The One-Loop Functional as a Berezinian*

H. Neufeld\textsuperscript{1,2}, J. Gasser\textsuperscript{3} and G. Ecker\textsuperscript{2}

\textsuperscript{1) CERN, CH-1211 Geneva 23, Switzerland}
\textsuperscript{2) Institut für Theoretische Physik, Universität Wien}
\hspace{1em} Boltzmannstrasse 5, A-1090 Vienna, Austria
\textsuperscript{3) Institut für Theoretische Physik, Universität Bern}
\hspace{1em} Sidlerstrasse 5, CH-3012 Berne, Switzerland

Abstract

We employ notions familiar from supersymmetry for constructing the one-loop functional of general quantum field theories with bosons and fermions (spin $\leq 1/2$). To demonstrate the advantages of such an approach for calculating one-loop divergences, we analyse a simple Yukawa theory with two different versions of a super-heat-kernel expansion. These methods also simplify the calculation of one-loop divergences for non-renormalizable meson–baryon Lagrangians occurring in chiral perturbation theory.

\textsuperscript{CERN-TH/98-197}
\textsuperscript{June 1998}

* Work supported in part by Schweizerischer Nationalfonds and by TMR, EC-Contract No. ERBFMRX-CT980169 (EURODAΦNE)
1. The one-loop functional of a general bosonic or fermionic theory can be expressed in terms of the determinant of a second-order differential operator. A common procedure in theories with both bosonic and fermionic degrees of freedom is to first integrate out the fermions and then treat the resulting bosonic theory.

The purpose of this note is to demonstrate the usefulness of treating bosons and fermions on the same footing. Although this has been common practice in supersymmetric theories from the early days of SUSY, we emphasize the advantages of such an approach also for non-supersymmetric quantum field theories with arbitrary numbers of bosonic and fermionic fields. Here, we consider only fields with spin $\leq 1/2$.

The method turns out to be especially useful for calculating the divergent parts of one-loop functionals occurring in the renormalization program of chiral perturbation theory with mesons and baryons [1, 3, 8, 9]. In the standard approach, the bosonic loop, the fermionic loop and the mixed loop (boson and fermion lines in the loop) are treated separately [5, 6]. This requires a cumbersome investigation of the singular behaviour of products of propagators, because the mixed loop does not have the form of a determinant, as the purely bosonic or fermionic loops. The SUSY-inspired treatment, on the other hand, reduces the problem to simple matrix manipulations, in complete analogy to the familiar heat-kernel expansion technique for bosonic or fermionic loops.

We demonstrate the simplicity of this approach in the case of a Yukawa theory, where the divergences can relatively easily be extracted by standard Feynman diagram techniques. With applications to baryon chiral perturbation theory in mind [7], we will actually analyse the Yukawa theory in two different ways. In the first method, we square the fermionic differential operator in the usual way to arrive at the superspace version of an elliptic second-order differential operator (in Euclidean space). In the second method, we set up the heat-kernel expansion directly for the original supermatrix. Since we are not aware of a discussion of this technique in the literature, we will be more explicit for the second method.

2. We start from a general Euclidean action

$$S[\varphi, \psi, \overline{\psi}] = \int d^d x \ L(\varphi, \psi, \overline{\psi})$$

for $n_B$ real scalar fields $\varphi_i$ and $n_F$ spin 1/2 fields $\psi_\alpha$. To construct the generating functional $Z$ of connected Green functions, we couple these fields to external sources $j_i$ ($i = 1, \ldots, n_B$), $\rho_\alpha$, $\overline{\rho}_\alpha$ ($\alpha = 1, \ldots, n_F$),

$$e^{-Z[j, \rho, \overline{\rho}]} = \int [d\varphi d\psi d\overline{\psi}] e^{-S[\varphi, \psi, \overline{\psi}]+j^T \varphi + \psi \rho + \overline{\psi} \overline{\rho}} ,$$

