Positive Operator Valued Measures and Feller Markov Kernels

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Abstract

A Positive Operator Valued Measure (POVM) is a map $F: \mathcal{B}(X) \rightarrow \mathcal{L}^+(\mathcal{H})$ from the Borel $\sigma$-algebra of a topological space $X$ to the space of positive self-adjoint operators on a Hilbert space $\mathcal{H}$. We assume $X$ to be Hausdorff, locally compact and second countable and prove that a POVM $F$ is commutative if and only if it is the smearing of a spectral measure $E$ by means of a Feller Markov kernel. Moreover, we prove that the smearing can be realized by means of a strong Feller Markov kernel if and only if $F$ is uniformly continuous. Finally, we prove that a POVM which is absolutely continuous with respect to a finite measure $\nu$ admits a strong Feller Markov kernel.

That provides a characterization of the smearing which connects a commutative POVM $F$ to a spectral measure $E$ and is relevant both from the mathematical and the physical viewpoint since smearings of spectral measures form a large and very relevant subclass of POVMs: they are paradigmatic for the modeling of certain standard forms of noise in quantum measurements [22, 21], they provide optimal approximators as marginals in joint measurements of incompatible observables [22], they are important for a range of quantum information processing protocols, where classical post-processing plays a role [34].

The mathematical and physical relevance of the results is discussed and particular emphasis is given to the connections between the Markov kernel and the imprecision of the measurement process.

1 Introduction

A Positive operator Valued measure (or POVM) is a map $F: \mathcal{B}(X) \rightarrow \mathcal{L}^+(\mathcal{H})$ from the Borel $\sigma$-algebra of a topological space $X$ to the space of positive self-adjoint operators on a Hilbert space $\mathcal{H}$. In the present paper we assume $X$ to be Hausdorff, locally compact and second countable. If $F(\Delta)$ is a projection operator for each
Δ ∈ ℘(X), $F$ is called Projection Valued measure (or PVM). If $X = \mathbb{R}$ we have real POVMs (or semispectral measures) and real PVMs (or spectral measures) respectively. Therefore, the set of PVMs is a subset of the set of POVMs and the set of spectral measures is a subset of the set of semispectral measures. Moreover, spectral measures are in one-to-one correspondence with self-adjoint operators (spectral theorem) [49] and are used in standard quantum mechanics to represent quantum observables. It was pointed out [11, 21, 24, 36, 48, 51] that PVMs are more suitable than spectral measures in representing quantum observables. The quantum observables described by POVMs are called generalized observables or unsharp observables and play a key role in quantum information theory, quantum optics, quantum estimation theory [21, 32, 36, 52] and in the phase space formulation of quantum mechanics [48, 52, 50, 24, 15, 16]. It is then natural to ask what are the relationships between POVMs and spectral measures. A clear answer can be given in the commutative case [11, 6, 7, 8, 9, 10, 11, 12, 13, 14, 35, 37]. Indeed [7, 37], a POVM $F$ is commutative if and only if there exist a bounded self-adjoint operator $A$ and a Markov kernel (transition probability) $\mu_{\Lambda}(\cdot): \sigma(A) \times ℘(X) \to [0, 1]$ such that

$$F(\Delta) = \int_{\sigma(A)} \mu_{\Lambda}(\lambda) dE_{\lambda}$$

where, $E$ is the spectral measure corresponding to $A$. In other words, $F$ is a smearing of the spectral measure $E$ corresponding to $A$.

Smearings of spectral measures form a large and very relevant subclass of POVMs and are paradigmatic for the modeling of certain standard forms of noise in measurements [21, 22]. They also provide optimal approximators as marginals in joint measurements of incompatible observables (for example, for position and momentum) as shown by Busch, Lahti, Werner in Ref. [22]. Moreover, they are important for a range of quantum information processing protocols, where classical post-processing plays a role [34]. Another relevant application of commutative POVMs is the smearing of incompatible observables in order to get compatible observables (see [23, 17]). All that explains the relevance of commutative POVMs both form the mathematical and the physical viewpoint. As a notable example we analyze (section 6.1) the unsharp position and momentum observables which are the marginals of a joint position momentum observable (see [51, 21, 22]).

Although, it is well known that $F$ can be interpreted as the smearing of $E$, no characterization of the smearing (the Markov kernel) is known. In the present paper such a characterization is given and its mathematical and physical implications are analyzed. That also provides a stronger characterization of commutative POVMs by means of Feller Markov kernels.

In order to outline some of the problems we deal with, it is helpful to consider the unsharp position observable in the interval $[0, 1]$. It can be represented as follows.

$$\langle \psi, Q^f(\Delta) \psi \rangle := \int_{[0,1]} \mu_{\Lambda}(x) d(\psi, Q_x \psi), \quad \Delta \in ℘(\mathbb{R}), \quad \psi \in L^2([0,1]), \quad (1)$$

$$\mu_{\Lambda}(x) := \int_\mathbb{R} \chi_{\Lambda}(x-y) f(y) dy, \quad x \in [0,1]$$
where, \( f \) is a positive, bounded, Borel function such that \( f(y) = 0, \ y \not\in [0,1], \) and \( \int_{[0,1]} f(y) \, dy = 1, \) while \( Q_x \) is the spectral measure corresponding to the position operator

\[
Q : L^2([0,1]) \to L^2([0,1])
\]

\[
\psi(x) \mapsto Q\psi(x)
\]

We recall that \( \langle \psi, Q(\Delta)\psi \rangle \) is interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in \( \Delta. \) Then, a possible interpretation of equation (1) is that \( Qf \) is a randomization of \( Q. \) Indeed \cite{48}, the outcomes of the measurement of the position of a particle depend on the measurement imprecision\footnote{There are other possible interpretations of the randomization. For example, it could be due to the existence of a no-detection probability depending on hidden variables \cite{28}.} so that, if the sharp value of the outcome of the measurement of \( Q \) is \( x \) then the apparatus produces with probability \( \mu_\Delta(x) \) a reading in \( \Delta. \)

It is worth noting that (see example 2 in section 5) the Markov kernel

\[
\mu_\Delta(x) := \int_\mathbb{R} \chi_\Delta(x-y)\ f(y) \, dy, \quad x \in [0,1]
\]

in equation (1) above is such that the function \( x \mapsto \mu_\Delta(x) \) is continuous for each \( \Delta \in \mathcal{B}(\mathbb{R}). \) The continuity of \( \mu_\Delta \) means that if two sharp values \( x \) and \( x' \) are very close to each other then, the corresponding random diffusions are very similar, i.e., the probability to get a result in \( \Delta \) if the sharp value is \( x \) is very close to the probability to get a result in \( \Delta \) if the sharp value is \( x'. \) That is quite common in important physical applications and seems to be reasonable from the physical viewpoint. It is then natural to look for general conditions which ensure the continuity of \( \lambda \mapsto \mu_\Delta. \) That is one of the aims of the present work. What we prove is that, in general, the continuity does not hold for all the Borel sets \( \Delta \) but only for a ring of subsets which generates the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}). \) (Anyway, that is sufficient to prove the weak convergence of \( \mu_{\epsilon_1}(x) \) to \( \mu_{\epsilon_1}(x'), \)) We also prove that the continuity for each Borel set is equivalent to the uniform continuity of \( F \) which in its turn is equivalent to require that the smearing in equation (1) can be realized by a strong Feller Markov kernel.

It is our opinion that, in the real case, the continuity of \( \mu_\Delta \) over a ring \( \mathcal{B} \) which generates the Borel \( \sigma \)-algebra of the reals could be helpful in dealing with problems connected to the characterization of functions of the kind

\[
G_f(x) = \int f(t) \, d\mu_t(x).
\]

A similar (but less general) problem arises in Ref. \cite{12} where the relationships between Naimark extension theorem and the characterization of commutative POVMs as smearing of spectral measures are analyzed. That is a second motivation for the analysis of the continuity properties of \( \mu_\Delta. \)

The results outlined above are a consequence of the two main theorems of the present work.

The first is a characterization of the smearing which connects a commutative POVM to a real PVM. In particular, we show (see theorems 4.1) that a POVM is commutative
if and only if there exist a spectral measure $E$ and a Feller Markov kernel $\mu(\cdot) : \Gamma \times \mathcal{B}(X) \to [0, 1]$, $\Gamma \subset \sigma(A)$, $E(\Gamma) = 1$, such that

$$F(\Delta) = \int_{\Gamma} \mu_\Delta(\lambda) \, dE_\lambda$$  \hspace{1cm} (2)$$

and $\mu_\Delta(\cdot)$ is continuous for each $\Delta \in \mathcal{B}$ where, $\mathcal{B} \subset \mathcal{B}(X)$ is a ring which generates the Borel $\sigma$-algebra $\mathcal{B}(X)$. Therefore, $F$ is commutative if and only if there exists a Feller Markov kernel $\mu$ such that equation (2) is satisfied. See section 4 for the definition of Feller Markov kernel. That provides a new and stronger characterization of commutative POVMs.

We also prove that the family of functions $\{\mu_\Delta\}_{\Delta \in \mathcal{B}(X)}$ separates the points of $\sigma(A)$ up to a null set (see theorems 3.1, and 4.1). In other words, the probability measures $\mu_{\cdot,\Delta}(\lambda)$ and $\mu_{\cdot,\Delta'}(\lambda')$ which represent the randomizations corresponding to the sharp values $\lambda$ and $\lambda'$ are different.

