Improved packings of \( n(n-1) \) unit squares in a square

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Abstract

Let \( s(n) \) be the side of the smallest square into which we can pack \( n \) unit squares. The purpose of this paper is to prove that \( s(n^2 - n) < n \) for all \( n \geq 12 \). Besides, we show that \( s(18^2 - 17) < 18, s(17^2 - 16) < 17, \) and \( s(16^2 - 15) < 16 \).

Mathematics Subject Classifications: 05B40, 52C15

1 Introduction

The problem of packing equal squares in a square has been around for some 40 years [1]. Let \( s(n) \) be the side of the smallest square into which we can pack \( n \) unit squares. Nagamochi [3] proved that \( s(n^2 - 2) = s(n^2 - 1) = n \). It follows from [1] that \( s(n^2 - O(n^{7/11})) < n \) for big \( n \). From [4] it follows that the 7/11 degree can be reduced to 5/8.

An important question is to find the minimum \( n \) for which \( s(n^2 - n) < n \). For small \( n \), only \( s(2) = 2 \) and \( s(6) = 3 \) have been proved, but we don’t even know the proof of \( s(12) = 4 \). It was proved in [2] that \( s(n^2 - n - 1) < n \) for \( 3 < n < 11 \). Due to Lars Cleemann it was known that \( s(17^2 - 17) < 17 \) [2]. Nagamochi in [3] mistakenly says that the following is proved in [2]

\[
s(n^2 - n) < n \quad \forall n \geq 17. (1)
\]

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The truth is that in [2] a sporadic squeezable packing of 272 unit squares in a square \((17,17)\) is given, proving that \(s(17^2 - 17) < 17\), but from this it does not follow that \(s(18^2 - 18) < 18\) etc. Thus, Nagamochi’s implicit conjecture (1) needs a proof.

We prove the conjecture and even more: \(s(n^2 - n) < n \ \forall n \geq 12\), and, moreover,

\[
\begin{align*}
s(18^2 - 17) &< 18, \\
s(17^2 - 16) &< 17, \\
s(16^2 - 15) &< 16.
\end{align*}
\]

2 Some squeezable packing of rectangles

Let a packing of \(m\) unit squares in a rectangle \(R = (R_x, R_y)\) be given. We assume that \((R_x - 1)(R_y - 1) < m < R_x R_y\) and we can’t pack a unit square in the waste area. This packing is called squeezable if both sides of a rectangle can be reduced, i.e., for some \(\delta > 0\) there exists a packing of \(m\) unit squares in a rectangle \((R_x - \delta, R_y - \delta)\). The maximum of such \(\delta > 0\) is called the value of squeezing and is denoted by \(\delta(R, m)\). We write \(\delta(R, m) = 0\) if the packing is not squeezable.

The property of squeezability of a packing for small parameters can be proved rather simply. However proving this property for large parameters is a non-trivial mathematical problem. The following obvious formula connects \(\delta(R, m)\) and \(s(n)\):

\[
s(n) = \lceil s(n) \rceil - \delta(\lceil s(n) \rceil, \lceil s(n) \rceil, n).
\]

If \(\delta((R_x, R_y), m) < 1\) then the fact that for integer \(R_x, R_y\)

\[
\delta((R_x, R_y), m) \leq \delta((R_x + 1, R_y), m + R_y - 1)
\]

can be proved by adding \(R_y - 1\) unit squares to the \(x\)-side of a rectangle \((R_x, R_y)\). Figure 1 shows the basic idea for efficiently packing unit squares in a square \(S\), where rectangles \(C\) and \(D\) are integer and the waste is in rectangles \(A\) and \(B\). It is easy to see that if the packing of unit squares in rectangles \(A, B\) is squeezable, then the packing of unit squares in \(S\) is squeezable and

\[
\delta(S, \cdot) \geq \min(\delta(A, \cdot), \delta(B, \cdot)).
\]

This bound can be increased if we note that after squeezing there is a little space between rectangles \(A, B\). We can give this space to a rectangle with minimal squeezing value in order to increase that value and thus to increase the evaluation of \(\delta(S, \cdot)\).

Let us consider a packing of 26 unit squares in a rectangle \((4, 8)\) (see Figure 2). This packing is centrally symmetric and the waste is equal to 6.

