Development of free surface flow between concentric cylinders with vertical axes

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Abstract. Numerical and experimental studies are conducted on flows developing between two concentric cylinders with vertical axes. The inner cylinder rotates and the outer and the lower end wall are fixed. The upper boundary is a free surface. The flow is at rest in an initial state, and the inner cylinder impulsively begins to rotate or its rotation speed linearly increases to a prescribed value. The acceleration rate of the inner cylinder changes the formation processes of flows and/or the final flow modes. Time-dependent flows appear at higher Reynolds numbers, and the numerical and experimental results of the power spectra show some agreements. It is suggested that critical Reynolds numbers appear, at which the fluctuations in the displacement of the free surface and the kinetic energy of a velocity component steeply increase.

1. Introduction

Rotating flows with free surfaces and/or interfaces appear many kinds of fluid machinery and chemical reactors, and the investigation of the physical and mathematical characteristics of these flows is important. Especially, Taylor-Couette flow is expected to present attractive phenomena when it has a free surface, even though it has a simple geometry.

Besides flow between cylinders of infinite lengths and flows between finite cylinders with fixed end walls, variety of topics of Taylor-Couette system have been proposed. Even in cases of finite cylinders, flows between a rotating end wall and a stationary end wall and the stability limit of flows between cylinders with one free-slip wall were investigated [1][2]. Abshagen et al. [3] rotated the end walls with a synchronized speed different from the speeds of the inner and outer cylinders, and they showed two-cell flows with inward or outward flows on the end walls.

When the flow is two phase, Campero and Vigil [4] observed the spatiotemporal flow patterns, and Gelfgat et al. [5] examined the deformation of a liquid and liquid interface. For flows with free surfaces, Michalland et al. [6] investigated a viscous fingering formed around a rotating cylinder with a horizontal axis. Nakamura et al. [7] and Toya et al. [8] showed their interest in free surface flows between rotating and stationary cylinders with vertical axes. They estimated the effect of the free surface on the height of vortex cells and classified the flows into normal modes and anomalous modes. At relatively small Reynolds number, circular waves appear on the free surface and traveling waves emerge at higher Reynolds numbers.
Figure 1. Taylor-Couette flow with a free surface. The axes of coaxial cylinders are parallel to the direction of the gravitational force $g$. The inner cylinder rotates, and the outer cylinder and the bottom end wall are fixed.

In this paper, we carry out numerical study and experimental measurement on Taylor-Couette flow with a free surface (figure 1), and examine its dynamical phenomena. In the followings, we give our numerical formulation and an explanation of the experimental method in section 2. In section 3 we present experimental results of the flow patterns and the power spectra of the surface displacement, and the numerical results of the flow developments, power spectra and the dependencies of bulk flows on the Reynolds number. Finally we give some conclusions.

2. Formulation and experimental apparatus

2.1 Formulation

In Taylor-Couette system investigated in this study, the outer cylinder remains stationary and the inner cylinder rotates, and their axes are parallel to the direction of the gravitational force. The lower boundary is a fixed end wall, and the upper boundary is a free surface between a working fluid and the air. The reference length $D$ is the gap width between the coaxial cylinders and the reference velocity $U$ is the characteristic velocity of the inner cylinder in a steady state. The reference time is the advection time scale given by the fraction of the reference length to the reference velocity. Dimensionless quantities are evaluated by these reference values.

The Reynolds number $Re$, the Froude number $Fr$ and the Weber number $We$ are given by

$$ Re = \frac{UL}{\nu}, \quad Fr = \frac{U}{\sqrt{g|D|}}, \quad We = \frac{\rho DU^2}{\sigma}, \quad (1) $$

where $\nu$ is the kinematic viscosity, $\rho$ is the density and $\sigma$ is the surface tension of the working fluid. The gravitational acceleration is given by $g$. The aspect ratio $\Gamma$ is the fraction of the height of the quiescent working fluid to the gap width $D$, and the radius ratio $\eta$ is the fraction of the inner cylinder radius to the outer cylinder radius.