where we have used the notation

$$j^T \varphi + \overline{\psi} \rho + \overline{\rho} \overline{\psi} := \int d^d x \ (j_i \varphi_i + \overline{\psi}_\alpha \rho_\alpha + \overline{\rho}_\alpha \overline{\psi}_\alpha) .$$
The normalization of the functional integral is determined by the condition $Z[0, 0, 0] = 0$. We denote by $\varphi_{cl}$, $\psi_{cl}$ the solutions of the classical equations of motion
\[ \frac{\delta S}{\delta \varphi_i} = j_i, \quad \frac{\delta S}{\delta \psi_\alpha} = \rho_\alpha, \quad \frac{\delta S}{\delta \bar{\psi}_\alpha} = -\overline{\rho}_\alpha. \] (4)

With fluctuation fields $\xi, \eta$ defined by
\[ \varphi_i = \varphi_{cl,i} + \xi_i, \quad \psi_\alpha = \psi_{cl,\alpha} + \eta_\alpha, \] (5)

the integrand in (2) is expanded in terms of $\xi, \eta, \overline{\eta}$. The resulting loop expansion of the generating functional
\[ Z = Z_{L=0} + Z_{L=1} + \ldots \]
starts with the classical action in the presence of external sources:
\[ Z_{L=0} = S[\varphi_{cl}, \psi_{cl}, \overline{\psi}_{cl}] - j^T \varphi_{cl} - \overline{\psi}_{cl} \rho - \overline{\rho} \psi_{cl}, \] (6)

where the classical fields are fixed by the external sources through (4). The one-loop term $Z_{L=1}$ is given by a Gaussian functional integral
\[ e^{-Z_{L=1}} = \int [d\xi d\eta d\overline{\eta}] e^{-S^{(2)}[\varphi_{cl}, \psi_{cl}, \overline{\psi}_{cl}; \xi, \eta, \overline{\eta}]}, \] (7)

where
\[ S^{(2)}[\varphi_{cl}, \psi_{cl}, \overline{\psi}_{cl}; \xi, \eta, \overline{\eta}] = \int d^d x \mathcal{L}^{(2)}(\varphi_{cl}, \psi_{cl}, \overline{\psi}_{cl}; \xi, \eta, \overline{\eta}) \] (8)
is quadratic in the fluctuation variables. Employing the notation introduced in (3), $S^{(2)}$ takes the general form
\[ S^{(2)} = \frac{1}{2} \xi^T A \xi + \overline{\eta} B \eta + \xi^T \Gamma \eta + \bar{\eta} \Gamma^T \xi \]
\[ = \frac{1}{2} \left( \xi^T A \xi + \pi B \eta - \eta^T B^T \overline{\eta} + \xi^T \Gamma \eta - \eta^T \Gamma^T \xi + \bar{\eta} \Gamma \xi - \xi^T \Gamma^T \eta \right), \] (9)

where $A, B, \Gamma$ are operators in the respective spaces; $A = A^T$ and $B$ are bosonic differential operators, whereas $\Gamma$ and $\Gamma^T$ are fermionic (Grassmann) operators. They all depend on the classical solutions $\varphi_{cl}, \psi_{cl}$. The second explicitly symmetric form of $S^{(2)}$ in (9) will be used for the definition of the supermatrix operator $K$ in (11).