The second theorem is a characterization of the POVMs which admit a strong Feller Markov kernel, i.e., a Markov kernel $\mu$ such that the function $\lambda \mapsto \mu_\Delta(\lambda)$ is continuous for each $\Delta \in \mathcal{B}(X)$. In particular, we prove (see theorem 5.1) that a POVM $F$ admits a strong Feller Markov kernel if and only if it is uniformly continuous. As an example, we develop the details for the unsharp position observable defined in equation (1) above. Finally, we prove (see section 6) that a POVM $F$ which is absolutely continuous with respect to a regular finite measure $\nu$ is uniformly continuous (theorem 6.1). We give some examples of absolutely continuous POVMs (see example 4) and analyze the unsharp position observable which is obtained as the marginal of a phase space observable (see section 6.1).

2 Some preliminaries about POVMs

In what follows, we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of a topological space $X$, by $\mathbf{0}$ and $\mathbf{1}$ the null and the identity operators, by $L^+_s(\mathcal{H})$ the space of all bounded self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$, by $L^+_s(\mathcal{H})$ the subspace of all positive, bounded self-adjoint operators on $\mathcal{H}$, by $\mathcal{E}(\mathcal{H}) \subset L^+_s(\mathcal{H})$ the subspace of all projection operators on $\mathcal{H}$.

**Definition 1.** A Positive Operator Valued measure (for short, POVM) is a map $F : \mathcal{B}(X) \to L^+_s(\mathcal{H})$ such that:

$$F\left( \bigcup_{n=1}^{\infty} \Delta_n \right) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = 1$$

**Definition 2.** A POVM is said to be commutative if

$$[F(\Delta_1), F(\Delta_2)] = 0, \ \forall \Delta_1, \Delta_2 \in \mathcal{B}(X).$$ \hspace{1cm} (3)
Definition 3. A POVM is said to be orthogonal if

\[ F(\Delta_1)F(\Delta_2) = 0 \text{ if } \Delta_1 \cap \Delta_2 = \emptyset. \]  

(4)

Definition 4. A Projection Valued measure (for short, PVM) is an orthogonal, nor-
malized POVM.

It is simple to see that for a PVM \( E \), we have \( E(\Delta) = E(\Delta)^2 \), for any \( \Delta \in \mathcal{B}(X) \). Then, \( E(\Delta) \) is a projection operator for every \( \Delta \in \mathcal{B}(X) \), and the PVM is a map \( E : \mathcal{B}(X) \to \mathcal{D}(\mathcal{H}) \).

In quantum mechanics, non-orthogonal normalized POVM are also called generalised or unsharp observables and PVM standard or sharp observables.

Definition 5. The spectrum \( \sigma(F) \) of a POVM \( F \) is the set of points \( x \in X \) such that \( F(\Delta) \neq 0 \), for any open set \( \Delta \) containing \( x \).

The spectrum \( \sigma(F) \) of a POVM \( F \) is a closed set since its complement \( X - \sigma(F) \) is the union of all the open sets \( \Delta \subset X \) such that \( F(\Delta) = 0 \).

Definition 6. The von Neumann algebra \( \mathcal{A}^W(F) \) generated by the POVM \( F \) is the von Neumann algebra generated by the set \( \{F(\Delta)\}_{\Delta \in \mathcal{B}(X)} \).

In the following we use the symbols \( w - \lim \) and \( u - \lim \) to denote the limit in the weak operator topology and the limit in the uniform operator topology respectively.

Definition 7. A POVM is regular if for any Borel set \( \Delta \),

\[ w - \lim_{i \to \infty} F(G_j) = F(\Delta) = w - \lim_{i \to \infty} F(O_j) \]

where, \( \{G_j\}_{j \in \mathbb{N}}, \Delta \subset G_j \), is a decreasing family of open sets and \( \{O_j\}_{j \in \mathbb{N}}, O_j \subset \Delta \), is an increasing family of compact sets.

We recall [43] that a topological space \( (X, \tau) \) is second countable if it has a countable basis for its topology \( \tau \); i.e., if there is a countable subset \( \mathcal{B} \) of \( \tau \) such that each member of \( \tau \) is the union of members of \( \mathcal{B} \).

Proposition 1. A POVM defined on a Hausdorff locally compact, second countable space \( X \) is regular.

Proof. A locally compact Hausdorff space is regular. (See Ref. [46], page 205). By the Urysohn’s theorem, a second countable regular space is metrizable (see [46], page 215). Moreover, the second countability implies the \( \sigma \)-compactness (46, page 289). In a metrizable \( \sigma \)-compact space the ring of Borel sets coincides with the ring of Baire sets (see page 25 in [19]) and the thesis comes from the fact that each Baire POVM is regular (see Theorem 18 in [19]).

In what follows, we use the term “measurable” for the Borel measurable functions. For any vector \( \psi \in \mathcal{H} \) the map

\[ \langle F(\cdot)\psi, \psi \rangle : \mathcal{B}(X) \to \mathbb{R}, \quad \Delta \mapsto \langle F(\Delta)\psi, \psi \rangle, \]
is a measure. In the following we will use the symbol $d\langle F \cdot \psi, \psi \rangle$ to mean integration with respect to the measure $\langle F(\cdot) \psi, \psi \rangle$. We shall say that a measurable function $f : N \subset X \to f(N) \subset \mathbb{R}$ is almost everywhere (a.e.) one-to-one with respect to a POVM $F$ if it is one-to-one on a subset $N' \subset N$ such that $N - N'$ is a null set with respect to $F$. We shall say that a function $f : X \to \mathbb{R}$ is bounded with respect to a POVM $F$, if it is equal to a bounded function $g$ a.e. with respect to $F$, that is, if $f = g$ a.e. with respect to the measure $\langle F(\cdot) \psi, \psi \rangle$, $\forall \psi \in \mathcal{H}$. For any real, bounded and measurable function $f$ and for any POVM $F$, there is a unique [19] bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle B \psi, \psi \rangle = \int f(x) d\langle F \cdot \psi, \psi \rangle, \quad \text{for each } \psi \in \mathcal{H}. \quad (5)$$

If equation (5) is satisfied, we write $B = \int f(x) dF$, or $B = \int f(x) F(dx)$ equivalently.

By the spectral theorem [27, 49], there is a one-to-one correspondence between real PVMs $E$ and self-adjoint operators $B$, the correspondence being given by

$$B = \int_{\mathbb{R}} \lambda dE^B_{\lambda}. \quad (6)$$

Notice that the spectrum of $E^B$ coincides with the spectrum of the corresponding self-adjoint operator $B$. Moreover, in this case a functional calculus can be developed. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [49]

$$f(B) = \int_{\mathbb{R}} f(\lambda) dE^B_{\lambda} \quad (6)$$

where, $E^B$ is the PVM corresponding to $B$. If $f$ is bounded, then $f(B)$ is bounded [49]. Equation (6) cannot be extended to the case of non orthogonal POVMs.

In the following we do not distinguish between real PVMs and the corresponding self-adjoint operators.

Let $\Lambda$ be a subset of $\mathbb{R}$ and $\mathcal{B}(\Lambda)$ the corresponding Borel $\sigma$-algebra.

**Definition 8.** A real Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu(\cdot) (\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

**Definition 9.** Let $\nu$ be a measure on $\Lambda$. A map $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ is a weak Markov kernel with respect to $\nu$ if:

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $0 \leq \mu_X(\lambda) \leq 1, \quad \nu - a.e.,$
3. $\mu_X(\lambda) = 1, \mu_\emptyset(\lambda) = 0, \quad \nu - a.e.,$
4. for any sequence $\{\Delta_i\}_{i \in \mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,

$$\sum_i \mu_{\Delta_i}(\lambda) = \mu_{\emptyset \cup \Delta_i}(\lambda), \quad \nu - a.e.
Definition 10. The map \( \mu : \Lambda \times \mathcal{B}(X) \to [0,1] \) is a weak Markov kernel with respect to a PVM \( E : \mathcal{B}(\Lambda) \to \mathcal{S}(\mathcal{H}) \) if it is a weak Markov kernel with respect to each measure \( \nu \psi(\cdot) := \langle E(\cdot) \psi, \psi \rangle, \psi \in \mathcal{H} \).

In the following, by a weak Markov kernel \( \mu \) we mean a weak Markov kernel with respect to a PVM \( E \). Moreover the function \( \lambda \mapsto \mu_\Delta(\lambda) \) will be denoted indifferently by \( \mu_\Delta \) or \( \mu_\Delta(\cdot) \).

Definition 11. A POVM \( F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H}) \) is said to be a smearing of a POVM \( E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H}) \) if there exists a weak Markov kernel \( \mu : \Lambda \times \mathcal{B}(X) \to [0,1] \) such that,

\[
F(\Delta) = \int_\Lambda \mu_\Delta(\lambda) dE_\lambda, \quad \Delta \in \mathcal{B}(X).
\]

Example 1. In the standard formulation of quantum mechanics, the operator \( Q : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \)

\[
\psi(x) \in L^2(\mathbb{R}) \mapsto Q\psi := x\psi(x)
\]

is used to represent the position observable. A more realistic description of the position observable of a quantum particle is given by a smearing of \( Q \) as, for example, the optimal position POVM

\[
F^Q(\Delta) = \frac{1}{l \sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_\Delta e^{-\frac{(x-y)^2}{2l^2}} dy \right) dE^Q_x = \int_{-\infty}^{\infty} \mu_\Delta(x) dE^Q_x
\]

where,

\[
\mu_\Delta(x) = \frac{1}{l \sqrt{2\pi}} \int_\Delta e^{-\frac{(x-y)^2}{2l^2}} dy
\]

defines a Markov kernel and \( E^Q \) is the spectral measure corresponding to the position operator \( Q \).

In the following, the symbol \( \mu \) is used to denote both Markov kernels and weak Markov kernels. The symbols \( A \) and \( B \) are used to denote self-adjoint operators.

Definition 12. Whenever \( F, A, \) and \( \mu \) are such that \( F(\Delta) = \mu_\Delta(A), \Delta \in \mathcal{B}(X) \), we say that \( \langle F, A, \mu \rangle \) is a von Neumann triplet.