In Figure 2 we see one of the main ideas for packing unit squares: using of stacks \((4, 1)\) tilted by an angle \(\alpha = \arcsin(8/17)\). The main idea for squeezing a packing follows from it: tilting stacks \((4, 1)\) by an angle \(\alpha + \varepsilon\) so that the stack \((4, 1)\) is located in a vertical strip of width \(4 - \delta\), where \(\varepsilon\) and \(\delta\) are sufficiently small. Hereinafter we determine the orientation of a unit square by a unit vector \((x, y)\) with \(x > 0, y \geq 0, x^2 + y^2 = 1\) directed along the side of this unit square. If the bottom vertex of the unit square is at the origin then the three other vertices have coordinates \((x, y), (x - y, y + x), (-y, x)\). Note that if
two points $P_t, P_b$ are taken on the top side and the bottom side of this unit square then the scalar product $\langle P_t - P_b, (x, y) \rangle$ is equal to 1.

Continuing with the example in Figure 2, after increasing the tilt the stack $(4, 1)$ in a vertical strip of width $4 - \delta$ has orientation $(x_1, y_1), x_1 > 0, y_1 \geq 0$ satisfying the system of equations

$$4x_1 + y_1 = 4 - \delta, x_1^2 + y_1^2 = 1.$$ 

To evaluate the squeezing value $\delta((4, 8), 26)$, we use the bisection method. The packing remains centrally symmetric. The distance between the point $P = (P_x, P_y) = (1 - \delta/2, 2 - \delta/2)$ and the upper side of the square $S_2$ intersecting the line $x = 1 - \delta/2$ in the point $P_1 = (P_{1x}, P_{1y}) = (1 - \delta/2, (1 - \delta/2)x_1 + \frac{1}{x_1} + \frac{1-x_1}{x_1y_1})$ is critical. For $\delta = 0.01$ we have $x_1 = .877695\ldots, y_1 = .479219\ldots, P_y - P_{1y} = 0.021604 > 0$. For $\delta = 0.02 x_1 = .87312663\ldots, y_1 = .48749347\ldots, P_y - P_{1y} = -0.0061309\ldots < 0$. The bisection method gives evaluation $\delta((4, 8), 26) > 0.0177702$.

Figure 3 shows a more complex example, a centrally symmetric squeezable packing of
64 unit squares in a rectangle (6,12). Four unit squares: \( S_3, S_6 \) and their symmetric ones have not the orientation \( (\frac{35}{37}, \frac{12}{37}) \) nor \( (1,0) \). Hereinafter we denote points and squares by the same indices in different figures without losing accuracy.

In this packing the left vertex of \( S_2 \) is on a side of \( S_1 \). The square \( S_3 \) is placed so that the right vertices of squares \( S_2, S_5 \), and the top vertex of \( S_4 \) are on the sides of \( S_3 \). Vertices of the squares \( S_8, S_7, S_9 \) are on sides of \( S_6 \). Calculations show that there is a small distance between \( S_3 \) and \( S_6 \), which guarantees squeezability of the given packing.

To calculate the squeezing value \( \delta((6,12), 64) \), take \( \delta = 0.004 \) and define the existence of a packing 64 unit squares in a rectangle \((6-\delta, 12-\delta)\). The distance between the right vertex of \( S_3 \) and the top side of \( S_6 \) should be not less than 1.

Table 1 contains calculations with \( \delta = 0.004 \).

Calculations with \( \delta = 0.005 \) give \( \langle P_8 - P_5, (x_2, y_2) \rangle = 0.999617371807702270 \), i.e., the squares \( S_3, S_6 \) intersect. The bisection method gives evaluation \( \delta((6,12), 64) > .00490823 \).

