SPLIT CASIMIR OPERATOR AND SOLUTIONS OF THE YANG–BAXTER EQUATION FOR THE \( \text{osp}(M|N) \) AND \( \text{s}\ell(M|N) \) LIE SUPERALGEBRAS, HIGHER CASIMIR OPERATORS, AND THE VOGEL PARAMETERS

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We find the characteristic identities for the split Casimir operator in the defining and adjoint representations of the \( \text{osp}(M|N) \) and \( \text{s}\ell(M|N) \) Lie superalgebras. These identities are used to build the projectors onto invariant subspaces of the representation \( T \otimes T \) of the \( \text{osp}(M|N) \) and \( \text{s}\ell(M|N) \) Lie superalgebras in the cases where \( T \) is the defining or adjoint representation. For the defining representation, the \( \text{osp}(M|N) \)- and \( \text{s}\ell(M|N) \)-invariant solutions of the Yang–Baxter equation are expressed as rational functions of the split Casimir operator. For the adjoint representation, the characteristic identities and invariant projectors obtained are considered from the standpoint of a universal description of Lie superalgebras by means of the Vogel parameterization. We also construct a universal generating function for higher Casimir operators of the \( \text{osp}(M|N) \) and \( \text{s}\ell(M|N) \) Lie superalgebras in the adjoint representation.

Keywords: invariant subspace, projector, simple Lie superalgebra, split Casimir operator, Vogel parameters, generating function

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1. Introduction

It is known that the split Casimir operator \( \hat{C} \) (see the definition in Sec. 2, and also see [1]) plays an important role in the description of Lie algebras and superalgebras as well as in the study of their representations. Furthermore, the operator \( \hat{C} \) is used to construct solutions of the semiclassical and quantum Yang–Baxter equations that are invariant under the action of Lie algebras and superalgebras in various representations (see, e.g., [2], [3]).

In this paper, we use the operator \( \hat{C} \) to construct a system of projectors onto invariant subspaces of the representations \( T \otimes T \) of the complex Lie superalgebras \( \text{osp}(M|N) \) and \( \text{s}\ell(M|N) \) in the cases where \( T = T_f \) is the defining representation and where \( T = \text{ad} \) is the adjoint representation.

The idea to construct projectors onto invariant subspaces of representations of Lie algebras and superalgebras by means of invariant operators is not new. For example, the invariant projectors that act on the tensor product of the \( \text{s}\ell(N) \) Lie algebra defining representations are called the Young symmetrizers and are

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constructed as images of specific elements of the group algebra $\mathbb{C}[S_r]$ of the symmetric group $S_r$. The algebra $\mathbb{C}[S_r]$ centralizes the action of the algebra $sl(N)$ in the representation $T_f^\otimes r$. For the $so(N)$ and $sp(N)$ Lie algebras (where $N = 2n$ is even), there exists an analogous statement: the action of those algebras on the representation $T_f^\otimes r$ is centralized by the Brauer algebra $B_{r_1}^r(\omega)$. The aforementioned properties of the $sl(N)$, $so(N)$, and $sp(N)$ Lie algebras are carried over to the case of Lie superalgebras: in [4] and [5], the method of describing subrepresentations of $T^\otimes r$ by means of Young symmetrizers was generalized to encompass the $sl(M|N)$ Lie superalgebras, and in [6] an analogous result was obtained for the $osp(M|N)$ Lie superalgebras. Here, we consider a decomposition of the representation $T \otimes T$ into subrepresentations by using the operator $\hat{C}$ defined uniformly for all Lie superalgebras with a nondegenerate Cartan–Killing metric. Within this approach, the $sl(M|N)$ and $osp(M|N)$ Lie superalgebras are described in a similar fashion.

In the case where $T = ad$ is the adjoint representation, the construction of projectors onto invariant subspaces of the representation $T \otimes T = ad \otimes ad$ by using $\hat{C}$ is significant from yet another standpoint. It is related to the notion of the *Universal Lie algebra*, which was introduced by Vogel in [7] (see also [8], [9]). The Universal Lie algebra was supposed to be a model of all complex simple Lie algebras, in addition embracing some Lie superalgebras. For example, many quantities that characterize the Lie algebra $\mathfrak{g}$ in different (possibly reducible) representations $T\lambda$ that participate in the decomposition $ad^\otimes k = \sum \lambda T\lambda$, where $k \geq 1$, are expressed as rational functions of the three Vogel parameters (see their definition in 5). These parameters take specific values for all complex simple Lie algebras as well as for all basic classical Lie superalgebras (see, e.g., [10] and Sec. 5 below). In particular, it was shown for Lie algebras that using the Vogel parameters allows expressing the dimensions of the representations $T\lambda$ when $k = 2, 3$ [7], the dimensions of the $ad$-series representations, i.e., the representations $T\lambda\nu$ with the highest weight $\lambda' = k\lambda_{ad}$, where $\lambda_{ad}$ is the highest weight of a given Lie algebra [11], the dimensions of the representations of $X_2$-series [12], and the values of higher Casimir operators in the adjoint representation of a given Lie algebra [10]. Furthermore, it was shown in [13], [14] that the universal description of complex simple Lie algebras allows formulating some types of knot polynomials as a single function for all simple Lie algebras.

This paper is organized as follows. In Sec. 2, we recall the main notions of the Lie superalgebra theory and introduce some necessary conventions that we use throughout the paper. Sections 3 and 4 are dedicated to calculating the characteristic identity for the split Casimir operator in the defining $T_f$ and adjoint $ad$ representations of the $osp(M|N)$ and $sl(M|N)$ Lie superalgebras and to constructing projectors onto invariant subspaces of these representations. We also show that the results obtained are in full correspondence with the conclusions in [15], [16], where analogous calculations were carried out for Lie algebras. In Sec. 5, we write characteristic identities for the symmetric part of the split Casimir operator and the corresponding projectors onto symmetric invariant subspaces, uniformly (in accordance with a universal way) for both the $osp(M|N)$ and $sl(M|N)$ Lie algebras by using the Vogel parameters. In Sec. 6, following the approach in [1], [10], we find a universal form of the generating function of the higher Casimir operators of the $osp(M|N)$ and $sl(M|N)$ Lie superalgebras in the adjoint representation.

2. Background on Lie superalgebras

In this section, we briefly discuss the main definitions and conventions from the theory of Lie superalgebras (see, e.g., [17], [18]) and introduce the notation to be used in what follows.

2.1. Lie superalgebras and associative algebras. A linear superspace (or $\mathbb{Z}_2$-graded space) over the field $\mathbb{C}$ is a linear space $V = V_0 \oplus V_1$ that is a direct sum of the linear spaces $V_0$ and $V_1$ over the field $\mathbb{C}$. The spaces $V_0$ and $V_1$ are respectively called even and odd. The vectors from $V$ that lie in the even subspace $V_0$ are called even, and those lying in the odd space $V_1$ are called odd. Those vectors that
are either even or odd are called homogeneous. The grading of an arbitrary homogeneous vector \( v \in V \) is denoted by \( \text{deg}(v) \equiv [v] \in \mathbb{Z}_2 \), i.e., \([v] = 0, 1 \pmod{2}\). We write \( V_{(M|N)} \) for the linear superspace \( \dim V_{\bar{0}} = M \) and \( \dim V_{\bar{1}} = N \). The superdimension of the space \( V_{(M|N)} \) is defined by \( \text{sdim}(V_{(M|N)}) = M - N \). In the rest of this paper, we always assume the basis \( \{ e_a \}_{a=1}^{M+N} \) of \( V_{(M|N)} \) to be homogeneous, with the first \( M \) of its elements being even and the last \( N \) odd. We let \([a]\) denote the grading of a basis element \( e_a \). We note that in our convention, the grading is carried by the basis vectors of the space \( V_{(M|N)} \); for an example of another (but equivalent) convention whereby the grading is carried by the coordinates, see, e.g., [19], [20].

Let \( g \) be a Lie superalgebra over the field \( \mathbb{C} \) with a Lie superbracket \([\cdot, \cdot] : g \times g \to g\). For arbitrary homogeneous vectors \( X, Y, Z \in g \), the following two properties must be satisfied (see, e.g., [17]):

\[
[X, Y] \in g_{\text{even}}, \quad [X, Y] = (-1)^{|X||Y|}[Y, X], \quad (2.1)
\]

\[
(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [X, Z]] + (-1)^{|Z||X|}[Z, [X, Y]] = 0. \quad (2.2)
\]

If \( \{ X_i \}, i = 1, \ldots, \dim g \) is a homogeneous basis of \( g \), then

\[
[X_i, X_j] = X_k X^{ij}_k, \quad (2.3)
\]

where the numbers \( X^{ij}_k \) are the structure constants of \( g \). Clearly, \( X^{ij}_k = 0 \) if \((|i| + |j| + |k|) \pmod{2} \neq 0\), and \( X^{ij}_k = (-1)^{|i||j|}X^{ji}_k \).

We consider an associative superalgebra \( A = A_0 \oplus A_1 \). For any two homogeneous elements \( A, B \in A \), we can define the bracket \([\cdot, \cdot]\) as

\[
[A, B] = AB - (-1)^{|A||B|}BA, \quad (2.4)
\]

together with relations (2.1) and (2.2). The algebra \( A \) can then be viewed as a Lie superalgebra with respect to bracket (2.4). Following [17], we let this Lie superalgebra be denoted by \((A)_L\).

A representation of a Lie superalgebra \( g \) is a homomorphism \( T : g \to (\text{End}(V_{(M|N)}))_L \), i.e.,

\[
T([X, Y]) = [T(X), T(Y)] \quad \forall X, Y \in g, \quad (2.5)
\]

where the Lie superbracket in the right-hand side is defined in (2.4). In this paper, the key role is played by the adjoint representation \( \text{ad} : g \to (\text{End}(g))_L \), which is defined by the formula

\[
\text{ad}(X) \cdot Y = [X, Y] \quad (2.6)
\]

for arbitrary vectors \( X, Y \in g \equiv V_{ad} \). It follows from (2.3) and (2.6) that the entries of the matrix of the operator \( \text{ad}(X_i) \) in a homogeneous basis \( \{ X_j \} \) are equal to the structure constants of \( g \):

\[
\text{ad}(X_i)^k_j = X^{ij}_k. \quad (2.7)
\]

We consider a linear superspace \( V_{(M|N)} \) and an operator \( A : V_{(M|N)} \to V_{(M|N)} \) with the components \( A^a_b \) in some homogeneous basis \( \{ e_a \}_{a=1}^{M+N} \) of \( V_{(M|N)} \). We recall that the supertrace of \( A \) is defined as \( \text{str} A = (-1)^{|a|} A^a_a \) and has the important property

\[
\text{str}([A, B]) = 0 \quad \forall A, B \in \text{End}(V_{(M|N)}). \quad (2.8)
\]

The Cartan–Killing metric of a Lie superalgebra \( g \) is defined in the standard way,

\[
g_{ij} = \text{str}(\text{ad}(X_i) \text{ad}(X_j)) = (-1)^{|m|} X^{im}_j X^{jm}_i. \quad (2.9)
\]
and has the properties
\[ g(X, Y) = (-1)^{[X][Y]} g(Y, X) \quad \forall X \in g_{\bar{\alpha}}, \forall Y \in g_{\bar{\beta}}, \quad \bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2, \quad (2.10) \]
\[ g([X, Y], Z) = g(X, [Y, Z]) \quad \forall X, Y, Z \in g, \quad (2.11) \]

and
\[ g_{ij} = (-1)^{[i][j]} g_{ji} = (-1)^{[i]} g_{ji} = (-1)^{[j]} g_{ij}, \quad (2.12) \]
\[ g_{ij} = 0, \quad \text{if} \quad [i] + [j] \neq 0 \quad \text{(mod 2)}. \quad (2.13) \]

In the case of a nondegenerate Cartan–Killing metric, we also introduce the inverse Cartan–Killing metric with the components \( \bar{g}^{ij} \) given by
\[ \bar{g}^{ij}g_{jk} = \delta^i_k, \quad \bar{g}_{ij}\bar{g}^{jk} = \delta_i^k. \quad (2.14) \]

The metric \( \bar{g}^{ij} \) has the same properties (2.12) with respect to index permutations as \( g_{ij} \). The metrics \( \bar{g}^{ij} \) and \( g_{ij} \) can be used to lower and raise the indices of vectors and covectors in \( g \). We note that the raising of indices in \( g_{ij} \) yields \( g^{ij} = \bar{g}^{ik} \bar{g}^{jm} g_{km} = \bar{g}^{ij} \), and hence the metric tensor with the upper indices \( g^{ij} \) does not coincide with the inverse matrix \( \bar{g}^{ij} \). From now on, we only use \( \bar{g}^{ij} \).

We next introduce the structure constants of \( g \) with the lower indices:
\[ X_{kij} \equiv g_{km}X^m_{ij}. \quad (2.15) \]

From (2.7), (2.11), and (2.12) we deduce the following properties of \( X_{ijk} \) with respect to index permutations:
\[ X_{kji} = -(-1)^{[i][j]} X_{kij}, \]
\[ X_{jik} = -(-1)^{[k]+[j]} X_{kij}, \]
\[ X_{ikj} = -(-1)^{[i][k]} X_{kij}. \quad (2.16) \]

2.2. The split Casimir operator and comultiplication for Lie superalgebras. Let \( \mathcal{U}(g) \) denote the enveloping algebra of the Lie superalgebra \( g \). We consider the quadratic Casimir operator
\[ C_2 = \bar{g}^{ij}X_iX_j \in \mathcal{U}(g). \quad (2.17) \]

In view of (2.13), the operator \( C_2 \) is even and commutes with all generators \( X_k \) of \( \mathcal{U}(g) \) with respect to the bracket (2.4). Therefore, \( C_2 \) belongs to the center of \( \mathcal{U}(g) \).

We consider two associative superalgebras \( A \) and \( B \). The graded tensor product of \( A \) and \( B \) is the associative superalgebra \( A \otimes B \) that as a linear space coincides with the tensor product of the spaces \( A \) and \( B \), and where the product of arbitrary homogeneous vectors \( A, A' \in A \) and \( B, B' \in B \) is defined as
\[ (A \otimes B) \cdot (A' \otimes B') = (-1)^{[A][B]} AA' \otimes BB'. \quad (2.18) \]

Here and below, we always understand “tensor product” to mean the graded tensor product for which (2.18) holds. The matrix of the operator \( A \otimes B \) in the basis \( \{e_i \otimes e_\alpha\} \) of \( V \otimes V' \) is given by
\[ (A \otimes B)(e_i \otimes e_\alpha) = \begin{cases} e_k \otimes e_\beta (A \otimes B)^{k\beta}_{i\alpha}, \\ (-1)^{[B][i]} (Ae_i) \otimes (B e_\alpha) = (-1)^{[B][i]} (e_k A^k) \otimes (e_\beta B^\beta_\alpha), \end{cases} \]
whence in the case of a homogeneous $B$ we obtain
\[(A \otimes B)^{k_1 \otimes \alpha} = (-1)^{[B][\alpha]} A^{k_1} B^{\beta}. \tag{2.19}\]
Formula (2.19) can be generalized to arbitrary (not necessarily homogeneous) operators $B \in \text{End}(V')$ as
\[(A \otimes B)^{k_1 \otimes \alpha} = (-1)^{[\alpha]+[\beta][\alpha]} A^{k_1} B^{\beta}. \tag{2.20}\]
Generally, for the matrix of the operator $(A \otimes B \otimes C \otimes \cdots \otimes E)$ acting in $V_1 \otimes V_2 \otimes V_3 \otimes \cdots \otimes V_n$, we have
\[(A \otimes B \otimes C \otimes \cdots \otimes E)^{k_1 \cdots k_n} \otimes_{i_1 \cdots i_n} \equiv (A)^{k_1}_{i_1} \otimes (B)^{k_2}_{i_2} \otimes \cdots \otimes (C)^{k_3}_{i_3} \cdots \cdots \otimes (E)^{k_n}_{i_n}, \tag{2.21}\]
For an arbitrary $A: V \to V$, we define the operators $A_1, A_2: V^\otimes 2 \to V^\otimes 2$ by
\[A_1 \equiv A \otimes I, \quad A_2 \equiv I \otimes A, \tag{2.22}\]
where $I: V \to V$ is the identity operator. Using (2.21), we have
\[A_1 \cdot B_2 = (A \otimes I)(I \otimes B) = A \otimes B, \quad B_2 \cdot A_1 = (I \otimes B)(A \otimes I) = (-1)^{[\alpha][\beta]} A \otimes B, \tag{2.23}\]
and hence $A_1 \cdot B_2 \neq B_2 \cdot A_1$ in general.

The notation in (2.21) can be generalized to the case of any operator,
\[A = \hat{A}^{i_1 \cdots i_r}_{j_1 \cdots j_s} e_{i_1}^{j_1} \otimes \cdots \otimes e_{i_r}^{j_r} \in \text{End}(V^\otimes r), \tag{2.24}\]
where $e_{i}^{j}$ are the matrix identities defined as operators that act on the space $V$ with a basis $\{e_a\}$ such that
\[e_{i}^{j} \cdot e_a = e_b (e_{i}^{j})^b_a = e_i \delta_a^b \quad \Leftrightarrow \quad (e_{i}^{j})^b_a = \delta_i^a \delta_a^b. \tag{2.25}\]
Let $s > r$ and $1 \leq \alpha_1 < \cdots < \alpha_r \leq s$. We define $A_{\alpha_1 \cdots \alpha_r} \in \text{End}(V^{\otimes s})$ as
\[A_{\alpha_1 \cdots \alpha_r} = \hat{A}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \otimes \cdots \otimes I \otimes e_{i_1}^{j_1} \otimes \cdots \otimes I \otimes e_{i_r}^{j_r} \otimes I \otimes \cdots \otimes I, \tag{2.26}\]
where each of the matrix identities $e_{i_1}^{j_1}, k = 1, \ldots, r$, stands at the $\alpha_k$th place in the tensor product in the right-hand side, while all other places are occupied by the identity operators $I$ on $V$. For instance, given $A = \hat{A}^{i_1 k m}_{j_1 m} e_{i_1}^{k} \otimes e_{j_1}^{m} \otimes I$, the operators $A_{13}, A_{12}, A_{23}: V^\otimes 4 \to V^\otimes 4$ are defined as
\[A_{13} = \hat{A}^{i_1 k m}_{j_1 m} e_{i_1}^{k} \otimes I \otimes e_{j_1}^{m} \otimes I, \quad A_{12} = \hat{A}^{i_1 k m}_{j_1 m} (e_{i_1}^{k} \otimes e_{j_1}^{m} \otimes I \otimes I), \tag{2.27}\]
\[A_{23} = \hat{A}^{i_1 k m}_{j_1 m} (I \otimes e_{i_1}^{k} \otimes e_{j_1}^{m} \otimes I). \tag{2.28}\]
In what follows, we need the superpermutation operator
\[\mathcal{P} = (-1)^{[\alpha]} e_{i}^{j} \otimes e_{j}^{i} \quad \implies \quad \mathcal{P}^{k_1 k_2}_{m_1 m_2} = (-1)^{[k_1][k_2]} \delta_{m_1}^{k_1} \delta_{m_2}^{k_2}. \tag{2.29}\]
We note that the operators $\mathcal{P}_{\alpha, \alpha+1}$, $\alpha = 1, \ldots, s - 1$, constructed in accordance with (2.24) define a representation $\tau: S_s \to \text{End}(V^{\otimes s})$ of the symmetric group $S_s$ with the generators $\sigma_\alpha$,
\[\sigma_\alpha \sigma_{\alpha+1} \sigma_\alpha = \sigma_{\alpha+1} \sigma_\alpha \sigma_{\alpha+1} \quad \forall \alpha = 1, \ldots, s - 2, \tag{2.30}\]
\[\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha \quad \forall \alpha, \beta = 1, \ldots, s - 1, \quad |\alpha - \beta| > 1, \tag{2.31}\]
\[\sigma_\alpha^2 = e \quad \forall \alpha = 1, \ldots, s - 1, \tag{2.32}\]
where $e$ is the identity of $S_s$ and
\[\tau(\sigma_\alpha) = \mathcal{P}_{\alpha, \alpha+1}, \quad \tau(e) = I^{\otimes s}. \tag{2.33}\]
By direct calculations using the definition of $P_{\alpha, \alpha+1}$, we prove the following statement.

