LEBESGUE-FOURIER ALGEBRA OF A HYPERGROUP

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ABSTRACT. Let $\mathcal{L}A(H)$ be the Lebesgue-Fourier space of a hypergroup $H$ considered as a Banach space on $H$. In addition $\mathcal{L}A(H)$ is a Banach algebra with the multiplication inherited from $L^1(H)$. Moreover if $H$ is a regular Fourier hypergroup, $\mathcal{L}A(H)$ is a Banach algebra with pointwise multiplication. We study the amenability and character amenability of these two Banach algebras.

The theory of locally compact hypergroups in harmonic analysis was initiated independently by Dunkl [4] and Jewett [11] in the early 1970s with small differences. In 1968 Pym [17] also considered convolution structures which are close to this theory. A nice exposition of the subject can be found in [18]. We use the term hypergroup to refer to the locally compact hypergroups defined by Jewett [11].

Fourier algebras over hypergroups have not been subject of much attention, since they do not need to form an algebra with pointwise multiplication. In his recent work, Muruganandam [15] studies several hypergroups whose Fourier space forms a Banach algebra with pointwise multiplication. He defines a class of hypergroups called regular Fourier hypergroups where the associated Fourier space is a Banach algebra with its norm.

Lebesgue-Fourier algebras of locally compact groups were studied extensively by Ghahramani and Lau in [8]. Lebesgue-Fourier algebras are not only Segal algebras with convolution but also abstract Segal algebras with respect to the Fourier algebra of a locally compact group.

After an overview of the preliminaries in section 1, in section 2, we define Lebesgue-Fourier space over a hypergroup. We extend the definition and the main ideas of [8] to hypergroup $H$: accordingly, we consider $\mathcal{L}A(H)$ with the induced multiplication as a dense ideal of the algebra of integrable functions, $L^1(H)$. Moreover, we show that the amenability of $\mathcal{L}A(H)$ with the multiplication induced from $L^1(H)$ leads to the amenability and discrete being of $H$; also, $\alpha$-amenability of $H$ is equivalent to the $\phi_\alpha$-amenability of $\mathcal{L}A(H)$ for each $\alpha \in \hat{H}$. Section 3 is devoted to regular Fourier hypergroups $H$ when Lebesgue-Fourier space with pointwise multiplication is a dense ideal of the Fourier algebra $A(H)$ and therefore is a Banach algebra with pointwise multiplication, and as a result, amenability of $\mathcal{L}A(H)$ with the pointwise product leads to compactness of $H$ and amenability of $A(H)$. Eventually, we show that $A(H)$ and $\mathcal{L}A(H)$ are $\phi_x$-amenable for each $x \in H$.

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1. Preliminaries

Let \( H \) be a (locally compact Hausdorff) hypergroup which admits a left Haar measure \( m_H \). Here we use \( dt \) instead of \( dm_H(t) \). We follow the definition of a hypergroup in \( [11] \). For every \( x \in H \), suppose \( \delta_x \) denotes the point measure at \( x \). We show the probability measure \( \delta_x \ast \delta_y \) simply by \( xy \) and \( \hat{x} \) denotes the involution of \( x \in H \). Also \( C(H) \) is the space of all continuous functions on \( H \) and for all \( \lambda \in \pi \) one considers representations \( \pi \). We note that the left regular representation of \( \pi \) is given by \( \lambda(x)f := f(x) \) for each \( \lambda \in \pi \). The Banach space \( L^1(H) \) is a Banach algebra with \( f \ast g := lfg \) when \( f, g \in L^1(H) \).

Let \( \Sigma \) denote the set of equivalence classes of representations of \( H \) and \( \lambda \) denote the left regular representation of \( H \) on \( L^2(H) \) given by \( \lambda(x)f(y) = f(xy) \) for all \( x, y \in H \) and for all \( f \in L^2(H) \).

