Basic properties of magnetohydrodynamic turbulence in the inertial range

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ABSTRACT
We revisit the issue of the spectral slope of magnetohydrodynamic (MHD) turbulence in the inertial range and argue that the numerics favour a Goldreich–Sridhar \(-5/3\) slope rather than \(-3/2\). We also perform precision measurements of the anisotropy of MHD turbulence and determine the anisotropy constant \(C_A = 0.34\) of Alfvénic turbulence. Together with the previously measured Kolmogorov constant \(C_K = 4.2\), or 3.3 for a purely Alfvénic case, it constitutes a full description of the MHD cascade in terms of spectral quantities, which is of high practical value for astrophysics.

Key words: MHD – turbulence.

1 INTRODUCTION
Astrophysical plasmas are often described as an ideal magnetohydrodynamic (MHD) fluid – a perfectly conducting, inviscid fluid described by MHD equations. The use of the continuous fluid approach and the absence of dissipation terms is motivated by the fact that astrophysical scales are typically huge compared with molecular and plasma mean free paths and microscopic dissipation scales. MHD can be successfully applied to all phases of the interstellar medium (ISM), large scales of molecular clouds, intracluster medium (ICM), stellar interiors and stellar outflows, etc. The Reynolds numbers of astrophysical flows are typically very high, which makes turbulence ubiquitous. Initially unmagnetized well-conductive turbulent fluid generates its own magnetic field, which is dynamically important on almost all scales. In spiral galaxies, a large-scale dynamo generates ordered fields on the scale of the galactic disc, while in more symmetric objects such as galaxy clusters a small-scale dynamo operates and generates fields on scales up to 20 kpc.

The inertial range of turbulence was introduced by Kolmogorov (1941) as a range of spatial scales on which driving and dissipation are unimportant and perturbations exist due to energy transfer from one scale to another. In the inertial range of MHD turbulence, perturbations of both velocity and magnetic field will be much smaller than the local Alfvénic velocity \(v_A = B/\sqrt{4\pi\rho}\) due to the turbulence spectrum being steeper than \(k^{-1}\), and therefore the local mean magnetic field will strongly affect dynamics in this range (Iroshnikov 1964; Kraichnan 1965). As in the case of hydrodynamics, the study of MHD turbulence began with weakly compressible and incompressible cases, which are directly applicable to many environments such as stellar interiors, the ICM and hot phases of the ISM. Later it was realized that many features of incompressible MHD turbulence are still present even in supersonic dynamics, due to the dominant effect of Alfvénic shearing (Cho & Lazarian 2003; Beresnyak, Lazarian & Cho 2005). It had been pointed out by Goldreich & Sridhar (1995) that strong mean-field incompressible turbulence is split into a cascade of Alfvénic modes, described by reduced MHD (RMHD: Kadomtsev & Pogutse 1974; Strauss 1976), and a passive cascade of slow (pseudo-Alfvén) modes. In the strong mean-field case it was sufficient to study only the Alfvénic dynamics, as it will determine all statistical properties of the turbulence, such as the spectrum or anisotropy. This decoupling was also observed in numerics. Luckily, as it is the limit of the case of a very strong mean field, RMHD has a precise two-parametric symmetry similar to that in incompressible hydrodynamics (see Section 2), which, under certain conditions, makes a universal cascade with a power-law energy spectrum possible.

Interaction of Alfvénic perturbations propagating in a strong mean field is unusual due to a peculiar dispersion relation of Alfvénic modes, \(\omega = k\cdot v_A\), where \(k\) is a wavevector parallel to the mean magnetic field. This results in a tendency of MHD turbulence to create a ‘perpendicular cascade’, where the flux of energy is preferentially directed perpendicular to the magnetic field. This tendency enhances the non-linearity of the interaction, described by \(\xi = \delta B_k \cdot /\nu_{\alpha} k,\) which is the ratio of the mean-field term to the non-linear term and results in the development of essentially strong turbulence. As the turbulence becomes marginally strong, \(\xi \sim 1\), the cascading time-scales become close to the dynamical time-scales, \(\tau_{\text{casc}} \sim \tau_{\text{dyn}} = 1/\nu_{\alpha} k,\) and the perturbation frequency \(\omega\) has a lower bound due to an uncertainty relation \(\tau_{\text{casc}} \omega > 1\) (Goldreich & Sridhar 1995). This makes turbulence become ‘stuck’ in the \(\xi \sim 1\) regime, which is known as ‘critical balance’. Note that the above argument is not the only lower bound on \(\omega,\) with another bound being due to the directional uncertainty of \(v_A,\) which was discovered by Beresnyak & Lazarian (2008). In this paper we will mostly consider so-called ‘balanced’ turbulence, in which both uncertainties coincide. We refer the reader to Beresnyak & Lazarian (2008, 2009a) and the references therein for the more general imbalanced case.

