Rooted-tree network for optimal non-local gate implementation

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Abstract A general quantum network for implementing non-local control-unitary gates, between remote parties at minimal entanglement cost, is shown to be a rooted-tree structure. Starting from a five-party scenario, we demonstrate the local implementation of simultaneous class of control-unitary (Hermitian) and multiparty control-unitary gates in an arbitrary n-party network. Previously, established networks are turned out to be special cases of this general construct.

Keywords Gate teleportation · Local operation and classical communication (LOCC) · Control–unitary gates · Rooted tree

1 Introduction

Apart from a host of other quantum communication protocols [1–4], remote implementation of general quantum gates is crucial for quantum computation, as it enables one to non-locally design useful quantum circuits. The task is to perform a multiqubit quantum gate (that cannot be decomposed into single qubit gate) where the qubits are

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possessed by spatially separated parties. This involves multiparty operations, making essential use of entanglement [5,6], local operation and classical communication (LOCC). Designing a quantum network for implementing gate teleportation with minimal entanglement cost has been of deep interest to both theoretical and experimental community in recent years.

In the pioneering work, Eisert et al. [7] proposed a method for optimal local implementation of non-local control-unitary gates. Subsequently, a wide range of works [8–21] are provided for efficient implementation of non-local quantum gates under various scenarios. Moreover, teleportation of control-unitary gates has been realized in laboratory [22–25]. The proposal of Eisert et al., to perform an arbitrary control-unitary gate, involved a network in a parallel configuration, where each control party (having control qubit) shares a pair of entangled state with the target party (having the target qubit) and no entangled state is shared between control parties. The target party, therefore, has to deal with large number of entangled pairs, making this protocol inherently complicated to realize. To avoid this problem, another configuration has been considered in [21] which uses a linear entangled channel that can be visualized as a series connection. Here, each control party shares entanglement with the adjacent one and only one of them is entangled with the target party. As each party deals with only two entangled pairs, the linear network has an advantage over the parallel one, although the classical communication cost increases significantly. However, parallel and linear are not the most general network when the entanglement cost is minimal.

In this paper, we present the generalized network for gate teleportation, which is a branched tree network channel, known as rooted-tree network in graph theory parlance. We explicate the proposed network by implementing two types of non-local gates. Firstly, we consider the scenario where each control party wants to apply a class of control-unitary(Hermitian) (\(CH\)) gate simultaneously on a target qubit. Here, class of control-unitary(Hermitian) gate is defined as control-unitary gate in which the target unitary gate is also Hermitian. Further, we study the implementation of multiparty control-unitary (\(CU\)) gate, a generalized form of the Toffoli gate. The rooted-tree network opens up the possibility to construct a quantum network with minimal entanglement cost in many ways, optimizing several aspects of computation. In other words, it enables us to implement a multiparty gate, according to the maintenance capacity of the entangled channels shared by each remote party, simultaneously reduce the classical communication cost and the number of implementation steps, as compared to the linear network. The above-mentioned two networks are found to be the special cases of the rooted-tree structure.

In the following section, we establish the rooted-tree network as the most general entanglement distribution structure among remote parties, which makes optimal use of entanglement resources. We restrict ourselves with the bipartite entanglement resources (like Bell states). In the case of multipartite correlations, the corresponding graph will have hyper edges and subsequently a lot more possibilities. In the subsequent two sections, the protocol for implementation of class of control-unitary(Hermitian) gate simultaneously and multiparty control-unitary gate is carried out. For explicitness, we demonstrate the protocol in five-party system, with four control parties and one target party which is then generalized for arbitrary control parties. The required entanglement and classical communication resources, as well as the number of imple-
2 Rooted-tree network

As mentioned earlier, quantum computation in a distributed scenario and implementation of desired tasks at remote locations requires implementation of non-local quantum control gates. In our proposed construct, there are \((n - 1)\) remote control parties who want to implement control-unitary gates on one target party. In the graph theory framework, each remote party is represented by a vertex and each edge connecting two vertices depicts one Bell state. Since each remote agent is connected to at least one other, the graph representing the quantum network is connected. A graph is connected, if all the vertices are joined to each other by a path through edges. To ensure an optimal entanglement cost, the number of edges of the relevant graph should be \((n - 1)\) [7]. It can be shown that a connected graph of \(n\) vertices with \((n - 1)\) edges has to be a tree. For this to occur, the graph should be devoid of any circuit, which connects a vertex with itself through edges in a closed path. It is evident that if there exists a circuit in the network, then by removing one edge one can still make a connected graph. However, the fact that a connected graph of \(n\) vertices has at least \((n - 1)\) edges contradicts that a connected graph with \((n - 1)\) edges has circuit [26,27].

