A mathematical framework for determining the stability of steady states of reaction-diffusion equations with periodic source terms

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We develop a mathematical framework for determining the stability of steady states of generic nonlinear reaction-diffusion equations with periodic source terms, in one spatial dimension. We formulate an a priori condition for the stability of such steady states, which relies only on the properties of the steady state itself. The mathematical framework is based on Bloch’s theorem and a generalization of Poincaré’s inequality for mean-zero periodic functions. Our framework can be used for stability analysis to determine the regions in an appropriate parameter space for which steady-state solutions are stable.

Keywords: Bounds, Reaction-Diffusion equations

I. INTRODUCTION AND PROBLEM STATEMENT

We introduce a mathematical framework for analyzing the stability of equilibrium solutions of generic reaction-diffusion equations with a periodic source term. Nonlinear reaction-diffusion equations occur in the context of pattern formation, chemical reactions, mathematical biology, and phase separation of binary alloys [1–3]). The addition of a forcing term in such systems of equations can be used effectively to drive a chemical reaction to a desired outcome [4]. Equally, a source term of travelling-wave type can be used to control the naturally-occurring travelling waves in such systems [5]. Motivated by these applications, in this article we consider the generic nonlinear reaction-diffusion equation

$$\frac{\partial C}{\partial t} = \sigma(x) + N(C) + \frac{\partial^2 C}{\partial x^2}, \quad x \in (-\infty, \infty), \quad t > 0,$$

where $N(C)$ is a smooth nonlinear function of the variable $C$, and $\sigma(x)$ is a periodic source term, with $\sigma(x + L) = \sigma(x)$. We seek periodic steady-state solutions $C_0(x)$ that satisfy

$$0 = \sigma(x) + N(C_0) + \frac{\partial^2 C_0}{\partial x^2}, \quad x \in (0, L), \quad C_0(x + L) = C_0(x).$$

If such solutions can be found, it is of interest to classify their stability according to linear stability theory. As such, in this article we consider a solution

$$C(x, t) = C_0(x) + \delta C(x, t),$$

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where \( \delta C(x, t) \) is a small perturbation. By substituting Equation (3) into Equation (1) and linearizing the nonlinear term \( N(C) \), a linearized partial differential equation for the perturbation \( \delta C(x, t) \) is obtained, which is valid provided the magnitude of the perturbation remains small in the sense that \( |\delta C| \ll \frac{|N''(C_0)|}{N'(C_0)} \), for all \( x \) and \( t > 0 \) (here, \( N' \) denotes the derivative of \( N \) with respect to its argument and \( N'' \) denotes the second derivative). As such, the following linearized partial differential equation for \( \delta C \) is obtained:

\[
\frac{\partial}{\partial t} \delta C = N'(C_0) \delta C + \frac{\partial^2}{\partial x^2} \delta C, \quad x \in (-\infty, \infty), \quad t > 0.
\]

The boundary conditions for Equation (4) are provided based on physical intuition, namely that the perturbations should vanish as \( |x| \to \infty \). In this article, we introduce a mathematical framework that enables us to derive sufficient conditions such that the solution \( C_0(x) \) is stable, i.e. such that \( \lim_{t \to \infty} \delta C(x, t) = 0 \). As such, we emphasize that the purpose of this article is not to construct base-state solutions \( C_0(x) \) corresponding to Equation (2), but rather to determine \textit{a priori} the stability of such solutions, once \( C_0(x) \) is known.

This article is organized as follows. In Section II we reduce Equation (4) to an eigenvalue problem, and we characterize the eigenvalues and eigenfunctions. We formulate the condition for linear stability of the solution \( C_0(x) \) in terms of eigenvalues of a certain linear operator. In Section III we derive certain estimates inspired by Poincaré's inequality for mean-zero periodic functions [6]. Using these estimates, in Section IV we formulate sufficient conditions such that \( C_0(x) \) is a stable equilibrium solution of Equation (1). Concluding remarks are presented in Section V.

