Dynamics and Adaptive Control for Spacecraft Relative Motion with Disturbances and Parametric Uncertainties*

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A nonlinear relative motion dynamics model in the presence of disturbances and parametric uncertainties is presented for the high precision relative motion of a spacecraft. The disturbances include the Earth’s oblateness, atmospheric drag, and thrust error. The parametric uncertainties in the atmospheric drag coefficients and thrust alignments are considered. To minimize fuel cost $\Delta V$ while keeping the desired relative orbit, a relative $J_2$-invariant dynamics model is also designed. For spacecraft relative motion tracking maneuver, an adaptive backstepping sliding mode control law under limited low thrust is developed. This control law combines the advantages of adaptive backstepping and sliding mode control, where knowledge of the upper bounds of parametric uncertainties and disturbances is not required. Within the Lyapunov framework, the proposed control law is proved to guarantee global asymptotic convergence to the desired states. Numerical simulation results show the effectiveness of the nonlinear relative motion dynamics model and proposed control law.

Key Words: Relative Motion, Adaptive Backstepping Sliding Mode Control, $J_2$, Thrust Error, Atmospheric Drag

Nomenclature

- $a$: semi-major axis of the reference spacecraft
- $e$: eccentricity of the reference spacecraft
- $f$: true anomaly of the reference spacecraft
- $\Omega$: right ascension of ascending node of the reference spacecraft
- $\theta$: argument of latitude of the reference spacecraft
- $i$: orbit inclination of the reference spacecraft
- $\mu$: Earth’s gravitational constant
- $J_2$: second zonal harmonic coefficient of the Earth gravitational potential model
- $R_e$: equatorial radius of the Earth
- $r_i$: orbital radius of the reference spacecraft
- $\dot{r}_i$: time derivative of $r_i$
- $c_d$: aerodynamic drag coefficient
- $A/m$: area to mass ratio
- $\rho_0$: atmospheric density
- $h$: angular momentum
- $\Delta a_{2B}$: relative gravity acceleration due to the two-body gravitational field
- $\Delta a_{J_2}$: relative acceleration due to $J_2$ effect between the target and chaser in the rotating reference frame
- $\Delta a_{AD}$: relative acceleration due to air drag between the target and chaser in the rotating reference frame
- $\Delta a_{MA}$: acceleration due to thrust error of the chaser
- $T_m$: nominal thrust magnitude
- $C_{Bi}$: direction cosine matrix from a body-fixed to a rotating frame
- $C_{Ti}$: direction cosine matrix from the inertial to rotating frames
- $\alpha$: thrust azimuth direction angle
- $\beta$: thrust elevation direction angle
- $\kappa(t)$: small random variable
- $\Delta \alpha$: thrust azimuth error
- $\Delta \beta$: thrust elevation error
- $\Delta T_m$: thrust magnitude error
- $\Lambda$: positive constant
- $u^\text{max}$: upper bound of control input $u$
- $x_1$: relative chaser position vector in LVLH $[x, y, z]^T$
- $x_2$: relative chaser velocity vector in LVLH

Subscripts

- $t$: target satellite of the orbital reference frame
- $c$: chaser satellite of the orbital reference frame
- $m_1$: desired target
- $m_2$: desired chaser

1. Introduction

In recent years, significant effort has been directed toward modeling and controlling the relative motion between two or more flying objects. Relative motion between two satellites is exemplified by the scenario in which a servicing satellite (chaser) periodically flies around an objective (target) satellite. Thus, managing relative motion plays an important role in saving time and lowering the cost for spacecraft formation flying, spacecraft rendezvous and docking, etc.1) This research has focused on major issues including dynamics modeling and control.2) The linear and nonlinear dynamics models have been used to design a spacecraft relative motion. Hill’s equation is a linear dynamics model for a circular reference orbit.3) Control designs based on Hill’s equation consume large amounts of fuel, and still may not guarantee relative motion tracking over long durations. It may also result in wide separation of spacecraft because it does not include nonlinear terms.

To reduce excessive fuel consumption related to the use of linear dynamics, various nonlinear dynamics modeling and
control designs have been attempted. Among nonlinear dynamics including external disturbances, Kechichian developed a full set of exact nonlinear differential equations including air drag and $J_2$ perturbations on the basis of Newtonian mechanics to describe the relative motion.\textsuperscript{4)} For the nonlinear control designs, Liu and Wang\textsuperscript{5)} proposed a slide mode control law to achieve formation flying maneuvers by providing global asymptotic stability with low-thrust under disturbance due to $J_2$ and bounded uncertainties. To enhance the tracking performance, the control law design requires the exact knowledge of external disturbances such as the space environment. However, it is difficult to obtain the exact information because they are time-varying in practice. For low Earth orbit (LEO), atmospheric drag is one of the dominant perturbing forces. Its density and coefficient of the upper atmosphere for spacecraft employ approximate values rather than exact ones.\textsuperscript{6)} Even in the case of an electric propulsion system in satellite, an ion thruster may not control nominally by unknown thrust magnitude error and misalignment (i.e., thrust magnitude error to a maximum absolute error of 0.5 mN, and misalignment in a range from 0.5 to 5 deg).\textsuperscript{7)} These parametric uncertainties may degrade the performances of spacecraft relative motion. To achieve the relative motion tracking more accurately, these unknown parameters (i.e., thrust misalignment, atmospheric density, and drag coefficient) should be estimated. To estimate these parametric uncertainties, Queiroz et al.\textsuperscript{8)} proposed a nonlinear adaptive control law for relative position tracking of multiple satellites. Lim and Bang\textsuperscript{9)} developed a thrust error model for a single thruster with misalignment and proposed an adaptive backstepping control law to handle the relative position tracking problem assuming the presence of thrust misalignment and disturbances.\textsuperscript{9)} Even though adaptive control or adaptive backstepping control can estimate unknown parameters, they also require prior knowledge of the upper bounds of parametric uncertainties and external disturbances.

Furthermore, to minimize the fuel consumption of spacecraft relative motion for long flight times, the design of the desired reference orbit should be considered. Pan et al.\textsuperscript{10)} proposed a relative motion model for elliptical reference orbits based on differences between an initial instantaneous element and true anomaly by accommodating secular relative motion drift caused by $J_2$ perturbation; and by using the transition between instantaneous and mean orbital elements (MOE) under $J_2$ perturbation. Gim and Alfriend\textsuperscript{11)} proposed a state transition matrix of relative motion with MOE under $J_2$ perturbation using a geometric method. Schaub and Alfriend\textsuperscript{12)} worked on establishing a $J_2$-invariant relative orbit with differences in MOE using a matching method and the average drift rates of the relative orbit as the desired relative orbit. These analytical kinematics models for the $J_2$-invariant orbits provide only relative position and velocity vectors. Thus, the need for a relative acceleration vector makes it rather difficult to achieve the control goal.

