WHAT IS THE GEOMETRY OF SUPERSPACE? *

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Abstract
We investigate certain properties of the Wheeler-DeWitt metric (for constant lapse) in canonical General Relativity associated with its non-definite nature.

1. Introduction
As is well known, the dynamics of General Relativity can be formulated in terms of a constrained Hamiltonian system, with the configuration space for pure gravity being given by the space of all Riemannian metrics on a 3-dimensional manifold $\Sigma$ of fixed but arbitrary topology. We call this space $Q(\Sigma)$ to indicate its dependence upon the choice of $\Sigma$. In this Hamiltonian picture, space-time is looked upon as a history of dynamically evolving geometries on $\Sigma$ represented by a path $g_{ab}(s)$ in $Q(\Sigma)$. In the special gauge where the lapse function $N = 1$ and the shift vector $N^a = 0$, the vacuum Einstein equations without cosmological constant decompose into the dynamical part (in units where $16\pi G/c^4 = 1$; a dot means differentiation with respect to the parameter $s$).

$$\ddot{g}_{ab} + \Gamma_{ab}^{ijkl} \dot{g}_{ij} \dot{g}_{kl} = -2(R_{ab} - \frac{1}{4}g_{ab}R),$$ (1)

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and the constraint part

\[ G^{ab\ cd}g_{ab}\dot{g}_{cd} - 4\sqrt{g}R = 0 \quad \text{Hamiltonian Constraint} \quad (2) \]

\[ G^{ab\ cd}\nabla_b\dot{g}_{cd} = 0 \quad \text{Momentum Constraint.} \quad (3) \]

\( R_{ab} \) and \( R \) are the Ricci-tensor and Ricci scalar of the metric \( g_{ab} \). \( G^{ab\ cd} \) is the DeWitt metric (DeWitt 1967) on the space of symmetric positive definite matrices (defined below as \( G_{\beta}^{ab\ cd} \) for \( \beta = 1 \)). The \( \Gamma \)-symbols in (1) are the Christoffel symbols for the DeWitt metric. If (2)(3) are satisfied initially it follows from (1) that they continue to be satisfied throughout the evolution. (1) and (2) have an obvious geometric interpretation, whereas (3) says that the velocity must be orthogonal to the orbits of the diffeomorphism group. This is explained in more detail below.

Due to diffeomorphism invariance, \( Q(\Sigma) \) is endowed with an action of the Diffeomorphism group \( D(\Sigma) \) of \( \Sigma \): each point of \( Q(\Sigma) \) is a Riemannian metric on \( \Sigma \) which is acted upon by a diffeomorphism via pull-back. To different metrics which are connected by a diffeomorphism in such a way are considered to be physically indistinguishable. Redundancies of this sort are avoided by going to the quotient \( S(\Sigma) := Q(\Sigma)/D(\Sigma) \), called the superspace associated to \( \Sigma \). It represents the space of geometries rather than metrics on \( \Sigma \). Although superspace now faithfully labels physical configurations, paths in superspace do not faithfully represent space-times. Two different paths of geometries may be obtained by “waving” \( \Sigma \) differently through the same space-time. But not every path in \( S(\Sigma) \) can be obtained by appropriately “waving” \( \Sigma \) through a given space-time. The former freedom is precisely the freedom in the choice of the lapse function.

The existence of some geometric structures of superspace is implicit in many of the investigations into the dynamical structure of General Relativity. So for example in John Wheeler’s view of General Relativity as Geometrodynamics (Wheeler 1968) and the associated quantization programme, where superspace serves as domain for the quantum mechanical state functional. The equations to be satisfied by this state functional, the Wheeler-DeWitt equations, explicitly refer to the metric (DeWitt 1967 and Wheeler 1968), just like the classical equation (1). Julian Barbour sees the fulfilment of the Machian requirement on General Relativity in a successfull formulation of dynamics solely within superspace (Barbour 1993). The dynamical principle envisaged is a kind of geodesic equation with respect to some generalized metric on superspace (Barbour 1993). All these attempts motivate to have a closer look at some of the metric structures of superspace.
So we first ask: “What is the geometry of \( Q(\Sigma) \)?” Mathematically there is a variety of possibilities to endow \( Q(\Sigma) \) with a geometry. On the other hand, the laws of General Relativity select a family of such metrics, one for each choice of the lapse function \( N \). For the particular choice \( N = 1 \) this is displayed in equations (1)-(3). They define a metric on \( Q(\Sigma) \):

