Braneworld Cosmological Models

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Abstract

The main cosmological models on the brane are presented. A generic equation is given, from which the Friedmann equations of the Randall-Sundrum, induced gravity, Gauss-Bonnet and the combined induced gravity and Gauss-Bonnet cosmological models are obtained. We discuss the modifications they bring to the standard cosmology and the main features of their inflationary dynamics.

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1 Introduction

The recent observation data from the Wilkinson Microwave Anisotropy Probe (WMAP) [1], show strong support for the standard inflationary predictions of a flat Universe with adiabatic density perturbations in agreement with the simplest class of inflationary models [2]. In particular, these observations had significantly narrowed the parameter space of slow-roll inflationary models. We are entering an area where the physics in the early universe can be probed by upcoming high-precision observational data.

In light of these developments, it is important to understand further the inflationary scenario from the theoretical point of view and also from a more phenomenological approach. A new idea that was put forward is, that our universe lies in a three-dimensional brane within a higher-dimensional bulk spacetime and this idea may have important consequences to our early time Universe cosmology. The most successful model that incorporates this idea is the Randall-Sundrum model of a single brane in an AdS bulk [3]. There are also other brane cosmological models which give novel features compared to standard cosmology. These models are mainly generalizations of the Randall-Sundrum model. The induced gravity cosmological model [4, 5] arises when we add to the brane action, the generated four-dimensional scalar curvature term by localized matter fields on the brane. The Gauss-Bonnet cosmological model [6, 7, 8, 9, 10] arises when we include a Gauss-Bonnet correction term to the five-dimensional action. Finally, if both terms are included in the action, the combined cosmological model [11] describes their cosmological evolution.

In this talk we will give a brief account of these models, describing the modifications they bring to the standard cosmology and discussing their inflationary dynamics. We will follow a general way of presentation writing the five-dimensional action that includes a Gauss-Bonnet term in the bulk and a four-dimensional scalar curvature term on the brane. Then, we will derive a generic cubic equation in $H^2$, from which taking appropriate limits of the parameters, we will derive the Friedmann equations of the above cosmological models.
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Consider the five-dimensional gravitational action

\[
S_{\text{grav}} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-(5)g} \left\{ (5)R - 2\Lambda_5 + \alpha \left[ (5)R^2 - 4 (5)R_{AB} (5)R^{AB} + (5)R_{ABCD} (5)R^{ABCD} \right] \right. \\
\left. + \frac{r}{2\kappa_5^2} \int_{y=0} d^4x \sqrt{-(4)g} \left[ (4)R - 2\Lambda_4 \right] \right. ,
\]

(1)

where \( \alpha \) is the Gauss-Bonnet coupling with dimensions \((\text{length})^2\) which is defined by

\[
\alpha = \frac{1}{8g_s^2},
\]

(2)

with \( g_s \) the string energy scale. The induced-gravity is specified by the crossover length scale

\[
r = \frac{\kappa_5^2}{\kappa_4^2} = \frac{M_4^2}{M_5^3}.
\]

(3)

Here, the fundamental \((M_5)\) and the four-dimensional \((M_4)\) Planck masses are given by

\[
\kappa_5^2 = 8\pi G_5 = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_4 = M_4^{-2}.
\]

(4)

We assume there are no sources in the bulk other than \( \Lambda_5 \). We assume there are no sources in the bulk other than \( \Lambda_5 \). Varying Eq. (1) with respect to the bulk metric \((5)g_{AB}\), we obtain the field equations:

\[
(5)G_{AB} - \frac{\alpha}{2} \left[ (5)R^2 - 4 (5)R_{CD} (5)R^{CD} + (5)R_{CDEF} (5)R^{CDEF} \right] (5)g_{AB} \\
+ 2\alpha \left[ (5)R (5)R_{AB} - 2 (5)R_{AC} (5)R_{BC} \right. \\
- 2 (5)R_{ACBD} (5)R^{CD} + (5)R_{ACDE} (5)R_B^{CDE} \left. \right] \\
= -\Lambda_5 (5)g_{AB} + \kappa_5^2 (\text{loc}) T_{AB} \delta(y),
\]

(5)
where \(^{(4)}g_{AB} = (5)g_{AB} - n_A n_B\) is the induced metric on the hypersurfaces \(\{y = \text{constant}\}\), with \(n^A\) the normal vector. The localized energy-momentum tensor on the brane is

\[(\text{loc})T_{AB} \equiv (4)T_{AB} - \lambda (4)g_{AB} - \frac{r}{\kappa_5^2} (4)G_{AB},\]

where \(\lambda\) is the brane tension and we have used the normalized Dirac delta function, \(\hat{\delta}(y) = \sqrt{(4)g/(5)g} \delta(y)\). Note that the last term in (6) is due to the presence of the scalar curvature term on the brane.

