WEAK SEMIPROJECTIVITY FOR
PURELY INFINITE $C^*$-ALGEBRAS

JACK SPIELBERG

Abstract. We prove that a separable, nuclear, simple purely infinite, $C^*$-algebra
satisfying the universal coefficient theorem is weakly semiprojective if and only if its
$K$-groups are direct sums of cyclic groups.

Introduction

The first definition of semiprojectivity for $C^*$-algebras was given by Effros and
Kaminker in the context of noncommutative shape theory ([3]). A more restrictive
definition was given by Blackadar in [1]. Loring introduced a third definition,
which he termed weak semiprojectivity, in his investigations of stability problems
for $C^*$-algebras defined by generators and relations ([9]). Recently Neubüser has
introduced a slew of variants, the most important being what he called asymptotic
semiprojectivity ([10]). Using the authors’ initials to represent the above notions,
the implications among them are: $B \Rightarrow N \Rightarrow EKL$.

All versions of semiprojectivity are of the following form: $*$-homomorphisms into
inductive limit $C^*$-algebras can be lifted (in some sense) to a finite stage of the
limit (the precise definitions may be found in section 1, and in the references). As
a consequence, among the first (and easiest) examples for which semiprojectivity
was established are the Cuntz-Krieger algebras. This drew attention to the class
of separable, nuclear, simple purely infinite $C^*$-algebras, now commonly referred
to as Kirchberg algebras ([12]). Kirchberg, and independently Phillips, have shown
that in the presence of the universal coefficient theorem, $K$-theory is a complete
invariant for Kirchberg algebras ([6], [11]). Blackadar proved in [2] that for such
algebras, finitely generated $K$-theory is necessary for semiprojectivity in the sense of
[3]. He conjectured that for these algebras finitely generated $K$-theory is sufficient
for semiprojectivity in the sense of [1], and proved this for the case of free $K_0$
and trivial $K_1$. Szymański extended this to the case where rank $K_1 \leq$ rank $K_0$
([18]), and in [15] semiprojectivity was proved whenever $K_1$ is free. The conjecture
remains open in the case that $K_1$ has torsion. The methods used in all work
on the conjecture rely upon explicit models for these algebras, constructed from
directed graphs. In another direction, Neubüser used abstract methods to show
that (for the algebras under consideration) finitely generated $K$-theory is equivalent
to asymptotic semiprojectivity.

In this paper we study weak semiprojectivity for UCT-Kirchberg algebras. We
prove that such an algebra is weakly semiprojective if and only if its $K$-groups are

1991 Mathematics Subject Classification. Primary 46L05, 46L80. Secondary 22A22.
Key words and phrases. Kirchberg algebra, weak semiprojectivity, graph $C^*$-algebra.

Typeset by $\LaTeX$
direct sums of cyclic groups. The key difficulty lies in dealing with torsion in \( K_1 \), where we are forced to use tensor products of known semiprojectives. Semiprojectivity is badly behaved with respect to tensor products, and we rely on Neubüser’s result to get started. Our contribution is thus in extending to the case where the \( K \)-theory is not finitely generated. Another crucial technical aid is an alternate characterization of weak semiprojectivity, due to Eilers and Loring ([4]).

Our method of proof uses explicit models for the \( C^* \)-algebras constructed from a hybrid object which is partly a directed graph and partly a 2-graph (in the sense of [7]). The construction of this object, and the proof that it defines a UCT-Kirchberg algebra having the desired \( K \)-theory and given by suitable generators and relations, appears in [16].

The outline of the paper is as follows. In section 1 we prove the necessity in the main theorem. This involves a kind of finite approximation property for abelian groups. In section 2 we prove the main theorem. During the final stages of writing an earlier draft of this paper we learned of Huaxin Lin’s preprint [8], where the same theorem is proved by different means.

The figures in this paper were prepared with X Y-pic.

1. Direct Sums of Cyclic Groups

The definition of weak semiprojectivity that follows is not Loring’s original one, but was proved to be equivalent to it in [4], Theorem 3.1.

