RAMIFICATION FILTRATION OF THE GALOIS GROUP OF A LOCAL FIELD VIA DEFORMATIONS

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ABSTRACT. Let $K$ be a field of formal Laurent series with coefficients in a finite field of characteristic $p$, $G_{<p} = \text{Gal}(K_{\text{sep}}/K)$ of period $p$ and nilpotent class $< p$ and $\{G_{<p}^{(v)}\}_{v \geq 0}$ — its filtration by ramification subgroups in the upper numbering. Let $G_{<p} = G(L)$ be the identification of nilpotent Artin-Schreier theory: here $G(L)$ is the group obtained from a suitable profinite Lie $\mathbb{F}_p$-algebra $L$ via the Campbell-Hausdorff composition law. We develop a new technique to characterize the ideals $L^{(v)}$ such that $G(L^{(v)}) = G_{<p}^{(v)}$ and to find their generators. Given $v_0 > 0$ we construct epimorphism of Lie algebras $\tilde{\eta}^\dagger : L \rightarrow \bar{L}^\dagger$ and the action $\Omega_U$ of the formal group $\alpha_p = \text{Spec} \mathbb{F}_p[U], U^p = 0$, of order $p$ on $\bar{L}^\dagger$. Suppose $d\Omega_U = B^I U$, where $B^I \in \text{Diff} \bar{L}^\dagger$, and $\bar{L}^\dagger[v_0]$ is the ideal of $\bar{L}^\dagger$ generated by the elements of $B^I(\bar{L}^\dagger)$. The main result of the paper states that $L^{(v_0)} = (\tilde{\eta}^\dagger)^{-1} \bar{L}^\dagger[v_0]$. In the last sections we relate this result to the explicit construction of generators of $L^{(v_0)}$ obtained earlier by the author and develop its more efficient version.

INTRODUCTION

Let $K$ be a complete discrete valuation field of characteristic $p$ with finite residue field $k \simeq \mathbb{F}_{p^{N_0}}, N_0 \in \mathbb{N}$. Let $K_{<p}$ be a maximal $p$-extension of $K$ with the Galois group $\text{Gal}(K_{<p}/K) := G_{<p}$ of nilpotence class $< p$ and exponent $p$. The group $G_{<p}$ represents sufficiently large non-abelian quotient of the absolute Galois group of $K$ but has definite advantage due to the following fact: any $p$-group $G$ of nilpotence class $s_0 < p$ and exponent $p$ can be presented in the form $G(L)$, where $L$ is a Lie $\mathbb{F}_p$-algebra of nilpotence class $s_0$ and the set $G(L) := L$ is provided with a group structure via the Campbell-Hausdorff composition law, cf. Sect.1.2.

Consider the decreasing filtration by ramification subgroups in the upper numbering $\{G_{<p}^{(v)}\}_{v \geq 0}$ of $G_{<p}$. These subgroups reflect arithmetic structure on $G_{<p}$, cf. motivation in [7]. First results about these ramification subgroups were obtained by the author in [1]. This approach included:

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a) a construction of the identification $\mathcal{G}_{<p} = G(\mathcal{L})$, where $\mathcal{L}$ is explicitly defined Lie $\mathbb{F}_p$-algebra (nilpotent Artin-Shreier theory);  

b) a specification of ideals $\mathcal{L}^{(v)}$ such that $\mathcal{G}^{(v)} = G(\mathcal{L}^{(v)})$: we constructed explicit elements $\mathcal{F}_{a,-N} \in \mathcal{L}_k := \mathcal{L} \otimes k, \alpha > 0$, such that $\mathcal{L}^{(v)}$ appears as the minimal ideal in $\mathcal{L}$ such that for all $\alpha \geq v$, $\mathcal{F}_{a,-N} \in \mathcal{L}_k ^{(v)} := \mathcal{L}^{(v)} \otimes k$ (here $N = N(v)$ is a sufficiently large fixed natural number).

For a generalization of these results, cf. [2, 3], and their application to an analogue of the Grothendieck conjecture, cf. [4, 5]. Recently, we applied them to the study of the structure of an analogue $\Gamma_{<p} = G(L)$ of the group $\mathcal{G}_{<p}$ for complete discrete valuation fields $K$ of mixed characteristic containing non-trivial $p$-th roots of unity, cf. [8, 9]. As a result, we described the corresponding ramification ideals $L^{(v)}$ and related them to the image of the Demushkin relation in $\Gamma_{<p}$. In these papers we developed a new technique (a linearization procedure) which allowed us to work with arithmetic properties of local fields in terms of Lie algebras. The statement of final results in terms of Lie algebras looks quite natural and, it seems impossible to express them in a reasonable way just in terms of involved groups. In some sense this could be considered as an evidence of the existence of a hidden “analytic structure” on the Galois side which is reflected on the level of Lie algebras in our case. However, when obtaining these mixed characteristic results we used quite substantially the characteristic $p$ results from the papers [1], [2] and [3]. It should be pointed out that in [1] the proof of the main result was not done completely in terms of Lie algebras: the verification of the criterion which describes the ramification ideals $\mathcal{L}^{(v)}$ was not linearized and required heavy and highly non-trivial calculations in the enveloping algebra of $\mathcal{L}$. Later in the papers [2] and [3] these calculations were generalized to the case of groups of period $p^M, M > 1$ (but still of nilpotence class $< p$) but it became clear that it would be unlikely to continue in such a way in more complicated situations, say, in the case of higher local fields, cf. e.g. [10].

In this paper we develop a linearization procedure which allows us to obtain the results from [1] exclusively in terms of Lie theory. For a given $v_0 > 0$, we characterize the ramification ideal $\mathcal{L}^{(v_0)}$ in terms of deformations of some auxiliary Lie $\mathbb{F}_p$-algebra $\mathcal{L}^+$ with a suitably chosen module of coefficients. This algebra is provided with an action of a formal group of order $p$ which comes from a derivation of a higher order. The appearance of such derivations is quite a new phenomenon. Note that in [8, 9] we also used the action of formal group of order $p$ but it came from usual derivations.

Let us sketch briefly the main steps of our approach.

We start with a choice of an (sufficiently general) epimorphism $\eta_e: \mathcal{G} \longrightarrow G(\mathcal{L})$ which induces identification $\mathcal{G}_{<p} \simeq G(\mathcal{L})$ given by the nilpotent Artin-Shreier theory. Here $\mathcal{L}$ is a profinite Lie $\mathbb{F}_p$-algebra.
such that its extension of scalars $L_k := L \otimes k$ has a fixed set of profinite generators. The map $\eta_e$ depends on a choice of an element $e \in L_k := L \otimes K$ specified below.

Choose $v_0 \in \mathbb{R}$, $v_0 > 0$. We aim to characterize the ideal $L^{(v_0)} \subset L$ such that $\eta_e(G^{(v_0)}) = L^{(v_0)}$. For this reason we:

a) introduce a decreasing central filtration of $L$ by its ideals $L = L(1) \supset \cdots \supset L(s) \supset \cdots$, and set $\bar{L} = L/L(p)$ with the induced filtration $\{L(s)\}_{s \geq 1}$ (note that $\bar{L}(p) = 0$);

b) introduce a lift $V : \bar{L}^\dagger \rightarrow \bar{L}$ where $\bar{L}^\dagger$ is a Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$ together with its central filtration $L^\dagger(s)$ such that $V(L^\dagger(s)) = L(s)$ and $L^\dagger(p) = 0$;

c) specify a group epimorphism $\eta_{\bar{L}^\dagger} : G \rightarrow G(\bar{L}^\dagger)$ such that

$$
\forall \eta_{\bar{L}^\dagger} : \eta_e := \eta_e \mod G(L(p));
$$

d) introduce the actions $\Omega_\gamma : \bar{L}^\dagger \rightarrow \bar{L}^\dagger$ of the elements $\gamma \in \mathbb{Z}/p$;

e) introduce the ideal $\bar{L}[v_0]$ in $\bar{L}$ as the minimal ideal such that for any $\gamma \in \mathbb{Z}/p$, $\mathcal{V}^{-1}\bar{L}[v_0] \supset \Omega_\gamma(\text{Ker}\mathcal{V})$ (this condition is not easy to study because the action of $\mathbb{Z}/p$ appears in terms of complicated Campbell-Hausdorff group law);

f) establish that the actions $\Omega_\gamma$ can be defined in terms of some coaction $\Omega_U : \bar{L}^\dagger \rightarrow \bar{L}^\dagger \otimes \mathbb{F}_p[U]$ of the formal group scheme $\alpha_p = \mathbb{F}_p[U]$, $U^p = 0$, with coaddition $\Delta U = U \otimes 1 + 1 \otimes U$;

g) if $d\Omega_U = B^\dagger U$ is the differential of $\Omega_U$ (here $B^\dagger \in \text{Diff} \bar{L}^\dagger$) then $\bar{L}[v_0]$ appears as the minimal ideal in $\bar{L}$ containing $\mathcal{V}B^\dagger(\bar{L}^\dagger)$;

h) verify that $L^{(v_0)} = \bar{p_r}^{-1}\bar{L}[v_0]$, where $\bar{p_r}$ is the natural projection from $L$ to $\bar{L}$.

The above characterization of $L^{(v_0)}$ can be used for a considerable simplification of the process of recovering explicit generators in [1]. These generators appeared in [1] as “linear” components of some elements from $L^{(v_0)}$. Our characterization allows us to skip the verification that these linear components generate the ideal $L^{(v_0)}$.

The methods of this paper admit a generalization to the Galois groups of period $p^M$ as well as to the case of higher dimensional local fields in the characteristic $p$ case. In particular, the “$p^M$-version” [3] of [1] required much more complicated study of “non-linear” components which can be now avoided due to our approach (the paper in preparation). This also will provide us with much better background for the papers [8, 9] and their upcoming “$p^M$-versions” including the case of higher dimensional local fields.

**Notation.** Suppose $s \in \mathbb{N}$. For any topological group $G$, we denote by $C_s(G)$ the closure of the subgroup of $G$ generated by the commutators of order $\geq s$. If $L$ is a topological Lie algebra then $C_s(L)$ is the
closure of the ideal generated by commutators of degree \(\geq s\). For any topological \(A\)-modules \(M\) and \(B\) we use the notation \(M_B := M \hat{\otimes}_A B\).

1. Preliminaries

Suppose \(K\) is a field of characteristic \(p\), \(K_{\text{sep}}\) is a separable closure of \(K\) and \(G = \text{Gal}(K_{\text{sep}}/K)\). We assume that \(G\) acts on \(K_{\text{sep}}\) as follows: if \(g_1, g_2 \in G\) and \(a \in K_{\text{sep}}\) then \(g_1(g_2a) = (g_1g_2)a\). Denote by \(\sigma\) the morphism of taking \(p\)-th power in \(K\).

In [1, 2] we developed a nilpotent analogue of the classical Artin-Schreier theory of cyclic field extensions of characteristic \(p\). We are going to use the covariant analog of this theory, cf. the discussion in [7], for explicit description of the group \(\mathcal{G}_{<p} = G/\mathcal{G}^pC_p(G)\) as follows.

1.1. Lie algebra \(\mathcal{L}\). Suppose \(K = k((t))\) where \(t\) is a fixed uniformizer and \(k \simeq \mathbb{F}_pN_0\) with \(N_0 \in \mathbb{N}\). Fix \(\alpha_0 \in k\) such that \(\text{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1\).

Let \(\mathbb{Z}^+(p) = \{a \in \mathbb{N} \mid \gcd(a, p) = 1\}\) and \(\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}\).

Let \(\tilde{\mathcal{L}}\) be a profinite free Lie \(\mathbb{F}_p\)-algebra with the (topological) module of generators \(\mathcal{K}^*/\mathcal{K}^p\) and \(\mathcal{L} = \tilde{\mathcal{L}}/C_p(\tilde{\mathcal{L}})\). We can obtain the set
\[
\{D_0\} \cup \{D_n \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}
\]
of topological generators of \(\mathcal{L}_k\) via the following identifications:
\[
(\mathcal{K}^*/\mathcal{K}^p)\hat{\otimes}\mathbb{F}_p k = \text{Hom}_{\mathbb{F}_p}(\mathcal{K}/(\sigma - \text{id})\mathcal{K}, k) = \prod_{a \in \mathbb{Z}^+(p)} \text{Hom}_{\mathbb{F}_p}(kt^{-a}, k) \times kD_0
\]
and \(\text{Hom}_{\mathbb{F}_p}(kt^{-a}, k) = \prod_{n \in \mathbb{Z}/N_0} kD_{an}\), where for any \(\alpha \in k\) and \(a, b \in \mathbb{Z}^+(p)\), \(D_{an}(t^{-a}t^{-b}) = \delta_{ab} \sigma^n(\alpha)\). Note also that the first identification uses the Witt pairing [11, 6] and \(D_0\) comes from \(t \otimes 1 \in (\mathcal{K}^*/\mathcal{K}^p)\hat{\otimes}\mathbb{F}_p k\).

For any \(n \in \mathbb{Z}/N_0\), set \(D_{an} = t \otimes (\sigma^n\alpha_0) = (\sigma^n\alpha_0)D_0\).

1.2. Groups and Lie algebras of nilpotent class \(< p\). The basic ingredient of the nilpotent Artin-Schreier theory is the equivalence of the category of \(p\)-groups of nilpotent class \(s_0 < p\) and the category of Lie \(\mathbb{Z}_p\)-algebras of the same nilpotent class \(s_0\), [14, 13]. In the case of objects killed by \(p\), this equivalence can be explained as follows.

Let \(L\) be a Lie \(\mathbb{F}_p\)-algebra of nilpotent class \(< p\), i.e. \(C_p(L) = 0\).

Let \(A\) be an enveloping algebra of \(L\). Then there is a natural embedding \(L \subset A\), the elements of \(L\) generate the augmentation ideal \(J\) of \(A\) and we have a morphism of algebras \(\Delta : A \rightarrow A \otimes A\) uniquely determined by the condition \(\Delta(l) = l \otimes 1 + 1 \otimes l\) for all \(l \in L\).

Applying the Poincare-Birkhoff-Witt Theorem as in [1] Sect. 1.3.3, we obtain that:
- \(L \cap J^p = 0\);
- \(L \mod J^p = \{a \mod J^p \mid \Delta(a) \equiv a \otimes 1 + 1 \otimes a \mod (J \otimes 1 + 1 \otimes J)^p\}\);
the set \( \exp(L) \bmod J^p \) is identified with the set of all ”diagonal elements modulo degree \( p \), i.e. with the set of \( a \in 1 + J \bmod J^p \) such that \( \Delta(a) \equiv a \otimes a \bmod (J \otimes 1 + 1 \otimes J)^p \). (Here \( \exp(x) = \sum_{0 \leq i < p} x^i/i! \) is the truncated exponential.)

In particular, there is a natural embedding \( L \subset A/J^p \) and in terms of this embedding the Campbell-Hausdorff formula appears as

\[
(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \ldots, \quad l_1, l_2 \in L,
\]

where \( \exp(l_1)\exp(l_2) \equiv \exp(l_1 \circ l_2) \bmod J^p \). This composition law provides the set \( L \) with a group structure and we denote this group by \( G(L) \). Note that a subset \( I \subset L \) is an ideal in \( L \) if and only if \( G(I) \) is a normal subgroup in \( G(L) \). Clearly, \( G(L) \) has exponent \( p \) and nilpotent class \( s < p \). Then the correspondence \( L \mapsto G(L) \) is the above mentioned equivalence of the categories of \( p \)-groups of exponent \( p \) and nilpotent class \( s < p \) and Lie \( \mathbb{F}_p \)-algebras of the same nilpotent class \( s \). This equivalence can be naturally extended to the categories of pro-finite Lie algebras and pro-finite \( p \)-groups.

1.3. Epimorphism \( \eta_e : \mathcal{G} \to G(\mathcal{L}) \). Let \( L \) be a finite Lie \( \mathbb{F}_p \)-algebra of nilpotent class \( < p \) and set \( L_{\text{sep}} := L_{K_{\text{sep}}} \). Then the elements of \( \mathcal{G} = \text{Gal}(K_{\text{sep}}/K) \) and \( \sigma \) act on \( L_{\text{sep}} \) through the second factor, \( L_{\text{sep}}|_{\sigma = \text{id}} = L \) and \( (L_{\text{sep}})^\sigma = L_K \). The covariant nilpotent Artin-Schreier theory states that for any \( e \in G(L_K) \), the set

\[
\mathcal{F}(e) = \{ f \in G(L_{\text{sep}}) \mid \sigma(f) = e \circ f \}
\]

is not empty and for any fixed \( f \in \mathcal{F}(e) \), the map \( \tau \mapsto (-f) \circ \tau(f) \) is a continuous group homomorphism \( \pi_f(e) : \mathcal{G} \to G(L) \). The correspondence \( e \mapsto \pi_f(e) \) has the following properties:

a) if \( f' \in \mathcal{F}(e) \) then \( f' = f \circ l \), where \( l \in G(L) \), and \( \pi_f(e) \) and \( \pi_{f'}(e) \) are conjugated via \( l \);

b) for any continuous group homomorphism \( \pi : \mathcal{G} \to G(L) \), there are \( e \in G(L_K) \) and \( f \in \mathcal{F}(e) \) such that \( \pi_f(e) = \pi \);

c) for appropriate elements \( e, e' \in G(L_K) \) and \( f, f' \in G(L_{\text{sep}}) \), we have \( \pi_f(e) = \pi_{f'}(e') \) iff there is an \( x \in G(L_K) \) such that \( f' = x \circ f \) and, therefore, \( e' = \sigma(x) \circ e \circ (-x) \).

In [1, 2, 3] we applied this theory to the Lie algebra \( \mathcal{L} \) from Sect.1.1 via a special choice of \( e \in L_K \). Now we just assume that

\[
e \equiv \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0} \bmod C_2(L_K).
\]

Under this assumption the map \( \pi_f(e) \bmod G^p C_2(\mathcal{G}) \) induces a group isomorphism of \( \mathcal{G}^{ab} \otimes \mathbb{F}_p \) and \( G(\mathcal{L})/C_2(G(\mathcal{L})) = L^{ab} = K^*/K^{*p} \), which
coincides with the inverse to the reciprocity map of local class field theory, cf. [6]. This also implies that \( \pi_f(e) \) (when taken modulo \( G^pC_p(G) \)) induces a group isomorphism \( G_{<p} \cong G(L) \). We agree to fix a choice of \( f \in F(e) \) and use the notation \( \eta_e = \pi_f(e) \). So, at this stage, \( \eta_e \) is just an arbitrary lift of the canonical isomorphism of local class field theory.

1.4. Auxiliary fields \( K'_\gamma \). Our approach to the ramification filtration in \( G_{<p} \) substantially uses the construction of a totally ramified extension \( K' \) of \( K \) such that \( [K' : K] = q \) and the Herbrand function \( \varphi_{K'/K} \) has only one edge point \((r^*, r^*)\). Here \( q = p^N^* \) with \( N^* \in \mathbb{N} \), and \( r^* = b^*/(q - 1) \), where \( b^* \in \mathbb{Z}^+(p) \). For simplicity, we assume that \( N^* \equiv 0 \mod N_0 \), i.e. \( \sigma_{N^*} \) acts as identity on the residue field \( k \) of \( K \). More substantial restrictions on these parameters will be introduced in Sect.2.1.

For a detailed explanation of the construction of \( K' \) cf. e.g. [3], Sect.1.5. We just recall that if \( r^* = m/n \) with coprime \( m, n \in \mathbb{N} \), then \( K' = K(U^n) \subset K(u(U)) \), where \( u^n = t \) and \( U^q + r^*U = u^{-m} \). We can apply Hensel’s Lemma to choose a uniformizer \( t_1 \) in \( K' \) such that \( t = t_1^q E(t_1^{r^*})^{-1}, \) where \( E(X) = \exp(X + X^p/p + \cdots + X^{p^n}/p^n + \cdots) \in \mathbb{Z}_p[[X]] \) is the Artin-Hasse exponential.

