Entropy production and folding of the phase space in chaotic dynamics

Eugen Mihailescu

Abstract

We study the entropy production of Gibbs (equilibrium) measures for chaotic dynamical systems with folding of the phase space. The dynamical chaotic model is that generated by a hyperbolic non-invertible map $f$ on a general basic (possibly fractal) set $\Lambda$; the non-invertibility creates new phenomena and techniques than in the diffeomorphism case. We prove a formula for the entropy production, involving an asymptotic logarithmic degree, with respect to the equilibrium measure $\mu_{\phi}$ associated to the potential $\phi$. This formula helps us calculate the entropy production of the measure of maximal entropy of $f$. Next for hyperbolic toral endomorphisms, we prove that all Gibbs states $\mu_{\phi}$ have non-positive entropy production $e_f(\mu_{\phi})$. We study also the entropy production of the inverse Sinai-Ruelle-Bowen measure $\mu^{-}$ and show that for a large family of maps, it is strictly negative, while at the same time the entropy production of the respective (forward) Sinai-Ruelle-Bowen measure $\mu^{+}$ is strictly positive.

Mathematics Subject Classification 2000: 37D35, 37D45, 82C05, 37A60, 82B05.

Keywords: Folding entropy, Jacobian of an invariant probability, Gibbs states for hyperbolic non-invertible maps, entropy production, SRB and inverse SRB measures, stationary states.

1 Entropy production. Outline of main results.

In statistical mechanics, one concerns himself with the stationary (steady) states, which are probability measures on the phase space, invariant under time evolution. The study of such states can be done with the help of dynamical systems and ergodic theory (for instance [2], [3], [4], [18], [20], etc.) The fundamental postulate of statistical mechanics (see [3]) says that a physical system in thermodynamical equilibrium is described by Gibbs measures. In the nonequilibrium scenario, once a system is kept out of equilibrium and subjected to non-Hamiltonian forces, the energy is in general not conserved ([18]); so we couple it with a large external system called a thermostat, and record the entropy changes. It is thus justified to have a notion of entropy production, as a measure of the average differences in the entropy of the system over time. Certain nonequilibrium steady states are described also by Gibbs states, but for a different problem (see [18]). From a mathematical point of view, Ruelle identifies in [20] (see also [19] and [18]) several types of entropy productions given by: i) a diffeomorphism $f$ of a manifold $M$; ii) an endomorphism $f$ on $M$, i.e a non-invertible smooth map $f$; here the folding of $M$ by $f$ will itself contribute to the entropy
a diffusion model given by a map $f$ restricted to a neighbourhood of a compact invariant set $X \subset M$. Whether entropy production of a state is positive or not, is not clear a priori.

In this paper we are concerned with the case when $f$ is a smooth endomorphism on a manifold $M$, having a compact invariant set $\Lambda \subset M$. Since hyperbolicity plays an important role in modelling time evolutions in statistical mechanics (for instance \cite{18}, \cite{20}, etc.), we shall assume that the endomorphism $f$ is hyperbolic in the sense of \cite{21}, i.e that there exists a continuous splitting of the tangent bundle over the inverse limit $\hat{\Lambda}$ ($\hat{\Lambda}$ being the space of past trajectories of points in $\Lambda$), into stable and unstable directions; $f$ is not assumed expanding. This implies that we have stable directions and local stable manifolds of type $W^s_r(x), x \in \Lambda$, and unstable directions and local unstable manifolds of type $W^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$. Thus through a given point $x$ there may pass many (even uncountably many) local unstable manifolds corresponding to different prehistories of $x$ in $\hat{\Lambda}$. This follows from the non-invertibility of $f$.

For systems given by Anosov diffeomorphisms, or for diffeomorphisms having a hyperbolic attractor, we have the existence of Sinai-Ruelle-Bowen (SRB) measures, which are natural invariant measures in the sense that they describe the distribution of trajectories of Lebesgue-almost all points in a neighbourhood of the attractor. As was shown by Sinai, the SRB measure of an Anosov diffeomorphism $f$, is in fact the Gibbs state of a Holder potential $\Phi^u$, where $\Phi^u = -\log |\det Df_u|$ (the unstable potential).

For a $C^2$ diffeomorphism $f$, the entropy production of an arbitrary $f$-invariant probability measure $\mu$ is defined (see \cite{20}) as $e_f(\mu) = -\int \log |\det (Df)(x)|d\mu(x)$. If $\mu$ is an SRB state, then Ruelle proved in \cite{20} that $e_f(\mu) \geq 0$; moreover if the SRB state $\mu$ has no vanishing Lyapunov exponents and if $e_f(\mu) = 0$, then $\mu$ must be absolutely continuous with respect to the Lebesgue measure (see \cite{20}, \cite{18}; and \cite{8} for a characterization of invariant absolutely continuous measures). SRB measures exist also for diffeomorphisms having Axiom A attractors, as shown by Ruelle (see for instance \cite{4}). Moreover SRB measures do exist also for Anosov endomorphisms and for endomorphisms with hyperbolic attractors, and they are equal to the equilibrium measures of the respective unstable potentials on the inverse limit spaces (see \cite{16}).

For a non-invertible smooth map $f$ on a Riemannian manifold $M$ and an $f$-invariant probability $\mu$ on $M$, Ruelle defined in \cite{20} the entropy production of $\mu$ by:

$$e_f(\mu) := H_f(\mu) - \int \log |\det (Df)(x)|d\mu(x),$$

where $H_f(\mu)$ is called the folding entropy of $\mu$ with respect to $f$. $H_f(\mu)$ is defined as the conditional entropy $H_{\mu}(\epsilon/f^{-1}\epsilon)$, of $\epsilon$ with respect to $f^{-1}\epsilon$, where $\epsilon$ is the single point partition.

For example we can obtain stationary measures $\mu$, with respect to $f$, as weak limits (when $n \to \infty$) of averages of type

$$\frac{1}{n} \sum_{k=0}^{n-1} f^k \rho,$$

where $\rho$ is an absolutely continuous probability with respect to Lebesgue measure, with density $\bar{\rho}$. For such $f$-invariant limit measures $\mu$, Ruelle showed that the entropy production is non-negative (\cite{20}). There do exist in fact dynamical systems from physics presenting non-invertibility (see for
example [4], [20], [18]). Hyperbolicity also appears to have physical meaning and it may be used as an approximation for certain physical phenomena (see for example [18]). The study of dynamics of hyperbolic endomorphisms, and of their characteristics different from diffeomorphisms, appeared also in [1], [21], [10], [12], [15], etc.