The standard procedure for the evaluation of (7) is to integrate first over the fermion fields $\eta, \overline{\eta}$ to yield the bosonic functional integral
\[ e^{-Z_{L=1}} = \det B \int [d\xi] e^{-\frac{1}{2} \xi^T (A - \Gamma^T B^{-1} \Gamma + \Gamma^T B^{-1} \Gamma^T) \xi}. \]
In this way, we obtain the familiar result

\[
Z_{L=1} = \frac{1}{2} \left[ \ln \det(A - \Gamma B^{-1} \Gamma + \Gamma^T B^{-1T} \Gamma^T) - \ln \det A_0 \right] - (\ln \det B - \ln \det B_0)
\]

\[
= \frac{1}{2} \mathrm{Tr} \ln \frac{A}{A_0} - \mathrm{Tr} \ln \frac{B}{B_0} + \frac{1}{2} \mathrm{Tr} \ln(1 - A^{-1} \Gamma B^{-1} \Gamma + A^{-1} \Gamma^T B^{-1T} \Gamma^T)
\]

\[
= \frac{1}{2} \mathrm{Tr} \ln \frac{A}{A_0} - \mathrm{Tr} \ln \frac{B}{B_0} - \sum_{n=1}^{\infty} \frac{1}{2n} \mathrm{Tr} \left( A^{-1} \Gamma B^{-1} \Gamma - A^{-1} \Gamma^T B^{-1T} \Gamma^T \right)^n , \tag{10}
\]

where

\[ A_0 := A|_{j=\rho=\bar{\rho}=0}, \quad B_0 := B|_{j=\rho=\bar{\rho}=0} \]

denote the free-field limit of \( A \) and \( B \), respectively. Recalling that \( A^{-1}, B^{-1} \) are the scalar and fermion matrix propagators in the presence of external sources, the one-loop functional \( Z_{L=1} \) is seen to be a sum of the bosonic one-loop functional \( \frac{1}{2} \mathrm{Tr} \ln(A/A_0) \), the fermion-loop functional \( -\mathrm{Tr} \ln(B/B_0) \) and a mixed one-loop functional where scalar and fermion propagators alternate. In order to determine the ultraviolet divergences that occur in the last term in (10), the calculational inconveniences mentioned in paragraph 1 are encountered.

In the remainder of this letter, we discuss a procedure that allows us to identify the singular pieces in \( Z_{L=1} \) more directly. Inspired by supersymmetric quantum field theories, we reorganize the three parts of \( Z_{L=1} \) into a more compact form, using the notion of supermatrices, supertraces, etc. (cf., e.g., Refs. [8, 9, 10]).

Assembling the bosonic and fermionic fluctuation fields into a multicomponent field

\[
\lambda^T = \left( \xi^T, \eta^T, \eta \right),
\]

\( S^{(2)} \) in (9) can be written as

\[
S^{(2)} = \frac{1}{2} \lambda^T K \lambda ,
\]

where \( K \) is the supermatrix operator

\[
K = \begin{pmatrix} A & \Gamma & -\Gamma^T \\ -\Gamma^T & 0 & -B^T \\ \Gamma & B & 0 \end{pmatrix} . \tag{11}
\]

For a general supermatrix of the form

\[
M = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} ,
\]

where \( a, b (\alpha, \beta) \) are bosonic (fermionic) variables, one defines the supertrace \( \text{str} \ M \) and the Berezinian \( \text{sdet} \ M \) as [8, 9, 11]

\[
\text{str} \ M := \text{tr} a - \text{tr} b ,
\]

\[
\text{sdet} \ M := \det(a - \alpha b^{-1} \beta) / \det b .
\]
These definitions give rise to the relation
\[ \text{sdet} \ M = \exp (\text{str} \ \ln M) \],
(12)
in analogy to the one for ordinary matrices.

Comparing this with Eq. (10), we obtain
\[ Z_{L=1} = \frac{1}{2} \text{Str} \ \ln \frac{K}{K_0} = \frac{1}{2} \text{Str} \ \ln \frac{K'}{K_0'} \]
(13)
with
\[ K' = \begin{pmatrix} A & \sqrt{\mu} \Gamma & -\sqrt{\mu} \Gamma^T \\ \sqrt{\mu} \Gamma & \mu B & 0 \\ \sqrt{\mu} \Gamma^T & 0 & \mu B^T \end{pmatrix} \].
(14)

With our notation
\[ \text{Str} \ O = \int d^d x \ \text{str} \langle x | O | x \rangle \]
we are distinguishing supertraces with and without integration over Euclidean space. In (14) we have introduced a mass parameter \( \mu \) that guarantees equal dimensions for all entries in \( K' ([K'] = [A] = 2) \). Although this quantity does, of course, not appear in any final result, it turns out to be quite helpful for the inspection of expressions at intermediate stages of calculations.

