Operational extreme points of unital completely positive maps

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Abstract

Two notions for linear maps (operational convex combinations and operational extreme points) are introduced. The set $S$ of ucp maps on $M_n(\mathbb{C})$ is the operational convex combinations of the identity map. An operational extreme point of $S$ is an extreme point of $S$ but the converse does not hold, and every automorphism is an operational extreme point of $S$.

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1 Introduction

From a viewpoint of von Neumann entropy for states of $M_n(\mathbb{C})$, we gave some characterization for unital positive Tr-preserving maps of the algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$. That is, a positive unital Tr-preserving map $\Phi$ of $M_n(\mathbb{C})$ preserves the von Neumann entropy of a given state $\phi$ if and only if $\Phi$ plays a role of an automorphism for $\phi$.

In this note, we pick up the set of unital completely positive (called "ucp" for short) maps of $M_n(\mathbb{C})$. That is, for a unital linear map of $M_n(\mathbb{C})$, we replace the property "Tr-preserving and positive" to the property "completely positive", and investigate that what kind of position the automorphisms stand in ucp maps.
The set of ucp maps is a convex subset of linear maps of $M_n(\mathbb{C})$. The notion of convex sets begins with basic definition of the linear concepts of addition and scalar multiplication.

Here, we shall consider the notion of convexity not only scalar multiplication but also the multiplication via operators and generalize the notion of convexity, i.e., we introduce the notion of operational convex combination.

The motivation for the terminology "operational convex combinations" comes from the following two definitions: One is Lindblad’s "operational partition" in [5] (cf. [6] or [7]) and the other is Cuntz’s canonical endomorphism $\Phi_n$ in [4]. It seems to be natural for treating the set of ucp maps as the set of operational convex combinations of automorphisms of $M_n(\mathbb{C})$.

We also introduce the notion of operational extreme point. Since an automorphism of $M_n(\mathbb{C})$ is an extreme point of the set of positive maps (cf. [9]), any automorphism can not be expressed as a convex combination of two different automorphisms. However, it is possible to be expressed as an operational convex combination of two different automorphisms. By our definition, operational extreme points are extreme points, but the converse is not true as we show in Example 3.6, and we show that automorphisms are operational extreme points in the set of ucp maps of $M_n(\mathbb{C})$.

2 Preliminaries

Here we summarize notations, terminologies and basic facts.

2.1 Finite partition of unity

What we need to define a convex sum? In usual, we need a probability vector $\lambda = (\lambda_1, \cdots, \lambda_n)$: $\lambda_i \geq 0, \sum_{i} \lambda_i = 1$.

Given a finite subset $x = \{x_1, \ldots, x_n\}$ of a vector space $X$, the vector $\sum_i \lambda_i x_i$ is called a convex sum of $x$ via $\lambda$.

Now, we consider such a $\lambda$ as a "finite partition of 1".

Two generalized notions of finite partition of 1 are given in the framework of the non-commutative entropy as follows:

Let $A$ be a unital $C^*$-algebra.

(1) A finite subset $\{x_1, \ldots, x_k\}$ of $A$ is called a finite partition of unity by Connes-Størmer ([3]) if they are nonnegative operators which satisfy that $\sum_{i=1}^{n} x_i = 1_A$.
(2) A finite subset \( \{x_1, \ldots, x_k\} \) of \( A \) is called a finite operational partition in \( A \) of unity of size \( k \) by Lindblad (3) if \( \sum_i x_i^* x_i = 1_A \).

Our main target in this note is a finite subset \( \{v_1, \ldots, v_k\} \) of non-zero elements in \( A \) such that \( \{v_1^*, \ldots, v_k^*\} \) is a finite operational partition of unity so that \( \sum_i v_i v_i^* = 1_A \). We call such a set \( \{v_1, \ldots, v_k\} \) a finite operational partition of unity of size \( k \) in \( A \), and denote the set of all finite operational partition of unity in \( A \) by \( FOP(A) \):

\[
FOP(A) = \{ \{v_1, \ldots, v_k\} \mid 0 \neq v_i \in A, \forall i, \sum_i v_i v_i^* = 1_A, k = 1, 2, \ldots \} \tag{2.1}
\]

We denote by \( U(A) \) the set of all unitaries in \( A \). Clearly, \( U(A) \) is the set of the most trivial finite operational partition of unity with the size 1.

