A THEOREM OF LEVINSON FOR RIEMANNIAN SYMMETRIC SPACES OF NONCOMPACT TYPE

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ABSTRACT. A classical result of N. Levinson characterizes the existence of a nonzero integrable function vanishing on a nonempty open subset of the real line in terms of the pointwise decay of its Fourier transform. We prove an analogue of this result for Riemannian symmetric spaces of noncompact type.

1. INTRODUCTION

It is a well known fact in harmonic analysis that if the Fourier transform of an integrable function on $\mathbb{R}$ is very rapidly decreasing then the function cannot vanish on a nonempty open subset of $\mathbb{R}$ unless it vanishes identically. A manifestation of this fact is as follows; let $f \in L^1(\mathbb{R})$ and $a > 0$ be such that

$$|\mathcal{F}f(\xi)| \leq Ce^{-a|\xi|}, \quad \text{for all } \xi \in \mathbb{R},$$

where,

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx,$$

is the usual Fourier transform. If $f$ vanishes on a nonempty open set on $\mathbb{R}$ then $f$ is identically zero. This is due to the fact that the very rapid decay of the Fourier transform extends the function as a holomorphic function in $\{z \in \mathbb{C} \mid |\Im z| < a\}$. This initial observation motivates one to endeavour for a more optimal decay of the Fourier transform $\mathcal{F}f$ for such a conclusion. For instance, we may ask: if for an increasing function $\psi$ on $[1, \infty)$, the Fourier transform $\mathcal{F}f$ decays faster than $e^{-\psi(|x|)}$ for large $|x|$, can $f$ vanish on a nonempty open set without being identically zero? For example, one can take $\psi(x) = x(1 + \log x)^{-1}$ which clearly imposes a slower decay on the Fourier transform compared to (1.1). The answer to the above question is in the negative and follows from certain results of Levinson proved in [22] [23]. Analogous problems have been studied by Paley-Wiener, Ingham and Hirschman ([25], Theorem II; [26], Theorem XII, P.16, [18], [17]). All these results can be grouped under the so called uncertainty principles of harmonic analysis which says that both the function and its Fourier transform cannot be sharply localized (see [7], [11]). In the context of the present paper localization of the function can be interpreted as the smallness of the support and that of the Fourier transform can be interpreted in terms of its decay at infinity.

We now state the relevant result of Levinson whose extension to Riemannian symmetric spaces of noncompact type is the main topic of this paper.

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Theorem 1.1 ([22], Theorem II). Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing function with \( \lim_{r \to \infty} \psi(r) = \infty \) and
\[
I = \int_1^\infty \frac{\psi(\xi)}{\xi^2} d\xi.
\]

(a) Suppose \( f \in L^1(\mathbb{R}) \) and
\[
|\mathcal{F} f(\xi)| \leq Ce^{-\psi(\xi)}, \quad \text{for all } \xi > 1,
\]
for some \( C \) positive. If the integral \( I \) is infinite then \( f \) cannot vanish on any nonempty open interval unless it is identically zero over \( \mathbb{R} \).

(b) If \( I \) is finite then there exists a nontrivial \( f \in C_c(\mathbb{R}) \) satisfying the estimate (1.2).

It is the sharpness of the condition on \( \psi \) which makes the theorem interesting to us. It was later proved by Beurling [20] that if the function satisfies the condition given in the theorem above then it cannot even vanish on a set of positive Lebesgue measure without being identically zero. However, these results of Levinson and Beurling are available only for the circle group and real line. Coming back to Levinson’s theorem, Levinson proved his theorem by reducing matters to a theorem of Paley and Wiener ([20], P. 16). This method of proof seems to be very special to \( \mathbb{R} \) and is hard to push through for other spaces. However, a different proof of Levinson’s theorem was obtained later ([20], Chapter VII, P. 248) which we find more illuminating. It was proved in [20] that Levinson’s theorem is actually related to completeness of exponential functions in certain normed linear space of continuous functions on \( \mathbb{R} \). It is this approach we are going to adopt to obtain a version of Levinson’s theorem for Riemannian symmetric spaces of noncompact type. The main approximation result we prove is Theorem 4.4 which, in turn, is an analogue of Lemma 2.2.

If \( X \) is a Riemannian symmetric space of noncompact type then \( X \) can be viewed as a quotient space \( G/K \) where \( G \) is a connected, noncompact, semisimple Lie group with finite center and \( K \) a maximal compact subgroup of \( G \). For integrable functions \( f \) on \( G/K \) one can talk about an appropriate analogue of Fourier transform denoted by \( \tilde{f} \). It is natural to ask about an extension of Levinson’s theorem in terms of the Fourier transform \( \tilde{f} \) for functions defined on \( X \). The following is the main theorem of this paper which can be viewed as an analogue of Levinson’s theorem for a Riemannian symmetric space of noncompact type.

Theorem 1.2. Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing function with \( \lim_{r \to \infty} \psi(r) = \infty \) and
\[
I = \int_{\mathfrak{a}^*_+ \setminus \{ \lambda \in \mathfrak{a}^*_+ \mid \| \lambda \|_B \geq 1 \}} \frac{\psi(\| \lambda \|_B)}{\| \lambda \|_B^{d+1}} d\lambda,
\]
where \( d = \text{rank}(X) \).

(a) Suppose \( f \in L^1(X) \) and its Fourier transform \( \tilde{f} \) satisfies the estimate
\[
\int_{\mathfrak{a}^*_+ \times K} |\tilde{f}(\lambda, k)| \ e^{\psi(\| \lambda \|_B)} |c(\lambda)|^{-2} d\lambda \ dk < \infty,
\]
where \( |c(\lambda)|^{-2} d\lambda \ dk \) denotes the Plancherel measure for \( L^2(X) \). If \( f \) vanishes on a nonempty open set in \( X \) and \( I \) is infinite then \( f = 0 \).

(b) If \( I \) is finite then there exists a nontrivial \( f \in C_c^\infty(X) \) satisfying the estimate (1.3).

We refer the reader to Section 3 for meanings of symbols used in the theorem above. For discussions on certain consequences and variants of Theorem 1.2 see Theorem 4.7, Remark 3.6 and Remark 4.8.
This paper is organised as follows; in Section 2 we prove some results on Euclidean spaces $\mathbb{R}^d$ which will be used for the proof of Theorem 1.2. The main results of this section are Lemma 2.2 and Lemma 2.3. In Section 3 we recall the required preliminaries regarding harmonic analysis on Riemannian symmetric spaces of noncompact type. In Section 4 we first prove an approximation result (Theorem 4.4) which we then apply to prove Theorem 1.2.

2. SOME RESULTS ON EUCLIDEAN SPACES

In this section, our aim is to prove certain approximation results for $\mathbb{R}^d$, $d \geq 1$, which will be needed later on. We start by describing certain relevant function spaces. Throughout this article $\psi$ will stand for a function satisfying the following: $\psi : [0, \infty) \to [0, \infty)$ is increasing and $\psi(r)$ goes to infinity as $r$ goes to infinity. We consider the following space of functions

$$C_\psi(\mathbb{R}^d) = \left\{ \phi : \mathbb{R}^d \to \mathbb{C} \mid \phi \text{ is continuous and } \lim_{\|x\| \to \infty} \frac{\phi(x)}{e^{\psi(\|x\|)}} = 0 \right\},$$

$$\|\phi\|_\psi = \sup_{x \in \mathbb{R}^d} \frac{|\phi(x)|}{e^{\psi(\|x\|)}}, \quad \phi \in C_\psi(\mathbb{R}^d).$$

Clearly, $(C_\psi(\mathbb{R}^d), \|\cdot\|_\psi)$ is a normed linear space. The following lemma follows by the usual technique of multiplying by a cut off function.

**Lemma 2.1.** $C_\psi(\mathbb{R}^d)$ is dense in $(C_\psi(\mathbb{R}^d), \|\cdot\|_\psi)$.

For a positive real number $L$, we denote by $\mathcal{E}_L$ the set of bounded, complex-valued functions on $\mathbb{R}^d$ which has an entire extension to $\mathbb{C}^d$ with exponential type at most $L$. That is,

$$\mathcal{E}_L = \left\{ f : \mathbb{R}^d \to \mathbb{C} \mid f \text{ is bounded on } \mathbb{R}^d, \text{ extends to an entire function on } \mathbb{C}^d \text{ and } |f(z)| \leq C e^{L\|z\|}, \epsilon > 0 \right\}.$$

A standard application of Phragmén-Lindelöf theorem shows that $\mathcal{E}_L$ has the following alternative description ([10], Lemma 2)

$$\mathcal{E}_L = \left\{ f : \mathbb{R}^d \to \mathbb{C} \mid f \text{ is bounded, extends to an entire function on } \mathbb{C}^d \text{ and } |f(z)| \leq C e^{\epsilon \|z\|}, \ z \in \mathbb{C}^d \right\}.$$

Since elements of $\mathcal{E}_L$ are bounded continuous functions on $\mathbb{R}^d$ and $\psi(\|x\|)$ goes to infinity, it follows that $\mathcal{E}_L \subseteq C_\psi(\mathbb{R}^d)$. For $\lambda \in \mathbb{R}^d$, we consider the exponential functions $e_\lambda : \mathbb{R}^d \to \mathbb{C}$ given by

$$e_\lambda(x) = e^{i\lambda \cdot x},$$

where $\lambda \cdot x$ denotes the usual Euclidean inner product. Since $e_\lambda$ is a bounded continuous function it belongs to $C_\psi(\mathbb{R}^d)$, for all $\lambda \in \mathbb{R}^d$. Let $Q(0, L)$ denote the cube centered at zero and sides parallel to the coordinate axes with side length $2L/\sqrt{d}$,

$$Q(0, L) = \left\{ x = (x_1, \cdots, x_d) \in \mathbb{R}^d \mid |x_j| < \frac{L}{\sqrt{d}}, \ 1 \leq j \leq d \right\}.$$

Let,

$$\Phi_L(\mathbb{R}^d) = \text{Span}\{e_\lambda : \lambda \in Q(0, L)\}.$$

Clearly, if $f \in \Phi_L(\mathbb{R}^d)$ then its natural extension to $\mathbb{C}^d$ is contained in $\mathcal{E}_L$. The following results constitute the heart of the matter for a proof of Levinson’s theorem on $\mathbb{R}$. For $d = 1$, it was proved in [20], Ch VII, P. 243; [20], Ch VI, P. 171 that

1. $\Phi_L(\mathbb{R})$ is dense in $(\mathcal{E}_L, \|\cdot\|_\psi)$. 


(2) $\mathcal{E}_L$ is dense in $(C_\psi(\mathbb{R}), \| \cdot \|_\psi)$ if

\begin{equation}
\int_1^\infty \frac{\psi(r)}{r^2} \, dr = \infty.
\end{equation}

It follows from above that for every positive real number $L$ the space $\Phi_L(\mathbb{R})$ is dense in $(C_\psi(\mathbb{R}), \| \cdot \|_\psi)$ if (2.1) holds. It is crucial for us to be able to extend these results to $\mathbb{R}^d$, $d > 1$.

