LARGE-DEGREE ASYMPTOTICS OF RATIONAL PAINLEVÉ-IV FUNCTIONS
ASSOCIATED TO GENERALIZED HERMITE POLYNOMIALS

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Abstract. The Painlevé-IV equation has three families of rational solutions generated by the generalized Hermite polynomials. Each family is indexed by two positive integers \( m \) and \( n \). These functions have applications to nonlinear wave equations, random matrices, fluid dynamics, and quantum mechanics. Numerical studies suggest the zeros and poles form a deformed \( n \times m \) rectangular grid. Properly scaled, the zeros and poles appear to densely fill certain curvilinear rectangles as \( m, n \to \infty \) with \( r := m/n \) a fixed positive real number. Generalizing a method of Bertola and Bothner [2] used to study rational Painlevé-II functions, we express the generalized Hermite rational Painlevé-IV functions in terms of certain orthogonal polynomials on the unit circle. Using the Deift-Zhou nonlinear steepest-descent method, we asymptotically analyze the associated Riemann-Hilbert problem in the limit \( n \to \infty \) with \( m = r \cdot n \) for \( r \) fixed. We obtain an explicit characterization of the boundary curve and determine the leading-order asymptotic expansion of the functions in the pole-free region.

1. Introduction

Rational solutions of the Painlevé-IV equation
\[
(1-1) \quad w_{yy} = \frac{(w_y)^2}{2w} + \frac{3}{2} w^3 + 4yw^2 + 2(y^2 - \alpha)w + \frac{\beta}{w}, \quad w : \mathbb{C} \to \mathbb{C} \text{ with parameters } \alpha, \beta \in \mathbb{C}
\]
arise in the study of steady-state distributions of electric charges for a two-dimensional Coulomb gas in a parabolic potential [27]; rational solutions of the defocusing nonlinear Schrödinger equation [10], the Boussinesq equation [11], the classical Boussinesq system [12], and the point vortex equations with quadrupole background flow [13]; rational-logarithmic solutions of the dispersive water wave equation and the modified Boussinesq equation [15]; rational extensions of the harmonic oscillator and related exceptional orthogonal polynomials [28, 29]; and the recurrence coefficients for polynomials orthogonal to the weight \( e^{-x^2} |x|^n \) and Gaussian Unitary Ensemble matrices with repeated eigenvalues [6]. The fact that these functions have interesting mathematical properties in their own right is suggested by plots of the zeros and poles. Indeed, as \( \alpha \) and \( \beta \) vary along certain sequences, the zeros and poles (when appropriately scaled) appear to form strikingly regular patterns in the complex plane that densely fill out curvilinear rectangles (for the rational functions that can be expressed in terms of generalized Hermite polynomials; see Figures 1–2) and curvilinear rectangles with equilateral curvilinear triangles attached to the edges (for the rational solutions expressed in terms of generalized Okamoto polynomials) [7]. In this work we explicitly determine the boundary curves for the rational Painlevé-IV functions associated to the generalized Hermite polynomials, and derive the leading-order asymptotic expansions of these rational functions in the exterior of the zero/pole region.

Various other geometric patterns are also seen in the plots of poles and zeros of rational solutions of the Painlevé-II equation and equations in the Painlevé-II hierarchy [14], the Painlevé-III equation [8], systems of the symmetric Painlevé-IV hierarchy [19], and the Painlevé-V equation [9], as well as certain Wronskians of Hermite polynomials that are extensions of the generalized Hermite polynomials and have connections to Young diagrams [18]. Recently, significant progress has been made in understanding the rational solutions of the Painlevé-II equation, which can be indexed by a single integer \( m \). As \( m \to \infty \), appropriately scaled zeros and poles of these rational functions densely fill a region \( T \) bounded by a curvilinear triangle. By analyzing a Riemann-Hilbert problem derived from the Garnier-Jimbo-Miwa Lax pair, the large-\( m \) behavior of these functions (and certain functions arising in the study of critical behavior in the semiclassical sine-Gordon equation whose logarithmic derivatives are the rational Painlevé-II functions [3]) was rigorously calculated with error terms outside \( T \) in terms of elementary functions, inside \( T \) in terms of Riemann theta functions, along edges of \( T \) in terms of trigonometric functions, and at corners of \( T \) in terms of the tritronquée Painlevé-I solution [4, 5]. In a later work, Bertola and Bothner [2] reproduced part of these results, in particular the
Figure 1. The zeros of $H_{m,n}(m^{1/2}x)$ in the complex $x$-plane for $(m,n,r) = (50, 5, 10)$ (left), $(m,n,r) = (40, 10, 4)$ (center), and $(m,n,r) = (30, 15, 2)$ (right), along with the boundary of the elliptic region $E_r$ that depends only on $r = m/n$.

Figure 2. The zeros of $H_{m,n}(m^{1/2}x) = H_{m,n}(n^{1/2}x)$ in the complex $x$-plane for $(m,n,r) = (20, 20, 1)$, along with the boundary of the elliptic region $E_r$ that depends only on $r = m/n$.

Figure 3. The zeros of $H_{m,n}(n^{1/2}x)$ in the complex $x$-plane for $(m,n,r) = (15, 30, \frac{1}{2})$ (left), $(m,n,r) = (10, 40, \frac{1}{2})$ (center), and $(m,n,r) = (5, 50, \frac{1}{10})$ (right), along with the boundary of the elliptic region $E_r$ that depends only on $r = m/n$.

equation for the boundary of $T$ and information about the location of the zeros and poles, by deriving a new determinantal formula for the squares of the associated Yablonskii-Vorob’ev polynomials and applying Riemann-Hilbert analysis to a related family of orthogonal polynomials. Joint with Balogh, they also used their method to obtain the boundary of the zero region for the generalized Yablonskii-Vorob’ev polynomials.
associated to the Painlevé-II hierarchy [1]. Miller and Sheng [30] have recently shown that, for monodromy data corresponding to rational solutions, the Riemann-Hilbert problem associated to the Flaschka-Newell Painlevé-II Lax pair is equivalent to the Riemann-Hilbert problem for orthogonal polynomials studied by Bertola and Bothner.

In this work we use the Bertola-Bothner orthogonal polynomial approach to analyze the rational Painlevé-IV functions associated to the generalized Hermite polynomials. Set

\begin{align}
\alpha_{m,n}^{(I)} &:= 2m + n + 1, & \beta_{m,n}^{(I)} &:= -2n^2, & \mathcal{P}_{m,n}^{(I)} &= \{(\alpha_{m,n}^{(I)}, \beta_{m,n}^{(I)}): m \geq 0, n \geq 1\}, \\
\alpha_{m,n}^{(II)} &:= -(m + 2n + 1), & \beta_{m,n}^{(II)} &:= -2m^2, & \mathcal{P}_{m,n}^{(II)} &= \{(\alpha_{m,n}^{(II)}, \beta_{m,n}^{(II)}): m \geq 1, n \geq 0\}, \\
\alpha_{m,n}^{(III)} &:= n - m, & \beta_{m,n}^{(III)} &:= -2(m + n + 1)^2, & \mathcal{P}_{m,n}^{(III)} &= \{(\alpha_{m,n}^{(III)}, \beta_{m,n}^{(III)}): m, n \in \mathbb{N}_0\}, \\
\alpha_{j,k}^{(Oka)} &:= j, & \beta_{j,k}^{(Oka)} &:= -2(2k - j + \frac{3}{2})^2, & \mathcal{P}_{j,k}^{(Oka)} &= \{(\alpha_{j,k}^{(Oka)}, \beta_{j,k}^{(Oka)}): j, k \in \mathbb{Z}\},
\end{align}

where \(\mathbb{N}_0\) denotes the nonnegative integers. It is known that the Painlevé-IV equation (1-1) has a rational solution if and only if \((\alpha, \beta) \in \mathcal{P}_{-1/2}^{(I)} \cup \mathcal{P}_{-1/2}^{(II)} \cup \mathcal{P}_{-2}^{(III)}\). Furthermore, for fixed \((\alpha, \beta)\) this rational solution is unique when it exists [31, 22, 32]. The families of rational solutions to (1-1) corresponding to \(\mathcal{P}_{-1/2}^{(I)} \cup \mathcal{P}_{-1/2}^{(II)} \cup \mathcal{P}_{-2}^{(III)}\), and \(\mathcal{P}_{j,k}^{(Oka)}\), are referred to as the \(-1/2\), \(-2\), and \(-2/3\) hierarchies, respectively. The rational functions corresponding to \(\mathcal{P}_{j,k}^{(Oka)}\) can be constructed from the generalized Okamoto polynomials. The rational solutions of (1-1) for \((\alpha, \beta) \in \mathcal{P}_{-1/2}^{(I)} \cup \mathcal{P}_{-1/2}^{(II)} \cup \mathcal{P}_{-2}^{(III)}\) can be constructed from generalized Hermite polynomials. We will analyze these rational solutions in the remainder of this work.

The generalized Hermite polynomials \(H_{m,n}(y)\) are defined for \(m, n \in \mathbb{N}_0\) by the recurrence relations

\begin{align}
2mH_{m+1,n}H_{m-1,n} &= H_{m,n}(H_{m,n}'' - (H_{m,n}')^2) + 2mH^2_{m,n}, \\
2nH_{m,n+1}H_{m,n-1} &= -H_{m,n}(H_{m,n}'' + (H_{m,n}')^2) + 2nH^2_{m,n}
\end{align}

and the initial conditions

\begin{align}
H_{0,0} &= H_{1,0} = H_{0,1} = 1, & H_{1,1} &= 2y.
\end{align}

The name arises from the fact that

\begin{align}
H_{m,1}(y) &= H_m(y) \quad \text{and} \quad H_{1,n}(y) = i^{-n}H_n(iy),
\end{align}

where for \(m \in \mathbb{N}_0\), \(H_m(y)\) is the standard Hermite polynomial defined by the generating function

\begin{align}
e^{y^2-s^2} = \sum_{n=0}^{\infty} \frac{H_n(y)s^n}{n!}.
\end{align}

The generalized Hermite polynomials also have the symmetry

\begin{align}
H_{m,n}(iy) &= i^{mn}H_{m,n}(y).
\end{align}

While we will not use them, it is interesting to note that their zeros satisfy various sum relations that generalize the Stieltjes relations for the zeros of Hermite polynomials [25]. The connection to the rational Painlevé-IV functions is that

\begin{align}
w^{(I)}_{m,n} := \frac{d}{dy} \log \left( \frac{H_{m+1,n}(y)}{H_{m,n}(y)} \right)
\end{align}

solves the Painlevé-IV equation (1-1) with parameters \((\alpha, \beta) = (\alpha_{m,n}^{(I)}, \beta_{m,n}^{(I)})\),

\begin{align}
w^{(II)}_{m,n} := \frac{d}{dy} \log \left( \frac{H_{m,n+1}(y)}{H_{m,n}(y)} \right)
\end{align}

solves (1-1) with parameters \((\alpha, \beta) = (\alpha_{m,n}^{(II)}, \beta_{m,n}^{(II)})\), and

\begin{align}
w^{(III)}_{m,n} := -2y + \frac{d}{dy} \log \left( \frac{H_{m,n+1}(y)}{H_{m,n+1}(y)} \right) &= -2y - w^{(I)}_{m,n}(y) - w^{(II)}_{m,n}(y)
\end{align}

solves (1-1) for \((\alpha, \beta) = (\alpha_{m,n}^{(III)}, \beta_{m,n}^{(III)})\).
1.1. Outline and results. Our starting point is the known identity (2-2) expressing the generalized Hermite polynomial $H_{m,n}$ in terms of a Hankel determinant of Hermite polynomials. In Lemma 1 we rewrite this as a Hankel determinant of certain moments (defined in (2-4)) of a measure supported on the unit circle. This establishes a connection to the associated orthogonal polynomials on the unit circle (see (2-17)), and we write the rational Painlevé-IV functions in terms of these orthogonal polynomials and their normalization constants in (2-20) and (2-21) (see also (1-10)). We write down the standard Riemann-Hilbert problem associated to the orthogonal polynomials, and show how to directly extract the rational Painlevé-IV functions from the Riemann-Hilbert problem in Lemmas 2 and 3.