It is seen that the three types of one-loop functionals (bosonic, fermionic and mixed) in Eq. (11) are not distinguished any more in the representation (13), where \( Z_{L=1} \) is expressed in terms of a Berezinian. Generalizing the heat-kernel expansion to supermatrix operators, the determination of the divergent part of \( Z_{L=1} \) reduces to matrix algebra, in complete analogy to the purely bosonic or fermionic loop functionals.

3. As long as we are only interested in those parts of the one-loop functional that are at most bilinear in fermion fields, we can reduce the supermatrix \( K' \) to the simpler form
\[ K'' = \begin{pmatrix} A & \sqrt{2\mu} \Gamma \\ \sqrt{2\mu} \Gamma & \mu B \end{pmatrix}, \]
(15)
such that the one-loop functional can be written as
\[ Z_{L=1} = \frac{1}{2} \text{Str} \ \ln \frac{K''}{K_0''} - \frac{1}{2} \text{Tr} \ \ln \frac{B}{B_0} + \ldots \]
(16)
The terms omitted are at least quartic in the fermion fields.

The representation (16) can be written in a more compact form by “squaring” the fermionic differential operator \( B \). We first note that
\[ \text{Tr} \ \ln \frac{B}{\mu} = \ln \det \frac{B}{\mu} = \ln \det \gamma_5 \frac{B}{\mu} \gamma_5 = \text{Tr} \ \ln \gamma_5 \frac{B}{\mu} \gamma_5 = -\text{Str} \ \ln \begin{pmatrix} 1 & 0 \\ 0 & \frac{B}{\mu} \gamma_5 \end{pmatrix}. \]
Suppressing terms of higher than second order in the fermion fields in the rest of this paper, we get

$$Z_{L=1} = \frac{1}{2} \text{Str} \ln \frac{\Delta}{\Delta_0},$$

(17a)

with

$$\Delta = \left( \frac{A}{\sqrt{2}\mu} \Gamma \right) \sqrt{\frac{2/\mu}{B\gamma_5B\gamma_5}}.$$  

(17b)

In many cases (as the Yukawa theory to be considered below), this procedure brings the supermatrix differential operator $\Delta$ into the form

$$\Delta = -D_\mu D_\mu + Y,$$

(18)

where $D_\mu = \partial_\mu + X_\mu$ and $X_\mu, Y$ are matrices in superspace. Generalizing the ordinary heat-kernel expansion to superspace, we will extract the second Seeley–DeWitt coefficient of the generic operator (18) to obtain the one-loop divergences including the terms bilinear in fermion fields.

In order to avoid the pitfalls with $\gamma_5$ in $d$ dimensions, the steps leading from Eq. (16) to (18) may be carried through by using an intermediate regularization that allows the evaluation of determinants in four dimensions. An example of such a regularization is provided by Eq. (19), where the integration over the parameter $\tau$ may be cut off at the lower end. In the following, we assume that $X_\mu, Y$ are independent of $\gamma_5$, as is the case in the Yukawa theory, and we return to dimensional regularization for ease of comparison with the standard methods to evaluate the loop integrals. In cases where $\gamma_5$ pertains (as in pion–nucleon effective theories), one may stick to the regularization just mentioned.

