EXCEPTIONAL COLLECTIONS OF LINE BUNDLES ON
PROJECTIVE HOMOGENEOUS VARIETIES

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Abstract. We construct new examples of exceptional collections of line bundles
on the variety of Borel subgroups of a split semisimple linear algebraic group G
of rank 2 over a field. We exhibit exceptional collections of the expected length
for types $A_2$ and $B_2 = C_2$ and prove that no such collection exists for type $G_2$.
This settles the question of the existence of full exceptional collections of line
bundles on projective homogeneous $G$-varieties for split linear algebraic groups
$G$ of rank at most 2.

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Introduction

The existence question for full exceptional collections in the bounded derived
category of coherent sheaves $\mathcal{D}^b(X)$ of a smooth projective variety $X$ goes back to
the foundational results of Beilinson [Bei78], [Bei84] and Bernstein–Gelfand–Gelfand
[BGG78] for $X = \mathbb{P}^n$. The works of Kapranov [Kap83], [Kap84], [Kap86], [Kap88]
suggested that the structure of projective homogeneous variety on $X$ should imply
the existence of full exceptional collections.

Conjecture. Let $X$ be a projective homogeneous variety of a split semisimple linear
algebraic group $G$ over a field of characteristic zero. Then there exists a full
exceptional collection of vector bundles in $\mathcal{D}^b(X)$.

2010 Mathematics Subject Classification. Primary 14F05, Secondary 14M15, 17B22, 20G15.
Version of May 2, 2014.
This conjecture remains largely unsolved; see [KP11, §1.1] for a recent survey of known results. The bounded derived category $\mathcal{D}^b(X)$ has come to be understood as a homological replacement for the variety $X$; exceptional collections provide a way to break up $\mathcal{D}^b(X)$ into simple components. Such decompositions of the derived category can be seen as analogous to decompositions of the motive of $X$, a relationship that has been put into a conjectural framework by Orlov [Orl05]. As an example, the existence of a full exceptional collection implies a splitting of the Chow motive $M(X)_Q$ into twists of Lefschetz motives [MM12]. We remark that such a motivic decomposition is already known for projective homogeneous varieties of split linear algebraic groups [Koe91].

Let $X$ be a variety over a field $k$. An object $E$ of $\mathcal{D}^b(X)$ is called exceptional if $\text{Ext}^i(E, E) = k$, cf. [GR87, Def. 1.1]. Let $W$ be a finite set and let $\mathcal{P}$ be a partial order on $W$. An ordered set (with respect to $\mathcal{P}$) of exceptional objects $\{E_w\}_{w \in W}$ in $\mathcal{D}^b(X)$ is called a $\mathcal{P}$-exceptional collection if

$$\text{Ext}^i(E_w, E_{w'}) = 0 \text{ for all } w \prec_P w'.$$

If $\mathcal{P}$ is a total order, then a $\mathcal{P}$-exceptional collection is simply called an exceptional collection. A $\mathcal{P}$-exceptional collection $\{E_w\}_{w \in W}$ is called full if the smallest triangulated category containing $\{E_w\}_{w \in W}$ is $\mathcal{D}^b(X)$ itself. Finally, a $\mathcal{P}$-exceptional collection of vector bundles $\{E_w\}_{w \in W}$ is said to be of the expected length if the cardinality of $W$ equals the minimal number of generators of the abelian group $K_0(X)$. Note that any full $\mathcal{P}$-exceptional collection of vector bundles is of the expected length. It is expected that the converse also holds [Kuz09, Conj. 9.1].

In the present paper we address the following closely related question:

**Question.** Let $X$ be the variety of Borel subgroups of a split semisimple linear algebraic group $G$ and fix a partial order $\mathcal{P}$ on the Weyl group $W$ of $G$. Does $\mathcal{D}^b(X)$ have a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles?

On the one hand, the question strengthens the conjecture by requiring the collection to consist of line bundles. On the other hand, it weakens the conjecture by allowing partial orders (such as the weak or strong Bruhat orders) instead of a total order and allowing the collection to merely generate $K_0(X)$.

So far, a natural way to propagate known exceptional collections of line bundles is to use the result of Orlov [Orl93, Cor. 2.7], that $\mathcal{D}^b(X)$ has a full exceptional collection of line bundles if there exists a (Zariski locally trivial) projective bundle $X \to Y$ such that $\mathcal{D}^b(Y)$ has a full exceptional collection of line bundles. More generally, $\mathcal{D}^b(X)$ has a full exceptional collection of line bundles if $X$ is the total space of a smooth Zariski locally trivial fibration, whose fiber and base both have derived categories with full exceptional collections of line bundles [CRM11]. We remark that the result of Orlov on semiorthogonal decompositions of projective bundles (hence that of Beilinson and Bernstein–Gelfand–Gelfand on $\mathbb{P}^n$) holds over an arbitrary noetherian base scheme, cf. [Wal03, §11]. Using such techniques, one immediately answers the question positively for all projective homogeneous varieties $X$ associated to split semisimple groups of rank 2, except in the following cases: type $B_2 = C_2$ and $X$ a 3-dimensional quadric; and type $G_2$ and all $X$ (this includes the case of 5-dimensional quadrics). These exceptions can be viewed as key motivating examples for the present paper.

It is not necessarily expected that a $\mathcal{P}$-exceptional collection of line bundles of the expected length exists for every projective homogeneous variety. For instance, certain
Grassmannians of type $A_n$ have full exceptional collections of vector bundles, but not of line bundles. This distinction is highlighted in [CRM11, Problem 1.2]. Observe also that if $K_0(X)$ is not generated by line bundles, then $D^b(X)$ can not possess a $P$-exceptional collection of line bundles of the expected length for obvious reasons. For example, we prove such a result in Proposition 4.1. Finally, $P$-exceptional collections may exist, while exceptional collections may not.

We will now outline our main results. The paper consists of two parts. In Part I, using $K_0$ techniques, we introduce a new purely combinatorial algorithm for constructing $P$-exceptional collections of line bundles based on a different description of the Steinberg basis [Ste75] obtained in [Ana12]. We apply it to show the following:

A. Theorem. Let $X$ be the variety of Borel subgroups of a split semisimple linear algebraic group $G$ of rank 2 over a field of characteristic zero. Then $D^b(X)$ has a $P$-exceptional collection of the expected length consisting of line bundles, for $P$ a partial order isomorphic to the left weak Bruhat order on the Weyl group of $G$.

For example, $D^b(X)$ possesses such a $P$-exceptional collection for the variety $X$ of complete flags of a split group of type $G_2$.

In §4, using combinatorial and geometric arguments we settle the question of the existence of full exceptional collections of line bundles on projective homogeneous $G$-varieties for every split semisimple $G$ of rank $\leq 2$ over an arbitrary field. The crux case here is that of type $G_2$, to which the entirety of Part II is devoted:

B. Theorem. None of the three non-trivial projective homogeneous varieties of a simple algebraic group of type $G_2$ have an exceptional collection of the expected length consisting of line bundles.

Acknowledgments. We are grateful to Alexander Kuznetsov for comments and for explaining us some facts concerning exceptional collections. This research was partially supported by NSF grant DMS-0903039 (Auel); NSA grant H98230-11-1-0178 and the Charles T. Winship Fund (Garibaldi); and NSERC Discovery 385795-2010, Accelerator Supplement 396100-2010, and the Early Researcher Award grants (Zainoulline).

Part I. Preliminaries and existence results

1. Weights and line bundles

In the present section we recall several basic facts concerning root systems, weights, associated line bundles, and the Grothendieck group $K_0$; see [Bou05], [Dem74], [FH91], [Pan94].

1.1. Let $G$ be a split simple simply connected linear algebraic group of rank $n$ over a field $k$. We fix a split maximal torus $T$ and a Borel subgroup $B$ such that $T \subset B \subset G$.

Let $\Lambda$ be the weight lattice of the root system $\Phi$ of $G$. Observe that $\Lambda$ is the group of characters of $T$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots and let $\{\omega_1, \ldots, \omega_n\}$ be the respective set of fundamental weights (a basis of $\Lambda$), i.e., $\alpha_i^\vee(\omega_j) = \delta_{ij}$. Let $\Phi^+$ denote the set of all positive roots and let $\Lambda^+$ denote the cone of dominant weights.

