying the conditions of 1.4, the $I$-adic filtration. We will observe that the comultiplication on the coordinate algebra of a formal group preserves this filtration, so that the formal group structure lifts uniquely as well.

### 1.2 Deformation to the normal cone

Our next order of business is to study a deformation to the normal cone construction in the setting of derived algebraic geometry. In essence this takes a closed immersion $X \rightarrow Y$ of classical schemes and gives a $\mathbb{G}_m$ equivariant family of formal schemes over $\mathbb{A}^1$, generically equivalent to the formal completion $\widehat{Y}_X$ which degenerate to the normal bundle of $N_{X|Y}$ formally completed at the identity section. When applied to a formal group $\widehat{G}$ produces a $\mathbb{G}_m$-equivariant 1-parameter family of formal groups over the affine line.

**Theorem 1.5.** Let $f : \text{Spec}(k) \rightarrow \widehat{G}$ be the unit section of a formal group $\widehat{G}$. Then there exists a stack $\text{Def}_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G}) \rightarrow \mathbb{A}^1/\mathbb{G}_m$ such that there is a map

$$X \times \mathbb{A}^1/\mathbb{G}_m \rightarrow \text{Def}_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})$$

whose fiber over $1 \in \mathbb{A}^1/\mathbb{G}_m$ is

$$\text{Spec}k \rightarrow \widehat{G}$$

and whose fiber over $0 \in \mathbb{A}^1/\mathbb{G}_m$ is

$$\text{Spec}k \rightarrow \widehat{T}_{\widehat{G}/k} \simeq \widehat{g}_a,$$

the formal completion of the tangent Lie algebra of $\widehat{G}$.

We would like to point out that the constructions occur in the derived setting, but the outcome is a degeneration between formal groups, which belongs to the realm of classical geometry. One may then apply the aforementioned filtered Cartier duality to this construction to obtain a group scheme $\text{Def}_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})^\vee$ over $\mathbb{A}^1/\mathbb{G}_m$, thereby equipped with a canonical filtration on the cohomology of the (classical) Cartier dual $\widehat{G}^\vee$.

By [Mou19, Proposition 7.3], $\mathcal{O}(\text{Def}_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G}))$ acquires the structure of a complete filtered algebra. We have the following characterization of the resulting filtration on $\mathcal{O}(\text{Def}_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G}))$ relating the deformation to the normal cone construction with the $I$-adic filtration of Theorem 1.4.
Corollary 1.6. Let \( \hat{G} \) be a formal group over \( k \). Then there exists a unique filtered formal group with \( \mathcal{O}(\hat{G}) \) as its underlying object. In particular, there is an equivalence

\[
\mathcal{O}(\text{Def}_{A^1/G_m}(\hat{G})) \simeq F^*_\text{ad} A
\]

of abelian cogroup objects in \( \text{CAlg}(\hat{\text{Fil}}_k) \).

Hence, the deformation to the normal cone construction applied to a formal group \( \hat{G} \) produces a filtered formal group.

Next, we specialize to the case of the formal multiplicative group \( \hat{G}_m \). By applying Theorem 1.5 to the unit section \( \text{Spec} k \to \hat{G}_m \), we recover the filtration on the group scheme

\[
\text{Fix} := \text{Ker}(F - 1 : \mathbb{W}(-) \to \mathbb{W}(-))
\]

of Frobenius fixed points on the Witt vector scheme and show that this filtration arises via Cartier duality precisely from a certain \( \mathbb{G}_m \)-equivariant family of formal groups over \( \mathbb{A}^1 \). As a consequence, the formal group defined is precisely an instance of the deformation to the normal cone of the unit section \( \text{Spec} k \to \hat{G}_m \).

Theorem 1.7. Let \( H \to A^1/G_m \) be the filtered group scheme of \( [MRT19] \). This arises as the Cartier dual \( \text{Def}_{A^1/G_m}(\hat{G}_m)^\vee \) of the deformation to the normal cone of the unit section \( \text{Spec} k \to \hat{G}_m \). Namely, there exists an equivalence of group schemes over \( A^1/G_m \)

\[
\text{Def}_{A^1/G_m}(\hat{G})^\vee \to H.
\]

Putting this together with Corollary 1.6 we obtain the following curious characterization of the HKR filtration on Hochschild homology studied in \( [MRT19] \):

Corollary 1.8. The HKR filtration on Hochschild homology is functorially induced by way of filtered Cartier duality, by the \( I \)-adic filtration on \( \mathcal{O}((\hat{G}_m)) \simeq k[[t]] \).

1.3 Filtration on \( \hat{G} \)-Hochschild homology

One may of course apply the deformation to the normal cone construction to an arbitrary formal group of height \( n \) over any base commutative ring. As a consequence, one obtains a canonical filtration on the aforementioned \( \hat{G} \)-Hochschild homology

Corollary 1.9. (cf. 7.3) Let \( \hat{G} \) be an arbitrary formal group. The functor

\[
\text{HH}\hat{G}(\cdot) : \text{sCAlg}_R \to \text{Mod}_R
\]

admits a refinement to the \( \infty \)-category of filtered \( R \)-modules

\[
\text{HH}\hat{G}(\cdot) : \text{sCAlg}_R \to \text{Mod}_R^{\text{filt}},
\]

such that

\[
\text{HH}\hat{G}(\cdot) \simeq \text{colim}_{(Z, \leq)} \text{HH}\hat{G}(\cdot)
\]

In other words, \( \text{HH}\hat{G}(A) \) admits an exhaustive filtration for any formal group \( \hat{G} \) and simplicial commutative algebra \( A \).
1.4 A family of group schemes over the sphere

We now shift our attention over to the topological context. In [Lur18], Lurie defines a notion of formal groups intrinsic to the setting of spectral algebraic geometry. We explore a weak notion of Cartier duality in this setup, between formal groups over an $E_\infty$-ring and affine group schemes, interpreted as group like commutative monoids in the category of spectral schemes. Leveraging this notion of Cartier duality, we demonstrate the existence a family of spectral group schemes for each height $n$. Since Cartier duality is compatible with base-change, one rather easily sees that these spectral schemes provide lifts of various affine group schemes.

**Theorem 1.10.** Let $\widehat{G}$ be a formal group over $\text{Spec} \ k$, for $k$ a finite field of height $n$. Let $\text{Fix}_{\widehat{G}} := \widehat{G}^\vee$ be its Cartier dual affine group scheme. Then there exists a functorial lift $\text{Fix}_{\widehat{G}}^\text{un} \to \text{Spec} R^\text{un}_{\widehat{G}}$ giving the following Cartesian square of affine spectral schemes:

$$
\begin{array}{ccc}
\text{Fix}_{\widehat{G}} & \xrightarrow{p'} & \text{Fix}_{\widehat{G}}^\text{un} \\
\phi \downarrow & & \phi \\
\text{Spec}(\mathbb{F}_p) & \xrightarrow{p} & \text{Spec}(R^\text{un}_{\widehat{G}})
\end{array}
$$

Moreover, $\text{Fix}_{\widehat{G}}^\text{un}$ will be a group-like commutative monoid object in the $\infty$-category of spectral stacks $s\text{Stk}_{R^\text{un}_{\widehat{G}}}$ over $R^\text{un}_{\widehat{G}}$.

The spectral group scheme of the theorem arises as the weak Cartier dual of the universal deformation of the formal group $\widehat{G}$; this naturally lives over the spectral deformation ring $R^\text{un}_{\widehat{G}_0}$. This $E_\infty$ ring studied in [Lur18], corepresents the formal moduli problem sending a complete (noetherian) $E_\infty$ ring $A$ to the space of deformations of $\widehat{G}_0$ to $A$ and is a spectral enhancement of the classical deformation rings of Lubin and Tate. A key such example arises from the restriction to $\mathbb{F}_p$ of the subgroup scheme $\text{Fix}$ of of fixed points on the Witt vector scheme, in height one.

1.5 Liftings of $\widehat{G}$-twisted Hochshild homology

Finally we study an $E_\infty$ (as opposed to simplicial commutative) variant of $\widehat{G}$-Hochshild homology. For an $E_\infty$ $k$-algebra, this will be defined in an analogous manner to $\text{HH}\widehat{G}(A)$ (see Definition 9.1). We conjecture that for a simplicial commutative algebra $A$ with underlying $E_\infty$-algebra, denoted by $\theta(A)$, this recovers the underlying $E_\infty$ algebra of the simplicial commutative algebra $\text{HH}\widehat{G}(A)$. In the case of the formal multiplicative group $G_m$, we verify this to be true, so that one recovers Hochschild homology.

These theories now admit lifts to the associated spectral deformation rings:

**Theorem 1.11.** Let $\widehat{G}$ be a height $n$ formal group over a finite field $k$ of characteristic $p$, and let $R^\text{un}_{\widehat{G}}$ be the associated spectral deformation $E_\infty$ ring. Then there exists a functor

$$
\text{THH}\widehat{G} : \text{CAlg}_{R^\text{un}_{\widehat{G}}} \to \text{CAlg}_{R^\text{un}_{\widehat{G}}}
$$

defined as

$$
\text{THH}\widehat{G}(A) := R\Gamma(\text{Map}(B\text{Fix}_{\widehat{G}}^\text{un}, \text{Spec}A), O)
$$

This lifts $\widehat{G}$-Hochshild homology in the sense that if $A$ is a $k$-algebra for which there exists a $R^\text{un}_{\widehat{G}}$-algebra lift $\widetilde{A}$ with

$$
\widetilde{A} \otimes_{R^\text{un}_{\widehat{G}}} k \simeq A
$$

5
then there is a canonical equivalence, cf Theorem 9.7

\[ \text{THH}^\hat{G}(\hat{A}) \otimes R^\text{un}_\hat{G} k \simeq \text{HH}^\hat{G}(A) \]

We tie the various threads of this work together in the speculative final section where we discuss the question of lifting the filtration on \( \text{HH}^\hat{G}(-) \), defined in section 7 as a consequence of the degeneration of \( \hat{G} \) to \( A^1/G_m \), to a filtration on the topological lift \( \text{THH}^\hat{G}(-) \).

1.6 Future work

We work over a ring of integers \( O_K \) in a local field extension \( K \supset \mathbb{Q}_p \) of degree one obtains a formal group, known as the Lubin-Tate formal group. This is canonically associated to a choice of uniformizer \( \pi \in O_K \). In future work, we investigate analogues of the construction of \( \mathbb{H} \) in [MRT19], which will be related by Cartier duality to this Lubin-Tate formal group. By the results of this paper, these filtered group schemes will have a canonical degeneration arising from the deformation to the normal cone construction of the Cartier dual formal groups.

In another vein, we expect the study of these spectral lifts \( \text{THH}^\hat{G}(-) \) to be an interesting direction. For example, there is the question of filtrations, and to what extent they lift to \( \text{THH}^\hat{G}(-) \).

One could try to base-change this along the map to the orientation classifier

\[ R^\text{un}_\hat{G} \to R^\text{cor}_\hat{G}, \]

cf. [Lur18]. Roughly, this is a complex orientable \( E_\infty \) ring with the universal property that it classifies oriented deformations of the spectral formal group \( \hat{G}^\text{un} \); these are oriented in that they coincide with the formal group corresponding to a complex orientation on the underlying \( E_\infty \) algebra of coefficients. For example, one obtains \( p \)-complete \( K \)-theory in height one. It is conceivable questions about filtrations and the like would be more tractable over this ring.

Outline We begin in section 2 with a short overview of the perspective on formal groups which we adopt. In section 3 we describe some preliminaries from derived algebraic geometry. In section 4, we construct the deformation to the normal cone and apply it to the case of the unit section of a formal group. In section 5 we apply this construction to the formal multiplicative group \( \hat{G}_m \) and relate the resulting degeneration of formal groups to constructions in [MRT19]. In section 6, we study resulting filtrations on the associated \( \hat{G} \)-Hochshild homologies. We begin section 7 with a brief overview of the ideas which we borrow from [Lur18] in the context of formal groups spectral algebraic geometry; and lift describe a family of spectral group schemes that arise in this setting that correspond to height \( n \) formal groups over characteristic \( p \) finite fields. In section 8, we study lifts \( \text{THH}^\hat{G}(-) \) of \( \hat{G} \)-Hochschild homology to the sphere, with a key input the group schemes of the previous section. Finally, we end with a short speculative discussion in section 9 about potential filtrations on \( \text{THH}^\hat{G}(-) \).

Conventions We often work over the \( p \)-local integers \( \mathbb{Z}_{(p)} \), and so we typically use \( k \) to denote a fixed commutative \( \mathbb{Z}_{(p)} \)-algebra. If we use the notation \( R \) for a ring or ring spectrum, then we are not necessarily working \( p \)-locally. In another vein, we work freely in the setting of \( \infty \)-categories and higher algebra from [Lur]. We would also like to point out that our use of the notation \( \text{Spec}(-) \) depends on the setting; in particular when working with spectral schemes, \( \text{Spec}(A) \) denotes the spectral scheme corresponding to the \( E_\infty \)-algebra \( A \). Finally, we will always be working in the commutative setting, so we implicitly assume all relevant algebras, coalgebras, formal groups, etc. are (co)commutative.
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2 Basic notions from derived algebraic geometry

In this section we review some of the relevant concepts that we shall use from the setting of derived algebraic geometry. We recall that there are two variants, one whose affine objects are connective $E_\infty$-rings, and one where the affine objects are simplicial commutative rings. We review parallel constructions from both simultaneously, as we will switch between both settings.

Let $\mathcal{C} = \{\text{CAlg}^{cn}_R, \text{sCAlg}_R\}$ denote either of the $\infty$-category of connective $R$-algebras or the $\infty$-category of simplicial commutative algebras. Recall that the latter can be characterised as the completion via sifted colimits of the category of (discrete) free $R$-algebras. Over a commutative ring $R$, there exists a functor

$$\theta : \text{sCAlg}_R \rightarrow \text{CAlg}^{cn}_R,$$

which takes the underlying connective $E_\infty$-algebra of a simplicial commutative algebra. This preserves limits and colimits so is in fact monadic and comonadic.

In any case one may define a derived stack via its functor of points, as an object of the $\infty$-category $\text{Fun}(\mathcal{C}, S)$ satisfying hyperdescent with respect to a suitable topology on $\mathcal{C}^{op}$, e.g. the étale topology. Throughout the sequel we distinguish the context we are work in by letting $\text{dStk}_R$ denote the $\infty$-category of derived stacks and let $\text{sStk}_R$ denote the $\infty$-category of “spectral stacks”.

In either cases, one obtains an $\infty$-topos, which is Cartesian closed, so that it makes sense to talk about internal mapping objects: given any two $X, Y \in \text{Fun}(\mathcal{C}, S)$, one forms the mapping stack $\text{Map}_\mathcal{C}(X, Y)$. In various cases of interest, if the source and/or target is suitably representable by a derived scheme or a derived Artin stack, then this is the case for $\text{Map}_\mathcal{C}(X, Y)$ as well.

There is a certain type of base-change result that we will use, cf. [HLPT14 Proposition A.1.5] [Lur16 Proposition 9.1.5.7].

Proposition 2.1. Let $f : X \rightarrow \text{Spec} R$ be a geometric stack over $\text{Spec} R$. Assume that one of the two conditions hold:

- $X$ is a derived scheme
- The morphism $f$ is of finite cohomological dimension over $\text{Spec} R$, so that the global sections functor sends $\text{QCoh}(X)_{\geq 0} \rightarrow (\text{Mod}_R)_{\geq -n}$ for some positive integer $n$.

Then, for $g : \text{Spec} R' \rightarrow \text{Spec} R$, the following diagram of stable $\infty$-categories

$$\begin{array}{ccc}
\text{Mod}_R & \xrightarrow{f^*} & \text{QCoh}(X) \\
\downarrow{g^*} & & \downarrow{g^{f^*}} \\
\text{Sp} & \xrightarrow{f'^*} & \text{QCoh}(X_{R'})
\end{array}$$

is right adjointable, and so, the Beck-Chevalley natural transformation of functors $g^* f_* \simeq f'^* g'^* : \text{QCoh}(X) \rightarrow \text{Mod}_{R'}$ is an equivalence.
2.1 Formal algebraic geometry and derived formal descent

In this paper, we shall often find ourselves in the setting of formal algebraic geometry and formal schemes. Hence we recall some basic notions in this setting. We end this subsection with a notion of formal descent which is intrinsic to the derived setting. This phenomenon will be exploited in Section 5.

A (underived) formal affine scheme corresponds to the following piece of data:

**Definition 2.2.** We define an adic $R$-algebra to be an $R$-algebra $A$ together with an ideal $I \subset A$ endowing a topology on $A$.

