Beyond Quantum Computation and 
Towards Quantum Field Computation

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Abstract

Because the subject of relativistic quantum field theory (QFT) contains all of non-relativistic quantum mechanics, we expect quantum field computation to contain (non-relativistic) quantum computation. Although we do not yet have a quantum theory of the gravitational field, and are far from a practical implementation of a quantum field computer, some pieces of the puzzle (without gravity) are now available. We consider a general model for computation with quantum field theory, and obtain some results for relativistic quantum computation. Moreover, it is possible to see new connections between principal models of computation, namely, computation over the continuum and computation over the integers (Turing computation). Thus we identify a basic problem in QFT, namely Wightman’s computation problem for domains of holomorphy, which we call WHOLO. Inspired by the same analytic functions which are central to the famous CPT theorem of QFT, it is possible to obtain a computational complexity structure for QFT and shed new light on certain complexity classes for this problem WHOLO.

1 Introduction

Feynman believed [28] that his greatest research contributions were in the area of quantum electrodynamics (QED). QED is a relativistic theory of photons and electrons and is a discipline contained within a wider subject of quantum field theory. In later life Feynman originated a fertile new subject of quantum computation. It would be nice to relate these two areas of research, i.e. quantum computation and quantum field theory computation (quantum field computation), in directions of his main interests. It is possible to attack some of these complicated problems even now, as this article addresses. New results

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are obtained. There is a further motivation, since people are now trying to relate Feynman’s space-time path integrals \[16, 17\] to computational methods in quantum field theory. Our approach to computation of quantum transition amplitudes is motivated through the analytic functions \[31\] which are the basis for the famous CPT theorem of Pauli, Lüders and Jost.

The two great physical theories of the twentieth century were *quantum theory* and *relativity*, both of which generalize classical Newtonian mechanics. Quantum mechanics and classical mechanics are special limiting cases of quantum field theory. By taking the limit as the velocity of light \(c \to \infty\) we expect to get non-relativistic quantum mechanics. The limit as Planck’s constant \(\hbar \to 0\) gives classical mechanics \[16\]. Quantum mechanics by itself did not correctly predict small experimentally observed deviations like the Lamb shift. This necessitated evaluating radiation corrections to quantum mechanics \[3\] and these endeavors established remarkable successes in quantum field theory. Neither quantum theory nor relativity can be ignored. Quantum field theory is a logical and natural result of combining quantum theory and relativity.

In the same twentieth century, the mathematical theory of *computation* was developed and the *electronic computer* was invented. Gate implementation for the standard classical computer or *Turing machine* can be based on classical mechanics; in the sense that the gates, ideally in the absence of perturbations (depicted by the electrical engineering term *noise*), could even consist of perfect billiard balls, in an extreme case.

Meanwhile, we have become increasingly dependent on computing machines, and there are more models of computation. By Church’s thesis, to state it simply, all *reasonable* models of computation are equivalent. At the present time, there appear to be three principal models of computation, which have grown largely independently. The models are, *quantum field computation, real computation* and *quantum computation*. It is proposed and argued here that these three models of computation are indeed significantly related. Hopefully, from a unified point of view, there is much more to be learned about computation.

Unity of these computational models is not surprising because physicists generally believe that quantum field theory contains quantum mechanics, which in turn contains classical mechanics.

Correspondingly, we present the (quantum field computation) thesis:

A. *Quantum field computation contains quantum computation as a proper subset.*

B. *Quantum field computation involves more elaborate computational tools than quantum computation (e.g. infinite dimensional Hilbert spaces, some methods of real computation, and holomorphic functions).*

On the other hand it might be argued that quantum methods may not be suitable for certain real computing problems. A simple argument is that mathematics is a much wider subject than physics; not all mathematics is necessarily applicable to physics. But in a practical implementation of quantum computation, it is the physics that applies to the quantum machine. Quantum computation can take care of small errors, using appropriate error correction schemes.
At present, quantum field theory is regarded only as an asymptotically valid theory. Some of the difficulties, although not so bad as in classical mechanics, are inherited from classical mechanics itself, for example, the infinite self-energy of the electron. Because of present unknowns in quantum field theory, we regard this thesis as not completely in the realm of provability. For example, problems of infinities and renormalizability extend to many areas of physics. We expect that well developed future generalizations of physics theories, which include non-abelian gauge theory, general relativity (Einstein gravitation), and perhaps supersymmetry, will replace quantum field theory in the above thesis, with additional and more elaborate computational tools brought into play.

1.1 Quantum Computation

Based on quantum theory, Feynman proposed quantum computation. In quantum computation, as opposed to Turing computation, qubits (quantum binary digits) are used in place of classical bits. A bit could be in one of two discrete states, 0 or 1. A qubit, on the other hand, corresponds to a 2 level quantum system, like a spin $1/2$ state of an electron. We can have a complex linear superposition of wave functions (eigenstates) of a 2 level quantum system so that two complex number amplitudes (in $\mathbb{C}^2$) are involved in each qubit. Feynman hoped to exploit the quantum system itself by making it do the computation, but practically, decoherence noise is a serious implementation problem even with very much less than 10 qubits. Yet, considerable progress is being continually made on practical applications.