In the proper-time formulation, the one-loop functional assumes the form

$$Z_{L=1} = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{Str} \left( e^{-\tau\Delta} - e^{-\tau\Delta_0} \right)$$

$$= -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^4x \text{str} \langle x|e^{-\tau\Delta} - e^{-\tau\Delta_0}|x \rangle.$$  

(19)

To extract the divergences, we need the coefficient $a_2(x, x)$ in the heat-kernel expansion

$$\langle x|e^{-\tau\Delta}|x \rangle = (4\pi\tau)^{-d/2} \sum_{n=0}^\infty \tau^na_n(x, x).$$

(20)

The divergent part $Z_{L=1}^{\text{div}}$ of the one-loop functional is then given by

$$Z_{L=1}^{\text{div}} = \frac{1}{(4\pi)^2(d-4)} \int d^4x \text{str} \left[ a_2(x, x) - a_2^0(x, x) \right].$$

(21)

The derivation of the diagonal super-Seeley–DeWitt coefficient $a_2(x, x)$ is completely analogous to the ordinary case. Employing the method of Ball [11], we write

$$\langle x|e^{-\tau\Delta}|x \rangle = \int d^dk \langle x|e^{-\tau\Delta}|k \rangle \langle k|x \rangle = \int \frac{d^dk}{(2\pi)^d} e^{-ikx} e^{-\tau\Delta} e^{ikx}$$

$$= \int \frac{d^dk}{(2\pi)^d} e^{-\tau(k^2-2ik\mu D_\mu+\Delta)} \mathbf{1}. $$

(22)
With the dimensionless integration variable \( l = \sqrt{\tau k} \), we obtain from (20)

\[
a_2(x, x) = \int \frac{d^4l}{\pi^2} e^{-l^2} \left[ \frac{1}{2} \Delta^2 + \frac{2}{3} l_{\mu} l_{\nu} (\Delta D_{\mu} D_{\nu} + D_{\mu} \Delta D_{\nu} + D_{\mu} D_{\nu} \Delta) \right. \\
+ \left. \frac{2}{3} l_{\mu} l_{\rho} l_{\sigma} D_{\mu} D_{\nu} D_{\rho} D_{\sigma} \right] 1 
\]

and therefore, after integration over \( l \),

\[
a_2(x, x) = \frac{1}{12} X_{\mu\nu} X_{\mu\nu} + \frac{1}{2} Y^2 + \left( -\frac{1}{6} D^2 Y - \frac{1}{6} Y D^2 + \frac{1}{3} D_{\mu} Y D_{\mu} \right) 1 ,
\]

(24)

with

\[
X_{\mu\nu} = [D_{\mu}, D_{\nu}] = \partial_{\mu} X_{\nu} - \partial_{\nu} X_{\mu} + [X_{\mu}, X_{\nu}] .
\]

By partial integration, the last term does not contribute to the divergence functional (21) and we arrive at the final result

\[
\int d^4x \text{ str } a_2(x, x) = \int d^4x \text{ str } \left( \frac{1}{12} X_{\mu\nu} X_{\mu\nu} + \frac{1}{2} Y^2 \right) .
\]

(25)

4. We illustrate the formalism with a specific case, the Yukawa theory of one scalar and one fermion field. We consider the following Lagrangian in Euclidean space:

\[
\mathcal{L} = \frac{1}{2} \varphi (-\partial^2 + M^2) \varphi + \frac{\lambda}{4!} \varphi^4 + \bar{\psi} (\gamma_{\mu} \partial_{\mu} + m - g \varphi) \psi - j \varphi - \bar{\psi} \rho - \bar{\varphi} \psi .
\]

(26)

The one-loop divergences of this theory are due to the Feynman diagrams in Fig. 1: the first two diagrams are contained in the bosonic one-loop functional, the next four are included in the fermionic counterpart, whereas the last two belong to the mixed functional, with both bosons and fermions running in the loop.

