Five Lectures on Khovanov Homology

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Abstract. These five lectures provide an introduction to Khovanov homology covering the basic definitions, important properties, a number of variants and some applications. At the end of each lecture the reader is referred to the relevant literature for further reading.

About these lectures
These lectures were designed for the summer school Heegaard-Floer homology and Khovanov homology in Marseilles, 29th May - 2nd June, 2006.

The intended audience is graduate students with some minimal background in low-dimensional and algebraic topology. I hesitated to produce lecture notes at all, since much of the literature in the subject is very well written, but decided in the end that notes could serve some purpose.

In order to keep the narrative flowing I found it convenient to delay all attributions of credit until the end of each lecture. I have attempted to do this as accurately as possible and if I have failed to properly attribute a certain piece of work or omitted to mention someone in a particular context, my apologies to the injured party in advance.

At the present time the pace of development of the subject is very rapid and the reader is encouraged to consult math/GT for the latest developments.

Many thanks F. Costantino, M. Mackaay and P. Vaz for their comments on a draft version and to D. Matignon for organising a very stimulating summer school.

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CHAPTER 1

Lecture One

In this lecture we begin with a very brief introduction to the subject, followed by some recollections about the Jones polynomial. We then define the main object of interest: the Khovanov complex of an oriented link diagram.

1. What is it all about?

Given an oriented link diagram, $D$, Khovanov constructs in a purely combinatorial way a bi-graded chain complex $C^{*,*}(D)$ associated to $D$.

$$
D \xrightarrow{\text{Khovanov}} C^{*,*}(D)
$$

Given a chain complex we can apply homology to it and for $C^{*,*}(D)$ this results in the Khovanov homology, $KH^{*,*}(D)$, of the diagram $D$.

$$
C^{*,*}(D) \xrightarrow{\text{Homology}} KH^{*,*}(D)
$$

The following properties are satisfied:

1. If $D$ is related to another diagram $D'$ by a sequence of Reidemeister moves then there is an isomorphism
   
   $$
   KH^{*,*}(D) \cong KH^{*,*}(D').
   $$

2. The graded Euler characteristic is the unnormalised Jones polynomial i.e.
   
   $$
   \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim(KH^{i,j}(D)) = \hat{J}(D).
   $$

You should think, by way of analogy, of the relationship of the ordinary Euler characteristic to homology. For a space $M$ (there are some restrictions on $M$, say, a finite CW complex) we can assign a numerical invariant, the Euler characteristic $\chi(M) \in \mathbb{Z}$. This can be calculated by a simple algorithm based on some combinatorial information about the space (e.g. the CW structure, a triangulation etc). On the other hand, homology assigns a graded vector space to $M$ and is related to the Euler characteristic by the formula:

$$
\sum_{i \in \mathbb{Z}} (-1)^i \dim(H_i(M; \mathbb{Q})) = \chi(M).
$$

In this way homology categorifies the Euler characteristic: a number gets replaced by a (graded) vector space whose (graded) dimension gives back the number you started with.

In the case of links we can assign quantum invariants such as the Jones polynomial. These can be calculated by a simple algorithm based on some combinatorial information about the link (e.g. a diagram). Khovanov homology is the analogue of homology and categorifies the Jones polynomial.

Homology has many advantages over the Euler characteristic. For example

- homology is a stronger invariant than the Euler characteristic,
- homology reveals richer information e.g. torsion,
- homology is a functor.

As we will see soon, Khovanov homology has similar advantages over the Jones polynomial.
2. Recollections about the Jones polynomial

Let $L$ be an oriented link and $D$ a diagram for $L$ with $n$ crossings. Suppose that $n_-$ of these are negative crossings (like this: $\backslash\backslash$) and $n_+$ are positive (like this: $\backslash\backslash$). The Kauffman bracket of the diagram $D$, written $\langle D \rangle$ is the Laurent polynomial in a variable $q$ (i.e. $\langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$) defined recursively by:

\begin{align*}
\langle \backslash\backslash \rangle &= \langle \backslash \rangle - q \langle \rangle \\
\langle k \text{ circles in the plane} \rangle &= (q + q^{-1})^k
\end{align*}

(Beware: this is not the usual normalisation).

The Kauffman bracket is not a link invariant but by defining

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

one gets a genuine link invariant i.e. something invariant under all the Reidemeister moves. The Jones polynomial is given by

$$J(D) = \frac{\hat{J}(D)}{q + q^{-1}}.$$ 

(The usual formula for the Jones polynomial involves a variable $t$. Substitute $q = -t^{\frac{2}{2}}$ to make the descriptions match). The polynomial $\hat{J}(D)$ is known as the unnormalised Jones polynomial.

Equation (1) reduces the number of crossings at the expense of twice as many terms on the right hand side. For a diagram $D$ with $n$ crossings we can apply this equation $n$ times to end up with $2^n$ pictures on the right hand side each of them consisting of a collection of circles in the plane which we then evaluate using Equation (2).

To do this in a systematic way let us agree that given a crossing (looking like this: $\backslash\backslash$) we will call the two pictures on the right of Equation (1) the 0-smoothing (looking like this: $\backslash\backslash$) and the 1-smoothing (looking like this: $\backslash\backslash$).

Thus, if we number the crossings of $D$ by $1, 2, \ldots, n$ then each of the $2^n$ pictures can be indexed by a word of $n$ zeroes and ones i.e. an element of $\{0, 1\}^n$. We will call a picture in which each crossing has been resolved (in one of the two ways above) a smoothing. Thus a diagram $D$ has $2^n$ smoothings indexed by $\{0, 1\}^n$. The set $\{0, 1\}^n$ is the vertex set of a hyper-cube as shown in Figure 1, with an edge between words differing in exactly one place.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

It is convenient to think of the smoothings as living on the vertices of this cube. For $\alpha \in \{0, 1\}^n$ we will denote the associated smoothing (the collection of circles in the plane) by $\Gamma_\alpha$. Given $\alpha \in \{0, 1\}^n$ we define

$$r_\alpha = \text{the number of 1's in } \alpha$$

and

$$k_\alpha = \text{the number of circles in } \Gamma_\alpha.$$ 

We can now use Equations (1) and (2) to write down a state-sum expression for $\hat{J}(D)$.

$$\hat{J}(D) = \sum_{\alpha \in \{0, 1\}^n} (-1)^{r_\alpha} q^{r_\alpha - n_-} (q + q^{-1})^{k_\alpha}.$$
3. The definition of the Khovanov complex of a link diagram

For the time being we will work over \( \mathbb{Q} \), so “vector space” means “vector space over \( \mathbb{Q} \)”. Khovanov’s idea is to assign a cochain complex \((C^*((D), d)\) to a link diagram \(D\). The homotopy type of this complex will turn out to be an invariant and its graded Euler characteristic the unnormalised Jones polynomial.

Before getting to the definition let’s recall a few things about finite dimensional graded vector spaces.

1. The graded (or quantum) dimension, \( \text{qdim} \), of a graded vector space \( W = \bigoplus_m W^m \) is the polynomial in \( q \) defined by
   \[
   \text{qdim}(W) = \sum_m q^m \text{dim}(W^m).
   \]

2. The graded dimension satisfies
   \[
   \text{qdim}(W \otimes W') = \text{qdim}(W)\text{qdim}(W'),
   \]
   \[
   \text{qdim}(W \oplus W') = \text{qdim}(W) + \text{qdim}(W').
   \]
(3) For a graded vector space $W$ and an integer $l$ we can define a new graded vector space $W\{l\}$ (a shifted version of $W$) by

$$W\{l\}^m = W^{m-l}$$

Notice that $\text{qdim}(W\{l\}) = q^l \text{qdim}(W)$.

Now we turn to the definition of the Khovanov complex, $C^{*,*}(D)$, of an oriented link diagram $D$. An important role is played by the following two-dimensional graded vector space. Let $V = \mathbb{Q}\{1, x\}$ (the $\mathbb{Q}$-vector space with basis 1 and $x$) and grade the two basis elements by $\deg(1) = 1$ and $\deg(x) = -1$.

**Exercise 3.1.** Show that $\text{qdim}(V \otimes_k) = (q + q^{-1})^k$.

Recall that we have $2^n$ smoothings of our diagram. To each $\alpha \in \{0, 1\}^n$ now associate the graded vector space

$$V_\alpha = V \otimes_k \{r_\alpha + n_+ - 2n_\}$$

and define

$$C^{i,*}(D) = \bigoplus_{\alpha \in \{0, 1\}^n \atop r_\alpha = i + n_-} V_\alpha.$$ 

The internal grading comes from the fact that each $V_\alpha$ is a graded vector space. Note that the vector spaces $C^{i,*}(D)$ are trivial outside the range $i = -n_-, \ldots, n_+.$

Recall that in the last section we arranged the $2^n$ smoothings of the diagram on a cube with $2^n$ vertices indexed by $\{0, 1\}^n$. The above definition means that we now replace the smoothing $\Gamma_\alpha$ with the vector space $V_\alpha$ so that the space $C^{i,*}(D)$ is the direct sum of vector spaces in column $i + n_-$ of the cube as indicated in Figure 3.

![Figure 3](image-url)

**Example 3.2.** For the Hopf link we have the cube in Figure 4.

An element of $C^{i,j}(D)$ is said to have homological grading $i$ and $q$-grading $j$. If $v \in V_\alpha \subset C^{*,*}(D)$ with homological grading $i$ and $q$-grading $j$ then it is useful to remember that

$$i = r_\alpha - n_-$$

$$j = \deg(v) + i + n_+ - n_-$$
What we need now is a differential $d$ turning $(C^\ast, (D), d)$ into a complex. Recall that we have a smoothing $\Gamma_\alpha$ (i.e. a collection of circles) associated to each vertex $\alpha$ of the cube $\{0, 1\}^n$. Now to each edge of the cube we associate a cobordism (i.e. an (orientable) surface whose boundary is the union of the circles in the smoothings at either end).

