Asymptotic Freedom and Compositeness

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Abstract

We compute the phase and the modulus of an energy- and pressure-free, composite, adjoint, and inert field $\phi$ in an SU(2) Yang-Mills theory at large temperatures. This field is physically relevant in describing part of the ground-state structure and the quasiparticle masses of excitations in the electric phase of the theory. The field $\phi$ possesses nontrivial $S^1$-winding on the group manifold $S^3$. Even at asymptotically high temperatures, where the theory reaches its Stefan-Boltzmann limit, the field $\phi$, though strongly power-suppressed, is conceptually relevant: its presence resolves the infrared problem of thermal perturbation theory.
1 Introduction

In [1] one of us has put forward an analytical and nonperturbative approach to SU(N) Yang-Mills thermodynamics. This approach assumes the existence of a composite, adjoint Higgs field $\phi$, describing part of the thermal ground state, that is, the BPS saturated topologically nontrivial sector of the theory. The field $\phi$ is generated by noninteracting trivial-holonomy SU(2) calorons [16] which are embedded in SU(N). Besides its very existence a nontrivial, periodic dependence of $\phi$’s phase on the Euclidean time $\tau$ was assumed in [1]. The ‘condensation’\(^1\) of trivial-holonomy SU(2) calorons into the field $\phi$ must take place at an asymptotically high temperature [1]. For any physics model formulated in terms of an SU(N) Yang-Mills theory this is to say that caloron ‘condensation’ takes place at $T \sim M_P$ where $M_P$ denotes the Planck scale. Since $|\phi| \sim \sqrt{\frac{\Lambda^3}{2\pi T}}$ topological defects only very marginally deform the ideal-gas expressions for thermodynamical quantities at $T \gg \Lambda$. Here $\Lambda$ denotes the Yang-Mills scale. Every contribution to a thermodynamical quantity, which arises from the topologically nontrivial sector, is power suppressed in temperature. As a consequence, the effective theory is asymptotically free and exhibits the same infrared-ultraviolet decoupling property [1] that is seen in renormalized perturbation theory [2]. Asymptotic freedom is a conceptually very appealing property of SU(N) Yang-Mills theories. It first was discovered in perturbation theory [3].

In the effective thermal theory the effects of the gluon exchange between trivial-

\(^1\)By ‘condensation’ we mean the correlating effects of the classical, BPS saturated, trivial-holonomy configurations in singular gauge, see (9).
holonomy calorons in the ground state are taken into account by obtaining a pure-gauge solution to the classical equations of motion for the topologically trivial sector in the (nonfluctuating and nonbackreacting) background $\phi$. Thus the partition function of the fundamental theory in its electric phase is evaluated in three steps: (i) integrate over a dilute-gas ensemble of trivial-holonomy calorons and anticalorons to derive the (classical) thermodynamics of an adjoint scalar $\phi$, (ii) establish the quantum mechanical and statistical inertness of $\phi$ and use it as a thermal background for integrating over the topologically trivial sector (including the pure-gauge ground-state portion of nontrivial holonomy and the integral over tree-level quasiparticle fluctuations), and (iii) impose thermodynamical selfconsistency to correct for the fluctuation-free treatment of the nontrivial sector by deriving an evolution equation for the effective gauge coupling $e$.

A dynamically generated, nontrivial holonomy of the ground state, expressed in terms of the above pure-gauge configuration, indicates the existence of isolated\(^2\) magnetic charges generated by dissociating nontrivial-holonomy calorons $[4, 5, 6, 7, 8, 9, 10]$. The latter are, in turn, excited by gluon exchanges between trivial-holonomy calorons and anticalorons. A network of (propagating) magnetic monopoles and antimonopoles generates a negative equation of state for the thermal ground state, $\rho^{g.s.} = -P^{g.s.}$ where $\rho^{g.s.} = 4\pi\Lambda^3 T$ for SU(2). A negative equation of state is a familiar feature in polymer physics (see for example $[11]$). Some polymers can effectively be treated in terms of a Chaplygin gas which, in turn, can be employed to describe

\(^2\)For $T \gg \Lambda$ magnetic monopoles are spatially separated, for $T \sim \Lambda$ they overlap.
cosmological evolution [12]. Caloron-induced tree-level masses for gauge-field modes decay as $1/\sqrt{T}$ when heating up the system. Due to the linear rise of $\rho^{g.s.}$ with $T$ the thermodynamics of the ground state is thus subdominant at large temperatures$^3$.

The main purpose of the present work is to compute and to discuss the dynamical

$^3$Excitations are free at large temperatures [13] and contribute to the total pressure and the total energy density like $\sim T^4$. The small residual interactions, which peak close to a 2nd-order transition to the magnetic phase at $T_{c,E}$, are likely to explain the large-angle anomaly in the CMB power spectrum [13, 17]. When cooling the system, monopoles and antimonopoles start to overlap at a temperature $T_o$ slightly higher than $T_{c,E}$. At $T_o$ the total pressure starts to be negative in the electric phase. Naively seen, negative pressure corresponds to an instability of the system causing it to collapse. We usually imagine a contracting system in terms of a decrease of the mean interparticle distance while tacitly assuming the particles to be structure-free. Despite an overall negative pressure contraction in the above sense does not occur in an SU(2) Yang-Mills theory. This can be understood as follows: The charge radius of an isolated monopole is $R = e\sqrt{\frac{2\pi T}{\Lambda^3}}$, and the effective gauge coupling $e$ is constant if monopoles do not overlap, that is, for $T \gg T_o$ (magnetic charge conservation!). At $T_o$ the mean distance between the monopoles and antimonopoles becomes comparable to $R$, and thus annihilation takes place. At the same time new dipoles are created by the dissociation of nontrivial-holonomy calorons. The latter are generated in a large abundance due to the increased strength of the gluon-to-caloron coupling in comparison to the large-temperature situation. Once a dipole is generated monopole and antimonopole quickly separate back-to-back and increase their charge radii until overlap and annihilation take place. On the one hand, the process of separation generates an overall negative pressure (polymer). On the other hand, the process of charge-radius-growth dynamically stabilizes the infall of monopoles until complete annihilation when the very notion of a local collapse ceases to make sense since the associated, infalling particles cease to exist.
generation of an adjoint and composite scalar field $\phi$. This is a first-principle analysis of the ground-state structure in the electric phase of an SU(2) Yang-Mills theory. The paper is organized as follows: In Sec. 2 we write down and discuss a nonlocal definition relevant for the determination of $\phi$’s phase in terms of a spatial and scale-parameter average over an adjointly transforming series of $n$-point functions. This series needs to be evaluated on trivial-holonomy caloron and anticaloron configurations at a given time $\tau$. It is very likely that the series is an asymptotic expansion in a dimensionless parameter $\xi$, which, however, is irrelevant for our analysis. In Sec. 3 we perform the average for the two-point case and discuss the occurrence of a global gauge freedom in $\phi$’s phase having a geometrical interpretation. In Sec. 4 we show how the derived information about a nontrivial $S^1$ winding of the field $\phi$ together with analyticity of the right-hand side of the associated BPS equation can be used to uniquely construct a potential determining $\phi$’s thermodynamics. In Sec. 5 we semi-quantitatively discuss the ground state structure for the magnetic phase of an SU(2) Yang-Mills theory where magnetic monopoles are condensed. In Sec. 6 we summarize and discuss our results and give an outlook on future research.

