On some Schwarz type inequalities

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Abstract
First, we establish some Schwarz type inequalities for mappings with bounded Laplacian, then we obtain boundary versions of the Schwarz lemma.

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1 Introduction and preliminaries
Motivated by the role of the Schwarz lemma in complex analysis and numerous fundamental results, see for instance [16, 19] and references therein, in 2016, the first author [1](a) has posted on ResearchGate the project “Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis”. Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question “What are the most recent versions of the Schwarz lemma?”[1](b). In this project and in [16], cf. also [13], we developed the method related to holomorphic mappings with strip codomain (we refer to this method as the approach via the Schwarz–Pick lemma for holomorphic maps from the unit disc into a strip). It is worth mentioning that the Schwarz lemma has been generalized in various directions; see [2, 4, 7, 8, 13, 14, 18, 21] and the references therein.

Recently Wang and Zhu [20] and Chen and Kalaj [5] have studied boundary Schwarz lemma for solutions of Poisson’s equation. They improved Heinz’s theorem [10] and Theorem A below. We found that Theorem A is a forgotten result of Hethcote [11], published in 1977.

Note that previously Burgeth [3] improved the above result of Heinz and Theorem A for real-valued functions (it is easy to extend his result for complex-valued functions; see below) by removing the assumption $f(0) = 0$ but it is overlooked in the literature. Recently, Mateljević and Svetlik [18] proved a Schwarz lemma for real harmonic functions with values in $(−1, 1)$ using a completely different approach than Burgeth [3] and showed that the inequalities obtained are sharp.

In this paper, we further develop the method initiated in [18]. More precisely, we show that, if $U$ denotes the open unit disc and $f : U \to (−1, 1)$, $f \in C^2(U)$ and is continuous on $\overline{U}$, and $|\Delta f| \leq c$ on $U$ for some $c > 0$, then the mapping $u = f \pm \frac{\xi}{2}(1 − |z|^2)$ is subharmonic or superharmonic and we estimate the harmonic function $P[u^*]$; see Theorem 2.
we extend the previous result to complex-valued functions; see Corollary 1. As an application, we provide an elementary proof of a theorem of Chen and Kalaj [5] giving an estimate of the solutions of the Poisson equations. Finally, we establish Schwarz lemmas at the boundary for solutions of $|\Delta f| \leq c$. Our results are generalizations of Theorem 1.1 [20] and Theorem 2 [5].

The proofs are mainly based on two ingredients, the first of which is a sharp Schwarz lemma for real harmonic functions with values in $(-1, 1)$, see Theorem B, and the second is the principle of harmonic majoration, which is a consequence of the maximum principle for subharmonic functions.

### 1.1 Notations and background

In this paper $\mathbb{T}$ denotes the unit circle.

Recall that a real-valued function $u$, defined in an open subset $D$ of the complex plane $\mathbb{C}$, is harmonic if it satisfies Laplace’s equation $\Delta u = 0$ on $D$.

A real-valued function $u \in C^2(D)$ is called subharmonic if $\Delta u(z) \geq 0$ for all $z \in D$.

Let $P$ be the Poisson kernel, i.e., the function

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},$$

and let $G$ be the Green function on the unit disc, i.e., the function

$$G(z, w) = \frac{1}{2\pi} \log \left| \frac{1 - zw}{z - w} \right|, \quad z, w \in \mathbb{U}, z \neq w.$$  

Let $\phi \in L^1(\mathbb{T})$ be an integrable function on the unit circle. Then the function $P[\phi]$ given by

$$P[\phi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta})\phi(e^{i\theta}) \, d\theta$$

is harmonic in $\mathbb{U}$ and has a radial limit that agrees with $\phi$ almost everywhere on $\mathbb{T}$.

For $g \in C(\overline{\mathbb{U}})$, let

$$G[g](z) = \int_{\mathbb{U}} G(z, w)g(w) \, dm(w),$$

$|z| < 1$ and let $dm(w)$ denote the Lebesgue measure in $\mathbb{U}$.

