Longitudinal Mapping Knot Invariant for SU(2)

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Abstract

The knot coloring polynomial defined by Eisermann for a finite pointed group is generalized to an infinite pointed group as the longitudinal mapping invariant of a knot. In turn this can be thought of as a generalization of the quandle 2-cocycle invariant for finite quandles. If the group is a topological group then this invariant can be thought of a topological generalization of the 2-cocycle invariant. The longitudinal mapping invariant is based on a meridian-longitude pair in the knot group. We also give an interpretation of the invariant in terms of quandle colorings of a 1-tangle for generalized Alexander quandles without use of a meridian-longitude pair in the knot group. The invariant values are concretely evaluated for the torus knots $T(2,n)$, their mirror images, and the figure eight knot for the group SU(2).

1 Introduction

In this paper all knots will be oriented and we write equality for orientation preserving ambient isotopy. For a knot $K$ we write $r(K)$ for $K$ with orientation reversed and $m(K)$ for the mirror image of $K$. It is known that the knot quandle $Q(K)$ distinguishes distinct oriented knots $K_1$ and $K_2$ if and only if $K_2 \neq rm(K_1)$ (see, e.g., [13, 17]). The knot group $\pi_K$ cannot distinguish $K$ from $s(K)$ for any $s \in \{r, m, rm\}$ (see, e.g., [2]). It follows that neither the set of (quandle) homomorphisms $\text{Hom}_{\text{Qnd}}(Q(K), Q)$ from $Q(K)$ to a quandle $Q$ nor the set $\text{Hom}_{\text{Grp}}(\pi_K, G)$ of (group) homomorphisms from $\pi_K$ to a group $G$ is a complete invariant of oriented knots.

In the case of quandles a stronger invariant (the 2-cocycle invariant or 2-cocycle state-sum invariant) was obtained in [3] using a 2-cocycle $\varphi$ for a finite quandle $Q$ with coefficients in an abelian group $\Lambda$. One defines a mapping

$$B_{\varphi} : \text{Hom}_{\text{Qnd}}(Q(K), Q) \to \Lambda : \rho \mapsto B_{\varphi}(\rho)$$

whose fibers determine a partition of $\text{Hom}_{\text{Qnd}}(Q(K), Q)$ indexed by $\Lambda$. Since $Q$ is finite this partition can be expressed as an element $\Phi^\varphi_Q(K)$ of the group ring $\mathbb{Z}[\Lambda]$. See [5] for evidence that the 2-cocycle invariant might be a complete invariant for oriented knots.

In the case of groups the knot group peripheral system $(\pi_K, m_K, l_K)$, where $(m_K, l_K)$ is a meridian-longitude pair, is a complete invariant of oriented knots (see [2]). Using this, Eisermann [9] defined the knot coloring polynomial for a pointed finite group $(G, x)$ corresponding to a peripheral system $(\pi_K, m_K, l_K)$ as

$$P^x_G(K) = \sum_{\rho} \rho(\rho)$$
where the sum is taken over all homomorphism \( \rho : \pi_K \to G \) with \( \rho(m_K) = x \). It turns out that longitude images lie in \( \Lambda = C(x) \cap G' \) and hence \( P^G_2(K) \) is an element of the group ring \( \mathbb{Z}[\Lambda] \). Eismann shows in [9] that when \( G \) is finite and \( \Lambda \) is abelian a knot coloring polynomial can be expressed as a 2-cocycle invariant over the conjugation quandle \( x^G \) and conversely a 2-cocycle invariant for a finite quandle \( Q \) is a specialization of a knot coloring polynomial for \( G = \text{Inn}(\bar{Q}) \) where \( \bar{Q} \) is the abelian extension corresponding to the given 2-cocycle. In particular, any knots distinguishable by 2-cocycle invariants are distinguishable by knot coloring polynomials. We note however that in general the price one pays for this is a group much larger than the quandle.

In case \( G \) is infinite the coefficients of \( P^G_2(K) \) may be infinite, then we replace it by the longitudinal mapping

\[
\mathcal{L}^G_2(K) : \text{Hom}_{\text{gp}}(\pi_K, m_K; G, x) \to \Lambda, \quad \rho \mapsto \rho(l_K),
\]

where \( \text{Hom}_{\text{gp}}(\pi_K, m_K; G, x) \) is the set of homomorphisms \( \rho : \pi_K \to G \) with \( \rho(m_K) = x \). If \( G \) is a topological group, \( \mathcal{L}^G_2(K) \) may be thought of as a topological analogue of the 2-cocycle invariant or the knot coloring polynomial. This is the invariant we examine for the case \( G = SU(2) \) in this paper. We find \( \mathcal{L}^G_2(K) \) when \( K \) is a torus knot \( T(2, n) \) for odd \( n \geq 3 \) and when \( K \) is the figure eight knot \( 4_1 \).

Let \( Q \) be any quandle (possibly infinite) and let \( T \) be a 1-tangle diagram whose closure is the knot \( K \). Denote the initial arc of \( T \) by 0 and the terminal arc by \( n \). For arbitrary fixed \( e \in Q \) let \( \text{Col}_Q^e(T) \) denote the set of colorings of \( T \) by quandle \( Q \) such that \( C(0) = e \). Furthermore, by Lemma 2.2 in [5], for \( C \in \text{Col}_Q^e(T), b = C(n) \) satisfies \( R_b = R_e \). That is, \( b \) lies the the fiber \( F_e = \text{inn}^{-1}(R_e) \). We define the mapping

\[
\Psi^e_Q(K) : \text{Col}_Q^e(T) \to F_e, \quad C \mapsto C(n).
\]

In the appendix we show that if \( Q \) is the generalized Alexander Quandle \( \text{GAlex}(G', f_x) \) constructed from the pointed group \( (G, x) \) where \( f_x(u) = x^{-1}ux \), then \( \mathcal{L}^G_2(K) \) is equivalent to \( \Psi^e_Q(K) \) where \( e = 1 \). This gives a way to construct the longitudinal mapping without use of a meridian-longitude pair.

## 2 Basic Definitions

In this section we briefly review some definitions and examples. More details can be found, for example, in [4].

If \( X \) is a set with a binary operation \( \ast \) the right translation \( R_a : X \to X, \) by \( a \in X, \) is defined by \( R_a(x) = x \ast a \) for \( x \in X \). The magma \( (X, \ast) \) is a quandle if each right translation \( R_a \) is an automorphism of \( (X, \ast) \) and every element of \( X \) is idempotent. A quandle homomorphism between two quandles \( X, Y \) is a map \( f : X \to Y \) such that \( f(x \ast_X y) = f(x) \ast_Y f(y) \), where \( \ast_X \) and \( \ast_Y \) denote the quandle operations of \( X \) and \( Y \), respectively. A quandle isomorphism is a bijective quandle homomorphism, and two quandles are isomorphic if there is a quandle isomorphism between them. The set of quandle homomorphisms from \( X \) to \( Y \) is denoted by \( \text{Hom}_{\text{quad}}(X, Y) \). A quandle epimorphism \( f : X \to Y \) is a covering if \( f(x) = f(y) \) implies \( a \ast x = a \ast y \) for all \( a, x, y \in X \).

