Pinching surface groups in complex hyperbolic plane

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Abstract

We construct first examples of discrete geometrically finite subgroups of $PU(2,1)$ which contain parabolic elements, and are isomorphic to surface groups of genus $\geq 2$.

1 Introduction

Even though complex hyperbolic Kleinian groups lack the flexibility of their real hyperbolic cousins, they do come in many shapes and sizes, and there is no structure theory in sight. At this stage it seems useful to work out individual examples, such as the ones considered in this paper.

Let $M$ be a closed orientable surface of genus $g \geq 2$. In this note we construct new examples of discrete embeddings of $\pi = \pi_1(M)$ into $PU(2,1)$, the full group of biholomorphic isometries of the complex hyperbolic plane $H^2_C$. The group $\pi$ can be realized as a lattice in the subgroups $SO(2,1)$ and $SU(1,1)$ of $PU(2,1)$, so there are two obvious discrete embeddings $\rho_r$ and $\rho_c$ of $\pi$ into $PU(2,1)$. The group $\rho_r(\pi)$ stabilizes a totally real plane in $H^2_C$, and the quotient complex hyperbolic surface $H^2_C/\rho_r(\pi)$ is the total space of the tangent bundle to $M$. Similarly, $\rho_c(\pi)$ preserves a complex geodesic, and $H^2_C/\rho_c(\pi)$ is the square root of the tangent bundle to $M$ [GKL01].

These two representations are also distinguished by the so-called Toledo invariant $\tau$, which associates to a representation $\rho \in \text{Hom}(\pi_1(M), PU(2,1))$ the (normalized) integral over $M$ of the pullback of the Kähler form via a section of the flat $H^2_C$-bundle corresponding to $\rho$. In
fact, $\tau(\rho_r) = 0$ and $\tau(\rho_c) = \pm \chi(M)$. According to [Tol89, GKL01], $\tau$ is a $\frac{2}{3}\mathbb{Z}$-valued locally constant function on the representation space $\text{Hom}(\pi_1(M), \text{PU}(2,1))$ satisfying $|\tau| \leq 2g - 2$. E. Xia showed [Xia00] that $\tau$ classifies the connected components of $\text{Hom}(\pi_1(M), \text{PU}(2,1))$, and D. Toledo proved that $|\tau(\rho)| = 2g - 2$ iff $\rho$ is an isomorphism onto a cocompact lattice in the stabilizer of a complex geodesic.

By amalgamating the representations $\rho_r, \rho_c$, W. Goldman, M. Kapovich, and B. Leeb [GKL01] showed that each even value of $\tau$ is realized by a faithful discrete representations $\rho$ such that the complex hyperbolic surface $\mathbb{H}^2_r/\rho(\pi)$ is an oriented $\mathbb{R}^2$-bundle over $M$ with Euler number $2g - 2 + |\tau(\rho)|/2$.

In all the above examples, the group $\rho(\pi)$ is geometrically finite without parabolics. Goldman asked if this is always the case for faithful discrete representations. We provide a negative answer as follows.

**Theorem 1.1.** Let $M$ be a closed orientable surface of genus $g \geq 2$, and let $\gamma \in \pi_1(M) = \pi$ be a nontrivial element represented by a simple closed curve that separates $M$. Then there exists a faithful discrete representation $\rho: \pi \to \text{PU}(2,1)$ such that $\rho(\pi)$ is geometrically finite, any maximal parabolic subgroup of $\rho(\pi)$ is generated by a conjugate of $\rho(\gamma)$, the quotient $\mathbb{H}^2_r/\rho(\pi)$ is diffeomorphic to the tangent bundle of $M$, and $\tau(\rho) = 0$.

Loosely speaking, any nontrivial element $\gamma \in \pi_1(M)$ represented by a simple closed curve that separates $M$ can be pinched (i.e. made parabolic) by some faithful discrete representation $\rho$. The element $\rho(\gamma)$ is called an accidental parabolic. A simple modification of our construction yields faithful discrete representations with several conjugacy classes of accidental parabolics, however the result we get is not optimal, so do not write down the details.

We do not know which nonzero values of the Toledo invariant can be realized by faithful discrete representations with accidental parabolics. As we mentioned above, the components of $\text{Hom}(\pi_1(M), \text{PU}(2,1))$ with $|\tau| = 2g - 2$ cannot contain such representations.

The group $\rho(\pi)$ in Theorem [1.1] is obtained by amalgamating two discrete groups, each being a noncocompact lattice in the stabilizer of some totally real plane, along a common cyclic parabolic subgroup. Discreteness of the amalgamated product is proved using the topological version of the Maskit combination theorem.

