The graded Lie algebra of general relativity

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Abstract We construct a graded Lie algebra \( E \) in which the Maurer-Cartan equation is equivalent to the vacuum Einstein equations. The gauge groupoid is the groupoid of rank 4 real vector bundles with a conformal inner product, over a 4-dimensional base manifold, and the graded Lie algebra construction is a functor out of this groupoid. As usual, each Maurer-Cartan element in \( E^1 \) yields a differential on \( E \). Its first homology is linearized gravity about that element. We introduce a gauge-fixing algorithm that generates, for each gauge object \( G \), a contraction to a much smaller complex whose modules are the kernels of linear, symmetric hyperbolic partial differential operators. This contraction opens the way to the application of homological algebra to the analysis of the vacuum Einstein equations. We view general relativity, at least at the perturbative level, as an instance of ‘homological PDE’ at the crossroads of algebra and analysis.

Important note: This paper considerably extends and simplifies paper [8] of the same title. Some readers may still want to consult [8] since it presents some things differently or in more detail.

1 Introduction

The moduli space of solutions to the vacuum Einstein equations is naively the set of Ricci-flat metrics of signature \(-+++\) modulo the action of the gauge groupoid \( \text{diffGrpd} \) of diffeomorphisms. For simplicity, we restrict the discussion to manifolds diffeomorphic to \( \mathbb{R}^4 \). In this paper, we realize this moduli space as the set of Maurer-Cartan (MC) elements in a graded Lie algebra (gLa) \( E \) modulo automorphisms induced by the gauge groupoid. Informally,

\[
\frac{\text{Ricci-flat metrics}}{\text{diffeomorphisms}} = \frac{\text{nondegenerate MC-elements in } E^1}{\sim} \tag{1}
\]

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Approximately, an element of $\mathcal{E}$ is a pair of a conformal orthonormal frame and a connection-like object, in the jargon of general relativity.

The bijection (1) sets the moduli problem for the vacuum Einstein equations, especially at the formal perturbative level, into the homological algebraic framework of differential graded Lie algebras (dgLa) and $L_\infty$-algebras. To exploit this, we provide two tools. An algebraic tool, namely a contraction of the complex associated to an MC-element to a much smaller complex. And an analytical tool, namely a way of formulating the MC-equation as a symmetric hyperbolic PDE. Both tools rely on a new gauge fixing algorithm, formulated using Clifford modules.

The power of the present formalism is in its homological nature combined with the gauge-fixing algorithm. There are innumerable reformulations and reinterpretations of the vacuum Einstein equations. We distinguish ours with concrete applications in forthcoming papers to, for instance, the so-called Belinskii-Khalatnikov-Lifshitz (BKL) proposal for spatially inhomogeneous singular spacetimes [9].

This gLa is used in [10] to study tree scattering amplitudes for general relativity about Minkowski spacetime, using $L_\infty$ homotopy transfer.

Let $\text{gaugeGrpd}$ be the category in which an object is a real, rank 4 vector bundle with a conformal inner product of signature $-+++,$ over a manifold $M$ diffeomorphic to $\mathbb{R}^4.$ Let $W$ be the free $C^\infty$-module of smooth sections. Here, $C^\infty$ are the real smooth functions on $M.$ A morphism in $\text{gaugeGrpd}$ is a vector bundle isomorphism that preserves the conformal inner product. The gauge groupoid $\text{gaugeGrpd}$ is deliberately bigger than the groupoid $\text{diffGrpd}$ that underlies the metric formalism. The construction of $E$ is a functor $E$ into the category of real, graded Lie algebras, $E : \text{gaugeGrpd} \to \text{gLa}$

More precisely, $\mathcal{E}$ is also a graded Lie algebroid over $\wedge W.$

A module derivation of $W$ is a pair of maps $C^\infty \to C^\infty$ and $W \to W$ that satisfy the Leibniz rule for both of the multiplications $C^\infty \times C^\infty \to C^\infty$ and $C^\infty \times W \to W.$ The module derivations $\text{MDer}_{C^\infty}(W)$ constitute a Lie algebroid over $C^\infty.$ Let $\text{CDer}(W)$ be the sub Lie algebroid of module derivations that preserve the conformal inner product. See Definition 2 for Lie algebroids. Set $\mathcal{L} = \wedge W \otimes \text{CDer}(W).$ All tensor products are over $C^\infty.$ The tensor product $\mathcal{L}$ is naturally a graded Lie algebroid over the graded commutative algebra $\wedge W$ with the bracket

$$[\omega \delta, \omega' \delta'] = \omega \omega'[\delta, \delta'] + (\omega \lambda(\delta)(\omega'))\delta' - (\lambda(\delta')(\omega)\omega')\delta$$

where $\lambda : \text{CDer}(W) \to \text{Der}^0(\wedge W)$ is the canonical $C^\infty$-Lie algebroid morphism. The anchor map is of type $\mathcal{L} \to \text{Der}(\wedge W).$

Let $\text{MC} : \text{gLa} \to \text{Set}$ be the Maurer-Cartan functor.

**Theorem (Vacuum Einstein equations as Maurer-Cartan equations – informal)**

*There is a $\wedge W$-graded Lie algebroid ideal $\mathcal{I} \subseteq \mathcal{L},$ supported in degrees 2 and higher, such that the quotient $\mathcal{E} = \mathcal{L} / \mathcal{I}$ realizes the bijection (1). Concretely,

$$\text{MC}(\mathcal{E}) = \{x \in \mathcal{E}^1 \mid [x, x] = 0\}$$

\[1\] The category can be refined by introducing orientations and or time orientations.
and \( \sim \) is equivalence under automorphisms of \( \mathbf{MC}(\mathcal{E}) \) that are in the image of \( \mathbf{MC} \circ \mathbf{E} \).

See Section 5 for Ricci-flatness.

This theorem places us in a standard homological algebraic framework.

The nondegeneracy referred to in (1) is detailed in Definition 6. Degenerate elements are an important feature. Since degenerate MC-elements can be easier to analyze, it is natural to attempt to perturb degenerate elements into nondegenerate ones, a strategy that we pursue elsewhere [9] to study the BKL proposal.

Our construction is based on \( \wedge \mathcal{W} \) and this is essential. If one replaces this by the algebra of differential forms, the algebraic structure falls apart. Differential forms are used, for instance, to formulate the Yang-Mills equations.

Associated to every \( x_0 \in \mathbf{MC}(\mathcal{E}) \) is the moduli space of formal perturbations

\[
\frac{\{x_0 + x \in \mathbf{MC}(\mathcal{E})[[s]] \mid x \in s\mathcal{E}^1[[s]]\}}{\exp(s\mathcal{E}^0[[s]])}
\]

with \( \mathcal{E}[[s]] \) the gLa of formal power series in the symbol \( s \) with coefficients in \( \mathcal{E} \), and the denominator is a group by the Baker-Campbell-Hausdorff formula. One can check that the formal moduli space (2) is a formal version of (1), see also Remark 1.

As usual, if \( x \in \mathbf{MC}(\mathcal{E}) \) then the differential \( d = [x, -] \) turns \( \mathcal{E} \) into a dgLa. The first homology \( H^1(d) \) is interpreted as linearized gravity about \( x \). The second homology \( H^2(d) \) is the obstruction space to deformations of \( x \) within \( \mathbf{MC}(\mathcal{E}) \). If the obstruction space vanishes, then the formal moduli space (2) admits a nonlinear parametrization by \( H^1(d)[[s]] \), as reviewed in Section 11.

Gauges are an integral part of this formalism. By a gauge we mean a comprehensive homological object. Here we discuss properties that all gauges will have, a construction is in Section 7. Technically it is a graded \( C^\infty \)-submodule \( \mathcal{E}_G \subseteq \mathcal{E} \) together with certain bilinear forms, see Definition 8 for details. An element of \( \mathcal{E}^1 \) is gauged if and only if it belongs to \( \mathcal{E}_G^1 \). For every future timelike \( w \in \mathcal{W} \) there is a splitting

\[
\mathcal{E} = \mathcal{E}_G \oplus w\mathcal{E}_G
\]

where multiplication by \( w \) is a map of degree one, injective as a map \( \mathcal{E}_G \rightarrow \mathcal{E} \). So, \( \mathcal{E}_G \) is a half-ranked direct summand. Our construction of gauges uses \( \mathbb{Z}_2 \)-graded filtered Clifford modules, and a homological unitarity trick, based on averaging over the finite Clifford group. We also give an algorithm that generates such a gauge \( \mathcal{E}_G \) for every choice of a Hermitian inner product on a rank 18 complex vector bundle.

We convey the analytical content of a homological gauge with informal statements about three systems of PDE. These are local statements. Fix a nondegenerate \( x_0 \in \mathcal{E}_{-1} \), so that \( x_0 + x \) is still nondegenerate for all small \( x \in \mathcal{E}^1 \). Then,

(0) For each small \( x \in \mathcal{E}^1 \) the equation \( E(\phi)(x_0 + x) = x_0 \) mod \( \mathcal{E}_G^1 \) is quasilinear symmetric hyperbolic for an unknown automorphism \( \phi \equiv \mathbb{I} \) in gaugeGrpd.

(1) The equation \( [x_0 + x, x_0 + x] = 0 \) mod \( \mathcal{E}_G^2 \) is a quadratically nonlinear, quasilinear symmetric hyperbolic system for an unknown small \( x \in \mathcal{E}_G^1 \).

(2) For each solution \( x \) to (1), the equation \( [x_0 + x, u] = 0 \) mod \( \mathcal{E}_G^3 \) is a linear symmetric hyperbolic system for an unknown \( u \in \mathcal{E}_G^2 \).
To illustrate the utility of these systems, imagine that one attempts to locally construct $x_0 + x \in \text{MC}(\mathcal{E})$ with small unknown $x \in \mathcal{E}$. Then, (0) justifies restricting to $x \in \mathcal{E}^G$. (1) tells us that at least the MC-equation modulo $\mathcal{E}^G$ is hyperbolic and can be locally solved by standard methods; and (2) is a tool to show that the remainder $u = [x_0 + x, x_0 + x] \in \mathcal{E}^G$, which solves $[x_0 + x, u] = 0$ by a Jacobi identity, vanishes. In [5] we followed this route, unaware of the homological framework, to demonstrate the dynamical formation of trapped spheres in solutions to the vacuum Einstein equations, simplifying earlier work by Christodoulou.

The systems in (0) and (1) are nonlinear. The analogous linear statements are simpler, and can be made globally.

**Theorem (Contraction and quasiisomorphism)** Suppose $x \in \text{MC}(\mathcal{E})$ is globally hyperbolic, see Definition 6. Define $d = [x, -]$ and the composition

$$K : \mathcal{E}^G \hookrightarrow \mathcal{E} \xrightarrow{d} \mathcal{E} \twoheadrightarrow \mathcal{E}^G,$$

Then $K$ is a linear symmetric hyperbolic operator, and there is a contraction, hence quasiisomorphism, of complexes from $(\mathcal{E}, d)$ down to $(\ker K, d|_{\ker K})$.

Informally, this complex is much smaller because it lives in three dimensions, $\ker K$ being the space of homogeneous solutions to a linear symmetric hyperbolic system. The contraction is a tool for calculating the homology of $d$. At the nonlinear level, when also applied to the bracket, the contraction yields an $L_\infty$ algebra, that is, higher many-to-one brackets on $\ker K$, most concretely using the homological perturbation lemma also known as ‘homotopy transfer’ [12, 13].