We need the following generalization of the construction of \( K' \).

For \( \gamma \in \mathbb{Z}/p \setminus \{0\} \), let the field \( K'_\gamma = k((t_\gamma)) \) be such that:

a) \([K'_\gamma : K] = q;\]

b) \( \varphi_{K'_\gamma/K}(x) \) has only one edge point \((r^*, r^*)\);

c) \( K'_\gamma = k((t_\gamma)), \) where \( t = t_\gamma^n E(\gamma t_\gamma^{r^*})^{-1}. \)

The fields \( K'_\gamma \) appear in the same way as the field \( K' \). More precisely, \( K'_\gamma = K(U^n_\gamma) \subset K(u(U_\gamma)) \), where \( u^n = t \) and \( U^q_\gamma + \gamma r^*U_\gamma = u^{-m}. \) Note that \( K'_\gamma \) is separable over \( K \).

1.5. The criterion. Suppose \( K'_\gamma \) is the field from Sect.1.4. Consider the field isomorphism \( \iota_\gamma : K \longrightarrow K'_\gamma \) such that \( \iota_\gamma : t \mapsto t_\gamma \) and \( \iota_{\gamma|k} = \text{id}_k \).

Let \( e_\gamma = (\text{id}_E \otimes \iota_\gamma)e. \) Then \( \sigma_{N^*} e_\gamma(t_\gamma) = e(t_\gamma^n). \)

Choose \( f_\gamma \in F(e_\gamma) \) and consider \( \pi_{f_\gamma}(e_\gamma) : \text{Gal}(K_{\text{sep}}/K'_\gamma) \longrightarrow G(L). \)

For \( Y \in L_{\text{sep}} \) and an ideal \( I \) in \( L \), define the field of definition of \( Y \mod I_{\text{sep}} \) over, say, \( K \) as

\[ K(Y \mod I_{\text{sep}}) := K_{\text{sep}}^H, \]

where \( H = \{ g \in G \mid (\text{id}_E \otimes g)Y \equiv Y \mod I_{\text{sep}} \}. \)

For any field extension \( E'/E \) in \( K_{\text{sep}} \), define the biggest ramification number

\[ v(E'/E) = \max \{ v \mid \text{Gal}(K_{\text{sep}}/E)^{(v)} \text{ acts non-trivially on } E' \}. \]

The methods from [1, 2, 3] are based on the following criterion.
Suppose $v_0 > 0$, $r^* < v_0$ and the auxiliary fields $\mathcal{K}'_\gamma$ correspond to the parameters $r^*$ and $N^*$ (with $q = p^{N^*}$).

**Proposition 1.1.** Suppose $f = X_\gamma \circ \sigma^{N^*}(f_\gamma)$. Then $\mathcal{L}^{(\text{vo})}$ is the minimal ideal in the family of all ideals $\mathcal{I}$ of $\mathcal{L}$ such that

$$v(\mathcal{K}'_\gamma(X_\gamma \mod \mathcal{I}_{\text{sep}})/\mathcal{K}'_\gamma) < qv_0 - b^*.$$ 

The proof goes along the lines of the proof for $\gamma = 1$, cf. e.g. [3], Sect.1.6. It is based just on the following elementary properties of the upper ramification numbers:

— if $v = v(\mathcal{K}(f \mod \mathcal{I}_{\text{sep}})/\mathcal{K})$ then:

- $v(\mathcal{K}'_\gamma(f_\gamma \mod \mathcal{I}_{\text{sep}})/\mathcal{K}'_\gamma) = v$;
- $v(\mathcal{K}'_\gamma(f_\gamma \mod \mathcal{I}_{\text{sep}})/\mathcal{K}) = \varphi_{\mathcal{K}'_\gamma/\mathcal{K}}(v)$;
- if $v > r^*$ then $\varphi_{\mathcal{K}'_\gamma/\mathcal{K}}(v) = r^* + (v - r^*)/q < v$.

Note that $f = X_\gamma \circ \sigma^{N^*} f_\gamma$ implies that $e(t) = \sigma X_\gamma \circ \sigma^{N^*} e_\gamma \circ (-X_\gamma)$.

Inversely, suppose $X \in \mathcal{L}_{\text{sep}}$ and

$$e(t) = \sigma X \circ \sigma^{N^*} e_\gamma \circ (-X).$$

Then $l = (-\sigma^{N^*} f_\gamma) \circ (-X) \circ f \in \mathcal{L}_{\text{sep}}|_{\sigma = \text{id}} = \mathcal{L}$ and replacing $f_\gamma$ by $f_\gamma \circ l \in \mathcal{F}(e_\gamma)$ we obtain $f = X \circ \sigma^{N^*} f_\gamma$. Therefore, in Prop.1.1 we can use identity (1.2) instead of the identity $f = X_\gamma \circ \sigma^{N^*} f_\gamma$.

Note that for any $\gamma$, there is a unique field isomorphism $\iota'_\gamma : \mathcal{K}'_\gamma \longrightarrow \mathcal{K}$ such that $\iota'_\gamma(t_\gamma) = t$ and $\iota'_\gamma|_k = \text{id}$. Therefore, if we set $e^{(q)} := e(t^q)$ and $\gamma * e^{(q)} := e(t^q E(\gamma t^b)^{-1})$ then Prop.1.1 can be stated in the following equivalent form.

**Proposition 1.2.** If $X_\gamma \in \mathcal{L}_{\text{sep}}$ is such that

$$\gamma * e^{(q)} = \sigma X_\gamma \circ e^{(q)} \circ (-X_\gamma)$$

then $\mathcal{L}^{(\text{vo})}$ is the minimal ideal in the set of all ideals $\mathcal{I}$ of $\mathcal{L}$ such that

$$v(\mathcal{K}(X_\gamma \mod \mathcal{I}_{\text{sep}})/\mathcal{K}) < qv_0 - b^*. $$

Suppose $\tilde{\mathcal{F}} \subset \mathcal{L}$ is a closed ideal and $\pi : \mathcal{L} \longrightarrow L := \mathcal{L}/\tilde{\mathcal{F}}$ is a natural projection. Then we can use $e_L = \pi_K(e) \in L_K$, $f_L := \pi_{\text{sep}}(f) \in L_{\text{sep}}$, $\eta_{e L} = \pi_{\eta e} : G \longrightarrow G(L)$ and $X_{\gamma L} := \pi_{\text{sep}}(X_\gamma)$ to state the following analog of Prop.1.2.

**Proposition 1.3.** $L^{(\text{vo})} := \eta_{e L}(G^{(\text{vo})})$ is the minimal ideal in the set of all ideals $\mathcal{I}$ of $L$ such that $v(\mathcal{K}(X_{\gamma L} \mod \mathcal{I}_{\text{sep}})/\mathcal{K}) < qv_0 - b^*$. 

1.6. Lie algebra $\tilde{\mathcal{L}}$ and epimorphism $\eta_{\ell}$. Introduce a weight function $\operatorname{wt} : \mathcal{L}_k \to \mathbb{N}$ on $\mathcal{L}_k$ by setting on its generators $\operatorname{wt}(D_{an}) = s$ if $(s - 1)v_0 \leq a < sv_0$. We obtain a decreasing central filtration by the ideals $\mathcal{L}(s) = \{ t \in \mathcal{L} \mid \operatorname{wt}(t) \geq s \}$ of $\mathcal{L}$ such that $\mathcal{L}(1) = \mathcal{L}$. This weight function gives us also a decreasing filtration of ideals $\mathcal{J}(s)$ in the enveloping algebra $\mathcal{A}$ such that $\mathcal{J}(1) = \mathcal{J}$ and for any $s$, $(\mathcal{J}(s) + \mathcal{J}^p) \cap \mathcal{L} = \mathcal{L}(s)$ (use the Poincare-Birkhoff-Witt theorem).

Consider a $k$-submodule $\mathcal{N}$ in $\mathcal{L}_k$ generated by all $t^{-b}l$, where for some $s \geq 1$, $t \in \mathcal{L}(s)_k$ and $b < sv_0$. Then $\mathcal{N}$ has a natural structure of Lie algebra over $k$. For any $i \geq 0$, let $\mathcal{N}(i)$ be the $k$-submodule in $\mathcal{L}_k$ generated by all $t^{-b}l$ where $t \in \mathcal{L}(s)$ and $b < (s - i)v_0$. Then $\mathcal{N}(i)$ is ideal in $\mathcal{N}$.

Let $\bar{\pi} : \mathcal{L} \to \tilde{\mathcal{L}} := \mathcal{L}/\mathcal{L}(p)$ be a natural projection. Then $\tilde{\mathcal{L}}(s) = \bar{\pi}(\mathcal{L}(s))$ is a decreasing central filtration in $\tilde{\mathcal{L}}$ such that $\tilde{\mathcal{L}}(p) = 0$. Let $\bar{\mathcal{N}} \subset \tilde{\mathcal{L}}_k$ be an analog of $\mathcal{N}$ (where the algebra $\tilde{\mathcal{L}}$ is used instead of $\mathcal{L}$).

For $i \geq 0$, let $\bar{\mathcal{N}}(i)$ be the appropriate ideals in $\bar{\mathcal{N}}$. Note that $\bar{\mathcal{N}}(p - 1) \subset \tilde{\mathcal{L}}_m$, where $m = tk[\lbrack [t] \rbrack]$ (use that $\tilde{\mathcal{L}}(p) = 0$), and introduce the Lie algebra $\bar{\mathcal{N}} = \bar{\mathcal{N}}/\bar{\mathcal{N}}(p - 1)$.

From now on we assume (in addition to (1.2)) that:

$$e \in \mathcal{N}$$

(now $\eta_{\ell}$ is not an arbitrary lift of the reciprocity map of class field theory but it is still quite general).

Let $\bar{e} := \bar{\pi}_K e \in \bar{\mathcal{N}}$ and $\bar{f} := (\bar{\pi}_\text{sep}) f \in \tilde{\mathcal{L}}_{\text{sep}}$. If $\eta_{\ell} := \bar{\pi} \cdot \eta_{\ell}$ then for any $\tau \in \mathcal{G}$, $\eta_{\ell}(\tau) = (-\bar{f}) \circ \tau \bar{f}$. Verify that $\eta_{\ell}$ depends only on $\bar{e} := \bar{e} \bmod \bar{\mathcal{N}}(p - 1) \in \bar{\mathcal{N}}$.

**Proposition 1.4.** Let $\bar{\epsilon}' \in \tilde{\mathcal{L}}_k$ and $\bar{\epsilon}' \equiv \bar{e} \bmod \bar{\mathcal{N}}(p - 1)$. Then there is a unique $\bar{f}' \in \tilde{\mathcal{L}}_{\text{sep}}$ such that $\sigma \bar{f}' = \bar{\epsilon}' \circ \bar{f}'$ and $\bar{f}' \circ (-\bar{f}) \in \bar{\mathcal{N}}(p - 1)$.

**Proof.** We can use that $\sigma$ is topologically nilpotent on $\bar{\mathcal{N}}(p - 1) \subset \tilde{\mathcal{L}}_m$ to obtain a unique $\bar{x} \in \bar{\mathcal{N}}(p - 1)$ such that $\bar{\epsilon}' = (\sigma \bar{x}) \circ \bar{e} \circ (-\bar{x})$:

— indeed, apply induction on $s \geq 1$ modulo the ideals $\tilde{\mathcal{L}}(s)_K$ as follows:

if $s = 1$ there is nothing to prove;

if $s \geq 1$ and $\bar{x}_s \in \bar{\mathcal{N}}(p - 1)$ is such that $\bar{\epsilon}' = (\sigma \bar{x}_s) \circ \bar{e} \circ (-\bar{x}) + A_s$ with $A_s \in \tilde{\mathcal{L}}(s)_K$, then $\bar{x}_s \in \bar{\mathcal{N}}(p - 1) \cap \tilde{\mathcal{L}}(s)_K$. If $\delta = -\sum_{m \geq 0} \sigma^m(A_s)$ then $\bar{x}_{s + 1} := \bar{x}_s + \delta \in \bar{\mathcal{N}}(p - 1) \cap \tilde{\mathcal{L}}(s)_K$, $\sigma \delta - \delta = A_s$ and

\[ \bar{\epsilon}' \equiv (\sigma \bar{x}_{s + 1}) \circ \bar{e} \circ (-\bar{x}_{s + 1}) \bmod \tilde{\mathcal{L}}(s + 1)_K. \]

Clearly, $\bar{f}' = \bar{x} \circ \bar{f} \in \tilde{\mathcal{L}}_{\text{sep}}$ with $\bar{x} := \bar{x}_p$ satisfies the requirements of proposition. If $\bar{f}'' \in \tilde{\mathcal{L}}_{\text{sep}}$ also has such properties then $\bar{f}'' \circ (-\bar{f}') \in \bar{\mathcal{N}}(p - 1) \cap \tilde{\mathcal{L}} = 0$ and $\bar{f}'' = \bar{f}'$. $\square$

2. Lie algebra $\tilde{\mathcal{L}}^\dagger$ and ideal $\tilde{\mathcal{L}}[v_0] \subset \tilde{\mathcal{L}}$

In this section we introduce the Lie $\mathbb{F}_p[\ell]$-algebra $\tilde{\mathcal{L}}^\dagger$ together with the epimorphism of Lie algebras $\mathcal{V} : \tilde{\mathcal{L}}^\dagger \to \tilde{\mathcal{L}}$ and its section $(j_0)^{-1} : \tilde{\mathcal{L}} \simeq$
2.1. Parameters $r^*$ and $N^*$. Fix $u^* \in \mathbb{N}$ and $w^* > 0$. (Below we will specify $u^* = (p-1)(p-2)+1$ and $w^* = (p-1)v_0$.)

For $1 \leq s < p$, denote by $\delta_0(s)$ the minimum of positive values of

$$v_0 - \frac{1}{s}(a_1 + a_2/p^{n_2} + \cdots + a_u/p^{n_u}),$$

where $u \leq u^*$, all $n_i \in \mathbb{Z}_{\geq 0}$ and $a_i \in [0, w^*) \cap \mathbb{Z}$. The existence of such $\delta_0(s)$ can be proved easily by induction on $u$ for any fixed $s$.

Set $\delta_0 := \min\{\delta_0(s) \mid 1 \leq s < p\}$.

Let $r^* \in \mathbb{Q}$ be such that $r^* = b_0/(q_0^* - 1)$, where $q_0^* = p^{N_0^*}$ with $N_0^* \geq 2$, $b_0 \in \mathbb{N}$ and $\gcd(b_0, p(q_0^* - 1)) = 1$. The set of such $r^*$ is dense in $\mathbb{R}_{> 0}$ and we can assume that $r^* \in (v_0 - \delta_0, v_0)$.

For $1 \leq u \leq u^*$, introduce the following subsets in $\mathbb{Q}$:

- $A[u]$ is the set of all

$$a_1 p^{-m_1} + a_2 p^{-m_2} + \cdots + a_u p^{-m_u},$$

where $0 = n_1 \leq \cdots \leq n_u$, all $a_i \in [0, w^*) \cap \mathbb{Z}$. If $M \in \mathbb{Z}_{\geq 0}$ we denote by $A[u, M]$ the subset of $A[u]$ consisted of the elements satisfying the additional restriction $n_u \leq M$. Note that $A[u, M]$ is finite.

- $B[u]$ is the set of all numbers

$$r^*(b_1 p^{-m_1} + b_2 p^{-m_2} + \cdots + b_u p^{-m_u}),$$

where all $0 = m_1 \leq \cdots \leq m_u$, $b_i \in \mathbb{Z}_{\geq 0}$, $b_i \neq 0$ and $b_1 + \cdots + b_u < p$. (In particular, $0 \notin B[u]$.) For $M \in \mathbb{Z}_{\geq 0}$, $B[u, M]$ is the subset of $B[u]$ consisted of the elements satisfying the additional restrictions $m_u \leq M$. The set $B[u, M]$ is also finite.

**Lemma 2.1.** For any $u$, $A[u] \cap B[u] = \emptyset$.

**Proof.** Note that $A[u] \subset \mathbb{Z}[1/p]$. Prove that $B[u] \cap \mathbb{Z}[1/p] = \emptyset$.

It will be sufficient to verify that for any $n_1, \ldots, n_u, b_1, \ldots, b_u \in \mathbb{Z}_{\geq 0}$ such that $0 < b_1 + \cdots + b_u < p$, we have

$$b_1 p^{n_1} + \cdots + b_u p^{n_u} \not\equiv 0 \mod (q_0^* - 1).$$

Since $q_0^* \equiv 1 \mod (q_0^* - 1)$ we can assume that all $n_i < N_0^*$. But then $0 < b_1 p^{n_1} + \cdots + b_u p^{n_u} \leq (p-1)p^{N_0^* - 1} < q_0^* - 1$. The lemma is proved.

For $\alpha, \beta \in \mathbb{Q}$, set $\rho(\alpha, \beta) = |\alpha - \beta|$. 

Lemma 2.2. If $\alpha \notin B[u]$ then
\[\rho(\alpha, B[u]) := \inf \{ \rho(\alpha, \beta) \mid \beta \in B[u] \} \neq 0.\]

Proof. Use induction on $u$.
If $u = 1$ there is nothing to prove because $B[1]$ is finite.
Suppose $u \geq 1$ and $\rho(\alpha, B[u]) > 0$.
Choose $M_u \in \mathbb{Z}_{\geq 0}$ such that $r^*(p - 1)/p^{M_u + 1} < \rho(\alpha, B[u])/2$.
If $\beta \in B[u + 1] \setminus B[u + 1, M_u]$ then there is $\beta' \in B[u]$ such that $\rho(\beta, \beta') < \rho(\alpha, B[u])/2$. Then
\[\rho(\alpha, \beta) \geq \rho(\alpha, \beta') - \rho(\beta', \beta) \geq \rho(\alpha, B[u]) - \rho(\alpha, B[u])/2 = \rho(\alpha, B[u])/2,\]
and we obtain
\[\rho(\alpha, B[u + 1]) \geq \min \{ \rho(\alpha, B[u + 1, M_u]), \rho(\alpha, B[u])/2 \} > 0.\]
The lemma is proved. \qed

Lemma 2.3. If $\beta \notin A[u]$ then $\rho(\beta, A[u]) \neq 0$.

Proof. The proof is similar to the proof of above Lemma 2.2. \qed

Lemma 2.4. For all $u_1, u_2 \leq u^*$, $\rho(A[u_1], B[u_2]) > 0$.

Proof. If $u_1 = 1$ this follows from Lemma 2.2 because $A[1]$ is finite.
Suppose $u_1 \geq 1$ and $\rho(A[u_1], B[u_2]) = \delta > 0$.
Choose $M_1 \in \mathbb{Z}_{\geq 0}$ such that $w^*/p^{M_1} < \delta/2$.
If $\alpha \in A[u_1 + 1] \setminus A[u_1 + 1, M_1]$ then there is $\alpha' \in A[u_1]$ such that $\rho(\alpha, \alpha') < \delta/2$. Then for any $\beta \in B[u_2]$, we have
\[\rho(\alpha, \beta) \geq \rho(\alpha', \beta) - \rho(\alpha, \alpha') > \delta/2.\]
Therefore, for any $\alpha \in A[u_1 + 1]$,
\[\rho(\alpha, B[u_2]) \geq \min \{ \rho(A[u_1 + 1, M_1], B[u_2]), \delta/2 \} > 0.\]
Lemma is proved. \qed

Fix the values $u^* = (p - 1)(p - 2) + 1$ and $w^* = (p - 1)v_0$ (since $u^* > p - 1$, $B[u^*] = B[p - 1]$).

