Abstract. For a real number $k$, define $\pi_k(x) = \sum_{p \leq x} p^k$. When $k > 0$, we prove that

$$\pi_k(x) - \pi(x^{k+1}) = \Omega_{\pm} \left( \frac{x^{\frac{k}{k+1}+k}}{\log x \log \log \log x} \right)$$

as $x \to \infty$, and we prove a similar result when $-1 < k < 0$. This strengthens a result in a paper by J. Gerard and the author and it corrects a flaw in a proof in that paper. We also quantify the observation from that paper that $\pi_k(x) - \pi(x^{k+1})$ is usually negative when $k > 0$ and usually positive when $-1 < k < 0$.

For a real number $k$, define

$$\pi_k(x) = \sum_{p \leq x} p^k, \quad \psi_k(x) = \sum_{n \leq x} \Lambda(n)n^k,$$

where the first sum runs over prime numbers and $\Lambda(n) = \log p$ if $n = p^m$ (with $m \geq 1$) is a prime power, and $\Lambda(n) = 0$ otherwise. Then $\pi_0(x) = \pi(x)$, the number of primes less than or equal to $x$, and $\psi_0(x) = \psi(x)$ is the Chebyshev function (Note: There are other uses of the notation $\psi_k$ in the literature that are different from the present one). As usual, we use the notation $f(x) = \Omega_{\pm}(g(x))$ to indicate that there exists a constant $c > 0$ such that $f(x) > cg(x)$ for a sequence of $x \to \infty$ and $f(x) < -cg(x)$ for a sequence of $x \to \infty$. Our goal is to prove the following:

**Theorem 1.** Let $\theta_0$ be the supremum of the real parts of the zeros of $\zeta(s)$, and let $\epsilon > 0$. As $x \to \infty$,

$$\pi_k(x) - \pi(x^{k+1}) = \begin{cases} \Omega_{\pm}(x^{\theta_0+k-\epsilon}) & \text{if } k > 0, \\ \Omega_{\pm}(x^{(k+1)(\theta_0-\epsilon)}) & \text{if } -1 < k < 0 \text{ and } \theta_0 < 1. \end{cases}$$

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If the Riemann Hypothesis is true,

\[
\pi_k(x) - \pi(x^{k+1}) = \begin{cases} 
\Omega \left( \frac{x^{k+\frac{1}{2}} \log \log \log x}{\log x} \right) & \text{if } k > 0, \\
\Omega \left( \frac{x^{1/2} \log \log x}{\log x} \right) & \text{if } k < 0.
\end{cases}
\]

We are not able to treat the case where \( \theta_0 = 1 \) and \( k < 0 \).

A paper of J. Gerard and the author [2] showed that \( \pi_k(x) \) is asymptotic to \( \pi(x^{k+1}) \) (see also [4] and [8]) and proved the first half of the theorem. Unfortunately, this proof was based on an incorrect formula. When the correct formula is used, the proof in [2] is valid only when the Riemann Hypothesis is false. The main work of the present paper uses a technique of Littlewood to establish the second half of the theorem, namely under the assumption that the Riemann Hypothesis is true. The second half implies the first half when \( \theta_0 = 1/2 \).

In [2], a heuristic explanation was given for why \( \pi_k(x) - \pi(x^{k+1}) \) is usually negative when \( k > 0 \) and usually positive when \( -1 < k < 0 \). In Sections 2 and 3 of the present paper, the following more quantitative results are proved. The methods in these sections were inspired by [5].

**Theorem 2.**

\[
\int_1^\infty \frac{\pi_k(t) - \pi(t^{k+1})}{t^{k+2}} dt = -\frac{1}{k+1} \log(k+1) \begin{cases} 
< 0 & \text{if } k > 0, \\
> 0 & \text{if } -1 < k < 0.
\end{cases}
\]

**Theorem 3.** Let \( 0 < k \leq 10.32 \). The following are equivalent:

(1) The Riemann Hypothesis.