In author’s opinion this method of proving discreteness has some advantages over the complex hyperbolic version of the Poincaré polyhedron theorem which was recently developed in [FZ99, GP]. Namely, the Poincaré theorem usually requires explicit knowledge of the geometry of the fundamental polyhedron. By contrast, the Maskit combination theorem is stated in purely topological terms, in particular, one does not need the fundamental polyhedron to be bounded by bisectors, or any other special hypersurfaces. This soft nature of the combination theorems makes them easier to use.

This paper is a revised version of the preprint [Bel] written in 1995 when the author was a student at the University of Maryland. It is a pleasure to thank Bill Goldman for numerous helpful discussions, and Robert Miner for comments on the first version of the paper.
2 Vertical and horizontal translations in the Siegel domain

In this section we set up notations and collect some elementary facts about geometry of the complex hyperbolic plane. The reader is referred to [Go99] for more information. One of the standard models for the complex hyperbolic plane is the Siegel domain

\[ \mathbb{H}^2_C = \{(w_1, w_2) \in \mathbb{C}^2 : w_1 \overline{w_1} < w_2 + \overline{w_2}\} . \]

The real hypersurface

\[ \{(w_1, w_2) \in \mathbb{C}^2 : w_1 \overline{w_1} = w_2 + \overline{w_2}\} \]

corresponds to the sphere at infinity with one point removed; the point is denoted \( \infty \). We identify the stabilizer of \( \infty \) in \( \text{PU}(2,1) \) with the Heisenberg group \( \mathcal{H} \). One can introduce horospherical coordinates in \( \mathbb{H}^2_C \) by

\[ (x, y, u, v) = (z, u, v) \in \mathbb{C} \times (0, \infty) \times \mathbb{R} \]

where \( z = x + iy = w_1 \) and \( u + iv = 2\overline{w_2} - w_1\overline{w_1} \).

For every fixed \( u > 0 \), the real hypersurface \( \{(z, u, v) : z \in \mathbb{C}, v \in \mathbb{R}\} \) is a horosphere centered at the point \( \infty \). The group \( \mathcal{H} \) acts simply transitively on each horosphere, so that every horosphere gets a Heisenberg space structure given by

\[ (x_1, u_1, v_1) + (x_2, u_2, v_2) = (x_1 + x_2, u_1 + u_2, v_1 + v_2 + 2\text{Im}(z_1z_2)) . \]

Similarly, the hypersurface \( u = 0 \), that corresponds to \( \partial_\infty \mathbb{H}^2_C \setminus \{\infty\} \), has a simply transitive \( \mathcal{H} \)-action, and the \( \mathcal{H} \)-action on \( Y = \mathbb{H}^2_C \cup \partial_\infty \mathbb{H}^2_C \setminus \{\infty\} \) is smooth, free, and proper.

The real hyperbolic plane \( \mathbb{H}^2_R \) sits inside \( \mathbb{H}^2_C \) as a totally real 2-plane \( \{(x, 0, u, 0) \in \mathbb{H}^2_C\} \). (In this paper we always think of \( \mathbb{H}^2_R \) as an upper half plane and think of \( \mathbb{H}^2_C \) as the Siegel domain). The orthogonal projection \( \mathbb{H}^2_C \to \mathbb{H}^2_R \) is a \( \text{SO}(2,1) \)-equivariant smooth 2-plane bundle where the fibers are totally real 2-planes orthogonal to \( \mathbb{H}^2_R \). The projection extends to a \( \text{SO}(2,1) \)-equivariant smooth map \( \mathbb{H}^2_C \cup \partial_\infty \mathbb{H}^2_C \to \mathbb{H}^2_R \cup \partial_\infty \mathbb{H}^2_R \) which is the identity on \( \partial_\infty \mathbb{H}^2_R \) and is a closed disk bundle away from \( \partial_\infty \mathbb{H}^2_R \). Since any totally real 2-plane is \( \text{PU}(2,1) \)-equivalent to \( \mathbb{H}^2_R \), the orthogonal projection to any totally real 2-plane enjoys the same properties.

Let \( \langle H_r \rangle \) be the group of all horizontal translation \( H_r(x, y, u, v) = (x + r, y, u, v - 2ry) \), and let \( \langle V_t \rangle \) be the group of all vertical translations \( V_t(x, y, u, v) = (x, y, u, v + t) \). Thus \( \langle V_t \rangle \) and \( \langle H_r \rangle \) are Lie subgroups of \( \mathcal{H} \), each isomorphic to the group of additive reals.

One can check that the quotient map \( \Pi : Y \to Y/\langle H_r \rangle \) is the smooth trivial principal \( \langle H_r \rangle \)-bundle, and the restriction \( \Pi|_P : P \to Y/\langle H_r \rangle \) of \( \Pi \) to the plane

\[ P = \{(x, y, u, v) \in Y : x = 0\} \]

is a diffeomorphism, so that \( Y/\langle H_r \rangle \) is diffeomorphic to \( \mathbb{R}^2 \times [0, \infty) \).

