A NOTE ON ALGEBRAIC RICCATI EQUATIONS ASSOCIATED
WITH REDUCIBLE SINGULAR M-MATRICES

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ABSTRACT. We prove a conjecture about the minimal nonnegative solutions
of algebraic Riccati equations associated with reducible singular M-matrices.
The result enhances our understanding of the behaviour of doubling algorithms
for finding the minimal nonnegative solutions.

1. INTRODUCTION

For the algebraic Riccati equation
\[ XCX - XD - AX + B = 0, \]
where \( A, B, C, D \) are real matrices of sizes \( m \times m, m \times n, n \times m, n \times n \), respectively,
a systematic study was done in [1] when
\[ K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \]
is a nonsingular M-matrix or an irreducible singular M-matrix. The study was re-
cently extended in [2] to reducible singular M-matrices under suitable assumptions.

A real square matrix \( A \) is called a Z-matrix if all its off-diagonal elements are
nonpositive, so any Z-matrix \( A \) can be written as \( sI - B \) with \( B \geq 0 \). A Z-matrix \( A \)
is called an M-matrix if \( s \geq \rho(B) \), where \( \rho(\cdot) \) is the spectral radius; it is a singular
M-matrix if \( s = \rho(B) \) and a nonsingular M-matrix if \( s > \rho(B) \).

Some regularity assumption is needed to guarantee the existence of a solution of
the equation (1) associated with the M-matrix \( K \). An M-matrix \( A \) is said to be
regular if \( Av \geq 0 \) for some \( v > 0 \).

The following result is proved in [2].

**Theorem 1.** Suppose the matrix \( K \) in (2) is a regular M-matrix. Then (1) has a
minimal nonnegative solution \( \Phi \) and \( D - C\Phi \) is a regular M-matrix, and the dual
equation
\[ YBY - YA - DY + C = 0, \]
has a minimal nonnegative solution \( \Psi \) and \( A - B\Psi \) is a regular M-matrix. Moreover,
\( I_m - \Phi\Psi \) and \( I_n - \Psi\Phi \) are both regular M-matrices.

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Council of Canada.
Associated with the matrix $K$ in (2) is the matrix

\[
H = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} K = \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}.
\]

When the matrix $K$ in (2) is regular singular $M$-matrix, the following assumption has been introduced in [2].

**Assumption 1.** The matrix $H$ in (4) has only one linearly independent eigenvector corresponding to the zero eigenvalue of multiplicity $r \geq 1$.

It is known [1] that the assumption is satisfied with $r = 2$ when $K$ is an irreducible singular $M$-matrix.

Under Assumption [1] there are nonnegative nonzero vectors \( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \) and \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \), where $u_1, v_1 \in \mathbb{R}^n$ and $u_2, v_2 \in \mathbb{R}^m$, such that

\[
K \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0, \quad [u_1^T \ u_2^T]K = 0.
\]

They are each unique up to a scalar multiple [2].

The purpose of this note is to provide an affirmative answer to a conjecture in [2], regarding the matrices $I_m - \Phi \Psi$ and $I_n - \Psi \Phi$.

2. The result

The following result was conjectured to be true in [2], and was proved under the restrictive assumption that at least one of $\Phi$ and $\Psi$ is positive.

**Theorem 2.** Let $K$ be a regular singular $M$-matrix with Assumption [2]. If $u_1^T v_1 \neq u_2^T v_2$, then $I_m - \Phi \Psi$ and $I_n - \Psi \Phi$ are nonsingular $M$-matrices.

**Proof.** By Theorem [1] $I_m - \Phi \Psi$ and $I_n - \Psi \Phi$ are both $M$-matrices. So we just need to show they are nonsingular when $u_1^T v_1 \neq u_2^T v_2$. Since $I_n - \Psi \Phi$ is nonsingular if and only if $I_m - \Phi \Psi$ is nonsingular, we only need to show $I_n - \Psi \Phi$ is nonsingular.

