Global optimization of Lipschitz functions

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Abstract
The goal of the paper is to design sequential strategies which lead to efficient optimization of an unknown function under the only assumption that it has a finite Lipschitz constant. We first identify sufficient conditions for the consistency of generic sequential algorithms and formulate the expected minimax rate for their performance. We introduce and analyze a first algorithm called LIPO which assumes the Lipschitz constant to be known. Consistency, minimax rates for LIPO are proved, as well as fast rates under an additional Hölder like condition. An adaptive version of LIPO is also introduced for the more realistic setup where the Lipschitz constant is unknown and has to be estimated along with the optimization. Similar theoretical guarantees are shown to hold for the adaptive LIPO algorithm and a numerical assessment is provided at the end of the paper to illustrate the potential of this strategy with respect to state-of-the-art methods over typical benchmark problems for global optimization.

Keywords: global optimization, Lipschitz constant, statistical analysis, convergence rate bounds

1. Introduction
In many applications such as complex system design or hyperparameter calibration for learning systems, the goal is to optimize the output value of an unknown function with as few evaluations as possible. Indeed, in such contexts, evaluating the performance of a single set of parameters often requires numerical simulations or cross-validations with significant computational cost and the operational constraints impose a sequential exploration of the solution space with small samples. Moreover, it can generally not be assumed that the function has good properties such as linearity or convexity. This generic problem of sequentially optimizing the output of an unknown and potentially nonconvex function is often referred to as global optimization (Pintér (1991)), black-box optimization (Jones et al. (1998)) or derivative-free optimization (Rios and Sahinidis (2013)). In particular, there is a large number of algorithms based on various heuristics which have been introduced in order to address this problem, such as genetic algorithms, model-based methods or Bayesian optimization. We focus here on the smoothness-based approach to global optimization. This approach is based on the simple observation that, in many applications, the system presents some regularity with respects to the input. In particular, the use of the Lipschitz constant, first proposed in the seminal works of Shubert (1972) and Piyavskii (1972), initiated an
active line of research and played a major role in the development of many efficient global optimization algorithms such as DIRECT (Jones et al. (1993)), MCS (Huyer and Neumaier (1999)) or more recently SOO (Preux et al. (2014)). Convergence properties of global optimization methods have been developed in the works of Valko et al. (2013) and Munos (2014) under local smoothness assumptions, but, up to our knowledge, such properties have not been considered in the case where only the global smoothness of the function can be specified. An interesting question is how much global assumptions on regularity which cover in some sense local assumptions may improve the convergence of the latter. In this work, we address the following questions: (i) find the limitations and the best performance that can be achieved by any algorithm over the class of Lipschitz functions and (ii) design efficient and optimal algorithms for this class of problems. More specifically, our contribution with regards to the above mentioned works is twofold. First, we introduce two novel algorithms for global optimization which exploit the global smoothness of the unknown function and display good performance in typical benchmarks for optimization. Second, we show that these algorithms can achieve faster rates of convergence on globally smooth problems than the previously known methods which only exploit the local smoothness of the function. The rest of the paper is organized as follows. In Section 2, we introduce the framework and provide generic results about the convergence of global optimization algorithms. In Section 3, we introduce and analyze the LIPO algorithm which requires the knowledge of the Lipschitz constant. In Section 4, the algorithm is extended to the case where the Lipschitz constant is not assumed to be previously known. Finally, the adaptive version of the algorithm is compared to other global optimization methods in Section 5. All proofs are postponed to the Appendix Section.

2. Setup and preliminary results

2.1 Setup and notations

Setup. Let \( \mathcal{X} \subseteq \mathbb{R}^d \) be a compact and convex set with non-empty interior and let \( f : \mathcal{X} \to \mathbb{R} \) be an unknown function which is only supposed to admit a maximum over its input domain. The goal in global optimization consists in finding some point

\[
x^* \in \arg\max_{x \in \mathcal{X}} f(x)
\]

with a minimal amount of function evaluations. The standard setup involves a sequential procedure which starts by evaluating the function \( f(X_1) \) at an initial point \( X_1 \) and iteratively selects at each step \( t \geq 1 \) an evaluation point \( X_{t+1} \in \mathcal{X} \) which depends on the previous evaluations \( (X_1, f(X_1)), \ldots, (X_t, f(X_t)) \) and receives the evaluation of the unknown function \( f(X_{t+1}) \) at this point. After \( n \) iterations, we consider that the algorithm returns an evaluation point \( X_{\hat{n}} \) with \( \hat{n} \in \arg\min_{i=1,\ldots,n} f(X_i) \) which has recorded the highest evaluation. The performance of the algorithm over the function \( f \) is then measured after \( n \) iterations through the difference between the value of the true maximum and the value of the highest evaluation observed so far:

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1,\ldots,n} f(X_i).
\]
Global optimization of Lipschitz functions

The analysis provided in the paper considers that the number \( n \) of evaluation points is not fixed and it is assumed that function evaluations are noiseless. Moreover, the assumption made on the unknown function \( f \) throughout the paper is that it has a finite Lipschitz constant \( k \), i.e.

\[
\exists k \geq 0 \text{ s.t. } |f(x) - f(x')| \leq k \cdot \|x - x'\|_2 \ \forall (x, x') \in \mathcal{X}^2.
\]

Before starting the analysis, we point out that similar settings have also been studied in the works of Munos (2014) and Malherbe et al. (2016) and that Valko et al. (2013) and Grill et al. (2015) considered the noisy scenario.

Notations. For all \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we denote by \( \|x\|_2^2 = \sum_{i=1}^{d} x_i^2 \) the standard \( \ell_2 \)-norm and by \( B(x, r) = \{x' \in \mathbb{R}^d : \|x - x'\|_2 \leq r \} \) the ball centered in \( x \) of radius \( r \geq 0 \). For any bounded set \( \mathcal{X} \subset \mathbb{R}^d \), we define its inner-radius as \( \text{rad}(\mathcal{X}) = \max\{r > 0 : \exists x \in \mathcal{X} \text{ such that } B(x, r) \subseteq \mathcal{X}\} \), its diameter as \( \text{diam}(\mathcal{X}) = \max_{(x,x') \in \mathcal{X}^2} \|x - x'\|_2 \) and we denote by \( \mu(\mathcal{X}) \) its volume where \( \mu(\cdot) \) stands for the Lebesgue measure. In addition, \( \text{Lip}(k) := \{f : \mathcal{X} \to \mathbb{R} \text{ s.t. } |f(x) - f(x')| \leq k \cdot \|x - x'\|_2, \ \forall (x, x') \in \mathcal{X}^2\} \) denotes the class of \( k \)-Lipschitz functions defined on \( \mathcal{X} \) and \( \bigcup_{k \geq 0} \text{Lip}(k) \) denotes the set of Lipschitz continuous functions. Finally, \( \mathcal{U}(\mathcal{X}) \) stands for the uniform distribution over a bounded measurable domain \( \mathcal{X} \), \( \mathcal{B}(p) \) for the Bernoulli distribution of parameter \( p \), \( \mathbb{I}\{\cdot\} \) for the standard indicator function taking values in \( \{0, 1\} \) and the notation \( X \sim \mathcal{P} \) means that the random variable \( X \) has the distribution \( \mathcal{P} \).

2.2 Preliminary results

In order to design efficient procedures, we first investigate the best performance that can be achieved by any algorithm over the class of Lipschitz functions.

Sequential algorithms and optimization consistency. We first describe the sequential procedures that are considered here and the corresponding concept of consistency in the sense of global optimization.

Definition 1 (Sequential Algorithm) The class of optimization algorithms we consider, denoted in the sequel by \( \mathcal{A} \), contains all the algorithms \( A = \{A_t\}_{t \geq 1} \) completely described by:

1. A distribution \( A_1 \) taking values in \( \mathcal{X} \) which allows to generate the first evaluation point, i.e. \( X_1 \sim A_1 \);

2. An infinite collection of parametric distributions \( \{A_t\}_{t \geq 2} \) taking values in \( \mathcal{X} \) and based on the previous evaluations which define the iteration loop, i.e. \( X_{t+1} | X_1, \ldots, X_t \sim A_{t+1}( (X_1, f(X_1)), \ldots, (X_t, f(X_t)) ) \).

Note that this class of algorithms also includes the deterministic methods in which case the distributions \( \{A_t\}_{t \geq 1} \) are degenerate. The next definition introduces the notion of asymptotic convergence.
Definition 2 (Optimization Consistency) A global optimization algorithm $A$ is said to be consistent over a set $\mathcal{F}$ of real-valued functions admitting a maximum over their input domain $\mathcal{X}$ if and only if

$$\forall f \in \mathcal{F}, \quad \max_{i=1}^{n} f(X_i) \xrightarrow{p} \max_{x \in \mathcal{X}} f(x)$$

where $X_1, \ldots, X_n$ denotes a sequence of $n$ evaluations points generated by the algorithm $A$ over the function $f$.

Asymptotic performance. We now investigate the minimal conditions for a sequential algorithm to achieve asymptotic convergence. Of course, it is expected that a global optimization algorithm should be consistent at least for the class of Lipschitz functions and the following result reveals a necessary and sufficient condition (NSC) in this case.

Proposition 3 (Consistency NSC) A global optimization algorithm $A$ is consistent over the set of Lipschitz functions if and only if

$$\forall f \in \bigcup_{k \geq 0} \text{Lip}(k), \quad \sup_{x \in \mathcal{X}} \min_{i=1}^{n} \|X_i - x\|_2 \xrightarrow{p} 0.$$  

A crucial consequence of the latter proposition is that the design of any consistent method ends up to covering the whole input space regardless of the function values. The example below introduces the most popular space-filling method which will play a central role in our analysis.

Example 1 (Pure Random Search) The Pure Random Search (PRS) consists in sequentially evaluating the function over a sequence of points $X_1, X_2, X_3, \ldots$ uniformly and independently distributed over the input space $\mathcal{X}$. For this method, a simple union bound indicates that for all $n \in \mathbb{N}^*$ and $\delta \in (0, 1)$, we have with probability at least $1 - \delta$ and independently of the function values,

$$\sup_{x \in \mathcal{X}} \min_{i=1}^{n} \|X_i - x\|_2 \leq \text{diam}(\mathcal{X}) \cdot \left(\frac{\ln(n/\delta) + d \ln(d)}{n}\right)^{\frac{3}{2}}.$$  

In addition to this result, we point out that the covering rate of any method can easily be shown to be at best of order $\Omega(n^{-1/d})$ and thus subject to to the curse of dimensionality by means of covering arguments. Keeping in mind the equivalence of Proposition 3, we may now turn to the nonasymptotic analysis.

Finite-time performance. We investigate here the best performance that can be achieved by any algorithm with a finite number of function evaluations. We start by casting a negative result stating that any algorithm can suffer, at any time, an arbitrarily large loss over the class of Lipschitz functions.

Proposition 4 Consider any global optimization algorithm $A$. Then, for any constant $C > 0$ arbitrarily large, any $n \in \mathbb{N}^*$ and $\delta \in (0, 1)$, there exists a function $\tilde{f} \in \bigcup_{k \geq 0} \text{Lip}(k)$ only depending on $(A, C, n, \delta)$ for which we have with probability at least $1 - \delta$,

$$C \leq \max_{x \in \mathcal{X}} \tilde{f}(x) - \max_{i=1}^{n} \tilde{f}(X_i).$$
This result might not be very surprising since the class of Lipschitz functions includes functions with finite, but arbitrarily large variations. When considering the subclass of functions with fixed Lipschitz constant, it becomes possible to derive finite-time bounds on the minimax rate of convergence.

**Proposition 5 (Minimax rate), adapted from Bull (2011).** For any Lipschitz constant \( k \geq 0 \) and any \( n \in \mathbb{N}^* \), the following inequalities hold true:

\[
c_1 \cdot k \cdot n^{-\frac{1}{d}} \leq \inf_{A \in A} \sup_{f \in \text{Lip}(k)} \mathbb{E} \left[ \max_{x \in \mathcal{X}} f(x) - \max_{i=1}^n f(X_i) \right] \leq c_2 \cdot k \cdot n^{-\frac{1}{d}}
\]

where \( c_1 = \text{rad}(\mathcal{X}) / (8\sqrt{d}) \), \( c_2 = \text{diam}(\mathcal{X}) \times d! \) and the expectation is taken over a sequence \( X_1, \ldots, X_n \) of \( n \) evaluation points generated by the algorithm \( A \) over \( f \).

We stress that this minimax convergence rate of order \( \Theta(n^{-1/d}) \) can still be achieved by any method with an optimal covering rate of order \( O(n^{-1/d}) \). Observe indeed that since \( \mathbb{E} [\max_{x \in \mathcal{X}} f(x) - \max_{i=1}^n f(X_i)] \leq k \times \mathbb{E} [\sup_{x \in \mathcal{X}} \min_{i=1}^n \|x - X_i\|_2] \) for all \( f \in \text{Lip}(k) \), then an optimal covering rate necessarily implies minimax efficiency. However, as it can be seen by examining the proof of Proposition 5 provided in the Appendix Section, the functions constructed to prove the limiting bound of \( \Omega(n^{-1/d}) \) are spikes which are almost constant everywhere and do not present a large interest from a practical perspective. In particular, we will see in the sequel that one can design:

I) An algorithm with fixed constant \( k \geq 0 \) which achieves minimax efficiency and also presents exponentially decreasing rates over a large subset of functions, as opposed to space-filling methods (LIPO, Section 3).

II) A consistent algorithm which does not require the knowledge of the Lipschitz constant and presents comparable performance as when the constant \( k \) is assumed to be known (AdaLIPO, Section 4).

### 3. Optimization with fixed Lipschitz constant

In this section, we consider the problem of optimizing an unknown function \( f \) given the knowledge that \( f \in \text{Lip}(k) \) for a given \( k \geq 0 \).

#### 3.1 The LIPO Algorithm

The inputs of the LIPO algorithm (displayed in Figure 1) are a number \( n \) of function evaluations, a Lipschitz constant \( k \geq 0 \), the input space \( \mathcal{X} \) and the unknown function \( f \in \text{Lip}(k) \). At each iteration \( t \geq 1 \), a random variable \( X_{t+1} \) is sampled uniformly over the input space \( \mathcal{X} \) and the algorithm decides whether or not to evaluate the function at this point. Indeed, it evaluates the function over \( X_{t+1} \) if and only if the value of the upper bound on possible values \( UB_{k,t} : x \mapsto \min_{i=1}^t f(X_i) + k : \|x - X_i\|_2 \) evaluated at this point and computed from the previous evaluations, is at least equal to the value of the best evaluation observed so far \( \max_{i=1}^t f(X_i) \). To illustrate how the decision rule operates in practice, an example of the computation of the upper bound can be found in Figure 2.
Input: $n \in \mathbb{N}^*, k \geq 0, \mathcal{X} \subset \mathbb{R}^d, f \in \text{Lip}(k)$

1. Initialization: Let $X_1 \sim U(\mathcal{X})$
   Evaluate $f(X_1)$, $t \leftarrow 1$

2. Iterations: Repeat while $t < n$:
   Let $X_{t+1} \sim U(\mathcal{X})$
   If $\min_{i=1 \ldots t} (f(X_i) + k \cdot \|X_{t+1} - X_i\|_2) \geq \max_{i=1 \ldots t} f(X_i)$ \{Decision rule\}
   Evaluate $f(X_{t+1})$, $t \leftarrow t + 1$

3. Output: Return $X_{\hat{i}_n}$ where $\hat{i}_n \in \arg\max_{i=1 \ldots n} f(X_i)$

Figure 1: The LIPO algorithm

More formally, the mechanism behind the decision rule can be explained using the active subset of consistent functions previously considered in active learning (see, e.g., Dasgupta (2011) or Hanneke (2011)).

Definition 6 (CONSISTENT FUNCTIONS) The active subset of $k$-Lipschitz functions consistent with the unknown function $f$ over a sample $(X_1, f(X_1)), \ldots, (X_t, f(X_t))$ of $t \geq 1$ evaluations is defined as follows:

$$\mathcal{F}_{k,t} := \{g \in \text{Lip}(k) : \forall i \in \{1 \ldots t\}, g(X_i) = f(X_i)\}.$$  

One can indeed recover from this definition the subset of points which can actually maximize the target function $f$.

