Traveling Wave Solutions for Integro-Difference Systems

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Abstract

This paper is concerned with the traveling wave solutions for integro-difference systems of higher order. Using the generalized upper and lower solutions, the existence of positive traveling wave solutions is proved. Then the asymptotic behavior of traveling wave solutions is established by the idea of contracting rectangles. To illustrate our results, the traveling wave solutions of three systems are considered, which improves/completes some known results.

Keywords: generalized upper and lower solutions; contracting rectangle; asymptotic behavior.

AMS Subject Classification (2000): 45C05; 45M05; 92D40.

1 Introduction

In this paper, we consider the existence and asymptotic behavior of traveling wave solutions of the following integro-difference system

\[ u_{n+1}^i(x) = \int_{\mathbb{R}} P_i[u_{n-k+1}^1(y), \ldots, u_1^1(y), u_{n-k+1}^2(y), \ldots, u_1^m(y)]k_i(x-y)dy, \]  

(1.1)
in which \( n \in \mathbb{N} \cup \{0\}, m, k \in \mathbb{N}, i \in I =: \{1, 2, \ldots, m\} \), \( x, y \in \mathbb{R}, u_n^i \in \mathbb{R}, P_i : \mathbb{R}^{m \times k} \to \mathbb{R}, \)

\( k_i : \mathbb{R} \to \mathbb{R}^+ \) is a probability function or kernel function, \( i \in I \). Moreover, \( P_i \) satisfies the following assumptions:

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(P1) there exists $M = (M_1, M_2, \cdots, M_m)$ such that $[0, M]$ is an invariant region of the corresponding difference system of (1.1), i.e.,

$$0 \leq P_i[h_{11}, h_{12}, \cdots, h_{mk}] \leq M_i$$

with

$$0 \leq h_{ij} \leq M_i, i \in I, j \in J = \{1, 2, \cdots, k\};$$

(P2) there exists $L > 0$ such that

$$|P_i[h_{11}, h_{12}, \cdots, h_{mk}] - P_i[f_{11}, f_{12}, \cdots, f_{mk}]| \leq L \sum_{i \in I, j \in J} |h_{ij} - f_{ij}|$$

for any $h_{ij}, f_{ij} \in [0, M_i], i \in I, j \in J$;

(P3) $P_i[0, 0, \cdots, 0] = 0$ and there exists $E = (E_1, E_2, \cdots, E_m)$ such that

$$P_i \left[ E_1, \cdots, E_k, E_2, \cdots, E_2, E_3, \cdots, E_n \right] = E_i, i \in I;$$

(P4) $0 \ll E \leq M$.

Moreover, the probability function $k_i$ satisfies the following conditions:

(k1) $k_i : \mathbb{R} \to \mathbb{R}^+$ is Lebesgue measurable;

(k2) $k_i : \mathbb{R} \to \mathbb{R}^+$ satisfies $k_i(x) = k_i(-x)$;

(k3) $\int_{\mathbb{R}} k_i(y) dy = 1$ and $\int_{\mathbb{R}} k_i(y)e^{\lambda y} dy < \infty$ for all $\lambda > 0$.

If $m = 1$ and $k = 1$, then (1.1) becomes

$$v_{n+1}(x) = \int_{\mathbb{R}} b(v_n(y))k(x-y)dy,$$  \hspace{1cm} (1.2)

in which $b : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and continuous and $b(0) = 0$, $k : \mathbb{R} \to \mathbb{R}^+$ satisfies (k1)-(k3). In the past three decades, the traveling wave solutions of (1.2) have been widely studied, we refer to Creegan and Lui [1], Hsu and Zhao [2], Kot [3], Kot et al. [4], Liang and Zhao [6], Lui [11–14], Neubert and Caswell [15], Wang et al. [18], Weinberger [19,20], Weinberger et al. [21] and Weinberger et al. [22]. In these papers, the (local) monotonicity of $b$ plays a very important role.

If $k = 1$, Liang and Zhao [6] and Weinberger et al. [21] investigated the propagation modes of (1.1). Similar to the study of scalar equations, the monotonicity of semiflows is
the most essential assumption. Recently, Lin and Li [8] and Lin et al. [9] considered the existence of traveling wave solutions of a competitive system by a cross iteration scheme. If \( k = 2, m = 1 \), and the birth function is (locally) monotone, the traveling wave solutions and asymptotic spreading were studied by Lin and Li [7] and Pan and Lin [16].

When \( k > 1 \), it is possible that standard upper and lower solutions make no sense in the study of traveling wave solutions of (1.1). For example, let us consider the following model

\[
v_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+\lambda)v_n(y)}{1 + \lambda(v_n(y) + av_{n-1}(y))} k(x - y)dy, \tag{1.3}
\]

in which \( \lambda > 0, a \geq 0 \) are constants. If \( a = 0 \), then the propagation modes of (1.3) have been investigated by Kot [3]. For \( a > 0 \), it is clear that

\[
\int_{\mathbb{R}} \frac{(1+\lambda)v_n(y)}{1 + \lambda(v_n(y) + av_{n-1}(y))} k(x - y)dy
\]

is monotone increasing in \( v_n(y) \) while decreasing in \( v_{n-1}(y) \) such that the standard upper and lower solutions cannot be applied. Moreover, the equation does not satisfy the local monotonicity in Pan and Lin [16]. In this paper, we shall introduce the generalized upper and lower solutions to overcome the difficulty that arises from the deficiency of classical comparison principle. Combining the generalized upper and lower solutions with Schauder’s fixed point theorem, we deduce the existence of positive traveling wave solutions of (1.1) to the existence of generalized upper and lower solutions.

The asymptotic behavior of traveling wave solutions is very important since it reflects the transition between different steady states. If a system can generate monotone semiflows, then the behavior can be proved by the monotonicity of traveling wave solutions. Otherwise, the study will be very hard. In [8, 9], we obtained the asymptotic behavior by constructing very precise upper and lower solutions of a competitive system with special kernel functions. If the kernel functions are different from those in [8, 9], then the study is still very complex. Furthermore, if a system, e.g., (1.3), does not admit classical comparison principle, then it is difficult to obtain the asymptotic behavior of traveling wave solutions by constructing auxiliary systems and functions (e.g., upper and lower solutions). Recently, Lin and Ruan [10] established the asymptotic behavior of traveling wave solutions of some non-monotonic delayed reaction-diffusion systems via contracting rectangles. Motivated by the idea in [10], we shall consider the asymptotic behavior of traveling wave solutions of (1.1) by the contracting rectangles of the corresponding difference system.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, the existence of positive traveling wave solutions is studied by
generalized upper and lower solutions. In Section 4, the asymptotic behavior of traveling wave solutions is investigated by contracting rectangles. In the last section, we consider some examples to illustrate our results, which completes some known results.

