Two sum rules involving leading vector form factors in hyperon semileptonic decays

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Abstract. When the weak and electromagnetic currents are considered to be in the same SU(3) octet, it is possible to derive two sum rules involving leading vector form factors in hyperon semileptonic decays in the limit of exact flavor SU(3) symmetry. According to the Ademollo-Gatto theorem, deviations of these sum rules from this limit are expected to occur due to second-order flavor SU(3) symmetry-breaking effects, which are evaluated within the 1/Nc expansion of QCD. Accordingly, one sum rule acquires no deviations whereas the other one obtains contributions coming from the 10 + 10 representation. The sum rules are validated using results obtained in heavy baryon chiral perturbation theory. Up to order O(p^2) in the chiral expansion, the sum rules are fulfilled.

1. Introduction
Despite the enormous progress achieved in the understanding of the fundamental interactions with the Standard Model, first principles calculations of the properties of hadrons are not possible because the theory is strongly coupled at low energies. In order to resort this situation, several methods have been introduced to understand the low-energy QCD hadron dynamics and the 1/Nc expansion [1] has played a crucial role. This method promotes QCD to a SU(Nc) non-Abelian gauge theory, where Nc is the number of colors.

In this tenor, hyperon semileptonic decays (HSD) offer some challenges until today. In a recent paper [2], two sum rules involving the leading vector form factors f1 were derived by considering that the weak currents and the electromagnetic current are members of the same SU(3) octet. According to the Ademollo-Gatto theorem [3], the leading vector form factors f1 are protected against SU(3) symmetry breaking (SB) corrections to lowest order in m_s - m, where m denotes the mean mass of the up and down quarks. The purpose of the present paper is to provide some generalities about the method used to obtain those sum rules and to discuss some important implications.

The organization of this paper is as follows. In section II, the two sum rules involving leading vector form factors are derived by taking the matrix element of the vector current between eigenstates of the V2 operator. The resulting expressions are valid in the exact flavor SU(3) symmetry limit. Subsequently, the two sum rules are modified to account for SB. In section III, a brief review about how Ademollo and Gatto came up with their result on the nonrenormalization of the electric charge serve as an important factor to understand which
operator coefficients (since they are the ones responsible for SB effects) will be present in the expressions for $f_1$. Collecting all partial findings, the operators responsible for SB in the sum rules are properly identified. Next, results obtained in the frame of chiral perturbation theory are used to validate the sum rules. Up to $O(p^2)$ in the chiral expansion, the sum rules are satisfied. A validation beyond this chiral order is not currently possible because there are some partial results available. Finally, a numerical evaluation of the sum rules is also performed through a least-squares fit to data. Results are encouraging. Some closing remarks are given in section VI.

2. Sum rules for baryon vector form factors

The Cabibbo model for weak hadronic currents based on $SU(3)$ symmetry has been a key approach to describe HSD [4]. Since flavor $SU(3)$ symmetry is broken, its exact symmetry limit has been scrutinized using several methods in order to find discrepancies between theory and experiment and the leading weak form factors $f_1$ in HSD are the usual probes.

The starting point of the present analysis is the observation made by Pham about the matrix elements of any $SU(3)$ baryon octet into eigenfunctions of $V^2$ yields

\begin{equation}
\langle I = 1, I_3 = 0 | \pi \Gamma^n d | I = 1, I_3 = 1 \rangle = - \langle I = 1, I_3 = -1 | \pi \Gamma^n d | I = 1, I_3 = 0 \rangle,
\end{equation}

and

\begin{equation}
\langle I = 0, I_3 = 0 | \pi \Gamma^n d | I = 1, I_3 = 1 \rangle = \langle I = 1, I_3 = -1 | \pi \Gamma^n d | I = 0, I_3 = 0 \rangle,
\end{equation}

the rotated $V$-spin versions of the above relations become

\begin{equation}
\langle V = 1, V_3 = 0 | \pi \Gamma^n s | V = 1, V_3 = 1 \rangle = - \langle V = 1, V_3 = -1 | \pi \Gamma^n s | V = 1, V_3 = 0 \rangle,
\end{equation}

