SELF-ADJOINT ELEMENTS IN THE PSEUDO-UNITARY GROUP $U(p, p)$

SACHIN MUNSHI* AND RONGWEI YANG

Abstract. The pseudo-unitary group $U(p, q)$ of signature $(p, q)$ is the group of matrices that preserve the indefinite pseudo-Euclidean metric on the vector space $\mathbb{C}^{p\times q}$. The goal of this paper is to describe the set $U_s(p, p)$ of Hermitian, or, self-adjoint elements in $U(p, p)$.

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1. Introduction

The pseudo-unitary group $U(p, q)$ is also often called an indefinite unitary group. It is a connected, non-compact Lie group defined by

$$U(p, q) := \{ M \in \text{GL}(n, \mathbb{C}) : M^* I_{p,q} M = I_{p,q} \},$$

with $I_{p,q} := \text{diag}(I_p, -I_q)$, where $I_p$ is the identity matrix of size $p \times p$, and $n = p+q$.

It is the complex analogue of the pseudo-orthogonal group

$$O(p, q) := \{ M \in \text{GL}(n, \mathbb{R}) : M^T I_{p,q} M = I_{p,q} \}.$$ 

In particular, $O(1, 3)$ is the Lorentz group, i.e. the group of Lorentz transformations associated to the Minkowski metric $\eta := \text{diag}\{1, -1, -1, -1\}$ for special relativity. So then $U(1, 3)$ may be viewed as the complex Lorentz group (13), and $U(p, q)$ as the generalized complex Lorentz group.

$U(p, q)$ also contains certain subgroups that are of particular interest to researchers in modern math and physics. One of them is the maximal compact subgroup $U(p) \oplus U(q)$ (11). As we shall see, this subgroup is precisely the set of unitary elements inside the pseudo-unitary group. Another subgroup, which is extensively studied in the physics literature, is

$$SU(p, q) := U(p, q) \cap \text{SL}(n, \mathbb{C}).$$

In particular, $SU(1, 1)$ is important for the coherent states of the harmonic oscillator and the Coulomb problem, in the study of path integrals in quantum mechanics (5). Moreover, $SU(2, 2)$ is the symmetry group of twistor space developed by Roger Penrose, and it is locally isomorphic to $SO(4, 2)$ (11). For more information about $U(p, q)$, and $U(p, p)$ in particular, we refer readers to 6, 11, 12, 16.

This paper aims to describe the set $U_s(p, q)$ of self-adjoint (Hermitian) elements in $U(p, q)$ for the case $p = q$. By the Cartan decomposition theorem (11, Theorem 3.4), every matrix $M \in U(p, q)$ is of the form $M = US$, where $U \in U(p) \oplus U(q)$

* Corresponding author.
and $S$ is self-adjoint. It is easy to check that $S \in \mathbf{U}_s(p, q)$, and hence there is the following factorization:

$$
\mathbf{U}(p, q) = (\mathbf{U}(p) \oplus \mathbf{U}(q)) \cdot \mathbf{U}_s(p, q).
$$

This indicates that the structure of $\mathbf{U}(p, q)$ is closely linked to that of $\mathbf{U}_s(p, q)$. However, self-adjoint elements in the classical Lie groups have not been explicitly described in any mathematics literature to date.

This paper is organized in the following manner. Section 2 consists of preliminary information and facts related to $\mathbf{U}(p, q)$; for instance, it leaves invariant particular Hermitian bilinear and quadratic forms ([9]). For this reason, we are primarily interested in the Hermitian elements inside the pseudo-unitary group. Therefore, in Section 3, we introduce the set $\mathbf{U}_s(p, q)$ of self-adjoint (Hermitian) matrix elements in $\mathbf{U}(p, q)$, and study the structure of these matrices through vector analysis and unitary transformations. As the main result of this paper, Theorem 3.7 gives a complete description for $\mathbf{U}_s(p, p)$. In Section 4, we study $\mathbf{U}_s(p, q)$ from the perspective of Lie algebras, and determine the range of the exponential map, again taking particular care of the case $p = q$. We give our concluding remarks in Section 5.