For a quandle \((X, \ast)\), since \( R_a \) for each \( a \in X \) is an automorphism, one may define the binary operation \( \hat{\ast} \) by \( x \hat{\ast} y = R_{y^{-1}}^{-1}(x) \). This gives a quandle structure on \( X \), called the dual quandle. The subgroup of \( \text{Sym}(X) \) generated by the permutations \( R_a, a \in X \), is called the inner automorphism
group of \( X \), and is denoted by \( \text{Inn}(X) \). The map \( \text{inn} : X \to \text{inn}(X) \subset \text{Inn}(X) \) (which is a quandle under conjugation) defined by \( \text{inn}(x) = R_x \) is called the inner representation. An inner representation is a covering.

A quandle is indecomposable if \( \text{Inn}(X) \) acts transitively on \( X \). We use indecomposable here rather than connected to avoid confusion with the topological sense of the word. A quandle is faithful if the mapping \( \text{inn} : X \to \text{Inn}(X) \) is an injection.

As in Joyce [13], given a group \( G \) and and \( f \in \text{Aut}(G) \), a quandle operation is defined on \( G \) by \( x \ast y = f(xy^{-1})y, \ x, y \in G \). We call such a quandle a generalized Alexander quandle and denote it by \( \text{GAlex}(G, f) \). If \( G \) is abelian, such a quandle is known as an Alexander quandle or affine quandle.

Let \( D \) be a diagram of a knot \( K \), and \( \mathcal{A}(D) \) be the set of arcs of \( D \). A coloring of a knot diagram \( D \) by a quandle \( X \) is a map \( C : \mathcal{A}(D) \to X \) satisfying the condition depicted in Figure 1 at every positive (left) and negative (right) crossing, respectively. The set of colorings of \( D \) by \( X \) is denoted by \( \text{Col}_X(D) \). There is a bijection from \( \text{Hom}_{\text{Qu}}(\mathcal{Q}(K), X) \) to \( \text{Col}_X(D) \). The cardinality \( |\text{Col}_X(D)| \) is a knot invariant (e.g. see [4]).

A 1-tangle, or a long knot, is a properly embedded arc in a 3-ball, and the equivalence of long knots is defined by ambient isotopies of the 3-ball fixing the boundary. A diagram of a 1-tangle is defined in a manner similar to a knot diagram, from a regular projection to a disk by specifying crossing information. An orientation of a 1-tangle is specified by an arrow on a diagram. A knot diagram is obtained from a 1-tangle diagram by closing the end points by a trivial arc outside of a disk. This procedure is called the closure of a 1-tangle. If a 1-tangle is oriented, then the closure inherits the orientation. Two diagrams of the same 1-tangle are related by Reidemeister moves. There is a bijection between knots and 1-tangles for classical knots, and invariants of 1-tangles give rise to invariants of knots, see, for example, [10,18].

A quandle coloring of an oriented 1-tangle diagram is defined in a manner similar to those for knots. We do not require that the end points receive the same color for a quandle coloring of 1-tangle diagrams. However this will be the case for a conjugation quandle. For a quandle \( Q \) and \( x \in Q \), denote by \( \text{Col}_Q^x(T) \) the set of colorings of a 1-tangle \( T \) by \( Q \) with the initial arc colored by \( x \).

### 3 Computation of the Longitudinal Mapping

For convenience we often identify the diagram of a tangle \( T \) with the tangle itself.
Definition 3.1. (Wirtinger code) Eisermann [10] Label the arcs of a 1-tangle $T$ by integers, $A(T) = \{0, \ldots, n\}$, such that 0 and $n$ are the initial and terminal arcs, respectively, and the remaining arcs are labeled in order when traveled along the tangle from 0 to $n$. At the end of arc number $i - 1$, we undercross arc $\kappa(i) = ki$ and continue on arc number $i$. Let $\epsilon(i) = \epsilon i$ be the sign of crossing $i$. Note that these are maps $\kappa : \{1, \ldots, n\} \to \{0, \ldots, n\}$ and $\epsilon : \{1, \ldots, n\} \to \{1, -1\}$. The pair $(\kappa, \epsilon)$ is called the Wirtinger code of the diagram $T$.

The 1-tangle group $\pi_T$ with diagram $T$ and Wirtinger code $(\kappa, \epsilon)$ allows the presentation

$$\pi_T = \langle x_0, x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle,$$

where $r_i$ is the relation $x_i = x_{\kappa i}^{-\epsilon i} x_{i-1} x_{\kappa i}^{\epsilon i}$.

As in [9] we choose the meridian

$$m_T = x_0$$

and the (preferred) longitude

$$l_T = x_0^{-w(T)} x_{\kappa 1}^{\epsilon 1} x_{\kappa 2}^{\epsilon 2} \cdots x_{\kappa n}^{\epsilon n}.$$ 

See Remark 3.13 of [2] for this form of the longitude. The knot group $\pi_K$ is isomorphic to $\pi_T$.

For a pointed finite group $(G, x)$, Eisermann defined the knot coloring polynomial of $K$ to be

$$P_G^K(K) = \sum_{\rho} \rho(l_T),$$

where the sum is taken over all homomorphisms $\rho : \pi_T \to G$ with $\rho(m_T) = x$. It turns out (see [9]) that the values $\rho(l_T)$ lie in the longitudinal group $\Lambda = C(x) \cap G'$ where $C(x)$ is the centralizer of $x$ and $G'$ is the commutator subgroup of $G$. Thus $P_G^K(K)$ lies in the group ring $\mathbb{Z}[\Lambda]$.

Let $\text{Rep}_G^K(T)$ be the set of homomorphisms $\rho : \pi_T \to G$ with $\rho(m_T) = x$, and $\text{Col}_Q^K(T)$ be the set of colorings $C$ by a quandle $Q$ such that $C(0) = x$, where 0 is the initial arc of $T$. There is a bijection between $\text{Rep}_G^K(T)$ and $\text{Col}_Q^K(T)$ where $Q$ is the conjugacy class $x^G$ of $x$ under the product $a \ast b = b^{-1}ab$.

We wish to extend Eisermann’s knot coloring polynomial to groups not necessarily finite.

Definition 3.2. Let $(G, x)$ be any pointed group. Let $K$ be a knot and $T$ be a 1-tangle corresponding to $K$. We define the knot invariant

$$\mathcal{L}_G^K(K) : \text{Rep}^x_G(T) \to \Lambda, \quad \rho \mapsto \rho(l_T).$$

We call it the longitudinal mapping. When there is no chance of confusion we write $\mathcal{L}$ in place of $\mathcal{L}_G^K(K)$. We shall say that two such longitudinal mappings $\mathcal{L}_1 : \text{Rep}^x_G(T_1) \to \Lambda$ and $\mathcal{L}_2 : \text{Rep}^x_G(T_2) \to \Lambda$ are equivalent if there is a bijection $\beta : \text{Rep}^x_G(T_1) \to \text{Rep}^x_G(T_2)$ such that $\mathcal{L}_1 = \mathcal{L}_2 \beta$. Clearly the longitudinal mapping $\mathcal{L} : \text{Rep}^x_G(T) \to \Lambda$ is a knot invariant up to equivalence of mappings and if $G$ is a topological group $\mathcal{L}$ is continuous. In this case $\beta$ must be a homeomorphism. See Rubinsztein [20] for the topology on $\text{Col}_Q^K(T)$.