We have not emphasized spinors in this paper, to avoid an extra layer of notation, but they can be extremely useful. Section 12 is included as a succinct discussion of the spinor functor $\text{Spinor} : \text{gaugeGrpd}_{\text{spinor}} \rightarrow \text{gaugeGrpd}$.

We have consciously kept this paper minimalist. Some constructions and statements generalize to other dimensions; other signatures; topologically nontrivial base manifolds; more general commutative rings as base rings.

2 Related work and acknowledgments

In general relativity, our work is related to the Newman-Penrose orthonormal frame formalism [3] which puts the vacuum Einstein equations in quadratically nonlinear form, though not in Maurer-Cartan form; and examples of gauge-fixing for the vacuum Einstein equations to symmetric hyperbolic systems by H. Friedrich [1]. The concept of symmetric hyperbolicity is due to K.O. Friedrichs [2, 11]. The novelty of our work is in its homological nature; functorial constructions; and a comprehensive concept for gauges. In algebra, our work is related to deformation and obstruction theory, see Gerstenhaber [4, 16]. Applications are bound to yield $L_\infty$-algebras and homotopy transfer [12–14]. Our own precursors include [6, 7] and parts of [5].

We thank J. Stasheff for emphasizing the role of obstructions in formal perturbation theory, and for pointing out that our contraction can be used to run $L_\infty$ homotopy

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2 Use the fact that ‘a solution to a homogeneous hyperbolic equation that is zero initially is identically zero’. This reduces showing that $u$ vanishes to showing that it vanishes initially, the ‘constraint equations’.
transfer. We thank T. Willwacher for conceptual clarifications, and for pointing us to Lie algebroids and base change techniques, which we have now fully adopted.

3 Preliminaries and conventions

It is implicitly assumed, throughout this paper, that an object of $\text{gaugeGrpd}$ is given. The notation $M, C^\infty, W$ refers to such an object. So $M \cong \mathbb{R}^3$ is a manifold, $C^\infty$ is the algebra of real smooth functions on it, and $W$ is a free $C^\infty$-module of rank 4 with fiberwise a conformal inner product of signature $-+++$. Tensor products are over $C^\infty$ whenever this makes sense, $\otimes = \otimes_{C^\infty}$. Otherwise, tensor products are over $\mathbb{C}$. So $\Lambda W$ is constructed from the tensor algebra over $C^\infty$. The tensor product of elements is often denoted by juxtaposition.

3.1 Graded Lie algebroids and representations

The following definitions use base field $\mathbb{C}$ for simplicity, similar for $\mathbb{R}$. A grading is a $\mathbb{Z}$-grading; it induces a $\mathbb{Z}_2$-grading. Ungraded means concentrated in degree zero. If $x, y$ are homogeneous then in the notation $(-1)^{xy}$ the exponent is the product of the degrees. By $\text{Hom}_k$ we mean morphisms that raise the degree by $k \in \mathbb{Z}$.

**Definition 1 (dgLa and $MC$-functor)** A real differential graded Lie algebra (dgLa) is a real graded vector space $g$ with a $d \in \text{End}^1(g)$ and a $[\cdot, \cdot] \in \text{Hom}^0(g \otimes g, g)$ that satisfy $[x,y] = -(-1)^{xy}[y,x]$ and $d^2 = 0$ and $d[x,y] = [dx,y] + (-1)^x[x,dy]$ and $[x,[y,z]] + (-1)^{x(y+z)}[y,[z,x]] + (-1)^{z(x+y)}[z,[x,y]] = 0$ for all homogeneous $x, y, z \in g$. A real graded Lie algebra (gLa) is a dgLa with $d = 0$. A real Lie algebra (La) is an ungraded gLa. Let $\text{La} \hookrightarrow \text{gLa} \hookrightarrow \text{dgLa}$ be the corresponding categories. The Maurer-Cartan functor $MC: \text{dgLa} \rightarrow \text{Set}$ is, on objects,

$$MC(g) = \{ x \in g^1 \mid dx + \frac{1}{2}[x,x] = 0 \}$$

For every graded real vector space $X$, the graded vector space $\text{End}(X)$ is a gLa using the graded commutator. These are the prototypical examples, so a representation of a gLa $g$ is by definition a gLa morphism $g \rightarrow \text{End}(X)$ for some $X$.

A graded Lie algebroid has more structure than a gLa. To define algebroids, let $A$ be a unital associative graded commutative $\mathbb{R}$-algebra. Graded commutative means $ab = (-1)^{ab}ba$ for all homogeneous $a, b \in A$. A $\delta \in \text{End}_\mathbb{R}(A)$ is called a derivation if the Leibniz rule $\delta(ab) = \delta(a)b + (-1)^a \delta(b)$ holds for homogeneous elements. The derivations $\text{Der}(A)$ are a graded $A$-module and gLa using the graded commutator. For a graded $A$-module, scalar multiplication must respect the grading. We will apply this with $A = C^\infty$ and more interestingly $A = \Lambda W$. 

**Definition 2** (A-gLaoid) Suppose $A$ is a unital associative graded commutative $\mathbb{R}$-algebra. An $A$-graded Lie algebroid (A-gLaoid) is a triple $(\mathfrak{g}, [-, -], \rho)$ with $\mathfrak{g}$ a graded $A$-module; $(\mathfrak{g}, [-, -])$ a gLa; the 'anchor' $\rho: \mathfrak{g} \rightarrow \text{Der}(A)$ is $A$-linear and a gLa morphism; and $[x, ay] = \rho(x)(a)y + (-1)^{ax}a[x, y]$ for all homogeneous $x, y \in \mathfrak{g}$ and $a \in A$. An A-gLaoid morphism must be a gLa morphism, and intertwine anchors. An A-gLaoid ideal must be a gLa ideal, an A-submodule, and be contained in the kernel of the anchor. There is a forgetful functor $g\text{Laoid}_A \rightarrow g\text{La}$.

The quotient by an ideal is an A-gLaoid. Note that $\text{Der}(A)$ is an A-gLaoid.

**Lemma 1** (A-module derivations) With $A$ as above and $X$ a graded $A$-module, denote by $\text{MDer}_A(X) \subseteq \text{Der}(A) \oplus \text{End}_A(X)$ the elements $\delta = \delta_A \oplus \delta_X$ for which $\delta_X(ax) = \delta_A(a)x + (-1)^{\delta_A}a\delta_X(x)$ for all homogeneous elements. It is canonically an A-gLaoid, with bracket the graded commutator and anchor $\delta \mapsto \delta_A$.

**Proof** Omitted.

When clear from context, we write $\delta$ for either $\delta_A$ or $\delta_X$. If $X$ is a faithful module, which it always is in our applications, then $\delta_X$ determines $\delta_A$.

**Definition 3** (A-gLaoid representation) A representation of an A-gLaoid $\mathfrak{g}$ is an A-gLaoid morphism $\mathfrak{g} \rightarrow \text{MDer}_A(X)$ for some graded A-module $X$. The trivial representation is the anchor map $\mathfrak{g} \rightarrow \text{Der}(A)$ itself.

The quotient by a $g$-invariant $A$-submodule of $X$ is a new A-gLaoid representation. The adjoint representation $\mathfrak{g} \rightarrow \text{MDer}_A(\mathfrak{g})$ is not in general $A$-linear, so not an A-gLaoid representation. We now define base change along a morphism $A \hookrightarrow B$. A simple example is $\mathbb{R} \hookrightarrow C^\infty$, but our main application is base change along $C^\infty \hookrightarrow \wedge W$.

**Lemma 2** (Base change) Suppose $A \hookrightarrow B$ is an injective morphism of unital associative graded commutative $\mathbb{R}$-algebras, with $A$ ungraded. If $X$ is an $A$-module then $X' = B \otimes_A X$ is a graded $B$-module. If $\mathfrak{g}$ is an A-Laoid and $\lambda : \mathfrak{g} \rightarrow \text{Der}^B(B)$ is an A-Laoid morphism then $\mathfrak{g}' = B \otimes_A \mathfrak{g}$ is a B-gLaoid with bracket

$$[by, b'y'] = bb'[y, y'] + (b\lambda(y)(b'))y' - (\lambda(y')(b)b')y \quad (3)$$

and anchor $\mathfrak{g}' \rightarrow \text{Der}(B), by \mapsto (b' \mapsto b\lambda(y)(b'))$. In this situation, given an A-Laoid representation $\Delta : \mathfrak{g} \rightarrow \text{MDer}_A(X)$, then a B-gLaoid representation is given by

$$\mathfrak{g}' \rightarrow \text{MDer}_B(X'), \; by \mapsto (b'x \mapsto b\lambda(y)(b')y + bb'y')(\Delta(x))$$

**Proof** Omitted. Among other things, one has to check that the various assignments are well-defined, so consistent with the tensor product over $A$; and that the representation intertwines the (omitted) anchors.

Beware that while $X'$ can be viewed as just a graded $A$-module, in general one cannot view $\mathfrak{g}'$ as just an A-gLaoid because there is no induced anchor $\mathfrak{g}' \rightarrow \text{Der}(A)$. 
3.2 Conformal structure of $W$

As always, $W$ is the module coming with an object of gaugeGrpd.

**Definition 4 (Conformally orthonormal module derivations)** A conformal inner product is an equivalence class of inner products $\langle \cdot, \cdot \rangle \in \text{Hom}_{\mathbb{C}^\infty}(W \otimes W, \mathbb{C}^\infty)$, two being equivalent iff multiples by a positive function. Let $\text{CDer}(W) \subseteq \text{MDer}_{\mathbb{C}^\infty}(W)$ be the sub $\mathbb{C}^\infty$-module of all $\delta$ such that for every representative $\langle \cdot, \cdot \rangle$ there is an $f \in \mathbb{C}^\infty$ with $\delta((x,y)) = \langle \delta(x), y \rangle + \langle x, \delta(y) \rangle + f(x,y)$ for all $x,y \in W$.

By a basis for $W$ we always mean a conformally orthonormal basis, as in the next definition. The terms basis and frame are synonymous in this context.

**Definition 5 (Conformally orthonormal basis for $W$)** These are elements $\theta_0, \ldots, \theta_3$ that give a direct sum decomposition $W = \mathbb{C}^\infty \theta_0 \oplus \mathbb{C}^\infty \theta_1 \oplus \mathbb{C}^\infty \theta_2 \oplus \mathbb{C}^\infty \theta_3$ such that a representative $\langle \cdot, \cdot \rangle$ of the conformal inner product is given with $i,j = 1,2,3$

\[
\langle \theta_0, \theta_0 \rangle = -1 \quad \langle \theta_0, \theta_i \rangle = 0 \quad \langle \theta_i, \theta_j \rangle = \delta_{ij}
\]

Such a basis determines:

- Elements $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_{23}, \sigma_{31}, \sigma_{12} \in \text{End}_{\mathbb{C}^\infty}(W) \cap \text{CDer}(W)$ by $\sigma_i(\theta_i) = \theta_i$ and $\sigma_i(\theta_j) = \delta_{ij}\theta_0$ and $\sigma_j(\theta_i) = 0$ and $\sigma_{ij}(\theta_k) = \delta_{jk}\theta_i - \delta_{ik}\theta_j$ and $\sigma_{ik}(\theta_0) = \theta_0$ and $\sigma_{ij}(\theta_0) = \theta_0$ for all $i,j,k = 1,2,3$. Note that $\sigma_{ij} = -\sigma_{ji}$.

- A map $\text{Der}(\mathbb{C}^\infty) \to \text{CDer}(W)$ producing elements that annihilate $\theta_0, \theta_1, \theta_2, \theta_3$.

