Entanglement between two qubits one of which interacts with a thermal field

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Abstract

In this paper, we have investigated the entanglement between two dipole coupled two-level atoms. The model, in which only one atom is trapped in an lossless cavity and interacts with single-mode thermal field, and the other one can be spatially moved freely outside the cavity has been carried out. We have considered the effect of the atomic coherence on the entanglement behavior. We have shown that a thermal field might cause high entanglement between the atoms both for coherent and incoherent initial atomic states only for small values of the cavity mean photon number. In the considered model the atoms would get entangled even when both atoms are initially in the excited state. We have also derived that the degree of entanglement is weakly dependent on the strength of dipole-dipole interaction for coherent initial states.

Keywords: Entanglement, Cavity QED, Negativity, Thermal field, Atomic coherence

1. Introduction

Entanglement between separate quantum systems is one of the key problem in quantum mechanics. It plays a central role in quantum information, quantum computation and communication, and quantum cryptography. Several methods of creating entanglement have been proposed involving trapped...
and cooled ions or neutral atoms in cavities, superconducting circuits, spins in solids etc. In order to function optimally various applications require maximally entangled states. Because of decoherence, which is generally related to noise, there is great difficulty in generating and keeping the integrity of a pure entangled states. Although the interaction between the environment and quantum systems can lead to decoherence, it may also be associated with the formation of non-classical effects such as entanglement. Thus, understanding and investigating entanglement of mixed states becomes one of the actual problem of quantum information. Recently, Bose et al. have shown that entanglement can always arise in the interaction of an arbitrary large system in any mixed state with a single qubit in a pure state, and illustrated this using the Jaynes-Cummings interaction of a two-level atom in a pure state with a field in a thermal state at an arbitrary high temperature. Kim et al. have investigated the atom-atom entanglement in the system of two identical two-level atoms with one-photon transition induced by a single-mode thermal field. They showed that a chaotic field with minimal information can entangled atoms which were prepared initially in a separable state. Zhang directly generalizes Kim’s study to the case when the atoms are slightly detuned from the thermal field, and study how the detuning would affect atom-atom entanglement. Zhou et al. have considered the same problem for nonidentical atoms with different couplings. The entanglement between two identical two-level atoms through nonlinear two-photon interaction with one-mode thermal field has been studied by Zhou et al. They showed that atom-atom entanglement induced by nonlinear interaction is larger than that induced by linear interaction. In has discovered that two atoms can be entangled also through nonlinear nondegenerate two-photon interaction with two-mode thermal field. The influence of dipole-dipole interaction on entanglement between two cubits induced by one-mode and two-mode thermal field has been investigated in.

The problem of creating or controlling the atomic entanglement is greatly related to the atomic coherence of population between different levels. The authors of papers have shown that the entanglement between two atoms
induced by one-mode or two-mode thermal field can be manipulated by changing
the initial parameters of the atoms, such as the superposition coefficients and
the relative phases of the initial atomic coherent state and the mean photon
number of the cavity field. They have also discovered that entanglement may be
greatly enhanced due to dipole-dipole interaction in the presence of the atomic
coherence 17.

While, as has been mentioned in 19, the practical applications in quan-
tum information processing require engineering entangled atoms so this expects
operable atoms which can be moved to distance without losing of information.
Recently many schemes have been proposed to realize the engineering entangled
atoms 20-25. Guo and coauthors have proposed a simple scheme to realize
an easily engineered two-atom entangled state. The advantage of this scheme
is only one atom is trapped in a cavity, and the other one can be spatially
moved freely outside the cavity. But the authors have investigated the atom-
atom entanglement only for vacuum initial cavity field. In this paper we study
the entanglement dynamics for model supposed in 19 for thermal cavity field
taking into account the initial atomic coherence.

