Chromatic Topological Indices of Certain Cycle Related Graphs

Smitha Rose
Department of Mathematics
Christ University
Bengaluru, Karnataka, India.
smitha.rose@res.christuniversity.in

Sudev Naduvath
Centre for Studies in Discrete Mathematics,
Vidya Academy of Science & Technology,
Thrissur-680501, Kerala, India.
sudevnk@gmail.com

Abstract
Topological indices are real numbers invariant under graph isomorphisms. Chromatic analogue of topological indices have been introduced recently in literature in 2017. Mainly, chromatic versions of Zagreb indices are studied lately. This paper discusses the notion of chromatic topological and irregularity indices of certain cycle related graphs.

Key words: Chromatic Zagreb index, chromatic irregularity index
Mathematics Subject Classification 2010: 05C15, 05C38.

1 Introduction
Chemical graph theory finds a variety of application in today’s world especially in the field of pharmaceuticals. It concerns itself mainly with the mathematical modelling of the chemical phenomenon and the gaining of valuable insights into chemical behaviour. A topological index of a graph $G$ is a real number preserved under isomorphisms, which constitutes one of the basic notions of molecular descriptors in chemical graph theory. Chromatic topological indices of a graph $G$ is a term that has recently been coined [12] to designate a new colouring version to these indices which embrace both proper colouring and topological indices. Here, the vertex degrees are interchanged with minimal colouring, keeping up the additional colouring conditions of proper colouring. The graphs discussed in this paper
are finite, non-trivial, undirected, connected and without loops or multiple edges. For notation and terminology not explicitly defined here see [9 3 5 17].

Generally, graph colouring is referred to as an assignment of colours, labels or weights to the vertices of a graph under consideration subject to certain conditions. A **proper vertex colouring** of a graph \( G \) is an assignment \( \varphi : V(G) \to \mathcal{C} \) of the vertices of \( G \), where \( \mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\} \) is a set of colours such that adjacent vertices of \( G \) are given different colours. The minimum number of colours required to apply a proper vertex colouring to \( G \) is called the **chromatic number** of \( G \) and is denoted \( \chi(G) \). The set of all vertices of \( G \) which have the colour \( c_i \) is called the **colour class** of that colour \( c_i \) in \( G \) and may be denoted by \( \mathcal{C}_i \). The **strength** of the colour class, denoted by \( \theta(c_i) \), is the cardinality of each colour class of colour \( c_i \).

A proper vertex colouring consisting of the colours having minimum subscripts may be called a **minimum parameter colouring** (see [12]). If we colour the vertices of \( G \) in such a way that \( c_1 \) is assigned to maximum possible number of vertices, then \( c_2 \) is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called \( \varphi^- \)-**colouring** of \( G \). In a similar manner, if \( c_\ell \) is assigned to maximum possible number of vertices, then \( c_{(\ell-1)} \) is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called \( \varphi^+ \)-**colouring** of \( G \) (see [12]).

For computational convenience, we define function \( \zeta : V(G) \to \{1, 2, 3, \ldots, \ell\} \) such that \( \zeta(v_i) = s \iff \varphi(v_i) = c_s, c_s \in \mathcal{C} \). The total number of edges with end points having colours \( c_t \) and \( c_s \) is denoted by \( \eta_{ts} \), where \( t < s, 1 \leq t, s \leq \chi(G) \).

Analogous to the definitions of Zagreb and irregularity indices of graphs (see [1 8 18 19]), the notions of different chromatic Zagreb indices and chromatic irregularity indices have been introduced in [12] as follows:

**Definition 1.0.1.** [12] Let \( G \) be a graph and let \( \mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\} \) be a proper colouring of \( G \) such that \( \varphi(v_i) = c_s, 1 \leq i \leq n, 1 \leq s \leq \ell \). Then for \( 1 \leq t \leq \ell \!),

(i) The **first chromatic Zagreb index** of \( G \), denoted by \( M_1(x)(G) \), is defined as

\[
M_1(x)(G) = \sum_{i=1}^{n} (\zeta(v_i))^2 = \sum_{j=1}^{\ell} \theta(c_j) \cdot j^2.
\]

(ii) The **second chromatic Zagreb index** of \( G \), denoted by \( M_2(x)(G) \), is defined as

\[
M_2(x)(G) = \sum_{j=1}^{n-1} \sum_{j=2}^{n} (\zeta(v_i) \cdot \zeta(v_j)), \ v_iv_j \in E(G).
\]

(iii) The **chromatic irregularity index** of \( G \), denoted by \( M_3(x)(G) \), is defined as

\[
M_3(x)(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)|, \ v_iv_j \in E(G).
\]

**Definition 1.0.2.** [12] The chromatic total irregularity index of a graph \( G \) corresponding to a proper colouring \( \varphi : V(G) \to \mathcal{C} = \{c_1, c_2, \ldots, c_\ell\} \) is defined as

\[
M_4(x)(G) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)|, \ v_i, v_j \in V(G).
\]
In view of the above notions, the minimum and maximum chromatic Zagreb indices and the corresponding irregularity indices are defined in [12] as follows.

\[ M_1^{\phi^-(G)} = \min\{M_1^{\phi^t}(G) : 1 \leq t \leq \ell\}, \text{for } 1 \leq i \leq 4 \]

\[ M_1^{\phi^+(G)} = \max\{M_1^{\phi^t}(G) : 1 \leq t \leq \ell\}, \text{for } 1 \leq i \leq 4. \]

## 2 Chromatic Topological Indices of Flower Graphs

A flower graph \( F_n \) is a graph which is obtained by joining the pendant vertices of a helm graph \( H_n \) to its central vertex. The following theorem discusses the chromatic topological indices of the flower graph \( F_n \).

**Theorem 2.1.** For a flower graph \( F_n \), we have

(i) \( M_1^{\phi^-}(F_n) = \begin{cases} 5n + 9; & \text{if } n \text{ is even;} \\ 5n + 21; & \text{if } n \text{ is odd.} \end{cases} \)

(ii) \( M_2^{\phi^-}(F_n) = \begin{cases} 13n; & \text{if } n \text{ is even;} \\ 13n + 16; & \text{if } n \text{ is odd.} \end{cases} \)

(iii) \( M_3^{\phi^-}(F_n) = \begin{cases} 5n; & \text{if } n \text{ is even;} \\ 5(n + 1); & \text{if } n \text{ is odd.} \end{cases} \)

(iv) \( M_4^{\phi^-}(F_n) = \begin{cases} \frac{n^2 + 3n}{2}; & \text{if } n \text{ is even;} \\ \frac{n^2 + 7n - 2}{2}; & \text{if } n \text{ is odd.} \end{cases} \)

**Proof.** The chromatic number of a flower graph \( F_n \) is 3 when \( n \) is even and is 4 when \( n \) is odd. Let \( v_1, v_2, \ldots, v_n \) be the non-adjacent vertices around \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of \( C_n \) on the rim of the wheel and \( u \) be the central vertex. In order to calculate the minimum values of chromatic Zagreb indices, we apply the \( \phi^- \) colouring pattern to \( F_n \) as described below.

