Squeezed States of the Generalized Minimum Uncertainty State for the Caldirola-Kanai Hamiltonian

Sang Pyo Kim

Department of Physics, Kunsan National University, Kunsan 573-701, Korea and Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea

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Abstract

We show that the ground state of the well-known pseudo-stationary states for the Caldirola-Kanai Hamiltonian is a generalized minimum uncertainty state, which has the minimum allowed uncertainty $\Delta q \Delta p = \hbar \sigma_0 / 2$, where $\sigma_0 (\geq 1)$ is a constant depending on the damping factor and natural frequency. The most general symmetric Gaussian states are obtained as the one-parameter squeezed states of the pseudo-stationary ground state. It is further shown that the coherent states of the pseudo-stationary ground state constitute another class of the generalized minimum uncertainty states.

*Electronic address: sangkim@kunsan.ac.kr
I. INTRODUCTION

The Hamiltonian for a harmonic oscillator with an exponentially increasing mass has been introduced by Caldirola and Kanai [1] and the corresponding Lagrangian by Bateman [2]. The fact that its classical motion describe a damping motion has motivated the investigation of the Caldirola-Kanai (CK) Hamiltonian as a quantum damped system [3]. The pseudo-stationary states of the CK Hamiltonian have been found in many different ways [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In particular, the invariant operator method provides a convenient tool to find exact wave functions for such time-dependent oscillators [18]. However, there have been debates whether this quantum oscillator genuinely describes a dissipative system or not [4, 6, 7, 19].

In this paper we show that all the Gaussian states of the CK Hamiltonian with $\langle \hat{q} \rangle = 0 = \langle \hat{p} \rangle$ satisfy the generalized minimum uncertainty relation

$$\Delta q \Delta p \geq \frac{\hbar}{2} \sigma_0, \quad (\sigma_0 \geq 1),$$

(1)

where $\sigma_0 = 1/(1 - \gamma^2/4\omega_0^2)^{1/2}$ is a constant depending on the damping factor $\gamma$ and the natural frequency $\omega_0$. It is shown that the pseudo-stationary ground state is the generalized minimum uncertainty state (GMUS), a generalization of the minimum uncertainty state with $\sigma_0 = 1$ [20]. Using the linear invariant operators [21, 22, 23], we find the most general Gaussian states for the CK Hamiltonian, which have the zero moment of position and momentum, and show that the pseudo-stationary ground state is, in fact, a GMUS. The GMUS that is symmetric about the origin is interpreted as the vacuum state of time-dependent oscillator in Ref. [22]. We further show that the coherent states of the pseudo-stationary ground state are also the GMUS’s.

II. SQUEEZED STATES OF PSEUDO-STATIONARY STATES

The harmonic oscillator with an exponentially increasing mass $m = m_0 e^{\gamma t}$ has the CK Hamiltonian

$$\hat{H}(t) = \frac{1}{2m_0} e^{-\gamma t} \hat{p}^2 + \frac{m_0 \omega_0^2}{2} e^{\gamma t} \hat{q}^2.$$  

(2)

The Hamilton equations describe a classical damped motion

$$\ddot{u} + \gamma \dot{u} + \omega_0^2 u = 0.$$  

(3)
Now we use the invariant operator method to find exact quantum states of the time-dependent CK Hamiltonian. For each complex solution \( u(t) \) of Eq. (3), one can introduce a pair of linear invariant operators \[ \hat{a}(t) = \frac{i}{\sqrt{\hbar}}[u^*(t)\hat{p} - m_0e^{\gamma t}\dot{u}^*(t)\hat{q}], \]
\[ \hat{a}^\dagger(t) = -\frac{i}{\sqrt{\hbar}}[u(t)\hat{p} - m_0e^{\gamma t}\dot{u}(t)\hat{q}]. \] (4)