\section{2. Preliminaries}

Throughout this paper, $I_{p,q} = \text{diag}\{I_p, -I_q\}$ denotes the diagonal block matrix of size $n \times n$, $n = p + q$, where $I_p$ is the identity matrix of size $p \times p$, and $\mathbf{U}(p)$ is the group of unitary matrices of size $p \times p$, i.e., $U \in \mathbf{U}(p)$ if and only if $U^*U = I_p$, where $U^*$ is the conjugate transpose (or adjoint) of $U$. We now give the definition of the pseudo-unitary group of signature $(p, q)$.

**Definition 2.1.** The \textit{pseudo-unitary group} of signature $(p, q)$ is given by

$$
\mathbf{U}(p, q) = \{ M \in \text{GL}(p + q, \mathbb{C}) : M^* I_{p,q} M = I_{p,q} \},
$$

where $p + q = n$.

Note that $I_{p,q}^2 = I_n$ and for $M \in \mathbf{U}(p, q)$, we have $|\det M| = 1$. Moreover, $\mathbf{U}(p, 0) \cong \mathbf{U}(p)$, $\mathbf{U}(0, q) \cong \mathbf{U}(q)$, and $\mathbf{U}(q, p) \cong \mathbf{U}(p, q)$. A discussion on the structure of special types of matrices, including pseudo-unitary ones, can be found in [7]. For a viewpoint on the pseudo-unitary group that is more inclined toward pseudo-Euclidean geometry, see [11]. Without loss of generality, we shall assume $p \leq q$ in this paper.

**Definition 2.2.** The \textit{special pseudo-unitary group} of signature $(p, q)$ is given by

$$
\text{SU}(p, q) = \{ M \in \mathbf{U}(p, q) : \det M = 1 \}.
$$

We now mention a few facts about $\mathbf{U}(p, q)$ and $\text{SU}(p, q)$.

**Fact 2.3.** For $p + q = n$, we have that

$$
\mathbf{U}(p, q) \cap \mathbf{U}(n) = \mathbf{U}(p) \oplus \mathbf{U}(q).
$$

**Proof.** It is easy to see that $\mathbf{U}(p) \oplus \mathbf{U}(q) \subset \mathbf{U}(p, q) \cap \mathbf{U}(n)$. 

Writing \( M = \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right) \in U\,(p,q) \), where \( M_{11} \) and \( M_{22} \) are matrices of size \( p \times p \) and \( q \times q \), respectively, and using the identity \( M^*I_{p,q}M = I_{p,q} \) we have that
\[
\begin{align*}
M_{11}^*M_{11} - M_{21}^*M_{21} &= I_p, \\
M_{21}^*M_{21} - M_{22}^*M_{22} &= -I_q, \\
M_{11}^*M_{12} - M_{12}^*M_{22} &= M_{21}^*M_{11} - M_{22}^*M_{21} = 0.
\end{align*}
\]
Using identities in (2.5), one verifies that
\[
M^{-1} = \left( \begin{array}{cc} M_{11}^* & -M_{12}^* \\ -M_{21}^* & M_{22}^* \end{array} \right).
\]
If \( M \) is in the unitary group \( U\,(n) = U\,(p+q) \), then \( M^* = M^{-1} \). Therefore, it follows that \( M_{12} = M_{21} = 0 \), and hence, \( M \in U\,(p) \oplus U\,(q) \) by the first two equations in (2.4).

**Fact 2.4.** For any \( z, w \in \mathbb{C}^{p,q} \), both \( U\,(p,q) \) and \( SU\,(p,q) \) leave the following indefinite Hermitian bilinear and quadratic forms invariant:
\[
\begin{align*}
B_{p,q}(z,w) &= \sum_{i=1}^{p} \bar{z}_i w_i - \sum_{j=p+1}^{n} \bar{z}_j w_j, \quad (2.6) \\
Q_{p,q}(z) &= \sum_{i=1}^{p} |z_i|^2 - \sum_{j=p+1}^{n} |z_j|^2. \quad (2.7)
\end{align*}
\]

**Fact 2.5.** \( SU\,(1,1) \cong SL\,(2,\mathbb{R}) \) and moreover, \( \mathbb{D} \cong SU\,(1,1)/U\,(1) \), where \( \mathbb{D} \subset \mathbb{C} \) is the unit disk.