A packing of 58 unit squares in a rectangle \((6,11-2/35)\) can be obtained by removing one stack \((6,1)\) in Figure 3 and lifting up by \( 37/35 \) all the squares that are below this
Table 1: Calculations with $\delta = 0.004$

| Object | Formulae or system of equations | Numerical value |
|--------|---------------------------------|-----------------|
| $\delta$ | $y_1^2 + x_1^2 = 1, 6y_1 + x_1 = 6 - \delta$ | 0.004 |
| Orientation ($(x_1, y_1)$) of stack (6,1) | $P_0 = (-2 + \delta/2, (2 - \delta/2)x_1/y_1 + \frac{2}{y_1} + \frac{1-y_1}{x_1y_1})$ | (-1.998,2.989621361) |
| | $P_1 = P_0 + (x_1 + y_1, y_1 - x_1)$ | (-0.725282311,3.6062165968) |
| | $P_2 = (\delta/2 - 1.4 - \delta/2)$ | (-0.998,3.998) |
| | $P_3 = (3 - 3y_1 - \frac{\delta}{2}, -(3 - 3y_1 - \frac{\delta}{2})x_1/y_1 + \frac{x_1}{y_1})$ | (1.1640306130, 4.177378839) |
| Orientation $(x_2, y_2)$ of $S_3$ | $x_2^2 + y_2^2 = 1,$ $(P_2 - P_3, (-y_2, x_2)) = 1$ | (3.90085325,9207787136) |
| | $P_4 = (P_1, (x_2, y_2)) \cdot (x_2, y_2) + (P_2, (y_2, -x_2)) \cdot (y_2, -x_2)$ | (-1.09722313,76378828) |
| | $P_5 = P_4 + (x_2 + y_2, y_2 - x_2)$ | (0.213640902,4.29448167498) |
| | $P_6 = (\frac{1}{2}\delta, 5 - \frac{1}{2}\delta)$ | (0.002,4.998) |
| | $P_7 = (3 - \delta/2, -3 - \delta/2)x_1/y_1 + 5(0,1/y_1) + 2(-y_1, x_1)$ | (1.108687,4.9079035) |
| | $P_8 = (1 - \delta/2, 5 - \delta/2)$ | (0.998,4.998) |
| Orientation $(x_3, y_3)$ of $S_6$ | $x_3^2 + y_3^2 = 1,$ $(P_6 - P_7, (-y_3, x_3)) = 1$ | (5.062565099,862382946) |
| Distance between $P_5$ and top side of $S_6$ | $(P_8 - P_5, (x_3, y_3)) = 1$ | 1.00378910536129684 |

Consider a more difficult problem of a squeezable packing of 43 unit squares in a rectangle $(5,10)$. In Figure 4 six unit squares $S_1, S_4, S_9, S_{10}, S_{11}, S_{12}$ have not the orientation $\langle 0,1 \rangle$ nor $(1,0)$. The square $S_1$ has a vertex on the side of the rectangle $(5,10)$, one on a side of $S_2$, and one on a side of $S_3$. The right vertex of $S_1$ is on the bottom side of $S_4$. $S_4$ is tilted so that the bottom right vertex of $S_3$ is on the left side of $S_4$ and the top vertex of the stack $(3, 1)$ is on the right side of $S_4$. The left vertex of $S_5$ is on the side of $S_6$. The squares $S_9$, $S_{10}$ are tilted by the same angle so that the vertex of $S_8$ is on the side of $S_9$, the vertex of $S_5$ is on the bottom side of $S_9$, and the vertex of $S_7$ is on the bottom side of $S_{10}$. The squares $S_{11}, S_{12}$ form a rectangle $(2,1)$. The right vertex of $S_{12}$ is on the right side of a rectangle $(5,10)$. The vertex of $S_{13}$ is on the top side of $S_{11}$. The bottom sides of $S_{11}$ and $S_{12}$ are parallel to the line connecting the right vertices of $S_9$ and $S_{10}$. The vertex of $S_{14}$ is on the bottom side of $S_{15}$. Calculations show that there is a small distance 0.0055111... between the bottom side of the rectangle $(2,1) = S_{11} \cup S_{12}$ and the line connecting the
Figure 4: Squeezable packing of 43 unit squares in a rectangle (5,10)

right vertices of $S_9$ and $S_{10}$. This guarantees squeezability of the given packing.

Calculation of the squeezing value $\delta((5,10), 43)$ gives the evaluation $\delta((5,10), 43) > 0.0009652493$. This packing plays an important role in the squeezable packing of 132 unit squares in a square (12,12). Below we show the evaluation of $\delta((12,12), 132)$. From this evaluation one can obtain the evaluation of $\delta((5,10), 43)$. Analogous calculations give the evaluation of the squeezing value $\delta((5,9), 38) > 0.020403$.