**Construction 2.3.** Let $A$ be an adic commutative ring having a finitely generated ideal of definition $I \subseteq \pi_0 A$. Then there exists a tower $\cdots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1$ with the properties that

1. each of the maps $A_{i+1} \rightarrow A_i$ is a surjection with nilpotent kernel.
2. the canonical map $\text{colim} \text{Map}_{\text{CAlg}}(A_n, B) \rightarrow \text{Map}_{\text{CAlg}}(A, B)$ induces an equivalence of the left hand side with the summand of $\text{Map}_{\text{CAlg}}(A, B)$ consisting of maps $\phi : A \rightarrow B$ annihilating some power of the ideal $I$.
3. Each of the rings $A_i$ is finitely projective when regarded as an $A$-module.

One now defines $\text{Spf} A$ to be the filtered colimit of

$$\text{colim}_i \text{Spec} A_i$$

in the category of locally ringed spaces. In fact, is is the left Kan extension of the $\text{Spec}(-)$ functor along the inclusion $\text{CAlg} \rightarrow \text{CAlg}^{\text{ad}}$.

**Definition 2.4.** A formal scheme over $R$ is a functor $X : \text{CAlg}_{R}^0 \rightarrow \text{Set}$ which is Zariski locally of the above form. A (commutative) formal group is an abelian group object in the category of formal schemes. By remark 3.3, this consists of the data of a formal scheme $\hat{G}$ which takes values in groups, which commutes with direct sums.

There is a rather surprising descent statement one can make in the setting of derived algebraic geometry. For this we first recall the notion of formal completion. We remark that in this section we are always working in the locally Noetherian context.

**Definition 2.5.** Let $f : X \rightarrow Y$ be a closed immersion of schemes. We define the formal completion to be the following stack $\hat{Y}_X$ whose functor of points is given by

$$\hat{Y}_X(R) = Y(R) \times_{Y(R_{\text{red}})} X(R_{\text{red}})$$

where $R_{\text{red}}$ denotes the reduced ring $(\pi_0 R)_{\text{red}}$.

Although defined in this way as a stack, this is actually representable by an object in the category of formal schemes, commonly referred to as the formal completion of $Y$ along $X$.

We form the nerve $N(f)_\bullet$ of the map $f : X \rightarrow Y$, which we recall is a simplicial object that in degree $n$ is the $(n+1)$-fold product

$$N(f)_n = X \times_Y X \cdots \times_Y X$$

The augmentation map of this simplicial object naturally factors through the formal completion (by the universal property the formal completion satisfies). We borrow the following key proposition from [Toe14]:
Theorem 2.6. The augmentation morphism \( N(f) \to \hat{Y}_X \) displays \( \hat{Y}_X \) as the colimit of the diagram \( N(f) \) in the category of derived schemes: this gives an equivalence
\[
\text{Map}_{dStk}(\hat{Y}_X, Z) \simeq \lim_{n \in \Delta} \text{Map}_{dSch}(N(f)_n, Z)
\]
for any derived scheme.

Remark 2.7. At its core, this is a consequence of [Car08, Theorem 4.4] on derived completions in stable homotopy, giving a model for the completion of a \( A \)-module spectrum along a map of ring spectra \( f: A \to B \) to be the totalization of a certain cosimplicial diagram of spectra obtained via a certain co-Nerve construction.

2.2 Tangent and Normal bundles

Let \( X \) be a derived stack, and \( E \in \text{Perf}(X) \) a perfect complex of Tor amplitude concentrated in degrees \([0, n] \) Then the we have the following notion, cf [Toe14, Section 3]:

Definition 2.8. We defined to linear stack associated to \( E \) to be the functor \( V(E) \) sending \( (\text{Spec} A \to X) \mapsto \text{Map}_{\text{Mod}_A}(u^*(E), A) \)

Example 2.9. Let \( O[n] \in \text{Perf}(X) \) be a shift of the structure sheaf. Then \( V(O[n]) \) is simply \( K(\mathbb{G}_a, -n) \). For a general perfect complex \( E \), this \( V(E) \) may be obtained by taking various twisted forms and finite limits of these \( K(\mathbb{G}_a, -n) \).

Definition 2.10. Let \( f: X \to Y \) be a map of derived stacks. We define the normal bundle stack to be \( V(T_X|_Y[1]) \). This will be a derived stack over \( X \); if \( f \) is a closed immersion of classical schemes then this will be representable by a derived scheme.

Example 2.11. Let \( i: \text{Spec}k \to \hat{G} \) be the unit section of a formal group. This is a lci closed immersion, hence the cotangent complex is concentrated in (homological) degree 1; thus the tangent complex is just \( k \) in degree \(-1 \). It follows that the normal bundle \( V(T_{k[\hat{G}][1]}) \) is just \( K(\mathbb{G}_a, 0) = \mathbb{G}_a \), the additive group. In fact we may identify the normal bundle with the tangent Lie algebra of \( \hat{G} \).

3 Formal groups and Cartier duality

In this section we review some ideas pertaining to the theory of (commutative) formal groups which will be used throughout this paper. In particular we carefully review the notion of Cartier duality as introduced by Cartier in [Car62], and also described in [Haz78, Section 37].

There are several perspectives one may adopt when studying formal groups. In general, one may think of them as an abelian group object in the category of formal schemes or representable formal moduli problems. In this paper we will be focusing on the somewhat restricted setting of formal groups which arise from certain types of Hopf algebras. In this setting one has a particularly well behaved duality theory which we shall exploit. Furthermore it is this structure which has been generalized by Lurie in [Lur18] to the setting of spectral algebraic geometry.

3.1 Abelian group objects

We start off with the notions of abelian group and commutative monoid objects in an arbitrary \( \infty \)-category and review their distinction.
Definition 3.1. Let $\mathcal{C}$ be an $\infty$-category which admits finite limits. A commutative monoid object is a functor $M : \text{Fin}_* \to \mathcal{C}$ with the property that for each $n$, the natural maps $M(\rho(n)) \to M(\rho(1))$ induce equivalences $M(\rho(n)) \simeq M((1))^n$ in $\mathcal{C}$.

In addition, a commutative monoid $M$ is grouplike if for every object $C \in \mathcal{C}$, the commutative monoid $\pi_0 \text{Map}(C, M)$ is an abelian group.

We now define the somewhat contrasting notion of abelian group object. This will be part of the relevant structure on a formal group in the spectral setting.

Definition 3.2. Let $\mathcal{C}$ be an $\infty$-category. Then the $\infty$-category of abelian objects of $\mathcal{C}$, $\text{Ab}(\mathcal{C})$ is defined to be $\text{Fun}^\times(\text{Lat}^{\text{op}}, \mathcal{C})$, the category of product preserving functors from the category Lat of finite rank abelian groups into $\mathcal{C}$.

Remark 3.3. Let $\mathcal{C}$ be a small category. Then an abelian group object $A$ is such that its representable presheaf $h_A$ takes values in abelian groups. Furthermore, in this setting, the two notions of abelian groups and grouplike commutative monoid objects coincide.

3.2 Formal groups and Cartier duality over a field

Before setting the stage for the various manifestations of Cartier duality to appear we say a few things about Hopf algebras, as they are central to this work. We begin with a brief discussion of what happens over a field $k$.

Definition 3.4. For us, a (commutative, cocommutative) Hopf algebra $H$ over $R$ is an abelian group object in the category of coalgebras over $k$.

Unpacking the definition, and using the fact the category of coalgebras is equipped with a Cartesian monoidal structure (it is the opposite category of a category of commutative algebra objects), we see that this is just another way of identifying bialgebra objects $H$ with an antipode map

$$i : H \to H;$$

this arises from the “abelian group structure” on the underlying coalgebra.

Construction 3.5. Let $H$ be a Hopf algebra. Then one may define a functor

$$\text{coSpec}(H) : \text{CAlg} \to \text{Ab}, \quad R \mapsto \text{Gplike}(H \otimes_k R) = \{x | \Delta(x) = x \otimes x\},$$

assigning to a commutative ring $R$ the set of grouplike elements of $R \otimes_k H$. The Hopf algebra structure on $H$ endows these sets with an abelian group structure, which is what makes the above an abelian group object-valued functor. In fact, this will be a formal scheme and there will be an equivalence

$$\text{coSpec}(H) \simeq \text{Spf}(H^\vee)$$

where $H^\vee$, the linear dual of $H$ is an $R$-algebra, complete with respect to an $I$-adic topology induced by an ideal of definition $I \subset R$. Hence we arrive at our first interpretation of a formal group; these correspond precisely to Hopf algebras.
**Construction 3.6.** Let us unpack the previous construction from an algebraic vantage point. Over a field \( k \), there is an equivalence

\[
cCAlg_k \simeq \text{Ind}(cCAlg_k^{fd})
\]

where \( cCAlg_k^{fd} \) denotes the category of coalgebras whose underlying vector space is finite dimensional. By standard duality, there is an equivalence between

\[
\text{Ind}(cCAlg_k^{fd}) \simeq \text{Pro}(CAlg_k^{fd})
\]

where we remark that \( cCAlg_k^{fd} \simeq (CAlg_k^{fd})^{op} \). This may then be promoted to a duality between abelian group/cogroup objects:

\[
\text{Hopf}_k := \text{Ab}(cCAlg_k) \simeq \text{coAb}(\text{Pro}(CAlg_k^{fd})) \quad (3.7)
\]

**Remark 3.8.** The interchange of display (3.7) is precisely the underlying idea of Cartier duality of formal groups and affine group schemes. Recall that Hopf algebras correspond contravariantly via the Spec(−) functor to affine group schemes. Hence one has

\[
\text{AffGp}_k^{op} \simeq \text{Hopf}_k \simeq \text{FG}_k,
\]

where the left hand side denotes the category of affine group schemes over \( k \). The functor on the right is given by the functor coSpec(−) described above. We remark that in this setting, the category of Hopf algebras over the field \( k \) is actually abelian, hence the categories of formal groups and affine group schemes are themselves abelian.

**3.3 Formal groups and Cartier duality over a commutative ring**

Over a general commutative ring \( R \), the duality theory between formal groups and affine group schemes isn’t quite as simple to describe. In practice, one restricts to certain subcategories on both sides, which then fit under the Ind-Pro duality framework of Construction 3.6. This will be achieved by imposing a condition on the underlying coalgebra of the Hopf algebras at hand.

**Remark 3.9.** We study coalgebras following the conventions of [Lur18, Section 1.1]. In particular, if \( C \) is a coalgebra over \( R \), we always require that the underlying \( R \)-module of \( C \) is flat. This is done as in [Lur18], to ensure that \( C \) remains a coalgebra in the setting of higher algebra. Furthermore, we implicitly assume that all coalgebras appearing in this text are (co)commutative.

To an arbitrary coalgebra, one may functorially associate a presheaf on the category of affine schemes given by the cospectrum functor

\[
\text{coSpec} : cCAlg_R \to \text{Fun}(\text{CAlg}_R, \text{Set}).
\]

**Definition 3.10.** Let \( C \) be a coalgebra. We define \( \text{coSpec}(C) \) to be the functor

\[
\text{coSpec}(C) : \text{CAlg}_R \to \text{Set}
\]

sending \( R \mapsto \text{Gplike}(C \otimes_k R) = \{x | \Delta(x) = x \otimes x\} \)

The \( \text{coSpec}(-) \) functor is fully faithful when restricted to a certain class of coalgebras. We borrow the following definition from [Lur18]. See also [Str99] for a related notion of coalgebra with good basis.
Definition 3.11. Fix $R$ and let $C$ be a (co-commutative) coalgebra over $R$. We say $C$ is smooth if its underlying $R$-module is flat, and if it is isomorphic to the divided power coalgebra

$$\Gamma^n_R(M) := \bigoplus_{n \geq 0} \Gamma^n_R(M)$$

for some projective $R$-module $M$. Here, $\Gamma^n_R(M)$ denotes the invariants for the action of the symmetric group $\Sigma_n$ on $M^\otimes n$.

Given an arbitrary coalgebra $C$ over $R$, the linear dual $C^\vee = \text{Map}(C, R)$ acquires a canonical $R$-algebra structure. In general $C$ cannot be recovered from $C^\vee$. However, in the smooth case, the dual $C$ acquires the additional structure of a topology on $\pi_0$ giving it the structure of an adic $R$-algebra. This allows us to recover $C$, via the following proposition, c.f. [Lur18, Theorem 1.3.15]:

Proposition 3.12. Let $C, D \in \text{cCAlg}^\text{sm}_R$ be smooth coalgebras. Then $R$-linear duality induces a homotopy equivalence

$$\text{Map}_{\text{cCAlg}_R}(C, D) \simeq \text{Map}_{\text{CAlg}_R}(C^\vee, D^\vee).$$

Remark 3.13. One can go further and characterize intrinsically all adic $R$-algebras that arise as duals of smooth coalgebras. These will be equivalent to $\text{Sym}^\ast(M)$, the completion along the augmentation ideal $\text{Sym}_{\geq 1}(M)$ for some $M$ a projective $R$-module of finite type.

Remark 3.14. Fix $C$ a smooth coalgebra. There is always a canonical map of stacks $\text{coSpec}(C) \to \text{Spec}(A)$ where $A = C^\vee$, but it is typically not an equivalence. The condition that $C$ is smooth guarantees precisely that there is an induced equivalence $\text{coSpec}(C) \to \text{Spf}(A) \subseteq \text{Spec}A$, where $\text{Spf}(A)$ denotes the formal spectrum of the adic $R$ algebra $A$. In particular $\text{coSpec}(C)$ is a formal scheme in the sense of [Lur16, Chapter 8].

Proposition 3.15 (Lurie). Let $R$ be an commutative ring. Then the construction $C \mapsto \text{cSpec}(C)$ induces a fully faithful embedding of $\infty$-categories

$$\text{cCAlg}^\text{sm}_R \to \text{Fun}(\text{CAlg}_R^0, \mathcal{S})$$

Moreover this comutes with finite products and base-change.

Proof. This essentially follows from the fact that a smooth coalgebra can be recovered from its adic $E_\infty$-algebra. \hfill \square

Construction 3.16. As a consequence of the fact that the $\text{coSpec}(-)$ functor preserves finite products, this can be upgraded to a fully faithful embedding of abelian group objects in smooth coalgebras

$$\text{Ab}(\text{cCAlg}) \to \text{Ab}(f \text{Sch})$$

into formal groups. Unless otherwise mentioned we will focus on formal groups of this form. Hence, we use the notation $\text{FG}_R$ to denote the category of adic algebraic formal groups over $R$. We refer to this equivalence as Cartier duality.

We would like to interpret the above correspondence geometrically. Let $\text{AffGrp}^b_R$ be the subcategory of affine group schemes, corresponding via the $\text{Spec}(-)$ functor to the category $\text{Hopf}^\text{sm}$, which we use to denote the category of Hopf algebras whose underlying coalgebra is smooth. Meanwhile, a cogroup object $\tilde{H}$ in the category of adic algebras corepresents a functor

$$F : \text{CAlg}^\text{ad} \to \text{Grp}, \ R \mapsto \text{Hom}_{\text{CAlg}^\text{ad}}(\tilde{H}, R),$$
where the group structure arises from the co-group structure on $H$. Essentially by definition, this is exactly the data of a formal group, so we may identify the category of formal groups with the category $\text{coAb}(\text{CAlg}^{ad})$.

We have identified the categories in question as those of affine group schemes and formal groups respectively; one can further conclude that these dualities are representable by certain distinguished objects in these categories.

**Proposition 3.17.** cf [Haz78, Proposition 37.3.6, 37.3.11] There exist natural bijections

$$\text{Hom}_{\text{Hopf}}(A[t, t^{-1}], C) \cong \text{Hom}_{\text{CAlg}^{ad}}(D(C), A)$$

$$\text{Hom}_{\text{CoAb}(\text{CAlg}^{ad})}(B[[T]], A) \cong \text{Hom}_{\text{CAlg}}(D^T(A), B).$$

Here, for a coalgebra $C$, $D(C)$ is the linear dual and for a topological algebra $A$, $D^T(A) = \text{Map}_{\text{cont}}(A, R)$ is the continuous dual

One can put this all together to see that there are duality functors which are moreover represented by the multiplicative group and the formal multiplicative group respectively.

One has the following expected base-change property:

**Proposition 3.18.** Let $\hat{G}$ be a formal group over $\text{Spec} R$, and suppose there is a map $f : R \to S$ be a map of commutative rings. Let $\hat{G}_S$ denote the formal group over $\text{Spec} S$ obtained by base change. Then there is a natural isomorphism

$$D^T(\hat{G} | S) \simeq D^T(\hat{G})_S$$

of affine group schemes over $\text{Spec} S$.