**Statement 1.** Let $A$ be operator (2.22) acting on $V^{\otimes r}$, and $A_{\alpha_1...\alpha_r}$ be operator (2.24) that acts on $V^{\otimes s}$, $s > r$. If $\alpha_p + 1 < \alpha_{p+1}$ for $p < r$, or $\alpha_p + 1 \leq s$ for $p = r$, then

$$P_{\alpha_p, \alpha_p+1}A_{\alpha_1...\alpha_p...\alpha_r}P_{\alpha_p, \alpha_p+1} = A_{\alpha_1...\alpha_p+1...\alpha_r}. \quad (2.29)$$

That is, if the above conditions are satisfied, the superpermutation $P_{\alpha_p, \alpha_p+1}$ moves the nontrivial factor at the $\alpha_p$th position in $A_{\alpha_1...\alpha_r}$ to the adjacent position $\alpha_p + 1$ on the right, while the identity $I$ standing at the $(\alpha_p + 1)$th place in $A_{\alpha_1...\alpha_r}$ is moved to the $\alpha_p$th position.

In particular, Statement 1 implies that the operators $A_{\alpha_1, \alpha_2}: V^{\otimes 4} \rightarrow V^{\otimes 4}$ defined in (2.25) satisfy the relations

$$A_{13} = P_{23}A_{12}P_{23} = P_{12}A_{23}P_{12} \Rightarrow P_{13} = P_{23}P_{12}P_{23} = P_{12}P_{23}P_{12}, \quad (2.30)$$

where the chain of equalities is in accordance with (2.27). Besides, it follows from (2.29) that for any $A: V^{\otimes r} \rightarrow V^{\otimes r}$, we have

$$A_{\sigma(1)...\sigma(r)} = \tau(\sigma)A_{1...r}\tau(\sigma)^{-1}, \quad (2.31)$$

where $\sigma \in S_r$, $s \geq r$, and $\tau(\sigma)$ is its image in representation (2.28) constructed as a product of the $P_{\alpha, \alpha+1}$.

For $U(g)$, we define a homomorphic map $\Delta: U(g) \rightarrow U(g) \otimes U(g)$. It acts on the generators $X_i$ of $U(g)$ as

$$\Delta X_i = X_i \otimes I + I \otimes X_i. \quad (2.32)$$

The map $\Delta$ is called the comultiplication of $U(g)$. Acting by $\Delta$ on the quadratic Casimir operator $C_2$ (2.17) yields

$$\Delta(C_2) = C_2 \otimes I + I \otimes C_2 + 2\hat{C}, \quad (2.33)$$

where $\hat{C} \in U(g) \otimes U(g)$ is called the split Casimir operator. Explicitly,

$$\hat{C} = g^{ij}X_i \otimes X_j. \quad (2.34)$$

The operator $\hat{C}$ has the ad-invariance property, i.e., for all generators $X_i \in U(g)$, in accordance with (2.33) we have

$$[\hat{C}, \Delta X_i] = \frac{1}{2}\Delta([C_2, X_i]) - \frac{1}{2}[C_2, X_i] \otimes I - \frac{1}{2}I \otimes [C_2, X_i] = 0. \quad (2.35)$$

It is convenient to use the notion of a tensor product of the enveloping superalgebras $U(g)$ and the comultiplication of $U(g)$ to define the tensor product $T \otimes T'$ of representations $T: g \rightarrow (\text{End}(V))_L$ and $T': g \rightarrow (\text{End}(V'))_L$ of a Lie superalgebra $g$. For an arbitrary homogeneous vector $X \in g$, we define $(T \otimes T')(X)$ as

$$(T \otimes T')(X) \equiv (T \otimes T')(\Delta(X)) = T(X) \otimes T'(I) + T(I) \otimes T'(X), \quad (2.36)$$

and hence for any homogeneous vectors $v \in V$ and $u \in V'$ we have

$$(T \otimes T')(X) \cdot (v \otimes u) = (T(X) \cdot v) \otimes u + (-1)^{|X||v|}v \otimes (T'(X) \cdot u). \quad (2.37)$$

The map thus defined $(T \otimes T'): U(g) \rightarrow \text{End}(V \otimes V')$ is indeed a homomorphism and hence a representation of $U(g)$ on $V \otimes V'$.
From (2.35) and (2.36), we can infer that for any representations \( T \) and \( T' \), the operator \( (T \otimes T')(\hat{C}) \) commutes with \( (T \otimes T')(X) \) for all \( X \in \mathfrak{g} \). We recall that by Schur’s lemma (more precisely, by its generalization to the case of Lie superalgebras), for each irreducible representation \( \hat{T} \) of any complex Lie superalgebra \( \mathcal{A} \), an even operator \( A \) that commutes with all elements of \( \mathcal{A} \) in a representation \( \hat{T} \) must be proportional to the identity operator, i.e., \( A = \lambda \mathbb{I} \), where \( \lambda \in \mathbb{C} \). Thus, if an irreducible representation \( \hat{T} \) of \( \mathcal{U}(\mathfrak{g}) \) is contained in \( (T \otimes T') \), we have \( \hat{T}(\hat{C}) \sim I_{\hat{T}} \). The following corollary of Schur’s lemma is central for our work: if a representation \( T \otimes T \) of a Lie superalgebra \( \mathfrak{g} \) on the space \( V_T \otimes V_T \) is completely reducible, then \( V_T \otimes V_T \) can be decomposed into a direct sum of invariant eigenspaces of \( (T \otimes T)(\hat{C}) \). From now on, we let this operator be denoted by \( \hat{C}_T \). If the operator \( \hat{C}_T \) satisfies the characteristic identity

\[
(\hat{C}_T - a_1 I_{\hat{T}}^{\otimes 2})(\hat{C}_T - a_2 I_{\hat{T}}^{\otimes 2}) \ldots (\hat{C}_T - a_p I_{\hat{T}}^{\otimes 2}) = 0,
\]

where \( I_{\hat{T}}^{\otimes 2} \) is the identity operator on \( V_T \otimes V_T \), all complex numbers \( a_1, a_2, \ldots, a_p \) are different, and omitting any of the parentheses in the left-hand side of (2.38) breaks the identity, then the numbers \( a_1, a_2, \ldots, a_p \) are the eigenvalues of \( \hat{C}_T \), and the projector onto the eigenspace of \( \hat{C}_T \) corresponding to the eigenvalue \( a_j \) is given by

\[
P_j \equiv P_{a_j} = \prod_{\substack{i=1 \atop i \neq j}}^p \frac{\hat{C}_T - a_i I_{\hat{T}}^{\otimes 2}}{a_j - a_i}.
\]

Moreover, \( P_k P_j = \delta_{kj} P_j \) and \( \sum_{j=1}^p P_j = I_{\hat{T}}^{\otimes 2} \). We emphasize that the spaces extracted by the projectors \( P_j \) are not necessarily spaces of irreducible representations of \( \mathfrak{g} \), because there may exist other nontrivial invariant operators on \( V_T \otimes V_T \) that cannot be expressed as polynomials in \( \hat{C}_T \). Therefore, the spaces \( P_j(V_T \otimes V_T) \) can in principle be further decomposed into direct sums of nontrivial invariant subspaces.

In contrast to Lie algebras, reducible representations of simple Lie superalgebras are not always completely reducible. Accordingly, the Casimir operators in such representations are not always diagonalizable, and we encounter this situation in this paper. The operator \( \hat{C}_T \) is not diagonalizable if and only if it does not satisfy any identity of form (2.38) with pairwise different \( a_i \). In this case, \( \hat{C}_T \) must satisfy

\[
(\hat{C}_T - a_1 I_{\hat{T}}^{\otimes 2})^{k_1}(\hat{C}_T - a_2 I_{\hat{T}}^{\otimes 2})^{k_2} \ldots (\hat{C}_T - a_p I_{\hat{T}}^{\otimes 2})^{k_p} = 0,
\]

where all \( k_i \in \mathbb{Z}_{\geq 1} \) are minimal, i.e., subtracting 1 from any of them breaks the identity and, as previously, the \( a_i \) are pairwise different. Instead of projectors onto eigenspaces of \( \hat{C}_T \), we can construct projectors onto its generalized eigenspaces (weight spaces):

\[
P_j \equiv P_{a_j} = I_{\hat{T}}^{\otimes 2} - \left( I_{\hat{T}}^{\otimes 2} - \prod_{\substack{i=1 \atop i \neq j}}^p \frac{\hat{C}_T - a_i I_{\hat{T}}^{\otimes 2}}{a_j - a_i} \right)^{k_j}.
\]

If \( k_i = 1 \) for some \( i \), then the image of \( P_i \) is an eigenspace of \( \hat{C}_T \). If \( k_i > 1 \), then \( P_i \) projects onto a generalized eigenspace. We note that if \( k_1 = k_2 = \cdots = k_p = 1 \), then (2.40) and (2.41) turn into respective relations (2.38) and (2.39).

2.3. The split Casimir operator for simple complex Lie superalgebras with a nondegenerate Cartan–Killing metric in the adjoint representation. Hereinafter, we consider only those simple Lie superalgebras for which the Cartan–Killing metric \( g_{ab} \) is nondegenerate, i.e., there must be the inverse metric \( \tilde{g}^{ab} \) satisfying (2.14). This also implies that the structure constants \([X^k]_{ij}\) satisfy the relation

\[
\text{str} \left( \text{ad}(X_i) \right) = (-1)^k X^k_{ik} = 0 \quad \forall i.
\]

\(^1\)By weight spaces, we mean spaces on which \( \hat{C}_T \) acts by a Jordan cell with the corresponding eigenvalue.
Using (2.7), (2.19), and (2.34), we can find the components of the split Casimir operator in the adjoint representation with respect to a basis \( \{ X_a \otimes X_i \} \) of \( V_{ad} \otimes V_{ad} \) \((V_{ad} \text{ coincides with } g \text{ as a vector space and is defined to be the adjoint representation space} )\):

\[
(\hat{C}_{ad})^{i_1 i_2, j_1 j_2} = (-1)^{[a_2][i_1]} \tilde{g}^{a_1 a_2} X^{i_1 a_1 j_1} X^{i_2 a_2 j_2}.
\]  

(2.43)

Here, we need three more ad-invariant operators in \( V_{ad} \otimes V_{ad} \) with the components

\[
(I)^{i_1 i_2, j_1 j_2} = \delta_{j_1}^{i_1} \tilde{g}_{j_2}^{i_2}, \quad (P)^{i_1 i_2, j_1 j_2} = (-1)^{[i_1][i_2]} \delta_{j_2}^{i_1} \delta_{j_1}^{i_2}, \quad (K)^{i_1 i_2, j_1 j_2} = \tilde{g}^{i_1 i_2} g_{j_1 j_2}.
\]  

(2.44)

Therefore, \( P \) is the operator of superpermutation of the two spaces \( V_{ad} \). It can be easily verified that for any \( X \in g \),

\[
[ad^2(\Delta X), P] = 0.
\]  

(2.45)

Because \( P \) is invariant under the adjoint action of \( g \), then so are its eigenspaces, which are extracted by the projectors \((I + P)/2\) and \((I - P)/2\).

Using the component form of \( \hat{C}_{ad}, I, P, \) and \( K \), given by (2.43) and (2.44), we can verify the identities

\[
P \cdot P = I, \quad P \hat{C}_{ad} P = \hat{C}_{ad}, \quad P \cdot K = K \cdot P = K,
\]  

(2.46)

\[
K \cdot K = \text{sdim } g \cdot K.
\]  

(2.47)

Applying the projectors

\[
P_{\pm}^{(ad)} = \frac{1}{2}(I \pm P)
\]  

(2.48)

to \( \hat{C}_{ad} \), we define the symmetric and antisymmetric parts of the split Casimir operator as

\[
\hat{C}_{\pm} = P_{\pm}^{(ad)} \hat{C}_{ad} = \hat{C}_{ad} P_{\pm}^{(ad)},
\]  

(2.49)

where the last equality is verified by using (2.46). From (2.49) and the relations

\[
P_{+}^{(ad)} + P_{-}^{(ad)} = I, \quad P_{+}^{(ad)} P_{-}^{(ad)} = P_{-}^{(ad)} P_{+}^{(ad)} = 0,
\]  

(2.50)

which follow from the first formula in (2.46) and (2.48), we instantly deduce that

\[
\hat{C}_{ad} = \hat{C}_{+} + \hat{C}_{-}, \quad \hat{C}_{+} \hat{C}_{-} = \hat{C}_{-} \hat{C}_{+} = 0.
\]  

(2.51)

Using symmetry properties (2.16) of the structure constants of \( g \) with respect to index permutations, the graded Jacobi identity (2.2), and the identity

\[
(-1)^{[i_1][i_2]} X^{i_1 i_2 a} X^{b}_{i_1 i_2} = -\delta_{a}^{b} \Leftrightarrow (-1)^{[i_1][i_2]} X^{i_1 i_2 a} X_{i_1 i_2 b} = -g_{ab},
\]  

(2.52)

where \( X^{i_1 i_2 a} = \tilde{g}^{i_2 j_2} X^{i_1 j_2 a} \), we obtain a convenient form of \( \hat{C}_{-} \):

\[
(\hat{C}_{-})^{i_1 i_2, j_1 j_2} = \frac{1}{2} (-1)^{[i_1][i_2]} X^{i_1 i_2 a} X^{a}_{j_1 j_2}.
\]  

(2.53)

From (2.9), (2.16), (2.52), and identities (2.53), we can derive the equality

\[
\hat{C}_{-}^2 = -\frac{1}{2} \hat{C}_{-}.
\]  

(2.54)
In addition, it follows from (2.12), (2.16), and (2.53) that

\[ \tilde{C}_{ad}K = K\tilde{C}_{ad} = -K, \quad \tilde{C}_{-}K = K\tilde{C}_{-} = 0, \quad \tilde{C}_{+}K = K\tilde{C}_{+} = -K, \]  

(2.55)

where the last equality is a consequence of the other two and of the first relation in (2.51).

Supertraces of the operators \( I, P, K, \tilde{C}_{ad}, \tilde{C}_{+}, \) and \( \tilde{C}_{-} \), as well as of some of their powers can be obtained by using (2.42)–(2.44), (2.53), and (2.54):

\[
\begin{align*}
\text{str}(\tilde{C}_{ad}) &= 0, \\
\text{str}(\tilde{C}_{\pm}) &= \pm \frac{1}{2} \text{sdim}\, g, \\
\text{str}(\tilde{C}_{ad}^2) &= \frac{1}{4} \text{sdim}\, g, \\
\text{str}(\tilde{C}_{-}^2) &= \frac{1}{8} \text{sdim}\, g, \\
\text{str}(\tilde{C}_{+}^3) &= \frac{3}{8} \text{sdim}\, g.
\end{align*}
\]

(2.56)

Here, \( \text{str} = \text{str}_1 \text{str}_2 \) is the trace in \( V_{ad} \otimes V_{ad} \), and the indices 1 and 2 in \( \text{str}_1 \) and \( \text{str}_2 \) refer to the tensor components of \( V_{ad} \otimes V_{ad} \).

3. The \( \mathfrak{osp}(M|N) \) Lie superalgebra

There exist various conventions on how to define the orthosymplectic Lie superalgebras \( \mathfrak{osp}(M|N) \) (see, e.g., [17]–[24]), all of which are equivalent. In this section, we fix the definition of these algebras that was formulated in [19], [20], as the one most convenient for our purpose. For this, we introduce a scalar product \( \varepsilon \) on the space \( V_{(M|N)} \) (where \( N = 2n \) is even). Its components \( \varepsilon_{ab} \equiv \varepsilon(e_{a}, e_{b}) \) in some homogeneous basis \( \{e_{a}\}_{a=1}^{M+N} \) of \( V_{(M|N)} \) are given by the matrix

\[
\varepsilon = \begin{pmatrix} I_{M} & 0 \\ 0 & J_{N} \end{pmatrix},
\]

(3.1)

where, \( I_{M} \) is the \( M \times M \) identity matrix and the antisymmetric matrix \( J_{N} \) is

\[
J = \begin{pmatrix} 0 & I_{n} \\ -I_{n} & 0 \end{pmatrix}.
\]

(3.2)

The definition (3.1) of \( \varepsilon \) implies the following relations for its components:

\[
\varepsilon_{ba} = (-1)^{|a|} \varepsilon_{ab} = (-1)^{|b|} \varepsilon_{ab} = (-1)^{|a|+|b|} \varepsilon_{ab}.
\]

(3.3)

The metric \( \varepsilon \) can be used to raise and lower indices,

\[
\varepsilon^{ab} = \varepsilon_{ab}, \quad \varepsilon^{a\ldots}_{b\ldots} = \varepsilon_{a\ldots}^{\phantom{a\ldots}b\ldots} = \varepsilon_{a\ldots}^{\phantom{a\ldots}b\ldots},
\]

(3.4)

where \( \varepsilon^{ab} \) are the components of the matrix \( \varepsilon^{-1} \):

\[
\varepsilon^{ab}\varepsilon_{bc} = \delta_{c}^{a}, \quad \varepsilon_{ab}\varepsilon^{bc} = \delta_{a}^{c}.
\]

(3.5)

By (3.4), we have \( \varepsilon^{ab} = \varepsilon^{ac}\varepsilon^{bd}\varepsilon_{cd} = \varepsilon_{ba} \), i.e., the matrix of the metric tensor with the upper indices \( \varepsilon^{ab} \) does not coincide with the inverse matrix \( \varepsilon^{ab} \). From now on, we only use the matrix \( \varepsilon^{ab} \).
We define the $osp(M|N)$ Lie superalgebra as the algebra of operators $A: V_{(M|N)} \to V_{(M|N)}$ that leave the metric $\varepsilon$ invariant. For a homogeneous $A$ this means
\[ \varepsilon(A(u), v) + (-1)^{|A|} \varepsilon(u, A(v)) = 0, \] (3.6)
for any homogeneous $u, v \in V_{(M|N)}$. If $A = A_0 + A_1$ is not homogeneous, with $A_0$ and $A_1$ being its even and odd parts, we require Eq. (3.6) to hold simultaneously for $A_0$ and $A_1$. Expanding $u = u^ae_a$ and $v = v^be_b$ over the basis $\{e_a\}$, using the definitions of the components $\varepsilon_{ab} = \varepsilon(e_a, e_b)$ of the metric $\varepsilon$ and of the operator $A$: $A(e_a) = e_bA^a$, as well as the fact that the grading of a homogeneous operator $A$ with nonzero components $A^b_a$ equals $[a] + [b]$, we can rewrite (3.6) in the component form [19], [20]
\[ A^c_a\varepsilon_{cb} + (-1)^{|a|+|b|}\varepsilon_{ac}A^b = 0. \] (3.7)
We note that for the left-hand side of (3.7) to be well-defined, $A$ does not have to be homogeneous; thus, (3.7) can be viewed as the definition of the invariance of a metric under the action of an arbitrary operator $A$. Multiplying (3.7) by $(-1)^{|a|+|b|}$ yields
\[ (-1)^{|a|+|b|}A^c_a\varepsilon_{cb} + \varepsilon_{ac}A^b = 0. \] (3.8)
To draw an analogy between (3.8) and the definition of the matrix Lie algebras $so(M)$ and $sp(N)$, we introduce the operation of supertransposition of any $A: V_{(M|N)} \to V_{(M|N)}$, applying which gives an operator $A^T: \overline{V}_{(M|N)} \to \overline{V}_{(M|N)}$, where $\overline{V}_{(M|N)}$ is the dual space of $V_{(M|N)}$. The components $(A^T)_b^a$ of $A^T$, which are defined by $A^T(e^a) = \epsilon^b(A^T)_b^a$, with $\{e^a\}$ being the dual basis of $\{e_a\}$ in $\overline{V}_{(M|N)}$, are given explicitly by
\[ (A^T)_b^a = (-1)^{|a|+|b|}A^b_a. \] (3.9)
Thus, for the $[(M + N) \times (M + N)]$ matrix of $A^T$, we have
\[ A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \implies A^T = \begin{pmatrix} X^t & Z^t \\ -Y^t & W^t \end{pmatrix}, \] (3.10)
where $X, Y, Z,$ and $W$ are respective $M \times M$, $M \times N$, $N \times M$, and $N \times N$ matrices, and $t$ denotes the usual matrix transposition.