The following theorem establishes a relationship between commutative POVMs and spectral measures and gives a characterization of the former. Other characterizations and an analysis of the relationships between them can be found in Refs. [1, 35, 4, 38].

Theorem 2.1 ([7, 37]). A POVM \( F \) is commutative if and only if there exist a bounded self-adjoint operator \( A \) and a Markov kernel (weak Markov kernel) \( \mu \) such that \( \langle F, A, \mu \rangle \) is a von Neumann triplet.

Corollary 1. A POVM \( F \) is commutative if and only if it is a smearing of a real PVM \( E \) with bounded spectrum.

Definition 13. If \( A \) and \( F \) in theorem 2.1 generate the same von Neumann algebra then \( A \) is named the sharp version of \( F \).

Theorem 2.2. [7] The sharp version \( A \) is unique up to almost everywhere bijections.
3 On the separation properties of $\mu$

In the following, we assume $X$ to be Hausdorff, locally compact and second countable. The symbol $\mathcal{S}$ denotes a countable basis for the topology of $X$. The symbol $\mathcal{B}(\mathcal{S})$ denotes the ring generated by $\mathcal{S}$. Notice that by theorem c, page 24, in Ref. [31], $\mathcal{B}(\mathcal{S})$ is countable too. Moreover, $\mathcal{B}(\mathcal{S})$ generates the Borel $\sigma$-algebra $\mathcal{B}(X)$.

A weak Markov kernel $\mu$ such that $(F,A,\mu)$ is a von Neumann triplet, separates the point of $\Gamma \subset \sigma(A)$ if the family of functions $\{\mu_\lambda\}_{\Delta \in \mathcal{S}(X)}$ separates the points of $\Gamma$ or, in other words, if $\lambda \neq \lambda'$ implies $\mu_{\lambda'}(\lambda) \neq \mu_{\lambda'}(\lambda')$. It is then natural to ask if in general $\mu$ has that property. The following theorem answers in the positive.

**Theorem 3.1.** Let $(F,A,\mu)$ be a von Neumann triplet and suppose that $A$ is a sharp version of $F$. Then, there exists a set $\Gamma \subseteq \sigma(A)$, $E^A(\Gamma) = 1$, such that the family of functions $\{\mu_\lambda(\cdot)\}_{\Delta \in \mathcal{S}(X)}$ separates the points of $\Gamma$.

**Proof.** In the following, $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\lambda)\}_{\Delta \in \mathcal{S}(X)}$, $O_2 := \{F(\lambda)\}_{\Delta \in \mathcal{S}(\mathcal{S})}$ and $\mathcal{A}^C(O_2)$ is the $C^*$-algebra generated by $O_2$. The von Neumann algebra generated by $\mathcal{A}^C(O_2)$ coincides with $\mathcal{A}^W(F)$ (see appendix A). Moreover, $\mathcal{A}^W(F) = \mathcal{A}^W(A)$ since $A$ is the sharp version of $F$ and generates $\mathcal{A}^W(F)$. By the Gelfand-Naimark theorem [27][47], there is a * isomorphism $\phi$ between $\mathcal{A}^C(O_2)$ and the algebra of continuous functions $\mathcal{C}(\Lambda_2)$ where $\Lambda_2$ is the spectrum of $\mathcal{A}^C(O_2)$. Moreover,

$$ f \in \mathcal{C}(\Lambda_2) \mapsto \phi(f) = \int_{\Lambda_2} f(\lambda) \, d\tilde{E}_\lambda $$

where, $\tilde{E}$ is the spectral measure from the Borel $\sigma$ algebra $\mathcal{B}(\Lambda_2)$ to $\mathcal{B}(\mathcal{S})$ whose existence is assured by theorem 1, page 895, in Ref. [27]. The Gelfand-Naimark isomorphism $\phi$ can be extended to a homomorphism between the algebra of the Borel functions on $\Lambda_2$ and the von Neumann algebra $\mathcal{A}^W(F)$ and the von Neumann algebra $\mathcal{A}^W(F)$ is the von Neumann algebra $\mathcal{A}^C(O_2)$ (see Ref. [26], page 360, section 3). Therefore, there is a Borel function $h$ such that

$$ A = \int_{\Lambda_2} h(\lambda) \, d\tilde{E}_\lambda \quad (7) $$

Let $\mathcal{S}$ be a countable basis for the topology of $X$. Let $\{\Delta_i\}_{i \in \mathbb{N}}$ denote an enumeration of the set $\mathcal{B}(\mathcal{S})$. Since $\mathcal{A}^C(O_2)$ is the smallest uniform closed algebra containing $\{F(\Delta_i)\}_{i \in \mathbb{N}}$, $\mathcal{C}(\Lambda_2)$ is the smallest uniform closed algebra of functions containing $\{v_{\Delta_i} := \phi^{-1}(F(\Delta_i))\}_{i \in \mathbb{N}}$. In other words $\{v_{\Delta_i}\}_{i \in \mathbb{N}}$ generates $\mathcal{C}(\Lambda_2)$. The Stone-Weierstrass theorem [27] assures that $\{v_{\Delta_i}\}_{i \in \mathbb{N}}$ separates the points in $\Lambda_2$. On the other hand, the fact that $(F,A,\mu)$ is a von Neumann triplet, implies that, for each $\Delta_i \in \mathcal{B}(\mathcal{S})$, there is a Borel function $\mu_{\Delta_i}$ such that

$$ \int_{\Lambda_2} v_{\Delta_i}(\lambda) \, d\tilde{E}_\lambda = F(\Delta_i) = \mu_{\Delta_i}(A) = \int_{\Lambda_2} \mu_{\Delta_i}(h(\lambda)) \, d\tilde{E}_\lambda. $$

Then, for each $\Delta_i \in \mathcal{B}(\mathcal{S})$, there is a set $M_i \subseteq \Lambda_2$, $\tilde{E}(M_i) = 1$, such that

$$ \mu_{\Delta_i}(h(\lambda)) = v_{\Delta_i}(\lambda), \quad \lambda \in M_i. \quad (8) $$

8
Let \( M := \cap_{i=1}^{\infty} M_i \). Then,

\[
\tilde{E}(M) = \lim_{n \to \infty} \tilde{E}(\cap_{i=1}^{n} M_i) = \lim_{n \to \infty} \prod_{i=1}^{n} \tilde{E}(M_i) = 1
\]

and, for each \( i \in \mathbb{N} \),

\[
(\mu_{\Delta_i} \circ h)(\lambda) = v_{\Delta_i}(\lambda), \quad \lambda \in M \subseteq \Lambda_2.
\]

Since \( \{v_{\Delta_i}\}_{i \in \mathbb{N}} \) separates the points in \( \Lambda_2 \), it separates the points in \( M \). Then, equation (9) implies that \( \{\mu_{\Delta_i}\}_{i \in \mathbb{N}} \) separates the points in \( \Gamma := h(M) \). Moreover\(^2\),

\[
E^A(\Gamma) = E^A(h(M)) = \tilde{E}[h^{-1}(h(M))] = 1
\]

where, \( E^A \) is the spectral measure defined by the relation

\[
E^A(\Delta) = \tilde{E}(h^{-1}(\Delta))
\]

and such that,

\[
A = \int x \, dE^A_x
\]

while, \( h^{-1}(h(M)) \) is a Borel set containing \( M \).

We have proved that the set of functions \( \{\mu_{\Delta_i}\}_{i \in \mathbb{N}} \) separates the points of \( \Gamma \) and that \( E^A(\Gamma) = 1 \). In other words,

\[
\mu(t) \neq \mu(t'), \quad t \neq t', \quad t, t' \in \Gamma.
\]

\( \square \)

4 Characterization of Commutative POVMs by means of Feller Markov kernels

In the present section we introduce the concept of strong Markov kernel, i.e., a weak Markov kernel \( \mu(\cdot, \cdot) : \Lambda \times \mathcal{B}(X) \to [0, 1] \) with respect to a PVM \( E : \mathcal{B}(\Lambda) \to \mathcal{B}(\mathcal{H}) \) such that \( \mu(\cdot, \lambda) \) is a probability measure for each \( \lambda \in \Gamma \subseteq \Lambda, E(\Gamma) = 1 \). Then, we prove (theorem 4.1) that \( F \) is commutative if and only if there exists a bounded self-adjoint operator \( A \) and a Feller Markov kernel \( \mu \) such that

\[
F(\Delta) = \int_{\Gamma} \mu(\lambda) \, dE_\lambda
\]

Moreover, we prove that \( \mu_\Delta \) is continuous for each \( \Delta \in \mathcal{R} \), where \( \mathcal{R} \) is a ring which generates \( \mathcal{B}(X) \), and the family of functions \( \{\mu_\Delta\}_{\Delta \in \mathcal{R}} \) separates the points in \( \Gamma \) (see theorems 3.1 and 4.1).

In order to prove the main theorem we need the following definitions.

\(^2\) Notice that \( h(M) \) is a Borel set. In order to prove that, we first recall that \( \Lambda_2 \) is a Polish space (that is, a complete, separable, space \(^{42}\)). Indeed, by theorem 11, page 871, in Ref. \(^{27}\), it is homeomorphic to a closed subspace of the Cartesian product \( \prod_{i=1}^{\infty} \sigma(F(\Delta_i)) \), where \( \sigma(F(\Delta_i)) \) is a complete separable metric space, and by theorem 2, page 406, and theorem 6, page 156, in Ref. \(^{43}\), it is complete and separable. Moreover, \( h \) is measurable and injective on \( M \). Therefore, Soulsin’s theorem (see theorem 9 page 440 and Corollary 1 page 442 in Ref. \(^{42}\)) assures that \( h(M) \) is a Borel set.
Moreover, theorem c, page 24, in Ref. [31].

