Streami ng Submodular Maximization with Fairness Constraints

Yanhao Wang  
University of Helsinki  
yanhao.wang@helsinki.fi

Francesco Fabbri  
Universitat Pompeu Fabra  
francesco.fabbri@upf.edu

Michael Mathioudakis  
University of Helsinki  
michael.mathioudakis@helsinki.fi

ABSTRACT

We study the problem of extracting a small subset of representative items from a large data stream. Following the convention in many data mining and machine learning applications such as data summarization, recommender systems, and social network analysis, the problem is formulated as maximizing a monotone submodular function subject to a cardinality constraint – i.e., the size of the selected subset is restricted to be smaller than or equal to an input integer $k$. In this paper, we consider the problem with additional fairness constraints, which takes into account the group membership of data items and limits the number of items selected from each group to a given number. We propose efficient algorithms for this fairness-aware variant of the streaming submodular maximization problem. In particular, we first provide a $(\frac{1}{2} - \epsilon)$-approximation algorithm that requires $O(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon})$ passes over the stream for any constant $\epsilon > 0$. In addition, we design a single-pass streaming algorithm that has the same $(\frac{1}{2} - \epsilon)$ approximation ratio when unlimited buffer size and post-processing time is permitted.

CCS CONCEPTS

• Information systems → Data stream mining; • Theory of computation → Streaming, sublinear and near linear time algorithms.

KEYWORDS

algorithmic fairness, approximation algorithm, data summarization, submodular maximization

ACM Reference Format:

Yanhao Wang, Francesco Fabbri, and Michael Mathioudakis. 2020. Streaming Submodular Maximization with Fairness Constraints. In Proceedings of ACM Conference (Conference'17). ACM, New York, NY, USA, 7 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

A crucial task in modern data-driven applications, ranging from influence maximization [22] and group recommendation [33], to nonparametric learning [16] and coverage problems [32], is to extract concise summaries from large datasets. In all aforementioned applications, this task is formulated as selecting a subset of items to maximize a utility function that quantifies the “representativeness” (or “utility”) of the selected subset. Often times, the objective function satisfies submodularity, a notion of “diminishing returns” stating that adding an item to a smaller set always leads to a larger increase in utility than adding it to a larger set. Consequently, maximizing submodular set functions subject to cardinality constraints (i.e., when the size of the selected subset is restricted to a given $k$) is general enough to model many practical problems in data mining and machine learning.

The classic approach to the cardinality-constrained submodular maximization problem is the Greedy algorithm proposed by Nemhauser et al. [30], that achieves an approximation factor of $1 - \frac{1}{e}$ that is NP-hard to improve [13]. In many real-world scenarios, the data becomes too large to fit in memory or arrives incrementally at a high rate. In such cases, the Greedy algorithm becomes very inefficient because it requires $k$ repeated sequential scans over the whole dataset. Therefore, streaming algorithms for submodular maximization problems have received much attention recently [2, 3, 16, 20, 31]. They require only one or a few passes over the dataset, store a very small portion of items in memory, and compute a solution more efficiently than the Greedy algorithm at the expense of slightly lower quality.

Despite the extensive studies on streaming submodular maximization, unfortunately, it seems that none of the existing methods consider fairness of the subset extracted from a data stream. In fact, recent studies [5, 10, 11, 19] revealed that the data summaries automatically generated by algorithms might be biased with respect to sensitive attributes such as gender, race or ethnicity, and the biases in summaries could be passed to data-driven decision-making processes in education, recruitment, banking, and judiciary systems. Thus, it is necessary to introduce fairness constraints into submodular maximization problems so that the selected subset can fairly represent each sensitive attribute in the dataset. Towards this end, we consider that the data stream $V$ comprises $|V|$ groups $V_1, V_2, \ldots, V_I$ defined by some sensitive attribute. For example, groups may correspond to demographic groups according to sex or age. We define the fairness constraint by assigning a cardinality constraint $k_i$ to each group $V_i$ with $\sum_{i=1}^{I} k_i = k$. Then, our goal is to maximize the submodular objective function under the constraint that the selected subset contains $k_i$ items from $V_i$. Note that our fairness constraint can express different concepts of fairness by assigning different values of $k_1, k_2, \ldots, k_I$. For example, one can extract a summary that approximately represents the proportion of each group in the dataset by setting $k_i = \frac{|V_i|}{|V|}k$. As another example, one can also enforce a balanced representation of each group by setting $k_i = \frac{k}{I}$. 

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

Conference’17, July 2017, Washington, DC, USA
© 2020 Association for Computing Machinery.
ACM ISBN 978-x-xxxx-xxxx-x/YM/MM... $15.00
https://doi.org/10.1145/nnnnnnn.nnnnnnn
They also proposed streaming algorithms with approximation function efficiently in practice. However, it is still inefficient for data streams. In addition, the state-of-the-art streaming algorithms [6, 8, 14] for matroid-constrained submodular maximization are only $\frac{1}{2}$-approximate and do not provide solutions of the same quality as the Greedy algorithm efficiently in practice.

In this paper, we investigate the problem of streaming submodular maximization with fairness constraints. Our main contributions are summarized as follows.

- We first formally define the fair submodular maximization (FSM) problem and show its NP-hardness. We also describe the $\frac{1}{2}$-approximation Greedy algorithm for the FSM problem in the offline setting and discuss why it cannot work efficiently in data streams. (Section 3)
- We propose a multi-pass streaming algorithm MP-FSM for the FSM problem. Theoretically, MP-FSM requires $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ passes over the dataset, stores $O(k)$ items in memory, and has an approximation ratio of $\frac{1}{2} - \varepsilon$ for any constant $\varepsilon > 0$. (Section 4.1)
- We propose a single-pass streaming algorithm OP-FSM for the FSM problem, which requires only one pass over the data stream and offers the same approximation ratio as MP-FSM when an unbounded buffer size is permitted. We further discuss how to adapt OP-FSM heuristically to limit the buffer size to $O(k)$. (Sections 4.2 & 4.3)

2 RELATED WORK

There has been a large body of work on submodular optimization for its wide applications in data mining and machine learning, including influence maximization [22], facility location [26, 27], non-parametric learning [3, 16], and group recommendation [33]. We refer interested readers to [24] for a survey.

