Modal Strong Structural Controllability for Networks with Dynamical Nodes

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Abstract—In this article, a new notion of modal strong structural controllability is introduced and examined for a family of linear time-invariant (LTI) networks. These networks include structured LTI subsystems (or dynamical nodes), whose system matrices have the same zero/nonzero/arbitrary pattern. An eigenvalue (or the corresponding mode) associated with a system matrix is controllable if it can be directly influenced by the control inputs. We consider an arbitrary set \( \Delta \subseteq \mathbb{C} \), and we refer to a network as modal strongly structurally controllable with respect to \( \Delta \) if, for all systems in a specific family of LTI networks, every \( \lambda \in \Delta \) is a controllable eigenvalue. For this family of LTI networks, not only is the zero/nonzero/arbitrary pattern of system matrices available, but also for a given \( \Delta \), there might be extra information about the intersection of the spectrum associated with some subsystems and \( \Delta \). For instance, for a \( \Delta \) defined as the set of all complex numbers in the closed right-half plane, we may know that some structured subsystems are stable and have no eigenvalue in \( \Delta \). In this case, modal strong structural controllability is equivalent to strong structural stabilizability. Given a set \( \Delta \), we first define a \( \Delta \)-network graph, and by introducing a coloring process of this graph, we establish a correspondence between the set of control subsystems and the so-called zero forcing sets. For networks of one-dimensional subsystems, it is shown that the graph-theoretic condition is necessary and sufficient. We also demonstrate how with \( \Delta = \{0\} \) or \( \Delta = \mathbb{C} \setminus \{0\} \), existing results on strong structural controllability can be derived through our approach. In fact, compared to existing works on strong structural controllability, a more restricted family of LTI networks is considered in this work, and for this family, the derived controllability conditions are less conservative.

Index Terms—Modal strong structural controllability, Network controllability, Pattern matrices, Strong structural stabilizability, Dynamical nodes.

I. INTRODUCTION

In the last two decades, a surge of interest in studying networks of dynamical systems has arisen in both the control and network communities [1]. Many features of networks of dynamical systems can be interpreted in terms of properties of the corresponding graph that captures the network structure. One of the key features of large-scale networks is controllability [2]. In fact, controllability analysis of networks can shed light on their understanding from a topological (structural) point of view. There are two lines of research on controllability of networks with linear time-invariant (LTI) dynamics. In the first approach, a network with predetermined weights of interconnections is assumed [3]. In this setting, the system matrix can be considered as the adjacency or the Laplacian matrix of the network [4–8]. In the second approach, the controllability of a network with unspecified link weights is studied [9–11].

When the interaction strengths along the edges of a network are unknown, classical controllability tests cannot be utilised. However, in many practical cases, the underlying pattern of interconnections between the nodes of a network is available, so controllability can be examined in a structural framework. In this framework, the two notions of weak and strong structural controllability have been introduced in the literature. A zero/nonzero pattern for the system matrices, an LTI system is called weakly structurally controllable if for almost all values of the nonzero parameters, the system is controllable [9], [10], [12], [13]. Strong structural controllability has been introduced to ensure that for all nonzero weights of interconnections, the network remains controllable. Accordingly, in the structural framework, the network is viewed in terms of the pattern of zero and nonzero values of system matrices, and controllability is examined through combinatorial and graph-theoretic conditions [14].

This paper deals with a new notion of modal strong structural controllability of LTI networks. Almost all literature on strong structural controllability deals with the controllability of all systems with the same zero/nonzero pattern. However, in this work, we take a different approach and focus on the controllability of specific eigenvalues (corresponding modes) for a specific family of LTI networks with the same pattern. An eigenvalue of a system matrix associated with an LTI system is called controllable if it can be directly influenced by the control input, and by applying an appropriate control signal, it can be moved throughout the complex plane. The controllability of an eigenvalue can be checked through the Popov-Belevitch-Hautus (PBH) test [15]. Now, consider an arbitrary subset of complex numbers \( \Delta \subseteq \mathbb{C} \), that can be either discrete or continuous. We call a network modal strongly structurally controllable with respect to \( \Delta \) if for all systems in a specific family of LTI networks, whose system matrices have some eigenvalue \( \lambda \in \Delta \), \( \lambda \) is a controllable eigenvalue.

Modal strong structural controllability, where strong structural controllability is examined with respect to particular subspaces associated with the network, is of importance in numerous applications. For instance, for \( \Delta = \{y \in \mathbb{C} | \)
The minimal rank problem was motivated by the minimum rank problem. A zero forcing set is a subset of black nodes in a graph that can force its other nodes to be black through applying a coloring rule as many times as possible. The minimum rank problem was motivated by the inverse eigenvalue problem of a graph, whose goal is to obtain information about the possible eigenvalues of a family of patterned matrices; the first step towards this aim is to find the maximum multiplicity of an arbitrary eigenvalue of all matrices in this family. With this in mind, using the notion of zero forcing sets, [37] and [38] have presented upper bounds on the maximum multiplicity of the zero eigenvalue for loop-free undirected and loop directed graphs, respectively.

Most of the works on weak and strong structural controllability have been dedicated to the networks, whose any node state is scalar. However, recently, the problem of weak structural controllability has been studied for networks with dynamical nodes, where every node is made of a linear dynamic system [39]. Moreover, in [40], the strong structural controllability of a network of single-input single output structured subsystems has been investigated, and it has been shown that an LTI network is strongly structurally controllable if and only if an associated network of an order at most twice the number of the included subsystems is strongly structurally controllable.

B. Main Contributions

In this paper, we introduce a new notion of modal strong structural controllability and analyze it for a family of LTI networks, which include structured LTI subsystems. We assume that, other than the zero/nonzero/arbitrary pattern of system matrices, some restrictive information about the subsystems included in the LTI network may be available. In fact, based on our assumption, we might have information about the intersection of a given subset of complex numbers $\Delta$ and the spectra of some of the subsystems. In this direction, a $\Delta$-characteristic vector is defined, which captures the information about the intersection of the spectrum associated with any subsystem and $\Delta$. Moreover, a $\Delta$-specified pattern class associated with an LTI network includes all system matrices of the same zero/nonzero/arbitrary pattern, which have the extra properties described by the $\Delta$-characteristic vector. For instance, let $\Delta$ be the set of all eigenvalues in the closed right-half complex plane, and assume that some of the subsystems are stable, in the sense that all of their associated eigenvalues are in the open left-half plane. Therefore, the specific family of LTI networks includes all systems of the same zero/nonzero/arbitrary pattern, with this extra property that some given subsystems are stable.

We also introduce a notion of $\Delta$-network graph associated with a $\Delta$-characteristic vector and propose a coloring process applied to this graph. The main contributions of this work are:
1) We develop a correspondence between zero forcing sets of a Δ-network graph and sets of control subsystems, rendering it modal strongly structurally controllable (Theorem 1). Since the family of LTI networks in this work is more specified than the family of networks with only the same structure, the set of control subsystems obtained through our approach can be of a smaller cardinality. Moreover, given a Δ that includes at least one real number, we establish a necessary and sufficient condition for modal strong structural controllability of N1DSs (Theorem 2). As a particular case, we show how the derived graph-theoretic conditions can be utilised for the analysis of the strong structural stabilisability of LTI networks.

2) Finding the maximum geometric multiplicity of an arbitrary eigenvalue for all matrices of the same pattern is of interest as one goal of the minimum rank problem. In this direction, we take a step forward, and for any given set Δ, we provide an upper bound on the maximum geometric multiplicity of all eigenvalues in Δ associated with the system matrices in a Δ-specified pattern class of an LTI network (Proposition 2). We also derive combinatorial conditions, under which, no matrix in a Δ-specified pattern class associated with an N1DS has any eigenvalue in Δ (Theorem 5). For example, through a graph-theoretic condition, one can see whether any system matrix associated with an N1DS is stable or not.

3) Although there exist results in the literature on the full rankness of a pattern matrix with a zero/nonzero [37], [38] or zero/nonzero/arbitrary structure [20], we provide an equivalent combinatorial condition, which can be tested by considering the corresponding bipartite graph (Proposition 3). This can facilitate forming of Δ-network graph associated with a Δ-specified pattern class from the corresponding global graph.

