A Liouville Theorem for Mean Curvature Flow

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Abstract
Ancient solutions arise in the study of parabolic blow-ups. If we can categorize ancient solutions, we can better understand blow-up limits. Based on an argument of Giga and Kohn in [4], we give a Liouville-type theorem restricting ancient, type-I, non-collapsing two-dimensional mean curvature flows to either spheres or cylinders.

0 Introduction
We study ancient solutions to mean curvature flow. Let $F : \mathcal{M} \times \mathbb{R}^{-} \to \mathbb{R}^{N+1}$ be a family of smooth embeddings $F(\cdot, t) = M(t)$, where $\mathcal{M}$ is a closed $N$-dimensional manifold. We say that $M = \{M(t)\}_{t \in [0,T)}$ is a mean curvature flow if
\[ \partial_t F = -H\nu, \]
where $H$ is the scalar mean curvature, $\nu$ is the outward unit normal, and $-H\nu$ is the mean curvature vector.

We call a mean curvature flow ancient if it is defined for all negative time. Ancient solutions arise as blow-ups of singularities (see the discussion after Definition 0.9 for rescaling below for one way this can be done). Daskalopoulos, Hamilton, and Šešum completely classified ancient convex solutions for embedded curves in $\mathbb{R}^2$ in [2]. Here our goal is to further the classification to two dimensions for mean-convex, type-I, non-collapsed flows. At any point in time, an ancient solution has had an arbitrarily long amount of time for diffusion to take place, so we expect it to be highly regular and symmetric. We see this in the work of Huisken and Sinestrari in [8] where they show, assuming convexity and compactness, a number of conditions equivalent to
the flow being a shrinking sphere. This is similar to our result here, so we emphasize that although we impose other restrictions, we allow for compactness or noncompactness. (Haslhofer and Kleiner show in [6] that ancient mean-convex, non-collapsing solutions are convex anyway.)

In the theorem, we do assume some regularity to begin with. In the spotlight are the type-I curvature bound and the non-collapsing condition. With the type-I assumption, we show that an eternal solution for the rescaled flow, as in Definition 0.9 (see [7]), all orders of curvature are bounded in time. The non-collapsing condition prevents sheeting, thereby preserving embeddings as $t \to -\infty$. This is important for integral convergence if one intends to integrate on the embedded hypersurface itself, rather than a background manifold. Both assumptions are rather strong, but since we have in mind ancient solutions which arise from blow-ups at singularities of type-I, mean-convex, compact flows, both are quite reasonable.

There are examples of ancient solutions that do not satisfy the conclusions of our main theorem. The paperclip solution, one of the two classes in [2], converges to two parallel lines as $t \to -\infty$, but behaves like the grim reaper solution at either end. This was generalized in a sense by White in [13] to higher dimensions, but was studied in more detail by Haslhofer and Hershkovitz in [5]. The paperclip, however, is neither type-I, nor non-collapsing, as $t \to -\infty$.

The method here is inspired by that of Giga and Kohn in [4]. There they show that the rescaled limits as $t \to -\infty$ and $t \to +\infty$ are the same. They then classify self-similar solutions to find that the forward and backward limits of the rescaled solution must have the same energy. The energy they use is decreasing, so once they relate it to the time derivative of the solution, they can integrate across time to show the solution is constant in time.

We can build off the work of Huisken in [7] or White in [13] to classify the forward limit, and the work of Haslhofer and Kleiner in [6] to classify the backward limit. However, the geometric nature of the flow adds a complication: there are different self-similar solutions that can arise as blow-ups and blow-downs, and they have different energies. We calculate the energy (Huisken’s Gaussian area functional defined in [7]) explicitly in each case. The fact that energy is decreasing means that the backward limit cannot
have a lower energy than the forward limit, but this does not cover the case when the backward limit has a strictly higher energy than that of the forward limit. We see in the proof of Proposition 3.3 that the only case in which the monotonicity does not help is a noncompact backward limit with a compact forward limit. This case is ruled out rather directly in Lemma 1.4, since the rescaled evolution equation tends to expand the hypersurface.

We now give some definitions so we can state the main theorem (Theorem 0.6).

