Kodaira dimension of algebraic fiber spaces over surfaces

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Abstract

In this short note we prove the Iitaka $C_{nm}$ conjecture for algebraic fiber spaces over surfaces.

1. Introduction

Let $p : X \to Y$ be a fibration between two projective manifolds. A central problem in birational geometry is the Iitaka conjecture, stating that

$$\kappa(X) \geq \kappa(Y) + \kappa(X/Y) \quad (1.0.1)$$

where $\kappa(X)$ is the Kodaira dimension of $X$ and $\kappa(X/Y)$ is the Kodaira dimension of the generic fiber.

In this note, we prove that the log-version of Iitaka conjecture holds true, provided that the base $\dim Y \leq 2$; this generalizes a result obtained by C. Birkar in [Bir09, Thm 1.4] and a result of Y. Kawamata in [Kaw82a]. More precisely, we have the following statement.

1.1. Theorem. Let $p : X \to Y$ be a fibration between two projective manifolds. Let $F$ be the generic fiber and let $\Delta$ be a $\mathbb{Q}$-effective klt divisor on $X$. Set $\Delta_F := \Delta|_F$. If $\dim Y \leq 2$, then we have

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y). \quad (1.1.1)$$

We will next explain the main steps of the proof. Since $\dim Y \leq 2$, we can assume that $K_Y$ is nef, by using the MMP. Three cases as follows might occur, according to the Kodaira dimension of $Y$.

If $\kappa(Y) \geq 1$, the inequality (1.1.1) is quickly verified by using [Kaw82a].

If $\kappa(Y) = 0$, by the classification theory we know that $Y$ is a torus or a K3 surface, modulo a finite étale cover. If $Y$ is a torus, (1.1.1) is proved in [CH11] for the absolute case (i.e., $\Delta = 0$) and in [CP17] for the klt pair case. Therefore, to prove (1.1.1), it is enough to assume that $Y$ is a K3 surface. In order to treat this case, we rely on two main ingredients, namely the positivity of the direct images $p_*(mK_{X/Y} + m\Delta)$ and the geometry of orbifold Calabi-Yau surfaces. Different

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aspects of the first topic were extensively studied in [Gri70, Fuj78, Kaw82a, Kaw82b, Kol85, Vie95, Ber09, BP10, PT14, BC15, Fuj16, HPS16, KP17], among many other articles. In our set-up, this implies that \( \det p_*(mK_{X/Y} + m\Delta) \) is pseudo-effective (by using [PT14]). As \( Y \) is a K3 surface, the numerical dimension of \( \det p_*(mK_{X/Y} + m\Delta) \) coincides with its Iitaka dimension.

If the numerical dimension \( \text{nd}(\det p_*(mK_{X/Y} + m\Delta)) \geq 1 \), we achieve our goals by standard arguments. If \( \text{nd}(\det p_*(mK_{X/Y} + m\Delta)) = 0 \), we can show that there exists a finite set of exceptional curves \( [C_i] \) on \( Y \) such that \( p_*(mK_{X/Y} + m\Delta) \) is hermitian flat on \( Y \setminus (\cup C_i) \), by using the results in [PT14, CP17, HPS16]. At this point we use the second ingredient, namely the uniformization theorem for the compact Kähler orbifolds with trivial first Chern class, cf. [Cam04a]. We thus infer that the fundamental group of \( Y \setminus (\cup C_i) \) is almost-abelian. Therefore we can construct sufficient elements in \( H^0(X, mK_{X/Y} + m\Delta) \) by using parallel transport, and (1.1.1) is proved.

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2. Preparation

In this section, we will recall the uniformization theorem for the compact Kähler orbifolds with trivial first Chern class [Cam04a] (cf. also [CC14, GKP15]) as well as the results concerning singular metrics on vector bundles and the positivity of direct images, cf. [BP08, PT14, Rau15, Pau16, HPS16] for more details.

First of all, we recall a few basic definitions concerning compact Kähler orbifolds by following [Cam04a].

2.1. Definition. [Cam04a, Definition 3.1, Definition 5.1]

(i) A compact Kähler orbifold is a compact Kähler normal variety with only quotient singularities, i.e., for every point \( a \in X \), we can find a neighbourhood \( U \) of \( a \), and a biholomorphism \( \psi : U \to \tilde{U}/G \) where \( \tilde{U} \) is an open set in \( \mathbb{C}^n \) and \( G \subset GL(n, \mathbb{C}) \) is a finite subgroup acting on \( \tilde{U} \) with \( \psi(a) = 0 \). For every \( g \in G \), the set of the fixed points of \( g \) is of codimension at least 2.

(ii) Let \( X^* \) be the smooth locus of a compact Kähler orbifold \( X \). We say that \( X \) is simply connected in the sense of orbifolds if \( X^* \) is simply connected.

(iii) A holomorphic morphism between two Kähler orbifolds \( r : X' \to X \) is said to be an orbifold cover, if it satisfies the following two conditions:

- The restriction of \( r \) to \( r^{-1}(X^*) \) is an étale cover.
- For every \( a \in X \) with its neighbourhood \( \tilde{U}/G \) (cf. (i)), each component of \( r^{-1}(\tilde{U}/G) \) is of the form \( \tilde{U}'/G' \) for some subgroup \( G' \) of \( G \), and the restricted morphism \( r|_{\tilde{U}'/G'} \) is nothing but the natural quotient morphism \( \tilde{U}'/G' \to \tilde{U}/G \).

(iv) A \( m \)-dimensional compact Kähler orbifold \( X \) is called Calabi-Yau (resp. Hyperkähler), if it is simply connected in the sense of orbifold (cf. (ii)) and it admits a Ricci-flat Kähler metric.
such that the holonomy (when restricted to $X^*$) is $SU(m)$ (resp. $Sp(m/2)$).

We state here the uniformization theorem for the compact Kähler orbifolds with trivial first Chern class, established in [Cam04a]. The statement parallels the classical case of smooth Kähler manifolds with trivial first Chern class.

2.2. Theorem. [Cam04a, Thm 6.4] Let $X$ be a compact Kähler orbifold with $c_1(X) = 0$. Then $X$ admits a finite orbifold cover $\overline{X} = \overline{C} \times \overline{S} \times T$, where $\overline{C}$ (resp. $\overline{S}$) is a finite product of Calabi-Yau Kähler orbifold (resp. Hyperkähler) and $T$ is a complex torus.

