Superfluid properties of BPS monopoles.

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Abstract

This paper is devoted to demonstrating manifest superfluid properties of the Minkowskian Higgs model with vacuum BPS monopole solutions at assuming the "continuous" \( \sim S^2 \) vacuum geometry in that model.

It will be also argued that point hedgehog topological defects are present in the Minkowskian Higgs model with BPS monopoles.

It turns out, and we show this, that the enumerated phenomena are compatible with the Faddeev-Popov "heuristic" quantization of the Minkowskian Higgs model with vacuum BPS monopoles, coming to fixing the Weyl (temporal) gauge \( A_0 = 0 \) for gauge fields \( A \) in the Faddeev-Popov path integral.

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1 Introduction.

In the series of recent works [1, 2, 3, 4, 5] there was worked out the model of the physical "Yang-Mills-Higgs" (YMH) vacuum in the Minkowski space involving BPS monopole solutions.

The base of the approach [1, 2, 3, 4, 5] to constructing the Minkowskian Higgs model was the Dirac fundamental quantization [6] of that model, coming to the constraint-shell reduction of the theory in terms of topological Dirac variables $A^D_i (i = 1, 2, 3)$ [1, 2, 3] (in the YM sector of the considered Minkowskian YMH model), manifestly gauge invariant (G-invariant), relativistic covariant (S-covariant) and transverse.

The brief analysis of the Dirac fundamental quantization method [6] was performed recently in Ref. [7].

The historical retrospective development the Minkowskian Higgs model quantized by Dirac [6] was given in [7], and principal results got in [1, 2, 3, 4, 5] about the Dirac fundamental quantization of the Minkowskian Higgs model involving vacuum BPS monopoles were looked into.

Also the principal distinctions between the Dirac fundamental quantization method [6] and the Faddeev-Popov (FP) ”heuristic” gauge fixing method [8] were pointed out in [7].

Repeating the arguments [9], there was demonstrated that these come, basically, to violating the gauge equivalence (independence) theorem [10, 11] in the cases of collective vacuum excitations and bound states.

On the other hand, as it was noted already in [7] (towards the end of the discussion therein), the investigations about the Minkowskian Higgs model quantized by Dirac and involving vacuum BPS monopoles are far today for their finishing, and there lot of job is in prospect in order to specify details the model constructed in [1, 2, 3, 4, 5] and to made new observations.

Just specifying details of the Dirac fundamental quantization [6] for the Minkowskian Higgs model with BPS monopoles will be devoted the series of papers we now begin.

In the present paper we shall occupied ourselves with the more profound analysis of the Minkowskian Higgs model with BPS monopoles, digressing for some time from assuming about the Dirac fundamental quantization [6] of that model.

We continue to study the "classical" Minkowskian Higgs model with BPS monopoles, the job beginning in Refs. [4, 5].

Herewith speaking about the "classical" Minkowskian Higgs model with BPS monopoles, we mean results got without solving the Gauss law constraint, issuing only from the action functional of the Minkowskian Higgs model before fixing any gauge.

A good analysis of the "classical" Minkowskian Higgs model with BPS monopoles was performed in Refs. [12, 13, 14] (this analysis was reproduced partially in [4, 5]).

The important premise for the "classical" Minkowskian Higgs model [12, 13, 14] with BPS monopoles is assuming about the "continuous"

$$R \equiv SU(2)/U(1) \simeq S^2$$
vacuum geometry.

Repeating the reasoning [12], we show in Section 1 that such "continuous" vacuum geometry implies point (hedgehog) topological defects presenting in the "classical" Minkowskian Higgs model [12, 13, 14] with vacuum BPS monopole solutions.

The next important topic of the present study will be demonstrating manifest superfluid properties the "classical" Minkowskian Higgs model with BPS monopoles. This will be done in Section 2.

These properties, unique for the "classical" Minkowskian Higgs model [12, 13, 14] with vacuum BPS monopole solutions, are induced by the Bogomol’nyi equation [12]

\[ B = \pm D\Phi, \]

giving the relation between the vacuum "magnetic" field \( B \) and the Higgs isomultiplet \( \Phi \).

We argue that the Bogomol’nyi equation is just the potentiality condition for the vacuum of the "classical" Minkowskian Higgs model with BPS monopoles.

The transparent parallel of the Higgs model [12, 13, 14] and the liquid helium II (at rest) theory [15] will prove to be helpful for us in this argumentation.

The enumerated properties of the "classical" Minkowskian Higgs model with BPS monopoles (we mean point hedgehog topological defects and manifest superfluid properties inherent in that model) are compatible with the FP "heuristic" quantization [8] of that model.

We demonstrate (and this will be the one of most important topics of the present study) that the FP "heuristic" quantization [8] of the Minkowskian Higgs model with vacuum BPS monopole solutions comes to fixing the gauge \( A_0 = 0 \) for YM fields in the appropriate FP path integral.

2 Point hedgehog topological defects always accompany "continuous" vacuum geometry.

Let us denote as \( G \) the initial symmetry group in a gauge (Minkowskian) model.

If this model implicates the spontaneous breakdown of the initial gauge symmetry group \( G \) (as a rule, this is associated with Higgs modes), we shall denote as \( H \) the appropriate residual gauge symmetry group.

For instance, in the Minkowskian YMH model

\[ G \equiv SU(2), \quad H \equiv U(1), \]

respectively.

