RIEMANN’S HYPOTHESIS AND SOME INFINITE SET OF MICROSCOPIC UNIVERSES OF THE EINSTEIN’S TYPE IN THE EARLY PERIOD OF THE EVOLUTION OF THE UNIVERSE

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Abstract. We obtain in this paper, as a consequence of the Riemann hypothesis, certain class of topological deformations of the graph of the function $|\zeta(\frac{1}{2} + it)|$. These are used to construct an infinite set of microscopic universes (on the Planck’s scale) of the Einstein type.

Dedicated to the 90th anniversary of the A.S. Edington’s book The mathematical theory of relativity.

1. Introduction and the main Result

1.1. Let (see [13], pp. 79, 329)

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right),$$

$$\vartheta(t) = -\frac{t}{2}\ln\pi + \Im\ln\Gamma\left(\frac{1}{4} + \frac{i}{2}t\right) =$$

$$= \frac{t}{2}\ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O\left(\frac{1}{t}\right).$$

We denote by $\{\gamma\}$ the sequence of the roots of the equation

$$Z(t) = 0,$$

and, further, we denote by $\{t_0\}$ the sequence of the roots of the equation

$$Z'(t) = 0, \; t_0 \neq \gamma; \; Z(t_0) \neq 0.$$

Remark 1. On the Riemann hypothesis, the points of the sequences $\{\gamma\}, \{t_0\}$ are separated, i.e.

$$\gamma' < t_0 < \gamma'',$$

where $\gamma', \gamma''$ are neighboring points of the sequence $\{\gamma\}$, (comp. [7]).

We have proved in our paper [9] the following theorem: on the Riemann hypothesis we have

$$\frac{Q(t_0)}{m(t_0)} < t_0\ln^2t_0 \ln_2 t_0 \ln_3 t_0, \; t_0 \to \infty,$$

where

$$\ln_2 t_0 = \ln \ln t_0, \ldots,$$

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(1.3) \[ Q(t_0) = \max\{\gamma'' - t_0, t_0 - \gamma'\}, \quad m(t_0) = \min\{\gamma'' - t_0, t_0 - \gamma'\}. \]

Remark 2. The sequence \(\{t_0\}\) oscillates in a complicated manner around the sequence \(\{\gamma\}\). Consequently, the quotient

\[ \frac{Q(t_0)}{m(t_0)} \]

characterizes the asymmetry of the point \(t_0\) relatively to the points \(\gamma', \gamma''\).

The estimate (1.2) follows from the formula

\[ \frac{\pi}{4} \sim \sum_{\gamma} \frac{t_0}{\gamma^2 - t_0^2}, \quad t_0 \to \infty, \]

(see [7]) and this is the conjugate formula to the Riemann formula

\[ c + 2 - \ln 4\pi = \sum_{\gamma} \frac{1}{\frac{1}{4} + \gamma^2}, \]

where \(c\) is the Euler constant.

1.2. In this paper we will study the following quotient

\[ 0 < \frac{Z(t)}{Z(t_0)}, \quad t \in (\gamma', \gamma''), \quad \gamma' < t_0 < \gamma'', \quad \gamma' \to \infty, \]

where \(Z(t_0)\) is the local maximum or local minimum of the function \(Z(t)\). We make use the estimate (1.2) to prove the following theorem.

**Theorem.** If

\[ \Omega(t_0) = t_0 \ln^4 t_0 + \ln t_0 \ln w(t_0), \]

\[ w(t_0) = \max\left\{ \frac{2}{\omega(t_0)} t_0 \ln^3 t_0, 2 \frac{m(t_0)}{\omega(t_0)} \right\}, \quad \omega(t_0) \in (0, 1), \]

\[ \Delta_1(t_0) = \{1 - \omega(t_0)\}(t_0 - \gamma'), \quad \Delta_2(t_0) = \{1 - \omega(t_0)\}(\gamma'' - t_0), \]

\[ J(t_0) = [t_0 - \Delta_1(t_0), t_0 + \Delta_2(t_0)], \]

then, on the Riemann hypothesis, we have

\[ e^{-\Omega(t_0)} < \frac{Z(t)}{Z(t_0)} \leq 1, \quad t \in J(t_0) \subset (\gamma', \gamma''), \]

where, of course, the lower estimate in (1.5) is the nontrivial result.