The line of research that is the most relevant to this work is streaming algorithms for submodular maximization. The seminal work of Fisher, Nemhauser, and Wolsey [15, 30] showed that the Greedy algorithm required $k$ passes over the dataset and gave approximation ratios of $1 - \frac{1}{e}$ and $\frac{1}{2}$ for maximizing monotone submodular functions with cardinality and matroid constraints, respectively. Then, a series of recent studies [3, 16, 20, 25] proposed multi- or single-pass streaming algorithms for maximizing monotone submodular functions subject to cardinality constraints with the same approximation ratio of $\frac{1}{2} - \varepsilon$. Furthermore, Norouzi-Fard et al. [31] showed that any single-pass streaming algorithm must use $\Omega(\frac{1}{\varepsilon})$ memory to achieve an approximation ratio of over $\frac{1}{2}$. They also proposed streaming algorithms with approximation factors better than $\frac{1}{2}$ by assuming that items arrive in random order or running in multiple passes. Alahb et al. [2] proposed a 0.2779-approximation streaming algorithm for the case of non-monotone submodular functions. Moreover, streaming submodular maximization was also studied in different models, e.g., the sliding-window model [12, 35] where only recent items within a time window are available for selection, the time-decay model [36] where the weights of old items decrease over time, and the deletion-robust [21, 28, 29] model where existing items might be removed from the stream. However, all above streaming algorithms are specific for the cardinality constraint and cannot be directly used for the fairness constraint (a case of matroid) in this paper. We note that fairness for submodular maximization was also studied in [21]. However, they considered that sensitive items might be removed from the dataset for ensuring fairness, which is different from the fairness constraints in this paper.

Chakrabarti and Kale [6] proposed a $\frac{1}{2}$-approximation single-pass streaming algorithms for maximizing monotone submodular functions with the intersections of $p$ matroid constraints. Chekuri et al. [8] generalized the algorithm in [6] to the case of non-monotone submodular functions. Both algorithms have a $\frac{1}{2}$-approximation for the FSM problem. Chan et al. [7] improved the approximation ratio for partition matroids to 0.3178 via randomization and relaxation. Feldman et al. [14] introduced a subsampling method to accelerate the algorithm of [6, 8] while preserving a $\frac{1}{2}$-approximation (in expectation). Very recently, Huang et al. [17] proposed an $O(\frac{1}{\varepsilon})$-pass $\frac{1}{\varepsilon}$-approximation algorithm for monotone submodular maximization with matroid constraints. The algorithms in [6, 8, 14, 17] are implemented as baselines and compared with our proposed algorithms in the experiments. We do not implement the algorithm in [7] since it is not scalable to large-scale data.

Another line of research related to this work is fair data summarization. Fair k-center for data summarization was studied in [9, 18, 23]. Celis et al. [5] proposed a determinantal point process (DPP) based sampling method for fair data summarization. Dash et al. [11] considered the fairness issue on summarizing user-generated textual content. Although these studies adopt similar fairness constraints to ours, the proposed methods cannot be applied to the FSM problem since the problems considered are different from submodular optimization.

3 PROBLEM DEFINITION

We consider the problem of selecting a subset of representative items from a dataset $V$ of size $n$. Our goal is to maximize a non-negative set function $f : 2^V \to \mathbb{R}_+$, where, for any subset $S \subseteq V$, $f(S)$ quantifies the utility of $S$, i.e., how well $S$ represents $V$ according to some objective. In many data mining and machine learning applications (e.g., [3, 4, 12, 16, 22, 26, 27, 33]), the utility functions satisfy an intuitive diminishing returns property called submodularity. Formally, we define the marginal gain $\Delta_f(v|S) := f(S \cup \{v\}) - f(S)$ as the increase in utility when an item $v$ is added to a set $S$. A set function $f$ is submodular iff $\Delta_f(v|A) \geq \Delta_f(v|B)$ for any $A \subseteq B \subseteq V$ and $v \in V \setminus B$. This means that adding an item $v$ to a set $A$ leads to at least as much utility gain as adding $v$ to a superset $B$ of $A$. Additionally, a submodular function $f$ is monotone iff $\Delta_f(v|S) \geq 0$ for any $S \subseteq V$ and $v \in V \setminus S$, i.e., adding a new item $v$ will not decrease the utility of $S$. In this work, we assume that function $f$ is both monotone and submodular. Moreover, following most existing works [3, 8, 12, 14, 16, 20, 26, 31], we assume that the
utility $f(S)$ of any set $S \subseteq V$ is given by a value oracle i.e., the value of $f(S)$ is retrieved in constant time.

Let us consider the following canonical optimization problem: given a monotone submodular set function $f$ and a dataset $V$, find a subset of size $k$ from $V$ that maximizes the function $f$, i.e.,

$$\max_{S \subseteq V} f(S) \quad \text{s.t.} \quad |S| = k.$$ \hfill (1)

The problem in Eq. 1 is referred to as the cardinality-constrained submodular maximization (CSM) problem and proven to be NP-hard [13] for many classes of submodular functions. The well-known greedy algorithm of Nemhauser et al. [30], which iteratively adds items with the maximum marginal gains to a solution, achieves a $(1 - 1/e)$-approximation for the CSM problem.

Now let us introduce fairness into the CSM problem. Suppose that the dataset $V$ is partitioned into $I$ (disjoint) groups, each of which corresponds to a sensitive class, and $V_i$ is the set of items from the $i$-th group in $V$ with $\bigcup_{i=1}^I V_i = V$. Then, for each group, we restrict that the solution $S$ must contain $k_i$ items from $V_i$ with $\sum_{i=1}^I k_i = k$. Formally, we define the fair submodular maximization (FSM) problem as follows:

$$S^* = \arg \max_{S \subseteq V} f(S) \quad \text{s.t.} \quad |S \cap V_i| = k_i, \forall i \in [I]$$ \hfill (2)

where $S^*$ and $OPT = f(S^*)$ denote the optimal solution and its utility. The values of $k_1, \ldots, k_I$ are given as input to the problem and can be determined according to different notions of fairness. For example, one can set $k_i = \frac{n_i}{|V|}$ where $n_i = |V_i|$ to obtain a proportional representation. As another example, an equal representation can be acquired by setting $k_i = \frac{k}{I}$ for all $i \in [I]$.