Definition 7 (POTENTIAL MAXIMIZERS) Using the same notations as in Definition 6, we define the subset of potential maximizers estimated over any sample $t \geq 1$ evaluations with a constant $k \geq 0$ as follows:

$$\mathcal{X}_{k,t} := \{x \in \mathcal{X} : \exists g \in \mathcal{F}_{k,t} \text{ such that } x \in \arg\max_{x \in \mathcal{X}} g(x)\}.$$  

We may now provide an equivalence which makes the link with the decision rule of the LIPO algorithm.

Lemma 8 If $\mathcal{X}_{k,t}$ denotes the set of potential maximizers defined above, then we have the following equivalence:

$$x \in \mathcal{X}_{k,t} \iff \min_{i=1 \ldots t} f(X_i) + k \cdot \|x - X_i\|_2 \geq \max_{i=1 \ldots t} f(X_i).$$  

Hence, we deduce from this lemma that the algorithm only evaluates the function over points that still have a chance to be a maximizer of the unknown function.

Remark 9 (ADAPTATION TO NOISY EVALUATIONS) In addition to these definitions, we point out that the LIPO algorithm could be extended to settings with noisy evaluations.
by slightly adapting the ideas developed in Dasgupta (2011) and Hanneke (2011). More specifically, when considering a sample \((X_1, Y_1), \ldots, (X_n, Y_n)\) of \(n \geq 1\) noisy observations where \(Y_i = f(X_i) + \sigma \epsilon_i\) and \(\epsilon_i \sim \mathcal{N}(0, 1)\) and observing that the empirical mean-squared error \(R_n(f) := \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2 = \frac{\sigma^2}{n} \sum_{i=1}^{n} \epsilon_i^2\) evaluated in \(f\) is distributed as a chi-square, a possible approach would consist in using a relaxed version of the active subset \(\mathcal{F}_{k, \delta, t} := \{g \in \text{Lip}(k) : R_n(g) \leq \left(\frac{\sigma^2}{n}\right) \chi_{1-\delta,n}^2\}\) of Definition 6 where \(\chi_{1-\delta,n}^2\) denotes the \(1 - \delta\) quantile of the chi-squared distribution with \(n\) degrees of freedom.

**Remark 10** (Extension to other smoothness assumptions) Additionally, it is also important to note the proposed optimization scheme could easily be extended to a large number of classes of globally and locally smooth functions by slightly adapting the decision rule. For instance, when \(\mathcal{F}_\ell := \{f : \mathcal{X} \rightarrow \mathbb{R} | \text{x* is unique and } \forall x \in \mathcal{X}, f(x*) - f(x) \leq \ell(x*, x)\}\) denotes the set of functions previously considered in Munos (2014) which are locally smooth around their maxima with regards to a given semi-metric \(\ell : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+\), a straightforward derivation of Lemma 8 directly gives that the decision rule applied in \(X_{t+1}\) would simply consists in testing whether \(\max_{i=1\ldots t} f(X_i) \leq \min_{i=1\ldots t} f(X_i) + \ell(X_{t+1}, X_i)\).

However, since the purpose of this work is to design fast algorithms for Lipschitz functions, we will only derive convergence results for the version of the algorithm stated above.

### 3.2 Convergence analysis

We start by casting the consistency property of the algorithm.

**Proposition 11** (Consistency) For any Lipschitz constant \(k \geq 0\), the LIPO algorithm tuned with a parameter \(k\) is consistent over the set \(k\)-Lipschitz functions, i.e.

\[
\forall f \in \text{Lip}(k), \quad \max_{i=1\ldots n} f(X_i) \overset{P}{\rightarrow} \max_{x \in \mathcal{X}} f(x).
\]

The next result shows that the value of the highest evaluation observed by the algorithm is always superior or equal in the usual stochastic ordering sense to the one of a Pure Random Search.

**Proposition 12** (Faster than pure random search) Consider the LIPO algorithm tuned with any constant \(k \geq 0\). Then, for any \(f \in \text{Lip}(k)\) and \(n \in \mathbb{N}^*\), we have that \(\forall y \in \mathbb{R}\),

\[
P\left(\max_{i=1\ldots n} f(X_i) \geq y\right) \geq P\left(\max_{i=1\ldots n} f(X'_i) \geq y\right)
\]
where \( X_1, \ldots, X_n \) is a sequence of \( n \) evaluation points generated by LIPO and \( X'_1, \ldots, X'_n \) is a sequence of \( n \) independent random variables uniformly distributed over \( X \).

Based on this result, one can easily derive a first finite-time bound on the difference between the value of the true maximum and its approximation.

**Corollary 13** (Upper bound) For any \( f \in \text{Lip}(k) \), any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), we have with probability at least \( 1 - \delta \),

\[
\max_{x \in X} f(x) - \max_{i=1}^n f(X_i) \leq k \cdot \text{diam}(X) \cdot \left( \frac{\ln(1/\delta)}{n} \right)^{\frac{1}{2}}.
\]

This bound which proves the minimax optimality of LIPO stated in Proposition 5 once integrated does however not show any improvement over PRS and it cannot be significantly improved without making any additional assumption as shown below.

**Proposition 14** For any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), there exists a function \( \tilde{f} \in \text{Lip}(k) \), only depending on \( n \) and \( \delta \), for which we have with probability at least \( 1 - \delta \):

\[
k \cdot \text{rad}(X) \cdot \left( \frac{\delta}{n} \right)^{\frac{1}{2}} \leq \max_{x \in X} \tilde{f}(x) - \max_{i=1}^n \tilde{f}(X_i).
\]

As announced in Section 2.2, one can nonetheless get tighter polynomial bounds and even an exponential decay by using the following condition which describes the behavior of the function around its maximum.

**Condition 1** (Decreasing rate around the maximum) A function \( f : X \to \mathbb{R} \) is \((\kappa, c_{\kappa})\)-decreasing around its maximum for some \( \kappa \geq 0 \), \( c_{\kappa} \geq 0 \) if:

1. The global optimizer \( x^* \in X \) is unique;
2. For all \( x \in X \), we have that:

\[
f(x^*) - f(x) \geq c_{\kappa} \cdot \|x - x^*\|_2^n.
\]

This condition, already considered in the works of Zhigljavsky and Pintér (1991) and Munos (2014), captures how fast the function decreases around its maximum. It can be seen as a local one-sided Hölder condition that can only be met for \( \kappa \geq 1 \) when \( f \) is assumed to be Lipschitz. As an example, three functions satisfying this condition with different values of \( \kappa \) are displayed on Figure 3.2.

**Theorem 15** (Fast rates) Let \( f \in \text{Lip}(k) \) be any Lipschitz function satisfying Condition 1 for some \( \kappa \geq 1 \), \( c_{\kappa} > 0 \). Then, for any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), we have with probability at least \( 1 - \delta \),

\[
\max_{x \in X} f(x) - \max_{i=1}^n f(X_i) \leq k \cdot \text{diam}(X) \times \begin{cases} \exp \left\{ - C_{k,\kappa} \cdot \frac{n \ln(2)}{\ln(n/\delta) + 2(2\sqrt{d})^d} \right\}, & \kappa = 1, \\ \frac{2^\kappa}{2} \left( 1 + C_{k,\kappa} \cdot \frac{n(2^d(\kappa - 1) - 1)}{\ln(n/\delta) + 2(2\sqrt{d})^d} \right)^{-\frac{\kappa}{\kappa - 1}} \Bigg\}, & \kappa > 1, \end{cases}
\]

where \( C_{k,\kappa} = (c_{\kappa} \max_{x \in X} \|x - x^*\|_{\kappa - 1} / 8k)^d \).
We point out that the polynomial bound can be slightly improved and shown to be of order $O_p^*(n^{-1+\epsilon/(\kappa c+1)})$ in the case where the function is locally equivalent to $\|x^*-x\|_2^k$ (i.e., when $\exists c_\kappa, c_2 > 0$, $c_\kappa \|x^*-x\|_2^k \leq f(x^*) - f(x) \leq c_2 \|x^*-x\|_2^k$). The last result we provide states an exponentially decreasing lower bound.

**Theorem 16 (Lower Bound)** For any $f \in \text{Lip}(k)$ satisfying Condition 1 for some $\kappa \geq 1$, $c_\kappa > 0$ and any $n \in \mathbb{N}^*$ and $\delta \in (0,1)$, we have with probability at least $1 - \delta$, 

$$c_\kappa \cdot \text{rad}(\mathcal{X})^\kappa \cdot e^{-\frac{n}{2} (n+2\sqrt{\ln(1/\delta)}+\ln(1/\delta))} \leq \max_{x \in \mathcal{X}} f(x) - \max_{i=1\ldots n} f(X_i).$$

The next section provides an explicit derivation of the fast rate on some toy examples and a discussion on these results can be found in Section 3.4 where LIPO is compared with similar algorithms.

### 3.3 Examples

The next examples consider that the optimization is performed over the hypercube $\mathcal{X} = [-R, R]^d$ for some fixed $R > 0$ and $d \geq 1$.

**Sphere function.** The sphere function $f(x) = 1 - \|x\|_2$ is the canonical example of Lipschitz function. For this function, the Lipschitz continuity is a direct consequence of the triangle inequality: $\forall (x, y) \in \mathcal{X}^2$, we have that $|f(x) - f(y)| = \|x\|_2 - \|y\|_2 = \|y + x - y\|_2 = \|x - y\|_2$ by assuming w.l.o.g. that $\|x\|_2 \geq \|y\|_2$. Observing now that $x^* = 0$ and $f(x^*) - f(x) = \|x^* - x\|_2$ for all $x \in \mathcal{X}$, it is easy to see that Condition 1 is satisfied with $\kappa = 1$ and $c_\kappa = 1$. Hence, running LIPO tuned with any $k \geq 1$ would provide an exponentially decreasing rate of order $O_p^*(e^{-n/2(16k\sqrt{d})^d})$.

**Linear slope.** The second class of functions we consider are the linear functions of the form $f(x) = 1 - \langle w, x \rangle$ with weight vectors $w \in \mathbb{R}^d$. Applying Cauchy-Schwartz inequality directly gives the Lipschitz continuity: $\forall (x, y) \in \mathcal{X}^2$, $|f(x) - f(y)| = |\langle w, x - y \rangle| \leq \|w\|_2 \cdot \|x - y\|_2$. In the case of non-zero weights, $x^* = -R \times (\text{sgn}(w_1), \ldots, \text{sgn}(w_d))$ and we have for all $x \in \mathcal{X}$, $f(x^*) - f(x) = \langle w, x^* - x \rangle \geq \min_{i=1\ldots d} |w_i| \cdot \|x^* - x\|_2$. Therefore, Condition 1 is satisfied with $\kappa = 1$ and $c_\kappa = \min_{i=1\ldots d} |w_i|$ and we deduce that running LIPO with any $k \geq \|w\|_2$ would provide a decay of order $O_p^*(e^{-n/2(16k\sqrt{d}\min|w_i|)^d})$.

**Largest coordinate.** The last function we consider is the largest coordinate function $f(x) = 1 - \max\{|x_1|, \ldots, |x_d|\}$. Taking any $(x, y) \in \mathcal{X}^2$ and denoting by $i(x)$ and $i(y)$ the indexes of (one of) their largest absolute coordinate we obtain that $|f(x) - f(y)| = \max_{i \in \{1, \ldots, d\} \setminus \{i(x), i(y)\}} |x_i| + |y_i| - |x_{i(x)}| - |y_{i(y)}|$. We show that for any $|x_{i(x)}| \neq |y_{i(y)}|$, the bound above can be interpreted as a sum of $d - 2$ similar functions of the type $f(x) = 1 - \max\{|x_1|, \ldots, |x_{i(x)}|\}$.

![Figure 3](image-url) Three one-dimensional functions satisfying Condition 1 with $\kappa = 1/2$ (Left), $\kappa = 1$ (Middle) and $\kappa = 2$ (Right).
\[ |x_i(x)| - |y_i(y)| \leq |x_i(x)| - |y_i(x)| \leq |x_i(x) - y_i(x)| \leq \|x - y\|_2 \]

by assuming w.l.o.g. that \[ x_i(x) \geq y_i(y). \]
Notice that \[ x^* = 0 \] and that \[ f(x^*) - f(x) = \max_{i=1...d} |x_i| \geq \|x^* - x\|_2 / \sqrt{d} \]
for all \[ x \in \mathcal{X} \], we deduce that Condition 1 is satisfied with \[ \kappa = 1 \] and \[ c_{\kappa} = 1 / \sqrt{d} \]. Thus, running LIPO with any \[ k \geq 1 \] would provide an exponential decay of order \[ O_{\kappa}(e^{-n/2(16kd)^d}) \].

Note that the polynomial bounds could also be derived from the previous examples by slightly adapting the functions. For instance, it is easy to see that the function \[ f(x) = 1 - \|x\|_2^\kappa \] satisfies Condition 1 with \[ \kappa \geq 0 \] and \[ c_{\kappa} = 1 \].

### 3.4 Comparison with existing works

**Algorithms.** The Piyavskii algorithm (Piyavskii (1972)) is a sequential algorithm which requires the knowledge of the Lipschitz constant \[ k \geq 0 \] and iteratively evaluates the function over a point \[ X_{i+1} = \arg \max_{x \in \mathcal{X}} UB_{k,t}(x) \] which maximizes the upper bound on possible values \[ UB_{k,t}(x) := \min_{i=1...t} f(X_i) + k \cdot \|x - X_i\| \]. Munos (2014) also proposed a similar algorithm (DOO) that requires the knowledge of a semi-metric \[ \ell : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \] for which the function is at least locally smooth (\[ i.e., \forall x \in \mathcal{X}, f(x^*) - f(x) \leq \ell(x, x^*) \]) and a hierarchical partitioning of the space \[ \mathcal{X} \] in order to sequentially expand and evaluate the function over the center of a partition which has recorded the highest upper bound computed according to the semi-metric \[ \ell \]. With regards to those works, the LIPO algorithm aims at optimizing globally Lipschitz functions and combines space-filling and exploitation rather than pure exploitation. Recall indeed that LIPO evaluates the function over any point which might improve the function values (see Lemma 8) while DOO and the Piyavskii algorithm sequentially select among a restricted set of points the next evaluation point which have recorded the highest upper bound on possible values.

**Results.** To the best of our knowledge, only the consistency of the Piyavskii algorithm was proven in Mladineo (1986) and Munos (2014) derived finite-time upper bounds for DOO with the use of weaker local smoothness assumptions. To cast their results into our framework, we thus considered DOO tuned with the semi-metric \[ \ell(x, x') = k \cdot \|x - x'\|_2 \] over the domain \[ \mathcal{X} = [0, 1]^d \] partitioned into a \[ 2^d \]-ary tree of hypercubes and with \[ f \] belonging to the sets of globally smooth functions: (a) \[ \text{Lip}(k) \], (b) \[ \mathcal{F}_\kappa = \{ f \in \text{Lip}(k) \text{ satisfying Condition 1 with } c_{\kappa}, \kappa \geq 1 \} \] and (c) \[ \mathcal{F}_\kappa' = \{ f \in \mathcal{F}_\kappa : \exists c_2 > 0, f(x^*) - f(x) \leq c_2 \|x - x^*\|_2^\kappa \} \]. The results of the comparison can be found in Table 2. In addition to the novel lower bounds and the rate over \[ \text{Lip}(k) \], we were able to obtain similar upper bounds as DOO over \[ \mathcal{F}_\kappa \], uniformly better rates for the functions in \[ \mathcal{F}_\kappa' \] locally equivalent to \[ \|x^* - x\|_2^\kappa \] with \[ \kappa > 1 \] and up to a constant factor a similar exponential rate when \[ \kappa = 1 \]. Hence, when \[ f \] is only known to be \[ k \]-Lipschitz, one thus should expect the algorithm exploiting the global smoothness (LIPO) to perform asymptotically better or at least similarly to the one using the local smoothness (DOO) or no information (PRS). However, keeping in mind that the constants are not necessarily optimal, it is also interesting to note that the term \[ (k \sqrt{d}/c_{\kappa})^d \] appearing in both the fast rates of LIPO and DOO tends to suggest that if \[ f \] is also known to be locally smooth for some \[ k_\ell \ll k \], then one should expect the algorithm exploiting the local smoothness \[ k_\ell \] to be asymptotically faster than the one using the global smoothness \[ k \] in the case where \[ \kappa = 1 \].
Global optimization of Lipschitz functions

Algorithm | DOO | LIPO | Piyavskii | PRS
--- | --- | --- | --- | ---

\( f \in \text{Lip}(k) \)

Consistency
✓ ✓ ✓ ✓
Upper Bound
- \( O_P(n^{\frac{1}{2}}) \)
- \( O_P(n^{\frac{1}{2}}) \)

\( f \in \mathcal{F}_{\mathcal{K}}, \mathcal{K} > 1 \)

Upper bound
\( O(n^{\frac{n}{4(k+1)}}) \)
\( O_P(n^{\frac{n}{4(k+1)}}) \)
\( - \)
\( O_P(n^{\frac{1}{2}}) \)
Lower bound
- \( \Omega_P(e^{-\frac{n}{2}}) \)
- \( \Omega_P(n^{\frac{1}{2}}) \)

\( f \in \mathcal{F}_{\mathcal{K}'}, \mathcal{K} > 1 \)

Upper bound
\( O(n^{\frac{n}{4(k+1)}}) \)
\( O_P(n^{\frac{n}{4(k+1)}}) \)
\( - \)
\( O_P(n^{\frac{1}{2}}) \)
Lower bound
- \( \Omega_P(e^{-\frac{n}{2}}) \)
- \( \Omega_P(n^{\frac{1}{2}}) \)

\( f \in \mathcal{F}_{\mathcal{K}'}, \mathcal{K} = 1 \)

Upper bound
\( O(e^{-\frac{n \ln(2)}{4k\sqrt{d/c_n}}}) \)
\( O_P(e^{-\frac{n \ln(2)}{4k\sqrt{d/c_n}}}) \)
\( - \)
\( O_P(n^{\frac{1}{2}}) \)
Lower bound
- \( \Omega_P(e^{-\frac{n}{2}}) \)
- \( \Omega_P(n^{\frac{1}{2}}) \)

Table 1: Comparison of the results reported over the difference \( \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \) in global optimization literature. Dash symbols are used when no results could be found.