The contributions of this paper can be summarized as follows: first, we derive a set of nonlinear relative dynamics model incorporating $J_2$ perturbation, atmospheric drag, and thrust error. The dynamic differential equations by Kechichian are too complicated to apply and thrust error is not included. This is achieved by simplifying and extending the equations developed by Kechichian.\textsuperscript{4)} This derived dynamics model can be applied to design a general, elliptical orbit for satellite relative motion in LEO without any approximation. Second, we design a $J_2$-invariant relative dynamics model to determine the desired elliptical reference orbit. This is designed with MOE and differences of MOE using the matching method,\textsuperscript{12)} on the basis of the direction cosine matrix in terms of the Euler angles. For robust control, this $J_2$-invariant dynamics model can provide not only the position, velocity, and acceleration directly, but also the desired orbit keeping for a long period of time while reducing fuel consumption. Third, we propose a robust control scheme for the relative motion. It uses adaptive backstepping sliding mode control (ABSMC) enabled by continuous low thrust in the presence of disturbances and parametric uncertainties. This is to combine adaptive backstepping control and sliding mode control. The adaptive backstepping control is used to handle parametric uncertainties, and the sliding mode control is used to suppress bounded disturbances. The adaptive law in this control law is used to estimate the parametric uncertainties and bounds of randomly bounded external disturbances. Thus, this proposed control law can manage unmatched uncertainties to achieve the desired states without requiring prior knowledge of the bounds of uncertainty.\textsuperscript{13,14)}

The convergence of the proposed control law is proved with the Lyapunov stability theorem.

This paper is organized as follows. The nonlinear relative dynamics model is derived in Section 2. The $J_2$-invariant dynamics model for the desired orbit is designed in Section 3. The adaptive backstepping sliding mode control law is proposed in Section 4. Numerical simulation results are provided to illustrate various features of the proposed algorithms in Section 5. Finally, conclusions are presented in Section 6.

2. System Dynamics Model

2.1. Equation of relative motion with disturbances

In this section, three coordinate systems are used. The Earth-centered inertial (ECI) frame, reference orbital frame, and spacecraft body-fixed frame are illustrated in Fig. 1. Let $[X Y Z]$ be the inertial coordinate system. The origin of the inertial frame is at the mass center of the Earth. The origin of the reference orbital frame (local vertical and local horizontal, LVLH) is at the mass center of the reference spacecraft (target). The $x$-axis points in the radial direction, the $y$-axis points to the along track direction, and the $z$-axis points to the direction of the orbital angular momentum vector. The origin of the reference spacecraft body-fixed frame is at the mass center of the reference spacecraft, whereas it is assumed that the thruster is tilted over the nominal thrust direction with small constant angles $\Delta \alpha$ and $\Delta \beta$, and the thrust is always activated at the mass center of the satellite.

The relative position vector of a chaser defined in the tar-
The time rate of change in the angular velocity of the reference frame is denoted by \( \dot{\varepsilon} \) and the angular velocity vector of the rotating orbital frame is denoted as \( \omega = [\omega_x, \omega_y, \omega_z]^T \). The angular velocity components are derived such that:

\[
\begin{align*}
\omega_x &= \dot{\Omega} \sin i \sin \theta + i \cos \theta = - \frac{1}{h} \frac{K_{J_2}}{r_i^2} \sin 2i \sin \theta \\
\omega_y &= \dot{\Omega} \sin i \cos \theta - i \sin \theta = 0 \\
\omega_z &= \dot{\Omega} \cos i + \dot{\theta} = \frac{h}{r_i^2}
\end{align*}
\]

where \( K_{J_2} \) is a constant determined such that:

\[
K_{J_2} = \frac{3}{2} \mu J_2 R_e^2
\]

The time rate of change in the angular velocity of the reference spacecraft including the perturbations due to air drag and gravitational perturbation vector is given by:

\[
\begin{align*}
\dot{\omega}_x &= -\frac{K_{J_2}}{h^2 r_i^4} \sin^3 i \cos^2 \theta \sin 2i \sin \theta - \frac{K_{J_2}}{r_i^2} \sin 2i \cos \theta \\
+ & \frac{3}{r_i^2} K_{J_2} \sin 2i \sin \theta r_i - \frac{K_{J_2} c_A \rho A}{2 hr_i^4} \sin 2i \sin \theta \\
\dot{\omega}_y &= 0 \\
\dot{\omega}_z &= - \frac{1}{2} \frac{A}{r_i^3 m} c_A \rho A \frac{h}{r_i} - \frac{K_{J_2}}{r_i^2} \sin^2 i \sin 2\theta - \frac{2hr_i}{r_i^2}
\end{align*}
\]

where the general expressions \( \hat{a}, \hat{e}, \) and \( \hat{f} \) are in the following Lagrange-Gauss equations:

\[
\begin{align*}
\hat{a} &= \frac{a(1 - e^2)}{1 + e \cos f} \\
\hat{e} &= \frac{r_i}{a} - \frac{\mu}{h^2} \dot{r} \left( 2ae + r_i \cos f \right) + \frac{r_i^2 e}{h^2} \sin f \dot{f} \\
\hat{f} &= \frac{2a^2}{h} \left( e \sin f (P_{r2} + P_{r10}) + \frac{p}{r_i} (P_{r2} + P_{r10}) \right)
\end{align*}
\]

Note that the angular momentum vector \( \mathbf{h} \) and the time rate of change in the angular momentum vector \( \dot{\mathbf{h}} \) in the rotating reference frame, \( \dot{\mathbf{h}} \) and \( \dot{\mathbf{h}} \), are unit vectors of \( \mathbf{h} \) and \( \dot{\mathbf{h}} \) respectively.

\[
\mathbf{h} = r_i \times \mathbf{v}_i = r_i^2 \omega_i^2 \mathbf{h}
\]

Then,

\[
\begin{bmatrix}
P_{r2} \\
P_{r12} \\
P_{p2} \\
P_{p12}
\end{bmatrix} = \begin{bmatrix}
-\frac{K_{J_2}}{r_i^2} (1 - 3 \sin^2 i \sin^2 \theta) \\
-\frac{K_{J_2}}{r_i^2} \sin^2 i \sin 2\theta \\
-\frac{K_{J_2}}{r_i^2} \sin 2\theta \\
-\frac{1}{2} \frac{A}{m} c_A \rho A \frac{h}{r_i} \\
-\frac{1}{2} \frac{A}{m} c_A \rho A \frac{h}{r_i}
\end{bmatrix}
\]

Thus, substituting perturbation vector \( \mathbf{P} \) into Eq. (8)

\[
\dot{\mathbf{h}} = \frac{1}{2} \frac{A}{m} c_A \rho A \left( \frac{h}{r_i} \right)
\]

Furthermore, assume thrust error in the case of an ion propulsion system using electrical power. This thrust error includes thrust magnitude error and misalignment. The real thrust acceleration with magnitude error and misalignment is expressed as the sum of nominal and thrust error terms in the body-fixed frame. Lim and Bang developed a thrust error model for a single thruster with misalignment and magnitude errors. The thrust error model is expressed as:

\[
\Delta a_{MA} = T_m C_b C_i C_b^i \begin{bmatrix}
- \sin \alpha \cos \beta & - \cos \alpha \sin \beta \\
\cos \alpha \cos \beta & - \sin \alpha \sin \beta \\
o & \cos \beta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix} + \kappa(t) T_m C_b C_i C_b^i \begin{bmatrix}
\sin \alpha \cos \beta \\
\sin \beta
\end{bmatrix}
\]

where each parameter in \( T_m \) is given in the Appendix; \( C^R \in R^{3\times3} \) is the matrix to rotate a thruster to the desired di-
rection in the LVLH frame and is realized by attitude control in practical implementation. The rotation matrix \( C_B^R \) can be constructed by applying the condition of minimal and optimal proper pointing. All of this produces a unique rotation matrix.\(^{15} \) In this study, it is assumed that attitude maneuver is executed by an inner control loop that makes the thruster point in the control direction in real-time. The first term of Eq. (10) is thrust misalignment and the second term corresponds to thrust magnitude error.