Let \( C^*(H) \) and \( C^*_\lambda(H) \) represent the full and reduced \( C^* \)-algebras of \( H \) respectively. The Von Neumann algebra associated to \( \lambda \) of \( H \), namely, the bicommutant of \( \lambda(L^1(H)) \) in \( B(L^2(H)) \) is called the Von Neumann algebra of \( H \) and is denoted by \( VN(H) \) (see \([13, 16] \)). For any \( f \in L^1(H) \) the norm of \( C^*(H) \) is given by

\[
\|f\|_{C^*(H)} = \sup_{\pi \in \Sigma} \|\pi(f)\|_{B(H_\pi)}
\]

when \( H_\pi \) is the Hilbert space related to each \( \pi \in \Sigma \); accordingly we have

\[
\|f\|_{C^*_\lambda(H)} = \|\lambda(f)\|_{B(L^2(H))}.
\]

The Banach space dual of \( C^*(H) \) is called the Fourier-Stieltjes space and is denoted by \( B(H) \). Let \( B_\lambda(H) \) denote the Banach space dual of the reduced \( C^* \)-algebra \( C^*_\lambda(H) \). In fact, \( B_\lambda(H) \) can be realized as a closed subspace of \( B(H) \). The closed subspace spanned by \( \{f \ast \hat{f} : f \in C^*_\lambda(H)\} \) in \( B_\lambda(H) \) is called the Fourier space of \( H \) and is denoted by \( A(H) \) \([13, 16] \). The Banach space dual of \( A(H) \) can be identified with the Von Neumann algebra of \( H \) in the following way. For every \( T \in VN(H) \) there exists a unique continuous linear functional \( \varphi_T \) on \( A(H) \) satisfying

\[
\varphi_T((f \ast \hat{g})) = \langle T(f), g \rangle_{L^2(H)} \text{ for all } f, g \in L^2(H).
\]

The mapping \( T \to \varphi_T \) is a Banach space isomorphism between \( VN(H) \) and \( A(H)^* \). By an abuse of notation, we use \( T \) to represent \( \varphi_T \).
2. Lebesgue-Fourier algebra for general hypergroups

Let $H$ be a hypergroup and let us define

$$\mathcal{L}A(H) := L^1(H) \cap A(H)$$

and

$$|||f||| = ||f||_1 + ||f||_{A(H)}$$

for each $f \in \mathcal{L}A(H)$. Similar to [15] lemma 2.1, we have

**Lemma 2.1.** Given a hypergroup $H$, $\mathcal{L}A(H)$ is a dense left ideal of $L^1(H)$.

**Proof.** By [15] corollary 2.12, we know that

$$\left\{ \sum_{i=1}^{n} \phi_i * \tilde{\varphi}_i \mid \phi_i, \varphi_i \in L^2(H), n \in \mathbb{N} \right\}$$

is dense in $A(H)$. But $g * \phi$ is an element of $L^2(H)$ for each $g \in L^1(H)$ and $\phi \in L^2(H)$, so $\sum_{i=1}^{n} g * (\phi_i * \tilde{\varphi}_i) = \sum_{i=1}^{n} (g * \phi_i) * \tilde{\varphi}_i$ belongs to $\mathcal{L}A(H)$ for each $g \in L^1(H)$ and $\sum_{i=1}^{n} \phi_i * \tilde{\varphi}_i \in \mathcal{L}A(H)$.

To show $\mathcal{L}A(H)$ is a dense set in $L^1(H)$ we refer to [15] proposition 2.22 and [20] lemma 2.1 to generate a left bounded approximate identity for $L^1(H)$ whose elements belong to $\mathcal{L}A(H)$.

**Proposition 2.2.** Let $H$ be a hypergroup. Then $(\mathcal{L}A(H), ||| \cdot |||)$ with the induced multiplication from $L^1(H)$ is a Banach algebra.

**Proof.** Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}A(H)$ which $\cdot |||_1$-converges to some $f \in L^1(H)$ and $\cdot |||_{A(H)}$-converges to some $f' \in A(H)$. There exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which pointwise converges to $f$ almost everywhere. Moreover, based on [15] remark 2.9, $||f||_{\infty} \leq ||f||_{A(H)}$ for each $f \in \mathcal{L}A(H)$; therefore $f = f'$ almost everywhere. So $||f_n - f||_1 \to 0$.

Since $||g * \phi||_2 \leq ||g||_1 ||\phi||_2$ for each $g \in L^1(H)$ and $\phi \in L^2(H)$,

$$||g * (\phi * \tilde{\varphi})||_{A(H)} \leq ||g||_1 ||\phi * \tilde{\varphi}||_{A(H)}$$

for $g \in L^1(H)$ and $\phi, \varphi \in L^2(H)$. Consequently, we have $||g * f|| \leq ||g||_1 ||f||$ for each $f \in \mathcal{L}A(H)$. Hence,

$$||g * f|| \leq ||g||_1 ||f||$$

when $f, g \in \mathcal{L}A(H)$.

