Approximate solutions of linear time-fractional differential equations

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Abstract

In this research work, the numerical scheme for obtaining the linear time-fractional differential equations was considered and the nature of these time-fractional differential equations are in sense of Caputo. A theorem was proved to show the Kamal transform of nth order Caputo derivatives. Finally, three problems were considered regarding the linear time-fractional differential equations which presented that the convergence of the scheme provided in the research are of high accuracy for solving and linear fractional differential equations.

Keywords: Kamal transform, adomian polynomial, linear time-fractional differential equations

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1. Introduction

The interest of researchers in handling fractional differential equations has greatly increased due to their applications in the field of science and engineering. Some of these applications are found in Physics, Chemistry, Signal and Image processing, Economics, Biology and so on [4, 7, 18, 19, 27, 29, 30]. Recently, it was investigated that fractional differential equations are used to model real world problems in accurate way as when compared with classical order and these has motivated researchers to be looking for a way to handle these fractional order differential equations, this is because there is no precise method that will yield an exact solution for fractional differential equation, it is only approximate solutions that would be derived in solving it [2, 20, 23, 24, 36].

Several methods has been used to obtain the approximate analytical solution of these so called fractional differential equation. Some of the methods used include: Shehu decomposition and variational iteration [6], Laplace modified Adomian decomposition method [3], Laplace Adomian decompositon method [2, 4, 16, 17, 29, 30], Adomian decomposition method [10, 11, 20, 22–24, 27, 28, 33, 34], homotopy analysis method [13], Laplace decomposition method [12, 21], variational iteration method [35], homotopy perturbation method [5], iterative Laplace transform method [32], and multi-stem Laplace decomposition method [36].

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2. Definitions

The Caputo definition has the advantage of dealing properly with the initial value problems, in which the initial conditions are given in terms of the field variables and their integral order, which is the case in most physical process.

**Definition 2.1.** The Riemann-Liouville fractional derivative of order $\alpha$ is defined as \[10, 24\]

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t).$$

where $\alpha > 0$, $[0, t]$ is the interval, $\Gamma(\cdot)$ denotes the Gamma function and $*$ is the convolution operator.

**Definition 2.2.** The Caputo fractional derivative of order $\alpha$ is defined as \[20, 27\]

$$D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+n-1}} ds,$$

where $n - 1 < \alpha \leq n$, and $n$ is an integer.

3. The concept of Kamal transform

Kamal transform is defined for the function of exponential order \[1, 25, 26\]. Consider the functions in the set $S$ defined by

$$S = \{ f(t) : k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_2}}, t \in (-1)^j \times [0, \infty) \},$$

where $M$ is a constant which must be a finite number, $k_1, k_2$ are also constants which can be finite or infinite. Kamal transform is denoted by the operator $K(\cdot)$ which is defined as:

$$K[f(t)] = \int_0^\infty f(t)e^{-t/v} dt = G(v), \quad t \geq 0, \quad k_1 \leq v \leq k_2.$$

**Theorem 3.1.** Let $G(v)$ be a Kamal transform of $f(t)$, where $K[f(t)] = G(v)$, then

1. $K[f'(t)] = \frac{1}{v} G(v) - f(0);$ 
2. $K[f''(t)] = \frac{1}{v^2} G(v) - \frac{1}{v} f(0) - f'(0);$ 
3. $K[f^n(t)] = \frac{1}{v^n} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0).$

The proof of Theorem 3.1 is in \[25\].

**Theorem 3.2.** Given the $n^{th}$ order Caputo fractional derivative, then the Kamal transform is given by

$$K [c^D_0^\alpha f(x)] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} f^k(0) \quad n - 1 < \alpha < n, \quad n \in \mathbb{N}.$$