**Definition 0.1** (Singular Point). We say \( x \in \mathbb{R}^{N+1} \) is a singular point if there is a sequence \((p_i, t_i) \in \mathcal{M} \times \mathbb{R}^-\) with \( t_i \not\to 0 \) such that \( F(p_i, t_i) \to x \) and \( |A(p_i, t_i)| \to \infty \) as \( i \to \infty \).

All mentions of singularities are at the first singular time \( t = 0 \).

**Definition 0.2** (Type-I Flow). Let \( M \) be a mean curvature flow for times \( t \in \mathbb{R}^- \). Let \( \lambda(t) = (-2t)^{-\frac{1}{2}} \), and write \( A \) for the second fundamental form. We say \( M \) is type-I if there is a \( C_0 > 0 \) so that
\[
\max_{x \in M(t)} |A(x, t)| \leq C_0 \lambda(t) \quad \text{for} \quad t \in \mathbb{R}^-.
\]

**Remark 0.3.** The type-I condition is typically employed in discussions of blow-ups at singularities. However we apply the condition to the entirety of an ancient flow, meaning curvature decays as \( t \searrow -\infty \) as well.

**Definition 0.4** (Polynomial Volume Growth). We say a surface \( \Sigma \in \mathbb{R}^3 \) has polynomial volume growth if \( \text{Vol}(B_R(0) \cap \Sigma) \) is bounded by some polynomial \( P(R) \). (Volume here refers to the intrinsic volume, in this case area.)

We say a mean curvature flow \( M \) has uniform polynomial volume growth if, for every \( t \) that \( M \) is defined, \( M(t) \) has polynomial volume growth, where the polynomial \( P(R) \) is independent of \( t \).

**Definition 0.5** (Non-Collapsing Condition). From Definition 1 of [1]: We say a mean-convex hypersurface \( M_0 \) bounding an open region \( \Omega \) in \( \mathbb{R}^{N+1} \) is \( \alpha \)-non-collapsed if, for every \( x \in M_0 \), there exists a sphere of radius \( \frac{\alpha}{H(x)} \) contained in \( \text{Cl}(\Omega) \), and another contained in \( \Omega^c \), tangent to \( M_0 \) at \( x \). (See Figure 1).
**Theorem 0.6** (Main Theorem). Let $M(t)$ be a smooth, properly embedded, complete, ancient, type-I, mean-convex, $\alpha$-non-collapsed, two-dimensional mean curvature flow in $\mathbb{R}^3$ with first singular point $x$ at time $t = 0$. Further assume that $M(t)$ has uniform polynomial volume growth on $\mathbb{R}^-$. Then $M(t)$ is either a sphere or cylinder, shrinking homothetically until it vanishes at time $t = 0$.

**Remark 0.7.** The assumption that $N = 2$ is necessary to restrict the topologies of blow-ups at a singular point. See Proposition 3.3 and the discussion before it for further explanation.

**Remark 0.8.** Of course, for manifolds without boundary, properly embedded implies completeness. Furthermore, we have by Corollary 1.6 of [12], that $\alpha$-non-collapsed implies properly embedded and uniform polynomial volume growth.

We also need the following definitions for the proof.

**Definition 0.9.** (Gaussian Area) For a flow $M(t)$ of surfaces in $\mathbb{R}^3$, define

$$E_{(x_0,t_0)}(t) = \int_{M(t)} \rho_{(x_0,t_0)}(x,t) \, d\mu,$$

where

$$\rho_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0 - t))^\frac{3}{2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}.$$

We will mostly be assuming $(x_0,t_0) = (0,0)$. In that case, we omit the subscript. That is, $E := E_{(0,0)}$ and $\rho := \rho_{(0,0)}$.

**Definition 0.10** (Rescaled Flow). Let $F$ be a parameterization of a mean curvature flow. Let $\lambda(t) = (2(-t))^{-\frac{1}{2}}$, $\xi = \lambda(t)x$, and $s = -\frac{1}{2} \log(-t)$.
Without loss of generality, we assume throughout the paper that the singular point is 0. So define the rescaled flow
\[ \tilde{F}(p, s) := \lambda(t)F(t). \]
Further define
\[ \tilde{E}(s) = \int_{M(s)} \tilde{\rho}(x) \, d\tilde{\mu}(s), \]
where
\[ \tilde{\rho}(x) = e^{-\frac{|x|^2}{2^2}}. \]
As introduced in [7], the new flow satisfies the equation
\[ \partial_s \tilde{F} = \tilde{F} - \tilde{H}\nu, \tag{2} \]
where \( \tilde{H} = \tilde{H}(\tilde{F}(p, s)) \) and \( \tilde{\nu} = \tilde{\nu}(\tilde{F}(p, s)) \). We will assume for simplicity that the singular point in question is the origin, so we only need to rescale around 0. Notice that \( e^s = \sqrt{2}\lambda(t) \).