**Proof.** See [4, 9].

**Remark 2.6.** More quantum dynamical applications for general pseudo-unitary operators can be found in [10] and references therein.

### 3. Self-adjoint Elements in \( U\,(p,p) \)

Suppose we have a complex Hilbert space \( V \) with an inner product \( \langle \cdot, \cdot \rangle \). Let \( L : V \to V \) be a bounded linear operator. Then its adjoint \( L^* : V \to V \) is defined through the equation
\[
\langle Lu, v \rangle = \langle u, L^*v \rangle \forall u, v \in V. \quad (3.8)
\]
\( L \) is said to be Hermitian, or self-adjoint, whenever \( \langle Lu, v \rangle = \langle u, Lv \rangle \), which means that \( L = L^* \). For \( V \) finite-dimensional with a given orthonormal basis, this is the same as saying that the matrix associated to \( L \) is self-adjoint, i.e. equal to its conjugate transpose. We define \( U_s\,(p,q) \) to be the set
\[
U_s\,(p,q) = \{ M \in U\,(p,q) : M = M^* \}. \quad (3.9)
\]
Note that \( M \in U_s\,(p,q) \) in (3.9) satisfies the equation \( M^*I_{p,q}M = I_{p,q} \). Moreover, \( U_s\,(p,q) \) itself is not a group as the product of two Hermitian matrices is not necessarily Hermitian, unless the matrices commute. But, as we shall see, it contains
a nontrivial abelian group. The group $U(p) \oplus U(q)$ acts on $U_s(p, q)$. Indeed, if $U \in U(p), V \in U(q)$, then for every $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22}^* \end{pmatrix}$ in $U_s(p, q)$, we have

$$\hat{M} := \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22}^* \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} U^* M_{11} U & U^* M_{12} V \\ V^* M_{12}^* U & V^* M_{22} V \end{pmatrix}, \quad (3.10)$$

which is obviously self-adjoint. Moreover, one computes easily that

$$\hat{M} I_{p,q} \hat{M} = \text{diag}\{U^*, V^*\} M \text{diag}\{U, V\} I_{p,q} \text{diag}\{U^*, V^*\} M \text{diag}\{U, V\} = \text{diag}\{U^*, V^*\} M I_{p,q} \text{diag}\{U, V\} = I_{p,q}.$$ 

Moreover, if $M \in U_s(p, q)$, then clearly $-M \in U_s(p, q)$, hence the group $\{\pm 1\}$ acts on $U_s(p, q)$ as well. These observations justify the following.

**Definition 3.1.** Two elements $M_1, M_2 \in U(p, q)$ are said to be equivalent, and denoted by $M_1 \sim M_2$, if $M_1 = \pm M_2$ or $M_1 = Q M_2 Q$ for some $Q \in U(p) \oplus U(q)$.

To gain a better sense of the general case, we first look at the example of $U_s(1, 1)$.

**Example 3.2.** $p = q = 1$. Assume $M = \begin{pmatrix} m_{11} & m_{12} \\ \bar{m}_{12} & m_{22} \end{pmatrix} \in U_s(1, 1)$. Then by (2.5), we have

$$m_{11}^2 - |m_{12}|^2 = 1, \quad |m_{12}|^2 - m_{22}^2 = -1, \quad m_{12}(m_{11} - m_{22}) = 0. \quad (3.11)$$

Clearly, if $m_{12} = 0$, then $M = \text{diag}\{\pm 1, \pm 1\}$. If $m_{12} \neq 0$, then $m_{11} = m_{22}$, and we can write

$$M = \begin{pmatrix} c & s \\ \bar{s} & c \end{pmatrix},$$

where $s \neq 0$ and $c \in \mathbb{R}$ with $c^2 = 1 + |s|^2$. Up to an action by $U(1) \oplus U(1)$ as in (3.10), we may assume $s$ is real and $c \geq 1$. Letting $c = \cosh t$ and $s = \sinh t$, we have

