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Some new results on the self-dual $[120,60,24]$ code

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Abstract

The existence of an extremal self-dual binary linear code of length 120 is a long-standing open problem. We continue the investigation of its automorphism group, proving that automorphisms of order 30 and 57 cannot occur. Supposing the involutions acting fixed point freely, we show that also automorphisms of order 8 cannot occur and the automorphism group is of order at most 120, with further restrictions. Finally, we present some necessary conditions for the existence of the code, based on shadow and design theory.

Keywords: Self-dual code, extremal code, automorphism group

1. Introduction

In coding theory, binary self-dual codes play a central role: they are linear codes with a rich algebraic structure, good decoding properties and relations with other areas of mathematics, such as group theory, lattice theory and design theory. For example, this class includes the binary extended Golay code, whose automorphism group is the sporadic simple group $M_{24}$ and which is related to the Leech lattice.

Gleason, Pierce and Turyn showed (see [3]) that if a natural number $r > 1$ divides the weight of all codewords of a binary self-dual code, then $r = 2$ (even code) or $r = 4$ (doubly-even code). Every binary self-dual code is even. If a binary self-dual code is even but not doubly-even (singly-even code), then it is called a Type I code, while if a binary self-dual code is doubly-even, then it is called a Type II code. Type II codes exist only for lengths which
are multiples of 8 and Mallows and Sloane showed in \[27\] that they have minimum distance bounded by \(4\left\lfloor n/24 \right\rfloor + 4\), where \(n\) is the length. A type II code attaining this bound is called extremal code. Among extremal codes, those of length a multiple of 24 are particularly interesting: Assmus-Mattson’s theorem \([2]\) guarantees that the supports of their codewords of a fixed nonzero weight form a 5-design. Moreover, they have relations, as mentioned above, with simple groups and extremal lattices. Zhang proved in \([32]\) that their length is at most 3672.

Despite their theoretical importance, only two extremal codes of length a multiple of 24 are known, namely the famous binary extended Golay code, the unique up to equivalence of length 24, and the extended quadratic residue code of length 48, which is the unique up to equivalence of this length. In 1973 Sloane \([30]\) posed explicitly the question: is there a self-dual \([72,36,16]\) code? Since then, multiple attempts to establish the non existence of such a code or to present a construction have been done, till now unsuccessfully. The problem is still open for all lengths from 72 to 3672 and many investigations have been also done for the cases of length 96 and 120.

This paper focuses on the last one, i.e. on the study of a self-dual \([120,60,24]\) code. In particular, in Section 2 we will collect, for the reader’s convenience, all the definitions and the known results which will be used in the following. In Section 3 we prove new properties about the automorphism group of a self-dual \([120,60,24]\) code. In particular we exclude the existence of automorphisms of order 30 and 57 and we investigate the structure of the automorphism group, in the case that involutions act fixed point freely (see the introduction of Subsection 3.3 for a motivation of this choice), proving that it is either trivial or isomorphic to a group of order at most 120, with further restrictions. Finally, in Section 4 we give necessary conditions for the existence of the code, based on shadow and design theory.

2. Background

In this section we collect some classical results of coding theory which are useful in the rest of the paper.

2.1. Gleason’s theorem and the shadow of a code

For the whole subsection, let \(C\) be a binary code of length \(n\), i.e. a subspace of \(\mathbb{F}_2^n\). We recall that a \([n,k,d]\) code is a code of length \(n\), dimension \(k\) and minimum distance \(d\).
Definition 1. The weight distribution of \( C \) is the sequence \((A_0(C), \ldots, A_n(C))\), where \( A_i(C) \) is the number of codewords of \( C \) of weight \( i \), for every \( i \in \{1, \ldots, n\} \). The polynomial \( W_C(y) := \sum_{i=0}^{n} A_i(C) y^i = \sum_{c \in C} y^{\text{wt}(c)} \in \mathbb{Z}[x] \) is called the weight enumerator of \( C \) and the polynomial \( W_C(x, y) := x^n W\left(\frac{y}{x}\right) \in \mathbb{Z}[x, y] \), is the homogeneous weight enumerator of \( C \).

Definition 2. The dual of \( C \) is \( C^\perp := \{ v \in \mathbb{F}_2^n | \langle v, c \rangle = 0, \forall c \in C \} \). If \( C = C^\perp \), we say that \( C \) is self-dual. If \( C \) and \( C^\perp \) have the same weight enumerator, \( C \) is called a formally self-dual code.

Theorem 3 \([24]\). Let \( g_1(x, y) := y^2 + x^2 \), \( g_2(x, y) := x^2y^2(x^2 - y^2)^2 \), and \( g_3(x, y) := y^{24} + 759x^8y^{16} + 2576x^{12}y^{12} + 759x^{16}y^8 + x^{24} \).

(a) If \( C \) is formally self-dual and even,
\[
W_C(x, y) = \sum_{i=0}^{[n/8]} a_i g_1(x, y)^{2i} g_2(x, y)^i.
\]

(b) If \( C \) is formally self-dual and doubly-even,
\[
W_C(x, y) = \sum_{i=0}^{[n/24]} a_i g_2(x, y)^{2i} g_3(x, y)^i.
\]

In all cases, every \( a_i \in \mathbb{Q} \) and \( \sum_i a_i = 1 \).

Let \( C \) be a self-dual code and let \( C_0 \) be the subset consisting of all codewords in \( C \) whose weights are multiples of 4. If \( C \) is of type II then \( C_0 = C \), while \( C_0 \) is a subcode of index 2 of \( C \) if \( C \) is of type I.

Definition 4. The shadow of \( C \) is the set
\[
S := \begin{cases} 
C_0^\perp \setminus C, & \text{if } C \text{ is of type I} \\
C, & \text{if } C \text{ is of type II}.
\end{cases}
\]

Let \( C \) be a type I code. Since \( C_0 \) is of index 2, then \#(\( C_0^\perp / C_0 \)) = 4. Hence there are three cosets \( C_1, C_2, C_3 \) of \( C_0 \) in \( C_0^\perp \) such that \( C_0^\perp = C_0 \cup C_1 \cup C_2 \cup C_3 \), where \( C = C_0 \cup C_2 \) and \( S = C_1 \cup C_3 = C_0^\perp \setminus C \) is the shadow of \( C \) (see \([17]\), Theorem 5)).
Theorem 5 ([17]). Let $S$ be the shadow of $C$, code of type I.