In §3 we compute the so-called $g$-function, a standard tool used to regularize the Riemann-Hilbert problem and turn oscillatory jumps into constants. By studying topological changes in the level lines of the related phase function $\varphi$, we derive an explicit form of the boundary curve, which we now state. Fix $r \in [1, \infty)$. Let $x_c(r)$ be the unique value of $x$ satisfying

$$
(1-11) \quad r^4 x^8 - 24r^2 (r^2 + r + 1)x^4 + 32r(2r^3 + 3r^2 - 3r - 2)x^2 - 48(r^2 + r + 1)^2 = 0
$$

with $\Re(x_c) > 0$ and $\Im(x_c) > 0$. The four points $\{\pm x_c, \pm x_c\}$ will be the four corners of the boundary of the elliptic region as well as the four branch points of a function $Q$ we will define shortly. While it is possible to solve (1-11) exactly since it is a quartic in $x^2$, we simply note that for $r = 1$ the corner points are the four $x$ values satisfying

$$
(1-12) \quad x^4 = 36 - 24\sqrt{3} \quad (r = 1),
$$

so that $x_c(1) \approx 1.086 + 1.086i$ (compare Figure 2). Now define $Q(x;r)$ as the unique function satisfying

$$
(1-13) \quad 3(1 + r)^2 Q + 8(1 + r)^{1/2} x Q^3 + 4(r - 1 + rx^2) Q^2 - 4 = 0
$$

such that $Q(x;r) = -x + \mathcal{O}(x^{-2})$ as $x \to +\infty$ and cut as shown in Figure 4. Also define

$$
\begin{align*}
\overline{Q}(x;r) & := (1 + r)Q(x;r)^3 + 2r^{1/2}xQ(x;r)^2, \\
S(x;r) & := (1 + r)Q(x;r)^3 + 2r^{1/2}xQ(x;r)^2.
\end{align*}
$$

Then let $a(x;r)$ and $b(x;r)$ be the two values of $z$ satisfying

$$
(1-15) \quad z^2 - S(x;r)z + Q(x;r)^2 = 0.
$$

For definiteness we choose $\Im(a) < \Im(b)$ for $\arg(x_c) \leq \arg(x) \leq \arg(x_c)$ and $\Re(a) > \Re(b)$ for $\arg(x_c) \leq \arg(x) \leq \arg(-x_c)$. Throughout we restrict our analysis to $\arg(x_c) \leq \arg(x) \leq \arg(-x_c)$, which is sufficient due to the symmetry (1-7). We now specify a contour $\Sigma$ connecting $a$ and $b$. Define

$$
(1-16) \quad \tilde{R}(z;x,r) := (z^2 - S(x;r)z + Q(x;r)^2)^{1/2}
$$

with $\tilde{R}(z) = z + \mathcal{O}(1)$ as $z \to \infty$ and branch cut chosen as the straight line segment between $a$ and $b$. Now define $\varphi(z;x,r) \equiv \varphi(z)$ by

$$
(1-17) \quad \varphi(z) := \frac{\tilde{R}(z)}{Qz^2} + \left(1 + r - \frac{S}{2Q} \right) \frac{\tilde{R}(z)}{z} - (1 + r) \log(2z + 2\tilde{R}(z) - S) + (r - 1) \log \left( \frac{2Q \tilde{R}(z) - Sz + 2Q^2}{z} \right) + \log(S^2 - 4Q^2) - (1 + r)i\pi.
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{The branch cuts for $Q(x;r)$.}
\end{figure}
Here all logarithms are chosen with principal branches (as we will only need the real part of \( \tilde{\varphi} \) the particular choice is unimportant). There is a level line of \( \Re(\tilde{\varphi}(z)) \) connecting \( a \) and \( b \) traveling in the clockwise direction around the origin; we call this bounded contour \( \Sigma \). Now set \( R(z; x, r) \) to be the function satisfying
\[
R(z; x, r)^2 = z^2 - S(x; r)z + Q(x; r)^2
\]
that is analytic for \( z \notin \Sigma \) and satisfies \( R(z) = z + O(1) \) as \( z \to \infty \). Note we have the useful relations
\[
S = a + b, \quad Q = R(0), \quad Q^2 = ab.
\]
Also define
\[
R_r(x; r) = R_{c} := \frac{(1 + r)^2 Q^4 + 2(1 + r)QS + 4)^{1/2}}{(1 + r)Q},
\]
with the choice of branch inherited from \( R(z) \). Then we have the following definitions of the elliptic region in which the zeros and poles of the rational Painlevé-IV functions lie (at least asymptotically) and the complementary genus-zero region. See Figures 1 and 2.

**Definition 1.** Fix \( r \in [1, \infty) \). Then the **elliptic region** \( E_r \) is the bounded domain of the complex plane defined by the curves
\[
\Re \left\{ \frac{(1 + r)^{1/2}x}{2} R_{c} - (1 + r) \log \left( 2r_c - \frac{4}{(1 + r)Q} - S \right) + (r - 1) \log \left( (1 + r)Q^3 + (1 + r)Q^2 r_c + S \right) + \log \left( S^2 - 4Q^2 \right) \right\} = 0.
\]
The **genus-zero region** is the complement of the closure of the elliptic region.

In §4 we carry out the Deift-Zhou nonlinear steepest-descent analysis [17] of the Riemann-Hilbert problem for the orthogonal polynomials. This consists of several standard steps:

1. Conjugating the jump matrices by a matrix involving the \( g \)-function, which identifies the contours that will contribute to the leading-order solution.
2. Opening lenses so all jumps are constants or decaying to the identity as \( n \to \infty \).
3. Solving the model problem obtained by disregarding jumps close to the identity.
4. Controlling the errors and showing that the model solution is a good approximation to the exact problem.

Following this procedure, we obtain the following asymptotic formulas for the rational Painlevé-IV functions valid in the genus-zero region.

**Theorem 1.** Fix \( p, q \in \mathbb{N} \) (the positive integers) with \( p \geq q \) and set \( r := p/q \). Fix \( x \) in the genus-zero region as defined in Definition 1. Then as \( m, n \to \infty \) along the sequence \( \{m, n\} = \{jp, jq\} \) for \( j \in \mathbb{N} \) we have
\[
\frac{1}{n^{1/2}} w^{(f)}_{m,n}(m^{1/2} x) = -\frac{1}{Q(x, r)} - \frac{S(x, r)}{2Q(x, r)^2} + O \left( \frac{1}{n} \right).
\]
Theorem 1 is illustrated in Figure 5.

**Theorem 2.** Fix \( p, q \in \mathbb{N} \) with \( p \geq q \) and set \( r := p/q \). Fix \( x \) in the genus-zero region as defined in Definition 1. Then as \( m, n \to \infty \) along the sequence \( \{m, n\} = \{jp, jq\} \) for \( j \in \mathbb{N} \) we have
\[
\frac{1}{n^{1/2}} w^{(II)}_{m,n}(m^{1/2} x) = \frac{1}{Q(x, r)} - \frac{S(x, r)}{2Q(x, r)^2} + O \left( \frac{1}{n} \right).
\]
Theorem 2 is illustrated in Figure 6. Finally, combining these two theorems with (1-10) immediately gives the following.

**Theorem 3.** Fix \( p, q \in \mathbb{N} \) with \( p \geq q \) and set \( r := p/q \). Fix \( x \) in the genus-zero region as defined in Definition 1. Then as \( m, n \to \infty \) along the sequence \( \{m, n\} = \{jp, jq\} \) for \( j \in \mathbb{N} \) we have
\[
\frac{1}{n^{1/2}} w^{(II)f}_{m,n}(m^{1/2} x) = -2r^{1/2} x + \frac{S(x, r)}{Q(x, r)^2} + O \left( \frac{1}{n} \right).
\]
**Remark 1.** To understand the behavior of the rational Painlevé-IV functions as \( m, n \to \infty \) with \( r = m/n \) fixed, it is sufficient to consider the case \( r \geq 1 \) due to the symmetry (1-7). In the case \( 0 < r < 1 \) (see Figure 3) the natural variable is \( \chi := n^{-1/2} y \), since the zeros of \( H_{m,n}(n^{1/2} \chi) \) are bounded in the \( \chi \) plane as \( n \to \infty \).
A comment on the literature. Before beginning our analysis we make a few remarks regarding a recent paper by Novokshenov and Schelkonogov [33] that concerns some of the same questions we address here. In particular, they are interested in the distribution of the zeros of $w_{n,n}^{(III)}$ for large $n$. The proposed strategy is intriguing: determine a Riemann-Hilbert problem for $w_{0,0}^{(III)}$ and then apply Schlesinger/Bäcklund transformations to obtain Riemann-Hilbert problems for $w_{n,n}^{(III)}$. Unfortunately, [33, Equation (22)] expressing $w_{n,n}^{(III)}$ (or, in their notation, $u_{n,n}$) in terms of the solution of the Riemann-Hilbert problem in [33, Equation (21)] is not correct. This means that the subsequent asymptotic results for the rational Painlevé-IV functions are also incorrect, including [33, Equation (37)] and [33, Equation (38)] describing the asymptotic behavior of $w_{n,n}^{(III)}$ and [33, Equation (41)] for the location of the zeros. In fact, it is not possible to extract any information about $w_{n,n}^{(III)}$ from the Riemann-Hilbert problem in [33, Equation (21)]. In their notation, this
problem is to find a matrix $Y(\xi)$ analytic for $\xi \notin \mathbb{R}$ satisfying

$$
Y_+(\xi) = Y_-(\xi) \begin{bmatrix} 2\pi i e^{-n(\xi^2-x^2)} & 1 \\ 0 & 1 \end{bmatrix} \text{ for } \xi \in \mathbb{R}; \quad Y(\xi) = (I + O(\xi^{-1})) \begin{bmatrix} \xi^{2n} & 0 \\ 0 & \xi^{-2n} \end{bmatrix} \text{ as } \xi \to \infty
$$

(here the parameter $x$ is, after scaling, the independent variable for the Painlevé-IV functions and is the same as our $x$ defined in (2-26) if $m = n$). Then the function $(2\pi i e^{nx^2})^{-\sigma_3/2}Y(\xi)(2\pi i e^{nx^2})^{\sigma_3/2}$ satisfies a Fokas-Its-Kitaev Riemann-Hilbert problem [20] for the (standard) Hermite polynomials. The solution to this problem can be written exactly in terms of $H_n$, $H_n-1$, and their Cauchy transforms, which is not enough information to construct $H_{n,n}$ or $w_{n,n}$.  

1.3. Notation. We denote the positive integers by $\mathbb{N}$ and the nonnegative integers by $\mathbb{N}_0$. If $f$ is a function defined on a specified oriented contour, then $f_+$ ($f_-$) denotes the boundary value taken from the left (right). Matrices are denoted by bold capital letters, with the exception of the $2 \times 2$ identity matrix $I$ and the Pauli matrix

$$(1-26) \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

The $(jk)$-entry of a matrix $M$ is denoted by $[M]_{jk}$.