**Definition 1.1.** The \( C^* \)-algebra \( A \) is called *weakly semiprojective* if given a \( C^* \)-algebra \( B \) with ideals \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I = \bigcup_k I_k \), a \(*\)-homomorphism \( \pi : A \to B/I \), a finite set \( M \subseteq A \), and \( \epsilon > 0 \), there exists \( n \) and a \(*\)-homomorphism \( \phi : A \to B/I_n \) such that

\[
\|\pi(x) - \nu_n \circ \phi(x)\| < \epsilon \quad \text{for } x \in M,
\]

where \( \nu_n : B/I_n \to B/I \) is the quotient map.

It is sometimes convenient to replace the increasing sequence of ideals by a directed family.

We remark that if \( M \) and \( \epsilon \) are omitted, and it is required that \( \pi = \nu_n \circ \phi \), then we recover Blackadar’s definition of semiprojectivity. Neubüser’s definition of asymptotic semiprojectivity can be obtained by omitting \( M \) and \( \epsilon \), and replacing \( \phi \) by a point-norm continuous path \( \phi_t \) such that for every \( x \in A \), \( \lim_t \|\pi(x) - \nu_n \circ \phi_t(x)\| = 0 \).

**Definition 1.2.** An abelian group \( G \) has Property C (for cyclic — see Proposition 1.5 below) if for every finite set \( F \subseteq G \) there exists a finitely generated abelian group \( K \), and homomorphisms \( \alpha : G \to K \), \( \beta : K \to G \) such that \( \beta \circ \alpha(x) = x \) for all \( x \in F \).

**Lemma 1.3.** Let \( A \) be a UCT-Kirchberg algebra. If \( A \) is weakly semiprojective, then \( K_*(A) \) has Property C.

**Proof.** By [6], \( A = \bigcup A_n \), \( A_n \subseteq A_{n+1} \), where each \( A_n \) is a UCT-Kirchberg algebra with finitely generated \( K \)-theory. We modify the mapping telescope construction
slightly (see, e.g., [9]). Let
\[ B = \left\{ f \in C([0,1], A) \mid f(t) \in A_n \text{ for } t \geq \frac{1}{n} \right\}, \]
\[ J_n = \left\{ f \in B \mid f\big|_{[0,1/n]} = 0 \right\}, \]
\[ J = \bigcup J_n = \left\{ f \in B \mid f(0) = 0 \right\}. \]

Then
\[ B/J_n \cong \left\{ f \mid f\big|_{[0,1/n]} = 0 \right\}, \]
\[ B/J \cong A. \]

Now let \([x_1], \ldots, [x_k] \in K_*(A).\) Then by weak semiprojectivity there are \(n\) and \(\phi : A \to B/J_n\) such that
\[ \|\phi(x_i)(0) - x_i\| < 1, \quad 1 \leq i \leq k. \]

Since \(\phi(x_i)(1/n)\) is homotopic to \(\phi(x_i)(0)\), we have that
\[ [x_i] = [\phi(x_i)(1/n)], \quad 1 \leq i \leq k. \]

Let \(\alpha : K_*(A) \to K_*(A_n)\) be given by
\[ \alpha([x]) = [\phi(x)(1/n)], \]
and let \(\beta : K_*(A_n) \to K_*(A)\) be induced from the inclusion. Then \(\beta \circ \alpha([x_i]) = [x_i]\) for \(1 \leq i \leq k.\)

**Lemma 1.4.** Let \(G\) be a countable abelian group with Property \(C\). Then \(G/G_{\text{tor}}\) is free.

**Proof.** By [5], Exercise 52, it suffices to show that every finite rank subgroup of \(G/G_{\text{tor}}\) is free. So let \(H \subseteq G/G_{\text{tor}}\) be a subgroup of finite rank. Put \(\overline{H} = \pi^{-1}(H)\), where \(\pi\) is the quotient map of \(G\) onto \(G/G_{\text{tor}}\). Let \(e_1, \ldots, e_r\) be a basis for \(H\). Then we may write \(\mathbb{Z}^r \subseteq H \subseteq \mathbb{Q}^r\) (relative to this basis). Let \(\overline{e_1}, \ldots, \overline{e_r} \in \overline{H}\) with \(\pi(\overline{e_i}) = e_i, 1 \leq i \leq r.\) Let \(K, \alpha : G \to K\) and \(\beta : K \to G\) be as in Property \(C\), with \(\beta \circ \alpha(\overline{e_i}) = \overline{e_i}\) for \(1 \leq i \leq r.\)

