Let $x$ be a nilpotent element of an infinite ring $R$ (not necessarily with 1). We prove that $A(x)$—the two-sided annihilator of $x$—has a large intersection with any infinite ideal $I$ of $R$ in the sense that $\text{card}(A(x) \cap I) = \text{card}I$. In particular, $\text{card}A(x) = \text{card}R$; and this is applied to prove that if $N$ is the set of nilpotent elements of $R$ and $R \neq N$, then $\text{card}(R \setminus N) \geq \text{card}N$.

For an element $x$ of a ring $R$, let $A_\ell(x)$, $A_r(x)$, and $A(x)$ denote, respectively, the left, right and two-sided annihilator of $x$ in $R$. For a set $X$, we denote $\text{card}X$ by $|X|$; and say that a subset $Y$ of $X$ is large in $X$ if $|Y| = |X|$. We prove that if $x$ is any nilpotent element and $I$ is any infinite ideal of $R$, then $A(x) \cap I$ is large in $I$, and in particular $|A_\ell(x)| = |A_r(x)| = |A(x)| = |R|$. The last result is applied to obtain a generalization of a result of Putcha and Yaqub [2] which shows that an infinite nonnil ring has infinitely many nonnilpotent elements. A short proof of their result is given in [1]. We prove a much stronger result showing that the set of nonnilpotent elements of a nonnil ring is at least as large as is its set of nilpotent elements. The following lemma is simple but crucial.

**Lemma 1.** Let $R$ be an infinite ring, $(S, +)$ an infinite subgroup of $(R, +)$, and $x$ an element of $R$. Then either $|Sx| = |S|$ or $|A_\ell(x) \cap S| = |S|$, and similarly $|xS| = |S|$ or $|A_r(x) \cap S| = |S|$.

**Proof.** Consider the map $y \mapsto xy$ from $(S, +)$ onto $(Sx, +)$. The kernel is $A_\ell(x) \cap S$, so $|S| = |Sx||A_\ell(x) \cap S|$ and the result follows since $S$ is infinite. \hfill $\square$

A subset of a ring $R$ is said to be root closed if whenever it contains a power of an element, it also contains the element itself.

**Theorem 2.** Let $R$ be an infinite ring and $\alpha$ an infinite cardinal. Then, the following hold.

(i) For any left (right) ideal $I$ of $R$, the set $\{x \in R \mid |A_r(x) \cap I| = \alpha\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = \alpha\}$) is root closed. In particular, if $I$ is infinite, $\{x \in R \mid |A_r(x) \cap I| = |I|\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = |I|\}$) is root closed, so it contains the set $N$ of nilpotent elements of $R$. 

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International Journal of Mathematics and Mathematical Sciences 2005:21 (2005) 3517–3519
DOI: 10.1155/IJMMS.2005.3517
(ii) For any ideal $I$ of $R$, $\{x \in R \mid |A(x) \cap I| = \alpha\}$ is root closed. In particular, if $I$ is infinite, $\{x \in R \mid |A(x) \cap I| = |I|\}$ is root closed, so it contains $N$.

**Proof.** (i) Let $|A_r(x^n) \cap I| = \alpha$ for some $n \geq 2$ and consider $x^{n-1}(A_r(x^n) \cap I)$. By Lemma 1, either $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$ or $|A_r(x^{n-1}) \cap I| = |A_r(x^{n-1}) \cap (A_r(x^n) \cap I)| = \alpha$. Now $x^{n-1}(A_r(x^n) \cap I) \subseteq A_r(x) \cap I \subseteq A_r(x^{n-1}) \cap I \subseteq A_r(x^n) \cap I$, so $|A_r(x^{n-1}) \cap I| = \alpha$ even when $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$. It follows by induction that $|A_r(x) \cap I| = \alpha$.