1.2. Consider the integral group ring $\mathbb{Z}[\Lambda]$; its elements are finite linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in \Lambda$, $a_i \in \mathbb{Z}$. Observe that $\mathbb{Z}[\Lambda]$ can be identified with the representation ring of $T$. Let $X = G/B$ denote the variety of Borel subgroups of $G$, i.e., the variety of subgroups conjugate to $B$. 
Consider the characteristic map for $K_0$,

$$
c : \mathbb{Z}[\Lambda] \to K_0(X)
$$
defined by sending $e^\lambda$ to the class of the associated homogeneous line bundle $\mathcal{L}(\lambda)$ over $X$; see [Dem74, §2.8]. It is a surjective ring homomorphism with kernel generated by augmented invariants. More precisely, if $\mathbb{Z}[\Lambda]^W$ denotes the subring of $W$-invariant elements and $c : \mathbb{Z}[\Lambda] \to \mathbb{Z}$, $e^\lambda \mapsto 1$ is the augmentation map, then $\ker c$ is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ such that $c(x) = 0$. In particular, the Picard group $\text{Pic}(X)$ coincides with the set of homogeneous line bundles $\{\mathcal{L}(\lambda)\}_{\lambda \in \Lambda}$.

1.3. The Weyl group $W$ acts linearly on $\Lambda$ via simple reflections $s_i$ as

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \quad \lambda \in \Lambda.$$

Let $\rho$ denote the half-sum of all positive roots; it is also the sum of the fundamental weights [Bou02, VI.1.10, Prop. 29].

Following [Ana12], for each $w \in W$ consider the cones $\Lambda^+$ and $w^{-1}\Lambda^+$. Let $H_w$ denote the hyperplane orthogonal to a positive root $\alpha \in \Phi^+$. We say that $H_w$ separates $\Lambda^+$ and $w^{-1}\Lambda^+$ if

$$\Lambda^+ \subset \{\lambda \in \Lambda \mid \alpha(\lambda) \geq 0\}$$

and

$$w^{-1}\Lambda^+ \subset \{\lambda \in \Lambda \mid \alpha(\lambda) \leq 0\}.$$

or, equivalently, if $\alpha(w^{-1}\rho) < 0$. Let $H_w$ denote the union of all such hyperplanes, i.e.,

$$H_w = \bigcup_{\alpha(w^{-1}\rho) < 0} H_\alpha.$$

Consider the set $A_w = w^{-1}\Lambda^+ \setminus H_w$ consisting of weights $\lambda \in w^{-1}\Lambda^+$ separated from $\Lambda^+$ by the same set of hyperplanes as $w^{-1}\rho$. By [Ana12, Lem. 6] there is a unique element $\lambda_w \in A_w$ such that for each $\mu \in A_w$ we have $\mu - \lambda_w \in w^{-1}\Lambda^+$. In fact, the set $A_w$ can be viewed as a cone $w^{-1}\Lambda^+$ shifted to the vertex $\lambda_w$.

1.4. Example. In particular, for the identity $1 \in W$ we have $\lambda_1 = 0$. Let $w = s_j$ be a simple reflection, then

$$w^{-1}\Lambda^+ = N_0s_j(\omega_j) \oplus \bigoplus_{i \neq j} N_0\omega_i = N_0(\omega_j - \alpha_j) \oplus \bigoplus_{i \neq j} N_0\omega_i$$

and $A_w = (\omega_j - \alpha_j) + w^{-1}\Lambda^+$. Hence, in this case we have $\lambda_w = \omega_j - \alpha_j$.

For the longest element $w_0 \in W$ we have $\lambda_{w_0} = -\rho$.

1.5. By [Ana12, Thm. 2], the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\lambda_w}\}_{w \in W}$. As there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} = \mathbb{Z}[\Lambda]/\ker e \simeq K_0(X),$$

classes of the associated homogeneous line bundles $c(e^{\lambda_w}) = [\mathcal{L}(\lambda_w)]$ for $w \in W$ generate $K_0(X)$.

2. $\mathcal{P}$-EXCEPTIONAL COLLECTIONS AND SCALAR EXTENSION

In this section, we assemble some results concerning the interaction between flat base change and semioriental decompositions in order to reduce questions concerning exceptional collections on smooth projective varieties over $k$ to over the algebraic closure $\overline{k}$. If $X$ is a variety over $k$, write $\overline{X}$ for the base change $X \times_k \overline{k}$, and similarly for complexes of sheaves on $X$.

Let $W$ be an finite set and $\mathcal{P}$ be a partial order on $W$. 
2.1. **Proposition.** Let $X$ be a smooth projective variety over a field $k$ and let \( \{E_w\}_{w \in W} \) be a $\mathcal{P}$-ordered set of line bundles on $X$. Then \( \{E_w\} \) is a $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(X)$ if and only if \( \{\overline{E}_w\} \) is a $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(\overline{X})$. Moreover, \( \{E_w\} \) is full if and only if \( \{\overline{E}_w\} \) is full.

**Proof.** For any coherent sheaves $E$ and $F$ on $X$, we have $\text{Ext}^*(E, F) \otimes_k \overline{E} \cong \text{Ext}^*(\overline{E}, \overline{F})$ by flat base change. In particular, $E$ is an exceptional object of $\mathcal{D}^b(X)$ if and only if $\overline{E}$ is an exceptional object of $\mathcal{D}^b(\overline{X})$. Also, for each $w <_\mathcal{P} w'$, we have that $\text{Ext}^*(\overline{E}_w, \overline{E}_{w'}) = 0$ if and only if $\text{Ext}^*(E_w, E_{w'}) = 0$. Thus \( \{E_w\} \) is a $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(X)$ if and only if \( \{\overline{E}_w\} \) is a $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(\overline{X})$.

Suppose that \( \{E_w\} \) is a full $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(X)$. As $X$ is smooth, $\mathcal{D}^b(X)$ is equivalent to the derived category $\mathcal{D}^\text{perf}(X)$ of perfect complexes on $X$, and similarly for $\overline{X}$. As \( \{E_w\} \) are line bundles, they are perfect complexes and \( \{E_w\} \) is a $\mathcal{P}$-exceptional collection of $\mathcal{D}^\text{perf}(X)$. The main results of [Kuz11] imply that \( \{\overline{E}_w\} \) is a full $\mathcal{P}$-exceptional collection of $\mathcal{D}^\text{perf}(\overline{X}) = \mathcal{D}^b(\overline{X})$. Indeed, $\text{Spec} \overline{E} \to \text{Spec} k$ is faithful and the proof in [Kuz11, Prop. 5.1] immediately generalizes to the $\mathcal{P}$-exceptional setting, showing that \( \{\overline{E}_w\} \) is a full $\mathcal{P}$-exceptional collection.

Now, suppose that \( \{\overline{E}_w\} \) is a full $\mathcal{P}$-exceptional collection of $\mathcal{D}^b(\overline{X})$. Let $\mathcal{E}$ be the triangulated subcategory of $\mathcal{D}^b(X)$ generated by the $\mathcal{P}$-exceptional collection \( \{E_w\} \) and $\mathcal{J}$ be the orthogonal complement of $\mathcal{E}$, i.e., there is a semiorthogonal decomposition $\mathcal{D}^b(X) = (\mathcal{E}, \mathcal{J})$. By [Kuz08, Prop. 2.5], the projection functor $\mathcal{D}^b(X) = \mathcal{D}^\text{perf}(X) \to \mathcal{J}$ has finite cohomological amplitude, hence by [Kuz11, Thm. 7.1], is isomorphic to a Fourier–Mukai transform for some object $K$ in $\mathcal{D}^b(X \times_k \overline{X})$. However, $\mathcal{J} = 0$ by assumption, hence $\overline{K} = 0$. This implies that $K = 0$, otherwise, $K$ would have a nonzero homology group, which would remain nonzero over $\overline{k}$ by flat base change. Thus $\mathcal{J} = 0$ and so \( \{E_w\} \) generates $\mathcal{D}^b(X)$. \( \square \)

In this paper, we are concerned with the case where $X$ is a projective homogeneous variety under a linear algebraic group $G$ that is split semisimple or has type $G_2$. Under this hypothesis, the pull back homomorphism $K_0(X) \to K_0(\overline{X})$ is an isomorphism [Pan94]. Together with Proposition 2.1, this reduces the question of (non)existence of an exceptional collection of expected length on $X$ to the same question on $\overline{X}$.

3. $\mathcal{P}$-exceptional collections on Borel varieties

Consider the variety $X$ of Borel subgroups of a split semisimple simply connected linear algebraic group $G$ over a field $k$. Let $\mathcal{L}(\lambda)$ be the homogeneous line bundle over $X$ associated to the weight $\lambda$. Recall that $\lambda$ is singular if $\alpha^\vee(\lambda) = 0$ for some root $\alpha$.

