Locally covariant charged fields and background independence

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Abstract

We discuss gauge background independence at the example of the charged Dirac field. We show that a perturbative version of background independence, termed perturbative agreement by Hollands and Wald, can be fulfilled, and discuss some of its consequences.

1 Introduction

The framework of locally covariant field theory [1] has proved very fruitful for quantum field theory (QFT) on curved spacetimes. The central idea is to define a quantum field theory simultaneously on all spacetimes, in a coherent way. Given a locally covariant field theory specified by a Lagrangean $L$, one may then wonder about background independence, i.e., is the field theory on a spacetime $M$ for the Lagrangean $L$ in some sense equivalent to a field theory defined on $M'$, if one adds $L_M - L_{M'}$ as an interaction term to the latter. This question was investigated for the scalar field in [2], and the requirement of background independence was formulated as the principle of perturbative agreement. It takes the form of a renormalization condition, and it was shown that this principle can indeed be fulfilled for spacetime dimension $n > 2$.

Recently, it was proposed to generalize the framework of locally covariant field theory to also accommodate for fields charged under a gauge group, in the presence of a background connection [3]. In this setting, the background connection and gauge transformations are treated on equal footing with the background metric and its isometries. It is then natural to ask for background independence w.r.t. changes in the background connection.

[1] I am very grateful to K. Rejzner for proposing this question and helpful discussions on this topic.
The main result of the present work is that this background independence can indeed be fulfilled for the Dirac field for spacetime dimension $n \leq 4$. As a byproduct, we formulate the principle of perturbative agreement in the framework of locally covariant field theory, combined with that of perturbative algebraic QFT [4], i.e., in terms of functionals and natural transformations. Most of our results are in close analogy to those of [2], but we correct some minor mistakes in the proofs given there. We also present an application of perturbative agreement: We show that the fermionic contribution to the one-loop renormalization group flow of Yang–Mills theories can be obtained from the nontrivial scaling of Wick squares, i.e., of the parametrix. This is in the spirit of the background field method and shows the connection to heat kernel methods.

The article is structured as follows: In the next section, we introduce the setup. We review the definition of the charged locally covariant Dirac field given in [3]. Subsection 2.4 deals with the question of how to relate functionals defined on different backgrounds, an issue that is crucial for our discussion. In Section 3 we first formulate the principle of perturbative agreement. Then, in Subsection 3.1, we show that if the principle of perturbative agreement for variations of the background connection is fulfilled, then the current is conserved, for semisimple gauge groups. We also show that current conservation can be ensured by a suitable choice of the parametrix. The background field method is discussed in Subsection 3.2. Finally, in Subsection 3.3 we show that the principle of perturbative agreement can be fulfilled by a suitable redefinition of time-ordered products. In an appendix, we prove a technical lemma used in the proof of the fulfillment of perturbative agreement.

2 Setup

We review the definition of the charged locally covariant Dirac field given in [3]. For further details, we refer to this reference and the ones cited therein, especially [5] for a detailed description of the definition of the spinor and Dirac bundles.

In the following, $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$. The symbol $\equiv$ stands for the definition of the left hand side by the right hand side. $\Gamma_{(c)}(M,B)$ denotes the space of (compactly supported) smooth sections of the bundle $B$ over $M$. The causal future/past of a region $K$ is denoted by $J^{\pm}(K)$. The total diagonal of $M^k$ is denoted by $D_k$.

We use the following categories:

\textbf{Vec}_{(i)}: The objects are locally convex vector spaces over $\mathbb{C}$. The morphisms are continuous linear (injective) maps.

\footnote{We do not expect obstructions to generalize this to arbitrary dimensions.}
**Alg:** The objects are topological unital *-algebras. The morphisms are continuous injective *-algebra homomorphisms.

**GSpMan:** The objects are quintuples \((SM, P, \bar{g}, \bar{A}, \bar{m})\), where \(SM\) is a spin structure over \((M, \bar{g})\), which is an oriented, time-oriented globally hyperbolic \(n\)-dimensional manifold. \(P\) is a principal \(G\) bundle over \(M\), \(\bar{A}\) a connection on \(P\), and \(\bar{m} \in C^\infty(M)\). A morphism \(\chi : (SM, P, \bar{g}, \bar{A}, \bar{m}) \rightarrow (SM', P', \bar{g}', \bar{A}', \bar{m}')\) is given by \((\chi_{SM}, \chi_P)\), where \(\chi_{SM(P)}\) is a principal Spin\(_0\) \((G)\) bundle morphism, \(\chi_P\) cover the same orientation, time-orientation and causality preserving isometric embedding \(\psi : (M, \bar{g}) \rightarrow (M', \bar{g}')\), which is a diffeomorphism on its range. Furthermore, \(\pi_S \circ \chi_{SM} = \psi_* \circ \pi_S\), where \(\pi_S\) is the spin projection from \(SM\) to the (time-) oriented frame bundle \(FM\), and \(\bar{A} = \chi_P^\ast \bar{A}', \bar{m} = \psi^\ast \bar{m}'\).

In the following, the background fields \((\bar{g}, \bar{A}, \bar{m})\) are often subsumed in the symbol \(\bar{X}\).

We recall that the spin connection on \(SM\) and the connection \(\bar{A}\) on \(P\) induce a unique connection on the principal Spin\(_0\) \times \(G\) bundle \(SM + P\) over \(M\), which is obtained by taking the direct product bundle \(SM \times P\) and restricting to the diagonal \([6]\).

Given a finite-dimensional representation \(\rho\) of \(G\) on a \(\mathbb{C}\) vector space \(V\), we define the Dirac bundle

\[ D_\rho M = (SM + P) \times_{(\sigma, \rho)} (\mathbb{C}^{2^{\lceil n/2 \rceil}} \otimes V), \]

where \(\sigma\) is the spinor representation. The dual bundle is denoted by \(D^\ast_\rho M\) and the double Dirac bundle is the direct sum

\[ D^{\oplus}_\rho M = D_\rho M \oplus D^\ast_\rho M. \]

Then we have a contravariant functor \(\mathfrak{E}^{\oplus}\) from \textbf{GSpMan} to \textbf{Vec}, by assigning to \((SM, P)\) the vector space of smooth sections,

\[ \mathfrak{E}^{\oplus}(SM, P) \doteq \Gamma^\infty(M, D^{\oplus}_\rho M), \]

and to a morphism of \textbf{GSpMan} the pullback. Similarly, we define \(\mathfrak{D}^{\oplus}\) as a covariant functor \textbf{GSpMan} to \textbf{Vec}, by assigning to \((SM, P)\) the vector space of compactly supported smooth sections,

\[ \mathfrak{D}^{\oplus}(SM, P) \doteq \Gamma^c_\infty(M, D^{\oplus}_\rho M), \]

and to a morphism of \textbf{GSpMan} the pushforward. Introducing a sesquilinear form on \(V\) such that \(\rho\) is unitary, one obtains a conjugation \(+ : V \rightarrow V^*\). Together with the Dirac conjugation, this induces conjugations \(+ : D_\rho M \rightarrow D^\ast_\rho M\), \(+ : D^\ast_\rho M \rightarrow D_\rho M\), which in turn yields a conjugation of \(D^{\oplus}_\rho M\) by

\[ (u, v)^\ast = (v^+, u^+), \]
where \( u \in D_pM|_x \), \( v \in D^*_pM|_x \) for some \( x \in M \). Using this pointwise definition, one defines the involution \( * \) on \( \mathcal{D}^\oplus(SM,P) \) and \( \mathcal{E}^\oplus(SM,P) \).

On the vector bundle \( D^\oplus_pM \), there is a bundle metric, induced by the pairing
\[
\langle [p, (y \otimes v, y' \otimes v')], [p, (z \otimes w, z' \otimes w')] \rangle \doteq \langle y', z \rangle \langle v', w \rangle + \langle z', y \rangle \langle w', v \rangle,
\]
where \( p \in SM + P, v, w \in V, y, z, \bar{v}, \bar{z} \in \mathbb{C}^{2^{n/2}} \) and the primed elements in the corresponding duals. As usual, the square brackets denote the equivalence class. This induces a pairing \( \mathcal{E}^\oplus(SM,P) \times \mathcal{E}^\oplus(SM,P) \rightarrow C^\infty(M) \).

The connection on \( SM + P \) induces a covariant derivative \( \bar{\nabla}^\oplus = \bar{\nabla} \oplus \bar{\nabla}^* \) on the associated vector bundle \( \mathcal{E}^\oplus(SM,P) \), which in turn induces the double Dirac operator
\[
D^\oplus \doteq D \oplus -D^* \doteq (-\gamma^\mu \nabla_\mu + \bar{m}) \oplus (-\gamma^\mu \nabla^*_\mu - \bar{m}),
\]
where \( \bar{m} \) is the smooth function specified in the object of \( \text{GSpMan} \). There is a corresponding causal propagator \( S^\oplus = S^\oplus_{\text{ret}} - S^\oplus_{\text{adv}} \), where \( S^\oplus_{\text{ret/adv}} \) is the retarded/advanced propagator. The double Dirac operator \( D^\oplus \), and hence also \( S^\oplus_{\text{ret/adv}} \), anticommutes with the involution.

In order to describe the coupling of Dirac fields to gauge fields, we also introduce the following vector bundles over \( M \):
\[
E^0 \doteq (SM + P) \times_{(1, \text{ad})} \mathfrak{g}, \quad E^k \doteq E^0 \otimes \Omega^k(M).
\]
Here 1 denotes the trivial representation of \( \text{Spin}_0 \) and in the definition of \( E^k \), we take the tensor product of vector bundles. The representation \( \rho \) and the spin projection induces an action of \( E^0 \) and \( E^1 \) on sections of the Dirac bundle (and hence also on the double Dirac bundle), fiberwisely given by
\[
([p, \xi], [p, z \otimes v]) \mapsto [p, z \otimes \rho(\xi)v],
\]
\[
([p, \xi] \otimes \omega, [p, z \otimes v]) \mapsto [p, \omega \gamma^\mu z \otimes \rho(\xi)v].
\]
There is also a product \( \wedge : \Gamma^\infty(M, E^k) \times \Gamma^\infty(M, E^l) \rightarrow \Gamma^\infty(M, E^{k+l}) \), fiberwisely defined by
\[
[p, \xi] \otimes \omega \wedge [p, \eta] \otimes \nu \doteq [p, [\xi, \eta]] \otimes \omega \wedge \nu,
\]
\[
p \in SM + P, \xi, \eta \in \mathfrak{g}, \omega, \nu \in \Omega^k_{\pi(p)}, \nu \in \Omega^l_{\pi(p)}.
\]
Finally, there is a map \( \bar{d} : \Gamma^\infty(M, E^k) \rightarrow \Gamma^\infty(M, E^{k+1}) \) given by
\[
\bar{d}(a\Xi) = \bar{\nabla}_\mu adx^\mu \wedge \Xi + ad\Xi, \quad a \in \Gamma^\infty(M, E^0), \Xi \in \Gamma^\infty(M, \Omega^k),
\]
where \( \bar{\nabla} \) is the covariant derivative induced by the background connection on \( E^0 \), cf. [7] for more details on the construction.
The vector space of test tensors is now defined as (this is a generalization of the definition used in [3])

$$T_c(SM, P) \cong \Gamma_c(\infty, M, \bigotimes D_\rho M \otimes \bigotimes TM \otimes \bigotimes E^0 \otimes \bigotimes E^1),$$

where $\bigotimes$ denotes the direct sum of the tensor product bundles of all orders. This defines a covariant functor from $\text{GSpMan}$ to $\text{Vec}_c$.

### 2.1 Functionals

In the functional approach [4, 8, 9], one considers the algebra of functionals on a vector space of configurations, in the present case

$$\wedge \mathcal{E}^\otimes(SM, P) \cong \bigoplus_{k=0}^{\infty} \wedge^k \mathcal{E}^\otimes(SM, P),$$

with

$$\wedge^k \mathcal{E}^\otimes(SM, P) \cong \{ B \in \Gamma^\infty(M^k, (D_\rho M)^k) | B \text{ antisymmetric} \}.$$

This space is equipped with its natural topology (uniform convergence of all derivatives on compact subsets). For an element $B \in \wedge \mathcal{E}^\otimes(SM, P)$, we denote by $B_k$ its component in $\wedge^k \mathcal{E}^\otimes(SM, P)$. On $\wedge \mathcal{E}^\otimes(SM, P)$, the conjugation $^*$ acts by fiberwise conjugation and reversal of the order of the arguments.

