Non dissipative decoherence of Rabi oscillations

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We present a simple theoretical description of two recent experiments where damping of Rabi oscillations, which cannot be attributed to dissipative decoherence, has been observed. This is obtained considering the evolution time or the Hamiltonian as random variables and then averaging the usual unitary evolution on a properly derived, model-independent, probability distribution.

Even though decoherence is a very general phenomenon, it is very difficult to verify it experimentally because most often the physical nature of the environmental degrees of freedom responsible for the decoherence process remains unknown. The only controlled experimental verification of decoherence has been given by the experiment of Ref. 4, in which the progressive transformation of a linear superposition of two coherent states of a microwave cavity mode into the corresponding statistical mixture has been monitored. In this case, the environmental decoherence has been checked with no fitting parameters because its physical origin, i.e., photon leakage out of the cavity, was easily recognizable and measurable. In this case, it is even possible to control decoherence, i.e., to considerably suppress its effects, for example by using appropriately designed feedback schemes.

In some cases however, the mechanisms responsible for decoherence are not easily individuated and examples are provided by two recent experiments which observed Rabi oscillations between two circular states of a Rydberg atom in a high-Q cavity 5, and of two internal states of a 9Be + ion coupled with the vibrations in a trapping potential 6. In both cases one observes damped oscillations to a steady state in which the population of each of the two levels approaches 1/2. A number of candidates have been already considered as possible physical sources of decoherence in these cases. In the trapped ion case of Ref. 6, fluctuations of classical parameters such as the intensity of the laser beams used to couple internal and vibrational degrees of freedom, or the voltage and the frequency of the trapping potentials have been suggested. In the case of the Rydberg atom in a high-Q cavity, dark counts of the atomic detectors, dephasing collisions with background gas or stray magnetic fields within the cavity have been proposed as possible sources of decoherence. Despite this, a complete quantitative explanation of the observed decay rate of the Rabi oscillations (see Ref. 7) in the two experiments is still lacking. The only established fact is that, differently from Ref. 4, in both cases, decoherence has a non-dissipative origin. In fact, the observed decay of the Rabi oscillations is much faster than the energy relaxation rate in these experimental configurations. Moreover, the fact that in both cases the population of each of the two levels asymptotically approaches 1/2 cannot be explained in terms of dissipative mechanisms as the photon leakage out of the cavity.

A different approach to decoherence has been proposed in 8, where a model-independent formalism has been derived to describe decoherence. Here we shall adopt a more pragmatic point of view and we shall use this formalism to explain in simple terms both Rabi oscillation experiments, even though they are realized in different physical situations. The idea underlying the approach of Ref. 8 is the fact that the interaction time, i.e., the time interval in which the effective Hamiltonian evolution takes place, is a random variable. This randomness can have different origins depending on the studied system. For example, in the case of the Rydberg atom experiment 5, the interaction time is determined by the transit time of the velocity-selected atom through the high-Q microwave cavity. This interaction time is random, due to fluctuations of the atomic velocities. This randomness implies having random phases $e^{-iE_n t/\hbar}$ in the energy eigenstates basis. The experimental results unavoidably average over these random phases and this leads to decoherence, i.e., to the decay of off-diagonal matrix elements of the density operator in the energy basis. Notice however that one would have the same phase fluctuations if the Hamiltonian (and therefore the eigenvalues $E_n$) fluctuates instead of the interaction time. Therefore, as we shall see, our approach will give a generalized phase-destroying master equation, able to describe many situations in which decoherence is associated with random phases, originating for example from some frequency or interaction time fluctuations.

Let us consider an initial state $\rho(0)$ and consider the case of a random evolution time. The experimentally observed state is not described by the usual density matrix of the whole system $\rho(t)$, but by its time averaged counterpart

$$\bar{\rho}(t) = \int_0^\infty dt' P(t,t') \rho(t') ,$$

(1)