If \( n \) is even, then the sets \( S_1 = \{v_1, v_3, \ldots, v_{n-1}, u_2, u_4, \ldots, u_n\} \) and \( S_2 = \{v_2, v_4, \ldots, v_n, u_1, u_3, \ldots, u_{n-2}\} \) form the two maximum independent sets with same cardinality \( n \). We colour them with minimum colours \( c_1 \) and \( c_2 \) respectively. The remaining central vertex \( u \) is coloured with \( c_3 \). If \( n \) is odd, then the maximum independent sets \( S_1 = \{v_1, v_3, \ldots, v_n, u_2, u_4, \ldots, u_{n-1}\} \), \( S_2 = \{v_2, v_4, \ldots, v_{n-1}, u_1, u_3, \ldots, u_{n-2}\} \) have cardinality \( n \) and \( n - 1 \) and coloured with \( c_1 \) and \( c_2 \) respectively. Also, the vertices \( u_n \) and \( u \) are coloured with \( c_4 \) and \( c_3 \) respectively. Then,

**Part (i):** In order to calculate \( M_1^{\phi^-} \) of \( F_n \), we first colour the vertices as mentioned above and then proceed to consider the following cases.

**Case-1:** Let \( n \) be even, then we have \( \theta(c_1) = \theta(c_2) = n \) and \( \theta(c_3) = 1 \). Therefore, the corresponding chromatic Zagreb index is given by

\[ M_1^{\phi^-}(F_n) = \sum_{i=1}^{3} (\zeta(v_i))^2 = 5n + 9. \]
Case-2: Let \( n \) be odd. Then, we have \( \theta(c_1) = n, \theta(c_2) = n - 1 \) and \( \theta(c_3) = \theta(c_4) = 1 \). Now, by the definition of first chromatic Zagreb index, we have
\[
M_1^\varphi^-(F_n) = \sum_{i=1}^{4} (\zeta(v_i))^2 = 5n + 21.
\]

Part (ii): To compute \( M_2^\varphi^- \), we first colour the vertices as per the instructions in introductory part for both cases of \( n \). Now, consider the following cases:

Case-1: If \( n \) is even, then we observe that \( \eta_{12} = 2n, \eta_{23} = \eta_{13} = n \). The definition of second chromatic Zagreb index gives the sum
\[
M_2^\varphi^-(F_n) = \sum_{1 \leq t,s \leq \chi(F_n)} ts\eta_{ts} = 4n + 3n + 6n = 13n.
\]

Case-2: Let \( n \) be odd. Here we see that \( \eta_{12} = 2n - 3, \eta_{13} = n, \eta_{23} = n - 1, \eta_{14} = 2, \eta_{34} = \eta_{24} = 1 \). Hence, we have the sum
\[
M_2^\varphi^-(F_n) = \sum_{1 \leq t,s \leq \chi(F_n)} ts\eta_{ts} = 13n + 16.
\]

Part (iii): We calculate the minimum irregularity measurement by considering the following cases:

Case-1: Let \( n \) be even. Here, \( \eta_{12} + \eta_{23} = 3n \) edges contribute the distance 1 to the total summation, while \( \eta_{13} = n \) contribute the distance 2. The result follows from the following calculations:
\[
M_3^\varphi^-(F_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 5n.
\]

Case-2: Let \( n \) be odd. Here we see that \( \eta_{12} + \eta_{23} + \eta_{34} \) edges contribute 1 to the colour distance, \( \eta_{13} + \eta_{24} \) edges contribute 2, while \( \eta_{14} \) edges contribute 3. Then the result follows from the following calculations:
\[
M_3^\varphi^-(F_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 5(n + 1).
\]

Part (iv): To calculate the chromatic total irregularity indices of flower graphs, we have to consider all the possible vertex pairs and all colour combinations contributing non zero distances are considered according to the following two cases:

Case-1: Let \( n \) be even. The combinations possible are charted as \( \{1, 2\}, \{2, 3\} \) contributing a distance 1 and \( \{1, 3\} \) contributing 2. Observe that \( \theta(c_1) = \theta(c_2) = n \) and \( \theta(c_3) = 1 \). Thus, we have
\[
M_4^\varphi^-(F_n) = \frac{1}{2} \sum_{u,v \in V(F_n)} |\zeta(u) - \zeta(v)|
\]
Case 2: Let \( n \) be odd. Here, the possible combinations which contribute to the colour distances are \{1, 2\}, \{2, 3\}, \{3, 4\} contributing 1, \{1, 3\}, \{2, 4\} contributing 2 and \{1, 4\} contributing 3. We calculate the chromatic total irregularity as given below:

\[
M^\phi_4(F_n) = \frac{1}{2} \sum_{u,v \in V(F_n)} |\zeta(u) - \zeta(v)|
\]

\[
= \frac{n^2 + 7n - 2}{2}
\]

Instead of \( \phi^- \) colouring, one can also work with \( \phi^+ \) colouring of flower graphs using minimum parameter colouring. The results obtained are charted below as next theorem.

**Theorem 2.2.** For a flower graph \( F_n \), we have

(i) \[
M^\phi_1(F_n) = \begin{cases} 
13n + 1; & \text{if } n \text{ is even} \\
25n - 4; & \text{if } n \text{ is odd}; 
\end{cases}
\]

(ii) \[
M^\phi_2(F_n) = \begin{cases} 
17n; & \text{if } n \text{ is even} \\
38n - 29; & \text{if } n \text{ is odd}; 
\end{cases}
\]

(iii) \[
M^\phi_3(F_n) = \begin{cases} 
5n; & \text{if } n \text{ is even} \\
5n + 5; & \text{if } n \text{ is odd}; 
\end{cases}
\]

(iv) \[
M^\phi_4(F_n) = \begin{cases} 
\frac{n^2 + 3n}{2}; & \text{if } n \text{ is even} \\
\frac{n^2 + 7n - 2}{2}; & \text{if } n \text{ is odd}. 
\end{cases}
\]

**Proof.** Here we follow \( \phi^+ \) colouring of flower graphs to obtain desired results.

When \( n \) is even, the vertices \( S_1 = \{v_1, v_3, \ldots, v_{n-1}, u_2, u_4, \ldots, u_n\} \) and \( S_2 = \{v_2, v_4, \ldots, v_n, u_1, u_3, \ldots, u_{n-1}\} \) forms the two maximum independent sets with same cardinality \( n \). We colour them with maximum colours \( c_3 \) and \( c_2 \) respectively. The remaining central vertex \( u \) is coloured with \( c_1 \). Then \( \eta_{12} = \eta_{13} = n \) and \( \eta_{23} = 2n \).