Let $Y$ be a K3 surface and let $\cup C_i$ be a set of exceptional curves on $Y$. By Grauert’s criterion [BHPV, III, Thm 2.1], there is a contraction morphism $\tau : Y \to Y_{\text{can}}$ which contracts all $C_i$ to some points $p_i$ in a normal space $Y_{\text{can}}$. As $K_Y$ is trivial, we know that $Y_{\text{can}}$ is in fact a compact Kähler orbifold ([KM98 Def 4.4, Rk 4.21], [Cam04a Example 3.2]) with $c_1(Y_{\text{can}}) = 0$.

As a corollary of Theorem 2.2, we have.

2.3. Proposition. [Cam04a, Cor 6.7] Let $Y$ be K3 surface and let $\cup C_i$ be some exceptional curves on $Y$. Then $\pi_1(Y \setminus (\cup C_i))$ is almost-abelien.

Moreover, let $\tau : Y \to Y_{\text{can}}$ be the morphism which contracts the exceptional curves $C_i$ to some points $p_i \in Y_{\text{can}}$. If $\pi_1(Y \setminus (\cup C_i))$ is not finite, there exists a finite orbifold cover from a complex torus $T$ to $Y_{\text{can}}$:

$$\sigma : T \to Y_{\text{can}}.$$ 

In particular, $\sigma$ is a non-ramified cover over $Y_{\text{can}} \setminus (\cup p_i)$ and $\sigma^{-1}(p_i)$ is of codimension 2 for every $i$.

The following non-vanishing property for pseudo-effective line bundles on K3 surfaces is an immediate consequence of the abundance theorem (which holds true in dimension two).

2.4. Proposition. Let $Y$ be a K3 surface (in the smooth sense) and let $L$ be a pseudo-effective line bundle on $Y$. Then $L$ is $\mathbb{Q}$-effective.

Proof. Since $L$ is pseudo-effective, by using Zariski decomposition for surface [Fuj79 Thm 1.12], we know that

$$L \equiv_{\mathbb{Q}} \sum_{i=1}^{p} a_i[C_i] + M,$$

where $a_i \in \mathbb{Q}^+$, $C_i$ are negative intersection curves, $M$ is nef and $M \cdot C_i = 0$ for every $i$. Since $Y$ is K3, all nef line bundles on $Y$ are effective (cf. [BHPV VIII, Prop 3.7]). Therefore $L$ is $\mathbb{Q}$-effective.

2.5. Remark. It is well-known that a nef line bundle $M$ on a K3 surface is semiample. If its numerical dimension $\text{nd}(M)$ is equal to one, then it induces an elliptic fibration over $\mathbb{P}^1$.

In the second part of this section we will recall a few definitions and results about the singular metrics on vector bundles and the positivity of direct images. We refer to [BP08, Rau15, PT14, Pau16, HPS16] for more details.
2.6. **Definition.** Let $E \to X$ be a holomorphic vector bundle on a manifold $X$ (which is not necessary compact). Locally, a singular hermitian metric $h_E$ on $E$ is a measurable map from $X$ to the space of non-negative Hermitian forms on the fibers. We say that $(E, h_E)$ is negatively curved, if $0 < \det h_E < +\infty$ almost everywhere and

$$ x \to \ln |u|_{h_E}(x) \quad x \in X $$

is a psh function, for any choice of a holomorphic local section $u$ of $E$.

We say that the pair $(E, h_E)$ is positively curved, if the dual $(E^*, h_E^*)$ is negatively curved. We note it by $i\Theta_{h_E}(E) \geq 0$.

When $h_E$ is smooth, ”positively curved” is nothing but the classical Griffiths semi-positivity. The following result proved in [PT14] plays an important role in this article.

2.7. **Theorem.** [PT14 Thm 5.1.2] Let $p : X \to Y$ be a fibration between two projective manifolds and let $L$ be a line bundle on $X$ with a possibly singular metric $h_L$ such that $i\Theta_{h_L}(L) \geq 0$. Let $m \in \mathbb{N}$ such that the multiplier ideal sheaf $\mathcal{I}(h_L^m|_{X_y})$ is trivial over a generic fiber $X_y$, namely $\int_{X_y} |e_L|_{h_L}^2 < +\infty$, where $e_L$ is a basis of $L$.

Let $Y_1$ be the locally free locus of $p_*(mK_{X/Y} + L)$. Then the vector bundle $p_*(mK_{X/Y} + L)$ over $Y_1$ admits a possibly singular hermitian metric $h$ such that $i\Theta_h(p_*(mK_{X/Y} + L)) \geq 0$ on $Y_1$. Moreover, $h$ induces a possibly singular metric $\det h$ on the line bundle $\det p_*(mK_{X/Y} + L)$ over $Y$ such that

$$ i\Theta_{\det h}(\det p_*(mK_{X/Y} + L)) \geq 0 \quad \text{on } Y $$

in the sense of current.

2.8. **Remark.** Let us recall briefly the construction of the metric $h$: Let $h_B$ be the $m$-relative Bergman kernel metric on $K_{X/Y} + \frac{1}{m}L$ constructed in [BP10, A.2]. Set $L_1 := (m-1)K_{X/Y} + L$ and $h_{L_1} := (m-1)h_B + \frac{1}{m}h_L$. Thanks to [BP10, A.2], we know that

$$ i\Theta_{h_{L_1}}(L_1) \geq 0 \quad \text{on } X \quad (2.8.1) $$

in the sense of current. Now $h_{L_1}$ induces a Hodge type metric $h$ on $p_*(mK_{X/Y} + L)$ on the smooth locus $Y_0$ of $\pi$ as follows: let $X_y$ be a smooth fiber and let $f \in H^0(X_y, mK_{X/Y} + L)$. As $mK_{X/Y} + L = K_{X/Y} + L_1$, the norm

$$ ||f||^2_h := \int_{X_y} |f|_{h_{L_1}}^2 $$

is well defined. Since $h_L$ is not necessarily smooth, $h$ is a possibly singular hermitian metric on $(\pi_*(mK_{X/Y} + L), Y_0)$. Thanks to [Ber09, BP08], we can prove that $(p_*(mK_{X/Y} + L), h)$ is positively curved on $Y_0$. By studying the comportment of $h$ near $Y_1 \setminus Y_0$, [PT14] proved finally that $h$ can be extended as a possibly singular hermitian metric on $Y_1$ with positive curvature in the sense of Definition 2.6.

The following proposition comes from the standard extension theorem.

