FACTORIZATION HOMOLOGY OF POLYNOMIAL ALGEBRAS

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ABSTRACT. We compute the factorization homology of a polynomial algebra over a compact and closed manifold with trivialized tangent bundle up to weak equivalence in a new way. This calculation is based on the model of a graph complex and an explicit morphism into the codomain, which makes it possible to twist the algebra with a Maurer-Cartan element and potentially apply other deformations.

1. INTRODUCTION

Factorization homology, also known as manifoldic homology, is an invariant of topological n-manifolds. In this paper we present a new and more transparent method of its computation. Based on the work of Beilinson and Drinfeld on factorization algebras the notion of factorization homology generalizes the topological chiral homology theory by Lurie with coefficients in n-disks algebras as well as labeled configuration spaces from Salvatore and Segal. With origins in conformal field theory as well as the configuration space models of mapping spaces it creates an overlap of various fields of interest and contributes for example to topological quantum field theory as an algebraic model for observables in [Sch14] and others. Applications can be found numerously in the literature.

Given an \(e_n\)-algebra \(A\), the factorization homology \(\int_M A\) for a framed \(n\)-manifold \(M\) may be defined by a colimit. Unfortunately, this is notoriously hard to compute and usually avoided by employing methods of excision. The necessity of other means of computation is evident by the large number of applications and affiliated concepts. Markarian has computed in [Mar16, Proposition 4] the factorization homology of a polynomial algebra using directly its definition as a colimit. The main result of this paper is an alternative computation of the factorization homology of the same polynomial algebra. It relies instead on a graph complex model introduced by Campos and Willwacher in [CW16]. This model for the \(n\)-disks operad allows us to perform computations with graphs on a combinatorial level. In choosing this accessible and easily manipulable approach, we enable further generalizations and modifications such as twistings of the coefficient algebra, which leads to the following improved result.

**Theorem 4.1.** Let \(M\) be a compact and oriented manifold with trivialized tangent bundle. Let moreover \(V\) be the shifted cotangent bundle \(V = T^*[1-n]R^N\) and consider the polynomial algebra \(O = O(V)\). Then the factorization homology \(\int_M O[h]\) of \(M\) with coefficients in the twisted polynomial algebra \(O[h] = \bigwedge^{m_1 + \cdots + m_j} T^*[1-n]R^N\) is weakly equivalent to the algebra of twisted polynomials \(S(H(M) \otimes V)\).

The computation consists of rewriting the derived composition product with a cofibrant replacement of the graph complex model and the explicit construction of a map into the desired codomain. Subsequently it is only left to show that this map is a weak equivalence using multiple nested spectral sequences. In the course of setting up these spectral sequences, we make use of the Lambrech- Stanley model and results from Sinha in [Sin10].

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2. BACKGROUND

For the sake of introducing the proper framework for our computation and presenting notational conventions, we briefly recall the construction of the essential tools used in this paper.
2.1. Operads. For the notational conventions of operads we will follow Fresse’s book [Fre07]. Let \((\mathcal{C}, \otimes, 1)\) be a symmetric monoidal category. A \(\Sigma_n\)-object in \(\mathcal{C}\) is a collection \((C(0), C(1), \ldots)\) of \(\Sigma_n\)-modules \(C(n) \in \text{Obj}(\mathcal{C})\) for \(n \in \mathbb{N}\). To a \(\Sigma_n\)-object we associate the endofunctor \(S(C) \in \text{End}(\mathcal{M})\) of a symmetric monoidal category \(\mathcal{M}\) over \(\mathcal{C}\), where \(S(C)\) applied to \(M \in \text{Obj}(\mathcal{M})\) is defined to be

\[
S(C, M) = \bigoplus_r (C(r) \otimes M^\otimes r)_{\Sigma_r}
\]

Denote by \(\mathcal{C}^{\Sigma_r}\) the category of \(\Sigma_r\)-objects in \(\mathcal{C}\) where a morphism \(f : C \to D\) consists of a collection of \(\Sigma_r\)-equivariant morphisms \(f : C(n) \to D(n)\) in \(\text{Hom}(\mathcal{C})\). Note that the map \(S : \mathcal{C}^{\Sigma_r} \to \text{End}(\mathcal{M})\) is functorial. With the composition of \(\Sigma_r\)-objects \(C\) and \(D\) given by

\[
C \circ D = S(C, D)
\]

and the unit object

\[
I(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}
\]

the category \((\mathcal{C}^{\Sigma_r}, \circ, I)\) obtains the structure of a monoidal category over \(\mathcal{C}\) and makes \(S : (\mathcal{C}^{\Sigma_r}, \circ, I) \to (\text{End}(\mathcal{M}), \circ, \text{Id})\) into a monoidal functor.

An operad in \(\mathcal{C}\) comprises a \(\Sigma_r\)-object \(\mathcal{P} \in \mathcal{C}^{\Sigma_r}\) together with a composition morphism \(\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}\) and a unit morphism \(\eta : I \to \mathcal{P}\), satisfying the usual monoidal unit and associativity condition. The composition morphism of an operad can be described by morphisms

\[
\mu_r : \mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r)
\]

or equivalently by partial composites

\[
o : \mathcal{P}(k) \otimes \mathcal{P}(l) \to \mathcal{P}(k + l - 1)
\]

Let us furthermore use for an operad \(\mathcal{P}\) the notation \(\mathcal{P}^\vee\) for its Koszul dual operad and \(\mathcal{P}^*\) for its linear dual operad.

2.2. Examples of Operads. Commonly used operads are the commutative operad \(\text{Com}\) and the Lie operad \(\text{Lie}\), which carry the structure of the commutative product \(\wedge\) and Lie bracket \(\{,\}\) respectively. The (framed) embedding spaces of \(r\) disks \(D^n\) into another disk \(D^n\)

\[
\text{Disks}^\text{(fr)}_n(r) = \text{Emb}^\text{(fr)}(D^n \times \{1, \ldots, r\}, D^n)
\]

assemble with the composition of (framed) embeddings the (framed) little \(n\)-disks operad \(\text{Disks}^\text{(fr)}_n\) in the category of topological spaces. The homology of the little \(n\)-disks operad inherits the operadic structure and is denoted by \(e_n^\text{fr} = H_*(\text{Disks}^\text{(fr)}_n)\). The \(e_n\) operad can be identified with the operadic composition \(\text{Com} \circ \text{Lie}[n - 1]\) and is therefore generated by the commutative product and the Lie bracket in arity two. It’s Koszul dual is \(e_n^* = e_n^\text{fr}[n] = \text{Com}^*[n] \circ \text{Lie}[1]\).

2.3. Algebras and Modules over Operads. An algebra over the operad \(\mathcal{P}\), or short \(\mathcal{P}\)-algebra, is an object \(A \in \text{Obj}(\mathcal{C})\) with a morphism \(\lambda : S(\mathcal{P} \circ A) \to A\) such that the natural unit, associativity and equivariance relations are satisfied. This morphism can equivalently be expressed by morphisms \(\lambda_r : \mathcal{P}(r) \otimes A^\otimes r \to A\). Denote the category of \(\mathcal{P}\)-algebras by \(\text{alg}_\mathcal{P}\); where the morphisms are morphisms in \(\text{Hom}(\mathcal{C})\) that intertwine with the evaluation morphisms.