As a result of Feynman’s proposal there has been an enormous amount of research, not only on quantum computation, but also on quantum cryptography. The efforts in this regard are to seek improved ways of performing computations, including a refinement of Church’s thesis by Deutsch to tackle quantum computation, or building new types of computing machines. It is hoped that not only exponentially faster computation will be achieved, but that better understanding of computational complexity will come about.

1.2 Real Computation

In another direction, the classical discrete digital Turing computer has been generalized to include the possibility of computing over the continuum. This generalization is called real computation. The need for doing this is because computing over the continuum is more appropriate to the way we do analysis, physics, numerical analysis and engineering problems. Accordingly, the classic logic theory of computation was enhanced with analysis, topology and algebraic geometry.

Until recently, it was considered unthinkable to speak of computing over a continuum, for example, over the infinite number of points in the real interval $[0, 1]$, without approximating at a finite number of points. But Tarski, in a little known paper, proved completeness over the reals for elementary algebra and geometry. The complexity was extremely high (exponential), but Smale et
have rectified that situation. Tarski’s result is in contrast to Gödel’s famous theorem of incompleteness of arithmetic over the integers $\mathbb{Z}$, and to Turing’s theorem of undecidability of the Halting problem for computation over the integers $\mathbb{Z}$.

A question was raised by Penrose as to whether the Mandelbrot set was an (albeit beautiful, picturesque) example of an undecidable set (i.e. a recursively enumerable set that is not recursive). It was concluded that it was not possible to answer this question because there was no proper definition for computing over the continuum. One problem is: how does one feed a real number, consisting of an infinitely long sequence of bits, into a computing machine in finite time? A proper definition was indeed given in the work of Smale et al. on real computation, and the question on the Mandelbrot set was answered in the affirmative. (The proof hinges on the fact that the Hausdorff-Besicovitch dimension of the boundary of the Mandelbrot set is indeed equal to 2.)

In fact, Tarski hoped to build a machine which would compute over the reals. But it is now possible to do some simple real computation even on a Turing machine. Our thesis on quantum field computation relies heavily, not only on the fact that quantum field theory generalizes quantum theory, but also on possibilities of computing over the continuum.

Real computation is a computing model that is based on classical mechanics and classical dynamical systems. But classical mechanics could also be extended to include relativity, resulting in relativistic mechanics.

1.3 Quantum Field Computation

In studying atomic phenomena, classical mechanics has been replaced by quantum mechanics. Correspondingly the classical computer could be improved with a quantum computer. But we could also think of more general models of computation based on adding relativity to quantum theory to get relativistic quantum field theory, and consider appropriate quantum field computation models.

In an approach to the central computer science problem of the P (Polynomial time) versus NP (Nondeterministic Polynomial time) complexity classes, a quantum field computer has been proposed. Under consideration were topological quantum field theories, and physical systems which contained non-Abelian gauge terms in the Lagrangian. The initial preparation of states was supposed to be consistent with knot types.

Of course, in a general situation, as in non-Abelian gauge theories, string theories (including general relativity), superstring theories, or topological field theories, quantum field computation would be an immensely difficult undertaking. Although it has not been possible to obtain stronger computation in this manner, this work has provided tools for quantum field computation.

Moreover, due to the work of Wightman on relativistic quantum field theory (incorporating Einstein’s special relativity and employing analytic functions of

\footnote{\textsuperscript{1}The user-friendly, interactive and animated color graphics “SnapPea” program for creating knots and studying hyperbolic 3-manifolds is available at: \url{http://thames.northnet.org/weeks/index/SnapPea.html}.}
several complex variables) many of the components for some quantum field computation are already available \[31, 16\]. Some additional computation methods are described in \[22, 23\].

In this article the approach is based on mathematical physics but the results also impact computer science.

In section 2 we discuss relationships between the main computation models. Next, in section 3, we consider the particular quantum field theory enhancements to the quantum computational model, that we will need. Analog and symbolic computation are related in section 4. Section 5 deals with uniformity of computation over different values of the function index and in different space-time dimensions. Section 6 discusses complexity classes and section 7 concludes with an outlook for the future.

2 Relationships Between Computation Models

There is a remarkable relationship between quantum field computation and real computation. Computation over the continuum appears in quantum field computation as well as in real computation. In the former, it is already possible to compute over \textit{cells} which are actually certain chunks of the continuum space \(C^n\) of \(n\) complex variables.

We might say this comes about because it is natural to consider a physical or quantum system in the continuum limit. In fact Isaac Newton, when studying gravitation, found it natural to consider a continuous distribution of matter to model the earth’s gravitational action at external points. From continuum quantum mechanics, by combining relativity, we have quantum field theory, a system with an infinite number of degrees of freedom. The development in real computation of the Newton endomorphism method in numerical analysis follows naturally from Newton’s continuum limit.