4. Optimization with unknown Lipschitz constant

In this section, we consider the problem of optimizing any unknown function \( f \) in the class \( \bigcup_{k \geq 0} \text{Lip}(k) \).

4.1 The adaptive algorithm

The AdaLIPO algorithm (displayed in Figure 4) is an extension of LIPO which involves an estimate of the Lipschitz constant and takes as input a parameter \( p \in (0,1) \) and a nondecreasing sequence of Lipschitz constant \( k_i \in \mathbb{Z} \) defining a meshgrid of \( \mathbb{R}^+ \) \((i.e. \text{ such that } \forall x > 0, \exists i \in \mathbb{Z} \text { with } k_i \leq x \leq k_{i+1})\). The algorithm is initialized with a Lipschitz constant \( \hat{k}_1 \) set to 0 and alternates randomly between two distinct phases: exploration and exploitation. Indeed, at step \( t < n \), a Bernoulli random variable \( B_{t+1} \) of parameter \( p \) which drives this trade-off is sampled. If \( B_{t+1} = 1 \), then the algorithm explores the space by evaluating the function over a point uniformly sampled over \( \mathcal{X} \). Otherwise, if \( B_{t+1} = 0 \), the algorithm exploits the previous evaluations by making an iteration of the LIPO algorithm with the smallest Lipschitz constant of the sequence \( \hat{k}_t \) which is associated with a subset of Lipschitz functions that probably contains \( f \) (step abbreviated in the algorithm by \( X_{t+1} \sim \mathcal{U}(\mathcal{X}_{k_{t+1}}) \)). Once an evaluation has been made, the Lipschitz constant estimate \( \hat{k}_t \) is updated.

**Remark 17** (Examples of meshgrids) Several sequences of Lipschitz constants with various shapes could be considered such as \( k_i = |i|^{\text{sgn}(i)}, n(1 + |i|^{\text{sgn}(i)}) \) or \( (1 + \alpha)^i \) for some \( \alpha > 0 \). It should be noticed in particular that the computation of the estimate is
**Input:** \( n \in \mathbb{N}^*, k_{i \in \mathbb{Z}}, \mathcal{X} \subset \mathbb{R}^d, f \in \bigcup_{k \geq 0} \text{Lip}(k) \)

1. **Initialization:** Let \( X_1 \sim \mathcal{U}(\mathcal{X}) \) 
   Evaluate \( f(X_1) \), \( t \leftarrow 1 \), \( \hat{k}_1 \leftarrow 0 \)

2. **Iterations:** Repeat while \( t < n \)
   Let \( B_{t+1} \sim \mathcal{B}(p) \)
   If \( B_{t+1} = 1 \) (Exploration)
   Let \( X_{t+1} \sim \mathcal{U}(\mathcal{X}) \)
   If \( B_{t+1} = 0 \) (Exploitation)
   Let \( X_{t+1} \sim \mathcal{U}(\mathcal{X}_{k_{t}, t}) \) where \( \mathcal{X}_{k_{t}, t} \) denotes the set of potential maximizers of Definition 7 computed with \( k \) set to \( \hat{k}_t \)
   Evaluate \( f(X_{t+1}) \), \( t \leftarrow t + 1 \)
   Let \( \hat{k}_t := \inf \left\{ k_{i \in \mathbb{Z}} : \max_{i \neq j} \frac{|f(X_i) - f(X_j)|}{\|X_i - X_j\|_2} \leq k_i \right\} \)

3. **Output:** Return \( X_{\hat{t}_n} \) where \( \hat{t}_n \in \arg\max_{i=1 \ldots n} f(X_i) \)

**Figure 4:** The AdaLIPO algorithm

straightforward with these sequences. For instance, when \( k_i = (1+\alpha)^i \), we have \( \hat{k}_t = (1+\alpha)^{i_t} \) where \( i_t = \lceil \ln(\max_{i \neq j} |f(X_j) - f(X_i)|/\|X_j - X_i\|_2)/\ln(1 + \alpha) \rceil \).

**Remark 18 (Alternative Lipschitz constant estimate)** Due to the genericity of the algorithm, we point out that any Lipschitz constant estimate such as the one proposed in Wood and Zhang (1996) or Bubeck et al. (2011) could also be considered to implement the algorithm. However, as the analysis requires the estimate to be universally consistent (see Section below), we will only consider the proposed one that presents such a property.

**4.2 Convergence analysis**

**Lipschitz constant estimate.** Before starting the analysis of the algorithm, we first provide a control on the Lipschitz constant estimate based on a sample of random evaluations that will be useful to analyse its performance. More precisely, the next result illustrates the purpose of using a discretization of Lipschitz constant instead of a raw estimate of the maximum slope by showing that, given this estimate, a small subset of functions containing the unknown function can be recovered in a finite-time.

**Proposition 19** Let \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \) be any non-constant Lipschitz function. Then, if \( \hat{k}_t \) denotes the Lipschitz constant estimate of Algorithm 2 computed with any increasing sequence \( k_{i \in \mathbb{Z}} \) defining a meshgrid of \( \mathbb{R}^+ \) over a sample \((X_1, f(X_1)), \ldots, (X_t, f(X_t))\) of \( t \geq 2 \) evaluations where \( X_1, \ldots, X_t \) are uniformly and independently distributed over \( \mathcal{X} \), we have that

\[
P \left( f \in \text{Lip}(\hat{k}_t) \right) \geq 1 - (1 - \Gamma(f, k_{i_{t-1}}))^{[t/2]} \]
Global optimization of Lipschitz functions

where the coefficient

\[ \Gamma(f, k_{i^*}) := \mathbb{P} \left( \frac{|f(X_1) - f(X_2)|}{\|X_1 - X_2\|_2} > k_{i^*} \right) > 0 \]

with \( i^* = \min\{i \in \mathbb{Z} : f \in \text{Lip}(k_i)\} \), is strictly positive.

The following remarks provide some insights on the quantities involved in the bound.

**Remark 20** (Measure of Global Smoothness) The coefficient \( \Gamma(f, k_{i^*}) \) which appears in the lower bound of Proposition 19 can be seen as a measure of the global smoothness of the function \( f \) with regards to \( k_{i^*} \). Indeed, observing that \( (1/[t/2]) \cdot \sum_{i=1}^{[t/2]} \mathbb{I}\{|f(X_i) - f(X_{i+[t/2]})| > k_{i^*} \|X_i - X_{i+[t/2]}\|_2\} \overset{p}{\rightarrow} \Gamma(f, k_{i^*}) \), it is easy to see that this coefficient records the ratio of volume the product space \( \mathcal{X} \times \mathcal{X} \) where \( f \) is witnessed to be at least \( k_{i^*} \)-Lipschitz.

**Remark 21** (Density of the Sequence of Lipschitz Constants) As a consequence of the previous remark, we point out that the density of the sequence of Lipschitz constants \( k_i \in \mathbb{Z} \), captured here by \( \alpha = \sup_{i \in \mathbb{Z}} (k_{i+1} - k_i)/k_i \), has opposite impacts on the maximal deviation of the estimate and its convergence rate. Indeed, as \( \alpha \) is involved in both the following upper bounds on the deviation and on the coefficient Gamma: \( \lim_{t \to \infty} k_i - k^* \)/\( k_i \) \( \leq \alpha \) and \( \Gamma(f, k_{i^*}) \leq \Gamma(f, k^*/(1 + \alpha)) \) where \( k^* = \sup\{k \geq 0 : f \notin \text{Lip}(k)\} \) denotes the optimal Lipschitz constant, we deduce that using a dense sequence of Lipschitz constants with a small \( \alpha \) reduces the bias but also the convergence rate through a small coefficient \( \Gamma(f, k_{i^*}) \).

**Remark 22** (Impact of the Dimensionality) Last, we provide a simple result which illustrates the fact that, independently of the function, the task of estimating the Lipschitz constant becomes harder as the dimensionality grows large. Let \( f : [0,1]^d \to \mathbb{R} \) be any \( k \)-Lipschitz function for some \( k \geq 0 \) with regards to the Euclidean distance. Then, for all \( t > 0 \),

\[ \mathbb{P} \left( \frac{|f(X) - f(X')|}{\|X - X'\|_2} > tk \right) \leq 5e^{-cdt^2} \]

where \( X \) and \( X' \) are uniformly distributed over \([0,1]^d\) and \( c > 0 \) is some absolute constant. Combined with the naive bound \( \mathbb{P}(f \in \text{Lip}(k_i)) \leq t^2 \Gamma(f, k_{i^*}) \), this result shows that one should collect at least \( \Omega_p(e^{cdk/k_{i^*}}) \) evaluation points to successfully estimate the constant \( k_{i^*} \).

Equipped with this result, we may now turn to the analysis of AdaLIPO.

**Analysis of AdaLIPO.** Given the consistency equivalence of Proposition 3, one can directly obtain the following asymptotic result.

**Proposition 23** (Consistency) The AdaLIPO algorithm tuned with any parameter \( p \in (0,1) \) and any sequence of Lipschitz constant \( k_i \in \mathbb{Z} \) defining a meshgrid of \( \mathbb{R}^+ \) is consistent over the set of Lipschitz functions, i.e.,

\[ \forall f \in \bigcup_{k \geq 0} \text{Lip}(k), \max_{i=1,...,n} f(X_i) \overset{p}{\rightarrow} \max_{x \in \mathcal{X}} f(x). \]
The next result provides a first finite-time bound on the difference between the maximum and its approximation.

**Proposition 24 (Upper Bound)** Consider AdaLIPO tuned with any \( p \in (0, 1) \) and any sequence \( k_i \in \mathbb{Z} \) defining a meshgrid of \( \mathbb{R}^+ \). Then, for any non-constant \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \), any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), we have with probability at least \( 1 - \delta \),

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \leq k_{i^*} \times \text{diam}(\mathcal{X}) \times \left( \frac{5}{p} + \frac{2 \ln(\delta/3)}{p \ln(1 - \Gamma(f, k_{i^*} - 1))} \right)^{\frac{1}{3}} \times \left( \frac{\ln(3/\delta)}{n} \right)^{\frac{1}{3}}
\]

where \( \Gamma(f, k_{i^*} - 1) \) and \( i^* \) are defined as in Proposition 19 and \( \ln(0) = -\infty \) by convention.

This result might be misleading since it advocates that doing pure exploration gives the best rate (i.e., when \( p \to 1 \)). As Proposition 19 provides us with the guarantee that \( f \in \text{Lip}(\hat{k}_t) \) within a finite number of iterations where \( \hat{k}_t \) denotes the Lipschitz constant estimate, one can however recover faster convergence rates similar to the one reported for LIPO where the constant \( k \) is assumed to be known.

**Theorem 25 (Fast Rates)** Consider the same assumptions as in Proposition 24 and assume in addition that the function \( f \) satisfies Condition 1 for some \( \kappa \geq 1 \), \( c_\kappa \geq 0 \). Then, for any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), we have with probability at least \( 1 - \delta \),

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \leq k_{i^*} \times \text{diam}(\mathcal{X}) \times \exp \left( \frac{2 \ln(\delta/4)}{p \ln(1 - \Gamma(f, k_{i^*} - 1))} + \frac{7 \ln(4/\delta)}{p(1-p)^2} \right) \times \left\{ \begin{array}{ll}
\exp \left\{ -C_{k_{i^*},\kappa} \cdot \frac{n (1-p) \ln(2)}{2 \ln(n/\delta) + 4(2 \sqrt{d})^d} \right\}, & \kappa = 1 \\
2^\kappa \left( 1 + C_{k_{i^*},\kappa} \cdot \frac{n (1-p) (2^{d(k-1)} - 1)}{2 \ln(n/\delta) + 4(2 \sqrt{d})^d} \right)^{-\frac{\kappa}{\sigma(k-1)}}, & \kappa > 1
\end{array} \right.
\]

where \( C_{k_{i^*},\kappa} = (c_\kappa \max_{x \in \mathcal{X}} \|x - x^*\|_2^{\kappa-1} / 8 k_{i^*})^d \).

This bound shows the precise impact of the parameters \( p \) and \( k_i \in \mathbb{Z} \) on the convergence of the algorithm. It illustrates the complexity of the exploration/exploitation trade-off through a constant term and a convergence rate which are inversely correlated to the exploration parameter and the density of the sequence of Lipschitz constants. Recall also that as these bounds are of the same order as when \( k \) is known, the examples of Section 3.3 still remain valid. We may now compare our results with existing works.

### 4.3 Comparison with previous works

**Algorithms.** The DIRECT algorithm (Jones et al. (1993)) is a Lipschitz optimization algorithm where the Lipschitz constant is unknown. It uses a deterministic splitting technique of the search space in order to sequentially divide and evaluate the function over a subdivision of the space that have recorded the highest upper bound among all subdivisions of similar size for at least a possible value of \( k \). Munos (2014) generalized
DIRECT in a broader setting by extending the the DOO algorithm to any unknown and arbitrary local semi-metric under the name SOO. With regards to these works, we proposed an alternative stochastic strategy which directly relies on the estimation of the Lipschitz constant and thus only presents guarantees for globally Lipschitz functions.