This map is given by $X \mapsto (f_0 \theta_0 + \ldots + f_3 \theta_3 \mapsto X(f_0) \theta_0 + \ldots + X(f_3) \theta_3)$. So as $\mathbb{C}^\infty$-modules we obtain $\text{CDer}(W) \simeq \text{Der}(\mathbb{C}^\infty) \oplus \mathbb{C}^\infty \sigma_0 \oplus \ldots \oplus \mathbb{C}^\infty \sigma_{12}$.

In the vernacular of relativity, $\sigma_0$ generates dilations; $\sigma_1, \sigma_2, \sigma_3$ generate boosts; $\sigma_{23}, \sigma_{31}, \sigma_{12}$ generate rotations.

3.3 Isotypic decomposition under $\mathfrak{so}(W)$

Define $\mathfrak{so}(W) = \{ \delta \in \text{End}_{\mathbb{C}^\infty}(W) \cap \text{CDer}(W) \mid \text{tr} \delta = 0 \}$, a $\mathbb{C}^\infty$-Lie algebra with each fiber non-canonically isomorphic to $\mathfrak{so}(1,3)$, and a subalgebra and ideal of $\text{CDer}(W)$.

It consists of only vertical elements, meaning elements that are $\mathbb{C}^\infty$-linear.

**Lemma 3** Suppose $X$ is a finite free $\mathbb{C}^\infty$-module. Then every $\mathbb{C}^\infty$-module $\text{CDer}(W) \to \text{MDer}_{\mathbb{C}^\infty}(X)$ has a canonical $\mathfrak{so}(W)$-isotypic decomposition. Each $\mathfrak{so}(W)$-isotypic component is invariant under $\text{CDer}(W)$.

**Proof** We use the fiberwise isomorphism with $\mathfrak{so}(1,3)$; the choice of the isomorphism including orientation is irrelevant. Define the complex isotypic projections using $\mathbb{C} \otimes \mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Relative to a basis for $X$, these are projection matrices with entries in $\mathbb{C} \otimes \mathbb{C}^\infty$. By pairing them up if necessary we get the real isotypic projections. The isotypic projections commute with $\text{CDer}(W)$ because $\mathfrak{so}(W)$ is an ideal and its infinitesimal automorphisms, that is derivations, are inner. □

Label the characters of $\mathfrak{su}(2)$ by half-integers $p \geq 0$ with dimension $2p + 1$, hence those of $\mathbb{C} \otimes \mathfrak{so}(1,3)$ by pairs of half-integers $(p,q)$. The $\mathfrak{so}(1,3)$-isotypic components are labeled by $(p,p)$ respectively $(p,q) \oplus (q,p)$ with $p \neq q, p + q \in \mathbb{Z}$. 
4 The graded Lie algebra $\mathcal{B} = \mathcal{L} / \mathcal{I}$

Canonically $\text{MDer}_c(W) \simeq \text{Der}^0(\wedge W)$ as $C^\infty$-Laoids; this is actually an alternative definition of module derivations of $W$. By restriction we get a $C^\infty$-Laoid morphism $\lambda : \text{CDer}(W) \to \text{Der}^0(\wedge W)$ used for the base change in the next lemma.

**Lemma 4 (The gLaoid $\mathcal{L}$)** Consider the $C^\infty$-Laoid $\text{CDer}(W)$. A base change along $C^\infty \hookrightarrow \wedge W$, using $\lambda$ given above, yields the $\wedge W$-gLaoid

$$\mathcal{L} = (\wedge W) \otimes \text{CDer}(W)$$

with bracket given by (3). As a $\wedge W$-module it is finite free.

**Proof** Use Lemma 2. □

Let $m$ be the kernel of the anchor map $\mathcal{L} \to \text{Der}(\wedge W)$. The adjoint representation $\mathcal{L} \to \text{MDer}_W(\mathcal{L})$ is not an algebroid representation, but $\mathcal{L} \to \text{MDer}_W(m)$ is. It restricts to a $C^\infty$-Laoid representation $\mathcal{L}^0 = \text{CDer}(W) \to \text{MDer}_C(m)$.

**Lemma 5 (The ideal $\mathcal{I}$)** Let $\mathcal{I} \subseteq m^2$ be the so$(W)$-isotypic component

$$\langle 2, 0 \rangle \oplus \langle 0, 2 \rangle$$

Let $\mathcal{I} = (\wedge W)\mathcal{I}^2$. Then $\mathcal{I} \subseteq \mathcal{L}$ is a $\wedge W$-gLaoid ideal.

**Proof** Clearly $\mathcal{I}$ is a $\wedge W$-submodule and $\mathcal{I} \subseteq m$. We have $[\mathcal{L}^0, \mathcal{I}^2] \subseteq \mathcal{I}^2$ since $\mathcal{I}^2$ is isotypic, see Lemma 3, and now $[\mathcal{L}, \mathcal{I}] \subseteq \mathcal{I}$ using the gLaoid-axioms and $\mathcal{I} \subseteq m$. This proof would have gone through for any isotypic component of $m^2$. □

**Lemma 6 (The gLaoid $\mathcal{B}$)** The quotient $\mathcal{B} = \mathcal{L} / \mathcal{I}$ is a $\wedge W$-gLaoid.

**Proof** Clear. □

**Remark 1** If in a groupoid the automorphisms of every object form a Lie group, then one can associate to every object the Lie algebra of that Lie group. The construction of $\mathcal{L}^0 = \mathcal{B}^0$ does morally just that in the infinite-dimensional context of gaugeGrpd. This relates the non-formal and the formal moduli spaces, in Section 1.

The main goal of this section was the construction of $\mathcal{B}$. In the remainder of this section we give more information about, and alternative definitions of, the ideal $\mathcal{I}$. For example, we have yet to establish that $\mathcal{I} \neq 0$.

**Lemma 7 (Isotypic components)** We have $\mathcal{I} = \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \mathcal{I}^4$ where $\mathcal{I}^3 \subseteq m^3$ is the isotypic component $\langle \frac{1}{2}, \frac{1}{2} \rangle \oplus \langle \frac{1}{2}, \frac{1}{2} \rangle$ and $\mathcal{I}^4 \subseteq m^4$ is the component $\langle 1, 0 \rangle \oplus \langle 0, 1 \rangle$. The isotypic component has multiplicity one for each of $\mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4$. So

$$\text{rank}_{C^\infty} \mathcal{I}^2, \mathcal{I}^3, \mathcal{I}^4 = 10, 16, 6$$

**Proof** Use $m \subseteq (\wedge W) \otimes n$, with $n \subseteq \mathcal{L}^0$ the kernel of the anchor map $\mathcal{L}^0 \to \text{Der}(C^\infty)$. The unique isotypic component of $\wedge^k W$ is $\langle 1, 0 \rangle \oplus \langle 0, 1 \rangle$ if $k = 2$: $\langle \frac{1}{k}, \frac{1}{k} \rangle$ if $k = 1, 3$; and $\langle 0, 0 \rangle$ if $k = 0, 4$. The components of $n$ are $\langle 0, 0 \rangle$ and $\langle 1, 0 \rangle \oplus \langle 0, 1 \rangle$. All have multiplicity one. The multiplication table for so$(1, 3)$ implies the claim. □
Lemma 8 (A basis for $J^2$) For every choice of a conformally orthonormal basis for $W$, the module $J^2 \subseteq L^2$ is generated over $C^\infty$ by:

$$
\text{Re} \begin{bmatrix}
\theta_0 \theta_1 + i \theta_2 \theta_3 \\
\theta_0 \theta_2 + i \theta_3 \theta_1 \\
\theta_0 \theta_3 + i \theta_1 \theta_2
\end{bmatrix}^T S
\begin{bmatrix}
\sigma_1 + i \sigma_{23} \\
\sigma_2 + i \sigma_{31} \\
\sigma_3 + i \sigma_{12}
\end{bmatrix}
$$

where $S \in \mathbb{C}^{3 \times 3}$ runs over all symmetric traceless matrices. Here $\theta_0 \theta_1 = \theta_0 \wedge \theta_1$.

Proof Direct computation. These elements annihilate $C^\infty$, and by the properties of $S$ they annihilate $W$, hence they are in $m^2$. Roughly, the two given vectors are separately $(1,0)$-representations, and the construction picks out the desired isotypic component for $J^2$. This is related to what is called, in the vernacular of relativity, the electromagnetic decomposition of the Weyl tensor. 

We give further independent constructions of $J$:

- Using a deformation argument, in Section 6.
- Using spinors, in Section 12.

5 Theorems relating to Ricci-flatness

An object of $\text{gaugeGrpd}$ is fixed, in particular we have a manifold $M \simeq \mathbb{R}^4$ and we have constructed a $\wedge_W$-gLaoid $\mathcal{E}$. Let $\Omega = \wedge \Omega^1$ be the graded commutative $C^\infty$-algebra of differential forms on $M$. Base change yields an $\Omega$-gLaoid

$$
\mathcal{Y} = \Omega \otimes \text{MDer} C^\infty(\Omega^1)
$$

Definition 6 (Affine, nondegenerate, globally hyperbolic) We use the fact that $\Omega^1$ and $\text{Der}(C^\infty)$ are canonically dual as $C^\infty$-modules. We say:

- $\nabla \in \mathcal{Y}^1$ is affine iff $\nabla|_{C^\infty} \in \Omega^1 \otimes \text{Der}(C^\infty)$ is the identity $\Omega^1 \to \Omega^1$.
  Affine $\nabla$ are in canonical one-to-one correspondence with affine connections.
- $x \in \mathcal{E}^1$ is nondegenerate iff $x|_{C^\infty} \in W \otimes \text{Der}(C^\infty)$ is an isomorphism $\Omega^1 \to W$.
  Such an isomorphism can be interpreted as a frame for the tangent bundle.
- $x \in \mathcal{E}^1$ is globally hyperbolic iff there exists a diffeomorphism $M \to \mathbb{R}^4$ such that, with $t, \xi_1, \xi_2, \xi_3 \in C^\infty$ the four coordinate functions, we have $x(t + \sum n^i \xi_i) \in W_+$ for all constants $(n^1, n^2, n^3) \in \mathbb{R}^3$ with $(n^1)^2 + (n^2)^2 + (n^3)^2 \leq 1$.

We do not need globally hyperbolic in this section. Our definition of global hyperbolicity is stronger than necessary, for the sake of simplicity. The way we have defined it, globally hyperbolic does not imply nondegenerate.

Theorem 1 (From an MC-element to a Ricci-flat metric) For a nondegenerate $x \in \mathcal{E}^1$ denote by $i : W \to \Omega^1$ the inverse of $x|_{C^\infty}$. It induces a gLa map $\mathcal{L} \to \mathcal{Y}$. Define $\nabla$ to be the image of $x$ under

$$
\mathcal{E}^1 = \mathcal{L}^1 \to \mathcal{Y}^1
$$
Then $\nabla$ is affine, and compatible with the $i$-induced conformal metric on $M$, so every representative metric $g \in S^2\Omega^1$ satisfies $\nabla g \in \Omega^1 g$. If in addition $x \in MC(\mathcal{E})$, then $\nabla$ is torsion-free; there is a unique representative $g$, up to a positive multiplicative constant, that satisfies $\nabla g = 0$; and this metric $g$ of signature $-+++$ is Ricci-flat.