Algorithm 1: Greedy

| Input: | Dataset $V$, groups $V_1, \ldots, V_I \subseteq V$, size constraint $k \in \mathbb{N}$, group size constraints $k_1, \ldots, k_I \in \mathbb{N}$ |
| Output: | Solution $S$ for the FSM problem on $V$ |

1. Initialize the solution $S \leftarrow \emptyset$;
2. for $j \leftarrow 1, \ldots, k$ do
3.   for $i \leftarrow 1, \ldots, I$ do
4.     if $|S \cap V_i| < k_i$ then
5.       Pick an item $v^*_i \leftarrow \arg \max_{v \in V_i \setminus S} \Delta f(v | S)$;
6.       else
7.         $v^*_i \leftarrow \text{NULL}$;
8.   end if
9.   Select an item $v^* \leftarrow \arg \max_{i \in [I]} : v^*_i \neq \text{NULL} \; \Delta f(v^*_i | S)$;
10.  $S \leftarrow S \cup \{v^*\}$, $V \leftarrow V \setminus \{v^*\}$;
11. return $S$;

The FSM problem in Eq. 2 is still NP-hard because the CSM problem in Eq. 1 is its special case when $I = 1$. Nevertheless, a generalized Greedy algorithm first proposed in [15] provides a $\frac{1}{2}$-approximate solution for the FSM problem, since the fairness constraint we consider is a special case of the partition matroid constraint. The procedure of Greedy is described in Algorithm 1. Starting from $S = \emptyset$, it iteratively adds an item $v^*$ with the maximum utility gain $\Delta f(v^* | S)$ to the current solution $S$. To guarantee that solution $S$ satisfies the fairness constraint, it excludes all items of $V_i$ from consideration once there have been $k_i$ items from $V_i$ in $S$, i.e., $|S \cap V_i| = k_i$. The solution $S$ after $k$ iterations is returned for the FSM problem. The running time of Greedy is $O(nk)$ because it requires $k$ passes through the dataset and evaluates the value of $f$ at most $n$ times per pass for identifying $v^*$. Therefore, Greedy becomes very inefficient when the dataset size is large; even worse, Greedy cannot work in the one-pass streaming setting if the dataset does not fit in the memory. In this paper, we investigate the FSM problem in streaming settings.

4 OUR ALGORITHMS

In this section, we present our proposed algorithms for the fair submodular maximization problem in data streams. Firstly, we propose a multi-pass streaming algorithm called MP-FSM. For any parameter $\epsilon \in (0, 1)$, MP-FSM requires $O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$ passes over the dataset, stores $O(k)$ items in memory, and provides a $\frac{1}{2}(1-\epsilon)$-approximate solution for the FSM problem. Secondly, we propose a one-pass streaming algorithm called OP-FSM on the top of MP-FSM. OP-FSM has an approximation ratio of $\frac{1}{2} - \epsilon$ and sublinear update time per item. But it might keep $O(n)$ items in a buffer for post-processing in the worst case, and thus its space complexity is $O(n)$. Therefore, we further discuss how to manage the buffer when the memory space is limited and how the approximation ratio of OP-FSM is affected by the buffer size.

4.1 The Multi-Pass Streaming Algorithm

In this subsection, we present our multi-pass streaming algorithm called MP-FSM for the FSM problem. In general, MP-FSM adopts a thresholding framework similar to existing streaming algorithms for the CSM problem [3, 20, 25, 31]. The high-level idea of the thresholding framework is to process items in a data stream sequentially with a threshold $\tau$: For each item $v$ received from the stream, it will accept $v$ into a solution $S$ if $\Delta f(v | S)$ reaches $\tau$ and discard $v$ otherwise. But different from most thresholding algorithms [3, 20, 25] for the CSM problem, which run in only one pass and use a fixed threshold for each candidate solution, MP-FSM scans the dataset in multiple passes using a decreasing threshold to determine whether to include an item in each pass so that the solution has a constant approximation ratio while satisfying the fairness constraint. We present the detailed procedure of MP-FSM in Algorithm 2. In the first pass, it finds the item $v_{\text{max}}$ with the maximum utility $\delta_{\text{max}} = f(v_{\text{max}})$ among all items in the dataset $V$. The purpose of finding $v_{\text{max}}$ is to determine the range of thresholds to be used in subsequent passes. Meanwhile, it keeps a random sample $R$ of $k_i$ items uniformly from $V_i$ for each $i \in [I]$, which will be used for post-processing to guarantee that the solution satisfies the fairness constraint. Then, it initializes a solution $S$ containing only $v_{\text{max}}$ and a threshold $\tau = (1 - \epsilon) \cdot \delta_{\text{max}}$ for the second pass. After that, it scans the dataset $V$ sequentially in multiple passes. In each pass, it decreases the threshold $\tau$ by $(1 - \epsilon)$ times and adds an item $v \in V_i$ to the current solution $S$ if the marginal gain of $v$ w.r.t. $S$ reaches $\tau$ and there are less than $k_i$ items in $S$ from $V_i$. When the solution $S$ has contained $k_i$ items or the threshold $\tau$ has been decreased to be lower than $\frac{1}{2} \cdot \delta_{\text{max}}$, no more passes are needed. Finally, if the solution $S$ does not satisfy the fairness constraint, it will add items from random samples to $S$ for ensuring its validity.
Algorithm 2: MP-FSM

**Input**: Dataset $V$, groups $V_1, \ldots, V_l \subseteq V$, size constraint $k_i \in \mathbb{N}$, group size constraints $k_1, \ldots, k_l \in \mathbb{N}$, parameter $\epsilon \in (0, 1)$