**Results.** Up to our knowledge, only the consistency property of DIRECT was shown in Finkel and Kelley (2004) and Munos (2014) derived convergence rates for SOO using weaker local smoothness assumptions. To compare our results, we considered SOO tuned with the depth function \( h_{\text{max}}(t) = \sqrt{t} \) providing the best rate, over the domain \( X = [0,1]^d \) partitioned into a \( 2^d \)-ary tree of hypercubes and with \( f \) belonging to the following sets of globally Lipschitz functions: (a) \( \bigcup_{k \geq 0} \text{Lip}(k) \), (b) \( F_\kappa = \{ f \in \bigcup_{k \geq 0} \text{Lip}(k) \text{ satisfying Condition 1 with } c_\kappa, \kappa \geq 1 \} \) and (c) \( F'_\kappa = \{ f \in F_\kappa : \| f(x^*) - f(x) \| \leq c_2 \| x - x^* \|_2^\kappa \} \). The result of the comparison can be found in Table 2. In addition to the novel rate over the class of Lipschitz functions, we were also able to obtain a faster polynomial rate than SOO over the set \( F_\kappa \). However, SOO achieves its best rate of order \( O(e^{-c\sqrt{n}}) \) over the whole set \( \{ F'_\kappa, \kappa \geq 1 \} \) by adapting similarly to any function locally equivalent to \( \| x^* - x \|_2^\kappa \) while AdaLIPO achieves a slower polynomial rate in the case where \( \kappa > 1 \) but an even faster exponential rate of order \( O(e^{-cn}) \) when \( \kappa = 1 \). Hence, by exploiting the global smoothness of the function (AdaLIPO), we were able to derive a faster convergence rate than the best one reported for the algorithm exploiting the local smoothness (SOO) which however remains valid over a larger subset of functions.

| Algorithm | AdaLIPO | DIRECT | PRS | SOO |
|-----------|---------|--------|-----|-----|
| \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \) | ✓ | ✓ | ✓ | ✓ |
| Consistency | | | | |
| Upper Bound | \( O_2^* (n^{-\frac{\kappa}{2}}) \) | - | \( O_2 (n^{-\frac{\kappa}{2}}) \) | - |
| \( f \in F_\kappa, \kappa > 1 \) | | | | |
| Upper Bound | \( O_2^* (n^{-\frac{\kappa}{2(\kappa - 1)}}) \) | - | \( O_2 (n^{-\frac{\kappa}{2}}) \) | \( O(n^{-\frac{\kappa}{2(\kappa - 1)}}) \) |
| Lower Bound | - | - | \( \Omega_2 (n^{-\frac{\kappa}{2}}) \) | - |
| \( f \in F'_\kappa, \kappa > 1 \) | | | | |
| Upper Bound | \( O_2^* (n^{-\frac{\kappa x}{2(\kappa - 1)}}) \) | - | \( O_2 (n^{-\frac{\kappa}{2}}) \) | \( O(e^{-\frac{\sqrt{\pi} \ln(2)}{(\sqrt{2d}(c_2/c_1)^{1/\kappa})^d})} \) |
| Lower Bound | - | - | \( \Omega_2 (n^{-\frac{\kappa}{2}}) \) | - |
| \( f \in F'_\kappa, \kappa = 1 \) | | | | |
| Upper Bound | \( O_2^* (e^{-\frac{n(1-p) \ln(2)}{4(16k^2 - 1)\sqrt{d/c_1}^\pi})} \) | - | \( O_2 (n^{-\frac{\kappa}{2}}) \) | \( O(e^{-\frac{\sqrt{\pi} \ln(2)}{(c_2/2d/c_1)^{1/\kappa})^d}) \) |
| Lower Bound | - | - | \( \Omega_2 (n^{-\frac{\kappa}{2}}) \) | - |

Table 2: Comparison of the results reported over the difference \( \max_{x \in X} f(x) - \max_{i=1...n} f(X_i) \) in global optimization literature. Dash symbols are used when no results could be found.
5. Numerical experiments

In this section, we compare the empirical performance of AdaLIPO to existing state-of-the-art global optimization methods on real and synthetic optimization problems.

**Algorithms.**  
Six different types of algorithms developed from various approaches of global optimization were considered in addition to AdaLIPO:

- **BayesOpt** (Martinez-Cantin (2014)) is a Bayesian optimization algorithm. It uses a distribution over functions to build a surrogate model of the unknown function. The parameters of the distribution are estimated during the optimization process.

- **CMA-ES** (Hansen (2006)) is an evolutionary algorithm. It samples the next evaluation points according to a multivariate normal distribution with mean vector and covariance matrix computed from the previous evaluations.

- **CRS** (Kaelo and Ali (2006)) is a variant of PRS which includes local mutations. It starts with a random population and evolves these points by an heuristic rule.

- **DIRECT** (Jones et al. (1993)) is the Lipschitz optimization algorithm with unknown Lipschitz introduced in Section 4.3.

- **MLSL** (Kan and Timmer (1987)) is a multistart algorithm. It performs a series of local optimizations starting from points randomly chosen by a clustering heuristic that helps to avoid repeated searches of the same local optima.

- **PRS** is the standard random covering method described in Section 2 (see Example 1).

For a fair comparison, the tuning parameters were all set to default and AdaLIPO was constantly used with a parameter $p$ set to 0.1 and a sequence $k_i = (1 + 0.01/d)^i$ fixed by an arbitrary rule of thumb.

**Data sets.** Following the steps of (Malherbe and Vayatis (2016)), we considered a series of global optimization problems involving real data sets and a series of synthetic problems:

I) First, we studied the task of estimating the regularization parameter $\lambda$ and the bandwidth $\sigma$ of a Gaussian kernel ridge regression minimizing the empirical mean squared error of the predictions over a 10-fold cross validation with real data sets. The optimization was performed over $(\ln(\lambda), \ln(\sigma)) \in [-3, 5] \times [-2, 2]$ with five data sets from the UCI Machine Learning Repository Lichman (2013): **Auto-MPG**, **Breast Cancer Wisconsin (Prognostic)**, **Concrete slump test**, **Housing** and **Yacht Hydrodynamics**.

II) Second, we compared the algorithms on a series of five synthetic problems commonly met in standard optimization benchmark taken from (Jamil and Yang (2013); Surjanovic and Bingham (2013)): **HolderTable**, **Rosenbrock**, **Sphere**, **LinearSlope** and **Deb N.1**. This series includes multimodal and non-linear functions as well as ill-conditioned and well-shaped functions with a dimensionality ranging from 2 to 5.

---

1. In Python 2.7 from the libraries: *BayesOpt Martinez-Cantin (2014), †CMA 1.1.06 Hansen (2011) and ‡NLOpt Johnson (2014).
A complete description of the test functions of the benchmark can be found in Table 3.

| Problem   | Objective function                                                                 | Domain               | Local max. |
|-----------|-----------------------------------------------------------------------------------|----------------------|------------|
| Auto MPG  | \[ \frac{1}{10} \sum_{k=1}^{10} \sum_{i \in D_k} (\hat{f}_k(x_i) - Y_i)^2 \]   | \([-2,4] \times [-5,5]\) | -          |
| Breast Cancer | where: \[ \hat{f}_k \in \arg\min_{f \in H_{\sigma}} \sum_{i \in D_k} (f(x_i) - Y_i)^2 + \lambda \|f\|_{H_{\sigma}} \]  | \([-2,4] \times [-5,5]\) | -          |
| Concrete  | \(-\hat{f}_k \in \arg\min_{f \in H_{\sigma}} \sum_{i \in D_k} (f(x_i) - Y_i)^2 + \lambda \|f\|_{H_{\sigma}} \)  | \([-2,4] \times [-5,5]\) | -          |
| Yacht     | \(-\|f\|_{H_{\sigma}} \) is the corresponding norm                                 | \([-2,4] \times [-5,5]\) | -          |
| Housing   | \(-\sigma = 10^4\)                                                                | \([-2,4] \times [-5,5]\) | -          |
| Holder Table | \[|\sin(x_1)| \times |\cos(x_2)| \times \exp(|1-(x_1^2 + x_2^2)^{1/2}/\pi|)\] | \([-10,10]^2\) | 36         |
| Rosenbrock | \(-\sum_{i=1}^{2}[100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]\)                       | \([-2.048, 2.048]^3\) | -          |
| Sphere    | \(-\left(\sum_{i=1}^{4}(x_i - \pi/16)^2\right)^{1/2}\)                          | \([0,1]^4\)          | 1          |
| Linear Slope | \[\sum_{i=1}^{4}10^{(i-1)/4}(x_i-5)\]                                           | \([-5,5]^4\)         | 1          |
| Deb N.1   | \[\frac{1}{5} \sum_{i=1}^{5}\sin^6(5\pi x_i)\]                                  | \([-5,5]^5\)         | 36         |

Table 3: Description of the test functions of the benchmark. Dash symbols are used when a value could not be calculated.

**Protocol and performance metrics.** For each problem and each algorithm, we performed \(K=100\) distinct runs with a budget of \(n=1000\) function evaluations. For each target parameter \(t = 90\%, 95\%\) and \(99\%\), we have collected the stopping times corresponding to the number of evaluations required by each method to reach the specified target

\[
\tau_k := \min\{i = 1, \ldots, n : f(X_i^{(k)}) \geq f_{\text{target}}(t)\}
\]

where \(\min\{\emptyset\} = 1000\) by convention, \(\{f(X_i^{(k)})\}_{i=1}^n\) denotes the evaluations made by a given method on the \(k\)-th run with \(k \leq K\) and the target value is set to

\[
f_{\text{target}}(t) := \max_{x \in \mathcal{X}} f(x) - \left(\max_{x \in \mathcal{X}} f(x) - \int_{x \in \mathcal{X}} f(x) \, dx / \mu(\mathcal{X})\right) \times (1 - t).
\]
The normalization of the target to the average value prevents the performance measures from being dependent of any constant term in the unknown function. In practice, the average was estimated from a Monte Carlo sampling of $10^6$ evaluations and the maximum by taking the best value observed over all the sets of experiments. Based on these stopping times, we computed the average and standard deviation of the number of evaluations required to reach the target, i.e.

$$\bar{\tau}_K = \frac{1}{K} \sum_{k=1}^{K} \tau_k \quad \text{and} \quad \hat{\sigma}_\tau = \sqrt{\frac{1}{K} \sum_{k=1}^{K} (\tau_k - \bar{\tau}_K)^2}.$$ 

Results. Results are collected in Figure 5. Our main observations are the following. First, we point out that the proposed method displays very competitive results over most of the problems of the benchmark (except on the non-smooth DebN.1 where most methods fail). In particular, AdaLIPO obtains several times the best performance for the target 90% and 95% (see, e.g., BreastCancer, HolderTable, Sphere) and experiments Linear Slope and Sphere also suggest that, in the case of smooth functions, it can be robust against the dimensionality of the input space. However, in some cases, the algorithm can be witnessed to reach the 95% target with very few evaluations while getting more slowly to the 99% target (see, e.g., Concrete, Housing). This problem is due to the instability of the Lipschitz constant estimate around the maxima but could certainly be solved with the addition of a noise parameter that would allow the algorithm to be more robust against local perturbations. Additionally, investigating better values for $p$ and $k_i$ as well as alternative covering methods such as LHS Stein (1987) could also be promising approaches to improve its performance. However, an empirical analysis of the algorithm with these extensions is beyond the scope of the paper and will be carried out in a future work.

6. Conclusion

We introduced two novel strategies for global optimization: LIPO which requires the knowledge of the Lipschitz constant and its adaptive version AdaLIPO which estimates the constant during the optimization process. A theoretical analysis is provided and empirical results based on synthetic and real problems have also been obtained demonstrating the performance of the adaptive algorithm with regards to existing state-of-the-art global optimization methods.

Appendix A. Preliminary results

We provide here two geometric results (Corollary 28 and Lemma 29) and a stochastic result (Proposition 30) that are used repeatedly in the computations. We start with the definition of covering numbers.

**Definition 26 (Covering number and $\epsilon$-cover)** For any compact and convex set $X \subset \mathbb{R}^d$ and any $\epsilon > 0$, we say that a sequence $x_1, \ldots, x_n$ of $n$ points in $X$ defines an $\epsilon$-cover of $X$ if and only if $X \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon)$. The covering number $N_\epsilon(X)$ of $X$ is then defined as the
Global optimization of Lipschitz functions

| Problem   | Auto-MPG  | BreastCancer | Concrete | Housing | Yacht |
|-----------|-----------|--------------|----------|---------|-------|
| AdaLIPO   | 14.6 (±09)| 05.4 (±03)   | 04.9 (±02)| 05.4 (±04)| 25.2 (±21)|
| BayesOpt  | 10.8 (±03)| 06.8 (±04)   | 06.4 (±03)| 07.5 (±04)| 13.8 (±20)|
| CMA-ES    | 29.3 (±25)| 11.1 (±09)   | 10.4 (±08)| 12.4 (±12)| 29.6 (±25)|
| CRS       | 28.7 (±14)| 08.9 (±08)   | 10.0 (±09)| 13.8 (±10)| 32.6 (±15)|
| DIRECT    | 11.0 (±00)| 06.0 (±00)   | 06.0 (±00)| 06.0 (±00)| 11.0 (±00)|
| MLSL      | 13.1 (±15)| 06.6 (±03)   | 06.1 (±04)| 07.2 (±03)| 14.4 (±13)|
| PRS       | 65.1 (±62)| 10.6 (±10)   | 09.8 (±09)| 11.5 (±10)| 73.3 (±72)|

| AdaLIPO   | 17.7 (±09)| 06.6 (±04)   | 06.4 (±04)| 17.9 (±25)| 33.3 (±26)|
| BayesOpt  | 12.2 (±06)| 08.4 (±03)   | 07.9 (±03)| 13.9 (±22)| 15.9 (±21)|
| CMA-ES    | 42.9 (±31)| 13.7 (±10)   | 13.5 (±10)| 23.0 (±16)| 40.5 (±30)|
| CRS       | 35.8 (±13)| 13.6 (±10)   | 14.6 (±11)| 22.8 (±12)| 38.3 (±31)|
| DIRECT    | 11.0 (±00)| 11.0 (±00)   | 11.0 (±00)| 19.0 (±00)| 27.0 (±00)|
| MLSL      | 15.0 (±15)| 07.6 (±03)   | 07.3 (±04)| 16.3 (±10)| 16.3 (±13)|
| PRS       | 139 (±131)| 17.7 (±17)   | 14.0 (±12)| 39.6 (±39)| 247(±249)|

| AdaLIPO   | 32.6 (±16)| 34.1 (±36)   | 70.8 (±58)| 65.4 (±62)| 61.7 (±39)|
| BayesOpt  | 14.0 (±07)| 31.0 (±51)   | 28.2 (±34)| 17.9 (±22)| 18.5 (±22)|
| CMA-ES    | 73.7 (±49)| 35.1 (±20)   | 46.3 (±29)| 61.5 (±85)| 70.9 (±50)|
| CRS       | 48.5 (±16)| 34.8 (±12)   | 36.6 (±15)| 43.7 (±14)| 52.9 (±18)|
| DIRECT    | 47.0 (±00)| 27.0 (±00)   | 37.0 (±00)| 41.0 (±00)| 49.0 (±00)|
| MLSL      | 20.6 (±17)| 12.8 (±03)   | 14.7 (±10)| 16.3 (±10)| 21.4 (±14)|
| PRS       | 747 (±330)| 145(±124)    | 176(±148)| 406(±312)| 779 (±334)|

Figure 5: Results of the numerical experiments. The tables display the number of evaluations required by each method to reach the specified target (mean ± standard deviation). In bold, the best result obtained in terms of average of function evaluations.

minimal size of a sequence defining an \( \epsilon \)-cover of \( \mathcal{X} \), i.e.

\[
\mathcal{N}_\epsilon(\mathcal{X}) := \inf \left\{ n \in \mathbb{N}^* : \exists (x_1, \ldots, x_n) \in \mathcal{X}^n \text{ s.t. } \mathcal{X} \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon) \right\}.
\]
The next result provides an upper bound on the covering numbers of hypercubes.

**Proposition 27 (Covering number of hypercubes)** Let \([0, R]^d\) be an hypercube of dimensionality \(d \geq 1\) whose side has length \(R > 0\). Then, for all \(\epsilon > 0\), we have that

\[
\mathcal{N}_\epsilon([0, R]^d) \leq (\sqrt{dR}/2\epsilon)^d \vee 1.
\]

**Proof** Observe first that since \([0, R]^d \subseteq B(c, \sqrt{dR}/2)\) where \(c\) denotes the center of the hypercube, then the result trivially holds for any \(\epsilon \geq \sqrt{dR}/2\). Fix any \(\epsilon < \sqrt{dR}/2\), set \(N_\epsilon = \lceil \sqrt{dR}/2\epsilon \rceil\) and define for all \(I \in \{0, \ldots, N_\epsilon - 1\}^d\) the series \(H_I = I \times R/N_\epsilon + [0, R/N_\epsilon]^d\) of \(N_\epsilon\) hypercubes which cover \([0, R]^d := \bigcup_{I \in \{0, \ldots, N_\epsilon - 1\}^d} H_I\). Denoting by \(c_I\) the center of \(H_I\) and since \(\max_{x \in H_I} \|x - c_I\|_2 \leq \epsilon\), it necessarily follows that \(H_I \subseteq B(c_I, \epsilon)\) which implies that \([0, R]^d \subseteq \bigcup_{I \in \{0, \ldots, N_\epsilon - 1\}^d} B(c_I, \epsilon)\) and proves that \(\mathcal{N}_\epsilon([0, R]^d) \leq N_\epsilon^d \leq (\sqrt{dR}/2\epsilon)^d\). \(\Box\)

This result can be extended to any compact and convex set of \(\mathbb{R}^d\) as shown below.