2 Definition of $\phi$’s phase

In this section we discuss the BPS saturated, topological part of the ground-state physics in the electric phase of an SU(2) Yang-Mills theory. According to the approach in [1] the adjoint scalar $\phi$ is an energy- and pressure-free field if interactions
between trivial-holonomy calorons, mediated by the topologically trivial sector of
the theory, are neglected. To describe the thermal ground state these interactions
are taken into account in an exact way by a pure-gauge solution to the classical,
trivial-topology gauge-field equations in the background $\phi$. This is consistent since
$\phi$’s quantum mechanical and statistical inertness can be established. Without as-
suming the existence of a Yang-Mills scale $\Lambda$ only $\phi$’s phase, that is $\frac{\phi}{|\phi|}$, can be
computed from first principles. A computation of $\phi$ itself requires the existence of $\Lambda$. As we shall see, the information about the $S^1$ winding of $\phi$’s phase together with
the analyticity of the right-hand side of $\phi$’s BPS equation uniquely determines $\phi$’s modulus in terms of $\Lambda$ and $T$.

Let us first set up some prerequisites. We consider periodic-in-Euclidean-time $\tau$ ($0 \leq \tau \leq \frac{1}{T}$) and BPS saturated solutions of trivial holonomy to the Yang-Mills
equation

$$D_\mu F_{\mu\nu} = 0$$

(1)
with topological charge $4 \pm 1$. They are [16]

$$A^C_\mu(\tau, x) = \tilde{\eta}_{\mu
u} \frac{\lambda^a}{2} \partial_\nu \ln \Pi(\tau, x) \quad \text{or}$$

$$A^A_\mu(\tau, x) = \eta_{\mu
u} \frac{\lambda^a}{2} \partial_\nu \ln \Pi(\tau, x) \quad (2)$$

where the 't Hooft symbols $\eta^a_{\mu\nu}$ and $\tilde{\eta}^a_{\mu\nu}$ are defined as

$$\eta_{\mu\nu} = \epsilon_{\mu\nu} + \delta_{\mu a} \delta_{\nu 4} - \delta_{\nu a} \delta_{\mu 4}$$

$$\tilde{\eta}_{\mu\nu} = \epsilon_{\mu\nu} - \delta_{\mu a} \delta_{\nu 4} + \delta_{\nu a} \delta_{\mu 4} \quad (3)$$

The solutions in Eq. (2) (the superscript (A)C refers to (anti)caloron) are generated by a temporal mirror sum of the 'pre'potential $\Pi$ of a single (anti)instanton in singular gauge [15]. They have the same color orientation as the 'seed' instanton or 'seed' antiinstanton. In Eq. (2) $\lambda^a, (a = 1, 2, 3)$, denote the Pauli matrices.

The 'nonperturbative' definition of the gauge field is used were the gauge coupling constant $g$ is absorbed into the field.

The scalar function $\Pi(\tau, x)$ is given as [16]

$$\Pi(\tau, x) = \Pi(\tau, r) \equiv 1 + \frac{\pi \rho^2}{\beta r} \sinh \left( \frac{2\pi r}{\beta} \right) \cosh \left( \frac{2\pi r}{\beta} \right) - \cos \left( \frac{2\pi r}{\beta} \right), \quad (4)$$

4Configurations with higher topological charge and trivial holonomy have been constructed, see for example [18, 19]. They should, in principle, contribute to the ground-state thermodynamics of the theory in terms of additional adjoint scalar fields, but they are also power suppressed in the small parameter $\exp[-8\pi^2 g^2]$, where $g$ at most is order unity [14], as compared to the charge-one case. We thus do not consider higher topological charges here but leave their investigation to future research.
where \( r \equiv |\mathbf{x}|, \beta \equiv 1/T, \) and \( \rho \) denotes the scale parameter. At a given \( \rho \) the solutions in Eq. (2) can be generalized by shifting the center from \( z = 0 \) to \( z = (\tau_z, \mathbf{z}) \) by the (quasi) translational invariance of the classical action\(^5\). If a spatial integral of a local density is considered the choice \( z = 0 \) is no restriction of generality. From the BPS saturation

\[
F_{\mu\nu}[A^{(C,A)}] = (+, -) \tilde{F}_{\mu\nu}[A^{(C,A)}] \tag{5}
\]

it follows that the (Euclidean) energy-momentum tensor \( \theta_{\mu\nu} \), evaluated on \( A_{\mu}^{(C,A)} \), vanishes identically

\[
\theta_{\mu\nu}[A^{(C,A)}] \equiv 0. \tag{6}
\]

This property translates to the macroscopic field \( \phi \) with energy-momentum tensor \( \bar{\theta}_{\mu\nu} \) in an effective theory if \( \phi \) is obtained by an ensemble average over calorons and anticalorons neglecting their interactions

\[
\bar{\theta}_{\mu\nu}[\phi] \equiv 0. \tag{7}
\]

Since \( \phi \) is spatially homogeneous, that is, describing a thermal ground state, Eq. (7) is equivalent to \( \phi \) being BPS saturated in its time dependence. Thus \( \phi \) solves the first-order equation

\[
\partial_\tau \phi = V^{(1/2)} \tag{8}
\]

where \( V^{(1/2)} \) denotes the 'square-root' of a suitable potential\(^6\) \( V \equiv \text{tr} V^{(1/2)} V^{(1/2)} \).

In Eq. (8) the right-hand side is determined only up to a global gauge rotation, \( \tau_z \) needs to be restricted as \( 0 \leq \tau_z \leq \beta \).