(ii) Let $|A(x^n) \cap I| = \alpha$ for some $n \geq 2$. Since $A_r(x^n) \cap I$ is a left ideal and $|A_r(x^n) \cap (A_r(x^n) \cap I)| = |A(x^n) \cap I| = \alpha$, it follows by (i) that $|A_r(x) \cap (A_r(x^n) \cap I)| = \alpha = |A_r(x^n) \cap (A_r(x) \cap I)|$; and since $A_r(x) \cap I$ is a right ideal, we get, again by (i), that $|A_r(x) \cap (A_r(x) \cap I)| = \alpha$, namely $|A(x) \cap I| = \alpha$. $\square$

Applying the previous theorem for $I = R$, we obtain the following corollary.

**Corollary 3.** Let $x$ be a nilpotent element of an infinite ring $R$, then $|A_r(x)| = |A(x)| = |A(x)|$.

The previous corollary will be applied in the proof of the above-mentioned generalization of a result of Putcha and Yaqub [2]. We also need the following result.

**Lemma 4.** Let $b$ be a nonnilpotent element of an infinite ring $R$. If $R \setminus N$ is infinite, then $|A_r(b)| \leq |R \setminus N|$ and $|A_r(b)| \leq |R \setminus N|$.

**Proof.** Let $x \in A_r(b) \cap N$, then $xb = 0$ and $x^n = 0$ for some $n \geq 1$. Let $m \geq n$, then $(b + x)^m = bm + b^{m-1}x + \cdots + bx^{m-1}$. Since $(bm-1x + \cdots + bx^{m-1})^2 = 0$ and $b \notin N$, $b^{2m} \neq 0$ and $(b + x)^m \neq 0$, so $b + x \notin N$. Hence, the map $x \mapsto b + x$ is $1 - 1$ from $A_r(b) \cap N$ into $R \setminus N$ and therefore $|A_r(b) \cap N| \leq |R \setminus N|$. Since $R \setminus N$ is infinite, we get that $|A_r(b)| = |A_r(b) \setminus N| + |A_r(b) \cap N| \leq |R \setminus N| + |R \setminus N| = |R \setminus N|$. $\square$

In a ring with 1, the map $x \mapsto 1 + x$ from $N$ into $R \setminus N$ is $1 - 1$, so $|R \setminus N| \geq |N|$. The next theorem shows that the same result holds for any nonnil ring. In particular, we get the result of Putcha and Yaqub [2] stating that $R$ is finite when $R \setminus N$ is finite and not empty.

**Theorem 5.** Let $R$ be a nonnil ring, then $|R \setminus N| \geq |N|$.

**Proof.** We start with $R$ infinite. Suppose $|R \setminus N| < |N|$, then $|N| = |R|$ and $|R \setminus N| < |R|$. By the previous lemma, if $b \in R \setminus N$, $|A_r(b)| \leq |R \setminus N|$, so $|A_r(b)| < |R|$ and by Lemma 1, $|Rb| = |R|$. Now $|R| = |Rb| \leq |N|b + |(R \setminus N)b|$ and $|R \setminus N|b \leq |R \setminus N| < |R|$, so $|N|b = |R|$. Therefore, $|\{b + xb \mid x \in N\}| = |R|$, so since $|R \setminus N| < |R|$, there exists $x \in N$ such that $b + xb \notin R \setminus N$, namely $b + xb \in N$. Since $x \in N, b + x$ is formally invertible, so $A_r(b + xb) = A_r(b)$. By Corollary 3, $|A_r(b + xb)| = |R|$ and by Lemma 4, $|A_r(b)| \leq |R \setminus N| < |R|$, a contradiction.

Now let $R$ be finite and let $J$ be its radical. Since $J$ is nilpotent, if $a \in R$, $a + J$ is nilpotent in $R/J$ if and only if $a$ is nilpotent, and if $a \notin N$, $(a + J) \cap N = \emptyset$. Since $R/J$ is a finite semisimple ring, it has 1, so at least half of its elements are nonnilpotent, hence at least half of the distinct cosets $a + J, a \in R$, do not intersect $N$, and therefore at least half of the elements of $R$ are not nilpotent, so $|R \setminus N| \geq |N|$. $\square$
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