Choose $N^* \in \mathbb{N}$ satifying the following conditions:

C1) $N^* \equiv 0 \mod N_0^*$;

C2) $p^{N^*} \rho(A[u^*], B[u^*]) \geq 2r^*(p - 1)$;

C3) $r^*(1 - p^{-N^*}) \in (v_0 - \delta_0, v_0)$.

Introduce $q = p^{N^*}$ and $b^* = b_0^*(q - 1)/(q_0 - 1) \in \mathbb{N}$.
Note that $r^* = b^*/(q - 1)$ and $b^* \in \mathbb{Z}^+(p)$.

Proposition 2.5. If $\alpha \in A[u^*]$ and $\beta \in B[u^*]$ then
\[q | q\alpha - (q - 1)\beta | > b^*(p - 1).\]
Proof. Indeed, the left-hand side of our inequality equals
\[ q |qa - (q - 1)\beta| = q^2 |\alpha - \beta + \beta/q| \geq q^2 |\alpha - \beta| - \beta q \geq q^2 q(A[u^\ast], B[u^\ast]) \]
\[ -r^*(p - 1)q \geq 2r^*(p - 1)q - r^*(p - 1)q = r^*(p - 1)q > b^*(p - 1). \]
\[ \square \]

2.2. The set \( \mathfrak{A}^0 \). Use the above parameters \( r^*, N^*, q = p^N^* \).

Definition. \( \mathfrak{A}^0 \) is the set of all \( \iota = p^m(q\alpha - (q - 1)\beta) \), where \( m \in \mathbb{Z}_{\geq 0}, \alpha \in A[u^\ast, m], \beta \in B[u^\ast, m] \cup \{0\} \) and \( |\iota| \leq b^*(p - 1) \). (Note that \( p^m\alpha \in \mathbb{Z}_{\geq 0} \) and \( p^m\beta/r^* \in \mathbb{N} \))

Let \( \mathfrak{A}^0_0 := \{ \iota \in \mathfrak{A}^0 \mid \beta = 0 \} \).

Lemma 2.6. Suppose \( \iota = p^m(q\alpha - (q - 1)\beta) \in \mathfrak{A}^0 \). Then:

a) \( \mathfrak{A}^0_0 = \{ qa \mid a \in [0, (p - 1)v_0) \} \);  
b) if \( \beta \neq 0 \) then \( m < N^* \) (in particular, \( \mathfrak{A}^0 \) is finite);  
c) the integers \( p^m\alpha \) and \( p^m\beta/r^* \) do not depend on the presentation of \( \iota \) in the form \( p^m(q\alpha - (q - 1)\beta) \) from the definition of \( \mathfrak{A}^0 \).

Proof. a) Here \( 0 \leq p^m\alpha/(p - 1) \leq b^*/q \in (v_0 - \delta_0, v_0) \). By the choice of \( \delta_0 \) in Sect.2.1, the inequalities \( p^m\alpha/(p - 1) < v_0 \) and \( p^m\alpha/(p - 1) \leq v_0 - \delta_0 \) are equivalent.

b) If \( \beta \in B[u^\ast, m] \) and \( m \geq N^* \) then by Prop.2.5, \( |\iota| > b^*(p - 1) \) i.e. \( \iota \notin \mathfrak{A}^0_0 \).

c) If \( \iota = p^{m'}(q\alpha' - (q - 1)\beta') \) is another presentation of \( \iota \) then \( p^{m'}\beta/r^* \) and \( p^{m'}\beta'/r^* \) are non-negative congruent modulo \( q \) integers and the both are smaller than \( q \). Indeed, if \( \beta/r^* = b_1 + b_2 p^{-m_2} + \cdots + b_u p^{-m_u} \), where all \( 0 \leq m_i \leq m \) and \( u \leq u^* \), then
\[ p^{m'}\beta/r^* \leq p^{m'}(b_1 + \cdots + b_u) \leq p^{m'}(p - 1) < p^{m'+1} \leq q \]
because \( m < N^* \). Similarly, \( p^{m'}\beta'/r^* < q \). Therefore, they coincide and this implies also that \( p^m\alpha = p^{m'}\alpha' \). \( \square \)

Corollary 2.7. Suppose that \( \iota = p^m(q\alpha - (q - 1)\beta) \in \mathfrak{A}^0 \). Then the sum of the \( \beta/r^* \) digits \( b_1 + \cdots + b_u \) of the appropriate \( \beta/r^* \) depends only on \( \iota \).

Definition. \( \text{ch}(\iota) := b_1 + \cdots + b_u \).

In the notation from Sect.2.2 suppose \( \iota = p^m(q\alpha - (q - 1)\beta) \in \mathfrak{A}^0 \). By Lemma 2.6 \( p^m\alpha \) depends only on \( \iota \) and can be presented (non-uniquely) in the form \( a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_u p^{n_u} \), where all coefficients \( a_i \in [0, (p - 1)v_0) \cap \mathbb{Z}, 0 \leq n_i \leq m, n_1 = m \) and \( u \leq u^* \).

Definition. \( \kappa(\iota) \) is the maximal natural number such that for any above presentation of \( p^m\alpha, \kappa(\iota) \leq u \).
Remark. a) If $\iota \in \mathfrak{A}^0$ then $\kappa(\iota) \leq u^*$ and $\text{ch}(\iota) \leq p - 1$;
   b) if $\iota \in \mathfrak{A}^0_0$ then $\text{ch}(\iota) = 0$;
   c) if $\iota \in \mathfrak{A}^0_0$ and $\iota \neq 0$ then $\kappa(\iota) = 1$.

2.3. Lie algebras $\mathcal{L}^\dagger$ and $\bar{\mathcal{L}}^\dagger$. Suppose $\iota = p^m(q_\alpha - (q - 1)\beta) \in \mathfrak{A}^0$ is given in the standard notation from Sect.2.2. Let $w^0(\iota)$ be the minimal natural number such that $\iota < w^0(\iota)b^*$.

Definition. The subset $\mathfrak{A}^+(p)$ consists of $\iota \in \mathfrak{A}^0$ such that

- $\iota > 0$;
- $\gcd(p^m\alpha, p^n\beta/r^*, p) = 1$;
- $w^0(\iota) + \text{ch}(\iota) \leq p - 1$;
- $\kappa(\iota) \leq (p - 2)\text{ch}(\iota) + w^0(\iota)$.

Remark. For any $\iota \in \mathfrak{A}^+(p)$, $(p-2)\text{ch}(\iota) + w^0(\iota) \leq (p-2)^2 + p - 1 = u^*$.

The elements of $\{t^{-\iota} \mid \iota \in \mathfrak{A}^+(p)\}$ behave “well” modulo $(\sigma - \text{id})\mathcal{K}$, i.e. the natural map $\sum_{\iota \in \mathfrak{A}^+(p)} k t^{-\iota} \rightarrow \mathcal{K}/(\sigma - \text{id})\mathcal{K}$ is injective. This is implied by the following proposition.

Proposition 2.8. Let $v_p$ be the $p$-adic valuation such that $v_p(p) = 1$.

a) Then all $\iota p^{-v_p(\iota)}$, where $\iota \in \mathfrak{A}^+(p)$, are pairwise different.

b) If $\iota \in \mathfrak{A}^+(p)$ and $\text{ch}(\iota) = 1$ then $\iota p^{-v_p(\iota)} > qv_0 - b^*$.

Proof. a) Suppose $\iota = p^m(q_\alpha - (q - 1)\beta) \in \mathfrak{A}^+(p)$.

If $\text{ch}(\iota) = 0$ then $\iota \mapsto \iota p^{-v_p(\iota)}$ identifies $\{\iota \in \mathfrak{A}^+(p) \mid \text{ch}(\iota) = 0\}$ with $\mathbb{Z}^+(p) \cap [0, (p-1)v_0)$, cf. Lemma 2.6a).

Remark. For similar reasons, if $1 \leq s < p$ and $a \in \mathbb{Z}^+(p) \cap [0, (p-1)v_0)$ then $a < sv_0$ iff $qa < sb^*$.

If $\text{ch}(\iota) \geq 1$ then $\iota p^{-m} \notin p\mathbb{N}$, i.e. $m \geq v_p(\iota)$.

Indeed, $\iota p^{-m} = q_\alpha - (q - 1)\beta \in p\mathbb{N}$ implies (use that $q_\alpha \in p\mathbb{N}$ because $m < N^*)$ that $p^{-m}(b_1 + b_2p^{m_2} + \cdots + b_wp^{m_w}) \in p\mathbb{N}$ where all $m_i \in [0, m]$. But this number is $\leq \text{ch} \iota < p$. The contradiction.

Then by Prop.2.5, $\iota p^{-v_p(\iota)} \geq \iota p^{-m} = |q \alpha - (q - 1)\beta| > (b^*/q)(p-1) = r^*(1 - q^{-1})(p-1) > (v_0 - \delta_0)(p-1)$ (use property C3 from Sect.2.1).

Finally, if $\iota \in \mathfrak{A}^0_0$ then $\iota p^{-v_p(\iota)} = a < (p - 1)v_0$ implies that $a < (v_0 - \delta_0)(p-1)$ by the choice of $\delta_0$, cf. Sect.2.1. On the other hand, for all $\iota \in \mathfrak{A}^+(p)$ with $\text{ch}(\iota) \geq 1$, the values $\iota p^{-v_p(\iota)}$ are different (use that $\gcd(p^m\alpha, p^n\beta/r^*) \neq 0 \mod p$) and bigger than $(v_0 - \delta_0)(p-1)$.

b) Here $\iota p^{-v_p(\iota)} = \iota p^{-m} = qa - b^*$. If $\alpha \geq v_0$ then $\iota^{-v_p(\iota)} \geq qv_0 - b^*$. If $\alpha < v_0$ then $\iota p^{-v_p(\iota)} \leq q(v_0 - \delta_0 - r^*(q-1)/q)) < 0$, cf. condition C3) from Sect.2.1. The contradiction.

The proposition is completely proved. □
Definition. $\mathfrak{A}^0(p) = \mathfrak{A}^+(p) \cup \{0\}$.

Let $\tilde{L}^\dagger_k$ be the Lie algebra over $k$ with the set of free generators

$$\{ D_{in}^\dagger \mid \iota \in \mathfrak{A}^+(p), n \in \mathbb{Z}/N_0 \} \cup \{D_n^\dagger\}. $$

Set (compare with Sect.1.1) $D_{on}^\dagger = \sigma^n(\alpha_0) D_0^\dagger$, use the notation $\sigma$ for the $\sigma$-linear $k$-automorphism of $\tilde{L}_k^\dagger$ such that for all $\iota \in \mathfrak{A}^0(p)$, $\sigma : D_{en}^\dagger \mapsto D_{e,n+1}^\dagger$, and introduce the Lie $\mathbb{F}_p$-algebras $\tilde{L}^\dagger := \tilde{L}_k^\dagger|_{\sigma=\text{id}}$ and $\mathcal{L}^\dagger = \tilde{L}^\dagger/C_p(\tilde{L}^\dagger)$. Note that $\mathcal{L}^\dagger \otimes k = \tilde{L}^\dagger_k$, and this matches the agreement about extension of scalars from the end of Introduction.

For $\iota \in \mathfrak{A}^0(p)$, set $w^\dagger(\iota) = w^0(\iota) + \text{ch}(\iota)$.

**Remark.** Now the last inequality from the above definition of $\iota \in \mathfrak{A}^+(p)$ can be written as $\kappa(\iota) \leq (p-3)\text{ch}(\iota) + w^\dagger(\iota)$.

Introduce the $\dagger$-weights, $w^\dagger(D_{in}^\dagger) := w^\dagger(\iota)$.

Denote by $\{\mathcal{L}^\dagger(s)\}_{s \geq 1}$ the minimal central filtration of $\mathcal{L}^\dagger$ such that all $D_{in}^\dagger$ with $w^\dagger(D_{in}^\dagger) \geq s$ belong to $\mathcal{L}^\dagger(s)_k$. This means that $\mathcal{L}^\dagger(s)_k$ is an ideal in $\mathcal{L}^\dagger_k$ generated as $k$-module by all $[\ldots[D_{i_1n_1}, D_{i_2n_2}], \ldots, D_{i_rn_r}]$ such that $w^\dagger(\iota_1) + \cdots + w^\dagger(\iota_r) \geq s$. Note that $C_s(\mathcal{L}^\dagger) \subseteq \mathcal{L}^\dagger(s)$.

Let $\tilde{p}^\dagger : \mathcal{L}^\dagger \mapsto \mathcal{L}^\dagger := \mathcal{L}^\dagger/\mathcal{L}^\dagger(p)$ be the natural projection. Then $\mathcal{L}^\dagger$ is provided with the induced decreasing central filtration by the ideals $\mathcal{L}^\dagger(s) = \tilde{p}^\dagger(\mathcal{L}^\dagger(s)), s \geq 1,$ and $\mathcal{L}^\dagger(p) = 0$.

Let $\mathcal{A}^\dagger$ be an enveloping algebra for $\mathcal{L}^\dagger$.

For $m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{A}^\dagger[m]_k$ be the $k$-submodule in $\mathcal{A}^\dagger_k$ generated by all monomials $D_{i_1n_1}^\dagger D_{i_2n_2}^\dagger \cdots D_{i_rn_r}^\dagger$ such that $\text{ch}(\iota_1) + \cdots + \text{ch}(\iota_r) = m$.

By setting $\mathcal{A}^\dagger[m] = \mathcal{A}^\dagger \cap \mathcal{A}^\dagger[m]_k$ we obtain a grading in the category of $\mathbb{F}_p$-algebras $\mathcal{A}^\dagger = \oplus_{m \geq 0} \mathcal{A}^\dagger[m]$ and the induced grading $\mathcal{L}^\dagger = \oplus_{m \geq 0} \mathcal{L}^\dagger[m]$ in the category of Lie algebras.

For $s \geq 1$, set $\mathcal{L}^\dagger(s)[m] = \mathcal{L}^\dagger(s) \cap \mathcal{L}^\dagger[m]$ and $\mathcal{L}^\dagger[m] = \mathcal{L}^\dagger[m]/\mathcal{L}^\dagger(p)[m]$.

Note that $\mathcal{L}^\dagger[p] = 0$ (use that for any $l \in \mathcal{L}^\dagger[p], w^\dagger(l) \geq p$).

This gives the induced gradings $\mathcal{L}^\dagger = \oplus_{m \geq 0} \mathcal{L}^\dagger[m]$ and $\mathcal{L}^\dagger(s) = \oplus_{m \geq 0} \mathcal{L}^\dagger(s)[m]$, where $\mathcal{L}^\dagger(s)[m] := \mathcal{L}^\dagger(s) \cap \mathcal{L}^\dagger[m]$ and $\mathcal{L}^\dagger(s)[s] = 0$.

**Definition.** If $l \in \mathcal{L}^\dagger[m]_k, l \neq 0$, we set $\text{ch}(l) = m$.

Note that $\mathcal{L}^\dagger[m]_k$ is a $k$-submodule of $\mathcal{L}^\dagger_k$ generated by all commutators $[\ldots[D_{i_1n_1}^\dagger, \ldots, D_{i_rn_r}^\dagger]$ such that $\sum_{1 \leq j \leq r} \text{ch}(\iota_j) = m$.

Clearly, for any $m_1, m_2$, $[\mathcal{L}^\dagger[m_1], \mathcal{L}^\dagger[m_2]] \subset \mathcal{L}^\dagger[m_1 + m_2]$.

**2.4. Lie algebra $\widetilde{\mathcal{N}}^{sp}$.** Let $\mathcal{N}^\dagger$ be the $k$-submodule in $\mathcal{L}^\dagger_k$ generated by the elements of the form $t^{-b}l$, where $l \in \mathcal{L}^\dagger(s)_k$ and $b < sb^*$. Then $\mathcal{N}^\dagger$ is a Lie $k$-subalgebra in $\mathcal{L}^\dagger_k$ and for $j \geq 0$, $t^{j^b} \mathcal{N}^\dagger$ are ideals in $\mathcal{N}^\dagger$. 
Lemma 2.9. Condition b) means that

Lemma 2.9. Condition b) means that

Definition. Let $\tilde{\mathcal{N}}^{sp}$ be the $k$-submodule in $\tilde{\mathcal{N}}^{\dagger}$ generated by the elements of the form $t^{-i}l$ with $l \in \tilde{\mathcal{L}}^{l}(s)[m]_{k}$ such that:

a) $\iota + \text{ch}(\iota)b^{*} \in sl^{*}$

b) $\iota + \text{ch}(\iota)b^{*} \in sp^{*}$

c) $\text{ch}(\iota) \geq m, \kappa(\iota) \leq (p - 3)m + s.$

Remark. Condition b) means that $t^{-i}l \in t^{\text{ch}(\iota)b^{*}}\tilde{\mathcal{N}}^{\dagger}.$

Lemma 2.9. a) $\tilde{\mathcal{N}}^{sp}$ is a Lie subalgebra in $\tilde{\mathcal{N}}^{\dagger}.$

b) For any $j \geq 0$, $t^{\text{ch}(\iota)b^{*}}\tilde{\mathcal{N}}^{sp}$ is an ideal in $\tilde{\mathcal{N}}^{sp}.$

Proof. a) Suppose $w_{1} = t^{-i}l_{1}$ and $w_{2} = t^{-j}l_{2}$ belong to $\tilde{\mathcal{N}}^{sp}.$ We must prove that $w = [w_{1}, w_{2}] \in \tilde{\mathcal{N}}^{sp}.

We can assume that for $j = 1, 2, \iota = \text{ch}(\iota)_{j} \in \tilde{\mathcal{L}}^{l}(s)[m]_{k},$ where $s_{j} = \text{wt}(l_{j})$ and $m_{j} = \text{ch}(l_{j}).$ We can assume also that $s := s_{1} + s_{2} < p$ (otherwise, $\text{wt}(l_{1}l_{2}) \geq p$ and $[w_{1}, w_{2}] = 0).$ Then $m := m_{1} + m_{2} < p$ (because, $m_{1} \leq s_{1}$ and $m_{2} \leq s_{2}).$ This implies that $l := [l_{1}, l_{2}] \in \tilde{\mathcal{L}}^{l}(s)[m]_{k}.$

Let $\iota := \iota_{1} + \iota_{2}.$ Then

$$\iota + (\text{ch}(\iota_{1}) + \text{ch}(\iota_{2}))b^{*} \in sp^{*} \subseteq (p - 1)b^{*}.$$ 

This implies that $\iota < (p - 1)b^{*}.$ If $-\iota > (p - 1)b^{*}$ then $t^{-i}l \in t^{(p-1)b^{*}}\tilde{\mathcal{N}}^{\dagger} = 0.$ Therefore, we can assume that $|\iota| < (p - 1)b^{*}$.