(a) If we write

$$W_C(x, y) = \sum_{j=0}^{n/8} a_j(x^2 + y^2)^{\frac{n}{2} - 4j}(x^2y^2(x^2 - y^2)^2)^j,$$

for suitable rationals $a_j$, then

$$W_S(x, y) = \sum_{j=0}^{n/8} (-1)^ja_j2^{\frac{n}{2} - 6j}(xy)^{\frac{n}{2} - 4j}(x^4 - y^4)^2j.$$

(b) Writing $W_S(x, y) = \sum_{i=0}^n B_ix^{n-i}y^i$, we have

(i) $B_i = B_{n-i}$ for all $i$.

(ii) $B_i = 0$, unless $i \equiv n/2 \mod 4$.

(iii) $B_0 = 0$.

(iv) $B_i \leq 1$, for $i < d/2$.

(v) at most one $B_i$ is nonzero for $i < (d + 4)/2$.

Definition 6. If $C$ is a self-dual $[n, n/2, d]$ code with $d > 2$, pick two positions and consider the $(n/2 - 1)$-dimensional subcode $C'$ of $C$ with either two 0s or two 1s in these positions. If we puncture $C'$ on these positions, we obtain a self-dual code $C''$ of length $n - 2$; $C''$ is called a child of $C$ and $C$ is called a parent of $C''$.

Theorem 7 ([20]). Let $m \geq 1$ be an integer. If $C$ is a $[24m - 2, 12m - 1, 4m + 2]$ type I code whose shadow has minimum distance $4m + 3$, then $C$ is a child of a $[24m, 12m, 4m + 4]$ type II code.

Lemma 8 ([20]). If $C$ is a child of an extremal type II code with shadow $S = C_1 \cup C_3$, then $W_{C_1}(y) = W_{C_3}(y)$.

Lemma 9 ([4]). Let $C$ be a type I code of length $n$ with the shadow $S = C_1 \cup C_3$. Suppose that $n \equiv 2 \mod 4$. Let $C^*$ be the code of length $n + 2$ obtained by extending $C_1^*$ as follows:

$$(0, 0, C_0), (1, 0, C_2), (0, 1, C_1), (1, 1, C_3).$$

If $W_{C_1}(y) = W_{C_3}(y)$, then $C^*$ is a formally self-dual code with weight enumerator

$$W_{C_0}(y) + y(W_{C_1}(y) + W_{C_2}(y)) + y^2W_{C_3}(y).$$
Definition 10. Two self-dual codes of length $n$ are *neighbors* if their intersection is a code of dimension $n/2 - 1$.

2.2. Automorphism group of binary codes

The symmetric group $S_n$ acts on $\mathbb{F}_2^n$ by the group action $v\sigma := (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$, where $v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n$ and $\sigma \in S_n$.

Definition 11. Let $C$ and $C'$ be two codes of the same length $n$. We say that $C$ and $C'$ are *equivalent* and denote $C \sim C'$ if only if $C\sigma = C'$ where $\sigma \in S_n$. If $v\sigma \in C$ for all $v \in C$, then $\sigma$ is an *automorphism* of $C$. The set of all automorphisms of $C$ is a group, denoted $\text{Aut}(C)$.

Definition 12. Let $C$ be a binary code of length $n$ and $\sigma \in \text{Aut}(C)$.

(a) If $\sigma$ is of prime order $p$, we say that $\sigma$ is of *type* $p$-$(c; f)$ if it has $c$ cycles of length $p$ and $f$ fixed points.

(b) If $\sigma$ is of order $p \cdot r$, where $p, r$ are distinct primes, then we say that $\sigma$ is of *type* $p \cdot r$-$(s_1, s_2, s_3; f)$ if $\sigma$ has $s_1$ $p$-cycles, $s_2$ $r$-cycles, $s_3$ $pr$-cycles and $f$ fixed points.

Remark 13. In order to simplify the notation, if $\sigma$ is an automorphism of composite order $r$ and has $c r$-cycles and $f$ fixed points with $n = c \cdot r + f$, then we say that the cycle structure of $\sigma$ is $r$-$(c; f)$.

Let us first prove a result which is useful in the following sections.

Lemma 14. Let $C$ be a code of length $n$, such that all automorphisms of prime order $p$ act fixed point freely. If $|\text{Aut}(C)| = p^a m$, with $(p, m) = 1$, then $a \leq \max\{ r \in \mathbb{Z} : p^r \mid n \}$.

Proof. Suppose $a > \max\{ r \in \mathbb{Z} : p^r \mid n \}$. By Sylow’s theorem, there exists a subgroup $H \leq \text{Aut}(C)$ with $|H| = p^a$. The group $H$ acts on the set $\{1, \ldots, n\}$. Since all automorphisms of order $p$ act fixed point freely, then each orbit has $p^a$ elements. Therefore $p^a \mid n$, a contradiction. \qed

Definition 15. Let $\sigma \in \text{Aut}(C)$. The *fixed code* of $\sigma$ is

$$F_\sigma(C) := \{ v \in C \mid v\sigma = v \}.$$  

Let $\Omega_1, \ldots, \Omega_c$ be the cycle sets and let $\Omega_{c+1}, \ldots, \Omega_{c+f}$ be the fixed points of $\sigma$. Clearly $v \in F_\sigma(C)$ if and only if $v \in C$ and $v$ is constant on each cycle. Let $\pi_\sigma : F_\sigma(C) \to \mathbb{F}_2^{c+f}$ denotes the *projection map* defined by $\pi_\sigma(v|_{\Omega_i}) = v_j$ for some $j \in \Omega_i$ and $i\{1, \ldots, c + f\}$.
A useful result, which is a reformulation of a very classical result about group actions, is the following.

Lemma 16. If $\sigma \in \text{Aut}(C)$, $W_C(y) = \sum A_i y^i$ and $W_{F_\sigma(C)}(y) = \sum A_i^F y^i$, then $A_i \equiv A_i^F \mod p$.

Finally, let us introduce a classical decomposition of a code with an automorphism of prime order, which comes from Maschke’s theorem. Let $p$ be an odd prime and $\sigma$ is an automorphism of type $p-(c, f)$. Let

$$E_\sigma(C) := \{v \in C \mid \text{wt}(v|_{\Omega_i}) \equiv 0 \mod 2, \ i = 1, \ldots, c + f\},$$

where $v|_{\Omega_i}$ is the restriction of $v$ on $\Omega_i$.

