Cartan normal conformal connections from pairs of second-order PDEs

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Abstract
We explore the different geometric structures that can be constructed from the class of pairs of second-order PDEs that satisfy the condition of a vanishing generalized Wünschmann invariant. This condition arises naturally from the requirement of a vanishing torsion tensor. In particular, we find that from this class of PDEs we can obtain all four-dimensional conformal Lorentzian metrics as well as all Cartan normal conformal $O(4,2)$ connections. To conclude, we briefly discuss how the conformal Einstein equations can be imposed by further restricting our class of PDEs to those satisfying additional differential conditions.

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1. Introduction
This work, which deals with general relativity from a non-conventional perspective, has several independent objectives.

One of them is to further demonstrate and develop the rich geometric structures that are buried in a large class of differential equations. Some of these structures have been known for a long time [1–3], while others are new. Here we will concentrate on the geometry associated with pairs of second-order PDEs with two independent and one dependent variables. Via the emerging structure, the discussion will then be narrowed down to a large special class of equations referred to as the generalized Wünschmann class. We will describe the differential geometry that is induced by the differential equations of this class on the four-dimensional solution space of the PDEs. They include the existence of all Lorentzian conformal geometries as well as Cartan normal conformal connections. Since all Lorentzian conformal metrics can be constructed from some pair of second-order PDEs, this means, in particular, that all metrics that are conformally related to vacuum Einstein metrics are included in this context or discussion.
However, in order to gain perspective on this work, we point out and summarize similar work that was done on a considerably simpler problem, namely the study of the class of all third-order ODEs. Cartan [4–7] and Chern [8] showed how to construct, from the differential equation, a triad and its associated connection on the three-dimensional solution space of a generic third-order ordinary differential equation of the form

\[ u''' = F(u, u', u'', s). \] (1)

The starting problem was to study the equivalence class of equations under the group of contact transformations on the \((u, u', s)\) space. One remarkable result that follows from the equivalence problem is that the equivalence classes of third-order ODEs split into two major classes: those with a vanishing Wünschmann invariant [9] and those with a non-vanishing invariant. The Wünschmann invariant, \(I[F]\), a differential expression involving \(F\) and its derivatives in all four variables, was discovered by early workers in the theory of differential equations and extensively used by Chern [8]. More specifically, when a third-order ODE satisfies \(I[F] = 0\), one shows that the solution space, i.e., the three-dimensional space of constants of integration, \((x^a)\), possesses, directly from the differential equation, equation (1), a conformal Lorentzian metric with the level surfaces of the solutions themselves, \(u = z(x^a, s)\), forming a one-parameter family of null surfaces. All members of the equivalence class, under contact transformations, yield the same conformal metric. The converse statement is also true; namely, given a three-dimensional conformal Lorentzian spacetime, from any complete solution \((u = U(x^a, s))\) of the eikonal equation, \(g^{ab} \partial_a U \partial_b U = 0\), one can obtain a third-order ODE (by differentiating with respect to \(s\) three times and eliminating the \(x^a\)) all belonging to the same equivalence class [11]. It was then shown by two different methods [4–8, 12] that from the third-order ODE with vanishing Wünschmann invariant one could generate on the solution space a Cartan normal conformal connection with group \(O(2, 3)\). The present work generalizes these ideas from third-order ODEs to pairs of second-order PDEs.

A second objective of this work is to prepare the basic structures needed to code the Einstein equations (or more accurately, the conformal Einstein equations) into the formalism of pairs of PDEs. We first point out that there are two uses to the term ‘conformal Einstein equations’; in one case there are differential equations solely for conformal classes of Lorentzian metrics such that there exists a conformal factor whose choice converts the class into a single vacuum metric that is a solution of the vacuum Einstein equations, and in the other case, the differential equations involve both the metric and the needed conformal factor. Our interest, at present, is only in the first case.

More than 20 years ago, the null surface formulation of GR [10, 12–15] was introduced as an alternative tool to study GR and in particular to capture the conformal degrees of freedom. The novelty of the approach consisted in using null surfaces as the main variables for the theory. The surfaces themselves were obtained as solutions of an integrable pair of second-order PDEs for sections of a line-bundle over the 2-sphere with a complex function \(\Lambda_1\) playing an analogous role to the above \(F\). (Here, we use \(S\) instead of \(\Lambda_1\).) The four-dimensional solution space of these equations emerged as the spacetime itself. An explicit algebraic method for constructing the conformal metric, on the solution space, was derived provided a certain differential condition on \(\Lambda_1\) was imposed. This condition, referred to then as the metricity condition, written as \(M = M[\Lambda]\), was essential for the formulation of the NSF. At the time we were not aware of the different geometric meanings of this newly discovered expression (even in the non-vanishing case) as a generalized Wünschmann invariant for the second-order PDEs under consideration. Explicitly, in terms of \(\Lambda_1\), we were able to construct, in addition to the conformal metric, objects such as the Weyl tensor, the Riemann tensor, the Levi-Civita connection, etc. Although it was not very elegant to mix the non-conformally
invariant connections with null surfaces for the construction of these objects, at that time it was not clear how to produce a completely conformally invariant formulation of conformal GR.

In this work we try to fill this gap. Starting with the pair of PDEs that define our main variables and without prior assumptions of a spacetime, we construct what is called a conformal Cartan connection on the solution space of these equations. A tetrad (that becomes null), defined from the PDEs, is introduced on the solution space. The requirement of a vanishing torsion uniquely fixes the associated connection and imposes a differential condition on the class of considered PDEs for our main variable. Further, it guarantees the existence of a metric on the solution space. The formalism is conformally covariant by construction; the non-trivial part of its curvature is the Weyl and generalized Cotton–York tensors. The Ricci tensor is coded into the Cartan normal conformal connection 1-form.

The conformal Einstein equations, in this new language, are to be differential conditions imposed on the pair of PDEs that define the characteristic surfaces. They are to be found by imposing conditions on the Cartan conformal curvature; both a cubic algebraic condition and a differential condition that is equivalent to the vanishing Bach tensor.

In section 2, we describe certain preliminary ideas and review earlier results. Our main results are presented in section 3 where we show how, from the first Cartan structure equation, we find a conformal connection with the restriction on the class of considered PDEs to the so-called generalized Wünschmann class. Many of the explicit detailed expressions and proofs are relegated to appendices due to their length. In section 4, we discuss the second Cartan structure equation and the Cartan curvature tensors. For clarity, in section 5 we give a brief synopsis of the earlier sections. Section 6 unifies the earlier material into a Cartan normal conformal connection. In section 7, we discuss the issues of obtaining the conformal Einstein equations.

2. Preliminaries

On a two-dimensional space with coordinates \((s, s^*)\), we consider the following PDEs

\[
Z_{ss} = S(Z, Z_s, Z_{s*}, Z_{ss*}, s, s^*), \quad Z_{s* s*} = S^*(Z, Z_s, Z_{s*}, Z_{ss*}, s, s^*),
\]

where the subscripts denote partial derivatives and \(Z\) is a real function of \((s, s^*)\). Though it would have been equally possible to treat \((s, s^*)\) as a pair of real variables it turns out to be more useful to consider them as a complex-conjugate pair. In that case the second equation is simply the complex-conjugate of the first equation. In the following, \((\,)^*\) will denote the complex-conjugate. By assumption, the functions \(S\) and \(S^*\) satisfy the integrability conditions. Solutions, \(Z = Z(s, s^*)\), are 2-surfaces in the six-dimensional space, \(J^6\), with coordinates

\[
(Z, W, W^*, R, s, s^*) \equiv (Z, Z_s, Z_{s*}, Z_{ss*}, s, s^*).
\]

For an arbitrary function \(H = H(Z, W, W^*, R, s, s^*)\), the total derivatives in the \(s\) and \(s^*\) are

\[
\frac{dH}{ds} \equiv DH \equiv H_s + W H_z + S H_W + R H_{W^*} + T H_R,
\]

\[
\frac{dH}{ds^*} \equiv D^* H \equiv H_{s^*} + W^* H_Z + R H_W + S^* H_{W^*} + T^* H_R,
\]

where

\[
T = D^* S, \quad T^* = DS^*.
\]
The $T$ and $T^*$ are explicit functions of $(Z, W, W^*, R, s, s^*)$ that are obtained in the
following way. Letting $H = S^*$ in equation (4) and $H = S$ in equation (5), we get two
equations containing $T$ and $T^*$. From them, we find

$$T = \frac{S_r + W*S_Z + RS_W + S^*S_W^*}{1 - S_R S_R^*} + \frac{S_R (S_r + W S_Z + S S_W^*) + R S_W^*}{1 - S_R S_R^*}.$$  (7)

Note that the $D$ and $D^*$ are actually the coordinate vectors $e_r$ and $e_r^*$, respectively. Hence,

$$e_r \equiv D = \frac{d}{dx} = \frac{\partial}{\partial s} + W \frac{\partial}{\partial Z} + S \frac{\partial}{\partial W} + R \frac{\partial}{\partial W^*} + T \frac{\partial}{\partial R},$$
$$e_r^* \equiv D^* = \frac{d}{dx^*} = \frac{\partial}{\partial s^*} + W^* \frac{\partial}{\partial Z} + S^* \frac{\partial}{\partial W} + S^* \frac{\partial}{\partial W^*} + T^* \frac{\partial}{\partial R}.$$  (8)

Often, for detailed calculations, the following identities are very useful. For $H = H(Z, W, W^*, R, s, s^*)$
and $y \in \{Z, W, W^*, R, s, s^*\}$

$$D(H_y) = (DH)_y - (S_y H_W + T_y H_R + \delta_{W,y} H_Z + \delta_{R,y} H_W),$$
$$D^*(H_y) = (D^* H)_y - (S_y^* H_W^* + T_y^* H_R + \delta_{W^*,y} H_Z + \delta_{R,y} H_W),$$  (9)

where $\delta_{\cdot,y}$ is the Kronecker symbol.

