The Chern-Simons invariant as the natural time variable for classical and quantum cosmology

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(Revised and extended)

January 5, 2022

ABSTRACT

We propose that the Chern-Simons invariant of the Ashtekar-Sen connection is the natural internal time coordinate for classical and quantum cosmology. The reasons for this are a number of interesting properties of this functional, which we describe here. 1) It is a function on the gauge and diffeomorphism invariant configuration space, whose gradient is orthogonal to the two physical degrees of freedom, in the metric defined by the Ashtekar formulation of general relativity. 2) The imaginary part of the Chern-Simons form reduces in the limit of small cosmological constant, \( \Lambda \), and solutions close to DeSitter spacetime, to the York extrinsic time coordinate. 3) Small matter-field excitations of the Chern-Simons state satisfy, by virtue of the quantum constraints, a functional Schroedinger equation in which the matter fields evolve on a DeSitter background in the Chern-Simons time. We then propose this is the natural vacuum state of the theory for \( \Lambda \neq 0 \). 4) This time coordinate is periodic on the configuration space of Euclideanized spacetimes, due to the large gauge transformations, which means that physical expectation values for all states in non-perturbative quantum gravity will satisfy the KMS condition, and may then be interpreted as thermal states. 5) Forms for the physical Hamiltonians and inner product which support the proposal are suggested, and a new action principle for general relativity, as a geodesic principle on the connection superspace, is found.

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1 Introduction

To construct a single unified physical theory from general relativity and quantum field theory we must be able to extend quantum theory to the universe as a whole. But efforts to accomplish this have so far failed at least in part because of the problem of time[1]. This stems from the apparent conflict between, on the one hand, the quantum theory’s need to refer to a preferred clock when defining the notions of evolution and exclusive outcomes that are essential for the probability interpretation and, on the other hand, the diffeomorphism invariance of general relativity, which forbids description in terms of a fixed or absolute notion of time, external to the universe. One proposal that has been made about this problem is that we might be able to identify a degree of freedom of the gravitational field that could serve as a clock, with respect to which the evolution of both matter and the other gravitational degrees of freedom might be measured[2, 3, 4].

In this letter we would like to propose that there is a particularly natural choice for such an “internal” clock made from the geometry of spacetime, it is the the Chern-Simons invariant[5] of the Ashtekar-Sen connection, $A_{ai}$. Our proposal applies both to physical, Lorentzian spacetimes and to their Euclidean extensions, which are expected to be useful for the quantum theory.

Before stating our proposal, there is an important point about the specification of the time function that must be stressed. In some treatments of the problem of time, it is stated that what is required is a time function on the spacetime manifold itself[1]. This must be a scalar function on spacetime, satisfying certain properties. This may be a useful thing for the classical theory, but it suffers from a severe limitation when we attempt to approach the problem of time in the quantum theory, which is that, in general, the spacetime has no meaning in the quantum theory. Instead, the natural arena for the quantum theory is the configuration space. As is emphasized in many applications of quantum mechanics, the configuration space is the place on which the quantum states are defined, and in which their evolution takes place. More than this, spacetime can be expected to play no fundamental role in the interpretation of quantum gravity, just as trajectories of particles play no role in the interpretation of ordinary quantum mechanics. It may emerge in the classical limit, but for the exact, non-perturbative theory, quantum states will exist on the configuration space of general relativity, and they will not normally have a simple interpretation in terms of spacetime.
For this reason, what is wanted to address the problem of time in the quantum theory is a time function on the configuration space. Now, it might seem at first that it is possible to have both, that is to have a local function on spacetime that, given any slicing into three dimensional surfaces, can be integrated to give a time function on the space of configurations. However, a little thought leads to the conclusion that this is not possible, for the simple reason that if a function is going to be integrated over a three manifold to give a time function on the configuration space, it must be a density on that three surface. In that case it cannot come from a function on spacetime without the specification of additional information. Given an appropriate spatial density, one can make such an association, but as any such density must be, in a diffeomorphism invariant field theory, a dynamical degree of freedom, the difference is significant.

Thus, we have to make a choice as to whether the object we are investigating is a time function on the configuration space or a time function on a spacetime. For the reasons stated, we choose the former. Finally, we may emphasize that from the point of view of the quantum theory, one does not need to have specified a slicing condition to speak of the configuration space in terms of functions on an abstract three manifold. Instead, the slicing condition is to be seen as merely a gauge condition that helps, when appropriate, to translate the fundamental dynamics on the configuration space into statements about spacetimes\(^1\).

Having specified the context, we may now discuss what we claim and do in this paper. The main result of the Ashtekar formalism, and all the work done using it, is to reinforce the idea that it is useful to think of the configuration space of general relativity as built from an \(SO(3)\) connection on an abstract three manifold. The main idea of this paper is that it may be useful for certain purposes to conceive of time as being measured by a particular function on this configuration space, which is the Chern-Simons invariant of that connection.

\(^1\)Of course, there are cases where it is interesting to employ a gauge fixing as an auxiliary device to describe a particular spacetime in terms of a (gauge dependent) trajectory in the configuration space, rather than in terms of a gauge-equivalent class of trajectories. In these cases a gauge condition that chooses the slicings must be specified. This is completely compatible with what we do here. In such cases our proposal becomes a way to label the slices picked out by the slicing condition. For some applications this may not be necessary, but if one is interested in studying the dynamics of the gravitational field, it may be natural to use a label on the slices that measures dynamical information about the gravitational field, rather than simply the gauge dependent information coded in the slicing condition.
The evidence we have for this conjecture comes from both the physical regime of the theory where the metric has a Minkowskian signature and the Euclidean regime, which is relevant for the path integral formulation of the theory and for the discussion of the thermodynamics of the gravitational field. In the latter case, the Ashtekar-Sen connection, $A_{ai}$, is real and we propose to take
\[ \tau_{CS} = \int_{\Sigma} Y_{CS}(A) \]
(1)
as a measure of the Euclideanized time. Here $Y_{CS}(A) = \frac{1}{2}(A^i \wedge dA^i + \frac{1}{3}\epsilon_{ijk}A^i \wedge A^j \wedge A^k)$ is the Chern-Simons form, which is integrated over a spatial three manifold, $\Sigma$, which, as we are studying cosmology, we take to be compact.

As we shall see, the evidence that the Chern-Simons invariant plays the role of a time coordinate is, in the Euclidean case, rather direct, and comes from an analysis of both the geometry of the configuration space and the hamiltonian formulation of the theory. In the physical, Minkowskian case, the situation is more complicated because the Ashtekar-Sen connection, $A^i_{a}$, is complex. This means that the whole Chern-Simons invariant is complex, and cannot directly serve as a measure of time, in the classical theory, in the same way it appears to in the Euclidean case. However, we are able to show that in the semiclassical limit, and in the case of a non-vanishing cosmological constant, the imaginary part of the Chern-Simons invariant
\[ \tau_{ICS} = \Im \int_{\Sigma} Y_{CS}(A), \]
(2)
does play the role of a time coordinate for the theory.

The evidence for our proposal, which is described in the remainder of this paper, may then be summarized as follows.

1) In the configuration space of left-handed spin connections appropriate to the Ashtekar formulation of general relativity, the Chern-Simons invariant, $\int Y_{CS}$, is a natural time coordinate in that its gradient is both timelike in connection superspace (for Euclidean signature spacetimes), and orthogonal to variations in the two physical degrees of freedom, with respect to

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2We use the notation in which $a, b, c, \ldots$ are spatial indices and $i, j, k$ are internal $SO(3)$ indices that label the frame fields of space. We may note that we are taking here the Ashtekar connection to have natural dimensions of inverse length, which means it differs from the convention usually employed by a factor of Newton’s constant, $G$. We will use units here in which $\hbar = 1$, but $G$ is written explicitly, so that $G$ has dimensions of $(\text{length})^2$, while the cosmological constant $\Lambda$ has dimensions of $(\text{length})^{-4}$. The combination $\lambda = G^2 \Lambda$, where $G$ is Newton’s constant, is then dimensionless.
the natural metric on the configuration space[^1]. This natural metric arises from the Hamiltonian constraint of the Ashtekar formulation. In Sections 2-4 we proceed to investigate and substantiate this claim, for the case of vanishing cosmological constant. We show in Section 2 that the geometry of the connection superspace can be understood in a simple way in terms of structures derived from the Chern-Simons functional. This leads us, in Section 3, to the discovery of a simple action principle, which expresses the fact, previously known, that for vanishing cosmological constant the classical spacetimes correspond to a certain class of null geodesics of the metric on the connection configuration space. We find that the Chern-Simons functional is the natural candidate for the parametrization of trajectories and null geodesics in the arena of connection superspace. We then find, in Section 4, and again for the case of vanishing cosmological constant, that the Hamiltonian which evolves the gravitational field in the Chern-Simons form has a simple form as the square root of a positive functional. This allows us to derive a simple Schrödinger equation for the evolution of the quantum state on configuration space in the Chern-Simons time[^2]. It also suggests that “stationary states” with respect to the Chern-Simons time may exist in non-perturbative quantum gravity.