and

\begin{equation}
\langle V = 0, V_3 = 0 | \pi \Gamma^n s | V = 1, V_3 = 1 \rangle = \langle V = 1, V_3 = -1 | \pi \Gamma^n s | V = 0, V_3 = 0 \rangle,
\end{equation}

where the bilinear forms $\bar{q}_1 \Gamma^n q_2$ are given in terms of quark fields $q_k$ and the matrices $\Gamma^n_{\alpha \beta}$ are the usual ones that appear in applications of Dirac theory.

Decomposing the $SU(3)$ baryon octet into eigenfunctions of $V^2$ yields

\begin{equation}
| \Xi^- \rangle = | V = 1, V_3 = 1 \rangle,
\end{equation}

\begin{equation}
\left| \frac{1}{2} \Sigma^0 + \frac{\sqrt{3}}{2} \Lambda \right\rangle = | V = 1, V_3 = 0 \rangle,
\end{equation}

\begin{equation}
| p \rangle = | V = 1, V_3 = -1 \rangle,
\end{equation}

\begin{equation}
\left| \frac{\sqrt{3}}{2} \Sigma^0 - \frac{\Lambda}{2} \right\rangle = | V = 0, V_3 = 0 \rangle.
\end{equation}

Relations (2) are general enough to apply to matrix elements of any $SU(3)$ octet $\Delta S = 1$ operator. The Gell-Mann–Okubo mass formula is obtained from (2a) by relating $\langle B_2 | \pi LS | B_1 \rangle$ to $[f_1^{SU(3)}]_{B_1 B_2} (M_{B_2} - M_{B_1})$, where $f_1^{SU(3)}$ stands for the leading vector form factor at zero recoil in the limit of exact flavor $SU(3)$ symmetry and $M_{B_i}$ is the mass of the $B_i$ baryon. Applications to the axial-vector to vector form factor ratios $q_1/f_1$ are obtained when $\Gamma^n = \gamma_5 \gamma_\mu$ [5].

The vector current can be straightforwardly analyzed for $\Gamma^n = \gamma_\mu$. Consequently, there are two nontrivial expressions that relate the matrix elements of the vector current between baryon octet states, namely:

\begin{equation}
\left\langle \frac{1}{2} \Sigma^0 + \frac{\sqrt{3}}{2} \Lambda | \pi \gamma_\mu s \right| \Xi^- \rangle = - \left\langle p | \pi \gamma_\mu s | \frac{1}{2} \Sigma^0 + \frac{\sqrt{3}}{2} \Lambda \right\rangle,
\end{equation}

(7a)
Thus read are formally of order of the strangeness-violating vector currents \([3]\).

Following the assumption that the vector and electromagnetic currents are members of the same unitary octet and that the breaking of the unitary symmetry behaves as the eighth component \(\Sigma\), M. Ademollo and R. Gatto set up an important theorem on the nonrenormalization of an octet, namely, \(8\) requires \(k\) SB, which arise from SB, are formally of order \(O(\epsilon^2)\) according to the Ademollo-Gatto theorem. Obtaining \(\delta^{SB}\) requires and extra effort and it will be performed within the \(1/N_c\) expansion.

### 3. Some generalities about the Ademollo-Gatto theorem

Following the assumption that the vector and electromagnetic currents are members of the same unitary octet and that the breaking of the unitary symmetry behaves as the eighth component of an octet, M. Ademollo and R. Gatto set up an important theorem on the nonrenormalization of the strangeness-violating vector currents \([3]\).

In the Ademollo and Gatto formalism, the \(a\)th component of the current \(J^a\) to first-order in SB is written as

\[
\mathcal{J}^a + \epsilon \delta \mathcal{J}^a = a_0 \text{Tr}(\overline{B} B \lambda^a) + b_0 \text{Tr}(\overline{B} \lambda^a B) + c_0 \text{Tr}(\overline{B} \lambda^a B^8) + \ldots
\]
where $B$ represents the baryon matrix, $\lambda^a$ stand for the Gell-Mann matrices and $a_0$, $b_0$, $\ldots$, $h$ are coupling constants. The parameter $\epsilon$ is a reminder of the order SB is accounted for. At the end $\epsilon$ can be set to one.