From the above, it is then justified to study the entropy production of equilibrium measures of Holder potentials for non-invertible smooth maps \( f \) on basic sets \( \Lambda \) on which \( f \) is hyperbolic and transitive; here by basic set (or locally maximal set [6]) we mean a compact \( f \)-invariant set \( \Lambda \) s.t \( \Lambda = \cap_{n \in \mathbb{Z}} f^n(U) \), for a neighbourhood \( U \) of \( \Lambda \). Our endomorphism \( f \) is not assumed expanding. The main results of the paper are the following:

In Theorem 1 we give a precise estimate for the Jacobian (in the sense of Parry, [14]) of the equilibrium measure \( \mu_\phi \) associated to an arbitrary Holder potential \( \phi \), with respect to the iterate \( f^n \). This estimate is independent of \( n \) and will allow us to express the folding entropy of \( \mu_\phi \) with respect to \( f \). Next we will describe the folding entropy of \( \mu_\phi \) as the limit of the weighted integral, of the logarithm of the degree function of \( f^n \) with respect to \( \mu_\phi \) on \( \Lambda \). In this way in Theorem 2 we give a formula for the entropy production of \( \mu_\phi \) in terms of an ”asymptotic logarithmic degree” (with respect to \( \mu_\phi \)) minus the integral of the Jacobian with respect to the Riemannian metric; the asymptotic logarithmic degree takes into consideration only those \( n \)-preimages (i.e preimages with respect to \( f^n \)) which behave well with respect to \( \phi \). In Corollary 2 we will use the formula proved in Theorem 2 in order to calculate the folding entropy of the measure of maximal entropy.

We investigate next the case of a hyperbolic toral endomorphism on \( \mathbb{T}^k \) and its Gibbs measures associated to various Holder potentials. We prove in Corollary 1 that in this setting, the entropy production of any equilibrium measure of a Holder potential is non-positive.

In [11], we introduced an inverse SRB measure \( \mu^- \) which has physical relevance since it gives the distribution of past trajectories with respect to the endomorphism \( f \), for Lebesgue almost all points in a neighbourhood of a hyperbolic repellor. This unique inverse SRB measure is not just the SRB measure for \( f^{-1} \), since our map \( f \) is non-invertible in general. We proved that in fact \( \mu^- \) is the equilibrium measure of the stable potential (with respect to the forward system), and that it is the only invariant probability having absolutely continuous conditional measures on local stable manifolds. Here we will show in Theorem 3 that for perturbations of hyperbolic toral endomorphisms, the entropy production of \( \mu^- \) is strictly negative unless \( \mu^- \) is equal to the (forward) SRB measure \( \mu^+ \) of the endomorphism \( f \), in which case both are absolutely continuous. In Corollary 4 a) we show that most maps in a neighbourhood of a hyperbolic toral endomorphism have inverse SRB measures with negative entropy production. And we actually construct in Corollary 4 b) a family of perturbations of hyperbolic toral endomorphisms, whose respective inverse SRB measures have negative entropy production. In particular an endomorphism with negative entropy production, will not be a stationary measure obtained as a weak limit of averages of iterates of absolutely continuous measures as in [2]. In this way we find certain chaotic (hyperbolic) systems with folding of phase space, and Gibbs states for them having negative entropy production.

Several interesting and important results from statistical physics point towards the profound relationship between entropy production and the time arrow/irreversibility, and also the possibility
of negative entropy production on short time scales (for exp. [3], [5], [7], [18], [23], etc.) However our results are abstract mathematical ones, and we do not investigate here possible physical implications, if any.

2 Main results and proofs.

For the rest of the paper let us fix a smooth (say $C^2$) non-invertible map $f : M \to M$ defined on a compact Riemannian manifold and let $\Lambda$ be a fixed basic set of $f$, i.e there exists some neighbourhood $U$ of $\Lambda$ with $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$. Assume also that $f$ is transitive and hyperbolic on $\Lambda$. Sometimes the set $\Lambda$ may be the whole manifold as in the case of Anosov endomorphisms (for example for hyperbolic toral endomorphisms). However in general $\Lambda$ may not be totally invariant, i.e we do not always have $f^{-1}(\Lambda) = \Lambda$.

Here hyperbolicity is understood in the sense of endomorphisms (i.e non-invertible maps) (see [21]), i.e there exists a continuous splitting of the tangent bundle into stable and unstable directions, over the inverse limit $\hat{\Lambda}$ consisting of sequences of consecutive preimages,

$$\hat{\Lambda} = \{ \hat{x} = (x, x_{-1}, x_{-2}, \ldots) \text{ with } x_{-i} \in \Lambda, f(x_{-i}) = x_{-i+1}, i \geq 1 \}$$

For any element $\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{\Lambda}$ we have a stable direction $E^s_{\hat{x}}$ (which depends only on $x$) and an unstable direction $E^u_{\hat{x}}$. Consequently there exists a small $r > 0$ so that we can construct local stable and local unstable manifolds, $W^s_r(x)$ and $W^u_r(\hat{x})$ for any $\hat{x} \in \hat{\Lambda}$. We shall also denote

$$Df_s(x) := Df|_{E^s_x}, x \in \Lambda \text{ and } Df_u(\hat{x}) := Df|_{E^u_{\hat{x}}}, \hat{x} \in \hat{\Lambda} \quad (3)$$

The endomorphism $f$ is assumed to have stable directions too, so it is non-expanding. More about hyperbolicity for endomorphisms can be found in [21], [13], etc. When the map is not invertible, there appear significantly different phenomena and different techniques than in the case of diffeomorphisms (as for example in [1], [18], [10], [12], etc.)

We will use in the sequel the notions of Jacobian of an invariant measure introduced by Parry in [14]. Let $f : M \to M$ be a smooth endomorphism on the manifold $M$ and $\mu$ an $f$-invariant probability on $M$ (whose support may be smaller than $M$); assume also that $f$ is at most countable-to-one. Then as shown by Rohlin ([17], [14]), there exists a measurable partition $\xi = (A_0, A_1, \ldots)$ so that $f$ is injective on each $A_i$. It was proved that the push-forward measure $((f|_{A_i})^{-1})_*\mu$ is absolutely continuous on $A_i$ with respect to $\mu$; so it makes sense to define (as in [14]) the respective Radon-Nykodim derivative, which will be called the Jacobian of $\mu$ with respect to $f$:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{\mu}(x), \mu - \text{a.e on } A_i, i \geq 0$$

Notice that $J_f(\mu)(x) \geq 1, \mu - \text{a.e } x$. We have also a Chain Rule when dealing with a composition of maps, namely

$$J_{f \circ g}(\mu) = J_f(g_*\mu)J_g(\mu)$$
**Definition 1.** Given two positive quantities \( Q_1(n, x), Q_2(n, x) \), we will say that they are **comparable** if there exists a positive constant \( C \) so that \( \frac{1}{C} \leq \frac{Q_1(n, x)}{Q_2(n, x)} \leq C \) for all \( n, x \).