## 4 Filtered formal groups

We define here a notion of a filtered formal group, along with Cartier duality for these. We discuss here only (“underived”) formal groups over discrete commutative rings but we conjecture that these notions generalize to the case where $R$ is a connective $E_\infty$ ring.

### 4.1 Filtrations and $\mathbb{A}^1/\mathbb{G}_m$

We first recall a few preliminaries about filtered objects.

**Definition 4.1.** Let $R$ be an $E_\infty$-ring. We set

$$\text{Fil}_R := \text{Fun}(\mathbb{Z}^{op}, \text{Mod}_R),$$

where $\mathbb{Z}$ is viewed as a category with morphisms given by the partial ordering and refer to this as the $\infty$-category of filtered $R$-modules.

**Remark 4.2.** The $\infty$-category $\text{Fil}_R$ is symmetric monoidal with respect to the Day convolution product.

**Definition 4.3.** There exist functors

$$\text{Und} : \text{Fil}_R \to \text{Mod}_R \quad \text{gr} : \text{Fil}_R \to \text{Gr}_R,$$

such that to a filtered $R$-module $M$, one associates its underlying object $\text{Und}(M) = \text{colim}_{n \to \infty} M_n$ and $\text{gr}(M) = \oplus_n \text{cofib}(M_{n+1} \to M_n)$ respectively.
Example 4.4. Let $A$ be a commutative ring, and $I \subseteq A$ be an ideal of $A$. We define a filtration $F_I^*(A)$ with

$$F_I^n(A) = \begin{cases} A, & n \leq 0 \\ I^n, & n \geq 1 \end{cases}$$

This is the $I$-adic filtration on $A$.

Definition 4.5. There exists a notion of completeness in the setting of filtrations. We say a filtered $R$-module $M$ is complete if

$$\lim_n M_n \simeq 0$$

Alternatively, $M$ is complete if $\lim_{n \to -\infty} M_n \simeq M_{-\infty} = \text{Und}(M)$. We denote that $\infty$-category of filtered modules which are complete by $\hat{\text{Fil}}_R$. This will be a localization of $\hat{\text{Fil}}_R$ and will come equipped with a completed symmetric monoidal, such that the completion functor

$$(\_ ) : \text{Fil}_R \to \hat{\text{Fil}}_R$$

is symmetric monoidal.

Construction 4.6. The category of filtered $R$-modules, as a $R$-linear stable $\infty$-category can be equipped with several different $t$-structures. We will occasionally work with the neutral $t$-structure on $\text{Fil}_R$, defined so that $F^*(M) \in (\text{Fil}_R)_{\geq 0}$ if $F^n(M) \in (\text{Mod}_k)_{\geq 0}$ for all $n \in \mathbb{Z}$. Similarly, $F^*(M) \in (\text{Fil}_R)_{\leq 0}$ if $F^n(M) \in (\text{Mod}_R)_{\leq 0}$ for all $n \in \mathbb{Z}$.

We remark that the standard $t$-structure on $\text{Mod}_R$ is compatible with sequential colimits (cf. [Lur, Definition 1.2.2.12]. This has the consequence that if $F^*(M) \in \text{Fil}_R$ then

$$\colim_{n \to -\infty} F^n(M) = \text{Und}(F^*(M)) \in \text{Mod}^\heartsuit_k.$$ 

We occasionally refer to filtered $R$-modules with are in the heart of this $t$-structure as discrete.

We now briefly recall the description of filtered objects in terms of quasi-coherent sheaves over the stack $\mathbb{A}^1/\mathbb{G}_m$. This quotient stack may be defined as the quotient of $\mathbb{A}^1 = \text{Spec}(R[t])$ by the canonical $\mathbb{G}_m = \text{Spec}(R[t, t^{-1}])$ action induced by the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ arrow of group schemes. This comes equipped with two distinguished points

$$0 : \text{Speck} \cong \mathbb{G}_m/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$$

$$1 : B\mathbb{G}_m = \text{Speck}/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$$

which we often refer to in this work as the generic and special/closed point respectively. We remark that the quotient map $\pi : \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$ is a smooth (and hence fppf) atlas for $\mathbb{A}^1/\mathbb{G}_m$, making $\mathbb{A}^1/\mathbb{G}_m$ into an Artin stack.

Theorem 4.7. There exists a symmetric monoidal equivalence

$$\text{Fil}_R \to \text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$$

Furthermore, under this equivalence, one may identify the underlying object and associated graded functors with pullbacks along $1$ and $0$ respectively.
4.2 Formal algebraic geometry over $\mathbb{A}^1/\mathbb{G}_m$

We propose in this section the rough heuristic that an affine formal scheme over $\mathbb{A}^1/\mathbb{G}_m$ should be interpreted as none other than a complete filtered algebra. We justify this by showing that a complete filtered algebra quasi-coherent, as sheaf over $\mathbb{A}^1/\mathbb{G}_m$ satisfies a form of completeness directly related to the standard notion of $t$-adic completeness for a $R[t]$-algebra $A$. This may then be pulled back along the atlas $\mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$ to an adic commutative $R[t]$-algebra, which is complete with respect to multiplication by $t$.

**Construction 4.8.** Recall, e.g., by [Lur15], that there is an equivalence

$$\text{Fil}_R \cong \text{Mod}_{R[t]}(\text{Gr}_R),$$

where $R[t]$ is given the grading such that $t$ sits in weight $-1$. More precisely it is the graded $E_\infty$ algebra given by

$$R[t] = \begin{cases} R, & n \leq 0, \\ 0, & n > 0 \end{cases}$$

One has a map

$$R[t] \to \text{Map}_{\text{gr}}(X, X)$$

in $\text{Gr}_R$ making $X \in \text{Gr}_R$ into a $R[t]$-module. There is an equivalence of $E_1$-algebras $R[t] \cong \text{Free}_{E_1}(R(1))$ making $R[t]$ expressible as the free $E_1$ algebra on $R(1)$. Unpackaging all this, we obtain a map

$$R \to \text{Map}_{\text{gr}}(X, X) \otimes R(-1)$$

which precisely singles out the structure maps of the filtration on $X$.

**Definition 4.9.** We say a graded $R[t]$-module $X \in \text{Mod}_{R[t]}(\text{Gr}_R)$ is $(t)$-complete if and only if the limit of the following sequence of multiplication by $t$

$$\cdots \to X \otimes R(n + 1) \xrightarrow{t} X \otimes R(n) \xrightarrow{t} X \otimes R(n - 1) \xrightarrow{t} \cdots$$

vanishes, where the product here is the Day convolution symmetric monoidal structure on $\text{Gr}_R$

It is immediately clear that the above agrees with the notion of completeness in the sense of Definition 4.5. Namely $X \in \text{Fil}_R$ is complete if it is complete in the sense of Definition 4.9 when viewed as an object of $\text{Mod}_{R[t]}(\text{Gr}_R)$.

We would like to use this observation to show that completeness may further be checked after “forgetting the grading”, i.e, upon pullback of the associated quasi-coherent sheaf on $\mathbb{A}^1/\mathbb{G}_m$ along $\tau : \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$. First, recall the relevant (unfiltered/ ungraded) classical notion of $t$-completeness:

**Definition 4.10.** Fix $R[t]$, the polynomial algebra in one generator (with no additional structure of a grading). An $R[t]$-module $M$ is $t$-complete if the limit of the tower

$$\cdots \to M \xrightarrow{t} M \xrightarrow{t} M \xrightarrow{t} \cdots$$

vanishes. By [Lur16 8.2.4.15], there is an equivalence

$$\text{QCoh}(\hat{\mathbb{A}}^1) \cong \text{Mod}^{\text{Cpl}(t)}$$

where the right hand side denotes $t$-complete $R[t]$-modules and the left hand side denotes the $R$-linear $\infty$-category of quasi-coherent sheaves on the formal completion of the affine line $\hat{\mathbb{A}}^1 = \text{Spf} R[[t]]$. 

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Now we use this to show that completeness can be tested upon pullback to $\mathbb{A}^1$.

**Proposition 4.11.** Let $X \in \Fil_R \in \QCoh(\mathbb{A}^1/\mathbb{G}_m)$ be a filtered $R$-module. Then $X$ is complete as a filtered object if and only if its pullback $\pi^*(X) \in \QCoh(\mathbb{A}^1)$ is complete, as an $R[t]$-algebra.

**Proof.** By the above discussion, we express completeness as the property that

$$\lim(... \xrightarrow{t} X \otimes R(n) \xrightarrow{t} X \otimes R(n-1) \xrightarrow{t} ...)$$

vanishes in the $\infty$-category

$$\Gr_R \simeq \Fun(\mathbb{Z}, \Mod_R)$$

of of graded $R$-modules, where $\mathbb{Z}$ is viewed as discrete $E_\infty$-space. We would like to show that the limit vanishes upon applying

$$\bigoplus : \Gr_R \to \Mod_R \quad (X)_n \mapsto \bigoplus_n X_n$$

By [Mon19, Proposition 4.2] this functor will preserve the limit, as it satisfies the equivalent conditions for the comonadic Barr-Beck theorem, so that the limit vanishes in $\Mod_R$. Conversely, suppose $X$ is a filtered $R$-module which has the property that $\bigoplus_{n \in \mathbb{Z}} X_n$ is complete as an $R[t]$-module. This means that the limit along multiplication by $t$ in $\Mod_R$ vanishes. However, we may apply [Mon19, Proposition 4.2] again to see that this limit is actually created in $\Gr_R$, and moreover the functor $\bigoplus$ preserves this limit. In particular, this means that $\lim(... \xrightarrow{t} X \otimes R(n) \xrightarrow{t} X \otimes R(n-1) \xrightarrow{t} ...) \text{ vanishes in } \Gr_R$, as we wanted to show. □

**Remark 4.12.** We see therefore that if $A$ is a complete filtered algebra over $R$, then it gives rise to a commutative algebra $\pi^*(A) \in \QCoh(\mathbb{A}^1/\mathbb{G}_m) \simeq \Mod_R[t]$, which can be endowed with a topology with respect to the ideal $(t)$ with respect to which it is complete. By [Lur16, Proposition 8.1.2.1, 8.1.5.1], algebras of this form embed fully faithfully into the $\infty$-category of spectral stacks over $\mathbb{A}^1_R$, with essential image being precisely the formal affine schemes over $\mathbb{A}^1_R$.

### 4.3 Filtered Cartier duality

We adopt the approach to formal groups in [Lur18], described above where they are in particular smooth coalgebras $C$ with

$$C = \bigoplus_{i \geq 0} \Gamma^i(M)$$

where $M$ is a (discrete) projective module of finite type. Here, $\Gamma^n$ for each $n$ denotes the $n$th divided power functor, which for a dualizable module $M$, can be alternatively defined as

$$\Gamma^n(M) := \text{Sym}^n(M^\vee)^\vee,$$

that is to say as the dual of the symmetric powers functor

**Construction 4.13.** By the results of [BM19, Rak20], these can be extended to the $\infty$-categories $\Mod_k$, $\Gr(\Mod_R)$, $\Fil(\Mod_k)$ of $R$-modules, graded $R$-modules and filtered $R$-modules, respectively. These are referred to as the derived symmetric powers

In particular, the $n$th (derived) divided power functors

$$\Gamma^n_{gr} : \Gr_R \to \Gr_R \quad \Gamma^n_{fil} : \Fil_R \to \Fil_R$$

make sense in the graded and filtered contexts as well.
**Definition 4.14.** Let $M$ be a filtered $R$-module whose underlying object is a discrete projective $R$-module of finite type such that $\text{gr}(M)$ is concentrated in non-positive weights. A smooth filtered coalgebra is a coalgebra of the form

$$C = \bigoplus_{n \geq 0} \Gamma^R_{\text{fil}}(M)$$

Note that this acquires a canonical coalgebra structure, as in [Lur18, Construction 1.1.11]. Indeed if we apply $\Gamma^*$ to $M \oplus M$, we obtain compatible maps

$$\Gamma^{n'+n''}(M \oplus M) \to \Gamma^{n'}(M) \otimes \Gamma^{n''}(M)$$

where this is to be interpreted in terms of the Day convolution product. As in the unfiltered case in [Lur18, Construction 1.1.11], these assemble to give equivalences

$$\Gamma^*(M \oplus M) \simeq \Gamma^*(M) \otimes \Gamma^*(M)$$

Via the diagonal map $M \to M \oplus M$ (recall $\text{Fil}(\text{Mod}_k)$ is stable), this gives rise to a map

$$\Delta : \Gamma^*(M) \to \Gamma^*(M \oplus M) \simeq \Gamma^*(M) \otimes \Gamma^*(M)$$

which one can verify exhibits $\Gamma^*(M)$ as a coalgebra in the category of filtered $k$-modules.

**Proposition 4.15.** Let $M$ be a dualizable filtered $R$-module. Then the formation of divided powers is compatible with the associated graded and underlying object functors.

**Proof.** Let $\text{Und} : \text{Fil}_R \to \text{Mod}_R$ and $\text{gr} : \text{Fil}_R \to \text{Gr}_R$ denote the underlying object and associated graded functors respectively. Each of these functors commute with colimits and are symmetric monoidal. Thus, we are reduced to showing that each of these functors commutes with the divided power functor

$$\Gamma^R_{\text{fil}}(M) = \text{Sym}^n(M^\vee)^\vee$$

The statement now follows from the fact that $\text{Und}$ and $\text{gr}$, being symmetric monoidal, commute with dualizable objects and that they commute with $\text{Sym}^n$, which follows from the discussion in [Rak20, 4.2.25].

**Definition 4.16.** The category of smooth filtered coalgebras $\text{cCAlg}(\text{Fil}_k)^{\text{sm}}$ is the full subcategory of filtered coalgebras generated by objects of this form. Namely, $C \in \text{cCAlg}(\text{Fil}_R)^{\text{sm}}$ if there exists a filtered module $M$ which is dualizable, discrete and zero in positive degrees for which

$$C \simeq \bigoplus_{n \geq 0} \Gamma^R_{\text{fil}}(M)$$

**Remark 4.17.** The filtered module $M$ in the above definition is of the form

$$\ldots \supset M_{-2} \supset M_{-1} \supset M_0 \supset 0 \ldots$$

which is eventually constant.

We now give the first definition of a filtered formal group:

**Definition 4.18.** A filtered formal group is an abelian group object in the category of smooth coalgebras. That is to say it is a product preserving functor

$$F : \text{Lat}^{\text{op}} \to \text{CAlg}(\text{Fil}_R)^{\text{sm}}$$
**Construction 4.19.** Let $M \in \mathrm{Fil}_R$ be a filtered $R$-module. We denote the (weak) dual by $\overline{\text{Map}}_{\mathrm{Fil}}(M, R)$. Note that if $M$ has a commutative coalgebra structure, then this acquires the structure of a commutative algebra.

**Example 4.20.** Let $C = \bigoplus \Gamma^n_{jfil}(M)$. Then one has an equivalence

$$C^\vee \simeq (\bigoplus \Gamma^n(M))^\vee \simeq \prod_n \text{Sym}^n(M^\vee)$$

This is a complete filtered algebra.

**Proposition 4.21.** Let $C$ be a filtered smooth coalgebra, and let $C^\vee$ denote its (filtered) dual. Then

at the level of the underlying object there is an equivalence

$$\text{Und}(C^\vee) \simeq \prod \text{Sym}^n(N)$$

for some projective module $N$ of finite type.

**Proof.** We unpack what the weak dual functor does on the $n$th filtering degree of a filtered $R$-module. If $M \in \mathrm{Fil}_R$, then this may be described as

$$M_n^\vee = \overline{\text{Map}}_{\mathrm{Fil}}(M, R)_n \simeq \text{fib}(M_n^\vee \to M_{1-n}^\vee)$$

where $M_n^\vee$ is the dual of the underlying $R$-module. Now let $M = C$ be a smooth coalgebra, so that $C = \bigoplus \Gamma^n(N)$ for $N$ as in Definition 4.14. Then $\Gamma^n(N)$ for each $n$ will be concentrated in negative filtering degrees so that $C_{1-n}^\vee \simeq 0$ for all $n$ where $C_n$ is nontrivial. Hence we have the following description for the underlying object of $C^\vee$:

$$\text{Und}(C^\vee) \simeq \colim_n \text{fib}(C_n^\vee \to C_{1-n}^\vee) \simeq \text{fib} \colim_n (C_n^\vee \to C_{1-n}^\vee) = \colim_n C_n^\vee.$$ 

In particular, since $C_{1-n}$ eventually vanishes, we obtain the colimit of the constant diagram associated to $C_\infty^\vee$. Hence

$$\text{Und}(C^\vee) \simeq \text{Und}(C)^\vee \simeq \prod_{m \geq 0} \text{Sym}^m_R(N)$$

This shows in particular that weak duality of these smooth filtered coalgebras commutes with underlying object functor.

