Threshold for Non-Thermal Stabilization of Open Quantum Systems

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We generally study whether or not the information of an open quantum system could be totally erased by its surrounding environment in the long time. For a harmonic oscillator coupled to a bath of a spectral density with zero-value regions, we quantitatively present a threshold of system-bath coupling \( \eta_c \), above which the initial information of the system can remains partially as its long time stabilization deviates from the usual thermalization. This non-thermal stabilization happens as a non-Markovian effect.

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Introduction. —Thermalization is a dynamic process of an open system reaching the thermal equilibrium at the same temperature \( T \) as its surrounding heat bath. From the point of view of the information theory, thermalization is regarded as an information erasure process.1 The open system initially prepared in an arbitrary state will relax to a thermal state after a long-time Markov process. This steady state is irrelevant to the system’s initial state at all, but it carries partial bath’s information characterized its temperature \( T \). Thus, the conventional thermalization plays a necessary role in the initializations of computation or thermodynamic cycle.2–4 This perspective results in a comprehensive understanding for Landauer’s erasure principle.1,2

Thermalization is dynamically associated with the Markovian processes2 and also can be described by the Langevin equation under the Wigner-Weyl approximations.3–8 However, it was found that a strong system-bath coupling might result in a non-Markovian process when the interaction spectral density has zero-value regions.3–4,9,10 Two questions naturally follow for further investigation: (1) to what extent the strength of system-bath coupling increases so that the system’s stabilization largely deviates from the usual thermalization? (2) how much information of the initial state is left in the final stable state for a non-Markovian process?

To answer these questions generally, we revisit the “standard model” of open quantum system,3–6,9,10 a harmonic oscillator (HO) coupled to a bath of HO’s with a spectral density with zero-value regions. We analytically examine the mean occupation number of the system through the density with zero-value regions.9–14 Two questions naturally follow for stabilization largely deviates from the usual thermalization?

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The “standard model” of open quantum system. —We consider an open system consisting of a harmonic oscillator interacting with its environment (or bath). The environment is modeled as a collection of harmonic oscillators with linear coupling to the system. This has been extensively studied in numerous literatures as a “standard model” of open quantum system, since it can be universally utilized to reveal the core spirit of quantum dissipation process according to Caldeira and Leggett.7

The total Hamiltonian of our model reads

\[
H = \Omega a^\dagger a + \sum_l \omega_l b_l^\dagger b_l + \sum_l \left( \eta_l a^\dagger b_l + \eta_l^* b_l^\dagger a \right),
\]

where \( a(a^\dagger) \) and \( b_l(b_l^\dagger) \) are the annihilation(creation) operators of the system and the \( l \)-th mode of the environment, respectively. The corresponding Heisenberg equation has the formal solution3–6,9,10

\[
a(t) = u(t)a + \sum_l u_l(t)b_l.
\]

The coefficient \( u(t) \) is governed by the following differential-integral equation,

\[
\frac{du_l(t)}{dt} + i\Omega u_l(t) + \int_0^t G(t-\tau) u(\tau) d\tau = 0,
\]

with the initial condition \( u(0) = 1 \). Here, the integral kernel

\[
G(t) = \mathcal{F}[J(\omega)] \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} J(\omega) e^{-i\omega t} d\omega,
\]

is the Fourier transformation of the system-bath interaction spectral density \( J(\omega) \equiv 2\pi \sum_l |\eta_l|^2 \delta(\omega - \omega_l) \), which is usually taken as \textit{a priori} microscopic knowledge. The other coefficients \( u_l(t) \) are given by

\[
u_l(t) = -i\eta_l \int_0^t u(t-\tau) e^{-i\omega_l \tau} d\tau.
\]
When the system and the bath are initially in the direct product state \( \rho(0) = \rho_S(0) \otimes \rho_E(0) \), where \( \rho_E(0) \) is the thermal equilibrium state of the bath at temperature \( T \) and \( \rho_S(0) \) is an arbitrary initial state of the system, the system’s mean occupation number is obtained as \[ \langle \hat{n}(t) \rangle = \text{Tr} \langle \hat{n}(t) \rangle_{S} + \sum_{l} \langle \hat{n}(t) \rangle_{l}, \tag{6} \]
where \( \langle \cdots \rangle_{S(E)} = \text{Tr}(S(E)[\rho_{S(E)} \cdots] \rangle \) means the average over the state \( \rho_{S(E)} \). The mean occupation number \( \langle \hat{n}(t) \rangle \) is divided into two parts: the first part, which vanishes in a long-time Markov process, only depends on the system’s initial condition. The second part, which usually leads to the thermalization of the system in the weak-coupling case \[15\], characterizes the contribution from the thermal bath. In this letter, it will be shown that the first part describes the dynamic process of erasing or preserving the system’s initial information, while the second part describes how the bath’s information is inherited by the system.