Since vertical and horizontal translations commute, \( \langle V_t \rangle \) naturally acts on \( Y/\langle H_r \rangle \) so that the diffeomorphism \( \Pi|_P \) is \( \langle V_t \rangle \)-equivariant. Thus since \( \langle V_t \rangle \) acts on \( P \) properly, it acts properly on \( Y/\langle H_r \rangle \). In particular, if \( U \) and \( U' \) are subsets of \( Y \) such that \( \Pi(U) \) and \( \Pi(U') \) are precompact in \( Y/\langle H_r \rangle \), then \( \{t \in \mathbb{R} : V_t(U) \cap U' \neq \emptyset\} \) is precompact.
For $v \in \mathbb{R}$, let $\Sigma_v$ be the totally real 2-plane in $H_2^2$ given by $\{(x, 0, u, v) \in H_2^2\}$. Let $\pi_v : H_2^2 \to \Sigma_v$ be the orthogonal projection, and let $\tilde{\pi}_v : H_2^2 \cup \partial_{\infty} H_2^2 \to \Sigma_v$ be the extension of $\pi_v$ as above. Since $\Sigma_v$ is $(H_v)$-invariant, $\pi_v$ is $(H_v)$-equivariant. One can check that for any horoball $B \subset \Sigma_v$ centered at $\infty$, the set $\Pi(\tilde{\pi}_v^{-1}(\Sigma_v \setminus B))$ is precompact in $Y/(H_v)$. Finally, it is easy to see that if $l \subset \Sigma_v$ is a geodesic passing through $\infty$ and $L = \pi_v^{-1}(l)$, then the restriction $\Pi|_L : L \to H_2^2/(H_v)$ of $\Pi$ to $L$ is a diffeomorphism. Thus $L$ can be thought of as a smooth section of the bundle $\Pi$.

### 3 Maskit combination theorem

One of the common methods of producing new discrete groups is the so-called combination theorems. Here we state a combination theorem for groups acting by homeomorphisms on an arbitrary topological space due to B. Maskit [Mas88, VII.A].

Let $X$ be a topological space and $\Gamma$ be a subgroup of $\text{Homeo}(X)$. A subspace $Y \subset X$ is called precisely invariant with respect to a subgroup $H \leq \Gamma$ if $Y$ is $H$-invariant, and $\gamma(Y) \cap Y = \emptyset$ for $\gamma \in \Gamma \setminus H$. A subset $F \subset X$ is called a fundamental set for the $\Gamma$-action on $X$ if $F$ contains no $\Gamma$-equivalent points and intersects every $\Gamma$-orbit. We say that $\Gamma$ acts discontinuously on $X$ if it has a fundamental set with nonempty interior.

Let $\Gamma_1, \Gamma_2$ be subgroups of $\text{Homeo}(X)$ and let $J$ be a subgroup of $\Gamma_1 \cap \Gamma_2$. Following Maskit, a pair $(X_1, X_2)$ of disjoint, nonempty, $J$-invariant subsets of $X$ is called proper interactive if, for each $m \in \{1, 2\}$ every element of $\Gamma_m \setminus J$ maps $X_m$ into $X_{3-m}$, and for some $m \in \{1, 2\}$ there is a point in $X_{3-m}$ that is not $\Gamma_m$-equivalent to any point of $X_m$.

**Theorem 3.1.** (Maskit) Let $\Gamma_1$ and $\Gamma_2$ be subgroups of $\text{Homeo}(X)$, and let $J$ be a subgroup of $\Gamma_1 \cap \Gamma_2$ such that there exists a proper interactive pair $(X_1, X_2)$. Assume that for each $m \in \{1, 2\}$ there is a fundamental set $\Phi_m$ for the $\Gamma_m$-action on $X$ such that $\Phi_m$ has nonempty interior, $\Phi_m \cap X_m$ is a fundamental set for the $J$-action on $X_m$, and every element of $\Gamma_m$ maps $\Phi_m \cap X_{3-m}$ into $X_{3-m}$.

Then the group $\Gamma = (\Gamma_1, \Gamma_2)$ is isomorphic to $\Gamma_1 \ast_J \Gamma_2$, and the set $\Phi = (\Phi_1 \cap X_2) \cup (\Phi_2 \cap X_1)$ is precisely invariant under the identity in $\Gamma$. In particular, if $\Phi$ has nonempty interior, then $\Gamma$ acts discontinuously on $X$.

### 4 Main construction

For the rest of the paper we fix a closed orientable surface $M$ and a nontrivial element $\gamma \in \pi_1(M)$ represented by a simple closed loop $\tilde{\gamma}$ that separates $M$ into two connected compact surfaces $M_1, M_2$ with boundary $\tilde{\gamma}$. Thus $\pi_1(M)$ can be written as the amalgamated product $\pi_1(M_1) \ast_{\langle \gamma \rangle} \pi_1(M_2)$.