A left \(\mathcal{P}\)-module \(M\) is in a similar fashion an object \(M \in \text{Obj}(\mathcal{C}^{\Sigma_r})\) with an \(\Sigma_r\)-equivariant left \(\mathcal{P}\)-action \(\lambda : \mathcal{P} \circ M \to M\) compatible with the operads associativity and unity relations. Left \(\mathcal{P}\)-modules together with morphisms \(\text{Hom}(\mathcal{C}^{\Sigma_r})\) that intertwine with \(\lambda\) form the category \(\text{mod}_\mathcal{P}\). There is a functor

\[
\text{alg}_\mathcal{P} \to \text{mod}_\mathcal{P}
\]

given by \(A \mapsto A^\otimes = (0, A, 0, 0, \ldots)\).

Right \(\mathcal{P}\)-modules are constructed in the same way with a right \(\mathcal{P}\)-action, but note that due to the lack of symmetry in the operadic composition left and right \(\mathcal{P}\)-modules are entirely different objects. The composition product \(M \circ \mathcal{P} \circ N\) of a right \(\mathcal{P}\)-module \(M\) and a left \(\mathcal{P}\)-module \(N\) is defined through the pushout diagram

\[
\begin{array}{ccc}
M \circ \mathcal{P} \circ N & \rightarrow & M \circ N \\
\downarrow & & \downarrow \\
M \circ N & \rightarrow & M \circ \mathcal{P} \circ N
\end{array}
\]
When taking the composition product of a right \( \mathcal{P} \)-module and a \( \mathcal{P} \)-algebra \( A \), we associate to \( A \) its left \( \mathcal{P} \)-module to use the composition product of \( \mathcal{P} \)-modules. The derived composition product \( M^\wedge \circ \mathcal{P} A := \widehat{M} \circ \mathcal{P} A \) is the composition product of a cofibrant replacement \( \widehat{M} \) with \( A \). For example, given a (framed) manifold \( M \) the spaces \( \text{Disks}^{(fr)}_n(r) = \text{Emb}^{(fr)}(D^n \times \{1, \ldots, r\}, M) \) from a right \( \text{Disks}^{(fr)}_n \)-module, where the action of \( \text{Disks}^{(fr)}_n \) is given by a (framed) embedding.

2.4. The Graph Complex \( \text{Graphs}_M \). In their paper [CW16] Campos and Willwacher build \( \text{Graphs}_M \) as a model for the configuration space of a sufficiently well behaved manifold, which we will denote from here on by \( M \). That means that \( M \) has to be compact, closed and its tangent bundle \( T M \) is trivialized. The construction of this model is summarized in the following.

Define for \( r \geq 0 \) and \( n \in \mathbb{N} \) the space \( \text{Gra}_n(r) \) to be the quotient of the free graded commutative algebra generated by elements \( s_{ij} \) of degree \( n-1 \), by the relations \( s_{ij} = (-1)^{n}s_{ji} \), where \( 1 \leq i \neq j \leq r \). Elements \( \Gamma \in \text{Gra}_n(r) \) can be pictured as the linear combination of graphs with \( r \) vertices. The spaces \( \text{Gra}_n(r) \) for \( r \in \mathbb{N} \) form the operad \( \text{Gra}_n \) for which the partial operadic composition \( \circ_i \) of two graphs is declared to be the insertion of the second graph into the \( i \)th vertex of the first graph with summation over all possibilities of reconnecting the edges.

The degree of a graph \( \Gamma \in \text{Graphs}_M \) is

\[
\deg(\Gamma) = n \cdot s - (n-1) \cdot e + \sum_{\alpha} \deg_{H^*}(\alpha)
\]

where \( s \) is the number of internal vertices and \( e \) the number of edges. Due to the symmetry properties of edges and vertices we may define a notion of orientation for the cases of \( n \) odd and even. In the case that \( n \) is odd the orientation is given by the order of external vertices, a direction of every edge and an order of all odd decorations. In the case that \( n \) is even the orientation is given by an order of edges and an order of all odd decorations. One term of the differential on this module is \( \delta_{\text{split}} \), which splits an internal vertex out of both internal and external vertices and sums over all possible ways of reconnecting the incident edges.

The other term \( \delta_{\text{pair}} \) pairs any two complementary decorations \( \alpha \) and \( \beta \) as \( \langle \alpha, \beta \rangle = (-1)^{|\alpha|} \int_M \alpha \wedge \beta \) and connects the vertices to which they are attached to by an edge. The label or direction of the new edge is subject to conventions.

\[
\delta_{\text{split}} \Gamma = (-1)^{|\Gamma|} \sum_{e \in \Gamma} \Gamma \circ_v o \bigcirc
\]

\[
\delta_{\text{pair}} \begin{pmatrix} 1 \cdot \alpha^1 \\ 2 \cdot \beta^2 \end{pmatrix} = \langle \alpha, \beta \rangle \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
Denote the sum of these two differentials by $\delta = \delta_{\text{split}} + \delta_{\text{pair}}$. The union of two graphs $\Gamma$ and $\Sigma$ with the orientation given by concatenating the order of vertices or edges from left to right is denoted by $\Gamma \cdot \Sigma$. The failure of $\delta_{\text{pair}}$ to be a derivation with respect to this product is measured by the Lie bracket

$$\{\Gamma, \Sigma\}_G = \delta_{\text{pair}}(\Gamma \cdot \Sigma) - \delta_{\text{pair}}(\Gamma) \cdot \Sigma - (-1)^{|\Gamma|} \Gamma \cdot \delta_{\text{pair}}(\Sigma)$$

In the literature a linear combination of possibly decorated graphs which only contain internal vertices is of interest. This so called partition function $z_M$ satisfies the Maurer-Cartan equation in $\text{Graphs}_M$

$$\delta z_M + \frac{1}{2} [z_M, z_M]_G = 0$$

Such an element $z_M$ then defines a twisting differential $d^M = \{z_M, \cdot \}_G$. The full differential on $\text{Graphs}_M$ is $\delta + d^M$.

2.5. The Polynomial Algebra $O$. Consider for two fixed positive integers $n$ and $N$ the the polynomial algebra $O$ over the shifted cotangent bundle $V = T^n[1−n]\mathbb{R}^N$ of Euclidean space.

$$O = O(T^n[1−n]\mathbb{R}^N)$$

Given a basis $x_1, \ldots, x_N$ of $\mathbb{R}^N$, we may write $O$ as the free commutative algebra in coordinates $p_1, \ldots, p_N$ of degree $(1−n)$ and $x_1, \ldots, x_N$ of degree 0, also denoted by $y_1, \ldots, y_{2N}$ for convenient notation. The graded Poisson bracket on $O$ is uniquely defined by the relations

$$\{p_i, x_j\} = \delta_{ij}$$

$$\{p_i, p_j\} = \{x_i, x_j\} = 0$$

and the symmetry property $\{f, g\} = (-1)^{|f||g|}\{g, f\}$.