3.1. **Proposition.** Let $W$ be a finite set endowed with a partial order $\mathcal{P}$ and let \( \{\lambda_w\}_{w \in W} \) be a set of weights indexed by $W$. The statements:

1. \( \{\mathcal{L}(\lambda_w)\}_{w \in W} \) is a $\mathcal{P}$-exceptional collection of line bundles on the Borel variety $X$.
2. $\lambda_w - \lambda_w + \rho$ is a singular weight for every $w <_\mathcal{P} w'$.

are equivalent if $\text{char} \ k = 0$. If $\text{char} \ k > 0$, then (1) implies (2).

**Proof.** Since $X$ is smooth, proper, and irreducible, we have $\text{Hom}(\mathcal{L}', \mathcal{L}) = k$ for any line bundle $\mathcal{L}$ over $X$. Furthermore, for any weights $\lambda, \lambda'$, we have

\[
\text{Ext}^i(\mathcal{L}(\lambda), \mathcal{L}(\lambda')) = H^i(X, \mathcal{L}(\lambda)^\vee \otimes \mathcal{L}(\lambda')) = H^i(X, \mathcal{L}(\lambda' - \lambda)).
\]
In particular, since \( \omega \) is a dominant weight, Kempf’s vanishing theorem [Jan03, Prop. II.4.5] implies that \( \mathcal{L}(\lambda) \) is an exceptional object for every weight \( \lambda \).

Equation (3.1) says that (1) is equivalent to: \( H^*(X, \mathcal{L}(\lambda_{w'} - \lambda_w)) = 0 \) for all \( w <_p w' \). If \( \text{char} k = 0 \), this is equivalent to (2) by the Borel-Weil-Bott Theorem as in [FH91, p.392] or [Jan03, II.5.5]. If \( \text{char} k = p > 0 \), then \( X \) and every line bundle \( \mathcal{L}(\mu) \) is defined over the field \( \mathbb{F}_p \) and can be lifted to a line bundle over \( \mathbb{Q} \), via a smooth projective model \( X \to \text{Spec} \mathbb{Z} \) defined in terms of the corresponding Chevalley group schemes. Semicontinuity of cohomology shows that vanishing of \( H^1(\mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_p, \mathcal{L}(\lambda_{w'} - \lambda_w)) \) implies the vanishing of the analogous cohomology group over \( \mathbb{Q} \), which implies (2) by the characteristic zero case. □

3.2. Remark. The reverse implication statement of Proposition 3.1 in \( \text{char} k > 0 \) does not hold in general. Indeed, take \( G = \text{SL}_3 \) over a field \( k \) of prime characteristic \( p \). Write \( \omega_1, \omega_2 \) for the fundamental dominant weights and put \( \mu = -(p+2)\omega_1 + p\omega_2 \). For the partially ordered set of line bundles \( \mathcal{L}(0) < \mathcal{L}(\mu) \), (2) obviously holds, but (1) fails because both \( H^1(\mu) \) and \( H^2(\mu) \) are nonzero by [Gri80, Cor. 5.1].

3.3. Definition. A collection of weights \( \{\lambda_w\}_{w \in W} \) is called \( \mathcal{P} \)-exceptional (resp. of the expected length) if the corresponding collection of line bundles \( \{\mathcal{L}(\lambda_w)\}_{w \in W} \) is thus.

The proof of Theorem A consists of two steps. First, we find a maximal \( \mathcal{P} \)-exceptional subcollection of weights among the weights \( \lambda_w \) constructed in §1.3. This is done by direct computations using Proposition 3.1(2). Then we modify the remaining weights to fit in the collection, i.e., to satisfy Proposition 3.1(2) and to remain a basis. This last point is guaranteed, since we modify the weights according to the following fact.

3.4. Lemma. Let \( \mathcal{B} \) be a basis of \( \mathbb{Z}[\Lambda] \) over \( \mathbb{Z}[\Lambda]^W \) and let \( e^\lambda \in \mathcal{B} \) be such that for some \( W \)-invariant set \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) we have \( e^{\lambda + \lambda_i} \in \mathcal{B} \) for all \( i < k \) and \( e^{\lambda + \lambda_k} \notin \mathcal{B} \). Then the set

\[
\left( \mathcal{B} \cup \{e^{\lambda + \lambda_i}\} \right) \setminus \{e^{\lambda + \lambda_1}\}
\]

is also a basis of \( \mathbb{Z}[\Lambda] \) over \( \mathbb{Z}[\Lambda]^W \).

Proof. Indeed, there is a decomposition

\[
e^{\lambda + \lambda_k} = (e^{\lambda_1} + e^{\lambda_2} + \cdots + e^{\lambda_k})e^\lambda - e^{\lambda + \lambda_1} - e^{\lambda + \lambda_2} - \cdots - e^{\lambda + \lambda_{k-1}}
\]

with the coefficients from \( \mathbb{Z}[\Lambda]^W \) and an invertible coefficient at \( e^{\lambda + \lambda_1} \). □

We can give a geometric description of this fact. For instance, in type \( B_2 \), the rule says that if we have a square (shifted orbit of a fundamental weight) where the center and three vertices are the basis weights, then replacing one of these basis weights by the missing vertex gives a basis; see Figure 1. For \( G_2 \) we use a hexagon (shifted orbit of a fundamental weight) instead of the square, where the center and all but one vertex are the basis weights.

Proof of Theorem A. We use the following notation: a product of simple reflections \( w = s_{i_1}s_{i_2}\cdots s_{i_k} \) is denoted by \( [i_1, i_2, \ldots, i_k] \); the identity is denoted by \( [.] \). Given a presentation \( \lambda = a_1\omega_1 + \cdots + a_n\omega_n \) in terms of fundamental weights we denote \( \lambda \) by \( (a_1, \ldots, a_n) \). We write \( \mathcal{P} \) for the left weak Bruhat order on the Weyl group \( W \).
Type $A_2$: The Weyl group $W$ of type $A_2$ consists of the following elements:

$$W = \{ [], [1], [2], [1, 2], [1, 2, 1] \}.$$

The respective basis weights $\{ \lambda_w \}_{w \in W}$ of §1.3 are given by (here the $i$-th weight is indexed by the $i$-th element of the Weyl group):

$$\{ (0, 0), (-1, 1), (1, -1), (-1, 0), (0, -1), (-1, -1) \}.$$

Direct computations using Proposition 3.1(2) show that $\{ \lambda_w \}_{w \in W}$ is a $P$-exceptional collection.

Using Lemma 3.4 we can modify the weights $\{ \lambda_w \}$ to obtain the following exceptional collection of weights (there are no Ext's from left to the right):

$$\{ (0, 0), (-1, 0), (-2, 0), (1, -1), (0, -1), (-1, -1) \}.$$

Type $A_1 \times A_1$: The Weyl group $W$ of type $A_1 \times A_1$ is the Klein four-group; the root system has orthogonal simple roots $\alpha_1, \alpha_2$ which may be taken to have square-length 2 and equal the fundamental weights. The procedure from §1.3 gives the list of basis weights $0, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2$, which is an exceptional collection.

Type $B_2$ or $C_2$: The Weyl group $W$ consists of the following elements:

$$W = \{ [], [1], [2], [1, 2], [1, 2], [1, 2, 1], [1, 2, 2], [1, 2, 1, 2] \}.$$

The respective basis weights $\{ \lambda_w \}_{w \in W}$ of §1.3 are given by:

$$\{ (0, 0), (-1, 1), (2, -1), (-2, 1), (1, -1), (-1, 0), (0, -1), (-1, -1) \}.$$

(here $\omega_1 = (e_1 + e_2)/2$ and $\omega_2 = e_2$). Direct computations show that $\{ \lambda_w \}_{w \in W}$ is a $P$-exceptional collection except of the weight $\lambda_{[2, 1]} = (-2, 1)$: indeed, the property
in Proposition 3.1(2) fails only for the weights 0 and \((-2, 1)\), i.e., \(\lambda_{[2,1]} + \rho\) is regular or, equivalently, \([1] \ast \lambda_{[2,1]} = 0\) is dominant.