Therefore, the nonlinear relative dynamic equation in the rotating reference frame including perturbations due to \( J_2 \) effect, atmospheric drag and thrust error, is as follows:

\[
\dot{x} = \left( \frac{h}{r_i^2} \right)^2 - \left[ \frac{1}{2} \frac{1}{r_i^2} C_d \rho_m \frac{A_v}{m} + K_{J_2} \sin^2 i \sin 2\theta + \frac{2h}{r_i^2} \right] y + \Delta a_{2B} + \Delta a_{J_2} + \Delta a_{AD} + \Delta a_{MA},
\]

\[
\dot{y} = \left[ \frac{h}{r_i^2} \right]^2 \sin 2i \sin \theta_x + \frac{2}{r_i^2} \frac{K_{J_2}}{h} \sin 2i \sin \theta + \Delta a_{2B_i} + \Delta a_{J_2} + \Delta a_{AD} + \Delta a_{MA},
\]

\[
\dot{z} = \frac{K_{J_2}}{r_i^2} \sin 2i \sin \theta_x \sin \theta + \frac{2}{r_i^2} \left\{ \frac{K_{J_2}}{h} \sin 2i \sin \theta_x + \frac{2h}{r_i^2} \right\} y + \Delta a_{2B} + \Delta a_{J_2} + \Delta a_{AD} + \Delta a_{MA},
\]

\[
\Delta a_{2B} = -\mu \left[ \frac{r + x}{\| r + \rho \|^3} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

\[
\Delta a_{J_2} = -\frac{3 \mu J_2 R_z^2}{2r_i^2} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

where Eq. (16) is the \( J_2 \) acceleration in the inertial frame, while Eq. (17) represents relative acceleration due to \( J_2 \) between the target and chaser in the rotating reference frame.

\[
\Delta a_{J_2} = C_{J_2}^F [a_{J_2}(r_i, \theta_i) - a_{J_2}(r_i, \theta_i)]
\]

The atmospheric drag in the ECI coordinate frame is expressed as follows:

\[
a_{AD} = -\frac{1}{2} \rho_d \left( \frac{c_d A}{m} \right) \| V_r \| V_r
\]

where \( V_r \) is the velocity vector relative to the rotating atmosphere.

\[
V_r = \frac{dr}{dt} - \omega_0 \times r
\]

Then, the relative acceleration \( \Delta a_{AD} \) due to air drag between the target and chaser in the rotating reference frame can be written as

\[
\Delta a_{AD} = C_{J_2}^F [a_{AD}(\rho_d, c_d, A_m, m_c, V_r)] - a_{AD}(\rho_d, c_d, A_m, m_c, V_r)]
\]

We assume that the target is a passive satellite without control and that the chaser is an active satellite with a constrained low thrust.

2.2. Relative motion dynamics model with disturbances and parametric uncertainties

In this study, it is assumed that the thrust model and atmospheric drag model have uncertainties. Atmospheric drag is one of the predominant perturbations in LEO, and is perhaps the most difficult parameter to determine. \( c_d \) is a dimensionless quantity that reflects the susceptibility of a satellite to drag forces. The drag coefficient for a satellite in the upper atmosphere involves the use of approximate values rather than exact ones.\(^6 \) Thus, we need to estimate the unknown parameters \( c_d \) and \( \rho_d \) from Eq. (19). Spacecraft relative motion in this study is assumed to operate in a close relative distance where the air density of the two spacecraft has no difference. Thus, Eq. (19) can be rearranged into Eq. (20).

\[
\Delta a_{AD} = \rho_d \Delta a_{AD}'
\]
where \( \rho \) is an unknown parameter to be estimated. Then,

\[
\Delta a_{AD} = C_f[a_{AD}(A_i, m_i, V_e) - a_{AD}(A_t, m_r, V_e)]
\]

The ion propulsion system using electrical power is assumed as the thrust model. For the thrust uncertainties, it is also assumed that the chaser has a single thruster fixed in the chaser’s body frame and is acting at the center of the mass. The mass of the chaser is assumed to be known as constant. In Eq. (10), if the thrust error is proportional to thrust magnitude, thrust misalignment error (angle \( \Delta \alpha \)) and thrust elevation error (angle \( \Delta \beta \)) in Fig. 1. Each term of Eq. (10) can be rearranged such that

\[
\Delta a_{MA} = \Delta a_{MA}^u + \Delta a_{MA}^w
\]

First term of Eq. (21) with parametric uncertainties can be rewritten as

\[
\Delta a_{MA}^u = \Delta a_{MA}^u \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}
\]

In contrast to thrust misalignment, thrust magnitude error is considered as the known disturbance in this problem.

Finally, substituting Eqs. (20) and (21a) into Eqs. (12)–(14), the relative motion dynamics model with disturbances and parametric uncertainties can be rearranged and simplified.

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x_1, x_2) + H(t)\Theta
\end{aligned}
\]

where \( f(x_1, x_2) \in \mathbb{R}^{3 \times 6} \) is the nonlinear relative dynamic equation including the disturbances with known parameters such as \( J_2 \) and thrust magnitude error from Eqs. (12)–(14).

The transformation matrix \( \Theta \) is an unknown parameter set, and \( H(t) \) is a time-varying matrix including unknown parameters. Note that \( f(x_1, x_2) \) can be reformulated as

\[
f(x_1, x_2) = 
\begin{bmatrix}
 f'_{11} & f'_{12} & f'_{13} & f'_{14} & f'_{15} & f'_{16} \\
 f'_{21} & f'_{22} & f'_{23} & f'_{24} & f'_{25} & f'_{26} \\
 f'_{31} & f'_{32} & f'_{33} & f'_{34} & f'_{35} & f'_{36}
\end{bmatrix}
\]

where each parameter in \( f(x_1, x_2) \) is given in the Appendix, and the function of time-varying matrix is formulated in the following form.

\[
H = 
\begin{bmatrix}
 H_{11,AD} & H_{12,AD} & H_{13,AD} \\
 H_{21,AD} & H_{22,AD} & H_{23,AD} \\
 H_{31,AD} & H_{32,AD} & H_{33,AD}
\end{bmatrix}
\]

where each parameter of \( H \) is also described in the Appendix.

3. Desired Reference Dynamics Model for the Reference Orbit

For spacecraft relative motion such as formation flying, formation keeping is primarily intended to maintain the desired reference orbit for a long period of time. The desired reference orbits should be designed to reduce fuel consumption allowing the formation to be maintained for as long as possible. Reference orbits, including the \( J_2 \) perturbation effect, have been studied to reduce fuel consumption and maintain the desired orbit. In this section, a \( J_2 \)-invariant relative dynamics model for the desired elliptical reference orbit is designed with the mean orbital elements (MOE) and MOE differences based on the direction cosine matrix in terms of Euler angles. A schematic layout of the desired reference orbit design is presented in Fig. 2.