Using (3.9) and (3.3), we can rewrite (3.8) as
\[ (A^T)_b^a\varepsilon_{cb} + \varepsilon_{ac}A^c_b = 0. \] (3.11)
The matrix form of (3.11) is
\[ A^T\varepsilon + \varepsilon A = 0. \] (3.12)
The form of (3.12) coincides with that of an analogous expression used in the definitions of the matrix algebras $so(M)$ and $sp(N)$ ($N = 2n$) with the appropriate choice of the metric $\varepsilon$ and supertransposition replaced with the regular transposition. Using (3.10), where $Y, Z,$ and $W$ are viewed as block matrices, together with (3.12), we infer the explicit form of the matrix of $A$:
\[ A = \begin{pmatrix} X & Q \\ -S^t & E \\ Q^t & G & -E^t \end{pmatrix}. \] (3.13)
Here, \( X \) is a matrix of dimension \( M \times M \) (as in (3.10)), \( M \times n \) blocks \( Q \) and \( S \) form an \( M \times 2n \) matrix \( Y = (Q,S) \), and the \( n \times n \) matrices \( E, F, \) and \( G \) comprise \( W \). Furthermore, \( X, F, \) and \( G \) satisfy the relations \( X^t = -X, F^t = F, \) and \( G^t = G \).

We note that supertransposition does not possess some properties intrinsic to the usual transposition. In general, for some \( A, B : V_{(M|N)} \rightarrow V_{(M|N)} \), it may be that \( (A^T)^T \neq A \) and \( (AB)^T \neq B^T A^T \). Nevertheless, by (2.4) and (3.9),

\[
[A, B]^T = -[A^T, B^T],
\]

(3.14)

whence it follows that the vector space of all operators \( A \) satisfying (3.11) (or, equivalently, (3.12)) is closed under Lie bracket (2.4). Therefore, it forms a Lie superalgebra.

To find all solutions of (3.7), we introduce the operators acting on the space \( \text{End}(V_{(M|N)}) \) as

\[
P_{\pm}(E)^a_c = \frac{1}{2}(E^c_e \pm (-1)^{|c|+|a|} \varepsilon_{cb} E^b_d \varepsilon^{da}),
\]

(3.15)

where \( E \in \text{End}(V_{(M|N)}) \) and \( E^a_b \) is its matrix. It is easy to verify that the operators \( P_{\pm} \) satisfy

\[
P_A P_B = P_A \delta_{AB} \quad (A, B = +, -), \quad P_+ + P_- = I_{M+N},
\]

(3.16)

and therefore constitute a full system of mutually orthogonal projectors on \( \text{End}(V_{(M|N)}) \).

In terms of \( P_{\pm} \), Eq. (3.7) becomes

\[
(P_+ A)^a_b = 0 \iff P_+ A = 0 \iff P_- A = A.
\]

(3.17)

The last condition is satisfied if and only if \( A \) lies in the image of \( P_- \). Hence, the matrix of any operator \( A \in \text{osp}(M|N) \) is of the form

\[
A^a_b = E^a_b - (-1)^{|b|+|a|} \varepsilon_{bc} E^c_d \varepsilon^{da},
\]

(3.18)

where \( ||E^a_b|| \in \text{Mat}_{M+N}(C) \) is an arbitrary matrix.

The basis elements \( M_{ij} \in \text{osp}(M|N) \) in the defining representation are realized as matrices \( (M_{ij})^a_b \) obtained from (3.18) by the substitution \( E^a_b \rightarrow (e_{ij})^a_b \),

\[
(M_{ij})^a_b = (e_{ij})^a_b - (-1)^{|b|+|a|} \varepsilon_{bc} (e_{ij})^c_d \varepsilon^{da},
\]

(3.19)

where \( e_{ij} \) are the matrix identities on \( V_{(M|N)} \), \( (e_{ij})^a_b = \delta^a_i \delta^b_j \) (see (2.23)). Lowering the index \( j \) in (3.19) via the metric \( \varepsilon \) given in (3.1), we obtain the final form of the matrices of the \( \text{osp}(M|N) \) Lie superalgebra basis elements \( M_{ij} \):

\[
(M_{ij})^a_b = \varepsilon_{jb} \delta^a_i - (-1)^{|a+b|} \varepsilon_{ib} \delta^a_j = \varepsilon_{jb} \delta^a_i - (-1)^{|i|+|j|} \varepsilon_{ib} \delta^a_j.
\]

(3.20)

The degree of \( M_{ij} \) is \([i] + [j]\). Moreover,

\[
M_{ij} = -(-1)^{|i|+|j|} M_{ji}.
\]

(3.21)

Taking this condition into account, we require the components of any vector (or covector) from the \( \text{osp}(M|N) \) Lie superalgebra to satisfy the same index permutation symmetry, that is, \( X^{ij} = -(-1)^{|i|+|j|} X^{ji} \) and \( Y_{ij} = (-1)^{|i|+|j|} Y_{ji} \), where \( X^{ij} \) are the coordinates of an arbitrary vector from \( \text{osp}(M|N) \) and \( Y_{ij} \) are the coordinates of an arbitrary covector. This requirement ensures uniformity in assigning coordinates to vectors of \( \text{osp}(M|N) \) in basis (3.20) (and to covectors in the dual basis to (3.20)).
The Lie superbracket (2.4) of $M_{ij}$ is

\[ [M_{ij}, M_{km}] = \varepsilon_{jk} M_{im} - (-1)^{|k||m|} \varepsilon_{jm} M_{ik} - (-1)^{|l||m|} \varepsilon_{ik} M_{jm} + (-1)^{|l||m| + |k||m|} \varepsilon_{im} M_{jk}. \]  

(3.22)

The structure constants $X_{i_1i_2,j_1j_2}$ of $osp(M|N)$ are defined by

\[ [M_{i_1i_2}, M_{j_1j_2}] = X_{i_1i_2,j_1j_2} k_1 k_2 M_{k_1k_2}, \]  

(3.23)

and are explicitly given by

\[ X^{k_1 k_2}_{i_1 i_2, j_1 j_2} = \varepsilon_{i_1 j_1} \delta^{(k_1)}_{i_2} \delta^{(k_2)}_{j_2} - (-1)^{|j_1||j_2|} \varepsilon_{i_2 j_2} \delta^{(k_1)}_{i_1} \delta^{(k_2)}_{j_1} - (-1)^{|i_1||i_2|} \varepsilon_{i_1 j_1} \delta^{(k_1)}_{i_2} \delta^{(k_2)}_{j_2} + (-1)^{|i_1||i_2| + |j_1||j_2|} \varepsilon_{i_2 j_2} \delta^{(k_1)}_{i_1} \delta^{(k_2)}_{j_1}, \]  

(3.24)

where

\[ A^{(k_1 k_2)} = \frac{1}{2} (A^{k_2 k_1} - (-1)^{|k_1||k_2|} A^{k_1 k_2}). \]  

(3.25)

The Cartan–Killing metric (2.9) of $osp(M|N)$ in the basis (3.20) is

\[ g_{i_1 i_2,j_1 j_2} = (-1)^{|m_1| + |m_2|} X^{m_1 m_2}_{i_1 i_2, j_1 j_2} X^{k_1 k_2}_{j_1 j_2, m_1 m_2} = 2(\omega - 2) \varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2} - (-1)^{|j_1||j_2|} \varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2}, \]  

(3.26)

where $\omega \equiv M - N$. For $\omega = 2$, metric (3.26) is degenerate and hence this case is omitted in what follows. The identity operator $\hat{1}$ that acts on the algebra $osp(M|N)$ (which is considered here as a vector space embedded into $V^{\otimes 2}_{(M|N)}$) has the following components in basis (3.20):

\[ \hat{1}^{i_1 i_2,j_1 j_2} = \frac{1}{2} (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - (-1)^{|i_1||i_2|} \delta^{i_2}_{j_1} \delta^{i_1}_{j_2}), \]  

(3.27)

and turns out to be a projector onto (super)antisymmetric second-rank tensors (see Sec. 3.1). The components of the inverse Cartan–Killing metric are defined by (2.14):

\[ g^{i_1 i_2,j_1 j_2} g_{j_1 j_2,k_1 k_2} = \hat{1}^{i_1 i_2,j_1 j_2} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \delta^{k_1}_{k_1} \delta^{k_2}_{k_2} = \hat{1}^{i_1 i_2,j_1 j_2} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2}. \]  

(3.28)

Direct calculation yields their explicit form

\[ g^{i_1 i_2,j_1 j_2} = \frac{1}{8(\omega - 2)} (\varepsilon^{i_1 j_1} \varepsilon^{i_2 j_2} - (-1)^{|i_1||i_2|} \varepsilon^{i_1 j_1} \varepsilon^{i_2 j_2}). \]  

(3.29)

### 3.1. Projectors onto invariant subspaces of the tensor product of two defining representations

Using (3.20), (3.29), and (3.24), we write the split Casimir operator in the tensor product of two defining representations as $[19], [20]$:

\[ \hat{C}^{k_1 k_2}_{m_1 m_2} = \frac{1}{2(\omega - 2)} (-1)^{|k_1||k_2|} \delta^{k_1}_{m_1} \delta^{k_2}_{m_2} - \varepsilon^{k_1 k_2} \varepsilon^{m_1 m_2}. \]  

(3.30)

We define the operators $1, \mathcal{P}, \mathcal{K} : V^{\otimes 2}_{(M|N)} \to V^{\otimes 2}_{(M|N)}$ (the matrix forms of these operators are given on the right of these formulas):

\[ 1 = e_i \otimes e_j \quad \implies \quad 1^{k_1 k_2}_{m_1 m_2} = \delta^{k_1}_{m_1} \delta^{k_2}_{m_2}, \]  

(3.31)

\[ \mathcal{P} = (-1)^{|j|} e_i \otimes e_j \quad \implies \quad \mathcal{P}^{k_1 k_2}_{m_1 m_2} = (-1)^{|k_1||k_2|} \delta^{k_1}_{m_1} \delta^{k_2}_{m_2}, \]  

(3.32)

\[ \mathcal{K} = (-1)^{|i||j|} \varepsilon^{i j} e_k \otimes e_j \quad \implies \quad \mathcal{K}^{k_1 k_2}_{m_1 m_2} = \varepsilon^{k_1 k_2} \varepsilon^{m_1 m_2}. \]  

(3.33)
Here, 1 is the identity operator, \( \mathcal{P} \) is a superpermutation, and \( e_i^j \) are the matrix identities defined in (2.23), which act on \( V_{(M|N)} \). The operators 1, \( \mathcal{P} \), and \( \mathcal{K} \) have the properties
\[
\mathcal{P}^2 = 1, \quad \mathcal{K}^2 = \omega \mathcal{K}, \quad \mathcal{P} \mathcal{K} = \mathcal{K} \mathcal{P} = \mathcal{K}.
\] (3.34)

In terms of \( \mathcal{P} \) and \( \mathcal{K} \), the split Casimir operator in the defining representation can be written as [19], [20]
\[
\hat{C}_f = \frac{1}{2(\omega - 2)}(\mathcal{P} - \mathcal{K}).
\] (3.35)

The characteristic identity for \( \hat{C}_f \) has degree three:
\[
\hat{C}_f^3 + \frac{\omega - 1}{2(\omega - 2)} \hat{C}_f^2 - \frac{1}{4(\omega - 2)^2} \hat{C}_f - \frac{\omega - 1}{8(\omega - 2)^3} = 0.
\] (3.36)

Formula (3.36) can easily be verified using (3.35) and the relations
\[
\hat{C}_f^2 = \frac{1}{4(\omega - 2)^2} \mathbf{1} + \frac{\omega - 1}{4(\omega - 2)} \mathcal{K}, \quad \hat{C}_f^3 = \frac{1}{8(\omega - 2)^3} [\mathcal{P} - (\omega^2 - 3\omega + 3) \mathcal{K}].
\] (3.37)

The left-hand side of (3.36) can be factored, which results
\[
\left( \frac{\hat{C}_f - \frac{1}{2(\omega - 2)}}{\hat{C}_f + \frac{1}{2(\omega - 2)}} \right) \left( \hat{C}_f + \frac{1}{2(\omega - 2)} \right) = 0.
\] (3.38)

Using this factored form of the characteristic identity for \( \hat{C}_f \) and (2.39), where we must set \( p = 3 \) and fix the roots
\[
a_1 = \frac{1}{2(\omega - 2)}, \quad a_2 = -\frac{1}{2(\omega - 2)}, \quad a_3 = \frac{1 - \omega}{2(\omega - 2)},
\]
and also using (3.35) and the left hand-side of (3.37) for \( \hat{C}_f \) and \( \hat{C}_f^2 \), we obtain three projectors onto invariant subspaces of \( V_{(M|N)}^{\otimes 2} \):
\[
P_1 = \frac{1}{2}(1 + \mathcal{P}) - \frac{1}{\omega} \mathcal{K}, \quad P_2 = \frac{1}{2}(1 - \mathcal{P}), \quad P_3 = \frac{1}{\omega} \mathcal{K}.
\] (3.39)

We note that the substitutions \( \omega \rightarrow M \) and \( \omega \rightarrow -N \) turn projectors (3.39) into the corresponding projectors onto invariant subspaces of the representation \( T_{(M|N)}^\otimes 2 \) of the \( so(M) \) and \( sp(N) \) Lie algebras (for the explicit formulas, see, e.g., [25], [26]).

To conclude this subsection, we show that given the split Casimir operator (3.35), we can write the solution \( R^{i_1j_1k_1k_2}(u) \) of the graded Yang–Baxter equation [27] (see also [19], [28])
\[
R^{i_1j_1j_2j_3}(u)(-1)^{|i_1||j_2||j_3|}R^{i_1j_2k_1k_3}(u + v)(-1)^{|k_1||j_2||j_3|}R^{j_1j_2k_1k_3}(v) = R^{i_1j_3j_2j_3}(v)(-1)^{|i_1||j_3||j_2|}R^{i_1j_1j_2k_3}(u + v)(-1)^{|j_1||j_2||j_3|}R^{i_1j_2k_1k_3}(u),
\] (3.40)
which is invariant under the action of \( osp(M|N) \) in the defining representation. We recall that this solution can be written in several equivalent ways [19], [20], [28]:
\[
R(u) = \frac{1}{1 - u} \left( u + \mathcal{P} - \frac{u}{u + \omega / 2 - 1} \mathcal{K} \right) = 1 + \frac{u}{1 - u} P_1 - P_2 + \frac{\omega / 2 - 1 - u}{\omega / 2 - 1 + u} P_3.
\] (3.41)

The important point to note here is that this solution can be written as a rational function of \( \hat{C}_f \):
\[
R(u) = \frac{(\omega - 2) \hat{C}_f + 1/2 + u}{(\omega - 2) \hat{C}_f + 1/2 - u}.
\] (3.42)

This form of the \( R \)-matrix generalizes that obtained in [15] for the \( so(M) \) and \( sp(2n) \) Lie superalgebras in the defining representation. We note that the solution in (3.41) and (3.42) is unitary: \( \mathcal{P} R(u) \mathcal{P} R(-u) = 1 \).
3.2. Projectors onto invariant subspaces of the tensor product of two adjoint representations. To find a characteristic identity for the split Casimir operator in the adjoint representation \( \hat{C}_{ad} \), we first write the components of the basis elements of \( osp(M|N) \) in this representation. By (2.7), they coincide with the structure constants (3.24) of \( osp(M|N) \):

\[
(M_{i_1i_2})^{k_1k_2}_{j_1j_2} = \varepsilon_{i_2j_1} \delta_{i_1}^{(k_1} \delta_{j_2}^{k_2)} - (-1)^{|i_1||j_2|}\varepsilon_{i_2j_1} \delta_{i_1}^{(k_1} \delta_{j_2}^{k_2)} - (-1)^{|i_1||j_2|}\varepsilon_{i_1j_2} \delta_{i_1}^{(k_1} \delta_{j_2}^{k_2)} + (-1)^{|i_1||j_2|+|i_1||j_2|}\varepsilon_{i_1j_2} \delta_{i_1}^{(k_1} \delta_{j_2}^{k_2)}. \tag{3.43}
\]

A comparison of (3.20) and (3.43) suggests a convenient relation between the components of the basis elements \( M_{ij} \) of \( osp(M|N) \) in the adjoint and defining representations,

\[
(M_{i_1i_2})^{k_1k_2}_{j_1j_2} = 2(M_{i_1i_2})^{(k_1}_{j_1} \delta_{j_2)}_{2} = 4 \text{Sym}_{1+2} \varepsilon_{i_2j_1} \delta_{i_1}^{k_1} \delta_{j_2}^{k_2}, \tag{3.44}
\]

where \( \text{Sym}_{1+2} \) denotes (anti)symmetrization over the pairs of indices \((i_1,i_2), (j_1,j_2)\) and \((k_1,k_2)\). Using (3.44), we can express the following form of the components of \( \hat{C}_{ad} \):

\[
(\hat{C}_{ad})^{k_1k_2k_3k_4}_{m_1m_2m_3m_4} = g^{i_1i_2j_1j_2}(M_{i_1i_2} \otimes M_{j_1j_2})^{k_1k_2k_3k_4}_{m_1m_2m_3m_4} = \\
= (-1)^{|i_1||j_2|+|i_2||j_1|}(d_{i_1i_2j_1j_2}(M_{i_1i_2})^{k_1k_2}_{m_1m_2}(M_{j_1j_2})^{k_3k_4}_{m_3m_4} = \\
= 4(-1)^{|i_1||j_2|+|i_2||j_1|}(d_{i_1i_2j_1j_2}(M_{i_1i_2})^{(k_1}_{m_1} \delta_{m_2}^{k_2})_{(j_1)} \delta_{j_2)}_{(m_3)} \delta_{m_4)}. \tag{3.45}
\]

As a result, we find a connection between the components of the split Casimir operator in the adjoint and defining representations,

\[
(\hat{C}_{ad})^{k_1k_2k_3k_4}_{m_1m_2m_3m_4} = 4(\hat{C}_f)^{(k_1}_{k_2}k_4}_{(m_1m_2)(m_3m_4)}, \tag{3.46}
\]

where

\[
(\hat{C}_f)^{(k_1}_{k_2}k_4}_{(m_1m_2)(m_3m_4)} = g^{i_1i_2j_1j_2}(M_{i_1i_2} \otimes I \otimes M_{j_1j_2} \otimes I)^{k_1k_2k_3k_4}_{m_1m_2m_3m_4} = \\
= (-1)^{|i_1||j_2|+|i_2||j_1|}(d_{i_1i_2j_1j_2}(M_{i_1i_2})^{k_1}_{m_1} \delta_{m_2}^{k_2})_{(j_1)} \delta_{j_2)}_{(m_3)} \delta_{m_4)}. \tag{3.47}
\]

The lower index 13 of \( \hat{C}_f \) is defined in accordance with (2.25).

In what follows, we need the operators \( P_{\alpha\beta} \) and \( K_{\alpha\beta} : V^4_{(M|N)} \to V^4_{(M|N)} \), \( \alpha, \beta = 1, \ldots, 4, \alpha \neq \beta \), that are built from (3.32) and (3.33) by (2.24). Direct calculations show that \( P_{\alpha,\alpha+1} \) and \( K_{\alpha,\alpha+1} \) satisfy the relations between the generators \( \sigma_\alpha \) and \( \kappa_\alpha \) of the Brauer algebra \( Br_4(\omega) \):

\[
\sigma_\alpha^2 = I, \quad \kappa_\alpha^2 = \omega \kappa_\alpha, \quad \sigma_\alpha \kappa_\alpha = \kappa_\alpha \sigma_\alpha = \kappa_\alpha, \\
\sigma_\alpha \kappa_\alpha = \kappa_\alpha \sigma_\alpha = \kappa_\alpha, \quad \alpha = 1, \ldots, 3, \\
\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha, \quad \kappa_\alpha \kappa_\beta = \kappa_\beta \kappa_\alpha, \quad \sigma_\alpha \kappa_\beta = \kappa_\beta \sigma_\alpha, \quad |\alpha - \beta| > 1, \tag{3.48}
\]

\[
\sigma_\alpha \sigma_{\alpha+1} \sigma_\alpha = \sigma_{\alpha+1} \sigma_\alpha \sigma_{\alpha+1}, \quad \kappa_\alpha \kappa_{\alpha+1} \kappa_\alpha = \kappa_{\alpha+1} \kappa_\alpha \kappa_{\alpha+1} = \kappa_{\alpha+1}, \\
\sigma_\alpha \kappa_{\alpha+1} \kappa_\alpha = \sigma_{\alpha+1} \kappa_\alpha, \quad \kappa_{\alpha+1} \sigma_{\alpha+1} \sigma_\alpha = \kappa_{\alpha+1} \sigma_{\alpha+1} \sigma_\alpha, \quad \alpha = 1, \ldots, 3.
\]

Thus, \( P_{\alpha,\alpha+1} = \tau(\sigma_\alpha) \) and \( K_{\alpha,\alpha+1} = \tau(\kappa_\alpha) \), where \( \tau \) is a representation of \( Br_4(\omega) \) in the space \( V^4_{(M|N)} \).