We claim that \(\ker(\alpha|_{\overline{H}}) \subseteq G_{\text{tor}}.\) To see this, let \(y \in \ker(\alpha|_{\overline{H}}).\) Choose \(N \in \mathbb{Z}\) such that \(N\pi(y) \in \mathbb{Z}^r.\) We may write
\[ \pi(Ny) = \sum_{i=1}^r c_i e_i, \quad c_i \in \mathbb{Z}. \]

Then
\[ z = Ny - \sum_{i=1}^r c_i \overline{e_i} \in \ker \pi = G_{\text{tor}}. \]

Thus
\[ 0 = N\beta \circ \alpha(y) = \beta \circ \alpha(Ny) = \beta \circ \alpha(z) + \sum_{i=1}^r c_i \overline{e_i}. \]
But since $\beta \circ \alpha(G_{\text{tor}}) \subseteq G_{\text{tor}}$, we may apply $\pi$ to the last equation to get

$$0 = \sum_{i=1}^{r} c_i e_i.$$ 

It follows that $c_i = 0$ for all $i$, so that $Ny = z \in G_{\text{tor}}$. Hence $y \in G_{\text{tor}}$.

Next we claim that $\ker((\pi_K \circ \alpha)\mid H) = G_{\text{tor}}$, where $\pi_K$ is the quotient map of $K$ onto $K/K_{\text{tor}}$. To see this, first note that the containment $\supseteq$ is obvious. For the other containment, let $y \in \ker((\pi_K \circ \alpha)\mid H)$. Then $\alpha(y) \in K_{\text{tor}}$, so $N\alpha(y) = 0$ for some $N \in \mathbb{Z} \setminus \{0\}$. Then $Ny \in \ker(\alpha\mid H)$, so $Ny \in G_{\text{tor}}$ by the previous claim. Hence $y \in G_{\text{tor}}$.

Finally, it follows from the last claim that $\pi_K \circ \alpha\mid H$ induces an injection $H \to K/K_{\text{tor}}$, which implies that $H$ is free.

**Proposition 1.5.** Let $G$ be a countable abelian group. Then $G$ has Property C if and only if $G$ is a direct sum of cyclic groups.

**Proof.** It is clear that a direct sum of cyclic groups has Property C. Conversely, by Lemma 1.4, $G \cong G_{\text{tor}} \oplus G/G_{\text{tor}}$, where $G/G_{\text{tor}}$ is free, and hence a direct sum of (infinite) cyclic groups. Since $G_{\text{tor}} = \bigoplus_p G_p$, where $G_p$ is the $p$-primary component of $G_{\text{tor}}$, it suffices to prove that $G_p$ is a direct sum of cyclic groups. By [5], Theorem 11, it suffices to prove that $G_p$ contains no element of infinite height. To see this, let $x \in G_p \setminus \{0\}$. Choose $K, \alpha : G \to K$, and $\beta : K \to G$ as in Property C so that $\beta \circ \alpha(x) = x$. We have $\alpha(x) \in K_p$, the $p$-primary component of $K$. Let $n$ be the maximal height of elements of $K_p$. Now if $x = p^j y$ in $G$, then

$$x = \beta \circ \alpha(x)$$
$$= \beta \circ \alpha(p^j y)$$
$$= \beta(p^j \alpha(y))$$
$$= 0, \quad \text{if } j > n.$$ 

Therefore $j \leq n$, and so $x$ is of finite height. 

**Corollary 1.6.** Let $A$ be a UCT-Kirchberg algebra. If $A$ is weakly semiprojective, then $K_*(A)$ is a direct sum of cyclic groups.