**Remark 3.** Since (see (1.4))

\[ \mes\{(\gamma', \gamma'') \setminus J(t_0)\} = \omega(t_0)(\gamma'' - \gamma'), \]

then the interval \(J(t_0)\) is the essential part of the interval \((\gamma', \gamma'')\) if \(\omega(t_0)\) is sufficiently small, for example \(\omega(t_0) = 10^{-58}, \ldots \)
1.3. Now we define the sequence \( \{ \alpha(t_0) \} \) by the following condition
\[
\alpha(t_0) = \frac{\omega^4(t_0)m^4(t_0)}{t_0\Omega(t_0)}, \quad t_0 > K
\]
for sufficiently big \( K > 0 \). Since the Littlewood estimate (see \[5\], p. 237)
\[
\gamma'' - \gamma' < \frac{A}{\ln \ln \gamma'}, \quad \gamma' \to \infty
\]
holds true on the Riemann hypothesis, then by (1.4), (1.7)
\[
\alpha(t_0)\Omega(t_0) < \frac{B}{t_0(\ln \ln t_0)^2}, \quad t_0 \to \infty.
\]
Hence, we have from (1.6) by (1.4), (1.8)
\[
\alpha(t_0)\Omega(t_0) = 1 + O\left(\frac{1}{t_0}\right), \quad t \in J(t_0), \quad t_0 \to \infty.
\]
\[\text{i.e. we obtain from (1.5) by (1.9)}\]
\[
\sum_{\gamma} \frac{\Omega(t)}{Z(t_0)} = 1 + O\left(\frac{1}{t_0}\right), \quad t \in J(t_0), \quad t_0 \to \infty.
\]
\[\text{i.e. we have the following}\]

**Corollary 1.** On the Riemann hypothesis the following asymptotic formula

\[
\sum_{\gamma} \frac{\Omega(t)}{Z(t_0)} = 1 + O\left(\frac{1}{t_0}\right), \quad t \in J(t_0), \quad t_0 \to \infty.
\]

2. **Formulae for the logarithmic derivatives of the function \( Z(t) \) and some lemmas**

2.1. The following main formula follows from the Riemann hypothesis (see \[7\], (1))
\[
-\frac{d}{dt} \left\{ \frac{Z'(t)}{Z(t)} \right\} = \sum_{\gamma} \frac{1}{(t - \gamma)^2} + O\left(\frac{1}{t}\right), \quad t \neq \gamma.
\]
Since the series in (2.1) is uniformly convergent on \( J(t_0) \) and \( Z'(t_0) = 0 \), then we obtain by integration of (2.1) in the limits \( t_0, t \in J(t_0) \) (comp. \[8\], (7)) the following

**Formula 1.** We have on the Riemann hypothesis
\[
-\frac{Z'(t)}{Z(t)} = (t - t_0) \sum_{\gamma} \frac{1}{(t - \gamma)(t_0 - \gamma)} + O\left(\frac{|t - t_0|}{t_0}\right), \quad t \in J(t_0).
\]

Since
\[
(t - t_0) \sum_{\gamma} \frac{1}{(t - \gamma)(t_0 - \gamma)} = \sum_{\gamma} \left(\frac{1}{t_0 - \gamma} - \frac{1}{t - \gamma}\right)
\]
then we obtain by integration of (2.2) in the limits \( t_0, t \in J(t_0) \) the following

**Formula 2.** On the Riemann hypothesis
\[
-\ln \frac{Z(t)}{Z(t_0)} = \sum_{\gamma} \left\{ \frac{t - t_0}{t_0 - \gamma} - \ln \left| \frac{t - \gamma}{t_0 - \gamma} \right| \right\} + O\left(\frac{(t - t_0)^2}{t_0}\right), \quad t \in J(t_0).
\]
2.2. Next, the following lemmas hold true.

Lemma 1. \[ \sum \frac{1}{(t - \gamma)^2} = O \left\{ \frac{\ln t_0}{\omega^2(t_0)m^2(t_0)} \right\}, \quad t \in J(t_0). \]

Lemma 2. On the Riemann hypothesis we have \[ \left\{ \frac{Z'(t)}{Z(t)} \right\}^2 = O \left\{ \frac{\ln^2 t_0}{\omega^4(t_0)m^4(t_0)(\ln \ln t_0)^2} \right\}, \quad t \in J(t_0). \]

Since \( \omega(t_0), m(t_0) \in (0, 1), \Omega(t_0) > t_0 \ln^4 t_0, t_0 \to \infty \) (see (1.4), (1.8), then we obtain from (2.4), (2.5) by (1.6) the following