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The Goldreich–Sridhar model predicts a $k^{-5/3}$ energy spectrum with anisotropy described as $k_1 \sim k_2^{1/3}$. Numerical studies by Cho & Vishniac (2000) and Maron & Goldreich (2001) confirmed the steep spectrum and scale-dependent anisotropy, but Maron & Goldreich (2001) and Müller & Grappin (2005) claimed a shallower than $-5/3$ spectral slope in the strong mean-field case, which was close to $-3/2$. This motivated adjustments to the Goldreich–Sridhar model (Boldyrev 2005; Galtier, Pouquet & Mangeney 2005; Gogoberidze 2007). A model with so-called ‘dynamic alignment’ (Boldyrev 2005, 2006) became popular after this alignment was discovered in numerical simulations (Beresnyak & Lazarian 2006). Boldyrev model is based on the idea that the alignment between velocity and magnetic perturbations decreases the strength of the interaction scale dependingly, relying on the alignment being a power-law function of scale. This would, as he argued modify the spectral slope of MHD turbulence from the $-5/3$ Kolmogorov slope to the observed $-3/2$ slope. It was also claimed that there is a self-consistent turbulent mechanism that produces such an alignment. In this paper we examine both the alignment and the spectrum. It turns out that the alignment is not universal and is tied to the driving scale. Also, spectral resolution studies at high numerical resolutions favour a $-5/3$ spectral slope more than a $-3/2$ slope. It seems that earlier measurements of the MHD slope were premature and were not conducted with enough rigour.

In this paper we (a) support our earlier claim of $-5/3$ scaling and the measurement of the Kolmogorov constant of Beresnyak (2011) and (b) perform a measurement of the anisotropy constant. These two measurements allow us to predict the local properties of MHD turbulence on any scale $l$, given only the dissipation rate $\epsilon$ and Alfvénic velocity $v_A$. Such a straightforward prediction is of great practical use for astrophysics.

## 2 BASIC EQUATIONS

The ideal MHD equations describe the dynamics of an ideally conducting inviscid fluid with a magnetic field and can be written in Heaviside and $c = 1$ units as

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\rho (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + j \times \mathbf{B},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

with current $j = \nabla \times \mathbf{B}$ and vorticity $\omega = \nabla \times \mathbf{v}$. This should be supplanted with an energy equation and a prescription for pressure $P$. The incompressible limit assumes that the pressure is so high that the density is constant and the velocity is purely solenoidal ($\nabla \cdot \mathbf{v} = 0$). This does not necessarily refer to the ratio of outer-scale kinetic pressure to molecular pressure, but could be interpreted as a scale-dependent condition. Indeed, if we go to the frame of the fluid, local perturbations of velocity will diminish with scale and will be much smaller than the speed of sound. In this situation it will be possible to decompose the velocity into low-amplitude sonic waves and an essentially incompressible component of $\mathbf{v}$, as long as we are not in the vicinity of a shock. The incompressible component, bound by $\nabla \cdot \mathbf{v} = 0$, will be described by much simpler equations:

$$\partial_t \mathbf{v} = S(-\omega \times \mathbf{v} + \mathbf{j} \times \mathbf{b}),$$

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}),$$

where we have renormalized the magnetic field to velocity units $\mathbf{b} = \mathbf{B}/\rho^{1/2}$ (the absence of $4\pi$ is due to the Heaviside units) and used the solenoidal projection operator $\hat{S} = (1 - \nabla \Delta^{-1} \nabla)$ to get rid of pressure. Finally, in terms of Elsässer variables $\mathbf{w} = \mathbf{v} \pm \mathbf{b}$ this could be rewritten as

$$\partial_t \mathbf{w}^\pm + (\mathbf{v}_A \cdot \nabla) \mathbf{w}^\pm + \hat{S}(\delta \mathbf{w}^\pm \cdot \nabla)\mathbf{w}^\pm = 0.$$  (1)

This equation resembles the incompressible Euler’s equation. Indeed, hydrodynamics is just a limit of $b = 0$ in which $\mathbf{w}^\pm = \mathbf{v}^\pm$. This resemblance, however, is misleading, as the local magnetic field could not be excluded by the choice of reference frame and, as we noted earlier, will strongly affect the dynamics on all scales. We can explicitly introduce the local mean field as $v_A$, assuming that it is constant, so that $\delta \mathbf{w}^\pm = \mathbf{w} \pm v_A$.

$$\partial_t \mathbf{w}^\pm + (\mathbf{v}_A \cdot \nabla) \mathbf{w}^\pm + \hat{S}(\delta \mathbf{w}^\pm \cdot \nabla)\mathbf{w}^\pm = 0.$$  (2)

In the linear regime of small $\delta w$ this represents perturbations propagating along and against the direction of the magnetic field, with a non-linear term describing their interaction. As we noted earlier, due to the resonance condition of Alfvénic perturbations they tend to create more perpendicular structure, making MHD turbulence progressively more anisotropic. This was empirically known from tokamak experiments and was used in the so-called RMHD approximation, which neglected parallel gradients in the non-linear term (Kadomtsev & Pogutse 1974; Strauss 1976). Indeed, if we denote $\parallel$ and $\perp$ as directions parallel and perpendicular to $\mathbf{v}_A$, the mean-field term $(\mathbf{v}_A \nabla)\mathbf{w}^\pm$ is much larger than $\delta \mathbf{w}^\pm \nabla \delta \mathbf{w}^\pm$ and the latter could be ignored in the inertial range where $\delta \mathbf{w}^\pm \ll v_A$.