**Corollary 28 (Covering number of a convex set)** For any bounded compact and convex set \(\mathcal{X} \subset \mathbb{R}^d\), we have that for all \(\epsilon > 0\),

\[
\mathcal{N}_\epsilon(\mathcal{X}) \leq (\sqrt{d\text{diam}(\mathcal{X})}/\epsilon)^d \vee 1.
\]

**Proof** First, we show that \(\mathcal{N}_\epsilon(\mathcal{X}) \leq \mathcal{N}_\epsilon([0, 2\text{diam}(\mathcal{X})]^d)\) and then, we use the bound of Proposition 27 to conclude the proof. By definition of \(\text{diam}(\mathcal{X})\), we know that there exists some \(x \in \mathbb{R}^d\) such that \(\mathcal{X} \subseteq x + [0, 2\text{diam}(\mathcal{X})]^d\). Hence, we know from Proposition 27 that there exists a sequence \(c_1, \ldots, c_{N_\epsilon}\) of \(N_\epsilon\) \(\mathcal{N}_\epsilon([0, 2\text{diam}(\mathcal{X})]^d)\) points in \([0, 2\text{diam}(\mathcal{X})]^d\) forming an \(\epsilon\)-cover of \(\mathcal{X}\):

\[
\mathcal{X} \subseteq [0, 2\text{diam}(\mathcal{X})]^d \subseteq \bigcup_{i=1}^{N_\epsilon} B(c_i, \epsilon).
\]

However, we do not have the guarantee at this point that the centers \(c_1, \ldots, c_{N_\epsilon}\) belong to \(\mathcal{X}\). To build an \(\epsilon\)-cover of \(\mathcal{X}\), we project each of those centers on \(\mathcal{X}\). More precisely, we show that \(\mathcal{X} \subseteq \bigcup_{i=1}^{N_\epsilon} B(\Pi_\mathcal{X}(c_i), \epsilon)\) where \(\Pi_\mathcal{X} : x \in \mathbb{R}^d \mapsto \arg\min_{x' \in \mathcal{X}} \|x - x'\|_2 \in \mathcal{X}\) denotes the projection over the compact and convex set \(\mathcal{X}\). Starting from (1), it sufficient to show that \(B(c_i, \epsilon) \cap \mathcal{X} \subseteq B(\Pi_\mathcal{X}(c_i), \epsilon)\), for all \(i \in \{1, \ldots, N_\epsilon\}\) to prove that

\[
\mathcal{X} \subseteq \bigcup_{i=1}^{N_\epsilon} B(c_i, \epsilon) \cap \mathcal{X} \subseteq \bigcup_{i=1}^{N_\epsilon} B(\Pi_\mathcal{X}(c_i), \epsilon).
\]

Pick any \(c \in \{c_1, \ldots, c_{N_\epsilon}\}\) and consider the following cases on the distance \(\|c - \Pi_\mathcal{X}(c)\|_2\) between the center and its projection: (i) if \(\|c - \Pi_\mathcal{X}(c)\|_2 = 0\), then \(c = \Pi_\mathcal{X}(c)\) and we have \(B(c, \epsilon) \cap \mathcal{X} \subseteq B(\Pi_\mathcal{X}(c), \epsilon)\), (ii) If \(\|c - \Pi_\mathcal{X}(c)\|_2 > \epsilon\), then \(\mathcal{X} \cap B(c, \epsilon) = \emptyset\), and we have \(\mathcal{X} \cap B(c, \epsilon) \subseteq B(\Pi_\mathcal{X}(c), \epsilon)\). We now consider the non-trivial case where \(\|c - \Pi_\mathcal{X}(c)\|_2 \in (0, \epsilon)\). Pick any \(x \in B(c, \epsilon) \cap \mathcal{X}\) and note that since \(x \in B(c, \epsilon)\), then

\[
\epsilon^2 \geq \|x - c\|_2^2
\]

\[
= \|x - \Pi_\mathcal{X}(c) + \Pi_\mathcal{X}(c) - c\|_2^2
\]

\[
= \|x - \Pi_\mathcal{X}(c)\|_2^2 + \|c - \Pi_\mathcal{X}(c)\|_2^2 + 2 \cdot \langle x - \Pi_\mathcal{X}(c), \Pi_\mathcal{X}(c) - c \rangle
\]
which combined with the fact that \( \|c - \Pi_{X}(c)\|_{2}^{2} \geq 0 \) gives

\[
\|x - \Pi_{X}(c)\|_{2}^{2} \leq \epsilon^{2} - 2 \cdot \langle x - \Pi_{X}(c), \Pi_{X}(c) - c \rangle.
\] (2)

We will simply show that the inner product \( \langle x - \Pi_{X}(c), \Pi_{X}(c) - c \rangle \) cannot be strictly negative to prove that \( \|x - \Pi_{X}(c)\|_{2} \leq \epsilon \). Assume by contradiction that \( \langle x - \Pi_{X}(c), \Pi_{X}(c) - c \rangle < 0 \). Since \( \Pi_{X}(c) \in X \) and \( x \in \mathcal{X} \), it follows the convexity of \( \mathcal{X} \) implies that \( \forall \lambda \in [0, 1] \),

\[
x_{\lambda} = \Pi_{X}(c) + \lambda \cdot (x - \Pi_{X}(c)) \in \mathcal{X}.
\]

However, for all \( \lambda \in (0, 1) \) we have that

\[
\|x_{\lambda} - c\|_{2}^{2} = \|\Pi_{X}(c) - c + \lambda \cdot (x - \Pi_{X}(c))\|_{2}^{2}
= \|\Pi_{X}(c) - c\|_{2}^{2} + \lambda^{2} \|x - \Pi_{X}(c)\|_{2}^{2} + 2\lambda \cdot \langle \Pi_{X}(c) - c, x - \Pi_{X}(c) \rangle
= \|\Pi_{X}(c) - c\|_{2}^{2} + \lambda \cdot \|x - \Pi_{X}(c)\|_{2}^{2} + 2 \cdot \langle \Pi_{X}(c) - c, x - \Pi_{X}(c) \rangle.
\]

Therefore, taking any \( 0 < \lambda^{*} < \|\Pi_{X}(c) - c, x - \Pi_{X}(c)\| / \|\Pi_{X}(c) - c\|_{2} \) and \( 1 \) so that the second term of the right hand term of the previous equation is strictly negative gives that \( \|x_{\lambda^{*}} - c\|_{2}^{2} < \|\Pi_{X}(c) - c\|_{2}^{2} \) leads us to the following contradiction \( \min_{x \in \mathcal{X}} \|x - c\|_{2} \leq \|x_{\lambda^{*}} - c\|_{2} < \|\Pi_{X}(c) - c\|_{2} = \min_{x \in \mathcal{X}} \|x - c\|_{2} \). Hence, \( \langle x - \Pi_{X}(c), \Pi_{X}(c) - c \rangle \geq 0 \) and we deduce from (2) that \( \mathcal{X} \cap B(c, \epsilon) \subseteq B(\Pi_{X}(c), \epsilon) \), which completes the proof. \( \Box \)

The next inequality will be useful to bound to bound volume of the intersection of a ball and a convex set.

\textbf{Lemma 29} \( \text{ (From Zabinsky and Smith (1992), see Appendix Section therein). For any compact and convex set } \mathcal{X} \subseteq \mathbb{R}^{d} \text{ with non-empty interior, we have that for any } x^{*} \in \mathcal{X} \text{ and } \epsilon \in (0, \text{diam}(\mathcal{X})), \)

\[
\frac{\mu(B(x^{*}, \epsilon) \cap \mathcal{X})}{\mu(\mathcal{X})} \geq \left( \frac{\epsilon}{\text{diam}(\mathcal{X})} \right)^{d}.
\]

\textbf{Proof} We point out that a detailed proof of this result can be found in the Appendix Section of (Zabinsky and Smith (1992)). Nonetheless, we provide here a proof with less details for completeness. Introduce the similarity transformation \( S : \mathbb{R}^{d} \to \mathbb{R}^{d} \) defined by

\[
S : x \mapsto x^{*} + \frac{r}{\text{diam}(\mathcal{X})} (x - x^{*})
\]

and let \( S(\mathcal{X}) := \{S(x) : x \in \mathcal{X}\} \) be the image of \( \mathcal{X} \) by \( S \). Since \( x^{*} \in \mathcal{X} \) and \( \max_{x \in \mathcal{X}} \|x - x^{*}\|_{2} \leq \text{diam}(\mathcal{X}) \) by definition, it follows from the convexity of \( \mathcal{X} \) that \( S(\mathcal{X}) \subseteq B(x^{*}, r) \cap \mathcal{X} \) which implies that \( \mu(B(x^{*}, r) \cap \mathcal{X}) \geq \mu(S(\mathcal{X})) \). However, as \( S \) is a similarity transformation conserves the ratios of the volumes before/after transformation, we thus deduce that

\[
\frac{\mu(B(x^{*}, r) \cap \mathcal{X})}{\mu(\mathcal{X})} \geq \frac{\mu(S(\mathcal{X}))}{\mu(\mathcal{X})} = \frac{\mu(S(B(x^{*}, \text{diam}(\mathcal{X}))))}{\mu(B(x^{*}, \text{diam}(\mathcal{X})))} = \frac{\mu(B(x^{*}, r))}{\mu(B(x^{*}, \text{diam}(\mathcal{X})))}
\]

and the result follows using the fact that \( \forall r \geq 0, \mu(B(x^{*}, r)) = \pi^{d/2} r^{d} / \Gamma(d/2 + 1) \) where \( \Gamma(\cdot) \) stands for the standard gamma function. \( \Box \)
Proposition 30 (Pure Random Search) Let \( \mathcal{X} \subset \mathbb{R}^d \) be a compact and convex set with non-empty interior and let \( f \in \text{Lip}(k) \) be a \( k \)-Lipschitz functions defined on \( \mathcal{X} \) for some \( k \geq 0 \). Then, for any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), we have with probability at least \( 1 - \delta \),

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \leq k \cdot \text{diam}(\mathcal{X}) \cdot \left( \frac{\ln(1/\delta)}{n} \right)^{1/2}
\]

where \( X_1, \ldots, X_n \) denotes a sequence of \( n \) independent copies of \( X \sim \mathcal{U}(\mathcal{X}) \).

Proof Fix any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), let \( \epsilon = k \cdot \text{diam}(\mathcal{X}) \cdot (\ln(1/\delta)/n)^{1/d} \) be the value of the upper bound and \( \mathcal{X}_\epsilon = \{ x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \epsilon \} \) the corresponding level set. As the result trivially holds whenever \( n \leq \ln(1/\delta) \), we consider that \( n > \ln(1/\delta) \). Observe now that since \( f \in \text{Lip}(k) \), then for any \( x^* \in \text{arg max}_{x \in \mathcal{X}} f(x) \), we have that \( \mathcal{X} \cap B(x^*, \epsilon/k) \subseteq \mathcal{X}_\epsilon \) since \( |f(x) - f(x^*)| \leq k \cdot \|x - x^*\|_2 = \epsilon \) for all \( x \in B(x^*, \epsilon/k) \cap \mathcal{X} \). Therefore, by picking any \( x^* \in \text{arg max}_{x \in \mathcal{X}} f(x) \), one gets

\[
\mathbb{P}\left( \max_{i=1 \ldots n} f(X_i) \geq \max_{x \in \mathcal{X}} f(x) - \epsilon \right) = \mathbb{P}\left( \bigcup_{i=1}^{n} \{ X_i \in \mathcal{X}_\epsilon \} \right) \quad \text{(def. of \( \mathcal{X}_\epsilon \))}
\]

\[
= 1 - \mathbb{P}(X_1 \notin \mathcal{X}_\epsilon)^n \quad \text{(i.i.d. r.v.)}
\]

\[
\geq 1 - \mathbb{P}(X_1 \notin \mathcal{X} \cap B(x^*, \epsilon/k))^n \quad \text{\( \mathcal{X} \cap B(x^*, \epsilon/k) \subseteq \mathcal{X}_\epsilon \)}
\]

\[
= 1 - \left( 1 - \frac{\mu(\mathcal{X} \cap B(x^*, \epsilon/k))}{\mu(\mathcal{X})} \right)^n \quad \text{(\( \mathcal{X} \sim \mathcal{U}(\mathcal{X}) \))}
\]

\[
\geq 1 - \left( 1 - \frac{\epsilon}{k \cdot \text{diam}(\mathcal{X})} \right)^n \quad \text{(Lemma 29)}
\]

\[
= 1 - \left( 1 - \frac{\ln(1/\delta)}{n} \right)^n \quad \text{(def. of \( \epsilon \))}
\]

\[
\geq 1 - \delta. \quad \text{(1 + x \leq e^x)}
\]

\[\square\]

Appendix B. Proofs of Section 3

In this section, we provide the proofs of Propositions 3, 4, 5 and Example 1.

Proof of proposition 3. \((\Leftarrow)\) Let \( A \) be any global optimization algorithm such that \( \forall f \in \bigcup_{k \geq 0} \text{Lip}(k), \sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \|X_i - x\|_2 \geq \delta > 0 \). Pick any \( \epsilon > 0 \), any \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \) and let \( \mathcal{X}_\epsilon = \{ x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \epsilon \} \) be the corresponding level set. As \( \mathcal{X}_\epsilon \) is non-empty, there necessarily exists some \( x_\epsilon \in \mathcal{X} \) and \( r_\epsilon > 0 \) such that \( B(x_\epsilon, r_\epsilon) \cap \mathcal{X} \subseteq \mathcal{X}_\epsilon \). Thus, if \( X_1, \ldots, X_n \) denotes a sequence a sequence of \( n \) evaluation points generated by \( A \)
Global optimization of Lipschitz functions

over $f$, we directly obtain from the convergence in probability of the mesh grid that

$$
P\left( \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) > \epsilon \right) = P\left( \bigcap_{i=1}^{n} \{ X_i \notin \mathcal{X} \} \right)
\leq P\left( \bigcap_{i=1}^{n} \{ X_i \notin B(x_\epsilon, r_\epsilon) \} \right)
= P\left( \min_{i=1 \ldots n} \| X_i - x_\epsilon \|_2 > r_\epsilon \right)
\leq P\left( \sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \| X_i - x \|_2 > r_\epsilon \right) \xrightarrow{n \to \infty} 0.

(\Rightarrow) Let $A$ be any global optimization algorithm consistent over the set of Lipschitz functions and assume by contradiction that there exists some $f^* \in \bigcup_{k \geq 0} \text{Lip}(k)$ such that $\sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \| x - X_i \|_2 \xrightarrow{p} 0$. The implication is proved in two steps: first, we show that there exists a ball $B(c^*, \epsilon)$ for some $c^* \in \mathcal{X}$ which is almost never hit by the algorithm and second, we build a Lipschitz function which admits its maximum over this ball.