Lemma 3. On the Riemann hypothesis we have \[ \sum \frac{1}{(t - \gamma)^2} \left\{ \frac{Z'(t)}{Z(t)} \right\}^2 = O \left\{ \frac{\ln^2 t_0}{\omega^4(t_0)m^4(t_0)} \right\}, \quad t \in J(t_0). \]

\[ \alpha(t_0) \sum \frac{1}{(t - \gamma)^2}, \quad \alpha^2(t_0) \left\{ \frac{Z'(t)}{Z(t)} \right\}^2 = O \left( \frac{1}{t_0} \right), \quad t \in J(t_0), \quad t_0 \to \infty. \]

3. **Main equations of the relativistic cosmology and their incompleteness**

3.1. Let us remind the Einstein’s equations for the gravitation

\[ R^\mu{}^\nu - \frac{1}{2} g^\mu{}^\nu R + g^\mu{}^\nu \Lambda = -\kappa c^2 T^\mu{}^\nu, \]

where \( T^\mu{}^\nu = \left( \rho + \frac{p}{c^2} \right) u^\mu u^\nu - g^\mu{}^\nu \frac{p}{c^2}, \)

is the energy-momentum tensor. In the case

\[ ds^2 = dt^2 - \frac{R^2(t) \, dv^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{(1 + kr^2)^2}, \quad k = -1, 0, 1, \]

\[ u^1 = u^2 = u^3 = 0, \quad u^4 = 1 \]

we obtain the fundamental equations of the relativistic cosmology (comp. [6], p. 209)

\[ \kappa c^2 \rho = \frac{3}{R^2} (k c^2 + R^2) - \Lambda, \quad R' = \frac{dR}{dt}, \ldots \]

\[ \kappa p = -\frac{2R^2}{R} - \frac{R c^4 R^2}{R^2} - k c^2 + 2 R^2 + \Lambda. \]

In (3.1) we have: \( R = R(t) \) is the radius of the Universe, \( \rho = \rho(t) \) and \( p = p(t) \) denote the density and the pressure of the cosmic matter, and \( \Lambda \) stands for the cosmological constant, \( \kappa \) is the Einstein’s gravitational constant, and finally \( c \) is the velocity of the light in the vacuum.
Remark 4. It is clear that the system of equations (3.1) is incomplete. Namely, to make it complete we have to postulate some state equation

\[(3.2) \quad G(\kappa_c^2 \rho, \kappa p) = 0; \quad \kappa p = g(\kappa c^2 \rho), \]

and after this we can solve the system of equations (3.1).

For example, when we postulate the state equation

\[\kappa p = (\kappa c^2 \rho)^{3/17}; \quad \left(\frac{3}{17} \to a \in (0, 1)\right)\]

then we obtain from (3.1) the following agreeable differential equation for the function \(R(t)\) (comp. (3.2)

\[(3.3) \quad \left(- \frac{2R''}{R} - \frac{R'^2}{R^2} - \frac{k c^2}{R^2} + \Lambda\right)^{17} = \left\{ \frac{3}{R^2} (k c^2 + R'^2) - \Lambda \right\}^3.\]

3.2. For the purpose of this paper we will suppose in (3.1) that \(k = 1\) (the spherical geometry) and \(\Lambda > 0\). Consequently, we will study the equations

\[(3.4) \quad k\rho = 3 \left(\frac{R'}{R}\right)^2 + 3 \frac{c^2}{R^2} - \Lambda,\]

\[\kappa p = - \frac{2R''}{R} - \left(\frac{R'}{R}\right)^2 - \frac{c^2}{R^2} + \Lambda.\]

Remark 5. In this paper:

(a) we will postulate (instead of (3.2) infinite set of the lines

\[(3.5) \quad R(t) = h \left(\zeta \left(\frac{1}{2} + it\right)\right), \quad t \to \infty,\]

(b) after this we will define the physical domain for the function \(R(t)\),

(c) finally, we will study the set of the corresponding state equations

\[\kappa p = g(\kappa c^2 \rho),\]

(comp. (2) and the papers [10] – [12]).

Remark 6. These are the reasons for our postulate (3.5):

(a) the aesthetic criterion based on the internal connection of the Riemann ideas with itself

\[\text{Riemann} \quad \begin{cases} \text{Riemann’s geometry} & \rightarrow \text{Einstein’s theory} \\
\text{Riemann’s zeta-function} & \rightarrow \text{Riemann’s hypothesis} \\
\end{cases}\]

(i. e. we wish to find some binding ↑).