If \( n \) is odd, we have chromatic number 4. Here the maximum independent sets \( S_1 = \{v_1, v_3, \ldots, v_n, u_2, u_4, \ldots, u_{n-1}\}, S_2 = \{v_2, v_4, \ldots, v_{n-1}, u_1, u_3, \ldots, u_{n-2}\} \) have cardinality \( n \) and \( n - 1 \) and coloured with \( c_1 \) and \( c_2 \) respectively. Also, the vertices \( u_n \) and \( u \) are coloured with \( c_4 \) and \( c_3 \) respectively, to get maximum values. Thus \( \eta_{12} = \eta_{13} = 1, \eta_{14} = 2, \eta_{23} = n - 1, \eta_{24} = n \) and \( \eta_{34} = 2n - 3 \).

The remaining part of the proof follows in the same way as mentioned in the proof of Theorem 2.1. \( \square \)
3 Chromatic Topological Indices of Sunflower Graphs

A sunflower graph $SF_n$ is a graph obtained by replacing each edge of the rim of a wheel graph $W_n$ by a triangle such that two triangles share a common vertex if and only if the corresponding edges in $W_n$ are adjacent in $W_n$. The following result discusses the chromatic topological indices of a sunflower graph by using $\varphi$-colouring.

**Theorem 3.1.** For a sunflower graph $SF_n$, we have

$$
(i) \quad M_1^{\varphi^-}(SF_n) = \begin{cases} \frac{15n+2}{2}, & \text{if } n \text{ is even} \\ \frac{15n+21}{2}, & \text{if } n \text{ is odd} \end{cases} 
$$

$$
(ii) \quad M_2^{\varphi^-}(SF_n) = \begin{cases} \frac{27n}{2}, & \text{if } n \text{ is even} \\ \frac{27n+25}{2}, & \text{if } n \text{ is odd} \end{cases} 
$$

$$
(iii) \quad M_3^{\varphi^-}(SF_n) = \begin{cases} \frac{11n}{2}, & \text{if } n \text{ is even} \\ \frac{11n+11}{2}, & \text{if } n \text{ is odd} \end{cases} 
$$

$$
(iv) \quad M_4^{\varphi^-}(SF_n) = \begin{cases} \frac{3n^2+6n}{8}, & \text{if } n \text{ is even} \\ \frac{3n^2+16n+1}{8}, & \text{if } n \text{ is odd} \end{cases} 
$$

**Proof.**

As we know, the sunflower graph $SF_n$ has chromatic number 3 when $n$ is even and chromatic number 4 when $n$ is odd. Let $v_1, v_2, \ldots, v_n$ be the non-adjacent vertices on the outer cycle, $u_1, u_2, \ldots, u_n$ be the vertices on the wheel and $u$ be the central vertex. To obtain the minimum values of the chromatic topological indices we follow the $\varphi$-colouring pattern to $SF_n$ as described below.

When $n$ is even, we can find the maximum independent sets $S_1 = \{v_1, v_2, \ldots, v_n, u\}$ with cardinality $n + 1$, $S_2 = \{u_1, u_3, \ldots, u_{n-1}\}$, $S_3 = \{u_2, u_4, \ldots, u_n\}$ with same cardinality $\frac{n}{2}$. We colour them with minimum colours $c_1$, $c_2$ and $c_3$ respectively. If $n$ is odd, then $S_1 = \{v_1, v_2, \ldots, v_n, u\}$ with cardinality $n + 1$ and we colour it with $c_1$. Now, $S_2 = \{u_1, u_3, \ldots, u_{n-2}\}$, $S_3 = \{u_2, u_4, \ldots, u_{n-1}\}$ are next maximum independent sets with same cardinality $\frac{n-1}{2}$. So we colour it with $c_2$ and $c_3$ respectively and the colour $c_4$ is assigned to vertex $u_n$.

**Part (i):** In order to find $M_1^{\varphi^-}$ of $SF_n$, we first colour the vertices as mentioned above and then proceed to consider the following cases:

*Case-1:* Let $n$ be even. Then, we have $\theta(c_1) = n + 1$ and $\theta(c_2) = \theta(c_3) = \frac{n}{2}$. Therefore, the corresponding chromatic Zagreb index is given by

$$
M_1^{\varphi^-}(SF_n) = \sum_{i=1}^{3} (\xi(v_i))^2 = (n + 1) + \frac{4n}{2} + \frac{9n}{2} = \frac{15n + 2}{2}.
$$

*Case-2:* Let $n$ be odd. Then, we have $\theta(c_1) = n + 1$, $\theta(c_2) = \theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = 1$. Now, by the definition of first chromatic Zagreb index, we have

$$
M_1^{\varphi^-}(SF_n) = \sum_{i=1}^{4} (\xi(v_i))^2 = (n + 1) + 2(n - 1) + \frac{9(n - 1)}{2} + 16 = \frac{15n + 21}{2}.
$$
Part (ii): We colour the vertices as per the instructions in introductory part for even and odd cases of \( n \). Now consider the following cases:

Case-3: Let \( n \) be even. In this case, we see that \( \eta_{23} = n, \eta_{12} = \eta_{13} = \frac{3n}{2} \). The definition of second chromatic Zagreb index gives the sum

\[
M^{\phi - 2}_{\eta}(SF_n) = \sum_{1 \leq t, s \leq \chi(SF_n)} ts \eta_{ts} = \frac{6n}{2} + \frac{9n}{2} + 6n = \frac{27n}{2}.
\]

Case-4: Let \( n \) be odd. Here we see that \( \eta_{12} = \eta_{13} = \frac{3(n-1)}{2}, \eta_{23} = n-2, \eta_{14} = 3, \eta_{34} = \eta_{24} = 1 \). Hence, we have the sum

\[
M^{\phi - 2}_{\eta}(SF_n) = \sum_{1 \leq t, s \leq \chi(SF_n)} ts \eta_{ts} = 3(n-1) + \frac{9(n-1)}{2} + 6(n-2) + 32 = \frac{27n + 25}{2}.
\]

