THE SPECTRAL NORM OF RANDOM INNER-PRODUCT KERNEL MATRICES

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Abstract. We study an “inner-product kernel” random matrix model, whose empirical spectral distribution was shown by Xiuyuan Cheng and Amit Singer to converge to a deterministic measure in the large \(n\) and \(p\) limit. We provide an interpretation of this limit measure as the additive free convolution of a semicircle law and a Marcenko-Pastur law. By comparing the tracial moments of this random matrix to those of a deformed GUE matrix with the same limiting spectrum, we establish that for odd kernel functions, the spectral norm of this matrix converges almost surely to the edge of the limiting spectrum. Our study is motivated by the analysis of a covariance thresholding procedure for the statistical detection and estimation of sparse principal components, and our results characterize the limit of the largest eigenvalue of the thresholded sample covariance matrix in the null setting.

1. Introduction

Let \(X \in \mathbb{R}^{p \times n}\) be a random matrix with independent entries of mean 0 and variance 1, and let \(\hat{\Sigma} = n^{-1}XX^T\) be the sample covariance. Define a matrix \(K(X) \in \mathbb{R}^{p \times p}\) entrywise as

\[
K(X)_{ii'} = \begin{cases} \frac{1}{\sqrt{n}} k(\sqrt{n}\hat{\Sigma}_{ii'}) & i \neq i' \\ 0 & i = i' \end{cases}
\]

(1)

where \(k : \mathbb{R} \to \mathbb{R}\) is a (nonlinear) “kernel” function. In this paper, we study the spectral norm \(\|K(X)\|\) in the asymptotic regime \(n, p \to \infty\) such that \(p/n \to \gamma \in (0, \infty)\), when \(k\) is a fixed function independent of \(n\) and \(p\).

Our study of this model is motivated by the analysis of a covariance thresholding procedure proposed in [36] and subsequently analyzed in [22] for the sparse PCA problem in statistics. In the simplest setting, this problem may be formulated as follows:

1.1. Sparse PCA. Consider a data matrix \(X \in \mathbb{R}^{p \times n}\) with independent columns distributed as \(\mathcal{N}(0, \Sigma)\), where \(\Sigma\) is a \(p \times p\) covariance matrix of the “spiked model” form

\[
\Sigma = \text{Id} + \lambda vv^T
\]

(2)

with \(\lambda > 0\) a constant and \(v \in \mathbb{R}^p\) a vector of unit Euclidean norm. Assume further that \(\|v\|_0 \ll p\) where \(\|v\|_0\) denotes the number of nonzero entries of \(v\), and (for simplicity of discussion) that each such nonzero entry equals \(\pm 1/\sqrt{\|v\|_0}\). Based on observing \(X\), we would like to detect the spike (i.e. distinguish this from the null model \(\Sigma = \text{Id}\)) and to recover the support of \(v\) [1, 5].

As \(n, p \to \infty\) with \(p/n \to \gamma \in (0, \infty)\), in the “supercritical” regime \(\lambda > \lambda^*\) where \(\lambda^* := \sqrt{\gamma}\), the largest eigenvalue \(\lambda_{\text{max}}(\hat{\Sigma})\) separates from the bulk, and the corresponding eigenvector \(\hat{v}\) partially aligns with \(v\). Consequently, consistent spike detection and support recovery may be performed

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the support of $\lambda$ conjectured that no polynomial-time algorithm can consistently detect the spike or recover $\|v\|_0 = 0.3\sqrt{n}$. The asymptotic prediction $\|\mu_{a,v,\gamma}\| + 1$ is shown as the red curve, where $a := \mathbb{E}[\xi k_\tau(\xi)]$, $\nu := \mathbb{E}[k_\tau(\xi)^2]$, and $\gamma := p/n = 1$.

The threshold function $k_\tau$ here is a smoothed soft-threshold, defined by $k_\tau(x) = 0$ for $|x| \leq 0.8\tau$, $k_\tau(x) = \text{sign}(x)(|x| - \tau)_+$ for $|x| \geq 1.2\tau$, and quadratic interpolation in between.\footnote{The proof of our main result requires a technical condition that $k(x)$ is continuously differentiable. Oftentimes threshold functions used in practice are not smooth in this sense, but the same qualitative phenomena regarding detection and support recovery should hold for both smooth and non-smooth thresholds.}

In $[22]$, it was shown that for any $\lambda > 0$, spike detection and support recovery based on $\lambda_{\text{max}}(M_\tau(X))$ cannot distinguish the null and spiked models, and furthermore no test using only the eigenvalues of $\hat{\Sigma}$ can distinguish the null and spiked models, and furthermore no test using only the eigenvalues of $\hat{\Sigma}$ can distinguish the models with probability approaching one $[32, 3, 4, 44, 42, 43]$. In this regime, $[36]$ proposed to exploit the sparsity of $v$ by applying a thresholding operation $x \mapsto k_\tau(\sqrt{n}x)/\sqrt{n}$ entrywise to $\hat{\Sigma}$ to yield a matrix $M_\tau(X)$, and then performing a spectral decomposition of $M_\tau(X)$. Here, $\tau > 0$ is a constant and $k_\tau : \mathbb{R} \rightarrow \mathbb{R}$ is a threshold function satisfying $k_\tau(x)/x \rightarrow 1$ as $x \rightarrow \pm\infty$ and $k_\tau(x) = 0$ for $|x| \leq \tau$, so that entries of $\hat{\Sigma}$ of magnitude less than $\tau/\sqrt{n}$ are set to 0 while large entries are essentially preserved. (The matrix $K(X)$ in $[4]$ when $k := k_\tau$ is precisely $M_\tau(X)$ with diagonal set to 0.)

The choice of threshold level $\tau/\sqrt{n}$ is motivated by the following consideration: For $\lambda < \lambda^\ast$, it is in fact conjectured that no polynomial-time algorithm can consistently detect the spike or recover the support of $v$ if $\|v\|_0 \gtrsim n^{1/2+\varepsilon}$; for any $\varepsilon > 0$ $[3] [36]$. Hence it is believed that the most difficult setting which permits a computationally tractable solution to these problems is when $\|v\|_0 \asymp \sqrt{n}$. In this setting, both the non-zero off-diagonal entries of $\Sigma$ (the “signal”) and the fluctuations of the entries of $\hat{\Sigma}$ (the “noise”) are of order $1/\sqrt{n}$, so the threshold must also be of order $1/\sqrt{n}$ to preserve the signal while reducing the noise.

In $[22]$, it was shown that for any $\lambda > 0$, spike detection and support recovery based on $\lambda_{\text{max}}(M_\tau(X))$ and the corresponding eigenvector can succeed with probability approaching 1 when $\|v\|_0 \leq c\sqrt{n}$, for some constants $c := c(\lambda) > 0$ and $\tau := \tau(c, \lambda) > 0$. This phenomenon is illustrated in Figure $[1]$, which shows that for a range of thresholds $\tau$, there is a difference between the values of $\lambda_{\text{max}}(M_\tau(X))$ under the null model $\Sigma = \text{Id}$ and under a spiked alternative with $\lambda < \lambda^\ast$ and sparsity $\|v\|_0 \asymp \sqrt{n}$. The main result of this paper strengthens the non-asymptotic analysis in $[22]$ under the null model $\Sigma = \text{Id}$ to establish an exact asymptotic value for $\lambda_{\text{max}}(M_\tau(X))$ in terms of

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Largest eigenvalue of the thresholded covariance matrix $M_\tau(X)$ for a threshold function $k_\tau$ under $\Sigma = \text{Id}$ (red circles) and $\Sigma = \text{Id} + \lambda vv^T$ (blue triangles), for $n = p = 2000$, $\lambda = 0.9$, and $\|v\|_0 = 0.3\sqrt{n}$. The asymptotic prediction $\|\mu_{a,v,\gamma}\| + 1$ is shown as the red curve, where $a := \mathbb{E}[\xi k_\tau(\xi)]$, $\nu := \mathbb{E}[k_\tau(\xi)^2]$, and $\gamma := p/n = 1$. The threshold function $k_\tau$ here is a smoothed soft-threshold, defined by $k_\tau(x) = 0$ for $|x| \leq 0.8\tau$, $k_\tau(x) = \text{sign}(x)(|x| - \tau)_+$ for $|x| \geq 1.2\tau$, and quadratic interpolation in between.\footnote{The proof of our main result requires a technical condition that $k(x)$ is continuously differentiable. Oftentimes threshold functions used in practice are not smooth in this sense, but the same qualitative phenomena regarding detection and support recovery should hold for both smooth and non-smooth thresholds.}}
\end{figure}
$k_\tau$. Procedurally, this indicates the point above which this method should reject the null model in favor of a spiked alternative. We are not aware of a similar analytic characterization of the value of $\lambda_{\text{max}}(M_\tau(X))$ under the alternative model; such a characterization may yield insight on the exact critical sparsity level $c^\ast(\lambda)$ and optimal choices of $\tau$ and $k_\tau$ for spike detection using this method to succeed.

In the null model of this example, since all diagonal entries of $\hat{\Sigma}$ concentrate around 1 and thresholding essentially preserves the diagonal, the thresholded sample covariance $M_\tau(X)$ satisfies $\|M_\tau(X) - (K(X) + \text{Id})\| \to 0$, where $K(X)$ is as in (1) for $k := k_\tau$. Hence the largest eigenvalue limit of $M_\tau(X)$ is simply that of $K(X)$ translated by 1. For odd and increasing threshold functions, the condition of Corollary 1.5 below is satisfied, so the largest eigenvalue of $K(X)$ equals its spectral norm.

1.2. Properties of the limit measure. For the model (1), the weak limit of the empirical spectral measure $p^{-1} \sum_i \delta_{\lambda_i(K(X))}$ of $K(X)$ was characterized by Cheng and Singer [20] Theorem 3.4 and Remark 3.2. We restate this result in the following form:

**Theorem 1.1** (Cheng, Singer). Let $X \in \mathbb{R}^{p \times n}$ have entries $x_{ij} \sim \mathcal{N}(0, 1)$. For $\xi \sim \mathcal{N}(0, 1)$, suppose $\mathbb{E}[k(\xi)] = 0$, $\mathbb{E}[k(\xi)^2] < \infty$, and $\int k(x)^2 q_n(x) - q(x) dx \to 0$ as $n \to \infty$ where $q$ and $q_n$ are the density functions of the laws of $\xi$ and $\sqrt{n}\Sigma_{12}$. Then, denoting $a := \mathbb{E}[k(\xi)]$ and $\nu := \mathbb{E}[k(\xi)^2]$, as $n, p \to \infty$ with $p/n \to \gamma \in (0, \infty)$,

$$
\frac{1}{p} \sum_{i=1}^{p} \delta_{\lambda_i(K(X))} \Rightarrow \mu_{a,\nu,\gamma}
$$

weakly almost surely, where $\mu_{a,\nu,\gamma}$ is a deterministic measure whose Stieltjes transform $m : \mathbb{C}^+ \to \mathbb{C}^+$ is the unique solution (in $\mathbb{C}^+$, for any $z \in \mathbb{C}^+$) to the equation

$$
-\frac{1}{m(z)} = z + a \left(1 - \frac{1}{1 + a\gamma m(z)}\right) + \gamma(\nu - a^2)m(z).
$$

This result was generalized by Do and Vu to the setting of non-Gaussian entries $x_{ij}$ in [23].

Before stating our main results, let us discuss some basic properties of this limit measure: For a linear kernel function $k(x) = ax$, $\mu_{a,\nu,\gamma}$ is a translation and rescaling of the Marcenko-Pastur law. Interestingly, it was observed in [20] that for kernel functions for which $a = 0$, $\mu_{a,\nu,\gamma}$ is a Wigner semicircle law. In fact, the measure $\mu_{a,\nu,\gamma}$ in general is the additive free convolution (in the sense of Voiculescu [57]) of these two laws.

**Proposition 1.2.** Let $\mu_{\text{sc}}$ be the semicircle law supported on $[-2, 2]$ and let $\sqrt{\gamma(\nu - a^2)}\mu_{\text{sc}}$ denote the law of $\sqrt{\gamma(\nu - a^2)}y$ for $y \sim \mu_{\text{sc}}$. Let $\mu_{\text{MP},\gamma}$ be the standard Marcenko-Pastur law that is the limiting spectral measure of $n^{-1}XX^T$ when $X \in \mathbb{R}^{p \times n}$ and $p/n \to \gamma$, and let $a(\mu_{\text{MP},\gamma} - 1)$ denote the law of $a(y - 1)$ for $y \sim \mu_{\text{MP},\gamma}$. Then

$$
\mu_{a,\nu,\gamma} = a(\mu_{\text{MP},\gamma} - 1) \boxplus \sqrt{\gamma(\nu - a^2)}\mu_{\text{sc}}.
$$

Proof. By (3), the measure $\mu_{a,\nu,\gamma}$ has $\mathcal{R}$-transform

$$
\mathcal{R}(z) = -a \left(1 - \frac{1}{1 + a\gamma z}\right) + \gamma(\nu - a^2)z.
$$

It is easily verified that $-a(1 - 1/(1 - a\gamma z))$ is the $\mathcal{R}$-transform of $a(\mu_{\text{MP},\gamma} - 1)$ and $\gamma(\nu - a^2)z$ is the $\mathcal{R}$-transform of $\sqrt{\gamma(\nu - a^2)}\mu_{\text{sc}}$, and the result follows from additivity of $\mathcal{R}$-transforms under additive free convolution [57].

Recalling that the additive free convolution of semicircle laws is itself a semicircle law, Proposition 1.2 implies the following further decomposition of $\mu_{a,\nu,\gamma}$: Let $\{h_d\}_{d=0}^\infty$ denote any orthonormal basis
of functions $f : \mathbb{R} \to \mathbb{R}$ with respect to the inner product $\langle f, g \rangle_{\xi} := \mathbb{E}[f(\xi)g(\xi)]$ when $\xi \sim \mathcal{N}(0, 1)$, where $h_0(x) = 1$ and $h_1(x) = x$. Consider the corresponding orthogonal decomposition of the kernel function

$$k(x) = \sum_{d=1}^{\infty} a_d h_d(x)$$

(where $a_0 = 0$ because $\mathbb{E}[k(\xi)] = 0$), and the decomposition

$$K(X) = \sum_{d=1}^{\infty} K_d(X)$$

where $K_d(X)$ is the matrix (1) with kernel function $a_d h_d(x)$. Letting $\mu_d$ denote the limiting spectral measure of $K_d(X)$, which is $a(\mu_{\text{MP}, \gamma} - 1)$ for $d = 1$ and $|a_d|^{1/2} \mu_{\text{sc}}$ for $d \geq 2$, the limiting spectral measure of $K(X)$ is given by

$$\mu_{a, \nu, \gamma} = \mu_1 \boxplus \mu_2 \boxplus \mu_3 \boxplus \ldots$$

In the proof of our main result, we will apply such a decomposition of the kernel matrix when each $h_d$ is the degree-$d$ Hermite polynomial.

Proposition 1.2 implies, via the general analysis of [7], that $\mu_{a, \nu, \gamma}$ is compactly supported, has one interval of support when $\gamma \leq 1$ and at most two intervals of support when $\gamma > 1$, and (except for the singularity at 0 in the Marcenko-Pastur case $\nu = a^2$ and $\gamma > 1$) admits a density on all of $\mathbb{R}$ that is analytic in the interior of the support. The following may also be deduced from the $\mathcal{R}$-transform:

**Proposition 1.3.** Let $\text{supp}(\mu_{a, \nu, \gamma})$ denote the support of $\mu_{a, \nu, \gamma}$. If $a \geq 0$, then

$$\max\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\} \geq -\min\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\},$$

and if $a \leq 0$, then

$$\max\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\} \leq -\min\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\}.$$

**Proof.** Replacing $k(x)$ by $-k(x)$, it suffices to consider $a \geq 0$. The $\mathcal{R}$-transform (4) admits the series expansion

$$\mathcal{R}(z) = \gamma \nu z + \sum_{l \geq 2} a^{l+1} \gamma^l z^l$$

around $z = 0$, implying that the free cumulants of $\mu_{a, \nu, \gamma}$ are given by $\kappa_1 = 0$, $\kappa_2 = \gamma \nu$, and $\kappa_l = a^l \gamma^{l-1}$ for $l \geq 3$ [51]. The moments of $\mu_{a, \nu, \gamma}$ are then

$$\int x^l \mu_{a, \nu, \gamma}(dx) = \sum_{\pi \in \text{NC}_l} \prod_{S \in \pi} \kappa_{|S|},$$

where $\text{NC}_l$ denotes the set of all non-crossing partitions of $\{1, \ldots, l\}$ [51]. In particular, when $a \geq 0$, all moments of $\mu_{a, \nu, \gamma}$ are non-negative, whereas if $\max\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\} < -\min\{x : x \in \text{supp}(\mu_{a, \nu, \gamma})\}$, then the $l$th moment must be negative for a sufficiently large odd integer $l$. \qed

The support of $\mu_{a, \nu, \gamma}$ is easily numerically computed, as (3) is a cubic equation in $m(z)$, and $\text{supp}(\mu_{a, \nu, \gamma})$ is the set of $z \in \mathbb{R}$ for which this cubic equation has an imaginary root. The explicit form for the density function of $\mu_{a, \nu, \gamma}$ was provided in [20, Appendix A].
1.3. Main results. Denoting \( \| \mu_{a,\nu,\gamma} \| = \max \{|x| : x \in \text{supp}(\mu_{a,\nu,\gamma})\} \), the following is the main result of this paper:

**Theorem 1.4.** Suppose \( k : \mathbb{R} \to \mathbb{R} \) is odd (i.e., \( k(-x) = -k(x) \)) and continuously differentiable, with \( |k'(x)| \leq A e^{\beta|x|} \) for some constants \( A, \beta > 0 \) and all \( x \in \mathbb{R} \). Let \( X \in \mathbb{R}^{p \times n} \) have entries \( x_{ij} \overset{\text{IID}}{\sim} \mathcal{N}(0, 1) \). Then with \( \mu_{a,\nu,\gamma} \) as defined in Theorem 1.1, almost surely as \( n, p \to \infty \) with \( p/n \to \gamma \in (0, \infty) \),

\[
\| K(X) \| \to \| \mu_{a,\nu,\gamma} \|. 
\]

Proposition 1.3 yields the following corollary:

**Corollary 1.5.** Under the conditions of Theorem 1.4, if \( a := \mathbb{E}[k(\xi)] \geq 0 \), then almost surely

\[
\lambda_{\text{max}}(K(X)) \to \max \{x : x \in \text{supp}(\mu_{a,\nu,\gamma})\}. 
\]

(It may be verified, cf. our proof of the above corollary in Section 2 that any kernel function \( k \) satisfying the conditions of Theorem 1.4 also satisfies the conditions of Theorem 1.1.)

We will prove Theorem 1.4 via the following two auxiliary results, the first giving a non-asymptotic concentration bound on \( \| K(X) \| \) that is of constant order when \( n \asymp p \), and the second providing an asymptotically tight bound in the case where \( k(x) \) is a polynomial function:

**Theorem 1.6.** Suppose \( k : \mathbb{R} \to \mathbb{R} \) is odd, continuous, and differentiable almost everywhere with \( |k'(x)| \leq A e^{\beta|x|} \) for some \( A, \beta > 0 \) and all \( x \in \mathbb{R} \). Let \( X \in \mathbb{R}^{p \times n} \) have entries \( x_{ij} \overset{\text{IID}}{\sim} \mathcal{N}(0, 1) \). Then, for any \( \alpha > 0 \), there exist constants \( C, C' > 0 \) depending only on \( A, \beta \), and \( \alpha \) such that

\[
P \left( \| K(X) \| > C \max \left( \frac{p}{n}, \sqrt{\frac{p}{n}} \right) \right) \leq C'(p^{-\alpha} + p e^{-\alpha n}).
\]

**Theorem 1.7.** Let \( k \) be a polynomial function such that \( \mathbb{E}[k(\xi)] = 0 \) when \( \xi \sim \mathcal{N}(0, 1) \), and let \( a_2 := \frac{1}{\sqrt{2}} \mathbb{E}[k(\xi)(\xi^2 - 1)] \). Let \( X \in \mathbb{R}^{p \times n} \) have IID entries that are symmetric in law \( (x_{ij} \overset{\text{L}}{=}-x_{ij}) \) and satisfy \( \mathbb{E}[x_{ij}^2] = 1 \) and

\[
\mathbb{E}[|x_{ij}|^k] \leq k^{\alpha k}
\]

for all \( k \geq 2 \) and some \( \alpha > 0 \). Then

\[
K(X) = \hat{K}(X) + \hat{R}(X)
\]

where \( \hat{K}(X) \) and \( \hat{R}(X) \) are such that, as \( n, p \to \infty \) with \( p/n \to \gamma \in (0, \infty) \),

1. \( \| \hat{K}(X) \| \to \| \mu_{a,\nu,\gamma} \| \) almost surely, and
2. \( \hat{R}(X) = 0 \) if \( a_2 = 0 \), and otherwise \( \hat{R}(X) \) is of rank at most two, with non-zero eigenvalues converging to \( \pm a_2 \sqrt{(\mathbb{E}[x_{ij}^4] - 1)/2} \).

The precise form of the rank-two matrix \( \hat{R}(X) \) is given by (9) in Section 2. We make the trivial observation that \( a_2 = 0 \) and \( \hat{R}(X) = 0 \) if the polynomial \( k(x) \) is an odd function.

Theorem 1.4 follows from Theorems 1.6 and 1.7 via a polynomial approximation argument, which we present in Section 2. The assumption that \( k(x) \) is odd, or more specifically that \( a_2 = 0 \), is important: Figure 2 displays the simulated spectrum of \( K(X) \) for a kernel function where \( a_2 \neq 0 \), in which we see that \( \hat{R}(X) \) contributes two spike eigenvalues to \( K(X) \) that fall outside of \( \text{supp}(\mu_{a,\nu,\gamma}) \). In the covariance thresholding application of Section 1.1, commonly-used threshold functions are indeed odd. We recommend caution if using a non-odd threshold function, as the possible presence of these spurious spike eigenvalues may lead to the incorrect inference that \( \Sigma \) has non-trivial spike eigenvectors, even in this null setting where \( \Sigma = \text{Id} \).
Figure 2. Simulated spectrum of $K(X)$ when $k(x) := h_2(x) + h_3(x) = \frac{1}{\sqrt{2}}(x^2 - 1) + \frac{1}{\sqrt{6}}(x^3 - 3x)$, $n = 1000$, and $p = 10000$. The semicircle limit for the spectral distribution is superimposed in black, and the locations of two observed outlier eigenvalues of $K(X)$ are indicated with red arrows.

1.4. Further related literature. The off-diagonal entries of $K(X)$ are the evaluations of a symmetric kernel $f(u, v) := k(u^T v / \sqrt{n}) / \sqrt{n}$ on pairs of rows of $X$. Such matrices for general kernels $f(u, v)$ are used in “kernel methods” in statistics and machine learning, such as SVM classifiers [10] and kernel PCA [17]. Koltchinskii and Giné [35] studied the spectra of kernel matrices in a regime where each row of $X$ is sampled from a probability distribution over a fixed space (for example $\mathbb{R}^n$ for fixed $n$), showing that under suitable conditions, as $p \to \infty$, the spectrum converges to that of a limiting infinite-dimensional operator. El Karoui [25] studied kernel matrices in the regime $n, p \to \infty$ with $p/n \to \gamma \in (0, \infty)$ under the alternative scaling $f(u, v) := k(u^T v / n)$, showing that under mild conditions, the matrix is asymptotically equivalent to a linear combination of $XX^T$, the all-1’s matrix, and the identity, and hence the limiting spectrum is Marcenko-Pastur. The scaling in (1) is different from the regime considered in [25]; each off-diagonal entry of $\hat{\Sigma}$ has typical size $1/\sqrt{n}$, and hence (1) applies the nonlinearity $k$ to values of size $O(1)$ rather than $O(1/\sqrt{n})$. This and more general scalings were studied probabilistically in [20], and the results were further generalized in [23]. Let us remark that [35, 25] considered distributions for the rows of $X$ where the entries are not necessarily IID, but that the extension of our result to more general covariances $\Sigma \in \mathbb{R}^{p \times p}$ for the application of Section 1.1 will require the study of a model in which the columns (rather than the rows) of $X$ are independent with this covariance.

