The enriched phase structure of black branes in canonical ensemble

J. X. Lu\textsuperscript{a}, Shibaji Roy\textsuperscript{b} and Zhiguang Xiao\textsuperscript{a,3}

\textsuperscript{a} Interdisciplinary Center for Theoretical Study
University of Science and Technology of China, Hefei, Anhui 230026, China

\textsuperscript{b} Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta-700 064, India

Abstract

It is found that a necessary completion of phase structure of $D$-dimensional charged black $p$-brane ($p > 0$) in a cavity requires two additional thermodynamical phases, the so-called “bubble of nothing” and/or the extremal brane, in canonical ensemble. This finding resolves the puzzle about the missing phases which are needed for the underlying phase diagram when $\tilde{d} = D - p - 3 \leq 2$ and gives a new (bubble) phase which can become globally stable when $\tilde{d} > 2$. An analog of Hawking-Page transition is also found among other new phase transitions, giving a complete phase structure in this setup.

\textsuperscript{1}E-mail: jxlu@ustc.edu.cn
\textsuperscript{2}E-mail: shibaji.roy@saha.ac.in
\textsuperscript{3}E-mail: xiaozg@ustc.edu.cn
1 Introduction

Understanding the nature of black hole thermodynamics may teach us lessons about quantum gravity. The underlying phase structure can be useful not only in this regard but for other purposes as well. For example, with the advent of AdS/CFT correspondence, the known phase structure of large black holes in asymptotically Anti-de Sitter (AdS) space can be used to understand various physical phenomena in other branches of physics. An illustration of this is that the Hawking-Page transition for AdS black hole ‘evaporating’ into regular “hot empty AdS space” at certain temperature [1] can enhance understanding of the confinement-deconfinement phase transition in large $N$ gauge theory [2].

A large part of the phase structure of an AdS black hole [3, 4] is actually not unique to the black hole in asymptotically AdS space but shared universally by suitably stabilized black holes/branes, say, in asymptotically flat space [5, 6, 7, 8], even in the presence of a charge $q$. For example, a chargeless (suitably stabilized) asymptotically flat $p$-brane can also undergo a Hawking-Page transition at certain temperature, now evaporating into a regular ‘hot flat space’ instead. When $q \neq 0$, there exists also a critical charge $q_c$ and for $q < q_c$, the phase diagram universally contains a van der Waals-Maxwell liquid-gas type phase structure along with a first-order phase transition line ending at a second-order critical point with a universal exponent for the specific heat as $-2/3$ when $q = q_c$. This universal phase structure may hint holography even in asymptotically flat space, as pointed out in [5].

However, unlike an AdS black hole, an isolated asymptotically flat black hole/brane is unstable due to its Hawking radiation and needs to be stabilized first before one can discuss the equilibrium thermodynamics. To establish its stability, the standard practice is to place such a system inside a finite spherical cavity [9] with its surface temperature fixed. In other words, a thermodynamical ensemble is considered which can be either canonical or grand canonical, depending on whether the charge inside the cavity or the potential at the surface of the cavity is fixed [10]. In this paper, our focus is the canonical ensemble, i.e., the charge inside the cavity is fixed, and in particular our main interest is to study the phase structure when the flux/charge inside the cavity is fixed but non-zero.

When the ensemble temperature drops below a certain minimum value, there appears a puzzle of missing phases, in certain cases, as it is not clear where the system would be in the absence of globally stable phases (No such issues arise in grand canonical ensemble, however, [5, 8].). Furthermore, it is not known whether there exists a new globally stable phase in the present setup other than what have been discussed in the literature so far [5, 6, 7]. This was noticed in [5, 6] for the charged black hole and in [7] for the charged
black $p$-brane when $\tilde{d} \leq 2$.

In this paper, we will resolve the above puzzle and give the new phase(s) for the $D$-dimensional asymptotically flat stabilized compact black $p$-brane (for $p > 0$)\(^4\) by finding two missing phases, namely, the regular ‘hot bubble’, due to the existence of “bubble of nothing” \cite{11}, and the extremal brane, each carrying the same flux/charge as the black $p$-brane. The bubble or the extremal brane \cite{12, 13, 14, 15} each can have an arbitrary period $\beta$ in Euclidean time, analogous to the ‘hot flat space’ in the chargeless case. As such the black $p$-brane can make a transition to this bubble or the extremal brane, depending on whose free energy is smaller, giving an analog of Hawking-Page transition. As a result, the underlying phase structure is greatly enriched and many new phase transitions between black branes, bubbles and extremal brane are revealed, giving a rather complete phase structure in this setup.

This paper is organized as follows. In section 2, we will present the basic setup for phase structure of black branes which will be discussed in the following sections. In section 3, we discuss the phase structure for the special zero flux/charge case as a warm up exercise. Here we see how the inclusion of bubble phase will enrich the previously known phase structure. Section 4 is the main focus of this paper and we will resolve the aforementioned puzzle for $\tilde{d} \leq 2$ and find a new global stable (bubble) phase for $\tilde{d} > 2$, therefore giving a necessary completion of underlying phase structure and various new phase transitions including the analog of Hawking-Page transition among other things. We discuss the results obtained in this paper and conclude in section 5.

