Stabilized plethysms for the classical Lie groups

Cédric Lecouvey
Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville
B.P. 699 62228 Calais Cedex
Cedric.Lecouvey@lmpa.univ-littoral.fr

Abstract

The plethysms of the Weyl characters associated to a classical Lie group by the symmetric functions stabilize in large rank. In the case of a power sum plethysm, we prove that the coefficients of the decomposition of this stabilized form on the basis of Weyl characters are branching coefficients which can be determined by a simple algorithm. This generalizes in particular some classical results by Littlewood on the power sum plethysms of Schur functions. We also establish explicit formulas for the outer multiplicities appearing in the decomposition of the tensor square of any irreducible finite dimensional module into its symmetric and antisymmetric parts. These multiplicities can notably be expressed in terms of the Littlewood-Richardson coefficients.

1 Introduction

This paper is concerned with the plethysms of the Weyl characters associated to classical Lie groups by the symmetric functions. Let $g$ be a classical Lie group with rank $n$, and $\lambda$ a partition. We denote by $s_\lambda^g$ the Weyl character of the $g$-module $V^g(\lambda)$ (see Section 2). Consider $f$ a symmetric function of degree $d$ and suppose $n > dl(\lambda)$ where $l(\lambda)$ is the number of non-zero parts of $\lambda$. It follows from results by Littlewood [14] that the plethysm $f \circ s_\lambda^g$ of the Weyl character $s_\lambda^g$ by $f$ decomposes on the basis $\{s_\mu^g \mid \mu \in P_n\}$ with coefficients which do not depend on $n$. When $f = p_\ell$ is the power sum of degree $\ell$, we establish that the coefficients so obtained are branching coefficients corresponding to the restriction to certain Levi subgroups (Theorem 4.5.1). Suppose $n > \ell l(\lambda)$ and set

$$p_\ell \circ s_\lambda^g = \sum_\mu a_{\lambda,\mu}^{g,\ell} s_\mu^g.$$ 

For $g = gl_n$, it is well known, by an algorithm due to Littlewood [12], that the coefficients $a_{\lambda,\mu}^{g,\ell}$ can, up to a sign, be expressed as a sum of products of Littlewood-Richardson coefficients. They are then obtained from the $\ell$-quotient of the partition $\mu$. We give a similar algorithm for computing the coefficients $a_{\lambda,\mu}^{g,\ell}$ when $g = so_{2n+1}$, $sp_{2n}$ or $so_{2n}$. This algorithm was originally introduced in [11] to decompose the plethysms $p_\ell \circ s_\lambda^{so_{2n+1}}$ on the basis of Weyl characters for any integers $n \geq 2$ and $\ell \geq 1$ (that is, with no restrictive conditions on the rank $n$). Although similar procedures also exist for $g = sp_{2n}$ or $so_{2n}$ when $\ell$ is odd, our method failed for the even power sum plethysms on the Weyl characters of type $C_n$ or $D_n$. In the present paper, we show that this difficulty can be overcome by considering stabilized power sum plethysms, i.e. by assuming that $n > \ell l(\lambda)$. Under this hypothesis, one has indeed $a_{\lambda,\mu}^{so_{2n+1},\ell} = a_{\lambda,\mu}^{so_{2n},\ell}$ and $a_{\lambda,\mu}^{sp_{2n},\ell} = (-1)^{|\lambda|}(-1)^{\ell-1}a_{\lambda,\mu}^{so_{2n+1},\ell}$. So it suffices
to consider the coefficients $a^{g_{02n+1}}_{\lambda,\mu}$ for which there exists an algorithm in both cases $\ell$ even and $\ell$ odd.

In Proposition 5.2.1 we use our expression of the coefficients $a^{g_{02}}_{\lambda,\mu}$ as branching coefficients, to derive explicit formulas giving the decompositions of the symmetric and antisymmetric parts of $V^{g_{0}}(\lambda)^{\otimes 2}$ in their irreducible components when $n > 2(\lambda)$. The corresponding multiplicities can then be expressed in terms of the Littlewood-Richardson coefficients and give an alternative to analogous formulas introduced without a complete proof by Littlewood in [14].

The paper is organized as follows. In Section 2, we recall some basics on the representation theory of the classical Lie groups. Section 3 is concerned with plethysms $f \circ s^{g}_{\lambda}$ and their stabilization in large rank. Most of the material of this section can be found in [12], [13], [14] and [15]. In Section 4, we describe the algorithm of [11] which permits to compute the plethysms $p_{\ell} \circ s^{g_{02n+1}}_{\lambda}$ for any positive integer $\ell$. We then state Theorem 4.5.1 Finally, in Section 5, we express the multiplicities $a^{g_{02}}_{\lambda,\mu}$ in terms of the Littlewood-Richardson coefficients.

Acknowledgments: The author wants to thank the anonymous referees for having pointed out some mistakes and inaccuracies in a previous version of this paper. In particular, the stabilization phenomenon explained in Section 4.1.2 emerge now naturally from the algorithms of Section 4.2 and 4.3. This yields to stabilization conditions stated in Theorem 4.4.2.

2 Background on classical Lie groups

2.1 Root systems and Weyl groups

In the sequel $G$ is one of the complex Lie groups $SP_{2n}$, $SO_{2n+1}$ or $SO_{2n}$ and $\mathfrak{g}$ is its Lie algebra. We follow the convention of [9] to realize $G$ as a subgroup of $GL_{n}$ and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_{n}$ where

$$N = \begin{cases}
  n & \text{when } G = GL_{n} \\
  2n & \text{when } G = SP_{2n} \\
  2n+1 & \text{when } G = SO_{2n+1} \\
  2n & \text{when } G = SO_{2n}.
\end{cases}$$

Let $d_{N}$ be the linear subspace of $\mathfrak{gl}_{N}$ consisting of the diagonal matrices. For any $i \in I_{n} = \{1, ..., n\}$, write $\varepsilon_{i}$ for the linear map $\varepsilon_{i} : d_{N} \to \mathbb{C}$ such that $\varepsilon_{i}(D) = \delta_{n-i+1}$ for any diagonal matrix $D$ whose $(i,i)$-coefficient is $\delta_{i}$. Then $(\varepsilon_{1}, ..., \varepsilon_{n})$ is an orthonormal basis of the Euclidean space $h_{R}$ (the real part of $h$). Let $(\cdot,\cdot)$ be the corresponding nondegenerate symmetric bilinear form defined on $h_{R}$.

Write $R$ for the root system associated to $G$. For any $\alpha \in R$ we set $\alpha^{\vee} = \frac{\alpha}{(\alpha,\alpha)}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g} = h \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$. We take for the set of positive roots:

$$R^{+} = \{ \varepsilon_{j} - \varepsilon_{i} \text{ with } 1 \leq i < j \leq n \} \text{ for the root system } A_{n-1}$$

$$R^{+} = \{ \varepsilon_{j} - \varepsilon_{i}, \varepsilon_{j} + \varepsilon_{i} \text{ with } 1 \leq i < j \leq n \} \cup \{ \varepsilon_{i} \text{ with } 1 \leq i \leq n \} \text{ for the root system } B_{n}$$

$$R^{+} = \{ \varepsilon_{j} - \varepsilon_{i}, \varepsilon_{j} + \varepsilon_{i} \text{ with } 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_{i} \text{ with } 1 \leq i \leq n \} \text{ for the root system } C_{n}.$$

$$R^{+} = \{ \varepsilon_{j} - \varepsilon_{i}, \varepsilon_{j} + \varepsilon_{i} \text{ with } 1 \leq i < j \leq n \} \text{ for the root system } D_{n}$$

For any $i \in I_{n}$, we write $\overline{i}$ for $-i$. The Weyl group $W$ of the Lie group $G$ is the subgroup of the permutation group of the set $J_{n} = \{1, ..., \overline{2}, \overline{1}, 1, 2, ..., n\}$ generated by the permutations

$$\begin{cases}
  s_{i} = (i, i+1) \text{ for the root systems } B_{n} \text{ and } C_{n} \\
  s_{i} = (i, i+1) \text{ for the root system } D_{n}.
\end{cases}$$

$$\begin{cases}
  s_{i} = (i, i+1) \text{ for the root systems } B_{n} \text{ and } C_{n} \\
  s_{i} = (i, i+1) \text{ for the root system } D_{n}.
\end{cases}$$
where for \( a \neq b \), \((a, b)\) is the simple transposition which switches \( a \) and \( b \). We identify the subgroup of \( W \) generated by \( s_i = (i, i + 1)(i, i + 1) \), \( i = 1, \ldots, n - 1 \) with the symmetric group \( S_n \). We denote by \( l \) the length function corresponding to the above set of generators. For any \( w \in W \), we set \( \varepsilon(w) = (-1)^{l(w)} \). The action of \( w \in W \) on \( \beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}_R^* \) is defined by

\[
w \cdot (\beta_1, \ldots, \beta_n) = (\beta_1^{w^{-1}}, \ldots, \beta_n^{w^{-1}})\]

where \( \beta_i^w = \beta_{w(i)} \) if \( w(i) \in \{1, \ldots, n\} \) and \( \beta_i^w = -\beta_{w(i)} \) otherwise. We denote by \( \rho \) the half sum of the positive roots of \( R^+ \). For any \( x \in J_n \), we set \( \overline{x} = x \) and \( |x| = x \) if \( x \) is unbarred, \( |x| = \overline{x} \) otherwise.