It is convenient to extend the limit of integration of \( \tau \) in Eq. \[3\] from \([0, t]\) to \((-\infty, t]\) by defining \( u(t)_{t<0} = 0 \), and then the differential-integral equation \[3\] changes into \[18\]
\begin{equation}
\frac{du(t)}{dt} + i\Omega u(t) + \int_{-\infty}^{t} d\tau G(t-\tau)u(\tau) = \delta(t). \tag{7}
\end{equation}

It is obvious that Eq. \[7\] is exactly equivalent to Eq. \[3\] in the time domain \((0, \infty)\). A formal solution of \( u(t) \) is obtained via the Fourier transformation as \[11\]
\begin{equation}
u(t) = -\frac{1}{2\pi} \int P\frac{J(\omega')d\omega'}{\omega - \omega' + i\epsilon} + \frac{i}{2}J(\omega) + i\epsilon, \tag{9}\end{equation}
where the denominator in the integral is
\[F(\omega) \equiv \omega - \Omega - \frac{1}{2\pi} \int P\frac{J(\omega')d\omega'}{\omega - \omega'} + \frac{i}{2}J(\omega) + i\epsilon, \tag{9}\]
and \( \epsilon \) is an infinitesimal positive constant. For some special spectrum, e.g. a Lorentzian-type spectrum, the above integral can be carried out analytically \[11\].

For \( t \to \infty \), we assume an asymptotic solution of Eq. \[3\] \( u(t) \sim A \exp(-i\omega_0 t) \), which oscillates with a single frequency \( \omega_0 \) and amplitude \( A \). Due to the linearity of Eq. \[3\], the superposition of such single-mode solutions is also an asymptotic solution of Eq. \[3\]. Therefore, we only need to investigate the existence conditions and the properties of the single-mode case. We first let \( \tilde{u}(t) \equiv \exp(i\omega_0 t)u(t) \), which satisfies the integral-differential equation similar to Eq. \[3\] with modified frequency \( \tilde{\Omega} \equiv \Omega - \omega_0 \) and modified kernel \( \tilde{G}(t) \equiv F[J(\omega + \omega_0)] \) \[18\]. The steady asymptotic solution is determined by \( \left. \frac{d\tilde{u}(t)}{dt} \right|_{t=\infty} = 0 \), or
\begin{equation}
\left[ i(\Omega - \omega_0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega + \omega_0)}{\omega} d\omega + \frac{i}{2}J(\omega_0) \right] A = 0. \tag{10}\end{equation}

For the case with \( A \neq 0 \), the above equation gives the criteria for existence of nonvanishing solution of Eq. \[3\] about a real oscillating frequency \( \omega_0 \): \[11\]
\begin{align}
J(\omega_0) &= 0, \tag{11a} \\
\Omega - \omega_0 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(\omega)}{\omega - \omega_0} d\omega. \tag{11b}
\end{align}

**Criteria for non-thermal stabilization.**—In the conventional thermalization process, \( u(t) \) decays to 0 as \( t \to \infty \). This effect implies that the system’s initial information will be totally erased. However, there exist some clues reminding us that \( u(t) \) may not vanish at long time \[9–11\]. Now, we explicitly present the criteria for the occurrence of such non-thermal stabilization.

According to Ref. \[11\], the non-thermal stabilization firstly requires the spectrum \( J(\omega) \) to have at least one zero-value region. Thus, the non-thermal stabilization would never happen if the spectrum were of Lorentzian-type. Eq. \[11b\] must have at least one solution in theses zero regions. If Eq. \[11b\] has more than one solution in the zero-value regions of \( J(\omega) \), the general solution of Eq. \[3\] will be the superposition of these single-mode solutions.

Let us consider a specific kind of spectrum that possesses a half-side configuration \( J(\omega)|_{\omega<0} = 0 \) [see Fig. \[11a\]a]. It should be emphasized that the interaction spectrum of a bosonic bath is always of half-side form, otherwise the total Hamiltonian will have no lower bound. Then, we consider whether there exists a solution \( \omega_0 \) (\( \omega_0 < 0 \)) satisfying Eq. \[11b\]. The l.h.s. of Eq. \[11b\] is a monotonically increasing function of \( -\omega_0 \) and has no upper limit, while the r.h.s. is a monotonically decreasing function of \( -\omega_0 \) (see Fig. \[2\]). Thus, there is no more than one solution for Eqs. \[11a\] and \[11b\]. Moreover, the criteria for the non-thermal stabilization reduces to
\[1\]
\begin{equation}
\frac{1}{2\pi} \int_{0}^{\infty} \frac{J(\omega)}{\omega} d\omega \geq \Omega. \tag{12}\end{equation}

Usually, the spectral density can be rewritten as \( J(\omega) = \eta J_0(\omega) \), where \( \eta \) characterizes the system-bath interac-
When the coupling strength is so strong that it is tautomatically described as the sentence “coupling is weak enough.” Consequently, the Markov approximation can be approximated as an integral around $\omega = 0$ according to Eq. (31). There exists one solution if $\eta > \eta_c$. In this case, there always exists a solution as long as the system couples to the bath.