\[
G(h, k) := \int_{\Sigma} G^{abcd} h_{ab} k_{cd} d^3 x ,
\]

which we call the Wheeler-DeWitt (WDW) metric. In this article we investigate some properties of this particular metric connected with its indefinite nature.

Note that due to the constraint (3), General Relativity only uses the WDW metric to calculate inner products on the subspace of tangent vectors satisfying (3), which requires those vectors to be WDW-orthogonal to the directions of the diffeomorphisms. We call the diffeomorphism directions vertical and the WDW-orthogonal directions horizontal. Due to the indefinite nature of the WDW metric, the horizontal subspace might also contain vertical directions. When this is not the case, the WDW metric restricted to the horizontal subspace defines a metric on the quotient space \( S(\Sigma) \). But what generally happens is that in different regions of superspace this quotient-space metric has different signatures. Such signature changes are precisely signalled by non-trivial intersections of vertical with horizontal subspaces. To clarify the WDW geometry of superspace would mean to: 1) characterize the singular set in \( Q(\Sigma) \) which consists of those points where horizontal and vertical subspaces intersect non-trivially, and 2) study the restriction of the WDW-metric to the horizontal subspaces. Only partial results are known so far. Note that we do not consider the constraint equation (2) in the same way as we did with (3). This would select a non-linear subspace of vectors and thus prevent us from having a pseudo-Riemannian structure. In this respect we differ from the approach taken by Barbour (Barbour 1993).

What we wish to show here is that the WDW metric has rather special properties. This we do by introducing a 1-parameter family of fiducial metrics of which the WDW metric is one member. The parameter will be called \( \beta \) and the WDW metric is obtained for \( \beta = 1 \).
2. Ultralocal Metrics

In order to do differential geometry on $Q(\Sigma)$ we heuristically assume that $Q(\Sigma)$ is a differentiable manifold with tangent space $T_g(Q)$ and cotangent space $T^*_g(Q)$ at the metric $g_{ab} \in Q$ (we shall sometimes drop the reference to $\Sigma$). Elements of $T_g(Q)$ are any symmetric covariant tensor field and elements of $T^*_g(Q)$ are any symmetric contravariant tensor density of weight one on $\Sigma$. Suppose we want to define a metric, i.e. a non-degenerate bilinear form in each $T_g(Q)$. Then up to an overall constant there is a unique 1-parameter family of ultralocal metrics (i.e. depending locally on $g_{ab}$ but not on its derivatives) defined in the following way: take $h, k \in T_g(Q)$, then

$$G_{\beta}(h, k) := \int_{\Sigma} G_{ab \ cd}^{\beta} h_{ab} k_{cd} \, d^3x, \quad (5)$$

where $G_{ab \ cd}^{\beta} = \frac{\sqrt{g}}{2} (g^{ac} g^{bd} + g^{ad} g^{bc} - 2\beta g^{ab} g^{cd})$. \quad (6)

The WDW metric, introduced in (4), is just $G_1$. Given $p, q \in T^*_g(\Sigma)$, the “inverse” metric, $G_{\beta}^{-1}$, is

$$G_{\beta}^{-1}(p, q) := \int_{\Sigma} G_{ab \ cd}^{\beta} p^{ab} q^{cd} \, d^3x, \quad (7)$$

where $G_{ab \ cd}^{\beta} = \frac{1}{2\sqrt{g}} (g_{ac} g_{bd} + g_{ad} g_{bc} - 2\alpha g_{ab} g_{cd})$ \quad (8)

with $\alpha + \beta = 3\alpha \beta$, so that $G_{\beta}^{ab \ nm} G_{\cd \ nm}^{\beta} = \frac{1}{2} (\delta_{c}^{a} \delta_{d}^{b} + \delta_{d}^{a} \delta_{c}^{b})$. \quad (9)

