Quantum Distributions for the Electromagnetic Field

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The coherence properties of the classical waves are discussed in terms of the Cauchy problem for the wave equation, and of a discrete representation by an ensemble of Hamiltonian systems. Wave quanta are related to specific "action fields", and phase-space distributions of phonons and photons are obtained by Wigner transform. For photons in a thermal environment, the proposed Wigner function evolves towards the Planck equilibrium distribution. It is shown that the free electromagnetic field can also be found in states of definite helicity, described by a complex vector potential.
1 Introduction

The great discoveries of X-rays, with particle-like properties (Röntgen, 1895), and of the energy quanta, explaining the spectral distribution of thermal radiation (Planck, 1900), soon after the discovery of radio waves (Hertz, 1887), have re-opened the old debate on the particle-wave nature of light, with partial solutions, and updates, until today. Now QED is a well established theory [1], and an example of accuracy for its predictions at the atomic level, but still, there is no clear relationship between some of its basic elements, like the photons, and the classical theory [2]. A detailed discussion of optical phenomena such as diffraction and refraction of light, using the Feynman’s path integrals, can be found in [3], while various aspects concerning a photon wave function defined by a complex linear combination of the classical fields \( E, B \) are presented in [4, 5]. Though, these fields are not canonical variables [6], and the outcome provides energy distributions [7] rather than probability density.

This work is an attempt to describe the wave quanta (phonons and photons) by specific ”action waves”, and distributions on a phase-space with granular structure. However, instead of considering the limit of vanishing rest mass in the relativistic Schrödinger equation for massive particles [8], the starting point here is the real, observable field. The basic elements of the present approach are introduced in Section 2, using the simple example of a 1-dimensional lattice of coupled harmonic oscillators. In Section 3, after a brief recall of the geometric framework behind the Maxwell equations, the photon wave function is defined, and its phase-space representation, obtained by the Wigner transform, is discussed both at zero and finite temperature. Concluding remarks are summarized in Section 4.

2 The phonon wave modes and phase-space for a 1d lattice

Let us consider the phase space \( M = \mathbb{Z} \times \mathbb{R}^2 \), parameterized by the real canonical variables \( \{ u_n, v_n \} \in \mathbb{R}^2, n \in \mathbb{Z} \), with the Poisson brackets \( \{ u_n, v_{n'} \} = \delta_{n n'}, \{ u_n, u_{n'} \} = 0, \{ v_n, v_{n'} \} = 0 \). For a system of coupled harmonic oscillators, described by the Hamiltonian [9]

\[
H = \sum_{n \in \mathbb{Z}} \frac{v_n^2}{2m} + \frac{m \omega_0^2}{2} u_n^2 + \frac{m \kappa}{2} \left( u_n - u_{n-1} \right)^2, \quad \kappa > 0,
\]

the equations of motion \( \dot{u}_n = \{ u_n, H \}, \dot{v}_n = \{ v_n, H \}, \) lead to

\[
\dot{u}_n = -\omega_0^2 u_n + \kappa (u_{n+1} + u_{n-1} - 2u_n), \quad n \in \mathbb{Z}.
\]

With respect to a unit of length \( \ell \in \mathbb{R}^+ \), (the lattice constant), one can define a real index \( x = n \ell \in \ell \mathbb{Z} \), and new coordinates \( u_{n\ell} = u_n/\sqrt{\ell}, v_{n\ell} = v_n/\sqrt{\ell} \), such that \( \{ u_{n\ell}, v_{n'\ell} \} = \delta_{n n'} \ell^{-1} \).

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1. For the electromagnetic field in a medium there is also the open problem of the energy-momentum tensor \( T_{ik} \), usually presented in the Minkowski (standard), or Abraham (symmetric) form [7].

