Solutions of Higher Order Linear Differential Equations

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Abstract. We show that the higher order linear differential equation possesses all solutions of infinite order under certain conditions by extending the work of authors about second order differential equation [7].

1. Introduction

For entire functions $A_{m-1}(z), \ldots, A_0(z)$ and $H(z)$, the differential equation

$$f^{(m)} + A_{m-1}(z)f^{(m-1)} + \ldots + A_0(z)f = H(z), m \geq 3$$

has entire functions as its solutions, where $A_0(z), H(z) \neq 0$. If functions $A_{m-1}(z), \ldots, A_0(z)$ are polynomials and $H(z)$ is an entire function of finite order then all solutions of equation (1) have finite order. Therefore, if at least one of the coefficients is transcendental entire then a solution of infinite order of equation (1) exists. The associated homogeneous linear differential equation

$$f^{(m)} + A_{m-1}(z)f^{(m-1)} + \ldots + A_0(z)f = 0$$

has all non-trivial solutions of finite order if and only if all coefficients are polynomials [9]. It is well known that a solution of equation (1) is related to solution of equation (2). The aim of this article is to find a necessary condition for the non-existence of solutions of finite order of equation (1). Wang and Laine [11] proved that solutions of equation (1) are of infinite order when orders of coefficients $A_{m-1}(z), \ldots, A_0(z)$ are all equal. The authors have established certain conditions under which the associated homogeneous differential equation of (1) possesses all solutions having infinite order [5]. The main result of this paper is a generalization of Theorem 2 in [7] to higher order linear differential equations which we state below. We follow the notations $\rho(f)$, $\lambda(f)$ and $\rho_2(f)$ for order of growth, exponent of convergence and hyper-order of growth of entire function $f$ respectively, as used in [4, 5, 6, 7].

Theorem 1. Suppose that there exists a fixed integer $j \in \{1, 2, \ldots, m - 1\}$ such that $\lambda(A_j) < \rho(A_j)$, $A_0(z)$ is a transcendental entire function satisfying $\rho(A_0) \neq \rho(A_j)$ and $\max\{\rho(A_k) : k = 1, 2, \ldots, m - 1, k \neq j\} < \rho(A_0)$. Also, suppose that $H(z)$ is an entire function. If $\lambda(A_j) < \rho(A_j)$ and $A_0(z)$ is a transcendental entire function satisfying $\rho(A_0) \neq \rho(A_j)$ and $\max\{\rho(A_k) : k = 1, 2, \ldots, m - 1, k \neq j\} < \rho(A_0)$. Also, suppose that $H(z)$ is an entire function.
function with \( \rho(H) < \max\{\rho(A_0), \rho(A_j)\} \). Then all transcendental solutions \( f \) of equation (1) satisfies

(a) \( \rho(f) = \infty \)
(b) \( \lambda(f) = \infty \)
(c) \( \rho_2(f) = \max\{\rho(A_0), \rho(A_j)\} \) where \( \max\{\rho(A_0), \rho(A_j)\} \) is a finite quantity.
(d) For every \( c \in \mathbb{C} \), \( \delta(c, f) = 0 \) and therefore, \( f \) has no finite deficient value.

**Remark 1.** Under the hypothesis of Theorem 1, equation (1) may possesses non-constant polynomial solutions. Also, by order consideration of an entire function, we obtain that all non-constant polynomial solutions of equation (1) are of degree less than \( j \), where \( j \in \{1, 2, \ldots, m - 1\} \) is fixed in Theorem 1. However, when \( j = 1 \), then equation (1) has no polynomial solution.

The following examples justify that the conditions in the hypothesis of Theorem 1(a) cannot be relaxed.

**Example 1.** The finite order function \( f(z) = e^{-z^2} \) satisfies the linear differential equation

\[
f''' + e^z f'' - f' - (e^z - 1)f = e^{-z^2}
\]

Here we have, \( \rho(A_k) < \rho(A_0) = \rho(A_j) \) and \( \rho(H) = \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 1 \) and \( j = 2 \), which shows that hypothesis in Theorem 1 are necessary.