Remark 3.3. See the appendix for a definition of $\mathcal{L}$ that doesn’t depend on the meridian-longitude pair $(m_T, l_T)$.
For a finite group $G$ the knot coloring polynomial is $P^G_K(K) = \sum_{v \in \Lambda} |\mathcal{L}^{-1}(v)|v$. Thus $\mathcal{L}$ can be seen as an analogue of the knot coloring polynomial defined for topological quandles. Since the knot coloring polynomial is a generalization of the quandle 2-cocycle invariant (see Theorem 3.24 in [9]), the invariant $\mathcal{L}$ is a generalization of the quandle 2-cocycle invariant. See [5,9] for more details of relations among these invariants. A similar but different invariant using longitudes was considered in [18].

**Remark 3.4.** Note that the group $\Lambda$ acts on the set of homomorphisms $\rho : \pi_K \rightarrow G$ with $\rho(m_K) = x$ by setting $\rho^\varphi(a) = g^{-1}\rho(a)g$ for $a \in G$. Since $g \in C(x)$ it follows that $\rho^\varphi(m_K) = x$. Hence if $\Lambda$ is abelian, then $\mathcal{L}$ is constant on the orbits of this action by $\Lambda$. In our application, $\Lambda$ is abelian. Thus for example for a two-bridge knot with diagram $K$ and the values of $\rho$ by setting $\rho(a) = g^{-1}\rho(a)g$, $g \in C(x)$.

**Proposition 3.5.** $\mathcal{L}^G_L(rm(K))(\rho) = \mathcal{L}^G_L(K)(\rho)^{-1}$ for all $\rho \in \text{Rep}^G_L(T)$.

**Proof.** This is immediate from the fact that if $(\pi_K, m_K, l_K)$ is a peripheral system for knot $K$ then $(\pi_K, m_K, l_K^{-1})$ is a peripheral system for the knot $rm(K)$ ([14], Chapter 6). \qed

4 Background for SU(2)

For the remainder of the paper, we examine the invariant $\mathcal{L}$ for $(G, x) = (SU(2), x)$ with various choices of $x$. We represent $SU(2)$ by the group of unit quaternions, that is,

$$SU(2) = \{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}.$$ 

The group $SO(3)$ will also be of use. Elements of $SO(3)$ will be denoted by $\text{Rot}_\theta(v)$, $\theta \in \mathbb{R}$, $v \in S^2$. If $u \in \mathbb{R}^3$, $u\text{Rot}_\theta(v)$ is the vector obtained by rotating $u$ about $v$ by $\theta$ radians using the right-hand rule.

We represent elements of $\mathbb{R}^3$ as pure quaternions $u = u_1i + u_2j + u_3k$ and we identify the set of pure unit quaternions with the sphere $S^2$. Then each element of $SU(2)$ can be represented the form

$$e^{\theta u} = \cos(\theta) + \sin(\theta)u, \quad u \in S^2, \quad 0 \leq \theta < 2\pi.$$ 

Note that a pure quaternion $u$ satisfies $u^2 = -1$ and hence the quaternions $e^{\theta u}$ for fixed $u$ behave just like complex numbers $e^{\theta i} = \cos(\theta) + \sin(\theta)i$.

From [7] (Section 1.2) the conjugacy classes of $SU(2)$ are given by

$$\tilde{C}_\theta = \{e^{\theta u} : u \in S^2\},$$

for $0 \leq \theta \leq \pi$. In this case $\tilde{C}_0 = \{1\}$, $\tilde{C}_\pi = \{-1\}$ and for $0 < \theta < \pi$, $\tilde{C}_\theta$ is a sphere. This also follows from Lemma 4.1 below.

It is known (see for example [16], Theorem 5.1) that for $u, v \in S^2$ and $\theta \in \mathbb{R}$ that

$$e^{\theta u}v e^{-\theta u} = v\text{Rot}_\theta(u).$$

The double covering homomorphism $\phi : SU(2) \rightarrow SO(3)$ may be defined by

$$v\phi(q) = q^{-1}vq.$$ 

In this case if \( q = e^{\theta u} \), then \( \phi(q) = \text{Rot}_{-2\theta}(u) \), the rotation by \(-2\theta\) radians about the unit vector \( u \). We must take \( \phi(q) \) to be \( q^{-1}vq \) instead of \( qvq^{-1} \) since we write the rotation operator on the right of the argument.

**Lemma 4.1.** For fixed \( \theta, \beta \in \mathbb{R} \) and \( u, v \in S^2 \) we have
\[
e^{-\beta v}e^{\theta u}e^{\beta v} = e^{\theta w}.
\]
where \( w = u\text{Rot}_{-2\beta}(v) \).

**Proof.** We compute:
\[
e^{-\beta v}e^{\theta u}e^{\beta v} = e^{-\beta v}(\cos(\theta) + \sin(\theta)u)e^{\beta v} = \cos(\theta) + \sin(\theta)e^{-\beta v}ue^{\beta v} = \cos(\theta) + \sin(\theta)u\text{Rot}_{-2\beta}(v) = e^{\theta w},
\]
where \( w = u\text{Rot}_{-2\beta}(v) \).

Since \( SO(3) \) acts transitively on \( S^2 \) from Lemma 4.1 we have:

**Corollary 4.2.** The conjugacy class of \( x = e^{\theta u} \) has the form
\[
x^{SU(2)} = \{ e^{\theta v} : v \in S^2 \}.
\]

**Definition 4.3.** For \( 0 < \psi < 2\pi \) we denote by \( S^2_\psi \) the quandle with underlying set \( S^2 \) and product \( u*v = u\text{Rot}_\psi(v) \), for \( u, v \in S^2 \). We call this a spherical quandle.

**Lemma 4.4.** For \( 0 < \theta < \pi \) the mapping \( u \mapsto e^{\theta u} \) is an isomorphism from quandle \( S^2_\psi \) with \( \psi = 2\pi - 2\theta \) to the conjugacy class \( \tilde{C}_\theta = \{ e^{\theta u} : u \in S^2 \} \) considered as a quandle under conjugation: \( p*q = q^{-1}pq \).

**Proof.** The result follows from Lemma 4.1 by taking \( \beta = \theta \). □

**Lemma 4.5.** \( SU(2) \) is a perfect group, that is, it is its own commutator subgroup.

**Proof.** By [19], Prop. 10.24 every unit quaternion \( q \) has the form \( q = aba^{-1}b^{-1} \) for non-zero quaternions \( a \) and \( b \). The same holds if we normalize \( a \) and \( b \). □

**Lemma 4.6.** If \( x = e^{\theta u} \) for \( 0 < \theta < \pi \) then the centralizer \( C(x) \) is the circle group:
\[
\{ e^{\beta u} : 0 \leq \beta < 2\pi \}.
\]

Hence, the longitudinal group for \( (SU(2), x) \) is given by
\[
\Lambda = C(x) \cap SU(2)' = C(x) = \{ e^{\beta u} : 0 \leq \beta < 2\pi \}.
\]

**Proof.** This follows from Lemma 4.1 and the fact that for \( 0 < \beta < \pi \), \( u\text{Rot}_\beta(v) = u \), with \( u, v \in S^2 \) if and only if \( v = \pm u \) together with the fact that
\[
\{ e^{\beta u} : 0 \leq \beta < 2\pi \} = \{ e^{\beta(-u)} : 0 \leq \beta < 2\pi \}.
\]
Remark 4.7. It is easy to see that for $x = e^{\theta u}$ the conjugacy classes $x_{SU(2)} = \tilde{C}_\theta$ and $(-x)_{SU(2)} = \tilde{C}_{\theta + \pi}$ are isomorphic via $q \mapsto -q$ as conjugation quandles. Note also that $u \mapsto -u$ leaves the longitude invariant. Thus for our purposes it suffices to consider only those $x = e^{\theta u}$ for $0 < \theta < \pi$. Note that $\text{Rot}_\psi(v) = \text{Rot}_{-\psi}(-v)$. It follows that $S^2_\psi$ is isomorphic to $S^2_{-\psi}$ via $u \mapsto -u$. Thus when coloring knots by the family of quandles $S^2_\psi$ we may restrict $\psi$ to the interval $(0, \pi]$. And for the quandles $\tilde{C}_\theta$ we may restrict $\theta$ to the interval $[\pi/2, \pi)$.