2. In this limit the Compton wavelength, which sets the scale for space discretization, becomes infinite. Besides, wave quanta may have \( E/p \ll c \).
δ_{nn’}/ℓ. Presuming that in the limit ℓ → ℓ₀ > 0, x can be considered as a continuous variable, then \{u_ξ, v_ξ’\} → δ(x - x’), ℓ \sum_n \to f dx, and

\[ H \to \int dx \left[ \frac{v_x^2}{2m} + \frac{mω_0^2}{2} u_x^2 + \frac{mv^2}{2} (\partial_x u)^2 \right], \tag{3} \]

where \( v = ℓ₀\sqrt{k}, \partial_x \equiv \partial/∂x \). In the continuous limit (2), becomes \( δ_x^2 u + ω_0^2 u = v^2 δ_x^2 u \), as \((u_{x+\ell} + u_{x-\ell} - 2u_x)/\ell^2 \) is the finite differences expression of \( δ_x^2 u \). When \( ω_0 = 0 \) this reduces to the plane wave equation without sources, \( δ_x^2 u = v^2 δ_x^2 u \), having the known solution

\[ u(t) = E\hat{i}u^0 + \partial_t E\hat{i}u^0, \tag{4} \]

where \( E \) is the fundamental solution of \((δ_x^2 - v^2 δ_x^2)E = δ(x)δ(t), \hat{∗} \) denotes the convolution product, and \( u^0 \equiv δu|_{t=0}, u^0 \equiv u|_{t=0}, \) are the initial conditions of the Cauchy problem. An alternative formulation can be given by a partial decoupling of the oscillators using the complex coordinates

\[ u_α' = \frac{1}{\sqrt{2π}} \sum_{n∈Z} e^{iαn} u_n , \quad v_α' = \frac{1}{\sqrt{2π}} \sum_{n∈Z} e^{-iαn} v_n , \tag{5} \]

where \( α \in [-π, π] \) is an angle variable, or \( u_α' \equiv \sqrt{2}u_α', v_α' \equiv \sqrt{2}v_α', \) with \( k = α/ℓ, \) \((u_α')^* = u_{-α}, (v_α')^* = v_{-α}, \) \{u_α', v_α'\} = δ(k - k'). Because

\[ u_{nℓ} = \frac{1}{\sqrt{2π}} \int_{-π/ℓ}^{π/ℓ} dk e^{-ikℓn} u_k' , \quad v_{nℓ} = \frac{1}{\sqrt{2π}} \int_{-π/ℓ}^{π/ℓ} dk e^{ikℓn} v_k' , \tag{6} \]

for the interaction term in (1) one obtains

\[ \ell \sum_{n∈Z} \frac{mκ}{2} (u_n - u_{n-1})^2 = 2mκ \int_{-π/ℓ}^{π/ℓ} dk \sin^2 \left( \frac{kℓ}{2} \right) |u_k'|^2 , \]

so that

\[ H = \int_{-π/ℓ}^{π/ℓ} dk H_k' , \quad H_k' = \frac{v_{-k}' v_k'}{2m} + \frac{mω_0^2}{2} u_k' u_{-k}' , \tag{7} \]

where \( ω_k^2 = ω_0^2 + 4κ \sin^2 (kℓ/2) = ω_0^2 + v_k^2 4κ^2 (kℓ/2), v_ℓ = ℓ\sqrt{κ}. \) The equations of motion

\[ u_k' = v_{-k}'/m , \quad v_k' = -ω_k^2 u_{-k}' , \tag{8} \]

show that there is a "pure" oscillator mode at \( k = 0, \) while for \( k \neq 0 \) one can find both oscillator modes, such as \( u_k'(t) = \cos ω_k t \; u_k'^0, (u_k'^0)^* = u_{-k'}^0, \) and "plane rotator" modes [10], \( u_k^±(t) = e^{±iσ_kω_k t} u_k'^0, \) \( σ_k = k/|k|, \) of constant \(|u_k^±(t)|. \) At \( ω_0 = 0, \) or intermediate wavelengths \( λ = 2π/k, 2ℓ ≪ λ ≪ 2πν_ℓ/ω_0, \) such that \( ω_k ≈ ν_ℓ|k|, \) the "rotational" amplitudes \( u_k^±(t) \) provide by (6)

