Analyzing propagation of low-frequency dissipative oscillations in the upper atmosphere

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Abstract
At a horizontally homogeneous isothermal atmosphere approximation, we derive an ordinary six-order differential equation describing linear disturbances with consideration for heat conductivity and viscosity of medium. The wave problem may be solved analytically by representing the solution through generalized hypergeometric functions only at a nonviscous heat-conducting isothermal atmosphere approximation. The analytical solution may be used to qualitatively analyze propagation of acoustic and internal gravity waves (AGWs) in the real atmosphere: a) to classify waves of different frequencies and horizontal scales according to a degree of attenuation and thus according to their ability to appear in observations and in general dynamics of the upper atmosphere; b) to describe variations in amplitude and phase characteristics of disturbances propagating in a height region with dominant dissipation; c) to analyze applicability of quasi-classical wave description to a medium with exponentially growing dissipation. In this paper, we also present wave and quasi-classical methods for deriving waveguide solutions (dissipative ones corresponding to a range of internal gravity waves (IGWs)) with consideration of wave leakage into the upper atmosphere. We propose a qualitative scheme which formally connects the wave leakage solution to the wave solution in the upper dissipative atmosphere. Spatial and frequency characteristics of dissipative disturbances generated by a waveguide leakage effect in the upper atmosphere are demonstrated to agree well with observed characteristics of middle-scale traveling ionospheric disturbances (TIDs).

Keywords: Dissipative waves, upper atmosphere, IGW waveguide, TIDs

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1. Introduction

Acoustic gravity waves (AGWs) in the upper atmosphere cannot be described without an accurate account of dissipative characteristics of the environment. The dissipation effect on structure and propagation of disturbances in the upper atmosphere plays a very special role due to an exponential growth of its kinematic value with height caused by an exponential decrease in density. There is nothing to restrain the growth in wave equations, which causes the dissipation to become, sooner or later, dominant in a wave process, changing its physical properties radically. Such a dissipation effect severely restricts the use of geometrical optics (quasiclassical, WKB) approximation, the most effective means of describing disturbances in stratified media. This approximation is conditionally valid only to a "critical" height $z_c$ at which ordinary nondissipative terms of a wave system of equations with dissipative terms are quantitatively compared. At heights of order of $z_c$ and higher, the quasiclassical description is impossible at all. As a result, a large number of disturbances in the upper atmosphere and their associated ionospheric effects cannot be described and understood in WKB. Thus, in response to dissipation, disturbances with small wave vertical scales should be trivially reduced to vanishingly small values at a height of order of $z_c$, and should not significantly contribute to the atmospheric dynamics above this height. In contrast, typical vertical scales of disturbances in the upper atmosphere are generally comparable with the height of the atmosphere. Their occurrence at large heights may be understood only in the context of a rigorous wave description. Being able to penetrate deeply into the upper atmosphere, such disturbances may serve as agents transforming wave energy from lower atmospheric layers into heat energy and thus may contribute greatly not only to the dynamics but also to the general heat balance of the upper atmosphere.

This paper relies on a description of wave disturbances in the dissipative atmosphere in the form of solutions of a geometrical optics equation. Fundamental aspects of this theoretical description have been put forward before in Lyons and Yanovitch (1974); Yanovitch (1967a); Yanovitch (1967b) and have been stated in the most common form in Rudenko (1994a); Rudenko (1994a). The approach in hand ignores many aspects of the real atmosphere: horizontal irregularity, wind motions, nonisothermality; simultaneous overall consideration of dissipation (thermal conductivity and viscosity) is impossible. The quasiclassical approach does not have such limitations, but it is valid, as indicated above, only to certain heights. Despite its idealization, the
rigorous wave description allows us to understand the nature of dissipative disturbances in the upper atmosphere, at least qualitatively.

The structure of this paper is as follows. Section 2 gives a detailed derivation of a general wave equation in the form of a six-order ordinary differential equation with one variable comprising all types of dissipation: thermal conductivity and two components of viscosity. The equation describes a vertical structure of linear oscillations in the isothermal dissipative atmosphere for arbitrary values of frequency $\omega$ and horizontal wave number $k_x$. At low heights, where dissipation can be ignored, this equation describes either acoustic or internal-gravity oscillations, depending on the wave parameters. In particular, this general equation readily yields all particular types of wave description reducing to hypergeometric equations discussed in Lyons and Yanowitch (1974); Yanowitch (1967a); Yanowitch (1967b); Rudenko (1994a). The general equation is not hypergeometric but, theoretically, can be solved numerically or used for analyzing the asymptotic behavior of its solutions (see Rudenko (1994b)). Section 3 discusses a fundamental hypergeometric equation derived in Rudenko (1994a). It is a particular case of the general equation in which the first and second viscosities are set equal to zero. In this case, the order of the general equation decreases by 2, and its solutions are represented by four independent generalized hypergeometric Meijer $G$-functions. In this section, asymptotic properties of independent solutions are used to find a unique meaningful solution satisfying a finite condition in the upper half-space. Formally, the solution describes full continua of acoustic and gravity oscillations in a thermal conductivity medium. In the real atmosphere, the Prandtl number is not sufficiently small to justify the exclusion of dissipative terms of viscosity from consideration. Nevertheless, the generality of the solution and the rigor of the description totally justify the use of this approximation for analyzing dissipative propagation of real disturbances. Section 4 analyzes general properties of penetration of waves with various periods and spatial scales into the upper atmosphere and the degree of their attenuation there. We present the $z_c$ dependence on an oscillation period in the real atmosphere. This height conditionally divides the space into two parts. In the lower half-space, the WKB wave description is possible. In the upper half-space, dissipation is a main agent determining a wave process. Seeing that dissipation strongly suppresses waves with low vertical scales (relative to the scale of the height of the atmosphere), $z_c$ may also be considered a threshold height above which the said waves may be ignored in physical considerations. To estimate the ability of arbitrary
acoustic and internal gravity waves to penetrate into the upper atmosphere (above $z_c$), we propose to use the dependence of characteristic of the degree of attenuation on wave period and length. We show that the characteristics of the degree of attenuation obtained from analytical and WKB solutions at the limit of their co-validity correspond to each other. Section 5 explores the possibility of using the analytical solution, derived in Section 3, for interpreting upper-atmosphere disturbances of special range which can manifest themselves in observations as traveling ionospheric disturbances (TIDs) owing to ion-neutral collisions. Upward propagating disturbances are considered to result from penetration of a horizontally propagating normal IGW mode, located in the stratosphere, from the waveguide. Using the NRLMSISE-2000 particular model atmosphere, we demonstrate that there may be only one 0-mode located at stratospheric heights. The mode dispersive dependence of its period on a horizontal wavelength can be found in two ways: through the WKB approximation and through the numerical solution of a boundary wave problem. We estimate the total change of wave amplitude during wave propagation through a barrier opaque region and a propagation region to $z_c$ with due regard to dissipation. The structure of the amplitude and phase wave characteristics above $z_c$ is represented by an analytical dissipative solution in the isothermal atmosphere. Section 6 summarizes findings of this study and discusses possibilities of their application.