We recall that by convention (2.31), the operators \( P_{\alpha,\beta} \) for \( \beta > \alpha + 1 \) can be obtained from \( P_{\alpha,\beta} \) by a consecutive action of adjacent transpositions \( P_{\gamma,\gamma+1} \). For instance,

\[
P_{14} = P_{34} P_{13} P_{34} = P_{34} P_{23} P_{12} P_{23} P_{34} = \tau(\sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3). \tag{3.49}
\]
Besides, being even (or, alternatively by virtue of (3.48)), the operators $\mathcal{P}_{\alpha_1\alpha_2}$ and $\mathcal{P}_{\beta_1\beta_2}$ commute for $\alpha_1 \neq \beta_1, \beta_2$ and $\alpha_2 \neq \beta_1, \beta_2$, and so do $\mathcal{K}_{\alpha_1\alpha_2}$ and $\mathcal{K}_{\beta_1\beta_2}$:

$$\mathcal{P}_{\alpha_1\alpha_2}\mathcal{P}_{\beta_1\beta_2} = \mathcal{P}_{\beta_1\beta_2}\mathcal{P}_{\alpha_1\alpha_2}, \quad \mathcal{K}_{\alpha_1\alpha_2}\mathcal{K}_{\beta_1\beta_2} = \mathcal{K}_{\beta_1\beta_2}\mathcal{K}_{\alpha_1\alpha_2}. \quad (3.50)$$

We define the antisymmetrizer $\mathcal{P}^- : V^{\otimes 2}_{(M|N)} \to V^{\otimes 2}_{(M|N)}$ by

$$\mathcal{P}^- = \frac{1}{2}(1 - \mathcal{P}), \quad (3.51)$$

where $1$ and $\mathcal{P}$ are given in (3.31) and (3.33). Then, by (3.46) and (3.35),

$$\tilde{C}_{\text{ad}} = 4\mathcal{P}^-_{12}\mathcal{P}^-_{34}(\tilde{C}_f)_{13}\mathcal{P}^-_{12}\mathcal{P}^-_{34} = \frac{2}{\omega - 2}\mathcal{P}^-_{12}\mathcal{P}^-_{34}(\mathcal{P}_{13} - \mathcal{K}_{13})\mathcal{P}^-_{12}\mathcal{P}^-_{34}. \quad (3.52)$$

We define the space $V_{\text{ad}}$ of the adjoint representation of the $osp(M|N)$ Lie superalgebra by $V_{\text{ad}} = \mathcal{P}^- V^{\otimes 2}_{(M|N)}$. The algebra $osp(M|N)$ coincides with $V_{\text{ad}}$ as a vector space. We now introduce the following operators that act on $V^{\otimes 2}_{(M|N)}$:

$$\mathcal{I} = \mathcal{P}^-_{12}\mathcal{P}^-_{34} = \mathcal{P}^-_{12,34}; \quad \mathcal{P} = \mathcal{P}^-_{12,34}\mathcal{P}_{13}\mathcal{P}_{24}\mathcal{P}^-_{12,34}; \quad \mathcal{K} = \mathcal{P}^-_{12,34}\mathcal{K}_{13}\mathcal{K}_{24}\mathcal{P}^-_{12,34}; \quad (3.53)$$

where $\mathcal{P}^-_{12,34} = \mathcal{P}^-_{12}\mathcal{P}^-_{34}$. By (2.24), operators (3.53) (in a way similar to $\mathcal{P}$ and $\mathcal{K}$) define a representation of the Brauer algebra $Br_s(\omega)$ in $V^{\otimes s}_{(M|N)}$.

The following relations hold for $\mathcal{I}, \mathcal{P},$ and $\mathcal{K}$ introduced in (3.53):

$$\mathcal{I} = \mathcal{I}\mathcal{P}_{12}\mathcal{P}_{34} = \mathcal{P}_{12}\mathcal{P}_{34}\mathcal{I}, \quad \mathcal{P} = \mathcal{P}_{12}\mathcal{P}_{34}\mathcal{P} = \mathcal{P}\mathcal{P}_{12}\mathcal{P}_{34}, \quad (3.54)$$

$$\mathcal{P}^2 = \mathcal{I}, \quad \mathcal{K}\mathcal{P} = \mathcal{P}\mathcal{K} = \mathcal{K}, \quad \mathcal{K}^2 = \frac{\omega(\omega - 1)}{2}\mathcal{K}, \quad (3.55)$$

$$\tilde{C}_{\text{ad}}\mathcal{P} = \mathcal{P}\tilde{C}_{\text{ad}}, \quad \tilde{C}_{\text{ad}}\mathcal{K} = \mathcal{K}\tilde{C}_{\text{ad}} = -\mathcal{K}. \quad (3.56)$$

Operators (3.53) are invariant with respect to the $osp(M|N)$ Lie superalgebra in the adjoint representation (the definition of ad-invariance is given in (2.35)). Comparing the last formula in (3.55) with (2.47), shows that

$$\text{sdim} \mathfrak{g} = \frac{\omega(\omega - 1)}{2}. \quad (3.57)$$

To find the characteristic identity for $\tilde{C}_{\text{ad}}$, it is convenient to introduce the symmetric $\tilde{C}_+$ and antisymmetric $\tilde{C}_-$ projections of $\tilde{C}_{\text{ad}}$:

$$\tilde{C}_+ = \frac{1}{2}(\mathcal{I} \pm \mathcal{P})\tilde{C}_{\text{ad}}, \quad (3.58)$$

which satisfy the relations

$$\tilde{C}_-\tilde{C}_+ = 0, \quad \mathcal{P}\tilde{C}_\pm = \pm \tilde{C}_\pm, \quad \mathcal{K}\tilde{C}_- = \tilde{C}_-\mathcal{K} = 0, \quad \mathcal{K}\tilde{C}_+ = \tilde{C}_+\mathcal{K} = -\mathcal{K}. \quad (3.59)$$

We note that formulas (3.55), (3.56), and (3.59) were derived for all Lie superalgebras with a nondegenerate Cartan–Killing metric in Sec. 2.3. Substituting (3.52) and (3.53) in (3.58) gives explicit formulas for the antisymmetric and symmetric parts of $\tilde{C}_{\text{ad}}$:

$$\tilde{C}_- = \frac{1}{\omega - 2}\mathcal{P}^-_{12,34}(\mathcal{K}_{13}\mathcal{P}_{24} - \mathcal{K}_{13})\mathcal{P}^-_{12,34}, \quad (3.60)$$

$$\tilde{C}_+ = \frac{1}{\omega - 2}\mathcal{P}^-_{12,34}(2\mathcal{P}_{24} - \mathcal{K}_{13}\mathcal{P}_{24})\mathcal{P}^-_{12,34}. \quad (3.61)$$
Statement 2. The antisymmetric \( \hat{C}_- \) and symmetric \( \hat{C}_+ \) parts of the split Casimir operator of the \( \text{osp}(M|N) \) Lie superalgebra for \( M - N \equiv \omega \neq 0, 1, 2, 4, 8 \) satisfies the relations

\[
\hat{C}_-^2 = -\frac{1}{2} \hat{C}_- \quad \iff \quad \hat{C}_- \left( \hat{C}_- + \frac{1}{2} \right) = 0, \tag{3.62}
\]

\[
\hat{C}_+^3 = -\frac{1}{2} \hat{C}_+^3 - \frac{\omega - 8}{2(\omega - 2)^2} \hat{C}_+ + \frac{\omega - 4}{2(\omega - 2)^3} (\mathbf{I} + P - 2\mathbf{K}), \tag{3.63}
\]

\[
\hat{C}_+^4 + 3 \hat{C}_3 + \left( \frac{\omega + 1}{2} - 2 \right) \hat{C}_+^2 + \frac{\omega^2 - 12\omega + 24}{2(\omega - 2)^2} \hat{C}_+ - \frac{\omega - 4}{2(\omega - 2)^3} (\mathbf{I} + P) = 0, \tag{3.64}
\]

\[
\hat{C}_+ (\hat{C}_+ + 1) \left( \hat{C}_+ - \frac{1}{\omega - 2} \right) \left( \hat{C}_+ + \frac{2I}{\omega - 2} \right) \left( \hat{C}_+ + \frac{(\omega - 4)I}{2(\omega - 2)} \right) = 0. \tag{3.65}
\]

The split Casimir operator \( \hat{C}_{\text{ad}} = \hat{C}_- + \hat{C}_+ \) for \( \omega \neq 0, 1, 2, 4, 6, 8 \) satisfies the relation

\[
\hat{C}_{\text{ad}} \left( \hat{C}_{\text{ad}} + \frac{1}{2} \right) (\hat{C}_{\text{ad}} + 1) \left( \hat{C}_{\text{ad}} - \frac{1}{\omega - 2} \right) \left( \hat{C}_{\text{ad}} + \frac{2}{\omega - 2} \right) \left( \hat{C}_{\text{ad}} + \frac{\omega - 4}{2(\omega - 2)} \right) = 0. \tag{3.66}
\]

**Proof.** For \( M - N \equiv \omega = 2 \), the Cartan–Killing metric (3.26) of \( \text{osp}(M|N) \) is degenerate, and hence this case is excluded from consideration. The special cases of \( \omega = 0, 1, 4, 6, 8 \) are considered later.

Identity (3.62) for \( \text{osp}(M|N) \) is a special case of (2.54), which holds for all Lie superalgebras with a nondegenerate Cartan–Killing metric. We also note a useful consequence of (3.62):

\[
\hat{C}_-^k = \left( -\frac{1}{2} \right)^{k-1} \hat{C}_-, \quad k \geq 1. \tag{3.67}
\]

Using explicit formula (3.61) for \( \hat{C}_+ \), we can directly calculate an expression for \( \hat{C}_+^2 \):

\[
\hat{C}_+^2 = \frac{1}{(\omega - 2)^2} (\mathbf{I} + \mathbf{P} + \mathbf{K}) - \frac{1}{2} \hat{C}_+ + \frac{(\omega - 8)}{2 (\omega - 2)^2} \mathcal{P}_{12.34} (K_{13} P_{24} + K_{13} P_{12.34}). \tag{3.68}
\]

If \( \omega = 8 \), then the last term in the right-hand side of (3.68) vanishes and the characteristic identity for \( \hat{C}_+ \) takes the form

\[
\hat{C}_+^2 = -\frac{1}{6} \hat{C}_+ + \frac{1}{36} (\mathbf{I} + \mathbf{P} + \mathbf{K}). \tag{3.69}
\]

If \( \omega \neq 2, 8 \), then multiplying (3.68) by \( \hat{C}_+ \) yields the third-degree identity (3.63). We note that for \( \omega = 4 \), the last term in (3.63) is zero, and hence in this case (3.63) is the characteristic identity for \( \hat{C}_+ \), which has the explicit form

\[
\hat{C}_+^3 = \frac{1}{2} \hat{C}_+^2 + \frac{1}{2} \hat{C}_+. \tag{3.70}
\]

To obtain a characteristic identity for \( \hat{C}_+ \) when \( \omega \neq 2, 4, 8 \), we eliminate \( \mathbf{P} \) and \( \mathbf{K} \) from (3.63). Multiplying (3.63) by \( \hat{C}_+ \) and using (3.59), we can express \( \mathbf{K} \) in terms of \( \hat{C}_+ \):

\[
\mathbf{K} = \hat{C}_+^4 + \frac{1}{2} \hat{C}_+^3 + \frac{(\omega - 8)}{2(\omega - 2)^2} \hat{C}_+ - \frac{\omega - 4}{2(\omega - 2)^3} \hat{C}_+. \tag{3.71}
\]

Substituting \( \mathbf{K} \) from (3.71) in (3.63) gives (3.64). Multiplying both sides of (3.71) by \( (\hat{C}_+ + \mathbf{I}) \) and using the last relation in (3.59), we obtain the characteristic identity for \( \hat{C}_+ \),

\[
\hat{C}_+^5 + \frac{3}{2} \hat{C}_+^4 + \frac{(\omega + 1)(\omega - 4)}{2(\omega - 2)^2} \hat{C}_+^3 + \frac{(\omega^2 - 12\omega + 24)}{2(\omega - 2)^2} \hat{C}_+^2 - \frac{\omega - 4}{(\omega - 2)^3} \hat{C}_+ = 0, \tag{3.72}
\]
which can be rewritten in the form
\[
\hat{C}_+^6 = -\frac{3}{2} \hat{C}_+^4 - \frac{(\omega + 1)(\omega - 4)}{2(\omega - 2)^2} \hat{C}_+^3 - \frac{\omega^2 - 12\omega + 24}{2(\omega - 2)^3} \hat{C}_+^2 + \frac{\omega - 4}{(\omega - 2)^3} \hat{C}_+.
\] (3.73)

Now, (3.65) is the result of factoring (3.72). For further calculations, we also need an expression for \(\hat{C}_+^6\), which can be derived by multiplying (3.73) by \(\hat{C}_+\) and using the known polynomial for \(\hat{C}_+^3\) from (3.73):
\[
\hat{C}_+^6 = \frac{7\omega^2 - 30\omega + 44}{4(\omega - 2)^2} \hat{C}_+^4 + \frac{3\omega^3 - 17\omega^2 + 30\omega - 24}{4(\omega - 2)^3} \hat{C}_+^3 + \frac{3\omega^2 - 32\omega + 56}{4(\omega - 2)^4} \hat{C}_+^2 + \frac{3(\omega - 4)}{2(\omega - 2)^5} \hat{C}_+.
\] (3.74)

Our next goal is to find a characteristic polynomial for the split Casimir operator \(\hat{C}_{\text{ad}} = \hat{C}_+ + \hat{C}_-\) by using the expressions obtained. We seek such that polynomial as a degree-six polynomial in \(\hat{C}_{\text{ad}}\) with arbitrary coefficients \(\alpha_i\):
\[
\hat{C}_{\text{ad}}^6 + \alpha_5 \hat{C}_{\text{ad}}^5 + \alpha_4 \hat{C}_{\text{ad}}^4 + \alpha_3 \hat{C}_{\text{ad}}^3 + \alpha_2 \hat{C}_{\text{ad}}^2 + \alpha_1 \hat{C}_{\text{ad}} + \alpha_0 = 0.
\] (3.75)

We need to find \(\alpha_i\) that make (3.75) vanish. The first formula in (3.59) implies \(\hat{C}_k = \hat{C}_+^k + \hat{C}_-^k\), and hence equating polynomial (3.75) to zero yields the equation
\[
\hat{C}_+^6 + \alpha_5 \hat{C}_+^5 + \alpha_4 \hat{C}_+^4 + \alpha_3 \hat{C}_+^3 + \alpha_2 \hat{C}_+^2 + \alpha_1 \hat{C}_+ + \alpha_0 = 0.
\]

Using the expressions for \(\hat{C}_+^5, 6\) in terms of \(\hat{C}_+^{4,3,2,1}\) given by (3.73) and (3.74) as well as the expressions for \(\hat{C}_-^{5,4,3,2,1}\) in terms of \(\hat{C}_-\) from (3.67) and setting the coefficients of those operators to zero, we obtain the \(\alpha_i\) as
\[
\alpha_0 = 0, \quad \alpha_1 = -\frac{\omega - 4}{2(\omega - 2)^3}, \quad \alpha_2 = \frac{\omega^2 - 16\omega + 40}{4(\omega - 2)^3},
\]
\[
\alpha_3 = \frac{\omega^3 - 3\omega^2 - 22\omega + 56}{4(\omega - 2)^4}, \quad \alpha_4 = \frac{5\omega^2 - 18\omega + 4}{4(\omega - 2)^2}, \quad \alpha_5 = 2.
\]

Thus, the characteristic identity for \(\hat{C}_{\text{ad}}\) takes the form
\[
\hat{C}_{\text{ad}}^6 + 2\hat{C}_{\text{ad}}^5 + \frac{5\omega^2 - 18\omega + 4}{4(\omega - 2)^2} \hat{C}_{\text{ad}}^4 + \frac{\omega^3 - 3\omega^2 - 22\omega + 56}{4(\omega - 2)^3} \hat{C}_{\text{ad}}^3 + \frac{\omega^2 - 16\omega + 40}{4(\omega - 2)^3} \hat{C}_{\text{ad}}^2 - \frac{\omega - 4}{2(\omega - 2)^5} \hat{C}_{\text{ad}} = 0.
\] (3.76)

The roots of the polynomial in the left-hand side of (3.76) can be found explicitly:
\[
a_1 = 0, \quad a_2 = -\frac{1}{2}, \quad a_3 = -1, \quad a_4 = \frac{1}{\omega - 2}, \quad a_5 = \frac{-2}{\omega - 2}, \quad a_6 = \frac{4 - \omega}{2(\omega - 2)}.
\] (3.77)

We note that degenerate roots appear for the following values of \(\omega\):
\[
\omega = 0 \implies a_2 = a_4 = -\frac{1}{2}, \quad a_3 = a_6 = -1, \quad \omega = 1 \implies a_3 = a_4 = -1,
\]
\[
\omega = 4 \implies a_1 = a_6 = 0, \quad a_3 = a_5 = -1, \quad \omega = 6 \implies a_2 = a_5 = -\frac{1}{2},
\] (3.78)
\[
\omega = 8 \implies a_5 = a_6 = -\frac{1}{3}.
\]

Therefore, the cases \(\omega = 0, 1, 4, 6, 8\) are to be considered separately (see Remark 1 below).

Using the roots (3.77) of the characteristic polynomial in the left-hand side of (3.76), we can rewrite characteristic identity (3.76) for \(\omega \neq 0, 1, 2, 4, 6, 8\) as (3.66).
Remark 1. To derive the characteristic identities for $\hat{C}_\text{ad}$ with $\omega = 0, 1, 4, 6, 8$, we need to retain a single parenthesis in (3.66) among all those that correspond to degenerate roots (3.78). This operation turns the left-hand side of (3.66) into the following polynomials in $\hat{C}_\text{ad}$:

\[
\begin{align*}
\omega = 0: & \quad \hat{C}_\text{ad} \left( \hat{C}_\text{ad} + \frac{1}{2} \right) (\hat{C}_\text{ad} + 1) (\hat{C}_\text{ad} - 1) = \frac{1}{2} K \neq 0, \\
\omega = 1: & \quad \hat{C}_\text{ad} \left( \hat{C}_\text{ad} + \frac{1}{2} \right) (\hat{C}_\text{ad} + 1) (\hat{C}_\text{ad} - 2) (\hat{C}_\text{ad} + \frac{3}{2}) = -\frac{3}{2} K \neq 0, \\
\omega = 4: & \quad \hat{C}_\text{ad} \left( \hat{C}_\text{ad} + \frac{1}{2} \right) (\hat{C}_\text{ad} + 1) (\hat{C}_\text{ad} - \frac{1}{2}) = 0, \\
\omega = 6: & \quad \hat{C}_\text{ad} \left( \hat{C}_\text{ad} + \frac{1}{2} \right) (\hat{C}_\text{ad} + 1) (\hat{C}_\text{ad} - \frac{1}{6}) (\hat{C}_\text{ad} + \frac{1}{3}) = 0, \\
\omega = 8: & \quad \hat{C}_\text{ad} \left( \hat{C}_\text{ad} + \frac{1}{2} \right) (\hat{C}_\text{ad} + 1) (\hat{C}_\text{ad} - \frac{1}{6}) (\hat{C}_\text{ad} + \frac{1}{3}) = 0.
\end{align*}
\]

(3.79)

The equalities in the right-hand sides of (3.79) are obtained by substituting $\hat{C}_\text{ad} = \hat{C}_+ + \hat{C}_-$ and using (3.67). It follows from (3.79) that for $\omega = 0, 1$, the characteristic polynomial for $\hat{C}_\text{ad}$ of form (2.38) does not exist, and hence $\hat{C}_\text{ad}$ is not diagonalizable. Accordingly, the representation $\text{ad}^{\otimes 2}$ of $\text{osp}(M|N)$ in this case is not completely reducible. This can also be deduced from the fact that for $\omega = 0, 1$ the ad-invariant operator $K$ is nilpotent (because $K^2 = 0$) and therefore not diagonalizable. The cases $\omega = 0, 1$ are considered later.