2) When the cosmological constant is non-vanishing the Hamiltonian constraint is cubic in the momentum conjugate to the Ashtekar-Sen connection. As a result, the use of the Chern-Simons functional as an exact choice for “time” may no longer be valid. In Sections 5 and 6, we discuss how the Chern-Simons functional can still emerge as a natural choice for “time”, both in the presence and absence of matter, when the cosmological constant $\lambda$ is non-vanishing but small.

In Section 5 we consider the semiclassical regime of the theory. By considering states of the form $\Psi[A, \phi] = \psi_\text{CS}[A] \chi[A, \phi]$, where $\phi$ is a matter field and $\psi_\text{CS}$, the Chern-Simons state (eqn. 37, below), is an exact quantum state in the absence of matter[^1], we show that in the limit of small $\lambda$ and solutions that are close to DeSitter spacetime, the quantum constraints reduce to a Schrödinger equation in which $\chi$ evolves in a time given by $K$, the trace of the extrinsic curvature $K_{ai}$. Further, in this same limit it is easy to show that

$$Y_{\text{CS}} = \frac{i}{3G} \frac{\lambda}{\sqrt{\text{det}(g)}} K + O(\sqrt{\lambda})$$ \hspace{1cm} (3)

[^1]: We find also an alternative form of the time parameter, as a kind of averaged Chern-Simons time (eq. 35), which gives an extremely simple form for a functional Schrödinger equation.
so that the times as measured by \( \sqrt{gK} \) and by the imaginary part of the Chern-Simons form coincide. Furthermore, the real part of the \( Y_{CS} \) is of higher order, so that it becomes purely imaginary in this limit. It is as a result of this that we are able to assert that in the semiclassical limit of the Minkowskian signature theory, the imaginary part of the Chern-Simons invariant is playing the role of the time coordinate. We may recall that the trace of the extrinsic curvature was proposed some time ago by York to be the internal time of general relativity[3] and was studied by Kuchar[4] and others in the context of several models as well as in the semiclassical limit. We believe that \( \tau_{ICS} \) offers a natural way to preserve what is useful about the York time coordinate, while extending it in a way that gives a simple description of the non-perturbative dynamics of the theory.

3) This choice of a physical time then leads to a particular form for the physical inner product, described in Section 7, which may be computable in terms of a power series in \( K_{ai} \), when use is made of Witten’s discovery[12] of the connection between Chern-Simons theory and the Jones polynomial of knot theory.

4) Because it takes the imaginary part, \( \tau_{ICS} \) is invariant under large gauge transformations, which modify only the real part of the Chern-Simons functional. However, if we continue to the Euclidean theory then, due to the large gauge transformations, \( \tau_{CS} = \int_{\Sigma} Y_{CS} \) becomes a periodic variable on the now real connection configuration space. In Section 8, we argue that, as a result, expectation values defined in terms of path integral over the connection-configuration space will satisfy the KMS condition[13], so that all states of quantum cosmology must be thermal. This extends the results found earlier for the semiclassical theory around classical spacetime solutions[14, 15], and may offer a useful perspective on the link between “time” and thermodynamics in non-perturbative quantum gravity.

The paper closes with a concluding section that summarizes the different results found here.

2 The geometry of the connection-configuration space

We begin by describing the geometry of the kinematical connection configuration space which is the space of connections, \( \mathcal{A} \), modulo the space of all \( SO(3) \)-valued small gauge transformations, \( \mathcal{G} \),

\[
\mathcal{C}_{kin} = \mathcal{A}/\mathcal{G}
\]  

(4)
and the related diffeomorphism invariant configuration space

\[ C_{\text{diff}} = \mathcal{C}_{\text{kin}} / \text{Diff}(\Sigma). \]  

(5)

We will take the viewpoint here that classical general relativity as a dynamical theory is best understood in terms of trajectories in \( C_{\text{diff}} \). It is then important to use whatever information can be gained about the geometry of this space. Here we describe what we know about the natural metric structure of this space from previous work\(^8\) and extend that analysis. We may note that many things we would like to know are not yet well studied, among these are many interesting global questions that we will not be able to address. The results we describe here for connection superspace may be compared with the geometry of the configuration space of three metrics studied first by DeWitt\(^\[8\]\).

For simplicity we also restrict our discussion of the geometry of \( C_{\text{diff}} \) to the Euclidean signature case. We also assume throughout this paper that the spatial manifold \( \Sigma \) is compact.

We may note that we do not know good coordinates for the configuration space \( C_{\text{diff}} \). For this reason we begin using the coordinates on the space of connections \( \mathcal{A} \) on \( \Sigma \), and then investigate explicitly how to go down to the moduli spaces \( \mathcal{C}_{\text{kin}} \) and \( \mathcal{C}_{\text{diff}} \).

Our starting point is to notice that \( \tau_{\text{CS}} \) does describe a functional on \( C_{\text{diff}} \) and to investigate the consequences of choosing it for a time function. We then seek to decompose the geometry of the infinite dimensional configuration spaces locally by making a splitting of the geometry according to this time function, analogous to the usual 3+1 splitting we make in classical relativity.

Thus, the gradient of the time function is given by\(^4\)

\[ \tilde{\tau}^{ai}(x) \equiv \frac{\delta \tau_{\text{CS}}}{\delta A_{ai}(x)} = \frac{1}{2} \tilde{\epsilon}^{abc} F_{bc}^i \]  

(6)

where \( F_{bc}^i \) is the curvature of the Ashtekar-Sen connection. Much of the simplicity of what follows is a consequence of this fact that the left-handed spacetime curvature may be interpreted to be precisely the gradient of a

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\(^4\)As usual, densities are often denoted by tildes. We may note that as we are on the space of connections, the indices are reversed, so that an abstract cotangent space index on \( \mathcal{A} \) is composed of a spatial coordinate, an internal \( SO(3) \) index and a spatial one form index.
natural time function on the configuration space $\mathcal{A}$. There is a natural contravariant metric on $\mathcal{C}_{\text{kin}}$, which is

$$G_{ai, bj} = \frac{1}{\tilde{b}} \varepsilon_{abc} \varepsilon_{ijk} \tilde{\tau}^{ck}.$$  \hspace{1cm} (7)

Here, $\tilde{b} = (\text{det} \tilde{\tau}^{ai})^{1/2}$ is necessary so that $G_{ai, bj}$ has the appropriate density weight on $\Sigma$ to be the contravariant metric on $\mathcal{C}_{\text{kin}}$. We note that when $\tilde{b} = 0$ the metric on $\mathcal{C}_{\text{kin}}$ becomes degenerate. For the purposes of this paper we assume we are away from such points, which are in any case non-generic.

We may note, further, that (7) is special, in that it is diagonal in the points of the spatial manifold $\Sigma$. A general contravariant metric on the configuration space would be of the form of $G_{ai, bj}(x, y)$. In these terms we might write (7) as $G_{ai, bj}(x, y) = \tilde{b}^{-1} \varepsilon_{abc} \varepsilon_{ijk} \tilde{\tau}^{ck} \delta^3(y, x)$.

Using the contravariant metric, we may contract with the index of the gradient of the time function, to find the unit time vector field,

$$\tilde{\tau}_{ai} \equiv \frac{1}{6\tilde{b}} G_{ai, bj} \tilde{\tau}^{bj} = \frac{1}{6\tilde{b}^2} \varepsilon_{abc} \varepsilon_{ijk} \tilde{\tau}^{bj} \tilde{\tau}^{ck}.$$  \hspace{1cm} (8)

It follows directly that at each point of $\Sigma$, $\tilde{\tau}_{ai} \tilde{\tau}^{ai} = 1$. Note that we choose this definition for the unit time vector field because our aim is to describe structures that are local in space as well as in the configuration space.