\(^5\)The fact that an ordinary and not a covariant derivative appears in Eq. (8) is, of course, tied to our specific gauge choice. If we were to leave the (singular) gauge for the (anti)instanton, in which
Figure 1: Possible directions of winding of $\phi_{|\phi|}(\tau)$ around the group manifold $S^3$ of SU(2). The angles $\gamma, \delta$ are arbitrary but constant. They are determined by the choice of plane in which angular regularization is carried out. The angle $\alpha(\tau)$ parametrizes the $S^1$ winding of $\phi_{|\phi|}$.

see Fig. 1. It is important to note at this point already that the Yang-Mills scale parametrizes the potential $V$ and thus also the classical solution to Eq. (8). In the absence of trivial-topology fluctuations it is, however, invisible, see Eq. (7). Only after the macroscopic equation of motion for the trivial-topology sector is solved for a pure-gauge configuration in the background $\phi$ does the existence of a Yang-Mills scale become manifest by a nonvanishing ground-state pressure and a nonvanishing ground-state energy density [1]. Hence the trace anomaly $\tilde{\theta}_{\mu\nu} \neq 0$ for the total energy-momentum tensor $\tilde{\theta}_{\mu\nu} \equiv \tilde{\theta}_{\mu\nu}^{g.s.} + \theta_{\mu\nu}^{\text{fluc}}$ which includes the effects of trivial-the solutions of Eq. (2) are constructed, by a time-dependent gauge rotation $\tilde{\Omega}(\tau)$ then a pure-gauge configuration $A_{\mu}^{p,g.}(\tau) = i\delta_{\mu4}\tilde{\Omega}^{\dagger}\partial_{\tau}\tilde{\Omega}$ would appear in a covariant derivative on the left-hand side of Eq. (8).
topology fluctuations. Since $\bar{\theta}_{\mu\nu} = 4\pi T \Lambda^3 \delta_{\mu\nu}$ and $\theta_{\mu\nu} \propto T^4$ for $T \gg \Lambda$ the trace anomaly dies off as $\Lambda^3 T^3$.

The crucial question is how uniquely determined $V$ is. In [1] uniqueness was obtained if one insists on a nontrivial periodicity (a winding) $\phi(\tau = 0) = \phi(\tau = \beta)$, such that the modulus $|\phi|$ (a gauge invariant quantity which determines the quasiparticle spectrum and the ground-state energy-density of the theory) is spacetime independent, and on an analytical dependence of $V^{(1/2)}$ on $\phi$. Here we will show that the constraint of nontrivial periodicity is redundant by computing the two-point contribution to the $\tau$ dependence of a function that satisfies the same homogeneous equation of motion as $|\phi|$ itself, and demanding BPS saturation of the solution. The latter requirement is an immediate consequence of Eq. (7).
We define:

\[
\frac{\phi^a}{|\phi|}(\tau) \sim \text{tr} \left[ \right.
\beta^0! \int d^3x \int d^3y \int d^3z \int d^3\rho \\
\left. \frac{\lambda^a}{2} F_{\mu\nu}[A_\alpha(\rho, \beta)] ((\tau, 0)) \{(\tau, 0), (\tau, x)\} [A_\alpha(\rho, \beta)] \times \\
F_{\mu\nu}[A_\alpha(\rho, \beta)] ((\tau, x)) \{(\tau, x), (\tau, 0)\} [A_\alpha(\rho, \beta)] + \\
\beta^{-1!} \int d^3x \int d^3y \int d^3z \int d^3\rho \\
\frac{\lambda^a}{2} F_{\mu\lambda}[A_\alpha(\rho, \beta)] ((\tau, 0)) \{(\tau, 0), (\tau, x)\} [A_\alpha(\rho, \beta)] \times \\
F_{\mu\nu}[A_\alpha(\rho, \beta)] ((\tau, x)) \{(\tau, x), (\tau, y)\} [A_\alpha(\rho, \beta)] \times \\
F_{\nu\mu}[A_\alpha(\rho, \beta)] ((\tau, y)) \{(\tau, y), (\tau, 0)\} \right].
\]

\[\text{(9)}\]

In (9) the \(\sim\) sign indicates that both left- and right-hand sides satisfy the same homogeneous evolution equation in \(\tau\)

\[
\mathcal{D} \left[ \frac{\phi}{|\phi|} \right] = 0.
\]

\[\text{(10)}\]

Here \(\mathcal{D}\) is an operator such that Eq. (10) represents a homogeneous differential equation. As it will turn out, Eq. (10) is a linear second-order equation which, up to
Figure 2: The (asymptotic) series for a function satisfying the same homogeneous evolution equation as $\phi$’s phase does.

global gauge rotations, determines the first-order or BPS equation whose solution $\phi$’s phase is.

The dots in (9) stand for the contributions of higher $n$-point functions and for reducible, that is factorizable, contributions as far as the spatial integrations are concerned. The situation is depicted diagrammatically in Fig. 2. Each term in the series is understood as a sum over the two solutions in Eq. (2), that is, $A_\alpha = A^C_\alpha$ or $A_\alpha = A^A_\alpha$. Notice that the power $\beta^{-(n-2)}$ in the prefactor of the integral over the $n$-point function is canceled by a power $\beta^{(n-2)}$ arising from the respective integral. Together this corresponds to a power $\xi^{(n-2)}$ of a dimensionless parameter $\xi$. Thus the normalization of the $\tau$ dependence in the right-hand side of (9) is independent of $\beta$. There is no apparent reason why the factorially rising multiplicity of diagrams, expressed by the prefactor $(n-1)!$ in (9), can be compensated for by the dependence on $n$ of the corresponding integrals. We would then conclude that the expansion in
powers of $\xi$ in (9) is asymptotic at best. This, however, is irrelevant for our analysis, since we are only interested in the operator $D$ which annihilates the right-hand side of (9). Since $D$ is linear an annihilation must take place for each power in $\xi$ separately, and thus we can learn about $D$ by an analysis of the contribution arising from 2-point correlations only, that is, the first diagram in Fig. 2. In contrast to the higher $n$-point correlations the evaluation of the 2-point case is computationally feasible.

For (9) the following definitions apply:

\[
|\phi| \equiv \frac{1}{2} \text{tr} \phi^2,
\]

\[
\{(\tau, 0), (\tau, x)\} [A_\alpha] \equiv \mathcal{P} \exp \left[ i \int_{(\tau,0)}^{(\tau,x)} dy_\beta A_\beta(y, \rho) \right],
\]

\[
\{(\tau, x), (\tau, 0)\} [A_\alpha] \equiv \mathcal{P} \exp \left[ -i \int_{(\tau,0)}^{(\tau,x)} dy_\beta A_\beta(y, \rho) \right]. \tag{11}
\]

The integral in the Wilson lines in Eqs. (11) is along a straight line connecting the points $(\tau, 0)$ and $(\tau, x)$, and $\mathcal{P}$ denotes the path-ordering symbol.