Recall also (for example from [6]) that, given an expansive homeomorphism \( f : X \to X \) on a compact metric space, having the specification property, the equilibrium measure \( \mu_\phi \) of the Holder potential \( \phi \) satisfies
\[
A_\varepsilon e^{S_{n\phi}(x) - nP(\phi)} \leq \mu_\phi(B_n(x, \varepsilon)) \leq B_\varepsilon e^{S_{n\phi}(x) - nP(\phi)},
\]
where \( B_\varepsilon \) and \( B_\varepsilon \) are independent of \( x, n \). The general homeomorphism framework above allows us to apply this result to equilibrium measures on the inverse limit \( \hat{\Lambda} \). If \( \pi : \hat{\Lambda} \to \Lambda \), \( \pi(\hat{x}) := x, \hat{x} \in \hat{\Lambda} \) is the **canonical projection** and if \( \phi \) is a Holder potential on \( \Lambda \), then \( \mu_\phi \) is the unique equilibrium measure for \( \phi \) on \( \Lambda \) if and only if
\[
\mu_\phi = \pi_* \mu_{\phi \circ \pi},
\]
where \( \mu_{\phi \circ \pi} \) is the unique equilibrium measure of \( \phi \circ \pi \) on the compact metric space \( \hat{\Lambda} \); here the homeomorphism \( \hat{f} : \hat{\Lambda} \to \hat{\Lambda} \) is the shift map defined by \( \hat{f}(x, x_1, x_2, \ldots) = (f(x), x_1, x_2, \ldots) \). So we obtain for the non-invertible map \( f \) and the equilibrium measure \( \mu_\phi \) the same estimate as above:
\[
A_\varepsilon e^{S_{n\phi}(x) - nP(\phi)} \leq \mu_\phi(B_n(x, \varepsilon)) \leq B_\varepsilon e^{S_{n\phi}(x) - nP(\phi)},
\]
with positive constants \( A_\varepsilon, B_\varepsilon \) independent of \( n, x \).

**Theorem 1.** Let \( f \) be a smooth hyperbolic endomorphism on a folded basic set \( \Lambda \), which has no critical points in \( \Lambda \); let also \( \phi \) a Holder continuous potential on \( \Lambda \) and denote by \( \mu_\phi \) the unique equilibrium measure of \( \phi \) on \( \Lambda \). Then for all \( m \geq 1 \), the Jacobian of \( \mu_\phi \) w.r.t. \( f^m \) is comparable to the ratio
\[
\frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(\zeta)}}{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(x)}},
\]
i.e there exists a comparability constant \( C > 0 \) (independent of \( m, x \)) s.t. for \( \mu_\phi \)-a.e. \( x \in \Lambda \):
\[
C^{-1} \cdot \frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(\zeta)}}{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(x)}} \leq J_{f^m}(\mu_\phi)(x) \leq C \cdot \frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(\zeta)}}{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_{m\phi}(x)}}, \tag{4}
\]

**Proof.** We know from definition that the Jacobian \( J_{f^m}(\mu_\phi) \) is the Radon-Nikodym derivative of \( \mu_\phi \) with respect to \( \mu_\phi \) on sets of injectivity for \( f^m \). In order to estimate the Jacobian of \( \mu_\phi \) with respect to \( f^m \), we have to compare the measure \( \mu_\phi \) on different components of the preimage set \( f^{-m}(B) \), for a small borelian set \( B \), where \( m \geq 1 \) is fixed. Let us consider two subsets \( E_1, E_2 \) of \( \Lambda \) so that \( f^m(E_1) = f^m(E_2) \subset B \) and \( E_1, E_2 \) belong to two disjoint balls \( B_m(y_1, \varepsilon), B_m(y_2, \varepsilon) \). This happens if the diameter of \( B \) is small enough, since \( f \) has no critical points in \( \Lambda \) and thus there exists a positive distance \( \varepsilon_0 \) between any two different preimages from \( f^{-1}(y) \) for \( y \in \Lambda \).

As in [6], since the borelian sets with boundaries of measure zero form a sufficient collection, we can assume that each of the sets \( E_1, E_2 \) have boundaries of \( \mu_\phi \)-measure zero. We recall that \( f^m(E_1) = f^m(E_2) \). But as in [6], \( \mu_\phi \) is the limit of the sequence of measures:
\[
\hat{\mu}_n := \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_{n\phi}(x)} \delta_x,
\]
where $P(f, \phi, n) := \sum_{x \in \text{Fix}(f^n) \cap \Lambda} e^{S_n\phi(x)}, n \geq 1$. So we obtain

$$
\hat{\mu}_n(E_1) = \frac{1}{P(f, \phi, n)} \sum_{x \in \text{Fix}(f^n) \cap E_1} e^{S_n\phi(x)}, n \geq 1
$$

Let us now consider a periodic point $x \in \text{Fix}(f^n) \cap E_1$; it follows that $f^m(x) \in \text{Fix}(E_1)$, so there exists a point $y \in E_2$ such that $f^m(y) = f^m(x)$. However the point $y$ is not necessarily periodic. Hence we will use the Specification Property ([6], [2]) on hyperbolic locally maximal sets in order to approximate $y$ with a periodic point whose orbit follows that of $y$ for sufficiently long time. Indeed if $\varepsilon > 0$ is fixed, there exists a constant $M_{\varepsilon} > 0$ such that for all $n > M_{\varepsilon}$, there is a point $z \in \text{Fix}(f^n) \cap \Lambda$ which $\varepsilon$-shadows the $(n - M_{\varepsilon})$-orbit of $y$. In particular $z \in B_m(y_2, 2\varepsilon)$, since $E_2 \subset B_m(y_2, \varepsilon)$.

Let now $V \subset B_m(y_2, \varepsilon)$ be an arbitrary neighbourhood of the set $E_2$. Let us take two points $x, x' \in \text{Fix}(f^n) \cap E_1$ and assume the same periodic point $z \in V \cap \text{Fix}(f^n)$ corresponds to both of them through the previous shadowing procedure. Thus the $(n - M_{\varepsilon} - m)$-orbit of $f^m(z)$ $\varepsilon$-shadows the $(n - M_{\varepsilon} - m)$-orbit of $f^m(x)$ and also the $(n - M_{\varepsilon} - m)$-orbit of $f^m(x')$. Thus the $(n - M_{\varepsilon} - m)$-orbit of $f^m(x')$ $2\varepsilon$-shadows the $(n - M_{\varepsilon} - m)$-orbit of $f^m(x')$. But recall that we took $x, x' \in E_1 \subset B_m(y_1, \varepsilon)$, so $x' \in B_m(x, 2\varepsilon)$ and hence from above, $x' \in B_{n-M_{\varepsilon}}(x, 2\varepsilon)$. We will partition now the set $B_{n-M_{\varepsilon}}(x, 2\varepsilon)$ in at most $N_{\varepsilon}$ smaller Bowen balls of type $B_n(\zeta, 2\varepsilon)$. In each of these $(n, 2\varepsilon)$-Bowen balls we may have at most one fixed point for $f^n$. Indeed, fixed points for $f^n$ are solutions to the equation $f^n\xi = \xi$ and $Df^n$ does not have unitary eigenvalues. Then if $d(f^i\xi, f^i\zeta) < 2\varepsilon, i = 0, \ldots, n - 1$ and if $\varepsilon$ is small enough, we can apply the Inverse Function Theorem at each step, and thus there exists only one fixed point for $f^n$ in the Bowen ball $B_n(\zeta, 2\varepsilon)$. So there may exist at most $N_{\varepsilon}$ periodic points in $\Lambda$ from $\text{Fix}(f^n) \cap E_1$ having the same point $z \in V \cap \text{Fix}(f^n)$ associated to them by the above shadowing correspondence.