These are non-degenerate bilinear forms for $\beta \neq 1/3$ (we exclude $\beta = 1/3$), positive definite for $\beta < 1/3$ and of mixed signature for $\beta > 1/3$ with infinitely many plus as well as minus signs. Because they are ultralocal, they arise from metrics on the space $S^+_3$ of symmetric positive definite matrices – which is diffeomorphic to the homogeneous space $GL(3, R)/SO(3) \cong R^6$ – carrying the metric $G_{\beta}$. One has $GL(3, R)/SO(3) \cong SL(3, R)/SO(3) \times R^+ \cong R^5 \times R^+$ and with respect to this decomposition the metric has a simple warped-product form

$$G_{ab \ cd}^{\beta} dg_{ab} \otimes dg_{cd} = -\epsilon d\tau \otimes d\tau + \frac{r^2}{c^2} tr(r^{-1} dr \otimes r^{-1} dr), \quad (10)$$

with $c^2 = 16|\beta - 1/3|$, $\tau = cg^{1/2}$, $r_{ab} = g^{-\frac{1}{3}} g_{ab}$, $\epsilon = \text{sign}(\beta - 1/3)$. \quad (11)
The matrices $r_{ab}$ are just the coordinates on $SL(3, R)/SO(3)$ and the trace in (10) is just the left-$SL(3, R)$ invariant metric on this space. This gives rise to 8 Killing vectors of $G_\beta$. An additional homothety is generated by the multiplicative action of $R^+$ on the $\tau$ coordinate. Moreover, geodesics in this metric can be explicitly determined (DeWitt 1967). If we now regard $Q(\Sigma)$ as a mapping space, i.e. as the space of all smooth mappings from $\Sigma$ into $S^+_3$, endowed with the metric (5), then, due to its ultralocal nature, geometric structures like Killing fields, homotheties and geodesics of the “target” metric (10) are inherited by the full metric (5). For example, dragging the maps $g_{ab}(x)$ along a Killing flow in $S^+_3$ produces a Killing flow in $Q(\Sigma)$. In this way, some geometry of the infinite dimensional $Q(\Sigma)$ can be studied by looking at the 6-dimensional $S^+_3$.

Note also that expression (5) is invariant under diffeomorphisms of $\Sigma$. An infinitesimal diffeomorphism is represented by a vector field $\xi$ on $\Sigma$ and gives rise to a vector field $X^\xi$ on $Q(\Sigma)$:

$$X^\xi_{ab} = \nabla_a \xi_b + \nabla_b \xi_a,$$  \hspace{1cm} (12)

which is a Killing field of the metric (5). The totality of vectors of the form (12) at $g \in Q(\Sigma)$ span what we call the vertical vector space $V_g \subset T_g(Q)$. With respect to $G_\beta$ we can define the orthogonal complement to $V_g$ which we call the horizontal vector space $H_\beta^g \subset T_g(Q)$. From (5)(6) and (12) we have

$$k_{ab} \in H_\beta^g \Leftrightarrow \nabla^a (k_{ab} - \beta g_{ab} k^c_c) = 0.$$  \hspace{1cm} (13)

Under the isometric action of $D(\Sigma)$ on $Q(\Sigma)$ horizontal spaces are clearly mapped into horizontal spaces.

If we set $\beta = 0$, the metric (5) is positive definite such that orthogonality also implies transversality, i.e. $V_g \cap H^0_g = \{0\}$. It is in fact true that the tangent space splits into the direct sum of closed orthogonal subspaces: $T_g(\Sigma) = V_g \oplus H^0_g$. This allows to define a Riemannian geometry on the quotient space $S(\Sigma)$ by identifying its tangent spaces with the horizontal spaces in $T(Q)$ (Ebin 1970). (Here we pretend $S(\Sigma)$ being a genuine manifold). This works for all $\beta < 1/3$. We are, however, interested in the range $1/3 \leq \beta \leq 1$ with special attention paid to the transition $\beta < 1$ to $\beta = 1$.