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2. The Associated Orthogonal Polynomials

To analyze the asymptotic behavior of these functions we will use a determinantal formula. Define $\tau_{m,n}(y)$ by $\tau_{m,0}(y) := 1$ and by the $n \times n$ Hankel determinant

$$(2-1) \quad \tau_{m,n}(y) := \begin{vmatrix} H_m(y) & H_{m+1}(y) & \cdots & H_{m+n-1}(y) \\ H_{m+1}(y) & H_{m+2}(y) & \cdots & H_{m+n}(y) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n-1}(y) & H_{m+n}(y) & \cdots & H_{m+2n-2}(y) \end{vmatrix}_{n \times n}$$

for $n \geq 1$. Then $\tau_{m,n}$ is related [22, 32] to the generalized Hermite polynomial $H_{m,n}$ by

$$(2-2) \quad \tau_{m,n}(y) = (-1)^{\lceil \frac{(n-1)}{2} \rceil} \left( \prod_{k=0}^{n-1} [k]^{2k} \right) H_{m,n}(y),$$

where $[\cdot]$ denotes the ceiling function. We rewrite $\tau_{m,n}$ in terms of certain moments as follows. Let the contour $C$ be the unit circle with clockwise orientation. For $\zeta \in C$, define the measure

$$(2-3) \quad d\nu_m(\zeta; y) := \exp \left( \frac{2y}{\zeta} - \frac{1}{\zeta^2} \right) \zeta^m \frac{d\zeta}{2\pi i \zeta}.$$ 

Define the moments

$$(2-4) \quad \mu_k^{(m)}(y) := -\oint_C \zeta^k d\nu_m(\zeta; y).$$

Now, via the generating function (1-6), the Cauchy integral formula for derivatives, and the change of variables $s = \zeta^{-1}$, we see we can write the standard Hermite polynomials as

$$(2-5) \quad H_{m+j}(y) = \frac{d^{m+j}}{d^{m+j} s} \left( e^{2sy-s^2} \right) \bigg|_{s=0} = -\frac{(m+j)!}{2\pi i} \oint_C e^{2sy-s^2} \frac{ds}{s^{m+j+1}} = -(m+j)! \oint_C \zeta^j d\nu_m(\zeta; y) = (m+j)! \mu_j^{(m)}(y).$$
In particular, this means we can write

\[
(2-6) \quad \tau_{m,n}(y) = \begin{vmatrix}
  m!\mu_0^{(m)}(y) & (m+1)!\mu_1^{(m)}(y) & \cdots & (m+n-1)!\mu_{n-1}^{(m)}(y) \\
  (m+1)!\mu_0^{(m)}(y) & (m+2)!\mu_1^{(m)}(y) & \cdots & (m+n)!\mu_{n-1}^{(m)}(y) \\
  \vdots & \vdots & \ddots & \vdots \\
  (m+n-1)!\mu_0^{(m)}(y) & (m+n)!\mu_1^{(m)}(y) & \cdots & (m+2n-2)!\mu_{2n-2}^{(m)}(y)
\end{vmatrix}_{n \times n}.
\]

Define the related \(n \times n\) Hankel determinant

\[
(2-7) \quad T_{m,n}(y) := \left[ \mu_{j+k-2}^{(m)}(y) \right]_{j,k=1}^{n} = \begin{vmatrix}
  \mu_0^{(m)}(y) & \mu_1^{(m)}(y) & \cdots & \mu_{n-1}^{(m)}(y) \\
  \mu_1^{(m)}(y) & \mu_2^{(m)}(y) & \cdots & \mu_n^{(m)}(y) \\
  \vdots & \vdots & \ddots & \vdots \\
  \mu_{n-1}^{(m)}(y) & \mu_n^{(m)}(y) & \cdots & \mu_{2n-2}^{(m)}(y)
\end{vmatrix}_{n \times n}.
\]

Certain ratios of these determinants can be expressed in terms of normalization constants for a family of orthogonal polynomials (see (2-19) below). We now show how to relate \(\tau_{m,n}\) with \(T_{m,n}\) (with shifted indices), thus providing a bridge between the rational Painlevé-IV functions and the orthogonal polynomials.

Lemma 1.

\[
(2-8) \quad \tau_{m,n}(y) = \left( \prod_{k=0}^{n-1} [(m+k)!2^k] \right) \cdot T_{m-n+1,n}(y).
\]

Proof. We start by writing the right-hand side of (2-8) in terms of Hermite polynomials:

\[
(2-9) \quad \left( \prod_{k=0}^{n-1} [(m+k)!2^k] \right) \cdot T_{m-n+1,n} = \prod_{k=1}^{n-1} 2^k \begin{vmatrix}
  m! & \frac{(m+1)!}{(m-n+1)!} & \cdots & \frac{(m+n)!}{(m-n+1)!} \\
  \frac{(m+1)!}{(m-n+2)!} & \frac{(m+2)!}{(m-n+2)!} & \cdots & \frac{(m+n+1)!}{(m-n+2)!} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{(m+n)!}{(m-n)!} & \frac{(m+n+1)!}{(m-n)!} & \cdots & \frac{(m+n+n)!}{(m-n)!}
\end{vmatrix}.
\]

Our goal is to manipulate \(\tau_{m,n}\) into this form. We start by completely reversing the order of the rows:

\[
(2-10) \quad \tau_{m,n} = \prod_{k=1}^{n-1} (-1)^k \begin{vmatrix}
  H_{m+n-1} & H_{m+n} & \cdots & H_{m+n-2} \\
  H_{m+n-2} & H_{m+n-1} & \cdots & H_{m+n-3} \\
  H_{m+n-3} & H_{m+n-2} & \cdots & H_{m+n-4} \\
  \vdots & \vdots & \ddots & \vdots \\
  H_{m} & H_{m+1} & \cdots & H_{m+n}
\end{vmatrix}.
\]

Note that the \(n\)th row is in the desired form (up to the overall constant). We now perform a set of operations on the first \(n-1\) rows that will leave the \((n-1)\)st row in the desired form. Repeating this set of operations on the first \(n-2\) rows, then the first \(n-3\) rows, and so on, will establish the identity. The Hermite polynomials satisfy the recursion relation

\[
(2-11) \quad H_{m+1}(y) = 2yH_{m}(y) - 2mH_{m+1}(y).
\]

Using this in the top row gives

\[
(2-12) \quad \tau_{m,n} = \prod_{k=1}^{n-1} (-1)^k \begin{vmatrix}
  2yH_{m+n-2} - 2(m+n-2)H_{m+n-3} & \cdots & 2yH_{m+n-2} - 2(m+n-3)H_{m+n-4} \\
  H_{m+n-2} & \cdots & H_{m+n-3} \\
  H_{m+n-3} & \cdots & H_{m+n-4} \\
  \vdots & \ddots & \vdots \\
  H_{m+n-1} & \cdots & H_{m+n}
\end{vmatrix}.
\]
Note that we can eliminate the terms proportional to \( y \) by subtracting a multiple of the second row from the first row. We can then pull out the common \(-2\) factor from the first row, and subtract a multiple of the third row from the first row to change the coefficients in front of the Hermite polynomials. The result is

\[
\tau_{m,n} = -2 \prod_{k=1}^{n-1} (-1)^k \begin{vmatrix}
  mH_{m+n-3} & \cdots & (m + n - 1)H_{m+2n-4} \\
  H_{m+n-2} & \cdots & H_{m+2n-3} \\
  H_{m+n-3} & \cdots & H_{m+2n-4} \\
  \vdots & \ddots & \vdots \\
  H_{m+1} & \cdots & H_{m+n} \\
  H_m & \cdots & H_{m+n-1}
\end{vmatrix}
\]

(2.13)

We now carry out the same procedure on rows 2, 3, \ldots, \( n-1 \): apply the recursion relation, use the next row to remove terms proportional to \( y \), and then use the subsequent row to change the coefficient of the first entry to \( m \). (For row \( n-1 \) the leading coefficient in column 1 is already \( m \) once the \( y \)-terms are removed.) Once every row has been modified in this way we obtain

\[
\tau_{m,n} = 2^{n-1} \prod_{k=1}^{n-2} (-1)^k \begin{vmatrix}
  mH_{m+n-3} & \cdots & (m + n - 1)H_{m+2n-4} \\
  mH_{m+n-4} & \cdots & (m + n - 1)H_{m+2n-5} \\
  mH_{m+n-5} & \cdots & (m + n - 1)H_{m+2n-6} \\
  \vdots & \ddots & \vdots \\
  mH_{m-n-1} & \cdots & (m + n - 1)H_{m+n-2} \\
  H_m & \cdots & H_{m+n-1}
\end{vmatrix}
\]

(2.14)

This fixes the last two rows. We now repeat this procedure on rows 1, \ldots, \( n-2 \), the only difference being that we change the leading coefficients in column 1 to \( m(m-1) \). The result is

\[
\tau_{m,n} = \prod_{k=2}^{n-1} 2^k \prod_{j=1}^{n-3} (-1)^j \begin{vmatrix}
  m(m-1)H_{m+n-5} & \cdots & (m + n - 1)(m + n - 2)H_{m+2n-6} \\
  m(m-1)H_{m+n-6} & \cdots & (m + n - 1)(m + n - 2)H_{m+2n-7} \\
  \vdots & \ddots & \vdots \\
  m(m-1)H_{m-n-2} & \cdots & (m + n - 1)(m + n - 2)H_{m+n-3} \\
  mH_{m-n-1} & \cdots & (m + n - 1)H_{m+n-2} \\
  H_m & \cdots & H_{m+n-1}
\end{vmatrix}
\]

(2.15)

Note that now the final three rows have the intended form. Repeating this procedure \( n-3 \) more times, each time involving one less row than before and modifying the leading coefficient appropriately (i.e. so the last row changed has the correct coefficient), yields the form (2.9), as desired.

\[\square\]

**Remark 2.** We observe that the result of Lemma 1 can be written in terms of Hermite polynomials as

\[
|H_{m+j+k-2}(y)|_{j,k=1}^n = \prod_{k=0}^{n-1} 2^k \cdot \left| \frac{(m + k - 1)!}{(m - n + j + k - 1)!} H_{m-n+j+k-1}(y) \right|_{j,k=1}^n.
\]

(2.16)

Hankel determinants of orthogonal polynomials such as the expression on the left-hand side are known as *Turánians*. The Hermite Turánian can be expressed as a Wronskian for general \( m \) [23, Equation (18.2)] and evaluated in closed form for \( m = 0 \) [24, Equation (3.55)]. For more background and references on Turánians see [21].

For fixed \( m \in \mathbb{N}_0 \), define the monic orthogonal polynomials \( \psi^{(m)}_n \), \( n \geq 0 \), by

\[
\int_C \psi^{(m)}_n(\zeta; y) \zeta^j d\nu_m(\zeta; y) = \delta_{jn} h^{(m)}_n(y), \quad j = 0, \ldots, n,
\]

(2.17)

where \( \delta_{jn} \) is the Kronecker delta function and \( h^{(m)}_n(y) \) is the normalization constant (that is, constant in \( \zeta \) but with parametric dependence on \( y \)). Then (see, for example, [2, 1]) the value of the orthogonal polynomials evaluated at \( \zeta = 0 \) can be expressed in terms of determinants via

\[
\psi^{(m)}_n(0; y) = (-1)^n \frac{T_{m+1,n}(y)}{T_{m,n}(y)}.
\]

(2.18)
and the normalization constant $h_n^{(m)}$ can be expressed as
\begin{equation}
(2-19) \quad h_n^{(m)}(y) = -\frac{T_{m,n+1}(y)}{T_{m,n}(y)}.
\end{equation}

Note that $(2-18)$ and $(2-19)$ provide ways to shift the two indices of $T_{m,n}(y)$. Applying $(2-2)$, $(2-8)$, $(2-18)$, and $(2-19)$ to $(1-8)$–$(1-9)$ gives
\begin{equation}
(2-20) \quad w_{m,n}^{(I)}(y) = \frac{d}{dy} \log \left( \frac{\tau_{m+1,n}(y)}{\tau_{m,n}(y)} \right) = \frac{d}{dy} \log \left( \frac{T_{m+2,n}}{T_{m+1,n}} \right) = \frac{\partial}{\partial y} \log \left( \psi_n^{(m+1)}(0;y) \right)
\end{equation}

and
\begin{equation}
(2-21) \quad w_{m,n}^{(II)}(y) = \frac{d}{dy} \log \left( \frac{\tau_{m,n}(y)}{\tau_{m,n+1}(y)} \right) = \frac{d}{dy} \log \left( \frac{T_{m,n+1}}{T_{m,n+1}} \right) = \frac{\partial}{\partial y} \log \left( \frac{\psi_n^{(m-1)}(0;y)}{h_n^{(m-1)}(y)} \right).
\end{equation}

Note that $w_{m,n}^{(III)}(y)$ can also be expressed in terms of the orthogonal polynomials and their normalization constants through the previous two equations and $(1-10)$. We now introduce the Fokas-Its-Kitaev Riemann-Hilbert problem \cite{20} in order to analyze the large-degree behavior of the orthogonal polynomials.

**Riemann-Hilbert Problem 1** (Unscaled orthogonal polynomial problem). Fix $y \in \mathbb{C}$ and $m, n \in \mathbb{N}$. Seek a $2 \times 2$ matrix $M_{m,n}(\zeta;y)$ with the following properties:

- **Analyticity**: $M_{m,n}(\zeta;y)$ is analytic for $\zeta \in \mathbb{C}$ except on $C$ (the unit circle oriented clockwise) with Hölder-continuous boundary values.

- **Jump condition**: The boundary values taken by $M_{m,n}(\zeta;y)$ on $C$ are related by the jump condition

\begin{equation}
(2-22) \quad M_{m,n+1}(\zeta;y) = M_{m,n-1}(\zeta;y) \begin{bmatrix} 1 & \frac{1}{2\pi i} \exp \left( \frac{2y}{\zeta} - \frac{1}{\zeta^2} + m \log \zeta \right) \\ 0 & 1 \end{bmatrix}, \quad \zeta \in C.
\end{equation}

- **Normalization**: As $\zeta \to \infty$, the matrix $M_{m,n}(\zeta;y)$ satisfies the condition

\begin{equation}
(2-23) \quad M_{m,n}(\zeta;y) = (I + O(\zeta^{-1})) \zeta^{\sigma_3}
\end{equation}

with the limit being uniform with respect to direction.