2.9. **Proposition.** In the setting of Theorem 2.7, we suppose moreover that there exists a fibration $q : Y \to Z$ to some projective manifold $Z$. Let $H$ be a pseudo-effective line bundle on $Y$ with a possible singular metric $h_H$ such that $i\Theta_{h_H}(H) \geq 0$ in the sense of current. Let $A_Z$ be an ample line bundle on $Z$. Then for $c \in \mathbb{N}$ large enough (depending only on $A_Z$ and $Z$), the following extension property holds:
Let $z \in Z$ be a generic point and let $X_z$ (resp. $Y_z$) be the fiber of $p \circ q$ (resp. $q$) over $z$. Let
\[ e \in \mathcal{O}_{Z,z}(c \cdot A_Z) \] and let $s \in H^0(Y_z, K_Y \otimes H \otimes p_*(mK_{X/Y} + L))$ such that
\[ \int_{Y_z} |s|_{h_{u},h}^2 < +\infty, \tag{2.9.1} \]
where $h$ is the metric on $p_*(mK_{X/Y} + L)$ in Theorem 2.7. Then there exists a section
\[ S \in H^0(Y, K_Y \otimes H \otimes p_*(mK_{X/Y} + L) \otimes q^*(cA_Z)) \]
such that $S|_{Y_z} = s \otimes q^*e$.

**Proof.** Let $(L_1,h_1)$ be the line bundle constructed in Remark 2.8. Then $s$ induces a section
\[ u \in H^0(X_z, K_X + H + L_1) \] and (2.9.1) implies that
\[ \int_{X_z} |s|_{h_{u},h_1}^2 < +\infty. \]
For $c \in \mathbb{N}$ large enough (depending only on $Z$ and $A_Z$), by the standard Ohsawa-Takegoshi extension theorem (cf. for example [Dem12, Chapter 13]), we can find a
\[ U \in H^0(X, K_X + H + L_1 + (p \circ q)^*cA_Z) \]
such that $U|_{X_z} = u \otimes (p \circ q)^*e$. Then $U$ induces a section
\[ S \in H^0(Y, K_Y \otimes H \otimes p_*(mK_{X/Y} + L) \otimes q^*(cA_Z)) \]
such that $S|_{Y_z} = s \otimes q^*e$ and the proposition is proved. \hfill \Box

As another direct consequence of the Ohsawa-Takegoshi extension, the following proposition is will be important for us.

**2.10. Proposition.** [BP10, A.2] In the setting of Theorem 2.7, let $U$ be a small Stein open subset of $X$ and let $V \subseteq U$ be some open set of compact support in $U$. Let $e$ be a basis of $mK_{X/Y} + L$ over $U$. Then there exists a uniform constant $C(U,V,e)$ depending only on $U,V,e$ such that for every $t \in \pi(V)$ and every $s \in H^0(X_t, mK_{X/Y} + L)$, we have
\[ \|s\|_e \leq C(U,V,e) \cdot \|s\|_h. \]

**Proof.** As explained in Remark 2.8, the line bundle $L_1 := (m - 1)K_{X/Y} + L$ can be equipped with a possibly singular metric $h_{L_1}$ such that
\[ i\Theta_{h_{L_1}}(L_1) \geq 0 \quad \text{on } X. \]
Since $U$ is a small open set, we can find a Stein open set $B \subseteq Y$ such that $U \subseteq p^{-1}(B)$. As $mK_{X/Y} + L = K_{X/Y} + L_1$, by applying the Ohsawa-Takegoshi extension theorem to the fibration $p^{-1}(B) \to B$, we can find a $\tilde{s} \in H^0(p^{-1}(B), K_X + L_1)$ such that
\[ \int_{p^{-1}(B)} |\tilde{s}|_{h_{L_1}}^2 \leq C \int_{X_t} |s|_{h_{L_1}}^2 = C \cdot \|s\|_h^2 \tag{2.10.1} \]
and
\[ \tilde{s}|_{X_t} = s \wedge p^*(e_B), \tag{2.10.2} \]
where $e_B$ is a basis of $K_Y$ over $B$.

As $q^*K_Z$ is a trivial bundle on $Y_z$, modulo this trivial bundle, $|s|_{h_{u},h}^2$ can be seen as a volume form on $Y_z$. Therefore the integral (2.9.1) is well-defined.
On the open set $U$, $\tilde{s}$ can be written as $\tilde{s} = \tilde{w} \cdot e \wedge p^*(e_B)$ for some holomorphic function $\tilde{w}$ on $U$. Note that $V \subseteq p^{-1}(B)$, (2.10.1) implies thus that
\[ \|\tilde{w}\|_{C^0(V)} \leq C(U,V,e) \cdot \|s\|_h \]
for some constant $C(U,V,e)$ depending only on $U, V$ and $e$. Thanks to (2.10.2), we have $\tilde{w}|_{X_t} = \tilde{e}$. Therefore
\[ \|\tilde{e}\|_{C^0(V \cap X_t)} \leq C(U,V,e) \cdot \|s\|_h. \quad (2.10.3) \]
The proposition is proved.

The last result of this section concerns the regularity of the metric $h$.

2.11. Proposition. [CP17, Cor 2.8] Let $E \to X$ be a holomorphic vector bundle on a manifold $X$ (which is not necessary compact). Let $h_E$ be a possibly singular hermitian metric on $E$ such that $(E,h_E)$ is positively curved. Let $U$ be a topological open set of $X$. If
\[ i\Theta_{h_E}(\det E) \equiv 0 \quad \text{on } U, \]
then $h_E$ is a smooth metric on $E|_U$, and $(E|_U,h_E)$ is hermitian flat.

3. Proof of the main theorem

We now prove the main theorem of the article.

3.1. Theorem. [=Theorem 1.1] Let $p : X \to Y$ be a fibration between two projective manifolds. Let $F$ be the generic fiber and let $\Delta$ be a $\mathbb{Q}$-effective klt divisor on $X$. Set $\Delta_F := \Delta|_F$. If $\dim Y \leq 2$, then
\[ \kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y). \quad (3.1.1) \]

Proof. Since $Y$ is of dimension 2, we can consider its minimal model and assume that $Y$ is a smooth projective surface with nef canonical bundle. We show next that it will be enough to treat the case where $Y$ is a K3 surface.

Indeed, if $\kappa(Y) \geq 1$, as the klt version of $C_{n,1}$ is known (cf. [Kaw82a, CP17]), we have thus (3.1.1). We refer to Proposition 4.2 in the appendix for a detailed proof. If $\kappa(Y) = 0$, by using the classification of minimal surface [BHPV, Thm 1.1], we have $c_1(Y) = 0 \in H^{1,1}_{\mathbb{Q}}(Y)$. After a finite étale cover, the base $Y$ is either a torus or a K3 surface. If $Y$ is a torus, [CP17, Thm 1.1] implies (3.1.1). We assume in this way for the rest of our proof that $Y$ is a K3 surface.