2 The basic setup

For the purpose of this paper, let us consider the $D$-dimensional black $p$-brane metric in Euclidean signature as \cite{16, 17},

$$ds^2_{bl} = \Delta_+ \Delta_- d^{d-2}dt^2 + \Delta_+ d^{d-2}(dx^1)^2 + \Delta_+ d^{d-2} \sum_{i=2}^{P} (dx^i)^2 + \Delta_+^{-1} \Delta_+^{-\frac{2}{\tilde{d}} - 1} d\rho^2 + \rho^2 \Delta_+^{-\frac{2}{\tilde{d}}} d\Omega_\tilde{d+1}^2 \quad (1)$$

where $\Delta_\pm = 1 - (r_\pm/\rho)^{\tilde{d}}$, with $r_\pm (r_+ \geq r_-)$ related to the mass and the charge of the black $p$-brane. The horizon occurs at $\rho = r_+$ while the curvature singularity at $\rho = r_-$.\(^4\)

\(^4\)For $p = 0$, there is no bubble phase. The only phases are non-extremal brane and extremal one, and the phase relation between the two for $p > 0$ discussed in the text will hold true also for this case. For simplicity, we focus from now on only for $p > 0$ case.
Here $d = 1 + p$, $\tilde{d} = D - d - 2$ and ‘$a’$ is the dilaton coupling defined by

$$a^2 = 4 - \frac{2dd}{(D - 2)},$$

for supergravity with maximal supersymmetry. To have a large but finite Euclidean action for the black brane, the brane directions $x^i$, with $i = 1, \ldots, p$ should be compact (In the metric, $x^1$ coordinate is explicitly isolated for the purpose of constructing bubble solution later). For the metric (1) to be free from conical singularity at $\rho = r_+$, ‘$t’$ coordinate must be periodic with periodicity

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left( 1 - \frac{r^d_+}{r^d} \right)^{1/\tilde{d} - 1/2},$$

the inverse of the black brane temperature at $\rho = \infty$. With this, the inverse of the local temperature at $\rho$ is

$$\beta(\rho) = \Delta^{1/2}_+ \Delta^{-1/\tilde{d}}_+ \frac{4\pi \tilde{r}_+}{\tilde{d}} \left( 1 - \frac{\tilde{r}^d_+}{\tilde{r}^d} \right)^{1/\tilde{d} - 1/2}.$$ (3)

However, for the corresponding extremal brane, ‘$t’$ coordinate can have an arbitrary period\footnote{Though this provides a simple means to obtain a bubble solution, it can also be obtained by other means, for example, by solving the underlying equations of motion. In other words, once the solution is obtained, we can forget about its connection with the original black brane. Even in the present context of considering possible allowed phases with the same boundary data, the parameters $r_+$ and $r_-$, characterizing the solution, can be different in both the bubble solution and the black brane. As such, the period of $x^1$ (or $t$) in the bubble case is not necessarily the same as that of $t$ (or $x^1$) in the black brane case. Only when the $r_+$ and $r_-$ are set to be the same in both cases, the two periods will be the same which is just a special case. This recognition, as discussed later in the text, is crucial for the role of the bubble phase in the phase diagram uncovered in this paper.} and this is crucial for the phase transition we will discuss later. Here the physical radius $\bar{\rho} \equiv \Delta^{a^2/4\tilde{d}}_+ \rho$ can be read from the metric (1) and so the physical parameters $\bar{r}_\pm = \Delta^{a^2/4\tilde{d}}_\pm r_\pm$. On the other hand, the coordinate $x^1$, like other compact coordinates, has arbitrary local periodicity. By renaming the coordinates $x^1 \rightarrow t$ and $t \rightarrow -x^1$ in the black $p$-brane configuration given above, we can obtain the bubble carrying the same flux/charge in Euclidean signature with its metric given as

$$ds^2_{bb} = \Delta^{d}_- \frac{d}{(D - 2)} dt^2 + \Delta_+ \Delta^{-d}_- (dx^1)^2 + \Delta_+ \sum_{i=2}^p (dx^i)^2 + \Delta^{-1}_+ \Delta^{a^2}_- d\rho^2 + \rho^2 \Delta^{a^2}_- d\Omega^2_{d+1},$$ (5)
which is defined only for \( \rho \geq r^+ \) and is regular if \( x^1 \) has a period of

\[
R^* = \frac{4\pi r^+}{d} \left( 1 - \frac{r_d}{r^+} \right)^{1/d-1/2},
\]

at \( \rho = r^+ \). However, in this case the periodicity of the coordinate ‘\( t \)’ as well as the other spatial compact coordinates remain arbitrary and so it can be in thermal equilibrium with the cavity in any temperature.

Now in order to study equilibrium thermodynamics \[18\] in canonical ensemble, the allowed configuration (black brane or extremal brane or bubble or coexistence of two or more of them) should be placed in a cavity \[9\] with fixed physical radius \( \bar{\rho}_B \). The other fixed quantities are the cavity temperature \( 1/\beta \), the physical periodicity of \( x^1 \), i.e., \( R \) (also the physical sizes of other compact directions), the dilaton value \( \bar{\phi} \) and the charge/flux enclosed in the cavity \( \bar{Q}_d \). In equilibrium, these fixed values are set equal to the corresponding ones of the allowed configuration enclosed in the cavity. For example, we set the charge

\[
\bar{Q}_d = Q_d \equiv \frac{i}{\sqrt{2\kappa}} \int e^{-a(d)\phi} \times F^{[p+2]} = \frac{\Omega_{d+1}\bar{\phi}}{\sqrt{2\kappa}} e^{-a\phi/2}(\bar{r}_+\bar{r}_-)^{d/2},
\]

where \( \ast \) denotes the Hodge duality and \( F^{[p+2]} = i\bar{d}((\bar{r}^+\bar{r}^-)^{d}/\bar{\rho}^{d+1})d\bar{\rho} \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^p \), the field strength for the configuration considered. In the charge expression \( \Omega_n \) denotes the volume of a unit \( n \)-sphere, \( \kappa \) is a constant with \( 1/(2\kappa^2) \) appearing in front of the Hilbert-Einstein action in canonical frame but containing no asymptotic string coupling \( g_s \). We have also set \( e^{\bar{\phi}} = e^{\phi(\bar{\rho}_B)} \equiv g_s\Delta^{d/2} \). In canonical ensemble, it is the Helmholtz free energy which determines the stability of the equilibrium states and is related to the Euclidean action by \( F = I_E/\beta \) in the leading order approximation. So, in order to understand the phase structure we will evaluate the action for the black (non-extremal) brane, the extremal brane and the bubble, with the above mentioned boundary data. Note that specifying the boundary data does not necessarily mean that they have the same \( \bar{r}_+ \). For example, in the extremal case, \( \bar{r}_+ = \bar{r}_- \) are completely fixed by the given charge.