2 Preliminaries

In this paper, we shall use the standard partial ordering in \( \mathbb{R}^m \). Define \( C \) by

\[
C = \{ U(x) | U(x) : \mathbb{R} \to \mathbb{R}^m \text{ is uniformly continuous and bounded} \}
\]

equipped with the compact open topology. Moreover, if \( A \leq B \in \mathbb{R}^m \), then

\[
C[A,B] = \{ U : U(x) \in C \text{ and } A \leq U(x) \leq B \text{ for all } x \in \mathbb{R} \}.
\]

The definition of traveling wave solutions is given as follows.

**Definition 2.1** A traveling wave solution of (1.1) is a special solution of the form

\[
u_n^i(x) = \phi_i(\xi), \xi = x + cn,
\]

where \( \Phi = (\phi_1, \phi_2, \cdots, \phi_m) \in C(\mathbb{R}, \mathbb{R}^m) \) is the wave profile that propagates through the one-dimensional spatial domain \( \mathbb{R} \) at the constant wave speed \( c > 0 \).

By the above definition, \( \phi_i \) and \( c \) must satisfy

\[
\phi_i(\xi) = \int_{-\infty}^{\xi} P_i[\phi_1(s - kc + c), \cdots, \phi_i(s), \phi_2(s - kc + c), \cdots, \phi_m(s)]k_i(\xi - s - c)ds (2.1)
\]

for \( i \in I, \xi \in \mathbb{R} \). Moreover, we also require that

\[
\lim_{\xi \to -\infty} \phi_i(\xi) = 0, \lim_{\xi \to \infty} \phi_i(\xi) > 0, i \in I.
\]

(2.2)

Then the traveling wave solutions satisfying (2.1), (2.2) could formulate the invasion process in population dynamics. In particular, we also consider the following asymptotic boundary conditions

\[
\lim_{\xi \to -\infty} \phi_i(\xi) = 0, \lim_{\xi \to \infty} \phi_i(\xi) = E_i, i \in I.
\]

(2.3)

Let \( \| \cdot \| \) be the supremum norm in \( \mathbb{R}^m \) and \( \mu > 0 \) be a constant. Define

\[
B_\mu(\mathbb{R}, \mathbb{R}^2) = \left\{ \Phi \in C : \sup_{x \in \mathbb{R}} \| \Phi(x)\| e^{-\mu|x|} < \infty \right\}
\]

and

\[
|\Phi|_\mu = \sup_{x \in \mathbb{R}} \| \Phi(x)\| e^{-\mu|x|}.
\]

Then \( (B_\mu(\mathbb{R}, \mathbb{R}^m), |\cdot|_\mu) \) is a Banach space.
3 Generalized Upper and Lower Solutions

In this section, we establish the existence of positive solutions of (1.3). The definition of generalized upper and lower solutions is given as follows.

**Definition 3.1** \( \Phi(\xi) = (\bar{\phi}_1(\xi), \bar{\phi}_2(\xi), \ldots, \bar{\phi}_m(\xi)) \) and \( \Phi(\xi) = (\underline{\phi}_1(\xi), \underline{\phi}_2(\xi), \ldots, \underline{\phi}_m(\xi)) \in C_{[0,M]} \) are generalized upper and lower solutions of (1.3) if

\[
\bar{\phi}_i(\xi) \geq \int_{\mathbb{R}} P_i[\psi_{11}(s-kc+c), \ldots, \psi_{1k}(s), \psi_{21}(s-kc+c), \ldots, \psi_{mk}(s)]k_i(\xi-s-c)ds \tag{3.1}
\]

and

\[
\underline{\phi}_i(\xi) \leq \int_{\mathbb{R}} P_i[\rho_{11}(s-kc+c), \ldots, \rho_{1k}(s), \rho_{21}(s-kc+c), \ldots, \rho_{mk}(s)]k_i(\xi-s-c)ds \tag{3.2}
\]

for all \( i \in I, \xi \in \mathbb{R} \) and

\[
\phi_j(\xi) \leq \psi_{ij}(\xi), \rho_{ij}(\xi) \leq \bar{\phi}_i(\xi), j \in J.
\]

**Remark 3.2** The definition implies that \( \Phi(\xi) \geq \Phi(\xi) \).

If \( 0 \leq \psi_{ij}(\xi) \leq M_i \) is continuous for \( \xi \in \mathbb{R}, i \in I, j \in J \), then

\[
Q_i(\xi) = \int_{\mathbb{R}} P_i[\psi_{11}(s-kc+c), \ldots, \psi_{1k}(s), \psi_{21}(s-kc+c), \ldots, \psi_{mk}(s)]k_i(\xi-s-c)ds
\]

is uniform continuous in \( \xi \in \mathbb{R} \) by (k1).

**Lemma 3.3** If \( \underline{\phi}_i(\xi) \leq \psi_{ij}(\xi) \leq \bar{\phi}_i(\xi) \), then \( \underline{\phi}_i(\xi) \leq Q_i(\xi) \leq \bar{\phi}_i(\xi) \).

The lemma is clear by Definition 3.1 and we omit the proof.

Let

\[
\Gamma = \{ \Phi(\xi) \in C : \Phi(\xi) \leq \Phi(\xi) \leq \Phi(\xi), \xi \in \mathbb{R} \},
\]

and define

\[
F(\Phi)(\xi) = (F_1(\Phi)(\xi), F_2(\Phi)(\xi), \ldots, F_m(\Phi)(\xi))
\]

by

\[
F_i(\Phi)(\xi) = \int_{\mathbb{R}} P_i[\phi_1(s-kc+c), \ldots, \phi_1(s), \phi_2(s-kc+c), \ldots, \phi_m(s)]k_i(\xi-s-c)ds
\]

with \( \Phi = (\phi_1, \phi_2, \ldots, \phi_m) \in \Gamma \).