2.6. Twisting. $(O, \{ , \})$ is in particular a dg Lie algebra with the zero differential. Therefore the Maurer-Cartan equation for $m \in O$ simplifies to $\{m, m\} = 0$. Any such $m$ satisfying the Maurer-Cartan equation gives rise to a twisting differential

$$d^m = -\{m, \cdot \}$$

which is a derivation with respect to the product in $O$. The twisted polynomial algebra is given by $O^m = (O, \{ , \}, d^m)$. For the sake of simplicity, let us define as a first step $O[h] = O^m$ for $m = hm_1 + h^2m_2 + \cdots$ with $h > 0$ and $m_i \in \text{MC}(O)$.

2.7. Decorated Polynomials. A slight modification of the algebra $O$ introduced above is

$$S = S(H(M) \otimes V)$$

the polynomial algebra in formal variables $p_1, \ldots, p_N$ and $x_1, \ldots, x_N$, each one tensored with a cohomology class $\alpha \in H^*(M)$. To make the distinction to polynomials in $O$ as apparent as possible, we will denote polynomials $F \in S$ by capital letters. The degree of a decorated polynomial $F$ is iteratively defined by the degree of its variables

$$\text{deg}(\alpha \otimes y_k) = \text{deg}(\alpha) + \text{deg}(y_k)$$

Polynomials $F, G \in S$ satisfy the graded commutativity relation

$$F \cdot G = (-1)^{|F||G|} G \cdot F$$

We consider the differential $\Delta$ on $S$ that acts in the following way on decorated polynomials.

$$\Delta(F) = \sum_{\substack{j=1, \ldots, n \in \text{basis of } H^* \wedge \omega}} \frac{\partial}{\partial (\omega \otimes p_j)} \frac{\partial}{\partial (\omega^* \otimes x_j)} (F)$$

The Poincaré duality pairing $\langle \alpha, \beta \rangle$ is non degenerate, because $M$ is compact and oriented. With this and the Poisson bracket on $O$ we obtain a Poisson bracket on $S$.

$$\{\alpha \otimes y_k, \beta \otimes y_l\}_S = \langle \alpha, \beta \rangle \{y_k, y_l\}$$

for $y_k, y_l \in \{p_1, \ldots, p_N, x_1, \ldots, x_N\}$. This bracket may be expressed as

$$\{F, G\}_S = \sum_{\substack{j=1, \ldots, n \in \text{basis of } H^* \wedge \omega}} \frac{\partial F}{\partial (\omega \otimes p_j)} \frac{\partial G}{\partial (\omega^* \otimes x_j)} + (-1)^n \frac{\partial G}{\partial (\omega \otimes p_j)} \frac{\partial F}{\partial (\omega^* \otimes x_j)}$$
2.8. Factorization Homology. For a \( \text{Disks}_n \)-algebra \( A \), we define its algebraic factorization homology over a manifold \( M \) with trivialized tangent bundle to be the derived composition product

\[
\int_M A := C_\bullet (\text{Disk}_M) \odot_{e_n} A
\]

For compact and closed manifolds Campos and Willwacher have shown that \( \text{Graphs}_M \) is a model for chains \( C_\bullet (\text{Disks}_M) \). That is, the pair of right module and operad \( (\text{Graphs}_M \odot e_n) \) is weakly equivalent to the pair \( (C_\bullet (\text{Disks}_M) \odot e_n) \). We can therefore rewrite for an \( e_n \)-algebra \( A \) its factorization homology as

\[
\int_M A \simeq \text{Graphs}_M \odot_{e_n} A
\]

To compute the derived composition product we choose a cofibrant replacement of \( \text{Graphs}_M \) using the bar-cobar construction of right \( e_n \) modules

\[
\widehat{\text{Graphs}}_M = \text{Free}_{e_n} \left( \text{coFree}_{e_n} (\text{Graphs}_M) \right) \\
= \text{Graphs}_M \circ e_n^\nu \circ e_n
\]

3. Computation of Factorization Homology

Now that the framework is set, we can prove the main theorem, which computes the factorization homology of the polynomial algebra introduced above.

**Theorem 3.1.** Let \( M \) be a compact and oriented manifold \( M \) with trivialized tangent bundle. Let moreover \( V \) be the shifted cotangent bundle \( V = T^* [1 - n] \mathbb{R}^N \) and consider the the polynomial algebra \( O = O(V) \). Then the factorization homology \( \int_M O \) of \( M \) with coefficients in the polynomial algebra \( O \) is weakly equivalent to the algebra of twisted polynomials \( S (H(M) \otimes V) \).

3.1. Map Construction. To perform the calculation, we start by rewriting the derived composition product with a cofibrant replacement of \( \text{Graphs}_M \).

\[
\int_M O = \widehat{\text{Graphs}}_M \circ e_n, O \\
= \text{Free}_{e_n} \left( \text{coFree}_{e_n} (\text{Graphs}_M) \right) \circ e_n, O \\
= \text{coFree}_{e_n} (\text{Graphs}_M) \circ O \\
= \text{Graphs}_M \circ e_n^\nu \circ O \\
= \text{Graphs}_M \circ \text{Com}^* [n] \circ \text{Lie}^* [1] \circ \text{Com} \circ V
\]

We denote this complex by \( D \) as this will be the domain of the map we are constructing in this section. It is equipped with the differential

\[
\partial = \delta_{\text{split}} + \delta_{\text{pair}} + \delta_{\text{alg}} + \delta_{\text{mod}} + \delta_{\text{com}} + \delta_{\text{lie}}
\]

The differentials of the form \( \delta_{\text{mod}} \) are induced by twisting morphisms from the (co)operad \( B \) into the algebra or module specified by \( B \). The action of \( \delta_{\text{com}} \) is explained in more depth in the proof of lemma \( (3.10) \). \( D \) inherits a Lie bracket from \( \text{Graphs}_M \)

\[
\{ \Gamma \otimes f_I, \Sigma \otimes f_J \}_D = \{ \Gamma, \Sigma \}_G \otimes f_{IJ}
\]

With the composition of operads, we can moreover rewrite the codomain as \( S = \text{Com} \circ H^* (M) \circ V \). Let us first consider the maps

\[
\varphi_r : \text{Graphs}_M(r) \otimes O^{\otimes r} \to S \\
\Gamma \otimes f_1 \otimes \cdots \otimes f_r \mapsto \mu_{\text{mult}} \left( \prod_{(i, j)} \Xi_{ij} \left( f_1 \otimes \cdots \otimes f_r \right) \right)
\]

where every edge \( (i, j) \) and decoration \( (i, \alpha) \) is associated to an operator \( \Xi_{ij} \) or \( \Xi_{i\alpha} \) respectively, acting on the \( i \)th and/or \( j \)th factor in the tensor product.

\[
\Xi_{ij} = \sum_{k=1}^{2N} \left[ y_k, \gamma_k \right] \left( \frac{\partial}{\partial y_k} \right)_{(i)} \left( \frac{\partial}{\partial y_k} \right)_{(j)} \\
\Xi_{i\alpha} = \sum_{k=1}^{2N} \left[ \alpha \otimes y_k \right] \left( \frac{\partial}{\partial y_k} \right)_{(i)}
\]