For $m \in \{1, 2\}$, we identify the interior of $M_m$ with the finite volume hyperbolic surface $H_2^2/\Gamma_m$ where $\Gamma_m$ is a noncocompact lattice in $\text{SO}(2, 1)$. We can assume that $\Gamma_m$ has a fundamental polyhedron with finitely many sides, and exactly two sides passing through the point infinity in the upper half plane $H_2^\infty$. We call these two sides unbounded and
all the other sides \emph{bounded}. After removing some boundary points of the polyhedron, we get a fundamental set \( F_m \) for \( \Gamma_m \) in \( H^2_C \). Slightly abusing notations, we usually do not distinguish between \( F_m \) and the original polyhedron, in particular, we often talk about sides of \( F_m \).

Let \( \Sigma_{v_1}, \Sigma_{v_2} \) be totally real 2-planes as in section \([2]\). The stabilizer of \( \Sigma_{v_m} \) in \( PU(2,1) \) is isomorphic to \( SO(2,1) \) so we can make \( \Gamma_m \) act on \( H^2_C \) stabilizing \( \Sigma_{v_m} \). Furthermore, we identify \( F_m \) with a polyhedron in \( \Sigma_{v_m} \) such that unbounded sides of \( F_m \) pass through \( \infty \) in the Siegel domain \( H^2_C \).

As the projection \( \pi_{v_m} : H^2_C \to \Sigma_{v_m} \) is \( \Gamma_m \)-equivariant, \( \Gamma_m \) acts properly discontinuously on \( H^2_C \) with a fundamental set \( \Psi_m = \pi_{v_m}^{-1}(F_m) \). Since \( F_m \) is the intersection of finitely many halfplanes, \( \Psi_m \) is the intersection of finitely many \emph{halfspaces}. Each halfspace is bounded by a smooth hypersurface which we call \emph{hyperplane}. We say a hyperplane is \emph{unbounded} if it passes through \( \infty \). Otherwise, a hyperplane is \emph{bounded}.

Let \( d_m \in \Gamma_m \) be a parabolic element pairing the unbounded sides of \( F_m \). Then \( d_m \) acts as a horizontal translation preserving \( \Sigma_{v_m} \), that is, \( d_m = H_{r_m} \) for some \( r_m \in \mathbb{R} \). We now assume that \( v_1 = v_2 \), i.e. \( d_1 = d_2 \), and we denote this horizontal translation by \( d \).

Let \( \beta_m \subset b_m \subset B_m \) be three concentric horoballs in the plane \( \Sigma_{v_m} \) centered at \( \infty \). For each \( m \), choose \( B_m \) small enough so that \( \Sigma_{v_m} \setminus B_m \) contains all bounded sides of \( F_m \) (and, hence, \( \pi_{v_m}^{-1}(\Sigma_{v_m} \setminus B_m) \) contains all bounded hyperplanes of \( \Psi_m \)).

By section \([2]\), \( \Pi \) maps the complements of \( \pi_{v_m}^{-1}(\beta_m) \) and \( \pi_{v_3-m}^{-1}(\beta_3-m) \) onto precompact subsets of \( Y/H_r \), hence we can choose \( t = v_2 - v_1 \) so large that the subsets are disjoint. In fact, we can assume that the subsets lie on the different sides of a properly embedded hypersurface \( H \subset Y/H_r \) which becomes a linear half plane under the identification \( Y/H_r \simeq \mathbb{R}^2 \times [0, \infty). \)

The hypersurface \( \Pi^{-1}(H) \) divides \( H^2_C \) into two connected components. We let \( X_1 \) be the component containing the bounded hyperplanes of \( \Phi_2 \) and let \( X_2 \) be the closure of the other component. Thus \( X_2 \) contains all bounded hyperplanes of \( \Phi_1 \) and \( X_1 \cup X_2 = H^2_C \).

Let \( D_1 = \Pi(\pi_{v_2}^{-1}(\beta_2)) \) and \( D_2 = \Pi(\pi_{v_3}^{-1}(\Sigma_{v_3} \setminus b_2)) \); these are disjoint domains in \( H^2_C/H_r \). Consider the unbounded hyperplanes \( S_m, d(S_m) \) of \( \Psi_m \). By section \([2]\), \( S_m \) can be thought of as a smooth section of the bundle \( \Pi : H^2_C \to H^2_C/H_r \). Using bump functions, we construct a smooth section \( S \) of the bundle whose restriction to \( D_m \) is \( S_m \). Notice that \( S \) splits \( H^2_C \) into two connected components; we call the component that contains \( d(S) \) a \emph{halfspace associated to} \( S \). Similarly, \( d(S) \) splits \( H^2_C \) in two components and we call the component that contains \( S \) a \emph{halfspace associated to} \( d(S) \). Let \( \Phi_m \) be the intersection of all the bounded halfspaces of \( \Psi_m \) and the halfspaces associated to \( S \) and \( d(S) \).