Fix $\theta \in (0, \pi)$ and $x = x_\theta = e^{\theta i}$ where $i = (1, 0, 0)$ we are interested in computing $\mathcal{L}_\theta = \mathcal{L}^{x_\theta}_{SU(2)}$.

5 Knot Colorings by the Spherical Quandles $S^2_\psi$

Knot group representations in SU(2) were studied in Klassen [15], in particular for all torus knots and twist knots. We present explicit colorings of torus knots $T(2, n)$ and the figure eight knot in this section and we compute the longitudinal mappings of these knots in the next section.

Fix $\psi \in [0, 2\pi]$ and as above denote by $\text{Rot}_\psi(v)$ the rotation by $\psi$ about $v$. Then the quandle structure on $Q = S^2_\psi$ is as defined in Definition 4.3, for $u, v \in S^2$, by $u \ast v = u\text{Rot}_\psi(v)$ with right action of the rotation. Denote by $(u, v)$ the inner product of $u, v$ in $\mathbb{R}^3$, so that $S^2_\psi = \{u \in \mathbb{R}^3 \mid \langle u, u \rangle = 1\}$. We also denote the length of the shortest spherical geodesic segment between two unit vectors $u, v, w \in S^2_\psi$ by $\angle(uvw) = \psi$ if $w = u\text{Rot}_\psi(v)$ and $0 \leq \psi < 2\pi$.

Let $\kappa : \{1, \ldots, n\} \to \{0, \ldots, n\}$ and $\epsilon : \{1, \ldots, n\} \to \{1, -1\}$ be the Wirtinger code of a tangle diagram $T$ as described in Definition 3.1. We observe that the coloring condition depicted in Figure 1 is formulated as follows. Let $Q = S^2_\psi$. Then a coloring $\rho \in \text{Col}_Q(T)$ corresponds to a sequence of points

$$(\rho(0), \ldots, \rho(n)) \in (S^2_\psi)^{n+1}$$

satisfying

$$\rho(i) = \rho(i-1)\text{Rot}_\psi(\rho(\kappa(i)))^{\epsilon(i)}, \quad i \in \{1, \ldots, n\}.$$ 

Thus we have the following, as stated in [15]:

Lemma 5.1. For a coloring of a knot diagram by $Q = S^2_\psi$, consider a crossing with the colors $(a, b)$ as depicted in Figure 2. Then $\epsilon = a \ast b$ if and only if $d(a, b) = d(b, c)$ and $\angle(abc) = \psi$. In particular, any orientation preserving isometry of the sphere takes a coloring to a coloring.

Corollary 5.2. For any coloring $\rho \in \text{Col}_Q(T)$ such that $\rho(0), \ldots, \rho(n)) \in (S^2_\psi)^{n+1}$,

$$(\rho(0)\text{Rot}_\phi(x), \ldots, \rho(n)\text{Rot}_\phi(x))$$

defines a coloring in $\text{Col}_Q(T)$ for all $\phi \in [0, 2\pi]$.

Remark 5.3. As $\psi$ varies, we have a continuous family $\{S^2_\psi : \psi \in (0, 2\pi)\}$ of quandles. This leads to continuous family of knot colorings by $\tilde{C}_\theta$, where $\theta = \pi - \psi/2$. The longitudinal mapping invariant, then, can be seen as a continuous family of invariants $\mathcal{L}^{x_\theta}_{SU(2)}$ over $\theta$. 

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Let $T$ be a tangle corresponding to a 2-bridge knot $K$. Then we may choose a diagram of $T$ to be a diagram with two bridges, i.e., there are two arcs $x_0$ and $x_1$ such that $x_0$ is the initial arc of $T$, and the colors of $x_0$ and $x_1$ uniquely determine a color of all arcs of $T$.

Let $Q = S^2_\psi$, and we fix $x = (1,0,0)$. Thus for all elements $\rho \in \text{Col}_Q^r(T)$, we have $\rho(x_0) = x$ as $x_0$ is the initial arc of $T$. Let $E \subset S^2_\psi$ be half of the equator, 

$$E = \{(\cos \phi, \sin \phi, 0) : 0 \leq \phi \leq \pi\}.$$

**Lemma 5.4.** Let $Q = S^2_\psi$ and let $x \in SU(2)$, $T$, $x_0$, and $x_1$ be as above. Suppose that the number $h$ of elements $\rho \in \text{Col}_Q^r(T)$ such that $\rho(x_1) \in E$ and $\rho(x_1) \neq \rho(x_0)$ is finite. Then $\text{Col}_Q^r(T)$ is homeomorphic to $h$ copies of $S^1$.

**Proof.** This follows from Corollary 5.2.

**Remark 5.5.** In [15], non-abelian representations of knot groups in $SU(2)$ for torus knots and twist knots up to conjugation action were determined by Klassen. For each $\psi$, $\text{Col}_Q^r(T) \cap E$, $Q = S^2_\psi$ corresponds to Klassen’s representation. Thus the sets $\text{Col}_Q^r(T)$ are known from the paper [15]. We determine explicit colorings of $T(2,n)$ and the figure 8 knot by $S^2_\psi$ in the next two subsections and compute the longitudinal mappings for these knots in the next section.

### 5.1 Colorings of the torus knots $T(2,n)$ by $S^2_\psi$

Let $n = 2k + 1$ and we label the arcs of $T(2,n)$ by $u_i$ as in Figure 2. For later convenience in computing the longitude, we use the notation $u_i = q_{2i}$ and $u_{k+i} = q_{2i-1}$ for $i = 0, \ldots, k$ as depicted in Figure 2. Note that the subscripts on the $u_i$’s correspond to the labeling of the Wirtinger code (Definition 3.1).

![Figure 2: Arc labeling diagram for $T(2,n)$](image)

Let $p_i$, $i = 0, \ldots, n-1$ (subscripts taken modulo $n$), be a set of points on $S^2$ that are the vertices of a spherical regular $n$-gon arranged in counterclockwise order, for example,

$$p_i = (\sqrt{1 - r^2 \cos((2\pi/n)i)}, \sqrt{1 - r^2 \sin((2\pi/n)i)}, r)$$

where $r \in (-1,1)$. Then the side lengths $d(p_i, p_{i+1})$ and the angles $\angle p_{i-1}p_ip_{i+1}$ are constant.

**Lemma 5.6.** Let $n = 2k + 1$. Let $C_h$ be the map $A(T(2,n)) \rightarrow S^2_\psi$ defined by $C_h(q_i) = p_{hi}$ where the subscripts are taken modulo $n$. If $\psi = \angle p_{(i-1)h}p_ip_{(i+1)h}$, then $C_h$ defines a coloring of $T(2,n)$.