2. Equation of dissipative linear oscillations in the isothermal atmosphere

Consider the isothermal atmosphere with constant coefficients of thermal conductivity and viscosity, which is formally determined in the whole space:

\[
\begin{align*}
p_0(z) &= p_0(z_r)e^{-\frac{z-z_r}{H}}, \\
\rho_0(z) &= \rho_0(z_r)e^{-\frac{z-z_r}{H}}, \\
H &= \frac{RT_0}{g}, \kappa = \text{const}, \nu_1 = \text{const}, \nu_2 = \text{const}.
\end{align*}
\]

Here $p_0, \rho_0$ are undisturbed density and pressure of a medium; $H$ is the height of the atmosphere; $R$ is the gas constant; $g$ is the free fall acceleration; $\kappa, \nu_1, \nu_2$ are dynamic coefficients of thermal conductivity, first and second viscosities respectively; $z_r$ is the reference height with specified undisturbed pressure and density. Without loss of generality, we will assume that all disturbances are independent of the y-coordinate and take the form:
\( Q' = Q'(z)e^{-i\omega t}. \) For the convenience of the wave description, we will exploit a completely dimensionless form of its representation, i.e. coordinates, time, wave parameters, and disturbance function will be represented by corresponding dimensionless values:

\[
\begin{align*}
 r^* &\equiv (x^*, y^*, z^*) = r/H \equiv (x, y, z)/H, t^* = t\sqrt{g/H}, \\
 k &\equiv k_x H, \sigma = \omega \sqrt{H/g}, \\
 n &\equiv \rho'/\rho_0, f = p'/p_0, \Theta = T'/T_0, \\
 u &\equiv v_x/\sqrt{Hg}, w = v_y/\sqrt{Hg}, v = v_z/\sqrt{Hg}.
\end{align*}
\]

(2)

To bring the disturbance equations to dimensionless form, we will use dimensionless expressions of kinematic dissipative values:

\[
\begin{align*}
 s(z) &= \frac{\kappa}{\sigma c_v H \sqrt{gH} \rho_0^{-1}}, \\
 \mu(z) &= \frac{\nu}{\sigma H \sqrt{gH} \rho_0^{-1}}, \\
 q(z) &= \frac{\nu^2}{\sigma H \sqrt{gH} \rho_0^{-1}}.
\end{align*}
\]

(3)

Here \( \gamma \) is the adiabatic index; \( c_v \) is the specific heat capacity at constant volume. By applying introduced determinations in Eqs. (1)-(3) to linearized equations of state, continuity, moment, and entropy change, we obtain a complete system of equations for disturbances of density, pressure, velocity, and temperature:

\[
\begin{align*}
 a) \quad & \quad \Theta = f - n, \\
 b) \quad & -i\sigma n + \Psi - v = 0, \\
 c) \quad & -i\sigma f - v + \gamma \Psi = \sigma \gamma s \Delta \Theta, \\
 d) \quad & -i\sigma u + ikf = \sigma \mu \Delta u + ik\sigma q \Psi, \\
 e) \quad & -i\sigma v + \dot{f} - f + n = \sigma \mu \Delta v + \sigma q \dot{\Psi}, \\
 f) \quad & \quad i\sigma w = \sigma \mu \Delta w.
\end{align*}
\]

(4)

Here, the dot is the derivative of a function with respect to \( z^* \) argument; \( \Psi = \dot{v} +iku \) is the dimensionless divergence of velocity disturbance; \( \Delta = d^2/dz^{*2} - k^2 \) is the dimensionless Laplacian. Eq. (4f) describes an independent trivial viscous solution unrelated to the disturbances we are interested in. Thus, from now on we set \( w = 0 \) and will consider the system of five Eqs. (4a)-(4e) with unknowns \( (\Theta, n, f, u, v) \). The kinematic, dissipative coefficients \( s, \mu, \) and \( q \) are functions which grow exponentially with height. At low heights, where these coefficients may be ignored, equation system of Eqs. (4) describes ordinary classical acoustic and gravitational oscillations. In the real
atmosphere, thermal conductivity exceeds viscosity. Hence, with an increase in height, thermal-conductivity dissipation does occur first, i.e. where the dimensionless function $s$ becomes of order of unity. A corresponding height in the real atmosphere may be determined using first formula of Eqs. (3) from the condition

$$\frac{\kappa}{\omega \gamma c_v (H(z_c))^2 \rho_0(z_c)} = 1.$$  

(5)

Here, the height dependence of the height of the atmosphere is taken into account. In what follows, we show that the presence of the wave frequency $\omega$ in this condition is due to properties of wave solutions in which lower and upper asymptotic behaviors depend on the ratio of $s$ to unity. The value of $z_c$ may be found by solving implicit Eq. (5). Reference to Eq. (5) shows that with an increase in frequency of oscillations the critical height should also increase. Next, we will assume for convenience that the point of reference of the dimensionless coordinate $z$ corresponds to $z_c$:

$$z^* = (z - z_c)/H.$$  

(6)

Accordingly, $p_0(z_r)$ and $\rho_0(z_r)$ in Eq. (1) are determined by values at $z_c = z_r$ and, therefore, are implicit functions of frequency of oscillations too. From Eqs. (5) and (6), expressions for $s$, $\mu$, and $q$ take the following form:

$$s = e^{z^*},$$

$$\mu = \frac{\nu_1}{\kappa c_v} e^{z^*},$$

$$q = \frac{\nu_1/3 + \nu_2}{\kappa} c_v e^{z^*}. $$  

(7)

Equation system (4a)-(4e) allows reducing to one six-order ordinary differential equation with one variable $\Theta$. To derive this equation, we exclude variables in the following sequence. First, we exclude variables $n$ and $f$ from Eqs. (4a), (4b):

$$n = \frac{1}{i \sigma} (\Psi - v), \quad f = \Theta + n = \Theta + \frac{1}{i \sigma} (\Psi - v)$$  

(8)
Using (8), we obtain the following expressions:

\( a \) \( \Psi = \frac{\sigma}{\gamma - 1}(\gamma s \Delta \Theta + i \Theta) \),
\( b \) \( - i \sigma u + \frac{k}{\sigma} v + \sigma \mu \Delta u = ik \Theta - ik \left( \sigma q - \frac{1}{i \sigma} \right) \Psi \),
\( c \) \( - i \sigma v + \frac{1}{i \sigma} \dot{v} + \sigma \mu \Delta v = \dot{\Theta} - \Theta - \left( \sigma q - \frac{1}{i \sigma} \right) \Psi \),
\( d \) \( \Psi = \dot{\Psi} + i k u \),
\( e \) \( \theta = \dot{u} - ik v \),
\( f \) \( \Delta u = \dot{\theta} + ik \Psi \),
\( g \) \( \Delta v = \dot{\Psi} + ik \theta \).