The Euclidean action for the black \( p \)-brane has been evaluated in \[7\] by using standard technique and is given as,

\[
I^{\text{bl}}_E = \frac{\beta RV^{p-1}_B}{2\kappa^2} \bar{\rho}_B \left[ (2 + \bar{d}) \left( \frac{\Delta^+}{\Delta^-} \right)^{1/2} + \bar{d}(\Delta^-\Delta^+)^{1/2} - 2(\bar{d} + 1) \right] - \frac{4\pi RV^{p-1}_B}{2\kappa^2} \bar{r}_+^{d+1} \Delta^{-\frac{1}{d}-\frac{1}{2}} \left( 1 - \frac{\bar{r}_-}{\bar{r}_+} \right)^{\frac{1}{2}+\frac{1}{d}},
\]

where

\[
\Delta^\pm = \Delta^- \pm \bar{\rho}_B 
\]

and

\[
\Delta^{-\frac{1}{d}-\frac{1}{2}} = \left( 1 - \frac{\bar{r}_-}{\bar{r}_+} \right)^{\frac{1}{2}+\frac{1}{d}}.
\]
with $\Delta_{\pm}$ taking their respective value at $\bar{\rho} = \bar{\rho}_B$. Since the Helmholtz free energy is given as $F_{bl} = E_{bl} - TS_{bl}$, so we have $I_{E}^{bl} = \beta E_{bl} - S_{bl}$, where $E_{bl}$ is the internal energy and $S_{bl}$ is the entropy of the black $p$-brane. Thus we identify the internal energy of the black brane on dividing the first term in (8) by $\beta$ and can be checked to match the ADM mass per unit volume of the black brane [19] as $\bar{\rho}_B\to\infty$. The second term in (8) is the entropy of the black brane. Note that we have written the usual compact brane volume $V_p = RV_{p-1}$.

The action for the bubble $I_{E}^{bb}$ can be obtained simply from (8) by making the change $\beta \leftrightarrow R$ and is given as

$$I_{E}^{bb} = \frac{\beta RV_{p-1}\Omega_{d+1}}{2\kappa^2} \bar{\rho}_B \left( 2 + \tilde{d} \right) \left( \frac{\Delta_{+}}{\Delta_{-}} \right)^{1/2} + \tilde{d} \Delta_{-} \Delta_{+}^{1/2} - 2(\tilde{d} + 1)$$

As for the bubble since there is no entropy, we have $I_{E}^{bb} = \beta E_{bb}$ and so we can identify the internal energy of the bubble on dividing its action by $\beta$ and again can be checked to have the correct ADM mass per unit volume of the bubble as $\bar{\rho}_B\to\infty$. However, for the black brane in canonical ensemble the local temperature $1/\beta(\bar{\rho}_B)$ is determined by $r_+, Q_d, \bar{\rho}_B$ as given before and the periodicity of $x^1$ is arbitrary whereas for the bubble the local periodicity of $x^1$, i.e. $R(\bar{\rho}_B)$, is similarly determined by the corresponding quantities and the temperature is now arbitrary. On-shell, they are all set equal to the corresponding fixed boundary values.

Now for convenience, we will work instead with the reduced action defined as $\bar{I}_{E}^{bl, bb}(z) = 2\kappa^2 I_{E}^{bl, bb}/((4\pi)^2 \bar{\rho}_B^{\tilde{d}+2} V_{p-1}\Omega_{d+1}) = G_q(z)$ with $G_q(z)$ defined as,

$$G_q(z) = -\bar{b}\bar{R} \left( \tilde{d} + 2 \right) \left( \frac{1 - z}{1 - \frac{q^2}{z}} \right)^{1/2} + \tilde{d} \left( 1 - z \right)^{1/2} \left( 1 - \frac{q^2}{z} \right)^{1/2} - 2(\tilde{d} + 1)$$

$$-\bar{l} z^{1+1/\tilde{d}} \left( \frac{1 - q^2}{1 - \frac{q^2}{z}} \right)^{1/2+1/\tilde{d}},$$

where we have defined $z = (\bar{r}_+ / \bar{\rho}_B)^{\tilde{d}} < 1$ and $z = x$ for the black brane and $z = y$ for the bubble since the two need not have the same $\bar{r}_+$. Further we have defined $\bar{b} = \beta/(4\pi\bar{\rho}_B)$, $\bar{R} = R/(4\pi\bar{\rho}_B)$ and $\bar{l} = \bar{R}$ for the black brane and $\bar{l} = \bar{b}$ for the bubble. Also, $q = (Q_d / \bar{\rho}_B)^{\tilde{d}}$, where $Q_d^{\tilde{d}} = ((\sqrt{2}\kappa Q_d)/(\Omega_{d+1}\tilde{d}))^{1/\tilde{d}}$. In terms of these new fixed parameters we have,