Finally, we note that the strong structural controllability of LTI networks including LTI subsystems has been investigated in [40] as well; however, our approach differs from [40], mainly because other than the zero/nonzero/arbitrary pattern of the system matrices, we assume that other restrictive information about the subsystems may be available. In fact, a set of system matrices represented by a Δ-specified pattern class is smaller and more restrictive than a set of system matrices with only the same zero/nonzero/arbitrary pattern. Thus, the controllability conditions that we derive are less conservative than the existing results on strong structural controllability. Additionally, we show how our results can extend the existing knowledge on strong structural controllability of LTI networks. We also provide numerous examples to better illustrate definitions and the obtained results.

C. Outline

The paper is organized as follows. Preliminaries, including all necessary definitions and problem formulation, are presented in Section II. In Section III we introduce a Δ-network graph associated with a given set Δ. Moreover, the definition of the coloring process and zero forcing sets are presented in this section. In Section IV the main results are established. Finally, Section V concludes the paper.

II. Preliminaries

The set of real and complex numbers are denoted by ℝ and ℂ, respectively. The i-th element of the vector v is designated by v(i), and M(i, j) is the entry in row i and column j of M. Moreover, for i2 ≥ i1 and j2 ≥ j1, M(i1 : i2, j1 : j2) is a submatrix of M formed from the successive rows i1, i2, ..., and successive columns j1, j2, ..., j2. A subvector v(X) is comprised of v(i), for i ∈ X, ordered lexicographically. I_n denotes the vector of all ones in ℝ^n. We denote the n × n identity matrix by I_n and represent its j-th column by e_j. The cardinality of a set S is designated by |S|.

A. Definitions

Pattern Matrices: A pattern matrix A is a matrix whose entries are chosen from the set of symbols {0, *, ?}. A pattern class of a q × p pattern matrix A, denoted by P(A), is defined as P(A) = {A ∈ ℝ^{q×p} | A(i, j) = 0 if A(i, j) = 0, and A(i, j) ≠ 0 if A(i, j) = *}. By this definition, if A(i, j) = ?, A(i, j) can be any arbitrary real number, including zero and nonzero. We say that a q × p pattern matrix A has full row rank if every A ∈ P(A) has full row rank.

Graphs: Let A ∈ {0, *, ?}^{q×p} be a pattern matrix, where l = max(q, p). Then, we define an associated graph G(A) = (V, E) with the node set V = {v_1, ..., v_l} and edge set E ⊆ V × V. We have (v_i, v_j) ∈ E if and only if A(i, j) ≠ 0. Then, there is an edge from node v_j to node v_i. In this case, node v_i (respectively, node v_j) is said to be an out-neighbor (respectively, in-neighbor) of node v_j (respectively, node v_i). We denote by N_out(v_j) the set of out-neighbors of node v_j. For an undirected graph, (v_i, v_j) ∈ E if and only if (v_j, v_i) ∈ E, and thus the associated pattern matrix A should be symmetric in this case. Note that a graph G(A) can contain loops as (v_i, v_i) (i.e., self-loops) for some v_i ∈ V. Let E = E^* ∪ E^?, where (v_i, v_j) ∈ E^* if and only if A(i, j) = *, and (v_i, v_j) ∈ E^? if and only if A(i, j) = ?. If (v_i, v_j) ∈ E^* (respectively, (v_j, v_i) ∈ E^?), v_j is called a strong (respectively, weak) out-neighbor of v_j. To distinguish the edges in E^* and E^?, we show them by solid and dotted arrows, respectively.

Bipartite graphs: Given a pattern matrix A ∈ {0, *, ?}^{q×p}, one can associate a bipartite graph as G_b(A) = (V_r, V_c, E_b) with V_r = {v^r_1, ..., v^r_q} and V_c = {v^c_1, ..., v^c_p}. The set E_b is a subset of edges from nodes of V_r to nodes of V_c, where (v^r_i, v^c_j) ∈ E_b if and only if A(i, j) ≠ 0. We let E_b = E_b^* ∪ E_b^?, where (v^r_i, v^c_j) ∈ E_b^* if and only if A(i, j) = *, and (v^r_i, v^c_j) ∈ E_b^? if and only if A(i, j) = ?. The edges in E^* and E^? are shown by solid and dotted arrows, respectively.

Network graphs: Consider a network N with n nodes. An associated network graph G_N is denoted by G_N = (V_N, E_N), where V_N = {1, 2, ..., n} is the node set, and E_N ⊆ V_N × V_N is the edge set of the graph. In the next section of the paper, we provide a detailed description of the edge set of a network graph regarding our problem. Essentially, we assume that a set of numbers Δ is given, and we demonstrate how one can form an associated Δ-network graph G_N^Δ.
**Node graphs:** Every node $i$ in a network graph $G_N$, $i = 1, 2, \ldots, n$, is indeed a “super node”, in the sense that it can represent a system itself. Thus, it has $l_i$ internal vertices and is illustrated by a node graph $G_i = (V_i, E_i)$, where $V_i = \{v^1_i, \ldots, v^{l_i}_i\}$. To make a distinction, we refer to $V_i$ as the set of “vertices” of $G_i$, while $V_N$ is the set of “nodes” of $G_N$. For a subset of nodes $Z \subseteq V_N$, we define the set of vertices of $Z$ as $\text{Ver}(Z) = \bigcup_{i \in Z} V_i$.

**Eigenvalues:** Let $\Lambda(A)$ denote the spectrum or the set of eigenvalues of matrix $A$. The geometric multiplicity of eigenvalue $\lambda \in \Lambda(A)$, which is denoted by $\psi_A(\lambda)$, is the dimension of the subspace $V_A(\lambda) = \{v \in \mathbb{R}^n \mid v^T A = \lambda v^T\}$. For a subset $M \subseteq \Lambda(A)$, the maximum geometric multiplicity of the eigenvalues of $A$ belonging to $M$ is defined as $\Psi_M(A) = \max\{\psi_A(\lambda) \mid \lambda \in M\}$.

**System matrices:** For $i = 1, 2, \ldots, n$, let $A_{ii} \in \{0, *, ?\}^{l_i \times l_i}$ be a pattern matrix. Now, consider a network $N$, including some LTI subsystem $i$ with system matrix $A_{ii} \in \mathcal{P}(A_i)$, $i = 1, 2, \ldots, n$. Note that for some $1 \leq i \leq n$, we may have $l_i = 1$. Now, let $N = \sum_{i=1}^{n} l_i$. The global system matrix associated with this network is

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \in \mathbb{R}^{N \times N},$$

(1)

and we let $A \in \mathcal{P}(A)$, for a pattern matrix $A \in \{0, *, ?\}^{N \times N}$. Note that for all $1 \leq i, j \leq n$, we have $A_{ij} \in \mathbb{R}^{l_i \times l_j}$, and $A_{ij} \in \mathcal{P}(A_{ij})$, where $A_{ij} \in \{0, *, ?\}^{l_i \times l_j}$. For $i = 1, 2, \ldots, n$, node $i$ is a super node, representing the subsystem $i$. Node graph $G_i = G(A_{ii}) = (V_i, E_i)$ illustrates the internal dynamics of node $i$. On the other hand, graph $G = G(A) = (V_G, E_G)$, with $V_G = \bigcup_{i=1}^{n} V_i$, is called the **global graph**, representing the structure of the entire system.

**Example 1:** Consider an LTI network $N$, including subsystems 1, 2, 3, and 4, which are depicted in Fig. 1(a). The node graphs $G_1$, $G_2$, $G_3$, and $G_4$ of size $l_1 = 3$, $l_2 = 4$, $l_3 = 2$, and $l_4 = 2$, respectively, are shown in Fig. 1(a). The global graph $G$ of size $N = 11$ is also illustrated in Fig. 1(b). For instance, the corresponding pattern matrices $A_{21}$ and $A_{11}$, where $G_1 = G(A_{11})$, are:

$$A_{21} = \begin{bmatrix} 0 & 0 & * \\ 0 & ? & * \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} * & 0 & 0 \\ * & ? & * \\ ? & ? & 0 \end{bmatrix}.$$

**B. LTI Networks**

Consider an LTI network $N$, including LTI subsystem $i$ with the system matrix $A_{ii} \in \mathbb{R}^{l_i \times l_i}$, $i = 1, 2, \ldots, n$. If all the subsystems are one-dimensional, that is, $l_i = 1$, for all $1 \leq i \leq n$, we have a network of one-dimensional subsystems (NIDS).