(9)

Here, Eq. (9a) results from the subtraction of Eq. (4b) from Eq. (4c) with the use of Eq. (4a); Eqs. (9b) and (9c) result from the substitution of Eq. (8) into Eq. (4d) and Eq. (4e) respectively; Eq. (9d) is the determination of divergence; Eq. (9e) is a new auxiliary function of current; Eq. (9f) and Eq. (9g) are auxiliary identities evident from the determinations of divergence and current function. Eq. (9) gives us an explicit expression of divergence through temperature disturbance \( \Theta \). The current function may also be explicitly expressed through \( \Theta \) by summing up differentiated Eq. (9c) and Eq. (9b) multiplied by \( ik \):

\[
\theta = \sigma \frac{1}{k + i \sigma^2 \mu} \left\{ \hat{L} \left[ \sigma (\mu + q) \Psi - \Theta - \frac{1}{i \sigma} \Psi \right] + i \sigma \Psi \right\},
\]

(10)

where \( \hat{L} = \Delta - \frac{d}{dz^*} \). By differentiating Eq. (9b) and subtracting Eq. (9c) multiplied by \( ik \), we derive a differential equation expressed through one unknown function \( \Theta \):

\[
\left( 1 - i \hat{L} \mu \right) \theta + k (\mu + q) \Psi - \frac{k}{\sigma} \Theta = 0,
\]

(11)

or in more detail

\[
\left( 1 - i \hat{L} \mu \right) \frac{\hat{L} \left[ \sigma (\mu + q) \Psi - \Theta - \frac{1}{i \sigma} \Psi \right] + i \sigma \Psi}{1 + i \sigma^2 \mu} + \frac{k^2}{\sigma} (\mu + q) \Psi - \frac{k^2}{\sigma^2} \Theta = 0.
\]

(12)

Other values are represented by \( \Theta \) the following set of expressions:

\( v = \frac{\sigma^2}{\sigma^4 - k^2} \times 
\left[ \left( 1 - \frac{d}{dz^*} - \frac{k^2}{\sigma^2} \right) (\Psi + i \sigma \Theta) + i \sigma^2 (\mu + q) \left( \frac{d}{dz^*} + \frac{k^2}{\sigma^2} \right) \Psi - k \mu \left( \frac{d}{dz^*} + \sigma^2 \right) \theta \right],
\)

\( f = \Theta + \dot{\theta} (v - \Psi),\)

\( u = \frac{k}{\sigma} f + i \mu \dot{\theta} - k (\mu + q) \Psi,\)

\( n = f - \Theta\)

(13)
Expressions Eq. (12) and Eqs. (13) completely describe the wave disturbance in the selected model of medium. Eq. (12) does not have analytical solutions, but it may be used for analyzing the asymptotic behavior of solutions at large and small $z^*$, or for numerical solution (see Rudenko (1994a); Rudenko (1994b)). Unlike the classical dissipationless solution, Eq. (12) allows the solution without an infinite increase in amplitude of relative values of disturbances; i.e., in the whole space, the solution may satisfy a linear approximation (Rudenko (1994a)).

3. Rigorous solution for AGV in the isothermal heat-conducting atmosphere

The most interesting possibility of deriving an analytical form of dissipative solutions which describes disturbances of acoustic and gravitational ranges is provided by a model of viscosity-free heat-conducting medium ($\mu = q = 0$). In this case, Eqs. (12) and (13) take the following form:

\[
\begin{align*}
- \hat{L} \left( \Theta + \frac{1}{i\sigma} \Psi \right) + i\sigma \Psi - \frac{k^2}{\sigma^2} \Theta &= 0; \\
\Psi &= \frac{\sigma}{\gamma - 1} \left( \gamma e^{z^*} \Delta \Theta + i\Theta \right), \\
v &= \frac{\sigma^2}{\sigma^2 - k^2} \left( 1 - \frac{d}{dz^*} - \frac{k^2}{\sigma^2} \right) (\Psi + i\sigma \Theta), \\
f &= \Theta + \frac{1}{\sigma} (v - \Psi), \\
u &= \frac{k}{\sigma} f, \\
n &= f - \Theta.
\end{align*}
\]

Introducing a new variable

\[
\xi = \exp \left( -z^* + i\pi \frac{3}{2} \right)
\]

allows us to represent Eq. (14) in the canonical form of generalized hypergeometric equation

\[
\left[ \xi \prod_{j=1}^{2} (\delta - a_j + 1) - \prod_{i=1}^{4} (\delta - b_i) \right] \Theta = 0,
\]

where $\delta = \xi d/d\xi$, $a_{1,2} = \frac{1}{2} \pm iq$, $b_{1,2} = \pm k, b_{3,4} = 1/2 \pm \alpha$

\[
q = \sqrt{-\frac{1}{4} + \frac{\gamma - 1}{\gamma} k^2 + \frac{\sigma^2}{\gamma} - k^2}, \\
\alpha = \sqrt{\frac{1}{4} + k^2 - \sigma^2}.
\]
Eq. (17) has two singular points: \( \xi = 0 \) (the regular singular point, \( z^* = +\infty \)) and \( \xi = \infty \) (the irregular singular point, \( z^* = -\infty \)). The fundamental system of solutions of Eq. (17) may be expressed by four linearly independent generalized Meijer functions (Luke (1975)):

\[
a) \quad \Theta_1 = G_{2,4}^{4,1}(\xi e^{-i\pi/2} | a_1, a_2 b_1, b_2, b_3, b_4), \\
b) \quad \Theta_2 = G_{2,4}^{4,1}(\xi e^{-i\pi/2} | a_2, a_1 b_1, b_2, b_3, b_4), \\
c) \quad \Theta_3 = G_{2,4}^{4,0}(\xi e^{-2i\pi} | a_1, a_2 b_1, b_2, b_3, b_4), \\
d) \quad \Theta_4 = G_{2,4}^{4,0}(\xi | a_1, a_2 b_1, b_2, b_3, b_4).
\]

Desired meaningful solution of Eq. (14) may be found from known properties of asymptotic behaviors of \( \Theta_i \)-functions at two singular points in Eq. (17).