Using these functions we may then find the metric on the slices of $\mathcal{C}_{\text{kin}}$ of constant $\tau_{CS}$, for example the projection operator into the slices of constant $\tau_{CS}$ is

$$H_{ai}^{\; bj} \equiv \tilde{\tau}_{ai} \tilde{\tau}^{bj} - \delta_a^b \delta_i^j.$$  \hspace{1cm} (9)

while lowering with $G_{ai, bj}$ gives

$$H_{ai, bj} \equiv 6\tilde{b} \tilde{\tau}_{ai} \tilde{\tau}^{bj} - G_{ai, bj}.$$  \hspace{1cm} (10)

The full configuration space metric may then be written

$$ds^2_{\text{superspace}} = \int_\Sigma G_{ai, bj} \delta A_{ai} \delta A_{bj} = \int_\Sigma \left( \frac{1}{6\tilde{b}} (\tilde{\tau}^{ai} \delta A_{ai})^2 - \tilde{H}_{ai, bj} \delta A_{ai} \delta A_{bj} \right),$$  \hspace{1cm} (11)

where the covariant connection superspace metric is

$$G_{ai, bj}(x) = \frac{1}{\tilde{b}} \left( \frac{1}{2} \tilde{\tau}^{ai} \tilde{\tau}^{bj} - \tilde{\tau}^{ba} \tilde{\tau}^{aj} \right).$$  \hspace{1cm} (12)
Note that $\int \tilde{\tau}^a_i \delta A_{ai} = \delta \int Y_{CS}$, so that
\[
\tilde{H}^{ai,bj} = \frac{1}{6\tilde{b}} \tilde{\tau}^a_i \tilde{\tau}^b_j - \tilde{G}^{ai,bj}
\] (13)
is the metric on constant $\tau_{CS}$ surfaces.

Now, from previous work [8], we know that for Euclidean spacetimes, $\tilde{H}^{ai,bj}$ has signature $(5,3) \times \infty^3$, where the three negative directions at each point correspond to changes of the connections under spatial diffeomorphisms, and the space spanned by the five positive directions at each point include three gauge degrees which correspond to $SO(3)$ gauge transformations. Thus, after gauge fixing, $\tilde{G}^{ai,bj}$ may be pulled back to a signature $(1,2) \times \infty^3$ metric on the diffeomorphism invariant configuration space, $C_{diff}$. We see that the gradient of $\tau_{CS}$ spans the $\infty^3$ “timelike” directions. The remaining $2 \times \infty^3$ “spacelike” directions, orthogonal to $\tilde{\tau}^a_i$, must then be considered to be the variations in the two physical degrees of freedom of the gravitational field.

We may note that when the connection is complex, the same decomposition of the degrees of freedom may be done. However it no longer is meaningful to talk of the signature of the configuration space metric. Whether it is meaningful when the Lorentzian reality conditions are imposed, which involve relations between configuration and momenta variables, is presently an open problem.

3 Einstein’s theory from a geodesic principle on the connection configuration space

We may note that the usual Ashtekar form of the Hamiltonian constraint is
\[
\mathcal{H} = \tilde{b} G^{ai,bj} \tilde{E}^a_i \tilde{E}^b_j + \frac{\lambda}{3G} \text{det}(\tilde{E}^{ai}) = 0
\] (14)

Note that when $\lambda = 0$ what is relevant is only the “conformal class” of metrics on $C_{kin}$ that differ from $\tilde{G}^{ai,bj}$ by the multiplication by a free, non-vanishing function, $\Omega$, of $\Sigma$. Thus, for $\lambda = 0$, physical spacetimes are defined by the common geodesics of the conformal class of metrics $\Omega \tilde{G}^{ai,bj}$ on $C_{kin}$. These may be called the “locally-null” geodesics of $\tilde{G}^{ai,bj}$. Another way to say this is that a simple action principle for general relativity, written in
terms of connections is $\mathcal{S}_{GR}[A, dA, N] \equiv \frac{1}{16\pi G} \int dt \int_{\Sigma} \frac{1}{N} \tilde{G}^{ai, bj} \dot{A}_a \dot{A}_b$ \hspace{1cm} (15)

From this action, the momentum conjugate to $A_{ai}$ is given by

$$\tilde{E}_{ai}/(8\pi G) = \tilde{G}^{ai, bj} \dot{A}_b/(NG)$$ \hspace{1cm} (16)

Rewritten in the canonical form, the action is

$$\mathcal{S}_{GR}[A, \tilde{E}, N] \equiv \frac{1}{8\pi G} \int dt \int_{\Sigma} \tilde{E}_{ai} \dot{A}_a - (N/2) \tilde{G}_{ai, bj} \tilde{E}^a \tilde{E}^b$$ \hspace{1cm} (17)

Variation with respect to $N$ results in the locally null geodesic constraint

$$\tilde{G}^{ai, bj} \dot{A}_a \dot{A}_b = 0$$ \hspace{1cm} (18)

or equivalently (as we have said earlier, we assume $\tilde{b} \neq 0$)

$$\tilde{G}_{ai, bj} \tilde{E}^a \tilde{E}^b = 0$$ \hspace{1cm} (19)

All of the dynamics and constraints of general relativity are derived from this simple functional (expression (15)) of the left-handed spin connection. By taking the Poisson bracket of the “scalar” (superhamiltonian) constraint (19) with itself, the “vector” (supermomentum) constraint emerges as a secondary constraint. The Poisson bracket of the vector constraints then yields Gauss’ Law as a further constraint. Thus the locally null geodesic constraint reproduces the rest of the Ashtekar constraints of 3d-diffeomorphism and $SO(3)$ gauge invariance through secondary constraints and the requirement of closure. With the action (15), under $t$-reparametrization $t \mapsto T$, $1/N$ scales by $dt/dT$. Since $1/N$ is a Lagrange multiplier, the physics is $t$-reparametrization invariant. A natural and explicitly gauge and diffeomorphism invariant choice for $T$ as a parametrization of null geodesics in connection-superspace is precisely $\int_{\Sigma} Y_{CS}$.

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This is related to the Capovilla-Dell-Jacobson action[\textsuperscript{19}]. It is interesting to note that, in contrast to the Barlein, Sharp and Wheeler form of the action[\textsuperscript{2}], the shifts do not appear explicitly. Instead the diffeomorphism constraints appear by requiring the closure of the algebra generated by the Hamiltonian constraint.

\textsuperscript{6} Equivalently, we may write the action principle in a way that is explicitly invariant under reparametrizations of the $t$ coordinate on the trajectories in the configuration space.
Finally, before we close this section, it is interesting to note that this construction generalizes to give a connection between a large class of topological field theories in three dimensions and a class of gravitational theories in 3 + 1 dimensions. Because any three manifold supplies us with a natural $\tilde{\epsilon}^{abc}$, any functional, $f(A)$ on $C_{\text{diff}}$ determines an inverse metric on that space, and hence by the geodesic principle on $C_{\text{diff}}$ a dynamics for the gravitational field. Our construction may then be generalized by replacing $\tau_{CS}$ by any functional $f(A)$ on a space of connections on a three manifold. The result is that any topological field theory on a three manifold, specified by an action $S(A)$, then corresponds through eqn. (7) to a gravitational theory given by the action (15).

This may be related to the theories studied in [18].

4 A canonical transformation and a hamiltonian

Given the proposal that the Chern-Simons functional is to be regarded as labeling surfaces of constant “time” in the configuration space, as well as a natural parametrization of trajectories there, we would like to ask if it is possible to make a coordinate transformation to exhibit explicitly a decomposition of the directions in the configuration space into “timelike” and physical degrees of freedom. If this can be done we would further like to know if we can perform a canonical transformation on the phase space to exhibit that splitting. What we will show here is that, at least in the Euclidean case, this can be done locally, in the phase space. One byproduct

Just as we do for ordinary geodesics on finite dimensional manifolds, we may write the action principle as

$$S'_{GR}[A, dA, \Omega] \equiv \frac{1}{16\pi G} \int dt \sqrt{\int \Sigma \tilde{G}^{ai, bj} \dot{A}_a \dot{A}_b}, \quad (20)$$

Of course, this variational principle is subject to the same difficulty of the standard variation principle for null geodesics, which is that the canonical momenta diverge when the equations of motion (18) are satisfied. However, it is still the case that the constraint (19) is satisfied, as may be seen by rescaling the momenta by the square root of the constraint (19). Once this is done, one finds that variation of this action by $\Omega$ then leads to the locally null geodesic constraint (18), and the Hamiltonian constraint (19), and hence to the full Hamiltonian formulation of general relativity.

Note that when the group is larger than $SO(3)$ the condition that the algebra of constraints closes requires that the inverse metric (7) may be written in the Peldan form [18] in which $\epsilon^{ijk} \rightarrow \epsilon_{abc} \tilde{B}^{ai} \tilde{B}^{bj} \tilde{B}^{ck} / \det(\tilde{B})$. 

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of this will be that this canonical transformation will yield, at least for the case of vanishing cosmological constant, a Hamiltonian that generates the evolution of the gravitational degrees of freedom in $\tau_{CS}$.