$$M_t := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}. \quad (3.12)$$

Using identities of cosh and sinh functions, it is easy to check that

$$M_t M_{t'} = M_{t+t'}.$$

Hence, $G := \{M_t : t \in \mathbb{R}\}$ is a nontrivial abelian one-parameter group isomorphic to $(\mathbb{R}, +)$. $G$ can be viewed as a hyperbolic group that is studied in [2] [17]. It is easy to check that $M_{-t} = I_{11} M_t I_{11}$, hence $M_{-t} \sim M_t$. Let $G_+ = \{M_t : t \geq 0\}$. Then $G_+$ is an abelian semigroup. Moreover, since the trace $\text{Tr}(M_t) = 2 \cosh t$ is a strictly increasing function on the interval $[0, \infty)$, we see that no two distinct elements in $G_+$ are equivalent. Therefore, we have $U_s(1, 1) / \sim = G_+ \cup \{I_{11}\}$.

Further, if we let

$$U_{s+}(1, 1) := \{M \in U_s(1, 1) : \text{Tr}(M) \geq 2\},$$
then the above observations indicate that $U_s(1, 1)$ is the connected component of $U_s(1, 1)$ that contains the identity matrix $I_2$. Moreover, we have the disjoint decomposition
\[
U_s(1, 1) = U_s(1, 1) \cup -U_s(1, 1) \cup \{\pm I_{1,1}\}.
\]
Furthermore, it is not hard to see that $SU_s(1, 1)$ is also invariant under the multiplication by $\{\pm 1\}$ and the action by $U(1) \oplus U(1)$ as defined in (3.10). Hence, the quotient $SU_s(1, 1)/\sim$ is well-defined. Since $\det I_{1,1} = -1$ and $\det M = 1$ for all $M \in G_+$, we see that $SU_s(1, 1)/\sim = G_+$.

Now we move on to the general case. Consider an arbitrary $M \in U_s(p, q)$. The fact $MI_{p,q}M = I_{p,q}$ is equivalent to the following equation:
\[
(M - I_{p,q}) I_{p,q} (M + I_{p,q}) = 0. \tag{3.13}
\]
The Sylvester rank inequality then implies the following.

**Corollary 3.3.** If $M \in U_s(p, q)$, then $\text{rank}(M - I_{p,q}) + \text{rank}(M + I_{p,q}) \leq p + q$.

Without loss of generality, we shall assume $\text{rank}(M - I_{p,q}) \geq \text{rank}(M + I_{p,q})$, as otherwise we may work with $-M$. Then $k := \text{rank}(M + I_{p,q}) \leq q$. Since $M + I_{p,q}$ is Hermitian, using spectral decomposition as in [5], we may write
\[
M + I_{p,q} = \sum_{i=1}^{k} \lambda_i z_i z_i^*, \tag{3.14}
\]
where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the nonzero eigenvalues of $M + I_{p,q}$ whose eigenvectors $z_i$ are assumed orthonormal throughout this paper, i.e. $\langle z_i, z_j \rangle = z_i^* z_j = \delta_{ij}$. With slight algebraic tweaking we see that $MI_{p,q}M = I_{p,q}$ if and only if
\[
\sum_{i,j=1}^{k} \lambda_i \lambda_j (z_i^* I_{p,q} z_j) z_j^* - 2 \sum_{j=1}^{k} \lambda_j z_j z_j^* = 0. \tag{3.15}
\]
Since $\{z_j : 1 \leq j \leq k\}$ is an orthonormal set, applying the standard inner product on $\mathbb{C}^n$ with $z_j$ from the right to (3.15), we see that (3.14) is equivalent to
\[
\sum_{i=1}^{k} \lambda_i \lambda_j z_i (z_i^* I_{p,q} z_j) - 2 \lambda_j z_j = 0, \quad \forall 1 \leq j \leq k, \tag{3.16}
\]
and this is true if and only if
\[
\begin{cases}
z_i^* I_{p,q} z_j = 0; & i \neq j, \\
\lambda_j^2 z_j^* I_{p,q} z_j - 2 \lambda_j z_j = 0; & i = j.
\end{cases} \tag{3.17}
\]
Note that $z_i^* I_{p,q} z_j$ in (3.17), regardless of whether $i = j$ or not, is a scalar, not a matrix. From the second equation in (3.17), the nonzero eigenvalues of $M + I_{p,q}$ are $\lambda_j = \frac{2}{z_j^* I_{p,q} z_j}$.1