Let us notice also that if $x, x' \in \text{Fix}(f^n) \cap E_1$ have the same point $z \in V$ attached to them, then as seen before, $x' \in B_{n-M_{\varepsilon}}(x, 2\varepsilon)$ and then, from the Holder continuity of $\phi$,

$$
|S_n\phi(x) - S_n\phi(x')| \leq \tilde{C}_\varepsilon,
$$

for some positive constant $\tilde{C}_\varepsilon$ depending on $\phi$ (but independent of $n, m, x$). This can be used then in the estimate for $\hat{\mu}_n(E_1)$, from [5]. Notice also that, if $z \in B_{n-M_{\varepsilon}}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_{\varepsilon}-m}(f^m(x), \varepsilon)$. Thus from the Holder continuity of $\phi$ and the fact that $x \in E_1 \subset B_m(y_1, \varepsilon)$, it follows that there exists a positive constant $\tilde{C}_\varepsilon'$ satisfying:

$$
|S_n\phi(z) - S_n\phi(x)| \leq |S_m\phi(y_1) - S_m\phi(y_2)| + \tilde{C}_\varepsilon', \text{ for } n > n(\varepsilon, m).
$$

Then from [6], [13], and since there are at most $N_{\varepsilon}$ points $x \in \text{Fix}(f^n) \cap E_1$ having the same $z \in V \cap \text{Fix}(f^n) \cap \Lambda$ corresponding to them, we obtain that there exists a constant $C_{\varepsilon} > 0$ s.t:

$$
\hat{\mu}_n(E_1) \leq C_{\varepsilon}\hat{\mu}_n(V) \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}},
$$

6
where we recall that $E_1 \subset B_m(y_1, \varepsilon), E_2 \subset B_m(y_2, \varepsilon)$ and $f^m(E_1) = f^m(E_2)$. But $\partial E_1, \partial E_2$ were assumed of $\mu_\phi$-measure zero, hence:

$$\mu_\phi(E_1) \leq C \varepsilon \mu_\phi(V) \cdot e^{S_m \phi(y_1)}/e^{S_m \phi(y_2)}.$$  

Recall now that $V$ was chosen arbitrarily as a neighbourhood of $E_2$, and by applying the same procedure for $E_1$ instead of $E_2$ we obtain the estimates:

$$\frac{1}{C} \mu_\phi(E_2) e^{S_m \phi(y_1)}/e^{S_m \phi(y_2)} \leq \mu_\phi(E_1) \leq C \mu_\phi(E_2) e^{S_m \phi(y_1)}/e^{S_m \phi(y_2)},$$

where $C > 0$ does not depend on $m, E_1, E_2$.

Now the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to $\mu_\phi$ on sets of injectivity for $f^m$, hence

$$\mu_\phi(f^m(D)) = \int_D J_{f^m}(\mu_\phi)(x) d\mu_\phi(x),$$

for any borelian set $D$ on which $f^m$ is injective. And on the other hand from the invariance of $\mu_\phi$, we have $\mu_\phi(f^m(D)) = \mu_\phi(f^{-m}(f^m D))$. Thus from (8), the fact that $|S_m \phi(\zeta) - S_m \phi(y)| \leq \tilde{C}_\varepsilon$ for $\zeta \in B_m(y, \varepsilon)$ and from the Lebesgue Derivation Theorem, it follows that the Jacobian of $\mu_\phi$ satisfies:

$$J_{f^m}(\mu_\phi)(x) \approx \frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}}, \mu_\phi - \text{a.e } x \in \Lambda,$$

where the comparability constant $C > 0$ is independent of $m > 1, x \in \Lambda$.

Let us give now the definition of the folding entropy and the entropy production according to Ruelle, [20].

**Definition 2.** Let $f : M \to M$ be a smooth endomorphism and $\mu$ an $f$-invariant probability on $M$, then the **folding entropy** $F_f(\mu)$ of $\mu$ is the conditional entropy:

$$F_f(\mu) := H_\mu(\epsilon|f^{-1}\epsilon),$$

where $\epsilon$ is the partition into single points. Also define the **entropy production** of $\mu$ by:

$$e_f(\mu) := F_f(\mu) - \int \log |\det Df(x)| d\mu(x).$$

From [17] it follows that we can use the measurable single point partition $\epsilon$ in order to desintegrate the invariant measure $\mu$ into a canonical family of conditional measures $\mu_x$ supported on the finite fiber $f^{-1}(x)$ for $\mu$-a.e $x$. Thus the entropy of the conditional measure of $\mu$ restricted to $f^{-1}(x)$ is $H(\mu_x) = -\sum_{y \in f^{-1}(x)} \mu_x(y) \log \mu_x(y)$. From [14] we have also

$$J_f(\mu)(x) = \frac{1}{\mu_f(x)(x)}, \mu - \text{a.e } x,$$
hence we obtain that

$$F_f(\mu) = \int \log J_f(\mu)(x) d\mu(x) \tag{9}$$

Let us return now to the case of a hyperbolic basic set $\Lambda$ for a smooth endomorphism $f$ and consider a Hölder potential $\phi$ on $\Lambda$, with its unique equilibrium measure $\mu_\phi$. We will give a formula for the folding entropy of the equilibrium measure $\mu_\phi$ in terms of an ”asymptotic logarithmic degree” with respect to $\mu_\phi$. This will take into account the $n$-preimages of points which behave well (are generic) with respect to $\mu_\phi$. To this end, for an $f$-invariant probability (borelian) measure $\mu$ on $\Lambda$ let us define, for any small $\tau > 0$, $n > 0$ integer and $x \in \Lambda$ the set

$$G_n(x, \mu, \tau) := \{ y \in f^{-n}(f^n x) \cap \Lambda, \text{s.t. } |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau \}, \tag{10}$$

where $S_n\phi(y) := \phi(y) + \ldots + \phi(f^{n-1}y), y \in \Lambda$ is the consecutive sum of $\phi$ on $y$.

**Definition 3.** In the above setting, denote by $d_n(x, \mu, \tau) := \text{Card} G_n(x, \mu, \tau)$, $x \in \Lambda$, $n > 0$, $\tau > 0$. The function $d_n(\cdot, \mu, \tau)$ is measurable, nonnegative and finite on $\Lambda$.

**Theorem 2.** Let $f : M \to M$ be a smooth endomorphism and $\Lambda$ a basic set for $f$ so that $f$ is hyperbolic on $\Lambda$ and does not have critical points in $\Lambda$. Let also $\phi$ a Hölder continuous potential on $\Lambda$ and $\mu_\phi$ the equilibrium measure associated to $\phi$. Then we have the following formula for the folding entropy of $\mu_\phi$:

$$F_f(\mu_\phi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) d\mu_\phi(x)$$

**Proof.** First let us recall formula (9) for an arbitrary $f$-invariant measure $\mu$, namely

$$F_f(\mu) = \int_\Lambda \log J_f(\mu)(x) d\mu(x)$$

From the Chain Rule for Jacobians, $J_{f^n}(\mu)(x) = J_f(\mu)(x) \ldots J_f(\mu)(f^{n-1}(x))$ $\mu$-a.e, for any $n \geq 1$. On the other hand, since $\mu$ is $f$-invariant, we have that

$$\int \log J_f(\mu)(x) d\mu(x) = \int \log J_f(\mu)(f(x)) d\mu(x) = \int \log J_f(\mu)(f^k x) d\mu(x),$$

for all $k \geq 1$. These facts imply that for any $n \geq 1$,

$$F_f(\mu) = \frac{1}{n} \int \log J_{f^n}(\mu)(x) d\mu(x) \tag{11}$$

Therefore from Theorem 1 since the constant $C$ is independent of $n$ we obtain that:

$$F_f(\mu_\phi) = \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \sum_{y \in f^{-n}(f^n x) \cap \Lambda} \frac{e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{12}$$

Now since $\Lambda$ is compact, each point $x \in \Lambda$ has only finitely many $f$-preimages in $\Lambda$, i.e there exists a positive integer $d$ s.t. $\text{Card}(f^{-1}x) \leq d, x \in \Lambda$. 