where $\rho(t') = \exp(-iLt')\rho(0)$ is the usual unitarily evolved density operator from the initial state and $L \ldots = ...$
\[ H, \ldots ]/\hbar. \] Hence one can write
\[ \dot{\rho}(t) = V(t)\rho(0), \quad (2) \]
where \( V(t) = \int_0^\infty dt' P(t, t') e^{-iLt'}. \) In Ref. 3, the function \( P(t, t') \) has been determined so to satisfy the following conditions: i) \( \dot{\rho}(t) \) must be a density operator, i.e. it must be self-adjoint, positive-definite, and with unit-trace. This leads to the condition that \( P(t, t') \) must be non-negative and normalized, i.e. a probabil-
ity density in \( t' \) so that Eq. (2) is a completely posi-
tive mapping. ii) \( V(t) \) satisfies the semigroup property \( V(t_1 + t_2) = V(t_1)V(t_2) \), with \( t_1, t_2 \geq 0 \). These require-
ments are satisfied by 
\[ V(t) = (1 + iL \tau)^{-t/\tau} \quad (3) \]
\[ P(t, t') = \frac{e^{-t'/\tau}(t'/\tau)^{t(\tau)-1}}{\Gamma(t/\tau)} \quad (4) \]
so that \( V(t) \) and \( P(t, t') \) are connected by the so-called \( \Gamma \)-function integral identity [9,10]. The parameter \( \tau \) char-
acterizes the strength of the evolution time fluctuations. When \( \tau \to 0 \), \( P(t, t') \to \delta(t - t') \) so that \( \dot{\rho}(t) = \rho(t) \) and \( V(t) = \exp\{-iLt\} \) is the usual unitary evolution. How-
ever, for finite \( \tau \), the evolution operator \( V(t) \) of Eq. (2) 

describes a decay of the off-diagonal matrix elements in the energy represen-
tation, whereas the diagonal matrix elements remain constant, i.e. the energy is still a con-
stant of motion. In fact, by differentiating with respect to \( t \) and using Eq. (4), one gets the follow-
ing master equation for \( \dot{\rho}(t) \)
\[ \dot{\rho}(t) = \frac{1}{\tau} \log (1 + iL \tau) \dot{\rho}(t) \quad (5) \]
If one expands the logarithm at second order in \( \tau \), one obtains
\[ \dot{\rho}(t) = \frac{i}{\hbar} [H, \rho(t)] - \frac{\tau}{\hbar} [H, [H, \rho(t)]] \quad (6) \]
which is the well-known phase-destroying master equation [11]. Hence Eq. (2) appears as a gen-
eralized phase-
destroying master equation taking into account higher order terms in \( \tau \). Notice, however, that the present ap-
proach is different from the usual master equation ap-
proach in the sense that no perturbative and specific statistical assumptions are made.

We now apply this formalism to the two experiments of Refs. 3,13. In the experiment of Ref. 3, the reso-
nant interaction between a quantized mode in a high-Q microwave cavity (with annihilation operator \( a \)) and two circular Rydberg states \( \langle e | \) and \( \langle g | \) of a Rb atom is studied. This interaction is well described by the usual Jaynes-Cummings [12] model, in which the interaction picture reads
\[ H = \hbar \Omega_R (|e\rangle\langle g| a + |g\rangle\langle e| a^\dagger) \quad (7) \]
where \( \Omega_R \) is the Rabi frequency. The Rabi oscilla-
tions describing the exchange of excitations between atom and cavity mode are studied by injecting the velocity-selected Rydberg atom, prepared in the excited state \( |e\rangle \), in the high-Q cavity and measuring the population of the lower atomic level \( g \), \( P_{eg}(t) \) as a function of the interaction time \( t \), which is varied by changing the Rydberg atom velocity. In the case of vacuum state induced Rabi osci-
lations, the decoherence effect is particularly evident and the Hamiltonian evolution according to Eq. (2) predicts
\[ P_{eg}(t) = \frac{1}{2} (1 - \cos (2\Omega_R t)) \quad (8) \]
Experimentally instead, damped oscillations are observed, which are well fitted by
\[ P_{exp}(t) = \frac{1}{2} (1 - e^{-\gamma t} \cos (2\Omega_R t)) \quad (9) \]
where the decay time fitting the experimental data is \( \gamma^{-1} = 40 \mu s [3] \) and the corresponding Rabi frequency is \( \Omega_R /2\pi = 25 \text{ KHz} \). This decay of quantum coherence cannot be associated with photon leakage out of the cavity because the cavity relaxation time is larger (220 \( \mu s \)) and also because in this case one would have an asymp-
totic limit \( P_{exp}(\infty) = 1 \). The damped behavior of Eq. (9) is instead easily obtained if one applies the approach described above. In fact, from the linearity of Eq. (5), one has that the time averaging procedure is also valid for mean values and matrix elements of each subsystem. Therefore one has
\[ \bar{P}_{eg}(t) = \int_0^\infty dt' P(t, t') P_{eg}(t') \quad (10) \]
Using Eqs. (2), (3), (4) and (5), Eq. (10) can be rewritten in the same form of Eq. (9)
\[ \bar{P}_{eg}(t) = \frac{1}{2} (1 - e^{-\gamma t} \cos (\nu t)) \quad (11) \]
where
\[ \gamma = \frac{1}{2\pi} \log \left(1 + 4\Omega_R^2 \tau^2 \right) \quad (12) \]
\[ \nu = \frac{1}{\tau} \arctg (2\Omega_R \tau) \quad (13) \]
We note that in general the time averaging procedure introduces not only a damping of the probability osci-
lations but also a frequency shift. However, if the charac-
teristic time \( \tau \) is sufficiently small, i.e. \( \Omega_R \tau \ll 1 \), there is no phase shift, \( \nu \approx 2\Omega_R \), and
\[ \gamma = 2\Omega_R^2 \tau \quad (14) \]
The fact that in Ref. 3 the Rabi oscillation frequency essentially coincides with the theoretically expected one, suggests that the time \( \tau \) characterizing the fluctuations of the interaction time is sufficiently small so that it is reason-
able to use Eq. (14). Using the above values for \( \gamma \) and \( \Omega_R \), one can derive an estimate for \( \tau \), so to get \( \tau \approx 0.5 \)
μsec. This estimate is consistent with the assumption \( \Omega R \tau \ll 1 \) we have made, but, more importantly, it turns out to be comparable to the experimental value of the uncertainty in the interaction time. In fact, the fluctuations of the interaction time are mainly due to the experimental uncertainty of the atomic velocity \( v \). In fact, one has \( t = \sqrt{\pi} w / v \), where \( w \) is the cavity mode waist. Since \( w = 0.6 \) cm, the mean velocity is \( \bar{v} \approx 300 \) m/sec and the velocity uncertainty is \( \delta v / v = 1\% \) (see Ref. [8]), one has \( t = \sqrt{\pi} w / \bar{v} \approx 50 \mu sec \) and \( \tau \approx \delta t = \delta v / v = 0.5 \mu sec \), which is just the estimate we have derived from the experimental values. This simple argument supports the interpretation that the decoherence observed in [9] is essentially due to the randomness of the interaction time.