With this definition in mind, we can see one way to arrive at an ancient solution in the study of blow-ups. For a non-ancient solution of mean curvature flow defined for times starting at \( t = 0 \) and first singularity at \((x, t) = (0, T)\), the solution rescaled around the singularity is defined for \( s \in [s_0, \infty) \), where \( s_0 = -\frac{1}{2} \log T \). If we define \( \tilde{M}_n(s) = \tilde{M}(s + s_n) \) with \( s_n \nearrow \infty \), a limit solution is obtained given enough curvature control (as one would have if the flow is type-I).

Remark 0.11. If \( M \) is type-I, then \( \tilde{H} \) uniformly bounded for all time, even if \( M \) is ancient.

Definition 0.12 (Local Graph Convergence). Assume \( k \geq 1 \). Let \( \Sigma \) and \( \Sigma_n \) be \( k \)-smooth, properly embedded hypersurfaces in \( \mathbb{R}^{N+1} \). Assume \( \Sigma \) is oriented by a smooth normal vector field \( \nu \). We say \( \Sigma_n \) converges to \( \Sigma \) locally in the graph sense to order \( k \) if the following holds:

For every open ball \( B \subset \mathbb{R}^{N+1} \), there is \( n_0 > 0 \) so that whenever \( n \geq n_0 \)

i) The limit set \( \Sigma \) is the set of all accumulation points of \( \Sigma_n \). That is \( \Sigma \) is the set of all \( x \in \mathbb{R}^{N+1} \) such that there is a sequence of points \( x_n \in M_n \) with \( x_n \to x \).
1 SOME TECHNICAL LEMMAS

ii) If $\Sigma \cap B$ is nonempty, the nearest point map

$$
\pi^B_n : \Sigma_n \cap B \to \Sigma
$$

is a well-defined diffeomorphism onto its image $V^B_n \subset \Sigma$.

iii) For $y \in M_n \cap B$, write $x = \pi^B_n(y)$. Then define $g^B_n : V^B_n \to \mathbb{R}$ to be the height function

$$
g^B_n(x) = (y - x) \cdot \nu(x)
$$

over $V^B_n \subset \Sigma$ so that

$$
(\pi^B_n)^{-1}(x) = x + g^B_n(x) \nu(x)
$$

so that $g^B_n$ is the signed height of $V^B_n \subset \Sigma_n$ over $\Sigma \cap B$. Then for every $k \in \mathbb{N}$,

$$
\|g^B_n\|_{C^k(V^B_n)} \longrightarrow 0.
$$

1 Some Technical Lemmas

1.1 mean curvature flow background

Lemma 1.1 (Proposition 2.3 of [7]). Given $s_0 \in \mathbb{R}$ (and corresponding $t_0$), for each $m > 0$, there is $C(m) < \infty$, such that $|\nabla^m \tilde{A}|^2 < C(m)$ holds on $\tilde{M}(s)$ uniformly in $s$, where $C(m)$ depends on $N$, $m$, $C_0$, and $M(t_0)$.

This phrasing is changed slightly to accommodate ancient solutions by choosing $M(t_0)$ as "initial data".

Lemma 1.2 (Corollary 3.2 of [7]). For the rescaled flow $\tilde{M}$,

$$
\partial_s \tilde{E}(s) = \int_{\tilde{M}(s)} \left| \tilde{F}^+ - \tilde{H} \tilde{\nu} \right|^2 \tilde{\rho} \, d\tilde{\mu}.
$$

Lemma 1.3. For a mean curvature flow $M$ and rescaled flow $\tilde{M}$, $\tilde{E}(s) = (2\pi)^{\frac{N}{2}} E(t)$.

The proof is a direct calculation.
1.2 Some Calculus

Lemma 1.4 (Backwards Compactness Preservation). If $M$ is a compact, type-I mean curvature flow, then $\tilde{M}(s)$ is uniformly bounded for all times $s \leq s_0$.