It starts from the assumption that the desired initial MOE is already known. Mean secular rates by \( J_2 \) are computed from the desired initial MOE and are then propagated to time-explicit MOE. The MOE differences are chosen with constraints for the \( J_2 \)-invariant orbit as described by Schaub and Alfriend, and then added to the MOE. The inertial position, velocity, and acceleration vectors for each desired target and chaser are obtained from the direction cosine matrix in terms of Euler angles. Finally, the desired relative position, velocity, and acceleration vector are translated from each inertial position, velocity, and acceleration vector with the direction cosine matrix.

The transformation matrix between the inertial and the rotating frames of the desired target is

\[
T_1 = T_1^i(O_m, m, \theta_{m}) = 
\begin{bmatrix}
 A_m & B_m & E_m \\
 C_m & D_m & F_m \\
 G_m & H_m & K_m
\end{bmatrix}
\]

where \( T_1 \) is a transformation matrix from the inertial to the reference orbital frames in the 3-1-3 Euler rotation sequence of the desired target with MOE.

The inertial position vector of the desired target \( r_{m} = [X_m, Y_m, Z_m]^T \) can be obtained from Eq. (25).

\[
r_{m} = r_{m}A_m \hat{X} + r_{m}B_m \hat{Y} + r_{m}E_m \hat{Z}
\]
The components of \( \mathbf{r}_{m1} \) are given by
\[
\begin{align*}
X_{m1} &= r_{m1} A_{m1} \\
Y_{m1} &= r_{m1} B_{m1} \\
Z_{m1} &= r_{m1} E_{m1}
\end{align*}
\] (26a)
where \( r_{m1} \) is a position of the desired target, \( \mathbf{r}_{m1} \) is a radial direction unit vector in the desired target orbital frame \( \left[ \hat{r}_{m1} \hat{\theta}_{m1} \hat{h}_{m1} \right]^T \), and \( \left[ \hat{X} \hat{Y} \hat{Z} \right]^T \) is an unit vector of the inertial frame (Fig. 1).

\[
\begin{pmatrix}
\hat{r}_{m1} \\
\hat{\theta}_{m1} \\
\hat{h}_{m1}
\end{pmatrix}
= 
\begin{pmatrix}
A_{m1} & B_{m1} & E_{m1} \\
C_{m1} & D_{m1} & F_{m1} \\
G_{m1} & H_{m1} & K_{m1}
\end{pmatrix}
\begin{pmatrix}
\hat{X} \\
\hat{Y} \\
\hat{Z}
\end{pmatrix}
\]

The inertial velocity vector \( \mathbf{v}_{m1} = \left[ \mathbf{X}_{m1} \mathbf{Y}_{m1} \mathbf{Z}_{m1} \right]^T \) and acceleration vector \( \mathbf{a}_{m1} = \left[ \mathbf{\dot{X}}_{m1} \mathbf{\dot{Y}}_{m1} \mathbf{\dot{Z}}_{m1} \right]^T \) of the desired target can be obtained from Eq. (26a)
\[
\begin{align*}
\mathbf{X}_{m1} &= \mathbf{r}_{m1} A_{m1} - 2r_{m1} \mathbf{B}_{m1} \Omega_{m1} + r_{m1} \mathbf{C}_{m1} \hat{\theta}_{m1} \\
\mathbf{Y}_{m1} &= \mathbf{r}_{m1} \mathbf{B}_{m1} + r_{m1} \mathbf{A}_{m1} \Omega_{m1} + r_{m1} \mathbf{D}_{m1} \hat{\theta}_{m1} \\
\mathbf{Z}_{m1} &= \mathbf{r}_{m1} \mathbf{E}_{m1} + r_{m1} \mathbf{F}_{m1} \hat{\theta}_{m1}
\end{align*}
\] (27)
and the acceleration vector becomes
\[
\begin{align*}
\mathbf{\dot{X}}_{m1} &= \mathbf{r}_{m1} A_{m1} - 2r_{m1} \mathbf{B}_{m1} \Omega_{m1} + 2r_{m1} \mathbf{C}_{m1} \hat{\theta}_{m1} \\
&+ r_{m1} \mathbf{D}_{m1} \hat{\theta}_{m1} \\
\mathbf{\dot{Y}}_{m1} &= \mathbf{r}_{m1} \mathbf{B}_{m1} + 2r_{m1} \mathbf{A}_{m1} \Omega_{m1} + 2r_{m1} \mathbf{D}_{m1} \hat{\theta}_{m1} \\
&- r_{m1} \mathbf{B}_{m1} \mathbf{\dot{\Omega}}_{m1} + 2r_{m1} \mathbf{C}_{m1} \mathbf{\dot{\Omega}}_{m1} \hat{\theta}_{m1} \\
&+ r_{m1} \mathbf{D}_{m1} \mathbf{\dot{\theta}}_{m1} \\
\mathbf{\dot{Z}}_{m1} &= \mathbf{r}_{m1} \mathbf{E}_{m1} + 2r_{m1} \mathbf{F}_{m1} \hat{\theta}_{m1} - r_{m1} \mathbf{F}_{m1} \mathbf{\dot{\theta}}_{m1} + r_{m1} \mathbf{F}_{m1} \mathbf{\dot{\theta}}_{m1}
\end{align*}
\] (28)
where each parameter of the velocity and acceleration in Eqs. (27) and (28) is presented in Appendix.

To construct the desired relative state vector, one has to find the inertial position, velocity, and acceleration vectors of the desired chaser. \( \mathbf{r}_{m2} = \left[ X_{m2} Y_{m2} Z_{m2} \right]^T \) can be obtained by adding the MOE of the desired target and MOE differences. MOE differences are as follows,
\[
\Delta \mathbf{e}_{m} = [\Delta a_m, \Delta e_m, \Delta i_m, \Delta \Omega_m, \Delta a_{\Omega_m}, \Delta f_m]
\] (29)
where \( \Delta a_m, \Delta i_m, \) and \( \Delta e_m \) are derived with constraints for a \( J_2 \)-invariant orbit by Schaub and Alfriend.\(^{12}\)
\[
\Delta a_m = 2D a \Delta \eta, \quad \Delta e_m = \frac{-y}{e} \Delta \eta, \quad \Delta \eta = \frac{y}{4} \tan i_m \Delta i_m
\]
and the parameters are defined as
\[
\eta = \sqrt{(1 - e^2)} \quad L = \frac{m}{R_e} \quad D = \frac{J_3}{4L^2 \eta^2} \quad (4 + 3 \eta)(1 + 5 \cos^2 i_m)
\]
Thus, the MOE of the desired chaser is
\[
\mathbf{e}_{m2} = [a_{m2} + \Delta a_m, e_{m2} + \Delta e_m, i_{m2} + \Delta i_m, \Omega_{m2} + \Delta \Omega_m, \omega_{m2} + \Delta \omega_m, f_{m2} + \Delta f_m]
\] (30)
Then, \( T_2, \mathbf{r}_{m2} = \left[ X_{m2} Y_{m2} Z_{m2} \right]^T, \mathbf{r}_{m2} = \left[ \mathbf{\dot{X}}_{m2} \mathbf{\dot{Y}}_{m2} \mathbf{\dot{Z}}_{m2} \right]^T, \) and \( \mathbf{r}_{m2} = \left[ \mathbf{X}_{m2} \mathbf{Y}_{m2} \mathbf{Z}_{m2} \right]^T \) can be obtained by the same procedure like the desired target. Therefore, it becomes simple to construct the desired relative state vectors with MOE by using the direction cosine matrix.
\[
\mathbf{x}_d = T_1 (\mathbf{r}_{m2} - \mathbf{r}_{m1})
\] (31)
\[
\mathbf{\dot{x}}_d = T_1 (\mathbf{\dot{r}}_{m2} - \mathbf{\dot{r}}_{m1})
\] (32)
\[
\mathbf{\ddot{x}}_d = T_1 (\mathbf{\ddot{r}}_{m2} - \mathbf{\dot{r}}_{m1}) + T_1 (\mathbf{\dot{r}}_{m2} - \mathbf{r}_{m1})
\] (33)
where the parameters in \( T_1 \) and \( T_1 \) are given in Appendix.