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1In the standard Euclidean inner product on $\mathbb{C}^n$
If we write vectors \( z_i = \left( \frac{z_i^+}{z_i^-} \right), \) where \( z_i^+ \in \mathbb{C}^p, z_i^- \in \mathbb{C}^q, \) then for \( i \neq j, \) from the first equation in (3.17), we have that
\[
\left( \frac{z_i^+}{-z_i^-} \right) \cdot \left( \frac{z_j^+}{z_j^-} \right) = \langle z_i^+, z_j^+ \rangle - \langle z_i^-, z_j^- \rangle = 0,
\]
where \( \langle , \rangle \) stands for the Euclidean inner product on \( \mathbb{C}^n. \) It follows that \( \langle z_i^+, z_j^+ \rangle = 0 = \langle z_i^-, z_j^- \rangle, \) so \( z_i^+, z_j^+ \) and \( z_i^-, z_j^- \) are orthogonal pairs for \( i \neq j. \) This means that \( \{z_j^+\}_{j=1}^k, \{z_j^-\}_{j=1}^k \) are orthogonal sets in \( \mathbb{C}^p, \mathbb{C}^q, \) respectively. Note that it is possible that for some \( j \) we have \( z_j^+ = 0 \) or \( z_j^- = 0. \) We summarize the above observations as follows.

**Lemma 3.4.** \( M \in U_s(p,q) \) if and only if there are orthogonal sets \( \{z_j^+\}_{j=1}^k \) and \( \{z_j^-\}_{j=1}^k \) in \( \mathbb{C}^p \) and \( \mathbb{C}^q, \) respectively, with \( k \leq q, \|z_j^+\|^2 + \|z_j^-\|^2 = 1, \) and \( \|z_j^+\|^2 \neq \|z_j^-\|^2 \) such that either \( M \) or \( -M \) is of the form
\[
\sum_{j=1}^k \lambda_j z_j z_j^* - I_{p,q},
\]
where \( z_j = \left( \frac{z_j^+}{z_j^-} \right) \) and \( \lambda_j = \frac{2}{\|z_j^+\|^2 - \|z_j^-\|^2}. \)

The following corollary is immediate.

**Corollary 3.5.** If \( M \in U_s(p,q) \) and \( \lambda \neq 0 \) is an eigenvalue of \( M + I_{p,q}, \) then \( |\lambda| \geq 2. \)

**Example 3.6.** We now look at the case \( p = 1, q = 2. \) Assume \( M \in U_s(1,2). \) Then there are four cases. If \( \text{rank}(M + I_{1,2}) = 0, \) then clearly \( M = -I_{1,2}. \) Next, \( \text{rank}(M + I_{1,2}) = 1 \) if and only if
\[
M = \lambda z z^* - I_{1,2},
\]
where \( z = \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) \) such that \( \|z\| = 1, \) and
\[
\lambda z^* I_{1,2} z = \lambda \left( |z_1|^2 - |z_2|^2 - |z_3|^2 \right) = 2.
\]
If \( \text{rank}(M + I_{1,2}) = 2, \) then by Corollary 3.3, \( \text{rank}(M - I_{1,2}) \leq 1, \) and hence, \( -M \) fits into the above discussion. Finally, if \( \text{rank}(M + I_{1,2}) = 3, \) then we have that \( \text{rank}(M - I_{1,2}) = 0, \) and hence, \( M = I_{1,2}. \)

Now we are in the position to state and prove the main theorem of this paper.

**Theorem 3.7.** For every integer \( p \geq 1 \) we have that
\[
U_s(p,p)_{\sim} = \left( G_+ \cup \{I_{1,1}\} \right) \oplus \left( G_+ \cup \{I_{1,1}\} \right) \oplus \cdots \oplus \left( G_+ \cup \{I_{1,1}\} \right).
\]
Proof. We show $M \in U_s(p, p)$ if and only if there are $M_j \in U_s(1, 1), 1 \leq j \leq p$, such that

$$M \sim M_1 \oplus M_2 \oplus \cdots \oplus M_p.$$ 

The theorem then follows from the discussion in Example 3.2.