We now consider functionals on $\wedge \mathcal{E}^\otimes(SM, P)$, i.e., linear maps from this space into the complex numbers. The restriction of a functional $F$ to $\wedge^k \mathcal{E}^\otimes(SM, P)$ is denoted by $F_k$, and the grade $|F_k|$ of $F_k$ is $k$. The regular functionals, $\mathcal{F}_{\text{reg}}(SM, P)$, are those of the form

$$F_k(B) = \int \langle f_k, B_k \rangle (x_1, \ldots, x_k) d\bar{g}x_1 \ldots d\bar{g}x_k,$$

with $f_k \in \Gamma^\infty(M^k, (D_\rho M)^k)$, $f_k$ antisymmetric. Here $d\bar{g}x$ is the volume element corresponding to the background metric. $f_k$ is called the kernel of $F_k$. Here we used the obvious generalization of the pairing of sections of the double Dirac bundle. The support of a functional is defined as the support of its kernel,

$$\text{supp} F_k \equiv \text{supp}_M f_k \equiv \{ x \in M | (x, x_2, \ldots, x_k) \in \text{supp} f_k \text{ for some } x_i \}.$$

On $\mathcal{F}_{\text{reg}}(SM, P)$, one introduces an antisymmetric product $\wedge$, by defining the kernel of the product $F \wedge H$ as

$$(f \wedge h)_k(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{l=0}^{k} \sum_{\pi \in S_k} (-1)^{|\pi|} f_{\pi(1)}(x_{\pi(1)}, \ldots, x_{\pi(l)}) h_{k-l}(x_{\pi(l+1)}, \ldots, x_{\pi(k)})$$

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An involution on $\mathfrak{F}_{\text{reg}}(SM, P)$ is defined as

$$F^*(B) = \overline{F(B^*)}.$$ 

Finally, we equip $\mathfrak{F}_{\text{reg}}(SM, P)$ with the topology induced from the standard locally convex topology on $\Gamma_{\infty}^c(M^k, D^\oplus P^k)$ (uniform convergence of all derivatives on compact sets), the space of the kernels. The assignment $(SM, P) \mapsto \mathfrak{F}_{\text{reg}}(SM, P)$ is then a covariant functor from $GSp\text{Man}$ to $\text{Alg}$, where

$$\mathfrak{F}_{\text{reg}}(\chi)(F)(B) = F(\chi^* B) \quad (3)$$

for a morphism $\chi$.

The regular functionals do not allow for the description of local interactions or nonlinear observables, such as the current. In order to cure this, one allows for more general kernels $f_k$, namely compactly supported distributions fulfilling the wave front set condition

$$WF(f_k) \cap (\bar{V}_+^k \cup \bar{V}_-^k) = \emptyset, \quad (4)$$

where $\bar{V}_\pm$ is the closure of the dual of the forward/backward light cone. These are called the microcausal functionals. They also form an algebra $\mathfrak{F}(SM, P)$. It can be equipped with a topology such that it is a nuclear, locally convex vector space [4, 10], see also [11] for a detailed discussion of this topology. $\mathfrak{F}$ is then also a covariant functor from $GSp\text{Man}$ to $\text{Alg}$, with the action on morphisms as in (3).

The subspace $\mathfrak{F}_{\text{loc}}(SM, P)$ of $\mathfrak{F}(SM, P)$ in which the $f_k$’s are localized on the total diagonal $D_k$ with $WF(f_k) \perp TD_k$ is the space of local functionals. It is a covariant functor from $GSp\text{Man}$ to $\text{Vec}_i$. We note that by [12], Thm. 2.3.5, the kernels are then of the form

$$f_k(x_1, \ldots, x_k) = \int f(x)\delta^{\alpha_1}(x, x_1)\ldots\delta^{\alpha_k}(x, x_k)d\bar{g}_x, \quad (5)$$

where $f$ is a compactly supported smooth section and the $\alpha_i$ are multiindices.

We denote by $\mathfrak{F}_0(SM, P, X)$ the ideal of functionals that vanish on all on-shell configurations, i.e., on configurations fulfilling $D^\oplus B = 0$, where $D^\oplus$ acts on an arbitrary coordinate. We define the on-shell functionals as $\mathfrak{F}^{o.s.}(SM, P, X) = \mathfrak{F}(SM, P)/\mathfrak{F}_0(SM, P, X)$. This amounts to identifying two functionals if they agree on all on-shell configurations. This is also a covariant functor from $GSp\text{Man}$ to $\text{Alg}$.

Remark 2.1. $F \in \mathfrak{F}_0(SM, P, X)$ implies that $f_k = \sum_j D^\oplus_j g^j_k$, where the $g^j_k$ are compactly supported distributional sections on $M^k$ fulfilling the wave front set condition [11] and $D^\oplus_j$ acts on the $j$th coordinate. To see this for $k = 1$, choose a compactly supported smooth function $\chi$ such that $\chi = 1$ on supp $f_1$. Given $B_1$, define $\hat{B}_1 = S^\oplus_{\text{ret}}(\chi D^\oplus B_1)$. Then
\(D^\oplus (B_1 - \tilde{B}_1)|_{\text{supp} f_1} = 0,\) so that \(F(B) = F(\tilde{B})\). It follows that \(f_1\) can be written as \(f_1 = D^\oplus (\chi_{\text{adv}}(f_1))\). This straightforwardly generalizes to \(k > 1\).

Functional derivatives are defined as follows \([9]\):

\[
F^{(1)}(B)(u) = F(u \wedge B), \quad B \in \wedge D^\oplus, \quad u \in \mathcal{C}^\oplus.
\]

Hence, \(F^{(1)}(B)\) can be interpreted as a compactly supported distributional section of \(D^\oplus \rho_M\). We denote its integral kernel by \(F^{(1)}(B)(x)\). For \(F \in \mathcal{F}_{\text{reg}},\) this is even a smooth section. The functional \(B \mapsto F^{(1)}(B)(u)\) will in the following be denoted by \(F^{(1)}(u)\).

### 2.2 Quantization

Quantization in the sense of deformation quantization \([8]\) is straightforward for regular functionals, by defining the \(\star\) product via functional derivatives and convolution with \(S^\oplus\). In order to proceed to microcausal functionals, one uses Hadamard two-point functions:

**Definition 2.2.** A Hadamard two-point function is a distributional section \(\omega \in \Gamma^\infty_c(M^2, D^\oplus M^2)'\) fulfilling

\[
\omega(D^\oplus u, v) = 0, \quad \omega(u, v) + \omega(v, u) = iS^\oplus(u, v), \quad \omega(u, v) = \omega(u^*, u^*) , \quad \WF(\omega) \subset C_+ ,
\]

where \(u, v \in \Gamma^\infty_c(M, D^\oplus M)\) and

\[
C_\pm = \{(x_1, x_2; k_1, -k_2) \in T^* M^2 \setminus \{0\} | (x_1; k_1) \sim (x_2; k_2), k_1 \in V_{x_1}^\pm \}.
\]

Here \((x_1; k_1) \sim (x_2; k_2)\) if there is a lightlike geodesic joining \(x_1\) and \(x_2\) to which \(k_1\) and \(k_2\) are co-parallel and co-tangent. For \(x_1 = x_2, k_1, k_2\) are lightlike and coinciding.

Note that \((6)\) and \((7)\) imply that \(\omega\) is in fact a bi-solution.

The existence of Hadamard two-point functions for arbitrary backgrounds was proven in \([3]\). For a Hadamard two-point function \(\omega\), one defines

\[
F \star_\omega G = \wedge \exp(h\Gamma^\otimes_\omega) F \otimes G,
\]

where \(F, G \in \mathfrak{g}(SM, P)\), the wedge denotes the wedge product, \(\wedge(F \otimes G) = F \wedge G\), and

\[
\Gamma^\otimes_\omega(F \otimes G) = (-1)^{|F|+1} \int F^{(1)}(x) \otimes G^{(1)}(y) \omega(x, y) d_3 x d_3 y.
\]
Due to Remark 2.1 and the fact that $\omega$ is a bi-solution, this is well-defined also on $\mathfrak{F}^{\alpha\cdot s}(SM, P, \tilde{X})$. In order to make covariance explicit, consider the set $\text{Had}(SM, P, \tilde{X})$ of all Hadamard two-point functions on the background $(SM, P, \tilde{X})$, and define the space $\mathfrak{A}(SM, P, \tilde{X})$ of quantum functionals as the space of families

$$ F = \{ F_\omega \}_{\omega \in \text{Had}(SM, P, \tilde{X})}, \quad F_\omega \in \mathfrak{F}(SM, P)[[\hbar]] $$

fulfilling

$$ F_{\omega'} = \exp(\hbar \Gamma_{\omega' - \omega}) F_\omega, \quad (11) $$

where

$$ \Gamma_\omega F = \int \omega(x, y) F^{(2)}(x, y) d\tilde{g} x d\tilde{g} y. \quad (12) $$

An element $F$ of $\mathfrak{A}(SM, P, \tilde{X})$ is entirely specified by (11) and $F_\omega$ for a single $\omega \in \text{Had}(SM, P, \tilde{X})$. We equip $\mathfrak{A}(SM, P, \tilde{X})$ with the product

$$ (F * G)_\omega = F_\omega \tau_\omega G_\omega, $$

and the involution

$$ (F^*)_\omega = (F_\omega)^*. $$

The condition (5) ensures that this is consistent.

The assignment $(SM, P, \tilde{X}) \mapsto (\mathfrak{A}(SM, P, \tilde{X}), *)$ is a covariant functor from $\text{GSpMan}$ to $\text{Alg}$, with

$$ (\mathfrak{A}(\chi)F)_{\omega'} \doteq \mathfrak{F}[[\hbar]](\chi)(F_{\chi^* \omega'}), $$

with $\mathfrak{F}(\chi)$ defined by (3). We define the algebra $\mathfrak{A}^{\alpha\cdot s}(SM, P, \tilde{X})$ of on-shell functionals analogously to $\mathfrak{F}^{\alpha\cdot s}(SM, P, \tilde{X})$. The local elements $\mathfrak{A}_{\text{loc}}(SM, P, \tilde{X})$ of $\mathfrak{A}(SM, P, \tilde{X})$ are defined as those for which $F_\omega \in \mathfrak{F}_{\text{loc}}(SM, P)[[\hbar]]$ for one (and hence all) $\omega$. Again, $\mathfrak{A}_{\text{loc}}$ is a covariant functor from $\text{GSpMan}$ to $\text{Vec}$.

By evaluating a state on $\mathfrak{A}^{\alpha\cdot s}(SM, P, \tilde{X})$ on products of linear functionals, i.e., functionals of the form (2) with $k = 1$, multiplied with the $*$ product, one obtains the $n$-point functions of the state. The truncated $n$-point functions are defined as usual.

**Definition 2.3.** A **Hadamard state** is a state whose two-point function is a Hadamard two-point function and whose truncated $n$-point functions are smooth.

The importance of Hadamard states is that the set of all Hadamard states is identical to the set of continuous functionals on $\mathfrak{A}^{\alpha\cdot s}(SM, P, \tilde{X})$ [13].
2.3 Fields

In the framework of locally covariant field theory \[1\], fields allow to define functionals on different backgrounds in a coherent way. A field \( \Phi \) is a natural transformation from \( T_c \) to \( \mathfrak{F}_{\text{loc}} \), i.e., to each background \((SM,P,\bar{X})\) it associates a linear map \( \Phi_{(SM,P,\bar{X})}(t) : T_c(SM,P) \to \mathfrak{F}_{\text{loc}}(SM,P) \), such that, for a morphism \( \chi : (SM,P,\bar{X}) \to (SM',P',\bar{X}') \),

\[
\mathfrak{F}(\chi)\Phi_{(SM,P,\bar{X})}(t) = \Phi_{(SM',P',\bar{X}')}((T_c(\chi)t)). \tag{13}
\]

Leaving the test tensor open, we can then interpret the kernel of \( \Phi_{(SM,P,\bar{X})}(\cdot)_k \) as a distributional section on \( M^{k+1} \). We require that it is supported on the diagonal \( D_{k+1} \), has finite order, and fulfills

\[
\text{WF}(\Phi_{(SM,P,\bar{X})}(\cdot)_k) \perp TD_{k+1}.
\]

Hence, it can be written in a form analogous to \[15\], but with \( f \) not compactly supported. A further condition implying a smooth and (if applicable) analytic dependence on the background data will be given in Section 2.4 below.

A \( k \)-local field is a natural transformation from the functor \( T^\otimes_k \) to the functor \( \mathfrak{F}_{\text{loc}} \) (interpreted as a functor from \( \text{GSpMan} \) to \( \text{Vec} \)), preserving the support. \( k \)-local fields for \( k > 1 \) are also called multilocals. A quantum field is a natural transformation from the functor \( T_c \) to the functor \( A_{\text{loc}} \), which preserves the support, and analogously for multilocal quantum fields. Similarly, on-shell (quantum) (multilocal) fields are natural transformations to the corresponding on-shell functors.