In the present case with one target party and \((n - 1)\) control parties, one can define a direction for each edge directed away from the root. The network has a designated vertex called the root, representing the target party. This general entanglement distribution between remote parties forms a rooted-tree structure, as illustrated in Fig. 1.

One defines the depth \((d)\) of a control party as the number of edges present in the unique path from the target party. The height \((h)\) of the rooted tree is the largest depth of the tree, and the control party having an outward edge or possessing only one edge
is called a leaf. Here, the number of control parties of the same depth is denoted as $n_d$:

$$n = \sum_{d=1}^{h} n_d + 1,$$

where $n$ is the total number of parties. The rooted-tree network is the most general network structure under the condition of optimal entanglement cost, which is $(n - 1)$ ebits for $n$ remote parties. We recall that the parallel configuration [7] is a network where each party shares a pair of entangled state with target party and no entangled state is shared between control parties, whereas in linear configuration [21] each party shares entanglement with the adjacent one and only one of them is entangled with the target party. Thus, the parallel network and linear network are special case of rooted-tree structure where $h = 1$ and $h = n - 1$, respectively. We now proceed to the details of the protocol of implementing non-local gates.

3 Simultaneous implementation of class of control-unitary(Hermitian) gate

The simultaneous implementation of class of control-unitary(Hermitian) gate is a realization of a number of class of control-unitary(Hermitian) gates on a common target qubit in a quantum network. We emphasize that control-unitary(Hermitian) gates form a subset of control-unitary gates in which the unitary gate has an additional property of hermiticity. If the quantum state $|\psi\rangle_{12...n}$ connects $n$ remote parties, possessing a single qubit each, with the $n$th qubit as the target, then the desired state with the gate teleportation protocol will be $C^H_{n-1}C^H_{n-2}...C^H_1|\psi\rangle_{12...n}$. Here, $C^H_i$ denotes the control-unitary(Hermitian) gate where $i$ is the control qubit and $j$ is the target. Note

![Fig. 2](image)

**Fig. 2**. A rooted-tree network comprising of four control parties, $S_{d_j}$ (where $d$ represents the depth and $j$ the location in that depth), and the target party $T$. In the figure, each party is represented by a system of numbered qubits enclosed in the rectangular shape.
that the target Hermitian gate can be arbitrary but same for all control parties. Although the protocol does not allow one to simultaneously implement all control-unitary gate, it is worth mentioning that many of important gates in quantum computation belong to this category.

For the sake of explicitness, we first consider a five-party scenario, maintaining the entangled channel in a rooted-tree pattern, as depicted in Fig. 2. Let $S_{11}, S_{12}, S_{21}$ and $S_{22}$ be the four control parties (each control party is denoted as $S_{dj}$, where $d$ represents the depth and $j$ the location in that depth) want to simultaneously implement class of control-unitary(Hermitian) gate on a target party ($T$).

Let the five parties possess qubits 1, 3, 7, 9, 13 of an arbitrary state $|\psi\rangle_{1,3,7,9,13}$. $S_{21}$ and $S_{22}$ both share Bell states $|\Phi\rangle_{2,5} = |\Phi\rangle_{4,6} = |\Phi\rangle_{8,11} = |\Phi\rangle_{10,12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (2)

The protocol to implement simultaneous class of control-unitary(Hermitian) gate, describing each implementation step, is given below. Here, the implementation step is considered as a successive operation, which may be a quantum measurement or some other quantum operation or classical communication, as required. The following steps enumerate the implementation of protocol.

**Step 1** All the leaves, i.e., $S_{12}, S_{21}$ and $S_{22}$ first apply control-Not ($CN_{ji}$, $i$ is the control and $j$ is the target) gate on their respective qubits $CN_{21}^8$, $CN_{43}^8$, $CN_{109}^8$.