Let $m \in \mathbb{N}$ be sufficiently divisible and let $Y_1$ be the locally free locus of the direct image sheaf $p_*(mK_{X/Y} + m\Delta)$. By using Theorem 2.7 there exists a possibly singular hermitian metric $h$ on $(p_*(mK_{X/Y} + m\Delta), Y_1)$ such that
\[ i\Theta_h(p_*(mK_{X/Y} + m\Delta)) \geq 0 \quad \text{on } Y_1, \quad (3.1.2) \]
and $h$ induces a hermitian metric $\det h$ on $(\det p_*(mK_{X/Y} + m\Delta), Y)$ such that
\[ i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta)) \geq 0 \quad \text{on } Y \]
in the sense of current. In particular, the bundle $\det p_*(mK_{X/Y} + m\Delta)$ is pseudo-effective.
By Proposition 2.4, we have a Zariski decomposition
\[ \det p_*(mK_{X/Y} + m\Delta) \equiv q \sum_{i=1}^{s} a_i[C_i] + L_m, \tag{3.1.3} \]
where \( a_i \in \mathbb{Q}^+ \), \([C_i]\) are negative intersection curves, \( L_m \) is nef and \( L_m \cdot C_i = 0 \) for every \( i \). Let \( \text{nd}(L_m) \) be the numerical dimension of \( L_m \). We will distinguish next among three cases, according to the numerical dimension of \( L_m \).

**Case 1:** The numerical dimension of \( L_m \) equals two

Then we infer that the bundle \( \det p_*(mK_{X/Y} + m\Delta) \) is big on \( Y \), and (3.1.1) is thus proved by using \[\text{Cam04b}\] or \[\text{CP17 Thm 5.1}\].

**Case 2:** The numerical dimension of \( L_m \) equals one

Thanks to Remark 2.5, \( L_m \) is semiample. Then \( L_m \) induces a fibration \( \pi : Y \to \mathbb{P}^1 \). As \( L_m \cdot [C_i] = 0 \) for every \( i \), we have
\[ L_m \cdot \det p_*(mK_{X/Y} + m\Delta) = 0. \tag{3.1.4} \]
By using \[\text{Vie83 Lemma 7.3}\], we can find a birational morphism \( Y' \to Y \) from a projective manifold \( Y' \), and a desingularisation \( X' \) of \( Y' \times_Y X \) satisfying:
\[
\begin{array}{ccc}
X' & \xrightarrow{\pi_X} & X \\
\downarrow p' & & \downarrow p \\
Y' & \xrightarrow{\pi_Y} & Y \\
\downarrow \pi & & \downarrow \pi_Y \\
& & \mathbb{P}^1
\end{array}
\]
each divisor \( W \subset X' \) such that \( \text{codim}_{Y'} p'(W) \geq 2 \) is \( \pi_X \)-contractible. Since \( \Delta \) is klt, we can find a klt \( \mathbb{Q} \)-effective divisor \( \Delta' \) on \( X' \) and some effective \( \pi_X \)-exceptional divisor \( D' \) such that
\[ \pi_X^*(K_X + \Delta) + D' = K_{X'} + \Delta'. \tag{3.1.5} \]

**Claim.** The bundle
\[ \det p'_*(mK_{X'/Y'} + m\Delta') - c(\pi_Y \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1) \]
is pseudo-effective on \( Y' \) for some constant \( c > 0 \).

We will verify this claim later; for now we finish the proof of the theorem. By using \[\text{CP17 Thm 3.4}\], the claim implies the existence of a divisor \( E \subset X' \) such that \( \text{codim}_{Y'} p'(E) \geq 2 \) and
\[ D := K_{X'/Y'} + \Delta' + E - \epsilon(p' \circ \pi_Y \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1) \tag{3.1.6} \]
is \( \mathbb{Q} \)-pseudo-effective on \( X' \) for some \( \epsilon > 0 \).

Let \( m_1 \gg m_2 \gg 1 \). Thanks to (3.1.6), we have
\[ (m_1 + m_2)(K_{X'/Y'} + \Delta' + E) \]
\[ = m_1(K_{X'/Y'} + \Delta' + \frac{m_2}{m_1}D + E) + \epsilon m_2(p' \circ \pi_Y \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1). \tag{3.1.7} \]
As \( m_1 \gg m_2 \), we can apply Theorem 2.7 to \( m_1(K_{X'/Y'} + \Delta' + \frac{m_2}{m_1}D + E) \). In particular, we can find a possibly singular metric \( h_{m_1} \) on

\[
V_1 := p'_*(m_1(K_{X'/Y'} + \Delta' + \frac{m_2}{m_1}D + E))
\]
such that \( i\Theta_{h_{m_1}}(V_1) \geq 0 \). Set \( T := i\Theta_{det h_{m_1}}(det V_1) \). Then \( T \geq 0 \) in the sense of current. Let \( Y'_0 \) be a generic fiber of \( \pi \) and

If \( T_{|Y'_0} \) is not identically 0, as \( Y'_0 \) is of dimension 1, \( T_{|Y'_0} \) is strictly positive at a generic point of \( Y'_0 \). Together this with (3.1.7), \( det p'_*((m_1 + m_2)(K_{X'/Y'} + \Delta' + E)) \) is big on \( Y'_0 \). By applying [CP17], we get

\[
\kappa(X', K_{X'} + \Delta' + E) \geq \kappa(F, K_F + \Delta_F) + 2. \tag{3.1.8}
\]

As \( E \) and \( D' \) are \( \pi_X \)-contractible, (3.1.5) and (3.1.8) imply (3.1.1).