\[ u_{nℓ}^±(t) = \frac{1}{\sqrt{2π}} \int_{-π/ℓ}^{π/ℓ} dk e^{-ik(ℓn+ℓt)} u_k'^0 , \tag{9} \]
or, in the continuous limit, $u^\pm(x,t) = u^0(x \mp vt)$. In this case, the initial conditions $u^0, \dot{u}^0$ in (10) are not independent anymore, but related by $(\dot{u}^0)^\pm = \mp v \dot{u}^0$, $\dot{u}u^0(x) \equiv \dot{u}^0/dx$, such that

$$u^\pm(t) = \mp v \xi \dot{u}^0 + \partial_t \xi \dot{u}^0 = G^\pm \dot{u}^0,$$

where $G^\pm = (\partial_t \mp v \partial_x) \xi$. Because $G^\pm$ is the fundamental solution of the transport equation $(\partial_t \pm v \partial_x)G^\pm = \delta(x)\delta(t)$, linear in $\partial_t$, the functions $u^\pm(t)$ may also be called phonon wave modes.

For the further study, a more suitable set of coordinates is provided by the functionals $\psi^\prime_k, \psi^{* \prime}_k$ of $u, v$,

$$\psi^\prime_k = \sqrt{\frac{m \omega_k}{2}} (u^*_{k} + \frac{i \nu^\prime_k}{m \omega_k}) , \{\psi^*_{k}, \psi^{\prime \ast}_{k}\} = i \delta(k - k') ,$$

adapted to the solutions $u^\pm$ as $\psi^\prime_k \equiv \psi^\prime_k|_{u^\pm} = \sqrt{m \omega_k/2}(1 \pm \sigma_k)(u^\pm_k)^*$ is independent of $v$. In these variables $H'_\ell = \omega_k(|\psi^\prime_k|^2 + |\psi^{\prime \ast}_k|^2)/2$, such that $\dot{i} \psi^\prime_k = \omega_k \psi^{\prime \ast}_k$. The relationship $i \psi^\prime_k = \sqrt{m \omega_k}e^{ikx}$ to the "harmonic oscillator" action-angle variables $(\eta_k, \varphi_k)$ shows that during time evolution $\eta_k = |\psi^\prime_k|^2$ is a constant, while $\varphi_k(t) = - \omega_k t + \varphi^\prime_k$. In terms of $\psi^\prime_k, \psi^{* \prime}_k$ the canonical symplectic form $\Omega_M$ on $M$,

$$\Omega_M = \sum_{n \in \mathbb{Z}} du_n \wedge dv_n = \ell \sum_{n \in \mathbb{Z}} du_{n \ell} \wedge dv_{n \ell} ,$$

becomes

$$\Omega_M = \frac{1}{\pi} \int_{-\pi/\ell}^{\pi/\ell} dk \, dv_{k} \wedge dv^{\ast}_{k} = \frac{1}{2\pi} \int_{-\pi/\ell}^{\pi/\ell} dk \, d\eta_k \wedge d\varphi_k ,$$

so that $A_D = \int_D \Omega_M = 2\pi \int_{-\pi/\ell}^{\pi/\ell} dk \, \eta_k$ is the phase-space area of the invariant subset $D \subset M$, bounded by the "wave-orbit" cylinder $\partial D = \{(\varphi_k \in [-\pi, \pi], \eta_k = \text{constant}), k \in [-\pi/\ell, \pi/\ell]\}$.

The coordinates $\psi^\prime_k$ can also be represented by functions $\psi^\prime_{n \ell}$ on $\ell \mathbb{Z}$,

$$\psi^\prime_{n \ell} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\ell}^{\pi/\ell} dk \, e^{ik\ell \eta} \psi^\prime_k \, , \langle \psi | \psi \rangle = \ell \sum_{n \in \mathbb{Z}} \psi^{* \prime}_{n \ell} \psi^\prime_{n \ell} = \frac{A_D}{2\pi} .$$