With these definitions of $D$ and $D^*$ the integrability conditions of (2) are

$$D^2 S^* = D^2 S.$$  (10)

In addition, the functions $S$ and $S^*$ are assumed to satisfy the weak inequality

$$1 - S_R S_R^* > 0.$$  (11)

From this inequality and the Frobenius theorem one can show [11] that the solutions depend
on four parameters, namely $x^a$, defining the spacetime manifold, $\mathbb{R}^4$, as the solution space of the
PDEs. We can thus write

$$Z = Z(x^a, s, s^*), \quad W = W(x^a, s, s^*), \quad W^* = W^*(x^a, s, s^*), \quad R = R(x^a, s, s^*).$$  (12)

Remark 1. The space $J^6$ is foliated by the integral curves of $D$ and $D^*$ which are labelled by $x^a$. The above
relations can be interpreted as $(s, s^*)$ dependent coordinate transformation between the $(Z, W, W^*, R)$ and the $x^a$.

The exterior derivatives of (12),

$$dZ = Z_a dx^a + W ds + W^* ds^*, \quad dW = W_a dx^a + S ds + R ds^*,$$
$$dW^* = W^*_a dx^a + R ds + S^* ds^*, \quad dR = R_a dx^a + T ds + T^* ds^*,$$  (13)

can be re-written as the Pfaffian system of four 1-forms

$$\beta^0 \equiv dZ - W ds - W^* ds^* = Z_a dx^a,$$
$$\beta^- \equiv dW^* - R ds - S^* ds^* = W^*_a dx^a,$$
$$\beta^+ \equiv dW - S ds - R ds^* = W_a dx^a,$$  (14)
$$\beta^1 \equiv dR - T ds - T^* ds^* = R_a dx^a.$$

The vanishing of the four $\beta^i$ is equivalent to the PDEs of equations (2), which motivates their definitions. For later use, we choose the equivalent set of 1-forms,

$$\theta^0 = \Phi \beta^0, \quad \theta^* = \Phi \alpha (\beta^* + b \beta^-),$$
$$\theta^- = \Phi \alpha (\beta^- + b^* \beta^+), \quad \theta^1 = \Phi (\beta^1 + a \beta^* + a^* \beta^- + c \beta^0).$$  (15)

We refer to the set $(a, b, b^*, a, a^*, c)$ as tetrad parameters and $\Phi$ as a conformal parameter. For
the moment they are undetermined functions of $(S, S^*)$ and their derivatives. Later, we
uniquely determine \((a, b, b^*, a, a^*, c)\) explicitly in terms of \((S, S^*)\) and impose conditions on \(\Phi\).

**Remark 2.** Note that one could generalize the \(\theta^i\) by including more parameters, i.e., by taking linear combinations of the \(\theta^i\). We will not do so, as the above definitions of the \(\theta^i\) are sufficient for our purposes. For the study of Cartan’s equivalence problem for pairs of second-order PDEs, the other parameters are needed. In either case, however, \(\bar{\beta}^0\) must be preserved up to scale. We will return to the issue of the other parameters in section 5 in relationship to the Cartan [8] normal conformal connection.

**Remark 3.** We impose two different conditions on \(\Phi_1\): (1) for intermediate or transitional use in the display of complicated expressions in the appendices we use \(\Phi_1 = 1\) and refer, in this case, to the \(\hat{\theta}^i\) as \(\hat{\theta}^i\); (2) a more basic choice is for \(\Phi_1\) to satisfy a certain differential equation that simplifies the structure of the conformal metric defined below. (For this choice see section 3.) We thus have that

\[
\theta^i = \Phi_1 \hat{\theta}^i, \tag{16}
\]

for the non-trivial \(\Phi_1\).

From equation (15), the dual basis vectors, \(e_i\), are

\[
e_0 = \Phi^{-1} \left( \partial_Z - c \partial_R \right), \quad e_+ = \Phi^{-1} \frac{\partial_W - b^* \partial_{W^*} - (a - a^* b^*) \partial_R}{\alpha(1 - bb^*)}, \tag{17}
e_- = \Phi^{-1} \frac{\partial_{W^*} - b \partial_W - (a^* - ab) \partial_R}{\alpha(1 - bb^*)}, \quad e_1 = \Phi^{-1} \partial_R.
\]

From equation (16),

\[
e_i = \Phi^{-1} \hat{e}_i. \tag{18}
\]

By adding the forms

\[
\theta^i \equiv \delta s, \quad \theta^s \equiv \delta s^*, \tag{19}
\]

which are the duals to the vectors \(e_i\), equation (8), to the four \(\theta^i\) defined above, we have a basis of 1-forms on the six-dimensional space \((Z, W, W^*, R, s, s^*)\). We will refer to \(\theta^0, \theta^*, \theta^-\) and \(\theta^1\) as the spacetime set of 1-forms, and we will refer to \(\theta^s\) and \(\theta^s^*\) as fibre 1-forms. We denote the spacetime 1-forms with the lower-case \(i, j\), etc, and denote all six 1-forms with the upper-case indices \(I, J\), etc. Thus,

\[
\theta^i \in \{\theta^0, \theta^*, \theta^-, \theta^1\}, \quad \theta^s \in \{\theta^0, \theta^*, \theta^- \theta^1, \theta^s, \theta^s^*\}. \tag{20, 21}
\]

Note that, in general, a \(p\)-form with tetrad indices will have components in all six dimensions. For example, the 1-form \(\Pi^i_j\) and the 2-form \(\Upsilon^i\) will have the respective expansions

\[
\Pi^i_j = \Pi^i_{jk} \theta^K = \Pi^i_{jk} \theta^k + \Pi^i_{js} \theta^s + \Pi^i_{js^*} \theta^{s^*}, \quad \Upsilon_i = \frac{1}{2} \Upsilon^i_j \theta^j \wedge \theta^K, \tag{22}
\]

\[
= \frac{1}{2} \Upsilon^i_{jk} \theta^j \wedge \theta^k + \Upsilon^i_{js} \theta^j \wedge \theta^s + \Upsilon^i_{js^*} \theta^j \wedge \theta^{s^*} + \Upsilon^i_{js^*} \theta^{s^*} \wedge \theta^s.
\]

For later use, we construct a metric to make \(\theta^i\) a null tetrad

\[
g(Z, W, W^*, R, s, s^*) = \theta^0 \otimes \theta^1 + \theta^1 \otimes \theta^0 - \theta^* \otimes \theta^- - \theta^- \otimes \theta^*, \tag{23}
\]

\[= \eta_{jj} \theta^j \otimes \theta^j.\]
This defines the constant flat metric \( \eta_{ij} \) as
\[
\eta_{ij} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\] (24)

From equation (11), it follows [10, 11] that the metric \( g \) is Lorentzian.

In addition, we also define the symmetric tensors \( G_{ij} \) and \( G^*_{ij} \) from the Lie derivatives of the metric in the \( e_s \) and \( e_s^* \) directions:
\[
\mathcal{L}_{e_s} g = G_{ij} \theta^i \otimes \theta^j, \quad \mathcal{L}_{e_s^*} g = G^*_{ij} \theta^i \otimes \theta^j,
\] (25)
and the exterior derivatives of the 1-form basis by
\[
d\theta^i = \frac{1}{2} \Delta^i_{JK} \theta^J \wedge \theta^K.
\] (26)

From this and
\[
\mathcal{L}_{e_s} \theta^i = e_s \lrcorner d\theta^i,
\] (27)
we have that the \( G_{ij} \) and the \( \Delta^i_{JK} \) are related by
\[
G_{ij} = -2 \Delta_{(ij)K}, \quad G^*_{ij} = -2 \Delta_{(ij)K^*},
\] (28)
where
\[
\Delta_{sJK} = \eta_{il} \Delta^l_{JK}.
\] (29)
The expressions for both \( G_{ij} \) and \( \Delta^i_{JK} \), in terms of \( S, S^*, \Phi \), the tetrad parameters and their derivatives, are given in appendix B.

3. The first structure equation

We begin by inserting our 1-forms, \( \theta^i \in \{ \theta^0, \theta^+, \theta^-, \theta^1 \} \), into Cartan’s torsion-free first structure equation,
\[
d\theta^i + \omega^j \wedge \theta^i = 0.
\] (30)

Our goal now is to solve this equation for the connection 1-forms, \( \omega^j \). To do so, write
\[
d\theta^i = \frac{1}{2} \Delta^i_{JK} \theta^J \wedge \theta^K,
\] (31)
\[
\omega^i = \omega_{ij} \theta^j,
\] (32)
defining the \( \Delta^i_{JK} \) and \( \omega_{ij} \). Note that
\[
\omega^i_k = \eta^{ij} \omega_{jk},
\] (33)
where \( \eta_{ij} \) is the flat metric defined in equation (24). The structure equation then becomes
\[
\frac{1}{2} \Delta^i_{JK} \theta^J \wedge \theta^K + \eta^{ij} \omega_{jk} \theta^j \wedge \theta^K = 0.
\] (34)

Since we are interested in the conformal geometry contained in the structure equation, we require that the connection 1-forms be generalized Weyl connections (’generalized’ because of the extra degrees of freedom in the fibre directions, \( s \) and \( s^* \)):
\[
\omega^i = \omega_{ij} + \omega_{i(j)}, \quad \omega_{i(j)} = \eta_{ij} A,
\] (35)
where the 1-form
\[
A = A_i \theta^i = A_t \theta^t + A_s \theta^s + A_{s^*} \theta^{s^*}
\] (36)
is the (generalized) Weyl 1-form.
In equations (15) we expressed our spacetime tetrad, $\theta^i$, in terms of $S$, $S^*$, and the unspecified tetrad parameters, $(a, b, b^*, a^*, c)$ and $\Phi$. Thus, we can explicitly compute the $\Delta'_{jkK}$ in terms of $S$, $S^*$, the tetrad parameters, $\Phi$ and their derivatives. (The explicit expressions for the $\Delta'_{jkK}$, as we said earlier, are given in appendix B.) Therefore, we will use equation (34) to solve for the connection coefficients, $\omega_{ijK}$, in terms of the $\Delta'_{jkK}$ and the undetermined $A_j$. In doing so, we will find several things: (i) the four spacetime components of the Weyl 1-form, $A_i$, remain arbitrary; (ii) the skew-symmetric part of the connection, $\omega_{[ij]}$, and the fibre parts of the Weyl 1-form, $A_s$ and $A_s^*$, are uniquely determined functions of $S$, $S^*$, $A_i$ and $\Phi$; (iii) the tetrad parameters are uniquely determined functions of $S$ and $S^*$ and (iv) there must be restrictions on the class of second-order PDEs to which $S$ and $S^*$ belong. The conditions of (iv) are known as the (generalized) Wünschmann condition and its complex-conjugate. (Note that when we say ‘Wünschmann condition’, we often mean both the condition and its conjugate.) These conditions are complex differential equations in all six variables of our six-dimensional space ($Z, W, W^*, R, s, s^*$).