Definition 14. Let $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ be a PVM. The map $\mu_{\mu}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0, 1]$ is a strong Markov kernel with respect to $E$ if it is a weak Markov kernel with respect to $E$ and there exists a set $\Gamma \subset \Lambda$, $E(\Gamma) = 1$, such that $\mu_{\mu}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0, 1]$ is a Markov kernel. A strong Markov kernel is denoted by the symbol $(\mu, E, \Gamma \subset \Lambda)$.

Definition 15. A Feller Markov kernel is a Markov kernel $\mu_{\mu}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that the function

$$G(\lambda) = \int_X f(x) \, d\mu_x(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever $f$ is continuous and bounded.

Theorem 4.1. A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if, there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$ and a strong Markov Kernel $(\mu, E, \Gamma \subset \sigma(A))$ such that:

1) $\mu_\Delta(\cdot) : \sigma(A) \to [0, 1]$ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$,

2) $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda) \, dE_\lambda$, $\Delta \in \mathcal{B}(X)$.

3) $\mathfrak{A}^W(F) = \mathfrak{A}^W(A)$.

4) $\mu$ separates the points in $\Gamma$.

Moreover, $\mu : \Gamma \times \mathcal{B}(X) \to [0, 1]$ is a Feller Markov kernel.

Proof. Let $\mathfrak{A}^W(F)$ be the von Neumann algebra generated by $F$. $\mathfrak{A}^W(F)$ coincides with the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in \mathcal{R}(\mathcal{S})}$ where, $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(X)$ is the ring generated by $\mathcal{S}$, the countable sub-basis for the topology of $X$ (see appendix A for the proof). We recall that both $\mathcal{S}$ and $\mathcal{B}(\mathcal{S})$ are countable (see theorem c, page 24, in Ref. [31]).

Now, we proceed to the proof of the existence of $A$. Let $\{\Delta_i\}_{i \in \mathbb{N}}$ be an enumeration of the set $\mathcal{R}(\mathcal{S})$ and $O_2 := \{F(\Delta_i)\}_{\Delta_i \in \mathcal{R}(\mathcal{S})}$. Let $E(i)$ denote the spectral measure corresponding to $F(\Delta_i) \in O_2$. We have $F(\Delta_i) = \int_x dx E(i)_x$. Therefore, for each $i, k \in \mathbb{N}$ there exists a division $\{\Delta_{j}^{(i, k)}\}_{j=1, \ldots, m_{i, k}}$ of $[0, 1]$ such that

$$\left\| \sum_{j=1}^{m_{i, k}} x_{j}^{(i, k)} E(i)(\Delta_{j}^{(i, k)}) - F(\Delta_i) \right\| \leq \frac{1}{k}. \quad (10)$$

By the spectral theorem [27] the von Neumann algebra $\mathfrak{A}^W(F)$ contains all the projection operators in the spectral resolution of $F(\Delta)$, $\Delta \in \mathcal{B}(X)$. Therefore, the von Neumann algebra $\mathfrak{A}^W(D)$ generated by the set $D := \{E(i)(\Delta_j^{(i)})\}_{j \leq m_{i, k}, i, k \in \mathbb{N}}$ is contained in $\mathfrak{A}^W(F)$ and then

$$\mathfrak{A}^W(D) \subset \mathfrak{A}^W(F) = \mathfrak{A}^W(O_2). \quad (11)$$

Moreover, the $C^*$-algebra $\mathfrak{A}^C(D)$ generated by $D$ contains the $C^*$-algebra $\mathfrak{A}^C(O_2)$ generated by $O_2$ (see equation (10)). Summing up the preceding observations, we have

$$\mathfrak{A}^C(O_2) \subset \mathfrak{A}^C(D) \subset \mathfrak{A}^W(F).$$
By the double commutant theorem \cite{40},
\[
\mathcal{A}^W(F) = [\mathcal{A}^C(O_2)]'' \subset [\mathcal{A}^C(D)]'' = \mathcal{A}^W(D)
\]
so that (see equation \ref{11}),
\[
\mathcal{A}^W(D) = \mathcal{A}^W(F).
\] (12)

By theorem 11, page 871 in Ref. \cite{27}, the spectrum \( \Lambda \) of \( \mathcal{A}^C(D) \) is homeomorphic to a closed subset of \( \prod_{i=1}^{\infty} \{0,1\} \). Let \( \pi : \Lambda \to \prod_{i=1}^{\infty} \{0,1\} \) denote the homeomorphism between the two spaces.

Now, if we identify \( \Lambda \) with a closed subset of \( \prod_{i=1}^{\infty} \{0,1\} \), we can prove the existence of a continuous function distinguishing the points of \( \Lambda \). Indeed, let \( \pi(\lambda) = \tilde{x} := (x_1, \ldots, x_n, \ldots) \in \prod_{i=1}^{\infty} \{0,1\} \). The function
\[
f(\lambda) = \sum_{i=1}^{\infty} \frac{x_i}{3^i}
\]
is continuous and injective and then it distinguishes the points of \( \Lambda \). Moreover, since \( \Lambda \) and \( [0,1] \) are Hausdorff, the map \( f : \Lambda \to f(\Lambda) \) is a homeomorphism.

By theorem 1, page 895, in Ref. \cite{27}, there exists a spectral measure \( \tilde{E} : \mathcal{B}(\Lambda) \to \mathcal{B}(\mathcal{H}) \) such that the map
\[
T : \mathcal{C}(\Lambda) \to B(\mathcal{H})
\]
\[
g \mapsto T(g) = \int_{\Lambda} g(\lambda) \tilde{d}E_\lambda
\] (13)
defines an isometric \(^*\)-isomorphism between \( \mathcal{A}^C(D) \) and \( \mathcal{C}(\Lambda) \).

The fact that \( f \) distinguishes the points of \( \Lambda \), implies that the self-adjoint operator
\[
A = \int_{\Lambda} f(\lambda) \tilde{d}E_\lambda
\]
is a generator of the von Neumann algebra \( \mathcal{A}^W(D) = \mathcal{A}^W(F) \). Indeed, by the Stone-Weierstrass theorem, \( \mathcal{C}(\Lambda) \) is singly generated, in particular \( f \) is a generator. Then, the isomorphism between \( \mathcal{A}^C(D) \) and \( \mathcal{C}(\Lambda) \) assures that \( \mathcal{A}^C(D) \) is singly generated and that \( A \) is a generator. Hence, \( \mathcal{A}^W(F) = \mathcal{A}^W(D) = [\mathcal{A}^C(D)]'' \) is singly generated.

In particular, \( A \) generates \( \mathcal{A}^W(F) \), i.e., \( \mathcal{A}^W(F) = \mathcal{A}^W(A) \).

Now, we proceed to the proof of the existence of the weak Markov kernel \( \tilde{\nu} \) such that \( (F,A,\tilde{\nu}) \) is a von Neumann triplet.

By \ref{13}, for each \( \Delta \in \mathcal{R}(\mathcal{H}) \), there exists a continuous function \( \gamma_\Delta \in \mathcal{C}(\Lambda) \) such that
\[
F(\Delta) = \int_{\Lambda} \gamma_\Delta(\lambda) \tilde{d}E_\lambda.
\]
Now, we show that, for each \( \Delta \in \mathcal{R}(\mathcal{H}) \), there is a continuous function \( \nu_\Delta : \sigma(A) \to [0,1] \) from the spectrum of \( A \) to the interval \([0,1]\) such that \( \nu_\Delta(f(\lambda)) = \gamma_\Delta(\lambda), \lambda \in \Lambda \), and \( F(\Delta) = \nu_\Delta(A) \).

To prove this, let us consider the function
\[
\nu_\Delta(t) := (\gamma_\Delta \circ f^{-1})(t), \quad \Delta \in \mathcal{R}(\mathcal{H}).
\]
Indeed, by the change of measure principle (page 894, ref. [27]),

\[
\nu_\lambda(f(\lambda)) = \gamma_\lambda(f^{-1}(f(\lambda))) = \gamma_\lambda(\lambda).
\]

we have,

\[
\nu_\lambda(A) = F(\Delta), \quad \forall \Delta \in \mathcal{B}(\mathcal{F}).
\]

Indeed, by the change of measure principle (page 894, ref. [27]),

\[
F(\Delta) = \int_{\lambda} \gamma_\lambda(\lambda) dE_\lambda = \int_{\lambda} \gamma_\lambda(f^{-1}(f(\lambda))) dE_\lambda = \int_{\sigma(A)} \gamma_\lambda(f^{-1}(t)) dE_t = \int_{\sigma(A)} \nu_\lambda(t) dE_t = \nu_\lambda(A)
\]

where \(\sigma(A) = f(\Lambda)\) is the spectrum of \(A\) and \(E\) is the spectral measure corresponding to \(A\) defined by the relation \(E(\Delta) = E(f^{-1}(\Delta)), \Delta \in \mathcal{B}(\sigma(A))\) (see corollary 10, page 902, in Ref. [27]).

For each \(\lambda \in \sigma(A)\), the map \(\nu_\lambda : \mathcal{B}(\mathcal{F}) \to [0,1]\) defines an additive set function. Indeed, let \(\Delta \in \mathcal{B}(\mathcal{F})\) be the disjoint union of the sets \(\Delta_1, \Delta_2 \in \mathcal{B}(\mathcal{F})\). Then,

\[
\int \nu_{(\Delta_1 \cup \Delta_2)}(\lambda) dE_\lambda = F(\Delta_1 \cup \Delta_2) = F(\Delta_1) + F(\Delta_2)
\]

and the continuity of the functions \(\nu_{(\Delta_1)}(\lambda)\) and \(\nu_{(\Delta_2)}(\lambda)\), we get (see theorem 1, page 895, in Ref. [27])

\[
\nu_{(\Delta_1)}(\lambda) + \nu_{(\Delta_2)}(\lambda) = \nu_{(\Delta_1 \cup \Delta_2)}(\lambda), \quad \forall \lambda \in \sigma(A).
\]

Now, we extend \(\nu\) to all \(\mathcal{B}(\mathcal{F})\).