In addition, it will be sufficient to consider the case $m' := \text{ch}(\iota_{1}) + \text{ch}(\iota_{2}) < p.$ (Otherwise, for $j = 1, 2$ $w_{j} \in t^{\text{ch}(\iota)b^{*}}\tilde{\mathcal{N}}^{\dagger}$ and $w = [w_{1}, w_{2}] \in t^{m'b^{*}\tilde{\mathcal{N}}^{\dagger}} = 0.$)

Now notice that $\kappa(\iota) \leq \kappa(\iota_{1}) + \kappa(\iota_{2}) \leq (p - 3)m + s < u^{*},$ i.e. $\iota \in \mathfrak{A}^{0}.$ Finally, $\text{ch}(\iota) = m' \geq m$ and $w \in \tilde{\mathcal{N}}^{sp}.$

b) It will be sufficient to verify that $t^{\text{ch}(\iota)b^{*}}w \in \tilde{\mathcal{N}}^{sp},$ where $w = t^{-i}l$ with $l \in \tilde{\mathcal{L}}^{l}(s)[m]_{k},$ $\text{ch}(\iota) \geq m,$ $\text{ch}(\iota)b^{*} \in sp^{*}$ and $\kappa(\iota) \leq (p - 3)m + s.$

Suppose $\text{ch}(\iota) + j < p.$

In this case $\iota' = \iota - jb^{*} \in \mathfrak{A}^{0}.$ Indeed, $\text{ch}(\iota') = \text{ch}(\iota) + j < p$ and $\kappa(\iota') = \kappa(\iota) < u^{*}.$ Therefore, $t^{\text{ch}(\iota)b^{*}}w = t^{-i}l \in \tilde{\mathcal{N}}^{sp},$ because $\iota' + \text{ch}(\iota')b^{*} = \iota - jb^{*} + \text{ch}(\iota) + j < u^{*}$ and $\kappa(\iota') \leq (p - 2)m + s.$

If $\text{ch}(\iota) + j \geq p$ then (as earlier) $t^{\text{ch}(\iota)b^{*}}w \in t^{(\text{ch}(\iota) + j)b^{*}}\tilde{\mathcal{N}}^{\dagger} = 0.$

The lemma is proved.

\qed
Clearly, we have the induced grading \( \tilde{N}^{sp} = \oplus_{m \geq 0} \tilde{N}^{sp}[m] \), where \( \tilde{N}^{sp}[p-1] = 0 \). Any element from \( \tilde{N}^{sp}[m] \) appears as a sum of elements of the form \( t^{-l}l \), where for some \( s \geq 1 \), \( l \in \mathcal{L}[s][m] \), \( \iota + ch(\beta) < sb^* \), \( ch(\iota) \geq m \) and \( \kappa(\iota) \leq (p-3)m + s \).

**Definition.** For \( j \geq 0 \) and \( s \geq 1 \), let:

a) \( \tilde{N}^{sp}(j) \) be the \( k \)-submodule in \( \tilde{N}^{sp} \) generated by all \( t^{-l}l \in \tilde{N}^{sp} \) such that for some \( m \geq 0 \), \( t^{-l}l \in \tilde{N}^{sp}[m] \) and \( ch(\iota) \geq m + j \);

b) \( \tilde{N}^{sp}(s, j) \) be the submodule in \( \tilde{N}^{sp}(j) \) generated by \( t^{-l}l \) (in the above notation) such that \( l \in C_s(\mathcal{L}_k^\dagger) \).

Note that:

- \( \tilde{N}^{sp}(0) = \tilde{N}^{sp}(1, 0) = \tilde{N}^{sp} \);
- all \( \tilde{N}^{sp}(j) \) and \( \tilde{N}^{sp}(s, j) \) are ideals in \( \tilde{N}^{sp} \);
- for all \( j_1, j_2 \) and \( s_1, s_2 \), \( [\tilde{N}^{sp}(j_1), \tilde{N}^{sp}(j_2)] \subset \tilde{N}^{sp}(j_1 + j_2) \) and \( [\tilde{N}^{sp}(s_1, j_1), \tilde{N}^{sp}(s_2, j_2)] \subset \tilde{N}^{sp}(s_1 + s_2, j_1 + j_2) \);
- \( \tilde{N}^{sp}(p - 1) = 0 \);
- for any \( \iota \in \mathfrak{A}^0(p) \), \( t^{-1}D_{10}^\dagger \in \tilde{N}^{sp} \).

2.5. **The action** \( \Omega_\gamma \). Suppose \( \gamma \in \mathbb{Z}/p \). If \( \iota = p^n(q\alpha - (q - 1)\beta) \in \mathfrak{A}^0 \) and \( t^{-l}l \in \tilde{N}^{sp} \), where \( l \in \mathcal{L}_k^\dagger \), then by Lemma 2.9

\[
\Omega_\gamma(t^{-1}l) := t^{-1}\exp(\gamma(p^n\alpha)b^*)l \in \tilde{N}^{sp}.
\]

If \( w \in \tilde{N}^{sp} \) then there is a unique presentation \( w = \sum_{\iota \in \mathfrak{A}^0} t^{-1}l_\iota \), where all \( t^{-1}l_\iota \in \tilde{N}^{sp} \), and we set

\[
\Omega_\gamma(w) = \sum_{\iota \in \mathfrak{A}^0} \Omega_\gamma(t^{-1}l_\iota) .
\]

The correspondence \( w \mapsto \Omega_\gamma(w) \) is a well-defined action of the elements \( \gamma \) of the (additive) group \( \mathbb{Z}/p \) on the Lie algebra \( \tilde{N}^{sp} \). This action is unipotent because for any \( n \in \tilde{N}^{sp}(j) \), \( \Omega_\gamma(n) \equiv n \mod\tilde{N}^{sp}(j + 1) \).

Choose \( \tilde{e}^{sp} \in \tilde{N}^\dagger \) satisfying the following two conditions:

\[
\tilde{e}^{sp} = \sum_{\iota \in \mathfrak{A}^0(p)} t^{-1}D_{10}^\dagger \mod C_2(\mathcal{L}_k) ;
\]

\[
\tilde{e}^{sp} := \tilde{e}^{sp} \mod t^{(p-1)b^*} \tilde{N}^\dagger \in \tilde{N}^{sp} .
\]

A choice of \( \tilde{e}^{sp} \) allows us to associate to the above defined actions \( \Omega_\gamma \) the “conjugated” actions of \( A_1^\dagger \) on \( \tilde{L}^\dagger \) as follows.
Proposition 2.10. For any \( \gamma \in \mathbb{Z}/p \), there are unique \( \widetilde{c}_\gamma \in \widetilde{N}^s p(1) \) and \( A_1^\gamma \in \text{Aut}_{\text{L}} \tilde{L}^1 \) such that

a) \( \sigma \tilde{c}_\gamma \in \widetilde{N}^s p(1) \) and \( \Omega_\gamma(\tilde{c}^s p) = (\sigma \tilde{c}_\gamma) \circ (A_1^\gamma \otimes \text{id}_{\tilde{L}}) \tilde{c}^s p \circ (-\tilde{c}_\gamma) \);

b) for any \( \iota \in \mathfrak{A}^0(p) \), \( A_1^\gamma (D_{t_0}^1) - D_{t_0}^1 \in \oplus_{m<\text{ch}(\iota)} \tilde{L}^1[m]_k \).

Proof. We need the following lemma.

Lemma 2.11. Suppose \( j, s \geq 1 \) and \( n \in \widetilde{N}^s p(s, j) \). Then there are unique \( S(n), R(n) \in \widetilde{N}^s p(s, j) \) such that

a) \( R(n) = \sum_{l \in \mathbb{A}^+(p)} t^{-\iota} l \) with all \( l \in C_s(L^1)_k \) (if \( \text{ch}(\iota) < j \) then \( l \equiv 0 \));

b) \( n = R(n) + (\sigma - \text{id}) S(n) \).

Proof of lemma. Note that any \( n \in \widetilde{N}^s p(s, j) \) appears as a sum of elements of the form \( t^{-\iota} l \), where for some \( m^0 \) and \( s^0 \), it holds \( l \in L^1(s^0)[m^0]_k \cap C_s(L^1)_k \), \( \iota + \text{ch}(s^0) < s^0 \), \( \text{ch}(\iota) \geq m^0 + j \) and \( \kappa(\iota) \leq (p - 3)m^0 + s^0 \). When proving the existence of \( S(n) \) and \( R(n) \) we can assume that \( n = t^{-\iota} l \).

— Let \( \iota < 0 \).

Set \( R(n) = 0 \) and \( S(n) = -\sum_{m \geq 0} t^{-mp^m} \sigma^m l \).

If \( -mp^m \geq b^*(p - 1) \) then \( t^{-mp^m} \sigma^m l \in t^{b^*(p - 1)} \widetilde{N}^1 = 0 \).

If \( -mp^m < b^*(p - 1) \) then:

— \( mp^m + \text{ch}(\iota p^m)b^* \leq \iota + \text{ch}(\iota)b^* < s^0 b^* = \text{wt}^1(\sigma^m l)b^* \);

— \( m^0 = \text{ch}(l) = \text{ch}(\sigma^m l) \) and \( \text{ch}(\iota p^m) = \text{ch}(\iota) \geq m^0 + j \);

— \( \kappa(\iota p^m) = \kappa(\iota) \).

Therefore, if \( \iota < 0 \) then both \( R(n), S(n) \in \widetilde{N}^s p(s, j) \).

— Let \( \iota > 0 \).

Suppose \( p^{m(\iota)} \) is the maximal power of \( p \) such that \( \iota = p^{m(\iota)} \iota_1 \) and \( \iota_1 \in \mathfrak{A}^0 \). Then \( \iota_1 \in \mathfrak{A}^+(p) \). Indeed, we must verify just the last inequality for \( \kappa(\iota) \) from the definition of \( \mathfrak{A}^+(p) \) in Sect.2.3. Using that \( t^{-\iota} l \in \widetilde{N}^s p, w^0(\iota) \geq 1 \) and \( \text{ch}(\iota) \geq m^0 + 1 \) we obtain that

\[ \kappa(\iota) \leq (p - 3)m^0 + s^0 \leq (p - 3)\text{ch}(\iota) + 2 \leq (p - 2)\text{ch}(\iota) + w^0(\iota) \].

Then we set

\[ R(n) = t^{-\iota_1} \sigma^{-m(\iota)} l, \quad S(n) = \sum_{0 \leq m < m(\iota)} \sigma^m(R(n)) \].

Finally, if \( 0 \leq m \leq m(\iota) \) then \( \sigma^m R(n) \in \widetilde{N}^s p(j) \). Indeed,

— \( \iota p^m + \text{ch}(\iota_1 p^m)b^* \leq \iota + \text{ch}(\iota)p^m < s^0 b^* = \text{wt}^1(\sigma^{-m(\iota)} + m) l)b^* ;

— \( \text{ch}(\iota_1 p^m) = \text{ch}(\iota) \geq m^0 + j , \sigma^{-m(\iota) + m} l \in \tilde{L}^1(s^0)[m^0]_k \cap C_s(L^1)_k ; \)
The lemma is proved. □

and the appropriate $t^{-i}$ are independent modulo $(\sigma - \text{id}) \mathcal{K}$, cf. Prop. 2.8. The lemma is proved.

Continue the proof of Prop. 2.10.

Use induction on $i \geq 1$ to prove the proposition modulo \( \tilde{N}^{sp}(i, i) \).

— If $i = 1$ take $\tilde{c}_\gamma = 0$, $A_1^{\gamma} = \text{id}$ and use $\Omega_{\gamma}(\tilde{c}^{sp}) - \tilde{c}^{sp} \in \tilde{N}^{sp}(1, 1)$.

— Assume $1 \leq i < p$ and for $\tilde{c}_\gamma \in \tilde{N}^{sp}(1, 1)$ and $A_i^{\gamma} \in \text{Aut}_{\text{Lie}}(\tilde{L}^i)$,

\[
H = \Omega_{\gamma}\tilde{c}^{sp} - (\sigma\tilde{c}_\gamma) \circ (A_{i}^{\gamma} \otimes \text{id}_K)\tilde{c}^{sp} \circ (-\tilde{c}_\gamma) \in \tilde{N}^{sp}(i, i).
\]

Then $\mathcal{R}(H), \mathcal{S}(H) \in \tilde{N}^{sp}(i, i)$. Set $\mathcal{R}(H) = \sum_{\text{ch}(i) \geq i + m} t^{-i} H_{im}$, where all $H_{im} \in \tilde{L}^i[m]_k \cap C_i(\tilde{L}^i)_k$. Introduce $A_{i}^{\gamma} \in \text{Aut}_{\text{Lie}}(\tilde{L}^i)$ by setting for all involved $i$ and $m$, $A_{i}^{\gamma}(D_{i0}) = A_{i}^{\gamma}(D_{i0}) - \sum_m H_{im}$. Set also $\tilde{c}_i^{\gamma} = \tilde{c}_\gamma - \mathcal{S}(H)$. Then

\[
\Omega_{\gamma}\tilde{c}^{sp} \equiv (\sigma\tilde{c}_1^{\gamma}) \circ (A_{1}^{\gamma} \otimes \text{id}_K)\tilde{c}^{sp} \circ (-\tilde{c}_1^{\gamma}) \bmod \tilde{N}^{sp}(i + 1, i + 1).
\]

The uniqueness follows similarly by induction on $i$ and the uniqueness part of Lemma 2.11.

The proposition is proved. □

We have obviously the following properties.

**Corollary 2.12.** For any $\gamma, \gamma_1 \in \mathbb{Z}/p$,

a) $A_{1}^{\gamma+\gamma_1} = A_1^{\gamma} A_1^{\gamma_1}$;

b) $\Omega_{\gamma}(\tilde{c}_\gamma) \circ (A_{1}^{\gamma} \otimes \text{id}_K)\tilde{c}_\gamma = \tilde{c}_{\gamma+\gamma_1}$;

c) if $l \in \tilde{L}^i[m]$ then $A_1^{\gamma}(l) - l \in \oplus_{m' < m} \tilde{L}^i[m']$, e.g. $A_1^{\gamma}|_{\tilde{L}^i[1]} = \text{id}.

2.6. **The action** $\Omega_U$. Let $A_1^{\gamma} := A_1^{\gamma}|_{\gamma=1}$. Then for any $\gamma = n \bmod p$, $A_1^{\gamma} = A_1^{\gamma|p}$, in particular, $A_1^{p|} = \text{id}_{\tilde{L}^i}$. By part c) of the above corollary, for all $m \geq 0$,

\[
(A_1^{\gamma} - \text{id}_{\tilde{L}^i})(\oplus_{m' < m} \tilde{L}^i[m']) \subset \oplus_{m' < m} \tilde{L}^i[m'].
\]

Therefore, there is a differentiation $B_1^{\gamma} \in \text{End}_{\text{Lie}}(\tilde{L}^i)$ such that for all $m \geq 0$, $B_1^{\gamma}(\tilde{L}^i[m]) \subset \oplus_{m' < m} \tilde{L}^i[m']$ and for all $\gamma \in \mathbb{Z}/p$, $A_1^{\gamma} = \exp(\gamma B_1^{\gamma})$.

Applying the methods from [8], Sect.3 we obtain the existence of $\tilde{c}_\gamma \in \tilde{N}^{sp}(j)$ with $1 \leq j < p$, such that:

if $\tilde{c}_U = \tilde{c}^1 U + \cdots + \tilde{c}^{p-1} U^{p-1} \in \tilde{N}^{sp} \otimes \mathbb{F}_p[U]$, where $U^p = 0$, then for all $\gamma \in \mathbb{Z}/p$, $\tilde{c}_U|_{U=\gamma} = \tilde{c}_\gamma$. (Here also all $\sigma\tilde{c}_\gamma \in \tilde{N}^{sp}(j).$)
As a result, we obtain the action $\Omega_U$ of the formal group scheme $\alpha_p = \text{Spec} \mathbb{F}_p[U]$ (here $U^p = 0$ and $\Delta : U \mapsto U \otimes 1 + 1 \otimes U$) on the Lie algebra $\tilde{N}^{sp}$. In particular,

\begin{equation}
(2.3) \quad \Omega_U(\tilde{c}^{sp}) = \sigma(\tilde{c}_U) \circ (A_U^{\dagger} \otimes \text{id})\tilde{c}^{sp} \circ (-\tilde{c}_U).
\end{equation}

where $A_U^\dagger|_{U=\gamma} = A^\dagger_\gamma$.

This formalism allows us to recover uniquely the action $\Omega_U$ from its differential $d\Omega_U := \Omega^1U$, where $\Omega_U = \sum_{0 \leq i < p} \Omega^iU^n$ (here $\Omega^0 = \text{id}$). Similarly, the cocycle $\tilde{c}_U$ is determined uniquely by its linear part $\tilde{c}^{\dagger}$.

2.7. Ideals $\tilde{L}^{\dagger}[v_0]$ and $\tilde{L}[v_0]$.

Recall that $\tilde{L}^{\dagger}[0]$ is the minimal Lie subalgebra of $\tilde{L}^{\dagger}$ such that $\tilde{L}^{\dagger}[0]_k$ contains all $D_{an}^{\dagger}$ with $i \in \mathfrak{A}_0^0(p) = \{i \in \mathfrak{A}_0^0(p) \mid \chi(i) = 0\}$. Then $\tilde{L}^{\dagger}[0]$ has the induced filtration $\{\tilde{L}^{\dagger}(s)[0]\}_{s \geq 1}$ and there is epimorphism of filtered Lie algebras $\mathcal{V}^0 : \tilde{L}^{\dagger} \longrightarrow \tilde{L}[0]$ such that $D_{an}^{\dagger} \mapsto D_{an}^{\dagger}$ if $i \in \mathfrak{A}_0^0(p)$ and $D_{an}^{\dagger} \mapsto 0$, otherwise.

By Lemma 2.6, $\mathfrak{A}_0^0(p) = \{qa \mid a \in [0, (p-1)v_0) \cap \mathbb{Z}^+(p)\}$. By Remark from the proof of Proposition 2.8a), the correspondences $D_{an}^{\dagger}$ establish isomorphism of filtered Lie algebras $j^0 : \tilde{L}^{\dagger}[0] \longrightarrow \tilde{L}$.

Let $\mathcal{V} := j^0\mathcal{V}^0 : \tilde{L}^{\dagger} \longrightarrow \tilde{L}$.

Define the ideal $\tilde{L}^{\dagger}[v_0]$ as the minimal ideal in $\tilde{L}^{\dagger}$ containing all $A^{\dagger}_\gamma(\text{Ker} \mathcal{V})$, $\gamma \in \mathbb{Z}/p$. Set $\tilde{L}[v_0] = \mathcal{V}(\tilde{L}^{\dagger}[v_0])$. Then $\tilde{L}[v_0]$ is the minimal ideal in $\tilde{L}$ such that $\mathcal{V}^{-1}(\tilde{L}[v_0])$ is invariant with respect to all $A^{\dagger}_\gamma$.

**Proposition 2.13.** If $l \in \tilde{L}^{\dagger}$ and $\gamma \in \mathbb{Z}/p$ then

$$\mathcal{V}(A^{\dagger}_\gamma l) \equiv \mathcal{V}(l) \mod \tilde{L}[v_0].$$

**Proof.** a) Let $l' = \mathcal{V}^0(l)$. Then $l \in l' + \text{Ker} \mathcal{V}$ and, therefore,

$$A^{\dagger}_\gamma(l) \in A^{\dagger}_\gamma(l') + A^{\dagger}_\gamma(\text{Ker} \mathcal{V}) \subset l' + \mathcal{V}^{-1}\tilde{L}[v_0].$$

It remains to apply $\mathcal{V}$ to this embedding. (Use that $A^{\dagger}_\gamma|_{\text{Im} \mathcal{V}^0} = \text{id}$.)

The ideal $\tilde{L}[v_0]$ can be also defined in terms related to the action $\Omega_U$. If $B^{\dagger}$ is the differentiation from Sect.2.6 then $\tilde{L}[v_0]$ appears as the minimal ideal in $\tilde{L}$ such that $\tilde{L}[v_0]_k$ contains all the elements $\mathcal{V}B^{\dagger}_k(D^{\dagger}_0)$, where $i \in \mathfrak{A}_0^0(p)$ and $\chi(i) \geq 1$ (if $i \in \mathfrak{A}_0^0(p)$ then $B^{\dagger}(D^{\dagger}_0) = 0$). This is implied by the following proposition.