We know that $\mathcal{L}A(H)$ has an approximate identity whose elements are bounded in $|| \cdot ||_1$ [20] lemma 2.1. Accordingly, in the following proposition we study the existence of an approximate identity for $\mathcal{L}A(H)$ bounded in $|| \cdot ||$.

**Proposition 2.3.** Let $H$ be a hypergroup. The followings are equivalent:

(a) $H$ is discrete;

(b) $\mathcal{L}A(H) = L^1(H)$;

(c) $\mathcal{L}A(H)$ has a bounded approximate identity.
Proof. It is easy see (a) ⇒ (b) and (a) ⇒ (c).

(b) ⇒ (a). We know \( L^1(H) \subseteq A(H) \subseteq C_0(H) \). We define \( \iota : L^1(H) \to C_0(H) \) where \( \iota(f) \) is the function in \( C_0(H) \) and in the equivalence class of \( f \). We will show that \( \iota \) is continuous. Let \( \{f_n\} \) be a sequence in \( L^1(H) \) converging to \( f \in L^1(H) \); as a result there exists a subsequence \( \{f_{n_k}\} \) that converges to \( f \) pointwise almost everywhere. If \( \{\iota(f_n)\} \) converges to \( f' \in C_0(H) \), it follows \( f' = \iota(f) \) almost everywhere. So by closed graph theorem, \( \iota \) is a continuous map. Let \( H \) be a non-discrete hypergroup, then \( m_H(\{e\}) = 0 \) by [11] theorem 7.1B. By an argument similar to the proof of [8] proposition 2.3, there is a bounded net \( \{f_\gamma\} \) in \( L^1(H) \) when \( \gamma \to \infty \), which contradicts the continuity of \( \iota \) as a linear map.

(c) ⇒ (b). By assumption, there is a bounded right approximate identity \( (u_\gamma) \) such that \( (u_\gamma) \subseteq L(A(H)) \) and \( \|u_\gamma\| \leq K \) for all \( \gamma \). Since \( |||f * u_\gamma||| \leq |||f||| \|u_\gamma\| \) for all \( f \in L(A(H)) \) and \( \gamma \), we have \( |||f||| \leq K |||f||| \). On the other hand, \( |||f||| \leq |||f||| \). Thus the two norms \( ||| \cdot |||_1 \) and \( ||| \cdot ||| \) are equivalent on \( L(A(H)) \), so \( L(A(H)) = L^1(H) \) by lemma [2,1].

**Corollary 2.4.** Let \( H \) be a hypergroup. If \( \mathcal{L}A(H) \) with the multiplication induced from \( L^1(H) \) is amenable then \( H \) is discrete and amenable.

Proof. If \( \mathcal{L}A(H) \) is an amenable algebra, then based on [19] proposition 2.2.1, it has a bounded approximate identity. So by proposition [2,3] we have that \( H \) is discrete. Also since \( \mathcal{L}A(H) = l^1(H) \), by [20] proposition 4.9 \( H \) is amenable. □

The converse of the preceding corollary is not true in general, because the amenability of \( H \) does not show the amenability of \( L^1(H) \) (see [20] and to consider a counter example see [11]).

Let \( H \) be a commutative hypergroup. The dual space of the hypergroup algebra, \( L^1(H) \), can be identified with the usual Banach space \( L^\infty(H) \), and its structure space is homomorphic to the character space of \( H \), i.e.

\[
\mathcal{A}^b(H) := \{ \alpha \in C^b(H) : \alpha(e) = 1, \alpha(xy) = \alpha(x)\alpha(y), \text{ for all } x, y \in H \}
\]

equipped with the compact-open topology. \( \mathcal{A}^b(H) \) is a locally compact Hausdorff space. Let \( \hat{H} \) denote the set of all hermitian characters \( \alpha \) in \( \mathcal{A}^b(H) \), i.e. \( \alpha(x) = \overline{\alpha(x)} \) for every \( x \in H \) with a Plancharer measure \( \pi_H \). Note that \( \hat{H} \) in general may not have the dual hypergroup structure and a proper inclusion in \( supp(\pi_H) \subseteq \hat{H} \subseteq \mathcal{A}^b(H) \) is possible.

The Fourier-Stieltjes transform of \( \mu \in M(H) \), \( \hat{\mu} \in C^b(\hat{H}) \), is given by \( \hat{\mu}(\alpha) := \int_{H} \alpha(x)d\mu(x) \). Its restriction to \( L^1(H) \) is called the Fourier transform. We have \( \hat{f} \in C_0(\hat{H}) \) for \( f \in L^1(H) \), and the map that takes \( \alpha \) to \( \mathcal{I}(\alpha) = ker(\phi_\alpha) \) is a bijection of \( \hat{H} \) onto the space of all maximal ideals of \( L^1(H) \), where \( ker(\phi_\alpha) \) denotes the kernel of the homomorphisms \( \phi_\alpha(f) = \hat{f}(\alpha) \) on \( L^1(H) \) (see [3]).