The electromagnetic current is defined in the usual way as

$$J_{em} = J^Q \equiv J^3 + \frac{1}{\sqrt{3}} J^8,$$

so the baryon charges can be obtained from (12) without any difficulties. For instance, the neutron charge, including first-order SB is

$$Q_n + \epsilon \delta Q_n = -\frac{2}{3} a_0 - \frac{2}{3} b_0 + \frac{4}{3\sqrt{3}} \epsilon a - \frac{8}{3\sqrt{3}} \epsilon b + \frac{2}{\sqrt{3}} \epsilon c + \frac{2}{\sqrt{3}} \epsilon g + \frac{\sqrt{3}}{10} \epsilon h.$$

Similar relations can be obtained for the remaining baryons. Therefore, there are eight equations (one for each baryon charge) and seven parameters to be found. However, the isospin relation

$$\frac{1}{2} (Q_{\Sigma^+} + Q_{\Sigma^-}) = Q_{\Sigma^0},$$

leads to seven linear independent equations. Solving the system for the seven unknowns yields

$$a_0 = -\frac{1}{2}, \quad b_0 = \frac{1}{2}, \quad a = b = c = g = h = 0.$$

The solution of the system shows that the electromagnetic charge acquires no corrections to first-order SB (as it is expected since the electric charge remains unrenormalized to all orders in perturbation theory). As a direct consequence, the vector current is also protected against first-order SB. This is the original conclusion reached by Ademollo and Gatto [3].

4. Generalities on the $1/N_c$ expansion

The formalism on the $1/N_c$ expansion for baryons can be found in Ref. [8], so a few salient facts will be repeated. In the large-$N_c$ limit, the baryon sector has a contracted $SU(2N_f)$ spin-flavor symmetry, where $N_f$ is the number of light quark flavors [7]. The $1/N_c$ expansion is given in terms of $1/N_c$-suppressed operators with well-defined spin-flavor transformation properties. For $N_f = 3$, the lowest-lying baryon states fall into a representation of the spin-flavor group $SU(6)$.

The $1/N_c$ expansion of any QCD operator transforming according to a given $SU(2) \otimes SU(N_f)$ spin-flavor representation is expressed in terms of $n$-body operators $O_n$ as [8]

$$O_{QCD} = \sum_n c_n \frac{1}{N_n!} O_n,$$

where the operator coefficients $c_n$ are undetermined and have power series expansions in $1/N_c$, beginning at order one. The operators $O_n$ are polynomials in the spin-flavor generators $J^k$, $T^c$, and $G^{kc}$, which are 1-body operators acting on the baryon states. They read

$$J^k = \sum_{\alpha} q^\dagger_\alpha \left( \frac{\sigma^k}{2} \otimes I \right) q_\alpha,$$

$$T^c = \sum_{\alpha} q^\dagger_\alpha \left( I \otimes \lambda^c \right) q_\alpha,$$

$$G^{kc} = \sum_{\alpha} q^\dagger_\alpha \left( \frac{\sigma^k}{2} \otimes \lambda^c \right) q_\alpha.$$

Operators $q^\dagger_\alpha$ and $q_\alpha$ create and annihilate states in the fundamental representation of $SU(6)$ and $\sigma^k$ and $\lambda^c$ are the Pauli spin and Gell-Mann flavor matrices, respectively.
5. The leading baryon vector form factor in the $1/N_c$ expansion

In this section a brief description of the baryon vector current will be provided. The operator is constructed [2] in such a way that its matrix elements give the actual values of the leading vector form factor as defined in HSD.