2. The model

We consider two identical two-level atoms and one-mode quantum electro-
magnetic cavity field. The first atom is trapped in a lossless microcavity and
resonantly interacts with the cavity field of the frequency $\omega$. The second atom
lies beside the first atom out of the cavity. We assume that the distance between
atoms can compare with a wavelength on working transition. In this case the
dipole-dipole interaction should be included. The Hamiltonian of this system
can be written as

$$H = (1/2)\hbar \omega (\sigma^z_1 + \sigma^z_2) + \hbar \omega a^+ a + \hbar g(a^+ \sigma^+_1 a + \sigma^-_1 a^-) + \hbar J(\sigma^+_1 \sigma^-_2 + \sigma^-_1 \sigma^+_2),$$

(1)

where $(1/2)\sigma^z_i$ is the inversion operator for the $i$th atom ($i = 1, 2$), $\sigma^+_i =
|+\rangle_{ii} \langle -|$, and $\sigma^-_i = |\rangle_{ii} \langle +|$ are the transition operators between the excited
|+⟩_i and the ground |−⟩_i states in the i-th atom, a^+ and a are the creation and the annihilation operators of photons of the cavity mode, g is the coupling constant between atom and the cavity field and J is the coupling constant of the dipole interaction between the atoms and |+⟩ and |−⟩ are the excited and the ground states of a single two-level atom. The two-atom wave function can be expressed as a combination of state vectors of the form |v_1, v_2⟩ = |v_1⟩|v_2⟩, where v_1, v_2 = +, −.

We consider that the initial states of atoms are the coherent superposition of the two levels, that is,

|Ψ_1(0)⟩ = \cos θ_1|+⟩_1 + e^{iϕ_1} \sin θ_1|−⟩_1,

|Ψ_2(0)⟩ = \cos θ_2|+⟩_1 + e^{iϕ_2} \sin θ_2|−⟩_2. \tag{2}

Here θ_1 and θ_2 denote the amplitudes of the polarized atoms, and ϕ_1 and ϕ_2 are relative phases of two atoms, respectively. So the initial density matrix for two-atom system can be written as

ρ_A(0) = |Ψ_1(0)⟩⟨Ψ_2(0)|⟨Ψ_1(0)|⟨Ψ_2(0)|.

The initial cavity mode state are assumed to be the thermal one-mode state

ρ_F(0) = \sum_n p_n |n⟩⟨n|.

The weight functions are

p_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}},

where \bar{n}_i is the mean photon number in the i-th cavity mode,

\bar{n}_i = (\exp[\hbar ω_i/k_BT] − 1)^{-1},

k_B is the Boltzmann constant and T is the equilibrium cavity temperature.

The initial density matrix of the whole system is

ρ(0) = ρ_A(0)ρ_F(0) = \sum_n p_n |Ψ_1(0)⟩⟨Ψ_2(0)|⟨Ψ_1(0)|⟨Ψ_2(0)|n⟩⟨n|, \tag{3}
Before considering the dynamics of the system for thermal initial cavity field, it is straightforward to first study the case when the trapped two-level atom interacts with Fock state. Suppose that the excitation number of the atom-field system is \( n \) \((n \geq 0)\) the evolution of the system is confined in the subspace \(|-,-,n+2\rangle, |+,-,n+1\rangle, |-,+,n+1\rangle, |+,+,n\rangle\). On this basis, the eigenfunctions of the Hamiltonian (1) can be written as

\[
|\Phi_{in}\rangle = \xi_{in}(X_{i1n}|-,,-,n+2\rangle + X_{i2n}|+,-,n+1\rangle + X_{i3n}|-,+,n+1\rangle + X_{i4n}|+,+,n\rangle) \quad (i = 1, 2, 3, 4),
\]

where

\[
\xi_{in} = \frac{1}{\sqrt{|X_{i1n}|^2 + |X_{i2n}|^2 + |X_{i3n}|^2 + |X_{i4n}|^2}}
\]

and

\[
X_{i1n} = 1, \quad X_{i2n} = (-1)^{i+1} \frac{S_n}{\sqrt{n+2}},
\]

\[
X_{i3n} = \frac{S_n^2 - (n+2)}{(\sqrt{n+2})\alpha}, \quad X_{i4n} = (-1)^{i+1} \frac{S_n(S_n^2 - (n+2) - \alpha^2)}{(\sqrt{n+1}\sqrt{n+2})\alpha}.
\]

Here \( S_n = A_n \) for odd \( i \) and \( S_n = B_n \) for even \( i \). The corresponding eigenvalue are

\[
E_{1n}/\hbar = (n+1)\omega + A_n, \quad E_{2n}/\hbar = (n+1)\omega - A_n,
\]

\[
E_{3n}/\hbar = (n+1)\omega + B_n, \quad E_{4n}/\hbar = (n+1)\omega - B_n.
\]

Here

\[
A_n = \sqrt{W_n + V_n}/2, \quad B_n = \sqrt{W_n - V_n}/2
\]

and

\[
W_n = 4n + 6 + 2\alpha^2, \quad V_n = 2\sqrt{4(n+1)\alpha^2 + (\alpha^2 + 1)},
\]

where \( \alpha = J/g \).