**Step 2** In the second step, both $S_{21}$ and $S_{22}$ measure their qubits 2 and 4 in the computational basis.

**Step 3** They then convey the outcomes of the measurements to $S_{11}$ using classical resources.

**Step 4** Depending on the outcome, $S_{11}$ performs the local operations as listed in Table 1, where $\sigma_{x}^i, \sigma_{z}^i, \sigma_{y}^i$ are Pauli operators, with superscript ‘$i$’ indicating the operating qubit.

**Step 5** $S_{11}$ and $S_{12}$ measure their qubits 8 and 10 in the computational basis.

**Step 6** Next, they convey the outcomes to target party $T$.

**Step 7** Based on their outcome $T$ then performs the local operations as described in Table 2,
Table 2  Local operations performed by target party $T$, based on the outcomes of qubits 8 and 10 measured by $S_{11}$ and $S_{12}$, respectively

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|0\rangle_8 | 0\rangle_{10}$ | $C_{H}^{13} C_{H}^{12}$ |
| $|1\rangle_8 | 0\rangle_{10}$ | $C_{H}^{13} C_{H}^{12} \sigma_{x}^{11}$ |
| $|0\rangle_8 | 1\rangle_{10}$ | $C_{H}^{13} C_{H}^{12} \sigma_{x}^{12}$ |
| $|1\rangle_8 | 1\rangle_{10}$ | $C_{H}^{13} C_{H}^{12} \sigma_{x}^{11} \sigma_{x}^{12}$ |

Fig. 3  Simultaneous class of control-unitary(Hermitian) gate implementation through rooted-tree network. Here the relevant party possesses a qubit given in first bracket. $H$ and $H$ denote Hadamard and Hermitian gates, respectively; the dotted line represents Bell state. All the measurements shown here are in computational basis.

Step 8  Subsequently, qubits 5 and 6 are measured in the Hadamard basis by $S_{11}$ and qubits 11 and 12 by $T$.

Step 9  $T$ conveys the measurement outcomes of qubit 11 to $S_{11}$, $S_{21}$, $S_{22}$ and qubit 12 to $S_{12}$. $S_{11}$ conveys the outcomes of qubits 5 and 6 to $S_{21}$ and $S_{22}$, respectively.

Step 10  Correspondingly, the control parties perform following unitary operations to achieve the desired state shared by the parties,

The quantum circuit for the simultaneous implementation of $C_H$ gate as described by the above protocol is presented in Fig. 3. The entanglement resources (ebits) and classical communication resources (cbits) required to obtain the desired state are 4 and 10, respectively. It is worthy to point out that the necessity of class of control-unitary(Hermitian) gate is reflected in steps 7 and 8. A unitary, as well as Hermitian operation, has involution property, i.e., $H = H^{-1}$. We can check that unless this property the desired gate teleportation cannot be achieved (Table 3).
Table 3 Unitary operations performed by control parties to achieve the desired state

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|++-+\rangle_{5,6,11,12}$ | $I$ |
| $|++-\rangle_{5,6,11,12}$ | $\sigma_z^9$ |
| $|++-+\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^3 \sigma_z^7$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^1$ |
| $|+-+-\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^3 \sigma_z^7 \sigma_z^9$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^3$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^9$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^7$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^3 \sigma_z^9$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^1 \sigma_z^3 \sigma_z^9$ |
| $|--+\rangle_{5,6,11,12}$ | $\sigma_z^7 \sigma_z^9$ |