We begin by splitting the structure equation into its fibre–fibre, tetrad–fibre and tetrad–tetrad components.

A. The fibre–fibre component contains no information since the term $\omega_{ijK} \theta_K \wedge \theta_j$ has no fibre–fibre parts and (using direct calculation and integrability conditions)

$$\Delta'_{ss^*j} \equiv 0.$$  (37)

B. The tetrad–fibre parts of the structure equation are

$$\omega_{ij} = \Delta_{ij}, \quad \omega_{ij^*} = \Delta_{ij^*}. \quad (38)$$

An important observation to make is that

$$\Delta_{ij} = \hat{\Delta}_{ij} - \eta_{ij} \Phi^{-1} D \Phi, \quad (39)$$

where $\hat{\Delta}_{ij}$ is defined by

$$d \hat{\theta}^i = \frac{1}{2} \hat{\Delta}_{jk} \hat{\theta}^j \wedge \hat{\theta}^k. \quad (40)$$

Symmetrizing on $(i, j)$ in equation (38) and using equations (35) and (39) yields

$$\eta_{ij} A_s = \Delta_{(ij)s} = \hat{\Delta}_{(ij)s} - \eta_{ij} \Phi^{-1} D \Phi, \quad \eta_{ij} A_{s^*} = \Delta_{(ij)s^*} = \hat{\Delta}_{(ij)s^*} - \eta_{ij} \Phi^{-1} D^* \Phi, \quad (41)$$

while the skew-symmetric parts give

$$\omega_{(ij)j^*} = \Delta_{(ij)j^*}, \quad \omega_{(ij)j^*} = \Delta_{(ij)j^*}. \quad (42)$$

Equations (41), uniquely determine $A_s$ and $A_{s^*}$ in terms of $S$, $S^*$ and $\Phi$ as

$$A_s = \frac{1}{2} \Delta^k_{ks} = \hat{A}_s - \Phi^{-1} D \Phi, \quad (43)$$

$$A_{s^*} = \frac{1}{2} \Delta^k_{ks^*} = \hat{A}_{s^*} - \Phi^{-1} D^* \Phi, \quad (44)$$

In addition, the trace-free part of equations (41),

$$\Delta_{(ij)s} - \frac{1}{4} \eta_{ij} \Delta^k_{ks} = \hat{\Delta}_{(ij)s} - \frac{1}{4} \eta_{ij} \hat{\Delta}^k_{ks} = 0,$$

$$\Delta_{(ij)s^*} = \frac{1}{4} \eta_{ij} \Delta^k_{ks^*} = \hat{\Delta}_{(ij)s^*} - \frac{1}{4} \eta_{ij} \hat{\Delta}^k_{ks^*} = 0, \quad (45)$$

imposes conditions on the $S$, $S^*$ and determines uniquely the tetrad parameters while $\Phi$ remains undetermined. Alternatively, from equation (28), i.e., $G_{ij} = -2 \Delta_{(ij)s}$, and by the definition

$$\hat{G}_{ij} = -2 \hat{\Delta}_{(ij)s}, \quad (46)$$
we have
\[ G_{ij} = \hat{G}_{ij} + 2\eta_{ij}\Phi^{-1}D\Phi, \]
from which we easily see that
\[ G^{TF}_{ij} = \hat{G}^{TF}_{ij} = 0, \]
where TF denotes the trace-free part.

The details for analysing equations (48) are quite involved and will be given in appendix C.

**Theorem 1.** From equations (41) and the relationship between the \( \Delta^k_{jk} \) and the \( G_{ij} \), i.e., equations (28), we have the results that (1) \( \mathcal{L}_{e_s} g = -2A_s g, \quad \mathcal{L}_{e_s^*} g = -2A_s^* g \)
and (2) the degenerate six-dimensional metric defined in equation (23) yields a conformal 4-metric in the solution space, with the motion along either \( e_s \) or \( e_s^* \) generating the conformal re-scaling of the metric.

**Corollary 2.** Up to this point \( \Phi \) has remained undetermined. There is however, via equation (43), a canonical way to chose it, namely by,
\[ D\Phi - \frac{1}{4}\Delta^k_{jk}\Phi = 0, \quad D^*\Phi - \frac{1}{4}\Delta^k_{jk}\Phi = 0, \]
where \( (s, s^*) \) is an arbitrary function on the space of fibres (the solution space, \( M^4 \)). The solution is thus of the form \( \Phi = \tilde{\Phi}_0 \) where
\[ \Phi_0 = \Phi_0[S, S^*] = \exp \left( \frac{1}{4} \left( \int \Delta^k_{jk} ds + \Delta^k_{jk} ds^* \right) \right), \]
where the integral is taken along an arbitrary path from any initial point to the final point \( (s, s^*) \). The needed integrability conditions are satisfied. Multiplication of the metric by \( \Phi^2 \) is the ordinary conformal freedom, \( g \Rightarrow \Phi^2 g \).

C. Returning to the tetrad–tetrad parts of the structure equations, we have that
\[ \Delta^i_{mn} + \eta^{ij}(\omega_{jmn} - \omega_{jnm}) = 0, \]
or
\[ \omega_{i[jk]} = \frac{1}{2}\Delta_{ijk}. \]
From the tensor identity
\[ \omega_{i[jk]} = \omega_{ijk} - \omega_{ijk} + \omega_{k[ij]} + \omega_{i[jk]} - \omega_{k[ij]} + \omega_{j[ik]}, \]
and equations (35) and (53), we obtain the tetrad–tetrad coefficients of the connection;
\[ \omega_{ijk} = \eta_{ij}A_k - \eta_{jk}A_i + \eta_{ki}A_j + \frac{1}{2}(\eta_{mi}\Delta^m_{jk} - \eta_{mk}\Delta^m_{ij} + \eta_{mj}\Delta^m_{ki}). \]
This decomposes naturally into a Levi-Civita part \( \gamma_{ijk} \) (which is independent of \( A_i \)) plus a ‘Weyl’ part, \( \tilde{\omega}_{ijk} \), i.e.,
\[ \omega_{ijk} = \gamma_{ijk} + \tilde{\omega}_{ijk}, \]
\[ \gamma_{ijk} = \frac{1}{2}\left(\eta_{mi}\Delta^m_{jk} - \eta_{mk}\Delta^m_{ij} + \eta_{mj}\Delta^m_{ki}\right). \]
\[ \tilde{\omega}_{ijk} = \eta_{ij} A_k - \eta_{jk} A_i + \eta_{ki} A_j. \]  
(58)

For clarity of exposition we point out where different variables are hidden. First note that \( \Delta_{imn} \) can be decomposed, as in equation (39), as
\[ \Delta_{imn} = \hat{\Delta}_{imn} \Phi^{-1} + 2 \varepsilon_{[m} \Phi \cdot \eta_{n]} \Phi^{-2}, \]  
(59)
where \( \hat{\Delta}_{imn} \) is the same expression as \( \Delta_{imn} \) but with \( \Phi = 1 \). The \( \hat{\Delta}_{imn} \) depends only on the \((S, S^*)\). This means that \( y_{ijk} \), equation (57), also decomposes into terms that depend on the \((S, S^*)\) and the \( \Phi \). Since \( \Phi = \sigma \Phi_0[S, S^*] \), with \( \sigma = \sigma(x^a) \), an arbitrary function of \( x^a \), carried along. This gauge or conformal freedom will be discussed later.

In summary, we have shown that equations (38) and (55) completely determine the \( \omega_{ijK} \) in terms of the \( \Delta_{iJK} \) and the undetermined spacetime parts of \( A_i \), i.e., \( A_i \).

3.1. A theorem

To conclude this section we return to the vanishing of the trace-free part of \( \Delta_{(ij)s} \) or the trace-free part of \( \hat{\Delta}_{(ij)s} \), i.e., to equation (45). They are nine complex equations for the determination of the tetrad parameters, \((a, a^*, b, b^*, c)\). Thus, either there must be several identities and/or conditions to be imposed on the \( S \) and \( S^* \). By explicitly solving these equations (see appendix C), the results can be summarized in the following theorem:

**Theorem 3.** The torsion-free condition on the connection:

1. uniquely determines the connection \( \omega_{ij} \), via equations (55) and (38).
2. uniquely determines the tetrad parameters in terms of \( S \) and \( S^* \) (see below).
3. imposes a (complex) condition, the vanishing of the Wünschmann invariant, on \( S \) and \( S^* \) (see below), with the tetrad parameters given by

\[ b = -1 + \sqrt{1 - S_R S_R^*}/S_R^*, \]  
(60)
\[ a^2 = 1 + bb^*/(1 - bb^*)^2, \]  
(61)
\[ a = b^{-1} b^* - (1 - bb^*)^{-2} (1 + bb^*) \{ b^* (-Db + b S_W - S_{W^*}) + b (-D^* b^* + b^* S_{W^*} - S_{W^*}) \}, \]  
(62)
\[ c = -\frac{D a + D^* a^* + T_W + T_{W^*}}{4} = \frac{aa^* (1 + 6bb^* + b^2 b^*^2)}{2(1 + bb^*)^2} + \frac{(1 + bb^*)(b S_Z^* + b^* S_{Z^*})}{2(1 + bb^*)^2} + \frac{a(2ab - b^* S_{W^*}) + a^* (2a^* b^* - b S_{W^*})}{2(1 + bb^*)}, \]  
(63)

and the differential (Wünschmann) condition imposed on \( S \) and \( S^* \) is
\[ M \equiv \frac{Db + b D^* b + S_{W^*} - b S_W + b^2 S_{W^*} - b^3 S_{W^*}}{1 - bb^*} = 0. \]  
(64)
4. The Cartan curvatures

In the previous section, we used the first structure equation, equation (30), to algebraically solve for the components of a torsion-free connection defined with the symmetry

$$\omega_{ij} = \omega^{ij} + \eta_{ij} A,$$

(65)

uniquely in terms of \((S, S^*)\) and the undetermined \(A_i\) and \(\sigma\).