Lemma 17 ([25]). If $p$ is odd, then $C = F_\sigma(C) \oplus E_\sigma(C)$. Moreover, if $C$ is self-dual, then

(a) the code $\pi_\sigma(F_\sigma(C)) \leq \mathbb{F}_2^{c+f}$ is self-dual and, if $C$ is doubly even and $p \equiv 1 \mod p$, then $\pi_\sigma(F_\sigma(C))$ is doubly even.

(b) $\dim E_\sigma(C) = \frac{(p-1)^c}{2}$.

2.3. Designs and codes

In this section we briefly recall the main definitions of design theory and its relationship with coding theory.

Definition 18. A $t-(v, k, \lambda)$ design, or briefly a $t$-design, is a pair $D = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of $v$ elements, called points, and $\mathcal{B}$ is a collection of distinct subsets of $\mathcal{P}$ of size $k$, called blocks, such that every subset of points of size $t$ is contained in precisely $\lambda$ blocks.

If $D$ is a $t-(v, k, \lambda)$ design, it is also an $i$-$(v, k, \lambda_i)$ design for all $i \in \{0, \ldots, t\}$, where $\lambda_i$ is given by $\lambda_i = \lambda \binom{v-i}{k-t-i}$.

Definition 19. Let $D = (\mathcal{P}, \mathcal{B})$ be a design with $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$.

(a) If we list the points $\{p_1, p_2, \ldots, p_v\}$ and the blocks $\{B_1, B_2, \ldots, B_b\}$, then we define the incidence matrix of $D$ as a $b \times v$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } p_j \in B_i \\ 0, & \text{if } p_j \notin B_i. \end{cases}$$
(b) The code \(C(D)\) over \(\mathbb{F}_2\) which is generated by the rows of \(A\) is called the code of the design \(D = (\mathcal{P}, \mathcal{B})\).

Although there are several incidence matrices, depending on the choice of the order of the points and of the blocks, for a given design the codes generated by these matrices are equivalent: changing the order of the points is equivalent to permute the coordinates, while a reordering of the blocks does not change the code (see [1, p.41]).

The following theorem, due to Assmus and Mattson, establishes a relationship between coding theory and design theory.

**Theorem 20** ([2]). Let \(C\) be a binary \([n, k, d]\) code. Suppose \(C^\perp\) has minimum weight \(d^\perp\). Suppose that \(A_0, \ldots, A_n\) and \(A_0^\perp, \ldots, A_n^\perp\) are the weight distributions of \(C\) and \(C^\perp\), respectively. Fix a positive integer \(t < d\), and let \(s\) be the number of \(i\) with \(A_i^\perp \neq 0\) for \(i \in \{0 \ldots n - t\}\). Suppose \(s \leq d - t\).

(a) The vectors of weight \(i\) in \(C\) form a \(t\)-design provided \(A_i \neq 0\) and \(d \leq i \leq n\).

(b) The vectors of weight \(i\) in \(C^\perp\) form a \(t\)-design provided \(A_i^\perp\) and \(d^\perp \leq i \leq n - t\).

3. The automorphism group of an extremal \([120, 60, 24]\) code

For the whole section, let \(C\) be an extremal \([120, 60, 24]\) code. By Theorem 3(b), we can easily deduce (see [27]) that

\[
W_C(1, y) = 1 + 39703755y^{24} + 6101289120y^{28} + 475644139425y^{32} + \ldots \quad (1)
\]

Knowledge of the existence of a non-trivial automorphism group \(G\) is very useful in constructing the code, since in this case the code has the structure of a \(\mathbb{F}_2G\)-module. For this reason, there is an intensive research on the automorphism group of extremal codes.

**Remark 21.** Concerning the code of length 120, the following results on the automorphism group \(G\) of \(C\) are known (see [7, 13, 14, 18, 19]):

(a) The order of \(G\) divides \(2^a \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23\) for a non-negative integer \(a\) (which is at most 116, since \(G \subseteq S_{120}\)).
(b) If $\sigma$ is an automorphism of $C$ of prime order $p$ then its cycle structure is

| $p$  | number of $p$-cycles | number of fixed points |
|------|----------------------|------------------------|
| 2    | 48, 60               | 24, 0                  |
| 3    | 40                   | 0                      |
| 5    | 24                   | 0                      |
| 7    | 17                   | 1                      |
| 19   | 6                    | 6                      |
| 23   | 5                    | 5                      |

(c) If $\sigma$ is an automorphism of $C$ of odd composite order $r$, then the cycle structure of $\sigma$ is either $15$-$\langle 8; 0 \rangle$, $3 \cdot 19$-$\langle 2, 0, 2; 0 \rangle$ or $5 \cdot 23$-$\langle 1, 0, 1; 0 \rangle$.

Moreover, if all involutions act fixed point freely, the following conditions hold:

(d) If $\sigma$ is an automorphism of $C$ of even composite order not divisible by 8, then the cycle structure of $\sigma$ is either $4$-$\langle 30; 0 \rangle$, $6$-$\langle 20; 0 \rangle$, $10$-$\langle 12; 0 \rangle$, $12$-$\langle 10; 0 \rangle$, $20$-$\langle 6; 0 \rangle$, $30$-$\langle 4; 0 \rangle$, $60$-$\langle 2; 0 \rangle$.

(e) The order of $G$ is in $\{7, 19, 23, 38, 56, 57, 114, 115, 552, 2760\}$ or $G$ is a $\{2, 3, 5\}$-group of order dividing 120.

Remark 22. Condition (d) is not stated explicitly in any of the above references, but it is an easy consequence of the results in [7]. Furthermore, we give only the structure of the automorphisms of even order not divisible by 8 because we prove in the following that an automorphism of order 8 cannot exist under the hypothesis that involutions act fixed point freely.

Remark 23. Condition (e) corrects a mistake in Proposition 15 b) of [14], where “$|G| = 2^3 \cdot 5^c \cdot 23$” should have been “$|G| = 2^3 \cdot 3 \cdot 5^c \cdot 23$”. Moreover, it gives a preciser statement about $\{2, 3, 5\}$-groups, based on Lemma [14].