Under a microscopic gauge transformation $\Omega(y)$ the following relations hold:

\[
\{(\tau, 0), (\tau, x)\} [A_\alpha] \rightarrow \Omega^\dagger((\tau, 0)) \{(\tau, 0), (\tau, x)\} [A_\alpha] \Omega((\tau, x)),
\]

\[
\{(\tau, x), (\tau, 0)\} [A_\alpha] \rightarrow \Omega^\dagger((\tau, x)) \{(\tau, x), (\tau, 0)\} [A_\alpha] \Omega((\tau, 0)),
\]

\[
F_{\mu\nu}[A_\alpha] ((\tau, x)) \rightarrow \Omega^\dagger((\tau, x)) F_{\mu\nu}[A_\alpha](\{(\tau, x), (\tau, 0)\}) \Omega((\tau, x)),
\]

\[
F_{\mu\nu}[A_\alpha] ((\tau, 0)) \rightarrow \Omega^\dagger((\tau, 0)) F_{\mu\nu}[A_\alpha](\{(\tau, 0), (\tau, x)\}) \Omega((\tau, 0)). \tag{12}
\]

As a consequence of Eq. (12) the right-hand side of (9) transforms as

\[
\frac{\phi^a}{|\phi|}(\tau) \rightarrow R_{ab}(\tau) \frac{\phi^b}{|\phi|}(\tau) \tag{13}
\]
where the SO(3) matrix $R_{ab}(\tau)$ is defined as

$$R^{ab}(\tau)\lambda^b = \Omega((\tau,0)) \lambda^a \Omega^\dagger((\tau,0)). \quad (14)$$

Thus we, indeed, have defined an adjointly transforming scalar in (9). Moreover, we have just shown that only the time-dependent part of a microscopic gauge transformation survives on the thermodynamical level$^7$.

Since one of the two available length scales $\rho$ and $\beta$ parametrizing the caloron or the anticaloron is integrated over in (9) the only scale responsible for a nontrivial $\tau$ dependence of $\phi_a$ is $\beta$. Notice that a Yang-Mills scale arises due to trivial-topology fluctuations on top of BPS saturated backgrounds [15]. The weight for the $\rho$-integration in (9) is trivial because $\rho$ is a modulus of the classical caloron or anticaloron action. Except for translations in $\tau$, which must not be integrated over, we remark the following for the integration over the other moduli of the solutions in Eq. (2): The integration over spatial translations is trivial in the functional average over the moduli space since the right-hand side of (9) is translation-invariant. For gauge invariant (nonlocal or local) densities the functional integral over global color rotations generates a factor unity. For the gauge variant density in (9) an average over global color rotations would yield zero and thus is forbidden$^8$. The definition

$^7$Shifting the spatial part of the argument $(\tau,0) \rightarrow (\tau,z)$ in (9) introduces a finite parameter $z$ to the gauge rotation $R^{ab}: R^{ab}(\tau) \rightarrow R^{ab}(\tau,z)$. This is nothing but a global gauge rotation acting on $R^{ab}(\tau)$ as defined in Eq. (14). We will show below that only the asymptotic behaviour ($r \rightarrow \infty$) is relevant for deducing the operator $\mathcal{D}$ from (9), and thus we do not expect a shift $(\tau,0) \rightarrow (\tau,z)$ to be of physical significance.

$^8$The 'naked' gauge charge is needed for a coupling of the trivial topology sector to the ground-
(9) exploits the correlations inherent in the classical configurations (2).

3 Computation of two-point correlation

Before we perform the actual calculation let us stress some simplifying properties of the solutions $A^{(C,A)}_\mu$ in Eq. (2).

The path-ordering prescription for the Wilson lines $\{(\tau, 0), (\tau, x)\}$ and $\{(\tau, 0), (\tau, x)\}^*$ in Eq. (11) can actually be omitted if we truncate the series in (9) on the level of two-point correlations. To see this, we first consider the quantity $P^{(C,A)}(\tau, r st)$ defined as

$$P^{(C,A)}(\tau, r st) \equiv A^{(C,A)}_i(\tau, s r t) t_i \quad (15)$$

where $0 \leq r \leq \infty$, $0 \leq s \leq 1$, $(i = 1, 2, 3)$. The vector $t$ denotes the unit line-tangential along the straight line connecting the points $(\tau, 0)$ and $(\tau, x) \equiv (\tau, rt)$.

We have

$$\{(\tau, 0), (\tau, rt)\}^{(C,A)} = \mathcal{P} \exp \left[ i r \int_0^1 ds \ P^{(C,A)}(\tau, s rt) \right] \quad (16)$$

where

$$P^{(C,A)}(\tau, s rt) = \mp \frac{1}{2} t \cdot \lambda \partial_4 \ln \Pi(\tau, sr) \quad (17)$$

Thus the path-ordering symbol can, indeed, be omitted in Eq. (16). The field state generating (i) quasiparticle masses and (ii) finite values of the ground-state energy density and the ground-state pressure [1].
strength $F^C_{a\mu\nu}$ on the caloron solution in Eq. (2) is

$$F^C_{a\mu\nu} = \bar{\eta}_{a\mu\nu} \frac{(\partial_\kappa \Pi)(\partial_\kappa \Pi)}{\Pi^2} + \bar{\eta}_{a\mu\nu} \frac{\Pi (\partial_\nu \partial_\kappa \Pi) - 2 (\partial_\kappa \Pi) (\partial_\nu \Pi)}{\Pi^2}
- \bar{\eta}_{a\kappa\nu} \frac{\Pi (\partial_\mu \partial_\kappa \Pi) - 2 (\partial_\kappa \Pi) (\partial_\mu \Pi)}{\Pi^2}$$

(18)

where $\Pi$ is defined in Eq. (4). For the anticaloron one replaces $\bar{\eta}$ by $\eta$ in Eq. (18).

Using Eqs. (9), (16), and (18), we obtain the following expression for the two-point contribution $\frac{\phi^a}{|\phi|}\bigg|_{\text{2-point},C}$ arising from calorons:

$$\frac{\phi^a}{|\phi|}\bigg|_{\text{2-point},C} \sim i \int d\rho \int d^3x \frac{x^a}{r} \left[ \frac{(\partial_4 \Pi(\tau,0))^2}{\Pi^2(\tau,0)} - \frac{2 \partial_4^2 \Pi(\tau,0)}{3 \Pi(\tau,0)} \right] \times \left\{ (1 - 3 \cos(2g(\tau, r))) \left[ \frac{\partial_\tau \partial_4 \Pi(\tau, r)}{\Pi(\tau, r)} - \frac{2 (\partial_\tau \Pi(\tau, r)) (\partial_4 \Pi(\tau, r))}{\Pi^2(\tau, r)} \right] + \sin(2g(\tau, r)) \left[ 4 \frac{(\partial_4 \Pi(\tau, r))^2 - (\partial_\tau \Pi(\tau, r))^2}{\Pi^2(\tau, r)} + 2 \frac{\partial_\tau^2 \Pi(\tau, r) - \partial_4^2 \Pi(\tau, r)}{\Pi(\tau, r)} \right] \right\}$$

(19)

where

$$g(\tau, r) \equiv \int_0^1 ds \frac{r}{2} \partial_4 \ln \Pi(\tau, sr)$$

(20)

and

$$\frac{(\partial_4 \Pi(\tau, 0))^2}{\Pi^2(\tau, 0)} - \frac{2 \partial_4^2 \Pi(\tau, 0)}{3 \Pi(\tau, 0)} = -\frac{16}{3} \frac{\rho^2}{\beta} \frac{\pi^2 \rho^2 + \beta^2 \left( 2 + \cos \left( \frac{2\pi \rho}{\beta} \right) \right)}{2\pi^2 \rho^2 + \beta^2 \left( 1 - \cos \left( \frac{2\pi \rho}{\beta} \right) \right)^2}. \quad (21)$$

The dependences on $\rho$ and $\beta$ are suppressed in the integrands of (19) and Eq. (20). It is worth mentioning that the integrand in Eq. (20) is proportional to $\delta(s)$ for $r \gg \beta$. 