**Example 2.** The differential equation

\[
f''' - 2z f'' + e^z f' - (2ze^z - 1)f = (8z + 1)e^{z^2}
\]
is satisfied by the finite order function \( f(z) = e^{z^2} \).

Here we have, \( \rho(A_k) < \rho(A_0) = \rho(A_j) \) and \( \rho(H) > \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 2 \) and \( j = 1 \), which also implies that hypothesis of Theorem 1 are necessary.

**Example 3.** The linear differential equation

\[
f^{(iv)} + f''' - e^z f'' - f' + (2e^z - 1)f = 1
\]
has a finite order solution \( f(z) = e^{-z^2} \), where \( \rho(A_k) < \rho(A_0) = \rho(A_j) \) and \( \rho(H) < \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 1, 3 \) and \( j = 2 \).

**Example 4.** The finite order function \( f(z) = e^{-z^2} \) is a solution of linear differential equation

\[
f''' + (e^z - 1)f'' + e^z f' + e^z f = 1 - e^{z^2 - z}
\]
where \( \rho(A_j) \neq \rho(A_0) = \rho(A_k) \) and \( \rho(H) = \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 2 \) and \( j = 1 \).

**Example 5.** The differential equation

\[
f''' + (e^{z^2} + 1)f'' - e^z f' - (e^{z^2} - e^z)f = 2
\]
has a finite order solution \( f(z) = e^{-z^2} \), where \( \rho(A_j) \neq \rho(A_0) = \rho(A_k) \) and \( \rho(H) < \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 2 \) and \( j = 1 \).

**Example 6.** The differential equation

\[
f''' + f'' + e^z f' + \cos z^2 f = (6 + 6z + 3z^2 e^z + z^3 \cos z^2)
\]
is satisfied by the polynomial \( f(z) = z^2 \) and \( \rho(A_k) < \rho(A_0) \) for \( k = 2 \) and \( \rho(A_0) \neq \rho(A_j) \) and \( \rho(H) = \max\{\rho(A_0), \rho(A_j)\} \) for \( j = 1 \).
Example 7. The function \( f(z) = e^{z^2} \) is a finite order solution of the differential equation

\[
 f''' + e^{-z} f'' + f' - (4z^2 + 2)e^{-z} f = (14z + 8z^3)e^{z^2}
\]

where \( \rho(A_k) < \rho(A_0) = \rho(A_j) \) and \( \rho(H) > \max\{\rho(A_0), \rho(A_j)\} \) for \( k = 1 \) and \( j = 2 \).

2. Auxiliary Results

This section is devoted to the known results which will be useful in proving the main theorem. For a subset \( E \subset (1, \infty) \), \( m(E), m_l(E), \log \text{dens}(E) \) and \( \log \text{dens}_{\alpha}(E) \) denotes the linear measure, logarithmic measure, upper logarithmic density and lower logarithmic density respectively.

The following lemma of Gundersen [3] provides estimates for a meromorphic function outside a set of finite logarithmic measure.

**Lemma 1.** Let \( f \) be a meromorphic function and let \( \Gamma = \{(k_1, j_1), \ldots, (k_p, j_p)\} \) be the set of distinct pairs of integers such that \( k_i > j_i \geq 0 \) for \( t = 1, 2, \ldots, p \). Let \( \alpha > 1 \) and \( \epsilon > 0 \) be given real constants. Then there exists \( E \subset (1, \infty) \) satisfying \( m_l(E) < \infty \) and a constant \( c > 0 \) depending on \( \alpha \) and \( \Gamma \) such that

\[
 \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq c \left( \frac{T(\alpha r, f)}{r} \right)^{(k-j)} - \log^\alpha r \log T(\alpha r, f)
\]

Moreover, if \( f(z) \) is of finite order then \( f(z) \) satisfies:

\[
 \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\epsilon)}
\]

for all \( z \) satisfying \( |z| \notin E \cup [0,1] \) and \( |z| \geq R_0 \) and for all \( (k, j) \in \Gamma \).

The next lemma is used to establish estimates for a transcendental entire function.