**Lemma 2.2.** The space $\Phi_L(\mathbb{R}^d)$ is dense in $(C_\psi(\mathbb{R}^d), \| \cdot \|_\psi)$ if $\psi$ satisfies (2.1).

**Proof.** We know that the result is true for $d = 1$. Our method is to reduce the problem to the case $d = 1$ and then apply the available results. We define

$$\psi_0(s) = \frac{\psi(s)}{d}, \quad s \in [0, \infty),$$

and consider the following spaces of functions

$$\mathcal{P}C_c(\mathbb{R}^d) = \text{span}\{ f : \mathbb{R}^d \to \mathbb{C} \mid f(x_1, \ldots, x_d) = f_1(x_1) \cdots f_d(x_d), f_j \in C_c(\mathbb{R}), \quad 1 \leq j \leq d \} \subseteq C_c(\mathbb{R})^d.$$ 

$$\mathcal{P}\Phi_L(\mathbb{R}^d) = \text{span}\{ f : \mathbb{R}^d \to \mathbb{C} \mid f(x_1, \ldots, x_d) = f_1(x_1) \cdots f_d(x_d), f_j \in \Phi_L(\mathbb{R}), \quad 1 \leq j \leq d \} \subseteq \Phi_L(\mathbb{R}^d).$$ 

$$\mathcal{P}C_\psi(\mathbb{R}^d) = \text{span}\{ f : \mathbb{R}^d \to \mathbb{C} \mid f(x_1, \ldots, x_d) = f_1(x_1) \cdots f_d(x_d), f_j \in C_\psi(\mathbb{R}), \quad 1 \leq j \leq d \}.$$ 

By a standard application of Stone-Weierstrass theorem, it follows that $\mathcal{P}C_c(\mathbb{R}^d)$ is dense in $(C_c(\mathbb{R}^d), \| \cdot \|_\infty)$. Since $\| \phi \|_\psi$ is smaller than $\| \phi \|_\infty$, we get that $\mathcal{P}C_c(\mathbb{R}^d)$ is dense in $(C_c(\mathbb{R}^d), \| \cdot \|_\psi)$. Lemma 2.1 now implies that $\mathcal{P}C_c(\mathbb{R}^d)$ is dense in $(C_\psi(\mathbb{R}^d), \| \cdot \|_\psi)$. Next, we notice that

$$\mathcal{P}C_c(\mathbb{R}^d) \subseteq \mathcal{P}C_\psi(\mathbb{R}^d) \subseteq C_\psi(\mathbb{R}^d).$$

The first inclusion follows straightforward from the definitions involved. It suffices to check the second inclusion for functions of the form

$$\phi(x_1, \ldots, x_d) = \phi_1(x_1) \cdots \phi_d(x_d),$$

where $\phi_j \in C_\psi(\mathbb{R})$, $1 \leq j \leq d$. As $\psi$ (and hence $\psi_0$) is an increasing function we get that

\begin{equation}
\frac{\| \phi(x) \|}{e^{\psi(\|x\|)}} = \frac{\| \phi_1(x_1) \| \cdots \| \phi_d(x_d) \|}{e^{\psi(\|x\|)}} \leq \frac{\| \phi_1(x_1) \|}{e^{\psi_0(\|x_1\|)}} \cdots \frac{\| \phi_d(x_d) \|}{e^{\psi_0(\|x_d\|)}}.
\end{equation}

From the definition of $C_\psi_0(\mathbb{R})$, it follows that

$$\lim_{|x_j| \to \infty} \frac{\phi_j(x_j)}{e^{\psi_0(|x_j|)}} = 0, \quad 1 \leq j \leq d.$$ 

In particular, the functions

$$s \to \frac{\phi_j(s)}{e^{\psi(|s|)}}, \quad s \in \mathbb{R},$$

are bounded for all $j \in \{1, \ldots, d\}$. If the norm of $x$ goes to infinity then at least one of the coordinates $x_j$ of $x$ must go to infinity. Hence, we conclude from (2.3) that

$$\lim_{\|x\| \to \infty} \frac{\| \phi(x) \|}{e^{\psi(\|x\|)}} = 0.$$
It now follows from (2.2) that $\mathcal{PC}_{\psi_0}(\mathbb{R}^d)$ is dense in $(C_\psi(\mathbb{R}^d), \| \cdot \|_\psi)$. As $\mathcal{P}\Phi_L(\mathbb{R}^d)$ is contained in $\Phi_L(\mathbb{R}^d)$, it suffices for us to prove that $\mathcal{P}\Phi_L(\mathbb{R}^d)$ is dense in $(\mathcal{PC}_{\psi_0}(\mathbb{R}^d), \| \cdot \|_\psi)$. This is where we are going to use the case $d = 1$. It is enough for us to prove that functions of the form

$$f(x_1, \ldots, x_d) = f_1(x_1) \cdots f_d(x_d),$$

$f_j \in C_{\psi_0}(\mathbb{R}), 1 \leq j \leq d$, can be approximated by elements of $\mathcal{P}\Phi_L(\mathbb{R}^d)$ in $\| \cdot \|_\psi$ norm. Now, given any $\epsilon \in (0, 1)$, by the case $d = 1$, there exists $g_j \in \Phi_{L/\sqrt{\pi}}(\mathbb{R}), 1 \leq j \leq d$, such that

$$\sup_{s \in \mathbb{R}} \frac{|f_j(s) - g_j(s)|}{e^{\psi_0(|s|)}} < \epsilon.$$

By triangle inequality we have

$$\sup_{s \in \mathbb{R}} \frac{|g_j(s)|}{e^{\psi_0(|s|)}} \leq 1 + \|f_j\|_{\psi_0}, \quad 1 \leq j \leq d.$$

We now define

$$g(x) = g_1(x_1) \cdots g_d(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Clearly, $g \in \mathcal{P}\Phi_L(\mathbb{R}^d)$. By defining

$$g_0(y) = e^{\psi_0(|y|)} = f_{d+1}(y), \quad y \in \mathbb{R},$$

and using

$$\psi(|x|) \geq \psi(|x_j|), \quad 1 \leq j \leq d$$

(as $\psi$ is increasing) we have for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$

$$\frac{|f(x) - g(x)|}{e^{\psi(|x|)}} \leq \frac{|f_1(x_1) \cdots f_d(x_d) - g_1(x_1) \cdots g_d(x_d)|}{e^{\psi_0(|x_1|) \cdots e^{\psi_0(|x_d|)}}}$$

$$\leq \sum_{k=1}^d \frac{|f_k(x_k) - g_k(x_k)|}{e^{\psi_0(|x_k|)}} \left( \prod_{j=k+1}^{d+1} \frac{|f_j(x_j)|}{e^{\psi_0(|x_j|)}} \right) \left( \prod_{j=0}^{k-1} |g_j(x_j)| \right)$$

$$\leq \epsilon d \prod_{j=1}^d (1 + \|f_j\|_{\psi_0}).$$

This completes the proof. \qed

**Remark 2.3.** Since $\Phi_L(\mathbb{R}^d) \subseteq \mathcal{E}_L(\mathbb{R}^d)$ it follows from the above lemma that $\mathcal{E}_L(\mathbb{R}^d)$ is also dense in $(C_\psi(\mathbb{R}^d), \| \cdot \|_\psi)$, if $\psi$ satisfies (2.1).

Our next result can be viewed as an approximation theorem on $\mathbb{R}^d$ which will play a fundamental role in the proof of our main theorem.

**Lemma 2.4.** Let $\mu$ be a Radon measure on $\mathbb{R}^d$ and $f \in C_c(\mathbb{R}^d)$ with supp $f \subset B(0, L)$, for some given positive number $L$. Suppose $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is such that

i) $|g(x, \lambda)| \leq 1$, for all $x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d$.

ii) For each $\lambda \in \mathbb{R}^d$, the function $g(\cdot, \lambda)$ is smooth.

iii) For all $x \in B(0, L)$ and $\lambda$ in any compact subset $K$ of $\mathbb{R}^d$,

$$\left| \frac{\partial}{\partial x_j} g(x, \lambda) \right| \leq M_K, \quad 1 \leq j \leq d.$$
If
\[ F(\lambda) = \int_{B(0,L)} f(x)g(x,\lambda) \, d\mu(x), \quad \lambda \in \mathbb{R}^d, \]
then for any given \( \epsilon \) and \( \tau \) positive, there exists \( \{v_1, \cdots, v_N\} \subset B(0,L) \) and \( C_{v_j} \in \mathbb{C}, \ j = 1, \cdots, N, \) such that
\[
\left| F(\lambda) - \sum_{j=0}^{N} C_{v_j} g(v_j,\lambda) \right| < \epsilon, \quad \text{for all } \lambda \in B(0,\tau),
\]
and
\[
\sum_{j=0}^{N} C_{v_j} g(v_j,\lambda) \leq \int_{B(0,L)} |f(x)| \, dx, \quad \text{for all } \lambda \in \mathbb{R}^d.
\]

**Remark 2.5.**
1) The lemma basically says that the function \( F \) can be uniformly approximated on compact sets by finite linear combinations of functions of the form \( g(v_j,\cdot) \).

2) As \( g \) is complex valued we interpret \( \frac{\partial g}{\partial x_j} \) as \( \frac{\partial u}{\partial x_j} + i \frac{\partial v}{\partial x_j} \) where \( u \) and \( v \) are real and imaginary parts of \( g \) respectively.