**Lemma 3.4** \( F : \Gamma \to \Gamma \) is complete continuous with respect to the decay norm \( | \cdot |_{\mu} \).
Proof. According to Lemma 3.3, $F : \Gamma \to \Gamma$ is true. We now prove that the mapping is complete continuous. Let

$$\Phi = (\phi_1, \phi_2, \ldots, \phi_m) \in \Gamma, \Psi = (\psi_1, \psi_2, \ldots, \psi_m) \in \Gamma,$$

then

$$|F_i(\Phi)(\xi) - F_i(\Psi)(\xi)| = \left| \int_{\mathbb{R}} P_i[\phi_1(s - kc + c), \ldots, \phi_1(s), \phi_2(s - kc + c), \ldots, \phi_2(s - kc + c), \ldots, \phi_m(s)]k_i(\xi - s - c)ds - \int_{\mathbb{R}} P_i[\psi_1(s - kc + c), \ldots, \psi_1(s), \psi_2(s - kc + c), \ldots, \psi_2(s - kc + c), \ldots, \psi_m(s)]k_i(\xi - s - c)ds \right| \leq L \sum_{t \in I, j \in J} \int_{\mathbb{R}} |\phi_t(s - jc + c) - \psi_t(s - jc + c)| k_i(\xi - s - c)ds$$

and

$$|F_i(\Phi)(\xi) - F_i(\Psi)(\xi)| e^{-\mu|\xi|} = e^{-\mu|\xi|} \left| \int_{\mathbb{R}} P_i[\phi_1(s - kc + c), \ldots, \phi_1(s), \phi_2(s - kc + c), \ldots, \phi_2(s - kc + c), \ldots, \phi_m(s)]k_i(\xi - s - c)ds - \int_{\mathbb{R}} P_i[\psi_1(s - kc + c), \ldots, \psi_1(s), \psi_2(s - kc + c), \ldots, \psi_2(s - kc + c), \ldots, \psi_m(s)]k_i(\xi - s - c)ds \right| \leq Le^{-\mu|\xi|} \sum_{t \in I, j \in J} \int_{\mathbb{R}} |\phi_t(s - jc + c) - \psi_t(s - jc + c)| k_i(\xi - s - c)ds \leq Lm|\Phi - \Psi|_{\mu} \sum_{j \in J} \int_{\mathbb{R}} e^{\mu j c}e^{-\mu|\xi|} k_i(\xi)d\xi,$$

which further implies that

$$\sup_{\xi \in \mathbb{R}} |F_i(\Phi)(\xi) - F_i(\Psi)(\xi)| e^{-\mu|\xi|} \leq Lm|\Phi - \Psi|_{\mu} \sum_{j \in J} \int_{\mathbb{R}} e^{\mu j c}e^{-\mu|\xi|} k_i(\xi)d\xi$$

and the continuous is proved.

Moreover, if $\Phi \in \Gamma$, then

$$|F_i(\Phi)(\xi_1) - F_i(\Phi)(\xi_2)| \leq M_i \int_{\mathbb{R}} |k_i(\xi_1) - k_i(\xi_2)| ds$$

and the equicontinuity is true. For any $\epsilon > 0$, let $B > 0$ such that

$$Me^{-\mu|\xi|} < \epsilon, |\xi| > B.$$  \hfill (3.3)

Moreover, the equicontinuity implies that there exist $P \in \mathbb{N}$ and

$$\{\Phi^i(\xi)\}_{i=1}^P \in \{F(\Phi)(\xi) : \Phi(\xi) \in \Gamma\}$$
such that \( \{ \Phi^i(\xi) \}_{i=1}^P \) is a finite \( \epsilon \)-net of \( \{ F(\Phi)(\xi) : \Phi(\xi) \in \Gamma \} \) for \( |\xi| \leq B \). From (3.3), \( \{ \Phi^i(\xi) \}_{i=1}^P \) is a finite \( \epsilon \)-net of \( \{ F(\Phi)(\xi) : \Phi(\xi) \in \Gamma \} \) for \( \xi \in \mathbb{R} \), which implies the compactness. The proof is complete. \( \square \)

**Theorem 3.5** Assume that \( \Phi(\xi), \Phi(\xi) \) are generalized upper and lower solutions of (1.3). Then (1.3) admits a positive solution \( \Phi(\xi) \) such that \( \Phi(\xi) \leq \Phi(\xi) \leq \Phi(\xi) \).

**Proof.** By Schauder’s fixed point theorem, \( F \) has a fixed point

\[
(\phi^*_1(\xi), \phi^*_2(\xi), \cdots, \phi^*_m(\xi)) \in \Gamma,
\]

which is also a solution of (1.3). The proof is complete. \( \square \)

### 4 Asymptotic Behavior via Contracting Rectangles

In this section, we shall consider the asymptotic behavior of traveling wave solutions. For the purpose, we first study the long time behavior of the following difference system

\[
u_{n+1}^i = P_i[u_{n-k+1}^1, \cdots, u_{n-k+1}^1, \cdots, u_{n-k+1}^m],
\]

in which \( P_i \) satisfies (P1)-(P4).

**Definition 4.1** Let \( R(s), T(s) \) be two vector functions of \( s \in [0, 1] \) and take the form as follows

\[
R(s) = (r_1(s), r_2(s), \cdots, r_m(s)) \in \mathbb{R}^m, \quad T(s) = (t_1(s), t_2(s), \cdots, t_m(s)) \in \mathbb{R}^m.
\]

Then \( [R(s), T(s)] \) is a contracting rectangle of (4.1) if

(C1) \( r_i(s), t_i(s) \) are continuous in \( s \in [0, 1], i \in I \);

(C2) \( r_i(s) \) is strictly increasing in \( s \) while \( t_i(s) \) is strictly decreasing in \( s, i \in I \);

(C3) \( 0 \leq R(0) < R(1) = E = T(1) < T(0) \leq M \);

(C4) for each \( s \in (0, 1) \) and \( i, j \in I, l \in J \),

\[
r_i(s) < P_i[u_{l}^1, \cdots, u_{l}^1, u_{l}^2, \cdots, u_{l}^m]
\]

for any \( u_l^j \in [r_j(s), t_j(s)] \), and

\[
t_i(s) > P_i[v_{l}^1, \cdots, v_{l}^1, v_{l}^2, \cdots, v_{l}^m]
\]

for any \( v_l^j \in [r_j(s), t_j(s)] \).
Theorem 4.2 Assume that $[R(s), T(s)]$ is a contracting rectangle of (4.1) and there exists $s_0 \in (0, 1)$ such that $r_i(s_0) \leq u_{i' - j} \leq t_i(s_0)$ (4.2) for some $n' \in \mathbb{N}$ and all $i \in I, j \in J$. Then

$$\lim_{n \to \infty} u_n^i = E_i, i \in I.$$ \(\square\)

Proof. By the definition of contracting rectangle, (4.2) implies that $r_i(s_0) \leq u_n^i \leq t_i(s_0), n \geq n'$.

By the boundedness, there exist $\underline{u}_i, \overline{u}_i$ such that

$$\liminf_{n \to \infty} u_n^i = \underline{u}_i, \limsup_{n \to \infty} u_n^i = \overline{u}_i, i \in I.$$ \(\square\)

We now give the main result of this section.