Part (iii): To find the minimum irregularity measurement, consider the following cases:

Case-5: Let \( n \) be even. Here \( \eta_{12} + \eta_{23} = \frac{6n}{2} \) edges contribute the distance 1 to the total summation while \( \eta_{13} = \frac{3n}{2} \) contribute the distance 2. The result follows from the following calculations:

\[
M^{\phi - 3}_{\eta}(SF_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} \mid \zeta(v_i) - \zeta(v_j) \mid = \frac{3n}{2} + \frac{6n}{2} + n = \frac{11n}{2}.
\]

Case-6: Let \( n \) be odd. Here we see that, \( \eta_{12} + \eta_{23} + \eta_{34} \) edges contribute 1 to the colour distance, \( \eta_{13} + \eta_{24} \) edges contribute 2, while \( \eta_{14} \) edges contribute 3. Then, the result follows from the following calculations:

\[
M^{\phi - 3}_{\eta}(SF_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} \mid \zeta(v_i) - \zeta(v_j) \mid = \frac{3(n-1)}{2} + 3(n-1) + (n-2) + 12 = \frac{11n + 11}{2}.
\]

Part (iv): To calculate the total irregularity of \( SF_n \), all the possible vertex pairs from \( SF_n \) have to be considered and their possible colour distances are determined. The possibility of the vertex pairs which contribute to the colour distance can be classified according to the following two cases.

Case-7: Let \( n \) be even. The combinations possible are charted as \( \{1, 2\}, \{2, 3\} \) contributing 1 and \( \{1, 3\} \) contributing 2. Observe that \( \theta(c_2) = \theta(c_3) = \frac{n}{2} \) and \( \theta(c_1) = n + 1 \). Thus, we have

\[
M^{\phi - 4}_{\eta}(SF_n) = \frac{1}{2} \sum_{u, v \in V(SF_n)} \mid \zeta(u) - \zeta(v) \mid = \frac{7n^2 + 6n}{8}.
\]

Case-8: Let \( n \) be odd. Here, the possible combinations which contributes to the colour distances are \( \{1, 2\}, \{2, 3\}, \{3, 4\} \) contributing 1, \( \{1, 3\}, \{2, 4\} \) contributing 2 and \( \{1, 4\} \) contributing 3. We calculate the total irregularity as given below:
Using the minimum parameter colouring we can also work on $\varphi_+$ colouring of sunflower graphs. Next theorem deals with this matter.

**Theorem 3.2.** For a sun flower graph $SF_n$, we have

(i) $M_1^{\varphi_+}(SF_n) = \begin{cases} \frac{2n^2+18}{2}, & \text{if } n \text{ is even} \\ \frac{4n^2+21}{2}, & \text{if } n \text{ is odd} \end{cases}$

(ii) $M_2^{\varphi_+}(SF_n) = \begin{cases} 17n, & \text{if } n \text{ is even} \\ 36n - 25, & \text{if } n \text{ is odd} \end{cases}$

(iii) $M_3^{\varphi_+}(SF_n) = \begin{cases} \frac{11n}{2}, & \text{if } n \text{ is even} \\ \frac{11n+11}{2}, & \text{if } n \text{ is odd} \end{cases}$

(iv) $M_4^{\varphi_+}(SF_n) = \begin{cases} \frac{7n^2+6n}{8}, & \text{if } n \text{ is even} \\ \frac{7n^2+16n+1}{8}, & \text{if } n \text{ is odd}. \end{cases}$

**Proof.** Here we follow $\varphi_+$ colouring of sun flower graphs to obtain desired results. When $n$ is even, we have $\theta(c_3) = n+1$ and $\theta(c_2) = \theta(c_1) = \frac{n}{2}$. Then $\eta_{23} = \eta_{13} = \frac{3n}{2}$ and $\eta_{12} = n$.

Let $n$ be odd, we have chromatic number 4. Here the cardinality of the colour classes are $\theta(c_4) = n + 1$, $\theta(c_2) = \theta(c_3) = \frac{n-1}{2}$ and $\theta(c_1) = 1$. Thus we have $\eta_{12} = \eta_{13} = 1, \eta_{14} = 3, \eta_{34} = \eta_{24} = \frac{3(n-1)}{2}$ and $\eta_{23} = n - 2$.

The remaining part of the proof follows in a similar manner as mentioned in the proof of Theorem 3.1.

### 4 Chromatic Topological Indices of Closed Sunflower Graphs

A closed sunflower graph $CSF_n$ is a graph obtained by joining the independent vertices of a sunflower graph $SF_n$, which are not adjacent to its central vertex so that these vertices induce a cycle on $n$ vertices. The following result provides the expressions for the topological indices of a closed sunflower graph.

**Theorem 4.1.** For a closed sunflower graph $CSF_n$, we have
\[ M_1^\varphi^{-1}(CSF_n) = \begin{cases} 
\frac{28n+48}{3}; & n \equiv 0 \, (\text{mod } 3) \\
\frac{28n+143}{3}; & n \equiv 1 \, (\text{mod } 3) \\
\frac{28n+109}{3}; & n \equiv -1 \, (\text{mod } 3)
\end{cases} \]

\[ M_2^\varphi^{-1}(CSF_n) = \begin{cases} 
\frac{68n}{3}; & n \equiv 0 \, (\text{mod } 3) \\
\frac{74n+136}{3}; & n \equiv 1 \, (\text{mod } 3) \\
\frac{68n+134}{3}; & n \equiv -1 \, (\text{mod } 3)
\end{cases} \]

\[ M_3^\varphi^{-1}(CSF_n) = \begin{cases} 
\frac{25n+8}{3}; & n \equiv 0 \, (\text{mod } 3) \\
\frac{22n+10}{3}; & n \equiv 1 \, (\text{mod } 3)
\end{cases} \]

\[ M_4^\varphi^{-1}(CSF_n) = \begin{cases} 
\frac{16n^2+74n-32}{18}; & n \equiv 0 \, (\text{mod } 3) \\
\frac{16n^2+4n-92}{18}; & n \equiv 1 \, (\text{mod } 3)
\end{cases} \]

Proof. A closed sunflower graph $CSF_n$ has chromatic number 4 when $n \equiv 0 \, (\text{mod } 3)$ and has chromatic number 5 otherwise. Let $v_1, v_2, \ldots, v_n$ be the vertices of the inner wheel and $u_1, u_2, \ldots, u_n$ be the vertices on the rim of the outer wheel and $v$ be the central vertex. In order to calculate the minimum values of chromatic Zagreb indices we apply the $\varphi^{-1}$ colouring pattern to $CSF_n$ as described below:

Let $n \equiv 0 \, (\text{mod } 3)$ be assumed. When $n \equiv 0 \, (\text{mod } 3)$, obeying the rules of minimum colouring, we can find three colour classes with same cardinality $\frac{2n}{3}$ and we colour them with minimal colours $c_1, c_2, c_3$. The central vertex $v$ is coloured with $c_4$.