This will result in equation (2) being split into

$$\partial_t \delta \mathbf{w}^\parallel + (\mathbf{v}_A \cdot \nabla) \delta \mathbf{w}^\parallel + \hat{S}(\delta \mathbf{w}^\parallel \cdot \nabla)\delta \mathbf{w}^\parallel = 0,$$  (3)

$$\partial_t \delta \mathbf{w}^\perp + (\mathbf{v}_A \cdot \nabla) \delta \mathbf{w}^\perp + \hat{S}(\delta \mathbf{w}^\perp \cdot \nabla)\delta \mathbf{w}^\perp = 0,$$  (4)

which physically represents the limit of a very strong mean field, where $\delta \mathbf{w}^\perp$ is a slow (pseudo-Alfvén) mode and $\delta \mathbf{w}^\parallel$ is the Alfvén mode and equation (3) describes the passive dynamics of the slow mode sheared by the Alfvén mode, while equation (4) describes the essentially non-linear dynamics of the Alfvén mode and is known as RMHD. For our purposes, to figure out the asymptotic behaviour in the inertial range, it is sufficient to study Alfvénic dynamics and the slow mode can always be added later, because it will have the same statistics.

It turns out that RMHD is often applicable beyond the incompressible MHD limit in highly collisionless environments, such as tokamaks or the solar wind. This is due to the fact that the Alfvén mode is transverse and does not require pressure support. Indeed, Alfvénic perturbations rely on magnetic tension as a restoring force and it is sufficient that charged particles be tied to magnetic field lines to provide inertia (Schekochihin et al. 2009).

A remarkable property of RMHD is that it has a precise two-parametric symmetry with respect to the anisotropy and the strength of the mean field: $\mathbf{w} \rightarrow \mathbf{w}A$, $\lambda \rightarrow \lambda B$, $t \rightarrow t/B/A$, $A \rightarrow A/B/A$, which is similar to hydrodynamic symmetry. Here $\lambda$ is a perpendicular scale, $A$ is a parallel scale and $A$ and $B$ are arbitrary parameters of the transformation. It is due to this precise symmetry and the absence of any designated scale that we can hypothesize a universal regime, similar to the hydrodynamic cascade of Kolmogorov (1941). In nature, the universal regime for MHD can be achieved with $\delta \mathbf{w}^\parallel \ll v_A$. In numerical simulations we can directly solve RMHD equations, which have precise symmetry already built in. From a practical viewpoint, the statistics from the full MHD simulation with $\delta \mathbf{w}^\parallel \sim 0.1v_A$ is virtually indistinguishable from RMHD statistics and even $\delta \mathbf{w}^\parallel \sim v_A$ is still fairly similar to the strong mean-field case (Beresnyak & Lazarian 2009a).
3 BASIC SCALINGS

As was shown in a rigorous perturbation study of weak MHD turbulence, it has a tendency to become stronger on smaller scales (Galtier et al. 2000). Indeed, if $k_\perp$ is constant and $k_\parallel$ is increasing, $\xi = \delta w_\perp / \nu k_\parallel$ will increase due to $\delta w \sim k^{-1/2}_\perp$ in this regime. This will naturally lead to strong turbulence, where $\xi$ will become stuck around unity due to two competing processes: (1) increasing interaction by the perpendicular cascade and (2) decrease of interaction due to the uncertainty relation $\tau_{\text{cas}} \omega > 1$, where $\tau_{\text{cas}}$ is a cascading time-scale. Therefore, MHD turbulence will always be marginally strong in the inertial range, which means that the cascading time-scale is associated with the dynamical time-scale, $\tau_{\text{dyn}} \sim \tau_{\text{cas}} = 1/\delta w k_\perp$. (Goldreich & Sridhar 1995). In this case, assuming that energy transfer is local in scale and therefore depends only on the perturbation amplitude on each scale, we can write the Kolmogorov-type phenomenology as

$$\epsilon^+ = (\delta w_\perp^2 / \nu) \lambda, \quad \epsilon^- = (\delta w_\parallel^2 / \nu) \lambda,$$

where $\epsilon^\pm$ is an energy flux of each of the Ehlers variables and $\delta w_\pm$ is a characteristic perturbation amplitude on a scale $\lambda$. Such an amplitude can be obtained by Fourier filtering with a dyadic filter in $k$-space; see e.g. Beresnyak (2012).

Since we consider the so-called balanced case with both $w$ having the same statistical properties and energy fluxes, one of these equations is sufficient. This will result in $\delta w \sim \lambda^{1/3}$, where $\lambda$ is a perpendicular scale, or, in terms of the energy spectrum $E(k)$,

$$E(k) = C_v k^{5/3} / \nu^{5/3},$$

where $C_v$ is known as the Kolmogorov constant. We are interested in the Kolmogorov constant for MHD turbulence. This scaling is supposed to work until dissipation effects kick in. In our further numerical argumentation the dissipation scale will play a big role, not from a physical but rather from a formal point of view. We will introduce an idealized scalar dissipation term on the right-hand side of equation (1) as $-v_\parallel (-\nabla)^2 w = \nu$, where $\nu$ is an order of viscosity and $n = 2$ corresponds to normal Newtonian viscosity, while for $n > 2$ it is called hyperviscosity. The dissipation scale for this Goldreich–Sridhar model is the same as the one for the Kolmogorov model, i.e. $\eta = (v_\parallel^2 / \nu)^{1/(n-2)}$. This is a unique combination of $v_\parallel$ and $\nu$ that has units of length. Note that the Reynolds number, estimated as $vL/\nu$, where $L$ is an outer scale of turbulence, is around $(L/\eta)^{3/2}$.