(b) the almost random distribution of the members of the sequence \(\{\gamma'' - \gamma'\}\),

(comp., for example, the graph of the function \(Z(t)\) in the neighborhood of the first Lehmer pair of the zeroes, [3], p. 296),

(c) the Eddington discovery of the instability of the Einstein’s spherical world.
3.3. In the case of the Einstein’s universe (1917) – the first cosmological application of the Einstein’s theory of gravitation – we have (comp. \(3.2\))

\[ p(t) = 0, \quad R(t) = R_0, \]

and, from \(3.3\) we obtain

\[ R(t) = R_0 = \frac{c}{\sqrt{\Lambda}}, \quad k c^2 \rho(t) = 2\Lambda. \]  

Consequently, the Einstein’s universe is described by the following triple

\[ \{ R(t), k c^2 \rho(t), k \rho(t) \} = \left\{ \frac{c}{\sqrt{\Lambda}}, 2\Lambda, 0 \right\}. \]

Remark 7. Let us remind the instability of the Einstein’s universe. This important fact was discovered by Eddington in 1930, (see \[2\], comp. \[1\], pp. 463-479). Namely, from \(3.3\), in the case \( p(t) = 0 \) (see \(3.7\)) we obtain

\[ 6R'' = (2\Lambda - k c^2 \rho)R. \]

Consequently, if we have a small perturbation of the density \( \rho(t) \) in \(3.7\) such that

\[ k c^2 \rho < 2\Lambda, \]

then the expansion (see \(3.3\)) of the universe follows and, if we have the small perturbation such that

\[ k c^2 \rho > 2\Lambda, \]

then the contraction of the universe follows.

4. A NEW CLASS OF MATHEMATICAL UNIVERSES; SOME KINDRED OF THE EINSTEIN’S UNIVERSE

4.1. In this paper we use the following postulate (see \(1.11\), \(3.5\), comp. \[10\], \[11\])

\[ R(t) = R(t; t_0, \Lambda, \mu) = \mu \frac{c}{\sqrt{\Lambda}} \left\{ \frac{Z(t)}{Z(t_0)} \right\}^{\alpha(t_0)} = \]

\[ = \mu \frac{c}{\sqrt{\Lambda}} \left( 1 + \mathcal{O} \left( \frac{1}{t_0} \right) \right), \quad t \in J(t_0), \quad \mu > 0. \]

Since by \(4.8\)

\[ \frac{R'}{R} = \alpha(t_0) \frac{Z'(t)}{Z(t)} - \frac{dt}{dt} \left( \frac{R'(t)}{R(t)} \right) = \alpha(t_0) \frac{dt}{dt} \left\{ \frac{Z'(t)}{Z(t)} \right\}, \]
then from (3.4) by (1.11), (2.6), (4.1), (4.2) we have

\[
\kappa c^2 \rho(t) = 3 \alpha^2(t_0) \left( \frac{Z'(t)}{Z(t)} \right)^2 + \frac{3 \Lambda}{\mu^2} \left( \frac{Z(t_0)}{Z(t)} \right)^{2\alpha(t_0)} - \Lambda =
\]

\[
\left( \frac{3}{\mu^2} - 1 \right) \Lambda + O \left( \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right),
\]

\[
\kappa p(t) = \alpha(t_0) \sum_{\gamma} \frac{1}{(t-\gamma)^2} - 3 \alpha^2(t_0) \left( \frac{Z'(t)}{Z(t)} \right)^2 - \frac{\Lambda}{\mu^2} \left( \frac{Z(t_0)}{Z(t)} \right)^{\alpha(t_0)} + \Lambda + O \left( \frac{1}{t_0} \right) -
\]

\[
\left( 1 - \frac{1}{\mu^2} \right) \frac{1}{t_0} + O \left( \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right).
\]

Consequently, we have the following.

**Corollary 2.** On the Riemann hypothesis we have the following infinite set of a mathematical universes

\[
R(t; t_0, \Lambda, \mu) = \mu \frac{c}{\sqrt{\Lambda}} + O \left( \frac{1}{t_0} \right),
\]

\[
(4.3)
\]

\[
\kappa c^2 \rho(t; t_0, \Lambda, \mu) = \left( \frac{3}{\mu^2} - 1 \right) \Lambda + O \left( \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right),
\]

\[
\kappa p(t; t_0, \Lambda, \mu) = \left( 1 - \frac{1}{\mu^2} \right) \Lambda + O \left( \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right),
\]

\[
t \in J(t_0), \; \gamma' < t_0 < \gamma'', \; \mu > 0, \; t_0 \to \infty.
\]

4.2. The extremal state equations like these

\[
p(t) = c^2 \rho(t), \quad p(t) = -c^2 \rho(t)
\]

are also used in the relativistic cosmology. We will define the **physical domain** of the universe (4.3) by means of the inequality (comp. [10] – [12])

\[
|p(t)| \leq c^2 \rho(t).
\]

**Definition.** Let

\[
E_1(t; t_0, \Lambda, \mu) = \kappa c^2 \rho - \kappa p,
\]

\[
E_2(t; t_0, \Lambda, \mu) = \kappa c^2 \rho + \kappa p.
\]

Then we will call the set

\[
F(t_0, \Lambda, \mu) = \{ t \in (\gamma', \gamma'') : E_1(t) \geq 0, \; E_2(t) \geq 0, \; \rho(t) > 0 \}, \; t_0 \to \infty
\]

the **physical domain** of the universe (4.3).