Let $n \equiv 1 \, (\text{mod } 3)$ be assumed. Here we form three colour classes with the maximum independent sets having the same cardinality $\frac{2(n-1)}{3}$ and we colour them with minimal colours $c_1, c_2, c_3$. Also the vertices $u_1$ and $v_n$ are coloured with $c_4$ and the central vertex $v$ is coloured with $c_5$.

Let $n \equiv -1 \, (\text{mod } 3)$ be assumed. Here we form three colour classes with the maximum independent sets having the same cardinality $\frac{2n-1}{3}$ and we colour them with minimal colours $c_1, c_2, c_3$. Also the balance two vertices $v$ and $v_n$ are coloured with $c_4$ and $c_5$ respectively. Now we can deal with the parts of the proof.

Part (i): In order to calculate $M_1^\varphi^{-1}$ of $CSF_n$, we first colour the vertices as mentioned above and then proceed to consider the following cases.

Case-1.1: Let $n \equiv 0 \, (\text{mod } 3)$. Then, we have $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2n}{3}$ and $\theta(c_4) = 1$. Therefore, the corresponding chromatic Zagreb index is given by

\[ M_1^\varphi^{-1}(CSF_n) = \sum_{i=1}^{4} (\zeta(v_i))^2 = \frac{2n}{3} + \frac{8n}{3} + \frac{18n}{3} + 16 = \frac{28n + 48}{3}. \]

Case-1.2: Let $n \equiv 1 \, (\text{mod } 3)$. Then, we have $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2(n-1)}{3}$, $\theta(c_4) = 2$ and $\theta(c_5) = 1$. Therefore, the corresponding chromatic Zagreb index is
given by
\[ M_1^\alpha (CSF_n) = \sum_{i=1}^{5} (\xi(v_i))^2 = 25 + 32 + \frac{18(n-1)}{3} + \frac{8(n-1)}{3} + \frac{2(n-1)}{3} = \frac{28n + 143}{3}. \]

Case-1.3: Let \( n \equiv -1 \mod 3 \). Then we have \( \theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2n-1}{3} \) and \( \theta(c_3) = \theta(c_4) = 1 \). Therefore, the corresponding chromatic Zagreb index is given by
\[ M_1^\alpha (CSF_n) = \sum_{i=1}^{5} (\xi(v_i))^2 = \frac{28(n-1)}{3} + 41 = \frac{28n + 109}{3}. \]

Part (ii): We colour the vertices as per the instructions in introductory part for different cases of \( n \). Now consider the following cases:

Case-2.1: Let \( n \equiv 0 \mod 3 \). Here, we see that \( \eta_{12} = \eta_{23} = \eta_{13} = \frac{4n}{3} \) and \( \eta_{14} = \eta_{24} = \eta_{34} = \frac{n}{3} \). The definition of second chromatic Zagreb index gives the sum
\[ M_2^\alpha (CSF_n) = \sum_{1 \leq t,s \leq \chi(CSF_n)} ts\eta_{ts} = \frac{44n}{3} + \frac{24n}{3} = \frac{68n}{3}. \]

Case-2.2: Let \( n \equiv 1 \mod 3 \). Here, we see that \( \eta_{12} = \eta_{23} = \frac{4n-7}{3}, \eta_{13} = \frac{4n-10}{3}, \eta_{14} = \eta_{25} = \eta_{35} = \frac{n-1}{3}, \eta_{14} = \eta_{34} = 3, \eta_{24} = 2 \) and \( \eta_{45} = 1 \). The definition of second chromatic Zagreb index gives the sum
\[ M_2^\alpha (CSF_n) = \sum_{1 \leq t,s \leq \chi(CSF_n)} ts\eta_{ts} = \frac{8(4n-7)}{3} + \frac{3(4n-10)}{3} + \frac{30(n-1)}{3} + 84 = \frac{74n + 136}{3}. \]

Case-2.3: Let \( n \equiv -1 \mod 3 \). Here we see that \( \eta_{12} = \frac{4n-2}{3}, \eta_{23} = \eta_{13} = \frac{4n-5}{3}, \eta_{14} = \eta_{24} = \frac{n-2}{3}, \eta_{34} = \frac{n+1}{3}, \eta_{35} = 2 \) and \( \eta_{45} = 1 \). The definition of second chromatic Zagreb index gives the sum
\[ M_2^\alpha (CSF_n) = \sum_{1 \leq t,s \leq \chi(CSF_n)} ts\eta_{ts} = \frac{2(4n-2)}{3} + \frac{9(4n-5)}{3} + \frac{12(n-2)}{3} + \frac{12(n+1)}{3} + 55 = \frac{68n + 134}{3}. \]

Part (iii): To find the minimum irregularity measurement, we consider the following cases, after applying the colouring mentioned above:

Case-3.1: Let \( n \equiv 0 \mod 3 \). Here, the result obviously follows from the following calculations:
\[ M_3^\alpha (CSF_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\xi(v_i) - \xi(v_j)| = \frac{16n}{3} + \frac{6n}{3} = \frac{22n}{3}. \]

Case-3.2: Let \( n \equiv 1 \mod 3 \). In this case, the result follows from the following calculations:
\[ M_3^\alpha (CSF_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\xi(v_i) - \xi(v_j)| = \frac{2(4n-7)}{3} + \frac{2(4n-10)}{3} + \frac{9(n-1)}{3} + 17 = \frac{25n + 8}{3}. \]
**Case-3.3:** Let \( n \equiv -1 \pmod{3} \). Here the result follows from the following calculations:

\[
M_3^\varphi (CSF_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = \frac{(4n - 2)}{3} + \frac{3(4n - 5)}{3} + \frac{5(n - 2)}{3} + \frac{n + 1}{3} + 9 = \frac{22n + 10}{3}.
\]

**Part (iv):** To calculate the total irregularity of \( CSF_n \), all the possible vertex pairs from \( CSF_n \) have to be considered and their possible colour distances are determined. The possibility of the vertex pairs which contribute to the colour distance can be classified according to the following three cases.