Sparse PCA has been widely studied in statistics for both the “single spike” model (2) as well as multi-spike models. Computationally-efficient procedures for estimating sparse principal components include diagonal thresholding [31, 32, 9], $\ell_1$- and model-selection-penalization approaches [33, 32, 21, 49, 60, 12], iterative thresholding via the QR method [40], approximate message passing [22], and covariance thresholding as discussed in Section 1.1 [36, 22]. From the theoretical perspective, both exact [1, 36] and approximate [32, 9, 40, 59, 12, 13] sparsity models have been considered, and a major focus has been on rate-optimal recovery of the sparse eigenvectors, their spanned subspace, and/or the sparse covariance [9, 40, 12, 59, 13]. Support recovery and spike detection in the specific model (2) were considered in [1, 5, 6, 36, 22]. In this setting, non-polynomial-time algorithms can detect the spike and recover the support even when $v$ has sparsity near-linear in $n$ [1, 5, 13], but it is conjectured that polynomial-time methods require the higher sparsity levels $\|v\|_0 \lesssim \sqrt{n}$. This problem is closely related to the planted clique problem in computer science [51, 6] upon which this conjecture is based, with a vector $v$ of sparsity $\|v\|_0 \asymp \sqrt{n}$ corresponding to a planted clique of size $k \asymp \sqrt{n}$ in a graph of $n$ vertices.
Consistency of elementwise hard-thresholding for estimating sparse covariance matrices was studied in [8, 24]. Optimal rates of convergence under various matrix norms and sparsity models were established in [15, 14], and generalizations to other thresholding functions and to “adaptive” entry-specific thresholds were studied respectively in [46] and [11]. Many analyses assume that each row of \( \Sigma \) contains \( \ll \sqrt{n} \) non-zero elements and perform thresholding at the level \( \sqrt{(\log p)/n} \) or higher, which does not apply to the regime of interest discussed in Section 1.1 where \( v \) has sparsity \( \|v\|_0 \asymp \sqrt{n} \) and non-zero elements of size \( n^{-1/4} \). Thresholding at more general levels, including \( 1/\sqrt{n} \), was studied in [22], which established a special case of Theorem 1.6 for the soft-thresholding kernel function \( k \). The proof of [22] may be extended to globally Lipschitz functions \( k \), but we require the application of such a bound when \( k \) is the difference of the (possibly Lipschitz) kernel function of interest and a polynomial approximation to this function. This difference may increase at any polynomial rate as \( |x| \to \infty \), hence requiring new ideas in the proof of Theorem 1.6 to extend beyond Lipschitz kernels. Other spectral norm bounds for polynomial kernels were derived in [20] and [34], but they do not yield the desired bound of constant order when restricted to our setting.

In the context of random matrix theory, convergence of the extremal eigenvalues of \( K(X) \) was posed as an open question in [20]. For linear kernels \( k \), \( K(X) \) is equivalent to a translation and rescaling of the sample covariance \( \hat{\Sigma} \), and almost-sure convergence of the extremal eigenvalues follows from [29, 61, 2]. Proposition 1.2 implies that in the general case, \( K(X) \) has the same limiting spectrum as a deformed Wigner matrix \( W + V \) where \( W \) is Wigner and \( V \) is deterministic with spectral measure converging to \( a(\mu_{MP, \gamma} - 1) \) [58]. When \( W \) is GUE and \( V \) has no spike eigenvalues, the results of [16, 41] imply that the eigenvalues of \( W + V \) stick to the limiting support, and the fluctuations of the eigenvalues at the edges of the support are also understood in various settings [48, 17, 39]. The proof of our main result leverages the connection between these models.

Our proof uses the moment method and is different from the resolvent analysis of [20], although the decomposition of \( k(x) \) in the Hermite polynomial basis plays an important role in both analyses. While the resolvent method has been successful in establishing many properties of Wigner and covariance matrices (see e.g. [50, 30, 55, 26, 45] as well as the recent work of [52, 19] in a non-independent setting), the model [1] for nonlinear kernels does not have the same independence structure as these models, and it is also not a sum of rank-one updates. These difficulties were overcome in [20] via Gaussian conditioning arguments, but strengthening the bounds of [20] to yield finer control of the Stieltjes transform \( m(z) \) near the real axis does not seem (in our viewpoint) more straightforward than our moment-based approach. We believe that our combinatorial estimates and moment-comparison argument, in the simpler setting of a fixed moment \( l \) not varying with \( n \), are sufficient to yield an alternative proof of Theorem 1.1 and also to establish asymptotic freeness of the matrices \( K_1(X), K_2(X), \ldots \) leading to the decomposition (5). For brevity, we will not discuss this in the current paper.

### 1.5. Notation

\( \|v\| = (\sum_i v_i^2)^{1/2} \) denotes the Euclidean norm for vectors. \( \|X\| = \max_{\|v\|=1} \|Xv\| \) denotes the spectral norm (i.e. \( \ell_2 \)-operator norm) for matrices. \( X_i \) denotes the \( i \)-th row of \( X \). If \( X \in \mathbb{R}^{p \times p} \) is symmetric, \( \lambda_{\max}(X) := \lambda_1(X) \geq \ldots \geq \lambda_p(X) \) denote the ordered eigenvalues of \( X \). \( \text{supp}(\mu) \) denotes the support of a measure \( \mu \), and \( \|\mu\| \) denotes \( \max\{|x| : x \in \text{supp}(\mu)\} \).

In an asymptotic setting, for positive (\( n, p \)-dependent) quantities \( a \) and \( b \), \( a \asymp b \) means \( ca \leq b \leq Ca \) for constants \( C, c > 0 \), \( a \sim b \) means \( a/b \to 1 \), \( a \ll b \) means \( a/b \to 0 \), and \( a \lesssim b \) means \( a \leq Cb \) for a constant \( C > 0 \).

We will use \( i, i', i_1, i_2, \ldots \) for indices in \( \{1, \ldots, p\} \), and \( j, j', j_1, j_2, \ldots \) for indices in \( \{1, \ldots, n\} \).

### 2. Overview of proof

In this section, we summarize the high-level proof ideas for Theorems 1.6 and 1.7, and we establish Theorem 1.4 and Corollary 1.5 using these results.
The proof of Theorem 1.6 uses a covering net argument:

$$
\|K(X)\| \leq C \sup_{y \in D^p_2} y^T K(X)y,
$$

for a constant $C > 0$ and a finite covering net $D^p_2$ of the unit ball $\{y \in \mathbb{R}^p : \|y\| \leq 1\}$. We use a particular construction of a covering net due to Latala [37]:

**Definition 2.1.** For $m = \lceil \log_2 p \rceil$, let

$$
D^p_2 = \left\{ y \in \mathbb{R}^p : \|y\| \leq 1, \ y_i^2 \in \{0, 1, 2^{-1}, 2^{-2}, \ldots, 2^{-(m+3)}\} \text{ for all } i \right\}.
$$

For each $l = 0, 1, \ldots, m + 3$, let $\pi_l : D^p_2 \to D^p_2$ be defined by $(\pi_l(y))_i = y_i1\{y_i^2 \geq 2^{-l}\}$, and let $\pi_{l\setminus l-1} : D^p_2 \to D^p_2$ be defined by $(\pi_{l\setminus l-1}(y))_i = y_i1\{y_i^2 = 2^{-l}\}$.

Corresponding to the identity $y = \sum_{l=0}^{m+3} \pi_{l\setminus l-1}(y)$ for any $y \in D^p_2$, $y^T K(X)y$ may be decomposed as

$$
y^T K(X)y = \sum_{l=0}^{m+3} \pi_l(y)^T K(X) \pi_{l\setminus l-1}(y) + \sum_{l=1}^{m+3} \pi_{l\setminus l-1}(y)^T K(X) \pi_{l\setminus l-1}(y).
$$

Each of the terms

$$
\sup_{y \in D^p_2} \pi_l(y)^T K(X) \pi_{l\setminus l-1}(y), \quad \sup_{y \in D^p_2} \pi_{l\setminus l-1}(y)^T K(X) \pi_{l\setminus l-1}(y)
$$

may be bounded via a standard union bound, with quantities of the form $F_{y,z}(X) := y^T K(X)z$ controlled by bounding the gradient $\|\nabla_X F_{y,z}(X)\|$ and applying Gaussian concentration of measure for Lipschitz functions. The key idea of the construction of $D^p_2$ and the decomposition (7) is that for each $l$, the union bound may be applied over $y \in \pi_l(D^p_2)$, which has smaller cardinality for smaller $l$. For larger $l$, the entries of $\pi_{l\setminus l-1}(y)$ are smaller, which we will show implies stronger control of the gradient $\|\nabla_X F_{y,z}(X)\|$ over a high-probability set $X \in \mathcal{G}$. The moment generating function of $F_{y,z}(X)$ may be controlled using the integration argument of Maurey and Pisier, by extending this high-probability set to pairs of matrices $(X, X')$ in such a way that we remain in this set along the entire integration path. The cardinality of $\pi_l(D^p_2)$ balances the moment generating function bound thus obtained for each $l$, yielding Theorem 1.6. Details of this argument are given in Section 3.

The proof of Theorem 1.7 uses the moment method and a moment comparison with a deformed GUE matrix. We first define the orthonormal Hermite polynomials, which play a central role in our proof (as well as in the proof in [20] for Theorem 1.1):

**Definition 2.2.** Let $\{h_d\}_{d=0}^\infty$ denote the orthonormal Hermite polynomials with respect to the inner product $\langle f, g \rangle = \mathbb{E}[f(\xi)g(\xi)]$ when $\xi \sim \mathcal{N}(0, 1)$, i.e. $h_d$ is of degree $d$ and $\langle h_d, h_{d'} \rangle_\xi = 1\{d = d'\}$.

The first few such polynomials are given by $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$, and $h_3(x) = \frac{1}{\sqrt{6}}(x^3 - 3x)$.

Our proof of Theorem 1.7 follows three high-level steps:

1. For IID random variables $z_1, \ldots, z_n$ with $\mathbb{E}[z_i] = \mathbb{E}[z_i^2] = 0$ and $\mathbb{E}[z_i^3] = 1$, we show that

$$
\sqrt{d!}h_d \left( \frac{\sum_{i=1}^n z_i}{\sqrt{n}} \right) \approx \sqrt{\frac{1}{n^d}} \sum_{j_1, \ldots, j_d}^{n} \prod_{i=1}^{d} z_{j_i}.
$$

(The summation on the right side is over all tuples of distinct indices $j_1, \ldots, j_d \in \{1, \ldots, n\}$.) Each $h_d$ has leading coefficient $1/\sqrt{d!}$, so $\sqrt{d!}h_d(x) = x^d + \text{lower degree terms}$. Replacing $\sqrt{d!}h_d(x)$ with $x^d$ on the left side would yield the right side of (8) without the restriction that the indices of summation $j_1, \ldots, j_d$ are distinct; (8) states that the terms of this summation in which the indices $j_1, \ldots, j_d$ are not distinct are essentially cancelled out by the lower
degree terms of $\sqrt{d}h_d(x)$. We prove this approximation in Section 4 by induction on $d$, using the three-term recurrence for Hermite polynomials. The right side of (8) is of typical size $O(1)$, and we also quantify the error of the approximation by computing a second-order term, which is of typical size $O(n^{-1/2})$, and showing that the third and higher-order terms in this approximation are of typical size $O(n^{-1})$.

(2) Since $k$ is a polynomial such that $E[k(\xi)] = 0$, we may write

$$k(x) = \sum_{d=1}^{D} a_d h_d(x)$$

where $D < \infty$ is the degree of $k$. Applying the approximation in step (1) above to each $h_d$, we obtain a decomposition

$$K(X) = Q(X) + R(X) + S(X),$$

where $Q$, $R$, and $S$ correspond to the first-order, second-order, and third-and-higher-order terms of these approximations (each summed over all $d = 1, \ldots, D$). We establish for the first-order matrix $Q(X)$ that

$$\limsup_{n,p \to \infty} \|Q(X)\| \leq \|\mu_{a,\nu,\gamma}\|$$

almost surely, via a moment comparison argument: For an even integer $l \asymp \log n$, we apply the standard moment method bound $\|Q(X)\|^l \leq \text{Tr} Q(X)^l$ \cite{25,29}. By \cite{8}, the non-diagonal entries of $Q(X)$ are given by

$$Q(X)_{ij'} = \sum_{d=1}^{D} a_d n^{-d/2} \sum_{j_1, \ldots, j_d = 1}^{n} \prod_{s=1}^{d} x_{i j_s} x_{i' j_s}$$

We expand the trace $\text{Tr} Q(X)^l$ and interpret the terms of the resulting sum as labelings of a certain graph. We then consider a deformed GUE matrix $M = W + V$ having the same limiting spectrum as $K(X)$, and employ a combinatorial argument to upper-bound $E[\text{Tr} Q(X)^l]$ using $E[\text{Tr} M^l]$. We conclude the proof by using the known convergence result $\|M\| \to \|\mu_{a,\nu,\gamma}\|$ from \cite{16} and a concentration of measure argument to bound $E[\text{Tr} M^l]$. We present the main ideas of this step in Section 5 with details deferred to Appendices A and B.

(3) Finally, we analyze the remainder matrices $R(X)$ and $S(X)$ from the decomposition in step (2) above. It is easily shown that $\|S(X)\| \to 0$. For $R(X)$, we may write

$$R(X) = \sum_{d=2}^{D} R_d(X),$$

where $R_d(X)$ is the contribution from the Hermite polynomial $h_d$. (The linear polynomial $h_1$ does not have such a remainder term in the decomposition.) We show $\|R_d(X)\| \to 0$ for each $d \geq 3$, and $\|R_2(X) - \tilde{R}(X)\| \to 0$ where

$$\tilde{R}(X) = \frac{a_2}{n\sqrt{2}} (v(X)1^T + 1v(X)^T),$$

for $1 = (1, \ldots, 1) \in \mathbb{R}^p$, and $v(X) \in \mathbb{R}^p$ has entries $(v(X))_i = \sum_{j=1}^{n} (x_{ij}^2 - 1)/\sqrt{n}$. Noting that $\tilde{R}(X)$ is a rank-two matrix, this yields Theorem 1.7 upon setting $\tilde{K}(X) = K(X) - \tilde{R}(X)$. This argument and the conclusion of the proof of Theorem 1.7 are presented in Section 6.

Let us now prove Theorem 1.4 and Corollary 1.5 using Theorems 1.6 and 1.7. We approximate the derivative of the kernel function by a polynomial using the following result:
Theorem 2.3 (Carleson [18]). Suppose \( w(x) \) is an even, lower semi-continuous function on \( \mathbb{R} \) with \( 1 \leq w(x) < \infty \), such that \( \log w(x) \) is a convex function of \( \log x \). Let \( C_w \) be the class of continuous functions on \( \mathbb{R} \) such that \( \lim_{|x| \to \infty} f(x)/w(x) = 0 \) for all \( f \in C_w \), and suppose \( C_w \) contains all polynomial functions. If \( \int_1^\infty (\log w(x))/x^2 \, dx = \infty \), then for any \( f \in C_w \) and \( \epsilon > 0 \), there exists a polynomial \( P \) such that \( |f(x) - P(x)| < \epsilon w(x) \) for all \( x \in \mathbb{R} \).

Proof of Theorem 1.4. By the given conditions, there exists \( \beta > 0 \) such that \( \lim_{|x| \to \infty} |k'(x)|/e^{\beta|x|} = 0 \). Applying Theorem 2.3 with \( w(x) = e^{\beta|x|} \), for any \( \epsilon > 0 \), there exists a polynomial \( \hat{q} \) such that \( |k'(x) - \hat{q}(x)| < \epsilon e^{\beta|x|} \) for all \( x \in \mathbb{R} \). As \( k \) is an odd function, \( k' \) is even, so we may take \( \hat{q} \) to be an even polynomial function. (Otherwise, take the polynomial to be \( \frac{1}{2}(q(x) + q(-x)) \).) Let \( q(x) = \int_0^x \hat{q}(x) \, dx \) for all \( x \in \mathbb{R} \), and let \( r(x) = k(x) - q(x) \). Then \( q \) is an odd polynomial function, \( r \) is hence also an odd function, and \( |r(x)| < \epsilon e^{\beta|x|} \) by construction. Let \( Q(X) \) be the kernel matrix \([1]\) with kernel function \( q(x) \), and let \( R(X) \) be the kernel matrix \([1]\) with kernel function \( r(x) \), so that \( K(X) = Q(X) + R(X) \). (These matrices \( Q(X) \) and \( R(X) \) are not related to the matrices \( Q, R, \) and \( S \) in the above proof outline.)

Applying Theorem 1.6 with \( \alpha = 2 \) to \( R(X) \), \( \limsup_{n,p \to \infty} \|R(X)\| < \epsilon C_{\beta,\gamma} \) almost surely for some constant \( C_{\beta,\gamma} > 0 \). On the other hand, if \( q(x) = a_{0,\epsilon} + a_{1,\epsilon} h_1(x) + \ldots + a_{D,\epsilon} h_D(x) \) where \( h_1, \ldots, h_D \) are the orthonormal Hermite polynomials of Definition 2.2, then \( a_{j,\epsilon} = 0 \) for all even \( j \) (since \( q \) is an odd function), and Theorem 1.7 implies \( \|Q(X)\| \to \|\mu_{a,\nu,\gamma}\| \) where \( a = a_{1,\epsilon} \) and \( \nu = \sum_{d=1}^D a_{d,\epsilon}^2 \). Hence, almost surely,

\[
\|\mu_{a,\nu,\gamma}\| - \epsilon C_{\beta,\gamma} < \liminf_{n,p \to \infty} \|K(X)\| \leq \limsup_{n,p \to \infty} \|K(X)\| < \|\mu_{a,\nu,\gamma}\| + \epsilon C_{\beta,\gamma}
\]

for any \( \epsilon > 0 \). Note that \( |k(x) - q(x)| \leq \frac{\xi}{2} e^{\beta|x|} \) for all \( x \in \mathbb{R} \), so by dominated convergence \( \lim_{\epsilon \to 0} \mathbb{E}[(k(\xi) - q(\xi))^2] = 0 \) for \( \xi \sim \mathcal{N}(0,1) \). Then \( a_\epsilon \to a \) and \( \nu_\epsilon \to \nu \) as \( \epsilon \to 0 \), where \( a := \mathbb{E}[k(\xi)] \) and \( \nu := \mathbb{E}[k(\xi)^2] \). As \( \|\mu_{a,\nu,\gamma}\| \) is continuous in \( a, \nu, \) and \( \gamma \), \( \lim_{\epsilon \to 0} \|\mu_{a,\nu,\gamma}\| \to \|\mu_{a,\nu,\gamma}\| \), and hence taking \( \epsilon \to 0 \) yields \( \lim_{n,p \to \infty} \|K(X)\| = \|\mu_{a,\nu,\gamma}\| \) almost surely. \( \Box \)

Proof of Corollary 1.5. We verify the conditions of Theorem 1.1. The kernel function is odd and bounded as \( |k(x)| \leq C e^{\beta|x|} \) for a constant \( C := C_{A,\beta} > 0 \), so \( \mathbb{E}[k(\xi)] = 0 \) and \( \mathbb{E}[k(\xi)^2] < \infty \). Writing \( y := \sqrt{n} \tilde{\Sigma}_{12} \), for any \( R > 0 \)

\[
\mathbb{E}[k(y)^2 \mathbb{I}\{|y| \geq R\}] \leq C^2 \mathbb{E}[e^{4\beta|y|}]^{1/2} \mathbb{P}|\{y| \geq R\}|^{1/2}.
\]

Note that \( \mathbb{E}[e^{4\beta y}] = \mathbb{E}[e^{-4\beta y}] = (1 - 16\beta^2/n)^{-n/2} \) for all \( n > 16\beta^2 \), so \( \mathbb{E}[e^{4\beta|y|}] \) is bounded by a constant for all large \( n \). By Lemma C.4 of [20], \( \mathbb{P}|\{y| \geq R\}|^{1/2} \to 0 \) as \( R \to \infty \) uniformly in \( n \). Then Lemma C.5 of [20] implies that the remaining technical condition of Theorem 1.1 holds. Theorems 1.1 and 1.3 then together imply

\[
\max\{x : x \in \text{supp}(\mu_{a,\nu,\gamma})\} \leq \liminf_{n,p \to \infty} \lambda_{\text{max}}(K(X)) \leq \limsup_{n,p \to \infty} \lambda_{\text{max}}(K(X)) \leq \|\mu_{a,\nu,\gamma}\|,
\]

and the result follows as the left and right sides coincide by Proposition 1.3. \( \Box \)

3. Proof of Concentration Inequality

In this section, we prove Theorem 1.6 following the outline sketched in Section 2. By rescaling \( k(x) \), we may assume without loss of generality \( A = 1 \). We denote by \( X_i \) the \( i^{th} \) row of \( X \).

Lemma 3.1. For any \( \alpha, \beta > 0 \), there exist constants \( C, C' > 0 \) depending only on \( \alpha \) and \( \beta \) such that the following holds: Define \( \mathcal{G}(\alpha, \beta) \subset \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times n} \) as the set of pairs \( (X, X') \) of \( p \times n \) matrices
such that \( \|X\| \leq \sqrt{p} + (1 + \sqrt{2\alpha})\sqrt{n} \), \( \|X'\| \leq \sqrt{p} + (1 + \sqrt{2\alpha})\sqrt{n} \), and for each \( l = 1, \ldots, p \)

\[
\frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i^T X_l| \right) \leq C, \quad \frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i'X_l'| \right) \leq C,
\]

\[
\frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i'^T X_l| \right) \leq C, \quad \frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i^T X_l'| \right) \leq C.
\]

If \( X, X' \in \mathbb{R}^{p \times n} \) are random and independent with \( x_{ij}, x_{ij}' \) IID \( \mathcal{N}(0,1) \), then

\[
\mathbb{P}[ (X, X') \notin \mathcal{G}(\alpha, \beta) ] \leq C' \left( p^{-\alpha} + pe^{-\alpha n} \right).
\]

**Proof.** By Corollary 5.35 of [56], \( \mathbb{P}[ \|X\| > \sqrt{p} + (1 + \sqrt{2\alpha})\sqrt{n} ] \leq 2e^{-\alpha n} \), and similarly for \( X' \).

For \( \xi \sim \mathcal{N}(0,1) \) and any \( u > 0 \), \( \mathbb{E}[e^{u|\xi|}] \leq \mathbb{E}[e^{u\xi}] + \mathbb{E}[e^{-u\xi}] = 2e^\frac{u^2}{2} \) and \( \text{Var}[e^{u|\xi|}] \leq \mathbb{E}[e^{2u|\xi|}] \leq 2e^{2u^2} \). Let \( C(\alpha, u) \) and \( c(\alpha, u) \) denote large and small constants that may change from instance to instance. Defining \( f(\xi) = e^{u|\xi|} - \mathbb{E}[e^{u|\xi|}] \), \( \mathbb{E}[|f(\xi)|^2] \leq C(\alpha, u) \). Then for \( \xi_1, \ldots, \xi_p \) IID \( \mathcal{N}(0,1) \), applying Corollary 4 of [27] with \( t = \alpha + 2 \),

\[
\mathbb{P} \left[ \frac{1}{p} \sum_{i=1}^{p} e^{u\xi_i} > 3e^{u^2} \right] \leq \mathbb{P} \left[ \sum_{i=1}^{p} f(\xi_i) > pe^{u^2} \right] \leq C(\alpha, u)p^{-\alpha-1}.
\]

For any \( i \neq l \), \( (X_i^T X_l, X_l) \overset{L}{=} ((X_l, \xi_i, X_l) \) where \( \xi_i \sim \mathcal{N}(0,1) \) is independent of \( X_l \). Hence

\[
\mathbb{P} \left[ \frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i^T X_l| \right) > 3e^{\frac{128\beta^2|X_i||X_l|^2}{n}} \right] \leq C \left( \alpha, \frac{\|X_l\|}{\sqrt{n}} \right) p^{-\alpha-1},
\]

and

\[
\mathbb{P} \left[ \frac{1}{p} \sum_{i=1 \atop i \neq l}^{p} \exp \left( \frac{16\beta}{\sqrt{n}} |X_i^T X_l| \right) > C(\alpha, \beta) \right] \leq C'(\alpha, \beta) p^{-\alpha-1}
\]

for some constants \( C(\alpha, \beta) \) and \( C'(\alpha, \beta) \). Lemma 1 of [38] implies the chi-squared tail bound \( \mathbb{P}[\|X_l\|^2 > (1 + 2\alpha + 2\sqrt{\alpha})n] \leq e^{-\alpha n} \). The same argument holds for the analogous sums with \( X_i^T X_l, X_i^T X_l' \), and \( X_i^T X_l' \) in place of \( X_i^T X_l \), and the result follows by a union bound over \( l \).