If we consider the function

$$u(z) := P[\phi](z) - G[g](z),$$

then $u$ satisfies the Poisson equation

$$\begin{align*}
\Delta u &= g \quad \text{on the disc } \mathbb{U}, \\
\lim_{r \to 1^-} u(re^{i\theta}) &= \phi(e^{i\theta}) \quad \text{a.e. on the circle.}
\end{align*}$$

One can easily see that the previous equation has a non-unique solution. Indeed, the Poisson kernel $P(z) = \frac{1 - |z|^2}{|1 - z|^2}$ is a harmonic function on the unit disc and $\lim_{r \to 1^-} P(re^{i\theta}) = 0$ a.e., but $P \neq 0$. 


It is well known that, if $\phi$ is continuous on the unit circle, then the harmonic function $P[\phi]$ extends continuously on $\mathbb{T}$ and equals $\phi$ on $\mathbb{T}$; see Hörmander [12].

The following is a consequence of the maximum principle for subharmonic functions.

**Theorem (Harmonic majoration)** Let $u$ be a subharmonic function in $C^2(U) \cap C(\overline{U})$. Then

$$u \leq P[u|_{\mathbb{T}}] \text{ on } \mathbb{U}.$$  

2 The Schwarz lemma for harmonic functions

In [10], Heinz proved that, if $f$ is a harmonic mapping $f$ from the unit disc into itself such that $f(0) = 0$, then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|.$$  

Moreover, this inequality is sharp for each point $z \in \mathbb{U}$.

This inequality for functions from the unit disk to unit ball of $\mathbb{C}^n$ are discussed in [9] to establish Landau’s theorem for $p$-harmonic mappings in several variables.

Later, in 1977, Hethcote [11] improved the above result of Heinz by removing the assumption $f(0) = 0$ and showed the following.

**Theorem A ([11])** If $f$ is a harmonic mapping from the unit disc into itself, then

$$|f(z) - 1 - |z|^2 f(0)| \leq \frac{4}{\pi} \arctan |z|$$

holds for all $z \in \mathbb{U}$.

We remark that the estimate of Theorem A cannot be sharp for all values $z$ in the unit disc.

Recently, Mateljević and Sveltik [18] proved a Schwarz lemma for real harmonic functions with values in $(-1,1)$ using a completely different approach from Burgeth [3].

**Theorem B ([18])** Let $u : \mathbb{U} \rightarrow (-1,1)$ be a harmonic function such that $u(0) = b$. Then

$$m_b(|z|) \leq u(z) \leq M_b(|z|) \text{ for all } z \in \mathbb{U}.$$  

Moreover, this inequality is sharp for each $z \in \mathbb{U}$, where $M_b(r) := \frac{4}{\pi} \arctan \frac{ar}{1+ar}$, $m_b(r) := \frac{4}{\pi} \arctan \frac{a-r}{1-ar}$, and $a = \tan \frac{\pi}{4}$.

Clearly Theorem B improves Theorem A for real harmonic functions, as one can check the following elementary proposition.

**Proposition 2.1** Let $b$ be in $(-1,1)$ and $r \in [0,1)$. Then

1. $M_b(r) \leq \frac{1-r^2}{1+r^2} b + \frac{4}{\pi} \arctan r =: A_b(r)$ and $m_b(r) \geq \frac{1-r^2}{1+r^2} b - \frac{4}{\pi} \arctan r$.
2. The mapping $b \mapsto M_b(r)$ is increasing on $(-1,1)$.

Using a standard rotation, we can extend Theorem B for complex harmonic functions from the unit disc into itself.
**Theorem 1** Let \( f : \mathbb{U} \rightarrow \mathbb{U} \) be a harmonic function from the unit disc into itself. Then

\[
|f(z)| \leq M_{f(0)}(|z|)
\]

holds for all \( z \in \mathbb{U} \).

**Proof** Fix \( z_0 \) in the unit disc and choose unimodular \( \lambda \) such that \( \lambda f(z_0) = |f(z_0)| \).

Define \( u(z) = \Re(\lambda f(z)) \).

Hence, using Theorem B, we get

\[
|f(z_0)| = u(z_0) \leq M_u(0)(|z_0|) \leq M_{f(0)}(|z_0|),
\]

as the mapping \( b \mapsto M_b(|z_0|) \) is increasing.

\[\Box\]

### 3 Schwarz lemma for mappings with bounded Laplacian

The following theorem is our main result of this section.