At the irregular point \( \xi = \infty \) (\( z^* = -\infty \)), we have:

\[
a, b) \quad \Theta_{1,2}(\xi) \sim \Theta_{1,2}^\infty(\xi) = p_{1,2} \cdot e^{(1/2+iq)z^*} = \\
\frac{\Gamma(1/2+iq+k)\Gamma(1/2+iq-k)\Gamma(1+iq+\alpha)\Gamma(1+iq-\alpha)}{\Gamma(1+2iq)}e^{\frac{\pi}{4} - i\frac{\pi}{8}} \cdot e^{(1/2+iq)z^*}, \\
c, d) \quad \Theta_{3,4}(\xi) \sim \Theta_{3,4}^\infty(\xi) = \pi^{1/2}e^{-i\frac{\pi}{2}(1+2)} \cdot \varepsilon^{z^*/4}e^{\mp \sqrt{2}(1-i)e^{-z^*/2}}.
\]

From Eqs. (20a), (20b) follows that, at real \( q \), asymptotics \( \Theta_1 \) and \( \Theta_2 \) correspond to two differently-directed classical dissipationless waves. Here, we will assume that \( \Theta_1 \) corresponds to an upward propagating wave; \( \Theta_2 \), to a downward propagating wave. Then, for \( \sigma \) and \( k \) corresponding to acoustic waves, we set \( q < 0 \); for those corresponding to internal gravity waves, \( q > 0 \). Asymptotics \( \Theta_3 \) and \( \Theta_4 \) specified by Eqs. (20c), (20d) correspond to dissipative oscillations. These asymptotics are functions of an extremely rapid decrease and a downward increase in \( z^* \). Decreasing \( \Theta_4 \) leaves the physical scene very quickly at decreasing \( z^* \), and an asymptotic growth of \( \Theta_4 \) provokes the necessity to completely exclude \( \Theta_4 \) from the desired meaningful solution.

The behavior of other solutions \( \Theta_1, \Theta_2, \) and \( \Theta_3 \) nearby the regular point \( \xi = 0 \) (\( z^* = +\infty \)) is represented by the following expressions:

\[
\Theta_i \sim \Theta_i^0 = \sum_{j=1}^{4} t_{ij}e^{-b_jz^*} \quad (i = 1, 2, 3).
\]

Expressions for \( t_{ij} \) are derived from known asymptotics of the Meijer G-function at the regular singular point (Luke (1975)). Their explicit expressions are listed in ??.
When constructing the meaningful solution, we will assume that there should not be upward growing asymptotic terms \( \sim e^{-b_2 z^*} \) and \( \sim e^{-b_1 z^*} \). Besides, we set the incident wave amplitude = 1. Accordingly, the desired solution is found in the following form

\[
\Theta(\xi) = p_1^{-1}\Theta_1(\xi) + \alpha_2 \Theta_2(\xi) + \alpha_3 \Theta_3(\xi),
\]

where coefficients \( \alpha_2 \) and \( \alpha_3 \) are selected from the condition of elimination of growing asymptotics at the regular point of Eq. (17):

\[
\begin{align*}
\alpha_2 &= -p_1^{-1} e^{-2\pi q} \frac{\sin[\pi(\alpha-iq)]\cos[\pi(k-iq)]}{\sin[\pi(\alpha+iq)]\cos[\pi(k+iq)]}, \\
\alpha_3 &= 2\pi p_1^{-1} e^{-2\pi q} \left\{ \frac{\sin[\pi(\alpha-iq)]}{\sin[\pi(\alpha+iq)]} - \frac{\cos[\pi(k-\alpha)]}{\cos[\pi(k+iq)]} \right\}.
\end{align*}
\]

Eqs. (22) and (24a), (24b) give a strict analytical expression of the desired meaningful solution. For \( |\xi| < 1 \) \( (z^* > 0) \), solution of Eq. (22) may be expressed through generalized hypergeometric functions \( _mF_n \) which, in this region of acceptable arguments, are represented by simple convergent power series suitable for the numerical calculation. Such a representation of solution is obtained from the standard representation of the Meijer G-function.

\[
\Theta(z^* > 0) = \beta_0 \beta_1 e^{-k z^*} \times_2 F_3 \left( \begin{array}{c} 1/2 - k - i q, 1/2 + k + i q \\ 1 - 2 k, 1/2 + k + \alpha, 1/2 + k - \alpha \end{array} \right| i e^{-z^*} \right) + \\
\beta_0 \beta_2 e^{(1/2 + \alpha) z^*} \times_2 F_3 \left( \begin{array}{c} 1 + \alpha - i q, 1 + \alpha + i q \\ 1/2 + k, 1/2 - k + \alpha, 1 + 2 \alpha \end{array} \right| - i e^{-z^*} \right),
\]

where

\[
\beta_0 = \frac{\Gamma\left(\frac{1}{2} + i q + k\right)\Gamma\left(i q + \alpha\right)}{\Gamma(2i q)\Gamma\left(\frac{1}{2} + \alpha + k\right)} e^{-i \frac{\pi}{4} - \frac{\pi}{2}} \quad \text{and} \quad \beta_1 = \frac{\Gamma\left(\frac{1}{2} + \alpha - k\right)\Gamma\left(\frac{1}{2} + i q + k\right)}{\Gamma(1 + 2k)\Gamma(1 - i q + \alpha)} e^{-i \frac{\pi}{4} k},
\]

from the standard representation of the Meijer G-function.
\[ \beta_2 = \frac{\Gamma\left(-\frac{1}{2}-\alpha+k\right)\Gamma\left(1+\imath q+\alpha\right)}{\Gamma\left(\frac{1}{2}+\alpha+k\right)\Gamma\left(1+2\alpha\right)\Gamma\left(\frac{1}{2}-\imath q+k\right)}e^{-\frac{i\pi}{2} \left(\frac{1}{2}+\alpha\right)}. \]

Given \( z^* \to \infty \), the generalized hypergeometric functions \( F \) in Eq. (26) tend to unity, and the solution \( \Theta \) takes a simple asymptotic form with two exponentially decreasing terms.

The computed solution describes the incidence of internal gravity or acoustic wave with an arbitrary inclination to the dissipative region \( z^* > 0 \), its reflection from this region, and its penetration to this region with the transformation of it into a dissipative form. The complex coefficient of reflection can be expressed by a simple analytical expression:

\[ K = \alpha_2 p_2. \] (27)

In Rudenko (1994a), the reflection coefficient module is shown to take a value of order of unity, if the typical vertical scale of incident wave \( q^{-1} \gtrsim 1 \). Otherwise, the contribution of the reflected wave exponentially decreases with decreasing vertical scale. This behavior of reflection is similar to the ordinary wave reflection from an irregularity of a medium. In our case, the scale of the irregularity is the height of the atmosphere \( H \). It is worth introducing one more value characterizing a value of wave attenuation in the region \( z^* < 0 \):

\[ \eta = |\Theta(0)|. \] (28)

This value may be calculated using the form of solution of Eq. (26). In what follows, we will show that the behavior of \( \eta \), depending on a vertical wave scale, is similar to the behavior of reflection coefficient. The value \( \eta \) also characterizes the capability of a significant part of wave disturbance to penetrate to the upper region \( z^* > 0 \). It is reasonable that, if \( \eta \) is negligible, further wave disturbance propagation may be neglected too.