**Remark 4.22.** The above proposition justifies the definition 4.14 of smooth filtered coalgebras which we propose. In general it is not clear that weak duality commutes with the underlying object functor (although this of course hold true on dualizable objects).

**Proposition 4.23.** The assignment $c\text{CAlg}_{\text{sm}}^\text{em}(\mathrm{Fil}_R) \to \text{CAlg}(\widehat{\mathrm{Fil}}_R)$ given by

$$C \mapsto C^\vee = \text{Map}(C, R)$$

is fully faithful.
Proof. Let $D$ and $C$ be two arbitrary smooth coalgebras. We would like to display an equivalence of mapping spaces

$$\text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(D, C) \simeq \text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(C^{\vee}, D^{\vee}); \quad (4.24)$$

Each of $C$ and $D$ may be written as a colimit, internally to filtered objects,

$$C \simeq \text{colim} \ C_{k}, \quad D \simeq \text{colim} \ D_{m}$$

where

$$C_{k} = \bigoplus_{0 \leq i \leq k} \Gamma^{i}(M); \quad D_{m} = \bigoplus_{0 \leq i \leq m} \Gamma^{i}(N).$$

Hence the map (4.24) may be rewritten as a limit of maps of the form

$$\text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(D_{m}, C) \to \text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(C^{\vee}, D^{\vee}) \quad (4.25)$$

The left side of this may now be rewritten as

$$\text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(D_{m}, \text{colim} \ C_{k})$$

Now, the object $D_{m}$ will be compact by inspection (in fact, its underlying object is just a compact projective $k$-module) so that the above mapping space is equivalent to

$$\text{colim} \ \text{Map}_{c\text{CAlg}^{sm}(\text{Fil}_{R})}(D_{m}, C_{k})$$

We would now like to make a similar type of identification on the right hand side of the map (4.25). For this note that as a complete filtered algebra, $C^{\vee} \simeq \lim \ C_{k}^{\vee}$. Note that there is a canonical map

$$\text{colim} \ \text{Map}(C_{k}^{\vee}, D_{m}) \to \text{Map}(\lim \ C_{k}^{\vee}, D_{m})$$

By lemma 4.26 this is an equivalence. Each term $C_{k}^{\vee}$ as a filtered object is zero in high enough positive filtration degrees. As limits in filtered objects are created object-wise, one sees that the essential image of the above map consists of morphisms

$$\lim \ C_{k}^{\vee} \to C_{j}^{\vee} \to D_{m}$$

which factor through some $C_{j}^{\vee}$. Since $D_{m}$ is itself of the same form, then every map factors through some $C_{j}^{\vee}$. Hence we obtain the desired decomposition on the right hand side of (4.25). It follows that the morphism of mapping spaces (4.24) decomposes into maps

$$\text{Map}(D_{m}, C_{k}) \to \text{Map}(C_{k}^{\vee}, D_{m}^{\vee}).$$

These are equivalences because $D_{j}$ and $C_{k}$ are dualizable for every $j, k$, and the duality functor $(-)^{\vee}$ gives rise to an anti-equivalence between commutative algebra and commutative coalgebra objects whose underlying objects are dualizable. Assembling this all together we conclude that (4.24) is an equivalence.

Lemma 4.26. The canonical map of spaces

$$\text{colim} \ \text{Map}_{c\text{CAlg}(\text{Fil}_{R})}(C_{k}^{\vee}, D_{m}) \to \text{Map}_{c\text{CAlg}(\text{Fil}_{R})}(\lim \ C_{k}^{\vee}, D_{m})$$

induced by the projection maps $\pi_{k}: \lim \ C_{k} \to C_{k}$ is an equivalence.

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Proof. Fix an index $k$. We claim that the following is a pullback square of spaces:

$$
\begin{aligned}
\Map_{\CAlg(\Fil_R)}(C^\vee_k, D_m) & \xrightarrow{\Und} \Map_{\CAlg}(C^\vee_k, D_m) \\
\pi_k^* & \downarrow \quad \pi_k^*
\end{aligned}
$$

(4.27)

Note first that even though $\Und(-)$ does not generally preserve limits, it will preserve these particular limits by Proposition 4.21. To prove the claim, we see that the pullback

$$
\Map_{\CAlg(\Fil_R)}(\lim_k C^\vee_k, D_m) \times_{\Map_{\CAlg}(\lim_k C^\vee_k, D_m)} \Map_{\CAlg}(C^\vee_k, D_m)
$$

parametrizes, up to higher coherent homotopy, ordered pairs $(f, g)$ with

$$
f : \lim_k C^\vee_k \to D_m
$$
a map of filtered algebras and

$$
g_k : C^\vee_k \to D_m
$$
a map at the level of underlying algebras, such that there is a factorization of the underlying map

$$
\Und(f) \simeq \pi_k^*(g_k) = g_k \circ \pi_k
$$
along the map $\pi_k : \lim_k C^\vee_k \to C^\vee_k$. Recall that $\pi_k$ is also the underlying map of a morphism of filtered objects; since the composition $\Und(f) = g_k \circ \pi_k$ respects the filtration this means that $g_k$ itself must respect the filtration as well. This in particular gives rise to an inverse

$$
\Map_{\CAlg(\Fil_R)}(\lim_k C^\vee_k, D_m) \times_{\Map_{\CAlg}(\lim_k C^\vee_k, D_m)} \Map_{\CAlg}(C^\vee_k, D_m) \to \Map_{\CAlg(\Fil_R)}(C^\vee_k, D_m)
$$
of the canonical map

$$
\Map_{\CAlg(\Fil_R)}(C^\vee_k, D_m) \to \Map_{\CAlg(\Fil_R)}(\lim_k C^\vee_k, D_m) \times_{\Map_{\CAlg}(\lim_k C^\vee_k, D_m)} \Map_{\CAlg}(C^\vee_k, D_m)
$$
duced by the universal property of the pullback, which proves the claim. Now let $P_k$ denote the fiber of the left vertical map of 4.27. One sees that the fiber of the map

$$
\Map_{\CAlg(\Fil_R)}(\lim_k C^\vee_k, D_m) \times_{\Map_{\CAlg}(\lim_k C^\vee_k, D_m)} \Map_{\CAlg}(C^\vee_k, D_m)
$$
of the statement is $\colim P_k$. We would like to show that this is contractible. By the claim, this is equivalent to $\colim P_k^{und}$, where $P_k^{und}$ for each $k$ is the fiber of the right hand vertical map of 4.27. By [Lur16], this is contractible. We will be done upon showing that the essential image the map in the statement is all of $\Map_{\CAlg}(\lim_k C^\vee_k, D)$. To this end we see that the essential image consists of maps

$$
\lim_k C^\vee_k \to C^\vee_j \to D_m
$$

which factor through some $C^\vee_j$. However, since the underlying algebra of $D_m$ is nilpotent, every map factors through such a $C^\vee_j$. \qed

Remark 4.28. We remark that this is ultimately an example of the standard duality between ind and pro objects of an $\infty$-category $\mathcal{C}$. Indeed, one has a duality between algebras and coalgebras in $\Fil_k$ whose underlying objects are dualizable. The equivalence of proposition 4.23 is an equivalence between certain full subcategories of $\Ind(\CAlg^{\omega,fil})$ and $\Pro(\CAlg^{\omega,fil})$. 

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Definition 4.29. Let \( \mathcal{D} \) denote the essential image of the duality functor of Proposition 4.23. Then, we define the category of (commutative) cogroup objects \( \text{coAb}(\mathcal{D}) \) to just be an abelian group object of the opposite category (i.e. of the category of smooth filtered coalgebras. As \((-)^\vee\) is an anti-equivalence of \( \infty \)-categories, this implies that Cartesian products on \( \text{cCAlg}(\text{Fil}_k)^{\text{sm}} \) are sent to coCartesian products on \( \mathcal{D} \). Hence, this functor sends group objects to cogroup object. We refer to an object \( C \in \text{coAb}(\mathcal{D}) \) as a filtered formal group.

Remark 4.30. If \( C^\vee \) is discrete (which is the setting we are primarily concerned with for the moment) then a commutative cogroup structure on \( C \) is none other than a (co)commutative comonoid structure on \( C^\vee \), making it into a bialgebra in complete filtered \( R \)-modules.

Construction 4.31 (Cartier duality). Let
\[
(-)^\vee : \text{cCAlg}(\text{Fil}_R)^{\text{sm}} \to \mathcal{D}
\]
be the equivalence of Proposition 4.23. This may now be promoted to an equivalence
\[
(-)^\vee : \text{Ab}(\text{cCAlg}(\text{Fil}_R)^{\text{sm}}) \to \text{CoAb}(\mathcal{D})
\]
We refer to the correspondence which is implemented by this equivalence as filtered Cartier duality.

Remark 4.32. We explain our usage of the term filtered Cartier duality. As we saw in Section 3.2, classical Cartier duality gives rise to an (anti)-equivalence between formal groups and affine groups schemes, at least in the most well-behaved situation over a field. An abelian group object in smooth filtered coalgebras will be none other than a filtered Hopf algebra. This is due to the fact that we ultimately still restrict to the a \( 1 \)-categorical setting where remark 3.3 applies, so abelian group objects agree with grouplike commutative monoids. Out of this, therefore, one may extract an relative affine group scheme over \( \mathbb{A}^1/G_m \). Hence, 4.31 may be viewed as a correspondence between filtered formal groups and a full subcategory of relatively affine group schemes over \( \mathbb{A}^1/G_m \).

Next we prove a unicity result on complete filtered algebra structures with underlying object a commutative ring \( A \) and specified associated graded, [cf. Theorem 1.4].

Proposition 4.33. Let \( A \) be an commutative ring which is complete with respect to the \( I \)-adic topology induced by some ideal \( I \subset A \). Let \( A_n \in \text{CAlg}(\text{Fil}_R) \) be a (discrete) complete filtered algebra with underlying object \( A \). Suppose there is an inclusion
\[
A_1 \to I
\]
of \( A \)-modules inducing an equivalence
\[
\text{gr}(A_n) \simeq \text{gr}(F_I^*(A)) = \text{Sym}_{gr}(I/I^2)
\]
of graded objects, where \( I/I^2 \) is of pure weight 1. Then \( A_n = F_I^*(A) \), namely the filtration in question is the \( I \)-adic filtration.

Proof. Let \( A_n \) be a complete filtered algebra with these properties. The map
\[
A_1 \to I
\]
in the hypothesis extends by multiplicativity to a map
\[
A_n \to F_I^*(A).
\]
In degree 2 for example, being that $A_2 \to A_1$ is the fiber of the map $A_1 \to I/I^2$, there is an induced $A$-module map

$$A_2 \to I^2$$

fitting into the left hand column of the following diagram:

$$
\begin{array}{ccc}
A_2 & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
I^2 & \rightarrow & I \\
\downarrow & & \downarrow \\
I & \rightarrow & I/I^2 \\
\end{array}
$$

By assumption, one obtains an isomorphism of graded objects

$$\text{gr}(A_n) \cong \text{gr}(F^*_I(A))$$

after passing to the associated graded of this map. Since both filtered objects are complete, and since the associated graded functor when restricted to complete objects is conservative, we deduce that the map

$$A_n \to F^*_I(A)$$

is an equivalence of filtered algebras. In particular, this implies that the inclusion $A_1 \to I$ is surjective at the level of discrete modules, so that $A_1 = I$. We claim that this is enough to deduce that $A_n$ is the $I$-adic filtration, up to equality. For this, we need to show that there is an equality $A_n = I^n$ for every positive integer $n$ and that the structure maps $A_{n+1} \to A_n$ of the filtration are simply the inclusions. Indeed, in each degree, we now have equivalences

$$A_n \cong I^n$$

of $A$-modules, which moreover admit monomorphisms into $A$. The category of such objects is a poset category, and so any isomorphic objects are equal; hence we conclude $A_n = I^n$ for all $n$. \qed

**Remark 4.34.** In particular, we may choose $A_n \in D$, the image of the duality functor from smooth filtered coalgebras. In this case, $I = \text{Sym}^{\geq 1}(M)$, the augmentation ideal of $\widehat{\text{Sym}}(M)$ for $M$ some projective module of finite type.

Now let $G$ be a formal group law over $\text{Spec} k$, and let $O(G)$ be its complete adic algebra of functions. This acquires a comultiplication

$$O(G) \to O(G) \hat{\otimes} O(G)$$

and counit

$$\epsilon : O(G) \to R$$

making $O(G)$ into an abelian cogroup object in $D$. By Proposition 4.33, at the level of underlying $k$-algebras, there is a uniquely determined complete filtered algebra $F_{ad} A$ such that

$$\colim_{n \to -\infty} F_{ad}^n A \cong O(G)$$

We show that this inherits the cogroup structure as well:
Corollary 4.35. The comultiplication
\[ \Delta : \mathcal{O}(\hat{G}) \to \mathcal{O}(\hat{G}) \otimes \mathcal{O}(\hat{G}) \]
can be promoted to a map of filtered complete algebras. Thus, there is a unique filtered formal group, i.e. an abelian cogroup object in the category \( \mathcal{D} \) with associated graded free on a filtered module concentrated in weight one and with underlying object is \( \mathcal{O}(\hat{G}) \), which refines the comultiplication on \( \mathcal{O}(\hat{G}) \).

Proof. We need to show that the comultiplication
\[ \Delta : \mathcal{O}(\hat{G}) \to \mathcal{O}(\hat{G}) \otimes \mathcal{O}(\hat{G}) \]
preserves the adic filtration. Let us assume first that the formal group is 1-dimensional and oriented so that \( \mathcal{O}(\hat{G}) \cong \mathbb{R}[\![x]\!] \). We remark that every formal group is locally oriented. In this case, by the formal group law is given in coordinates by the power series
\[ f(x_1, x_2) = x_1 + x_2 + \sum_{i,j \geq 1} a_{i,j} x^i y^j \]
with suitable \( a_{i,j} \). In particular, the image of the ideal commensurate with the filtration is contained in \( I^{\otimes 2} = (x_1, x_2) \), the ideal commensurate with the filtration on \( \mathcal{O}(\hat{G}) \otimes \mathcal{O}(\hat{G}) \cong \mathbb{R}[\![x_1, x_2]\!] \). Note that this is itself the \( (x_1, x_2) \)-adic filtration on \( \mathbb{R}[\![x_1, x_2]\!] \). By multiplicativity, \( \Delta(I^n) \subset I^{\otimes 2n} \) for all \( n \). This shows that \( \Delta \) preserves the filtration, making giving \( F^* A \) a unique coalgebra structure compatible with the formal group structure on \( \hat{G} \). The same argument works in higher dimensions.

5 The deformation of a formal group

5.1 Deformation to the normal cone

To a pointed formal moduli problem (such as a formal group) one may associate an equivariant family over \( \mathbb{A}^1 \), whose fiber over \( \lambda \neq 0 \) recovers \( G \). We will use this construction in the sequel to produce filtrations on the associated Hochschild homology theories. The author would like to thank Bertrand Toën for the idea behind this construction, and in fact related constructions appear in \cite{Tou20}. A variant of this construction in the characteristic zero setting also appears in \cite[Chapter IV.5]{GR17}. We would also like to point out \cite{KR18}.

The construction pertains to more than just formal groups. Indeed let \( \mathcal{X} \to \mathcal{Y} \) be closed immersion of locally Noetherian schemes. We construct a filtration on \( \hat{\mathcal{Y}} \), the formal completion of \( \mathcal{Y} \) along \( \mathcal{X} \), with associated graded the shifted tangent complex \( T^{[\lambda]}_{\mathcal{X}|\mathcal{Y}}[1] \).

**Proposition 5.1.** There exists a filtered stack \( S^0_{\text{fil}} \to \mathbb{A}^1/\mathbb{G}_m \), whose underlying object is the constant stack \( S^0 = \text{Spec} \subset \text{Spec} \) and whose associated graded is \( \text{Spec}(k[\epsilon]/(\epsilon^2)) \).