Edges of the cube can be labelled by a string of zeroes and ones with a star ($\ast$) at the position that changes. For example the edge joining $0100$ to $0110$ is denoted $01 \ast 0$. We can turn edges into arrows by the rule: $\ast = 0$ gives the tail and $\ast = 1$ gives the head. For an arrow $\alpha \xrightarrow{\zeta} \alpha'$ note that the smoothings $\alpha$ and $\alpha'$ are identical except for a small disc (the changing disc) around the crossing that changes from a 0- to a 1-smoothing (the one marked by a $\ast$ in $\zeta$). For example the changing disc for the arrow $\zeta = 1\ast$ in the cube (square!) of the Hopf link above is shown in Figure 5.
The cobordism $W_\zeta$ associated to $\alpha \xrightarrow{\zeta} \alpha'$ is defined to be the following surface: outside the changing disc take the product of $\Gamma_\alpha$ with the unit interval and then plug the missing tube with the saddle $\text{\Large }\alpha$. Thus each $W_\zeta$ consists of a bunch of cylinders and one pair-of-pants surface ($\text{\Large }\cup$ or $\text{\Large }\cap$).

**Cobordism convention:** pictures of cobordisms go down the page.

Above we replaced the smoothing $\Gamma_\alpha$ by the vector space $V_\alpha$ and now we will replace the cobordism $W_\zeta$ associated to the edge $\alpha \xrightarrow{\zeta} \alpha'$ by a linear map $d_\zeta : V_\alpha \to V_{\alpha'}$. Since each circle in a smoothing has a copy of the vector space $V$ attached to it, to define $d_\zeta$ we only require two linear maps: one that fuses $m : V \otimes V \to V$ and one that splits $\Delta : V \to V \otimes V$. Then we can define $d_\zeta$ to be the identity on circles not entering the changing disc and either $m$ or $\Delta$ on the circles appearing in the changing disc (depending on whether the pair-of-pants has two or one input boundary circles).

We define $m : V \otimes V \to V$ by

$1^2 = 1, \quad 1x = x1 = x, \quad x^2 = 0$,

and $\Delta : V \to V \otimes V$ by

$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x$.

In fact by defining a unit $i(1) = 1$ and counit $\epsilon(1) = 0$ and $\epsilon(x) = 1$ we have endowed $V$ with the structure of a commutative Frobenius algebra. Isomorphism classes of commutative Frobenius algebras are in bijective correspondence with isomorphism classes of 1+1-dimensional topological quantum field theories so what we are really doing here is applying a TQFT (the one defined by $V$) to the cube of circles and cobordisms - more on this in Lecture 2.

We are finally ready to define $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$. For $v \in V_\alpha \subset C^{i,*}(D)$ set

$d^i(v) = \sum_{\zeta \text{ such that } \text{Tail}(\zeta) = \alpha} \text{sign}(\zeta)d_\zeta(v)$

where $\text{sign}(\zeta) = (-1)^{\text{number of 1's to the left of } \star \text{ in } \zeta}$.

**PROPOSITION.** $d^{i+1} \circ d^i = 0$.

**PROOF.** (sketch) The idea of the proof is that without the signs each face of the cube commutes. To see this one can either look at a number of cases and use the definition of the maps $m$ and $\Delta$ or (much better) begin to think geometrically: each of the two routes around a face gives the same cobordism (up to homeomorphism) and so applying the TQFT defined by $V$ gives the same linear map. Once all the non-signed faces commute then observe that the signs occur in odd numbers on every face, thus turning commutativity into anti-commutativity.

**EXERCISE 3.4.** Write out the above proof properly - you may wish to wait until after Lecture 2 where there is a more detailed discussion of Frobenius algebras and TQFTs.

**EXERCISE 3.5.** Check that $d$ has bi-grading $(1,0)$.

The graded Euler characteristic of this complex i.e.

$\sum (-1)^i q \dim(C^{i,*}(D)) \in \mathbb{Q}[q^{\pm1}]$

is nothing other than the unnormalised Jones polynomial.

**EXERCISE 3.6.** Convince yourself that the previous statement is true - this is simply a matter of unwinding the definitions and then comparing with the state-sum formula for the unnormalised Jones polynomial.
Later we will see that the homotopy type of \((C^{\ast,\ast}(D), d)\) is invariant under transformations by Reidemeister moves. For now, to end this lecture, let us perform a homology calculation.

**Example 3.7.** Let us compute the homology of \((C^{\ast,\ast}(\text{\textcircled{}})), \ast\). The complex has only three non-trivial terms:

\[
0 \longrightarrow C^{-2,\ast}(D) \xrightarrow{d} C^{-1,\ast}(D) \xrightarrow{d} C^{0,\ast}(D) \longrightarrow 0.
\]

More explicitly we have:

\[
\begin{align*}
V\{-3\} & \overset{m}{\longrightarrow} (V \otimes V)\{-4\} \oplus (V \otimes V)\{-2\} & \overset{-\Delta}{\longrightarrow} V\{-3\}
\end{align*}
\]

and based on this one can compute as follows.

| Homological degree | -2 | -1 | 0 |
|--------------------|----|----|---|
| Cycles             | \{1 \otimes x - x \otimes 1, x \otimes x\} | \{(1, 1), (x, x)\} | \{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\} |
| Boundaries         | -  | \{(1, 1), (x, x)\} | \{1 \otimes x + x \otimes 1, x \otimes x\} |
| Homology           | \{1 \otimes x - x \otimes 1, x \otimes x\} | - | \{1 \otimes 1, 1 \otimes x\} |
| q-degrees          | -4, -6 | 0, -2 |

The homology can be summarised as a table where the homological degree is horizontal and the q-degree vertical.

```
| j | i | -2 | -1 | 0 |
|---|---|----|----|---|
| 0 |   | Q  |    |   |
| -1|   |    | Q  |   |
| -2|   |    | Q  |   |
| -3|   |    |    |   |
| -4|   | Q  |    |   |
| -5|   |    |    |   |
| -6|   | Q  |    |   |
```

**Exercise 3.8.** Write out the cube for the trefoil explicitly (including the correct signs on the edges). Calculate the homology of the Khovanov complex. This involves a bit of work, but is a very good test to see if you have understood all the definitions.

### 4. Notes and further reading

The original paper by Khovanov in which he defines the complex and establishes the basic properties is [Khovanov1]. In this paper he starts out working over the ring \(\mathbb{Z}[c]\) and then sets \(c = 0\) to work over \(\mathbb{Z}\). We work over \(\mathbb{Q}\) because some things are a little simpler. We will say more about other coefficients in Lecture 3. Bar-Natan’s exposition of Khovanov’s work [Bar-Natan2] is extremely readable and has also been very influential.
CHAPTER 2

Lecture Two

The complex $C^*(D)$ for a link diagram $D$ defined in Lecture 1 depends very much on the diagram. However, it turns out that different diagrams for the same link give complexes which are homotopy equivalent. In this lecture we begin with an aside on Frobenius algebras and topological quantum field theories and after this discuss the homotopy invariance properties of the Khovanov complex concentrating on the first Reidemeister move. Finally in this lecture we define Khovanov homology and discuss some properties.

1. Frobenius algebras and TQFTs

Hidden in the background in Lecture 1, about to come to deserved prominence, are 1+1-dimensional topological quantum field theories and their algebraic counterparts, Frobenius algebras.

A commutative Frobenius algebra over $R$ (a commutative ring with unit) is a unital, commutative $R$-algebra $V$ which as an $R$-module is projective of finite type (if $R = \mathbb{Q}$ this just means a finite dimensional vector space over $\mathbb{Q}$), together with a module homomorphism, the counit, $\epsilon : V \to R$ such that the bilinear form $\langle -,- \rangle : V \otimes V \to R$ defined by $\langle v,w \rangle = \epsilon(vw)$ is non-degenerate i.e. the adjoint homomorphism $V \to V^*$ is an isomorphism. It is useful to define a coproduct $\Delta : V \to V \otimes V$ by $\Delta(v) = \sum_i v'_i \otimes v''_i$ being the unique element such that for all $w \in V$, $vw = \sum_i v'_i \langle v''_i, w \rangle$.

Frobenius algebras reflect the topology of surfaces. This statement is the rough equivalent of the more accurate:

\[
\{\text{Iso. classes of comm. Frobenius algebras} \} \leftrightarrow \{\text{Iso. classes of 1+1-dimensional TQFTs} \}
\]

Recall that a 1+1-dimensional TQFT is a monoidal functor $\text{Cob}_{1+1} \to \text{Mod}_R$ where $\text{Cob}_{1+1}$ is the category whose objects are closed, oriented 1-manifolds and where a morphism $\Gamma \to \Gamma'$ is an oriented surface $W$ with $\partial W = \Gamma \sqcup \Gamma'$ (here the overline means take the opposite orientation). In fact you have to be careful to get the details of all this right - see the references at the end of the lecture. The upshot is that a TQFT:

- assigns to each closed 1-manifold $\Gamma$, an $R$-module $V_\Gamma$ such that if $\Gamma = \Gamma_0 \sqcup \Gamma_1$ then $V_\Gamma = V_{\Gamma_0} \otimes V_{\Gamma_1}$ (this what the adjective “monoidal” refers to) and
- assigns to each cobordism $W : \Gamma \to \Gamma'$, an $R$-homomorphism $V_\Gamma \to V_{\Gamma'}$.

These assignments are subject to some axioms which, among other things, guarantee

- homeomorphic cobordisms induce the same homomorphism,
- gluing of cobordisms is well behaved, and
- $V_{\emptyset} = R$.