**Proof.** Recalled the fractional derivative $f(x)$ in the Caputo sense as given by \[14\], then,

$$K [c^D_0^\alpha f(x)] = K \left[ I_0^\alpha f^n(x) \right].$$

Here, we need to show that

$$K \left[ I_x^{\alpha-\alpha} f(x) \right] = v^\alpha F(v),$$
so that
\[ I_\alpha^x f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}f(t)\,dt, \quad x > 0, \quad \alpha > 0, \]
and by convolution theorem,
\[ f \ast g(x) = \int_0^x f(x-t) \ast g(t)\,dt, \]
and so,
\[ \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}f(t)\,dt = \frac{1}{\Gamma(\alpha)} (x^{\alpha-1} \ast f(x)). \tag{3.1} \]
Then, equation (3.1) becomes
\[ I_\alpha^x f(x) = \frac{1}{\Gamma(\alpha)} (x^{\alpha-1} \ast f(x)). \tag{3.2} \]
Taking the Kamal transform of equation (3.2) gives
\[ K[I_\alpha^x f(x)] = \frac{1}{\Gamma(\alpha)} K[x^{\alpha-1} \ast F(v)] = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) v^{\alpha} \ast F(v). \]
Therefore,
\[ K[I_\alpha^x f(x)] = v^{\alpha} \ast F(v). \]
Now, if \( f^n(x) = g(x) \), then we have
\[ K[I_\alpha^x f^n(x)] = K[I_\alpha^x g(x)] = v^{n-\alpha} G(v). \tag{3.3} \]
Applying Theorem 3.1 on equation (3.3) gives
\[ K[I_\alpha^x f^n(x)] = v^{n-\alpha} \left( \frac{1}{v^n} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0) \right). \]
Therefore the Kamal transform of the Caputo fractional derivative is defined as:
\[ K[c D_\alpha^x f(x)] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} f^k(0). \]

4. The Kamal decomposition transform method on fractional differential equations
Consider the general fractional differential equation given as ([4])
\[ D_\alpha^x u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \quad t > 0, \quad 0 < \alpha \leq 1, \]
\[ u^k(x, 0) = r(x), \quad \forall x \in \mathbb{R}, \quad k = 1, 2, 3, \ldots, \tag{4.1} \]
where \( D_\alpha^x u(x, t) \) is the Caputo fractional derivative of order \( \alpha \), \( Ru(x, t) \) is the linear operator with fractional derivative order less than \( \alpha \), \( Nu(x, t) \) is a nonlinear operator which also might include other derivative of order less than \( \alpha \), and \( g(x, t) \) is known as the analytical function.
Taking the Kamal transform of equation (4.1) gives
\[ \mathcal{K} [D_x^\alpha u(x,t)] + \mathcal{K} [Ru(x,t) + Nu(x,t)] = \mathcal{K} [g(x,t)]. \] (4.2)

Simplifying equation (4.2) gives
\[ v^{-\alpha} \mathcal{K} [u(x,t)] - \sum_{k=0}^{n-1} v^{k-\alpha+1} u^k(0) + \mathcal{K} [Ru(x,t) + Nu(x,t)] = \mathcal{K} [g(x,t)]. \] (4.3)

Thus, equation (4.3) becomes
\[ \mathcal{K} [u(x,t)] - v^{-\alpha} \mathcal{K} [u(x,t)] - n = 1 \sum_{k=0}^{n-1} v^{k-\alpha+1} u^k(0) + v^{\alpha} K [Ru(x,t) + Nu(x,t)] = v^{\alpha} \mathcal{K} [g(x,t)]. \] (4.4)

Introducing the inverse Kamal transform on equation (4.4) gives
\[ u(x,t) = K^{-1} \left[ v^{\alpha} \left( \sum_{k=0}^{n-1} v^{k-\alpha+1} u^k(0) + \mathcal{K} [g(x,t)] \right) \right] + K^{-1} [v^{\alpha} \mathcal{K} [Ru(x,t) + Nu(x,t)]] \] (4.5)

Given that
\[ u(x,t) = h(x,t) + K^{-1} [v^{\alpha} K [Ru(x,t) + Nu(x,t)]], \]
where \( h(x,t) \) denotes the expression that arises from the given initial condition and the source term after simplification. The solution will be in the form of infinite series given as
\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \]

Decomposing the nonlinear term in equation (4.2) gives
\[ Nu(x,t) = \sum_{n=0}^{\infty} A_n, \]
where \( A_n \) is defined by the Adomian polynomial which can be calculated with the aid of formula [25, 26]. That is
\[ A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, \ldots. \]

Then from equation (4.5),
\[ u_0(x,t) = K^{-1} \left[ v^{\alpha} \left( \sum_{k=0}^{n-1} v^{k-\alpha+1} u^k(0) + \mathcal{K} [g(x,t)] \right) \right], \]
and the recursive relation is given as
\[ u_{n+1} = K^{-1} [v^{\alpha} \mathcal{K} [Ru(x,t) + A_n]]. \]

The analytical solution \( u(x,t) \) can be approximated by a truncated series
\[ u(x,t) = \lim_{N \to \infty} \sum_{n=0}^{N} u_n(x,t). \]
5. Applications

Problem 5.1. Consider the linear time fractional Klein-Gordon equation ([4])

\[ D_\alpha^x - u_{xx} - u = 0, \quad 1 < \alpha \leq 2, \quad (5.1) \]

with initial condition \( u(x, 0) = 0, u_t(x, 0) = x \). Taking Kamal transform to both sides of equation (5.1) gives

\[ K[D_\alpha^x] = K[u_{xx} - u]. \quad (5.2) \]