If \( T_{|Y'_0} \equiv 0 \), thanks to Proposition 2.11, \( (V_1_{|Y'_0}, h_{m_1}) \) is hermitian flat on \( Y'_0 \). In particular, \( h_{m_1} \) is a smooth metric. Note that \( H^0(Y', K_{Y'}) \) is of dimension 1. It defines a canonical metric \( h_{Y'} \) on \( K_{Y'} \) and the restriction of \( h_{Y'} \) on \( Y'_0 \) is smooth. As a consequence, we have

\[
\int_{Y'_0} |s|^{2}_{(m_1-1)h_{Y'}, h_{m_1}} < +\infty \quad \text{for every } s \in H^0(Y'_0, K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes V_1).
\]

Combining this with Proposition 2.9, we get

\[
h^0(Y', K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes V_1 \otimes \varepsilon m_2(\pi_Y \circ \pi)^*\mathcal{O}_{\mathbb{P}^1}(1)) \geq h^0(Y'_0, K_{Y'} \otimes (m_1 - 1)K_{Y'} \otimes V_1) = h^0(X'_0, m_1(K_{X'_0} + \Delta' + \frac{m_2}{m_1}D + E)).
\]

Together with (3.1.7), we obtain

\[
h^0(X', m_1 \cdot (p')^* K_{Y'} + (m_1 + m_2)(K_{X'/Y'} + \Delta' + E)) \geq h^0(X'_0, m_1(K_{X'_0} + \Delta' + \frac{m_2}{m_1}D + E)). \tag{3.1.9}
\]

Finally, by applying [Kaw82a, CP17] to \( X'_0 \rightarrow Y'_0 \), we have

\[
\kappa(X'_0, K_{X'_0} + \Delta' + \frac{m_2}{m_1}D + E) \geq \kappa(F, K_F + \Delta_F).
\]

Together with (3.1.9) and the fact that \( K_{Y'} \) is \( \mathbb{Q} \)-effective, we obtain

\[
\kappa(X', K_{X'} + \Delta' + E) \geq \kappa(F, K_F + \Delta_F). \tag{3.1.10}
\]

As \( E \) and \( D' \) are \( \pi_X \)-contractible, (3.1.5) and (3.1.10) imply (3.1.1).

**Case 3: The numerical dimension of \( L_m \) equals zero**

Then \( L_m \) is trivial (as it is semiample) and we have

\[
det p_*(mK_{X/Y}) \equiv_{\mathbb{Q}} \sum_{i=1}^{s} a_i[C_i] \tag{3.1.11}
\]

---

3We take \( H = (m_1 - 1)K_{Y'} \) and \( h_H = (m_1 - 1)h_{Y'} \).
where \([C_i]\) are negative curves. As \(i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta))\) is a positive current in the same class of \(\sum_{i=1}^s a_i[C_i]\), we get

\[
i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta)) = \sum_{i=1}^s a_i[C_i] \quad \text{on } Y
\]

in the sense of current. In particular, we have

\[
i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta)) \equiv 0 \quad \text{on } Y \setminus (U_C).
\]

By using Proposition \(2.11\), \((p_*(mK_{X/Y} + m\Delta), h)\) is hermitian flat on \(Y_1 \setminus (U_C)\).

Let : \(\tau : Y \rightarrow Y_{\text{can}}\) be the morphism which contracts the negative curves \(U_C\). There are two possible cases: \(\pi_1(Y \setminus (U_C))\) is finite or infinite. We will analyze each possibility.

3.0.1 \textit{The fundamental group } \(\pi_1(Y \setminus (U_C))\) \textit{is finite.} As codim}_Y(Y \setminus Y_1) \geq 2, we know that \(\pi_1(Y_1 \setminus (U_C)) = \pi_1(Y \setminus (U_C))\) is finite. Let \(r\) be the number of elements of the finite group \(\pi_1(Y_1 \setminus (U_C))\). Fix a generic point \(y \in Y_1 \setminus (U_C)\). As the direct image vector bundle \((p_*(mK_{X/Y} + m\Delta), h)\) is hermitian flat on \(Y_1 \setminus (U_C)\), the parallel transport induces a representation

\[
\rho : \pi_1(Y_1 \setminus (U_C)) \rightarrow \text{Aut}(H^0(X_y, mK_{X/Y} + m\Delta)).
\]

Let \(f \in H^0(X_y, mK_{X/Y} + m\Delta)\) be an element with unit norm. Although the parallel transport of \(f\) cannot induce a global section over \(Y_1 \setminus (U_C)\), the corresponding parallel transport of

\[
\prod_{a \in \pi_1(Y_1 \setminus (U_C))} \rho(a)(f) \in H^0(X_y, mr(K_{X/Y} + \Delta))
\]

induces a section \(\tilde{f} \in H^0(p^{-1}(Y_1 \setminus (U_C)), mr(K_{X/Y} + \Delta))\).

We now prove that \(\tilde{f}\) can be extended to the total space \(X\). Let \(U\) be an arbitrary small Stein open subset of \(X\) and \(V \subset U\) be some arbitrary open set with compact support in \(U\). Let \(e\) be a basis of \(mK_{X/Y} + m\Delta\) on \(U\). We have \(\tilde{f} = \tilde{\ell} \cdot e^{\otimes r}\) for some holomorphic function

\[
\tilde{\ell} \in H^0(V \cap p^{-1}(Y_1 \setminus (U_C)), \mathcal{O}_{V \cap p^{-1}(Y_1 \setminus (U_C)))}.
\]

By construction, on every fiber \(X_t\), \(\tilde{f} = \prod_{i=1}^r f_i\) for some \(f_i \in H^0(X_t, mK_{X/Y} + m\Delta)\) with unit norm. Thanks to Proposition \(2.10\), the \(C^r\)-norm \(\|\tilde{f}\|_{C^0(V \cap X_t)}\) is bounded by a constant \(C(U, V, e)\) independent of \(t\). Therefore

\[
\|\tilde{f}\|_{C^0(V \cap X_t)} = \prod_{i=1}^r \|f_i\|_{C^0(V \cap X_t)} \leq C(U, V, e)^r.
\]

In particular, \(\|\tilde{f}\|\) is bounded on \(V \cap p^{-1}(Y_1 \setminus (U_C))\) and \(\tilde{f}\) can be thus extended as a holomorphic section on \(V\). Since \(V\) is an arbitrary small open set in \(X\), \(\tilde{f}\) can be extended to the total space \(X\).

In conclusion, for any element \(f \in H^0(X_y, mK_{X/Y} + m\Delta)\), we can find a

\[
\tilde{f} \in H^0(X, mr(K_{X/Y} + \Delta))
\]
such that \( \tilde{f}|_{\chi_y} = \prod_{a \in \pi_1(Y_1 \setminus (\cup C_t))} \rho(a)(f) \). In particular, we have
\[
\text{div}(\tilde{f}|_{\chi_y}) = \sum_{a \in \pi_1(Y_1 \setminus (\cup C_t))} \text{div}(\rho(a)(f)).
\]
Therefore, \( \kappa(K_X + \Delta) \geq 1 \) if \( \kappa(K_F + \Delta_F) \geq 1 \). In other words, we have
\[
\kappa(K_X + \Delta) \geq \min\{1, \kappa(K_F + \Delta_F)\}.
\]
Together with a standard argument (cf. Proposition 4.1 in the appendix), we get
\[
\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F)
\]
and the first subcase is completely proved.