**Proof of Lemma 2.4.** We fix \( n \in \mathbb{N} \) and for \( k = (k_1, \cdots, k_d) \in \mathbb{Z}^d \) consider the disjoint rectangles
\[
I^n_k = \prod_{j=1}^{d} \left[ \frac{k_j}{2^n}, \frac{k_j + 1}{2^n} \right),
\]
and
\[
A^n = \bigcup_{k \in \mathbb{Z}^d} \{ I^n_k : I^n_k \subseteq B(0,L) \}.
\]
We note that for small values of \( n \) the set \( A^n \) could be empty but for large values of \( n \), \( A^n \) is nonempty. From the definition (2.4) it is clear that \( A^n \subset A^{n+1} \), for all \( n \in \mathbb{N} \) and
\[
\bigcup_{n \in \mathbb{N}} A^n = B(0,L).
\]
As the sequence of sets \( A^n \) is increasing it follows that given any positive \( \epsilon \), there exists \( N_1 \in \mathbb{N} \) such that
\[
\mu \left( B(0,L) \setminus A^n \right) < \frac{\epsilon}{2}, \quad n \geq N_1,
\]
as \( \mu \) takes finite values on compact sets. Hence, for \( n \geq N_1 \)
\[
\left| \int_{B(0,L)} f(x)g(x,\lambda) d\mu(x) - \int_{A^n} f(x)g(x,\lambda) d\mu(x) \right|
\leq \int_{B(0,L) \setminus A^n} |f(x)g(x,\lambda)| d\mu(x)
\leq \int_{B(0,L) \setminus A^n} |f(x)| d\mu(x)
< \frac{\epsilon}{2} \|f\|_{L^\infty(B(0,L))}.
\]
For \( \lambda \in \mathbb{R}^d \), we define two sequences of functions
\[
F_n(\lambda) = \int_{A^n} f(x)g(x, \lambda)d\mu(x) = \sum_{k \in \mathbb{Z}^d, I_k^g \subseteq B(0,L)} \int_{I_k^g} f(x)g(x, \lambda)d\mu(x),
\]
and
\[
h_n(\lambda) = \sum_{k \in \mathbb{Z}^d, I_k^g \subseteq B(0,L)} g\left(\frac{k}{2^n}, \lambda\right) \int_{I_k^g} f(x)d\mu(x)
\]
\[(2.6)\]
\[
= \sum_{k \in \mathbb{Z}^d, I_k^g \subseteq B(0,L)} C_{k,n} g\left(\frac{k}{2^n}, \lambda\right),
\]
where
\[
C_{k,n} = \int_{I_k^g} f(x)d\mu(x).
\]

Let \( \tau \) be a positive real number. Now, using the mean value inequality for derivative (\[27\], Theorem 9.19) applied to the real and imaginary part of \( g \) we get that for all \( \lambda \) in \( B(0,\tau) \),
\[
\left| F_n(\lambda) - h_n(\lambda) \right| = \left| \sum_{k \in \mathbb{Z}^d, I_k^g \subseteq B(0,L)} \int_{I_k^g} f(x)g(x, \lambda) - f(x)g\left(\frac{k}{2^n}, \lambda\right) d\mu(x) \right|
\] \[
\leq \sum_{k \in \mathbb{Z}^d, I_k^g \subseteq B(0,L)} \int_{I_k^g} \left| f(x) \right| \left| g(x, \lambda) - g\left(\frac{k}{2^n}, \lambda\right) \right| d\mu(x)
\] \[
(2.7)
\]
\[
\leq C_M \tau \epsilon \frac{\sqrt{d}}{2^n} \left\| f \right\|_{L^1(\mathbb{R}^d)}.
\]

Therefore, for all \( \lambda \in B(0,\tau) \) and \( n \geq N_1 \) we have from (2.5) and (2.7) that
\[
|F(\lambda) - h_n(\lambda)| \leq \frac{\epsilon}{2} \left\| f \right\|_{L^\infty(B(0,L))} + C_M \tau \epsilon \frac{\sqrt{d}}{2^n} \left\| f \right\|_{L^1(\mathbb{R}^d)}.
\]

From the above inequalities, it follows that there exists \( N_2 \in \mathbb{N} \) sufficiently large such that for all \( n \geq N_2 \) and for all \( \lambda \in B(0,\tau) \)
\[
|F(\lambda) - h_n(\lambda)| < C_\tau \epsilon.
\]
It is also clear from the definition (2.6) of \( h_n \) that
\[
|h_n(\lambda)| \leq \int_{B(0,L)} |f(x)| \, dx, \quad \text{for all } \lambda \in \mathbb{R}^d.
\]
This completes the proof. \( \square \)

We end this section by recalling some standard facts regarding Radon transform on \( \mathbb{R}^d \) (see [16] for details). For \( \omega \in S^{d-1} \), the unit sphere in \( \mathbb{R}^d \), and \( s \in \mathbb{R} \), let
\[
H_{\omega,s} = \{ x \in \mathbb{R}^d \mid x \cdot \omega = s \}
\]
denote the hyperplane in \( \mathbb{R}^d \) with normal \( \omega \) and distance \( |s| \) from the origin. It is clear from the above definition that \( H_{\omega,s} = H_{-\omega,-s} \).
Definition 2.6. For \(f \in C_c(\mathbb{R}^d)\) the Radon transform \(\mathcal{R}f\) of the function \(f\) is defined by the integral
\[
\mathcal{R}f(\omega, s) = \int_{H_{\omega,s}} f(x) dm(x), \quad \omega \in S^{d-1}, s \in \mathbb{R},
\]
where \(dm(x)\) is the \(d-1\) dimensional Lebesgue measure on \(H_{\omega,s}\).

The one-dimensional Fourier transform of \(\mathcal{R}f(\omega, \cdot)\) and the \(d\)-dimensional Fourier transform of \(f\) are closely connected by the slice projection theorem (10, P. 4):
\[
\mathcal{F}f(\lambda \omega) = \mathcal{F}\mathcal{R}f(\omega, \cdot)(\lambda), \quad \text{for } \lambda \in \mathbb{R}, \ \omega \in S^{d-1},
\]
where on the right-hand side we have taken the one-dimensional Fourier transform of the function \(\mathcal{R}f(\omega, \cdot)\) on \(\mathbb{R}\). Clearly, if \(f\) is a radial function on \(\mathbb{R}^d\), then \(\mathcal{R}f(\omega, s)\) is independent of \(\omega\) and hence can be considered as an even function on \(\mathbb{R}\). Let \(C_c^\infty(\mathbb{R})_0\) denote the subspace of radial functions in \(C_c^\infty(\mathbb{R}^d)\) and \(C_c^\infty(\mathbb{R})_e\) be the subspace of even functions in \(C_c^\infty(\mathbb{R})\). By Theorem 2.10 of [10] it is known that
\[
\mathcal{R}: C_c^\infty(\mathbb{R}^d)_0 \rightarrow C_c^\infty(\mathbb{R})_e
\]
is a bijection with the property that if \(g \in C_c^\infty(\mathbb{R})_e\) with \(\text{supp } g \subseteq [-l, l]\) then there exists a unique \(f \in C_c^\infty(\mathbb{R}^d)_0\) with \(\text{supp } f \subseteq B(0,l)\) and \(\mathcal{R}f = g\).

3. Riemannian symmetric spaces of noncompact type

In this section we describe the necessary preliminaries regarding semisimple Lie groups and harmonic analysis on associated Riemannian symmetric spaces. These are standard and can be found, for example, in [9, 12, 13, 14]. To make the article self-contained, we shall gather only those results which will be used throughout this paper.

Let \(G\) be a connected, noncompact, real semisimple Lie group with finite centre and \(\mathfrak{g}\) its Lie algebra. We fix a Cartan involution \(\theta\) of \(\mathfrak{g}\) and write \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) where \(\mathfrak{k}\) and \(\mathfrak{p}\) are +1 and −1 eigenspaces of \(\theta\) respectively. Then \(\mathfrak{k}\) is a maximal compact subalgebra of \(\mathfrak{g}\) and \(\mathfrak{p}\) is a linear subspace of \(\mathfrak{g}\). The Cartan involution \(\theta\) induces an automorphism \(\Theta\) of the group \(G\) and \(K = \{g \in G \mid \Theta(g) = g\}\) is a maximal compact subgroup of \(G\). Let \(\mathfrak{a}\) be a maximal subalgebra in \(\mathfrak{p}\); then \(\mathfrak{a}\) is abelian. We assume that \(\dim \mathfrak{a} = d\), called the real rank of \(G\). Let \(B\) denote the Cartan Killing form of \(\mathfrak{g}\). It is known that \(B\mid_{\mathfrak{p} \times \mathfrak{p}}\) is positive definite and hence induces an inner product and a norm \(\| \cdot \|_B\) on \(\mathfrak{p}\). The homogeneous space \(X = G/K\) is a smooth manifold with \(\text{rank}(X) = d\). The tangent space of \(X\) at the point \(o = eK\) can be naturally identified to \(\mathfrak{p}\) and the restriction of \(B\) on \(\mathfrak{p}\) then induces a \(G\)-invariant Riemannian metric \(d\) on \(X\). For a given \(g \in G\) and a positive number \(L\) we define
\[
B(gK, L) = \{xK \mid x \in G, \ d(gK, xK) < L\}
\]
to be the open ball with center \(gK\) and radius \(L\). We can identify \(\mathfrak{a}\) with \(\mathbb{R}^d\) endowed with the inner product induced from \(\mathfrak{p}\) and let \(\mathfrak{a}^*\) be the real dual of \(\mathfrak{a}\). The set of restricted roots of the pair \((\mathfrak{g}, \mathfrak{a})\) is denoted by \(\Sigma\). It consists of all \(\alpha \in \mathfrak{a}^*\) such that
\[
\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X, \ \text{for all } Y \in \mathfrak{a}\}
\]
is nonzero with \(m_\alpha = \dim(\mathfrak{g}_\alpha)\). We choose a system of positive roots \(\Sigma_+\) and with respect to \(\Sigma_+\), the positive Weyl chamber \(\mathfrak{a}_+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \ \text{for all } \alpha \in \Sigma_+\}\). We denote by
\[
n = \oplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha.
Then \( n \) is a nilpotent subalgebra of \( g \) and we obtain the Iwasawa decomposition \( g = \mathfrak{f} \oplus a \oplus n \). If \( N = \exp n \) and \( A = \exp a \) then \( N \) is a Nilpotent Lie group and \( A \) normalizes \( N \). For the group \( G \), we now have the Iwasawa decomposition \( G = KAN \), that is, every \( g \in G \) can be uniquely written as

\[
g = \kappa(g) \exp H(g)\eta(g), \quad \kappa(g) \in K, H(g) \in a, \eta(g) \in N,
\]

and the map

\[
(k, a, n) \mapsto kan
\]
is a global diffeomorphism of \( K \times A \times N \) onto \( G \). Let \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \) be the half sum of positive roots counted with multiplicity. Let \( M' \) and \( M \) be the normalizer and centralizer of \( a \) in \( K \) respectively. Then \( M \) is a normal subgroup of \( M' \) and normalizes \( N \). The quotient group \( W = M'/M \) is a finite group, called the Weyl group of the pair \((g, \mathfrak{k})\). \( W \) acts on \( a \) by the adjoint action. It is known that \( W \) acts as a group of orthogonal transformation (preserving the Cartan-Killing form) on \( a \). Each \( w \in W \) permutes the Weyl chambers and the action of \( W \) on the Weyl chambers is simply transitive. Let \( A_+ = \exp a_+ \). Since \( \exp : a \to A \) is an isomorphism we can identify \( A \) with \( \mathbb{R}^d \). If \( \overline{A_+} \) denotes the closure of \( A_+ \) in \( G \), then one has the polar decomposition \( G = KAK \), that is, each \( g \in G \) can be written as

\[
g = k_1(\exp Y)k_2, \quad k_1, k_2 \in K, Y \in a.
\]

In the above decomposition, the \( A \) component of \( g \) is uniquely determined modulo \( W \). In particular, it is well defined in \( \overline{A_+} \). The map \((k_1, a, k_2) \mapsto k_1ak_2\) of \( K \times A \times K \) into \( G \) induces a diffeomorphism of \( K/M \times A_+ \times K \) onto an open dense subset of \( G \). We extend the inner product on \( a \) induced by \( B \) to \( a^* \) by duality, that is, we set

\[
\langle \lambda, \mu \rangle = B(Y_\lambda, Y_\mu), \quad \lambda, \mu \in a^*, \quad Y_\lambda, Y_\mu \in a,
\]

where \( Y_\lambda \) is the unique element in \( a \) such that

\[
\lambda(Y) = B(Y_\lambda, Y), \quad \text{for all} \ Y \in a.
\]