Equation (5.2) becomes

\[ K[u(x, t)] = v^\alpha \sum_{k=0}^{1} v^{k-\alpha+1} u^k(x, 0) - v^\alpha K \left[ \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) \right]. \quad (5.3) \]

Then, equation (5.3) becomes

\[ K[u(x, t)] = v^\alpha \left( v^{-\alpha+1} u(x, 0) + v^{2-\alpha} u'(x, 0) \right) - v^\alpha K \left[ \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) \right]. \]

Applying the initial conditions given, then we have

\[ K[u(x, t)] = v^\alpha \left( v^{2-\alpha} x \right) - v^\alpha K \left[ \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) \right]. \quad (5.4) \]

Introducing the inverse Kamal transform on equation (5.4) and simplifying gives

\[ u(x, t) = K^{-1} \left[ v^2 x \right] - k^{-1} \left[ \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) \right]. \quad (5.5) \]

From equation (5.5),

\[ u_0(x, t) = K^{-1} \left[ v^2 x \right] = tx. \]

Now the recursive relation is given as

\[ u_{n+1} = -k^{-1} \left[ v^\alpha K \left[ \frac{\partial^2 u_n(x, t)}{\partial x^2} + u_n(x, t) \right] \right]. \quad (5.6) \]

If \( n = 0 \) in equation (5.6), then

\[ u_1(x, t) = -k^{-1} \left[ v^\alpha K \left[ \frac{\partial^2 u_0(x, t)}{\partial x^2} + u_0(x, t) \right] \right]. \quad (5.7) \]

Simplifying equation (5.7) gives

\[ u_1(x, t) = -k^{-1} \left[ v^\alpha v^2 x \right] = v^\alpha tx = \frac{t^{\alpha+1} x}{\Gamma(\alpha+2)}. \]

If \( n = 1 \) in equation (5.6), then

\[ u_2(x, t) = -k^{-1} \left[ v^\alpha K \left[ \frac{\partial^2 u_1(x, t)}{\partial x^2} + u_1(x, t) \right] \right]. \quad (5.8) \]
Simplifying equation (5.8) gives

\[ u_2(x, t) = -k^{-1} \left[ \frac{v^\alpha(\alpha + 1)!xv^{\alpha+2}}{\Gamma(\alpha + 2)} \right] = \frac{v^\alpha t^{\alpha+1}x}{\Gamma(\alpha + 2)}. \]

Therefore

\[ u_2(x, t) = \frac{t^{2\alpha+1}x}{\Gamma(2\alpha + 2)}. \]

If \( n = 2 \) in equation (5.6), then

\[ u_3(x, t) = -k^{-1} \left[ v^\alphaK \left[ \frac{\partial^2 u_2(x, t)}{\partial x^2} + u_2(x, t) \right] \right]. \quad (5.9) \]

Simplifying equation (5.9) gives

\[ u_3(x, t) = -k^{-1} \left[ \frac{v^\alpha(2\alpha + 1)!xv^{2\alpha+2}}{\Gamma(2\alpha + 2)} \right] = \frac{-v^\alpha t^{2\alpha+1}x}{\Gamma(2\alpha + 2)}. \]

Therefore

\[ u_3(x, t) = -\frac{t^{3\alpha+1}x}{\Gamma(3\alpha + 2)}. \]

Thus, the series solution becomes

\[ u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots. \]

Then

\[ u(x, t) = xt - \frac{t^{\alpha+1}x}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}x}{\Gamma(2\alpha + 2)} - \frac{t^{3\alpha+1}x}{\Gamma(3\alpha + 2)} + \cdots. \quad (5.10) \]

If \( \alpha = 2 \) in equation (5.10), then the close solution becomes

\[ u(x, t) = x \sin t. \]

Figure 1: The left and right panel show the graphs of exact and Kamal transform method solution for \( \alpha = 2 \).
Problem 5.2. Consider the two-dimensional fractional wave equation ([15])

$$D_\alpha^x - \frac{y^2}{12} \frac{\partial^2 u}{\partial x^2} - \frac{x^2}{12} \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < \alpha \leq 2,$$

(5.11)

with initial condition $u(x, y, 0) = x^4, u_t(x, y, 0) = y^4$. Taking Kamal transform to both sides of equation (5.11) gives

$$K[D_\alpha^x] = K \left[ \frac{y^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{x^2}{12} \frac{\partial^2 u}{\partial y^2} \right].$$  