Real computation could also be regarded as a stepwise form of \textit{analog computation} working within a continuum.

Conversely, quantum (mechanics) computation would be suspected to be a discrete finite case of quantum field computation where the number of qubits is finite, and the corresponding Hilbert vector space is a finite dimensional vector space.

At the present time, in some approaches, quantum computation proceeds as a time evolution over a finite number of discrete time intervals, whereas time must be regarded as a continuous variable. (Continuous variable quantum computation has also been done \[7\].) Yet, space and time are interwoven in relativity, depending on the frame of reference: thus the need to handle the problem in a covariant manner. Also, because a quantum field computation model does exist, it is important to say that quantum computation can therefore benefit by including considerations of relativity, methods of computing over the continuum, and an unbounded number of qubits (infinite dimensional Hilbert space).
The important concepts for quantum computation are unitary transformations, finite superposition of states, entanglement, and quantum cryptography. Superposition is standard also in quantum field theory. Entanglement is a rather interesting form of superposition (which is of course available in quantum field theory too), with applications to quantum teleportation considerations and quantum cryptography \[6\] [1], and relies often on a basis of EPR (named after Einstein, Podolsky and Rosen) or Bell states. The EPR gedanken (thought) experiment itself, is now regarded as the first demonstration of a particularly strange form of non-local structure in quantum mechanics. At the present time we can say that quantum mechanics (without hidden variables) has been confirmed within experimentally available accuracies.

Just as the rotation group is of importance in non-relativistic quantum mechanics (with Euclidean geometry), the Lorentz group (with Minkowski geometry) is relevant to relativistic quantum mechanics. The Lorentz group contains the rotation group as a subgroup.

Thus a basic symmetry group in quantum mechanics is SU(2), the special (determinant = 1) unitary group of $2 \times 2$ complex matrices. This is also the universal covering group of the rotation group (real special orthogonal group) SO(3) in 3-dimensional space.

SU(2) is a proper subgroup of SL(2,\(\mathbb{C}\)), the universal covering group of the Lorentz group, which is the symmetry group for relativity in the usual 1-time and 3-space dimensions. Hence SL(2,\(\mathbb{C}\)) is the group appropriate for quantum field computation.

Consider, for example, the EPR states. One particular EPR state, based on electron spins, can be written as

$$\{ |01\rangle - |10\rangle \}/\sqrt{2},$$

or equivalently as

$$\{ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \}/\sqrt{2},$$

where, in the usual description, the first qubit refers to Alice and the second to Bob. They prepared the entangled state, perhaps on Earth when they were together, and now Bob could be in the Alpha Centauri system, at a space-like separation from Alice on Earth. This is simply the singlet state for adding two spins of 1/2, where we have a simultaneous eigenstate of the total spin angular momentum $S = 0$, and the total z-component of spin $S_z = 0$. (It is possible also to have entangled states for photons, which can have horizontal or vertical polarization. Entanglement produces quantum interference between photons.)

Addition of angular momentum of spins 1/2 appears here as $D^{1/2} \times D^{1/2} = D^1 + D^0$, in terms of decomposition of representations of the rotation group in 3-dimensional space. In quantum field computation, this group is enlarged to the group SL(2,\(\mathbb{C}\)), which covers the restricted (proper, orthochronous) Lorentz group. In general, one can build up from irreducible representations $D^{(j/2,k/2)}$ of SL(2,\(\mathbb{C}\)).

The concept of electron spin 1/2 is added on to (non-relativistic) quantum computation in an ad hoc fashion. But Dirac showed that electron spin natu-
rally follows from considerations of relativity, and the requirement of first order differential equations \[3\]. Entanglement of quantum states is also applicable to the relativistic theory, i.e., to quantum field theory. Indeed, experiments verify that quantum rather than classical field theory gives the correct results \[21, 36\]. Very recently \[13\] it has been shown that relativistic quantum entanglement is a most interesting subject. Spin and momentum entanglement are not separately invariant and become mixed when viewed by a moving observer. Quantum entanglement can be compared with topological entanglement \[18\]. With the analytic function approach studied in this article, there is also another type of entanglement, namely, analytic combinatorial entanglement \[23\].

3 Field Theory Enhancements to Quantum Computation

Quantum field theory not only includes all of quantum mechanics, and classical mechanics, but much more in the form of well-known results. Examples are, discrete anti-unitary symmetries, namely, CPT invariance; also the well-known spin statistics connection \[31\]. Our purpose here is to exploit results that enhance quantum computation, through working with a relativistic quantum field theory model.

Non-relativistic quantum mechanics is not complete because radiative corrections have to be made to it, using field theory. In dealing with a system corresponding to an infinite number of degrees of freedom, it is well known historically that formulations of quantum field theory like perturbation theory lead to infinities resulting in the need for renormalization. Nevertheless, quantum electrodynamics has turned out to be “the most accurate theory known to man” \[2\]. Dirac, Schwinger and Feynman are some of the principal contributors to quantum electrodynamics (the spectacular history of which is related in \[28\]), and hence to quantum field theory \[34\]. Relativistic covariance is of paramount importance in correctly performing the renormalization process.