Examples for fields are the monomials that map a test section \( t_k \in \Gamma^\infty_c(M,\wedge^k(D^\otimes \rho M \otimes \otimes TM)) \) to the functional

\[
\Phi_{(SM,P,\bar{X})}(t_k)(B) = \int (\gamma^\mu_1 \cdots \gamma^\mu_k, \nabla^{\oplus 1} \cdots \nabla^{\oplus k} B_k)(x, \ldots, x) d\bar{g}_x,
\]

where \( \mu_i \) is the multiindex corresponding to the \( \otimes TM \) part of the \( i \)th factor of \( \wedge^k(D^\otimes \rho M \otimes \otimes TM) \), and \( \nabla^{\oplus i} \) denotes the covariant derivative w.r.t. the \( i \)th coordinate. One can thus generate all local functionals by fields. Hence, when aiming at proving a statement for local functionals, one can equivalently prove it for fields.

An example for a field that we will encounter in the following is, for \( A \in \Gamma^\infty_c(M,E^1) \),

\[
j_{(SM,P,\bar{X})}(A)(B) \doteq \int (\gamma^\mu)^\alpha_\beta \rho(A_\mu(x))^\alpha_b(B_2)_{ab}^\beta(x, x) d\bar{g}_x. \tag{14}
\]

Here \( \alpha, \beta \) are double spinor indices, \( a, b \) are gauge indices, and we pick the component of \( B_2 \), whose first entry is in the dual Dirac bundle and the

\[\text{Note that here one has to generalize [12, Thm. 2.3.5] to non-compactly supported distributions, which is possible.}\]
second in the Dirac bundle. This is nothing but the Lie algebra valued current of Yang-Mills theories. In a succinct notation, we may write it as

\[ j^\mu_I = \psi^+ \gamma^\mu T_I \psi, \]

where \( I \) is a Lie algebra index and \( T_I \) the corresponding generator in the representation \( \rho \). As we will show below, the current can be obtained by differentiating the free Dirac Lagrangean \( S \), defined by

\[ S((SM, P, \bar{X}) (f))(B) = \int f(x)(D^2_{(SM, P, \bar{X})})^{\alpha \beta}(B_2)_{\alpha \beta}(x, x)d\bar{g}x, \]

w.r.t. the background connection. Here \( f \in C^\infty_c(M) \) and \( D^2 \) denotes the Dirac operator acting on the second coordinate.

**Remark 2.4.** It follows from (3) that, for a multilocal (quantum) field \( \Phi \) and a morphism \( \chi : (SM, P, \bar{X}) \to (SM', P', \bar{X}') \), we have

\[ \Phi((SM, P, \bar{X}))(\chi^* t_1, \ldots, \chi^* t_k)(B) = \Phi((SM, P, \bar{X}'))(t_1, \ldots, t_k)(\chi^* B). \]

As we have seen, the construction of fields is straightforward. However, the construction of quantum fields requires a parametrix:

**Definition 2.5.** A parametrix \( H \) is a quasi-covariant assignment \((SM, P, \bar{X}) \to H \in \Gamma^c_c(U, D^\rho M^2)'\), where \( U \) is a neighborhood of the diagonal of \( M^2 \), such that (7), (8), (9) hold. Quasi-covariance means that for \( \chi : D^\rho M \to D^\rho M' \) the bundle morphism corresponding to a morphism \((SM, P, \bar{X}) \to (SM', P', \bar{X}') \) we have that \( H - \chi^* H' \) is smooth on the common domain and vanishing at the diagonal, together with all the derivatives.

A construction prescription for parametrices was given in [3].

Given a parametrix \( H \), we may associate to a local functional \( F \in \mathcal{F}_{\text{loc}} \) an element of \( \mathcal{A}_{\text{loc}} \) by

\[ (\alpha_H(F))_\omega = \exp(h_{\omega - H})F, \]

where the operator \( \Gamma \) was defined in [12]. This is well-defined as \( H - \omega \) is smooth [3] and the values of all its derivatives on the diagonal are unambiguous. As we only act on local functionals, the expression is well-defined even though \( H \) is only defined in a neighborhood of the diagonal.

Let us now discuss scaling properties. One notices that on a scaled background, \( \bar{X}_\lambda \cong (\lambda^{-2} g, \bar{A}, \lambda \bar{m}) \), the Dirac operator also scales. It follows that there is a \(*\)-isomorphism \( \sigma_\lambda : \mathcal{A}(SM, P, \bar{X}_\lambda) \to \mathcal{A}(SM, P, \bar{X}) \), acting on linear fields as

\[ \sigma_\lambda(\psi(u))_\omega = \lambda^{-\frac{n+1}{2}} \psi(f)_\omega, \]

where \( \omega_\lambda(u, v) = \lambda^{-n-1}\omega(u, v) \), cf. [14] Lemma 4.2 for a proof in the scalar case. For a multilocal (on-shell) quantum field \( \Phi \), one defines another multilocal (on-shell) quantum field \( S_\lambda \Phi \) by

\[ (S_\lambda \Phi)_{(SM, P, \bar{X}_\lambda)}(t_1, \ldots, t_k) = \lambda^{nk} \sigma_\lambda(\Phi_{(SM, P, \bar{X}_\lambda)}(t_1, \ldots, t_k)). \]
The scaling dimension of a field $\Phi$ is defined as $\frac{n-1}{2}$ times the grade plus the scaling dimension of the geometric factors in $\Phi$ under the scaling $X \rightarrow \bar{X}_\lambda$.

2.4 Background variation

In the following, we will need a means to compare functionals defined on different backgrounds. Consider an object $(SM, P, \bar{X})$ of $\text{GSpMan}$. We may obtain another object $(SM, P, \bar{X} + X)$ by considering compactly supported perturbations $X = (g, A, m) \in \Psi(SM, P)$, where

$$\Psi(SM, P) = \Gamma_\infty^c(M, \text{Sym}^2 TM \oplus E^1 \oplus (M \times \mathbb{R})).$$

Here $g$ denotes the perturbation of the metric (which has to be chosen small enough in order to preserve the signature), $A$ denotes the variation of the gauge connection, and $m$ denotes the variation of the Yukawa potential. Note that the space of connections is an affine space, so indeed the deviations from a given connection can be parametrized by a vector space, the sections of $E_1$.

We will need to identify configurations or test tensors on $(SM, P, \bar{X})$ and $(SM, P, \bar{X} + X)$. For test tensors, i.e., elements of $\Sigma_c(SM, P)$, this is no problem for tangent vectors and one-forms, as they do not depend on the background structure. Only spinors may be problematic, as the spin structure depends on the metric. To deal with this, we proceed as follows: When changing $(SM, P, \bar{X})$ to $(SM, P, \bar{X} + X)$, we keep the spin and Dirac bundle, and just change the projection from the spin bundle $SM$ to the orthonormal frame bundle $FM$. For that, identify $FM$ with a principal $SO_0(n-1,1)$ bundle $LM$. To construct $(SM, P, \bar{X} + X)$, keep $LM$ and the projection from $SM$ to $LM$, but change the identification of $FM$ and $LM$. Fix a trivialization, so that $LM|_U \cong U \times SO_0(n-1,1)$, i.e., we are dealing with matrices $B$ such that $B^\sigma B^\mu = \eta$, where $\eta = \text{diag}(-1,1,\ldots,1)$. The identification of $LM$ and $FM$ for a metric $h$ is then a map $\pi_h$ such that $\pi_h(B)h\pi_h(B)^t = \eta$, i.e., it is given by a vielbein, $\pi_h(B)^a = B^a_b e^b$, with $h^{ab} = e^a_\mu \eta^{\mu\nu} e^b_\nu$. Changing the background metric then amounts to changing the vielbein. Infinitesimally, this change is given by $\delta e^a_\mu = -\frac{1}{2} e^a_\mu h^{\nu\lambda} \delta h_{\nu\lambda}$. This provides an identification of sections of the Dirac bundle on $(SM, P, \bar{X})$ and $(SM, P, \bar{X}')$. This also gives an identification of the corresponding test tensor spaces $\Sigma_c$ and configuration spaces $\wedge \mathfrak{e}^\otimes$. This procedure is the one used in [15] to compute the stress-energy tensor for Dirac fermions.

In the following, we will need to consider families of backgrounds $\bar{X}_s$, $s \in I$, where $I$ is an interval of $\mathbb{R}$. We will require that the modifications only affect a compact subset, i.e., $\bar{X}_s = \bar{X}_s'$ outside a compact set $K$ and for all $s, s'$. The family is assumed to be smooth in the sense that for any fixed $s_0$, $\bar{X}_s - \bar{X}_{s_0}$ is jointly smooth in $s$ and the spacetime point $x$. A compatible family of functionals is a family $F_s$, $s \in I$ such that $F_s \in \Psi(SM, P, \bar{X}_s)$. The
definition of compatible families of on-shell and/or quantum functionals is completely analogous.

**Definition 2.6.** Given \( F \in \mathfrak{F}(SM,P,\bar{X}_{s0}) \),

\[
\hat{F}_s(B) = i_{\bar{X}_s,\bar{X}_{s0}} F(B) \doteq F(i_{\bar{X}_{s0},\bar{X}} B)
\]
defines a compatible family of functionals, where \( i_{\bar{X}_{s0},\bar{X}} \) is the bijection from \( \wedge \mathfrak{C}(SM,P,\bar{X}_s) \) and \( \wedge \mathfrak{C}(SM,P,\bar{X}_{s0}) \) described above.

Note that this construction is not possible for on-shell or quantum functionals.

In order to have an identification of functionals on different backgrounds that is applicable also for on-shell or quantum functionals, we use Møller operators [16]. Consider two backgrounds \((SM,P,\bar{X})\) and \((SM,P,\bar{X}')\), where the background fields \( \bar{X} \) and \( \bar{X}' \) differ in some compact region \( K \). On \( \mathfrak{C}(SM,P) \), we define the retarded Møller operator as

\[
r_{\bar{X},\bar{X}'} u \doteq i_{\bar{X},\bar{X}'} u + S_{\text{ret}}^\oplus ([i_{\bar{X},\bar{X}'} \circ D^\oplus - D^\oplus \circ i_{\bar{X},\bar{X}'}) u].
\]

This is well defined, as the expression in square brackets is compactly supported. We have

\[
D^\oplus (r_{\bar{X},\bar{X}'} u) = i_{\bar{X},\bar{X}'} D^\oplus u, \quad \text{supp}(r_{\bar{X},\bar{X}'} u - i_{\bar{X},\bar{X}'} u) \subset J^+(K).
\] (19)

Obviously, this map is continuous, and commutes with the conjugation \( \ast \). It is invertible, the inverse being given by \( r_{\bar{X}',\bar{X}} \). We denote its transpose w.r.t. the pairing \( \langle \cdot, \cdot \rangle \) : \( \mathfrak{D}(SM,P) \times \mathfrak{C}(SM,P) \rightarrow \mathbb{C} \) by \( r_{\bar{X}',\bar{X}'} \).

On \( \wedge^k \mathfrak{C}(SM,P) \), the Møller operator is defined as the continuous linear operator which acts on elements \( u_1 \wedge \cdots \wedge u_k, u_i \in \mathfrak{C}(SM,P) \) as

\[
r_{\bar{X},\bar{X}'} (u_1 \wedge \cdots \wedge u_k) = (r_{\bar{X},\bar{X}'}, u_1) \wedge \cdots \wedge (r_{\bar{X},\bar{X}'}, u_k).
\]

The retarded Møller operator can be used to construct a \( \ast \)-isomorphism of the algebras \( \mathfrak{A}(SM,P,\bar{X}') \) and \( \mathfrak{A}(SM,P,\bar{X}) \):

**Definition 2.7.** For \( F \in \mathfrak{A}(SM,P,\bar{X}') \) we set

\[
(\tau_{\text{ret}}^\bar{X},\bar{X}') F_{\omega}(B) \doteq F_{\omega'}(r_{\bar{X}',\bar{X}} B),
\]

where \( \omega'(...) \doteq \omega(r_{\bar{X}',\bar{X}'}, r_{\bar{X}',\bar{X}'}, ...) \).

On elements of \( \mathfrak{F}(SM,P,\bar{X}') \), \( \tau_{\text{ret}} \) is defined in complete analogy.

**Remark 2.8.** To see that \( \omega' \) as defined in Definition 2.7 is a Hadamard two-point function, we first note that \( r_{\bar{X}',\bar{X}'} \circ D^\oplus = D^\oplus \circ i_{\bar{X},\bar{X}'} \), so that \( \omega' \) is a bi-solution. As the Møller operator commutes with the conjugation, also [8] holds. Furthermore, \( \omega' \) and \( \omega \) coincide outside the causal future of \( K \).
In particular, they coincide when restricted to a neighborhood of a Cauchy surface in the past of $K$, so that $\omega'$ fulfills (11) and (12) there. As $\omega'$ is a bisolution and has the correct Cauchy data, it fulfills (11) everywhere. That (12) holds everywhere follows from arguments given in [5, Sec. 4.2].