Since \(A\) is the generator of \(\mathcal{P}^W(F)\), for each \(\Delta \in \mathcal{B}(\mathcal{F})\), there exists a Borel function \(\omega_\lambda\) such that.

\[
F(\Delta) = \int_{\sigma(A)} \omega_\lambda(t) dE_t = \int_{\lambda} (\omega_\lambda \circ f)(\lambda) dE_\lambda
\]

Then, we can consider the map \(\tilde{\nu} : \sigma(A) \times \mathcal{B}(\mathcal{F}) \to [0,1]\) defined as follows

\[
\tilde{\nu}_\lambda = \begin{cases} 
\nu_\lambda \quad \text{if} \quad \Delta \in \mathcal{B}(\mathcal{F}) \\
\omega_\lambda \quad \text{if} \quad \Delta \notin \mathcal{B}(\mathcal{F}). 
\end{cases}
\]

Since \(\tilde{\nu}\) coincides with \(\nu\) on \(\mathcal{B}(\mathcal{F})\) it is additive on \(\mathcal{B}(\mathcal{F})\).

In order to prove that \(\tilde{\nu}\) is a weak Markov kernel, let us consider a set \(\Delta \in \mathcal{B}(\mathcal{F})\) which is the disjoint union of the sets \(\{\Delta_i\}_{i \in \mathbb{N}}, \Delta_i \in \mathcal{B}(\mathcal{F})\). Then,

\[
\int \tilde{\nu}_{(\cup_{i=1}^\infty \Delta_i)}(\lambda) dE_\lambda = \int \tilde{\nu}_\lambda dE_\lambda = F(\Delta) = \sum_{i=1}^\infty F(\Delta_i)
\]

\[
= \sum_{i=1}^\infty \int \tilde{\nu}_\lambda dE_\lambda = \int \sum_{i=1}^\infty \tilde{\nu}_\lambda dE_\lambda
\]

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so that, by Corollary 9, page 900, in Ref. [27],
\[
\sum_{i=1}^{\infty} \tilde{V}_\lambda(i) = \tilde{V}_\lambda, \quad E - a.e.,
\]
which implies that \( \tilde{V} : \sigma(A) \times \mathcal{B}(X) \to [0, 1] \) is a weak Markov kernel. In particular \((F,A,\tilde{V})\) is a von Neumann triplet.

Now, we proceed to prove the existence of the Markov kernel \( \mu : \Gamma \times \mathcal{B}(X) \to [0,1] \) such that items 1, 2, and 3 of the theorem are satisfied.

Since \( X \) is Hausdorff locally compact second countable, it is a Polish space (theorem 5.3 in [41]). Then, to each weak Markov kernel \( \tilde{V} : \sigma(A) \times \mathcal{B}(X) \to [0,1] \) such that \((F,A,\tilde{V})\) is a von Neumann triplet, there corresponds a Markov kernel \( \phi : \sigma(A) \times \mathcal{B}(X) \to [0,1] \) such that \((F,A,\phi)\) is a von Neumann triplet [37] [39] [7].

Then, for each \( \lambda \in \mathcal{B}(X) \),
\[
\int \tilde{V}_\lambda(\lambda) dE = F(\lambda) = \int \phi_\lambda(\lambda) dE,
\]
hence,
\[
\phi_\lambda(\lambda) = \tilde{V}_\lambda(\lambda), \quad E - a.e. \tag{15}
\]

Now, let \( \{\Delta_i\}_{i \in \mathbb{N}} \) be an enumeration of \( \mathcal{B}(\mathcal{I}) \). By equation (15), for each \( i \in \mathbb{N} \), there is a set \( N_i \subset \sigma(A) \), \( E(N_i) = 0 \), such that
\[
\phi_\lambda(\lambda) = \tilde{V}_\lambda(\lambda), \quad \lambda \in \sigma(A) - N_i. \tag{16}
\]

Then, for each \( i \in \mathbb{N} \),
\[
\phi_\lambda(\lambda) = \tilde{V}_\lambda(\lambda), \quad \lambda \in \sigma(A) - N \tag{17}
\]
where,
\[
N := \bigcup_{i=1}^{\infty} N_i, \quad E(N) = 0.
\]

Therefore, for almost all \( \lambda \in \sigma(A) \), \( \tilde{V}_\lambda(\lambda) \) is \( \sigma \)-additive on \( \mathcal{B}(\mathcal{I}) \).

Now, we can define the map
\[
\mu_\lambda(\lambda) = \begin{cases} 
\tilde{V}_\lambda(\lambda) & \lambda \in N \\
\phi_\lambda(\lambda) & \lambda \in \sigma(A) - N
\end{cases}
\]
If we put \( \Gamma = \sigma(A) - N \), we have that \( \mu_\lambda(\cdot) : \Gamma \times \mathcal{B}(X) \to [0, 1] \) is a Markov kernel. Therefore, \( \mu_\lambda(\cdot) : \sigma(A) \times \mathcal{B}(X) \to [0, 1] \) is a strong Markov kernel.

Notice that, for each \( \lambda \in \sigma(A) \),
\[
\mu_\lambda(\lambda) = \tilde{V}_\lambda(\lambda)
\]
so that, \( \mu_\lambda \) is continuous for each \( \lambda \in \sigma(A) \) and additive on \( \mathcal{B}(\mathcal{I}) \). We also have,
\[
\mu_\lambda(A) = \phi_\lambda(A) = F(\lambda), \quad \Delta \in \mathcal{B}(\mathcal{I})
\]
We have proved items 1, 2, and 3. Item 4 comes from theorem [3.1]
It remains to prove that $\mu$ is a Feller Markov kernel. By item 1, $\mu_\Delta$ is continuous for each $\Delta \in \mathcal{B}(S)$. Notice that for each open set $O \in \mathcal{B}(X)$, there is a countable family of sets $\Delta_i \in \mathcal{B}(S)$ such that $O = \bigcup_{i=1}^\infty \Delta_i$. Therefore, by theorem 2.2 in Ref. [20], and the continuity of $\mu_\Delta$ for each $\Delta \in \mathcal{B}(S)$, $\lim_{n \to \infty} \lambda_n = \lambda$ implies,

$$\lim_{n \to \infty} \int_X f(x) \, \mu_{\Delta_n}(x) = \int_X f(x) \, \mu_\Delta(x), \quad f \in \mathcal{C}_b(X)$$

where, $\mathcal{C}_b(X)$ is the space of bounded, continuous real functions. Than, $G(\lambda) := \int f(x) \, \mu_\Delta(x)$ is continuous whenever $f$ is continuous and $\mu$ is a Feller Markov kernel.

Finally, we note that $F(\Delta) = \mu_\Delta(A)$ implies the commutativity of $F$ and that ends the proof.

In the proof of theorem 4.1 we have also proved the following theorem.

**Theorem 4.2.** A POVM $F : \mathcal{B}(X) \to \mathcal{B}(\mathcal{H})$ is commutative if and only if, there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0,1]$ and a Markov Kernel $\mu : \mathcal{B}(X) \times \sigma(A) \to [0,1]$ such that

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathcal{B}(X).$$

**Proof.** We have already shown (see equation 14) the existence of a weak Markov kernel $\tilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ which is additive on the the ring $\mathcal{B}(\mathcal{S})$ and such that

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \forall \Delta \in \mathcal{B}(X).$$

Moreover, since $X$ is a Polish space, to each weak Markov kernel $\tilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ there corresponds a Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ such that $(F,A,\mu)$ is a von Neumann triplet [37, 39, 7].

5 Characterization of POVMs which admit strong Feller Markov Kernels

In the last section we proved that each commutative POVM admits a strong Markov kernel $\mu$ such that $\mu_\Delta$ is a continuous function for each $\Delta \in \mathcal{B}(\mathcal{S})$ where, $\mathcal{B}(\mathcal{S})$ is a ring which generates the Borel $\sigma$-algebra $\mathcal{B}(X)$.

In the present section we characterize the commutative POVMs for which the Markov kernel $\mu$, whose existence was proved in theorem 4.2 is such that $\mu_\Delta$ is continuous for each $\Delta \in \mathcal{B}(X)$. Whenever such a Markov kernel exists, we say that the POVM admits a strong Feller Markov kernel. In particular, we prove that a commutative POVM $F$ admits a strong Feller Markov kernel if and only if $F$ is uniformly continuous.
Definition 16. Let \( F : \mathcal{B}(X) \to \mathcal{L}_+^+(\mathcal{H}) \) be a POVM. \( F \) is said to be uniformly continuous at \( \Delta \) if, for any disjoint decomposition \( \Delta = \bigcup_{i=1}^{\infty} \Delta_i \),

\[
\lim_{n \to \infty} \sum_{i=1}^{n} F(\Delta_i) = F(\Delta)
\]

in the uniform operator topology. \( F \) is said uniformly continuous if it is uniformly continuous at each \( \Delta \in \mathcal{B}(X) \).

Notice that the term uniformly continuous derives from the fact that the \( \sigma \)-additivity of \( F \) in the uniform operator topology is equivalent to the continuity in the uniform operator topology. Analogously, the \( \sigma \)-additivity of \( F \) in the weak operator topology is equivalent to the continuity of \( F \) in the weak operator topology [19].

Proposition 2 ([18]). \( F \) is uniformly continuous if and only if,

\[
\lim_{i \to \infty} \|F(\Delta_i)\| = 0
\]

whenever \( \Delta_i \downarrow \emptyset \).