2. The Main Theorem

We now wish to prove the converse of Corollary 1.6, establishing weak semiprojectivity for any UCT-Kirchberg algebra whose $K$-theory is a direct sum of cyclic groups. To do this we will use the models for UCT-Kirchberg algebras constructed in [16]. For each $k \geq 2$ let $H_k$ be the directed graph shown in figure 1.

![Figure 1. $H_k$](image)
One easily checks that $K_{*}O(E_{k}) = (\mathbb{Z}/(k), 0)$ (see e.g. [17]). We let $H_{\infty}$ denote the usual directed graph of the Cuntz algebra $O_{\infty}$: one vertex with denumerably many loops. Finally we let $\mathcal{H}_{\infty}$ denote the graph shown in figure 2.

![Figure 2](image)

Again one can easily check that $K_{*}O(\mathcal{H}_{\infty}) = (0, \mathbb{Z})$. We remark that the graphs $H_{k}$, $H_{\infty}$ and $\mathcal{H}_{\infty}$ have a distinguished vertex emitting infinitely many edges, as required for the construction in [16].

We now construct a UCT-Kirchberg algebra having as $K$-theory a prescribed direct sum of cyclic groups. Let $G^{i} = (G_{0}^{i}, G_{1}^{i})$ for $i = 0, 1, \ldots$, where for each $i$, one of $G_{0}^{i}$, $G_{1}^{i}$ is a cyclic group and the other is the zero group. By the Künneth formula ([13]), $G^{i}$ is equal either to $K_{*}(O(H_{k_{i}}) \otimes O(H_{\infty}))$ or to $K_{*}(O(H_{k_{i}}) \otimes O(\mathcal{H}_{\infty}))$ for some $k_{i} \in \{2, 3, \ldots, \infty\}$. Let $E_{i} = H_{k_{i}}$ and $F_{i} = H_{\infty}$ or $\mathcal{H}_{\infty}$ so that $G^{i} = K_{*}(O(E_{i}) \otimes O(F_{i}))$. As in [16], we let $\Omega$ denote the hybrid object constructed from the product 2-graphs $E_{i} \times F_{i}$ and the connecting 1-graphs $D_{i}$. By Theorem 4.8 of [16], $C^{*}(\Omega)$ is a UCT-Kirchberg algebra with $K$-theory equal to $\oplus_{i}G^{i}$. Let $A = C^{*}(\Omega)$.

We briefly recall the definition of the $C^{*}$-algebra $C^{*}(\Omega)$ from [16]. First let us recall the definitions of the $C^{*}$-algebras of a directed graph in a form convenient for this purpose. A directed graph $E$ consists of two sets, $E^{0}$ (the vertices) and $E^{1}$ (the edges), together with two maps $o, t : E^{1} \to E^{0}$ (origin and terminus). We let $O(E)$ denote the $C^{*}$-algebra of $E$. It is the universal $C^{*}$-algebra defined by generators $\{P_{a} \mid a \in E^{0}\}$ and $\{S_{e} \mid e \in E^{1}\}$ with the Cuntz-Krieger relations:

- $\{P_{a} \mid a \in E^{0}\}$ are pairwise orthogonal projections.
- $P_{a}S_{e}S_{e}^{*}P_{a} = P_{t(e)}$, for $e \in E^{1}$.
- $S_{e}S_{f}^{*} \leq P_{o(e)}$, for $e, f \in E^{1}$ with $e \neq f$.
- $0 < \# E^{1}(a) < \infty \implies P_{a} = \sum S_{e}S_{e}^{*} \mid o(e) = a\}$, for $a \in E^{0}$,

where in the fourth relation we use the notation $E^{1}(a)$ to denote the set of edges with origin $a$. (These are a variant of the relations given in [14], Theorem 2.21.)