As it was shown in the monograph [16] (and these arguments were repeated in [4, 5]), in this case the initial gauge symmetry group \( G \) may be represented as

\[ G = H \oplus G/H. \quad (2.1) \]
Herewith the second item in (2.1), $R \equiv G/H$, is, from the geometrical viewpoint, is a space proving to be invariant under gauge transformations $H$.

Additionally, it may be supposed that

$$HR_i = R_i$$

for all the points $R_i$ of this space: in other words, that $H$ is the stationary subgroup of the point $R_i \in R$.

In particular, in the Minkowskian YMH model

$$R = SU(2)/U(1) \simeq S^2.$$  \hfill (2.2)

The space $R$ is called the vacuum manifold.

This term is justified by two considerations.

Firstly, vacuum manifolds in gauge models may be defined merely as those invariant with respect to appropriate (residual) gauge groups in these models.

Good explaining this fact was given in the monograph [16], in §8.1.

The above remark prompts the possibility to give an alternative interpretation of vacuum manifolds. Such interpretation was given in Ref. [12], in §Φ1.

It is quite correct to interpret the initial symmetry group $G$ as that does not change the energy functional (Hamiltonian) of the considered (e.g. non-Abelian) theory, while the residual symmetry group $H \subset G$ as that consisting of transformations that keep invariant a fixed equilibrium state.

All these states (at a fixed temperature $T$) form the so-called degeneration space (vacuum manifold), that we just denote as $R$ in the present study, following [12].

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1In this case the following general definition [12] of quantum fluctuations over a vacuum manifold $R$ may be given.

Such quantum fluctuations belong to a set $\mathcal{R}$ treated as an (infinitesimal) neighbourhood of the appropriate vacuum manifold $R$.

2If the initial symmetry $G$ in the considered gauge theory is violated up to its subgroup $H$, $T < T_c$, with $T_c$ being the Curie point in which the initial symmetry $G$ is violated and the second-order phase transition occurs.

It will be useful to adduce here some evidences in favour the fact that second-order phase transition occur indeed in Minkowskian Higgs models with YM fields (YMH models).

To do this, let us write down (following [5]) explicitly the appropriate action functional. This has the typical look

$$S = -\frac{1}{4g^2} \int d^4x F_{\mu\nu}^b F^{b\mu\nu} + \frac{1}{2} \int d^4x (D_\mu \Phi, D^\mu \Phi) - \frac{\lambda}{4} \int d^4x \left[ (\Phi^b)^2 - \frac{m^2}{\lambda} \right]^2,$$

with

$$D_\mu \Phi = \partial_\mu \Phi + g[A_\mu, \Phi]$$

being the covariant derivative an $g$ being the YM coupling constant.

The action functional (2.10) results the equations of motion [17]

$$(D_\nu F^{\mu\nu})_a = -g e_{abc} \phi^b (D_\mu \phi)^c,$$

$$(D^\mu D_\mu \Phi)_a = -\lambda \Phi_a (\Phi \cdot \Phi - a^2); \quad a^2 = m^2/\lambda.$$
The natural claim to this space in gauge theories is herewith to be topological.

Just the second of these equations of motion implies the second-order phase transition occurring in Minkowskian YMH models.

Issuing from this equation, one can demonstrate, repeating the arguments [18] (see §3.1 in this monograph) that the false vacuum $\Phi = 0$ may be linked in a continuous (although not smooth) wise with the true vacua $\Phi = \pm a$.

More exactly, at finite temperatures $T \neq 0$, if a shift of the Higgs field: $\Phi \rightarrow \Phi(T) + \delta \Phi$, is performed, it turns out that the equation of motion for $\delta \Phi$ can be recast to the look [18]

$$(D^\mu D_\mu \delta \Phi)_a - [m^2 + (\lambda/4)T^2] \delta \Phi_a = 0,$$

got in an infinitesimal neighbourhood of $\Phi(T) = 0$ at utilizing the relations

$$<\Phi^2> \sim T^2/12$$

for the Gibbs average of the bosonic field $\Phi$ squared, and

$$\Phi(T) = \sqrt{a^2 - T^2/4}.$$  

Such field is the one of solution (together with $\Phi(T) = 0$) of Eq. [18]

$$\Phi(T) [\lambda \Phi^2(T) - m^2 + (\lambda/4)T^2] = 0.$$  

In turn, this is the look of the general equation

$$(D^\mu D_\mu \Phi)_a(T) - [\lambda \Phi^2(T) - m^2 + (\lambda/4)T^2] \Phi(T) = 0,$$

in which a constant value $(D^\mu D_\mu \Phi = 0)$ is substituted.

Generally speaking, to get the equations of motion in Minkowskian YMH models at finite temperatures, the Gibbs average of these equations would be taken [18].

Above Eqs. are the particular case where this method is applied.

We can assert now that at finite temperatures $T \neq 0$ the Higgs field $\Phi$ acquires the effective mass

$$m' = -m^2 + (\lambda/4)T^2.$$

in the point $\Phi(T) = 0$; then [18]

$$(D^\mu D_\mu \delta \Phi)_a - m'^2 \delta \Phi_a = 0.$$  

The value $m'^2$ becomes negative at

$$T < T_c; \quad T_c = 2m/\sqrt{\lambda} \equiv 2a.$$  

Vice verse, it becomes positive at $T > T_c$.