**Proof.** From Figure 2 $C_h(q_i)$, $i = 0, \ldots, n-1$, gives rise to a non-trivial coloring if the following equations are satisfied: $C_h(q_{i-1}) \ast C_h(q_i) = C_h(q_{i+1})$ for all $i$, where the subscripts are taken modulo $n$. Since the lengths $d(p_{ih}, p_{(i+1)h})$ and the angles $\angle p_{(i-1)h}p_{ih}p_{(i+1)h}$ are constant, the conditions for a coloring in Lemma 5.1 are satisfied. \qed
Example 5.7. For \( n = 7 \) and \( h = 1, 2, 3 \) respectively, the points corresponding to the colorings are illustrated in Figure 3. For each \( h = 1, 2, 3 \), the ranges of \( \psi \) are computed from Lemma 5.8 below as \( (5/7)\pi < \psi < (9/7)\pi \), \( (3/7)\pi < \psi < (11/7)\pi \), and \( (1/7)\pi < \psi < (13/7)\pi \).

Lemma 5.8. Let \( n = 2k + 1 \). For \( h = 1, \ldots, k \), there exists a regular star \( n \)-gon with vertices \( p_{ih} \), \( i = 1, \ldots, n-1 \), with \( \psi = \angle p_{(i-1)h}p_{ih}p_{(i+1)h} \) if and only if

\[
(n - 2h)\pi/n < \psi < (n + 2h)\pi/n.
\]

Proof. Assume that there exists such a regular star \( n \)-gon with \( \psi = \angle p_{(i-1)h}p_{ih}p_{(i+1)h} \). The angle \( \angle p_{(i-1)h}p_{ih}p_{(i+1)h} \) is smaller as the length \( d(p_{ih}, p_{(i+1)h}) \) is smaller, and hence the lower bound of such \( \psi \) is computed as the corresponding angle \( \angle p_{(i-1)h}p_{ih}p_{(i+1)h} \) for a planar, infinitesimal regular \( n \)-gon formed by \( p_{ih} \).

For the planar regular \( n \)-gon with vertices \( p_{i} \), \( i = 0, \ldots, n-1 \) in this cyclic order, the angle \( \angle p_{i-1}p_{0}p_{i+1} \) equals \( [(n - 2)/n]\pi \) since there are \( n - 2 \) triangles in a regular \( n \)-gon. This angle \( \angle p_{0-1}p_{0}p_{1} \) at \( p_{0} \) is equally divided to the angle \( \angle p_{i}p_{0}p_{i+1} \) inscribed by \( p_{i} \) and \( p_{i+1} \) for each \( i \), hence \( \angle p_{0}p_{0}p_{i+1} = \pi/n \). The angle \( \angle p_{1}p_{0}p_{h} \) and \( \angle p_{k}p_{0}p_{n-1} \) consist of \( (h - 1) \) parts of \( \pi/n \). Hence the lower bound is computed as

\[
\angle p_{h}p_{0}p_{kh} = \angle p_{n-1}p_{0}p_{1} - (\angle p_{1}p_{0}p_{h} + \angle p_{kh}p_{0}p_{n-1}) = [(n - 2) - 2(h - 1)]\pi/n = (n - 2h)\pi/n.
\]

See Figure 4. Since the bounds are symmetric about \( \pi \), we obtain the upper bound of

\[
\pi + (\pi - (n - 2h)\pi/n) = (n + 2h)\pi/n
\]
as desired.

Corollary 5.9. For \( n = 2k + 1 \) there is a non-trivial coloring of \( T(2, n) \) by \( S_{\psi}^{2} \) if and only if

\[
(n - 2h)\pi/n < \psi < (n + 2h)\pi/n,
\]

for some \( h = 1, \ldots, k \).
Proof. Immediate from Lemma 5.6 and Lemma 5.8. \qed

Remark 5.10. For fixed \( n \) and \( h \) as \( \psi \) ranges over the interval \(( (n-2h)\pi/n, \pi ]\) continuously, the polygons formed by the lengths \( d(p_{ih}, p_{(i+1)h}) \) continuously change from an infinitesimal polygon to a polygon on the equator. As \( \psi \) approaches the lower bound \((n-2h)\pi/n\), the polygon converges to a planar polygon.

The coloring condition holds for the Euclidean rotational quandles investigated in [12], in which Inoue proved that there exists a non-trivial coloring by planar rotational quandles if and only if the Alexander polynomial has a root on the unit circle \( S^1 \subset \mathbb{C} \). The Alexander polynomial of \( T(2, n) \) is a factor of \( x^{2n} - 1 \).

Remark 5.11. In [15], SU(2) representations up to conjugacy are studied. Furthermore, in [11], under certain conditions satisfied by \( T(2, n) \) and twist knots, the representations are deformations of dihedral representations at \( \psi = \pi \).

These results are seen in the above continuous family of star polygons. They start from infinitesimal planar polygons and converge to the equatorial “polygons” that correspond to Fox colorings by dihedral quandles.

Proposition 5.12. Let \( Q = S^2_\psi \) and \( T \) be a tangle of \( T(2, n) \) as depicted in Figure 2. For \( n = 2k+1 \) and \( h = 1, \ldots, k \), if \((n-2h)\pi/n < \psi \leq (n-2h+2)\pi/n\) then

\[
\text{Col}^2_{S_\psi}(T) = \sqcup hS^1,
\]

\( h \) copies of disjoint circles.

Proof. By Lemma 5.8, if \( \psi \) is in the stated range, then for any \( h' \leq h \), \( h' \) satisfies the condition stated in Lemma 5.8. In Figure 2, the arcs \( q_0 \) and \( q_1 \) are taken as \( x_0 \) and \( x_1 \) in Lemma 5.4. Hence in the notation in Lemma 5.4, \( \text{Col}^2_{S_\psi}(T) \cap E \) consists of \( h \) points, and the result follows from Lemma 5.4. \qed

5.2 Colorings of the figure eight knot by \( S^2_\psi \)

In this subsection we describe the colorings of a figure eight knot by the spherical quandle \( S^2_\psi \).

Lemma 5.13. A sequence \( U = (u_0, u_1, u_2, u_3) \) defines a coloring if and only if the following conditions are satisfied in \( S^2_\psi \):

\[
d(u_1, u_2) = d(u_2, u_0) = d(u_0, u_3), \quad d(u_0, u_1) = d(u_1, u_3) = d(u_3, u_2),
\]

and

\[
\angle(u_0u_2u_1) = \angle(u_0u_1u_3) = \angle(u_2u_3u_1) = \angle(u_2u_0u_3) = \psi.
\]
Proof. Direct inspection of Figure 5 gives the following:

\[ u_0 * u_2 = u_1, \ u_0 * u_1 = u_3, \ u_2 * u_3 = u_1, \ u_2 * u_0 = u_3, \]

where \( u_4 = u_0 \) and the equalities are derived from the crossings. By Lemma 5.1 the statement follows.

Figure 5: Colorings of the figure eight knot

**Lemma 5.14.** For \( \psi = 2\pi/3 \) and \( \psi = 4\pi/3 \) there is a unique solution \( U \) to the equations in Lemma 5.13 such that

\[ u_0 = x_\psi = (1, 0, 0) = i \ and \ u_2 = (\cos(\beta), \sin(\beta), 0). \]

The solution \( U \) forms a regular spherical tetrahedron. In this case \( \beta = \arccos(-1/3) \).