## 5 Discreteness

In this section, we show that \( \Phi_m \) is a fundamental set for the \( \Gamma_m \)-action on \( H^2_C \), and \( \Phi = \Phi_1 \cap \Phi_2 \) is the fundamental set for the \( \Gamma \)-action on \( H^2_C \), where \( \Gamma = (\Gamma_1, \Gamma_2) \).

**Lemma 5.1.** For each \( m \in \{1, 2\} \), \( \Phi_m \) is a fundamental set for \( \Gamma_m \).
Proof. The intersection of the halfspaces associated to $S$ and $d(S)$ is a fundamental set for the group generated by $d$. By construction, $\Psi_m \subseteq \bigcup_{n \in \mathbb{Z}} d^n(\Phi_m)$. Hence $H^2_2 = \bigcup_{\gamma \in \Gamma_m} \Psi_m \subset \bigcup_{\gamma \in \Gamma_m} \Phi_m$. If $\Phi_m$ has $\Gamma_m$-equivalent points, they must be $\langle d \rangle$-equivalent since $\Phi_m \subseteq \bigcup_{n \in \mathbb{Z}} d^n(\Psi_m)$. By construction, $\Phi_m$ has no $\langle d \rangle$-equivalent points which completes the proof. 

Lemma 5.2. $(X_1, X_2)$ is a proper interactive pair.

Proof. Clearly, $X_m$ is $\langle d \rangle$-invariant, and $X_m \subseteq \bigcup_{n \in \mathbb{Z}} d^n(\Phi_m)$. So for any $g \in \Gamma_m \setminus \langle d \rangle$, 

$$g(X_m) \cap X_m \subseteq g\left(\bigcup_{n \in \mathbb{Z}} d^n(\Phi_m)\right) \cap \bigcup_{n \in \mathbb{Z}} d^n(\Phi_m) = \emptyset$$

since $\Phi_m$ is a fundamental set. Hence, $X_m$ is precisely invariant under $\langle d \rangle$ in $\Gamma_m$. Now take $g \in \Gamma_m \setminus \langle d \rangle$. Since $X_1$ is the complement of $X_2$, and $X_1$, $X_2$ are disjoint, $g(X_m) \cap X_m = \emptyset$ implies $g(X_m) \subseteq X_{3-m}$ as desired. It remains to find a point in $X_{3-m}$ that is not $\Gamma_m$-equivalent to any point of $X_m$. Take $x \in \Phi_1 \cap \Phi_2 \cap X_{3-m}$ and assume $g(x) \in X_m$ for some $g \in \Gamma_m$. Since $X_m \subseteq \bigcup_{n \in \mathbb{Z}} d^n(\Phi_m)$, we get $d^n g(x) \in \Phi_m$ for some $n$. Hence $g = d^{-n}$ and, therefore, $g(x) \in X_{3-m}$ because $X_{3-m}$ is $d$-invariant. Since $X_1, X_2$ are disjoint, we get a contradiction. 

Theorem 5.3. The group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is discrete and isomorphic to $\Gamma_1 *_{\langle d \rangle} \Gamma_2 \cong \pi_1(M)$. Moreover, $\Phi = \Phi_1 \cap \Phi_2$ is a fundamental set for $\Gamma$.

Proof. We want to apply Theorem 3.1. First, obviously, $\Phi_m \cap X_m$ is a fundamental set for the $\langle d \rangle$-action on $X_m$. Second, we check that every element of $\Gamma_m$ maps $\Phi_m \cap X_{3-m}$ into $X_{3-m}$. (If not, then there is $x \in g(\Phi_m \cap X_{3-m}) \cap X_m$. Then for some $n$, $d^n(x) \in \bigcup_{n \in \mathbb{Z}} d^n(\Phi_m) \cap (\Phi_m \cap X_m)$). Since $\Phi_m$ is a fundamental set, $g = d^{-n}$ which leads to a contradiction because $X_{3-m}$ is $d$-invariant). Thus, since $\Phi_1 \cap \Phi_2 = (\Phi_2 \cap X_1) \cup (\Phi_1 \cap X_2)$ has nonempty interior we can apply Theorem 3.1 to conclude $\Gamma$ is discrete, isomorphic to $\Gamma_1 *_{\langle d \rangle} \Gamma_2$ and $\Phi$ is precisely invariant under the identity in $\Gamma$.