This Riemann-Hilbert problem is solvable exactly when $\psi_n^{(m)}$ exists, and
\begin{equation}
(2-24) \quad \psi_n^{(m)}(\zeta;y) = [M_{m,n}(\zeta;y)]_{11}
\end{equation}
(that is, the 11-entry of $M$) while
\begin{equation}
(2-25) \quad h_n^{(m)}(y) = -2\pi i \lim_{\zeta \to \infty} \zeta \left[ M_{m,n}(\zeta;y) \zeta^{-\sigma_3} - I \right]_{12}.
\end{equation}

Motivated by the exponent in $(2-22)$, we define rescaled versions of $y$ and $\zeta$:
\begin{equation}
(2-26) \quad x := m^{-1/2} y, \quad z := n^{1/2} \zeta.
\end{equation}

These definitions suggest scaling the orthogonal polynomials as well. Define
\begin{equation}
(2-27) \quad \Psi_n^{(m)}(z;x) := n^{n/2} \psi_n^{(m)} \left( \frac{z}{\sqrt{n/2}; m^{1/2} x} \right), \quad \mathcal{H}_n^{(m)}(x) := n^{n/2} h_n^{(m)}(m^{1/2} x).
\end{equation}

These new polynomials satisfy the orthogonality relations
\begin{equation}
(2-28) \quad \int_C \Psi_n^{(m)}(z;x) z^j dV_m(z;x) = \delta_{jn} \mathcal{H}_n^{(m)}(x), \quad j = 0, \ldots, n, \quad dV_m := \exp \left( n \left[ \frac{2\pi i}{x^2} - \frac{1}{z^2} \right] \right) \frac{z^{r-n} dz}{2\pi i z},
\end{equation}

where $r = m/n$.

The desired rational functions can be expressed in terms of the scaled orthogonal polynomials as
\begin{equation}
(2-29) \quad m^{1/2} w_{m,n}^{(I)}(m^{1/2} x) = \frac{\partial}{\partial x} \log \left( \Psi_n^{(m-1)}(0;x) \right) = \frac{\partial}{\partial z} \Psi_n^{(m-1)}(0;x) \frac{\partial}{\partial x} \Psi_n^{(m-1)}(0;x)
\end{equation}

and
\begin{equation}
(2-30) \quad m^{1/2} w_{m,n}^{(II)}(m^{1/2} x) = \frac{\partial}{\partial x} \log \left( \frac{\Psi_n^{(m)}(0;x)}{\mathcal{H}_n^{(m-n)}(x)} \right).
\end{equation}
We now pose a Riemann-Hilbert problem for the orthogonal polynomials $\Psi_n^{(m-n+1)}(z;x)$.

**Riemann-Hilbert Problem 2** (Scaled orthogonal polynomial problem). Fix $x \in \mathbb{C}$ and $m, n \in \mathbb{N}$ with $m \geq n$ and set $r = m/n$. Find the unique $2 \times 2$ matrix $\mathbf{N}_{m,n}(z;x)$ with the following properties:

**Analyticity:** $\mathbf{N}_{m,n}(z;x)$ is analytic in $z$ except on $C$ (the unit circle oriented clockwise) with Hölder-continuous boundary values.

**Jump condition:** The boundary values taken by $\mathbf{N}_{m,n}(z;x)$ on $C$ are related by the jump condition

$$\mathbf{N}_{m,n}(z;x) = \mathbf{N}_{m,n}(\tilde{z};x) \begin{pmatrix} 1 & \frac{1}{2\pi i} e^{-\theta(z;x,r)} \\ 0 & 1 \end{pmatrix}, \quad z \in C,$$

where

$$\theta(z;x,r) := (1-r) \log z - \frac{2r^{1/2}x}{z} + \frac{1}{z^2}.$$  

**Normalization:** As $z \to \infty$, the matrix $\mathbf{N}_{m,n}(z;x)$ satisfies the condition

$$\mathbf{N}_{m,n}(z;x) = (I + \mathcal{O}(z^{-1})) z^{\sigma_3}$$

with the limit being uniform with respect to direction.

It is immediate that

$$\Psi_n^{(m-n+1)}(0;x) = [\mathbf{N}_{m,n}(0;x)]_{11}$$

and

$$\mathcal{H}_n^{(m-n+1)}(x) = -2\pi i \lim_{z \to \infty} z[\mathbf{N}_{m,n}(z;x)z^{-\sigma_3} - I]_{12}.$$  

In the next two lemmas we show how to extract $w_m^{(II)}$ and $w_m^{(I)}$ directly from the solution of the Riemann-Hilbert problem.

**Lemma 2.** Write the expansion of $\mathbf{N}_{m,n}(z;x)$ about $z = 0$ as

$$\mathbf{N}_{m,n}(z;x) = \mathbf{N}_0(x) + \mathbf{N}_1(x)z + \mathcal{O}(z^2),$$

where $\mathbf{N}_0(x)$ and $\mathbf{N}_1(x)$ are independent of $z$. Then

$$\frac{1}{n^{1/2}} w_m^{(II)}(m^{1/2}x) = ([\mathbf{N}_0(x)]_{11}[\mathbf{N}_0(x)]_{22} + [\mathbf{N}_0(x)]_{12}[\mathbf{N}_0(x)]_{21} - 1) \frac{[\mathbf{N}_1(x)]_{11}}{[\mathbf{N}_0(x)]_{11}} - 2[\mathbf{N}_0(x)]_{12}[\mathbf{N}_1(x)]_{21}. $$

**Proof.** From (2-34), we have

$$\Psi_n^{(m-n+1)}(0;x) = [\mathbf{N}_0(x)]_{11}.$$ 

Thus, from the last expression in (2-29) we merely need to express $\frac{\partial}{\partial z} \Psi_n^{(m-n+1)}(0;x) = \left[ \frac{\partial}{\partial z} \mathbf{N}_{m,n}(0;x) \right]_{11}$ in terms of (undifferentiated) entries of $\mathbf{N}_{m,n}$. Define

$$\mathbf{N}_{m,n}(z;x) := \mathbf{N}_{m,n}(z;x)e^{-\theta(z;x,r)\sigma_3/2}.$$  

This function is analytic in $\mathbb{C}\setminus\{0 \cup C\}$ with a jump discontinuity on $C$ that is independent of $x$ (and $z$). This means that $\frac{\partial}{\partial z} \mathbf{N}_{m,n}(z;x)$ has the same properties with the same jump on $C$. It follows that

$$\mathbf{W}_{m,n}(z;x) := \left( \frac{\partial}{\partial z} \mathbf{N}_{m,n}(z;x) \right) \mathbf{N}_{m,n}(z;x)^{-1}$$

is analytic in $\mathbb{C}\setminus0$. Inserting (2-39) into (2-40) gives

$$\mathbf{W}_{m,n}(z;x) = \left( \frac{\partial}{\partial z} \mathbf{N}_{m,n}(z;x) \right) \mathbf{N}_{m,n}(z;x)^{-1} + \frac{nr^{1/2}}{z} \mathbf{N}_{m,n}(z;x)\sigma_3 \mathbf{N}_{m,n}(z;x)^{-1}.$$ 

This shows that $\mathbf{W}_{m,n}(z;x)$ has a simple pole at $z = 0$ and, in particular, that $z\mathbf{W}_{m,n}(z;x)$ is entire in $z$. Inserting the large-$z$ expansion (2-33) into (2-40) (using (2-39)) shows that $\mathbf{W}_{m,n}(z;x) = \mathcal{O}(z^{-1})$ as $z \to \infty$. This demonstrates that $z\mathbf{W}_{m,n}(z;x)$ is bounded as $z \to \infty$. Therefore Liouville’s theorem tells us that $z\mathbf{W}_{m,n}(z;x)$ is a constant matrix (i.e. independent of $z$ with parametric dependence on $x$). This
constant can be determined by considering (2-41) and noting that the first summand on the right-hand side is bounded as \( z \to 0 \). Thus

\[
W_{m,n}(z; x) = \frac{n^r}{z} N_{m,n}(0; x) \sigma_3 N_{m,n}(0; x)^{-1}.
\]

Combining (2-41) and (2-42) gives

\[
\frac{\partial}{\partial x} N_{m,n}(z; x) = \frac{n^r}{z} \left( N_{m,n}(0; x) \sigma_3 N_{m,n}(0; x)^{-1} N_{m,n}(z; x) - N_{m,n}(z; x) \sigma_3 \right).
\]

Evaluating both sides at \( z = 0 \) (using the expansion (2-36) on the right-hand side) yields

\[
\frac{\partial}{\partial x} N_{m,n}(0; x) = n^r \left( N_0(x) \sigma_3 N_0(x)^{-1} N_1(x) - N_1(x) \sigma_3 \right).
\]

Therefore

\[
\frac{\partial}{\partial x} \psi_{n}^{(m-n+1)}(0; x) = \left[ \frac{\partial}{\partial x} N_{m,n}(0; x) \right]_{11} = n^r \left[ ([N_0(x)]_{11} [N_0(x)]_{22} + [N_0(x)]_{12} [N_0(x)]_{21} - 1) [N_1(x)]_{11} - 2[N_0(x)]_{11} [N_0(x)]_{12} [N_1(x)]_{21} \right].
\]

Combining (2-29), (2-38), and (2-45) finishes the proof. □

**Lemma 3.** Write the expansion of \( N_{m,n}(z; x) \) as \( z \to \infty \) as

\[
N_{m,n}(z; x) = \left( 1 + \frac{N_{-1}(x)}{z} + O\left( \frac{1}{z^2} \right) \right) z^{n \sigma_3}
\]

and recall the expansion (2-36) about \( z = 0 \). Then

\[
\frac{1}{n^{1/2}} w_{m+1,n}^{(I)}((m+1)^{1/2}) = \left( \frac{1}{n^{1/2}} w_{m,n}^{(I)}(m^{1/2}) + \frac{2[N_0(x)]_{11}[N_0(x)]_{12}}{[N_{-1}(x)]_{12}} \right) \left( 1 + O\left( \frac{1}{m} \right) \right).
\]

Here \( w_{m,n}^{(I)} \) can be expressed in terms of \( N_{m,n} \) via Lemma 2.

**Proof.** Starting from (2-30), we shift \( m \to m + 1 \) and use Lemma 2 to discover

\[
\frac{1}{n^{1/2}} w_{m+1,n}^{(I)}((m+1)^{1/2}) = \left( \frac{1}{n^{1/2}} \frac{\partial}{\partial x} \log(\psi_n^{(m-n+1)}(0; x)) - \frac{1}{n^{1/2}} \frac{\partial}{\partial x} \log(H_n^{(m-n+1)}(x)) \right) \frac{1}{(m+1)^{1/2}}
\]

\[
= \left( \frac{1}{n^{1/2}} w_{m,n}^{(I)}(m^{1/2}) - \frac{1}{n^{1/2}} \frac{\partial}{\partial x} \log(H_n^{(m-n+1)}(x)) \right) \frac{m^{1/2}}{(m+1)^{1/2}}
\]

\[
= \left( \frac{1}{n^{1/2}} w_{m,n}^{(I)}(m^{1/2}) - \frac{1}{n^{1/2}} \frac{\partial}{\partial x} \frac{H_n^{(m-n+1)}(x)}{H_n^{(m-n)}} \right) \left( 1 + O\left( \frac{1}{m} \right) \right).
\]

From (2-35) and (2-46) we have

\[
H_n^{(m-n+1)}(x) = -2\pi i [N_{-1}(x)]_{12}.
\]

We now express \( \frac{\partial}{\partial x} H_n^{(m-n+1)}(x) = -2\pi i \frac{\partial}{\partial x} [N_{-1}(x)]_{12} \) in terms of undifferentiated entries of \( N_{m,n} \). Insert the large-\( z \) expansion (2-46) into the expression (2-41) for \( W_{m,n} \):

\[
W_{m,n}(z; x) = \frac{1}{z} \left( \frac{\partial}{\partial x} N_{-1}(x) + n \cdot r^{1/2} \sigma_3 \right) + O\left( \frac{1}{z^2} \right).
\]

Recalling from the proof of Lemma (2) that \( z W_{m,n} \) is a constant matrix, the \( O(z^{-2}) \) terms must be identically zero. Combining this expression with (2-42) gives

\[
\frac{\partial}{\partial x} N_{-1}(x) + n \cdot r^{1/2} \sigma_3 = n^r N_0(x) \sigma_3 N_0(x)^{-1}.
\]

Taking the (12)-entry of both sides generates

\[
\frac{\partial}{\partial x} [N_{-1}(x)]_{12} = -2n^r [N_0(x)]_{11} [N_0(x)]_{12}.
\]
Using (2-49) and (2-52) in (2-48) completes the proof of the lemma. □

3. Determination of the boundary curve

We begin the Riemann-Hilbert analysis by finding the $g$-function and related phase function $\varphi$. This will be sufficient to specify the boundary of the elliptic region, which will be used in §4 to compute the asymptotics of the rational Painlevé-IV functions in the genus-zero region.