If there is some way we can avoid approximations due to series expansions of perturbation theory, and also avoid renormalization problems, at least up to our point of departure of computational enhancements, we should do so. Fortunately we can achieve this by working within the Wightman formulation \[31, 16\] of quantum field theory. We are dealing with fields in the Heisenberg picture without using perturbation theory nor any particular time frame related Hamiltonians. We recall, using Dirac’s quantum terminology, that in the Heisenberg picture (in contrast to the Schrödinger picture) the quantum state vectors (bras and kets) are stationary (do not vary with time) while the quantum field operators carry the full interaction. Heisenberg discovered quantum mechanics, in a form which was called matrix mechanics, and the Schrödinger equation (which is used often in quantum computation) came later. Dirac related these apparently different versions of quantum mechanics through his \textit{transformation}.

\[^2\]This statement is attributed to Feynman.
theory, so that there is either a Heisenberg picture or an alternate Schrödinger picture for viewing quantum mechanics. The Heisenberg picture is preferable in relativistic quantum mechanics because time is not separated from space nor treated in a preferred manner as it is in the Schrödinger picture.

The theory is in terms of analytic functions (Wightman functions) of several complex variables. These functions arise from their boundary values which are vacuum expectation values (in Dirac’s bra-ket notation) of the form

\[ W_m(x_1, x_2, \ldots, x_m) = \langle \Omega | \phi^{(1)}(x_1)\phi^{(2)}(x_2) \cdots \phi^{(m)}(x_m) | \Omega \rangle \]  

(1)

of products of \( m \) quantum field operators \( \phi \) in a separable Hilbert space. \( |\Omega\rangle \) denotes the unique vacuum state. These transition amplitudes \( W_m \) are called Wightman distributions. They are boundary values of analytic functions called Wightman functions, denoted by

\[ W_m(z_1, z_2, \ldots, z_m) \]

where each \( x_i = \Re z_i, i = 1, 2, \ldots, m \). We summarize the physical foundations for this in the next subsection.

Wightman reconstructs quantum fields uniquely from these analytic functions. This is called the reconstruction theorem.

3.1 Physical Requirements for Relativistic QFT

We recall some basics of the known and solid mathematical physics foundation [31, 16], due to Wightman et al, for what we need here.

The space-time metric (with no gravity) is in terms of a diagonal matrix

\[ G = [g_{\mu\nu}] \]

with entries \( \{1, -1, \ldots, -1\} \) for 1 time and \( s-1 \) space dimensions. The Lorentz invariant scalar product of space-time vectors \( x \) and \( y \) is

\[ g_{\mu\nu}x^\mu y^\nu \]

with Einstein summation convention understood on repeated indices. For example, in \( s = 4 \), Lorentz transformations can be denoted by \( x' = \Lambda x \) where \( \Lambda \) is a \( 4 \times 4 \) matrix satisfying \( \Lambda^T G \Lambda = G \). General \( s \) is treated in [16].

The Poincaré group, whose elements are of the form \( (\Lambda, a) \), extends the Lorentz group, with space-time translations \( a \). For \( s = 4 \) the restricted (proper, i.e. determinant = 1, orthochronous) Lorentz group is (universally) covered \( 2 \to 1 \) by the \( \text{SL}(2, \mathbb{C}) \) group which consists of \( 2 \times 2 \) complex matrices, denoted \( A \), of determinant = 1. The image of the covering homomorphism is denoted by \( \Lambda(A) \).

Irreducible matrix representations of \( A \) will be denoted by \( S(A) \). Hence [31] the standard spinor representations \( D^{(j/2, k/2)} \) of \( \text{SL}(2, \mathbb{C}) \) mentioned in Sec.2 are obtained. The Poincaré like group, called the inhomogeneous \( \text{SL}(2, \mathbb{C}) \) group, is the \( \text{SL}(2, \mathbb{C}) \) group together with translations \( a \), and its elements can be denoted by \( (A, a) \).

The basic physical requirements [31] are the relativistic transformation law for states, spectral conditions, the transformation laws for fields, and micro-causality.

States transform according to:

\[ |\Psi'\rangle = U(A, a)|\Psi\rangle \]

where \( U(A, a) \) denotes a continuous unitary representation of the inhomogeneous \( \text{SL}(2, \mathbb{C}) \) group; the vacuum state exists and is invariant up a constant phase factor.
The spectral conditions are: The mass spectrum is assumed to be reasonable in the sense that momentum vectors $p^\mu$ lie in the open forward light cone, with time component $p^0 > 0$, except for the unique vacuum state having $p = 0$. (The electromagnetic and neutrino fields are supposed to be treated with a small positive mass epsilon, with zero limit taken later.)