Table 2 contains the evaluations of the squeezing values of some rectangles.

| Rectangle $R$ | $n$  | $\delta(R, n)$              |
|---------------|------|-----------------------------|
| (4,8)         | 26   | $> 0.01777021751$           |
| (5,10)        | 43   | $> 0.0009652493$            |
| (5,9)         | 38   | $> 0.020403$                |
| (6,12)        | 64   | $> 0.004908231774819$       |
| (6,11)        | 58   | $> 0.01681735886$           |

Table 2. Evaluations of squeezing value of some rectangles

To prove conjecture (1), we need the following lemma.

**Lemma 1.** For any $k \geq 3$ there exists a squeezable packing of $4k^2 + 6k - 2$ unit squares in a rectangle $(2k, 2k + 4)$ (the waste is equal to $2k + 2$).
The proof is technically simple and can be understood from Figure 5, showing a centrally symmetric squeezable packing of 86 unit squares in a rectangle \((8, 12)\). For an arbitrary \(k \geq 3\), the centrally symmetric packing in the upper half of a rectangle \((2k, 2k + 4)\) consists of 2 staircases. A staircase with orientation \((1,0)\) having \(\frac{k(k+1)}{2}\) unit squares, and a staircase with orientation \((x_1, y_1) = \left(\frac{4k^2-1}{4k^2+1}, \frac{4k}{4k^2+1}\right)\) that has \(\frac{(3k-1)(k+2)}{2}\) unit squares. The top vertex of \(S_{k+1}\) has ordinate

\[ y_{k+1} = -\frac{4k^2}{4k^2 - 1} + (k + 2) \frac{4k^2 + 1}{4k^2 - 1} + (k - 1) \frac{4k}{4k^2 + 1} < \]

\[ < -\frac{4k^2}{4k^2 - 1} + (k + 2) \frac{4k^2 + 1}{4k^2 - 1} + (k - 1) \frac{4k}{4k^2 - 1} = k + 2 - \frac{2(k - 2)}{4k^2 - 1} < k + 2, \]

i.e., \(S_{k+1}\) is in rectangle \((2k, 2k + 4)\). The top vertex of \(S_0\) has ordinate

\[\frac{4k^2}{4k^2 - 1} + \frac{4k^2 - 1}{4k^2 + 1} = 2 + \frac{1}{4k^2 - 1} - \frac{2}{4k^2 + 1} < 2,\]

i.e., \(S_0\) does not intersect the staircase with orientation \((1,0)\). Each square \(S_j, 1 \leq j \leq k\) intersects the vertical line \(x = k - j\) in the point

\[(k - j, j \cdot \frac{1 - x_1}{x_1y_1} + (k - j) \frac{y_1}{x_1} + \frac{j}{x_1}).\]

The ordinate of this point satisfies

\[ j \cdot \frac{1 - x_1}{x_1y_1} + (k - j) \frac{y_1}{x_1} + \frac{j}{x_1} = 1 + j + \frac{1}{2} \cdot j \cdot \frac{(-4k^2 + 4k + 1) + 2k}{k(4k^2 - 1)} < 1 + j, \]

i.e., none of the \(S_j, 1 \leq j \leq k\) intersects the staircase with orientation \((1,0)\). We see that there is a positive distance between the two staircases. Therefore, this packing is squeezable. The lemma is proved.

### 3 Improved squeezable packing of some squares

As mentioned in the introduction, in [3] Nagamochi mistakenly says that in [2] it is proved that

\[ s(n^2 - n) < n \quad \forall n \geq 17. \]

Thus he implicitly formulates the conjecture (4). For the proof of this conjecture we use lemma 1 as follows.

For even \(n \geq 14\) we use Figure 1 with rectangles \(A = (12, 6), B = (n - 10, n - 6), C = (10, n - 6), D = (n - 12, 6)\).

For odd \(n \geq 13\) we use Figure 1 with rectangles \(A = (10, 5), B = (n - 9, n - 5), C = (9, n - 5), D = (n - 10, 5)\).

Thus the conjecture (4) is proved for \(n \geq 13\).

