Modelling the Nonlinear Response of Silicon Photomultipliers

Jaime Rosado

Abstract—A statistical model of the nonlinear response of silicon photomultipliers for light pulses of arbitrary shape is described. It allows the calculation of losses of both the photodetection efficiency and gain during pixel recovery periods as a function of the supplied overvoltage by means of a simple numerical integration. Analytical expressions for typical light pulse shapes and for continuous light are also provided. Effects due to correlated noise are included in a single fitting parameter that determines the effective gain of the detector in the linear region. The model has been validated for two different silicon photomultipliers using scintillation light pulses from a LYSO crystal as well as continuous light from a LED. Good agreement is found with experimental data at moderate nonlinearity. The model proves to be applicable to both types of light sources and to describe correctly the overvoltage dependence of the detector response.

Index Terms—Silicon photomultipliers, SiPM, nonlinearity, statistical model

I. INTRODUCTION

THE silicon photomultiplier (SiPM) is a high-sensitivity semiconductor photodetector that is increasingly becoming the best choice in many applications owing to its compactness, high gain, rapid response and exceptional photon-counting resolution. This device consists in an array of Geiger-mode avalanche photodiodes in the µm scale, hereafter called pixels, which are sensitive to single photons. The pixels are connected in parallel in such a way the output SiPM signal is the sum of the signals of individual pixels. In the absence of other effects, the mean output charge for a light pulse of n photons impinging on the SiPM photosensitive area is 

$$\langle Q \rangle = \varepsilon \cdot n \cdot q,$$

where $\varepsilon$ is the photodetection efficiency and $q$ is the mean charge released by a breakdown avalanche, which determines the gain of the device.

Both $\varepsilon$ and $q$ are functions of the supplied overvoltage $U$, defined as the reverse-bias voltage $V_{\text{bias}}$ minus the breakdown voltage $V_{\text{br}}$ of the SiPM. The mean avalanche charge $q$ is proportional to $U$ and it can be obtained from the distance between peaks in the output pulse charge spectrum at photon counting levels (see, e.g., [1]). On the other hand, $\varepsilon$ increases exponentially with $U$ as

$$\varepsilon = \varepsilon_{\text{max}} \cdot \left[ 1 - \exp \left( -\frac{U - U_0}{U_{\text{ch}}} \right) \right] \quad (1)$$

for $U \geq U_0$, otherwise $\varepsilon = 0$. Here $\varepsilon_{\text{max}}$ is given by the product of the quantum efficiency and the fill factor (i.e., the ratio of the photosensitive area of a pixel to the entire pixel area). The parameter $U_{\text{ch}}$ is a characteristic overvoltage of the SiPM that is dependent on the input wavelength spectrum [2]. The overvoltage shift $U_0$ is usually assumed to be zero, but experimental data of $\varepsilon$ are often better fitted by (1) with $U_0$ taking a positive value of a few tenths of a volt. This can be attributed to two distinct breakdown voltages for the avalanche gain and the trigger probability [3], [4].

However, the SiPM behaves nonlinearly at high photon density due to the fact that the number of pixels is finite (typically of the order of 1000, depending on the SiPM) and they take a few tens of ns to recharge after each breakdown avalanche. Besides, SiPMs have uncorrelated noise due to the thermal production of electron-hole pairs in silicon, as well as correlated noise caused by two processes called afterpulsing and crosstalk. Afterpulses are stochastic parasitic avalanches produced in the same pixel where a previous avalanche has been triggered, while the crosstalk effect is the production of parasitic avalanches in nearby pixels [1], [5].

The combination of all these features makes the dynamics of avalanche triggering and charge production very complex. As a consequence, an exact statistical description of the response of SiPMs is only possible via Monte Carlo simulations. H.T. van Dam et al. [6] developed a comprehensive statistical model of the SiPM response for exponentially decaying light pulses, accounting for both the correlated noise and the pixel recovery under some approximations. However it requires non-trivial numerical calculations that are also computational costly. A simpler statistical model was presented in [7], but it ignores the correlated noise and accounts of the recovery of pixels in an incomplete way. In addition, analytical expressions of the SiPM response in two limit situations are available. First, for light pulses much shorter than the pixel recovery time and in the absence of correlated noise, the mean output charge $\langle Q \rangle$ is given by the well-known expression [9]

$$\langle Q \rangle = N \cdot q \cdot \left[ 1 - \exp \left( -\frac{\varepsilon \cdot n}{N} \right) \right], \quad (2)$$

where $N$ is the number of pixels. Second, for continuous light and ignoring again the correlated noise, the output current intensity $I$ produced by the SiPM can be approximated by

$$I = \frac{\varepsilon \cdot r \cdot q}{1 + \varepsilon \cdot r \cdot t_{\text{dead}}/N}, \quad (3)$$

where $r$ is rate of impinging photons and $t_{\text{dead}}$ is a certain non-paralyzable dead time accounting for the pixel recovery [9]. Modified versions of the above expressions are being used to describe the response of SiPMs at particular conditions, e.g.,...
However, a general analytical model is not available yet.