**Lemma 3.2.** Let \( y, z \in \mathbb{R}^p \) satisfy \( \|y\| \leq 1 \) and \( \|z\| \leq 1 \). Under the setup of Theorem 3.4, let \( F(X) = z^T K_{\alpha,p}(X)y \) and define \( \mathcal{G}(\alpha, \beta) \) as in Lemma 3.1. Then for a constant \( C := C(\alpha, \beta) > 0 \) and any \( t > 0 \),

\[
\mathbb{E} \left[ e^{t(F(X) - F(X'))} \mathbb{1}\{ (X, X') \in \mathcal{G}(\alpha, \beta) \} \right] \leq 2 \exp \left( \frac{C\|y\|_{\infty} t^2 p^{1/2}(n + p)}{n^2} \right).
\]

**Proof.** Consider \( F \) as a function from \( \mathbb{R}^{pn} \) to \( \mathbb{R} \). The gradient with respect to column \( l \) of \( X \) is

\[
\nabla_{X_l} F(X) = \nabla_{X_l} \left( \sum_{i=1}^{p} \sum_{i' \neq l}^{p} \frac{1}{\sqrt{n}} \left( \frac{X_i^T X_{i'}}{\sqrt{n}} \right) z_i y_{i'} \right).
\]
where the last inequality applies \( v^T M w \leq \|v\|_1 M w \|_{\infty} \). Applying Cauchy-Schwarz and the bound \( \|y\|_4^2 \leq \|y\|_2 \|y\|_\infty \leq \|y\|_\infty \),

\[
\|\nabla F(X)\|_4^2 \leq \frac{4\|X\|_2^2 \|y\|_\infty}{n^2} \max_{i=1}^p \left( \sum_{l=1}^p k^l \left( \frac{X_l X_l}{\sqrt{n}} \right)^4 \right)^{1/2} \left( \sum_{i=1}^p \sum_{l \neq i} k^l \left( \frac{X_l X_l}{\sqrt{n}} \right)^2 \right)^{1/2}.
\]

We apply the integration argument of Maurey and Pisier: For each \( \theta \in [0, \frac{\pi}{2}] \), let \( X_\theta = X' \cos \theta + X \sin \theta \) and \( \tilde{X}_\theta = -X' \sin \theta + X \cos \theta \). Then

\[
\mathbb{E} \left[ e^{(F(X) - F(X'))} 1\{(X, X') \in \mathcal{G}(\alpha, \beta)\} \right]
\]

\[
= \mathbb{E} \left[ e^{\frac{x}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta} \right] \mathbb{E} \left[ e^{-c \theta} \left( \frac{\pi}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta \right) \right] 1\{(X, X') \in \mathcal{G}(\alpha, \beta)\}
\]

\[
= 2 \int_0^{\pi/2} \frac{d}{d\theta} \mathbb{E} \left[ e^{\frac{x}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta} \right] \mathbb{E} \left[ e^{-c \theta} \left( \frac{\pi}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta \right) \right] d\theta,
\]

where \( \nabla F(X_\theta)^T \tilde{X}_\theta \) represents the vector inner-product in \( \mathbb{R}^m \). Noting that \( X_\theta \) and \( \tilde{X}_\theta \) are independent and both equal in law to \( X \), we may first condition on \( X_\theta \) and use the Cauchy-Schwarz inequality and the bound \( \mathbb{E}[e^{c(X_\theta)}] \leq \mathbb{E}[e^{c(\tilde{X}_\theta)}] + \mathbb{E}[e^{-c(\tilde{X}_\theta)}] \leq 2e^{c^2/2} \) to obtain

\[
\mathbb{E} \left[ e^{(F(X) - F(X'))} 1\{(X, X') \in \mathcal{G}(\alpha, \beta)\} \right]
\]

\[
\leq \frac{2}{\pi} \int_0^{\pi/2} \mathbb{E} \left[ e^{\frac{x}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta} \right] \mathbb{E} \left[ e^{-c \theta} \left( \frac{\pi}{2} \int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta) d\theta \right) \right] d\theta
\]

\[
\leq \frac{4}{\pi} \int_0^{\pi/2} \mathbb{E} \left[ e^{\frac{x^2}{4} \|\nabla F(X_\theta)\|^2} \right] \mathbb{E} \left[ 1\{(X, X') \in \mathcal{G}(\alpha, \beta)\} \left| X_\theta \right|^{1/2} \right] d\theta
\]
Proof. For any symmetric matrix $M$, Lemma 3.3. Then for any $(X, X') \in G(\alpha, \beta)$, the result follows.

Let us now recall $D_2^p$, $\pi_{l-1}$, and $\pi_l$ from Definition 2.1.

Lemma 3.3. For any symmetric matrix $M \in \mathbb{R}^{p \times p}$, $\|M\| \leq 10 \sup_{y \in D_2^p} y^T M y$.

Proof. For any $x \in \mathbb{R}^p$ with $\|x\| < 1$, we may construct $y \in D_2^p$ such that

$$y_i = \begin{cases} 2^{-\frac{l}{2}} \text{sign}(x_i) & 2^{-l} \leq x_i^2 < 2^{-l+1} \\ 0 & x_i^2 < 2^{-l+3} \end{cases}.$$ 

Then $\|y\| \leq \|x\| < 1$ and, letting $c = (1 - 1/\sqrt{2})^2$, $\|x - y\|^2 = \sum_{i: x_i^2 \geq 2^{-l+3}} (x_i - y_i)^2 + \sum_{i: x_i^2 < 2^{-l+3}} x_i^2 \leq \sum_{i: x_i^2 \geq 2^{-l+3}} cx_i^2 + \sum_{i: x_i^2 < 2^{-l+3}} x_i^2 < c + (1 - c) \sum_{i: x_i^2 < 2^{-l+3}} x_i^2 < c + \frac{1 - c}{8} < (9/20)^2.$

The result then follows from Lemma 5.4 of [56].

Lemma 3.4. Let $m = \lfloor \log_2 p \rfloor$. For some $C > 0$ and all $l \in \{0, 1, \ldots, m + 3\}$,

$$\log |\{\pi_l(y) : y \in D_2^p\}| \leq C(m + 4 - l)2^l.$$
Proof. Let $C > 0$ denote a constant that may change from instance to instance. For any $l \in \{0, 1, \ldots, m\}$,
\[
|\{\pi_{l,l-1}(y) : y \in D^p_2\}| \leq \sum_{k=0}^{2^l} \binom{p}{k} 2^k,
\]
as there are at most $2^l$ non-zero entries of $\pi_{l,l-1}(y)$, and for each non-zero entry there are two choices of sign. Using $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, and noting that $k \mapsto (2ep)^k k^{-k}$ is monotonically increasing over $k \in [0, 2p]$ and that $2^l \leq 2p$ for $l \leq m$, this implies
\[
\log |\{\pi_{l,l-1}(y) : y \in D^p_2\}| \leq \log \left(1 + 2^l \left(2ep\right)^{2^l}\right) \leq \log \left(1 + 2^l \left(2e2^{m-l}\right)^{2^l}\right) \leq C(m + l2^l).
\]
For $l \in \{m + 1, m + 2, m + 3\}$, we use the bound $|\{\pi_{l,l-1}(y) : y \in D^p_2\}| \leq 3^p$, as each coordinate of $\pi_{l,l-1}(y)$ takes one of three values. Then
\[
\log |\{\pi_{l,l-1}(y) : y \in D^p_2\}| \leq C 2^m \leq C(m + 4 - l)2^l.
\]
Combining these bounds,
\[
\log |\pi_l(y)| \leq \sum_{j=0}^l \log |\pi_{j+1}(y)| \leq C \sum_{j=0}^l (m + 4 - j)2^j \leq C(m + 4 - l)2^l.
\]
\[\square\]

Lemma 3.5. Under the setup of Theorem 1.6, let $m = \lceil \log_2 p \rceil$ and let $\mathcal{G}(\alpha, \beta)$ be as in Lemma 3.4. Then there are constants $C, c > 0$ depending only on $\alpha$ and $\beta$ such that for any $l \in \{0, 1, \ldots, m+3\}$, $j = l$ or $j = l - 1$ (if $l \geq 1$), and $t > 0$,
\[
\mathbb{P} \left[ \sup_{y \in D^p_2} \pi_j(y)^T K(X) \pi_{l,l-1}(y) > t \text{ and } (X, X') \in \mathcal{G}(\alpha, \beta) \right] \leq 2 \exp \left( C(m + 4 - l)2^l - \frac{ct2^{l/2}n^2}{p^{1/2}n^2} \right).
\]

Proof. For notational convenience, define the event $\mathcal{E} := \{(X, X') \in \mathcal{G}(\alpha, \beta)\}$. Applying Lemma 3.4 and a union bound over $\{\pi_l(x) : x \in D^p_2\}$, for any $\lambda > 0$,
\[
\mathbb{P} \left[ \sup_{y \in D^p_2} \pi_j(y) K(X) \pi_{l,l-1}(y) > t \text{ and } \mathcal{E} \right] \leq e^{C(m+4-4t)2^l} \sup_{y \in \{\pi_l(x) : x \in D^p_2\}} \mathbb{P} \left[ \pi_j(y) K(X) \pi_{l,l-1}(y) > t \text{ and } \mathcal{E} \right] \leq e^{C(m+4-4t)2^l} e^{-\lambda t} \sup_{y \in \{\pi_l(x) : x \in D^p_2\}} \mathbb{E} \left[ e^{\lambda \pi_j(y) K(X) \pi_{l,l-1}(y)} 1 \{\mathcal{E}\} \right].
\]
Let $\Lambda$ be the set of all diagonal matrices in $\mathbb{R}^{p \times p}$ with all diagonal entries in $\{-1, 1\}$. Note that $(X, X') \in \mathcal{G}(\alpha, \beta)$ if and only if $(X, DX') \in \mathcal{G}(\alpha, \beta)$ for all $D \in \Lambda$. Then, conditional on $X$ and the event $\mathcal{E}$, $X'$ equals $DX'$ in law for $D$ uniformly distributed over $\Lambda$. Hence
\[
\mathbb{E} [K(X')|X, \mathcal{E}] = \mathbb{E} [K(DX')|X, \mathcal{E}] = \mathbb{E} [\mathbb{E} [K(DX')|X', X, \mathcal{E}]|X, \mathcal{E}] = 0,
\]
where the last equality follows from $\mathbb{E} [K(DX')|X'] = 0$ as the kernel function $k$ is odd. Then Jensen’s inequality yields, for any $y \in D^p_2$ and $\lambda > 0$,
\[
\mathbb{E} \left[ e^{-\lambda \pi_j(y) K(X') \pi_{l,l-1}(y)} \bigg| X, \mathcal{E} \right] \geq 1,
\]
and so
\[
\mathbb{E} \left[ e^{-\lambda \pi_j(y) K(X) \pi_{l,l-1}(y)} 1 \{\mathcal{E}\} \right] = \mathbb{E} \left[ e^{-\lambda \pi_j(y) K(X) \pi_{l,l-1}(y)} \bigg| \mathcal{E} \right] \mathbb{P} [\mathcal{E}]
\]

Hermite polynomial of degree $C$ for a constant $\lambda$. Recalling Proposition 4.1.

Let $E$ be an independent copy of $X$. Then by Lemma 3.5, for each $l = 0, \ldots, m+3$ and $j = l$ or $j = l - 1$,

$$
P \left[ \sup_{y \in D_2^p} \pi_j(y)^T K(X) \pi_{l-1}(y) > t_l \text{ and } (X, X') \in \mathcal{G}(\alpha, \beta) \right] \leq 2e^{-(C-c\lambda)(m+4)2^l}.
$$

Recalling $m = \lceil \log_2 p \rceil$, we may pick $C_0$ sufficiently large such that

$$
\sum_{l=0}^{m+3} 4e^{-(C-c\lambda)(m+4)2^l} \leq 4(m+4)e^{-(C-c\lambda)(m+4)} \leq C'p^{-\alpha}
$$

for a constant $C' := C'('\alpha, \beta)$. Then (7) and a union bound imply

$$
P \left[ \sup_{y \in D_2^p} y^T K(X)y > 2 \sum_{l=0}^{m+3} t_l \text{ and } (X, X') \in \mathcal{G}(\alpha, \beta) \right] \leq C'p^{-\alpha}.
$$

Finally, the bound

$$
2 \sum_{l=0}^{m+3} t_l < \frac{2C_0^{1/4}p^{1/4}(n+p)^{1/2}}{n} \sum_{l=0}^{m+3} (m+4-l)2^l
$$

$$
= \frac{2C_0^{1/2}p^{1/2}(n+p)^{1/2}}{n} \sum_{l=0}^{m+3} \sum_{j=0}^{l} 2^j \leq \frac{Cp^{1/4}(n+p)^{1/2}p^{1/4}}{n} \leq C \max \left( \frac{p}{n}, \sqrt{\frac{p}{n}} \right),
$$

the decomposition (7), and Lemmas 3.1 and 3.3 yield the desired result. \qed

4. Decomposition of Hermite polynomials of sums of IID random variables

In this section, we prove the approximation (8) formalized as the following proposition:

Proposition 4.1. Let $Z = (z_j : 1 \leq j \leq n) \in \mathbb{R}^n$, where $z_j$ are IID random variables such that $E[z_j] = E[z_j^3] = 0$, $E[z_j^2] = 1$, and $E|z_j| < \infty$ for each $l \geq 1$. Let $h_d$ denote the orthonormal Hermite polynomial of degree $d$. Define

$$
q_{d, n}(Z) = \sqrt{\frac{1}{n^d d!}} \sum_{j_1, \ldots, j_d=1 \atop j_1 \neq j_2 \neq \ldots \neq j_d}^{n} \prod_{i=1}^{d} z_{j_i},
$$

(11)
Then, for each $d \geq 1$ and any $\alpha, \beta > 0$, $P[|s_{d,n}(Z)| > n^{-1+\alpha}] < n^{-\beta}$ for all sufficiently large $n$ (i.e. for $n \geq N$ where $N$ may depend on $\alpha, \beta, d$, and the distribution of $z_j$).

The following lemma shows that $P[|q_{d,n}(Z)| > n^{\alpha}] < n^{-\beta}$ and $P[|r_{d,n}(Z)| > n^{-\frac{1}{2}+\alpha}] < n^{-\beta}$ for any $\alpha, \beta > 0$ and all sufficiently large $n$. Hence Proposition 4.1 may be interpreted as decomposing $h_d(n^{-1/2} \sum_{j=1}^n z_j)$ into the sum of an $O(1)$ term $q_{d,n}(Z)$, an $O(n^{-1/2})$ term $r_{d,n}(Z)$, and an $O(n^{-1})$ term $s_{d,n}(Z)$.

**Lemma 4.2.** Suppose $z_1, \ldots, z_n$ are IID random variables, with $E[|z_j|^l] < \infty$ for all $l \geq 1$. Let $p_1, \ldots, p_d : \mathbb{R} \to \mathbb{R}$ be any polynomial functions such that $E[p_i(z_j)] = 0$ for each $i = 1, \ldots, d$. Then for any $\alpha, \beta > 0$,\n
$$P \left[ n^{-d} \sum_{j_1, \ldots, j_d = 1}^n \prod_{i=1}^d p_i(z_{j_i}) > n^\alpha \right] < n^{-\beta}$$

for all sufficiently large $n$.

**Proof.** Fix $\alpha, \beta > 0$. Let

$$f(z_1, \ldots, z_n) = n^{-d} \sum_{j_1, \ldots, j_d = 1}^n \prod_{i=1}^d p_i(z_{j_i})$$

and let $l$ be an even integer such that $\alpha l > \beta$. Then

$$P[f(z_1, \ldots, z_n) > n^\alpha] \leq \frac{E[f(z_1, \ldots, z_n)^l]}{n^{\alpha l}}$$

and it suffices to show $E[f(z_1, \ldots, z_n)^l] \leq C$ for a constant $C$ independent of $n$. Note that

$$E[f(z_1, \ldots, z_n)^l] = n^{-ld} \sum_{j_1, \ldots, j_d = 1}^n \ldots \sum_{j_1, \ldots, j_d = 1}^n \mathbb{E} \left[ \prod_{i=1}^d \prod_{k=1}^l p_i \left( z_{j_i}^k \right) \right].$$

For each term of the above sum, if there is some $j$ such that $j = j_i$ for exactly one pair of indices $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, l\}$, then the expectation of that term is 0 as $E[p_i(z_j)] = 0$ and $z_j$ is independent of $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$. Hence, for terms in the sum with non-zero expectation, there are at most $\frac{ld}{2}$ distinct values of $j_i^k$. Then the number of such terms is at most $C n^{\frac{ld}{2}}$, and the magnitude of each such term is at most $C'$, for some constants $C, C'$ independent of $n$, establishing $E[f(z_1, \ldots, z_n)^l] \leq C$.\hfill \qed
Proof of Proposition 4.1. Let $S = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} z_j$. It will be notationally convenient to work with the monic Hermite polynomials $\tilde{h}_d = \sqrt{d!}h_d$. Let us accordingly define $\tilde{q}_{d,n} = q_{d,n} \sqrt{d!}$, $\tilde{r}_{d,n} = r_{d,n} \sqrt{d!}$, and $\tilde{s}_{d,n} = s_{d,n} \sqrt{d!}$. Then

$$
\tilde{h}_d(S) = \tilde{q}_{d,n}(Z) + \tilde{r}_{d,n}(Z) + \tilde{s}_{d,n}(Z),
$$

and we wish to show for any $\alpha, \beta > 0$, $\Pr[|\tilde{s}_d(Z)| > n^{-1+\alpha}] < n^{-\beta}$ for all sufficiently large $n$.

We proceed by induction on $d$. Note that $\tilde{h}_0(x) = 1$, $\tilde{h}_1(x) = x$, and $\tilde{h}_2(x) = x^2 - 1$. Then for $d = 1$, $\tilde{h}_1(S) = \tilde{S} = \tilde{q}_{1,n}(Z)$, and for $d = 2$,

$$
\tilde{h}_2(S) = S^2 - 1 = n^{-1} \left( \sum_{j_1,j_2=1 \atop j_1 \neq j_2}^{n} z_{j_1} z_{j_2} + \sum_{j=1}^{n} (z_j^2 - 1) \right) = \tilde{q}_{2,n}(Z) + \tilde{r}_{2,n}(Z).
$$

Hence the proposition holds with $\tilde{s}_{1,n}(Z) = \tilde{s}_{2,n}(Z) = 0$.

Let us assume by induction that the proposition holds for $d - 1$ and $d$. Recall that the monic Hermite polynomials satisfy the three-term recurrence $\tilde{h}_{d+1}(x) = x\tilde{h}_d(x) - d\tilde{h}_{d-1}(x)$ (c.f. eq. (5.5.8) of [53]). We may compute

$$
S\tilde{q}_{d,n}(Z) = n^{-\frac{d+1}{2}} \sum_{j=1}^{n} z_j \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} \prod_{i=1}^{d} z_{j_i} = n^{-\frac{d+1}{2}} \left( \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} \prod_{i=1}^{d+1} z_{j_i} - d \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} z_{j_1}^2 \prod_{i=2}^{d} z_{j_i} \right) = \tilde{q}_{d+1,n}(Z) + \frac{2}{d+1} \tilde{r}_{d+1,n}(Z) + \frac{d(n-d+1)}{n} \tilde{q}_{d-1,n}(Z),
$$

$$
S\tilde{r}_{d,n}(Z) = n^{-\frac{d+1}{2}} \left( \frac{d}{2} \right) \sum_{j=1}^{n} z_j \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} \left( z_{j_1}^2 - 1 \right) \prod_{i=2}^{d} z_{j_i} = n^{-\frac{d+1}{2}} \left( \frac{d}{2} \right) \left( \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} \prod_{i=2}^{d+1} z_{j_i} - \sum_{j_1,\ldots,j_d=1 \atop j_1 \neq \ldots \neq j_d}^{d} z_{j_1}^2 \prod_{i=3}^{d} z_{j_i} \right) + (d-2) \sum_{j_1,\ldots,j_{d-1}=1 \atop j_1 \neq \ldots \neq j_{d-1}}^{d} \left( z_{j_1}^2 - 1 \right) \prod_{i=3}^{d-1} z_{j_i} \right) = \frac{d-1}{d+1} \tilde{r}_{d+1,n}(Z) + n^{-\frac{d+1}{2}} \left( \frac{d}{2} \right) \sum_{j_1,\ldots,j_{d-1}=1 \atop j_1 \neq \ldots \neq j_{d-1}}^{d} \left( z_{j_1}^3 - z_{j_1} \right) \prod_{i=2}^{d-1} z_{j_i} + \frac{d(n-d+2)}{n} \tilde{r}_{d-1,n}(Z).
$$
Substituting these expressions into the three-term recurrence,
\[ \tilde{h}_{d+1}(S) = S(\tilde{q}_{d,n}(Z) + \tilde{r}_{d,n}(Z) + \tilde{s}_{d,n}(Z)) - d(\tilde{q}_{d-1,n}(Z) + \tilde{r}_{d-1,n}(Z) + \tilde{s}_{d-1,n}(Z)) \]
for
\[ \tilde{s}_{d+1,n}(Z) := -\frac{d(d-1)}{n}\tilde{q}_{d-1,n}(Z) + n^{-\frac{d+1}{2}}(d-2) \sum_{j_1,...,j_{d-1}=1}^{n} \left(\prod_{i=2}^{d-1} z_{j_i}\right) \]
\[ + n^{-\frac{d+1}{2}}(d-2) \sum_{j_1,...,j_{d-1}=1}^{n} \left(\prod_{i=3}^{d-1} z_{j_i}\right) - \frac{d(d-2)}{n}\tilde{r}_{d-1,n}(Z) \]
\[ + S\tilde{s}_{d,n}(Z) - d\tilde{s}_{d-1,n}(Z) =: I + II + III + IV + V + VI. \]
Fix \( \alpha, \beta > 0 \). Note that \( \mathbb{E}[z_j] = 0 \), \( \mathbb{E}[z_j^2 - 1] = 0 \), and \( \mathbb{E}[z_j^3 - z_j] = 0 \), so by Lemma 4.2
\[ \max\left(\mathbb{P}\left[|I| > n^{-1+\frac{\alpha}{2}}\right], \mathbb{P}\left[|II| > n^{-1+\frac{\alpha}{2}}\right], \mathbb{P}\left[|III| > n^{-1+\frac{\beta}{2}}\right], \mathbb{P}\left[|IV| > n^{-1+\frac{\beta}{2}}\right]\right) < n^{-2\beta} \]
for all large \( n \). By the induction hypothesis, \( \mathbb{P}(|\tilde{s}_{d,n}| > n^{-1+\frac{\alpha}{2}}) < n^{-2\beta}/2 \) for all large \( n \), and also \( \mathbb{P}(|S| > n^{\frac{\beta}{2}}) < n^{-2\beta}/2 \) for all large \( n \) by Lemma 4.2 (applied to the simple case where \( d = 1 \) and \( p_1(x) = x \)). Then \( \mathbb{P}(|V| > n^{-1+\frac{\beta}{2}}) < n^{-2\beta} \) for all large \( n \). Similarly, the induction hypothesis implies \( \mathbb{P}(|VI| > n^{-1+\frac{\beta}{2}}) < n^{-2\beta} \) for all large \( n \). Putting this together,
\[ \mathbb{P}\left[|\tilde{s}_{d+1,n}(Z)| > n^{-1+\alpha}\right] \leq \mathbb{P}\left[|I + II + III + IV + V + VI| > 6n^{-1+\frac{\beta}{2}}\right] < 6n^{-2\beta} < n^{-\beta} \]
for all large \( n \), completing the induction. \( \square \)

5. Bounding the dominant matrix

Consider the polynomial kernel matrix \( K(X) \) in Theorem 1.7. Throughout this section, we let \( D < \infty \) denote the (fixed) degree of the polynomial \( k \), and we write
\[ k(x) = \sum_{d=1}^{D} a_d h_d(x). \]
Corresponding to the decomposition of \( h_d \) given in Proposition 4.1, we consider the following decomposition of \( K(X) \):

**Definition 5.1.** Define \( Q(X) = (q_{ii'} : 1 \leq i, i' \leq p) \in \mathbb{R}^{p \times p} \) with entries
\[ q_{ii'} = \begin{cases} \frac{1}{\sqrt{n}} \sum_{d=1}^{D} a_d q_{d,n}(x_{i1}x_{i'1},\ldots,x_{in}x_{i'n}), & i \neq i' \\ 0, & i = i' \end{cases} \]
where \( q_{d,n} \) is as in (11). Define \( R(X) \in \mathbb{R}^{p \times p} \) and \( S(X) \in \mathbb{R}^{p \times p} \) analogously with \( r_{d,n} \) and \( s_{d,n} \) in place of \( q_{d,n} \), where \( r_{d,n} \) and \( s_{d,n} \) are as in (12) and (13).