To derive the full dynamics of our model one can consider also the basis states \(|-,+,1\rangle, |+,-,0\rangle, |-,+,0\rangle\). In this basis the eigenfunctions and eigenvalues of the Hamiltonian (1) are

\[
|\varphi_1\rangle = (\alpha^2/\Omega)[[-,-,1\rangle - (1/\alpha)|-,+,0\rangle], \quad E_1 = 0;
\]
\[ |\varphi_2\rangle = (1/\sqrt{2})[(1/\Omega)|-, -, 1\rangle + |+, -, 0\rangle + (\alpha/\Omega)|-, +, 0\rangle], \quad E_2/\hbar = \Omega; \]
\[ |\varphi_3\rangle = (1/\sqrt{2})[-(1/\Omega)|-, -, 1\rangle + |+, -, 0\rangle - (\alpha/\Omega)|-, +, 0\rangle], \quad E_2/\hbar = -\Omega, \]
where \( \Omega = \sqrt{1 + \alpha^2} \).

At last, the Hamiltonian (1) has one more eigenfunction
\[ \varphi_0 = |-, -, 0\rangle \]
which corresponds to energy \( E_0 = -\hbar \omega \).

Assume that the whole system is initially in the state \(|+, +, n\rangle\) \((n \geq 0)\), then at time \(t\), the whole system will evolve to
\[ |\Psi(t)\rangle = Z_{11,n}|-, -, n+2\rangle + Z_{21,n}|+, -, n+1\rangle + Z_{31,n}|-, +, n+1\rangle + Z_{41,n}|+, +, n\rangle. \quad (4) \]

Here
\[
Z_{11,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{41n} X_{11n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{42n} X_{21n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{43n} X_{31n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{44n} X_{41n}, \\
Z_{21,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{41n} X_{12n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{42n} X_{22n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{43n} X_{32n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{44n} X_{42n}, \\
Z_{31,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{41n} X_{13n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{42n} X_{23n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{43n} X_{33n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{44n} X_{43n}, \\
Z_{41,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{41n} X_{14n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{42n} X_{24n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{43n} X_{34n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{44n} X_{44n}, \\
\]
where \( Y_{ijn} = \xi_{jn} X^*_{jin} \).

Similarly, when the whole system is initially in the state \(|+, -, n+1\rangle\) \((n \geq 0)\), then at time \(t\), the whole system will evolve to
\[ |\Psi(t)\rangle = Z_{12,n}|-, -, n+2\rangle + Z_{22,n}|+, -, n+1\rangle + Z_{32,n}|-, +, n+1\rangle + Z_{42,n}|+, +, n\rangle. \quad (5) \]

Here
\[
Z_{12,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{21n} X_{11n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{22n} X_{21n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{23n} X_{31n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{24n} X_{41n}, \\
Z_{22,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{21n} X_{12n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{22n} X_{22n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{23n} X_{32n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{24n} X_{42n}, \\
Z_{32,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{21n} X_{13n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{22n} X_{23n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{23n} X_{33n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{24n} X_{43n}, \\
Z_{42,n} = e^{-iE_1n t/\hbar} \xi_{1n} Y_{21n} X_{14n} + e^{-iE_2n t/\hbar} \xi_{2n} Y_{22n} X_{24n} + \\
+ e^{-iE_3n t/\hbar} \xi_{3n} Y_{23n} X_{34n} + e^{-iE_4t n/\hbar} \xi_{4n} Y_{24n} X_{44n}, \\
\]
where \( Y_{ijn} = \xi_{jn} X^*_{ijn} \).
\[ Z_{22,n} = e^{-iE_1 t / \hbar} \xi_{1n} Y_{21n} X_{12n} + e^{-iE_2 t / \hbar} \xi_{2n} Y_{22n} X_{22n} + \]
\[ + e^{-iE_3 t / \hbar} \xi_{3n} Y_{23n} X_{32n} + e^{-iE_4 t / \hbar} \xi_{4n} Y_{24n} X_{42n}, \]
\[ Z_{32,n} = e^{-iE_1 t / \hbar} \xi_{1n} Y_{21n} X_{13n} + e^{-iE_2 t / \hbar} \xi_{2n} Y_{22n} X_{23n} + \]
\[ + e^{-iE_3 t / \hbar} \xi_{3n} Y_{23n} X_{33n} + e^{-iE_4 t / \hbar} \xi_{4n} Y_{24n} X_{43n}, \]
\[ Z_{42,n} = e^{-iE_1 t / \hbar} \xi_{1n} Y_{21n} X_{14n} + e^{-iE_2 t / \hbar} \xi_{2n} Y_{22n} X_{24n} + \]
\[ + e^{-iE_3 t / \hbar} \xi_{3n} Y_{23n} X_{34n} + e^{-iE_4 t / \hbar} \xi_{4n} Y_{24n} X_{44n}. \]