In this paper, I present a simple model that includes all the above features under some simplifications. A preliminary version of this model was first reported in [14], but significant upgrades have been made since then. The model has been validated against experimental data for scintillation light pulses from a LYSO crystal as well as for continuous light from a LED.

II. THE MODEL

A. Approach to the problem

In steady state, all the pixels of a SiPM are biased with the supplied overvoltage \( U \). However, just when a breakdown avalanche is triggered in a pixel, the instantaneous overvoltage \( u \) of that pixel drops to zero and then grows exponentially as

\[
u = U \cdot \left[ 1 - \exp \left( -\frac{s}{t_{\text{rec}}} \right) \right],
\]

where \( s \) is the delay time from the avalanche triggering and \( t_{\text{rec}} \) is the recovery time of the SiPM, which is of the order of tens of ns. During the recovery period, both the probability that a photon triggers a new avalanche in the pixel and the mean charge released by this avalanche (if triggered) depend on \( u \). If the pixel is hit by several photons, the time evolution of \( u \) may be very complicated, especially if the light pulse length is comparable to \( t_{\text{rec}} \).

Besides, each avalanche can induce correlated noise that also contributes to the total output charge. The probabilities of crosstalk and afterpulsing are proportional to the charge released by the primary avalanche, but also depend on the instantaneous overvoltages in the pixels where these secondary processes may take place. Taking into account that secondary avalanches can in turn induce correlated noise, this may result in a complex chain process that relies on the temporal and spatial characteristics of crosstalk and afterpulsing (see [1], [3] for details).

Despite of the complexity of the problem, the whole process of charge production for each impinging photon can be modelled in three steps:

i) The photon has a probability \( \varepsilon \) to produce an avalanche seed in the pixel that it hits, where \( \varepsilon \) is given by \( [1] \).

ii) If the seed is produced, it has a probability \( a(s) \) to develop into an avalanche with mean charge \( b(s) \cdot q \), where

\[
a(s) = \begin{cases} 0 & s < s_0 \\ \varepsilon \exp \left( -\frac{U \cdot b(s) - U_0}{t_{\text{ch}}} \right) & s \geq s_0 
\end{cases}
\]

\[
b(s) = 1 - \exp \left( -\frac{s}{t_{\text{rec}}} \right),
\]

with \( s \) being the elapsed time \( s \) from the last avalanche. Here the time shift \( s_0 \) is defined such that \( b(s_0) = U_0 / U \) and it can generally be assumed to be much lower than \( t_{\text{rec}} \). However, \( s_0 \) may become significant when \( U \) approaches to \( U_0 \), making \( a(s) \) to grow more slowly than \( b(s) \).

iii) If the avalanche is triggered, it may induce crosstalk and afterpulsing. For simplicity, these correlated processes are not considered to produce further seeds, but the summed charge from the primary avalanche and the ensuing secondary ones is associated to the same seed. The distribution of this net charge \( \theta \) of a seed has a certain probability density distribution \( f(s, \theta) \) with mean \( b(s) \cdot (1 + c) \cdot q \), where \( c \) accounts for the average relative contribution from correlated noise, including the development of chains of secondary avalanches as well as their possible interactions with avalanches from other seeds.

The so-defined avalanche seeds are independent of each other. Therefore, the number of seeds \( m \) has a Poisson distribution

\[
P(m) = \frac{(\varepsilon \cdot n)^m}{m!} \cdot e^{-\varepsilon \cdot n},
\]

with \( \varepsilon \cdot n \) being the mean number of seeds per light pulse. Under this scheme, the probability density function of the total output charge \( Q \) of the SiPM can be expressed as

\[
F(Q) = \sum_{m=0}^{n} P(m) \cdot F_m(Q),
\]

where \( F_m(Q) \) is the output charge density function in the case that exactly \( m \) seeds are produced upon a light pulse. Then, the mean output charge is given by

\[
\langle Q \rangle = \sum_{m=0}^{n} P(m) \cdot \langle Q \rangle_m,
\]

where

\[
\langle Q \rangle_m = \int_0^\infty F_m(Q) \cdot Q \cdot dQ.
\]

In this way, all the complex dynamics of pixel recovery and the interactions between seeds are enclosed in the function \( F_m(Q) \). Therefore, the objective is modelling \( F_m(Q) \) and \( \langle Q \rangle_m \) for an arbitrary number of seeds \( m \).