We show that $\Phi = \Phi_1 \cap \Phi_2$ is a fundamental set for $\Gamma$ by proving that the projection $p: \mathbb{H}^2_N / \Gamma = N$ maps $\Phi$ onto an open and a closed subset of $N$, so that $p(\Phi) = N$. We first show that $p(\Phi)$ is open. Since $p$ is an open map and $p(\Phi) = p(\bigcup_{g \in \Gamma} g(\Phi))$, it is enough to prove that $\bigcup_{g \in \Gamma} g(\Phi)$ is open. Any point of the set is $\Gamma$-equivalent to a point of $\Phi$, hence, it suffices to show that $\Phi$ has an open neighborhood in $\bigcup_{g \in \Gamma} g(\Phi)$. One easily sees that $\Phi$ lies in the interior of the set $\bigcup_{g \in S} g(\Phi)$ where $S = \{g \in \Gamma: \overline{\Phi} \cap g(\overline{\Phi}) \neq \emptyset\}$. Now prove that $p(\Phi)$ is closed in $N$. Let $x_n \in \Phi$ be an arbitrary sequence such that $p(x_n)$ converges to $y \in N$. If $x_n$ subconverges to $z \in \overline{\Phi}$, then $p(z) = y$ and we are done since $p(\Phi) = p(\overline{\Phi})$. It remains to consider the case when $x_n$ converges to $\infty$. Passing to a subsequence, we can assume that $x_n \in \pi_{\beta_2}^{-1}(\beta_2) \cap \Phi$, and there is a sequence $y_n$ with $p(y_n) = y$ and $\text{dist}(x_n, y_n) < 1$. Note that $\pi_{\beta_2}^{-1}(\beta_2) \cap \Phi = \pi_{\beta_2}^{-1}(\beta_2) \cap \Psi_1 \subset \pi_{\beta_1}^{-1}(F_1)$. Hence $\pi_{\beta_1}(x_n) \in F_1$ and since $x_n$ converges to $\infty$, we can assume $x_n \in F_1 \cap \beta_1$ and $\text{dist}(\pi_{\beta_1}(x_n), \partial \beta_1) > 1$. Since $\pi_{\beta_1}$ is distance nonincreasing, $\pi_{\beta_1}(y_n) \in \beta_1$. Thus, $y_n \in \pi_{\beta_1}^{-1}(\beta_1) \subset \bigcup_{n \in \mathbb{Z}} d^n(\Phi)$ so that $y \in p(\Phi)$. 


6 Geometrical finiteness

We refer \[\text{Bow95}\] for background on geometrical finiteness for manifolds of pinched negative curvature.

**Theorem 6.1.** The group \( \Gamma \) is a geometrically finite and any maximal parabolic subgroup of \( \Gamma \) is conjugate to the cyclic subgroup generated by \( \rho(\gamma) \).

**Proof.** One of the definitions of geometrical finiteness given in \[\text{Bow95}\] is that the quotient manifold has finitely many ends, and every end is standard parabolic. Each of the manifolds \( \Phi/\Gamma \) and \( \Psi_1/\Gamma_1 \) has exactly one end which is isometric to the quotient of

\[ \pi_{\nu_2}^{-1}(\beta_2) \cap \Phi = \pi_{\nu_2}^{-1}(\beta_2) \cap \Psi_1 \]

by the subgroup \( \langle d \rangle \). The group \( \Gamma_1 \) is of course geometrically finite as a subgroup of \( \text{SO}(2,1) \), hence \( \Gamma_1 \) is geometrically finite as a subgroup of \( \text{PU}(2,1) \) since geometric finiteness is encoded in the \( \Gamma \)-action on the limit set \[\text{Bow95}\]. Therefore, \( \Gamma \) is geometrically finite. \( \square \)

**Remark 6.2.** Since \( \Gamma \) is geometrically finite, the limit set \( \Lambda(\Gamma) \) of \( \Gamma \) consists of conical limit points and bounded parabolic points. The set of bounded parabolic points is precisely the \( \Gamma \)-orbit of \( \infty \). Being a closed surface group \( \Gamma \) acts on \( S^1 = \partial_\infty \mathbb{H}^2_\mathbb{C} \). Now Floyd’s theorem \[\text{Flo80}\] (adapted for the complex hyperbolic case) implies that there is a continuous \( \Gamma \)-equivariant map of \( S^1 \) onto \( \Lambda(\Gamma) \) such that every conical limit point has one preimage and every bounded parabolic point has exactly two preimages.

7 Quotients are tangent bundles

**Theorem 7.1.** The quotient manifold \( N = \mathbb{H}^2_\mathbb{C}/\Gamma \) is diffeomorphic to the total space of the tangent bundle to \( M \).