4. Classification of AGV in the real atmosphere by dissipative properties

The understanding of certain wave effects at heights of the upper atmosphere requires knowing possible modes of wave propagation which are associated with dissipation effect. Of particular interest first is to determine the typical height from which dissipation assumes a dominant control over the wave process, and second is the type of wave propagation, depending on the period and spatial scale of disturbance (whether it is an approximately dissipativeless propagation or propagation with dominant dissipation).
4.1. Classification of waves by the parameter of dissipation critical height

The dependence of $z_c$ on oscillation frequency enables a convenient clas-
sification of waves only by their oscillation periods. To make such a classifi-
cation, we will exploit the following model of medium:

- The vertical distribution of temperature $T_0(z)$ according to the NRLMSISE-
  2000 distribution with geographic coordinates of Irkutsk for the local noon of winter opposition;

\[- p_0(z) = p_0(0) \exp \left[-\frac{g}{R} \int_0^z \frac{1}{T(z')} dz' \right], \quad p_0(0) = 1.01 \text{Pa};\]

\[- \rho_0(z) = \rho_0(0) \exp \left[-\frac{g}{R} \int_0^z \frac{1}{T(z')} dz' \right], \quad \rho_0(0) = 287.0 \text{g/m}^3;\]

\[- g = 9.807 \text{m/s}^2, \quad R = 287 \text{J/(kg} \cdot \text{K)}, \quad \kappa = 0.026 \text{J/(K} \cdot \text{m} \cdot \text{s)}, \quad c_v = 716.72 \text{J/(kg} \cdot \text{K}).\]

Solving Eq. (5) for $z_c$ (the plot in Fig. 1) provides the following useful
information: a) the height of transformation of waves of the selec-
ted period into dissipative oscillations; b) the height above which waves of the selec-
ted period with vertical scales much less than the height of the atmosph-
eric should be heavily suppressed by dissipation (in fact, such waves should not appear
at these heights); c) the height limiting the applicability of the WKB ap-
proximation for the wave of the selected period.

4.2. Classification of AGV by damping value

An important characteristic of a wave with certain wave parameters $\sigma$
and $k$ is the ratio of the amplitude of the wave solution at $z_c$ to the amplit-
itude of the wave incident from minus infinity $\eta$, Eq. (28). This ratio gives
useful information about the ability of this wave to have physically mean-
ingful values at heights of order of and higher than $z_c$. A direct calculation of
the wave solution for the height region $z < z_c$ from general expression of Eq.
(22) cannot be technically feasible. The first term of asymptotic expansion
Eq. (20a) describes the solution only at a sufficiently large distance from $z_c$
and does not reflect the real behavior of the amplitude under the influence of
dissipation because this term has the form of ordinary propagating dissipa-
tionless wave. Only beginning from $z_c$, we can exactly calculate the solution
by power expansion Eq. (26). On the other hand, the characteristic we are
interested in may be compared with its values obtained in the WKB approxi-
mation for Eq. (17). The use of the WKB approximation also enables us
to obtain the qualitative behavior of the solution in the region $z \lesssim z_c$. For
simplicity, consider a case when a WKB estimated value $\eta$ may be expressed
in an analytical form. For this purpose, rewrite Eq. (17) for a new variable $\Theta_{ref} = \Theta e^{-\frac{1}{2} z^*}$:

$$\left\{ \frac{d^2}{dz^*} + i \left( \frac{d^2}{dz^*} - \frac{1}{4} - k^2 + \sigma^2 \right) \left[ \left( \frac{d}{dz^*} + \frac{1}{2} \right)^2 - k^2 \right] \right\} \Theta_{ref} = 0.$$  

(29)

In the WKB approximation, Eq. (29) yields a quartic algebraic equation for a complex value of a dimensionless vertical wave number $k^*_z$:

$$- k^*_z + q^2 + \frac{e^{z^*}}{i} \left( -k^*_z - \frac{1}{4} - k^2 + \sigma^2 \right) \left[ \left( ik^*_z + \frac{1}{2} \right)^2 - k^2 \right] = 0.$$  

(30)

Note that Eq. (30) can be precisely obtained from similar dispersion Eq. (19) from Vadas (2005) by excluding terms comprising first and second viscosities. For simplicity, consider IGW continuum waves with low frequencies $\sigma$ and $k^*_z \gg 1$. In this case, Equation

$$- k^*_z + q^2 + \frac{e^{z^*}}{i} \left( k^*_z - k^2 + \sigma^2 \right) \left[ \left( ik^*_z + \frac{1}{2} \right)^2 - k^2 \right] = 0.$$  

(31)

gives four simple roots, one of which corresponding to upward propagating IGW wave at $z^* \to -\infty$ will be interesting to us:

$$k^*_z = - \sqrt{\frac{2K^2}{(\sqrt{4ie^{z^*} K^2 + 1} + 1)}} - k^2.$$  

(32)

where $K = \sqrt{q^2 + k^2}$ is the full vector of a dissipationless wave. The dissipative wave attenuation at $z^*$ may be presented in the form:

$$\eta_{WKB} = \sqrt{\frac{q}{|k^*_z(z^*)|}} e^{-\text{Im} \left[ \int_{z^*}^{\infty} k^*_z(z'^*) dz'^* \right]} = \sqrt{\frac{q}{|k^*_z(z^*)|}} e^{-\Gamma}.$$  

(33)

Expression of Eq. (33) may be used for estimating wave attenuation in the real atmosphere. In the isothermal atmosphere approximation, the integral in the exponent may be presented in the analytical form:

$$\Gamma = \text{Re} \left[ i \sqrt{2K} \left( 2i \text{Ln} \frac{a + in}{b + a} + b \text{Ln} \frac{b - a}{b + a} - 2a - i \frac{\pi}{2\sqrt{2}} \right) \right].$$  

(34)
where \( n = \frac{k}{\sqrt{2K}}; \ a = \sqrt{\frac{1}{1+\sqrt{1+4iKe^*}} - n^2}; \ b = \sqrt{\frac{1}{2} - n^2}. \) In addition to the value determined by Formula of Eq. (33), the attenuation value versus the disturbance amplitude at the given lower height \( z_n^* \) may also be useful:

\[
\eta_{WKB}(z^*, z_n^*) = \frac{\eta_{WKB}(z^*)}{\eta_{WKB}(z_n^*)}.
\]

(35)

The wave attenuation \( \eta_{WKB}(0) \) expressed via Eq. (34) is comparable with similar value in Eq. (28) obtained from the analytical wave solution (Fig. 2); as function of \( z^* \), this value qualitatively describes the amplitude of the wave solution in the isothermal atmosphere below \( z_c \) (\( z^* < 0 \)) (Fig. 3).

