A SUFFICIENT CONDITION FOR $L^p$ REGULARITY OF THE BEREZIN TRANSFORM

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ABSTRACT. We prove that the Berezin transform is $L^p$ regular on a large class of domains in $\mathbb{C}^n$ and not $L^2$ regular on the Hartogs triangle.

Let $\Omega$ be a domain in $\mathbb{C}^n$ with boundary $b\Omega$. We denote the Bergman space by $A^2(\Omega)$ and the Bergman projection, the orthogonal projection, by $P_\Omega : L^2(\Omega) \to A^2(\Omega)$. The Bergman Kernel of $\Omega$ is denoted by $K_\Omega : \Omega \times \Omega \to \mathbb{C}$ and, in case $K_\Omega(z,z) \neq 0$ for all $z \in \Omega$,

$$k^\Omega_z(w) = \frac{K_\Omega(w,z)}{\sqrt{K_\Omega(z,z)}}$$

is called the normalized Bergman kernel as $\|k^\Omega_z\|_{L^2} = 1$ for all $z \in \Omega$ where $\|\cdot\|_{L^p}$ denotes the $L^p$ norm on $\Omega$. In this paper, $K_\Omega(z,z) \neq 0$ for all $z \in \Omega$ will be a running assumption.

We note that Boas in [Boa86] observed that the monomial $z_1^jz_2^k$ is square-integrable on the unbounded logarithmically convex Reinhardt domain $B = \{ (z_1,z_2) \in \mathbb{C}^2 : |z_2| < (1 + |z_1|)^{-1} \} \text{ if and only if } 0 \leq j < k$. Hence $K_B(0,0) = 0$. Later Engliš in [Eng07] constructed the unbounded pseudoconvex complete Reinhardt domain $E = \{ (z_1,z_2,z_3) \in \mathbb{C}^3 : |z_3|^2(|z_1|^2 + |z_2|^2)^2 < 1, |z_1|^2 + |z_2|^2 < 1 \}$ such that $A^2(E)$ is infinite dimensional, yet $|K_E(w,z)|^2 / K_E(z,z)$ is discontinuous at $z = 0$ for each fixed $w \neq 0$. Moreover, he showed that the Berezin transform (defined below) of $|z_1|^2$, a smooth bounded function on $E$, is discontinuous at the origin.

Next we will define the Berezin transform of a bounded operator $T : A^2(\Omega) \to A^2(\Omega)$. The Berezin transform $B_\Omega T(z)$ of $T$ at $z \in \Omega$ is defined as

$$B_\Omega T(z) = \langle Tk^\Omega_z, k^\Omega_z \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\Omega)$. For a function $\phi \in L^\infty(\Omega)$ we define its Berezin transform as

$$B_\Omega \phi(z) = \langle T_\phi k^\Omega_z, k^\Omega_z \rangle$$

where $T_\phi : A^2(\Omega) \to A^2(\Omega)$ is the Toeplitz operator with symbol $\phi$. Namely, $T_\phi f = P_\Omega(\phi f)$. We note that $B_\Omega \phi(z) = \int \phi(w) |k^\Omega_z(w)|^2 dV(w)$ for $\phi \in L^\infty(\Omega)$.

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The Berezin transform has been an important tool in operator theory. For instance, the Axler-Zheng theorem and its various extensions establish compactness of operators using boundary values of the Berezin transform (see, for instance, [AZ98, Suá07, Eng99, ČŞ13, ČŞZ18]). The Berezin transform is also related to Schatten norm of Hankel (and Toeplitz) operators (see [Zhu91, Li93, GŞ18]). We refer the reader to [Zhu07] for more details about the Berezin transform.