For the proof of this conjecture for \(n = 12\) see Figure 6.
Figure 5: Squeezable packing of $4k^2 + 6k - 2$ unit squares in a rectangle $(2k, 2k + 4)$

The packing in Figure 6 is obtained from the squeezable packing in rectangles $(8, 4)$, $(5, 10)$. In the packing in $(5, 10)$ we tilt the angular squares $S_1, S_2$ by an angle $\arcsin(10/26)$ so that the bottom vertex of $S_1$ has an integer $y$-coordinate and $S_2$ has intruded space in the rectangle $(8, 4)$. From the packing in $(8, 4)$ we remove two right top squares and move to the left by $1/20$ unit squares tilted by an angle $\arcsin(8/17)$ so that the bottom vertex of $S_3$ is on the side of $S_4$. The small distance between $S_2$ and $S_5$ makes the packing in Figure 6 squeezable.

Thus we have proved that

$$s(n^2 - n) < n \ \forall n \geq 12.$$  

To evaluate $\delta((12, 12), 132)$, take $\delta = 0.002$. The origin is in the right bottom vertex of the integer rectangle $(7, 8)$. The bottom side of $(12, 12)$ has $y$-coordinate $-4 + \delta$, the right side of $(12, 12)$ has $x$-coordinate $5 - \delta$.

Table 2 contains the calculations.
| Object                              | Formulae or system of equations                                                                 | Numerical value              |
|------------------------------------|-----------------------------------------------------------------------------------------------|------------------------------|
| Orientation \((x_1, y_1)\) of stack \((4, 1)\) | \(y_1^2 + x_1^2 = 1, y_1 + 4x_1 = 4 - \delta \)                                                  | \((0.881413748866,\)        |
|                                    | \(P_0 = (4/x_1 - 1/y_1 + x_1/y_1 - 5, 0)\)                                                   | \(0.4723450045357421)\)     |
| Orientation \((x_2, y_2)\) of stack \((5, 1)\) | \(y_2^2 + x_2^2 = 1, 5y_2 + x_2 = 5 - \delta \)                                              | \((0.386451637219073,...,\) |
|                                    | \(P_1 = (1.788247541, -1.998)\)                                                             | \(.9223096725561...)\)       |
| Orientation \((x_3, y_3)\) of square \(S_6\) | \(x_3 + y_3^2 = 1, \) \((x_3 + y_3^2 - 2 + \delta) + P_0 \)                                   | \((1.788247541,-1.998)\)    |
| Lower ordinate of intersection \(S_2\) with line \(x = 0\) | \(Y_1 = P_{1y} + P_{1x} \cdot \frac{x_2}{y_2} \)                                             | \(-1.248716749\)            |
| Orientation \((x_4, y_4)\) of square \(S_0\) | \(x_4^2 + y_4^2 = 1, \) \((1, \frac{6}{y_4} + (P_{1x} - 1) \frac{x_4}{y_4} - 2 + \delta + \frac{1-y_4}{x_2}/y_2)\) | \((0.9893426726...)\)       |
|                                    | \(P_2 = (x_3 + y_3^2 - 4 - x_3)\)                                                             | \((3.847656465)\)           |
|                                    | \((P_{3y} - 1) \frac{y_4}{y_2} + 2y_2, y_1 + \frac{5}{y_2} + y_2 - 2x_2)\)                  | \((2.321070982,4.321862330)\) |
| Orientation \((x_5, y_5)\) of squares \(S_{14}, S_{15}\) | \(x_5^2 + y_5^2 = 1, \) \((P_8 - (2, 6), (y_5 - x_5)) = 2\)                                 | \((2.30876131,5.5915134434)\) |
|                                    | \((P_8 - 2, 6) ) \cdot (x_5 - y_5)\)                                                          | \((4.608883990)\)           |
|                                    | \(P_7 = P_6 + (x_2 + y_2, y_2 - x_2)\)                                                        | \((3.2057967042318,5.14474202553891)\) |
|                                    | \(P_8 = P_3 + (2y_2, 2/y_2 - 2x_2)\)                                                          | \((4.075690327,5.717428128)\) |
|                                    | \(P_9 = P_8 + (2y_2, 2/y_2 - 2x_2)\)                                                          | \((4.235421115,9.05876415...)\) |
|                                    | \(P_0 = (P_{9x}, P_{9y})\) \((x_5, y_5)\) \cdot (y_5, x_5)\)                                | \((3.222807975740513)\)     |
|                                    | \((2, 6), (x_5, y_5)\) \cdot (x_5, y_5)                                                       | \((6.265585540152)\)        |
|                                    | \((x_5 + y_5, y_5 - x_5)\) \cdot (x_5 + y_5, y_5 - x_5)                                       | \((4.1243576636105)\)       |
|                                    | \((x_5 + y_5, y_5 - 2x_5)\) \cdot (x_5 + y_5, y_5 - 2x_5)                                    | \((5.81959314362795)\)      |
| Distance between \(P_{11}\) and segment \([P_3, P_10]\) | \(\frac{|(P_{11} - P_{10}) \cdot (P_{11} - P_{10})|}{\sqrt{(P_{11} - P_{10})^2 + (P_{11} - P_{10})^2}}\) | \((1.000648944...)\)       |