The relationship between the $C^{*}$-algebras of a graph and a subgraph are crucial to our methods. We refer to [14]. The results are as follows. Let $E$ be a graph and let $F$ be a subgraph of $E$. We let $S = S(F)$ be the set of vertices in $F^{0}$ that do not emit more edges in $E$ than in $F$. We let $TO(F, S)$ denote the relative Toeplitz Cuntz-Krieger algebra of $F$ in $E$. It is the universal $C^{*}$-algebra defined by generators $\{P_{a} \mid a \in F^{0}\}$ and $\{S_{e} \mid e \in F^{1}\}$ with the relations (as above) for $O(F)$, modified by requiring the fourth relation only if $a \in S$. Then $TO(F, S)$ is the $C^{*}$-subalgebra of $O(E)$ generated by the projections and partial isometries associated to the vertices and edges of $F$ ([14], Theorem 2.35).

The hybrid object $\Omega$ is constructed from a directed graph $D$ (see figure 3), and the sequence of product 2-graphs $E_{i} \times F_{i}$ ([17]).
We let $v_i$, respectively $w_i$, denote the distinguished vertex in $E_i$, respectively $F_i$, omitting infinitely many edges, and we form $\Omega$ by attaching $E_i \times F_i$ to $D$ by identifying $u_i$ with $(v_i, w_i)$. By a vertex of $\Omega$ we mean an element of

$$\cup_i(E_i^0 \times F_i^0) \cup D^0,$$

where we identify $u_i$ and $(v_i, w_i)$. By an edge we mean an element of

$$\left(\cup_i(E_i^1 \times F_i^0) \cup (E_i^0 \times F_i^1)\right) \cup D^1.$$

The $C^*$-algebra of $\Omega$ is defined by generators and relations as follows. We let $S$ denote the set of symbols

$$\{P_x \mid x \text{ is a vertex}\} \cup \{S_y \mid y \text{ is an edge}\}.$$

We let $R$ denote the following set of relations on $S$:

(i) $P_x$ is a projection for every vertex $x$, $S_y$ is a partial isometry for every edge $y$.

(ii) For every $a \in E_i^0$, the projections for $\{a\} \times F_i^0$ and the partial isometries for $\{a\} \times F_i^1$ satisfy the Cuntz-Krieger relations corresponding to the graph $F_i$ (see the discussion at the end of the section 1).

(ii') For every $b \in F_i^0$, the projections for $E_i^0 \times \{b\}$ and the partial isometries for $E_i^1 \times \{b\}$ satisfy the Cuntz-Krieger relations corresponding to the graph $E_i$.

(iii) The projections for $D^0$ and the partial isometries for $D^1$ satisfy the Toeplitz-Cuntz-Krieger relations corresponding to the graph $D$ and the vertices $\{a_0, a_1\}$.

(iv) If $\mu$ and $\nu$ are edges of types $D$ and $E_i \times F_i$, respectively, then $S^{*}_{\mu}S_{\nu} = 0$.

(v) For all $e \in E_i^1$ and $f \in F_i^1$ we have

$$S_{(o(e), f)}S_{(e, t(f))} = S_{(e, o(f))}S_{(t(e), f)}$$

$$S_{(t(e), f)}S^{*}_{(e, t(f))} = S^{*}_{(e, o(f))}S_{(o(e), f)}.$$
Lemma 2.1. The C*-algebra $A = C^*(\Omega)$ is weakly semiprojective.

Proof. Let $\Omega_n$ be the subobject of $\Omega$ consisting of $D_0, D_1, \ldots, D_n, E_0 \times F_0, \ldots, E_n \times F_n$. (We use parentheses in order to avoid confusion with the notation of [16]s.) Theorem 4.8 of [16] applies to $\Omega_n$, so that $A_n = C^*(\Omega_n)$ is a UCT-Kirchberg algebra with finitely generated $K$-theory. Note that the generators and relations defining $C^*(\Omega_n)$ are the same in $C^*(\Omega)$ (this is essentially because of the infinite valence of the vertex $u_{n+1}$). Thus $A_n$ is a $C^*$-subalgebra of $A$, and $A = \bigcup_n A_n$. By Satz 6.12 of [10], $A_n$ is uniformly asymptotically semiprojective. It follows from Theorem 3.1 of [4] that $A_n$ is weakly semiprojective.