Furthermore, the perturbations of $w$ will be strongly anisotropic and this anisotropy can be calculated from the critical balance condition $\xi \approx 1$, so that $k_\perp \sim k_\parallel^{1/3}$. Interestingly enough this could be obtained directly from units and the symmetry of RMHD equations from above. Indeed, in the RMHD limit, $k_\perp$ or $1/\Lambda$ must be in a product with $v_\parallel$, since only the product enters the original RMHD equations. We already assumed above that turbulence is local and each scale of turbulence has no knowledge of other scales but only the local dissipation rate $\epsilon$. In this case the only dimensionally correct combination for the parallel scale $\Lambda$ corresponding to perpendicular scale $\lambda$ is

$$\Lambda = C_\Lambda v_\parallel \lambda^{1/3} \epsilon^{-1/3},$$

where we have introduced a dimensionless 'anisotropy constant' $C_\Lambda$. Equations (6) and (7) roughly describe the spectrum and anisotropy of MHD turbulence. Note that Goldreich–Sridhar’s $-5/3$ is a basic scaling that should be corrected for intermittency. This correction is negative due to structure-function power-law exponents being a concave function of their order (see e.g. Frisch 1995) and is expected to be small in the three-dimensional case. This correction for hydrodynamic turbulence is around $-0.03$. Such a small deviation should be irrelevant in the context of debate between $-5/3$ and $-3/2$, which differ by about 0.17.

An alternative model was proposed by Boldyrev (2005, 2006), who suggested that these scalings are modified by a scale-dependent factor that decreases the strength of the interaction, so that the right-hand side of equation (5) is effectively multiplied by a factor of $(\nu L)^{5/3}$, where $L$ is an outer scale. We will discuss this hypothesis further in Section 8. In this case the spectrum will be expressed as $E(k) = C_v \nu L^{5/3} k^{-3/2} L^5$. Note that this spectrum is the only dimensionally correct spectrum with $k^{-3/2}$ scaling that does not contain the dissipation scale $\eta$. The absence of $L \eta$ is due to the so-called zeroth-law of turbulence, which states that the amplitude at the outer scale should not depend on the viscosity. This law follows from the locality of energy transfer and has been known empirically to hold very well. The dissipation scale of the Boldyrev model is different from that of the Goldreich–Sridhar model and can be expressed as $\eta' = (v_\parallel^2 / \nu)^{1/(3n-5)} L^{3/2}$. (3n-5).

4 THE SCALING ARGUMENT

Turbulence with a very long range of scales is common in astrophysics. Three-dimensional numerics, however, is unable to reproduce such ranges and normally struggles to obtain even a small inertial range. In this situation we have to use rigorous quantitative arguments in order to investigate asymptotic scalings.

Suppose we performed several simulations with different Reynolds numbers. If we believe that turbulence is universal and the scale separation between the forcing scale and the dissipation scale is large enough, the properties of small scales should not depend on how the turbulence was driven. This is because neither MHD nor hydrodynamic equations explicitly contains any scale, so a simulation with a smaller dissipation scale could be considered, due to the symmetries from Section 2, as a simulation with the same dissipation scale but a larger driving scale. For example, the small-scale statistics in a 1024 simulation should look the same as small-scale statistics in a 512 one, if the physical size of the elementary cell is the same and the dissipation scale is the same.

This scaling argument requires that the geometry of the elementary cells is the same and the actual numerical scheme used to solve the equations is the same. Also, numerical equations should not contain any scale explicitly, which is usually satisfied. The scaling argument does not require high precision of the dissipation scale or a particular form of dissipation, either explicit or numerical. This is because we only need the small-scale statistics to be similar in both simulations. This is achieved on one hand by applying the same numerical procedure and on the other hand by using a turbulence locality that ensures that outer-scale influence is small.

In practice, the scaling argument, also known as a resolution study, is performed in the following way: the averaged spectra in two simulations expressed in dimensionless units corresponding to the expected scaling, for example $E(k)k^{5/3} \nu^{-2/3}$ is used for hydrodynamics, and plotted versus dimensionless wavenumber $kn$, where the dissipation scale $\eta$ corresponds to the same model, e.g. $\eta = (v_\parallel^2 / \nu)^{1/4}$ is used for scalar second-order viscosity $v$ and Kolmogorov phenomenology. As long as the spectra are plotted this way and the scaling is correct, the curves obtained in simulations with different resolutions have to collapse to a single curve on small scales. Not only the spectrum but also any other statistical property of turbulence can be treated in this way. This method has been used in hydrodynamics for a long time, e.g. by Yeung & Zhou (1997),
Gotoh, Fukayama & Nakano (2002) and Kaneda et al. (2003). For hydrodynamics, good convergence on the dissipation scale has been obtained with rather moderate resolutions, which implies that the hydrodynamic cascade has a fairly narrow locality. When the resolution is increasing, the convergence is supposed to become better. This is because of the increased separation of scale between driving and dissipation. Better convergence means higher precision in discriminating between different scalings, something demonstrated by Kaneda et al. (2003), who directly measured the intermittency correction to the spectral slope. The optimum strategy for our MHD study is to perform the largest resolution simulations possible and to perform the scaling study as described in this section.