With the above definitions, \( K(X) = Q(X) + R(X) + S(X) \). In this section, we establish the following result:

**Proposition 5.2.** Under the conditions of Theorem 1.7, letting \( Q(X) \) be as in Definition 5.1, \( \limsup_{n,p \to \infty} \|Q(X)\| \leq \|\mu_{a,\nu,\gamma}\| \) almost surely.
Our proof uses the moment method and the moment comparison argument described in Section 2. The following definitions of an $l$-graph and a multi-labeling of such a graph will correspond to the primary combinatorial object of interest in the subsequent analysis.

**Definition 5.3.** For any integer $l \geq 2$, an $l$-graph is a graph consisting of a single cycle with $2l$ vertices and $2l$ edges, with the vertices alternatingly denoted as $p$-vertices and $n$-vertices.

We will consider the vertices of the $l$-graph to be ordered by picking an arbitrary $p$-vertex as the first vertex and ordering the remaining vertices according to a traversal along the cycle. A vertex $V$ “follows” or “precedes” another vertex $W$ if $V$ comes before or after $W$, respectively, in this ordering, and the last vertex of the cycle (which is an $n$-vertex) is followed by the first $p$-vertex.

**Definition 5.4.** A multi-labeling of an $l$-graph is an assignment of a $p$-label in \{1, 2, 3, \ldots\} to each $p$-vertex and an ordered tuple of $n$-labels in \{1, 2, 3, \ldots\} to each $n$-vertex, such that the following conditions are satisfied:

1. The $p$-label of each $p$-vertex is distinct from those of the two $p$-vertices immediately preceding and following it in the cycle.
2. The number $d_s$ of $n$-labels in the tuple for each $s^{th}$ $n$-vertex satisfies $1 \leq d_s \leq D$, and these $d_s$ $n$-labels are distinct.
3. For each distinct $p$-label $i$ and distinct $n$-label $j$, there are an even number of edges in the cycle (possibly 0) such that its $p$-vertex endpoint is labeled $i$ and its $n$-vertex endpoint has label $j$ in its tuple.

A $(p, n)$-multi-labeling is a multi-labeling with all $p$-labels in \{1, \ldots, p\} and all $n$-labels in \{1, \ldots, n\}.

A key bound on the number of possible distinct $p$-labels and $n$-labels that appear in a multi-labeling of an $l$-graph is provided by the following lemma. We will always consider $p$-labels to be distinct from $n$-labels, even though (for notational convenience) we use the same label set \{1, 2, 3, \ldots\} for both.

**Lemma 5.5.** Suppose a multi-labeling of an $l$-graph has $d_1, \ldots, d_l$ $n$-labels on the first through $l^{th}$ $n$-vertices, respectively, and suppose that it has $m$ total distinct $p$-labels and $n$-labels. Then $m \leq \frac{l + \sum_{s=1}^{l} d_s}{2} + 1$.

We defer the proof of Lemma 5.5 to Appendix A. Figure 3 shows an example of a multi-labeling of an $l$-graph for $l = 4$ and $D = 3$. In this multi-labeling, $\sum_{s=1}^{4} d_s = 3 + 3 + 1 + 1 = 8$ and the number of total distinct labels is $m = 3 + 4 = 7$, so Lemma 5.5 holds with equality.

The non-negative quantity \(\frac{l + \sum_{s=1}^{l} d_s}{2} + 1 - m\) appears in many of our combinatorial lemmas, and we give it a name:
Definition 5.6. Suppose a multi-labeling of an $l$-graph has $d_1, \ldots, d_l$ $n$-labels on the first through $l^{th}$ $n$-vertices, respectively, and suppose that it has $m$ total distinct $p$-labels and $n$-labels. The excess of the multi-labeling is $\Delta := \frac{l+\sum_{i=1}^{l} d_i}{2} + 1 - m$.

A high-level intuition, which we make precise in various ways in Appendix A, is that multi-labelings with zero or small excess satisfy many regularity properties. For example, we prove the following in Appendix A:

Lemma 5.7. Suppose a multi-labeling of an $l$-graph has excess $\Delta$. For each $i \in \{1, 2, 3, \ldots\}$ and $j \in \{1, 2, 3, \ldots\}$, let $b_{ij}$ be the number of edges in the $l$-graph such that the $p$-vertex endpoint is labeled $i$ and the $n$-vertex endpoint has label $j$ in its tuple. Then $\sum_{i,j : b_{ij} > 2} b_{ij} \leq 12\Delta$.

In particular, a multi-labeling with excess $\Delta = 0$ has either $b_{ij} = 2$ or $b_{ij} = 0$ for every label-pair $(i,j)$, by the above lemma and condition (3) of Definition 5.4. This indeed holds for the example of Figure 3.

Definition 5.8. Two multi-labelings of an $l$-graph are equivalent if there is a permutation $\pi_p$ of $\{1, 2, 3, \ldots\}$ and a permutation $\pi_n$ of $\{1, 2, 3, \ldots\}$ such that one labeling is the image of the other upon applying $\pi_p$ to all of its $p$-labels and $\pi_n$ to all of its $n$-labels. For any fixed $l$, the equivalence classes under this relation will be called multi-labeling equivalence classes.

The number of distinct $p$-labels, number of distinct $n$-labels, number of $n$-labels $d_1, \ldots, d_l$ on each of the $l$ $n$-vertices, and excess $\Delta$ are equivalence class properties, i.e. they are the same for all labelings in the same multi-labeling equivalence class. The connection between Definition 5.4 of a multi-labeling and our matrix of interest $Q(X)$ is provided by the following lemma:

Lemma 5.9. Let $Q(X)$ be as in Proposition 5.2, and let $l \geq 2$ be an even integer. Let $C$ denote the set of all multi-labeling equivalence classes for an $l$-graph. For each multi-labeling equivalence class $\mathcal{L} \in C$, let $\Delta(\mathcal{L})$ be the excess, $r(\mathcal{L})$ the number of distinct $p$-labels, and $d_1(\mathcal{L}), \ldots, d_l(\mathcal{L})$ the number of $n$-labels on the first to $l^{th}$ $n$-vertices, respectively. Then, for $\alpha > 0$ as in (6) and with the convention $0^0 = 1$,

$$\mathbb{E}[\text{Tr} Q(X)^{\alpha}] \leq n \sum_{\mathcal{L} \in C} \left( \frac{(12\Delta(\mathcal{L}))^{12\alpha}}{n} \right) \left( \frac{p}{n} \right)^{r(\mathcal{L})} \left( \prod_{s=1}^{l} \frac{|a_{d_s(\mathcal{L})}|}{(d_s(\mathcal{L})!)^{1/2}} \right).$$

Proof. By Definition 5.1, letting $i_{l+1} := i_1$ for notational convenience,

$$\mathbb{E}[\text{Tr} Q(X)^{\alpha}] = \sum_{i_1, \ldots, i_l=1 \atop i_1 \neq i_2, i_2 \neq i_3, \ldots, i_l \neq i_1} \mathbb{E} \left[ \prod_{s=1}^{l} q_{i_s i_{s+1}} \right]$$

$$= \sum_{i_1, \ldots, i_l=1 \atop i_1 \neq i_2, i_2 \neq i_3, \ldots, i_l \neq i_1} n^{-\frac{l}{2}} \prod_{s=1}^{l} \left( \sum_{d_s=1}^{D} \frac{1}{n^d d_s!} \sum_{j_1, \ldots, j_d=1 \atop j_1 \neq j_2 \neq \ldots \neq j_d}^{n} \prod_{a=1}^{d_s} x_{i_s j_a} x_{i_{s+1} j_a} \right)$$

$$= \sum_{i_1, \ldots, i_l=1 \atop i_1 \neq i_2, i_2 \neq i_3, \ldots, i_l \neq i_1} \sum_{d_1, \ldots, d_l=1}^{D} \sum_{j_1, \ldots, j_d=1 \atop j_1 \neq j_2 \neq \ldots \neq j_d}^{n} \sum_{j_1', \ldots, j_d'=1 \atop j_1' \neq j_2' \neq \ldots \neq j_d'}^{n} \prod_{s=1}^{l} \frac{a_{d_s}}{(d_s!)^{1/2}} \prod_{s=1}^{l} \prod_{a=1}^{d_s} x_{i_s j_a} x_{i_{s+1} j_a} \prod_{s=1}^{l} \prod_{a=1}^{d_s} x_{i_s j_a} x_{i_{s+1} j_a} \prod_{s=1}^{l} \prod_{a=1}^{d_s} x_{i_s j_a} x_{i_{s+1} j_a}$$

$$n^{-\frac{l+\sum_{s=1}^{l} d_s}{2}} \left( \prod_{s=1}^{l} \frac{a_{d_s}}{(d_s!)^{1/2}} \right) \mathbb{E} \left[ \prod_{s=1}^{l} \prod_{a=1}^{d_s} x_{i_s j_a} x_{i_{s+1} j_a} \right].$$
Note that as $x_{ij} \overset{L}{=} -x_{ij}$ by assumption, $\mathbb{E}[x_{ij}^2] = 0$ for any positive odd integer $c$. Hence, if any $x_{ij}$ appears an odd number of times in the expression $\prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s}$, then as the entries of $X$ are independent, $\mathbb{E} [\prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s}] = 0$. We identify the combination of sums above, over the remaining non-zero terms, as the sum over all possible $(p, n)$-multi-labelings of an $l$-graph. Here, the first sum over $i_1, \ldots, i_l$ is over all choices of $p$-labels, with condition (1) in Definition 5.4 corresponding to the constraints $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_l \neq i_1$ in the sum. The sum over $d_1, \ldots, d_l$ is over all choices of the number of $n$-labels in the tuple for each $n$-vertex, and the sum over $j_1^s, \ldots, j_l^s$ is over all choices of $d_s$ $n$-labels for the $s^{th}$ $n$-vertex, with condition (2) in Definition 5.4 corresponding to the constraint that $j_1^s, \ldots, j_l^s$ are distinct. The product expression $\prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s}$ then corresponds to a product, over all $n$-vertices, all $d_s$ $n$-labels for that $n$-vertex, and both $p$-vertices immediately preceding and immediately following that $n$-vertex, of $x_{ij}$, where $j \in \{1, \ldots, n\}$ is the $n$-label and $i \in \{1, \ldots, p\}$ is the $p$-label of the $p$-vertex. The condition that each $x_{ij}$ appears an even number of times so that this term has non-zero expectation is precisely condition (3) in Definition 5.4. Thus, to summarize,

$$\mathbb{E}[\text{Tr } Q(X)^l] = \sum_{\text{l-graph } (p,n)\text{-multi-labelings}} n^{-l+i-1} \prod_{s=1}^{l} a_{d_s} \left( \prod_{s=1}^{l} \prod_{a=1}^{d_s} x_{i_s j_s} x_{i_s+1 j_s} \right),$$

where $d_1, \ldots, d_l$ are the numbers of $n$-labels for the first through $l^{th}$ $n$-vertices, respectively.

Consider a fixed $(p, n)$-multi-labeling and write $\prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s} = \prod_{j=1}^{n} \prod_{i=1}^{p} x_{ij}$, where $b_{ij}$ is the number of times $x_{ij}$ appears as a term in this product. Note that each $b_{ij}$ is even (possibly 0). As $\mathbb{E}[x_{ij}^2] = 1$, $\mathbb{E}[|x_{ij}|^k] \leq k^{\alpha k}$, and the entries of $X$ are independent,

$$\mathbb{E} \left[ \prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s} \right] = \prod_{i,j : b_{ij} > 2} \mathbb{E} \left[ x_{ij}^2 \right] \leq \prod_{i,j : b_{ij} > 2} b_{ij} \leq \left( \sum_{i,j : b_{ij} > 2} b_{ij} \right)^{\alpha} \leq (12\Delta)^{12\Delta},$$

where the last inequality applies Lemma 5.7 and we use the convention $0^0 = 1$. (14) then follows upon noting that each $(p, n)$-multi-labeling with $r$ distinct $p$-labels and $m - r$ distinct $n$-labels has $\frac{p!}{(p-r)!} \cdot \frac{n!}{(n-m+r)!} \leq n^m \left( \frac{p}{n} \right)^r$ $(p, n)$-multi-labelings in its equivalence class, and

$$n^{-i+1} \prod_{s=1}^{l} x_{i_s j_s} x_{i_s+1 j_s} + m = n^{1-\Delta}.$$

We wish to compare the upper bound in (14) to an analogous quantity for a deformed GUE matrix:

**Definition 5.10.** For $\tilde{n}, \tilde{p} \geq 1$, let $W = (w_{ii'} : 1 \leq i, i' \leq \tilde{p}) \in \mathbb{C}^{\tilde{p} \times \tilde{p}}$ be distributed according to the GUE, i.e. $\{w_{ii} : 1 \leq i \leq \tilde{p}\} \cup \{\sqrt{2} \text{Re } w_{ii'}, \sqrt{2} \text{Im } w_{ii'} : 1 \leq i < i' \leq \tilde{p}\}$ are IID $\mathcal{N}(0,1)$, and $w_{ii'} = \overline{w_{i'i}}$ for $i > i'$. Let $V \in \mathbb{R}^{\tilde{p} \times \tilde{p}}$ be standard real Wishart-distributed with $\tilde{n}$ degrees of freedom and zero diagonal, i.e. $V = ZZ^T - \text{diag}(\|Z\|_2^2)$ where $Z = (z_{ij} : 1 \leq i \leq \tilde{p}, 1 \leq j \leq \tilde{n}) \in \mathbb{R}^{\tilde{p} \times \tilde{n}}$, $z_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, and $ZZ^T - \text{diag}(\|Z\|_2^2)$ denotes $ZZ^T$ with its diagonal set to 0. Take $V$ and $W$ to be independent, and define

$$M = \sqrt{\frac{\gamma (\nu - a^2)}{\tilde{p}}} W + \frac{a}{\tilde{n}} V \in \mathbb{C}^{\tilde{p} \times \tilde{p}}.$$

As $\tilde{n}, \tilde{p} \to \infty$ with $\tilde{p}/\tilde{n} \to \gamma$, the limiting spectral distribution of $M$ is also $\mu_{a,\nu,\gamma}$. It follows from the results of [16] that, in fact, a norm convergence result holds for $M$, i.e. $\lim_{\tilde{n}, \tilde{p} \to \infty} \|M\| = \|\mu_{a,\nu,\gamma}\|$, using which we may establish the following Proposition:
Proposition 5.11. Let $M$ be as in Definition 5.10. Suppose $l$ is an even integer and $\tilde{p}, \tilde{n}, l \to \infty$ with $\tilde{p}/\tilde{n} \to \gamma$ and $l \leq C \log \tilde{n}$ for some constant $C > 0$. Then, for any $\varepsilon > 0$ and all sufficiently large $\tilde{n}$,

$$\mathbb{E}[\| M \|^l] \leq (\| \mu_{a,\nu,\gamma} \| + \varepsilon)^l.$$ 

The proof of Proposition 5.11 is deferred to Appendix B. As $\tilde{p} - 1 \mathbb{E}[\text{Tr } M^l] \leq \mathbb{E}[\| M \|^l]$, our strategy for proving Proposition 5.2 will be to show that the upper bound in (14) can in turn be bounded above using the quantity $\mathbb{E}[\text{Tr } M^l]$, for some choices of $\tilde{p}$ and $\tilde{n}$. To analyze $\mathbb{E}[\text{Tr } M^l]$, we consider the following notion of a simple-labeling of an $l$-graph:

Definition 5.12. A simple-labeling of an $l$-graph is an assignment of a $p$-label in $\{1, 2, 3, \ldots\}$ to each $p$-vertex and either one $n$-label in $\{1, 2, 3, \ldots\}$ or the empty label $\emptyset$ to each $n$-vertex, such that the following conditions are satisfied:

1. The $p$-label of each $p$-vertex is distinct from those of the two $p$-vertices immediately preceding and following it in the cycle.
2. For each distinct $p$-label $i$ and distinct non-empty $n$-label $j$, there are an even number of edges in the cycle (possibly 0) such that its $p$-vertex endpoint is labeled $i$ and its $n$-vertex endpoint is labeled $j$.
3. For any two distinct $p$-labels $i$ and $i'$, the number of occurrences (possibly 0) of the three consecutive labels $i, \emptyset, i'$ on a $p$-vertex, its following $n$-vertex, and its following $p$-vertex is equal to the number of occurrences of the three consecutive labels $i', \emptyset, i$.

A $(p, n)$-simple-labeling is a simple-labeling with all $p$-labels in $\{1, \ldots, p\}$ and all non-empty $n$-labels in $\{1, \ldots, n\}$.

Analogous to Lemma 5.5, the following lemma provides a key bound on the number of possible distinct $p$-labels and $n$-labels that appear in a simple-labeling of an $l$-graph.

Lemma 5.13. Suppose a simple-labeling of an $l$-graph has $\tilde{k}$ $n$-vertices with non-empty label and $\tilde{m}$ total distinct $p$-labels and distinct non-empty $n$-labels. Then $\tilde{m} \leq \frac{l + \tilde{k}}{2} + 1$.

The proof of Lemma 5.13 is deferred to Appendix A. We may then define the excess of a simple-labeling, analogous to Definition 5.6, and note that the excess is always nonnegative.

Definition 5.14. Suppose a simple-labeling of an $l$-graph has $\tilde{k}$ $n$-vertices with non-empty label and $\tilde{m}$ total distinct $p$-labels and distinct non-empty $n$-labels. The excess of the simple-labeling is $\Delta := \frac{l + \tilde{k}}{2} + 1 - \tilde{m}$.

Figure 4 shows a simple-labeling of an $l$-graph for $l = 4$, with $\tilde{k} = 2$ $n$-vertices having non-empty label and $\tilde{m} = 3 + 1 = 4$ distinct $p$-labels and non-empty $n$-labels. Hence in this example, Lemma 5.13 holds with equality, and the excess is $\Delta = 0$. 

Figure 4. A simple labeling of an $l$-graph for $l = 4$. $p$-vertices are depicted with a circle and $n$-vertices are depicted with a square.
Definition 5.15. Two simple-labelings of an \(l\)-graph are **equivalent** if there is a permutation \(\pi_p\) of \(\{1, 2, 3, \ldots\}\) and a permutation \(\pi_n\) of \(\{1, 2, 3, \ldots\}\) such that one labeling is the image of the other upon applying \(\pi_p\) to all of its \(p\)-labels and \(\pi_n\) to all of its \(n\)-labels. (The empty \(n\)-label remains empty under any such permutation \(\pi_n\).) For any fixed \(l\), the equivalence classes under this relation will be called **simple-labeling equivalence classes**.

Motivation for Definition 5.12 of a simple labeling is provided by the following lemma, which gives a lower bound for the quantity \(\mathbb{E}[\text{Tr} M^l]\):

**Lemma 5.16.** Let \(M\) be as in Definition 5.10 and let \(l \geq 2\) be an even integer. Let \(\bar{C}\) denote the set of all simple-labeling equivalence classes for an \(l\)-graph. For each simple-labeling equivalence class \(\bar{C} \in \bar{C}\), let \(\bar{\Delta}(\bar{C})\) be its excess, \(\bar{k}(\bar{C})\) be the number of \(n\)-vertices with non-empty label, and \(\bar{r}(\bar{C})\) be the number of distinct \(p\)-labels. Then, with the convention \(0^0 = 1\),

\[
\mathbb{E}[\text{Tr} M^l] \geq \tilde{n} \left( \frac{\tilde{p} - 1}{\tilde{p}} \right)^l \left( \frac{\tilde{n} - l}{\tilde{n}} \right)^l \sum_{\bar{C} \in \bar{C}} \left( \frac{1}{\tilde{n}} \right)^{\bar{\Delta}(\bar{C})} \left( \frac{\tilde{p}}{\tilde{n}} \right)^{\bar{r}(\bar{C}) - \frac{l - \bar{k}(\bar{C})}{2}} |\bar{a}|^{\bar{k}(\bar{C})} (\gamma - a^2)^{\frac{l - \bar{k}(\bar{C})}{2}}. \tag{15}
\]

**Proof.** By Definition 5.10 letting \(i_{s+1} := i_1\) for notational convenience,

\[
\mathbb{E}[\text{Tr} M^l] = \mathbb{E} \left[ \text{Tr} \left( \sqrt{\frac{\gamma - a^2}{\tilde{p}}} W + \frac{a}{\tilde{n}} V \right)^l \right] = \sum_{i_1, \ldots, i_l = 1}^{\tilde{p}} \mathbb{E} \left[ \prod_{s=1}^l \left( \sqrt{\frac{\gamma - a^2}{\tilde{p}}} w_{i_s i_{s+1}} + \frac{a}{\tilde{n}} v_{i_s i_{s+1}} \right) \right] = \sum_{S \subseteq \{1, \ldots, l\}}^{\tilde{p}} \sum_{i_1, \ldots, i_l = 1}^{\tilde{p}} \mathbb{E} \left[ \prod_{s=1}^l \frac{(a_n)^{|S|}}{|S|} \left( \frac{\gamma - a^2}{\tilde{p}} \right)^{\frac{l - |S|}{2}} \mathbb{E} \left[ \prod_{s \in S} v_{i_s i_{s+1}} \right] \mathbb{E} \left[ \prod_{s \notin S} w_{i_s i_{s+1}} \right] \right] = \sum_{S \subseteq \{1, \ldots, l\}}^{\tilde{p}} \sum_{i_1, \ldots, i_l = 1}^{\tilde{p}} \mathbb{E} \left[ \prod_{s=1}^l \frac{(a_n)^{|S|}}{|S|} \left( \frac{\gamma - a^2}{\tilde{p}} \right)^{\frac{l - |S|}{2}} \mathbb{E} \left[ \prod_{s \in S} z_{i_s j_s} z_{i_{s+1} j_s} \right] \mathbb{E} \left[ \prod_{s \notin S} w_{i_s i_{s+1}} \right] \right].
\]

In the fourth line above, we restricted the summation to \(i_s \neq i_{s+1} \forall s \in S\), as \(v_{i_s} = 0\) for each \(i = 1, \ldots, \tilde{p}\) by Definition 5.10.