**Output**: Solution $S$ for the FSM problem on $V$

1. $v_{\text{max}} \leftarrow \arg\max_{v \in V} f(v)$ and $\delta_{\text{max}} \leftarrow f(v_{\text{max}})$
2. Keep a random sample $k_l$ of $k_l$ items uniformly from $V_l$ for each $i \in [l]$ via reservoir sampling [34];
3. Pass 2 to $p$: Compute solution $S$ /
4. while $\tau > \frac{\epsilon}{1-\epsilon} \cdot \delta_{\text{max}}$
do
5. foreach item $o \in V \setminus S$
do
6. if $o \in V_i$ and $|S \cap V_i| < k_i$ and $\Delta_f(o|S) \geq \tau$
do
7. $S \leftarrow S \cup \{o\}$;
8. if $|S| = k$ then
9. break;
10. else
11. $\tau \leftarrow (1-\epsilon) \cdot \tau$;
do
/* Post processing: Ensure fairness */
do
12. while $\exists i \in [l] : |S \cap V_i| < k_i$
do
13. Add items in $R_l$ to $S$ until $|S \cap V_i| = k_i$;
14. return $S$;

In the following, we will provide some theoretical analyses for the MP-FSM algorithm. First of all, we give the approximation ratio of MP-FSM in Theorem 4.1. Then, we provide the complexity of MP-FSM in Theorem 4.2.

**Theorem 4.1.** For any parameter $\epsilon \in (0, 1)$, MP-FSM in Algorithm 2 is a $\frac{1}{1-\epsilon}$-approximation algorithm for the fair submodular maximization (FSM) problem.

**Proof.** Let $O$ be the optimal solution for the FSM problem on dataset $V$ and $O_i = O \cap V_i$ be the intersection of $O$ and $V_i$ for each $i \in [l]$. In addition, we consider that MP-FSM runs in $p$ passes and $S^{(i)}_j$ (1 $\leq j \leq p$) is the partial solution of MP-FSM after $j$ passes. For any subset $O_i$ of $O$ and the solution $S^{(p)}_j$ before post-processing, we have either (1) $|S^{(p)}_j \cap V_i| = k_i$ or (2) $|S^{(p)}_j \cap V_i| < k_i$. If $|S^{(p)}_j \cap V_i| = k_i$, we further consider two cases for the item $o \in O_i$, i.e., $\{1.1\) $o \in S^{(p)}_j$ and $\{1.2\) $o \not\in S^{(p)}_j$. In Case (1.1), we have $\Delta_f(o|S^{(p)}_j) = 0$. In Case (1.2), we compare $o$ with an item $s$ from $V_i$ added to the solution during the $j$-th pass. Since both $o$ and $s$ cannot be added in the $(j-1)$-th pass and $|S^{(j-1)} \cap V_i| < k_i$, it is safe to say that the marginal gains of $o$ and $s$ w.r.t. $S^{(j-1)}$ do not reach the threshold $\tau$ of the $(j-1)$-th pass. As $s$ is added in the $j$-th pass, we have $\Delta_f(s|S') = \tau_j$ where $S' \subseteq S^{(j)}$ is the partial solution before $s$ is added. Therefore, we have the following sequence of inequalities:

$$\Delta_f(o|S^{(p)}_j) \leq \Delta_f(o|S^{(j-1)}) < \tau_j - \tau = \frac{\tau_j}{1-\epsilon} \leq \frac{\Delta_f(s|S')}{1-\epsilon}$$

Then, if $|S^{(p)}_j \cap V_i| < k_i$, we also consider two cases for $o \in O_i$, i.e., (2.1) $o \in S^{(p)}_j$ and (2.2) $o \not\in S^{(p)}_j$. Case (2.1) is exactly the same as Case (1.1). In Case (2.2), we have:

$$\Delta_f(o|S^{(p)}_j) < \tau(p) \leq \frac{\epsilon}{k(1-\epsilon)} \cdot \delta_{\text{max}}$$

where $\tau(p)$ is the threshold of the $p$-th pass.

Next, we divide $O$ into two disjoint subsets $O'$ and $O''$ as follows: $O' = \bigcup_r O_r$ where $|S^{(p)}_j \cap V_r| = k_r$, i.e., all items in $O'$ from groups satisfying Case (1), and $O'' = O \setminus O'$, i.e., all items in $O$ from groups satisfying Case (2). We define an injection $\pi : O' \rightarrow S^{(p)}_j$ that maps each item in $O'$ to an item in $S^{(p)}_j$ as follows: If $o \in S^{(p)}_j$, then $\pi(o) = o$; Otherwise, $\pi(o)$ will be an arbitrary item $s \in S^{(p)}_j$ from the same group as $o$ and $s \not\in O'$. Based on the result of Eq. 3, we can get the following result for $O'$:

$$\sum_{o \in O'} \Delta_f(o|S^{(p)}_j) \leq \sum_{o \in O'} \frac{\Delta_f(o|S')}{1-\epsilon} \leq \frac{f(S^{(p)}_j)}{1-\epsilon}$$

Here, $S'$ denotes the partial solution before $\pi(o)$ is added and the second inequality is acquired from the fact that $f(S^{(p)}_j) = \sum_{o \in O'} f(S^{(p)}_j)$. Then based on the result of Eq. 4, we have the following result for $O''$:

$$\sum_{o \in O''} \Delta_f(o|S^{(p)}_j) \leq \frac{f(O \setminus S^{(p)}_j)}{1-\epsilon} \cdot \Delta_{\text{max}} \leq \frac{\epsilon}{1-\epsilon} \cdot f(S^{(p)}_j)$$

because $|O''| < k$ and $\Delta_{\text{max}} \leq f(S^{(p)}_j)$. Finally, we have the following sequence of inequalities from Eqs. 5 and 6:

$$f(O \setminus S^{(p)}_j) = f(S^{(p)}_j) = \sum_{o \in O''} \Delta_f(o|S^{(p)}_j) + \sum_{o \in O'} \Delta_f(o|S^{(p)}_j) \leq \frac{1}{1-\epsilon} \cdot f(S^{(p)}_j) + \frac{\epsilon}{1-\epsilon} \cdot f(S^{(p)}_j) = \frac{1}{1-\epsilon} \cdot f(S^{(p)}_j)$$

Since $\text{OPT} = f(O) \leq f(O \setminus S^{(p)}_j)$, we have $\text{OPT} \leq f(S^{(p)}_j) + \frac{1-\epsilon}{1-\epsilon} \cdot f(S^{(p)}_j)$. Finally, we conclude the proof from the fact that $f(S) \geq f(S^{(p)}_j) \geq \frac{1}{1-\epsilon} \cdot \text{OPT}$. $\square$

**Theorem 4.2.** MP-FSM in Algorithm 2 requires $O\left(\frac{k}{\epsilon} \cdot \log \frac{1}{\epsilon} \right)$ passes over the dataset $V$, stores at most $O(k)$ items, and has $O\left(\frac{k}{\epsilon} \cdot \log \frac{1}{\epsilon} \right)$ time complexity.