II. EIGENVALUE ANALYSIS

We consider Equation (4), written here in abstract terms as follows:

\[
\frac{\partial \phi}{\partial t} = s(x) \phi + \frac{\partial^2 \phi}{\partial x^2}, \quad x \in (-\infty, \infty), \quad t > 0,
\]

with smooth square-integrable initial condition

\[
\phi(x, t = 0) = \phi_0(x), \quad \phi_0 \in L^2(-\infty, \infty),
\]

and where \( s(x) \) is an \( L \)-periodic function, with \( s(x + L) = s(x) \). We further specify the behavioral boundary condition

\[
\phi(x) \to 0 \text{ as } |x| \to \infty.
\]

We apply separation of variables to Equation (5a), with \( \phi(x, t) = e^{-\lambda t} \psi(x) \). This yields

\[
-\lambda \psi(x) = \left[ s(x) + \frac{\partial^2}{\partial x^2} \right] \psi(x),
\]
The linear operator $L = s(x) + \partial_{xx}$ is translation-invariant under the group operation $x \to x + L$. The corresponding group action on functions can be written in terms of a translation operator $T_L f(x) = f(x + L)$, for a generic function $f(x)$. Thus, $L$ and $T_L$ commute, and are therefore simultaneously diagonalizable. As such, the eigenfunctions of $L$ can be written as

$$\psi(x) = \psi_{pn}(x) = e^{ipx} u_{pn}(x), \quad p \in [-\kappa/2, \kappa/2], \quad n \in \{0, 1, \ldots\},$$

(7)

where $u_{pn}(x)$ is an $L$-periodic function and solves the self-adjoint problem

$$-\lambda_{pn} u_{pn} = \left[s(x) - p^2 + 2ipx \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\right] u_{np},$$

(8)

**Remark 1** Equation (7) is a particular application of Bloch’s Theorem.

By self-adjointness of the operator in Equation (8) the eigenfunctions $u_{pn}$ can be normalized to satisfy

$$\int_0^L u_{np}^*(x) u_{n'p'}(x) \, dx = (L/2\pi) \delta_{nn'},$$

(9)

where the star indicates complex conjugation. Hence also,

$$\int_{-\infty}^{\infty} \psi_{np}^*(x) \psi_{n'p'}(x) \, dx = \sum_{j=-\infty}^{\infty} \int_{jL}^{(j+1)L} e^{i(p-p')x} u_{np}^*(x) u_{n'p'}(x) \, dx,$$

$$= \sum_{j=-\infty}^{\infty} e^{i(jL)(p-p')} \int_0^L e^{i(p-p')x} u_{np}^*(x) u_{n'p'}(x) \, dx,$$

$$= (2\pi/L) \delta(p - p') \int_0^L e^{i(p-p')x} u_{np}^*(x) u_{n'p'}(x) \, dx,$$

$$= \delta(p - p') \delta_{nn'},$$

which is a completeness relation for the Bloch eigenfunctions. As such,

$$\phi_0(x) = \int_{-\kappa/2}^{\kappa/2} \left[ \sum_{n=0}^{\infty} (\psi_{np}, \phi_0) \psi_{np}(x) \right] \, dp,$$

where

$$\langle \psi_{np}, \phi_0 \rangle = \int_{-\infty}^{\infty} \left[ e^{ipx} u_{np}(x) \right]^* \phi_0(x) \, dx$$

denotes the inner product of $\psi_{np}$ with $\phi_0$ (the integral makes sense because $\phi_0(x) \in L^2(-\infty, \infty)$). Hence, the general solution of Equation (5) reads

$$\phi(x, t) = \int_{-\kappa/2}^{\kappa/2} \left[ \sum_{n=0}^{\infty} (\psi_{np}, \phi_0) \psi_{np}(x) e^{-\lambda_{np}t} \right] \, dp.$$

Hence, in order for the solution $\phi(x, t)$ to remain bounded for all time, we require $\lambda_{np} \geq 0$, for all $p \in [-\kappa/2, \kappa/2]$ and all $n \in \{0, 1, \ldots\}$. 

III. THE POINCARÉ INEQUALITY FOR BLOCH FUNCTIONS

In this section, we generalize Poincaré’s inequality, with a view to applying the generalized inequality to analyze the eigenvalues of Equation (8). We begin by recalling Poincaré’s inequality for purely periodic mean-zero functions \( f(x + L) = f(x) \). To fix ideas, we assume that \( f(x) \) is continuously differentiable, however, this assumption may be replaced with something less restrictive if required. Hence, \( f(x) \) may be written as a Fourier series

\[
f(x) = a_0 + \sum_{j=-\infty}^{\infty} a_j e^{i\kappa jx} \quad \kappa = \frac{2\pi}{L}. \tag{10}\]

We define the mean of \( f(x) \),

\[
\langle f \rangle = \frac{1}{L} \int_0^L f(x) dx,
\]

hence \( a_0 = L\langle f \rangle \). We work with mean-zero functions \( \langle f \rangle = 0 \), hence \( a_0 = 0 \) in Equation (10). We further define the \( L^2 \) norm

\[
\|f\|_2^2 = \int_0^L |f|^2 dx, \tag{11}
\]

and for all \( f \in L^2([0,L]) \). By Parseval’s identity, we have

\[
\|df/dx\|_2^2 = \kappa^2 L \sum_j' j^2 |a_j|^2 \geq L\kappa^2 \sum_j' |a_j|^2 = \kappa^2 \|f\|_2^2. \tag{12}
\]