**Proof** Let $h \in S^2W$ be the inverse of a representative of the conformal inner product on $W$. Then $xh \in W h$ in $W \otimes S^2W$, by the definition of $CDer(W)$. The $i$-induced conformal metric satisfies $\nabla g \in \Omega^1 g$. The choice of $i$ makes $\nabla$ affine, so it corresponds to an affine connection and one can speak about torsion. Then $\nabla$ is torsion-free iff $[\nabla, \nabla] \mid C^\infty = 0$. And if $\nabla$ is torsion-free then $[\nabla, \nabla]: E \to \Omega^2 \otimes E$ is $C^\infty$-linear and it is the curvature of the connection $\nabla: E \to \Omega^1 \otimes E$ induced on (the only two cases we need) $E = C^\infty g$ and $E = \Omega^1$. If $x \in MC(\mathcal{E})$ then $[\nabla, \nabla] \in \mathcal{X}$ with $\mathcal{X}$ the image of $\mathcal{F}$ under $\mathcal{L}^2 \to \mathcal{Y}^2$. The definition of $\mathcal{F}$ and hence $\mathcal{X}$ implies that $\nabla$ is torsion-free; that $\nabla$ has vanishing curvature as a connection on the rank one module $E = C^\infty g$ which yields existence and uniqueness of a new representative $g$ as stated with $\nabla$ its Levi-Civita connection; that this new $g$ is Ricci-flat using $E = \Omega^1$. $\square$

Let $\text{diffGrpd}$ be the category of manifolds diffeomorphic to $\mathbb{R}^4$ and diffeomorphisms. There is a forgetful functor $\text{Forget}: \text{gaugeGrpd} \to \text{diffGrpd}$. By assigning to every manifold the set of Ricci-flat metrics of signature $-+++$ over it, we get a functor $\text{RicciFlat}: \text{diffGrpd} \to \text{Set}$. Let $E: \text{gaugeGrpd} \to gLa$ be the construction of $E$.

**Theorem 2 (Equivalent moduli spaces)** For all $X \in \text{obj}(\text{gaugeGrpd})$: 

$$\text{RicciFlat}(\text{Forget}(X)) \sim \text{nondegenerate elements in } MC(E(X))$$

where quotient by $\sim$ means modulo automorphisms in the image of $\text{RicciFlat}$ respectively in the image of $MC \circ E$. The bijection $\simeq$ is induced by Theorem 1.

**Proof** Omitted. $\square$

###6 Clifford modules as deformations

Some constructions in this paper, perhaps more than we are aware, are naturally stated using Clifford algebras and modules [17]. Clifford algebras are $\mathbb{Z}$-filtered and $\mathbb{Z}_2$-graded. Accordingly, Clifford modules can be filtered or $\mathbb{Z}_2$-graded. In Section 9 we review the highly constrained structure of such Clifford modules. Beware that by a graded or filtered module we mean one where module multiplication respects the grading or filtration respectively.

This section has two goals. One is to show that $\mathcal{E}$ is free as an unfiltered Clifford module, which we need for gauges in Section 7. The other is a new and clean definition of the ideal $\mathcal{I} \subseteq \mathcal{L}$ using a deformation argument.

Suppose $R$ is a filtered ring and $\text{Gr}R$ is its associated graded ring. Then $\text{Gr}$ is also a functor from filtered $R$-modules to graded $\text{Gr}R$-modules. We will use this intuitive fact: If $f: A \to B$ is a morphism of filtered $R$-modules, then by ‘semicontinuity’ the kernel of $\text{Gr}f: \text{Gr}A \to \text{Gr}B$ should not be smaller than the kernel of $f$. A rigorous
version is that if \( k : K \to A \) satisfies \( f \circ k = 0 \) and if \( k \) is a split monomorphism, then \( \text{Gr} f \circ \text{Gr} k = 0 \) and \( \text{Gr} k : \text{Gr} K \to \text{Gr} A \) is a split monomorphism. Here \( k \) being a split monomorphism means equivalently that it has a left-inverse \( A \to K \) that is also a map of filtered modules, equivalently that \( K \) (via \( k \)) a direct summand of \( A \) as a filtered module. So being a split monomorphism is stronger than being injective.

To define the Clifford algebra \( \text{Cl}(W) \) we need a representative \((-,-)\) of the conformal inner product on \( W \), though the dependence on the representative is minor. A basis \( \theta_0, \ldots, \theta_3 \in W \) is always understood to be orthonormal for this representative.

**Definition 7 (Clifford algebra)** For a chosen representative \((-,-)\), let \( \text{Cl}(W) \) be the free associative \( C^\infty \)-algebra generated by \( W \) modulo the two-sided ideal generated by \( vw + vw + 2 \langle v, w \rangle \) for all \( v, w \in W \). It has a canonical filtration \( \text{Cl}(W) \leq k \) and compatible \( \mathbb{Z}_2 \)-grading. We abbreviate \( \text{Cl} = \text{Cl}(W) \).

The map \( W \hookrightarrow \text{Cl} \) is injective, and \( \text{Gr} \text{Cl} = \wedge W \) as \( C^\infty \)-algebras. Hence \( \text{Gr} : F \to G \) with \( F \) the category of filtered and compatibly \( \mathbb{Z}_2 \)-graded \( \text{Cl} \)-modules; \( G \) the category of graded \( \wedge W \)-modules. If \( X \in \text{obj}(F) \) then we set \( X \leq k = X \leq k \cap X \text{odd} \) for \( k \) odd, \( X \leq k = X \leq k \cap X \text{even} \) for \( k \) even. Filtration and \( \mathbb{Z}_2 \)-grading being compatible means \( X \leq k = X \leq k \cap X \leq k-1 \), in particular \( \text{Gr} X = \oplus_k X \leq k / X \leq k-1 \simeq \oplus_k X \leq k / X \leq k-2 \).

**Lemma 9 (The Clifford module \( \mathcal{P} \))** Abbreviate \( \Omega = \wedge^4 W \). Define

\[ \mathcal{P} \leq k \subseteq \text{Hom}_R(C^\infty, \text{Cl} \leq k) \oplus \text{Hom}_R(W, \text{Cl} \leq k+1) \oplus \text{Hom}_R(\Omega, \text{Cl} \leq k \odot \Omega) \]

to be the elements \( \delta_{\wedge} + \delta_w + \delta_\Omega \) for which\(^3\)

\[
\begin{align*}
\delta_{\wedge}(ff') &= f'\delta_{\wedge}(f) + f\delta_{\wedge}(f') \\
\delta_w(fw) &= \delta_{\wedge}(f)w + f\delta_w(w) \\
\delta_\Omega(f\eta) &= \delta_{\wedge}(f) \otimes \eta + f\delta_\Omega(\eta)
\end{align*}
\]

Then \( \mathcal{P} \) is a \( \text{Cl} \)-module, and an object in \( F \). It is free of rank 9 if the filtration is ignored, \( \mathcal{P} \simeq \text{Cl}^0 \) as unfiltered \( \mathbb{Z}_2 \)-graded \( \text{Cl} \)-modules. The \( \wedge W \)-module \( \mathcal{P} = \text{Gr} \mathcal{P} \) is given in the same way, by syntactically replacing \( \text{Cl} \) by \( \wedge W \), and obvious grading. We have rank \( \mathcal{P} \leq 0, 1, 2, 3, 4 = 21, 48, 67, 72, 72 \).

**Proof** Let \( \text{Cl}' \) be the space \( \text{Cl} \) with opposite \( \mathbb{Z}_2 \)-grading, without filtration, then we have \( \text{Cl}' \simeq \text{Cl} \) as unfiltered \( \mathbb{Z}_2 \)-graded \( \text{Cl} \)-modules; this statement fails for \( \wedge W \). One shows that \( \mathcal{P} \simeq (\text{Cl} \odot \text{Der}(C^\infty)) \oplus \text{Hom}_\wedge(\Omega, \text{Cl} \odot \Omega) \oplus \text{Hom}_\wedge(\wedge W, \text{Cl}') \oplus \text{Hom}_\wedge(\Omega, \text{Cl} \odot \Omega) \), using a basis for \( W \) and the Leibniz rules defining \( \mathcal{P} \).

**Lemma 10 (The morphism \( f \))** Set \( \mathcal{L} = \text{Cl} \odot \text{CDer}(W) \in \text{obj}(F) \). In \( G \) we have a canonical \( \text{Gr} \mathcal{L} \simeq \mathcal{L} = \wedge W \odot \text{CDer}(W) \). There is a morphism in \( F \) given by

\[
f : \mathcal{L} \to \mathcal{P} \quad \omega \delta \mapsto \delta_{\wedge} + \delta_w + \delta_\Omega
\]

where \( \delta_{\wedge}(f) = \omega \delta(f) \) and \( \delta_w(w) = \omega \delta(w) \) and \( \delta_\Omega(\eta) = \omega \delta(\eta) \). Then:

\(^3\) Juxtaposition is multiplication in \( C^\infty \) or \( \text{Cl} \), or scalar multiplication for a \( C^\infty \)-module, and the injection \( W \hookrightarrow \text{Cl} \) is implicit.
Lemma 11 (Properties of $\mathcal{F}$). We have:

- $\mathcal{F}$ is a direct summand of $\mathcal{L}$ in $\mathcal{F}$, free unfiltered $\mathcal{C}$-module of $\mathcal{C}$-rank 32.

- $\mathcal{F}^0 = 0$ and the elements (4), now interpreted in $\mathcal{L}^\odot 2$, are a $\mathcal{C}$-basis of $\mathcal{F}^\odot 2$.

- $\mathcal{F} = \mathcal{C}/\mathcal{F}^\odot 2$, and rank $\mathcal{F}^\odot 0,1,2,3,4 = 0,0,10,16,16$.

- We have a split short exact sequence $0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{P} \to 0$ in $\mathcal{F}$.

Proof. Rank 32 since $f$ is surjective. By rank $\mathcal{F}^\odot 0,1,2,3,4 = 11,44,77,88,88$ we get rank $\mathcal{F}^\odot 2,3,4 = 10,16,16$. It suffices to check rank $\mathcal{F}^\odot 0,1 = 0,0$ which we omit. In the third claim, both sides are free $\mathcal{C}$-modules by Theorem 10, so their ranks are multiples of 16, evenly distributed on even and odd parts. Inclusion $\supseteq$ is clear, and $\subseteq$ follows from rank $\mathcal{F}^\odot 2 = 10 > 8$. \qed

Theorem 3 (Associated graded and new definition of $\mathcal{F}$) In $G$ the associated graded $G\mathcal{F}$ is a direct summand of $G\mathcal{L}$, where $G/\mathcal{F}$ has $\mathcal{C}$-basis (4), and $F = (\Lambda W)\mathcal{F}^\odot 2$. Then

- $G\mathcal{F}^0 = 0$ and $G\mathcal{F}^\odot 2$ has $\mathcal{C}$-basis (4), and $G\mathcal{F} = (\Lambda W)\mathcal{F}^\odot 2$.

- $\mathcal{F}$ is contained in the kernel of the anchor map $\mathcal{L} \to \text{Der}(\Lambda W)$ of $\mathcal{L}$.

- $[\mathcal{L}, \mathcal{F}] \subseteq \mathcal{F}$.

Define afresh, $\mathcal{G} = \mathcal{L}/\mathcal{F} \simeq G\mathcal{L}/G\mathcal{F} \simeq G\mathcal{F}$, and it is a $\Lambda W$-$g$Laoid.