The fact that \( \|k^\Omega_z\|_{L^2} = 1 \) for all \( z \in \Omega \) implies that \( \|B_\Omega\|_{L^\infty} \leq 1 \) on any domain \( \Omega \) with the property that \( K_\Omega(z,z) \neq 0 \) for all \( z \in \Omega \). Furthermore, \( \|B_\Omega\|_{L^\infty} = 1 \) if \( \Omega \) is bounded. Even though \( B_\Omega \) is defined on \( L^\infty(\Omega) \), on some domains, such as the unit ball and the polydisc, it extends as a bounded map on \( L^p(\Omega) \) for some \( p < \infty \). In this article, we want to know on what domains this is the case. This problem has been studied in few cases. Dostanić in [Dos08] showed that the Berezin transform is \( L^p \) regular on the unit disc and computed the norm \( \|B_\Omega|_{L^p} = \pi(p + 1)/(p^2 \sin(\pi/p)) \) for \( 1 < p \leq \infty \). Later Liu and Zhou in [LZ12] and Marković in [Mar15] generalized Dostanić’s result to the unit ball. On the bidisc \( \mathbb{D}^2 \), Lee in [Lee97, Proposition 3.2] showed that the Berezin transform is bounded on \( L^p(\mathbb{D}^2) \) for \( p > 1 \) and unbounded on \( L^1(\mathbb{D}^2) \). In this article we are interested in investigating \( L^p \) regularity of the Berezin transform on domains in \( \mathbb{C}^n \). However, we will not attempt to compute its norm.

Now we define the class of domains on which we will prove \( L^p \) regularity of the Berezin transform.

**Definition 1.** Let \( \Omega \) and \( K_\Omega \) be a domain in \( \mathbb{C}^n \) and the Bergman kernel of \( \Omega \), respectively. We say that \( \Omega \) satisfies property BR if \( K_\Omega(z,z) \neq 0 \) for all \( z \in \Omega \) and

\[
\sup \left\{ \frac{|K_\Omega(w,z)|}{K_\Omega(z,z)} : z, w \in \Omega \right\} < \infty.
\]

The infimum of \( \left\{ \frac{|K_\Omega(w,z)|}{K_\Omega(z,z)} : z, w \in \Omega \right\} \) is 0 whenever \( \Omega \) is a bounded domain in \( \mathbb{C}^n \). One can see this as follows: using a holomorphic affine change of coordinates, if necessary, we assume that \( \Omega \subset \{ z \in \mathbb{C}^n : \Re(z_n) > 0 \} \) and \( 0 \in b\Omega \). Next we define \( f(z) = \log(z_n) \). Then \( |f(z)| \to \infty \) as \( z \to 0 \) and one can show that \( f \in A^2(\Omega) \). That is, \( K_\Omega(z,z) \to \infty \) as \( z \to 0 \). Hence for a fixed \( w \in \Omega \) we have

\[
\frac{|K_\Omega(w,z)|}{K_\Omega(z,z)} \leq \frac{\sqrt{K_\Omega(z,z)K_\Omega(w,w)}}{K_\Omega(z,z)} = \frac{\sqrt{K_\Omega(w,w)}}{\sqrt{K_\Omega(z,z)}} \to 0 \text{ as } z \to 0.
\]

Therefore, \( \inf \left\{ \frac{|K_\Omega(w,z)|}{K_\Omega(z,z)} : z, w \in \Omega \right\} = 0 \).

Property BR is a biholomorphic invariant in the following sense: Assume that \( \Omega_1 \) satisfies property BR and \( F : \Omega_1 \to \Omega_2 \) is a biholomorphism with the property that there exists \( 1 < C < \infty \) such that \( C^{-1} < |\det J_Cf(z)| < C \) for all \( z \in \Omega_1 \) where \( J_Cf \) denotes the holomorphic Jacobian matrix of \( F \). Then \( \Omega_2 \) satisfies property BR. One can easily see this using
the transformation formula for the Bergman kernel (see, for instance, [Kra01, Proposition 1.4.12], [JP13, Proposition 12.1.10], [Ran86, Theorem 4.9]).

Domains with property BR are not necessarily bounded or pseudoconvex. For example, one can show that the upper half plane in \( \mathbb{C} \) and \( \{ z \in \mathbb{C}^2 : 1 < |z| < 2 \} \), a bounded non-pseudoconvex domain, have property BR. On the other hand, not every pseudoconvex domain satisfies property BR. For instance, the Hartogs triangle,

\[ \mathcal{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < |z_1| < 1\}, \]

does not satisfy property BR. One can show this as follows. Using the fact that \( (\mathbb{D} \setminus \{0\}) \times \mathbb{D} \) is biholomorphic to \( \mathcal{H} \) one can compute that the Bergman kernel of \( \mathcal{H} \) (see, for instance, [Edh16, EM16])

\[ K_\mathcal{H}(w, z) = \frac{w_1 \bar{z}_1}{\pi^2(w_1 \bar{z}_1 - w_2 \bar{z}_2)^2(1 - w_1 \bar{z}_1)^2}. \]