(5.12)

Equation (5.12) becomes

$$K[u(x, y, t)] = v^\alpha \sum_{k=0}^{1} v^{k-\alpha+1} u^k(x, y, 0) + v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u}{\partial y^2} \right].$$  

(5.13)

Then, equation (5.13) becomes

$$K[u(x, y, t)] = v^\alpha (v^{-\alpha+1} u(x, y, 0) + v^{2-\alpha} u'(x, y, 0)) + v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u}{\partial y^2} \right].$$

Applying the initial conditions given, then we have

$$K[u(x, y, t)] = (vx^4 + v^2y^4) + v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u}{\partial y^2} \right].$$  

(5.14)

Introducing the inverse Kamal transform on equation (5.14) and simplifying gives

$$u(x, y, t) = K^{-1} [vx^4 + v^2y^4] + K^{-1} \left[ v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u}{\partial y^2} \right] \right].$$  

(5.15)

From equation (5.15),

$$u_0(x, y, t) = K^{-1} [vx^4 + v^2y^4] = x^4 + ty^4.$$

Now the recursive relation is given as

$$u_{n+1} = K^{-1} \left[ v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u_n(x, y, t)}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u_n(x, y, t)}{\partial y^2} \right] \right].$$  

(5.16)
If \( n = 0 \) in equation \( (5.16) \), then
\[
 u_1(x, y, t) = K^{-1} \left[ v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u_0(x, y, t)}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u_0(x, y, t)}{\partial y^2} \right] \right]. \tag{5.17}
\]
Simplifying equation \( (5.17) \) gives
\[
 u_1(x, y, t) = K^{-1} \left[ v^\alpha K \left[ x^4 + ty^4 \right] \right] = K^{-1} \left[ v^\alpha K \left[ x^4 \right] \right] + K^{-1} \left[ v^\alpha K \left[ ty^4 \right] \right]. \tag{5.18}
\]
Then equation \( (5.18) \) becomes
\[
 u_1(x, y, t) = v^\alpha x + v^\alpha y^4 t = \frac{x^4 t^\alpha}{\Gamma(\alpha + 1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha + 2)}.
\]
If \( n = 1 \) in equation \( (5.16) \), then
\[
 u_2(x, y, t) = K^{-1} \left[ v^\alpha K \left[ \frac{x^2}{12} \frac{\partial^2 u_1(x, y, t)}{\partial x^2} + \frac{y^2}{12} \frac{\partial^2 u_1(x, y, t)}{\partial y^2} \right] \right]. \tag{5.19}
\]
Simplifying equation \( (5.19) \) gives
\[
 u_2(x, y, t) = K^{-1} \left[ v^\alpha K \left[ \frac{x^4 t^\alpha}{\Gamma(\alpha + 1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha + 2)} \right] \right]. \tag{5.20}
\]
Further simplification of equation \( (5.20) \) gives
\[
 u_2(x, y, t) = K^{-1} \left[ \frac{v^\alpha x^4 t^{\alpha+1}}{\Gamma(\alpha + 1)} + \frac{v^\alpha y^4 (\alpha + 1)! t^{\alpha+2}}{\Gamma(\alpha + 2)} \right] \tag{5.21}
\]
Then
\[
 u_2(x, y, t) = \frac{x^4 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{y^4 t^{2\alpha+1}}{\Gamma(2\alpha + 2)}.
\]
Thus, the series solution becomes
\[
 u(x, y, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots.
\]
Then
\[
 u(x, y, t) = x^4 + y^4 t + \frac{x^4 t^\alpha}{\Gamma(\alpha + 1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{x^4 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{y^4 t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \cdots. \tag{5.22}
\]
If \( \alpha = 2 \) in equation \( (5.21) \), then the close solution becomes
\[
 u(x, y, t) = x^4 \cosh t + y^4 \sinh t.
\]

**Problem 5.3.** Consider the three-dimensional diffusion equation \([31]\)
\[
 D_\alpha^x u(x, y, z, t) = \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2}, \quad 0 < \alpha \leq 1, \tag{5.22}
\]
with initial condition \( u(x, y, z, 0) = \sin x \sin y \sin z \). Taking Kamal transform to both sides of equation \( (5.22) \) gives
\[
 K \left[ D_\alpha^x u(x, y, z, t) \right] = K \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right]. \tag{5.23}
\]
Equation (5.23) becomes
\[
K [u(x, y, z, t)] = v^\alpha \sum_{k=0}^{0} v^{k-\alpha+1} u^k(x, y, z, 0) \\
+ v^\alpha K \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right].
\] (5.24)