From the action (1), for a homogeneous and isotropic brane at fixed coordinate position \(y = 0\) in the bulk, we get the cubic equation in \(H^2\),

\[
\frac{4}{r^2} \left[1 + \frac{8}{3} \alpha \left(H^2 + \frac{k}{a^2} + \Phi_0\right)\right] \left(H^2 + \frac{k}{a^2} - \Phi_0\right) = \left[H^2 + \frac{k}{a^2} - \frac{\kappa_5^2}{3} (\rho + \lambda)\right]^2,
\]

where \(\Phi_0 = \Phi(t, 0)\) and \(\Phi\) is a solution of the equation \(\Phi + 2\alpha \Phi^2 = \Lambda_5/6 + C/a^4\) with \(C\) an integration constant, known as the ‘dark’ radiation term and in \(\Lambda_5\) the coupling \(\kappa_5^2\) has been absorbed.

The fundamental parameters appearing in (7) are: three energy scales, i.e. the fundamental Planck mass \(M_5\), the induced-gravity crossover energy scale \(r^{-1}\), and the Gauss-Bonnet coupling energy scale \(\alpha^{-1/2}\), and two vacuum energies, i.e. the bulk cosmological constant \(\Lambda_5\) and the brane tension \(\lambda\). The parameters \(r^{-1}\) and \(\alpha^{-1/2}\) are independent to each other. The crossover scale \(r\) of the induced gravity appears in loops involving matter particles, and depending on the mass, it can be arbitrarily large. On the other hand, the Gauss-Bonnet coupling \(\alpha\) arises from integrating out massive string modes, and depending on the scale of the theory, it can also be arbitrarily large. From the generic cubic equation (7), taking the appropriate limits of the above parameters, we will generate the Friedmann equations of the various known cosmological models.

### 2.1 The Randall-Sundrum Model

Taking the limits \(r \to 0\) and \(\alpha \to 0\) in (7), we get the Friedmann equation of the Randall-Sundrum model \(\text{[12, 13]}\)

\[
H^2 + \frac{k}{a^2} = \frac{\kappa_5^2}{36} \rho^2 + \frac{C}{a^4} + \frac{\kappa_5^2}{18} \lambda \rho + \frac{\Lambda_5}{6} + \frac{\kappa_5^2}{36} \Lambda^2.
\]
In the early universe $\rho >> \lambda$, and the $\rho^2$ term in (8) is dominant, giving a high energy modification of the standard Friedmann equation. To recover the late time cosmology we define the four-dimensional Newton’s constant as $8\pi G = \kappa_5^4 \lambda / 6$. Note that if $\lambda = 0$ we cannot recover the late time cosmology in this model. There is also an effective cosmological constant given by the last two terms of (8)

$$\Lambda_{\text{eff}} = \Lambda_5 + \frac{\kappa_5^4}{36} \lambda^2.$$  

(9)

Hence in principle, it is easy to have an accelerating phase in the Randall-Sundrum model. The second term in (8) is also a new brane-effect term called ‘dark’ radiation, because it scales as $a^{-4}$. The constant $C$ is having the information of the bulk and it is proportional to the mass of the black hole in the bulk.

If we assume that the early accelerating phase is governed by a scalar field, we can set $\Lambda_{\text{eff}}$ of (9) equal to zero. Then, the Randall-Sundrum Friedmann equation (8) becomes

$$H^2 = \kappa_5^2 \rho \left(1 + \frac{\rho}{2\Lambda_5}\right).$$

(10)

Applying the inflationary formalism to this Friedmann equation it was found, that the $\rho^2$ term acts as friction term which damps the rolling of the scalar field, allowing steeper potentials [14, 15, 16]. It was also found that this damping effect can bring the value of the inflaton field below the Planck mass, giving a solution to one of the basic problems of the chaotic inflationary scenario [14].