The field operators, whose components are $\phi_\alpha$, transform according to appropriate unitary spin representations of the inhomogeneous $SL(2, \mathbb{C})$ group, for $3 + 1$ space-time dimensions; and generally in $s$-dimensional space-time the field operators transform as spinors in $s$-dimensions.

Thus $U \phi_\alpha(x) U^{-1} = S_\alpha^\beta(A^{-1}) \phi_\beta(x')$ where $U$ abbreviates $U(A, a)$, $x' = A x + a$ and $(i)$ distinguishes field types, which are not regarded as indices and over which there is no summation.

Because of translational invariance, the Wightman distributions are distributions in the set of difference (vector) coordinates:

$$W_m(x_1, x_2, \ldots x_m) = W_{m-1}(\xi_1, \xi_2, \ldots \xi_{m-1})$$

where $\xi_i = x_i - x_{i+1}$, $i = 1, 2, \ldots m - 1$.

It helps not to factor out (i.e. separate) the translational invariance; instead, we can use all $m$ vector coordinates, rather than the $m-1$ difference coordinates. Thus we introduce the complex space-time vectors $z_i, i = 1, 2, \ldots z_m$ such that $\Re z_i = x_i$ and $-\Im (z_i - z_{i+1})$ lie in the open forward light cone, $V_+$. Then because of the spectral conditions, the Wightman distributions $W_m$ in (1) are boundary values of Wightman functions $W_m$ as all $-\Im (z_i - z_{i+1}) \to 0^+$. Since the $\Re z$ are unrestricted and $\Im z$ are restricted, this domain for $(z_1, z_2, \ldots z_m)$ is called a tube domain, namely $T_m$.

Because these analytic functions are fundamental to the theory, one is led to computations of holomorphy domains for these functions over the space of several complex variables, $\mathbb{C}^n$. (The mathematical foundations for time-ordered and retarded transition amplitude functions are not as well established as for the vacuum expectation values at the present time.)

By the deep Hall-Wightman theorem the functions $W_m$, which are initially holomorphic in the tube domains $T_m$, can be analytically continued into what are called the extended tube domains $T'_m$. Thus $T'_m$ is obtained by applying all proper complex Lorentz transformations to the vector complex variables in the tube domain $T_m$. (Extended tubes are not tubes.) We note that the complex Lorentz group in $s$ dimensions is just a physical view of the same complex orthogonal group in $s$ dimensions.

We will call these extended tube domains $T'_m$, primitive domains of holomorphy, because they have been shown in the literature, in different ways for different $m$, to be also natural domains of holomorphy.

The primitive domains of holomorphy are basic to the proof of the CPT theorem [31]. The reason is that whereas there are no real points in the tube domains (because of the way tubes were defined), real points (called Jost points) do in fact exist in the extended tube domains. Assuming only what is called
weak local commutativity in a real neighborhood of a Jost point, we have the consequence of CPT invariance \[31\].

### 3.2 The problem WHOLO

With emphasis on the computational aspects, we will denote by WHOLO the (Wightman) problem for computing domains of holomorphy for \(W_m\), and we split the problem into a few different parts which we need in this article:

A. First characterize the extended tube domains in suitable ways, such as finding their boundaries, which are in the form of hypersurfaces; see Sec. 4.

B. Next one uses microcausality, and analytically continues \(W_m\) into the union of permuted extended tube domains, i.e. find new boundaries; see Sec. 4.1.

C. Finally one tries to find the envelope of holomorphy of the union of permuted extended tube domains, i.e. find the furthermost boundaries possible; see Sec. 4.2.

Parts A, B, C happen to be also inter-related through values of \(n\) (the function index) and \(s\) (the space-time dimension); see Sec. 5 which addresses uniformity of computation over these values.

Thus the many complex numbers (or amplitudes) that need to be handled in quantum field computation were indeed tamed as complex variables in analytic functions, i.e. the Wightman functions.

One might ask why there is only one time dimension. It was only recently known, how to physically understand concepts like closed time-like loops in more than one time dimension \[14\], where the second time dimension is in a tiny loop of a Kaluza-Klein type brane universe theory. However, on non-tiny time scales, the concept of more than one time dimensions is difficult to reconcile with causality and we will restrict ourselves here to the conventional single dimension in time \[15\].

We use a general space-time dimension \(s\) for the sake of considering uniformity of computation (to approach universality of computation), for what appear to be computational problems in their own right; whereas certain values of \(s\), such as 0, 1, 2, 3, 4, 5, 10, 11 and 26 have turned out to be more appropriate for purely physics problems. (0 and 1 are very special cases which we do not need).

We will now abbreviate the notation by suppressing the explicit spinor indices: Let the \((m\text{-point})\) Wightman function, which has \(m\) complex vector variables, each vector being of length \(s\), be denoted by \(W(n; z)\) where \(z\) denotes the set of \(n\) complex \(s\) dimensional vector variables lumped together. Thus \(n = sm\) where \(s \geq 2\) is the space-time dimension; space-time will consist of 1-time and \((s-1)\)-space dimensions. \(m\) is also called the function order (i.e. the number of fields in the vacuum expectation value in Eq.(1)), and \(n\) will be called the function index (the total number of complex variables).