\[
T_1 = \begin{pmatrix}
\hat{A}_{m1} & \hat{B}_{m1} & \hat{E}_{m1} \\
\hat{C}_{m1} & \hat{D}_{m1} & \hat{F}_{m1} \\
\hat{G}_{m1} & \hat{H}_{m1} & \hat{K}_{m1}
\end{pmatrix}
\]
\[
\hat{T}_1 = \begin{pmatrix}
\hat{A}_{m1} & \hat{B}_{m1} & \hat{E}_{m1} \\
\hat{C}_{m1} & \hat{D}_{m1} & \hat{F}_{m1} \\
\hat{G}_{m1} & \hat{H}_{m1} & \hat{K}_{m1}
\end{pmatrix}
\] (34)

4. Adaptive Backstepping Sliding Mode Controller Design

In this study, control and parameter update laws are proposed for the tracking problem of relative motion in the presence of disturbances and parametric uncertainties using adaptive backstepping sliding mode control (ABSMC). The advantage of this proposed control law does not require upper bounds for the unknown parameters and disturbances. The control law is implemented with normalization to avoid control input saturation. The nonlinear relative system dynamic model presented in Eq. (22) can be expressed as a state space method for the controller design:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f'(x_1, x_2) + H(t)\Theta + D(t) + u
\end{align*}
\] (35)
where \( D(t) = [D_1 D_2 D_3] \) represents unknown time-varying external disturbances.

The design procedure for the ABSMC in Fig. 3 is described in the following three steps.
Step 1:
For the position tracking problem, \( z_1 \) is selected as the trajectory tracking error and defined as
\[
z_1 = x_1 - x_d
\] (36)
Define a stabilizing function such that
\[
\alpha_1 = c_1 z_1
\] (37)
The tracking velocity error is defined as:
\[
z_2 = \dot{z}_1 + \alpha_1 = x_2 - \dot{x}_d + \alpha_1
\] (38)
Choose a Lyapunov function candidate \( V_1 \)
Thus, if the upper bound of parametric uncertainties and external disturbances are correctly known in advance, the above equation is negative semi-definite. However, this controller still requires prior knowledge of the upper bounds of the parametric uncertainties and external disturbances. To avoid the need for upper bounds, following adaptive algorithms are adopted.\(^{(17)}\)

Step 3:

The third Lyapunov function candidate is defined as

\[
V_3 = V_2 + \frac{1}{2} \hat{\Theta}' \Gamma^{-1} \Theta' + \frac{1}{2} \hat{\Psi}' \psi^{-1} \hat{\Psi} \tag{46}
\]

where the unknown parametric estimate error is

\[
\hat{\Theta} = \Theta - \Theta_0 = [\hat{\theta}_1 \hat{\theta}_2 \hat{\theta}_3]' \tag{47}
\]

The estimation error of the upper bound of the unknown external disturbance error is

\[
\hat{D} = \bar{D} - \bar{D}_n = [\bar{D}_1 \bar{D}_2 \bar{D}_3]' \tag{48}
\]

The positive-definite constant diagonal design matrix is \(F \in R^{3 \times 3}\) and \(\psi\) is a positive constant. Then, the time derivative of \(V_3\) becomes

\[
\dot{V}_3 = \dot{V}_2 + \hat{\Theta}' \Gamma^{-1} \dot{\hat{\Theta}} + \hat{D}' \psi^{-1} \dot{\hat{\Psi}} \tag{49}
\]

To design only the backstepping control law, uncertainties have to be assumed to be bounded (i.e., \(|\theta_i| \leq \hat{\theta}_i\), \(i = 1, 2, 3\) and \(|D_i| \leq \bar{D}_n\), \(i = 1, 2, 3\)). The second Lyapunov function \(V_2\) was chosen as follows:

\[
V_2 = V_1 + \frac{1}{2} s' s \tag{50}
\]

where \(s\) is a sliding surface with the form

\[
s = k z_1 + z_2 \tag{51}
\]

Then, the time derivative of \(s\) is

\[
\dot{s} = k_1(z_2 - c_1 z_1) + f' + u + H \Theta + D - \dot{x}_d \tag{52}
\]

where \(k_1 = (k + c_1), k_1 \geq 0,\) and the time derivative of \(V_2\) can be derived as

\[
\dot{V}_2 = \dot{V}_1 + s'^ T \dot{s} = \dot{z}_1'^ T A z_2 - \dot{z}_1'^ T c_1 A z_1 + s'^ T \dot{s} \tag{53}
\]

To satisfy \(\dot{V}_2 \leq 0\) and stabilize the second sub-system, the controller can be designed as\(^{(14)}\)

\[
\dot{u} = -k_1(z_2 - c_1 z_1) - f' - \dot{z}_1 - H \Theta + \dot{x}_d - \text{diag}(\text{sgn}(s)) \hat{D} - c_2 s \tag{54}
\]

where \(c_2\) is a positive constant and \(\text{diag}(\text{sgn}(s))\) is as follows

\[
\text{diag}(\text{sgn}(s)) = \begin{bmatrix}
\text{sgn}(s_1) & 0 & 0 \\
0 & \text{sgn}(s_2) & 0 \\
0 & 0 & \text{sgn}(s_3)
\end{bmatrix} \tag{55}
\]

Substituting Eq. (54) into Eq. (53), the following results are obtained:

\[
\dot{V}_2 = -z_1'^ T A k_1 z_1 - s'^ T c_2 s + s'^ T (D - \text{diag}(\text{sgn}(s)) \hat{D}) \leq -z_1'^ T A k_1 z_1 - s'^ T c_2 s + s'^ T [(D - \hat{D})] \leq 0 \tag{56}
\]

Fig. 3. Schematic diagram of the whole control process.
porate the saturation constraint. The actual control input can be expressed as \(^{19–21}\):

\[
\mathbf{u} = \text{satc}(\mathbf{u})
\]

\[
= \begin{cases} 
\text{diag}(\text{sgn}(\mathbf{u}))\mathbf{u}_{\text{max}} & \text{if } \mathbf{u}_{i} > \mathbf{u}_{\text{max}} (i = x, y, z) \\
\mathbf{u}_{i} & \text{if } \mathbf{u}_{i} \leq \mathbf{u}_{\text{max}} 
\end{cases}
\] (51)

Note that the thrust misalignment and drag disturbance with unknown parameter \(H\mathbf{\bar{\Theta}}\), and the external disturbance \(\mathbf{D}\) are put into the input \(\mathbf{u}\).