Proof. The isomorphism $i$ is induced from the isomorphism $G\mathcal{C} \to \Lambda W$. By construction and semicontinuity, the newly defined $\mathcal{F}$ is contained in the kernel of $Gf$ hence in the kernel of the anchor map $\mathcal{L} \to \text{Der}(\Lambda W)$. It is not difficult to see that $[\mathcal{L}^\odot 0, \mathcal{F}^\odot 2] \subseteq \mathcal{F}^\odot 2$. Together it follows that $[\mathcal{L}, \mathcal{F}] \subseteq \mathcal{F}$. The rest is omitted. \qed

The definition of $\mathcal{F}$ in Theorem 3 matches the old one, in Section 4. The point of the new definition is that one can prove all the main properties independently.

Theorem 4 (Freeness as unfiltered Clifford modules) There are $\mathcal{C}$-submodules $A \subseteq \mathcal{L}^\odot 0$ and $B \subseteq \mathcal{F}^\odot 2 \subseteq \mathcal{L}^\odot 2$, of rank 9 and 2 respectively, such that

$\mathcal{L} \simeq \mathcal{C} \odot (A \oplus B)$, $\mathcal{F} \simeq \mathcal{C} \odot B$ and $\mathcal{G} \simeq \mathcal{C} \odot A$ as $\mathbb{Z}_2$-graded $\mathcal{C}$-modules (not necessarily in $\mathcal{F}$) with isomorphism $\mathcal{A}x \leftrightarrow \omega \odot x$.

Proof. Use the morphism $f$ and Lemma 9, or use Theorem 10. Explicitly, one can take $A = \text{Der}(\mathcal{C}^\odot) \odot \text{span}_\mathbb{C}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_3 \}$. \qed
The freeness of $\mathcal{E}$ is exploited in Section 7. Beware that $\mathcal{E}$ is not free over $\wedge W$, indeed a free $X$ must necessarily satisfy $(\wedge^4 W)X \simeq X / W X$, whereas $\mathcal{E}^4 \not\simeq \mathcal{E}^0$.

As algebras $\text{Gr} \, \text{Cl} = \wedge W$, but we have so far consciously suppressed the well-known fact that as $C^\infty$-modules there is even a canonical $\text{Cl} \simeq \wedge W$, Lemma 12. Hence $\text{Cl}$ acquires a module $\mathbb{Z}$-grading, and if $\text{Cl}_1$ and $\text{Cl}_2$ are defined using two representatives of the conformal inner product, then there is still a canonical $C^\infty$-module isomorphism $\text{Cl}_1 \simeq \wedge W \simeq \text{Cl}_2$.

**Lemma 12** A $\text{Cl}$-module structure on $\wedge W$ is induced by $W \to \text{End}(\wedge W), w \mapsto e_w + i_w = c_w$ where $e_w \in \text{End}^1(\wedge W)$ is multiplication by $w \in W$ and $i_w \in \text{End}^{-1}(\wedge W)$ is defined by $i_w e_v + e_w i_v = -\langle v, w \rangle$ for all $v \in W$. As $\text{Cl}$-modules, $\text{Cl} \simeq \wedge W$.

**Proof** We have $e_w e_v + e_v e_w = i_w i_v + i_v i_w = 0$, hence $c_w e_v + c_v e_w + 2\langle v, w \rangle = 0$. □

In an orthonormal basis, the identification is $\theta_{i_1} \cdots \theta_{i_k} \mapsto \theta_{\text{odd}} \wedge \theta_{\text{even}}$ for $i_1 < \cdots < i_k$.

### 7 Gauges, definition and construction

We start with a purely algebraic definition of a gauge, Definition 8. These are comprehensive gauges in all degrees of $\mathcal{E}$, suitable for homology, and designed for compatibility with the PDE concept of symmetric hyperbolicity, see Section 10.

To show that gauges as in Definition 8 actually exist, we use the Clifford module $\mathcal{E}$ from Section 6. This entire section only depends on the fact that $\mathcal{E}$ is a filtered $\mathbb{Z}_2$-graded $\text{Cl}$-module, meaning $\mathcal{E}$ is in $F$, free as an unfiltered $\mathbb{Z}_2$-graded $\text{Cl}$-module, and $\mathcal{E} = \text{Gr} \mathcal{E}$ as graded $\wedge W$-modules, so in $G$. As before, $\text{Cl} = \text{Cl}(W)$. When using Clifford modules we implicitly use a representative $(\langle -,-\rangle)$ of the conformal inner product, but the dependence on it is completely minor; we will not dwell on this. The account given here is a consolidated one, based on [7, 8].

Let $W_+ \subseteq W$ be the nonempty set of all elements that are everywhere future timelike, this requires the choice of a time direction. For example, using a conformally orthonormal basis, $W_+ = \{ \sum w_i \theta_i \mid \text{w} > (w_1^2 + w_2^2 + w_3^2)^{1/2} \in C^\infty \}$. Set $\text{Hom} = \text{Hom}_{\mathbb{C}}$; continue to set $\otimes = \otimes_{\mathbb{C}}$; and let $\otimes^2$ be the symmetric tensor product over $C^\infty$.

**Definition 8 (Gauge)** A gauge is a pair $(\mathcal{E}_G, B)$. A graded finite free $C^\infty$-submodule $\mathcal{E}_G \subseteq \mathcal{E}$ such that for every $w \in W_+$, left-multiplication $w : \mathcal{E}_G \to \mathcal{E}$ is injective and

$$\mathcal{E} = \mathcal{E}_G \oplus w \mathcal{E}_G$$

so necessarily $\mathcal{E}_G$ must have half the rank of $\mathcal{E}$, and $\mathcal{E}^0_G = \mathcal{E}^0$ and $\mathcal{E}^k_G = 0$. And for every $k$ an element $B^k \in \text{Hom}(\mathcal{E}^k_G \otimes \mathcal{E}^{k+1}_G, C^\infty)$ with:

(a) $B^k (\langle -,- \rangle | \mathcal{E}_G, \mathcal{E}_G) \in \text{Hom}(\otimes^2 \mathcal{E}_G, C^\infty)$ for all $w \in W$, a symmetry requirement.

(b) This is positive definite whenever $w \in W_+$.

(c) $\mathcal{E}^{k+1}_G = \{ x \in \mathcal{E}^{k+1} | B^k (\mathcal{E}^k_G, x) = 0 \}$.

We take $b \in \text{Hom}(\otimes^2 \mathcal{E}, C^\infty)$ to mean $b(\mathcal{E}^\text{odd}, \mathcal{E}^\text{odd}) = b(\mathcal{E}^\text{even}, \mathcal{E}^\text{even}) = 0$. Multiplication by $w$ uses the $\wedge W$-module structure in (a), the $\text{Cl}$-module structure in (i).

**Theorem 5 (Sufficient linear problem)** Suppose $b \in \text{Hom}(\otimes^2 \mathcal{E}, C^\infty)$ satisfies:
(i) \(b(-, w-) \in \text{Hom}^{\text{even}}(S^2 \mathcal{F}, \mathcal{C}^\infty)\) for all \(w \in W\), a symmetry requirement.

(ii) This is positive definite whenever \(w \in W_+\).

Define
\[
\mathcal{F}^k_G = \{ x \in \mathcal{F}^{<k} | b(x, \mathcal{F}^{<k-1}) = 0 \}
\]

Then for every \(w \in W_+\), Clifford left-multiplication \(w : \mathcal{F}^{<k-1} \to \mathcal{F}^{<k}\) is injective and
\[
\mathcal{F}^{<k} = \mathcal{F}^k_G \oplus w\mathcal{F}^{<k-1}
\]

The map \(b\) induces a map
\[
b^k \in \text{Hom}(\mathcal{F}^k_G \otimes (\mathcal{F}^{<k+1} / \mathcal{F}^{<k-1}), \mathcal{C}^\infty)
\]

We have \(\mathcal{F}^k_G \cap \mathcal{F}^{<k-2} = 0\). Let \(\mathcal{F}_G^{\pm}\) be the isomorphic image of \(\mathcal{F}_G^k\) under the canonical surjection \(p_k : \mathcal{F}^{<k} \to \mathcal{F}^{\pm} / \mathcal{F}^{<k-2} \cong \mathcal{F}^k\). Let \(B^k \in \text{Hom}(\mathcal{F}_G^k \otimes \mathcal{F}_G^{k+1}, \mathcal{C}^\infty)\) be the map corresponding to \(b^k\). Then this defines a gauge as in Definition 8.

Proof Clifford left-multiplication by \(w \in W_+\) is injective since \(w^2\) is a nonzero multiple of the identity. To prove (6) show that the intersection of the summands vanishes using (ii) and make a rank argument again using (ii). Fix a \(w \in W_+\). If \(x \in \mathcal{F}^k_G \cap \mathcal{F}^{<k-2}\) then \(b(x, wx) = 0\), so by (ii) we get \(x = 0\). Applying (6) twice gives (5), because
\[
\mathcal{F}^{<k} = \mathcal{F}^k_G \oplus w\mathcal{F}^{k-1} \oplus w^2\mathcal{F}^{<k-2}
\]

and \(w^2\) is a nonzero multiple of the identity. Note that \(p^{k+1}w = wp^k\) as maps \(\mathcal{F}^{<k} \to \mathcal{F}^{k+1}\). For every \(x \in \mathcal{F}^k\) let \(x' = p^k x\), so \(x \mapsto x'\) is bijective as a map \(\mathcal{F}^k \to \mathcal{F}^k\). For \(x \in \mathcal{F}^k_G\) and \(y \in \mathcal{F}^k\) we have \(B^k(x', wy') = B^k(x, wy') = b^k(x, (wy)') = b^k(x, wy)\), then restrict to \(y \in \mathcal{F}^k_G\), to get (a) and (b), and (c) by a rank argument. \(\Box\)

Remark 2 Condition (i) would be easy to satisfy if it was only required for a single \(w\), say for \(w = \theta_0\). In fact, there is a bijection between:

- The set of \(b' \in \text{Hom}^{\text{odd}}(S^2 \mathcal{F}, \mathcal{C}^\infty)\) for which \(b'(-, \theta_0-) \in \text{Hom}^{\text{even}}(S^2 \mathcal{F}, \mathcal{C}^\infty)\).
- The set of \(b'' \in \text{Hom}(S^2 \mathcal{F}^{\text{even}}, \mathcal{C}^\infty)\).

Furthermore \(b'(-, \theta_0-)\) is positive definite if and only if \(b''\) is positive definite. The map \(b' \mapsto b''\) is given by \(b''(x,y) = b'(x, \theta_0 y)\) for even \(x,y\). The inverse \(b'' \mapsto b'\) is given by \(b'(x,y) = b''(y,x) = b''(x, \theta_0 y)\) for even \(x\), odd \(y\). Use \((\theta_0)^2 = 1\) in Cl.

The following theorem can be used to construct \(b\) that satisfy (i) and that partially satisfy (ii), in a way that is still useful.

**Theorem 6 (Invariant Clifford average, Clifford unitarity trick)** The invariant Clifford averaging element \(\pi \in S^2 \text{Cl}\) in Theorem 11 defines a \(\Pi \in \text{End}^{\text{even}}(S^2 \mathcal{F})\). Suppose \(b'\) as in Remark 2 with \(b'(-, \theta_0-)\) positive definite. Then
\[
b = b' \circ \Pi
\]
satisfies (i) and \(b(-, \theta_0-)\) is positive definite. And this is a projection, in the sense that \(b = b'\) if and only if \(b'\) already satisfied (i).
Proof. We have \( b \in \text{Hom}^{\text{odd}}(S^2 \mathcal{H}, C^\infty) \) since \( \pi \) is even. Use Theorem 11. Positivity since \( b(-, \theta_0) = \frac{1}{|\mathcal{E}|} \sum_{f \in \mathcal{E}} b'(f-, \theta_0 f-) \) is an average over signs. \( \square \)

We now parametrize more explicitly the space of \( b \) that satisfy the assumptions of Theorem 5. These assumptions are oblivious to the filtration of \( \mathcal{H} \), only its structure as a \( \mathbb{Z}_2 \)-graded \( \mathcal{H} \)-module counts, so we can use the isomorphism in Theorem 4. The rank of \( A \) plays a minor role in the following.