3.1. Fixed code of automorphism of prime order

In this subsection we present some preliminary results about the automorphisms of prime order. It is a hard problem to prove that the primes 3, 5, 7, 19 and 23 cannot occur as orders of an automorphism $\sigma$ of $C$: even
if we can completely determine the fixed code $F_\sigma(C)$, there are too many possibilities to check for the complement $E_\sigma(C)$ defined in Section 2. Also the case of the prime 2 is computationally hard and we do not even know the fixed code.

**Automorphism of order 2:** Let $\sigma \in \text{Aut}(C)$ be of order 2. Then $\sigma$ is either of type $2$-(48; 24) or of type $2$-(60; 0). In the second case, by Theorem 1.2 of [10], $\pi_\sigma(F_\sigma(C))$ is a self-dual $[60, 30, 12]$ code. Although some self-dual codes with these parameters are known, a complete classification is still unknown.

**Automorphism of order 3:** Let $\sigma \in \text{Aut}(C)$ be of order 3. Then $\sigma$ is of type $3$-(40; 0) and $\pi_\sigma(F_\sigma(C))$ is a self-dual doubly-even $[40, 20, 8]$ code. By [5], there are 16470 such codes up to equivalence.

**Automorphism of order 5:** Let $\sigma \in \text{Aut}(C)$ be of order 5. Then $\sigma$ is of type $5$-(24; 0) and $\pi_\sigma(F_\sigma(C))$ is a self-dual $[24, 12, 8]$ code. This implies that $\pi_\sigma(F_\sigma(C))$ is equivalent to the binary extended Golay code $G_{24}$.

**Automorphism of order 7:** Let $\sigma \in \text{Aut}(C)$ be of order 7. Then $\sigma$ is of type $7$-(17; 1) and $\pi_\sigma(F_\sigma(C))$ is a self-dual $[18, 9, 4]$ code. By [28], $\pi_\sigma(F_\sigma(C))$ is equivalent to $H_{18}$ or $I_{18}$.

A vector of weight 4 in $\pi_\sigma(F_\sigma(C))$ has to be a vector of weight 28 in $F_\sigma(C)$, i.e. all nonzero coordinates of vectors of weight 4 correspond to cycles. By the study of clusters (see [25]) we can easily prove that $H_{18}$ cannot occur. Moreover, with the same technique, we can prove that, up to equivalence,

$$
\text{gen}(F_\sigma(C)) = \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

where 1 is the all-one vector and 0 the zero-vector of length 7.

**Automorphism of order 19:** Let $\sigma \in \text{Aut}(C)$ be of order 19. Then $\sigma$ is of type $19$-(6; 6) and $\pi_\sigma(F_\sigma(C))$ is a self-dual $[12, 6, 4]$ code. By [28],
\( \pi_\sigma(F_\sigma(C)) \) is equivalent to \( B_{12} \).

By Lemma 16 and by (1), \( A_{24}^T \equiv 0 \mod 19. \) Therefore there are 6 mod 19 vectors of \( F_\sigma(C) \) of weight 24. If \( v \in F_\sigma(C) \) has weight 24, then \( \text{wt}(\pi_\sigma(v)) = 6. \) Suppose that \( v_1, v_2 \in F_\sigma(C) \) of weight 24 coincide in the coordinate corresponding to a cycle of length 19. Then \( \text{wt}(v_1 + v_2) \leq 2. \) Therefore \( v_1 = v_2 \) and there are exactly 6 vectors in \( F_\sigma(C) \) of weight \( \text{wt}(v) = 24. \) These vectors are linearly independent and so, up to a permutation of the last six columns,

\[
\text{gen}(F_\sigma(C)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix},
\]

where \( \mathbf{1} \) is the all-one vector and \( \mathbf{0} \) the zero-vector of length 19.

**Automorphism of order 23:** Let \( \sigma \in \text{Aut}(C) \) be of order 23. Then \( \sigma \) is of type 23-(5; 5) and \( \pi_\sigma(F_\sigma(C)) \) is a self-dual [10, 5, 2] code. So (see [31]), up to equivalence,

\[
\text{gen}(F_\sigma(C)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

### 3.2. Automorphisms of composite order

In this subsection we present some new results about automorphisms of composite order. The result for the automorphism of order 8 is a corollary of Theorem 1.2. in [10] while the main idea for the other orders is the following: if \( \sigma \in \text{Aut}(C) \) is an automorphism of order \( p \cdot q, \) then, in some cases, we can classify the possible sums \( F_{\sigma \tau}(C) + F_{\sigma \tau}(C). \) If no sum has minimum distance greater than or equal to 24, then an automorphism of this order cannot occur. Note that these methods are a simplified version of those in Section V of [6].

**Automorphism of order 8:** Let \( \sigma \in \text{Aut}(C) \) be a fixed point free automorphism of order 8. Then \( \sigma \) is of type 8-(15; 0). By Theorem 1.2. in
C is a free $\mathbb{F}_2(\sigma^4)$-module, so, by Chouinard’s Theorem \[16\], C is a free $\mathbb{F}_2(\sigma)$-module. This is impossible, since 8 does not divide 60.

**Theorem 24.** The automorphism group of a self-dual $[120,60,24]$ code does not contain fixed point free elements of order 8.

**Automorphism of order 30:** Let $\sigma \in \text{Aut}(C)$ be of order 30. Then $\sigma$ is of type 30-(4; 0). We can suppose, up to equivalence, that