3.0.2 The fundamental group \( \pi_1(Y \setminus (\cup C_t)) \) is not finite. As a consequence of Proposition 2.3, there exists an orbifold cover from a complex torus \( T \) to \( Y_{\text{can}} \):
\[
\tau_Y : T \to Y_{\text{can}}.
\]
Let \( X' \) be a desingularisation of \( X \times_{Y_{\text{can}}} T \). We have thus a commutative diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\tau_X} & X \\
\downarrow{p'} & & \downarrow{p} \\
T & \xrightarrow{\tau_Y} & Y_{\text{can}}
\end{array}
\]
Set \( T_1 := \tau_Y^{-1}(\tau(Y_1 \setminus (\cup C_t))) \), where \( \tau : Y \to Y_{\text{can}} \) is the contraction morphism. Thanks to Proposition 2.3, \( \tau_Y \) is a non-ramified cover on \( T_1 \) and
\[
\text{codim}_T(T \setminus T_1) \geq 2.
\]
As \( \Delta \) is klt, we can find a klt \( \mathbb{Q} \)-effective divisor \( \Delta' \) on \( X' \) and some \( \mathbb{Q} \)-divisor \( D' \) supported in \((p')^{-1}(T \setminus T_1)\) such that
\[
\pi_X^*(K_X + \Delta) + D' = K_{X'} + \Delta'. \tag{3.1.13}
\]
Since \( T \) is a torus, by applying \([CP17]\), we have
\[
\kappa(K_{X'} + \Delta') \geq \kappa(K_F + \Delta_F).
\]
Let \( m \in \mathbb{N} \) be a sufficiently divisible number and let \( s \in H^0(X', mK_{X'}/T + m\Delta') \). Thanks to (3.1.13) and the fact that \( D' \) is supported in \((p')^{-1}(T \setminus T_1)\), \( s \) induces an element
\[
s_T \in H^0(T_1, \tau_Y^*(p_*(mK_{X'/Y} + m\Delta))).
\]
Since \((p_*(mK_{X'/Y} + m\Delta), h)\) is hermitian flat on \( Y_1 \), \( \|s_T\|_{\tau_Y^*h(t)} \) is a psh function on \( t \in T_1 \). As \( \text{codim}_T(T \setminus T_1) \geq 2 \), \( \|s_T\|_{\tau_Y^*h(t)} \) is thus constant with respect to \( t \in T_1 \). Let \( r \) be the degree of the cover \( \tau_Y \). Since \( \tau_Y \) is a non-ramified cover on \( T_1 \), \( s_T \) induces an element \( \tilde{s} \in H^0(p^{-1}(Y_1 \setminus (\cup C_t)), mrK_{X'/Y} + mr\Delta) \). As \( \|s_T\|_{\tau_Y^*h(t)} \) is constant, by using the same argument as in the subcase 3.0.1, \( \tilde{s} \) can be extended to as an element in \( H^0(X, mrK_{X'/Y} + mr\Delta) \). (3.1.1) is thus proved by using the same argument as in the end of Subcase 3.0.1.

Our next job is to establish the claim used in the proof of our main result, which is a consequence of the volume estimate inequality (or the holomorphic Morse inequalities).
Proof of the claim. Thanks to [PT14], we know that $\det p'_* (mK_{X'/Y'} + m\Delta')$ is pseudo-effective on $Y'$. Let $A$ be the nef part of the Zariski decomposition of $\det p'_* (mK_{X'/Y'} + m\Delta')$. Set $B := (\pi_Y \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1)$. As $B$ is semiample, we have

$$0 \leq A \cdot B \leq c_1 (\det p'_* (mK_{X'/Y'} + m\Delta')) \cdot c_1 (B)$$

$$= (\pi_Y)_* (c_1 (\det p'_* (mK_{X'/Y'} + m\Delta'))) \cdot \pi^* c_1 (\mathcal{O}_{\mathbb{P}^1}(1))$$

$$= c_1 (\det p_*(mK_{X/Y} + m\Delta)) \cdot \pi^* c_1 (\mathcal{O}_{\mathbb{P}^1}(1)) = 0,$$

where the last equality a consequence of (3.1.4). Then we have

$$A \cdot B = 0. \quad (3.1.14)$$

Let $L$ be an ample line bundle on $Y'$ and set $c := L \cdot A < \infty$. For any $\tau \in \mathbb{Q}^+$ small enough, thanks to (3.1.14) and the choice of $c$, the basic volume estimate (cf. for example [Dem12, 8.4] or [Laz04, Thm 2.2.15]) implies that

$$\text{vol}(A + \tau L - cB) \geq (A + \tau L)^2 - 2c(A + \tau L) \cdot B$$

$$\geq 2\tau (L \cdot A - cL \cdot B) + o(\tau) > 0.$$  

Therefore $A + \tau L - cB$ is big for any $\tau \in \mathbb{Q}^+$. Letting $\tau \to 0^+$, $A - cB$ is pseudoeffective. Then $\det p'_* (mK_{X'/Y'} + m\Delta') - cB$ is pseudo-effective and the claim is proved.

4. Appendix

In this appendix, we will gather two standard results which should be well-known to experts.

4.1. Proposition. [Kaw82, CH11] Let $p : X \to Y$ be a fibration from a $n$-dimensional projective manifold to a K3 surface, and let $\Delta$ be an effective klt $\mathbb{Q}$-divisor on $X$. Assume that Theorem 1.1 holds for $\dim X \leq n - 1$. If $\kappa(K_X + \Delta) \geq 1$, then

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F),$$

where $F$ is the generic fiber of $p$ and $\Delta_F = \Delta|_F$.

Proof. We use here the argument in [CP17, Prop 3.7]. Modulo desingularization, we can assume that the Iitaka fibration of $K_X + \Delta$ is a morphism between two projective manifolds $\varphi : X \to W$.

$$\begin{array}{ccc}
X & \xrightarrow{\varphi} & W \\
\bigwedge & \downarrow p & \bigwedge \\
& \downarrow & \\
& Y & \\
\end{array}$$

Let $G$ be the generic fiber of $\varphi$ and set $\Delta_G := \Delta|_G$. Then

$$\kappa(K_G + \Delta_G) = 0. \quad (4.1.1)$$

Let $p : G \to p(G)$ be the restriction of $p$ on $G$. We will analyze next among three cases which may occur.