3.1. Construction of the $g$-function. Suppose two complex numbers $a = a(x, r)$ and $b = b(x, r)$ are given, along with an oriented contour $\Sigma = \Sigma(x, r)$ from $a$ to $b$ (specifying these quantities is part of the process of defining the $g$-function). The genus-zero $g$-function is determined via the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3 (The $g$-function). Fix $x \in \mathbb{C}$ and $r \in [1, \infty)$ and find $g(z) = g(z; x, r)$ such that

Analyticity: $e^{g(z; x, r)}$ is analytic for $z \in \mathbb{C}$ except on $\Sigma$, where it attains Hölder-continuous boundary values at all interior points. The function $g(z; x, r)$ also has a logarithmic branch cut that will play no role since $g$ only appears exponentiated.

Jump condition:

\begin{equation}
   \varphi^+(z) + \varphi^-(z) = \theta(z) + \ell, \quad z \in \Sigma
\end{equation}

for some constant $\ell = \ell(x; r)$.

Normalization:

\begin{equation}
   g(z) = \log z + O \left( \frac{1}{z} \right), \quad z \to \infty.
\end{equation}

There are some values of $x$ for which it is not possible to pick a single connected contour $\Sigma$ such that this Riemann-Hilbert problem is solvable. When it is possible, then the resulting outer model problem (see Riemann-Hilbert Problem 7 below) has jumps on a single band and the associated Riemann surface is genus zero. As a result, we dub the region where the Riemann-Hilbert problem where $g$ is solvable the genus-zero region (see Definition 1). We then show that the Painlevé-IV functions are (asymptotically) free of zeros and poles in this region.

Given $g(z)$ and $\ell$, we could define a function $\varphi$ by

\begin{equation}
   \varphi(z; x, r) = \theta(z; x, r) - 2g(z; x, r) + \ell.
\end{equation}

In actuality, we will work in the opposite order, first determining $\varphi'(z)$, integrating to find $\varphi(z)$, and then using (3-3) to find the explicit formula for $g(z)$. Note $\varphi'(z)$ is specified by the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 4 (The phase function $\varphi$). Fix $x \in \mathbb{C}$ and $r \in [1, \infty)$ and find $\varphi'(z) = \varphi'(z; x, r)$ such that

Analyticity: $\varphi'(z; x, r)$ is analytic for $z \in \mathbb{C}$ except at $z = 0$ and on $\Sigma$, where it attains Hölder-continuous boundary values at all interior points.

Jump condition:

\begin{equation}
   \varphi'_+(z) + \varphi'_-(z) = 0, \quad z \in \Sigma.
\end{equation}

Pole at $z = 0$:

\begin{equation}
   \varphi'(z) = \theta'(z) + O(1) = -\frac{2}{z^3} + \frac{2r^{1/2}x}{z^2} + \frac{1 - r}{z} + O(1), \quad z \to 0.
\end{equation}

Normalization:

\begin{equation}
   \varphi'(z) = -\frac{1 + r}{z} + O \left( \frac{1}{z^2} \right), \quad z \to \infty.
\end{equation}
We now see how the defining relations (1-13) and (1-14) for $Q$ and $S$ arise. If we momentarily assume $a(x; r)$, $b(x; r)$, and $\Sigma$ are known, then we can define $R(z; x, r)$ by (1-18). Furthermore, writing $a + b$ as $S$ and $R(0)$ as $Q$, then we can see that in order to satisfy the analyticity, jump, and normalization conditions in Riemann-Hilbert Problem 4, we can choose $\varphi'(z)$ to have the form

$$
\varphi'(z) = - \frac{R(z)}{(1 + r)z + \frac{2}{Q} R(z) z^3}.
$$

Now for $\varphi'(z)$ to satisfy the pole condition (3-5) at $z = 0$, $S$ and $Q$ must satisfy the moment conditions

$$
\frac{(1 + r)Q^3 - S}{2Q^2} = - r^{1/2}x, \quad \frac{4Q^2 - 2(1 + r)SQ^3 - S^2}{8Q^4} = r - \frac{1}{2}.
$$

Solving the first equation for $S$ yields the relation (1-14). Plugging that into the second yields the quartic equation (1-13) for $Q$. The specific sheet so that $\varphi(\infty) = 0$, we can choose $\Sigma$ to be the contour connecting $a$ and $b$ connecting $a$ and $b$ connecting $a$ and $\varphi \equiv \varphi(z)$ as defined in (1-17) is an antiderivative of (3-7) (with $R$ replaced with $\tilde{R}$). The integration constant is chosen so $\varphi(a) = 0$. Now $\Re(\varphi(z)) = \Re(\varphi(b))$, and for $|z|$ sufficiently large there are two contours connecting $a$ and $b$ that do not pass through $z = 0$ (in fact, the existence of both of these contours is equivalent to being in the genus-zero region – see Lemma 4). We choose $\Sigma$ to be the contour connecting $a$ to $b$ when traveling clockwise around the origin. Now that $\Sigma$ is defined, we can define $R(z)$ by (1-18) (which amounts to a deformation of the branch cut for $\tilde{R}(z)$), and define $\varphi(z)$ and $\varphi(z)$ via

$$
\varphi(z) := \frac{R(z)}{Q z^2} + \left(1 + r - \frac{S}{2Q^3}\right) \frac{R(z)}{z} - (1 + r) \log(2z + 2R(z) - S) + (r - 1) \log\left(\frac{2QR(z) - Sz + 2Q^2}{z}\right) + \log(S^2 - 4Q^2) - (1 + r)i\pi.
$$

Here the branches of the logarithms are chosen so $\varphi_+(z) + \varphi_-(z) = 0$ for $z \in \Sigma$, a choice that depends on both $x$ and $r$. The behavior of the Riemann-Hilbert problem is controlled by $\Re(\varphi(z))$ (see Figures 7 and 8).
Now we can set
\begin{equation}
(3-14) \quad g(z; x, r) := \frac{1}{2} \theta(z; x, r) - \frac{1}{2} \phi(z; x, r) + \frac{\ell(x; r)}{2},
\end{equation}
where only $\ell$ remains unspecified. The role of $\ell$ is to ensure the normalization (3-2) for $g(z)$, so we choose
\begin{equation}
(3-15) \quad \ell(x; r) := 2 \lim_{z \to \infty} \left( \log z - \frac{1}{2} \theta(z; x, r) + \frac{1}{2} \phi(z; x, r) \right).
\end{equation}
While $\ell(x; r)$ can be computed in terms of elementary functions, we will not need its explicit form.

**Figure 7.** Signature charts of $\Re(\phi(z))$ in the complex $z$-plane with $r = 1$ for different values of $x$ in the genus-zero region and on the boundary of the elliptic region. The band $\Sigma$ and the band endpoints $a$ and $b$ are indicated, as is $c$ when it lies on a zero-level line of $\Re(\phi(z))$. The topology of the zero-level lines is similar for other values of $r$ (see Figure 8 for $r = 10$). Bottom left: The boundary of the elliptic region in the complex $z$-plane, along with the values of $x$ corresponding to the signature charts.
3.2. The boundary and corners of the elliptic region. For generic values of $x$ and $r$ the function $\varphi'(z)$ (recall (3-7)) has three distinct zeros at

$$a(x;r), \quad b(x;r), \quad \text{and} \quad c(x;r) := -\frac{2}{(1+r)Q(x;r)}.$$  

The transition from the genus-zero region to the elliptic region occurs when one of the zero-level lines of $\Re(\varphi)$ crosses $c$, i.e. $\Re(\varphi(c)) = 0$. See the plots with $x \approx 1.0253$ and $x \approx 1.0253i$ in Figure 7 and the plot with $x \approx 1.2953$ in Figure 8. This condition can be written in the more explicit form (1-21), where $R_c = R(c)$. It is important to note that the boundary of the curvilinear rectangles illustrated in Figures 1–3 are not the only curves along which $\Re(\varphi(c)) = 0$. There are four additional curves that start at the four corners and tend to infinity (see Figure 9). The signature chart of $\Re(\varphi(z))$ along one of these lines is illustrated in the plot with $x = 1.2 + 1.2i$ in Figure 7. Nevertheless, the genus-zero Riemann-Hilbert analysis in §4 will go through without change along these curves, so they are part of the genus-zero region.

**Remark 3.** As illustrated in Figure 7, the breaking mechanism at the boundary of the elliptic region depends on whether $\text{arg}(x_c) < \text{arg}(x) < \text{arg}(x_c)$ or $\text{arg}(x_{-c}) < \text{arg}(x) < \text{arg}(-x_c)$. In the first case, a region in which $\Re(\varphi(z)) > 0$ is pinched off, as in the plot with $x \approx 1.0253$ in Figure 7. Looking ahead to Figure 10, this means it is no longer possible to pass the gap contour $\Gamma$ through this region in which its jump is exponentially close to the identity, and it is necessary to open a second band to control the Riemann-Hilbert problem once $x$ has moved into the elliptic region. On the other hand, for $\text{arg}(x_c) < \text{arg}(x) < \text{arg}(-x_c)$ (see the plot with
$x \approx 1.0253i$ in Figure 7), it is a region in which $\Re(\varphi(z)) < 0$ that is pinched off. In this case the gap $\Gamma$ remains controlled, and the necessary modification occurs on the band $\Sigma$. We conjecture that, as $x$ enters the elliptic region from the top boundary, a second band opens up directly on $\Sigma$ and then moves closer to the origin as $\Im(x)$ decreases. This gives a consistent picture in which, just inside the boundary, there is one small and one large band. As $x$ moves clockwise, the larger band rotates clockwise in the $z$-plane while the small band rotates counterclockwise. The small band is near an endpoint of the large band exactly when $x$ is near a corner of the boundary region. We emphasize the Riemann-Hilbert analysis in §4 goes through uniformly for all $x$ in the genus-zero region as long as $x$ stays bounded away from the boundary curve.

We now identify the corner points. These are the values of $x$ for which $c(x) = a(x)$ or $c(x) = b(x)$ (see the plot with $x = x_c$ in Figure 7, as well as [5] for a similar analysis for the Painlevé-II equation). In either case we have $c^2 - Sc + Q^2 = 0$ from (1-15). Using (1-14) and (3-16) to express $S$ and $c$ in terms of $Q, x$, and $r$ yields

$$
(3-17) \quad 3(1+r)^2Q^4 + 4(1+r)r^{1/2}xQ^3 + 4 = 0.
$$

Adding this to (1-13) gives

$$
(3-18) \quad 6(1+r)^2Q^4 + 12(1+r)r^{1/2}xQ^3 + 4(r - 1 + rx^2)Q^2 = 0,
$$

which is equivalent to (3-9), the derivative of (1-13) with respect to $Q$. Once (3-9) holds, the analysis following that equation used to determine the branch points of $Q$ also holds, and so the corner points must satisfy (1-11). While there are eight solutions to that equation, only four of them are off the coordinate axes, and so the geometry of the boundary shows that the corners are $\{\pm x_c, \pm 3x_c\}$.

4. Asymptotic expansion of the rational Painlevé-IV functions

We now apply the Deift-Zhou nonlinear steepest-descent method to obtain an approximation of $N_{m,n}(z; x)$. We perform a series of transformations

$$
N_{m,n}(z; x) \rightarrow O_{m,n}(z; x) \rightarrow P_{m,n}(z; x) \rightarrow Q_{m,n}(z; x) \approx R_{m,n}(z; x).
$$

The first transformation (to $O_{m,n}$) deforms the jump contours away from the unit circle and onto $\Sigma \cup \Gamma$, where $\Gamma$ lies in a region where $\Re(\varphi) > 0$. The second transformation (to $P_{m,n}$) introduces the $q$-function to regularize the jump matrices. In the third transformation (to $Q_{m,n}$) we open lenses, which replaces rapidly oscillating jump matrices with ones that are approximately constant. The associated Riemann-Hilbert problem is then replaced with a constant-jump problem that can be solved exactly for $R_{m,n}$. A key point is that the error in approximating $Q_{m,n}$ with $R_{m,n}$ can be controlled, as we will show in Lemma 4.