Our next goal is to compute the curvature 2-forms, \(\Theta_{ij}\), defined by the second structure equation:

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Theta^i_j = \frac{1}{2} \Theta^i_{jLM} \theta^L \wedge \theta^M.$$

(66)

By taking the exterior derivative of the first structure equation, equation (30), with the second structure equation, equation (66), we obtain the first Bianchi identity:

$$\Theta_{ij} \wedge \theta^j = 0 \iff \Theta_{ij [LM]} = 0.$$

(67)

Splitting this into its tetrad–tetrad, tetrad–fibre and fibre–fibre parts, we have

$$\Theta_{ijkm} + \Theta_{ikmj} + \Theta_{imjk} = 0,$$

(68)

$$\Theta_{ij[k]} = 0,$$

(69)

$$\Theta_{ij[s]} = 0.$$  

(70)

The last two relations are due to the fact that the first two indices of \(\Theta_{ijLM}\) are only four dimensional, whereas the last two are six dimensional.

We calculate the \(\Theta_{ij}\) as explicit functions of \((S, S^*)\) and the undetermined \(A_i\) and \(\sigma\).

First, note that from equations (65) and (66), it is straightforward to see that the \(\Theta_{ij}\) inherits the symmetry of the \(\omega_{ij}\), and thus can be written as

$$\Theta_{ij} = \Theta^{ij} + \eta_{ij} dA,$$

(71)

with

$$dA = \frac{1}{2} (dA)_{LM} \theta^L \wedge \theta^M.$$

(72)

(This defines the components \((dA)_{LM}\).)

Next, we split the components \(\Theta_{ijLM}\) into tetrad–tetrad parts, \(\Theta_{ijkm}\), and tetrad–fibre parts, \(\Theta_{ij[s]}\). (The fibre–fibre parts are identically zero from the first Bianchi identity). We first calculate the \(\Theta_{ijkm}\), which can be split into terms arising from the Levi-Civita part of the connection and terms arising from the Weyl part of the connection. These are denoted respectively by \(\mathcal{R}_{ijkm}\) and \(\tilde{\Theta}_{ijkm}\).

$$\Theta_{ijkm} = \mathcal{R}_{ijkm} + \tilde{\Theta}_{ijkm},$$

(73)

The \(\mathcal{R}_{ijkm}\) are the components of the standard Riemann tensor of the (Levi-Civita) \(\gamma_{ijk}\) connection.

The \(\tilde{\Theta}_{ijkm}\) depends on \(A\) and its derivatives. Denoting the covariant derivative associated with the Levi-Civita connection \(\gamma_{ijk}\), by \(\nabla_i\), we have

$$\nabla_i A_j = e_i (A_j) - \gamma_{ijk} A_k,$$

(74)

and

$$dA_{ij} = 2\nabla_i [A_j].$$

(75)
\( \bar{\Theta}_{ij[km]} \) can, thus, be written as

\[
\frac{1}{2} \bar{\Theta}_{ij[km]} = \eta_{ij[k} \nabla_{\ell} A_{m]} - \eta_{ij[k} \nabla_{m] A_{\ell]} + A^2 \eta_{ij[k} \eta_{m]}} + A_{j} \eta_{ij[k} \eta_{m]} - A_{i} \eta_{ij[k} A_{m]}. \tag{76}
\]

where \( A^2 \equiv A^\mu A_\mu \).

Defining

\[
R_{jm} \equiv \eta^{ik} \Theta_{ijkm}, \tag{77}
\]

and using equations (73) and (76), we obtain

\[
R_{jm} = \Re_{(jm)} - \eta_{jm} \nabla_{p} A^p - 2 \{ \nabla_{(m} A_{j]} + \eta_{jm} A^2 - A_{j} A_{m} \} + 4 \nabla_{(j} A_{m)}, \tag{78}
\]

where the \( \Re_{(im)} \) are the components of the Ricci tensor of the \( \gamma_{jk} \).

If we also let

\[
R \equiv \eta^{im} R_{jm}, \tag{79}
\]

then from equation (78), we obtain

\[
R = \Re - 6 \{ \nabla_{p} A^p + A^2 \}, \tag{80}
\]

where \( \Re \) is the standard Ricci scalar.

The tetrad–fibre part of \( \Theta_{ij} \), which is

\[
\Theta_{ij[ks} = \eta_{ij} (dA)_{ks} + \eta_{ik} (dA)_{js} - \eta_{jk} (dA)_{is}, \tag{81}
\]

will be derived below.

### 4.1. The first Cartan curvature

The Cartan first curvature 2-form is given by

\[
\Omega_{ij} = \Theta_{ij} + \Psi_{i} \wedge \eta_{jk} \theta^{k} + \eta_{ik} \theta^{j} \wedge \Psi_{j} - \eta_{ij} \Psi_{k} \wedge \theta^{k}, \tag{82}
\]

where the (Ricci) 1-forms \( \Psi_{i} \) are appropriately chosen so that

\[
\Omega_{ij} = \frac{1}{2} \Omega_{ijLM} \theta^{L} \wedge \theta^{M}, \tag{83}
\]

satisfies the following conditions:

\[
\Omega_{ijkm} = \Omega_{ij[jkm}. \tag{84}
\]

\[
\eta^{ik} \Omega_{ijkm} = 0, \tag{85}
\]

\[
\Omega_{ijk} = 0. \tag{86}
\]

Note, from its definition, that \( \Omega_{ij} \) also satisfies the first Bianchi identity, equation (67), i.e.,

\[
\Omega_{ij} \wedge \theta^{j} = 0. \tag{87}
\]

It is straightforward to show that the conditions, equations (84), (85) and (86), are satisfied uniquely by the 1-form

\[
\Psi_{i} = \Psi_{iK} \theta^{K} = \Psi_{ij} \theta^{j} + \Psi_{is} \theta^{s} + \Psi_{i*} \theta^{*}, \tag{88}
\]

with

\[
\Psi_{ij} = - \frac{1}{2} R_{[ij]} - \frac{1}{4} \left( R_{[ij]} - \frac{1}{6} R \eta_{ij} \right), \tag{89}
\]

and

\[
\Psi_{i*} = -(dA)_{i*}, \quad \Psi_{is} = -(dA)_{is}. \tag{90}
\]
From equations (90), (82) and (86), we find equation (81), i.e.,
\[ \Theta_{ij} = \eta_{ij} (dA)_k + \eta_{ik} (dA)_j - \eta_{jk} (dA)_i . \]
Using equations (78) and (80), we obtain
\[ \Psi_{ij} = 3_{ij} - \nabla_i A_j - 2 \left\{ \nabla_i A_j + \frac{1}{2} \eta_{ij} A^2 - A_i A_j \right\} , \tag{91} \]
with
\[ 3_{ij} = - \frac{1}{2} \left( \Re_{ij} - \frac{1}{6} \Re \eta_{ij} \right) . \tag{92} \]
By using equation (91), we can insert the above expression into equation (82), yielding
\[ \Omega_{ijkm} = \Re_{ijkm} - \eta_{kj} \Im_{im} + \eta_{ki} \Im_{jm} - \eta_{mi} \Im_{jk} + \eta_{mj} \Im_{ik} , \tag{93} \]
which is manifestly independent of the \( A_i \). Furthermore, using equation (92), we recover the standard definition of the Weyl tensor,
\[ \Omega_{ijkm} = C_{ijkm} . \tag{94} \]

4.2. Second Cartan curvature

Finally, we define the second Cartan curvature (with the covariant exterior derivative) of the 1-form \( \Psi_i \) as
\[ \Omega_i = d \Psi_i + \Psi_k \wedge d \theta^k \equiv \Psi_i , \]
\[ = \frac{1}{2} \Omega_{iJK} \theta^J \wedge \theta^K . \tag{95} \]
Using equation (88) in the above, we obtain, after a lengthy calculation, the simple results
\[ \Omega_{imn} = \nabla^j C_{ijmn} + A^j C_{ijmn} , \tag{96} \]
and
\[ \Omega_{ims} = 0 , \quad \Omega_{iss} = 0 . \tag{97} \]
\( \nabla^j \) again is the Levi-Civita covariant derivative.

In section 6, the two Cartan curvatures will be used to construct the curvature of a conformal normal Cartan connection.

5. Synopsis

Since so many different quantities and their symbols have been introduced, we have added a few essentially pedagogical remarks concerning the placement of different variables and where the conformal transformation acts.

First we return to the definitions of \( \theta^i \), i.e., equation (15) and write them (and their related duals) as
\[ \theta^i = \Phi \tilde{\theta}^i , \tag{98a} \]
\[ e_i = \Phi^{-1} \tilde{e}_i , \tag{98b} \]
and observe that \( \Delta^i_{JK} \) and \( \tilde{\Delta}^i_{JK} \) and their relationship, equations (39) and (59), arise from
\[ d \tilde{\theta}^i = \frac{1}{2} \tilde{\Delta}^i_{JK} \tilde{\theta}^j \wedge \tilde{\theta}^K , \quad dd \tilde{\theta}^i = \frac{1}{2} \tilde{\Delta}^i_{JK} \tilde{\theta}^j \wedge \tilde{\theta}^K . \tag{99} \]
The action of \( \Phi \), taking \( \tilde{\theta}^i \Rightarrow \theta^i \), takes the metric
\[ \tilde{g} = \eta_{ij} \tilde{\theta}^i \otimes \tilde{\theta}^j \Rightarrow g = \Phi^2 \tilde{g} , \tag{100} \]
where \( g \) and \( \hat{g} \) both depend, in general, on \((s, s^*)\). However when we make the special choice \( \Phi = \sigma \Phi_0 \), equation (50), the resulting \( g \) is then a function on \( \mathbb{M}^4 \) alone. What remains is the standard conformal freedom that is given by the choice of \( \sigma(x^a) \). Whenever, in any expression, \( \sigma(x^a) \) is changed,

\[
\sigma(x^a) \Rightarrow f(x^a)\sigma(x^a),
\]

(101)

that change constitutes the effect of the conformal transformation.