**Proof.** First of all, since the threshold $\tau$ is decreased by $1-\epsilon$ times after one pass, $\tau^{(2)} = (1-\epsilon) \cdot \Delta_{\text{max}}$, and $\tau^{(p)} \geq \frac{1}{p} \cdot \Delta_{\text{max}}$ we get $(1-\epsilon)^{p-1} \geq \frac{1}{p}$. Taking the logarithm on both sides of the last inequality and the Taylor expansion of $\log(1-\epsilon)$, we have $p \leq 1 + \frac{1}{\log(1-\epsilon)} \cdot \log \frac{1}{\epsilon} \leq 1 + \frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}$ and thus the number $p$ of passes in MP-FSM is $O\left(\frac{k}{\epsilon} \cdot \log \frac{1}{\epsilon} \right)$. Furthermore, MP-FSM only needs to store items in the current solution and random samples for post-processing, both of which contain at most $k$ items. Hence, MP-FSM stores at most $O(k)$ items. Finally, because MP-FSM evaluates the value of function $f$ at most $n$ times per pass, the total number of function evaluations is $O\left(\frac{k}{\epsilon} \cdot \log \frac{1}{\epsilon} \right)$. $\square$

### 4.2 The One-Pass Streaming Algorithm

In this subsection, we present our one-pass streaming algorithm called OP-FSM for the FSM problem. Generally, OP-FSM is based on a similar thresholding framework to MP-FSM. But several extensions are required so that OP-FSM can provide an approximate solution in only one pass over the dataset. First of all, because $v_{\text{max}}$...
and $\delta_{\text{max}}$ are unknown in advance, OP-FSM should keep track of them from received items, dynamically decide a sequence of thresholds based on the observed $\delta_{\text{max}}$, and maintain a candidate solution for each threshold (instead of keeping only one solution over multiple passes in MP-FSM). Furthermore, as only one pass is permitted, an item will be missing forever once it is discarded.

To provide a theoretical guarantee for the quality of solutions in adversary settings, OP-FSM keeps a buffer to store items that are neither included into solutions nor safely discarded. Finally, whenever a solution is requested during the stream, OP-RSM will reconsider the buffered items for post-processing by trying to add them greedily to candidate solutions. We will show that OP-FSM has an approximation ratio of $\frac{1}{e} - \epsilon$ with a judicious choice of parameters when the buffer size is unlimited.

**Algorithm 3: OP-FSM**

**Input:** Data stream $V$, groups $V_1, \ldots, V_j \subseteq V$, size constraint $k \in N$, size constraint $k_1, \ldots, k_t \in N$, parameters $\alpha, \beta \in (0, 1)$

**Output:** Solution $S$ for the FSM problem on $V$

1. $\delta_{\text{max}} \leftarrow 0$, $LB \leftarrow 0$, $B \leftarrow \emptyset$, and $R_i \leftarrow \emptyset$ for each $i \in [I]$; /* Stream processing */

2. **foreach** item $v \in V_i$ received from $V$ **do**
   3. $\delta_{\text{max}} \leftarrow \max(\delta_{\text{max}}, f(\{v\}))$;
   4. Update $R_i$ w.r.t. $v$ using reservoir sampling [34];
   5. $T \leftarrow \{(1 + \alpha)^j | j \in \mathbb{Z}, \max(\delta_{\text{max}}/LB, 2k) \leq (1 + \alpha)^j \leq \delta_{\text{max}}\}$;
   6. Discard $S_f$ for all $r \in T$;
   7. Initialize $S_f \leftarrow \emptyset$ for each $r$ newly added to $T$;

8. **foreach** $r \in T$ **do**
   9. **if** $|S_f \cap V_i| < k_i$ **then**
      10. **if** $\Delta_f(v|S_f) \geq \tau$ **then**
         11. $S_f \leftarrow S_f \cup \{v\}$;
      12. **else if** $\Delta_f(v|S_f) \geq \frac{\beta \cdot LB}{\alpha}$ **then**
         13. $B \leftarrow B \cup \{v\}$;

14. $LB \leftarrow \max_{r \in T} f(S_f)$; /* Post processing */

15. **if** $r'$ be the smallest $r \in T$ such that $|S_f \cap V_i| < k_i$ for each $i \in [I]$ or the largest $r \in T$ if there exists some $i$ such that $|S_f \cap V_i| = k_i$ for every $S_i$ **then**

16. **else if** $r \leq r'$ **in** $T$ **do**
   17. Run GREEDY in Algorithm 1 to add items from buffer $B$
   18. and samples $R_i (i \in [I])$ to $S_k$ until $|S_k| = k$;

19. **return** $S$ as $\arg\max_{r \in T} f(S_f)$;