Using the correspondence between Frobenius algebras and 1+1-dimensional TQFTs one can use the topology to prove algebraic statements. For example, for any Frobenius algebra one can check that $m(\Delta(v)) = m(m(\Delta(1)), v)$ for all $v \in V$. Checking this algebraically is a bit of a pain, but geometrically it is a triviality: the two surfaces in Figure 1 are homeomorphic and therefore correspond to the same homomorphism of Frobenius algebras.

In Lecture 1 we defined a particular two dimensional Frobenius algebra $V$ which, by the above, defines a 1+1-dimensional TQFT. Given a link diagram we considered the cube $\{0,1\}^n$ and associated to each vertex $\alpha$ a collection of circles $\Gamma_\alpha$ (a smoothing) and to each edge $\zeta$ a cobordism $W_\zeta$. In order
to get a complex we then replaced each collection of circles by a vector space and each cobordism by a linear map. This last step is nothing other than applying the TQFT defined by $V$.

**Exercise 1.1.** A TQFT associates to the empty manifold the ground ring $R$ and thus a closed cobordisms gives an element of $R$ (a closed cobordism is a cobordism $\emptyset$ to $\emptyset$ and hence induces a map $R \to R$ which you evaluate at $1 \in R$). Compute the value of the torus for the TQFT defined by the Frobenius algebra $V$ of Lecture 1.

### 2. Reidemeister invariance

Recall that complexes $A^*$ and $B^*$ are homotopy equivalent if there are chain maps $F: A^* \to B^*$ and $G: B^* \to A^*$ such that $GF - Id_{A^*}$ and $FG - Id_{B^*}$ are null-homotopic.

**Proposition 2.1.** If $D'$ is a diagram obtained from $D$ by the application of a Reidemeister move then the complexes $(C^*,*(D),d)$ and $(C^*,*(D'),d')$ are homotopy equivalent.

We are not going to provide a complete proof of this by any means. The aim is to give you an idea of how things go. Let us look at the first Reidemeister move for a positive twist, so that diagrams $D$ and $D'$ are identical except within a small region where they are shown in Figure 2.

We need to define chain maps $F: C^*,*(D') \to C^*,*(D)$ and $G: C^*,*(D) \to C^*,*(D')$ such that $GF - I$ and $FG - I$ are null-homotopic. The first thing is to notice that we can split the vector space $C^*,*(D)$ as

$$C^*,*(D) = C^*,*(D_0) \oplus C^*,*(D_1)$$

where $D_0$ and $D_1$ are diagrams identical to $D$ except within the small region where they are shown in Figure 3.
EXERCISE 2.2. Understand why there is the shift for $D_1$ in the above decomposition. What happens to the $q$-grading?

The differential $d$ can be written with respect to this splitting as a matrix \( \begin{pmatrix} d_0 & 0 \\ \delta & d_1 \end{pmatrix} \).

EXERCISE 2.3. Describe the map $\delta : C^{i,*}(D_0) \to C^{i-1+1,*}(D_1)$ both algebraically and geometrically.

To define $F : C^{*,*}(D') \to C^{*,*}(D)$ we need to define two coordinate maps $F_0 : C^{*,*}(D') \to C^{*,*}(D_0)$ and $F_1 : C^{*,*}(D') \to C^{*,*}(D_1)$ and then set $F = (F_0, F_1)$. How should we go about constructing maps $C^{*,*}(D') \to C^{*,*}(D_0)$? Given a smoothing $\alpha'$ of $D'$ there is a corresponding one $\alpha$ of $D_0$ (the one that resolves the crossings in the same way). These smoothings look identical outside the small region above. We can construct a cobordism from $\alpha'$ to $\alpha$ by taking a product with $I$ outside the small region and inserting in the missing tube. By applying the TQFT to this cobordism we get a map $V_{\alpha'} \to V_{\alpha}$. As we can do this for each smoothing these maps assemble into a map $C^{*,*}(D') \to C^{*,*}(D_0)$.

We can define another such map by gluing into the missing tube. The map $F_0$ is the difference between the two maps just defined. In pictures $F_0$ is the map defined by

We take $F_1 = 0$ and set $F = (F_0, F_1)$.

As with Frobenius algebras we could write out $F$ more algebraically if we wanted to. For each smoothing $\alpha$ of $D_0$ the vector space $V_{\alpha}$ is of the form $V_{\alpha} = Y_{\alpha} \otimes V \otimes V$ - the last copy of $V$ for the separate circle we see in the picture and the other copy of $V$ for the other circle appearing. The corresponding smoothing of $D'$ has associated to it the vector space $Y_{\alpha'} \otimes V$ - the copy of $V$ for the circle which enters the region shown. In this language $F_0 : C^{*,*}(D') \to C^{*,*}(D_0)$ is the map

$$F_0(y \otimes v) = y \otimes v \otimes 2x - y \otimes \Delta(v)$$

(Remember here that $x$ is the degree -1 generator of $V$).

EXERCISE 2.4. Show that the bi-degree of $F$ is $(0, 0)$.

Now we turn to $G$ where we can be briefer. Define $G_0 : C^{*,*}(D_0) \to C^{*,*}(D')$ by Figure 4 (using the method above) and let $G_1 = 0$.
**Exercise 2.5.** So far $F$ and $G$ are maps of vector spaces: check they are chain maps.

We now claim that $G$ and $F$ are part of a homotopy equivalence i.e. that $GF - I$ is null homotopic and $FG - I$ is null homotopic.

For the first of these we claim $GF = I$ (showing $GF - I$ is null homotopic via a trivial homotopy). This is where using pictures comes into its own: the picture for the composition $GF$ is simply gotten by placing one picture on top of the other as seen in Figure 5.

$$GF = {\begin{array}{c} \includegraphics[width=0.2\textwidth]{figure5a} \\ \includegraphics[width=0.2\textwidth]{figure5b} \end{array}} = 2 \quad {\begin{array}{c} \includegraphics[width=0.2\textwidth]{figure5c} \end{array}} = I$$

**Figure 5**

Next we claim there is a map $H : C^{*,*}(D) \to C^{*,*}(D)$ such that $FG - I = H d + d H$. Using the splitting above $H$ is the matrix $\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}$ where $-h : C^{*,*}(D_1) \to C^{*,*}(D_0)$ is the map gotten by the method above using the picture

We compute $H d + d H = \begin{pmatrix} h \delta & h d_1 + d_0 h \\ 0 & \delta h \end{pmatrix}$. Thus we need to show

(3) $h \delta = F_0 G_0 - I$
(4) $h d_1 + d_0 h = 0$
(5) $\delta h = -I$

Equation (5) is easy: just compose pictures as shown in Figure 6.

**Figure 6**

Equation (4) is essentially just due to the fact that $d_0$ and $d_1$ are defined using the same cobordism (for $d_0$ there is an extra cylinder) and by then looking carefully at the signs in the definition of the differential.

**Exercise 2.6.** Check the assertions of the previous sentence.
Pictorially (3) is shown in Figure 7. I know of no enlightening way to do it, but it is a simple matter to see that this holds for $V$. (Remember the cobordism is the identity outside the region so we need to check the equality for maps $V \otimes V \to V \otimes V$.)

This concludes invariance under Reidemeister I positive twist: we have produced $F$ and $G$ such that $FG - I$ and $GF - I$ are null-homotopic, thus demonstrating that there is a homotopy equivalence $C^{*,\ast}(D') \simeq C^{*,\ast}(D)$.

The above essentially follows Bar-Natan’s proof - though Bar-Natan is cleverer still: in his set-up one constructs a geometric complex and works with tangles. One proves invariance without ever applying a TQFT. This gives rise to a universal theory - more on this in the next lecture. Refer to the end of the lecture for further remarks and a reference.

3. Khovanov homology

Given an oriented link diagram $D$ we now define the Khovanov homology of the diagram $D$ by

$$KH^{*,\ast}(D) = H(C^{*,\ast}(D), d).$$

By the previous section if $D$ is related to $D'$ by a series of Reidemeister moves then there is an isomorphism $KH^{*,\ast}(D) \cong KH^{*,\ast}(D')$. Thus if $L$ is an oriented link it makes sense to talk about the Khovanov homology of the link $L$ (defined up to isomorphism as the Khovanov homology of any diagram representing it).

**Proposition 3.1.**

$$\sum (-1)^i \text{qdim}(KH^{i,\ast}(L)) = \hat{J}(L)$$

**Proof.** It is an exercise in linear algebra to show that

$$\sum (-1)^i \text{qdim}(KH^{i,\ast}(D)) = \sum (-1)^i \text{qdim}(C^{i,\ast}(D))$$

and we have already observed that the right-hand side is $\hat{J}(L)$. 

Khovanov homology is a stronger invariant than the Jones polynomial as the following example illustrates.