Let \( \mathcal{A} \) be a Banach algebra and \( \sigma(\mathcal{A}) \) be the set of all non-zero characters on \( \mathcal{A} \). Kaniuth, Lau and Pym [12,13] introduced and studied the concept of \( \phi \)-amenability for Banach algebras as a generalization of left amenability of Lau algebras when \( \phi \in \sigma(\mathcal{A}) \). \( \mathcal{A} \) is \( \phi \)-amenable if there exists a bounded net \( \{a_\gamma\} \) in \( \mathcal{A} \) such that
φ(aγ) → 1 and ∥aαγ − φ(a)aγ∥ → 0 for all a ∈ A. Any such net is called a bounded approximate φ-mean.

The notion of α-amenable hypergroups was introduced and studied in [6]. As shown in [2], H is α-amenable if and only if L1(H) is φα-amenable. In the following theorem we explore the connection between α-amenable of H and φα-amenable of LA(H).

**Theorem 2.5.** Let H be a commutative hypergroup and let α ∈ ˆH be real-valued. Then the Lebesgue-Fourier algebra, LA(H), is φα-amenable if and only if H is α-amenable.

**Proof.** Suppose H is α-amenable. Then L1(H) is φα-amenable by [2, Theorem 1.1]. Thus there is a bounded approximate φα-mean in L1(H), say (fγ). Fix h0 ∈ LA(H) such that φα(h0) = 1 and set hγ = fγ * h0 ∈ LA(H) for all γ, and consequently, for each h ∈ LA(H) we have

\[ |||h * hγ − φα(h)hγ||| \leq ||h * fγ − φ(h)fγ||_1 |||h0||| \to 0 \]

and φα(hγ) = φα(fγ) → 1. Since (fγ) is ∥ ∥1-bounded, it follows that (hγ) is || ∥ ∥-bounded. Thus LA(H) is φα-amenable.

Conversely, suppose that LA(H) is φα-amenable. Then there is a bounded approximate φα-mean (hγ) in LA(H). Fix h0 ∈ LA(H) such that φα(h0) = 1 and set fγ = h0 * hγ for all γ. Since LA(H) is a left ideal in L1(H), we have

\[ ||f * fγ − φα(f)fγ||_1 = ||f * h0 * hγ − φ(f)h0 * hγ||_1 \]

\[ \leq ||f * h0 * hγ − φα(f)φα(h0)hγ||_1 + ||φα(f)φα(h0)hγ − φ(f)h0 * hγ||_1 \]

\[ \leq |||f * h0 * hγ − φα(F * h0)hγ||| + ||φ(f)|| |||φα(h0)hγ − h0 * hγ||| \to 0, \]

and φα(fγ) = φα(hγ) → 1 for each f ∈ L1(H). Since || ∥ ∥1 ≤ || ∥ ∥, it follows that (fγ) is a ∥ ∥1-bounded approximate φα-mean in L1(H), and L1(H) is φα-left amenable. Thus H is α-amenable by [2, Theorem 1.1]. □

### 3. Lebesgue-Fourier algebra for regular Fourier hypergroups

Let H be a commutative hypergroup, we define

\[ S = \{α ∈ ˆH | |ˆμ(α)| ≤ |λ(μ)| for all μ ∈ M(H)\} \]

A non empty closed subset of ˆH, see [15]. Muruganandam has defined (F) condition as following.

**Definition.** Let H be a commutative hypergroup. We say H satisfies (F) condition if there exists M > 0 satisfying the following

For every pair α, α′ ∈ S, αα′ belongs to B_λ(H) and ∥αα′∥_{B(H)} ≤ M.

Some interesting results for commutative hypergroups which satisfy (F) condition have been obtained in [15]. We quote the following corollary.
Corollary 3.1. Let $H$ be a commutative hypergroup satisfying condition (F). Then the Fourier space $A(H)$ is an algebra under pointwise product. Moreover, 

$$\|f \cdot g\|_{A(H)} \leq M\|f\|_{A(H)} \|g\|_{A(H)}$$

for $f, g \in A(H)$.

In particular if $M = 1$, then $A(H)$ forms a Banach algebra. Similar results hold for $B_\lambda(H)$.