Hence, \( \|df/dx\|_2^2 \leq \rho \|f\|_2^2 \) for mean-zero periodic functions, with \( \rho = (2\pi/L)^2 \). This is the standard Poincaré inequality for mean-zero periodic functions, and is well studied. This result is typically used in applications where idealized problems may be posed on periodic domains – for instance, in the analysis of the Navier–Stokes equations \( \square \), and in fluid mixing \( \square \). In the periodic case, the optimal value of the Poincaré constant is realised for mean-zero functions of the form \( f(x) = A \cos[2\pi/L x + \varphi] \), where \( A \) and \( \varphi \) are real positive numbers corresponding respectively to an amplitude and a phase. It is straightforward to extend this result to functions of the type \( e^{i\eta x} f(x) \), where \( f(x) \) is again a mean-zero \( L \)-periodic function:

**Theorem 1** Let \( f(x) \) be an \( L \)-periodic mean-zero function. Then,

\[
\|\frac{d}{dx} [e^{i\eta x} f(x)]\|_2^2 \geq \rho_p \|f\|_2^2, \quad \rho_p = (|\eta| - \kappa)^2. \tag{13}
\]

**Proof** The proof follows by direct computation analogous to Equation (12).

**Remark 2** The bound in Equation (13) is sharp for \( \eta \leq 0 \), and \( f(x) = Ae^{i(\kappa x+\varphi)} \), where \( A \) and \( \varphi \) are real positive numbers.
Remark 3 The derivative term in Equation (13) can also be written as
\[ \parallel \frac{d}{dx} \left[ e^{ipx} f(x) \right] \parallel_2^2 = p^2 \parallel f \parallel_2^2 - 2ip(\frac{d}{dx} f, \frac{df}{dx}) + \parallel \frac{d}{dx} e^{ipx} f \parallel_2^2, \] (14)
where the angle brackets \( \langle \cdot, \cdot \rangle \) denote the usual \( L^2 \) norm,
\[ \langle f, g \rangle = \int_0^L (f^*) g \, dx, \quad f, g \in L^2([0, L]). \]
This can be verified by direct computation.

IV. SUFFICIENT CONDITIONS FOR STABILITY

We return to the eigenvalue analysis of Equation (8), to determine sufficient conditions for \( \lambda_{np} \geq 0 \). As such, we multiply Equation (8) by \( u_{np}^* \) and integrate from \( x = 0 \) to \( x = L \), applying the periodic boundary conditions to \( u_{np}(x) \). We obtain
\[ \lambda_{np} \parallel u_{np} \parallel_2^2 = -\langle u_{np}, s u_{np} \rangle + \parallel \frac{d}{dx} e^{ipx} u_{np} \parallel_2^2. \] (15)
Clearly, if \( s(x) \leq 0 \) for all \( x \in [0, L] \), then \( \lambda_{np} \geq 0 \). This is certainly a sufficient condition for stability, however, it is highly prescriptive. As such, we further explore the possibility that \( \langle s \rangle < 0 \) but that \( \max_{[0,L]} s(x) = s_0 \geq 0 \). We therefore derive constraints on \( \langle s \rangle \) and \( s_0 \) such that \( \lambda_{np} \geq 0 \). We summarize the results here:

**Theorem 2** The eigenvalues \( \lambda_{np} \) of Equation (8) are all positive if

- \( s(x) \leq 0 \) for all \( x \in [0, L] \), or less restrictively,
- If \( s_0 = \max_{[0,L]} s(x) \) is positive, but
  \[ s_0 \leq \max_{p \in [-\kappa/2,\kappa/2]} \rho_p, \] (16a)
  \[ \parallel \delta s \parallel_2^2 \leq \min_{p \in [-\kappa/2,\kappa/2]} (\rho_p - s_0) \left( (\langle s \rangle) + p^2 \right). \] (16b)

Here, \( \langle s \rangle = L^{-1} \int_0^L s(x) \, dx \), and \( \delta s = s - \langle s \rangle \).