We modify the basis weights using Lemma 3.4: An element \(e^{(-2,0)}\) has the following representation with respect to the initial basis:

\[
e^{(-2,0)} = (e^{(1,0)} + e^{(1,-1)} + e^{(-1,1)} + e^{(-1,0)})e^{(-1,0)} - e^{(0,0)} - e^{(-2,1)} - e^{(0,-1)}
\]

where \((e^{(1,0)} + e^{(1,-1)} + e^{(-1,1)} + e^{(-1,0)}) \in \mathbb{Z}[\Lambda]^{W}\). Hence, we can substitute \(e^{(-2,1)}\) by \(e^{(-2,0)}\). Figure 1 illustrates our arguments. Finally, after reindexing we obtain a \(P\)-exceptional collection

\[
\{(0, 0), (-1, 1), (2, -1), (-1, 0), (1, -1), (-2, 0), (0, -1), (-1, -1)\}.
\]

Repeating Lemma 3.4 we obtain the following exceptional collection of weights

\[
\{(1, 0), (0, 0), (-1, 0), (-2, 0), (2, -1), (1, -1), (0, -1), (-1, -1)\}.
\]

**Type** \(G_2\): The Weyl group \(W\) of type \(G_2\) consists of the following 12 elements:

\[
W = \{[, [1], [2], [2, 1], [1, 2], [1, 2, 1], [2, 1, 2], [2, 1, 2, 1, 2], 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]\}.
\]

The respective basis weights \(\{\lambda_{w}\}_{w \in W}\) of \(\Lambda^W\) are given by:

\[
\{(0, 0), (-1, 1), (3, -1), (-3, 2), (2, -1), (-2, 1), (3, -2), (-3, 1), (1, -1), (0, -1), (0, 1), (1, -1), (-2, 0), (0, -1), (-3, 0), (2, -2), (-1, -1)\}.
\]

Using Lemma 3.4 we obtain the following \(P\)-exceptional collection

\[
\{(0, 0), (-1, 1), (3, -1), (-3, 1), (2, -1), (-1, 0), (1, -1), (-2, 0), (0, -1), (-3, 0), (2, -2), (-1, -1)\}.
\]

Indeed, the difference between the initial basis and the modified one consists in the substitution of \(e^{(-3,2)}, e^{(-2,1)}, e^{(3,-2)}\) by \(e^{(-3,0)}, e^{(-2,0)}, e^{(2,-2)}\). We proceed in several steps using the same reasoning as in the \(B_2\)-case. Denote

\[
A = e^{(1,0)} + e^{(2,-1)} + e^{(1,-1)} + e^{(-1,0)} + e^{(-2,1)} + e^{(-1,1)},
\]

\[
B = e^{(0,1)} + e^{(3,-1)} + e^{(3,-2)} + e^{(0,-1)} + e^{(-3,1)} + e^{(-3,2)},
\]

the sums of the elements corresponding to the orbits of the fundamental weights. Note that \(A, B \in \mathbb{Z}[\Lambda]^{W}\). We have the following decompositions with respect to the initial basis (coefficients belong to \(\mathbb{Z}[\Lambda]^{W}\)):

\[
e^{(2,-2)} = Ae^{(1,-1)} - e^{(0,0)} - e^{(2,-1)} - e^{(3,-2)} - e^{(0,-1)} - e^{(-1,0)},
\]

\[
e^{(-4,2)} = Ae^{(-2,1)} - e^{(0,0)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-3,2)} - e^{(-1,1)}.
\]

Hence we can substitute \(e^{(2,-2)}\) for \(e^{(3,-2)}\) and \(e^{(-4,2)}\) for \(e^{(-3,2)}\). Using the decomposition

\[
e^{(-2,0)} = Ae^{(-1,0)} - e^{(0,0)} - e^{(1,-1)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-2,1)},
\]

we substitute \(e^{(-2,0)}\) for \(e^{(-2,1)}\). Then, using

\[
e^{(-4,1)} = Be^{(-1,0)} - e^{(-4,2)} - e^{(-1,1)} - e^{(2,-1)} - e^{(2,-2)} - e^{(-1,-1)}
\]

substitute \(e^{(-4,1)}\) for \(e^{(-4,2)}\). At last, using

\[
e^{(-3,0)} = Ae^{(-2,0)} - e^{(-1,0)} - e^{(0,0)} - e^{(-1,-1)} - e^{(-4,1)} - e^{(-3,1)}
\]

we substitute \(e^{(-3,0)}\) for \(e^{(-4,1)}\), obtaining the required basis. \(\square\)
4. Full exceptional collections of line bundles in rank \( \leq 2 \)

We now provide the answer to the question

\[(4.1) \text{ Does } X \text{ have a full exceptional collection of line bundles?} \]

for each projective homogeneous \( G \)-variety \( X \), where \( G \) is split semisimple of rank \( \leq 2 \). In type \( B_2 = C_2 \), we assume that the base field has characteristic \( \neq 2 \).

**Types \( A_1 \) and \( A_2 \).** In case \( G \) has type \( A \), it is isogenous to \( SL_2 \) or \( SL_3 \). In all but one case, \( X \) is projective space \( \mathbb{P}^1 \) or \( \mathbb{P}^2 \) and the answer to \((4.1)\) is “yes” by Beilinson [Bei78] and Bernstein–Gelfand–Gelfand [BGG78] (which holds over any field, cf. [Wal03, §11]). In the remaining case, \( G \) is isogenous to \( SL_3 \) and \( X \) is the variety of Borel subgroups of \( G \); then \( X \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \) and the answer is “yes” by [Orl93, Cor. 2.7] (which also holds over any field).

**Type \( A_1 \times A_1 \).** In this case, \( G \) is isogenous to \( SL_2 \times SL_2 \), and \( X \) is either \( \mathbb{P}^1 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \). In both cases, the answer to \((4.1)\) is “yes” since we know the answers for projective space and products of projective spaces, as in the previous paragraph.

**Type \( G_2 \).** For \( G \) of type \( G_2 \), the answer is always “no” by Theorem B, which holds over any field.

**Type \( B_2 = C_2 \).** In this case, \( G \) is isogenous to \( SO_4 \) and to \( Sp_4 \). Two of the possibilities for \( X \) are \( \mathbb{P}^3 \) (for which the answer is “yes” as in the type \( A_2 \) case) and the Borel variety \( X \) (which is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^3 \), so that the answer is again “yes” as in the type \( A_1 \) and \( A_2 \) case). The only remaining possibility for \( X \) is a 3-dimensional quadric, for which the answer is “no” by the following:

**4.1. Proposition.** Let \( X \) be a smooth 3-dimensional quadric defined over a field of characteristic \( \neq 2 \). Then \( K_0(X) \) is not generated by line bundles. In particular, \( X \) does not have a \( P \)-exceptional collection of the expected length consisting of line bundles for any partially ordered set.

**Proof.** We may assume that \( G \) is simply connected; we write \( \tilde{G} \) for the adjoint group. Put \( P \) for a standard parabolic subgroup of \( G \) so that \( X \simeq G/P \). We use the identification

\[ Z \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[\Lambda]^W_P = K_0(X) \]

where \( \Lambda \) is the weight lattice, \( W_P \) is the Weyl of the Levi part of \( P \) and \( \mathbb{Z}[\Lambda]^W_P \) and \( \mathbb{Z}[\Lambda]^W \) are the representation rings of \( P \) and \( G \) respectively.

For sake of contradiction, suppose \( K_0(X) \) is generated by line bundles. Since \( \text{Pic}(X) = Z \) is generated by \( \mathcal{L}(\omega) \), where \( \omega_1 \) is the first fundamental weight (numbered as in [Bou02]), all such bundles are powers of \( \mathcal{L}(\omega_1) \), i.e., are of the form \( \mathcal{L}(n\omega) \) for some \( n \in \mathbb{Z} \).

There is a surjective homomorphism \( \pi : K_0(X) \to K_0(\tilde{G}) \) induced by \( \Lambda \to \Lambda/\Lambda_r = \mathbb{Z}^r/2\mathbb{Z} = \langle \sigma \rangle \), where \( \Lambda_r \) is the root lattice. By [Zai, Example 3.7], \( K_0(\tilde{G}) = \mathbb{Z}[y]/(y^2 - 2y, 4y) \), where \( y = 1 - e^\sigma \).

Now take the vector bundle \( E \) corresponding to the \( W_P \)-orbit of the second fundamental weight \( \omega_2 \), i.e., to \( e^\omega_2 + e^{\omega_2 - \alpha_2} \) (here \( \alpha_2 \) is the second simple root), and observe that \( \pi(\mathcal{L}(n\omega_1)) = 1 \) and \( \pi(E) = 2 - 2y \). This is a contradiction, as \( \pi(E) \) can’t be written as a linear combination of 1’s. \( \square \)
Part II. Exceptional collection of line bundles on $G_2$-varieties

In this part, we study exceptional collections of line bundles on the Borel variety $X$ of a group $G$ of type $G_2$ over an arbitrary field.