B. Exact solution for a simple case

Before going to the general situation, let us consider first \( m \leq 2 \) and no correlated noise (i.e., \( c = 0 \)). It is clear that \( F_1(Q) \) corresponds to the single-avalanche charge density function for a steady pixel with \( u = U \), and that \( \langle Q \rangle_1 = q \). For \( m = 2 \), the two seeds are generally produced in two different pixels and thus they develop into two independent avalanches with charge density functions \( F_1(\theta) \) and \( F_1(\theta') \). In this case, the density function of the sum output charge is

\[
F_2(Q) = \int_0^Q F_1(\theta) \cdot F_1(Q - \theta) \cdot d\theta,
\]

accounting for all the possible combinations of the charges \( \theta \) and \( \theta' \) adding up to \( Q \), and the mean output charge is simply \( \langle Q \rangle_2 = 2 \cdot q \).

However, in the unlikely event that both seeds are produced in the same pixel at times \( t \) and \( t' \), the first one develops into an avalanche with charge density function \( F_1(\theta) \) and mean charge \( q \), whereas the second one has a probability \( a(s) \) to trigger an avalanche and, if triggered, the avalanche charge distribution has a certain probability density \( f(s, \theta) \) with mean \( b(s) \cdot q \), where \( a(s) \) and \( b(s) \) are given by (5) and (6), respectively, for \( s = |t - t'| \).

Now, let us consider a certain arrival photon time distribution \( p(t) \), which is just the normalized light pulse shape. For
two seeds produced at random times in the same pixel, the probability that the second seed develops into an avalanche is
\[
\alpha = 2 \cdot \int_0^\infty \int_0^{s_0} p(t) \cdot p(t+s) \cdot a(s) \cdot ds \cdot dt , \tag{12}
\]
where the factor 2 accounts for both possibilities \( t < t' \) and \( t' < t \). Here, the pulse width is assumed to be larger than \( s_0 \), otherwise \( \alpha = 0 \). Similarly, it can be defined a time-averaged charge density function for the second seed as
\[
\phi(t) = 2 \cdot \int_0^\infty \int_{s_0}^{\infty} p(t) \cdot p(t+s) \cdot a(s) \cdot f(s,t) \cdot ds \cdot dt . \tag{13}
\]
This density function already includes the avalanche triggering probability so that \( \int_0^\infty \phi(t) \cdot dx = \alpha \) and \( \int_0^\infty \phi(t) \cdot \theta \cdot d\theta = \gamma \cdot q \), where the parameter \( \gamma \) is given by
\[
\gamma = 2 \cdot \int_0^\infty \int_{s_0}^{\infty} p(t) \cdot p(t+s) \cdot a(s) \cdot b(s) \cdot ds \cdot dt . \tag{14}
\]
From this result, in the absence of correlated noise, the probability density function of the sum charge for two seeds randomly distributed over time and over the SiPM photosensitive area is
\[
F_2(Q) = \frac{1}{N} \cdot F_1(\theta) \cdot F_1(Q-\theta) \cdot d\theta + \frac{1 - \alpha}{N} \cdot F_1(Q) + \int_0^Q \int_{s_0}^{\infty} \phi(t) \cdot F_1(Q-t) \cdot d\theta . \tag{15}
\]
The three terms of the right-hand side of this equation arise from the following contributions: i) the two seeds develop into two avalanches in different pixels, i) both seeds are produced in the same pixel and only the first one develops into an avalanche, iii) both seeds are produced in the same pixel and develop into two avalanches. By substituting (15) into (10), results in
\[
\langle Q \rangle_2 = 2 \cdot q \cdot \left( 1 - \frac{1}{N} \right) + \frac{1 + \gamma}{N} \cdot q . \tag{16}
\]
C. General case

It becomes unwieldy to use the above procedure for arbitrary \( m \) and including correlated noise. Nevertheless, (15) and (16) can be generalized under some simplifications:

i) For a total output charge \( Q \), the mean number of fired pixels is \( Q/q \) to first-order approximation, which is equivalent to assume that each avalanche is triggered in a different pixel and has charge \( q \). In actuality, several avalanches (both primary and secondary ones) may be produced in the same pixel, but this is partly compensated by the fact that the greater the number of avalanches in a pixel, the smaller the mean charge per avalanche as a consequence of the reduction of the pixel gain during recovery periods. Under this approximation, the probability that a seed is produced in a \textit{busy pixel} is \( (Q-\theta)/(N \cdot q) \), where \( \theta \) is the net charge contribution (if any) from this seed and \( Q - \theta \) is that from all other seeds. Otherwise, the seed is considered to be produced in a \textit{free pixel} with \( u = U \).