**Theorem 2** Let \( f \) be a \( C^2(\mathbb{U}) \) real-valued function, continuous on \( \mathbb{U} \) and \( f^* = f|_\mathbb{R} \). Let \( b = P[f^*](0), c \in \mathbb{R} \) and \( K \) be a positive number such that \( K \geq \|P[f^*]\|_\infty \).

(i) If \( f \) satisfies \( \Delta f \geq -c \), then

\[
f(z) \leq KM_{b/K}(|z|) + \frac{c}{4} (1 - |z|^2)
\]

holds for all \( z \in \mathbb{U} \).

(ii) If \( f \) satisfies \( \Delta f \leq c \), then

\[
f(z) \geq K m_{b/K}(|z|) - \frac{c}{4} (1 - |z|^2)
\]

holds for all \( z \in \mathbb{U} \).

**Proof** (i) Define \( f^0(z) = f(z) + \frac{c}{4}(|z|^2 - 1) \), and set \( P[f^*](0) = b \). Then \( f^0 \) is subharmonic and \( f^0 \leq P[f^*] \). As \( \frac{1}{K} P[f^*] \) is a real harmonic function with codomain \((-1,1)\), by Theorem B, we obtain \( P[f^*](z) \leq KM_{b/K}(|z|) \). Thus

\[
f(z) \leq KM_{b/K}(|z|) + \frac{c}{4} (1 - |z|^2), \quad \text{for all } z \in \mathbb{U}.
\]

(ii) If \( f \) satisfies \( \Delta f \leq c \), then define \( f_0(z) = f(z) - \frac{c}{4}(|z|^2 - 1) \), and set \( P[f^*](0) = b \). In a similar way, we show that the inequality

\[
f(z) \geq K m_{b/K}(|z|) - \frac{c}{4} (1 - |z|^2)
\]

holds for all \( z \in \mathbb{U} \). \[\Box\]

For complex-valued functions with bounded Laplacian from the unit disc into itself, we prove the following.
Corollary 1 Suppose that $f : \mathbb{U} \to \mathbb{U}$, $f \in C^2(\mathbb{U})$ and continuous on $\overline{\mathbb{U}}$, and $|\Delta f| \leq c$ on $\mathbb{U}$ for some $c > 0$. Then

$$|f(z)| \leq M_b(|z|) + \frac{c}{4}(1 - |z|^2)$$

holds for all $z \in \mathbb{U}$, where $b = |P[f^*](0)|$.

Proof Fix $z_0$ in the unit disc and choose $\lambda$ such that $\lambda f(z_0) = |f(z_0)|$. Define $u(z) = \Re(\lambda f(z))$ (we say that $u$ is a real-valued harmonic associated to complex-valued harmonic $f$ at $z_0$). We have $\Delta u = \Re(\lambda \Delta f)$. As $u$ is a real function with codomain $(-1, 1)$ satisfying $|\Delta u| \leq c$, by Theorem 2, we get

$$|u(z)| \leq M_{b_1}(|z|) + \frac{c}{4}(1 - |z|^2), \quad \text{where } b_1 = |P[u^*](0)|.$$ 

We have $b_1 = |P[u^*](0)| = \Re(\lambda |P[f^*](0)|) \leq |P[f^*](0)|$. Hence

$$|f(z_0)| \leq M_b(|z_0|) + \frac{c}{4}(1 - |z_0|^2),$$

where $b = |P[f^*](0)|$, as the mapping $b \mapsto M_b(|z_0|)$ is increasing. \hfill \Box

Under the conditions of the previous theorem and using Proposition 2.1 we obtain

$$\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} P[f^*](0) \right| \leq \frac{4}{\pi} \arctan |z| + \frac{c}{4}(1 - |z|^2). \quad (3.1)$$

3.1 Applications

For a given continuous function $g : G \to \mathbb{C}$, Chen and Kalaj [5] established some Schwarz type Lemmas for mappings $f$ in $G$ satisfying the Poisson equation $\Delta f = g$, where $G$ is a subset of the complex plane $\mathbb{C}$. Then they applied these results to obtain a Landau type theorem, which is a partial answer to the open problem in [6].

We provide a different and an elementary proof of Theorem C, giving a Schwarz type lemma for mappings satisfying Poisson’s equations.