4.2. Example. Suppose that $G$ is split and char $k = 0$. One of the projective homogeneous $G$-varieties is a 5-dimensional quadric $Y$ and $X \to Y$ is a $\mathbb{P}^1$-bundle. An exceptional collection of vector bundles on $Y$ described in [Kap86] and [Kap88] includes 5 line bundles. These yield an exceptional collection of line bundles on $X$ of length 10 by [Orl93, Cor. 2.7].

As $K_0(X)$ is a free module of rank 12 (the order of the Weyl group), the collection provided in the preceding example is not of expected length. Nonetheless, we prove that it is “best possible”:

4.3. Theorem. Every exceptional collection of line bundles on the Borel variety of a group of type $G_2$ has length $\leq 10$.

The proof will occupy the rest of the paper. But for now, we note that this theorem is sufficient to prove Theorem B.

Proof of Theorem B. By Proposition 2.1, we may assume that the group $G$ of type $G_2$ is split. For $X$ the variety of Borel subgroups of $G$, $K_0(X)$ is isomorphic to $\mathbb{Z}^{12}$ and Theorem 4.3 says that there does not exist an exceptional collection of the expected length consisting of line bundles. Any other projective homogeneous variety $Y$ for $G$ can be displayed as the base of a $\mathbb{P}^1$-bundle $X \to Y$ and $K_0(Y)$ is a free $\mathbb{Z}$-module of rank 6. Hence any exceptional collection of the expected length consisting of line bundles on $Y$ lifts to an exceptional collection of the expected length consisting of line bundles on $X$ by [Orl93, Cor. 2.7] (which holds over any noetherian base scheme, cf. [Wal03, §11]).

The group $K_0(X)$ depends neither on the base field nor on the particular group $G$ of type $G_2$ under consideration. Combining this with Proposition 2.1, in order to prove Theorem 4.3 we may assume that the base field is algebraically closed and hence that $G$ is split. Proposition 3.1 reduces the proof to a computation with totally ordered lists of weights, which we write in ascending order. We will use without much comment that, for any exceptional collection $\lambda_1, \ldots, \lambda_n$ and any weight $\mu$, the lists

\begin{align*}
\lambda_1 - \mu, \lambda_2 - \mu, \ldots, \lambda_n - \mu \quad \text{and} \quad -\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1
\end{align*}

are also exceptional collections. Thus, given an exceptional collection, we obtain another exceptional collection of the same length but with first entry $\lambda_1 = 0$. Note also the trivial fact that $\lambda_i \neq \lambda_j$ for all $i \neq j$, since $\text{Ext}^*(\mathcal{L}(\lambda_i), \mathcal{L}(\lambda_i)) = k$ for every weight $\lambda_i$ as in the proof of Proposition 3.1.

5. A dichotomy

The crab is the collection of weights $\lambda$ of $G_2$ such that $\lambda + \rho$ is singular. The crab consists of weights lying on 6 crab lines. Any pair of lines meets only at $-\rho$, and $-\rho$ lies on all 6 crab lines. The weight zero is not on any crab line. See Figure 2 for a picture of the crab.

5.1. Definition. In Figure 2, we find 20 weights on the intersection of the crab lines and the singular lines, i.e., there are 20 weights $\lambda$ such that $\lambda$ and $\lambda + \rho$ are both singular. We call them the 20 weights.
Figure 2. The crab lines (solid), the weights of the crab (circles), the singular lines (dashed lines), and the 20 weights (disks). All singular lines meet at 0 and all crab lines meet at $-\rho$. The simple roots are $\alpha_1 = (1, 0)$ and $\alpha_2 = (-3/2, \sqrt{3}/2)$.

We note for future reference that for each of the 20 weights $\lambda$, we have $||\lambda|| \leq 3\sqrt{3}$ and $||\lambda + \rho|| \leq 3\sqrt{3}$.

5.2. Lemma. Suppose $0, b, c$ is an exceptional collection.

(1) If $b$ and $c$ lie on the same crab line, then $c - b$ is one of the 20 weights and it is on the singular line parallel to that crab line.

(2) If $c - b$ and $c$ lie on the same crab line, then $b$ is one of the 20 weights and it is on the singular line parallel to that crab line.

Proof. We prove (2) first. The weight $b$ is in the crab because $0, b$ is exceptional. Further, $c - b = tc + (1 - t)(-\rho)$ for some $t \in \mathbb{R}$, hence $b = (1 - t)(c + \rho)$, which is singular because $0, c$ is exceptional, so $b$ is one of the 20. If $x$ is a nonzero vector orthogonal to $c + \rho$ and $(c - b) + \rho$, then it is also orthogonal to their difference, $b$, which proves (2). Then (1) is deduced from (2) via (4.2). □

5.3. Corollary. In any exceptional collection $0, \lambda_2, \ldots, \lambda_n$, the distance between any pair of weights on the same crab line is at most $3\sqrt{3}$.

Proof. Fix a crab line of interest and let $0, \lambda_2, \ldots, \lambda_n$ be an exceptional collection. By restricting to a sub-list, we may assume that all of the weights $\lambda_2, \ldots, \lambda_n$ lie on that crab line. By Lemma 5.2(1), $\lambda_j - \lambda_2$ is one of the 20 weights for $j = 3, \ldots, n$, hence $||\lambda_j - \lambda_2|| \leq 3\sqrt{3}$, as claimed. □
5.4. **Lemma** (Trigonometry). If two weights on crab lines are each \( \geq R \) from \(-\rho\) and are closer than \(2(2 - \sqrt{3})R\) apart, then they are on the same crab line.

**Proof.** For sake of contradiction, suppose that the two weights are on different crab lines. The distance between the two weights is at least the length of the shortest line segment joining the two crab lines and meeting them at least \(R\) from \(-\rho\). This segment is the third side of an isosceles triangle with two sides of length \(R\) and internal angle \(2\theta\), hence it has length \(2R\tan\theta\), where \(2\theta\) is the angle between the two crab lines. As \(15^\circ \leq \theta \leq 75^\circ\), the minimum is achieved at \(\tan 15^\circ = 2 - \sqrt{3}\). \[\square\]

The following proposition says that a close-in weight early in the exceptional collection controls the distribution of far weights on crab lines coming later in the collection.

5.5. **Proposition** (Dichotomy). Let \(0, \mu, \lambda_2, \ldots, \lambda_n\) be an exceptional collection. Then exactly one of the following holds:

1. \(\mu\) is one of the 20 weights, and all the \(\lambda_j\) such that \(|\lambda_j + \rho| > 6\sqrt{3}\) lie on the crab line parallel to the singular line containing \(\mu\).
2. \(\mu\) is not one of the 20 weights and \(|\lambda_j + \rho| < 3|\mu|\) for all \(j = 3, \ldots, n\).

**Proof.** First suppose that \(\mu\) is not one of the 20 weights. As \(0, \lambda_j - \mu, \lambda_j\) is an exceptional collection, \(\lambda_j - \mu\) is on a crab line; by Lemma 5.2(2) it is a different crab line from \(\lambda_j\). By the Trigonometry Lemma,

\[
1.9|\mu| \geq \frac{||\lambda_j - (\lambda_j - \mu)||}{2(2 - \sqrt{3})} \geq \min\{||\lambda_j + \rho||, ||\lambda_j - \mu + \rho||\}.
\]

If \(|\lambda_j - \mu + \rho|| is the minimum, then \(|\lambda_j + \rho|| \leq ||\lambda_j - \mu + \rho|| + ||\mu|| \leq 2.9||\mu||. This proves (2).

Suppose that \(0, \mu, \lambda\) is an exceptional collection such that \(\mu\) is one of the 20 weights and \(\lambda\) does not lie on the crab line parallel to the \(\mu\) singular line. Translating \(0, \mu, \lambda\) by \(-\mu\) gives the exceptional collection \(0, \lambda - \mu\) so \(\lambda - \mu\) also belongs to the crab, i.e., \(\lambda\) lies in the intersection of the crab and the crab shifted by \(\mu\). We will show that this implies \(|\lambda + \rho|| \leq 6\sqrt{3}\), even ignoring questions of belonging to the weight lattice.