**Proposition 2.14.** Suppose $\mathcal{I}$ is an ideal in $\tilde{L}$. Then the following conditions are equivalent:

a) for any $\gamma \in \mathbb{Z}/p$, $A^{\dagger}_\gamma(\text{Ker} \mathcal{V}) \subset \mathcal{V}^{-1}(\mathcal{I})$;

b) $B^{\dagger}(\text{Ker} \mathcal{V}) \subset \mathcal{V}^{-1}(\mathcal{I})$.  

Proof. Part a) implies b) because for any \( l \in \bar{L}^1 \) we have a non-degenerate system of linear relations

\[
(A^l_\gamma - \text{id}_{\bar{L}^1})l \equiv \sum_{1 \leq s < p} \gamma^s B^{ls}(l)/s! \mod \mathcal{I}
\]

with \( \gamma = 1, \ldots, p - 1 \).

Vice versa, b) implies that for all \( s \geq 1, B^{ls}(\text{Ker} \, V) \subset B^1(\text{Ker} \, V) \). Indeed, \( \bar{L}^1 = \text{Ker} \, V \oplus \bar{L}^1[0] \) implies that \( V^{-1}(\mathcal{I}) = \text{Ker} \, V \oplus (j^0)^{-1}(\mathcal{I}) \). Therefore, \( B^{l2}(\text{Ker} \, V) \subset B^1(V^{-1}(\mathcal{I})) = B^1(\text{Ker} \, V) \) (use \( B^1|_{\bar{L}^1[0]} = 0 \)). It remains to use relations (2.3). Proposition is proved. \( \square \)

2.8. Lie algebras \( \mathcal{N}^{(q)}, \mathcal{N}^{(q)}_l \) and \( \bar{N}^{(q)}_l \).

Introduce an analogue \( \mathcal{N}^{(q)} \subset \mathcal{L}_K \) of \( \mathcal{N} \) as the \( k \)-module generated by all \( t^{-a} \), where for some \( s \geq 1, l \in \mathcal{L}(s)_k \) and \( a < sb^* \). It is a Lie \( k \)-algebra and \( e^{(q)} \) together with all \( \gamma \ast e^{(q)}, \gamma \in \mathbb{Z}/p \), cf. Sect.1.5, belong to \( \mathcal{N}^{(q)} \).

Similarly, introduce the Lie algebras \( \bar{N}^{(q)} \) (use the algebra \( \bar{L} \) instead of \( \mathcal{L} \)) and \( \bar{N}^{(q)}_l = \mathcal{N}^{(q)} / t^{(p-1)b^*} \mathcal{N}^{(q)} \). These algebras are related to \( \mathcal{N}^{(q)}_l \) via the natural projection \( \bar{p}_{\mathcal{K}} : \mathcal{L}_K \to \bar{L}_K \). The appropriate images of \( e^{(q)} \) in \( \bar{N}^{(q)} \) and \( \bar{N}^{(q)}_l \) will be denoted, resp., by \( \bar{e}^{(q)} \) and \( \bar{e}^{(q)}_l \). Note that there are natural identifications \( \mathcal{N}^{(q)} = \mathcal{V}_K(\bar{N}^{(q)}_l) \) and \( \bar{N}^{(q)} = \mathcal{V}_K(\bar{N}^{(q)}_l) \), where \( \bar{N}^{(q)}_l, \bar{N}^{(q)}_l \) and \( V_{\mathcal{K}} \) were defined in Sect.2.

2.9. Generators of \( \bar{L}[v_0] \). Introduce the following condition of compatibility

\[
\mathcal{V}_K(e^{sp}) = \bar{e}^{(q)}_l \quad .
\]

Recall that Prop.2.14 implies that \( \bar{L}[v_0] \) is the minimal ideal in \( \bar{L} \) such that for all \( \ell \in \mathfrak{A}^0(p) \) with \( \text{ch}(\ell) \geq 1 \), \( \mathcal{V}_K B^1_k(D_{a0}) \in [\bar{L}[v_0], \bar{L}] \). (Note that this implies \( V_{\mathcal{L}} B^1(C_2(\bar{L}^1)) \subset [\bar{L}[v_0], \bar{L}] \).

Proposition 2.15. If \( \text{ch}(\ell) \geq 2 \) then \( \mathcal{V}_K B^1_k(D_{a0}) \in [\bar{L}[v_0], \bar{L}]_k \).

Proof. Suppose \( \bar{e}^{(q)} = \sum_a t^{-qa} l^{(q)}_a \) and \( \bar{e}^{sp} = \sum t^{-i^{sp}}_a \), where all \( l^{(q)}_a \in \bar{L}_k \)

and \( l^{sp} \in \bar{L}^1_k \). Note that:

- if \( a \in \mathbb{Z}^0(p) \) then \( l^{(q)}_a = D_{a0} \mod C_2(\bar{L}_k) \);
- if \( a \notin \mathbb{Z}^0(p) \) then \( l^{(q)}_a \in C_2(\bar{L}_k) \);
- if \( \ell = qa \) with \( a \in \mathbb{Z} \), then \( \mathcal{V}_K(l^{sp}) = l^{(q)}_a \), otherwise \( \mathcal{V}_K(l^{sp}) = 0 \);
- if \( \ell \in \mathfrak{A}^0(p) \) then \( l^{sp} = D_{a0} \mod C_2(\bar{L}^1_k) \), otherwise, \( l^{sp} \in C_2(\bar{L}^1_k) \).

Applying formalism from Sect.2.6 we obtain

\[
\Omega_{\ell}(\bar{e}^{sp}) \equiv (U \text{sg}^1) \circ (\bar{e}^{sp} + U(B^1 \otimes \text{id}_K)\bar{e}^{sp}) \circ (-U \text{sg}^1) \mod U^2 \bar{N}^1 \quad .
\]
where $\tilde{c}^1, \sigma \tilde{c}^1 \in \tilde{N}^{sp}(1)$. Note that

$$\Omega_U(\tilde{e}^{sp}) \equiv \tilde{e}^{sp} + U \sum_l (p^m \alpha)_l t^{-\ell+b} l^{sp} \text{ mod } U^2,$$

where $\ell = q(p^m \alpha)_1 - (q-1)(p^m \beta)_1$.

Applying $\mathcal{V}_k$ to relation (2.6) and setting $\tilde{x} := \mathcal{V}_k \tilde{c}^1$ we obtain

\begin{equation}
(2.7) \quad \tilde{c}^{(q)} + U \sum_a at^{-qa+b^*} f_a^{(q)} \equiv \nonumber \end{equation}

\begin{equation*}
(U\sigma \tilde{x}) \circ \left( \tilde{c}^{(q)} + U \sum_l t^{-\ell} \mathcal{V}_k B^1_k(l^{sp}) \right) \circ (-U \tilde{x}) \text{ mod } U^2 \tilde{N}^{(q)}.
onumber \end{equation*}

Let $\tilde{f}_1, \tilde{f}_2 \in \tilde{N}^{(q)}$ be such that

\begin{equation*}
(U\sigma \tilde{x}) \circ \tilde{c}^{(q)} \equiv \tilde{c}^{(q)} + U(\sigma \tilde{x} + \tilde{f}_1) \text{ mod } U^2 \nonumber \end{equation*}

\begin{equation*}
\tilde{c}^{(q)} \circ (-U \tilde{x}) \equiv \tilde{c}^{(q)} + U(-\tilde{x} + \tilde{f}_2) \text{ mod } U^2.
onumber \end{equation*}

There are explicit formulas for $\tilde{f}_1$ and $\tilde{f}_2$, cf. e.g. Sect.3.2 of [8], but we need only that they are just $\mathbb{F}_p$-linear combinations of the commutators $[\ldots [\sigma \tilde{x}, \tilde{c}^{(q)}], \ldots, \tilde{c}^{(q)}]$ and, resp., $\ldots [\tilde{x}, \tilde{c}^{(q)}], \ldots, \tilde{c}^{(q)}]$.

Comparing the coefficients for $U$ in (2.7) we obtain

\begin{equation}
(2.8) \quad \sum_a at^{-qa+b^*} f_a^{(q)} = \sigma \tilde{x} - \tilde{x} + \sum_l t^{-\ell} \mathcal{V}_k B^1_k(l^{sp}) + \tilde{f}_1 + \tilde{f}_2. \nonumber \end{equation}

Note that:

a) $\tilde{c}^1, \sigma \tilde{c}^1 \in \tilde{N}^{sp}(1)$ implies that $\tilde{x}, \sigma \tilde{x} \in \sum_{\text{ch}(i) \geq 1} t^{-i} \tilde{L}_k$;

b) $\{ y \in \sum_{\text{ch}(i) \geq 1} t^{-i} \tilde{L}_k | \sigma y = y \} = 0$;

c) if $\ell \notin \mathbb{N}(p)$ then $\mathcal{V}_k B^1_k(l^{sp}) \in [\tilde{L}[v_0], \tilde{L}]_k$;

d) if $\ell \in \mathbb{N}(p)$ then $\mathcal{V}_k B^1_k(l^{sp}) = \mathcal{V}_k B^1_k(D^1_{i,0}) \text{ mod } [\tilde{L}[v_0], \tilde{L}]_k$.

Let $\tilde{x} = \sum_l t^{-i} x_i$ and $\tilde{f}_1 + \tilde{f}_2 = \sum_l t^{-l} f_l$, where $x_i, f_l \in \tilde{L}_k$ and the both sums are taken for $\ell \in \mathbb{N}$ such that $\text{ch}(i) \geq 1$. Let $\tilde{x}[m]$ be a part of the first sum containing all the summands $t^{-i} x_i$ with $\text{ch}(i) = m$. Similarly, define a part $\tilde{f}[m]$ of the second sum. Note that $\tilde{f}[m]$ is a linear combination of the commutators $[\ldots [\sigma (\tilde{x}[m]), \tilde{c}^{(q)}], \ldots, \tilde{c}^{(q)}]$ and $[\ldots [\sigma (\tilde{x}[m]), \tilde{c}^{(q)}], \ldots, \tilde{c}^{(q)}]$.

Then (2.8) and above congruence d) imply that for any $m \geq 2$,

$$\mathcal{H}[m] \equiv -\tilde{f}[m] \in \mod [\tilde{L}[v_0], \tilde{L}]_k,$$

where $\mathcal{H}[m] := \sigma (\tilde{x}[m]) - \tilde{x}[m] + \sum_{\text{ch}(i) = m} t^{-i} \mathcal{V}_k B^1_k(D^1_{i,0})$.

Let $\tilde{D}(s) := [\tilde{L}[v_0], \tilde{L}] + \tilde{L}(s)$. 

Prove by induction on \( s \geq 1 \) that \( \tilde{x}[m] \in \tilde{D}(s)_K \) and \( V_k B_k^1(D_{i0}) \in \tilde{D}(s)_k \) (here \( \text{ch}(\iota) = m \)).

If \( s = 1 \) there is nothing to prove.
Suppose it is proved for \( s < p \).

Then \( f'[m] \in \tilde{D}(s+1)_K \) and, therefore, \( H[m] \in \tilde{D}(s+1)_K \). Then analog of Lemma 2.11 implies that \( \tilde{x}[m] \) and

\[
\sum_{\text{ch}(\iota) = m} t^{-1} V_k B_k^1(D_{i0})
\]

to \( \tilde{D}(s+1)_K \). In particular, all \( V_k B_k^1(D_{i0}) \in \tilde{D}(s+1)_k \).

The proposition is proved because \( D(p) = [\mathcal{L}[v_0], \mathcal{L}] \).

\[ \square \]

Corollary 2.16. \( \mathcal{L}[v_0] \) is the minimal ideal in \( \mathcal{L} \) such that for all \( \iota \in \mathfrak{A}^0(p) := \{ \iota \in \mathfrak{A}(p) \mid \text{ch}(\iota) = 1 \} \), \( V_k B_k^1(D_{i0}) \in \mathcal{L}[v_0]_k \).

3. Application to the ramification filtration

3.1. Statement of the main result. Recall that in Sect.1 we fixed an element \( e \in \mathcal{L}_K \) satisfying conditions (1.1) and (1.3). We also fixed \( f \in \mathcal{L}_{\text{sep}} \) such that \( \sigma f = e \circ f \) and introduced epimorphism \( \eta_e = \pi_f(e) : G \rightarrow G(L) \) which induces identification \( G_{<p} \simeq G(L) \). Conditions (1.1) and (1.3) mean that \( \eta_e \) is a “sufficiently good” lift of the reciprocity map of class field theory. We are going to describe the ideal \( \mathcal{L}^{(\nu)} \) of \( \mathcal{L} \) such that \( \eta_e(G^{(\nu)}) = \mathcal{L}^{(\nu)} \) under additional condition of compatibility (2.5).

In the next section we will use a special case of this assumption to obtain the explicit description of the ideal \( \mathcal{L}^{(\nu)} \) from the paper [1].

Remark. Condition (2.5) states that there is \( \bar{e}^{np} \in \bar{N}^{t} \) such that \( V_K(\bar{e}^{np}) = \bar{e}^{(q)}; \) this can be done if \( e = \sum_{a \geq 0} t^{-a} l_a \), where all \( l_a \in \mathcal{L}_k \).

Consider the parameters \( r^*, N^*, q \) from Sect.2 (they depend on the original \( v_0 \) and the appropriate \( \delta_0 > 0 \) chosen there). Note that if \( e^{(q)} = \sigma^{N^*}(e) \) and \( f^{(q)} = \sigma^{N^*}(f) \) then the appropriate morphism \( \pi_f^{(q)}(e^{(q)}) \) coincides with \( \eta_e \).

Recall that in Sect.2 we introduced the ideal \( \bar{N}[v_0] \) of \( \bar{N} \) defined in the terms of action of the formal group \( \alpha_p \) on \( \bar{e}^{np} = \bar{e}^{np} \mod t^{-1} \bar{N}^{t} \in \bar{N}^{np} \subset \bar{N}^{t} \) which satisfies assumption (2.1) and the compatibility condition (2.5).

Theorem 3.1. \( \mathcal{L}^{(\nu)} = \bar{p}t^{-1} \bar{N}[v_0] \).

3.2. Inductive assumption. Prove theorem by induction on \( s_0 \geq 1 \) in the following form (the statement of theorem appears with \( s_0 = p \))

\[
\mathcal{L}^{(\nu)} + C_{s_0}(\mathcal{L}) = \bar{p}t^{-1}(\bar{N}[v_0]) + \mathcal{L}(s_0).
\]

It is obviously true for \( s_0 = 1 \) because \( C_1(\mathcal{L}) = \mathcal{L}(1) = \mathcal{L} \).

Suppose (3.1) holds for some \( 1 \leq s_0 < p \).

Note that for any \( \gamma \in \mathbb{Z}/p \), the image \( \gamma * \bar{e}^{(q)} \) of \( \gamma * \bar{e}^{(q)} \) in \( \bar{N}^{(q)} \) coincides with \( V_k \Omega_\gamma(\bar{e}^{np}) \).
Proposition 3.2. There is $x _ { \gamma } \in t ^ { b _ { * } } \sum _ { 1 \leq s < s _ { 0 } } t ^ { - s b _ { * } } L ( s ) _ { m } \subset N ( q )$ such that

$$\gamma \ast e ( q ) \equiv ( \sigma x _ { \gamma } ) \circ e ( q ) \circ ( - x _ { \gamma } ) \mod ( L ( v _ { 0 } ) + C _ { s _ { 0 } } ( L ) ) _ { K } .$$

Proof. From Prop.2.10 a) it follows that

$$\gamma \ast \bar{e} ( q ) = ( \sigma \bar{x} _ { \gamma } ) \circ ( V A _ { 0 } ^ { \dag } \otimes id _ { K } ) \bar{e} ^ { sp } \circ ( - \bar{x} _ { \gamma } ) ,$$

where $\bar{x} _ { \gamma } = V _ { K } ( \bar{c} _ { \gamma } ) \in V _ { K } ( N ( sp ) \langle 1 \rangle ) \subset V _ { K } ( t ^ { b _ { * } } \bar{N} ( q ) ) = t ^ { b _ { * } } \bar{N} ( q )$ (use Remark before Lemma 2.9).

Recall that $\bar{e} ^ { sp } \in \bar{N} ^ { \dag }$ is a lift of $\bar{e} ^ { sp } \in \bar{N} ^ { sp } \subset \bar{N} ^ { \dag }$ such that $V _ { K } ( \bar{e} ^ { sp } ) = \bar{e} ( q )$ and $\sigma$ is nilpotent on the kernel of the projection $\bar{N} ^ { \dag } \rightarrow \bar{N} ^ { \dag }$. Therefore, proceeding similarly to the proof of Prop.1.4 we can establish the existence of a unique lift $\bar{x} _ { \gamma } \in t ^ { b _ { * } } \bar{N} ( q )$ of $\bar{x} _ { \gamma }$ such that

$$(3.2) \quad \gamma \ast \bar{e} ( q ) = ( \sigma \bar{x} _ { \gamma } ) \circ ( V A _ { 0 } ^ { \dag } \otimes id _ { K } ) \bar{e} ^ { \dag } \circ ( - \bar{x} _ { \gamma } ) .$$

Prop.2.13 implies that $( V A _ { 0 } ^ { \dag } \otimes id _ { K } ) \bar{e} ^ { \dag } \equiv \bar{e} ( q ) \mod \bar{L} [ v _ { 0 } ] _ { K }$ and we obtain the following congruence

$$(3.3) \quad \gamma \ast e ( q ) \equiv ( \sigma x _ { \gamma } ) \circ e ( q ) \circ ( - x _ { \gamma } ) \mod pr ^ { - 1 } \bar{L} [ v _ { 0 } ] _ { K } ,$$

where $x _ { \gamma } \in L _ { K }$ is any lift of $\bar{x} _ { \gamma }$.

We can choose $x _ { \gamma } \in t ^ { b _ { * } } \sum _ { 1 \leq s < s _ { 0 } } t ^ { - s b _ { * } } L ( s ) _ { m }$ when taking this congruence modulo the ideal $( pr ^ { - 1 } \bar{L} [ v _ { 0 } ] + L ( s _ { 0 } ) ) _ { K }$. It remains to use the inductive assumption. The proposition is proved.

Remark. a) Due to the criterion from Sect.1.5 congruence (3.3) already implies that $L ( v _ { 0 } ) \subset pr ^ { - 1 } \bar{L} [ v _ { 0 } ]$ (use that all $x _ { \gamma }$ are defined over $K$).

b) In the above proof we have automatically that $\sigma \bar{x} _ { \gamma } \in t ^ { b _ { * } } \bar{N} ( q )$ and $\sigma x _ { \gamma } \in t ^ { b _ { * } } \sum _ { 1 \leq s < s _ { 0 } } t ^ { - s b _ { * } } L ( s ) _ { m }$.

For (non-commuting) variables $U$ and $V$ from some Lie $F _ { p }$-algebra $L$ of nilpotent class $< p$, let $\delta ^ { 0 } ( U, V ) := U \circ V - ( U + V )$. Note, if $U$ and $V$ are well-defined modulo $C _ { s _ { 0 } } ( L )$ then $\delta ^ { 0 } ( U, V )$ is well-defined modulo $C _ { s _ { 0 } + 1 } ( L )$.

Let $y _ { \gamma } = \gamma \ast e ( q ) - e ( q ) + \delta ^ { 0 } ( \gamma \ast e ( q ), x _ { \gamma } ) - \delta ^ { 0 } ( \sigma x _ { \gamma }, e ( q ) )$.