This corollary led Muruganandam to define (regular) Fourier hypergroups in [15].

Definition. A hypergroup $H$ is called a Fourier hypergroup if

1. The Fourier space $A(H)$ forms an algebra with pointwise product.
2. There exists a norm on $A(H)$ which is equivalent to the original norm with respect to which $A(H)$ forms a Banach algebra.

A hypergroup is called a regular Fourier hypergroup if $A(H)$ is a Banach algebra with its original norm and pointwise product.

As we have seen in corollary 3.1 all commutative hypergroups which satisfy (F) for some $M > 0$ are Fourier hypergroups. If $M = 1$ then $H$ is a regular Fourier hypergroup. Consequently, several (regular) Fourier hypergroups have been introduced in [15] section 4.1. We will scope on regular hypergroups to pursue results for $LA(H)$.

Proposition 3.2. Let $H$ be a regular Fourier hypergroup. Then $LA(H)$ is a dense ideal in $A(H)$.

Proof. For each $f \in LA(H)$ and $\phi \in A(H)$, $\phi \cdot f$ is in $L^1(H)$, since $f$ belongs to $C_0(H)$. Moreover, regular Fourier hypergroup property of $H$ implies $\phi \cdot f \in A(H)$. Because $A(H) \cap C_c(H)$ is dense in $A(H)$, $LA(H)$ is dense in $A(H)$, see [15] corollary 2.12. □

Proposition 3.3. Let $H$ be a regular Fourier hypergroup. Then $(LA(H), \|\cdot\|)$ with the pointwise multiplication is a Banach algebra.

Proof. As we have seen in the proof of proposition 2.2, $LA(H)$ is a Banach space. Since $\|\cdot\|_\infty \leq \|\cdot\|_{A(H)}$, we have

$$\|\phi \cdot f\|_1 \leq \|\phi\|_{A(H)} \|f\|_1$$

for each $\phi \in LA(H)$ and $f \in L^1(H)$. Hence

$$\|g \cdot f\|_1 \leq \|g\|_1 \|f\|_1$$

for each $f, g \in LA(H)$. □

Proposition 3.4. Let $H$ be a regular Fourier hypergroup. The followings are equivalent:

(a) $H$ is compact;
(b) $LA(H) = A(H)$;
(c) $LA(H)$ with the pointwise product has a bounded approximate identity.
Proof. Clearly \((a) \Rightarrow (b)\) and \((a) \Rightarrow (c)\).

\((b) \Rightarrow (a)\). We know \(A(H) \subseteq L^1(H)\). We define \(i : A(H) \to L^1(H)\) where \(i(f)\) is the equivalent class of \(f\) in \(L^1(H)\). We show that \(i\) is continuous. Let \(\{f_n\}\) be a convergent sequence in \(A(H)\) to some \(f \in A(H)\) which implies \(f_n \to f\) uniformly. If \(\{i(f_n)\}\) converges to \(f' \in L^1(H)\), and as a result there is a subsequence \(i(f_{n_k})\) which converges to \(f'\) pointwise almost everywhere. Therefore \(f' = i(f)\) almost everywhere. So by closed graph theorem, \(i\) is a continuous map. Let \(H\) be a non-compact hypergroup, then \(m_H(H) = \infty\) by \([11]\) theorem 7.2B. By the same argument as in the proof of \([8,\ \text{proposition}\ 2.6]\) we can make a bounded net \(\{f_n\}\) in \(A(H)\) when \(\|i(f_n)\|_1 \to \infty\), which contradicts the continuity of \(i\) as a linear map.

\((c) \Rightarrow (b)\). By assumption, there is a bounded approximate identity, say \((u_n) \subseteq \mathcal{L}A(H)\), with \(\|u_n\| \leq K\) for all \(\gamma\). Since \(\|u_n\| \leq \|f\|_{A(H)}\|u_n\|\) for all \(f \in \mathcal{L}A(H)\) and \(\gamma\), it follows that \(\|f\| \leq K\|f\|_{A(H)}\). On the other hand \(\|f\|_{A(H)} \leq \|f\|\), Thus the two norms \(\|\cdot\|_{A(H)}\) and \(\|\cdot\|\) are equivalent on \(\mathcal{L}A(H)\), and so \(\mathcal{L}A(H) = A(H)\); this is because \(\mathcal{L}A(H)\) is dense in \(A(H)\) under \(\|\cdot\|_{A(H)}\) ·

Corollary 3.5. Let \(H\) be a regular Fourier hypergroup and \(\mathcal{L}A(H)\) with the pointwise product be amenable. Then \(H\) is compact, and the Fourier algebra \(A(H)\) is amenable.