Proof. From the type-I bound, we know $|\tilde{H}| \leq C_0$. Going back in time,

\[
\partial_s|\tilde{F}|^2 = -\partial_s|\tilde{F}|^2 = -2\tilde{F} \cdot \partial_s\tilde{F} = -2\tilde{F} \cdot (\tilde{F} - \tilde{H}\nu) \\
= -2\tilde{F} \cdot \tilde{F} + 2\tilde{F} \cdot \tilde{H}\nu \leq -2|\tilde{F}|^2 + 2C_0|\tilde{F}|.
\]

So $\partial_s|\tilde{F}|^2$ is strictly negative whenever $|\tilde{F}| > C_0$. Let

\[
\Lambda := \max \left\{ C_0, \max_{\tilde{M}(s_0)} |\tilde{F}| \right\}.
\]

Thus, going back in time, $\tilde{M}(s)$ cannot escape the ball $B_{2\Lambda}(0)$.

Corollary 1.5. Assume $M$ has a singular point at 0. Then $\tilde{M}(s) \cap B_N(0)$ is nonempty for every $s \in \mathbb{R}$.

Proof. Assume without loss of generality that 0 is a singular point. By Proposition 2.2.6 [9], $M(t) \cap Cl(B_{\sqrt{\frac{2}{\lambda(t)}}}(0))$ is nonempty for all time. Rescaling by $\lambda(t) = (-2t)^{-\frac{1}{2}}$, we find that $\tilde{M}(s) \cap Cl(B_{\sqrt{\lambda}})$. Our conclusion immediately follows.

2 Regularity

We will need two time derivatives of $E$ later, which involves fourth order terms, so we need high-regularity control to properly manage convergence. Huisken takes care of this forward in time in [7], but the proof relies on a maximum principle. We need to prove bounds for $|\tilde{\nabla}^m A|$ backward as well. We refer to a parabolic regularity result in [3].

Lemma 2.1 (Proposition 3.22 of [3]). Let $(M_t)$ be a smooth, properly embedded solution of mean curvature flow in $B_{\rho}(x_0) \times (t_0 - \rho^2, t_0)$

\[
|A(x)|^2 \leq \frac{c_0}{\rho^2}
\]
for all $t \in (t_0 - \rho^2, t_0)$ and $x \in M_t \cap B_\rho(x_0)$. Then for every $m \in \mathbb{N}$ there is a constant $c_m = c_m(N, m, c_0)$ such that for all $x \in M_t \cap B_{\frac{\rho}{2}}(x_0)$ and $t \in (t_0 - \frac{\rho^2}{4})$, 

\[
|\nabla^m A(x)|^2 \leq \frac{c_m}{\rho^{2(m+1)}},
\]

Now we want to use the above lemma to get a bound on covariant derivatives in the rescaled flow, and we want to do so for all time. For $s > 0$, Huisken did this in Proposition 2.3 of [7] (Lemma 1.1 of this work). For $s < 0$, we take advantage of the type-I bound. In the nonrescaled setting, going farther back in time forces the curvature to decay. This allows us to choose larger $\rho$ for more control on $|\nabla A|$.

**Lemma 2.2** (Ancient Regularity). Let $M$ be a smooth, properly embedded, ancient, type-I mean curvature flow. Then for $m \in \mathbb{N}$, there is $c_m > 0$ so that for each $t \in \mathbb{R}^-$,

\[
|\nabla^m A| \leq \sqrt{c_m} \lambda^{m+1}(t) = \sqrt{c_m}(-2t)^{-\frac{m+1}{2}}
\]

uniformly over $M(t)$.

**Proof.** Let $t \in \mathbb{R}^-$, and $x \in M(t)$. With the goal of applying Lemma 2.1, choose $\rho = \lambda^{-1}(t) = (-2t)^{-\frac{1}{2}}$, $x_0 = x$, and $t_0 = t + \frac{\rho^2}{8} = \frac{3}{4}t < 0$. That puts our point of interest, $(x_0, t) = (x, t)$, at the center of the inner cylinder, with $t_0$ at the top of the cylinders.
Due to the type-I bound, $|A(y, \tau)| \leq C_0 \lambda(t_0)$ for every $(y, \tau)$ in the outer cylinder since $\tau \leq t_0$. Now setting $c_0 = \frac{2}{\sqrt{3}} C_0$,

$$|A(y, \tau)| \leq C_0 \lambda(t_0) = C_0 \frac{2}{\sqrt{3}} \lambda(t) = \frac{c_0}{\rho}.$$