**Case-1:** Let \( n \equiv 0 \pmod{3} \). The combinations possible are charted as \( \{1, 2\}, \{2, 3\}, \{3, 4\} \) contributing 1, \( \{1, 3\}, \{2, 4\} \) contributing 2 and \( \{1, 4\} \) contributing 3. Observe that \( \theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2n}{3} \) and \( \theta(c_4) = 1 \). Thus, we have

\[
M_4^\varphi (CSF_n) = \frac{1}{2} \sum_{u,v \in V(CSF_n)} |\zeta(u) - \zeta(v)| = \frac{16n^2 + 36n}{18}.
\]

**Case-2:** Let \( n \equiv 1 \pmod{3} \). Here the possible combinations which contributes to the colour distances are \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\} \) contributing 1, \( \{1, 3\}, \{2, 4\}, \{3, 5\} \) contributing 2, \( \{1, 4\}, \{2, 5\} \) contributing 3 and \( \{1, 5\} \) contributing 4. We calculate the total irregularity as given below:

\[
M_4^\varphi (CSF_n) = \frac{1}{2} \sum_{u,v \in V(CSF_n)} |\zeta(u) - \zeta(v)| = \frac{16n^2 + 94n - 32}{18}.
\]

**Case-3:** Let \( n \equiv -1 \pmod{3} \). Here the possible combinations which contributes to the colour distances are \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\} \) contributing 1, \( \{1, 3\}, \{2, 4\}, \{3, 5\} \) contributing 2, \( \{1, 4\}, \{2, 5\} \) contributing 3 and \( \{1, 5\} \) contributing 4. We calculate the total irregularity as given below:

\[
M_4^\varphi (CSF_n) = \frac{1}{2} \sum_{u,v \in V(CSF_n)} |\zeta(u) - \zeta(v)| = \frac{16n^2 + 74n - 32}{18}.
\]

Using the minimum parameter colouring we can also work on \( \varphi_+ \) colouring of closed sunflower graphs. Next theorem deals with this matter.

**Theorem 4.2.** For a closed sunflower graph \( CSF_n \), we have
\begin{align*}
(i) \ M_1^{\varphi^-}(CSF_n) &= \begin{cases} 
\frac{58n+3}{3} ; & n \equiv 0 \pmod{3} \\
\frac{100n-73}{3} ; & n \equiv 1 \pmod{3} \\
\frac{100n-35}{3} ; & n \equiv -1 \pmod{3} ; 
\end{cases} \\
(ii) \ M_2^{\varphi^-}(CSF_n) &= \begin{cases} 
\frac{113n}{3} ; & n \equiv 0 \pmod{3} \\
\frac{200n-188}{3} ; & n \equiv 1 \pmod{3} \\
\frac{212n-154}{3} ; & n \equiv -1 \pmod{3} ; 
\end{cases} \\
(iii) \ M_3^{\varphi^-}(CSF_n) &= \begin{cases} 
\frac{22n}{3} ; & n \equiv 0 \pmod{3} \\
\frac{25n+8}{3} ; & n \equiv 1 \pmod{3} \\
\frac{22n+10}{3} ; & n \equiv -1 \pmod{3} ; 
\end{cases} \\
(iv) \ M_4^{\varphi^-}(CSF_n) &= \begin{cases} 
\frac{16n^2+36n}{18} ; & n \equiv 0 \pmod{3} \\
\frac{16n^2+94n-92}{18} ; & n \equiv 1 \pmod{3} \\
\frac{16n^2+74n-32}{18} ; & n \equiv 1 \pmod{3} ; 
\end{cases}
\end{align*}

**Proof.** A closed sunflower graph \(CSF_n\) has chromatic number 4 when \(n \equiv 0 \pmod{3}\) and has chromatic number 5 otherwise. Let \(v_1, v_2, \ldots, v_n\) be the vertices of the inner wheel and \(u_1, u_2, \ldots, u_n\) be the vertices on the rim of the outer wheel and \(v\) be the central vertex. In order to calculate the minimum values of chromatic Zagreb indices we apply the \(\varphi^-\) colouring pattern to \(CSF_n\) as described below.

Let \(n \equiv 0 \pmod{3}\) be assumed. When \(n \equiv 0 \pmod{3}\), obeying the rules of maximum colouring, we can find three colour classes with same cardinality \(\frac{2n}{3}\) and we colour them with maximal colours \(c_4, c_3, c_2\). The central vertex \(v\) is coloured with \(c_1\). So we have \(\theta(c_4) = \theta(c_3) = \theta(c_2) = \frac{2n}{3}\) and \(\theta(c_1) = 1\). Also, \(\eta_{24} = \eta_{23} = \eta_{34} = \frac{4n}{3}\) and \(\eta_{12} = \eta_{13} = \eta_{14} = \frac{n}{3}\).

Let \(n \equiv 1 \pmod{3}\) be assumed. Here we form three colour classes with the maximum independent sets having the same cardinality \(\frac{2(n-1)}{3}\) and we colour them with maximal colours \(c_3, c_4, c_3\). Also the vertices \(u_1\) and \(v_n\) are coloured with \(c_2\) and the central vertex \(v\) is coloured with \(c_1\). So here we have the values, \(\theta(c_3) = \theta(c_4) = \theta(c_3) = \frac{2(n-1)}{3}, \theta(c_2) = 2, \theta(c_1) = 1\) and \(\eta_{45} = \eta_{34} = \frac{4n-7}{3}, \eta_{35} = \frac{4n-10}{3}, \eta_{13} = \eta_{14} = \eta_{15} = \frac{n-1}{3}, \eta_{23} = \eta_{25} = 3, \eta_{24} = 2\) and \(\eta_{12} = 1\).

Let \(n \equiv -1 \pmod{3}\) be assumed. Here we form three colour classes with the maximum independent sets having the same cardinality \(\frac{2(n-1)}{3}\) and we colour them with maximal colours \(c_5, c_4, c_3\). Also the balance two vertices \(v\) and \(v_n\) are coloured with \(c_2\) and \(c_1\) respectively. Thus the following are the values: \(\theta(c_5) = \theta(c_4) = \theta(c_3) = \frac{2n-1}{3}, \theta(c_1) = \theta(c_2) = 1, \eta_{45} = \frac{4n-2}{3}, \eta_{34} = \frac{4n-3}{3}, \eta_{35} = \frac{4n-5}{3}, \eta_{25} = \eta_{24} = \frac{n-2}{3}, \eta_{23} = \frac{n+1}{3}, \eta_{13} = 2\) and \(\eta_{12} = \eta_{14} = \eta_{15} = 1\).