**Lemma 2.** [1] Let \( A(z) = v(z)e^{P(z)} \) be an entire function, where \( P(z) \) is a polynomial of degree \( n \) and \( v(z) \) is an entire function of order less than \( n \). Then for every \( \epsilon > 0 \) there exists \( E \subset [0, 2\pi) \) of linear measure zero such that

(i) for \( \theta \in [0, 2\pi) \) with \( \delta(P, \theta) > 0 \), there exists \( R > 1 \) satisfying

\[
 \exp\{(1-\epsilon)\delta(P, \theta)r^n\} \leq |A(re^{\theta})| \leq \exp\{(1+\epsilon)\delta(P, \theta)r^n\}
\]

for \( r > R \);

(ii) for \( \theta \in [0, 2\pi) \) with \( \delta(P, \theta) < 0 \), there exists \( R > 1 \) satisfying

\[
 \exp\{(1+\epsilon)\delta(P, \theta)r^n\} \leq |A(re^{\theta})| \leq \exp\{(1-\epsilon)\delta(P, \theta)r^n\}
\]

for \( r > R \).

The following lemma gives upper bound for solutions of equation (2).

**Lemma 3.** [2] Suppose that \( \rho(A_k) \leq \rho' < \infty \) for all \( k = 0, 1, \ldots, m-1 \). If \( f \) is a solution of equation (2) then \( \rho_z(f) \leq \rho' \).

The next lemma provides a lower bound for modulus of an entire function in a neighborhood of a particular \( \theta \in [0, 2\pi) \).

**Lemma 4.** [10] Suppose \( f(z) \) is an entire function of finite order \( \rho(f) \) and \( M(r, f) = |f(re^{i\theta})| \) for every \( r \). Given \( \zeta > 0 \) and \( 0 < C(\rho(f), \zeta) < 1 \) there exists \( 0 < l_0 < 1/2 \) and a
set $S \subset (1, \infty)$ with $\log \text{dens}(S) \geq 1 - \zeta$ such that

$$e^{-5\pi} M(r, f)^{1-C} \leq |f(re^{i\theta})|$$

for all sufficiently large $r \in S$ and for all $\theta$ satisfying $|\theta - \theta_r| \leq l_0$.

The following result is from [9] and includes the central index of an entire function.

**Lemma 5.** Let $f$ be a transcendental entire function, $\delta \in (0, 1/4)$ and $z$ be such that $|z| = r$ and that

$$|f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{2} + \delta}$$

holds. Then there exists a set $F \subset (1, \infty)$ with $m_1(F) < \infty$ such that

$$f^{(p)}(z) = \left(\frac{\nu(r, f)}{z}\right)^p (1 + o(1)) f(z)$$

holds for all non-negative integers $p$ and for all $r \notin F$.

**Remark 2.** If $|f(re^{i\theta})| = M(r, f)$ then equation [8] holds and there exists $F \subset (1, \infty)$ with $m_1(F) < \infty$ such that

$$\left|\frac{f^{(p)}(re^{i\theta})}{f(re^{i\theta})}\right| = \left(\frac{\nu(r, f)}{r}\right)^p (1 + o(1))$$

for all non-negative integers $p$ and for all $r \notin F$. We know that the central index of a transcendental entire function $f$ satisfies $\nu(r, f) \geq 1$, as a result we have

$$\left|\frac{f^{(p)}(re^{i\theta})}{f(re^{i\theta})}\right| \geq \frac{1}{rp}(1 + o(1))$$

holds for all non-negative integers $p$ and $r \notin F$.

The following result [9, Proposition 9.3.2] provides relation between the proximity function of $1/(f - c)$ and characteristic function of $f$.

**Proposition 1.** Let $P(z, f)$ be a polynomial in $f$ and its derivatives with meromorphic coefficients $a_\kappa, \kappa \in I$. Suppose that $f$ is a transcendental meromorphic function solution of $P(z, f) = 0$ and $c$ is a complex number. If $P(z, c) \neq 0$ then

$$m\left(r, \frac{1}{f - c}\right) = O\left(\sum_{\kappa \in I} T(r, a_\kappa)\right) + S(r, f).$$

The next three lemmas provides relation between maximum modulus and characteristic functions of two entire functions under certain conditions.