Now recall $(x, t)$ is in the inner cylinder. Then since $|A| \leq \frac{c_0}{\rho}$ in the outer cylinder, Lemma 2.1 says that for every $m \in \mathbb{N}$, there is $c_m$ so $|\nabla^m A|^2 \leq \frac{c_m}{\rho^2 (m+1)}$ in the inner cylinder. Rather,

$$|\nabla^m A(x, t)| \leq \sqrt{c_m} \rho = \sqrt{c_m} \lambda^{m+1}(t).$$

Corollary 2.3 (Eternal Regularity). Let $M$ be a smooth, properly embedded, ancient, type-I mean curvature flow. Then for $m \in \mathbb{N}$, there is $C_m > 0$ so that

$$\sup_{\xi \in \tilde{M}(s), s \in \mathbb{R}} |\tilde{\nabla}^m \tilde{A}| \leq C_m.$$

Proof. Recall $\lambda(t) = \frac{e^t}{\sqrt{2}}$, so that we have from Lemma 2.2,

$$|\tilde{\nabla}^m \tilde{A}| = \lambda^{1-m} |\nabla^m A| \leq \sqrt{c_m} \lambda^{1-m} \lambda^{m+1} = \sqrt{c_m} \lambda^2 = \frac{\sqrt{c_m}}{2} e^{2s}.$$

Now, for $s \in (-\infty, 0)$, $|\tilde{\nabla}^m \tilde{A}| \leq \frac{\sqrt{c_m}}{2}$. Then Lemma 1.1 provides a $C_m \geq \frac{\sqrt{c_m}}{2}$ for which $|\tilde{\nabla}^m \tilde{A}| \leq C_m$ for $s \in (0, \infty)$. Therefore,

$$|\tilde{\nabla}^m \tilde{A}| \leq C_m$$

for all time. \qed

3 Proving the Main Theorem

Theorem 3.1 (Subsequential Limits). Let $M$ be a smooth, properly embedded, ancient, type-I, mean-convex, two-dimensional mean curvature flow with uniform polynomial growth. Assume $M$ has a singular point at the origin at time $t = 0$. 

\boxed{\begin{align*}
\end{align*}}
Then for every sequence of rescaled times $s_i \searrow \infty$, there is a subsequence \( \{ s_{i_j} \} \) so that \( \lim_{j \to \infty} \widetilde{M}(s_{i_j}) \) converges to some \( \widetilde{M}_{-\infty} \) in \( C^2_{\text{loc}} \) in the graph sense. Furthermore, \( \widetilde{M}_{-\infty} \) is either a plane passing through 0, a cylinder centered at 0 with radius 1, or a sphere centered at 0 with radius \( \sqrt{2} \).

All the same can be said of some sequence \( s_i \nearrow \infty \) and a limit \( \widetilde{M}_{+\infty} \). Although in that case we can rule out the plane.

**Proof.** Since \( |\nabla^m A| \leq C_m \) by Corollary 2.3 and \( \widetilde{M}(s) \cap B_N(0) \) is nonempty by Corollary 1.5, \( \widetilde{M}_{-\infty} \) exists by Corollary 1.6 of [12]. We know from (5) of [11] that \( \widetilde{M}_{-\infty} \) is a tangent flow, or blowdown soliton. Therefore Theorem 1.11 of [6] says that \( \widetilde{M}_{-\infty} \) is either a plane, cylinder, or sphere.

Again, \( \widetilde{M}_{+\infty} \) exists due to Corollary 1.6 of [12]. Since \( \widetilde{M}_{+\infty} \) is a tangent flow, we know it is either a plane, cylinder, or sphere by Theorem 1 of [13]. However, Corollary 1.8 of [10] rules out the plane for tangent flows at first singularities for mean-convex flows.

It follows from (2) that for a stationary sphere or cylinder, \( \widetilde{H} = \widetilde{F} \cdot \widetilde{v} \). The necessary radii follow directly from there.

\[ \square \]

**Lemma 3.2.** The limits \( \widetilde{E}_{\pm \infty} := \lim_{s \to \pm \infty} \widetilde{E}(s) \) exist. Furthermore, the limits \( \widetilde{E}_{\pm \infty} \) are equal to the Gaussian areas of \( \widetilde{M}_{\pm \infty} \).