Indeed, the crab and the crab shifted by \(\mu\) give a picture as in Figure 3. For each weight on the intersection of two dashed lines, we find a triangle where a side of length \(|\mu|| is bracketed by angles \(\alpha, \beta\) that are multiples of \(30^\circ\) and \(\alpha + \beta \leq 150^\circ\). Using the Law of Sines, we find that the length of the longest of the other two sides of such a triangle is

\[
\frac{\sin(\max\{\alpha, \beta\})}{\sin(180^\circ - \alpha - \beta)}||\mu||.
\]

Plugging in all possibilities for \(\alpha\) and \(\beta\), we find that the fraction has a maximum of 2. The length of \(\mu\) is at most \(3\sqrt{3}\), whence the claim. \[\square\]

6. Three weights on two crab lines

We now examine the possibilities for exceptional collections \(0, \lambda_2, \lambda_3, \lambda_4\) such that two of the \(\lambda_j\) lie on one crab line and the third lies on a different crab line; there are three possible such permutations, which we label BAA, AAB, and ABA.

6.1. **Definition.** A weight \(a\) is near if \(|a + \rho|| \leq 42\). It is far if \(|a + \rho|| > 42 + 3\sqrt{3}\).
6.2. **Lemma** (BAA). Suppose that $0, b, a_1, a_2$ is an exceptional collection such that $a_1, a_2$ are on the same crab line and $b$ is neither on that crab line nor on the parallel singular line. Then $b, a_1, a_2$ are all near weights (and in fact are within $21.1$ of $-\rho$).

*Proof.* Translating the exceptional collection, we find the collection $0, a_1 - b, a_2 - b$, so $a_1 - b$ and $a_2 - b$ belong to the crab. Now, $a_1, a_2$ are on a crab line (call it $A$) and $b$ is not on the parallel singular line (i.e., $b$ is not parallel to $a_2 - a_1$), therefore $a_1 - b$ and $a_2 - b$ are not on the $A$ crab line. Furthermore, because the direction $a_2 - a_1 = (a_2 - b) - (a_1 - b)$ characterizes the $A$ line, we conclude that $a_1 - b$ and $a_2 - b$ lie on different crab lines.

However, 

$$||a_2 - b - (a_1 - b)|| = ||a_2 - a_1|| \leq 3\sqrt{3}$$

by Corollary 5.3, hence 

$$||a_j - b + \rho|| \leq \frac{3\sqrt{3}}{2(2 - \sqrt{3})}$$

by the Trigonometry Lemma 5.4. By the triangle inequality 

$$||a_j - b|| \leq ||a_j - b + \rho|| + ||\rho|| \leq \frac{3\sqrt{3}}{2(2 - \sqrt{3})} + \sqrt{7} < 12.4.$$ 

As $a_j$ and $b$ are on different crab lines, the argument in the Trigonometry Lemma 5.4 gives that $||a_j + \rho||$ and $||b + \rho||$ are at most 

$$\frac{3\sqrt{3}}{2(2 - \sqrt{3})} + \sqrt{7} < 21.04.$$ 

Here are two corollaries from the Trigonometry Lemma 5.4.
6.3. Corollary. If \( \lambda \) and \( \lambda + \rho \) are on crab lines and \( \| \lambda + \rho \| > 7.7 \), then \( \lambda \) and \( \lambda + \rho \) are on the same crab line.

Proof. We use the triangle inequality to bound the distance of \( \lambda + \rho \) from \( -\rho \):

\[
\| \lambda + 2\rho \| \geq \| \lambda + \rho \| - \| -\rho \| > 7.7 - \sqrt{7} > 5.05.
\]

The distance between \( \lambda \) and \( \lambda + \rho \) is \( \| \rho \| = \sqrt{7} < 2(2 - \sqrt{3})5.05 \), so taking \( R = 5.05 \) in the Trigonometry Lemma 5.4 gives the claim. \( \square \)

6.4. Corollary. If \( a \) and \( b - a \) both lie on some crab line \( A \), then so does \( b + \rho \). If furthermore \( \| b + \rho \| > 7.7 \), then \( b \) also lies on \( A \).

Proof. Let \( x \) be a nonzero vector orthogonal to \( a + \rho \) and \( b - a + \rho \). Then \( x \) is also orthogonal to \( (b - a + \rho) + a + \rho = (b + \rho) + \rho \); this proves the first claim. For the second claim, we apply Corollary 6.3. \( \square \)

6.5. Lemma (AAB). If \( 0, a_1, a_2, b \) is an exceptional collection where \( a_1, a_2 \) are on one crab line and \( b \) is on a different crab line, then at least one of \( a_1, a_2, b \) is near.

Proof. For sake of contradiction, suppose all three nonzero weights are at least 42 from \( -\rho \). Translating, we find an exceptional collection \( 0, a_2 - a_1, b - a_1 \), where for \( j = 1, 2 \) we have \( ||b - a_j|| > 2(2 - \sqrt{3})42 \). Further,

\[
||b - a_j + \rho|| \geq ||b - a_j|| - ||-\rho|| > 2(2 - \sqrt{3})42 - \sqrt{7} > 19.8.
\]

Now

\[
||(b - a_2) - (b - a_1)|| = ||a_1 - a_2|| \leq 3\sqrt{3} < 2(2 - \sqrt{3})19.8,
\]

so by the Trigonometry Lemma 5.4 \( b - a_2 \) and \( b - a_1 \) lie on the same crab line.

As \( a_1, a_2 \) also lie on one crab line, we can find nonzero vectors \( x, y \) such that \( x \) is orthogonal to \( b - a_j + \rho \) and \( y \) is orthogonal to \( a_j + \rho \) for \( j = 2, 3 \). It follows that

\[
a_1 - a_2 = (a_1 + \rho) - (a_2 + \rho) = (b - a_2 + \rho) - (b - a_1 + \rho)
\]

is orthogonal to both \( x \) and \( y \). As \( a_1 - a_2 \neq 0 \), it follows that the four weights \( a_j, b - a_j \) for \( j = 2, 3 \) all lie on one crab line. Corollary 6.4 gives that \( b + \rho \) lies on this same line. As \( b \) is also on a crab line and \( b \) is at least 42 from \( -\rho \), Corollary 6.3 gives that \( b \) and \( b + \rho \) are on the same crab line. This contradicts the hypothesis that \( a_1, b \) are on different crab lines. \( \square \)

We now prepare for ABA, the most complicated of the three configurations.

6.6. Definition. The mirror 20 weights consist of the intersection of the crab with the crab shifted to \( -\rho \). A weight \( \mu \) is one of the mirror 20 weights if both \( \mu \) and \( \mu + \rho \) are in the crab. The lines through \( -2\rho \) parallel to the crab lines will be called the mirror singular lines.

Now we need a “mirror” version of Proposition 5.5(2).

6.7. Proposition. If \( 0, \lambda, \mu \) is an exceptional collection with \( \| \lambda \| \geq 2.9\|\mu + \rho\| + 7.6 \), then \( \mu \) is one of the mirror 20 weights on the mirror singular line parallel to \( \lambda \). In particular, in this case \( \| \mu + \rho \| \leq 3\sqrt{3} \).

Proof. As \( \lambda \) is in the crab, \( \lambda + \rho \) is singular, thus \( -2\rho - \lambda + \rho = -\rho - \lambda = -(\lambda + \rho) \) is singular, hence \( -2\rho - \lambda \) is in the crab. Also \( \mu - \lambda \) is in the crab by exceptionality. We will show that \( \mu - \lambda \) and \( -2\rho - \lambda \) are on the same crab line, hence that \( (\mu - \lambda) - (-2\rho - \lambda + \rho) = \mu + \rho \) is also on the crab and thus \( \mu \) is one of the mirror 20 weights.
We have
\[ \| (\mu - \lambda) - (-2\rho + \lambda) \| = \| \mu + 2\rho \| \leq \| \mu + \rho \| + \sqrt{7} \]
by the triangle inequality. By the Trigonometry Lemma 5.4, \( \mu \) will then be one of the mirror 20 weights as long as \( \mu - \lambda \) and \(-2\rho - \lambda\) are a distance
\[ \frac{\| \mu + \rho \| + \sqrt{7}}{2(2 - \sqrt{3})} < 1.87\| \mu + \rho \| + 4.94 \]
from \( -\rho \). But indeed, by hypothesis, we have
\[ \| \mu - \lambda + \rho \| \geq \| \lambda \| - \| \mu + \rho \| \geq 1.9\| \mu + \rho \| + 7.6 \]
and
\[ \| -2\rho - \lambda + \rho \| \geq \| \lambda \| - \| \rho \| \geq 2.9\| \mu + \rho \| + 7.6 - \sqrt{7} > 2.9\| \mu + \rho \| + 4.96. \]
The final claims are apparent. \( \square \)

6.8. Lemma (ABA). Suppose that \( 0, a_1, b, a_2 \) is an exceptional collection of far weights such that \( a_1, a_2 \) are on the same crab line and \( b \) is on a different crab line. Then the collection \( 0, a_1, b, a_2 \) is maximal.