Let us write \(\prod_{s \in S} z_{i_s j_s} z_{i_{s+1} j_s} = \prod_{i=1}^{\tilde{p}} \tilde{n}^{-1} \tilde{p}^{-1} z_{i_{j}}\), where \(z_{i_{j}}\) is the number of times \(z_{i_{j}}\) appears in this product, and let us write \(\prod_{s \notin S} w_{i_s i_{s+1}} = \prod_{i=1}^{\tilde{p}} \tilde{n}^{-1} \tilde{p}^{-1} a_i^{a_i} a_i^{a_i} w_{i_i} w_{i_i}^b\), where \(a_i^{a_i}\) and \(b_i^{a_i}\) are the numbers of times \(w_{i_i}^a\) and \(w_{i_i}^b\) appear in this product, respectively. \(\mathbb{E}\left[\prod_{i=1}^{\tilde{p}} \tilde{n}^{-1} \tilde{p}^{-1} z_{i_{j}}\right] \neq 0\) only if each \(z_{i_{j}}\) is even (possibly zero), in which case this quantity is at least 1. Similarly, note that if \(w = re^{i\theta}\) such that \(\sqrt{2} \text{Re}(w), \sqrt{2} \text{Im}(w) \sim N(0, 1)\), then \(r\) and \(\theta\) are independent with \(\theta \sim \text{Unif}(0, 2\pi)\). Then \(\mathbb{E}[w^a w^b] = \mathbb{E}[r^{a+b}] \mathbb{E}[e^{i(a-b)\theta}]\) for all nonnegative integers \(a, b\), and this is 0 if \(a \neq b\) and at least 1 if \(a = b \geq 0\). Hence \(\mathbb{E}[\prod_{i=1}^{\tilde{p}} w_{i_i}^{a_i} \prod_{1 \leq i < i' \leq \tilde{p}} a_i^{a_i} a_i^{a_i} w_{i_i} w_{i_i}^{b_i}] = 0\) unless \(a_i^{a_i} = b_i^{a_i}\) for each \(i' > i\) and \(a_i^{a_i}\) is even (possibly zero) for each \(1 \leq i \leq \tilde{p}\), in which case this quantity is also at least 1.
The above arguments imply, in particular, that \( \mathbb{E} \left[ \prod_{s \neq S} w_{is,i+s+1} \right] = 0 \) unless \( l - |S| \) is even. As \( l \) is even by assumption, then \( |S| \) must also be even, in which case \( a^{\left| S \right|} = |a|^{|S|} \geq 0 \). Hence each term of the sum in the above expression for \( \mathbb{E}[\text{Tr} M^l] \) is nonnegative, so a lower bound is obtained if we further restrict the summation to \( i_s \neq i_{s+1} \forall s \in \{1, \ldots, l\} \) (rather than just \( \forall s \in S \)), i.e.

\[
\mathbb{E}[\text{Tr} M^l] \geq \sum_{S \subseteq \{1, \ldots, l\}} \sum_{i_1 \neq i_2, i_3 \neq \ldots, i_l \neq i_1} \sum_{(j_s : s \in S) \in \{1, \ldots, \tilde{n}\}^{|S|}} \tilde{n}^{-\frac{l+|S|}{2}} \left( \frac{\tilde{p}}{\tilde{n}} \right)^{-\frac{l}{2}} |a|^{|S|}(\nu - a^2)^{-\frac{l}{2}} \mathbb{E} \left[ \prod_{s \in S} z_{is} z_{i_{s+1}s} \right] \mathbb{E} \left[ \prod_{s \notin S} w_{is,i_{s+1}} \right].
\]

We identify the combination of these sums as a sum over all \((\tilde{p}, \tilde{n})\)-simple-labelings of an \( l \)-graph. Here, the first sum over \( S \) is over all choices of the subset of \( n \)-vertices having non-empty label. The second sum over \( i_1, \ldots, i_l \) is over all choices of \( p \)-labels, with condition (1) in Definition 5.12 corresponding to the constraints \( i_1 \neq i_2, i_2 \neq i_3, \ldots, i_l \neq i_1 \). The last sum over \( (j_s : s \in S) \) is over all choices of \( n \)-labels for the \( n \)-vertices that have nonempty label. The product expression \( \prod_{s \in S} z_{is} z_{i_{s+1}s} \) then corresponds to a product over all \( n \)-vertices with non-empty label and both \( p \)-vertices immediately preceding and following that \( n \)-vertex, and the condition that each \( z_{ij} \) appears an even number of times corresponds to condition (2) in Definition 5.12. Similarly, the product expression \( \prod_{s \notin S} w_{is,i_{s+1}} \) corresponds to a product over all \( n \)-vertices with empty label, and the condition that each \( w_{ii} \) appears the same number of times as \( w_{ii} \) is precisely condition (3) in Definition 5.12. (By restricting the sum to \( i_s \neq i_{s+1} \) for all \( s \), no diagonal terms \( w_{ii} \) appear in this product.) Applying the bound \( \mathbb{E}[\prod_{s \in S} z_{is} z_{i_{s+1}s}] \mathbb{E}[\prod_{s \notin S} w_{is,i_{s+1}}] \geq 1 \) whenever this quantity is nonzero,

\[
\mathbb{E}[\text{Tr} M^l] \geq \sum_{l \text{-graph } (\tilde{p}, \tilde{n}) \text{-simple-labelings}} \tilde{n}^{-\frac{l-k}{2}} \left( \frac{\tilde{p}}{\tilde{n}} \right)^{-\frac{k}{2}} |a|^k(\nu - a^2)^{-\frac{k}{2}},
\]

where \( k = |S| \) is the number of \( n \)-vertices in the simple-labeling with non-empty label. Any simple labeling with \( \tilde{r} \) distinct \( p \)-labels and at most \( \tilde{m} - \tilde{r} \) distinct non-empty \( n \)-labels has at most \( \frac{n!}{(n-m+r)!} \frac{p^r}{(p-r)!} \geq \tilde{m}^\tilde{r} \left( \frac{\tilde{p}}{\tilde{n}} \right)^\tilde{r} \left( \frac{n-l}{p} \right)^l \) labelings in its equivalence class (where we have used \( \tilde{m} - \tilde{r} \leq l \) and \( \tilde{r} \leq l \)). The desired result then follows upon identifying \( \tilde{n}^{1-\tilde{\Delta}} = \tilde{n}^{-\frac{l-k}{2}} + \tilde{m} \).

The remainder of the proof of Proposition 5.2 involves a comparison of the upper bound in (14) and the lower bound in (15). The intuition for the comparison is the following: The dominant contributions to the sums in (14) and (15) come from labelings with small excess. Focusing on labelings with excess 0, if we take any multi-labeling equivalence class \( \mathcal{L} \) with \( \Delta(\mathcal{L}) = 0 \) and replace the labels of \( n \)-vertices having more than one \( n \)-label with \( \emptyset \), then it may be shown that we obtain a valid simple-labeling equivalence class \( \tilde{\mathcal{L}} \) with \( \tilde{\Delta}(\tilde{\mathcal{L}}) = 0 \). For example, the multi-labeling of Figure 2 is mapped to the simple labeling of Figure 4 under this procedure. Furthermore, for any \( \tilde{\mathcal{L}} \) with \( \tilde{\Delta}(\tilde{\mathcal{L}}) = 0 \), we may show

\[
\sum_{\mathcal{L} : \mathcal{L} \text{ maps to } \tilde{\mathcal{L}}} \prod_{s=1}^l \left| a_{d_s}(\mathcal{L}) \right| (d_s(\mathcal{L}))^{1/2} = |a|^k(\nu - a^2)^{-\frac{k}{2}}.
\]

(The arguments that establish these claims are a specialization of our combinatorial lemmas in Appendix A to the cases of \( \Delta = 0 \) and \( \Delta = 0 \).) Hence, this mapping yields an exact correspondence between terms in (14) with excess \( \Delta(\mathcal{L}) = 0 \) and terms in (15) with excess \( \tilde{\Delta}(\tilde{\mathcal{L}}) = 0 \).
As we must consider \( l \asymp \log n \) to establish a tight bound in spectral norm, we need to also handle terms in \([14]\) where \( \Delta(\mathcal{L}) \neq 0 \). We do so by extending the above mapping to all multi-labeling equivalence classes \( \mathcal{L} \), in the case \( a \neq 0 \). The properties of this mapping that we will need are summarized in the following proposition.

**Proposition 5.17.** Suppose \( a \neq 0 \) and \( l \geq 2 \). Let \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \) denote the set of all multi-labeling and simple labeling equivalence classes of an \( l \)-graph, respectively. For \( \mathcal{L} \in \mathcal{C} \), let \( \Delta(\mathcal{L}) \) be its excess and \( r(\mathcal{L}) \) be the number of distinct \( p \)-labels, and for \( \mathcal{L} \in \tilde{\mathcal{C}} \), let \( \tilde{\Delta}(\mathcal{L}) \) be its excess, \( \tilde{r}(\mathcal{L}) \) be the number of distinct \( p \)-labels, and \( \tilde{k}(\tilde{\mathcal{L}}) \) be the number of \( n \)-vertices with non-empty label. Then there exists a map \( \varphi: \mathcal{C} \to \tilde{\mathcal{C}} \) such that, for some constants \( C_1, C_2, C_3, C_4 > 0 \) depending only on \( D \),

1. For all \( \mathcal{L} \in \mathcal{C} \), \( r(\mathcal{L}) = \tilde{r}(\tilde{\mathcal{L}}) \),
2. For all \( \mathcal{L} \in \mathcal{C} \), \( \tilde{\Delta}(\varphi(\mathcal{L})) \leq C_1 \Delta(\mathcal{L}) \), and
3. For any \( \mathcal{L} \in \tilde{\mathcal{C}} \) and \( \Delta_0 \geq 0 \),

\[
\sum_{\mathcal{L}\in\varphi^{-1}(\tilde{\mathcal{L}})} \prod_{s=1}^{l} \left( \frac{|a_d(\mathcal{L})|}{(d_s(\mathcal{L})!)^{1/2}} \right) \leq \left( \frac{\sqrt{\nu}}{|a|} \right)^{C_2 \Delta_0} |a|^\Delta(\tilde{\mathcal{L}})(\nu - a^2)^{\frac{\Delta_0}{4}} (1 + \frac{2}{\Delta_0})^{l} C_2 C_4 \Delta_0.
\]

(16)

The proof of this proposition and the explicit construction of the map \( \varphi \) require some detailed combinatorial arguments, which we defer to Appendix A. Using this result, we may complete the proof of Proposition 5.2 in the case \( a \neq 0 \).

**Proof of Proposition 5.2 (Case \( a \neq 0 \).** For any \( \varepsilon > 0 \) and even integer \( l \geq 2 \),

\[ \mathbb{P}[\|Q(X)\| > (1 + \varepsilon)\|\mu_{a,\nu,\gamma}\|] \leq \mathbb{P}\left[ \text{Tr} Q(X)^l > ((1 + \varepsilon)\|\mu_{a,\nu,\gamma}\|)^l \right] \leq \frac{\mathbb{E}[\text{Tr} Q(X)^l]}{(1 + \varepsilon)^l \|\mu_{a,\nu,\gamma}\|}. \]

By Lemma 5.9, Definition 5.6 and Proposition 5.17,

\[
\mathbb{E}[\text{Tr} Q(X)^l] \leq n \sum_{\mathcal{L} \in \mathcal{C}} \left( \frac{(12 \frac{l+DL}{2})^{12a}}{n} \right)^{\Delta(\mathcal{L})} \left( \frac{p}{n} \right)^{r(\mathcal{L})} \left( \prod_{s=1}^{l} \frac{|a_d(\mathcal{L})|}{(d_s(\mathcal{L})!)^{1/2}} \right) \\
= n \sum_{\mathcal{L} \in \mathcal{C}} \sum_{\Delta_0 = \left[ \frac{\tilde{\Delta}(\mathcal{L})}{\nu_{2}} \right]} \sum_{\mathcal{L} \in \varphi^{-1}(\tilde{\mathcal{L}})} \left( \frac{(6l + 6DL)^{12a}}{n} \right)^{\Delta_0} \left( \frac{\sqrt{\nu}}{|a|} \right)^{C_2 \Delta_0} |a|^\Delta(\tilde{\mathcal{L}})(\nu - a^2)^{\frac{\Delta_0}{4}} (1 + \frac{2}{\Delta_0})^{l} C_2 C_4 \Delta_0 \\
\leq n \sum_{\mathcal{L} \in \mathcal{C}} \left( \frac{p}{n} \right)^{\tilde{r}(\tilde{\mathcal{L}})} \left( \frac{(6l + 6DL)^{12a}}{n} \right)^{\Delta_0} \left( \frac{\sqrt{\nu}}{|a|} \right)^{C_2 \Delta_0} |a|^\Delta(\tilde{\mathcal{L}})(\nu - a^2)^{\frac{\Delta_0}{4}} (1 + \frac{2}{\Delta_0})^{l} C_2 C_4 \Delta_0 \\
\leq n l^{C_3} \left( \frac{l+DL}{2} + 1 \right) \sum_{\mathcal{L} \in \mathcal{C}} \left( \frac{p}{n} \right)^{\tilde{r}(\tilde{\mathcal{L}})} \left( \frac{\sqrt{\nu}}{|a|} \right)^{C_2 \frac{\Delta_0}{C_1}} (6l + 6DL)^{\frac{12a}{C_1}} \left( \frac{\sqrt{\nu}}{|a|} \right)^{C_2} l^{C_4} \Delta(\tilde{\mathcal{L}}),
\]

where the last line holds for all sufficiently large \( n \) if \( l \asymp \log n \). Let

\[
\tilde{n} = \left[ \frac{n^{\frac{1}{C_1}}}{(6l + 6DL)^\frac{12a}{C_1}} \right],
\]
and let \( \tilde{p} = \lfloor \frac{np}{\tilde{r}} \rfloor \). Then for all sufficiently large \( n \) and \( l \geq \log n \), \( nC_{\delta} \left( \frac{l+Dl}{2} + 1 \right) \leq n^2 \), and also \( \frac{p}{n} \tilde{r} \tilde{L} \leq \left( \frac{\tilde{p}}{n} \right) \tilde{r} \tilde{L} (1 + \frac{\epsilon}{2})^l \) (as \( \tilde{r} \tilde{L} \leq l, p/n \rightarrow \gamma \), and \( \tilde{p}/n \rightarrow \gamma \)). Then

\[
E[\text{Tr} \, Q(X)^l] \leq n^2 (1 + \frac{\epsilon}{2})^l \sum_{L \in \tilde{C}} \left( \frac{1}{n} \right) \frac{\Delta(L)}{\tilde{r} \tilde{L}} \left( \frac{\tilde{p}}{n} \right) a \left| \tilde{k} \tilde{L} (\nu - a^2) \right|^l \frac{l-\tilde{r} \tilde{L}}{2}.
\]

On the other hand, by Lemma 5.16

\[
(1 - \frac{\epsilon}{2})^l \tilde{n} \sum_{L \in \tilde{C}} \frac{1}{n} \frac{\Delta(L)}{\tilde{r} \tilde{L}} \left( \frac{\tilde{p}}{n} \right) a \left| \tilde{k} \tilde{L} (\nu - a^2) \right|^l \frac{l-\tilde{r} \tilde{L}}{2} \leq E \left[ \text{Tr} \, M^l \right]
\]

for all sufficiently large \( n \). Since \( \tilde{p}/n \rightarrow \gamma \) and \( l \sim BC_{\delta} \log n \) if \( l \sim B \log n \), Proposition 5.11 implies \( E[\text{Tr} \, M^l] \leq \tilde{p} E[\| M^l \|^l] \leq \tilde{p} (\| \mu_{\alpha,\nu,\gamma} \| (1 + \frac{\epsilon}{2}))^l \) for all large \( n \). Thus

\[
P(\| Q(X) \| > (1 + \epsilon)\| \mu_{\alpha,\nu,\gamma} \|) \leq n e^{\frac{2}{n}} \left( \frac{(1 + \frac{\epsilon}{2})^2}{(1 - \frac{\epsilon}{2}) (1 + \epsilon)} \right)^l.
\]

Taking \( l \sim B \log n \) with \( B > 0 \) sufficiently large such that \( B \log \frac{(1 + \frac{\epsilon}{2})^2}{(1 - \frac{\epsilon}{2}) (1 + \epsilon)} < -4 \) (which is possible for any sufficiently small \( \epsilon > 0 \)), this implies \( P(\| Q(X) \| > (1 + \epsilon)\| \mu_{\alpha,\nu,\gamma} \|) < n^{-2} \) for all large \( n \). Then \( \limsup_{n,p \rightarrow \infty} \| Q(X) \| \leq (1 + \epsilon)\| \mu_{\alpha,\nu,\gamma} \| \) almost surely, and taking \( \epsilon \rightarrow 0 \) concludes the proof. \( \square \)

As \( \| \mu_{\alpha,\nu,\gamma} \| \) is continuous in \( a, \nu, \) and \( \gamma \), Proposition 5.2 in the case \( a = 0 \) may be established via a continuity argument:

**Proof of Proposition 5.2 (Case \( a = 0 \)).** For any \( a > 0 \), let \( k_a(x) = k(x) + ax \), and let \( Q_a(X) \) be the matrix as defined in Definition 5.1 for the kernel function \( k_a \). Then \( Q_a(X) = Q(X) + \frac{ax}{n} V(X) \), where \( V(X) \) has zero diagonal and equals \( XXT^T \) off of the diagonal. By Proposition 5.2 for the \( a \neq 0 \) case, established above, \( \limsup_{n,p \rightarrow \infty} \| Q_a(X) \| \leq \| \mu_{\alpha,\nu,\gamma} \| \). By standard results for covariance matrices (see e.g. [29]), \( \limsup_{n,p \rightarrow \infty} \| \frac{1}{n} V(X) \| \leq C_\gamma \) almost surely under the assumption \( (6) \), for a constant \( C_\gamma > 0 \). This implies \( \limsup_{n,p \rightarrow \infty} \| Q(X) \| \leq \| \mu_{\alpha,\nu,\gamma} \| - aC_\gamma \) for any \( a > 0 \), and the desired result follows by taking \( a \rightarrow 0 \). \( \square \)

6. **Analyzing the remainder matrices**

To conclude the proof of Theorem 1.7, we analyze in this section the remainder matrices \( R(X) \) and \( S(X) \) of Definition 5.1.

**Lemma 6.1.** As \( n, p \rightarrow \infty \) with \( p/n \rightarrow \gamma \in (0, \infty) \), \( \| S(X) \| \rightarrow 0 \) almost surely.

**Proof.** Note that \( \| S(X) \| \leq \| S(X) \|_F \leq p \max_{1 \leq i, i' \leq p} |s_{ii'}| \) where \( \| \cdot \|_F \) is the Frobenius norm. By Definition 5.1 and Proposition 4.1, for any \( 1 \leq i, i' \leq p \) and \( \alpha > 0 \), \( |s_{ii'}| \leq n^{\frac{1}{2} + \alpha} \sum_{d=1}^{D} |a_d| \) with probability at least \( 1 - n^{-4} \), for all large \( n \). Then \( p \max_{1 \leq i, i' \leq p} |s_{ii'}| \leq Cpn^{\frac{1}{2} + \alpha} \) with probability at least \( 1 - p^{2}n^{-4} \). Taking any \( \alpha < 1/2 \) yields the desired result. \( \square \)

**Definition 6.2.** For \( d \geq 2 \), define \( R_d(X) = (r_{ii'} : 1 \leq i, i' \leq p) \in \mathbb{R}^{p \times p} \) with entries

\[
r_{ii'} = \begin{cases} \left( \frac{d}{2} \right)^{d-1} n^{d+1} \sum_{j_1 \ldots j_{d-1}=1}^{n} \left( x_{i,j_1} x_{i,j_1}^2 - 1 \right) \prod_{a=2}^{d-1} x_{i,a} x_{i',a} \right) & i \neq i' \\ 0 & i = i'. \end{cases}
\]

Note that \( R(X) \) in Definition 5.1 is given by \( R(X) = \sum_{d=2}^{D} a_d R_d(X) \).
Lemma 6.3. As \( n, p \to \infty \) with \( p/n \to \gamma \in (0,\infty) \), \( \| R_d(X) \| \to 0 \) almost surely for any \( d \geq 3 \).

Proof. Letting \( i_7 := i_1 \) for notational convenience, note that

\[
\mathbb{E} \left[ \text{Tr} \ R_d(X)^6 \right] = \sum_{i_1, \ldots, i_6 = 1}^{P} \mathbb{E} \left[ \prod_{s=1}^{6} r_{i_s i_{s+1}} \right] = \frac{(d/2)^6}{(d!)^3} n^{-3(d+1)} V_d
\]

where we set

\[
V_d := \sum_{i_1, \ldots, i_6 = 1}^{P} \sum_{i_1 \neq i_2, i_2 \neq i_3, \ldots, i_6 \neq i_1}^{n} \sum_{j_1, j_1 \neq j_2, \ldots, j_6 = 1}^{1} X(i,j)
\]

and

\[
X(i,j) := \mathbb{E} \left[ \prod_{s=1}^{6} \left( x_{i_s j_s} x_{i_{s+1} j_s} - 1 \right) \prod_{a=2}^{d-1} x_{i_s j_a} x_{i_{s+1} j_a} \right].
\]

If there is some \( j^* \in \{1, \ldots, n\} \) such that \( j^*_a = j^* \) for exactly one pair of indices \((s,a) \in \{1, \ldots, 6\} \times \{1, \ldots, d-1\} \), then \( X(i,j) = 0 \). Hence, we may restrict the sum in (17) to terms where each index \( j^*_a \) equals some other index \( j'^*_a \). Then the number of distinct indices \( j^*_a \) is at most \( 6(d-1)/2 = 3d - 3 \). Furthermore, it is clear that \( |X(i,j)| \leq C \) always, for a constant \( C \) independent of \( n \) and \( p \).

We now consider several cases for a nonzero term \( X(i,j) \), depending on the number of distinct indices among \( \{i_1, \ldots, i_6\} \):

Case 1: \( |\{i_1, \ldots, i_6\}| \leq 4 \). Letting \( V_{d,4} \) denote the sum over terms of (17) belonging to this case, the above implies \( V_{d,4} \leq C p^n n^{3d-3} \) for a constant \( C \) independent of \( n \) and \( p \).

Case 2: \( |\{i_1, \ldots, i_6\}| = 5 \). Then either \( i_s = i_{s+2} \) or \( i_s = i_{s+3} \) for some \( s \) (where \( s+2 \) and \( s+3 \) are taken modulo 6), with the remaining indices all distinct. Suppose without loss of generality that \( i_2 \) and \( i_3 \) are distinct from each other and from \( \{i_1, i_4, i_5, i_6\} \). Let \( i^* = i_2 \) and \( j^* = j_2^* \) (which exists when \( d \geq 3 \)). By the distinctness conditions in (17), \( j^* \neq j^*_a \) for all \( a \neq 2 \). If furthermore \( j^* \neq j^*_a \) for all \( a \in \{1, \ldots, d-1\} \), then \( x_{i_2 x_{i_{2} x_{i_{2}}} x_{i_{2} x_{i_{2}}} x_{i_{2}}} \) appears exactly once in (18), so \( X(i,j) = 0 \). If \( j^* = j^*_1 \), then \( x_{i_2 x_{i_{2}}} \) appears twice, once as the term \( x_{i_2 x_{i_2}} \) and once in the term \( x_{i_2 x_{i_2}} - 1 \). The product of these terms is \( x_{i_2 x_{i_{2}}} x_{i_2 x_{i_{2}}} - x_{i_2 x_{i_{2}}} \), and as \( \mathbb{E}[x_{i_2 x_{i_{2}}} x_{i_2 x_{i_{2}}} x_{i_{2}}] = 0 \) and \( \mathbb{E}[x_{i_2 x_{i_{2}}} x_{i_{2}}] = 0 \), this also implies \( X(i,j) = 0 \). Hence we must have \( j^* = j^*_1 = j^*_a \) for some \( a \geq 2 \). The same argument applied to \( i^* := i_3 \) and \( j^* := j^*_3 \) shows that we must have \( j^* = j^*_a = j^*_a \) for some \( a' \geq 2 \). Then \( j^*_2 = j^*_3 = j^*_a \), so there cannot be exactly 3d - 3 distinct indices \( j^*_a \). Then there are at most 3d - 4 such distinct indices, and letting \( V_{d,5} \) denote the sum over terms of (17) belonging to this case, we obtain \( V_{d,5} \leq C p^n n^{3d-4} \) for a constant \( C \).

Case 3: \( |\{i_1, \ldots, i_6\}| = 6 \). Then all indices \( i_1, \ldots, i_6 \) are distinct. Applying the argument of Case 2, there exists \( a \geq 2 \) such that \( j^*_1 = j^*_a \). There exists further \( a' \geq 2 \) such that \( j^*_a = j^*_a \), \( a'' \geq 2 \) such that \( j''_a = j''_a \), etc., and for each \( s = 1, \ldots, 6 \) we obtain some \( a \geq 2 \) such that \( j^*_a = j^*_a \). Then the number of distinct indices \( j^*_a \) is at most \( \frac{6(d-1)-6}{2} + 1 = 3d - 5 \). Letting \( V_{d,6} \) denote the sum over terms of (17) belonging to this case, we obtain \( V_{d,6} \leq C p^n n^{3d-5} \).

Putting the cases together,

\[
\mathbb{E} \left[ \text{Tr} \ R_d(X)^6 \right] \leq C n^{-3(d+1)} (V_{d,4} + V_{d,5} + V_{d,6}) \leq C n^{-2}
\]

for a constant \( C > 0 \) and all large \( n \) and \( p \). Then for any \( \varepsilon > 0 \),

\[
\mathbb{P} \left[ \| R_d(X) \| > \varepsilon \right] \leq \frac{\mathbb{E}[\text{Tr} \ R_d(X)^6]}{\varepsilon^6} \leq \frac{C}{\varepsilon^6 n^{2d}},
\]

so \( \limsup_{n,p \to \infty} \| R_d(X) \| \leq \varepsilon \) almost surely, and the result follows by taking \( \varepsilon \to 0 \). \( \square \)
Lemma 6.4. Let $R_d(X)$ be as in Definition 6.2 and let $\tilde{R}(X)$ be as in (9). As $n,p \to \infty$ with $p/n \to \gamma \in (0,\infty)$, $\|a_2 R_2(X) - \tilde{R}(X)\| \to 0$ almost surely.