5 NUMERICAL METHODS

We used a pseudospectral dealiased code to solve the RMHD equations. The same code was used earlier for RMHD, incompressible MHD and incompressible hydrodynamic simulations. The right-hand side of equation (4) was complemented by an explicit dissipation term $-w_{,n}(-\nabla^2)^{\nu/2}w$ and forcing term $f$. The code and the choice for numerical resolution, driving, etc., were described in great detail in our earlier publications (Beresnyak & Lazarian 2009a,b, 2010; Beresnyak 2011). Table 1 shows the parameters of the simulations. The Kolmogorov scale is defined as $\eta = (\nu/\epsilon)^{1/(3-\nu)}$, while the integral scale is $L = 3\pi/4E \int_{k_0}^{\infty} k^{-3}E(k) \, dk$ (which was approximately 0.79 for R1–3). Dimensionless ratio $L/\eta$ could serve as a ‘length of the spectrum’, although the spectrum is actually significantly shorter for $n = 2$ viscosity and somewhat shorter for $n = 6$ viscosity.

Since we would like to use this paper to illustrate the resolution-study argument, we used a variety of resolution, dissipation and driving schemes. There are four schemes, presented in Table 1 and used in simulations R1–3, R4–5, R6–7 and R8–9. In some of the simulations the resolution in the direction parallel to the mean magnetic field, $n_y$, was reduced by a factor compared with the perpendicular resolution. This was deemed possible due to an empirically known lack of energy in the parallel direction in $k$-space and has been used before (see e.g. Müller & Grappin 2005). The R4–5 and R8–9 groups of simulations were fully resolved in the parallel direction. One would expect that roughly the same resolution will be required in parallel and perpendicular directions (Beresnyak & Lazarian 2009a). In all simulation groups, the time step was strictly inversely proportional to the resolution, so that we can utilize the scaling argument.

Driving had a constant energy injection rate for all simulations except R6–7, which had fully stochastic driving. All simulations except R8–9 had Elsässer driving, while R8–9 had velocity driving. All simulations were well-resolved and R6–7 was overresolved by a factor of 6.6 in scale (a factor of 2 in $Re$). The anisotropy of driving was that of a box, while the injection rate was chosen so that the amplitude was around unity on the outer scale; this roughly corresponds to critical balance on the outer scale. Indeed, as we will show in subsequent sections, since the anisotropy constant is smaller than unity our driving with $\lambda \sim \Lambda \sim 1$ and $\delta w \sim 1$ on the outer scale is somewhat overcritical, so $\Lambda$ decreases below the driving scale to satisfy the uncertainty relation (see Fig. 5 later). This is good for maintaining critical balance over a wide range of scales, as it eliminates the possibility of weak turbulence.

In presenting four groups of simulations, with different geometries of elementary cell, different dissipation terms and different driving, our intention is to show that the scaling argument works irrespective of numerical effects, but rather relies on scale separation and the assumption of universal scaling. Simulations R1–3 are the same as those presented in Beresnyak (2011). The largest simulation R3 with $3.1 \times 10^7$ points used about 0.85 million CPU hours on the Texas Advanced Computing Center (TACC) Ranger.

### Table 1. Three-dimensional RMHD simulations.

| Run | $n_x \times n_y \times n_z$ | Dissipation $(\epsilon)$ | $L/\eta$ |
|-----|-----------------------------|--------------------------|----------|
| R1  | $256 \times 768^2$          | $-6.82 \times 10^{-14}k^6$ | 0.073    | 200     |
| R2  | $512 \times 1536^2$         | $-1.51 \times 10^{-15}k^6$ | 0.073    | 400     |
| R3  | $1024 \times 3072^2$        | $-3.33 \times 10^{-17}k^6$ | 0.073    | 800     |
| R4  | $768^3$                     | $-6.82 \times 10^{-14}k^6$ | 0.073    | 200     |
| R5  | $1536^3$                    | $-1.51 \times 10^{-15}k^6$ | 0.073    | 400     |
| R6  | $384 \times 1024^2$         | $-1.70 \times 10^{-4}k^2$  | 0.081    | 280     |
| R7  | $768 \times 2048^2$         | $-6.73 \times 10^{-5}k^2$  | 0.081    | 560     |
| R8  | $768^3$                     | $-1.26 \times 10^{-4}k^2$  | 0.073    | 350     |
| R9  | $1536^3$                    | $-5.00 \times 10^{-5}k^2$  | 0.073    | 700     |

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Figure 1. Numerical convergence of spectra in all simulations. Two upper rows are used to study convergence assuming a Boldyrev model and the two bottom rows assuming a Goldreich–Sridhar model. Note that the definition of dissipation scale $\eta$ depends on the model (see Section 3); this difference is tiny in hyperviscous simulations R1–5 but significant in viscous simulations R6–9. Numerical convergence requires that spectra will be similar on small scales, including the dissipation scale; see e.g. Gotoh et al. (2002). As we see from the plots, numerical convergence is absent for the Boldyrev model. For the Goldreich–Sridhar model convergence is reached only at the dissipation scale. Higher resolution simulations are required to demonstrate convergence in the inertial range.

wavenumber of the driving, then the change of circulation within a loop of size $l < 2\pi/k_{\text{max}}$ associated with forcing is going to be small, emulating the situation of real turbulence, where the forces exerted on the boundaries of a fluid element will actually preserve the circulation along any loop within this element.