3.1 Protocol for $n$-party simultaneous implementation of class of control-unitary(Hermitian) gate

We now generalize the above protocol for $n$ remote parties, sharing an unknown $n$-qubit state $|\psi\rangle$ and maintaining an arbitrary rooted-tree entangled network. At first, all the control parties, located at leaves, apply a control-Not ($CN$) gate on their respective pair of qubits taking the target qubit as the entangled qubit shared with another control party at the successive upper depth. In the next step, all the leaves, which are at the maximum depth ($d = h$), measure their target qubit in computational basis and convey the outcomes to the successive control party located at $d = (h - 1)$. Based on the outcome of measurements, these control parties at $d = (h - 1)$ depth perform local operations, i.e., if the outcome is $|1\rangle$, they apply $CN_p^q \sigma_p^x$, otherwise $CN_p^q \sigma_q^x$, for every $p$ and $q$, where $p$ is the entangled qubit shared with the control party at $d = h$ depth, $q$ is the entangled qubit with the qubit of control party at $d = (h - 2)$ depth and $l$ is the unknown qubit of the state $|\psi\rangle$ possessed by that party. These control parties then measure their target qubit $q$ in computational basis and convey the outcomes to the successive control party at $d = (h - 2)$. Similarly on the basis of outcomes, they perform local operations, measure their target qubit in computational basis and convey to the next control party. This process continues till the control parties at $d = 1$ depth is reached. The control parties at depth $d = 1$ measure their shared qubit and convey the outcomes to the target party $T$. If the outcome is $|1\rangle$, then $T$ performs local operations $\mathcal{CH}_k^l \sigma_k^x$, otherwise only $\mathcal{CH}_k^l$ for every $k$, where $k$ is the entangled qubit shared with the control party at $d = 1$ depth and $l$ is the unknown qubit of the type.
$(|\psi\rangle$ possessed by the target party. Subsequently, all the entangled qubits shared by the target party are measured in Hadamard basis and the outcomes of the measurements are conveyed to all the control parties, starting from the respective control party with whom the entangled state has been shared. The other control parties, except the ones in the leaves, measure the shared entangled qubit with the control party at successive lower depth in the Hadamard basis and convey the outcomes to them. Finally, all the control parties perform $\sigma_z$ on their possessed qubit of the state $|\psi\rangle$, if odd number of parties (including all the control parties with higher depth and the target party) get the measurement outcome $|1\rangle$.

Thus, for $n$-party simultaneous implementation of a $CH$ gate, the total number of cbits required is,

$$\sum_{d=1}^{h} n_d + \sum_{d=1}^{h} d.n_d = \sum_{d=1}^{m} n_d(d + 1),$$

and the number of implementation steps is $3h + 4$.

### 4 Implementation of multiparty control-unitary gate

We now proceed to describe the implementation of multiparty control-unitary gate on the rooted-tree network. The multiparty control-unitary is defined as a generalized form of a Toffoli gate for multiqubit system, with one target party, where an arbitrary unitary gate acts on the target qubit, only if all the control qubits shared by the remote parties are $|1\rangle$. Thus, given an arbitrary quantum state $|\psi\rangle_{12...n}$, shared by $n$ remote parties, where $n\text{th}$ qubit is the target, the desired state which will be realized by LOCC is given by, $CU^n_{1,2,...,(n-1)}|\psi\rangle_{12...n}$. Here $CU^t_{i,j,k}$ is denoted as the multiparty control-unitary gate, where $i, j, k$ are the control qubits and $t$ is the target.

To demonstrate the implementation, first we consider the same five-party scenario as shown in Fig. 2. The qubit distribution is described in the earlier section, where the four control parties want to implement a control-unitary gate $CU^3_{1,3,7,9}$ on an unknown state $|\psi\rangle_{1,3,7,9,13}$. The protocol for such implementation is described below:

**Step 1** $S_{21}, S_{22}$ and $S_{12}$, respectively, apply $CN^2_{1}, CN^4_{3}, CN^9_{9}$.

**Step 2** Then, $S_{21}$ and $S_{22}$ measure qubits 2 and 4, respectively, in computational basis.

**Step 3** Following that, they convey the outcomes, using classical communication resource to $S_{11}$.

**Step 4** Now, $S_{11}$ applies the following local operations as described in Table 4, depending on the measurement outcomes:

- **Step 5** $S_{11}$ and $S_{12}$ measure qubits 8 and 10, respectively, in the computational basis.

- **Step 6** They convey the outcomes to $T$.

- **Step 7** $T$ applies the following unitary operation as shown in Table 5, where $U$ is the multiparty gate.

- **Step 8** $T$ measures qubits 11 and 12 in Hadamard basis.

- **Step 9** Next $T$ conveys the outcomes of qubits 11 and 12 to $S_{11}$ and $S_{12}$, respectively.