If the initial state of considered system is \(|+, -, 0\rangle\), the time dependent wave function takes the form

\[ |\Psi(t)\rangle = Z_{12}|-, -, 1\rangle + Z_{22}|+, -, 0\rangle + Z_{32}|-, +, 0\rangle, \]

(6)

where

\[ Z_{12} = -\frac{i}{\Omega} \sin \Omega t, \quad Z_{22} = \cos \Omega t, \quad Z_{32} = -\frac{i\alpha}{\Omega} \sin \Omega t. \]

For initial state \(|-, +, n + 1\rangle\ (n \geq 0)\) the time-dependent wave function is

\[ |\Psi(t)\rangle = Z_{13,n}|-, -, n + 2\rangle + Z_{23,n}|+, -, n + 1\rangle + Z_{33,n}|-, +, n + 1\rangle + Z_{43,n}|+, +, n\rangle. \]

(7)
If the initial state of considered system is $|-, +, 0\rangle$, the time dependent wave function of the whole system takes the form

$$|\Psi(t)\rangle = Z_{13}|-, -, 1\rangle + Z_{23}|+, -, 0\rangle + Z_{33}|-, +, 0\rangle,$$

where

$$Z_{13} = -\frac{\alpha}{\Omega^2}(\cos \Omega t - 1), \quad Z_{23} = -\frac{\alpha}{\Omega} \sin \Omega t, \quad Z_{33} = -\frac{1}{\Omega^2}(1 + \alpha^2 \cos \Omega t).$$

For initial state $|-, -, n + 2\rangle$ ($n \geq 0$) we have at time $t$ that

$$|\Psi(t)\rangle = Z_{14,n}|-, -, n+2\rangle + Z_{24,n}|-, -, n+1\rangle + Z_{34,n}|-, +, n+1\rangle + Z_{44,n}|+, +, n\rangle.$$

Here

$$Z_{14,n} = e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{41n} X_{11n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{42n} X_{21n} +$$

$$+ e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{43n} X_{31n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{44n} X_{41n},$$

$$Z_{24,n} = e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{41n} X_{12n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{42n} X_{22n} +$$

$$+ e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{43n} X_{32n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{44n} X_{42n},$$

$$Z_{42,n} = e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{41n} X_{13n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{42n} X_{23n} +$$

$$+ e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{43n} X_{33n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{44n} X_{43n},$$

$$Z_{44,n} = e^{-iE_{1n}t/\hbar} \xi_{1n} Y_{41n} X_{14n} + e^{-iE_{2n}t/\hbar} \xi_{2n} Y_{42n} X_{24n} +$$

$$+ e^{-iE_{3n}t/\hbar} \xi_{3n} Y_{43n} X_{34n} + e^{-iE_{4n}t/\hbar} \xi_{4n} Y_{44n} X_{44n}. $$

If the initial state of considered system is $|-, -, 1\rangle$, the time dependent wave function of the whole system takes the form

$$|\Psi(t)\rangle = Z_{14}|-, -, 1\rangle + Z_{24}|+, -, 0\rangle + Z_{34}|-, +, 0\rangle,$$

where

$$Z_{14} = \frac{1}{\Omega}(\alpha^2 + \cos \Omega t), \quad Z_{24} = -\frac{\alpha}{\Omega} \sin \Omega t, \quad Z_{34} = -\frac{\alpha}{\Omega^2}(1 - \cos \Omega t).$$