The global dynamics of an LTI network is described by:

$$\dot{x} = Ax + Bu,$$

(2)

where $A \in \mathcal{P}(A)$ is described in Fig. 1(a). Let $x_i = [x^1_i \ldots x^{l_i}_i]^T \in \mathbb{R}^{l_i}$ be the state vector of subsystem $i$. Then, $x = [x_1^T \ldots x_n^T]^T \in \mathbb{R}^N$ is the aggregated state vector of the global system. Now, let $A_{kj} = [a_{kj}^1 \ldots a_{kj}^{l_k}]$, where $a_{kj}^i \in \mathbb{R}, k = 1, 2, \ldots, m$, is the vector of control signals injected into vertices of node graph $G_{jk}$. Moreover, the input matrix $B = [B_1 \ldots B_m] \in \mathbb{R}^{N \times M}$ is a binary matrix, where for $k = 1, 2, \ldots, m$, $B_k = [e_{jk}^1 \ldots e_{jk}^{l_k}] \in \mathbb{R}^{N \times l_k}$. In this case, the subsystems $j_k$, $k = 1, 2, \ldots, m$, are referred to as control subsystems, and we define $V_C = \{j_1, \ldots, j_m\}$.

**Example 2:** Consider an LTI network $N$ with dynamics (2), whose global graph $G$ is depicted in Fig. 1(b). Let $x = [x_1^T \ldots x_4^T]^T$, where for instance, $x_1 = [x_1^T \ x_2^T \ x_3^T \ x_4^T]^T$. Assume that we choose subsystem 1 as the control subsystem, i.e., $V_C = \{1\}$. Then, we have...
such that for some $f$, given the sequence

$$\Delta \subseteq \mathbb{C},$$

the PBH test is used for checking the controllability of $L$ given $\Delta$. For instance, for some $A$, $\Delta = \{\mu\}$ and $A_{kk} = \mu I_k$. In other words, if $f_{\Delta}^L(k) = 0$, $\Delta$ should be a singleton with only member $\mu$, and $A_{kk}$ is a diagonal matrix whose all diagonal entries are equal to $\mu$. Finally, $f_{\Delta}^L(k) = ?$ implies that $\Lambda(A_{kk}) \cap \Delta$ can be empty or nonempty, or if we have $\Lambda(A_{kk}) \cap \Delta \neq 0$, then either $|\Delta| > 1$, or for $\Delta = \{\mu\}$, $A_{kk}$ should not necessarily equal $\mu I_k$.

Now, in the reverse direction, assume that a set of system matrices $S^\ast$ along with a nonempty set $\Delta \subseteq \mathbb{C}$ are given, and we aim to define a corresponding pattern matrix $A$ and a $\Delta$-characteristic vector $f_{\Delta}^L$, where $S^\ast \subseteq \mathcal{P}(A)$ and $S^\ast \subseteq S(f_{\Delta}^L, A)$. One can define $A$ and $f_{\Delta}^L$ as follows. For all $1 \leq i, j \leq N$, one has:

$$A(i, j) = \begin{cases} *, & \text{if } A(i, j) \neq 0, \forall A \in S^\ast, \\ 0, & \text{if } A(i, j) = 0, \forall A \in S^\ast, \\ ?, & \text{otherwise.} \end{cases}$$

Moreover, for $k = 1, 2, \ldots, n$, one can define:

$$f_{\Delta}^L(k) = \begin{cases} *, & \text{if } \Lambda(A_{kk}) \cap \Delta = \emptyset, \forall A \in S^\ast, \\ 0, & \text{if } \Delta = \{\mu\} \& A_{kk} = \mu I_k, \forall A \in S^\ast, \\ ?, & \text{otherwise.} \end{cases}$$

Example 3: Let $\Delta = \{y \in \mathbb{C} \mid \mathbb{R}(y) \geq 0\}$. Now, assume that for some $J \subseteq \{1, 2, \ldots, n\}$, we have $f_{\Delta}^L(J) = *$ if and only if $k \in J$. Moreover, since $|\Delta| > 1$, we let $f_{\Delta}^L(k) = ?$ for all $k \in \{1, 2, \ldots, n\} \setminus J$. Therefore, $S(f_{\Delta}^L, A)$ is the set of all matrices $A \in \mathcal{P}(A)$, where for all $k \in J, A_{kk}$ is stable, in the sense that all of its eigenvalues are in the open left-half plane; moreover, for $k \in \{1, 2, \ldots, n\} \setminus J$, either we do not have any information about the stability of $A_{kk}$ or we know that it is not stable. Now, consider an NIDS, where $I_i = 1$, for all $1 \leq i \leq n$. In this case, $\Lambda(A_{kk}) \cap \Delta = \emptyset$ implies that $A_{kk} = A(k, k) < 0$. Hence, $S(f_{\Delta}^L, A)$ is the set of all $A \in \mathcal{P}(A)$, where $A(k, k) < 0$, for every $k \in J$. Therefore, besides the zero/nonzero/arbitrary pattern of the matrix $A$, we have extra information about the sign of some of its diagonal entries.

Example 4: In this example, consider an NIDS, and for some $a, b \in \mathbb{R}$, where $a < b$, assume that $\Delta = [a, b] = \{y \in \mathbb{R} \mid a \leq y \leq b\}$. Now, suppose that for some $J \subseteq \{1, 2, \ldots, n\}$, $f_{\Delta}^L(J) = *$ if and only if $k \in J$. Thus, one can conclude that $k \in J$, then either $A(k, k) < a$ or $A(k, k) > b$. Otherwise, for $k \notin J$, either there is no information about the value of $A(k, k)$ or we know that it can take values from the interval $[a, b]$.

Example 5: Let $\Delta = \{0\}$, and consider an NIDS. For a given pattern matrix $A \in \{0, *, ?\}^{N \times N}$, assume that the set of system matrices $S^\ast$ is the same as the set $\mathcal{P}(A)$. In this case, $\Lambda(A_{kk}) \cap \Delta = \emptyset$ implies that $A(k, k) \neq 0$, for all $A \in S^\ast$. Thus, 5 leads to $A(k, k) = *$. Moreover, from 6, for $k = 1, 2, \ldots, n$, $f_{\Delta}^L(k) = *$ if $A(k, k) = *$, and $f_{\Delta}^L(k) = 0$ if $A(k, k) = 0$. Otherwise, if $A(k, k) = ?$, we have $f_{\Delta}^L(k) = ?$.

Example 6: Let $\Delta = \mathbb{C} \setminus \{0\}$, and consider an NIDS. One can conclude that $\Lambda(A_{kk}) \cap \Delta = \emptyset$ implies that $A(k, k) = 0$. Accordingly, $S(f_{\Delta}^L, A)$ is the set of all matrices $A \in \mathbb{R}^{N \times N}$ which have the same pattern $A$; moreover, $\Delta$ and the spectrum of $A_{kk}$ should have no member in common if $f_{\Delta}^L(k) = *$. Furthermore, $f_{\Delta}^L(k) = 0$ implies that for some $\mu \in \mathbb{R}$, $\Delta = \{\mu\}$ and $A_{kk} = \mu I_k$. In other words, if $f_{\Delta}^L(k) = 0$, $\Delta$ should be a singleton with only member $\mu$, and $A_{kk}$ is a diagonal matrix whose all diagonal entries are equal to $\mu$.
Note that since $|\Delta| > 1$, $f^k_\Delta \in \{\ast, ?\}^n$. Hence, considering $S^* = \mathcal{P}(A)$, for a given pattern matrix $A \in \{0, *, ?\}^{n \times n}$ and for $k = 1, 2, \ldots, n$, we have $f^k_\Delta(k) = \ast$ if $A(k, k) = 0$, and $f^k_\Delta = ?$ otherwise.