In this representation $H = \ell^2 \sum_{n, n'} \psi^\prime_{n \ell} \hat{H}_{n n'} \psi^\prime_{n' \ell} \equiv \langle \psi | \hat{H} | \psi \rangle$, with

$$\hat{H}_{n n'} = \frac{1}{2\pi} \int_{-\pi/\ell}^{\pi/\ell} dk \, \omega_k e^{ik(n-n')} ,$$

and $i \partial_t \psi = \hat{H} \psi$. Because $A_D$ in (14) is a positive scalar, a constraint $A_D = Nh$, $N \in \mathbb{N}$, to integer multiples of the Planck’s constant $\hbar$ [11], allows the interpretation of $\psi/\sqrt{N}$ as amplitude for linear density/probability distributions of $N$ action quanta $\hbar$ (phonons)

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3The normalization of the 1-phonon (RPA) excited states, by "action integrals" quantization, was noticed before in the case of the constrained quantum dynamics on coherent states manifolds [12, 13].
along the coordinate \( x = n \ell \in \ell \mathbb{Z} \). However, a phonon momentum \( p \), conjugate to \( x \), cannot be defined like in the case of the inertial motion, because when \( \ell \mathbb{Z} \) is taken as a configuration space, the tangent space does not exist\(^4\). An alternative is to define \( p = \hbar k \), where \( k \) is the Fourier dual variable to \( x \), known to appear in linear combinations with the mechanical momentum\(^5\) at the massive particles \([10]\). For consistency, the obtained phase-space \( M_\ell = \ell \mathbb{Z} \times h[-\pi/\ell, \pi/\ell] \) carries indeed an intrinsic ”granular structure”, with \( h = \ell \cdot 2\pi \hbar/\ell \), as the \( \ell \)-independent area of an ”elementary cell”, until the continuous limit, \( M_\ell \to \mathbb{R}^2 \). A phase-space distribution function \( f_N \) of the \( N \) phonons can be further defined by the Wigner transform of \( \psi/\sqrt{\hbar} \),

\[
f_N(x, p) = \frac{1}{2\pi \hbar} \int dk \ e^{-ikx_0} \psi^*_{x, p} e^{ikx} = \frac{1}{2\pi \hbar} \int dk \ e^{ikx_0} \psi^*_{x, p} e^{-ikx},
\]

such that \( \int dx dp f_N(x, p) = A_D / \hbar = N \). Considering here \( \psi_k(t) = \sqrt{\eta_k} e^{-i\omega t}, \) \((\varphi_k^0 = 0)\), and \( \omega_a - \omega_b \approx (a - b)v_g \), where \( v_g = d\omega_k/dk_p \) denotes the group velocity at \( k_p = (a + b)/2 = p/\hbar \), \( f_N \) becomes

\[
f_N(x, p, t) = \frac{1}{2\pi \hbar^2} \int dk \ e^{ik(x-v_g t)} \sqrt{\eta_k + \hbar^2 \eta_k - \hbar^2}.
\]

In the case of a Gaussian function \( \eta_k^G = C_N \sqrt{g/\pi} \ e^{-g(k-k_0)^2} \), parameterized by \( k_0, \ g, \) and \( C_N = A_D / 2\pi \hbar \), one obtains

\[
f_N^G(x, p, t) = \frac{N}{\pi \hbar} e^{-g(p-hk_0)^2/\hbar^2 - (x-v_g t)^2/g},
\]

corresponding to an ensemble of \( N \) phonons having the average momentum \( p_0 = h k_0 \), moving coherently along the x-axis with the velocity \( v_g \) (e.g. as observed in \([15]\)).

Similarly to \( \psi \) of \((14)\), one can define a function \( \Phi \),

\[
\Phi_{n\ell} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\ell}^{\pi/\ell} \! dk \ e^{ik \ell \eta} \Phi_k^* \equiv \frac{i}{\sqrt{2m}} (\varphi_k^0 + \frac{i\varphi_k'}{\omega_k})
\]

and in the limit \( M_\ell \to \mathbb{R}^2 \), the quasi-energy density \( f_E \),

\[
f_E(x, p) = \frac{1}{2\pi} \int dk' e^{-ik'p} \Phi^*_{x, p} \sqrt{\omega_k} \psi_k^* = \frac{1}{2\pi} \int \! dx dp \ f_E(x, p) = E
\]

resembling the photon phase-space distribution presented in \([4]\).