In the case where $\omega \neq 0, 1, 2, 4, 6, 8$, the characteristic identity in form (3.66) allows constructing projectors onto invariant subspaces of $V_\text{ad} \otimes V_\text{ad}$ by (2.39), where $p = 6$ and $a_i$ are the roots (3.77) of characteristic equation (3.66). Using (3.59), (3.63), (3.67), and (3.73), we find explicit expressions for the projectors (2.39) in terms of $I$, $P$, $K$, $\hat{C}_+$, and $\hat{C}_-$:

\[
\begin{align*}
P_1 &= \frac{1}{2} (I - P) + 2 \hat{C}_-, \quad P_2 = -2 \hat{C}_-, \quad P_3 = \frac{2K}{(\omega - 1)\omega}, \\
P_4 &= \frac{2}{3} (\omega - 2) \hat{C}_+^2 + \frac{\omega}{3} \hat{C}_+ + \frac{(\omega - 4)(I + P)}{3(\omega - 2)} - \frac{2(\omega - 4)K}{3(\omega - 2)(\omega - 1)}, \\
P_5 &= \frac{2(\omega - 2)^2}{3(\omega - 8)} \hat{C}_+^3 - \frac{(\omega - 2)(\omega - 6)}{3(\omega - 8)} \hat{C}_+^2 + \frac{\omega - 4)(I + P)}{6(\omega - 8)} + \frac{2K}{3(\omega - 8)}, \\
P_6 &= \frac{4(\omega - 2)}{\omega - 8} \hat{C}_+^2 + \frac{4}{\omega - 8} \hat{C}_+ - \frac{4(I + P)}{(\omega - 2)(\omega - 8)} - \frac{8(\omega - 4)K}{\omega(\omega - 2)(\omega - 8)}).
\end{align*}
\]

(3.80)

The images of $P_1$ and $P_2$ are contained in the antisymmetric part $P_-(V_\text{ad}^{\otimes 2})$, while the images of $P_i$, $i = 3, \ldots, 6$, lie within the symmetric part $P_+(V_\text{ad}^{\otimes 2})$ of $V_\text{ad}^{\otimes 2}$, where $P_\pm = (I \pm P)/2$. We note that for $\omega = 4, 6$, all projectors (3.80) are well defined and constructed from the ad-invariant operators $I$, $P$, $K$, $\hat{C}_-$, $\hat{C}_+$, and $\hat{C}_+^2$; hence, they are projectors onto invariant subspaces of the $\text{ad}^{\otimes 2}$ representation of $\text{osp}(M|N)$. Besides, although $P_5$ and $P_6$ are not formally defined for $\omega = 8$, substituting the explicit expressions for $I$, $P$, $K$, $\hat{C}_+$, and $\hat{C}_+^2$ in (3.80) leads to a cancellation of the pole at $\omega = 8$, with the result

\[
\begin{align*}
P_5 &= \frac{1}{6} (1 - (P_{14} + P_{23} + P_{13} + P_{24}) + P_{12}P_{24})P_{12,34}, \\
P_6 &= \frac{4}{\omega - 2} P_{12,34}K_{13} \left[ \frac{1}{2}(1 + P_{24}) - \frac{1}{\omega}K_{24} \right] P_{12,34}.
\end{align*}
\]

(3.81)

(3.82)

It is worth pointing out that expression (3.81) for the projector $P_5$ is independent of $\omega$ and coincides with the full (anti)symmetrizer of $V_{(M|N)}^{\otimes 4}$.
To build projectors onto the generalized eigenspaces of \( \hat{C}_{ad} \) for \( \omega = 0, 1 \), we use characteristic identity (2.40). For \( \omega = 0 \), it takes the form

\[
\hat{C}_{ad}\left( \hat{C}_{ad} + \frac{1}{2} \right)\left( \hat{C}_{ad} + 1 \right)^2\hat{C}_{ad} - 1 = 0,
\]

and hence we need to set \( a_1 = 1, a_2 = -1/2, a_3 = -1, a_4 = 1, k_1 = k_2 = k_4 = 1, \) and \( k_3 = 2 \) in (2.40). Then (2.41) gives the projectors onto the generalized eigenspaces of \( \hat{C}_{ad} \):

\[
P_1 = \frac{1}{2}(I - P) + 2\hat{C}_-,
\]
\[
P_2 = -2\hat{C}_- + \frac{2}{3}(I + P) + \frac{4}{3}K - \frac{4}{3}\hat{C}_+^2,
\]
\[
P_3 = -\frac{1}{4}(I + P) - \frac{5}{4}K - \frac{1}{2}\hat{C}_+ + \hat{C}_+^2,
\]
\[
P_4 = \frac{1}{12}(I + P) - \frac{1}{12}K + \frac{1}{2}\hat{C}_+ + \frac{1}{3}\hat{C}_+^2.
\]

(3.83)

Here, the operators \( P_1, P_2, \) and \( P_4 \) extract eigenspaces of \( \hat{C}_{ad} \), while \( P_3 \) projects \( V_{ad}^{\otimes 2} \) onto a generalized eigenspace of \( \hat{C}_{ad} \). We thus conclude that the restriction of the representation \( \text{ad}^{\otimes 2} \) to \( P_3(V_{ad}^{\otimes 2}) \) is reducible but not completely reducible. It is also worth noting that \( P_2 \) given in (3.83) is a linear combination of symmetric operators \( (I + P) \), \( K \), and \( \hat{C}_- \), and the antisymmetric operator \( \hat{C}_- \). As noted in Sec. 2.3, the symmetric and antisymmetric parts of \( V_{ad}^{\otimes 2} \) are invariant under the action of any Lie superalgebra \( g \) in the \( \text{ad}^{\otimes 2} \) representation, and hence \( P_2 \) can be invariantly split into its symmetric and antisymmetric parts:

\[
P_2^{(\pm)} = \frac{2}{3}(I + P) + \frac{4}{3}K - \frac{4}{3}\hat{C}_+, \quad P_2^{(-)} = -2\hat{C}_-.
\]

(3.84)

Finally, for \( osp(M|N) \) in the case \( \omega \equiv M - N = 0 \), the required projectors are

\[
P_1 \equiv P_1^{(-)} = \frac{1}{2}(I - P) + 2\hat{C}_-,
\]
\[
P_2^{(-)} = -2\hat{C}_-,
\]
\[
P_3 \equiv P_3^{(\pm)} = -\frac{1}{4}(I + P) - \frac{5}{4}K - \frac{1}{2}\hat{C}_+ + \hat{C}_+^2,
\]
\[
P_4 \equiv P_4^{(\pm)} = \frac{1}{12}(I + P) - \frac{1}{12}K + \frac{1}{2}\hat{C}_+ + \frac{1}{3}\hat{C}_+^2.
\]

If \( \omega = 1 \), then the characteristic identity (2.40) with \( \hat{C}_{ad} \) is

\[
\hat{C}_{ad}\left( \hat{C}_{ad} + \frac{1}{2} \right)\left( \hat{C}_{ad} + 1 \right)^2\hat{C}_{ad} - 2\left( \hat{C}_{ad} + \frac{3}{2} \right) = 0,
\]

whence \( a_1 = 0, a_2 = -1/2, a_3 = -1, a_4 = 2, a_5 = -3/2, k_1 = k_2 = k_4 = k_5 = 1, \) and \( k_3 = 2 \) in (2.40). The projectors onto generalized eigenspaces of \( \hat{C}_{ad} \) then become by (2.41):

\[
P_1 = \frac{1}{2}(I - P) + 2\hat{C}_-,
\]
\[
P_3 = I + P - \frac{10}{3}K + \frac{1}{3}\hat{C}_+ + \frac{2}{3}\hat{C}_+^2,
\]
\[
P_2 = -2\hat{C}_-,
\]
\[
P_4 = \frac{1}{14}(I + P) - \frac{2}{21}K + \frac{5}{21}\hat{C}_+ + \frac{2}{21}\hat{C}_+^2,
\]
\[
P_5 = -\frac{4}{7}(I + P) + \frac{24}{7}K - \frac{4}{7}\hat{C}_+ + \frac{4}{7}\hat{C}_+^2.
\]

(3.83)

Here, the operators \( P_1, P_2, P_4, \) and \( P_5 \) extract eigenspaces of \( \hat{C}_{ad} \) and \( P_3 \) projects \( V_{ad}^{\otimes 2} \) onto a generalized eigenspace of \( \hat{C}_{ad} \). We can then conclude that \( P_3(V_{ad}^{\otimes 2}) \) is not a space of an irreducible or completely reducible representation of \( osp(M|N) \) with \( \omega = 1 \).
To find the dimensions of the invariant subspaces, we need to calculate the traces and supertraces of $P_1, \ldots, P_6$. First, we calculate the following auxiliary traces and supertraces (where we also use the notation $\xi = M + N$):

\[
\begin{align*}
\text{tr} I &= \frac{1}{4}(\xi^2 - \omega)^2, & \text{str} I &= \frac{1}{4}\omega^2(\omega - 1)^2, \\
\text{tr} P &= \frac{1}{2}\omega(\omega - 1), & \text{str} P &= \frac{1}{2}\omega(\omega - 1), \\
\text{tr} K &= \frac{1}{2}\omega(\omega - 1), & \text{str} K &= \frac{1}{2}\omega(\omega - 1), \\
\text{tr} \hat{C}_- &= -\frac{1}{4}(\xi^2 - \omega), & \text{str} \hat{C}_- &= -\frac{1}{4}\omega(\omega - 1), \\
\text{tr} \hat{C}_+ &= \frac{1}{4}(\xi^2 - \omega), & \text{str} \hat{C}_+ &= \frac{1}{4}\omega(\omega - 1), \\
\text{tr} \hat{C}_+ &= \frac{2\xi^4 + (\omega^2 - 16\omega + 20)\xi^2 + \omega^3 + 4\omega^2 - 12\omega}{8(\omega - 2)^2}, & \text{str} \hat{C}_+ &= \frac{3}{8}\omega(\omega - 1).
\end{align*}
\]

Here, the formulas for str in the second and fourth rows correspond to general formulas (2.56) in accordance with (3.57). From (3.80) and (3.85), we deduce the traces

\[
\begin{align*}
\text{tr} P_1 &= \frac{1}{8}(\xi^4 - 2\xi^2(\omega + 2) - \omega(\omega - 6)), & \text{tr} P_4 &= \frac{1}{12}(\xi^4 - 10\xi^2 + 3\omega(\omega - 2)), \\
\text{tr} P_2 &= \frac{1}{2}\xi^2 - \omega, & \text{tr} P_5 &= \frac{1}{24}(\xi^4 + 2\xi^2(3\omega - 4) + 3\omega(\omega - 2)), \\
\text{tr} P_3 &= 1, & \text{tr} P_6 &= \frac{1}{2}(\xi^2 + \omega - 2)
\end{align*}
\]

and supertraces of the $P_i$,

\[
\begin{align*}
\text{str} P_1 &= \frac{1}{8}\omega(\omega - 1)(\omega + 2)(\omega - 3), & \text{str} P_4 &= \frac{1}{12}\omega(\omega + 1)(\omega + 2)(\omega - 3), \\
\text{str} P_2 &= \frac{1}{2}\omega(\omega - 1), & \text{str} P_5 &= \frac{1}{24}\omega(\omega - 1)(\omega - 2)(\omega - 3), \\
\text{str} P_3 &= 1, & \text{str} P_6 &= \frac{1}{2}(\omega - 1)(\omega - 2),
\end{align*}
\]

for $\omega \neq 0, 1$. The dimension $\dim_0 V_i$ of the even part of the invariant subspace $V_i = V_{i \bar{0}} \oplus V_{i \bar{1}} \subseteq V_{(M|N)}^{\otimes 4}$ extracted by $P_i$ is $\dim_0 V_i = (\text{tr} P_i + \text{str} P_i)/2$, and the dimension of the odd part $V_{i \bar{1}}$ of $V_i$ is $\dim_1 V_i = (\text{tr} P_i - \text{str} P_i)/2$. Using (3.86) and (3.87), and substituting $\omega = M - N$ and $\xi = M + N$, we obtain the following values of the dimensions of invariant subspaces:

\[
\begin{align*}
\dim_0 V_1 &= \frac{1}{8} M(M - 1)(M + 2)(M - 3) + \frac{1}{8} N(N + 1)(N - 2)(N + 3) + \frac{1}{4} MN(3MN + M - N + 1), \\
\dim_0 V_2 &= \frac{1}{2} M(M - 1) + \frac{1}{2} N(N + 1), & \dim_0 V_3 &= 1, \\
\dim_0 V_4 &= \frac{1}{12} M(M + 1)(M + 2)(M - 3) + \frac{1}{12} N(N - 1)(N - 2)(N + 3) + \frac{1}{2} MN(MN - 1), \\
\dim_0 V_5 &= \frac{1}{24} M(M - 1)(M - 2)(M - 3) + \frac{1}{24} N(N + 1)(N + 2)(N + 3) + \frac{1}{4} MN(M - 1)(N + 1), \\
\dim_0 V_6 &= \frac{1}{2} (M - 1)(M + 2) + \frac{1}{2} N(N - 1)
\end{align*}
\]
and
\[
\dim_1 V_1 = \frac{1}{2} MN(M(M - 1) + (N - 1)(N + 2)), \quad \dim_1 V_2 = MN,
\]
\[
\dim_1 V_3 = 0, \quad \dim_1 V_4 = \frac{1}{3} MN(M^2 + N^2 - 5),
\]
\[
\dim_1 V_5 = \frac{1}{6} MN((M - 1)(M - 2) + (N + 1)(N + 2)), \quad \dim_1 V_6 = MN.
\]  
(3.89)

We note that substituting \( \omega = M \) (which implies \( N = 0 \)) in identities (3.62)–(3.66) yields analogous identities for the \( so(M) \) Lie algebra, which are given in [15], [16], and dimensions (3.88) of \( osp(M|N) \)-invariant subspaces in the \( \text{ad}^{\otimes 2} \) representation transform into the corresponding dimensions of the invariant subspaces of \( so(M) \). Analogously, the substitution \( \omega = -N \) (which implies \( M = 0 \)) transforms identities (3.62)–(3.66) into analogous identities for the \( sp(N) \) algebra and (3.88) gives the dimensions of the invariant subspaces of the \( sp(N) \)-representation \( \text{ad}^{\otimes 2} \). Indeed, the dimensions of the odd parts of invariant subspaces (3.89) vanish after the substitutions \( M = 0 \) and \( N = 0 \), which corresponds to the transition from \( osp(M|N) \) to the \( so(M) \) or \( sp(N) \) Lie algebras.

4. The \( s\ell(M|N) \) Lie superalgebra

The \( s\ell(M|N) \) Lie superalgebra (where \( M \neq N \)) is defined as the algebra of operators \( A : V(M|N) \to V(M|N) \) that satisfy
\[
\text{str} A = (-1)^{|a|} A^a_a = 0,
\]  
(4.1)

with the Lie superbracket given by (2.4). It is known that \( s\ell(N,N) \) is not simple, and \( s\ell(M|N) \cong s\ell(N|M) \), and we therefore restrict ourselves to \( \omega = M - N > 0 \).

To build a basis of \( s\ell(M|N) \), we use the same method as in the \( osp(M|N) \) case, that is, we build a projector onto the space of solutions of (4.1). We consider the operators \( P_0 \) and \( P_I \) that act on \( \text{End}(V(M|N)) \) as
\[
P_0(E) = E - \frac{\text{str} E}{M - N} I, \quad P_I(E) = \frac{\text{str} E}{M - N} I,
\]  
(4.2)

for any \( E \in \text{End}(V(M|N)) \) and \( I \) being the identity operator on \( V(M|N) \). Clearly,
\[
\text{str} P_0(E) = 0, \quad \text{str} P_I(E) = \text{str} E,
\]  
(4.3)

whence \( P_A P_B = \delta_{AB} P_A \). Therefore, \( P_0 \) and \( P_I \) comprise a full system of projectors in \( \text{End}(V(M|N)) \). Analogously to the \( osp(M|N) \) case, Eq. (4.1) can be rewritten in terms of \( P_A \) as \( P_I(A) = 0 \) or \( P_0(A) = A \), which yields the general solution of (4.1) in the form
\[
A = P_0(E) \quad \Rightarrow \quad A^a_c = E^a_c - \frac{(-1)^{|b|} E^b_b}{M - N} \delta^a_c,
\]  
(4.4)

where \( E \) is an arbitrary element of \( \text{End}(V(M|N)) \).

We consider the matrix identities \( e_{ij} : V(M|N) \to V(M|N) \) with the components
\[
(e_{ij})^a_b = \delta^a_a \delta_{jb}.
\]  
(4.5)

Unlike the matrix identities \( e_{ij} \) in the \( osp(M|N) \) case, both indices of \( e_{ij} \) are lower. The matrices of the basis elements \( (T_{ij})^a_b \) of \( s\ell(M|N) \) in the defining representation are obtained by substituting \( E^a_b \to (e_{ij})^a_b \) in (4.4):
\[
(T_{ij})^a_b = (e_{ij})^a_b - \frac{(-1)^{|ij|} E^b_b}{M - N} \delta_{ij} \delta^a_c.
\]  
(4.6)
The degree of $T_{ij}$ coincides with that of $e_{ij}$ and equals $[i] + [j]$. Besides, we have the following equality for $T_{ij}$:

$$\text{Tr}(T) \equiv T_{ii} = 0. \quad (4.7)$$

For the vectors $X = X^{ij}T_{ij}$ of the algebra $\mathfrak{sl}(M|N)$ to correspond uniquely to their coordinates $X^{ij}$, we require the numbers $X^{ij}$ to satisfy the condition

$$(-1)^{[i]}X^{ii} = 0, \quad (4.8)$$

which has the following advantage: for any $X = X^{ij}T_{ij} \in \mathfrak{sl}(M|N)$, we have $X = X^{ij}e_{ij} \in \mathfrak{sl}(M|N)$ in the defining representation. If the vector $X \in \text{End}(V_{(M|N)})$ is expanded over the basis $\{e_{ij}\}_{i,j=1}^{M+N}$, Eq. (4.8) means that $X$ lies in $\mathfrak{sl}(M|N)$. We also impose the conditions analogous to (4.7) on the coordinates $Y_{ij}$ of the covectors $Y$ in the dual basis to (4.6):

$$\text{Tr}(Y) = Y_{ii} = 0. \quad (4.9)$$

Calculating Lie superbrackets (2.4) of $T_{ij} \in \mathfrak{sl}(M|N)$ defined in (4.6), we obtain

$$[T_{i_1i_2}, T_{j_1j_2}] = \delta_{j_1j_2} T_{i_1j_2} - (-1)^{([i_1]+[i_2])([j_1]+[j_2])}\delta_{i_1j_2} T_{j_1i_2} = T_{i_1k_2} X^{k_1k_2}_{i_1i_2,j_1j_2}, \quad (4.10)$$

where the structure constants $X^{k_1k_2}_{i_1i_2,j_1j_2}$ are written explicitly as

$$X^{k_1k_2}_{i_1i_2,j_1j_2} = \delta_{j_1j_2} \delta^{k_1}_{i_1} \delta^{k_2}_{i_2} - (-1)^{([i_1]+[i_2])([j_1]+[j_2])}\delta_{i_1j_2} \delta^{k_1}_{j_1} \delta^{k_2}_{j_2}. \quad (4.11)$$

It can be verified that the pairs of indices $(i_1, i_2)$, $(j_1, j_2)$, and $(k_1, k_2)$ in (4.11) satisfy respective relations (4.7), (4.9), and (4.8).