4.1. Initial deformation of the contours ($N_{m,n} \rightarrow O_{m,n}$). The first step is to deform the jump contours away from the unit circle $C$. Define a smooth, non-self-intersecting contour $\Gamma$ starting at $b$ and ending at $a$ whose interior is entirely in the region in which $\Re(\varphi(z)) > 0$ (see Figure 10). The existence of $\Gamma$ in the genus-zero region is shown below in Lemma 4. Then $\Sigma \cup \Gamma$ is a topological deformation of $C$, as shown in Figure 10. Define $D_{\text{in}}$ to be the region in the interior of the unit circle but the exterior of $\Sigma \cup \Gamma$, and $D_{\text{out}}$ to be the region in the exterior of the unit circle but the interior of $\Sigma \cup \Gamma$ (again see Figure 10). It is possible one of these regions may be empty. Then define

$$
(4-1) \quad O_{m,n}(z; x) := \begin{cases} 
N_{m,n}(z; x) \left[ \begin{array}{c} 1 \\ \frac{1}{2\pi i} e^{-n\theta(z;x,r)} \\ 0 \\ 1 \end{array} \right], & z \in D_{\text{in}}, \\
N_{m,n}(z; x) \left[ \begin{array}{c} 1 \\ \frac{-1}{2\pi i} e^{-n\theta(z;x,r)} \\ 0 \\ 1 \end{array} \right], & z \in D_{\text{out}}, \\
N_{m,n}(z; x), & z \in \mathbb{C}\backslash(D_{\text{in}} \cup D_{\text{out}}).
\end{cases} 
$$

Now $O_{m,n}(z; x)$ satisfies exactly the same Riemann-Hilbert problem as $N_{m,n}(z; x)$ (i.e. Riemann-Hilbert Problem 2) with $C$ replaced by $\Sigma \cup \Gamma$. 
We introduce the lens regions $\Omega_{\pm}$. Recall that $\phi$.

4.2. Introduction of the $g$-function ($O_{m,n} \to P_{m,n}$). Define
\begin{equation}
P_{m,n}(z;x) := e^{-n\ell\sigma_3/2}O_{m,n}(z;x)e^{-n(g(z;x,r)-\ell/2)\sigma_3}.
\end{equation}

The jump for $z \in C$ is
\begin{equation}
V^{(P)}_{m,n} = P_{m,n}^{-1}P_{m,n+} = \begin{bmatrix}
e^{-n(g_+-g_-)} & \frac{1}{2\pi i}e^{-n(g_+(z;x,r)-\ell/2)} \\ 0 & e^{n(g_+-g_-)}\end{bmatrix}.
\end{equation}

Recall that $\varphi(z;x,r)$ is defined in (3-3). Note from (3-1) that $g_+(z) - g_-(z) = -\varphi_+(z) = \varphi_-(z)$ for $z \in \Sigma$. Also taking into account the asymptotic behavior (3-2), we are led to the following Riemann-Hilbert problem.

**Riemann-Hilbert Problem 5** (Introduction of $\varphi$). Fix a complex number $x$ in the genus-zero region and $m,n \in \mathbb{N}$ with $m \geq n$ and set $r = m/n$. Determine the unique $2 \times 2$ matrix $P_{m,n}(z;x)$ with the following properties:

**Analyticity:** $P_{m,n}(z;x)$ is analytic for $z \in \mathbb{C}$ except on $\Sigma \cup \Gamma$ where it achieves Hölder-continuous boundary values. See Figure 10.

**Jump condition:** The boundary values taken by $P_{m,n}(z;x)$ are related by the jump conditions
\begin{equation}
P_{m,n+}(z;x) = P_{m,n-}(z;x)V^{(P)}_{m,n}(z;x), \text{ where}
\end{equation}
\begin{equation}
V^{(P)}_{m,n+}(z;x) = \begin{cases}
e^{n\varphi_+(z;x,r)} & \frac{1}{2\pi i}e^{n\varphi_-(z;x,r)} \\ 0 & e^{n\varphi_-(z;x,r)}\end{cases}, & z \in \Sigma,
\end{equation}
\begin{equation}
V^{(P)}_{m,n+}(z;x) = \begin{cases}
1 & \frac{1}{2\pi i}e^{-n\varphi_+(z;x,r)} \\ 0 & 1\end{cases}, & z \in \Gamma.
\end{equation}

**Normalization:** As $z \to \infty$, the matrix $P_{m,n}(z;x)$ satisfies the condition
\begin{equation}
P_{m,n}(z;x) = I + O(z^{-1})
\end{equation}
with the limit being uniform with respect to direction.

4.3. Opening of the lenses ($P_{m,n} \to Q_{m,n}$). On $\Sigma$, the jump matrix $V^{(P)}_{m,n}$ has the factorization
\begin{equation}
\begin{bmatrix}
e^{n\varphi_+} & 1 \\ 0 & e^{-n\varphi_-}\end{bmatrix} = \begin{bmatrix}
1 & 0 \\ 2\pi i e^{n\varphi_-} & 1\end{bmatrix}\begin{bmatrix}
0 & \frac{1}{2\pi i} \\ -2\pi i & 0\end{bmatrix}\begin{bmatrix}
1 & 0 \\ 2\pi i e^{n\varphi_+} & 1\end{bmatrix}.
\end{equation}

We introduce the lens regions $\Omega_{\pm}$ and the lens boundaries $L_{\pm}$ as shown in Figure 11. The boundaries $L_{\pm}$
are taken to lie inside the regions in which $\Re \varphi(z) < 0$ and be such that $0 \notin (\Omega_+ \cup \Omega_-)$. Make the change of variables

$$Q_{m,n}(z; x) := \begin{cases} \begin{bmatrix} 1 & 0 \\ -2\pi i e^{n\varphi(z;x,r)} & 1 \end{bmatrix}, & z \in \Omega_+, \\ \begin{bmatrix} 1 & 0 \\ 2\pi i e^{-n\varphi(z;x,r)} & 1 \end{bmatrix}, & z \in \Omega_-, \\ P_{m,n}(z; x), & \text{otherwise}. \end{cases}$$

We have the following Riemann-Hilbert problem.

**Riemann-Hilbert Problem 6** (Lens-opened problem). Fix a complex number $x$ in the genus-zero region and $m, n \in \mathbb{N}$ with $m \geq n$, and set $r = m/n$. Determine the unique $2 \times 2$ matrix $Q_{m,n}(z; x)$ with the following properties:

- **Analyticity:** $Q_{m,n}(z; x)$ is analytic for $z \in \mathbb{C} \setminus \{\Sigma \cup \Gamma \cup L_+ \cup L_-\}$ with Hölder-continuous boundary values. See Figure 11.

- **Jump condition:** The boundary values taken by $Q_{m,n}(z; x)$ are related by the jump condition $Q_{m,n+}(z; x) = Q_{m,n-}(z; x)V_{m,n}(Q_{m,n})(z; x)$, where

$$V_{m,n}(Q_{m,n})(z; x) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -2\pi i & 0 \end{bmatrix}, & z \in \Sigma, \\ \begin{bmatrix} 1 & 0 \\ 2\pi i e^{n\varphi(z;x,r)} & 1 \end{bmatrix}, & z \in L_+, \\ \begin{bmatrix} 1 & 0 \\ 2\pi i e^{-n\varphi(z;x,r)} & 1 \end{bmatrix}, & z \in \Gamma. \end{cases}$$

- **Normalization:** As $z \to \infty$, the matrix $Q_{m,n}(z; x)$ satisfies the condition

$$Q_{m,n}(z; x) = I + \mathcal{O}(z^{-1})$$

with the limit being uniform with respect to direction.

4.4. **The model and error problems.** The jumps for $Q_{m,n}(z)$ decay to the identity matrix except for $z \in \Sigma$ (although this decay is not uniform near $a$ and $b$). We now define a model solution $R_{m,n}(z)$ that is a good approximation for $Q_{m,n}(z)$ (up to $\mathcal{O}(n^{-1})$) everywhere in the complex plane. We begin by defining the outer model Riemann-Hilbert problem, which is obtained by neglecting all decaying jumps.
Riemann-Hilbert Problem 7 (The outer model problem). Fix a complex number $x$ in the genus-zero region and $m,n \in \mathbb{N}$ with $m \geq n$ and set $r = m/n$. Determine the unique $2 \times 2$ matrix $R_{m,n}^{(\text{out})}(z;x)$ with the following properties:

**Analyticity:** $R_{m,n}^{(\text{out})}(z;x)$ is analytic in $z$ except on $\Sigma$ with Hölder-continuous boundary values in the interior of $\Sigma$ and at worst quarter-root singularities at the endpoints.

**Jump condition:** The boundary values taken by $R_{m,n}^{(\text{out})}(z;x)$ on $\Sigma$ are related by the jump condition

$$R_{m,n}^{(\text{out})}(z;x) = R_{m,n}^{(\text{out})}(z;x) \begin{bmatrix} 0 & \frac{1}{2\pi i} \\ -2\pi i & 0 \end{bmatrix}. \tag{4-10}$$

**Normalization:** As $z \to \infty$, the matrix $R_{m,n}^{(\text{out})}(z;x)$ satisfies the condition

$$R_{m,n}^{(\text{out})}(z;x) = 1 + O(z^{-1}) \tag{4-11}$$

with the limit being uniform with respect to direction.

This constant-jump problem can be solved in a standard way by diagonalizing the matrix (thereby reducing the problem to two scalar problems) and then using the Plemelj formula. Alternately, it is straightforward to check that Riemann-Hilbert Problem 7 is satisfied by

$$R_{m,n}^{(\text{out})}(z;x) := \begin{bmatrix} \gamma(z;x,r) + \gamma(z;x,r)^{-1} & \gamma(z;x,r) - \gamma(z;x,r)^{-1} \\ \pi(\gamma(z;x,r) - \gamma(z;x,r)^{-1}) & \frac{4\pi}{2} \gamma(z;x,r) + \gamma(z;x,r)^{-1} \end{bmatrix}, \tag{4-12}$$

where

$$\gamma(z;x,r) := \left(\frac{z-a}{z-b}\right)^{1/4} \tag{4-13}$$

is analytic for $z \notin \Sigma$ and satisfies $\lim_{z \to \infty} \gamma(z) = 1$.