The detailed procedure of OP-FSM is presented in Algorithm 3. Here, $\delta_{\text{max}}$ keeps the maximum utility of any single item among all items received so far, $LB$ maintains the lower bound of $\text{OPT}$ estimated from candidate solutions, $B$ stores the buffered items, and $R_i$ is a set of $k_i$ items sampled uniformly from all received items in $V_i$. In addition, two parameters $\alpha$ and $\beta$ are used to control the number of candidate solutions and the number of buffered items, respectively. In general, larger $\alpha$ means bigger gaps between neighboring thresholds and thus fewer candidates while larger $\beta$ means more rigorous conditions for adding an item to the buffer and naturally smaller buffer sizes. The procedure for stream processing of OP-FSM is given in Line 2–14. For each item $v \in V_i$ received from $V$, it first updates the value of $\delta_{\text{max}}$ and the sample $R_i$ w.r.t. $v$ accordingly. Then, it maintains a sequence $T$ of thresholds picked from a geometric progression $(1 + \alpha)^j | j \in \mathbb{Z}$ and a candidate solution $S_r$ for each $r \in T$. Specifically, the upper bound of the threshold is set to $\delta_{\text{max}}$ since $S_f = \emptyset$ for any $r > \delta_{\text{max}}$; the lower bound of the threshold is set to $\frac{\text{max}(\delta_{\text{max}}, LB)}{2k}$ because any candidate with a threshold lower than $\frac{\text{OPT}}{2k}$ is safe to be discarded (as shown in our theoretical analysis later) and $\text{max}(\delta_{\text{max}}, LB)$ is the lower bound of $\text{OPT}$. After maintaining the thresholds and their corresponding candidates, OP-FSM evaluates the marginal gain $\Delta_f(v|S_f)$ of $v$ for each candidate $S_f$ with threshold $r \in T$. Similar to MP-FSM, it will add $v$ to $S_f$ if $\Delta_f(v|S_f)$ reaches $r$ and $|S_f \cap V_i| < k_i$; Additionally, it will add $v$ to the buffer $B$ if $\Delta_f(v|S_f)$ is at least $\frac{\text{OPT}}{2k}$ but less than $r$. Finally, LB is updated to the utility value of the best solution found so far. The procedure for post-processing of OP-FSM is shown in Lines 15–17. It first finds out the smallest $r \in T$ such that $|S_f \cap V_i| < k_i$ for each $i \in [I]$ as $r^*$. If such $r$ does not exist, i.e., there exists some $i$ such that $|S_f \cap V_i| = k_i$ for every $S_f$, the largest $r \in T$ is used as $r^*$. For each $r \leq r^* \in T$, it runs GREEDY in Algorithm 1 to reevaluate the items in $B$ and $R_i (i \in [I])$ and add them to $S_k$ until $|S_k| = k$. Lastly, the candidate solution with the maximum utility after post-processing is returned as the final solution.

Next, we will provide the theoretical analyses for the OP-FSM algorithm. In Lemma 4.3, we first analyze the special cases when the solution returned after stream processing (without post-processing) can achieve a good approximation ratio.

**Lemma 4.3.** Assume that $\frac{\text{OPT}}{2k} \leq \tau \leq \frac{(1 + \alpha) \cdot \text{OPT}}{2k}$. If either $|S_f| = k$ or $|S_f \cap V_i| < k_i$ for all $i \in [I]$, then $f(S_f) \geq \frac{1}{e} \cdot \text{OPT}$.

**Proof.** First of all, when $|S_f| = k$, it holds that $f(S_f) \geq k \cdot \frac{\text{OPT}}{2k} = \frac{1}{2} \cdot \text{OPT} \geq \frac{1}{3} \cdot \text{OPT}$. Then, when $|S_f \cap V_i| < k_i$ for all $i \in [I]$, we have $\Delta_f(v|S_f) < \tau$ for any $v \in V \setminus S_f$. Let $O$ be the optimal solution for the FSM problem on $V$. We can acquire that

$$f(O \cup S_f) - f(S_f) \leq \sum_{v \in O \setminus S_f} \Delta_f(v|S_f) < k \tau$$

$$\leq k \cdot \frac{(1 + \alpha) \cdot \text{OPT}}{2k} = (1 + \alpha) \cdot \frac{\text{OPT}}{2}$$

Therefore, we have $f(S_f) \geq f(O \cup S_f) - (1 + \alpha) \cdot \frac{\text{OPT}}{2k} \geq \text{OPT} - (1 + \alpha) \cdot \frac{\text{OPT}}{2k} = \text{OPT} - \frac{\text{OPT}}{2k} \cdot \frac{(1 + \alpha) \cdot \text{OPT}}{2}$. Finally, we conclude the proof by considering both cases collectively.

**□**

Lemma 4.3 is useful because one of the thresholds $r \in T$ of OP-FSM (Line 5 of Algorithm 3) must fall in the range $\left[\frac{\text{OPT}}{2k}, \frac{(1 + \alpha) \cdot \text{OPT}}{2} \right]$. This is because $T$ is a geometric progression with a scale factor of $(1 + \alpha)$ and spans the range $[\frac{\text{max}(\delta_{\text{max}}, LB)}{2k}, \delta_{\text{max}}]$, as well as $\text{max}(\delta_{\text{max}}, LB) \leq \text{OPT} \leq k \cdot \delta_{\text{max}}$.

This implies that, if the remaining conditions of Lemma 4.3 were satisfied as well, the solution of OP-FSM after stream processing would have the strong approximation guarantee given by Lemma 4.3. Intuitively, this would be the case when the utility distribution of items was generally "balanced" among groups, so that either all
or none of the group budgets would be exhausted by the end of the stream processing procedure. However, in case that the utilities are highly imbalanced among groups, the approximation ratio would become significantly lower. On the other hand, OP-FSM might miss high-utility items in some groups from the stream because the threshold is too low and the solution has been filled by earlier items with lower utilities in these groups. On the other hand, OP-FSM cannot include enough items from the other groups because the threshold is too high for them. Note that, for \( \frac{OPT}{2k} \leq \tau \leq \frac{(1+\varepsilon)OPT}{2k} \), Lemma 4.3 allows the approximation factor of \( S_r \) to drop to \( \min_{i \in [k]} \frac{k_i \tau}{\min_{i \in [k]} k_i} \geq \frac{1}{\tau} \geq \frac{1}{\tau} \) when some group budgets are exhausted but the others are not.

Therefore, we further include the buffer and post-processing procedures in OP-FSM so that it still achieves a constant approximation independent of \( k \) for an arbitrary group size constraint. In Lemma 4.4, we analyze the approximation ratio of the solution returned by OP-FSM after post-processing.

**Lemma 4.4.** Let \( r' \) be chosen according to Line 15 of Algorithm 3. It holds that \( f(S_r') \geq \frac{1-\beta}{\beta}OPT \) after post-processing.