Lemma 3.3. For any $\gamma \in \mathbb{Z} / p$, there is $X _ { \gamma } \in L _ { sep }$ such that

a) $\gamma \ast e ( q ) \equiv ( \sigma X _ { \gamma } ) \circ e ( q ) \circ ( - X _ { \gamma } ) \mod ( [ L ( v _ { 0 } ), L ] + C _ { s _ { 0 } + 1 } ( L ) ) _ { sep } ;$

b) $X _ { \gamma } \equiv x _ { \gamma } \mod ( L ( v _ { 0 } ) + C _ { s _ { 0 } } ( L ) ) _ { sep } .$

Proof of lemma. Prop.3.2 implies that

$$y _ { \gamma } \equiv \sigma x _ { \gamma } - x _ { \gamma } \mod ( L ( v _ { 0 } ) + C _ { s _ { 0 } } ( L ) ) _ { K } .$$

Therefore, there is $X _ { \gamma } \in L _ { sep }$ such that $\sigma X _ { \gamma } - X _ { \gamma } = y _ { \gamma }$ and $X _ { \gamma }$ satisfies the congruence b).

It remains to note that a) is equivalent to the following congruence

$$\sigma X _ { \gamma } - X _ { \gamma } \equiv \gamma \ast e ( q ) - e ( q ) + \delta ^ { 0 } ( \gamma \ast e ( q ), X _ { \gamma } ) - \delta ^ { 0 } ( \sigma X _ { \gamma }, e ( q ) )$$
modulo \((\mathcal{L}_{k}^{(a)}, \mathcal{L}) + C_{s_{0}+1}(\mathcal{L}))_{\text{sep}}\) and by the same modulo we have 
\(d^{0}(\gamma * e^{(q)}, X_{\gamma}) \equiv d^{0}(\gamma * e^{(q)}), x_{\gamma})\) and 
\(d^{0}(\sigma x_{\gamma}, e^{(q)}) \equiv d^{0}(\sigma x_{\gamma}, e^{(q)}).\) \(\square\)

The element \(y_{\gamma}\) can be uniquely written as

\[
y_{\gamma} = \sum_{m \geq 0} t^{-aq^{m}}l_{am} + l_{0},
\]

where all \(l_{am} \in \mathcal{L}_{k}\) and \(l_{0} \in \mathcal{L}_{O}\) (and \(O = k[[t]] \subset K\)). By Prop.1.3
the ideal \(\mathcal{L}_{k}^{(a)}, \mathcal{L}) + C_{s_{0}+1}(\mathcal{L})\) appears as the minimal ideal in the set of all ideals \(\mathcal{I}\) such that:

- \(\mathcal{I} \supset [\mathcal{L}_{k}^{(a)}, \mathcal{L}) + C_{s_{0}+1}(\mathcal{L})];\)
- if \(a \in \mathbb{Z}^{+}(p)\) and \(a \geq qv_{0} - b^{*}\) then \(l^{(a)} := \sum_{m \geq 0} \sigma^{-m}l_{am} \in \mathcal{I}_{k}.\)

**Proposition 3.4.** \(\mathcal{L}(s_{0}+1) \subset \mathcal{L}_{k}^{(a)} + C_{s_{0}+1}(\mathcal{L}),\) or (equivalently) if \(a \geq s_{0}v_{0}\) then all 
\(D_{am} \in \mathcal{L}_{k}^{(a)} + C_{s_{0}+1}(\mathcal{L}_{k}).\)

**Proof.** We have \(e^{(q)}, \gamma * e^{(q)} \in \mathcal{N}^{(q)}\) and \(\gamma * e^{(q)} - e^{(q)} \in t^{b^{*}}\mathcal{N}^{(q)}.\) Then from Prop.3.2 we obtain that \(y_{\gamma} \equiv \gamma * e^{(q)} - e^{(q)}\) modulo

\[
[t^{b^{*}}\mathcal{N}, \mathcal{N}] \subset (t^{b^{*}}\mathcal{N}) \cap C_{2}(\mathcal{L})_{k} \subset \sum_{2 \leq s_{1} + s_{2} \leq s_{0}} \mathcal{L}(s_{1}, \mathcal{L}(s_{2}))m^{t^{-(s_{1}+s_{2}+1)b^{*}}} + \mathcal{L}(s_{0} + 1)_{k} \cap C_{2}(\mathcal{L})_{k}
\]

\(\subset t^{-(s_{0}+1)b^{*}} \mathcal{L}_{m} + \mathcal{L}(s_{0} + 1)_{k} \cap C_{2}(\mathcal{L})_{k}.
\)

**Lemma 3.5.** \(\mathcal{L}(s_{0}+1) \cap C_{2}(\mathcal{L}) \subset \mathcal{L}_{k}^{(a)} \cap C_{2}(\mathcal{L}) + C_{s_{0}+1}(\mathcal{L}).\)

**Proof of lemma.** From the definition of \(\mathcal{L}(s_{0}+1)\) it follows that the \(k\)-module \(\mathcal{L}(s_{0}+1)_{k} \cap C_{2}(\mathcal{L})_{k}\) is generated by the commutators

\[
[\ldots [D_{a_{1}n_{1}}, D_{a_{2}n_{2}}], \ldots, D_{a_{n}n_{r}}]
\]

such that \(r \geq 2\) and \(\text{wt}(D_{a_{1}n_{1}}) + \cdots + \text{wt}(D_{a_{n}n_{r}}) \geq s_{0}+1.\)

Here for \(1 \leq i \leq r, \text{wt}(D_{a_{i}n_{i}}) = s_{i},\) where \((s_{i} - 1)v_{0} \leq a_{i} < s_{i}v_{0}.\) Hence, if \(s_{i} := \min\{s_{i}, s_{0}\}\) then \(\sum_{s} s_{i} \geq s_{0} + 1\) (use that \(r \geq 2).\)

By inductive assumption all \(D_{a_{i}n_{i}} \subseteq \mathcal{L}(s_{i})_{k} \subset \mathcal{L}_{k}^{(a)} + C_{s_{i}}(\mathcal{L})_{k}\) and, therefore, our commutator belongs to \(\mathcal{L}_{k}^{(a)} + C_{s_{0}+1}(\mathcal{L})_{k}.\) It remains to note that \((\mathcal{L}_{k}^{(a)} + C_{s_{0}+1}(\mathcal{L})) \cap C_{2}(\mathcal{L}) = \mathcal{L}_{k}^{(a)} \cap C_{2}(\mathcal{L}) + C_{s_{0}+1}(\mathcal{L}).\)

The lemma is proved. \(\square\)

Lemma 3.5 implies that for \(a \geq (s_{0} - 1)b^{*},\) all \(l^{(a)}\) modulo the ideal 
\(\mathcal{L}_{k}^{(a)} \cap C_{2}(\mathcal{L})_{k} + C_{s_{0}+1}(\mathcal{L})_{k}\) appear as linear combinations of the linear terms of \(\gamma * e^{(q)} - e^{(q)}.\) More precisely, this can be stated as follows.

Let

\[
\sum_{a \in \mathbb{Z}^{+}(p)} t^{-aq}(E(at^{b^{*}}) - 1)D_{a_{0}} = \sum_{a', u} \alpha(a', u)t^{-qa^{*}+ub^{*}} D_{a'_{0}},
\]
where \( a' \) and \( u \) run over \( \mathbb{Z}^+(p) \) and \( \mathbb{N}, \) resp., and all \( \alpha(a', u) \in \mathbb{F}_p \) (note that \( \alpha(a', 1) = a' \)).

**Lemma 3.6.** If \( a \geq (s_0 - 1)b^* \) then

\[
 l^{(a)} = \sum_{m \geq 0} \alpha(a', u) D_{a',-m} \mod (\mathcal{L}_{k}^{(v_0)} \cap C_2(\mathcal{L})_k + C_{s_0+1}(\mathcal{L})_k).
\]

**Proof of Lemma.** Suppose \( a_0 \in \mathbb{Z}^0(p) \) satisfies the following inequality \( a_0 \geq s_0v_0 \). Then \( a = qa_0 - b^* \geq (s_0 - 1)b^* \) and \( l^{(a)} \) is congruent to \( a_0D_{a_0} + \{k\text{-}(pro)linear \ combination \ of \ D_{a'} \ \text{with} \ a' > a_0 \} \).

Since all such \( l^{(a)} \) should belong to \( \mathcal{L}^{(v_0)} + C_{s_0+1}(\mathcal{L}) \), this implies that all \( D_{a_0} \) with \( a_0 \geq s_0v_0 \) (or, equivalently, with the weight \( \geq s_0 + 1 \)) must belong to \( \mathcal{L}^{(v_0)} + C_{s_0+1}(\mathcal{L})_k \).

Proposition is proved.

### 3.3. Interpretation in \( \tilde{\mathcal{L}}^1 \)

It remains to prove that in \( \tilde{\mathcal{L}} \) we have \( \tilde{\mathcal{L}}^{(v_0)} + C_{s_0+1}(\tilde{\mathcal{L}}) = \tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0 + 1) \). By Prop.3.4 and Remark from Sect.3.2 it will be sufficient to establish that

\[
 \tilde{\mathcal{L}}^{(v_0)} + \tilde{\mathcal{L}}(s_0 + 1) \supset \tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0) + 1).
\]

We can use the inductive assumption in the following form

\[
 \tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0) = \tilde{\mathcal{L}}^{(v_0)} + \tilde{\mathcal{L}}(s_0).
\]

By the definition of \( \tilde{\mathcal{L}}[v_0] \) and Prop.2.15 the ideal \( \tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0 + 1) \) appears as the minimal ideal in the set of all ideals \( \mathcal{I} \) of \( \tilde{\mathcal{L}} \) such that:

- \( \mathcal{I} \supset \tilde{\mathcal{D}}(s_0 + 1) := [\tilde{\mathcal{L}}[v_0], \tilde{\mathcal{L}}] + \tilde{\mathcal{L}}(s_0 + 1) = [\tilde{\mathcal{L}}^{(v_0)}, \tilde{\mathcal{L}}] + \tilde{\mathcal{L}}(s_0 + 1); \)
- if \( \iota \in \mathfrak{g}_0^0(p) \) then \( \mathfrak{v}_k B_{k}^1(D_{d_0}) \in \mathcal{I}_k \).

Prove that for any \( \iota \in \mathfrak{g}_0^0(p) \), \( \mathfrak{v}_k B_{k}^1(D_{d_0}) \in \tilde{\mathcal{L}}^{(v_0)} + \tilde{\mathcal{L}}(s_0 + 1)_k \) or, equivalently, for any \( \gamma 
eq 0 \),

\[
 \mathfrak{v}_k (A_{\gamma}^\dagger - \text{id} \mathcal{L}_x)(D_{d_0}^\dagger) \in \tilde{\mathcal{L}}^{(v_0)} + \tilde{\mathcal{L}}(s_0 + 1)_k.
\]

Fix \( \gamma 
eq 0 \) and consider equality (3.2).

By Prop.2.13, \( (\mathcal{V} A_{\gamma}^\dagger \otimes \text{id}_K)^{e_{\gamma}^\dagger} \equiv \tilde{e}^{(q)} \mod (\tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0))_K \). Hence, there is \( Z_\gamma \in (\tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0))_{\text{sep}} \) such that

\[
 \sigma Z_\gamma - Z_\gamma = (\mathcal{V} A_{\gamma}^\dagger \otimes \text{id}_K)^{e_{\gamma}^\dagger},
\]

and we obtain (use that \( (\tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0 + 1)) \mod \tilde{\mathcal{D}}(s_0 + 1) \) is abelian)

\[
 \gamma \ast \tilde{e}^{(q)} = \sigma (\bar{x}_\gamma \circ Z_\gamma) \circ \tilde{e}^{(q)} \circ (-(\bar{x}_\gamma \circ Z_\gamma)) \mod \tilde{\mathcal{D}}(s_0 + 1)_{\text{sep}}.
\]

Therefore, the ideal \( \tilde{\mathcal{L}}^{(v_0)} + \tilde{\mathcal{L}}(s_0 + 1) \supset \tilde{\mathcal{D}}(s_0 + 1) \) is the minimal in the family of all ideals \( \mathcal{I} \) such that

- \( \tilde{\mathcal{L}}[v_0] + \tilde{\mathcal{L}}(s_0 + 1) \supset \mathcal{I} \supset \tilde{\mathcal{D}}(s_0 + 1); \)
\[ v(Z_{\gamma} \mod \mathcal{I}_{\text{sep}}/K) < qv_0 - b^* \] (use that \( \bar{x}_{\gamma} \) is defined over \( K \)).

By Prop. 2.15 we have the following congruence modulo \( \bar{D}(s_0 + 1)_K \)
\[ (\mathcal{V}(A_1^t - \text{id}_{L^t}) \otimes \text{id}_K) e^\dagger \equiv \sum_{\text{ch}(i)=1} t^{-i} \mathcal{V}(A_1^t - \text{id}_{L^t}) D_{0i}^t. \]

For any \( t \in \mathbb{N}_0(p) \), consider \( W_{\gamma_t} := \mathcal{V}_k(A_1^t - \text{id}_{L^t})k D_{0i}^t \in \bar{L}_k \).

Recall that \( \bar{L}[v_0] + \bar{L}(s_0 + 1)_k \) is generated by all \( W_{\gamma_t} \) and the elements of \( \bar{D}(s_0 + 1)_k \). Then
\[ Z_{\gamma_t} \equiv \sum_{\text{ch}(i)=1} Z_{\gamma_t} \mod \bar{D}(s_0 + 1)_{\text{sep}}, \]
where \( Z_{\gamma_t} \in (\bar{L}[v_0] + \bar{L}(s_0 + 1)/\bar{D}(s_0 + 1))_{\text{sep}}, s Z_{\gamma_t} - Z_{\gamma_t} = t^{-i} W_{\gamma_t} \) and \( W_{\gamma_t} \mod \bar{D}(s_0 + 1)_k \in (\bar{L}[v_0] + \bar{L}(s_0 + 1)/\bar{D}(s_0 + 1))_k \).

Note that all above \( Z_{\gamma_t} \) come essentially from elementary Artin-Schreier equations. More precisely, suppose \( \{ \omega_j \} \) is a (finite) \( F_p \)-basis of \( (\bar{L}[v_0] + \bar{L}(s_0 + 1)/\bar{D}(s_0 + 1)) \). Then for some \( w_{\gamma_t-j} \in k \), \( W_{\gamma_t} = \sum w_{\gamma_t-j} \omega_j \) and \( Z_{\gamma_t} = \sum z_{\gamma_t} \omega_j \), where \( z_{\gamma_t-j} - z_{\gamma_t} = w_{\gamma_t-j} t^{-i} \). In particular, for any fixed \( t \) (and \( \gamma \)), \( \mathcal{K}(Z_{\gamma_t}) \) is a composit of all \( \mathcal{K}(z_{\gamma_t}) \). Therefore, \( \mathcal{K}(Z_{\gamma_t})/\mathcal{K} \) is an elementary abelian \( p \)-extension, which is either trivial or has only one ramification number \( \ell p^{-v_{p(i)}} \).

This immediately implies that

— \( \mathcal{K}(Z_{\gamma_t} \mod \mathcal{I}_{\text{sep}})/\mathcal{K} \) coincides with the composite of all \( \mathcal{K}(Z_{\gamma_t} \mod (\mathcal{I}/\bar{D}(s_0 + 1))_{\text{sep}})/\mathcal{K} \) (use that for different \( t \) these extensions are linearly disjoint because by Prop. 2.8a) their ramification numbers are different).

In particular,

— if \( W_{\gamma_t} \notin \mathcal{I}_k \) then the field \( \mathcal{K}(Z_{\gamma_t} \mod (\mathcal{I}/\bar{D}(s_0 + 1))_{\text{sep}})/\mathcal{K} \) is a finite abelian \( p \)-extension with only one ramification number \( \ell p^{-v_{p(i)}} \);

— by Prop. 2.8a), the ramification numbers of different non-trivial extensions \( \mathcal{K}(Z_{\gamma_t} \mod (\mathcal{I}/\bar{L}^*(s_0 + 1))_{\text{sep}})/\mathcal{K} \) are different.

As a result, the biggest upper ramification number of the field extension \( \mathcal{K}(Z_{\gamma_t} \mod \mathcal{I}_{\text{sep}})/\mathcal{K} \) coincides with \( \max \{ \ell p^{-v_{p(i)}} \mid W_{\gamma_t} \notin \mathcal{I}_k \} \).

By Prop. 2.8b), if \( t \in \mathbb{N}_0(p) \) then \( \ell p^{-v_{p(i)}} \geq qv_0 - b^* \). This implies that the biggest upper ramification number \( v(\mathcal{K}(Z_{\gamma_t} \mod \mathcal{I}_{\text{sep}})/\mathcal{K}) < qv_0 - b^* \) if and only if all \( W_{\gamma_t} \notin \mathcal{I}_k \), i.e. \( \mathcal{I} = \bar{L}[v_0] + \bar{L}(s_0 + 1) \).

Theorem 3.1 is completely proved.
4. CONSTRUCTION OF EXPLICIT GENERATORS OF $\hat{L}[v_0]$

4.1. Choice of $e \in L_K$. In [1, 2, 3] we fixed the group isomorphism $G_{<p} \simeq G(\mathcal{L})$ induced by the epimorphism $\eta_e = \pi_f(e) : \mathcal{G} \to G(\mathcal{L})$ via a special choice of $e \in L_K$. In this paper we use more general element $e$ by assuming that $e$ satisfies the following identities:

(Io) $\eta(a_1) = 1$;

(IIe) if $0 \leq s_1 \leq s < p$ then

$$\eta(a_1, \ldots, a_{s_1})\eta(a_{s_1+1}, \ldots, a_s) = \sum_{\pi \in I_{s,s}} \eta(a_{\pi(1)}, \ldots, a_{\pi(s)}),$$

where $I_{s,s}$ consists of all permutations $\pi$ of order $s$ such that the sequences $\pi^{-1}(1), \ldots, \pi^{-1}(s_1)$ and $\pi^{-1}(s_1+1), \ldots, \pi^{-1}(s)$ are increasing (i.e. $I_{s,s}$ is the set of all “insertions” of the ordered set $\{1, \ldots, s_1\}$ into the ordered set $\{s_1+1, \ldots, s\}$).

Assumption Io means that $e$ satisfies (1.1) from Sect.1. Assumption Ile means that

$$\Delta(\exp(e)) \equiv \exp(e) \otimes \exp(e) \mod (J_K \otimes 1 + 1 \otimes J_K)^p,$$

i.e. $\exp(e)$ is diagonal modulo degree $p$. This means that $e$ is a $k$-linear combination of the commutators $t^{-(a_1+\cdots+a_s)}[\ldots[D_{a_1,0}, \ldots], D_{a_s,0}]$. In particular, $e$ satisfies the assumption (1.3) from Sect.1 and the compatibility (2.5) can be easily satisfied. Therefore, we can use Theorem 3.1 to obtain generators of the ramification ideal $L[v_0]$. Note that in the majority of applications of the results from [1, 2, 3] we used the simplest choice $e = \sum_{a \in \mathbb{Z}^0(p)} t^{-a}D_{a,0}$, where all $\eta(a_1, \ldots, a_s) = 1/s!$.

4.2. Statement of the main result. For $\bar{a} = (a_1, \ldots, a_s)$ with all $a_i \in \mathbb{Z}^0(p)$, set $\eta(\bar{a}) = \eta(a_1, \ldots, a_s)$.