Example 7: For an NIDS, let $\Delta = \mathbb{C}$. Since $|\Delta| > 1$, one should have $f^k_\Delta \in \{\ast, ?\}^n$. Moreover, for all $1 \leq k \leq n$, $\Lambda(A_{kk}) \cap \Delta \neq \emptyset$. Now, for a given pattern matrix $A \in \{0, *, ?\}^{n \times n}$, let $S^* = \mathcal{P}(A)$. Then, from (6), one has $f^k_\Delta(k) = \ast$, for all $1 \leq k \leq n$.

Example 8: As the last example in this part, for an NIDS, assume that $A \in \{0, *, ?\}^{n \times n}$ is a pattern matrix, whose all diagonals are ?. This pattern matrix is called an arbitrary-diagonal pattern matrix. Now, consider three sets $\Delta_1 = \{1\}$, $\Delta_2 = \mathbb{C} \setminus \{0\}$, and $\Delta_3 = \mathbb{C}$, and let $S^* = \mathcal{P}(A)$. In this case, one can observe that for every $k$, $1 \leq k \leq n$, since $A(k, k) = ?$, there is some $A \in \mathcal{P}(A)$ with $A_{kk} = 0$. Thus, for $i = 1, 2, \lambda(A_{kk}) \cap \Delta_i \neq \emptyset$. Moreover, there is some $\Delta' \in \mathcal{P}(A)$ with $\lambda'(k, k) = 0$. Therefore, for every $k$, $1 \leq k \leq n$, there is some matrix $A' \in S^*$, where $A'_{kk} \neq 0$, and $A'_{kk} \cap \Delta_3 = \emptyset$. Accordingly, considering (6), one can conclude that $f^k_\Delta(k) = ?$, for all $k = 1, 2, \ldots, n$.

D. Modal Strong Structural Controllability

Let $\Delta \subseteq \mathbb{C}$ be a nonempty discrete or continuous set. For a given pattern matrix $A \in \{0, *, ?\}^{n \times n}$, let $A \in \mathcal{P}(A)$, defined in (1), be a system matrix associated with an LTI network $N$, and let $\mathcal{L} = (l_1, \ldots, l_n)$ be a sequence including the dimensions of submatrices $A_{ii}$, $i = 1, 2, \ldots, n$. Assume that a $\Delta$-characteristic vector $f^k_\Delta$, defined in (6), is given, which represents some information about $\Lambda(A_{ii}) \cap \Delta$, $i = 1, 2, \ldots, n$. Let $S(f^k_\Delta, A)$, defined in (4), be a $\Delta$-specified pattern class of $f^k_\Delta$ and $A$.

Definition 1: An LTI network $N$ with dynamics (2) is (modal) strongly structurally controllable with respect to $\Delta$ if for every $\lambda \in \Delta$ and for all $A \in S(f^k_\Delta, A)$ that $\lambda \in \Lambda(A)$, $\lambda$ is a controllable eigenvalue. We refer to this network as $\Delta$-SSC.

Based on the classical definition of strong structural controllability in the literature, a network is called strongly structurally controllable if it is $\mathbb{C}$-SSC. Now, let $\Delta' = \emptyset \setminus \Delta$ (note that for an undirected network, we define $\Delta' = \mathbb{R} \setminus \Delta$). Then, based on Definition 1, an LTI network $N'$ with dynamics (2) is strongly structurally controllable if it is both $\Delta$-SSC and $\Delta'$-SSC. In a more general case, for some $k \geq 1$ and the disjoint sets $\Delta_1, \ldots, \Delta_k$, where $\mathbb{C} = \bigcup_{k=1}^{k} \Delta_k$, an LTI network $N$ is strongly structurally controllable if for every $1 \leq i \leq k$, it is $\Delta_i$-SSC. Notice that for a larger $k$, a less conservative condition can be obtained for strong structural controllability of networks.

Now, consider Example 3 where for a given pattern matrix $A \in \{0, *, ?\}^{N \times N}$, $A \in \mathcal{P}(A)$. Moreover, for all $k \in J$, the subsystem $k$ is stable. An LTI network $N$ is called strongly structurally stabilizable if for $\Delta = \{y \in \mathbb{C} | \Re(y) \geq 0\}$, the network is $\Delta$-SSC.

We note that if there is no constraint on the system matrix $A$ other than $A \in \mathcal{P}(A)$, then from Theorem 14 of [20], an LTI network is strongly structurally controllable if and only if it is strongly structurally stabilizable; however, since in Example 3 we consider a smaller set of system matrices $S(f^k_\Delta, A)$, where $S(f^k_\Delta, A) \subseteq \mathcal{P}(A)$, and we assume that for every $k \in J$, all eigenvalues of $A_{kk}$ are in the open left-half plane, the strong structural controllability and stabilizability will not necessarily be equivalent.

E. Problem Formulation

Consider an LTI network with dynamics (2), which includes $n$ LTI structured subsystems, and its system matrix $A \in \mathcal{P}(A)$ is described in (1). Given a nonempty set $\Delta \subseteq \mathbb{C}$, our focus in this work is on the combinatorial characterizations of modal strong structural controllability of an LTI network with respect to $\Delta$. Therefore, the main problem that we aim to investigate in this paper is the following.

Problem 1: Given an arbitrary $\Delta \subseteq \mathbb{C}$, find graph-theoretic conditions under which an LTI network with dynamics (2) is strongly structurally controllable with respect to $\Delta$.

Now, let us interpret this problem on an applied example and discuss how the existing results on strong structural controllability are too conservative in this case. In this example, we let $\Delta = \{y \in \mathbb{C} | \Re(y) \geq 0\}$. The LTI network, which can be, for example, a network of robots or a platoon of heterogeneous vehicles moving along a ring road, includes stable subsystems. Due to the physical constraints, it is known that all subsystems are stable and have no associated eigenvalues in the closed right-half plane.

Example 9: Let $\gamma_{11}$ and $\gamma_{22}$ be some nonzero real parameters, and assume that $\beta_{11}, \beta_{22}, \beta_{33} > 0$, $i = 2, 3, \ldots, n - 1$. Now, consider an LTI network $N$, whose system matrix $A$, described in (1), is defined as:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1n} \\ 0 & A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & A_{n-1,n} & A_{nn} \end{bmatrix},$$ (7)

where we have

$$A_{ii} = \begin{bmatrix} 0 & \gamma_{11} \\ 0 & \gamma_{22} \end{bmatrix}, \quad 1 \leq i \leq n,$$ (8)

$$A_{ii} = \begin{bmatrix} 0 & -\beta_{11} \\ -\beta_{22} & -\beta_{33} \end{bmatrix}, \quad 1 < i < n.$$ (8)

Then, besides the zero/nonzero/arbitrary pattern of any matrix $A_{ii}, i = 2, 3, \ldots, n - 1$, we know the sign of its nonzero parameters. Now, one can see that for all numerical realizations of system matrix $A$, any subsystem $i, i = 2, 3, \ldots, n - 1$, is stable (because the eigenvalues of any subsystem $i$ are the roots of the characteristic equation $\lambda^2 + \beta_{33}\lambda + \beta_{11}\beta_{22} = 0$). Now, let $\Delta' \in \{?\}^2 \times 2$ be a $2 \times 2$ pattern matrix with all entries equal to ?. We let $A_{11}, A_{nn} \in \mathcal{P}(A')$. Thus, there is no available information about the zero/nonzero/arbitrary pattern of $A_{11}$ and $A_{nn}$. However, we know that these matrices are also associated with stable subsystems. Thus, with $\Delta = \{y \in \mathbb{C} | \Re(y) \geq 0\}$, we have $f^k_\Delta(i) = \ast$, for all $i = 1, 2, \ldots, n$. 