Let us now consider the case of the trapped ion experiment of Ref. [4], in which the interaction between two internal states (|↑⟩ and |↓⟩) of a Be ion and the center-of-mass vibrations in the \( z \) direction, induced by two driving Raman lasers is studied. In the interaction picture with respect to the free vibrational and internal Hamiltonian, this interaction is described by the following Hamiltonian [10]:

\[
H = \hbar \Omega |↑⟩⟨↓| \exp \left\{ i \left[ \eta \left( a e^{-i \omega_z t} + a^\dagger e^{i \omega_z t} \right) - \delta t + \phi \right] \right\} + H.C.,
\]

where \( a \) denotes the annihilation operator for the vibrations along the \( z \) direction, \( \omega_z \) is the corresponding frequency and \( \delta \) is the detuning between the internal transition and the frequency difference between the two Raman lasers. The Rabi frequency \( \Omega \) is proportional to the two Raman laser intensities, and \( \eta \) is the Lamb-Dicke parameter [10,11]. When the two Raman lasers are tuned to the first blue sideband, i.e. \( \delta = \omega_z \), Hamiltonian (15) predicts Rabi oscillations between \( |↓, n⟩ \) and \( |↑, n+1⟩ \). (\( n \) is a vibrational Fock state) with a frequency [10]

\[
\Omega_n = \Omega \sqrt{n+1} \eta \frac{L_n^1(\eta^2)}{\sqrt{n+1}},
\]

where \( L_n^1 \) is the generalized Laguerre polynomial. These Rabi oscillations have been experimentally verified by preparing the initial state \( | ↓, n⟩ \), (with \( n \) ranging from 0 to 16) and measuring the probability \( P_n(t) \) as a function of the interaction time \( t \), which is varied by changing the duration of the Raman laser pulses. Again, as in the cavity QED experiment of [4], the experimental Rabi oscillations are damped and well fitted by [10,11]

\[
P_n(t) = \frac{1}{2} \left( 1 + e^{-\gamma_n t} \cos(2\Omega_n t) \right),
\]

where the measured oscillation frequencies \( \Omega_n \) are in very good agreement with the theoretical prediction (16) corresponding to the measured Lamb-Dicke parameter \( \eta = 0.202 \). As concerns the decay rates \( \gamma_n \), the experimental values are fitted in [10] by

\[
\gamma_n = \gamma_0 (n+1)^{0.7}
\]

where \( \gamma_0 = 11.9 \) KHz. This power-law scaling has been investigated in Refs. [13,14], but a clear explanation of this behavior of the decay rates is still lacking. On the contrary, the scaling law (18) can be accounted for in the previous formalism if we consider the small \( \tau \) limit of Eq. (14), which is again suggested by the fact that the experimental and theoretical predictions for the frequencies \( \Omega_n \) agree. In fact, the \( n \)-dependence of the theoretical prediction of Eq. (14) for \( \eta = 0.202 \) is well approximated, within 10 \%, by the power law dependence