At last, if the initial state is $|-, -, 0\rangle$, the time dependent wave function will evolve to

$$|\Psi(t)\rangle = e^{-i\omega t}|-, -, 0\rangle.$$
Now we go back to the theme of this paper. If the initial state for two atoms is (2), then using the equations (4)-(11) one can obtain the density operator for the whole system. Taking a partial trace over the heat bath variable one can obtain that the reduced atomic density operator in the two-atom basis evolves to

$$\rho_A(t) = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{12}^* & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{13}^* & \rho_{23}^* & \rho_{33} & \rho_{34} \\
\rho_{14}^* & \rho_{24}^* & \rho_{34}^* & \rho_{44}
\end{pmatrix}. \quad (11)$$

The matrix elements of (11) are given in Appendix.

For two-qubit system described by the density operator $\rho_A(t)$, a measure of entanglement or negativity can be defined in terms of the negative eigenvalues $\mu_i^-$ of partial transpose of the reduced atomic density matrix $\rho_A^{T_1}$ [26], [27]. The partial transpose of the reduced atomic density matrix (11) is

$$\rho_A^{T_1}(t) = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13}^* & \rho_{14}^* \\
\rho_{12}^* & \rho_{22} & \rho_{23}^* & \rho_{24}^* \\
\rho_{13} & \rho_{14} & \rho_{33} & \rho_{34} \\
\rho_{23} & \rho_{24} & \rho_{34} & \rho_{44}
\end{pmatrix}.$$ 

The negativity is

$$\varepsilon = -2 \sum \mu_i^- . \quad (12)$$

When $\varepsilon = 0$ two qubits are separable and $\varepsilon > 0$ means the atom-atom entanglement. The case $\varepsilon = 1$ indicates maximum entanglement.

The results of calculations of entanglement parameter (12) for initial atomic states (2) are shown in Fig. 1-3. For all figures we put $\varphi_1 = \varphi_2 = 0$. 

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3. Results

In Fig. 1 we plot the negativity as a function of $gt$ for coherent and three incoherent atomic states described by formula (2). The curves correspond to a fixed value of dipole strength $\alpha = 0.5$ and weak thermal two-mode field with $\bar{n} = 0.01$. From these figure we can find that, firstly, entanglement can be induced by thermal field for all initial atomic state. The more significant result is that the atomic entanglement may be induced by thermal field when both the atoms are prepared in their excited states. By contrast, the atomic entanglement does not induced by thermal field when both atoms are trapped in cavity and prepared in their excited states [5]. Secondly, the maximum degree of entanglement is slightly enhanced owing to the atomic coherence comparing the curves in Fig.1(c) and Fig.1(d). With increasing of the mean photon number (see Figs.2(a) and Fg2(b)) the value of atom-atom negativity decreases. But for coherent states this decreasing is much sharper. Such entanglement behavior essentially differs from that for model when both atoms are trapped in cavity and interact with a thermal field. In the last case [16] the entanglement is greatly enhanced due to the initial atomic coherence for thermal cavity with large mean photon numbers. Fig. 3 displays how the evolution of entanglement depends on the dipole-dipole strength for $\bar{n} = 1$. From these figures, we find that entanglement revealed by the solid lines ($\alpha = 0.1$) is stronger than that by the dotted lines ($\alpha = 1$) at most time. The value of atom-atom negativity is larger when the dipole interaction between the atoms increases. This dependence is much sharp for incoherent initial atomic state.

4. Conclusion

In this paper we have investigated the effect of the atomic coherence on the entanglement of two dipole coupled two-level atoms when only one atom is trapped in a lossless cavity and interacts with one-mode thermal field, and the other one can be spatially moved freely outside the cavity. We have shown that thermal field can produce atom-atom entanglement for all pure initial atomic
states. The results also show that the atom-atom entanglement can be controlled by changing the system parameters, such as the amplitudes of the polarized atoms, the mean photon numbers of thermal field, and the strength of the dipole interaction (or distance between atoms). We have derived that for considered model the dipole-dipole interaction can produce the appreciable amount of entanglement only for small thermal noise if the atoms are prepared in the coherent states.