**Example 3.2.**

$$D_1 = \quad D_2 =$$
\begin{align*}
\text{Unnormalised Jones polynomial:} & \quad \hat{J}(D_1) = q^{-3} + q^{-5} + q^{-7} - q^{-15} \\
\text{Khovanov homology } KH^{i,j}(D_1): & \\
\begin{array}{|c|c|c|c|c|c|c|}
\hline
j & i & -5 & -4 & -3 & -2 & -1 & 0 \\
\hline
-3 &    &    &    &    & Q &    & \\
-5 &    &    &    & Q &    &    & \\
-7 & Q &    &    &    &    &    & \\
-9 &    &    &    &    &    &    & \\
-11 & Q & Q &    &    &    &    & \\
-13 &    &    &    &    &    &    & \\
-15 & Q &    &    &    &    &    & \\
\hline
\end{array}
\end{align*}

\begin{align*}
\text{Unnormalised Jones polynomial} & \quad \hat{J}(D_2) = q^{-3} + q^{-5} + q^{-7} - q^{-15} \\
\text{Khovanov homology } KH^{i,j}(D_2): & \\
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
j & i & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
\hline
-1 &    &    &    &    &    &    & Q & Q & \\
-3 &    &    &    &    &    & Q &    & Q & \\
-5 &    &    &    &    & Q &    & Q &    & Q & \\
-7 &    &    &    &    &    &    & Q &    & \\
-9 &    &    &    &    &    &    &    & Q & Q & Q & \\
-11 & Q & Q &    &    &    &    &    &    & Q & Q & Q & \\
-13 &    &    &    &    &    &    &    &    &    & Q & Q & Q & \\
-15 & Q &    &    &    &    &    &    &    &    &    & Q & Q & Q & \\
\hline
\end{array}
\end{align*}

The missing rows in the above table (those with even \( q \)-degree) are all trivial. In fact this is a more general phenomenon.

\textbf{Proposition 3.3.} If a link \( L \) has an odd number of components then \( KH^{*,\text{even}}(L) = 0 \). If \( L \) has an even number of components then \( KH^{*,\text{odd}}(L) = 0 \).

\section*{4. Notes and further reading}

To find out more about Frobenius algebras and TQFTs two good places to start are \cite{Kock} and \cite{Abrams}. For a more general treatment of TQFTs consult \cite{Turaev}.

Reidemeister invariance was first proved by Khovanov in his original paper \cite{Khovanov} and proofs can also be found in \cite{Bar-Natan2} which also contains computations of the Khovanov homology of prime knots with diagrams with up to ten crossings. This is where the computation in Example 3.2 is taken from. There are some simple examples in \cite{Wehrli} showing that there exist mutant links (and so having the Jones polynomial) which are separated by Khovanov homology. At the time of writing it is unknown whether mutant \textit{knots} can be separated by Khovanov homology.

The proof of invariance under Reidemeister move I presented above is closer to the proof found in \cite{Bar-Natan1}. In this paper Bar-Natan considers the cube (with smoothings at vertices and cobordisms on edges) as a geometric complex (i.e. a complex in an abelianized category of cobordisms). In order to prove invariance (in the homotopy category of these geometric complexes) one needs to take a quotient of the cobordism category by the relations shown in Figure 8. It is the 4-Tu relation which is used in

\begin{align*}
S &= 0 \\
T &= 2 \\
+ &= + \\
4-\text{Tu} &
\end{align*}

\textbf{Figure 8}

Figure 7. Things are better even than this: the theory is completely local and one works with tangles (we had pictures of tangles, but kept in our minds the fact that these were part of a larger diagram).
CHAPTER 3

Lecture Three

In this lecture we begin by looking at a long exact sequence in Khovanov homology. Then we
examine the kind of functoriality present and briefly discuss the invariants of embedded surfaces in $\mathbb{R}^4$
thus defined. We end with a look at theories defined over different base rings.

1. A long exact sequence

In algebraic topology there are many theoretical tools for computation such as long exact se-
quences, spectral sequence and so on. In Khovanov homology there is less available in the arsenal,
but there is one useful long exact sequence which we now discuss.

If we choose a crossing of a diagram $D$ we can resolve it in the two possible ways to give two new
diagrams $D_0$ and $D_1$ as in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

Ignoring gradings for a moment there is a decomposition of (ungraded) vector spaces
\[ C(D) = C(D_0) \oplus C(D_1). \]

In fact $C(D_1)$ is a sub-complex and there is a short exact sequence
\[ 0 \to C(D_1) \to C(D) \to C(D_0) \to 0. \]

Putting the gradings back in requires a bit of care though it is not hard.

\textbf{Case I:} the selected crossing of $D$ is a negative crossing. In this case $D_1$ inherits an orientation
from $D$ (the 1-smoothing is the orientation preserving smoothing). For $D_0$ there is no orientation
consistent with $D$ so just orient it as you please. Let
\[ c = \text{number of negative crossings in } D_0 - \text{number of negative crossings in } D. \]

Then for each $j$ there is a short exact sequence
\[ 0 \to C^{i,j+1}(D_1) \to C^{i,j}(D) \to C^{i-c,j-3c-1}(D_0) \to 0 \]

and hence a long exact sequence
\[ \delta_* \to KH^{i,j+1}(D_1) \to KH^{i,j}(D) \to KH^{i-c,j-3c-1}(D_0) \to \delta_* \to KH^{i+1,j+1}(D_1) \to \cdots. \]
If we write the differential of $C^\ast,\ast(D)$ as a matrix $\begin{pmatrix} d_0 & 0 \\ \delta & d_1 \end{pmatrix}$ then the boundary map in the long exact sequence is $\delta$, i.e. the map induced in homology (suitably shifted to take account of the new orientation of $D_0$).

**Case II:** the selected crossing of $D$ is a positive crossing. In this case $D_0$ inherits an orientation from $D$ (this time the 0-smoothing is the orientation preserving smoothing). For $D_1$ there is no orientation consistent with $D$ so just orient it as you please. Let $c = \text{number of negative crossings in } D_1 - \text{number of negative crossings in } D$.

Then for each $j$ there is a short exact sequence

$$0 \to C^{\ast-c-1,j-3c-2}(D_1) \to C^{\ast,j}(D) \to C^{\ast,j-1}(D_0) \to 0$$

and hence a long exact sequence

$$\begin{array}{cccccccccccccc}
\delta_\ast & \to & KH^{i-c-1,j-3c-2}(D_1) & \to & KH^{i,j}(D) & \to & KH^{i,j-1}(D_0) & \to & \delta_\ast & KH^{i-c,j-3c-2}(D_1) & \to \\
\end{array}$$

**Exercise 1.1.** Check the gradings in the long exact sequences above.

**Example 1.2.** At the end of Lecture 1 we computed the Khovanov homology of the Hopf link. Let us re-do this calculation using the Khovanov homology of the unknot and the long exact sequence. We choose to resolve the top crossing (a negative crossing) thus giving $D_0$ and $D_1$ as shown in Figure 2

![Figure 2](image)

Here we have $c = -2$ so the long exact sequence is

$$\begin{array}{cccccccccccccc}
\delta_\ast & \to & KH^{i,j+1}(D_1) & \to & KH^{i,j}(D) & \to & KH^{i+2,j+5}(D_0) & \to & \delta_\ast & KH^{i+1,j+1}(D_1) & \to \\
\end{array}$$

Since $D_0$ and $D_1$ are both the unknot they only have non-trivial homology in homological degree 0 (where there are generators in $q$-degree +1 and −1). Thus the long exact sequence breaks up and there are two interesting pieces

(6) $$0 \to KH^{0,j+1}(D_1) \to KH^{0,j}(D) \to 0 \to 0$$

(7) $$0 \to 0 \to KH^{-2,j}(D) \to KH^{0,j+5}(D_0) \to 0$$

From (6) we see that all groups are zero unless $j = 0, -2$ from which we conclude $KH^{0,0}(D) \cong KH^{0,-2}(D) \cong \mathbb{Q}$. Similarly, from (7) we see that all groups are zero unless $j = -4, -6$ from which we conclude $KH^{-2,-4}(D) \cong KH^{-2,-6}(D) \cong \mathbb{Q}$. This result is happily in agreement with the computation at the end of Lecture 1.
2. Functorial properties

It is convenient to study links by projecting onto the plane and studying link diagrams instead. The diagrammatic representation of a given link is far from unique, but this is well understood: two diagrams represent the isotopic links if and only if they are related by Reidemeister moves.

Something similar is true for link cobordisms (recall that a link cobordism $(Σ, L_0, L_1)$ is a smooth, compact, oriented surface $Σ$ generically embedded in $\mathbb{R}^3 \times I$ such that $\partial Σ = L_0 \sqcup L_1$ with $\partial Σ \subset \mathbb{R}^3 \times \{0, 1\}$). A link cobordism can be represented by a sequence of oriented link diagrams - the first in the sequence being a diagram $D_0$ for $L_0$ and the last being a diagram $D_1$ for $L_1$. Two consecutive diagrams in this sequence must be related by a small set of allowable moves which are

1. Reidemeister I, II or III moves,
2. Morse 0-, 1- or 2-handle moves.

The Reidemeister moves are just the usual ones and the Morse moves are shown in Figure 3.

![Figure 3](image)

0–handle | 1–handle | 2–handle

Geometrically the Morse moves are shown in Figure 4.

![Figure 4](image)

0–handle | 1–handle | 2–handle

Such a sequence of diagrams is known as a movie.

**Example 2.1.** Figure 5 is a movie representing a cobordism from the Hopf link to the empty cobordism (drawn across the page rather than down to save space).

A movie representation of a link cobordism is not unique: there may be many different movies of the same cobordism. However, again this is well understood: two movies represent isotopic link cobordisms if and only if they are related by a series of movie moves or by interchanging the levels of distant critical points. Each movie move replaces a small clip of the movie by a different clip. We will not go into this in more detail here.
A movie \((M, D_0, D_1)\) induces a map on Khovanov homology
\[ \phi_M : KH^{*,*}(D_0) \to KH^{*,*+\chi}(D_1) \]
in the following way. (Here \(\chi\) is the number of Morse 0- and 2-handle moves minus the number of Morse 1-handle moves). We will define a map between each two consecutive frames of the movie and then compose all of these to get \(\phi_M\). For Reidemeister moves we have already argued that there is a homotopy equivalence of chain complexes which gives a map in homology and it is this map we take. For a 0-handle move, if the before-frame consists of a link diagram \(D\) then the after frame consists of \(D \sqcup \text{unknot}\). Since \(KH^{*,*}(D \sqcup \text{unknot}) = KH^{*,*}(D) \otimes V\) we take the map \(KH^{*,*}(D) \to KH^{*,*}(D) \otimes V\) to be \(\text{Id} \otimes i\) where \(i : Q \to V\) is the unit of the Frobenius algebra \(V\). Since \(1 \in V\) has \(q\)-degree 1 this map increments \(q\)-degree by one. For the 2-handle move we do a similar thing using the counit of the Frobenius algebra.