Let $B$ be a $C^*$-algebra with ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I = \bigcup_k I_k$, and let $\pi : A \rightarrow B/I$ be a $\ast$-homomorphism. Let $M \subseteq A$ be a finite set, and let $\epsilon > 0$. Choose $n$, and a finite set $M' \subseteq A_{n-1}$, such that $d(x, M') < \epsilon/2$ for all $x \in M$. Since $A_n$ is weakly semiprojective there is $k$, and a $\ast$-homomorphism $\phi_0 : A_n \rightarrow B/I_k$, such that

$$
\|\pi(x') - \nu_k \circ \phi_0(x')\| < \epsilon/2 \quad \text{for} \ x' \in M',
$$

where $\nu_k : B/I_k \rightarrow B/I$ is the quotient map. We will construct a $\ast$-homomorphism $\phi : A \rightarrow B/I_k$ extending $\phi_0|_{A_{n-1}}$. Then it will follow that

$$
\|\pi(x) - \nu_k \circ \phi(x)\| < \epsilon \quad \text{for} \ x \in M,
$$

concluding the proof.

Let $p = P_{u_{n+1}}$ and $q = P_{u_n}$, the projections in $A$ corresponding to the vertices $u_{n+1}$ and $u_n$. The hereditary subalgebra $pAp$ of $A$ contains a hereditary subalgebra, $C$, isomorphic to $A$. (This follows easily from the pure infiniteness of $A$. See, e.g., the proof of Theorem 3.12 in [15].) Let $\psi_1 : A \rightarrow C$ be a $\ast$-isomorphism. Since the inclusion of $C$ into $A$ induces the identity in $K$-theory, it follows that $\psi_1$ is an automorphism of $K_\ast(A)$. It follows from Theorem 4.2.1 of [11] that there is a $\ast$-automorphism, $\alpha$, of $A$ with $\alpha_\ast = \psi_1$. Let $\psi_2 = \psi_1 \circ \alpha^{-1}$. Then $\psi_2 : A \rightarrow C$ is a $\ast$-isomorphism, and $\psi_2$ is the identity in $K$-theory. Let $x \in A$ be a partial isometry with $x^*x = q$ and $xx^* = \psi_2(q)$. Increasing $k$ if necessary we may find a partial isometry $z \in B/I_k$ with $z^*z = \phi_0(q)$ and $zz^* = \phi_0 \circ \psi_2(q)$. We define $\phi : A \rightarrow B/I_k$ by defining it on the generators $S$ of $A$ (Definition 3.3 of [16]):

$$
\phi(s_y) = \begin{cases} 
\phi_0(s_y), & y \in \Omega_{n-1} \\
\phi_0 \circ \psi_2(s_y), & y \notin \Omega_{n-1} \text{ and } o(y), t(y) \neq u_n \\
(\phi_0 \circ \psi_2(s_y))^*, & y \notin \Omega_{n-1} \text{ and } t(y) = u_n, o(y) \neq u_n \\
z(\phi_0 \circ \psi_2(s_y)), & y \notin \Omega_{n-1} \text{ and } o(y) = u_n, t(y) \neq u_n \\
z(\phi_0 \circ \psi_2(s_y))^*, & y \notin \Omega_{n-1} \text{ and } o(y) = t(y) = u_n.
\end{cases}
$$

It is easy to see that the elements $\phi(s_y)$ satisfy the relations $R$ of [16], Theorem 3.3, and hence $\phi$ defines a $\ast$-homomorphism.

Theorem 2.2. Let $A$ be a UCT-Kirchberg algebra. Suppose that $K_\ast(A)$ is a direct sum of cyclic groups. Then $A$ is weakly semiprojective.

Proof. As in the proof of Theorem 3.12 of [15], it suffices to prove that if $A$ is unital and $K \otimes A$ is weakly semiprojective, then $A$ is weakly semiprojective. (We are relying on the classification theory of [6] and [11], as well as the theorem of [19]
that nonunital separable, simple, purely infinite $C^*$-algebras are stable.) Let $u_1$, $u_2, \ldots \in A$ with $u^*_iu_j = \delta_{ij}$. Put
\[ A_0 = \text{span}\{u_1Au_1^*\}. \]
Then $A_0$ is isomorphic to $K \otimes A$.