This inner product induces a norm, again denoted by \( \| \cdot \|_{B} \), on \( a^* \),

\[
\|\lambda\|_B = \langle \lambda, \lambda \rangle^{1/2}, \quad \lambda \in a^*.
\]

The elements of the Weyl group \( W \) acts on \( a^* \) by the formula

\[
sY_\lambda = Y_{s\lambda}, \quad s \in W, \ \lambda \in a^*.
\]

Let \( a^*_C \) denote the complexification of \( a^* \), that is, the set of all complex-valued real linear functionals on \( a \). If \( \lambda : a \to \mathbb{C} \) is a real linear functional then \( \Re \lambda : a \to \mathbb{R} \) and \( \Im \lambda : a \to \mathbb{R} \), given by

\[
\Re \lambda(Y) = \text{Real part of } \lambda(Y), \quad \text{for all} \ Y \in a,
\]

\[
\Im \lambda(Y) = \text{Imaginary part of } \lambda(Y), \quad \text{for all} \ Y \in a,
\]

are real-valued linear functionals on \( a \) and \( \lambda = \Re \lambda + i\Im \lambda \). The usual extension of \( B \) to \( a^*_C \), using conjugate linearity is also denoted by \( B \). Hence \( a^*_C \) can be naturally identified with \( \mathbb{C}^d \) such that

\[
\|\lambda\|_B = \left(\|\Re \lambda\|_{B}^2 + \|\Im \lambda\|_{B}^2\right)^{1/2}, \quad \lambda \in a^*_C.
\]

Through the identification of \( A \) with \( \mathbb{R}^d \), we use the Lebesgue measure on \( \mathbb{R}^d \) as the Haar measure \( da \) on \( A \). As usual on the compact group \( K \), we fix the normalized Haar measure \( dk \) and \( dn \) denotes a Haar measure on \( N \). The following integral formulae describe the Haar
measure of \( G \) corresponding to the Iwasawa and polar decomposition respectively. For any \( f \in C_c(G) \),

\[
\int_G f(g)dg = \int_K \int_{\tilde{a}} \int_N f(k\exp Yn) e^{2\rho(Y)} \, dn \, dY \, dk \\
= \int_K \int_{A_+} \int_K f(k_1ak_2) J(a) \, dk_1 \, da \, dk_2,
\]

where \( dY \) is the Lebesgue measure on \( \mathbb{R}^d \) and

\[
J(\exp Y) = c \prod_{\alpha \in \Sigma_+} (\sinh \alpha(Y))^{m_{\alpha}}, \quad \text{for } Y \in \overline{a_+},
\]
c being a normalizing constant. It follows that

\[
(3.1) \quad J(\exp Y) \leq Ce^{2\|\rho\|_B \|Y\|_B}, \quad \text{for all } Y \in \overline{a_+}.
\]

If \( f \) is a function on \( X = G/K \) then \( f \) can be thought of as a function on \( G \) which is right invariant under the action of \( K \). It follows that on \( X \) we have a \( G \) invariant measure \( dx \) such that

\[
\int_X f(x) \, dx = \int_{K/M} \int_{A_+} \int_K f(k\exp Y) J(\exp Y) \, dY \, dk_M,
\]

where \( dk_M \) is the \( K \)-invariant measure on \( K/M \). We shall also need the following integral formula ([14], Chapter 1, Lemma 5.19): if \( F \in L^1(K) \) and \( g \in G \) then

\[
(3.2) \quad \int_K F(\kappa(g^{-1}k)) \, dk = \int_K F(k) e^{-2\rho(H(gk))} \, dk.
\]

In [14] this was proved for \( F \in C(K) \) but the proof works for \( F \in L^1(G) \) as well.

For a sufficiently nice function \( f \) on \( X \), its Fourier transform \( \tilde{f} \) is a function defined on \( a_+^* \times K \) given by

\[
(3.3) \quad \tilde{f}(\lambda,k) = \int_G f(g) e^{(i\lambda - \rho) H(g^{-1}k)} \, dg, \quad \lambda \in a_+^*, \ k \in K,
\]

whenever the integral exists ([13], P. 199). As \( M \) normalizes \( N \) the function \( k \mapsto \tilde{f}(\lambda,k) \) is right \( M \)-invariant. It is known that if \( f \in L^1(X) \) then \( \tilde{f}(\lambda,k) \) is a continuous function of \( \lambda \in a_+^* \), for almost every \( k \in K \). If in addition \( \tilde{f} \in L^1(a_+^* \times K, |c(\lambda)|^{-2} \, d\lambda \, dk) \) then the following Fourier inversion holds,

\[
(3.4) \quad f(gK) = |W|^{-1} \int_{a_+^* \times K} \tilde{f}(\lambda,k) e^{-(i\lambda + \rho)H(g^{-1}k)} |c(\lambda)|^{-2} \, d\lambda \, dk,
\]

for almost every \( gK \in X \) ([13], Theorem 1.8, Theorem 1.9). Here \( c(\lambda) \) denotes Harish-Chandra’s \( c \)-function and \( |W| \) is the number of elements in the Weyl group. Moreover, \( f \mapsto \tilde{f} \) extends to an isometry of \( L^2(X) \) onto \( L^2(a_+^* \times K, |c(\lambda)|^{-2} \, d\lambda \, dk) \) ([13], Theorem 1.5).

**Remark 3.1.** It is known that ([1], P. 394, [3], P. 117) for all \( \|\lambda\|_B \geq 1, \ \lambda \in a_+^* \) there exists a positive number \( C \) such that

\[
(3.5) \quad |c(\lambda)|^{-2} \leq C \|\lambda\|_B^{\dim a}.
\]

If \( \text{rank}(X) = 1 \), then a similar lower estimate holds ([2], P. 653), that is, there exist two positive numbers \( C_1 \) and \( C_2 \) such that for all \( \lambda \geq 1 \)

\[
(3.6) \quad C_1 \lambda^{\dim a} \leq |c(\lambda)|^{-2} \leq C_2 \lambda^{\dim a}.
\]
We now specialize to the case of $K$-biinvariant function $f$ on $G$. Using the polar decomposition of $G$ we may view a $K$-biinvariant function $f$ on $G$ as a function on $A_+$, or by using the inverse exponential map we may also view $f$ as a function on $a$ solely determined by its values on $a_+$. Henceforth, we shall denote the set of $K$-biinvariant functions in $L^1(G)$ by $L^1(K\backslash G/K)$. If $f \in L^1(K\backslash G/K)$ then the Fourier transform $\hat{f}$ takes a special form. It can be easily shown that in this case

$$(3.7) \quad \hat{f}(\lambda, k) = \int_G f(g) \phi_{-\lambda}(g) \, dg,$$

where

$$(3.8) \quad \phi_{\lambda}(g) = \int_K e^{-(i\lambda + \rho)(H(g^{-1}k))} \, dk,$$

for $\lambda \in \mathfrak{a}_+^*$, is Harish Chandra’s elementary spherical function.

Let $\mathcal{U}(g)$ be the universal enveloping algebra of $G$. The elements of $\mathcal{U}(g)$ act on $C^\infty(G)$ as differential operators on both sides. We shall write $f(E : g : D)$, for the action of $(E, D) \in \mathcal{U}(g) \times \mathcal{U}(g)$ on $f \in C^\infty(G)$ at $g \in G$. Precisely, if $E = E_1 E_2 \cdots E_l, D = D_1 D_2 \cdots D_q$ then

$$(3.9) \quad f(E : g : D) = \left( \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_l} \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_q} \right) \bigg|_{t_1=\cdots=t_l=s_1=\cdots=s_q=0} f(\exp s_1 D_1 \cdots \exp s_q D_q g \exp t_1 E_1 \cdots \exp t_l E_l).$$

We now list down some well known properties of the elementary spherical functions which are important for us ([9], Prop 3.1.4 and Chapter 4, §4.6; [13], Lemma 1.18, P. 221).

**Theorem 3.2.**

1) $\phi_{\lambda}(g)$ is $K$-biinvariant in $g \in G$ and $W$-invariant in $\lambda \in \mathfrak{a}_+^*$.

2) $\phi_{\lambda}(g)$ is $C^\infty$ in $g \in G$ and holomorphic in $\lambda \in \mathfrak{a}_+^*$.

3) For all $\lambda \in \mathfrak{a}_+^*$ we have

$$(3.10) \quad |\phi_{\lambda}(g)| \leq \phi_0(g) \leq 1.$$

4) For all $Y \in \mathfrak{a}_+$ and $\lambda \in \mathfrak{a}_+^*$

$$(3.11) \quad 0 < \phi_{i\lambda}(\exp Y) \leq e^{\lambda(Y)} \phi_0(\exp Y).$$

5) Given $E, D \in \mathcal{U}(g)$ there exists a positive constant $C_{D,E}$ such that

$$|\phi_{\lambda}(E : g : D)| \leq C_{D,E} (1 + \|\lambda\|_B)^{\deg E + \deg D} \phi_0(g), \quad \lambda \in \mathfrak{a}^*.$$

6) For $\lambda \in \mathfrak{a}^*$

$$\phi_{-\lambda}(hg) = \int_K e^{(i\lambda - \rho)(H(g^{-1}k))} e^{-(i\lambda + \rho)(H(hk))} \, dk, \quad g, h \in G,$$

**Remark 3.3.** If $\lambda \in \mathfrak{a}^*$ and $f \in L^1(X)$ then

$$\int_K |\hat{f}(\lambda, k)| \, dk = \int_K \left| \int_X f(g) e^{(i\lambda - \rho)(H(g^{-1}k))} \, dg \right| \, dk \leq \int_X |f(g)| \phi_0(g) \, dg < \infty,$$

by Theorem **3.2.** 3). Hence, the function $k \mapsto \hat{f}(\lambda, k)$ is integrable on $K$.