$$
\sigma = (1, \ldots, 30)(31, \ldots, 60)(61, \ldots, 90)(91, \ldots, 120).
$$

Let $\sigma_3 := \sigma^{10}$ and $\sigma_5 := \sigma^6$. Then $\sigma_3$ is of type 3-(40; 0) and $\sigma_5$ is of type 5-(24; 0). Since $\sigma$ is in the centralizer of both $\sigma_3$ and $\sigma_5$ in $S_{120}$, it acts on $\pi_{\sigma_3}(F_{\sigma_3}(C))$ as an automorphism, say $\pi_{\sigma_3}(\sigma)$, of type 10-(4; 0) and on $\pi_{\sigma_5}(F_{\sigma_5}(C))$ as an automorphism, say $\pi_{\sigma_5}(\sigma)$, of type 6-(4; 0). Among the 16470 self-dual $[40,20,8]$ codes, only 28, say $D_1, \ldots, D_{28}$, have an automorphism of this type, for a total of 69 conjugacy classes. So, up to a permutation in $C_{S_{120}}(\pi_{\sigma_3}(\sigma))$, $\pi_{\sigma_3}(C)$ belong to a set, say $\mathcal{D}$, of 69 elements. On the other hand, the extended binary Golay code has only one conjugacy class of elements of type 6-(4; 0). If $E_0$ is an extended binary Golay code with automorphism $\pi_{\sigma_3}(\sigma)$, then the orbit, say $\mathcal{E}$, of $E_0$ under the action of $C_{S_{24}}(\pi_{\sigma_3}(\sigma))$ has 1296 elements. The code $\pi_{\sigma_3}(F_{\sigma_3}(C))$ belongs to $\mathcal{E}$. With Magma \[12\] we check that all the codes in $\mathcal{C} := \{\pi_{\sigma_3}^{-1}(D) + \pi_{\sigma_3}^{-1}(E) \mid D \in \mathcal{D}, E \in \mathcal{E}\}$ have minimum distance less than 24. Since $F_{\sigma_3}(C) + F_{\sigma_5}(C) \subseteq C$ would have to belong to $\mathcal{C}$, this implies the following result.

**Theorem 25.** The automorphism group of a self-dual $[120,60,24]$ code does not contain elements of order 30.

This theorem implies that also automorphism of order 60 cannot occur in $\text{Aut}(C)$.

**Automorphism of order 57:** Let $\sigma \in \text{Aut}(C)$ be of order 57. Then $\sigma$ is of type $3 \cdot 19-(2,0,2;0)$. We can suppose, up to equivalence, that

$$
\sigma = (1, \ldots, 57)(58, \ldots, 114)(115, 116, 117)(118, 119, 120).
$$

Let $\sigma_3 := \sigma^{19}$ and $\sigma_{19} := \sigma^3$. Then $\sigma_3$ is of type 3-(40; 0) and $\sigma_{19}$ is of type 19-(6; 6). Since $\sigma$ is in the centralizer of both $\sigma_3$ and $\sigma_{19}$ in $S_{120}$, it acts on $\pi_{\sigma_3}(F_{\sigma_3}(C))$ as an automorphism, say $\pi_{\sigma_3}(\sigma)$, of type 19-(2; 2) and
on $\pi_{\sigma_{19}}(F_{\sigma_{19}}(C))$ as an automorphism, say $\pi_{\sigma_{19}}(\sigma)$, of type 3-(4; 0). Among the 16470 self-dual $[40, 20, 8]$ codes, only 3, say $D_1, D_2$ and $D_3$, have an automorphism of this type, for a total of 396 conjugacy classes. So, up to a permutation in $C_{S_{40}}(\pi_{\sigma_{19}}(\sigma))$, $\pi_{\sigma_{19}}(\sigma)$ belong to a set, say $\mathcal{D}$, of 396 elements. On the other hand, the code $B_{12}$ has only one conjugacy class of elements of type 3-(4; 0). If $E_0$ is a $B_{12}$ code with automorphism $\pi_{\sigma_{19}}(\sigma)$, then the orbit, say $\mathcal{E}$, of $E_0$ under the action of $C_{S_{12}}(\pi_{\sigma_{19}}(\sigma))$ has 27 elements. The code $\pi_{\sigma_{19}}(F_{\sigma_{19}}(C))$ belongs to $\mathcal{E}$. With Magma [12], we check that all the codes in $\mathcal{C} := \{\pi_{\sigma_{3}}^{-1}(D) + \pi_{\sigma_{19}}^{-1}(E) \mid D \in \mathcal{D}, E \in \mathcal{E}\}$ have minimum distance less than 24. Since $F_{\sigma_{3}}(C) + F_{\sigma_{19}}(C) \subseteq C$ would have to belong to $\mathcal{C}$, this implies the following result.

**Theorem 26.** The automorphism group of a self-dual $[120, 60, 24]$ code does not contain elements of order 57.

**Other orders:** in the case of automorphisms of order 12 (fixed point free), 15, 20 (fixed point free) and 115 we do not get a contradiction on the minimum distance, while in the case of the automorphism of order $2 \cdot p$, with $p$ prime, we cannot use the method above, since we do not have a classification of the fixed code by the automorphism of order 2.

### 3.3. Structure of the automorphism group in the fixed point free case

In this subsection we present a theorem on the structure of the automorphism group of a self-dual $[120, 60, 24]$ code as in Section 6 of [11] for the self-dual $[72, 36, 16]$. Note that in [13] it is proved that involutions acting on extremal codes of length $24m$ with $m > 1$ are always fixed point free, except for $m = 5$, i.e. our case. It seems to be very difficult, although very interesting, to exclude this exceptional case. Allowing fixed points increases enormously the number of possible automorphism groups and we cannot get nice results. Therefore we decided, as in [14], to restrict our attention to the fixed point free case in order to get, at least under this hypothesis, a stronger result.

**Theorem 27.** If all the involutions act fixed point freely, the automorphism group $G$ of a self-dual $[120, 60, 24]$ code is trivial or isomorphic to one of the following 64 groups:
Proof. All assertions about groups of order less than or equal to 552 make use of the library SmallGroups of Magma \cite{12}. Condition (e) of Remark 21 implies that the order of $G$ is in \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 15, 19, 20, 23, 24, 30, 38, 40, 56, 57, 60, 114, 115, 120, 552, 2760\}. Moreover, by Remark \cite{21} and by the previous results, the order of every element in $G$ is in $O := \{1, 2, 3, 4, 5, 6, 7, 10, 12, 15, 19, 20, 23, 115\}$.