**Proof.** Our first goal is to define a \( \Gamma \)-equivariant \( \mathbb{R}^2 \)-bundle structure on \( \overline{\Phi} \) (in this proof all closures are taken in \( \mathbb{H}^2_\mathbb{C} \)). The set \( \overline{\Phi} = \overline{\Phi}_1 \cap \overline{\Phi}_2 \) is the union of three pieces:

\[ \overline{\Phi}_1 \setminus \pi_1^{-1}(b_1), \quad \overline{\Phi}_2 \setminus \pi_2^{-1}(b_2) \quad \text{and} \quad \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2) \cap \overline{\Phi} \]

where \( \overline{\Phi}_1 \setminus \pi_1^{-1}(b_1) \) and \( \overline{\Phi}_2 \setminus \pi_2^{-1}(b_2) \) are disjoint. Moreover, the intersection of \( \overline{\Phi}_m \setminus \pi_m^{-1}(b_m) \) and \( \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2) \cap \overline{\Phi} \) is the set \( \pi_m^{-1}(B_m \setminus b_m) \cap \overline{\Phi} \). The map

\[ \pi_{vm} : \overline{\Phi}_m \setminus \pi_m^{-1}(b_m) \to \overline{F}_m \setminus b_m \]

is a smooth \( \Gamma \)-equivariant \( \mathbb{R}^2 \)-bundle whose fibers are totally real 2-planes orthogonal to \( \Sigma_{vm} \). We now define a smooth \( \Gamma \)-equivariant \( \mathbb{R}^2 \)-bundle structure on \( \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2) \cap \overline{\Phi} \) in such a way that on the overlap \( \pi_{vm}^{-1}(B_m \setminus b_m) \cap \overline{\Phi} \) it coincides with

\[ \pi_{vm} : \pi_{vm}^{-1}(B_m \setminus b_m) \cap \overline{\Phi} \to (B_m \setminus b_m) \cap \overline{F}_m. \]
Find a smooth proper embedding \( f: \mathbb{R} \to \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2) \cap S \) such that the intersection of \( f(\mathbb{R}) \) and \( \pi_1^{-1}(B_m \setminus b_m) \cap S \) is the interval \( B_m \setminus b_m \cap S \). It is easy to construct a smooth \( \mathbb{R}^2 \)-bundle \( \pi_{v_m}: \pi^{-1}_v(B_m \setminus b_m) \cap S \to (B_m \setminus b_m) \cap S \).

This defines an \( (H_r) \)-equivariant smooth \( \mathbb{R}^2 \)-bundle structure on \( \pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2) \) over the \( (H_r) \)-orbit of \( f(\mathbb{R}) \) that extends the bundle \( \pi_{v_m}: \pi^{-1}_v(B_m \setminus b_m) \to B_m \setminus b_m \).

Let \( E \) be the intersection of \( \Phi \) and the \( (H_r) \)-orbit of \( f(\mathbb{R}) \), and let \( D = (\tilde{F}_1 \setminus b_1) \cup E \cup (\tilde{F}_2 \setminus b_2) \). We have just constructed a smooth \( \Gamma \)-equivariant \( \mathbb{R}^2 \)-bundle \( \Phi \to D \). Let \( \tilde{D} \) be the \( \Gamma \)-orbit of \( D \).

It is immediate to check that \( \tilde{D} \) is a smooth submanifold of \( H_2^2 \) and the \( \Gamma \)-action on \( \tilde{D} \) is smooth, free, and properly discontinuous so that \( \tilde{D}/\Gamma \) is diffeomorphic to \( M \). Therefore, we get a smooth \( \Gamma \)-equivariant \( \mathbb{R}^2 \)-bundle \( H_2^2 \to \tilde{D} \). Passing to quotients yields a smooth \( \mathbb{R}^2 \)-bundle \( N = H_2^2/\Gamma \to \tilde{D}/\Gamma = M \).

Choose orientations on \( \Sigma_{v_1}, \Sigma_{v_2} \) so that \( \Gamma_m \) preserves the orientation on \( \Sigma_{v_m} \) and the vertical translation \( V_t: \Sigma_{v_1} \to \Sigma_{v_2} \) is orientation preserving. Since \( \Sigma_{v_m} \) is totally real, the complex structure \( \mathbb{J} \) defines a \( \text{SO}(2,1) \)-equivariant isomorphism between the normal and tangent bundles to \( \Sigma_{v_m} \). The above orientation on \( \Sigma_{v_m} \) together with its \( \mathbb{J} \)-image defines an orientation on \( H_2^2 \), which coincides with the canonical orientation of \( H_2^2 \). This defines the orientations on \( M, N \), and the bundle \( N \to M \).