For the 1-handle move let \(D\) and \(D'\) be the before- and after-frames of the move. We construct a map \(C^{*,*}(D) \to C^{*,*-1}(D')\) by using the geometric techniques at the beginning of Lecture 2. For each smoothing \(\alpha\) of \(D\) there is a corresponding one \(\alpha'\) of \(D'\) different only in the small region in which the move takes place. A cobordism can be constructed from \(\alpha\) to \(\alpha'\) by taking a product with \(I\) outside the small region and inserting a saddle in the missing tube. Do this for each smoothing, apply the TQFT, assemble the resulting maps and take homology to get a map \(KH^{*,*}(D) \to KH^{*,*+1}(D')\).

**Proposition 2.2.** If \((M, D_0, D_1)\) is related to \((M', D_0, D_1)\) by a sequence of movie moves or interchanging the levels of distant critical points then \(\phi_{M'} = \pm \phi_M\).

We will not prove this theorem. The sign discrepancy is annoying but some movie moves (though not all!) change the sign. In order to say that \(KH^{*,*}(-)\) is a functor you therefore need to projectivize the target category.

### 3. Numerical invariants of closed surfaces

Using the above one can define a numerical invariant of closed oriented surfaces smoothly embedded in \(\mathbb{R}^4\). Such a surface \(\Sigma\) may be regarded as a link cobordism between the empty link and the empty link. Thus, representing \(\Sigma\) by a movie \(M\) and noting that \(KH^{*,*}([0]) = Q\), the above discussion gives us a map \(\phi_M : Q \to Q\). The **Khovanov-Jacobsson number**, of the embedded surface \(\Sigma\) is defined to be \(KJ_\Sigma = |\phi_M(1)|\).

Unfortunately, these numbers are rather disappointing. If \(\chi(\Sigma)\) is non-zero then \(KJ_\Sigma = 0\) since \(\phi_M\) shifts the \(q\)-degree by \(\chi(\Sigma)\) (and both the source and target of \(\phi_M\) are non-zero only in bi-degree \((0, 0))\). For embedded tori there is the following result.

**Proposition 3.1.** If \(\Sigma\) is a smoothly embedded torus in \(\mathbb{R}^4\) then \(KJ_\Sigma = 2\).

### 4. Coefficients and Torsion

So far we have been working over the rational numbers, but nothing we have said so far really relies on this. Everything remains valid (the construction of a complex, the proofs of invariance etc) replacing \(Q\) by any commutative ring with unit. Instead of “vector space” you need to write “projective.
In particular you can work over the integers and ask if, like for ordinary homology of spaces, the interesting phenomenon of torsion emerges. It does.

The first place this is seen is for the trefoil $\bigcirc$. The cube of this trefoil (as requested in the exercise at the end of Lecture 1) is given in Figure 6.

In bi-degree $(-2, -7)$ the cycles are generated by
\[
\{z_1 = (x \otimes x, 0, 0), z_2 = (0, x \otimes x, 0), z_3 = (0, 0, x \otimes x)\}
\]
and in bi-degree $(-3, -7)$ the chains are generated by
\[
\{c_1 = (1, x, x), c_2 = (x, 1, x), c_3 = (x, x, 1)\}.
\]
Recalling the definition of the differential one easily sees
\[
d(c_1) = z_1 + z_3 \quad d(c_2) = z_2 + z_3 \quad d(c_3) = z_1 + z_2
\]
Thus in homology we have $[z_1] = [z_2] = [z_3]$. Note also that
\[
d(c_1 + c_3 - c_2) = z_1 + z_3 + z_1 + z_2 - z_2 - z_3 = 2z_1.
\]
Thus, rationally $z_1$ is a boundary (it is hit by $\frac{1}{2}(c_1 + c_3 - c_2)$) and our potential homology class above is trivial. Over the integers $[z_1]$ is a non-trivial homology class, but $2[z_1]$ is trivial, thus in homology we have a copy of $\mathbb{Z}/2$.

For the record the full integral homology of the trefoil $\bigcirc$ is given below.

| $j$ | $i$ | -3 | -2 | -1 | 0 |
|-----|-----|----|----|----|---|
| -1  |     | $\mathbb{Z}$ |   |    |   |
| -3  |     | $\mathbb{Z}$ |   |    |   |
| -5  |     | $\mathbb{Z}$ |   |    |   |
| -7  |     | $\mathbb{Z}/2$ | |    |   |
| -9  |     | $\mathbb{Z}$ |   |    |   |

In fact torsion abounds as shown in the following result (which we do not prove):
PROPOSITION 4.1. The integral Khovanov homology of every alternating link, except the trivial knot, the Hopf link and their connected sums and disjoint unions, has torsion of order two.

In order to describe Khovanov homology with coefficients in a ring \( R \) in terms of integral Khovanov homology we apply a standard result in homological algebra: the universal coefficient theorem. Since \( C^{*,*}(D; R) = C^{*,*}(D; \mathbb{Z}) \otimes_{\mathbb{Z}} R \) the universal coefficient theorem tells us that there is a short exact sequence

\[
0 \longrightarrow KH^{i,j}(D; \mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow KH^{i,j}(D; R) \longrightarrow \text{Tor}(KH^{i+1,j}(D; \mathbb{Z}), R) \longrightarrow 0.
\]

EXERCISE 4.2. Use the short exact sequence above to compute \( KH^{*,*}(\bigotimes \mathbb{Z}/2) \).

Closely related to this is the Künneth formula, which we can use to compute the Khovanov homology of a disjoint union. Given two link diagrams \( D_1 \) and \( D_2 \) then \( C^{*,*}(D_1 \sqcup D_2) \cong C^{*,*}(D_1) \otimes C^{*,*}(D_2) \) or more precisely:

\[
C^{i,j}(D_1 \sqcup D_2) \cong \bigoplus_{p+q=i} C^{p,s}(D_1) \otimes C^{q,t}(D_2).
\]

Thus the Künneth formula gives us a split short exact sequence

\[
0 \to \bigoplus_{p+q=i} KH^{p,s}(D_1; R) \otimes KH^{q,t}(D_2; R) \to KH^{i,j}(D_1 \sqcup D_2; R)
\]

\[
\to \bigoplus_{p+q=i+1} \text{Tor}_1^R(KH^{p,s}(D_1; R), \otimes KH^{q,t}(D_2; R)) \to 0.
\]

Over \( \mathbb{Q} \) the Tor group is always trivial so we have

\[
KH^{i,j}(D_1 \sqcup D_2; \mathbb{Q}) \cong \bigoplus_{p+q=i} KH^{p,s}(D_1; \mathbb{Q}) \otimes KH^{q,t}(D_2; \mathbb{Q}).
\]

5. Notes and further reading

The long exact sequence is implicit in Khovanov’s original paper, but appeared in a slightly different form in [Viro]. Lee used the singly graded version in [Lee]. It has appeared in a variety of places since then and with gradings as we have given them in [Rasmussen]. This is an interesting survey paper in its own right discussing parallels between Khovanov homology and knot Floer homologies.

You can read about cobordisms and their representations as movies in [CarterSaito].

Khovanov conjectured the functoriality properties in his original paper [Khovanov1]. This was then proved in [Jacobsson] and independently in [Khovanov2]. Using his geometric techniques Bar-Natan proved functoriality (in more generality) in [Bar-Natan].

The proposition about Khovanov-Jacobsson numbers has been proved for a certain class of torus embeddings in [CarterSaitoSatoh] and then for all torus embeddings in [Tanaka] and independently using different techniques in [Rasmussen2].

Torsion in Khovanov homology has been studied in a number of papers. The best places to start would be [Shumakovitch] and [AsaedaPrzytycki]. Both of these prove Proposition 4.1 concerning 2-torsion and the former has a number of interesting conjectures about torsion. One of these conjectures that all torsion is 2-torsion, which is now known to be false. Bar-Natan’s computer program calculates \( KH^{22,73}(T(8,7)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/7 \), where \( T(8,7) \) is the torus link with 7 strands and 8 positive twists.
1. A family of Khovanov-type theories

An obvious question to ask is: can we replace the Frobenius algebra $V$ used to construct the Khovanov complex by some other Frobenius algebra and still get a link homology theory (i.e., an invariant with nice functorial properties)? Or, re-phrased, what conditions must a Frobenius algebra $A$ satisfy to give a link homology theory?

For simplicity let us again take $\mathbb{Q}$ as the base ring. It is relatively easy to see that we must have $\dim(A) = 2$. Consider the two representations of the unknot below.

The first gives a complex

$$0 \rightarrow A \rightarrow 0$$

and the second a complex

$$0 \rightarrow A \rightarrow A \otimes A \rightarrow 0.$$  

In the first, the copy of $A$ is in degree zero and in the second in degree -1. Since both diagrams represent the unknot we require these two complexes to be homotopy equivalent. The Euler characteristic of homotopy equivalent complexes must be equal, hence

$$\dim(A) = -\dim(A) + \dim(A \otimes A)$$

so $2\dim(A) - \dim(A)^2 = 0$ and we conclude $\dim(A) = 2$.