### 2.2 Unital completely positive (ucp) map \( \Phi \)

Let \( M_n(\mathbb{C}) \) be the \( C^* \)-algebra of \( n \times n \) matrices over the complex field \( \mathbb{C} \). A linear map \( \Phi \) on a unital \( C^* \)-algebra \( A \) is positive iff \( \Phi(a) \) is positive for all positive \( a \in A \) and completely positive iff \( \Phi \otimes 1_k \) is positive for all positive integer \( k \), where the map \( \Phi \otimes 1_k \) is the map on \( A \otimes M_k(\mathbb{C}) \) defined by \( \Phi \otimes 1_k(x \otimes y) = \Phi(x) \otimes y \) for all \( x \in A \) and \( y \in M_k(\mathbb{C}) \).

We restrict the unital \( C^* \)-algebra \( A \) to \( M_n(\mathbb{C}) \).

In [2, Theorem 2], Choi gave the following characterization: a linear map \( \Phi \) of \( M_n(\mathbb{C}) \) is completely positive iff \( \Phi \) is of the form \( \Phi(x) = \sum_{i=1}^m v_i x v_i^* \) for all \( x \in M_n(\mathbb{C}) \) by some \( \{v_i\}_{i=1}^m \subset M_n(\mathbb{C}) \). Moreover, for \( \{v_i\}_{i=1}^m \) inducing the form \( \Phi(x) = \sum_{i=1}^m v_i x v_i^* \), we may require that \( \{v_i\} \) is linearly independent so that in the form the number \( m \) is uniquely determind. Such a form was called a ‘canonical’ expression for \( \Phi \) (see [2, Remark 4]).

Let us call the uniquely determind number \( m \) the size of the \( \Phi \).

Now we pick up the case where \( \Phi \) is a unital completely positive (called “ucp” for short) map of \( M_n(\mathbb{C}) \). Then the \( \{v_1, \ldots, v_m\} \subset M_n(\mathbb{C}) \) used in the form \( \Phi(x) = \sum_{i=1}^m v_i x v_i^* , (x \in M_n(\mathbb{C})) \) satisfies that \( \sum_{i=1}^m v_i v_i^* = 1 \). This means that each ucp map \( \Phi \) of \( M_n(\mathbb{C}) \) is induced some \( \{v_1, \ldots, v_m\} \) in \( FOP(M_n(\mathbb{C})) \).

Given an operator \( v \in M_n(\mathbb{C}) \), the map \( \text{Adv} \) on \( M_n(\mathbb{C}) \) is given by \( \text{Adv}(x) = v x v^* , (x \in M) \). Then the group \( \text{Aut}(M_n(\mathbb{C})) \) of all automorphisms of \( M_n(\mathbb{C}) \) is written by the form \( \text{Aut}(M_n(\mathbb{C})) = \{ \text{Ad}_u \mid u \in U(M_n(\mathbb{C})) \} \).
where $U(M_n(\mathbb{C}))$ is the group of all unitaries in $M_n(\mathbb{C})$. Similarly, the set $UCP(M_n(\mathbb{C}))$ of all ucp maps on $M_n(\mathbb{C})$ is written by the following form:

$$UCP(M_n(\mathbb{C})) = \left\{ \sum_{i=1}^{m} \text{Adv}_{v_i} | \{v_i\}_{i=1}^{m} \in FOP(M_n(\mathbb{C})), \ m = 1, 2, \ldots \right\}$$

(2.2)

### 3 Operational Convex Combination

#### 3.1 Operational convexity

**Definition 3.1.** Let $\{\Phi_i\}_{i=1}^{m}$ be a set of linear maps on $M_n(\mathbb{C})$ and $\{v_i\}_{i=1}^{m} \in FOP(M_n(\mathbb{C}))$. We call $\sum_{i=1}^{m} \text{Adv}_{v_i} \circ \Phi_i$ an operational convex combination of $\{\Phi_i\}_{i=1}^{m}$ with an operational coefficients $\{v_i\}_{i=1}^{m}$. We also say that a subset $S$ of linear maps on $M_n(\mathbb{C})$ is operational convex if it is closed under all operational convex combinations.