To start with, let $V_{0c}$ denote the flavor octet baryon charge [9]

$$V_{0c} = \left\langle B_2 \left| \left( \frac{q_0}{2} \lambda^c \right)_{\mathrm{QCD}} \right| B_1 \right\rangle,$$

(18)

where the subscript QCD indicates that the quark fields are QCD quark fields [9]. $V_{0c}$ is a spin-0 and a flavor octet; it transforms as a $(0^+,8)$ under SU$(2) \otimes SU(N_f)$. The actual values of the leading vector form factors $f_i^{SU(3)}$ at zero recoil can be easily obtained by computing the appropriate matrix elements between baryon octet states for $c = 4 \pm i 5$.

In QCD, flavor SB is due to the strange quark mass $m_s$ and transforms as the eight component of an octet [8]. First-order SB corrections arise from the tensor product $(0,8) \otimes (0,8)$ so that the $SU(3) \otimes SU(3)$ representations to be included are $(0,1)$, $(0,8)$, $(0,10 + \bar{10})$ and $(0,27)$ [10].

Now let $V^c + \epsilon \delta V^c$ be the most general $(0,8)$ operator containing first-order SB. Its $1/N_c$ expansion is then

$$V^c + \epsilon \delta V^c = c_{(1)}^{8} T^c + c_{(2)}^{8} \frac{1}{N_c} \{ J^r, G^{rc} \} + \epsilon N_c Q_{(1)}^{8} \delta q^8 + \epsilon N_c q_{(1)}^{8} d^{8e} T^c + c_{(2)}^{8} \frac{1}{N_c} d^{8e} \{ J^r, G^{rc} \}$$

$$+ \frac{1}{N_c^2} \left[ \delta, \{ J^r, G^{rc} \} \right]$$

$$+ \epsilon a_{(3)}^{10 + \bar{10}} \frac{1}{N_c} \left[ \{ T^c, T^8 \} - \frac{N_f - 2}{2 N_f N_f - 1} N_c(N_c + 2 N_f) \delta q^8 - \frac{2}{N_f^2 - 1} \delta q^8 J^2$$

$$- \frac{N_f - 4}{N_f^2 - 4}(N_c + N_f) d^{8e} T^c - \frac{2 N_f}{N_f^2 - 4} d^{8e} \{ J^r, G^{rc} \} \right]$$

$$+ \epsilon a_{(3)}^{27} \frac{1}{N_c} \left[ \{ T^c, \{ J^r, G^{rc} \} \} + \{ T^8, \{ J^r, G^{rc} \} \} - \frac{4}{N_f(N_f + 1)}(N_c + N_f) \delta q^8 J^2$$

$$- \frac{2}{N_f + 2}(N_c + N_f) d^{8e} \{ J^r, G^{rc} \} - \frac{2}{N_f + 2} d^{8e} \{ J^r, T^c \} \right].$$

(19)

The expansion (19) has been truncated at the physical value $N_c = 3$, so up to three-body operators are retained. Also, singlet and octet representations are explicitly subtracted off from the two- and three-body operators of the $(0,27)$ representation to keep only the truly $(0,27)$ contributions. Finally, operator coefficients $c_{(n)}^{(rep)}$ accompany operators that belong to the exact flavor $SU(3)$ limit whereas $q_{(n)}^{(rep)}$ accompany $n$-body operators of the representation $\text{rep}$ that explicitly breaks flavor symmetry. Higher-order operators are constructed by anticommuting $J^2$ with the already existing ones as $O_{n+2} = \{ J^2, O_n \}$. These contributions are not included since they can be accounted for by redefining the operator coefficients.