Computation over \(\mathbb{C}^n\) is common also in real computation. But, in the Wightman model, we could possibly have a deeper understanding of computation because of the use of holomorphic functions (over \(\mathbb{C}^n\)) of several complex
variables. Not only the physics of quantum theory and special relativity, but also microcausality is utilized.

4 Analog Computation and Symbolic Computation

It is interesting that the strengths of analog and symbolic computation come into play as quantum field computation supplements and enhances quantum computation. We think that, it is sometimes debatable as to what is symbolic or analog computation when it comes to computing over the continuum. In real computation there seems to be a subtle re-emergence of the old analog computer in a new and powerful form. This new form is effectively digitally clamped to avoid noise problems (such as voltage drifts in potentiometers) which plagued the old analog computer.

When \( s = 2 \), i.e. in 1-dimensional space and 1-dimensional time, deterministic exact analog computation \(^{[23]}\) (computation over \( \text{cells} \) in the continuum of \( \mathbb{C}^n \)) is used to obtain what are called primitive extended tube domains of holomorphy for \( W(n; z) \). The computation can be done with essentially reversible logic, as a Horn clause satisfiability problem (HORNSAT), and with simulation on a Turing machine. But HORNSAT is in the complexity class \( P \) (polynomial time) \(^{[26]}\). This is now a deterministic problem of complexity \( P \), but (in a further problem) also implies non-deterministic polynomial time computation, in the complexity class \( \text{NP} \), as discussed below, in Sec. 4.2.

We note a couple of points in this connection. First, we rely here on the soundness theorem and the converse theorem, namely, Gödel’s theorem of completeness of first order predicate calculus \(^{[26]}\). Secondly, reversibility of computation is an asset because information content is maximized, or equivalently, the entropy increase is minimized.

Just as the classical computer, Turing machine, computes over \( \mathbb{Z} \) or (up to polynomial time) equivalently over \( \mathbb{Z}_2 \) (the classical bit representation of numbers), we now have what can be called a complex Turing machine, in fact, a severally complex Turing machine.

The primitive extended tube domains are bounded by analytic hypersurfaces, namely several Riemann cuts, and other analytic hypersurfaces of types denoted by \( S \) and \( F \), which too play a role. These domains are in the form of semi-algebraic sets in the language used in real computation. Since the computation is symbolic, it is also exact, which is important in handling holomorphic functions.

Because of Lorentz invariance properties of the physics involved, the domains have a structure referred to as Lorentz complex projective spaces. (These Lorentz complex spaces are different, but physical, “non-Euclidean” views of complex projective spaces which are well known in mathematics.) Related to this invariance are certain continuum \( \text{cells} \) over which the computation occurs. Thus this computation is also like analog computation which would otherwise be regarded as impossible to do exactly.
In this simple case, it is possible to think that (suitably encoded) pieces or whole continuous group orbits are being fed into the Turing machine. Hopefully there will be more possibilities like this in the future.

4.1 Analytic Extensions

In relativistic quantum field theory it is possible to implement the physical requirement of microcausality. There exists quantum microcausality i.e. field operators commute or anti-commute at space-like separations:

\[ [\phi_{\alpha}(x), \phi_{\alpha}(y)] = 0, \]

for \( (x - y)^2 < 0 \), where the \(-\) and \(+\) signs stand for commutation and anti-commutation respectively. The famous theorem on the connection of spin and statistics states that we have commutation for Boson fields and anti-commutation for Fermion fields \[31\].

Together with the consequence of permutation invariance of the domains, the so-called edge-of-the-wedge theorem provides enlargements of the original primitive domains of analyticity into analyticity in unions of permuted primitive domains.

Mapping these union domains creates some Boolean satisfiability problems. In fact, the novel methods of computation raise interesting issues of computability and complexity. Domains of analyticity are subject to a different type of entanglement which we can call analytic combinatorial entanglement \[23\].

4.2 Non-deterministic Holomorphic Extensions

By the nature of analytic domains in more than one complex variable, it is in general possible to further enlarge these domains (unions of permuted extended tubes) towards the maximal enlarged domains called envelopes of holomorphy.

The reasons are as follows: If we have a domain of analyticity for functions of one complex variable, there is always a function which is analytic inside the domain but is singular everywhere on the boundary of the domain, and so cannot be continued outside the domain. Thus every analyticity domain in \( \mathbb{C}^1 \), i.e. in one complex variable, is a natural domain of holomorphy. But for more than one complex variable, this is not true. We can have domains of analyticity in \( \mathbb{C}^n \) for \( n > 2 \), for which every function analytic in the domain can be analytically continued beyond the domain. This is the situation for the union of permuted extended tubes; it is not a natural domain of holomorphy.