We begin by looking for coordinates on the constant $\tau_{CS}$ sections of $C_{\text{kin}}$. To find these, it is natural to split the variations in the connection $A_{ai}$ into components that are parallel to, and orthogonal to, the “timelike” direction $\tilde{\tau}_{ai}$. We then define,

$$\delta A_{ai} = \tilde{\tau}_{ai}\delta \tilde{A}^\parallel - \delta A_{ai}^\perp$$

(21)

where,

$$\delta \tilde{A}^\parallel \equiv \tilde{B}^{ai}\delta A_{ai}$$

(22)

and

$$\delta A_{ai}^\perp \equiv H_{ai}^{\ bj}\delta A_{bj} = -\delta A_{ai} + \tilde{\tau}_{ai}\tilde{\tau}_{bj}\delta A_{bj}$$

(23)

Note that here we use the common shorthand, $\tilde{B}^{ai} = \frac{1}{2}\epsilon^{abc}F_{bc}$. It of course follows that $\tilde{\tau}_{ai}\delta A_{ai}^\perp = 0$.

It would be very useful to be able to integrate these equations, to define $\tilde{A}^\parallel$ and $A_{ai}^\perp$ as coordinates on the configuration space. We have not investigated the questions of what global obstructions there may be to the definition of such coordinates, or the related question of what initial conditions may be used to define solutions to the differential relations (22-23). However, even without this, certain things can be said. First, it follows that,

$$(\ast1)\delta \tilde{A}^\parallel = (\delta Y_{CS}) - \frac{1}{2}d(\delta A^i \wedge A^i)$$

(24)

where $(\ast1)$ is the volume element. So

$$\int_\Sigma \delta \tilde{A}^\parallel(\ast1) = \int_\Sigma \delta Y_{CS}$$

(25)

from which it follows that up to an overall constant

$$\int_\Sigma \tilde{A}^\parallel(\ast1) = \int_\Sigma Y_{CS}$$

(26)

$^8$It is interesting to note that $A^\perp$ must transform as a connection under the restricted algebra of $\tau_{CS}$-time independent gauge transformations, which are defined by gauge parameters $\Lambda$ that preserve its orthogonality to $\tau$ so that $0 = \tilde{\tau}^{bc}\delta_\Lambda A_{bc} = \tilde{\tau}^{bc}/b\Lambda_j$. It may then be useful to express the theory in terms of loop variables associated with this new connection.
Even without information on the global existence of the coordinates \( \tilde{A}^\| \) and \( A_{ai}^\perp \), we can go ahead and construct a canonical transformation to define momenta conjugate to them. Thus, we define a canonical transformation such that,

\[
\begin{align*}
A_{ai} & \rightarrow (\tilde{A}^\|, A_{ai}^\perp) \\
\tilde{E}^\| & \rightarrow (p^\|, \tilde{p}^\|)
\end{align*}
\] (27)

It will be important to note in what follows that it is \( \tilde{A}^\| \) and \( \tilde{p}^\| \) that are the densities. The transformation that accomplishes this may be found from the condition that it must leave the symplectic structure on the phase space fixed. In particular, we have

\[
-\frac{1}{8\pi G} \int_{\Sigma} d^3 x \tilde{E}^\| \delta A_{ai} = \int_{\Sigma} d^3 x \left( p^\|(x) \tilde{\tau}^{ai} \delta A_{ai}(x) + \tilde{p}^\|_{ai}(x) \delta A_{ai}^\perp(x) \right)
\] (28)

One then finds directly that,

\[
-\left( \frac{1}{8\pi G} \right) \tilde{E}^\| (x) = p^\|(x) \tilde{\tau}^{ai} + \tilde{p}^\perp_{ci} H_{ci}^{ai}
\] (29)

with \( p^\| = -\tau_{ai} \tilde{E}^\| / (8\pi G) \) and \( \tilde{p}^\|_{ai} = -H_{ai}^{bj} \tilde{E}^\| / (8\pi G) \).

We may also write down directly the momentum conjugate to the Chern-Simons invariant \( \tau_{CS} = \int_{\Sigma} Y_{CS} \). It is

\[
p_{CS} \equiv \frac{1}{B} \int_{\Sigma} \tilde{b} p^\|
\] (30)

where \( B = \int_{\Sigma} \tilde{b} \). It may be checked directly that \( \{ \tau_{CS}, p_{CS} \} = 1 \).

We may now proceed to construct the Hamiltonian conjugate to \( \tau_{CS} \). We plug (29) into the Hamiltonian constraint (14), to find,

\[
0 = \mathcal{H} = (8\pi G)^2 [(p^\|)^2 \tilde{b}^2 - \tilde{p}^\|_{ai} \tilde{p}^\|_{bj} \tilde{H}_{ai,bj} \tilde{b}^2] + (16\pi \lambda) \left( (p^\|)^3 \tilde{b}^2 + \frac{1}{2} p^\|_{ai} \tilde{H}_{ai,bj} \tilde{p}^\|_{bj} \tilde{p}^\|_{ck} + \text{det}(\tilde{p}^\|_{ck}) \right)
\] (31)

To proceed we must consider separately the cases in which there is or is not a cosmological constant. For the remainder of this section we set \( \lambda \) to zero, postponing to the next two sections the case of nonvanishing cosmological constant. With no \( \lambda \) terms, we may solve (31) to find a hamiltonian that generates evolution in \( \tau_{CS} \). First, we find directly that

\[
p^\|_{ai}(x) = \sqrt{\tilde{b}^{-1} H_{ai,bj}(x) \tilde{p}^\|_{ai}(x) \tilde{p}^\|_{bj}(x)}
\] (32)
To find the Hamiltonian conjugate to $\tau_{CS}$ we must integrate. We find

$$p_{CS} = \frac{1}{B} \int_{\Sigma} \sqrt{b H_{ai,bj} \tilde{p}_{ai} \tilde{p}_{bj}} \equiv h_{CS} \quad (33)$$

This suggests that a quantization could be developed along the lines of [24] in which a quantum state, $\Psi$ of the gravitational field, expressed either in the connection or the loop representation, evolves according to

$$i \frac{\partial \Psi}{\partial \tau_{CS}} = \hat{h}_{CS} \Psi \quad (34)$$

We may note that as we have defined the Hamiltonian without breaking spatial diffeomorphism invariance, it may be possible to implement the corresponding time evolution equation quantum mechanically as an operator on diffeomorphism invariant states, using the techniques developed in [24]. Furthermore, we may note that as $H_{ai,bj}$ is positive definite, the square root in (33), and hence the Hamiltonian, may be defined over all regions of the configuration space $C_{diff}$ on which $\tilde{b}$ is also positive definite everywhere on $\Sigma$. The only difficulties with the definition of this Hamiltonian then come through the possibility of vanishing or complex $\tilde{b}$. This is a better situation than the Hamiltonian obtained in [24] which is the square root of an expression that is not positive definite at generic points on the configuration space. This is at least one advantage of using a degree of freedom of the gravitational field, rather than that of a matter field, as a clock. Eqn.(34) suggests that for $\lambda = 0$, there may be “stationary states” of the form $\Psi[A] = e^{c \int_{\Sigma} \chi_{CS} \chi_{\lambda}[A^\perp]}$, with $c$ being dimensionless constants which correspond to the “spectrum” of $h_{CS}$.

Finally, we may note that an even simpler form for the Hamiltonian is obtained if we scale the time coordinate by $B$, so that we evolve the state not in $\tau_{CS}$, but in

$$\tau' = \frac{\tau_{CS}}{B} \quad (35)$$

The hamiltonian should then be proportional to $p' = \int_{\Sigma} \tilde{b} p_{CS}$, which is simply,

$$p' = \int_{\Sigma} \sqrt{\tilde{b} H_{ai,bj} \tilde{p}_{ai} \tilde{p}_{bj}} \quad (36)$$

Now we turn, in the next two sections, to the case of finite cosmological constant.
5 The physical interpretation of the Chern-Simons state

The case of finite cosmological constant, is more difficult, because of the term in (31) proportional to the cube of \( p_{||} \). This means that the quantum hamiltonian constraint equation contains third time derivatives in the apparent local time variable \( \tilde{A}_{||} \). The presence of these third time derivatives poses a difficulty for the interpretation of the dynamics of the theory, as they could be indications that the theory has instabilities or runaway solutions, of the kind that infest the Lorentz-Dirac formulation of the relativistic electrodynamics of point particles. We may note, in this connection, that working in perturbation theory, Tsamis and Woodard have found evidence that quantum general relativity with a finite cosmological constant is infrared unstable\[^{24}\]. Thus, we should be cautious about the interpretation of a quantum theory for nonvanishing \( \lambda \).

On the other hand, it may be that in spite of these cubic terms, the quantum theory with a cosmological constant is still well defined, at least for sufficiently small \( \lambda \). We may note that exponential growth is a property of the cosmological solutions associated with the presence of a cosmological constant, thus third time derivatives may be necessary if the quantum theory with a cosmological constant is to have a good semiclassical limit.