The matrix elements of the operator $V^Q + \epsilon \delta V^Q$ between $SU(6)$ symmetric baryon states give the actual values for the baryon charges $Q_B$ including first-order SB. For example, at the physical values $N_c = N_f = 3$, the proton charge reads

$$Q_p + \epsilon \delta Q_p = -\frac{2}{3} a_0 + \frac{4}{3} b_0 + \frac{4}{3 \sqrt{3}} e a + \frac{4}{3 \sqrt{3}} e b - \frac{2}{\sqrt{3}} e c + \frac{2}{\sqrt{3}} e g - \frac{7\sqrt{3}}{10} e h.$$

(20)
Similar expressions are found for the other members of the baryon octet [2]. When analyzing the expressions for the full baryon charges, it is straightforward to notice that the coefficients of the \((0, 27)\) representation are not independent and a new coefficient involving \(a_{(2)}^{27}\) and \(a_{(3)}^{27}\) can be introduced, namely,

\[
x_{(2)}^{27} = a_{(2)}^{27} + \frac{2}{3} a_{(3)}^{27}.
\]  

(21)

Since the new coefficient \(x_{(2)}^{27}\) appears in the eight expressions for the baryon charges this reduces the number of unknowns in the linear system by one. Solving the system of linear equations yields

\[
c_{(1)}^8 = 1, \ c_{(2)}^8 = a_{(0)}^1 = a_{(1)}^8 = a_{(2)}^8 = a_{(3)}^{10+\bar{10}} = x_{(2)}^{27} = 0.
\]  

(22)

The above nicely reproduces the Ademollo and Gatto results using the \(1/N_c\) expansion formalism. Operator coefficients of (19) are related to the coupling constants introduced in (11) by

\[
a_0 = -\frac{1}{2} c_{(1)}^8 + \frac{1}{12} c_{(2)}^8,
\]  

(23a)

\[
b_0 = \frac{1}{2} c_{(1)}^8 + \frac{5}{12} c_{(2)}^8,
\]  

(23b)

\[
a = -\frac{1}{4} a_{(1)}^8 + \frac{1}{24} a_{(2)}^8,
\]  

(23c)

\[
b = \frac{1}{4} a_{(1)}^8 + \frac{5}{24} a_{(2)}^8,
\]  

(23d)

\[
c = \frac{1}{6} a_{(3)}^{10+\bar{10}}
\]  

(23e)

\[
g = \frac{3}{2} a_{(0)}^1
\]  

(23f)

\[
h = -\frac{1}{6} x_{(2)}^{27}.
\]  

(23g)

Notice a matching between the various coefficients and the representations they belong to.

The next step is to incorporate flavor SB at second-order into the \(1/N_c\) expansion for the
baryon charge operator. The most general expression is [2]

\[ e^2 \delta V^c = e^2 b_{(0)}^{(0)} N_c d^{88} + e^2 b_{(1)}^{(1)} N_c d^{88} T^8 + e^2 c_{(1)}^{8e} f^{8e} f^{8eg} T^g + e^2 g_{(1)}^{8e} d^{8eg} d^{8eg} T^g \]
\[ + e^2 h_{(1)}^{(1)} \left( i f^{ceg} d^{8eg} T^g - i d^{ceg} f^{8eg} T^g - i f^{ceg} d^{8eg} T^g \right) + e^2 b_{(2)}^{(2)} \frac{1}{N_c} \delta^{88} \{ J^c, G^{(8)} \} \]
\[ + e^2 c_{(2)}^{8e} f^{8e} f^{8eg} \frac{1}{N_c} \{ J^c, G^{(g)} \} + c^2 g_{(2)}^{8e} d^{8eg} d^{8eg} \frac{1}{N_c} \{ J^c, G^{(g)} \} \]
\[ + e^2 h_{(2)}^{(2)} \left( i f^{ceg} d^{8eg} - i d^{ceg} f^{8eg} - i f^{ceg} d^{8eg} \right) \{ J^c, G^{(g)} \} \]
\[ + e^2 b_{(3)}^{(3)} \frac{1}{N_c} \epsilon d^{8e} \left\{ \{ T^e, \{ J^c, G^{(8)} \} \} - \{ T^g, \{ J^c, G^{(8)} \} \} \right\} \]
\[ + e^2 c_{(3)}^{27} \frac{1}{N_c} \left[ d^{8e} \{ T^c, T^8 \} - \frac{N_f - 4}{N_f^2 - 4} (N_c + N_f) d^{8e} d^{8eg} T^g \right] \]
\[ = \frac{2}{N_f + 2} (N_c + N_f) d^{8e} d^{8eg} \{ J^c, G^{(g)} \} - \frac{2}{N_f + 2} d^{8e} d^{8eg} \{ J^2, T^g \} \]
\[ + e^2 b_{(4)}^{(4)} \frac{1}{N_c} \left[ \{ T^c, \{ T^8, T^8 \} \} - \frac{N_f - 2}{N_f (N_f^2 - 1)} N_c (N_c + 2 N_f) \delta^{88} T^c \right] \]
\[ - \frac{1}{2} f^{8e} f^{8eg} T^g \]
\[ - \frac{N_f - 4}{2 (N_f^2 - 4)} (N_c + N_f) d^{8e} \{ T^c, T^8 \} \]
\[ - \frac{N_f - 4}{2 (N_f^2 - 4)} (N_c + N_f) d^{8e} \{ T^c, T^e \} - \frac{2}{N_f^2 - 1} \delta^{88} \{ J^2, T^c \} \]
\[ - \frac{N_f}{N_f^2 - 4} d^{8e} \{ T^8, \{ J^c, G^{(8)} \} \} - \frac{N_f}{N_f^2 - 4} d^{8e} \{ T^c, \{ J^c, G^{(8)} \} \} \right] \] (24)