(2)

\[
\int_1^x \left( \pi_k(t) - \pi(t^{k+1}) \right) dt < 0 \quad \text{for all } x \text{ sufficiently large.}
\]

1. **Proof of Theorem 1**

*Proof.* When \( k < 0 \) and \( \theta_0 < 1 \), the result was proved in [2, p. 174]. It was also proved that

\[
\pi_k(x) - \pi(x^{k+1}) = -E(x^{k+1}) + x^k E(x) - k \int_2^x t^{k-1} E(t) dt + O(1),
\]

where \( E(x) = \pi(x) - \text{li}(x) \). When the Riemann Hypothesis holds, the estimate \( E(y) = O(y^{1/2} \log y) \) (see [6, Theorem 13.1]), combined with Littlewood’s oscillation result (see below, or [6, p. 479]), yields the
stronger statement given in the second half of the theorem:

\[
\pi_k(x) - \pi(x^{k+1}) = \Omega_\pm \left( \frac{x^{k+1} \log \log x}{\log x} \right) + x^k O \left( x^{1/2} \log x \right) \\
+ O \left( \int_2^x t^{k-1/2} \log t \, dt \right) + O(1) \\
= \Omega_\pm \left( \frac{x^{k+1} \log \log x}{\log x} \right),
\]

since \((k + 1)/2 > k + \frac{1}{2}\) when \(k < 0\).

In the above, we have used the following well-known lemma. Since we also use it several times in the following, we state it explicitly.

**Lemma 1.** Let \(r\) be a real number (positive or negative) and let \(\ell > -1\). Then

\[
\int_2^x t^\ell \log^r \! t \, dt = \frac{1}{\ell + 1} x^{\ell+1} \log^r x + O \left( x^{\ell+1} \log^{r-1} x \right)
\]

as \(x \to \infty\).

**Proof.**

\[
\int_2^x t^\ell \log^r \! t \, dt = \frac{1}{\ell + 1} x^{\ell+1} \log^r x + O(1) - \frac{1}{\ell + 1} \int_2^x r t^\ell \log^{r-1} \! t \, dt.
\]

When \(\ell > 0\) and \(t\) is sufficiently large, \(t^\ell \log^{r-1} \! t\) is increasing, so

\[
\int_2^x t^\ell \log^{r-1} \! t \, dt \leq x^{\ell+1} \log^{r-1} \! x,
\]

for large \(x\).

Now suppose \(-1 < \ell \leq 0\). Choose \(0 < \alpha < 1\) and choose \(\epsilon\) with \(0 < \epsilon < (\ell + 1)(1 - \alpha)/\alpha\). Then \(\alpha(\ell + 1 + \epsilon) < \ell + 1\), so when \(x\) is sufficiently large,

\[
\int_2^x t^\ell \log^{r-1} \! t \, dt = \int_2^{x^\alpha} t^\ell \log^{r-1} \! t \, dt + \int_{x^\alpha}^x t^\ell \log^{r-1} \! t \, dt \\
= O \left( \int_2^{x^\alpha} t^{\ell+\epsilon} \, dt \right) + O \left( (\log^{r-1} \! x) \int_{x^\alpha}^x t^\ell \, dt \right) \\
= O \left( x^{\alpha(\ell+1+\epsilon)} \right) + O \left( x^{\ell+1} \log^{r-1} \! x \right) \\
= O \left( x^{\ell+1} \log^{r-1} \! x \right).
\]

\(\square\)
For the remainder of this section, we are interested in \( k > 0 \), although we state the lemmas in forms that hold for \( k > -1 \).

Let

\[
\begin{align*}
(1) \quad \Pi_k(x) &= \pi_k(x) + \frac{1}{2} \pi_{2k}(x^{1/2}) + \frac{1}{3} \pi_{3k}(x^{1/3}) + \frac{1}{4} \pi_{4k}(x^{1/4}) + \cdots \\
(2) \quad &= \sum_{n \leq x} \frac{\Lambda(n)n^k}{\log n}.
\end{align*}
\]

In [2], the formula (1) was given incorrectly (it used \( \pi_k(x^{1/2}), \pi_k(x^{1/3}), \) etc. instead of \( \pi_{2k}(x^{1/2}), \pi_{3k}(x^{1/3}), \) etc.).