Fig. 2 demonstrates convergence of the wave attenuation dependences at decreasing vertical wave scales. This comparison justifies validity of the introduced analytical measure of attenuation. The fact that the introduced new measure is determined without limitations for any wave parameters warrants its use as a universal wave characteristic. Alternatively, the obtained convergence in inverse limit (large scales) may justify applying the WKB approximation to amplitude estimates of real long-wave disturbances, though formally this approximation is not valid.

Fig. 3 illustrates the quite predictable typical behavior of the vertical distribution of wave amplitudes below \( z_c \). It is obvious that the dissipation effect begins several scales of the height of the atmosphere to the critical height. The additional characteristic \( \eta_{WKB,rel} = e^{\frac{1}{2}z^*} \eta_{WKB} \) demonstrates the required dissipation-provoked suppression of the exponential growth of the amplitude of relative oscillations. The suppression of the exponential growth provides, in turn, a possibility of satisfying the linearity of disturbance at all heights.

A general picture of distribution of universal characteristic of attenuation for internal gravity and acoustic waves is given in Fig. 4 which shows levels of constant values \( \eta \) in the plane of wave parameters \((k, \sigma)\). For convenience of comparison, Fig. 5 presents levels of constant values of the vertical wave number \( q \) in the same plane. Fig. 4 and Fig. 5 classify waves of weak, moderate, and strong attenuation according to \( k \) and \( \sigma \) and allow us to estimate the possibility of their penetration to heights above \( z_c \).

5. Long-distance disturbance propagation in the upper atmosphere

This section addresses a special type of long-period disturbances which occur at ionospheric heights far from their sources. Seeing that such waves
can not be captured on their own in the upper atmosphere (approximately isothermal), the only way to explain the observation is to consider the waves as a result of propagation from a waveguide located at lower heights. A rigorous description of waveguide propagation in the real atmosphere (without accounting for dissipation) may be obtained, in principle, from the solution of a boundary problem for a wave equation which more completely accounts for the stratification of the real atmosphere (Ostashev (1997); Ponomarev et. al (2006)). For simplicity, we will restrict ourselves to wave Eq. (3) from Ponomarev et. al (2006) in its short form for the windless atmosphere.

\[ \Psi'' + U \Psi = 0, \]

\[ U(z) = \frac{1}{2} \left( \frac{\rho_0}{\gamma p_0} - Q \right)' - \frac{1}{4} \left( \frac{g \rho_0}{\gamma p_0} - k_x^2 \right) \left[ 1 - \left( \ln p_1^{1/\gamma} \right)' \frac{g}{\omega^2} \right], \tag{36} \]

Here, wave function \( \Psi \) is related to the pressure disturbance through

\[ p = \Psi \left( \rho_0 p_0^{-1/\gamma} \left[ \omega^2 - \left( \ln p_0^{1/\gamma} \right)' \frac{g}{\omega^2} \right] \right)^{1/2} \exp \left( -\frac{1}{2} \int_0^z \frac{g \rho_0}{\gamma p_0} d\xi \right) e^{-i\omega t + ik_x x}. \]

The possibility of waveguide propagation depends on the presence of limited height intervals with positive values of \( U \)-function which may be interpreted as squared vertical wave vector \( k_z \) in the WKB approximation. Here, we will compare two methods for describing wave solutions with penetration, using the rigorous solution of the boundary wave problem for Eq. (36) and the WKB approximation. The WKB description for the wave phenomena with vertical scales comparable with atmospheric irregularities is, strictly speaking, rather conditional. Nevertheless, we will demonstrate that this description is quite sufficient to obtain dispersive properties of waveguide solutions and is more convenient for their physical interpretation. From Eq. (36) follows that constructing the \( U \)-function requires continuity of not only a temperature function but also of its first and second derivatives. The initial temperature distribution (Fig. 1) has a certain amount of points with derivative discontinuities that leads to unphysical \( U \)-function jumps. For the sequel, we will therefore approximate the temperature dependence to a set
of reference functions satisfying necessary conditions of smoothness:

\[
T(z > 430) = 944.4,
\]
\[
T(95.3 < z \leq 430) = \left( \left[ \cos \left( \frac{\pi}{2} \left( \frac{430-z}{430-95.3} \right)^6 \right) \right]^3 - 1 \right) (944.4 - 185.4) - 944.4,
\]
\[
T(46 < z \leq 95.3) = \left( \left[ \cos \left( \frac{\pi}{2} \left( \frac{95.3-z}{95.3-46} \right)^2 \right) \right]^3 - 1 \right) (257 - 185.4) - 257,
\]
\[
T(20 < z \leq 46) = \left( \left[ \cos \left( \frac{\pi}{2} \left( \frac{z-20}{46-20} \right)^2 \right) \right]^3 - 1 \right) (215.1 - 257) - 215.1,
\]
\[
T(0 < z \leq 20) = 2 \left( \left[ \cos \left( \frac{20-z}{20} \right)^2 \arccos \left( \frac{0.5^{3/2}}{2} \right) \right]^3 - 1 \right) (215.1 - 270.1) - 215.1.
\]

The comparison of the new temperature dependence on height with the basic dependence is shown in Fig. 6.

Discuss the U-function profile (Fig. 6) for arbitrarily selected wave parameters \( \omega \) and \( k \) \((T_w = 2\pi/\omega = 90 \text{ min}; \lambda_{hor} = 2\pi/k_x = 1390 \text{ km})\). It is evident that the waveguide may be located below \( z_1 \). If the waveguide solution is implemented by the \( U \)-profile, the upper locking wall of the waveguide is a region of negative \( U \)-values: \( z_1 < z < z_2 \). The same region, in turn, is also a barrier through which a part of wave energy may escape. Above \( z_2 \) is again a region of wave propagation extending up to the heights we are interested in. A distinguishing characteristic of the problem in hand is a strong variation in form and \( U \)-values, depending on wave parameters. Values of \( z_1 \) and \( z_2 \) change too.

5.1. Boundary wave problem (BWP) for wave modes

It is most convenient to solve the boundary problem for Eq. (36) via the corresponding nonlinear Riccati equation:

\[
G' - G^2U - 1 = 0,
\]

Where \( G \) is related to \( \Psi \) through

\[
G \Psi' = \Psi.
\]
The $G$-function of the waveguide solution must meet top and bottom boundary conditions. At the top ($z = z_\infty \to +\infty$), the $G$-function must fit an upward IGW:

$$G(z_\infty) = 1, \quad G'(z_\infty) = i/\sqrt{u(z_\infty)}. \quad (40)$$

In the numerical implementation, $z_\infty$ was taken to be 270 km above which, in our model, the $U$-function is constant. At the bottom ($z = 0$), we impose a requirement:

$$G(0) = 0. \quad (41)$$

This requirement, according to Eq. (39), is equivalent to the condition of equality to zero of the $\Psi$-function.