**Proposition 2.9.** The map $\tau_{ret}^{X,X'}$ is a $*$-isomorphism, which restricts to a $*$-isomorphism of the algebras of on-shell (quantum) functionals.

**Proof.** The homomorphism property follows from the form (11) of the $\star_\omega$ product and the definition of $\omega'$. The isomorphism property follows from the invertibility of the Møller operator, and the $*$ property by the fact that the involution $^*$ commutes with the Møller operator. Continuity follows from the Møller operator being continuous. The last statement follows from the fact that $D^{\Theta} (r^{X',X} u) = 0$ if and only if $D^{\Theta} u = 0$. \hfill $\Box$

The restriction of $\tau_{ret}$ to on-shell quantum functionals coincides with the retarded variation employed in [2].

**Proposition 2.10.** If $(SM, P, \bar{X})$ and $(SM, P, \bar{X}')$ as above are related by a morphism $\chi : (SM, P, \bar{X}') \to (SM, P, \bar{X})$ of $\text{GSpMan}$ and $\Phi$ is a multilocal on-shell (quantum) field, then

$$\tau_{ret}^{\bar{X}, \bar{X}'} (\Phi_{(SM,P,\bar{X})}(t_1, \ldots, t_k)) = \Phi_{(SM,P,\bar{X})}(\chi_* t_1, \ldots, \chi_* t_k).$$

**Proof.** As a straightforward consequence of (19) and the uniqueness of the Cauchy problem, $r^{\bar{X}',\bar{X}} u = \chi_* u$ for $D^{\Theta} u = 0$. Hence, also $\omega' = \chi_* \omega$. The claim then follows from Remark [2,4]. \hfill $\Box$

It is convenient to have an infinitesimal version of $\tau_{ret}$:

**Definition 2.11.** Let $X \in \mathfrak{P}(SM, P)$. Choose a smooth family $\bar{X}_s$ such that $\bar{X}_0 = \bar{X}$ and $\partial_s \bar{X}_s|_{s=0} = X$. Given a compatible family $F_s$ of (on-shell) (quantum) functionals, define

$$\delta_{ret}^X F \overset{13}{=} \frac{1}{ds} \tau_{ret}^{X,X} F_s|_{s=0}.$$ As a straightforward consequence of Proposition 2.9 it follows that $\delta_{ret}$ fulfills, for compatible families $F,G$ of quantum functionals,

$$\delta_{ret}^X (F \star G) = \delta_{ret}^X F \star G + F \star \delta_{ret}^X G,$$

$$\delta_{ret}^X F^* = (\delta_{ret}^X F)^*,$$

where

$$(F \star G)_s \overset{13}{=} F_s \star_s G_s, \quad (F^*)_s \overset{13}{=} F^*_s.$$
For fields, yet another way to construct a compatible family of functionals is possible: Given a $k$-local (on-shell) (quantum) field $\Phi$ and a set of corresponding test tensors $t_k \in \mathcal{T}_c(SM, P, \bar{X}_s)$, we may define
\[
\tilde{\Phi}(t_1, \ldots, t_k)_{s} \doteq \Phi_{(SM, P, \bar{X}_s)}(i_{\bar{X}_s, \bar{X}_0} t_1, \ldots, i_{\bar{X}_s, \bar{X}_0} t_k).
\] (22)

A straightforward consequence of Proposition 2.10 is then:

**Lemma 2.12.** Let $\Phi$ be a multilocal on-shell (quantum) field and $\chi_s$ be a family of morphisms $(SM, P, \bar{X}_s) \to (SM, P, \bar{X})$ with $\bar{X} = X_0$ and $\bar{X}_s - \bar{X}$ compactly supported. Let $X = \partial_s \bar{X}_s|_{s=0}$. Then
\[
\delta_{\text{ret}} X \tilde{\Phi}(t_1, \ldots, t_k) = \sum_j \Phi_{(SM, P, \bar{X}_s)}(t_1, \ldots, \mathcal{L}_X t_j, \ldots t_k),
\] where $\mathcal{L}_X t \doteq d\left. d_{s=0} \chi_s(i_{\bar{X}_s, \bar{X}} t)|_{s=0} \right.$.

In the following, we require a smooth dependence of fields on the background. Denoting by $\tilde{\Phi}(x)$ the distributional kernel of $\Phi(\cdot)$, consider the kernel of $(i_{\bar{X}_s, X_0} \tilde{\Phi}(x))_k$ as a distributional section on $I \times M^{k+1}$. We require that its wave front set is contained in
\[
\{(s, x, x_1, \ldots, x_k; \sigma, \xi, \xi_1, \ldots, \xi_k) \in \mathcal{T}^*(I \times M^{k+1}) | x = x_1 = \cdots = x_k, \xi + \sum_i \xi_i = 0, (\xi, \xi_1, \ldots, \xi_k) \neq 0 \}.
\]
In the case of an analytic family of analytic backgrounds, the same should be true for the analytic wave front set, cf. [12, 17].

Knowing that our fields are smooth w.r.t. variations of the background, we may consider the derivative of fields w.r.t. to such variations:

**Definition 2.13.** Let $X \in \Psi(SM, P)$. Choose a smooth family $\bar{X}_s$ such that $\bar{X}_0 = \bar{X}$ and $\partial_s \bar{X}_s|_{s=0} = X$. Then the functional differentiation of a field $\Phi$ w.r.t. the background is defined as
\[
\Phi^{(1)}_{(SM, P, \bar{X})}(X, t) = \left. \partial_s (i_{\bar{X}_s, X_0} \tilde{\Phi}(t)|_{s=0}) \right.
\] (23)

By definition we have that $\partial_s (i_{\bar{X}_s, \bar{X}} \tilde{\Phi}_{(SM, P, X)}(t)|_{s=0}) = 0$, cf. Definition 2.6

An obvious consequence is that
\[
\partial_s \left( i_{\bar{X}_s, \bar{X}_s} \left( \tilde{\Phi}(t) - \tilde{\Phi}_{(SM, P, X)}(t)|_{s=0} \right) \right)|_{s=0} = \Phi^{(1)}_{(SM, P, \bar{X})}(X, t).
\] (24)

We will impose one more condition on the smoothness of fields: Leaving the two slots for test tensors open, we may consider the kernel of $\Phi^{(1)}_{(SM, P, \bar{X})}(\cdot, \cdot)_k$ as a distributional section on $M^{k+2}$. From [13], it is clear that it has support on the total diagonal $D_{k+2}$. We also require it to have
finite order and wave front set orthogonal to $TD_{k+2}$. In particular, $\Phi^{(1)}$ can be seen as a field, where the test tensor is some linear combination of $\nabla_{(\lambda_1 \cdots \nabla_{\lambda_r})}X \otimes \nabla_{(\rho_1 \cdots \nabla_{\rho_s})}t$. We denote this new test tensor by $X \ast t$ in the following, so that, when we wish to emphasize that $\Phi^{(1)}$ is a field, we write $\Phi^{(1)}(SM,P,X)(X \ast t)$ instead of $\Phi^{(1)}(SM,P,X)(X,t)$.

Definition 2.14. The current is given by

$$j_{(SM,P,\bar{X})}(A) = \delta^{(1)}_{(SM,P,\bar{X})}(A,f),$$

with $f$ chosen to be identical to $1$ on the support of $A$. By the above, this defines a field. This coincides with (14).

Finally, we want to consider how $\delta^{X}_{ret}$ behaves under scaling, cf. (17). From the definition of $r^{X,X'}$ and the fact that the Dirac operator scales under the scaling transformation $\bar{X} \rightarrow \lambda \bar{X} = (\lambda^{-2} \bar{g}, \bar{A}, \lambda \bar{m})$, we conclude that $r^{X,X'} = r^{\lambda X,\lambda X'}$, and hence

$$\delta^{X}_{ret} \sigma_{\lambda} = \sigma_{\lambda} \delta^{X_{\lambda}}$$

on a family of (on-shell) (quantum) functionals compatible with a family of backgrounds $\bar{X}$, with $\bar{X}_0 = \bar{X}$ and $\partial_s \bar{X}_s|_{s=0} = X_{\lambda} = (\lambda^{-2} g, A, \lambda m)$.

### 2.5 Time-ordered products

In order to describe interacting fields, one has to introduce time-ordered products. These are natural transformations from $F^{\otimes k}_{loc}$ to $A^o$, interpreted as functors from $GSpMan$ to $Vec_i$, for arbitrary $k$, fulfilling certain axioms.

By concatenation with fields, the time-ordered products are thus multilocal on-shell quantum fields. The axioms for time-ordered products are [17]:

#### Starting element:
On a c-number functional, i.e., a functional satisfying $F(B) = F_0$ for all $B \in \wedge E$, $T$ acts as the identity.

#### Symmetry:
Time-ordered products are graded symmetric, i.e.,

$$T(F_1, \ldots, F_i, F_{i+1}, \ldots, F_k) = (-1)^{|F_i||F_{i+1}|} T(F_1, \ldots, F_{i+1}, F_i, \ldots, F_k).$$

#### Support:
$\text{supp } T(F_1, \ldots, F_k) \subset \bigcup_i \text{supp } F_i$.

#### Causal factorization:
Let $F_i, G_j$ be such that $\text{supp } F_i \cap J^- (\text{supp } G_j) = \emptyset$ for all $i, j$. Then

$$T(F_1, \ldots, F_k, G_1, \ldots, G_l) = T(F_1, \ldots F_k) \ast T(G_1, \ldots G_l).$$

For some purposes, it is convenient to consider time-ordered products that map into the off-shell quantum fields, cf. [13], for example. For our purposes, the on-shell quantum fields are sufficient.
Scaling: The time-ordered products scale almost homogeneously, i.e., if applied to fields $\Phi_i$ we have
\[
\lambda^{-d_\Phi} S_\lambda T(\Phi_1(t_1), \ldots, \Phi_k(t_k)) = \\
\sum_{I_0 \sqcup \cdots \sqcup I_j} (-1)^{\Pi} T(\Phi_{I_0}(t_{I_0}), r_\lambda(\Phi_{I_1}(t_{I_1})), \ldots, r_\lambda(\Phi_{I_j}(t_{I_j}))).
\] (27)

Here the sum is over all partitions of $\{1, \ldots, k\}$ into disjoint subsets, with $I_i \neq \emptyset$ for all $i \geq 1$. $\Phi_{I_i}$ stands for the collection of $\Phi_i$ with $i \in I$, and $\Pi$ is a combinatorial factor which takes the grades and permutations of the $\Phi_i$ into account. $d_\Phi$ is the sum of the scaling dimensions of the $\Phi_i$. Finally, $r_\lambda$ are natural transformations from $F \otimes_{\text{loc}}$ to $F_{\text{loc}}$, which fulfill the properties of renormalization maps discussed in Remark 2.15 below, and which are polynomials in $\log \lambda$.

Microlocal spectrum condition: Let $\omega$ be a Hadamard state. Then for all fields $\Phi_i$ the distributional section $\omega(T(\Phi_1(x_1), \ldots, \Phi_k(x_k)))$ has a wave front set contained in $C^k_T \subset T^* M^k$, defined through decorated graphs, cf. [17, 19].

Smoothness: The time-ordered products depend smoothly on the background fields [17]. Thus, let $\tilde{X}_s$ depend smoothly on a parameter $s \in \mathbb{R}$. Let $\omega^{(s)}$ be a family of Hadamard states on $\mathfrak{A}(SM, P, \tilde{X}_s)$, smoothly depending on $s$. One then requires that, for all fields $\Phi_i$,
\[
\WF\left(\omega^{(s)} \left( T^{(s)}(\tilde{\Phi}_1(x_1)_s, \ldots, \tilde{\Phi}_k(x_k)_s) \right) \right) \\
\subset \left\{ (s, \sigma; \{x_i, \xi_i\}) \in T^*(\mathbb{R} \times M^k)(\{x_i, \xi_i\}) \in C^k_T(s) \right\},
\]
where we used (22).

Analyticity: In the case of an analytic spacetime, the time-ordered products depend analytically on the background fields. This is made precise by a condition analogous to the one for smoothness, cf. [17].