Table 2: Calculations for \(\delta = 0.002\).
Figure 6: Squeezable packing of 132 unit squares in a square (12,12)

Calculations with $\delta = .0021$ give the distance 0.9999866543 between the bottom left vertex of $S_{18}$ and the segment $[P_9, P_{10}]$. The bisection method gives the evaluation $\delta((12,12), 132) > 0.00209798269$, i.e., $s(132) < 11.99790201731$.

Analogous calculations give evaluations

\[\delta((5,10), 43) > 0.0009652493, \delta((5,9), 38) > 0.020403\]

\[\delta((13,13), 156) > 0.0059576, s(156) < 12.9940424.\]

Calculations with $C = (10,8), D = (3,6), A = (11,6), B = (4,8)$ in Figure 1 give

\[\delta((14,14), 182) > 0.01681735886, s(14^2 - 14) < 13.98318264114.\]

For the square (15,15) we have $\delta((15,15), 210) \geq \min(\delta((5,9), 38), \delta((11,6), 58)) > 0.01681735886$, i.e., $s(210) < 14.98318264114$.

For the square (16,16) we have $\delta((16,16), 241) > \min(\delta((5,10), 43), \delta((12,6), 64)) > 0.0009652493$, i.e., $s(16^2 - 15) < 15.9990347507$.

More careful analysis when we use the space between rectangles (5,10) and (12,6) gives $\delta((16,16), 241) > 0.00404996$, i.e., $s(16^2 - 15) < 15.99595004$.

Calculations with $A = (12,6), B = (6,11), C = (11,11), D = (5,6)$ give

\[\delta((17,17), 17^2 - 16) > 0.0049082317748, s(17^2 - 16) < 16.9950017682252.\]
Notice that this squeezable packing of a square (17,17) contains one unit square more than in [2].

Calculations with $A = (13, 6), B = (6, 12), C = (12, 12), D = (5, 6)$ give

$$\delta((18, 18), 18^2 - 17) \geq 0.0049082317748, s(18^2 - 17) < 17.9950917682252.$$ 

Table 4 contains the evaluations of the squeezing values and the upper bounds of $s(n)$ for new $n$.

| $n$  | $s(n)$          | $\delta(\lceil s(n) \rceil, \lceil s(n) \rceil, n)$          |
|------|----------------|-------------------------------------------------------------|
| 132  | $s(12^2 - 12) < 11.99790201731$ | $\delta((12, 12), 132) > 0.00209798269$                    |
| 156  | $s(13^2 - 13) < 12.9940424$     | $\delta((13, 13), 156) > 0.0059576$                        |
| 182  | $s(14^2 - 14) < 13.98318264114$ | $\delta((14, 14), 182) > 0.01681735886$                    |
| 210  | $s(15^2 - 15) < 14.98318264114$ | $\delta((15, 15), 210) > 0.01681735886$                    |
| 241  | $s(16^2 - 16) < 15.99595004$    | $\delta((16, 16), 241) > 0.00404996$                       |
| 273  | $s(17^2 - 16) < 16.9950917682252$ | $\delta((17, 17), 17^2 - 16) > 0.0049082317748$           |
| 307  | $s(18^2 - 17) < 17.9950917682252$ | $\delta((18, 18), 18^2 - 17) > 0.0049082317748$           |

Table 4. Evaluations of squeezing values and upper bounds of $s(n)$ for new $n$

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