The equations of motion are

\[
(-\partial^2 + M^2) \varphi_{cl} = j - \frac{\lambda}{3!} \varphi_{cl}^3 + g \bar{\psi}_{cl} \psi_{cl} , \\
(\partial + m) \psi_{cl} = \rho + g \varphi_{cl} \psi_{cl} .
\]

In terms of the fluctuation fields defined in Eq. (3), the fluctuation Lagrangian is

\[
\mathcal{L}^{(2)} = \frac{1}{2} \xi (-\partial^2 + \tilde{M}^2) \xi + \bar{\eta} (\partial + \tilde{m}) \eta - g \xi (\bar{\psi}_{cl} \eta + \bar{\eta} \psi_{cl}) ,
\]

(27)

\[
\tilde{M}^2 = M^2 + \frac{\lambda}{2} \varphi_{cl}^2 , \quad \tilde{m} = m - g \varphi_{cl} .
\]

Thus, the three entries of the supermatrix \( K'' \) in Eq. (15) are

\[
A = -\partial^2 + \tilde{M}^2 , \quad B = \bar{\partial} + \tilde{m} , \quad \Gamma = -g \psi_{cl} .
\]
Figure 1: One-loop Feynman diagrams with divergent parts for the Yukawa Lagrangian \(^{(20)}\). Bosons (fermions) are denoted by dashed (full) lines.

The supermatrix \(\Delta\) in Eq. \((17b)\) has the structure \((18)\) with

\[
X_\mu = \begin{pmatrix}
0 & -\frac{g}{\sqrt{2\mu}} \psi_{\text{cl}} \gamma_\mu \\
0 & 0
\end{pmatrix}, \quad Y = \begin{pmatrix}
\tilde{M}^2 & -\frac{g}{\sqrt{2\mu}} \psi_{\text{cl}} (\tilde{\Theta} + 2\tilde{m}) \\
-\sqrt{2\mu} g \psi_{\text{cl}} & \sqrt{2\mu} \psi_{\text{cl}} \tilde{m} + \tilde{m}^2
\end{pmatrix}. \quad (28)
\]

In our simple example, without vector fields and without derivative interactions, the curvature \(X_{\mu\nu}\) does not contribute because \(X_\mu X_\nu = 0\). The divergence Lagrangian takes the form

\[
\mathcal{L}^{\text{div}} = \frac{1}{2(4\pi)^2(d-4)} \text{str} \ Y^2 ,
\]

reducing the whole problem to a simple matrix multiplication. The final result in Euclidean space is

\[
\mathcal{L}^{\text{div}} = \frac{1}{(4\pi)^2(d-4)} \left\{ \frac{1}{2} (\tilde{M}^4 - M^4) - 2(\tilde{m}^4 - m^4) - 2g^2 \partial_\mu \varphi_{\text{cl}} \partial_\mu \varphi_{\text{cl}} \\
+ g^2 \psi_{\text{cl}} (- \Theta + 2\tilde{m}) \psi_{\text{cl}} \right\} . \quad (29)
\]

An explicit calculation of the diagrams in Fig. 1 reproduces exactly these divergences. The same result can be obtained with the background field method \([12]\).

5. We now present an alternative method for calculating the one-loop functional by applying the super-heat-kernel expansion directly to the operator \(K'\) in \((14)\) or to \(K''\)
in (13). For the Yukawa theory we are considering here, the previous method is simple enough. However, for non-renormalizable Lagrangians with derivative couplings occurring in chiral perturbation theory, the expressions can become more compact in the following approach. In addition, we wish to demonstrate with an explicit example that the superheat-kernel expansion is perfectly well defined also in the linear version.