**Proof.** Morally one should think of this as families of two points degenerating into each other over the special fiber. For a more rigorous construction, one may begin with the nerve of the unit map of commutative algebra objects in the \( \infty \)-category \( \text{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \):
\[ \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \to 0_* (\mathcal{O}_{BG_m}), \] (5.2)
where \( 0 : BG_m \to \mathbb{A}^1/\mathbb{G}_m \) is the closed point. This gives rise to a groupoid object (cf. \cite[Section 6.1.2]{Lur09}) in the \( \infty \)-category \( \text{CAlg}(\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)) \).
We now give a more explicit description of this groupoid object. The structure sheaf $\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ may be identified with the graded polynomial algebra $k[t]$, where $t$ is of weight 1. In degree 1 one obtains the following fiber product
\[
\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \times_{\mathcal{O}_{\mathbb{B}\mathbb{G}_m}} \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}
\]
which may be thought of as the graded algebra
\[
k[t_1, t_2]/(t_1 + t_2)(t_1 - t_2).
\]
viewed as an algebra over $k[t]$. If we apply the Spec functor relative to $\mathbb{A}^1/\mathbb{G}_m$, we obtain the scheme corresponding to the union of the diagonal and antidiagonal in the plane. The pullback of this fiber product to $\text{Mod}_k$ is
\[
k \times_{1^* \mathcal{O}_{\mathbb{B}\mathbb{G}_m}} k \simeq k \times 0 k = k \oplus k
\]
The pullback to $\text{QCoh}(B\mathbb{G}_m)$ is $k[\epsilon]/\epsilon^2$, the trivial square-zero extension of $k$ by $k$. To see this we pull back the fiber product (5.3) to $\text{QCoh}(B\mathbb{G}_m)$, which gives the following homotopy cartesian square
\[
\begin{array}{ccc}
k[\epsilon]/(\epsilon^2) & \to & k \\
\downarrow & & \downarrow \\
k & \to & k \oplus k[1]
\end{array}
\]
in this category. Hence, we may define
\[
S^0_{fil} := \text{Spec}_{\mathbb{A}^1/\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \times_{\mathcal{O}_{\mathbb{B}\mathbb{G}_m}} \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m})
\]
as the relative spectrum (over $\mathbb{A}^1/\mathbb{G}_m$).

By construction, this admits a map
\[
S^0_{fil} \to \mathbb{A}^1/\mathbb{G}_m
\]
making it into a filtered stack, with generic fiber and special fiber described in the above proposition. We remark that we may think of $S^0_{fil}$ as the degree 1 part of a cogroupoid object $S^0_{fil}$ in the $\infty$-category of (derived) schemes over $\mathbb{A}^1/\mathbb{G}_m$; indeed we may apply $\text{Spec}(\quad)$ to the entire Cech nerve of the map 5.2. We can then take mapping spaces out of this cogroupoid to obtain a groupoid object.

Now let $X \to Y$ be as above. We will focus our attention on the following derived mapping stack, defined in the category $d\text{Stk}_{Y \times \mathbb{A}^1/\mathbb{G}_m}$ of derived stacks over $Y \times \mathbb{A}^1/\mathbb{G}_m$:
\[
\text{Map}_{Y \times \mathbb{A}^1/\mathbb{G}_m}(S^0_{fil}, X \times \mathbb{A}^1/\mathbb{G}_m)
\]
By composing with the projection map $Y \times \mathbb{A}^1/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$, we obtain a map,
\[
\text{Map}_{y \times \mathbb{A}^1/\mathbb{G}_m}(S^0_{fil}, X) \to \mathbb{A}^1/\mathbb{G}_m
\]
allowing us to view this as a filtered stack. The next proposition identifies its fiber over $1 \in \mathbb{A}^1/\mathbb{G}_m$:
Proposition 5.4. There is an equivalence

\[ 1^*(\text{Map}(S^0_{fil}, X)) \simeq X \times_Y X, \]

Proof. By formal properties of base change of mapping objects of \( \infty \)-topoi, there is an equivalence

\[ 1^*(\text{Map}(S^0_{fil}, X)) \simeq \text{Map}_Y(1^*S^0_{fil}, 1^*(X \times \mathbb{A}^1/G_m)) \]

The right hand side is the mapping object out of a disjoint sum of final objects, and therefore is directly seen to be equivalent to \( X \times_Y X \)

Next we identify the fiber over the “closed point” \( 0 : B\mathbb{G}_m \to \mathbb{A}^1/G_m \).

Proposition 5.5. There is an equivalence of stacks

\[ 0^*(\text{Map}(S^0_{fil}, X)) \simeq T_{X/Y}, \]

where \( T_{X/Y} \) denotes the relative tangent bundle of \( X \to Y \).

Proof. We base change along the map \( \text{Spec}k \to B\mathbb{G}_m \to \mathbb{A}^1/G_m \).

Invoking again the standard properties of base change of mapping objects we obtain the equivalence

\[ 0^*(\text{Map}(S^0_{fil}, X)) \simeq \text{Map}_Y(0^*S^0_{fil}, 0^*(X \times \mathbb{A}^1/G_m)). \]

By construction, we may identify \( 0^*S^0_{fil} \) with \( \text{Spec}(k[\epsilon]/\epsilon^2) \). Of course, this means that the right hand side of the above display is precisely the relative tangent complex \( T_{X/Y} \).

To summarize, we have constructed a cogroupoid object in the category of schemes over \( \mathbb{A}^1/G_m \), whose piece in cosimplicial degree 1 is \( S^0_{fil} \), and formed the derived mapping stack

\[ \text{Map}_Y \times \mathbb{A}^1/G_m(S^0_{fil}, X \times \mathbb{A}^1/G_m), \]

which will in turn be the degree one piece of a groupoid object in derived schemes over \( \mathbb{A}^1/G_m \).

Construction 5.6. Let \( M_\bullet := \text{Map}_Y \times \mathbb{A}^1/G_m(S^0_{fil}, X \times \mathbb{A}^1/G_m) \). Note that we can interpret the degeneracy map

\[ X \times \mathbb{A}^1/G_m \to \text{Map}_Y \times \mathbb{A}^1/G_m(S^0_{fil}, X \times \mathbb{A}^1/G_m) \]

as the “inclusion of the constant maps”. We reiterate that this is a groupoid object in the \( \infty \)-category of derived schemes over \( \mathbb{A}^1/G_m \). We let

\[ \text{Def}_{\mathbb{A}^1/G_m}(X/Y) := \text{colim}_\Delta M_\bullet \]

denote the colimit of this groupoid object. Note that the colimit is taken in the \( \infty \)-category of derived schemes over \( \mathbb{A}^1/G_m \) (as opposed to all of derived stacks).

By construction, \( \text{Def}_{\mathbb{A}^1/G_m}(X/Y) \) is a derived scheme over \( \mathbb{A}^1/G_m \). The following proposition identifies its “generic fiber” with the formal completion \( \hat{Y}_X \) of \( Y \) in \( X \).

Proposition 5.7. There is an equivalence

\[ 1^*\text{Def}_{\mathbb{A}^1/G_m}(X/Y) \simeq \hat{Y}_X \]
Proof. As pullback commutes with colimits, this amounts to identifying the delooping in the category of derived schemes over \(Y\). Note again that all objects are schemes and not stacks so that this statement makes sense. By the above identifications, delooping the above groupoid corresponds to taking the colimit of the nerve \(N(f)\) of the map \(f : \mathcal{X} \to \mathfrak{y}\), a closed immersion. Hence, it amounts to proving that

\[
\text{colim}_{\Delta^p} N(f) \simeq \hat{\mathfrak{y}_X}
\]

This is precisely the content of Theorem 2.6. \(\square\)

A consequence of the above proposition is that the resulting object is pointed by \(\mathcal{X}\) in the sense that there is a well defined map \(\mathcal{X} \to \hat{\mathfrak{y}_X}\), arising from the structure map in the associated colimit diagram. This map is none other than the “inclusion” of \(\mathcal{X}\) into its formal thickening.

Our next order of business is somewhat predictably at this point, to identify the fiber over \(BG_m\) of \(\text{Def}_{\mathbb{A}^1/G_m}(\mathcal{X}/\mathfrak{y})\) with the normal bundle of \(\mathcal{X}\) in \(\mathfrak{y}\).

Proposition 5.8. There is an equivalence

\[
0^* \text{Def}_{\mathbb{A}^1/G_m}(\mathcal{X}/\mathfrak{y}) \simeq \mathcal{V}(T_{\mathcal{X}/\mathfrak{y}}[1]) =: \hat{N}_{\mathcal{X}/\mathfrak{y}}
\]

in the \(\infty\)-category of derived schemes over \(BG_m\) of our stack \(\text{Def}_{\mathbb{A}^1/G_m}(\mathcal{X}/\mathfrak{y})\).

Proof. As in the proof of the previous proposition, it amounts to understanding the pull-back along \(\text{Spec} k \to BG_m \to \mathbb{A}^1/G_m\) of the groupoid object \(M_{\bullet}\). This is given by

\[
\mathcal{X} \Rightarrow T_{\mathcal{X}/\mathfrak{y}}...
\]

where we abuse notation and identify \(T_{\mathcal{X}/\mathfrak{y}}\) with \(\mathcal{V}(T_{\mathcal{X}/\mathfrak{y}}[1])\). Note that \(T_{\mathcal{X}/\mathfrak{y}} \simeq \Omega_X(T_{\mathcal{X}/\mathfrak{y}}[1])\) and so, we may identify the above colimit diagram with the simplicial nerve \(N(f)\) of the unit section \(\mathcal{X} \to T_{\mathcal{X}/\mathfrak{y}}[1] \simeq N_{\mathcal{X}/\mathfrak{y}}\). The result now follows from another application of Theorem 2.6. \(\square\)

The following statement summarizes the above discussion:

Theorem 5.9. Let \(f : \mathcal{X} \to \mathfrak{y}\) be a closed immersion of schemes. Then there exists a filtered stack \(\text{Def}_{\mathbb{A}^1/G_m}(\mathcal{X}/\mathfrak{y}) \to \mathbb{A}^1/G_m\) (making it into a relative scheme over \(\mathbb{A}^1/G_m\)) with the property that there exists a map

\[
\mathcal{X} \times \mathbb{A}^1/G_m \to \text{Def}_{\mathbb{A}^1/G_m}(\mathcal{X}/\mathfrak{y})
\]

whose fiber over \(1 \in \mathbb{A}^1/G_m\) is

\[
\mathcal{X} \to \hat{\mathfrak{y}_X}
\]

and whose fiber over \(0 \in \mathbb{A}^1/G_m\) is

\[
\mathcal{X} \to \hat{N}_{\mathcal{X}/\mathfrak{y}},
\]

the formal completion of the unit section of \(\mathcal{X}\) in its normal bundle.

5.2 Deformation of a formal group to its normal cone

Fix a (classical) formal group \(\hat{G}\). We now apply the above construction to the unit section of the formal group, \(\iota : \text{Spec} k \to \hat{G}\). Note that \(\hat{G}\) is already formally complete along \(\iota\). We set

\[
\text{Def}_{\mathbb{A}^1/G_m}(\hat{G}) := \text{Def}_{\mathbb{A}^1/G_m}(\text{Spec} k/\hat{G})
\]

This will be a relative scheme over \(\mathbb{A}^1/G_m\).
Proposition 5.10. Let $\text{Speck} \to \widehat{G}$ be the unit section of a formal group. Then, the stack $\text{Def}_{A^1/G_m}(\widehat{G})$ of Construction 5.6 is a filtered formal group.

Proof. We will show that there exists a filtered dualizable (and discrete) $R$-module $M$ for which

$$\mathcal{O}(\text{Def}_{A^1/G_m}(\widehat{G})) \simeq \Gamma_{fil}(M)^\vee \simeq \text{Sym}^*_{fil}(M^\vee).$$

As was shown above, there is an equivalence of stacks over $B\mathbb{G}_m$. We note that the right hand side may indeed be viewed as a stack over $B\mathbb{G}_m$, arising from the weight $-1$ action of $\mathbb{G}_m$ by homothety on the fibers. This is the $\mathbb{G}_m$ action which will be compatible with the grading on the dual numbers $k[\epsilon]$ (which appears in Proposition 5.1) such that $\epsilon$ is of weight one. In particular, since $\widehat{G}$ is a one dimensional formal group, it follows that the associated graded is none other than

$$\text{Sym}^*_{gr}(M(1))$$

the graded symmetric algebra on the graded $k$-module $M(1)$ which is $M$ concentrated in weight 1. Putting this all together we see that at the level of filtered objects, there is an equivalence

$$\mathcal{O}(\text{Def}_{A^1/G_m}(\widehat{G})) \simeq \text{Sym}_{fil}(M^f(1)),$$

where $M^f(1)$ is the filtered $k$-module

$$M^f(1) = \begin{cases} M^f(1)_n = 0, & n > 1 \\ M^f(1)_n = M, & n \leq 1 \end{cases}$$

Recall the deformation to the normal cone will be equipped with a “unit” map

$$A^1/G_m \to \text{Def}_{A^1/G_m}(\widehat{G}).$$

By passing to functions, we deduce from this map that the degree 1 piece of the filtration on $\mathcal{O}(\text{Def}_{A^1/G_m}(\widehat{G}))$ is a submodule of the augmentation ideal of $\text{Sym}(M)$. Thus, the conditions of Proposition 4.3 are satisfied here, so we conclude that this filtration is none other than the adic filtration of $\text{Sym}(\widehat{M})$ with respect to the augmentation ideal. Finally by Corollary 4.35 this acquires a canonical abelian cogroup structure which is a filtered enhancement of that of $\widehat{G}$, making $\text{Def}_{A^1/G_m}(\widehat{G})$ into a filtered formal group.

Now we combine this construction with the $A^1/G_m$-parametrized Cartier duality of Section 4.

Corollary 5.11. Let $\widehat{G}$ be a formal group over $\text{Speck}$, and let $\widehat{G}^\vee$ denote its Cartier dual. Then the cohomology $R\Gamma(\widehat{G}^\vee, \mathcal{O})$ acquires a canonical filtration.

Proof. By Construction 4.31 the coordinate algebra $\mathcal{O}(\text{Def}_{A^1/G_m}(\widehat{G}))$ corresponds via duality to an abelian group object in smooth filtered coalgebras. As we are in the discrete setting, this is equivalent to the structure of a grouplike commutative monoid in this category. In particular, this is a filtered Hopf algebra object, so it determines a group stack $\text{Def}_{A^1/G_m}(\widehat{G})^\vee$ over $A^1/G_m$. □
6 The deformation to the normal cone of $\hat{G}_m$

By the above, given any formal group $\hat{G}$, one may define a filtration on its Cartier dual $\hat{G}^\vee = \text{Map}(\hat{G}, \hat{G}_m)$ in the sense of [Mon19]. In the case of the formal multiplicative group, this gives a filtration on its Cartier dual $D(\hat{G}_m) = \text{Fix}$. In [MRT19], the authors defined a canonical filtration on this affine group scheme (defined over a $\mathbb{Z}(p)$-algebra $k$) given by a certain interpolation between the kernel and fixed points on the Frobenius on the Witt vector scheme. We would like to compare the filtration on $\text{Map}(\hat{G}_m, \hat{G}_m)$ with this construction.

Corollary 6.1. The filtration defined on $\text{Fix}$ is Cartier dual to the $(x)$-adic filtration on

$$O(\hat{G}_m) \simeq k[[x]].$$

Furthermore, this filtration corresponds to the deformation to the normal cone construction $\text{Def}_{\mathbb{A}^1/\hat{G}_m}(\hat{G}_m)$ on $\hat{G}_m$.

Proof. Let

$$\mathcal{G}_t = \text{Spec} k[X, 1/1 + tX]$$

This is an affine group scheme; one sees by varying the parameter $t$ that this is naturally defined over $\mathbb{A}^1$. If $t$ is invertible, then this is equivalent to $\hat{G}_m$; if $t = 0$, this is just the formal additive group $\hat{G}_a$. If we take the formal completion of this at the unit section, we obtain a formal group $\hat{G}_t$, with corresponding formal group law

$$F(X, Y) = X + Y + tXY$$

which we may think of as a formal group over $\mathbb{A}^1$. In [SS01] the authors describe the Cartier dual of the resulting formal group, for every $t \in k$, as the group scheme

$$\ker(F - t^{p-1} \text{id} : \mathbb{W}_p \to \mathbb{W}_p)$$

These of course assemble, by way of the natural $\hat{G}_m$ action on the Witt vector scheme $\mathbb{W}$, to give the filtered group scheme $\mathbb{H} \to \mathbb{A}^1/\hat{G}_m$ of [MRT19], whose classifying stack is the filtered circle.