$$\Delta_{+} = 1 - z, \quad \Delta_{-} = 1 - q^2/z, \quad 1 - \frac{\bar{r}_d}{\bar{r}_d^+} = 1 - q^2/z^2,$$
and they have been used in [10]. In writing $\Delta$ we have used the fact that since $Q_d$ is fixed, $\bar{r}_-$ is not an independent parameter but can be expressed in terms of $\bar{r}_+$. Given $(Q_d q)^2 / \bar{r}_+ = \bar{r}_- / \bar{r}_+ \leq 1$, so $z \geq q$. For extremal branes, $\bar{r}_- = \bar{r}_+$ and so $z = q$. The reduced action for extremal branes is now

$$I^\text{extremal} = b \bar{R} \bar{d} q,$$

(12)
determined completely by the boundary data. However, for non-extremal brane or bubble, we have a variable $z$ lying between $q < z < 1$. Now $dG_q(z)/dz|_{z = \bar{z}} = 0$ gives the equation of state $\bar{m} = m_q(\bar{z})$, where,

$$m_q(z) = \frac{1}{d} \frac{z^{1/d}(1 - z)^{1/2}}{\left(1 - \frac{q^2}{z^2}\right)^{1/2} \left(1 - \frac{q^2}{z}\right)^{1/d}}.$$

(13)

If

$$d^2 G_q(z)/dz^2|_{z = \bar{z}} \propto - dm_q(z)/dz|_{z = \bar{z}} > 0,$$

(14)
at $z = \bar{z}$, where $\bar{z}$ is determined from $\bar{m} = m_q(\bar{z})$, we get a local minimum of free energy. So the negative slope of $m_q(z)$ determines the local stability of the underlying system. In the above $\bar{m} = \bar{b}$, $m_q(z = x) = b_q(x)$ for the black brane and $\bar{m} = R$, $m_q(z = y) = R_q(y)$ for the bubble (Note that we always have $\bar{m} \bar{l} = b \bar{R}$).

The phase structure of $G_q(z)$ corresponding to the black $p$-brane has been analyzed in [7]. The bubble also has exactly similar phase structure, however, the relevant quantities here are $\bar{R}$ and $R_q(y)$ instead. For now, we just need to compare the free energies among the black brane, the bubble, at their respective global minimum, and the extremal brane, all with the same boundary data. The analog of Hawking-Page transition in either case and the final stable state will be determined by the smallest free energy of these phases. For this purpose we need the respective on-shell reduced free energy explicitly. It is

$$\tilde{F}^\text{bl, bb}(\bar{z}) \equiv \frac{I^\text{bl, bb}(\bar{z})}{\bar{b}} = - R^2 F_q(\bar{z}),$$

(15)

with

$$F_q(z) = 2 \left(\frac{1 - z}{1 - \bar{z}}\right)^{1/2} + \bar{d} \left(\frac{1 - q^2}{1 - z}\right)^{1/2} + \bar{d} (1 - z)^{1/2} \left(1 - \frac{q^2}{z}\right)^{1/2} - 2(\bar{d} + 1).$$

(16)

In the above, we have used the on-shell condition $\bar{m} = m_q(\bar{z})$ with $\bar{z}$ lying between $q < \bar{z} < 1$. Given the boundary data $\bar{b}$ and $\bar{R}$, the on-shell free energy of non-extremal brane is $\tilde{F}^\text{bl}(\bar{x}) = - R^2 F_q(\bar{x})$, with $\bar{x}$ determined by $\bar{b} = b_q(\bar{x})$, while the on-shell free energy
of bubble is \( \tilde{F}^{bb}(\bar{y}) = -\tilde{R}F_q(\bar{y}) \), but now with \( \bar{y} \) determined by \( \tilde{R} = R_q(\bar{y}) \). So \( \tilde{F}^{bl}(\bar{x}) \) and \( \tilde{F}^{bb}(\bar{y}) \) actually have different dependence on their respective on-shell variable even though both have the same functional form \( -\tilde{R}F_q(z) \) in appearance. So their profiles are in general different. For example, \( \tilde{F}^{bl}(\bar{x} = q) = \tilde{R}\tilde{d}q \) (giving the extremal brane free energy) and \( \tilde{F}^{bl}(\bar{x} \to 1) \to -\infty \) while \( \tilde{F}^{bb}(\bar{y} \to q) \to \infty \) and \( \tilde{F}^{bb}(\bar{y} \to 1) \to -1 \). Note that at the two ends \( F_q(q) = -\tilde{d}q \) and \( F_q(1) = \infty \). In spite of their difference, both \( \tilde{F}^{bl}(\bar{x}) \) and \( \tilde{F}^{bb}(\bar{y}) \) have the same characteristic behavior as that of \( m_q(\bar{z}) \), and this can be understood from the following relation (note its difference from (14)),

\[
\frac{d\tilde{F}^{bl,bb}(\bar{z})}{d\bar{z}} \propto \frac{dm_q(\bar{z})}{d\bar{z}}, \tag{17}
\]

where we have used (13), (15) and the above \( F_q(z) \) (In the case of bubble, we also need to consider the contribution from \( d\tilde{R}/d\bar{y} = dR_q(\bar{y})/d\bar{y} \)). For example, the maximum or minimum (if exists at all) for each of these functions occurs at the same \( z_{\text{max}} \) or \( z_{\text{min}} \). The above enables us, in the present context, to make use of the extrema (if any) and behavior of \( m_q(z) \) studied in [7] to compare the free energies at the respective global minimum.