All the further geometric quantities (the connection and different curvatures) developed and defined via the structure equations contain the following quantities: our basic variables \((S, S^*)\), the four arbitrary spacetime components of the Weyl 1-form \( A = A_i\theta^i \) where \( A_i = 0 \), and the conformal factor \( \sigma(x^a) \).

We give a brief survey of where these quantities appear:

(a) All \( \hat{\Delta}_{ijK} \) and \( \Delta_{ij}s \) depend only on \((S, S^*)\), while \( \Delta_{ijk} \) depends on \((S, S^*, \sigma)\). It is an easy task to see how \( \Delta_{ijk} \) transforms when \( \sigma \) is changed.

(b) Since \( \omega_{ij}s = \Delta_{ij}s \), it thus depends only on \((S, S^*)\).

(c) All the quantities \((S, S^*, A_i, \sigma)\) appear in the connection 1-forms \( \omega_{ijk} \), which can be split into

\[
\omega_{ijk} = \gamma_{ijk} + \tilde{\omega}_{ijk},
\]

(102)

where the Levi-Civita part, \( \gamma_{ijk} \), depends only on \((S, S^*, \sigma)\) while \( \tilde{\omega}_{ijk} \) depends only on the \( A_i \).

(d) The curvature \( \Theta_{ij[km]} \) splits into two parts

\[
\Theta_{ij[km]} = \mathfrak{H}_{ij[km]} + \tilde{\Theta}_{ij[km]},
\]

(103)

where the (standard) Riemann curvature \( \mathfrak{H}_{ij[km]} \) depends on \((S, S^*, \sigma)\) and \( \tilde{\Theta}_{ij[km]} \) depends on everything.

(e) The first Cartan curvature 2-form, \( \Omega_{ij} \), is the Weyl tensor, \( C_{ijmn} \), and depends only on \((S, S^*, \sigma)\).

(f) The second Cartan curvature 2-form,

\[
\Omega_i = \frac{1}{2} (\nabla^m C_{imijk} + A^m C_{imjki}) \theta^j \wedge \theta^k,
\]

(104)

depends on everything, though the \( A_i \) appears explicitly just in the linear term.

(g) Though the Ricci 1-forms,

\[
\Psi_i = \Psi_i\theta^j + \Psi_{is}\theta^s + \Psi_{irs}\theta^r,
\]

(105)

depend on everything, their separate parts do not. \( \Psi_{is} \) depends only on the \( A_i \). From

\[
\Psi_{ij} = \mathcal{C}_{ij} - \nabla^i A_j - 2 \left( \nabla_j A_i + \frac{1}{2} \eta_{ij} A^2 - A_i A_j \right),
\]

(106)

we have that \( \mathcal{C}_{ij} \) depend only on \((S, S^*, \sigma)\) while the remaining terms depend on everything. Whenever a geometric quantity depended only on \((S, S^*, \sigma)\), we could have said it depended on the Levi-Civita connection obtained from the metric \( g = \sigma^2 \Phi_0^2 \hat{g} \), equation (23), and referred to it as a conformal covariant.
6. Unification: Cartan’s normal conformal connection

From a pair of second-order PDEs satisfying the Wünschmann condition, we derived a rich geometric structure on the four-dimensional solution space of the PDEs. This structure includes: a conformal metric, a torsion-free connection and several different curvature tensors. Though it is not obvious, and the starting point of view is quite different, we are following Kobayashi’s [16] development of Cartan’s theory of normal conformal connections via the three structure equations, equations (30), (66) and (95), and the Ricci 1-forms, equation (88).

We now show that, basically, we have, in fact, recovered a (15-dimensional) principle bundle \( P \) over \( \mathbb{M}^4 \) with group \( H = CO(1,3) \oplus T^* \) and a Cartan normal conformal connection with values in the Lie algebra of \( O(4,2) \). The group \( H = CO(1,3) \oplus T^* \) is a 11-dimensional subgroup of \( O(4,2) \) [16] with \( T^* \) being a four-dimensional translation group. More specifically we have recovered a six-dimensional subbundle, \( J^6 \), of this 15-dimensional Cartan bundle. The base space is the four-dimensional solution space \( \mathbb{M}^4 \) and the two-dimensional fibres are formed by the integral curves of \( D \) and \( D^* \).

We began with the pair of second-order PDEs (that satisfy the Wünschmann condition) and a set of four associated 1-forms, \( \theta^i \), plus the two fibre 1-forms, \((ds, ds^*)\), on the six-dimensional space \( J^6 \). We then found the connection \( \omega_{ij} = \omega_{i(j)} + A_{ij} \), satisfying the first (torsion-free) structure equation
\[
d\theta^i + \omega^j_i \wedge \theta^j = 0, \tag{107a}
\]
where \( A_i = A_j = 0 \) and the four \( A_i \) are arbitrary. The first Cartan curvature 2-form was found via the second structure equation
\[
\Omega_{ij} = d\omega_{ij} + \eta^k_i \omega_k \wedge \omega_{ij} + \eta^j_i \theta^j \wedge \psi_j + \psi_i \wedge \theta^j \eta^j_l \eta^l \psi_k \wedge \theta^k = \frac{1}{2} C_{ijkl} \theta^j \wedge \theta^m, \tag{107b}
\]
with the proper choice of the Ricci 1-forms \( \psi_i \), equation (88). Finally the last structure equation and second Cartan curvature 2-form were introduced by
\[
\Omega_i = \psi_i = d\psi_i + \eta^k i \psi_j \wedge \omega_{ki} = \frac{1}{2} (\nabla^j C_{jmn} + A_j C_{jmn}) \theta^m \wedge \theta^n. \tag{109}
\]

The question or issue is what is the meaning of these resulting structures?

These equations can be unified in the following fashion: we first group together the 15 1-forms
\[
\omega = (\theta^i, \omega_{i(j)}, A, \psi_j), \tag{110}
\]
as the ‘Cartan connection’, represented by the \( 6 \times 6 \) matrix of 1-forms,
\[
\omega_B^A = \begin{bmatrix}
-A & \psi_i & 0 \\
\theta^i & \eta^k \omega_k & \eta^j \psi_j \\
0 & \eta^j \theta^j & A
\end{bmatrix}, \tag{111}
\]
and the curvature 2-forms (\( T^j \) is the vanishing torsion),
\[
R = (T^j = 0, \Omega^j, \Omega_i), \tag{112}
\]
as the 'Cartan curvature', represented by the $6 \times 6$ matrix of 2-forms,

$$R^A_B = \begin{bmatrix}
0 & \Omega_i & 0 \\
0 & \Omega^i_j & \eta^i_j \Omega^j_i \\
0 & 0 & 0
\end{bmatrix}. \tag{113}$$

One can then see, by a straightforward calculation, that remarkably the three structure equations, equations (107a), (107b) and (108), are all encompassed in the single Cartan structure equation

$$R^A_B = d\omega^A_B + \omega^A_C \wedge \omega_C^B. \tag{114}$$

One finds that the 1- and 2-form matrices $\omega^A_B$ and $R^A_B$ take their values in the Lie algebra of the 15-parameter group $O(4, 2)$ [16], though as forms they are in the six-space $J^6$. The $O(4, 2)$ Lie algebra is graded as

$$O(4, 2) = g_{-1} + g_0 + g_1,$$

with

$$\begin{align*}
\{ \theta^j \in g_{-1} \\
(A, \omega^k_i) & \in g_0 \\
\Psi_i & \in g_1
\end{align*} \tag{115}$$

Apart from the fact that the dimension count is not correct, i.e., the fibres are two dimensional and are not the required 11 dimensions, we have all the conditions for a Cartan $O(4, 2)$ normal conformal connection [16]. In addition to the correct Lie algebra, we have: the three structure equations, equations (107a), (107b) and (108), zero torsion, a trace-free first Cartan curvature $\Omega^i_j$ (the Weyl tensor) and a second Cartan curvature with the correct structure, i.e., vanishing fibre parts. It is clear that we are dealing with a six-dimensional subbundle of the full 15-dimensional bundle. The fibres should be coordinatized by the 11-dimensional subgroup $H = CO(3, 1) \otimes T^{*4}$ of $O(4, 2)$. The question is where and what are the missing coordinates needed to describe $H$?

The 11 coordinates (or parameters) must be such that when acting on the conformal metric, equation (23), it is left conformally unchanged. Actually we already have seven parameters, namely ($s, s^*, \varphi, A_i$); we have from equation (51) that variations in $s$ and $s^*$ or multiplication by $\varphi$ leaves equation (23) conformally unchanged. The $A_i$ have no relationship with the metric.

The other four parameters could be chosen as follows:

(a) Rescaling $\theta^*$ and $\theta^-$ respectively, by $e_{i\psi}$ and $e^{-i\psi}$ leaves equation (23) unchanged. (The parameters ($s, s^*, \varphi, A_i$) describe $O(3)$ transformations.)

(b) One could take three-parameter, $(\gamma, \gamma^+, \mu)$, linear combinations of the four $\theta^i$ that constitute Lorentz transformations with no change in the metric. For example, we could have the $(\theta^0, \theta^1)$ boost,

$$\theta^0' = \mu \theta^0, \quad \theta^1' = \mu^{-1} \theta^1, \tag{116}$$

and the null rotations given by,

$$\begin{align*}
\theta^0 &= \theta^0, \tag{117} \\
\theta^* &= \theta^* + \gamma \theta^0, \tag{118} \\
\theta^- &= \theta^- + \gamma^* \theta^0, \tag{119} \\
\theta^1 &= \theta^1 + \gamma^\theta^- + \gamma^* \theta^+ + \gamma^* \theta^0 + \gamma^* \theta^0. \tag{120}
\end{align*}$$
The seven \((s, s^*, \psi, \gamma, \gamma^*, \mu, \omega)\) parametrize the conformal Lorentz group, \(CO(3, 1)\) while \(A_i\) parametrize \(T^{\ast 4}\). With the exception of \((s, s^*)\) all the remaining parameters, \((\psi, \gamma, \gamma^*, \mu, \omega, A_i)\), are chosen as arbitrary functions on \(\mathfrak{M}\).