15
A useful set of identities is

\[ F^C_{\mu\nu}(\tau, x) = F^A_{\mu\nu}(\tau, -x) \]

\[ \{(\tau, 0), (\tau, x)\}^C = \{(\tau, x), (\tau, 0)\}^A\]

\[ \{(\tau, 0), (\tau, -x)\}^A = \{(\tau, -x), (\tau, 0)\}^A\]. \hspace{1cm} (22)

Eqs. (22) state that the integrand for \( \frac{\phi^a}{|\phi|} \big|_{\text{2-point}, A} \) can be obtained by a parity transformation \( x \rightarrow -x \) of the integrand for \( \frac{\phi^a}{|\phi|} \big|_{\text{2-point}, C} \). Since the latter changes its sign, see (19), one naively would conclude that

\[ \frac{\phi^a}{|\phi|} \big|_{\text{2-point}} = \frac{\phi^a}{|\phi|} \big|_{\text{2-point}, C} + \frac{\phi^a}{|\phi|} \big|_{\text{2-point}, A} = 0. \] \hspace{1cm} (23)

This, however, would only be the case if no ambiguity in evaluating the integral in both cases existed. But such an ambiguity does occur! To see this, we need to investigate the convergence properties of the radial integration in (19). It is easily checked that all terms give rise to a converging \( r \) integration except for the following one:

\[ 2 \frac{x^a}{r} \sin(2g(\tau, r)) \frac{\partial^2 \Pi(\tau, r)}{\Pi(\tau, r)}. \] \hspace{1cm} (24)

Namely, at \( r > R \gg \beta \) (24) goes over in

\[ 4 t^a \frac{\pi \rho^2 \sin(2g(\tau, r))}{\beta r^3}. \] \hspace{1cm} (25)

Thus the \( r \)-integral of the term in (24) is logarithmically divergent in the infrared\(^9\):

\[ 4 t^a \frac{\pi \rho^2}{\beta} \int_{R}^{\infty} \frac{dr}{r} \sin(2g(\tau, r)) \]. \hspace{1cm} (26)

\(^9\)The integral converges for \( r \rightarrow 0 \).
Recall, that \( g(\tau, r) \) behaves like a constant in \( r \) for \( r > R \). The angular integration, on the other hand, would yield zero if the radial integration was regular. Thus a logarithmic divergence can be cancelled by the angular integral to yield some finite and real answer. To investigate this, both angular and radial integration need to regularized.

We may regularize the \( r \) integral in (26) by prescribing

\[
\int_R^\infty \frac{dr}{r} \to \beta^\epsilon \int_R^\infty \frac{dr}{r^{1+\epsilon}}. \tag{27}
\]

with \( \epsilon > 0 \). We have

\[
\beta^\epsilon \int_R^\infty \frac{dr}{r^{1+\epsilon}} = \beta^\epsilon \int_0^\infty \frac{dr}{(r+R)^{1+\epsilon}} = \frac{1}{\epsilon} - \log \left( \frac{R}{\beta} \right) + \frac{1}{2} \epsilon \log^2 \left( \frac{R}{\beta} \right) + \cdots. \tag{28}
\]

Away from the pole at \( \epsilon = 0 \) this is regular. For \( \epsilon < 0 \) Eq. (28) can be regarded as a legitimate analytical continuation. The ambiguity inherent in Eq. (28) relates to how one circumvents the pole in the smeared expression

\[
\frac{1}{2\eta} \int_{-\eta}^\eta d\epsilon \left( \frac{1}{\epsilon \pm i0} - \log \left( \frac{R}{\beta} \right) + \frac{1}{2} \epsilon \log^2 \left( \frac{R}{\beta} \right) + \cdots \right)
= \mp \frac{\pi i}{2\eta} - \log \left( \frac{R}{\beta} \right) + \cdots, \quad (\eta > 0, \eta \ll 1). \tag{29}
\]

As for the regularization of the angular integration we may introduce defect (or surplus) angles \( 2\eta' \) in the \( \theta \) integration as

\[
\int_0^{\pi} d\omega \sin \omega \int_0^{2\pi} d\theta \to \int_0^{\pi} d\omega \sin \omega \int_{\alpha_0 \pm \eta'}^{\alpha_0 + 2\pi + \eta'} d\theta. \tag{30}
\]

In Eq. (30) \( \alpha_0 \) is a constant angle with \( 0 \leq \alpha_0 \leq 2\pi \) and \( \eta' > 0, \eta' \ll 1 \). Obviously,
this regularization singles out the $x_1x_2$ plane. As we shall show below, the choice of regularization plane translates into a global gauge choice for $\phi$’s phase and thus is physically irrelevant. The value of $\alpha_0$ is determined by a (physically irrelevant) initial condition. We have

$$
\int_0^{\pi} d\omega \sin \omega \int_{\alpha_0 \pm \eta'}^{\alpha_0 + 2\pi \pm \eta'} d\theta t^a \sim \mp \pi \eta' (\delta_{a1} \cos \alpha_0 + \delta_{a2} \sin \alpha_0).
$$

To see what is going we fix for the time being the ratio $\frac{\eta'}{\eta}$ to a finite and positive but otherwise arbitrary constant $\Xi$ when sending $\eta$ and $\eta'$ to zero in the end of the calculation:

$$
\lim_{\eta, \eta' \to 0} \frac{\eta'}{\eta} = \Xi.
$$
Combining Eqs. (26),(29),(31), (21), expression (19) reads:

\[
\frac{\phi^a}{|\phi|}_{\text{2-point}, C} \sim \pm \frac{32}{3} \pi^7 \frac{\delta_1 \cos \alpha_0 + \delta_2 \sin \alpha_0}{\beta^3} \int d\rho \left[ \lim_{\rho \to \infty} \sin(2g(\tau, r)) \right] \times
\]

\[
\rho^4 \left[ \frac{\pi^2 \rho^2 + \beta^2 \left( 2 + \cos \left( \frac{2\pi \tau}{\beta} \right) \right)}{\left[ 2\pi^2 \rho^2 + \beta^2 \left( 1 - \cos \left( \frac{2\pi \tau}{\beta} \right) \right) \right]^2} \right]
\]

\[
\equiv \pm \Xi \left( \delta_1 \cos \alpha_0 + \delta_2 \sin \alpha_0 \right) A \left( \frac{2\pi \tau}{\beta} \right)
\]  

(33)

where \( A \) is a dimensionless function of its dimensionless argument. The sign ambiguity in (33) arises from the ambiguity associated with the way how one circumvents the pole in Eq. (29) and whether one introduces a surplus or a defect angle in (30).