Case 1: We assume that $p(G)$ projects onto $Y$; then we argue as follows. Let $\widetilde{p} : G \to \widetilde{Y}$ be
the Stein factorization of $p : G \to Y$:

$$
\begin{array}{ccc}
G & \xrightarrow{p} & Y \\
\downarrow \bar{p} & \quad & \downarrow s \\
\bar{Y} & \quad & Y \\
\end{array}
$$

After desingularization $\bar{p}$, we can assume that $\bar{Y}$ is smooth. Let $G_t$ be the generic fiber of $\bar{p}$. By assumption, Theorem 1.1 holds for $G \to \bar{Y}$. Therefore (4.1.1) implies that

$$
\kappa(K_{G_t} + \Delta_{G_t}) = 0. \quad (4.1.2)
$$

We estimate next the dimension of $G$. Let $F$ be the generic fiber of $p : X \to Y$. By restricting $\varphi$ on $F$, we obtain a morphism

$$
\varphi_t : F \to V
$$

where $V$ is a subvariety of $W$. Let $\tilde{V} \to V$ be the Stein factorization of $\varphi_t$.

$$
\begin{array}{ccc}
F & \xrightarrow{\varphi_t} & V \\
\downarrow \tilde{\varphi}_t & \quad & \downarrow \tilde{V} \\
\end{array}
$$

Since $G$ is generic, we infer that the generic fiber of $\bar{p}$ coincides with the generic fiber of $\tilde{\varphi}_t$. Combining this with (4.1.2), then [Uen75, Thm 5.11] implies that

$$
\kappa(K_F + \Delta_F) \leq \dim \tilde{V} = \dim F - \dim G_t.
$$

Therefore we have

$$
\dim G_t \leq \dim F - \kappa(K_F + \Delta_F)
$$

and thus we infer that

$$
\dim G = \dim G_t + \dim \bar{Y} \leq \dim F - \kappa(K_F + \Delta_F) + \dim Y = \dim X - \kappa(K_F + \Delta_F).
$$

Finally, by construction of the Iitaka fibration, $\dim G = \dim X - \kappa(K_X + \Delta)$; we obtain the inequality

$$
\dim X - \kappa(K_X + \Delta) \leq \dim X - \kappa(K_F + \Delta_F),
$$

and in conclusion $\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F)$.

**Case 2:** We assume that the image $p(G)$ has dimension zero. Since $G$ is connected, $p(G)$ is a point in $Y$. This means that we can define a map $W \to Y$, which can be assumed to be regular by blowing up $W$. We have thus the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & W \\
\downarrow p & \quad & \downarrow q \\
Y & \quad & \\
\end{array}
$$

Set $t := p(G)$. Let $F$ be the fiber of $p$ over $t$. Then $F$ is a generic fiber of $p$ and $G$ is a generic fiber of

$$
\varphi : F \to \varphi(F),
$$

and by [Uen75, Thm 5.11] we infer that

$$
\kappa(K_F + \Delta_F) \leq \kappa(K_G + \Delta_G) + \dim \varphi(F) = \dim \varphi(F).
$$
Note that \( \varphi(F) \) is the fiber of \( q \) over \( t \in Y \). We have \( \dim W = \dim \varphi(F) + \dim Y \). Therefore \( \dim W \geq \kappa(K_F + \Delta_F) + \dim Y \). Combining this with the fact that \( \varphi \) is the Iitaka fibration, we have thus
\[
\kappa(K_X + \Delta) = \dim W \geq \kappa(K_F + \Delta_F) + \dim Y,
\]
and we are done.

**Case 3: The remaining case: \( p(G) \) is a proper subvariety of \( Y \).**

Let \( p(G)' \) be the normalization of \( p(G) \). If \( p(G)' \) is a curve of general type, then \( \kappa(K_G + \Delta_G) \geq 1 \) and we get a contradiction with (4.1.1). If \( p(G)' = \mathbb{P}^1 \), as \( G \) is generic, \( Y \) is thus covered by rational curves. We get a contradiction with the assumption that \( Y \) is K3. As a consequence, \( p(G)' \) is a torus. Then \( \kappa(K_G + \Delta_G) \geq 1 \) and we get a contradiction with (4.1.1). If \( p(G)' = \mathbb{P}^1 \), as \( G \) is generic, \( Y \) is thus covered by rational curves. We get a contradiction with the assumption that \( Y \) is K3. As a consequence, \( p(G)' \) is a torus. Then \( p(G) \) is a generic fiber of a fibration \( \pi: Y \to \mathbb{P}^1 \).

4.2. **Proposition.** Let \( p: X \to Y \) be a fibration between two projective manifolds. Let \( F \) be the generic fiber and let \( \Delta \) be a \( \mathbb{Q} \)-effective klt divisor on \( X \). Set \( \Delta_F := \Delta|_F \). If \( \dim Y = 2 \) and \( \kappa(Y) \geq 1 \), then
\[
\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y).
\] (4.2.1)

**Proof.** Since \( Y \) is of dimension 2, we can consider its minimal model and assume that \( Y \) is smooth with semi-ample canonical bundle.

If \( \kappa(Y) = 2 \), then \( K_Y \) is big and it is known that (4.2.1) holds.

If \( \kappa(Y) = 1 \), we can suppose that \( K_Y \) is semi-ample. Then \( K_Y \) induces a fibration
\[
\pi: Y \to Z
\]
to a 1-dimensional variety \( Z \) and \( K_Y = \pi^*A \) for some ample line bundle \( A \) on \( Z \). Let \( Y_z \) be a generic fiber of \( \pi \). Then \( Y_z \) is a 1-torus. Let \( m \in \mathbb{N} \) be a number sufficiently large and let \( h \) be the possibly singular hermitian metric on \( p_*((mK_{X/Y} + m\Delta)) \) defined in Theorem 2.7. There are two cases.
Case 1. \(i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta))|_{Y_z} \equiv 0\). Thanks to Proposition\ref{prop:11}, the vector bundle \((p_*(mK_{X/Y} + m\Delta)|_{Y_z}, h)\) is hermitian flat. Therefore
\[
\int_{Y_z} |s|^2 h < +\infty \quad \text{for every } s \in H^0(Y_z, p_*(mK_{X/Y} + m\Delta)).
\]
As \(K_Y = \pi^*A\) for some ample line bundle on \(Z\), Proposition\ref{prop:9} and the \(L^2\)-condition \ref{eq:2.2} imply that
\[
\kappa(X, K_X + \Delta) \geq \kappa(X_z, K_{X/Y} + \Delta|_{X_z}) + 1.
\]
Moreover, by applying \cite{CP17, Kaw82a} to \(X_z \to Y_z\) and the fact that \(Y_z\) is a torus, we have
\[
\kappa(X_z, K_{X/Y} + \Delta|_{X_z}) \geq \kappa(K_F + \Delta_F).
\]
Together with \ref{eq:2.3}, \ref{eq:2.1} is proved.