The $\mathfrak{sl}(M|N)$ Cartan–Killing metric (2.9) in basis (4.6) can be calculated using (2.9),

$$g_{i_1i_2,j_1j_2} = 2\omega \left((-1)^{[i_1][j_2]}\delta_{j_1j_2}\delta_{i_1j_2} - \frac{(-1)^{[i_1][j_2]}}{\omega}\delta_{i_1j_2}\delta^{j_1j_2} \right), \quad (4.12)$$

where $\omega = M - N$, as in the $\mathfrak{osp}(M|N)$ case. For $\omega = 0$, metric (4.12) is degenerate: $g_{i_1i_2,j_1j_2} = -2(-1)^{[i_1]+[j_2]}\delta_{i_1j_2}\delta_{j_1j_2}$. However, as we have mentioned, this case is omitted in our paper. We also note that for $(i_1, i_2)$ and $(j_1, j_2)$ in (4.12), the condition (4.9) holds.

We introduce the projector $\mathcal{I}$ that acts on $V^{\otimes 2}_{(M|N)}$, and is defined by the matrix

$$\mathcal{I}^{i_1i_2}_{j_1j_2} = \delta_{j_1j_2} \delta^{i_1}_{i_2} - \frac{(-1)^{[j_1][j_2]}}{\omega}\delta^{i_1}_{j_1} \delta_{j_1j_2} \Rightarrow \mathcal{I}^2 = \mathcal{I}. \quad (4.13)$$

It maps an arbitrary $Y \in V^{\otimes 2}_{(M|N)}$, with components $Y^{ik}$ to $X \in V^{\otimes 2}_{(M|N)}$ with the components $X^{ik} = \mathcal{I}^{ik}_{jk} Y^{jk}$, which satisfy (4.8). If we identify the space $\mathfrak{sl}(M|N)$ with $\mathcal{I}(V^{\otimes 2}_{(M|N)}) \subset V^{\otimes 2}_{(M|N)}$, then $\mathcal{I}$ can be viewed as the identity operator $\mathcal{I}$ acting on the algebra $\mathfrak{sl}(M|N)$.

The components of the inverse Cartan–Killing metric in basis (4.6) are defined by (2.14):

$$g^{i_1i_2,j_1j_2} g_{j_1j_2,k_1k_2} = \mathcal{I}^{i_1i_2}_{k_1k_2}, \quad g_{i_1i_2,j_1j_2} g^{j_1j_2,k_1k_2} = \mathcal{I}^{k_1k_2}_{i_1i_2}. \quad (4.14)$$

Direct calculations yield

$$g^{i_1i_2,j_1j_2} = \frac{1}{2\omega} \left((-1)^{[j_1][j_2]}\delta^{i_1i_2} \delta_{i_1j_2} - \frac{1}{\omega}\delta^{i_1i_2} \delta_{j_1j_2} \right), \quad (4.15)$$

where the pairs of indices $(i_1, i_2)$ and $(j_1, j_2)$ satisfy (4.8), as expected.
4.1. Projectors onto invariant subspaces of the tensor product of two defining representations. Using (4.6), (4.15), (2.20), and (2.34), we can write the matrix of the split \( sl(M|N) \) Casimir operator in the defining representation as

\[
\hat{C}_f^{k_1 k_2}_{m_1 m_2} = \frac{1}{2\omega} \left( \begin{pmatrix} -1 \\ 0 \end{pmatrix}^{[m_1]}^{[m_2]} \delta_{m_2}^{k_1} \delta_{m_1}^{k_2} - \frac{1}{\omega} \delta_{m_1}^{k_1} \delta_{m_2}^{k_2} \right). \tag{4.16}
\]

We define the identity operator \( 1 \) and the superpermutation \( \mathcal{P} \) acting on \( V_{(M|N)}^{\otimes 2} \),

\[
1 = e_{ii} \otimes e_{jj} \implies 1^{k_1 k_2}_{m_1 m_2} = \delta_{m_1}^{k_1} \delta_{m_2}^{k_2}, \tag{4.17}
\]

\[
\mathcal{P} = (-1)^{[\beta]} e_{ij} \otimes e_{ji} \implies \mathcal{P}^{k_1 k_2}_{m_1 m_2} = (-1)^{[k_1]} \delta_{m_1}^{k_1} \delta_{m_2}^{k_2}, \tag{4.18}
\]

where \( e_{ij} \) are matrix identities (4.5) acting on \( V_{(M|N)} \). The matrices of \( 1 \) and \( \mathcal{P} \) are presented in the right-hand sides of these equalities. By (4.18), \( \mathcal{P}^2 = 1 \). A comparison of (4.16) with (4.17) and (4.18) shows that \( \hat{C}_f \) can be written in terms of \( 1 \) and \( \mathcal{P} \) as

\[
\hat{C}_f = \frac{1}{2\omega} \left( \mathcal{P} - \frac{1}{\omega} 1 \right). \tag{4.19}
\]

Using this formula, we obtain a second-degree characteristic identity for \( \hat{C}_f \):

\[
\hat{C}_f^2 + \frac{1}{\omega^2} \hat{C}_f - \frac{\omega^2 - 1}{4\omega^2} 1 = 0. \tag{4.20}
\]

The left-hand side of this equality can be factored, which yields

\[
\left( \hat{C}_f - \frac{\omega - 1}{2\omega^2} 1 \right) \left( \hat{C}_f + \frac{\omega + 1}{2\omega^2} 1 \right) = 0. \tag{4.21}
\]

The projectors \( P_+ \) and \( P_- \) onto the eigenspaces of \( \hat{C}_f \) that correspond to the roots \( a_+ = (\omega - 1)/2\omega^2 \) and \( a_- = -(\omega + 1)/2\omega^2 \) of (4.21) are built by (2.39) with \( p = 2 \):

\[
P_{\pm} = \pm \left( \omega \hat{C}_f + \frac{1 \pm \omega}{2\omega} \right) = \frac{1}{2} (1 \pm \mathcal{P}). \tag{4.22}
\]

Thus, \( P_+ \) and \( P_- \) turn out to be the supersymmetrizer and the superantisymmetriser on \( V_{(M|N)}^{\otimes 2} \).

In the defining representation, the solution of the \( sl(M|N) \)-invariant graded Yang–Baxter equation (3.40) can be written in several equivalent ways,

\[
R(u) = \frac{u + \mathcal{P}}{1 - u} = \frac{1 + u}{1 - u} P_+ - P_-; \tag{4.23}
\]

where \( u \) is the spectral parameter. Similarly to the \( osp(M|N) \) case, this solution can be expressed as a rational function of \( \hat{C}_f \):

\[
R(u) = \frac{P_+ + u}{P_+ - u} = \frac{\omega \hat{C}_f + (1 + \omega)/(2\omega) + u}{\omega \hat{C}_f + (1 + \omega)/(2\omega) - u}. \tag{4.24}
\]

We note that the solution \( R(u) \) is unitary: \( \mathcal{P} R(u) \mathcal{P} R(-u) = 1 \). It is defined up to multiplication by an arbitrary function \( f(u) \) such that \( f(u)f(-u) = 1 \).
4.2. Projectors onto invariant subspaces of the tensor product of two adjoint representations. To make the subsequent calculations more concise, we introduce an operator $K$ acting in $V_{(M|N)}^\otimes 2$, defined by its components in the homogeneous basis $\{e_i, e_j\}$ of this space:

$$K^{i_1i_2, j_1j_2} = (-1)^{[j_1][j_2]}\delta^{i_1i_2}\delta^{j_1j_2}. \quad (4.25)$$

We use the following identities for the operators $K_{ab}$ and $P_{ab}$, $a, b = 1, \ldots, 4$, acting in $V_{(M|N)}^\otimes 4$:

$$
\begin{align*}
K_{ab}K_{ab} &= \omega K_{ab}, \\
P_{ab}K_{ad}K_{bc} &= P_{ab}K_{ad}K_{bc}, \\
K_{ad}K_{bc}P_{ab} &= K_{ad}K_{bc}P_{cd}, \\
K_{ab}P_{ab}K_{bc} &= K_{ab}P_{ab}K_{bc} = K_{ab}P_{bc}K_{bc}.
\end{align*}
\quad (4.26)
$$

($P_{ab}$ and $K_{ab}$ are defined in (2.24)). It is worth pointing out that $\tilde{I}$ defined in (4.13) can be written as $\tilde{I} = 1 - K/\omega$, where 1 is given in (4.17).

The components of the basis elements $T_{i_1i_2} \in \ell(M|N)$ in the adjoint representation are precisely the structure constants (4.11):

$$(T_{i_1i_2})^{k_1k_2, j_1j_2} = \delta_{j_1i_2}\delta^{k_1i_1}\delta^{k_2j_2} - (-1)^{[i_1][i_2][j_1][j_2]}\delta_{j_1i_2}\delta^{k_1i_1}\delta^{k_2j_2}. \quad (4.27)$$

Using (4.15), (4.27), and (2.34), we now find an explicit form of $\hat{C}_{ad}$. In terms of operators (4.18) and (4.25), it can be written as

$$\hat{C}_{ad} = \frac{1}{2\omega}(P_{13} + P_{24} - K_{32} - K_{41}). \quad (4.28)$$

In what follows, we need three more operators: $K$ and $P_{ad}$ acting in $V_{ad}^\otimes 2$ and $P$ acting in $V_{(M|N)}^\otimes 4$. First, $K$ is defined as

$$K^{i_1i_2i_3i_4, j_1j_2j_3j_4} = g^{i_1i_2i_3i_4}g_{j_1j_2j_3j_4}, \quad (4.29)$$

and with the help of (4.18) and (4.25) can also be written as

$$K = K_{32}K_{14} - \frac{1}{\omega}P_{13}K_{12}K_{34} - \frac{1}{\omega}P_{13}K_{32}K_{14} + \frac{1}{\omega^2}K_{12}K_{34}. \quad (4.30)$$

Furthermore, $K^2 = (\omega^2 - 1)K$. We recall that $K_{32} = P_{23}K_{23}P_{23}$, $K_{14} = P_{34}P_{23}K_{12}P_{23}P_{34}$, etc. Next, $P$ permutes the first factor with the third, and the second factor with the fourth in $V_{(M|N)}^\otimes 4$ and is defined by

$$P = P_{13}P_{24} \implies (P)^{i_1i_2i_3i_4, j_1j_2j_3j_4} = (-1)^{[i_1][i_3] + [i_1][i_4] + [i_2][i_3] + [i_2][i_4]}\delta^{j_1i_1}\delta^{j_2i_2}\delta^{j_3i_3}\delta^{j_4i_4}, \quad (4.31)$$

whence $P^2 = I$, where $I$ is the identity operator on $V_{(M|N)}^\otimes 4$. Finally,

$$P^{(ad)} = I_{12}I_{34}P, \quad (4.32)$$

where $I$ is given in (4.13), is the permutation operator in the space $V_{ad} \otimes V_{ad} \subset V_{(M|N)}^\otimes 4$.

We note that $P$ commutes with both $\hat{C}_{ad}$ and $K$:

$$PK = KP = K = KP. \quad (4.33)$$

Using $P$ and $P^{(ad)}$, we define the symmetrizer $P_{+}^{(ad)}$ and the antisymmetrizer $P_{-}^{(ad)}$ on $V_{ad}^\otimes 2$,

$$
\begin{align*}
P_{+}^{(ad)} &= \frac{1}{2}(I \pm P^{(ad)}) = \frac{1}{2}(I \pm P)I_{12}I_{34} = \frac{1}{2}I_{12}I_{34}(I \pm P) \implies \\
P_{-}^{(ad)} &= \frac{1}{2}(I - P_{13}P_{24}) \left( I - \frac{1}{\omega}(K_{12} + K_{34}) \right), \\
P_{+}^{(ad)} &= \frac{1}{2}(I + P_{13}P_{24}) \left( I - \frac{1}{\omega}(K_{12} + K_{34}) + \frac{1}{\omega^2}K_{12}K_{34} \right), \\
\end{align*}
\quad (4.34)
$$

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where \( I = J_{12} J_{34} \) is the identity operator acting on \( V^\otimes 2 \), which, in particular, satisfies

\[
IP^{(ad)} = P^{(ad)} = P^{(ad)}I, \quad IK = K = KI, \quad IC_{ad} = \tilde{C}_{ad} = \tilde{C}_{ad} I.
\] (4.35)

We now define the symmetric and antisymmetric parts of split Casimir operator (4.28):

\[
\tilde{C}_+ = p^{(ad)} \tilde{C}_{ad} = \frac{1}{2} (I + P) \tilde{C}_{ad} = \frac{1}{4\omega}(2P_{13} + 2P_{24} - (I + P_{13} P_{24})K_{32} - (I + P_{13} P_{24})K_{14}),
\]

\[
\tilde{C}_- = p^{(ad)} \tilde{C}_{ad} = \frac{1}{2} (I - P) \tilde{C}_{ad} = \frac{1}{4\omega}(P_{13} P_{24} - I)(K_{14} + K_{32}) = \frac{1}{4\omega}(K_{14} + K_{32})(P_{13} P_{24} - I).
\] (4.36)

By (4.26) and (4.28), the following relations hold for \( \tilde{C}_-, \tilde{C}_+, \) and \( K \):

\[
\tilde{C}_+ + \tilde{C}_- = \tilde{C}_{ad}, \quad P\tilde{C}_\pm = \tilde{C}_\pm P = \pm \tilde{C}_\pm, \quad \tilde{C}_+ \tilde{C}_- = \tilde{C}_- \tilde{C}_+ = 0, \quad K\tilde{C}_- = \tilde{C}_- K = 0, \quad K\tilde{C}_+ = \tilde{C}_+, \quad K = -K, \quad K\tilde{C}_{ad} = \tilde{C}_{ad} K = -K.
\] (4.37)

**Statement 3.** The antisymmetric \( \tilde{C}_- \) and symmetric \( \tilde{C}_+ \) parts of the split Casimir operator of the \( sl(M|N) \) Lie superalgebra with \( \omega \neq 0, 1, 2 \) satisfy the relations

\[
\tilde{C}_+^2 = -\frac{1}{2} \tilde{C}_- \iff \tilde{C}_- \left( \tilde{C}_- + \frac{1}{2} I \right) = 0,
\] (4.38)

\[
\tilde{C}_+^3 = -\frac{3}{2} \tilde{C}_+^2 + \frac{1}{4\omega^2} \tilde{C}_+ + \frac{1}{4\omega^2}(I + P^{(ad)} - 2K),
\] (4.39)

\[
\tilde{C}_+^4 = -\frac{3}{2} \tilde{C}_+^3 - \frac{\omega^2}{2\omega^2} \tilde{C}_+^2 + \frac{3}{2\omega^2} \tilde{C}_+ + \frac{1}{4\omega^2}(I + P^{(ad)}),
\] (4.40)

\[
\tilde{C}_+(\tilde{C}_+ + I) \left( \tilde{C}_+ - \frac{1}{\omega} I \right) \left( \tilde{C}_+ + \frac{1}{\omega} I \right) \left( \tilde{C}_+ + \frac{1}{2} I \right) = 0.
\] (4.41)

The split Casimir operator \( \tilde{C}_{ad} = \tilde{C}_- + \tilde{C}_+ \) with \( \omega \neq 0, 1, 2 \) satisfies the relation

\[
\tilde{C}_{ad}(\tilde{C}_{ad} + I) \left( \tilde{C}_{ad} - \frac{1}{\omega} I \right) \left( \tilde{C}_{ad} + \frac{1}{\omega} I \right) \left( \tilde{C}_{ad} + \frac{1}{2} I \right) = 0.
\] (4.42)

**Proof.** Identity (4.38) for \( sl(M|N) \) is a special case of (2.54), which holds for all Lie superalgebras with a nondegenerate Cartan–Killing metric.

By (4.26) and (4.28), we obtain

\[
\tilde{C}_+^2 = \frac{1}{8\omega^2}(I + P_{13} P_{24})(4I + 2K_{32}K_{14} + \omega K_{32} + \omega K_{14}) - \frac{1}{4\omega^2}(P_{13} + P_{24})(K_{12} + K_{34} + K_{32} + K_{14}).
\] (4.43)

Multiplying (4.43) by \( \tilde{C}_+ \) yields (4.39). Multiplying (4.39) once more by \( \tilde{C}_+ \) and using (4.37), we obtain

\[
\tilde{C}_+^4 = \frac{1}{2} \tilde{C}_+^3 + \frac{1}{\omega^2} \tilde{C}_+^2 + \frac{1}{2\omega^2} \tilde{C}_+ + \frac{1}{2\omega^2} K.
\] (4.44)
Now, we express $K$ from (4.44) and substitute the result in (4.39), which gives (4.40). Multiplying both sides of (4.44) by $(\hat{C}_+ + 1)$ and using the second identity in the second row of (4.37), or multiplying both sides of (4.40) by $\hat{C}_+$ and using the second identity in the first row of (4.37) yields

$$\hat{C}_+^5 = \frac{3}{2} \hat{C}_+^4 - \frac{\omega^2 - 2}{2\omega^2} \hat{C}_+^3 - \frac{3}{2\omega^2} \hat{C}_+^2 - \frac{1}{2\omega^2} \hat{C}_+,$$

(4.45)

which can be rewritten as

$$\hat{C}_+^5 + \frac{3}{2} \hat{C}_+^4 + \frac{\omega^2 - 2}{2\omega^2} \hat{C}_+^3 - \frac{3}{2\omega^2} \hat{C}_+^2 - \frac{1}{2\omega^2} \hat{C}_+ = 0.\quad (4.46)$$

Relation (4.46) is the characteristic identity for $\hat{C}_+$. The roots of the polynomial in the left-hand side of (4.46) are

$$a_1 = 0, \quad a_2 = -1, \quad a_3 = \frac{1}{\omega}, \quad a_4 = -\frac{1}{\omega}, \quad a_5 = -\frac{1}{2},\quad (4.47)$$

and hence characteristic identity (4.46) takes form (4.41). We note that for $\omega = 1, 2$, we have degenerate roots, and these cases are considered separately below.

Because $\hat{C}_+ = \hat{C}_+ P_{\text{ad}}$ and $\hat{C}_+ = \hat{C}_+ P^{\text{ad}}_+$, it follows from (4.38) and (4.40) that

$$\hat{C}_- \left( \hat{C}_+ + \frac{1}{2} I \right) P_{\text{ad}} = 0, \quad (4.48)$$

$$\left( \hat{C}_+ + I \right) \left( \hat{C}_+ - \frac{1}{\omega} I \right) \left( \hat{C}_+ + \frac{1}{\omega} I \right) \left( \hat{C}_+ + \frac{1}{2} I \right) P^{\text{ad}}_+ = 0. \quad (4.49)$$

These can be viewed as characteristic identities for $\hat{C}_-$ and $\hat{C}_+$ respectively restricted to $P_{\text{ad}}(V_{\text{ad}}^\otimes 2)$ and $P^{\text{ad}}_+(V_{\text{ad}}^\otimes 2)$. In this sense, the roots of the characteristic polynomial in the right-hand side of (4.49) are

$$a'_1 = -1, \quad a'_2 = \frac{1}{\omega}, \quad a'_3 = -\frac{1}{\omega}, \quad a'_4 = -\frac{1}{2}.\quad (4.50)$$

To find the characteristic polynomial of $\hat{C}_{\text{ad}}$, we substitute $\hat{C}_{\text{ad}}$ for $\hat{C}_+$ in (4.41) and use $\hat{C}_{\text{ad}} = \hat{C}_+ + \hat{C}_-$ and $\hat{C}_+ \hat{C}_- = 0,$

$$\hat{C}_{\text{ad}}(\hat{C}_{\text{ad}} + I) \left( \hat{C}_{\text{ad}} - \frac{1}{\omega} I \right) \left( \hat{C}_{\text{ad}} + \frac{1}{\omega} I \right) \left( \hat{C}_{\text{ad}} + \frac{1}{2} I \right) =$$

$$= \hat{C}_+ \left( \hat{C}_+ + I \right) \left( \hat{C}_+ - \frac{1}{\omega} I \right) \left( \hat{C}_+ + \frac{1}{\omega} I \right) \left( \hat{C}_+ + \frac{1}{2} I \right) +$$

$$+ \hat{C}_- \left( \hat{C}_- + I \right) \left( \hat{C}_- - \frac{1}{\omega} I \right) \left( \hat{C}_- + \frac{1}{\omega} I \right) \left( \hat{C}_- + \frac{1}{2} I \right) = 0,$$  

(4.51)

where the last relation holds by (4.38) and (4.41). Therefore, the characteristic identity for $\hat{C}_{\text{ad}}$ is given by (4.42).