For \( 2\pi/3 < \psi < 4\pi/3 \), there are two nontrivial solutions \( U \) to the equations in Lemma 5.13 such that

\[ u_0 = x_\psi = (1, 0, 0) = i \ and \ u_2 = (\cos(\beta), \sin(\beta), 0). \]

The solutions are determined by the two values of \( u_2 \), \( u_2 = (\cos(\beta_i), \sin(\beta_i), 0) \), where for \( i = 1, 2 \),

\[ \beta_1 = \pi - \arccos(-1 + \sqrt{4 \cos^2(\psi) - 4 \cos(\psi) - 3})/2 (\cos(\psi) - 1), \]
\[ \beta_2 = \arccos(1 + \sqrt{4 \cos^2(\psi) - 4 \cos(\psi) - 3})/2 (\cos(\psi) - 1). \]

Proof. This comes directly from Maple computations. The Maple worksheets can be found at [6].

Remark 5.15. Note that by Lemma 5.2 it suffices to restrict \( \beta \) to the interval \((0, \pi]\).

Remark 5.16. Maple computations give the above exact solutions. It was also pointed out by Shin Satoh (via personal communication) that the spherical laws of sine and cosine, together with the area formula that a spherical triangle with angles \( \alpha, \beta, \gamma \) has area \( \alpha + \beta + \gamma - \pi \), yield the solutions.

Remark 5.17. The solutions for \( \beta_i \) for \( i = 1, 2 \) in Lemma 5.14 are plotted in Figure 6 for \( \psi \in [2\pi/3, 4\pi/3] \). Each angle \( \beta_i \) is 0 outside of this interval. Hence the colorings are trivial for \( \psi \) outside this interval.
Remark 5.18. The solutions $U$ in Lemma 5.14 for $\psi = 2\pi/3, 7\pi/9, 19\pi/20$ and $\pi$ form vertices of spherical tetrahedra as depicted in Figure 7.

We recall that the figure eight knot is non-trivially colorable by the tetrahedral quandle (the solution $U$ at $\psi = 2\pi/3$) and the dihedral quandle $R_5$ (Fox 5-colorable). Note also that since the minimal diagram in Figure 5 has only four arcs, four colors in $R_5$ are used for non-trivial colorings. Up to mirror symmetry, there are two choices of elements of $u_2$ from $R_5$ for a fixed element for $u_0$. As in Remark 5.11, there are continuous family of solutions as $\psi$ varies from $2\pi/3$ to $\pi$. A single regular tetrahedral coloring bifurcates to two branches of solutions as in Lemma 5.14, and converges to the two solutions of Fox colorings, as described in [11]. Animations of this situation can be found at [http://shell.cas.usf.edu/~saito/SphericalQuandle/](http://shell.cas.usf.edu/~saito/SphericalQuandle/).

Remark 5.19. More generally, Klassen [15] described the representations of knot groups in SU(2) for twist knots $\text{Tw}_m$, $m > 0$, and proved that up to conjugation it consists of $m/2$ circles if $m$ is even, and $\lfloor m/2 \rfloor$ circles and a single open arc if $m$ is odd. The cases $m = 1$ and $m = 2$ correspond to the trefoil and the figure eight knot, respectively.

Remark 5.20. It is well known that the Alexander polynomial of $\text{Tw}_m$ for odd $m$ is given by $\Delta_{\text{Tw}_m}(t) = (m+1)t^2 - 2mt + (m+1)$. Direct calculations show that $\Delta_{\text{Tw}_m}(t)$ has roots on $S^1 \subset \mathbb{C}$, and by [12], there is a nontrivial coloring by planar rotational quandle for $\psi = \arg(\alpha)$, where $\alpha$ is its root. Let $\alpha$ be the root with smaller argument. Then for odd $m$ there is a non-trivial coloring of $\text{Tw}_m$ by $S_\psi^2$, for $\arg(\alpha) < \psi < \arg(\bar{\alpha})$.

6 Longitudinal Mapping Invariant Values

In this section we determine the invariant values $\mathcal{L}_\theta$ for the torus knots $T(2, n)$ and the figure eight knot.
Figure 7: Colorings for the figure eight knot by $S_\psi^2$ for $\psi = 2\pi/3, 7\pi/9, 19\pi/20$ and $\pi$.

6.1 Torus knots $T(2, n)$

We used the labeling of the diagram of $T(2, n)$ in Figure 2 where $n = 2k + 1$ is odd.

Lemma 6.1. for $n = 2k + 1$ and $h = 1, \ldots, k$, $T(2, n)$ is non-trivially colored by $\tilde{\mathcal{C}}_\theta$ if and only if

$$\frac{(n - 2h)\pi}{2n} < \theta < \frac{(n + 2h)\pi}{2n}.$$ 

Proof. By Lemma 4.4 for $0 < \theta < \pi$ the quandle $S_\psi^2$, $\psi = 2\pi - 2\theta$, is isomorphic to the conjugacy class $\tilde{\mathcal{C}}_\theta = \{e^{i\theta}u : u \in S^2\}$ considered as a quandle under conjugation: $p * q = q^{-1}pq$. Clearly the isomorphism $u \mapsto e^{i\theta}u$ takes a coloring to a coloring. By Corollary 5.9 for $n = 2k + 1$ there is a non-trivial coloring of $T(2, n)$ by $S_\psi^2$ if and only if for some $h = 1, \ldots, k$ we have

$$(n - 2h)\pi/n < \psi < (n + 2h)\pi/n,$$

since $\psi = 2\pi - 2\theta$ this is equivalent to

$$\frac{(n - 2h)\pi}{2n} < \theta < \frac{(n + 2h)\pi}{2n}.$$
**Lemma 6.2.** Let \( n = 2k + 1, k \geq 1 \). Let \( G \) be a group. Let \( q_i, i = 0, 1, \ldots, n - 1 \), be the colors of the arcs, as depicted in Figure 2 of a coloring of the diagram by \( G \). Then \( q_i \) satisfy \( q_{i+1} = q_{i-1}^{-1}q_i \) for \( i = 1, \ldots, n - 1 \) and \( q_n = q_0 \).

We thank Razvan Teodorescu for the idea of the following proof.

**Lemma 6.3.** Let \( n = 2k + 1, k \geq 1 \), and \( G \) be a group. For a coloring \( C \) of the diagram of \( T(2, n) \) in Lemma 6.2, let \( q = q_0q_1 \). Then the longitude is given by \( \mathcal{L}(C) = q_0^{-2n}q^n \).

**Proof.** By Lemma 6.2 we have \( q_iq_{i+1} = q_{i+1}q_{i+2} \) for \( i = 0, \ldots, n - 2 \), and \( q_{n-1}q_0 = q_0q_1 \). Note that \( q = q_0q_{i+1} \) for all \( i \).