Expansion: The time ordered product commutes with functional differentiation, i.e.,
\[
T(F_1, \ldots, F_k)^{(1)}(x) = \\
\sum_{i=1}^k (-1)^{\sum_{i=1}^{i-1} |F_i|} T(F_1, \ldots, F_i^{(1)}(x), \ldots, F_k).
\] (28)

Unitarity: We have
\[
T(F_1, \ldots, F_k)^* = \\
\sum_{I_0 \sqcup \cdots \sqcup I_j} (-1)^{k+j+\Pi} T(F_{I_1}^*, \cdots, F_{I_j}^*),
\] (29)
where $I_1 \sqcup \cdots \sqcup I_j$ denotes all partitions of $\{1, \ldots, k\}$ into nonempty, pairwise disjoint subsets. $\mathcal{T}(F^*_i)$ stands for $\mathcal{T}(F^*_{i(1)}, \ldots, F^*_{i(j)})$ and $\Pi$ denotes a combinatorial factor, depending on the grades of the $F_i$ and the partition, which accounts for the reordering of the $F_i$ on the right hand side.

**Source term:** For a linear functional $F(B) = \int \langle f, B \rangle \, dg \, x$, with $f \in \mathcal{D}^\otimes(\mathcal{SM}, P)$, we have

$$\mathcal{T}(F, F_1, \ldots, F_k) = F \ast \mathcal{T}(F_1, \ldots, F_k)$$

$$+ i\hbar \sum_j (-1)^{\sum_i |I_i|} \Pi |I_j| \mathcal{T}(F_1, \ldots, F_j^{(1)}(S^{\otimes \rho}_r f), \ldots, F_k). \quad (30)$$

Given a parametrix $H$, the map $\alpha_H$ defined in (16) defines time-ordered products for $k = 1$.

We note that condition (30) corresponds to axiom T11a of [2]. It implies the axiom of the field equation, which was used in [3]. The proof that it can be fulfilled by a redefinition of the time-ordered products proceeds as in the scalar case, cf. [2, Section 6.1].

**Remark 2.15.** The time-ordered products are not unique. Given time-ordered products $\mathcal{T}$, one can define new ones, $\mathcal{T}'$, by

$$\mathcal{T}'(F_1, \ldots, F_k) = \sum_{I_0 \sqcup \cdots \sqcup I_j} (-1)^{\Pi} \mathcal{T}(F_{I_0}, r^{I_1}(F_{I_1}), \ldots, r^{I_j}(F_{I_j})), \quad (27)$$

where we used the same notation as in (27) and $r^k, k \leq 1$ are natural transformations from $\mathcal{F}^{\otimes k}_{\text{loc}}$ to $\mathcal{F}_{\text{loc}}$. They are at least of first order in $\hbar$, lower the degree in the total grade by even numbers, and fulfill Symmetry, Support, Scaling, Expansion. As we can see a local functional as a field, we may write, for arbitrary fields $\Phi_i$,

$$r^k(\Phi_1(x_1), \ldots, \Phi_k(x_k)) = \int C_k(x, x_1, \ldots, x_k) \Phi(x) \, dg \, x,$$

where $\Phi$ is a field and $C$ a distributional section on $M^{k+1}$, supported on $\mathcal{D}_{k+1}$. Of course both $C$ and $\Phi$ depend on the $\Phi_i$. In order to ensure the **Microlocal spectrum condition**, we require

$$\text{WF}(C_k) \perp TD_{k+1}.$$  

In order to ensure **Smoothness**, one requires that for smooth variations $\bar{X}_s$ of the background field,

$$\text{WF}(C^s_k(x, x_1, \ldots, x_k)) \perp T(I \times \mathcal{D}_{k+1}),$$

17
where $I$ is the interval on which $s$ is varied. Analogously, one defines the **Analyticity** condition. To ensure **Starting element**, $r^1$ has to vanish on c-number functionals, and to ensure **Unitarity** one requires

$$r^k(F_1, \ldots, F_k)^* = (-1)^{k+1} r^k(F_k^*, \ldots, F_1^*).$$

Finally, to preserve **Source term**, one requires $r^k$ to vanish if one of its entries is a linear functional. This exhausts the renormalization freedom of time-ordered products [14].

Given an interaction Lagrangean $S^{\text{int}}$, i.e., a field $S^{\text{int}}_{(SM,P,\bar{X})}(f)$ for $f \in C^\infty_c(M)$, one can define interacting (multilocal) quantum fields as follows: One first restricts attention to an open causally convex region $O \subset M$ and to test sections $t$ supported inside $O$. One then chooses a test function $h$, which is identical to 1 on the closure of $O$, and sets, for arbitrary fields $\Phi$,

$$(\Phi_{S^{\text{int}}(h)}(O))_{(SM,P,\bar{X})}(t) = \sum_{k=0}^{\infty} \frac{i^k}{k!} R(\Phi_{S^{\text{int}}(h)}(t) ; S^{\text{int}}_{(SM,P,\bar{X})}(h), \ldots, S^{\text{int}}_{(SM,P,\bar{X})}(h)), \tag{31}$$

where the **retarded product** $R$ is defined by

$$R(e^{iF}; e^{iG}) \equiv \mathcal{T}(e^{i(F+G)}) \mathcal{T}(e^{i(G-F)}), \quad \mathcal{T}(e^{iF}) \equiv \sum_{k=0}^{\infty} \frac{i^k}{k!} \mathcal{T}(F, \ldots, F). \tag{31}$$

The definition [31] has to be understood in the sense of formal power series in $F$ and $G$. The interacting quantum field is a formal power series in $S^{\text{int}}$ (and also $\hbar$). The algebra $\mathfrak{A}_{S^{\text{int}}}(O)$ generated by the $(\Phi_{S^{\text{int}}(h)}(O))_{(SM,P,\bar{X})}(t)$’s does not depend on the choice of the cut-off function $h$. By taking the inductive limit $O \to M$, one obtains the full interacting algebra, cf. [19] for details. This is the **algebraic adiabatic limit**. Time-ordered products taking values in $\mathfrak{A}_{S^{\text{int}}}(O)$ can be defined by

$$\mathcal{T}_{S^{\text{int}}}(F_1, \ldots, F_k) = \sum_{j=0}^{\infty} \frac{i^j}{h^j j!} R(F_1, \ldots, F_k ; S^{\text{int}}(h), \ldots, S^{\text{int}}(h)), \tag{31}$$

where the $F_i$ are supported in $O$.

### 3 The principle of perturbative agreement

The principle of perturbative agreement requires that it does not matter whether one puts parts of the free Lagrangean into the interaction Lagrangean. This means that, for arbitrary local functionals $F_i \in \mathfrak{F}_{\text{loc}}(SM, P, \bar{X})$,
and arbitrary background fields $\bar{X}$, $\bar{X}'$, differing in a compactly supported region, one has

$$
\tau_{\text{ret}}^{\bar{X},\bar{X}'} \mathcal{R}(SM, P, \bar{X}); e^{i(S_{\bar{X}' - S_{\bar{X}}}/\hbar)}.
$$

Here we used the identification of elements of $\mathfrak{g}(SM, P, \bar{X})$ and $\mathfrak{g}(SM, P, \bar{X}')$ introduced in Section 2.4.

As such, the above equation is not meaningful, as the r.h.s. is a formal power series in $S_{\bar{X}' - S_{\bar{X}}}$, whereas the l.h.s. is not. To cure this, we consider the infinitesimal version:

$$
\delta_{\text{ret}}^{\bar{X}} \mathcal{T}(\tilde{F}_1, \ldots, \tilde{F}_k) = i\hbar^{-1} \mathcal{R}(F_1, \ldots, F_k; S^{(1)}(X, f))
$$

where on the r.h.s. one chooses an arbitrary $f$ which is equal to 1 on $\text{supp } X$.

Using (24), we may write this for fields in a form similar to the one used in [2]:

$$
\delta_{\text{ret}}^{\bar{X}} \mathcal{T}(\Phi_1(t_1), \ldots, \Phi_k(t_k)) = i\hbar^{-1} \mathcal{R}(\Phi_1(t_1), \ldots, \Phi_k(t_k); S^{(1)}(X, f))
$$

$$
+ \sum_j \mathcal{T}(\Phi_1(t_1), \ldots, \Phi_j^{(1)}(X, t_j), \ldots, \Phi_k(t_k)).
$$

For variations of the gauge background, this means

$$
\delta_{\text{ret}}^{\bar{A}} \mathcal{T}(\Phi_1(t_1), \ldots, \Phi_k(t_k)) = i\hbar^{-1} \mathcal{R}(\Phi_1(t_1), \ldots, \Phi_k(t_k); j(A))
$$

$$
+ \sum_j \mathcal{T}(\Phi_1(t_1), \ldots, \Phi_j^{(1)}(A, t_j), \ldots, \Phi_k(t_k)).
$$

(33)

**Remark 3.1.** Often, we will be considering gauge transformations, i.e., $A = \bar{d}c, c \in \Gamma^\infty(M, E^0)$, and make use of Lemma 2.12. In that case, the family of morphisms maps the background connection $\bar{A} + s \bar{d}c$ to the background connection $\bar{A}$, i.e., $\chi_s$ performs the gauge transformation $\bar{A} + s \bar{d}c \to \bar{A}$. The corresponding push-forward action on test tensors is $\chi_s \ast t = t + s \rho(c)t$, i.e., we define $\mathcal{L}_e t \doteq \rho(c)t$. For a $k$-local field $\Phi$ we define

$$
\partial_s \Phi(t_1, \ldots, t_k) \doteq \Phi^{(1)}(\bar{d}c, t_1, \ldots, t_k),
$$

(34)

which is a $k + 1$-local field.

\footnote{For nonabelian gauge groups, the family $\chi_s$ of gauge transformations generated by $A = \bar{d}c$ is not affine linear in $s$. However, for the present purposes only the linear component matters, so we stick to this inaccurate notation.}
3.1 Current conservation

In [2] it was shown that the principle of perturbative agreement for variations of the metric background implies the conservation of the stress-energy tensor. We want to perform the corresponding analysis for the gauge current. We define the field

\[
(\bar{\delta}j)_{\{SM,P,\bar{X}\}}(c) = j_{\{SM,P,\bar{X}\}}(\bar{dc}), \quad c \in \Gamma_c^\infty(M, E^0).
\]  

(35)

It is easily checked that as an element of \( \mathfrak{F}(\{SM,P,\bar{X}\}) \), this vanishes on all on-shell configurations. As the grade of this functional is 2, it follows that its image under \( T \), i.e., \( \alpha_H \), cf. (16), is a c-number. The principle of perturbative agreement implies a Ward identity for \( \bar{\delta}j \):

**Proposition 3.2.** If \( G \) is semisimple and (33) holds, then for arbitrary fields \( \Phi_i \),

\[
i^{-1}T(\Phi_1(t_1), \ldots, \Phi_k(t_k), (\bar{\delta}j)(c)) = \sum_j T(\Phi_1(t_1), \ldots, (\Phi_j(\mathcal{L}_ct_j) - \Phi_j(1)(\bar{dc},t_j)), \ldots \Phi_k(t_k)).
\]

(36)

This also holds if \( G \) is not semisimple but

\[
T((\bar{\delta}j)(c)) = 0,
\]

(37)

for all \( c \in \Gamma_c^\infty(M, E^0) \).

**Proof.** The proof closely follows the proof of [2, Thm 5.1]. We begin by showing (37) for semisimple \( G \). We consider an infinitesimal gauge transformation \( A = \bar{dc}' \), with \( c' \in \Gamma_c^\infty(M, E^0) \). It follows from Lemma 2.12 Remark 3.1 and the fact that \( T \) applied to a field yields a quantum field that

\[
\delta^{dc'}_\text{ret} T((\bar{\delta}j)(c)) = T((\bar{\delta}j)(c' \wedge c)).
\]

On the other hand, from (33), we have

\[
\delta^{dc'}_\text{ret} T((\bar{\delta}j)(c)) = i^{-1}R((\bar{\delta}j)(c); (\bar{\delta}j)(c')) + T((\bar{\delta}j)(1)(\bar{dc}', c))
\]

We now interchange the role of \( c \) and \( c' \) and subtract the resulting identity. We obtain

\[
2T((\bar{\delta}j)(c' \wedge c)) = i^{-1}R((\bar{\delta}j)(c); (\bar{\delta}j)(c')) - i^{-1}R((\bar{\delta}j)(c'); (\bar{\delta}j)(c))
\]

\[
+ T((\bar{\delta}j)(1)(\bar{dc}', c)) - T((\bar{\delta}j)(1)(\bar{dc}, c'))
\]

(38)

For the first two terms on the r.h.s., we get

\[
i^{-1}[T((\bar{\delta}j)(c), T((\bar{\delta}j)(c'))]_r.
\]
This is a commutator of c-numbers, so it vanishes. The last two terms in (38) yield
\[ T(\partial_c \partial_{c'} S(f)) = T(\partial_{c',c} S(f)), \]
where we choose \( f \) to be equal to one on the intersection of the supports of \( c \) and \( c' \). We used the notation (34) and computed, for some functional \( F \) of the background connection, in local coordinates,
\[
\partial_c \partial_{c'} F[A_\mu] = \partial_c \frac{d}{dt} F[A_\mu + t \partial_\mu c + t[A_\mu, c]]|_{t=0} \\
= \frac{d}{ds} \frac{d}{dt} F[A_\mu + s \partial_\mu c' + s[A_\mu, c'] + t \partial_\mu c + t[A_\mu + s \partial_\mu c' + s[A_\mu, c']]|_{s=t=0}.
\]
Taking the difference with \( \partial_c \partial_{c'} F[A] \) yields \([\partial_{c'}, \partial_c] = \partial_{c',c} \). Hence, (38) gives
\[ T((\delta j)(c' \wedge c)) = 0. \]
As by assumption \( g \) is semisimple, (37) follows.