Then for \( 0 < \delta, \varepsilon < 1 \) we have

\[ \frac{|K_\mathcal{H}((\varepsilon, 0), (\delta, 0))|}{K_\mathcal{H}((\delta, 0), (\delta, 0))} = \frac{\delta(1 - \delta^2)^2}{\varepsilon(1 - \delta \varepsilon)^2}. \]

Now if we fix \( \delta \) and let \( \varepsilon \to 0 \) we see that the ratio \( |K_\mathcal{H}((\varepsilon, 0), (\delta, 0))|/K_\mathcal{H}((\delta, 0), (\delta, 0)) \to \infty \). Hence \( \mathcal{H} \) does not satisfy property BR.

Domains satisfying property BR include the domains with sharp \( B \)-type Bergman kernels (see Lemma 4) and their cross products such as the polydisc (see Corollary 1). Since it is technical, we will not give a precise definition of domains with sharp \( B \)-type Bergman kernels and we refer the reader to [KLT19]. However, we note that domains with sharp \( B \)-type Bergman kernels form a large class of domains in \( \mathbb{C}^n \) including: strongly pseudoconvex domains (such as the unit ball), pseudoconvex finite type domains in \( \mathbb{C}^2 \), finite type convex domains, decoupled domains of finite type, and pseudoconvex domains of finite type whose Levi-forms have only one degenerate eigenvalue or comparable eigenvalues. Domains with sharp \( B \)-type Bergman kernels were first introduced by McNeal and Stein in [MS94] in which they prove that if the Bergman kernel of a finite type convex domain is of \( B \)-type, then the absolute Bergman projection, defined below, is bounded on \( L^p \) for \( 1 < p < \infty \).

The absolute Bergman projection \( P^+_{\Omega} \) of \( \Omega \) is defined as

\[ P^+_{\Omega} f(z) = \int_{\Omega} |K_\Omega(z, w)| f(w) dV(w). \]

In the proposition below, we show that \( L^p \) regularity of the absolute Bergman projection implies \( L^p \) regularity of the Berezin transform for domains with property BR. The main
reason the absolute Bergman projection has been studied is the following simple fact: $L^p$-regularity of the absolute Bergman projection implies the $L^p$-regularity of the Bergman projection. For an incomplete list we refer the reader to [PS77, MS94, McN94, LS12, EM16, Huo18, HWW21] and references therein.

**Proposition 1.** Let $\Omega$ be a domain in $\mathbb{C}^n$ satisfying property BR and the absolute Bergman projection $P^+_\Omega : L^{p_0}(\Omega) \to L^{p_0}(\Omega)$ is bounded for some $1 < p_0 < \infty$. Then $B _\Omega : L^p(\Omega) \to L^p(\Omega)$ is bounded for all $p_0 \leq p \leq \infty$ and

$$\|B _\Omega\|_{L^{p_0}} \leq \sup \left\{ \frac{|K _\Omega(w,z)|}{K _\Omega(z,z)} : z, w \in \Omega \right\} \|P^+_\Omega\|_{L^{p_0}}.$$  

The absolute Bergman projections are $L^p$ regular on domains whose Bergman kernels are of sharp $B$-type (see Lemma 3, below). On products of such domains we have the following result.

**Theorem 1.** Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be a finite product of bounded domains whose Bergman kernels are of sharp $B$-type. Then the Berezin transform $B _\Omega : L^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 < p \leq \infty$. Furthermore,

$$\|B _\Omega\|_{L^p} \leq \sup \left\{ \frac{|K _\Omega(w,z)|}{K _\Omega(z,z)} : z, w \in \Omega \right\} \|P^+_\Omega_1\|_{L^p} \cdots \|P^+_\Omega_m\|_{L^p}$$

for $1 < p < \infty$ and where $P^+ _\Omega _j$ is the absolute Bergman projection of $\Omega_j$ for $1 \leq j \leq m$.

Our second result is about the $L^2$ irregularity of the Berezin transform on the Hartogs triangle.

**Theorem 2.** Let $H = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < |z_1| < 1\}$ be the Hartogs triangle. Then the Berezin transform $B _H$ is not a bounded mapping on $L^2(H)$.

**Proofs of Proposition 1 and Theorem 1**

We start this section with the proof of Proposition 1.

**Proof of Proposition 1.** First, as mentioned in the introduction, $B _\Omega : L^\infty(\Omega) \to L^\infty(\Omega)$ is bounded and $\|B _\Omega\|_{L^\infty} \leq 1$.