The matrix elements of \( e^2 \delta V^Q \) between \( SU(6) \) symmetric baryon octet states give second-order corrections to the baryon octet charges. For the proton the contribution reads

\[ e^2 \delta Q_p = -e^2 b_{(0)}^{1} + e^2 b_{(1)}^{8} + \frac{1}{12} e^2 b_{(2)}^{8} + \frac{1}{3} e^2 g_{(1)}^{8} + \frac{1}{6} e^2 g_{(2)}^{8} - \frac{1}{9} e^2 b_{(3)}^{10+10} + \frac{1}{15} e^2 b_{(2)}^{27} - \frac{2}{45} e^2 b_{(3)}^{27} + \frac{13}{120} e^2 b_{(4)}^{27} \] (25)

Similar expressions for the remaining baryon octet charges are found [2]. Again, the operator coefficients of the \( (0, 27) \) representation are not independent and a new coefficient that includes \( b_{(3)}^{27} \) and \( b_{(3)}^{27} \) can be defined as

\[ 2 b_{(2)}^{27} = b_{(2)}^{1} + 2 \frac{1}{3} b_{(3)}^{27} \] (26)

Only eight out of twelve operator coefficients that appear in (24) make up the octet baryon charges. Again, using the argument that the electric charge remains unrenormalized to all orders in perturbation theory, it is possible to solve the resulting linear system for the eight unknowns,
which yields

\[ b_1^{(0)} = b_2^{(0)} = g_8^{(2)} = y_8^{27} = 0, \]

\[ g_8^{(1)} = -\frac{5}{24} b_1^{10+\overline{10}}, \]

\[ b_8^{(1)} = \frac{13}{72} b_1^{10+\overline{10}}, \]

\[ b_6^{(3)} = \frac{5}{2} b_1^{10+\overline{10}}. \]

Hence, following the above results, the vector current at second-order in flavor SB can be written in terms of eight operators and five operator coefficients.

The matrix elements of \( V^{4\pm i8} + e^2 \delta V^{4\pm i8} \) between \( SU(6) \) baryon states yields the actual expressions for the leading vector form factors at second-order in flavor SB. For the observed \( \Delta S = 1 \) transitions, one has:

\[
\left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Lambda^p} = 1 + \frac{3}{4} e_8^{(1)} + \frac{3}{8} e_8^{(2)} + \frac{1}{18} b_1^{10+\overline{10}} - \frac{1}{10} y_8^{27},
\]

\[
\left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-n}} = 1 + \frac{3}{4} e_8^{(1)} - \frac{1}{8} e_8^{(2)} - \frac{1}{18} b_1^{10+\overline{10}} - \frac{1}{30} y_8^{27} + \frac{1}{30} b_6^{(3)},
\]