Let $A = B/I$, where $I$ is the closure of a directed family of ideals, $\mathcal{L}$, of $B$. (Since $A$ is simple we may dispense with the homomorphism $\pi$ of Definition 1.1.) We let $\pi : B \to B/I$, $\pi_J : B \to B/J$ for $J \in \mathcal{L}$, denote the quotient maps. Put $B_0 = \pi^{-1}(A_0)$. We will use [4], Theorem 3.1. So let $F \subseteq A$ be a finite set, and let $\epsilon > 0$. We may assume that $\epsilon < 1$ and that $1 \in F$. Choose $\gamma < 1$ such that $3\gamma \|x\| < \epsilon$ for all $x \in F$. Since $A_0$ is weakly semiprojective by hypothesis, there is a *-homomorphism $\psi_{00} : A_0 \to B/J$ such that
\[ \|\pi \circ \psi_{00}(x) - x\| < \gamma \quad \text{for } x \in u_1Fu_1^*. \]
In particular, we have
\[ \|\pi \circ \psi_{00}(u_1u_1^*) - u_1u_1^*\| < \gamma. \]
Choose $v \in B$ with $\pi(v) = u_1$. By increasing $J$, if necessary, we may assume that $\pi_J(v)$ is an isometry. Since
\[ \|\pi(vv^*) - \pi \circ \psi_{00}(u_1u_1^*)\| < \gamma, \]
we may assume, again by increasing $J$ if necessary, that
\[ \|\pi_J(vv^*) - \psi_{00}(u_1u_1^*)\| < \gamma. \]
Then there exist $s, t \in B$ such that
\[
\begin{align*}
\pi_J(s^*s) &= \pi_J(vv^*) \\
\pi_J(ss^*) &= \psi_{00}(u_1u_1^*) \\
\pi_J(tt^*) &= 1 - \pi_J(vv^*) \\
\|\pi_J(s) - \pi_J(vv^*)\| &< \gamma \\
\|\pi_J(t) - (1 - \pi_J(vv^*))\| &< \gamma.
\end{align*}
\]
Let $z = s + t$. Then $\pi_J(z)$ is unitary, and $\|\pi_J(z) - 1\| < \gamma$. Define $\psi_0 : A_0 \to B/J$ by
\[ \psi_0(x) = \pi_J(z)^*\psi_{00}(x)\pi_J(z). \]
Then $\psi_0$ is a *-homomorphism, and
\[
\begin{align*}
\|\pi \circ \psi_0(x) - x\| &= \|\pi(z)^*\pi \circ \psi_{00}(x)\pi(z) - x\| \\
&\leq 2\|\pi(z) - 1\| \|x\| + \|\pi \circ \psi_{00}(x) - x\| \\
&\leq 2\gamma \|x\| + \gamma \\
&< \epsilon, \quad \text{for } x \in u_1Fu_1^*,
\end{align*}
\]
\[ \psi_0(u_1u_1^*) = \pi_J(z)^*\psi_{00}(u_1u_1^*)\pi_J(z) = \pi_J(s)^*\psi_{00}(u_1u_1^*)\pi_J(s) = \pi_J(vv^*). \]
Now define $\psi : A \to B/J$ by

$$\psi(x) = \pi_J(v^*\psi_0(u_1xu_1^*)\pi_J(v)).$$

By (**) we have that $\psi$ is a $*$-homomorphism. For $x \in F$, we have

$$\|\pi \circ \psi(x) - x\| = \|u_1^*\pi \circ \psi_0(u_1xu_1^*)u_1 - x\|
= \|u_1^*(\pi \circ \psi_0(u_1xu_1^*) - u_1xu_1^*)u_1\|
\leq \|\pi \circ \psi_0(u_1xu_1^*) - u_1xu_1^*\|
< \epsilon,$$

by (*).

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Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804

E-mail address: jack.spielberg@asu.edu