ii) A seed produced in a \textit{free pixel} develops into an avalanche with mean charge \( q \) that may induce correlated noise. In this case, the net charge distribution of the seed is supposed to have a density function \( F_1(\theta) \) and mean \( \langle Q \rangle_1 \), that is, \( c \) is a constant such that \((1 + c) \cdot q = \langle Q \rangle_1 \) for all seeds. This assumption involves that the charge contribution from secondary avalanches associated to a seed is unaffected by the presence of other seeds. Note, however, that the correlated noise does contribute to nonlinearity, because it reduces the number of free pixels via the first point.

iii) For a seed produced in a \textit{busy pixel}, the probability that it develops into an avalanche and the net charge distribution are assumed to be given respectively by (12) and (13), with the only difference that \( f(s,\theta) \) and \( \phi(\theta) \) include the contribution from correlated noise so that \( \int_0^\infty \phi(\theta) \cdot \theta \cdot d\theta = \gamma \cdot \langle Q \rangle_1 \), where \( \gamma \) is calculated by (14) and \( \langle Q \rangle_1 \) is the same to that for a \textit{free pixel}.

This is a simplistic way to account for complex interactions between avalanches (e.g., those involving afterpulses with a certain delay time distribution). Nevertheless, it is clear that this model approaches to the exact solution when both correlated noise and nonlinear effects are small. With these ingredients, the following recurrence equation is built for \( m > 1 \):
\[
F_m(Q) = \int_0^Q \left( 1 - \frac{Q - \theta}{N \cdot q} \right) \cdot F_{m-1}(Q-\theta) \cdot F_1(\theta) \cdot d\theta \\
+ \frac{1 - \alpha}{N} \cdot F_{m-1}(Q) \\
+ \int_0^Q \frac{(Q - \theta)}{N \cdot q} \cdot F_{m-1}(Q-\theta) \cdot \phi(\theta) \cdot d\theta ,
\tag{17}
\]
where the three terms of the right-hand side of this equation have a similar meaning to those of (15). For the \( m \)-th seed of the sequence, this equation evaluates the probability that the seed is produced in either a \textit{free} or \textit{busy} pixel and its contribution to the total output charge. It can easily demonstrated that (17) leads to
\[
\langle Q \rangle_m = \frac{N \cdot q}{1 - \gamma} \cdot \left[ 1 - \left( 1 - \frac{(1 - \gamma) \cdot \langle Q \rangle_1}{N \cdot q} \right)^m \right] ,
\tag{18}
\]
which in combination with (7) results in
\[
\langle Q \rangle = \frac{N \cdot q}{1 - \gamma} \cdot \left[ 1 - \exp \left( - \varepsilon \cdot n \cdot (1 - \gamma) \cdot \langle Q \rangle_1 \right) \right] .
\tag{19}
\]

In Fig. 1 the properties of (19) are illustrated in a plot of the normalized mean charge \( \langle Q \rangle/(N \cdot q) \) versus \( (\varepsilon \cdot n)/N \) for different values of \( \langle Q \rangle_1 \) and \( \gamma \). The slope in the linear region is \( \langle Q \rangle_1/q \), where \( \langle Q \rangle_1 \) can be regarded as the effective gain of the SiPM and it approaches to \( q \) when the correlated noise is low. A theoretical determination of \( \langle Q \rangle_1 \) is difficult because it would involve describing the development of chains of secondary avalanches. Nevertheless, this parameter can easily be determined experimentally using low-intensity light pulses for which the SiPM behaves linearly, that is, \( \langle Q \rangle \approx \varepsilon \cdot n \cdot \langle Q \rangle_1 \). It should be clarified that, although the integration time window is large enough to contain the full signal pulse, the measured value of \( \langle Q \rangle_1 \) may not include small contributions from afterpulses with long delay (see, e.g., [1], [15]).

For \( (\varepsilon \cdot n)/N \gg 1 \), the curves shown in Fig. 1 saturate to \( 1/(1 - \gamma) \). The parameter \( \gamma \) quantifies the nonlinearity due
to pixel recovery periods during the light pulse. For a light pulse much longer than the recovery time, \( \gamma \) approaches to unity. This makes that \( \gamma \) is approximately linear even for \( (\varepsilon \cdot n)/N > 1 \), since each pixel may be fired and recovered several times during the light pulse. In the opposite case where the light pulse is much shorter than \( t_{\text{rec}} \), one gets \( \gamma \approx 0 \) and the saturation level is unity. If, in addition, \( (Q) = q \) can be assumed, then [19] reduces to [2], which corresponds to the limit situation for instantaneous light pulses and no correlated noise.