**Proof** The starting-point for the proof is Equation (8) and its averaged version, recalled here as
\[ -\lambda u_p = su_p + \mathcal{D} u_p, \]
\[ -\lambda(u_p) = \langle su_p \rangle + \langle \mathcal{D} u_p \rangle, \]
where we have suppressed subscripts and where \( \mathcal{D} = -p^2 + 2ip \partial_x + \partial_{xx} \). We further decompose \( s \) and \( u_p \) in terms of mean components and fluctuations:
\[ s = \langle s \rangle + \delta s, \quad u_p = \langle u_p \rangle + \delta u_p. \]
Hence,
\[ -\lambda(u_p) = -\langle s \rangle (u_p) - \langle \delta s \delta u_p \rangle - p^2 (u_p). \]
We identify two cases:
• If \( \lambda + \langle s \rangle - p^2 = 0 \), then \( \lambda = p^2 - \langle s \rangle = p^2 + |\langle s \rangle| \geq 0 \) and the eigenvalues \( \lambda \) are all non-negative.

• Otherwise, \( \lambda + \langle s \rangle - p^2 \neq 0 \).

If Case 1 pertains, the eigenvalues are all positive, and we are done. We therefore assume that Case 2 pertains, hence

\[
\langle u_p \rangle = -\frac{\langle \delta s \delta u_p \rangle}{\lambda + \langle s \rangle - p^2}. \tag{17}
\]

We now further rewrite the eigenvalue problem (8) in terms of \( \delta u_p \) and \( \delta s \):

\[-\lambda \delta u_p = \langle s \rangle \delta u_p + \langle u_p \rangle \delta s + \delta s \delta u_p - \langle \delta s \delta u_p \rangle + \mathcal{D} \delta u_p.\]

We multiply both sides by \( \delta u_p^* \) and integrate from \( x = 0 \) to \( x = L \). We obtain

\[- \lambda \| \delta u_p \|^2_2 = \langle s \rangle \| \delta u_p \|^2_2 + \langle u_p \rangle (\delta u_p, \delta s) + (\delta u_p, \delta s \delta u_p) + (\delta u_p, \mathcal{D} \delta u_p). \tag{18}\]

This can furthermore be written as

\[- \lambda \| \delta u_p \|^2_2 = \langle s \rangle \| \delta u_p \|^2_2 + \langle u_p \rangle (\delta u_p, \delta s) + (\delta u_p, \mathcal{D} \delta u_p). \tag{19}\]

We use Equation (17) to eliminate \( \langle u_p \rangle \) from Equation (18):

\[\lambda \| \delta u_p \|^2_2 = -\langle s \rangle \| \delta u_p \|^2_2 + \langle u_p \rangle (\delta u_p, \delta s) + \frac{\langle \delta s \delta u_p \rangle}{\lambda + \langle s \rangle - p^2} (\delta u_p, \delta s).\]

Since \( s(x) \) is real-valued, we have

\[\langle \delta s \delta u_p \rangle (\delta u_p, \delta s) = |\langle \delta u_p, \delta s \rangle|^2,\]

hence

\[\lambda \| \delta u_p \|^2_2 = -\langle s \rangle \| \delta u_p \|^2_2 + \langle u_p \rangle (\delta u_p, \delta s) + \frac{|\langle \delta s \delta u_p \rangle|^2}{\lambda + \langle s \rangle - p^2}. \tag{20}\]

In what follows, it will be necessary to have \( Q \geq 0 \). We now formulate conditions on \( s(x) \) such that \( Q \geq 0 \). We have

\[
Q \overset{\text{Eq. (14)}}{=} -\langle s \rangle \| \delta u_p \|^2_2 - \langle \delta u_p, \mathcal{D} \delta u_p \rangle, \\
\geq -s_0 \| \delta u_p \|^2_2 + \| \frac{d}{dx} e^{ipx} \delta u_p \|^2_2, \\
\geq -s_0 \| \delta u_p \|^2_2 + \| \frac{d}{dx} e^{ipx} \delta u_p \|^2_2.
\]