If we agree upon circumventing the pole for the anticaloron contribution in the opposite way as compared to the caloron contribution and on using the same angular regularization in both cases then we have

\[
\frac{\phi^a}{|\phi|}_{\text{2-point}} = \frac{\phi^a}{|\phi|}_{\text{2-point}, C} + \frac{\phi^a}{|\phi|}_{\text{2-point}, A} = 2 \times \frac{\phi^a}{|\phi|}_{\text{2-point}, C}
\]

where \( \frac{\phi^a}{|\phi|}_{\text{2-point}, C} \) is given in (33). Eq. (34) is the basis for fixing the operator \( D \) in Eq. (10).

To evaluate (33) numerically, we introduce a cutoff for the \( \rho \) integration as follows:

\[
\int d\rho \to \int_0^{\zeta \beta} d\rho , \quad (\zeta > 0).
\]

(34)

This introduces an additional dependence of \( A \) on \( \zeta \). In Fig. 4 the \( \tau \) dependence of \( A \) for various values of \( \zeta \) is depicted. It can be seen that

\[
A \left( \frac{2\pi \tau}{\beta}, \zeta \to \infty \right) \to 272 \zeta^3 \sin \left( \frac{2\pi \tau}{\beta} \right).
\]

(35)
Figure 4: $A$ as a function of $\frac{2\pi}{\beta} \tau$ for $\zeta = 1, 2, 10$. For each case the dashed line is a plot of $\max A \times \sin \left( \frac{2\pi}{\beta} \tau \right)$. We have fitted the asymptotic dependence on $\zeta$ of the amplitude of $A$ as $A \left( \frac{2\pi}{\beta} \tau = \frac{\pi}{2}, \zeta \right) = 272 \zeta^3$, $(\zeta > 10)$. The fit is stable under variations of the fitting interval. For the case $\zeta = 10$ the difference between the two curves cannot be resolved anymore.
Therefore we have

$$\frac{\phi^a}{|\phi|_{\text{2-point}}} \sim \pm 544 \zeta^3 \Xi (\delta_{a1} \cos \alpha_0 + \delta_{a2} \sin \alpha_0) \sin \left(\frac{2\pi \beta \tau}{\beta}\right) \equiv \pm 544 \zeta^3 \Xi \hat{\phi}^a. \quad (36)$$

The number $\zeta^3 \Xi$ in (36) is undetermined. This reflects the fact that on the classical level the theory is scale invariant. To give a meaning to this number, a mass scale needs to be generated dynamically. This, however, can only happen due to dimensional transmutation, which is known to be an effect induced by trivial-topology fluctuations [3]. The result in (36) is highly nontrivial since it is obtained only after an integration over the entire admissible part of the moduli spaces of (anti)calorons is performed.

Let us now discuss the physical content of (36). For a fixed value of $\alpha_0$ the right-hand side resembles a fixed linear polarization in adjoint color space. Since $\zeta^3 \Xi$ is determined by an initial condition to an evolution equation and since this initial condition is unknown we need to consider the evolution equation itself. This equation derives as follows. The function $\hat{\phi}$ as defined in (36) reads

$$\hat{\phi} = \pm \exp (-i\alpha_0 \lambda_3) \lambda_1 \sin \left(\frac{2\pi}{\beta} \frac{\beta \tau}{\beta}\right)$$

$$\equiv \pm \exp (-i\alpha_0 \lambda_3) \left(\hat{\phi}^+ + \hat{\phi}^\mp\right) \quad \text{(37)}$$

where

$$\hat{\phi}^\pm \equiv \pm \frac{\lambda_2}{2} \exp \left(\pm \frac{2\pi i}{\beta} \lambda_3 \tau\right). \quad \text{(38)}$$

The functions $\hat{\phi}^+$ and $\hat{\phi}^-$ obey the following equations

$$\partial_\tau \hat{\phi}^\pm = \pm \frac{2\pi i}{\beta} \lambda_3 \hat{\phi}^\pm. \quad \text{(39)}$$
From Eqs. (39) it follows that
\[ \partial_\tau \left( \hat{\phi}^+ \pm \hat{\phi}^- \right) = \frac{2\pi i}{\beta} \lambda_3 \left( \hat{\phi}^+ \mp \hat{\phi}^- \right). \] (40)

Differentiating the + case in Eq. (40) and substituting the − case into the result yields
\[ \partial_\tau^2 \left( \hat{\phi}^+ + \hat{\phi}^- \right) = \left( \frac{2\pi}{\beta} \right)^2 \left( \hat{\phi}^+ + \hat{\phi}^- \right) \] or
\[ \partial_\tau^2 \hat{\phi} = \left( \frac{2\pi}{\beta} \right)^2 \hat{\phi}. \] (41)

Eq. (41) is a second-order differential equation. What we need to assure the validity of Eq. (7) is a BPS saturation\(^{10}\) of the solution to Eq. (41). Thus we need to find first-order equations whose solutions solve the second-order equation (41). The relevant two first-order equations are
\[ \partial_\tau \hat{\phi} = \pm \frac{2\pi i}{\beta} \lambda_3 \hat{\phi}. \] (42)

Obviously, Eqs. (42) are subject to a global gauge ambiguity, see Fig. 1. Moreover, a solution to either of the two equations (42) also solves Eq. (41). Solutions to Eqs. (42) are given as
\[ \hat{\phi} = C \lambda_1 \exp \left( \pm \frac{2\pi i}{\beta} \lambda_3 \tau + i\alpha_0 \right) \] (43)

where \( C \) and \( \alpha_0 \) denote dimensionless and real integration constants. If, on the one hand, no mass scale is provided by the effects of trivial-topology fluctuations then \( C \) is undetermined. On the other hand, the value of \( \alpha_0 \) reflects our choice of the axis

---

\(^{10}\)The modulus of \( \phi \) must not depend on \( \tau \) in thermal equilibrium.
for angular regularization and thus is arbitrary but finite. The solutions in Eq. (43) indicate that $\phi$ itself winds along an $S^1$ on the group manifold $S^3$ of SU(2). Both winding senses appear but can not be distinguished physically [1].