If $|G| = 2760$, there exists either one 23-Sylow or 24 23-Sylow subgroups. In the first case, the 23-Sylow is normal and its product with a 2-Sylow subgroup is a subgroup of order 184. All groups of order 184 contain an element of order 46. In the second case, $G$ acts on the 24 23-Sylow subgroups and $G_H$ is of order 115 for every 23-Sylow subgroup $H$. Therefore $G_H$ is cyclic. Let $K$ be the only subgroup of $G_H$ of order 23. This acts on 23 groups (all except $H$) and so it has 23 fixed points. Then $K$ is contained in $G_H$ for every $H$ and it should be the unique group of order 23 which is

| Order | Groups | Order | Groups |
|-------|--------|-------|--------|
| 2     | $C_2$  | 23    | $C_{23}$ |
| 3     | $C_3$  | 24    | $Dic_{24}, S_3 \times C_4, D_{12}, Dic_{12} \times C_2, C_3 \times D_4, C_6 \times C_4, D_4 \times C_3, S_4, A_4 \times C_2, D_6 \times C_2, C_2 \times C_2 \times C_2 \times C_2 \times C_3$ |
| 4     | $C_4, C_2 \times C_2$ | 30    | $D_{15}, C_5 \times S_3, C_3 \times D_5$ |
| 5     | $C_5$  | 38    | $D_19$ |
| 6     | $C_6, S_3$ | 40    | $C_2 \times C_2, D_5 \times C_4, C_5 \times (C_4 \times C_2), D_{20}, C_5 \times D_4, D_4 \times C_5, GA(1, 5) \times C_2, D_5 \times C_2 \times C_2, C_2 \times C_2 \times C_2 \times C_2 \times C_5$ |
| 7     | $C_7$  | 56    | $(C_2 \times C_2 \times C_2) \times C_7$ |
| 8     | $C_4 \times C_2, C_2 \times C_2 \times C_2, D_4$ | 57    | $C_{19} \times C_3$ |
| 10    | $C_{10}, D_5$ | 60    | $A_5, D_5 \times C_6, C_{15} \times C_4, D_{15} \times C_2, A_4 \times C_5$ |
| 12    | $C_{12}, C_6 \times C_2, D_6, A_4, Dic_{12}$ | 114   | $C_{19} \times C_6$ |
| 15    | $C_{15}$ | 115   | $C_{115}$ |
| 19    | $C_{19}$ | 120   | $S_5, A_5 \times C_2, S_4 \times C_5, A_4 \times D_5, A_4 \times D_5$ |
| 20    | $C_{20}, C_{10} \times C_2, D_{10}, Dic_{20}, GA(1, 5)$ | | |
contained in $G_H$ for all $H$. This is not possible, since every $H$ is contained in $G_H$, so we have a contradiction.

The quaternion group $Q_8$ cannot occur, again by Chouinard’s Theorem (see the proof above for the element of order 8).

The group $C_5 \rtimes (S_3 \times C_4)$, of order 120, is not possible, since if $\sigma$ is the element of order 5, then in the automorphism group of $\pi_\sigma(F_\sigma(C))$ (which is an extended binary Golay code) there should be a subgroup isomorphic to $S_3 \times C_4$ acting fixed point freely, and this is not the case.

Finally, all the other groups are excluded by verifying that they have elements of order which is not in $O$, or a subgroup isomorphic to $Q_8$ or to $C_5 \rtimes (S_3 \times C_4)$. \hfill \Box

**Remark 28.** It would be interesting to exclude other non-abelian groups, using methods similar to those in [8], or elementary abelian groups, using methods similar to those in [9]. However, for a lack of classification of smaller codes, this seems to be still computationally impossible. It would be also interesting to get similar results without the hypothesis of the fixed point free action, but this seems to make the number of possibilities grow enormously. Finally, another direction of further research can be to get a similar result for the extremal code of length 96, which is studied in [15, 20, 22], but this is beyond the aim of this paper.

4. Some necessary conditions for the existence of a self-dual extremal [120, 60, 24] code

In this section we establish some necessary conditions for the existence of an extremal [120, 60, 24] code. Similar conditions are given in [23] and [4] for an extremal [72, 36, 16] and [96, 48, 20] code.

Let $C$ be a [118, 59, 22] type I code. By Theorem 3 (a) we have

$$W_C(y) = \sum_{j=0}^{14} a_j (1 + y^2)^{59-4j} (y^2(1 - y^2)^2)^j = a_0 + (59a_0 + a_1)y^2 + (1711a_0 + 53a_1 + a_2)y^4 + (32509a_0 + 1376a_1 + 47a_2 + a_3)y^6 + (455126a_0 + 23320a_1 + 1077a_2 + 41a_3 + a_4)y^8 + \ldots$$

with $a_j \in \mathbb{Q}$ for $j = 0, \ldots, 14$. Since the minimum distance of $C$ is 22, we get $a_0 = 1$, $a_1 = -59$, $a_2 = 1416$, $a_3 = -17877$, $a_4 = 128679$, $a_5 = -538375$, $a_6 = 1472509$, $a_7 = -3610308$, $a_8 = 22990664$, $a_9 = -124747072$, $a_{10} = 748297056$, $a_{11} = -748297056$, $a_{12} = 22990664$, $a_{13} = -3610308$, $a_{14} = 2147483648$. The coefficients of $y^j$ for $j = 0, \ldots, 14$ are given in the table below.

| $j$ | $a_j$ |
|-----|-------|
| $0$ | $1$   |
| $1$ | $-59$ |
| $2$ | $1416$|
| $3$ | $-17877$|
| $4$ | $128679$|
| $5$ | $-538375$|
| $6$ | $2147483648$|

Using these coefficients, we can verify that the necessary conditions for the existence of a self-dual extremal [120, 60, 24] code are satisfied.
Let $S$ be the shadow of $C$. Then by Theorem 5 (a) we have

$$W_S(y) = \frac{1}{33554432}a_{14}y^3 + \left(\frac{1}{524288}a_{13} - \frac{7}{8388608}a_{14}\right)y^7 + \left(\frac{189}{16777216}a_{14} + \frac{1}{8192}a_{12} + \frac{13}{262144}a_{13}\right)y^{11} + \left(-\frac{1}{819}\frac{a_{14}}{8388608} - \frac{1}{128}a_{11} - \frac{3}{2576}a_{13}\right)y^{15} + \left(\frac{325}{65536}a_{13} + \frac{69}{2048}a_{12} + \frac{11}{64}a_{11} + \frac{20475}{33554432}a_{14}\right)y^{19} + \left(-\frac{231}{128}a_{11} - \frac{7475}{262144}a_{13} - \frac{12285}{4194304}a_{14} - \frac{253}{1024}a_{12} + 12811968\right)y^{23} + \ldots$$

Let $W_S(y) = \sum_{i=0}^{118} B_i y^i$. By Theorem 5 (b) we have $B_i \in \{0, 1\}$ for $i = 3, 7$ and at most one $B_i$ is nonzero for $i \leq 11$. Therefore there are four possibilities:

- If $B_3 = 1$, $B_7 = 0$, $B_{11} = 0$, then $a_{14} = 33554432$, $a_{13} = -14680064$, $a_{12} = 2867200$. Since $B_{15} = -2576 - \frac{1}{128}a_{11} \geq 0$, we have $B_{19} = 44275 + \frac{11}{164}a_{11} < 0$, a contradiction.