In a point $\Phi(T) \neq 0$ (in an infinitesimal neighbourhood of $\Phi(T) = 0$), $m'^2(\Phi, T) > 0$ for $\Phi(T) = \sqrt{a^2 - T^2/4}$ [18]:

$$m'^2(\Phi, T) = 3\lambda \Phi^2(T) - m^2 + (\lambda/4)T^2 = 2\lambda \Phi^2(T).$$

To derive this Eq., it is necessary to take account of $<\Phi> = 0$ [18]. Then $\delta(\Phi^3) = 3\Phi^2 \delta \Phi$ would be substituted for deriving $m'^2(\Phi, T)$.

Thus the solution $\Phi(T) = \sqrt{a^2 - T^2/4}$ to the YMH equations of motion at finite temperatures is steady at $T < T_c$ and vanishes (more precisely, becomes complex, losing thus its physical sense) at $T > T_c$, in the moment when the solution $\Phi = 0$ (the false vacuum solution) becomes steady.
The structure of a degeneration space may be investigated with the aid of the Landau theory of second-order phase transitions (see e.g. §142 in [20]).

An equilibrium state is determined by the condition for the free energy of the given system to be minimal.

In the Landau theory of second-order phase transitions (the pattern of which is the Minkowskian Higgs model) one supposes that an equilibrium state may be found at minimizing the free energy of the given system by the set of states specified by a finite number of parameters (called order parameters), but not by the set of all the states.

Return to the case of the Minkowskian YMH model, when

\[ R = SU(2)/U(1) \simeq S^2. \]  

(2.3)

In this case, obviously,

\[ \pi_2(R) = \pi_2 S^2 = \pi_1(H) = \pi_1 U(1) = Z. \]  

(2.4)

Generally speaking, repeating the arguments [12] (§Φ1), there may be shown that the isomorphism (2.4) predestines the existence of point topological defects in a gauge theory.

This means that a phase transition occurs at the temperature (Curie point) \( T = T_c \) implicating restoring the (initial) \( SU(2) \) gauge symmetry.

The results have been got may be illustrated graphically.

It turns out [18] that the lines \( \Phi(T) = 0 \) (coinciding with the abscises axis \( T \)) and \( \Phi(T) = \sqrt{a^2 - T^2}/4 \) are linked in a continuous (although not smooth) wise.

This just corresponds to the second-order phase transition occurring in the Minkowskian YMH model.

Note that first-order phase transitions take place in some (Minkowskian) Higgs models: in particular in the Minkowskian Abelian Higgs model and in GUT.

This was demonstrated in [18]. The principal argument that was suggested in [18] in favour of first-order phase transitions occurring in the mentioned models are gaps that appropriate plots \( \Phi(T) \) suffer (more precisely, it is impossible to link in a continuous wise the false vacuum \( \Phi(T) = 0 \) and other vacua in the quested models).

For example, in the Minkowskian Abelian Higgs model with the Lorentz gauge \( \partial_\mu A^\mu \) fixed the first-order phase transitions occurs when \( \lambda \leq e \) (with \( e \) being the elementary charge).

In this case it can be shown [18] that three solutions to appropriate equations of motion exist, including the false vacuum \( \Phi(T) = 0 \), in the temperatures interval \( T_{c_1} < T < T_{c_2} \).

As a result, the false vacuum \( \Phi(T) = 0 \) becomes a metastable state at \( T > T_{c_1} \), while there exists also the instable solution \( \Phi_2 \) corresponding to the local maximum of the appropriate potential \( V(\Phi, T) \) and \( \Phi_1 \) corresponding to the "true" vacuum, i.e. to the (global) minimum of \( V(\Phi, T) \).

Herewith the phase transition from the \( \Phi_1 \) to the \( \Phi(T) = 0 \) state, accompanied by restoring the \( U(1) \) gauge symmetry, begins at a temperature \( T_c \) at which [18]

\[ V(\Phi_1(T_c), T_c) \sim V(0, T_c). \]

Graphically [18], there is however a gap between the both potentials at \( T = T_c \) corresponding to the gap \( \Delta F(T) = \Delta V(\Phi, T) \) (the latent heat) in the free energy \( F(T) \). It is just the sign of the first-order phase transition occurring in the Minkowskian Abelian Higgs model.

On the other hand, the metastability of the false vacuum \( \Phi(T) = 0 \) implies the supercooling phenomenon: the system of fields remains in this state, coexisting simultaneously with the "true" vacuum \( \Phi_1 \) even when \( T < T_{c_1} \), coexisting herewith simultaneously with the "true" vacuum \( \Phi_1 \).

Vice verse, when \( T > T_c \), the latent heat is liberated as \( \Delta F(T) \). This phenomenon is referred to as reheating [18] [19].
with the spontaneous breakdown of the initial gauge symmetry $G$ down to its subgroup $H$, involving the vacuum manifold $R = G/H$.

The ground cause of these topological defects is violating the thermodynamic equilibrium over an (infinitesimal) neighbourhood $U$ of a point in the coordinate (e.g. in the Minkowski) space.

Such neighbourhood is topologically equivalent to the two-sphere $S^2$.

Already mentioned violating the thermodynamic equilibrium in a neighbourhood $U \simeq S^2$ (is the case of point topological defects inside the vacuum manifold $R$) is always associated with gaps (singularities) which order parameters inherent in appropriate gauge models suffer over such infinitesimal neighbourhoods $U$ that may be contracted into points.

As the typical order parameter in Minkowskian Higgs models involving vacuum monopoles (the model [12, 13, 14] with BPS monopoles is the one of such models), the vacuum "magnetic" field $B$ appears.