**Lemma 2.** Let \( k > -1 \). Then

\[
\Pi_k(x) = \pi_k(x) + \frac{1}{2} \pi_{2k}(x^{1/2}) + O(x^{k+\frac{1}{2}}) + O(\log x)
\]

and

\[
\Pi_k(x) = \pi_k(x) + O\left(\frac{x^{k+\frac{1}{2}}}{\log x}\right) + O(\log x).
\]

**Proof.** There are \( O(\log x) \) positive integers \( \ell \) such that \( x^{1/\ell} \geq 2 \), so the sum in (1) has \( O(\log x) \) nonzero terms. When \( k \geq 0 \), we have

\[
\pi_k(y) \leq \pi(y)y^k \leq 1.3 \frac{y^{k+1}}{\log y}
\]

(the first inequality is the trivial estimate and the second is in [7]). Therefore,

\[
\frac{1}{3} \pi_{3k}(x^{1/3}) + \frac{1}{4} \pi_{4k}(x^{1/4}) + \cdots = O\left(\frac{x^{k+\frac{1}{2}}}{\log x}\log x\right) = O(x^{k+\frac{1}{2}}).
\]

Now suppose \( -1 < k < 0 \). Let \( n \) be the largest positive integer such that \( nk \geq -1 \). Since \( (n+1)k < -1 \), we have \( \pi_{mk}(y) \leq \pi_{(n+1)k}(y) = O(1) \) for all \( m \geq n+1 \), and at most \( O(\log x) \) terms in (1) are nonzero.
Moreover, \( \pi_{2k}(y) \sim \pi(y^{2k+1}) \), from which it follows that
\[
\Pi_k(x) = \pi_k(x) + \frac{1}{2} \pi_{2k}(x^{1/2}) + \frac{x^{k+\frac{1}{3}}}{(3k+1) \log(x)} + o\left(\frac{x^{k+\frac{1}{3}}}{(3k+1) \log(x)}\right)
\]
\[
+ \cdots + \frac{x^{k+\frac{1}{n}}}{(nk+1) \log(x)} + o\left(\frac{x^{k+\frac{1}{n}}}{(nk+1) \log(x)}\right) + O(\log x)
\]
\[
= \pi_k(x) + \frac{1}{2} \pi_{2k}(x^{1/2}) + O\left(\frac{x^{k+\frac{1}{2}}}{\log x}\right) + O(\log x),
\]
as desired (the implied constants depend on \( k \), but not on \( x \)).

The second equality of the lemma follows from the first equality and the fact that \( \pi_{2k}(y) \sim \pi(y^{2k+1}) \). □

In [2], it was proved that \( \Pi_k(x) - \Pi_0(x^{k+1}) = \Omega_{\pm}(x^\theta_{0+k-\epsilon}) \). When \( k > 0 \), Lemma 2 allows us to change \( \Pi_k(x) \) and \( \Pi_0(x^{k+1}) \) to \( \pi_k(x) \) and \( \pi(x^{k+1}) \) with errors of \( o(x^{k+\frac{1}{2}}) \) and \( o(x^{\frac{k+1}{2}}) \), respectively. These are dominated by the oscillation term if the Riemann Hypothesis is false.

Henceforth, we assume the Riemann Hypothesis (RH) is true and deduce Theorem 1 in this case.