It turns out that additionally one needs $\epsilon(1) = 0$ and this is all that is required for a Frobenius algebra to give rise to a link homology theory. To understand why this is the case one needs to use Bar-Natan’s geometric theory. As explained in the “Further reading” of Lecture 2, Bar-Natan imposes three relations on the category of cobordisms: $S, T$ and $4-Tu$. If we wish to apply a TQFT to his geometric setting, the underlying Frobenius algebra $A$ must satisfy these relations (or at least their algebraic counterparts). The relation $S$ just says $\epsilon(1) = 0$ and $T$ is the condition $\dim(A) = 2$. While the $4-Tu$ relation is necessary geometrically it is automatically satisfied for two dimensional Frobenius algebras satisfying $\epsilon(1) = 0$. This is an indication of the power of Bar-Natan’s geometric approach - things are proved in terms of complexes of cobordisms and so applying a different TQFT (satisfying the necessary conditions) simply gives another theory with no additional effort.

As a vector space we may as write $A = \mathbb{Q}\{1, x\}$ as before. Then normalising so that $\epsilon(x) = 1$ we have a family of theories one for each pair $(h, t) \in \mathbb{Q} \times \mathbb{Q}$. The Frobenius algebra $A_{h,t}$ has
multiplication given by

\[ 1^2 = 1 \quad 1x = x1 = x \quad x^2 = hx + tl \]

and comultiplication given by

\[ \Delta(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1 \quad \Delta(x) = x \otimes x + t1 \otimes 1. \]

The unit and counit are

\[ i(1) = 1 \quad \epsilon(1) = 0 \quad \epsilon(x) = 1. \]

A link homology theory is obtained by carrying out exactly the same construction as outlined in Lecture 1, but replacing the Frobenius algebra \( V \) with \( A_{h,t} \). When \( h = t = 0 \) then you get the original theory of Lecture 1.

The first such variant to be studied was the case \((h, t) = (0, 1)\) by E.S. Lee giving a theory now known as Lee Theory - the topic of the next section.

In fact, up to isomorphism Lee theory is the only other rational theory in the family under discussion.

**Proposition 1.1.** If \( h^2 + 4t = 0 \) then the resulting theory is isomorphic to the original Khovanov homology and if \( h^2 + 4t \neq 0 \) then the resulting theory is isomorphic to Lee theory.

2. Lee theory

The alert reader will have noticed a minor problem in carrying out the construction in Lecture 1 using the Frobenius algebra \( A_{h,t} \). This is that one loses the \( q \)-grading. The degree of \( x^2 = hx + tl \) is not even homogeneous if \( h \) and \( t \) are both non-zero. In fact the only case where a second grading exists is \( h = t = 0 \). For now let us simply ignore the \( q \)-grading: all theories will be singly graded by the homological grading. In the next section we will see that in fact that we do not need to completely throw away the second grading - it just gets replaced by a filtration instead.

Lee theory \((h = 0, t = 1)\) was the first variant of Khovanov homology to appear and remarkably it can be computed explicitly.

**Proposition 2.1.** The dimension of \( \text{Lee}^*(L) \) is \( 2^k \) where \( k \) is the number of components in \( L \).

Things are even better still and there are explicit generators whose construction is as follows. There are \( 2^k \) possible orientations of \( L \). Given an orientation \( \theta \) there is a canonical smoothing obtained by smoothing all positive crossings to 0-smoothings and all negative crossings to 1-smoothings. For this smoothing one can divide the circles into two disjoint groups, Group 0 and Group 1 as follows. A circle belongs to Group 0 (Group 1) if it has the counter-clockwise orientation and is separated from infinity by an even (odd) number of circles or if it has the clockwise orientation and is separated from infinity by an odd (even) number of circles. Figure 1 shows an orientation of the Borromean rings, its canonical smoothing and division into groups.

![Figure 1](image)

Now consider the element in the chain complex for \( L \) defined by labelling each circle from Group 0 with \( x + 1 \) and each circle from Group 1 with \( x - 1 \). It turns out that this defines a cycle, \( s_\theta \), and the homology class thus defined is a generator. Moreover all generators are obtained this way and one has:

\[ \text{Lee}^*(L) \cong \mathbb{Q}\{ [s_\theta] \mid \theta \text{ is an orientation of } L \} \]
This is not supposed to be obvious - read Lee's paper to find out why.

It is also possible to determine the degree of the generators in terms of linking numbers. Let \( L_1, \ldots, L_k \) denote the components of \( L \). Recalling that \( L \) is oriented from the start, if we are given another orientation of \( L \), say \( \theta \), then we can obtain \( \theta \) by starting with the original orientation and then reversing the orientation of a number of strands. Suppose that for the orientation \( \theta \) the subset \( E \subset \{1, 2, \ldots, k\} \) indexes this set of strands to be reversed. Let \( \overline{E} = \{1, \ldots, k\} \setminus E \). The degree of the corresponding generator \( [s_\theta] \) is then given by

\[
\deg([s_\theta]) = 2 \times \sum_{l \in E, m \in \overline{E}} \text{lk}(L_l, L_m)
\]

where \( \text{lk}(L_l, L_m) \) is the linking number (for the original orientation) between component \( L_l \) and \( L_m \).

**Exercise 2.2.** Compute \( \text{Lee}^*(\overline{\bigcirc}) \).

Since Lee theory is a link homology theory (and so has nice functorial properties) one can ask how canonical generators behave under cobordisms.

**Proposition 2.3.** Let \( (\Sigma, L_0, L_1) \) be a cobordism presented by a movie \( (M, D_0, D_1) \). Suppose that every component of \( \Sigma \) has a boundary component in \( L_0 \). Then the induced map \( \phi_M : \text{Lee}^*(D_0) \to \text{Lee}^*(D_1) \) has the property that \( \phi_M([s_{\theta_0}]) \) is a non-zero multiple of \( [s_{\theta_1}] \), where \( s_{\theta_0} \) and \( s_{\theta_1} \) are the orientations induced by the orientation of \( \Sigma \).

3. **Rasmussen’s invariant of knots**

Even though Lee theory is only singly graded it possesses a filtration which can be used to define a new concordance invariant of knots. Recall that originally we defined the \( q \)-grading of a chain \( v \in C^i(D) \) by

\[
q(v) = \deg(v) + i + n_+ - n_-
\]

In Lee theory we end up with elements that are not homogeneous with respect to \( q \)-degree. However, for any monomial \( w \) the quantity \( q(w) \) still makes sense and for an arbitrary element \( v \in C^i(D) \) which can be written as a sum of monomials \( v = v_1 + \cdots v_l \) we set

\[
q(v) = \min\{q(v_i) \mid i = 1, \ldots, l\}.
\]

This defines a decreasing filtration on \( C^*(D) \) by setting

\[
F^k C^*(D) = \{v \in C^*(D) \mid q(v) \geq k\}.
\]

The differential in \( C^*(D) \) is a filtered map and thus Lee theory is a *filtered* theory.

Passing to homology we define for \( \alpha \in \text{Lee}^*(D) \)

\[
s(\alpha) = \max\{q(v) \mid [v] = \alpha\}
\]

i.e. look at all representative cycles of \( \alpha \) and take their maximum \( q \)-value. Now for a knot \( K \) define

\[
s_{\min}(K) = \min\{s(\alpha) \mid \alpha \in \text{Lee}^0(K), \alpha \neq 0\},
\]

\[
s_{\max}(K) = \max\{s(\alpha) \mid \alpha \in \text{Lee}^0(K), \alpha \neq 0\}
\]

and finally *Rasmussen’s s-invariant of K* is

\[
s(K) = \frac{s_{\min}(K) + s_{\max}(K)}{2}.
\]

It turns out that \( s_{\max}(K) = s_{\min}(K) + 2 \) and so \( s(K) \) is always an integer. We have the following properties.

1. \( s(K) \) is an invariant of the concordance class of \( K \),
2. \( s(K_1 \# K_2) = s(K_1) + s(K_2) \),
3. \( s(K^\dagger) = -s(K) \), where \( K^\dagger \) is the mirror image of \( K \).
These are not obvious, but we refer the reader to the original reference for a proof.

In general it is hard to calculate the $s$-invariant of a knot. For positive knots (one which has a diagram with only positive crossings) it is easy.

**Example 3.1.** Let $K$ be a positive knot and $D$ a diagram for $K$. Since all the crossings are positive, there is only one smoothing making up homological degree zero: the canonical smoothing. Thus the canonical generator from the given orientation lies in degree zero. Since $C^{-1}(D) = 0$, the only representative of $[s_\theta]$ is $s_\theta$ itself so $s([s_\theta]) = q(s_\theta)$.

The minimum possible $q$-value in degree zero is when each circle of the canonical smoothing is labelled with $x$, and this does occur: as a monomial in $s_\theta$. Thus $s_{\min}(K) = s([s_\theta]) = q(s_\theta) = -r + n$ where $r$ is the number of circles in the canonical smoothing. Thus $s(K) = -r + n + 1$.

**Exercise 3.2.** Show that the $s$-invariant of the $(p, r)$-torus knot is $(p - 1)(r - 1)$.

One of the most interesting properties of $s$ is that it provides a lower bound for the slice genus (also known as the 4-ball genus). Recall that the slice genus $g^*(K)$ is the minimum possible genus of a smooth surface-with-boundary smoothly embedded in $B^4$ with $K \subset \partial B^4$ as its boundary.

**Proposition 3.3.** $|s(K)| \leq 2g^*(K)$

We will prove this as it is relatively easy and is a great demonstration of the usefulness of functoriality of link homology. Let $\Sigma$ be a smooth surface of genus $g$ smoothly embedded in $B^4$ with boundary the knot $K$. We can remove a small disc from $\Sigma$ to get a smooth cobordism from $K$ to the unknot $U$. We can represent this cobordism by a movie $(M, D, U)$ (here $D$ is a diagram for $K$). Since $\text{Lee}^*(-)$ has the functorial property described in Lecture 3 there is a map

$$\phi_M : \text{Lee}^*(D) \to \text{Lee}^*(U) = \mathbb{Q}\{1, x\}.$$  

It turns out this map has filtered degree $\chi(\Sigma) = -2g$.