We use the Poincaré-type estimate from Section III to write

\[Q \geq [-s_0 + (|P| - \kappa)^2] \| \delta u_p \|^2_2, \quad p \in [-\kappa/2, \kappa/2].\]
Hence, we require
\[ s_0 \leq (|p| - \kappa)^2, \quad \text{for all } p \in [-\kappa/2, \kappa/2], \]
hence
\[ s_0 \leq \frac{1}{4} \kappa^2. \]
Hence, Equation (20) can be written as
\[ \lambda = a + \frac{b}{\lambda - c}, \quad (21) \]
where \( a, b, \) and \( c \) are positive real numbers:
\[
a = -\frac{\langle \delta u_p, s \delta u_p \rangle + \langle \delta u_p, D \delta u_p \rangle}{\| \delta u_p \|_2^2} - \frac{\langle \delta u_p, s \delta u_p \rangle}{\| \delta u_p \|_2^2} + \frac{\| \frac{d}{dx} e^{ipx} \delta u_p \|_2^2}{\| \delta u_p \|_2^2},
\]
and
\[ b = \frac{|\langle \delta u_p, \delta s \rangle|^2}{\| \delta u_p \|_2^2}, \quad c = |\langle s \rangle| + p^2. \]
Equation (21) has solutions
\[ \lambda = \frac{a + c}{2} \left[ 1 \pm \sqrt{1 - \frac{4(ac - b)}{(a + c)^2}} \right]. \quad (22) \]
Since Equation (8) is a self-adjoint problem, the eigenvalues \( \lambda \) are real-valued. Furthermore, to make \( \lambda \geq 0 \) for both signs in Equation (22), it suffices to make \( b \leq ac \), hence
\[ \frac{|\langle \delta u_p, \delta s \rangle|^2}{\| \delta u_p \|_2^2} \leq \left[ -\frac{\langle \delta u_p, s \delta u_p \rangle}{\| \delta u_p \|_2^2} + \frac{\| \frac{d}{dx} e^{ipx} \delta u_p \|_2^2}{\| \delta u_p \|_2^2} \right] \left( |\langle s \rangle| + p^2 \right). \]
A sufficient condition for this to be true for eigenfunctions \( \delta u_p \) is if
\[ \sup_{\phi \neq 0} \frac{|\langle \phi, \delta s \rangle|^2}{\| \phi \|_2^2} \leq \inf_{\phi \neq 0} \left[ -s_0 + \frac{\| \frac{d}{dx} e^{ipx} \phi \|_2^2}{\| \phi \|_2^2} \right] \left( |\langle s \rangle| + p^2 \right), \]
where the sup and inf are taken over all mean-zero differentiable functions on \([0, L]\).
We therefore again apply the Poincaré-type estimate (13) to write this condition as
\[ \| \delta s \|_2^2 \leq \left[ -s_0 + (|p| - \kappa)^2 \right] \left( |\langle s \rangle| + p^2 \right). \]
This should be true for all \( p \):
\[ \| \delta s \|_2^2 \leq \min_{p \in [-\kappa/2, \kappa/2]} \left[ -s_0 + (|p| - \kappa)^2 \right] \left( |\langle s \rangle| + p^2 \right). \]
This concludes the proof.
To illustrate how the conditions in Theorem 2 may apply, we consider the model source term

\[ s = -\alpha + \beta \cos(\kappa x), \]

where \( \alpha \) and \( \beta \) are positive constants. (This corresponds to some notional base state \( C_0(x) \) in Equation (2).) By Equation (16), we require

\[
\frac{1}{2} \beta^2 \leq \min_{p \in [-\kappa/2, \kappa/2]} \left[ -\left( \beta - \alpha \right) + (|p| - \kappa)^2 \right] \left( \alpha + p^2 \right). \tag{23}
\]

The results of applying Equation (23) are shown graphically in Figure 1. The shaded regions correspond to positive eigenvalues, and hence, linear stability of the basic equation (4). The region with light shading in the figure corresponds to the region with \( s(x) \leq 0 \) (Case 1 of Theorem 2). The region with dark shading corresponds to Equation (23) and hence, Case 2 of Theorem 2. Outside of the shaded regions, the \textit{a priori} stability analysis is inconclusive, the eigenvalues may have either sign, and the basic equation (4) may be stable or unstable.

![FIG. 1. The region defined by the constraints in Equation (23) (taking \( L = 1 \))](image)

V. CONCLUSIONS

Summarizing, we have formulated a model reaction-diffusion equation, posed on the entire real line, in the presence of a periodic source term. We have introduced steady
periodic solutions of the reaction-diffusion equation (the steady periodic solutions are referred to as the ‘base state’). We have performed a linear stability analysis of the base state. In principle, the linear stability analysis relies on the solution of a linearized reaction-diffusion equation, with the base state appearing parametrically therein. However, by formulating the linear-stability analysis in terms of Bloch functions, we have outlined an \textit{a priori} criterion for the base state to be stable, based only on the base-state solution itself, i.e. without needing to solve the linearized equation explicitly. We anticipate that this approach will be of use in the study of reaction-diffusion equations where the base state contains multiple parameters (reaction rate, diffusivity, source amplitude, source lengthscale), since the \textit{a priori} stability criterion will then provide immediate answers as to which parameter values give rise to a stable base state.

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