The direct computations have been performed in Ref. 13 for the Minkowskian Higgs model with vacuum BPS monopoles results the $O(r^{-2})$ behaviour of the vacuum "magnetic" field $B$ at the origin of coordinates.

The same $O(r^{-2})$ behaviour of the vacuum "magnetic" field $B$ at the origin of coordinates can be observed in two another very important Minkowskian models with monopoles.

There are the Wu-Yang model [21] (analysed in detail in Refs. 3, 4, 5) and the 't Hooft-Polyakov model [22, 23].

The said about the $O(r^{-2})$ behaviour at the origin of coordinates of the vacuum "magnetic" field $B$, treated as the order parameter in the enumerated Minkowskian models with vacuum monopole solutions, just testifies in favour of point topological defects inside appropriate vacuum manifolds (topologically equivalent to the two-sphere $S^2$).

Indeed, a two-sphere $S^2$ may be always picked out inside such $U$: $S^2 \in U$. In this case, following [12], let us denote simultaneously as $f$ two maps.

These are, firstly, the map $f : U \to R$ and, secondly, the map $f : S^2 \to R$. Latter one is treated as the restriction of the map $f : U \to R$ onto the sphere $S^2$.

If the map $f : S^2 \to R$ is not homotopical to zero ($\pi_2(R) \neq 0$), this map cannot be continue onto the map $D^3 \to R$, with $D^3$ being the ball restricted by the sphere $S^2$ (the complete proof of the latter fact was given in Ref. 12, in §T1).

Just the said means that there is a point topological defect inside the sphere $S^2$.

Indeed, as it was explained in Refs. 3, 4, 5. Wu-Yang monopoles $\Phi_i$ arisen in the Minkowskian model [21] are solutions to the classical equation of motion

$$D^a_k (\Phi_i) F^a_{bk} (\Phi_i) = 0$$

of the "pure" (Minkowskian) YM theory (without Higgs fields).

Unlike Wu-Yang monopoles [21], 't Hooft-Polyakov monopoles [22, 23] are specific vacuum solutions of the Minkowskian Higgs model in its YM and Higgs sectors (see §10.4 in [16]).

And furthermore, 't Hooft-Polyakov monopoles [22, 23] are associated with the Georgi-Glashow theory involving the initial $SO(3) \simeq SU(2)$ gauge symmetry violated then down to its $U(1)$ subgroup (this model was experimentally ruled out after the discovery of neutral-current phenomena.).
The kind of point topological defects located at the origin of coordinates is referred to as point hedgehog topological defects \[12\].

This terminology is connected historically with Polyakov hedgehogs \[16, 23\] and Higgs solutions in the 't Hooft-Polyakov model \[22, 23\]:

\[
\phi^a \sim \frac{x^a}{r} f(r, a),
\] (2.5)

involving a continuous function \(f(r, a)\) (with \(a\) being the radius of the two-sphere \(R \simeq S^2\); \[2.2\]) \[5\].

The vacuum monopole solutions in Minkowskian (Higgs) models us cited in the present study may be got issuing from constrained action functionals, i.e. before solving the YM Gauss law constraint \[7\]

\[
\frac{\partial W}{\partial A_0} = 0
\] (2.6)

with fixing a gauge.

More concretely, in the 't Hooft-Polyakov model \[22, 23\], Higgs and YM monopole solutions: there are Polyakov hedgehogs \[2.5\] in the Higgs sector of that model and \[16\]

\[
A_i^a = -\epsilon_{iab} \frac{r^b}{gr^2},
\] (2.7)

in the YM sector, are derived as solutions to the equations of motions \[16\]

\[
(D_{\nu}F^{\mu\nu})_a = -g\epsilon_{abc}(D_\mu \phi)^c,
\] (2.8)

\[
(D^\mu D_\nu \phi)_a = -\lambda \phi_a (\vec{\phi} \cdot \vec{\phi} - a^2).
\] (2.9)

(with \(\lambda\) being the Higgs selfinteraction constant); \(a = m/\sqrt{\lambda}\) (with \(m\) being the mass of the Higgs field) is the radius of the two-sphere \(R \simeq S^2\); \[2.3\]).

These, in turn, follow immediately from the standard action functional \[4, 5\]

\[
S = -\frac{1}{4g^2} \int d^4x F_{\mu\nu}^b F_b^{\mu\nu} + \frac{1}{2} \int d^4x (D_\mu \phi, D_\nu \phi) - \frac{\lambda}{4} \int d^4x \left[ (\phi^b)^2 - \frac{m^2}{\lambda} \right]^2
\] (2.10)

of the Minkowskian Higgs model.

In the Minkowskian Wu-Yang model \[21\] the equation of motion \[4, 5\]

\[
D^b_k (\Phi_i) F^{b\nu}_k (\Phi_i) = 0 \implies \frac{d^2 f}{dy^2} + \frac{f(f^2 - 1)}{y^2} = 0
\] (2.11)

of this purely YM model result Wu-Yang “ansatzes” \(f = \pm 1\) (at \(r \neq 0\)) corresponding to Wu-Yang monopoles \(\Phi_i\) with topological charges \(n = \pm 1\), respectively (hedgehog and antihedgehog in the terminology \[23\]).

\[5\]The look \(2.5\) for Polyakov hedgehogs was cited in the monograph \[18\] (p. p. 114- 116).
In conclusion, in the "classical" Minkowskian Higgs model \cite{12, 13, 14} with vacuum BPS monopole solutions, the latter are, indeed, solutions to the Bogomol’nyi equation \cite{4, 5, 9, 12, 13, 14}

\begin{equation}
\mathbf{B} = \pm D \Phi, \tag{2.12}
\end{equation}

derived (see §11 in \cite{12}) at evaluating the Bogomol’nyi bound

\begin{equation}
E_{\text{min}} = 4\pi m^a g, \quad a = \frac{m}{\sqrt{\lambda}}; \tag{2.13}
\end{equation}

with m denoting the magnetic charge, of the YMH field configuration energy.