The above arguments show that, if the coupling strength $\eta < \eta_c$, $u(t)$ asymptotically vanishes as $t \to \infty$. Previously, this condition for the analytic calculation was qualitatively described by the expression “coupling is weak enough”. When the coupling strength is so strong that $\eta \geq \eta_c$, the asymptotic value of $|u(t)| \neq 0$ and then the initial information of the system will not be totally erased even at long time. Consequently, the Markov approximation can not work well when $\eta > \eta_c$. When the half-side spectral density satisfies $\int_0^\infty J_0(\omega)/\omega d\omega = \infty$, the critical coupling strength becomes zero according to Eq. (13) [see Fig. 2(b)]. Thus, no matter how weak the system-bath interaction is, the stabilization is non-thermal and the Markov approximation or Wigner-Weisskopf approximation is not valid. In other words, such spectrum is born to be non-Markovian, e.g., the square spectrum.

Next, we show how to estimate the amplitude $A$ from the formally-exact solution of $u(t)$ in Eq. (11). For a given solution $\omega_0$ of the Eq. (11a) and (11b), $F(\omega_0)$ vanishes. Therefore, the integral around $\omega_0$ contributes most to the integration in Eq. (8) and $F(\omega)$ can be approximately replaced by $F(\omega_0) (\omega - \omega_0)$. According to the residue theorem, we have $u(t) \approx \exp(-i\omega_0 t)/F'(\omega_0)$. Then, the amplitude $A$ is approximated as $1/F'(\omega_0)$ [18], i.e.,

$$A \approx \left(1 + \frac{1}{2\pi} \int P\left(\frac{J(\omega)}{\omega - \omega_0}\right)^2\right)^{-1}. \quad (14)$$

Example of non-thermal stabilizations.—The first example of the non-thermal stabilization is the case with a symmetrical half-side spectrum that satisfies $J(\Omega - \omega) = J(\Omega + \omega)$ with respect to the resonance point $\omega = \Omega$ and $J(\omega)$ does not vanish if and only if $\omega \in (0, 2\Omega)$ (see Fig. 1). There exists a critical coupling strength $\eta_c$ determined by Eq. (13) [18]. In the non-thermal region $\eta > \eta_c$, the asymptotic solution of $u(t)$ is the superposition of two single-mode solutions. In the supplemental material, we examine two concrete examples, the triangular spectrum and the rectangular spectrum. The coincidence between the analytical calculations and numerical results implies that the criterion (11a) and the estimation of the amplitude Eq. (14) work well.

The second example is a more realistic one—Ohmic spectrum with the density distribution:

$$J(\omega) = 2\pi \eta \theta(\omega) \omega \exp(-\omega/\Omega_c). \quad (15)$$

Here, $\eta$ characterizes the coupling strength and $\Omega_c$ is the cutoff frequency. This spectrum is widely applied in open systems [19, 20]. There exists a critical coupling strength $\eta_c = \Omega_c/\Omega_c$ according to Eq. (13). As shown by the numerical simulation of $u(t)$ in Fig. 2(a-c), when $\eta < \eta_c$, $u(t)$ decays exponentially and the decay rate increases with $\eta$. On the other hand, when $\eta \geq \eta_c$, $|u(t)|$ has a non-vanishing asymptotic value $|A|$, which increases with the coupling strength $\eta$. The numerical results also confirms the former qualitative analysis from Eq. (11a) and Eq. (14). Sometimes, the spectrum that one meets in the experiment may be a modified one, such as sub-Ohmic or super-Ohmic spectrum. Our method can be applied to these cases straightforwardly [11].

In the third example, the we deal with the gapped-spectrum case. As shown in the supplementary, the density of the spectrum $J(\omega)$ vanishes if and only if $\omega \in (\omega_1, \omega_2)$ [see Fig. 4(d)]. One may meet this kind of spectrum when the heat bath has a band gap, i.e., photonic crystal system [12, 21]. One question arises that whether there exists a solution $\omega_0 \in (\omega_1, \omega_2)$ satisfying
the criterion [1]. After an argument similar to the half-side spectrum case, we find that the criteria require the frequency of the system $\Omega$ to satisfy $\Omega \in [\Omega_1, \Omega_2]$, where

$$\Omega_i \equiv \omega_i - \frac{1}{2\pi} \int_{-\infty}^{\omega_i} J(\omega) \, d\omega + \frac{1}{2\pi} \int_{\omega_i}^{\infty} J(\omega) \, d\omega,$$