**Lemma 6.** [7] Suppose $f(z)$ is an entire function with $\rho(f) \in (0, \infty)$. Then for each $\epsilon > 0$, there exists a set $S \subset (1, \infty)$ that satisfies $\log \text{dens}(S) > 0$ and

$$M(r, f) \geq \exp\{\rho(f) - \epsilon\}$$

for all $r$ sufficiently large and $r \in S$.

**Lemma 7.** [7] Let $f(z)$ and $g(z)$ be two meromorphic functions satisfying $\rho(g) < \rho(f)$. Then there exists a set $S \subset (1, \infty)$ with $\log \text{dens}(S) > 0$ such that

$$T(r, g) = o(T(r, f))$$

for sufficiently large $r \in S$. 
Lemma 8. Suppose \( f(z) \) and \( g(z) \) be two entire functions satisfying \( \rho(g) \leq \rho(f) \). Then for \( 0 < \epsilon \leq \min\{3\rho(f)/4, (\rho(f) - \rho(g))/2\} \), there exists \( S \subset (1, \infty) \) with \( \log \text{dens}(S) = 1 \) satisfying

\[
|g(z)| = o(M(|z|, f))
\]

for sufficiently large \( |z| \in S \).

3. Proof of Main Theorem

We state and prove a lemma which will be used in the proof of Theorem 1.

Lemma 9. Suppose \( A_{m-1}(z), \ldots, A_0(z) \) and \( H(z) \) are entire functions and there is an integer \( j \in \{1, 2, \ldots, m - 1\} \) such that \( \rho(A_j) \neq \rho(A_0) \), \( \max\{\rho(A_k) : k = 1, 2, \ldots, m - 1, k \neq j\} \) and \( \rho(H) < \max\{\rho(A_j), \rho(A_0)\} \). Then all transcendental solutions \( f \) of equation (11) of finite order satisfies \( \rho(f) \geq \max\{\rho(A_j), \rho(A_0)\} \).

Proof. Suppose that \( \rho(A_j) < \rho(A_0) \). Then using equation (11), first fundamental theorem of Nevanlinna theory, lemma of logarithmic derivatives and Lemma 7 we have

\[
m(r, A_0) \leq m \left( r, \frac{f^{(m)}}{f} \right) + m \left( r, \frac{f^{(m-1)}}{f} \right) + \ldots + m \left( r, \frac{f^1}{f} \right) + m \left( r, \frac{A_{m-1}}{f} \right) + \ldots + m \left( r, A_1 \right) + m \left( r, \frac{H}{f} \right)
\]

\[
T(r, A_0) \leq O(\log r) + T(r, A_{m-1}) + \ldots + T(r, A_1) + T(r, f) + T(r, H) = O(\log r) + o(T(r, A_0)) + T(r, f)
\]

for all \( r \geq R \) and \( r \in S \) where \( \log \text{dens}(S) > 0 \). Combining the equations, we obtain \( \rho(A_0) \leq \rho(f) \). Similarly, when \( \rho(A_0) < \rho(A_j) \) then using equation (11) we have

\[
|A_j(z)| \leq \left| \frac{f^{(m)}}{f(z)} \right| + \left| A_{m-1}(z) \right| \left| \frac{f^{(m-1)}}{f(z)} \right| + \ldots + \left| A_1(z) \right| \left| \frac{f^{(1)}}{f(z)} \right| + \left| A_{j-1}(z) \right| \left| \frac{f^{(j-1)}}{f(z)} \right| + \ldots \left| A_0(z) \right| \left| \frac{f^0}{f(z)} \right| + \left| A_{j+1}(z) \right| \left| \frac{f^{(j+1)}}{f(z)} \right|
\]

This will imply

\[
m(r, A_j) \leq m \left( r, \frac{f^{(m)}}{f} \right) + m \left( r, \frac{f^{(m-1)}}{f} \right) + \ldots + m \left( r, \frac{f^{(j+1)}}{f} \right) + m \left( r, \frac{f^j}{f} \right) + \ldots
\]

\[
+ m \left( r, A_{m-1} \right) + \ldots + m \left( r, A_0 \right) + m \left( r, \frac{H}{f} \right)
\]
Now using first fundamental theorem of Nevanlinna theory, lemma of logarithmic derivatives, Lemma 5 and 7 we obtain

\[ T(r, A_j) \leq O(\log r) + o(1) + \sum_{k=0, k \neq j}^{m-1} T(r, A_k) + T(r, H) + T(r, f) \]

\[ = O(\log r) + o(1) + o(T(r, A_j)) + T(r, f) \]

for sufficiently large \( r \in S \setminus F \). This will imply that \( \rho(A_j) \leq \rho(f) \).