We are interested in the region for which the corresponding free energy will be at least locally stable, i.e., \( dm_q(z)/dz|_{z=\bar{z}} < 0 \). Equation (17) immediately tells us that the on-shell \( \tilde{F}^{bl}(\bar{z}) \) and \( \tilde{F}^{bb}(\bar{z}) \) along with \( m_q(\bar{z}) \) will decrease in this region. For \( m_q(\bar{z}) \), \( \bar{b} = b_q(\bar{x}) \) and \( \tilde{R} = R_q(\bar{y}) \) then imply that in each of the cases \( \bar{b} > \tilde{R} \), \( \bar{b} = \tilde{R} \) and \( \bar{b} < \tilde{R} \) we will get \( \bar{x} < \bar{y} \), \( \bar{x} = \bar{y} \) and \( \bar{x} > \bar{y} \), respectively. Further, we can use the property of either \( \tilde{F}^{bl}(\bar{x}) \) or \( \tilde{F}^{bb}(\bar{y}) \) as a decreasing function to determine which of the phases – the brane or the bubble have smaller free energy for given \( \bar{b} \) and \( \tilde{R} \). Let us take a particular case for illustration, say \( \bar{b} > \tilde{R} \), implying \( \bar{x} < \bar{y} \). So we have \( \tilde{F}^{bl}(\bar{x}) > \tilde{F}^{bl}(\bar{y}) = \tilde{F}^{bb}(\bar{y}) \), which says now that the bubble has smaller free energy. Alternatively, this can also be determined using \( \tilde{F}^{bb} \) as follows: \( \tilde{F}^{bb}(\bar{y}) < \tilde{F}^{bb}(\bar{x}) = -\tilde{b}F_q(\bar{x}) < -RF_q(\bar{x}) = \tilde{F}^{bl}(\bar{x}) \). In the following two sections, we will use \( \tilde{F}^{bl}(\bar{z}) \) as a decreasing function in the region of interest to determine the global stable phase, which appears slightly more straightforward than using \( \tilde{F}^{bb}(\bar{z}) \) instead.

With the above preparation, we are now ready to discuss the phase structure for various cases as promised in the Introduction.
3 The zero flux/charge case

Let us discuss the zero charge/flux case first. As discussed in Ref. [7], the function $m_0(z) = 0$ at the two ends $z = 0, 1$ (look at eq. (13)) and has a maximum $m_{\text{max}} = \frac{1}{\sqrt{2d}} \left( \frac{2}{d+2} \right)^{\frac{\frac{1}{2}+\frac{1}{d}}{}}$ in between at $z_{\text{max}} = \frac{2}{(\tilde{d}+2)}$ (see the second graph in Fig. 1). So, above $m_{\text{max}}$ there is no black brane or bubble phase and the system will be in ‘hot flat space’ phase (with zero free energy). But below $m_{\text{max}}$, the $\tilde{m} = m_0(\tilde{z})$ always gives a local minimum of free energy at each $\tilde{z}_2$ lying in $z_{\text{max}} < \tilde{z}_2 < 1$ since there $dm_0(z)/dz|_{z=z_2} < 0$, implying a locally stable black brane or bubble. The locally stable configuration becomes globally stable when

$$\tilde{m} < m_g = \frac{1}{d+2} \left( \frac{4(d+1)}{(d+2)^2} \right)^{1/d} ,$$

(i.e, $b_g = R_g = m_g$) at $\tilde{z}_2$ lying in $1 > \tilde{z}_2 > z_g$ with

$$z_g = \frac{4(d+1)}{(d+2)^2} ,$$

since its free energy is now negative. Here $z_g$ is determined by setting the free energy to zero at $\tilde{z}_2 = z_g > z_{\text{max}}$. In other words, the free energy has its positive maximum value at $z_{\text{max}}$, decreases from there to zero at $z_g(z > z_{\text{max}})$, then continues to decrease to negative and finally reaches $-\infty$ at $\tilde{z}_2 = 1$ (see the first graph for black brane in Fig. 1). So from
\( \tilde{z}_2 = z_{\text{max}} = 2/(\tilde{d} + 2) \) to \( \tilde{z}_2 = 1 \), the free energy is a monotonically decreasing function and so, greater \( \tilde{z}_2 \) (i.e., \( \tilde{x}_2 \) or \( \tilde{y}_2 \)) will give smaller free energy for the black brane or the bubble. Keeping this in mind we consider the following cases: 1) If \( \tilde{R}, \tilde{b} > b_g = R_g \), the black brane or bubble phase is at most locally stable with a positive free energy and the locally stable phase will make a Hawking-Page phase transition to the globally stable ‘hot flat space’ phase. If \( \tilde{R} > \tilde{b} = b_g \) (or \( \tilde{b} > \tilde{R} = R_g \)), the bubble (or the black brane) is at most locally stable and the locally stable phase will make also a Hawking-Page transition to the globally stable phase which is now a coexisting phase of both the black brane (or the bubble) and the ‘hot flat space’. 2) If \( \tilde{R} = \tilde{b} = b_g = R_g \), the three phases – the bubble, the black brane and the ‘hot flat space’ coexist and the transition between any two of them is a first order one since the first derivative of free energy has a discontinuity when the transition occurs. 3) If \( \tilde{R} < R_g \) and \( \tilde{R} < \tilde{b} \), the bubble is the globally stable phase. If \( \tilde{b} < b_g \) and \( \tilde{b} < \tilde{R} \), black brane is the globally stable phase instead. If \( \tilde{b} = \tilde{R} < b_g = R_g \), the bubble and the black brane coexist and are globally stable.