The algebra of functions $O(\mathbb{H})$ acquires is a comultiplication; by results of [Mon19], we may think of this as a filtered Hopf algebra.

Let us identify this filtered Hopf algebra a bit further, which by abuse of notation, we refer to as $O(\mathbb{H})$. After passing to underlying objects, it is the divided power coalgebra $\bigoplus \Gamma^n(k)$. The algebra structure on this comes from the multiplication on $\hat{G}_m$, via Cartier duality. On the graded side, we have the coordinate algebra of $\text{Ker}$, which by [Dri20, Lemma 3.2.6], is none other than the free divided power algebra

$$k(x) \cong k[x, \frac{x^2}{2!}, ...]$$

One gives this the grading where each $\frac{x^n}{n!}$ is of pure weight $-n$. The underlying graded smooth coalgebra is

$$\bigoplus_n \Gamma_{gr}(k(-1))$$

We deduce by weight reasons that there is an equivalence of filtered coalgebras

$$O(\mathbb{H}) \simeq \bigoplus_n \Gamma^n(k_f(-1))$$
where \( k^{fil}(-1) \) is trivial in filtering degrees \( n > 1 \) and equal to \( k \) otherwise.

The consequence of the analysis of the above paragraph is that the Hopf algebra structure on \( \mathcal{O}(\mathbb{H}) \) corresponds to the data of an abelian group object in smooth filtered coalgebras, cf. section 4. In particular, it corresponds to a coAbelian group object structure on the dual, \( \text{Sym}_f(k^f(1)) \). This is a complete filtered algebra satisfying the conditions of Proposition 4.33 and thus coincides with the adic filtration on \( k[[x]] \). The corresponding filtered coalgebra structure is the unique one commensurate with the adic filtration, since by corollary 4.35 the comultiplication preserves the adic filtration. Thus, there exists a unique filtered formal group which recovers \( \hat{G}_m \) and \( \hat{G}_a \) upon taking underlying objects and associated gradeds respectively. In the setting of the filtered Cartier duality of Section 4, this must then be dual to the specified abelian group object structure on \( \mathcal{O}(\mathbb{H}) \).

Finally we relate this to the deformation to the normal cone construction applied to \( \hat{G}_m \), which also outputs a filtered formal group. Indeed by the reasoning of Proposition 5.10 this filtered formal group is itself given by the adic filtration on \( k[[t]] \) together with the filtered coalgebra structure uniquely determined by the group structure on \( \hat{G}_m \).

7 \( \hat{G} \)-Hochschild homology

As an application to the above deformation to the normal cone constructions associated to a formal group, we further somewhat the following proposal of [MRT19] described in the introduction.

**Construction 7.1.** Let \( k \) be a \( \mathbb{Z}_{(p)} \)-algebra. Let \( \hat{G} \) be a formal group over \( k \). Its Cartier dual \( \hat{G}^\vee \) is an affine commutative group scheme We let \( B\hat{G}^\vee \) denote the classifying stack of the group scheme \( \hat{G}^\vee \). Let \( X = \text{Spec}A \) be an affine derived scheme, corresponding to a simplicial commutative ring \( A \). One forms the derived mapping stack

\[
\text{Map}_{dStk}(B\hat{G}^\vee, X).
\]

If \( \hat{G} = \hat{G}_m \), then by the affinization techniques of [Toë06, MRT19], one recovers, at the level of global sections

\[
R\Gamma(\text{Map}_{dStk}(B\hat{G}_m^\vee, X), \mathcal{O}) \simeq \text{HH}(A),
\]

the Hochschild homology of \( A \) as the global sections of this construction. Following this example one can make the following definition (cf. [MRT19 Section 6.3])

**Definition 7.2.** Let \( \hat{G} \) be a formal group over \( k \). Let

\[
\text{HH}^\hat{G} : \text{sCAlg}_k \to \text{Mod}_k
\]

be the functor defined by

\[
\text{HH}^\hat{G}(A) := R\Gamma(\text{Map}_{dStk}(B\hat{G}^\vee, X), \mathcal{O})
\]

As was shown in section 5.2, given a formal group \( \hat{G} \) over a commutative ring \( R \), one can apply a deformation to the normal cone construction to obtain a formal group \( \text{Def}_{\mathbb{A}^1/G_m}(\hat{G}) \) to obtain a formal group over \( \mathbb{A}^1/G_m \). By applying \( \mathbb{A}^1/G_m \)-parametrized Cartier duality, one obtains a group scheme over \( \mathbb{A}^1/G_m \).

**Theorem 7.3.** Let \( \hat{G} \) be an arbitrary formal group. The functor

\[
\text{HH}^\hat{G}(-) : \text{sCAlg}_R \to \text{Mod}_R
\]
admits a refinement to the $\infty$-category of filtered $R$-modules

$$\widetilde{\text{HH}}(-) : s\text{CAlg}_R \to \text{Mod}_R^{filt},$$

such that

$$\widetilde{\text{HH}}(-) \simeq \text{colim}_{(Z, \leq)} \widetilde{\text{HH}}(-)$$

Proof. Let $Def_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})$ be the Cartier dual of the deformation to the normal cone $Def_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})$. Form the mapping stack

$$\text{Map}_{dStk_{/\mathbb{A}^1/\mathbb{G}_m}}(BDef_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})^\vee, X \times \mathbb{A}^1/\mathbb{G}_m).$$

This base-changes along the map

$$1 : \text{Spec} \to \mathbb{A}^1/\mathbb{G}_m$$

to the mapping stack

$$\text{Map}_{dStk_k}(B\widehat{G}^\vee, X),$$

which gives the desired geometric refinement. The stack $\text{Map}_{dStk_{/\mathbb{A}^1/\mathbb{G}_m}}(BDef_{\mathbb{A}^1/\mathbb{G}_m}(\widehat{G})^\vee, X \times \mathbb{A}^1/\mathbb{G}_m)$ is a derived scheme relative to the base $\mathbb{A}^1/\mathbb{G}_m$. Indeed, it is nilcomplete, infinitesimally cohesive and admits an obstruction theory by the arguments of [TV08, Section 2.2.6.3]. Finally its truncation is the relative scheme $t_0 X \times \mathbb{A}^1/\mathbb{G}_m$ over $\mathbb{A}^1/\mathbb{G}_m$ - this follows from the identification

$$t_0 \text{Map}(B\widehat{G}^\vee, X) \simeq t_0 \text{Map}(B\widehat{G}^\vee, t_0 X)$$

and from the fact that there are no nonconstant (nonderived) maps $BG \to t_0 X$ for $G$ a group scheme.

Hence we conclude by the criteria of [TV08, Theorem C.0.9] that this is a relative affine derived scheme. By Proposition 2.1 we conclude that $L_{\text{fil}}^\mathbb{G}(X) \to \mathbb{A}^1/\mathbb{G}_m$ is of finite cohomological dimension and so $\widetilde{\text{HH}}(\mathbb{G})$ defines an exhaustive filtration on $\text{HH}^\mathbb{G}(A)$.

Remark 7.4. In characteristic zero, all one-dimensional formal groups are equivalent to the additive formal group $\widehat{\mathbb{G}_a}$, via an equivalence with its tangent Lie algebra. In particular the above filtration splits canonically, one one obtains an equivalence of derived schemes

$$\text{Map}_{dStk}(B\widehat{G}^\vee, X) \simeq \mathbb{T}_{X|R}[-1]$$

In positive or mixed characteristic this is of course not true. However, one can view all these theories as deformations along the map $BG_m \to \mathbb{A}^1/\mathbb{G}_m$ of the de Rham algebra $DR(A) = \text{Sym}(L_{A|k}[1])$

8 Liftings to spectral deformation rings

In this section we lift the above discussion to the setting of spectral algebraic geometry over various ring spectra that parametrize deformations of formal groups. These are defined in [Lur18] in the context of the elliptic cohomology theory. As we will be switching gears now and working in this setting, we will spend some time recalling and slightly clarifying some of the ideas in [Lur18]. Namely, we introduce a correspondence between formal groups over $E_\infty$-rings, and spectral affine group schemes, and show it to be compatible with Cartier duality in the classical setting. We stress that the necessary ingredients already appear in [Lur18].
8.1 Formal groups over the sphere

We recall various aspects of the treatment of formal groups in the setting of spectra and spectral algebraic geometry. The definition is based on the notion of smooth coalgebra studied in Section 3.

**Definition 8.1.** Fix an arbitrary $E_\infty$ ring $R$, and let $C$ be a coalgebra over $R$. Recall that this means that $C \in \text{CAlg}(\text{Mod}_R^{op})^{op}$. Then $C$ is smooth if it is flat as an $R$-module, and if $\pi_0 C$ is smooth as a coalgebra over $\pi_0(R)$, as in Definition 3.11.

Given an arbitrary coalgebra $C$ over $R$, the linear dual $C^\vee = \text{Map}(C, R)$ acquires a canonical $E_\infty$-algebra structure. In general $C$ cannot be recovered from $C^\vee$. However, in the smooth case, the dual $C$ acquires the additional structure of a topology on $\pi_0$ giving it the structure of an adic $E_\infty$ algebra. This allows us to recover $C$, via the following proposition, c.f. [Lur18, Theorem 1.3.15]:

**Proposition 8.2.** Let $C, D \in \text{cCAlg}^{sm}_R$ be smooth coalgebras. Then $R$-linear duality induces a homotopy equivalence

$$\text{Map}_{\text{cCAlg}} R(C, D) \simeq \text{Map}_{\text{CAlg}}^\text{cont} R(C^\vee, D^\vee).$$

**Remark 8.3.** One can go further and characterize intrinsically all adic $E_\infty$ algebras that arise as duals of smooth coalgebras. These (locally) have underlying homotopy groups a formal power series ring.

**Construction 8.4.** Given a coalgebra $C \in \text{cCAlg}_R$, one may define a functor

$$c\text{Spec}(C) : \text{CAlg}_R^{cn} \to \mathcal{S};$$

this associates, to a connective $R$-algebra $A$, the space of grouplike elements:

$$\text{GLike}(A \otimes_R C) = \text{Map}_{\text{cCAlg}} A(A, A \otimes_R C).$$

**Remark 8.5.** Fix $C$ a smooth coalgebra. There is always a canonical map of stacks $\text{coSpec}(C) \to \text{Spec}(A)$ where $A = C^\vee$, but it is typically not an equivalence. The condition that $C$ is smooth guarantees precisely that there is an induced equivalence $\text{coSpec}(C) \to \text{Spf}(A) \subseteq \text{Spec}A$, where $\text{Spf}(A)$ denotes the formal spectrum of the adic $E_\infty$ algebra $A$. In particular $\text{coSpec}(C)$ is a formal scheme in the sense of [Lur16, Chapter 8].

One has the following proposition, to be compared with Proposition 3.15:

**Proposition 8.6** (Lurie). Let $R$ be an $E_\infty$-ring. Then the construction $C \mapsto c\text{Spec}(C)$ induces a fully faithful embedding of $\infty$-categories

$$\text{cCAlg}^{sm} \to \text{Fun}(\text{CAlg}_R^{cn}, \mathcal{S})$$

This facilitates the following definition of a formal group in the setting of spectral algebraic geometry.

**Definition 8.7.** A functor $X : \text{CAlg}_R^{cn} \to \mathcal{S}$ is a formal hyperplane if it is in the essential image of the $\text{coSpec}$ functor. We now define a formal group to be an abelian group object in formal hyperplanes, namely an object of $\text{Ab}(\text{HypPlane})$.

As is evident from the thread of the above construction, one may define a formal group to be a certain type of Hopf algebra, but in a somewhat strict sense. Namely we can define a formal group to be an object of $\text{Ab}(\text{cCAlg}^{sm})$; namely an abelian group object in the $\infty$-category of smooth coalgebras.
Remark 8.8. The monoidal structure on $cCAlg_R$ induced by the underlying smash product of $R$-modules is Cartesian; in particular it is given by the product in this $\infty$-category. Hence, a “commutative monoid object” in the category of $R$-coalgebras will be coalgebras which are additionally equipped with an $E_\infty$-algebra structure. In particular, they will be bialgebras.

Construction 8.9. Let $\hat{G}$ be a formal group over an $E_\infty$-algebra $R$. Let $\mathcal{H}$ be a strict Hopf algebra $\mathcal{H}$ in the above sense, for which $\text{coSpec}\, \mathcal{H} = \hat{G}$.

Let $U : Ab(cCAlg_R) \to \text{CMon}(cCAlg_R)$ be the forgetful functor from abelian group objects to commutative monoids. Since the monoidal structure on $cCAlg_R$ is cartesian, the structure of a commutative monoid in $cCAlg_R$ is that of a commutative algebra on the underlying $R$-module, and so we may view such an object as a bialgebra in $\text{Mod}_R$. Finally, applying $\text{Spec}(-)$ (the spectral version) to this bialgebra to obtain a group object in the category of spectral schemes. This is what we refer to as the Cartier dual $\hat{G}^\vee$ of $\hat{G}$.

Remark 8.10. The above just makes precise the association, for a strict Hopf algebra $\mathcal{H}$, (i.e. an abelian group object) the association

$$\text{Spf}(H^\vee) \simeq \text{coSpec}(H) \mapsto \text{Spec}(H)$$

Unlike the 1-categorical setting studied so far, there is no equivalence underlying this, as passing between abelian group objects to commutative monoid objects loses information; hence this is not a duality in the precise sense. In particular, it is not clear how to obtain a spectral formal group from a grouplike commutative monoid in schemes, even if the underlying coalgebra is smooth.

Proposition 8.11. Let $R \to R'$ be a morphism of $E_\infty$-rings and let Let $\hat{G}$ be a formal group over $\text{Spec}R$, and $\hat{G}_{R'}$ its extension to $R'$. Then Cartier duality satisfies base-change, so that there is an equivalence

$$D(\hat{G}_{R'}|_{R}) \simeq D(\hat{G})|_{R}$$

Proof. Let $\hat{G} = \text{Spf}(A)$ be a formal group corresponding to the adic $E_\infty$ ring $A$. Then the Cartier dual is given by $\text{Spec}(\mathcal{H})$ for $\mathcal{H} = A^\vee$, the linear dual of $A$ which is a smooth coalgebra. The linear duality functor $(-)^\vee = \text{Map}_R(-, R)$-for example by [Lur18, Remark 1.3.5] - commutes with base change and is an equivalence between smooth coalgebras and their duals. Moreover it preserves finite products and so can be upgraded to a functor between abelian group objects.

8.2 Deformations of formal groups

Let us recall the definition of a deformation of a formal group. These are all standard notions.

Definition 8.12. Let $\hat{G}_0$ be formal group defined over a finite field of characteristic $p$. Let $A$ be a complete Noetherian ring equipped with a ring homomorphism $\rho : A \to k$, further inducing an isomorphism $A/m \cong k$. A deformation of $\hat{G}_0$ along $\rho$ is a pair $(\hat{G}, \alpha)$ where $\hat{G}$ is a formal group over $A$ and $\alpha : \hat{G}_0 \to \hat{G}|_k$ is an isomorphism of formal groups over $k$.

The data $(\hat{G}, \alpha)$ can be organized into a category $\text{Def}_{\hat{G}_0}(A)$. The following classic theorem due to Lubin and Tate asserts that there exists a universal deformation, in the sense that there is a ring which corepresents the functor $A \mapsto \text{Def}_{\hat{G}_0}(A)$. 

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Theorem 8.13 (Lubin-Tate). Let $k$ be a perfect field of characteristic $p$ and let $\hat{G}_0$ be a one dimensional formal group of height $n < \infty$ over $k$. Then there exists a complete local Noetherian ring $R^cl_{\hat{G}}$ a ring homomorphism

$$\rho : R^cl_{\hat{G}} \to k$$

inducing an isomorphism $R^cl_{\hat{G}}/m \cong k$, and a deformation $(\hat{G}, \alpha)$ along $\rho$ with the following universal property: for any other complete local ring $A$ with an isomorphism $A \cong A/m$, extension of scalars induces an equivalence

$$\text{Hom}_k(A_n, A) \cong \text{Def}_{\hat{G}_0}(A, \rho)$$

(here, we regard the right hand side as a category with only identity morphisms)

For the purposes of this text, we can interpret the above as saying that every formal group over a complete local ring $A$ with residue field $k$ can be obtained from the universal formal group over $A_0$ by base change along the map $A_0 \to A$. We let $G^\text{un}$ denote the universal formal group over this ring.

Remark 8.14. As a consequence of the classification of formal groups due to Lazard, one has a description

$$A_0 \cong W(k)[[v_1, ..., v_{n-1}]],$$

where the map $\rho : W(k)[[v_1, ..., v_{n-1}]] \to k$ has kernel the maximal ideal $m = (p, v_1, ..., v_{n-1})$.