Theorem 4.3 Assume that $\Phi(\xi) = (\phi_1, \cdots, \phi_m)$ is a positive solution of (1.3) and $[R(s), T(s)]$ is a contracting rectangle of (4.1). If

$$R(s_1) \leq \liminf_{\xi \to \infty} \Phi(\xi) \leq \limsup_{\xi \to \infty} \Phi(\xi) \leq T(s_1)$$

with some $s_1 \in (0, 1)$, then $\lim_{\xi \to \infty} \Phi(\xi) = E$.

Proof. The proof is similar to that of Theorem 4.2. Were the statement false. Let

$$\liminf_{\xi \to \infty} \phi_i = \underline{\phi}_i, \limsup_{\xi \to \infty} \phi_i = \overline{\phi}_i, i \in I.$$ \(\square\)

Without loss of generality, we assume that $s_2 \in [s_1, 1)$ such that

$$\underline{\phi}_i = r_1(s_2)$$
and
\[ r_i(s_2) \leq \phi_i \leq t_i(s_2), \quad i \in I. \]

By \( \lim \inf \), there exists \( \xi_n, n \in \mathbb{N} \) with \( \xi_n \to \infty, n \to \infty \) such that
\[ \phi_1(\xi_n) \to \phi_1 = r_1(s_2), \quad n \to \infty. \]

From the dominated convergence theorem (let \( n \to \infty \) in \( F \)) and properties of continuous functions on bounded closed interval (see (P1)-(P2)), we see that
\[ \phi_1 = P_1[u_1^1, \ldots, u_1^k, u_2^1, \ldots, u_2^m] \]
with some \( u_j^i \in [r_l(s_2), t_l(s_2)] \). By the definition of contracting rectangles, we obtain a contradiction and complete the proof. \( \square \)

5 Applications

In this section, we shall study the traveling wave solutions of several models. To use our conclusions, we first introduce some classical theory of asymptotic spreading. Consider
\[
\begin{aligned}
    v_{n+1}(x) &= \int_{\mathbb{R}} b(v_n(y))k(x-y)dy, x \in \mathbb{R}, n = 0, 1, 2, \ldots , \\
    v_0(x) &= \varphi(x), x \in \mathbb{R},
\end{aligned}
\tag{5.1}
\]
in which \( k \) satisfies (k1)-(k3) and
\[
\begin{aligned}
(b1) \quad & b(0) = 0, b(v^*) = v^* \text{ for some } v^* > 0, \text{ and there exists } \overline{v} \geq v^* \text{ such that } [0, \overline{v}] \text{ is an} \\
\text{invariant region of } v_{n+1} = b(v_n); \\
(b2) \quad & 0 < b(v) < b'(0)v, v \in (0, \overline{v}); \\
(b3) \quad & b(v) > v, v \in (0, v^*); \\
(b4) \quad & b(v) \text{ is continuous in } v \in [0, \overline{v}]; \\
(b5) \quad & \text{there exists } L > 0 \text{ such that } \\
    & 0 \leq b'(0)v - b(v) \leq Lv^2, v \in [0, \overline{v}].
\end{aligned}
\]

Moreover, from (b1)-(b5), we can define
\[
\begin{aligned}
\overline{b}(v) &= \sup_{u \in (0, v)} b(u), \\
\underline{b}(v) &= \inf_{u \in (v, \overline{v})} b(u).
\end{aligned}
\]
Then there exist $0 < v_1 \leq \varphi \leq v_2$ such that
\[
\overline{b}(v_2) = v_2, \underline{b}(v_1) = v_1.
\]

By Hsu and Zhao \cite{2}, the following result is true.

**Lemma 5.1** Assume that $0 < \varphi(x) \leq \overline{\varphi}$ holds and $\varphi(x)$ is continuous in $x \in \mathbb{R}$. Then
\[
v_1 \leq \liminf_{n \to \infty} v_n(x) \leq \limsup_{n \to \infty} v_n(x) \leq v_2
\]
and the limit is locally uniform in $x \in \mathbb{R}$.

**5.1 A Scalar Equation**

In this part, we consider the traveling wave solutions of (1.2) with (b1)-(b5). It should be noted that the results in this part have been proved by Hsu and Zhao \cite{2}. We add this part in order to show that our theory can be applied to some well studied models, and to display some preliminaries.

Let $v_n(x) = \varphi(x + cn)$ be a traveling wave solution of (1.2), then $\varphi(\xi)$ satisfies
\[
\varphi(\xi) = \int_{\mathbb{R}} b(\varphi(\xi - c + y))k(y)dy, \xi \in \mathbb{R}.
\] (5.2)

To continue our discussion, we need some constants and define
\[
\Delta_1(\lambda, c) = b'(0) \int_{\mathbb{R}} e^{\lambda y - \lambda c}k(y)dy
\]
for $\lambda > 0, c > 0$.

**Lemma 5.2** There exists $c_1 > 0$ such that $\Delta_1(\lambda, c) = 1$ has two real positive roots $\lambda_1(c) < \lambda_2(c)$ for any $c > c_1$ and $\Delta_1(\lambda, c) < 1$ for all $\lambda \in (\lambda_1(c), \lambda_2(c))$. Moreover, if $c < c_1$, then $\Delta_1(\lambda, c) > 1$ for all $\lambda > 0$.

The lemma can be proved by the properties of convex functions, and we omit it here (see Liang and Zhao \cite[Lemma 3.8]{6}). By these constants, we state the following results on the existence of positive traveling wave solutions of (1.2).

**Theorem 5.3** Assume that $c \geq c_1$ is true. Then (5.2) has a positive solution $\varphi(\xi)$ such that $\lim_{\xi \to -\infty} \varphi(\xi) = 0$ and
\[
v_1 \leq \liminf_{\xi \to \infty} \varphi(\xi) \leq \limsup_{\xi \to \infty} \varphi(\xi) \leq v_2.
\] (5.3)

Moreover, if $c < c_1$ and $\varphi(x) > 0$, then $u_n(x)$ defined by (5.1) satisfies
\[
\liminf_{n \to \infty} \inf_{|x| < cn} u_n(x) > 0.
\]
Proof. We first consider \( c > c_1 \). Define

\[
\varphi(\xi) = \min\{e^{\lambda_1(c)\xi}, v_2\}, \psi(\xi) = \max\{e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}, 0\}
\]

with

\[
q > 1, \eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\right\}\right).
\]

If \( q > 1 \) is large enough, then we can verify that \( \varphi(\xi), \psi(\xi) \) are generalized upper and lower solutions of (5.2) and Theorem 3.5 implies that (5.2) has a positive solution. Applying Lemma 5.1 and the invariant of traveling wave solutions, we obtain (5.3).

