Mesons From String Theory

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A brief historical synopsis of the connection between gauge theories and string theory is given. Meson configurations known as $k$-strings are examined from string theory via the gauge/gravity correspondence. Backgrounds dual to $k$-strings in both $2 + 1$ and $3 + 1$ are discussed. The energy of $k$-strings to lowest order consists of a tension term, proportional to the length, $L$, of the $k$-string, i.e., the size of the mesons in the configuration. The first quantum correction is a Coulombic $1/L$ correction, known as a Lüscher term, plus a constant. Acquiring tensions and Lüscher terms via the gauge/gravity correspondence is discussed.

I. INTRODUCTION

From its inception, string theory has long thought to have a deep connection with the strong nuclear force. From a series of calculations by t’Hooft, Green, Schwarz, Maldacena, Herzog, Klebanov, et. al. [1, 2, 3, 4, 5, 6], today we see the connection through the gauge/gravity correspondence. As string theory can describe gauge theories in the case of open strings, and supergravity (SUGRA) in the case of closed strings, we can hope to find a map from SUGRA to gauge theories, known as the gauge/gravity correspondence. The specific case we will investigate is performing calculations done near the horizon of a SUGRA solution, and relating it to a strongly coupled gauge theory calculation.

The strongly coupled gauge theory example we use is the $k$-string, which we find particular SUGRA solutions to be dual to in either $2 + 1$ or $3 + 1$ dimensions. The $k$-string is an assemblage of fundamental strings, where the fundamental string is a quark and an anti-quark source connected by a color flux tube of length $L$, as described in [7]. For large quark separations, $L$, the energy of $k$-strings is dominated by the tension term, a term proportional to $L$. The lowest order correction is a Coulombic term, proportional to $1/L$. Both of these terms can be found from supergravity duals of $k$-strings. For $2 + 1$ $k$-strings, we will demonstrate this specifically in the background of Cvetic, Gibbons, Lu, and Pope (CGLP) [8]. We will compare this tension to preliminary results for another background dual to $2 + 1$ $k$-strings: the background of Maldacena and Nastase (MNa) [9]. We will also show results of $3 + 1$ $k$-string calculations from the background of Klebanov and Strassler (KS) [10].

II. SU($N$) k-STRINGS

A. Lüscher’s Fundamental String

Lüscher’s picture of the fundamental string is as in Figure 1. Lüscher found the energy of this configuration to be of the form

$$E = TL + \beta - \frac{\pi(d-2)}{24L} + \mathcal{O}(L^{-2}),$$

where $T$ is known as the tension, and $d$ is the dimension of spacetime. The $1/L$ correction term is the so-called, Lüscher term.

B. The $k$-string

Here is a quick overview of $k$-strings. A more complete review can be found in [11]. SU($N$) $k$-strings are an assemblage of fundamental strings, a close distance $d << L$ next to each other as in Figure 2 where $k = |l - m|$, $l$ being the number of quarks on one side of the $k$-string and $m$ being the number of anti-quarks on that same side.

For large $L$, $k$-strings exhibit $k$-ality: the tension vanishes whenever $k = N$. This $k$-ality is exhibited in models from lattice gauge theory [12, 13, 14], Hamiltonian methods [15, 16, 17], and supergravity calculations using the gauge/gravity correspondence [2, 18].
Two possible forms of the $k$-string tension, $T_k$, are

$$T_k \propto \begin{cases} N \sin \frac{k\pi}{N} & \text{sine law} \\ \frac{k}{N} & \text{casimir law} \end{cases}$$

where clearly either law exhibits $k$-ality. Table I compares $k$-strings tensions in $2 + 1$ dimensions calculated from various methods. From this data, it would seem that in $2 + 1$ dimensions, the casimir law is more appropriate. It also shows that the supergravity calculations in $2 + 1$ done here more closely align with the anti-symmetric quark representations.

**TABLE I:** Comparison of $k$-string tensions from various methods. The values quoted are $T_k/T_f$, where $T_k$ is the $k$-string tension, and $T_f$ is the fundamental string tension, i.e., $k = 1$. The CGLP tension is calculated from the transcendental Eqs. (26,27); MNa(Sine) and Casimir from Eq.(2). Data in quark representations: $S =$symmetric [17], $A =$antisymmetric [17], $M =$mixed [17], $*$ =antisymmetric [14]