Similarly, Alfvén’s theorem is a conservation of the magnetic flux through a given loop and this implies that the magnetic field lines are frozen into the fluid. It can be broken by an external EMF, however. If such an EMF is restricted to low wavenumbers in a simulation with a periodic box, this will effectively emulate an external EMF action on the boundaries of the fluid element. There is a full analogy between Alfvén’s theorem and Kelvin’s theorem, since we expect neither significant external forces nor external EMF on the inertial range scales. Also, it turns out that there is an analogy between the breakdown of Kelvin’s theorem and Alfvén’s theorem in turbulent flows (Eyink & Aluie 2006; Eyink 2007).

To summarize, the driving for a simulation with a strong mean field or RMHD simulation reproducing the inertial range must have Elsässer driving, i.e. independent driving of $w^+$ and $w^-$. This will supply a supply of Elsässer energies from larger scales. If the turbulence is strong, a pure velocity driving is also possible due to the quick non-linear decorrelation of $w^+$ and $w^-$. Volumetric Elsässer driving does not honour Kelvin’s or Alfvén’s theorems; however, by restricting driving to large scales we effectively emulate the action of external forces, which conserve fluxes and circulations. On the dissipation scales Kelvin’s or Alfvén’s theorems are broken by viscosity and magnetic diffusivity and in the inertial range they will be broken by turbulence. Therefore, there is an analogy between forced viscous hydrodynamic simulations in a periodic box and forced dissipative MHD simulations.

7 SPECTRA

Fig. 1 presents a resolution study of all simulations. The upper rows assume Boldyrev scaling, while the bottom rows assume Goldreich–Sridhar scaling. Reasonable convergence on small scales was achieved only for Goldreich–Sridhar scaling. The normalized amplitude at the dissipation scale for the two upper rows of plots systematically decreases with resolution, suggesting that $-3/2$ is not an asymptotic scaling. The flat part of the normalized spectrum on the R1–3 plots was fitted to obtain a Kolmogorov constant $C_{KA} = 3.27 \pm 0.07$, which was reported in Beresnyak (2011). The total Kolmogorov constant for both Alfvén and slow mode in the above paper was estimated as $C_K = 4.2 \pm 0.2$ for the case of isotropically driven turbulence with zero mean field, where the energy ratio of slow and Alfvén modes $C_s$ is between 1 and 1.3. This larger value $C_K = C_{KA}(1 + C_s)^{1/3}$ is due to the slow mode being passively advected and not contributing to non-linearity. The measurement of $C_{KA}$ had relied on an assumption that the region around $k\eta \approx 0.07$ represents the asymptotic regime. It is possible, though unlikely, that $C_{KA}$ is slightly underestimated due to this region being somewhat lower than the asymptotic regime, owing to an ‘antibottleneck’ effect exacerbated by a reduced parallel resolution. The antibottleneck effect, however, is typically much smaller than the bottleneck effect, so this correction is probably within the stated error bars. The next simulation in the R4–5 series (3072$^3$)
should make this clear. As far as simulations with normal viscosity R6–9 go, it seems impossible to approach a good inertial range until resolutions of at least 4096^3 are reached.

8 DYNAMIC ALIGNMENT

Boldyrev (2005) suggested that w^+ and w^- eddies are systematically aligned. This was investigated numerically by Beresnyak & Lazarian (2006) and no significant alignment was found for the averaged angle between w^+ and w^-. AA = ⟨|δw^+_n × δw^-_n|/|δw^+_n||δw^-_n|⟩, but when this angle was weighted with the amplitude PI = ⟨|δw^+_n × δw^-_n|/|δw^+_n||δw^-_n|⟩, this was termed polarization intermittency (PI), some alignment was found. Then Boldyrev (2006) suggested an alignment between v and b and Mason, Cattaneo & Boldyrev (2006) suggested a particular amplitude-weighted measure, dynamic alignment DA = ⟨|δv_x × δb_y|/|δv_x||δb_y|⟩. We note that DA is similar to PI but contains two effects: alignment and local imbalance. The latter could be measured with imbalance measure (IM) = ⟨|δ(w^+_n)^2 − δ(w^-_n)^2|/|δ(w^+_n)^2 + δ(w^-_n)^2⟩ (Beresnyak & Lazarian 2009b).

In this section we check the assertion of Boldyrev (2005, 2006) that alignment depends on the scale as λ^{1/4}, by using DA which is, for some reason, favoured by the aforementioned group. We carried out a resolution study of DA, assuming the suggested scaling, which is presented in Fig. 2. Convergence was absent in all simulations. Note that previous studies that claimed that there is good correspondence with the Boldyrev model did not perform the scaling study, therefore these claims are not well substantiated. A result from a single isolated simulation could be easily contaminated by the effects of the outer scale, since it is not known a priori how local MHD turbulence is and what resolution is sufficient to get rid of such effects. In contrast, the resolution study offers a systematic approach to this problem.

Fig. 3 shows the ‘dynamic alignment’ slope for all simulations. Although there is no convergence as in the previous plot, it is interesting to note that the alignment slope decreases with resolution. This suggests that most likely the asymptotic state for the alignment slope is zero, i.e. the alignment is scale-independent and the Goldreich–Sridhar model is recovered. Also, the alignment from simulations R1–5 seems to indicate that the maximum of the alignment slope is tied to the outer scale and therefore alignment is a transitional effect.