First step. Let $\{ X_i \}_{i \in \mathbb{N}^*}$ be a sequence of evaluation points generated by $A$ over $f^*$. Observe first that since for all $\epsilon > 0$, the series $n \in \mathbb{N}^* \mapsto P(\sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \| x - X_i \|_2 > \epsilon)$ is non-increasing, then the contradiction assumption necessarily implies that

$$\exists \epsilon_1, \epsilon_2 > 0 \text{ such that } \forall n \in \mathbb{N}^*, \ P\left( \sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \| x - X_i \|_2 > \epsilon_1 \right) > \epsilon_2. \quad (3)$$

Consider now any sequence $c_1, \ldots, c_{N_1}$ of $N_1 = \mathcal{N}_\epsilon(\mathcal{X})$ points in $\mathcal{X}$ defining an $\epsilon_1$-cover of $\mathcal{X}$ and suppose by contradiction that

$$\forall c \in \{ c_1, \ldots, c_{N_1} \}, \ \exists n_c \in \mathbb{N}^* \text{ such that } P\left( \bigcap_{i=1}^{n_c} \{ X_i \notin B(c, \epsilon_1) \cap \mathcal{X} \} \right) \leq \frac{\epsilon_2}{2N_1}$$

which gives by setting $N_2 = \max_{c \in \{ c_1, \ldots, c_{N_1} \}} n_c$ that

$$\forall c \in \{ c_1, \ldots, c_{N_1} \}, \ P\left( \bigcap_{i=1}^{N_2} \{ X_i \notin B(c, \epsilon) \cap \mathcal{X} \} \right) \leq \frac{\epsilon_2}{2N_1}.$$
However, as \( c_1, \ldots, c_{N_1} \) form an \( \epsilon_1 \)-cover of \( \mathcal{X} \), it follows that
\[
\mathbb{P} \left( \sup_{x \in \mathcal{X}} \min_{i=1 \ldots N_2} \| x - X_i \|_2 \leq \epsilon_1 \right) \geq \mathbb{P} \left( \bigcap_{j=1}^{N_1} \bigcup_{i=1}^{N_2} \{ X_i \in B(c_j, \epsilon_1) \cap \mathcal{X} \} \right) 
= 1 - \mathbb{P} \left( \bigcup_{j=1}^{N_1} \bigcap_{i=1}^{N_2} \{ X_i \notin B(c_j, \epsilon_1) \cap \mathcal{X} \} \right) 
\geq 1 - \sum_{j=1}^{N_1} \mathbb{P} \left( \bigcap_{i=1}^{N_2} \{ X_i \notin B(c_j, \epsilon) \cap \mathcal{X} \} \right) 
\geq 1 - N_1 \times \frac{\epsilon_2}{2N_1} 
= 1 - \frac{\epsilon_2}{2}
\]
which contradicts (3). Hence, we deduce that
\[
\exists c^* \in \{ c_1, \ldots, c_{N_1} \} \text{ such that } \forall n \in \mathbb{N}^*, \quad \mathbb{P} \left( \bigcap_{i=1}^{n} \{ X_i \notin B(c^*, \epsilon_1) \cap \mathcal{X} \} \right) \geq \frac{\epsilon_2}{2N_1}.
\]

**Second Step.** Based on this center \( c^* \in \mathcal{X} \), one can introduce the function \( \tilde{f} : \mathcal{X} \rightarrow \mathbb{R} \) defined for all \( x \in \mathcal{X} \) by
\[
\tilde{f}(x) = \begin{cases} 
    f^*(x) + 3 \left( 1 - \frac{\|c^* - x\|_2}{\epsilon_1} \right) \times (\max_{x \in \mathcal{X}} f^*(x) - \min_{x \in \mathcal{X}} f^*(x)) & \text{if } x \in B(c^*, \epsilon_1) \\
    f^*(x) & \text{otherwise}
\end{cases}
\]
which is maximized over \( B(c^*, \epsilon_1) \) and Lipschitz continuous as both \( f^* \) and \( x \mapsto \|c^* - x\|_2 \) are Lipschitz. However, since \( \tilde{f} \) and \( f^* \) can not be distinguished over \( \mathcal{X} / B(c, \epsilon_1) \), we have that \( \forall n \in \mathbb{N}^* \),
\[
\mathbb{P} \left( \max_{x \in \mathcal{X}} \tilde{f}(x) - \max_{i=1 \ldots n} \tilde{f}(X_i') > \max_{x \in \mathcal{X}} f(x) \right) \geq \mathbb{P} \left( \bigcap_{i=1}^{n} \{ X_i' \notin B(c, \epsilon_2) \cap \mathcal{X} \} \right) 
= \mathbb{P} \left( \bigcap_{i=1}^{n} \{ X_i \notin B(c, \epsilon_2) \cap \mathcal{X} \} \right) 
\geq \epsilon_2 / (2N_1) > 0
\]
where \( X_1', \ldots, X_n' \) denotes a sequence of evaluation points generated by \( A \) over \( \tilde{f} \), and we deduce that there exists \( \tilde{f} \in \bigcup_{k \geq 0} \text{Lip}(k) \) such that \( \max_{i=1 \ldots n} \tilde{f}(X_i') \not\rightarrow \max_{x \in \mathcal{X}} \tilde{f}(x) \). Hence, it contradicts the fact that \( A \) is consistent over \( \bigcup_{k \geq 0} \text{Lip}(k) \) and we deduce that, necessarily, \( \sup_{x \in \mathcal{X}} \min_{i=1 \ldots n} \| X_i - x \|_2 \not\rightarrow 0 \) for all \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \). \( \square \)

**Proof of Example 1.** Fix any \( n \in \mathbb{N}^* \), set \( \delta \in (0,1) \), define \( \epsilon = \text{diam}(\mathcal{X}) \cdot ((\ln(n/\delta) + d \ln(d))/n)^{1/d} \) and let \( X_1, \ldots, X_n \) be a sequence of \( n \) independent copies of
\( X \sim \mathcal{U}(\mathcal{X}) \). Since the result trivially holds whenever \((\ln(n/\delta) + d \ln(d))/n \geq 1\), we consider the case where \((\ln(n/\delta) + d \ln(d))/n < 1\). From Proposition 28, we know that there exists a sequence \(x_1, \ldots, x_{N_\epsilon} \) of \(N_\epsilon = N_\epsilon(\mathcal{X})\) points in \(\mathcal{X}\) such that \(\mathcal{X} \subseteq \bigcup_{j=1}^{N_\epsilon} B(x_j, \epsilon)\). Therefore, using the bound on the covering number \(N_\epsilon(\mathcal{X})\) of Corollary 28, we obtain that

\[
\mathbb{P}\left( \sup_{x \in \mathcal{X}} \min_{i=1}^{n} \|x - x_i\|_2 \leq \epsilon \right) \geq \mathbb{P}\left( \bigcap_{j=1}^{N_\epsilon} \bigcup_{i=1}^{n} \{x_i \in B(x_j, \epsilon) \cap \mathcal{X}\} \right) \\
= 1 - \mathbb{P}\left( \bigcup_{j=1}^{N_\epsilon} \bigcap_{i=1}^{n} \{x_i \notin B(x_j, \epsilon) \cap \mathcal{X}\} \right) \\
\geq 1 - \sum_{j=1}^{N_\epsilon} \mathbb{P}\left( \bigcap_{i=1}^{n} \{x_i \notin B(x_j, \epsilon) \cap \mathcal{X}\} \right) \\
\geq 1 - N_\epsilon \times \max_{j=1 \ldots N_\epsilon} \mathbb{P}(X_1 \notin B(x_j, \epsilon) \cap \mathcal{X})^n \\
= 1 - N_\epsilon \times \max_{j=1 \ldots N_\epsilon} \left( 1 - \frac{\mu(\mathcal{X} \cap B(x_j, \epsilon))}{\mu(\mathcal{X})} \right)^n \\
\geq 1 - N_\epsilon \times \left( 1 - \left( \frac{\epsilon}{\text{diam}(\mathcal{X})} \right)^d \right)^n \\
\geq 1 - \left( \frac{\sqrt{d} \cdot \text{diam}(\mathcal{X})}{\epsilon} \right)^d \times \left( 1 - \left( \frac{\epsilon}{\text{diam}(\mathcal{X})} \right)^d \right)^n \\
\geq 1 - \delta
\]

and the proof is complete. \(\square\)

**Proof of Proposition 4.** The proof heavily builds upon the arguments used in the proof of the Theorem 1 in (Bull (2011)). Pick any algorithm \(A \in A\) and any constant \(C > 0\). Fix any \(n \in \mathbb{N}^*\) and \(\delta \in (0, 1)\) and set \(N_\delta = \lceil (n/\delta)^{1/d} \rceil\). By definition of \(\text{rad}(\mathcal{X})\), we know there exists some \(x \in \mathcal{X}\) such that \(x + [0, 2 \text{rad}(\mathcal{X})/\sqrt{d}] \subseteq \mathcal{X}\). One can then define for all \(I \in \{1, \ldots, N_\delta\}^d\), the centers \(c_I\) of the hypercubes \(H_I\) whose side are equal to \(D = 2 \text{rad}(\mathcal{X})/(\sqrt{d}N_\delta)\) and cover \(\mathcal{X}\), i.e., \(\bigcup_{I} H_I = x + [0, 2 \text{rad}(\mathcal{X})/\sqrt{d}] \subseteq \mathcal{X}\). Now, let \(X_1, \ldots, X_n\) be a sequence of \(n\) evaluation points generated by the algorithm \(A\) over the constant function \(f_0 : x \in \mathcal{X} \mapsto 0\) and define for all \(I \in \{1, \ldots, N_\delta\}^d\) the event

\[
E_I = \bigcap_{i=1}^{n} \{X_i \notin \text{Int}(H_I)\}.
\]

As the interiors of the \(N_\delta^d\) hypercubes are disjoint and we have \(n\) points, it necessarily follows that

\[
N_\delta^d \times \max_I \mathbb{P}(E_I) \geq \sum_I \mathbb{P}(E_I) = \mathbb{E}\left[ \sum_I \mathbb{I}(E_I) \right] \geq N_\delta^d - n.
\]
Hence, there exists some fixed $I^*$ only depending on $\mathcal{A}$ which maximizes the above probability and thus satisfies

$$\mathbb{P}(E_{I^*}) \geq \frac{N^d - n}{N^d} = 1 - \frac{n}{[(n/d)^{1/d}]^d} \geq 1 - \delta.$$  

Now, using the center $c_{I^*}$ of the hypercube $H_{I^*}$, one can then introduce the function $\tilde{f} \in \bigcup_{k \geq 0} \text{Lip}(k)$ defined for all $x \in \mathcal{X}$ by

$$\tilde{f}(x) = \begin{cases} C \times (1 - 2 \|c_{I^*} - x\|_2 / D) & \text{if } \|c_{I^*} - x\|_2 \leq D/2 \\ 0 & \text{otherwise.} \end{cases}$$

However, since the functions $\tilde{f}$ and $f_0$ cannot be distinguished over $\mathcal{X}/H_{I^*}$, we have that

$$\mathbb{P}\left(\max_{x \in \mathcal{X}} \tilde{f}(x) - \max_{i=1,\ldots,n} \tilde{f}(X'_i) \geq C \right) \geq \mathbb{P}\left(\bigcap_{i=1}^{n} \{X'_i \notin \text{Int}(H_{I^*})\}\right) = \mathbb{P}(E_{I^*}) \geq 1 - \delta$$

where $X'_1, \ldots, X'_n$ denotes a sequence of evaluation points generated by $\mathcal{A}$ over $\tilde{f}$, which proves the result. \hfill \Box

**Proof of Proposition 5. (Lower bound).** Pick any $n \in \mathbb{N}^*$ and set $D = 2 \text{rad}(\mathcal{X})/\sqrt{d [(2n)^{1/d}]}$. It can easily be shown by reproducing the same steps as in the proof of Proposition 4 with $\delta$ set to $1/2$, that for any global optimization algorithm $\mathcal{A}$, there exists a function $\tilde{f}_A \in \text{Lip}(k)$ defined by

$$\tilde{f}_A(x) = \begin{cases} kD/2 - k \cdot \|c_A - x\|_2 & \text{if } \|c_A - x\|_2 \leq D/2 \\ 0 & \text{otherwise,} \end{cases}$$

for some center $c_A \in \mathcal{X}$ only depending on $\mathcal{A}$, for which we have $\mathbb{P}(\max_{x \in \mathcal{X}} \tilde{f}_A(x) - \max_{i=1,\ldots,n} \tilde{f}_A(X'_i) \geq k \cdot D/2) \geq 1/2$ where $X'_1, \ldots, X'_n$ is a sequence of $n$ evaluation points generated by $\mathcal{A}$ over $\tilde{f}_A$. Therefore, using the definition of the supremum and Markov’s inequality gives that $\forall \mathcal{A} \in \mathcal{A}$:

$$\sup_{f \in \text{Lip}(k)} \mathbb{E}\left[\max_{x \in \mathcal{X}} f(x) - \max_{i=1,\ldots,n} f(X_i)\right] \geq \mathbb{E}\left[\max_{x \in \mathcal{X}} \tilde{f}_A(x) - \max_{i=1,\ldots,n} \tilde{f}_A(X_i)\right] \geq kD/2 \times \mathbb{P}\left(\max_{x \in \mathcal{X}} \tilde{f}_A(x) - \max_{i=1,\ldots,n} \tilde{f}_A(X_i) \geq k \cdot D/2\right) \geq k \cdot \text{rad}(\mathcal{X}) \cdot 8\sqrt{d} \cdot n^{-\frac{1}{d}}.$$

As the previous inequality holds true for any algorithm $\mathcal{A}$, the proof is complete.

**Upper bound.** Sequentially using the fact that (i) the infinimum minimax loss taken over all algorithms is necessarily upper bounded by the loss suffered by a Pure Random
Search, (ii) for any positive random variable, \( \mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t) dt \), (iii) Proposition 30 and (iv) the change of variable \( u = n(t/d \text{diam}(\mathcal{X}))^{1/d} \), we obtain that

\[
\inf_{A \in A} \sup_{f \in \text{Lip}(k)} \mathbb{E} \left[ \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \right] \leq \sup_{f \in \text{Lip}(k)} \mathbb{E} \left[ \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X'_i) \right] \\
\leq \int_0^\infty \exp \left\{ -n(t/k \cdot \text{diam}(\mathcal{X}))^{1/d} \right\} dt \\
= k \cdot \text{diam}(\mathcal{X}) \cdot n^{-d} \cdot d \cdot \int_0^\infty u^{d-1} e^{-u} du \\
= k \cdot \text{diam}(\mathcal{X}) \cdot n^{-d} \cdot d \cdot \Gamma(d)
\]

where \( X'_1, \ldots, X'_n \) denotes a sequence of \( n \) independent copies of \( X' \sim \mathcal{U}(\mathcal{X}) \) and \( \Gamma(\cdot) \) the Euler’s Gamma function. Recalling that \( \Gamma(d) = (d-1)! \) for all \( d \in \mathbb{N}^* \) completes the proof. □

Appendix C. Proofs of Section 3

In this section, we provide the proofs for Lemma 8, Proposition 11, Proposition 12, Corollary 13, Proposition 14, Theorem 15 and Theorem 16.

Proof of Lemma 8. The first implication \( (\Rightarrow) \) is a direct consequence of the definition of \( \mathcal{X}_{k,t} \). Noticing that the function \( \hat{f}: x \mapsto \min(\max_{i=1 \ldots t} f(X_i), \min_{i=1 \ldots t} f(X_i) + k \|x - X_i\|_2) \) belongs to \( \mathcal{F}_{k,t} \) and that \( \arg \max_{x \in \mathcal{X}} \hat{f}(x) = \{ x \in \mathcal{X} : \min_{i=1 \ldots t} f(X_i) + k \|x - X_i\|_2 \geq \max_{i=1 \ldots t} f(X_i) \} \) proves the second implication. □

Proof of Proposition 11. Fix any \( f \in \text{Lip}(k) \), pick any \( n \in \mathbb{N}^* \), set \( \epsilon > 0 \) and let \( \mathcal{X}_\epsilon = \{ x \in \mathcal{X} : f(x) \geq \max_{x \in \mathcal{X}} f(x) - \epsilon \} \) be the corresponding level set. Denoting by \( X'_1, \ldots, X'_n \) a sequence of \( n \) random variable uniformly distributed over \( \mathcal{X} \) and observing that \( \mu(\mathcal{X}_\epsilon) > 0 \), we directly obtain from Proposition 12 that

\[
\mathbb{P} \left( \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) > \epsilon \right) \leq \mathbb{P} \left( \max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X'_i) > \epsilon \right) \\
\leq \mathbb{P} \left( \bigcap_{i=1}^n \{ X'_i \notin \mathcal{X}_\epsilon \} \right) \\
\leq \left( 1 - \frac{\mu(\mathcal{X}_\epsilon)}{\mu(\mathcal{X})} \right)^n \xrightarrow{n \to \infty} 0.
\]

□

Proof of Proposition 12. The proof is similar to the one of Proposition 12 in (Malherbe and Vayatis (2016)). □

Proof of Corollary 13. Combining Proposition 12 and Proposition 30 stated at the beginning of the Appendix Section gives the result. □

Proof of Proposition 14. Fix any \( \delta \in (0,1) \), set \( n \in \mathbb{N}^* \) and let \( r_{\delta,n} = \text{rad}(\mathcal{X}) (\delta/n)^{\frac{1}{d}} \) be
the value of the lower bound divided by \( k \). As \( \text{rad}(\mathcal{X}) > 0 \), there necessarily exists some point \( x^* \in \mathcal{X} \) such that \( B(x^*, \text{rad}(\mathcal{X})) \subseteq \mathcal{X} \). Based on this point, one can then introduce the function \( \tilde{f} \in \text{Lip}(k) \) defined for all \( x \in \mathcal{X} \) by

\[
\tilde{f}(x) = \begin{cases} k \cdot r_{\delta,n} - k \cdot \|x - x^*\|_2 & \text{if } x \in B(x^*, r_{\delta,n}) \\ 0 & \text{otherwise.} \end{cases}
\]