D. Effect of uncorrelated noise

In the above discussion, the uncorrelated noise has been ignored. Nevertheless, its effect can be readily accounted for by supposing a low rate \( r_{\text{bg}} \) of background photons in addition to those from the light pulse. This can be characterized by measuring the background output charge \( \langle Q \rangle_{\text{bg}} \) in a given integration time window \( T \). Assuming linearity, it is obtained that

\[
\langle Q \rangle_{\text{bg}} = \varepsilon \cdot r_{\text{bg}} \cdot T \cdot \langle Q \rangle_1 ,
\]

where \( \varepsilon \cdot r_{\text{bg}} \) can be regarded as a continuous rate of background seeds, each one contributing with a mean net charge \( \langle Q \rangle_1 \). It should be clarified that the dark count rate \( r_{\text{dc}} \), usually given in datasheet specifications, includes afterpulses, so that \( r_{\text{dc}} \) is somewhat larger than \( \varepsilon \cdot r_{\text{bg}} \).

The effect of the uncorrelated noise is twofold. In the first place, the effective number of photons \( n_{\text{eff}} = n + r_{\text{bg}} \cdot T \) should be substituted for \( n \) in [19], where \( T \) is assumed to be set large enough to contain the full pulse. In the second place, the effective photon arrival time distribution is

\[
p_{\text{eff}}(t) = (1 - \chi) \cdot p(t) + \chi \cdot \frac{1}{T} ,
\]

where \( \chi = (r_{\text{bg}} \cdot T)/n_{\text{eff}} \). Using this, an effective parameter \( \gamma_{\text{eff}} \) can be defined as

\[
\gamma_{\text{eff}} = (1 - \chi)^2 \cdot \gamma + \frac{2 \cdot (1 - \chi) \cdot \chi}{T} .
\]
This parameter \( t_{\text{dead}} \) decreases with \( U \) and approaches to \( t_{\text{rec}} \) when \( U \) is large enough so that \( s_0 \ll t_{\text{rec}} \) and \( \varepsilon_{\text{max}}/\varepsilon \approx 1 \).

The above result for very long rectangular pulses can also be used to describe the SiPM response for continuous light by making the substitutions \( r = n/T \) and \( I = \langle Q \rangle/T \), where \( r \) is the rate of impinging photons and \( I \) is the output current intensity. Using (26) in (19) results in

\[
I = \frac{N \cdot q}{2 \cdot t_{\text{dead}}} \left[ 1 - \exp\left( -\frac{2 \cdot \varepsilon \cdot r \cdot t_{\text{dead}} \cdot \langle Q \rangle}{N \cdot q} \right) \right].
\] (28)

Notice that, in the absence of correlated noise, i.e., \( \langle Q \rangle = q \), the expansion of this equation to first order in \( (r \cdot t_{\text{dead}})/N \) is equivalent to (3) for a non-paralyzable detector with dead time \( t_{\text{dead}} \).

**B. Double exponential pulses**

In many cases, the light pulse shape can be modelled as a double exponential function

\[
p(t) = \frac{1}{t_2 - t_1} \left[ \exp\left( -\frac{t}{t_2} \right) - \exp\left( -\frac{t}{t_1} \right) \right]
\] (29)

with \( t \geq 0 \) and \( t_1 < t_2 \). The rise time and the fall time of the pulse are approximately given by \( t_1 \) and \( t_2 \), respectively, when \( t_1 \ll t_2 \).

Substituting (29) into (14) results in

\[
\gamma = \frac{1}{t_2 - t_1} \left( t_2^2 \cdot \gamma_2 - t_1^2 \cdot \gamma_1 \right),
\] (30)

where

\[
\gamma_i = \frac{1}{t_i} \int_{s_0}^{\infty} \exp\left( -\frac{s}{t_i} \right) \cdot a(s) \cdot b(s) \, ds \quad (i = 1, 2).
\] (31)

Here it has been assumed that the integration time window \( T \) is large enough so that \( p(T) \approx 0 \).

For \( t_1 \ll t_{\text{rec}} \), (31) reduces to

\[
\gamma_i \approx \frac{\varepsilon_{\text{max}} \cdot U}{\varepsilon \cdot U_{\text{ch}}} \cdot \frac{\tau_i \cdot (2 \cdot \tau_i + s_0)}{t_{\text{rec}}} \cdot \exp\left( -s_0/\tau_i \right).
\] (32)

On the other hand, for \( t_{\text{rec}} \ll \tau_i \), (31) reduces to

\[
\gamma_i \approx 1 - \frac{t_{\text{dead}}}{\tau_i},
\] (33)

where it has been used that \( 1 - a(s) \cdot b(s) \) goes to zero much faster than \( \exp\left( -s/\tau_i \right) \). Therefore, for a very long pulse such that \( t_{\text{rec}} \ll \tau_1 < \tau_2 \), one gets

\[
\gamma \approx 1 - \frac{t_{\text{dead}}}{\tau_1 + \tau_2}.
\] (34)

The particular case of \( \tau_1 = 0 \) in (29) corresponds to a single exponential pulse with decay time \( \tau = \tau_2 \), for which \( \gamma \) reduces to \( \gamma_2 \) as given by (31). If the integration time window is large so that the uncorrelated noise is appreciable, the parameter \( \gamma_{\text{eff}} \) can be approximated in this case by

\[
1 - \gamma_{\text{eff}} \approx (1 - \gamma) \cdot \left[ (1 - \chi)^2 + 4 \cdot \chi \cdot \frac{T}{\tau} - 2 \cdot \chi^2 \cdot \frac{T}{\tau} \right],
\] (35)

where \( \tau \gg t_{\text{rec}} \) is assumed.