In our earlier studies (Beresnyak & Lazarian 2006, 2009b) we measured several types of alignment and found no evidence that all alignment measures follow the same scaling; see e.g. Fig. 4. As one alignment measure, PI, was already known to be scale-dependent (Beresnyak & Lazarian 2006) prior to DA, it appears that a particular measure of the alignment in Mason et al. (2006) was picked as being the most scale-dependent and no thorough explanation was given as to why it was preferred. Furthermore, it was claimed that DA has an asymptotic scaling of ~⟨(IL)^1/4⟩ and at the same time is an interaction-weakening factor that will result in a ~1/2 spectrum. This is unlikely, since the interaction-weakening factor will appear from a third-order structure function, while DA is a correction factor based on a ratio of second-order structure functions. Intermittency corrections are supposed to be small, however. More importantly, a physical justification of DA and its preference over other measures such as PI or IM that differ from DA quite significantly was lacking.

We are not aware of any convincing physical arguments explaining why alignment should necessarily be a power law of scale. Boldyrev (2006) argues that alignment will tend to increase, but will be bounded by field wandering, i.e. the alignment on each scale will be created independently of other scales and will be proportional to the relative perturbation amplitude δB/B. However, this violates the two-parametric symmetry of RMHD equations mentioned above, which suggests that field wandering cannot destroy alignment or imbalance. Indeed, a perfectly aligned state, e.g., with δw^- = 0, is a precise solution of the MHD equations and is not destroyed by its own field wandering. The alignment measured in simulations of strong MHD turbulence with different values of δB/Bo showed very little or no dependence on this parameter (Beresnyak & Lazarian 2009b).

Why alignment measures are scale-dependent quantities over about one order of magnitude in scale is an interesting question. The most plausible explanation is that because MHD turbulence is much less local than hydrodynamic turbulence (Beresnyak & Lazarian 2009b, 2010; Beresnyak 2011) and because driving does not mimic the properties of the inertial range, the transition to asymptotic statistics is very wide and many quantities appear as scale-dependent, while they are simply adjusting to the asymptotic regime.

The contribution to the energy flux from different k wavebands is important to understand, since most cascade models assume locality, or rather the very term ‘cascade’ assumes locality. An analytical
upper bound on locality suggests that the width of the energy-transfer window can scale as $C_k^{9/4}$ (Beresnyak 2012); however, in practice turbulence can be more local. The observation of Beresnyak & Lazarian (2009b) that MHD simulations normally lack a bottleneck effect even with high-order dissipation, while hydrodynamic simulations always have a bottleneck which is especially dramatic with high-order dissipation, is consistent with the above conjecture on locality, since the bottleneck effect relies on locality of energy transfer. The locality constraint depends on the efficiency of the energy transfer, so that efficient energy transfer must be local, while inefficient transfer could be non-local (Beresnyak & Lazarian 2010; Beresnyak 2011, 2012). As we observe a larger $C_k$ in MHD turbulence compared with hydrodynamic turbulence, the former could be less local than the latter, which is consistent with our earlier findings.

9 ANISOTROPY

In Section 3 we suggested that anisotropy should be universal in the inertial range and expressed as $\Lambda = C_A k \lambda^{2/3} \epsilon^{-1/3}$, where $C_A$ is an anisotropy constant to be determined from numerical experiments or observations. Note that both Alfvénic and slow modes should have the same anisotropy. This is because they have the same ratio of propagation to non-linear time-scales. Fig. 5 shows the anisotropy for the two-best-resolved groups R1–3 and R4–5. We used a model-independent method of the minimum parallel structure function, described in detail in Beresnyak & Lazarian (2009a). Alternative definitions of local mean field give comparable results, as long as they are reasonable. The convergence of anisotropy curves at the dissipation scale is not as good as the convergence of spectra in R1–5 on the dissipation scale. This is due to this measure being calculated from structure functions that are less local in scale than 3D spectra, i.e. the dissipation scale is still being somewhat affected by transition to the asymptotic regime. From R1–3 we obtain $C_A = 0.34$. Note that the conventional definition of critical balance involves the amplitude, rather than $(\epsilon/\lambda)^{1/3}$, so the constant in this classical formulation will be $C_A C_k^{1/2} \approx 0.63$, which is closer to unity. Together with the energy spectrum this is a full description of a universal axisymmetric two-dimensional spectrum of MHD turbulence in the inertial range.

10 SUMMARY AND DISCUSSION

In this paper we argued that the properties of Alfvén and slow components of MHD turbulence in the inertial range will be determined only by the Alfvén speed $v_A$, dissipation rate $\epsilon$ and scale of interest $\lambda$. The energy spectrum and anisotropy of the Alfvén mode will be expressed as

$E(k) = C_k k^{2/3} \lambda^{-5/3}$,

$\Lambda/\lambda = C_A v_A (\epsilon \lambda)^{-1/3}$,

with $C_k = 3.3$ and $C_A = 0.34$. If the slow mode is present, its anisotropy will be the same and it will contribute to both energy and dissipation rate. Assuming the ratio of slow to Alfvén energies is between 1 and 1.3 – the latter was observed in a statistically isotropic high-resolution MHD simulation with no mean field – we can use $C_k = 4.2$ for the total energy spectrum (Beresnyak 2011).