The essential point deriving the Bogomol’nyi equation (2.12) is \cite{4, 5, 12} going over to the Bogomol’nyi-Prasad-Sommerfeld (BPS) limit \(\lambda \to 0, \quad m \to 0\) : \(\frac{1}{\epsilon} \equiv \frac{gm}{\sqrt{\lambda}} \neq 0\). \(\tag{2.14}\)

All these Minkowskian models with monopoles may be described with the aid of the FP path integrals formalism \cite{8}.

Herewith it is enough \cite{12} to fix the temporal (Weyl) gauge \(A_0 = 0\) for YM fields via the Dirac delta-function \(\delta(A_0)\) in the appropriate FP path integrals.

This just corresponds to \(F_{0i}^a = 0\) for "electric" fields in the enumerated Minkowskian models with stationary monopole solutions.

The analysis has been performed in the papers \cite{1, 2, 3, 4, 5, 7}, devoted to the Dirac fundamental quantization \cite{6} of Minkowskian (Higgs) models, shows that just nonzero vacuum "electric" fields \(F_{0i}^a\) in the Minkowskian YMH model \cite{1, 2, 3, 4, 5} quantized by Dirac are associated with a nontrivial (topological) dynamics inherent in that model.

Herewith mentioned nonzero vacuum "electric" fields \(F_{0i}^a\) in the Minkowskian YMH model \cite{1, 2, 3, 4, 5} quantized by Dirac take the shape of vacuum "electric" monopoles, directly proportional to the topological dynamical variable \(\dot{N}(t)\), the time derivative of \(N(t)\), the noninteger degree of the map referring to the \(U(1) \to SU(2)\) embedding:

\begin{equation}
F_{0i}^a = \dot{N}(t) D^{ac}_i (\Phi_{(0)}) \Phi_{(0)c}(\mathbf{x}). \tag{2.15}
\end{equation}

Topologically trivial vacuum "electric" monopoles \(F_{0i}^a\) implicate Higgs vacuum BPS monopole solutions \(\Phi_{(0)a}(\mathbf{x})\).

As it was explained in Ref. \cite{24}, this is the immediate result of choosing temporal components of YM vacuum modes in the Minkowskian YMH model \cite{1, 2, 3, 4, 5} quantized by Dirac to be proportional to Higgs vacuum BPS monopoles \(\Phi_{(0)a}(\mathbf{x})\): with \(\dot{N}(t)\) playing the role of the proportionality coefficient:

\begin{equation}
Z^a = \dot{N}(t) \Phi^{(0)a}(\mathbf{x}). \tag{2.16}
\end{equation}

This choice of YM vacuum modes is dictated just by the specific of the Dirac fundamental scheme \cite{6}, in which \(Z^a\) are (general) solutions to the YM Gauss law constraint

\begin{equation}
(D^2)^{ab} \Phi_{b(0)} = 0. \tag{2.17}
\end{equation}
This, in turn, is the look of the YM Gauss law constraint (2.10):

$$\partial W/\partial A_0 = 0 \iff [D^2(A)]^{ac} A_{0c} = D_i^{ac} A_0 A^i_c,$$

resolved with the Coulomb covariant gauge \[1, 3, 7\]

$$D_i^{ac} A_0 A^i_c = 0$$ \hfill (2.19)

for YM fields \(A\).

As it was explained in the papers \[1, 2, 3, 4, 5, 7\], the Coulomb covariant gauge (2.19) may be satisfied by topological Dirac variables \[1, 2, 3, 5, 7\]

$$\hat{A}^D_i(x, t) := v^T(n)(x, t)(\hat{A}_i + \partial_i(v^T(n))^{-1}(x, t)), \quad \hat{A}_i = g\tau^a A_{ai}; \quad n \in \mathbb{Z};$$ \hfill (2.20)

implicating Gribov topological multipliers \(v^T(n)(x, t)\) (in the Minkowskian Higgs model with vacuum BPS monopole solutions, these topological multipliers depend explicitly on a scalar combination of Higgs BPS monopoles via \(\tau^a \Phi_a\); \(\tau^a (a = 1, 2, 3)\) are Pauli matrices).

The functionals \(\hat{A}^D_i(x, t)\) specified in such a wise prove to be gauge invariant and transverse fields:

$$\partial_0 D_i \hat{A}^D_i(x, t) = 0; \quad u_i(x, t) \hat{A}^D_i(x, t) u_i(x, t)^{-1} = \hat{A}^D_i(x, t) \quad (2.21)$$

for gauge matrices \(u(x, t)\).

As it was discussed in the review \[7\] (repeating the said in Refs. \[1, 3, 4, 5\]), vacuum "electric" monopoles (2.15) involve collective "solid" rotations of the Minkowskian physical YMH BPS monopole vacuum.