Any graph that contains one or more internal vertices gets send to zero by this map. Subsequently \( \mu_{\text{mult}} : f_1 \otimes \cdots \otimes f_k \mapsto f_1 \cdots f_k \) replaces the tensor product with the product in \( S \). Combined this gives the map

\[
\varphi' : \bigoplus_r \text{Graphs}_M(r) \otimes O^{\otimes r} \to S (H(M) \otimes V)
\]
Example 3.2. These definitions should become clearer by looking at an example of a graph in arity three acting on functions \( f, g, h \in \mathcal{O} \) in the case that \( n \) is odd.

\[
\varphi'(\begin{array}{c}
\alpha \cdots \beta \\
\circ f \circ g \circ h
\end{array}) = \sum_{j,k, l=1}^{2N} \{y_k, \tilde{y}_k\} \{y_l, \tilde{y}_l\} (\alpha \otimes y_j) \frac{\partial f}{\partial y_j} \frac{\partial^2 g}{\partial y_k \partial y_l} \frac{\partial h}{\partial y_l}
\]

Next, we need to check that this map passes on to the composition of Graphs\(_M\) and \( \mathcal{O} \) as \( e_n \)-modules. For this it is necessary for the action of the generators of \( e_n \) on the module and the algebra to coincide when \( \varphi' \) is applied. The commutative product and Lie bracket includes into Graphs\(_M\) as \((\circ \circ) \) and \((\circ - \circ) \) respectively.

Remark 3.3. In the following, calculations edges and decorations exhibit the same behavior, it is therefore possible to reduce these calculations without loss of generality to non decorated graphs. Starting with the simplest case of only one edge in a graph, say between the vertices 1 and 2, the product rule implies

\[
\Xi_{12} (f_a f_b \otimes f_2 \cdots \otimes f_r) = \sum_{k=1}^{N} \{y_k, \tilde{y}_k\} \frac{\partial f_a}{\partial y_k} \otimes \frac{\partial f_b}{\partial y_k} \otimes \cdots \otimes f_r
\]

For an index set \( J = \{j_1, \ldots, j_k\} \) we write \( I^C \) for the compliment of an index subset \( I \subset J \). Abbreviate moreover for such an index set the products \( \Xi_{i,j} = \Xi_{ij_1} \cdots \Xi_{ij_k} \) and \( f_I = f_{j_1} \otimes \cdots \otimes f_{j_k} \). This amounts to the more general case where we take into account multiple edges at one vertex.

\[
\Xi_{i,j} (f_a f_b \otimes f_2 \otimes \cdots \otimes f_r) = \sum_{I \subset J} \Xi_{i,j} \Xi_{I^C} (f_a \otimes f_b \otimes f_2 \otimes \cdots \otimes f_r)
\]

Take \( J \) to be the index set of all vertices connected to 1. Finally this leads to

\[
\varphi' \left( \Gamma \otimes f_a f_b \cdots \otimes f_r \right) = \mu_{\text{mult}} \left( \prod_{(i,j)} \Xi_{ij} \left( f_a f_b \otimes \cdots \otimes f_r \right) \right)
\]

The calculation for the generator \( \{ , \} \) is very similar, since we can write \( \{f_a, f_b\} = \mu_{\text{mult}} (\Xi_{ab} (f_a \otimes f_b)) \). This additional factor \( \Xi_{ab} \) in the calculation above corresponds then to an additional edge between the vertices which are being inserted in \( \Gamma \). Therefore we need to compose \( \Gamma \) with \((\circ - \circ) \) instead of \((\circ \circ) \). This means we have

\[
\varphi' \left( \Gamma \otimes \{f_a, f_b\} \otimes f_2 \otimes \cdots \otimes f_r \right) = \varphi' \left( \left( \Gamma \circ_1 (\circ - \circ) \right) \otimes f_a \otimes f_b \otimes \cdots \otimes f_r \right)
\]

Hence in the following diagram the induced quotient map \( \tilde{\varphi} \) is well defined and we can establish the desired morphism \( \varphi = \tilde{\varphi} \circ \pi \) as a composition.
For the sake of simplicity we will abbreviate \( \varphi(\Gamma \otimes f_I) = \Gamma \circ f_I \)

**Proposition 3.4.** The map \( \varphi : (D, \partial, \{, \}_D) \to (S, \Delta, \{, \}_S) \) is a morphism of dg Lie modules over \( e_n \). More specifically, \( \varphi_{\text{pair}} = \Delta \varphi \) whereas all other terms of \( \partial \) vanish on \( S \).

**Proof.** Due to the commutativity of the partial derivatives it is sufficient to verify the claim on the following simple case. We are using here that \( \Delta \) applied to monomials in \( S \) vanishes.

\[
(\Delta \varphi) \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} \otimes f \otimes g = \Delta \left( \sum_{k,l} (\alpha \otimes y_k) \frac{\partial f}{\partial y_k} (\beta \otimes y_l) \frac{\partial g}{\partial y_l} \right)
\]

\[
= \sum_{k,l} (\alpha \otimes y_k, \beta \otimes y_l) S \left( \frac{\partial f}{\partial y_k} \frac{\partial g}{\partial y_l} \right)
\]

\[
= \sum_{k,l} (\alpha, \beta) \{y_k, y_l\} \frac{\partial f}{\partial y_k} \frac{\partial g}{\partial y_l}
\]

\[
= \varphi \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} \begin{pmatrix}
1 \\
2
\end{pmatrix} \otimes f \otimes g
\]

\[
= \varphi \begin{pmatrix}
\delta_{\text{pair}} \\
\alpha
\end{pmatrix} \begin{pmatrix}
1 \\
2
\end{pmatrix} \otimes f \otimes g
\]

As \( \varphi \) sends graphs containing any internal vertices to zero and both of \( \delta_{\text{split}} \) and \( \delta_{\text{str}} \) result in a graph with internal vertices, these differentials correspond to zero. The compatibility of \( \varphi \) with the brackets follows then immediately from their relation to \( \delta_{\text{pair}} \) and \( \Delta \). We may write this property as

\[
\{ \Gamma \circ f_I, \Sigma \circ f_I \}_D = \{ \Gamma, \Sigma \}_G \circ f_{IJ}
\]

\[\Box\]

**3.2. Filtrations.** In this section we will build on the constructions in the previous section to prove that the constructed morphism \( \varphi \) is indeed a quasi-isomorphism. This procedure will heavily rely on the following two Lemmas.

**Lemma 3.5.** [Wei03, Mapping Lemma 5.2.4] Let \( f : \{E^r_{pq}\} \to \{F^r_{pq}\} \) be a morphism of spectral sequences, that is, a family of maps \( f_{pq}^r : E^r_{pq} \to F^r_{pq} \) satisfying

\[
d^r f^r = f^{r+1} d^r
\]

\[
f^{r+1} = H(f^r)
\]

such that moreover for some fixed \( r \), \( f^r : E^r_{pq} \simeq F^r_{pq} \) is an isomorphism for all \( p \) and \( q \). The 5-Lemma implies that \( f^s : E^s_{pq} \simeq F^s_{pq} \) for all \( s \geq r \) as well.