But perhaps the best evidence that the theory may be sensible in the presence of a cosmological constant is that we know that in that case there is an exact quantum state of the theory, which is the Chern-Simons state discovered by Kodama\[^{9}\]. This is an exact solution to all the constraints of quantum gravity in the connection representation for \( \lambda \neq 0 \), which is given by\[^{10}\],

\[
\psi_{CS}[A] \equiv e^{\frac{3}{16\pi} \int_{\Sigma} Y_{CS}[A]}
\]  

(37)

This state has been much studied in quantum gravity, and it is known that a class of states with very interesting properties can be constructed by transforming it to the loop representation\[^{11}\].

The question of whether the quantum theory with a finite \( \lambda \) can be sensible then depends to some extent on the interpretation of this state. While we

\[^{9}\]We may note that the Chern-Simons state may be multiplied by any topological invariant, \( I \), of \( A \) on \( \Sigma \). We may note that, as \( I \) may be chosen arbitrarily without affecting the demonstration that \( \psi_{CS} \) solves the constraints, there could actually be a number of such exact Chern-Simons states which depend on the topology of \( \Sigma \). We do not here develop this very interesting fact.
are not able to settle this question here, we are able to discover a significant piece of evidence that the Chern-Simons state might, at least for sufficiently small $\lambda$, be interpreted as the ground state of the theory. This evidence is gotten by coupling the theory to matter, and then studying the behavior of perturbations of this state involving the matter degrees of freedom. We find that these excitations behave, to leading order in $\lambda$, exactly like excitations of a quantum field on the background of a DeSitter spacetime. Furthermore, we find that in this description the natural time coordinate in which these matter states evolve is the Chern-Simons time.

We then consider an excitation of a matter field, $\phi$, defined by a state,

$$
\Psi[A, \phi] = \psi_{CS}[A]|\chi[A, \phi]
$$

(38)

For concreteness, we will take the matter field to be a single massless free scalar field, although the results are independent of the choice. We then apply the Hamiltonian constraint to this state, using a regularization and an ordering in which $\psi_{CS}[A]$ is an exact solution[9, 10, 11] and find, using $\tilde{E}^{ai}(x) = -(8\pi G)\delta/\delta A_{ai}(x)$, that,

$$
0 = \int_{\Sigma} N \left\{ \frac{3G}{2\lambda} \varepsilon_{abc} \varepsilon_{ijk} \tilde{B}^{ck} \tilde{B}^{bj} \frac{\delta}{\delta A_{ai}} + \mathcal{H}_{\text{matter}}[\tilde{E}^{ai} = -\frac{3G}{\lambda} \tilde{B}^{ai}] \right\} \chi[A, \phi] \\
+ \int_{\Sigma} N \left\{ (\partial_a \phi)(\partial_b \phi) \left( \frac{3G}{\lambda} \tilde{B}^{(a)}_i (8\pi G) \frac{\delta}{\delta A_{bj}} + (\partial_a \phi)(\partial_b \phi)(8\pi G)^2 \frac{\delta}{\delta A^{a}_i} \frac{\delta}{\delta A_{bi}} \right) \chi[A, \phi] \\
- G \frac{\chi[A, \phi]}{6} \int_{\Sigma} N (8\pi)^2 \varepsilon^{ijk} \varepsilon_{abc} \frac{\delta}{\delta A_{ai}} \frac{\delta}{\delta A_{bj}} \frac{\delta}{\delta A_{ck}} \chi[A, \phi] + ...
$$

(39)

To analyze the meaning of this equation, let us assume that $\chi[A, \phi]$ is peaked around self-dual spacetimes, and has only a slow dependence on $A_{ai}$ compared to the leading exponential term in the Chern-Simons state. In this case, the terms in the second and third lines of (39) are of lower order compared to the first. Using (22), we then have, to leading order

$$
\frac{3G}{2\lambda} \int_{\Sigma} N \delta_{ai} \frac{\delta \chi[A, \phi]}{\delta A_{ai}} = \frac{3G}{2\lambda} \int_{\Sigma} N \delta_{ai} \frac{\delta \chi[A, \phi]}{\delta A^{ai}} = \int_{\Sigma} N \mathcal{H}_{\text{matter}}[\tilde{E}^{ai}] = -\frac{3G}{\lambda} \tilde{B}^{ai}|\chi[A, \phi]
$$

(40)

Now, let us assume that the dependence on $\chi[A, \phi]$ on $A^{ai}$ is holomorphic, following the usual analogy between the Ashtekar connection and the
Bargmann quantization of the harmonic oscillator. Then, we may write

$$\frac{\delta \chi}{\delta A^\| (x)} = \frac{i}{\delta \text{Im} A^\| (x)}$$  \hspace{1cm} (41)

It is also possible to show that in the limit of small $\lambda$ with $\tilde{E}^{ai}$ approaching $\delta^{ai}$, on the L.H.S. of the Schroedinger equation,

$$\frac{\delta \chi}{\delta \text{Im} A^\| (x)} = \frac{\delta \chi}{\delta \text{Im} Y (x)} + O(\lambda) = \frac{3G}{\lambda} \frac{\delta \chi}{\delta \tilde{K}} + O(\lambda)$$  \hspace{1cm} (42)

where $\tilde{K} = \sqrt{q}K$ is the densitized trace of the extrinsic curvature, and we have used (3).

Thus we have a local Schwinger-Tommonoga equation of the form

$$\frac{i}{2} \left( \frac{3G}{\lambda} \right)^2 \frac{\delta \chi}{\delta \tilde{K}} = \tilde{b} - 2H_{\text{matter}}[\tilde{E}^{ai} = -\frac{3G}{\lambda} \tilde{B}^{ai}] + O(\lambda)$$  \hspace{1cm} (43)

Alternatively, we can find a single Schroedinger equation that governs the propagation of the state in the Chern-Simons time. To find this, we integrate (40), with $N = \tilde{b}^{-1}$, to find, again using (3),

$$\frac{i}{\lambda^2 V(q)} \int_{\Sigma} \sqrt{q}H_{\text{matter}} \chi$$  \hspace{1cm} (44)

where $H_{\text{matter}} = \tilde{q}^{-1}H_{\text{matter}}$ is the undensitized matter hamiltonian and $V(q) = \int_{\Sigma} \sqrt{q} \approx (3G/\lambda)^{3/2} \int_{\Sigma} \tilde{b}$ is the spatial volume of the universe.

These equations tell us that for small cosmological constant, the leading term of the Hamiltonian constraint can be interpreted as the quantum field theory for the matter fields evolving with respect to the Chern-Simons time $\tau_{ICS}$ on a classical background manifold which obeys $\tilde{E}^{ai} = -\frac{3G}{\lambda} \tilde{B}^{ai}$. But the reality conditions then imply that the background must be a DeSitter spacetime. Thus, in the limit of small $\lambda$, and excitations energies small in Planck units, the excitations of the exact state may be interpreted as describing quantum fields evolving on a background DeSitter spacetime. Further, the natural time coordinate on that DeSitter spacetime that emerges from this limit of the Hamiltonian constraint is precisely $\tau_{ICS}$.

In closing this section, we note that one further comment may be made about the hypothesis that the Chern-Simons state describes the vacuum of the theory. We may note that it is indeed remarkable that a state that is an
exact eigenstate of momentum such as is \( \psi_{CS}[A] \) can have an interpretation of an exact vacuum of a quantum field theory. It is indeed for this reason that there has been a suspicion that the Chern-Simons state is unphysical, for such is certainly the case for the corresponding solution of Yang-Mills theory. The difference between these two cases lies in the fact that in quantum gravity the inner product reflects the dynamics of the theory, and must be imposed on the space of solutions to the theory. If the Chern-Simons invariant is time, and if the inner product is taken at fixed time, as we will shortly propose, then there is a possibility that this state may be normalizable. This, and the fact that it is only the inner product that will impose the reality conditions, means that a state may be simultaneously an exact solution to the constraints in an ordering that is, as we have shown here, consistent with the semiclassical limit and a Hamiltonian-Jacobi function for a sector of solutions of the theory.

Further, we note that because the Chern-Simons state is simultaneously a semiclassical state and an exact state, the semiclassical expansion is cleaner than in the conventional formulation\(^{25}\), so that there is no DeWitt-Morette-Van Vleck determinant. The Chern-Simons state must then already contain information about the linearized virtual excitations of the gravitational field.

This means that the Chern-Simons state corresponds to a universe in which only the left-handed curvature is virtually excited. This is sensible because, as the reality conditions are imposed by the inner product, it need not be satisfied by the virtual quantum excitations. This suggests that the Chern-Simons state describes a vacuum at the Planck scale that is naturally chiral, corresponding to a condensate of left-handed nonlinear\(^{26}\) gravitons, while at the same time reproducing the physics of classical general relativity in the semiclassical limit. This suggests that quantum gravity may naturally have \( CP \) violating effects at short distances. Further evidence for this conclusion is in \(^{27}\).