\[ \square \]

It is to be noted that hypothesis of Lemma 9 are only necessary and not sufficient. Examples 1, 2 and 7 justifies that hypothesis of Lemma 9 are not sufficient. Also, Examples 4 - 6 justifies that hypothesis of Lemma 9 are necessary.

**Proof of Theorem 1**: (a) Suppose there is a transcendental solution \( f \) of equation (1) having finite order. From Lemma 1 there exists a set \( E \subset (1, \infty) \) satisfying \( m_1(E) < \infty \) such that

\[ \left| \frac{f^{(k)}(z)}{f^{(l)}(z)} \right| \leq |z|^{|\rho(f)|, l < k = 1, 2, \ldots, m - 1} \quad (13) \]

for all \( z \) satisfying \( |z| = r \notin E \cup [0, 1] \) and \( |z| \geq R \). Then Lemma 6 implies that there exists \( S_1 \subset (1, \infty) \) satisfying \( 0 < \log \text{dens}(S_1) = \delta \) such that

\[ M(r, A_0) \geq \exp \left( r^{\rho(A_0) - \epsilon} \right) \quad (14) \]

for all \( r \in S_1 \) and \( r > R \). We suppose that \( |f(re^{\theta r})| = M(r, f) \) for each \( r \). From Lemma 4 for \( \delta > 0 \) and \( C \in (0, 1) \), there exists \( l_0 \in (0, 1/2) \) and \( S_2 \subset (1, \infty) \) with \( \log \text{dens}(S_2) \geq 1 - \delta/2 \) such that

\[ e^{-5\pi} M(r, f)^{(1-C)} \leq |f(re^{\theta r})| \]

for all sufficiently large \( r \in S_2 \) and \( \theta \) such that \( |\theta - \theta_0| \leq l_0 \). Using Lemma 5, \( \rho(f) \geq \max\{\rho(A_j), \rho(A_0)\} \) and hence Lemma 6 implies

\[ \frac{|H(z)|}{M(r, f)} \to 0 \quad (15) \]

as \( r \to \infty \) where \( r \in S_3 \subset (1, \infty) \) and \( \log \text{dens}(S_3) = 1 \). We know that

\[ \chi_{S_1 \cap S_2} = \chi_{S_1} + \chi_{S_2} - \chi_{S_1 \cup S_2} \]

and \( \log \text{dens}(S_1 \cup S_2) \leq 1 \) therefore,

\[ \log \text{dens}(S_1 \cap S_2) \geq \log \text{dens}(S_1) + \log \text{dens}(S_2) - \log \text{dens}(S_1 \cup S_2) \]

\[ \geq \delta + 1 - \frac{\delta}{2} - 1 = \frac{\delta}{2} \]

Also,

\[ \log \text{dens}(S_1 \cap S_2 \cap S_3) \geq \log \text{dens}(S_1 \cap S_2) + \log \text{dens}(S_3) - \log \text{dens}(S_1 \cup S_2 \cup S_3) \]

\[ \geq \frac{\delta}{2} + 1 - 1 = \frac{\delta}{2} > 0. \]
As $n_0(E) < \infty$, this gives $\log \text{dens}(S_1 \cap S_2 \cap S_3 \setminus E) > 0$. Hence we can choose $z_q = r_q e^{\theta_q}$ with $r_q \to \infty$ such that
\[
r_q \in (S_1 \cap S_2 \cap S_3 \setminus E), \quad |f(r_q e^{\theta_q})| = M(r_q, f).
\]
We may suppose that there exists a subsequence $(\theta_q)$ such that
\[
\lim_{q \to \infty} \theta_q = \theta_0.
\]
We have $\lambda(A_j) < \rho(A_j)$ therefore, $A_j(z) = v(z)e^{P(z)}$, where $v(z) < \rho(e^{P(z)}) = n \in \mathbb{N}$.