If \( c = c_1 \), then we can obtain the existence of (5.2) by passing to a limit function (see [2, Theorem 3.2]). For \( c < c_1 \), the result has been proved by Hsu and Zhao [2]. The proof is complete. \( \square \)

If \( b(v) \) is monotone for \( v \in [0, v^*] \), then the limit behavior of traveling wave solutions has been obtained by the monotonicity of traveling wave solutions [2]. If \( b(v) \) is not monotone for \( v \in [0, v^*] \), then the limit behavior should be further investigated. In 2008, Hsu and Zhao [2] gave some sufficient conditions on the topic. Using the contracting rectangle, we may prove some results on the limit behavior although the verification is technical. For example, let

\[
b(v) = 3v(1 - v), \quad (5.4)
\]

then [2] has shown that a positive traveling wave solution of (5.2) satisfies \( \lim_{\xi \to \infty} \varphi(\xi) = \frac{2}{3} \). We now verify the limit by contracting rectangle.

By Theorem 5.3, we obtain

\[
\frac{9}{16} \leq \liminf_{\xi \to \infty} \varphi(\xi) \leq \limsup_{\xi \to \infty} \varphi(\xi) \leq \frac{3}{4}.
\]

It is clear that

\[
b\left(\frac{9}{16}\right) < \frac{3}{4}, b^2\left(\frac{9}{16}\right) > \frac{9}{16}.
\]

then

\[
\frac{9}{16} < \liminf_{\xi \to \infty} \varphi(\xi) \leq \limsup_{\xi \to \infty} \varphi(\xi) < \frac{3}{4}
\]

by dominated convergence theorem. In particular, we also have

\[
b^2(v) < v, v \in \left(\frac{2}{3}, \frac{3}{4}\right); \quad b^2(v) > v, v \in \left[\frac{9}{16}, \frac{2}{3}\right).
\]

(5.5)

Let

\[
r(s) = \frac{9}{16} + \frac{5s}{48}, \quad t(s) = b(r(s)) + \epsilon(b^2(r(s)) - r(s))
\]

for some \( \epsilon > 0 \).
Lemma 5.4 If $\epsilon > 0$ is small enough, then $[r(s), t(s)]$ defines a contracting rectangle of $v_{n+1} = 3v_n(1 - v_n)$.

Proof. If $s \in (0, 1)$ and $\epsilon > 0$, then $r(s) \in \left(\frac{9}{16}, \frac{3}{4}\right)$. By (5.5), we obtain

$$t(s) > b(u(s)), u(s) \in [r(s), t(s)]$$

To prove that

$$r(s) < b(u(s)), u(s) \in [r(s), t(s)],$$

we first assume that $\epsilon > 0$ is small such that

$$\frac{2}{3} < t(s) < \frac{3}{4}, s \in (0, 1).$$

Then the monotonicity of $b(u), u \in \left(\frac{1}{2}, 1\right)$ implies that we only need to verify that

$$r(s) < b(t(s)).$$

Note that $\sup_{v \in \left[\frac{9}{16}, \frac{3}{4}\right]} |3 - 6u| < 2$ such that

$$|b(u) - b(v)| < 2|u - v|, u, v \in \left[\frac{9}{16}, \frac{3}{4}\right].$$

We further have

$$b(t(s)) - r(s)
\begin{align*}
    &= b(b(r(s)) + \epsilon(b^2(r(s)) - r(s))) - r(s) \\
    &> b^2(r(s)) - r(s) - 2\epsilon(b^2(r(s)) - r(s)) \\
    &> 0, s \in (0, 1),
\end{align*}$$

if $\epsilon \in (0, \frac{1}{2})$. The proof is complete. □

Using Theorem 4.3, we give the following result.

Theorem 5.5 $\lim_{\xi \to \infty} \varphi(\xi) = \frac{2}{3}$ holds if $b(v)$ is defined by (5.4).

5.2 A Competitive System of Two Species

In this part, we consider

$$
\begin{align*}
    p_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1 + d_1)p_n(x-y)}{1 + d_1(p_n(x-y) + b_1p_{n-1}(x-y) + a_1q_n(x-y))} k_1(y)dy, \\
    q_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1 + d_2)q_n(x-y)}{1 + d_2(q_n(x-y) + b_2q_{n-1}(x-y) + a_2p_n(x-y))} k_2(y)dy,
\end{align*}
$$

(5.6)
in which \( n \geq 0, x \in \mathbb{R} \), all the parameters are nonnegative, and \( k_1(y), k_2(y) \) are probability functions describing the migration of the individuals and satisfy (k1)-(k3). In particular, we assume that

\[
d_1 > 0, \ d_2 > 0, \ 1 + b_1 > a_2, \ 1 + b_2 > a_1
\]

such that (5.6) admits a spatially homogeneous steady state

\[
(p^*, q^*) = \left( \frac{1 + b_1 - a_2}{(1 + b_1)(1 + b_2) - a_1 a_2}, \frac{1 + b_2 - a_1}{(1 + b_1)(1 + b_2) - a_1 a_2} \right) \gg (0, 0).
\]

If \( b_1 = b_2 = 0 \), then (5.6) is the model in Lewis et al. [5], Lin et al. [9], Zhang and Zhao [24]. More precisely, Lewis et al. [5] investigated the propagation modes between a resident and an invader, Zhang and Zhao [24] proved the existence and stability of bistable traveling wave solutions. Recently, Lin and Li [8] and Lin et al. [9] obtained the existence of traveling wave solutions describing the coinvasion of two competitors if \( k_i \) is the Gaussian or admits compact support. In this paper, we assume that \( k_1(y), k_2(y) \) satisfy (k1)-(k3) and establish the existence/nonexistence of traveling wave solutions, which covers and completes the corresponding results in [8, 9].

If \( a_1 = a_2 = 0 \), then (5.6) becomes two uncoupled integro-difference equations of second order (see (1.3)). It should be noted that (1.3) does not satisfy monotone conditions in Lin and Li [7] and Pan and Lin [16]. Moreover, since the delay is a constant, we cannot obtain the existence of traveling wave solutions by exponential order [17, 23]. In this paper, the corresponding conclusions on traveling wave solution of (1.3) will be obtained by generalized upper and lower solutions and contracting rectangles.