\[
\left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Xi^{-\Lambda}} = 1 + \frac{3}{4} e_8^{(1)} + \frac{5}{8} e_8^{(2)} - \frac{1}{18} b_1^{10+\overline{10}} + \frac{1}{30} y_8^{27} + \frac{1}{30} b_6^{(3)},
\]

\[
\left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Xi^{-\Sigma^0}} = 1 + \frac{3}{4} e_8^{(1)} - \frac{5}{8} e_8^{(2)} - \frac{1}{18} b_1^{10+\overline{10}} - \frac{1}{30} y_8^{27} + \frac{1}{30} b_6^{(3)},
\]

where the \( \tilde{e}_{(n)} \) coefficients are written in terms of those coefficients that come along with \( n \)-body octet operators.

Substituting the obtained expressions for the leading vector form factors (28)-(31) in (9a) and (9b) yields

\[ \delta_1^{SB} = \frac{1}{6} y_8^{27}, \quad \delta_2^{SB} = \frac{1}{6} b_1^{10+\overline{10}} + \frac{1}{20} b_6^{(3)}. \]

With the results obtained in (27), sum rules (9a) and (9b) become

\[
\frac{1}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-\Sigma^0}} + \frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-\Lambda}} - \frac{1}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-n}} - \frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Lambda^p} = 0,
\]

and

\[
\frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-\Sigma^0}} - \frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-\Lambda}} + \frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Sigma^{-n}} - \frac{3}{4} \left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Lambda^p} = -\frac{1}{24} b_1^{10+\overline{10}}.
\]

There are some interesting results here: Sum rule (33a) vanishes even in the presence of second-order SB, which is equivalent to state that corrections to it arise at third-order SB and should be, in principle, highly suppressed. On the other hand, sum rule (33b) gets corrections mainly from the \( 10+\overline{10} \) flavor representation.

In order to be conclusive about the above findings, sum rules (33) can be tested against theoretical results found in the literature. Several methods have been used to evaluate the
effects of SB in the leading vector form factors of HSD. One of the earliest is baryon chiral perturbation theory (BChPT). In this context, Krause [11] included calculations in relativistic BChPT to order $O(p^2)$ and Anderson and Luty [12] used heavy baryon chiral perturbation theory (HBChPT) up to (partially complete) order $O(p^3)$ in the chiral expansion. In both works, only spin-1/2 octet baryons were included as baryonic degrees of freedom. In a later work, Villadoro [13] computed (partially) complete corrections up to order $O(p^3)$ corresponding to subleading in $1/M_B$ terms; octet and decuplet baryon degrees of freedom were included in this work. Also, in the frame of covariant BChPT, Lacour, Kubis, and Meissner [14] used infrared regularization whereas Geng, Camalich, and Vicente-Vacas [15] used the extended on-mass-shell (EOMS) renormalization scheme. Both works included corrections to order $O(p^3)$; while the former included only spin-1/2 as baryonic degrees of freedom, the latter included both octet and decuplet baryons. Finally, another calculation within large-$N_c$ baryon chiral perturbation theory to order $O(p^2)$ has been provided in Ref. [16]. In this approach loop graphs with octet and decuplet intermediate states are systematically incorporated into the analysis because both spin-1/2 and spin-3/2 baryons together form an irreducible representation of spin-flavor symmetry.

A comparison with all the different analytical results mentioned above can readily be performed. Thus, sum rule (33a) is by the $f_1/f_1^{SU(3)}$ ratios given in Eqs. (65)–(68) of Ref. [16]. It is interesting to notice that the sum rule is also satisfied when the decuplet fields are not explicitly retained in the effective theory. Therefore, it is fulfilled by all the expressions for the $f_1/f_1^{SU(3)}$ ratios obtained within (heavy) baryon chiral perturbation theory to order $O(p^3)$ of [11], [13], [14], and [15]. The only exception is found in Ref. [12] where there is a wrong sign in one of the loop diagrams. Terms of subleading order in $1/M_B$ of Refs. [13], [14] and [15] also satisfy sum rule (33a). As for higher-order terms, for now it is not possible to perform a detailed analysis because only partial results are available.