We confine ourselves to the mixed functional in this case, i.e. to the part bilinear in the external fermion fields. Going through essentially the same steps as in paragraph 3 and specializing immediately to the Yukawa Lagrangian (26), we obtain

\[ Z_{L=1} = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^4x \int \frac{d^dl}{(2\pi \sqrt{\tau})^d} \text{str} \left( e^{V+W}1 \right) + \ldots \]  

(30)

with

\[ V = -\text{diag} \left( (l - i\sqrt{\tau}\partial)^2 + \tau \hat{M}^2, \mu [i\sqrt{\tau} I + \tau (\hat{\theta} + \hat{m})] \right), \]

\[ W = \sqrt{2\mu \tau g} \begin{pmatrix} 0 & \bar{\psi}_c \\ \psi_c & 0 \end{pmatrix}. \]  

(31)

The appropriate decomposition of the exponential in (30) can be performed by using Feynman’s “disentangling” theorem [13]:

\[ \exp(V + W) = \exp V \, P_s \exp \int_0^1 ds \, \bar{W}(s) \]  

(32)

with

\[ \bar{W}(s) := e^{-sV} W e^{sV} \]

and

\[ P_s \exp \int_0^1 ds \, \bar{W}(s) := \sum_{n=0}^\infty \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \bar{W}(s_1) \bar{W}(s_2) \ldots \bar{W}(s_n). \]  

(33)

Since we are only interested here in terms with the structure \( \bar{\psi}_c \ldots \psi_c \), we pick out the part bilinear in \( W \):

\[ \text{str} \left( e^{V+W}1 \right) = \int_0^1 ds \int_0^s ds' \text{str} \left[ e^{(1-s)V} W e^{(s-s')V} W e^{s'V}1 \right] + \ldots \]  

(34)

Manipulating this expression further, we will freely use partial integration and shifts in \( l \) keeping in mind that (34) appears in the one-loop functional (30). Moreover, to avoid keeping track of irrelevant terms, we use power counting to argue [cf. Eq. (29)] that the divergences bilinear in the fermion fields cannot contain any derivatives of \( \hat{M}^2 \) or \( \hat{m} \) so that we can treat \( \hat{M}^2, \hat{m} \) as constants. Of course, \( \hat{M}^2 \) will not appear at all in the final

9
result. With these qualifications, we obtain for the terms of $O(\bar{\psi}\psi)$

\[
\text{str} \left( e^{V+W} 1 \right) = 2\mu^2 g^2 \int_0^1 ds \int_0^s ds' \left\{ e^{-(s-s')/(l^2+\tau \hat{M}^2)} \psi_{cl} e^{-(s-s')\mu[i\sqrt{\tau}l+(\theta+m)]} \psi_{cl} \right. \\
- \left. \text{tr} \left[ e^{-(s-s')\mu(i\sqrt{\tau}l+\hat{m})} \psi_{cl} e^{-(s-s')[(l+i\sqrt{\tau}\partial)^2+\tau \hat{M}^2]} \psi_{cl} \right] \right\} \\
= 2\mu^2 g^2 \int_0^1 ds \int_0^s ds' \psi_{cl} \left\{ e^{-(s-s')\mu(i\sqrt{\tau}l+\hat{m})} e^{-(s-s')[(l+i\sqrt{\tau}\partial)^2+\tau \hat{M}^2]} \psi_{cl} \\
+ e^{-(s-s')\mu(i\sqrt{\tau}l+\hat{m})} e^{-(s-s')[(l+i\sqrt{\tau}\partial)^2+\tau \hat{M}^2]} \psi_{cl} \right\} \\
= 2\mu^2 g^2 \int_0^1 dz \psi_{cl} e^{-z\mu(i\sqrt{\tau}l+\hat{m})} e^{-(1-z)[(l+i\sqrt{\tau}\partial)^2+\tau \hat{M}^2]} \psi_{cl}.
\]

(35)

After integration over $z$, the one-loop functional bilinear in the external fermion fields is found to be

\[
Z_{L=1}^{\bar{\psi}\psi} = \mu g^2 \int d^4x \int_0^\infty \frac{d\tau}{\tau} \tau^{2-\frac{d}{2}} \int \frac{d^d l}{(2\pi)^d} \frac{\tau^{2-\frac{d}{2}} e^{-\tau \hat{M}^2} e^{-i\tau}}{\left( l + i\sqrt{\tau}\partial \right)^2 + l^2 \hat{M}^2 - \mu(i\sqrt{\tau}l + \hat{m})} \psi_{cl}.
\]

(36)

Except for derivatives of $\hat{M}^2$ and $\hat{m}$, this is still the complete one-loop functional bilinear in the fermion fields.