For $f \in L^1(K\backslash G/K)$, we define its spherical Fourier transform $\hat{f}(\lambda)$ by the integral

$$\hat{f}(\lambda) = \int_G f(g) \phi_{-\lambda}(g) \, dg.$$
If \( f \) is \( K \)-bi-invariant then by (3.7) the Fourier transform \( \hat{f} \) coincides with the spherical Fourier transform \( \hat{\tilde{f}} \). If \( F \in L^1(G/K) \) and \( f \in L^1(K \backslash G/K) \) then it is easy to see that \( F * f \in L^1(G/K) \) and the following holds

\[
(F * \tilde{f})(\lambda, k) = \hat{\tilde{f}}(\lambda)\hat{F}(\lambda, k).
\]  

(3.11)

We shall now state the Paley-Wiener theorem for the spherical Fourier transform. For a positive real number \( L \) let \( \mathcal{H}^L(a_C^*)_W \) be the space of \( W \)-invariant, entire functions \( h \) on \( a_C^* \) such that for each \( N \in \mathbb{N} \)

\[
|h(\lambda)| \leq C_N \frac{e^{L\|\lambda\|_B}}{(1 + \|\lambda\|_B)^N}, \quad \lambda \in a_C^*,
\]

and

\[
\mathcal{H}(a_C^*)_W = \bigcup_{L>0} \mathcal{H}^L(a_C^*)_W.
\]

**Theorem 3.4** ([8], [14], Theorem 7.1). The spherical Fourier transform \( f \mapsto \hat{\tilde{f}} \) is a bijection from \( C_c^\infty(K \backslash G/K) \) onto \( \mathcal{H}(a_C^*)_W \) and \( \text{supp } f \subset \overline{B}(0, L) \) if and only if \( \hat{\tilde{f}} \in \mathcal{H}^L(a_C^*)_W \).

One observes that \( \mathcal{H}(a_C^*)_W \) is also the image of \( C_c^\infty(a)_W \) under the Euclidean Fourier transform, where

\[
C_c^\infty(a)_W = \{ f \in C_c^\infty(a) \mid f(w \cdot Y) = f(Y), \text{ for all } Y \in a, w \in W \}.
\]

This is related to the fact that the spherical Fourier transform and the Euclidean Fourier transform on \( a \) are related by the so-called Abel transform. For \( f \in L^1(K \backslash G/K) \) its Abel transform \( A f \) is defined by the integral

\[
A f(Y) = e^{\rho(Y)} \int_N f((\exp Y)n) \, dn, \quad Y \in a,
\]

([9], P. 107, [15], P. 27). We will need the following theorem regarding Abel transform ([9], Prop 3.3.1, Prop 3.3.2), which is an analogue of the relation (2.8).

**Theorem 3.5.** The map \( A : C_c^\infty(K \backslash G/K) \to C_c^\infty(a)_W \) is a bijection. If \( f \in C_c^\infty(K \backslash G/K) \) then

\[
\mathcal{F}(Af)(\lambda) = \hat{\tilde{f}}(\lambda), \quad \lambda \in a^*,
\]

where \( \mathcal{F}(Af) \) denotes the Euclidean Fourier transform on \( a \cong \mathbb{R}^d \).

**Remark 3.6.** It is easy to see that a special case of Theorem 1.2 namely when \( f \in C_c^\infty(K \backslash G/K) \), can be proved simply by using the slice projection theorem (2.8) for the Euclidean Radon transform \( \mathcal{R} \) and the relation (3.12) for the Abel transform (see [5] for a more general result). However, this approach cannot be used to prove Theorem 1.2. The reason is that if an integrable \( K \)-bi-invariant function \( f \) vanishes on an open set then it is not necessarily true that \( Af \) also vanishes on an open subset of \( a \).

We end this section with the notion of heat kernel \( h_t \) on \( X = G/K \) (see [3] for details). There exists a unique family \( \{h_t\}_{t>0} \subset C_c^\infty(K \backslash G/K) \) which solves the heat equation and satisfy the following properties

a) For each \( t > 0 \), \( h_t \) is positive with \( \|h_t\|_{L^1(G)} = 1 \) and \( h_{t+s} = h_t * h_s \) for positive \( t \) and \( s \).

b) If \( f \in L^2(X) \) then for each \( t > 0 \), the function \( f * h_t \) is real analytic on \( X \) (see [21]).

c) The Spherical Fourier transform of \( h_t \) is given by

\[
\hat{h}_t(\lambda) = e^{-t(\|\lambda\|_B^2 + \|\rho\|_B^2)}, \quad \lambda \in a^*.
\]
Remark 3.7. From a) and b) we observe that if \( f \in L^1(X) \) then \( f \ast h_t = (f \ast h_{t/2}) \ast h_{t/2} \) is also real analytic as \( f \ast h_{t/2} \in L^2(X) \). In particular, if \( f \in L^1(X) \) is nonzero then \( f \ast h_t \) (for any fixed \( t \in (0, \infty) \)) cannot vanish on any nonempty open subset of \( X \). This follows from (3.11), the Fourier inversion (3.4) and the fact that \( \widehat{h_t} \) is nonzero everywhere on \( \mathfrak{a}^* \).

4. Levinson’s theorem on Riemannian symmetric spaces of noncompact type

We start by defining certain function spaces which are analogous to the function spaces described in Section 2. Let \( \psi \) be as in Section 2 and \( L \) be a given positive number. We define the following spaces of functions;

\[
C_{\psi}(\mathfrak{a}^*) = \{ f : \mathfrak{a}^* \to \mathbb{C} \mid f \text{ is continuous, } \lim_{|\lambda|_B \to \infty} \frac{f(\lambda)}{e^{\psi(|\lambda|_B)}} = 0 \},
\]

\[
E_L(\mathfrak{a}^*) = \{ f : \mathfrak{a}^* \to \mathbb{C} \mid f \text{ is bounded on } \mathfrak{a}^*, \text{ and has an entire extension to } \mathfrak{a}^*_C \text{ with } |f(\lambda)| \leq C e^{L |\lambda|_B}, \lambda \in \mathfrak{a}^*_C \},
\]

\[
\Phi_L(\mathfrak{a}^*) = \text{span} \{ \chi_x : \mathfrak{a}^* \to \mathbb{C} \mid x \in \mathcal{B}(o, L), \chi_x(\lambda) = \phi_\lambda(x), \lambda \in \mathfrak{a}^* \},
\]

As before, we define

\[
\|f\|_\psi = \sup_{\lambda \in \mathfrak{a}^*} \frac{|f(\lambda)|}{e^{\psi(|\lambda|_B)}} \quad f \in C_{\psi}(\mathfrak{a}^*).
\]

Clearly, \((C_{\psi}(\mathfrak{a}^*), \| \cdot \|_\psi)\) is a normed linear space.

Remark 4.1.

1) It is clear that \( E_L(\mathfrak{a}^*) \subseteq C_{\psi}(\mathfrak{a}^*) \). From the expression of \( \widehat{h_t} \) it is also clear that \( \widehat{h_t} \in C_{\psi}(\mathfrak{a}^*) \) for all \( t \in (0, \infty) \).

2) It follows from Theorem 3.2 that \( \Phi_L(\mathfrak{a}^*) \subseteq E_L(\mathfrak{a}^*) \). In fact, writing \( \lambda = \Re \lambda + i \Im \lambda \in \mathfrak{a}^*_C \) and taking \( x = k_1 \exp(Y)K \in \mathcal{B}(o, L), Y \in \mathfrak{a}^*_+ \), \( k_1 \in K \) we get by (3.9) and (3.10) that

\[
|\phi_{\Re \lambda + i \Im \lambda}(x)| \leq C e^{L |\Re \lambda|_B} e^{\Im \lambda Y} \leq C e^{L |\lambda|_B}.
\]

As \( x \in \mathcal{B}(o, L) \) it follows that

\[
|\phi_{\Re \lambda + i \Im \lambda}(x)| \leq C e^{L |\lambda|_B}.
\]

Since for each \( x \in X \), the function \( \lambda \mapsto \phi_\lambda(x) \) is holomorphic in \( \mathfrak{a}^*_C \) (Theorem 3.2 2)) the conclusion follows.

3) The Paley Wiener theorem (Theorem 3.4) tells us that if \( \phi \in C^\infty(\mathcal{K} \backslash G/K) \) with \( \text{supp } \phi \subseteq \mathcal{B}(o, L) \), then \( \widehat{\phi} \in E_L(\mathfrak{a}^*) \). However, not all elements of \( E_L(\mathfrak{a}^*) \) are of this form. This is because elements of \( E_L(\mathfrak{a}^*) \) may not have polynomial decay on \( \mathfrak{a}^* \).

Because of the identification of \( \mathfrak{a}^* \) with \( \mathbb{R}^d \) and \( \mathfrak{a}^*_C \) with \( \mathbb{C}^d \) the following lemma follows straightway from Lemma 2.2 and Remark 2.3

Lemma 4.2. \( E_L(\mathfrak{a}^*) \) is dense in \((C_{\psi}(\mathfrak{a}^*), \| \cdot \|_\psi)\), for each positive number \( L \) if

\[
\int_1^{\infty} \frac{\psi(r)}{r^2} dr = \infty.
\]

We now consider the following Weyl group invariant subspaces of \( C_{\psi}(\mathfrak{a}^*) \) and \( E_L(\mathfrak{a}^*) \).

\[
C_{\psi}(\mathfrak{a}^*)_W = \{ f \in C_{\psi}(\mathfrak{a}^*) \mid f(w \cdot \lambda) = f(\lambda), \text{ for all } w \in W, \lambda \in \mathfrak{a}^* \},
\]

\[
E_L(\mathfrak{a}^*)_W = \{ f \in E_L(\mathfrak{a}^*) \mid f(w \cdot \lambda) = f(\lambda), \text{ for all } w \in W, \lambda \in \mathfrak{a}^* \},
\]

Lemma 4.3. \( E_L(\mathfrak{a}^*)_W \) is dense in \((C_{\psi}(\mathfrak{a}^*)_W, \| \cdot \|_\psi)\), for each positive number \( L \) if \( \psi \) is as in Lemma 4.2.
Proof. If \( f \in C_\psi(a^*)_W \) then by Lemma 4.2 there exists a sequence \( \{f_n\} \) in \( E_L(a^*) \) such that
\[
\lim_{n \to \infty} \|f_n - f\|_\psi = 0.
\]
We now consider the averaging operator
\[
Tf_n(\lambda) = \frac{1}{|W|} \sum_{w \in W} f_n(w \cdot \lambda), \quad \text{for all } \lambda \in a^*.
\]
Clearly \( Tf_n \) is \( W \)-invariant and bounded for each \( n \in \mathbb{N} \). As each \( f_n \) extends to an entire function of exponential type \( L \) so does \( Tf_n \). This proves that \( Tf_n \in E_L(a^*)_W \), for each \( n \in \mathbb{N} \). The proof now follows by observing that
\[
\|Tf_n - f\|_\psi = \|T(f_n - f)\|_\psi \leq \|f_n - f\|_\psi.
\]
\[\square\]

The following theorem is an analogue of Lemma 2.2 and constitutes the main step for the proof of Theorem 1.2.

**Theorem 4.4.** For any given positive number \( L \) the space \( \Phi_L(a^*) \) is dense in \( (E_L(a^*)_W, \|\cdot\|_\psi) \). If in addition
\[
\int_1^\infty \frac{\psi(r)}{r^2} \, dr = \infty,
\]
then \( \Phi_L(a^*) \) is dense in \( (C_\psi(a^*)_W, \|\cdot\|_\psi) \).