Let $F$ give a value of the fractional volume of the fluid occupying a numerical cell surrounded by calculation grid lines. The governing equations are the unsteady axisymmetric Navier-Stokes equations, the equation of continuity and the conservation equation of $F$:

$$ \frac{\partial u}{\partial t} + \nabla \cdot (uu) = -\nabla p + \frac{1}{Re} \nabla^2 u + \frac{g}{Fr^2}, \quad (2) $$

$$ \nabla \cdot u = 0, \quad (3) $$

$$ \frac{\partial F}{\partial t} + \nabla \cdot (uF) = 0. \quad (4) $$

Each equation is expressed in the dimensionless cylindrical coordinate system $(r, \theta, z)$, where the direction of $z$ is upward. The vector $u$ is the velocity vector with components $(u, v, w)$, $p$ is the pressure and $g$ is the vector of the gravitational acceleration with its components $(0, 0, g)$. The time
\( t \) is made dimensionless by the advection time scale \( D/U \). The viscous time \( t_{vis} \) is also introduced, which is made dimensionless by the viscous time scale \( D^2/\nu \), and \( t_{vis} = t/Re \).

The free surface is modeled by the volume-of-fluid method. A five-point formula [9] is used to obtain \( z_s(r) \) which is a smoothed profile of the free surface. The first partial derivative and the second partial derivative of the free surface profile with respect to \( r \) are evaluated [10], and the effect of the surface tension is considered. The curvature of the profile of the free surface is

\[
\kappa = \frac{\partial^2 z_s / \partial r^2}{(1 + (\partial z_s / \partial r)^2)^{3/2}},
\]

and the increment in the pressure, \( \Delta p_s \), is given by

\[
\Delta p_s = -\frac{\kappa}{W_e}.
\]

The values of \( \partial z_s / \partial r \) on the inner cylinder wall and the outer cylinder wall are set to be zero. While not only the curvature in the radial direction but the curvature in the direction orthogonal to the radial direction appears [11], we assume that the effect of the additional curvature is small.

On the inner and outer cylinder walls and the bottom end wall, the boundary condition of the pressure is the Neuman condition estimated from the momentum equations, and the no-slip condition is imposed on the velocity components.

In the momentum equations, the convection terms are discretized by the third-order upward difference method and another terms are formulated by the second-order central difference method. The first-order upward difference method is used for the conservation equation of \( F \). The typical number of grid points in the radial direction is 81, and the number of points in the axial direction is 80 per a unit aspect ratio. The time interval \( \Delta t \) is 0.0001. We confirmed that halving the spatial and temporal intervals did not make any recognizable differences in numerical results. The result of the steady flow showed that the torque acting on the inner cylinder was equal to the sum of torques on the outer cylinder and the bottom end wall, and the kinetic balance was established.

2.2 Experimental apparatus

The experimental apparatus consists of cylinders with radii of 40 mm and 60 mm, and the radius ratio \( \eta \) is 0.667. The inner cylinder is made by alumina and the outer cylinder is acrylic. The inner cylinder is driven by a DC servo motor. The outer cylinder and the lower end wall are fixed. The working fluid is silicon oil, and a small amount of aluminum flakes is used for visualization.

A video camera and a still camera are used to observe flows in an azimuthal section illuminated by a slit light beam, as well as flows in a view from the top. The variation of the displacement of the free surface is measured by an ultrasonic sensor with measuring length from 30 mm to 70 mm. Though this sensor is not a velocimeter, it has a resolution of 0.2 mm and a high response time of 2 ms. The analog data obtained by the sensor is digitalized and processed by a personal computer.

2.3 Conditions

The surface tension \( \sigma \) and the kinematic viscosity \( \nu \) of the silicon oil largely depend on the temperature and others. In this experiment, the average values of them are \( 2.02 \times 10^{-3} \) N/m and 9.54 mm²/s, respectively, and the specific density \( \rho \) is 0.935. The gravitational acceleration has a standard value of -9.8 m/s². When the comparison between numerical results and experimental results is required, these physical values are used to convert dimensional and dimensionless values.

Numerical and experimental studies have been carried out in the same geometrical conditions. The aspect ratio \( \Gamma \) is fixed to be 5.7 and the radius ratio \( \eta \) is 0.667. Our experiment has confirmed that flows at this aspect ratio show their stability and the probability of the appearance of a unique flow is relatively high. The critical Reynolds number for the onset of Taylor vortex between infinite cylinders is estimated to be 76.3 at \( \eta = 0.667 \) [12].
In the initial state, the flow is at rest. The rotation speed of the inner cylinder is impulsively increased or linearly accelerated to a prescribed rotation speed. The time is measured from the point when the inner cylinder begins to rotate.