A special case of this Lemma is
Lemma 3.6. Let \( f : V \to W \) be a map of chain complexes \( V \) and \( W \) and equip both with a bounded above and complete filtration \( F^p \) compatible with \( f \), that is, \( f(F^p V) \subset F^p W \). Then \( f : V \to W \) is a quasi-isomorphism if \( \text{gr} f : \text{gr} V \to \text{gr} W \) is one.

The latter lemma can be applied to \( \varphi : D \to S \) with a suitable filtration on both sides because proposition 3.4 implies that it is a chain map. By mimicking the construction from [CW16] we will construct different filtrations to sort out certain differentials on both complexes and thereby simplify the verification of \( \varphi \) being a quasi-isomorphism.

A generic element \( T \) in the complex \( D \) consists of a graph \( \Gamma \in \text{Graphs}_{\text{alg}} \), elements \( c^* \) and \( l^* \) of \( \text{Com}^* \) and \( \text{Lie}^* \) respectively, as well as polynomials \( f_1, \ldots, f_r \in \mathcal{O} \)

![Diagram of a graph with vertices and edges, representing a generic element in the complex D.](image)

Let \( l \) be the number of \( \mathcal{C}^i \) generators in this element and \( \text{deg}_S = \sum_{i=1}^{r} \text{deg}_S(f_i) \) the sum of the degrees of all attached polynomials. This degree is well defined as \( \text{deg}_S(f_i) = \text{deg}_S(f_i) + \text{deg}_O(f_j) \). Notice that on \( D \) the degree

\[
\text{deg}^1 = r - \text{deg} \quad \text{deg}^2 = e - v
\]

is raised by the differentials \( d_{\text{mod}}^{\text{lie}} \) and \( d_{\text{alg}}^{\text{com}} \) by one and left unchanged by all other parts. Hence on \( \text{gr}^1 D \), the graded complex associated to the filtration by this degree, the remaining differential is

\[
\delta_{\text{split}} + \delta_{\text{pair}} + \delta_{\text{mod}}^{\text{com}} + d_{\text{alg}}^{\text{lie}}
\]

A polynomial \( F \in S \) can equivalently be represented as a graph \( \Gamma \) with multiple connected components. Each one consists of a single vertex decorated with a cohomology class and an element \( v_i \in V \) and represents a variable \( \alpha \otimes v_i \).

\[
\begin{align*}
\alpha & \quad \beta & \quad \gamma \\
\vdots & \quad \vdots & \quad \vdots \\
v_1 & \quad v_2 & \quad v_k
\end{align*}
\]

Hence for the image of an element \( T \in D \) \( \text{deg}^1 \) vanishes and we therefore construct the trivial filtration by the zero-degree, in which \( D \) is concentrated in degree zero. Thereby we obtain that \( \varphi \) is compatible with the filtrations and can reduce our problem to \( \text{gr}^1 \varphi : \text{gr}^1 D \to \text{gr}^1 S \). To move on, let us define another degree on \( D \), which is the number of edges minus the number of vertices of the graph \( \Gamma \).

\[
\text{deg}^2 = e - v
\]

All differentials either increase of leave this degree constant, so we can filter the complex by \( \text{deg}^2 \). On the associated graded complex \( \text{gr}^2 \text{gr}^1 D \) the \( \delta_{\text{pair}} \) term in the induced differential vanishes completely. For \( \delta_{\text{mod}}^{\text{com}} \) the terms that contribute are the terms attaching components with \( e \) edges and \( s = e + 1 \) internal vertices. On \( S \) we choose the corresponding degree to be minus the polynomial degree

\[
\text{deg} = -\text{deg}_S
\]

such that \( \varphi \) is again compatible with these filtrations. Notice that this eliminates the differential \( \Delta \) on \( S \) and we are left with the zero differential on \( \text{gr}^2 \text{gr}^1 S \). All remaining parts of the differential on \( D \) exclusively either increase the number of internal vertices \( s \) by at least one, decrease the number of \( \text{Com}^* \) elements, which we denote by \( c \), by at least one or don’t affect any of these degrees at all. So it is well defined to filter by the degree

\[
\text{deg}^3 = s + (n - 1) \cdot c - \text{deg}
\]

where \( \text{deg} \) is the degree of \( \Gamma \) defined in equation (11). On the associated graded complex \( \text{gr}^3 \text{gr}^2 \text{gr}^1 D \) only the differentials \( \delta_{\text{split}}, \delta_{\text{mod}}^{\text{com}} \) and \( d_{\text{alg}}^{\text{lie}} \) remain unchanged. By construction only the terms of \( \delta_{\text{split}} \), which add exactly
one edge and one internal vertex remain. That is either the term replacing a decoration by an internal vertex with the same decoration or connecting an internal vertex to an edge. In the latter case the vertex can be considered to be decorated with the unit element $I \in H(M)$. Both terms appear as well in $\delta_{\text{split}}$ with the opposite sign and therefore cancel out. Because an element of $S$, as illustrated above, has neither edges nor internal vertices or $\text{Com}^*$ elements, we can once again use the trivial filtration on $S$ to achieve compatibility of $\varphi$.

Lastly we use the filtration by the degree $\deg^4 = -c$ to construct on $D$ the naturally determined spectral sequence $E_{pq}$, starting with $E^0_{pq} = F^p D_{p+q}/F^{p-1} D_{p+q}$. This splits the differential in the following way:

$$d = \delta_{\text{split}} + d_{\text{alg}}^* + d_{\text{mod}}^* = d^*$$

One easily checks that for the trivial spectral sequence $F_{pq}$ in which $S$ is concentrated in one column $p = 0$ we obtain that the collection of maps $\varphi_{pq}^0: E^0_{pq} \rightarrow F^0_{pq}$ is a morphism between the $0^{th}$ page of these spectral sequences. To compute the first page $E^1_{pq}$ of this spectral sequence, notice that the terms of the differential on the zeroth page act in the following way:

$$\delta_{\text{split}} \quad \text{Graphs}_M \circ \text{Com}^*[n] \circ \text{Lie}^*[1] \circ \text{Com} \circ V$$

In this case we can use the commutativity of the operadic composition and the homology functor $H_*$.

$$H_* \left( \text{Graphs}_M \circ \text{Com}^*[n] \circ \text{Lie}^*[1] \circ \text{Com} \circ V, \delta_{\text{split}} + d_{\text{alg}}^* \right)$$

The latter term of the homology can be directly simplified, as this is the Harrison complex of $O = \text{Com} \circ V$, which is acyclic. For the other term we need to introduce more tools and construct a nested spectral sequence.