Denoting now by \( X_1, \ldots, X_n \) a sequence of \( n \) evaluation points generated by LIPO tuned with a parameter \( k \) over \( \tilde{f} \) and observing that (i) \( X_1 \) is uniformly distributed over \( \mathcal{X} \) and (ii) \( X_{i+1} \) is also uniformly distributed over \( \mathcal{X} \) for \( i \geq 1 \) as soon as only constant evaluations have been recorded (i.e. \( \mathcal{X}_{k,i+1} = \mathcal{X} \) on the event \( \bigcap_{t \leq i} \{ X_t \notin B(x^*, r_{\delta,n}) \} \)), we have that

\[
P\left( \max_{x \in \mathcal{X}} \tilde{f}(x) - \max_{i=1 \ldots n} \tilde{f}(X_i) \geq k \cdot r_{\delta,n} \right) \geq P\left( \bigcap_{i=1}^n \{ X_i \notin B(x^*, r_{\delta,n}) \} \right)
= \left[ P(X_1 \notin B(x^*, r_{\delta,n})) \times \prod_{i=1}^{n-1} P\left( X_{i+1} \notin B(x^*, r_{\delta,n}) \mid \bigcap_{t=1}^i \{ X_t \notin B(x^*, r_{\delta,n}) \} \right) \right]
= \left( 1 - \frac{\mu(B(x^*, r_{\delta,n}) \cap \mathcal{X})}{\mu(\mathcal{X})} \right)^n
\geq \left( 1 - \frac{r_{\delta,n}}{\text{rad}(\mathcal{X})} \right)^n
= \left( 1 - \frac{\delta}{n} \right)^n
\geq 1 - \delta.
\]

**Proof of Theorem 15.** Pick any \( n \in \mathbb{N}^* \), fix any \( \delta \in (0,1) \) and let \( X_1, \ldots, X_n \) be a sequence of \( n \) evaluation points generated by the LIPO algorithm over \( f \) after \( n \) iterations. To clarify the proof, we set some specific notations: let \( D = \max_{x \in \mathcal{X}} \|x - x^*\|_2 \), set

\[
M = \begin{cases} \left( \frac{c_\kappa}{8k} \right)^d \cdot \frac{n}{\ln(n/\delta) + 2(2\sqrt{2})^d} & \text{if } \kappa = 1 \\ \frac{1}{\ln(2d(\kappa-1))} \ln \left( 1 + \left( \frac{c_\kappa D^\kappa}{8KD^\kappa} \right)^d \cdot \frac{n(2d(\kappa-1))}{\ln(n/\delta) + 2(2\sqrt{2})^d} \right) & \text{otherwise,} \end{cases}
\]

define for all \( m \in \{1 \ldots M\} \) the series of integers:

\[
N_m := \left[ \sqrt{d} \cdot \left( \frac{8kD^\kappa}{c_\kappa D^\kappa} \right) \cdot 2^{m(\kappa-1)} \right]^d \quad \text{and} \quad N'_m := \left[ \ln(M/\delta) \cdot \left( \frac{8kD^\kappa}{c_\kappa D^\kappa} \right)^d \cdot 2^{md(\kappa-1)} \right]
\]

28
and let $\tau_0, \ldots, \tau_M$ be the series of stopping times initialized by $\tau_0 = 0$ and defined for all $m \geq 1$ by

$$
\tau_m := \inf \left\{ t \geq \tau_{m-1} \mid \sum_{i=\tau_{m-1}+1}^{t} I\{X_i \in B(x^*, 2 \cdot D \cdot 2^{-m})\} = N'_m \right\}.
$$

The stopping time $\tau_m$ correspond to the time after $\tau_{m-1}$ where we have recorded at least $N'_m$ random evaluation points inside the ball $B(x^*, 2 \cdot D \cdot 2^{-m})$. To prove the result, we show that each of the following events:

$$
E_m := \left\{ \max_{i=1 \ldots \tau_m} f(X_i) \geq \max_{x \in \mathcal{X}} f(x) - \frac{c_\kappa}{2} \left( \frac{D}{2^m} \right)^\kappa \right\} \cap \left\{ \tau_m \leq N'_1 + \sum_{l=1}^{m-1} (N'_{l+1} + N_l) \right\}
$$

holds true with probability at least $1 - \delta/M$ on the event $\bigcap_{l=1}^{m-1} E_l$ for all $m \in \{2, \ldots, M\}$ so that:

$$
P(E_M) \geq P(E_1) \times \prod_{m=1}^{M-1} P\left( E_{m+1} | \bigcap_{l=1}^{m} E_l \right) \geq \left( 1 - \frac{\delta}{M} \right)^M \geq 1 - \delta \tag{4}
$$

that will leads us to the result by analyzing $E_M$.

**Analysis of $P(E_1)$.** Observe first that since $\mathcal{X} \subseteq B(x^*, D)$, then $\tau_1 = N'_1$. Using now the fact that (i) the algorithm is faster than a Pure Random Search (Proposition 12) and (ii) the bound of Proposition 30, we directly get that with probability at least $1 - \delta/M$,

$$
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots \tau_1} f(X_i) \leq k \cdot 2D \cdot \left( \frac{\ln(M/\delta)}{N'_1} \right)^{\frac{1}{\kappa}} \\
\leq k \cdot 2D \cdot \left( \frac{\ln(M/\delta)}{\ln(M/\delta)2^{d(\kappa-1)}} \left( \frac{c_\kappa D^\kappa \cdot d}{8kD} \right) \right)^{\frac{1}{\kappa}} \\
= \frac{c_\kappa}{2} \left( \frac{D}{2} \right)^\kappa
$$

which proves that $P(E_1) \geq 1 - \delta/M$.

**Analysis of $P(E_{m+1} | \bigcap_{l=1}^{m} E_l)$**. To bound this term, we use (i) a deterministic covering argument to control the stopping time $\tau_{m+1}$ (Lemma 31 and Corollary 32) and (ii) a stochastic argument to bound the maximum $\max_{i=1 \ldots \tau_{m+1}} f(X_i)$ (Lemma 33 and Corollary 34). The following lemma states that after $\tau_m$ and on the event $E_m$ there will be at most $N_m$ evaluation points that will fall inside the area $B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m})$.

**Lemma 31** For all $m \in \{1, \ldots, M - 1\}$, we have on the event $E_m$,

$$
\sum_{l=\tau_m+1}^{n} I\{X_l \in B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m})\} \leq N_m.
$$
Proof Fix \( m \in \{1, \ldots, M-1\} \) and assume that \( E_m = \{ \max_{i=1 \ldots m} f(X_i) \geq \max_{x \in \mathcal{X}} f(x) - c_k/2 \cdot (D/2^m)^\kappa \} \) holds true. Setting \( N = \lceil \sqrt{d} k D 2^m (\kappa - 1) / (c_k D^\kappa) \rceil \) and observing that \( B(x^*, 2D \cdot 2^{-m}) \subseteq x^* + 2D \cdot 2^{-m} \cdot [-1,1]^d \), one can then introduce the sequence \( H_I \), with \( I \in \{1, \ldots, N\}^d \), of \( N_d = N_m \) hypercubes whose side have length \( 4D \cdot 2^{-m} / N \) and cover \( x^* + 2D \cdot 2^{-m} \times [-1,1]^d \), so that
\[
B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m}) \subseteq x^* + 2D \cdot 2^{-m} \cdot [-1,1]^d = \bigcup_I H_I.
\]

Based on these hypercubes, one can define the set
\[
I_t = \{ I \in \{1, \ldots, N\}^d : H_I \cap B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m}) \cap X_{k,t} \neq \emptyset \}
\]
which contains the indexes of the hypercubes that still intersect the set of potential maximizers \( X_{k,t} \) at time \( t \) and the target area \( B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m}) \). We show by contradiction that there cannot be more than \( N_d = N_m \) evaluation points falling inside this area, otherwise it would be empty. Suppose that, after \( \tau_m \), there exists a sequence
\[
\tau_m < t_1 < t_2 < \cdots < t_{N_d+1} \leq n
\]
of \( N_d + 1 \) strictly increasing indexes for which the evaluation points \( X_{t_j}, j \geq 1 \), belong to the target area, i.e.,
\[
\forall j \in \{1, \ldots, N_d + 1\}, \ X_{t_j} \in B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m}).
\]

Fix any \( j \geq 1 \) and observe that since \( X_{t_j} \notin B(x^*, D \cdot 2^{-m}) \), then we have from Condition 1 that (i) \( f(X_{t_j}) < \max_{x \in \mathcal{X}} f(x) - c_k \cdot (D \cdot 2^{-m})^\kappa \). Moreover, as \( X_{t_j} \in X_{k,t_j-1} \cap B(x^*, 2D \cdot 2^{-m})/B(x^*, D \cdot 2^{-m}) \), it necessarily follows from the definition of the algorithm that (ii) there exists an index \( I^* \in I_{t_j-1} \) such that \( X_{t_j} \in H_{I^*} \). Therefore, combining (i) and (ii) with \( E_m \), gives that \( \forall x \in H_{I^*} \):
\[
\begin{align*}
f(x) & \leq f(X_{t_j}) + k \cdot \| X_{t_j} - x \|_2 \\
& \leq f(X_{t_j}) + k \cdot \max_{(x,x') \in H_{I^*}^2} \| x - x' \|_2 \\
& = f(X_{t_j}) + k \cdot \sqrt{d} \cdot 4D \cdot 2^{-m} / N \\
& \leq f(X_{t_j}) + \frac{c_k}{2} \cdot (D \cdot 2^{-m})^\kappa \\
& < \max_{x \in \mathcal{X}} f(x) - c_k \cdot (D \cdot 2^{-m})^\kappa + \frac{c_k}{2} \cdot (D \cdot 2^{-m})^\kappa \\
& \leq \max_{i=1 \ldots \tau_m} f(X_i) \\
& \leq \max_{i=1 \ldots t_j} f(X_i).
\end{align*}
\]

It has been shown that if \( X_{t_j} \) belongs to the target area then \( f(x) < \max_{i=1 \ldots t_j} f(X_i) \) for all \( x \in H_{I^*} \), which combined with the definition of the set of potential maximizers \( X_{k,t_j} \) at time \( t_j \) implies that \( H_{I^*} \notin X_{t_j} \). Hence, once an evaluation has been made in \( H_{I^*} \), there will not be
any future evaluation point falling inside this cube. We thus deduce that $|I_{t_j}| \leq |I_{t_{j-1}}| - 1$ for all $j \geq 1$ which leads us to the following contradiction:

$$0 \leq |I_{t_{N^d+1}}| = |I_{t_m}| + \sum_{j=\tau_{m+1}}^{t_{N^d+1}} |I_{t_j}| - |I_{t_{j-1}}| \leq |I_{t_m}| - (N^d + 1) \leq N^d - (N^d + 1) < 0$$

and proves the statement.

Based on this lemma, one might then derive a bound on the stopping time $\tau_{m+1}$.

**Corollary 32** For all $m \in \{1, \ldots, M - 1\}$, we have on the event $\bigcap_{i=1}^m E_i$ that

$$\tau_{m+1} \leq N'_1 + \sum_{l=1}^m (N'_{l+1} + N_l).$$

**Proof** The result is proved by induction. We start with the case where $m = 1$. Assuming that $E_1$ holds true and observing that (i) $\tau_1 = N'_1$ and (ii) $X \subseteq B(x^*, D) = B(x^*, D/2) \cup B(x^*, D)/B(x^*, D/2)$, one can then write:

$$\tau_2 = \tau_1 + \sum_{i=\tau_1+1}^{\tau_2} \mathbb{I}\{X_i \in B(x^*, D)\}$$

$$= N'_1 + \sum_{i=\tau_1+1}^{\tau_2} \mathbb{I}\{X_i \in B(x^*, D/2)\} + \sum_{i=\tau_1+1}^{\tau_2} \mathbb{I}\{X_i \in B(x^*, D)/B(x^*, D/2)\}.$$ 

However, since (i) $\sum_{i=\tau_1+1}^{\tau_2} \mathbb{I}\{X_i \in B(x^*, D/2)\} = N'_2$ by definition of $\tau_2$ and (ii) $\sum_{i=\tau_1+1}^{\tau_2} \mathbb{I}\{X_i \in B(x^*, D)/B(x^*, D/2)\} \leq N_1$ by Lemma 31, the result holds true for $m = 1$. Consider now any $m \geq 2$ and assume that the statement holds true for all $l < m$. Again, observing that $X \subseteq B(x^*, D \cdot 2^{-m}) \cup \bigcup_{l=1}^m B(x^*, D \cdot 2^{-(l-1)})/B(x^*, D \cdot 2^{-l})$ and keeping in mind that the stopping times are bounded by the induction assumption, one can write

$$\tau_{m+1} = \tau_m + \sum_{i=\tau_m+1}^{\tau_{m+1}} \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-m})\}$$

$$+ \sum_{i=\tau_m+1}^{\tau_{m+1}} \sum_{l=1}^m \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-(l-1)})/B(x^*, D \cdot 2^{-l})\}.$$ 

Now, combining the telescopic representation $\tau_{m+1} = \tau_1 + \sum_{l=1}^m (\tau_{l+1} - \tau_l)$ with the previous decomposition gives that

$$\tau_{m+1} = \tau_1 + \sum_{l=1}^m \sum_{\tau_l+1}^{\tau_{l+1}} \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-l})\}$$

$$+ \sum_{l=1}^m \sum_{\tau_l+1}^{\tau_{l+1}} \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-(l-1)})/B(x^*, D \cdot 2^{-l})\}.$$
However, since (i) \( \tau_1 = N_1' \), (ii) \( \sum_{i=1}^{\tau_m+1} \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-l})\} = N_m' \), for all \( l \geq 1 \) by definition of the stopping times and (iii) \( \sum_{i=1}^{\tau_m+1} \mathbb{I}\{X_i \in B(x^*, 2^{-(l+1)})/B(x^*, 2^{-l})\} \leq N_i \), for all \( l \geq 1 \) on the event \( \bigcap_{l=1}^{m} E_l \), from Lemma 31, we finally get that

\[
\tau_{m+1} \leq N_m' + \sum_{l=1}^{m} (N_{l+1} + N_l). 
\]


\( \Box \)

As Corollary 32 gives the desired bound on \( \tau_{m+1} \), it remains to control the maximum \( \max_{i=1...\tau_{m+1}} f(X_i) \). The next lemma shows that i.i.d. results can actually be used to bound this term.

**Lemma 33** For all \( m \in \{1,...,M-1\} \), we have that \( \forall y \in \text{Im}(f) \),

\[
\mathbb{P} \left( \max_{i=1...\tau_{m+1}} f(X_i) \geq y \mid \bigcap_{l=1}^{m} E_l \right) \geq \mathbb{P} \left( \max_{i=1...N_{m+1}'} f(X_i') \geq y \right).
\]

where \( X_1' \ldots X_{N_{m+1}'} \) denotes a sequence \( N_{m+1}' \) i.i.d. copies of \( X' \sim U(X \cap B(x^*, D \cdot 2^{-m})) \).

**Proof** From Corollary 32, we know that on the event \( \bigcap_{l=1}^{m} E_l \) the stopping time \( \tau_{m+1} \) is finite. Moreover, as \( \sum_{i=1}^{\tau_{m+1}} \mathbb{I}\{X_i \in B(x^*, D \cdot 2^{-m})\} = N_{m+1}' \) by definition of \( \tau_{m+1} \), it can then easily be shown by reproducing the same steps as in the proof of Proposition 12 with the evaluations points falling into \( B(x^*, D \cdot 2^{-m}) \) after \( \tau_m \) that the algorithm is faster than a Pure Random Search performed over the subspace \( X \cap B(x^*, D \cdot 2^{-m}) \), which proves the result.

As a direct consequence of this lemma, one can get the desired bound on the maxima as shown in the next corollary.

**Corollary 34** For all \( m \in \{1,...,M-1\} \), we have that

\[
\mathbb{P} \left( \max_{i=1...\tau_{m+1}} f(X_i) \geq \max_{x \in X} f(x) - \frac{c_k}{2} \cdot \left( \frac{D}{2^{m+1}} \right)^\kappa \mid \bigcap_{l=1}^{m} E_l \right) \geq 1 - \delta/M.
\]

**Proof** Omitting the conditionning upon \( \bigcap_{l=1}^{m} E_l \), we obtain from the combination of Lemma 33 and Proposition 30 that with probability at least \( 1 - \delta/M \):

\[
\max_{x \in X} f(x) - \max_{i=1...\tau_{m+1}} f(X_i) \leq k \cdot 2D \cdot 2^{-m} \cdot \left( \frac{\ln(M/\delta)}{N_{m+1}'} \right)^{\frac{1}{2}} \leq k \cdot 2D \cdot 2^{-m} \cdot \left( \frac{\ln(M/\delta)}{\ln(M/\delta)2^{d(m+1)(\kappa-1)}(c_k D^\kappa / (8k D))} \right)^{\frac{1}{2}} = \frac{c_k}{2} \cdot \left( \frac{D}{2^{m+1}} \right)^\kappa.
\]

32
Global optimization of Lipschitz functions

At this point, we know from the combination of Corollary 32 and Corollary 34 that

\[ \forall m \in \{1, \ldots, M - 1\}, \quad P\left( E_{m+1} \bigcap_{l=1}^{m} E_l \right) \geq 1 - \delta/M \]

which proves from (4) that \( P(E_M) \geq 1 - \delta \).