Let $C _\Omega = \sup \left\{ \frac{|K _\Omega(w,z)|}{K _\Omega(z,z)} : z, w \in \Omega \right\} < \infty$ and $\phi \in L^\infty(\Omega)$. Then

$$|B _\Omega \phi(z)| \leq \int \frac{|K _\Omega(w,z)|^2}{K _\Omega(z,z)} |\phi(w)| dV(w)$$

$$\leq C _\Omega \int |K _\Omega(w,z)||\phi(w)| dV(w)$$

$$= C _\Omega P^+_\Omega |\phi(z)|.$$
Then
\[ \|B_\Omega \phi\|_{L^p_0} \leq C_\Omega \|P_\Omega^+ \phi\|_{L^p_0} \leq C_\Omega \|P_\Omega^+\|_{L^p_0} \|\phi\|_{L^p_0}. \]

Therefore, using the fact that measurable bounded compactly supported functions are dense in \( L^p_0(\Omega) \), we conclude that
\[ \|B_\Omega\|_{L^p_0} \leq C_\Omega \|P_\Omega^+\|_{L^p_0}. \]

Then Riesz-Thorin Interpolation Theorem (see, for instance, [Fol99, Section 6.5]) implies that \( B_\Omega : L^p(\Omega) \rightarrow L^p(\Omega) \) is bounded for all \( p_0 \leq p \leq \infty \).

\[ \square \]

Remark 1. We note that the Berezin transform is not a self-adjoint operator, in general. Indeed one can compute that
\[ \langle B_\Omega \phi, \psi \rangle = \int \int \phi(w) \frac{|K_\Omega(w,z)|^2}{K_\Omega(z,z)} \psi(z) dV(w) dV(z) \]
\[ = \int \phi(w) K_\Omega(w,w) \left( \int \frac{|K_\Omega(z,w)|^2}{K_\Omega(w,w)} \psi(z) dV(z) \right) dV(w) \]
\[ = \langle \phi, M_{K_\Omega} B_\Omega M_{\frac{1}{K_\Omega}} \psi \rangle \]
for \( \phi, \psi \in C_0^\infty(\Omega) \) where \( M \) denotes the multiplication operator. That is \( B_\Omega^* = M_{K_\Omega} B_\Omega M_{\frac{1}{K_\Omega}} \).

Then, on the unit disc we have \( B_D 1 = 1 \) while
\[ B_D^* 1(0) = \int_D \frac{|K_D(0,w)|^2}{K_D(w,w)} dV(w) = \frac{1}{\pi} \int_D (1-|w|^2)^2 dV(w) = 2 \int_0^1 (1-r^2)^2 r dr = \frac{1}{3}. \]

Hence \( B_D^* \neq B_D \).

We note that, since \( B_\Omega \) is not necessarily self-adjoint, we cannot use \( L^p \) regularity of \( B_\Omega \) for \( p_0 \leq p \leq \infty \) in the proof of Proposition 1 to conclude \( L^p \) regularity for \( 1 < p < p_0/(p_0 - 1) \).

We will skip the proof of the following lemma as it is simply a consequence of the fact that the Bergman kernel of the product of domains is the product of the Bergman kernel of each factor.

Lemma 1. Let \( \Omega = \Omega_1 \times \cdots \times \Omega_m \) be a product domain such that each \( \Omega_j \) is a domain satisfying property BR. Then \( \Omega \) satisfies property BR.

Lemma 2. Let \( \Omega = \Omega_1 \times \cdots \times \Omega_m \) be a product domain such that each \( P_{\Omega_j}^+ : L^p(\Omega_j) \rightarrow L^p(\Omega_j) \) is bounded for some \( 1 \leq p < \infty \). Then \( P_{\Omega}^+ : L^p(\Omega) \rightarrow L^p(\Omega) \) is bounded and
\[ \|P_{\Omega}^+\|_{L^p} = \|P_{\Omega_1}^+\|_{L^p(\Omega_1)} \cdots \|P_{\Omega_m}^+\|_{L^p(\Omega_m)}. \]