**IV. EXPERIMENTAL VERIFICATION**

The model was validated against experimental data for two SiPMs of the series 13360 from Hamamatsu, which is characterized by a very low correlated noise [16]. The detectors were chosen to have the same photosensitive area but different pixel size, in such a way that nonlinearity is stronger (\( N \) is smaller) in one SiPM than the other. Their characteristics are summarized in Table I. Values of \( t_{\text{rec}} \) were taken from [1]. The breakdown voltage \( V_{\text{br}} \) and the recommended overvoltage were provided by the manufacturer for a temperature of 25°C. The dark count rate \( r_{\text{dc}} \) and the mean avalanche charge \( q \) are given at the recommended overvoltage in the datasheet of the devices [16]. The parameters \( U_{\text{ch}}, U_0, \varepsilon_{\text{max}} \) and \( q_{\text{ch}} \) were obtained by fitting (1) to data for 450 nm photons and at 25°C also available in the datasheet. In addition, I measured the pulse charge spectrum at photon counting levels for the two tested SiPMs, obtaining values of \( q \) consistent within 10% with those shown in the table.

It is worth noting that the larger the pixel, the larger the junction capacitance and thus the greater both \( t_{\text{rec}} \) and \( q \). The correlated noise also depends on the pixel size. As an example, the probability of crosstalk and afterpulsing of the 1325 SiPM at \( U = 5 \) V are about 0.4% and 3%, respectively, whereas these probabilities are about 3% and 9% for the 1350 SiPM at the same overvoltage [1].

The SiPMs were biased using a Hamamatsu C12332 driver circuit that incorporates a compensation system for the temperature dependence of the SiPM gain. The output signal was registered with a digital oscilloscope Tektronix TDS5032B with math functions, including pulse integration and histogramming.

Two types of light sources were used. In the first place, the SiPM was coupled to a scintillation crystal irradiated by \( \gamma \) rays. In the second place, the SiPM was illuminated with continuous light from a LED.

**A. Scintillation light pulses**

I used a LYSO(Ce) scintillation crystal of \( 3 \times 3 \) mm\(^2\) base and 20 mm length, with BaSO\(_4\) reflector. Silicone grease was used for the optical coupling to the SiPM. Different radioactive sources (\(^{22}\)Na, \(^{60}\)Co, \(^{137}\)Cs and \(^{226}\)Ra) were used, providing an ample range of \( \gamma \)-ray energies from 300 to 2100 keV. An integration time window of 400 ns was set on the oscilloscope, which was large enough to contain the full pulse (see Fig. 2), but still small so that contributions from uncorrelated noise can be neglected (\( r_{\text{dc}} \cdot T < 0.1 \)). Then, pulse charge spectra

**TABLE I: Characteristics of the Tested SiPMs.**

| Parameters                  | S13360-1325CS | S13360-1350CS | Unit         |
|-----------------------------|---------------|---------------|--------------|
| Photosensitive area         | 1.3 \times 1.3 | -             | mm\(^2\)     |
| Pixel pitch                 | 25            | 50            | \( \mu \)m   |
| Number of pixels            | 2668          | 667           | -            |
| Recovery time \( t_{\text{rec}} \) | 17            | 29            | ns           |
| Breakdown voltage \( V_{\text{br}} \) | 51.80         | V             |              |
| Rec. overvoltage            | 5.00          | 3.00          | V            |
| Dark count rate             | \(< 2 \times 10^4 \) | \(< 2 \times 10^7 \) | kcps         |
| Avalanche charge \( q \)    | 0.7           | 1.7           | \( \times 10^4 \) e |
| Ch. overvoltage             | 2.69          | 2.68          | -            |
| Overvoltage shift \( t_{\text{shift}} \) | 0.66          | 0.00          | V            |
| Maximum PDE                 | 0.327         | 0.597         | -            |
for the different radioactive sources and at several overvoltage values were obtained for the two tested SiPMs.

In Fig. 3 the positions of the recognizable photopeaks in the spectra are represented against the γ-ray energy $E$. It is shown the normalized mean output charge $\langle Q \rangle / (N \cdot q)$ to ease the comparison between both SiPMs. The estimated systematic uncertainty was 10% due to assessment of $q$. Nonlinearity is more apparent in the 1350 SiPM, because it has less pixels per unit area.