for $i = 1, 2$. When the spectral density is discontinuous at $\omega_1$ and $\omega_2$, for example,

$$J(\omega) = \theta(\omega_1 - \omega)\eta_1 e^{\gamma_1 \omega} + \theta(\omega - \omega_2)\eta_2 e^{-\gamma_2 \omega},$$

then $\Omega_1 = -\infty$ and $\Omega_2 = +\infty$. In this case, the criterion $\Omega \in [\Omega_1, \Omega_2]$ always holds. Thus, no matter how weak the coupling strength is, such spectrum is always associated with a non-vanishing asymptotic solution of $u(t)$ with oscillation frequency $\omega_0 \in [\omega_1, \omega_2]$. Practically, it is useful to judge the existence and the location of the spectral gap by measuring the frequency $\omega_0$.

The information from the bath inherited by system.— We have described how the first part of the system’s mean occupation number can represent the residual information of the system’s initial state. Now, we turn attention to the second part, which depends on the population distribution of the bath.

The second part of the system’s mean occupation number in Eq. (6) is re-written as

$$\sum_i |u_i(t)|^2 \langle b_i^\dagger b_i \rangle = \int p(\omega) f_\beta(\omega) d\omega,$$

where $p(\omega) = \sum_i |u_i(t)|^2 \delta(\omega - \omega_i)$ and $f_\beta(\omega) = 1/[\exp(\beta \omega) - 1]$ with $\beta = 1/(k_B T)$. Actually, this part can be viewed as the information ‘written’ into the system by the bath and $p(\omega)$ is the distribution function. According to Eq. (6), $u_i(t)$ is determined by an integral of $u(t)$ over the time domain $[0, t]$. As shown in the former sections, $u(t)$ decays exponentially in short time and relaxes to an asymptotic form $A \exp(-i\omega_0 t)$ at long time. In order to calculate the second part of Eq. (6), we consider two special cases: (1) $A = 0$ for small $\eta$, and (2) $A \neq 0$ for large $\eta$.

The first case has been well studied [13]. In this case, $u_i(t)$ is dominated by the short-time behaviour of $u(t)$ and the distribution function $p(\omega)$ is approximated by a Lorentzian-type distribution

$$p(\omega) = \frac{1}{2\pi} \frac{2\gamma}{(\omega - \Omega')^2 + \gamma^2},$$

where the two parameters $\Omega'$ and $\gamma$ can be calculated via the Wigner-Weisskopf approximation as shown in the supplemental material. Because $A = 0$, the first part of the system’s mean occupation number vanishes. In the weak coupling limit, $p(\omega) \rightarrow \delta(\omega - \Omega')$, which leads to $n(T) \approx f_\beta(\Omega')$. This implies that the system’s mean occupation number actually inherits the population of the environment mode with the renormalized mode frequency $\Omega'$.

In the second case, $u_i(t)$ is dominated by the long-time behavior of $u(t)$. If $u(t)$ has a single-mode asymptotic solution with oscillating frequency $\omega_0$ when $t \rightarrow \infty$, the distribution function is approximated as

$$p(\omega) \approx \frac{A^2}{2\pi} \frac{2J(\omega)}{(\omega - \omega_0)^2}. \quad (18)$$

This distribution is totally different from that in the weak coupling case as the two following ways: 1. It is no longer normalized to unity. This is a natural result since the system’s mean occupation number now depends on both the bath and its own initial value. Second, it is a widespread distribution instead of a sharp one, which implies that the information written by the environment becomes more complicated. However, it should be emphasized that when temperature is low enough ($f_\beta(\omega) \rightarrow 0$), the second term in Eq. (6) will be small compared to the first term. Thus, in this situation, one may physically observe the non-thermal stabilization effect by measuring the system’s mean occupation number [22].

Remarks and conclusion.— We have studied a non-thermal stabilization phenomenon by calculating the open system’s mean occupation number. The criteria for this non-Markovian effect was presented with a quantitative threshold $\eta_c$ for most system-bath interaction spectra. In the non-thermal region, $\eta \geq \eta_c$, the system’s initial information of the system is no longer totally erased by the bath.

Actually, $\eta_c$ explicitly provides the Markovian approximation of quantum open system with a quantitative upper limit. Our investigation undoubtedly clarified the misunderstanding that the Markovian approximation is valid only when the coupling strength is small enough, which is closely dependent of the structure of spectral density. In this sense the non-thermal stabilization effect due to the non-Markovian process above the threshold $\eta_c$ provides us with a new fashion to understand the information lost in open systems.

Apparently, our approach is universal and can be applied to the Fermion case. Then, the first open question is whether this non-thermal stabilization could happen for a Fermion like system, such as two level atom coupling to some bath. It is also worthy of discussing the impact of the non-thermal stabilization on the entanglement evolution [23, 24].

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