Proof. For \( 1 > \varepsilon > 0 \) we choose \( \phi_{j,\varepsilon} \in L^p(\Omega_j) \) such that \( \|\phi_{j,\varepsilon}\|_{L^p(\Omega_j)} = 1 \) and
\[ \|P_{\Omega_j}^+ \phi_{j,\varepsilon}\|_{L^p(\Omega_j)} \geq \|P_{\Omega_j}^+\|_{L^p} - \varepsilon. \]
for $j = 1, \ldots, m$. Then for $\phi_\varepsilon = \phi_{1, \varepsilon} \cdots \phi_{m, \varepsilon}$ we have $\|\phi_\varepsilon\|_{L^p(\Omega)} = 1$ and
\[
\left(\|P^+_{\Omega_1}\|_{L^p(\Omega_1)} - \varepsilon\right) \cdots \left(\|P^+_{\Omega_m}\|_{L^p(\Omega_m)} - \varepsilon\right) \leq \|P^+_{\Omega_1}\phi_{1, \varepsilon}\|_{L^p(\Omega_1)} \cdots \|P^+_{\Omega_m}\phi_{m, \varepsilon}\|_{L^p(\Omega_m)}
\]
\[
= \|P^+_{\Omega}\phi_\varepsilon\|_{L^p(\Omega)} 
\leq \|P^+_{\Omega}\|_{L^p(\Omega)}.
\]

Then by letting $\varepsilon \to 0$ we get
\[
\|P^+_{\Omega_1}\|_{L^p(\Omega_1)} \cdots \|P^+_{\Omega_m}\|_{L^p(\Omega_m)} \leq \|P^+_{\Omega}\|_{L^p(\Omega)}.
\]

To prove the converse, we first assume that $\Omega = \Omega_1 \times \Omega_2$. Let $\phi \in L^p(\Omega)$. Then using Fubini’s Theorem below we have
\[
\|P^+_{\Omega}\phi\|^p_{L^p} 
\leq \int_{\Omega_2} \int_{\Omega_1} \left( \int_{\Omega_1} |K_{\Omega_1}(z_1, w_1)| \int_{\Omega_2} |K_{\Omega_2}(z_2, w_2)| |\phi(w_1, w_2)| dV(w_2) dV(w_1) \right)^p dV(z_1) dV(z_2)
\leq \|P^+_{\Omega_1}\|_{L^p(\Omega_1)} \int_{\Omega_2} \int_{\Omega_1} \left( \int_{\Omega_2} |K_{\Omega_2}(z_2, w_2)| |\phi(z_1, w_2)| dV(w_2) \right)^p dV(z_1) dV(z_2)
\]
\[
= \|P^+_{\Omega_1}\|_{L^p(\Omega_1)} \int_{\Omega_2} \int_{\Omega_1} \left( \int_{\Omega_2} |K_{\Omega_2}(z_2, w_2)| |\phi(z_1, w_2)| dV(w_2) \right)^p dV(z_2) dV(z_1)
\]
\[
\leq \|P^+_{\Omega_1}\|^p_{L^p(\Omega_1)} \|P^+_{\Omega_2}\|^p_{L^p(\Omega_2)} \int_{\Omega} |\phi(z)|^p dV(z).
\]

Hence $\|P^+_{\Omega}\|_{L^p} \leq \|P^+_{\Omega_1}\|_{L^p(\Omega_1)} \|P^+_{\Omega_2}\|_{L^p(\Omega_2)}$. Then we use an induction argument to conclude that $\|P^+_{\Omega}\|_{L^p} \leq \|P^+_{\Omega_1}\|_{L^p(\Omega_1)} \cdots \|P^+_{\Omega_m}\|_{L^p(\Omega_m)}$. \qed

The following proposition is an easy consequence of [KLT19, Proposition 2.4]. We sketch the proof here for the convenience of the reader.

**Lemma 3.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ whose Bergman kernel is of $B$-type. Then the absolute Bergman projection $P^+_{\Omega} : L^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 < p < \infty$.

**Proof.** Let $\rho(z)$ denote the distance from $z$ to the boundary of $\Omega$. Then [KLT19, Proposition 2.4] implies that for $0 < \varepsilon < 1$ there exists $c_\varepsilon$ such that $P^+_{\Omega}\rho^{-\varepsilon} \leq c_\varepsilon \rho^{-\varepsilon}$ on $\Omega$. Then a standard argument (see, for instance, [MS94, pg 184]) using the Schur’s test implies that $P^+_{\Omega} : L^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 < p < \infty$. \qed

Next we prove the fact that domains whose Bergman kernels are of sharp $B$-type satisfy property BR.