\[
\Omega_n \approx \Omega_0 (n+1)^{0.35},
\]

so that, using Eq. (14) with \( \Omega_R \) replaced by \( \Omega_n \), one has immediately the power law dependence \( (n+1)^{0.7} \) of Eq. (18). The value of the parameter \( \tau \) can be obtained by matching the values corresponding to \( n = 0 \), and using Eq. (14), that is \( \tau = \gamma_0 / 2 \Omega_0^2 \approx 1.5 \cdot 10^{-8} \) sec, where we have used the experimental value \( \Omega_0 / 2\pi = 94 \) KHz. However, this value of the parameter \( \tau \) cannot be explained in terms of some interaction time uncertainty, such as the time jitter of the Raman laser pulses, which is experimentally found to be much smaller [13]. In this case, instead, the observed decoherence can be attributed, as already suggested in [13,14], to the fluctuation of the Raman laser intensities, yielding a fluctuating Rabi frequency parameter \( \Omega(t) \) of the Hamiltonian (13). In this case the evolution is driven by a fluctuating Hamiltonian

\[
H(t) = \hbar \Omega(t) \hat{H}, \quad \hat{H} = H / \Omega \quad \text{in Eq. (13),}
\]

so that

\[
\rho(t) = \exp \left\{ -i \hat{L} \int_0^t d\xi \Omega(\xi) \right\} \rho(0) = e^{-i \hat{A}(t)} \rho(0)
\]

where \( \hat{L} = [\hat{H}, \ldots] / \hbar \) and we have defined the positive dimensionless random variable \( \hat{A}(t) = \int_0^t \xi d\Omega(\xi) \), which is proportional to the pulse area. It is now easy to understand that the physical situation is analogous to that characterized by a random interaction time considered above, with \( L \) replaced by \( \hat{L} \) and \( t' \) by \( A(t) \). One has again phase fluctuations in the energy basis representation and, in analogy with Eq. (4), one considers an averaged density matrix

\[
\tilde{\rho}(t) = \int_0^\infty d\hat{A}(t) \rho(0)
\]

Imposing again that \( \tilde{\rho}(t) \) must be a density operator and the semigroup property, one finds results analogous to Eqs. (3) and (4)

\[
\tilde{V}(t) = \left( 1 + i \hat{L} \Omega \tau \right)^{-t / \tau}
\]

\[
\tilde{P}(t, A) = \frac{e^{-A / \Omega \tau} (A / \Omega \tau)^{(t / \tau) - 1}}{\Omega \tau^{(t / \tau) - 1}}
\]

where, the parameter \( \Omega \) of Eq. (13) plays now the role of a mean Rabi frequency. In fact, consistently with the probability distribution of Eq. (23), one has \( \Omega = (A / t) \).
The scaling time $\tau$ characterizes in this case the strength of the pulse area fluctuations, since from Eq. (23), one has $\sigma^2(A) = \langle A^2 \rangle - \langle A \rangle^2 = \Omega^2 \tau t$. The estimated value of $\tau$ is reasonable since it corresponds to a fractional error of the pulse area $\sqrt{\sigma^2(A)}/\langle A \rangle = \sqrt{\tau/t}$ of 10% for a pulse duration of $t = 1$ $\mu$sec, and which is decreasing for increasing pulse durations.

The present analysis shows many similarities with that of Ref. [13] which also tries to explain the decay of the Rabi oscillations in the ion trap experiments of [5] in terms of laser intensity fluctuations. The authors of Ref. [13] in fact use a phase destroying master equation approximated by the power law (19) but its Lamb-Dicke limit $\Omega = \Omega_0 (n + 1)^{0.5}$, which is valid only when $n \ll 1$. There is however another, more fundamental, difference of Ref. [13] with the identifications $G \leftrightarrow H/h$ and $\Gamma \leftrightarrow \tau$ and moreover derive the same numerical estimate for the Rabi oscillations in the ion trap experiments of [5] in fact use a phase destroying master equation is given by Eq. (5) (with $\Omega \equiv \Omega_0$), whose predictions significantly depart from its second order expansion (6) (corresponding to the gaussian limit) as soon as $\tau$ becomes comparable with the typical timescale of the system under study, which, in the present case, is the inverse of the Rabi frequency.

In conclusion, we have presented a model-independent theory for non-dissipative decoherence, able to provide a simple and unified description of the same decoherence phenomenon observed in two Rabi oscillations experiments which were performed under different situations. A simple way to test experimentally our prediction is to check that the dependence of the decay rate as a function of the Rabi frequency is given by Eq. (12). One should observe a transition from a quadratic dependence to a logarithmic dependence, increasing the value of the Rabi frequency, or of $\tau$.

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