First we consider $\rho(\lambda)$.

(i) if $\delta(P, \theta_0) > 0$, then since $\delta(P, \theta)$ is a continuous function we have,

\[
\frac{1}{2}\delta(P, \theta_0) < \delta(P, \theta_m) < \frac{3}{2}\delta(P, \theta_0)
\]

for all sufficiently large $m \in \mathbb{N}$. From part (i) of Lemma 2 we have

\[
\exp \left((1 - \epsilon)\frac{1}{2}\delta(P, \theta_0)r_q^m\right) \leq |A_j(z)| \leq \exp \left((1 + \epsilon)\frac{3}{2}\delta(P, \theta_0)r_q^m\right)
\]

for sufficiently large $m \in \mathbb{N}$. Using equations (1), (13), (14), (15) and (17) we get

\[
\exp \left(r_q^{\rho(A_0) - \epsilon}\right) \leq M(r, A_0)
\]

\[
\leq \frac{|f^{(m)}(z)|}{f(z)} + |A_{m-1}(z)| \frac{|f^{(m-1)}(z)|}{f(z)} + \cdots + |A_1(z)| \frac{|f'(z)|}{f(z)} + \frac{H(z)}{f(z)}
\]

\[
\leq r^{m\rho(f)}q (1 + |A_{m-1}(z)| + \cdots + |A_j(z)| + \cdots + |A_1(z)|) + o(1)
\]

\[
\leq r^{m\rho(f)}q \left(1 + \exp \left((1 + \epsilon)\frac{3}{2}\delta(P, \theta_0)r_q^m\right) + (m - 2) \exp r_q^m\right) + o(1)
\]

where $\max\{\rho(A_k) : k = 1, 2, \ldots, m - 1, k \neq j\} < \eta < \rho(A_0)$. But this is a contradiction for sufficiently large $r_q$, as $\rho(A_0) > \rho(A_j) = n$.

(ii) If $\delta(P, \theta_0) < 0$, then since $\delta(P, \theta)$ is a continuous function therefore,

\[
\frac{3}{2}\delta(P, \theta_0) < \delta(P, \theta_0) < \frac{1}{2}\delta(P, \theta_0)
\]

for sufficiently large $q \in \mathbb{N}$. Using part (ii) of Lemma 2 we get

\[
\exp \left((1 + \epsilon)\frac{3}{2}\delta(P, \theta_0)r_q^m\right) \leq |A(z)| \leq \exp \left((1 - \epsilon)\frac{1}{2}\delta(P, \theta_0)r_q^m\right)
\]

for sufficiently large $m \in \mathbb{N}$. From equations (1), (13), (14), (15) and (18) we have

\[
\exp \left(r_q^{\rho(A_0) - \epsilon}\right) \leq M(r, A_0)
\]

\[
\leq \frac{|f^{(m)}(z)|}{f(z)} + |A_{m-1}(z)| \frac{|f^{(m-1)}(z)|}{f(z)} + \cdots + |A_1(z)| \frac{|f'(z)|}{f(z)} + \frac{H(z)}{f(z)}
\]

\[
\leq r^{m\rho(f)}q (1 + |A_{m-1}(z)| + \cdots + |A_j(z)| + \cdots + |A_1(z)|) + o(1)
\]

\[
\leq r^{m\rho(f)}q \left(1 + \exp \left((1 - \epsilon)\frac{1}{2}\delta(P, \theta_0)r_q^m\right) + (m - 2) \exp r_q^m\right) + o(1)
\]

which will be a contradiction to the fact that $\rho(A_0) > 1$. 