By maximal we mean that there is no exceptional collection \( 0, a_2, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) that contains \( 0, a_1, b, a_2 \) as a sub-collection.

Proof. We have a number of cases, each of which we will deal with by contradiction.

Case 1: Consider extending the collection as \( 0, \mu, a_1, b, a_2 \). If \( \mu \) is on the same crab line as \( a_1, a_2 \), then the AAB Lemma 6.5 applied to the exceptional collection \( 0, \mu, a_1, b \) together with the fact that \( a_1, b \) are far, implies that \( \mu \) is near. But this is impossible since \( \mu \) and \( a_1 \) can be at most \( 3\sqrt{3} \) apart. If \( \mu \) is on a different crab line than \( a_1, a_2 \), then \( 0, \mu, a_1, a_2 \) is exceptional, which contradicts the BAA Lemma 6.2 (since \( a_1, a_2 \) are far) unless \( \mu \) is on the singular line parallel to the crab line containing \( a_1, a_2 \). But \( 0, \mu, b \) is exceptional, \( \mu \) is one of the 20 weights, and \( b \) is far, hence \( b \) is on the crab line parallel to the singular line containing \( \mu \). This is impossible since \( b \) and \( a_1, a_2 \) are on different crab lines.

Case 2: Similarly, consider extending the collection as \( 0, a_1, b, a_2, \mu \). If \( \mu \) is on the same crab line as \( a_1, a_2 \), then the exceptionality of \( 0, b, a_2, \mu \) contradicts the BAA Lemma 6.2 since \( b, a_2 \) are far (\( b \) cannot be on the singular line parallel to the crab line containing \( a_2, \mu \) because it is far). If \( \mu \) is on a different crab line than \( a_1, a_2 \), then by the AAB Lemma 6.5 and the fact that \( a_1, a_2 \) are far, we have that \( \mu \) is near. By Proposition 6.7 applied to \( 0, a_1, \mu \) and \( 0, b, \mu \), we have that \( \mu \) is one of the mirror 20 weights on the mirror singular line parallel to the crab lines containing \( a_1 \) and \( b \), hence \( \mu = -2\rho \), contradicting the hypothesis that \( \mu \) is in the crab.

Case 3: Now consider extending the collection as \( 0, a_1, \mu, b, a_2 \). If \( \mu \) is on the same crab line as \( a_1, a_2 \), then the AAB Lemma 6.5 applied to the exceptional collection \( 0, a_1, \mu, b \) together with the fact that \( a_1, b \) are far, implies that \( \mu \) is near. But this is impossible since \( \mu \) and \( a_1 \) can be at most \( 3\sqrt{3} \) apart. Similarly, if \( \mu \) is on the same crab line as \( b \), then the AAB Lemma 6.5 applied to the exceptional collection \( 0, \mu, b, a_2 \) together with the fact that \( b, a_2 \) are far, implies that \( \mu \) is near. But this is impossible since \( \mu \) and \( b \) can be at most \( 3\sqrt{3} \) apart.

Thus \( \mu \) is not on the crab line containing any of \( a_1, b, a_2 \). As in (4.2), the collection
\[ 0, a_2 - b, a_2 - \mu, a_2 - a_1, a_2 \]
is exceptional. By Lemma 5.2, \(a_2 - a_1\) is one of the 20 weights, in particular \(\|a_2 - a_1 + \rho\| \leq 3\sqrt{3}\). Since \(b, a_2\) are far and on different crab lines, \(\|b + \rho\|, \|a_2 + \rho\| \geq 42 + 3\sqrt{3}\), hence by the Trigonometry Lemma 5.4, we have

\[\|a_2 - b\| \geq 2(2 - \sqrt{3})(42 + 3\sqrt{3}) > 2.9 \cdot 3\sqrt{3} + 7.6.\]

Applying Proposition 6.7 to the exceptional collection \(0, a_2 - b, a_2 - a_1\), we have that \(a_2 - a_1\) is one of the mirror 20 weights on the mirror singular line parallel to the crab line containing \(a_2 - b\). In particular, \(\|a_2 - a_1 + \rho\| \leq \sqrt{3}\).

The first option of the Dichotomy Proposition 5.5 applied to the exceptional collection \(0, \mu, b, a_2\) is impossible since \(b, a_2\) are far and on different crab lines. Hence by Dichotomy, \(\|\mu\| > \frac{1}{3}(42 + 3\sqrt{3}) > 15\). In particular \(\|\mu + \rho\| > 15 - \sqrt{7}\), so by the Trigonometry Lemma 5.4 (using that \(a_2\) is far), we have \(\|a_2 - \mu\| > 2(2 - \sqrt{3})(15 - \sqrt{7}) > 7 > 3\sqrt{3}\), in particular \(a_2 - \mu\) is not one of the 20 weights.

It follows that the second option in Dichotomy holds for \(0, a_2 - \mu, a_2\) and \(42 + 3\sqrt{3} < \|a_2 + \rho\| < 3\|a_2 - \mu\|\). But then we have

\[\|a_2 - \mu\| > \frac{1}{3}(42 + 3\sqrt{3}) > 15 > 2.9 \cdot \sqrt{3} + 7.6\]

so that we can apply Proposition 6.7 to the exceptional collection \(0, a_2 - \mu, a_2 - a_1\). We conclude that \(a_2 - a_1\) is one of the mirror 20 weights on the mirror singular line parallel to the crab line containing \(a_2 - \mu\).

Since \(b\) is far and \(\mu\) is on a different crab line, the Trigonometry Lemma 5.4 says that

\[\|\mu - b\| > \frac{15 - \sqrt{7}}{2(2 - \sqrt{3})} > 3\sqrt{3},\]

so \(a_2 - \mu\) and \(a_2 - b\) cannot lie on the same crab line. But then it is impossible for \(a_2 - a_1\) to be on the mirror singular lines parallel to the crab lines of both \(a_2 - \mu\) and \(a_2 - b\). Therefore no such \(\mu\) can exist.

**Case 4:** Finally, consider extending the collection as \(0, a_1, b, \mu, a_2\). If \(\mu\) is far, then by interchanging the roles of \(\mu\) and \(b\), we can use the previous argument. Hence we can assume \(\mu\) is not far. If \(\mu\) is on the same crab line as \(a_1, a_2\), then the exceptionality of \(0, b, \mu, a_2\) contradicts the BAA Lemma 6.2 since \(b, a_2\) are far (in particular, \(b\) cannot be on the singular line parallel to the crab line containing \(a_2, \mu\)). Similarly, if \(\mu\) is on the same crab line as \(b\), then the exceptionality of \(0, a_1, b, \mu\) contradicts BAA Lemma 6.2. Thus \(\mu\) is not on the crab line containing any of \(a_1, b, a_2\). If \(\mu\) is one of the 20 weights, then we can apply Proposition 6.7 to the exceptional collections \(0, a_1, \mu\) and \(0, b, \mu\) (since \(42 + 3\sqrt{3} \geq 2.9 \cdot 3\sqrt{3} + 7.6\)), concluding that \(\mu\) is also one of the mirror 20 weights contained on the mirror singular lines parallel to crab lines containing \(a_1, b\), which is impossible. Hence as before, the Dichotomy Proposition 5.5 implies that \(\|a_2 - \mu\| > 7\). As in (4.2), the collection \(0, a_2 - \mu, a_2 - b, a_2 - a_1, a_2\) is exceptional and we can use the previous argument.

We have thus ruled out all possible exceptional extensions of \(0, a_1, b, a_2\).

**6.9. Proposition.** Suppose that \(a_1, a_2, b\) are far weights such that \(a_1, a_2\) lie on the same crab line and \(b\) lies on a different crab line. Then:

1. Neither \(0, b, a_1, a_2\) nor \(0, a_1, a_2, b\) are exceptional collections.
2. If \(0, a_1, b, a_2\) is an exceptional collection, then it is maximal.
7. Computer calculations

Our proof of Theorem 4.3 makes use of the following concrete facts, which can be easily verified by computer:

7.1. **Fact.** Every exceptional collection \(0, \lambda_2, \ldots, \lambda_n\) with all \(\lambda_j\) non-far has \(n \leq 10\).

7.2. **Fact.** Let \(A\) be a crab line and \(S\) the set of weights on the union of the singular line and the mirror singular line parallel to \(A\). Then every exceptional collection \(0, \lambda_2, \ldots, \lambda_n\) of weights in \(S\) has \(n \geq 5\). (Note that as the \(\lambda_j\)’s belong to the crab, they are all selected from the union of the 20 weights and the mirror 20 weights.)