4 How to obtain $\phi$’s modulus

Here we show how the information about $\phi$’s phase in Eq. (43) can be used to infer its modulus. Let us assume that a scale $\Lambda$ is externally given which characterizes this modulus at a given temperature $T$. We thus have

$$\phi = \phi \left( \beta, \Lambda, \frac{T}{\beta} \right).$$

(44)

In order to reproduce the phase in Eq. (43) a linear dependence on $\phi$ must appear on the right-hand side of the BPS equation (8). Moreover, this right-hand side ought not depend on $\beta$ explicitly and must be analytic$^{11}$ in $\phi$. Only the two following

---

$^{11}$The former requirement derives from the fact that $\phi$ and its potential $V$ are obtained by functionally integrating over dilute and noninteracting calorons and anticalorons. The associated part of the partition function does not exhibit an explicit $\beta$ dependence since the action is $\beta$ independent on the corlon and anticaloron moduli spaces. A $\beta$ dependence of $V$ or $V^{(1/2)}$ can thus only be generated via the periodicity of $\phi$ itself. The latter requirement derives from the demand that the thermodynamics at temperature $T + \delta T$ to any given accuracy must be derivable from the thermodynamics at temperature $T$ for $\delta T$ sufficiently small provided no phase transition occurs at $T$. This is done by a Taylor expansion of the right-hand side of the BPS equation (finite radius of convergence) which, in turn, is the starting point for a perturbative treatment with expansion parameter $\frac{\delta T}{T}$. 

---
possibilities exist:

\[ \partial_\tau \phi = \pm i \Lambda \lambda_3 \phi \]  

(45)

or

\[ \partial_\tau \phi = \pm i \Lambda^3 \lambda_3 \phi^{-1} \]  

(46)

where \( \phi^{-1} \equiv \frac{\phi}{|\phi|^2} \). Recall that

\[ \phi^{-1} = \phi_0^{-1} \sum_{n=0}^{\infty} (-1)^n \phi_0^n (\phi - \phi_0)^n \]  

(47)

has a finite radius of convergence. According to Eq. (43) we may write

\[ \phi = |\phi|(\beta, \Lambda) \times \lambda_1 \exp \left( \frac{2\pi i}{\beta} \lambda_3 \tau + i\alpha_0 \right). \]  

(48)

Substituting Eq. (48) into Eq. (45) yields

\[ \Lambda = \frac{2\pi}{\beta} \]  

(49)

which is unacceptable since \( \Lambda \) is a constant scale. On the other hand, we obtain

\[ |\phi|(\beta, \Lambda) = \frac{\sqrt{\beta \Lambda^3}}{2\pi} = \frac{\Lambda^3}{2\pi T} \]  

(50)

when substituting Eq. (48) into Eq. (46). This is acceptable and indicates that at \( T \gg \Lambda \phi \)'s modulus is small. In Eqs. (45) and (46) the existence of the mass scale \( \Lambda \) (the Yang-Mills scale) was assumed. One attributes the generation of a mass scale to the topologically trivial sector which, however, was assumed to be switched off so far. How can a contradiction be avoided? The answer to this question is that the scale \( \Lambda \) remains hidden as long as topologically trivial fluctuations are switched off, see Eq. (7). Only after switching on gluon exchanges within the ground state can \( \Lambda \)
be seen [1]. Let us repeat the derivation of this result: In [1] we have shown that the mass of $\phi$-field fluctuations, $\partial^2_{|\phi|} V(|\phi|)$, is much larger than the compositeness scale $|\phi|$. Moreover $\partial^2_{|\phi|} V(|\phi|)$ is much larger than $T$ for all temperatures $T \geq T_{c,E}$ where $T_{c,E}$ denotes the critical temperature for the electric-magnetic transition. Thus $\phi$ is quantum mechanically and statistically inert: It can be used as a (nonbackreacting and undeformable) source for the following equation of motion

$$D_\mu G_{\mu\nu} = 2ie[\phi, D_\nu \phi]. \quad (51)$$

Here $e$ denotes the effective gauge coupling of the adjoint Higgs model, $G_{\mu\nu}$ is a field strength on topologically trivial configurations, and $D_\mu$ stands for the adjoint covariant derivative. A nontrivial-holonomy and pure-gauge solution to Eq. (51), describing the ground state together with $\phi$, is

$$a_{\rho}^{bg} = \frac{\pi}{e} T \delta_{\rho 4} \lambda_3. \quad (52)$$

As a consequence of Eq. (52) we have $D_\mu \phi \equiv 0$, and thus a ground-state pressure $P^{g.s} = -4\pi \Lambda^3 T$ and a ground-state energy-density $\rho^{g.s} = 4\pi \Lambda^3 T$ are generated in the electric phase: The so-far hidden scale $\Lambda$ becomes visible.

5 The ground state in the magnetic phase

The exponential in the phase of the complex scalar field $\varphi$, describing a condensate of magnetic monopoles within a small interval $\Delta T = \frac{T_{c,E} - T_{c,M}}{T_{c,E}} \sim 10\%$, is defined as
In Eq. (53) \( \tilde{G}_{\mu\nu} \) refers to the 't Hooft tensor defined as

\[
\tilde{G}_{\mu\nu} = \frac{\phi^a \tilde{G}^a_{\mu\nu}}{|\phi|} - \frac{\epsilon_{abc}}{c|\phi|^3} \tilde{\phi}_a (D_\mu \tilde{\phi})_b (D_\nu \tilde{\phi})_c
\]  

(54)

where \( \tilde{G}^a_{\mu\nu} \) and \( \tilde{\phi}^a \) are the field strength and the Higgs field of a BPS monopole [21].

The surface integration in Eq. (53) is over a sphere \( S^2 \) with infinite radius. If we consider the phase of the theory which may possess isolated magnetic monopoles (the electric phase) then the right-hand side of Eq. (53) is precisely zero, and thus \( \phi \) does not wind as a function of \( \tau \). If, however, we consider a phase, where monopoles are massless and thus condensed, the charge of each monopole is smeared over the entire Universe, and thus monopoles overlap substantially. In this case the right-hand side of Eq. (53) is not zero, and may exhibit a nontrivial time dependence.

How does one see this? Imagine a massive, isolated and static BPS monopole (deep inside the electric phase). In this case the behavior of \( \tilde{G}_{\mu\nu} \) for \( r \equiv |\mathbf{x}| \to \infty \) is [20]

\[
\tilde{G}_{ij,a} \to -\frac{\epsilon_{ijc} \hat{x}_c \hat{x}_a}{er^2}
\]

(55)

where \( \hat{x} \) denotes a spatial unit vector. Moreover, \( \mathcal{D}_i \tilde{\phi} \) vanishes more rapidly than \( 1/r^2 \) [20]. Thus the magnetic flux \( \frac{4\pi}{e} \) is generated by the first term in Eq. (54).

This flux precisely cancels with that of an associated but isolated antimonopole\(^{12}\).