For any coloring \( C \), from Figure 2 we compute the longitude as
\[
\mathcal{L}(C) = q_0^{-n} (q_1q_3 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k}).
\]
To evaluate this, we compute
\[
q_0^{2n}\mathcal{L}(C) = q_0^n (q_1q_3 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k}).
\]
Since \( q_0q_1 = q_1q_2 \), we have
\[
q_0^{2n}\mathcal{L}(C) = (q_0 \cdots q_0) (q_0q_1) (q_3 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k})
= (q_0 \cdots q_0) (q_1q_2) (q_3 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k}).
\]
Further applying \( q_0q_1 = q_1q_2 \) and \( q_2q_3 = q_3q_4 \), we obtain
\[
= (q_0 \cdots q_0) (q_0q_1) (q_2q_3) (q_5 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k})
= (q_0 \cdots q_0) (q_1q_2) (q_3q_4) (q_5 \cdots q_{2k-3}) (q_0q_2 \cdots q_{2k}).
\]
Inductively we obtain
\[
(q_0 \cdots q_0) (q_1q_2q_3q_4 \cdots q_{2k}) (q_0q_2 \cdots q_{2k}).
\]
There are \( k + 1 \) copies of \( q_0 \) in the first factor, \( (q_1)^{2k}_{i=1} \) in the second factor, and consecutive even terms in the third factor. Then we continue with
\[
= (q_0 \cdots q_0) (q_1q_2q_3q_4 \cdots q_{2k-1}) (q_0q_1) (q_2 \cdots q_{2k})
= (q_0 \cdots q_0) (q_1q_2q_3q_4 \cdots q_{2k-1}) (q_0q_1) (q_2 \cdots q_{2k})
= (q_0 \cdots q_0) (q_1q_2q_3q_4 \cdots q_{2k-1}) (q_0q_1q_2) (q_3 \cdots q_{2k}) \cdots .
\]
In the last line, the left consecutive sequence keeps shifting to the left, as the middle pair \( (q_0q_1) \) shifts to the left. Inductively, we obtain \( q_0^{2n}\mathcal{L}(C) = (\prod_{i=0}^{n-1} q_i)^2 = q^n \). Hence we obtain \( \mathcal{L}(C) = q_0^{-2n}q^n \). \( \square \)

**Remark 6.4.** In the proof of Lemma 6.3, once the computation of \( \mathcal{L}(C) = q_0^{-2n}(\prod_{i=0}^{n-1} q_i)^2 \) is obtained, we found a diagrammatic method of obtaining the same formula. Specifically, from the diagram in Figure 8, we can read off the longitude directly as \( q_0^{2n}\mathcal{L}(C) = (\prod_{i=0}^{n-1} q_i)^2 \).

It is noteworthy that in the following theorem, the longitudinal mapping depends only on \( \theta \), and not on the different colorings \( C \) corresponding to \( \theta \).
Figure 8: Colored diagram for $T(2,n)$ with loops

**Theorem 6.5.** For any non-trivial coloring $C$ of $T(2,n)$, the value of the longitudinal mapping for $(SU(2), x)$ where $x = e^{θi}$ is given by

$$L(C) = e^{(π − 2nθ)i} = −\cos(2nθ) + \sin(2nθ)i.$$  

and for the mirror image $m(T(2,n))$ the value of the longitudinal mapping is given by

$$L(C) = e^{(2nθ − π)i} = −\cos(2nθ) − \sin(2nθ)i.$$  

**Proof.** In the case of $G = SU(2)$ in Lemma 6.3, we show that $q^n = −1$, where $q = q_0q_1 = q_iq_{i+1}$ for all $i$. Since

$$q^{-1}q_iq = (q_iq_{i+1})^{-1}q_i(q_iq_{i+1}) = q_{i+2},$$

we have $q^{-n}q_iq^n = q_i$ for every $i$. Then $q^n$ is in $C(q_i)$ for every $i$. For a non-trivial coloring, there are at least two $q_i$ and $q_j$ that do not commute, hence by Lemma 4.6, $C(q_i) ∩ C(q_j) = \{±1\}$, so that $q^n = ±1$.

For each $θ$, we have $q^n = ±1$, and $q^n$ is continuous with respect to $θ$. By Corollary 4.2 for $θ = π/2$, we have $S_π^2$ isomorphic to the conjugacy class $\hat{C}_{π/2}$. In this case, the colorings by $S_π^2$ up to the action of rotations about $x$ (cf. Corollary 5.2) are equivalent to Fox colorings by a dihedral quandle $R_m$ for some $m$. In [15], it was shown that the non-abelian representations of knot groups of torus knots $T(r,s)$ up to conjugacy consist of $(r−1)(s−1)/2$ open arcs. In our case the result implies that the set of non-trivial colorings of $T(2,n)$ consists of $n − 1$ open arcs each of which contains a coloring by the dihedral quandle $R_n$. Hence the fact $q^n = −1$ follows if it is proved for colorings by $\hat{C}_{π/2}$.

Let $θ = π/2$, then $q_0 = e^{πi} = i$. In this case $q_1 = \cos(2πm/n)i + \sin(2πm/n)j$ for some $m$. Then we compute

$$q = q_0q_1 = −\cos(2πm/n) + \sin(2πm/n)k = e^{(π−2πm/n)k}.$$  

Hence we obtain

$$q^n = e^{(π−2πm/n)nk} = e^{π(n−2m)k} = −1$$

since $n$ is odd, as desired.

The result for $m(T(2,n))$ follows immediately from the result for $T(2,n)$ via Proposition 3.5 and the known fact that $r(T(2,n)) = T(2,n)$.

6.2 Figure eight knot

The following Lemma is immediate from Lemma 5.14 and the fact that $S^2_π$ is isomorphic to $\hat{C}_θ$ when $ψ = 2π − 2θ$ and the fact that the isomorphism $u \mapsto e^{θu}$ takes a coloring to a coloring.
Lemma 6.6. The figure 8 knot is non-trivially colored by $\tilde{C}_\theta$ if and only if

$$\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}.$$ 

In which case there are two solutions for each $\theta \in (\frac{\pi}{3}, \frac{2\pi}{3})$, corresponding to the values of $\beta_1$ and $\beta_2$ in Lemma 5.14. The colorings for $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are the same.

Let $C(i) = u_i$ be a coloring for the figure 8 knot by $\tilde{C}_\theta$ for $\theta \in [\pi/3, 2\pi/3]$, as shown in Figure 5. Then from the definition of the longitude we obtain the following.

Lemma 6.7. $L_\theta(C) = u_2u_3^{-1}u_0u_1^{-1}$.

Maple computations give the following.

Proposition 6.8. If $C$ is a coloring of the figure 8 knot by $\tilde{C}_\theta$ then

$$L_\theta(C) = (\cos (4 \theta) - \cos (2 \theta) - 1) \pm \sqrt{-1 + 2 \cos (4 \theta) - 4 \cos (2 \theta) (\sin (2 \theta))} i.$$ 

The sign $\pm$ depends on the choice of $\beta_i$, $i = 1, 2$, in Lemma 5.14.

The longitude $L_\theta(C)$ may be written as $e^{i\phi}$ where $\phi$ is given in terms of the two argument arctan by

$$\phi = \arctan \left( \pm \sqrt{\frac{4 \cos (2 \theta)^2 - 4 \cos (2 \theta) - 3 (\sin (2 \theta))}{2 (\cos (2 \theta))^2 - \cos (2 \theta) - 2}} \right).$$

The graph of $\phi$ as a function of $\theta$ is given in Figure 9.