Anisotropy of MHD turbulence is an important property that affects such processes as interaction with cosmic rays; see e.g. Yan & Lazarian (2002). Since cosmic-ray pressure in our Galaxy is of the same order as dynamic pressure, their importance should not be underestimated. Another process affected is the three-dimensional turbulent reconnection; see e.g. Lazarian & Vishniac (1999). Previous measurements of the energy slope relied on the highest-resolution simulation and fitted the slope in the fixed $k$ range close to the driving scale, typically between $k = 5$ and $k = 20$. We argue that such a fit is unphysical unless numerical convergence has been demonstrated. We can plot the spectrum versus the dimensionless $k \eta$ and if we clearly see a converged dissipation range and a bottleneck then we can assume that larger scales in terms of $k \eta$ represent the inertial range. In fitting a fixed $k$ range at low $k$ we will never get rid of the influence of the driving scale. In fitting a fixed $k \eta$ range, the effects of the driving will diminish with increasing resolution.

Since we still have trouble transitioning into the inertial range in large mean-field simulations, for now it is impossible to demonstrate the inertial range in statistically isotropic simulations similar to the ones presented in Müller & Grappin (2005). This is because we do not expect a universal power-law scaling in the trans-Alfvénic regime, due to the absence of appropriate symmetries, and the transition to the sub-Alfvénic regime, where such scaling is possible, will require some extra scale separation. These two transitions require a numerical resolution that is even higher than the highest
resolution presented in this paper and for now seems computationally impossible.

Full compressible MHD equations contain extra degrees of freedom, which, in a weakly compressible case, entail an additional cascade of the fast MHD mode, possibly of a weak nature. Supersonic simulations with moderate Mach numbers (Cho & Lazarian 2003) show that the Alfvénic cascade is pretty resilient and is not much affected by compressible motions. Models of the ‘universal’ supersonic turbulence covering supersonic large scales and effectively subsonic small scales are based mainly on simulations with limited resolution and are unlikely to hold true. This is further reinforced by the results of this paper, which demonstrated that even a much simpler case of sub-Alfvénic turbulence requires fairly high resolutions to obtain an asymptotic scaling.

In this paper we treated the so-called balanced case with \( \delta w^+ \approx \delta w^\perp \). A more general imbalanced case has been discussed in Beresnyak & Lazarian (2008, 2009a, 2010); see also the references therein.

### 10.1 Results of Grappin and Müller

Grappin & Müller (2010) observed that MHD turbulence has scale-independent anisotropy, in contrast with the Goldreich–Sridhar model and our own results. We believe that this measurement is correct; in fact, similar 2D spectra have been reported in our study (Beresnyak & Lazarian 2009a). However, these are 2D spectra obtained with respect to the global mean magnetic field. A trivial exercise (Lazarian & Vishniac 1999; Cho & Vishniac 2000; Cho, Lazarian & Vishniac 2002; Beresnyak & Lazarian 2009b) shows that measuring anisotropy with respect to the global field will destroy scale-dependent anisotropy, even in the case of a very strong field. Indeed, even if we have \( \delta B_l/B_0 \ll 1 \), the anisotropy in the global frame will be limited from above by \( B_l/\delta B_l \) due to field-wandering, while the Goldreich–Sridhar anisotropy on the scale \( l \) will be much higher, \( \sim B_0/\delta B_l \) by a factor of \( B_l/B_0 \). In particular, each individual volume on scale \( l \) will have anisotropy with respect to the mean field averaged on the same volume; however, the direction of this field will deviate statistically from the global mean field, with a root-mean-square deviation of around \( \sim \delta B_l/B_0 \). This directional deviation will easily destroy an anisotropy \( \sim B_l/\delta B_l \) if we average spectra measured with respect to \( B_0 \). Therefore, one must measure anisotropy with respect to the local mean field, which was realized in Cho & Vishniac (2000). From this argument, we see that the local mean field must be averaged on scales smaller than or equal to \( l \), a scale at which spectrum or structure function is measured, and \( l \) itself is a preferred averaging scale. Also, it is the anisotropy with respect to the local mean field which is important for cosmic-ray dynamics (Yan & Lazarian 2002, 2004).

### 10.2 Results of Mason et al.

Mason et al. (2011) conducted a series of low-resolution simulations \((256^3, 512^3)\) studying the effect on dynamic alignment by the numerics on a viscous scale, e.g. by changing the resolution with constant \( \Reou \) or changing elementary cell geometry such as the ratio of parallel to perpendicular resolution. Their conclusion was that alignment is, indeed, somewhat influenced by the numerics.

As we explained previously, numerical effects on a viscous scale are irrelevant for scaling studies, if the latter are conducted properly. Therefore it was quite puzzling that Mason et al. (2011) claimed that their results invalidate those of Beresnyak (2011), who performed a proper scaling study with high-resolution simulations. Furthermore, it is also puzzling that Mason et al. (2011) claimed an ‘extended scaling law’ for alignment (the previously mentioned \( \Lambda^{1/4} \)), despite the fact that they did not even attempt a proper scaling study for this quantity and did not demonstrate convergence.

Although studying obscure effects on viscous scales is quite common in modern numerical studies of hydrodynamic turbulence, we will argue that such studies in MHD are of limited value for astrophysics, since the dissipation mechanism of astrophysical plasma is, typically, not a scalar second-order diffusion.

In contrast, asymptotic scalings of the inertial range of MHD turbulence are of high value. However, such scalings could only be demonstrated by a proper rigorous scaling study in the spirit of Yeung & Zhou (1997) and Gotoh et al. (2002), and not by arbitrarily assigning the ‘inertial range’ to a fixed range in wavenumbers.

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