8
Since $\mu_\phi$ is an ergodic measure and from Birkhoff Ergodic Theorem we obtain that $\mu_\phi(x \in \Lambda, |S_n \phi(x) - \int \phi d\mu| > \tau / 2) \to 0$, for any small $\tau > 0$. Thus for any $\eta > 0$ there exists a large integer $n(\eta)$ s.t for $n \geq n(\eta)$,

$$\mu_\phi(x \in \Lambda, |S_n \phi(x) - \int \phi d\mu| > \tau / 2) < \eta \quad (13)$$

Let us now take a point $x \in \Lambda$ with $|S_n \phi(x) - \int \phi d\mu| < \tau$. From Definition we have

$$\frac{e^{n(f \phi d\mu_\phi - \tau)}}{e^{n(f \phi d\mu_\phi + \tau)}} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau) \leq \sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)} \leq \frac{e^{n(f \phi d\mu_\phi + \tau)}}{e^{n(f \phi d\mu_\phi - \tau)}} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau) \quad (14)$$

where $r_n(x, \mu_\phi, \tau)$ is the remainder

$$\sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)}.$$ In order to simplify notation, we will also denote $r_n(x, \mu_\phi, \tau)$ by $r_n$ when no confusion can arise.

Given $n$ large, let us consider now a partition $(A^n)_{1 \leq i \leq K}$ of $\Lambda$ (modulo $\mu_\phi$) so that for each $0 \leq i \leq K$, there exists a point $z_i \in A^n_i$ so that for any $n$-preimage $\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, 1 \leq j \leq d_n, i$, we have $A^n_i \subset f^n(B_n(\xi_{ij}, \varepsilon))$, $1 \leq j \leq d_n, i \leq i \leq K$. For the above partition, let us denote by $A^n_{ij}$ the part of the $n$-preimage of $A^n_i$ which belongs to the Bowen ball $B_n(\xi_{ij}, \varepsilon)$, i.e $A^n_{ij} := f^{-n}(A^n_i) \cap B_n(\xi_{ij}, \varepsilon), 1 \leq j \leq d_n, i \leq i \leq K$. Since the sets $A^n_i$ were chosen disjoint, also the pieces of their preimages, namely $A^n_{ij}, i,j$, are mutually disjoint.

We will decompose the integral in $(12)$ over the sets $A^n_{ij}$. Notice that if $y, z \in A^n_{ij}$, then since $\phi$ is Holder continuous and $A^n_{ij} \subset B_n(\xi_{ij}, \varepsilon)$, it follows that we have

$$|S_n \phi(y) - S_n \phi(z)| \leq C(\varepsilon), \quad (15)$$

where $C(\varepsilon)$ is a positive function with $C(\varepsilon) \to 0$. So we will obtain now:

$$\int_{\Lambda} \frac{\sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) = \sum_{0 \leq j \leq d_n} \int_{A^n_{ij}} \frac{\sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) \quad (16)$$

Let us now denote by $R_n(i, \mu_\phi, \tau)$ the set of preimages $\xi_{ij}$ with $\xi_{ij} / G_n(\xi_{ij}, \mu_\phi, \tau)$, and denote simply by $R_n(i)$ the set of indices $j, 1 \leq j \leq d_n, i$ with $\xi_{ij} \in R_n(i, \mu_\phi, \tau)$ for every $1 \leq i \leq K$. Now in the decomposition from $(16)$ we notice that the integral over those sets $A^n_{ij}$ with $j \in R_n, i$ will not matter significantly. Indeed as $\text{Card}(f^{-1}x \cap \Lambda) \leq d, x \in \Lambda$ and since $-M \leq \phi(x) \leq M, x \in \Lambda$ we have

$$1 \leq \sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)} \leq d^n e^{2nM}$$

Now recall that each $A^n_{ij} \subset B_n(\xi_{ij}, \varepsilon)$ and the sets $A^n_{ij}, i,j$ are mutually disjoint (with respect to $\mu_\phi$). Hence by using inequalities $(13)$ and $(15)$ and the fact that $\xi_{ij} / G_n(\xi_{ij}, \mu_\phi, \tau)$ whenever $j \in R_n(i)$, we obtain:

$$\sum_{0 \leq i \leq K, j \in R_n(i)} \frac{1}{n} \int_{A^n_{ij}} \frac{\sum_{y \in f^{-n}(f_\phi(x) \cap \Lambda)} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) \leq \frac{1}{n} \log(d^n e^{2nM}) \cdot \eta = \eta(\log d + 2M) \quad (17)$$
But by using the comparison between different parts of the \( n \)-preimage of a small set from the proof of Theorem 1 (see (8)), we deduce that the last term of formula (16) is comparable to

\[
\sum_{i,j} \mu_\phi(A^n_{ij}) \log \frac{d_n(z_i, \mu_\phi, \tau) \mu_\phi(A^n_{ij}) + \tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})},
\]

(18)

where \( \tilde{r}_n(z_i, \mu, \tau) := \sum_{\xi_{ij} \in f^{-1}(z) \cap A, \xi_{ij} \notin G_n(\xi_{i0}, \mu, \tau)} \mu_\phi(A^n_{ij}) \).

Hence from (8), (17) and (18) we obtain:

\[
\frac{1}{n} \sum_{i,j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log d_n(z_i, \mu_\phi, \tau) + \frac{1}{n} \sum_{i,j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log (1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})}) - \delta(\tau) - \eta C' \leq
\]

\[
\leq \int_{\Lambda} \frac{1}{n} \sum_{y \in f^{-n} f^n x \cap \Lambda} \frac{\log d_n(z_i, \mu_\phi, \tau) \log \sum_{\xi_{ij} \in (\xi_{i0}, \mu, \tau) \cap \phi} \mu_\phi(A^n_{ij}) + \delta(\tau) + \eta C',
\]

(19)

with \( C' = \log d + 2M \) being the constant found in (17), and where the positive constant \( \delta(\tau) \) comes from the uniformly bounded variation of \( \frac{1}{n} S_n(\phi(x) \times R_{n,i}) = K, j \notin R_{n,i} \) vary; clearly we have \( \delta(\tau) \to 0 \).