In fact, these operators can be made the time-dependent annihilation and creation operators satisfying the standard commutation relation at equal time \[ [\hat{a}(t), \hat{a}^\dagger(t)] = 1, \] (5)
by imposing the Wronskian condition
\[ m_0e^{\gamma t}[u(t)\dot{u}^*(t) - u^*(t)\dot{u}(t)] = i. \] (6)

We note that the eigenfunctions of \( \hat{a}^\dagger(t)\hat{a}(t) \), another invariant operator, are the exact quantum state of the Schrödinger equation. Hence the task to find the general wave functions is equivalent to finding the general solutions to Eq. (3). Our stratagem is to select a complex solution \( u_0(t) \) satisfying Eq. (6) and, as Eq. (3) is linear, to find the general solution as a linear superposition of \( u_0(t) \) and \( u_0^*(t) \).

For the underdamped motion \( (\omega_0 > \gamma/2) \), we select the solution
\[ u_0(t) = \frac{e^{-\gamma t/2}}{\sqrt{2m_0\omega}} e^{-i\omega t}, \quad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}. \] (8)

Then the wave functions of number states with the solution (8) substituted into Eq. (7) yield the pseudo-stationary states
\[ \Psi_n(q, t) = \frac{1}{\sqrt{2^n n!\sqrt{2\pi hu^*u}}} e^{-\gamma t/2} e^{-i\omega t}\left(\frac{u}{\sqrt{u^*u}}\right)^{n+1/2} \left(\frac{q}{\sqrt{2h u^*u}}\right) \exp\left[i m_0 e^{\gamma t} \frac{u^*}{\sqrt{2h u^*u}} q^2\right]. \] (9)

Now, the general complex solutions satisfying the quantization condition (6) are written as
\[ u_\gamma(t) = \mu u_0(t) + \nu u_0^*(t), \] (10)
where

\[ |\mu|^2 - |\nu|^2 = 1. \quad (11) \]

The complex \( \mu \) and \( \nu \) have four real parameters, one of which is constrained by Eq. (11), and the other of which can be absorbed into the overall phase of \( u_r \) and hence does not change the wave functions. However, the relative phase between \( \mu \) and \( \nu \) is not determined by constraints. The squeezing parameters \( r \) and \( \phi \) in the form

\[ \mu = \cosh r, \quad \nu = e^{i\phi} \sinh r, \quad (12) \]

are, in fact, two integration constants of the second order equation (3). Conversely, given any complex solution \( u \) satisfying Eq. (6), we can find the corresponding parameters \( \mu \) and \( \nu \) or \( r \) and \( \phi \). Therefore, the most general solution to Eq. (3) can be written as

\[ u_{r\phi}(t) = (\cosh r)u_0(t) + (e^{i\phi} \sinh r)u_0^*(t). \quad (13) \]

That \( r \) and \( \phi \) are the squeezed parameters is understood from the Bogoliubov transformation

\[ \hat{a}_{r\phi}(t) = \mu^*\hat{a}_0(t) - \nu^*\hat{a}_0^\dagger(t), \]
\[ \hat{a}_{r\phi}^\dagger(t) = \mu\hat{a}_0^\dagger(t) - \nu\hat{a}_0(t), \quad (14) \]

which is obtained by substituting Eq. (13) into Eq. (4). The Bogoliubov transformation is a unitary transformation of \( \hat{a}_0(t) \) and \( \hat{a}_0^\dagger(t) \):

\[ \hat{a}_{r\phi}(t) = \hat{U}(z, t)\hat{a}_0(t)\hat{U}^\dagger(z, t), \]
\[ \hat{a}_{r\phi}^\dagger(t) = \hat{U}(z, t)\hat{a}_0^\dagger(t)\hat{U}^\dagger(z, t), \quad (15) \]

where

\[ \hat{U}(t, z) = \exp\left[\frac{1}{2} \left( z\hat{a}_0^2(t) - z^*\hat{a}_0^2(t) \right) \right], \quad z = e^{i(\phi+\pi)}r, \quad (16) \]

is the squeeze operator [20].