(iii) Finally, suppose \( \delta(P, \theta_0) = 0 \). We know that \(|\theta_q - \theta_0| \leq l_0\) for sufficiently large \( q \in \mathbb{N} \). Choose \( \theta^*_q \) such that \( l_0/3 \leq \theta^*_q - \theta_q \leq l_0 \) and \( \theta^*_q \to \theta_0 \) as \( q \to \infty \), we have

\[
\theta_q + \frac{l_0}{3} \leq \theta^*_q \leq \theta_q + l_0
\]

which implies \( \theta_0 + \frac{l_0}{3} \leq \theta^*_q \leq \theta_0 + l_0 \)

as \( q \to \infty \). We may assume without loss of generality that \( \delta(P, \theta^*_0) > 0 \) then as done in case (a), we obtain

\[
\exp \left( (1 - \epsilon)\frac{1}{2}\delta(P, \theta^*_0)r_q^n \right) \leq |A_j(z^*_q)| \leq \exp \left( (1 + \epsilon)\frac{3}{2}\delta(P, \theta^*_0)r_q^n \right).
\]

for sufficiently large \( q \in \mathbb{N} \). Using equations (11), (13), (14), (15) and (19) we get a contradiction as in case (a). Similarly if \( \delta(P, \theta^*_0) < 0 \) then we get contradiction as in case (a).

Now consider \( \rho(A_0) < \rho(A_j) \) and following cases:

(I) if \( \delta(P, \theta_0) > 0 \) then using equation (11), (10), (13), (15) and (17) we have

\[
\exp \left( (1 - \epsilon)\frac{1}{2}\delta(P, \theta_0)r_q^n \right) \leq |A_j(z_q)|
\]

\[
\leq \left| \frac{f^{(m)}(z_q)}{f^{(j)}(z_q)} \right| + |A_{m-1}(z_q)| \left| \frac{f^{(m-1)}(z_q)}{f^{(j)}(z_q)} \right| + \ldots
\]

\[
+ |A_{j-1}(z_q)| \left| \frac{f^{(j-1)}(z_q)}{f^{(j)}(z_q)} \right| + |A_{j+1}(z_q)| \left| \frac{f^{(j+1)}(z_q)}{f^{(j)}(z_q)} \right|
\]

\[
+ \ldots + |A_0(z_q)| \left| \frac{f(z_q)}{f^{(j)}(z_q)} \right| + \frac{H(z_q)}{f^{(j)}(z_q)}
\]

\[
\leq \left| \frac{f^{(m)}(z_q)}{f^{(j)}(z_q)} \right| + |A_{m-1}(z_q)| \left| \frac{f^{(m-1)}(z_q)}{f^{(j)}(z_q)} \right| + \ldots
\]

\[
+ \left| \frac{f(z_q)}{f^{(j)}(z_q)} \right| \left| \frac{f^{(j-1)}(z_q)}{f(z_q)} \right| + \ldots + |A_{j-1}(z_q)| \left| \frac{f^{(j-1)}(z_q)}{f(z_q)} \right|
\]

\[
+ |A_{j+1}(z_q)| \left| \frac{f^{(j+1)}(z_q)}{f(z_q)} \right| + \ldots + |A_0(z_q)| + \frac{H(z_q)}{f(z_q)}\}
\]

\[
\leq r^{mp(f)} + |A_{m-1}(z_q)|r^{mp(f)} + \ldots + r^{mp(f)}(1 + o(1)) \}
\]

\[
(|A_{j-1}|r^{mp(f)} + |A_{j+1}|r^{mp(f)} + \ldots + |A_0(z_q)| + o(1)) \}
\]

\[
\leq r^{2mp(f)}(1 + o(1)) ((m - 1) \exp (r^n) + o(1))
\]

where \( \max\{\rho(A_k) : k = 1, 2, \ldots, m - 1, k \neq j\} < \rho(A_0) < \eta < \rho(A_j) \). But this gives a contradiction to the fact that \( \rho(A_j) > \rho(A_0) \).

(II) When \( \delta(P, \theta_0) < 0 \) or \( \delta(P, \theta_0) = 0 \), then as done in earlier cases, we obtain a contradiction.

Thus all solutions of equation (11) are of infinite order.