Now we know that in general \( \log(1 + x) \leq x, \frac{1}{n} \log(1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})}) \leq \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})}, i, j \) and hence in (19) we have, for \( n \) large enough that:

\[
\sum_{i,j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \log (1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})}) \leq \sum_{i,j \notin R_{n,i}} \mu_\phi(A^n_{ij}) \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})} = \]

\[
\sum_{1 \leq i \leq K} \tilde{r}_n(z_i, \mu_\phi, \tau) \leq \eta,
\]

(20)

where we used that by definition, there are \( d_n(z_i, \mu_\phi, \tau) \) indices \( j \) in \( \{1, \ldots, d_{n,i}\} \setminus R_{n,i} \) for any \( 1 \leq i \leq K \). Therefore from the last displayed inequality and from (19) we obtain, for \( n \geq n(\eta) \), that:

\[
\frac{1}{n} \int_{\Lambda} \log \frac{\sum_{y \in f^{-n} f^n x \cap \Lambda} \mu_\phi(A^n_{ij}) \log d_n(z_i, \mu_\phi, \tau) \log (1 + \frac{\tilde{r}_n(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})})}{\eta C'} \leq \delta(\tau) + \eta,
\]

(21)

where \( \delta(\tau) \to 0 \). Then by taking \( n \to \infty \) and \( \tau \to 0 \), we will obtain the conclusion of the Theorem from (12) and (21), namely that:

\[
F_\phi(\mu_\phi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_{\Lambda} \log d_n(x, \mu_\phi, \tau) d\mu_\phi(x).
\]

\[\square\]
Corollary 1. a) Let $f : \mathbb{T}^{m} \to \mathbb{T}^{m}, m \geq 2$ be a hyperbolic toral endomorphism, and $\phi$ be an arbitrary Holder continuous potential on $\mathbb{T}^{m}$, with its associated equilibrium measure $\mu_{\phi}$. Then the entropy production of $\mu_{\phi}$ is non-positive, i.e
\[ e_{f}(\mu_{\phi}) \leq 0 \]
In the same setting the entropy production of the Haar (Lebesgue) measure is equal to 0.

b) The same conclusions as above hold also for any Anosov endomorphism $f : \mathbb{T}^{m} \to \mathbb{T}^{m}$ with constant Jacobian with respect to the Riemannian metric, i.e for which $\det Df$ is constant on $\mathbb{T}^{m}$.

Proof. a) In the case of a toral endomorphism $f$ given by the integer-valued matrix $A$, the determinant of the derivative $\det Df$ is constant and equal to $\det A$. Thus
\[ \int_{\mathbb{T}^{m}} \log |\det Df| d\mu_{\phi} = \log d, \]
where $d := |\det A|$. On the other hand, by looking at the area of $f(I \times \ldots \times I)$, it is easy to see that $d$ is exactly the number of $f$-preimages that any point from $\mathbb{T}^{m} = I \times \ldots \times I$ (m times) has. Therefore, by taking $\Lambda = \mathbb{T}^{m}$ and by recalling Definition 3, one obtains that
\[ d_{n}(x, \mu_{\phi}, \tau) \leq d^{n}, \forall x \in \mathbb{T}^{m}, n > 0, \tau > 0 \]
Hence from Theorem 2 it follows that
\[ e_{f}(\mu_{\phi}) \leq 0 \]
For the last statement of a), we have that $f$ invariates the Lebesgue measure $m$, that $|\det Df|$ is constant and equal to $d$ and that $d_{n}(x, m, \tau)$ is constant in $x$ and equal to $d$ since the Lebesgue (Haar) measure is the unique measure of maximal entropy. Therefore the entropy production of the Lebesgue measure $m$ with respect to $f$ is equal to 0.

The last statement of a) can also be obtained from the fact that the entropy production of invariant absolutely continuous measures is non-negative (from [20]), combined with the first part of the proof.

b) The argument is the same as for a), namely if $\det Df$ is constant, then $f$ invariates the Lebesgue measure $m$, and it is $d$-to-1, for $d = |\det Df|$. Then $d_{n}(x, \mu_{\phi}, \tau) \leq d$ for any $x, \tau, n$ and $e_{f}(\mu_{\phi}) \leq 0$.

However we will see later that Corollary 1 is no longer true for perturbations of a toral endomorphism $f$, and that there exist equilibrium measures of Holder potentials which have in certain cases positive entropy production.

Theorem 2 also helps us calculate the folding entropy of the measure of maximal entropy for a general hyperbolic (hence non-expanding) endomorphism. Then by knowing this, one can calculate the entropy production of the measure of maximal entropy, from Definition 2.
Corollary 2. In the setting of Theorem 2, denote by $\mu_0$ the unique measure of maximal entropy for $f$ on $\Lambda$. If $d_n(x)$ denotes the cardinality of $f^{-n}(f^n x) \cap \Lambda$ for $n \geq 1$, then we have:

$$F_f(\mu_0) = \lim_{n \to \infty} \frac{1}{n} \int_{\Lambda} \log d_n(x) d\mu_0(x)$$

In particular if $f$ is $d$-to-1 on $\Lambda$, then $F_f(\mu_0) = \log d$.

Let us now recall the notion of inverse SRB measure, introduced in [11]. These measures exist in the case of hyperbolic repellers (and in particular in the case of Anosov endomorphisms) and are physically relevant since they describe the past trajectories of Lebesgue almost all points in a neighbourhood of the repellor. We will show that there exist Anosov endomorphisms, whose respective inverse SRB measures have negative entropy production.

Let $\Lambda$ be a connected hyperbolic repeller for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold $M$, and assume $f$ has no critical points in $\Lambda$. Let $V$ be a neighbourhood of $\Lambda$ in $M$ and for any $z \in V$ define the measures

$$\mu_n^z := \frac{1}{n} \sum_{y \in f^{-n} z \cap V} \frac{1}{d(f(y)) \ldots d(f^n(y))} \sum_{i=1}^n \delta_{f^i y},$$

where $d(y)$ is the number of $f$-preimages belonging to $V$ of a point $y \in V$ ($d(\cdot)$ is called also the degree function).

Then we proved in [11] that there exists an $f$-invariant measure $\mu^-$ on $\Lambda$, a neighbourhood $V$ of $\Lambda$ and a borelian set $A \subset V$ with $m(V \setminus A) = 0$ (where $m$ is the Lebesgue measure on $M$) and a subsequence $n_k \to \infty$ such that for any $z \in A$,

$$\mu_{n_k}^z \to \mu^- \quad k \to \infty \quad (23)$$

The measure $\mu^-$ is called the inverse SRB measure of the hyperbolic repellor. We showed in [11] that $\mu^-$ is the equilibrium measure of the stable potential $\Phi^s(x) := \log |\det Df_s(x)|, x \in \Lambda$, with respect to $f$ (where we recall the notation from (3)). The difficulty is that the map $f$ is non-invertible, hence $\mu^-$ is not simply the SRB measure for the inverse $f^{-1}$. Moreover from the hyperbolicity condition in the case of endomorphisms, the unstable manifolds may intersect each other both in $\Lambda$ and outside $\Lambda$ and through any point of $\Lambda$ there may pass infinitely many (even uncountably many, as shown in [10]) unstable manifolds.