It would have been possible to start with a generalized version of the \(\theta^i\) (see remark 2) so that all 11 parameters in our equations appeared at the start. In the present work, we have effectively taken a six-dimensional subbundle (two-dimensional fibres) by choosing \(\gamma = \gamma^* = \psi = 0, \mu = 1\), with \(\omega\) being an arbitrary but given function on \(\mathfrak{M}\) and \(A_i\) being four arbitrary functions on \(J^6\). Since the \(A_i\) are arbitrary it is possible (and probably more attractive) to choose them to also be functions just on \(\mathfrak{M}\). Only the \((s, s^*)\) can vary on each fibre.

7. Conclusion

The work presented here addresses the issue of how four-dimensional differential geometry can be coded into pairs of second-order PDEs. More specifically, we have shown how all Cartan \(O(4, 2)\) normal conformal connections can be so coded. Our ultimate goal, however, is a step beyond this; we want to know how to code the conformal Einstein equations into such pairs of second-order PDEs. This would mean further conditions, over and above the Wünschmann condition, on the choice of the \(S\) and \(S^*\). Though at present we do not know the details of these ‘further conditions’. Nevertheless, there is a clear strategy for their determination.

It appears that they can be expressed as the vanishing of two (or three) different functionals of \(S(Z, Z_s, Z_s^*, s, s^*)\) and \(S^*(Z, Z_s, Z_s^*, Z_s, s, s^*)\). These functionals can be derived from the vanishing of the Bach tensor \([17]\) and an algebraic restriction on the Cartan curvature \(R^{ab}\) \([18]\), i.e., from

\[
\begin{align*}
\nabla^m \nabla^n C_{mabn} + \frac{1}{2} R^{mn} C_{mabn} &= 0, \\
C_{efgh} [C_{efgh} \nabla^d C_{cdab} - 4 \nabla^d C_{efgd} C_{chab}] &= 0.
\end{align*}
\]

It is known that both the vanishing of the Bach tensor, equation (121), and this restriction on the Cartan curvature, equation (122), yield metrics that are conformally related to vacuum Einstein metrics. It seems very likely that in the context of the Cartan connection these tensor equations, with their many components, can be reduced to simply two (or three) equations for the \(S\) and \(S^*\).

At the moment, the problem appears to be algebraically quite formidable, however, with computer algebra available, it seems to be manageable. Work on the problem has begun.

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Appendices

In appendix A, we give the relationship between the hatted and unhatted quantities and then explicitly display the hatted quantities in appendix B. In appendix C, we find the tetrad parameters as functions of \((S, S^*)\).
Appendix A

The expressions for the $\Delta$, $G$ and $\omega$ are most easily expressed through their hatted counterparts. The relationship between the hatted and unhatted quantities are given by

\[
\begin{align*}
\Delta_{ijs} &= \hat{\Delta}_{ijs} - \eta_{ij} \Phi^{-1} D \Phi, \\
\Delta_{ijk} &= \hat{\Delta}_{ijk} \Phi^{-1} + 2e_{ij} \Phi \eta_{ik} \Phi^{-2}, \\
G_{ij} &= \hat{G}_{ij} + 2\eta_{ij} \Phi^{-1} D \Phi, \\
\omega_{ijk} &= \hat{\omega}_{ijk} \Phi^{-1} + 2\hat{e}_{ij} \Phi \cdot \eta_{ik} \Phi^{-2}, \\
\omega_{ijs} &= \hat{\omega}_{ijs} - \eta_{ij} \Phi^{-1} D \Phi, \\
\omega_{ijs}^* &= \hat{\omega}_{ijs}^* - \eta_{ij} \Phi^{-1} D \Phi^*, \\
\hat{\Delta}_s &= -\frac{1}{2} S_W + \frac{b^* S_{W^*} - 2ab}{(1 + bb^*)} + \frac{a'(1 + 6bb^* + b^2b^*)}{2(1 + bb^*)^2} \Rightarrow \Delta_s = 0, \\
A_i &= \hat{A}_i \Phi^{-1}. 
\end{align*}
\]

They are easily derived from their definitions (sections 2 and 3) using the two different choices of $\Phi \ (\Phi = 1$ and $\Phi = \sigma \Phi_0)$. Note that from the differential equation for $\Phi$, equation (50), we have that

\[
\Phi^{-1} D \Phi = \frac{1}{2} \hat{\Delta}_{sk}. 
\]

In the next appendix, the hatted quantities ($\hat{\Delta}, \hat{G}, \hat{\omega}$) are explicitly displayed as functions of $(S, S^*, \hat{A}_i)$, leading to the expressions for $(\Delta, G, \omega)$ in terms of $(S, S^*, \hat{A}_i, \sigma)$.

Appendix B

Defining the quantities

\[
\begin{align*}
\gamma &\equiv 1 - bb^*, \\
\sigma &\equiv a - b^* a^*, \\
\zeta &\equiv a_R - b^* a_R^*, \\
\hat{h}_+ &= \hat{e}_+ + b^* \hat{e}_-, \\
\hat{h}_o &= \hat{e}_o + b^* \hat{e}_o, \\
\hat{h}^*_+ &= \hat{e}_+^* + b^* \hat{e}_-, \\
\hat{h}^*_o &= \hat{e}_o^* + b^* \hat{e}_o.
\end{align*}
\]

and

\[
\begin{align*}
\hat{\Delta}_{0}^0_{0} &= \hat{\Delta}_{0}^0_{0} = \hat{\Delta}_{0}^0_{0} = \hat{\Delta}_{0}^0_{0} = \hat{\Delta}_{0}^0_{0} = \hat{\Delta}_{0}^0_{0} = \hat{\Delta}_{0}^0_{0} = 0, \\
\hat{\Delta}_{0}^0_{1} &= \hat{\Delta}_{0}^0_{1} = \hat{\Delta}_{0}^0_{1} = \hat{\Delta}_{0}^0_{1} = \hat{\Delta}_{0}^0_{1} = 0, \\
\hat{\Delta}_{0}^0_{-s} &= \hat{\Delta}_{0}^0_{-s} = \frac{1}{a \gamma}, \\
\hat{\Delta}_{0}^0_{s} &= \frac{b^*}{a \gamma}, \\
\hat{\Delta}_{0}^0_{-s} &= \frac{b}{a \gamma},
\end{align*}
\]
\[ \tilde{\alpha}^*_{0+} = \tilde{\alpha}_0(\ln \alpha) - \frac{\tilde{b}_0^\gamma \tilde{c}(b)}{\gamma}, \]
\[ \tilde{\alpha}^*_{0-} = \frac{\tilde{e}_0(b)}{\gamma}, \]
\[ \tilde{\alpha}^*_{01} = 0, \]
\[ \tilde{\alpha}^*_{0 x} = \alpha(bc - \tilde{e}_0(S)), \]
\[ \tilde{\alpha}^*_{0 x} = \alpha(c - b\tilde{e}_0(S^\gamma)), \]
\[ \tilde{\alpha}^*_{1+} = \frac{\alpha b^\gamma b_R - \gamma \alpha R}{\alpha \gamma}, \]
\[ \tilde{\alpha}^*_{1+} = -D(\ln \alpha) + \frac{b^\gamma Db - \alpha \gamma \tilde{e}_s(S) + b\sigma}{\gamma}, \]
\[ \tilde{\alpha}^*_{1+} = -D^\gamma(\ln \alpha) + \frac{b^\gamma D^\gamma b - \alpha \gamma b_\gamma \tilde{e}_s(S^\gamma) + \sigma}{\gamma}, \]
\[ \tilde{\alpha}^*_{-1} = -\frac{\gamma}{\gamma}, \]
\[ \tilde{\alpha}^*_{-1} = -D\tilde{b} - \alpha \gamma \tilde{e}_\gamma(S) + b\sigma^\gamma, \]
\[ \tilde{\alpha}^*_{-1} = -D^\gamma b - \alpha \gamma b_\gamma \tilde{e}_\gamma(S^\gamma) + \sigma^\gamma, \]
\[ \tilde{\alpha}^*_{1 s} = -\alpha(b + S_R), \]
\[ \tilde{\alpha}^*_{1 s} = -\alpha(1 + bS_R^\gamma), \]
\[ \tilde{\alpha}^*_{1 s} = 0. \]

(B.5)
\[ \hat{\Delta}^1_{x-} = - (\hat{\varepsilon}_w(T) + a \hat{\varepsilon}_w(S)) + \frac{b(Da + c) - Da^w + a^w \sigma^w}{\alpha \gamma}, \]
\[ \hat{\Delta}^1_{x^w} = - (\hat{\varepsilon}_w(T^w) + a^w \hat{\varepsilon}(S^w)) + \frac{bD^w a - D^w a^w - c + a^w \sigma^w}{\alpha \gamma}, \]
\[ \hat{\Delta}^1_{1R} = -(T_R + a^R + aS_R), \]
\[ \hat{\Delta}^1_{y^w} = -(T_R^w + a + a^w S_R^w), \]
\[ \hat{\Delta}^1_{ss^w} \equiv 0. \] (B.6)

II. The \( \hat{G} \):
\[ \hat{G}_{00} = 2(Dc + aS_Z + T_Z - c(aS_R + T_R + a^R)), \]
\[ \hat{G}_{0w} = a^{-1}(1 - bb^w)^{-1} [(a^2(1 - bb^w)c(1 + b^w S_R) - b^w S_Z]
- b^w(Da^w - a^w S_R + aS_W - (a^w)^2 + T_W - a^w T_R)
+ Da + c + T_W + aS_W - aT_R - a^w S_R - a^w T_R), \]
\[ \hat{G}_{0-} = a^{-1}(1 - bb^w)^{-1} [(a^2(1 - bb^w)c(1 + b^w S_R) - S_Z]
- b(c + Da - aT_R - a^w S_R + aS_W + T_W)
+ Da^w + T_W - aS_W - a^w T_R - a^w S_R - (a^w)^2), \]
\[ \hat{G}_{01} = a^w + aS_R + T_R, \] (B.7)
\[ \hat{G}_{+-} = -2(1 - bb^w)^{-1} [b^w(a^w - b^w S_W + a^w b^w S_R - aS_R + S_W) + Db^w - a], \]
\[ \hat{G}_{++} = a^{-1}(1 - bb^w)^{-1} [(a^2(1 - bb^w)c(1 + b^w S_R) - 2Da + aD(bb^w)), \]
\[ \hat{G}_{w1} = a^{-1}(1 - bb^w)^{-1} [1 - a^2(1 - bb^w)(1 + b^w S_R)], \]
\[ \hat{G}_{--} = -2(1 - bb^w)^{-1} [b(ab - a^2 + aS_R - S_W) + Db - a^w S_R + S_W], \]
\[ \hat{G}_{--} = a^{-1}(1 - bb^w)^{-1} [-b - a^2(1 - bb^w)(b + S_R)], \]
\[ \hat{G}_{11} = 0. \]