### 3.3. The Homology of Graphs$_M$. In order to compute the homology of $(\text{Graphs}_M, \delta_{\text{split}})$, we introduce the following cdga model which was similarly presented by Lambrechts and Stanley in [LS08]. While the relations (1)-(3) where already discovered by Cohen, this model was first found independently by Kriz [Kri94] and Totaro [Tot93] in different ways. A Poincaré duality algebra $A$ of formal dimension $n$ is a graded commutative $k$-algebra over a field $k$ with a non degenerated pairing $A^k \otimes A^{n-k} \rightarrow k$. Choose a basis $\{\alpha^j\}_{j=1}^N$ with its Poincaré dual basis $\{\alpha^*\}_{j=1}^N$.

**Definition 3.7.**

1. For a Poincaré duality algebra $A$ of formal dimension $n$, the $k$-configuration model $(F^*_A(k), d^*_\varphi)$ is the free graded commutative differential graded algebra generated by

$$\omega_{ab} \quad 1 \leq a \neq b \leq r \quad \text{of degree } (n-1)
$$

$$\alpha^*_a \quad \{\alpha^*\} \text{ basis of } A$$

subject to the relations

(i) skew symmetry

$$\omega_{ab} = (-1)^{a+b} \omega_{ba}$$

(ii) Arnold identity

$$\omega_{ab} \omega_{bc} + \omega_{bc} \omega_{ca} + \omega_{ca} \omega_{ab} = 0$$

(iii) move decorations

$$\omega_{ab} \alpha^*_b = \omega_{ca} \alpha^*_a$$

(iv) multiply decorations

$$\alpha^*_a \alpha^*_b = (\alpha^* \cdot \alpha^*)_a$$

with the differential $d^*_\varphi \omega_{ab} = \sum_{j=1}^N \alpha^*_a \alpha^*_b \omega_{aj}^*$.  

2. The cdga’s $F^*_A(k)$ assemble together to the $e^*_n$-module $F^*_A$.

Using the paper [Sin10] by Sinha, we will now shed some light on this construction. Disregarding for a moment the generators $\alpha^*$ and the relations (iii) and (iv), we obtain the complex

$$\text{Siop}^n = \ast \text{Gr}^n \otimes_k \left( \text{Skew} + \text{Arnold} \right)$$
where an element $\omega_{ab}$ represents an edge between the vertices $a$ and $b$. These relations are graphically represented by

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\bullet \rightarrow \bullet
\end{array}
\end{array}
+ (-1)^n \begin{array}{c}
\begin{array}{c}
\bullet \rightarrow \bullet
\end{array}
\end{array}
= 0
\end{align*}
\]

Furthermore, we define

\[
\text{Pois}^n = \text{n-forests} / \langle \text{Skew + Jacobi} \rangle
\]

the quotient of the free module spanned by n-forests, that is, a collection of trees with univalent or trivalent vertices whose set of leaves is in bijection with the integers \{1, \ldots, n\}, by skew-symmetry and the Jacobi identity. Here every branch point stands for the Lie bracket evaluated on its leaves. \text{Pois}^n has a basis consisting of n-forests with tall trees (the distance between the leaf with the minimal label and the root in each tree is maximal) and \text{Siop}^n one consisting of long graphs (each component is a linear graph starting with the minimal label). Consider the map $\beta_{r,F} : E_1 \rightarrow I_F$ from the set of edges of an n-graph $\Gamma$ to the set of internal bifurcations of the trees in an n-forest $F$ that sends an edge directed from $i$ to $j$ to the lowest branch point on the shortest path from the $i^{th}$ to the $j^{th}$ leaf (if these leaves are attached to the same tree). With this map one can introduce a pairing between n-graphs and n-forests

\[
\langle \Gamma, F \rangle = \begin{cases} 
+1 & \text{if $\beta_{r,F}$ is a bijection} \\
0 & \text{otherwise}
\end{cases}
\]

where the sign depends on the ordering of the leaves on the tree. This pairing passes on to a perfect pairing between the quotients Pois $^n$ and Siop $^n$. Hence the induced map $\tilde{\beta} : \text{Pois}^n \xrightarrow{\sim} (\text{Siop}^n)^*$ is an isomorphism and we conclude from $\text{Pois}^n = \text{Com} \circ \text{Lie}[n-1]$ that $\text{Siop}^n = \text{Com}^* [n] \circ \text{Lie}^*[1]$. An element $\alpha$ decorates the vertex $a$ with the cohomology class $\alpha$. Due to relation (iii) one can move a decoration between vertices connected by an edge and by relation (iv) multiply two such decorations attached to the same vertex. Taking the Poincaré duality algebra to be $A = H_*(M)$, this implies that an element of $F_{H_*(M)}^*$ represents a graph without internal vertices, decorated with cohomology classes at each connected component. However, we will work with the dual model $F_{H_*(M)} = (F_{H_*(M)}^*)^*$ which is the complex of decorated n-forests, so we obtain the $e_n$-module

\[
F_{H_*(M)} = \text{Com} \circ H^*(M) \circ \text{Lie}[n-1]
\]

with the usual composition of trees and the dual differential $d_F = (d_\Gamma)^*$.  

**Lemma 3.8.** The model $F_{H_*(M)}$ satisfies the recursive identity

\[
F_{H_*(M)}(k) = F_{H_*(M)}(k-1) \otimes H_*(M) \otimes F_{H_*(M)}(k-1)[n-1]^\oplus(k-1)
\]

**Proof.** We will show this property in the dual setting, which will imply the claim about $F_{H_*(M)}$. One can split $U(r) := F_{H_*(M)}(r)$ into a direct sum

\[
U(r) = U_0(r) \oplus U_1(r) \oplus U_2(r)
\]

where $U_i$ contains all graphs in which the vertex 1 has degree $i$, counting decorations. For the case of an isolated vertex, it can be easily seen that $U_0(r) \simeq U(r-1)$. If we allow vertex 1 to have one edge, this can either be a decoration, then we obtain a term isomorphic to $U(r-1) \otimes \overline{\mathcal{H}}_*(M)$, or an edge to the remaining part of the graph, in which case the term is isomorphic to $U(r-1)[n-1]^{\oplus(r-1)}$. In the basis of long graphs the first vertex can have at most valence 2 of which one is due to a decoration. However, this case is equivalent to $U_1$, since one can move the decoration with relation (iii) in definition (3) along a connected component. In total this amounts to

\[
U(r) = U(r-1) \oplus U(r-1) \otimes \overline{\mathcal{H}}_*(M) \otimes U(r-1)[n-1]^{\oplus(r-1)}
\]

\[
= U(r-1) \otimes H_*(M) \otimes U(r-1)[n-1]^{\oplus(r-1)}
\]

Now we are able to conclude the desired result of this subsection, which is to express the homology of Graphs$_M$ up to quasi-isomorphism by the model $F_{H_*(M)}$. Since its differential $d_\Delta$ would naturally correspond to $\delta_{\text{pair}}$ which however vanished through the second filtration, we consider $F_{H_*(M)}$ without any differential.