4 The non-zero flux/charge case

We now move on to the case when the charge enclosed in the cavity is fixed but non-zero. As shown in [7], when the charge is non-zero, there exists a critical charge \( q_c \) at which the first and second derivatives of \( m_q(z) \) with respect to \( z \) at \( z = z_c \) vanish and these will determine completely \( q_c, z_c \) and \( m_c \). The underlying phase structure crucially depends on whether \( q > q_c, q = q_c \) and \( q < q_c \), so we discuss them in turns in the following for \( \tilde{d} > 2 \) (The \( \tilde{d} \leq 2 \) cases are different and will be discussed afterwards.): i) For \( q > q_c \), there is no extrema for either \( m_q(z) \) or free energy \( \tilde{F}^{\text{bl, bb}}(\tilde{z}) \), with each decreasing monotonically in the region of \( q < z < \tilde{z} < 1 \) (see the first graph in Fig. 2 for \( m_q(z) \) and in Fig. 3 for \( \tilde{F}^{\text{bl, bb}}(\tilde{z}) \)). The slope of \( m_q(z) \) is always negative and therefore the on-shell \( \tilde{F}^{\text{bl, bb}}(\tilde{z}) \) is a local minimum. So, we have the following cases to consider. If \( \tilde{b} > \tilde{R} \), the respective \( \tilde{m} = m_q(\tilde{z}) \) implies \( \tilde{x} < \tilde{y} \) and so, \( \tilde{F}^{\text{bl}}(\tilde{x}) > \tilde{F}^{\text{bb}}(\tilde{y}) \), i.e., bubble is the globally stable phase. In other words, an analog of Hawking-Page transition will take the locally stable black brane phase to the globally bubble phase. Similarly, if \( \tilde{b} < \tilde{R} \), the black brane is the globally stable phase. But if \( \tilde{b} = \tilde{R} \), the bubble and the black brane phases coexist and the transition between the two is a first order one due to the change of entropy involved. ii) For \( q = q_c \), we have \( \tilde{b}_c = \tilde{R}_c \) and \( x_c = y_c \), and the two phases are both globally stable and can coexist at the critical point and the transition between the two is also a first order one. iii) For \( q < q_c \), the situation is a bit involved. As noted in [7], in this case, the function \( m_q(z) \) does not decrease monotonically in the region \( q < z < 1 \), but in between there is a
minimum $m_{\text{min}}$ at $z = z_{\text{min}}$ and a maximum $m_{\text{max}}$ at $z = z_{\text{max}}$ (see the second graph in Fig. 2). Since $m_q(z)$ starts at infinity at $z = q$ and goes to zero at $z = 1$ (see eq. (13)), so $z_{\text{min}} < z_{\text{max}}$. Then as shown in [7], in the range $q < \bar{z} < z_{\text{min}}$, and $z_{\text{max}} < \bar{z} < 1$ the black branes or the bubbles are locally stable since each $\bar{z}$ gives a local minimum of free energy, whereas in $z_{\text{min}} < \bar{z} < z_{\text{max}}$ they are unstable since each $\bar{z}$ gives a maximum of free energy. So, for $m_{\text{min}} < \bar{m} < m_{\text{max}}$, $\bar{m} = m_q(\bar{z})$ gives three black brane or bubble phases with three solutions, say, $\bar{z}_1 < \bar{z}_2 < \bar{z}_3$, where $\bar{z}_1$ (small black brane or bubble) and $\bar{z}_3$ (large black brane or bubble) correspond to the locally stable phases and $\bar{z}_2$ corresponds to unstable phase (see the second graph in Fig. 2). Among the locally stable phases the system would prefer to be in the phase of lowest free energy. Now as demonstrated in [7], in between $m_{\text{max}}$ and $m_{\text{min}}$ and for given $q$, there exists a $\bar{m}$ denoted by $m_t(q, \bar{d})$ (here $b_t = R_t$), a function of $\bar{d}$ and charge $q$ only, where the large black brane (bubble) of size $x_{3t}$ ($y_{3t}$) has the same free energy ($\tilde{F}^{\text{bl,bb}}(z_{3t}) = \tilde{F}^{\text{bl,bb}}(z_{1t})$) as the small black brane (bubble) of size $x_{1t}$ ($y_{1t}$) and so they coexist. Above $m_t$ and below $m_{\text{max}}$ the small black brane (bubble) is globally stable and below $m_t$ and above $m_{\text{min}}$, the large black brane (bubble) is the globally stable phase if the black brane or the bubble is assumed to be the only phase. Given what has been said about either black branes or bubbles, we now determine the globally stable phase with the lowest free energy between black branes and bubbles with the same boundary data. Given that $\tilde{F}^{\text{bl,bb}}(\bar{z})$ decreases monotonically in the range $q < \bar{z} < z_{\text{min}}$ and $z_{\text{max}} < \bar{z} < 1$ (see the second graph in Fig. 3), so when $\bar{b}, \bar{R} > b_t = R_t$ or $\bar{b}, \bar{R} < b_t = R_t$, which phase is globally stable can be discussed following $q > q_c$ case and will not be repeated here. When $\bar{b} = \bar{R} = b_t = R_t$, all four phases i.e., the small black brane, the large black brane, the small bubble and the large bubble coexist. The transition

Figure 2: The typical behavior of $m_q(z)$ vs $z$ for $q > q_c$ and $q < q_c$
Figure 3: The typical behavior of reduced free energy $\tilde{F}^{bl}(\bar{z})$ vs $\bar{z}$ for $q > q_c$ and $q < q_c$.

from one phase to the other is a first order one (when it is between small and large black branes or between small and large bubbles) ending at a second order critical point when $q = q_c$ and a first order one (when it is between a black brane and a bubble) even at $q = q_c$ (due to a change of entropy there). The reason that this transition is always a first order one when $q < q_c$ is again due to the discontinuity of the first derivative of free energy when the transition occurs. When $\bar{b} > b_t = R_t$ and $\bar{R} < b_t = R_t$, $\bar{x} < x_{1t} < x_{3t} = y_{3t} < y$ and so, $\tilde{F}^{bl}(\bar{x}) > \tilde{F}^{bl}(x_{1t}) = \tilde{F}^{bl}(x_{3t}) = \tilde{F}^{bb}(y_{3t}) > \tilde{F}^{bb}(\bar{y})$, therefore, the large bubble is the globally stable phase. Similarly, when $\bar{b} < b_t = R_t$ and $\bar{R} > b_t = R_t$, the large black brane is the globally stable phase.