\section*{6 Gravitational perturbations with respect to the conformally self-dual sector}

In the previous sector we saw that, in the presence of a cosmological constant, it is possible to make the hypothesis that the ground state of quantum gravity is given by the Chern-Simons state. To further investigate this hypothesis, it would be necessary to study the higher order corrections to the
WKB limit we have just discussed. While we are not yet able to do this, we make some preliminary remarks in this section.

As we noted, the Chern-Simons state can be understood as being the exponential of a Hamilton-Jacobi function for the self-dual sector. Thus, if we want to understand how to study perturbations of the quantum theory from this vacuum, we must first understand how to describe perturbations away from self-dual sector in the classical theory. In this section we study this question by studying the perturbations of the classical Hamiltonian constraint away from the self-dual sector. To do this, it is important to observe that there are two dimensionless parameters of interest,\[ \tilde{q}^a_i \equiv \tilde{E}^a_i + \frac{3G}{\lambda} \tilde{B}^a_i \] and \( \lambda \), the cosmological constant. \( \tilde{q}^a_i \) vanishes for conformally self-dual solutions, such as the DeSitter solution, and thus describes fluctuations away from the conformally self-dual sector. Thus, to expand around the DeSitter solution, for universes large relative to the Planck scale, we may expand simultaneously in small \( \tilde{q}^a_i \) and small \( \lambda \).

We may then proceed by separating the terms of different orders of \( \lambda \) and \( \tilde{q}^a_i \) in the action of the classical Hamiltonian constraint. We consider first, the case with matter, in the form of a scalar field, as in the previous section. We may note that the classical hamiltonian constraint can be written in terms of the fluctuations \( \tilde{q}^a_i \) as

\[
0 = \epsilon_{abc} \epsilon_{ijk} \left[ \frac{\lambda}{G} \tilde{q}^a_i \tilde{q}^b_j \tilde{q}^c_k - 6 \tilde{B}^a_i \tilde{q}^b_j \tilde{q}^c_k + \frac{9G}{\lambda} \tilde{B}^a_i \tilde{B}^b_j \right] \\
- \left(16\pi G\right) \left[ \frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\tilde{q}^a_i \tilde{q}^b_i - 6G \tilde{B}^a_i \tilde{q}^b_i) + \frac{9G^2}{\lambda^2} \tilde{B}^a_i \tilde{B}^b_i \right] \partial_a \phi \partial_b \phi \]

(46)

First, we may expect that for small \( \tilde{q}^a_i \), the first and last terms in the matter hamiltonian are dominant (\( \frac{G\tilde{B}}{\lambda} \) is of order one). For small \( \tilde{q}^a_i \), the dominant contribution to the pure gravity hamiltonian comes from the last term which is first order in \( \tilde{q}^a_i \). Therefore, to leading order, in the presence of matter we have precisely

\[
\frac{9G}{\lambda} \epsilon_{abc} \epsilon_{ijk} \tilde{B}^a_i \tilde{B}^b_j \tilde{q}^c_k = \left(16\pi G\right) \left[ \frac{1}{2} \tilde{\pi}^2 + \frac{9G^2}{2\lambda^2} \tilde{B}^a_i \tilde{B}^b_i \partial_a \phi \partial_b \phi \right].
\]

(47)

But this, evaluated at \( \tilde{E}^a_i \approx -\frac{3G}{2\lambda} \epsilon_{abc} F^i_{bc} \), corresponds to the semiclassical limit of the Wheeler-DeWitt equation, as we found in the last section. It tells
us how the leading order deviations from the conformally self-dual sector are driven by the matter fields.

What happens in the absence of matter? Here the circumstance is somewhat different, as the leading gravitational term cannot be balanced against the matter Hamiltonian. Instead, we may seek solutions in which the different terms in the gravitational perturbations are balanced against each other.

To find such solutions, let us note that for small \( \lambda \) and \( \tilde{q}^{ai} \), we may neglect the term which is third order in \( \tilde{q}^{ai} \) in (46). The resultant Hamiltonian constraint up to terms quadratic in \( \tilde{q}^{ai} \) is then of the form

\[
q^2 - \frac{9}{2}q + \frac{3\lambda^2}{2G^2b} H_{ai,bj} \tilde{q}^{ai}_\perp \tilde{q}^{bj}_\perp = 0 \tag{48}
\]

where \( q \equiv \frac{3\lambda}{G} \tilde{q}^{ai} \tau_{ai} \) and \( \tilde{q}^{ai}_\perp \equiv q^{ai} - \frac{G}{3\lambda} \tau^{ai} q \). Note that \( \tau_{ai} \tilde{q}^{ai}_\perp = 0 \), and \( q \) is conjugate to \( \int_\Sigma Y_{CS} \). We may then solve this to find

\[
q = \mathcal{H}_\perp (\tilde{\mathcal{B}}, \tilde{q}_\perp) = \frac{9}{2} \left( 1 - \sqrt{1 - \frac{8\lambda^2}{27G^2b} H_{ai,bj} \tilde{q}^{ai}_\perp \tilde{q}^{bj}_\perp} \right) \tag{49}
\]

It is interesting to note that for small perturbations, this has the form of

\[
q = \mathcal{H}_\perp = \frac{2\lambda^2}{3G^2b} H_{ai,bj} \tilde{q}^{ai}_\perp \tilde{q}^{bj}_\perp + O(\tilde{q}^4_\perp) \tag{50}
\]

This shows us that, for small \( \lambda \), the gravitational perturbations of conformally self-dual solutions are stable, at least for the Euclidean sector in which \( \tilde{b}^2 > 0 \). This is a necessary, but of course, not sufficient, condition for the demonstration of the stability of the theory for nonvanishing cosmological constant.

7 Proposal for an inner product

One of the most difficult problems in non-perturbative quantum gravity is the construction of the inner product. While this, also, is a question that we cannot completely settle here, we are able to note that the hypothesis that the Chern-Simons invariant gives us a natural notion of time on the configuration space of general relativity leads to a suggestion for a form of the inner product.
This suggestion is, in principle, very straightforward: if there is a natural time coordinate on the configuration space, then the inner product may be chosen by integrating on a constant time surface of the configuration space. A related suggestion has recently been put forward by Moncrief[21], who suggested that an algebra of physical observables can be constructed from the values of physical fields on the $K = 0$ surface of each spacetime. More particularly, he noted that, at least for a large region of the physical phase space which corresponds to solutions that have $K = 0$ surfaces, a set of physical observables is given by a complete set of spatially diffeomorphism invariant functions of the fields on that surface. These functions have an algebra which is easy to compute, for one can simply evaluate their Poisson brackets, at each solution in the physical phase space, in terms of the Poisson brackets of the fields on the $K = 0$ surface of that spacetime.

This proposal has been explored in a study of a nonperturbative quantization of the Bianchi IX cosmologies[22]. Here we would like to show that a rather suggestive form for the inner product emerges if we take Moncrief’s suggestion seriously, but with two modifications. First, we take the special surface to be identified on the configuration space rather than on each spacetime. This is in line with our philosophy that the configuration space, rather than the spacetime, is the proper arena for the quantum theory. Second, we take the surface of initial time on which the inner product will be defined to be the surface in the configuration space where the imaginary part of the Chern-Simons invariant vanishes. Of course, by (3) this is closely related to the condition that $K$ vanishes. However, taking the condition in terms of an invariant on the configuration space means that a gauge condition must be fixed to identify which surfaces in a particular solution correspond to the $Im \int Y = 0$ surface of configuration space. This allows us to write the inner product in a more gauge-invariant form.