**Proposition 3.9.** The map

\[
F_{H_*(M)} \xrightarrow{\sim} H_*(\text{Graphs}_M, \delta_{\text{split}})
\]

is an isomorphism of graded $e_n$-modules.
Proof. The embedding $\mathcal{F}_{H^*(M)} \hookrightarrow \text{Graphs}_M$ defined on the generators by
\[
\{ \ , \ \} \mapsto \circ \circ, \quad \wedge \mapsto \circ \circ
\]
is a morphism of operadic $e_n$-modules. We will prove the claim by induction on the arity $r$. The result holds trivially for the case $\mathcal{F}_{H^*(M)}(0) \simeq H_*(\text{Graphs}_M(0), \delta_{\text{split}})$. Split the complex $\text{Graphs}_M(r)$ into $C(r) = C_0 \oplus C_1 \oplus C_{\geq 2}$ in the same way as $\mathcal{F}_{H^*(M)}$ was split in equation (3). For $C_0$ we observe $C_0 \simeq C/(n-1)$. The differential $\delta_{\text{split}}$ on $\text{Graphs}_M$ has the following components.

\[
\begin{array}{ccc}
C_0 & \oplus & C_1 & \oplus & C_{\geq 2} \\
\delta_{\text{split}}^0 \\
\end{array}
\]

With the descending filtration $F^p C(r) = \{ \Gamma \in C_{\geq 2} \mid \deg(\Gamma) \geq k \} \oplus \{ \Gamma \in C_1 \mid \deg(\Gamma) \geq k + 1 \}$ on $C(r)$ we set up a spectral sequence in which the part of the differential on the $0^{th}$ page is $\delta_{\text{split}}^0$. Since this differential is injective on $C_{\geq 2}$, the kernel is exactly $\ker(\delta_{\text{split}}^0) = C_1$. The next page of the spectral sequence is therefore
\[
E^1(C(r)) = H(\text{gr} C(r)) = C_0 \oplus \frac{\ker(\delta_{\text{split}}^0)}{\text{im}(\delta_{\text{split}}^0)} = \frac{C_1}{\text{im}(\delta_{\text{split}}^0)} = \text{coker}(\delta_{\text{split}}^0)
\]

Only two kinds of graphs can appear in this quotient, namely a graph with
(a) an isolated vertex 1 that is decorated
(b) a vertex 1 connected to another vertex that is not decorated

By the same argument as in lemma (3.9) we obtain for both cases one term in the following sum
\[
E^1(C(r)) = C_0 \oplus C(r-1) \oplus H^*(M) \oplus C(r-1)[n-1] \oplus \cdots
= C(r-1) \oplus H^*(M) \oplus C(r-1)[n-1] \oplus \cdots
\]

The differential on this page coincides with the differential on $C(r-1)$. This spectral sequence is a double complex concentrated on a double column and therefore it collapses on the second page. Hence we obtain the recursive identity
\[
H(C(r)) = H(C(r-1)) \oplus H^*(M) \oplus H(C(r-1))[n-1] \oplus \cdots
\]

which is the same as for $\mathcal{F}_{H^*(M)}$. By the induction hypothesis this concludes the proof.

With this result we continue the computation from equation (2) and obtain on the first page.
\[
E^1 = H_* (\text{Graphs}_M, \delta_{\text{split}}) \circ \text{Com}^*[n] \circ V
\]

\[
\simeq \mathcal{F}_{H^*(M)} \circ \text{Com}^*[n] \circ V
\]

\[
= \text{Com} \circ H^*(M) \circ \text{Lie}[n-1] \circ \text{Com}^*[n] \circ V
\]

The differential on this page is $d^1 = d_{\text{mod}}^{\text{com}}$. To proceed we verify that $d_{\text{mod}}^{\text{com}}$ corresponds to a canonically induced twisting differential $d_\kappa$.

Lemma 3.10. The isomorphism from lemma (3.9) induces an isomorphism of differential graded $e_n$-modules
\[
(\mathcal{F}_{H^*(M)} \circ \text{Com}^*[n] \circ V, d_\kappa) \simeq (H_* (\text{Graphs}_M, \delta_{\text{split}}) \circ \text{Com}^*[n] \circ V, d_{\text{mod}}^{\text{com}})
\]

Proof. One needs to verify that the map is compatible with the differentials. We will omit the terms $\text{Com} \circ M$ in $\mathcal{F}_{H^*(M)}$ as only the terms $\text{Lie} \circ \text{Com}^*$ play an important role. Since $\text{Com}^* = \text{Lie}^\vee$ is the Koszul dual operad of $\text{Lie}$ there exists a natural twisting morphism
\[
\kappa : \text{Com}^*[n] \to \text{Lie}[n-1]
\]
of degree $-1$, sending the generator of $\text{Com}^*$ to the Lie bracket, the generator of $\text{Lie}$. With this we construct the unique differential $d_\kappa$ on the Koszul complex $(\text{Lie} \circ \text{Com}^*, d_\kappa)$ as
\[
d_\kappa : \text{Lie} \circ \text{Com}^* \xrightarrow{\text{id}_{\text{Lie}} \circ \kappa} \text{Lie} \circ (\text{Com}^*; \text{Com}^* \circ \text{Com}^*)
\]
\[
\simeq (\text{Lie} \circ (\text{Com}^*) \circ \text{Com}^* \circ \text{id}_{\text{Com}^*}) \circ \text{Com}^* \circ \mu(1) \circ \text{id}_{\text{Com}^*}
\]

\[
\text{Lie} \circ \text{Com}^* 
\]
where Δ denotes only in this proof the decomposition map of the cooperad Com. The symbols \( \circ_{(1)} \), \( \mu_{(1)} \) and \( \circ' \) refer to the infinitesimal composite, the infinitesimal composition map and the infinitesimal composite of morphisms respectively, which are explained in chapter 6 of Loday’s and Valette’s book [LV12]. On Graphs\(_M \circ \text{Com}^* \langle n \rangle \) the differential \( d_{\text{Com}^*}^{\text{mod}} \) is constructed in a very similar fashion using instead of the twisting morphism \( \kappa \) the composition \( \kappa \) with the inclusion from equation (4). This time we represent this mechanism graphically.

\[
\begin{array}{c}
\text{Graphs}\_M \\
\text{Com}^* \text{Com}^* \text{Com}^* \\
\end{array} \xrightarrow{id_{\text{Graphs}\_M} \circ' \Delta} \begin{array}{c}
\text{Graphs}\_M \\
\text{Com}^* \text{Com}^* \text{Com}^* \\
\end{array}
\]

Thus using the fact that \( \iota \) and id_{Com} are morphisms of operads and cooperads respectively, we can verify the claim directly

\[
(\iota \circ \text{id}_{\text{Com}}) d_{\kappa} = (\iota \circ \text{id}_{\text{Com}}) \left( \left( \mu_{(1)} \circ \text{id}_{\text{Com}} \right) \left( \text{id}_{\text{Lie}} \circ' \left[ (\kappa \circ \text{id}_{\text{Com}}) \Delta \right] \right) \right)
\]

\[
= (\mu_{(1)} \circ \text{id}_{\text{Com}}) \left( (\iota \circ_{(1)} \iota) \circ \text{id}_{\text{Com}} \right) \left( \text{id}_{\text{Lie}} \circ_{(1)} \iota \right) \left( \text{id}_{\text{Lie}} \circ' \Delta \right)
\]

\[
= (\mu_{(1)} \circ \text{id}_{\text{Com}}) \left( (\iota \circ_{(1)} \iota) \circ \text{id}_{\text{Com}} \right) \left( (\iota \circ_{(1)} \iota) \circ \text{id}_{\text{Com}} \right) \left( \text{id}_{\text{Lie}} \circ' \Delta \right)
\]

\[
= \iota_{\text{Com}} \left( \iota \circ \text{id}_{\text{Com}} \right)
\]

We therefore denote (by abuse of notation) from now on both differentials by \( d_{\text{Com}, \text{mod}} \).