Since for

\[
t \in J(t_0), \; t_0 > K > 0,
\]

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where $K$ is sufficiently big, we have (see (4.3), (4.6))

$$E_1 = 2 \left( \frac{2}{\mu^2} - 1 \right) \Lambda + O \left\{ \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right\},$$

$$E_2 = 2 \frac{\mu}{\Lambda} + O \left\{ \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right\},$$

$$\kappa c^2 \rho = \left( \frac{3}{\mu^2} - 1 \right) \Lambda + O \left\{ \left( 1 + \frac{1}{\mu^2} \right) \frac{1}{t_0} \right\},$$

and the inequalities are fulfilled for $\mu \in (0, \sqrt{2})$, then we have the following

**Corollary 3.** On the Riemann hypothesis

(4.7) \quad J(t_0) \subset F(t_0, \Lambda, \mu), \quad \mu \in [\epsilon, \sqrt{2} - \epsilon],

where $\epsilon$ is an arbitrarily small fixed number.

**Remark 8.** The essential part of the interval $(\gamma', \gamma'')$, $\gamma' < t_0 < \gamma''$ belongs to the physical domain $F(t_0)$, (see (4.7) and the Remark 3).

4.3. Next, we obtain from (4.3), (4.7) the following

**Corollary 4.** On the Riemann hypothesis

(4.8) \quad \frac{p(t; t_0, \Lambda, \mu)}{c^2 \rho(t; t_0, \Lambda, \mu)} \sim \frac{\mu^2 - 1}{3 - \mu^2}, \quad \mu \in [\epsilon, \sqrt{2} - \epsilon], \quad t \in J(t_0), \ t_0 \to \infty.

**Remark 9.** Consequently, on the Riemann hypothesis, we have obtained the continuum set of the asymptotically linear state equations (4.8). For example:

- $\mu = \epsilon \rightarrow p \sim -\frac{1}{3} (1 - \epsilon') c^2 \rho$,
- $\mu = \frac{1}{\sqrt{2}} \rightarrow p \sim \frac{1}{3} c^2 \rho$,
- $\mu = 1 \rightarrow p \sim 0$,

in this case we have the so-called incoherent dust (of galaxies), the small generalization of the Einstein’s $p = 0$,

- $\mu = \sqrt{\frac{6}{5}} \rightarrow p \sim \frac{1}{9} c^2 \rho$,
- $\mu = \sqrt{\frac{3}{2}} \rightarrow p \sim \frac{1}{3} c^2 \rho$,

in this case we have the universe filled by the photon gas,

- $\mu = \sqrt{2} - \epsilon \rightarrow p \sim (1 - \epsilon'') c^2 \rho$,

where $0 < \epsilon', \epsilon''$ are arbitrarily small values.
5. An infinite subset of microscopic universes of the Einstein’s type and the condition for the inflationary expansion of Universe

5.1. In the case

\[ \mu = \epsilon \]

with \( \epsilon \) being an arbitrarily small fixed value, we obtain from (5.3) the following infinite subset of the universes

\[
R(t; t_0, \Lambda, \epsilon) = \epsilon \frac{c}{\sqrt{\Lambda}} + O \left( \frac{1}{t_0} \right),
\]

\[
\kappa c^2 \rho(t; t_0, \Lambda, \epsilon) = \left( \frac{3}{c^2} - 1 \right) \Lambda + O \left\{ \left( 1 + \frac{1}{c^2} \right) \frac{1}{t_0} \right\},
\]

\[
\kappa p(t; t_0, \Lambda, \epsilon) = \left( \frac{1}{c^2} - 1 \right) \Lambda + O \left\{ \left( 1 + \frac{1}{c^2} \right) \frac{1}{t_0} \right\},
\]

\[ t \in J(t_0), \gamma' < t_0 < \gamma'', \ t_0 \to \infty. \]