Figure 9: The graph of $\phi$ where $L_\theta(C) = e^{i\phi}$ for the figure 8 knot.
7 Concluding Remarks

In this paper, the knot coloring polynomial defined by Eisermann [9] with finite quandles is generalized to topological quandles as the longitudinal mapping invariant of long knots, which in turn can be thought of as a generalization of the quandle 2-cocycle invariant defined in [3] for finite quandles. Such generalizations for topological quandles have long been called for, and we propose one in this paper. The invariant values are concretely evaluated for torus knots of closed 2-braids $T(2, n)$ and the figure eight knot.

The following questions, for example, remain to be investigated: determine the coloring spaces for other knots, in particular knots with more than 2 bridges; determine the $\theta$-values with non-trivial colorings; determine the invariant values; relations to other invariants; investigate continuous cohomology theories of topological quandles, and relate it to the invariant discussed in this paper.

APPENDICES

A Eisermann quandles and generalized Alexander quandles

For an alternative description of the invariant $L$, we focus on the following quandles found in Lemma 25 and Remark 27 of [10].

Definition A.1. Let $G$ be a group and $x \in G$ such that conjugacy class $x^G$ generates $G$. The conjugacy class $x^G$ is a quandle under conjugation $a \ast b = b^{-1}ab$ and $a \bar{\ast} b = bab^{-1}$. Let $G'$ be the commutator subgroup of $G$. Define the set

$$Eis(G, x) = \{(a, g) \in x^G \times G' \mid a = x^g\}.$$ 

This set becomes an indecomposable quandle under the operations

$$(a, g) \ast (b, h) = (a \ast b, x^{-1}gb), \quad (a, g) \bar{\ast} (b, h) = (a \bar{\ast} b, xgb^{-1}),$$

We call this the Eisermann quandle given by the pair $(G, x)$. We write

$$p : Eis(G, x) \to x^G, \quad (a, g) \mapsto a,$$

for the projection onto $x^G$.

Eisermann [10] wrote $\tilde{Q}(G, x)$ for what we call here $Eis(G, x)$. Furthermore as he pointed out that this definition is tailor-made to capture the longitude information we need for the proof of Lemma B.3.

Lemma A.2. If $G$ is a group that is generated by the conjugacy class $x^G$ then $x^{G'} = x^G$, $Eis(G, x)$ is an indecomposable quandle and the projection

$$p : Eis(G, x) \to x^G, \quad (a, g) \mapsto a$$

is a quandle epimorphism that is equivalent to

$$\text{inn} : Eis(G, x) \to \text{inn}(Eis(G, x)).$$

The fiber $p^{-1}(x)$ is $C(x) \cap G'$ where $C(x)$ is the centralizer of $x$ in $G$. If $C(x) \cap G'$ is abelian then $p : Eis(G, x) \to x^G$ is an abelian extension.

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Proof. See Lemma 25 in [10] and Appendix B in [5].

As noted by Eisermann, Eis(G, x) has an alternative description as a generalized Alexander quandle GAlex(G', f_x) where f_x is the inner automorphism f_x(g) = x^{-1}gx, g ∈ G. Since G' is a normal subgroup, f_x is an automorphism of G' and so GAlex(G', f_x) is well-defined.

Lemma A.3. For x an element of a group G the quandles Eis(G, x) and GAlex(G', f_x) are isomorphic.

Proof. It is easy to check that the mapping : (a, g) → g is the desired isomorphism.

Remark A.4. The Eisermann quandle Eis(G, x) does not determine G since there are many groups in general with the same commutator subgroup. On the other hand every indecomposable generalized Alexander quandle Q = GAlex(G, f) determines the group G, namely G = Inn(Q)', and determines the automorphism f ∈ Aut(G) up to conjugacy in Aut(G). Moreover if Q = GAlex(G, f) is indecomposable and e ∈ Inn(Q) then Q ∼= Eis(Inn(Q), R_e) as noted in Corollary B.3 of [5].

B Alternative interpretation of ℒ for Eisermann and Alexander quandles

We recall the following two lemmas.

Lemma B.1 (Eisermann [10], Theorem 30). Let p : ˜Q → Q be a covering such that p(q) = x, q ∈ ˜Q, and let T be a 1-tangle diagram. Then the mapping ˜C ↦ p ˜C is a bijection from Col_{Q}(T) to Col_{Q}(T).

Lemma B.2 ([5]). Let C : A(T) → Y be a coloring of a 1-tangle diagram T by a quandle X. For the initial and terminal arcs 0 and n of T, respectively, let x_0 = C(0) and x_1 = C(n). Then inn(x_0) = R_{x_0} = R_{x_1} = inn(x_1).

Now let ˜Q = Eis(G, x) and Q = x^G and p : ˜Q → Q as in Lemma A.2 so that p(x, 1) = x. Let ˜C ∈ Col_{Q}^{(x, 1)} and C = p ˜C as in Lemma B.1. Let ℒ(C) be as defined above.

Proposition B.3. In the notation above let ˜C be the unique lifting of the coloring C ∈ Col_{Q}^{(x, 1)}(T) to Col_{Q}^{(x, 1)}(T). Then ˜C(n) = (x, ℒ(C)).

Proof. Let w(i) = \sum_{h=1}^{i} \epsilon(h) be the writhe counted along the tangle from the initial arc 0 along the tangle up until one reaches at the arc i. By Lemma B.1 we know that the coloring C ∈ Col_{Q}^{x}(T) lifts to a unique coloring ˜C ∈ Col_{Q}^{(x, 1)}(T). Write u_i = C(i) for i = 0, . . . , n. Thus we have ˜C(i) = (u_i, g_i) for i = 0, . . . , n. By Lemma B.2 we have u_n = x. Assume inductively that g_i = x^{-w(i)} \prod_{h=1}^{i} u_{\epsilon(h)}^{\epsilon(h)}. One computes using x = x^1 and ˜x = x^{-1}:
\[ \tilde{C}(i + 1) \]
\[ = \left( u_{i+1}, g_{i+1} \right) \]
\[ = \left( u_i, g_i \right)^{\epsilon(i+1)} \left( u_{\kappa(i+1)}, g_{\kappa(i+1)} \right) \]
\[ = \left( u_{i+1}, x^{-\epsilon(i+1)} g_i \epsilon(i+1) \right) \]
\[ = \left( u_{i+1}, x^{-\epsilon(i+1)} x^{-w(i)} \left( \prod_{h=1}^{i} u_{\kappa(h)}^\epsilon(h) \right) u_{\kappa(i+1)}^\epsilon(i+1) \right) \]
\[ = \left( u_{i+1}, g_{i+1} \right). \]

Taking \( i = n \) we see that the Proposition holds.

**Theorem B.4.** In the notation above and let \( \tilde{C} \) be the unique lifting of the coloring \( C \in \text{Col}_Q^*(T) \) to \( \text{Col}^1_{\text{GAlex}(G', f_x)}(T) \). Then \( \tilde{C}(n) = L(C) \).

**Remark B.5.** Each element of \( \text{SU}(2) \) is a commutator (\[19\] Prop. 10.24) so \( \text{SU}(2) \) is equal to its own commutator subgroup. Since \( \text{SO}(3) \) is a simple group (\[1\]), and the center of \( \text{SU}(2) \) is \( \{1, -1\} \) it follows that if \( x \neq \pm 1 \) then the conjugacy class \( x^{\text{SU}(2)} \) generates \( \text{SU}(2) \). Thus given any \( x \in \text{SU}(2) \) with \( x \neq \pm 1 \) we may apply the results of Appendix A to \( (G, x) = (\text{SU}(2), x) \).

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