- **Step 10** $S_{11}$ and $S_{12}$ apply following gates, as shown in Table 6, according to the outcomes ($CZ$ stands for control-$\sigma_z$ gate).
Table 4  Operations performed by $S_{11}$ depending upon the outcomes of measurement of qubits 2 and 4 measured by $S_{21}$ and $S_{22}$

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|0\rangle_2|0\rangle_4$   | $CN_{5,6,7}^8$                   |
| $|1\rangle_2|0\rangle_4$   | $CN_{5,6,7}^8 \sigma_x^5$        |
| $|0\rangle_2|1\rangle_4$   | $CN_{5,6,7}^8 \sigma_y^6$        |
| $|1\rangle_2|1\rangle_4$   | $CN_{5,6,7}^8 \sigma_x^5 \sigma_y^6$ |

Table 5  Local operations performed by $T$ which depends on the outcome of the measurement of qubit 8 and 10 measured by $S_{11}$ and $S_{12}$, respectively

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|0\rangle_8|0\rangle_{10}$ | $CU_{11,12}^{13}$            |
| $|1\rangle_8|0\rangle_{10}$ | $CU_{11,12}^{13} \sigma_x^{11}$ |
| $|0\rangle_8|1\rangle_{10}$ | $CU_{11,12}^{13} \sigma_y^{12}$ |
| $|1\rangle_8|1\rangle_{10}$ | $CU_{11,12}^{13} \sigma_x^{11} \sigma_y^{12}$ |

Table 6  Operations performed by $S_{11}$ and $S_{12}$ according to the outcomes conveyed by $T$

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|+\rangle_{11}|+\rangle_{12}$ | $\sigma_z^9$                |
| $|+\rangle_{11}|-\rangle_{12}$ | $I$               |
| $|-\rangle_{11}|+\rangle_{12}$ | $CZ_{5,6}^2 \sigma_z^9$          |
| $|-\rangle_{11}|-\rangle_{12}$ | $CZ_{5,6}^2$            |

Table 7  Operations performed by $S_{21}$ and $S_{22}$ to obtain the desired state, depending on the outcome conveyed by $S_{11}$

| Outcome of measurements | Local operation after measurements |
|-------------------------|-----------------------------------|
| $|+\rangle_5|+\rangle_6$   | $\sigma_z^3$                   |
| $|+\rangle_5|-\rangle_6$   | $I$                   |
| $|-\rangle_5|+\rangle_6$   | $\sigma_z^1 \sigma_z^3$       |
| $|-\rangle_5|-\rangle_6$   | $\sigma_z^1$        |

Step 11 After that, $S_{11}$ measures qubits 5 and 6 in Hadamard basis.
Step 12 Next, $S_{11}$ conveys the outcomes of qubits 6 and 5 measurements to $S_{21}$ and $S_{22}$, respectively.
Step 13 Finally, $S_{21}$ and $S_{22}$ apply gates to obtain the desired state as described in Table 7 :
The pictorial representation of the above protocol is shown in Fig. 4. The number of ebits and cbits required to implement a control-unitary gate is 4 and 8, respectively.

4.1 Protocol for implementation of $n$-party control-unitary gate

The generalization of the above protocol, for the $n$ qubit unknown state $|\psi\rangle$ in an arbitrary rooted-tree network, is as follows. At first, all the control parties which are located on the leaf apply a control-Not ($CN$) gate on their qubits, taking the shared Bell state as the target qubit. In the next step, all the parties which are at maximum depth ($d = h$) measure their target qubit in computational basis and convey the outcome to the control party at $d = (h - 1)$ depth. Based on the outcomes of the measurements, these control parties at $d = (h - 1)$ depth perform local operation $CN_{p,l}^{q} \sigma_{p}^{x}$, when the outcome is $|1\rangle$, otherwise $CN_{p,l}^{q}$ for all $p$ and $q$, where $p$ is the entangled qubit shared with the control party at $d = h$ depth, $q$ is the entangled qubit shared with the control party at $d = (h - 2)$ depth and $l$ is the qubit of the state $|\psi\rangle$ possessed by that party. All these control parties then measure their target qubit $q$ in computational basis and convey the outcomes to the respective control party at $d = (h - 2)$ depth. This process continues up to the control parties at $d = 1$ depth, where the parties measure their entangled qubit and convey the outcomes to the target party. $T$ performs local operations $\sigma_{i}^{x}$, only if the outcome is $|1\rangle$ for every $i$, where $i$ is the entangled qubit shared with the control party at $d = 1$ depth. Then, $T$ applies $C\mathcal{H}_{i,j,...}$, where $l$ is the unknown qubit of the state $|\psi\rangle$ possessed by the target party and $i, j \ldots$ are entangled qubits shared with each of the control party at $d = 1$. After that, all the entangled qubits shared by the target party are measured in the Hadamard basis and the
Table 8 Comparison of classical communication cost and number of steps for implementing simultaneous class of control-unitary(Hermitian) gate on different entangled channels: $n$ is total number of remote parties, and $h$ is the height of the rooted tree