These may be described by the action functional \[1\]

$$W_{\text{coll}} = \int d^4 x \frac{1}{2}(F_{0i}^c)^2 = \int dt \frac{\dot{N}^2(t)}{2},$$ \hfill (2.22)

with

$$I = \int_V d^3 x (D_0^{ac} (\Phi_k) \Phi_{(0)} c)^2 = \frac{4\pi^2 \epsilon}{\alpha_s} = \frac{4\pi^2}{\alpha_s} \frac{1}{V < B^2 >}$$ \hfill (2.23)

being the rotary momentum of the Minkowskian YMH vacuum.

In Eq. (2.23), \(\Phi_k\) are vacuum BPS monopole modes; \(\alpha_s \equiv g^2/4\pi\) (with \(g\) being the YM coupling constant); \(V\) is the spatial volume.

The purely real spectrum

$$P_N \equiv \frac{\partial W_{\text{coop}}}{\partial N} \equiv \dot{N} I = 2\pi k + \theta,$$ \hfill (2.24)

with the \(\theta\)-angle chosen to vary in the interval \([-\pi, \pi]\) \[1\].
At the FP "heuristic" quantization [8] of Minkowskian Higgs models with monopoles, all these and similar vacuum rotary effects disappear with disappearing "electric" fields $F^a_{0b}$.

The nontrivial topological dynamics inherent in the Minkowskian Higgs model [1, 2, 3, 4, 5] (involving vacuum BPS monopole solutions) quantized by Dirac [6] draws a peculiar watershed between the FP "heuristic" [8] and "fundamental" [6] approaches to the quantization of gauge models.

This is associated [7, 9] with violating the gauge equivalence (independence) theorem [10, 11] in the case of collective vacuum excitations.

In the Minkowskian YMH model [1, 2, 3, 4, 5] quantized by Dirac [6], there exist such "collective vacuum excitations".

There are just zero mode solutions $Z^a$, (2.16), to the YM Gauss law constraint (2.17), (2.18), generating various vacuum rotary effects in the enumerated model.

### 3 Superfluid properties of Minkowskian Higgs model involving vacuum BPS monopole modes and "continuous" vacuum geometry.

The "classical" Minkowskian Higgs model [12, 13, 14] with vacuum BPS monopole solutions proves to be the unique model with monopoles in which the appropriate vacuum possesses the manifest superfluid properties.

Now we shall attempt to argue in favour of this statement.

In the recent paper [5] there was pointed out the role of the Bogomol'nyi equation (2.12) as the potentiality condition for the Minkowskian YMH vacuum involving BPS monopole solutions.

Let us interpret the latter assertion.

Mathematically, any potential field may be represented as grad $\Phi$ (to within a constant).

It is a good prompt for us.

In the Minkowskian YMH theory involving BPS monopole solutions (for instance, in the "classical" Minkowskian Higgs model [12, 13, 14] with BPS monopoles), there exists always such a scalar field. It is just the Higgs (world) scalar $\Phi$ represented as the Higgs BPS monopole in the vacuum sector of that theory.

Then it is easy to guess that the Bogomol'nyi equation (2.12), having the look (3.1), can be treated as the potentiality condition for the Minkowskian YMH vacuum involving vacuum BPS monopole solutions. It is so due to the Bianchi identity $DB = 0$ which can be represented as

$$\epsilon^{ijk} \nabla_i F^b_{jk} = 0$$
(at neglecting the items in $DB$ directly proportional to $g$ and $g^2$).

Indeed, there can be drawn a highly transparent parallel between the Minkowskian YMH vacuum involving vacuum BPS monopole solutions (say, [12, 13, 14]) and a liquid helium II specimen described in the Bogolubov-Landau model [15].

In the latter case, the potential motion is proper to the superfluid component in this liquid helium specimen.

The superfluid motion in a liquid helium II is the motion without a friction between the superfluid component and the walls of the vessel where a liquid helium specimen is contained.

Thus the viscosity of the superfluid component in a helium II is equal to zero, and vortices (involving $\text{rot } \mathbf{v} \neq 0$) are absent in the superfluid component of a helium.

As L. D. Landau showed [15], at velocities of the liquid exceeding a critical velocity $v_0 = \min (\epsilon/p)$ for the ratio of the energy $\epsilon$ and momentum $p$ for quantum excitations spectrum in the liquid helium II, the dissipation of the liquid helium energy occurs via arising excitation quanta with momenta $\mathbf{p}$ directed antiparallel to the velocity vector $\mathbf{v}$. Such dissipation of the liquid helium energy becomes advantageous [25] just at

$$\epsilon + p \cdot \mathbf{v} < 0 \implies \epsilon - p \cdot \mathbf{v} < 0.$$ 

From the above reasoning concerning properties of potential motions, it becomes obvious that the vector $\mathbf{v}_0$ of the critical velocity for the superfluid potential motion possesses the zero curl: $\text{rot } \mathbf{v}_0 = 0$.

In this case, according to (3.1), the critical velocity $\mathbf{v}_0$ of the superfluid potential motion in a liquid helium specimen may be represented [26] as

$$\mathbf{v}_0 = \frac{\hbar}{m} \nabla \Phi(t, \mathbf{r}), \quad (3.2)$$

where $m$ is the mass of a helium atom and $\Phi(t, \mathbf{r})$ is the phase of the complex-value helium Bose condensate wave function $\Xi(t, \mathbf{r}) \in C$.