Each pair of squeeze parameters \( r \) and \( \phi \) defines a family of the invariant number operators

\[ \hat{N}_{r\phi}(t) = \hat{a}_{r\phi}^\dagger(t)\hat{a}_{r\phi}(t). \quad (17) \]

The number states

\[ \hat{N}_{r\phi}(t)|n, r, \phi, t\rangle = n|n, r, \phi, t\rangle \quad (18) \]
lead to the exact wave functions (7) for the Schrödinger equation in the form

\[ \Psi_n(q, t, r, \phi) = \frac{1}{\sqrt{2^n n!}} \left( \frac{A_{r\phi}}{\sqrt{\pi}} \right)^{1/2} e^{-i\Theta_{r\phi}(n+1/2)} H_n(A_{r\phi}q)e^{-B_{r\phi}q^2}, \] (19)

where

\[ A_{r\phi} = \frac{1}{\sqrt{2\hbar u^*_{r\phi} u_{r\phi}}} = \sqrt{\frac{m_0\omega e^{\gamma t}}{\hbar}} \left[ \frac{1}{\cosh 2r + \sinh 2r \cos(2\omega t + \phi)} \right]^{1/2}, \]

\[ B_{r\phi} = \frac{im\dot{u}^*_{r\phi}}{2\hbar u_{r\phi}} = \frac{m_0\omega e^{\gamma t}}{2\hbar} \left[ \frac{\cosh r e^{i\omega t} - e^{-i\phi} \sinh r e^{-i\omega t}}{\cosh r e^{i\omega t} + e^{-i\phi} \sinh r e^{-i\omega t} + i\gamma/2\omega} \right], \]

\[ \Theta_{r\phi} = \tan^{-1} \left[ \frac{\sin \omega t - \tanh r \sin(\omega t + \phi)}{\cos \omega t + \tanh r \cos(\omega t + \phi)} \right]. \] (20)

Here \( \Theta_{r\phi} \) is the negative phase of \( u_{r\phi} \), that is, \( u_{r\phi} = \rho_r e^{-i\Theta_{r\phi}}. \) The wave functions (19), which are symmetric about the origin ((\( \langle \hat{q} \rangle = \langle \hat{p} \rangle = 0 \)), are the squeezed states of the pseudo-stationary states (6). Besides the zero squeezing parameter \( (r = 0) \) leading to the pseudo-stationary states, another interesting squeezing parameters

\[ \cosh 2r_0 = 1 + \frac{\gamma^2}{8\omega^2}, \quad \tan \phi_0 = \frac{4\omega}{\gamma} \] (21)

lead to the simple harmonic wave functions at \( t = 0 \):

\[ \Psi_n(q, t = 0, r_0, \phi_0) = \exp \left[ -i \frac{\gamma}{4\omega} \left( n + \frac{1}{2} \right) \right] \times \left\{ \frac{1}{\sqrt{2^n n!}} \left( \frac{m_0\omega}{\pi \hbar} \right)^{1/4} H_n \left( \sqrt{\frac{m_0\omega}{\hbar}} q \right) \exp \left[ -\frac{m_0\omega}{2\hbar} q^2 \right] \right\}. \] (22)

The wave functions (19), evolving the harmonic wave functions of an undamped \( (\gamma = 0) \) oscillator at \( t = 0 \), differ from those in Ref. [11] only by the constant phase factor in Eq. (22).

III. GENERALIZED MINIMUM UNCERTAINTY STATE

We now find the GMUS satisfying the equality in Eq. (1) among the wave functions (19), which are symmetric about the origin. The wave functions (19) have the uncertainty

\[ (\Delta q)_{r\phi}(\Delta p)_{r\phi} = \langle n, r, \phi, t | \hat{q}^2 | n, r, \phi, t \rangle^{1/2} \langle n, r, \phi, t | \hat{p}^2 | n, r, \phi, t \rangle^{1/2} = \hbar \sec \left( \frac{\varphi}{2} \right) \left\{ \frac{\cosh 2r + \sinh 2r \cos(2\omega t + \phi)}{2} \right\} \]