III. The \( \hat{\omega} \):
\[ \hat{\omega}_{0w} = \left\{ \hat{\varepsilon}_w(c) + \frac{b^w \hat{\varepsilon}(a^w) - \hat{\varepsilon}_w(a)}{\alpha \gamma} \right\} \hat{\gamma}^0 + \left\{ \hat{\Delta}_w + \frac{\zeta}{2 \alpha \gamma} \right\} \hat{\gamma}^1 + \frac{\hat{\varepsilon}_w(b^w)}{\gamma} \hat{\gamma}^w, \]
\[ + \left\{ \hat{\Delta}_0 + \frac{2b^w \hat{\varepsilon}_w(b^w)}{2 \alpha \gamma} \right\} \hat{\gamma}^0 + \frac{\gamma c - b^w S_Z(1 + bb^w)}{\alpha \gamma} \hat{\gamma}^1 = \frac{b^w \gamma c + S_Z^w(1 + bb^w)}{\alpha \gamma} \hat{\gamma}^w, \] (B.8)
\[ \hat{\omega}_{0-} = (\hat{\omega}_{0w})^*. \]
\[ \hat{\omega}_{01} = \hat{e}_R \hat{\gamma}^0 + \left\{ \hat{\Delta}_w + \frac{\zeta}{2 \alpha \gamma} \right\} \hat{\gamma}^1 + \left\{ \hat{\Delta}_w + \frac{\zeta}{2 \alpha \gamma} \right\} \hat{\gamma}^w - 2A \hat{\gamma}^1 + 2A \hat{\gamma}^w + 2\hat{\Delta}_w \hat{\gamma}^w, \]
\[ \hat{\omega}_{10} = -\hat{\omega}_{01} + 2\hat{\Delta}_w, \]
\[ \hat{\omega}_{+-} = \left\{ -A_0 + \frac{b_0 \hat{\varepsilon}(b) - b \hat{\varepsilon}_0(b^w)}{2 \alpha \gamma} \right\} \hat{\gamma}^0 + \frac{\alpha \gamma (\hat{\Delta}_w - \hat{\Delta}_w(a^w))}{2(1 + bb^w)} \hat{\gamma}^1 + \left\{ -A_1 + \frac{bb^w_0 - b^w bR}{2 \alpha \gamma} \right\} \hat{\gamma}^w - \frac{\hat{\Delta}_w(a^w)}{\gamma} + \frac{(3 + bb^w) \hat{\varepsilon}_w(bb^w)}{2 \alpha \gamma} \hat{\gamma}^w, \]
\[ \begin{align*}
&= - \left\{ \frac{\hat{h}_+(b)}{\gamma} + \frac{(3 + bb^*)\hat{\theta}_-(bb^*)}{2\gamma(1 + bb^*)} + 2\tilde{A}_- \right\} \hat{\theta}^0 \\
&= \left\{ \frac{\gamma(S_W + 2A_\tau) + a^*(3 + bb^*)}{4} - \frac{ab(1 + 3bb^*)}{2(1 + bb^*)} \right\} \hat{\theta}^i \\
&= \left\{ \frac{\gamma S_W^*(a - 2A_\tau)(3 + bb^*)}{4} - \frac{a^*b^*(1 + 3bb^*)}{2(1 + bb^*)} \right\} \hat{\theta}^i, \\
\hat{\omega}_{+-} = -\hat{\omega}_{+-} - 2\tilde{A},
\end{align*} \]

\[ \hat{\omega}_{++} = \left\{ -\frac{\zeta}{2\alpha\gamma} \right\} \tilde{\psi}^0 - \frac{b^R}{\gamma} \tilde{\psi}^\tau = \left\{ \hat{A}_+ + \frac{(bb^*)R}{\gamma(1 + bb^*)} \right\} \tilde{\psi}^r \\
+ \frac{\alpha\gamma}{1 + bb^*} \tilde{\psi}^0 - \frac{\alpha\gamma b^*}{1 + bb^*} \tilde{\psi}^r, \]

\[ \hat{\omega}_{-+} = (\hat{\omega}_{++})^*. \]

### Appendix C

In this appendix, which is long and very complicated but given for completeness, we obtain the tetrad parameters and conditions on \( S \) and \( S^* \) that uniquely determine our torsion-free connection. The vanishing of the trace-free part of \( \hat{\triangle}_{ij}^{(0)} \), i.e., conditions (48), gives

\[ \hat{G}_{01}^+ + \hat{G}_{01}^- = \hat{G}_{01}^* + \hat{G}_{01}^*^- = 0, \]

\[ \hat{G}_{ij} = \hat{G}_{ij}^* = 0, \quad \text{for } (i, j) \neq (0, 1), (+, -). \]  

The explicit expressions for \( \hat{G}_{ij} \) given in the previous appendix are used to (i) solve for the tetrad parameters and (ii) derive the Wünschmann condition.

In what follows, we will often encounter pairs of equations that are complex-conjugate to one another. In these instances, we will list only one of the equations and imply the other. When we want to refer to the conjugate of a listed equation, we will write the listed equation’s reference number with a superscript (*).

We start with the equations \( \hat{G}_{+1} = 0, \hat{G}_{-1} = 0, \hat{G}_{+1}^* = 0 \) and \( \hat{G}_{-1}^* = 0 \), which depend only on \( b, b^* \) and \( \alpha \). They are four equations for three variables that satisfy an identity. From \( \hat{G}_{+1} = 0 \) and \( \hat{G}_{-1} = 0 \), we have

\[ b^* S_R = b S_R^*. \]  

Next, using \( \hat{G}_{+1}^* = 0 \) and \( \hat{G}_{-1} = 0 \) to eliminate \( \alpha^2 \), we obtain

\[ b = -1 + \frac{\sqrt{1 - S_R S_R^*}}{S_R^*}. \]  

(We have chosen the positive root since we want \( b \) to vanish when \( S \) vanishes.) One sees that \( b^* \) is the complex conjugate of \( b \). It is useful to invert equations (C.4) and (C.4*), yielding

\[ S_R = \frac{-2b}{1 + bb^*}, \quad S_R^* = \frac{-2b^*}{1 + bb^*}. \]  

From equation (C.5) and \( \hat{G}_{+1} = 0 \), we find

\[ \alpha^2 = \frac{1 + bb^*}{(1 - bb^*)^2}. \]
All four equations, \([\mathcal{G}_{++} = 0, \mathcal{G}_{+-} = 0, \mathcal{G}_{+1}^* = 0, \mathcal{G}_{-1}^* = 0]\), are satisfied by equations (C.4), (C.4*) and (C.6).

Our next step is to determine \(a, a^*\) and the Wünschmann condition from the equations \(\mathcal{G}_{01} + \mathcal{G}_{+1}^* = 0, \mathcal{G}_{++} + \mathcal{G}_{--} = 0\) and their conjugates. From this set of six equations we will be able to solve for \((a, a^*)\), find restrictions on \((S, S^*)\) and obtain further identities.

We first state some useful relationships. Taking \(D\) of equation (C.6) we have, after some simplification,

\[
D a = \frac{\alpha D (bb^*)(3 + bb^*)}{2(1 + bb^*)(1 - bb^*)}
\]

(C.7)

Next (see equation (6)), we find \(T_R = T_R[Db, Db^*, b, S]\) and \(T_R^* = T_R^*[Db, Db^*, b, S]\). By first taking \(D^*\) of equation (C.5) and \(D\) of equation (C.5*), i.e.,

\[
D^*(S_R) = D^* \left( \frac{-2b}{1 + bb^*} \right), \quad D^*(S_R^*) = D \left( \frac{-2b^*}{1 + bb^*} \right)
\]

(C.8)

then using equation (9) to commute the \(R\)-derivative and the fibre derivative, we obtain two equations containing \(T_R\) and \(T_R^*\). After simplifying with equations (C.5), they become

\[
T_R = \frac{4b(Db^* - b^2 Db)}{(1 + bb^*)(1 - bb^*)^2} + \frac{2(b^2 D^* b^* - D^* b + 2b^2 S_W^*)}{(1 - bb^*)^2} + \frac{S_W(1 + bb^*)^2}{(1 - bb^*)^2} - \frac{2(1 + bb^*)(b^* S_{W^*} + b S_{W^*})}{(1 - bb^*)^2}
\]

(C.9)

We are now in a position to find \(a, a^*\) and the Wünschmann condition. First, from \(\mathcal{G}_{01} + \mathcal{G}_{+1}^* = 0\) and \(\mathcal{G}_{01}^* + \mathcal{G}_{+1} = 0\), we solve for \(a\) and \(a^*\). With the aid of equations (C.7), (C.9) and their conjugates, we find

\[
a = \frac{(1 + bb^*)(b^* Db + Db^* + D^*(bb^*)) + (1 - bb^*)(b^* S_{W^*} + b S_{W^*})}{(1 - bb^*)^3}
\]

(C.10)

When they are inserted into \(\mathcal{G}_{--} = 0\), we find that \(S\) must obey the differential condition

\[
M = \frac{Db + b D^* b + S_{W^*} - b S_W + b^2 S_{W^*}^* - b^3 S_{W^*}^*}{1 - bb^*} = 0,
\]

(C.11)

where \(b\) is the known expression in terms of \(S\) and \(S^*\). The expression \(M = M[Db, D^* b, b]\) is known as the generalized Wünschmann invariant. Its vanishing is the condition on the \(S\) and \(S^*\), i.e., on the original pair of PDEs, for the existence of a torsion-free connection.