| Network channel | Classical communication cost | No. of steps for implementation |
|----------------|----------------------------|--------------------------------|
| Rooted-tree    | $\sum_{d=1}^{h} n_d(d + 1)$ | $3h + 4$                      |
| Parallel ($h = 1$) | $2(n - 1)$ | $7$ |
| Linear ($h = n - 1$) | $\frac{n^2 + n - 2}{2}$ | $3n + 1$ |

Table 9 Comparison of classical communication cost and number of steps for implementing multiparty control-unitary gate on different entangled channels

| Network channel | Classical communication cost | No. of steps for implementation |
|----------------|----------------------------|--------------------------------|
| Rooted-tree    | $2(n - 1)$ | $6h + 1$ |
| Parallel ($h = 1$) | $2(n - 1)$ | $7$ |
| Linear ($h = n - 1$) | $2(n - 1)$ | $6n - 5$ |

outcomes of the measurement are conveyed to all the control parties at depth $d = 1$. These control parties at $d = 1$ apply $CZ_{a,b,...},$ only if the outcomes in the Bell shared qubit with target party is $|1\rangle$. Here $a, b, ...$ are entangled qubits shared with the control parties at $d = 2$ and $l$ is the qubit of the state $|\psi\rangle$ possessed by that party. Then, they measure $a, b, ...$ qubits in Hadamard basis and convey the outcomes to respective control parties at lower depth. This continues upto the leaves, where they perform $\sigma_z$ operation if the outcome of the shared entangled qubit by the control party in the successive upper depth is $|1\rangle$. This completes the protocol.

For implementation of $n$-party control-unitary gate, the total number of implementation steps is $6h + 1$ and the number of cbits required is,

$$2 \sum_{d=1}^{h} n_d = 2(n - 1).$$

It is mentioned in Sect. 2 that the particular two cases where $h = 1$ and $h = n - 1$ are referred as parallel and linear configurations, respectively. Using this fact, the implementations steps as well as classical communication cost can be obtained for these two networks. The classical communication cost for parallel and linear structure is given in [7,21] that can be trivially recovered from the generalized formalism described here. The comparison is summarized in Tables 8 and 9 given below.

5 Discussion

We have shown that a rooted-tree network is the most general entanglement distribution, which provides a fresh perspective for implementing non-local gates locally from several control parties to a target party, under the condition of optimal entanglement
The explicit protocols of simultaneous implementation of controlled-Hermitian gate and the multiparty controlled-unitary gate are described here in detail.

The earlier explored entangled networks, namely parallel and linear configurations, are specific cases of the rooted-tree network, where \( h = 1 \) and \( h = n - 1 \), respectively. For all the cases, the entanglement cost is \((n - 1)\) ebits, which is optimal. It can be seen that, for a given \( n \), the classical communication cost and the number of steps for implementing simultaneous class of control-unitary(Hermitian) gate increases as we go from parallel to linear network, through an arbitrary rooted-tree structure. For multiparty control-unitary gate, only the number of implementation steps increases. On the other hand, as the height of rooted-tree increases or equivalently as we go from linear to parallel, the number of entangled states maintained by a party decreases. For the simultaneously gate teleportation, the classical communication cost follows a different pattern, while for the Toffoli-like gate teleportation it is same with two times of the entanglement cost. Since in the former case, the desired state is influenced by each control party separately, the target party needs to communicate separately with all the control parties making the classical communication cost higher. So, the classical communication cost depends on the peculiarity of the gate we want to implement. For example, in the scenario consider in Fig. 2, the maximum number of entangled states maintained by a party is 3; it is 4 for parallel and 2 for linear network. Thus, the novelty of the rooted-tree entanglement distribution between \( n \) remote parties lies in the fact that it opens up the freedom of designing the entanglement network according to the capacity of maintaining entangled channels of each party, as well as the suitable number of implementation steps and the available classical communication resources. In future, it will be of interest to investigate whether the classical communication cost for simultaneous class of control-unitary(Hermitian) gate teleportation described here is optimal or not. Teleportation of other non-local gates using local operations and classical communication is also worth studying.

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