For $n = 6$, the global graph $G$ associated with this network is depicted in Fig. 2. The $\Delta$-specified pattern class $S(f^\Delta_{\mathcal{N}}, \mathcal{A})$ includes all system matrices of the same zero/nonzero/arbitrary structure, whose all subsystems are stable. In this example, we aim to examine the strong structural stabilizability of the entire network, that is, the strong structural controllability with respect to $\Delta = \{y \in \mathbb{C} \mid \Re(y) \geq 0\}$. Now, one can verify that by applying the existing results in the literature on strong structural controllability (see e.g. [12], [20]), a set of control nodes should include at least 7 vertices. However, we will show that one control subsystem, including two vertices, can render this network strongly structurally stabilizable.

In this paper, we will also study the maximum geometric multiplicities of eigenvalues of $A \in S(f^\Delta_{\mathcal{N}}, \mathcal{A})$, for all eigenvalues that belong to the set $\Delta$. Then, the next problems of this work can be stated as follows.

**Problem 2:** Provide an upper bound on the maximum geometric multiplicity of eigenvalues of all $A \in S(f^\Delta_{\mathcal{N}}, \mathcal{A})$ that belong to a set $\Delta \in \mathbb{C}$.

**Problem 3:** Given a nonempty set $\Delta \in \mathbb{C}$, find a combinatorial condition under which no matrix $A \in S(f^\Delta_{\mathcal{N}}, \mathcal{A})$ has any eigenvalue in $\Delta$.

### III. $\Delta$-Network Graphs and Coloring Process

In this section, we introduce some essential notions that are employed in providing the main results of this work. First, given a nonempty set $\Delta$, a $\Delta$-characteristic vector $f^\Delta_{\mathcal{N}}$, and the global graph of an LTI network, we discuss how the corresponding $\Delta$-network graph is formed. Next, a coloring process and zero forcing sets are introduced.

#### A. $\Delta$-network Graphs

Consider an LTI network $\mathcal{N}$, as described in [2], whose system matrix $A$ is defined in [1], and for some pattern matrix $\mathcal{A} \in \{0, *, ?\}^{N \times N}$, we have $A \in \mathcal{P}(\mathcal{A})$. Let $\Delta \subset \mathbb{C}$ be a nonempty set. Now, assume that we are either given a vector $f^\Delta_{\mathcal{N}} \in \{0, *, ?\}^{n \times n}$, representing the extra information about $\Lambda(A_{ii}) \cap \Delta$, for $i = 1, 2, \ldots, n$ (similar to Examples [3] and [4]), or we have the set of system matrices $S^\Delta$ associated with this network, and $f^\Delta_{\mathcal{N}}$ is defined according to [6] (see Examples [5] [6] [7] and [8]).

Now, having $f^\Delta_{\mathcal{N}}$ and $\mathcal{A}$, we associate to the network $\mathcal{N}$ a $\Delta$-network graph $G^\Delta_{\mathcal{N}} = (V_N, E_N)$, where $V_N = \{1, 2, \ldots, n\}$ and $E_N = E_N^* \cup E_N^\Delta$. The self-loops of this graph are defined based on the entries of the vector $f^\Delta_{\mathcal{N}}$ as follows. For $i = 1, 2, \ldots, n$, there exists a self-loop $(i, i) \in E_N$ if and only if $f^\Delta_{\mathcal{N}}(i) \neq 0$. Moreover, $(i, i) \in E_N^\Delta$ if and only if $f^\Delta_{\mathcal{N}}(i) = *$, and $(i, i) \in E_N^*$ if and only if $f^\Delta_{\mathcal{N}}(i) = ?$.

For $i \neq j$, we have $(j, i) \notin E_N$ if and only if $A_{ij} = 0$. Moreover, $(j, i) \in E_N^\Delta$ if and only if the pattern matrix $A_{ij}$ has full row rank, and $(j, i) \in E_N^*$ otherwise. In the next section of the paper, a graph-theoretic condition is presented; by checking this condition in the associated global graph, we can find whether a pattern matrix $A_{ij}$ has full row rank or not. We differentiate the edges in $E_N^\Delta$ and $E_N^*$ through the solid and dotted arrows, respectively.

Notice that we may have no information about the pattern matrix of any subsystem $k$, $k = 1, 2, \ldots, n$. In other words, for $k = 1, 2, \ldots, n$ and $1 \leq i, j \leq l$, we have $A_{kk}(i, j) = ?$. However, if $f^\Delta_{\mathcal{N}}$ is available, then one can describe the self-loops of the $\Delta$-network graph. For instance, for $\Delta = \{y \in \mathbb{C} \mid \Re(y) \geq 0\}$, if no information about the structure of the subsystems is available, but we know which subsystems are stable, then we can determine the set of nodes of the $\Delta$-network graph that have self-loops, which are drawn by a solid arrow.

**Example 10:** Consider the LTI network in Example [9] As mentioned before, all subsystem are stable. Now, let $\Delta = \{y \in \mathbb{C} \mid \Re(y) \geq 0\}$. Thus, we have $f^\Delta_{\mathcal{N}}(i) = *$, for all $i = 1, \ldots, n$. Therefore, all nodes in the $\Delta$-network graph have self-loops represented by solid arrows. It is also obvious that $A_{i,i-1}$, for $1 \leq i \leq n$, is rank-deficient. Thus, for $i = 1, \ldots, n$, there is a dotted edge from node $i - 1$ to node $i$ in the $\Delta$-network graph. For $n = 6$, the corresponding $\Delta$-network graph $G^\Delta_{\mathcal{N}}$ is depicted in Fig. 3.

**Example 11:** Consider an NIDS with dynamics (2), whose corresponding global graph is depicted in Fig. 2a, and assume that the set of system matrices $S^\Delta$ is the same as $\mathcal{P}(\mathcal{A})$. First, let $\Delta = \{0\}$. As discussed in Example [5] one can see that the $\Delta$-network graph is the same as the global graph. Now, let $\Delta = \mathbb{C} \setminus \{0\}$. In this case, as explained in

![Fig. 2. Global graph $\mathcal{G}$ for Example [9]](image)

![Fig. 3. $\Delta$-network graph $G^\Delta_{\mathcal{N}}$ associated with the network in Example [9]](image)
Example 12: Consider the graph in Fig. 4(a), and let \( Z = \{1\} \) be the set of initial black nodes. The steps of the coloring process are: (a) 4 \( \rightarrow \) 5, (b) 1 \( \rightarrow \) 2, (c) 2 \( \rightarrow \) 3, and 3 \( \rightarrow \) 4. Moreover, the coloring process and its successive steps in Fig. 4(b) are as follows: (a) 5 \( \rightarrow \) 4, (b) 4 \( \rightarrow \) 1, and 1 \( \rightarrow \) 2, (c) 2 \( \rightarrow \) 3. Thus, \( Z = \{1\} \) is a zero forcing set for both graphs in Figs. 4(a) and (b), while one can see that the coloring process cannot be initiated in Fig. 4(c).

IV. MAIN RESULTS

In this section, we discuss the main results of this paper.

A. Modal Strong Structural Controllability

Let \( \Delta \subseteq C \) be a given nonempty set. Now, consider an LTI network \( N \), with dynamics (2) and the pattern matrix \( A \in \{0,*,?\}^{N \times N} \). Let \( f_\Delta \in \{0,*,?\}^n \) be a given \( \Delta \)-characteristic vector. Accordingly, the corresponding \( \Delta \)-network graph \( G_\Delta = (V_\Delta, E_\Delta) \), with \( V_\Delta = \{1,2,...,n\} \), and \( E_\Delta = E_\Delta^1 \cup E_\Delta^2 \), can be obtained as discussed in Section III-A. We recall that every node \( i \in V_\Delta \) is a super node, which represents subsystem \( i \), and \( G_i = (V_i, E_i) \) is the \( i \)-th node graph, where \( V_i = \{i^1, ..., i^n\} \) is called the set of vertices of node \( i \). For a subset of nodes \( Z \subseteq V_\Delta \), \( Ver(Z) \) is the set of all vertices associated with the nodes in \( Z \).

We first study the controllability of any eigenvalue \( \nu \in \Delta \) for a family of LTI networks with \( A \in S(f_\Delta, A) \). Let \( Z \subseteq V_\Delta \) be a subset of nodes of the \( \Delta \)-network graph. Moreover, \( D(Z) \) is the derived set of \( Z \) in the \( \Delta \)-network graph. In this direction, we first show that if a left eigenvector \( \nu \in \mathbb{R}^N \) associated with an eigenvalue \( \lambda \in \Delta \) for some \( A \in S(f_\Delta, A) \) vanishes at entries indexed by all the vertices of \( Z \), this eigenvector has to vanish at every entry indexed by the vertices of \( D(Z) \).