\(^{12}\text{Recall, that a dipole with infinite spatial separation between monopole and antimonopole is generated by a dissociating caloron of sufficiently large holonomy deep in the electric phase of the theory [8].}\)
This is still true if a $\tau$ dependent phase of the asymptotic Higgs field is switched on due to thermal effects\(^{13}\). Thus the right-hand side of Eq. (53) is precisely zero deep in the electric phase. The situation changes if monopoles and antimonopoles become massless, thus their charges are spread out over the entire Universe, and, as a consequence, a substantial overlap between a monopole and its antimonopole in a given dipole exists\(^{14}\). Since the integration over an $S^2$ of infinite radius probes the core of both the monopole and the antimonopole in a dipole the asymptotic behavior in Eq. (55) no longer is applicable for a computation of the magnetic flux. The full expressions for the field strength $\tilde{G}_{ij,a}$ and $(\mathcal{D}_i\tilde{\phi})_a$ of a monopole (antimonopole) are needed in Eq. (54)\[^{21}\]:
as a function of $\tau$) deform those of the antimonopole (rotating in the opposite sense) and vice versa. Mathematically, this happens because in Eq. (56) the term $\propto G'$ in the expression for $\tilde{G}_{i,j,a}$ and also the terms with $(\mathcal{D}\bar{\phi})^2$ in Eq. (54) become important: The flux no longer is generated by a sum of isolated monopole and antimonopole contributions but by the core of a superposition of two non-hedgehog-like field configurations. Due to the appearance of the Kronecker symbols $\delta_{ia}$ in Eq. (56) the rotational symmetry of the isolated monopole or antimonopole is broken by a reference axis in color space upon superposition. This allows to ‘define’ a directed magnetic flux at infinity which is linearly rising in $\tau$, see Fig. 5. This flux, however, is meaningless if no propagating gauge mode is available to probe it. On the thermodynamical level this is expressed by the fact that Eq. (53) defines the phase of a field $\varphi$ satisfying a BPS equation

$$\partial_\tau \varphi = v^{(1/2)}(\bar{\varphi}).$$

(58)

In the absence of the topologically trivial sector $\varphi$ is pressure- and energy-free and thus invisible. Switching on the (dual) photons within an (exact) thermodynamical description of the ground-state physics [1] deforms the straight flux line. This makes the magnetic flux physical (due to flux conservation it must come in closed, finite-size loops) and $\varphi$ subject to $\tau$-dependent and periodic $U_D(1)$ gauge transformations. The dynamical compactification, that is, the reduction of fluctuating degrees of freedom, which takes place when cooling the system to temperatures\textsuperscript{15} below $T_{c,E}$, \textsuperscript{15}The massive gauge modes of the electric phase decouple at $T_{c,E}$ [1]
Figure 5: Schematic drawing of the magnetic flux situation in a hypothetic monopole condensate where monopoles do not interact. In this case the net flux, which is directed but not localized, is energy- and pressure-free. Under the influence of trivial-topology fluctuations the net flux localizes and cranks into a closed loop of finite size and thus carries energy and pressure.

thus introduces a new gauge symmetry $U_D(1)$ at $T_{c,E}$. This is the only point in the phase diagram of an SU(2) Yang-Mills theory where electric-magnetic duality, that is, a coexistence of two gauge symmetries $U(1)$ and $U_D(1)$ on either side of the phase boundary, takes place. At this point it is inessential which of the angles $\gamma, \delta$ in Fig. 1 is made $\tau$-dependent for a given $\tau$-dependence of $\alpha$. What matters is that an $S^2 \sim S^1 \times S^1 / 2$ submanifold of SU(2)'s $S^3$ is available to gauge transformations. This is reminiscent of Kaluza-Klein like geometrical compactifications where a low-energy description of a higher dimensional spacetime symmetry renders this symmetry a gauge symmetry. For $T_{c,M} < T < T_{c,E}$ $U_D(1)$ is spontaneously broken and $U(1)$ does not exist. A more quantitative investigation of Eq. (53) and also a microscopic
investigation of the ground state in the center phase is reserved for future research.

6 Summary and Outlook

Let us summarize our results. We have derived from first principles the phase and the modulus of a statistically and quantum mechanically inert adjoint scalar field $\phi$ which describes part of the thermal ground-state structure of an SU(2) Yang-Mills theory being in its electric phase. The existence of $\phi$ originates from the correlations inherent in BPS saturated, trivial-holonomy solutions to the classical Yang-Mills equations at finite temperature: the Harrington-Shepard solutions of topological charge one. To derive $\phi$’s phase these field configurations are, in a first step, treated as noninteracting when performing the partial functional average over the admissible parts of their moduli spaces. We have argued that adjoint scalar fields arising from configurations of higher topological charge are extremely suppressed.

The field $\phi$ possesses nontrivial $S^1$ winding on the group manifold $S^3$. The associated trajectory on $S^3$ becomes circular and thus a pure phase only after the integration over the entire admissible parts of the moduli spaces is carried out. Together with a pure-gauge configuration of nontrivial holonomy $\phi$ induces a linear-in-temperature dependence of the ground-state pressure and the ground-state energy-density. This pure-gauge configuration solves the Yang-Mills equations in the background $\phi$ and, on a thermodynamical level, describes the gluon exchanges between trivial holonomy calorons generating nontrivial holonomy. The pure-gauge
configuration also makes explicit that the electric phase is deconfining [1]. Since trivial-topology fluctuations may acquire quasiparticle masses on tree-level by the adjoint Higgs mechanism [1] the presence of $\phi$ resolves the infrared problem inherent in a naive perturbative loop expansion of thermodynamical quantities [13]. Since there are kinematical constraints for the maximal hardness of topologically trivial quantum fluctuations no renormalization procedure for the treatment of ultraviolet divergences is needed in the loop expansion of thermodynamical quantities [13]. These kinematical constraints arise from $\phi$’s compositeness. The usual assertion that the effects of the topologically nontrivial sector are extremely suppressed at high temperature - they turn out to be power suppressed in $T$ - is shown to be correct by taking this sector into account. The theory, indeed, has a Stefan-Boltzmann limit at asymptotically high temperatures. It turns out to be incorrect, however, to neglect the topologically nontrivial sector from the start: assuming $T \gg \Lambda$ to justify an omission of the topologically nontrivial sector before performing a (perturbative) loop expansion of thermodynamical quantities does not capture the thermodynamics of an SU(2) Yang-Mills theory and leads to the known problems in the infrared sector [22].

It should be straightforward to generalize our results to SU(N) [1]. A worthwhile objective for future research is to derive the phases and moduli of the complex scalar fields $\varphi$ and $\Phi$ describing (parts of) the ground-state structure of the magnetic and the center phases, respectively. While the magnetic phase possesses a massive but propagating dual photon - $\varphi$ describes a magnetic monopole condensate - the center
phase, whose excitations are (selfintersecting) center-vortex loops with fermionic statistics, has no propagating gauge-field modes. The field $\Phi$ is proportional to the expectation of the 't Hooft loop and describes a condensate of Cooper pairs of (nonintersecting) center-vortex loops. When cooling the system, the transition to the magnetic phase is smooth for SU(2) ($2^{\text{nd}}$ order transition) while the transition to the center phase is nonthermal (Hagedorn transition). Both the magnetic and the center phase confine fundamental test charges.

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