To make these analytic extensions of the union of permuted extended tube domains, one needs to identify points at the boundaries of the domains which cannot be points of singularity for any function analytic in the domain. This was first done by Källen and Wightman \[19\]. These points are found by looking at boundaries of the union of permuted extended tubes which are in the nature of hypersurfaces, i.e. semi-algebraic sets in the language of real computation. The process is non-deterministic because there is a guessing step at the beginning, as to what the analytic extension of the domain could be, and then one makes a deterministic verification of the guess. After the guessing step, the verification is by deterministic processes mentioned above (at the beginning of Sec. 4).
Historically, this method was used by Källén and Wightman in computation, for the first time, of the holomorphy envelope [19] for $m = 3$.

Because HORN$SAT$ is in $P$, [26], this results in an NP type problem, i.e. guessing the result and verifying in polynomial time. So this part of the problem (for $s = 2$) is in the complexity class NP. (We note also that HORN$SAT$ is P-complete.)

Built-in permutation invariance (because we have analyticity in the union of permuted extended tubes) has considerable power just as $n!$ rapidly dominates over $2^n$ for large $n$. In applying local commutativity, it might appear that we have to generate permutations of $m$ objects; in fact, no algorithm is known to do this in polynomial time. But this is not a problem for us. Because of the power of non-deterministic computation [26], we are allowed to guess a candidate for a permutation; and then we can verify, in polynomial time, whether the guess is indeed a permutation, throwing out the candidate in case it is not a permutation.

5 Uniformity of Computation

Uniformity in the direction of universal computation has been discussed [4], in different contexts, including numerical analysis. We do indeed have certain types of uniformity here.

First we note that the computation is independent of any particular form of Lagrangian or dynamics, and is uniform in $n$, qualifying for a universal quantum machine over $C^\infty$ which allows for an infinite number of complex variables [24].

5.1 Function Index Uniformity

When the logic program (mentioned in Sec.4) runs for $s = 2$, dynamic memory allocation is used through the operating system. Because $n$ can be input as a variable, only part of the whole memory management cost is outside the program. The program itself is independent of $n = sm$ and therefore is uniform in $n$, which is unbounded above. We can call this function index uniformity in $n^\infty$.

5.2 Space-time Dimension Uniformity

In addition, there is uniformity in the dimension $s \geq 2$ of space-time, in the following manner. Given a dimension $s \geq 2$ of space-time, looking at the semi-algebraic sets defining the primitive extended tube domains of holomorphy (with hypersurface boundaries) and at function orders, $m$, there are three different classes of orders. These classes comprise, a) lower order W functions, b) intermediate order W functions, and c) high order W functions [24]. Extended tube domains for all high order W functions have the same complicacy. For a) we have $m \leq s + 1$, and for c), $m > s(s - 1)/2 + 2$. The remaining cases lie in class b). For example, there is no class b) for $s = 2$ (i.e. class b) is empty), the most complicated primitive domain being for the 3-point function. If $s = 3$,
then $m = 5$ is the only case in class b). When $s = 4$, we have in class b), the cases, $m = 6, 7$ and 8.

Since $s \geq 2$ is unbounded above, we can call this space-time dimension uniformity in $s^\infty$.

### 5.3 Uniformity of WHOLO

We recall that although deterministic complexity classes are closed under complements, the non-deterministic complexity class NP is not necessarily closed under complements. In fact, it is known \cite{26} that the complexity class P is a subset of both complexity classes co-NP and NP. Also the problem PRIMES (“given an integer, is it a prime?”) belongs to both complexity classes co-NP and NP. But it is not known whether PRIMES belongs to the complexity class P i.e. no polynomial time algorithm is known for PRIMES. This lack of knowledge is the basis for the success of trapdoor cipher type encryption algorithms like RSA.

Let us consider again the Wightman problem of computing holomorphy envelopes, name, WHOLO. Thus we have seen above that the problem WHOLO has uniformity in $n^\infty$ and $s^\infty$.

The holomorphy envelopes for different orders $m$ of Wightman functions are related; the holomorphy envelope for order $m$ is contained in the intersection of holomorphy envelopes for lower order functions \cite{23, 24}.

For example, in $s = 2$, the 4-point function cannot be continued beyond the 2-point function Riemann cuts nor the (permuted) 3-point function Källén-Wightman domains of holomorphy.

This is a statement regarding analyticity that does not exist, and thus refers to the complements of domains of holomorphy; hence the use of the prefix co-. Because computations of analytic extensions of domains are non-deterministic (hence the notation $N$), we can say that we have co-$N$ uniformity over $s^\infty$, and in particular, co-NP complexity for $s = 2$.

In the case that the holomorphy domains are Schlicht (i.e. a several complex variable analog to single sheeted Riemann surfaces in one complex variable), which is the only case known at present in this quantum field model, then the domains of holomorphy under consideration \cite{24} are closed under complements. This implies, in $s = 2$ for the relevant part of the WHOLO problem, that the succinct certificates (or polynomial witnesses) of co-NP complexity for higher order functions are contained in those for lower order functions. This result could have implications regarding general problems which are in co-NP and not in NP.