In Lemma 3.4, since $\{z_j^+\}_{j=1}^k$ and $\{z_j^-\}_{j=1}^k$ are orthogonal sets, there exist orthonormal bases $\{e_j^+\}_{j=1}^p$ and $\{f_j^-\}_{j=1}^q$ for $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, such that

$$z_j^+ = \|z_j^+\| e_j^+, \quad z_j^- = \|z_j^-\| f_j^-, \quad 1 \leq j \leq k.$$ 

Then there exist $U \in U(p)$ and $V \in U(q)$ such that $U e_j^+ = e_j$ and $V f_j^- = f_j$, where

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^p, \quad f_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^q,$$

with 1 in the $j$-th position. Setting $\alpha_j = \|z_j^+\|$ and $\beta_j = \|z_j^-\|$, we have $\alpha_j^2 + \beta_j^2 = z_j^+ z_j^- = 1$. This allows us to consider $M + I_{p,q}$ under the equivalence $\sim$, i.e.

$$M + I_{p,q} \sim \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} (M + I_{p,q}) \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix}. \quad (3.21)$$

In view of Lemma 3.4, expanding out the RHS of (3.21), we have:

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} (M + I_{p,q}) \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} = \sum_{j=1}^k \lambda_j \begin{pmatrix} \alpha_j^2 e_j e_j^+ \\ \alpha_j \beta_j f_j e_j^+ \\ p \times p \\ \alpha_j \beta_j f_j e_j^+ \\ q \times q \end{pmatrix}.$$ 

(3.22)

Now we focus on the case $p = q$. Then $e_j = f_j$ so that, in view of (3.22), we have equations

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} (M + I_{p,p}) \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} = \sum_{j=1}^k \lambda_j \begin{pmatrix} \alpha_j^2 e_j e_j^+ \\ \alpha_j \beta_j e_j e_j^+ \\ p \times p \\ \alpha_j \beta_j e_j e_j^+ \\ q \times q \end{pmatrix} \quad (3.23)$$

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} I_{p,p} \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} = I_{p,p} = \sum_{j=1}^p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes e_j e_j^*, \quad (3.24)$$

where $U, V \in U(p)$. Note that the index for the summation in (3.23) is up to $k$, and that in (3.24) it is up to $p$. Since $P_j := e_j e_j^*$ is the rank 1 projection onto $\mathbb{C} e_j$, we have

$$\sum_{j=1}^P P_j = I_p, \quad P_i P_j = 0 \text{ for } i \neq j.$$

Letting $K_j = \begin{pmatrix} \alpha_j^2 & \alpha_j \beta_j \\ \alpha_j \beta_j & \beta_j^2 \end{pmatrix}$, and subtracting (3.21) from (3.23), we have

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} M \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} = \sum_{j=1}^k (\lambda_j K_j - I_{1,1}) \otimes P_j - \sum_{j=k+1} P_j.$$

(3.25)
Let
\[ K = \begin{pmatrix} \alpha^2 & \alpha \beta \\ \alpha \beta & \beta^2 \end{pmatrix}, \quad \lambda = \frac{2}{\alpha^2 - \beta^2}, \]
where \( \alpha, \beta \in \mathbb{R} \) satisfy \( \alpha^2 + \beta^2 = 1 \) and \( \alpha \neq \pm \beta \). Setting
\[ m_{11} = \frac{1}{2\alpha^2 - 1}, \quad m_{12} = \frac{2\alpha \beta}{\alpha^2 - \beta^2}, \quad m_{22} = \frac{1}{1 - 2\beta^2}, \]
one verifies that the matrix \( \lambda K - I_{1,1} = (m_{ij}) \) satisfies (3.11) and hence is in \( U_s(1,1) \). It then follows from (3.25) that
\[ M \sim \sum_{j=1}^{p} M_j \otimes P_j, \]
for some \( M_j \in U_s(1,1), 1 \leq j \leq p \).