Definition. Let $\bar{n} = (n_1, \ldots, n_s)$ with $s \geq 1$. Suppose there is a partition $0 = i_0 < i_1 < \cdots < i_r = s$ such that if $i_j < u \leq i_{j+1}$ then $n_u = m_{j+1}$ and $m_1 > m_2 > \cdots > m_r$. Then set

$$\eta(\bar{a}, \bar{n})_s = \sigma^{m_1}\eta(\bar{a}^{(1)})\cdots\sigma^{m_r}\eta(\bar{a}^{(r)}),$$

where $\bar{a}^{(j)} = (a_{i_j+1}, \ldots, a_{i_r})$. If such a partition does not exist we set $\eta(\bar{a}, \bar{n})_s = 0$. (If there is no risk of confusion we just write $\eta(\bar{a}, \bar{n})$ instead of $\eta(\bar{a}, \bar{n})_s$.)
If \( s = 0 \) we set \( \eta(\bar{a}, \bar{n})_s = 1 \).
For \( \bar{a} = (a_1, \ldots, a_s) \), \( \bar{n} = (n_1, \ldots, n_s) \), set \( D(\bar{a}, \bar{n}) = D_{a_1n_1} \cdots D_{a_sn_s} \).
Note, if \( e_{(N^s, 0)} : = \sigma^{N^s-1}(e) \circ \sigma^{N^s-2}(e) \circ \cdots \sigma(e) \circ e \) then
\[
\exp e_{(N^s, 0)} \equiv \sum_{\bar{a}, \bar{n}} \eta(\bar{a}, \bar{n})D(\bar{a}, \bar{n}) \mod J^p_k.
\]

For \( \alpha \geq 0 \) and \( N \in \mathbb{Z}_{\geq 0} \), introduce \( F^0_{\alpha,-N} \in \mathcal{L}_k \) such that
\[
F^0_{\alpha,-N} = \sum_{1 \leq i < p, a_i, n_i} a_1\eta(\bar{a}, \bar{n})[\ldots [D_{a_1n_1}, D_{a_2n_2}], \ldots, D_{a_sn_s}] .
\]

Here:
- \( \bar{a} = (a_1, \ldots, a_s) \), \( n_1 = 0 \) and all \( n_i \geq -N \);
- \( \alpha = \gamma(\bar{a}, \bar{n}) = a_1p^{n_1} + a_2p^{n_2} + \cdots + a_sp^{n_s} \).

Note that non-zero terms in the above expression for \( F^0_{\alpha,-N} \) can appear only if \( 0 = n_1 \geq n_2 \geq \ldots \geq n_s \) and \( \alpha \in A[p - 1, N] \).

Our result about explicit generators of \( \bar{L}[v_0] \) can be stated in the following form.
Let \( F^0_{\alpha,-N} \) be the image of \( F^0_{\alpha,-N} \) in \( \bar{L}[[v]]_k \).
If \( \iota = qp^m\alpha - p^n b^* \in \mathfrak{A}^0(p) \) is the standard presentation from Sect.2.2 we indicate the dependence of \( \alpha \) and \( m \) on \( \iota \) by setting \( \alpha = \alpha[\iota] \) and \( m = m[\iota] \). Recall that \( \alpha(\iota) \in A[p - 1, m[\iota]] \) and \( m[\iota] < N^* \).

Let \( m(\iota) \) be the maximal non-negative integer such that \( \iota p^{m(\iota)} \leq (p - 1)b^* \). For any \( \iota \in \mathfrak{A}^0(p) \), fix a choice of \( m_{\iota} \geq m(\iota) \).

**Theorem 4.1.** \( \bar{L}[v_0] \) is the minimal ideal in \( \bar{L} \) such that for all \( \iota \in \mathfrak{A}^0(p) \), \( F^0_{\alpha[\iota], -(m[\iota] + m_{\iota})} \in \bar{L}[v_0]_k \).

The proof is given in Sections 4.3-4.6 below.

### 4.3. Recurrent relation
We are going to carry out computations in the enveloping algebra \( \mathcal{A} \) of the Lie algebra \( \mathcal{L} \). Note that the natural embedding \( \mathcal{L}_k \subset \mathcal{A}_k \) remains still injective when taken modulo \( J^p_k \). This can be established similarly to the corresponding property for \( F^p_k \) Lie algebras from Sect.1.2. Using in addition universal properties of the enveloping algebra \( \mathcal{A}_k \) we obtain the following lemma. (We are going to use this properties slightly later below.)

**Lemma 4.2.** Suppose \( I \) is an ideal in the Lie algebra \( L \) of nilpotence class \( < p \). Let \( \mathcal{A} \) be an enveloping algebra of \( L \) and \( J_I := IA - (\text{the corresponding (two-sided)} \) ideal in \( \mathcal{A} \). Then:

a) \( (J_I + J^p) \cap L = I \);

b) \( (JJ_I + J_IJ + J^p) \cap L = [I, L] \).

Coming back to the proof of Theorem 4.1 consider relation (2.5) and choose \( \bar{e}^p = \sum_{i} t^{-i}l^p_i \) such that for all \( \iota \in \mathfrak{A}^0(p) \) with \( \text{ch}(\iota) \geq 1 \),
\[ l^p = D^p_{i0} \] if \( i \in \mathfrak{A}^0(p) \) and \( l^p = 0 \), otherwise. In other words, the part of \( \bar{e}^p \) which “disappears under \( V_K \)” coincides with \( \sum_{\text{ch}(i) \geq 1} t^{-i} D^1_{i0} \).

Note that \( \overline{\text{exp}}(U \ast \bar{e}^{(q)}) \equiv \overline{\text{exp}} \bar{e}^{(q)} + \bar{e}U \mod \tilde{A}_K U^2 \), where

\[
\bar{e} = \sum_{s \geq 1 \atop a_i \in \mathbb{Z}^0(p)} \eta(a_1, \ldots, a_s) t^{-(a_1 + \cdots + a_s)q + b^r} (a_1 + \cdots + a_s) D_{a_10} \cdots D_{a_s0}.
\]

Apply \( \overline{\text{exp}} \) to (2.7) and find a lift \( \bar{x} \) of \( x \) to \( \mathcal{N}^{(q)} \) such that

\[
\overline{\text{exp}} \bar{e}^{(q)} + \bar{e}U \equiv (1 + U \sigma \bar{x}) \left( \overline{\text{exp}} \bar{e}^{(q)} + U \sum_{\text{ch}(i) \geq 1} t^{-i} V_k B^1_k (D^1_{i0}) \right) (1 - U \bar{x})
\]

modulo \( \tilde{J}_K^p U + \tilde{A}_K U^2 \) by proceeding similarly to the proof of Prop.1.4.

Comparing the coefficients for \( U \) and setting \( V_k B^1_k (D^1_{i0}) = V_{i0} \) we obtain in \( \tilde{A}_K \) the following congruence modulo \( \tilde{J}_K^p \)

\[
(\sigma - 1) \bar{x} - \bar{x} + \sum \ell^{-i} V_{i0} \equiv \tilde{E} + (\overline{\text{exp}} \bar{e}^{(q)} - 1) \bar{x} - \sigma \bar{x} (\overline{\text{exp}} \bar{e}^{(q)} - 1).
\]

This equality gives a recurrent procedure to determine uniquely the elements \( \bar{x} \in \sum_{\text{ch}(i) \geq 1} t^{-i} \tilde{L}_k + t^{(p-1)\delta} \mathcal{N}^{(q)} \) and \( V_{i0} \in \tilde{L}_k \).

### 4.4. Some combinatorial identities.

Let

\[-e_{[0,N^s]} := (-e) \circ (-\sigma e) \cdots (-\sigma^{N^s-1} e)\]

and introduce the constants \( \eta^o(\bar{a}, \bar{n}) \in k \) by the following congruence

\[
\overline{\text{exp}}(-e_{[0,N^s]}) \equiv \sum \eta^o(\bar{a}, \bar{n}) D_{(\bar{a}, \bar{n})} \mod \tilde{J}_K^p.
\]

Set \( \eta^o(\bar{a}) := \eta^o(\bar{a}, 0) \).

It can be easily seen that if there is a partition from the definition of \( \eta \)-constants in Sect.4.2 such that \( m_1 < m_2 < \cdots < m_r \) then

\[
\eta^o(\bar{a}, \bar{n}) = \sigma^{m_1} \eta^o(\bar{a}^{(1)}) \sigma^{m_2} \eta^o(\bar{a}^{(2)}) \cdots \sigma^{m_r} \eta^o(\bar{a}^{(r)}).
\]

Otherwise, \( \eta^o(\bar{a}, \bar{n}) = 0 \).

If there is no risk of confusion we just write \( \eta(1, \ldots, s) \) instead of \( \eta(\bar{a}, \bar{n}) \) and use the similar agreement for \( \eta^o \). E.g. the equalities

\[
e_{[N^s, 0]} \circ (-e_{[0,N^s]}) = e_{[0,N^s]} \circ (-e_{[N^s, 0]}) = 0
\]

can be written as the following identities

\[
(\sum_{0 \leq s_1 \leq s} \eta(1, \ldots, s_1) \eta^o(s_1 + 1, \ldots, s) = \delta_{0s}
\]

(here \( \delta_{0s} \) is the Kronecker symbol).
For \(1 \leq s_1 \leq s < p\), consider the subset \(\Phi_{ss_1}\) of substitutions \(\pi\) of order \(s\) such that \(\pi(1) = s_1\) and for any \(1 \leq l \leq s\), the subset \(\{\pi(1), \ldots, \pi(l)\}\) of the segment \([1, s]\) is “connected”, i.e. there exists \(n(l) \in \mathbb{N}\) such that

\[
\{\pi(1), \ldots, \pi(l)\} = \{n(l), n(l) + 1, \ldots, n(l) + l - 1\}.
\]

By definition, we set \(\Phi_{s0} = \Phi_{ss+1} = \emptyset\).

Set \(B_{s1}(1, \ldots, s) = \sum_{\pi \in \Phi_{ss_1}} \eta(\pi(1), \ldots, \pi(s))\).

Note that:

- \(B_0(1, \ldots, s) = B_{s+1}(1, \ldots, s) = 0\);
- \(B_1(1, \ldots, s) = \eta(1, 2, \ldots, s)\);
- \(B_s(1, \ldots, s) = \eta(s, s - 1, \ldots, 1)\).

**Lemma 4.3.** Suppose \(0 \leq s_1 \leq s < p\). Then:

a) \(B_{s1}(1, \ldots, s) + B_{s+1}(1, \ldots, s) = \eta(s_1, \ldots, 1)\eta(s_1 + 1, \ldots, s)\);

b) \(\eta^0(1, \ldots, s) = (-1)^s\eta(s, s - 1, \ldots, 1)\);

c) for indeterminates \(X_1, \ldots, X_s\),

\[
\sum_{1 \leq i_1 \leq s} (-1)^{s_1 - 1}X_{\pi^{-1}(i_1)} \ldots X_{\pi^{-1}(i_s)} = [\ldots [X_1, X_2], \ldots, X_s].
\]

**Proof.** a) Use that all insertions of \((s_1, \ldots, 1)\) into \((s_1 + 1, \ldots, s)\) are “connected” and start either with \(s_1\) or \(s_1 + 1\).

b) Clearly, part a) implies that

\[
\sum_{0 \leq s_1 \leq s} (-1)^{s_1}\eta(s_1, \ldots, 1)\eta(s_1 + 1, \ldots, s) = \delta_{0s}.
\]

Then our statement follows from above relation (4.3).

c) Use that the right-hand side is a linear combination of the monomials \(X_{i_1} \ldots X_{i_s}\) such that for any \(l \geq 1\), \(\{j \mid i_j \in [1, l]\}\) is a “connected” segment of consecutive \(l\) integers. \(\Box\)

### 4.5. Lie elements \(\bar{\mathcal{F}}[t]\) and \(\bar{\mathcal{F}}[t_0]\)

Introduce the following notation:

- \(\bar{n} = (n_1, \ldots, n_s) \geq M\) means that all \(n_i \geq M\). Similarly, we interpret \(\bar{n} > M\), \(\bar{n} \leq M\) and \(\bar{n} < M\).

- \(\gamma(\bar{a}, \bar{n}) = a_1p^{n_1} + \cdots + a_sp^{n_s}\).

For \(1 \leq s_1 \leq s\), let \(\gamma_{[s_1, s]}^*(\bar{a}, \bar{n}) = \sum_{s_1 \leq u \leq s} a_up^{n_u}\) where \(n_u^* = 0\) if \(n_u = M(\bar{n}) := \max\{n_1, \ldots, n_s\}\) and \(n_u^* = -\infty\), otherwise (i.e. in this case \(p^{n_u^*} = 0\)).

For \(t \in \mathbb{Q}_1^0\), introduce

\[
\bar{\mathcal{F}}[t] = \sum_{(\bar{a}, \bar{n})} \sum_{1 \leq s_1 \leq s} \eta^0(1, \ldots, s_1 - 1)\eta(s_1, \ldots, s)\gamma_{[s_1, s]}^*(\bar{a}, \bar{n})D(\bar{a}, \bar{n}) \in \bar{\mathcal{A}}_k.
\]
Here the first sum is taken over all \((\bar{a}, \bar{n})\) of lengths \(1 \leq s < p\) such that \(\bar{n} \geq 0\) and \(\gamma(\bar{a}, \bar{n}) - p^{M(\bar{n})}b^* = \iota\).

Let \(\bar{F}[\iota]_0\) be a part of the above sum taken under the additional condition \(m(\bar{n}) := \min\{n_1, \ldots, n_s\} = 0\).

Note that for any \(\iota \in \mathfrak{A}_0^0\) and \(m \geq 0\),

\[
\sigma^m \bar{F}[\iota]_0 + \sigma^{m-1} \bar{F}[\iota p]_0 + \cdots + \bar{F}[\iota p^m]_0 = \bar{F}[\iota p^m].
\]

In particular, \(\bar{F}[\iota] = \sum_{\iota', m} \sigma^m \bar{F}[\iota']_0\) where the sum is taken over all \(\iota' \in \mathfrak{A}_0^0\) and \(m \geq 0\) such that \(\iota' p^m = \iota\).

**Proposition 4.4.** If \(\iota \in \mathfrak{A}_0^0\) is given in standard notation in the form \(\iota = qp^m\alpha - p^mb^*\) then \(\bar{F}[\iota p^n] = \sigma^{m+n} \bar{F}_0^{0, -(m+n)}\).

**Proof.** We have

\[
\sigma^{-(m+n)} \bar{F}[\iota p^n] = \sum_{1 \leq s_1 \leq s < p} \eta^s(1, \ldots, s_1 - 1) \eta(s_1, \ldots, s) \gamma^*_s(\bar{a}, \bar{n}) D(\bar{a}, \bar{n}),
\]

where the sum is taken for all \((\bar{a}, \bar{n})\) such that \(M(\bar{n}) = 0\) and \(\gamma(\bar{a}, \bar{n}) = \alpha\). By Lemma 4.2, it holds that

\[
\eta^s(1, \ldots, s_1 - 1) = (-1)^{s_1 - 1} \eta(s_1 - 1, \ldots, 1)
\]

and we obtain

\[
\sum_{1 \leq s_1 \leq s < p} (-1)^{s_1 - 1} (B_{s_1 - 1}(1, \ldots, s) + B_{s_1}(1, \ldots, s)) \gamma^*_s(\bar{a}, \bar{n}) D(\bar{a}, \bar{n}) =
\]

\[
\sum_{1 \leq s_1 \leq s < p} (-1)^{s_1 - 1} B_{s_1}(1, \ldots, s)(\gamma^*_s(\bar{a}, \bar{n}) - \gamma^*_s(\bar{a}, \bar{n})) D(\bar{a}, \bar{n}) =
\]

\[
\sum_{1 \leq s_1 \leq s < p} \sum_{1 \leq s_1 \leq s} (-1)^{s_1 - 1} B_{s_1}(1, \ldots, s) a_{s_1} p^n D(\bar{a}, \bar{n}) =
\]

\[
\sum_{1 \leq s_1 \leq s < p} \sum_{1 \leq s_1 \leq s} (-1)^{s_1 - 1} \eta(1, \ldots, \iota) \eta(\pi(1), \ldots, \pi(s)) a_{s_1} p^n D(\bar{a}, \bar{n}) =
\]

\[
\sum_{1 \leq s_1 \leq s < p} \sum_{1 \leq s_1 \leq s} \sum_{\pi \in \mathfrak{F}_{s_1}} (-1)^{s_1 - 1} D_{a_{s_1} b_{s_1}} \cdots D_{a_{s_1} b_{s_1}} =
\]

\[
\sum_{1 \leq s_1 \leq s < p} \sum_{\pi \in \mathfrak{F}_{s_1}} \eta(1, \ldots, s) a_{s_1} D_{a_{s_1} b_{s_1}} \cdots D_{a_{s_1} b_{s_1}} = \bar{F}_0^{0, -(m+n)}.
\]

Proposition is proved.

**Corollary 4.5.** All \(\bar{F}[\iota p^m]\) and \(\bar{F}[\iota p^m]_0\) belong to \(\mathcal{L}_k\).
4.6. Solving recurrent relation (4.2). For \( i \in \mathfrak{A}^0 \), let

- \( m(i) := \max\{m \mid ip^m \in \mathfrak{A}^0\} = \max\{m \mid |ip^m| \leq (p - 1)b^*\} \).

- \( \mathfrak{A}_1^{\text{prim}} = \mathfrak{A}_1^0 \setminus p\mathfrak{A}_1^0 \) (note that \( \mathfrak{A}_1^0(p) = \{i \in \mathfrak{A}_1^{\text{prim}} \mid i > 0\} \)).

As earlier, set \( \bar{D} := [\bar{L}[v_0], \bar{L}] \), \( \bar{L}[v_0](s) := \bar{L}[v_0] + \bar{L}(s) \) and \( \bar{D}(s) := \bar{D} + \bar{L}(s) \). Clearly, \( \bar{L}[v_0] = \bar{L}[v_0](p) \) and \( \bar{D} = \bar{D}(p) \).

**Proposition 4.6.** a) \( \bar{x} \equiv \sum_{i,m} \bar{F}[ip^m]t^{-ip^m} \mod \bar{L}[v_0] \), where the sum is taken over all \( i \in \mathfrak{A}_1^{\text{prim}} \) and \( m \geq 0 \);

b) if \( i \in \mathfrak{A}_1^0(p) \), then \( V_{i0} \equiv -\sigma^{-m(i)} \bar{F}[ip^m(i)] \mod \bar{D}_k \).

**Proof.** Apply induction on \( 1 \leq s_0 < p \) by assuming that a) holds modulo \( \mathfrak{L}[v_0](s_0) \) and deducing from this that a) and b) hold modulo the ideals \( \mathfrak{L}[v_0](s_0 + 1) \) and, resp., \( D(s_0 + 1) \).

Clearly, a) holds with \( s_0 = 1 \).

Suppose \( 1 \leq s_0 < p \) and part a) holds modulo \( \bar{L}[v_0](s_0) \). Applying this assumption to the right-hand side of (4.2) we obtain (use (4.3)) that

\[
\sigma \bar{x} - \bar{x} + \sum_i t^{-i}V_{i0} \equiv -\sum_{i,m} \bar{F}[ip^m]_0 t^{-ip^m} \mod (\bar{J}e_{s_0} + \bar{J}e_{s_0}F + \bar{J}^s) \]

modulo \( (\bar{J}e_{s_0} + \bar{J}e_{s_0}F + \bar{J}^s) \), cf. notation from Lemma 4.2. Here the right-hand sum is taken over all \( i \in \mathfrak{A}_1^{\text{prim}} \) and \( m \geq 0 \).

Since the both parts of congruence (4.4) belong to \( \mathfrak{L}_K \), part b) of Lemma 4.2 implies that (4.4) holds modulo \( \bar{L}[v_0](s_0) \) and \( \bar{L}[v_0](s_0 + 1) \).

**Remark.** Since \( \bar{x}, \sigma \bar{x} \in \bar{N}^{(q)}(\sigma) \), relation (4.4) implies that \( \bar{F}[ip^m]_0 \in \mathfrak{D}(s_0 + 1) \) if \( ip^m > s_0 b^* \).