This condition tells us that this invariant must vanish if we are to find a non-trivial torsion-free connection. By substituting \(Db^*\), from the Wünschmann invariant and its conjugate, into equations (C.10) and (C.10*), our expression for \(a\) becomes

\[
a = b^{-1} b^*-1 \left(1 + bb^*\right)^2 \left(1 + bb^*\right) \left(b^2 (M - Db + b S_W - S_{W^*}) + b (M^* - D^* b^* + b^* S_{W^*}^* - S_{W^*})\right).
\]

(C.12)

or, with \(M = M^* = 0\), we have

\[
a = b^{-1} \left(1 - bb^*\right)^2 \left(1 + bb^*\right) \left(b^2 (b S_W - Db - S_{W^*}) + b (b^* S_{W^*}^* - D^* b^* - S_{W^*})\right).
\]

(C.13)

Summarizing our results so far, we have obtained the five tetrad parameters, \((b, b^*, a, a, a^*)\), as well as the Wünschmann condition in terms of \(S\) and \(S^*\). The search for the last parameter, i.e., \(c\), is the most interesting and at the same time the most difficult part of the construction.
There are four equations, namely, \( \hat{G}_{0+} = 0, \hat{G}_{0-} = 0, \hat{G}_{1+} = 0 \) and \( \hat{G}_{1-} = 0 \), for \( c \). As we will see below, three of those equations become identities once we algebraically solve for \( c \). It is, however, instructive to keep the Wünschmann invariant different from zero when solving the equations. We then explicitly show how its vanishing yields a unique solution for \( c \), such that the remaining identities among the \( \hat{G}_{ij} \) are also satisfied. Thus, for the subsequent calculations, \( M \) is left in the equations. We begin by

\[
Db = M + bS_W - S_W^* + \frac{b(1 - bb^*)(ab - a^*)}{1 + bb^*}, \tag{C.14}
\]

\[
Db^* = b^*(b^*S_W^* - S_W) - bM^* + \frac{(1 - bb^*)(a - a^*b^*)}{1 + bb^*}. \tag{C.15}
\]

Next, we insert the left-hand sides of equations (C.14), (C.15) and their conjugates into equations (C.9) and (C.9*) to find \( T_R = T_R[a, b; M] \) and \( T_R^* = T_R^*[a, b; M] \). The result is

\[
T_R = (\tau + S_W) + \frac{2(3ab - b^*S_W)}{1 + bb^*} - \frac{2a^*(1 + 4bb^* + b^2b^{*2})}{(1 + bb^*)^2}, \tag{C.16}
\]

where \( \tau = \tau[M] \) (which vanishes with \( M \)) is given below.

Third, using the integrability condition, one shows that the vectors \( D \) and \( D^* \) commute. In particular,

\[
DD^*b = D^*Db, \quad DD^*b^* = D^*Db^*. \tag{C.17}
\]

Thus by taking the appropriate fibre derivatives of the four equations (C.14), (C.14*), (C.15), and (C.15*), simplifying with equations (9), (C.14), (C.15), and their conjugates, and by using equations (C.17), we obtain two equations containing \( Da, D^*a, Da^* \) and \( D^*a^* \). They can be solved for \( Da^* = Da[Da, D^*a^*] \) and \( D^*a = D^*a[Da, D^*a^*] \) to find

\[
Da^* = (\Upsilon + a^*S_W - a^2 - T_W^*) + \frac{S_Z(1 + b^2b^{*2}) + 2b^2S_Z^*}{(1 - bb^*)^2} + \frac{b(Da - D^*a^* + T_W - T_W^* + 4aa^*) - 2a^*b^*S_W}{(1 + bb^*)^2} - \frac{aS_W(1 + b^2b^{*2}) + 2b^2(2a^2 + a^*S_W^*)}{(1 + bb^*)^2}. \tag{C.18}
\]

The term \( \Upsilon = \Upsilon[DM, D^*M, M] \), which vanishes with \( M \), is given below. Finally, in addition to equation (C.18), we can use the integrability condition to derive another identity on the fibre derivatives of \( a \) and \( a^* \). We begin by taking \( D^* \) of equation (C.16):

\[
D^*(T_R) = D^*\left[(\tau + S_W) + \frac{2(3ab - b^*S_W)}{1 + bb^*} - \frac{2a^*(1 + 4bb^* + b^2b^{*2})}{(1 + bb^*)^2}\right]. \tag{C.19}
\]

On the left-hand side, we use equation (9*) to commute the \( R - \)derivative and the fibre derivative so that we obtain the term \( U_R = \partial_R(D^*T) \), where

\[
U \equiv D^*T = D^2S = D^2S^* = DT^* \tag{C.20}
\]

denotes the integrability condition. We can then solve this equation for \( U_R \). Refer to the \( U_R \) that we obtain in this manner as \( U_R^{(i)} \). In a similar manner, we can obtain \( U_R^{(2)} \) by taking \( D \) of equation (C.16). Then equate \( U_R^{(i)} \) and \( U_R^{(2)} \), from which we find an identity on \( Da \) and
$D^*a^*$. By using equations (9), (C.5), (C.14), (C.15), (C.18) and their conjugates this identity becomes
\[
\begin{align*}
(\Gamma - \Gamma^* + Da - D^*a^* + T_W - T_W^*) + & \frac{2(1 + bb^*)(bS_Z^* - b^*S_Z)}{(1 - bb^*)^2} \\
+ & \frac{4(a^*b^* - a^2b) + 2(ab^*S_{W^*} - a^*bS_W^*)}{1 + bb^*} = 0.
\end{align*}
\] (C.21)

The terms $\Gamma = [\Gamma], D^*M, D^*M, M]$ and its conjugate vanish with $M$ and are given below. We are now in a position to find $c$ from the four equations $\hat{G}_{00} = 0, \hat{G}_{00}^* = 0, \hat{G}_{\nu} = 0$ and $\hat{G}_{\nu}^* = 0$. We algebraically solve each of the four equations for $c$, calling each solution $c^{(i)}$. Next, we replace $S_\nu$, $\alpha^2$, $T_\nu$ and $Da^*$ by equations (C.5), (C.6), (C.16) and (C.18), and use equation (C.21) to simplify. Finally, we separate each $c^{(i)}$ into a piece that contains all terms with the Wünschmann condition and its fibre derivatives, namely $\xi^{(i)}$, and another piece that contains no Wünschmann terms, namely $C^{(i)}$, so that $c^{(i)}$ has the form
\[
c^{(i)} = C^{(i)} + \xi^{(i)}.
\]

for all $i$. It is straightforward to verify that the four $C^{(i)}$ are real and equal. Imposing the Wünschmann condition, $M = M^* = 0$, so that the $\xi^{(i)} = 0$, then $C^{(i)} = c^{(i)} = c$, and we have our final expression for $c$, namely
\[
c = -\frac{Da + D^*a^* + T_W + T_W^*}{4} - \frac{aa^*(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2} \\
+ \frac{(1 + bb^*)(bS_Z^* + b^*S_Z)}{2(1 - bb^*)^2} + \frac{a(2ab - b^*S_{W^*}) + a^*(2a^*b^* - bS_W^*)}{2(1 + bb^*)}. \tag{C.22}
\]

Had the Wünschmann invariant been non-vanishing the whole construction obviously would have failed. Having determined all the tetrad parameters, we still have to verify that $\hat{G}_{00} = 0$ and $\hat{G}_{00}^* = 0$. By inspection, we see that these equations contain fibre derivatives of $c$. In fact, by explicitly taking these fibre derivatives on $c$, we find that $\hat{G}_{00} = 0$ and $\hat{G}_{00}^* = 0$ are identically satisfied. We see this in the following fashion:

From equations (C.18), (C.18*), (C.21) and (C.22), we find
\[
Da^* = a^*S_W^* - aS_W - T_W^* + \frac{2a^*(2ab - b^*S_{W^*})}{1 + bb^*} + \frac{S_Z(1 + bb^*)^2}{(1 + bb^*)^2} - \frac{a^2(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2}, \tag{C.23}
\]
\[\begin{align*}
Da &= -(2c + T_W) + \frac{2a(2ab - b^*S_{W^*})}{1 + bb^*} + \frac{2b^*S_Z(1 + bb^*)}{(1 + bb^*)^2} - \frac{aa^*(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2}. \tag{C.24}
\end{align*}
\]

By taking $D^*$ of equation (C.23) and $D$ of equation (C.24*), subtracting them and using the commutativity of $D$ and $D^*$, we have
\[
Dc = cS_W - T_Z - aS_Z + \frac{2c(2ab - b^*S_{W^*})}{(1 + bb^*)} - \frac{ca^*(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2}, \tag{C.25}
\]
which is equivalent to $\hat{G}_{00} = 0$.

Note that in the above analysis we have used the following expressions, all of which vanish when $M = M^* = 0$:
\[
\begin{align*}
\tau &= 2(b^*v - b^2v^*), \\
\Upsilon &= 2\rho + 2(1 + bb^*)^{-1}[\mu(1 - bb^*) + v[a^*b^* + a(1 - bb^* - b^2b^{*2})] \\
&\quad + b^2\mu^*(1 - bb^*) + b^2v^*[ab + a^*(1 - bb^* - b^2b^{*2})]], \\
\Gamma &= -b^*[4\mu + 2v(2abb^* + a^*b^* + 3a)].
\end{align*}
\]
\[ \xi_1 = \xi_4 = b \left\{ \mu^* (1 - bb^*) + b^* \rho + \frac{1}{2} v^*[a^*(3 - 2b^2b^*2) - ab]\right\} + \frac{1}{2} b^* v(a - a^*b^*), \]

\[ \xi_2 = \xi_3 = \frac{1}{b} \left\{ \mu(1 - bb^*) + b \rho + \frac{1}{2} v[a(2 + bb^* - 2b^2b^*2) - a^*bb^*2]\right\} + \frac{1}{2} b^* v(a - ab), \]

where

\[ \mu \equiv 2^{-1}(1 - bb^*)^{-3}(1 + bb^*)((b^* DM + D^*M) + M(b^*^2S_{w^*} - 2b^*S_{w^*} - 2bS_{w^*}^w + S_{w^*}^w)), \]

\[ \rho \equiv -2^{-1}(1 - bb^*)^{-2}(1 + bb^*)MM^*, \]

\[ v \equiv (1 - bb^*)^{-1}(1 + bb^*)^{-1}M. \]

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