ON HERMITE HADAMARD-TYPE INEQUALITIES FOR STRONGLY $\varphi$-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce the notion of strongly $\varphi$-convex functions with respect to $c > 0$ and present some properties and representation of such functions. We obtain a characterization of inner product spaces involving the notion of strongly $\varphi$-convex functions. Finally, a version of Hermite Hadamard-type inequalities for strongly $\varphi$-convex functions are established.

1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [4], [6, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

\begin{equation}
\frac{f(a) + f(b)}{2} \leq \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\end{equation}

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see (3-8) and the references cited therein.

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [13]:

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$-convex on $[a, b]$ if for every two points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

\[ f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)). \]

In [2], Cristescu proved the following results for the $\varphi$-convex functions

Lemma 1. For $f : [a, b] \to \mathbb{R}$, the following statements are equivalent:

(i) $f$ is $\varphi$-convex functions on $[a, b]$;
(ii) for every $x, y \in [a, b]$, the mapping $g : [0, 1] \to \mathbb{R}$, $g(t) = f(t\varphi(x) + (1-t)\varphi(y))$ is classically convex on $[0, 1]$.

Obviously, if function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the $\varphi$-convex functions can be found, for instance, in [1], [2], [13].

2000 Mathematics Subject Classification. 26D10, 26A51, 46C15.
Key words and phrases. Hermite-Hadamard’s inequalities, $\varphi$-convex functions, strongly convex with modulus $c > 0$. 

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Recall also that a function \( f : I \to \mathbb{R} \) is called strongly convex with modulus \( c > 0 \), if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2
\]
for all \( x, y \in I \) and \( t \in (0, 1) \). Strongly convex functions have been introduced by Polyak in [9] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature see ([9]-[12]) and the references cited therein.

In this paper, we introduce the notion of strongly \( \varphi \)-convex functions defined in normed spaces and present some properties of them. In particular, we obtain a representation of strongly \( \varphi \)-convex functions in inner product spaces and, using the methods of [11] and [12], we give a characterization of inner product spaces, among normed spaces, that involves the notion of strongly \( \varphi \)-convex function. Finally, a version of Hermite–Hadamard-type inequalities for strongly \( \varphi \)-convex functions is presented. This result generalizes the Hermite–Hadamard-type inequalities obtained in [10] for strongly convex functions, and for \( c = 0 \), coincides with the classical Hermite–Hadamard inequalities, as well as the corresponding Hermite–Hadamard-type inequalities for \( \varphi \)-convex functions in [2].

2. Main Results

In what follows \((X, \| \cdot \|)\) denotes a real normed space, \( D \) stands for a convex subset of \( X \), \( \varphi : D \to D \) is a given function and \( c \) is a positive constant. We say that a function \( f : D \to \mathbb{R} \) is strongly \( \varphi \)-convex with modulus \( c \) if
\[
(2.1) \quad f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2
\]
for all \( x, y \in D \) and \( t \in [0, 1] \). We say that \( f \) is strongly \( \varphi \)-midconvex with modulus \( c \) if (2.1) is assumed only for \( t = \frac{1}{2} \), that is
\[
f\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \leq \frac{f(\varphi(x)) + f(\varphi(y))}{2} - c\frac{\|\varphi(x) - \varphi(y)\|^2}{4}, \text{ for } x, y \in D.
\]
The notion of \( \varphi \)-convex function corresponds to the case \( c = 0 \). We start with the following lemma which give some relationships between strongly \( \varphi \)-convex functions and \( \varphi \)-convex functions in the case where \( X \) is a real inner product space (that is, the norm \( \| \cdot \| \) is induced by an inner product: \( \| \cdot \| := < x | x > \)).

**Lemma 2.** Let \((X, \| \cdot \|)\) be a real inner product space, \( D \) be a convex subset of \( X \) and \( c \) be a positive constant and \( \varphi : D \to D \).

i) A function \( f : D \to \mathbb{R} \) is strongly \( \varphi \)-convex with modulus \( c \) if and only if the function \( g = f - c\| \cdot \|^2 \) is \( \varphi \)-convex.