3. Results

3.1 Overview

Nakamura et al. [7] and Toya et al. [8] carried out experimental studies on Taylor-Couette flow with a free surface. They determined bifurcation diagrams and found, at $\Gamma = 5.7$, the primary normal five-cell mode, the normal seven-cell mode, the anomalous four-cell mode with an anomalous cell on the bottom end wall, the anomalous six-cell mode with an anomalous cell on the bottom end wall, and the anomalous six-cell mode with an anomalous cell near the free surface.

In the present experiment, the flow of the normal five-cell mode is observed and measured. The experimental results show that the motion of the free surface is found at the Reynolds number above 300. At the Reynolds number below around 1000, the flow is an axisymmetric Taylor vortex flow and

![Flow developments](image)

(a) Flow with a sudden increase in the rotation speed of the inner cylinder. The final flow is a normal five-cell mode.

(b) Flow with a linear increase in the rotation speed of the inner cylinder. The acceleration terminates at $t = 164.8 \ (t_{\text{vis}} = 0.118)$. The final flow is a normal seven-cell mode.

Figure 2. Developments of a suddenly stated flow and a linearly accelerated flow at $Re = 1395, Fr = 1.503$ and $We = 409.9$ (numerical results). The inner cylinder is on the left-hand side and the outer cylinder is on the right-hand side, and the fixed bottom end wall is on the lower side and the free surface is on the upper side. The final dimensionless time $t = 400.0$ corresponds to the viscous time $t_{\text{vis}} = 0.287$. 
the flow has a weak sloshing on the free surface. When the Reynolds number is beyond 1060, circular and axisymmetric waves appear, where the wave just inside of the outer cylinder is higher than another inner waves. At higher Reynolds numbers, the axisymmetry of the free surface profile is disturbed. Then traveling waves appear on the free surface and the flow is highly deformed with the further increase in the Reynolds number. We will see the dependency of the free surface pattern on the Reynolds number in section 3.3. The observed critical Reynolds number for the onset of the wavy Taylor vortex flow with a bulk motion is 999, and this value is lower than the value at which the axisymmetry of the free surface is disturbed.

The present numerical results shows that the time variation of the flow can be found at the Reynolds number above 700, while the motion of the flow at the Reynolds number below 700 is small or negligible. When the acceleration rate of the inner cylinder is large, the major mode of flows developing from the initial state at rest is a normal seven-cell mode. As will be seen below, the final flow mode depends on the acceleration rate. In this numerical study, the normal five-cell mode and the anomalous six-cell mode with an anomalous cell on the bottom end wall, as well as the seven-cell mode, appear. The acceleration rate of the inner cylinder also has an effect on the time dependency of the final flow. That is, a flow with a relatively large time variation and a flow with a smaller time variation may appear even at a constant Reynolds number, and the extent of the variation is determined by the acceleration rate.

3.2 Formation processes of final flows

In Taylor-Couette system with fixed end walls, the final flow modes and the formation processes of the flows depend on the acceleration rate of the inner cylinder starting from rest [13]. Developments of flows with free surfaces at $Re = 1395$ are shown in figure 2 as variations of a topology of the Stokes’ stream function $\Psi(r,z)$ given by

$$u = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad w = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (7)$$

The Froude number $Fr$ and the Weber number $We$ are 1.503 and 409.9, respectively. In each figure, the rotating inner cylinder is on the left hand side and the stationary outer cylinder is on the right hand side, and the stationary fixed end wall is lower and the free surface is upper. A cluster of closed contour lines almost corresponds to a vortex cell which is observed by the visualization in experiment [13]. Figure 2 (a) gives the flow development when the rotation speed of the inner cylinder is impulsively increases to a prescribed value giving $Re = 1395$ at $t = 0.0$. Figure 2 (b) shows the result when the inner cylinder is linearly accelerated during 5 seconds, and the acceleration ends at $t = 164.8$ in the advection time scale, which is $t_{vis} = 0.118$ in the viscous time scale. While the flow with the impulse increase in the rotation speed contains seven cells in its intermediate state at about $t = 100$ ($t_{vis} = 0.0717$), its final state shows a normal five-cell mode. When the flow is accelerated gradually, the final flow is a normal seven-cell mode. These say that the final state has multiple flow modes even at a constant Reynolds number. This result agrees with the result obtained in Taylor-Couette flows with fixed end walls [13].