We note that the roots of the polynomial in the left-hand side of (4.42) coincide with (4.47), and we have degenerate roots for $\omega = 1, 2$:

$$\omega = 1 \implies a_2 = a_4 = -1, \quad \omega = 2 \implies a_4 = a_5 = -\frac{1}{2}.\quad (4.52)$$
For $\omega = 1$, keeping a single parenthesis in the polynomial in the left-hand side of (4.42), the one that correspond to degenerate roots (4.52), we have

$$
\hat{C}_{ad}(\hat{C}_{ad} + I)(\hat{C}_{ad} - I)\left(\hat{C}_{ad} + \frac{1}{2} I\right) = \frac{1}{2} K \neq 0,
$$

(4.53)

Therefore, the $\mathfrak{sl}(M|N)$ representation $\text{ad}^{\otimes 2}$ is not completely reducible for $\omega = 1$. For $\omega = 2$, we similarly obtain

$$
\hat{C}_{ad}(\hat{C}_{ad} + I)\left(\hat{C}_{ad} - \frac{1}{2} I\right)\left(\hat{C}_{ad} + \frac{1}{2} I\right) = \frac{1}{16}(P^{(ad)}_+ + K) - \frac{1}{4} \hat{C}_+^2.
$$

(4.54)

Using the component form of (4.54), it can be verified that for $M = 2$ and $N = 0$, this expression vanishes, while for $M = 3$ and $N = 1$ it does not. Therefore, the representation $\text{ad}^{\otimes 2}$ of $\mathfrak{sl}(M|N)$ for $\omega = 2$ is not completely reducible in general. The exceptional cases $\omega \neq 1, 2$ are to be considered later in this section.

To construct projectors onto invariant subspaces of the $\mathfrak{sl}(M|N)$-representation $\text{ad}^{\otimes 2}$ for $\omega \equiv M - N \neq 1, 2$, we use the fact that for an arbitrary Lie superalgebra, the symmetric and antisymmetric parts of $V^{\otimes 2}_{\text{ad}}$ are invariant (see Sec. 2.3 and Eq. (2.45)). The symmetric invariant subspaces of $V^{\otimes 2}_{\text{ad}}$ can be expressed as eigenspaces of $\hat{C}_+$, which can be viewed as acting on $P^{(ad)}_+ (V^{\otimes 2}_{\text{ad}})$. The role of the identity operator is here played by $P^{(ad)}_+$. By (2.39), where we set $p = 4$, $\hat{C}_T = \hat{C}_+$, and $I^{\otimes 2}_T = P^{(ad)}_+$ and where the $a_i$ are given in (4.50), we have

$$
P^{(+)\dagger}_{a_1} = \frac{1}{\omega^2 - 1} K,
$$

$$
P^{(+)\dagger}_{a_2} = -\frac{\omega}{(\omega + 1)(\omega + 2)} K + \frac{\omega^2}{\omega + 2} \hat{C}_+^2 + \frac{\omega}{2(\omega + 2)} P^{(ad)}_+,
$$

$$
P^{(+)\dagger}_{a_3} = \frac{\omega}{2(\omega - 1)(\omega - 2)} K - \frac{\omega^2}{\omega - 2} \hat{C}_+^2 - \frac{\omega}{2} \hat{C}_+ + \frac{\omega}{2(\omega - 2)} P^{(ad)}_+,
$$

$$
P^{(+)\dagger}_{a_4} = \frac{4}{\omega^2 - 4} (\omega^2 \hat{C}_+^2 - P^{(ad)}_+ - K).
$$

(4.55)

It can be easily verified that $P^{(+)\dagger}_{a_1} + P^{(+)\dagger}_{a_2} + P^{(+)\dagger}_{a_3} + P^{(+)\dagger}_{a_4} = P^{(ad)}_+$.

For $\omega = 1$, the characteristic identity for $\hat{C}_+$ is

$$
(\hat{C}_+ + I)^2(\hat{C}_+ - I)\left(\hat{C}_+ + \frac{1}{2} I\right)P^{(ad)}_+ = 0.
$$

(4.56)

Hence, we must set $a_1' = -1$, $a_2' = 1$, $a_3' = -1/2$, $k_1 = 2$, and $k_2 = k_3 = 1$ in (2.40). We build projectors onto symmetric generalized eigenspaces of $\hat{C}_+$ similarly to the general case considered above. By (2.41), where we substitute $P^{(ad)}_+$ for $I^{\otimes 2}_T$,

$$
P^{(+)\dagger}_{a_1} = \frac{1}{2} P^{(ad)}_+ - \frac{5}{4} K - \frac{1}{2} \hat{C}_+ + \hat{C}_+^2,
$$

$$
P^{(+)\dagger}_{a_2} = \frac{1}{6} P^{(ad)}_+ - \frac{1}{12} K + \frac{1}{2} \hat{C}_+ + \frac{1}{3} \hat{C}_+^2,
$$

$$
P^{(+)\dagger}_{a_3} = \frac{4}{3} P^{(ad)}_+ + \frac{4}{3} K - \frac{4}{3} \hat{C}_+^2.
$$

(4.57)

The operators $P^{(+)\dagger}_{a_2}$ and $P^{(+)\dagger}_{a_3}$ project onto the eigenspaces of $\hat{C}_+$, and $P^{(+)\dagger}_{a_1}$ extracts a generalized eigenspace. Thus, the $\mathfrak{sl}(M|N)$ action on $P^{(+)\dagger}_{a_1}(V^{\otimes 2}_{\text{ad}})$ for $\omega \equiv M - N = 1$ is reducible but not completely reducible. This is related to the fact $K^2 = 0$ for $\omega = 1$; being nilpotent, $K$ is not diagonalizable.
For $\omega = 2$, the characteristic identity for $\hat{C}_+$ is

$$
(\hat{C}_+ + I)\left(\hat{C}_+ - \frac{1}{2}I\right)\left(\hat{C}_+ + \frac{1}{2}I\right)^2 P_+^{(ad)} = 0,
$$

(4.58)

and hence we set $a'_1 = -1, a'_2 = 1/2, a'_3 = -1/2, k_1 = k_2 = 1, $ and $k_3 = 2$ in (2.40). The projectors onto the generalized eigenspaces of $\hat{C}_+$ are given by (2.41)

$$
P_{a'_1}^{(+)} = P_1^{(+)} = \frac{3}{4}K, \quad P_{a'_2}^{(+)} = P_2^{(+)} = \frac{1}{4}P_+^{(ad)} - \frac{1}{12}K + \hat{C}_+ + \hat{C}_+^2,
$$

$$
P_{a'_3}^{(+)} = P_3^{(+)} = \frac{3}{4}P_+^{(ad)} - \frac{1}{4}K - \hat{C}_+ - \hat{C}_+^2.
$$

(4.59)

The operators $P_1^{(+)}$ and $P_2^{(+)}$ project onto the eigenspaces of $\hat{C}_+$, and the image of $P_3^{(+)}$ is a generalized eigenspace of $\hat{C}_+$. Thus, the restriction of the representation $\text{ad}^{\otimes 2}$ to $F_3^{(+)}(V_{\text{ad}}^{\otimes 2})$ is neither irreducible nor completely reducible for $\omega \equiv M - N = 2$. For convenience in what follows, we set $P_4^{(+)} = 0$ for $\omega = 1, 2$.

In the $\text{ad}^{\otimes 2}$ representation of $\mathfrak{sl}(M|N)$, we use (2.39) to find projectors onto antisymmetric invariant subspaces that can be expressed as eigenspaces of $\hat{C}_-$ (which is viewed here as acting on $P_-^{(\text{ad})}(V_{\text{ad}}^{\otimes 2})$, where the role of the identity operator is played by $P_-^{(\text{ad})}$); in accordance with (4.48), we then set $p = 2, \hat{C}_T = \hat{C}_-, I_T^{\otimes 2} = P_-^{(\text{ad})}, a_1 = 0, $ and $a_2 = 1/2$:

$$
P_{a_1}^{(-)} = P_1^{(-)} = 2\hat{C}_- + P_-^{(\text{ad})}, \quad P_{a_2}^{(-)} = P_2^{(-)} = -2\hat{C}_-.
$$

(4.60)

Evidently, $P_1^{(-)} + P_2^{(-)} = P_-^{(\text{ad})}$, i.e.,

$$
P_1^{(-)} + P_2^{(-)} + P_3^{(+)} + P_4^{(+)} = P_-^{(\text{ad})} + P_+^{(\text{ad})} = I.
$$

As a result, we have the full system of projectors given by $P_1^{(-)}, P_2^{(-)}, P_3^{(+)}, P_4^{(+)}$. However, not all of these projectors are primitive. To show this, we use the following relation, which holds for the $\mathfrak{sl}(M|N)$-generators $T_{ij}$ in the defining representation $T_f$ (we set $T_f(T_{ij}) = T_{ij}$):

$$
T_{ij}T_{km} + (-1)^{(i[j][j][k][m]}T_{km}T_{ij} \equiv [T_{ij}, T_{km}]_+ = D^{rs}_{ij,km}T_{rs} + \alpha g_{ij,km}.
$$

(4.61)

Here,

$$
\alpha = \frac{2c_2(T_f)}{\text{sdim}(\mathfrak{sl}(M|N))} = \frac{1}{\omega^2},
$$

where $c_2(T_f) = (\omega^2 - 1)/2\omega^2$ is the value of the quadratic Casimir operator (2.17) in the defining representation $T_f$, and the numbers $D^{rs}_{ij,km}$ are called the $\mathfrak{sl}(M|N)$-structure constants symmetric with respect to the pairs of indices $(ij)$ and $(km)$. We define the operators $\hat{C}$ and $\hat{C}_-$ that act on $V_{\text{ad}}^{\otimes 2}$ as

$$
\hat{C}^{ij_1i_2i_3i_4}_{j_1j_2j_3j_4} = (-1)^{(i_1+j_2)(i_2+j_2)}g^{a_1a_2,h_1b_2}X^{i_1i_2}_{a_1a_2,j_1j_2}D^{i_3i_4}_{b_1b_2,j_3j_4},
$$

$$
\hat{C}_- = \frac{\omega}{4}(I - P_-^{(ad)})\hat{C}(I - P_-^{(ad)}),
$$

(4.62)

where $X^{i_1i_2}_{a_1a_2,j_1j_2}$ are the structure constants of $\mathfrak{sl}(M|N)$ given in (4.11) and $g^{a_1a_2,h_1b_2}$ is the inverse Cartan–Killing metric (4.15).
The images of $P$.

Besides, $\tilde{C}_-$ satisfies

\[
P^{(ad)} \tilde{C}_- \tilde{C}_- = 0, \quad \tilde{C}_- \tilde{C}_- = \tilde{C}_- \tilde{C}_- = 0,
\]

\[
\tilde{C}_- = 2\tilde{C}_- + P^{(ad)}_C, \quad \tilde{C}_-(\tilde{C}_- + 1)(\tilde{C}_--1) = 0.
\]

The last identity in the second row in (4.64) is the characteristic identity for $\tilde{C}_-$.

**Proof.** Direct calculations show that the symmetric structure constants $D^s_{ij,km}$ of $sl(M|N)$ satisfy the relations

\[
D^s_{ij,km} = \left(\delta^s_i \delta^s_j - \frac{1}{\omega} \kappa^s_{ijn} \right) D_{ij,km} l_n.
\]

Using (4.62) and the equalities $(I-P^{(ad)}) = (I-P) I_{12} I_{34}$ and $P = P_{13} P_{24}$, we obtain (4.63). Relation (4.64) can be verified by direct calculations using (4.34), (4.36), and (4.63).

Because $\tilde{C}_- = \tilde{C}_- P^{(ad)} = P^{(ad)}_C \tilde{C}_-$, the last equality in (4.64) can be rewritten as

\[
\tilde{C}_-(\tilde{C}_- + 1)(\tilde{C}_--1)P^{(ad)}_C = 0.
\]

It is the characteristic identity for $\tilde{C}_-$ restricted to the antisymmetric part $P^{(ad)}_C(V_{ad}^\otimes 2)$ of $V_{ad}^\otimes 2$. Using (4.67) and (2.39), we then obtain projectors onto the eigenspaces of $\tilde{C}_-$. The explicit formulas are

\[
\tilde{P}_0^{(-)} = -2\tilde{C}_-, \quad \tilde{P}_1^{(-)} = \tilde{C}_- + \frac{1}{2} P^{(ad)}_C - \frac{1}{2} \tilde{C}_-, \quad \tilde{P}_1^{(-)} = \tilde{C}_- + \frac{1}{2} P^{(ad)}_C + \frac{1}{2} \tilde{C}_-.
\]

In (4.68), the lower indices of the projectors equal the eigenvalues of $\tilde{C}_-$ on the corresponding eigenspaces.

We note that $\tilde{P}_1^{(-)} + \tilde{P}_1^{(-)} = \tilde{P}_1^{(-)}$, where $\tilde{P}_1^{(-)}$ is given in (4.60), and $\tilde{P}_0^{(-)} = P_2^{(-)}$, where $P_2^{(-)}$ is defined in (4.60). Thus, we have the following full system of mutually orthogonal projectors for $\omega \neq 0, 1, 2$:

\[
\tilde{P}_1^{(-)} = \tilde{C}_- + \frac{1}{2} P^{(ad)}_C - \frac{1}{2} \tilde{C}_-, \quad \tilde{P}_1^{(-)} = \tilde{C}_- + \frac{1}{2} P^{(ad)}_C + \frac{1}{2} \tilde{C}_-, \quad \tilde{P}_2^{(-)} = -2\tilde{C}_-, \quad \tilde{P}_1^{(+)}, \quad \tilde{P}_2^{(+)}, \quad \tilde{P}_3^{(+)}, \quad \tilde{P}_4^{(+)},
\]

The images of $\tilde{P}_1^{(-)}$ and $\tilde{P}_2^{(-)}$ lie within the antisymmetric part $P^{(ad)}_C(V_{ad}^\otimes 2)$ of the space $V_{ad}^\otimes 2$, while the images of $\tilde{P}_i^{(+)}$, $i = 2, \ldots, 5$, belong to its symmetric part $P^{(ad)}_+(V_{ad}^\otimes 2)$.
For $\omega = 1, 2$, the projectors are given by (the left and right columns respectively correspond to $\omega = 1$ and $\omega = 2$)

\[
\begin{align*}
\tilde{P}_{-1}^{(-)} &= \tilde{C}_- + \frac{1}{2} P^{(ad)} - \frac{1}{2} \tilde{C}_-, \\
\tilde{P}_1^{(-)} &= \tilde{C}_- + \frac{1}{2} P^{(ad)} + \frac{1}{2} \tilde{C}_-, \\
P_2^{(-)} &= -2 \tilde{C}_-, \\
P_1^{(+)} &= -\frac{1}{2} P_+^{(ad)} - \frac{5}{4} K - \frac{1}{2} \tilde{C}_+ + \tilde{C}_+^2, \\
P_2^{(+)} &= \frac{1}{6} P_+^{(ad)} - \frac{1}{12} K + \frac{1}{2} \tilde{C}_+ + \frac{1}{3} \tilde{C}_+^2, \\
P_3^{(+)} &= \frac{4}{3} P_+^{(ad)} + \frac{4}{3} K - \frac{4}{3} \tilde{C}_+^2,
\end{align*}
\]

To find the dimensions of the invariant subspaces extracted by projectors (4.69), we calculate the traces and supertraces of those projectors. First, we compute some auxiliary traces and supertraces (where we recall that $\xi = M + N$):

\[
\begin{align*}
\text{tr } I &= (\xi^2 - 1)^2, \\
\text{tr } P_+^{(ad)} &= \frac{1}{2} (\xi^2 - 1)^2 + \frac{1}{2} (\omega^2 - 1), \\
\text{tr } P_-^{(ad)} &= \frac{1}{2} (\xi^2 - 1)^2 - \frac{1}{2} (\omega^2 - 1), \\
\text{tr } K &= \omega^2 - 1, \\
\text{tr } \tilde{C}_+ &= \frac{1}{2} (\xi^2 - 1), \\
\text{tr } \tilde{C}_+^2 &= \frac{\xi^4}{2 \omega^2} + \frac{\xi^2}{4} - 2 \frac{\xi^2}{\omega^2} + \frac{5}{4}, \\
\text{tr } \tilde{C}_- &= -\frac{1}{2} (\xi^2 - 1), \\
\text{tr } \tilde{C}_- &= 0,
\end{align*}
\]

Using (4.69) and (4.70), we obtain the traces

\[
\begin{align*}
\text{tr } \tilde{P}_{-1}^{(-)} &= \frac{1}{4} ((\xi^2 - 2)^2 - \omega^2), \\
\text{tr } \tilde{P}_1^{(-)} &= \frac{1}{4} ((\xi^2 - 2)^2 - \omega^2), \\
\text{tr } P_2^{(-)} &= \xi^2 - 1, \\
\text{tr } P_1^{(+)} &= 1, \\
\text{tr } P_2^{(+)} &= \frac{1}{4} ((\xi^2 - 1)^2 + 2(\xi^2 + 1)(\omega - 1) + (\omega - 1)^2), \\
\text{tr } P_3^{(+)} &= \frac{1}{4} ((\xi^2 - 1)^2 - 2(\xi^2 + 1)(\omega + 1) + (\omega + 1)^2), \\
\text{tr } P_4^{(+)} &= \xi^2 - 1
\end{align*}
\]

and supertraces of projectors (4.69)

\[
\begin{align*}
\text{str } \tilde{P}_{-1}^{(-)} &= \frac{1}{4} (\omega^2 - 1)(\omega^2 - 4), \\
\text{str } \tilde{P}_1^{(-)} &= \frac{1}{7} (\omega^2 - 1)(\omega^2 - 4), \\
\text{str } P_2^{(-)} &= \omega^2 - 1, \\
\text{str } P_1^{(+)} &= 1, \\
\text{str } P_2^{(+)} &= \frac{1}{4} \omega^2 (\omega - 1)(\omega + 3), \\
\text{str } P_3^{(+)} &= \frac{1}{4} \omega^2 (\omega + 1)(\omega - 3), \\
\text{str } P_4^{(+)} &= \omega^2 - 1
\end{align*}
\]

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for $\omega \neq 0, 1, 2$. Similarly to the $osp(M|N)$ case, we find the dimensions of the even and odd parts of the invariant subspaces:

$$\dim_{\bar{V}} V_{1}^{(-)} = \frac{1}{4}(M^2 - 1)(M^2 - 4) + \frac{1}{4}(N^2 - 1)(N^2 - 4) + \frac{1}{2}(MN + 1)(3MN - 2),$$

$$\dim_{\bar{V}} V_{1}^{(+)} = \frac{1}{4}(M^2 - 1)(M^2 - 4) + \frac{1}{2}(N^2 - 1)(N^2 - 4) + \frac{1}{2}(MN + 1)(3MN - 2),$$

$$\dim_{\bar{V}} V_{2}^{(-)} = M^2 + N^2 - 1,$$

$$\dim_{\bar{V}} V_{2}^{(+)} = 1,$$

$$\dim_{\bar{V}} V_{3}^{(-)} = \frac{1}{4}M^2(M - 1)(M + 3) + \frac{1}{4}N^2(N + 1)(N - 3) + \frac{1}{2}MN(3MN - M + N - 1),$$

$$\dim_{\bar{V}} V_{3}^{(+)} = \frac{1}{4}M^2(M + 1)(M - 3) + \frac{1}{4}N^2(N - 1)(N + 3) + \frac{1}{2}MN(3MN + M - N - 1),$$

$$\dim_{\bar{V}} V_{4}^{(-)} = M^2 + N^2 - 1$$

and

$$\dim_{\bar{V}} \bar{V}_{1}^{(-)} = MN(M^2 + N^2 - 2),$$

$$\dim_{\bar{V}} \bar{V}_{1}^{(+)} = 0,$$

$$\dim_{\bar{V}} \bar{V}_{2}^{(-)} = MN(M^2 + N^2 - 2),$$

$$\dim_{\bar{V}} \bar{V}_{2}^{(+)} = MN(M + 1) + N(N - 1) - 2),$$

$$\dim_{\bar{V}} \bar{V}_{3}^{(-)} = 2MN,$$

$$\dim_{\bar{V}} \bar{V}_{3}^{(+)} = MN(M - 1) + N(N + 1) - 2),$$

$$\dim_{\bar{V}} \bar{V}_{4}^{(+)} = 2MN.$$

We note that after the substitutions $M = 0$ and $N = 0$, the dimensions of the odd parts of $\text{ad}^{\otimes 2}$-invariant subspaces of $\mathfrak{sl}(M|N)$ vanish, as they should. Besides, the mentioned substitutions turn (4.73) into the corresponding expressions for the dimensions of the invariant subspaces of the $\mathfrak{sl}(N)$ (or $\mathfrak{sl}(M)$) Lie algebra, which are given in [15].