Now let $\alpha \in \text{Lee}^0(K)$ be a non-zero element such that $s(\alpha) = s_{\max}(K)$. Again by applying Proposition 2.3 we get $\phi_M(\alpha)$ is non-zero in $\text{Lee}^0(U)$ so

$$1 = s_{\max}(U) \geq s(\phi_M(\alpha)) \geq s(\alpha) - 2g = s_{\max}(K) - 2g.$$

Thus since $s_{\max}(K) = s(K) + 1$ we have $s(K) \leq 2g$ and since this argument applies to any surface (including one of minimal genus) we get

$$s(K) \leq 2g^*(K).$$

Now we can run this entire argument for the mirror image $K^\vee$ giving $s(K^\vee) \leq 2g^*(K^\vee) = 2g^*(K)$. Using the properties of $s$ above this implies $-s(K) \leq 2g^*(K)$ so we conclude $|s(K)| \leq 2g^*(K)$ finishing the proof.

**Proposition 3.4.** The slice genus of the $(p, r)$-torus knot is $(p - 1)(r - 1)/2$.

Using the $s$-invariant the proof of this is now amazingly simple. It is clear that the smooth slice genus is less than or equal to the genus of any Seifert surface. Seifert’s algorithm produces a Seifert surface with Euler characteristic $p - (p - 1)r$, that is of genus $(p - 1)(r - 1)/2$. Thus

$$|s(T_{p, r})| \leq 2g^*(T_{p, r}) \leq (p - 1)(r - 1).$$

But by the exercise above $s(T_{p, r}) = (p - 1)(r - 1)$ and the result follows straight away.

The remarkable thing about this proof is that a combinatorially defined invariant can tell us something about a result which involves smoothness. This is also striking in the following application on exotic smooth structures.

If you want to prove existence of exotic smooth structure on $\mathbb{R}^4$ you can do this if you are in possession of a knot which is topologically slice but not smoothly slice (slice means zero slice genus). Freedman has a result stating that a knot with Alexander polynomial 1 is topologically slice. We now have an obstruction ($s$ being non-zero) to being smoothly slice. So armed with these results all you
need to do to calculate $s$ for those knots known to have Alexander polynomial $1$ hoping to reveal one where $s \neq 0$. The knot in Figure 2, the $(-3, 5, 7)$ pretzel knot has $s = -1$.

Figure 2

4. Notes and further reading

The family of theories discussed at the beginning of the lecture is essentially a by-product of Bar-Natan’s (geometric) universal theory [Bar-Natan1]. One applies a TQFT satisfying certain relations to his theory and thus one only needs to classify the Frobenius algebras corresponding to these theories. This among other related things can be found in [Khovanov4]. It is possible to work over other rings than $\mathbb{Q}$. In particular one can work over the ring $\mathbb{Z}[h, t]$ to get a universal theory. The theory defined over $\mathbb{Z}/2[h]$ (take $t = 0$) is known as Bar-Natan theory. Proposition 1.1 is not hard to show - see [MackaayTurnerVaz] for details.

The reference for Lee theory is [Lee]

We have mentioned that Lee theory is filtered rather than bi-graded. An alternative is to work over $\mathbb{Q}[t]$ where $\deg(t) = -4$ and we have $x^2 = t1$. This gives a genuine bi-graded theory again. By taking a limit over the “times $t$” map one gets back the theory Lee defined. On can compute the bi-graded Lee theory using a spectral sequence (see [Turner] for the analogous case of the bi-graded Bar-Natan theory).

The reference for Rasmussen’s invariant is [Rasmussen1] where a proof of Proposition 2.3 can also be found. Proposition 3.4 was a conjecture (by Milnor) for many years, finally proved in [KronheimerMrowka] using gauge theory. Rasmussen’s is the first combinatorial proof. In fact there is a more general proposition proved in Rasmussen’s paper: for positive knots the $s$-invariant is twice the slice genus.

It was thought for a while (conjectured in [Rasmussen1]) that the $s$-invariant might be equal to twice the $\tau$-invariant in Heegaard-Floer homology. This is now known to be false and a counter-example can be found in [HeddenOrding].
CHAPTER 5

Lecture Five

A natural question is: what else can one categorify? Other knot polynomials are good candidates. In this lecture we offer a brief discussion of Khovanov-Rozansky link homology which categorifies a specialisation of the HOMFLYPT polynomial. We then discuss the topic of graph homology which has its origins in categorifying graph polynomials. The “Notes and further reading” section gives a cursory look at a number of topics which might be covered in a hypothetical set of a further five (or more) lectures.

1. Khovanov-Rozansky homology

The idea here is to categorify (a specialisation of) the HOMFLYPT polynomial. The specialisation in question is the one corresponding to the representation theory of $sl(N)$ and the polynomial $P_N(D)$ is determined by the skein relation

$$q^N P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) - q^{-N} P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) = (q - q^{-1}) P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\end{tikzpicture}),$$

and normalised by

$$P_N(\text{Unknot}) = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

To compute the Jones polynomial one can use the Kauffman bracket which reduces everything to the values (polynomials) assigned to circles in the plane. For $P_N(D)$ things are not quite as simple, however one can reduce things to values assigned to certain planar graphs.

Murakami, Ohtsuki and Yamada have defined a polynomial, $P_N(\Gamma)$, for four-valent planar graphs $\Gamma$ locally modeled on $\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}$. (In fact they use 3-valent graphs with different types of edges: elongate the black blob in the picture in the previous sentence to get a three valent graph looking like $\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}$). This polynomial satisfies the following properties:

1. $P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) = [N]$
2. $P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\end{tikzpicture}) = [2]P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\end{tikzpicture})$
3. $P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) = [N - 1]P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\end{tikzpicture})$
4. $P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) = P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) + [N - 2]P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\end{tikzpicture})$
5. $P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) + P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) = P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\fill[draw=black,fill=white] (1,.5) circle (.5);
\fill[draw=black,fill=white] (.5,0) circle (.5);
\end{tikzpicture}) + P_N(\begin{tikzpicture}[baseline=-.5ex]
\fill[draw=black,fill=white] (0,.5) circle (.5);
\fill[draw=black,fill=white] (.5,.5) circle (.5);
\end{tikzpicture})$
In the above the square brackets refer to the quantum integer, i.e.

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}.$$ 

For an oriented link diagram $D$ resolve each crossing into a 0- or 1-smoothing as indicated in Figure 1.

![Figure 1](attachment:image.png)

There are $2^n$ possible smoothings each of which is a planar graph of the sort above. For $\alpha \in \{0, 1\}^n$ let $\Gamma_{\alpha}$ be the associated graph. The polynomial $P_N(D)$ can be written as a statesum in terms of the polynomials $P(\Gamma_{\alpha})$ as follows.

$$P_N(D) = \sum_{\alpha \in \{0, 1\}^n} \pm q^{h(\alpha)} P_N(\Gamma_{\alpha})$$

The numbers $h(\alpha)$ and the signs are not hard to determine, though we will not elaborate on this here.

The categorification of $P_N(D)$ proceeds in two steps: (1) categorify the polynomial $P(\Gamma)$ i.e. to each graph above assign a vector space and (2) perform the cube construction to get a polynomial associated to a link diagram. You need to carry out (1) in such a way that the (appropriately categorified) properties of $P_N(\Gamma)$ are satisfied and in such a way that allows you to define maps between two graphs that differ locally as 0- and 1-smoothings do. You then need the cube construction to produce a complex whose homology is invariant under the Reidemeister moves.

It is not surprising that some new ingredients are needed. One such ingredient is the notion of a matrix factorization. For a commutative ring $R$ and an element $w \in R$, an $(R, w)$-factorization consists of two free $R$-modules $M^0$ and $M^1$ together with module maps $d^0: M^0 \to M^1$ and $d^1: M^1 \to M^0$ such that

$$d^1 \circ d^0 = w \text{Id}_{M^0} \quad \text{and} \quad d^0 \circ d^1 = w \text{Id}_{M^1}.$$ 

Put differently, $M = M^0 \oplus M^1$ and $d: M \to M$ where

$$d = \begin{pmatrix} 0 & 0 \\ d^0 & d^1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d^2 = wI.$$ 

The element $w \in R$ is called the potential.

**Example 1.1.** Take $M = R \oplus R$ and define $d = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ for $a, b \in R$. This is an $(R, ab)$-factorization.

In fact we want to consider marked graphs: each arc has one or more marks on it. Given a marked four-valent planar graph, to each mark $i$ assign a variable $x_i$ of degree 2. An example is given in Figure 2.
We now assign certain matrix factorizations to local pieces of the graph, which are later tensored together to get something associated to the graph itself.

We indicate the type of local piece, the ring $R$ and the potential $w$ in the table below.

| Local piece | $R$ | $w$ |
|-------------|-----|-----|
| ![Local piece 1](image1) | $\mathbb{Q}[x_i, x_j, x_k, x_l]$ | $x_i^{N+1} + x_j^{N+1} - x_k^{N+1} - x_l^{N+1}$ |
| ![Local piece 2](image2) | $\mathbb{Q}[x_i, x_j]$ | $x_i^{N+1} - x_j^{N+1}$ |
| ![Local piece 3](image3) | $\mathbb{Q}[x_i]$ | 0 |

It is not particularly enlightening in such a short review to provide the factorizations explicitly - please refer to the original source. All these factorizations are then tensored together (over a variety of intermediate rings) to get a factorization $C(\Gamma)$ which, it turns out, is a factorization with $R = \mathbb{Q}[x_i | i \in \text{set of marks}]$ and $w = 0$. In other words we have a length two complex. The homology of this complex is $\mathbb{Z}/2 \oplus \mathbb{Z}$-graded, but is only non-zero in one of the $\mathbb{Z}/2$-gradings. We write $H^*(\Gamma)$ for the homology in this non-zero grading (so $H^*(\Gamma)$ is a graded $\mathbb{Q}$-vector space). The assignment $\Gamma \mapsto H^*(\Gamma)$ categorifies the polynomial $P(\Gamma)$.