Our proposal is then to consider an inner product of the form

$$< \Psi' | \Psi > = \int \prod ((dA)(d\tilde{A})\delta[\pi_{ICS}(A)]) \tilde{\Psi}'(A)\Delta[\tilde{A}, \tilde{A}]\Psi(A) \quad (51)$$

This is not a complete definition, for the expression $\Delta[\tilde{A}, A]$ must be chosen so that the physical Hamiltonian, defined appropriately on such states is Hermitian, and will also contain gauge fixing factors. The form of $\Delta[\tilde{A}, A]$ that satisfies this condition has not yet been found and involves difficult issues of ordering and regularization. However, we may notice that, even without a complete specification of the inner product, we may see that this general form leads to a suggestive formal expressions for the inner products
of certain states. For if we take seriously the suggestion that the ChernSimons state is be the vacuum of the theory, we may consider modified loop
states of the form

\[ \Psi_\gamma[A] \equiv e^{i \int Y_{CS}[A]} T_\gamma[A] \]

(52)

where \( T_\gamma[A] \) is the usual product of the traces of holonomies of loops. The
functional integral () defining the inner product may for such states be
expressed in terms of a functional integral involving the \( SO(3) \) connection \( \Gamma_{ai} \) and the extrinsic curvature, \( K_{ai} \).

\[
< \Psi' | \Psi > = \int \prod \left( (d\Gamma_{ai}) (dK_{ai}) \delta \left[ Im \int Y(\Gamma, K) \right] \right) \\
\times e^{i \int \Sigma Y_{CS}[\Gamma] - \bar{\epsilon}^{abc} D_{\ell}^{f} K_{ai}} \\
\times T_\gamma'[\Gamma, K] \Delta[A = \Gamma + iK] T_\gamma[\Gamma, K]
\]

(53)

This inner product thus defines a field theory for a multiplet of vector fields, \( K_{ai} \) interacting in a background given by the \( SO(3) \) gauge field \( \Gamma_{ai} \).
As the exponential in (53) is quadratic in \( K_{ai} \) one may develop the integral
over \( K_{ai} \) perturbatively by a power series expansion in \( K_{ai} \).
Meanwhile, as \( T_\gamma[A] \) is a product of holonomies of loops, the integral over \( \Gamma \) might be
defined by using Witten’s discovery that the Chern-Simons path integral
yields the Jones polynomial of knot theory \[12\]. If this can be done, it will
support the conjecture that \( \psi_{CS}[A] \) is indeed the physical ground state of
the theory, by giving meaning to its gravitational excitations of the form of
(52).

In closing this section, we may note that in Section 5 we found evidence
that the imaginary part of the Chern-Simons invariant is playing the role of
time only in the semiclassical limit. The evidence we have for the role of the
Chern-Simons invariant that is beyond the semiclassical limit comes from the
role of the real Chern-Simons invariant in the Euclideanized sector. Neither
completely justifies the step we are taking here, which is to hypothesize
that \( \tau_{ICS} \) plays the role of time for the full, non-perturbative theory. This
proposal can only be confirmed if it leads in fact to physically significant results.
8 The KMS condition for nonperturbative quantum gravity

In this section we would like to discuss a final remarkable consequence of the choice of the Chern-Simons invariant as the time function on the configuration space. This is that the full nonperturbative quantum theory may be intrinsically thermal, in that all expectation values of physical observables satisfy a KMS condition \[13\].

To see why, note that if we study the real section of \(C_{\text{kin}}\) corresponding to Euclidean spacetimes, \(\tau_{CS}\) becomes a periodic coordinate because under large gauge transformations of winding number \(n\), the Chern-Simons form is not invariant but transforms as \(\int Y_{CS} \rightarrow \int Y_{CS} + 8\pi^2 n\) \[5\]. This means that the real section of \(C_{\text{diff}}\) has the topology of \(S^1 \times S\), where \(S\) is the slice of \(C_{\text{diff}}\) defined by \(\int Y_{CS} = 0\). This has a direct consequence, which is the following.

Let us assume that a complete quantum theory of gravity has been defined as a quantum theory of motion on the configuration space \(C_{\text{diff eq}}\). This should allow us to calculate expectation values of the form

\[
< \Psi | \hat{O}_1 \hat{O}_2 ... | \Psi > \tag{54}
\]

where the \(O_i\) are physical, and hence gauge invariant, observables, the \(|\Psi >\)'s are physical states and the expectation value is defined in terms of a physical inner product. Now, the problem of how to construct physical observables that measure evolution in quantum gravity, is well understood, at least in principle (see, for example, \[28\]). What is clear is that such observables describe correlations between certain degrees of freedom, which we take as representing a measure of time, and other degrees of freedom of the theory. It then follows from this that given any choice, \(\tau\) for a measure of time on the physical configuration space it will be the case that for every diffeomorphism invariant function \(O[A,E]\) on the phase space there will be one parameter family of physical observables in the classical theory, \(O(t)\), that measure the value of \(O[A,E]\) averaged over the configurations for which \(t = \tau\). For example, such a function might correspond to the measurement of some spatially diffeomorphism invariant function, \(O\), of \(A_{ai}\) and \(\tilde{E}^{ai}\) on slices that satisfy a certain gauge condition, such as maximal slicing. The definition of the observable, as a function on the physical phase space or, equivalently, on the space of solutions, would be to find that slice in the slicing of each solution for which \(\tau = t\), and then measure \(O\) on that slice.
We may then assume that in a successful quantum theory of gravity there will be one parameter families of quantum observables, \( \hat{O}(t) \), corresponding to some subset of these classical observables. It is natural to assume also that in such a theory there will be a functional integral representation of physical expectation values in terms of paths on the diffeomorphism invariant configuration space \( \mathcal{C}_{\text{diffeo}} \). In this case, then we may expect that there will be a measure \( \mu(A) \) on \( \mathcal{C}_{\text{diff}} \), such that, at least for a preferred vacuum state \( \langle 0 | \),

\[
G^O(t) \equiv \langle 0 | \hat{O}_1(t) \hat{O}_2(0) | 0 \rangle = \int_{\mathcal{C}_{\text{diff}}} d\mu(A) \hat{O}_1(t) \hat{O}_2(0) \tag{55}
\]

Now, given that this will exist, it is natural also to imagine that in Euclidean quantum gravity the expectation values are to be measured in terms of the path integral (55) over Euclidean configurations. If we want to interpret these expectation values in terms of a Euclideanized time, we have to follow the same procedures as in the physical theory, which is to find a good time coordinate \( \tau_{\text{Euc}} \) on the physical configuration space, \( \mathcal{C}_{\text{diff}} \) and construct operators that measure correlations between this degree of freedom and other degrees of freedom of the theory.

Now, we have found in the preceding sections that there is a preferred choice for the Euclideanized time of the theory, which is \( \tau_{\text{CS}} \), the Chern-Simons invariant. It is therefore natural to investigate the behavior of the Euclideanized quantum field theory when the observables are parameterized by this notion of time. It is very interesting to note that it follows immediately, from the periodicity of the configuration space for Euclidean spacetimes \( \mathcal{C}_{\text{diffeo}} \) just mentioned that

\[
G^O_{\text{Euc}}(\tau_{\text{CS}}) = G^O_{\text{Euc}}(\tau_{\text{CS}} + 8\pi^2 n) \tag{56}
\]

Thus, from the geometry of the diffeomorphism invariant configuration space, it must follow that, to the extent that a quantum theory of gravity exists along the conventional lines we have assumed here, physical expectation values in the theory must satisfy the KMS condition, when correlation functions expressed in terms of \( \tau_{\text{CS}} \) are measured. This means that there is a sense in which quantum general relativity must be an intrinsically thermal theory.

We may note that it may also be the case that the Euclidean amplitude \( G^O_{\text{Euc}}[\tau_{\text{CS}}] \) is the analytic continuation of the physical amplitude \( G^O[\tau_{\text{ICS}}] \). However, given the well known difficulties about Euclideanization in quantum gravity, this should not be assumed. It is then important to emphasize
that it is not necessary that this be the case for it still to follow that the Euclideanized quantum theory satisfied a $KMS$ condition when expressed in terms of a natural notion of time for the Euclideanized theory.

Of course, this is a formal argument. However, it is interesting to note that it can be directly confirmed in the semiclassical limit. To show this we must begin by asking an important question: What is the temperature associated with the periodicity we have found? The answer is that, because the period is a dimensionless parameter, there is actually no dimensional temperature associated with the theory. Instead, a particular temperature can arise from the periodicity of the Euclidean signature Chern-Simons invariant only for the case of a semiclassical state. In that case the amplitudes are dominated by a particular periodic trajectory on the configuration space, corresponding to a Euclidean signature spacetime that is periodic in Euclidean time. Then the metric of this spacetime can provide a dimensional measure of the periodicity, which then provides a temperature for the fluctuations around it.