Oriented plane bundles over an oriented closed surface \( M \) are (smoothly) isomorphic iff their Euler numbers are equal, so it suffices to show that the Euler number of the bundle \( N \to M \) is \( \chi(M) \). The Euler number is equal to the self-intersection number of the “zero section” \( \sigma_0: M = D/\Gamma \to N \). The surface \( M = D/\Gamma \) is the union of three pieces

\[
M_1 = (\tilde{F}_1 \setminus b_1)/\Gamma, \quad T = E/\Gamma \quad \text{and} \quad M_2 = (\tilde{F}_2 \setminus b_2)/\Gamma.
\]

Choose a smooth section \( \sigma: M \to N \) which is transverse to \( \sigma_0 \). Since \( T \) is an annulus, we can assume that \( \sigma(T) \) is disjoint from \( \sigma_0(M) \). So the intersection number of \( \sigma \) and \( \sigma_0 \) is the sum of the intersection numbers \( i_1 \) and \( i_2 \) where \( i_m \) is intersection number of \( \sigma \) and \( \sigma_0 \) restricted to \( M_m \). By construction the bundle \( N \to M \) restricted to \( M_m \) is isomorphic to the tangent bundle of \( M_m \) (as the universal cover of \( M_m \) is “totally real”). It is now a standard computation to see that \( i_m = \chi(M_m) \). (Hint: double \( M_m \) along the boundary to produce a closed oriented surface \( S \) with \( \chi(S) = 2\chi(M_m) \). The intersection number of the double of \( \sigma_0|_{M_m} \) and the double of \( \sigma|_{M_m} \) is \( 2i_m \). On the other hand, the self-intersection number of the zero section of \( TS \) is \( \chi(S) \).) Thus the self-intersection number of \( \sigma_0 \) is \( i_1 + i_2 = \chi(M_1) + \chi(M_2) = \chi(M) \) as desired. \( \square \)

8 Computing the Toledo invariant

For background on Toledo invariant see [Tol89, GKL01, Xia00]. Here is a short account sufficient for our purposes. Let \( \rho: \pi_1(M) \to \text{PU}(2,1) \) be a discrete faithful representation. The Kähler form \( \omega \) on \( H_2^2 \) defines the Kähler form \( \omega_N \) on \( N = H_2^2/\Gamma \). Consider a smooth homotopy equivalence \( f: M \to N \) uniquely defined by \( \rho \) up to homotopy. The Toledo invariant of \( \rho \) is \( \tau(\rho) = \frac{1}{2\pi} \int_M f^*\omega_N \). It is proved in [GKL01] that \( \tau(\rho) \in \frac{1}{2}\mathbb{Z} \).
Theorem 8.1. If $\rho(\pi_1(M)) = \Gamma$, then $\tau(\rho) = 0$.

Proof. The surface $M \setminus T$ in $N$ is the union of two disjoint surfaces $M_1$ and $M_2$. The universal cover of each of them lies in a totally real subspace. Since the Kähler form $\omega$ vanishes when restricted to a totally real subspace, we get $\int_{M_m} f^* \omega_N = 0$. Therefore,

$$
\tau(\rho) = \frac{1}{2\pi} \int_{M_1} f^* \omega_N + \frac{1}{2\pi} \int_{M_2} f^* \omega_N + \frac{1}{2\pi} \int_T f^* \omega_N = \frac{1}{2\pi} \int_T f^* \omega_N = \frac{1}{2\pi} \int_E \omega.
$$

Consider the horizontal translation $d = H_r$ that identifies the boundary components of $E$. Consider the positive integer $n = 3|\tau(\rho)| + 1$, and let $s = \frac{r}{n}$. Let $e$ and $H_r(e)$ be boundary curves of $E$. Call $E_i$ the subsurface of $E$ bounded by $H_{is}(e)$ and $H_{s}(H_{is}(e))$ where $i = 0, \ldots, n - 1$. Since the form $\omega$ is $\text{PU}(2,1)$-invariant, we get

$$
\int_{E_i} \omega = \int_{H_{is}(E_0)} \omega = \int_{E_0} H_{is}^* \omega = \int_{E_0} \omega.
$$

Therefore,

$$
\tau(\rho) = \frac{n}{2\pi} \int_{E_0} \omega \quad \text{so that} \quad \left| \frac{1}{2\pi} \int_{E_0} \omega \right| < \frac{1}{3}.
$$

To prove $\tau(\rho) = 0$, it suffices to show that $\frac{1}{2\pi} \int_{E_0} \omega \in \frac{2}{3} \mathbb{Z}$. Notice that in the construction of $\Gamma = \Gamma_1 \ast_{(d)} \Gamma_2$ the parabolic element $d = H_r$ can be chosen arbitrarily. In particular, one can take $r = s$. This defines a new discrete faithful representation $\rho_s: \pi_1(M) \to \text{PU}(2,1)$. Repeating the construction of Theorem 7.1, one can choose the surface $E$ so that it coincides with the surface that was denoted $E_0$ above. Then $\frac{1}{2\pi} \int_{E_0} \omega$ is equal to the Toledo invariant of $\rho_s$ which lies in $\frac{2}{3} \mathbb{Z}$ as needed.

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