**Lemma 4.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ whose Bergman kernel is of sharp $B$-type. Then $\Omega$ satisfies property BR.
Proof. We use an argument similar to [KLT19, (2.5)] to conclude that there exists $0 < C < \infty$ such that
\[
\frac{|K_\Omega(w, z)|}{K_\Omega(z, z)} \leq \frac{C |r(z)|^2}{(|r(z)| + |r(w)|)^2} \leq C
\]
for all $z, w \in \Omega$. Therefore, $\sup \left\{ \frac{|K_\Omega(w, z)|}{K_\Omega(z, z)} : z, w \in \Omega \right\} < \infty$. □

Corollary 1. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be a product domain such that each $\Omega_j$ is a bounded domain whose Bergman kernel is of sharp $B$-type. Then
i. $\Omega$ satisfies property BR,
ii. the absolute Bergman projection $P_\Omega^+ : L^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 < p < \infty$ and
\[
\|P_\Omega^+\|_{L^p} = \|P_\Omega^+\|_{L^p(\Omega_1)} \cdots \|P_\Omega^+\|_{L^p(\Omega_m)}.
\]  
(1)

Proof. To prove i. we use the fact that the Bergman kernel of $\Omega$ is the product of the Bergman kernels of $\Omega_j$s together with Lemma 4 to show that
\[
\sup \left\{ \frac{|K_\Omega(w, z)|}{K_\Omega(z, z)} : z, w \in \Omega \right\} < \infty.
\]
Hence, $\Omega$ satisfies property BR.

ii. is simply is a consequence of Lemmas 2 and 3. □

Proof of Theorem 1. The proof is very similar to the proof of Proposition 1 except here we use (1) in Corollary 1. □

Proof of Theorem 2

Before we start the proof of Theorem 2, we note that $P_H^+$ is $L^p$ regular for $4/3 < p < 4$ (see proof of [EM16, Theorem 1.2]) and [EM16, Corollary 3.5]). However, we will show that the Berezin transform is not $L^2$ regular on the Bergman space on the Hartogs triangle. Therefore, $L^p$-regularity of the absolute Bergman projection is not a sufficient condition for the $L^p$-regularity of the Berezin transform.

Proof of Theorem 2. The set $\{z_1^n z_2^m : m \geq 0, n + m \geq -1\}$ forms an orthogonal basis for $A^2(H)$ (see, for example, [CZ16]). Let $f_\epsilon(w) = |w_1|^{-2+2\epsilon}$ for $\epsilon > 0$. Then $f_\epsilon \in L^2(H)$ because when $\epsilon > 0$, the following calculation holds
\[
\|f_\epsilon\|_{L^2}^2 = 4\pi^2 \int_0^1 \int_0^{t_1} r_1^{-3+4\epsilon} r_2 dr_2 dr_1 = 2\pi^2 \int_0^1 r_1^{-1+4\epsilon} dr_1 = \frac{\pi^2}{2\epsilon} < \infty,
\]
and when $\epsilon \leq 0$, the integral diverges. We can write the Bergman kernel $K_H$ in series form as follows
\[
K_H(w_1, w_2, z_1, z_2) = \sum_{m \geq 0, n + m \geq -1} a_{nm} w_1^n w_2^m \overline{z_1^n z_2^m}
\]
where

\[ a_{nm} = \frac{1}{\|z_1^n z_2^m\|_L^2} = \frac{1}{4\pi^2 \int_0^1 \int_0^{r_1} r_1^{2n+1} r_2^{2m+1} dr_2 dr_1} = \frac{(m+1)(n+m+2)}{\pi^2} \]

for \( m \geq 0 \) and \( n + m + 1 \geq 0 \). Next we compute

\[
\langle f_\varepsilon K_H(\cdot,z), K_H(\cdot,z) \rangle = \sum_{m \geq 0, n + m \geq -1} |a_{nm}|^2 |z_1|^{2n} |z_2|^{2m} 4\pi^2 \int_0^1 \int_0^{r_1} r_1^{2n+1+2\varepsilon} r_2^{2m+1} dr_2 dr_1
\]

\[ = \sum_{m \geq 0, n + m \geq -1} |a_{nm}|^2 |z_1|^{2n} |z_2|^{2m} \frac{\pi^2}{(m+1)(n+m+1+\varepsilon)}. \]