Proof. Let $T(X) = a_2 R_2(X) - \tilde{R}(X)$. Then $T(X)$ has entries
\[
1 \frac{a_2 n^{-\frac{3}{2}}}{\sqrt{2}} \sum_{j=1}^{n} \left((x_{ij} v_{ij}^2 - 1) - (x_{ij}^2 - 1) - (x_{ij}^2 - 1)\right) \quad i \neq i' \\
0 \quad i = i'.
\]
Thus, excluding the diagonal, $T(X)$ equals $\frac{a_2}{\sqrt{2}} n^{-\frac{3}{2}} Y Y^T$ where $Y = (y_{ij}) \in \mathbb{R}^{p \times n}$ and $y_{ij} = x_{ij}^2 - 1$.

We now conclude the proof of Theorem 1.7.

Proof of Theorem 1.7. Recall Definitions 5.1 and 6.2 and the decompositions $K(X) = Q(X) + R(X) + S(X)$ and $R(X) = \sum_{d=2}^{D} a_d R_d(X)$. Proposition 5.2 and Lemmas 6.1, 6.3, and 6.4 imply $\limsup_{n,p \to \infty} \|K(X) - R(X)\| \leq \|\mu_{a,v,\gamma}\|$.

To verify the claim regarding the non-zero eigenvalues of $\tilde{R}(X)$ in property (2) of Theorem 1.7, we compute from (9) $Tr R(X) = \frac{a_2}{\sqrt{2}} n^{\frac{3}{2}} v(X)^T 1$ and $Tr \tilde{R}(X)^2 = \frac{a_2}{2n}(v(X)^T 1)^2 + p \|v(X)\|^2$.

By the law of large numbers, $n^{-1} v(X)^T 1 \to 0$ and $(p/n^2) \|v(X)\|^2 \to \gamma^2 (\mathbb{E} x_{ij}^4 - 1)$ almost surely. Since the roots of a polynomial are continuous in its coefficients, the result follows.

Appendix A. Combinatorial results

This appendix contains the proofs of Lemmas 5.5, 5.7, and 5.13 used in Section 5 as well as the proof of Proposition 5.17 and the explicit construction of the map $\varphi$ in that proposition.

A.1. Proof of Lemmas 5.5, 5.7, and 5.13. We restate the lemmas using their original numbering.

Lemma 5.13. Suppose a simple-labeling of an $l$-graph has $k$ $n$-vertices with non-empty label and $\tilde{m}$ total distinct $p$-labels and distinct non-empty $n$-labels. Then $\tilde{m} \leq \frac{l+k+1}{2}$.

Proof. Let $I = \{1, \ldots, p\}$ and $J = \{1, \ldots, n\}$, and consider an undirected graph $G$ on the vertex set $I \cup J$ (the disjoint union of $I$ and $J$ with $n+p$ elements, treating elements of $I$ and the elements of $J$ as distinct). Let $G$ have an edge between $i, i' \in I$ if there are three consecutive vertices $(p, n, p)$ of the $l$-graph with the labels $i, \emptyset, i'$ or $i'$, $\emptyset$, $i$. Let $G$ have an edge between $i \in I$ and $j \in J$ if there are two consecutive vertices of the $l$-graph such that the $p$-vertex has label $i$ and the $n$-vertex has label $j$. The number of vertices of $G$ incident to at least one edge is $\tilde{m}$, and $G$ must
be connected, so it has at least \( \tilde{m} - 1 \) edges. An edge in \( G \) between \( i, i' \in I \) corresponds to at least two consecutive pairs of \( p \)-vertices in the \( l \)-graph having an \( n \)-vertex with empty label in between, by condition (3) of Definition 5.12, so the number of such edges is at most \( \frac{l-k}{2} \). Similarly, an edge in \( G \) between \( i \in I \) and \( j \in J \) corresponds to at least two pairs of consecutive \( n \) and \( p \)-vertices of the \( l \)-graph such that the \( n \)-vertex has non-empty label, by condition (2) of Definition 5.12, so the number of such edges is at most \( \frac{2k}{l} \). Then \( \tilde{m} - 1 \leq \frac{l+k}{2} \).

Turning now to multi-labelings, for each \( j \in \{1, 2, 3, \ldots \} \) and a given multi-labeling, let us denote throughout

\[
N_j := \text{number of appearances of } j \text{ as an } n\text{-label.}
\]

Then the following two lemmas hold:

**Lemma A.1.** In any multi-labeling of an \( l \)-graph, each \( j \) that appears as an \( n \)-label has \( N_j \geq 2 \).

*Proof.* Suppose that an \( n \)-label \( j \) appears only once. The two \( p \)-vertices preceding and following that \( n \)-vertex must have distinct labels, say \( i_1 \) and \( i_2 \), by condition (1) of Definition 5.4. Then exactly one edge in the \( l \)-graph has \( p \)-vertex endpoint labeled \( i_1 \) and \( n \)-vertex endpoint having label \( j \) (and similarly for \( i_2 \) and \( j \)), contradicting condition (3) of Definition 5.4.

**Lemma A.2.** Suppose a multi-labeling of an \( l \)-graph has at most \( \frac{l}{2} \) distinct \( p \)-labels. If this multi-labeling has excess \( \Delta \), then

\[
\sum_{j: N_j \geq 3} N_j \leq 6\Delta - 6.
\]

Consequently, the number of \( n \)-vertices having any label \( j \) for which \( N_j \geq 3 \) is also at most \( 6\Delta - 6 \).

*Proof.* Observe that if \( m \) total distinct \( p \)-labels and \( n \)-labels appear in the labeling, and at most \( \frac{l}{2} \) of these are \( p \)-labels, then the labeling has at least \( m - \frac{l}{2} - c \) distinct \( n \)-labels. Let \( c = |\{j : N_j = 2\}| \). Then Lemma A.1 implies \( 2c + 3 \left( m - \frac{l}{2} - c \right) \leq \sum_{s=1}^{l} d_s \) (where \( d_1, \ldots, d_l \) are the numbers of \( n \)-labels on the \( l \) \( n \)-vertices), so \( c \geq 3m - \frac{3l}{2} - \sum_{s=1}^{l} d_s \). Then the \( n \)-labels in \( \{j : N_j = 2\} \) account for at least \( 6m - 3l - 2\sum_{s=1}^{l} d_s \) of the \( \sum_{s=1}^{l} d_s \) total \( n \)-labels, implying that at most \( 3l + 3\sum_{s=1}^{l} d_s - 6m = 6\Delta - 6 \) total \( n \)-labels remain. This establishes the first claim, and the second follows directly from the first.

We will prove many subsequent claims regarding multi-labelings by induction on \( l \). The following two lemmas describe the base case of the induction and the basic inductive step.

**Lemma A.3.** Suppose \( l = 2 \) or \( l = 3 \). Then for any multi-labeling of the \( l \)-graph, all \( l \) \( p \)-labels are distinct, and all \( l \) \( n \)-vertices have the same tuple of \( n \)-labels, up to reordering.

*Proof.* That all \( l \) \( p \)-labels are distinct is a consequence of condition (1) of Definition 5.4. Then by conditions (2) and (3) of Definition 5.4 the \( n \)-vertices immediately preceding and following each \( p \)-vertex must have the same tuple of \( n \)-labels, up to reordering.

**Lemma A.4.** In a multi-labeling of an \( l \)-graph with \( l \geq 4 \), suppose a \( p \)-vertex \( V \) is such that its \( p \)-label appears on no other \( p \)-vertices. Let the \( n \)-vertex preceding \( V \) be \( U \), the \( p \)-vertex preceding \( U \) be \( T \), the \( n \)-vertex following \( V \) be \( W \), and the \( p \)-vertex following \( W \) be \( X \).

1. If \( T \) and \( X \) have different \( p \)-labels, then the graph obtained by deleting \( V \) and \( W \) and connecting \( U \) to \( X \) is an \((l-1)\)-graph with valid multi-labeling.
2. If \( T \) and \( X \) have the same \( p \)-label, then the graph obtained by deleting \( U, V, W, \) and \( X \) and connecting \( T \) to the \( n \)-vertex after \( X \) is an \((l-2)\)-graph with valid multi-labeling.
Proof. First consider case (1). As $T$ and $X$ have distinct $p$-labels, it remains true that no two consecutive $p$-vertices in the $(l - 1)$-graph have the same $p$-label, so condition (1) of Definition 5.4 holds. Condition (2) of Definition 5.4 clearly still holds as well. If $V$ has $p$-label $i$ and $W$ has $n$-labels $(j_1, \ldots, j_d)$, then $U$ has $n$-labels $(j_1, \ldots, j_d)$ as well, up to reordering, by conditions (2) and (3) of Definition 5.4 and the fact that $V$ is the only $p$-vertex with label $i$. Then in the $(l - 1)$-graph obtained by deleting $V$ and $W$, the number of edges with $p$-vertex endpoint labeled $i$ and $n$-vertex endpoint having label $j_s$ for any $s = 1, \ldots, d$ is zero, and the number of edges with $p$-vertex endpoint labeled $i'$ and $n$-vertex endpoint having label $j'$ is the same as in the original $l$-graph for all other pairs $(i', j')$. Thus condition (3) of Definition 5.4 still holds as well, so the $(l - 1)$-graph has a valid multi-labeling.

Now consider case (2). $X$ and the $p$-vertex after $X$ must have different $p$-labels in the original $l$-graph, by condition (1) of Definition 5.4. As $T$ and $X$ have the same $p$-label, this implies $T$ and the $p$-vertex after $X$ must have different $p$-labels, so condition (1) of Definition 5.4 still holds in the $(l - 2)$-graph. Condition (2) of Definition 5.4 clearly still holds in the $(l - 2)$-graph as well. Suppose $V$ has $p$-label $i_1$, $T$ and $X$ have $p$-label $i_2$, and $W$ has $n$-labels $(j_1, \ldots, j_d)$. As in case (1), $U$ must also have $n$-labels $(j_1, \ldots, j_d)$ up to reordering. Then in the $(l - 2)$-graph obtained by deleting $U$, $V$, $W$, and $X$, the number of edges with $p$-vertex endpoint labeled $i_1$ and $n$-vertex endpoint having label $j_s$ for any $s = 1, \ldots, d$ is zero, the number of edges with $p$-vertex endpoint labeled $i_2$ and $n$-vertex endpoint having label $j_s$ for any $s = 1, \ldots, d$ is two less than in the original $l$-graph, and the number of edges with $p$-vertex endpoint labeled $i'$ and $n$-vertex endpoint having label $j'$ is the same as in the original $l$-graph for all other pairs $(i', j')$. Hence condition (3) of Definition 5.4 still holds as well, so the $(l - 2)$-graph has a valid multi-labeling. □

Lemma 5.5. Suppose a multi-labeling of an $l$-graph has $d_1, \ldots, d_l$ $n$-labels on the first through $l^{th}$ $n$-vertices, respectively, and suppose that it has $m$ total distinct $p$-labels and $n$-labels. Then $m \leq \frac{t + \sum_{s=1}^{t} d_s}{2} + 1$.

Proof. We induct on $l$. For $l = 2$, a multi-labeling must have $d_1 = d_2$ and $m = d_1 + 2$, and for $l = 3$, a multi-labeling must have $d_1 = d_2 = d_3$ and $m = d_1 + 3$, by Lemma A.3. The result is then easily verified in these two cases.

Suppose by induction that the result holds for $l - 2$ and $l - 1$, and consider a multi-labeling of an $l$-graph with $l \geq 4$. If each distinct $p$-label appears at least twice, then there are at most $\frac{t}{2}$ distinct $p$-labels. Lemma A.1 implies there are at most $\frac{\sum_{s=1}^{t} d_s}{2}$ distinct $n$-labels, so $m \leq \frac{t + \sum_{s=1}^{t} d_s}{2}$, establishing the result.

Thus, suppose that some $p$-vertex $V$ has a label that appears exactly once, and let $T, U, W, X$ be as in Lemma A.4. If $T$ and $X$ have different $p$-labels, follow procedure (1) in Lemma A.4 to obtain a multi-labeling of an $(l - 1)$-graph. This multi-labeling now has $m - 1$ total distinct $p$-labels and $n$-labels, and so the induction hypothesis implies $m - 1 \leq \frac{t - 1 + \sum_{s=1}^{t+1} d_s - d}{2} + 1$ where $d$ is the number of $n$-labels of the deleted $n$-vertex $W$. Hence $m \leq \frac{t + \sum_{s=1}^{t+1} d_s}{2} - \frac{d + 1}{2} + 2 \leq \frac{t + \sum_{s=1}^{t} d_s}{2} + 1$.

If $T$ and $X$ have the same $p$-label, follow procedure (2) of Lemma A.4 to obtain a multi-labeling of an $(l - 2)$-graph. This multi-labeling has between $m - d - 1$ and $m - 1$ (inclusive) total distinct $p$-labels and $n$-labels, where $d$ is the number of $n$-labels of the deleted $n$-vertex $W$. The induction hypothesis implies $m - d - 1 \leq \frac{t - 2 + \sum_{s=1}^{t+1} d_s - 2d}{2} + 1$, so $m \leq \frac{t + \sum_{s=1}^{t+1} d_s}{2} + 1$. This completes the induction in both cases, establishing the desired result. □

Lemma 5.7. Suppose a multi-labeling of an $l$-graph has excess $\Delta$. For each $i \in \{1, 2, 3, \ldots\}$ and $j \in \{1, 2, 3, \ldots\}$, let $b_{ij}$ be the number of edges in the $l$-graph such that the $p$-vertex endpoint is labeled $i$ and the $n$-vertex endpoint has label $j$ in its tuple. Then $\sum_{i,j} b_{ij}^2 \leq 12 \Delta$.

Proof. We induct on $l$. For $l = 2$ or $3$, we must have $b_{ij} = 0$ or $2$ for all $(i, j)$ by Lemma A.3 and $\Delta \geq 0$ by Lemma 5.5 so the result holds.
Suppose the result holds for $l - 2$ and $l - 1$, and consider a multi-labeling of an $l$-graph with $l \geq 4$. If each distinct $p$-label appears at least twice, then there are at most $\frac{1}{2}$ distinct $p$-labels, so Lemma A.2 applies. For any $j$ with $N_j = 2$, we have $b_{ij} = 2$ or $b_{ij} = 0$ for all $i$, by conditions (1) and (3) of Definition 5.4. For any $j$ with $N_j \geq 3$, we apply the bound $\sum_{i:j} b_{ij} \leq 2N_j$. Then $\sum_{i:j} b_{ij} \leq 2(6\Delta - 6) \leq 12\Delta$ by Lemma A.2.

Now suppose that some $p$-vertex $V$ has a $p$-label appearing exactly once. Consider the $(l - 1)$-graph or $(l - 2)$-graph obtained by Lemma A.4. In the case of the $(l - 1)$-graph, it is easily verified that $\sum_{i:j} b_{ij} \leq 12 \left( \frac{l+1}{2} \sum_{i} d_{ij} - d + 1 - (m - 1) \right) \leq 12\Delta$, where $d \geq 1$ is the number of $n$-labels on the deleted $n$-vertex $W$.

In the case of the $(l - 2)$-graph, suppose the deleted $n$-vertex $W$ (and $U$) has $d$ $n$-labels, of which $d'$ also appear on an $n$-vertex different from $W$ and $U$. If $j$ does not appear on $W$ or $U$, then clearly $b_{ij}$ is the same in the $(l - 2)$-graph and the original $l$-graph for all $i$. If $j$ is one of the $d - d'$ $n$-label values appearing only on $W$ and $U$, then $b_{ij} = 0$ or $2$ in both the $(l - 2)$-graph and the original $l$-graph for all $i$. If $j$ is one of the other $d'$ $n$-label values appearing on $W$ and $U$, then in deleting $U$, $V$, $W$, and $X$, we may have reduced $b_{ij}$ by $2$ for at most two distinct values of $i$ (corresponding to the $p$-labels of $V$ and $X$). This implies that $\sum_{i:j} b_{ij}$ reduces by at most $8$ for this $j$, with the maximal reduction occurring if $b_{ij} = 4$ for both of these values of $i$ in the original $l$-graph. Then by the induction hypothesis, $\sum_{i:j} b_{ij} - 8d' \leq 12 \left( \frac{l+2}{2} \sum_{i} d_{ij} - 2d + 1 - (m - 1) - (d' - d) \right)$, as the $(l - 2)$-graph has $m - 1 - (d - d')$ total distinct $n$ and $p$-labels. Then $\sum_{i:j} b_{ij} \leq 12 \left( \frac{l+2}{2} \sum_{i} d_{ij} - 1 - m - d' \right) + 8d' \leq 12\Delta$, so the result holds in this case as well, completing the induction.

□

A.2. Construction of the map $\varphi$.

Definition A.5. In an $l$-graph with a multi-labeling, an $n$-vertex is single if it has only one $n$-label. It is a good single if it is single and if its $n$-label $j$ appears only on single $n$-vertices. Otherwise, it is a bad single.

Definition A.6. In an $l$-graph with a multi-labeling, a pair $(V, V')$ of distinct (not necessarily consecutive) $n$-vertices is a good pair if the following conditions hold:

1. $V$ and $V'$ have the same tuple of $n$-labels, up to reordering,
2. $V$ and $V'$ are not single, and
3. $N_j = 2$ for each $j$ appearing as an $n$-label on $V$ and $V'$ (i.e. this label $j$ appears on no other $n$-vertices).

If an $n$-vertex $V$ is not single and not part of any good pair, then $V$ is a bad non-single.

Thus, every $n$-vertex is either a good single, a bad single, a bad non-single, or part of a good pair. Conditions (1) and (3) of Definition 5.4 require that, if $(V, V')$ is a good pair, then the two (distinct) $p$-labels of the $p$-vertices preceding and following $V$ are the same as those of the $p$-vertices preceding and following $V'$ (but not necessarily in the same order).

Definition A.7. Suppose $(V, V')$ is a good pair of $n$-vertices. Let the $p$-vertices preceding and following $V$ be $U$ and $W$, respectively, and let the $p$-vertices preceding and following $V'$ be $U'$ and $W'$, respectively. Then the good pair $(V, V')$ is proper if $U$ has the same label as $W'$ and $U'$ has the same label as $W$, and it is improper if $U$ has the same label as $U'$ and $W$ has the same label as $W'$.

Definition A.8. The label-simplifying map is the map from $(p, n)$-multi-labelings of an $l$-graph to $(p, n + 1)$-simple-labelings of an $l$-graph, defined by the following procedure:
(1) While there exists an improper good pair of $n$-vertices $(V, V')$, iterate the following: Let $W$ be the $p$-vertex following $V$ and $W'$ be the $p$-vertex following $V'$, and reverse the sequence of vertices starting at $W$ and ending at $W'$ (together with their labels).

(2) For each $n$-vertex in a good pair, relabel it with the empty label.

(3) For each $n$-vertex that is a bad single or a bad non-single, relabel it with the single label $n + 1$.

**Remark A.9.** In the case where there are multiple improper good pairs in step (1) of this procedure, it will not be important for our later arguments in which order the pairs $(V, V')$ are selected and which vertex we choose as $V$ and which as $V'$. For concreteness, we may always select $\{V, V'\}$ to be the improper good pair whose sorted $n$-label-tuple is smallest lexicographically, and we may take $V$ to come before $V'$ in the $l$-graph cycle.

**Lemma A.10.** The following are true for the label-simplifying map in Definition A.8:

1. Step (1) of the procedure in Definition A.8 always terminates in a valid $(p, n)$-multi-labeling with no improper good pairs.
2. The image of any $(p, n)$-multi-labeling under the map is a valid $(p, n + 1)$-simple-labeling.
3. If two multi-labelings are equivalent, then their image simple-labelings are also equivalent.

**Proof.** Clearly each reversal in step (1) of the procedure preserves condition (2) of Definition 5.4 as well as the number of good pairs and $n$-labels of each good pair. As $W$ and $W'$ have the same $p$-label because $(V, V')$ is improper, it also preserves conditions (1) and (3) of Definition 5.4, so the resulting labeling is still a valid $(p, n)$-multi-labeling. Each time this reversal is performed, $V$ and $V'$ become consecutive $n$-vertices in the $l$-graph, and the pair $(V, V')$ becomes a proper good pair. As $V$ and $V'$ are consecutive, they must remain consecutive under each subsequent reversal, so their properness is preserved. Hence the procedure must terminate after a number of iterations at most the total number of good pairs in the multi-labeling, and the final multi-labeling is such that all good pairs are proper. This establishes (1).

To prove (2), note that the image labeling has either one $n$-label or the empty label for each $n$-vertex. Condition (1) of Definition 5.12 holds for the image labeling by condition (1) of Definition 5.4 as the $p$-labels are preserved. As all good pairs in the multi-labeling obtained after applying step (1) of the procedure are proper, and step (2) of the procedure maps their labels to the empty label, condition (3) of Definition 5.12 holds for the image labeling. Finally, note that if $j$ is an $n$-label appearing on good single vertices in the multi-labeling, then condition (2) of Definition 5.12 holds in the image labeling for this $j$ and all $p$-labels $i$ by condition (3) in Definition 5.4. For the new $n$-label $n + 1$ created in step (3) of the map, note that for each $i \in \{1, 2, 3, \ldots\}$ there must be an even number of edges in the $l$-graph with $p$-endpoint labeled $i$. Of these, there must be an even number with $n$-endpoint $j$ for any good single label $j$, by the above argument, and there must also be an even number with $n$-endpoint belonging to a good pair since these edges must come in pairs. Hence the number of remaining edges adjacent to any $p$-vertex with label $i$ must also be even. These are precisely the edges with $p$-endpoint labeled $i$ and $n$-endpoint labeled $n + 1$ in the image labeling, so condition (2) of Definition 5.12 holds for the new $n$-label $n + 1$ and all $p$-labels $i$ as well. Hence the image labeling is a valid $(p, n + 1)$-simple-labeling, establishing (2).

(3) is evident, as equivalent multi-labelings have the same proper and improper good pairs of $n$-vertices and the same good single $n$-vertices. □

**Definition A.11.** Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be the set of all multi-labeling equivalence classes and simple-labeling equivalence classes, respectively, of an $l$-graph. For $\mathcal{L} \in \mathcal{C}$ and any multi-labeling in $\mathcal{L}$, let $\tilde{\mathcal{L}} \in \tilde{\mathcal{C}}$ contain its image simple-labeling under the label-simplifying map of Definition A.8 and define $\varphi : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ by $\varphi(\mathcal{L}) = \tilde{\mathcal{L}}$. 
A.3. Verification of Proposition 5.17, properties (1) and (2). For the map \( \varphi \) of Definition A.1, property (1) of Proposition 5.17 is evident as the \( p \)-labels are preserved. We verify property (2) by bounding the number of bad non-single \( n \)-vertices.

For each pair \( i, i' \in \{1, 2, 3, \ldots\} \) with \( i < i' \), and for a given multi-labeling, let us denote \( P_{i,i'} := \text{number of appearances of } i, i' \text{ as the } p \text{-labels of two consecutive } p \text{-vertices (in some order)}. \)

Lemma A.12. Suppose a multi-labeling of an \( l \)-graph has excess \( \Delta \). Then

\[
\sum_{i < i': P_{i,i'} \geq 3} P_{i,i'} \leq 42\Delta.
\]

Proof. We induct on \( l \). For \( l = 2 \) and \( 3 \), \( P_{i,i'} = 0 \) or 1 for all pairs \( i < i' \), and \( \Delta \geq 0 \) by Lemma 5.5, so the result holds.

Suppose by induction that the result holds for \( l - 2 \) and \( l - 1 \), and consider a multi-labeling of an \( l \)-graph with \( l \geq 4 \). First suppose each distinct \( p \)-label appears at least twice, so there are at most \( \frac{l}{2} \) distinct \( p \)-labels. If an \( n \)-label \( j \) is such that \( N_j = 2 \), then the pairs of \( p \)-vertices before and after the two \( n \)-vertices with label \( j \) must have the same pairs of \( p \)-labels, by conditions (1) and (3) of Definition 5.4. Thus the number of pairs \( i < i' \) with \( P_{i,i'} = 1 \) is at most the number of \( n \)-vertices for which \( N_j \geq 3 \) for all of its \( n \)-labels \( j \). This is at most \( 6\Delta \) by Lemma A.2. On the other hand, the number of distinct \( p \)-labels is at most one more than the number of distinct pairs of consecutive \( p \)-labels. (This is easily seen by considering the undirected graph with vertices \( \{1, \ldots, p\} \) having an edge between \( i, i' \) if and only if some consecutive pair of \( p \)-vertices have labels \( i \) and \( i' \), and noting that this graph is connected.) Lemma A.1 implies there are at most \( \sum_{i=1}^{\ell} d_i \) distinct \( n \)-labels, and hence at least \( m - \sum_{i=1}^{\ell} d_i - 1 = l - \Delta \) distinct pairs \( i < i' \) of consecutive \( p \)-labels. At least \( \frac{l}{2} - 7\Delta \)

of these have \( P_{i,i'} \geq 2 \). If \( c \) of these have \( P_{i,i'} = 2 \), then \( 2c + 3 \left( \frac{l}{2} - 7\Delta - c \right) \leq l \), so \( c \geq \frac{l}{2} - 21\Delta \).