| Group | $k$ | CGLP | MNa(Sine) | Casimir | lattice | Karabali-Nair |
|-------|----|------|-----------|---------|---------|---------------|
| $SU(4)$ | 2  | 1.310 | 1.414 | 1.333 | 1.355(A) | 1.332(A) |
| | 2  | 1.466 | 1.618 | 1.5 | 1.528* | 1.529* |
| $SU(5)$ | 2  | 1.562 | 1.732 | 1.6 | 1.617(A) | 1.601(A) |
| | 3  | 1.744 | 2.0 | 1.8 | 1.808(A) | 1.800(A) |
| $SU(6)$ | 2  | 1.674 | 1.848 | 1.714 | 1.752* | 1.741* |
| | 3  | 2.060 | 2.414 | 2.143 | 2.174* | 2.177* |
| | 4  | 2.194 | 2.613 | 2.286 | 2.366* | 2.322* |
| $SU(8)$ | 2  | 1.450 | 1.600 | 1.500 | 1.555(A) | 1.512(A) |
| | 3  | 1.846 | 2.000 | 2.100 | 2.125(A) | 2.110(A) |
| | 4  | 2.214 | 2.500 | 2.300 | 2.400* | 2.367* |

Furthermore, lattice calculations done by Brugholtz and Teper [14] find a Lüscher term in $2 + 1$ dimensional $SU(5)$ and $SU(4)$ gauge theory to be close to

$$-\frac{\pi}{6L},$$

the same value we calculated with the CGLP SUGRA model in [19] which is dual to a $2 + 1$ $SU(N)$ $k$-string configuration. At first glance, it may be troubling that these results are different from Lüscher’s, as seen in Eq. [1]. However, Lüscher’s result was for the fundamental string, which we do not expect to exhibit the precise behavior of a $k$-string, which is a series of fundamental strings “glued together”, as in Fig. 2.

**III. $k$-STRINGS FROM SUPERGRAVITY DUAL THEORIES**

Through the gauge gravity correspondence, we expect a string theory embedded in a SUGRA background to be dual to a gauge theory with a large number, $N$, of colors. Investigating $k$-string dual SUGRA solutions, Herzog and Klebanov [6, 18] considered an embedding as in Figure 3, where a probe D$p$-brane, either electrically($Q$) or magnetically($M$) charged

$$F = dA = Q dt \wedge dx + M d\theta \wedge d\phi,$$

is embedded in a classical SUGRA background, typically of the form

$$ds_{10}^2 = H_3 \eta_{\mu\nu} dx^\mu dx^\nu + \dot{H}_3 ds_{10-d}^2,$$

sourced by

$$F_{n+1}(X^\mu) = dC_n(X^\mu), \quad \Phi(X^\mu),$$

$$H_3(X^\mu) = dB_2(X^\mu),$$

where $t, x, \theta, \phi$ are four of the D$p$-brane coordinates, $\zeta^a$, and $X^\mu$ are the bosonic SUGRA coordinates. It is important to note here that $d$ is the spacetime dimension of the Minkowski spacetime portion of the metric in Eq. [5] which will be the spacetime dimension in which the $k$-string will be embedded in the gauge dual theory.

**FIG. 3:** A probe D$p$-brane embedded in a SUGRA background.

The SUGRA coordinates become fields on the D$p$-brane, the field theory dynamics governed by the D$p$-brane action

$$S_p = -\mu_p \int d^{p+1}x e^{-\Phi} \sqrt{-\det(g_{ab} + F_{ab})} +$$
brane action, yielding the Hamiltonian:

\[ H = \sum_{n} C_n \wedge \mathcal{F} + S_f \]  

(7)

where

\[ \mathcal{F}_{ab} = B_{ab} + 2\pi \alpha' F_{ab}, \quad \mu_p = (2\pi)^p(\alpha')^{(p+1)/2} \]  

(8)

and \( S_f \), which is a functional solely of fermionic fields, \( \Theta \), on the Dp-brane, is classically set to zero \[19, 20\]. Considering classical solutions \((A_0, X_0)\) where the only field with dynamics is the electric field component of \( F \),

\[ S_p = \int \mathcal{L}(A_0, \dot{A}_0, X_0) \]  

(9)

we can apply the Legendre transformation to the Dp-brane action, yielding the Hamiltonian:

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial A} \dot{A} - \mathcal{L} \]  

(10)

Minimization of this Hamiltonian leads to the \( k \)-string tension \[6, 18\]

\[ \mathcal{H}_{min} = T_k L \]  

(11)

The first quantum corrections are found by fluctuating around the classical solution

\[ X^\mu = X_0^\mu + \delta X^\mu, A^\mu = A_0^\mu + \delta A^\mu, \Theta = 0 + \delta \Theta, \]  

(12)

expanding out the action to second order in these fluctuations

\[ S_p = S_p^{(0)} + S_p^{(1)} + S_p^{(2)}, \]  

(13)

and calculating the free energy of the one loop corrections through

\[ e^{F_1T} = Z_2 = \int DXDAD\bar{\Theta}D\Theta e^{iS_p^{(2)}}. \]  

(14)