We can consider $UCP(M_n(\mathbb{C}))$ as the set of all operator convex combinations of the group $Aut(M_n(\mathbb{C}))$. Moreover $UCP(M_n(\mathbb{C}))$ is represented as the set of all operational convex combinations of the identity $id$ of $M_n(\mathbb{C})$. We give some characterization for a role of $Aut(M_n(\mathbb{C}))$ in $UCP(M_n(\mathbb{C}))$ from a view point of extreme points.

**3.1.1 Cuntz’s canonical endomorphism as an example**

The Cuntz’s canonical endomorphism $\Phi_n$ ([4]) is an interesting example in unital completely positive maps of infinite dimensional simple $C^*$-algebras, which is given as an operational convex combination of the identity. That is, let $\{S_1, S_2, \ldots, S_n\}$ be isometries on an infinite dimensional Hilbert space $H$ such that $\sum_i S_i S_i^* = 1$. The Cuntz algebra $O_n$ is the $C^*$-algebra generated by $\{S_1, S_2, \ldots, S_n\}$. The map $\Phi_n$ is given as $\Phi_n(x) = \sum_i S_i x S_i^*$ for all $x \in O_n$.

So, in our notation, $\{S_1, S_2, \ldots, S_n\} \in FOP(O_n)$ and $\Phi_n \in UCP(O_n)$.

The left inverse $\Psi$ of $\Phi_n$ plays an important role in the theory of Cuntz algebras and it is given by the form $\Psi(x) = (1/n) \sum S_i^* x S_i$, $(x \in O_n)$.

We remark that $\Psi$ is also an operational convex combination of the identity and $\Psi \in UCP(O_n)$.

Later we discuss in another paper on the case of unital infinite dimensional $C^*$-algebras represented by $O_n$. 

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[4] Cuntz, J. (1973). On one-parameter semigroups of endomorphisms of $C^*$-algebras. Advances in Mathematics, 13, 157-175.
3.1.2 Operational extreme point

Now let us remember the notion of extreme points. Let $S$ be a convex set. Then a $z \in S$ is an extreme point in $S$ if $z$ cannot be the convex combination $\lambda x + (1 - \lambda)y$ of two points $x, y \in S$ with $x \neq y$ and $\lambda \in (0, 1)$, i.e., if $z = \lambda x + (1 - \lambda)y$, $(x, y \in S)$ then $x = y = z$.

In this note, we say this notion of extreme points an extreme point in the usual sense.

Remark 3.2. A related notion are investigated for positive maps on $C^*$-algebras in [9]. A positive map $\Phi$ on a $C^*$-algebra $A$ is extremal if the only positive maps $\Psi$ on $A$, such that $\Phi - \Psi$ is positive, are of the form $\lambda \Phi$ with $0 \leq \lambda \leq 1$. In the set of all positive maps on $B(H)$ for a Hilbert space $H$, the map $Ad u, (u \in B(H))$ is extremal [9 Proposition 3.1.3]. This implies that any automorphism of $M_n(\mathbb{C})$ can not be expressed as a convex combination of two different automorphisms. However if we replace a convex combination to an operational convex combination, then it is possible for each automorphism $\Phi$ of $M_n(\mathbb{C})$.

Example 3.3. Let $\Theta, \Phi$ and $\Psi$ be three different automorphisms of $M_n(\mathbb{C})$. Assume that $u, v$ and $w$ are unitaries in $M_n(\mathbb{C})$ such that $\Theta = Ad u, \Phi = Ad v$ and $\Psi = Ad w$. Put $a = \lambda^{1/2}uv^*$ and $b = (1 - \lambda)^{1/2}uw^*$ for some $\lambda \in (0, 1)$. Then $\{a, b\} \in FOP(M_n(\mathbb{C}))$, and the operational convex combination of $\Phi$ and $\Psi$ with the operational coefficients $\{a, b\}$ is the automorphism $\Theta$, i.e., $a\Phi(x)a^* + b\Psi(x)b^* = \Theta(x)$ for all $x \in M_n(\mathbb{C})$.