A partition of length \( m \) is a weakly increasing sequence of \( m \) nonnegative integers. Denote by \( \mathcal{P}_m \) the set of partitions with at most \( m \) parts. Given \( \lambda \in \mathcal{P}_m \), \( \lambda' \) is its conjugate partition and \( l(\lambda) \) the number of nonzero parts in \( \lambda \). Set \( \mathcal{P} = \cup_{m \geq 0} \mathcal{P}_m \). For \( G = \text{Sp}_{2n} \) or \( \text{SO}_{2n+1} \) and \( \lambda \in \mathcal{P}_n \), denote by \( V^\lambda(\gamma) \) the irreducible finite dimensional representation of \( G \) of highest weight \( \lambda \). For \( G = \text{SO}_{2n} \), we define \( V^{\text{so}_{2n}}(\lambda) \) similarly when \( \lambda_1 = 0 \) and we write \( V^{\text{so}_{2n}}(\lambda) \) for the direct sum of the two irreducible representations of highest weights \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) \) and \( \overline{\lambda} = (-\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) \) when \( \lambda_n \neq 0 \). This means that \( V^{\text{so}_{2n}}(\lambda) \) is in fact the irreducible representation of \( O_{2n} \) associated to the partition \( \lambda \).

We shall also need the irreducible rational representations of \( GL_n \). They are indexed by the \( n \)-tuples

\[
(\gamma^-, \gamma^+) = (-\gamma_1^- , \ldots, -\gamma_1^-, \gamma_1^+, \ldots, \gamma_p^+) \tag{1}
\]

where \( \gamma^+ = (\gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+) \) and \( \gamma^- = (\gamma_1^- , \ldots, \gamma_q^-) \) are partitions of length \( p \) and \( q \) such that \( p + q = n \). Write \( \mathcal{P}_n \) for the set of such \( n \)-tuples and denote also by \( V^{\lambda_1}(\gamma) \) the irreducible rational representation of \( gl_n \) of highest weight \( \gamma = (\gamma^-, \gamma^+) \in \mathcal{P}_n \). For any \( \gamma = (\gamma^-, \gamma^+) \in \mathcal{P}_n \), we set \( |\gamma| = \sum \gamma^- + \sum \gamma^+ \).

Write \( s^\lambda_\chi \) for the Weyl character (Schur function) of the finite-dimensional \( gl_n \)-module \( V^{\lambda_1}(\gamma) \) of highest weight \( \lambda \). The character ring of \( GL_n \) is \( \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}} \), the ring of symmetric functions in \( n \) variables.

For any \( \lambda \in \mathcal{P}_n \), we denote by \( s_\lambda^\theta \) the Weyl character of \( V^{\lambda}(\gamma) \). Let \( \mathcal{R}^\theta \) be the \( \mathbb{Z} \)-algebra with basis \( \{ s^\theta_\lambda | \lambda \in \mathcal{P}_n \} \).

Consider \( P \) a parabolic subgroup of \( G \) and \( L \) its Levi subgroup. Write \( I \) for the Levi algebra associated to \( L \). We denote by \( P^+_L \) the set of dominant weights corresponding to \( L \). For any partition \( \lambda \in \mathcal{P}_n \) and \( \gamma \in P^+_L \), write \([V^{\lambda}(\gamma) : V^I(\gamma)]\) for the branching coefficient giving the multiplicity of \( V^I(\gamma) \) (the irreducible representation of \( L \) of highest weight \( \gamma \)) in the restriction of \( V^{\lambda}(\gamma) \) to \( L \).

### 2.2 Universal characters

For each Lie algebra \( g = \mathfrak{so}_N \) or \( \mathfrak{sp}_N \) and any partition \( \nu \in \mathcal{P}_N \), we denote by \( V^{\mathfrak{gl}_N}(\nu) \downarrow^\mathfrak{gl}_N^\mathfrak{so}_N \) the restriction of \( V^{\mathfrak{gl}_N}(\nu) \) to \( g \). Set

\[
V^{\mathfrak{gl}_N}(\nu) \downarrow^\mathfrak{gl}_N^\mathfrak{so}_N = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{so}_N}(\lambda)^{\oplus_{\nu_\lambda}} \quad \text{and} \quad V^{\mathfrak{gl}_N}(\nu) \downarrow^\mathfrak{gl}_N^\mathfrak{sp}_N = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{sp}_N}(\lambda)^{\oplus_{\nu_\lambda}}.
\]
This makes in particular appear the branching coefficients \( b^{g_0 N}_{\nu, \lambda} \) and \( b^{sp_{2n}}_{\nu, \lambda} \). The restriction map \( r^g \) is defined by setting
\[
  r^g : \begin{cases} \mathbb{Z}[x_1, \ldots, x_N]^{sym} \to \mathcal{R}^g \\ s^g_{\nu} \mapsto \text{char}(\mathcal{V}^{g_N}(\nu) \downarrow^g_{\theta}) \end{cases}
\]
We have then
\[
r^g(s^g_{\nu}) = \begin{cases} s^g_{\nu}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) & \text{when } N = 2n \\ s^g_{\nu}(x_1, \ldots, x_n, 1, x_n^{-1}, \ldots, x_1^{-1}) & \text{when } N = 2n + 1 \end{cases}
\]
Let \( \mathcal{P}_n^{(2)} \) and \( \mathcal{P}_n^{(1,1)} \) be the subsets of \( \mathcal{P}_n \) containing the partitions with even length rows and the partitions with even length columns, respectively. When \( \nu \in \mathcal{P}_n \) we have the following formulas for the branching coefficients \( b^{g_0 N}_{\nu, \lambda} \) and \( b^{sp_{2n}}_{\nu, \lambda} \):

**Proposition 2.2.1 (see [13] appendix p 295)**

Consider \( \nu \in \mathcal{P}_n \). Then:

1. \( b^{g_0 N+1}_{\nu, \lambda} = b^{g_0 N}_{\nu, \lambda} = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c'_{\lambda, \gamma} \)
2. \( b^{sp_{2n}}_{\nu, \lambda} = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c'_{\lambda, \gamma} \)

where \( c'_{\lambda, \gamma} \) is the \( n \)-independent multiplicity of \( s^g_{\nu} \) in the Schur functions product \( s^g_{\lambda} s^g_{\gamma} \).

**Remarks:**

(i) : Note that the equality \( b^{g_0 N+1}_{\nu, \lambda} = b^{g_0 N}_{\nu, \lambda} \) becomes false in general when \( \nu \notin \mathcal{P}_n \).

(ii) : By the above proposition we have for any \( \nu \in \mathcal{P}_m \) with \( m \leq n \)

\[
r^g(s^g_{\nu}) = \sum_{\lambda \in \mathcal{P}_m} \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c'_{\lambda, \gamma} r_{sp_{2n}}(s^g_{\nu}) \quad \text{and} \quad r^{g_0 N}(s^g_{\nu}) = \sum_{\lambda \in \mathcal{P}_m} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c'_{\lambda, \gamma} r^{g_0 N}(s^g_{\nu}) \quad (2)
\]

By Proposition 1.5.3 in [8], one has also for any \( \lambda \in \mathcal{P}_m \)

\[
s^{sp_{2n}}_{\lambda} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{|\nu|-|\lambda|} \sum_{\alpha=(\alpha_1 > \cdots > \alpha_s > 0)} c^\lambda_{\nu, \Gamma(\alpha)} r^{sp_{2n}}(s^g_{\nu}) \quad (3)
\]

\[
s^{g_0 N}_{\lambda} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{|\nu|-|\lambda|} \sum_{\alpha=(\alpha_1 > \cdots > \alpha_s > 0)} c^\lambda_{\nu, \Gamma(\alpha)} r^{g_0 N}(s^g_{\nu})
\]

where \( \Gamma(\alpha) = (\alpha_1 - 1, \ldots, \alpha_s - 1 \mid \alpha_1, \ldots, \alpha_s) \) in the Frobenius notation for the partitions. Observe that the coefficients appearing in the decompositions (2) and (3) do not depend on the rank \( n \) considered. Moreover they coincide for the orthogonal types \( B_n \) and \( D_n \).