The adaptive law for \(\mathbf{\bar{\Theta}}\) and \(\mathbf{\hat{D}}\) is designed such that

\[
\mathbf{\dot{\bar{\Theta}}} = \Gamma H^{T}\mathbf{s}
\] (52)

\[
\mathbf{\dot{\hat{D}}} = \psi \text{diag}(\text{sgn}(\mathbf{s}))\mathbf{s}
\] (53)

Through Eqs. (50), (52) and (53), prior knowledge of the upper bounds of the parametric uncertainties and unknown disturbance are not required for the proposed ABSMC. Substituting Eqs. (48), (52) and (53) into Eq. (47), it follows that

\[
\mathbf{V}_{3} = -\mathbf{z}_{1}^{T}\mathbf{k}_{1}\mathbf{A}_{\mathbf{z}_{1}} - \mathbf{s}^{T}\mathbf{c}_{2}\mathbf{s} \leq 0
\] (54)

As a result, \(\mathbf{V}_{3}\) is negative semi-definite along the feedback system. This implies that \(\mathbf{V}_{3}(t) \leq \mathbf{V}_{3}(0)\); that is, \(\mathbf{z}_{1}\) and \(\mathbf{s}\) are bounded. According to Barbala's lemma,\(^{22}\) \(\mathbf{V}_{3}\) converges to zero as \(t \to \infty\), which gives \(\mathbf{z}_{1}(t), \mathbf{s} \to \mathbf{0}\), and therefore \(\mathbf{z}_{2}(t) \to \mathbf{0}\). Thus, the position and velocity errors are proven to be globally and asymptotically convergent.

5. Numerical Simulation and Results

In this section, the performance of the designed \(J_{2}\)-invariant orbits, the proposed ABSMC law, and the derived relative dynamics is evaluated by numerical simulations. MOE and the differences of the MOE for the \(J_{2}\)-invariant orbit designed for the desired orbit are listed in Table 1. The simulation result of the designed \(J_{2}\)-invariant orbits is shown in Fig. 4. As seen in Fig. 4, the solid line corresponds to the \(J_{2}\)-invariant reference orbit designed from Eq. (31), and the dashed line shows the \(J_{2}\)-perturbed orbit derived from Eq. (23) for 30 orbits. Figure 4 displays that the \(J_{2}\)-invariant orbits are designed well to maintain the desired orbits for a long time compared to the \(J_{2}\)-perturbed orbits. To obtain the \(J_{2}\)-perturbed orbit from Eq. (23), MOE in Table 1 is transformed into corresponding osculating orbit elements by Brouwer’s Theory.\(^{23}\)

To verify the performance of the proposed control and adaptive law, the system parameters are selected:

- \(k = c_{1} = 4 \times 10^{-3}\), \(c_{2} = 3 \times 10^{-3}\), \(A = 5 \times 10^{-6}\)
- \(\Gamma = \text{diag}[5.9 \times 10^{-15}, 1.6, 1.37]\)
- \(\psi = 2.7 \times 10^{-7}\)
- \(\varepsilon = 0.1\)

The initial simulation errors, constrained control input, and spacecraft information are as follows:

- \(x_{1} = [100, 100, 100]^{T}\) m, \(\dot{x}_{1} = [0, 0, 0]^{T}\) m/sec
- \(u_{\text{max}} = 5 \times 10^{-3}\) m/sec\(^{2}\)
- \(m_{t} = m_{e} = 170\) kg
- \(A_{t} = A_{e} = 2.25\) m\(^{2}\)
- \([\alpha, \beta] = [210, 210]\) deg, \([\Delta \alpha, \Delta \beta] = [1.5, -1.5] \) deg

The unknown time-varying external disturbances are assumed as uniformly distributed random noises that are bounded within \(\pm 5 \times 10^{-5}\) m/sec\(^{2}\).\(^{9,21}\) For this mission scenario, the full state of the spacecraft is assumed to be known through spacecraft onboard navigation techniques.

Simulation results of the proposed ABSMC are drawn in Figs. 5–9. Figures 5 and 6 show that the tracking trajectory of the proposed ABSMC converges to the reference trajectory well. Figure 7 describes the tracking error. In Fig. 7, the tracking error is bounded within around 0.06 m in three-dimensional space after 2,000 s. These three figures indicate that the proposed controller provides reasonable tracking performance even in the presence of parameter uncertainties and disturbances with unknown bounds. Figure 8 displays the magnitude of control accelerations with limited low thrust and no chattering phenomenon. As seen in Fig. 8, this controller does not exceed the constrained control input

Fig. 5. Relative tracking position in 3D.

Fig. 6. Real and desired trajectory of the chaser.
and then carries out trajectory tracking as planned. Figures 9 and 10 show the response curves of the estimation parameters for atmospheric density and thrust misalignments. In Figs. 9 and 10, these parameters converge near to their corresponding constants. Figure 11 exhibits the estimated upper bound of external disturbances with unknown bounds in the control process, which shows that $\dot{D}$ can be estimated. As shown in Eqs. (52) and (53), Figs. 9–11 describe that the estimated parameters and their upper bounds can converge to any value without the requirement of the upper bounds and without more update when tracking errors approach zero.

To verify the performance of the derived relative dynamics and designed reference trajectories about the proposed controller, two cases are considered. In Case 1, the derived dynamics model of Eq. (35) is considered. It includes a drag- and precessing rotating frame with perturbations due to air drag and $J_2$. Then, two types of reference trajectories are also considered: the designed $J_2$-invariant reference trajectories of Eqs. (31)–(33) and the relative two-body reference trajectories\(^{24}\) without $J_2$. In Case 2, a two-body relative equation of motion\(^9\) is considered. The reference trajectories are considered to be the same as Case 1. The simulation re-

### Table 1. Initial conditions for the desired reference orbit.

| Desired mean orbital elements of target | $a_{\text{rel}}$ (m) | $e_{\text{rel}}$ | $i_{\text{rel}}$ (deg) | $\Omega_{\text{rel}}$ (deg) | $\omega_{\text{rel}}$ (deg) | $M_{\text{rel}}$ (deg) |
|----------------------------------------|-----------------------|-----------------|------------------------|------------------------|------------------------|------------------------|
| 6978.13 $\times 10^3$                  | 0.05                  | 98.7            | 250                    | 0                      | 0                      |

| Desired mean orbital element differences | $\Delta a_{\text{rel}}$ (m) | $\Delta e_{\text{rel}}$ | $\Delta i_{\text{rel}}$ (deg) | $\Delta \Omega_{\text{rel}}$ (deg) | $\Delta \omega_{\text{rel}}$ (deg) | $\Delta M_{\text{rel}}$ (deg) |
|------------------------------------------|-----------------------------|-------------------------|----------------------------|----------------------------|-----------------------------|----------------------------|
| 7.0e-3                                   | -5.69e-6                    | 1.0e-5                  | -1.0e-2                  | 1.77e-4                   |                             |                           |