The ‘transpose’ \( x \mapsto x^T \) is the unique anti-automorphism of \( \mathcal{H} \) that acts as the identity on the image of \( W \hookrightarrow \mathcal{H} \). As a \( C^\infty \)-module, \( \mathcal{H} \) has a canonical \( \mathbb{Z}_2 \)-grading by Lemma 12, the degree \( k \) subspace having basis \( \{ \theta_{i_1} \cdots \theta_{i_k} \mid i_1 < \ldots < i_k \} \). Let

\[
\langle - \rangle_\theta : \mathcal{H} \to (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes C^\infty
\]

be the unique \( C^\infty \)-linear map that annihilates elements of even degree, and \( \langle \sigma \rangle_\theta = \sigma_i \) for \( i = 0 \ldots 3 \) and \( \langle \theta_1 \theta_2 \theta_3 \rangle_\theta = -i \sigma_i \), \( \langle \theta_1 \theta_2 \theta_3 \rangle_\theta = -i \sigma_i \) and \( \langle \theta_1 \theta_2 \theta_3 \rangle_\theta = -i \sigma_i \) where \( \sigma_i \in \text{Herm}(\mathbb{C}^2) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \) are the Pauli matrices. Below, \( \text{Herm}(\mathbb{C}^2 \otimes A) \) are the \( C^\infty \)-bilinear Hermitian forms, antilinear in the first argument.

Theorem 7 (Explicit construction of gauges). We use \( \mathcal{H} \cong \mathcal{H} \otimes A \) from Theorem 4. An isomorphism of \( C^\infty \)-modules

\[
\text{Herm}(\mathbb{C}^2 \otimes A) \to \left\{ b \in \text{Hom}^{\text{odd}}(S^2 \mathcal{H}, C^\infty) \text{ with } b(-, w-) \in \text{Hom}^{\text{even}}(S^2 \mathcal{H}, C^\infty) \text{ for all } w \in W \right\}
\]

is given by \( h \mapsto b_h \) where for all \( x, x' \in \mathcal{H} \) and \( a, a' \in A \):

\[
b_h(xa, x'a') = \text{Re} \left( h(- \otimes a, - \otimes a') \langle x^T T' \rangle_\theta \right)
\]

If \( h \) is positive definite, then \( b_h(-, w-) \) is positive definite for all \( w \in W_+ \).

Proof. Note that \( \langle x^T \rangle_\theta \) is the conjugate transpose of \( \langle x \rangle_\theta \) for all \( x \in \mathcal{H} \). Therefore \( (x^T x')^T = (x^T x)^T \) and \( (x^T x')^T = (x^T)^T x = (x^T)^T x \) imply the symmetry of \( b_h \) and \( b_h(-, w-) \) respectively. By a linear algebra computer calculation the map is an isomorphism, in particular the space of \( b \) and the space \( \text{Herm}(\mathbb{C}^2 \otimes A) \) have equal rank \( 324 = 18^2 \). We sketch how positivity is proved. By \( \text{SL}(\mathbb{C}^2) \)-symmetry, it suffices to check positivity for \( w = \theta_0 \). It suffices to check that \( f : \text{Cl}_{13} \times \text{Cl}_{13} \to \mathbb{C}^2 \otimes \mathbb{C}^2 \), \( (x, x') \mapsto \langle x^T T' \rangle_\theta \) is of the form \( (x, x') \mapsto \sum b_{x \otimes y} \otimes \mathbb{C}^2 \) for a finite set of \( B \in \text{Hom}_{\mathbb{R}}(\text{Cl}_{13}, \mathbb{C}^2) \) whose common kernel vanishes. Since \( f \) annihilates \( \text{(odd,even)} \), consider \( \text{(even,even)} \) only, \( \text{(odd,odd)} \) is similar. Parametrize \( u : \mathbb{R}^4 \otimes \mathbb{R}^4 \to \text{Cl}_{13}^{\text{even}} \), \( v \otimes w \mapsto w_0 + v_1 \theta_0 \theta_1 + v_2 \theta_0 \theta_2 + v_3 \theta_0 \theta_3 + w_1 \theta_1 \theta_1 + w_2 \theta_1 \theta_2 + w_3 \theta_1 \theta_3 - v_0 \theta_0 \theta_1 \theta_2 \). A calculation shows that \( f(u(v \otimes w), u(v' \otimes w')) \) equals \( \sum B_e(v - iw) \otimes \mathbb{C}^2 B_e(v' - iw') \) times a positive constant, with summation over the 16 elements \( e = (\pm 1 \pm i, \pm 1 \pm i) \in \mathbb{C}^2 \), and where \( B_e \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^4, \mathbb{C}^2), v \mapsto i v_0 \sigma_0 + v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3 \). \( \square \)
8 Gauges, usage

The concept of a gauge in Definition 8 can be applied at both (i) the linear and formal perturbative nonlinear level and (ii) the nonlinear level. At the level (i) we get a contraction for a dgLa that, via the machinery of L∞-homotopy transfer, is directly applicable at the formal perturbative nonlinear level. At the level (ii) we get local-in-time existence and uniqueness for the Einstein equations, a standalone alternative to the traditional approach using the harmonic gauge of Einstein and, rigorously, Y. Choquet-Bruhat. Here we limit ourselves to (i) because it relates to the homological framework, and because the same manipulations also yield (ii).

Definition 8 is purely algebraic, whereas symmetric hyperbolicity is usually defined using explicit matrix notation as in Section 10. The following theorem and proof show how they are brought together, via the anchor map. Recall \( \text{Hom} = \text{Hom}_{C^\infty} \).

**Theorem 8 (Linear symmetric hyperbolic system)** Suppose a gauge \((\mathcal{E}_G, B)\) is fixed, see Definition 8. Suppose an element \(x \in \mathcal{E}^1\) is fixed, and suppose it is globally hyperbolic in the sense of Definition 6. For every \(k\) define

\[
L^k : \mathcal{E}^k_G \to \text{Hom}(\mathcal{E}^k_G, C^\infty)
\]

\[
u \mapsto B^k(-, [x, \nu])
\]

Then for every fixed \(R \in \text{Hom}(\mathcal{E}^k_G, C^\infty)\), the equation \(L^k(u) = R\) is a linear symmetric hyperbolic PDE for the unknown \(u \in \mathcal{E}^k_G\) when written out in a suitable coordinate system \(M \simeq \mathbb{R}^4\), and relative to a \(C^\infty\)-basis for \(\mathcal{E}^k_G\). The map \(L^k\) is surjective, and the kernel of \(L^k\) is isomorphic to restrictions of elements of \(\mathcal{E}^k_G\) to \(t = 0\).

**Proof** We suppress the index \(k\), and we note that the right hand side \(R\) is irrelevant for symmetric hyperbolicity. The map \(L\) is a first order differential operator, in the sense that for every \(f \in C^\infty\) the map \(J_f(u) = L(fu) - fL(u)\) is \(C^\infty\)-linear,

\[
J_f \in \text{Hom}(\mathcal{E}_G, \text{Hom}(\mathcal{E}_G, C^\infty)) \simeq \text{Hom}(\mathcal{E}_G \otimes \mathcal{E}_G, C^\infty)
\]

In fact there is an \(a \in W \otimes \text{Der}(C^\infty)\) with \([x, fu] = a(f)u + f[x, u]\) for all \(f \in C^\infty\) and \(u \in \mathcal{E}\), a piece of the anchor map, so \(J_f = B(-, a(f)-)\) with \(a(f) \in W\). Definition 8 implies \(J_f \in \text{Hom}(S^2 \mathcal{E}_G, C^\infty)\), the symmetry condition for a symmetric hyperbolic equation. For the positivity condition, use the coordinate system \(M \simeq \mathbb{R}^4\) that yields global hyperbolicity in Definition 6, with \(t \in C^\infty\) the first coordinate. Then \(a(t) \in W_+\), and therefore \(J_f\) is positive definite by Definition 8. The surjectivity and kernel follow from global solvability of linear symmetric hyperbolic equations. \(\square\)

**Theorem 9 (Contraction)** With the assumptions of Theorem 8, in particular global hyperbolicity, the following composition is surjective for every \(k\):

\[
K : \mathcal{E}^k_G \hookrightarrow \mathcal{E}^k \xrightarrow{[x,-]} \mathcal{E}^{k+1} \to \mathcal{E}^{k+1} / \mathcal{E}^k_G
\]

If in addition \(x \in \text{MC}(\mathcal{E})\) and \(d = [x, -]\) the associated differential, then there is a contraction from \((\mathcal{E}, d)\) down to the subcomplex \((\text{ker} K, d|_{\text{ker} K})\). A homotopy giving the contraction is given by the composition

\[
\mathcal{E}^{k+1} \to \mathcal{E}^{k+1} / \mathcal{E}^k_G \to \mathcal{E}^k_G \hookrightarrow \mathcal{E}^k
\]
where the middle arrow is any $\mathbb{R}$-linear (not $C^\infty$-linear) right-inverse of $K$.

Proof Every $r \in \mathcal{P}_{k+1}/\mathcal{P}_3$ yields a well-defined $R = B^1(-, r) \in \text{Hom}(\mathcal{P}_G, C^\infty)$, so surjectivity follows from Theorem 8 and Definition 8. \hfill \Box

9 The constrained structure of $\mathbb{Z}_2$-graded Clifford modules

The Clifford algebra construction [17, 18] is a functor from finite-dimensional real inner product spaces to finite-dimensional unital associative real algebras with a distinguished subspace. Let $\text{Cl}_{pq}$ be the real Clifford algebra with $p$ respectively $q$ generators squaring to $+1$ respectively $-1$. The generators $e_i$ are understood to satisfy $(e_i)^2 = \pm 1$ and $e_i e_j + e_j e_i = 0$ if $i \neq j$. The distinguished subspace is the span of the $p + q$ generators. There is a canonical $\mathbb{Z}_2$-grading by declaring that the distinguished subspace be odd. The Clifford algebra has a canonical non-decreasing filtration.

In general $\text{Cl}_{pq}$ is not isomorphic to $\text{Cl}_{qp}$ as a real algebra, but this is inconsequential if one studies $\mathbb{Z}_2$-graded modules; all Clifford modules in this paper are. All modules are understood to be finitely generated, unital left modules.

Lemma 13 (Category of $\mathbb{Z}_2$-graded modules) The $\mathbb{Z}_2$-graded algebras $\text{Cl}_{pq}$ and $\text{Cl}_{qp}$ have the same categories of $\mathbb{Z}_2$-graded modules.