2.2 The Induced Gravity Model

Taking the limit $\alpha \to 0$ in (7), we get the Friedmann equation of the induced gravity cosmological model [17, 18, 19, 20, 21]

$$H^2 + \frac{k}{a^2} = \kappa_4^2 \left(\rho + \lambda\right) + \frac{2}{r^2}$$

$$\pm \frac{1}{\sqrt{3}r} \left[4\kappa_4^2 (\rho + \lambda) - 2\Lambda_5 + \frac{12}{r^2} - \frac{12C}{a^4}\right]^{1/2}.$$  

(11)

Because of the square root, in the early universe the linear term is dominant and therefore the induced gravity model in the high energy limit gives a correction to standard Friedmann equation, while at late times there are
significant modifications to the Friedmann equation, and in the limit \( a \rightarrow \infty \) a linearization of (11) gives again the conventional cosmology with an effective cosmological constant

\[
\Lambda_{\text{eff}} = \kappa_4^2 \lambda + \frac{6}{r^2} \pm \frac{\sqrt{6}}{r^2} \sqrt{(2\kappa_4^2 \lambda - \Lambda_5) r^2 + 6}.
\]  

(12)

The cosmological evolution described by the induced gravity model is a four dimensional evolution at high energies/early times followed by a five-dimensional and at low energies/late times again four-dimensional evolution [20]. Applying the slow-roll inflationary formalism [22, 23, 24, 25] setting \( \Lambda_{\text{eff}} \) of (12) equal to zero and redefining the high energy four-dimensional Newton’s constant \( \kappa_4^2 = \kappa_{\text{Planck}}^2 / \mu \), where \( \mu \) is a small number, it was found a better agreement of the chaotic inflationary scenario with observational data, with the value of the inflaton field well below the Planck mass [22].

### 2.3 The Gauss-Bonnet Model

Taking the limit \( r \rightarrow 0 \) in (17), we get the Friedmann equation of the Gauss-Bonnet cosmological model [7, 8]

\[
H^2 + \frac{k}{a^2} = \frac{1}{8\alpha} \left( -2 + \frac{64I^2}{J} + J \right).
\]  

(13)

where the dimensionless quantities \( I, J \) are given by

\[
I = \frac{1}{8} (1 + 4\alpha \Phi_0) = \pm \frac{1}{8} \left[ 1 + \frac{4}{3} \alpha \Lambda_5 + \frac{8\alpha C}{a^4} \right]^{1/2},
\]  

(14)

\[
J = \frac{\kappa_4^2 \sqrt{\alpha}}{\sqrt{2} (\rho + \lambda)} + \sqrt{\frac{\kappa_4^2 \alpha}{2} (\rho + \lambda)^2 + (8I)^3}^{2/3}.
\]  

(15)

The Gauss-Bonnet cosmological model like the Randall-Sundrum model gives modifications to the standard cosmology in the early universe. In the high energy limit of (13) the dominant contribution to the Friedmann equation is proportional to \( \rho^{2/3} \). The Friedmann equation (13) can be written in a simpler form [26]

\[
H^2 = \frac{1}{4\alpha} \left[ b^{1/3} \cosh \left( \frac{2x}{3} - 1 \right) \right],
\]  

(16)
where \( x \) is defined by
\[
\sigma = \left( \frac{2b}{\alpha \kappa_5^2} \right) \sinh x,
\]
(17)
and \( b \) is a constant. Applying the inflationary dynamics to the Friedmann equation (16), it was found that if the inflation is driven by an exponential inflaton field, the Gauss-Bonnet term allows the inflationary parameters to take values closer to the recent observational data \cite{26} and generally the presence of the Gauss-Bonnet term softens the Randall-Sundrum constraints on steep inflation \cite{27, 28}.