ii) A function \( f : D \to \mathbb{R} \) is strongly \( \varphi \)-midconvex with modulus \( c \) if and only if the function \( g = f - c\| \cdot \|^2 \) is \( \varphi \)-midconvex.
Proof. i) Assume that $f$ is strongly $\varphi$-convex with modulus $c$. Using properties of the inner product, we obtain

$$g(t\varphi(x) + (1-t)\varphi(y)) = f(t\varphi(x) + (1-t)\varphi(y)) - c\|t\varphi(x) + (1-t)\varphi(y)\|^2$$

$$\leq \ t f(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 - c\|t\varphi(x) + (1-t)\varphi(y)\|^2$$

$$\leq \ t f(\varphi(x)) + (1-t)f(\varphi(y)) - c\left( t(1-t) \left[\|\varphi(x)\|^2 - 2 < \varphi(x)\varphi(y) > + \|\varphi(y)\|^2 \right] \right.$$  

$$- \left[ t^2\|\varphi(x)\|^2 + 2t(1-t) < \varphi(x)\varphi(y) > +(1-t)\|\varphi(y)\|^2 \right)$$

$$= \ t f(\varphi(x)) + (1-t)f(\varphi(y)) - ct\|\varphi(x)\|^2 - c(1-t)\|\varphi(y)\|^2$$

$$= \ tg(\varphi(x)) + (1-t)g(\varphi(y))$$

which gives that $g$ is $\varphi$-convex function.

Conversely, if $g$ is $\varphi$-convex function, then we get

$$f(t\varphi(x) + (1-t)\varphi(y)) = g(t\varphi(x) + (1-t)\varphi(y)) + c\|t\varphi(x) + (1-t)\varphi(y)\|^2$$

$$\leq \ tg(\varphi(x)) + (1-t)g(\varphi(y)) + c\|t\varphi(x) + (1-t)\varphi(y)\|^2$$

$$= \ t \left[ g(\varphi(x)) + c\|\varphi(x)\|^2 \right] + (1-t) \left[ g(\varphi(y)) + c\|\varphi(y)\|^2 \right]$$

$$-ct(1-t) \left[\|\varphi(x)\|^2 - 2 < \varphi(x)\varphi(y) > + \|\varphi(y)\|^2 \right)$$

$$= \ tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2$$

which shows that $f$ is strongly $\varphi$-convex with modulus $c$.

ii) Assume now that $f$ is strongly $\varphi$-midconvex with modulus $c$. Using the parallelogram law, we have

$$g\left(\frac{\varphi(x) + \varphi(y)}{2}\right) = f\left(\frac{\varphi(x) + \varphi(y)}{2}\right) - c\left\|\frac{\varphi(x) + \varphi(y)}{2}\right\|^2$$

$$\leq \ \frac{f(\varphi(x)) + f(\varphi(y))}{2} - c\frac{\|\varphi(x) - \varphi(y)\|^2}{4} - c\frac{\|\varphi(x) + \varphi(y)\|^2}{4}$$

$$= \ \frac{f(\varphi(x)) + f(\varphi(y))}{2} - \frac{c}{4}\left( 2\|\varphi(x)\|^2 + 2\|\varphi(y)\|^2 \right)$$

$$= \ \frac{g(\varphi(x)) + g(\varphi(y))}{2}$$

which gives that $g$ is $\varphi$-midconvex function.
Similarly, if \( g \) is \( \varphi \)-midconvex function, then we get
\[
\begin{align*}
f\left(\frac{\varphi(x) + \varphi(y)}{2}\right) &= g\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + c \left\| \frac{\varphi(x) + \varphi(y)}{2} \right\|^2 \\
&\leq \frac{g(\varphi(x)) + g(\varphi(y))}{2} + \frac{c}{4} \left\| \frac{\varphi(x) + \varphi(y)}{2} \right\|^2 \\
&= \frac{g(\varphi(x)) + \left\| \varphi(x) \right\|^2}{2} + \frac{g(\varphi(y)) + \left\| \varphi(y) \right\|^2}{2} \\
&\quad + \frac{c}{4} \left( \left\| \varphi(x) + \varphi(y) \right\|^2 - 2 \left\| \varphi(x) \right\|^2 - 2 \left\| \varphi(y) \right\|^2 \right) \\
&= \frac{f(\varphi(x)) + f(\varphi(y))}{2} - \frac{c}{4} \left\| \varphi(x) - \varphi(y) \right\|^2.
\end{align*}
\]
This completes the proof. \( \square \)

The following example shows that the assumption that \( X \) is an inner product space is essential in the above lemma.