Proof To avoid misconceptions, $p \neq q$. Let $e_i$ be the generators of $\text{Cl}_{pq}$ and $f_i$ the generators of $\text{Cl}_{qp}$. Order them such that $(e_i)^2 = 1$ if and only if $(f_i)^2 = -1$. Let $M$ be a $\mathbb{Z}_2$-graded module of $\text{Cl}_{pq}$. Let $s \in \text{End}(M)$ be equal to 1 respectively $-1$ on the even respectively odd sector of $M$. Then $M$ becomes a $\mathbb{Z}_2$-graded module of $\text{Cl}_{qp}$ by representing $f_i$ as $e_i s$. To see this, observe that $s^2 = 1$ and, since the $e_i$ are represented as odd elements, $se_i + e_i s = 0$. We have only discussed the correspondence at the level of objects, but it is easily extended to morphisms. As a strict aside, by viewing $s$ as a new Clifford generator, this proof establishes an isomorphism $\text{Cl}_{p+1,q} \simeq \text{Cl}_{q+1,p}$. \hfill \Box

The structure of $\mathbb{Z}_2$-graded Clifford modules is highly constrained, much more so than for the exterior algebra. We only consider $\text{Cl}_{13}$. Its even subalgebra is isomorphic as just algebras to $\text{Cl}_{30}$. Let $S \in \text{Aut}_R(\text{Cl}_{30})$ be the outer real algebra automorphism that acts like minus the identity on the three generators of $\text{Cl}_{30}$.

Lemma 14 A $\text{Cl}_{30}$-module is free iff it extends to a module of the real algebra given by the presentation $\langle \text{Cl}_{30}, T | T^2 = 1, S(m) = TmT \text{ for all } m \in \text{Cl}_{30} \rangle$ where the symbol $T$ is a new generator.

Proof We have $\text{Cl}_{30} \cong M_2(\mathbb{C})$ as real algebras by mapping the three generators $x_1, x_2, x_3 \in \text{Cl}_{30}$ to the three Pauli matrices $\sigma_1, \sigma_2, \sigma_3 \in M_2(\mathbb{C})$. The automorphism $S$ corresponds to conjugating elements of $M_2(\mathbb{C})$ by the quaternionic matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, an involution. The algebra presented in the lemma is $M_2(\mathbb{H}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})J$. Modules of $M_2(\mathbb{H})$ are isomorphic to $(\mathbb{H}^2)^n$ for some $n$, with $\mathbb{H}^2$ the quaternionic column vectors, and $\mathbb{H}^2 \cong M_2(\mathbb{C})$ as $M_2(\mathbb{C})$-modules, so it is free. \hfill \Box

Let $e_0, e_1, e_2, e_3$ be the generators of $\text{Cl}_{13}$, in particular $(e_0)^2 = 1$. Let $P \in \text{Aut}_R(\text{Cl}_{13})$ be the algebra automorphism induced by $(e_0, e_1, e_2, e_3) \mapsto (e_0, -e_1, -e_2, -e_3)$. 
for every $i$ there is a unique character $\chi_i$. The proof of invariance is omitted, but the idea is that say $\chi_i$ are $\{\pm\}$ submonoid generated by $F$. Let $\pi$ be the module. An algebra isomorphism $\pi$ of $\Cl_{13}$-module is free if it extends to a $\Z_2$-graded module of the real algebra presented by $\langle T \mid T^2 = 1, P(m) = TmT \text{ for all } m \in \Cl_{13} \rangle$ with $T$ a new symbol.

Being free means the module is isomorphic to a power of $\Cl_{13}$ as a $\Z_2$-graded $\Cl_{13}$-module; the isomorphism need not encompass the filtrations if the module is filtered.

Beware that $T$ has to be even. Otherwise the existence of such an operator is trivial because $P$ is inner, $P(m) = e_0me_0$, yet not all $\Z_2$-graded modules are free, for instance $\Cl_{13}$ itself is a direct sum of two proper submodules as a module over itself.

All $\Cl_{13}$-modules in this paper are $\Z_2$-graded and naturally come with an operator $T$, and all morphisms respect this, so Theorem 10 is quite useful.

Proof. We only prove the ‘if’ claim. Let $M$ be the module. An algebra isomorphism $\Cl_{30} \to \Cl_{13}$ is defined by $x_i \mapsto e_0e_i$. View $N = M^{even}$ as a $\Cl_{10}$-module, note that $T(N) \subseteq N$ and use Lemma 14, so $N$ is free. We have $M \cong N \oplus N$ as $\Z_2$-graded $\Cl_{13}$-modules, with the opposite $\Z_2$-grading in the second direct summand, where the module structure of $N \oplus N$ is such that $\Cl_{30}$ acts diagonally, and $e_0$ exchanges summands, and recall $(e_0)^2 = 1$. Conclude that $N \oplus N \cong M$ is free. □

Every Clifford module defines, and is defined by, a representation of a finite group called the Clifford group. This allows one to bring finite group techniques to bear.

Lemma 15 (The finite Clifford group) For a choice of generators $\{e_i\} \subseteq \Cl_{pq}$, the submonoid generated by $\{\pm 1, e_i\}$ is a group $F$ of finite order $|F| = 2^{p+q+1}$. Each element is $\Z_2$-odd or $\Z_2$-even. Every $\Cl_{pq}$-module restricts to a real representation of $F$ that represents $-1$ as minus the identity, and this is a one-to-one correspondence. For every $i$ there is a unique character $\chi_i : F \to \{\pm 1\}$ defined by $f e_i = \chi_i(f)e_i$.

Proof. Omitted. □

The group $F$ depends on the choice of generators, but it allows us to define an object that does not, for $\Cl_{1q}$.

Theorem 11 (Invariant Clifford average in $\Cl_{1q}$) Define $\pi \in S^2 \Cl_{1q}$ by

$$\pi = \frac{1}{|F|} \sum_{f \in F} \chi_0(f) f \otimes f$$

Then $\pi$ is invariant in the sense that it is independent of the choice of generators used to define $F$. In the $\Z_2$-graded algebra $S^2 \Cl_{1q}$ we have:

- $\pi^2 = \pi$ and $\pi$ is even.
- $\pi(x \otimes 1) = \pi(1 \otimes x)$ for all $x$ in the distinguished subspace, $x \in R^{1+q} \subseteq \Cl_{1q}$.
- $\pi(1 \otimes e_0) = \frac{1}{|F|} \sum_{f \in F} f \otimes e_0f$ with $e_0$ the first basis element used to define $F$.

Proof. The proof of invariance is omitted, but the idea is that say $\sum \chi_0(e_i)e_i = e_0 \otimes e_0 - e_1 \otimes e_1 - \ldots - e_q \otimes e_q$ is invariant. By construction $\pi(f \otimes f) = \chi_0(f) \pi$ for all $f \in F$ which implies $\pi^2 = \pi$. Also $\pi(f \otimes f^2) = \chi_0(f) \pi(1 \otimes f)$. Set $f = e_i$ and note that we happen to have $(e_i)^2 = \chi_0(e_i)$ and therefore $\pi(e_i \otimes 1) = \pi(1 \otimes e_i)$, hence $\pi(x \otimes 1) = \pi(1 \otimes x)$ by linearity. The rest is clear. □
Remark 3 The Clifford algebra is filtered, \( \text{Gr} \mathbb{C}l_{13} \cong \wedge \mathbb{R}^4 \) as graded commutative algebras. The associated graded \( \text{Gr} \) is a functor from filtered \( \mathbb{Z}_2 \)-graded \( \mathbb{C}l_{13} \)-modules to graded \( \wedge \mathbb{R}^4 \)-modules. One can ask which \( \wedge \mathbb{R}^4 \)-modules and morphisms are in the image of the \( \text{Gr} \)-functor, and which \( \wedge \mathbb{R}^4 \)-modules are the associated graded of \( \mathbb{C}l_{13} \)-modules that as unfiltered modules are free as in Theorem 10. A necessary condition is that the real dimension has to be a multiple of \( \dim_{\mathbb{R}} \mathbb{C}l_{13} = 16 \). Though free \( \wedge \mathbb{R}^4 \)-modules are in the image, some non-free modules are too.

10 Symmetric hyperbolic systems

The theorem of Picard-Lindelöf gives local existence and uniqueness for ODE. There is a similar theorem for a class of PDE called quasilinear symmetric hyperbolic systems. We only discuss local control and hence use germs; global control requires a more problem specific analysis, just as it does for ODE.

Let \( x^\mu \) and \( \partial_\mu \) be the standard coordinates and partial derivatives on \( \mathbb{R}^n \). Denote by \( \text{Herm}_k \subseteq \mathbb{C}^{k \times k} \) the real subspace of Hermitian matrices.

**Theorem 12 (Local existence and uniqueness)** Suppose \( A^\mu \in C^\infty(\mathbb{R}^n \times \mathbb{C}^k, \text{Herm}_k) \) for \( \mu = 0, \ldots, n-1 \) and \( b \in C^\infty(\mathbb{R}^n \times \mathbb{C}^4, \mathbb{C}^k) \). Suppose \( A^0(0,0) \) is positive definite. Then there exists a unique \( u \in C^\infty(\text{germs at } 0) \) such that, as germs at \( x = 0 \),

\[
\begin{align*}
\sum_\mu A^\mu(x,u(x))(\partial_\mu u)(x) &= b(x,u(x)) \\
u|_{x=0} &= 0
\end{align*}
\]

**Proof** Omitted, see [2, 11]. Briefly, one derives a-priori energy estimates by applying the divergence theorem to \( \sum \partial_\mu (u^* A^\mu(x,u)u) \) and higher derivative expressions. □

Beware that even if \( A^0 = I \), the claim fails if the \( A^\mu \) are not in \( \text{Herm}_k \), see Lewy’s example. We have assumed that \( A^\mu \) and \( b \) are smooth and everywhere defined, that \( u \) satisfies trivial initial conditions at \( x^0 = 0 \), and so forth. This simplified statement implies more general statements, say by changing coordinates in \( x \) and \( u \).

ODE correspond to \( n = 1 \). An interesting example related to Maxwell’s equations is \( \sum_\mu A^\mu \partial_\mu = \partial_0 + i \text{curl} \) with \( n = 4, k = 3 \). Unlike parabolic equations, symmetric hyperbolic systems enjoy finite speed of propagation.

11 Elements of Maurer-Cartan perturbation theory

See Gerstenhaber [16]. We describe the unobstructed case, for any gLa free over \( \mathbb{R}[[s]] \). Here \( s \) is a symbol, analogous statements hold for several symbols.