Flow developments at $Re = 1436$ are shown in figure 3. Figure 3 (a) represents the case where the flow is impulsively accelerated and figure 3 (b) gives the flow accelerated to the prescribed Reynolds number during 1 second (33.9 in the advection time scale and 0.0236 in the viscous time scale). The final state of the suddenly started flow is an anomalous six-cell mode. The lowest cell is an anomalous cell and it has an outward flow on the end wall. We have confirmed that the anomalous cell is accompanied with extra cells which appear at the corner between the outer cylinder and the end wall and at the corner between the inner cylinder and the end wall [14]. When the rotation speed increases linearly, the final flow is a normal five-cell mode, while seven cells appear in the intermediate states at about $t = 140$ ($t_{vis} = 0.0975$).

Figure 4 shows stream patterns near the free surface of the flow at $Re = 1436$. A snapshot of an experimental observation of the fully developed flow with five cells is given in figure 4 (a). Figure 4 (b) represents vectors $(u, w)$ of the flow shown in figure 3 (b). As the rotation speed of the inner
cylinder linearly increases, a small vortex appears on the inner cylinder wall. After some vortices are induced, a vortex occupying the whole range of the gap width appears. In the final state, the flow at the free surface is outward and the position of the surface is higher at the outer side of the annulus. The agreement of the numerical result and the experimental observation is favourable.

3.3 Flows observed in top views

When the Reynolds number is low, a weak sloshing emerges or circular waves appear on the free surface. As the Reynolds number increases, the axisymmetric profile of the free surface disappears and waves traveling in the circumferential direction grow. Figure 5 shows the patterns on free surfaces in top views. The flow at \( Re = 783 \) had a weak surface motion. When the Reynolds number is less than 1260, the flow shows its almost perfect axisymmetry. At \( Re = 1697 \), though the flow party has small traveling waves, the deformation of the flow around the inner cylinder is not large. However, the flow at \( Re = 1887 \) has a distorted pattern in the circumferential direction and it includes

(a) Flow with a sudden increase in the rotation speed of the inner cylinder. The final flow is an anomalous six-cell mode, and the lowest cell has an outward flow on the end wall.

(b) Flow with a linear increase in the rotation speed of the inner cylinder. The acceleration terminates at \( t = 33.9 \) (\( t_{vis} = 0.0236 \)). The final flow is a normal five-cell mode.

Figure 3. Developments of a suddenly stated flow and a linearly accelerated flow at \( Re = 1436, Fr = 1.547 \) and \( We = 434.3 \) (numerical results). The final time \( t = 400.0 \) corresponds to the viscous time \( t_{vis} = 0.279 \).
Figure 4. Velocity pattern in the azimuthal section of the flow with a linear increase in the rotation speed of the inner cylinder. The acceleration terminates at $t = 33.9$ ($v_{vis} = 0.0236$) and the final flow is a normal five-cell mode. $Re = 1436$, $Fr = 1.547$ and $We = 434.3$.

Figure 5. Experimental observations of the free surface flows in top views. In each photo, the circular white plane in the center is the inner cylinder and the outer thin rim is the outer cylinder. The rotation direction of the inner cylinder is counterclockwise.
larger traveling waves. This result shows that the numerical analysis based on an axisymmetric formulation gives reasonable results at the Reynolds number up to one thousand and some hundreds.

3.4 Spectrum analysis of time-dependent phenomena

Power spectra of the displacement of the free surface are evaluated from numerical and experimental results, and power spectra of the volume-averaged energy of the axial velocity component are obtained from numerical results. The volume-averaged energy $E_w(t)$ is given by

$$E_w(t) = \frac{1}{Q} \int_0^Q \frac{w^2}{2} dq,$$

where $Q$ is the volume of the working fluid between the cylinders. Figure 6 shows the power spectra of the experimentally measured displacement, numerically predicted displacement and $E_w(t)$ at $Re = 714, 1395$ and $1550$.

One or more than one peaks appear in each power spectrum. For example, the experimental result at $Re = 714$ has its frequency peaks at 1.59Hz and 3.05 Hz, where the higher frequency peak appears

(a) Displacement of the free surface position (experimental results).