The outer model solution $R_{m,n}^{(\text{out})}(z)$ is a good approximation of $Q_{m,n}(z)$ for all $z$ except in small $n$-independent neighborhoods $\mathbb{D}_a$ and $\mathbb{D}_b$ of the band endpoints $a$ and $b$, respectively. Here the decay of the jumps on $L_{\pm}$ and $\Gamma$ to the identity is not uniform. However, it is possible to construct functions $R_{m,n}^{(a)}(z)$ and $R_{m,n}^{(b)}(z)$ in terms of Airy functions that solve the Riemann-Hilbert problem exactly in their respective neighborhood and closely match the outer parametrix $R_{m,n}^{(\text{out})}(z)$ on the boundaries. The construction of Airy parametrices is standard (see, for example, [16, 4]). Here we follow [2, §4.1]. First, we have the local expansions

$$\varphi(z) = C_a(z-a)^{3/2} + O((z-a)^{5/2}), \quad z \in \mathbb{D}_a, \tag{4-14}$$
$$\varphi(z) = 2\pi i + C_b(z-b)^{3/2} + O((z-a)^{5/2}), \quad z \in \mathbb{D}_b,$$

(for appropriate choices of the square roots) where $C_a$ and $C_b$ are nonzero and independent of $z$. Then define two local coordinates

$$s_a(z) := e^{\pi \left(\frac{3m}{4}\right)^{2/3} \phi(z)^{2/3}} \text{ for } z \in \mathbb{D}_a; \quad s_b(z) := \left(\frac{3m}{4}\right)^{2/3} (\phi(z) - 2\pi i)^{2/3} \text{ for } z \in \mathbb{D}_b \tag{4-15}$$

such that if $z \in \mathbb{D}_a$ then $\Gamma$ is mapped to the negative real axis, while if $z \in \mathbb{D}_b$ then $\Gamma$ is mapped to the positive real axis. Set $V := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and define the analytic prefactors

$$B_a(z) := R_{m,n}^{(\text{out})}(z)(2\pi i)^{-\sigma_a/2} \begin{bmatrix} -i & -i \\ 1 & 1 \end{bmatrix} (e^{-\pi i s_a(z)})^{\sigma_a/4}, \tag{4-16}$$
$$B_b(z) := R_{m,n}^{(\text{out})}(z)(2\pi i)^{-\sigma_a/2} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} s_b(z)^{-\sigma_b/4}.$$
Let $A(s)$ be the function defined in [2, (A.1)–(A.2)] and built out of Airy functions with jumps on $\arg(s) \in \{0, \pm \frac{2\pi}{3}, \pi\}$ as given in [2, Figure 19] and satisfying

$$
A(s) = \frac{s^{\sigma_3/4}}{2\sqrt{\pi}} \left[ -1 - i \left( \frac{1}{48s^{3/2}} \left[ \begin{array}{cc} 1 & 6i \\ 6i & -1 \end{array} \right] + O(s^{-3}) \right) \right] e^{-2s^{3/2}\sigma_3/3}, \quad s \to \infty.
$$

Also let $\tilde{A}(s)$ be the function defined in [2, (A.4)] and built out of Airy functions with jumps on $\arg(s) \in \{0, \pm \frac{2\pi}{3}, \pi\}$ as given in [2, Figure A.1] and satisfying

$$
\tilde{A}(s) = \frac{(e^{-i\pi}s)^{-\sigma_3/4}}{2\sqrt{\pi}} \left[ 1 - i \left( \frac{1}{48s^{3/2}} \left[ \begin{array}{cc} -1 & 6i \\ 6i & 1 \end{array} \right] + O(s^{-3}) \right) \right] e^{-2is^{3/2}\sigma_3/3}, \quad s \to \infty.
$$

Then the Airy parametrix are

$$
A^{(a)}_{m,n}(z) := i\sqrt{\pi}B_{m,n}(z)A(s_a(z))e^{2is_a(z)^{3/2}\sigma_3/3}(2\pi i)^{\sigma_3/2}, \quad z \in \mathbb{D}_a,
$$

$$
A^{(b)}_{m,n}(z) := -i\sqrt{\pi}B_{m,n}(z)A(s_b(z))e^{2is_b(z)^{3/2}\sigma_3/3}(2\pi i)^{\sigma_3/2}, \quad z \in \mathbb{D}_b.
$$

The explicit form of the parametrix is only necessary to recover the $O(n^{-1})$ terms in the solution of the Riemann-Hilbert problem. For us it suffices to know that $A^{(a)}_{m,n}(z) = A^{(a)}_{m,n}(z)$ satisfies the same jump conditions as $Q_{m,n}(z)$ for $z \in \mathbb{D}_a$, $A^{(b)}_{m,n}(z) = A^{(b)}_{m,n}(z)$ satisfies the same jump conditions as $Q_{m,n}(z)$ for $z \in \mathbb{D}_b$, and

$$
R^{(a)}_{m,n}(z) = R^{(out)}_{m,n}(z)(I + O(n^{-1})) \quad \text{for } z \in \partial\mathbb{D}_a; \quad R^{(b)}_{m,n}(z) = R^{(out)}_{m,n}(z)(I + O(n^{-1})) \quad \text{for } z \in \partial\mathbb{D}_b
$$

uniformly for $x$ in the genus-zero region bounded away from the corners of the elliptic region. At the corners one of the band endpoints collides with the third critical point and a different parametrix is required (see [5] for a related analysis for the rational Painlevé-II functions).

The global model solution is now defined as

$$
R_{m,n}(z;x) := \begin{cases} R^{(out)}_{m,n}(z;x), \quad z \in \mathbb{C}\setminus(\mathbb{D}_a \cup \mathbb{D}_b), \\ R^{(a)}_{m,n}(z;x), \quad z \in \mathbb{D}_a, \\ R^{(b)}_{m,n}(z;x), \quad z \in \mathbb{D}_b. 
\end{cases}
$$

The error or ratio function is

$$
S_{m,n}(z;x) := Q_{m,n}(z;x)R_{m,n}(z;x)^{-1}.
$$

It satisfies the following Riemann-Hilbert problem. Note in particular that $S_{m,n}(z)$ has no jump across $\Sigma$ or inside $\mathbb{D}_a$ or $\mathbb{D}_b$, but does have jumps across $\partial\mathbb{D}_a$ and $\partial\mathbb{D}_b$.

**Riemann-Hilbert Problem 8 (The error problem).** Fix a complex number $x$ in the genus-zero region and $m, n \in \mathbb{N}$ with $m \geq n$ and set $r = m/n$. Determine the unique $2 \times 2$ matrix $S_{m,n}(z;x)$ with the following properties:

- **Analyticity:** $S_{m,n}(z;x)$ is analytic in $z$ except on $J^{(S)} := \partial\mathbb{D}_a \cup \partial\mathbb{D}_b \cup ((L_+ \cup L_- \cup \Gamma) \cap (\mathbb{D}_a \cup \mathbb{D}_b)^c)$ with Hölder-continuous boundary values. We orient $\partial\mathbb{D}_a$ and $\partial\mathbb{D}_b$ clockwise. See Figure 12.
- **Jump condition:** The boundary values taken by $S_{m,n}(z;x)$ are related by the jump conditions $S_{m,n+}(z;x) = S_{m,n-}(z;x)V^{(S)}_{m,n}(z;x)$, where

$$
V^{(S)}_{m,n}(z;x) = \begin{pmatrix} 1 & 0 \\ 2\pi i e^{n\varphi(z;x,r)} & 1 \end{pmatrix} \begin{pmatrix} R^{(out)}_{m,n}(z;x) \\ 0 \end{pmatrix} R^{(out)}_{m,n}(z;x)^{-1}, \quad z \in L_+ \cap (\mathbb{D}_a \cup \mathbb{D}_b)^c,
$$

$$
\begin{pmatrix} 0 & 1 \\ 2\pi i e^{n\varphi(z;x,r)} & 0 \end{pmatrix} \begin{pmatrix} R^{(out)}_{m,n}(z;x) \\ 1 \end{pmatrix} R^{(out)}_{m,n}(z;x)^{-1}, \quad z \in \Gamma \cap (\mathbb{D}_a \cup \mathbb{D}_b)^c,
$$

$$
\begin{pmatrix} R^{(a)}_{m,n}(z;x) R^{(out)}_{m,n}(z;x)^{-1} \\ R^{(b)}_{m,n}(z;x) R^{(out)}_{m,n}(z;x)^{-1} \end{pmatrix}, \quad z \in \partial\mathbb{D}_a,
$$

$$
\begin{pmatrix} R^{(a)}_{m,n}(z;x) R^{(out)}_{m,n}(z;x)^{-1} \\ R^{(b)}_{m,n}(z;x) R^{(out)}_{m,n}(z;x)^{-1} \end{pmatrix}, \quad z \in \partial\mathbb{D}_b.
$$
Normalization: As \( z \to \infty \), the matrix \( S_{m,n}(z; x) \) satisfies
\[
S_{m,n}(z; x) = I + O(z^{-1})
\]
with the limit being uniform with respect to direction.

![Figure 12. The jump contours \( J^{(S)} \) for the error problem \( S_{m,n}(z; x) \) in the complex \( z \)-plane for \( r = 1 \) and \( x = 1.1 \).](image)

We now show that the jump matrices for the error solution \( S_{m,n} \) are small as \( n \to \infty \).

**Lemma 4.** Fix \( \delta > 0 \). Then for \( z \in J^{(S)} \)
\[
V_{m,n}^{(S)}(z; x) = I + O \left( \frac{1}{n} \right)
\]
with the error term uniform in \( x \) if \( \text{dist}(x, E_r) > \delta \).

**Proof.** For \( z \in \partial \mathbb{D}_a \cup \partial \mathbb{D}_b \), the necessary estimate is given by (4-20). What remains is to show that, in the genus-zero region, the signature chart of \( \Re(\varphi(z)) \) has the topology shown in the genus-zero plots in Figures 7 and 8. More specifically, we need to show that (except for the endpoints \( a \) and \( b \), \( L_{\pm} \) can be placed entirely in a region in which \( \Re(\varphi(z)) < 0 \), and \( \Gamma \) entirely in a region in which \( \Re(\varphi(z)) > 0 \). If so, then with these choices we find that \( V_{m,n}^{(S)}(z; x) \) is exponentially close to the identity on the relevant parts of \( L_{\pm} \) and \( \Gamma \), and so (4-25) holds.

As a level set of a function that is harmonic except on \( \Sigma \) and at \( z = 0 \), \( \{ z : \Re(\varphi(z)) = 0 \} \) consists of a finite number of smooth arcs. Local analysis at infinity shows there are no zero-level lines of \( \Re(\varphi(z)) \) there. The only points at which two or more zero-level lines can intersect are the critical points \( a \), \( b \), and \( c \) (see (3-16)) or the origin. A direct calculation shows that \( a \) and \( b \) are distinct and nonzero. We also saw in §3.2 that \( c \) cannot be zero since \( Q \) has no finite singularities. Therefore, we can assume all four points \( a \), \( b \), \( c \), and \( 0 \) are distinct. By construction, \( \Re(\varphi(a)) = \Re(\varphi(b)) = 0 \), and local analysis shows there are three zero-level lines of \( \Re(\varphi(z)) \) emanating from both \( a \) and \( b \). Similarly, local analysis at the pole \( z = 0 \) shows there are four zero-level lines of \( \Re(\varphi(z)) \) intersecting at the origin. As we have seen in §3.2, \( \Re(\varphi(c)) \) is generically nonzero, but there are four semi-infinite arcs in the complex \( z \)-plane along which \( \Re(\varphi(c)) = 0 \), in which case four zero-level lines of \( \Re(\varphi(z)) \) intersect at \( c \).

First, assume \( x \) is such that \( \Re(\varphi(c)) \neq 0 \). In this case we have three arcs each emerging from \( a \) and \( b \) and four from 0. Therefore not all the arcs from \( a \) and \( b \) can connect to the origin, and at least one must join \( a \) and \( b \). Closed contours that are level lines of harmonic functions must enclose singularities, and so there are two options: either a second arc connects \( a \) to \( b \) and passes around the opposite side of the origin from the first such arc, or the other four arcs from \( a \) and \( b \) all connect to the origin. We are in the first situation for \( x \) sufficiently large and either \( x \) purely real or purely imaginary (for illustration see the plots with \( x = 1.2 \) and \( x = 1.2i \) in Figure 7). In this case the signature chart necessarily has the form show in those plots since \( \Re(\varphi(z)) < 0 \) for \( z \) sufficiently large. The only allowable mechanism for the contour topology to change as \( x \) varies is for \( c \) to intersect a zero-level line of \( \Re(\varphi(z)) \), which we have seen only occurs on the semi-infinite
arcs. Therefore, off these four arcs we see that the contours $L_{\pm}$ and $\Gamma$ can be chosen appropriately in the exterior of the elliptic region.

We now consider $x$ such that $\Re(\varphi(z)) = 0$. The signature chart at a corner point can be seen to have the form shown in the plot with $x = x_0$ in Figure 7. It is possible to continuously vary $x$ to the value that interests us keeping $c$ on a zero-level curve of $\Re(\varphi(z))$. Therefore the signature chart of $\Re(\varphi(z))$ must (topologically) have the form illustrated in the plot with $x = 1.2 + 1.2i$ in Figure 7, from which it is clear the contours $L_{\pm}$ and $\Gamma$ can be chosen as needed. \hfill $\square$

We have finally arrived at a small-norm Riemann-Hilbert problem for $S_{m,n}(z;x)$, that is, one with jumps close to the identity. The following analysis is standard (see, for example, [17] or [4, Appendix B]). Recursively define the functions

$$U_0(z) := \mathbb{I}, \quad U_k(z) := -\frac{1}{2\pi i} \int_{J^{(S)}} U_{k-1}(u)(V_{m,n}^{(S)}(u) - 1) \frac{du}{z - u},$$

in which $J^{(S)}$ means the integration is performed along the minus-side of $J^{(S)}$. Then $S_{m,n}(z)$ is the sum of an infinite Neumann series:

$$S_{m,n}(z) = \mathbb{I} - \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_{J^{(S)}} U_{k-1}(u)(V_{m,n}^{(S)}(u) - 1) \frac{du}{z - u}.$$

This gives us the bound

$$S_{m,n}(z) = \left( \mathbb{I} + O\left(\frac{1}{(|z| + 1)n}\right) \right), \quad n \to \infty$$

that holds uniformly for $z \in \mathbb{C}\setminus J^{(S)}$ and for $x$ a fixed distance away from the elliptic region.