The numerical solution of Cauchy problem Eqs. (38), (40) in the $G$-function allows us to derive a complex dispersion equation

$$D(\omega, k_x) = G(0, \omega, k_x). \quad (42)$$

Formally, we can solve Eq. (42) by assuming the first or the second argument of dispersion $D$-function to be real. In the former case, we will have modes attenuating (owing to the nonhermiticity of the problem) in the horizontal direction of propagation; in the latter case, modes attenuating in time. In this paper, we analyze modes only with real values of frequency $\omega$. A vertical spatial structure of the mode (for the pair of dispersive values $\omega$ and $k_x$ satisfying Eq. (42)) can be obtained by solving numerically the Cauchy problem for Eq. (36) with the initial condition $\Psi = 0, \Psi' = 1$ corresponding to boundary condition of Eq. (41) for the $G$-function.

5.2. WKB analysis of waveguide solution

In the general case, the condition of waveguide locking with energy leakage in the WKB approximation may be represented by a modified Borel-Sommerfeld condition of quantization (MBSCQ) with complex turning points:

$$\int_C \sqrt{U(z)} dz \approx \pi \left( \frac{1}{2} + n \right) + i \exp \left[ -2 \int_{z_1}^{z_2} \sqrt{|U_0(z)|} dz \right] \equiv \pi \left( \frac{1}{2} + n \right) + i E, \quad n = 0, 1, \ldots . \quad (43)$$

Condition of Eq. (43) gives dispersion ratios between real frequencies $\omega$ and complex wave numbers $k_x$ with a small imaginary part accounting for the degree of horizontal attenuation of the waveguide mode $n$. The integration contour $C$ of the integral on the left-hand side of Eq. (43) begins from
and ends at a complex turning point \( z_{c1} \) close to the real turning point \( z_1(U(z_1, \omega, \text{Re} k_x)) \). Besides, with inner (complex) turning points, we assume that the \( C \)-contour also passes through these points (in our calculations, for simplicity, sections of the \( C \)-contour with \( \text{Re} U(z) < 0 \) are ignored). The integral in the exponent argument on the right-hand side of Eq. (43) is assumed to be real; the 0-index of the \( U \)-function in the integrand means that it is a function of the real part \( k_x \) and real \( z \): \( U_0(z) = U(z, \omega, \text{Re} k_x) \).

Solve Eq. (43), using the perturbation theory:

\[
\int_{z_0}^{z_1} |U_0(z)| dz - \pi \left( \frac{1}{2} + n \right) = 0 \tag{44}
\]

for the selected real \( \omega \), find a real root of \( k_{0x} \) in Eq. (43); next, insymbols

\[
k_x = k_{0x}(1 + i\delta), \quad U(z) = u_0(z) + k_{0x} i\delta \frac{\partial}{\partial k_x} U_0; \tag{45}
\]

By substituting Eq. (45) to Eq. (43) and accounting for the complexity of turning points \( z_{cj} \) of the integration contour of the integral on the left-hand side of Eq. (43), we obtain an equation for complex addition of the horizontal wave vector:

\[
\delta \frac{1}{2} \int_{z_0}^{z_1} \frac{k_{0x}}{\sqrt{U_0}} \left( \frac{\partial}{\partial k_x} U_0 \right) dz + \delta^{3/2} \frac{2}{3} e^{3i\pi/4} k_{0x}^{3/2} \sum_j \left[ \frac{\partial U_0/\partial k_x}{\partial U_0/\partial z} \right]_{z=z_j} = E. \tag{46}
\]

We have, thus, three algebraic roots of cubic Eq. (46) for \( \delta^{1/2} \). We choose one of them that reflects physical attenuation of the wave mode.

5.3. Numerical calculations of characteristics of internal gravity waveguide modes

First, we have established that there is only one nodeless waveguide mode (with \( n = 0 \)) in the selected model atmosphere in the frequency range that corresponds to large-scale TIDs. This was demonstrated by both the algorithms described in Subsections 5.1 and 5.2. The dispersion curve of the 0-mode is shown by a segment of the thick curve in the plane of wave dimensionless parameters \( (\sigma, k) \) in Figs. 4 and 5. For a more detailed analysis of propagation characteristics of the waveguide mode, Fig. 8 gives:

- the dispersion dependence of horizontal wavelength on oscillation period (the dotted curve is the dependence obtained from BWP; the solid curve is
the dependence obtained from MBSCQ);
– asymptotic characteristics of the wave leakage to the upper atmosphere: full phase velocity (dashed curve); vertical group velocity (dash-dotted curve); vertical wavelength (dash-dot-dotted curve).

Fig. 9 presents estimated characteristics of the horizontal attenuation: the dotted curve is the characteristics obtained from BWP; the solid curve, from MBSCQ.

Note the most important points:
– The case of model atmosphere, we analyzed, showed that there is only one mode. Seeing that the chosen time of the model for the geographic localization considered corresponds most often to moments of detection of ionospheric disturbances, we may assume that the implementation of the conditions (somewhere) for two or more modes is most likely to be extremely rare or impossible at all.
– It is important that the obtained dispersion dependence fairly faithfully reproduces the observed ratio of horizontal scales to periods of TIDs and gives reasonable estimates for their full phase velocities obtainable in measurements. This result furnishes convincing proof of their inextricable connection with the waveguide propagation of IGW.
– The two approaches (BWP and MBSCQ) show a perfect match for each other in phase characteristics and a quite satisfactory match in attenuation sufficient for very long-distance propagation of waveguide disturbances. The analysis of the structure of the rigorous waveguide solution (Fig. 10) demonstrates that, despite the complex behavior of the $U$-function, the height dependence of the solution is smooth with a typical scale of change far exceeding that of $U$ change. It is even more surprising that the quasiclassical description, despite its formal invalidity, provides results which are very close to the exact solution. Thus, we verify practicality for the quasiclassical description of the problem in hand.

It is interesting to note a property which is likely to be specific only for IGW waveguides. The dependence in Fig. 9 shows a high Q factor of oscillations relative to the characteristic of horizontal attenuation in spite of the fact that, in the opacity barrier $[z_1, z_2]$, the amplitude of the rigorous solution (Fig. 10) decreases slightly. The parameter $\sqrt{E}$, determined in Eq. (43), is of order of 0.41. For an acoustic waveguide, its horizontal attenuation characteristic would be estimated as $0.41^2$. For IGW, the multiplier of the first term in Eq. (46) gives a value of order of 30 (for AGW, it would be $\sim 1$) on the whole dispersion curve. This factor causes a very weak attenuation of
mode. On the other hand, the physical substantiation of this may be the low vertical group velocity of the upward propagating wave penetrating through the opaque region (Fig. 9). The slow energy leakage from a waveguide causes the slow waveguide attenuation.