The balance proof follows exactly like that of just previous theorem.

\(\Box\)

5 Chromatic Topological Indices of Blossom Graphs

A blossom graph \(Bl_n\) is the graph obtained by joining all vertices of the outer cycle of a closed sunflower graph \(CSF_n\) to its central vertex. The chromatic Zagreb...
indices of blossom graphs are determined in the following results.

**Theorem 5.1.** For the blossom graph $Bl_n =$, we have

\[
\begin{align*}
(i) & \quad M_1^{\varphi^-}(Bl_n) = \begin{cases} 
\frac{30n+25}{2}, & \text{if } n \text{ is even} \\
\frac{15n+40}{2}, & \text{if } n \text{ is odd}; 
\end{cases} \\
(ii) & \quad M_2^{\varphi^-}(Bl_n) = \begin{cases} 
\frac{95n}{2}, & \text{if } n \text{ is even} \\
\frac{99n-89}{2}, & \text{if } n \text{ is odd}; 
\end{cases} \\
(iii) & \quad M_3^{\varphi^-}(Bl_n) = \begin{cases} 
\frac{22n}{2}, & \text{if } n \text{ is even} \\
\frac{25n-25}{2}, & \text{if } n \text{ is odd}; 
\end{cases} \\
(iv) & \quad M_4^{\varphi^-}(Bl_n) = \begin{cases} 
\frac{5n^2+10n}{8}, & \text{if } n \text{ is even} \\
\frac{11n^2+40n+29}{8}, & \text{if } n \text{ is odd}; 
\end{cases}
\end{align*}
\]

**Proof.** It is so clear that the blossom graph $Bl_n$ has chromatic number 5. In $Bl_n$ let’s put $v_1, v_2, \ldots, v_n$ be the vertices on the outer cycle, $u_1, u_2, \ldots, u_n$ be the vertices on the inner cycle and $u$ be the central vertex. Now we apply the $\varphi^-$ colouring pattern to $Bl_n$ as described below.

When $n$ is even, the outer cycle can be coloured with $c_1$ and $c_2$ alternatively and the inner cycle can be coloured with $c_3$ and $c_4$ alternatively such that both colour classes have cardinality $\frac{n}{2}$. We colour the central vertex $u$ with colour $c_5$.

Now, let $n$ be odd. Here we colour the vertices $\{v_1, v_3, \ldots, v_{n-2}, u_{n-1}\}$ with colour $c_1$ and $\{v_2, v_4, \ldots, v_{n-1}, u_n\}$ with colour $c_2$. The vertices $\{u_1, u_3, \ldots, u_{n-2}, v_n\}$ with colour $c_3$ and $\{u_2, u_4, \ldots, u_{n-3}\}$ with colour $c_4$. The central vertex is coloured with colour $c_5$. Now we proceed to the four parts of the theorem.

**Part (i):** In order to find $M_1^{\varphi^-}$ of $Bl_n$, we first colour the vertices as mentioned above and then proceed to consider the following cases.

**Case-1:** Let $n$ be even, then we have $\theta(c_1) = \theta(c_2) = \theta(c_3) = \theta(c_4) = \frac{n}{2}$ and $\theta(c_5) = 1$. Therefore, the corresponding chromatic topological index is given by

\[
M_1^{\varphi^-}(Bl_n) = \sum_{i=1}^{5} (\zeta(v_i))^2 = 30n + 25.
\]

**Case-2:** Let $n$ be odd. Then, we have $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{n+1}{2}$, $\theta(c_4) = \frac{n-3}{2}$ and $\theta(c_5) = 1$. Now, by the definition of first chromatic Zagreb index, we have

\[
M_1^{\varphi^-}(Bl_n) = \sum_{i=1}^{5} (\zeta(v_i))^2 = 15n + 40.
\]

**Part (ii):** We colour the vertices as per the instructions in the introductory part for odd cases of $n$. For even case, the colouring is mentioned in Case 1. Now consider the following cases:

**Case- 1:** Let $n$ be even. In order to get the minimum value of all second Zagreb indices, we follow like this. The outer cycle can be coloured with $c_1$ and $c_4$ alternatively
and the inner cycle can be coloured with $c_2$ and $c_3$ alternatively such that both colour classes have cardinality $\frac{n}{2}$. We colour the central vertex $u$ with colour $c_5$. Here we see that $\eta_{14} = \eta_{23} = n$ and $\eta_{12} = \eta_{13} = \eta_{15} = \eta_{25} = \eta_{34} = \eta_{35} = \eta_{45} = \eta_{24} = \frac{n}{2}$. The definition of second chromatic Zagreb index gives the sum

$$M_2^{c^*}(Bl_n) = \sum_{1 \leq t, s \leq \chi(Bl_n)} t \eta_{ts} = 6n + 4n + \frac{75n}{2} = \frac{95n}{2}.$$  

**Case- 2**: Let $n$ be odd. Here we see that $\eta_{12} = n + 1, \eta_{13} = \eta_{23} = \frac{n+5}{2}, \eta_{34} = n - 3, \eta_{15} = \eta_{25} = \eta_{35} = \frac{n+1}{2}$ and $\eta_{14} = \eta_{24} = \frac{n-3}{2}$. Hence, we have the sum

$$M_2^{c^*}(Bl_n) = \sum_{1 \leq t, s \leq \chi(Bl_n)} t \eta_{ts} = \frac{99n - 89}{2}.$$  

**Part (iii)**: To find the minimum irregularity measurement, consider the following cases:

**Case- 1**: Let $n$ be even. Here we see that $\eta_{12} + \eta_{23} + \eta_{34} + \eta_{45}$ edges contributes the distance 1 to the total summation while $\eta_{13} + \eta_{24} + \eta_{35}$ contributes the distance 2, $\eta_{14} + \eta_{25}$ contributes the distance 3 and $\eta_{15}$ contributes the distance 4. The result follows from the following calculations:

$$M_3^{c^*}(Bl_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 2n + 9n = 11n.$$

**Case- 2**: Let $n$ be odd. Here also we have that $\eta_{12} + \eta_{23} + \eta_{34} + \eta_{45}$ edges contributes the distance 1 to the total summation while $\eta_{13} + \eta_{24} + \eta_{35}$ contributes the distance 2, $\eta_{14} + \eta_{25}$ contributes the distance 3 and $\eta_{15}$ contributes the distance 4. Then the result follows from the following calculations:

$$M_3^{c^*}(Bl_n) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = (n+1) + 3(n-2) + 4(n-3) + \frac{9(n+1)}{2} = \frac{25(n-1)}{2}.$$  

**Part (iv)**: To calculate the total irregularity of $Bl_n$, all the possible vertex pairs from $Bl_n$ have to be considered and their possible colour distances are determined. The possibility of the vertex pairs which contribute to the colour distance can be classified according to the following two cases.