**Proof.**

**The Limits \( \widetilde{E}_{\pm \infty} \) Exist** Since \( M(t) \) exhibits uniform polynomial volume growth, \( E(t) \) is bounded for \( t \in \mathbb{R}^- \). Then by Lemma 1.3, \( \widetilde{E}(s) \) is also bounded for all \( s \in \mathbb{R} \). We know from Lemma 1.2 that \( \widetilde{E} \) is decreasing in time and bounded below by 0. Therefore, its limits at times \( \pm \infty \) both exist. We denote them \( \widetilde{E}_{\pm \infty} \).

**Gaussian areas** We do the proof for \( \widetilde{M}_{-\infty} \), and the proof for \( \widetilde{M}_{+\infty} \) is identical. One will notice below that different radii \( R + \varepsilon \) and \( R \) are used in the domains for integrals. This is of little interest, but necessary to accomodate the normal vectors to \( \widetilde{M}_{-\infty} \cap B_R(0) \), which leave the ball near the boundary.
Let $0 < \varepsilon < 1$. By uniform polynomial volume growth, there exists $R > 0$ such that

$$\int_{\hat{M}(s_i) \setminus B_R(0)} \tilde{\rho} \, d\hat{\mu}_i < \varepsilon$$

for all $i$ and also for $\hat{M}(s_i)$ replaced by $\hat{M}_{-\infty}$. By Corollary 1.6 of [12], for large $i$ there are open $V_i \subset \hat{M}_{-\infty} \cap B_{R+\varepsilon}(0)$ and $f_i : V_i \to \mathbb{R}$ with $\|f_i\|_{C^1} < \varepsilon$ such that

$$\varphi_i(x) := x + f_i(x)\tilde{\nu}_{-\infty}(x)$$

is a diffeomorphism from $V_i$ onto $\hat{M}(s_i) \setminus B_R(0)$.

Then

$$\int_{\hat{M}(s_i) \cap B_R(0)} \tilde{\rho} \, d\hat{\mu}_i = \int_{\hat{M}_{-\infty} \cap B_{R+\varepsilon}(0)} \chi_{V_i} \tilde{\rho}(\varphi_i(x)) \sqrt{1 + |\tilde{\nabla}_{-\infty} f_i|^2} \, d\tilde{\mu}_{-\infty},$$

where the integrals now have a fixed domain, and $\chi_{V_i}$ is the characteristic function. The integrand is bounded by 2 and converges pointwise to $\tilde{\rho}$. Therefore we can apply dominated convergence. Taking $i$ large enough, and repeatedly absorbing $O(\varepsilon)$-terms, we write

$$\int_{\hat{M}(s_i)} \tilde{\rho} \, d\hat{\mu}_i = \int_{\hat{M}(s_i) \cap B_R(0)} \tilde{\rho} \, d\hat{\mu}_i + O(\varepsilon)$$

$$= \int_{\hat{M}_{-\infty} \cap B_{R+\varepsilon}(0)} \tilde{\rho} \, d\tilde{\mu}_{-\infty} + O(\varepsilon)$$

$$= \int_{\hat{M}_{-\infty}} \tilde{\rho} \, d\tilde{\mu}_{-\infty} + O(\varepsilon),$$

where we used

$$\int_{\hat{M}_{-\infty} \setminus B_{R+\varepsilon}(0)} \tilde{\rho} \, d\tilde{\mu}_{-\infty} \leq \int_{\hat{M}_{-\infty} \setminus B_R(0)} \tilde{\rho} \, d\tilde{\mu}_{-\infty}. \quad \square$$

The following result is where we really need $N = 2$. That is, if $\hat{M}_{-\infty}$ and $\hat{M}_{+\infty}$ can be generalized cylinders, our method does not prevent them from being generalized cylinders with different numbers of flat factors. Lemma 1.4 lets us handle the case where either limit is a (compact) sphere, and we are able to rule out planes altogether. Restricting our scope to surfaces means the only other possibility is cylinders with the known factorization $S^1 \times \mathbb{R}^1$. 