### Table 2. Performance comparisons of the proposed dynamics model and reference trajectories with the proposed controller.

| Desired mean orbital elements of target | $a_{\text{rel}}$ (m) | $e_{\text{rel}}$ | $i_{\text{rel}}$ (deg) | $\Omega_{\text{rel}}$ (deg) | $\omega_{\text{rel}}$ (deg) | $M_{\text{rel}}$ (deg) |
|----------------------------------------|-----------------------|-----------------|------------------------|------------------------|------------------------|------------------------|
| 6978.13 $\times 10^3$                  | 0.05                  | 98.7            | 250                    | 0                      | 0                      |

| Desired mean orbital element differences | $\Delta a_{\text{rel}}$ (m) | $\Delta e_{\text{rel}}$ | $\Delta i_{\text{rel}}$ (deg) | $\Delta \Omega_{\text{rel}}$ (deg) | $\Delta \omega_{\text{rel}}$ (deg) | $\Delta M_{\text{rel}}$ (deg) |
|------------------------------------------|-----------------------------|-------------------------|----------------------------|----------------------------|-----------------------------|----------------------------|
| 7.0e-3                                   | -5.69e-6                    | 1.0e-5                  | -1.0e-2                  | 1.77e-4                   |                             |                           |

### Table 2. Performance comparisons of the proposed dynamics model and reference trajectories with the proposed controller.

| Total $\Delta V$ (m/s) | Case 1  | Case 2 |
|------------------------|---------|--------|
| 1 period (97 min)      | 0.8366  | 0.9381 |
| 2 periods (194 min)    | 0.8369  | 0.9358 |
| 3 periods (291 min)    | 0.8638  | 1.0788 |
| 4 periods (388 min)    | 0.8907  | 1.0731 |

To verify the performance of the derived relative dynamics and designed reference trajectories about the proposed controller, two cases are considered. In Case 1, the derived dynamics model of Eq. (35) is considered. It includes a dragging and precessing rotating frame with perturbations due to air drag and $J_2$. Then, two types of reference trajectories are also considered: the designed $J_2$-invariant reference trajectories of Eqs. (31)–(33) and the relative two-body reference trajectories\(^{24}\) without $J_2$. In Case 2, a two-body relative equation of motion\(^9\) is considered. The reference trajectories are considered to be the same as Case 1. The simulation re-

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sults of the two cases are presented in Table 2. As seen in Table 2, the results of Case 1 show that the designed $J_2$-invariant reference trajectories are more effective in reducing total fuel consumption (total $\Delta V$) as time passes than the relative two-body reference trajectories. The results of Case 2 show that the derived dynamics model is also more effective in reducing total fuel consumption (total $\Delta V$) as time passes than two-body relative equation of motion.9) As a result, Table 2 shows that the proposed algorithms for the control of spacecraft relative motion have good tracking performance for reducing total fuel consumption (total $\Delta V$) as time passes despite actuator input constraints, model uncertainties and disturbances.

6. Conclusions

In this paper, a set of nonlinear relative motion dynamics incorporating $J_2$ perturbation, atmospheric drag, and thrust error are derived. It simplifies and extends the equations developed by Kechcician. The derived relative dynamics can be useful for the accurate design of general elliptic orbits and be applied for spacecraft relative motion without approximations.

Furthermore, a $J_2$-invariant dynamics model for the desired orbit was designed using mean orbital elements and mean orbital elements differences based on the direction cosine matrix in Euler angles. The mean orbital element differences used the constraints for the $J_2$-invariant orbit. The designed $J_2$-invariant relative dynamics can be used to establish the relative position, velocity, and acceleration directly, and be more efficient than the conventional nonlinear relative equations of motion for reducing fuel consumption to maintain the desired orbit for a long time.

Finally, an adaptive backstepping sliding mode control law was proposed for spacecraft relative motion tracking maneuver in the presence of $J_2$ perturbation, atmospheric drag and thrust error, as parametric uncertainties and external disturbances. The advantage of the proposed control law is that it does not require prior knowledge of the upper bounds of parametric uncertainties and external disturbances. For limited low thrust, input saturation of the control law was also considered. Within the framework of Lyapunov analysis, the adaptive backstepping sliding mode control law not only guarantees closed-loop stability, but also assures the boundedness of errors. Moreover, the proposed control law essentially ensures globally asymptotic convergence of the estimated parameters and state tracking errors. Simulation results demonstrated the effectiveness of the developed relative motion dynamics and proposed control law for desired relative orbit keeping.

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Appendix

In this paper, five appendices are given: first, the control input $u$ can be expressed as the nominal thrust acceleration
from Eq. (10)

\[ u = T_m C_B^l \xi, \]

where \( \xi = [\xi_1, \xi_2, \xi_3]^T = [\cos \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha]^T. \)

\( \dot{H} \dot{\Theta} \) in Eq. (50) can be rewritten in the following form;

\[ \dot{H} \dot{\Theta} = \begin{bmatrix} H_{11} & H_{21} & H_{31} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \end{bmatrix} + T_m C_B^l G \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \end{bmatrix} \]

\[ G = \begin{bmatrix} -\sin \alpha \cos \beta & -\cos \alpha \sin \beta & \cos \alpha \sin \beta \\ \cos \alpha \cos \beta & -\sin \alpha \sin \beta & -\sin \alpha \sin \beta \\ 0 & \cos \alpha \sin \beta & -\sin \alpha \sin \beta \end{bmatrix} \]

To compute \( T_m \), Eq. (50) can be transformed. \(^9\)

\[ T_m C_B^l = q - H_1 \dot{\Theta}_1 - T_m C_B^l G \dot{\Theta}_1, \quad i = 1, 2, 3, \quad j = 2, 3 \]

\[ T_m C_B^l (\xi + G \dot{\Theta}_j) = q - H_1 \dot{\Theta}_1 \]

\[ q = -K_{jz} (z_2 - c_1 z_1) - f' - c_2 \dot{x} - \Delta z_1 + \ddot{x}_d \]

\[ \text{diag} (\text{sat}(s)) \dot{\Theta}_1 \]

Thus, \( T_m \) is defined as follows:

\[ T_m = \left\| \frac{q - H_1 \dot{\Theta}_1}{\xi + G \dot{\Theta}_j} \right\| \]