\[ \times \left\{ \cosh 2r - \sinh 2r \cos(2\omega t + \varphi + \varphi_\gamma) \right\}^{1/2} \left( n + \frac{1}{2} \right), \] (23)
where
\[ \vartheta_\gamma = \sin^{-1}\left(\frac{\gamma}{1 + \frac{\gamma^2}{4\omega^2}}\right) = \cos^{-1}\left(\frac{1 - \frac{\gamma^2}{4\omega^2}}{1 + \frac{\gamma^2}{4\omega^2}}\right), \quad (\pi > \vartheta_\gamma \geq 0). \] (24)

Using Eq. (23), we find the condition leading to the minimum allowed uncertainty. First, from \((\Delta q)_n(\Delta p)_n = (\Delta q)_0(\Delta p)_0(n + 1/2)\), the ground state \((n = 0)\) has the lower uncertainty than other excited states \((n \geq 1)\). Second, for the zero squeezing parameter \((r = 0)\), the pseudo-stationary ground state \(\Psi_0(q,t)\) has the generalized minimum uncertainty at all times
\[ (\Delta q)_{00}(\Delta p)_{00} = \frac{\hbar}{2} \sec\left(\frac{\vartheta_\gamma}{2}\right), \] (25)
Thus the generalized minimum uncertainty \((1)\) is satisfied for
\[ \sigma_0 = \sec\left(\frac{\vartheta_\gamma}{2}\right) = \frac{1}{\left(1 - \frac{\vartheta_\gamma^2}{4\omega_0^2}\right)^{1/2}}. \] (26)

Note that the generalized minimum uncertainty approaches the usual minimum uncertainty \((\hbar/2)\) in the weak damping limit \((\gamma/\omega_0 \ll 1)\). Similarly the time averaged uncertainty is
\[ \overline{(\Delta q)}_{0r}(\Delta p)_{0r} = \frac{\hbar}{2} \sec\left(\frac{\vartheta_\gamma}{2}\right) \left(\cosh^2 r - \frac{\cos \vartheta_\gamma}{2} \sinh^2 r\right) \geq \frac{\hbar}{2} \sec\left(\frac{\vartheta_\gamma}{2}\right), \] (27)
where the equality holds for \(r = 0\). Third, in the case of zero damping \((\gamma = 0 = \vartheta_\gamma)\), the CK Hamiltonian \((2)\) is just a simple (time-independent) harmonic oscillator. Then the uncertainty relation of \(\hat{q}\) and \(\hat{p}\) in the state \((19)\) is given by
\[ (\Delta q)_{0r}(\Delta p)_{0r} = \frac{\hbar}{2} \left[\cosh^2(2r) - \sinh^2(2r) \cos^2(2\omega t + \phi)\right]^{1/2} \geq \frac{\hbar}{2} \] (28)
The generalized minimum uncertainty is achieved either for the zero squeezing \((r = 0)\) at all times or when \(\cos(2\omega t + \phi) = \pm 1\). Therefore, we conclude that the pseudo-stationary ground state, which is provided by the zero squeezing \((r = 0)\) solution \(u_0\) in Eq. \((8)\), gives rise to the GMUS with the center at the origin. In particular, this GMUS is interpreted as the vacuum state in Ref. \(22\). Finally we obtain the Hamiltonian expectation value
\[ \langle \hat{H} \rangle_{nr} = \frac{\hbar \omega}{2} \sec^2\left(\frac{\vartheta_\gamma}{2}\right) \left[\cosh 2r + \sinh 2r \sin\left(\frac{\vartheta_\gamma}{2}\right) \sin\left(2\omega t + \phi + \frac{\vartheta_\gamma}{2}\right)\right] \left(n + \frac{1}{2}\right). \] (29)
The time averaged $\langle H \rangle_{nr\phi}$ has the minimum value for $n = r = 0$, coinciding with the generalized minimum uncertainty.