As suggested by the above decompositions, the manipulation of the Weyl characters is simplified by working with infinitely many variables. In [8], Koike and Terada have introduced a universal character ring for the classical Lie groups. This ring can be regarded as the ring \( \Lambda = \mathbb{Z}[x_1, \ldots, x_n]^{sym} \) of symmetric functions in countably many variables. It is equipped with three natural \( \mathbb{Z} \)-bases indexed by partitions, namely

\[
  \mathcal{B}^{gl} = \{ s^g_{\lambda} \mid \lambda \in \mathcal{P} \}, \quad \mathcal{B}^{sp} = \{ s^{sp}_{\lambda} \mid \lambda \in \mathcal{P} \} \quad \text{and} \quad \mathcal{B}^{so} = \{ s^{so}_{\lambda} \mid \lambda \in \mathcal{P} \}.
\]

(4)
We have then

\[ s^{gl}_{\lambda} = \sum_{\gamma \in \mathcal{P}} \sum_{\nu \in \mathcal{P}(2)} c_{\lambda, \gamma}^{\nu} s^{\lambda}_{\nu} \quad \text{and} \quad s^{sp}_{\lambda} = \sum_{\gamma \in \mathcal{P}} \sum_{\nu \in \mathcal{P}(1,1)} c_{\lambda, \gamma}^{\nu} s^{\lambda}_{\nu} \]

\[ s^{so}_{\lambda} = \sum_{\nu \in \mathcal{P}, |\nu| \leq |\lambda|} (-1)^{|\nu|-|\lambda|} \sum_{\alpha = (\alpha_1 > \cdots > \alpha_s > 0)} c_{\lambda, \Gamma'(\alpha)}^{\nu} s^{gl}_{\nu} \]

\[ s^{sp}_{\lambda} = \sum_{\nu \in \mathcal{P}, |\nu| \leq |\lambda|} (-1)^{|\nu|-|\lambda|} \sum_{\alpha = (\alpha_1 > \cdots > \alpha_s > 0)} c_{\lambda, \Gamma'(\alpha)}^{\nu} s^{gl}_{\nu} \]

In the sequel we will write for short

\[ b^{so}_{\nu, \lambda} = \sum_{\gamma \in \mathcal{P}(2)} c_{\lambda, \gamma}^{\nu}, \quad b^{sp}_{\nu, \lambda} = \sum_{\gamma \in \mathcal{P}(1,1)} c_{\lambda, \gamma}^{\nu}, \quad r^{so}_{\lambda, \nu} = \sum_{\alpha} c_{\nu, \Gamma'(\alpha)}^{\lambda} \quad \text{and} \quad r^{sp}_{\lambda, \nu} = \sum_{\alpha} c_{\nu, \Gamma'(\alpha)}^{\lambda} \]

We denote by \( \omega \) the linear involution defined on \( \Lambda \) by \( \omega(s^{gl}_{\lambda}) = s^{gl}_{\lambda} \). Then we have by Theorem 2.3.2 of [8]

\[ \omega(s^{so}_{\lambda}) = s^{sp}_{\lambda}. \]

Write \( \pi_n : \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}} \rightarrow \mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}} \) for the ring homomorphism obtained by specializing each variable \( x_i, i > n \) at 0. Then \( \pi_n(s^{gl}_{\lambda}) = s^{gl}_{\lambda_n} \). Let \( s^{sp}_{2n} \) and \( s^{so}_{2n} \) be the specialization homomorphisms defined by setting \( s^{sp}_{2n} = c_{\lambda, \Gamma'(\alpha_1)}^{\nu} \) and \( s^{so}_{2n} = r^{so}_{\lambda, \nu} \). For any partition \( \lambda \in \mathcal{P}_n \) one has \( s^{sp}_{2n} = s^{sp}_{2n}(s^{sp}_{\lambda}) \) and \( s^{so}_{2n} = s^{so}_{2n}(s^{so}_{\lambda}) \). We shall also need the following proposition (see [7] and [8]).

**Proposition 2.2.2** Consider a Lie algebra \( \mathfrak{g} \) of type \( X_n \in \{ B_n, C_n, D_n \} \). Let \( \lambda \in \mathcal{P}_r \) and \( \mu \in \mathcal{P}_s \). Suppose \( n \geq r + s \) and set

\[ V^{\mathfrak{g}}(\lambda) \otimes V^{\mathfrak{g}}(\mu) = \bigoplus_{\nu \in \mathcal{P}_n} V^{\mathfrak{g}}(\nu)^{\otimes d^{\nu}_{\lambda, \mu}}. \]

Then the coefficients \( d^{\nu}_{\lambda, \mu} \) neither depend on the rank \( n \) of \( \mathfrak{g} \) nor on its type \( B, C \) or \( D \). More we have

\[ d^{\nu}_{\lambda, \mu} = \sum_{\xi, \sigma} c_{\xi, \sigma}^{\nu} c_{\xi, \sigma}^{\nu} \]

**Remarks:**

(i) : The previous proposition implies the decompositions \( s^{sp}_{\lambda} \otimes s^{sp}_{\mu} = \sum_{\nu \in \mathcal{P}} d^{\nu}_{\lambda, \mu} s^{sp}_{\nu} \) and \( s^{so}_{\lambda} \otimes s^{so}_{\mu} = \sum_{\nu \in \mathcal{P}} d^{\nu}_{\lambda, \mu} s^{so}_{\nu} \) for any \( \lambda, \mu \in \mathcal{P} \), in the ring \( \Lambda \).

(ii) : The analogous result for \( \mathfrak{g} = \mathfrak{gl}_n \) is well-known: the outer multiplicities \( c_{\xi, \sigma}^{\nu} \) appearing in the decomposition of \( V^{\mathfrak{gl}_n}(\lambda) \otimes V^{\mathfrak{gl}_n}(\mu) \) do not depend on \( n \) provided \( n \geq r + s \).

### 3 Plethysms and stabilized plethysms

#### 3.1 Plethysms on the Weyl characters

Consider \( f \in \Lambda \) and \( s^\mathfrak{g}_\lambda \) the Weyl character for \( \mathfrak{g} \) associated to \( \lambda \in \mathcal{P}_n \). Set \( s^\mathfrak{g}_\lambda = \sum_{\beta \in \mathbb{Z}^n} a_\beta x^\beta. \) As in the case of ordinary plethysms on symmetric functions (see [15] p 135), one defines the set of variables \( y_i \) such that

\[ \prod_i (1 + ty_i) = \prod_\beta (1 + tx^\beta)^{a_\beta}. \]
Then the plethysm of the Weyl character $s_\lambda^g$ by the symmetric function $f$ is defined by $f \circ s_\lambda^g = f(y_1, y_2, \ldots)$. In the sequel, we will focus on the power sum plethysms $\psi_\ell$ where $\ell$ is a positive integer. They are defined from the identity $\psi_\ell(s_\lambda^g) = p_\ell \circ s_\lambda^g = s_\lambda^g(x_1^\ell, \ldots, x_n^\ell)$. In particular, the map $\psi_\ell$ is linear on $R^g$. The characters of the symmetric and antisymmetric parts of $V^g(\lambda) \otimes 2$ can be expressed as plethysms by the complete and elementary symmetric functions $h_2$ and $e_2$. More precisely we have

$$h_2 \circ s_\lambda^g = \text{char}(S^2(V^g(\lambda)))$$
and $e_2 \circ s_\lambda^g = \text{char}(\Lambda^2(V^g(\lambda)))$.

From the identities $h_2 = \frac{1}{2}(c_1^2 + p_2)$ and $e_2 = \frac{1}{2}(c_1^2 - p_2)$, we derive the relations

$$h_2 \circ s_\lambda^g = \frac{1}{2}((s_\lambda^g)^2 + \psi_2(s_\lambda^g))$$
and $e_2 \circ s_\lambda^g = \frac{1}{2}((s_\lambda^g)^2 - \psi_2(s_\lambda^g))$.  

### 3.2 Stabilized plethysms on the Schur functions

Given $(\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ a $\ell$-tuple of partitions, we write $c^{(\ell)}_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})}$ for the $n$-independent coefficient of $s_{\mu^{(0)}}^g \cdot \cdots \cdot s_{\mu^{(\ell-1)}}^g$ in the product $s_{\lambda_1}^g \ldots s_{\lambda_n}^g$. For any partition $\lambda \in \mathcal{P}_n$, the plethysm $\psi_\ell(s_{\lambda_n}^g)$ decomposes on the basis of Schur functions on the form

$$\psi_\ell(s_{\lambda_n}^g) = \sum_{|\mu|=|\lambda|} \epsilon(\mu)c^{(\ell)}_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})} s_{\mu}^g.$$  

Here $\epsilon(\mu) \in \{-1, 0, 1\}$ and $\mu/\ell = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ are respectively the $\ell$-sign and the $\ell$-quotient of the partition $\mu$. We now briefly recall the algorithm which permits to obtain the sign $\epsilon(\mu)$ and the $\ell$-tuple of partitions $\mu/\ell$. Our description slightly differs from that which can be usually found in the literature (see [15] Example 8 p 12). This is because we have made our notation consistent with Section 4.

Set $\rho_n = (1, 2, \ldots, n)$ and $I_n = \{1, 2, \ldots, n\}$. For any $k \in \{0, \ldots, \ell - 1\}$ consider the sequences

$$I^{(k)} = \{i \in I_n \mid \mu_i + i \equiv k \mod \ell\} \text{ and } J^{(k)} = \{i \in I_n \mid i \equiv k \mod \ell\}$$
in which the entries occur in the increasing order. Set $r_k = \text{card}(I^{(k)})$ and write $I^{(k)} = (i_1^{(k)}, \ldots, i_{r_k}^{(k)})$.

1. If there exists $k \in \{0, \ldots, \ell - 1\}$ such that $\text{card}(I^{(k)}) \neq \text{card}(J^{(k)})$ then $\epsilon(\mu) = 0$.

2. Otherwise let $\sigma_0 \in S_n$ be the permutation mapping $I^{(k)}$ to $J^{(k)}$ for any $k = 0, \ldots, \ell - 1$. Then we have $\epsilon(\mu) = \epsilon(\sigma_0)$ and $\mu/\ell = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ where for any $k \in \{0, \ldots, \ell - 1\}$

$$\mu^{(k)} = \left( \frac{\mu_i + i + \ell - k}{\ell} \right) \mid i \in I^{(k)} \right) - (1, 2, \ldots, r_k) \in \mathbb{Z}^r_k$$

(see example below).