We first sketch the main idea behind the proof. It suffices to prove the first part of the theorem and then apply Lemma 4.3 to obtain the second part. The main idea of the proof is to first approximate a given \( f \in E_L(a^*)_W \) by a function \( \beta = \hat{F} \), for some \( F \in C_\infty(K\setminus G/K) \) with \( \text{supp} \, F \subseteq B(o, L) \) in \( \|\cdot\|_\psi \). This function \( \beta \) can then be approximated (in \( \|\cdot\|_\psi \)) by elements of \( \Phi_L(a^*) \) using Lemma 2.2. Now, given any \( f \in E_L(a^*)_W \) one can think of a function of the form \( f \cdot \hat{\phi} = \beta \), where \( \phi \in C_\infty(K\setminus G/K) \). The Paley-Wiener theorem then implies that \( \beta = f \cdot \hat{\phi} \) is the spherical Fourier transform of a function in \( C_\infty(K\setminus G/K) \). However, there are two immediate problems; the function \( f \cdot \hat{\phi} \) may not belong to \( E_L(a^*)_W \) and may not be close to \( f \) in \( \|\cdot\|_\psi \). In the following, it will be shown that both these problems can be tackled by suitably dilating \( f \) and \( \hat{\phi} \) on \( a^*_\mathbb{C} \).

**Proof of Theorem 4.4.** Let \( f \in E_L(a^*)_W \) and \( \epsilon \) be a given positive number. We claim that there exists \( \nu \in (0, 1) \) such that
\[
(4.2) \quad \sup_{\lambda \in a^*} \frac{|f(\lambda) - f_\nu(\lambda)|}{e^{\psi(\|\lambda\|_\mu)}} < \epsilon,
\]
where \( f_\nu(\lambda) = f(\nu \lambda) \). This follows due to the facts that \( f \) is bounded, uniformly continuous on compact subsets of \( a^* \) and \( \psi \) increases to infinity. Let us fix \( \nu \in (0, 1) \) so that \((4.2)\) holds. Suppose \( \phi \in C_\infty(K\setminus G/K) \) with \( \text{supp} \, \phi \subseteq B(o, 1) \) and \( \hat{\phi}(0) = 1 \). We claim that there exists a positive real number \( h \) such that
\[
(4.3) \quad \sup_{\lambda \in a^*} \frac{|f_\nu(\lambda) - f_\nu(\lambda)\hat{\phi}(h \lambda)|}{e^{\psi(\|\lambda\|_\mu)}} < \epsilon.
\]
As before, using boundedness of $f$ on $\mathfrak{a}^*$ and $\lim_{r \to \infty} \psi(r) = \infty$, we can choose $M \in (0, \infty)$ such that

$$\frac{|f_\nu(\lambda)|}{e^{\psi(\|\lambda\|_B)}} < \frac{\epsilon}{1 + \|\widehat{\phi}\|_{L^\infty(\mathfrak{a}^*)}}, \quad \text{for all } \|\lambda\|_B \geq M.$$ 

Hence, for all $h \in (0, \infty)$,

$$\frac{|f_\nu(\lambda) - f_\nu(\lambda) \widehat{\phi}(h\lambda)|}{e^{\psi(\|\lambda\|_B)}} \leq \frac{|f_\nu(\lambda)|}{e^{\psi(\|\lambda\|_B)}} (1 + \|\widehat{\phi}\|_{L^\infty(\mathfrak{a}^*)}) < \epsilon, \quad \text{for all } \|\lambda\|_B \geq M.$$ 

As $\widehat{\phi}$ is continuous at $\lambda = 0$, there exists $\delta$ positive such that, if $\|\lambda\|_B < \delta$, then

$$|1 - \widehat{\phi}(\lambda)| < \frac{\epsilon}{\|f\|_{L^\infty(\mathfrak{a}^*)}}.$$ 

If we choose $h < \min \{\delta/M, L(1 - \nu)\}$, then

$$\frac{|f_\nu(\lambda) - f_\nu(\lambda) \widehat{\phi}(h\lambda)|}{e^{\psi(\|\lambda\|_B)}} \leq \|f\|_{L^\infty(\mathfrak{a}^*)} |1 - \widehat{\phi}(h\lambda)| < \epsilon, \quad \text{for all } \|\lambda\|_B < M.$$ 

This proves the claim. Note that in the inequality above we have only used the assumption $h < \delta/M$. The second condition on $h$ will be used in the following step. We fix such an $h$ and define

$$g_1(\lambda) = f_\nu(\lambda) \widehat{\phi}(h\lambda), \quad \lambda \in \mathfrak{a}^*_c.$$ 

Rewriting (4.3) we have

$$|f_\nu - g_1|_\psi < \epsilon.$$ 

We observe that $f_\nu, g_1$ are $W$-invariant and for all $\lambda \in \mathfrak{a}^*_c$

$$|f_\nu(\lambda)| \leq C e^{L\|\nu\|_{\mathfrak{a}^*_c} B} < C e^{L\|\lambda\|_B},$$

$$|g_1(\lambda)| = |f_\nu(\lambda) \widehat{\phi}(h\lambda)| \leq C e^{(\nu L + h)\|\lambda\|_B} \leq C e^{L\|\lambda\|_B},$$

as $h$ is smaller than $L(1 - \nu)$. Hence, $f_\nu$ and $g_1$ both are elements of $E_L(\mathfrak{a}^*_c)$. Since $\phi \in C_c^\infty(K \backslash G/K)$, Theorem 3.4 implies that for all $N \in \mathbb{N}$,

$$|g_1(\lambda)| = |f_\nu(\lambda) \widehat{\phi}(h\lambda)| \leq C_{h,N} \frac{e^{L\|\lambda\|_B}}{(1 + \|\lambda\|_B)^N}, \quad \text{for all } \lambda \in \mathfrak{a}^*_c.$$ 

By another application of Theorem 3.4 we have $g_1 = \widehat{F}$, for some $F \in C_c^\infty(K \backslash G/K)$ with $\text{supp } F \subset B(o, L)$. Hence,

$$g_1(\lambda) = \int_{B(o, L)} F(x) \phi_{-\lambda}(x) \, dx$$

$$= \int_{\{Y \in \mathfrak{a}_+ : \|Y\|_B \leq L\}} F(\exp Y) \phi_{-\lambda}(\exp Y) J(\exp Y) \, dY,$$

the integrand being determined by its restriction on $\mathfrak{a}_+$. We now wish to apply Lemma 2.4 to the function $g_1$ with $g(Y, \lambda) = \phi_{-\lambda}(\exp Y)$ and $d\mu(Y) = J(\exp Y) \, dY$, using identification of $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}^d$. Let $\{E_j\}_{j=1}^d$ be an orthonormal basis of $\mathfrak{a}$ with respect to $B|_{\mathfrak{a} \times \mathfrak{a}}$, the restriction of the Cartan-Killing form $B$ on $\mathfrak{a} \times \mathfrak{a}$. Then every $Y \in \mathfrak{a}$ can be written uniquely as

$$Y = \sum_{j=1}^d Y_j E_j, \quad Y_j \in \mathbb{R}.$$
Viewing $E_j$ as left $G$-invariant differential operator we have
\[
(E_j\phi - \lambda)(\exp Y) = d \bigg|_{t=0} \phi - \lambda (\exp Y \cdot \exp tE_j) = \sum_{j=1}^N \frac{\partial}{\partial Y_j} \phi (x_j) \exp Y.
\]
It now follows from Theorem 3.2, 3) and 5) that for all $\lambda$ in a compact subset $K_1$ of $a^*$
\[
|\left(\frac{\partial}{\partial Y_j} \phi - \lambda\right)(\exp Y)| \leq C_j (1 + \|\lambda\|_B) \phi_0 (\exp Y) \leq M_{K_1},
\]
for all $j \in \{1, \cdots, d\}$. Therefore, we can apply Lemma 2.4. In this regard, we first choose a positive number $\tau$ such that
\[
e^{\psi (\|\lambda\|_B)} > \frac{\|g_1\|_{L^\infty(a^*)} + \|F\|_{L^1(G)}}{\epsilon}, \quad \text{for all } \|\lambda\|_B \geq \tau.
\]
By Lemma 2.4 we get a finite set $\{x_1, \cdots, x_N\} \subset \{\exp Y \mid Y \in a, \|Y\|_B < L\}$ and $C_{x_j} \in \mathbb{C}$, for $j = 1, \cdots, N$ such that
\[
|g_1(\lambda) - \sum_{j=1}^N C_{x_j} \phi - \lambda (x_j)| < \epsilon, \quad \text{for all } \|\lambda\|_B \leq \tau.
\]
If we define
\[
g_N(\lambda) = \sum_{j=1}^N C_{x_j} \phi - \lambda (x_j),
\]
then we have
\[
\|g_N\|_{L^\infty(a^*)} \leq \|F\|_{L^1(G)}.
\]
Therefore,
\[
\|g_1 - g_N\|_{\psi} \leq \sup_{\|\lambda\|_B \leq \tau} \frac{|g_1(\lambda) - g_N(\lambda)|}{e^{\psi (\|\lambda\|_B)}} + \sup_{\|\lambda\|_B > \tau} \frac{|g_1(\lambda) - g_N(\lambda)|}{e^{\psi (\|\lambda\|_B)}}
\]}
\[
< \epsilon + \frac{\epsilon}{(\|F\|_{L^1(G)} + \|g_1\|_{L^\infty(a^*)})} \left(\|F\|_{L^1(G)} + \|g_1\|_{L^\infty(a^*)}\right)
\]
\[
= 2\epsilon.
\]
Clearly $g_N \in \Phi_L(a^*)$ and by (4.2), (4.4) and (4.5)
\[
\|f - g_N\|_{\psi} \leq \|f - f_\nu\|_{\psi} + \|f_\nu - g_1\|_{\psi} + \|g_1 - g_N\|_{\psi} < 4\epsilon.
\]
This completes the proof.

For $f \in L^1(X)$ we define the $K$-biinvariant component $Sf$ of $f$ by the integral
\[
Sf(x) = \int_K f(kx) \, dk, \quad x \in X,
\]
and for $g \in G$ we define the left translation operator $l_g$ on $L^1(X)$ by
\[
l_g f(x) = f(gx), \quad x \in X.
\]
Remark 4.5. Usually one defines the operator \( l_g \) as left translation by \( g^{-1} \). The reason we have defined \( l_g \) as left translation by \( g \in G \) because then it follows that \( S(l_g f) = S(l_g, f) \) if \( gK = g_1 K \).

It is known that ([13], Chapter III, §2, P. 209) the Fourier transforms of \( f \) and \( l_g f \) are related by the formula
\[
(l_g f)(\lambda, k) = e^{(i\lambda - \rho)(H(gk))} f(\lambda, \kappa(gk)).
\]
For a nonzero integrable function \( f \), its \( K \)-biinvariant component \( S(f) \) may not be nonzero. However, the following lemma shows that there always exists \( g \in G \) such that \( S(l_g f) \) is nonzero.