In the case of operational convex combinations for linear maps $\Phi$ and $\Psi$ on $M_n(\mathbb{C})$ with an operational coefficient $\{a, b\} \in FOP(M_n(\mathbb{C}))$, the map $Ada \circ \Phi$ corresponds $\lambda x$ and the $Ad b \circ \Psi$ does $(1 - \lambda)y$. Putting this in mind, let us define as follows and show that an automorphism of $M_n(\mathbb{C})$ (i.e., the ucp maps with the size 1) is an operational extreme point.

Definition 3.4. Let $S$ be an operational convex subset of linear maps on $M_n(\mathbb{C})$. We say that a $\Phi \in S$ is an operational extreme point of $S$ if a representation of $\Phi$ that $\Phi = Ada \circ \Phi_1 + Adb \circ \Phi_2, (\Phi_i \in S, (i = 1, 2), \{a, b\} \in FOP(M_n(\mathbb{C}))$ implies that $aa^* = \lambda 1_{M_n(\mathbb{C})}, bb^* = (1 - \lambda)1_{M_n(\mathbb{C})}$ for some $\lambda \in (0, 1)$ so that $\lambda^{-1}Ada \circ \Phi_1 = \{1 - \lambda\}^{-1}Adb \circ \Phi_2 = \Phi$.

Remark 3.5. If $\Phi$ is an operational extreme point of a convex subset $S$ of linear maps on $M_n(\mathbb{C})$, then $\Phi$ is an extreme point of $S$ in the usual sense.
In fact if \( \Phi = \lambda \Phi_1 + (1 - \lambda) \Phi_2 \) (\( \Phi_i \in S \)) and if \( \Phi \) is an operational extreme point of \( S \). then \( \Phi_1 = \lambda^{-1} \Phi_1 = \Phi \) and \( \Phi_2 = (1 - \lambda)^{-1}(1 - \lambda) \Phi_2 = \Phi \) so that \( \Phi \) is an extreme point of \( S \) in the usual sense.

The converse of the above is not true in general as the following example shows:

**Example 3.6.** Here we give a ucp map which is an extreme but not operational extreme point.

Let \( \{e_{ij}\}_{i,j=1,2} \) be a matrix units of \( M = M_n(\mathbb{C}) \), and let \( V = \{v_1 = e_{11}, v_2 = e_{21}\} \in FOP(M) \). Let \( \Phi_V = \sum_{i=1}^2 \text{Ad} v_i \). Then the ucp map \( \Phi_V \) satisfies that \( \Phi_V(e_{11}) = 1_M \), and \( \Phi_V(e_{12}) = \Phi_V(e_{21}) = \Phi_V(e_{22}) = 0 \). Assume that \( \Phi_V \) has a form \( \Phi_V = \lambda \Phi_1 + (1 - \lambda) \Phi_2 \) by \( \Phi_i \in UCP(M), (i = 1, 2) \). Then
\[
0 = \Phi_V(e_{22}) = \lambda \Phi_1(e_{22}) + (1 - \lambda) \Phi_2(e_{22}).
\]
This implies that \( \Phi_i(e_{22}) = 0 \), \( (i = 1, 2) \) because \( \Phi_i(e_{22}) \) is positive for \( i = 1, 2 \) and that \( \Phi_1(e_{11}) = \Phi_1(1_M - e_{22}) = 1_M - 0 = 1_M \) for \( i = 1, 2 \). By using Kadison-Schwarz inequality (cf. \([6, 7]\)), we have that \( \Phi_i(e_{21}) \Phi_i(e_{21})^* \leq \Phi_i(e_{21} e_{21}^*) = \Phi_i(e_{22}) = 0 \) so that \( \Phi_i(e_{21}) = 0 \) for \( i = 1, 2 \).