Theorem C ([5]) Let $g \in C(\mathbb{U})$ and $\phi \in C(\mathbb{T})$. If a complex-valued function $f$ satisfies $\Delta f = g$ in $\mathbb{U}$ and $f = \phi$ in $\mathbb{T}$, then for $z \in \mathbb{U}$

$$\left| f(z) - P[\phi](0) \frac{1 - |z|^2}{1 + |z|^2} \right| \leq \frac{4}{\pi} \|P[\phi]\|_\infty \arctan |z| + \frac{1}{4} \|g\|_\infty (1 - |z|^2), \quad (3.2)$$

where $\|P[\phi]\|_\infty = \sup_{z \in \mathbb{U}} |P[\phi](z)|$ and $\|g\|_\infty = \sup_{z \in \mathbb{U}} |g(z)|$.

Now we show that Theorem 2 implies Theorem C.

We will consider first the case when $f$ is a real-valued $C^2(\mathbb{U})$ function, continuous on $\mathbb{U}$, satisfying $\Delta f = g$ and $f^* = \phi$. Let $K := \|P[\phi]\|_\infty$. By Theorem 2, we have

$$m_{b/K}(|z|)K - \|g\|_\infty \frac{(1 - |z|^2)}{4} \leq f(z) \leq M_{b/K}(|z|)K + \|g\|_\infty \frac{(1 - |z|^2)}{4},$$
where \( b = P[\phi](0) \). Using Proposition 2.1(1), we get
\[
M_{b;K}(|z|)K \leq \frac{1 - |z|^2}{1 + |z|^2} b + \frac{4K}{\pi} \arctan |z|
\]
and
\[
m_{b;K}(|z|)K \geq \frac{1 - |z|^2}{1 + |z|^2} b - \frac{4K}{\pi} \arctan |z|.
\]
Hence, the following inequality:
\[
\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} P[\phi](0) \right| \leq \frac{4}{K} \|P[\phi]\|_\infty \arctan |z| + \frac{1}{4} \|g\|_\infty (1 - |z|^2)
\]
holds for all \( z \in \mathbb{U} \).

If \( f \) is a complex-valued function, we may consider \( u = \Re(\lambda f) \), where \( \lambda \) is a complex number of modulus 1. Indeed, we have
\[
u(z) - \frac{1 - |z|^2}{1 + |z|^2} P[u^*](0) = \Re\left( \lambda \left( f(z) - \frac{1 - |z|^2}{1 + |z|^2} P[\phi](0) \right) \right),
\]
where \( u^* = \Re(\lambda \phi) \) on \( \mathbb{T} \). Now, one can choose \( \lambda \) such that
\[
\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} P[\phi](0) \right| = u(z) - \frac{1 - |z|^2}{1 + |z|^2} P[u^*](0).
\]

4 Boundary Schwarz lemmas

We establish Schwarz lemmas at the boundary for solutions of \( |\Delta f| \leq c \). Our results are generalizations of Theorem 1.1 [20] and Theorem 2 [5].

Theorem 3 Suppose \( f \in C^2(\mathbb{U}) \), continuous on \( \overline{\mathbb{U}} \) with codomain \((-1,1)\), such that \( \Delta f \geq -c \). If \( f \) is differentiable at \( z = 1 \) with \( f(1) = 1 \), then the following inequality holds:
\[
f_1(1) \geq \frac{2}{\pi} \tan \frac{\pi}{4}(1 - b) - \frac{c}{2},
\]
where
\[
b = P[f^*](0).
\]
Before giving the proof, one can easily show that
\[
M_r'(a) = \frac{4}{\pi} \left[ \frac{1 - a^2}{(a^2 + 1)r^2 + 4ar + a^2 + 1} \right].
\]
Hence
\[
M_r'(1) = \frac{2}{\pi} \left[ \frac{1 - a}{1 + a} \right] = \frac{2}{\pi} \tan \frac{\pi}{4}(1 - b),
\]
as \( a = \tan \frac{b \pi}{4} \).
Proof Since \( f \) is differentiable at \( z = 1 \), we know that

\[
f(z) = 1 + f_\xi(1)(z - 1) + f_\xi(1)(\bar{z} - 1) + o(|z - 1|).
\]