**Proof.** We consider two cases separately: (1) \( |S_r \cap V_i| < k_i \) for each \( i \in [l] \) or (2) \( r' \) is the maximum in \( T \). In Case (1), we divide the items in the optimal solution \( O \) into three disjoint sub-sets: \( O_1 = O \cap S_r \), i.e., items included in \( S_r \) during stream and post processing; \( O_2 = O \cap (B \setminus S_r) \), i.e., items stored in the buffer but not added to \( S_r \); \( O_3 = O \cap (V \setminus (B \cup S_r)) \), i.e., items discarded during stream processing. For each \( o \in O_2 \), we can always find an item \( s \in S_r \) from the same group as \( o \) such that \( \Delta_f(s|S^r) \geq \Delta_f(o|S^r) \geq \Delta_f(o|S_r) \) where \( S^r \subseteq S_r \) is the partial solution when \( s \) is added. This is because \( GREEDY \) always picks the item with the maximum marginal gain in each group. In addition, for each \( o \in O_3 \), we have \( \Delta_f(o|S_r) \leq \frac{1}{k} \leq \frac{1}{k} \).

Therefore, we have

\[
\begin{align*}
    f(O \cup S_r') - f(S_r') &\leq \sum_{o \in O \setminus S_r} \Delta_f(o|S_r) \\
    &= \sum_{o \in O_1} \Delta_f(o|S_r) + \sum_{o \in O_2} \Delta_f(o|S_r) \\
    &\leq \sum_{s \in S_r} \Delta_f(s|S^r) + \beta \cdot OPT \\
    &= f(S_r') + \beta \cdot OPT
\end{align*}
\]

where \( S^r \) is the partial solution when \( s \) is added to \( S_r \). And we conclude that \( f(S_r') \geq \frac{1-\beta}{\beta}OPT \) from the above inequalities. In Case (2), we have \( r' \) is the maximum in \( T \) and thus \( r' \in [\frac{2m}{lb}, \delta_{\max}] \).

We divide \( O \) into \( O_1, O_2, O_3 \) in the same way as Case (1). It is easy to see that the results for \( O_1 \) and \( O_2 \) are exactly the same as Case (1). The only difference is that there may exist some items in \( O_2 \) rejected by \( S_r \) because their groups have been filled in \( S_r \). For any \( o \in O_2 \), we have \( \Delta_f(o|S_r) \leq \delta_{\max} \leq (1+\varepsilon) \cdot r' \leq (1+\varepsilon) \cdot \Delta_f(s|S^r) \) where \( s \) is from the same group as \( o \) and \( S^r \) is the partial solution when \( s \) is added. Accordingly, we can get \( OPT - f(S_r) \leq (1 + \varepsilon) \cdot f(S_r') + \beta \cdot OPT \) and thus \( f(S_r') \geq \frac{1-\beta}{\beta}OPT \) in both cases. \( \square \)

Considering the results of Lemmas 4.3 and 4.4 collectively, we have Theorem 4.5 for the approximation ratio of OP-FSM. Then, we analyze the complexity of OP-FSM in Theorem 4.6.

**Theorem 4.5.** Assuming that \( \alpha, \beta = O(\varepsilon) \), OP-FSM in Algorithm 3 is a \( (\frac{1}{2} - \varepsilon) \)-approximation algorithm for the FSM problem.

**Theorem 4.6.** Assuming that \( \alpha, \beta = O(\varepsilon) \), OP-FSM in Algorithm 3 requires only one pass over the data stream \( V \), stores at most \( O\left(\frac{k \log k}{\varepsilon}\right) \) items, has \( O\left(\frac{k \log k}{\varepsilon}\right) \) update time per item for stream processing, and takes \( O\left(\frac{k \log k}{\varepsilon} \cdot (|B| + k)\right) \) time for post-processing.

**Proof.** First of all, the number \( |T| \) of thresholds maintained at any time satisfies that \( (1+\varepsilon)|T| \leq 2k \). Using the Taylor expansion of \( \log(1+\varepsilon) \leq \frac{\log 2k}{\log(1+\varepsilon)} \), we have \( |T| \leq \frac{\log 2k}{\log(1+\varepsilon)} \leq \frac{\log 2k}{\alpha} = O\left(\frac{k \log k}{\varepsilon}\right) \). Therefore, the number of function evaluations per time is \( O\left(\frac{k \log k}{\varepsilon}\right) \). Moreover, since each candidate solution contains at most \( k \) items, the total number of items stored in OP-FSM is \( O\left(\frac{k \log k}{\varepsilon} \cdot (|B| + k)\right) \). Finally, for each candidate solution \( S_r \), the post-processing procedure runs in \( (k - |S_r|) \) iterations and processes at most \( |B| + k \) items at each iteration. Therefore, it takes at most \( O\left(\frac{k \log k}{\varepsilon} \cdot (|B| + k)\right) \) time for post-processing. \( \square \)

### 4.3 OP-FSM with Bounded Buffer Size

From the above results, we can see that OP-FSM may store \( O(n) \) items in the buffer and take \( O\left(\frac{nk \log k}{\varepsilon}\right) \) time for post-processing in the worst case. In practice, a streaming algorithm is often required to process massive data streams with limited time and memory (sublinear to or independent of \( n \)). And it is not favorable for OP-FSM to store an unlimited number of items in the buffer \( B \). Therefore, we propose a simple strategy for OP-FSM to manage the buffered items so that he buffer size is always bounded at the expense of a lower approximation ratio in adversary settings.