For application of (19), $\varepsilon$ was obtained from the data shown in Table I and $\gamma$ was calculated as described later. The number of impinging photons per scintillation pulse is unknown, but it can be assumed to be proportional to $E$. Therefore, the substitution

$$\frac{\langle Q \rangle_1}{q} \cdot n = \frac{\langle Q \rangle_1}{q} \cdot C \cdot Y \cdot E = \kappa \cdot E,$$

was made in (19), where $C$ is the light collection efficiency, which should be the same for both SiPMs because they have equal photosensitive area, and $Y$ is the light yield of the scintillation crystal, which is 29 photons per keV for the LYSO(Ce) material according to manufacturer’s data [17]. In doing so, $\kappa$ is the only fitting parameter, which is basically determined from data in the linear region.

For the calculation of the parameter $\gamma$, the photon arrival time distribution $p(t)$ was approximated by the SiPM response function $p_{out}(t)$, which was obtained by averaging many output pulses and normalizing the integral to unity. The recorded pulses were checked to have essentially the same average shape regardless of their charge. As long as the applied overvoltage was not much higher than the recommended value, $p_{out}(t)$ was well fitted by a double exponential function [29] with $\tau_1 = 17$ ns and $\tau_2 = 45$ ns for the 1325 SiPM, and with $\tau_1 = 20$ ns and $\tau_2 = 60$ ns for the 1350 SiPM. However, the pulse length was found to increase rapidly with overvoltage when it exceeded a certain value, namely 20 V for the 1325 SiPM and 10 V for the 1350 SiPM, as illustrated for the latter SiPM in Fig. 2. This pulse widening coincides with a sudden increase in $\langle Q \rangle / (N \cdot q)$ at high overvoltage (see Fig. 3), which is attributed to the fact that the correlated noise is high enough to produce a self-sustained chain of secondary avalanches. The present model does not include the time distribution of afterpulses and thus it is not expected to describe data accurately at so high overvoltage. Nevertheless, using $p_{out}(t)$ for the determination of $\gamma$ is a way to roughly account for this effect. Therefore, $\gamma$ was calculated from (30) using the values of $\tau_1$ and $\tau_2$ obtained from the fit of (29) at each overvoltage value. Results are represented by filled circles connected by solid lines (in blue for the 1350 SiPM and in red for the 1325 SiPM) in Fig. 4a. Instead of $\gamma$, it is represented $1/(1 - \gamma)$, which corresponds to the saturation level of $\langle Q \rangle / (N \cdot q)$. Notice that $1/(1 - \gamma)$ increases smoothly with overvoltage until the correlated noise starts to become significant.

Strong lines in Fig. 4 represent the results from (19) when using these calculated values of $\gamma$ and fitting $\kappa$ to data. The model describes correctly the SiPM response and its dependence on overvoltage as long as nonlinearity is not very strong. However, it slightly underestimates the saturation level of the 1350 SiPM. The fitted values of $\kappa$ are represented by filled circles connected by solid lines in Fig. 4b. Uncertainties are smaller than the data point size. At low overvoltage where correlated noise is very small (i.e., $\langle Q \rangle_1 \approx q$), $\kappa$ is...
approximately 3.5 keV$^{-1}$ for both SiPMs, which involves $C \approx 12\%$. This a reasonable value of the light collection efficiency taking into account that the photosensitive area of the SiPMs covers only 18% of the crystal surface. In addition, $\kappa$ is expected to increase with overvoltage as the same rate as $\langle Q \rangle / q$. For the 1325 SiPM, $\kappa \approx 9$ keV$^{-1}$ at $U = 20$ V, which translates into $\langle Q \rangle / q \approx 2.3$, indicating a very high correlated noise. On the other hand, for the 1350 SiPM, $\kappa$ does not show the expected monotonic increase with overvoltage. This may be justified by the fact that $\varepsilon$ was not obtained for the specific emission spectrum of the LYSO crystal, but for 450 nm photons.

For consistency purposes, I also tried fitting the model by taking both $\gamma$ and $\kappa$ as a free parameters. The best fits are represented by dotted lines in Fig. 4 and the fitted values of $1/(1 - \gamma)$ and $\kappa$ are shown as open circles connected by dotted lines in Fig. 4. For the 1325 SiPM, this fitting gives almost identical results to the previous one, except for the fact that the values of $\gamma$ are loosely determined (uncertainties are about 20% or larger), since nonlinearity is very weak in this energy range. For the 1350 SiPM, the fitted values of $\gamma$ are about 3% larger than the ones calculated from (30), resulting in a slightly better agreement with experimental data. Nevertheless, discrepancies between the calculated and the fitted values of $\gamma$ are within experimental systematic uncertainties.