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Appendix: Expressions for density matrix Elements in Eq. (11)

The evident expressions for matrix elements in (11) are

\[ \rho_{11} = |a|^2 \sum_{n=0}^{\infty} p_n |Z_{41,n}|^2 + |b|^2 \sum_{n=1}^{\infty} p_n |Z_{42,n-1}|^2 + bc^* \sum_{n=1}^{\infty} p_n Z_{42,n-1} Z_{43,n-1}^* + \]

\[ + cb^* \sum_{n=1}^{\infty} p_n Z_{43,n-1} Z_{42,n-1}^* + |c|^2 \sum_{n=1}^{\infty} p_n |Z_{43,n-1}|^2 + |d|^2 \sum_{n=2}^{\infty} p_n |Z_{44,n-2}|^2, \]

\[ \rho_{12} = ab^* \sum_{n=1}^{\infty} p_n Z_{41,n} Z_{22,n-1}^* + ac^* \sum_{n=1}^{\infty} p_n Z_{41,n} Z_{23,n-1}^* + \]

\[ + d^* \sum_{n=2}^{\infty} p_n Z_{42,n-1} Z_{24,n-2}^* + c^* \sum_{n=2}^{\infty} p_n Z_{43,n-1} Z_{24,n-2}^* + \]

\[ + p_1(bd^* Z_{42,0} G_{24}^* + cd^* Z_{43,0} G_{24}^*) + p_0(ab^* Z_{41,0} G_{22}^* + ac^* Z_{41,0} G_{23}^*), \]

\[ \rho_{13} = ab^* \sum_{n=1}^{\infty} p_n Z_{41,n} Z_{32,n-1} + ac^* \sum_{n=1}^{\infty} p_n Z_{41,n} Z_{33,n-1}^* + \]

\[ + bd^* \sum_{n=2}^{\infty} p_n Z_{42,n-1} Z_{34,n-2}^* + c^* \sum_{n=2}^{\infty} p_n Z_{43,n-1} Z_{34,n-2}^* + \]

\[ + p_1(bd^* Z_{42,0} G_{34}^* + cd^* Z_{43,0} G_{34}^*) + p_0(ab^* Z_{41,0} G_{32}^* + ac^* Z_{41,0} G_{33}^*), \]

\[ \rho_{14} = ad^* \sum_{n=2}^{\infty} p_n Z_{41,n} Z_{14,n-2} + p_1 ad^* Z_{41,1} G_{14}^* + p_0 ad^* Z_{41,0} G_{14}^* + \]
\[
\rho_{22} = |a|^2 \sum_{n=0} p_n |Z_{21,n}|^2 + |d|^2 \sum_{n=2} p_n |Z_{24,n-2}|^2 + \\
+bc^* \sum_{n=1} p_n Z_{22,n-1} Z_{23,n-1}^* + cb^* \sum_{n=1} p_n Z_{23,n-1} Z_{22,n-1}^* + \\
+|c|^2 \sum_{n=1} p_n |Z_{23,n-1}|^2 + |b|^2 \sum_{n=1} p_n |Z_{22,n-1}|^2 + \\
+p_1 |d|^2 |G_{24}|^2 + p_0 (|b|^2 |G_{22}|^2 + bc^* G_{22} G_{23}^* + p_0 (cb^* G_{23} G_{22}^* |c|^2 |G_{23}|^2),
\]

\[
\rho_{23} = |a|^2 \sum_{n=0} p_n Z_{21,n} Z_{31,n}^* + |d|^2 \sum_{n=2} p_n Z_{24,n-2} Z_{34,n-2}^* + \\
+bc^* \sum_{n=1} p_n Z_{22,n-1} Z_{33,n-1}^* + cb^* \sum_{n=1} p_n Z_{23,n-1} Z_{32,n-1}^* + \\
+|c|^2 \sum_{n=1} p_n |Z_{23,n-1}|^2 + |b|^2 \sum_{n=1} p_n Z_{22,n-1} Z_{32,n-1}^* + \\
+p_1 |d|^2 |G_{24}|^2 + p_0 (|b|^2 |G_{22}|^2 + bc^* G_{22} G_{32}^* + cb^* G_{23} G_{32}^* + |c|^2 |G_{23}|^2 G_{32}^*),
\]