### 6 Discussion

We have not used the non-linear positive definiteness conditions for W- functions in Hilbert space. These conditions are required for the reconstruction theorem.
On the other hand, we want to exploit the complexity conditions for the linear-program problem as computational problems in their own right.

The original problem posed [11] for a quantum field computer, was motivated by the existence of a great deal of mathematical physics relating to the case $s = 3$. In this 3-dimensional space-time, space itself is 2-dimensional, and there are a host of fruitful statistical mechanics and field theory problems in this case [16]. For example, instead of particles having to be Bosons or Fermions as in $s = 4$, we have Anyons corresponding to braid-group statistics. (The knot problem and 3-dimensional manifolds studied as knot complements, show up here [18].) There is also the fractional quantum Hall effect, which not only has produced some of the most accurate experimental results to date, but is the fertile testing ground for new physical theories as well. In particular, Chern-Simons type gauge interaction terms in the Lagrangian [8] give more insight into field theories, including gravitation. In the future, we should expect such theories to be part of quantum field computation.

At the time of Turing, a computer was a human being doing calculations. In the present era, computers are machines on which humans are extremely dependent, not only for calculations but also for modeling natural phenomena. Quantum computers indeed have the potential of greater power than classical computers. Exploiting real computation methods and quantum field computation enhancements by invoking special relativity, gives an even deeper understanding of computational tools. In quantum cryptography, more powerful computation would mean stronger private code distribution and weaker public code methods. In the private code case, when Eve eavesdrops on the transmission of quantum information from Alice to Bob, the quantum data is disturbed so that Bob can decide it is so and discard those data items, requesting Alice to re-transmit. In the public code case, for example in the well-known RSA encryption and decoding algorithm, the code will be easier to break.

There is discrete translational invariance in quantum computation, compared to continuous translational invariance in quantum field computation. The discrete Fourier transform is of profound importance to the power of quantum computation. (See also the discussion of Lomonaco [20] on a continuous variable Shor algorithm.) In the early days of quantum field theory, it was usual to quantize over a finite, rather than an infinite, box. The finite box incorporates discrete translational invariance and allows discrete Fourier transforms.

Since, in quantum field theory, particles with arbitrary spins can be annihilated and created, we can talk about qubits, qutrits, ququads, ..., and in general, quspinors.

Relying on a fruitful set of models, we have related what appeared to be different models of quantum and classical computation based on relativistic and non-relativistic quantum mechanics and classical mechanics. Exact deterministic and non-deterministic computation over continuous domains appear naturally. Furthermore there is uniformity in computation over, unbounded above, or arbitrarily high index $n$ of $W(n; z)$ and arbitrarily high dimension $s$ of space-time.

It is good to break up a complex problem into several parts and analyze the
complexity of each part separately. Three parts of the problem WHOLO have been identified above. (There is a fourth part, namely, the representation of unions of domains, which has been possible to do only by human interaction.) In the case \( s = 2 \) the first part is in the complexity class \( P \) (and is \( P \)-complete), the second in \( NP \), and the third in \( co-NP \).

Identification, within quantum field computation, of these methods of computation raise interesting issues of computability and complexity, and possibly could shed more light, not only on computability, but also on the description of Nature by fundamental physics theories themselves.

7 Conclusion

By unity between computation models we mean that the models are actually parts of a whole, higher (or broader) model of computation. Viewed from such a broader perspective it should be possible to better understand how the different parts, namely different computation models, fit together. The situation here is quite analogous to the situation in physics theories, where quantum field theory is the higher model (in this article), which contains quantum mechanics. Correspondingly we have quantum field computation as the higher level model which contains quantum computation.

Although some parts of the Wightman model of quantum field theory are exploited here, and in fact the only way employed up to the present of connecting up with the real computational model, these parts of the Wightman model should not be regarded as the only possible way of thinking in the future. The higher level model in physics is now quantum field theory, but this model might need to be expanded later (by including more symmetry groups such as non-abelian gauge symmetry, general relativity, topological fields, etc).

Each mathematical physics theory could possibly have some interesting, novel, computational and complexity ramifications \([11]\). Accordingly, within quantum field theory we have identified \( P \) versus \( NP \) consequences and certain uniformities of computation. These uniformities are helpful in thinking of universality of computation, a hopeful problem for the future.

Through Einstein’s relativity, we have shown why there is unity between quantum field computation, real computation (computation over the continuum) and quantum computation. The Church Turing thesis for computation is supposed to be presently enhanced with the quantum field computation thesis we have proposed above. Thus the known ingenious methods in quantum computation, of dealing with discrete Fourier transforms, entangled states and fault-tolerant quantum error corrections could be profitably supplemented with concepts of infinite dimensional Hilbert spaces and (some) methods of computation over the continuum. Computation for the quantum field theory problem WHOLO is structured in layers, and each layer itself has a complexity structure.
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