**Lemma 3.** Let \( k \) be a real number and let \( c > k+1 \). If \( x > 0 \), then
\[
\int_2^x \psi_k(t) \, dt = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s-k)}{\zeta(s-k)} \frac{x^{s+1} ds}{s(s+1)}.
\]

*Proof.* The proof is identical to the proof in [3, pp. 31-32]. □

**Lemma 4.** Assume RH. Let
\[
A = \text{Res}_{s=0} \left( \frac{\zeta'(s-k)}{\zeta(s-k)} \right) \left( \frac{s}{s+1} \right), \quad B = \text{Res}_{s=-1} \left( \frac{\zeta'(s-k)}{\zeta(s-k)} \right) \left( \frac{s}{s+1} \right).
\]

If \( k > -1 \), then
\[
\int_2^x \psi_k(t) \, dt = \frac{x^{k+2}}{(k+1)(k+2)} - \sum_{\rho} \frac{x^{\rho+k+1}}{(\rho+k)(\rho+k+1)} - Ax + B + O(x^{k-1})
\]
as \( x \to \infty \). The sum is over the zeros \( \rho \) of \( \zeta(s) \) with \( \text{Re}(\rho) = 1/2 \), counted with multiplicity.

*Proof.* The proof proceeds by moving the line of integration to the left. The details are the same as in [3, pp. 73-74], where the proof is given when \( k = 0 \). □
Lemma 5. Assume RH and let \( k > -1 \). Then

\[
\int_2^x \psi_k(t) \, dt = \sum_{n \leq x} (x - n) \Lambda(n) n^k = \frac{x^{k+2}}{(k+1)(k+2)} + O(x^{k+\frac{3}{2}}) + O(x).
\]

Proof. The first equality is proved by integration by parts:

\[
\int_2^x \psi_k(t) \, dt = x \psi_k(x) - \int_2^x t \, d\psi_k(t) = x \sum_{n \leq x} \Lambda(n) n^k - \sum_{n \leq x} n^{k+1} \Lambda(n),
\]

which yields the result.

The second equality follows from Lemma 4 and the absolute convergence of \( \sum 1/\rho^2 \).

Lemma 6. Assume RH and let \( k > -1 \). Then

\[
\pi_k(x) - \pi(x^{k+1}) = \psi_k(x) - \frac{x^{k+1}}{1 + k} + O(x^{k+\frac{3}{2}} \log x) + O\left(\frac{x^{k+\frac{3}{2}}}{\log x}\right).
\]

Proof. We first prove the lemma with \( \Pi_k(x) \) in place of \( \pi_k(x) \):

\[
\Pi_k(x) = \int_2^x \frac{d\psi_k(t)}{\log t} = \psi_k(x) - x \log x + \int_2^x \psi_k(t) \, dt - \frac{t^{k+1}}{k+1} \left( 1 + \frac{2}{\log t} \right) dt
\]

Integation by parts yields

\[
I_1 = \frac{\sum_{n \leq x} (x - n) \Lambda(n) n^k}{x \log^2 x} + O(1)
\]

\[
+ \int_2^x \frac{\sum_{n \leq t} (t - n) \Lambda(n) n^k}{t^2 \log^2 t} \left( 1 + \frac{2}{\log t} \right) \, dt
\]

\[
= O\left(\frac{x^{k+\frac{3}{2}}}{\log^2 x}\right) + O(1) + \int_2^x \frac{O(t^{k+\frac{3}{2}})}{t^2 \log^2 t} + O(t) \, dt \quad \text{(by Lemma 5)}
\]

\[
= O\left(\frac{x^{k+\frac{3}{2}}}{\log^2 x}\right) + O(1).
\]
Integration by parts also yields
\[ I_2 = \frac{1}{k+1} \int_{x/2}^{x} \frac{t^{k+1}}{t \log^2 t} dt \]
\[ = -\frac{x^{k+1}}{(k+1) \log x} + \int_{x/2}^{x} \frac{t^k}{\log t} dt + O(1) \]
\[ = -\frac{x^{k+1}}{(k+1) \log x} + \int_{x/2}^{x} \frac{du}{\log u} + O(1) \quad \text{(substitute } u = t^{k+1}) \]
\[ = -\frac{x^{k+1}}{(k+1) \log x} + \pi(x^{k+1}) + O \left( x^{k+1/2} \log x \right). \]