5. Universal characteristic identities for $\hat{C}_+$ for the $osp(M|N)$ and $\mathfrak{sl}(M|N)$ Lie superalgebras

For the $osp(M|N)$ and $\mathfrak{sl}(M|N)$ Lie superalgebras (which are denoted by $\mathfrak{g}$ in this section), characteristic identities (3.63) and (4.39) for $\hat{C}_+$ in the adjoint representation can be written in the general form:

$$\hat{C}_+^3 + \frac{1}{2}\hat{C}_+^2 = \mu_1 \hat{C}_+ + \mu_2 (I + \text{P}^{(\text{ad})} - 2\mathcal{K}),$$

which coincides precisely with the form of the universal identity for $\hat{C}_+$ in the case of the classical Lie algebras [15]. The parameters $\mu_1$ and $\mu_2$ corresponding to the algebras $osp(M|N)$ and $\mathfrak{sl}(M|N)$ are given in Table 1.

The subsequent analysis of (5.1) mostly follows the treatment of an analogous identity for the classical Lie algebras in [15]. Multiplying both sides of (5.1) by $\mathcal{K}$ and using

$$\mathcal{K}(I + \text{P}^{(\text{ad})}) = 2\mathcal{K}, \quad \mathcal{K} \hat{C}_+ = -\mathcal{K}, \quad \mathcal{K} \cdot \mathcal{K} = \text{sdim}(\mathfrak{g})\mathcal{K},$$

we can express the superdimension of $\mathfrak{g}$ as a function of $\mu_1$ and $\mu_2$:

$$\text{sdim} \mathfrak{g} = \frac{2\mu_2 - \mu_1 + 1/2}{2\mu_2}.$$

$$\text{(5.3)}$$
Next, multiplying both sides of (5.1) by \( \tilde{C}_+ (\tilde{C}_+ + 1) \), we obtain the characteristic identity for \( \tilde{C}_+ \):\[
\tilde{C}_+ (\tilde{C}_+ + 1) \left( \tilde{C}_+^2 + \frac{\alpha}{2t} \tilde{C}_+ - \mu_1 \tilde{C}_+ - 2 \mu_2 I \right) = 0, \tag{5.4}
\]
which in the factored form becomes\[
\tilde{C}_+ (\tilde{C}_+ + 1) \left( \tilde{C}_+^2 + \frac{\alpha}{2t} \tilde{C}_+ + \frac{\beta}{2t} I \right) \left( \tilde{C}_+^2 + \frac{\gamma}{2t} I \right) = 0. \tag{5.5}
\]
Thus, the roots of the polynomial in the left-hand side of (5.4) are\[
a_1 = 0, \quad a_2 = -1, \quad a_3 = -\frac{\alpha}{2t}, \quad a_4 = -\frac{\beta}{2t}, \quad a_5 = -\frac{\gamma}{2t}, \tag{5.6}
\]
where the normalization parameter is \( t = \alpha + \beta + \gamma \), because, by (5.4) and (5.5),\[
\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t} = \frac{1}{2}. \tag{5.7}
\]
We choose \( t = h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( g \). The values of \( h^\vee \) (see, e.g., [29]) for \( sl(M|N) \) and \( osp(M|N) \) are presented in Table 2. As usual, \( \omega = 2m + 1 - N \) and \( \omega = 2m - N \) for \( g = osp(2m + 1|N) \) and \( g = osp(2m, N) \).

### Table 2. The dual Coxeter numbers for the Lie superalgebras \( sl(M|N) \) and \( osp(M|N) \)

|         | \( sl(M|N) \) | \( osp(2m + 1|N), \omega > 1 \) | \( osp(2m + 1|N), \omega \leq 1 \) | \( osp(2m|N), \omega > 0 \) | \( osp(2m|N), \omega \leq 0 \) | \( -(\omega - 2)/2 \) |
|---------|----------------|----------------------------------|----------------------------------|----------------|----------------|----------------|
| \( h^\vee \) | \( \omega \) | \( \omega - 2 \) | \( -1(\omega - 2)/2 \) | \( -1(\omega - 2)/2 \) | \( -1(\omega - 2)/2 \) | \( -1(\omega - 2)/2 \) |

The parameters \( \alpha, \beta, \) and \( \gamma \) were introduced by Vogel in [7]. The values of these parameters for \( osp(M|N) \) and \( sl(M|N) \) can be deduced from (3.65) and (4.41) and are given in Table 3.

A comparison of (5.4) and (5.5) shows that \( \mu_1 \) and \( \mu_2 \) can be expressed in terms of the Vogel parameters,\[
\mu_1 = -\frac{\alpha \beta + \alpha \gamma + \beta \gamma}{4t^2}, \quad \mu_2 = \frac{\alpha \beta \gamma}{16t^3}, \tag{5.8}
\]
while the superdimension (5.3) of \( g \) acquires the universal form\[
\text{sdim} g = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha \beta \gamma}. \tag{5.9}
\]
From (5.9) and (5.12), we obtain the superdimensions of the invariant subspaces $V_{(a_i)}$ of the symmetric space $P_+(V^\otimes 2)$,

$$P^{(+)}_{(-\alpha/2t)} = P^{(+)}(\alpha|\beta, \gamma), \quad P^{(+)}_{(-\beta/2t)} = P^{(+)}(\beta|\alpha, \gamma), \quad P^{(+)}_{(-\gamma/2t)} = P^{(+)}(\gamma|\alpha, \beta),$$

(5.10)

where we use the notation

$$P^{(+)}(\alpha|\beta, \gamma) = \frac{4t^2}{(\beta - \alpha)(\gamma - \alpha)} \left( \tilde{C}_+^2 + \left(1 - \frac{\alpha}{2t}\right) \tilde{C}_+ + \frac{\beta\gamma}{8t^2} \left( I + P^{(ad)} - \frac{2\alpha}{\alpha - 2t} K \right) \right).$$

(5.11)

By (2.56) and (5.9), the supertrace of $P^{(+)}(\alpha|\beta, \gamma)$ is

$$\text{str} P^{(+)}(\alpha|\beta, \gamma) = -\frac{(3\alpha - 2t)(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)t}{\alpha^2(\alpha - \beta)(\alpha - \gamma)\beta\gamma}.\quad (5.12)$$

From (5.9) and (5.12), we obtain the superdimensions of the invariant subspaces $V_{(-1)}$, $V_{(-\alpha/2t)}$, $V_{(-\beta/2t)}$, and $V_{(-\gamma/2t)}$ extracted by projectors (5.10):

$$\text{sdim} V_{(-1)} = \text{str} P^{(+)}_{(-1)} = 1,$n
$$\text{sdim} V_{(-\alpha/2t)} = \text{str} P^{(+)}_{(-\alpha/2t)} = -\frac{(3\alpha - 2t)(\beta - 2t)(\gamma - 2t)(\beta + t)(\gamma + t)t}{\alpha^2(\alpha - \beta)(\alpha - \gamma)\beta\gamma},$$

(5.13)

$$\text{sdim} V_{(-\beta/2t)} = \text{str} P^{(+)}_{(-\beta/2t)} = -\frac{(3\beta - 2t)(\alpha - 2t)(\gamma - 2t)(\alpha + t)(\gamma + t)t}{\beta^2(\beta - \alpha)(\beta - \gamma)\alpha\gamma},$$

$$\text{sdim} V_{(-\gamma/2t)} = \text{str} P^{(+)}_{(-\gamma/2t)} = -\frac{(3\gamma - 2t)(\beta - 2t)(\alpha - 2t)(\beta + t)(\alpha + t)t}{\gamma^2(\gamma - \beta)(\gamma - \alpha)\beta\alpha}.$$

It is worth noting that for Lie algebras, the vanishing of either $3\alpha - 2t$ or $3\beta - 2t$, or $3\gamma - 2t$ corresponds to the exceptional Lie algebras $g_2$, $f_4$, $e_6$, $e_7$, $e_8$, as well as $sl(3)$ and $so(8)$. In this sense, the “exceptional” basic classical Lie superalgebras (see their definition, e.g., in [24]) are $sl(M|N)$ for $M - N = 0, \pm 3$, $osp(M|N)$ for $M - N = -1, 8$, and $F(4)$.

6. Eigenvalues of the higher Casimir operators in the adjoint representation

In this section, we derive a formula for the eigenvalues of the higher Casimir operators in the adjoint representation in terms of the Vogel parameters. Our method for constructing higher Casimir operators for Lie superalgebras is based on the method proposed in [1] for Lie algebras (see also [30]).
Let \( \{ Y_A \} \) be a homogeneous basis of the enveloping algebra \( \mathcal{U}(g) \) of a Lie superalgebra \( g \). If \( \tilde{C} = D^{AB} Y_A \otimes Y_B \in \mathcal{U}(g) \otimes \mathcal{U}(g) \) is an ad-invariant operator, then for an arbitrary \( g \)-representation \( T: g \to (\text{End}(V))_L \), the operator
\[
C = \text{str}_2((\text{id} \otimes T) \tilde{C}) = D^{AB} Y_A \text{str}(T(Y_B))
\] (6.1)
lies in the center of \( \mathcal{U}(g) \). Here, \( \text{id} \) is the identity operator and \( \text{str}_2 \) denotes the supertrace in the second factor in \( \mathcal{U}(g) \otimes (\text{End}(V))_L \). In what follows, we are only interested in the operator \( C \) in a particular representation \( T' \). For \( T'(C) \), we have
\[
T'(C) = \text{str}_2((T' \otimes T) \tilde{C}) = D^{AB} T'(Y_A) \text{str}(T(Y_B)).
\] (6.2)

Evidently, for the simple Lie superalgebra \( g \) with \( \dim(g) = n \) and with a nondegenerate Cartan–Killing metric \( g_{ab} \), an arbitrary power \( \tilde{C}^k \) of \( \tilde{C} \), defined in (2.34), is ad-invariant. The explicit form of \( \tilde{C}^k \) is
\[
\tilde{C}^k = (-1)^{\sum_{i,j} |a_i| |a_j|} \tilde{g}^{a_1 b_1} \tilde{g}^{a_2 b_2} \cdots \tilde{g}^{a_n b_n} X_{a_1} \cdots X_{a_n} \otimes X_{b_1} \cdots X_{b_n}.
\] (6.3)
Substituting \( \tilde{C} = \tilde{C}^k \) and \( T' = T = \text{ad} \) in (6.2) yields a relation for the \( k \)th Casimir operator \( \text{ad}(C_k) \) in the adjoint representation,
\[
\text{ad}(C_k) = \text{str}_2(\text{ad}^{\otimes 2}(\tilde{C}^k)) = \text{str}_2(\tilde{C}_{ad}^k) = \tilde{g}^{a_1 \cdots a_n} \text{ad}(X_{a_1}) \cdots \text{ad}(X_{a_n}),
\] (6.4)
where we use the notation
\[
\tilde{g}^{a_1 \cdots a_2} = (-1)^{\sum_{i,j} |a_i| |a_j|} \tilde{g}^{a_1 b_1} \tilde{g}^{a_2 b_2} \cdots \tilde{g}^{a_n b_n} \text{str}(\text{ad}(X_{b_1}) \cdots \text{ad}(X_{b_n})).
\] (6.5)
and \( \tilde{C}_{ad} = \text{ad}^{\otimes 2}(\tilde{C}) \). Because the adjoint representation of a simple Lie superalgebra is irreducible, it follows from Schur’s lemma that \( \text{ad}(C_k) \) is a scalar operator with the eigenvalue \( c_k \), i.e., \( \text{ad}(C_k) = c_k I \), where \( I \) is the identity operator on \( V_{ad} \).

We introduce the generating function for \( c_k \) as
\[
c(z) = \sum_{p=0}^{\infty} c_p z^p.
\] (6.6)
By (6.4) and (2.51),
\[
c(z) \cdot I = \text{str}_2 \left( \sum_{p=0}^{\infty} \tilde{C}_{ad}^p z^p \right) = \text{str}_2 \left( \sum_{p=0}^{\infty} \tilde{C}_+^p z^p \right) + \text{str}_2 \left( \sum_{p=0}^{\infty} \tilde{C}_-^p z^p \right),
\] (6.7)
where we define \( \tilde{C}_{ad}^0 = P_{ad}^0 \) and \( \tilde{C}_{ad}^0 = I \). By (2.54), \( \tilde{C}_+^p = (-1/2)^{p-1} \tilde{C}_- \), whence
\[
\sum_{p=0}^{\infty} \tilde{C}_+^p z^p = P_{ad}^+ + \sum_{p=1}^{\infty} \left( -\frac{1}{2} \right)^{p-1} \tilde{C}_- z^p = P_{ad}^+ + \frac{z}{1+z/2} \tilde{C}_-.
\] (6.8)
We now express \( \tilde{C}_{ad}^p \) in terms of \( P^+(\alpha|\beta, \gamma), P^+(\beta|\gamma, \alpha), P^+(\gamma|\alpha, \beta), \) and \( P^-(\alpha|\beta, \gamma) \) that were defined in (5.10) and (5.11). Using the condition
\[
P_{ad}^+ = P^+(\alpha|\beta, \gamma) + P^+(\beta|\gamma, \alpha) + P^+(\gamma|\alpha, \beta) + P^-(\alpha|\beta, \gamma),
\] (6.9)
we obtain

\[ \tilde{C}_+^p = \tilde{C}_+^p (P^{(+)}(\alpha|\beta, \gamma) + P^{(+)}(\beta|\gamma, \alpha) + P^{(+)}(\gamma|\alpha, \beta) + P^{(+)}_{(-1)}) = \]

\[ = \left( -\frac{\alpha}{2t} \right)^P P^{(+)}(\alpha|\beta, \gamma) + \left( -\frac{\beta}{2t} \right)^P P^{(+)}(\beta|\gamma, \alpha) + \]

\[ + \left( -\frac{\gamma}{2t} \right)^P P^{(+)}(\gamma|\alpha, \beta) + (-1)^P P^{(+)}_{(-1)}, \]

where \((-\alpha/2t), (-\beta/2t), (-\gamma/2t)\) and \((-1)\) are the eigenvalues of \(\tilde{C}_+\) corresponding to the projectors mentioned. Therefore,

\[ \sum_{p=0}^{\infty} \tilde{C}_+^p z^p = \sum_{p=0}^{\infty} \left( -\frac{\alpha z}{2t + \alpha z} P^{(+)}(\alpha|\beta, \gamma) - \frac{\beta z}{2t + \beta z} P^{(+)}(\beta|\gamma, \alpha) - \right. \]

\[ \left. - \frac{\gamma z}{2t + \gamma z} P^{(+)}(\gamma|\alpha, \beta) - \frac{z}{1 + z} P^{(+)}_{(-1)} + P^{(ad)} \right). \] (6.10)

By (6.9), Eq. (6.10) can be rewritten as

\[ \sum_{p=0}^{\infty} \tilde{C}_+^p z^p = -\frac{\alpha z}{2t + \alpha z} P^{(+)}(\alpha|\beta, \gamma) - \frac{\beta z}{2t + \beta z} P^{(+)}(\beta|\gamma, \alpha) - \]

\[ - \frac{\gamma z}{2t + \gamma z} P^{(+)}(\gamma|\alpha, \beta) - \frac{z}{1 + z} P^{(+)}_{(-1)} + P^{(ad)}. \] (6.11)

Adding (6.8) and (6.11) leads to

\[ \sum_{p=0}^{\infty} \tilde{C}_+^p z^p = -\frac{\alpha z}{2t + \alpha z} P^{(+)}(\alpha|\beta, \gamma) - \frac{\beta z}{2t + \beta z} P^{(+)}(\beta|\gamma, \alpha) - \]

\[ - \frac{\gamma z}{2t + \gamma z} P^{(+)}(\gamma|\alpha, \beta) - \frac{z}{1 + z} P^{(+)}_{(-1)} + \frac{2z}{2 + z} \tilde{C}_- + I. \] (6.12)

To find \(c(z)\) from (6.7), we need to calculate the supertrace \(\text{str}_2\) of the right-hand side of (6.12). Using (2.14), (2.43), (2.44), and (2.49), we obtain the auxiliary supertraces

\[ \text{str}_2(I) = s\dim \mathfrak{g} \cdot I, \quad \text{str}_2(\tilde{C}_-) = -\frac{1}{2} I, \quad \text{str}_2(P^{(ad)}) = I, \]

\[ \text{str}_2(\tilde{C}_+) = \frac{1}{2} I, \quad \text{str}_2(K) = I, \quad \text{str}_2(C^2) = \frac{3}{4} I, \] (6.13)

which agree with (2.56). Next, by (6.13) and (5.11), we have

\[ \text{str}_2 P^{(+)}(\alpha|\beta, \gamma) = \frac{(3\alpha - 2t)(\beta + t)(\gamma + t)}{\alpha(\alpha - \beta)(\alpha - \gamma)(\alpha - 2t)} I. \] (6.14)

Using (5.9), (5.10), and (6.13), we see that

\[ \text{str}_2 P^{(+)}_{(-1)} = \frac{1}{s\dim \mathfrak{g}} \text{str}_2 K = \frac{\alpha \beta \gamma}{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)} I. \] (6.15)
Substituting (6.12) in (6.7) and using Eqs. (6.13)–(6.15) results in

\[ \sum_{p=0}^{\infty} C_{\text{ad}} z^p = \sum_{p=0}^{\infty} c_p z^p \cdot I = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha \beta \gamma} \cdot I + z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)} \cdot I, \]

(6.16)

where \( t_2 = \alpha^2 + \beta^2 + \gamma^2 \) and \( t_3 = \alpha^3 + \beta^3 + \gamma^3 \). Therefore, the generating function (6.6) for the eigenvalues of the higher Casimir operators for \( \text{osp}(M|N) \) and \( \text{sl}(M|N) \) in the adjoint representation is

\[ c(z) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha \beta \gamma} + z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}. \]

(6.17)

Formula (6.17) agrees with the results in [10], where this expression was found by using a formula for the values of the higher Casimir operators obtained in [1].

7. Conclusion

We have found explicit formulas for the projectors onto the invariant subspaces of the tensor product of two adjoint representations of the Lie superalgebras \( \text{osp}(M|N) \) with \( M - N \neq 0, 1, 2 \) and \( \text{sl}(M|N) \) with \( M - N \neq 0, \pm 1, \pm 2 \). This was done by finding the characteristic identities for the split Casimir operator of the corresponding algebras. In the case of the \( \text{sl}(M|N) \) Lie superalgebras, an additional ad-invariant operator was defined by means of the so-called symmetric structure constants of \( \text{sl}(M|N) \). It was also shown that the dimensions of the invariant subspaces and the values of the quadratic Casimir operator in those subspaces agree with [7]–[9], where these quantities are written in terms of the Vogel parameters in the context of the Universal Lie algebra. Furthermore, the generating function of the eigenvalues of the higher Casimir operators in the adjoint representation was found and expressed in terms of the Vogel parameters. The last result is in agreement with [10].

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Conflicts of interest. The authors declare no conflicts of interest.

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