It remains to prove (36). Using Lemma 2.12, Remark 3.1, and (33) for \( A = \bar{\partial} c \), we obtain
\[
\sum_j T(\Phi_1(t_1), \ldots, \Phi_j(\mathcal{L}_c t_j) - \Phi_j^{(1)}(\bar{\partial} c, t_j), \ldots, \Phi_k(t_k)) = i \hbar^{-1} R(\Phi_1(t_1), \ldots, \Phi_k(t_k); (\delta j)(c)),
\]
so that, using
\[
R(\Phi_1(t_1), \ldots, \Phi_k(t_k); \Psi(t)) = T(\Phi_1(t_1), \ldots, \Phi_k(t_k), \Psi(t)) \\
- T(\Psi(t)) \ast T(\Phi_1(t_1), \ldots, \Phi_k(t_k))
\]
and (37), we obtain (36). \( \square \)

Remark 3.3. We may define the gauge transformation on a functional \( F \) by
\[
(\mathcal{L}_c F)(B) = F(\mathcal{L}_c B),
\]
where, as in Remark 3.1 \( \mathcal{L}_c B = \rho(c) B \). For a field \( \Phi \), we have
\[
\Phi(\mathcal{L}_c t) + \mathcal{L}_c (\Phi(t)) - \Phi^{(1)}(\bar{\partial} c, t) = 0.
\]
Hence, we may write (36) in a form more similar to [2 Thm. 5.1] as
\[
i \hbar^{-1} T(\Phi_1(t_1), \ldots, \Phi_k(t_k), (\delta j)(c)) = \\
- \sum_j T(\Phi_1(t_1), \ldots, \mathcal{L}_c(\Phi_j(t_j)), \ldots \Phi_k(t_k)).
\]
As shown in [3], the parametrix can be chosen such that (37) holds. Then the remaining ambiguity in the parametrix consists of changes that modify $j$ by adding a locally and covariantly constructed covector $j'$ that is conserved, $\delta j' = 0$. The only such covector is the current $j$ responsible for the background field. In particular, the current is unique in the absence of background currents.

Analogously to the metric variations [2, Thm. 5.3], there is also a Ward identity for the interacting field. One defines the current as

$$j^\text{int}(S,M,P,\overline{X})^i = (S + S^\text{int}(1))(A,f)$$

with $A \in \Gamma^\infty_c(M,E^1)$ and $f = 1$ on supp $A$. Its divergence $\overline{\delta}j^\text{int}$ is defined as in (35). Then we have:

**Proposition 3.4.** If (33) and (37) holds, then for arbitrary fields $\Phi_i$,

$$i^\hbar -1 T_{S^\text{int}}(\Phi_1(t_1),\ldots,\Phi_k(t_k), (\overline{\delta}j^\text{int})(c)) = -\sum_{j} T_{S^\text{int}}(\Phi_1(t_1),\ldots,\mathcal{L}_c\Phi_j(t_j),\ldots,\Phi_k(t_k)).$$

**Proof.** We proceed as in [2, Thm. 5.3]. We first show the result for the case $k = 0$. However, to facilitate the proof for other $k$, we allow for localized sources in the interaction Lagrangean, i.e., we allow for

$$\hat{S}^\text{int}(f,t_1,\ldots,t_k) = S^\text{int}(f) + \Phi_1(t_1) + \cdots + \Phi_k(t_k),$$

where $S^\text{int}$ is as before and the test tensors $t_i$ are arbitrary. Note that $\hat{S}^\text{int}$ is a sum of fields. Let us for simplicity assume that $S^\text{int}$ and the $\Phi_i$ have even grade. The changes necessary to account for generic grades are straightforward. By Proposition 3.2, we have

$$i^\hbar -1 T_{\hat{S}^\text{int}}(\hat{S}^\text{int}(f,t),\ldots,\hat{S}^\text{int}(f,t), (\overline{\delta}j)(c)) =$$

$$- lT \left( \left( (\overline{\delta}j^\text{int})(c) - (\overline{\delta}j)(c) + \sum_{i} \mathcal{L}_c\Phi_i(t_i) \right), \hat{S}^\text{int}(f,t),\ldots,\hat{S}^\text{int}(f,t) \right) ,$$

where $\hat{S}^\text{int}(f,t)$ stands for $\hat{S}^\text{int}(f,t_1,\ldots,t_k)$. It follows that

$$\sum_{l=0}^{\infty} \frac{i^l}{l!^l} T_{((\overline{\delta}j^\text{int})(c), \hat{S}^\text{int}(f,t),\ldots,\hat{S}^\text{int}(f,t))} = - \sum_{l=0}^{\infty} \frac{i^l}{l!^l} \sum_{i} T_{\mathcal{L}_c\Phi_i(t_i), \hat{S}^\text{int}(f,t),\ldots,\hat{S}^\text{int}(f,t)},$$

With $k = 0$, this proves (40) for $k = 0$, namely the vanishing of the divergence of the interacting current. The general result follows from replacing $\Phi_i(t_i)$ by $\lambda_i \Phi_i(t_i)$ and evaluating the derivatives w.r.t. all $\lambda_i$ at zero.  \[\square\]
3.2 The background field method

Let us now discuss an application of perturbative agreement, namely the background field method. In [3] it was already used to compute the fermion contribution to the renormalization group flow at the one-loop level. Concretely, we split the gauge connection into a background $\bar{A}$ and a perturbation $A$. The Dirac Lagrangean at first order in $A$ is just $j(A)$, which is now seen as an interaction term for $A$. However, as we are only interested in the fermion contribution, we may consider $A$ as parameter, instead of a field. We compute the renormalization group flow of this term by the scaling behavior, cf. [20]. This gives the renormalization group flow at first order in $A$, which was computed to be

$$r_\lambda(j(A)) \propto \hbar \int \langle \bar{d}A, F \rangle(x) dg x. \quad (41)$$

Here $F$ is the curvature of the background connection, and the pairing on sections of $E^n$ is fiberwisely defined by

$$\langle [p, \xi] \otimes \omega, [p, \eta] \otimes \nu \rangle = \kappa(\xi, \eta) \langle \omega, \nu \rangle_{\bar{g}}, \quad p \in SM + P, \xi, \eta \in g, \omega, \nu \in \Omega^k_{\pi(p)}.$$ 

Here $\kappa$ is the Killing form and $\langle \cdot, \cdot \rangle_{\bar{g}}$ is the pairing of forms induced by the metric $\bar{g}$. Assuming that the flow does not depend on the splitting into $\bar{A}$ and $A$, i.e., only depends on $\bar{A} + A$, this gives the renormalization group flow to all orders of $A$, namely the Yang–Mills action. It remains to show that this assumption is justified. Not surprisingly, it is a consequence of perturbative agreement. To see this, we recall [20] that the renormalization group flow of the $S$ matrix for an interaction term $S_{\text{int}}$ is determined by

$$r_\lambda(e^{ij(A)/\hbar})_{(1)}(A') = i\hbar^{-1} \left( r_\lambda(e^{ij(A)/\hbar}, j(A')) - r_\lambda(j(A')) \right). \quad (42)$$

**Proof.** To see that $r_\lambda(j(A)^k)$ is a c-number field, recall that $r_\lambda$ fulfills the requirements discussed in Remark 2.15. In particular, it vanishes if one of the arguments is a linear field and fulfills Expansion. As $j(A)$ is quadratic, it follows that the functional derivative of $r_\lambda(j(A)^k)$ vanishes, i.e., it is a c-number.

In order to prove the second statement, we note that, as a consequence of the scaling formula (27), we have, with $A_\lambda = \lambda^n A$,

$$S_\lambda T \left( e^{ij(A)/\hbar} \right) = T \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) \right). \quad (43)$$

---

By this we mean, in the usual Feynman graph notation, that all internal lines are fermions.
Furthermore, by (33),

\[ \delta_{\text{ret}}^{A'} \mathcal{T} \left( e^{ij(A)/\hbar} \right) = i\hbar^{-1} \mathcal{R} \left( \exp \left( ij(A)/\hbar ; j(A') \right) \right), \]

where we used that the \( j(A) \) are independent of the background connection. Using (39) and

\[ \mathcal{T} \left( e^{ij(A)/\hbar} ; j(A') \right) = -i\hbar \frac{d}{ds} \mathcal{T} \left( e^{ij(A+sA')/\hbar} \right)|_{s=0}, \]

we obtain

\[ S_\lambda \mathcal{R} \left( e^{ij(A)/\hbar} ; j(A') \right) = \mathcal{T} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) ; j(A'_\lambda) \right) \]

\[ + \mathcal{T} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) , r_\lambda(e^{ij(A_\lambda)/\hbar}) , j(A'_\lambda) \right) \]

\[ - \left( \mathcal{T} (j(A'_\lambda)) + r_\lambda (j(A'_\lambda)) \right) \mathcal{T} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) \right), \]

and hence

\[ -i\hbar S_\lambda \delta_{\text{ret}}^{A'} \mathcal{T} \left( e^{ij(A)/\hbar} \right) = \mathcal{R} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) ; j(A'_\lambda) \right) \]

\[ + \left( r_\lambda(e^{ij(A_\lambda)/\hbar}) , j(A'_\lambda) - r_\lambda(j(A'_\lambda)) \right) \mathcal{T} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) \right). \]

(44)

On the other hand, due to (33) and (43), we have

\[ \delta_{\text{ret}}^{A'} S_\lambda \mathcal{T} \left( e^{ij(A)/\hbar} \right) = i\hbar^{-1} \mathcal{R} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) ; j(A'_\lambda) \right) \]

\[ + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) \mathcal{T} \left( \exp \left( ij(A_\lambda)/\hbar + r_\lambda(e^{ij(A_\lambda)/\hbar}) \right) \right). \]

(45)

It follows from (18) and (26) that

\[ \delta_{\text{ret}}^{A'} S_\lambda = S_\lambda \delta_A^{A'}, \]

which proves (42) upon comparison of (44) and (45).

Using the definition (24) of the derivative of a field w.r.t. the background geometry, we may write (42) as

\[ \frac{d}{ds} i\overset{\cdot}{\chi},X-A \partial_\lambda (e^{ij(A+sA')/\hbar})|_{s=0} = i\hbar^{-1} r_\lambda(j(A')). \]

Hence, we may compute the renormalization group flow of the interaction term \( j(A) \) due to the fermions as

\[ r_\lambda,\overset{\cdot}{A}(e^{ij(A)/\hbar}) = \int_0^1 \frac{d}{ds} i\overset{\cdot}{\chi},X+(1-t)A \partial_\lambda,\overset{\cdot}{A}+(1-t)A \left( e^{ij(tA)/\hbar} \right)|_{t=s} ds \]

\[ = i \int_0^1 ((\overset{\cdot}{d} + sA \wedge)A, F + s dA + \frac{s^2}{2} A \wedge A) ds, \]

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where we used \(41\) and \(F[\bar{A} + A] = \bar{F} + \bar{d}A + \frac{1}{2}A \wedge A\). Carrying out the integration, we obtain \(\frac{1}{n}(\langle F[\bar{A} + A], F[\bar{A} + A]\rangle - \langle \bar{F}, \bar{F} \rangle)\), i.e., the Yang-Mills action, up to an \(A\)-independent term. Hence, one can infer the fermion contribution to the renormalization group flow from the flow at first order in \(A\). We note that the latter is given by a coinciding point limit of Hadamard coefficients \([3]\). These also appear in the heat kernel expansion (as Seeley-de Witt coefficients), which establishes a relation to the heat kernel method.