For $\beta > 1/3$ the metric (5) is not definite anymore such that generally $V_g \cap H^\beta_g \neq \{0\}$ for such $\beta$. A simple example is the following: Take as $\Sigma$ a 3-manifold that
carries a flat metric $g$. In $T_g(\Sigma)$ consider the infinite dimensional vector subspace given by all vectors of the form $k_{ab} = \nabla_a \nabla_b \phi$, where $\phi$ is a smooth function on $\Sigma$. These vectors satisfy (13) for $\beta = 1$ and are therefore in $H^1_g$. But they are also of the form (12), with $2\xi_a = \nabla_a \phi$, and hence in $V_g$. Moreover, suppose the metric is only flat in an open subset $U \subset \Sigma$. Then we can repeat the argument but this time only using functions $\phi$ with compact support inside $U$. Again, these give rise to an infinite intersection $V_g \cap H^1_g$ for each such partially flat metric $g$. Clearly, vectors in $H^\beta_g \cap V_g$ are necessarily of zero $G^\beta$-norm.

3. Some Observations Concerning the WDW Metric

It follows from (12) and (13) that a vertical vector $X^\xi$ is horizontal, if and only if

\[ D_\beta \xi_a := -\nabla^b (\nabla_b \xi_a - \nabla_a \xi_b) - 2(1 - \beta)\nabla_a \nabla^b \xi_b - 2R^b_a \xi_b = 0, \]

where $R^b_a$ denote the mixed components of the Ricci-tensor. Killing vectors, if existent, are obvious solutions but these do not interest us since they correspond to zero $X^\xi$. For $0 \leq \beta < 1/3$ these are the only solutions. This implies that for $\beta > 1/3$ any non-Killing solution must have non-zero divergence, since for zero divergence fields the $\beta$ dependence in (14) drops out. A more elegant way to write $D_\beta$ is, using the exterior derivative $d$, its adjoint $\delta$ (given by minus the divergence on the first index) and writing $\text{Ric}$ for the map induced by $R^b_a$:

\[ D_\beta = \delta d + 2(1 - \beta)d\delta - 2\text{Ric}, \]

which also displays its formal self-adjointness. The $G_\beta$-norm of $X^\xi$ is given by

\[ G_\beta(X^\xi, X^\xi) = 2 \int_{\Sigma} \xi^a D_\beta \xi_a d^3x. \]

For $\beta \leq 1$ and $\text{Ric} < 0$ (i.e. strictly negative eigenvalues) this operator is manifestly positive and $G_\beta$ restricted to $V_g$ is thus positive definite. In particular, we have $V_g \cap H^\beta_g = \{0\}$ for all $g$ such that $\text{Ric} < 0$ and $\beta \leq 1$. Since it is known that any 3-manifold $\Sigma$ admits such Ricci-negative metrics (Gao and Yau 1986), this tells us that in every superspace there are open regions (the Ricci-negative geometries) with well defined WDW metric, given by the restriction of $G_1$ to $H^1_g$, whose signature has infinitely many plus and minus signs.
For a flat metric $g$ and values $\beta < 1$, $D_\beta$ is non-negative with kernel given by the covariantly constant $\xi$. Indeed, from (11) it follows that $\xi$ is curl- and divergence-free on a flat manifold, hence covariantly constant. But this also means that $\xi$ is Killing and therefore $X^\xi$ zero. So for $g$ flat we have $V_g \cap H^\beta_g = \{0\}$ for $\beta < 1$. On the other hand, for $\beta = 1$ and $g$ flat, we can only infer from (15) that $\xi$ must be closed, hence exact or harmonic. But harmonicity implies Killing so that all horizontal $X^\xi$ are given by the expressions anticipated in the previous section. As stated there, we can localize the construction and obtain an infinite subspace in the intersection $V_g \cap H^1_g$ for metrics $g$ which contain a flat region $U \subset \Sigma$. Clearly, any manifold admits such metrics. In particular, this tells us that in every superspace there are regions where no WDW metric is defined.