We obtain for the volume of the universes (5.1)

\[
V = V(t_0, \Lambda, \epsilon) = 2\pi^2 R^3 = \frac{2\pi^2 c^3}{\Lambda^{3/2}} \epsilon^3 + O \left( \frac{1}{t_0} \right).
\]

We can introduce also the local time \( \tau \) for \( J(t_0) \), namely

\[ \tau = \tau(t_0) = t - \{ t_0 + \Delta_1(t_0) \}, \]

where

\[ \tau \in [0, \Delta_1(t_0) + \Delta_2(t_0)], \]

and (see (1.4), (1.7))

\[
\Delta_1(t_0) + \Delta_2(t_0) = \{ 1 - \omega(t_0) \} (\gamma'' - \gamma') < \gamma'' - \gamma' < \frac{A}{\ln \ln \gamma'} \to 0
\]

as \( \gamma' \to \infty. \)

Next, we have from (5.1) that

\[
\frac{p}{c^2 \rho} = \frac{(1 - \frac{1}{c^2}) \Lambda + O \left\{ \left( 1 + \frac{1}{c^2} \right) \frac{1}{t_0} \right\}}{(\frac{3}{c^2} - 1) \Lambda + O \left\{ \left( 1 + \frac{1}{c^2} \right) \frac{1}{t_0} \right\}} =
\]

\[
= \frac{(c^2 - 1) \Lambda + O \left\{ \left( 1 + c^2 \right) \frac{1}{t_0} \right\}}{(3 - c^2) \Lambda + O \left\{ \left( 1 + c^2 \right) \frac{1}{t_0} \right\}} =
\]

\[
= \frac{1}{3} + \frac{2c^2}{9 - 3c^2} + O \left( \frac{1}{\Lambda t_0} \right).
\]

Hence, we have the following

**Corollary 5.** On the Riemann hypothesis there is an infinite set of the microscopic (see (5.2), (5.3)) universes (5.1) (a subset of the set (4.3)) such that the state equation (see (5.4))

\[
\frac{p(t; t_0, \Lambda, \epsilon)}{c^2 \rho(t; t_0, \Lambda, \epsilon)} = -\frac{1}{3} + \frac{2c^2}{9 - 3c^2} + O \left( \frac{1}{\Lambda t_0} \right), \ t \in J(t_0), \ t_0 \to \infty.
\]
Remark 10. Let us remind that (see [4], p.13)

\[ L_P = 8.10 \times 10^{-35} \text{cm} \]

is the Planck length, and

\[ T_P = 2.70 \times 10^{-43} \text{s} \]

is the Planck time. In the case

\[ \epsilon \leq \frac{\sqrt{\Lambda}}{c} L_P, \quad \Delta_1(t_0) + \Delta_2(t_0) \leq T_P, \]

we have

\[ R(t; t_0, \Lambda, \epsilon) \leq L_P, \quad \tau \leq T_P, \]

i. e. we have in this case the infinite subset of the universes (5.1) of the Planck scale.

5.2. Next, let us remind (see, for example, [4], p. 40) that the condition for the inflationary expansion of the universe can be formulated as follows

(5.6) \text{INFLATION } \iff \ c^2\rho + 3p < 0 \]

with

\[ \Lambda = 0, \]

otherwise \( \Lambda \) is absorbed into \( c^2\rho \) and \( p \).

Remark 11. Consequently, the era of the inflation (that ends at \( 10^{-43} \text{s} \)) is connected with the negative pressure \( p \) because \( \rho > 0 \) (always), and

\[ \frac{p}{c^2\rho} < -\frac{1}{3}. \]

After this we can make the concluding remark.

Remark 12. The following holds true for the negative pressures:

(a) if

\[ \frac{p}{c^2\rho} \in \left[-1, -\frac{1}{3}\right) \]

(for \(-1 \) see (4.4)) then we have the inflationary expansion of our universe,

(b) if (see (5.5))

\[ \frac{p}{c^2\rho} = -\frac{1}{3} + \frac{2\epsilon^2}{9 - 3\epsilon^2} + \mathcal{O}\left(\frac{1}{M_0}\right) \in \left(-\frac{1}{3}, -\frac{1}{3} + \delta\right), \]

i. e. for a small right \( \delta \)-neighborhood of the point \(-\frac{1}{3}\), we have, on the Riemann hypothesis, the infinite subset of the microscopic universes of the Einstein’s type.