Let

\[
\rho(\xi) = p_n(x), \ \varrho(\xi) = q_n(x), \ \xi = x + cn
\]

be a traveling wave solution of (5.6), then

\[
\begin{cases}
\rho(\xi + c) = \int_{\mathbb{R}} \frac{(1 + d_1)\rho(\xi - y) + b_1(\rho(\xi - y) - a\varrho(\xi - y))}{1 + d_1(\rho(\xi - y) + b_1(\rho(\xi - y) - a\varrho(\xi - y)))} k_1(y)dy, \\
\varrho(\xi + c) = \int_{\mathbb{R}} \frac{(1 + d_2)\varrho(\xi - y) + b_2(\varrho(\xi - y) - a\rho(\xi - y))}{1 + d_2(\varrho(\xi - y) + b_2(\varrho(\xi - y) - a\rho(\xi - y)))} k_2(y)dy.
\end{cases}
\]

To study the traveling wave solutions, we also define

\[
\Theta_1(\lambda, c) = (1 + d_1) \int_{\mathbb{R}} e^{\lambda y - \lambda c} k_1(y)dy, \ \Theta_2(\lambda, c) = (1 + d_2) \int_{\mathbb{R}} e^{\lambda y - \lambda c} k_2(y)dy
\]

for \( \lambda > 0, c > 0 \).

**Lemma 5.6** There exists a positive constant \( c^* > 0 \) such that \( \Theta_i(\lambda, c) = 1 \) has two distinct positive roots \( \lambda_{i1}(c) < \lambda_{i2}(c) \) for any \( c > c^* \) and each \( i = 1, 2 \). Moreover, if \( c < c^* \), then
\( \Theta_1(\lambda, c) > 1 \) for any \( \lambda \geq 0 \) or \( \Theta_2(\lambda, c) > 1 \) for any \( \lambda \geq 0 \). In addition, for any given \( c > c^* \), there exists \( \eta \in (1, 2) \) such that \( \eta \lambda_{i1}(c) < \lambda_{11}(c) + \lambda_{21}(c) \) and \( \Theta_i(\lambda_{i1}'(c), c) < 1 \) for all \( \lambda_{i1}'(c) \in (\lambda_{11}(c), \eta \lambda_{i1}(c)), i = 1, 2 \).

**Theorem 5.7** Assume that \( c > c^* \) holds. Then (5.8) has a positive solution \((\rho(\xi), \varrho(\xi))\) satisfying

\[
0 < \rho(\xi) < 1, 0 < \varrho(\xi) < 1.
\]

**Proof.** For \( q > 1 \), define

\[
\overline{\rho}(\xi) = \min\{e^{\lambda_{11}\xi}, 1\}, \underline{\rho}(\xi) = \min\{e^{\lambda_{21}\xi}, 1\}
\]

and

\[
\overline{\varrho}(\xi) = \max\{e^{\lambda_{11}\xi} - qe^{\eta \lambda_{11}\xi}, 0\}, \underline{\varrho}(\xi) = \max\{e^{\lambda_{21}\xi} - qe^{\eta \lambda_{21}\xi}, 0\}
\]

with \( \eta \) formulated by Lemma 5.6. If \( q > 1 \) is large enough, then we obtain a pair if generalized upper and lower solution of (5.8).

In fact, if \( \overline{\rho}(\xi) = e^{\lambda_{11}\xi} \), then

\[
\int_{\mathbb{R}} \frac{(1 + d_1)\rho(\xi - y)}{1 + d_1(\rho(\xi - y) + b_1 \rho(\xi - y - c) + a_1 \varrho(\xi - y))} k_1(y) dy 
\]

\[
\leq \int_{\mathbb{R}} (1 + d_1)\overline{\rho}(\xi - y)k_1(y) dy 
\]

\[
\leq \int_{\mathbb{R}} (1 + d_1)e^{\lambda_{11}(\xi - y)}k_1(y) dy 
\]

\[
= e^{\lambda_{11}(\xi + c)} = \overline{\rho}(\xi + c).
\]

If \( \overline{\rho}(\xi) = 1 \), then

\[
\int_{\mathbb{R}} \frac{(1 + d_1)\rho(\xi - y)}{1 + d_1(\rho(\xi - y) + b_1 \rho(\xi - y - c) + a_1 \varrho(\xi - y))} k_1(y) dy 
\]

\[
\leq \int_{\mathbb{R}} (1 + d_1)\rho(\xi - y)k_1(y) dy 
\]

\[
\leq 1 = \overline{\rho}(\xi + c).
\]

If \( \underline{\rho}(\xi) = 0 \), then the result is clear. Otherwise, we have

\[
\int_{\mathbb{R}} \frac{(1 + d_1)\rho(\xi - y)}{1 + d_1(\rho(\xi - y) + b_1 \rho(\xi - y - c) + a_1 \varrho(\xi - y))} k_1(y) dy 
\]

\[
\geq \int_{\mathbb{R}} (1 + d_1)\rho(\xi - y)[1 - d_1(\rho(\xi - y) + b_1 \rho(\xi - y - c) + a_1 \varrho(\xi - y))]k_1(y) dy
\]
\[
\geq \int_{\mathbb{R}} (1 + d_1) \rho(\xi - y) dy \\
- \int_{\mathbb{R}} d_1(1 + d_1)[(1 + b_1)\rho^2(\xi - y) + a_1\rho(\xi - y)\overline{\rho}(\xi - y)]k_1(y) dy \\
\geq e^{\lambda_1(\xi + c)} - q\Theta_1(\eta_1, c)e^{\eta_1(\xi + c)} \\
- d_1(1 + b_1)\Theta_1(2\lambda_1, c)e^{2\lambda_1(\xi + c)} - d_1a_1\Theta_1(\lambda_1 + \lambda_2, c)e^{(\lambda_1 + \lambda_2)(\xi + c)}.
\]

Let
\[
q > \frac{d_1(1 + b_1)\Theta_1(2\lambda_1, c) + d_1a_1\Theta_1(\lambda_1 + \lambda_2, c)}{1 - \Theta_1(\eta_1, c)} + 1,
\]
then
\[
\int_{\mathbb{R}} \frac{(1 + d_1)\rho(\xi - y)}{1 + d_1(\rho(\xi - y) + b_1(\rho(\xi - y) - c) + a_1g(\xi - y))} k_1(y) dy \geq \rho(\xi + c).
\]

Similarly, if
\[
q > \frac{d_2(1 + b_2)\Theta_2(2\lambda_2, c) + d_2a_2\Theta_2(\lambda_1 + \lambda_2, c)}{1 - \Theta_2(\eta_2, c)} + 1,
\]
we obtain the inequalities satisfied by \(g(\xi), \overline{\rho}(\xi)\).