Definition 17. A Markov kernel \( \mu(\cdot) : \Lambda \times \mathcal{B}(X) \to [0,1] \) is said to be strong Feller if \( \mu(\cdot) \) is a continuous function for each \( \Delta \in \mathcal{B}(X) \).

Definition 18. We say that a commutative POVM admits a strong Feller Markov kernel if there exists a strong Feller Markov kernel \( \mu(\cdot) \) such that \( F(\Delta) = \int \mu(\lambda) dE(\lambda) \), where \( E \) is the sharp version of \( F \).

In order to prove the main theorem of the section we need the following lemma.

Lemma 1. Let \( F \) be uniformly continuous. Let \( \mu \) be a weak Markov kernel and \( (F,A,\mu) \) a von Neumann triplet. Suppose that \( \mu(\cdot) \) is continuous for each \( \Delta \in \mathcal{B}(\mathcal{X}) \). Then, for each \( \lambda \in \sigma(A) \), \( \mu(\lambda) \) is \( \sigma \)-additive on \( \mathcal{R}(\mathcal{S}) \).

Proof. Let \( \Delta, \Delta_i \in \mathcal{B}(\mathcal{S}), \Delta_i \cap \Delta_j = \emptyset, \cup_{i=1}^{\infty} \Delta_i = \Delta \). Then,

\[
0 = u - \lim_{n \to \infty} (F(\Delta) - F(\bigcup_{i=1}^{n} \Delta_i)) = u - \lim_{n \to \infty} \int (\lambda) - \sum_{i=1}^{n} \mu(\lambda)) dE(\lambda).
\]

By the uniform continuity of \( F \) and theorem 1, page 895, in Ref. [27], it follows that, \( \forall \varepsilon > 0 \), there exists a number \( \bar{n} \in \mathbb{N} \), such that \( n > \bar{n} \) implies,

\[
\left\| \mu(\lambda) - \sum_{i=1}^{n} \mu(\lambda) \right\|_{\infty} = \left\| \int (\lambda) - \sum_{i=1}^{n} \mu(\lambda)) dE(\lambda) \right\|
\]

(18)

\[
= \|F(\Delta) - F(\bigcup_{i=1}^{n} \Delta_i)\| \leq \varepsilon.
\]

By equation (18),

\[
|\mu(\lambda) - \sum_{i=1}^{n} \mu(\lambda)| \leq \varepsilon, \quad \forall \lambda \in \sigma(A).
\]

\( \square \)
Theorem 5.1. A commutative POVM $F : \mathcal{B}(X) \to \mathcal{L}_q^+(\mathcal{H})$ admits a strong Feller Markov kernel if and only if it is uniformly continuous.

Proof. Suppose $F$ is uniformly continuous. By theorem 4.1 there is a weak Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that $\mu_A(\cdot)$ is continuous for every $A \in \mathcal{B}(\mathcal{H})$ and a self-adjoint operator $A$ such that $(F, A, \mu)$ is a von Neumann triplet. By lemma 11 $\mu$ is $\sigma$-additive on $\mathcal{B}(\mathcal{H})$. Therefore, Charadodory theorem 44 assures that the map $\mu : \sigma(A) \times \mathcal{B}(\mathcal{H}) \to [0, 1]$ can be extended to a map $\tilde{\mu} : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ whose restriction to $\mathcal{B}(\mathcal{H})$ coincides with $\mu$ and such that $\tilde{\mu}_A(\lambda)$ is a probability measure for each $\lambda \in \sigma(A)$. Now we prove that $\tilde{\mu}$ is a Markov kernel such that $F(\Delta) = \tilde{\mu}_\Delta(A)$ and that $\tilde{\mu}_A$ is continuous for each $A \in \mathcal{B}(X)$. We proceed by steps.

1) Open sets. Each open set $G$ is the union of a countable family of sets in $\mathcal{H}$, i.e., $G = \bigcup_{i=1}^\infty \Delta_i$, $\Delta_i \in \mathcal{H}$. Let us define the set $G_n := \bigcup_{i=1}^n \Delta_i$. Therefore, $G_n \uparrow G$. Moreover, $\mu_{G_n}$ is continuous for each $n \in \mathbb{N}$, and

$$u - \lim_{n \to \infty} F(G_n) = F(G).$$

Then,

$$F(G) = u - \lim_{i \to \infty} F(G_i) = u - \lim_{i \to \infty} \int \tilde{\mu}_{G_i}(\lambda) dE_\lambda.$$ 

By the uniform continuity of $F$, it follows that, $\forall \varepsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\|\tilde{\mu}_{G_n}(\lambda) - \tilde{\mu}_{G_m}(\lambda)\|_\infty = \|\int [\tilde{\mu}_{G_n}(\lambda) - \tilde{\mu}_{G_m}(\lambda)] dE_\lambda\|$$ \hspace{1cm} (19)

$$= \|F(G_n) - F(G_m)\| \leq \varepsilon.$$ 

By equation (19),

$$|\tilde{\mu}_{G_n}(\lambda) - \tilde{\mu}_{G_m}(\lambda)| \leq \varepsilon, \ \forall \lambda \in \sigma(A).$$ \hspace{1cm} (20)

Since $\tilde{\mu}_A(\lambda)$ is a probability measure,

$$\lim_{i \to \infty} \tilde{\mu}_{G_i}(\lambda) = \tilde{\mu}_G(\lambda), \ \forall \lambda \in \sigma(A).$$

By equation (20), the convergence is uniform and this proves the continuity of $\tilde{\mu}_G$. Moreover,

$$F(G) = \lim_{i \to \infty} F(G_i) = \lim_{i \to \infty} \int \tilde{\mu}_{G_i}(\lambda) dE_\lambda = \int \tilde{\mu}_G(\lambda) dE_\lambda = \tilde{\mu}_G(A).$$

2) $G_\delta$ sets. For each $G_\delta$ set there exists a family of open sets $\{G_i\}_{i \in \mathbb{N}}$, $G_\delta \subset G_i$, such that $\bigcap_{i=1}^\infty G_i = G_\delta$. Then, by the uniform continuity of $F$,

$$F(G_\delta) = F(\bigcap_{i=1}^\infty G_i) = u - \lim_{n \to \infty} F(\bigcap_{i=1}^n G_i) = u - \lim_{n \to \infty} F(G_n)$$

where, $G_n := \bigcap_{i=1}^n G_i$ and $G_n \uparrow G_\delta$.

By theorem 1, page 895, in Ref. [27], it follows that, $\forall \varepsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\|\mu_{G_n}(\lambda) - \mu_{G_m}(\lambda)\|_{\infty} = \|\int [\mu_{G_n}(\lambda) - \mu_{G_m}(\lambda)] dE_\lambda\| \leq \varepsilon.$$ \hspace{1cm} (21)
Since $\tilde{\mu}_G (\lambda)$ is a probability measure for each $\lambda \in \sigma(A)$,

$$\lim_{i \to \infty} \tilde{\mu}_{G_i} (\lambda) = \tilde{\mu}_{G_\delta} (\lambda).$$

By equation (21) the convergence is uniform and then $\tilde{\mu}_{G_\delta}$ is continuous. Moreover,

$$F(G_\delta) = \lim_{i \to \infty} F(\tilde{G}_i) = \lim_{i \to \infty} \int \tilde{\mu}_{G_\delta} (\lambda) dE_\lambda = \int \tilde{\mu}_{G_\delta} (\lambda) dE_\lambda = \tilde{\mu}_{G_\delta} (A).$$

3) Borel sets. We use transfinite induction \[42, 25\]. Let $G_0$ be the family of open sets in $X$, $\omega$ the first uncountable ordinal and $G_\alpha$, $\alpha < \omega_1$ the Borel hierarchy (see page 236 in Ref. \[42\]). In particular, $G_1 = G_\delta$, $G_2 = G_\delta \sigma$, $G_3 = G_\delta \sigma \delta \ldots$ and $G_\alpha = (\cup_{\beta < \alpha} G_\beta)_\sigma$ for each limit ordinal $\alpha$. By means of the same reasoning that we used in items 1 and 2, one can prove the continuity of $\tilde{\mu}_\lambda$ as well as that $\tilde{\mu}_\lambda (A) = F(\Delta)$ whenever $\Delta$ is of the kind $G_\delta, G_\delta \sigma, G_\delta \sigma \delta \ldots$. Analogously, if $\tilde{\mu}_\lambda$ is continuous for each $\Delta \in G_\alpha$ then, $\tilde{\mu}_\lambda$ is continuous for each $\Delta$ in $G_{\alpha+1}$ and $\tilde{\mu}_\lambda (A) = F(\Delta)$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in $G_\alpha$ and the reasoning in items 1 and 2 can be used. If $\alpha$ is a limit ordinal and $\tilde{\mu}_\lambda$ is continuous for each $\Delta \in G_\alpha$, $\beta < \alpha$, then, $\tilde{\mu}_\lambda$ is continuous for each $\Delta \in G_\alpha = (\cup_{\beta < \alpha} G_\beta)_\sigma$ and $\tilde{\mu}_\lambda (A) = F(\Delta)$. Indeed, each set in $G_\alpha$ is the countable union of sets in $\cup_{\beta < \alpha} G_\beta$ and the reasoning used in item 1 can be used. Therefore, by transfinite induction, $\tilde{\mu}_\lambda$ is continuous for each $\Delta \in \cup_{\alpha < \omega_1} G_\alpha = \mathcal{B}(X)$ \[42\] and $\tilde{\mu}_\lambda (A) = F(\Delta)$.