6. Proof of Lemma 1

Let

(6.1) \[ \sum_{\gamma} \frac{1}{(t - \gamma)^2} = \sum_{\gamma \leq \gamma'} + \sum_{\gamma \in (\gamma' - 1, \gamma' + 1)} + \sum_{\gamma' + 1 \leq \gamma}. \]

Since (see (1.4))

\[ t - \gamma' \geq t_0 - \Delta_1(t_0) - \gamma' = \omega(t_0)(t_0 - \gamma'), \]

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and, similarly, 
\[ \gamma'' - t \geq \omega(t_0)(\gamma'' - t_0), \]
then (see (1.4))
\[ |t - \gamma| \geq \omega(t_0)m(t_0), \quad t \in J(t_0), \; \gamma \in (\gamma' - 1, \gamma'' - 1). \]

Next, (comp. [13], p. 178)
\[ \sum_{\gamma \in (\gamma' - 1, \gamma'' + 1)} 1 = O(\ln t_0). \]
Then we obtain by (6.2), (6.3)
\[ \sum_{\gamma \in (\gamma' - 1, \gamma'' + 1)} \frac{1}{(t - \gamma)^2} = O\left(\frac{\ln t_0}{\omega^2(t_0)m^2(t_0)}\right), \quad t \in J(t_0). \]

Since 
\[ |t - \gamma| = \gamma - t > \gamma - \gamma'', \quad t \in J(t_0), \; \gamma'' + 1 \leq \gamma, \]
then (see (6.3), comp. [13], p. 184)
\[ \sum_{\gamma'' + 1 \leq \gamma} \frac{1}{(t - \gamma)^2} < A \ln(\gamma'' + n), \quad n \leq \gamma'' \ln 2, \quad \ln 2 < A \ln t_0, \quad t \in J(t_0). \]

By a similar way one can obtain the following estimate
\[ \sum_{\gamma \leq \gamma'' + 1} \frac{1}{(t - \gamma)^2} < A \ln t_0, \quad t \in J(t_0). \]

Hence, we obtain (2.4) from (6.1) by (6.4) – (6.6).

7. PROOF OF LEMMA 2

First of all, we have by (2.4)
\[ \left| \sum_{\gamma} \frac{1}{(t - \gamma)(t_0 - \gamma)} \right| \leq \left\{ \sum_{\gamma} \frac{1}{(t - \gamma)^2} \right\}^{1/2} \left\{ \sum_{\gamma} \frac{1}{(t_0 - \gamma)^2} \right\}^{1/2} < \frac{\ln t_0}{\omega^2(t_0)m^2(t_0)} \quad t \in J(t_0). \]

Next, by the Littlewood estimate (1.7) we have
\[ |t - t_0| < \gamma'' - \gamma' < \frac{A}{\ln \ln t_0}, \quad t \in J(t_0). \]

Then, by (2.2) we obtain the estimate
\[ \frac{Z'(t)}{Z(t)} = O\left(\frac{\ln t_0}{\omega^2(t_0)m^2(t_0) \ln \ln t_0}\right), \quad t \in J(t_0), \]
and the estimate (2.5) follows.
8. Proof of Theorem

Let

\[ \sum_{\gamma} \left\{ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| \right\} = \sum_{\gamma \leq \gamma' - 1} + \sum_{\gamma' + 1 \leq \gamma} + \sum_{\gamma \in (\gamma' - 1, \gamma'' + 1)} . \]

(A). If \( \gamma \leq \gamma' - 1 \), then we have for \( t \in J(t_0) \) by (1.7) that

\[ \left| \frac{t-t_0}{t_0-\gamma} \right| < \gamma'' < \frac{A}{\ln \ln t_0} . \]

Next,

\[ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| = \frac{t-t_0}{t_0-\gamma} - \ln \left( 1 + \frac{t-t_0}{t_0-\gamma} \right) = \]

\[ = \left( \frac{t-t_0}{t_0-\gamma} \right)^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k+2} \left( \frac{t-t_0}{t_0-\gamma} \right)^k , \]

\[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k+2} \left( \frac{t-t_0}{t_0-\gamma} \right)^k = \frac{1}{2} + O \left( \frac{1}{\ln \ln t_0} \right) . \]

Consequently,

\[ \sum_{\gamma \leq \gamma' - 1} \left\{ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| \right\} = \]

\[ = (t-t_0)^2 \sum_{\gamma \leq \gamma' - 1} \frac{1}{(t_0-\gamma)^2} \left\{ \frac{1}{2} + O \left( \frac{1}{\ln \ln t_0} \right) \right\} , \]

and from this (see (1.7), (6.6))

\[ 0 \leq \sum_{\gamma \leq \gamma' - 1} \left\{ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| \right\} < A \frac{\ln t_0}{(\ln \ln t_0)^2} , \quad t \in J(t_0) . \]