To extract the divergent parts, we change integration variables and decompose the functional (36) in the following way:

\[
Z_{L=1}^{\bar{\psi}\psi} = g^2 \int d^4x \int \frac{d^d l}{(2\pi)^d} \frac{\tau^{2-\frac{d}{2}} e^{-\tau \hat{M}^2} e^{-i\tau}}{\left( l + i\sqrt{\tau}\partial \right)^2 + l^2 \hat{M}^2 - \mu(i\sqrt{\tau}l + \hat{m})} \psi_{cl}.
\]

(37)

The first ($\mu$-dependent) term in this expression is cancelled by the $\mu$-dependent parts of the second one. The divergent parts are now easily isolated by expanding the second denominator (setting $\mu = 0$) in $t$. With $l := \sqrt{\tau}^2$, the divergent parts are given by

\[
Z_{L=1}^{\text{div}}^{\bar{\psi}\psi} = g^2 \int d^4x \int_0^\infty \frac{dt}{t} e^{-\hat{m}t^3} \int \frac{d^d l}{(2\pi)^d} \psi_{cl} \left( \frac{2t \text{ sin } l}{l^3} \frac{\phi - \cos l}{l^2} \right) \psi_{cl}
\]

\[
\overset{d \to 4}{\underset{4}{\frac{g^2}{(4\pi)^2(d-4)}}} \int d^4x \psi_{cl} \left( - \phi + 2\hat{m} \right) \psi_{cl}.
\]

(38)

For the final coefficients we have used the integrals

\[
\lim_{d \to 4} \int \frac{d^d l}{(2\pi)^d} \left( \frac{\cos l}{l^2}, \frac{\sin l}{l^3} \right) = \frac{1}{(4\pi)^2} (-2, 2).
\]

Comparing with the divergence Lagrangian (29), we find complete agreement with (38) in the fermionic part.
In summary, we have shown that the one-loop functional of a general quantum field theory, not necessarily of the supersymmetric type, can be written in terms of the Berezinian of a supermatrix operator. We have illustrated – for the case of a Yukawa theory – that the divergent parts of the one-loop diagrams may be worked out quite easily. The method simplifies in a significant manner calculations in more realistic non-renormalizable theories [7].

References

[1] J. Gasser, M.E. Sainio and A. Švarc, Nucl. Phys. B 307 (1988) 779.
[2] A. Krause, Helv. Phys. Acta 63 (1990) 3.
[3] E. Jenkins and A.V. Manohar, Phys. Lett. B 255 (1991) 558.
[4] V. Bernard, N. Kaiser, J. Kambor and U.-G. Meißner, Nucl. Phys. B 388 (1992) 315.
[5] G. Ecker, Phys. Lett. B 336 (1994) 508.
[6] G. Müller and U.-G. Meißner, Nucl. Phys. B 492 (1997) 379.
[7] H. Neufeld, work in progress.
[8] R. Arnowitt, P. Nath and B. Zumino, Phys. Lett. B 56 (1975) 81.
[9] S.J. Gates, Jr., M.T. Grisaru, M. Rocek and W. Siegel, “Superspace” (Benjamin/Cummings, Reading, 1983).
[10] F.A. Berezin, “Introduction to Superanalysis”, A.E. Kirillov, Ed. (D. Reidel Publishing Company, Dortrecht, 1987).
[11] R.D. Ball, Phys. Rep. 182 (1989) 1.
[12] G. ’t Hooft, Nucl. Phys. B 62 (1973) 444.
[13] R.P. Feynman, Phys. Rev. 84 (1951) 108.