**Example.** Let \( X = \mathbb{R}^2 \). Let us consider a function \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \), defined by \( \varphi(x) = x \) for every \( x \in \mathbb{R}^2 \) and \( \left\| x \right\| = \max \{|x_1|, |x_2|\} \) for \( x = (x_1, x_2) \). Take \( f = \left\| . \right\|^2 \). Then \( g = f - \left\| . \right\|^2 \) is \( \varphi \)-convex being the zero function. However, \( f \) is neither strongly \( \varphi \)-convex with modulus 1 nor strongly \( \varphi \)-midconvex with modulus 1. Indeed, for \( x = (1, 0) \) and \( y = (0, 1) \), we have
\[
f\left(\frac{x + y}{2}\right) = \frac{1}{2} \geq \frac{3}{4} = \frac{f(x) + f(y)}{2} - \frac{1}{4} \left\| x - y \right\|^2
\]
which this contradicts (2.1).

The assumption that \( X \) is an inner product space in Lemma 2 is essential. Moreover, it appears that the fact that for every \( \varphi \)-convex function \( g : X \to \mathbb{R} \) the function \( f = g + c \left\| . \right\|^2 \) is strongly \( \varphi \)-convex characterizes inner product spaces among normed spaces. Similar characterizations of inner product spaces by strongly convex and strongly \( h \)-convex functions are presented in [11] and [12].

**Theorem 1.** Let \((X, \left\| . \right\|)\) be a real normed space, \( D \) be a convex subset of \( X \) and \( \varphi : D \to D \). Then the following conditions are equivalent:

i) \((X, \left\| . \right\|)\) is a real inner product space;

ii) For every \( c > 0 \), \( f : D \to \mathbb{R} \) defined on a convex subset \( D \) of \( X \), the function \( f = g + c \left\| . \right\|^2 \) is strongly \( \varphi \)-convex with modulus \( c \);

iii) \left\| . \right\|^2 : X \to \mathbb{R} \) is strongly \( \varphi \)-convex with modulus 1.

**Proof.** The implication i)⇒ii) follows by Lemma 2. To see that ii)⇒iii) take \( g = 0 \). Clearly, \( g \) is \( \varphi \)-convex function, whence \( f = c \left\| . \right\|^2 \) is strongly \( \varphi \)-convex with modulus \( c \). Consequently, \( \left\| . \right\|^2 \) is strongly \( \varphi \)-convex with modulus 1. Finally, to prove iii)⇒i) observe that by the strongly \( \varphi \)-convexity of \( \left\| . \right\|^2 \), we obtain
\[
\left\| \frac{\varphi(x) + \varphi(y)}{2} \right\|^2 \leq \frac{\left\| \varphi(x) \right\|^2}{2} + \frac{\left\| \varphi(y) \right\|^2}{2} - \frac{1}{4} \left\| \varphi(x) - \varphi(y) \right\|^2
\]
and hence
\[
(2.2) \quad \left\| \varphi(x) + \varphi(y) \right\|^2 + \left\| \varphi(x) - \varphi(y) \right\|^2 \leq 2 \left\| \varphi(x) \right\|^2 + 2 \left\| \varphi(y) \right\|^2
\]
for all \( x, y \in X \). Now, putting \( u = \varphi(x) + \varphi(y) \) and \( v = \varphi(x) - \varphi(y) \) in (2.2), we have

\[
2\|u\|^2 + 2\|v\|^2 \leq \|u + v\|^2 + \|u - v\|^2
\]

for all \( u, v \in X \).

Conditions (2.2) and (2.3) mean that the norm \( \|\cdot\|^2 \) satisfies the parallelogram law, which implies, by the classical Jordan-Von Neumann theorem, that \( (X, \|\cdot\|) \) is an inner product space. This completes the proof. \( \square \)

Now, we give a new Hermite–Hadamard-type inequalities for strongly \( \varphi \)-convex functions with modulus \( c > 0 \) as follows:

**Theorem 2.** If \( f : [a, b] \to \mathbb{R} \) is strongly \( \varphi \)-convex with modulus \( c > 0 \) for the continuous function \( \varphi : [a, b] \to [a, b] \), then

\[
f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) + \frac{c}{12} (\varphi(a) - \varphi(b))^2
\]

\[
\leq \frac{1}{\varphi(b) - \varphi(a)} \int f(x)dx
\]

\[
\leq \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{c}{6} (\varphi(a) - \varphi(b))^2.
\]