(B). If \( \gamma'' + 1 \leq \gamma \), then we have for \( t \in J(t_0) \), (comp. (8.2))

\[ \left| \frac{t-t_0}{t_0-\gamma} \right| \leq \frac{|t-t_0|}{1 + \gamma'' - t_0} < \gamma'' < \frac{A}{\ln \ln t_0} , \]

and, similarly to (8.3) – (8.5), we obtain

\[ 0 \leq \sum_{\gamma'' + 1 \leq \gamma} \left\{ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| \right\} < A \frac{\ln t_0}{(\ln \ln t_0)^2} , \quad t \in J(t_0) . \]
C. Let
\[ \gamma \in (\gamma', 1, \gamma'' + 1), \quad t \in J(t_0) \]
and
\[ V = \sum_{\gamma \in (\gamma'-1, \gamma''+1)} \left\{ \frac{t-t_0}{t_0-\gamma} - \ln \left| \frac{t-\gamma}{t_0-\gamma} \right| \right\} = \sum_{\gamma \in (\gamma', 1, \gamma''+1)} \frac{t-t_0}{t_0-\gamma} + \sum_{\gamma \in (\gamma'-1, \gamma''+1)} \ln \left| \frac{t_0-\gamma}{t-\gamma} \right| = V_1 + V_2. \] (8.7)

Since
\[ \left| \frac{t-t_0}{t_0-\gamma} \right| \leq \frac{Q(t_0)}{m(t_0)}, \quad t \in J(t_0), \quad \gamma \in (\gamma'-1, \gamma''+1), \]
then (see (1.2), (6.3), (8.7))
\[ |V_1| \leq \sum_{\gamma \in (\gamma'-1, \gamma''+1)} \left| \frac{t-t_0}{t_0-\gamma} \right| < A \ln t_0 + t_0 \ln^3 t_0 = A t_0 \ln^4 t_0. \] (8.8)

Next, by (1.2), (6.2) we have
\[ \left| \frac{t_0-\gamma}{t-\gamma} \right| = \left| 1 + \frac{t-t_0}{t_0-\gamma} \right| \leq 1 + \left| \frac{t-t_0}{t_0-\gamma} \right| < 1 + \frac{Q(t_0)}{\omega(t_0) m(t_0)} < \frac{2 Q(t_0)}{\omega(t_0)^2} \ln^3 t_0, \] (8.9)

Since (see (1.4), (1.7))
\[ |t_0-\gamma| \geq m(t_0), \quad \gamma \in (\gamma'-1, \gamma''+1), \]
\[ |t-\gamma| = |\gamma - t < \gamma'' + 1 - \gamma' = \gamma'' - \gamma' + 1 < 2, \quad \gamma \in [\gamma'', \gamma'' + 1), \]
\[ |t-\gamma| = |t_0 - \gamma < \gamma'' - \gamma' + 1 < 2, \quad \gamma \in (\gamma', 1, \gamma'], \]
then (see (8.9), (8.10))
\[ \frac{1}{2} m(t_0) < \left| \frac{t_0-\gamma}{t-\gamma} \right| < \frac{2}{\omega(t_0)} t_0 \ln^3 t_0. \]

Consequently,
\[ -\ln \frac{2}{m(t_0)} < \ln \left| \frac{t_0-\gamma}{t-\gamma} \right| < \ln \left\{ \frac{2}{\omega(t_0)^2} t_0 \ln^3 t_0 \right\}, \]
(of course, \( m(t_0) \in (0, 1), \quad t_0 \to \infty \), and (see (1.4))
\[ \left| \ln \left| \frac{t_0-\gamma}{t-\gamma} \right| \right| < \ln w(t_0), \quad t \in J(t_0), \quad \gamma \in (\gamma'-1, \gamma''+1). \] (8.11)

Next, (see (6.3), (8.7), (8.11))
\[ |V_2| \leq \sum_{\gamma \in (\gamma'-1, \gamma''+1)} \left| \ln \left| \frac{t_0-\gamma}{t-\gamma} \right| \right| < A \ln t_0 \ln w(t_0). \] (8.12)

Next, (see (8.7), (8.8), (8.12))
\[ |V_1 + V_2| \leq |V_1| + |V_2| < A t_0 \ln^4 t_0 + A \ln t_0 \ln w(t_0), \]
and, of course,
\[ V_1 + V_2 > -A t_0 \ln^4 t_0 - A \ln t_0 \ln w(t_0). \] (8.14)

Finally, the estimate (1.5) follows from (2.3) by (8.1), (8.5), (8.6), (8.13) and (8.14).
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