(b) Now, from equation (11) we have

\[
\frac{1}{f} = -\frac{1}{H} \left( \frac{f^{(m)}}{f} + A_{m-1} \frac{f^{(m-1)}}{f} + \ldots + A_1 \frac{f'}{f} + A_0 \right)
\]
As a consequence of lemma of logarithmic derivatives and first fundamental theorem of Nevanlinna theory we have

\[ m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(m)}}{f}\right) + \ldots + m\left(r, \frac{f'}{f}\right) + m\left(r, A_{m-1}\right) + \ldots + m\left(r, A_1\right) + m\left(r, A_0\right) + m\left(r, \frac{1}{H}\right) \]

\[ \leq S(r, f) + o(T(r, f)) + m(r, H) + O(1) \]

\[ = S(r, f) + o(T(r, f)) + O(1) \]

Again applying first fundamental theorem of Nevanlinna theory, we get

\[ T(r, f) + O(1) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) \]

\[ \leq S(r, f) + N\left(r, \frac{1}{f}\right) + o(T(r, f)) + O(1) \]

From here, it is easy to conclude that \( \lambda(f) = \infty \).

(c) Using Lemma 1 there exists \( E \subset (1, \infty) \) satisfying \( m_1(E) < \infty \) such that

\[ \left| \frac{f^{(l)}(z)}{f^{(p)}(z)} \right| \leq c[T(2r, f)]^{2(l-p)} \quad (20) \]

where \( p < l \) are non-negative integers, \( c > 0 \) is a constant and \( z \) satisfies \( |z| = r \notin E \cup [0, 1] \). Let us suppose that \( \rho(A_j) < \rho(A_0) \). Then as in case (a), using equations (11), (14), (15) and (20) we get

\[ \exp\left(r^{\rho(A_0)-\epsilon}\right) \leq M(r, A_0) \]

\[ \leq \left| \frac{f^{(m)}(z_q)}{f(z_q)} \right| + |A_{m-1}(z_q)| \left| \frac{f^{(m-1)}(z_q)}{f(z_q)} \right| + \ldots + |A_1(z_q)| \left| \frac{f'(z_q)}{f(z_q)} \right| + \frac{H(z_q)}{f(z_q)} \]

\[ \leq c[T(2r, f)]^{m\rho(f)} (1 + |A_{m-1}(z_q)| + \ldots + |A_1(z_q)| + \ldots + |A_1(z_q)|) + o(1) \]

\[ \leq c[T(2r, f)]^{m\rho(f)} (1 + (m-1) \exp r^\eta) + o(1) \]

where \( \rho(A_k) < \eta < \rho(A_0) \) for all \( k = 1, 2, \ldots, m-1 \). This will imply that \( \rho_2(f) \geq \rho(A_0) \).

Now, if \( \rho(A_0) < \rho(A_j) \) then as done in case (a), using equations (11), (11), (14), (15) and (20) we conclude from here that \( \rho_2(f) \geq \rho(A_j) \).

We know that if \( f \) is a solution of equation (11) then

\[ f(z) = c_1(z)f_1(z) + \ldots + c_m(z)f_m(z) \quad (21) \]

where \( f_1, \ldots, f_m \) are linearly independent solutions of equation (22) and \( c'_i = \frac{H G_i(f_1, f_2, \ldots, f_m)}{W(f_1, f_2, \ldots, f_m)} \)

with \( G_i(f_1, f_2, \ldots, f_m) \) being a polynomial in \( f_1, f_2, \ldots, f_m \) and their derivatives and \( W(f_1, f_2, \ldots, f_m) \) being Wronskian of \( f_1, f_2, \ldots, f_m \). From equation (21) we obtain

\[ T(r, f) \leq d_1T(r, f_1) + d_2T(r, f_2) + \ldots + d_mT(r, f_m) + dT(r, H) + O(1) \quad (22) \]

where \( d, d_1, d_2, \ldots, d_m \) are positive integers. From equation (22) and Lemma 8 we conclude that \( \rho_2(f) \leq \rho = \max\{\rho(A_0), \rho(A_j)\} \).
(d) For every complex number $c$, we know that $f \equiv c$ is not a solution of (1) therefore, using Proposition 1 and Lemma 7 we have

$$m \left( r, \frac{1}{f - c} \right) = S(r, f)$$

for $r \in S$. Thus

$$\delta(c, f) = \lim_{r \to \infty} \frac{m \left( r, \frac{1}{f - c} \right)}{T(r, f)} = 0.$$ 

Therefore, $f$ has no finite deficient value.

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