### 2.4 The Combined Model

If both \( r \) and \( \alpha \) are non-zero, the single real solution of the cubic equation (17) which is compatible with the previously considered limits, gives the Friedmann equation of the combined cosmological model having both the induced and Gauss-Bonnet terms \cite{11}

\[
H^2 + \frac{k}{a^2} = \frac{4 - 3\beta}{12\beta \alpha} - \frac{2}{3\beta \alpha} \sqrt{P^2 - 6Q} \cos \left( \frac{\Theta \pm \pi}{3} \right),
\]
(18)
where
\[
P = 1 + 3\beta I,
\]
(19)
\[
Q = \beta \left[ \frac{1}{4} + I + \frac{k^2 \alpha}{3}(\rho + \lambda) \right],
\]
(20)
\[
\Theta(P, Q) = \frac{1}{3} \arccos \left[ \frac{2P^3 + 27Q^2 - 18PQ}{2(P^2 - 6Q)^{3/2}} \right],
\]
(21)
and the constant \( \beta \) is given by
\[
\beta = \frac{256\alpha}{9r^2}.
\]
(22)

All the solutions of the combined cosmological model are of finite density, independently of the spatial curvature of the universe and the equation of state. As we discussed, the Gauss-Bonnet model on its own dominates at
early times and does not remove the infinite-density singularity, while the induced gravity model on its own mostly affects the late-time evolution. However, the combination of these terms produces an “interaction” that is not obviously the superposition of their separate effects. In general terms, the early-universe behaviour is strongly modified by the effective coupling of the 5D curvature to the matter. The late cosmological evolution of the combined model follows the standard cosmology, even for zero brane tension, with a positive Newton constant for one of the two branches of the solutions and positive cosmological constant.

It was showed in [11] that a radiation brane can, for some parameter values, undergo accelerated expansion at and near the minimal scale factor, independently of the spatial curvature of the universe. When there is a black hole in the bulk, a subset of these solutions has infinite acceleration at $a_0$, which signals the “birth” of an accelerated universe at finite energy, but with a curvature singularity.

The Friedmann equation (18) can be rewritten in a more familial form

$$H^2 = \frac{k}{a^2} = \frac{64(4 - 3\beta)}{27\beta^2 r^2} - \frac{16\sqrt{2}\xi}{9\beta r} \times \sqrt{-4\kappa_4^2 (\rho + \lambda) + \frac{3}{8}\beta \Lambda_5 + \frac{512}{9\beta^2 r^2} \left(1 - \frac{3}{2}\beta + \frac{9}{64}\beta^2\right) + \frac{9\beta C}{4a^4}}$$

where $\xi = \cos(\Theta \pm \frac{\pi}{3})$, and its detailed early inflationary dynamics is under investigation [29].

## 3 Conclusions

We have presented the main cosmological models on the brane and discussed their inflationary dynamics. All of them are giving modifications to the standard Friedmann equation, mostly in the high energy/early time regime. The terms that modify the standard Friedmann equation reflect the fact, that these models are defined in more that four dimensions and therefore any agreement with future observational data will give not only information on the early cosmological evolution of our universe, but also information on the structure of spacetime.

So far there is no any evidence that these models describe correctly the early time cosmological evolution and all recent astronomical, astrophysical
and cosmological data can be described, to large extent, within the general relativity theory. Nevertheless, pursuing these theoretical ideas may help us to understand better the recent and future observational data, detect possible inconsistencies and hopefully find some agreement between observational data and theoretical predictions.

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**References**

[1] D. N. Spergel et al., Astrophys. J. Suppl. **148**, 175 (2003), astro-ph/0302209.

[2] A. H. Guth, Phys. Rev. **D23**, 347 (1981); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982); A. D. Linde, Phys. Lett. **108B**, 389 (1982); A. D. Linde, Phys. Lett. **129B**, 177 (1983); J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D28, 679 (1983).

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-th/9905221; Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.

[4] H. Collins and B. Holdom, Phys. Rev. **D62**, 105009 (2000), hep-ph/0003173.

[5] G. Dvali, G. Gabadadze and M. Porati, Phys. Lett. **485B**, 208 (2000), hep-th/0005016; G. Dvali and G. Gabadadze, Phys. Rev. **D63**, 065007 (2001), hep-th/0008054.

[6] N. Deruelle and T. Dolezel, Phys. Rev. **D62**, 103502 (2000), gr-qc/0004021; B. Abdesselam and N. Mohammedi, Phys. Rev. **D65**, 084018 (2002), hep-th/0110143.