Sum rule (33b), on the other hand, can be checked with the analytical expressions of Ref. [16]. Its right-hand-side yields

$$
\frac{3}{2} D^2 [H(m_K, m_\eta) - H(m_\pi, m_K)] - \frac{3}{8} C^2 [K(m_K, m_\eta, \Delta) - K(m_\pi, m_K, \Delta)],
$$

where $D$ and $C$ are the conventional $SU(3)$ invariant couplings, the functions $H(m_1, m_2)$ and $K(m_1, m_2, \Delta)$ arise from loop integrals and $\Delta$ is the decuplet-octet baryon mass difference. The numerical values of the quantities between square parentheses are small [16], of the order of 0.056 and 0.029, which in turn explains the smallness of the right-hand side of sum rule (33b).

Finally, an analysis of the available experimental data [17] can be performed by using the results obtained here. The experimentally measured quantities in HSD are the total decay rate $R$, angular correlation coefficients $\alpha_{e\nu}$, and angular spin-asymmetry coefficients $\alpha_e$, $\alpha_\nu$, $\alpha_B$, $A$, and $B$. Often, the data is presented in terms of $R$ and the ratio $g_1/f_1$ for the decay. The theoretical expressions for the total decay rates and angular coefficients can be found in Ref. [18], where radiative corrections and the four-momentum-transfer contribution to the form factors are also discussed.

For a numerical analysis, it is convenient to rewrite Eqs. (28)–(31) is a more compact way as

$$
\begin{align*}
\frac{f_1}{f_1^{SU(3)}}_{\Lambda p} &= 1 + \frac{3}{4} v_1 + \frac{3}{8} v_2 + \frac{11}{18} v_3, \\
\frac{f_1}{f_1^{SU(3)}}_{\Sigma^- n} &= 1 + \frac{3}{4} v_1 - \frac{1}{8} v_2 + \frac{1}{6} v_3, \\
\frac{f_1}{f_1^{SU(3)}}_{\Xi^- \Lambda} &= 1 + \frac{3}{4} v_1 + \frac{1}{8} v_2 + \frac{11}{18} v_3,
\end{align*}
$$

9
\[
\left[ \frac{f_1}{f_1^{SU(3)}} \right]_{\Xi - \Sigma} = 1 + \frac{3}{4} v_1 + \frac{5}{8} v_2 + \frac{1}{6} v_3,
\] (38)

where \( v_1 = \tilde{e}_{(1)} \) and \( v_2 = \tilde{e}_{(2)} \) and \( v_4 \) comprises the remaining coefficients from the \( 10 + \overline{10} \) operator coefficients.

Without further ado, a fit to data using information on \( R \) and \( g_1/f_1 \) yields

\[
v_1 = 0.002 \pm 0.036, \quad v_2 = -0.095 \pm 0.053, \quad v_3 = 0.91 \pm 1.01.
\] (39)

With these results sum rules (33) are well satisfied. However, future experimental measurements should provide a more stringent test of these sum rules.

6. Closing remarks

The main aspects of the derivation of two novel sum rules involving the ratios of the leading semileptonic vector form factors with SB effects to the ones in the symmetric limit, \( f_1/f_1^{SU(3)} \), have been presented. These sum rules, given in (33), reveal some interesting features. The first one, (33a), gets corrections at third-order in SB effects. This result is surprising, in view of the Ademollo-Gatto theorem. The second one, (33b), gets corrections mainly from the flavor \( 10 + \overline{10} \) representation. These sum rules can be tested with analytical expressions obtained within baryon chiral perturbation theory, at least at chiral order \( O(p^2) \) and some partial \( O(p^3) \) expressions available. The validity is fulfilled. A complete comparison with the full \( O(p^3) \) expressions is not possible for now. In the near future, lattice QCD results should also be available, and the sum rules could be checked.

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