It is more difficult to obtain general results for metrics which are neither Ricci-negative nor flat. For the very special class of non-flat Einstein metrics it is at least easy to see that for $\beta = 1$ $H^1_g \cap V_g$ is zero. Indeed, for $R_{ab} = \lambda g_{ab}$, where $\lambda \in R - \{0\}$, (15) implies $0 = \delta D_1 \xi = 2\lambda \delta \xi$, so that $\xi$ must be divergence free and hence $X^\xi$ zero. So there exists a WDW metric for non-flat Einstein geometries in $\mathcal{S}(\Sigma)$, given by the restriction of $\mathcal{G}_1$ to $H^1_g$. For the study of such metrics it is instructive to look at a particular example in detail to which we now turn.

As non-flat Einstein metric we take the standard round metric on the three-sphere with some unspecified radius. Here $Ric > 0$ and not much can be directly read off (15) for general $\beta$. But taking elements of $T_g(\mathbb{Q})$ as first order perturbations of $g$, and expanding them in terms of the well known complete set of tensor harmonics (Gerlach and Sengupta 1978) one can establish the following scenario: For $1/3 < \beta < 1$ the number of negative directions (i.e. the number of linearly independent vectors of negative $\mathcal{G}_\beta$-norm) is finite in $V_g$ and infinite in $H^\beta_g$. For the discrete values $\beta = \beta_n$, where

$$\beta_n := \frac{n^2 - 3}{n^2 - 1}, \quad n \in \{3, 4, 5, \ldots\},$$

the intersection $V_g \cap H^\beta_g$ is non-trivial and of some finite dimension $d_n > 0$. At other values of $\beta$ it is zero. What turns out to happen is that when $\beta$ passes the value $\beta_n$ from below, $d_n$ of the negative directions change from $H^\beta_g$ to $V_g$. Since the $\beta_n$ accumulate at 1, this happens infinitely often as we turn up $\beta$ to 1. At $\beta = 1$ only a single negative direction has remained in $H^1_g$ and infinitely many are now in $V_g$. The intersection $V_g \cap H^1_g$ is in fact zero, in accordance with the more general argument.
given above. $G_1$ restricted to $H^1_g$ is of Lorentzian signature $(-, +, +, +, \ldots)$. This is directly related to the statement made in quantum cosmology, that the Wheeler-DeWitt equation\(^2\) (for constant lapse) for perturbations around the three-sphere is hyperbolic (Halliwell and Hawking 1985). It follows from our considerations that this can at best be locally valid since the metric for constant lapse necessarily suffers from signature changes\(^3\). Note also how delicately the signature structure of $G_\beta$ restricted to $H^\beta_g$ depends on whether $\beta < 1$ or $\beta = 1$.

There are other interesting differences between $\beta < 1$ and $\beta = 1$. Quite striking is the existence of an infinite dimensional intersection $H^1_g \cap V_g$ for flat $g$. This means that $D_1$ cannot be an elliptic operator since these have finite dimensional kernels. And, in fact, calculating the the principal symbol for $D_\beta$ from (14), we obtain

$$\sigma_\beta(\zeta)_b^a = \|\zeta\|^2 \left( \delta_b^a + (1 - 2\beta) \frac{\zeta^a \zeta_b}{\|\zeta\|^2} \right). \quad (18)$$

This matrix is positive definite for $\beta < 1$, invertible but not positive definite for $\beta > 1$ and singular positive semi-definite for $\beta = 1$. Expressed in standard terminology this says that the operator $D_\beta$ is strongly elliptic in the first, elliptic but not strongly elliptic in the second, and degenerate elliptic but not elliptic in the third case. This relates to the problem of how one would actually calculate the metric on superspace at the regular points. Throughout we said that it would be obtained by restricting the metric $G_\beta$ to the horizontal spaces $H^\beta_g$. But this means that we have to explicitly calculate the projection $T_g(Q) \rightarrow H^\beta_g$. A general tangent vector $k_{ab} \in T_g(Q)$ is projected by adding a vertical vector $X \xi$ so that the sum is horizontal, i.e. satisfies (13). This is equivalent to solving