3.4. Conclusion. With the results from the previous section, we are now able to infer the proof.

Proof of Theorem 3.1. Applying once again Koszul duality results the next page of the spectral sequence from equation (5).

\[
E^2 = H_*(\text{Com} \circ H^*(M) \circ \text{Lie} [n-1] \circ \text{Com}^* \langle n \rangle \circ V, d_{\text{Com}, \text{mod}}^\text{mod})
\]

\[
\simeq \text{Com} \circ H^*(M) \circ V
\]

At this point we can conclude that the morphism \( \varphi^2 : E^2 \to F^2 \) on the second page is an isomorphism and therefore every induced \( \varphi^r \) for \( r \geq 2 \) will be an isomorphism by lemma (3.5) as well. Consequently, as these spectral sequences converge, given by the fact that they are bounded, \( H_*(\varphi) \) is an isomorphism and \( \varphi \) is a quasi-isomorphism as claimed.

4. The Twisted Polynomial Algebra

As already explained in section (2.3), one can twist the polynomial algebra \( \text{O} \) by a Maurer-Cartan element \( m \) to obtain \( \text{O}[h] \). In the following we will show how the above computation can be altered to comprise this broader case where the additional differential \( d^m \) on \( \text{O}[h] \) appears.

Theorem 4.1. Let \( M \) be a compact and oriented manifold \( M \) with trivialized tangent bundle. Let moreover \( V \) be the shifted cotangent bundle \( V = T^*[1-n] \mathbb{R}^N \) and consider the the polynomial algebra \( \text{O} = \text{O}(V) \). Then the factorization homology \( \int_M \text{O}[h] \) of \( M \) with coefficients in the twisted polynomial algebra \( \text{O}[h] = \text{O}^{\text{hm} + \kappa \text{m}_2 + \cdots} \) is weakly equivalent to the algebra of twisted polynomials \( S(H(M) \otimes V) \).

First we add the twisting differential \( d^m \) to the composition

\[
(D^m, \partial^m) := (\text{Graphs}_M, \delta) \circ (\text{O}[h], d^m) = (\text{Graphs}_M, \delta) \circ (\text{O}[h], d^m)
\]

The revised map \( \varphi^m : (D^m, \partial^m) \to (S, \Delta) \) now recognizes internal vertices and assigns the Maurer-Cartan element \( m \) to all of them when \( \varphi^m \) is applied. For convenience we use again the notation \( \varphi^m (\Gamma \otimes f_I) = \Gamma_{\circ m} f_I \).
Denote the image of the partition function as $\varphi^m(z_M) = Z_M^m$. In contrast to proposition (3.4), $\delta_{\text{split}}$ now corresponds to $d^m$ on $\mathcal{O}[h]$.

**Lemma 4.2.** It holds $(\delta_{\text{split}}|_1)^m, f_I = (-1)^{|I|+1} \Gamma \circ_d^m f_I$

*Proof. It is sufficient to prove this statement for the first summand of each differential.*

$$\Gamma \circ_d^m f_I = \Gamma \circ_d^m (d^m f_1 \otimes \cdots \otimes f_k)$$

$$= \Gamma \circ_d^m (-\{m, f_1\} \otimes \cdots \otimes f_k)$$

$$= -([\Gamma \circ_d^m \circ \bullet] \circ_d^m (f_1 \otimes \cdots \otimes f_k)$$

$$= (-1)^{|I|+1} (\delta_{\text{split}}|_1)^m, f_I$$

With this we can show

**Lemma 4.3.** A Maurer-Cartan element $m \in \mathcal{O}$, leads to a Maurer-Cartan element $Z_M^m \in S$.

*Proof. We combine the assumptions that $m$ satisfies the Maurer-Cartan equation $\{m, m\} = 0$ in $(\mathcal{O}, \{\cdot, \cdot\})$ and $z_M$ satisfies the Maurer-Cartan equation in $(\text{Graphs}_M, \delta, \{\cdot, \cdot\})$. Applying $\varphi^m$ to the latter one results

$$0 = \varphi^m \left( \delta_M + \frac{1}{2} \{z_M, z_M\} \right)$$

$$= \varphi^m \left( \delta_{\text{split}} + \delta_{\text{pair}} \right) \varphi^m \left( \delta_{\text{split}} + \delta_{\text{pair}} \right) \cdot \frac{1}{2} \{z_M, z_M\} \right)$$

$$= \varphi^m \left( \delta_{\text{split}} z_M \right) + \Delta(\varphi^m(z_M)) + \varphi^m \left( \frac{1}{2} \{z_M, z_M\} \right)$$

$$= \Delta(z_M^m) + \frac{1}{2} \{Z_M^m, Z_M^m\} \right)$$

where $\varphi^m$ is compatible with the brackets and $\varphi^m(\delta_{\text{split}} z_M)$ vanishes due to lemma (4.2):

$$\varphi^m(\delta_{\text{split}} z_M) = (-1)^{|Z_M^m|+1} \frac{1}{2} \{z_M^m, z_M^m\} = 0$$

As always, a Maurer-Cartan element gives rise to a twisted differential $\Delta^m = \Delta + \{z_M^m, \cdot\}$. With these modifications we can show analogous to proposition (3.4) that

**Proposition 4.4.** The map $\varphi^m : (D^m, \partial^m, \{\cdot, \cdot\}_D) \to (S, \Delta^m, \{\cdot, \cdot\})$ is a morphism of dg Lie $e_n$-modules.

*Proof. For $\Gamma \in \text{Graphs}_M$ and $f_J \in \mathcal{O}^\otimes$ we compute with the help of lemma (4.3)

$$\varphi^m(\partial(\Gamma \otimes f_J)) = (\delta_{\text{split}} + \delta_{\text{pair}} + \delta_{\text{pair}}^M) \Gamma \circ_d^m f_J = (-1)^{|\Gamma|} \Gamma \circ_d^m d^m f_J$$

$$= \delta_{\text{pair}}^M \Gamma \circ_d^m f_J$$

$$= \Delta(\Gamma \circ_d^m f_J) + [z_M^m, \Gamma] \circ_d^m f_J$$

$$= \Delta^m(\Gamma \circ_d^m f_J)$$

*Proof of Theorem (4.4).* We are now in the position to reduce this case to Theorem (3.1). Therefore we filter by the number of factors $h$ in $D^m$ and $S$ respectively. Only the differentials $d^m$ and $\{z_M^m, \cdot\}$ raise this degree by one and therefore vanish in the associated graded. All other differentials remain unchanged and we obtain as the associated graded complex the same complex $D$ as before in subsection (3.2).

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