These account for at least \( l - 42\Delta \) pairs of consecutive \( p \)-vertices, implying that at most \( 42\Delta \) pairs of consecutive \( p \)-vertices remain. This establishes the result in this case.

Now suppose that there is some \( p \)-vertex \( V \) whose \( p \)-label appears only once. Consider the \((l - 1)\)-graph or \((l - 2)\)-graph obtained by Lemma A.4. It is easily verified that \( \sum_{i < i': P_{i,i'} \geq 3} P_{i,i'} \) is the same in this graph as in the original \( l \)-graph, because if \( P_{i,i'} \geq 3 \) in the original \( l \)-graph, then neither \( i \) nor \( i' \) can be the \( p \)-label of \( V \). On the other hand, our proof of Lemma 5.5 verified that this \((l - 1)\)-graph or \((l - 2)\)-graph has excess at most that of the original \( l \)-graph, so the desired result follows from the induction hypothesis.

The next lemma bounds the number of bad non-single \( n \)-vertices, i.e. it shows that in any multi-labeling with small excess \( \Delta \), most of the non-single \( n \)-vertices must belong to a good pair.

Lemma A.13. Suppose a multi-labeling of an \( l \)-graph has excess \( \Delta \) and \( k \) single \( n \)-vertices. Then there are at least \( \frac{l-k}{2} - 48\Delta \) good pairs of \( n \)-vertices.

Proof. Let \( m \) be the number of distinct \( n \) and \( p \)-labels and let \( d_1, \ldots, d_l \) be the numbers of \( n \)-labels on the \( l \) \( n \)-vertices. We induct on \( l \). If \( l = 2 \), then Lemma A.3 implies \( d_1 = d_2, \ m = d_1 + 2, \) and \( \Delta = 0 \). If \( d_1 = d_2 = 1 \), then \( k = 2 \) and there are no good pairs, and if \( d_1 = d_2 \geq 2 \), then \( k = 0 \) and there is one good pair. Hence the result holds. If \( l = 3 \), then Lemma A.3 implies \( d_1 = d_2 = d_3, \ m = d_1 + 3, \) and \( \Delta = \frac{d_1 - 1}{2} \). If \( d_1 = d_2 = d_3 = 1 \), then \( k = 3, \Delta = 0 \), and there are no good pairs. If \( d_1 = d_2 = d_3 \geq 2 \), then \( k = 0, \Delta \geq \frac{1}{2} \), and there are still no good pairs. In either case, the result also holds.

Consider \( l \geq 4 \), and assume by induction that the result holds for \( l - 2 \) and \( l - 1 \). First suppose each distinct \( p \)-label appears at least twice, so there are at most \( \frac{l}{2} \) distinct \( p \)-labels. By Lemma A.2 there are at most \( 6\Delta \) \( n \)-vertices with some \( n \)-label \( j \) such that \( N_j \geq 3 \), so there are at least \( l - k - 6\Delta \) non-single \( n \)-vertices for which each of its \( n \)-labels \( j \) has \( N_j = 2 \). Let \( V \) be one such \( n \)-vertex. We consider three cases:
Case 1: V has two n-labels \( j_1 \) and \( j_2 \) that appear on two different other n-vertices \( W_1 \) and \( W_2 \). Then Definition [5.4] implies that the three pairs of consecutive p-vertices around \( V \), \( W_1 \), and \( W_2 \) must have the same pair of p-labels. By Lemma [A.12] there are at most \( 42\Delta \) such n-vertices \( V \).

Case 2: All n-labels of \( V \) appear on a single other n-vertex \( W_1 \), but \( W_1 \) has some additional n-label \( j \) not appearing on \( V \). Then either all such additional n-labels \( j \) have \( N_j \geq 3 \), or there is some such \( j \) with \( N_j = 2 \). In the former case, the number of such vertices \( W_1 \) is at most \( 6\Delta \) by Lemma [A.2]. As \( V \) is the unique n-vertex sharing an n-label \( j \) with \( W_1 \) for which \( N_j = 2 \), this implies the number of such vertices \( V \) is also at most \( 6\Delta \). In the latter case, \( j \) appears on a vertex \( W_2 \) distinct from \( V \) and \( W_1 \). Then the three pairs of p-vertices around \( V \), \( W_1 \), and \( W_2 \) must have the same pair of p-labels, and by Lemma [A.12] the number of such vertices \( V \) is at most \( 42\Delta \). Hence the number of n-vertices \( V \) belonging to this case is at most \( 48\Delta \).

Case 3: \( V \) forms a good pair with some other vertex \( V' \). By the bounds in cases 1 and 2, there are at least \( l - k - 96\Delta \) such vertices \( V \), hence at least \( \frac{l-k}{2} - 48\Delta \) good pairs, and the result holds.

Now suppose there is some p-vertex \( V \) whose p-label appears only once. Let \( T, U, W, X \) be as in Lemma [A.4] and recall that \( U \) and \( W \) have the same n-labels up to reordering. Consider four cases:

Case 1: T and X have different p-labels, and U and W are single. Lemma [A.4] yields an \((l-1)\)-graph with \( k-1 \) single n-vertices, \( \sum_{s=1}^{l} d_s - 1 \) total n-labels, and \( m-1 \) distinct p- and n-labels. By the induction hypothesis, this \((l-1)\)-graph has at least

\[
\frac{(l-1) - (k-1)}{2} - 48 \left( \frac{(l-1) + (\sum_{s=1}^{l} d_s - 1)}{2} + 1 - (m-1) \right) \geq \frac{l-k}{2} - 48\Delta
\]
good pairs, which are also good pairs in the \( l \)-graph.

Case 2: T and X have different p-labels, and U and W each have \( d \geq 2 \) n-labels. Lemma [A.4] yields an \((l-1)\)-graph with \( k \) single n-vertices, \( \sum_{s=1}^{l} d_s - d \) total n-labels, and \( m-1 \) distinct p- and n-labels. By the induction hypothesis, this \((l-1)\)-graph has at least

\[
\frac{(l-1) - k}{2} - 48 \left( \frac{(l-1) + (\sum_{s=1}^{l} d_s - d)}{2} + 1 - (m-1) \right) > \frac{l-k}{2} - 48\Delta + 1
\]
good pairs. It can have at most one more good pair than the original \( l \)-graph (which occurs if \( W \) has a tuple of n-labels appearing on exactly three different n-vertices in the \( l \)-graph).

Case 3: T and X have the same p-label, and U and W are single. Lemma [A.4] yields an \((l-2)\)-graph with \( k-2 \) single n-vertices, \( \sum_{s=1}^{l} d_s - 2 \) total n-labels, and either \( m-2 \) distinct p- and n-labels if U and W have an n-label appearing only those two times, or \( m-1 \) distinct p- and n-labels otherwise. Supposing the former, this \((l-2)\)-graph has at least

\[
\frac{(l-2) - (k-2)}{2} - 48 \left( \frac{(l-2) + (\sum_{s=1}^{l} d_s - 2)}{2} + 1 - (m-2) \right) = \frac{l-k}{2} - 48\Delta
\]
good pairs, and it has the same number of good pairs as the original \( l \)-graph. Supposing the latter, this \((l-2)\)-graph has at least

\[
\frac{(l-2) - (k-2)}{2} - 48 \left( \frac{(l-2) + (\sum_{s=1}^{l} d_s - 2)}{2} + 1 - (m-1) \right) \geq \frac{l-k}{2} - 48\Delta + 1
\]
good pairs, and it can have at most one more good pair than the original \( l \)-graph (which occurs if the \((l-2)\)-graph has a good pair containing the n-label of the removed vertices \( U \) and \( W \)).

Case 4: T and X have the same p-label, and U and W each have \( d \geq 2 \) n-labels. Lemma [A.4] yields an \((l-2)\)-graph with \( k \) single n-vertices, \( \sum_{s=1}^{l} d_s - 2d \) total n-labels, and between \( m-d-1 \)
and $m - 1$ (inclusive) distinct $p$- and $n$-labels. If it has exactly $m - d - 1$ distinct $p$- and $n$-labels, then we must have removed a good pair, and the $(l - 2)$-graph has at least
\[
\frac{(l - 2) - k}{2} - 48\left(\frac{(l - 2) + (s = 1^{l})d_s - 2d\right)}{2} + 1 - (m - d - 1) = \frac{l - k}{2} - 48\Delta - 1
\]
good pairs. If, instead, the $(l - 2)$-graph has $m - c - 1$ distinct $p$- and $n$-labels for $0 \leq c < d$, then $U$ and $W$ cannot be a good pair in the original $l$-graph as they have $d - c$ $n$-labels $j$ for which $N_j \geq 3$, and the $(l - 2)$-graph can have at most $d - c$ more good pairs than the $l$-graph, one for each such $j$. The $(l - 2)$-graph has at least
\[
\frac{(l - 2) - k}{2} - 48\left(\frac{(l - 2) + (s = 1^{l})d_s - 2d\right)}{2} + 1 - (m - c - 1) > \frac{l - k}{2} - 48\Delta + d - c
\]
good pairs. In all cases, we establish that the $l$-graph has at least $\frac{l - k}{2} - 48\Delta$ good pairs, completing the induction. \hfill \Box

Proof of Proposition 5.17, property (2). Let $\mathcal{L} \in \mathcal{C}$ be any multi-labeling equivalence class. Let $\phi(\mathcal{L})$ have $k$ $n$-vertices with non-empty label. This means $\mathcal{L}$ has $\tilde{k}$ $n$-vertices that do not belong to a good pair. These vertices have at least $\tilde{k}$ total $n$-labels in $\mathcal{L}$, implying that there are at most $\sum_{s=1}^{l}d_s - \tilde{k}$ total $n$-labels on the good pair vertices. These good pair vertices account for at most $\frac{\sum_{s=1}^{l}d_s - \tilde{k}}{2}$ distinct $n$-labels in $\mathcal{L}$, and these are mapped to the empty label under the label-simplifying map. Furthermore, by Lemma A.13, there are at most $96\Delta(\mathcal{L})$ bad non-single $n$-vertices, and these have at most $96D\Delta(\mathcal{L})$ additional distinct $n$-labels that are mapped to the new $n$-label $n + 1$. Any bad single $n$-vertex has an $n$-label that is the same as one of these $96D\Delta(\mathcal{L})$ distinct $n$-labels (otherwise it is a good single by definition), and the $n$-label of any good single $n$-vertex is preserved under the label-simplifying map. Hence, if $m$ is the number of total distinct $p$- and $n$-labels in $\mathcal{L}$ and $\tilde{m}$ is the number of total distinct $p$-labels and non-empty $n$-labels in $\phi(\mathcal{L})$, then $\tilde{m} \geq m - \sum_{s=1}^{l}d_s - \tilde{k} - 96D\Delta(\mathcal{L})$, so $\Delta(\phi(\mathcal{L})) = \frac{l + k}{2} + 1 - \tilde{m} \leq (96D + 1)\Delta(\mathcal{L})$. Hence property (2) holds. \hfill \Box

A.4. Verification of Proposition 5.17, property (3). Recall that we order the vertices of an $l$-graph according to a cyclic traversal starting from a (arbitrary) $p$-vertex.

Definition A.14. The canonical simple labeling in a simple labeling equivalence class $\mathcal{L}$ is the one in which each $i^{th}$ new $p$-vertex label that appears in the cyclic traversal is $i$, and each $j^{th}$ new non-empty $n$-vertex label is $j$.

The canonical multi-labeling in a multi-labeling equivalence class $\mathcal{L}$ is the one in which each $i^{th}$ new $p$-vertex label is $i$ and each $j^{th}$ new $n$-vertex label is $j$, with the new $n$-vertex labels in the label-tuple for each new $n$-vertex appearing in sorted order.

Each $\mathcal{L}$ has a unique canonical simple-labeling, which is an $(l, l)$-simple labeling, and each $\mathcal{L}$ has a unique canonical multi-labeling, which is an $(l, Dl)$-multi-labeling.

For each $\mathcal{L} \in \mathcal{C}$ and $\Delta_0 \geq 0$, property (3) of Proposition 5.17 is a bound on a certain weighted cardinality of the set
\[
S(\Delta_0, \mathcal{L}) := \phi^{-1}(\mathcal{L}) \cap \{\mathcal{L} : \Delta(\mathcal{L}) = \Delta_0\}.
\]
We describe a series of non-determined steps by which the mapping $\phi$ may be “inverted” to obtain the canonical multi-labeling $L$ of any $\mathcal{L} \in \phi^{-1}(\mathcal{L})$, given $\mathcal{L}$:

1. Choose a non-empty $n$-label value appearing in $\mathcal{L}$ to be “$n + 1$”, or assume there is no such label. (The $n$-vertices with empty label will be the good pairs, and the remaining $n$-vertices with label different from “$n + 1$” will be the good singles.)
(2) Choose a subset $S$ of $n$-vertices with label “n+1” to be the bad non-singles in $L$. (The remaining $n$-vertices with label “n+1” will be the bad singles.)

(3) For each $n$-vertex in $S$, choose the size of its $n$-label tuple in $L$ to be between $2$ and $D$ (inclusive), and pick $n$-labels from $\{1, \ldots, Dl\}$ for that tuple.

(4) For each $n$-vertex with label “n+1” not in $S$, pick a single value in $\{1, \ldots, Dl\}$ for its $n$-label in $L$.

(5) For all $n$-vertices with empty label in $\hat{L}$, pair them up into good pairs for $L$.

(6) For each good pair, choose the size of its $n$-label tuple in $L$ to be between $2$ and $D$ (inclusive), and choose a permutation of the second $n$-label tuple of the pair that matches the first.

(7) Let $G$ be the set of good pairs $(V, V')$ that are consecutive $n$-vertices in the $l$-graph and such that the $p$-label (in $\hat{L}$) of the $p$-vertex between them appears at least twice. Choose an ordered subset of $G$. For each $(V, V')$ in this subset, if $W$ is the $p$-vertex between $V$ and $V'$, choose some other $p$-vertex $W'$ having the same $p$-label as $W$, and reverse the sequence of vertices from $W$ to $W'$ or from $W'$ to $W$.

(8) Choose $p$-labels for $L$ such that the resulting labeling is canonical and two $p$-vertices have the same label if and only if they do in $\hat{L}$. Choose the remaining $n$-labels for $L$ (corresponding to the good pairs and good singles) such that the resulting labeling is canonical, the properties of Definitions $A.5$ and $A.6$ are satisfied, and two good single vertices have the same $n$-label if and only if they do in $\hat{L}$.

These steps are non-determined in the sense that each step may be performed in multiple ways, yielding many possible output multi-labelings $L$. They “invert” $\varphi$ in the following sense:

**Lemma A.15.** For any $\mathcal{L} \in \varphi^{-1}(\hat{\mathcal{L}})$, the canonical multi-labeling $L$ of $\mathcal{L}$ is a possible output of the above procedure.

**Proof.** Let $L^*$ denote the $(l, Dl)$-multi-labeling obtained by applying step (1) of the label-simplifying map in Definition $A.8$ to $L$. (It is an $(l, Dl)$-multi-labeling by Lemma $A.10$.)

$L$ may be obtained by the above procedures as follows: Perform steps (1) and (2) to correctly partition the $n$-vertices into the good pair, good single, bad single, and bad non-single $n$-vertices of $L^*$. Perform steps (3) and (4) to recover the $n$-labels in $L^*$ of the bad single and bad non-single $n$-vertices. Perform steps (5) and (6) to correctly identify the good pairs of $L^*$ and the permutation that maps the label-tuple of the second vertex to that of the first vertex in each pair. Perform step (7) to invert the reversals that mapped $L$ to $L^*$ (in the reverse order of how they were applied in the label-simplifying map): This is possible because each reversal in step (1) of the label-simplifying map causes an additional good pair $(V, V')$ of $n$-vertices to become consecutive in the $l$-graph, with the $p$-vertex between them having $p$-label appearing at least twice, and these three vertices remain consecutive after each subsequent reversal. Finally, perform step (8) to recover the $p$-labels and the good single and good pair $n$-labels of $L$, which is possible because (by assumption) $L$ is a valid canonical multi-labeling.

To obtain the desired weighted cardinality bound for $\mathcal{S}(\Delta_0, \hat{L})$, we bound the number of ways each of the above 8 steps may be performed such that the final output $L$ is the canonical multi-labeling for some $\mathcal{L} \in \mathcal{S}(\Delta_0, \hat{L})$. The bounds for all but steps (4) and (7) follow from our preceding combinatorial estimates. The following simple lemma will yield a bound for step (7):

**Lemma A.16.** Suppose a multi-labeling of an l-graph has excess $\Delta$. Then there are at most $2\Delta$ good pairs of $n$-vertices such that the two vertices in the pair are consecutive in the l-graph cycle and the p-label of the p-vertex between them appears at least twice in the labeling.

**Proof.** Call a $p$-vertex “sandwiched” if it is between two consecutive $n$-vertices that form a good pair. Let $i$ be a $p$-label appearing on a total of $b \geq 2$ $p$-vertices, of which $c \geq 1$ are sandwiched. If $b > c$, then change the $c$ appearances of $i$ on the sandwiched $p$-vertices to $c$ new $p$-labels not yet
appearing in the labeling. Otherwise if \( b = c \) (so \( c \geq 2 \)), then change \( c - 1 \) appearances of \( i \) on the sandwiched \( p \)-vertices to \( c - 1 \) new \( p \)-labels not yet appearing in the labeling. Do this for every such \( i \). Note that changing the \( p \)-label of any sandwiched \( p \)-vertex does not violate any of the conditions of Definition 5.4, so the resulting labeling is still a valid multi-labeling. If \( x \) is the number of good pairs originally satisfying the condition of the lemma, then we have added at least \( \frac{x}{2} \) new \( p \)-labels to the labeling. Hence Lemma 5.5 implies \( m + \frac{x}{2} \leq \left( \frac{1}{2} \sum_{i=1}^{d} d_i \right) + 1 \), so \( x \leq 2\Delta \).

The remaining challenge is to bound the number of ways of performing step (4). This bound is not straightforward because the number of bad singles is not necessarily small when \( \Delta \) is small. We instead show that the number of bad singles that we may “freely label” is small:

**Definition A.17.** In a multi-labeling of an \( l \)-graph, \( i \in \{1, 2, 3, \ldots\} \) is a **connector** if it appears as a \( p \)-label and, among all \( n \)-vertices that are adjacent to any \( p \)-vertex with label \( i \), exactly two are bad singles and none are bad non-singles; these two bad singles are **connected**. A sequence of bad singles \( W_1, \ldots, W_a \) is a **connected cycle** if \( W_1 \) is connected to \( W_2 \), \( W_2 \) is connected to \( W_3 \), etc., and \( W_a \) is connected to \( W_1 \).

Note that “connector” refers to a label \( i \), not to any specific \( p \)-vertex having \( i \) as its label, and two “connected” bad singles are adjacent to \( p \)-vertices having the connector label \( i \), but these \( p \)-vertices may be distinct in the \( l \)-graph. Each bad single \( n \)-vertex may be connected to at most two other bad single \( n \)-vertices (where the connectors are the \( p \)-labels of its two adjacent \( p \)-vertices), and hence this notion of connectedness partitions the set of bad single \( n \)-vertices into connected components that are either individual vertices, linear chains, or cycles.

Motivation for this definition comes from the observation that if two bad single \( n \)-vertices are connected, then they must have the same \( n \)-label, as follows from condition (3) of Definition 5.4 and the fact that \( n \)-labels appearing on good singles and good pairs must be distinct from those appearing on the remaining \( n \)-vertices.

**Lemma A.18.** Suppose a multi-labeling of an \( l \)-graph has excess \( \Delta \) and \( k \) single \( n \)-vertices, of which \( k' \) are good single and \( k - k' \) are bad single. Then at least \( k - k' - (288D + 2)\Delta \) distinct \( p \)-labels are connectors, and there are at most \((192D + 1)\Delta \) connected cycles of bad single \( n \)-vertices.

**Proof.** Suppose the multi-labeling is a \((p, n)\)-multi-labeling. Construct an undirected multi-graph \( G \) with vertex set \( \{1, \ldots, p\} \), where each edge of \( G \) has one label in \( \{1, \ldots, n\} \), as follows: For each \( n \)-vertex \( V \) in the \( l \)-graph and each \( n \)-label \( j \) of \( V \), if \( V \) is preceded and followed by \( p \)-vertices with labels \( i_1 \) and \( i_2 \), then add an edge \( i_1 \sim i_2 \) in \( G \) with label \( j \). (Thus \( G \) has \( \sum_{s=1}^{l} d_s \) total edges.) Condition (3) of Definition 5.4 implies for any \( j \), each vertex of \( G \) has even degree in the sub-graph consisting of only edges with label \( j \).

We will sequentially remove edges of \( G \) corresponding to good pairs and good singles, until only edges corresponding to bad singles and bad non-singles remain. At any stage of this removal process, let us call a vertex of \( G \) “active” if there is at least one edge still adjacent to that vertex. Let us define a “component” as the set of active vertices that may be reached by traversing the remaining edges of \( G \) from a particular active vertex. (Hence a component of \( G \) is a connected component, in the standard sense, that contains at least two vertices.) We will track the quantity

\[
M = \#\{\text{active vertices}\} + \#\{\text{distinct edge labels}\} - \#\{\text{components}\}.
\]

Initially, \( G \) has \( m \) active vertices plus distinct edge labels (where \( m \) is the number of distinct \( n \)- and \( p \)-vertices of the \( l \)-graph), and one component, so \( M = m - 1 \). Let us remove the edges of \( G \) corresponding to good pairs. If an \( n \)-vertex of a good pair has \( d \) \( n \)-labels, then the good pair corresponds to \( 2d \) edges between a single pair of vertices in \( G \) whose edge labels do not appear elsewhere in \( G \). Removing these \( 2d \) edges removes \( d \) distinct edge labels, and if this also changes the connectivity structure of \( G \), then either \#\{components\} increases by 1, \#\{active vertices\} decreases by 1, or \#\{components\} decreases by 1 and \#\{active vertices\} decreases by 2. In all
cases, $M$ decreases by at most $d + 1$. Then after removing all edges of $G$ corresponding to good pairs, $M \geq m - 1 - \left( \sum_{s=1}^{x} d_s - k \right) - \frac{(l-k)}{2} = k - \Delta$, as there are at most $\sum_{s=1}^{x} d_s - k$ distinct $n$-labels for the good pairs and at most $\frac{l-k}{2}$ good pairs.

Let us now remove the edges of $G$ corresponding to good singles. Let $j$ be an $n$-label of a good single, and consider removing the edges of $G$ with label $j$ one at a time. As each vertex of $G$ has even degree in the subgraph of edges with label $j$, when the first such edge is removed, the number of components and active vertices cannot change. Subsequently, the removal of each additional edge might increase $\{\text{components}\} - \{\text{active vertices}\}$ by 1 upon considering the same three cases as above. When the last such edge is removed, there are no longer any edges with label $j$ by the definition of a good single, so $\{\text{distinct edge labels}\}$ decreases by 1. Hence removing all edges with label $j$ decreases $M$ by at most the number of such edges, and $M \geq k - k' - \Delta$ after removing the edges corresponding to all $k'$ good singles.

Call the resulting graph $G'$. Every vertex of $G'$ still has even degree in the subgraph of edges with label $j$, for any $j$. In particular, every active vertex of $G'$ has degree at least two. By Definition A.17, $i \in \{1, \ldots, p\}$ is a connector if and only if $i$ has degree exactly two in $G'$, in which case the edges incident to $i$ in $G'$ must have the same label $j$, and the $n$-vertices with label $j$ in the $l$-graph are the bad singles connected by $i$. A connected cycle of bad singles corresponds to the edges of a cycle of (necessarily distinct) vertices in $G'$ with degree exactly two.