**Lemma 4.6.** If \( f \in L^1(X) \) is nonzero then for every \( r \) positive there exists \( g \in G \) with \( gK \in \mathcal{B}(o, r) \) such that \( S(l_g f) \) is nonzero.

**Proof.** Suppose the result is false. Then there exists a positive number \( r \) such that for all \( gK \in \mathcal{B}(o, r) \) the function \( S(l_g f) \) is zero. Hence, for all \( t \) positive we have
\[
\int_G S(l_g f)(x) h_t(x^{-1}) \, dx = 0.
\]
This implies that \( (f \ast h_t)(gK) \) is zero for all positive number \( t \). In fact,
\[
\int_G S(l_g f)(x) h_t(x^{-1}) \, dx = \int_G \left( \int_K l_g f(kx) \, dk \right) h_t(x^{-1}) \, dx
\]
\[
= \int_G l_g f(x) h_t(x^{-1}) \, dx
\]
(using change of variable \( kx \mapsto x \))
\[
= \int_G f(gx) h_t(x^{-1}) \, dx
\]
\[
= f \ast h_t(gK).
\]
It follows that \( f \ast h_t \) vanishes on the open ball \( \mathcal{B}(o, r) \), for all \( t \) positive. Remark 3.7 now implies that \( f \) is the zero function which contradicts our assumption. \( \square \)

We are now in a position to prove our main result.

**Proof of Theorem LEVINSON’S THEOREM** We first prove part (a). The following steps will lead to the proof.

**Step 1.** We first observe that it suffices to work under the assumption that \( f \) is continuous. To see this we assume that \( f \) vanishes on an open ball \( \mathcal{B}(g_0 K, L) \) for some positive number \( L \) and satisfies ([13]). Let \( \phi \in C_c^\infty(K \setminus G/K) \) with \( \text{supp } \phi \subseteq \mathcal{B}(o, L/2) \). Then \( f \ast \phi \in C(G/K) \cap L^1(G/K) \) and
\[
\int_{a^* \times K} |(f \ast \phi)(\lambda, k)| \, e^{\psi(||\lambda||_B)} \, |c(\lambda)|^{-2} \, d\lambda
\]
\[
= \int_{a^* \times K} \left| \widehat{f}(\lambda, k) \right| \left| \widehat{\phi}(\lambda) \right| \, e^{\psi(||\lambda||_B)} \, |c(\lambda)|^{-2} \, d\lambda < \infty.
\]
Moreover, \( f \ast \phi \) vanishes on \( \mathcal{B}(g_0 K, L/2) \). In fact, if \( g_1 K \in \mathcal{B}(g_0 K, L/2) \) then for all \( gK \in \mathcal{B}(o, L/2) \) it follows by using \( G \)-invariance of the Riemannian metric \( d \) that
\[
d(g_0 K, g_1 gK) \leq d(g_0 K, g_1 K) + d(g_1 K, g_1 gK)
\]
\[
< \frac{L}{2} + d(o, gK) < L,
\]
that is, $g_1gK \in B(g_0K, L)$. This implies that $f(g_1g)$ is zero for all $gK \in B(o, L/2) = \text{supp } \phi$ and hence

\[
(f \ast \phi)(g_1) = \int_G f(g_1g) \phi(g^{-1}) \, dg = \int_{\text{supp } \phi} f(g_1g) \phi(g^{-1}) \, dg = 0.
\]

It is easy to see that to prove $f$ is zero it suffices to prove that $f \ast \phi$ is zero. Indeed, if $f \ast \phi$ vanishes identically then so does $\hat{f} \ast \hat{\phi}$. But since $\hat{\phi}$ is nonzero almost everywhere (as $\phi \in C_c^\infty(K \setminus G/K)$) it would follow that $f$ vanishes almost everywhere on $a^* \times K$ implying that $f$ is zero. This completes step 1.

**Step 2.** In this step, we prove part (a) under the additional assumption that $f \in L^1(X) \cap C(X)$ is $K$-biinvariant and vanishes on the open set $B(o, L)$, for some positive number $L$. The spherical Fourier transform of $f$ then satisfies the condition

\[
(4.7) \quad \int_{a_+^*} |\hat{f}(\lambda)| \, e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} d\lambda < \infty,
\]

and the integral $I$ is infinite. By (4.7) it follows that $\hat{f} \in L^1(a_+^*, |c(\lambda)|^{-2}d\lambda)$ and hence by the Fourier inversion (3.4) restricted to $K$-biinvariant functions

\[
f(x) = \int_{a_+^*} \hat{f}(\lambda) \phi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda = 0,
\]

for all $x \in B(o, L)$. This implies that for all $u \in \Phi_L(a^*)$

\[
(4.8) \quad \int_{a_+^*} \hat{f}(\lambda) \, u(\lambda) \, |c(\lambda)|^{-2} \, d\lambda = 0.
\]

To show that $f$ vanishes identically it suffices for us to show that $f \ast h_t$ vanishes identically for all $t$ positive. Since $I$ is infinite we have

\[
\int_1^\infty \frac{\psi(r)}{r^2} dr = C \int_{\{\lambda \in a_+^* \mid ||\lambda||_B \geq 1\}} \frac{\psi(||\lambda||_B)}{||\lambda||_B^{d+1}} d\lambda = \infty.
\]

By Theorem 4.4 we can therefore approximate $\tilde{h}_t \in C_\psi(a^*)$ by elements of $\Phi_L(a^*)$ for all $t$ positive. It follows that given any $\epsilon$ and $t$ positive there exists $u_1 \in \Phi_L(a^*)$ such that

\[
||\tilde{h}_t - u_1||_\psi < \epsilon.
\]

Using boundedness of $\phi_\lambda(x)$ for all $\lambda \in a^*$ and $x \in X$ we get

\[
\left| \int_{a_+^*} \hat{h}_t(\lambda) \hat{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} \, d\lambda \right| \leq \int_{a_+^*} \left| \frac{\hat{h}_t(\lambda) - u_1(\lambda)}{e^{\psi(||\lambda||_B)}} \right| |\hat{f}(\lambda)| \, e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \, d\lambda
\]

\[
+ \int_{a_+^*} |\hat{f}(\lambda)| u_1(\lambda) |c(\lambda)|^{-2} \, d\lambda
\]

\[
< \epsilon \int_{a_+^*} |\hat{f}(\lambda)| e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} d\lambda,
\]

by (4.8) as the second integral in the right hand side is zero. The last integral is finite by our assumption (4.7). As $\epsilon$ is arbitrary, it follows from the Fourier inversion that $f \ast h_t$ vanishes
identically for all \( t \) positive. Hence \( f \) is the zero function.

**Step 3.** We shall now reduce the general case to the case of \( K \)-biinvariant functions by using the radialization operator \( \mathcal{S} \). If possible, let \( f \in L^1(X) \) be a nonzero function which vanishes on a nonempty open subset \( U \) of \( X \) and satisfies the estimate \( (1.3) \). We now choose \( gK \in U \) and consider the function \( l_{g,f} \). The function \( l_{g,f} \) then vanishes on the open set \( g^{-1}U \) which contains the identity coset \( eK \). Hence, there exists a positive number \( L \) such that \( l_{g,f} \) vanishes on the ball \( B(o, L) \). Using the fact that \( k \mapsto H(gK) \) is a continuous function on the compact set \( K \) it follows from Remark 3.3 the integration formula (3.2) and (4.6) that

\[
\int_{a^* \times K} \left| (l_{g,f})^*(\lambda, k) \right| e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk
\]

\[
= \int_{a^* \times K} \left| e^{i(\lambda - \rho)(H(gK))} \tilde{f}(\lambda, \kappa(gk)) \right| e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk
\]

\[
= \int_{a^* \times K} e^{-\rho(H(gk))} \left| \tilde{f}(\lambda, \kappa(gk)) \right| e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk
\]

\[
\leq C_g \int_{a^* \times K} \left| \tilde{f}(\lambda, \kappa(gk)) \right| e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk
\]

\[
= C_g \int_{a^* \times K} \tilde{f}(\lambda, k) e^{-\rho(B^{-1}k)} e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk
\]

\[
\leq C_g' \int_{a^* \times K} \tilde{f}(\lambda, k) e^{\psi(||\lambda||_B)} |c(\lambda)|^{-2} \ d\lambda \ dk < \infty.
\]

Hence, the function \( (l_{g,f})^* \) satisfies the estimate \((1.3)\). Therefore, it is enough for us to assume that \( f \) vanishes on an open ball of the form \( B(o, L) \), for some positive number \( L \). An application of Lemma 4.6 for \( r = L/2 \) shows that there exists \( g_0K \in B(o, L/2) \) such that \( \mathcal{S}(l_{g_0,f}) \) is nonzero. We now claim that \( \mathcal{S}(l_{g_0,f}) \) vanishes on \( B(o, L/2) \). Since \( f \) vanishes on \( B(o, L) \) it follows that \( l_{g_0,f} \) vanishes on \( B(o, L/2) \). In fact, if \( g_1K \in B(o, L/2) \) then

\[
d(eK, g_0g_1K) \leq d(eK, g_0K) + d(g_0K, g_0g_1K) = d(eK, g_0K) + d(eK, g_1K) < \frac{L}{2} + \frac{L}{2} = L,
\]

that is, \( g_0g_1K \in B(o, L) \) for all \( g_1K \in B(o, L/2) \). Consequently, \( \mathcal{S}(l_{g_0,f}) \) also vanishes on the ball \( B(o, L/2) \) and proves the claim. The spherical Fourier transform of the \( K \)-biinvariant function \( \mathcal{S}(l_{g_0,f}) \) is given by

\[
\mathcal{S}(l_{g_0,f})(\lambda) = \int_G \mathcal{S}(l_{g_0,f})(\phi, \lambda) \ d\phi
\]

\[
= \int_G \left( \int_K (l_{g_0,f})(kg) \ dk \right) \phi_{-\lambda}(g) \ dg
\]

\[
= \int_G f(g) \phi_{-\lambda}(g) \ dg
\]

\[
= \int_G f(g) \phi_{-\lambda}(g_0^{-1}g) \ dg.
\]

\[(4.9)\]
using change of variable \(kg \mapsto g\) and \(K\)-biinvariance of \(\phi_{-\lambda}\). Using the expression of \(\phi_{-\lambda}(hg)\) given in Theorem 3.2 it follows that from above that
\[
S(l_{g_0f})(\lambda) = \int_G \int_K f(g) e^{(i\lambda-\rho)(H(g^{-1}k))} e^{-(i\lambda+\rho)(H(g_0^{-1}k))} \, dk \, dg
\]
\[
= \int_K \tilde{f}(\lambda, k) e^{-(i\lambda+\rho)(H(g_0^{-1}k))} \, dk.
\]
It now follows from the hypothesis (1.3) that
\[
\int_{a^*} |S(l_{g_0f})(\lambda)| e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \, d\lambda
\]
\[
= \int_{a^*} \int_K \tilde{f}(\lambda, k) e^{-(i\lambda+\rho)(H(g_0^{-1}k))} \, dk \left| e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \right| d\lambda
\]
\[
\leq C_{g_0} \int_{a^*} \int_K |\tilde{f}(\lambda, k)| e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \, d\lambda \, dk < \infty.
\]
That is, the nonzero \(K\)-biinvariant function \(S(l_{g_0f})\) satisfies (4.7). By step 2 we now conclude that \(S(l_{g_0f})\) vanishes identically which contradicts our hypothesis that \(S(l_{g_0f})\) is nonzero. Hence, \(f\) is zero after all and this completes the proof of part a).