**Proposition 1.2.**

$$\sum_i q^i \dim(H^i(\Gamma)) = P_N(\Gamma)$$

We can now move on to defining a link homology theory. Given an oriented link diagram $D$ put one mark on each arc of the link and define 0- and 1-smoothings as in Figure 1. The $2^n$ smoothings $\Gamma_\alpha$ are now marked graphs which as usual we index by the vertices of the cube $\{0, 1\}^n$. 
Let $V_\alpha$ be an appropriately shifted version of $H^*(\Gamma_\alpha)$ and set
\[
C^{i,*}(D) = \bigoplus_{\alpha \in \{0,1\}^n} V_\alpha.
\]

We need a differential and the key thing is to define the “partial” derivatives along edges of the cube. Again we will skip all the details but it is possible to define maps of factorizations as indicated below.

If $\Gamma_0$ and $\Gamma_1$ are graphs that agree outside a small region in which they look like the left and right picture above then $\chi_0$ and $\chi_1$ induce maps $C(\Gamma_0) \to C(\Gamma_1)$ and $C(\Gamma_1) \to C(\Gamma_0)$ and hence maps $\chi_0: H^*(\Gamma_0) \to H^*(\Gamma_1)$ and $\chi_1: H^*(\Gamma_1) \to H^*(\Gamma_0)$. Thus, to each cube edge $\zeta: \Gamma \to \Gamma'$ we can produce a map $d_\zeta: H^*(\Gamma) \to H^*(\Gamma')$. As before set
\[
d = \sum_{\zeta \text{ such that } \text{Tail}(\zeta) = \alpha} \text{sign}(\zeta) d_\zeta.
\]

Miraculously this all works and the following proposition holds.

**Proposition 1.3.** (i) $H(C^{i,*}(D), d)$ is invariant under Reidemeister moves.
(ii) $\sum_{i,j} (-1)^i q^j \text{dim}(H^{i,j}(D)) = P_N(D)$.

Clearly there are many details to check - which is why the paper by Khovanov and Rozansky runs to over one hundred pages!

### 2. Graph homology

The idea of graph homology is to interpret graph polynomials, like the chromatic polynomial, Tutte polynomial etc., as the graded Euler characteristic of a bi-graded vector space. It is much simpler to do this than it is to work with links, since there is no Reidemeister invariance to check. None-the-less, graph homology is interesting in its own right and also serves as a “toy model” revealing the same sort of phenomena that arise in link homology. For example, torsion also occurs in graph homology and is much more abundant and easier to get hold of than in link homology.

Let us look at the example of the chromatic polynomial. Let $G$ be a graph with vertex set $\text{Vert}(G)$ and edge set $\text{Edge}(G)$. The **chromatic polynomial**, $P(G) \in \mathbb{Z}[\lambda]$ is a polynomial which when evaluated at $\lambda = m \in \mathbb{Z}$ gives the number of colourings of the vertices of $G$ by a palette of $m$ colours satisfying the property that adjacent vertices have different colourings.

There is a procedure to calculate $P(G)$ as follows. Number the edges of $G$ by $1, \ldots, n$ and note that there is a one-to-one correspondence between the subsets of edges of $G$ and the set $\{0,1\}^n$. (An edge of $G$ is labelled with 1 if it is present in the subset and 0 otherwise). For $\alpha \in \{0,1\}^n$ define $G_\alpha$ to be the graph with $\text{Vert}(G_\alpha) = \text{Vert}(G)$ and
\[
\text{Edge}(G_\alpha) = \{e_i \in \text{Edge}(G) \mid \text{the } i^{\text{th}} \text{ entry in } \alpha \text{ is a 1}\}.
\]

Now define $r_\alpha = \text{the number of 1's in } \alpha$.
and
\[ k_\alpha = \text{the number of components in } G_\alpha. \]

A state-sum formula for \( P(G) \) is given by
\[
P(G) = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha} \chi^{k_\alpha}.\]

**Exercise 2.1.** Stop reading here and try to categorify \( P(G) \).

To categorify \( P(G) \) we start with a graded algebra \( R \). For \( \alpha \in \{0,1\}^n \) let \( R_\alpha = R^{\otimes k_\alpha} \) and as usual form a cube: associate \( R_\alpha \) to the vertex \( \alpha \). A simple example is shown in Figure 3.

![Figure 3](image)

Now set
\[
C^{i,*}(G) = \bigoplus_{r_\alpha = i} R_\alpha.
\]

To define a differential we follow the usual procedure. For a cube edge \( \zeta : \alpha \rightarrow \alpha' \) note that \( G_{\alpha'} \) either has the same number of components as \( G_\alpha \) or one component less (two components are fused by the additional edge in \( G_{\alpha'} \)). Thus we define \( d_\zeta : R_\alpha \rightarrow R_{\alpha'} \) to be multiplication in \( R \) on copies of \( R \) corresponding to components that fuse (if such exist) and the identity elsewhere. We thus get a complex whose homology is the graph homology of \( G \).

**Proposition 2.2.**
\[
\sum_{i,j} (-1)^i q^j \dim(H^{i,j}(G)) = P(G)|_{\lambda \rightarrow \dim(R)}
\]

**Exercise 2.3.** Would taking \( d = 0 \) for the differential work as well?

There is a long exact sequence in graph homology which categorifies the deletion-contraction relation. Given an edge \( e \) we can form two new graphs \( G - e \) and \( G/e \) where the first has the edge \( e \) deleted and the second contracts it. There is a short exact sequence
\[
0 \rightarrow C^{i-1,j}(G/e) \rightarrow C^{i,j}(G) \rightarrow C^{i,j}(G - e) \rightarrow 0
\]

which gives a long exact sequence
\[
\rightarrow H^{i-1,j}(G/e) \rightarrow H^{i,j}(G) \rightarrow H^{i,j}(G - e) \rightarrow .
\]
3. Notes and further reading

The first graph polynomial to be categorified was the chromatic polynomial in [Helme-GuizonRong]. The dichromatic polynomial was studied in [Stosić]. Torsion in graph homology has been investigated in [Helme-GuizonPrzytyckiRong].

The reference for Khovanov-Rozansky theory is [KhovanovRozansky1]. While this theory is considerably harder to compute than Khovanov’s original homology some progress has been made. In [Rasmussen3] Rasmussen describes the Khovanov-Rozansky polynomial of 2-bridge knots in terms of the HOMFLYPT polynomial and signature. There is an analogue of Lee’s theory investigated by Gornik in [Gornik]. The polynomial of Murakami, Ohtsuki and Yamada is defined and its properties studied in [MurakamiOhtsukiYamada].

Khovanov and Rozansky followed up their paper with a sequel [KhovanovRozansky2] in which they consider the two variable HOMFLYPT polynomial. Prior to Khovanov and Rozansky’s first paper the case $N = 3$ had been treated in a somewhat different manner by Khovanov in [Khovanov5]. Recently, a link with Hochschild homology has been uncovered [Przytycki].

Link diagrams can also be drawn on surfaces and the Jones polynomial can be defined in this context. If the surface $Σ$ is part of the structure then the diagram represents a link in an $I$-bundle over $Σ$. Khovanov homology in this context has been studied in [AsaedaPrzytyckiSikora]. If the surface is not really part of the structure, but rather just a carrier for the diagram (so you can add/subtract handles away from the diagram) then equivalence classes of diagrams are known as virtual links. Khovanov homology of these has been studied in [Manturov] and [TuraevTurner].

The Jones polynomial corresponds to the 2-dimensional representation of $U_q(sl_2)$ and allowing other representations leads to the coloured Jones polynomial. A link homology theory categorifying this was defined in [Khovanov3].

One of the most interesting questions surrounding the subject is to uncover the geometry that lies behind Khovanov homology. A proposal for a framework unifying Khovanov-Rozansky homology and knot Floer homology has been put forward in [DunfieldGukovRasmussen].

In a different direction P. Seidel and I. Smith have constructed a homology theory for links using symplectic geometry [SeidelSmith]. This theory is conjectured to be isomorphic to Khovanov homology (after suitably collapsing the bi-grading into a single grading). Building on this Manolescu has constructed a similar theory for each $N$ and conjectured this to be isomorphic to Khovanov-Rozansky homology [Manolescu].

Another exciting direction is to try to give some “physical” interpretation for Khovanov homology (in the sense that Witten gave a physical interpretation of the Jones polynomial as the partition function of a quantum field theory). S. Gukov, A. Schwartz and C. Vafa have made an attempt in this direction [GukovSchwartzVafa] conjecturing a connection to string theory.

Last, but certainly not least, there has been a huge effort to write computer programs to calculate Khovanov homology groups. The first of these by Bar-Natan (using Mathematica) coped with links up to 11 or 12 crossings. This was improved on by Shumakovitch [ShumakovitchKhoHo] with a program using Pari. Bar-Natan now has a nice theoretical trick which speeds things up considerably. This has been implemented by Jeremy Green and you can download the package at from the (wonderful) knot atlas (set up by Dror Bar-Natan and Scott Morrison).

http://katlas.math.toronto.edu/wiki/
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