**Remark:** Set $n = q\ell + r$ where $q$ and $r$ are respectively the quotient and the rest of the division of $n$ by $\ell$. Then we have $\text{card}(J^{(k)}) = q + 1$ for any $k \in \{1, \ldots, r\}$ and $\text{card}(J^{(k)}) = q$ for any $k \in \{0, r + 1, \ldots, n\}$. Hence in (12), we have $r_k = q + 1$ for any $k \in \{1, \ldots, r\}$ and $r_k = q$ for any $k \in \{0, r + 1, \ldots, n\}$ (see also Remark just before Section 4.2).
Example 3.2.1
Consider $\mu = (1, 2, 3, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho_3 = (2, 4, 6, 8, 9, 10, 13, 14)$. Thus $I^{(0)} = (3, 5), I^{(1)} = (2, 6, 7), I^{(2)} = (1, 4, 8)$ and $J^{(0)} = (3, 6), J^{(1)} = (1, 4, 7), J^{(2)} = (2, 5, 8)$. Then $\mu^{(0)} = (1, 1), \mu^{(1)} = (1, 2, 2)$ and $\mu^{(2)} = (0, 1, 2)$. Moreover
\[\sigma_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 6 & 4 & 7 & 8 \end{pmatrix}.\]
Hence $\varepsilon(\mu) = -1$.

Proposition 3.2.2 Consider $\mu \in \mathcal{P}_n$ such that $\varepsilon(\mu) \neq 0$ and set $\mu/\ell = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$. Let $\nu \in \mathcal{P}_{n+1}$ be the partition obtained by adding in $\mu$ a part 0. Then $\varepsilon(\nu) = \varepsilon(\mu)$ and we have $\nu/\ell = (\nu^{(0)}, \ldots, \nu^{(\ell-1)})$ where $\nu^{(0)} = \mu^{(\ell-1)}, \nu^{(k)} = \mu^{(k-1)}$ for any $k \in \{2, \ldots, \ell - 1\}$ and $\nu^{(1)} = (0, \mu^{(0)})$ is obtained by adding a part 0 in $\mu^{(0)}$.

**Proof.** Let us slightly abuse the notation and write $I^{(k)}(\mu), J^{(k)}(\mu), I^{(k)}(\nu), J^{(k)}(\nu), k = 0, \ldots, \ell - 1$ for the sequences defined from $\mu$ and $\nu$ by applying the previous procedure. Then, we have
\[\begin{cases} I^{(1)}(\nu) = \{1\} \cup (I^{(0)}(\mu) + 1), & I^{(0)}(\nu) = (I^{(\ell-1)}(\mu) + 1) \\ I^{(k)}(\nu) = (I^{(k-1)}(\mu) + 1) & \text{for } k = 2, \ldots, \ell - 1. \end{cases}\]

(13)
Here by $(I^{(k-1)}(\mu) + 1)$, we mean the sequence obtained by adding 1 to the entries of $I^{(k-1)}(\mu)$. Set $n = q\ell + r$ as in the previous remark. We will assume that $r \neq \ell - 1$ so that $q$ and $r + 1$ are respectively the quotient and the rest of the division of $n + 1$ by $\ell$. The case $r = \ell - 1$ is similar. We have then $\text{card}(J^{(k)}(\mu)) = \text{card}(I^{(k)}(\mu)) + q + 1$ for $k \in \{1, \ldots, r\}$ and $\text{card}(J^{(k)}(\mu)) = \text{card}(I^{(k)}(\mu)) = q$ for $k \in \{0, r + 1, \ldots, n\}$. Now observe that $J^{(k)}(\nu) = J^{(k)}(\mu)$ for $k \neq r + 1$ and $J^{(r+1)}(\nu) = J^{(r+1)}(\mu) \cup \{n + 1\}$. This implies that $\text{card}(J^{(k)}(\nu)) = \text{card}(I^{(k)}(\nu)) = q + 1$ for $k \in \{1, \ldots, r + 1\}$ and $\text{card}(J^{(k)}(\mu)) = \text{card}(I^{(k)}(\mu)) = q$ for $k \in \{0, r + 2, \ldots, n\}$. Thus $\varepsilon(\nu) = \varepsilon(\mu)$.

We then easily deduce $\nu^{(0)}, \ldots, \nu^{(\ell-1)}$ from (12) and (13). ■

Remarks:
(i) : The decomposition (11) do not depend on the rank $n$ considered provided $n > \ell l(\lambda)$. Indeed, by Proposition 3.2.2 $\varepsilon(\mu)$ and the non-zero parts of the partitions $\mu^{(k)}$ of the above algorithm are not modified when empty parts are added to $\mu$.

(ii) : When $n > \ell l(\lambda)$, we write for short $a_{\lambda, \mu}^{\ell, gl} = \varepsilon(\mu) c_{\mu^{(0)}, \ldots, \mu^{(\ell - 1)}}^\lambda$. Then $a_{\lambda, \mu}^{\ell, gl} \neq 0$ only if $|\mu| = \ell |\lambda|$.

Proposition 3.2.3 Consider $f \in \Lambda$ with degree $d$ and $\lambda \in \mathcal{P}_n$. Then the coefficients of the expansion of $f \circ s^d_{\lambda}$ on the basis of Schur functions do not depend on $n$ provided $n > d l(\lambda)$.

**Proof.** By (11) and the previous remark, the proposition is true for the power sum plethysms $p_\ell \circ s^d_{\lambda}$. The map $g \mapsto g \circ s^d_{\lambda}$ is a ring homomorphism of $\Lambda_n$. The subspace $\Lambda_n^d$ of polynomials in $\Lambda_n$ with degree $d$ is generated by the Newton polynomials $p_\beta = p_{\beta_1} \cdots p_{\beta_k}$, such that $\beta_i \in \mathbb{N}$ and $\beta_1 + \cdots + \beta_k = d$. So it suffices to prove the proposition for $f = p_\beta$. We have $p_\beta \circ s^d_{\lambda} = p_{\beta_1} \circ s^d_{\lambda} \times \cdots \times p_{\beta_k} \circ s^d_{\lambda}$. Suppose $n > d l(\lambda)$. For any $i = 1, \ldots, k$, we have $n \geq \beta_i l(\lambda)$. Thus we can write $p_{\beta_i} \circ s^d_{\lambda} = \sum_{\mu^{(i)} \in \mathcal{P}_n} a_{\lambda, \mu^{(i)}}^{\beta_i, gl} s^d_{\mu^{(i)}}$. Moreover $a_{\lambda, \mu^{(i)}}^{\beta_i, gl} \neq 0$ only if $|\mu^{(i)}| = \beta_i |\lambda|$. By Remark (ii) following Proposition 2.2.2 we obtain that the coefficients of the decomposition of $p_\beta \circ s^d_{\lambda}$ on the basis of Schur functions do not depend on $n$ when $n \geq |\beta| l(\lambda)$. ■
3.3 Stabilized plethysms on the Weyl characters

Lemma 3.3.1 Consider $\lambda \in \mathcal{P}_m$, $\ell$ a positive integer and $\mathfrak{g}$ an orthogonal or symplectic Lie algebra with rank $n \geq m$.

- The coefficients of the expansion of the plethysm $p_\ell \circ s_\lambda^\mathfrak{g}$ on the basis of Weyl characters do not depend on $n$ provided $n > \ell \lambda$.
- In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.
- For any $n > \ell \lambda$, set

$$p_\ell \circ s_\lambda^{\mathfrak{so}} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\mathfrak{so}} s_\mu^{\mathfrak{so}}$$

and

$$p_\ell \circ s_\lambda^{\mathfrak{sp}} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\mathfrak{sp}} s_\mu^{\mathfrak{sp}}.$$ 

We have

$$a_{\lambda,\mu}^{\mathfrak{so}} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{|\lambda| - |\nu|} \sum_{\delta \in \mathcal{P}, |\delta| = \ell \nu} r_{\lambda,\nu}^{\mathfrak{so}} a_{\nu,\delta}^{\ell \mathfrak{gl}} b_{\delta,\mu}^{\mathfrak{so}},$$

and

$$a_{\lambda,\mu}^{\mathfrak{sp}} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{|\lambda| - |\nu|} \sum_{\delta \in \mathcal{P}, |\delta| = \ell \nu} r_{\lambda,\nu}^{\mathfrak{sp}} a_{\nu,\delta}^{\ell \mathfrak{gl}} b_{\delta,\mu}^{\mathfrak{sp}}.$$

Proof. We have $n > \ell \lambda$. Hence, the decomposition

$$s_\lambda^{\mathfrak{so}} = \sum_{\mu \in \mathcal{P}_m, |\mu| \leq |\lambda|} (-1)^{|\lambda| - |\mu|} r_{\lambda,\mu}^{\mathfrak{so}} s_\mu^{\mathfrak{so}} (s_\mu^{\mathfrak{gl}})$$

holds. Since $\psi_\ell$ and $r^{\mathfrak{so}}$ commute, this gives

$$p_\ell \circ s_\lambda^{\mathfrak{so}} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{|\lambda| - |\nu|} \sum_{\delta \in \mathcal{P}, |\delta| = \ell \nu} r_{\lambda,\nu}^{\mathfrak{so}} a_{\nu,\delta}^{\ell \mathfrak{gl}} s_\delta^{\mathfrak{so}} s_\mu^{\mathfrak{so}}.$$

This yields the desired expression for the coefficients $a_{\lambda,\mu}^{\mathfrak{so}}$. In particular they do not depend on $n$ and coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.