Hence \( \Phi_V = \Phi_1 = \Phi_2 \) and \( \Phi_V \) is an extreme point of \( UCP(M) \).

Now as a \( \{a, b\} \in FOP(M) \) we choose \( a = e_{11}, b = e_{22} \). As two ucp maps \( \Phi_3, \Phi_4 \) on \( M \), let \( \Phi_3 \) be the identity map and \( \Phi_4 \) be the Ad \( u \) where \( u = e_{12} + e_{21} \). Then \( \Phi_V = \text{Ad} a \circ \Phi_3 + \text{Ad} b \circ \Phi_4 \). This shows that \( \Phi_V \) is not an operational extreme point.

**Theorem 3.7.** If an automorphism \( \Theta \) of \( M_n(\mathbb{C}) \) is decomposed into an operational convex combination of the form that \( \Theta = \text{Ad} a \circ \Phi + \text{Ad} b \circ \Psi \) via \( \{a, b\} \in FOP(M_n(\mathbb{C})) \) and \( \Phi, \Psi \in UCP(M_n(\mathbb{C})) \), then there exist unitaries \( u_a, u_b \in M_n(\mathbb{C}) \) and \( \lambda \in (0, 1) \) such that
\[
a = \sqrt{\lambda} u_a, \quad b = \sqrt{1 - \lambda} u_b \quad \text{so that} \quad \Phi = \text{Ad} u_a^* u, \quad \Psi = \text{Ad} u_b^* u. \quad (3.1)
\]
Here \( u \) is a unitary with \( \Theta = \text{Ad} u \).

**Proof.** Since \( \Theta = \text{Ad} a \circ \Phi = \text{Ad} b \circ \Psi \) is positive and \( \Theta \) is extremal ([Proposition 3.1.3, \([9]\)]) there exists a \( \lambda \in [0, 1] \) such that \( \text{Ad} a \circ \Phi = \lambda \text{Ad} u \), which implies that \( \text{Ad} b \circ \Psi = (1 - \lambda) \text{Ad} u \). On the other hand, since \( \Phi \) and \( \Psi \) are unital, we have that \( a a^* = \text{Ad} a \circ \Phi(1) = \lambda \text{Ad} u(1) = \lambda 1 \) and \( b b^* = (1 - \lambda) 1 \). Hence we have unitaries \( u_a, u_b \in M_n(\mathbb{C}) \) such that \( a = \sqrt{\lambda} u_a \) and \( b = \sqrt{1 - \lambda} u_b \). These relations imply that \( \lambda u_a \Phi(x) u_a^* = \text{Ad} \circ \Phi(x) = \lambda u_x u_x^* \) so that \( \Phi = \text{Ad} u_a^* u \). Similarly, \( \Psi = \text{Ad} u_b^* u \). \( \square \)
Corollary 3.8. An automorphism on $M_n(\mathbb{C})$ is an operational extreme point of the set of the unital completely positive maps on $M_n(\mathbb{C})$.

Proof. Assume that $\Theta \in Aut(M_n(\mathbb{C}))$ is given as an operational combination $Ada \circ \Phi + Adb \circ \Psi = \Theta$ of $\Phi, \Psi \in UCP(M_n(\mathbb{C}))$ with an operational coefficient $\{a, b\} \in FOP(M_n(\mathbb{C}))$. Then by Theorem 3.7, there exist unitaries $v, w \in M_n(\mathbb{C})$ and a $\lambda \in (0, 1)$ such that $a = \sqrt{\lambda}uv^*$, $b = \sqrt{1-\lambda}uw^*$ and $\Phi = Adv, \Psi(x) = Adw$ for a unitary $u$ with $\Theta = Adu$. These condition imply that $aa^* = \lambda 1_{M_n(\mathbb{C})}$, $bb^* = (1-\lambda)1_{M_n(\mathbb{C})}$ and that $\lambda^{-1}Ada \circ \Phi = (1-\lambda)^{-1}Adb \circ \Psi = \Phi$ so that $\Theta$ satisfies the condition of operational extreme points. \qed

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