By Theorem 3.5, (5.8) has a solution \((\rho(\xi), g(\xi))\) satisfying
\[
0 \leq \rho(\xi) \leq 1, 0 \leq g(\xi) \leq 1
\]
and
\[
\lim_{\xi \to -\infty} \rho(\xi) = \lim_{\xi \to -\infty} g(\xi) = 0, \sup_{\xi \in \mathbb{R}} \rho(\xi) > 0, \sup_{\xi \in \mathbb{R}} g(\xi) > 0.
\]

If \(\rho(\xi) = 0\) for some \(\xi \in \mathbb{R}\), then the dominated convergence theorem implies that \(\rho(\xi) \equiv 0\), which is a contradiction. Similarly, we can obtain \(0 < \rho(\xi) < 1, 0 < g(\xi) < 1\) and the proof is complete. \(\square\)

In what follows, we consider the limit behavior of traveling wave solutions and further assume that
\[
a_1 + b_1 < 1, a_2 + b_2 < 1. \tag{5.9}
\]

**Lemma 5.8** Assume that (5.7) and (5.9) hold and define
\[
r_1(s) = sp^*, t_1(s) = sp^* + (1 + \epsilon)(1 - s),
\]
\[
r_2(s) = sq^*, t_2(s) = sq^* + (1 + \epsilon)(1 - s)
\]
with \(\epsilon > 0\) such that
\[
(b_1 + a_1)(1 + \epsilon) < 1, (b_2 + a_2)(1 + \epsilon) < 1.
\]
Then \([r_1(s), t_1(s)] \times [r_2(s), t_2(s)]\) is a contracting rectangle of

\[
\begin{align*}
p_{n+1} &= \frac{(1+d_1)p_n}{1+d_1(p_n+b_1 p_{n-1}+a_1 q_n)}, \\
q_{n+1} &= \frac{(1+d_2)q_n}{1+d_1q_n+b_2 q_{n-1}+a_2 p_n}.
\end{align*}
\] (5.10)

**Proof.** The continuity and monotonicity in these four functions are clear and we just verify (C4). Since

\[
\frac{(1+d_1)p_n}{1+d_1(p_n+b_1 p_{n-1}+a_1 q_n)}
\]

is increasing in \(p_n\) and decreasing in \(p_{n-1}, q_n\), then for the first equation in (5.10), it suffices to prove that

\[
r_1(s) < \frac{(1+d_1)r_1(s)}{1+d_1(r_1(s)+b_1 t_1(s)+a_1 t_2(s))}
\] (5.11)

and

\[
t_1(s) > \frac{(1+d_1)t_1(s)}{1+d_1(t_1(s)+b_1 r_1(s)+a_1 r_2(s))}
\] (5.12)

for \(s \in (0, 1)\). In fact, \((b_1+a_1)(1+\epsilon) < 1\) and \(s \in (0, 1)\) imply that

\[
\begin{align*}
\frac{1+d_1}{1+d_1(r_1(s)+b_1 t_1(s)+a_1 t_2(s))} &= \frac{1+d_1}{1+d_1} \\
&= \frac{1+d_1}{1+d_1} \\
&= \frac{1+d_1}{1+d_1} \\
&= 1 \\
&> \frac{1+d_1}{1+d_1} \\
&= 1
\end{align*}
\]

and (5.11) has been proved. Moreover, from

\[
\begin{align*}
\frac{1+d_1}{1+d_1(t_1(s)+b_1 r_1(s)+a_1 r_2(s))} &= \frac{1+d_1}{1+d_1} \\
&= \frac{1+d_1}{1+d_1} \\
&= \frac{1+d_1}{1+d_1} \\
&< \frac{1+d_1}{1+d_1} \\
&= 1,
\end{align*}
\] (5.12)

is true.
In a similar way, we have
\[ r_2(s) < \frac{(1 + d_2)r_2(s)}{1 + d_2(r_2(s) + b_2t_2(s) + a_2t_1(s))}, \]
\[ t_2(s) > \frac{(1 + d_2)t_2(s)}{1 + d_2(t_2(s) + b_2r_2(s) + a_2r_1(s))}, \]
for \( s \in (0, 1) \). The proof is complete. \( \square \)

We now give our main conclusion on the asymptotic behavior of traveling wave solutions of \((5.6)\).

**Theorem 5.9** Assume that \((\rho(\xi), \varrho(\xi))\) is given by Theorem 5.7. If \((5.7)\) and \((5.9)\) hold, then
\[ \lim_{\xi \to \infty} (\rho(\xi), \varrho(\xi)) = (p^*, q^*). \]

**Proof.** It is clear that \( p_n(x) = \rho(\xi) \) satisfies
\[ p_{n+1}(x) > \int_{\mathbb{R}} \frac{(1 + d_1)p_n(x - y)}{1 + d_1(p_n(x - y) + b_1 + a_1)} k_1(y) dy, \]
then the classical comparison principle and Lemma 5.1 imply that
\[ \liminf_{n \to \infty} p_n(x) \geq 1 - a_1 - b_1 \]
and the limit is locally uniform in \( x \), which further implies that
\[ \liminf_{\xi \to \infty} \rho(\xi) \geq 1 - a_1 - b_1 > 0. \]

In a similar way, we have
\[ \liminf_{\xi \to \infty} \varrho(\xi) \geq 1 - a_2 - b_2. \]
By what we have done, there exists \( s_0 \in (0, 1) \) such that
\[ r_1(s_0) < \liminf_{\xi \to \infty} \rho(\xi) \leq \limsup_{\xi \to \infty} \rho(\xi) < t_1(s_0) \]
and
\[ r_2(s_0) < \liminf_{\xi \to \infty} \varrho(\xi) \leq \limsup_{\xi \to \infty} \varrho(\xi) < t_2(s_0). \]
Using Theorem 4.3 and Lemma 5.8, we complete the proof. \( \square \)

We now consider the nonexistence of traveling wave solutions of \((5.6)\), and we first present the conclusion as follows.
**Theorem 5.10** Assume that \((5.7)\) and \((5.9)\) hold. If \(c < c^*\), then \((5.8)\) has no positive solutions satisfying

\[
\lim_{\xi \to -\infty} \rho(\xi) = \lim_{\xi \to -\infty} \varrho(\xi) = 0, \sup_{\xi \in \mathbb{R}} \rho(\xi) > 0, \sup_{\xi \in \mathbb{R}} \varrho(\xi) > 0. \quad (5.13)
\]

**Proof.** Without loss of generality, we assume that \(c^*\) satisfies

\[
\Theta_1(\lambda, c) > 1 \text{ for any } c < c^*, \lambda > 0.
\]

We first give three claims as follows:

(C1) if \(\rho(\xi), \varrho(\xi)\) satisfy \((5.8)\) and \((5.13)\), then

\[
0 < \rho(\xi), \varrho(\xi) < 1, \xi \in \mathbb{R};
\]

(C2) for any given \(c_1 < c^*\), there exists \(\epsilon_0 > 0\) such that

\[
\Theta_1^\epsilon(\lambda, c) > 1 \text{ for any } \epsilon \in (0, \epsilon_0), c < c_1, \lambda > 0,
\]

where

\[
\Theta_1^\epsilon(\lambda, c) = \frac{1 + d_1}{1 + d_1 \epsilon (a_1 + b_1)} \int_{\mathbb{R}} e^{\lambda y - \lambda_c} k_1(y) dy;
\]

(C3) for any given \(\epsilon > 0\), there exists \(M > 1\) such that

\[
b_1 \rho(\xi - y - c) + a_1 \varrho(\xi - y)) \leq \max\{\epsilon, (M - 1) \rho(\xi)\}, \xi \in \mathbb{R}.
\]

These claims are evident by the limit behavior \((5.13)\) and the properties of continuous functions on bounded interval, and we omit the verification here.