**Proof.** From the strongly \( \varphi \)-convexity of \( f \), we have

\[
f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) = f\left(\frac{t\varphi(a) + (1-t)\varphi(b)}{2} + \frac{(1-t)\varphi(a) + t\varphi(b)}{2}\right)
\]

\[
\leq \frac{1}{2} f (t\varphi(a) + (1-t)\varphi(b)) + \frac{1}{2} \left( f((1-t)\varphi(a) + t\varphi(b)) - \frac{c}{4} (1 - 2t)^2 (\varphi(a) - \varphi(b))^2 \right).
\]

By Lemma 1, it follows that the previous inequality can be integrated with respect to \( t \) over \([0, 1]\), getting

\[
f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) + \frac{c}{12} (\varphi(a) - \varphi(b))^2
\]

\[
\leq \frac{1}{2} \int_0^1 f (t\varphi(a) + (1-t)\varphi(b)) dt + \frac{1}{2} \int_0^1 f ((1-t)\varphi(a) + t\varphi(b)) dt.
\]

In the first integral, we substitute \( x = t\varphi(a) + (1-t)\varphi(b) \). Meanwhile, in the second integral we also use the substitution \( x = (1-t)\varphi(a) + t\varphi(b) \), we obtain

\[
f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) + \frac{c}{12} (\varphi(a) - \varphi(b))^2
\]

\[
\leq \frac{1}{2 (\varphi(a) - \varphi(b))} \int_{\varphi(b)}^{\varphi(a)} f(x)dx + \frac{1}{2 (\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x)dx
\]

\[
= \frac{1}{\varphi(b) - \varphi(a)} \int f(x)dx.
\]
In order to prove the second inequality, we start from the strongly \(\varphi\)-convexity of \(f\) meaning that for every \(t \in [0, 1]\) one has

\[
f(t\varphi(a) + (1-t)\varphi(b)) \leq tf(\varphi(a)) + (1-t)f(\varphi(b)) - ct(1-t) (\varphi(a) - \varphi(b))^2.
\]

Lemma [1] allows us to integrate both sides of this inequality on \([0, 1]\), we get

\[
\int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt \leq f(\varphi(a)) + \int_0^1 (1-t)dt - c \int_0^1 (\varphi(a) - \varphi(b))^2 dt.
\]

The previous substitution in the first side of this inequality leads to

\[
\left(\frac{1}{\varphi(a) - \varphi(b)}\right) \int \frac{\varphi(a)}{\varphi(b)} f(x) dx \leq \frac{f(\varphi(a))}{2} + \frac{f(\varphi(b))}{2} - \frac{c}{6} (\varphi(a) - \varphi(b))^2
\]

which gives the second inequality of (2.4). This completes to proof. \(\square\)

**Theorem 3.** If \(f : [a, b] \to \mathbb{R}\) is strongly \(\varphi\)-convex with modulus \(c > 0\) for the continuous function \(\varphi : [a, b] \to [a, b]\), then

\[
(2.5) \quad \frac{1}{\varphi(b) - \varphi(a)} \int f(x) f(a + b - x) dx \leq \frac{f^2(\varphi(x)) + f^2(\varphi(y))}{6} + \frac{2f(\varphi(x))f(\varphi(y))}{3} - \frac{c}{6} (\varphi(x) - \varphi(y))^2 + \frac{c^2}{30} (\varphi(x) - \varphi(y))^4.
\]

**Proof.** Since \(f\) is strongly \(\varphi\)-convex with respect to \(c > 0\), we have that for all \(t \in [0, 1]\)

\[
(2.6) \quad f(t\varphi(a) + (1-t)\varphi(b)) \leq tf(\varphi(a)) + (1-t)f(\varphi(b)) - ct(1-t) (\varphi(a) - \varphi(b))^2
\]

and

\[
(2.7) \quad f((1-t)\varphi(a) + t\varphi(b)) \leq (1-t)f(\varphi(a)) + tf(\varphi(b)) - ct(1-t) (\varphi(a) - \varphi(b))^2.
\]