Lemma 1: Let \( Z \subset V_\Delta \), and let \( A \in S(f_\Delta, A) \). Assume that \( \nu \in \mathbb{C}^N \) be a left eigenvector of \( A \) associated with some \( \lambda \in \Delta \). If \( \nu(k) = 0 \) for all \( k \in Ver(Z) \), then \( \nu(k) = 0 \) for all \( k \in Ver(D(Z)) \).

Proof. If \( Z = D(Z) \), the statement of the lemma follows. Otherwise, the coloring process can be applied to the \( \Delta \)-network graph. Thus, we have one of the following cases: (a) There exists a black node \( i \in V_\Delta \) with exactly one white out-neighbor \( j \), where \( (i,j) \in E_\Delta^1 \); (b) there exists a white node \( i \) which has no white out-neighbor except itself, and \( (i,i) \in E_\Delta^2 \); and (c) there exists a white node \( i \) with no self-loop (i.e., \( (i,i) \notin E_\Delta^1 \) and exactly one white out-neighbor \( j \), where \( (i,j) \in E_\Delta^1 \).

Consider matrix \( A \) as a block matrix described in (1), and let \( \nu^T = \left[\nu_1^T \ldots \nu_n^T\right] \), where, for \( i = 1,2,...,n \), \( \nu_i = \left[\nu_i^1 \ldots \nu_i^n\right]^T \in \mathbb{C}^n \). Now, one can write the matrix equation \( \nu^T A = \lambda \nu^T \) as:

\[
\begin{bmatrix}
\nu_1^T & \nu_2^T & \ldots & \nu_n^T
\end{bmatrix}
\begin{bmatrix}
\lambda I_n - A_{11} & -A_{12} & \cdots & -A_{1n} \\
-A_{21} & \lambda I_n - A_{22} & \cdots & -A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{n1} & -A_{n2} & \cdots & \lambda I_n - A_{nn}
\end{bmatrix}
= 0.
\]
For the first case, the $i$-th block column of equation (9) assumes the form,
\[ \nu_i^T (\lambda I_i - A_{ii}) + \sum_{k \in N_{\text{out}}(i), k \neq i} \nu_k^T A_{ki} = 0. \tag{10} \]

Since node $i$ is black, $i$ belongs to $Z$, and $\nu(k) = 0$, for all $k \in \text{Ver}(\{i\})$. Thus, we have $\nu_i = 0$. Likewise, all the out-neighbors of node $i$ except $j$ are black, implying that $\nu_j = 0$, for all $k \in \text{Ver}(N_{\text{out}}(i) \setminus \{j\})$. Thus, (10) reduces to $\nu_j^T A_{ji} = 0$. Since $(i, j) \in E_N$, the submatrix $A_{ji}$ has full row rank, and hence we have that $\nu_j = 0$.

Now, consider case (b). Note that in the $\Delta$-network graph $G_N^\Delta$, $(i, i) \in E_N^\Delta$ implies that $f^\Delta_A(i) = *$. Thus, from (6), $\Lambda(A_{ii}) \cap \Delta = \emptyset$, and since $\lambda \in \Delta$, $\lambda \notin \Lambda(A_{ii})$. Thus, one can conclude that the matrix $\Lambda(I_i - A_{ii})$ is nonsingular. Moreover, we have,
\[ \nu_i^T (\lambda I_i - A_{ii}) + \sum_{k \in N_{\text{out}}(i), k \neq i} \nu_k^T A_{ki} = 0. \tag{11} \]

Since all out-neighbors of node $i$ except itself are black, $\nu_k = 0$, for all $k \in N_{\text{out}}(i), k \neq i$. Hence, we have $\nu_j^T A_{ji} = 0$. In addition, based on the definition of a $\Delta$-network graph, $(i, j) \in E_N^\Delta$ implies that $A_{ji}$ has full row rank. Thereby, one has $\nu_j = 0$. Therefore, if node $j$ becomes black during the coloring process, the subvector of $\nu$ corresponding to the vertices of $j$, that is, $\nu_j$, should be zero. Hence, by the termination of the coloring process, we have $\nu_j = 0$, for all $k \in \text{Ver}(D(Z))$.

The next theorem, regarding the modal strong structural controllability, is one of the main results of this work.

**Theorem 1:** Given some $\Delta \subseteq \mathbb{C}$, an LTI network with dynamics (2) and the $\Delta$-network graph $G_N^\Delta = (V_N, E_N^\Delta)$ is $\Delta$-SSC if $V_C$ is a ZFS of $G_N^\Delta$.

**Proof.** Suppose that $V_C$ is a ZFS of $G_N^\Delta$, but there exists $A \in S(f^\Delta_A)$ with some uncontrollable eigenvalue in $\Delta$. Then, $A$ has a nonzero left eigenvector $\nu = [\nu_1^T \ldots \nu_n^T]^T \in \mathbb{C}^n$ associated with $\lambda \in \Delta$ such that $\nu^T B = 0$, or equivalently, $\nu(i) = 0$, for all $i \in \text{Ver}(V_C)$. However, since $D(V_C) = V_N^\Delta$, it follows from Lemma 7 that $\nu = 0$; which is a contradiction.

**Example 13:** Consider Example 9, where the associated $\Delta$-network graph $G_N^\Delta$ is shown in Fig. 3. In this example, one can see that every set $V_C = \{i\}$, $i = 1, 2, \ldots, n$, is a ZFS of $G_N^\Delta$, and thus renders the network $\Delta$-SSC.

Note that Theorem [1] provides a sufficient condition for modal strong structural controllability of an LTI network that includes $n$ subsystems with (probably) different dimensions. In the following, given a set $\Delta \subseteq \mathbb{C}$ that includes at least one real number, we show that if the LTI network is an N1DS (i.e., all its subsystems are single-state), then the controllability condition is necessary as well.

**Theorem 2:** Consider some $\Delta \subseteq \mathbb{C}$, where $\Delta \cap \mathbb{R} \neq \emptyset$. Then, an N1DS with dynamics (2) and the $\Delta$-network graph $G_N^\Delta = (V_N, E_N^\Delta)$ is $\Delta$-SSC if and only if $V_C$ is a ZFS of $G_N^\Delta$.

**Proof.** The sufficiency is proved by considering Theorem [1]. To prove the necessity, assume that every eigenvalue $\lambda \in \Delta$ of all $A \in S(f^\Delta_A)$ is controllable, but $V_C$ is not a ZFS of $G_N^\Delta$, that is, $D(V_C) \neq V_N^\Delta$. In the $\Delta$-network graph, let $\mathcal{B} = D(V_C)$, and $\mathcal{W} = V_N \setminus D(V_C)$. Without loss of generality, index the nodes of $\mathcal{B}$ first. Let $r = |\mathcal{B}|$. Then, the pattern matrix $A$ can be partitioned as,
\[ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \]
where $A_1 \in \{*, 0, ?\}^{r \times r}$, $A_2 \in \mathbb{R}^{r \times (n-r)}$, $A_3 \in \mathbb{R}^{(n-r) \times r}$, and $A_4 \in \{*, 0, ?\}^{(n-r) \times (n-r)}$. Let $\nu^T = [\nu_1^T \ldots \nu_n^T]^T$, and assume that $\mu \in \Delta \cap \mathbb{R}$. Note that no node in $V_N$ has exactly one strong out-neighbor in $\mathcal{W}$. Otherwise, the coloring process can be applied, contradicting the definition of the derived set of $V_C$. Therefore, either every node in $\mathcal{B}$ has no white out-neighbor, or it has at least one white weak or two white strong out-neighbors. Then, in every column of $A_3$, if all entries are not zero, then either there is at least one question mark $?$ (i.e., an arbitrary entry, which can be zero or nonzero) or there are at least two stars $*$ (i.e., nonzero entries). Thus, the nonzero and arbitrary entries of $A_3$ can be chosen in a way that $1_{2r, r}^{-}A_3 = 0$. Now, consider a node $i$ in $\mathcal{W}$. Then, we have one of the following cases, which are shown in Figs. 5(a)-(e), respectively: (a) $i$ has no white out-neighbor; (b) $i$ has at least two white strong out-neighbors $j$ and $k$ $(j, k \neq i)$; (c) $i$ has at least two white strong out-neighbors $j$ and $k$ $(j \neq k)$; (d) $i$ has at least one white weak out-neighbor $j$; (e) $i$ has at least one white weak out-neighbor $j$ $(j \neq i)$. Since node $i$ in $G_N$ has no self-loop in cases (a), (b), and (e), then $f^\Delta_A(i) = 0$, and one can conclude that $\Delta = \{\mu\}$, and $A(i, i) = \mu$, for all $A \in S(f^\Delta_A)$. Moreover, in case (b), for all $A \in S(f^\Delta_A)$, we have $A(i, j) \neq 0$ and $A(k, i) \neq 0$. Also, in case (e), $A(i, j)$ can be zero or nonzero. Consider now case (c), where $i$ is strong out-neighbor of itself. Then, since $f^\Delta_A(i) = \ast$, we have $A(i, i) \cap \Delta = \emptyset$. Then, because