In the last equality, we have used the fact [6, Theorem 13.1] that the Riemann Hypothesis implies \( \pi(y) = Li(y) + O(y^{1/2} \log y) \). Putting everything together yields
\[ \Pi_k(x) = \frac{\psi_k(x) - \frac{x^{k+1}}{k+1}}{\log x} + \pi(x^{k+1}) + O \left( x^{k+1/2} \log x \right) + O \left( \frac{x^{k+1/2}}{\log x} \right). \]

Since Lemma 2 tells us that
\[ \Pi_k(x) = \pi_k(x) + O \left( \frac{x^{k+1/2}}{\log x} \right) + O(\log x), \]
the result of the lemma follows. \( \square \)

Lemma 6 translates oscillations of \( \psi_k(x) \) into oscillations of \( \pi_k(x) - \pi(x^{k+1}) \). We now use a method of Littlewood, as modified by Ingham, to produce oscillations in \( \psi_k(x) \). The following lemma allows us to look at \( \psi_k(x) \) averaged over a small interval.

**Lemma 7.** Assume RH and let \( k > -1 \). Then, uniformly for \( x \geq 4 \) and \( \frac{1}{2x} \leq \delta \leq \frac{1}{2} \),
\[ \frac{1}{(e^\delta - e^{-\delta})x} \int_{e^{-\delta}x}^{e^\delta x} \left( \psi_k(u) - \frac{u^{k+1}}{k+1} \right) du \]
\[ = -2x^{k+\frac{1}{2}} \sum_{\gamma > 0} \frac{\sin(\gamma \delta) \sin(\gamma \log x)}{\gamma} + O(x^{\frac{3}{2} + k}). \]

**Proof.** The proof follows from Lemma 4, as in the proof of [6, Lemma 15.9]. \( \square \)

Finally, the proof of Theorem 1 can now be completed using a Diophantine approximation argument, as in the proof of [6, Theorem 15.11]. In particular, suitable large values of \( x \) coupled with small
values $\delta$ can be found to show that the sum of the right side of the formula in Lemma 7 is $\Omega_\pm (\log \log \log x)$. Since the left side is the average over an interval, we find that

$$\psi_k(x) - \frac{x^{k+1}}{k+1} = \Omega_\pm \left( x^{k+\frac{1}{2}} \log \log \log x \right).$$

When $k > 0$ in Lemma 6, the oscillation term dominates. This yields the desired result for $\pi_k(x) - \pi(x^{k+1})$ and completes the proof of Theorem 1.

\[\square\]

2. PROOF OF THEOREM 2

Proof. Recall that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + o(1),$$

where $B = 0.261497 \ldots$ is a constant [7].

The substitution $u = t^{k+1}$ yields

$$\int_1^x \frac{\pi(t^{k+1})}{t^{k+2}} dt = \frac{1}{k+1} \int_1^{x^{k+1}} \frac{\pi(u)}{u^2} du$$

$$= -\frac{1}{k+1} \frac{\pi(x^{k+1})}{x^{k+1}} + \frac{1}{k+1} \int_1^{x^{k+1}} \frac{1}{u} d\pi(u)$$

$$= -\frac{1}{k+1} \frac{\pi(x^{k+1})}{x^{k+1}} + \frac{1}{k+1} \sum_{p \leq x^{k+1}} \frac{1}{p}$$

$$= \frac{1}{k+1} \log \log x^{k+1} + \frac{B}{k+1} + o(1).$$

Similarly,

$$\int_1^x \frac{\pi_k(t)}{t^{k+2}} dt = -\frac{1}{k+1} \frac{\pi_k(x)}{x^{k+1}} + \frac{1}{k+1} \int_1^x \frac{1}{t^{k+1}} d\pi_k(t)$$

$$= -\frac{1}{k+1} \frac{\pi_k(x)}{x^{k+1}} + \frac{1}{k+1} \sum_{p \leq x^{k+1}} \frac{1}{p}$$

$$= \frac{1}{k+1} \log \log x + \frac{B}{k+1} + o(1).$$

Therefore,

$$\int_1^x \frac{\pi_k(t) - \pi(t^{k+1})}{t^{k+2}} dt = -\frac{1}{k+1} \log (k+1) + o(1).$$

This yields the theorem. \[\square\]
3. Proof of Theorem 3

Proof. Assume the Riemann Hypothesis is true. We need the following technical result:

**Lemma 8.** Assume RH and let \( k > 0 \). Then

\[
\left| \int_2^x \frac{\psi_k(t) - \frac{t^{k+1}}{(k+1)}}{\log t} \, dt \right| < 0.04621 \frac{x^{k+\frac{3}{2}}}{\log x}
\]

for all sufficiently large \( x \).