Conversely, if \( M_j \in U_s(1,1), 1 \leq j \leq p \), and
\[ M = \sum_{j=1}^{p} M_j \otimes P_j, \]
then using (3.24), it is easy to check that \( MI_{p,p}M = I_{p,p} \), i.e., \( M \in U_s(p,p) \). The theorem thus follows from the discussion in Example 3.2. \( \square \)

Since \( G \) is an abelian one-parameter group in \( U_s(1,1) \), the above observations also yield

**Corollary 3.8.** For every integer \( p \geq 1 \), the set \( U_s(p,p) \) contains an abelian \( p \)-parameter group that is isomorphic to \( G \oplus G \oplus \cdots \oplus G \).

Like that in Example 3.2, it is not hard to see that \( SU_s(2,2) \) is also invariant under the multiplication by \( \{ \pm 1 \} \) and the action by \( U(2) \oplus U(2) \) as defined in (3.10). Hence, the next corollary follows readily from Theorem 3.7.

**Corollary 3.9.** \( SU_s(2,2) \)/\( \sim = (G_+ \oplus G_+) \cup \{ I_{1,1} \oplus I_{1,1} \} \).

### 4. \( U_s(p,q) \), Lie Algebras, and Exponential Map

In this section we take a look at \( U_s(p,q) \) from a Lie algebra point of view. Consider the Lie algebra \( u(p,q) \) of the Lie group \( U(p,q) \) and the associated exponential map \( \exp: u(p,q) \to U(p,q) \). For every \( T \in u(p,q) \), we consider the one-parameter subgroup
\[ M(t) = \exp(tT), \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad t \in \mathbb{R}. \] (4.26)

Since we need \( M(t) \in U(p,q) \), we have that
\[ \exp(tT^*)I_{p,q}\exp(tT) = I_{p,q}. \] (4.27)

Differentiating (4.27) with respect to \( t \), we have
\[ T^* \exp(tT^*)I_{p,q}\exp(tT) + \exp(tT^*)I_{p,q}T\exp(tT) = 0. \]
Evaluating the above at $t = 0$, we obtain the linear equation
\[ T^* I_{p,q} + I_{p,q} T = 0. \] (4.28)

This shows that the Lie algebra
\[ u(p, q) = \{ T \in M_n(\mathbb{C}) : T^* I_{p,q} = -I_{p,q} T \}. \]

Clearly, $\exp(T)$ is self-adjoint if and only if $T$ is self-adjoint. Hence, we set
\[ u_s(p, q) := \{ T \in u(p, q) : T = T^* \}. \]

Then $\exp(T) \in U_s(p, q)$ if and only if $T \in u_s(p, q)$. Substituting $T$ from (4.26) into (4.28) and solving $T = T^*$, we see that $T_{11} = T_{22} = 0$ and $T_{21} = T_{12}^*$. Letting $\hat{T} = T_{12}$, we have the following.

**Corollary 4.1.** Let $T \in M_n(\mathbb{C})$. Then the following are equivalent.
(a) $\exp(T) \in U_s(p, q)$.
(b) $T \in u_s(p, q)$.
(c) $T$ is of the form
\[ \begin{pmatrix} 0 & \hat{T} \\ \hat{T}^* & 0 \end{pmatrix}, \quad \hat{T} \in M_{pq}(\mathbb{C}), \] where $M_{pq}(\mathbb{C})$ is the algebra of $p \times q$ complex matrices.

Corollary 4.1(c), in particular, implies that $u_s(p, q)$ is of complex dimension $pq$. Since $\exp$ is a homeomorphism from an open neighborhood of $0 \in u_s(p, q)$ to an open neighborhood of $I \in U_s(p, q)$, the generic dimension of $U_s(p, q)$ is $pq$ as well.

By direct computation we have
\[ M = \exp(T) = \begin{pmatrix} \cosh |\hat{T}| & U \sinh |\hat{T}| \\ \sinh |T|U^* & \cosh |T| \end{pmatrix}, \] (4.29)

where $|A|$ stands for $\sqrt{A^*A}$ for any matrix $A$ and $T = U|T|$ is the polar decomposition of $T$. In view of Example 3.2, where we considered the case $p = q = 1$, we have
\[ T = \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} \in u_s(1, 1), \quad \xi = |\xi|e^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \] (4.30)

and using (4.29), we obtain
\[ \exp(T) = M = \begin{pmatrix} \cosh |\xi| & e^{i\theta} \sinh |\xi| \\ e^{-i\theta} \sinh |\xi| & \cosh |\xi| \end{pmatrix}. \] (4.31)

This calculation and Example 3.2 lead to the following.