For example, to find the temperature associated with the Chern-Simons state (37), which we conjecture to be the vacuum, we use the fact established above that this state can be understood in the semiclassical limit as describing fluctuations around DeSitter spacetime. It then must follow that the periodicity of the real section of $C_{diff}$ must corresponds to the known periodicity of the Euclidean DeSitter spacetime. In fact, we can use the invariance of the theory under large gauge transformations to discover the periodicity of Euclidean DeSitter spacetime. For constant-$t$ foliations of De-Sitter spacetime, the relation between the Chern-Simons invariant and the spacetime coordinate $t$ is

$$\frac{\partial \tau_{CS}}{\partial t} = \int \Sigma dx \tilde{B}^{ai} \{A_{ni}, N \mathcal{H}\}_{P.B.} = \int \Sigma d^3x N \xi_{abc} \epsilon_{ijk} \tilde{B}^{ai} \tilde{E}^{bj} (\tilde{B}^{ck} + \frac{\lambda}{2G} \tilde{E}^{ck})$$

(57)

In arriving at expression (57), which is the general relation between the variable $t$ in a classical $t$-foliated spacetime and the Chern-Simons functional, we have used $\frac{\partial \tau_{CS}}{\partial A_{ai}} = \tilde{B}^{ai}$. The DeSitter metric may be written as

$$ds^2_{DeSitter} = (1 - \frac{\lambda r^2}{3G}) dt^2 + \frac{1}{(1 - \frac{\lambda r^2}{3G})} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

(58)

On substituting $\sqrt{det(E)}N$ for the lapse function, $\sqrt{1 - \frac{\lambda r^2}{3G}}$, and integrating
over $0 \leq \phi < 2\pi, 0 \leq \theta < \pi, 0 \leq r < \sqrt{\frac{3G}{\lambda}}$, we arrive at

$$\frac{d\tau_{CS}[A_{DeSitter}(t)]}{dt} = -4\pi \sqrt{\frac{\lambda}{3G}}$$

(59)

However, by integrating this we find that the invariance of the theory under large gauge transformations requires that for the Euclidean DeSitter manifold, $t$ must be periodic with period $2\pi \sqrt{3G/\lambda}$. This also gives us the scaling between the dimensionless period of $\tau_{CS}$ and a physical temperature, which then implies that the temperature associated with the Chern-Simons state is indeed,

$$T_{\text{cosmological}} = \frac{1}{2\pi} \sqrt{\frac{\lambda}{3G}}.$$  

(60)

This coincides with the periodicity and the Hawking temperature found by demanding that the metric of the Euclidean DeSitter solution have no conical singularity [15].

In the case of zero cosmological constant, an analogous calculation for the Euclidean Schwarzschild solution with metric

$$ds^2_{\text{Schwarzschild}} = (1 - \frac{2Gm}{r})dt^2 + \frac{1}{(1 - \frac{2Gm}{r})}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

(61)

gives a similar correspondence

$$\frac{d\tau_{CS}[A_{\text{Schwarzschild}}(t)]}{dt} = \frac{\pi}{Gm}$$

(62)

between the periodicity of the Euclidean Chern-Simons term of the Ashtekar-Sen connection and the periodicity of the Euclidean time coordinate with period $8\pi Gm$.

As a result, we may conclude that the use of the connection configuration space, together with the conjecture that the Chern-Simons invariant is a natural time coordinate, gives us a setting in which the connection between periodicity in the Euclidean sector and thermal states may be extended from the semiclassical theory to the full nonperturbative theory. We thus conjecture that quantum gravity may be an intrinsically thermal theory, not only at the semiclassical level [15, 23] but also at the full non-perturbative level.

As a final note to this section, we may observe that the idea that the absolute distinction between quantum and thermal fluctuations might break down in quantum gravity has been raised before [29]. Furthermore, for rather
different reasons, Connes and Rovelli have found independent reasons to conjecture that general states of quantum gravity satisfy a \( KMS \) condition\cite{30}. In their “Thermal Time Hypothesis” they propose that a natural notion of time may be derived from a large class of quantum states of a diffeomorphism invariant quantum field theory such as general relativity. It is worth exploring whether the Chern-Simons time may be a realization of their hypothesis.

9 Conclusions

The diffeomorphism invariance of classical general relativity means that the coordinates on any given spacetime manifold have no physical significance. Instead, time evolution is to be described, in the classical theory, in terms of correlations between different degrees of freedom of the theory. As any of the infinite degrees of freedom may, at least locally, be taken as a clock, classical general relativity then allows an infinitude of possible descriptions of a universe evolving in time.

The problem in time in quantum gravity, especially in the cosmological case, then has two aspects. First, is it possible in the quantum theory, as it is in the classical theory, to describe time evolution in terms of an infinite variety of possible clocks? Second, is it practical to do so? That is, even if it is possible to formally define the theory in such a way as to accommodate evolution by any possible clock, might it be more convenient to define the theory in a simpler way so that evolution with respect to some particular clock is described. Whatever the answer to the first question, which may be considered to be the deep problem of time, it still may be much more convenient to define the theory with respect to a particular notion of evolution, with respect to which the equations may be particularly simple, rather than in a form allowing interpretations in terms of all possible clocks.

We certainly did not answer the deep question of time here. But what was accomplished was to show that the classical configuration space of general relativity has on it a natural function, the Chern-Simons invariant, which may serve as a useful time coordinate for the theory. Furthermore, we saw that this function endows the configuration spaces \( C_{\text{kin}} \) and \( C_{\text{diff}} \) with certain structures which are very convenient for understanding the dynamics of general relativity. This structure, which we described in Sections 2 and 3 represents a certain special kind of geometry in which the metric and
geodesic structure of an infinite dimensional space are defined in terms of a simple function. As we saw in various guises in the course of this paper, it gives us a framework for analyzing dynamical questions in general relativity, which may be of use for understanding a variety of features of the dynamics of classical and quantum general relativity.

At the level of the classical theory, we have found that a study of the role that the Chern-Simons invariant plays in the geometry of the configuration space of general relativity leads to the following results:

i) A splitting of the physical degrees of freedom into gauge, physical, and time like degrees of freedom (Section 2).

ii) The discovery of simple forms for the action principle, as a geodesic principle on the configuration space $C_{\text{diff}}$ (Section 3).

iii) The existence of a canonical transformation that separates the time-like and physical degrees of freedom, and that leads to particularly simple forms for Hamiltonians that evolve the physical degrees of freedom of the gravitational field in terms of time parameters related to the Chern-Simons invariant (Sections 4, for vanishing cosmological constant and 6 for nonvanishing $\lambda$).

We may note that all of these results have been written here for the Euclidean signature case, which is the simplest when dealing with the Ashtekar formulation. If we want to discuss the Minkowskian case we must impose reality conditions. As these mix coordinates and momenta, they are not easily expressed in terms of the configuration space variables. The effect of the reality conditions on these results, in both the classical and quantum theory, is not yet well explored. We may note only that the expectation that quantum states will be, under a certain definition of complex structure, holomorphic in the connection, suggests that these and other results on the geometry of the real configuration space will be very relevant for the Minkowskian quantum theory.

Another key set of issues that has not been explored here are global questions. As quantum states will be functionals on the whole configuration space, it is essential that more be learned about the global topological and differential properties of $C_{\text{diff}}$. We may note that the existence of a preferred function such as $\int Y_{CS}$ on this space may play an important role in this analysis.

Given those properties of the Chern-Simons function as a coordinate on the configuration space that we were able to discover, we went on to study the question of whether the Chern-Simons functional might play a useful role as a time parameter in the quantum theory. The results we found here
are necessarily all, in one way or another incomplete. However, we found two ways in which the Chern-Simons function might play a significant role in the quantum theory.

iv) It emerges in the semiclassical limit of the Hamiltonian constraint as the natural time coordinate for quantum gravity coupled to matter, if the cosmological constant is nonvanishing and the vacuum is taken to be the Chern-Simons state (Section 5). It is interesting to note also that in this limit, the Chern-Simons form reduces to its imaginary part, which is in turn proportional to the trace of the extrinsic curvature. This suggest that in the physical, Minkowskian case, it is the imaginary part of the Chern-Simons invariant that is playing the role of time. However, unlike the case of the Euclideanized theory, we presently have good evidence for this only at the semiclassical level.

v) $\tau_{CS}$ provides a periodic coordinate on the Euclidean section of the configuration space. This means that to the extent that a quantum theory of gravity can be established as a theory of motion on this configuration space, the theory must be intrinsically thermal, in that physical expectation values of the Euclideanized theory must satisfy the KMS condition when expressed in terms of the Chern-Simons time.

In the end, even if it turns out to be possible to invent a form of quantum cosmology that makes sense when evolution is described with respect to any particular clock, it will certainly be the case that important features of the theory will be more transparent when studied with one choice of time than another. We believe that the results we have reported here suggest that whatever the outcome of the deep problem of time in quantum cosmology, the Chern-Simons invariant can play a significant role in our unraveling the dynamics of general relativity, at both the classical and quantum level.

Acknowledgements

We would like to thank Abhay Ashtekar, Lay Nam Chang, Louis Crane, Chris Isham, Ted Jacobson, Karel Kuchar, Guillermo A. Mena Marugan, Vince Moncrief, Jorge Pullin, Carlo Rovelli and Richard Woodard for helpful discussions. This work has been supported by the National Science Foundation under grant NSF-PHY93-96246 and research funds provided by the Pennsylvania State University.
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