Note that the latter one may serve as a complex order parameter in the Bogolubov-Landau model of the liquid helium [15]; its explicit look is [26]

$$\Xi(t, \mathbf{r}) = \sqrt{n_0(t, \mathbf{r})} \ e^{i\Phi(t, \mathbf{r})}, \quad (3.3)$$

with $n_0(t, \mathbf{r})$ being the number of particles in the ground energy state $\epsilon = 0$.

Thus the similar look for the vacuum ”magnetic” field $\mathbf{B}$ in the Minkowskian Higgs model involving BPS monopole solutions, generating by the Bogomol’nyi equation (2.12), and for the critical velocity $\mathbf{v}_0$ of the superfluid motion in a liquid helium II, given by Eq. (3.2), testifies in favour of the potential motions occurring therein.

In this case, drawing a highly transparent parallel between the Minkowskian YMH vacuum involving BPS monopole solutions [4, 5, 12, 13, 14] and a liquid helium II specimen described in the Bogolubov-Landau model [15], we can also conclude about manifest superfluid properties of the Minkowskian YMH vacuum involving BPS monopoles.
As in the Bogolubov-Landau model \[15\] of liquid helium II, the ground cause of the superfluid properties of the Minkowskian YMH vacuum with BPS monopole roots in long-range correlations of local excitations \[24\].

While in the Bogolubov-Landau model \[15\] of liquid helium II this comes to repulsion forces between helium atoms as the cause of superfluidity effects, in the Minkowskian YMH vacuum involving BPS monopole solutions, the cause of the superfluidity taking place is in the strong YMH coupling \(g\) (entering effectively the appropriate action functional \(2.10\)).

The chief thing in alike superfluid effects occurring in a liquid helium II specimen as well as in the Minkowskian YMH vacuum involving BPS monopoles is that these both physical systems are non-ideal gases.

In ideal gases no superfluidity phenomena are possible.

There can be demonstrated (see e.g. §4 of Part 6 in \[27\]) that in ideal gases a deal of particles is accumulated on the zero energy quantum level at temperatures \(T < T_0\); herewith the temperature \(T_0\)

\[
kT_0 = \frac{1}{(2.61)^{2/3}} \frac{h^2}{2\pi m} \left( \frac{N}{V} \right)^{2/3}
\]

(with \(k\) and \(h\) being, respectively, the Boltzmann and Planck constants; \(N\) being the complete number of particles; \(V\) being the volume occupied by the ideal Bose gas; \(m\) being the mass of a particle) is called the condensation temperature, while the above deal of particles is called the Bose condensate.

Now we should like argue in favour that only the Minkowskian Higgs model with vacuum BPS monopole solutions possesses the above described superfluid properties, distinguishing it from another Minkowskian Higgs models with vacuum BPS monopoles.

In our argumentation we shall follow the paper \[28\].

Consider again the ‘t Hooft-Polyakov model \[22, 23\].

As it is well known \[16\], \(D_i \phi^a \to 0\) as \(r \to \infty\) for a ‘t Hooft-Polyakov Higgs monopole \(\phi^a\) (having the look \(2.5\))

In this case, asymptotically,

\[
B_i^a \partial_i \phi_a = \partial_i (B_i^a \phi_a) = 0,
\]

because of the Bianchi identity

\[
D_i B_i^a = \frac{1}{2} \epsilon_{ijk} D_j F_{jk}^a = 0
\]

and the remark \[12\] that \(B_i^a \phi_a\) is a \(U(1) \subset SU(2)\) scalar; thus one can replace the covariant derivative \(D\) with the partial one, \(\partial\), for \(B_i^a \phi_a\).

In turn, the complete energy of the YMH configuration may be represented as \[12, 28\]

\[
E_{\text{compl}} = \int d^3 x \left[ \frac{1}{2} (D \phi_a \pm B_a)^2 + \frac{\lambda}{4} ((\phi^a)^2 - a^2) \right] + \frac{4\pi}{g^2} M_W.
\]
The last item in Eq. (3.6) involves the mass $M_W$ of the $W$-boson.

Such look of $E_{\text{compl}}$ originates from the paper [22] devoted to the 't Hooft-Polyakov model.

The connection between the energy integral $E_{\text{compl}}$ and the general action functional (2.10) [4, 5] of the Minkowskian Higgs model is given by the identity [28]

$$(D\phi_a)^2 + B_a^2 = (D\phi_a \pm B_a)^2 \mp 2B_aD\phi_a. \quad (3.7)$$

Herewith the last item on the right-hand side of (3.7) vanishes at the spatial infinity, as we have noted above [16].

Just from Eq. (3.6) one can read formally the Bogomol'nyi equation in the shape (2.12).

In the 't Hooft-Polyakov model [22, 23] the Bogomol'nyi equation (2.12) determines the Bogomol'nyi bound [28]

$$M_{\text{mon}} = \frac{4\pi}{g^2} M_W \quad (3.8)$$

for the complete energy $E_{\text{compl}}$, (3.6), of the YMH configuration at going over to the BPS limit (2.14) [13].

Then the asymptotic $D_i\phi^a \to 0$ as $r \to \infty$ [16] for 't Hooft-Polyakov monopoles [22, 23] forces to vanish identically the first item under the integral sign in $E_{\text{compl}} (|\bm{B}| = 0)$.