**B. Continuous light**

To characterize the SiPM response for continuous light, I used a green LED (Kingbright L-53GD) placed at a few centimeters from the SiPM. The mean output current intensity was measured as a function of the LED forward current $I_{\text{LED}}$ at different overvoltage values for the two tested SiPMs. Measurements were also made by moving the LED away from the SiPM to ensure a linear response, checking that the light output of the LED was proportional to $I_{\text{LED}}$. The correlated noise was found to be negligible compared to the LED light intensity.

Results are shown in figure 4 where it is represented the normalized output current $(I \cdot t_{\text{rec}}) / (N \cdot q)$ to ease the comparison between both SiPMs. The estimated systematic uncertainty was 10% due to assessment of both $q$ and $t_{\text{rec}}$. Notice that the response curves are of a very similar shape to those shown in Fig. 3 for scintillation light pulses.

Fig. 5: Normalized output current of the two tested SiPMs at several overvoltages for continuous light from a LED as a function of the LED forward current. Dotted and solid lines represent the two different fits of (28) described in the text.

Fig. 6: Values of $t_{\text{rec}}/(2 \cdot t_{\text{dead}})$ and $\lambda$ used in the two fits of (28) to data shown in Fig. 5.
The calculated values of \( t_{\text{rec}}/(2 \cdot t_{\text{dead}}) \) and the fitted values of \( \lambda \) for both SiPMs are represented by filled circles connected by solid lines in Fig. 5. As mentioned in section III-A, \( t_{\text{dead}} \) approaches to \( t_{\text{rec}} \) when \( U \) is large, so that the upper limit of \( t_{\text{rec}}/(2 \cdot t_{\text{dead}}) \) is 1/2. As expected, \( \lambda \) depends on overvoltage in a very similar way to \( \kappa \) for both SiPMs.

The model was also adjusted to data by fitting both \( t_{\text{dead}} \) and \( \lambda \). The resulting fits are shown by dotted lines in Fig. 5 and the fitted values of \( t_{\text{rec}}/(2 \cdot t_{\text{dead}}) \) and \( \lambda \) are represented by open circles connected by dotted lines in Fig. 6. As happened the 1350, the fitted values of \( t_{\text{rec}}/(2 \cdot t_{\text{dead}}) \) are systematically larger than the calculated ones by about 20%. Nevertheless, both the fitted and the calculated values show the same overvoltage dependence, except for \( U = 1 \) V where the SiPM response is nearly linear.

V. CONCLUSIONS

The nonlinear response of SiPMs was modelled accounting for losses of both the avalanche triggering efficiency and the gain of pixels during recovery periods as a function of the supplied overvoltage and the incident light pulse shape. The model also includes the contribution of correlated noise, i.e., crosstalk and afterpulsing. Under some simplifications, the simple expression (19) was obtained, which describes the mean output charge of a SiPM for light pulses of arbitrary shape as long as nonlinear effects and correlated noise are moderate. The modified expression (28) was also obtained for continuous light. The model was shown to include the well-known expressions (2) and (3) as limit situations for instantaneous light pulses and for continuous light, respectively, in the absence of correlated noise.

The model introduces the parameters \( \langle Q \rangle_1 \) and \( \gamma \) (or \( t_{\text{dead}} \) for continuous light), which account for the effects of the correlated noise and the pixel recovery, respectively. The parameter \( \langle Q \rangle_1 \) can be determined experimentally as the effective gain of the SiPM in the linear region, whereas the \( \gamma \) (or \( t_{\text{dead}} \)) is calculated knowing the light pulse shape, the recovery time and the photodetection efficiency. In addition, corrections due to the uncorrelated noise as well as simplified expressions of \( \gamma \) for both rectangular pulses and double exponential pulses were obtained. Alternatively, both \( \langle Q \rangle_1 \) and \( \gamma \) can be fitted to experimental data at a given conditions and then the model may be used to make predictions of the SiPM response when varying the experimental parameters.

The model was validated against experimental data of two different SiPMs at moderate nonlinearity using scintillation light pulses from a LYSO crystal. The SiPM response function was found to be significantly widened when the correlated noise becomes important, which also affects the maximum output charge released by the SiPM. Both effects were properly accounted for by using the average output pulse shape, instead of the scintillation light pulse shape, for the calculation of \( \gamma \). In doing so, (19) agrees with data at all overvoltage values within experimental uncertainties.

The model was also proved to describe adequately the SiPM response for continuous light, although the saturation level was underestimated by about 20% for one of the SiPMs. More tests are necessary to understand this discrepancy. Nevertheless, (28) adjusted very accurately to data when fitting both \( \langle Q \rangle_1 \) and \( t_{\text{dead}} \), which show the expected overvoltage dependence.

ACKNOWLEDGMENT

This work was supported by Spanish MINECO under the contract FPA2017-82729-C6-3-R.

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