Proof. From [1, Corollary 1], with the standard notation \( \rho = \frac{1}{2} + i\gamma \) for zeros of \( \zeta(s) \) in the critical strip, we know that

\[
\sum_{\rho} \frac{1}{(\rho + k)(\rho + k + 1)} < \sum_{\rho} \frac{1}{\gamma^2} < 0.04620999.
\]

Let \( D(x) = \psi_k(x) - \frac{x^{k+1}}{(k+1)} \). Integration by parts yields

\[
\int_2^x \frac{D(t)}{\log t} \, dt = \int_2^x \frac{D(t) \, dt}{t} + \int_2^x \frac{D(u) \, du}{t \log^2 t} \, dt
\]

\[
< 0.04620999 \frac{x^{k+\frac{3}{2}}}{\log x} + 0.04620999 \int_2^x \frac{t^{k+\frac{1}{2}}}{\log^2 t} \, dt + o(x) + o(x^{k-1})
\]

(from Lemma 8)

\[
< 0.04621 \frac{x^{k+\frac{3}{2}}}{\log x}
\]

when \( x \) is large.

We know that

\[
\Pi_k(x) = \pi_k(x) + \frac{x^{k+\frac{1}{2}}}{(2k+1) \log x} + o \left( \frac{x^{k+\frac{1}{2}}}{\log x} \right),
\]

which implies that

\[
\int_1^x (\Pi_k(t) - \pi_k(t)) \, dt = \int_2^x \frac{t^{k+\frac{1}{2}}}{(2k+1) \log t} + o \left( \int_2^x \frac{t^{k+\frac{1}{2}}}{\log t} \, dt \right)
\]

\[
= (1 + o(1)) \frac{x^{k+\frac{3}{2}}}{(k + \frac{3}{2})(2k + 1) \log x}
\]

as \( x \to \infty \). From Lemma 8 and (3),

\[
\int_1^x (\Pi_k(t) - \pi(t^{k+1})) \, dt \leq (0.04621 + o(1)) \frac{x^{k+\frac{3}{2}}}{(k + \frac{3}{2}) \log x}
\]
for $x$ sufficiently large. Therefore,

$$\int_1^x (\pi_k(t) - \pi(t^{k+1})) \, dt \leq \left( 0.04621 - \frac{1}{2k+1} + o(1) \right) \frac{x^{k+\frac{3}{2}}}{(k+\frac{3}{2}) \log x} < 0$$

for $x$ sufficiently large, when $k \leq 10.32$.

**Remark.** It would be possible to improve the upper bound 10.32 slightly by using knowledge of the first several values of $\gamma$ to estimate the early terms of $\sum 1/(\rho+k)(\rho+k+1)$, but it follows from a theorem of Lehman (see [1]) that this sum is approximately a constant times $\log(k)/k$, hence is larger than $1/(2k+1)$ for large $k$.

Now suppose the Riemann Hypothesis is false. Then $\theta_0 = \sup\{\Re(\rho) : \zeta(\rho) = 0\} > \frac{1}{2}$. Let $0 < \epsilon < \theta_0 - \frac{1}{2}$ and let

$$F(x) = \int_1^x (\Pi_k(t) - \Pi_0(t^{k+1})) \, dt.$$

Then $F(x) = o\left(x^{k+2}\right)$ as $x \to \infty$. Let $s \in \mathbb{C}$ with $\Re(s) > k+1$. Then

$$G(s) := \int_1^\infty \left( \frac{F(t) - t^{\theta_0+k+1-\epsilon}}{t^{s+2}} \right) dt = -\frac{F(t)}{(s+1)t^{s+1}} \bigg|_1^\infty + \frac{1}{s+1} \int_1^\infty \frac{\Pi_k(t) - \Pi_0(t^{k+1})}{t^{s+1}} dt + \frac{1}{\theta_0 + k - s - \epsilon} \left( \log \zeta(s-k) - \log \zeta\left(\frac{s}{k+1}\right) \right) + \frac{1}{\theta_0 + k - s - \epsilon}.$$