\[
\rho_{33} = |a|^2 \sum_{n=0} p_n |Z_{31,n}|^2 + |b|^2 \sum_{n=1} p_n |Z_{32,n-1}|^2 + |c|^2 \sum_{n=1} p_n |Z_{33,n-1}|^2 + \\
+bc^* \sum_{n=1} p_n Z_{33,n-1} Z_{32,n-1}^* + |b|^2 \sum_{n=1} p_n |Z_{34,n-2}|^2 + \\
+p_1 |d|^2 |G_{34}|^2 + p_0 (|b|^2 |G_{32}|^2 + bc^* G_{32} G_{33}^* + cb^* G_{33} G_{32}^* + |c|^2 |G_{33}|^2),
\]

\[
\rho_{34} = ab^* \sum_{n=1} p_n Z_{31,n} Z_{12,n-1} + ac^* \sum_{n=1} p_n Z_{31,n} Z_{13,n-1} + \\
+bd^* \sum_{n=2} p_n Z_{32,n-1} Z_{14,n-2} + cd^* \sum_{n=2} p_n Z_{33,n-1} Z_{14,n-2}^* + \\
+p_1 (bd^* Z_{32,0} G_{14} + cd^* Z_{33,0} G_{14}^*) + p_0 (ab^* Z_{31,0} G_{12}^* + ac^* Z_{31,0} G_{13}^*) + \\
+p_0 (bd^* G_{32} + cd^* G_{33},
\]

\[
\rho_{44} = |a|^2 \sum_{n=0} p_n |Z_{11}|^2 + |b|^2 \sum_{n=1} p_n |Z_{12,n-1}|^2 + |c|^2 \sum_{n=1} p_n |Z_{13,n-1}|^2 + \\
+bc^* \sum_{n=1} p_n Z_{13,n-1} Z_{12,n-1}^* + |b|^2 \sum_{n=2} p_n |Z_{14,n-2}|^2 + \\
+p_1 |d|^2 |G_{14}|^2 + p_0 (|b|^2 |G_{12}|^2 + bc^* G_{12} G_{13}^* + cb^* G_{13} G_{12}^* + |c|^2 |G_{13}|^2 + |d|^2),
\]

where

\[
a = \cos \theta_1 \cos \theta_2, \quad b = \cos \theta_1 \sin \theta_2 e^{i\varphi_2}, \quad c = \cos \theta_2 \sin \theta_1 e^{i\varphi_1}, \quad d = \sin \theta_1 \sin \theta_2 e^{i(\varphi_1 + \varphi_2)}.
\]
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Figure 1: The negativity as a function of $gt$ for the model with $\bar{n} = 0.01$ and $\alpha = 0.5$. The initial atomic state: a) $|\Psi(0)\rangle_A = |+, -\rangle$ (a), $|\Psi(0)\rangle_A = |-, -\rangle$ (b), $|\Psi(0)\rangle_A = |+, +\rangle$ (c) and $|\Psi(0)\rangle_1 = (1/\sqrt{2})(|+\rangle + |\rangle)$, $|\Psi(0)\rangle_2 = (1/\sqrt{2})(|+\rangle + |\rangle)$ (d).
Figure 2: The negativity as a function of $gt$ for the model with $\bar{n} = 1$ (solid) and $\bar{n} = 2$ (dashed). The initial atomic state: a) $|\Psi(0)\rangle_A = |+, -\rangle$ (a) and $|\Psi(0)\rangle_1 = (1/\sqrt{2})(|+\rangle + | -\rangle)$, $|\Psi(0)\rangle_2 = (1/\sqrt{2})(|+\rangle + | -\rangle)$ (b). The dipole strength $\alpha = 0.5$. 
Figure 3: The negativity as a function of $gt$ for the model with $\alpha = 0.1$ (solid) and $\alpha = 1$ (dashed). The initial atomic state: a) $|\Psi(0)\rangle_A = |+,-\rangle$ (a) and $|\Psi(0)\rangle_1 = (1/\sqrt{2})(|+\rangle + |\rangle$, $|\Psi(0)\rangle_2 = (1/\sqrt{2})(|+\rangle + |\rangle$ (b). The mean photon number $\bar{n} = 0.5$