**Case- 1**: Let $n$ be even. The combinations possible are charted as $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$ contributing 1, $\{1, 3\}, \{2, 4\}, \{3, 5\}$ contributing 2, $\{1, 4\}, \{2, 5\}$ contributing 3 and $\{1, 5\}$ contributing 4. Observe that $\theta(c_1) = \theta(c_2) = \theta(c_3) = \theta(c_4) = \frac{5}{2}$ and $\theta(c_5) = 1$. Thus, we have

$$M_4^{c^*}(Bl_n) = \frac{1}{2} \sum_{u,v \in V(Bl_n)} |\zeta(u) - \zeta(v)| = \frac{5n^2 + 10n}{8}.$$
Case- 2: Let \( n \) be odd. Here the possible combinations which contributes to the colour distances are \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\} contributing 1, \{1, 3\}, \{2, 4\}, \{3, 5\} contributing 2, \{1, 4\}, \{2, 5\} contributing 3 and \{1, 5\} contributing 4. We calculate the total irregularity as given below:

\[
M_{\varphi^+}^e(Bl_n) = \frac{1}{2} \sum_{u,v \in V(Bl_n)} |\zeta(u) - \zeta(v)| = \frac{11n^2 + 40n + 29}{8}
\]

Using the minimum parameter colouring we can also work on \( \varphi_+ \) colouring of blossom graphs. Next theorem deals with this matter.

**Theorem 5.2.** For a blossom graph \( Bl_n \), we have

(i) \( M_1^{\varphi^+}(Bl_n) = \begin{cases} 27n + 1; & \text{if } n \text{ is even} \\ 27n + 20; & \text{if } n \text{ is odd}; \end{cases} \)

(ii) \( M_2^{\varphi^+}(Bl_n) = \begin{cases} 63n; & \text{if } n \text{ is even} \\ \frac{111n + 91}{2}; & \text{if } n \text{ is odd}; \end{cases} \)

(iii) \( M_3^{\varphi^+}(Bl_n) = \begin{cases} \frac{19n}{2}; & \text{if } n \text{ is even} \\ 11n + 1; & \text{if } n \text{ is odd}; \end{cases} \)

(iv) \( M_4^{\varphi^+}(Bl_n) = \begin{cases} \frac{11n^2 + 12n}{8}; & \text{if } n \text{ is even} \\ \frac{10n^2 + 16n - 5}{8}; & \text{if } n \text{ is odd}. \end{cases} \)

**Proof.** The proof follows exactly as mentioned in the proof Theorem 5.1.

6 Conclusion

The topic discussed in this paper do find a variety of applications in chemical graph theory and distribution theory. An outline of chromatic Zagreb indices and irregularity indices of some cycle related graphs are provided in this paper. The study seems to be promising for further studies as these indices can be computed for many graph classes and classes of derived graphs. The chromatic topological indices can be determined for graph operations, graph products and graph powers. The study on the same with respect to different types of graph colourings also seem to be much promising. The concept can be extended to edge colourings and map colourings also. In Chemistry, some interesting studies using the above-mentioned concepts are possible if \( c(v_i) \) (or \( \zeta(v_i) \)) assumes the values such as energy, valency, bond strength etc. Similar studies are possible in various other fields. All these facts highlight the wide scope for further research in this area. Even the chromatic version of other topological indices gives new areas of research with massive applications.
Acknowledgement

The first author would like to acknowledge the academic helps rendered by Centre for Studies in Discrete Mathematics, Vidya Academy of Science and Technology, Thrissur, Kerala, India.

References

[1] H. Abdo, S. Brandt and D. Dimitrov, (2014). The total irregularity of a graph, *Discrete Math. Theor. Computer Sci.*, 16(1): 201–206.

[2] M.O. Alberton, (1997). The irregularity of a graph, *Ars Combin.*, 46: 219–225.

[3] J.A. Bondy, U.S.R. Murty, (2008). *Graph theory*, Springer, New York.

[4] A. Brandstädt, V.B. Le and J.P. Spinrad, (1999). *Graph classes: A survey*, SIAM, Philadelphia.

[5] G. Chartrand and L. Lesniak, (2000). *Graphs and digraphs*, CRC Press, Boca Raton.

[6] N. Deo, (1974). *Graph theory with application to engineering and computer science*, Prentice Hall of India, New Delhi.

[7] G. H. Fath-Tabar, Old and new Zagreb indices of graphs, *MATCH Commun. Math. Comput. Chem.*, 65: 79–84.

[8] I. Gutman and N. Trinajstic, (1972). Graph theory and molecular orbitals, total π electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, 17: 535–538, DOI:10.1016/0009-2614(72)85099-1.

[9] F. Harary, (2001). *Graph theory*, Narosa Publications, New Delhi.

[10] T.R. Jensen, B. Toft, (1995). *Graph colouring problems*, John Wiley & Sons, New York.

[11] M. Kubale, (2004). *Graph colourings*, American Math. Soc., Rhode Island.

[12] J.Kok, N.K. Sudev, U. Mary, (2017). On chromatic Zagreb indices of certain graphs, *Discrete Math. Algorithm. Appl.*, 9(1), 1-11, DOI: 10.1142/S1793830917500148.

[13] J. Kok, N. K. Sudev and K. P. Chithra, (2016). General colouring sums of graphs, *Cogent Math.*, 3(1),: 1–11.

[14] S. Rose and N.K. Sudev, On certain chromatic topological indices of some mycielski graphs, *J. Math. Fund. Sci.*, communicated.

[15] H. Timmerman, T. Roberto, V. Consonni, R. Mannhold and H. Kubinyi, (2002). *Handbook of molecular descriptors*, Wiley-VCH.
[16] E.W. Weisstein, (2011). *CRC concise encyclopedia of mathematics*, CRC Press, Boca Raton.

[17] D.B. West, (2001). *Introduction to graph theory*, Pearson Education, Delhi.

[18] B. Zhou, (2004). Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 52: 113–118.

[19] B. Zhou and I. Gutman, (2005). Further properties of Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 54: 233–239.