![Diagram](image-url)
Provided in the works [18], [19], [21]–[23], [26], [42] are results showing that the algebraic and graph-theoretic conditions on forming the controllability are based on the eigenvalues of any column in \(A\). Therefore, based on Theorem 2, given a \(G\), one can conclude that the \(A\)-network matrix \(A\) is the same as the graph \(G\). Now, consider the pattern matrix \(A\) in (13). As inferred from Theorem 2, the \(\Delta\)-network graph associated with an N1DS with the pattern matrix \(A\) is the same as the graph \(G\). In this work, we compare Theorem 2 with some existing results.

Example 15: Consider an N1DS with dynamics (2) and an arbitrary-pattern matrix \(A\). As discussed in Example 3, if \(i \neq j\), \(A(i,j) = \overline{A}(i,j)\), and for 1 \(\leq i \leq n\),
\[
A(i,i) = \begin{cases} s, & \text{if } A(i,i) = 0, \\ ?, & \text{otherwise.}
\end{cases}
\]

Then, the network is strongly structurally controllable if and only if \(V_C\) is a ZFS of both the graphs \(G_1 = G(A)\) and \(G_2 = G(\overline{A})\).

In order to compare Theorems 2 and 3, we note that a very special example of our result is an N1DS with \(\Delta = \{0\}\) and \(\Delta' = \mathbb{C} \setminus \{0\}\). From Example 5, one can conclude that the \(\Delta\)-network graph associated to an N1DS with the pattern matrix \(A\) is strongly structurally controllable if and only if \(V_C\) is a ZFS of the \(\Delta\)-network graph \(G_1\) (respectively, \(\Delta'\)-network graph \(G_2\), verifying Theorem 3).

Now, let us discuss the result of the work [18] as well. In this work, an N1DS with an arbitrary-diagonal pattern matrix \(A\) in \(\{*, 0, ?\}_{1}^{n \times n}\) has been investigated, where we have \(A(i,i) = ?\), for all 1 \(\leq i \leq n\). Then, a loop-free graph \(G_{\text{Li}}\) associated with \(A\) has been defined as \(G_{\text{Li}} = G_{\text{loop-free}}(A) = (V, E)\), where for \(i \neq j\), \((v_i, v_j) \notin E\) if and only if \(A(i,j) = 0\). However, based on our definition, one can see that \(G_{\text{Li}} = G(A)\) is a loop graph whose every node is a weak out-neighbor of itself.

In [18], an “ordinary coloring rule” is introduced, which states that a white node \(v\) can become black if it is the only white out-neighbor of a “black node” \(v\). Now, let \(Z\) be an initial set of black nodes in \(G_{\text{Li}}\). Set \(Z\) is called an “ordinary zero forcing set” if by the repeated application of the coloring rule in \(G_{\text{Li}}\), all nodes become black. The next result from [18] provides a graph-theoretic condition for strong structural controllability.

**Theorem 4:** An N1DS with dynamics (2) and an arbitrary-diagonal pattern matrix \(A\) is strongly structurally controllable if and only if \(V_C\) is an ordinary ZFS of \(G_{\text{Li}}\).

Considering Example 8, we note that in our framework, the \(\Delta\)-network graph for any nonempty \(\Delta \subseteq \mathbb{C}\) is the same as \(G_{\text{Li}} = G(A)\), where there is a self-loop on every node of the graph, and all self-loops are represented by dotted arrows. Therefore, based on Theorem 2, given a \(\Delta \subseteq \mathbb{C}\), where \(\Delta \cap \mathbb{R} \neq \emptyset\), this network is \(\Delta\)-SSC if and only if \(V_C\) is a ZFS of \(G_{\text{Li}}\). Moreover, in [19], [31], it has been demonstrated that \(V_C\) is an ordinary ZFS of \(G_{\text{Li}}\) if and only if it is a ZFS of \(G_{\text{Li}}\). Thus, one can see that our result is in line with the result of [18].

**Corollary 1:** Consider two arbitrary nonempty sets of numbers \(\Delta_1\) and \(\Delta_2\), where \(\Delta_1 \subseteq \mathbb{C}\) and \(\Delta_1 \cap \mathbb{R} = \emptyset, i = 1, 2\).

---

1 A loop-free graph does not contain any self-loop, while a loop graph can have self-loops on some of the nodes.
An N1DS with dynamics (1) and an arbitrary-diagonal pattern matrix is $\Delta_1$-SSC if and only if it is $\Delta_2$-SSC.

C. Maximum Geometric Multiplicity and Zero Forcing Sets

Consider a block matrix $\mathbf{A}$ in (1) with $A_{ij} \in \mathbb{R}^{l \times l_i}$. For a pattern matrix $\mathbf{A}$, a characteristic vector $f_{\mathbf{A}}^N$ and a nonempty set $\Delta \subseteq C$, let $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$. Now, consider the corresponding $\Delta$-network graph $G_{\mathbf{A}, \Delta}$. A minimal ZFS, denoted by $\text{ZFS}_{\text{min}}$, is a ZFS of $G_{\mathbf{A}, \Delta}$ with $\text{Ver}(\text{ZFS}_{\text{min}})$ has the minimum cardinality. The cardinality of the set of vertices of a minimal ZFS (i.e., $|\text{Ver}(\text{ZFS}_{\text{min}})|$) is called the zero forcing number and denoted by $Z(G_{\mathbf{A}}^N)$.

There is a relation between $Z(G_{\mathbf{A}}^N)$ and the maximum nullity of $\lambda I - \mathbf{A}$, for all $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$ and $\lambda \in \Delta$. By this relation, we can provide an upper bound on the maximum geometric multiplicity of eigenvalues of all $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$ that belong to the set $\Delta$.

Proposition 2: Given the $\Delta$-network graph $G_{\mathbf{A}}^N = (V_N, E_N)$ with the zero forcing number $Z(G_{\mathbf{A}}^N)$, we have $\Psi_{\Delta}(\mathbf{A}) \leq Z(G_{\mathbf{A}}^N)$, for all $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$.

Proof. Suppose that for some $\lambda \in \Delta$, $\psi_A(\lambda) = k$. Then, for every $X \subseteq V_N$ with $|\text{Ver}(X)| = k - 1$, there is a nonzero $\nu \in V_A(\lambda)$ such that $\nu_i = 0$, for all $i \in X$ (see proof of Proposition 2.2 in [37]). Now, let $\beta \in \Delta$ be an eigenvalue of some $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$, with $\psi_A(\beta) > Z(G_{\mathbf{A}}^N)$. Then, for a ZFS denoted by $Z$ with $|\text{Ver}(Z)| = Z(G_{\mathbf{A}}^N)$, there is a nonzero $\nu \in V_A(\beta)$ such that $\nu_i = 0$, for all $i \in Z$. Moreover, from Lemma 1 since $D(Z) = V_N$, we have $\nu = 0$, contradicting the assumption.

