Suggestions for classifying the fractals and their connections with the Cantor’s cardinal numbers

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Abstract. In this paper we shall give a classification of the fractals and their connection with the Cantor’s cardinal numbers.

1. Introduction
There is a great diversity of the fractal sets [1], which are usually classified into two main types: self-similar and self-affine fractals or by using their constructive process, they are classified into geometrical, algebraic and stochastic fractals. A better understanding of patterns of fractalic diversity and of the underlying nonlinear processes is important in itself, but it is also indispensable for the development of the applications of the nonlinear theory and fractals in Physics. We recall here the El Nasche’s concept of fractal space-time which seems to have applications ranging from particle physics to cosmology [2-7]. Based on the concept of hierarchical, Cantorian space-time, E-infinity theory is essentially linked to the nonlinear dynamics and to the fractal geometry [8-12].

This is the reason for which, a way to classify the fractals is suggested in the followings. With this end in mind, we firstly introduce a new concept: we shall say that \( P_3 \in M \) is an “exceptionat point” (a point which makes an exception) of the set \( M \) through the subset \( S \subset M \) if \( P_3 \) presents a peculiar property connected with \( S \), a property which is not possessed by the points \( P \) related to \( M \). In this form, the previous sentence is not a definition, but a description which needs specifications referring to what we can understand by the expression “peculiar property”. That is why we shall pass to defining the first type of “exceptionat point”.

2. The concept of the “exceptionat point”

Definition 1. We say that \( P_{S_1} \in M \), \( M \) being a metric space, is an exceptionat point of kind 1 of the set \( M \) through the subset \( S \subset M \) if, for any open \( U \) which contains \( P_{S_1} \), the subset \( U \cap S \) has a non-integer dimension (in any definition of the fractal dimension).

Definition 2. The exceptionat point of kind 1 is called fractal point of the first kind and will be denoted by \( P_{F_1} \). The set \( F_1 \) which has at least one such point \( P_{F_1} \) is called fractal of the first kind.

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It is not necessary for a fractal to be a continuous set. The most common example of a
 discontinuous fractal is the Cantor dust which, in fact, has all the fractal points of the first kind.

Contrarily, the spiral, continuous curve defined by \( r \phi = \text{const.} \), \( r \) and \( \phi \) having the significance of
the polar coordinates in a plane, has only one fractal point of the first kind, namely, the origin of the
coordinates, \( r = 0 \). This point has an interesting feature: it is a singular point in the sense that it has
no derivative of any kind. Indeed, if we consider the fraction

\[
\frac{\Delta y}{\Delta x} = \frac{\Delta (r \sin \phi)}{\Delta (r \cos \phi)}.
\]

at the limit \( r \to 0 \) it can have any one value; if we consider the passage to the limit for
\( \phi = \text{const.} \) we obtain any limit for various values of the constant, since \( \ln \phi \in (-\infty, +\infty) \). This
observation suggests us to state

**Definition 3.** If \( P_{S_2} \in M \), \( M \) being a metric space and the set \( S \) in the neighborhood of the point
\( P_{S_2} \) is continuous, then we say that it \( P_{S_2} \) is an exceptionat point of the second kind of the set \( M \)
through the subset \( S \), if \( P_{S_2} \) is an asymptotic point in the sense that it has no derivative of any kind.

**Definition 4.** The exceptionat point of the second type is called fractal point of the second kind and
will be noted with \( P_{F_2} \). The set \( F_2 \) which has at least one of such a point \( P_{F_2} \) is named fractal of the
second kind.

A fractal can be simultaneously of the first and second kind. That is why we shall call it a hybrid
fractal. This is true in the case of the spiral. An example of the fractal of the second kind is the Peano
curve which has the dimension 2, and therefore, it is not of the first kind.

3. **Consequences**

The set of the exceptionat points of a fractal can have any cardinal number. In the case of the sets with
a finite number of elements, we shall write that their cardinal number is \( N = \aleph_{-1} \). Thus, we can
classify the fractals \( F \) according to the cardinal number of the subset \( E \subseteq F \) of the exceptionat points
from \( F \) and we shall note \( F_{\aleph_{-1}} \) ( \( k = -1,0,1,... \) ) fractal with \( \text{card} \ E = \aleph_k \). Therefore, we can have \( F_{\aleph_{-1}} \),
or \( F_{2\aleph_{-1}} \), depending on the type of the fractal points. The fractals with \( E = F \) (i.e. all the fractal points
\( F \) are exceptionat points) will be denoted with \( \overline{F} \) — The Cantor dust is a fractal of the kind \( \overline{F}_{\aleph_{-1}} \),
and Peano’s curve is a fractal of the type \( \overline{F}_{2\aleph_{-1}} \) — The continuous fractal curves with the infinite
length on any segment are also of the kind \( \overline{F}_{\aleph_{0}} \) ( \( \alpha = 1,2 \).

We should also specify the fact that some series of fractals have the property of self-similarity.
This means that each subdivision of the fractal determined by its construction looks like the entire
one. In the case of the Cantor dust, each third of the left interval has exactly the same structure as the
entire one from which one it starts. The same happens in the case of the von Koch curve where each
segment obtained by construction is presented as the entire curve.

There are also important the cases in which the self-similarity is not perfect, such as the general
case of the fractal curves. In these cases we talk about almost self-similar fractals.

In the above mentioned cases we talk about self \( F \) fractals, or \( \overline{F} \), or a-self \( F \), i.e. \( \widetilde{F} \).
In what follows we wish to point out some particular structures which appear in various experiments or in nature. One knows that the leaf, especially the fern, has a self-similar structure. However, this self-similarity cannot go on indefinitely, because it is no longer valid at sub-cellular level. However, such a structure can be treated as a fractal. Similarly, we mention the case of a sum with a great number of terms which is best approximated if we suppose that we refer to the sum of a convergent series, because what we have added is negligible and its treatment is much easier.

Many organic structures are almost fractals and there are also phenomena which are related to fractals, such as the Brownian motion. To specify the difference between self-similar fractals and these structures, we shall call them pre-fractals. Instead of establishing up to what limit these structures have the property of self-similarity, we should better complete them theoretically unlimited and treat them as fractals.

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