[7] C. Germani and C. Sopuerta, Phys. Rev. Lett. **88**, 231101 (2002), hep-th/0202060.
[8] C. Charmousis and J. Dufaux, Class. Quantum Grav. 19, 4671 (2002), hep-th/0202107.

[9] S. Davis, Phys. Rev. D67, 024030 (2003), hep-th/0208205; E. Gravanis and S. Willison, Phys. Lett. 562B, 118 (2003), hep-th/0209076; N.E. Mavromatos and J. Rizos, Phys. Rev. D62, 124004 (2000), hep-th/0008074; Y.M. Cho, I.P. Neupane and P.S. Wesson, Nucl. Phys. B621, 388 (2002), hep-th/0104227.

[10] I. Low and A. Zee, Nucl. Phys. B585, 395 (2000), hep-th/0004124; J.E. Lidsey, S. Nojiri and S. Odintsov, JHEP 06, 026 (2002), hep-th/0202198; P. Binetruy, C. Charmousis, S.C. Davis and J-F. Dufaux, Phys. Lett. B544, 183 (2002), hep-th/0206089; J.P. Gregory and A. Padilla, hep-th/0304250; N. Deruelle and M. Sasaki, gr-qc/0306032; C. Barcelo, C. Germani and C.F. Sopuerta, gr-qc/0306072.

[11] G. Kofinas, R. Maartens and E. Papantonopoulos, JHEP 0310, 066 (2003), hep-th/0307138.

[12] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. 477B, 285 (2000), hep-th/9910219.

[13] C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. 462B, 34 (1999), hep-ph/9906513; J. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 4245 (1999), hep-ph/9906523.

[14] R. Maartens, D. Wands, B. Bassett and I. Heard, Phys. Rev. D62, 041301 (2000), hep-ph/9912464.

[15] A. R. Liddle and A. J. Smith, astro-ph/0307017.

[16] S. Tsujikawa and A. R. Liddle, astro-ph/0312162.

[17] C. Deffayet, Phys. Lett. 502B, 199 (2001), hep-th/0010186.

[18] Y. Shtanov, hep-th/0005193; S. Nojiri and S.D. Odintsov, JHEP 07, 049 (2000), hep-th/0006232; C. Deffayet, Phys. Lett. 502B, 199 (2001), hep-th/0010186; G. Kofinas, JHEP 08, 034 (2001), hep-th/0108013; N.J. Kim, H.W. Lee and Y.S. Myung, it Phys. Lett. 504B, 323 (2001), hep-th/0101091; C. Deffayet, G. Dvali and G. Gabadadze, Phys. Rev. D65, 044023 (2002), astro-ph/0105068; C. Deffayet, S.J. Landau, J.
Raux, M. Zaldarriaga and P. Astier, Phys. Rev. D66, 024019 (2002), astro-ph/0201164.

[19] K. Maeda, S. Mizuno and T. Torii, Phys. Rev. D68, 024033 (2003), gr-qc/0303039.

[20] E. Kiritsis, N. Tetradis and T.N. Tomaras, JHEP 03, 019 (2002), hep-th/0202037.

[21] V. Sahni and Y. Shtanov, Int. J. Mod. Phys. 11, 1 (2002), gr-qc/0205111. U. Alam and V. Sahni, astro-ph/0209443.

[22] E. Papantonopoulos and V. Zamarias, JCAP 10, 001 (2004) gr-qc/0403090.

[23] H. Zhang and R-G. Cai, hep-th/0403234.

[24] M. Bouhmadi-Lopez, R. Maartens and D. Wands, hep-th/0407162.

[25] M. Bouhmadi-Lopez and D. Wands, hep-th/0408061.

[26] J. E. Lidsey and N. J. Nunes, Phys. Rev. D67, 103510 (2003), astro-ph/0303168.

[27] S. Tsujikawa, M. Sami, R. Maartens, astro-ph/0406078.

[28] M. Sami, N. Savchenko, A. Toporensky, hep-th/0408140.

[29] E. Leeper, R. Maartens, E. Papantonopoulos, V. Zamarias, work in progress.