**Corollary 4.2.** $\exp(u_s(1, 1)) = U_{s+}(1, 1)$. Moreover, it follows that
\[ \exp(u_s(1, 1)) / \sim = G_+. \]

For the general case, using the singular value decomposition, for $\hat{T} \in M_{pq}(\mathbb{C})$, there exist unitary $U \in M_p(\mathbb{C})$ and $V \in M_q(\mathbb{C})$ such that
\[ U^* \hat{T} V = \text{diag} \{ s_1, s_2, \ldots, s_p \}, \]
where $s_j \in \mathbb{R}$ are singular numbers of $T$. Note that it is possible $s_j = 0$ for some $j$. Then, we have

$$
\begin{pmatrix}
U^* & 0 \\
0 & V^*
\end{pmatrix}
T
\begin{pmatrix}
U & 0 \\
0 & V
\end{pmatrix}
=
\begin{pmatrix}
0 & U^*TV \\
V^*TU & 0
\end{pmatrix}
=
\begin{pmatrix}
0 & \text{diag} \{s_1, \ldots, s_p\}^* \\
\text{diag} \{s_1, \ldots, s_p\} & 0
\end{pmatrix}.
$$

So in the case $p = q$, using the $P_j$s from (3.24), we have that

$$
T \sim \sum_{j=1}^{p} \begin{pmatrix} 0 & s_j \\ s_j & 0 \end{pmatrix} \otimes P_j.
$$

Exponentiating the above equivalence identity gives us

$$
\exp(T) \sim \prod_{j=1}^{p} \begin{pmatrix} \cosh s_j & \sinh s_j \\ \sinh s_j & \cosh s_j \end{pmatrix} \otimes P_j.
$$

This gives another view on Theorem 3.7, although the map $\exp$ is not surjective. The next corollary follows from Corollary 4.2.

**Corollary 4.3.** For every integer $p \geq 1$, we have

$$
\exp(u_s(p,p)) / \sim = G_+ \oplus \cdots \oplus G_+.
$$

5. **Concluding Remarks**

The pseudo-unitary group $U(p, q)$ is a subject of many studies as in [3, 7, 9, 10, 11]. However, the structure of Hermitian (or self-adjoint) matrix elements inside this group is not known to have been thoroughly investigated. Using techniques from linear algebra, vector analysis, and Lie theory, Theorem 3.7 and Corollary 4.3 give an explicit picture of $U_s(p, p)$. The proof of Theorem 3.7, which relies on (3.23) and (3.24), doesn’t seem to have an easy generalization to the $p \neq q$ case.

While this paper has not developed a full-scale classification scheme for describing all of $U_s(p, q)$, we are encouraged and motivated by the results obtained so far. As a possible application of this paper, self-adjoint elements in $U(p, p)$ could be useful in the study of Clifford algebras which are widely used in the physics literature (see [6, 12, 14, 16]). For instance, the group $CU(p, q)$ of $c$-unitary elements in the complex Clifford algebra $Cl(p, q)$ is isomorphic to the pseudo-unitary group $U(2^{n-1}, 2^{m-1})$ if $q \neq 0$ and $p + q = 2m$ ([16]). More recently, links have been established between Clifford algebras of signature $(p, q)$ associated with $O(p, q)$ and certain equations at the heart of modern physics such as the Yang-Mills equations and the Proca equation ([8, 15]).

Although technically more challenging, extending the study of this paper to self-adjoint elements in $U(p, q)$ is a promising next step. Furthermore, describing self-adjoint elements in some other classical groups such as the indefinite symplectic group $Sp(p, q)$ and the spin group $Spin(p, q)$ is also an appealing subject of study. What is even more interesting, however, is to find applications of this study to Clifford algebras, group representation theory, geometry, or even quantum physics.
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