Proposition 3.3.2 Consider $f \in \Lambda$ with degree $d$ and $\lambda \in \mathcal{P}_n$. Then the coefficients of the expansion of $f \circ s_\lambda^\mathfrak{g}$ on the basis of Schur functions do not depend on $n$ provided $n \geq d|\lambda|$. In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.

Proof. The proposition follows from Lemma 3.3.1 by similar arguments to those of Proposition 3.2.3.

According to the previous Lemma, we have the decompositions

$$p_\ell \circ s_\lambda^{\mathfrak{gl}} = \sum_{\mu} a_{\lambda,\mu}^{\ell \mathfrak{gl}} s_\mu^{\mathfrak{gl}},$$

$$p_\ell \circ s_\lambda^{\mathfrak{sp}} = \sum_{\mu} a_{\lambda,\mu}^{\ell \mathfrak{sp}} s_\mu^{\mathfrak{sp}}$$

and

$$p_\ell \circ s_\lambda^{\mathfrak{so}} = \sum_{\mu} a_{\lambda,\mu}^{\ell \mathfrak{so}} s_\mu^{\mathfrak{so}}.$$
• \( \omega(f \circ s^g_\lambda) = f \circ \omega(s^g_\lambda) \) if \( |\lambda| \) is even,
• \( \omega(f \circ s^g_\lambda) = \omega(f) \circ \omega(s^g_\lambda) \) if \( |\lambda| \) is odd.

**Proof.** From Example 1 page 136 of [15] we have for any positive integer \( \ell \), \( \omega(p_\ell \circ g) = p_\ell \circ \omega(g) \)
if \( g \) is homogeneous of even degree and \( \omega(p_\ell \circ g) = \omega(p_\ell) \circ \omega(g) \) if \( g \) is homogeneous of odd degree.
Since \( \psi_\ell \) is linear, this shows that \( \omega(p_\ell \circ s^g_\lambda) = p_\ell \circ \omega(s^g_\lambda) \) if \( |\lambda| \) is even and \( \omega(p_\ell \circ s^g_\lambda) = \omega(p_\ell) \circ \omega(s^g_\lambda) \)
if \( |\lambda| \) is odd. Indeed, according to (6), \( s^g_\lambda \) is a sum of homogeneous functions of degrees equal to \( |\lambda| \)
modulo 2. The Lemma then follows since the maps \( \omega \) and \( f \mapsto f \circ s^g_\lambda \) are ring homomorphisms of \( \Lambda \).

**Remarks:**
(i) : Since \( \omega(p_\ell) = (-1)^{\ell-1} p_\ell \), one has by the previous lemma \( a^{\ell,sp}_{\lambda,\mu} = a^{\ell,so}_{\lambda,\mu} \) if \( |\lambda| \) is even and
\( a^{\ell,sp}_{\lambda,\mu} = (-1)^{\ell-1} a^{\ell,so}_{\lambda,\mu} \) otherwise. This can also be verified by using the explicit formulas of Lemma 3.3.1.
(ii) : The coefficients \( a^{\ell,so}_{\lambda,\mu} \) are rather complicated to compute by using formulas of Lemma 3.3.1. We
are going to see in the following Section that they coincide with branching coefficients corresponding
to restriction to certain Levi subalgebras.

## 4 Power sum plethysms for Weyl characters of type \( B_n \)

### 4.1 Statement of the theorem

In Theorem 3.2.8 of [11], we have described an algorithm for computing the plethysms \( p_\ell \circ s^{\text{so}_{2n+1}}_\lambda \)
for any positive integer \( \ell \) and any rank \( n \). It notably permits to show that the decomposition of
\( p_\ell \circ s^{\text{so}_{2n+1}}_\lambda \) on the basis of Weyl characters makes appear branching coefficients corresponding
to the restriction to a Levi subgroup of \( \text{so}_{2n+1} \). Surprisingly, similar algorithms for \( \text{sp}_{2n} \) and \( \text{so}_{2n} \) only exists when \( \ell \) is odd. In particular, the coefficients of the decomposition of \( p_\ell \circ s^{\text{sp}_{2n}}_\lambda \) and \( p_\ell \circ s^{\text{so}_{2n}}_\lambda \)
on the basis of Weyl characters are not branching coefficients in general when \( \ell \) is even. As we are going to see, this is nevertheless the case for the stabilized forms of these plethysms.

Theorems 3.2.8 and 3.2.10 of [11] can be reformulated as follows:

**Theorem 4.1.1** For any partition \( \lambda \in \mathcal{P}_n \) and any positive integer \( \ell \) we have

\[
 p_\ell \circ s^{\text{so}_{2n+1}}_\lambda = \sum_{\mu \in \mathcal{P}_n} \epsilon(\mu)[V^{\text{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]s^{\text{so}_{2n+1}}_\mu
\]

where

• \( \epsilon(\mu) \in \{-1,0,1\} \),
• \( \mathfrak{g}_{\ell,\mu} \) is the Levi algebra of \( G_{\ell,\mu} \), a Levi subgroup of \( \text{SO}_{2n+1} \),
• \( \gamma_{\ell,\mu} \) is a dominant weight for \( G_{\ell,\mu} \).

Moreover, \( \epsilon(\mu), G_{\ell,\mu} \) and \( \gamma_{\ell,\mu} \) are determined from \( \mu \) and \( \ell \) by an algorithm which can be regarded
as an analogue in type \( B_n \) of the computation of the \( \ell \)-quotient \( \mu/\ell \).
We now recall the algorithm which permits to determinate \( \epsilon(\mu), G_{\ell,\mu} \) and \( \gamma_{\ell,\mu} \) in the above theorem. Set
\[
J_n = \{\overline{1}, \ldots, \overline{1}, 1, \ldots, n\} \quad \text{and} \quad L_n = \{\overline{n-\overline{1}}, \ldots, \overline{1}, 0, 1, \ldots, n\}.
\]
Let \( \eta \) be the bijection from \( J_n \) to \( L_n \) defined by \( \eta(x) = x + 1 \) if \( x < 0 \) and \( \eta(x) = x \) otherwise. For each element \( w \in W \) (the Weyl group of \( \mathfrak{so}_{2n+1} \)), denote by \( \tilde{w} \) the bijection from \( J_n \) to \( L_n \) defined by \( \tilde{w} = \eta \circ w \). This means that \( \tilde{w}(x) = w(x) \) if \( w(x) > 0 \) and \( \tilde{w}(x) = w(x) + 1 \) if \( w(x) < 0 \). In particular \( w \) is determined by \( \tilde{w} \). For any \( x \in L_n \), set \( x^* = \overline{x} + 1 \). The map \( x \mapsto x^* \) is involutive from \( L_n \) to itself. Since \( w(\overline{x}) = \tilde{w}(x)^* \), we have also
\[
\tilde{w}(\overline{x}) = \tilde{w}(x)^*.
\]
Hence, \( \tilde{w} \) is determined by the images of any subset \( U_n \subset J_n \) such that \( \text{card}(U_n) = n \) and \( x \in U_n \) implies \( \overline{x} \not\in U_n \).

For any \( k = 1, \ldots, \ell \) set
\[
I^{(k)}(x) = (i \in I_n \mid \mu_i + i \equiv k \text{ mod } \ell) \quad \text{and} \quad J^{(k)} = (x \in L_n \mid x \equiv k \text{ mod } \ell).
\]
Note that \( (J^{(k)})^* = J^{(\ell-k+1)} \).

**Remark:** Set \( n = q\ell + r \) where \( q \) and \( r \) are respectively the quotient and the rest of the division of \( n \) by \( \ell \). Then we have
\[
\text{card}(J^{(k)}) = \begin{cases} 
2q & \text{for } \min(r+1, \ell-r-1) \leq k \leq \max(r+1, \ell-r-1) \\
2q + 1 & \text{otherwise}
\end{cases} \quad \text{when } r \neq \frac{\ell}{2}.
\]
\[
\text{card}(J^{(k)}) = 2q + 1 \quad \text{for any } k \in \{1, \ldots, \ell\} \quad \text{when } r = \frac{\ell}{2}.
\]

### 4.2 The even case \( \ell = 2p \)

For any \( k = 1, \ldots, p \), set \( s_k = \text{card}(I^{(k)}) \), \( r_k = \text{card}(I^{(k)}) + \text{card}(I^{(\ell-k+1)}) \) and define \( X^{(k)} \) as the increasing reordering of \( I^{(k)} \cup I^{(\ell-k+1)} \). Set
\[
X^{(k)} = (i^{(k)}_{\overline{1}}, \ldots, i^{(k)}_{r_k}).
\]

1. If there exists \( k \in \{1, \ldots, p\} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \epsilon(\mu) = 0 \).

2. Otherwise we have \( \text{card}(J^{(\ell-k+1)}) = \text{card}(J^{(k)}) = r_k \) since \( (J^{(k)})^* = J^{(\ell-k+1)} \). Let \( w_0 \) be the unique element of \( W \) mapping \( X^{(k)} \) to \( J^{(\ell-k+1)} \) for any \( k = 1, \ldots, p \). Define \( \alpha_k = \frac{1}{\ell}(\max J^{(k)} - k) \). For any \( k = 1, \ldots, p \), consider \( \mu^{(k)}(x) \in \overline{P}_{r_k} \) defined by
\[
\mu^{(k)} = \left( \text{sign}(i) \frac{\mu_i |i| + |i| + \text{sign}(i)k - \frac{1 + \text{sign}(i)}{2}}{\ell} \mid i \in X^{(k)} \right) - (1, \ldots, r_k) + (\alpha_k + 1, \ldots, \alpha_k + 1).
\]
Remark: With \( q \) and \( r \) as in (17), we can write
\[
\alpha_k = \begin{cases} 
q - 1 \text{ for } \min(r + 1, \ell - r - 1) \leq k \leq \max(r + 1, \ell - r - 1) \\
q \text{ otherwise}
\end{cases}
\text{ when } r \neq \frac{\ell}{2},
\]
\[
\alpha_k = q \text{ for any } k \in \{1, \ldots, \ell\} \text{ when } r = \frac{\ell}{2}.
\]
Note also that in step 2, \( r_k \in \{2q, 2q + 1\} \) according to (17).