8.3 Deformations over the sphere

As it turns out the ring $A_0$ has the special property that it can be lifted to the $K(n)$-local sphere spectrum. To motivate the discussion, we restate a classic theorem attributed to Goerss, Hopkins and Miller. We first set some notation.

Definition 8.15. Let $FG$ denote the category with

- objects being pairs $(k, \hat{G})$ where $k$ is a perfect field of characteristic $p$, and $\hat{G}$ is a formal group over $k$
- A morphism from $(K, \hat{G})$ to $(k', \hat{G}')$ is a pair $(f, \alpha)$ where $f : k \to k'$ is a ring homomorphism, and $\alpha : \hat{G} \cong \hat{G}'$ is an isomorphism of formal groups over $k'$

Theorem 8.16 (Goerss-Hopkins-Miller). Let $k$ be a perfect field of characteristic $p > 0$, and let $\hat{G}_0$ be a formal group of height $n < \infty$ over $k$. Then there is a functor

$$E : FG \to \text{CAlg}, \ (k, \hat{G}) \mapsto E_{k, \hat{G}}$$

such that for every $(k, \hat{G})$, the following holds

1. $E_{k, \hat{G}}$ is even periodic and complex orientable.

2. the corresponding formal group over $\pi_0 E_{k, \hat{G}}$ is the universal deformation of $(k, \hat{G})$. In particular, $\pi_0 E_{k, \hat{G}} \cong A_0 \cong \mathbb{W}(k)[[v_1, ..., v_{n-1}]]$

If we set $E_{k, \hat{G}} = (F_p, \Gamma)$, where $\Gamma$ is the $p$-typical formal group of height $n$, we denote

$$E_n := E_{F_p, \Gamma};$$

this is the $n$th Morava $E$-theory.
Remark 8.17. The original approach to this uses Goerss-Hopkins obstruction theory. A modern account due to Lurie can be found in [Lur18, Chapter 5]

As it turns out, this ring can be thought of as parametrizing oriented deformations of the formal group $\hat{G}$. This terminology, introduced in [Lur18], roughly means that the formal group in question is equivalent to the Quillen formal group arising from the complex orientation on the base ring. However, there exists an $E_\infty$-algebra parametrizing unoriented deformations of the formal group over $k$.

**Theorem 8.18** (Lurie). Let $k$ be a perfect field of characteristic $p$, and let $\hat{G}$ be a formal group of height $n$ over $k$. There exists a morphism of connective $E_\infty$-rings

$$\rho : R^{\text{un}}_{\hat{G}} \to k$$

and a deformation of $\hat{G}$ along $\rho$ with the following properties

1. $R^{\text{un}}_{\hat{G}}$ is Noetherian, there is an induced surjection $\epsilon : \pi_0R^{\text{un}}_{\hat{G}} \to k$ and $R^{\text{un}}_{\hat{G}}$ is complete with respect to the ideal $\ker(\epsilon)$.

2. Let $A$ be a Noetherian ring $E_\infty$-ring for which the underlying ring homorphism $\epsilon : \pi_0(A) \to k$ is surjective and $A$ is complete with respect to the ideal $\ker(\epsilon)$. Then extension of scalars induces an equivalence

$$\text{Map}_{\text{CAlg}/k}(R^{\text{un}}_{\hat{G}}, A) \simeq \text{Def}_{\hat{G}}(A)$$

**Remark 8.19.** We can interpret this theorem as saying that the ring $R^{\text{un}}_{\hat{G}}$ corepresents the spectral formal moduli problem classifying deformations of $\hat{G}_0$. Of course this then means that there exists a universal deformation (this is non classical) over $R^{\text{un}}_{\hat{G}_0}$ which base-changes to any other deformation of $\hat{G}$.

**Remark 8.20.** This is actually proven in the setting of $p$-divisible groups over more general algebras over $k$. However, the formal group in question is the identity component of a $p$-divisible group over $k$; moreover, any deformation of the formal group will arise as the identity component of a deformation of the corresponding $p$-divisible group.(cf. [Lur18, Example 3.0.5])

Now fix an arbitrary formal group $\hat{G}$ of height $n$ over a finite field, and take its Cartier dual $\hat{\hat{G}} := \hat{G}^\vee$. From Construction 8.9 we see that this is an affine group scheme over $\text{Spec}k$.

**Theorem 8.21.** There exists a spectral scheme $\text{Fix}^{\text{un}}_{\hat{G}}$ defined over the $E_\infty$ ring $R^{\text{un}}_{\hat{G}}$, which lifts $\text{Fix}_{\hat{G}}$, giving rise to the following Cartesian diagram of spectral schemes:

\[
\begin{array}{ccc}
\text{Fix}_{\hat{G}} & \xrightarrow{\phi'} & \text{Fix}^{\text{un}}_{\hat{G}} \\
\phi' \downarrow & & \downarrow \phi \\
\text{Spec}(F_p) & \xrightarrow{p} & \text{Spec}(R^{\text{un}}_{\hat{G}})
\end{array}
\]

**Proof.** By Theorem 8.18 above, given a formal group $\hat{G}$ over a perfect field, the functor associating to an augmented ring $A \to k$ the groupoid of deformations $\text{Def}(A)$ is corepresented by the spectral (unoriented) deformation ring $R^{\text{un}}_{\hat{G}}$. Hence we obtain a map

$$R^{\text{un}}_{\hat{G}} \to F_p$$
of $E_\infty$-algebras over $k$. Over $\text{Spec}(R^\text{un}_{\hat{G}})$, one has the universal deformation $\hat{G}_{\text{un}}$. This base-changes along the above map to $\hat{G}$. By definition, this formal group is of the form $\text{coSpec}(\mathcal{H})$ for some $\mathcal{H} \in \text{Ab}(c\text{CAlg}^\text{sm}_{R^\text{un}_{\hat{G}}})$. Let

$$U : \text{Ab}(c\text{CAlg}^\text{sm}_{R^\text{un}_{\hat{G}}}) \to \text{CMon}^\text{op}(c\text{CAlg}^\text{sm}_{R^\text{un}_{\hat{G}}})$$

be the forgetful functor from abelian group objects to grouplike commutative monoid objects. We recall that the symmetric monoidal structure on cocommutative coalgebras is the cartesian one. Hence, grouplike commutative monoids will have the structure of $E_\infty$-algebras in the symmetric monoidal $\infty$-category of $R^\text{un}_{\hat{G}}$-modules. In particular we obtain a commutative and cocommutative bialgebra, so we can take $\text{Spec}(\mathcal{H})$; this will be a grouplike commutative monoid object in the category of affine spectral schemes over $\text{Spec}(R^\text{un}_{\hat{G}})$. Since Cartier duality commutes with base change (cf. Proposition 8.11), we conclude that $\text{Spec}(\mathcal{H})$ base-changes to $\text{Fix}_{\hat{G}}$ under the map $R^\text{un}_{\hat{G}} \to \tau_{\leq 0}R^\text{un}_{\hat{G}} \simeq \mathbb{Z}_p \to \mathbb{F}_p$.

**Example 8.22.** As a motivating example, let $\hat{G} = \hat{G}_m$, the formal multiplicative group over $\mathbb{F}_p$. As described in loc. cit, this formal group is Cartier dual to $\text{Fix} \subset \mathbb{W}_p$, the Frobenius fixed point subgroup scheme of the Witt vectors $\mathbb{W}_p(-)$. This lifts to $R^\text{un}_{\hat{G}_m}$, which in this case is none other than the $p$-complete sphere spectrum $S^p_\ast$. In fact, this object lifts to the sphere itself, by the discussion in [Lur18, Section 1.6]. Hence we obtain an abelian group object in the category $c\text{CAlg}_{S^p_\ast}$ of smooth coalgebras over the $p$-complete sphere. Taking the image of this along the forgetful functor

$$\text{Ab}(c\text{CAlg}_{S^p_\ast}) \to \text{CMon}(c\text{CAlg}_{S^p_\ast})$$

we obtain a grouplike commutative monoid $\mathcal{H}$ in $c\text{CAlg}_{S^p_\ast}$, namely a bialgebra in $p$-complete spectra. We set $\text{Spec}\mathcal{H} = \text{Fix}^\text{un}_{\hat{G}}$. Then base changing $\text{Fix}^3$ along the map

$$S^p_\ast \to \tau_{\leq 0}S^p_\ast \simeq \mathbb{Z}_p \to \mathbb{F}_p$$

recovers precisely the affine group scheme $\text{Fix}^\text{un}_{\hat{G}}$, by compatibility of Cartier duality with base change.

One may even go further and base-change to the orientation classifier (cf. [Lur18, Chapter 6])

$$S^p_\ast \simeq R^\text{un}_{\hat{G}_m} \to R^\text{or}_{\hat{G}_m} \simeq E_1$$

and recover height one Morava $E$-theory, a complex orientable spectrum. Moreover, in height one, Morava $E$-theory is the $p$-complete complex $K$-theory spectrum $KU^p_\ast$. Applying the above procedure, one obtains the Hopf algebra corresponding to

$$C_* (\mathbb{C}P^\infty, KU^p_\ast)$$

whose algebra structure is induced by the abelian group structure on $\mathbb{C}P^\infty$. We now take the spectrum of this bi-algebra; note that this is to be done in the nonconnective sense (see [Lur16]) as $KU^p_\ast$ is nonconnective. In any case, one obtains an affine nonconnective spectral group scheme

$$\text{Spec}(C_* (\mathbb{C}P^\infty, KU^p_\ast))$$

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which arises via base change $\text{Spec} KU_p \to \text{Spec} \hat{R}^\text{un}_G$. We summarize this discussion with the following diagram of pullback squares in the $\infty$-category of nonconnective spectral schemes:

![Diagram](attachment:image.png)

Note that we have the following factorization of the map

$$\hat{S}^p \to \hat{k}u^p \to \hat{KU}^p$$

through $p$-complete connective complex $K$-theory, so these lifts exists there as well.

### 9 Lifts of $\hat{G}$-Hochschild homology to the sphere

Let $\hat{G}$ be a height $n$ formal group over a perfect field $k$. We study a variant of $\hat{G}$-Hochschild homology which is more adapted to the tools of spectral algebraic geometry. Roughly speaking, we take mapping stacks in the setting of spectral algebraic geometry over $k$, instead of derived algebraic geometry.

**Definition 9.1.** Let $\hat{G}$ be a formal group. We define the $E_\infty$-$\hat{G}$ Hochschild homology to be the functor defined by

$$HH_{E_\infty}^\hat{G}(A) : \text{CAlg}^{cn}_k \to \text{CAlg}^{cn}_k, \quad HH_{E_\infty}^\hat{G}(A) = \text{Map}_{sStk_k}(B\hat{G}^\vee, \text{Spec}A),$$

where $\text{Map}_{sStk_k}(\cdot, \cdot)$ denotes the internal mapping object of the $\infty$-topos $sStk_k$.

It is not clear how the two notions of $\hat{G}$-Hochschild homology compare.

**Conjecture 9.2.** Let $\hat{G}$ be a formal group and $A$ a simplicial commutative $k$-algebra. Then there exists a natural equivalence

$$\theta(HH^\hat{G}(A)) \to HH_{E_\infty}^\hat{G}(\theta(A))$$

In other words, the underlying $E_\infty$ algebra of the $G$-Hochschild homology coincides with the $E_\infty - \hat{G}$-Hochschild homology of $A$, viewed as an $E_\infty$-algebra.

At least when $\hat{G} = \hat{\mathbb{G}}_m$, we know that this is true. In particular, this also recovers Hochschild homology, (relative to the base ring $k$)

**Proposition 9.3.** There is a natural equivalence

$$\text{HH}(A/k) \simeq HH_{E_\infty}^{\hat{\mathbb{G}}_m}(A)$$

of $E_\infty$ algebra spectra over $k$.

**Proof.** This is a modification of the argument of [MRT19]. We have the (underived) stack $\text{Fix} \simeq \hat{\mathbb{G}}_m^\vee$ and in particular a map

$$S^1 \to B\text{Fix} \simeq B\hat{\mathbb{G}}_m^\vee$$
This can also be interpreted, by Kan extension as a map of spectral stacks. This further induces a map between the mapping stacks

$$\text{Map}_{\text{Stk}}(S^1, X) \to \text{Map}_{\text{Stk}}(B\hat{G}_m, X)$$

Recall that all (connective) $E_\infty$ $k$-algebras may be expressed as a colimits of free algebras, and all colimits of free algebras may be expressed as colimits of the free algebra on one generator $k\{t\}$. This follows from \cite[Corollary 7.1.4.17]{Lur}, where it is shown that $\text{Free}(k)$ is a compact projective generator for $\text{CAlg}_k$. Hence, it is enough to test the above equivalence in the case where $X = \mathbb{A}^1_{sm}$; this is the ”smooth” affine line, i.e. $\mathbb{A}^1_{sm} = \text{Spec}(k)$, the spectrum of the free $E_\infty$ $k$-algebra on one generator. For this we check that there is an equivalence on functor of points

$$B \mapsto \text{Map}(B\hat{G}_m \times \mathbb{A}^1_{sm}, \mathbb{A}^1_{sm}) \simeq \text{Map}(S^1 \times B, \mathbb{A}^1_{sm})$$

for each $B \in \text{CAlg}^{cn}$. Each side may be computed as $\Omega^\infty(\pi_*O)$ where $\pi : B\hat{G}_m \times B \to \text{Spec}k$ denotes the structural morphism (where $G \in \{\mathbb{Z}, \hat{G}_m\}$). The result now follows from the following two facts:

- there is an equivalence of global sections $C^*(B\text{Fix}, O) \simeq k^S^1$ \cite[Proposition 3.3.2]{MRT19}.
- $B\text{Fix}$ is of finite cohomological dimension, cf. \cite[Proposition 3.3.7]{MRT19},

as we now obtain an equivalence on $B$-points

$$\Omega^\infty(\pi_*((B\hat{G}_m \times B)) \simeq \Omega^\infty(\pi_*((B\hat{G}_m \times B)) \otimes_k B) \simeq \Omega^\infty(\pi_*(S^1) \otimes_k B) \simeq \Omega^\infty(\pi_*(S^1 \times B)).$$

Note that the second equivalence follows from the finite cohomological dimension of $B\hat{G}_m$. Applying global sections $R\Gamma(\cdot, O)$ to this equivalence gives the desired equivalence of $E_\infty$-algebra spectra.

We show that $\hat{G}$-Hochschild homology possesses additional structure which is already seen at the level of ordinary Hochshchild homology. Recall that for an $E_\infty$ ring $R$, its topological Hochschild homology may be expressed as the tensor with the circle:

$$\text{THH}(R) \simeq S^1 \otimes_S R.$$

Thus, when applying the $\text{Spec}(\cdot)$ functor to the $\infty$-category of spectral schemes, this becomes a cotensor over $S^1$. In fact this coincides with the internal mapping object $\text{Map}(S^1, X)$, where $X = \text{Spec}R$. Furthermore, one has the the following base change property of topological Hochshild homology: for a map $R \to S$ of $E_\infty$ rings, there is a natural equivalence:

$$\text{THH}(A/R) \otimes_R S \simeq \text{THH}(A \otimes_R S/S)$$

In particular if $R$ is a commutative ring over $\mathbb{F}_p$ which admits a lift $\tilde{R}$ over the sphere spectrum, then one has an equivalence

$$\text{THH}(\tilde{R}) \otimes_S \mathbb{F}_p \simeq \text{HH}(R/\mathbb{F}_p)$$

This can be interpreted geometrically as an equivalence of spectral schemes

$$\text{Map}(S^1, \text{Spec}(\tilde{R})) \times \text{Spec} \mathbb{F}_p \simeq \text{Map}(S^1, \text{Spec}(R))$$

over $\text{Spec} \mathbb{F}_p$. We show that such a geometric lifting occurs in many instances in the setting of $\hat{G}$-Hochschild homology.
Let $\hat{G}$ be a height $n$ formal group over $F_p$ and let $R$ be an commutative $F_p$-algebra. Let $\hat{G}_{\text{un}}$ denote the universal deformation of $\hat{G}$, which is a formal group over the $R_{\hat{G}}^{\text{un}}$. As in section 8.3, we let $\text{Fix}^{\text{un}}_{\hat{G}}$ denote its Cartier dual over this $E_\infty$-ring.