We consider that the maximum buffer size is restricted to \( k' = O(k) \) and extra items should be dropped from \( B \) once its size exceeds \( k' \). The following rules are considered for buffer management. Firstly, since \( LB \) increases over time, it is safe to drop at any time during stream processing any item already in the buffer whose marginal gain is lower than \( \frac{LB}{k} \) for the current value of \( LB \), without affecting the theoretical guarantee. Secondly, to avoid duplications, if an item is added to some candidate solution but needs to be buffered for another, it is not necessary to add this item to the buffer because the algorithm has already stored this item. In this case, items stored in both candidates and the buffer should be considered for post-processing. Thirdly, as the buffer is used for storing high-utility items for post-processing, the items with larger marginal gains should have higher priorities to be stored. If the buffer size still exceeds \( k' \) after (safely) dropping items using the first two rules, it is required to sort the items in \( B \) in a descending order of marginal gain \( \delta(o) = \max_{r \in R} \Delta_f(o|S_r) \) and drop the item \( o \) with the lowest \( \delta(o) \) until \( |B| = k' \). Finally, considering the fairness constraint, it will not drop any item \( o \) from \( V_i \) anymore if \( |B \cap V_i| \leq k_i \) even if \( \delta(o) \) is among the lowest marginal gains. In this case, it will drop the item with the lowest \( \delta(o) \) from \( V_i \) with \( |B \cap V_i| > k_i \) instead.
The first two rules above have no effect on the theoretical guarantee on the approximation ratio of OP-FSM. The latter two rules will lower the approximation ratio of OP-FSM in some cases. Let $\phi'$ be the item with the largest $\delta(v)$ among all items dropped due to Rule (3) or (4). We can see that the approximation ratio drops to $\frac{1-2/e}{\phi'}$ where $\phi' = \frac{k \cdot \delta(v)}{lb}$. Once $\phi' \geq 1 - \frac{1}{k}$, the approximation ratio will become $\frac{1}{k}$ in the worst case.

5 CONCLUSION

We studied the problem of extracting a small subset of representative items from a large data stream. Following the convention in many data mining and machine learning applications such as data summarization, recommender systems, and social network analysis, the problem was formulated as maximizing a monotone submodular function subject to a cardinality constraint—i.e., the size of the selected subset was restricted to be smaller than or equal to an integer $k$. In this paper, we considered the problem with additional fairness constraints, which took into account the group membership of data items and limited the number of items selected from each group to a given number. We proposed efficient algorithms for this fairness-aware variant of the streaming submodular maximization problem. In particular, we first provided a $(\frac{1}{2} - \epsilon)$-approximation algorithm that required $O\left(\frac{1}{\epsilon} \cdot \log S\right)$ passes over the stream for any constant $\epsilon > 0$. In addition, we designed a single-pass streaming algorithm that had the same $(\frac{1}{2} - \epsilon)$ approximation ratio when unlimited buffer size and post-processing time were permitted.

REFERENCES

[1] Zeinab Abbassi, Yahab S. Mirrokh, and Maryur Thakur. 2013. Diversity maximization under matroid constraints. In KDD. 32–40.

[2] Aaoor Alafu, Alina Ene, Moran Feldman, Huy L. Nguyen, and Andrew Suh. 2020. Optimal Streaming Algorithms for Submodular Maximization with Cardinality Constraints. In ICALP. 6:1–6:19.

[3] Ashwinkumar Badanidhvuru, Baharan Mirzasoleiman, Amin Karbasi, and Andreas Krause. 2014. Streaming Submodular Maximization: Massive Data Summarization on the Fly. In KDD. 671–680.

[4] L. Elisa Celis, Lingxiao Huang, and Nisheeth K. Vishnoi. 2018. Multiturner Voting with Fairness Constraints. In JÇAA. 144–151.

[5] L. Elisa Celis, Vijay Keswani, Damian Straszak, Amit Deshpande, Tarun Kathuria, and Nisheeth K. Vishnoi. 2018. Fair and Diverse DPP-Based Data Summarization. In ICML. 715–724.

[6] Amit Chakrabarti and Sagar Kale. 2015. Submodular maximization meets streaming: matchings, matroids, and more. Math. Program. 154, 1-2 (2015), 225–247.

[7] T-H. H. Hubert Chan, Zhiyi Huang, Shaoqin H.-C. Jiang, Ning Kang, and Zhi-hao Gavin Tang. 2017. Online Submodular Maximization with Free Dispersion: Randomization Beats 1/k; for Partition Matroids. In SODA. 1204–1223.

[8] Chandra Chekuri, Shalmoli Gupta, and Kent Quanrud. 2015. Streaming Algorithms for Submodular Function Maximization. In ICALP. 318–330.

[9] Ashish Chiplunkar, Sagar Kale, and Sivaramakrishnan Narasajam Ramamorthy. 2020. How to Solve Fair k-Center in Massive Data Models. In IJCAI. 6887–6896.

[10] Amin Karbasi. 2019. Submodular Streaming in All Its Glory: Tight Approximation, Minimum Memory and Low Adaptive Complexity. In IJCAI. 3311–3320.

[11] Ehsan Kazemi, Morteza Zadimoghaddam, Silvano Lattanzi, and Amin Karbasi. 2019. Submodular Streaming in All Its Glory: Tight Approximation, Minimum Memory and Low Adaptive Complexity. In ICML. 3311–3320.

[12] Marshall L. Fisher, George L. Nemhauser, and Laurence A. Wolsey. 1978. An analysis of approximations for maximizing submodular set functions—II. In Polynomailly Combinatorics, Michel L. Balinski and Alan J. Hoffman (Eds.). Springer Berlin Heidelberg. 73–87.

[13] Ryan Gomes and Andreas Krause. 2010. Budgeted Nonparametric Learning from Data Streams. In ICML. 391–398.

[14] Matthew Jones, Huy L Nguyen, and Thy Nguyen. 2020. Fair k-Centers via Maximum Matching. In IJCAI. 7460–7469.

[15] Marshall L. Fisher, George L. Nemhauser, and Laurence A. Wolsey. 1978. An analysis of approximations for maximizing submodular set functions—II. In Polynomial Combinatorics, Michel L. Balinski and Alan J. Hoffman (Eds.). Springer Berlin Heidelberg. 73–87.

[16] Ryan Gomes and Andreas Krause. 2010. Budgeted Nonparametric Learning from Data Streams. In ICML. 391–398.

[17] Chien-Chung Huang, Theophile Thiery, and Justin Ward. 2020. Improved Multi-Swarm Submodular Maximization with Matroid Constraints. In APPROX/RANDOM. 62:1–62:19.

[18] Matthew Jones, Huy L Nguyen, and Thy Nguyen. 2020. Fair k-Centers via Maximum Matching. In IJCAI. 7460–7469.