A. The CGLP Supergravity Background

First, we briefly review the CGLP SUGRA background. The complete details can be found in the original paper \[8\]. The CGLP background is a type IIA supergravity background, sourced by

\[ F_4 = g_s^{-1}d^3x \wedge dH^{-1} + m(f_i \epsilon_{ijk} \mu^j \epsilon^k + f_2 X_2 + f_3 X_3) \]  

(16)

\[ e^\Phi = g_s H^{1/4} \]  

(18)

with all other type IIA supergravity sources set to zero. In the above, we have

\[ X_2 = \frac{1}{2} \epsilon_{ijk} \mu^j \epsilon^k, \quad J_2 = \mu^i J_i \]  

(19)

\[ D\mu^i = \epsilon^{ijk} \epsilon^j \mu_k \]  

(20)

\[ J^i = dA_i + \frac{1}{2} \epsilon^{ijk} A_j \wedge A_k. \]  

(21)

and \( J^i \) satisfies the algebra of unit quaternions. With these sources, the background takes the form

\[ ds^2 = H^{-1/4}dx^a dx^b \eta_{ab} + H^{1/4}l^2 [h^2 dr^2 + a^2 (D\mu^i)^2 + b^2 d\Omega_2^2] \]  

(22)

where the bosonic supergravity coordinates are the set

\[ X^\mu = (x^0, x^1, x^2, r, \mu^1, \mu^2, \mu^3, \psi, \chi, \theta, \phi) \]  

(23)

with the constraint \((\mu^i)^2 = 1\). In the above, \( H, h, a, b, f_1 \) are functions of \( r \), and \( m \) and \( l \) are constants. In fact

\[ m = 8\pi \alpha'^{3/2} g_s N \]  

(24)

where \( N \) is the number of parallel D4-branes sourcing the background and is also the number of colors in the gauge theory dual.

B. SU(N) K-string Tension and Lüsher Term from CGLP Background

Here we show a brief outline of the full calculation of the tension and Lüsher term, which can be found in \[19\]. We use a probe D4-brane embedded in the CGLP SUGRA background. As the Minkowski spacetime portion of the CGLP background is 2+1 dimensional, the dual SU(N) \( k \)-string will be in 2+1 dimensions.

The classical action for a probe D4-brane in the CGLP background is

\[ S^{(0)} = -\mu_4 \int d^5 \zeta e^{-\Phi} \sqrt{-\text{det}(g_{ab} + 2\pi \alpha' F_{ab})} + \mu_4 \int C_3 \wedge F \]  

(25)

IV. K-STRING FROM CGLP SUPERGRAVITY BACKGROUND

Following the outline of the previous section, we summarize the work of \[18, 19\], where the 2+1 dimensional \( k \)-string energy was calculated as the dual of a D4-brane embedded in the CGLP background.
where $F$ is electrically charged. Constructing the Hamiltonian and minimizing with respect to the bosonic SUGRA coordinate $\psi$ yields

$$H_{\text{min}} = \alpha NL \sin^2 \psi_0 \sqrt{\sin^2 \psi_0 + (3\alpha/q)^2 \cos^2 \psi_0}$$

subject to the constraint

$$\frac{4k}{3N} = \xi(\psi_0) + 3\frac{\alpha^2}{q^2} \sin^2 \psi_0 \cos \psi_0$$

with

$$\xi(\psi_0) = \int_0^{\psi_0} \sin^3 u du$$

and where $\alpha$ and $q$ are constants with $\alpha/q \approx 0.3083$ and $\psi_0$ is the classical value of $\psi$ whose solution is the solution to the constant Eq. [27].

When we fluctuate about this classical solution by the method outlined in Eqs. [12], [13], and [14], we find the one loop energy to be

$$E_1 = -\frac{\pi}{6L} + \beta_3$$

which contains a term constant of $L$, $\beta_3$, which arises from the massive modes, plus a Lüscher term from the massless modes, $-\pi/6L$, which is the same as that found by lattice calculations of Bringholtz and Teper [14]. We find we can group this new calculation of the Lüscher term together with a previous calculation dual to $3 + 1$ $k$-strings [22], into a single formula

$$V_{\text{Lüscher}} = -\frac{(d + p - 3)}{24L}$$

where $d$ is the dimension of the Minkowski space-time portion of the SUGRA and also the dimension in which the dual $k$-string lives and $p$ is the spatial dimension of the probe D$p$-brane.

**V. CONCLUSION**

We have given a brief summary of the gauge/gravity correspondence and shown the method for calculating $k$-string tensions and Lüscher terms from the SUGRA side of this correspondence. The tension of $k$-strings in $2 + 1$ calculated from the CGLP background seems to align well with anti-symmetric quark representations, as shown in Table [1]. The Lüscher term found from the CGLP dual theory, Eq. [30], aligns well with Lattice calculations of Bringholtz and Teper [14]. Furthermore, our current findings for the Lüscher terms found in $2 + 1$ and $3 + 1$ can succinctly be written as in Eq. [30].

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