Were the statement false, then there exists \(c_2 < c^*\) such that \((5.8)\) with \(c = c_2\) has a positive solution satisfying \((5.13)\). Let \(\epsilon_1 > 0\) be small such that

\[
\Theta_1^{2\epsilon_1}(\lambda, c) > 1 \text{ for any } \epsilon \in (0, \epsilon_0), c < \frac{c_2 + c^*}{2}, \lambda > 0.
\]

At the same time, (C3) indicates that there exists \(M > 1\) such that

\[
\rho(\xi + c_2) \geq \int_{\mathbb{R}} \frac{(1 + d_1) \rho(\xi - y) + \rho(\xi - y)}{1 + d_1 ((a_1 + b_1) \epsilon_1 + M \rho(\xi - y))} k_1(y) dy,
\]

which leads to

\[
P_{n+1}(x) \geq \int_{\mathbb{R}} \frac{(1 + d_1) P_n(x - y)}{1 + d_1 (M P_n(x - y) + (b_1 + a_1) \epsilon_1)} k_1(y) dy,
\]
and \( p_0(x) > 0, x \in \mathbb{R} \).

Let \( c_3 = \frac{2c_2 + c^*}{4} < \frac{c_2 + c^*}{2} \). Using the comparison principle and Theorem 5.3, we obtain

\[
\lim \inf_{n \to \infty} \inf_{|x| = c_3n} p_n(x) > 0.
\]

However, if \( -x = c_3n \), then \( \xi \to -\infty, n \to \infty \), and (5.8) implies that

\[
\lim \sup_{n \to \infty} \sup_{-x = c_3n} p_n(x) = 0,
\]

which implies a contradiction. The proof is complete. \( \square \)

Before ending this part, we also give the following remark.

**Remark 5.11** If \( m = 1 \) or \( a_1 = a_2 = 0 \), then (k3) can be replaced by: there exists \( \Lambda \leq \infty \) such that \( \int_{\mathbb{R}} k_i(y)dy = 1 \) and \( \int_{\mathbb{R}} k_i(y)e^{\lambda y}dy < \infty \) for all \( \lambda \in [0, \Lambda) \).

### 5.3 \( m \) Species Competition System

We now consider the traveling wave solutions of (1.1) if

\[
P_i = \frac{(1 + r_i) v^i_n}{1 + r_i \left( v^i_n + \sum_{j \in I} e^i_j v^i_{n-j} + \sum_{j \in J, k \in I, k \neq j} f^i_{kj} v^k_{n-j} \right)}, \tag{5.14}
\]

in which all the parameters are nonnegative and \( r_i > 0, i \in I, J' = J \cup \{0\} \).

Define

\[
\Lambda_i(\lambda, c) = (1 + r_i) \int_{\mathbb{R}} e^{\lambda y - \lambda c_i(y)}dy, i \in I.
\]

**Lemma 5.12** There exists \( C > 0 \) such that for each \( c < C \) such that the following conclusions hold.

(R1) if \( c > C \), then for each \( i \in I \), there exists \( \lambda_i(c) > 0 \) satisfying \( \Lambda_i(\lambda_i(c), c) = 1 \);

(R2) for each fixed \( c > C \), there exists \( \eta \in (1, 2) \) such that

\[
\Lambda_i(\lambda, c) > 1, \lambda \in (0, \lambda_i(c)) \text{ and } \Lambda_i(\eta \lambda, c) < 1, \lambda \in (\lambda_i(c), \eta \lambda_i(c));
\]

(R3) for \( c < C \), there exists \( i \in I \) such that \( \Lambda_i(\lambda, c) > 1, \lambda > 0 \).

**Theorem 5.13** If \( c > C \), then (1.1) with (5.14) has a positive traveling wave solution

\[
u^i_n(y) = \psi_i(t), t = x + cn,
\]

and \( \psi_i(t) \) satisfies

\[
0 < \psi_i(t) < 1, t \in \mathbb{R}, \lim_{t \to -\infty} \psi_i(t)e^{-\lambda_i(c)t} = 1.
\]
Proof. We just give the definition of upper and lower solutions. Define
\[ \psi_i(t) = \min\{e^{\lambda_i(c)t}, 1\}, \overline{\psi_i}(t) = \max\{e^{\lambda_i(c)t} - Ne^{\eta\lambda_i(c)t}, 0\}, \quad i \in I. \]
Similar to the proof of Theorem 5.7, we can verify that \( \psi_i(t), \overline{\psi_i}(t) \) are upper and lower solutions if \( N > 1 \) is large enough. The proof is complete. \( \square \)

Theorem 5.14 Assume that Theorem 5.13 holds. Further suppose that
\[ \sum_{j \in J} e_j^i + \sum_{j \in J', k \in I, k \neq j} f_{kj}^i < 1, \quad (5.15) \]
then (2.3) is true.

Proof. The proof is similar to that of Theorem 5.10 and we just give the expression of contracting rectangles. Define
\[ r_i(s) = sE_i, t_i(s) = sE_i + (1-s)(1+\epsilon), \quad i \in I \]
with \( \epsilon > 0 \) small enough. We can verify (C4) for \( s \in (0,1) \).
\( \square \)

Finally, we present the nonexistence of traveling wave solutions without proof.

Theorem 5.15 If \( c < C \), then (1.1) with (5.14) has not a positive traveling wave solution
\[ u_i^1(y) = \psi_i(t), \quad t = x + ct \]
such that
\[ 0 < \psi_i(t) < 1, \quad t \in \mathbb{R}, \liminf_{t \to \infty} \psi_i(t) > 0, \lim_{t \to -\infty} \psi_i(t) = 0, \quad i \in I. \]

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