7.3. **Fact.** If \(0, \lambda_2, \ldots, \lambda_n\) is an exceptional collection with \(n = 9\) or \(10\), with all weights non-far, with all crab lines containing at most 2 weights, and with one crab line containing no weights, then \(\|\lambda_j + \rho\| \leq 5\) for all \(j = 2, \ldots, n\).

We used Mathematica 8.0.4 to check these facts. We first wrote a function `IsSingular` that returns `True` if a weight is singular and `False` otherwise. With the following code, and lists of weights \(L1\) and \(L2\), the command `FindCollections[L1, L2]` will fill the global variable `collections` with a list of all of the maximal exceptional collections that begin with \(L1\) and such that all weights following \(L1\) come from \(L2\). In the code, \(\rho\) denotes the highest root written in terms of the fundamental weights. We omit the sanity checks that ensure that for the initial values of \(L1\) and \(L2\), appending each element of \(L2\) to \(L1\) results in an exceptional collection.

```mathematica
collections = {}; FindCollections[L1_, L2_] := Module[{tmpL1}, If[Length[L2] == 0, AppendTo[collections, L1], Do[ tmpL1 = Append[L1, L2[[i]]]; FindCollections[tmpL1, Select[Delete[L2, i], IsSingular[# - L2[[i]] + rho] &&], {i, 1, Length[L2]}]]];
```

For example, to check Fact 7.1, we constructed the list \(L2\) consisting of all 445 non-far weights in the crab and executed `FindCollections[{{0, 0}}, L2]` to obtain the list of the 160,017 maximal exceptional collections \(0, \lambda_2, \ldots, \lambda_n\) with all \(\lambda_j\) non-far. With this list in hand, it is not difficult to select out collections meeting the criteria of Facts 7.2 and 7.3.

8. Bounding exceptional collections

This section will complete the proof of Theorem 4.3.

8.1. **Lemma.** In any exceptional collection \(0, \lambda_2, \ldots, \lambda_n\), at most 5 of the \(\lambda_j\)’s lie on any given crab line.

**Proof.** Fix a crab line of interest and let \(0, \lambda_2, \lambda_3, \ldots, \lambda_n\) be an exceptional collection. By restricting to a sub-list, we may assume that all of the weights \(\lambda_2, \ldots, \lambda_n\) lie on that crab line. By Corollary 5.2, \(\lambda_j - \lambda_2\) is one of the 20 weights for \(j = 3, \ldots, n\), and Figure 2 shows that \(\lambda_3, \ldots, \lambda_n\) has length at most 5 corresponding to having 6 weights on the line of interest, and that the proof is complete except in the case where the crab line makes a 120° angle with the horizontal.

For that line, we must argue that \(\lambda_3 - \lambda_2, \ldots, \lambda_n - \lambda_2\) cannot be the 5 weights of the 20 depicted in the figure. Indeed, if they were, one could translate by \(\lambda_3 - \lambda_2\)
to transform this to an exceptional collections $0, \lambda_4 - \lambda_3, \ldots, \lambda_7 - \lambda_3$ and $\lambda_j - \lambda_3$ must belong to the crab for $j \geq 4$. But we can see from the figure that this does not happen for any of the 5 choices for $\lambda_3 - \lambda_2$, hence the claim. \hfill \square

8.2. Proposition. If $0, \lambda_2, \ldots, \lambda_n$ is an exceptional collection containing at least 3 weights on some crab line $A$, then $n \leq 10$ and all the weights off $A$ are near.

Proof. In the exceptional collection $0, \lambda_2, \ldots, \lambda_n$, let $a_1, a_2, a_3$ be three weights on $A$. Then every nonzero weight $b$ in the collection and off $A$ either precedes $a_2, a_3$ or follows $a_1, a_2$.

First suppose $b$ precedes $a_2, a_3$. Then by the BAA Lemma 6.2, either $b, a_2, a_3$ are all near or $b$ is on the singular line parallel to $A$. In the latter case, $b$ is one of the 20 weights and so is near.

Suppose that $b$ follows $a_1, a_2$, then shifting by $-a_1$ gives an exceptional collection $0, a_2 - a_1, b - a_1$ where $a_2 - a_1$ is one of the 20 weights. By the Dichotomy Proposition 5.5, we have two possibilities:

Case 1: We could have that $\|b - a_1 + \rho\| \leq 6\sqrt{3}$, but in that case we find that

$$\|b - a_1\| - |\rho| \leq \|b - a_1 + \rho\| \leq 6\sqrt{3},$$

hence $\|b - a_1\| \leq 6\sqrt{3} + \sqrt{7}$. But $b$ and $a_1$ lie on different crab lines, so by the Trigonometry Lemma 5.4, we find that $\min\{\|b + \rho\|, \|a_1 + \rho\|\}$ is at most $(6\sqrt{3} + \sqrt{7})/(2(2 - \sqrt{3})) < 25$. If $\|a + \rho\| < 25$, then

$$\|b + \rho\| \leq \|a_1 + \rho\| + \|b - a_1\| < 25 + 6\sqrt{3} + \sqrt{7} < 42$$

and $b$ is near. (Note that if $a_1$ is non-near, then we would have $\|b + \rho\| < 25$ and this case is impossible.)

Case 2: Alternately, $b - a_1$ could lie on $A$. In that case, as $b$ is not on $A$, Corollary 6.4 gives that $\|b + \rho\| \leq 7.7$. Thus all weights in the exceptional collection off $A$ are near.

It remains to argue that the collection has length $\leq 10$. If the weights on $A$ are non-far, then all the weights in the collection are non-far and we are done by Fact 7.1. Therefore, we may assume that some of the weights on $A$ are far, hence all weights on $A$ are non-far. Let $b$ be a nonzero weight in the collection that is off $A$. If it precedes $a_2, a_3$, then $b$ is one of the 20 weights (because $b, a_2, a_3$ cannot all be near). Otherwise, $b$ comes after $a_1, a_2$ and we are in Case 2 above, so $2.9\|b + \rho\| + 7.6 \leq 29.93$; by Proposition 6.7, $b$ is one of the mirror 20 weights and lies on the mirror singular line parallel to $A$. Fact 7.2 and Lemma 8.1 show that one cannot obtain an exceptional collection of length $> 10$. \hfill \square

We can now conclude the proof of the main theorem.

Proof of Theorem 4.3. For sake of contradiction, we suppose we are given an exceptional collection $0, \lambda_2, \ldots, \lambda_{11}$. By Proposition 8.2, no crab line contains more than 2 weights.

We claim that the number $F$ of far weights in the collection is 1 or 2. Indeed, by Fact 7.1, $F$ is positive. Suppose that it is at least 3. Then by Proposition 6.9, all far weights lie on different crab lines, leaving $6 - F$ crab lines for the remaining $10 - F$ nonzero weights; but the remaining crab lines can only hold $12 - 2F$ weights, which contradicts our hypothesis that $F \geq 3$; hence $F = 1$ or 2.
We will now pick a crab line $A$ and a subset $S$ of $\lambda_2, \ldots, \lambda_{11}$ containing all the far weights and all the weights on $A$, and such that $|S| = 1$ or $2$.

**Case $F = 1$:** If $F = 1$, we take $A$ to be the crab line containing the far weight and let $S$ be the set of $\lambda_j$’s lying on $A$; by hypothesis $|S| \leq 2$.

**Case $F = 2$:** If both of the far weights are on one crab line, then we take it to be $A$ and $S$ to be the set of far weights. Otherwise, the two far weights are on different crab lines. We claim that one of these crab lines, call it $A$, contains exactly one weight from the exceptional collection. Indeed, otherwise there would be two crab lines each containing two weights; as all of these are non-near by Corollary 5.3, this contradicts Proposition 6.9, verifying the claim. We take $S$ to be the far weights in the exceptional collection.

We have found $S$ as desired, and deleting it from the exceptional collection leaves one as in Fact 7.3 and we conclude that $|\lambda_j + \rho| \leq 5$ for all $\lambda_j$ not in $S$. If such a $\lambda_j$ precedes one of the far weights, then it is one of the $20$ weights by the Dichotomy Proposition 5.5; if it follows one of the far weights then it is one of the mirror $20$ weights by Proposition 6.7. But deleting $S$ from our exceptional collection leaves an exceptional collection starting with $0$ and containing at least $8$ nonzero, non-far weights all lying off $A$, which contradicts Fact 7.2.

\[\square\]

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