We have then with the above notation:
\[
\varepsilon(\mu) = \varepsilon(w_0), \quad G_{\ell, \mu} = GL_{r_1} \times \cdots \times GL_{r_p} \quad \text{and} \quad \gamma_{\ell, \mu} = (\mu^{(1)}, \ldots, \mu^{(p)}) \in P_{G_{\ell, \mu}}^+.
\]

Example 4.2.1 Put \( n = 6, \ell = 2 \) (thus \( p = 1 \)) and consider \( \mu = (2, 5, 6, 7, 9) \). Then \( \mu + \rho_6 = (3, 7, 8, 10, 12, 15) \). Hence \( I^{(2)} = (3, 4, 5) \) and \( I^{(1)} = (1, 2, 6) \). Moreover \( J^{(1)} = (\overline{5, 3, 1}, 1, 3, 5) \) and \( J^{(2)} = (\overline{7, 2, 1}, 0, 2, 4, 6) \). Then \( \tilde{w}_0 \) sends \( X^{(1)} = (\overline{5, 3, 1}, 3, 4, 5) \) on \( J^{(2)} \). This gives
\[
\tilde{w}_0 = \left(\begin{array}{cccccccc}
\overline{5} & \overline{5} & \overline{7} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\
\overline{4} & \overline{5} & \overline{3} & \overline{1} & \overline{2} & 0 & 1 & 3 & 2 & 4 & 6 & 5
\end{array}\right).
\]
by using (15). Hence
\[
w_0 = \left(\begin{array}{cccccccc}
\overline{5} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\
\overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 3 & 2 & 4 & 6 & 5
\end{array}\right).
\]
We have \( \varepsilon(\mu) = 1, \alpha_1 = 2 \) and \( \gamma_{\ell, \mu} = (\mu^{(1)}) \) where
\[
\mu^{(1)} = (-7, -3, -1, 4, 5, 6) - (1, 2, 3, 4, 5, 6) + (3, 3, 3, 3, 3, 3) = (-5, -2, -1, 3, 3, 3).
\]
Observe that \( G_{\ell, \mu} \simeq GL_6 \).

4.3 The odd case \( \ell = 2p + 1 \)

In addition to the sets \( X^{(k)}, k = 1, \ldots, p \) defined in (18), we have also to consider \( I^{(p+1)} \). Set \( r_{p+1} = \text{card}(I^{(p+1)}) \) and write \( I^{(p+1)} = \{i_1^{(p+1)}, \ldots, i_{r_{p+1}}^{(p+1)}\} \). Observe that \( (J^{(p+1)})^* = J^{(p+1)} \). Let \( X^{(p+1)} \) be the increasing reordering of \( I^{(p+1)} \) \( \cup J^{(p+1)} \).

1. If \( \text{card}(I^{(p+1)}) \neq \frac{1}{2} \text{card}(J^{(p+1)}) \) or if there exists \( k \in \{1, \ldots, p\} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \varepsilon(\mu) = 0 \).

2. Otherwise, we have \( \text{card}(J^{(p+1)}) = 2\text{card}(I^{(p+1)}) = 2r_{p+1} \). Let \( w_0 \) be the unique element of \( W \) mapping \( X^{(k)} \) to \( J^{(l-k+1)} \) for any \( k = 1, \ldots, p \) and \( X^{(p+1)} \) to \( J^{(p+1)} \). Define
\[
\mu^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \mid i \in I^{(p+1)}\right) - (1, \ldots, r_{p+1}) \in P_{r_{p+1}}^+
\]
and for any \( k = 1, \ldots, p, \mu^{(k)} \) as in the even case. Set \( \mathcal{I} = \{I^{(p+1)}, X^{(1)}, \ldots, X^{(p)}\} \). We have then with the above notation
\[
\varepsilon(\mu) = \varepsilon(w_0), \quad G_{\ell, \mu} = GL_{r_1} \times \cdots \times GL_{r_p} \times GL_{2r_{p+1} + 1} \quad \text{and} \quad \gamma_{\ell, \mu} = (\mu^{(p+1)}, \mu^{(1)}, \ldots, \mu^{(p)}) \in P_{G_{\ell, \mu}}^+.
\]
Remark: With \( q \) and \( r \) as in (17), we have \( r_{p+1} = \frac{1}{2} \text{card}(J^{(p+1)}) = q \) when 1 is satisfied.

**Example 4.3.1** Put \( n = 6, \ell = 3 \) (thus \( p = 1 \)) and consider \( \mu = (1, 5, 5, 6, 7, 9) \). We have \( \mu + \rho_0 = (2, 7, 8, 10, 12, 15) \). Thus \( X^{(1)} = (4, 5, 6), I^{(1)} = (2, 4), J^{(2)} = (1, 3) \) and \( J^{(3)} = (5, 1, 4), J^{(2)} = (4, 2, 5) \) and \( J^{(3)} = (3, 0, 3, 6) \). In particular \( \alpha_1 = 1 \). Then

\[
\mu^{(1)} = \left( -\frac{10}{3}, -1 + 2, -\frac{7}{3} - 2 + 2, \frac{12}{3} - 3 + 2, \frac{15}{3} - 4 + 2 \right) = (-2, -2, 3, 3)
\]

and \( \mu^{(2)} = (\frac{2+1}{3} - 1, \frac{8+1}{3} - 2) = (0, 1) \). Moreover, one has by using (15)

\[
\bar{w}_0 = \begin{pmatrix}
\bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 2 & 3 & 4 & 5 & 6 \\
\bar{5} & \bar{2} & \bar{3} & \bar{4} & 0 & \bar{1} & 2 & 1 & 5 & 4 & 3 & 6
\end{pmatrix}
\]

Hence

\[
w_0 = \begin{pmatrix}
\bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 2 & 3 & 4 & 5 & 6 \\
\bar{6} & \bar{3} & \bar{4} & \bar{5} & \bar{2} & \bar{2} & 2 & 1 & 5 & 4 & 3 & 6
\end{pmatrix}
\]

and \( \varepsilon(\mu) = 1 \). We have \( G_{\ell, \mu} \simeq GL_4 \times SO_5 \).

### 4.4 The stabilization phenomenon

We begin this paragraph with further remarks:

**Remarks:**

(i) Suppose \( \varepsilon(\mu) \neq 0 \). In the even case, we have \( G_{\ell, \mu} = GL_{r_1} \times \cdots \times GL_{r_p} \). In the odd case and \( \ell \geq p + 1 \), \( G_{\ell, \mu} \) is not a direct product of linear groups since \( G_{\ell, \mu} = GL_{r_1} \times \cdots \times GL_{r_p} \times SO_{2r_{p+1}} \).

(ii) When \( \ell = 2 \), we always \( \text{card}(X^{(1)}) = n = \text{card}(J^{(2)}) \). Hence \( \varepsilon(\mu) \neq 0 \) for all partitions \( \mu \). Observe that it does not mean that the expansion (14) is infinite. In fact most of the branching coefficients \( [V^g_{02n+1}(\lambda) : V^g_{\ell, \mu}(\gamma_{\ell, \mu})] \) vanishes in this situation. Note also that we have always \( G_{\ell, \mu} \simeq GL_n \) in this case.

(iii) We have seen that the non-zero parts of the \( \ell \)-quotient \( \mu/\ell \) does not depend on the number of zero parts in \( \mu \) (see Proposition 3.2.2). This notably implies the stability of the coefficients \( a_{\lambda, \mu}^{g_{0, \mu}} \). The situation is more subtle for the coefficients \( a_{\lambda, \mu}^{g_{0, g}} \). Indeed, the dominant weights \( \gamma_{\ell, \mu} \) given by the previous algorithm do not stabilize in general when the number of zero parts in \( \mu \) increases. Let us consider for example \( \mu = (1, 5, 6, 9) \) and \( \ell = 2 \). By adding parts 0 to \( \mu \), we obtain successively for the dominant weights

\[
(-1, 2, 4, 4, 5), \quad (-5, -4, -4, -2, -2, 1), \quad (-1, 2, 2, 2, 4, 4, 5) \quad \text{etc.} \tag{20}
\]

This is not incompatible with Proposition 3.3.2 which asserts that \( \psi_{\ell}(s_{\mu}^{g_{02n+1}}) \) stabilizes in large rank. In fact, this only means that, when no assumption is made on the size of \( n \), there can exist non zero coefficients \( a_{\lambda, \mu}^{g_{0, g_{02n+1}}(\mu)} \) in the decomposition

\[
\psi_{\ell}(s_{\mu}^{g_{02n+1}}) = \sum_{\mu \in \mathcal{P}_n} a_{\lambda, \mu}^{g_{0, g_{02n+1}}(\mu)} s_{\lambda}^{g_{02n+1}}
\]

such that \( a_{\lambda, \mu}^{g_{0, g_{02n+1}}(\mu)} = 0 \). This is because the coefficients \( a_{\lambda, \mu}^{g_{0, g_{02n+1}}(\mu)} \) coincide for \( s_{\mu}^{g_{02n+1}} \) and \( s_{\mu}^{g_{02n}} \) in large rank whereas \( \psi_{\ell}(s_{\mu}^{g_{02n+1}}) \neq \psi_{\ell}(s_{\mu}^{g_{02n}}) \) in general when no assumption is made on the size of \( n \). In the rest
of this paragraph, we are going to see that the dominant weights $\mu$ for which $\gamma_{\ell,\mu}$ do not stabilize are such that $a_{\lambda,\mu}^{\ell,\nu_0} = 0$, that is their contribution to $\psi_\ell(s^{\nu_0}_{2n+1})$ vanishes in large rank. Moreover, we are going to characterize precisely these weights.