(b) Displacement of the free surface position (numerical results).

(c) Volume-averaged kinetic energy of the axial velocity component (numerical results).

Figure 6. Power spectra of the time-dependent variations of surface displacement and the volume-averaged energy in fully developed flows. In the conditions shown in section 2.3, the rotation frequencies of the inner cylinder at $Re = 714, 1395$ and $1550$ are 1.35 Hz, 2.65 Hz and 2.94 Hz, respectively.
to be the second harmonic of the lower peak. The numerical result at $Re = 714$ gives the peaks of the
displacement spectrum at 0.665 Hz and 1.37 Hz, and the peaks of the energy spectrum at 1.37 Hz and
0.582 Hz. Some agreements are found between the numerical results and the experimental results.

In the experiment, we observed that time-dependent motion of the free surface appears at the
Reynolds number above 300.0. The numerical results suggested that the observable variation is found
at the Reynolds number above 700.0. The difference between the numerical and experimental results,
as well as the difference shown in figure 6, may be caused by numerical errors, imperfections of
experimental conditions and the finiteness of the measuring region of the ultrasonic sensor. Figure 7
shows the numerical results of the power spectra of the displacement at three radius positions. The
profile of the spectrum depends on the radius position. It can be said that experimental result obtained
from the finite measuring region is different from the numerical result evaluated at a pinpoint, though
some another problems must be considered.

3.4 Growth of disturbances

Variances of the time-varying values of the volume-averaged energy $E_w(t)$ and the free surface
displacement at $r = 2.5$ are evaluated from the numerical results to see the growth of disturbances
with the increase in the Reynolds number. As has been said, the flow mode depends on the history of
the acceleration of the inner cylinder, even though the final Reynolds number is fixed. In order to
estimate a unique growth of the disturbances, the flow mode at a constant Reynolds number is limited

![Figure 7](image7.png)

Figure 7. Dependency of the power spectrum of the surface displacement on the radius
position (numerical results). $Re = 1395$. The scale of ordinate is fixed. The inner cylinder
is at $r = 2.0$ and the outer cylinder is at $r = 3.0$.

![Figure 8](image8.png)

(a) Volume-averaged energy of the axial
velocity component.  
(b) Displacement of the free surface.

Figure 8. Variations in the variances of the volume-averaged energy of the axial velocity
component and the displacement of the surface with the Reynolds number (numerical
results). $r = 2.5$.  

to one. We adopted the following way to predict one flow at a Reynolds number $Re_1$. First, a fully developed and unsteady flow at a Reynolds number $Re_0$ is obtained. Then the Reynolds number is linearly changed from $Re_0$ to $Re_1$. Finally, a developed flow at the Reynolds number $Re_1$ is obtained after a sufficient long time. In this study, we set $Re_0$ to be 1000, at which the flow is a unsteady flow of the normal seven-cell mode. After the change of the Reynolds number, the reference velocity is evaluated by the rotation speed of the inner cylinder at $Re_1$.

Figure 8 shows the numerical result of the effects of the Reynolds number on the variances of the volume-averaged energy $E_w(t)$ and the variances of the displacement of the free surface. The profile of the energy has its maximum and minimum at about $Re = 1100$ and 1400, respectively. Beyond the Reynolds number of 1500, the values of the variances of the energy and the displacement increase considerably. The onset of these increases corresponds to the mergence of the circumferential deformation of flows shown in figure 5. Further study is necessary to determine what kind of bifurcation appears at this growth.

4. Conclusions
In Taylor-Couette system with a fixed bottom end wall and a upper free surface, numerical and experimental studies are conducted to examine the dynamic phenomena of the flow. Followings are the main conclusions obtained here.

The development of flows from rest is numerically predicted. It is shown that more than one flow mode appear at a constant Reynolds number and the final flow modes depend on the acceleration rate of the inner cylinder.

Power spectra are obtained from the experimental result of the displacement of the free surface and the numerical results of the displacement and the volume-averaged energy of the axial velocity component. Some of peaks in the profiles of numerical and experimental results show qualitative agreements with each other.

Time variations of the volume-averaged energy and the displacement of the free surface are estimated from the numerical results and the dependencies of their variances on the Reynolds number are evaluated. The values of the variances show rapid increase beyond some critical Reynolds numbers.

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