4.5. The asymptotic expansion. We now prove the main theorems.

Proof of Theorems 1, 2, and 3. Retracing the various transformations gives

$$N_{m,n}(z;x) = e^{ng(z;x,r) - 4\sigma_3/3} S_{m,n}(z;x) R_{m,n}(z;x) e^{ng(z;x,r) - 4\sigma_3}, \quad z \in \mathbb{C}\setminus\{\Omega_+ \cup \Omega_- \cup D_{in} \cup D_{out}\}.$$

We therefore have

$$N_{m,n}(z) = \left( \mathbb{I} + O\left(\frac{1}{(|z| + 1)n}\right) \right) \begin{bmatrix} \gamma(z) + \gamma(z)^{-1} e^{ng(z)} & \gamma(z) - \gamma(z)^{-1} e^{-n(g(z) - \ell)} \\ \pi(\gamma(z) - \gamma(z)^{-1}) e^{ng(z) - \ell} & \gamma(z) + \gamma(z)^{-1} e^{-ng(z)} \end{bmatrix},$$

$$z \in \mathbb{C}\setminus\{\Omega_+ \cup \Omega_- \cup D_{in} \cup D_{out} \cup D_{a} \cup D_{b} \cup J^{(S)}\}.$$

In particular, this expression holds for $z = 0$ and for $|z|$ sufficiently large. We expand $g(z)$ and $\gamma(z)$ about $z = 0$:

$$g(z;x,r) = g_0(x,r) + g_1(x,r)z + O(z^2), \quad \gamma(z;x,r) = \gamma_0(x,r) + \gamma_1(x,r)z + O(z^2),$$

wherein

$$\gamma_0 = \left(\frac{a}{b}\right)^{1/4}, \quad \gamma_1 = \frac{a - b}{4ab} \left(\frac{a}{b}\right)^{1/4}.$$

(interestingly, it will turn out that we will not need the explicit form of $g_0$ or $g_1$). Thus, recalling the expansion (2-36) for $N_{m,n}$, we compute

$$[N_0]_{11} = \gamma_0 + \frac{1}{2} \gamma_0^{-1} e^{ng_0}(1 + O(n^{-1})), \quad [N_0]_{12} = \gamma_0 - \frac{1}{4\pi} e^{-n(g_0 - \ell)}(1 + O(n^{-1})), \quad [N_0]_{21} = \pi(\gamma_0 - \gamma_0^{-1}) e^{ng_0}(1 + O(n^{-1})), \quad [N_0]_{22} = \gamma_0 + \frac{1}{2} \gamma_0^{-1} e^{-ng_0}(1 + O(n^{-1})).$$

$$[N_1]_{11} = \frac{1}{2} \left[ (\gamma_0 + \gamma_0^{-1})ng_1 + \gamma_1 - \frac{1}{\gamma_0} \right] e^{ng_0}(1 + O(n^{-1})), \quad [N_1]_{21} = \pi \left[ (\gamma_0 - \gamma_0^{-1})ng_1 + \gamma_1 + \frac{1}{\gamma_0} \right] e^{ng_0}(1 + O(n^{-1})).$$
Inserting these into (2-37) gives

\[ \frac{1}{n^{1/2}} w_{m,n}^{(II)}(m^{1/2}x) = \frac{2}{\gamma_0(x,r)} \left( \frac{1 - \gamma_0(x,r)^2}{1 + \gamma_0(x,r)^2} \right) + \mathcal{O}(n^{-1}). \]

Using the expressions (4-32) for \( \gamma_0 \) and \( \gamma_1 \) produces

\[ \frac{1}{n^{1/2}} w_{m,n}^{(II)}(m^{1/2}x) = \frac{1}{(a(x,r)b(x,r))^{1/2} - \frac{a(x,r) + b(x,r)}{2a(x,r)b(x,r)}} + \mathcal{O}(n^{-1}). \]

Finally, using the identities \( S = a + b \) and \( Q = -(ab)^{1/2} \) gives (1-22) in the genus-zero region. This completes the proof of Theorem 1. See also Figure 5.

Next, we compute the asymptotic expansion of \( w_{m,n}^{(II)} \), starting from Lemma 3. From (4-30) we have

\[ [N_{m,n}(z)]_{12} = \frac{\gamma(z) - \gamma(z)^{-1}}{4\pi} e^{-n(g(z) - \ell)} (1 + \mathcal{O}(n^{-1})) \]

for \( x \) in the genus-zero region. We expand \( \gamma(z) \) at infinity as

\[ \gamma(z) = 1 + \frac{b - a}{4z} + \mathcal{O}\left(\frac{1}{z^2}\right). \]

Using the last two equations along with \( g(z) = \log(z) + \mathcal{O}(z^{-1}) \) and the expansion (2-46) shows

\[ [N_{-1}]_{12} = \frac{(b - a)e^{nt}}{8\pi}(1 + \mathcal{O}(n^{-1})). \]

Taking this along with (4-33) and then (4-32) shows

\[ \frac{[N_0]_{11}[N_0]_{12}}{[N_{-1}]_{12}} = \frac{\gamma_0^2 - \gamma_0^{-2}}{b - a} = -\frac{1}{(ab)^{1/2}} = \frac{1}{Q}. \]

We now plug this and (1-22) into the result of Lemma 3 to see

\[ \frac{1}{n^{1/2}} w_{m+1,n}^{(II)}((m + 1)^{1/2}x) = \left( \frac{1}{Q(x,r)} - \frac{S(x,r)}{2Q(x,r)^2} + \mathcal{O}(n^{-1}) \right) (1 + \mathcal{O}(m^{-1})). \]

As long as we agree \( r = \frac{m}{n} \) is fixed, we can replace \( \mathcal{O}(m^{-1}) \) with \( \mathcal{O}(n^{-1}) \). Therefore, we have

\[ \frac{1}{n^{1/2}} w_{m,n}^{(II)}(m^{1/2}x) = \frac{1}{Q(x, \frac{m}{n}-1)} - \frac{S(x, \frac{m}{n}-1)}{2Q(x, \frac{m}{n}-1)^2} + \mathcal{O}(n^{-1}). \]

From the dependence of \( Q(x,r) \) and \( S(x,r) \) on \( r \), we can replace \( Q(x, \frac{m}{n}-1) \) and \( S(x, \frac{m}{n}-1) \) with \( Q(x,r) \) and \( S(x,r) \), respectively, at the price of an \( \mathcal{O}(n^{-1}) \) error, so we obtain our final result (1-23). This completes the proofs of Theorems 2 and 3 (as Theorem 3 follows immediately from Theorems 1 and 2). See Figure 6 for plots demonstrating the convergence for \( w_{m,n}^{(II)}. \)

\[ \square \]

REFERENCES

[1] F. Balogh, M. Bertola, and T. Bothner, Hankel determinant approach to generalized Vorob'ev-Yablonski polynomials and their roots, *Constr. Approx.* 44, 417–453 (2016).
[2] M. Bertola and T. Bothner, Zeros of large degree Vorob'ev-Yablonski polynomials via a Hankel determinant identity, *Int. Math. Res. Not. IMRN* 2015, 9330–9399 (2015).
[3] R. Buckingham and P. Miller, The sine-Gordon equation in the semiclassical limit: critical behavior near a separatrix, *J. Anal. Math.* 118, 397–492 (2012).
[4] R. Buckingham and P. Miller, Large-degree asymptotics of rational Painlevé-II functions: noncritical behaviour, *Nonlinearity* 27, 2489–2577 (2014).
[5] R. Buckingham and P. Miller, Large-degree asymptotics of rational Painlevé-II functions: critical behaviour, *Nonlinearity* 28, 1539–1596 (2015).
[6] Y. Chen and M. Feigin, Painlevé IV and degenerate Gaussian unitary ensembles, *J. Phys. A* 39, 12381–12393 (2006).
[7] P. Clarkson, The fourth Painlevé equation and associated special polynomials, *J. Math. Phys.* 44, 5350–5374 (2003).
[8] P. Clarkson, The third Painlevé equation and associated special polynomials, *J. Phys. A* 36, 9507–9532 (2003).
[9] P. Clarkson, Special polynomials associated with rational solutions of the fifth Painlevé equation, *J. Comput. Appl. Math.* 178, 111–129 (2005).
[10] P. Clarkson, Special polynomials associated with rational solutions of the defocusing nonlinear Schrödinger equation and the fourth Painlevé equation, *European J. Appl. Math.* 3, 293–322 (2006).
[11] P. Clarkson, Rational solutions of the Boussinesq equation, *Anal. Appl. (Singap.)* 6, 349–369 (2008).
[12] P. Clarkson, Rational solutions of the classical Boussinesq system, *Nonlinear Anal. Real World Appl.* 10, 3360–3371 (2009).
[13] P. Clarkson, Vortices and polynomials, Stud. Appl. Math. 123, 37–62 (2009).
[14] P. Clarkson and E. Mansfield, The second Painlevé equation, its hierarchy and associated special polynomials, Nonlinearity 16, R1–R26 (2003).
[15] P. Clarkson and B. Thomas, Special polynomials and exact solutions of the dispersive water wave and modified Boussinesq equations, in Proceedings of Group Analysis of Differential Equations and Integrable Systems IV, 62–76 (2009).
[16] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, Comm. Pure Appl. Math. 52, 1335–1425 (1999).
[17] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, Ann. of Math. (2) 137, 295–368 (1993).
[18] G. Felder, A. Hemery, and A. Veselov, Zeros of Wronskians of Hermite polynomials and Young diagrams, Physica D 241, 2131–2137 (2012).
[19] G. Filipuk and P. Clarkson, The symmetric fourth Painlevé hierarchy and associated special polynomials, Stud. Appl. Math. 121, 157–188 (2008).
[20] A. Fokas, A. Its, and A. Kitaev, Discrete Painlevé equations and their appearance in quantum gravity, Comm. Math. Phys. 142, 313–344 (1991).
[21] M. Ismail, Determinants with orthogonal polynomial entries. J. Comput. Appl. Math. 178 (2005), 255–266.
[22] K. Kajiwara and Y. Ohta, Determinant structure of the rational solutions for the Painlevé IV equation, J. Phys. A 31, 2431–2446 (1998).
[23] S. Karlin and G. Szegö, On certain determinants whose elements are orthogonal polynomials, J. Anal. Math. 8, 1–157 (1961).
[24] C. Krattenthaler, Advanced determinant calculus. Sm. Lothar. Combin. 42 (1999), Art. B42q.
[25] N. Kudryashov and M. Demina, Relations between zeros of special polynomials associated with the Painlevé equations, Phys. Lett. A 368, 227–234 (2007).
[26] N. Lukashevich, The theory of Painlevé’s fourth equation (Russian), Differential’nye Uравнения 3, 771–780 (1967).
[27] V. Marikhin, Representation of a Coulomb gas for rational solutions of the Painlevé equations, Theoret. and Math. Phys. 127, 646–663 (2001). Translated from Teoret. Mat. Fiz. 127, 248–303 (2001).
[28] I. Marquette and C. Quesne, Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators, J. Phys. A 46 155201 (2013).
[29] I. Marquette and C. Quesne, Connection between quantum systems involving the fourth Painlevé transcendent and k-step rational extensions of the harmonic oscillator related to Hermite exceptional orthogonal polynomial, J. Math. Phys. 57, 052101 (2016).
[30] P. Miller and Y. Sheng, Rational solutions of the Painlevé-II equation revisited, arXiv:1704.04851 (2017).
[31] Y. Murata, Rational solutions of the second and the fourth Painlevé equations, Funkcial. Ekvac. 28, 1–32 (1985).
[32] M. Noumi and Y. Yamada, Symmetries in the fourth Painlevé equation and Okamoto polynomials, Nagoya Math. J. 153, 53–86 (1999).
[33] V. Novokshenov and A. Schelkonogov, Distribution of zeroes to generalized Hermite polynomials, Ufa Math. J. 7, 54–66 (2015).

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