The extremal brane phase with the same boundary data has always the largest free energy in any of the above three cases and therefore can never be a globally stable phase.

We now move on to the $d \leq 2$ cases which have similarities with the zero charge case. Let us consider $d = 1$ case first. The phase analysis for this case can be discussed following the chargeless case if we replace the ‘hot flat space’ there by the present extremal brane phase and $1 > z > 0$ for $m_q(z)$ there by $1 > z > q$ for $m_q(z)$ here and will not be repeated here. The present $m_{\text{max}}$ and $z_{\text{max}}$ can be determined now from $m_q(z)$ similarly but they cannot be given analytically as demonstrated in [7]. Instead, we will give their numerical values for a few selected values of $q$ along with the values of $m_g$ and $z_g$ in the following table. The $z_g$ falling in the range $1 > z_g > z_{\text{max}}$ can be determined from the following equation,

$$4 - q = 2 \left( \frac{1 - \bar{z}}{1 - \bar{z}^2} \right)^{1/2} + \left( \frac{1 - \bar{z}^2}{1 - \bar{z}} \right)^{1/2} + (1 - \bar{z})^{1/2} \left( 1 - \frac{q^2}{\bar{z}} \right)^{1/2}, \tag{21}$$

which is obtained by equating the system free energy at $\bar{z} = z_g$ with the corresponding extremal brane free energy at $\bar{z} = q$, i.e., $F_q(\bar{z} = z_g) = F_q(\bar{z} = q) = q$. This is actually a
quartic equation but one can check, as expected, that $\bar{z} = q$ is a solution. So this equation can be reduced to a third-order equation as

$$9\bar{z}^3 - (5q^2 - q + 8)\bar{z}^2 + (3q + 4)q^2\bar{z} - 4q^3 = 0,$$

(22)

which can be solved analytically and has only one real positive solution, giving $z_g$. The solution for $z_g$ is complicated and not very illuminating and for this reason we will not give its explicit analytic expression here. As mentioned above, we will list instead its value for each given $q$ along with the corresponding $m_g$ for $q$ starting at $q = 0.1$ with an increment of 0.1 up to $q = 0.9$ in the table below.

| $q$  | $z_g$  | $m_g$  | $z_{\text{max}}$ | $m_{\text{max}}$ |
|------|--------|--------|-------------------|------------------|
| 0.1  | 0.878471 | 0.307757 | 0.668371 | 0.386344 |
| 0.2  | 0.870088 | 0.319918 | 0.673943 | 0.390683 |
| 0.3  | 0.864945 | 0.332758 | 0.673943 | 0.397903 |
| 0.4  | 0.864458 | 0.346219 | 0.684781 | 0.407890 |
| 0.5  | 0.870015 | 0.360200 | 0.731065 | 0.420323 |
| 0.6  | 0.882600 | 0.374565 | 0.769645 | 0.434666 |
| 0.7  | 0.902505 | 0.389161 | 0.817842 | 0.450290 |
| 0.8  | 0.929316 | 0.403845 | 0.873604 | 0.466636 |
| 0.9  | 0.962160 | 0.418491 | 0.934861 | 0.483295 |

From the above table, one can see clearly that $1 > z_g > z_{\text{max}}$ and $m_g < m_{\text{max}}$ as expected.

Let us now discuss $\tilde{d} = 2$ case. For this case, as shown also in [7], there exists not only (like $\tilde{d} > 2$ case) a critical charge with $q_c = 1/3$, but also (like $\tilde{d} = 1$ case) a maximum of $m_q(z)$ at $z_{\text{max}}$. For $q \geq q_c$, we have $m_{\text{max}} = m_q(q) = \sqrt{q}/2$ at $z_{\text{max}} = q$ (see the first graph in Fig. 4). When $\bar{m} > m_{\text{max}}$, the extremal brane is the global phase. But below $m_{\text{max}}$, this case is similar to the analysis of $\tilde{d} > 2$ and will not be repeated here. When $q < q_c = 1/3$, we have now

$$m_{\text{max}} = \frac{\left(1 + 3q^2 + \sqrt{(1 - q^2)(1 - 9q^2)}\right)^{1/2}}{8\sqrt{2}} \left(3\sqrt{1 - q^2} - \sqrt{1 - 9q^2}\right),$$

$$z_{\text{max}} = \frac{1 + 3q^2 + \sqrt{(1 - q^2)(1 - 9q^2)}}{4},$$

(23)