Second, each parameter in \( f'(x_1, x_2) \) of Eq. (23) can be rewritten as

\[ f_{11} = \left( \frac{h}{r_i^2} \right)^2 + \frac{1}{\tau} \left( \Delta a_{2\beta} + \Delta a_{1\beta} + \Delta a_{M \alpha}^e \right) \]

\[ f_{12} = -K_{jz} \sin^2 \theta \sin 2\theta + 2 \frac{h r_i}{r_i^3} \]

\[ f_{13} = K_{jz} \sin 2 \theta \sin \theta, \quad f_{14} = 0, \quad f_{15} = \left( \frac{2 h}{r_i^3} \right), \quad f_{16} = 0 \]

\[ f_{22} = \left( \frac{h}{r_i^2} \right)^2 + \frac{1}{\tau} \left( \frac{K_{jz}}{h} \sin 2 \theta \sin \theta \right)^2 \]

\[ + \frac{1}{\nu} \left( \Delta a_{2\beta} + \Delta a_{1\beta} + \Delta a_{M \alpha}^e \right) \]

\[ f_{23} = -8 K_{jz}^2 \sin^3 \theta \cos \theta \sin 2 \theta \sin \theta \]

\[ + 3 K_{jz} \sin 2 \theta \sin \theta \sin \theta_i \]

\[ f_{24} = -\left( \frac{2 h}{r_i^3} \right), \quad f_{25} = 0, \quad f_{26} = -2 \left( \frac{1}{h} \frac{K_{jz}}{r_i^3} \sin 2 \theta \sin \theta \right) \]

\[ f_{31} = \frac{K_{jz}}{r_i^3} \sin 2 \theta \sin \theta \]

\[ f_{32} = 8 \frac{K_{jz}^2}{h} \sin^3 \theta \cos \theta + \frac{K_{jz}}{r_i^3} \sin 2 \theta \cos \theta \]

\[ - 3 \frac{K_{jz}}{h r_i^2} \sin 2 \theta \sin \theta_i \]

\[ f_{33} = \left( \frac{1}{h} \frac{K_{jz}}{r_i^3} \sin 2 \theta \sin \theta \right)^2 + \frac{1}{\nu} \left( \Delta a_{2\beta} + \Delta a_{1\beta} + \Delta a_{M \alpha}^e \right) \]

\[ f_{34} = 0, \quad f_{35} = 2 \frac{K_{jz}}{h} \sin 2 \theta \sin \theta, \quad f_{36} = 0 \]

Third, each parameter in \( H \) of Eq. (24) can be rewritten as

\[ H_{11 AD} = \left[ -\left( \frac{1}{2} \frac{A}{r_i^2} \cos \theta \right) \sin \theta \right] + \Delta a_{1 A}^e \]

\[ H_{12 AD} = \left[ \left( \frac{1}{2} \frac{A}{r_i^2} \sin \theta \right) \cos \theta \right] + \Delta a_{1 A}^e \]

\[ H_{13 AD} = \left[ -\frac{K_{jz} v A}{2 h r_i^2} \sin 2 \theta \sin \theta \right] + \Delta a_{1 A}^e \]

\[ H_{21 AD} = \left[ -\left( \frac{1}{2} \frac{A}{r_i^2} \cos \theta \right) \sin \theta \right] + \Delta a_{1 A}^e \]

\[ H_{22 AD} = \left[ \left( \frac{1}{2} \frac{A}{r_i^2} \sin \theta \right) \cos \theta \right] + \Delta a_{1 A}^e \]

\[ H_{23 AD} = \left[ -\frac{K_{jz} v A}{2 h r_i^2} \sin 2 \theta \sin \theta \right] + \Delta a_{1 A}^e \]

Furthermore, each parameter in Eqs. (27) and (28) is defined as

\[ r_m = a_m (1 - e_m^2) \]

\[ \dot{r}_m = \frac{a_m v_m \sin f_m}{\sqrt{1 - e_m^2}} \dot{M}_m \]

\[ \dot{v}_m = a_m \cos f_m (1 + e_m \cos f_m)^2 \dot{M}_m \]

\[ n_m = \left( \frac{1}{a_m} \right), \quad p_m = a_m (1 - e_m^2) \]

\[ \dot{f}_m = \frac{(1 + e_m \cos f_m)^2}{(1 - e_m^2)^3} \dot{M}_m \]

\[ \dot{\dot{M}}_m = n_m + 0.75 J_2 \left( \frac{R_e}{p_m} \right)^2 \sin \left( \frac{1}{e_m^2} (3 \cos^2 i_m - 1) \right) \]

\[ \dot{\tilde{\omega}}_m = 0.75 J_2 \left( \frac{R_e}{p_m} \right)^2 \n_m (5 \cos^2 i_m - 1) \]

\[ \dot{\tilde{\Omega}}_m = -1.5 J_2 \left( \frac{R_e}{p_m} \right)^2 n_m \cos i_m \]

\[ \dot{\dot{\theta}}_m = \omega_m + \dot{f}_m \]

Finally, the parameters for \( T_1 \) and \( \tilde{T}_1 \) of Eq. (34), are given as

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\[
\dot{A}_{m1} = -B_{m1}\dot{\Omega}_{m1} + C_{m1}\dot{\theta}_{m1}, \quad \dot{B}_{m1} = A_{m1}\dot{\Omega}_{m1} + D_{m1}\dot{\theta}_{m1}
\]
\[
\dot{E}_{m1} = F_{m1}\dot{\theta}_{m1}
\]
\[
\dot{C}_{m1} = -D_{m1}\dot{\Omega}_{m1} - A_{m1}\dot{\theta}_{m1}, \quad \dot{D}_{m1} = C_{m1}\dot{\Omega}_{m1} - B_{m1}\dot{\theta}_{m1}
\]
\[
\dot{F}_{m1} = -E_{m1}\dot{\theta}_{m1}
\]
\[
\dot{G}_{m1} = -H_{m1}\dot{\Omega}_{m1}^2, \quad \dot{H}_{m1} = G_{m1}\dot{\Omega}_{m1}, \quad \dot{K}_{m1} = 0
\]
\[
\ddot{A}_{m1} = -A_{m1}\bigl(\dot{\Omega}_{m1}^2 + \dot{\theta}_{m1}^2\bigr) - 2D_{m1}\dot{\Omega}_{m1}\dot{\theta}_{m1} + C_{m1}\ddot{\theta}_{m1}
\]
\[
\ddot{B}_{m1} = -B_{m1}\bigl(\dot{\Omega}_{m1}^2 + \dot{\theta}_{m1}^2\bigr) + 2C_{m1}\dot{\Omega}_{m1}\dot{\theta}_{m1} + D_{m1}\ddot{\theta}_{m1}
\]
\[
\ddot{C}_{m1} = -C_{m1}\bigl(\dot{\Omega}_{m1}^2 + \dot{\theta}_{m1}^2\bigr) + 2B_{m1}\dot{\Omega}_{m1}\dot{\theta}_{m1} - A_{m1}\ddot{\theta}_{m1}
\]
\[
\ddot{D}_{m1} = -D_{m1}\bigl(\dot{\Omega}_{m1}^2 + \dot{\theta}_{m1}^2\bigr) - 2A_{m1}\dot{\Omega}_{m1}\dot{\theta}_{m1} - B_{m1}\ddot{\theta}_{m1}
\]
\[
\ddot{E}_{m1} = -E_{m1}\dot{\theta}_{m1} + F_{m1}\dot{\theta}_{m1}, \quad \ddot{F}_{m1} = -F_{m1}\dot{\theta}_{m1} - F_{m1}\ddot{\theta}_{m1}
\]
\[
\ddot{G}_{m1} = -G_{m1}\dot{\Omega}_{m1}^2, \quad \ddot{H}_{m1} = -H_{m1}\dot{\Omega}_{m1}, \quad \ddot{K}_{m1} = 0
\]

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