Analysis of \( E_M \). As \( \max_{i=1 \ldots \tau_M} f(X_i) \geq \max_{x \in X} f(x) - c_\kappa \cdot D^\kappa \cdot 2^{-M\kappa} \) and \( \tau_M \leq N'_1 + \sum_{l=1}^{M-1} \left( N'_{l+1} + N_l \right) \) on the event \( E_M \), it remains to show that \( N'_1 + \sum_{l=1}^{M-1} \left( N'_{l+1} + N_l \right) \leq n \) to conclude the proof. Consider first the case \( \kappa = 1 \). Setting \( C = (8k/c_\kappa)^d \) and observing that (i) \( N'_1 \leq \ln(M/\delta)C \leq 1 \), (ii) \( N_l \leq 2 \cdot C \cdot (2\sqrt{d})^d - 1 \) for all \( l \leq M \) and (iii) \( M \leq n \), one gets:

\[
N'_1 + \sum_{l=1}^{M-1} \left( N'_{l+1} + N_l \right) \leq C \cdot M \left( \ln(M/\delta) + 2(2\sqrt{d})^d \right)
\leq n \cdot \ln(M/\delta) + 2(2\sqrt{d})^d
\leq n.
\]

For \( \kappa > 1 \), since (i) \( M \) was chosen so that \( \frac{2d(\kappa-1)M - 1}{2^{d(\kappa-1)}-1} \leq C \cdot \frac{n}{\ln(n/\delta) + 2(2\sqrt{d})^d} \) and (ii) \( M \leq n \), we obtain:

\[
N'_1 + \sum_{l=1}^{M-1} \left( N'_{l+1} + N_l \right) \leq C \cdot \left( \ln(M/\delta) + 2(2\sqrt{d})^d \right) \sum_{l=1}^{M} \left( 2d(\kappa-1) \right)^l
\leq C \cdot \left( \ln(M/\delta) + 2(2\sqrt{d})^d \right) \cdot \frac{2d(\kappa-1)^M - 1}{2^{d(\kappa-1)}-1}
\leq n.
\]

Finally, using the elementary inequality \( |x| \geq x - 1 \) over \( M \) and the inequality \( c_\kappa D^\kappa \leq k \cdot \text{diam}(X) \) (by Condition 1) leads to the desired result and completes the proof. \( \Box \)

Proof of Theorem 16. (Lower bound) Pick any \( n \in \mathbb{N}^* \) and \( \delta \in (0, 1) \), set \( \epsilon = c_\kappa \cdot \text{rad}(X)^\kappa \cdot \frac{\delta^{\kappa/d}}{\exp(-\kappa(n - \sqrt{2n \ln(1/\delta)})/d)} \), let \( \mathcal{X}_\epsilon = \{ x \in X : f(x) \geq \max_{x \in X} f(x) - \epsilon \} \) be the corresponding level set. Observe first that since (i) \( \mathcal{X}_\epsilon = \{ x \in X : \epsilon \geq f(x^*) - f(x) \} \subseteq \{ x \in X : \epsilon \geq c_\kappa \| x^* - x \|^\kappa \} = X \cap B(x^*, (\epsilon/c_\kappa)^{1/\kappa}) \) and (ii) there exists \( x \in X \) such that \( B(x, \text{rad}(X)) \subseteq X \), then \( \mu(\mathcal{X}_\epsilon)/\mu(X) \leq ((\epsilon/c_\kappa)^{1/\kappa} / \text{rad}(X))^d = \delta e^{-n - \sqrt{2n \ln(1/\delta)}} \). It can then easily be shown by reproducing the same steps as in the proof of the Lower bound of
Theorem 17 in (Malherbe and Vayatis (2016)) that
\[ \mathbb{P}\left( \max_{i=1}^{n} f(X_i) \geq \max_{x \in \mathcal{X}} f(x) - \epsilon \right) = \mathbb{P}\left( \frac{\mu(\{x \in \mathcal{X} : f(x) \geq \max_{i=1}^{n} f(X_i)\})}{\mu(\mathcal{X})} \leq \frac{\mu(\mathcal{X}_i)}{\mu(\mathcal{X})} \right) \]
\[ \leq \mathbb{P}\left( \prod_{i=1}^{n} U_i \leq \frac{\mu(\mathcal{X}_i)}{\mu(\mathcal{X})} \right) \]
\[ \leq \mathbb{P}\left( \prod_{i=1}^{n} U_i \leq \delta \cdot e^{-n - \sqrt{2n \ln(1/\delta)}} \right) \]
\[ = \mathbb{P}\left( \sum_{i=1}^{n} - \ln(U_i) > n + \sqrt{2n \ln(1/\delta)} + \ln(1/\delta) \right) \]
\[ \leq \delta \]
where \( U_1, \ldots, U_n \) denotes a sequence of \( n \) i.i.d. copies of \( U \sim \mathcal{U}([0, 1]) \). We point out that a concentration results for gamma random variable was used on the last line (see Lemma 37 and Lemma 38 in (Malherbe and Vayatis (2016)) for more details). □

Appendix D. Analysis of AdaLipOpt (proofs of Section 4)

Proof of Proposition 19. Pick any \( t \geq 2 \), consider any non-constant \( f \in \bigcup_{k \geq 0} \text{Lip}(k) \) and set \( i^\ast = \min\{i \in \mathbb{Z} : f \in \text{Lip}(k_i)\} \). To prove the result, we decorrelate the sample and use the fact that \((X_1, X_{[t/2]+1}), \ldots, (X_{[t/2]}, X_{2[t/2]})\) forms a sequence of \([t/2]\) i.i.d. copies of \((X, X') \sim \mathcal{U}(\mathcal{X} \times \mathcal{X})\):
\[ \mathbb{P}\left( f \in \text{Lip}(k_{i^\ast}) \right) = \mathbb{P}\left( k_{i^\ast} = k_i \right) \]
\[ = \mathbb{P}\left( \bigcup_{i \neq j} \{ |f(X_i) - f(X_j)| > k_{i^\ast - 1} : \|X_i - X_j\|_2 \} \right) \]
\[ \geq \mathbb{P}\left( \bigcup_{i=1}^{[t/2]} \{ |f(X_i) - f(X_{[t/2]+i})| > k_{i^\ast - 1} : \|X_i - X_{[t/2]+i}\|_2 \} \right) \]
\[ = 1 - \mathbb{P}\left( \frac{|f(X_1) - f(X_2)|}{\|X_1 - X_2\|_2} \leq k_{i^\ast - 1} \right)^{[t/2]} \]
\[ = 1 - (1 - \Gamma(f, k_{i^\ast - 1}))^{[t/2]} . \]

It remains to show that \( \Gamma(f, k_{i^\ast - 1}) > 0 \). Observe first that since \( f \in \text{Lip}(k_{i^\ast}) \), then the function \( F : (x, x') \mapsto |f(x) - f(x')| - k_{i^\ast - 1} : \|x - x'\|_2 \) is also continuous. However, as \( f \notin \text{Lip}(k_{i^\ast - 1}) \), we know that there exists some \((x_1, x_2) \in \mathcal{X} \times \mathcal{X}\) such that \( F(x_1, x_2) > 0 \). Hence, it follows from the continuity of \( F \) that there necessarily exists some \( \epsilon > 0 \) such that \( \forall (x, x') \in B(x_1, \epsilon) \cap \mathcal{X} \times B(x_2, \epsilon) \cap \mathcal{X}, F(x, x') > 0 \) which proves the proof. □

Proof of Proposition 23. Combining the consistency equivalence of Proposition 3 with the upper bound on the covering rate obtained in Example 1 gives the result. □
Proof of Proposition 24. Fix any $\delta \in (0,1)$, set $N_1 = 2 + \lceil 2 \ln(\delta/3)/\ln(1 - \Gamma(f, k_{i-1})) \rceil$ and $N_2 = \lceil \left(\sqrt{\ln(3/\delta)/2} + 4N_1 p - \sqrt{\ln(3/\delta)/2}/2p\right)^2 \rceil$. Considering any $n > N_2$, we prove the result in three steps.

Step 1. As the constant $N_1$ and $N_2$ were chosen so that Hoeffding’s inequality ensures that with probability at least $(1 - \delta/3)$, we know that after $N_2$ iterations and with probability $1 - \delta/3$ we have collected at least $N_1$ evaluation points randomly and uniformly distributed over $\mathcal{X}$ due to the exploration step.

Step 2. Using Proposition 19 and the first $N_1$ evaluation points which have been sampled independently and uniformly over $\mathcal{X}$, we know that after $N_2$ iterations and on the event $\{\sum_{i=1}^{N_2} B_i \geq N_1\}$ the Lipschitz constant $k_{i*}$ has been estimated with probability at least $1 - \delta/3$, i.e., $\mathbb{P}\left(\forall t \geq N_2 + 1, \hat{k}_t = k_{i*} \mid \sum_{i=1}^{N_2} B_i \geq N_1\right) \geq 1 - \delta/3$.

Step 3. Finally, as the Lipschitz constant estimate $\hat{k}_t$ satisfies $f \in \text{Lip}(\hat{k}_t)$ for all $t \geq N_2 + 1$ on the above event, one can easily show by reproducing the same steps as in Proposition 12 that conditioned upon the event $\{\forall t \geq N_2 + 1, \hat{k}_t = k_{i*}\} \cap \{\sum_{i=1}^{N_2} B_i \geq N_1\}$ the algorithm is always faster or equal to a Pure Random Search ran with $n - N_2$ i.i.d. copies of $X' \sim \mathcal{U}(\mathcal{X})$. Therefore, using (i) the bound of Proposition 30, (ii) the elementary inequalities $[x] \leq x + 1$, $[x] \geq x - 1$, $\sqrt{x + y} - \sqrt{x} \leq \sqrt{y}$ and (iv) the definition of $N_2 < n$, we obtain that with probability at least $(1 - \delta/3)^3 \geq 1 - \delta$,

$$
\max_{x \in \mathcal{X}} f(x) - \max_{i=1\ldots n} f(X_i) \leq k_{i*} \cdot \text{diam}(\mathcal{X}) \cdot \left(\frac{\ln(3/\delta)}{n - N_2}\right)^{1/3} \\
= k_{i*} \cdot \text{diam}(\mathcal{X}) \cdot \left(\frac{n}{n - N_2}\right)^{1/3} \cdot \left(\frac{\ln(3/\delta)}{n}\right)^{1/3} \\
\leq k_{i*} \cdot \text{diam}(\mathcal{X}) \cdot (1 + N_2)^{1/3} \left(\frac{\ln(3/\delta)}{n}\right)^{1/3} \\
\leq k_{i*} \cdot \text{diam}(\mathcal{X}) \cdot \left(\frac{5}{p} + \frac{2\ln(\delta/3)}{p\ln(1 - \Gamma(f, k_{i-1}))}\right)^{1/3} \cdot \left(\frac{\ln(3/\delta)}{n}\right)^{1/3}.
$$

The result is extended to the case where $n \leq N_2$ by noticing that the bound is superior to $k_{i*} \cdot \text{diam}(\mathcal{X})$ in that case, and thus trivial. \(\square\)

Proof of Proposition 25. Fix $\delta \in (0,1)$, set $N_1 = 2 + \lceil 2 \ln(4/\delta)/\ln(1 - \Gamma) \rceil$ and $N_2 = \lceil \left(\sqrt{\ln(4/\delta)/2} + 4N_1 p - \sqrt{\ln(4/\delta)/2}/2p\right)^2 \rceil$ and let $N_3 = N_2 + \lceil 2 \ln(4/\delta)/(1 - p)^2 \rceil$. Picking any $n > N_3$, we proceed similarly as in the proof of Proposition 24 in four steps:

Steps 1 & 2. As in the above prove, by definition of $N_1$ and $N_2$ and due to Hoeffding’s inequality and Proposition 19, we know that the following event: $\{\forall t \geq N_2 + 1, \hat{k}_t = k_{i*}\} \cap \{\sum_{i=1}^{N_2} B_i \geq N_1\}$ holds true with probability at least $(1 - \delta/4)^2$.

Step 3. Again, using Hoeffding’s inequality and the definition of $N_2$ and $N_3$, we know that after the iteration $N_2 + 1$ we have collected with probability at least $1 - \delta/4$ at least
\[(1 - p)(n - N_3)/2\] exploitative evaluation points:

\[
\sum_{i=1}^{n} I\{B_i = 0\} \geq (1 - p)(n - N_2) - \sqrt{\frac{(n - N_2) \ln(4/\delta)}{2}} \geq \frac{1 - p}{2} \cdot (n - N_3).
\]

**Step 4.** Reproducing the same steps as in the proof of the fast rate of Theorem 15 with the \((1 - p) \cdot (n - N_3)/2\) previous exploitative points and putting the previous results altogether gives that with probability at least \((1 - \delta/4)^4 \geq 1 - \delta\),

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \leq k_{i^*} \times \text{diam}(\mathcal{X}) \times
\]

\[
\begin{cases}
\exp \left\{ -C_{k,\kappa} \cdot \frac{(1 - p)(n - N_3) \ln(2)}{2 \ln(4n/\delta) + 4(2\sqrt{d})^d} \right\}, & \kappa = 1 \\
\frac{2^\kappa}{2} \left( 1 + C_{k,\kappa} \cdot \frac{(1 - p)(n - N_3)(2^d(d-1) - 1)}{2 \ln(4n/\delta) + 4(2\sqrt{d})^d} \right)^{-\frac{\kappa}{d(k-1)}} , & \kappa > 1.
\end{cases}
\]

We now take out the term \(N_3\). Since \(C_{k_{i^*},\kappa}(1 - p) \leq 1\), when \(\kappa = 1\), we have

\[
\max_{x \in \mathcal{X}} f(x) - \max_{i=1 \ldots n} f(X_i) \leq k_{i^*} \cdot \text{diam}(\mathcal{X}) \cdot \exp(5N_3/2) \exp \left\{ -C_{k,\kappa} \cdot \frac{(1 - p)n \ln(2)}{2 \ln(4n/\delta) + 4(2\sqrt{d})^d} \right\}.
\]

For \(\kappa > 1\), setting \(C = C_{k_{i^*},\kappa}(1 - p)/(2 \ln(4n/\delta) + 4(2\sqrt{d})^d)\) and using the decomposition \(n = (n - N_3) + N_3\), we bound the ratio:

\[
\left( \frac{1 + Cn(2^d(d-1) - 1)}{1 + C(n - N_3)(2^d(d-1) - 1)} \right)^{\frac{\kappa}{d(k-1)}} \leq \left( 1 + \frac{Cn_3(2^d(d-1) - 1)}{1 + C(2^d(d-1) - 1)} \right)^{\frac{\kappa}{d(k-1)}}.
\]

In the case where \(\kappa/d(k-1) \leq 1\), one directly obtains

\[
\left( 1 + \frac{Cn_3(2^d(d-1) - 1)}{1 + C(2^d(d-1) - 1)} \right)^{\frac{\kappa}{d(k-1)}} \leq (1 + N_3)^{\frac{\kappa}{d(k-1)}} \leq (1 + N_3) \leq e^{N_3}
\]

Considering the case where \(\kappa/d(k-1) > 1\) and setting \(\kappa = 1 + \epsilon/d\) with \(\epsilon \in (0, 1)\), we obtain from the inequalities (i) \(\kappa \leq 1 + 1/d \leq 2\), (ii) \(\forall \epsilon \in (0, 1), 2^\epsilon - 1 \leq \epsilon\) and (iii) \(C \leq 1/2\) that

\[
\left( 1 + \frac{Cn_3(2^d(d-1) - 1)}{1 + C(2^d(d-1) - 1)} \right)^{\frac{n}{d(k-1)}} \leq (1 + CN_3(2^\epsilon - 1))^{\frac{\kappa}{2}} \leq (1 + CN_3\epsilon)^{\frac{\kappa}{2}} \leq e^{CN_3} \leq e^{N_3}
\]

Finally, using standard bounds on \(N_3\) and noticing that the previous bound is superior to \(k_{i^*} \cdot \text{diam}(\mathcal{X})\) whenever \(n \leq N_3\), the previous result remains valid for any \(n \in \mathbb{N}^*\). \(\square\)
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