Apply the operators \( \mathcal{S} \) and \( \mathcal{R} \) from Lemma 2.11 to recover the elements \( \sum_{i \in \mathfrak{A}_1^0(p)} t^{-i}V_{i0} \) and \( \bar{x} \) modulo \( \mathfrak{D}(s_0 + 1) \) as follows.

Let \( \bar{x} = \bar{x}^+ + \bar{x}^- \), where \( \bar{x}^+ \) (resp., \( \bar{x}^- \)) is the linear combination of elements of \( \mathfrak{L}_k \) with positive (resp., negative) powers of \( t \).

If \( t < 0 \) then \( \mathcal{S}(\bar{F}[ip^m]_0 t^{-ip^m}) = -\sum_{n \geq 0} \sigma^n \bar{F}[ip^m]_0 t^{-ip^{n+m}} \) and, therefore, \( \bar{x}^+ = \sum_{i,m} \bar{F}[ip^m]t^{-ip^m} \mod \mathfrak{D}(s_0 + 1) \), where the sum is taken over all \( i \in \mathfrak{A}_1^{\text{prim}} \) and \( m \geq 0 \). This gives part a) modulo \( \mathfrak{L}[v_0](s_0 + 1) \) at the level of positive powers of \( t \).

Let \( i \in \mathfrak{A}_1^0(p) \). Then \( \mathcal{R}(t^{-ip^m} \bar{F}[ip^m]_0) = t^{-i} \sigma^{-m} \bar{F}[ip^m]_0 \) and

\[
V_{i0}t^{-i} = -t^{-i} \sum_{0 \leq m \leq m(i)} \sigma^{-m} \bar{F}[ip^m]_0 = -t^{-i} \sigma^{-m(i)} \bar{F}[ip^m(i)] \mod \mathfrak{D}(s_0 + 1) \].

This gives part b).

As a result, we have the following congruences modulo \( \mathfrak{L}[v_0](s_0 + 1) \).

**Corollary.**}
\[ S(x^-) = - \sum_{i,m} S(t^{-\varphi m} \bar{F}[\varphi m]) = - \sum_{i,m} \sum_{0 \leq m_1 < m} t^{-\varphi m_1} \sigma^{-m+m_1} \bar{F}[\varphi m]_0 \]

\[ \equiv - \sum_{i,m} t^{-\varphi m} \sum_{m_1 > m} \sigma^{-m_1+m} \bar{F}[\varphi m]_0 \equiv - \sum_{i,m} t^{-\varphi m} \sum_{m_1+m_2 = m, m_2 < 0} \sigma^{m_2} \bar{F}[\varphi m]_0 \]

\[ \equiv \sum_{i,m} t^{-\varphi m} \sum_{m_1+m_2 = m, m_1, m_2 \geq 0} \sigma^{m_2} \bar{F}[\varphi m]_0 = \sum_{i,m} \bar{F}[\varphi m]t^{-\varphi m} \]

(here \( \varepsilon \) and \( m \) run over \( \mathbb{A}_1^0(p) \) and, resp., \( \mathbb{Z}_{>0} \)) because

\[ \sum_{m_2+m_1 = m} \sigma^{m_2} \bar{F}[\varphi m_1]_0 = \bar{F}[\varphi m] \equiv \sigma^{m-m(\varepsilon)} V_\varepsilon \mod \bar{L}[v_0](s_0 + 1) k. \]

This completes the induction step for part a).

\[ \square \]

**Corollary 4.7.** \( \bar{L}[v_0] \) is the minimal ideal in \( \bar{L} \) such that \( \bar{L}[v_0]_k \) contains all \( \bar{F}[\varphi m] \).

**Proof.** If \( m > m(\varepsilon) \) then \( \varphi m > (p-1)b^* \) and by remark in the proof of Prop. 4.6, \( \bar{F}[\varphi m]_0 \in D_k \). Therefore, \( \bar{F}[\varphi m]_0 \equiv \bar{F}[\varphi m(\varepsilon)] \mod D_k. \)

Theorem 4.1 is completely proved.

5. **Explicit boundaries**

Theorem 4.1 gives explicit generators for the ideal \( \bar{L}^{(v_0)} \) but these generators still depend on the choice of auxiliary parameters \( \delta_0, b^* \) and \( q = p^{n^*} \). At the very beginning we chose these parameters to perform linearization of the general criterion from Sect. 1.5. This allows us to simplify the proof but the final result still depends on the numbers \( m[\varepsilon] + m(\varepsilon) \) related to the construction of auxiliary coefficients \( t^{-\varepsilon}, \varepsilon \in \mathbb{A}^0 \).

As we have noticed in the Introduction, an analogue of Theorem 4.1 was obtained in [1] by different methods. The appropriate construction of generators \( \bar{F}_{\alpha,-N}^0 \) also involves a boundary \( \bar{N}(v_0) \) (such that \( N \geq \bar{N}(v_0) \), cf. below). The proof of existence of this boundary in [1], Sect.5 and [3], Sect.2 is combinatorial and is not completely constructive.

In Sect. 5.1 we will rely Theorem 4.1 to the main result of [1] and prove that it holds with the boundary \( \bar{N}(v_0) = N^* - 1 \). In Sect. 5.2 we obtain quite reasonable and natural upper estimate for this boundary.
5.1. Relation to the main result from [1].
In [1] (cf. also [3]) we proved the following theorem.

**Theorem 5.1.** There is \( \tilde{N}(v_0) \in \mathbb{N} \) such that if \( N \geq \tilde{N}(v_0) \) is any fixed natural number then \( \mathcal{L}^{(v_0)} \) is the minimal ideal in \( \mathcal{L} \) such that for all \( \alpha \geq v_0 \), \( \mathcal{F}^0_{\alpha,-N} \in \mathcal{L}^{(v_0)} \).

Below we deduce this result from Theorem 4.1 with \( \tilde{N}(v_0) = N^* - 1 \).

**Proof.** Let \( \mathcal{L}^*_N[v_0] \) be the minimal ideal in \( \mathcal{L} \) such that for all \( \alpha \geq v_0 \), \( \mathcal{F}^0_{\alpha,-N} \in \mathcal{L}^*_N[v_0] \). We are going to prove that if \( N \geq \tilde{N}(v_0) := N^* - 1 \) then \( \mathcal{L}^*_N[v_0] = \mathcal{L}^{(v_0)} \).

We can use induction on \( s \geq 1 \) to deduce for \( s \geq sv_0 \) from \( \mathcal{F}^0_{\alpha,-N} \in \mathcal{L}^*_N[v_0] \) that \( D_{a_0} \in \mathcal{L}^*_N[v_0] + \mathcal{L}(s+1)_k \), e.g. cf. Lemma 2.3 from [9]. This implies that \( \mathcal{L}(p) \subset \mathcal{L}^*_N[v_0] \).

Denote by \( \tilde{\mathcal{L}}^*_N[v_0] \) the image of \( \mathcal{L}^*_N[v_0] \) in \( \tilde{\mathcal{L}} = \mathcal{L}/\mathcal{L}(p) \).

Theorem 4.1 shows that \( \tilde{\mathcal{L}}_N[v_0] \) is already the minimal ideal in \( \tilde{\mathcal{L}} \) such that \( \{ \mathcal{F}^0_{\alpha[i],-N} : \alpha \in \mathfrak{A}^0(p) \} \subset \tilde{\mathcal{L}}[v_0] \) (use that \( m[i] + m(i) < N^* \)).

Therefore, it remains to show that for any \( \alpha \geq v_0 \), \( \mathcal{F}^0_{\alpha,-N} \in \tilde{\mathcal{L}}[v_0] \).

Note that \( \mathcal{F}^0_{\alpha,-N} \neq 0 \) implies that \( \alpha \in A[p-1,N] \). Then by Lemma 2.6, \( qp^{M+1}a - M^{p+1}b > (p-1)v_0 \). Therefore, out theorem is implied by the following lemma.

**Lemma 5.2.** Suppose \( M \geq 0 \) and \( qp^{M+1}a - M^{p+1}b > (p-1)b^* \). Then

a) \( \mathcal{F}^0_{\alpha,-M} \in \tilde{\mathcal{L}}[v_0] \);

b) if in addition \( qp^M a - pMb^* > (p-1)b^* \) then
\[
\mathcal{F}^0_{\alpha,-(M-1)} \equiv \mathcal{F}^0_{\alpha,-M} \mod D_k.
\]

**Proof of lemma.** Apply induction on \( M \geq 0 \).

Let \( M = 0 \), then \( \mathcal{F}^0_{\alpha,0} \) is a linear combination of the commutators
\[
[\ldots [D_{a'_i,0},D_{a'_j,0}],\ldots,D_{a'_r,0}],
\]
where \( a'_1 + \cdots + a'_r = \alpha \).

If \( qa - b^* > (p-1)b^* \) then \( \alpha > pb^*/q > (p-1)(v_0 - \delta_0) \) and this implies \( \alpha > (p-1)v_0 \), cf. Lemma 2.6a). Therefore, the above commutators belong to \( \tilde{\mathcal{L}}(p) = 0 \). Indeed, if \( (s_i-1)v_0 \leq a_i < s_iv_0 \) then \( \sum \text{wt}(D_{a_i,0}) = \sum s_i > \alpha/v_0 > p-1 \).

If \( qa - b^* \leq (p-1)b^* \) then \( \iota := qa - b^* \in \mathfrak{A}^0(p) \) and \( m(\iota) = 0 \). Then Theorem 4.1 implies \( \mathcal{F}^0_{\alpha,0} = \mathcal{F}[\iota] \in \tilde{\mathcal{L}}[v_0] \).

Suppose \( M \geq 1 \).

If \( qp^M a - p^Mb^* \leq (p-1)b^* \) then there is \( \iota \in \mathfrak{A}_1(p) \) and \( n \geq 0 \) such that \( \iota p^n = qp^M a - p^Mb^* \in \mathfrak{A}_1 \) and, therefore, \( m(\iota) = n \leq M \). Then by Theorem 4.1 with \( m = M - n \) we obtain \( \mathcal{F}^0_{\alpha,-M} = \sigma^{-M} \mathcal{F}[\iota p^n] \in \tilde{\mathcal{L}}[v_0] \).
Suppose now that \( qp^M \alpha - p^M b^* > (p - 1)b^* \).
We can prove now simultaneously the remaining case of a) and b).

By the induction assumption \( \tilde{F}^0_{a',-(M-1)} \in \tilde{L}[v_0]_k \).

Note that \( \tilde{F}^0_{a,-M} - \tilde{F}^0_{a',-(M-1)} \) is a linear combination of the terms of the form

\[
(5.1) \quad \ldots [\tilde{F}^0_{a'_{r},-(M-1)}, D_{a_{r-1},-M}], \ldots, D_{a_{r},-M}], 
\]

where \( \alpha = \alpha' + (a'_1 + \cdots + a'_r)/p, r \geq 1 \) and \( \alpha' \in A[p-1, M-1] \).

It remains to prove that (5.1) belongs to \( \mathcal{D}_k = [\tilde{L}[v_0], \tilde{L}]_k \).

Let \( s \in \mathbb{N} \) be such that \( sb^*/q > a'_1 + \ldots + a'_r \geq (s-1)b^*/q \).

Then \( a'_1 + \ldots + a'_r \geq (s-1)v_0 \) (cf. Sect. 2.1) and \( \sum \mathop{\text{wt}}(D_{a_{r-i},-M}) \geq s \).

If \( (s_i - 1)v_0 \leq a'_i < s_i v_0 \) then \( \sum s_i > (a'_1 + \ldots + a'_r)/v_0 \geq s - 1 \).

We can assume that \( s \leq p - 2 \) because, otherwise, (5.1) belongs to \( \tilde{L}(p)_k = 0 \). Now the inequality \( a'_1 + \ldots + a'_r < sb^*/q \) implies

\[
(p - 1)b^*/p^M < qa - b^* \leq qa' - b^* + sb^*/p^M 
\]

and, therefore,

\[
(5.2) \quad qp^M \alpha' - p^M b^* > (p - 1 - s)b^*. 
\]

If \( qp^M \alpha' - p^M b^* > (p - 1)b^* \) then by the induction assumption \( \tilde{F}^0_{a'_{r-1},-(M-1)} \in \tilde{L}[v_0]_k \) and (5.1) belongs to \( \mathcal{D}_k \).

If \( qp^M \alpha' - p^M b^* \leq (p - 1)b^* \) then \( \iota' := qp^M \alpha' - p^M b^* \in \mathcal{A}^0_1, m(\iota') = 0 \) (use that \( \iota' > b^* \)) and, therefore, \( \tilde{F}[\iota'] \in \tilde{L}[v_0]_k \). Then inequality (5.2) together with Remark from Sect. 4.6 imply that

\[
\tilde{F}[\iota''p^M-m]_0 \in \mathcal{D}_k + \tilde{L}(p-s)_k. 
\]

Note that \( \alpha' \in A[p-1, M-1] \) implies that \( p^M \alpha' \equiv 0 \mod p \) and \( \iota'/p \in \mathcal{A}^0_1 \). Now the identity \( \tilde{F}[\iota'] = \tilde{F}[\iota']_0 + \sigma \tilde{F}[\iota'/p] \) implies that

\[
\tilde{F}[\iota'/p] \in \tilde{L}[v_0]_k + \tilde{L}(p-s)_k. 
\]

It remains to note that \( \sigma^{M-1} \tilde{F}^0_{\alpha',-(M-1)} = \tilde{F}[\iota'/p] \), this implies that \( \tilde{F}^0_{\alpha',-(M-1)} \in \tilde{L}[v_0]_k + \tilde{L}(p-s)_k \) and, as a result, (5.1) belongs to \( \mathcal{D}_k \).

The lemma is proved.

5.2. Efficient boundaries.

Suppose \( v_0 < v_0 \) is such that for any \( \nu \in (v_0, v_0], G^{(\nu)} = G^{(v_0)} \). The existence of \( v_0 \) follows from the left-continuity property of ramification filtration.

Introduce the weight function \( \mathop{\text{wt}}^b \) on \( \mathcal{L}_k \) such that \( \mathop{\text{wt}}^b(D_0) = 1 \) and if \( s \in \mathbb{N} \) is such that \( (s-1)v_0 < a \leq sv^0_0 \) then \( \mathop{\text{wt}}^b(D_{an}) = s \). Introduce the minimal central filtration \( \{\mathcal{L}^b(s)\}_{s \geq 1} \) on \( \mathcal{L} \) such that \( D_{an} \in \mathcal{L}^b(s)_k \) iff \( \mathop{\text{wt}}^b(D_{an}) \geq s \).

**Proposition 5.3.** \( \mathcal{L}^b(p) \subset \mathcal{L}^{(v_0)} \).
**Proof.** Recall that the central filtration $\mathcal{L}(s)$, $s \geq 1$, from Sect. 1.6 is related to the weight function $w$ such that $w(D_{an}) = s$ iff $(s-1)v_0 \leq a < sv_0$. For any $v \in \{v_0', v_0\}$ let $w_v$ be the weight function such that $w_v(D_{an}) = s$ iff $(s-1)v \geq a < sv$. Denote by $\mathcal{L}_v(s)$ the corresponding central filtration on $L$. (Note that $\mathcal{L}(s) = \mathcal{L}_{v_0}(s)$.) Clearly, if $v_1 < v_2$ then $\mathcal{L}_{v_1}(s) \supset \mathcal{L}_{v_2}(s)$.

If $w_v(D_{an}) \geq s$ then $a > (s-1)v_0^p$; there is $v \in \{v_0', v_0\}$ such that $a \geq (s-1)v$ and, therefore, $w_v(D_{an}) \geq s$. This implies that $\mathcal{L}_v(p) \subset \bigcup_v \mathcal{L}_v(p)$.

Suppose the commutator $A = \ldots [D_{a_1n_1}, \ldots], D_{a,n}] \in \mathcal{L}_v(p)$. Then there are $s_i$ such that $(s_i-1)v \leq a_i < s_i v$ and $\sum_i s_i \geq p$. But then all non-zero $a_i > (s_i-1)v_0^p$ and $w(D_{a,n}) \geq s_i$. If $a_i = 0$ then $w(D_{a,n}) = w_v(D_{a,n}) = 1$, and this implies that $A \in \mathcal{L}_v(p)$.

As a result, $\mathcal{L}_v(p)$ coincides with the union of all $\mathcal{L}_v(p)$.

It remains to note that by Prop.3.4 for any $v \in \{v_0', v_0\}$, $\mathcal{L}_v(p) \subset \mathcal{L}^v = \mathcal{L}^{(v)}$.

For any $\alpha \geq v_0$ choose a minimal $M_\alpha \in \mathbb{Z}_{\geq 0}$ such that $p^{M_\alpha+1}(\alpha - v_0^p) > (p-1)v_0^p$.

Let $\mathcal{L}^{(v_0)}$ be the minimal ideal in $\mathcal{L}$ such that $\mathcal{L}^{(v_0)}[v_0]_k$ contains $\mathcal{L}^{(v)}(p)$ and all $\mathcal{F}_{\alpha-M_\alpha}$ with $\alpha \geq v_0$.

**Theorem 5.4.** $\mathcal{L}^{(v_0)} = \mathcal{L}^{(v_0)}$

**Proof.** By Theorem 5.1, $\mathcal{L}^{(v_0)}$ is the minimal ideal in $\mathcal{L}$ containing $\mathcal{L}_{v_0}(p)_k$ and all $\mathcal{F}_{\alpha, -(N^* - 1)}$, where $\alpha \in A[p-1, N^* - 1]$ and $\alpha \geq v_0$. The number of such $\alpha$ is finite. Therefore, there is $v \in \{v_0', v_0\}$ such that all $p^{M_\alpha+1}(\alpha - v) > (p-1)v$.

Let $\bar{\mathcal{L}}_v = \mathcal{L}/\mathcal{L}_v(p)$ and $\bar{\mathcal{L}}^{(v_0)}_v = \bar{\mathcal{L}}^{(v_0)}_v$ are the images of $\mathcal{L}^{(v)} = \mathcal{L}^{(v_0)}$ in $\bar{\mathcal{L}}_v$. By Lemma 5.2, the images of $\mathcal{F}_{\alpha-M_\alpha}$ and $\mathcal{F}_{\alpha, -(N^* - 1)}$ in $\bar{\mathcal{L}}_v$ are congruent modulo $[\bar{\mathcal{L}}^{(v_0)}_v, \bar{\mathcal{L}}_v]_k$. Therefore, $\bar{\mathcal{L}}^{(v_0)}_v$ is the minimal Lie algebra such that for all $\alpha \geq v_0$, the images of $\mathcal{F}_{\alpha-M_\alpha}$ belong to $\bar{\mathcal{L}}^{(v_0)}_v$.

It remains to note that $\mathcal{L}_v(p) \subset \mathcal{L}^{(v)}(p)$.

Theorem is proved. □

**Remark.** a) The ramification breaks (jumps) of the field extension $K_{<p}/K$ form a discrete subset on $\mathbb{R}_{\geq 1}$ there is only finitely many ramification jumps in $[0, v_0]$; if these jumps are known Theorem 5.4 can be applied to any two successive breaks $v_0^p$ and $v_0$.

b) From Theorem 5.1 it follows that all jumps of the ramification filtration $\{G^{(v)} \mid v < v_0\}$ belong to the set $\mathcal{C}$ consisting of all $a_1 + a_2/p^{n_2} + \cdots + a_{p-1}/p^{n_{p-1}} < V_0$, where all $a_1, n_i \in \mathbb{Z}_{\geq 0}$ and $a_1 + \cdots + a_{p-1} < (p-1)v_0$. Then $\mathcal{C}$ contains a maximal element, cf. Sect.2.1, and this element can be taken as $v_0^p$. 


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