In order to prove the second part of the theorem we show that the existence of a strong Feller Markov kernel implies the uniform continuity of $F$. Suppose that there exists a strong Feller Markov kernel $\mu$ such that $F(\Delta) = \mu_\lambda (\lambda)$. Since $\mu$ is a Markov kernel it is $\sigma$-additive. Then,

$$\lim_{n \to \infty} (\mu_\lambda (\lambda) - \sum_{i=1}^{n} \mu_\lambda (\lambda)) = 0, \quad \lambda \in \sigma(A).$$

where, $\Delta, \Delta_i \in \mathcal{B}(X)$, $\cup_{i=1}^{\infty} \Delta_i = \Delta$.

By hypothesis,

$$\mu_\lambda (\lambda) - \sum_{i=1}^{n} \mu_\lambda (\lambda) \in \mathcal{G}(\sigma(A)), \quad \forall n \in \mathbb{N}.$$

Then, by theorem \[B1\] in appendix B,

$$u - \lim_{n \to \infty} (\mu_\lambda (\lambda) - \sum_{i=1}^{n} \mu_\lambda (\lambda)) = 0.$$

By theorem 1, page 895, in Ref. \[27\], $\|F(\Delta)\| = \|\mu_\lambda\|_\infty$, hence

$$\lim_{n \to \infty} \|F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\| = \lim_{n \to \infty} \|\mu_\lambda - \sum_{i=1}^{n} \mu_\lambda\|_\infty = 0.$$

which proves that $F$ is uniformly continuous. \[\square\]
Example 2. Let us consider the following unsharp position observable

\[
Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x) dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}),
\]

\[
\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y) f(y) dy, \quad x \in [0,1]
\]

where, \( f \) is a bounded, continuous function such that \( f(y) = 0, \ y \notin [0,1] \) and

\[
\int_{[0,1]} f(y) dy = 1,
\]

and \( Q_x \) is the spectral measure corresponding to the position operator

\[
Q : L^2([0,1]) \rightarrow L^2([0,1])
\]

\[
\psi(x) \mapsto (Q\psi)(x) := x\psi(x)
\]

Notice that, for each \( \Delta \in \mathcal{B}(\mathbb{R}) \), \( \mu_\Delta : [0,1] \rightarrow [0,1] \) is continuous. Indeed, by the uniform continuity of \( f \), for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |x-x'| \leq \delta \) implies \( |f(x-y) - f(x'-y)| \leq \varepsilon \), for each \( y \). Therefore,

\[
|\mu_\Delta(x) - \mu_\Delta(x')| = \left| \int_{\mathbb{R}} \chi_\Delta(x-y) f(y) dy - \int_{\mathbb{R}} \chi_\Delta(x'-y) f(y) dy \right| \\
= \left| \int_{\Delta} [f(x-y) - f(x'-y)] dy \right| \leq \varepsilon \int_{\Delta} dy \leq 2\varepsilon
\]

By theorem \([5,1]\) and the continuity of \( \mu_\Delta \), \( \Delta \in \mathcal{B}(\mathbb{R}) \), \( Q^f \) is uniformly continuous. That can be proved as follows. Suppose \( \Delta_i \downarrow \Delta \) and \( f(y) \leq M, \ y \in \mathbb{R} \). Since, for each \( x \in [0,1] \),

\[
\mu_{\Delta_i-\Delta}(x) = \int_{\Delta_i-\Delta} f(x-y) dy \leq M \int_{(\Delta_i-\Delta)\cap[-1,1]} dx
\]

we have that, for each \( \psi \in \mathcal{H} \), \( |\psi|^2 = 1 \),

\[
\langle \psi, Q^f(\Delta_i-\Delta)\psi \rangle = \int_{[0,1]} \mu_{\Delta_i-\Delta}(x) |\psi|^2(x) dx \leq M \int_{(\Delta_i-\Delta)\cap[-1,1]} dx
\]

which proves the uniform continuity of \( Q^f \).

In the case of uniformly continuous POVMs, we can prove a necessary condition for the norm-1-property.

Definition 19 (\([33]\)). A POVM \( F \) has the norm-1-property if \( \|F(\Delta)\| = 1 \), for each \( \Delta \in \mathcal{B}(X) \) such that \( F(\Delta) \neq 0 \).

Theorem 5.2. Let \( F \) be uniformly continuous. Then, \( F \) has the norm-1-property only if \( \|F(\{x\})\| \neq 0 \) for each \( x \in \sigma(F) \).

Proof. We proceed by contradiction. Suppose that \( F \) has the norm-1 property and that there exists \( x \in \sigma(F) \), such that \( F(\{x\}) = 0 \). Let \( \Delta_i \) be a decreasing family of open sets such that, \( \cap_{i=1}^\infty \Delta_i = \{x\} \). The existence of such family is assured by the local compactness of \( X \). (See theorem 29.2 in \([46]\).) Since \( x \in \sigma(F) \) and \( x \in \Delta_i \),
we have $F(\Delta_i) \neq 0$ for any $i \in \mathbb{N}$ (see Definition 5) and, by the norm-1 property, $\|F(\Delta_i)\| = 1$. By the uniform continuity of $F$ and proposition 2,

\[
1 = \lim_{i \to \infty} \|F(\Delta_i)\| = \lim_{i \to \infty} \|F(\Delta_i) - F(\{x\}) + F(\{x\})\|
\leq \lim_{i \to \infty} \|F(\Delta_i - \{x\})\| + \|F(\{x\})\| = 0.
\]

\[\square\]

Example 3. Let $Q^f$ be as in example 2. Theorem 5.2 implies that $Q^f$ cannot have the norm-1 property. Indeed, for each $\lambda \in \mathbb{R}$,

\[
Q^f(\{\lambda\})\psi = \lim_{i \to \infty} Q^f(\{\lambda, \lambda_i\})\psi = \lim_{i \to \infty} \mu(\lambda, \lambda_i)(x)\psi(x) = 0, \quad \forall \psi \in \mathcal{H}
\]
where, $\lambda, \lambda_i \in \mathbb{R}, \lambda_i \to \lambda$.

We refer to [18] for an analysis of the relevance of theorem 5.2 to the problem of localization of massless relativistic particles.

6 Absolutely continuous POVMs

In the present section, we prove that absolutely continuous commutative POVMs admit a strong Feller Markov kernel. Then, we apply the result to the case of the unsharp position observable.

**Definition 20.** [51, 52] A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ is absolutely continuous with respect to a measure $\nu : \mathcal{B}(X) \to [0, 1]$ if there exists a positive number $c$ such that $\|F(\Delta)\| \leq c \nu(\Delta)$, for each $\Delta \in \mathcal{B}(X)$.

**Theorem 6.1.** Let $F$ be absolutely continuous with respect to a finite measure $\nu$. Then, $F$ is uniformly continuous.

**Proof.** Suppose $\Delta_i \downarrow \emptyset$. We have

\[
\lim_{i \to \infty} \|F(\Delta_i)\| \leq c \lim_{i \to \infty} \nu(\Delta_i) = 0.
\]

Proposition 2 ends the proof. \[\square\]

**Corollary 2.** Let $F$ be absolutely continuous with respect to a finite measure $\nu$. Then, $F$ is commutative if and only if there exist a self-adjoint operator $A$ and a strong Feller Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that:

\[
F(\Delta) = \mu_\Delta(A), \quad \Delta \in \mathcal{B}(X)
\]

**Proof.** By theorem 6.1 $F$ is uniformly continuous. Then, theorem 5.1 implies the thesis. \[\square\]
Example 4. Let us consider the unsharp position operator defined as follows.

\[ Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x) dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (24) \]

where, \( f \) is a positive, bounded, Borel function such that \( f(x) = 0, x \notin [0,1] \),

\[ \int_{[0,1]} f(x) dx = 1, \]

and \( Q_x \) is the spectral measure corresponding to the position operator

\[ Q : L^2([0,1]) \to L^2([0,1]) \]

\[ \psi(x) \mapsto Q\psi := x\psi(x) \]

\( Q^f \) is absolutely continuous with respect to the measure

\[ \nu(\Delta) = M \int_{\Delta \cap [-1,1]} dx. \]

Indeed, for each \( \psi \in \mathcal{H}, |\psi|^2 = 1 \),

\[ \langle \psi, Q^f(\Delta)\psi \rangle = \int_{[0,1]} \mu_\Delta(x) |\psi|^2(x) dx \leq M \int_{\Delta \cap [-1,1]} dx \]

where, the inequality

\[ \mu_\Delta(x) = \int_{\Delta} f(x-y) dy \leq M \int_{\Delta \cap [-1,1]} dx \]

has been used.

Therefore, by theorem 6.1, \( Q^f(\Delta) \) is uniformly continuous.

### 6.1 Unsharp Position Observable

In the present subsection, we study an important kind of absolutely continuous POVMs, the unsharp position observables obtained as the marginals of a covariant phase space observable.

In the following \( \mathcal{H} = L^2(\mathbb{R}) \), \( Q \) and \( P \) denote position and momentum observables respectively and \( * \) denotes convolution, i.e. \( (f \ast g)(x) = \int f(y)g(x-y)dy \).

Let us consider the joint position-momentum POVM [11, 21, 24, 30, 36, 48, 52, 53]

\[ F(\Delta \times \Delta') = \int_{\Delta \times \Delta'} U_{q,p} \gamma U_{q,p}^* dq dp \]

where, \( U_{q,p} = e^{-ipQ}e^{ipQ} \) and \( \gamma = |f\rangle\langle f|, f \in L^2(\mathbb{R}), \|f\|_2 = 1 \).

The marginal

\[ Q^f(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (1_\Delta \ast |f|^2)(x) dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (25) \]
is an unsharp position observable. Notice that the map \( \mu_\Delta(x) := 1_\Delta \ast |f(x)|^2 \) defines a Markov kernel.