Case 2. \(i\Theta_{\det h}(\det p_*(mK_{X/Y} + m\Delta))|_{Y_z} \geq 0\). As \(Y_z\) is of dimension 1, \(\det p_*(mK_{X/Y} + m\Delta)|_{Y_z}\) is ample on \(Y_z\). Since \(K_Y\) is semi-ample, we can find some \(\mathbb{Q}\)-div \(\Delta' \geq 0\) in the same class of \(c \cdot p^*K_Y\) for some \(c > 0\) small enough such that \(\Delta + \Delta'\) is klt. Then
\[
\det p_*(mK_{X/Y} + m\Delta + m\Delta') = \det p_*(mK_{X/Y} + m\Delta) + m\Delta'
\]
is big on \(Y\). By applying for example \cite{Cam04b, CP17}, we have
\[
\kappa(K_{X/Y} + \Delta + \Delta') \geq \kappa(K_F + \Delta_F) + 2.
\]
As \(c < 1\), we know that \(K_X + \Delta - (K_{X/Y} + \Delta + \Delta') = (1 - c) \cdot p^*K_Y\) is \(\mathbb{Q}\)-effective. Therefore
\[
\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + 2,
\]
and \ref{eq:2.1} is proved. \(\square\)

References

BHPV Barths, Wolf P. and Hulek, Klaus and Peters, Chris A. M. and Van de Ven, Antonius: Compact Complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. 4. Springer-Verlag, Berlin, 2004.

Ber09 Berndtsson, Bo: Curvature of vector bundles associated to holomorphic fibrations. Ann. of Math. (2) 169 (2009), no. 2, 531-560.

BP08 Berndtsson, Bo and P˘ aun, Mihai: Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J., 145(2):341-378, 2008.

BP10 Berndtsson, Bo and P˘ aun, Mihai: Bergman kernels and subadjunction. arxiv: 1002.4145v1

Bir09 Birkar, Caucher: The Iitaka conjecture \(C_{n,m}\) in dimension six. Compos. Math. 145 (2009), no. 6, 1442-1446.

BC15 Birkar, Caucher and Chen, Jungkai Alfred: Varieties fibred over abelian varieties with fibres of log general type. Adv. Math. 270 (2015), 206-222.

Cam04a Campana, Frédéric: Orbifolds premiere classe de Chern nulle (French. English summary) [Orbifolds of zero first Chern class] The Fano Conference, 339-351, Univ. Torino, Turin, 2004.

Cam04b Campana, Frédéric: Orbifolds, special varieties and classification theory. Ann. Inst. Fourier (Grenoble) 54 (2004), no. 3, 499-630.

CC14 Campana, Frédéric and Benoît, Claudon: Abelianity conjecture for special compact Khler 3-folds. (English summary) Proc. Edinb. Math. Soc. (2) 57 (2014), no. 1, 55-78.

CP17 Cao, Junyan and P˘ aun, Mihai: Kodaira dimension of algebraic fiber spaces over Abelian varieties Invent. Math. 207 (2017), no. 1, pp 345–387
CH11 Chen, Jungkai Alfred and Hacon, Christopher D.: *Kodaira dimension of irregular varieties* Invent. Math. 186, (2011), 481–500

Dem12 Demailly, Jean-Pierre *Analytic methods in algebraic geometry*. Surveys of Modern Mathematics, 1. International Press, Somerville, MA; Higher Education Press, Beijing, 2012. vii+231 pp.

Fuj16 Fujino, Osamu: *Direct images of relative pluricanonical bundles* Algebr. Geom. 3 (2016), no. 1, 50-62.

Fuj78 Fujita, Takao: *On Kähler fiber spaces over curves*. J. Math. Soc. Japan 30, 779–794 (1978)

Fuj79 Fujita, Takao: *On Zariski problem* Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), no. 3, 106110.

GKP15 Kebekus, Stefan; Greb, Daniel and Peternell, Thomas: *Singular spaces with trivial canonical class* To appear in: Minimal models and extremal rays proceedings of the conference in honor of Shigefumi Moris 60th birthday, Advanced Studies in Pure Mathematics, Kinokuniya Publishing House, Tokyo.

Hor10 Höring, Andreas: *Positivity of direct image sheaves: a geometric point of view* Enseign. Math. (2) 56 (2010), no. 1-2, 87-142.

KP17 Kovács, Sándor J. and Patakfalvi, Zsolt: *Projectivity of the moduli space of stable log-variants and subadditivity of log-Kodaira dimension*. J. Amer. Math. Soc. 30 (2017), no. 4, 959–1021.

Gri70 Griffiths, Phillip A.: *Periods of integrals on algebraic manifolds III*. Publ. Math. IHES, 38, 125–180 (1970)

HPS16 Hacon, Christopher; Popa, Mihnea and Schnell, Christian: *Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Paun* arXiv:1611.08768

Kaw82a Kawamata, Yujiro: *Kodaira dimension of algebraic fiber spaces over curves* Invent. Math. 66 (1982), no. 1, 57-71.

Kaw82b Kawamata, Yujiro *Characterization of Abelian varieties* Compositio Mathematica 43 (1981), no. 2, 253-276

Kol85 Kollár, János *Subadditivity of the Kodaira dimension: fibers of general type* Algebraic geometry, Sendai, 1985, 361-398, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

KM98 Kollár, János and Mori, Shigefumi: *Birational geometry of algebraic varieties* With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998. viii+254 pp.

Laz04 Lazarsfeld, Robert *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.

PT14 Păun, Mihai and Takayama, Shigeharu: *Positivity of twisted relative pluricanonical bundles and their direct images* arxiv 1409.5504

Pau16 Păun, Mihai: *Singular Hermitian metrics and positivity of direct images of pluricanonical bundles* arXiv:1606.00174

Rau15 Raufi, Hossein: *Singular hermitian metrics on holomorphic vector bundles*, Arkiv för Matematik, October 2015, Volume 53, Issue 2, pp 359–382

Uen75 Ueno, Kenji: *Classification theory of algebraic varieties and compact complex spaces*, Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. xix+278 pp.

Vie83 Viehweg, Eckart: *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*. Algebraic varieties and analytic varieties (Tokyo, 1981), 329-353, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.

Vie95 Viehweg, Eckart: *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 30. Springer-Verlag, Berlin, 1995. viii+320 pp.