5.4. Analyzing the height dependence of amplitude-phase characteristics of the wave penetrating to the upper "dissipative" atmosphere

For a qualitative relation of the classical upward propagating wave with its subsequent modification under the action of dissipation, we will consider two propagation regions: \( z_2 < z < z_c \) and \( z > z_c \). If the disturbance amplitude is set 1 at \( z = z_1 \), we can estimate the relation between the amplitude of the waveguide solution and the amplitude of the upward propagating wave at \( z_c \): If we ignore the multiplier of the exponential growth –

\[
\mu(z_c; \lambda_{gor}, T_w) = \sqrt{E(\lambda_{gor}, T_w)\eta_{WKB}(z_2, z_c; \lambda_{gor}, T_w)}; \quad (47)
\]

In view of the exponential growth –

\[
\mu_{rel}(z_c; \lambda_{gor}, T_w) = \mu(z_c; \lambda_{gor}, T_w) \cdot \left( \frac{\rho_0(z_2)}{\rho_0(z_c)} \right)^{1/2}. \quad (48)
\]

Here, \( \lambda_{gor} \) and \( T_w \) are respective horizontal wavelength and wave period on the dispersion curve of the waveguide 0-mode; the first multiplier on the right-hand side of Eq. (47) corresponds to the quasiclassical estimate of the decrease in amplitude in the opacity barrier according to definition Eq. (43); the second multiplier on the right-hand side of Eq. (47) gives a change of the amplitude under the action of dissipation according to dissipative WKB estimate Eqs. (43), (45). Corresponding plots of these values are given in Fig. 11.

As was shown in Section 3, at \( z > z_c \), we can find the exact dissipative wave solution from (26), (15). This is demonstrated by Fig. 12 and Fig. 13 for one pair of wave parameters \( \lambda_{gor}, T_w \) on the dispersion curve of 0-mode of a wave with \( T_w = 1.5 \) h. Amplitude characteristics of all wave physical quantities in Fig. 12 are normalized to the amplitudes in accordance with equality 1 of the amplitude of the relative disturbance of density at \( z = z_c \). For convenience, vertical lines show initial amplitudes of the corresponding classical dissipationless wave at \( z = z_c \). Real changes of classical dissipationless amplitudes of IGW are not presented here because of their unlimited
exponential growth (several orders of their values in the given height interval). We can see that the relations between dissipative amplitude values of the exact solution almost exactly coincide with the similar relations of the classical solution (unchanged with height) at \( z = z_c \). The relations between amplitudes of the dissipative solution change with height significantly and demonstrate the predicted stop of the exponential growth with height and their subsequent gradual decrease (the beginning of which can be observed for all amplitudes in the upper height range). A similar behavior can be observed in relations between phases of all values in Fig. 13 i.e. they change with height significantly, coinciding with relations between phases of the classical solution at \( z = z_c^* \). The fact that the relations between amplitudes and between phases correspond to the classical solution counts in favor of the dissipative WKB description up to \( z_c \).

6. Conclusion

We have demonstrated that the physical interpretation of disturbances in the upper atmosphere (especially at ionospheric heights) requires a rigorous wave description accounting for the dominant dissipation effect. Beginning from a critical height (\( z_c \)) depending on an oscillation period, behavior of wave disturbances cannot be described in any form of the WKB approximation. Below this height, this approximation can be successfully exploited in the part of the upper atmosphere with dissipative corrections and then, in its classical (dissipationless) form, can satisfactorily describe wave propagation in the real, significantly stratified atmosphere. In this case, for low-frequency oscillations of the IGW range, the WKB description remains valid (despite its formal incorrectness) for waves vertical scales of which are comparable to or larger than scales of vertical irregularity of a medium. This statement is based on the remarkable coincidence we obtained between results of calculations of the zero waveguide mode. Waves with vertical scales comparable with the height of the atmosphere were shown to be able to reach a critical height and propagate further to higher ionospheric heights. Under the action of dissipation, these waves are considerably transformed: an intense exponential growth stops followed by a slow amplitude decrease allowing the wave to retain its linear features. This conclusion is based on our analysis of the rigorous hypergeometric solution of the wave problem for the isothermal thermal-conductivity atmosphere. We have examined a phenomenological hybrid model for physical interpretation of formation and long-distance
propagation of TIDs. This model assumes that manifestations of ionospheric
disturbances are caused by the interaction between a charged component and
a wave disturbance in the neutral atmosphere, which penetrates from a lower
waveguide. The construction of appropriate waveguide solutions showed that
there may be only one nodeless 0-mode. We studied its dispersion features
and estimated horizontal attenuation demonstrating the possibility for long-
distance horizontal propagation of the mode. The obtained dispersion de-
pendence of horizontal wave scale on wave period agrees well with known
spatio-temporal characteristics of TIDs. The estimated phase velocities of
the upward propagating wave also closely match the observed phase PIV
velocities (Medvedev et. al (2009); Ratovsky et. al. (2008)).

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Figure 1: The dashed curve shows the height dependence of temperature in the selected model; the thick solid curve indicates the dependence of $z_c$ on the period of oscillations corresponding to IGW waves incident from the lower atmosphere; the solid curve indicates the dependence of $z_c$ on the period of oscillations corresponding to acoustic waves incident from the lower atmosphere.
Figure 2: The solid line is the dependence of $\eta$ on vertical length of the incident wave; the dashed line is the same for $\eta_{WKB}(0)$; the dash-dotted line is the dependence of horizontal wavelength on vertical length of the incident wave for the selected wave solutions (the left axis). The frequency $\sigma$ is constant (0.14 of the Vaisala-Brent frequency $\sigma_{VBF}$).
Figure 3: The height dependence of attenuation values $\eta_{WKB}$ and $\eta_{WKB,rel}$ at given wave parameters.
Figure 4: The levels of $\eta$ in the plane of $k$ and $\sigma$. 
Figure 5: The levels of $q$ in the plane of $k$ and $\sigma$. 
Figure 6: The approximate temperature dependence on height (dotted line); the basic dependence (solid line).

Figure 7: The characteristic height distribution of $U$-function.
Figure 8: The waveguide characteristics of 0-mode: horizontal wavelength from MBSCQ (solid line); horizontal wavelength from BWP (dotted line); full phase velocity of the upward propagating wave (dashed line); vertical group velocity of the upward propagating wave (dash-dotted line); vertical length of the upward propagating wave (dash-dot-dotted line).
Figure 9: The waveguide characteristics of 0-mode: horizontal attenuation characteristic from MBSCQ (solid line); horizontal attenuation characteristic from BWP.

Figure 10: An example of the vertical structure of the waveguide solution for 0-mode.
Figure 11: The characteristics of the ratio of the upward propagating wave at the critical height to the amplitude of the waveguide solution at the low height of the waveguide opacity.
Figure 12: The amplitude characteristics of wave oscillation above the critical height at dominant dissipation.
Figure 13: The phase characteristics of wave oscillation above the critical height at dominant dissipation.