### 3.3 Fulfillment of perturbative agreement

We now want to investigate whether the principle of perturbative agreement can be satisfied. We proceed analogously to \([2]\). There, one defines the deviation

\[
D_k(A, t_1, \ldots, t_k) = \delta_{\text{ret}}^A \mathcal{T}(\bar{\Phi}_1(t_1), \ldots, \bar{\Phi}_k(t_k)) - i\hbar^{-1} \mathcal{R}(\Phi_1(t_1), \ldots, \Phi_k(t_k); j(A)) - \sum_j \mathcal{T}(\Phi_1(t_1), \ldots, \Phi_j(t_j), A(t_j), \ldots, \Phi_k(t_k)).
\]

and then proceeds inductively in the total grade \(N\) of the fields \(\Phi_i\). For \(N = 0\), only \(k = 0\) is nontrivial, but \(D_0 = 0\) identically. One assumes that \(D_k = 0\) for all fields with total grade smaller than \(N\). The idea is then to modify the time-ordered product \(\mathcal{T}(\Phi_1(t_1), \ldots, \Phi_k(t_k); j(A))\) appearing in the second term on the r.h.s. such that the modified \(D_k\) vanishes. Such a modification is only possible if the \(D_k\) fulfill a couple of properties, cf. the discussion of the renormalization freedom in Remark \([2.15]\). We proceed in close analogy to \([2]\), only highlighting the differences.

As in \([2]\), one shows that \(D_k\) is a c-number supported on the total diagonal \(D_{k+1}\). It is local and covariant, scales almost homogeneously, and vanishes if one of the \(\Phi_i\) is a linear field. We also have

\[
D^*_k(A, t_1, \ldots, t_k) = (-1)^{k+1}D_k^*(A, t^*_k, \ldots, t^*_1),
\]

where \(D_k^*(A, t^*_k, \ldots, t^*_1)\) is defined as in \([40]\), but with \(\Phi_1(t_1), \ldots, \Phi_k(t_k)\) replaced by \(\Phi_k(t_k)^*, \ldots, \Phi_1(t_1)^*\). This follows from the commutation of \(\delta_{\text{ret}}^A\) with the adjoint, cf. Proposition \([2.9]\) **Unitarity** of the time-ordered products, and the fact that for lower total grade we have \(D_k = 0\).

The next step is to show that

\[
\text{WF}(D_k)|_{D_{k+1}} \perp T\mathcal{D}_{k+1}. \tag{47}
\]

Furthermore, \(D_k\) depends smoothly and (if applicable) analytically on the background fields, i.e., for a smooth (analytic) family \(P \ni s \mapsto X_s\) of backgrounds, with \(P\) a finite dimensional parameter space, we have

\[
\text{WF}(D_k)|_{P \times D_{k+1}} \perp T(P \times D_{k+1}). \tag{48}
\]
For the second and third term in $D_k$, this follows from the properties of the retarded product. For the first term, one restricts to a sufficiently small neighborhood of $D_{k+1}$, where one can write
\[
\mathcal{T}(\Phi_1(t_1), \ldots, \Phi_k(t_k)) = \int w(y_1, \ldots, y_k; x_1, \ldots, x_N) : \Psi(x_1) \ldots \Psi(x_N) : H \prod_i d\bar{g}x_i \prod_j t(y_j)d\bar{g}y_j,
\]
where $:\cdot:\cdot_H$ denotes Hadamard normal ordering [2,3], and $w$ is a distributional section. $\Psi$ stands for either $\psi$ or $\psi^+$ in the notation used in (15). The dependence on the background resides in $w$ and the Hadamard normal ordered product. The variation w.r.t. the background thus gives two terms, one from the variation of $w$ and one from the variation of the normal ordered product. Both fulfill the conditions (47) and (48) independently. Let us first discuss the variation of the normal ordered product. As the treatment in [2, Section 6.2.5] contains a few mistakes, we sketch how to correct these and how to implement the changes necessary to treat Dirac spinors. For the wave front set of $w$, one knows from the Microlocal spectrum condition that
\[
\text{WF}(w)|_{D_k} \subset \{(y, p_1; \ldots, y_k; x_1, k_1; \ldots, x_N, k_N)|x_i = y \forall i, \sum_i p_i + \sum_i k_j = 0\}. \quad (49)
\]
In order to treat the variation of the Hadamard normal ordered product, one evaluates it in a suitable family $\omega_s$ of quasi-free Hadamard states (whose two-point functions will be denoted by the same symbol). Pick a Hadamard two-point function $\omega$ for $s = 0$ and define the corresponding family $\omega_s$ as in Definition [2,7]. Define
\[
d_s = \omega_s - H_s,
\]
where $H_s$ is the parametrix for the background $\bar{X}_s$. Then
\[
\omega \left( \mathcal{T}_{\text{ret}} X_s : \Psi(x_1) \ldots \Psi(x_N) : H_s \right) = \prod_{\text{pairs } ij} d_s(x_i, x_j),
\]
so that the part of the variation of $\omega(\mathcal{T}(\Phi_1(t_1), \ldots, \Phi_k(t_k)))$ that comes from the variation of the Hadamard ordered products can be written as
\[
\int w_0(y_1, \ldots, y_k; x_1, \ldots, x_r) \partial_s \prod_{\text{pairs } ij} d_s(x_i, x_j)|_{s=0} \prod_i d\bar{g}x_i \prod_j t(y_j)d\bar{g}y_j.
\]
\footnote{For metric variations, also the volume elements have to be varied, leading to $\delta(y_j, y)$ or $\delta(x_i, y)$ above. Knowing the wave front set of $w$ and the Hadamard ordered product, this fulfills (47) and (48).}
If $X$ is the infinitesimal variation corresponding to $X_s$, then one defines

$$(\delta d)(X, u_1, u_2) = \partial_s d_s(u_1, u_2)|_{s=0}.$$ 

According to the above, the wave front set of $\delta d$ at the diagonal is crucial for the determination of the wave front set of $D_k$. It is characterized by the following lemma, which corresponds to [2, Lemma 6.2]:

**Lemma 3.6.** For a sufficiently small causal domain $U \subset M$, with the support $K$ of the background variation contained in $U$, $d_s(x_1, x_2)$ is jointly smooth in $(s, x_1, x_2)$ for $x_i \in U$. Furthermore,

$${\rm WF}(\delta d)|_{D_3} \subset \{(y, p; x_1, k_1; x_2, k_2)| y = x_1 = x_2, p + k_1 + k_2 = 0\}. \quad (50)$$

A causal domain is a globally hyperbolic open subset, whose closure is contained in a geodesically convex region. As the proof given in [2] contains a mistake, we provide a corrected proof in Appendix A. It also includes the changes that are necessary to treat the Dirac equation instead of the Klein-Gordon equation. With this lemma and the knowledge about the wave front set of $w_0$, cf. (49), it is then straightforward to prove (47) and (48). For the treatment of the variation of $w$, we refer to [2, Section 6.2.5], noting however that there equation (238) has to be corrected, analogously to the correction of equation (230) that we give in (72).

The next step is to show that the $D_k$ are symmetric if one of the fields is the current, i.e., for $\Phi_1 = j$,

$$D_k(A_1, A_2, t_2, \ldots, t_k) - D_k(A_2, A_1, t_2, \ldots, t_k) = 0. \quad (51)$$

In fact, it will turn out that the problem can be reduced to the case $k = 1$. We denote the l.h.s. of this equation for $k = 1$ by $E(A_1, A_2)$. We have

$$E(A_1, A_2) = \delta^A_{\text{ret}} T(j(A_2)) - \delta^A_{\text{ret}} T(j(A_1)) + i\hbar^{-1} [T(j(A_1)), T(j(A_2))], \quad (52)$$

where we used that $S^{(2)}(A_1, A_2, f)$ is symmetric in $A_1, A_2$ (in fact, for the Dirac field $S$ is linear in the connection, so the second derivative vanishes anyway).

**Proposition 3.7.** If $E$ vanishes and $D_j$ vanishes for all $j$ and all total grades $N \leq \sum_{i=1}^k |\Phi_i|$, then (51) is fulfilled for all $k$.

**Proof.** The statement is only nontrivial for $k \geq 2$. In that case, we have

$$(51) = \delta^A_{\text{ret}} T(j(A_2), \Phi_2(t_2), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2) \quad (53)$$

$$- i\hbar^{-1} R(j(A_2), \Phi_2(t_2), \ldots, \Phi_k(t_k); j(A_1)) - (1 \leftrightarrow 2) \quad (54)$$

$$- \sum_{j=2}^k T(j(A_2), \Phi_2(t_2), \ldots, \Phi_j(1)(A_1 \ast t_j), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2). \quad (55)$$

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where we again used that \( S^{(2)}(A_1, A_2, f) \) is symmetric in \( A_1, A_2 \). We may replace the second line by
\[
\eqref{60} = i\hbar^{-1} T(j(A_1)) \ast T(j(A_2), \Phi_2(t_2), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2). \tag{56}
\]

The first line may be rewritten as
\[
\eqref{56} = \delta_{\text{ret}}^{A_1} R(\tilde{\Phi}_2(t_2), \ldots, \tilde{\Phi}_k(t_k); j(A_2)) - (1 \leftrightarrow 2) \\
+ \delta_{\text{ret}}^{A_1} T(j(A_2)) \ast T(\tilde{\Phi}_2(t_2), \ldots, \tilde{\Phi}_k(t_k)) - (1 \leftrightarrow 2). \tag{57}
\]

Due to the assumption on the vanishing of \( D_j \), \eqref{57} can be rewritten as
\[
\eqref{57} = -i\hbar \delta_{\text{ret}}^{A_1} \delta_{\text{ret}}^{A_2} T(\tilde{\Phi}_2(t_2), \ldots, \tilde{\Phi}_k(t_k)) - (1 \leftrightarrow 2) \\
+ i\hbar \sum_{j=2}^{k} \delta_{\text{ret}}^{A_1} T(\Phi_2(t_2), \ldots, (A_2 \ast t_j), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2). \tag{60}
\]

Here \( \tilde{\Phi} \) denotes the two-parameter family corresponding to variations along \( A_1 \) and \( A_2 \). The retarded variations commute, so that the first line \eqref{60} vanishes. The line \eqref{58} may be expanded by the Leibniz rule \eqref{20} to
\[
\eqref{58} = \delta_{\text{ret}}^{A_1} T(j(A_2)) \ast T(\Phi_2(t_2), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2) \\
+ T(j(A_2)) \ast \delta_{\text{ret}}^{A_1} T(\Phi_2(t_2), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2). \tag{61}
\]

Using again the assumption on the \( D_j \), \eqref{62} can be written as
\[
\eqref{62} = i\hbar^{-1} T(j(A_2)) \ast R(\Phi_2(t_2), \ldots, \Phi_k(t_k); j(A_1)) - (1 \leftrightarrow 2) \\
+ T(j(A_2)) \ast \sum_{j=2}^{k} T(\Phi_2(t_2), \ldots, (A_1 \ast t_j), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2). \tag{63}
\]

Now \eqref{56} and \eqref{63} add up to
\[
\eqref{56} + \eqref{63} = -i\hbar^{-1} [T(j(A_2), T(j(A_1)) \ast T(\Phi_2(t_2), \ldots, \Phi_k(t_k))]. \tag{65}
\]

Again using the vanishing of the \( D_j \), \eqref{60} is
\[
\eqref{60} = -\sum_{j=2}^{k} R(\Phi_2(t_2), \ldots, (A_2 \ast t_j), \ldots, \Phi_k(t_k); j(A_1)) - (1 \leftrightarrow 2), \tag{66}
\]

where we used the commutativity of the derivative w.r.t. the background field. Now \eqref{60} and \eqref{64} add up to
\[
\eqref{60} + \eqref{64} = -\sum_{j=2}^{k} T(j(A_1), \Phi_2(t_2), \ldots, (A_2 \ast t_j), \ldots, \Phi_k(t_k)) - (1 \leftrightarrow 2),
\]

\[\tag{67}\]
which cancels (55). The remaining terms (61) and (65) give
\[ (51) = E(A_1, A_2) \star T(\Phi_2(t_2), \ldots, \Phi_k(t_k)), \]
which vanishes by the hypothesis.