Proof. Suppose that \(\mathcal{L}A(H)\) is an amenable algebra. Then by proposition 2.2.1 of \([19]\), it has a bounded approximate identity. So, by proposition 3.3 \(H\) is compact. Also since \(\mathcal{L}A(H) = A(H)\), the norms \(\|\cdot\|\) and \(\|\cdot\|_{A(H)}\) are equivalent by the open mapping theorem. So \(A(H)\) is amenable.

Let \(H\) be a regular Fourier hypergroup and let \(x \in H\). Then the functional \(\phi_x\) given by \(\phi_x(f) = f(x)\) for all \(f \in A(H)\) belongs to \(\sigma(A(H))\) by \([15]\) proposition 2.22.

Theorem 3.6. Let \(H\) be a regular Fourier hypergroup. Then \(A(H)\) is \(\phi_x\)-amenable for all \(x \in H\).

Proof. Fix \(x \in H\) and let \(\mathcal{U}\) denote a net of relatively compact neighborhoods of \(e\) in \(H\). For \(U \in \mathcal{U}\), define \(f_U \in A(H)\) by

\[
    f_U(y) = m_H(U)^{-1} \langle \lambda(y)1_{\bar{x}U}, 1_U \rangle_{L^2(H)} = m_H(U)^{-1} \int_U 1_{\bar{x}U}(\gamma) d\gamma = m_H(U)^{-1} m_H(y \bar{x}U \cap U)
\]

for all \(y \in H\), where \(\lambda\) is the left regular representation of \(H\) on \(L^2(H)\). Recall that \(V N(H)\) is canonically identified with the dual space \(A(H)^*\) by the paring \(\langle \lambda(y), f \rangle = f(y)\) (cf. \([14]\) proposition 2.21). Then

\[
    \|f_U\|_{A(H)} \leq m_H(U)^{-1} 1_{\bar{x}U}^2 \|1_U\|_2 = 1.
\]
Let $F$ be a weak* cluster point of $(f_U)_U$ in $A(H)^{**}$. Then
\[ F(\phi_x) = \lim_U (f_U, \phi_x) = f_U(x) = 1, \]
and moreover
\[ F(\lambda(y)) = \lim_U \frac{m_H(y\hat{x}U \cap U)}{m_H(U)} = 0 \]
for $y \neq x$. Therefore
\[
F(\lambda(y) \cdot f) = \lim_U (f_U, \lambda(y) \cdot f) \\
= \lim_U (\lambda(y), f f_U) \\
= f(y) \lim_U \frac{m_H(y\hat{x}U \cap U)}{m_H(U)} \\
= f(y) F(\lambda(y)) \\
= \phi_x(f) F(\lambda(y))
\]
for all $y \in H$ and $f \in A(H)$. Since \{\lambda(y) : y \in H\} generates $VN(H)$, we conclude that $F(T \cdot f) = \phi_x(f) F(T)$ for all $T \in VN(H)$ and $f \in A(H)$. This completes the proof.

**Corollary 3.7.** Let $H$ be a regular Fourier hypergroup and $\mathcal{L}A(H)$ is equipped with the pointwise product. Then $\mathcal{L}A(H)$ is $\phi_x$-amenable for all $x \in H$.

**Proof.** First note that since $\mathcal{L}A(H)$ is dense in $A(H)$, $\phi_x|\mathcal{L}A(H) \neq 0$ and $\phi_x|\mathcal{L}A(H)$ belongs to $\sigma(\mathcal{L}A(H))$. In addition, since $A(H)$ is $\phi_x$-amenable, there is a bounded approximate $\phi_x$-mean $(f_\gamma)$ in $A(H)$. Fix $h_0 \in \mathcal{L}A(H)$ such that $\phi_x(h_0) = 1$ and set $h_\gamma = f_\gamma h_0 \in \mathcal{L}A(H)$ for all $\gamma$. Thus for each $h \in \mathcal{L}A(H)$ we have
\[
|||hh_\gamma - \phi_x(h)h_\gamma|| \leq ||hf_\gamma - \phi_x(h)f_\gamma||_{A(H)} ||h_0|| \to 0
\]
and $\phi_x(h_\gamma) = \phi(f_\gamma) \to 1$. Thus $\mathcal{L}A(H)$ is $\phi_x$-amenable.

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