We shall now prove part (b) which can be deduced from Theorem 1.1, b) by using the Euclidean Radon transform \(\mathcal{R}\) and the Abel transform \(\mathcal{A}\). If \(I\) is finite then we have
\[
\int_1^{\infty} \frac{\psi(r)}{r^2} \, dr < \infty.
\]
Since \(\psi\) is nondecreasing by part b) of Theorem 1.1 there exists a nontrivial \(g_1 \in C_c(\mathbb{R})\) with \(\text{supp } g_1 \subseteq [-l/4, l/4]\) such that
\[
|\mathcal{F}g_1(\xi)| \leq Ce^{-\psi(\xi)}, \quad \text{for all } \xi \in \mathbb{R}.
\]
Here \(\mathcal{F}g_1\) is the one-dimensional Fourier transform of \(g_1\). By considering \(g = g_1 * \phi\) with a \(\phi \in C^\infty_c(\mathbb{R})\), \(\text{supp } \phi \subseteq [-l/4, l/4]\) we get that \(g \in C^\infty_c(\mathbb{R})\) with \(\text{supp } g \subseteq [-l/2, l/2]\) and
\[
(4.10) \quad |\mathcal{F}g(\xi)| \leq Ce^{-\psi(\xi)}, \quad \text{for all } \xi \in \mathbb{R}.
\]
If \(g\) turns out to be an even function then the function \(\mathcal{R}^{-1}(g) = h_0\) (well defined by (2.9)) is a nontrivial function in \(C^\infty_c(\mathbb{R}^d)\). By the slice projection theorem (2.8), it satisfies the estimate
\[
|\mathcal{F}h_0(\lambda)| \leq Ce^{-\psi(\lambda)}, \quad \text{for all } \lambda \in \mathbb{R}^d.
\]
If \(g\) is not even then we consider the translate \(\bar{g}(x) = g(x + l/2)\). Then \(\bar{g} \in C_c^\infty(\mathbb{R})\) with \(\text{supp } \bar{g} \subseteq [-l, 0]\) and hence \(\bar{g}\) can not be an odd function. It follows that \(\bar{g}\) has a nontrivial even part given by
\[
\tilde{g}_e(x) = \frac{\bar{g}(x) + \bar{g}(-x)}{2}, \quad x \in \mathbb{R},
\]
and \(\mathcal{F}\tilde{g}_e\) satisfies the estimate (4.10). We can now consider \(h_0 = \mathcal{R}^{-1}(\tilde{g}_e)\) and argue as before. Therefore, if \(I\) is finite and \(\psi\) is nondecreasing then there exists a nontrivial radial function \(h_0 \in C^\infty_c(\mathbb{R}^d)\) such that
\[
(4.11) \quad |\mathcal{F}h_0(\lambda)| \leq Ce^{-\psi(\|\lambda\|)}, \quad \lambda \in \mathbb{R}^d.
\]
Since \(h_0\) is a radial function on \(\mathbb{R}^d\), it can be thought of as a \(W\)-invariant function on \(A \cong \mathbb{R}^d\). So, by Theorem 3.5 there exists \(h \in C^\infty_c(K \backslash G / K)\) such that \(\mathcal{A}(h) = h_0\). For a nontrivial
φ ∈ \( C^\infty(K\backslash G/K) \) we consider the function \( f = h \ast \phi \in \( C^\infty_c(K\backslash G/K) \). Using the analogue of the slice projection theorem (Theorem 3.5) it follows from the estimate (4.11) that

\[
\int_{a_+^*} |\hat{f}(\lambda)| \ e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \ d\lambda
= \int_{a_+^*} |\hat{h}(\lambda)| \ |\hat{\phi}(\lambda)| \ e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \ d\lambda
\leq C \int_{a_+^*} |\hat{\phi}(\lambda)| \ |c(\lambda)|^{-2} \ d\lambda.
\]

(4.12)

Since, \( \hat{\phi} \in \mathcal{H}(a_+^*) \), it follows from the estimate (3.5) that the integral in (4.12) is finite and consequently, \( \hat{f} \) satisfies the estimate (1.3). This completes the proof of part (b). \( \square \)

One can use the above theorem to prove an analogous \( L^p \) version. To illustrate this we prove an \( L^\infty \) version of the above theorem which can thought of as an analogue of Theorem 1.1.

**Theorem 4.7.** Let \( \psi \) and \( I \) be as in Theorem 1.2

a) Let \( f \in L^1(X) \) satisfy the estimate

\[
|\hat{f}(\lambda,k)| \leq C e^{-\psi(\|\lambda\|_B)}, \quad \text{for all } \lambda \in \text{a}^*, k \in K,
\]

If \( f \) vanishes on a nonempty open subset of \( X \) and \( I \) is infinite, then \( f = 0 \).

b) If \( I \) is finite then there exists a nontrivial \( f \in C^\infty_c(K\backslash G/K) \) satisfying (4.13).

**Proof.** As in Theorem 1.2 it suffices to prove the theorem for \( f \in L^1(K\backslash G/K) \) vanishing on an open ball of the form \( B(o,L) \) such that \( \hat{f} \) satisfies the estimate

\[
|\hat{f}(\lambda)| \leq C e^{-\psi(\|\lambda\|_B)}, \quad \text{for all } \lambda \in \text{a}^*_+.
\]

We choose a nonzero \( \phi \in \mathcal{C}^\infty_c(K\backslash G/K) \) with \( \text{supp} \ \phi \subseteq B(o,L/2) \) and consider the function \( f \ast \phi \). Since \( f \) vanishes on \( B(o,L) \) and the support of the function \( \phi \) is contained in \( B(o,L/2) \) it follows as before that \( f \ast \phi \) vanishes on \( B(o,L/2) \). Now,

\[
\int_{a_+^*} |f \ast \phi(\lambda)| \ e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \ d\lambda
= \int_{a_+^*} |\phi(\lambda)| \ |f(\lambda)| \ e^{\psi(\|\lambda\|_B)} |c(\lambda)|^{-2} \ d\lambda
\leq C \int_{a_+^*} |\phi(\lambda)| \ |c(\lambda)|^{-2} \ d\lambda < \infty.
\]

It now follows from Theorem 1.2 that \( f \ast \phi \) is zero almost everywhere. Since \( \hat{\phi} \) is nonzero almost everywhere we conclude that \( \hat{f} \) vanishes almost everywhere on \( \text{a}^* \) and so does \( f \). To prove part b) we observe that if \( I \) is finite then the function \( h \) constructed in the proof of Theorem 1.2 b) satisfies the estimate (4.13). \( \square \)

**Remark 4.8.** (1) It is not hard to see that part a) of Theorem 1.2 remains true if the integral \( I \) is replaced by the integral

\[
\int_{\{\lambda \in \text{a}^*_+ \ | \ \|\lambda\|_B \geq 1\}} \frac{\psi(\|\lambda\|_B)}{\|\lambda\|^{n+1}_B} |c(\lambda)|^{-2} \ d\lambda.
\]
where \( \eta = d + \dim n \), is the dimension of the symmetric space \( X \). This follows from the estimate (3.5) of \( |c(\lambda)|^{-2} \) as

\[
\int_1^\infty \frac{\psi(r)}{r^2} dr = C \int_{\{\lambda \in a^*_+ | \|\lambda\|_B \geq 1\}} \frac{\psi(\|\lambda\|_B)}{\|\lambda\|_B^{d+1}} d\lambda \\
= C \int_{\{\lambda \in a^*_+ | \|\lambda\|_B \geq 1\}} \frac{\psi(\|\lambda\|_B)}{\|\lambda\|_B^{\dim X + 1}} d\lambda \\
\geq C \int_{\{\lambda \in a^*_+ | \|\lambda\|_B \geq 1\}} \frac{\psi(\|\lambda\|_B)}{\|\lambda\|_B^{\dim X + 1}} |c(\lambda)|^{-2} d\lambda = \infty.
\]

Moreover, because of the estimate (3.6), part b) of Theorem 1.2 also remains true in this case if \( \text{rank}(X) = 1 \).

(2) If \( \psi(r) = r^2 \) or \( r \) then one may appeal to a result of Kotake and Narasimhan [19] to conclude that \( f \) is real analytic. However, the same line argument does not seem to work for more general \( \psi \) as in Theorem 1.1. For example, it follows from Theorem 4.7 that if the spherical Fourier transform of a nonzero function \( f \in L^1(K \backslash G/K) \) satisfies the estimate

\[
|\hat{f}(\lambda)| \leq Ce^{-\frac{\|\lambda\|_B}{1+\log(\|\lambda\|_B)}}, \quad \text{for all } \|\lambda\|_B \geq 1, \lambda \in a^*_+,
\]

then \( f \) cannot vanish on a nonempty open subset of \( X \). However, there exists a nonzero \( f \in C_c^\infty(K \backslash G/K) \) such that

\[
|\hat{f}(\lambda)| \leq Ce^{-\frac{\|\lambda\|_B}{(1+\log(\|\lambda\|_B))^2}}, \quad \text{for all } \|\lambda\|_B \geq 1, \lambda \in a^*_+.
\]

It is not known at the moment whether there exists a nonzero function \( f \in L^1(K \backslash G/K) \) satisfying (4.14) which vanishes on a positive measure subset of \( X \).

(3) We note that in Theorem 1.1 the decay of the Fourier transform was assumed only in one direction, that is around infinity. But in Theorem 1.2 and Theorem 4.7 the decay of the Fourier transform is uniform in all directions. It is not clear to us whether it is possible to prove an analogue of Theorem 1.2 by assuming the decay of Fourier transform only in some directions. We refer the reader to [29], Theorem A', where an analogous issue has been addressed for the Euclidean spaces \( \mathbb{R}^d \).

(4) One cannot fail to observe that the exponential volume growth of the Riemannian symmetric space \( X \) of noncompact type does not play any role in Theorem 1.2. The reason seems to be that the dual \( a^*_+ \times K \) is essentially of polynomial growth. In view of this, the following seems to be an interesting question: can we characterize nonnegative functions \( \psi \) for which there exists a nonzero \( f \in L^2((K \backslash G/K) \) such that

\[
|f(x)| \leq Ce^{-\psi(d(o,x)}, \quad x \in X
\]

but \( \hat{f} \) vanishes on a nonempty open subset of \( a^*_+ \)?

(5) It would be interesting to see whether results analogous to Theorem 1.2 can be proved in other contexts as well (see [4, 24, 28]).

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