There is another class of GMUS’s. It is known that for a time-independent oscillator, the coherent states of the vacuum state also have the minimum uncertainty [20]. Now, for the CK Hamiltonian, we either follow the definition of coherent states [24, 25]

$$\hat{a}_{r\phi}(t)\alpha, r, \phi, t) = \alpha|\alpha, r, \phi, t),$$

for any complex $\alpha$ or apply the displacement operator to the ground state in Eq. (19)

$$|\alpha, r, \phi, t) = e^{\alpha \hat{a}^\dagger_{r\phi}(t) - \alpha^* \hat{a}_{r\phi}(t)|0, r, \phi, t).$$

Then the generalized coherent states have the expectation values

$$q_c(t) = \langle \alpha, r, \phi, t|\hat{q}|\alpha, r, \phi, t) = \sqrt{\hbar}(\alpha u_{r\phi} + \alpha^* u^*_{r\phi}),$$

$$p_c(t) = \langle \alpha, r, \phi, t|\hat{p}|\alpha, r, \phi, t) = \sqrt{\hbar}m_0 e^{\gamma t}(\alpha \dot{u}_{r\phi} + \alpha^* \dot{u}^*_{r\phi}).$$

Here $q_c$ and $p_c$ describe a trajectory in the phase space for each choice of $\alpha$ and $u_{r\phi}$. Replacing the complex $\alpha$ by two real variables $q_c$ and $p_c$, we obtain the wave functions for the coherent states

$$\Psi(q, r, \phi, q_c, p_c) = \left(\frac{A_{r\phi}}{\sqrt{\pi}}\right)^{1/2} F_{r\phi} e^{-\frac{i \Theta_{r\phi}}{2}} e^{-B_{r\phi}(q - q_c)^2} e^{ip_c q / \hbar},$$

where $A_{r\phi}, B_{r\phi},$ and $\Theta_{r\phi}$ are given in Eq. (20) and $F_{r\phi}$ is the additional phase factor

$$F_{r\phi} = \exp \left[ \frac{i}{2\hbar \dot{u}_{r\phi}^* u_{r\phi}^*} (u_{r\phi}^2 p_c^2 - 2 \dot{u}_{r\phi}^* u_{r\phi} p_c q_c) \right].$$

It then follows that any coherent state (33) has the same uncertainty as the general Gaussian state with $n = 0$ in Eq. (19):

$$\langle \Delta q \rangle_{or\phi} \langle \Delta p \rangle_{or\phi} = \langle \alpha, r, \phi, t|\hat{q} - \langle \hat{q} \rangle|^2|\alpha, r, \phi, t\rangle^{1/2}\langle \alpha, r, \phi, t|\hat{p} - \langle \hat{p} \rangle|^2|\alpha, r, \phi, t\rangle^{1/2} = \langle \Delta q \rangle_{or\phi} \langle \Delta p \rangle_{or\phi}.$$

This implies that the coherent states of the GMUS are also GMUS’s. Thus the coherent states of the pseudo-stationary ground state constitute a family of GMUS’s, which is the time-dependent generalization of time-independent oscillator [20].
IV. CONCLUSION

We have shown that the Caldirola-Kanai Hamiltonian satisfies the generalized minimum uncertainty \( \Delta q \Delta p \geq \hbar \sigma_0 / 2 \) for \( \sigma_0 = 1/(1 - \gamma^2/4\omega_0^2)^{1/2} \), where \( \gamma \) is the damping factor and \( \omega_0 \) is the natural frequency. It is found that the well-known pseudo-stationary ground state has in fact the generalized minimum uncertainty. As the generalized minimum uncertainty state is uniquely selected for the Caldirola-Kanai Hamiltonian, this pseudo-stationary ground state may be interpreted as the vacuum state [22]. One-parameter family of squeezed states of the pseudo-stationary states are obtained as the most general states with the zero moment of position and moment. Further, it is shown that the coherent states of the pseudo-stationary ground state are the generalized minimum uncertainty states.

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