In view of

\[
\begin{bmatrix} I_n & 0 \\ -\Phi & I_m \end{bmatrix} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} I_n & \Psi \\ 0 & I_m - \Phi \Psi \end{bmatrix},
\]

we need to show that the matrix

\[
\begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix}
\]

is nonsingular. Since $\Phi$ and $\Psi$ are solutions of (1) and (3), respectively, it is easily verified that [2]

\[
\begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix},
\]

where $R = D - C\Phi$ and $S = A - B\Psi$ are $M$-matrices. Therefore, the eigenvalues of $R$ and $S$ are all in the closed right half plane, with 0 being the only possible eigenvalue on the imaginary axis. When $u_1^T v_1 \neq u_2^T v_2$, we know from [2] that one of the matrices $R$ and $S$ is singular and the other is nonsingular. It follows that the matrices $R$ and $-S$ have no eigenvalues in common.

Let

\[
W = \text{Ker} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix}.
\]
We need to show $W = \{0\}$.

For any $x \in W$, post-multiplying (4) by $x$ shows that $Tx \in W$, where $T$ is the linear transformation from $\mathbb{C}^{m+n}$ to $\mathbb{C}^{m+n}$, defined by

$$T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $y_1 \in \mathbb{C}^n$ and $y_2 \in \mathbb{C}^m$. Thus $W$ is an invariant subspace of the linear transformation $T$. Suppose $W \neq \{0\}$. Then we have $0 \neq w \in W$ such that $Tw = \lambda w$ for some $\lambda \in \mathbb{C}$. Write $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, where $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^m$. We then have $Rw_1 = \lambda w_1$ and $-Sw_2 = \lambda w_2$. Since $R$ and $-S$ have no eigenvalues in common, one of $w_1$ and $w_2$ must be a zero vector. It then follows from

$$\begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

that $w_1$ and $w_2$ are both zero vectors. The contradiction shows that $W = \{0\}$. □

It has been explained in [2] that when $K$ is a regular singular $M$-matrix with Assumption 1 and $u_1^T v_1 \neq u_2^T v_2$, doubling algorithms (such as SDA in [4] and ADDA in [5]) can be used to find the minimal nonnegative solutions $\Phi$ and $\Psi$ simultaneously. But before Theorem 2 is proved in this note, there were a few subtle issues associated with the doubling algorithms. We now have the following modification of [2] Theorem 15 about the ADDA, which uses two parameters $\alpha$ and $\beta$. The ADDA is reduced to the SDA when $\alpha = \beta$.

**Theorem 3.** Let $K$ be a regular singular $M$-matrix with Assumption 1 and $u_1^T v_1 \neq u_2^T v_2$. Assume that $\alpha \geq \max a_{ii} > 0$ and $\beta \geq \max d_{ii} > 0$. Then the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular $M$-matrices for each $k \geq 0$. Moreover, $E_0 \leq 0$, $F_0 \leq 0$, $E_k \geq 0$, $F_k \geq 0$, $0 \leq H_{k-1} \leq H_k \leq \Phi$, $0 \leq G_{k-1} \leq G_k \leq \Psi$ for all $k \geq 1$, and

$$\limsup_{k \to \infty} \sqrt[k]{\|H_k - \Phi\|} \leq r(\alpha, \beta), \quad \limsup_{k \to \infty} \sqrt[k]{\|G_k - \Psi\|} \leq r(\alpha, \beta),$$

where $r(\alpha, \beta) = \rho \left( (R + \alpha I)^{-1} (R - \beta I) \right) \cdot \rho \left( (S + \beta I)^{-1} (S - \alpha I) \right) < 1$ with $R = D - C\Phi, S = A - B\Psi$.

Since we have now proved that $I - \Phi\Psi$ and $I - \Psi\Phi$ are nonsingular $M$-matrices we can use the approach in [3] to prove that the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular $M$-matrices for each $k \geq 0$ even when $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$. That $r(\alpha, \beta) < 1$ in Theorem 3 is already known in [2]. Thus, $H_k$ converges to $\Phi$ quadratically, and $G_k$ converges to $\Psi$ quadratically.

By [3] Theorem 2.3], the parameters $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$ minimize $r(\alpha, \beta)$ among all parameters $\alpha \geq \max a_{ii}$ and $\beta \geq \max d_{ii}$. Therefore, we should normally use the optimal values $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$ for the ADDA. Before Theorem 2 is proved, we avoid using $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$, to ensure that the ADDA is well defined.

The matrices $(I - G_k H_k)^{-1}$ and $(I - H_k G_k)^{-1}$ appear in the ADDA. With the proof of Theorem 2 we now know that, in Theorem 3 the matrices $I - G_k H_k$ and $I - H_k G_k$ will not converge to singular matrices. This is of course favorable for the ADDA.
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