The last equality follows from a second integration by parts plus a change of variables in the second part of the integral; see [2, p. 173]. The last line represents a function that is analytic for all real numbers $s > \theta_0 + k - \epsilon$.

Let $\rho$ be the zero of $\zeta(s)$ with $\Re(\rho) > \theta_0 - \epsilon$ with smallest positive imaginary part. Then $\log \zeta(s-k)$ is not analytic at $\rho+k$. Since $\theta_0 \leq 1$,

$$\theta_0 - \epsilon < \Re(\rho) \leq \Re\left(\frac{\rho + k}{k + 1}\right).$$

Since the imaginary part of $(\rho + k)/(k + 1)$ is less than the imaginary part of $\rho$, the choice of $\rho$ implies that $\zeta((\rho + k)/(k + 1)) \neq 0$ and therefore does not cancel the singularity at $\rho + k$. Therefore, $G(s)$ is not analytic at $\rho + k$. 


If \( F(x) - x^{\theta_0 + k + 1 - \epsilon} \leq 0 \) for all sufficiently large \( x \), Landau’s Theorem (see, for example, [2] or [3]) implies that \( G(s) \) is analytic for all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \theta_0 + k - \epsilon \). Since  
\[
\text{Re}(\rho + k) > \theta_0 + k - \epsilon,
\]
this is a contradiction. Therefore, there is a sequence of \( x \to \infty \) with  
\[
\int_1^x (\Pi_k(t) - \Pi_0(t^{k+1})) \, dt > x^{\theta_0 + k + 1 - \epsilon}.
\]

We now need to change from \( \Pi_k \) to \( \pi_k \). Since  
\[
\Pi_k(t) = \pi_k(t) + O(t^{k+1/2} / \log x)
\]
and  
\[
\Pi_0(t^{k+1}) = \pi(t^{k+1}) + O(t^{k+1/2} / \log t),
\]
we have  
\[
\left| \int_1^x (\Pi_k(t) - \Pi_0(t^{k+1})) \, dt - \int_1^x (\pi_k(t) - \pi_0(t^{k+1})) \, dt \right| = O(x^{k+3/2} / \log x).
\]
Since \( \theta_0 + k + 1 - \epsilon > k + 3/2 \), we find that there exists a sequence of \( x \to \infty \) such that  
\[
\int_1^x (\pi_k(t) - \pi_0(t^{k+1})) \, dt > 0.
\]

\[\square\]

We note that the proof of “(2) \implies (1)” in Theorem 3 works for all \( k > 0 \).

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**REFERENCES**

[1] R. Brent, D. Platt, and T. Trudgian, Accurate estimation of sums over zeros of the Riemann zeta-function., *Math. Comp.* 90 (2021), no. 332, 2923–2935.

[2] J. Gerard and L. C. Washington, Sums of powers of primes, *Ramanujan J.* 45 (2018), no. 1, 171–180.

[3] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, 1990.

[4] R. Jakimczuk, Desigualdades y formulas asintóticas para sumas de potencias de primos, *Bol. Soc. Mat. Mexicana* (3) 11 (2005), no. 1, 5–10.

[5] D. Johnston, On the average value of \( \pi(t) - li(t) \), *Canadian Math. Bulletin*, 2022.

[6] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press, 2007.

[7] B. Rosser and L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, *Illinois J. Math.* 6 (1962), 64–94.

[8] T. Šalát and Š Znám, On sums of the prime powers, *Acta Fac. Rerum Natur. Univ. Comenian. Math.* 21 (1968), 21–24 (1969).
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