Suppose first $\ell = 2p$ is even. Consider $\mu \in \mathcal{P}_m$ such that $\varepsilon(\mu) \neq 0$. Set $\gamma_{\ell,\mu} = (\mu^{(1)}, \ldots, \mu^{(p)})$. Write $\nu$ for the partition of $\mathcal{P}_{m+\ell}$ obtained by adding $\ell$ parts 0 in $\mu$. For any $k \in \{1, \ldots, p\}$, Set $\mu^{(k)} = (\mu^{(k)}_{-}, \mu^{(k)}_{+})$ where $\mu^{(k)}_{-}$ (resp. $\mu^{(k)}_{+}$) is the sequence formed by the $s_k$ leftmost (resp. $r_k - s_k$ rightmost) components of $\mu^{(k)}$ (see Section 4.2 for the notation).

**Lemma 4.4.1** We have $\varepsilon(\nu) = \varepsilon(\mu)$. Moreover if we set $\gamma_{\ell,\nu} = (\nu^{(1)}, \ldots, \nu^{(p)})$, we obtain

$$\nu^{(k)} = (\mu^{(k)}_{-}, \alpha_k + 1 - s_k, \alpha_k + 1 - s_k, \mu^{(k)}_{+})$$

for any $k \in \{1, \ldots, p\}$, that is $\nu^{(k)}$ is obtained by inserting in $\mu^{(k)}$ two components equal to $\alpha_k + 1 - s_k$. In particular

$$|\gamma_{\ell,\nu}| = |\gamma_{\ell,\mu}| + 2 \sum_{k=1}^{p} |\alpha_k + 1 - s_k|.$$  \hfill (21)

**Proof.** Let us slightly abuse the notation by writing $I^{(k)}(\mu)$, $J^{(k)}(\mu)$, $I^{(k)}(\nu)$, $J^{(k)}(\nu)$, $k = 1, \ldots, \ell$ and $X^{(k)}(\mu)$, $X^{(k)}(\nu)$, $k = 1, \ldots, p$ for the sequences defined from $\mu$ and $\nu$ by applying the procedure of Section 4.2. We define $\alpha_k(\mu)$ and $\alpha_k(\nu)$, $k = 1, \ldots, p$ similarly. We have $I^{(k)}(\nu) = \{k\} \cup (I^{(k)}(\mu) + 1)$ for $k = 1, \ldots, \ell$. Moreover $\text{card}(J^{(k)}(\nu)) = \text{card}(J^{(k)}(\mu)) + 2$ and $\alpha_k(\nu) = \alpha_k(\mu) + 1$ for any $k \in \{1, \ldots, \ell\}$. Thus $\text{card}(X^{(k)}(\nu)) = \text{card}(J^{(k)}(\nu))$ for any $k \in \{1, \ldots, p\}$ and thus, $\varepsilon(\nu) = \varepsilon(\mu)$. So it makes sense to consider $\gamma_{\ell,\nu} = (\nu^{(1)}, \ldots, \nu^{(p)})$. It then follows by a direct application of the formulas (19) that

$$\nu^{(k)} = (\mu^{(k)}_{-}, \alpha_k + 1 - s_k, \alpha_k + 1 - s_k, \mu^{(k)}_{+})$$

and thus $|\gamma_{\ell,\nu}| = |\gamma_{\ell,\mu}| + 2 \sum_{k=1}^{p} |\alpha_k + 1 - s_k|$. \hfill $\blacksquare$

When $\ell = 2p + 1$ is odd and $\gamma_{\ell,\mu} = (\mu^{(1)}, \ldots, \mu^{(p)}, \mu^{(p+1)})$, we can define $\nu$ similarly. Then, one proves that $\varepsilon(\nu) = \varepsilon(\mu)$. We have $\gamma_{\ell,\nu} = (\nu^{(1)}, \ldots, \nu^{(p)}, \nu^{(p+1)})$ with

$$\nu^{(k)} = (\mu^{(k)}_{-}, \alpha_k + 1 - s_k, \alpha_k + 1 - s_k, \mu^{(k)}_{+})$$

for any $k = 1, \ldots, p$ \hfill (22)

and $\nu^{(p+1)} = (0, \mu^{(p+1)})$. Hence (21) still holds. With the notation of Sections 4.2 and 4.3, we obtain the following stabilization theorem :

**Theorem 4.4.2** Consider $\mu$ a partition such that $\varepsilon(\mu) \neq 0$. Let $\ell$ be a positive integer. Then for any partition $\lambda$

1. $a_{\lambda,\mu}^{\ell,\nu_0} \neq 0$ only if $s_k = \alpha_k + 1$ for any $k = 1, \ldots, p$.

2. In this case we have $a_{\lambda,\mu}^{\ell,\nu_0} = |\mathcal{V}^{\nu_0}_{2n+1}(\lambda) : \mathcal{V}^{\nu_0}_{\nu_0}(\gamma_{\ell,\mu})|$ and the non-zero components of the dominant weight $\gamma_{\ell,\mu}$ do not depend on the number of parts 0 in $\lambda$ and $\mu$. 

13
Proof. Suppose there exists \( k \in \{1, \ldots, p\} \) such that \( s_k \neq \alpha_k + 1 \). Write \( \mu(a) \) for the partition obtained by adding \( a \ell \) components 0 to \( \mu \). By (21), we have then \( |\gamma_{\ell, \mu(a)}| \geq |\gamma_{\ell, \mu}| + 2a \). Thus, for \( a \) sufficiently large, one has \( |\gamma_{\ell, \mu(a)}| > |\lambda| \). For such \( a \), we will obtain \( [V^{\mathfrak{sp}_{2n+1}}(\lambda) : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] = 0 \).

Hence \( [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] \) does not coincide with a non-zero coefficient \( a_{\lambda, \mu}^{\ell, \mathfrak{so}} \). When \( s_k = \alpha_k + 1 \) for any \( k = 1, \ldots, p \), the second assertion of the theorem follows from (22).

Remarks:
(i) There exist very efficient procedures to compute the branching coefficients \( [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] \) (see [8]). By the previous theorem, they permit to derive the coefficients \( a_{\lambda, \mu}^{\ell, \mathfrak{so}} \).
(ii) One can check that condition 1 of Theorem 4.4.2 is satisfied in Example 4.2.1 but fails in (20) where \( s_1 = 1 \) and \( \alpha_1 = 2 \).

4.5 Coefficients \( a_{\lambda, \mu}^{\ell, \mathfrak{m}} \) and restriction to Levi subgroups

By combining the results of Sections 3 and 4, we derive the following theorem which expresses \( a_{\lambda, \mu}^{\ell, \mathfrak{so}} \) and \( a_{\lambda, \mu}^{\ell, \mathfrak{sp}} \) as branching coefficients corresponding to restrictions to Levi subgroups.

**Theorem 4.5.1** Consider \( \lambda \in \mathcal{P}_m \) and \( \ell \) a positive integer. Let \( \mathfrak{g} \) be a symplectic or orthogonal Lie group with rank \( n > \max(\ell(\lambda), l(\lambda')) \). Then we have:

1. \( a_{\lambda, \mu}^{\ell, \mathfrak{so}} = \varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] \) where \( \varepsilon(\mu) \), \( \mathfrak{g}_\ell, \mu \) and \( \gamma_{\ell, \mu} \) are determined by the algorithms of Section 4.

2. \( a_{\lambda, \mu}^{\ell, \mathfrak{sp}} = \varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] \) if \( |\lambda| \) is even and

3. \( a_{\lambda, \mu}^{\ell, \mathfrak{sp}} = (-1)^{\ell-1}\varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu'})] \) if \( |\lambda| \) is odd.

Proof. Assertion 1 follows from Proposition 3.3.2 and Theorem 4.1.1. By remark following Lemma 3.3.3, one has \( a_{\lambda, \mu}^{\ell, \mathfrak{sp}} = a_{\lambda', \mu'}^{\ell, \mathfrak{so}} \) if \( |\lambda| \) is even and \( a_{\lambda, \mu}^{\ell, \mathfrak{sp}} = (-1)^{\ell-1}a_{\lambda', \mu'}^{\ell, \mathfrak{so}} \) otherwise which proves assertion 2. Note that the assumption \( n > \max(\ell(\lambda), l(\lambda')) \) suffices to guarantee that \( \lambda' \) belongs to \( \mathcal{P}_n \).