For an N1DS, where $l_1 = \ldots = l_n = 1$, by using the notion of ZFS in the corresponding $\Delta$-network graph, one can further examine whether there is any $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$ that has any eigenvalue in $\Delta$ or not.

Theorem 5: Consider $\Delta \subseteq C$, where $\Delta \cap \mathbb{R} \neq \emptyset$, and let $G_{\mathbf{A}}^N = (V_N, E_N)$ be a $\Delta$-network graph associated with an N1DS. No matrix $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$ has any eigenvalue in $\Delta$ if and only if $Z(G_{\mathbf{A}}^N) = 0$.

Proof. The sufficiency part of the Theorem follows immediately from Proposition 2. To prove the necessity, assume that no $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$ has any eigenvalue in $\Delta$, but $Z(G_{\mathbf{A}}^N) > 0$. Then, for every subset $X \subseteq V_N$ such that $|X| = Z(G_{\mathbf{A}}^N) - 1$, one should have $D(X) \neq V_N$. Since we have $\Psi_{\Delta}(\mathbf{A}) \cap \Delta = \emptyset$, for all $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$, then based on Definition 1 an N1DS with dynamics (1) and any choice of the set $V_C$ is $\Delta$-SSC. Then, from Theorem 2 $V_C = X$ should be a ZFS, contradicting the assumption.

D. Full Rank Condition for Pattern Matrices

Consider an LTI network $\mathcal{N}$ with the system matrix $\mathbf{A}$ described in (1), where $\mathbf{A} \in S(f_{\mathbf{A}}^N, \mathbf{A})$. Let $\mathcal{G} = (V_G, E_G)$ and $G_{\mathbf{A}}^N = (V_N, E_N)$ be the global graph and the corresponding $\Delta$-network graph, respectively. As discussed in Section III-A, for $j \neq i$, in order to find whether $(j, i) \in E^*_N$, one should check if the pattern matrix $A_{ij} \in 0, 1, \ast \}^{l \times l}$ has full row rank or not, which is discussed in the following.

Consider a pattern matrix $A' \in \{0, 1, \ast\}^{q \times p}$, where $l = \max(q, p)$. Then, recall that one can associate to $A'$ a graph $G(A') = (V, E)$, where $V = \{v_1, \ldots, v_l\}$, and $(v_j, v_i) \in E$ if and only if $A'(i, j) \neq 0$. In addition, corresponding to $A'$, one can define a bipartite graph as $G_b(A') = (V_r, V_c, E_b)$ with $V_r = \{v_1', \ldots, v_q'\}$ and $V_c = \{v_1'', \ldots, v_p''\}$, and $(v_j', v_i'') \in E_b$ if and only if $A'(i, j) \neq 0$. Thus, $G_b$ can be considered as a bipartite graph equivalent to the graph $G(A')$.

Now, let $G_i = (V_i, E_i)$ be the node graph associated with subsystem $i$, $1 \leq i \leq n$, where $V_i = \{i^1, \ldots, i^n\}$. For $i \neq j$, consider the pattern matrix $A_{ij} \in \{0, 1, \ast\}^{l \times l}$, that describes the edges connecting the vertices of $V_j$ to the vertices of $V_i$. Let us define the bipartite graph $G_i \to j = G_b(A_{ij}) = (V_i, V_j, E_{ij-j})$. Similarly, let $G_{ij-j} = G_b(A_{ij}) = (V_j, V_i, E_{j-i})$. Now, as a subgraph of $\mathcal{G}$, we define the graph $G(i,j) = (V_i \cup V_j, E_{ij-j} \cup E_{j-i})$, whose edge set is the set of all edges in $E_C$ with one end node in $V_i$ and the other end node in $V_j$.

Now, assume that all nodes of $G(i,j)$ are initially white, and apply the coloring rule in $G(i,j)$ as many times as possible. In the following, we show that pattern matrix $A_{ij}$ has full row rank if and only if all vertices in $V_i$ become black.

Proposition 3: Consider the graph $G(i,j)$, and let $Z = 0$. Then, for $1 \leq i, j \leq n$, where $i \neq j$, $A_{ij}$ has full row rank if and only if all vertices in $V_i$ become black.

Proof. For $i_1 \leq i_2$, assume that all nodes of the graph $G(A_{i_2}) = (V, E)$ with $V = \{1, \ldots, v_i\}$ are initially white, and perform a coloring process in this graph. Considering Theorem 10 in [29], one can conclude that $A_{i_2}$ has full row rank if and only if all nodes $v_1, \ldots, v_l$ in $G(A_{i_2})$ become finally black. Now, consider the equivalent bipartite graph $G_{j-i} = G_b(A_{ij})$, and suppose that all vertices in $V_i$ and $V_j$ are initially white. Assume that in some step of the coloring process in the graph $G(A_{ij})$, a white node $v_k$ has only one white strong out-neighbor $v_r$, and thus $v_k \to v_r$. One can see that equivalently, in the bipartite graph $G_{j-i}$, there is a vertex $j^k \in V_j$ that has only a white strong out-neighbor, that is, $i^r \in V_i$, and we have $j^k \to i^r$. Thus, the nodes $v_1, \ldots, v_l$ in $G(A_{ij})$ become finally black if and only if the vertices $i^1, \ldots, i^n$ in $G_{j-i}$ are eventually black. Moreover, since the in-neighbors of any node in $V_i$ should belong to $V_j$, one can see that during the coloring process, any vertex $i^k$ in $G_{j-i}$ becomes black if and only if it becomes black in $G(i,j)$. Thus, $A_{ij}$ has full row rank if and only if by the termination of the coloring process in $G(i,j)$, all vertices in $V_i$ become black.

Example 16: Consider the global graph $\mathcal{G}$ in Fig. 1(b). As
an example, \( G_\Delta(A_{42}) \cup G_\Delta(A_{43}) \) is shown in Fig. 7(a). One can see that by applying the coloring process in this graph, \( V_4 = \{4^1, 4^2\} \) become black, while \( V_3 = \{3^1, 3^2\} \) cannot be black. Thus, based on Proposition 3, \( A_{42} \) has full row rank. Similarly, one can see that \( A_{42} = 0, A_{41} = 0, A_{32} = 0, A_{34} = 0, \) and \( A_{43} = 0. \) Moreover, \( A_{31} \) and \( A_{43} \) have full row rank, while the nonzero pattern matrices \( A_{21}, A_{13}, A_{14}, A_{23}, \) and \( A_{34} \) are rank-deficient. Now, let \( \Delta = \{ y \in C \mid \Re(y) \geq 0 \}, \) and assume that subsystems 2 and 4 are stable. Then, one can provide the corresponding \( \Delta \)-network graph \( G_\Delta \) as shown in Fig. 7(b). Note that since \( V_C = \{1\} \) is a ZFS of \( G_\Delta, \) the network is strongly structurally stabilizable.

V. Conclusions

In this work, for a given nonempty set \( \Delta \subseteq C, \) we derived a graph-theoretic condition for strong structural controllability of LTI networks, including dynamical subsystems, with respect to \( \Delta. \) The family of LTI networks consists of systems with the same zero/nonzero/arbitrary structure. However, in real applications, there might be some more information about the subsystems other than the zero/nonzero/arbitrary pattern of system matrices. For instance, one might know that some of the subsystems have no eigenvalue in \( \Delta. \) To deal with this more general case, we have defined a more restrictive family of LTI networks, for which the existing results on strong structural controllability seem conservative. Then, it was shown how one can provide a corresponding \( \Delta \)-network graph for a given set \( \Delta. \) In this setup, a correspondence between a set of control subsystems and a zero forcing set of a structural controllability was investigated as a special case. Finally, it has been shown over all its pattern matrices (with extra spectrum features). In conditions under which an N1DS admits no eigenvalue in \( s, \) it is sufficient. Along the way, we have also derived structural conditions under which an N1DS admits no eigenvalue in \( \Delta \) over all its pattern matrices (with extra spectrum features). In addition, the strong structural stabilizability of LTI networks was investigated as a special case. Finally, it has been shown how this work can generalize the previous works on strong structural controllability.

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