$$D_\beta \xi_b = \nabla^a (k_{ab} - \beta g_{ab} k^c_c) \quad (19)$$

as equation for $\xi$ and given right hand side. Uniqueness for $X^\xi$ is given at regular geometries, i.e. those for which the kernel of $D_\beta$ consists of Killing vectors only. Since the right hand side is orthogonal to Killing vectors, ellipticity (for $\beta < 1$) guarantees existence for any $k_{ab}$. It is not clear to us at this moment whether the failure of ellipticity for $\beta = 1$ can in fact imply any problem. For example, in the special cases where $g_{ab}$ is an Einstein metric, we can Hodge decompose $\xi$ and the right hand side of (19) into exact, co-exact and harmonic forms. The Einstein condition then prevents the Ricci-term in $D_1$ to couple these components,
so that (19) decomposes into 3 decoupled equations for the Hodge modes, two purely algebraic ones and an elliptic partial differential equation for the co-exact mode. In this case we can thus show existence by restricting to appropriate subspaces.

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Notes

1 In three dimensions an Einstein metric implies constant sectional curvature so that Σ is a space form. But not only is the topology of Σ severely restricted (e.g. its second homotopy group must be trivial). If Σ allows for Einstein metrics, they only form a finite dimensional subspace in superspace which is in fact of dimension one if the Einstein constant is non-zero. In these cases the only deformations are the constant rescalings of the metric. In this sense Einstein metrics are very special.

2 There is one Wheeler-DeWitt equation for each smearing function. If written without smearing functions (as a distribution), the Wheeler-DeWitt equations for pure gravity looks like an infinite number of six-dimensional Klein-Gordon equations, one per point \( x \in \Sigma \) for the six components \( \{ g_{ab}(x) \} \). If added together with a smearing function, the resulting equation is clearly ultrahyperbolic. Only if the directions of differentiation are restricted to lie in a horizontal subspace, or even further, like suggested by Hawking (Hawking 1983, chapter 5), one may be able to eliminate all but one of the negative directions. This particular Wheeler-DeWitt equation may then said to be locally hyperbolic on superspace.

3 In applications, the Wheeler-DeWitt equations have only been studied in neighbourhoods of highly symmetric metrics like the one on the three sphere considered here. It would be interesting to know how “far” from such a point one has to go in order to encounter singular regions and signature change. The regions \( Ric < 0 \) do not seem “close”, and the reason why the Wheeler-DeWitt equations have not been studied in neighbourhoods of those metrics seems to be the fact that \( Ric < 0 \) metrics do not allow for any metrics with symmetries.
References

Barbour, Julian B. (1993). See contribution to this Volume.

DeWitt, Bryce S. (1967). “Quantum Theory Gravity I. The Canonical Theory.” Physical Review 160: 1113-1148.

Ebin, David G. (1970). “The Manifold of Riemannian Metrics”. Proceedings of the American Mathematical Society, Symposia in Pure Mathematics, Global Analysis 15: 11-40.

Friedman, John L. and Higuchi, Atsushi (1990). “Symmetry and Time on the Superspace of Asymptotically Flat Geometries”. Physical Review D 41: 2479-2486.

Gerlach, Ulrich H. and Sengupta, Uday K. (1978). “Homogeneous Collapsing Star: Tensor and Vector Harmonics for Matter and Field Asymmetries”. Physical Review D 18: 1773-1784.

Gao, Zhiyong L., and Yau, Shing-Tung (1986). “The Existence of Ricci Negatively Curved Metrics on Three Manifolds”. Inventiones Mathematicae 85: 637-652.

Halliwell, Jonathan, J. and Hawking, Stephen, W. (1985). “The Origin of Structure in the Universe”. Physical Review D 31: 1777-1791.

Hawking, Stephen W. (1984). “The Quantum State of the Universe”. Nuclear Physics B 239: 257-276.

Wheeler, John A. (1968). “Superspace and the Nature of Quantum Geometrodynamics.” In Battelle Rencontres, 1967 Lectures in Mathematics and Physics. Cecile M. DeWitt and John A. Wheeler, ed. New York and Amsterdam: W.A. Benjamin Inc., pp. 242-307.