**Definition 9 (gLa free over \( \mathbb{R}[[s]] \))** We say that \( p \) is a gLa free over \( \mathbb{R}[[s]] \) if \( p \) is a gLa over \( \mathbb{R}[[s]] \) and there is a graded \( \mathbb{R} \)-vector space \( a \) and an isomorphism of graded \( \mathbb{R}[[s]] \)-modules \( p \simeq a[[s]] \). The induced \( p/sp \simeq a \) makes \( a \) a real gLa.
Theorem 13 (The unobstructed case) Suppose \( p \) is a gLa free over \( \mathbb{R}[s] \). Suppose \( x_0 \in \mathbf{MC}(p) \). Define the differential \( d = [x_0, -] \in \mathrm{End}^1(p/sp) \) and set
\[
\mathbf{MC}_{x_0}(p) = \{ x \in \mathbf{MC}(p) \mid x = x_0 \mod sp^1 \}
\]
Write \( H^k = H^k(d) \) for the \( k \)-th homology. Suppose \( H^2 = 0 \) (‘unobstructed’). Then:

- There exists a map \( \phi : H^1 \to \mathbf{MC}_{x_0}(p) \) of the form \( \phi(\xi) = x_0 + \sum_{k \geq 1} s^k \phi_k(\xi^{\otimes k}) \) where \( \phi_k \in \mathrm{Hom}_\mathbb{R}(((H^1)^{\otimes k}, p^1)) \) and \( \phi_1(\xi) \mod sp^1 \) is a representative of \( \xi \in H^1 \).
- Every such \( \phi \) extends, by \( \mathbb{R}[s] \)-multilinear extension of \( \phi_0 \), to \( H^1[\{s\}] \to \mathbf{MC}_{x_0}(p) \), and this map induces a bijection onto ‘the formal moduli space at \( x_0 \):’
\[
H^1[[s]] \to \mathbf{MC}_{x_0}(p) / \exp(sp^0)
\]

**Proof** Freeness is used whenever we invoke the isomorphism \( 1/s^K : s^Kp \to p \). By \( H^2 = 0 \) there is an \( h : a^2 \to a^1 \) with \( dh|_{a^1 \cap \mathrm{ker}d} = 1 \). Let \( i : a \hookrightarrow p \) and \( p : p \to a \) be the canonical maps, \( pi = 1 \). Let \( r : H^1 \to a^1 \) choose representatives. For each \( \xi \in H^1 \) we construct \( c_k \in p^1 \) such that \( e_k \in s^{k+1}p^2 \) for all \( K \), where by definition \( e_k = [1_{\leq K}, 1_{\leq K}] \) and \( 1_{\leq K} = [k_{\leq K}, e_k] \). Set \( c_0 = i_0, c_1 = -\frac{1}{2}ihp(e_0/s) + ir\xi \), and thereafter set \( c_{k+1} = -\frac{1}{2}ihp(e_k/s^{k+1}) \). We show by induction on \( K \):
\[
A_K : e_k \in s^{k+1}p^2 \quad B_K : dp(e_k/s^{k+1}) = 0 \quad C_K : dp c_{k+1} = -\frac{1}{2}p(e_k/s^{k+1})
\]
Here, \( C_0 \) holds by \( x_0 \in \mathbf{MC}(a) \); and \( A_K \) by \( e_k = e_{k-1} + 2s^K[c_0, e_k] \mod s^{k+1}p^2 \) and \( C_{K-1} \). Next, \( B_K \) by \( [c_0, e_k] = [c_0 - 1_{\leq K}, e_k] + [1_{\leq K}, e_k] \in s^{k+2}p^3 \) where the first term is in \( s^{k+2}p^3 \) by \( A_K \), the second is zero by a Jacobi identity. Finally \( C_K \) holds by \( B_K \) and the definition of \( h \); for \( C_0 \) use \( dr\xi = 0 \). This map \( \xi \mapsto \sum_{k \geq 0} s^k c_k \) is a map \( \phi \) of the desired kind; the \( c_k \) are not homogeneous in \( \xi \) but one can reorganize to extract homogeneous \( \phi_k(\xi^{\otimes k}) \). The \( \mathbb{R}[s] \)-multilinear extension of a given \( \phi \) is clear. \( \square \)

12 Spinor functor

By the ‘spinor functor’ we mean a functor from the groupoid of 2-dimensional complex vector spaces to the groupoid of 4-dimensional real vector spaces with a conformal inner product of signature \(-+++\); the morphisms are the structure-preserving isomorphisms, and a conformal inner product is one modulo \( \mathbb{R}_+ \). The spinor functor associates to the 2-dimensional complex \( V \) the 4-dimensional real subspace\(^4\)
\[
W_V \subseteq V \otimes \overline{\nabla}
\]
\(^4\) The conjugate \( \overline{\nabla} \) is a vector space together with \( \mathbb{C} \)-antilinear maps \( V \to \overline{\nabla} \) and \( \overline{\nabla} \to V \) that are mutual inverses; it exists and is unique up to isomorphism. Conjugation on \( V \otimes \overline{\nabla} \) is \( x \otimes y \mapsto \overline{\nabla} \otimes \overline{x} \).
with a representative \( S^2 W_V \to \mathbb{R} \) of the conformal inner product the restriction of the canonical \( S^2 (V \otimes V) \to (\wedge^2 V) \otimes (\wedge^2 V) \simeq \mathbb{C} \), which has the right signature.

Applying this fiberwise yields a corresponding ‘spinor functor’

\[
gaugeGrpd_{\text{spinor}} \to \text{gaugeGrpd}
\]

where on the left we have the groupoid of rank 2 complex vector bundles over a base manifold \( \simeq \mathbb{R}^4 \); the morphisms are the isomorphism of vector bundles. We denote by \( V \) the \( C^\infty_\mathbb{C} = \mathbb{C} \otimes C^\infty \)-module of sections of this vector bundle, by \( W_V \) the associated \( C^\infty \)-module of rank 4 with conformal inner product.

**Lemma 16 (Module derivations of \( V \))** Let \( \text{MDer}_{C^\infty}(V) \subseteq \text{Der}(C^\infty) \oplus \text{End}_{C^\infty}(V) \) be the module derivations of \( V \) as a \( C^\infty \)-module (not \( C^\infty_\mathbb{C} \)) that are \( \mathbb{C} \)-linear on \( V \). Then there are canonical \( C^\infty \)-Laoid morphisms:

- \( \text{MDer}_{C^\infty}(V) \to \text{MDer}_{C^\infty}(\overline{V}) \), actually an isomorphism.
- \( \text{MDer}_{C^\infty}(V) \to C\text{Der}(W_V) \), surjective with kernel the \( C^\infty \)-span of \( 0 \oplus iI \).

**Proof** The first map is \( \delta \mapsto \delta' = c \circ \delta \circ c \) where \( c \) is conjugation, and it is the identity on \( \text{Der}(C^\infty) \). The second is \( \delta \mapsto (x \otimes y \mapsto \delta x \otimes y + x \otimes \delta' y) \) which is well-defined with, for once, the tensor products over \( C^\infty_\mathbb{C} \).

We give an equivalent definition of the spinor functor using a basis.

**Lemma 17** If \( V = C^\infty_\mathbb{C}^n \oplus C^\infty_\mathbb{C}^m \) then a conformally orthonormal frame for \( W_V \) is

\[
\theta_0 = v^\top + w^\top \quad \theta_1 = v^\top - w^\top \quad \theta_2 = iw^\top + iv^\top \quad \theta_3 = v^\top - w^\top
\]

Define \( \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_{23}, \sigma_{31}, \sigma_{12} \in \text{End}_{C^\infty}(V) \cap \text{MDer}_{C^\infty}(V) \) by

\[
\begin{align*}
\sigma_0 &= \frac{1}{2}(1 1) \\
\sigma_1 &= \frac{1}{2}(1 0) \\
\sigma_2 &= \frac{1}{2}(0 1) \\
\sigma_3 &= \frac{1}{2}(1 0) \\
\sigma_{23} &= \frac{1}{2}(1 0) \\
\sigma_{31} &= \frac{1}{2}(0 1) \\
\sigma_{12} &= \frac{1}{2}(1 0)
\end{align*}
\]

relative to the basis \( v, w \). This is consistent, so under \( \text{MDer}_{C^\infty}(V) \to C\text{Der}(W_V) \) these elements map to elements of the same name in Definition 5.

**Proof** Omitted.

Clearly \( \mathcal{L}_V = (\wedge W_V) \otimes \text{MDer}_{C^\infty}(V) \) is a \( \wedge W_V \)-Laoid via base change \( C^\infty \to \wedge W_V \). Base change gives \( \wedge W_V \)-gLaoid representations \( \mathcal{L}_V \to \text{MDer}_{\wedge W_V}(M_k) \) where

\[
M_k = (\wedge W_V) \otimes (\wedge^k_{C^\infty} V) \simeq (\wedge^k_{C^\infty}(V \otimes_{C^\infty} \overline{V})) \otimes_{C^\infty} (\wedge^k_{C^\infty} V)
\]

We can use this to construct the ideal \( \mathcal{I} \subseteq \mathcal{L} \).

**Lemma 18** Set \( N = (\wedge W_V) N^2 \) with \( N^2 = \text{span}\{ (v^\top \wedge_{C^\infty} v^\top) \otimes_{C^\infty} v \mid v, x, y \in V \} \). Then \( N \subseteq M_k \) is an \( \mathcal{L}_V \)-invariant \( \wedge W_V \)-submodule, hence \( M_k/N \) a representation. Let \( \mathcal{I}_V \) be the kernel of the gLaoid representation \( \mathcal{L}_V \to \text{MDer}_{\wedge W_V}(M_0 \oplus M_k/N \oplus M_2) \). Then the image of \( \mathcal{I}_V \) under the surjection \( \mathcal{L}_V \to \mathcal{L} = (\wedge W_V) \otimes C\text{Der}(W_V) \) is \( \mathcal{I} \).

**Proof** Omitted.
References

1. Friedrich H., Proc. R. Soc. Lond. A 375, (1981) 169-184
   *On the Regular and the Asymptotic Characteristic Initial Value Problem for Einstein’s Vacuum Field Equations*

2. Friedrichs K.O., Comm. Pure and Appl. Math. 7, 2, (1954) 345-392
   *Symmetric hyperbolic linear differential equations*

3. Newmann E. and Penrose R.J., J. Math. Phys. 3, (1962) 566-578
   *An Approach to Gravitational Radiation by a Method of Spin Coefficients*

4. Nijenhuis A. and Richardson R.W. Jr., Bull. Amer. Math. Soc. 70, (1964) 406-411
   *Cohomology and deformations of algebraic structures*

5. Reiterer M. and Trubowitz E., Comm. Math. Phys., 307, 2, (2011) 275-313
   *Strongly Focused Gravitational Waves*

6. Reiterer M. and Trubowitz E., arxiv.org/abs/0910.4666
   *A formalism for analyzing vacuum spacetimes*

7. Reiterer M. and Trubowitz E., arxiv.org/abs/1104.4972
   *A class of gauges for the Einstein equations*

8. Reiterer M. and Trubowitz E., arxiv.org/abs/1412.5561
   *The graded Lie algebra of general relativity*

9. Reiterer M. and Trubowitz E., Filtered expansions in general relativity, to appear
   A precursor is arxiv.org/abs/1505.06662

10. Nütz M. and Reiterer M., arxiv.org/abs/1812.06454
    *Scattering amplitudes in 1M and GR as minimal model brackets and their recursive characterization*

11. Taylor M.E., Springer
    *Partial Differential Equations III, Nonlinear Equations*

12. Berglund A., Algebr. Geom. Topol. 14, 2511-2548 (2014)
    *Homological perturbation theory for algebras over operads*

13. Huebschmann J. and Stasheff J., Forum Mathematicum 14, 847-868 (2002)
    *Formal solution of the master equation via HPT and deformation theory*

14. Kontsevich M., Lett. Math. Phys. 66, 157-216 (2003)
    *Deformation quantization of Poisson manifolds*

15. Quillen D., Annals of Math. 90, 205-295 (1969)
    *Rational Homotopy Theory*

16. Gerstenhaber M., Ann. of Math. 79 (1964), 59-103
    *On the deformation of rings and algebras*

17. Atiyah M. F., Bott R. and Shapiro A., Topology (1964), 3, 3-38
    *Clifford modules*

18. Lawson H. B. and Michelsohn M. L (2016), Princeton university press
    *Spin Geometry (PMS-38)*

19. Belinskii V.A., Khalatnikov I.M. and Lifshitz E.M., Adv. Phys. 31 (1982), 639-667
    *A general solution of the Einstein equations with a time singularity*

20. Reiterer M. and Trubowitz E., arxiv.org/abs/1005.4908
    *The BKL Conjectures for Spatially Homogeneous Spacetimes*