6 However, now $z_c = q_c$, occurring at the small end, and therefore, unlike $\tilde{d} > 2$, there is now no critical behavior since we don’t have a stable small brane (or bubble) phase.
with \( m_{\text{max}} > m_q(q) = \sqrt{q/2} \) and \( z_{\text{max}} > q \) (see the second graph in Fig. 4). This subcase is however similar to the discussion of \( \tilde{d} = 1 \). Again when \( \bar{m} > m_{\text{max}} \), the extremal brane is the stable phase. When \( \bar{b} < b_{\text{max}} \) and \( \bar{R} > R_{\text{max}} \) (or \( \bar{b} > b_{\text{max}} \) and \( \bar{R} < R_{\text{max}} \)) there exists a locally stable black brane (or bubble). As in \( \tilde{d} = 1 \) case, we now need to determine \( x_g \) (or \( y_g \)) below which locally stable black brane (or bubble) becomes globally stable. Exactly as in \( \tilde{d} = 1 \) case, \( z_g \) is now determined by the following equation,

\[
3 - q = \left( \frac{1 - \frac{q^2}{\bar{z}^2}}{1 - \frac{q^2}{z}} \right)^{1/2} + \left( \frac{1 - \frac{q^2}{\bar{z}^2}}{1 - \frac{q^2}{z}} \right)^{1/2} + (1 - \bar{z})^{1/2} \left( 1 - \frac{q^2}{\bar{z}} \right)^{1/2} .
\]

This equation, unlike in the previous case, can be solved exactly and has apart from the trivial solution at \( \bar{z} = q \), two other real solutions. Only one of them lies in the range \( z_{\text{max}} < z_g < 1 \) and is given as

\[
z_g = \frac{3 - 2q + 3q^2 + (1 + q)\sqrt{3(3 - q)(1 - 3q)}}{8} , \tag{25}
\]

with \( 1 > z_g > z_{\text{max}} \). With this \( z_g \), we have

\[
m_g = \frac{\left[ 5 - 3q - \sqrt{3(3 - q)(1 - 3q)} \right] \left[ 3 - 2q + 3q^2 + (1 + q)\sqrt{3(3 - q)(1 - 3q)} \right]^{1/2}}{16\sqrt{2}} . \tag{26}
\]

Now the discussion for globally stable phase and the analog of Hawking-Page transition are identical to the chargeless case if we replace the ‘hot flat space’ there by the present extremal brane phase and therefore will not be repeated.
5 Discussion and conclusion

In this paper, we find that a necessary completion of phase structure of charged black $p$-branes in canonical ensemble requires two additional phases, namely, the bubble and the extremal brane. This finding solves the puzzle about the missing phase(s) for $\tilde{d} \leq 2$ and gives a new phase for $\tilde{d} > 2$. We also find an analog of Hawking-Page transition among many new phase transitions revealed, giving now a much enriched and complete underlying phase structure (assuming no other new phases present) which can have a coexistence of up to four individual phases. The phase transitions are always the first-order ones. We also find that the extremal brane cannot be a stable phase for $\tilde{d} > 2$ but is vital to the completion of phase diagram for $\tilde{d} \leq 2$. These results are obtained solely on the basis of the free energy criterion which is used to justify the underlying stability of the various phases.

Before closing this section, we need to clarify which spin structure the extremal brane should take so that no inconsistency arises. For this, let us be somewhat specific in the present context. We focus on the relevant $(t, x^1, \rho)$ directions. For the non-extremal branes, the topology is $R^2 \times S^1$ with the $S^1$ denoting the $x^1$-circle while for the bubble it is $S^1 \times R^2$ with now $S^1$ denoting the $t$-circle. Though the two have the same topology in Euclidean signature, they are quite different in Lorentzian signature since their spacetime structures are different. Also the former has a non-vanishing entropy while the latter has zero entropy. In spite of their differences, the two share one common feature that there is only one spin-structure allowed corresponding to the fermions being anti-periodic along their respective $S^1$. However, for the extremal brane, the topology is $S^1 \times S^1 \times R$ and in general there are two spin structures, one for fermions being periodic and the other for fermions being anti-periodic, along either circle. The extremal $p$-branes, when considered in isolation, are the $1/2$-BPS objects in string/M theory, which preserve one half of the spacetime supersymmetries. So the natural choice for the spin structure is with periodic fermions. If this were the case in the present context, one would have difficulty in understanding\footnote{No such issues would arise for $\tilde{d} > 2$ since there is no phase transition between the extremal brane and the non-extremal brane (or the bubble).} the phase transitions between the extremal brane and the non-extremal brane (or bubble) for $\tilde{d} \leq 2$, vital for the completion of phase diagram, since the two have different spin structure as discussed above. However, the present setup of placing the extremal brane in a cavity with a fixed finite temperature in canonical ensemble breaks all the underlying supersymmetries and therefore requires the spin structure with anti-
periodic fermions \cite{20} for the extremal brane instead. So spin structure wise, there is no paradox arising and the free energy criterion is as usual a suitable means to obtain the phase structure and phase transitions as given in this paper.

**Acknowledgements:**

JXL acknowledges support by grants from the Chinese Academy of Sciences, a grant from 973 Program with grant No: 2007CB815401 and a grant from the NSF of China with Grant No: 10975129.

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\footnote{The discrepancy in the entropy between the semi-classical calculation \cite{12, 13, 15} and the microstate counting in string theory has been addressed recently in \cite{21}. It was pointed out there that the non-vanishing of the entropy is due to a different reason and the semi-classical result of vanishing entropy for the extremal black hole may still hold. The associated issue as discussed in \cite{20} is that the tachyon condensation may modify the semi-classical topology for the extremal brane when the circle size is of the order of string scale for which the curvature can be large and the semi-classical analysis may break down. While these issues appear to be not completely settled at present, however, we may re-interpret the modification of the topology of the time circle (or the spatial circle) due to the tachyon condensation, as the underlying dynamical process of the phase transition from the extremal brane to the non-extremal brane (or the bubble) (for $d \leq 2$) if the free energy indeed favors this process. Though difficult to imagine, the reverse process (if the free energy favors) may give the transition from the non-extremal brane (or bubble) to the extremal brane.}
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