- If $B_3 = 0$, $B_7 = 1$, $B_{11} = 0$, then $a_{14} = 0$, $a_{13} = -524288$, $a_{12} = 212992$. Since $B_{15} = -299 - \frac{1}{128}a_{11} \geq 0$, we have $B_{19} = 4576 + \frac{11}{164}a_{11} < 0$. It is again a contradiction.

- If $B_3 = 0$, $B_7 = 0$, $B_{11} \neq 0$, then $a_{14} = a_{13} = 0$ and $a_{12} > 0$. Therefore the system of inequalities $B_{15} = -\frac{1}{128}a_{11} - \frac{3}{1024}a_{12} \geq 0$ and $B_{19} = \frac{69}{2048}a_{12} + \frac{11}{64}a_{11} \geq 0$ has no solutions, a contradiction.

Hence $B_3 = 0$, $B_7 = 0$, $B_{11} = 0$ and we obtain $a_{14} = a_{13} = a_{12} = a_{11} = 0$.

In conclusion the shadow has minimal distance 23 and we can calculate the weight enumerators $W_C(y)$ and $W_S(y)$ (see Table 1 and Table 2).

**Lemma 29.** Let $C_0$ be the subcode of $C$ containing all codewords whose weights are multiples of 4. Then the supports of all vectors of a given weight in $C_0$ and in $C_0^\perp$ form a 3-design.

**Proof.** Since $C_0$ is a [118, 58, 24] code and $C_0^\perp = C \cup S$, then $C_0^\perp$ is a [118, 60, 22] code. If $W_{C_0}(x, y) = \sum_{i=0}^{118} A_i x^{118-i} y^i$, then we have $|\{i \mid A_i \neq 0, 0 < i \leq 115\}| = 19 \leq d(C_0^\perp) - 3$. Therefore by Theorem 20 the supports of the vectors of weight $i$ in $C_0^\perp$ and in $C_0$ form a 3-design. QED

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Table 1: The weight enumerator of a self-dual [118, 59, 22] code

| $i$  | $A_i$        |
|------|-------------|
| 22   | 96          |
| 24   | 94          |
| 26   | 92          |
| 28   | 90          |
| 30   | 88          |
| 32   | 86          |
| 34   | 84          |
| 36   | 82          |
| 38   | 80          |
| 40   | 78          |
| 42   | 76          |
| 44   | 74          |
| 46   | 72          |
| 48   | 70          |
| 50   | 68          |
| 52   | 66          |
| 54   | 64          |
| 56   | 62          |
| 58   | 60          |

**Question 30.** By Assmus-Mattson theorem the supports of the codewords of minimal weight 22 in a self-dual [118, 59, 22] code build a 3-(118, 22, 8885) design $D$. Similarly the supports of the codewords of minimal weight 24 in a self-dual [120, 60, 24] code build a 5-(120, 24, 8855) design. In [21] the authors showed that if $D$ is a self-orthogonal 5-(120, 24, 8855) design, then $C(D)$ is a self-dual [120, 60, $d$] code with $d = 20$ or $d = 24$. Unfortunately it was not possible to exclude the case $d = 20$ and show that the code is extremal. A natural question, although very difficult, is: if $D$ is a self-orthogonal 3-(118, 22, 8885) design, then is $C(D)$ a self-dual extremal [118, 59, 22] code?

**Remark 31.** Since the shadow of a self-dual [118, 59, 22] code has minimal distance 23, the existence of a self-dual [120, 60, 24] code is equivalent to the existence of a self-dual [118, 59, 22] code $C$. (In general by [29] the existence of an extremal code of length $24m$ is equivalent to the existence of a self-dual code...
Table 2: The weight enumerator of the shadow of a self-dual [118, 59, 22] code

| $i$  | $B_i$          |
|------|---------------|
| 23 95| 12811968      |
| 27 91| 2201249408    |
| 31 87| 187592982720  |
| 35 83| 7972733942784 |
| 39 79| 178129081470720 |
| 43 75| 2168688143930880 |
| 47 71| 14778320201079552 |
| 51 67| 57459493525644288 |
| 55 63| 129133310381938304 |
| 59 | 169008544553322240 |

$[24m - 2, 12m - 1, 4m + 2]$ code). By Lemma 29 the supports of the vectors of weight $k$ in the shadow $S$ of $C$ form a 3-(118, $k$, $\lambda$) design. Therefore if there exists a [118, 59, 22] code and its shadow has enumerator weight $W_S(y) = \sum_{k=0}^{118} B_k y^k$, then the coefficients of the shadow must satisfy the following condition of divisibility

\[
\frac{(k-i)!}{k!} \cdot \frac{(118-i)!}{(118-i)!} \mid B_k,
\]

since the terms $\lambda_i = B_k \frac{k!}{(k-i)!} \frac{(118-i)!}{118!} \in \mathbb{N}_0$. Actually, all the coefficients $B_k$ satisfy this condition (see Table 3).

Now we have the following necessary condition on the existence of an extremal type II code of length 120.

**Theorem 32.** If no linear $[120, 60, 23]$ code with weight enumerator given in Table 4 exists, then there exists no self-dual $[120, 60, 24]$ code.