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Proposition 3.3 ($\tilde{M}_{-\infty} \cong \tilde{M}_{\infty}$). Let $M$ be a smooth, complete, properly embedded, ancient, type-I, mean-convex, two-dimensional mean curvature flow with uniform polynomial volume growth. Assume $M$ has a singular point at the origin at time $t = 0$.

Then $\tilde{M}_{-\infty}$ and $\tilde{M}_{\infty}$ are either both spheres or are both cylinders. They have the same radius, and are centered at the origin.

Proof. Recall from Theorem 3.1 we know that $\tilde{M}_{-\infty}$ is either a plane, cylinder, or sphere, and $\tilde{M}_{+\infty}$ is only a cylinder or sphere.

Now we turn our attention to determining possible shapes for $\tilde{M}_{-\infty}$. The strategy is to use the monotonicity of $\tilde{E}$ to rule out the plane, then show that $\tilde{M}_{-\infty}$ if and only if $\tilde{M}_{\infty}$. If $\tilde{E}_P$, $\tilde{E}_C$, and $\tilde{E}_S$ are the Gaussian areas for the plane, cylinder of radius 1, and sphere of radius $\sqrt{2}$ respectively, a direct calculation gives $\tilde{E}_P = 2\pi$, $\tilde{E}_C = 2\pi\sqrt{2}$, and $\tilde{E}_S = 2\pi\frac{4}{e}$. That is

$$\tilde{E}_P < \tilde{E}_S < \tilde{E}_C.$$  

First suppose $\tilde{M}_{-\infty}$ is a plane. We already know $\tilde{M}_{\infty}$ is a cylinder of radius 1 or a sphere of radius $\sqrt{2}$. However that would mean $\tilde{E}$ increased, which is a contradiction.

If either $\tilde{M}_{-\infty}$ or $\tilde{M}_{\infty}$ is a sphere, then there is $s \in \mathbb{R}$ so that $\tilde{M}(s)$ is compact. Thus by Lemma 1.4, $\tilde{M}$ is a compact flow. Therefore both $\tilde{M}_{-\infty}$ and $\tilde{M}_{\infty}$ must be the same sphere.

Now we have that $\tilde{M}_{+\infty}$ is a sphere if and only if $\tilde{M}_{-\infty}$ is a sphere. Then, by process of elimination, $\tilde{M}_{+\infty}$ is a cylinder if and only if $\tilde{M}_{-\infty}$ is a cylinder. Thus $\tilde{M}_{+\infty}$ must be isometric to $\tilde{M}_{-\infty}$, since Theorem 3.1 ensures they have the same radius. Due to the equations $\tilde{F}_{\pm\infty} \cdot \tilde{\nu}_{\pm\infty} = \tilde{H}_{\pm\infty}$, the sphere or cylinder must be centered around the origin. 

Finally, since $\tilde{M}_{+\infty}$ and $\tilde{M}_{-\infty}$ are isometric and both centered at 0, they have the same Gaussian area. However, the axis of $\tilde{M}_{-\infty}$ could depend on the subsequence. We address this issue in the following proposition.

Proposition 3.4. Let $M$ be as in Proposition 3.3. Then $\tilde{M}_{-\infty} \equiv \tilde{M}(s) \equiv \tilde{M}_{+\infty}$. 


Proof. From Proposition 3.3, $\tilde{M}_{-\infty}$ and $\tilde{M}_{+\infty}$ are isometric. Then we can write
\[
0 = \tilde{E}_{-\infty} - \tilde{E}_{\infty} = \int_{-\infty}^{-\infty} \int_{\tilde{M}(s)} \left| \tilde{F}^{\perp} - \tilde{H} \tilde{\nu} \right|^2 \tilde{\rho} d\tilde{\mu} ds
\]
Thus we conclude that $\left( \partial_s \tilde{F} \right)^\perp = \tilde{F}^{\perp} - \tilde{H} \tilde{\nu} = 0$ for all time. This means, up to tangential diffeomorphism, that $\tilde{M}(s)$ is stationary. Thus $\tilde{M}(s)$ is a fixed sphere or cylinder.

Proof of Main Theorem. Without loss of generality, assume $M$ has a singularity at $(0,0)$. By Proposition 3.3 and Proposition 3.4, $\tilde{M}(s)$ is either a stationary sphere or cylinder centered at the origin. This corresponds to a homothetically shrinking $M(t)$ that is a sphere or cylinder.

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