In the sequel, we will assume for simplicity \( \ell \geq 2 \) and \( n \geq \ell \) which implies the condition \( n > \max(\ell(\lambda), l(\lambda')) \).

5 Splitting \( V^g(\lambda) \otimes^2 \) into its symmetric and antisymmetric parts

5.1 Decomposition of the plethysms \( p_2 \circ s^\mathfrak{g}^\lambda \)

Consider a partition \( \lambda \in \mathcal{P}_m \). According to Theorem 4.5.1, we have with the notation of Sections 3.2 and 3.3:

\[
p_2 \circ s^\mathfrak{g}^\lambda = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu)c_{\lambda, \mu}^{\ell, \mathfrak{so}} s^\mathfrak{g}_\mu,
\]

\[
p_2 \circ s^\mathfrak{sp}^\lambda = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_\ell, \mu}(\gamma_{\ell, \mu})] s^\mathfrak{sp}_\mu
\]

and

\[
p_2 \circ s^\mathfrak{sp}^\lambda = (-1)^{|\lambda|}p_2 \circ s^\mathfrak{sp}_\lambda.
\]
for any $n \geq 2 |\lambda|$. Here we have written for short $\gamma_{\mu}$ for $\gamma_{2,\mu}$ and $V^{\mathfrak{gl}}(\gamma_{\mu})$ instead of $V^{\mathfrak{so}_{2m}}(\gamma_{\mu})$ (see Remark (ii) of Section 4.4). Since $n \geq m$ and $\gamma_{\mu} = (\gamma^{-}, \gamma^{+})$ belongs to $\mathcal{P}_{n}$ we have the following decomposition (see [6]):

$$[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}}(\gamma_{\mu})] = \sum_{\delta, \xi, \eta \in \mathcal{P}_{n}} c_{\delta, \xi}^{\lambda} c_{\xi, \eta}^{\epsilon_{\gamma^{-}}, \epsilon_{\gamma^{+}}}. \quad (24)$$

5.2 Symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$

Consider $\lambda \in \mathcal{P}_{m}$. By Propositions 3.2.3 and 3.3.2 for any rank $n \geq 2 |\lambda|$ the plethysms $h_{2} \circ s_{\lambda}^{\mathfrak{g}}$ and $e_{2} \circ s_{\lambda}^{\mathfrak{g}}$ stabilize. Set

$$h_{2} \circ s_{\lambda}^{\mathfrak{g}} = \sum_{\mu \in \mathcal{P}_{n}} m_{\lambda, \mu}^{\Phi, +} s_{\mu}^{\mathfrak{g}} \quad \text{and} \quad e_{2} \circ s_{\lambda}^{\mathfrak{g}} = \sum_{\mu \in \mathcal{P}_{n}} m_{\lambda, \mu}^{\Phi, -} s_{\mu}^{\mathfrak{g}}$$

where $\mathfrak{g} = \mathfrak{gl}, \mathfrak{so}$ or $\mathfrak{sp}$ respectively when $\mathfrak{g} = \mathfrak{gl}_{n}, \mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$. Recall that $h_{2} \circ s_{\lambda}^{\mathfrak{g}}$ and $e_{2} \circ s_{\lambda}^{\mathfrak{g}}$ are the characters of $S^{2}(V^{\mathfrak{g}}(\lambda))$ and $\Lambda^{2}(V^{\mathfrak{g}}(\lambda))$. By using (10) and Theorem 4.5.1 we obtain for any rank $n \geq 2 |\lambda|$

$$m_{\lambda, \mu}^{\mathfrak{gl}, \pm} = \frac{1}{2} (d_{\lambda, \mu}^{\mathfrak{g}, \pm} \pm \varepsilon(\mu) c_{(\mu(0), \mu(1))}^{\lambda}),$$

$$m_{\lambda, \mu}^{\mathfrak{so}, \pm} = \frac{1}{2} (d_{\lambda, \mu}^{\mathfrak{g}, \pm} \pm \varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}}(\gamma_{\mu})]),$$

$$m_{\lambda, \mu}^{\mathfrak{sp}, \pm} = \frac{1}{2} (d_{\lambda, \mu}^{\mathfrak{g}, \pm} \pm (-1)^{1 |\lambda|} \varepsilon(\mu') [V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{gl}}(\gamma_{\mu'})])$$

where the coefficients $d_{\lambda, \mu}^{\mathfrak{g}, \pm}$ are the multiplicities appearing in Proposition 2.2.2. Now these multiplicities can be expressed in terms of the Littlewood coefficients [5]. Namely we have $d_{\lambda, \mu}^{\mathfrak{g}, \pm} = \sum_{\delta, \xi, \eta} d_{\delta, \xi, \eta}^{\mu} c_{\delta, \eta}^{\mathfrak{g}, \pm} c_{\xi, \eta}^{\lambda}$. In particular we recover the equality $d_{\lambda, \lambda'}^{\mathfrak{g}, \pm} = d_{\lambda, \lambda}^{\mathfrak{g}, \pm}$ since $c_{\delta, \eta}^{\mathfrak{g}, \pm} = c_{\delta, \eta}'^{\mathfrak{g}, \pm}$ for any partitions $\delta, \eta$ and $\gamma$. By using (24), this thus permits to express the multiplicities appearing in the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ in terms of the Littlewood-Richardson coefficients. Note that formulas for computing the plethysms $h_{2} \circ s_{\lambda}^{\mathfrak{g}}$ and $e_{2} \circ s_{\lambda}^{\mathfrak{g}}$ were introduced without a complete proof by Littlewood in [14].

**Proposition 5.2.1** With the above notation we have for any rank $n \geq 2 |\lambda|$

$$m_{\lambda, \mu}^{\mathfrak{gl}, \pm} = \frac{1}{2} (c_{\delta, \xi}^{\mathfrak{g}, \pm} \pm \varepsilon(\mu) c_{(\mu(0), \mu(1))}^{\lambda}),$$

$$m_{\lambda, \mu}^{\mathfrak{so}, \pm} = \frac{1}{2} \left( \sum_{\delta, \xi, \eta \in \mathcal{P}_{n}} c_{\delta, \xi}^{\mathfrak{g}, \pm} c_{\xi, \eta}^{\lambda} \pm \varepsilon(\mu) \sum_{\delta, \xi \in \mathcal{P}_{n}} c_{\delta, \xi}^{\lambda} c_{\xi, \eta}^{\mathfrak{so,} \pm}) \right),$$

$$m_{\lambda, \mu}^{\mathfrak{sp}, \pm} = \frac{1}{2} \left( \sum_{\delta, \xi, \eta \in \mathcal{P}_{n}} c_{\delta, \xi}^{\mathfrak{g}, \pm} c_{\xi, \eta}^{\lambda} \pm (-1)^{1 |\lambda|} \varepsilon(\mu') \sum_{\delta, \xi \in \mathcal{P}_{n}} c_{\delta, \xi}^{\lambda} c_{\xi, \eta}^{\mathfrak{sp,} \pm}) \right)$$

where $\gamma_{\mu} = (\gamma^{-}, \gamma^{+})$ and $\gamma_{\mu'} = (\kappa^{-}, \kappa^{+})$.  

15
References

[1] C. Carré, B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, Journal of Algebraic Combinatorics, 4, 201-231 (1995).

[2] V. Deodhar, On some geometric aspect of the Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials, J. Algebra, 111, 483-506 (1987).

[3] W. Fulton, J. Harris, Representation theory, Graduate Texts in Mathematics, Springer-Verlag.

[4] G. Goodman, N. R Wallach, Representation theory and invariants of the classical groups, Cambridge University Press.

[5] R. C. King, Modifications rules and products of irreducible representations of the unitary, orthogonal and symplectic groups, J. Math. Phys. 12, 1588-1598 (1971).

[6] R. C. King, Branching rules for classical Lie groups using tensor and spinor methods, J. Phys A, 8 429-449 (1975).

[7] R. C. King, S-functions and characters of Lie algebras and superalgebras in Invariant Theory and Tableaux, Ed D. Stanton, IMA Vol 19, New York Springer Verlag, 1989, 226-261.

[8] K. Koike, I. Terada, Young diagrammatic methods for the representations theory of the classical groups of type $B_n, C_n$ and $D_n$, Journal of Algebra, 107, 466-511 (1987).

[9] K. Koike, I. Terada, Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank, Advances in Mathematics, 79, 104-135 (1990).

[10] B. Leclerc, J. Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials, Advance Studies in Pure Mathematics 28, Combinatorial Methods in representation Theory, 155-220 (2000).

[11] C. Lecouvey, Parabolic Kazhdan-Lusztig polynomials, plethysms and generalized Hall-Littlewood functions for classical types To appear in European Journal of Combinatorics, arXiv RT/0607038.

[12] D-E. Littlewood, Modular representations of symmetric groups, Proc. Roy. Soc. A 239, 387-417 (1944).

[13] D-E. Littlewood, The theory of group characters and matrix representations of groups, Oxford University Press, second edition (1958).

[14] D-E. Littlewood, Products and plethysms of characters with orthogonal, symplectic and symmetric groups, Can. J. Math 10, 17-32 (1958).

[15] I-G. Macdonald, Symmetric functions and Hall polynomials, Second edition, Oxford Mathematical Monograph, Oxford University Press, New York, (1995).