Then, equation (5.24) becomes
\[
K [u(x, y, z, t)] = v^\alpha (v^{-\alpha+1} u(x, y, z, 0)) + v^\alpha K \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right].
\]

Applying the initial conditions given, then we have
\[
K [u(x, y, z, t)] = v \sin x \sin y \sin z + v^\alpha K \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right].
\] (5.25)

Introducing the inverse Kamal transform on equation (5.25) and simplifying gives
\[
u (x, y, z, t) = K^{-1} [v \sin x \sin y \sin z] + K^{-1} \left[ v^\alpha K \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right] \right].
\] (5.26)
From equation (5.26),

\[ u_0(x, y, z, t) = K^{-1} [v \sin x \sin y \sin z] = \sin x \sin y \sin z. \]

Now the recursive relation is given as

\[ u_{n+1} = K^{-1} [v^\alpha K \left( \frac{\partial^2 u_n(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u_n(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u_n(x, y, z, t)}{\partial z^2} \right)]. \]  \hspace{1cm} (5.27)

If \( n = 0 \) in equation (5.27), then

\[ u_1(x, y, z, t) = K^{-1} [v^\alpha K [-3 \sin x \sin y \sin z]] = K^{-1} [v^\alpha (-3v \sin x \sin y \sin z)]. \]  \hspace{1cm} (5.28)

Simplifying equation (5.28) gives

\[ u_1(x, y, z, t) = (-3 \sin x \sin y \sin z) v^\alpha = \frac{-3t^\alpha}{\Gamma(\alpha + 1)} \sin x \sin y \sin z. \]

If \( n = 1 \) in equation (5.16), then

\[ u_2(x, y, z, t) = K^{-1} [v^\alpha K \left( \frac{9t^\alpha}{\Gamma(\alpha + 1)} \sin x \sin y \sin z \right)]. \]  \hspace{1cm} (5.29)

Simplifying equation (5.30) gives

\[ u_2(x, y, z, t) = K^{-1} [v^\alpha K \left( \frac{9t^\alpha}{\Gamma(\alpha + 1)} \sin x \sin y \sin z \right)]. \]  \hspace{1cm} (5.31)

Further simplification of equation (5.31) gives

\[ u_2(x, y, z, t) = K^{-1} [v^\alpha K \left( \frac{9t^\alpha \alpha + 1}{\Gamma(\alpha + 1)} \sin x \sin y \sin z \right)]. \]

Then

\[ u_2(x, y, z, t) = \frac{9t^\alpha}{\Gamma(2\alpha + 1)} \sin x \sin y \sin z. \]

Thus, the series solution becomes

\[ u(x, y, z, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots. \]

Then,

\[ u(x, y, z, t) = \sin x \sin y \sin z - \frac{3t^\alpha}{\Gamma(\alpha + 1)} \sin x \sin y \sin z + \frac{9t^{2\alpha}}{\Gamma(2\alpha + 1)} \sin x \sin y \sin z + \cdots. \]  \hspace{1cm} (5.32)

If \( \alpha = 1 \) in equation (5.32), then the close solution becomes

\[ u(x, y, z, t) = e^{-3t} \sin x \sin y \sin z. \]
6. Discussion of results

The new scheme for solving fractional differential equations which are of Caputo form has been derived and the scheme was used to solve some set of time-linear fractional differential equations. It was observed that when $\alpha = 2$ and when $\alpha = 1$ in Problems 5.1, 5.2, and 5.3, respectively, there is a close solution between the exact and Kamal transform method.

Figures 1, 3, and 5 show the shape of the travelling wave when the differential equation is in classical form and it was observed that there is no change in the shape of the graph between that of the exact and the approximate solution.

Figures 2, 4, and 6 show the shape of the travelling wave at each derivative of $\alpha$ which are in fractional form, it was observed that at each derivative of $\alpha$, the shape of the travelling wave changes as compared with the shape of the classical solution. The features of changes in shape the graphs at each derivative of $\alpha$ which are in form of fraction will help the scientists and engineers to know the model which fit in correctly in prediction stage.

7. Conclusion

In this article, we have proved how Kamal transform is used on Caputo derivative. A scheme was explained and analyzed, and the scheme was used to solve different set of Time-fractional differential
equations. The solutions for certain illustrative problems are explained using the proposed scheme. The Kamal transform method result is in close contact with the exact solution of the suggested examples. The present scheme also calculate the solutions of the problems with the fractional order derivatives. Therefore, the proposed scheme can be extended to solve other complicated fractional order partial differential equations.

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