Using \( k = n + m + 1 \) in the second equality below we get

\[
B_H f_\varepsilon(z) = \frac{1}{\pi^2 K_H(z,z)} \sum_{m \geq 0, n + m \geq -1} |z_1|^{2n} |z_2|^{2m} \frac{(m+1)(n+m+2)^2}{n+m+1+\varepsilon}
\]

\[ = \frac{1}{\pi^2 K_H(z,z)} \sum_{m,k=0}^{\infty} |z_1|^{2k-2m-2} |z_2|^{2m} \frac{(m+1)(k+1)^2}{k+\varepsilon}
\]

\[ = \frac{1}{\pi^2 |z_1|^2 K_H(z,z)} \left( \sum_{m=0}^{\infty} (m+1) \frac{|z_2|^{2m}}{|z_1|^{2m}} \right) \left( \sum_{k=0}^{\infty} \frac{(k+1)^2}{k+\varepsilon} |z_1|^{2k} \right). \]

Next we note that

\[ \frac{1}{\pi^2 |z_1|^2 K_H(z,z)} = \frac{(|z_1|^2 - |z_2|^2)^2 (1 - |z_1|^2)^2}{|z_1|^4} = \left( 1 - \left| \frac{z_2}{z_1} \right|^2 \right)^2 (1 - |z_1|^2)^2 \]

and \( \frac{1}{(1-x)^2} = \sum_{m=0}^{\infty} (m+1) x^m \) for \( |x| < 1 \). Therefore,

\[ B_H f_\varepsilon(z) = (1 - |z_1|^2)^2 \sum_{k=0}^{\infty} \frac{(k+1)^2}{k+\varepsilon} |z_1|^{2k}. \]

Then

\[ \|B_H f_\varepsilon\|_{L^2}^2 = \int_{\mathbb{H}} |1 - |z_1|^2|^4 \left| \sum_{k=0}^{\infty} \frac{(k+1)^2}{k+\varepsilon} |z_1|^{2k} \right|^2 dV(z) \]

\[ \geq \frac{1}{\varepsilon^2} \int_{\mathbb{H}} |1 - |z_1|^2|^4 dV(z). \]

Then, we have \( \|f_\varepsilon\|_{L^2} = \pi / \sqrt{2} \varepsilon \) while \( \|B_H f_\varepsilon\|_{L^2} \geq \varepsilon^{-1} \|(1 - |z_1|^2)^2\|_{L^2} \) for \( \varepsilon > 0 \). Hence

\[ \frac{\|B_H f_\varepsilon\|_{L^2}}{\|f_\varepsilon\|_{L^2}} \geq \frac{\sqrt{2} \|(1 - |z_1|^2)^2\|_{L^2}}{\pi \sqrt{\varepsilon}} \]

for \( \varepsilon > 0 \). Therefore, \( B_H \) is unbounded on \( L^2(\mathbb{H}) \). We note that, by interpolation, we also conclude that \( B_H \) is unbounded on \( L^p(\mathbb{H}) \) for \( p \leq 2 \). \( \square \)
Remark 2. Let $f(z) = z^{-1} \in A^2(H)$. Then
\[
\langle f, k_z^H \rangle = \frac{1}{z_1 \sqrt{K_H(z, z)}}.
\]
Then (2), for $z_1 = 1/j$ and $z_2 = 0$, implies that
\[
|\langle f, k_{(1/j,0)}^H \rangle| = \pi(1 - j^{-2}) \to \pi \text{ as } j \to \infty.
\]
Therefore, $\{k_{(1/j,0)}^H\}$ does not converge to 0 weakly.

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REFERENCES

[AZ98] Sheldon Axler and Dechao Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. 47 (1998), no. 2, 387–400.

[Boa86] Harold P. Boas, *Counterexample to the Lu Qi-Keng conjecture*, Proc. Amer. Math. Soc. 97 (1986), no. 2, 374–375.

[ČŠ13] Željko Ćučković and Sönmez Şahutoğlu, *Axler-Zheng type theorem on a class of domains in $\mathbb{C}^n$*, Integral Equations Operator Theory 77 (2013), no. 3, 397–405.

[ČŠZ18] Željko Ćučković, Sönmez Şahutoğlu, and Yunus E. Zeytuncu, *A local weighted Axler-Zheng theorem in $\mathbb{C}^n$*, Pacific J. Math. 294 (2018), no. 1, 89–106.