**Proof.** A self-dual $[120, 60, 24]$ code has a child $C$, which is a self-dual $[112, 59, 22]$ code. By Lemma 8 $W_{C_1}(y) = W_{C_3}(y)$. Therefore the code $C^*$ defined as in Lemma 9 is formally self-dual, and the theorem follows (the weight enumerator of the code $C^*$ is given in Table 4). It is calculated thanks to Lemma 9 knowing the weight enumerator of the [118, 59, 22] type I code and of its shadow, which are given in Table 1 and Table 2 respectively). □

**Theorem 33.** If no self-dual doubly-even $[120, 60, 4]$ code with weight enumerator given in Table 5 exists, then there exists no self-dual $[120, 60, 24]$ code.
Table 3: Parameters of the 3-designs

| \(k\) | \(\lambda_0 = B_k\) | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3 = \lambda\) |
|-------|------------------|-------------|-------------|-----------------|
| 23    | 12811968         | 2497248     | 469568      | 58008           |
| 27    | 2201249408       | 503675712   | 9327328     | 24122400        |
| 31    | 187592882720     | 49282902240 | 12636641600 | 3159160400      |
| 35    | 797273932784     | 2364793966080 | 687205084160 | 19549798080     |
| 39    | 178129081470720  | 58873170994560 | 19121200835840 | 609903714880 |
| 43    | 216688143930880  | 79028466218880 | 283691930170880 | 97774813276480 |
| 47    | 1477832020107552  | 588628080091008 | 2314263963112704 | 89774813276480 |
| 51    | 57459493525644288 | 24834187879727616 | 10612900803302400 | 4483035684153600 |
| 55    | 129133310381938304 | 60189254839727616 | 2779966007556480 | 12692429070831840 |
| 59    | 16908541455322240 | 8454272276611200 | 41891006769626880 | 20584372016109760 |
| 63    | 129133310381938304 | 60189254839727616 | 2779966007556480 | 12692429070831840 |
| 67    | 57459493525644288 | 24834187879727616 | 10612900803302400 | 4483035684153600 |
| 71    | 1477832020107552  | 588628080091008 | 2314263963112704 | 89774813276480 |
| 75    | 216688143930880   | 79028466218880 | 2836919301708800 | 97774813276480 |
| 79    | 178129081470720   | 58873170994560 | 191212008358400 | 609903714880 |
| 83    | 797273932784      | 2364793966080 | 6872050841600 | 19549798080 |
| 87    | 187592982720      | 503675712248 | 9327328000000 | 241224004000 |
| 91    | 2201249408        | 49282902240 | 12636641600 | 3159160400      |
| 95    | 12811968         | 2497248     | 469568      | 58008           |

Proof. Let \(C\) be a self-dual \([120, 60, 24]\) code, \(u \in \mathbb{F}_2^{120}\) of weight \(\text{wt}(u) = 4\) and \(D := C \cap \langle u \rangle\). Since \(D \perp = C \perp + \langle u \rangle\), then \(D \leq D \perp = \dim(C + \langle u \rangle) = 61\) i.e. \(D\) is a self-orthogonal \([120, 59]\) code. If \(N \triangleq \langle D, u \rangle\), then \(N = D \oplus \langle u \rangle\) and \(\dim(N) = 60\). Since \((d_1 + u) \cdot (d_2 + u) = 0\), for all \(d_1, d_2 \in D\), we have that \(N\) is self-orthogonal and therefore it is self-dual. On the other hand, \(\text{wt}(d + u) \equiv 0 \mod 4\), because \(d \cdot u = 0\) for all \(d \in D\). Therefore \(N\) is a self-dual doubly-even \([120, 60, 4]\) code. Since \(\dim(N \cap C) = 59\), we have \(N\) is neighbor of \(C\). The code \(N\) has only one vector of weight 4. To determine the number of vectors of weight 20 it is sufficient to calculate the number of vectors \(w \in C\) of weight 24 with \(|\text{supp}(w) \cap \text{supp}(u)| = 4\). Since the vectors of weight 24 in \(C\) form a 5-design, then this number is equal to \(\lambda_4 = 51359\) where \(\lambda_4\) is the number of blocks incident with 4 different points. If \(\sum A_i y^i\) is the weight enumerator of \(N\), then \(A_0 = 1, A_4 = 1, A_8 = 0, A_{12} = 0, A_{16} = 0\) and \(A_{20} = 51359\). Therefore by Theorem 3 we obtain the weight enumerator of the code \(N\), which is given in Table 5. This concludes the proof. \(\square\)
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Table 4: The weight enumerator of a formally self-dual $[120, 60, 23]$ code

| Weight | Weight distribution | Weight | Weight distribution |
|--------|---------------------|--------|---------------------|
| 0      | 1                   | 61     | 84504272276661120   |
| 23     | 1534767             | 63     | 72637487089840296   |
| 24     | 31763004            | 64     | 120053624495708267  |
| 25     | 6405984             | 65     | 6456655190969152    |
| 27     | 323009424           | 67     | 37017173713636224   |
| 28     | 4677654992          | 68     | 50277056834938752   |
| 29     | 1100624704          | 69     | 28729746762822144   |
| 31     | 33041945820         | 71     | 10929799315381752   |
| 32     | 348805702245        | 72     | 12212639610614352   |
| 33     | 93796491360         | 73     | 7389160100539776    |
| 35     | 1660986238080       | 75     | 1848313759032000    |
| 36     | 13177157488768      | 76     | 1697854533735360    |
| 37     | 3986366971392       | 77     | 1084344071965440    |
| 39     | 43420813336368      | 79     | 175902467952336     |
| 40     | 264967008687696     | 80     | 132485354071728     |
| 41     | 89064540735360      | 81     | 89064540735360      |
| 43     | 613510461769920     | 83     | 9190790517376       |
| 44     | 293265783097440     | 84     | 5647353209472       |
| 45     | 1084344071965440    | 85     | 3986366971392       |
| 47     | 4823479510074576    | 87     | 255009210885        |
| 48     | 18318959415921528   | 88     | 126383437180        |
| 49     | 7389160100539776    | 89     | 93796491360         |
| 51     | 21547310072116608   | 91     | 3577030288          |
| 52     | 65746920476458368   | 92     | 1423634128          |
| 53     | 28729746762822144   | 93     | 1100624704          |
| 55     | 55486969304739115   | 95     | 25357020            |
| 56     | 137204142280809448  | 96     | 7940751             |
| 57     | 6456655190969152    | 97     | 6405984             |
| 59     | 83095867738716768   |        |                    |
| 60     | 167600140015377888  |        |                    |
Table 5: The weight enumerator of a $[120, 60, 4]$ neighbor of an extremal $[120, 60, 24]$ code

| $i$ | $A_i$       |
|-----|------------|
| 0   | 1         |
| 4   | 1         |
| 20  | 51359     |
| 24  | 43481179  |
| 28  | 6539254776|
| 32  | 494044041905|
| 36  | 19178964940125|
| 40  | 400399951557816|
| 44  | 4639015235035296|
| 48  | 30526043817770504|
| 52  | 115980280893408771|
| 56  | 257259077150523955|
| 60  | 335272511326715600|