We also proved that this inverse SRB measure $\mu^-$ is the unique $f$-invariant measure $\mu$ satisfying an inverse Pesin entropy formula: in the case when $f$ is $d$-to-1 on $\Lambda$

$$h_\mu(f) = \log d - \int_{\Lambda} \sum_{i, \lambda_i(\mu, x) < 0} \lambda_i(\mu, x) m_i(\mu, x) d\mu(x),$$

where $\lambda_i(\mu, x)$ are the Lyapunov exponents of the measure $\mu$ at $x$ and $m_i(\mu, x)$ are the respective multiplicities of these Lyapunov exponents. In addition if $f$ is $d$-to-1 on the connected hyperbolic repellor $\Lambda$, then the inverse SRB measure $\mu^-$ has absolutely continuous conditional measures on local stable manifolds (see [11]).
Also for an Anosov endomorphism $f$ on $M$, we know from \cite{15,16} that there exists a unique SRB measure $\mu^+$ which satisfies a Pesin entropy formula and which is the projection $\pi_*$ of the equilibrium measure of the unstable potential $\Phi^u(\hat{x}) := -\log |\det Df_u(\hat{x})|, \hat{x} \in \hat{M}$ (with the notation for the unstable derivative from \cite{3}).

We prove now that the entropy production of the respective inverse SRB measure of a perturbation $g$ of a hyperbolic toral endomorphism, is less than or equal to $0$; we identify the cases when it is $0$ as exactly those cases when $\mu_g^-$ is absolutely continuous on $\mathbb{T}^m$.

**Theorem 3.** Let $f$ be a hyperbolic toral endomorphism on $\mathbb{T}^m$, $m \geq 2$ given by an integer-valued matrix $A$ without zero eigenvalues, and let $g$ be a $C^1$ perturbation of $f$. Consider $\mu_g^-$ the inverse SRB measure of $g$ and $\mu_g^+$ the (usual forward) SRB measure. Then:

a) $e_g(\mu_g^-) \leq 0$ and $F_g(\mu_g^-) = \log d$. Moreover $e_g(\mu_g^+) \geq 0$.

b) $e_g(\mu_g^-) = 0$ if and only if $|\det Dg|$ is cohomologous to a constant on $\mathbb{T}^m$. Same condition on $|\det Dg|$ holds if and only if $e_g(\mu_g^+) = 0$. In either case we obtain $\mu_g^- = \mu_g^+$, and the common value is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.

**Proof.** a) If $f$ is given by an integer valued matrix $A$, then $f$ is $d$-to-$1$ on $\mathbb{T}^m$, where $d = |\det A|$. If $g$ is a $C^1$ perturbation of the hyperbolic toral endomorphism $f$, then it is clear that $g$ is also hyperbolic on $\mathbb{T}^m$. Thus from \cite{16} we can construct the SRB measure of $g$, denoted by $\mu_g^+$, which is the projection by $\pi_*$ of the equilibrium measure of $\Phi^u(\hat{x}) = -\log |\det Dg_a(\hat{x})|, \hat{x} \in \hat{\mathbb{T}}^m$. In particular $\mu_g^+$ is ergodic, hence its Lyapunov exponents are constant $\mu_g^+$-a.e.

From the discussion above, since $f$ has no critical points, we can construct the inverse SRB measure $\mu_g^-$ which is the equilibrium measure of the stable potential $\Phi^s(x) = \log |\det Dg_s(x)|, x \in \mathbb{T}^m$; thus $\mu_g^-$ is ergodic too, and its Lyapunov exponents are constant $\mu_g^-$-a.e on $\mathbb{T}^m$.

Now since $g$ is a perturbation of $f$, it follows that every point in $\mathbb{T}^m$ has exactly $d g$-preimages, where $d = |\det A|$. Thus from \cite{11}, it follows that $\mu_g^-$ is the weak limit of a sequence of measures of type \cite{22}, where the degree function $d(\cdot)$ is constant and equal to $d$ everywhere on $\mathbb{T}^m$. This implies then that the Jacobian of $\mu_g^-$ is constant and equal to $d$, since for any small borelian set $B$, we have that a point $x \in g(B)$ if and only if there is exactly one $g$-preimage $x_{-1}$ of $x$ in $B$, and we use this fact in the above convergence \cite{23} of measures towards $\mu^-$. Hence

$$F_g(\mu_g^-) = \int \log J_g(\mu_g^-)(x)d\mu_g^-(x) = \log d$$

And from \cite{24} we have that

$$h_{\mu_g^-}(g) = \log d - \sum_{\lambda_i(\mu_g^-) < 0} \lambda_i(\mu_g^-)$$

Thus if $e_g(\mu_g^-) > 0$, it would follow that $F_g(\mu_g^-) > \int \log |\det Dg_d|d\mu_g^- = \frac{1}{n} \int \log |\det Dg^m|d\mu_g^-, n \geq 1$. Hence from the last displayed formula and Birkhoff Ergodic Theorem, we obtain $h_{\mu_g^-}(g) > \sum_{\lambda_i(\mu_g^-) > 0} \lambda_i(\mu_g^-)$, which gives a contradiction with Ruelle’s inequality. Therefore we have for any perturbation $g$,

$$e_g(\mu_g^-) \leq 0$$
Now for the SRB measure $\mu_g^+$: if the entropy production $e_g(\mu_g^+)$ were strictly negative, then $F_g(\mu_g^+) < \int \log |\det Dg| d\mu_g^+$. Since from \cite{16}, $h_{\mu_g^+}(g) \leq F_g(\mu_g^+) - \sum_{\lambda_i(\mu_g^+) > 0} \lambda_i(\mu_g^+)$, it would follow that $h_{\mu_g^+}(g) < \sum_{\lambda_i(\mu_g^+) > 0} \lambda_i(\mu_g^+)$, which is a contradiction to the fact that the SRB measure satisfies Pesin entropy formula. Consequently,

$$e_g(\mu_g^+) \geq 0$$

b) If $e_g(\mu_g^-) = 0$, then $F_g(\mu_g^-) = \int \log |\det Dg| d\mu_g^-$; hence from the Birkhoff Ergodic Theorem and \cite{9} we obtain:

$$h_{\mu_g^-}(g) = \int \log |\det Dg| d\mu_g^- - \sum_{\lambda_i(\mu_g^-) < 0} \lambda_i(\mu_g^-) = \sum_{\lambda_i(\mu_g^-) > 0} \lambda_i(\mu_g^-)$$

Therefore from the uniqueness of the $g$-invariant measure satisfying Pesin entropy formula, we obtain that $\mu_g^- = \mu_g^+$. 

Recalling from above that $\mu_g^-$ is the equilibrium measure of the stable potential $\Phi^s$ and $\mu_g^+$ is the equilibrium measure of the unstable potential $\Phi^u$, we see from Livshitz Theorem (see \cite{6}), that $\mu_g^- = \mu_g^+$ if and only if $\det Dg$ is cohomologous to a constant.

Assume now that $\mu_g^+ = \mu_g^-$; then since $\mu_g^+$ has absolutely continuous conditional measures associated to a partition subordinated to local unstable manifolds \cite{16, 15} and $\mu_g^-$ has absolutely continuous conditional measures associated to a partition subordinated to local stable manifolds \cite{11}, we obtain that $\mu_g^+$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.

\[\square\]

**Corollary 3.** In the setting of Theorem 3 let $g$ be a perturbation of the hyperbolic toral endomorphism $f$ s.t $|\det Dg|$ is not cohomologous to a constant. Then its unique inverse SRB measure $\mu_g^-$ is not a weak limit of a sequence of type (2).