**Theorem 9.5.** Let $\hat{G}$ be a height $n$ formal group over $F_p$ and let $X$ be an $F_p$ scheme. Suppose there exists a lift $\tilde{X}$ over the spectral deformation ring $R_{\hat{G}}^{\text{un}}$. Then there exists a homotopy pullback square of spectral algebraic stacks

$$
\begin{array}{ccc}
\text{Map}(B\text{Fix}_{\hat{G}}, X) & \xrightarrow{\phi'} & \text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X}) \\
\downarrow{\phi'} & & \downarrow{\phi} \\
\text{Spec}(F_p) & \xrightarrow{p} & \text{Spec}(R_{\hat{G}}^{\text{un}})
\end{array}
$$

displaying $\text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X})$ as a lift of $\text{Map}(B\text{Fix}_{\hat{G}}, X)$.

**Proof.** Given a map $p : X \to Y$ of spectral schemes, there is an induced morphism of $\infty$-topoi

$$p^* : \text{Shv}_{Y, \ell} \to \text{Shv}_{X, \ell}$$

This pullback functor is symmetric monoidal, and moreover behaves well with respect to the internal mapping objects. Now let $X = \text{Spec} F_p$ and $Y = \text{Spec} R_{\hat{G}}^{\text{un}}$ and let $p$ be the map induced by the universal property of the spectral deformation ring $R$. In this particular case, this means there will be an equivalence

$$p^* \text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X}) \simeq \text{Map}(p^* B\text{Fix}_{\hat{G}}^{\text{un}}, p^* \tilde{X}) \simeq \text{Map}(B\text{Fix}_{\hat{G}}, X)$$

since $\tilde{X} \times \text{Spec} F_p \simeq X$ and $p^* B\text{Fix}_{\hat{G}}^{\text{un}} \simeq B\text{Fix}_{\hat{G}}$. □

From this we conclude that the $\hat{G}$-Hochschild homology has a lift in the geometric sense, in that there is a spectral mapping stack over $\text{Spec} R_{\hat{G}}^{\text{un}}$ which base changes to $\text{Map}(B\hat{G}^{\vee}, X)$. We would like to conclude this at the level of global section $E_\infty$ algebras. This is not formal unless we have a more precise understanding of the regularity properties of $\text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, X)$ for an affine spectral scheme $X = \text{Spec} A$.

Indeed, there is a map

$$R\Gamma(\text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X}), \mathcal{O}) \otimes F_p \to R\Gamma(\text{Map}(p^* B\text{Fix}_{\hat{G}}^{\text{un}}, p^* \tilde{X}), \mathcal{O})$$

(9.6)

but it is not a priori clear that this is an equivalence. In particular, we have the following diagram of stable $\infty$-categories

$$
\begin{array}{ccc}
\text{Mod}_{R_{\hat{G}}^{\text{un}}} & \xrightarrow{\phi^*} & \text{Qcoh}(\text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X})) \\
\downarrow{p^*} & & \downarrow{p'^*} \\
\text{Mod}_{F_p} & \xrightarrow{\phi'^*} & \text{Qcoh}(\text{Map}(B\text{Fix}_{\hat{G}}^{\text{un}}, \tilde{X}))
\end{array}
$$

for which we would like to verify the Beck-Chevalley condition holds; i.e. that the following canonically defined map

$$\rho : p^* \circ \phi_\ast \to \phi'_\ast \circ p'^*$$
is an equivalence. Here $\phi_*$ and $\phi'_*$ are the right adjoints and may be thought of as global section functors. This construction applied to the structure sheaf $\mathcal{O}$ recovers the map $\mathcal{O}$.

This would follow from Proposition 2.1 upon knowing either that the spectral stack $\mathrm{Map}(B\overline{\mathrm{Fix}}^{un}_G, \tilde{X})$ is representable by a derived scheme or, more generally if it is of finite cohomological dimension. In fact it is the former:

**Theorem 9.7.** Let $\hat{G}$ be as above and let $X = \mathrm{Spec} A$ denote a spectral scheme. Then the mapping stack $\mathrm{Map}(B\overline{\mathrm{Fix}}^{un}_G, X)$ is representable by a spectral scheme.

**Proof.** This will be an application of the Artin-Lurie representability theorem, cf. [Lur16, Theorem 18.1.0.1]. Given spectral stacks $X,Y$, the derived spectral mapping stack $\mathrm{Map}(Y, X)$ is representable by a spectral scheme if and only if it is nilcomplete, infinitesimally cohesive and admits a cotangent complex and if the truncation $t_0(\mathrm{Map}(Y, X))$ is representable by a classical scheme.

By Proposition 5.10 of [HLP14] if $Y$ is of finite tor-amplitude and $X$ admits a cotangent complex, then so does the mapping stack $\mathrm{Map}(Y, X)$; in our case $X$ is an honest spectral scheme which has a cotangent complex. Note that the condition of being finite tor-amplitude is local on the source with respect to the flat topology (cf. [Lur16, Proposition 6.1.2.1]). Thus if there exists a flat cover $U \to Y$ such that the composition $U \to Y \to \mathrm{Spec} R$ is of finite tor amplitude, then $Y \to \mathrm{Spec} R$ itself has this property. Infinitesimal cohesion follows from [TV08, Lemma 2.2.6.13]. The following lemma takes care of nilcompleteness:

**Lemma 9.8.** Let $Y$ be a spectral stack over $\mathrm{Spec}(R)$ which may be written as a colimit of affine spectral schemes

$$Y \simeq \mathrm{colim} \ \mathrm{Spec} A_i$$

where each $A_i$ is flat over $R$ and let $X$ be a nilcomplete spectral stack. Then $\mathrm{Map}_{\mathrm{Stkr}}(Y, X)$ is nilcomplete.

**Proof.** The argument is similar to that of an analogous claim appearing in the proof of Theorem 2.2.6.11 in [TV08]. Let $Y$ be as above. Then

$$\mathrm{Map}(Y, X) \simeq \lim_i \mathrm{Map}(\mathrm{Spec} A_i, X)$$

and so it amounts to verify this for when $Y = \mathrm{Spec} A$ for $A_i$ flat. In this case we see that for $B \in \mathrm{CAlg}^{cn}$,

$$\mathrm{Map}(\mathrm{Spec} A, X)(B) \simeq X(A \otimes_R B).$$

The map

$$\mathrm{Map}(\mathrm{Spec} A, X)(B) \to \lim \mathrm{Map}(\mathrm{Spec} A, X)(\tau_{\leq n} B_n)$$

which we need to check is an equivalence now translates to a map

$$X(A \otimes_R B) \to X(\tau_{\leq n} B \otimes_R A)$$

(9.9)

We now use the flatness assumption on $A$. Using the general formula (cf. [Lur, Proposition 7.2.2.13]) in this case

$$\pi_n(A \otimes B) \simeq \mathrm{Tor}^0_{\pi_0(R)}(\pi_0 A, \pi_n B)$$

we conclude that $\tau_{\leq n}(A \otimes B) \simeq A \otimes \tau_{\leq n} B$. Thus (9.9) above becomes a map

$$X(A \otimes_R B) \to X(\tau_{\leq n}(B \otimes_R A))$$

which is an equivalence because $X$ was itself assumed to be nilcomplete. \qed
Finally we show that the truncation is an ordinary scheme. Note first of all that the truncation functor
\[ t_0 : SS tk \to Stk \]
preserves limits and colimits. It is induced from the Eilenberg Maclane functor
\[ H : CA lg^0 \to CA lg, \ A \mapsto HA \]
which is itself adjoint to the truncation functor on \( E_\infty \) rings. One sees that the truncation functor
\[ t_0 = H^* : SS tk \to Stk \]
will have as a right adjoint the functor
\[ \pi_0^* : Stk \to SS tk, \]
induced by the \( \pi_0 \) functor
\[ R \mapsto \pi_0 R \]
Thus it is right exact and preserves colimits. Hence if \( Y = BG \) for some spectral group scheme \( G \), then \( t_0 BG \cong Bt_0 G \). Now, one has the identification
\[ t_0 \text{Map}(Y, X) \cong \text{Map}(t_0 Y, t_0 X), \]
which in our situation becomes
\[ t_0 \text{Map}(B \text{Fix}_{un} \hat{G}, X) \cong \text{Map}(BG, t_0 X) \]
for some (classical) affine group scheme \( G \). Recall that the only classical maps between \( f : BG \to t_0 X \) between a classifying stack and a scheme \( t_0 X \) are the constant ones. Hence we conclude that the truncation of this spectral mapping stack is equivalent to the scheme \( t_0 X \), the truncation of \( X \). \[ \square \]

9.1 Topological Hochschild homology

As we saw, for a height \( n \) formal group \( \hat{G} \) over a finite field \( k \), there exists a lift \( \text{Fix}_{un} \hat{G} \) of the Cartier dual of \( \hat{G} \); this allows one to define a lift of \( \hat{G} \)-Hochschild homology. We show that when the formal group is \( \hat{G}_m \) this lift is precisely topological Hochschild homology, at least after \( p \)-completion, as one would expect. For the remainder of this section we let \( \hat{G} = \hat{G}_m \), the formal multiplicative group.

Let \( X \) be a fixed spectral stack. We remark that there exists an adjunction of \( \infty \)-topoi:
\[ S \rightleftarrows SS tk \]
where one has on the right hand side the \( \infty \)-category of spectral stacks over \( X \).

First, one has the following proposition; here we think of \( S^1 \) as a “constant stack induced by the adjunction
\[ \pi^* : S \rightleftarrows SS tk_R : \pi_* \]

**Proposition 9.10.** There exists a canonical map
\[ S^1 \to B \text{Fix}_{un} \hat{G} \]
of group objects in the \( \infty \)-category of spectral stacks over \( S_p \).
Proof. By [MRT19, Construction 3.3.1], there is a canonical map
\[ Z \to \text{Fix} \] (9.11)
in the category of fpqc abelian sheaves over $\text{Spec}\mathbb{Z}_p$. We claim that the (discrete) group scheme $\text{Fix}$ is none other than the truncation of the spectral group scheme
\[ \widehat{G}_\text{un}^\forall \to \text{Spec}\mathbb{S}_p \]
This follows from the fact that $\widehat{G}_\text{un}^\forall$ is flat over $\mathbb{S}_p$, as the corresponding Hopf algebra is flat. As a result, the base change of this spectral group scheme along the map
\[ \text{Spec}\mathbb{Z}_p \to \text{Spec}\mathbb{S}_p \]
is itself flat over $\mathbb{Z}_p$ and in particular is 0-truncated. By definition, this is $\text{Fix}$. Now, there is an adjunction
\[ i^* : S\text{Stk}_{\mathbb{S}_p} \rightleftarrows \text{Stk}_{\mathbb{Z}_p} : t_0 \]
against which the map (9.11) is lifted to a map
\[ \mathbb{Z} \to \widehat{G}_\text{un}^\forall. \]
in $S\text{Stk}_{\mathbb{S}_p}$. This will be a map of group objects, since the adjoint pair preserves the group structure. Delooping this, we obtain the desired map
\[ S^1 \simeq B\mathbb{Z} \to B\widehat{G}_\text{un}^\forall. \]

Let $X = \text{Spec}A$ be an affine spectral scheme. By taking mapping spaces, the above proposition furnishes a map
\[ \text{Map}(B\widehat{G}_\text{un}^\forall, X) \to \text{Map}(S^1, X); \]
applying global sections further begets a map
\[ f : \text{THH}(A) \to R\Gamma(\text{Map}(B\widehat{G}_\text{un}^\forall, X), \mathcal{O}) \]
of $E_\infty \mathbb{S}_p$-algebras.

**Theorem 9.12.** Let
\[ f_p : \text{THH}(A; \mathbb{Z}_p) \to R\Gamma(\text{Map}(B\widehat{G}_\text{un}^\forall, X), \mathcal{O})^\wedge_p \]
denote the $p$-completion of the above map. Then $f$ is an equivalence.

*Proof.* Since this is a map of $p$-complete spectra, it is enough to verify it is an equivalence upon tensoring with the Moore spectrum $\mathbb{S}_p/p$. In fact, since these are both connective spectra, one can go further and test this simply by tensoring with $\mathbb{F}_p$ (eg. by [Mao20, Corollary A.33]) Hence, we are reduced to showing that
\[ \text{THH}(A; \mathbb{Z}_p) \otimes_{\mathbb{S}_p} \mathbb{F}_p \to R\Gamma(\text{Map}(B\widehat{G}_\text{un}^\forall, X), \mathcal{O})^\wedge_p \otimes \mathbb{F}_p \]
is an equivalence of $E_\infty \mathbb{F}_p$-algebras. By generalities on topological Hochschild homology, we have the following identification of the left hand side:
\[ \text{THH}(A; \mathbb{Z}_p) \otimes_{\mathbb{S}_p} \mathbb{F}_p \simeq \text{HH}(A \otimes_{\mathbb{S}_p} \mathbb{F}_p / \mathbb{F}_p). \]
Now we can use Theorem 9.7 to identify the right hand side with the global sections of the following mapping stack

$$\text{Map}(B\hat{G}^\vee_{\text{un}}, X) \times \text{Spec} F_p \simeq \text{Map}(B\hat{G}^\vee_{\text{un}} \times \text{Spec} F_p, X \times \text{Spec} F_p)$$

By Proposition 9.3 this is precisely $\text{HH}(A \otimes_{S_p} F_p/F_p)$, whence the equivalence. □

10 Filtrations in the spectral setting

In Section 6 an interpretation of the HKR filtration on Hochshild homology was given in terms of a degeneration of $\hat{G}_{m}$ to $\hat{G}_{a}$. Moreover, this was expressed as an example of the deformation to the normal cone construction of section 5.

In Section 9, we further saw that these $\hat{G}$-Hochshild homology theories may be lifted beyond the integral setting. A natural question then arises: do the filtrations come along for the ride as well? Namely, does there exist a filtration on $\text{THH}(\hat{G})(-)$ which recovers upon base-changing along $R^\text{un}_{\hat{G}} \to k$, the filtered object corresponding to $\text{HH}(\hat{G})(-)$?

We will not seek to answer this question here. However we do give a reason why some negative results might be expected. As mentioned in the introduction, many of the constructions do work integrally. For example, one can talk about the deformation to the normal cone $\text{Def}_{A^1/\hat{G}_{m}}(\hat{G})$ of an arbitrary formal group over $\text{Spec} \mathbb{Z}$. If we apply this to $\hat{G}_{m}$ we obtain a degeneration of the formal multiplicative group to the formal additive group. We let $\text{Def}_{A^1/\hat{G}_{m}}(\hat{G}_{m})^\vee$ the Cartier dual, as in section 4. In [Toé19] the Cartier dual to $\hat{G}_{m}$ is described to be $\text{Spec}(\text{Int} \mathbb{Z})$, the spectrum of the ring of integer valued polynomials on $\mathbb{Z}$. Moreover it is shown that $B\text{Spec}(\text{Int} \mathbb{Z})$ is the affinization of $S^1$, hence one can recover (integral) Hochshild homology mapping out of this.

Let us suppose there exists a lift of $\text{Def}_{A^1/\hat{G}_{m}}(\hat{G}_{m})^\vee$ to the sphere spectrum, which we shall denote by $\text{Def}^{S}(\hat{G}_{m})^\vee$. This would allow us to define a mapping stack in the $\infty$-category $sStk_{A^1/\hat{G}_{m}}$ of spectral stacks over the spectral variant of $A^1/\hat{G}_{m}$. By the results of [Mon19], this comes equipped with a filtration on its cohomology, which we would like to think of as recovering topological Hochshild homology.

However, over the special fiber $BG_m \to A^1/\hat{G}_{m}$, we would expect that such a lift $\text{Def}^{S}(\hat{G}_{m})^\vee$ recovers the formal additive group $\hat{G}_{a}$. More precisely, we would get a formal group over the sphere spectrum $\hat{G} \to \text{Spec} S$ which pulls back to the formal additive group $G_{a}$ along the map $S \to \mathbb{Z}$. However, by [Lur18 Proposition 1.6.20], this can not happen. Indeed there it is shown that $\hat{G}_{a}$ does not belong to the essential image of $F\text{Group}(S) \to F\text{Group}(\mathbb{Z})$.

We summarize this discussion into the following proposition.

Proposition 10.1. There exists no lift of $\text{Def}_{A^1/\hat{G}_{m}}(\hat{G}_{m})$ over to the sphere spectrum. In particular, there exists no formal group $\hat{G}$ over $A^1/\hat{G}_{m}$, relative to $S$ such that $\hat{G} \times \text{Spec} \mathbb{Z} \simeq \text{Def}_{A^1/\hat{G}_{m}}(\hat{G}_{m})$.

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