[CZ16] Debraj Chakrabarti and Yunus E. Zeytuncu, *$L^p$ mapping properties of the Bergman projection on the Hartogs triangle*, Proc. Am. Math. Soc. 144 (2016), no. 4, 1643–1653.

[Dos08] Milutin Dostanić, *Norm of Berezin transform on $L^p$ space*, J. Anal. Math. 104 (2008), 13–23.

[Edh16] Luke D. Edholm, *Bergman theory of certain generalized Hartogs triangles*, Pac. J. Math. 284 (2016), no. 2, 327–342.

[EM16] L. D. Edholm and J. D. McNeal, *The Bergman projection on fat Hartogs triangles: $L^p$ boundedness*, Proc. Am. Math. Soc. 144 (2016), no. 5, 2185–2196.

[Eng99] Miroslav Engliš, *Compact Toeplitz operators via the Berezin transform on bounded symmetric domains*, Integral Equations Operator Theory 33 (1999), no. 4, 426–455.

[Eng07] Miroslav Engliš, *Singular Berezin transforms*, Complex Anal. Oper. Theory 1 (2007), no. 4, 533–548.

[Fol99] Gerald B. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication.

[GŞ18] Nihat Gökhan Göğüş and Sönmez Şahutoğlu, *Schatten class Hankel and $\overline{\partial}$-Neumann operators on pseudoconvex domains in $\mathbb{C}^n$*, Monatsh. Math. 187 (2018), no. 2, 237–245.

[Huo18] Zhenghui Huo, *$L^p$ estimates for the Bergman projection on some Reinhardt domains*, Proc. Amer. Math. Soc. 146 (2018), no. 6, 2541–2553.

[HWW21] Zhenghui Huo, Nathan A. Wagner, and Brett D. Wick, *Bekollé-Bonami estimates on some pseudoconvex domains*, Bull. Sci. Math. 170 (2021), Paper No. 102993, 36.

[JP13] Marek Jarnicki and Peter Pflug, *Invariant distances and metrics in complex analysis*, extended ed., De Gruyter Expositions in Mathematics, vol. 9, Walter de Gruyter GmbH & Co. KG, Berlin, 2013.
[KLT19] Tran Vu Khanh, Jiakun Liu, and Phung Trong Thuc, Bergman-Toeplitz operators on weakly pseudoconvex domains, Math. Z. 291 (2019), no. 1-2, 591–607.

[Kra01] Steven G. Krantz, Function theory of several complex variables, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1992 edition.

[Lee97] Jaesung Lee, On the Berezin transform on $D^n$, Commun. Korean Math. Soc. 12 (1997), no. 2, 311–324.

[Li93] Huiping Li, Schatten class Hankel operators on the Bergman spaces of strongly pseudoconvex domains, Proc. Am. Math. Soc. 119 (1993), no. 4, 1211–1221.

[LS12] Loredana Lanzani and Elias M. Stein, The Bergman projection in $L^p$ for domains with minimal smoothness, Illinois J. Math. 56 (2012), no. 1, 127–154 (2013).

[LZ12] Congwen Liu and Lifang Zhou, On the p-norm of the Berezin transform, Illinois J. Math. 56 (2012), no. 2, 497–505.

[Mar15] Marijan Marković, On the Forelli-Rudin projection theorem, Integral Equations Operator Theory 81 (2015), no. 3, 409–425.

[McN94] Jeffery D. McNeal, The Bergman projection as a singular integral operator, J. Geom. Anal. 4 (1994), no. 1, 91–103.

[MS94] J. D. McNeal and E. M. Stein, Mapping properties of the Bergman projection on convex domains of finite type, Duke Math. J. 73 (1994), no. 1, 177–199.

[PS77] D. H. Phong and E. M. Stein, Estimates for the Bergman and Szegő projections on strongly pseudo-convex domains, Duke Math. J. 44 (1977), no. 3, 695–704.

[Ran86] R. Michael Range, Holomorphic functions and integral representations in several complex variables, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986.

[Suá07] Daniel Suárez, The essential norm of operators in the Toeplitz algebra on $A^p(B_n)$, Indiana Univ. Math. J. 56 (2007), no. 5, 2185–2232.

[Zhu91] Kehe Zhu, Schatten class Hankel operators on the Bergman space of the unit ball, Am. J. Math. 113 (1991), no. 1, 147–167.

[Zhu07] , Operator theory in function spaces, second ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.

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