Moreover, \( Q^f \) is absolutely continuous with respect to the Lebesgue measure. Indeed,

\[
    Q^f(\Delta) = F(\Delta \times \mathbb{R}) = \int_{\Delta \times \mathbb{R}} U_{q,p} \, \gamma U_{q,p}^* \, dq \, dp = \int_{\Delta} dq \int_{\mathbb{R}} U_{q,p} \, \gamma U_{q,p}^* \, dp = \int_{\Delta} \hat{Q}(q) \, dq \leq \int_{\Delta} 1 \, dq
\]

where,

\[
    \hat{Q}(q) = \int_{\mathbb{R}} U_{q,p} \, \gamma U_{q,p}^* \, dp.
\]

Although \( Q^f \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \), it is not uniformly continuous. That does not contradict theorem 6.1 since the Lebesgue measure on \( \mathbb{R} \) is not finite. Anyway, \( Q^f \) is uniformly continuous on each Borel set \( \Delta \) with finite Lebesgue measure.

Now, we show that \( Q^f \) is not in general uniformly continuous. We give the details of the following particular case.

**Example 5** (Optimal Phase Space Representation). If we choose

\[
    f^2(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2l^2}}, \quad l \in \mathbb{R} - \{0\}.
\]

in (25), we get an optimal phase space representation of quantum mechanics [48]. In this case,

\[
    Q^f(\Delta) = \int_{-\infty}^{\infty} \left( \int_{\Delta} |f(x-y)|^2 \, dy \right) \, dQ_x = \frac{1}{l \sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{\Delta} e^{-\frac{(x-y)^2}{2l^2}} \, dy \right) \, dQ_x = \int_{-\infty}^{\infty} \mu_\Delta(x) \, dQ_x
\]

where,

\[
    \mu_\Delta(x) = \frac{1}{l \sqrt{2\pi}} \int_{\Delta} e^{-\frac{(x-y)^2}{2l^2}} \, dy
\]

defines a Markov kernel.

In order to prove that \( Q^f \) is not uniformly continuous we consider the family of sets \( \Delta_i = (-\infty, a_i) \), \( \lim_{i \to -\infty} a_i = -\infty \) such that \( \Delta_i \downarrow \emptyset \), and prove that \( \lim_{i \to -\infty} \|Q^f(\Delta_i)\| = 1 \). For each \( i \in \mathbb{N} \),

\[
    \lim_{x \to -\infty} \mu_\Delta(x) = \lim_{x \to -\infty} \frac{1}{l \sqrt{2\pi}} \int_{\Delta_i} e^{-\frac{(x-y)^2}{2l^2}} \, dy = \lim_{x \to -\infty} \frac{1}{l \sqrt{2\pi}} \int_{(-\infty, a_i-x)} e^{-\frac{y^2}{2l^2}} \, dy = \frac{1}{l \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2l^2}} \, dy = 1.
\]

Now, we prove that \( \|Q^f(\Delta_i)\| = 1 \), \( i \in \mathbb{N} \). Indeed, if

\[
    \psi_n = \chi_{[-n,-n+1]}(x),
\]
\[
\lim_{n \to \infty} (\psi_n, Q^f(\Delta_i) \psi_n) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \mu_{\Delta_i}(x) |\psi_n(x)|^2 \, dx \\
= \lim_{n \to \infty} \int_{-n-n+1}^{-n} \mu_{\Delta_i}(x) \, dx = 1.
\]

Since, for each \( \Delta \in \mathcal{B}(\mathbb{R}) \), \( \|Q^f(\Delta)\| \leq 1 \), equation (27) implies that \( \|Q^f(\Delta_i)\| = 1 \), for each \( i \in \mathbb{N} \). Hence, \( \lim_{n \to \infty} \|Q^f(\Delta_i)\| = 1 \) and \( Q^f \) cannot be uniformly continuous.

It is worth noticing that although \( Q^f \) is not uniformly continuous, \( \mu_{\Delta_i} \) is continuous for each interval \( \Delta \in \mathcal{B}(\mathbb{R}) \). Indeed,

\[
|\mu_{\Delta_i}(x) - \mu_{\Delta_i}(x')| = \frac{1}{\sqrt{2\pi}} \left| \int_{\Delta_i} e^{-\frac{(x-y)^2}{2\sigma^2}} \, dy - \int_{\Delta_i} e^{-\frac{(x'-y)^2}{2\sigma^2}} \, dy \right| \\
= \frac{1}{\sqrt{2\pi}} \left| \int_{\Delta_i} e^{-\frac{y^2}{2\sigma^2}} - \int_{\Delta_i} e^{-\frac{(y-x)^2}{2\sigma^2}} \, dy \right| \leq \frac{1}{\sqrt{2\pi}} \left| \int_{\Delta_i} e^{-\frac{(y-x)^2}{2\sigma^2}} \, dy \right|
\]

where,

\[\Delta_i = \{ z \in \mathbb{R} | z = y-x, y \in \Delta \}, \quad \Delta_i' = \{ z \in \mathbb{R} | z = y-x', y \in \Delta \}\]

and,

\[\overline{\Delta} = (\Delta_i - \Delta_i') \cup (\Delta_i' - \Delta_i)\]

Therefore, \( |x-x'| \leq \varepsilon \) implies,

\[
|\mu_{\Delta_i}(x) - \mu_{\Delta_i}(x')| \leq \frac{1}{\sqrt{2\pi}} \left| \int_{\Delta_i} e^{-\frac{y^2}{2\sigma^2}} \, dy \right| \leq \frac{1}{\sqrt{2\pi}} \int_{\Delta_i} dy = \frac{1}{\sqrt{\pi \varepsilon}}.
\]

7 Conclusions

We already pointed out that although the set of commutative POVMs is a particular subset of the set of POVMs, the commutative POVMs are relevant both from the mathematical and the physical viewpoint. It is well known that they can be interpreted as the smearing of real PVMs, \( E \), and that the smearing can be realized by means of Markov kernels, \( \mu \). Anyway no characterization of the smearing (the Markov kernel) is known. In the present paper such a characterization is given and its mathematical and physical implications are analyzed. For example we answered the following questions: 1) Can the smearing be realized by means of a Feller Markov kernel?, 2) What can we say about the continuity of the functions \( \mu_{\Delta_i} \)?, 3) Can the smearing be realized by means of a strong Feller Markov kernel?, 4) What is the physical interpretation of the smearing when it is realized by means of a strong Feller Markov kernel? 5) Is the smearing able to distinguish the points in the spectrum of the PVM \( E \)? 5) Are there physical examples that can be used as an illustration of items 1) to 5) above? In order to answer such questions, we had to provide a new and stronger characterization of a commutative POVM \( F \) as the smearing of a real PVM \( E \).
Appendix B: Sequences of continuous functions

The following theorem is due to Dini. We give a proof based on the use of sequences.

**Theorem B1.** Let \( \{f_n(\lambda)\}_{n \in \mathbb{N}} \) be a non increasing sequence of continuous functions defined on a compact set \( B \subset [0, 1] \) with values in \([0, 1]\) and such that \( f_n(\lambda) \to 0 \) point-wise. Then, \( f_n(\lambda) \to 0 \) uniformly.

**Proof.** Since \( f_{n+1}(\lambda) \leq f_n(\lambda) \) for each \( \lambda \in B \), we have \( \|f_{n+1}\|_\infty \leq \|f_n\|_\infty \). If \( \|f_n\|_\infty \to 0 \) clearly \( f_n(\lambda) \to 0 \) uniformly.

Then, suppose \( \|f_n\|_\infty \to a > 0 \). Since \( \|f_{n+1}\|_\infty \leq \|f_n\|_\infty \), we have \( \|f_n\|_\infty \geq a \), for each \( n \in \mathbb{N} \).

Let \( \lambda_n \) be such that \( f_n(\lambda_n) = \|f_n\|_\infty \). Since \( \{\lambda_n\} \) is a bounded sequence of real numbers, there exists a convergent subsequence \( \{\lambda_{n_k}\}_{k \in \mathbb{N}} \). Let \( \beta \) be its limit, i.e., \( \beta := \lim_{k \to \infty} \lambda_{n_k} \). The compactness of \( B \) assures that \( \beta \in B \). Moreover, \( \lim_{k \to \infty} f_{n_k}(\lambda_{n_k}) = a \).
Let us consider the sequence of numbers \( f_n(\beta) \). We prove that \( f_n(\beta) \geq a \) for each \( k \in \mathbb{N} \). We proceed by contradiction. Suppose that there exists \( \bar{k} \in \mathbb{N} \) such that \( f_{\bar{k}}(\beta) < a \). Then, there exists a neighborhood \( I(\beta) \) of \( \beta \) such that \( f_n(\lambda) < a \) for each \( \lambda \in I(\beta) \). Moreover, since \( \lambda_{\bar{k}} \to \beta \), there exists \( l \in \mathbb{N} \) such that \( k > l \) implies \( \lambda_{\bar{k}} \in I(\beta) \). Take \( k > \max\{\bar{k}, l\} \). Then, \( \lambda_{\bar{k}} \in I(\beta) \) and \( f_{\bar{k}}(\lambda) \leq f_n(\lambda) \), for each \( \lambda \in B \). Therefore,
\[
f_{\bar{k}}(\lambda_{\bar{k}}) \leq f_n(\lambda_{\bar{k}}) < a
\]
which contradicts the fact that \( f_n(\lambda_{\bar{k}}) = \|f_n\|_{\infty} \geq a \), for each \( k \in \mathbb{N} \).

We have proved that \( f_n(\beta) \geq a \), for each \( k \in \mathbb{N} \). This implies that \( \lim_{k \to \infty} f_n(\beta) \geq a \) and contradicts one of the hypothesis of the lemma, i.e., \( \lim_{n \to \infty} f_n(\lambda) = 0 \) for each \( \lambda \in B \).

\[\square\]

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