Multiplying both sides of (2.6) by (2.7), it follows that

\[
(2.8) \quad f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b)) \leq t(1-t) f^2(\varphi(a)) + (t^2 + (1-t)^2) f(\varphi(a))f(\varphi(b)) - ct(1-t) (\varphi(a) - \varphi(b))^2 [f(\varphi(a)) + f(\varphi(b))] + c^2 t^2 (1-t)^2 (\varphi(a) - \varphi(b))^4.
\]
Integrating the inequality (2.8) with respect to $t$ over $[0,1]$, we obtain

$$
\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b))dt
$$

$$
\leq \left[ f^2(\varphi(a)) + f^2(\varphi(b)) \right] \int_0^1 t(1-t)dt + f(\varphi(a))f(\varphi(b)) \int_0^1 (t^2 + (1-t)^2) dt
$$

$$
- c (\varphi(a) - \varphi(b))^2 \left[ f(\varphi(a)) + f(\varphi(b)) \right] \int_0^1 t(1-t)dt - c^2 (\varphi(a) - \varphi(b))^4 \int_0^1 t^2(1-t)^2 dt
$$

$$
= \frac{f^2(\varphi(a)) + f^2(\varphi(b))}{6} + \frac{2f(\varphi(a))f(\varphi(b))}{3}
$$

$$
- \frac{c}{6} (\varphi(a) - \varphi(b))^2 \left[ f(\varphi(a)) + f(\varphi(b)) \right] - \frac{c^2}{30} (\varphi(a) - \varphi(b))^4.
$$

If we change the variable $x := t\varphi(a) + (1-t)\varphi(b)$, $t \in [0,1]$, we get the required inequality in (2.9). This proves the theorem. \hfill \Box

**Theorem 4.** If $f, g : [a, b] \to \mathbb{R}$ is strongly $\varphi$-convex with modulus $c > 0$ for the continuous function $\varphi : [a, b] \to [a, b]$, then

$$
\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx
$$

$$
\leq \frac{M(a,b)}{3} + \frac{N(a,b)}{6} - \frac{c}{12} (\varphi(a) - \varphi(b))^2 S(a,b) + \frac{c^2}{30} (\varphi(a) - \varphi(b))^4
$$

where

$$
M(a,b) = f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b))
$$

$$
N(a,b) = f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))
$$

$$
S(a,b) = f(\varphi(a)) + f(\varphi(b)) + g(\varphi(a)) + g(\varphi(b)).
$$

**Proof.** Since $f, g : [a, b] \to \mathbb{R}$ is strongly $\varphi$-convex with modulus $c > 0$, we have

$$
f(t\varphi(a) + (1-t)\varphi(b)) \leq tf(\varphi(a)) + (1-t)f(\varphi(b)) - ct(1-t)(\varphi(a) - \varphi(b))^2
$$

$$
g(t\varphi(a) + (1-t)\varphi(b)) \leq tg(\varphi(a)) + (1-t)g(\varphi(b)) - ct(1-t)(\varphi(a) - \varphi(b))^2.
$$
Multiplying both sides of (2.10) by (2.11), it follows that
\[
\begin{align*}
  f(t\varphi(a) + (1-t)\varphi(b))g(t\varphi(a) + (1-t)\varphi(b)) \\
  \leq t^2 f(\varphi(a))g(\varphi(a)) + (1-t)^2 f(\varphi(b))g(\varphi(b)) \\
  + t(1-t) [f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))] \\
  - ct^2 (\varphi(a) - \varphi(b))^2 [f(\varphi(a)) + g(\varphi(a))] \\
  - ct (1-t)^2 (\varphi(a) - \varphi(b))^2 [f(\varphi(b)) + g(\varphi(b))] \\
  + c^2 t^2 (1-t)^2 (\varphi(a) - \varphi(b))^4.
\end{align*}
\]

By Lemma 1, it follows that the previous inequality can be integrated with respect to \( t \) over \([0,1]\), getting
\[
\begin{align*}
  \int_0^1 f(t\varphi(a) + (1-t)\varphi(b))g(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
  \leq f(\varphi(a))g(\varphi(a)) \int_0^1 t^2 \, dt + f(\varphi(b))g(\varphi(b)) \int_0^1 (1-t)^2 \, dt \\
  + [f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))] \int_0^1 t(1-t) \, dt \\
  - c (\varphi(a) - \varphi(b))^2 [f(\varphi(a)) + g(\varphi(a))] \int_0^1 t^2 (1-t) \, dt \\
  - c (\varphi(a) - \varphi(b))^2 [f(\varphi(b)) + g(\varphi(b))] \int_0^1 t (1-t)^2 \, dt \\
  + c^2 (\varphi(a) - \varphi(b))^4 \int_0^1 t^2 (1-t)^2 \, dt.
\end{align*}
\]

In the first integral, we substitute \( x = t\varphi(a) + (1-t)\varphi(b) \) and simple integrals calculated, we obtain the required inequality in (2.9). \( \square \)

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