In the light of the said above it becomes obvious that the vacuum "magnetic" field $\bm{B}$, playing the role of the (critical) velocity for the superfluid motion in the Minkowskian non-Abelian vacuum with BPS monopoles, actually approaches zero in the 't Hooft-Polyakov model [22, 23], involving the $D_i\phi^a \to 0$ as $r \to \infty$ asymptotic [16] for Higgs monopoles.

There is no superfluidity also in the Wu-Yang model [21].

The absence of Higgs vacuum modes in that "purely YM" Minkowskian model is the cause of such situation.

On the other hand, Wu-Yang monopoles [21] approximate good, at the spatial infinity, YM BPS monopole solutions [4, 5, 12, 13, 14], ensuring manifest superfluid properties of the appropriate YMH vacuum.

4 Discussion.

In the present study we have discussed the specific of various Minkowskian (Higgs) models involving vacuum monopole solutions.

Our initial premise was herewith assuming about the "continuous", $\sim S^2$, (2.3), vacuum geometry in these models.

There was demonstrated that this assuming confines itself with the FP "heuristic" quantization [8] of Minkowskian (Higgs) models involving vacuum monopoles.

\footnote{Indeed, Wu-Yang monopoles [21] approximate YM ones outside cores of latter monopoles [3, 4].}
Moreover, the absence of "electric" fields $F^a_0$ in these models prevents any (topologically nontrivial) dynamics via fixing the Weyl (temporal) gauge $A_0 = 0$ through the $\delta(A_0)$ multipliers in appropriate FP integrals.

The opposite situation can be observed at the Dirac fundamental quantization [6] of Minkowskian Higgs models, coming to the Gauss-shell reduction of these models with choosing a definite (say, rest [1, 3]) reference frame.

As it was discussed in Refs. [1, 3, 2, 4, 5, 7] with the example of the Minkowskian Higgs model involving vacuum BPS monopole solutions, solving the YM Gauss law constraint (2.6), (2.18) with the transverse Coulomb covariant gauge (2.19) implies temporal components $Z^a$ [24], (2.16), of YM fields (referring to the BPS monopole vacuum).

These, in turn, generate vacuum "electric" monopoles (2.15) and the action functional (2.22), describing correctly collective solid rotations of the Minkowskian YMH BPS monopole vacuum.

Utilizing the general theory of topological defects (stated good in the monograph [12]), we have detected point hedgehog topological defects in all the Minkowskian Higgs models involving vacuum monopoles and the continuous geometry (2.3) of appropriate vacuum manifolds.

There was also shown that the Minkowskian Higgs model involving BPS monopole solutions is the unique model in which superfluid vacuum phenomena are possible.

The manifest superfluid properties of that model are determined by the Bogomol'nyi equation (2.12), derived [12] at evaluating the (topologically degenerated) Bogomol'nyi bound $E_{\text{min}}$, (2.13), of the YMH energy in the Minkowskian Higgs model with BPS monopoles.

The essential point of that deriving was going over to the BPS limit (2.14) [12, 13]. The Bogomol'nyi equation (2.12) may be interpreted as the potentiality condition for the BPS monopole vacuum.

Herewith the transparent parallel between this vacuum and the superfluid component in a liquid helium II specimen [15] is on hand.

On the other hand, the Bogomol'nyi equation (2.12) and manifest superfluid properties of the Minkowskian Higgs model with BPS monopoles (generated by this equation) prove to be compatible with the Dirac fundamental quantization [6] as well as with the FP "heuristic" one [8] of that model.

In the next paper of the series we devote to remarks about the "Minkowskian Higgs model quantized by Dirac" we shall attempt to argue in favour of the latter statement.

Our principal idea will be here that manifest superfluid properties of the Minkowskian Higgs model with BPS monopoles quantized by Dirac are determined by the "potentiality condition" [24]

$$D^2 \Phi = 0$$

imposed onto the Higgs field $\Phi$ having the look of a vacuum BPS monopole.

But this "potentiality condition" comes to the Bogomol’nyi equation (2.12) due to the Bianchi identity

$$DB = 0.$$
It will be shown, repeating the arguments \cite{3, 4, 5, 9}, that just Eq. (4.1) specifies the ambiguity in the choice of the Coulomb covariant gauge (2.19) for (topologically trivial) YM fields (2.20), that are, indeed, topological Dirac variables: gauge invariant and transverse.

In \cite{3, 4, 5}, Eq. (4.1) was referred to as the Gribov ambiguity equation.

In this way, the connection between the Gauss-shall reduction of the Minkowskian Higgs model with BPS monopoles and manifest superfluid properties of that model will be ascertained.

The next important topic of the coming investigations will be specifying the explicit look of stationary Gribov topological multipliers

$$v^{T(n)}(x) = v^{T(n)}(x, t)|_{t=t_0} ,$$

entering topological Dirac variables (2.20) in the fixed (initial) time instant $t = t_0$.

Herewith we shall concentrate our efforts about the behaviour of Gribov topological multipliers $v^{T(n)}(x)$ at the spatial infinity ($|x| \to \infty$).

The principal result will be demonstrated is \cite{9, 24}

$$v^{T(n)}(x) \to 1 \quad \text{as} \quad |x| \to \infty .$$

We shall follow the paper \cite{29} at grounding this fact.

As it was noted in Refs. \cite{7, 29}, such asymptotical the behaviour of Gribov topological multipliers $v^{T(n)}(x)$ at the spatial infinity provides the infrared (topological) confinement of these multipliers in gluonic and fermionic (quark) Green functions in all the orders of the perturbation theory.

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