**Proof.** As was proved in \cite{20}, the entropy production of any limit of measures of type (2) is nonnegative. On the other hand, if $|\det Dg|$ is not cohomologous to a constant, then $e_g(\mu_g^-) < 0$. Thus in our case $\mu_g^-$ is not a weak limit of measures of type (2). \[\square\]

We show now that the set of maps with negative entropy production for their respective inverse SRB measures, is open and dense in a neighbourhood of a hyperbolic toral endomorphism $f$.

**Corollary 4.** a) Let $f$ be a hyperbolic toral endomorphism on $\mathbb{T}^m, m \geq 2$. Then there exists a neighbourhood $V$ of $f$ in $\mathcal{C}^1(\mathbb{T}^m, \mathbb{T}^m)$ and a set $W \subset V$ such that $W$ is open and dense in the $\mathcal{C}^1$ topology in $V$ and s.t for any $g \in W$ we have $e_g(\mu_g^-) < 0$.

b) Consider the hyperbolic toral endomorphism on $\mathbb{T}^2$ given by $f(x, y) = (2x + 2y, 2x + 3y) \pmod{1}$ and its smooth perturbation

$$g(x, y) = (2x + 2y + \varepsilon \sin 2\pi y, 2x + 3y + 2\varepsilon \sin 2\pi y) \pmod{1}$$
Then the inverse SRB measure of $g$ has negative entropy production, while the SRB measure of $g$ has positive entropy production, i.e.

$$e_g(\mu^-) < 0 \text{ and } e_g(\mu^+) > 0$$

Proof. a) If $f$ is a hyperbolic toral endomorphism on $\mathbb{T}^m$ then there exists a neighbourhood $V$ of $f$ in $C^1$ topology, so that any $g \in V$ is hyperbolic and $d$-to-1, where $d = |\det Df|$.

We showed in Theorem 3 that $e_g(\mu^-) < 0$ unless $|\det Dg|$ is cohomologous to a constant. But from the Livshitz Theorem (see for instance [6]) it follows that this is equivalent to the existence of a constant $c$ such that for any $n \geq 1$,

$$S_n(|\det Dg|(x)) = nc, \forall x \in \text{Fix}(g^n)$$

As the set of $g$’s not satisfying the above equalities is open and dense in $V$, we obtain the conclusion of part a).

b) First of all we notice that $f$ is given by an integer valued matrix $A$ which has one eigenvalue larger than 1 and another eigenvalue in $(0, 1)$, so $f$ is hyperbolic. Thus for $\varepsilon > 0$ small enough, we have that $g$ (which is well defined as an endomorphism on $\mathbb{T}^m$) is hyperbolic as well.

We calculate now the determinant of the derivative of $g$ as

$$\det Dg(x, y) = 2 + 4\pi \varepsilon \cos 2\pi y$$

Now, from Theorem 3 we see that $e_g(\mu^-) < 0$ if and only if the function $|\det Dg|$ is cohomologous to a constant. But this is equivalent from the Livshitz conditions ([6]) to the fact that there exists a constant $c$ such that

$$S_n(|\det Dg|(x)) = nc, x \in \text{Fix}(g^n), n \geq 1$$

In our case, notice that both $(0, 0)$ and $(0, \frac{1}{2})$ are fixed points for $g$. But $|\det Dg(0, 0)| = 2 + 4\pi \varepsilon$, whereas $|\det Dg(0, \frac{1}{2})| = 2 - 4\pi \varepsilon$. So the Livshitz condition above is not satisfied, and $|\det Dg|$ is not cohomologous to a constant. Hence according to Theorem 3 we obtain

$$e_g(\mu^-) < 0 \text{ and } e_g(\mu^+) > 0$$



Acknowledgements: This work was supported by CNCSIS - UEFISCDI, project PNII - IDEI 1191/2008.

References

[1] H. G. Bothe, Shift spaces and attractors in noninvertible horseshoes, Fundamenta Math., 152, no. 3, 1997, 267-289.

[2] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics, 470, Springer 1975.
[3] R. L. Dobrushin, Ya. G. Sinai, Yu. M. Sukhov, Dynamical Systems of Statistical Mechanics, in Dynamical Systems, Ergodic Theory and Applications, ed. Ya. G. Sinai, vol. 100, Encyclopaedia of Mathematical Sciences, Springer, 2000.

[4] J. P. Eckmann and D. Ruelle, Ergodic theory of strange attractors, Rev. Mod. Physics, 57, 1985, 617-656.

[5] D. Evans and D. Searles, The fluctuation theorem, Adv. in Physics, 51, no. 7, 2002, 1529-1585.

[6] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, London-New York, 1995.

[7] J. Lebowitz, Boltzmann’s entropy and time’s arrow, Physics Today, 46, 1993, 32-38.

[8] F. Ledrappier, Proprietes ergodiques des mesures de Sinai, Publ. Math. IHES, vol. 59, 1984, 163-188.

[9] P. D Liu, Invariant measures satisfying an equality relating entropy, folding entropy and negative Lyapunov exponents, Commun. Math. Physics, vol. 284, no. 2, 2008, 391-406.

[10] E. Mihailescu, Unstable directions and fractal dimension for a class of skew products with overlaps in fibers, Math. Zeitschrift 2010, DOI: 10.1007/s00209-010-0761-y.

[11] E. Mihailescu, Physical measures for multivalued inverse iterates near hyperbolic repellors, J. Statistical Physics, 139, 2010, 800-819.

[12] E. Mihailescu, Metric properties of some fractal sets and applications of inverse pressure, Math. Proceed. Cambridge, 148, 3, 2010, 553-572.

[13] E. Mihailescu, Unstable manifolds and Holder structures associated with noninvertible maps, Discrete and Cont. Dynam. Syst. 14, 3, 2006, 419-446.

[14] W. Parry, Entropy and generators in ergodic theory, W. A Benjamin, New York, 1969.

[15] M. Qian and Z. Shu, SRB measures and Pesin’s entropy formula for endomorphisms, Trans. Amer. Math. Soc., 354, 2002, 1453-1471.

[16] M. Qian, Z. Zhang, Ergodic theory for axiom A endomorphisms, Ergodic Th. and Dynam. Syst., 15, 1995, 161-174.

[17] V. A. Rokhlin, Lectures on the theory of entropy of transformations with invariant measures, Russian Math. Surveys, 22, 1967, 1-54.

[18] D. Ruelle, Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics, J. Statistical Physics 95, 1999, 393-468.

[19] D. Ruelle, Entropy production in nonequilibrium statistical mechanics, Commun. Math. Phys., 189, 1997, 365-371.
[20] D. Ruelle, Positivity of entropy production in nonequilibrium statistical mechanics, J. Statistical Physics 85, 1/2, 1996, 1-23.

[21] D. Ruelle, Elements of differentiable dynamics and bifurcation theory, Academic Press, New York, 1989.

[22] Y. Sinai, Gibbs measures in ergodic theory, Russian Math. Surveys, 27, 1972, 21-69.

[23] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Experimental demonstration of violations of the Second Law of Thermodynamics for small systems and short time scales, Physical Rev. Letters, 89, 050601, 2002.

**E-mail:** Eugen.Mihailescu@imar.ro

Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, RO 014700, Bucharest, Romania.

Webpage: www.imar.ro/~mihailes