Hence, it remains to show that \( E = 0 \), which turns out to be a consequence of current conservation:

**Proposition 3.8.** For \( n \leq 4 \), if \( T((\tilde{\delta}j)(c)) = 0 \), then \( E = 0 \).

**Proof.** Set \( A_1 = A, A_2 = \bar{d}c \). By Lemma 2.12, we have
\[ \delta_{\text{ret}}^A T(j(A)) = T(j(c \wedge A)). \]
Furthermore
\[ \delta_{\text{ret}}^\bar{d}c T(j(\bar{d}c)) = \delta_{\text{ret}}^A T((\tilde{\delta}j)(c)) - T(j(A \wedge c)). \]
If \( T((\tilde{\delta}j)(c)) = 0 \), then the first term on the r.h.s. vanishes. Also the commutator term in (52) vanishes, so that
\[ E(A, \bar{d}c) = 0 \quad \forall A \in \Gamma_c^\infty (M, E^1), c \in \Gamma_c^\infty (M, E^0). \] (67)
On the other hand, from the above we know that \( E \) is given by
\[ E(A_1, A_2) = \sum_{r=0}^{R} \int A_{1,\mu}(x) \bar{\nabla}_{(\lambda_1} \ldots \bar{\nabla}_{\lambda_r)} A_{2,\nu}(x) C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r}(x) d_2 x - (1 \leftrightarrow 2) \] (68)
for some locally and covariantly constructed \( C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r} \) of scaling dimension \( n \) and a finite \( R \). Here \( I, J \) are indices labelling a basis of \( g \). Without loss of generality, we may assume that \( C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r} \) is symmetric in the \( \lambda_i \) and that
\[ C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r} = (-1)^r C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r}, \] (69)
as, by starting at \( r = R \), we may recursively cancel the (anti-)symmetric component for even (odd) \( r \) by partial integration. And for \( r = 0 \), the symmetric part cancels anyway. Using (69), differentiation of (67) w.r.t. \( A^I_{\mu}(x) \) yields
\[ \sum_{r=0}^{R} \bar{\nabla}_{(\lambda_1} \ldots \bar{\nabla}_{\lambda_r)}(\bar{\nabla}_\nu C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r}(x)) \]
\[ + \bar{\nabla}_{(\lambda_1} \ldots \bar{\nabla}_{\lambda_r)}(\bar{\nabla}_\nu C^\mu_{IJ}{}_{\lambda_1}{}^{...}{}_{\lambda_r}(x)) = 0. \]

\(^8\)Note that on the l.h.s. the background connection in \( \tilde{\delta} \) is not varied, contrary to the first term on the r.h.s. This is corrected for by the second term on the r.h.s.
We may choose $c^J$ such that $\bar{\nabla}_{\rho_1} \ldots \bar{\nabla}_{\rho_l} c^J (x) = 0$ for all $l \leq R$, and the symmetric parts of $\bar{\nabla}_{\rho_1} \ldots \bar{\nabla}_{\rho_{R+1}} c^J (x)$ can be chosen independently. Hence,

$$C_{IJ}^{\mu(\nu_1 \ldots \nu_R)} = 0.$$  \hspace{1cm} (70)

In particular, this already excludes $R = 0$.

Let us discuss the possibility to construct such tensors $C$ for dimensions $n \leq 4$. As we have $2 + r$ upper indices, we need at least $1 + \lceil r/2 \rceil$ inverse metrics\(^9\) ($\lceil m \rceil$ denoting the smallest number greater or equal to $m$). These have at least scaling dimension $2(1 + \lceil r/2 \rceil)$. Hence, for $n = 2$, $R = 0$ is possible, which we already excluded. The same applies to $n = 3$. For $n = 4$, $R = 0$ is already excluded, and $R = 1$ would require the existence of a covariant tensor of rank 3 and scaling dimension 4, which does not exist. Hence, $R = 2$ remains. But due to the scaling condition, only the Killing form $\kappa_{IJ}$ and $g^{\mu\nu}$ can be used. Hence, the most general form of $C$ is

$$C_{IJ}^{\mu\nu\lambda_1\lambda_2} = \kappa_{IJ} \left( c_1 g^{\mu\nu} g^{\lambda_1\lambda_2} + c_2 g^{\mu\lambda_1} g^{\nu\lambda_2} + c_3 g^{\mu\lambda_2} g^{\nu\lambda_1} \right).$$

Now (69) requires $c_1 = 0$ and $c_2 = -c_3$. On the other hand, symmetry in the $\lambda$’s requires $c_2 = c_3$, so $C = 0$. \hfill \Box

Remark 3.9. In the case of metric variations, it is claimed in [2, Section 6.2.6] that the analogous statement is true independently of the dimension. However, the proof seems to contain a gap\(^10\) related to the fact that in (70), we can only make statements about the symmetrization in $(\nu_1 \ldots \nu_R)$ instead of symmetrization in $(\lambda_1 \ldots \lambda_R)$. A thorough investigation of the “background cohomology” introduced in [2] might be helpful in overcoming the restriction on the dimension.

It now remains to redefine the time-ordered product in order to achieve $D_k = 0$. Assume that $G$ is simple. We may also assume that $\rho$ is non-trivial, as otherwise all requirements are trivially fulfilled. In that case, the quadratic form $\langle \lambda, \lambda' \rangle = \text{tr}_V (\rho(\lambda) \rho(\lambda'))$ on $\mathfrak{g}$ is a multiple of the Killing form $\kappa$. Hence, we may set

$$\tau^{k+1} [\psi^{a+}_\alpha, \Phi_1 , \ldots , \Phi_k ] = ic_{\rho,n} r D_k (\Phi_1 , \ldots , \Phi_k ) \kappa^{I}J b_{a}^\nu \alpha^\mu \beta,$$

where $\alpha, \beta$ are spinor indices, $a, b$ gauge indices, $I, J$ Lie algebra indices, $T_J$ the generator in the representation $\rho$, and $\kappa^{IJ}$ is the inverse of the Killing form. $c_{\rho,n}$ is an appropriate normalization factor which depends on the representation $\rho$ and the dimension $n$. Here we used the symbolic notation employed in (15). By the above, we have fulfilled Starting element,
Symmetry, Support, Scaling, Source term, Unitarity, Microlocal spectrum condition, Smoothness, and Analyticity, cf. Remark 2.15. Expansion can be fulfilled by adapting the $r^{k+1}$ for higher grades, cf. [2]. The same procedure applies for $G = U(1)$, where $\kappa$ above is replaced by $\kappa(\lambda, \lambda') = \lambda \lambda'$ with $\lambda, \lambda' \in i\mathbb{R}$. Hence, we have shown the following:

**Proposition 3.10.** Let $n \leq 4$ and $G = G_1 \times \cdots \times G_r \times U(1)^l$, where the $G_i$ are simple. Then the time-ordered products can be defined such that (33) holds.

**Remark 3.11.** For chiral models, this does in general not hold, as it may not be possible to define the parametrix such that (37) holds, cf. [3]. If, however, the gauge group and representation is such that (37) can be fulfilled, then also the principle of perturbative agreement can be fulfilled, as at no other place we made any use of the fact that we were dealing with Dirac instead of chiral fields.$^{11}$

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**A A smoothness result**

**Proof of Lemma 3.6.** As $\omega_s$ is a bi-solution, we have

\[
\begin{align*}
  d_s(D^\square_s u_1, u_2) &= -H_s(D^\square_s u_1, u_2) = G^1_s(u_1, u_2), \\
  d_s(u_1, D^\square_s u_2) &= -H_s(u_1, D^\square_s u_2) = G^2_s(u_1, u_2), \\
  d_s(D^\square_s u_1, D^\square_s u_2) &= -H_s(D^\square_s u_1, D^\square_s u_2) = G^3_s(u_1, u_2).
\end{align*}
\]

As $H_s$ is a bi-solution modulo $C^\infty$, the $G^i_s$ are smooth in $x_1, x_2$. From the Hadamard recursion relations that determine $H$ up to smooth terms, cf. [3], it follows that they are jointly smooth in $(s, x_1, x_2)$. Furthermore, if $K$ is the $^{11}$In the proof of Lemma 3.6 we used the Dirac equation, but a similar argument should hold for any wave equation with a well-posed Cauchy problem. Furthermore, in the proof of Proposition 3.8 we excluded the antisymmetric tensor for simplicity, but at least for $n \leq 4$ its inclusion would not change the conclusion.
support of the perturbation, then $d_s$ is independent of $s$ on any geodesically convex open set which does not intersect $J^+(K)$, by construction. Now choose two Cauchy surfaces $S_{\pm}$ of $U$ such that $J^+(K) \cap S^\pm = \emptyset$. For $u_i \in D_\ominus$, supported in $N \supseteq J^+(S^-) \cap J^-(S^+)$, consider

$$d(u_1, u_2) = d(\chi_N u_1, \chi_N u_2) = d(\chi_N D\ominus S_{\text{adv}} u_1, \chi_N D\ominus S_{\text{adv}} u_2),$$

where $\chi_N$ is the characteristic function of $N$ and $d, D\ominus, S_{\text{adv}}$ depend on $s$. Using integration by parts, we obtain

$$d(u_1, u_2) = G^3(\chi_N S_{\text{adv}} u_1, \chi_N S_{\text{adv}} u_2) + G^1(\chi_N S_{\text{adv}} u_1, \delta_{S^-} - n \cdot \gamma S_{\text{adv}} u_2)$$

$$+ G^2(\delta_{S^-} - n \cdot \gamma S_{\text{adv}} u_1, \chi_N S_{\text{adv}} u_2) + d(\delta_{S^-} - n \cdot \gamma S_{\text{adv}} u_1, \delta_{S^-} - n \cdot \gamma S_{\text{adv}} u_2),$$

(71)

where $\delta_{S^-}$ denotes the restriction of the integration to $S^-$, and $n$ is the corresponding normal vector. As $d_s$ is jointly smooth in $(x_1, x_2)$ for fixed $s$ and independent of $s$ in a neighborhood of $S^- \times S^-$, the wave front set of $d$ (as a function of $(s, x_1, x_2)$) has no intersection with $S^- \times S^-$. It now follows from

$$WF(S_{\text{adv}}^s) \subset \{(s, t; x_1, k_1; x_2, k_2) | (x_1, k_1) \sim_s (x_2, -k_2), x_1 \in J^-_s(x_2)\}$$

that $WF(\chi_N(x_1) S_{\text{adv}}(x_1, x_2))$ does not contain elements with $k_1 = 0$, so with the smoothness of $G^2$ and the wave front set calculus [12] we conclude that $WF(d)$ is empty, so $d$ is jointly smooth in $(s, x_1, x_2)$.

In order to restrict the wave front set of $\delta d$, we first determine the wave front set of $\delta G_i$ (which is defined analogously to $bd$). Explicit calculation shows that $G_i(x_1, x_2)$ is a series in Hadamard coefficients $V_k(x_1, x_2)$, possibly acted upon with $D\oplus_i$ or $D\ominus_i$, and multiplied with nonnegative powers of the squared geodesic distance $\Gamma(x_1, x_2)$. For both $V_k$ and $\Gamma$, one has to integrate the geometric data along the unique geodesic connecting $x_1$ and $x_2$. Hence, for the wave front set of $\delta \Gamma, \delta V_k$, we have to determine the wave front set of

$$\int_0^1 \delta(y, z_{x_1,x_2}(s))ds,$$

where $z_{x_1,x_2} : [0, 1] \to M$ is the unique geodesic from $x_1$ to $x_2$. To do so, consider the wave front set of $\delta(y, z_{x_1,x_2}(s))$ as a distribution in $(s, y, x_1, x_2)$ and then use the wave front set calculus to determine the wave front set of its convolution with $\vartheta(s)\vartheta(1 - s)$, $\vartheta$ being the Heaviside distribution. We obtain, by employing Riemannian normal coordinates,

$$WF(\delta V_k) \subset W, \quad WF(\delta \Gamma) \subset W,$$

32
with

\[ W = \{ (y,p;x_1,k_1;x_2,k_2) | y = z_{x_1,x_2}(s), 0 \leq s \leq 1, p(\dot{z}_{x_1,x_2}(s)) = 0, \]
\[ k_1 = (s-1)\Pi_z p, k_2 = -s\Pi_z p \}
\[ \cup \{ (y,p;x_1,k_1;x_2,k_2) | y = x_1, p = -k_1, k_2 = 0 \}
\[ \cup \{ (y,p;x_1,k_1;x_2,k_2) | y = x_2, p = -k_2, k_1 = 0 \}, \]

where \( \Pi_z \) denotes the parallel transport of the cotangent vector along \( z \). The wave front set of the \( \delta \) distribution \( \delta(y - x_i) \) that is implied by \( \delta D_i \oplus \delta S_i \) (as the geometric data at \( x_i \) is contained in \( D_i \oplus S_i \)) is already contained in \( W \) (as the second and the third component), so that\(^{12}\)

\[ WF(\delta G_i) \subset W. \] (72)

Furthermore, from \( \delta S_{\mathrm{adv}}^\oplus = -S_{\mathrm{adv}}^\oplus \circ \delta D_i \oplus S_{\mathrm{adv}}^\oplus \), we have

\[ WF(\delta S_{\mathrm{adv}}^\oplus) \subset \{(y,p;x_1,k_1;x_2,k_2) | y \in J^+(x_1) \cap J^-(x_2), \]
\[ \exists q_i \in T^*_y M \text{ such that } (y,q_i) \sim (x_i,-k_i), p = q_1 + q_2 \}. \]

Combining this with (72) and (71), noting that \( \delta d = 0 \) on \( S^- \times S^- \), and using the wave front set calculus, we obtain (50). \( \square \)

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\(^{12}\) This corrects equation (230) in [2].
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