That is,

\[
f_\xi(1) = \lim_{r \to 1^-} \frac{f(r) - 1}{r - 1}.
\]

On the other hand, Theorem 2(i) leads to

\[
1 - f(r) \geq 1 - M_b(r) - \frac{c}{4} (1 - r^2).
\]

Dividing by \((1 - r)\) and letting \( r \to 1^- \), we get

\[
f_\xi(1) \geq M'_b(1) - \frac{c}{2}.
\]

Thus

\[
f_\xi(1) \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2}.
\]

Corollary 2 Suppose \( f \in C^2(U) \), with codomain \((-1, 1)\), is continuous on \( U \) and is differentiable at \( z = 1 \) with \( f(1) = 1 \).

(i) If \( \Delta f \geq -c \), then

\[
f_\xi(1) \geq \frac{2}{\pi} - b - \frac{c}{2}.
\]

(ii) If \( |\Delta f| \leq c \) and \( f(0) = 0 \), then \( |b| \leq \frac{c}{4} \) and

\[
f_\xi(1) \geq \frac{2}{\pi} - \frac{3}{4} c,
\]

where \( b = P[f^*](0) \).

Proof (i) Using the inequality \( M_b \leq A_b \) from Proposition 2.1 and \( M_b(1) = A_b(1) = 1 \), we get

\[
M'_b(1) \geq A'_b(1) = \frac{2}{\pi} - b.
\]

(ii) The estimate \( |b| \leq \frac{c}{4} \) follows directly from Theorem 2 using the assumption \( f(0) = 0 \). \( \square \)

Remark 4 One can also prove directly that \( M'_b(1) \geq A'_b(1) \), that is,

\[
\frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \geq \frac{2}{\pi} - b \quad \text{for } b \in [0, 1].
\]

Using the convexity of the tangent function, we get

\[
\tan x \geq 2 \left( x - \frac{\pi}{4} \right) + 1 \quad \text{for } x \in [0, \pi/2).
\]
For \( b \in [0, 1) \), let us substitute \( x \) by \( \frac{\pi}{4} (1 - b) \), we obtain

\[
\frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \geq \frac{2}{\pi} - b.
\]

The following theorem is a generalization of Theorem 2 in [5] where the authors proved a Schwarz lemma on the boundary for a function \( f \) satisfying \( \Delta f = g \) and under the assumption \( f(0) = 0 \).

**Theorem 5** Suppose that \( f \in C^2(U) \cap C(\overline{U}) \) is a function of \( U \) into \( U \) satisfying |\( \Delta f \)\| \leq c, where \( 0 \leq c < \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \). If, for some \( \xi \in T \), \( \lim_{r \to 1^-} |f(r\xi)| = 1 \), then

\[
\liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2},
\]

where \( b = |P[f^*]|(0)| \).

If, in addition, we assume that \( f(0) = 0 \), then

\[
|b| \leq \frac{c}{4}
\]

and

\[
\liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \frac{2}{\pi} - \frac{3}{4} \cdot c.
\]

**Proof** Using Corollary 1, we have

\[
|f(\xi) - f(r\xi)| \geq 1 - |f(r\xi)| \geq 1 - M_b(r) - \frac{c}{4} (1 - r^2).
\]

Thus

\[
\liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \lim_{r \to 1^-} \frac{1 - M_b(r) - \frac{c}{4} (1 - r^2)}{1 - r} = M_b'(1) - \frac{c}{2}.
\]

The conclusion follows as \( M_b'(1) = \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) \).

If in addition, we assume that \( f(0) = 0 \), using the inequality (3.1), we obtain \(|b| < \frac{c}{4}\). Hence

\[
\liminf_{r \to 1^-} \frac{|f(\xi) - f(r\xi)|}{1 - r} \geq \frac{2}{\pi} \tan \frac{\pi}{4} (1 - b) - \frac{c}{2} \geq \frac{2}{\pi} - b - \frac{c}{2} = \frac{2}{\pi} - \frac{3}{4} \cdot c.
\]

The second estimate follows from the inequality (4.2). \( \square \)

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All authors contributed equally to the manuscript read and approved the final manuscript.

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Endnotes
a Motivated by Krantz’ paper [15].
b The subject has been presented at Belgrade analysis seminar [17].

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