The number of distinct edge labels in $G'$ equals the number of distinct $n$-labels in the $l$-graph appearing on bad non-singles (as any $n$-label appearing on a bad single also appears on some bad non-single). By Lemma A.13, this is at most $96D\Delta$. Hence $\{\text{active vertices}\} - \{\text{components}\} \geq k - k' - (96D + 1)\Delta$ for $G'$. The number of total edges in $G'$ is at most $k - k' + 96D\Delta$, with $k - k'$ of them corresponding to bad singles and at most $96D\Delta$ corresponding to bad non-singles. Then the total vertex degree of $G'$ is at most $2(k - k' + 96D\Delta)$. As each active vertex in $G'$ has degree at least two, this implies $\{\text{active vertices}\} \leq k - k' + 96D\Delta$. Then $\{\text{components}\} \leq (192D + 1)\Delta$, so there are at most $(192D + 1)\Delta$ connected cycles of bad singles. Furthermore, if there are $x$ connectors (i.e. active vertices with degree exactly two), then since there are at least $k - k' - (96D + 1)\Delta$ active vertices, $2x + 4(k - k' - (96D + 1)\Delta - x) \leq 2(k - k' + 96D\Delta)$, so $x \geq k - k' - (288D + 2)\Delta$. $\square$

Proof of Proposition 5.17, property (3). Let $C$ denote a positive constant that may depend on $D$ and that may change from instance to instance. Fix $\Delta_0 \geq 0$ and $\hat{L}$. We upper bound the number of ways in which steps (1)–(8) of the inversion procedure may be performed, such that the resulting multi-labeling $L$ is canonical for some $L \in S(\Delta_0, \hat{L})$:

There are at most $l + 1$ ways of performing step (1).

By Lemma A.13, to yield $L$ with excess $\Delta_0$, there can be at most $C\Delta_0$ bad non-single $n$-vertices, and hence we must take $|S| \leq C\Delta_0$ in step (2).

To perform step (3), for each vertex in $S$, we may first choose the number of $n$-labels $d$ between 2 and $D$, and then there are at most $(Dl)^d$ ways of choosing the $n$-labels for that vertex.

For step (4), suppose $k'$ good single and $k - k'$ bad single $n$-vertices were identified in steps (1) and (2). By Lemma A.18, there are at least $k - k' - C\Delta_0$ connectors, and any two connected bad single $n$-vertices must be given the same $n$-label. (The $p$-labels of $\hat{L}$ are known and are preserved in $L$, so after steps (1) and (2) we know which labels are connectors and which bad singles must be connected in $L$.) Going through the connectors one-by-one, each successive connector constrains the $n$-label of one more bad single $n$-vertex, unless that connector closes a connected cycle. But as there are at most $C\Delta_0$ connected cycles by Lemma A.18, the number of bad single $n$-vertices that we can freely label at most $C\Delta_0$. Then there are at most $(Dl)^{C\Delta_0}$ ways to perform step (4).

For step (5), recall that the pairs of $p$-vertices surrounding the two $n$-vertices of a good pair must have the same pair of $p$-labels. By Lemma A.12, for all but at most $C\Delta_0$ of the $n$-vertices with
empty label, this pairing is uniquely determined, so there are at most \((C\Delta_0)^{C\Delta_0}\) ways of performing step (5).

For step (6), there are \((l - \tilde{k}(\mathcal{L}))/2\) good pairs, and for each pair we may choose the number of \(n\)-labels \(d\) between 2 and \(D\) and then one of \(d!\) permutations.

Lemma A.16 shows that \(|G| \leq 2\Delta_0\) for step (7). For each element that we add to the ordered subset of \(G\), there are at most \(2\Delta_0\) choices for this element and at most \(2l!\) ways of choosing \(W'\) and which half of the cycle to reverse, or we may choose to not add any more elements. We make such a choice at most \(2\Delta_0\) times, so there are at most \((4\Delta_0l + 1)^{2\Delta_0}\) ways of performing step (7).

Finally, there is at most one way to perform step (8), as the labels on the good single and good non-single \(n\)-vertices and each new \(n\)-label and \(p\)-label has a unique choice to make \(L\) canonical.

We may incorporate the product \(\prod_{s=1}^{l} |a_{d_s}(\mathcal{L})|/(d_s!)^{1/2}\) on the left side of (16) into the cardinality count by noting that this product contributes \(|a_d|/(d!)^{1/2}\) for each vertex in \(S\) having \(d\) \(n\)-labels, \(a^2_d/d!\) for each good pair having \(d\) \(n\)-labels per vertex of the pair, and \(|a_1|\) for each of the \(\tilde{k}(\mathcal{L}) - |S|\) single vertices in \(L\). Combining the above bounds then yields

\[
\sum_{\mathcal{L} \in \varphi^{-1}(\mathcal{L})} \prod_{s=1}^{l} \frac{|a_{d_s}(\mathcal{L})|}{(d_s!)^{1/2}} \leq (l + 1) \sum_S \left( \sum_{d=2}^{D} (DL)^d \frac{|a_d|}{(d!)^{1/2}} \right)^{|S|} (DL)^{C\Delta_0} (C\Delta_0)^{C\Delta_0} \\

\sum_{d=2}^{D} \frac{a^2_d}{d!} \leq (4\Delta_0l + 1)^{2\Delta_0} |a_1| \tilde{k}(\mathcal{L}) - |S| \\

\leq (l + 1)(CL)^{C\Delta_0} |a_1| \tilde{k}(\mathcal{L}) \left( \nu - a^2 \right)^{-\frac{l-\tilde{k}(\mathcal{L})}{2}} \sum_S |a|^{-|S|} \left( \sum_{d=2}^{D} (DL)^d \frac{|a_d|}{(d!)^{1/2}} \right)^{|S|} ,
\]

where \(\sum_S\) denotes the sum over all possible sets \(S\) selected by step (2), and the second line applies \(\Delta_0 \leq CL\) and \(\sum_{d=2}^{D} a^2_d = \nu - a^2\). As \(|S| \leq C\Delta_0\), this implies by Cauchy-Schwarz

\[
|a|^{-|S|} \left( \sum_{d=2}^{D} (DL)^d \frac{|a_d|}{(d!)^{1/2}} \right)^{|S|} \leq (CL)^{C\Delta_0} |a|^{-|S|} \left( \sum_{d=2}^{D} \frac{a^2_d}{d!} \right)^{\frac{|S|}{2}} \leq (CL)^{C\Delta_0} \left( \frac{\sqrt{\nu}}{|a|} \right)^{C\Delta_0}.
\]

The sum is over at most \(l^{C\Delta_0}\) possible sets \(S\), so this verifies condition (3) of the proposition upon noting that \((CL)^{C\Delta_0} \leq l^{C_3 + C_4\Delta_0}\) for some constants \(C_3, C_4 > 0\) and all \(l \geq 2\).

**APPENDIX B. MOMENT BOUND FOR A DEFORMED GUE MATRIX**

In this appendix, we prove Proposition 5.11. Recall Definition 5.10 of \(M\), \(W\), \(V\), and \(Z\), which implicitly depend on \(p\) and \(n\). Throughout this section, we will use \(p\) and \(n\) in place of \(\tilde{p}\) and \(\tilde{n}\).

**Lemma B.1.** Suppose \(n, p \to \infty\) with \(p/n \to \gamma\). Then \(\|M\| \to \|\mu_{a, \nu, \gamma}\|\) almost surely.

**Proof.** Recall \(M = \sqrt{\frac{\gamma(\nu - a^2)}{p}} W + \frac{\alpha}{n} V\), where \(V = ZZ^T - D\) and \(D = \text{diag}([|Z_i|^2])\). The empirical spectral distribution of \(\frac{1}{n} ZZ^T\) converges weakly almost surely to \(\mu_{\text{MP}, \gamma}\). By a chi-squared tail bound and a union bound, \(\|\frac{1}{n} D - \text{Id}\| \to 0\), so the empirical spectral distribution of \(\frac{\alpha}{n} V\) converges weakly almost surely to \(a(\mu_{\text{MP}, \gamma} - 1)\). Furthermore, the maximal distance between an eigenvalue of \(\frac{\alpha}{n} V\) and the support of \(a(\mu_{\text{MP}, \gamma} - 1)\) converges to 0 almost surely by the results of [61] and [2].

Let \(V = \Theta \Theta^T\) where \(\Theta\) is the real orthogonal matrix that diagonalizes \(V\). Then the spectrum of \(M\) is the same as that of \(\sqrt{\frac{\gamma(\nu - a^2)}{p}} \Theta^T WO + \frac{\alpha}{n} \Lambda\), and \(\Theta^T WO\) is still distributed as the GUE.
Conditional on $V$, the above arguments and Proposition 8.1 of [16] imply $\|\sqrt{\frac{\gamma (\nu - a^2)}{p}} OTWO + \frac{a}{n} \Lambda\| \to \|\mu_{a,\nu,\gamma}\|$ almost surely. As this convergence holds almost surely in $V$, it holds unconditionally as well. \hfill \Box

**Lemma B.2.** Suppose $n, p \to \infty$ with $p/n \to \gamma$, let $l := l(n)$ be such that $l(n)/n \to 0$, and let $B_n$ be any event. Then there exist positive constants $C := C_{a,\nu,\gamma}$ and $c := c_{a,\nu,\gamma}$ such that $\mathbb{E}[\|M\|^l \mathbb{1} \{B_n\}] \leq C \mathbb{P}[B_n] + e^{-cn}$ for all large $n$.

**Proof.** Note

$$\|M\| \leq \sqrt{\frac{\gamma (\nu - a^2)}{p}} \|W\| + \frac{|a|}{n} |ZZ^T| + \frac{|a|}{n} \max_{1 \leq i \leq p} \|Z_i\|^2.$$ 

Applying standard tail bounds (e.g. Corollary 2.3.5 of [54], Corollary 5.35 of [56], and Lemma 1 of [38]), there exist constants $C, \varepsilon > 0$ depending on $a, \nu, \gamma$ such that, for all $t \geq C$ and sufficiently large $n$, $\mathbb{P}[\|M\| > t] \leq e^{-\varepsilon tn}$. Then we may write

$$\mathbb{E}[\|M\|^l \mathbb{1} \{B_n\}] = \mathbb{E}[\|M\|^l \mathbb{1} \{B_n\} \mathbb{1} \{\|M\| \leq C\}] + \mathbb{E}[\|M\|^l \mathbb{1} \{B_n\} \mathbb{1} \{\|M\| > C\}] \leq C \mathbb{P}[B_n] + \int_{C}^{\infty} \mathbb{P}[\|M\|^l > t] dt = C \mathbb{P}[B_n] + \int_{C}^{\infty} \mathbb{P}[\|M\| > s] \cdot ls^{-1} ds \leq C \mathbb{P}[B_n] + C \int_{C}^{\infty} e^{-\varepsilon sn + (l-1) \log s} ds \leq C \mathbb{P}[B_n] + C \int_{C}^{\infty} e^{-\varepsilon n - (l-1) s} ds = e^{-\varepsilon n - l} \mathbb{P}[B_n] + C$$

for all large $n$. As $l = o(n)$, the result follows upon setting $c = C \varepsilon / 2$. \hfill \Box

**Lemma B.3.** Suppose $n, p \to \infty$ with $p/n \to \gamma$. Then $\mathbb{E}[\|M\|] \to \|\mu_{a,\nu,\gamma}\|$.

**Proof.** Lemma B.1 and Fatou’s lemma imply $\liminf \mathbb{E}[\|M\|] \geq \|\mu_{a,\nu,\gamma}\|$. For any $\varepsilon > 0$, let $B_n = \{\|M\| > \|\mu_{a,\nu,\gamma}\| + \varepsilon\}$. Then

$$\mathbb{E}[\|M\|] = \mathbb{E}[\|M\| \mathbb{1} \{B_n^c\}] + \mathbb{E}[\|M\| \mathbb{1} \{B_n\}] \leq \|\mu_{a,\nu,\gamma}\| + \varepsilon + \mathbb{E}[\|M\| \mathbb{1} \{B_n\}]$$

Lemma B.1 implies $\mathbb{P}[B_n] \to 0$, so Lemma B.2 (with $l = 1$) implies $\mathbb{E}[\|M\| \mathbb{1} \{B_n\}] \to 0$ as well. Then $\mathbb{E}[\|M\|] \leq \|\mu_{a,\nu,\gamma}\| + 2\varepsilon$ for all large $n$, and the result follows by taking $\varepsilon \to 0$. \hfill \Box

**Lemma B.4.** Suppose $F : \mathbb{R}^d \to \mathbb{R}$ is $L$-Lipschitz on a set $G \subseteq \mathbb{R}^k$, i.e. $|F(x) - F(y)| \leq L\|x - y\|_2$ for all $x, y \in G$. Let $\xi \sim N(0, I_d)$. Then there exists a function $\tilde{F} : \mathbb{R}^d \to \mathbb{R}$ such that $\tilde{F}(x) = F(x)$ for all $x \in G$, $|\tilde{F}(x) - \tilde{F}(y)| \leq L\|x - y\|_2$ for all $x, y \in \mathbb{R}^k$, and, for all $\Delta > 0$,

$$\mathbb{P}[|F(\xi) - \mathbb{E}F(\xi)| \geq \Delta + \|\mathbb{E}F(\xi) - \mathbb{E}\tilde{F}(\xi)\| \text{ and } \xi \in G] \leq e^{-\frac{2\Delta^2}{22L^2}}.$$ 

**Proof.** Let $\tilde{F}(x) = \inf_{x' \in G}(F(x') + L\|x - x'\|_2)$. Note that if $x \in G$, then $F(x) \leq F(x') + L\|x - x'\|_2$ for all $x' \in G$, so $\tilde{F}(x) = F(x)$. Also, for any $x, y \in \mathbb{R}^k$ and $\varepsilon > 0$, there exists $x' \in G$ such that $\tilde{F}(x) \geq F(x') + L\|x - x'\|_2 - \varepsilon$. Then by definition, $\tilde{F}(y) \leq F(x') + L\|y - x'\|_2$, so $\tilde{F}(y) - \tilde{F}(x) \leq L\|y - x\|_2 - L\|x - x'\|_2 + \varepsilon \leq L\|x - y\|_2 + \varepsilon$. Similarly, $\tilde{F}(x) - \tilde{F}(y) \leq L\|x - y\|_2 + \varepsilon$. This holds for all $\varepsilon > 0$, so $|\tilde{F}(x) - \tilde{F}(y)| \leq L\|x - y\|_2$. Finally, applying Gaussian concentration of measure for the Lipschitz function $\tilde{F}$,

$$\mathbb{P}[|F(\xi) - \mathbb{E}F(\xi)| \geq \Delta + \|\mathbb{E}F(\xi) - \mathbb{E}\tilde{F}(\xi)\| \text{ and } \xi \in G] \leq e^{-\frac{2\Delta^2}{22L^2}}.$$
By elementary calculations, and denoting 
Recall
Proof. 
Take $G$ given by Lemma B.4. Note
Lemma B.5. Suppose $n, p \to \infty$ with $p/n \to \gamma$, and let $\varepsilon > 0$. Then there exist $c := c_{a, \nu, \gamma} > 0$ and $N := N_{a, \nu, \gamma, \varepsilon} > 0$ and a set $G := G_{a, p} \subset \mathbb{R}^{p \times n}$ with $P[Z \in G] \geq 1 - 2e^{-\frac{c}{\gamma}}$, such that for all $t > \varepsilon$ and $n > N$,

$$P[\|M\| \geq \|\mu_{a, \nu, \gamma}\| + t \text{ and } Z \in G] \leq e^{-ctu^2}.$$ 

$$= \mathbb{P}[\hat{F}(\xi) \geq \Delta + |EF(\xi) - E\hat{F}(\xi)| + EF(\xi) \text{ and } \xi \in G]$$

$$\leq \mathbb{P}[\hat{F}(\xi) \geq \Delta + E\hat{F}(\xi)] \leq e^{-\frac{\Delta^2}{144}}.$$ 

Then, for any $v \in \mathbb{C}^p$ such that $\|v\|_2 = 1$,

$$\|\nabla f_v(W, Z)\|_2^2 = \frac{\gamma(v - a^2)}{p} \left( \sum_{i=1}^{p} |v_i|^4 + 2 \sum_{1 \leq i < j \leq p} |\overline{v}_i v_j|^2 \right) + \frac{4a^2}{n^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \|\text{Re}(\overline{v}_i v_j) Z_j\|^2$$

$$\leq \frac{\gamma(v - a^2)}{p} \left( \sum_{i=1}^{p} |v_i|^2 \right)^2 + \frac{4a^2}{n^2} \sum_{i=1}^{p} |v_i|^2 \|Z\|^2 \|v\|_2^2$$

$$= \frac{\gamma(v - a^2)}{p} + \frac{4a^2 \|Z\|^2}{n^2}.$$ 

Take $G = \{Z \in \mathbb{R}^{p \times n} : \|Z\| \leq 2\sqrt{n} + \sqrt{p}\}$. Then by Corollary 5.35 of [56], $P[Z \notin G] \leq 2e^{-\frac{c}{\gamma}}$. As $\mathbb{R}^p \times G$ is convex, the above inequality implies $f_v(W, Z)$ is $L$-Lipschitz on $\mathbb{R}^p \times G$ for $L = O(n^{-1/2})$. Then

$$f(W, Z) - f(W', Z') \leq \sup_{v \in \mathbb{C}^p : \|v\|_2 = 1} (|f_v(W, Z) - f_v(W', Z')|)$$

$$\leq \sup_{v \in \mathbb{C}^p : \|v\|_2 = 1} \|f_v(W, Z) - f_v(W', Z')\|_2 \leq L \|(W, Z) - (W', Z')\|_2$$

for all $W, W' \in \mathbb{R}^p$ and $Z, Z' \in G$, so $f$ is also $L$-Lipschitz on $\mathbb{R}^p \times G$.

Let $\tilde{f} : \mathbb{R}^{p^2 + np} \to \mathbb{R}$ be the $L$-Lipschitz extension of $f$ on $\mathbb{R}^p \times G$ given by Lemma B.4. Note that

$$|\mathbb{E}f(W, Z) - \mathbb{E}\tilde{f}(W, Z)| = |\mathbb{E}[(f(W, Z) - \tilde{f}(W, Z)) \mathbb{1}_{\{Z \notin G\}}]|$$

$$\leq \mathbb{E}|f(W, Z) \mathbb{1}_{\{Z \notin G\}}| + \mathbb{E}|\tilde{f}(W, Z) \mathbb{1}_{\{Z \notin G\}}|.$$
Lemma \[\text{B.2}\] (with \(l = 1\)) implies \(\mathbb{E}|f(W, Z)\mathbbm{1}\{Z \notin G\}| = \mathbb{E}[\|M\|\mathbbm{1}\{Z \notin G\}] = o(1)\). As \(\tilde{f}\) is \(L\)-Lipschitz,

\[
|\tilde{f}(W, Z)| \leq |\tilde{f}(0, 0)| + L\|(W, Z)\|_2 = |f(0, 0)| + L\|(W, Z)\|_2 = L\|(W, Z)\|_2.
\]

Let \(A_n = \{\|(W, Z)\|_2 \leq \sqrt{2(p^2 + np)}\}\). As \(\|(W, Z)\|_2^2\) is chi-squared distributed with \(p^2 + np\) degrees of freedom, a standard tail bound gives \(\mathbb{P}\left[\|(W, Z)\|_2^2 \geq p^2 + np + t\right] \leq e^{-\frac{t^2}{8(p^2 + np)}}\). Then

\[
\mathbb{E}\left[\|(W, Z)\|_2^2 \mathbbm{1}\{A_n^c\}\right] = \int_p^{\infty} \mathbb{P}\left[\|(W, Z)\|_2^2 \geq p^2 + np + t\right] dt \leq \int_p^{\infty} e^{-\frac{t^2}{8(p^2 + np)}} dt = 2\sqrt{p^2 + np} \int_{\sqrt{p^2 + np}}^{\infty} e^{-\frac{s^2}{2}} ds \sim 4e^{-\frac{p^2+np}{8}}.
\]

This implies

\[
\mathbb{E}|\tilde{f}(W, Z)\mathbbm{1}\{Z \notin G\}| \leq \mathbb{E}[|f(W, Z)|\mathbbm{1}\{Z \notin G\}\mathbbm{1}\{A_n\}] + \mathbb{E}[|\tilde{f}(W, Z)\mathbbm{1}\{Z \notin G\}\mathbbm{1}\{A_n^c\}\}] \\
\leq L\sqrt{2(p^2 + np)}\mathbb{P}[Z \notin G] + LE\left[\|(W, Z)\|_2^2 \mathbbm{1}\{A_n^c\}\right]^{1/2} = o(1).
\]

Then \(|\mathbb{E}f(W, Z) - \mathbb{E}\tilde{f}(W, Z)| = o(1)\), so Lemmas \[\text{B.3}\] and \[\text{B.4}\] imply, for all \(t > \varepsilon\) and all sufficiently large \(n\) (i.e. \(n > N_{a, \nu, \gamma, \varepsilon}\) independent of \(t\)),

\[
\mathbb{P}[\|M_{n,p}\| \geq \|\mu_{a, \nu, \gamma}\| + t \text{ and } Z \in G] \\
\leq \mathbb{P}\left[\|M_{n,p}\| - \mathbb{E}\|M_{n,p}\| \geq t - \frac{\varepsilon}{2} + |\mathbb{E}f(W, Z) - \mathbb{E}\tilde{f}(W, Z)| \text{ and } Z \in G\right] \\
\leq e^{-\frac{(t - \varepsilon/2)^2}{2L^2}} \leq e^{-\frac{t^2}{8L^2}}.
\]

The result follows upon noting that \(L = O(n^{-1/2})\).

**Proof of Proposition \[5.11\]** Let \(c>0\) and \(G \subset \mathbb{R}^{p \times n}\) be as in Lemma \[\text{B.5}\]. Then, for any \(\varepsilon > 0\),

\[
\mathbb{E}[\|M\|\mathbbm{1}\{Z \in G\}] \leq (\|\mu_{a, \nu, \gamma}\| + \varepsilon)^t + \mathbb{E}\left[\|M\|\mathbbm{1}\{\|M\| \geq \|\mu_{a, \nu, \gamma}\| + \varepsilon\} \mathbbm{1}\{Z \in G\}\right] \\
= (\|\mu_{a, \nu, \gamma}\| + \varepsilon)^t + \int_0^{\infty} \mathbb{P}[\|M\| \geq t \text{ and } Z \in G] dt \\
= (\|\mu_{a, \nu, \gamma}\| + \varepsilon)^t + \int_0^{\infty} \mathbb{P}[\|M\| \geq s \text{ and } Z \in G] \cdot ls^{t-1} ds \\
\leq (\|\mu_{a, \nu, \gamma}\| + \varepsilon)^t + \int_0^{\infty} e^{-cs^2}(\|\mu_{a, \nu, \gamma}\| + s)^{t-1} ds \\
\text{for all sufficiently large } n, \text{ where we have applied Lemma } \text{B.5}. \text{ Note that}
\]

\[
l \int_\varepsilon^{\infty} e^{-cs^2}(\|\mu_{a, \nu, \gamma}\| + s)^{t-1} ds \leq l \int_\varepsilon^{\infty} e^{-cs^2 + l(\|\mu_{a, \nu, \gamma}\| + s)} ds \\
= le^{l\|\mu_{a, \nu, \gamma}\| + \frac{l^2}{2cn}} \int_\varepsilon^{\infty} e^{-cn(s - \frac{l}{2cn})^2} ds \\
= le^{l\|\mu_{a, \nu, \gamma}\| + \frac{l^2}{2cn}} \int_\varepsilon^{\infty} e^{-(\varepsilon - \frac{l}{2cn})^2} dt \\
\sim \frac{le^{l\|\mu_{a, \nu, \gamma}\| + \frac{l^2}{2cn}}}{2cn (\varepsilon - \frac{l}{2cn})^2} \rightarrow 0
\]
for $l = O(\log n)$, so $E[\|M\|^2 1\{Z \in G\}] \leq (\|\mu_{a,\nu,\gamma}\| + \varepsilon)^l + o(1)$. On the other hand, $P[Z \notin G] \leq 2e^{-\frac{l}{2}}$ by Lemma B.5, so Lemma B.2 implies $E[\|M\|^2 1\{Z \notin G\}] = o(1)$ for $l = O(\log n)$. Hence $E[\|M\|^2] \leq (\|\mu_{a,\nu,\gamma}\| + \varepsilon)^l + o(1)$, and taking $\varepsilon \to 0$ concludes the proof.

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