3 Electromagnetic action waves and photon distributions

On the space-time manifold \( \mathbb{R}^4 \), with coordinates \((x_0, x)\), \( x_0 = ct \), \( x = (x_1, x_2, x_3) \), and metric \( ds^2 = -dt^2 + dx^2 \), the electromagnetic field \( E, B \) provides a 2-form \( \omega_f \in \wedge^2(\mathbb{R}^4) \),

\[
\omega_f = E \cdot ds_0 - B \cdot dS,
\]

\(^4\)A similar situation occurs in the more general case of discrete fractal sets \([14]\).

\(^5\)When \( p = \hbar k \), the scaled canonical 1-form \( pdx/\hbar \) represents the number of wavelengths \( \lambda \) in the interval \( dx \).
where $dS_0 \equiv dx_0 \wedge dx$, $dS \equiv (dx_2 \wedge dx_3, -dx_1 \wedge dx_3, dx_1 \wedge dx_2)$. This form has the Hodge dual to $\star \omega_f = -\mathbf{B} \cdot dS_0 - \mathbf{E} \cdot dS$, and the Maxwell equations can be derived from the "intrinsic" expressions

$$d\omega_f = J_m, \quad d\star \omega_f = J_e,$$

(with $\star$-duality [16] and $P_c$ t-symmetry [17]).

Due to the absence of magnetic monopoles in our 3d space, $J_m = 0$, and the first equation $d\omega_f = 0$, (or $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\partial_0 \mathbf{B}, \partial_0 \equiv \partial/\partial x_0$), ensures the existence of a potential 1-form $\theta_f = A_0 dx_0 + \mathbf{A} \cdot dx$, ($-A_0 = V$ is the Coulomb potential), defined up to a "gauge" term $\delta f = \partial_0 f dx_0 + \nabla f \cdot dx$, such that $\omega_f = -d\theta_f, (\mathbf{E} = -\partial_0 \mathbf{A} + \nabla A_0, \mathbf{B} = \nabla \times \mathbf{A})$. In the following, the gauge (the initial conditions), will be fixed to have $\nabla \cdot \mathbf{A} = 0$.

The potential $\mathbf{A} = (A_0, \mathbf{A})$ provides a more restricted set of "coordinate" variables for the field, because the second equation $d\star \omega_f = J_e$ can be obtained from the variational principle $\delta \mathbf{A} = 0$, where $\delta \mathbf{A}$ denotes the functional variation with respect to $\mathbf{A}$, $S_f = -(\omega_f, \omega_f)_{ip}/2c, S_{int} = -\langle \theta_f, \star J_e \rangle_{ip}/c$, and ($\alpha, \beta)_{ip} = \int_{\partial_4} \alpha \wedge \star \beta$ is the internal product.

The current $J_e$ can be decomposed as $J_e = J_f + d\omega_{pol}$, where $J_f$ is due to the free charges, and $\omega_{pol} = -\mathbf{M} \cdot dS_0 + \mathbf{P} \cdot dS$ is the polarization form, with $\mathbf{P} = \chi_e \mathbf{E} + \mathbf{P}_0, \mathbf{M} = \chi_m \mathbf{H} + \mathbf{M}_0$, the polarization, respectively magnetization vectors, having both induced ($\sim \chi$), and intrinsic [18] terms. Thus, the equation $d\star \omega_f = J_e$ becomes $d(\star \omega_f - \omega_{pol}) = J_f$, and further, in a homogeneous, isotropic medium, with no intrinsic polarization ($\mathbf{P}_0 = 0, \mathbf{M}_0 = 0$), it provides

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{H} = \partial_0 \mathbf{D} + \mathbf{j}_f/c,$$

where $\mathbf{D} = \epsilon \mathbf{E}, \mathbf{H} = \mathbf{B}/\mu$, and $\epsilon = 1 + \chi_e, \mu = 1 + \chi_m$ (usually denoted $\epsilon_r, \mu_r$). By introducing the complex vector $\mathbf{F} = \sqrt{\epsilon} \mathbf{E} + i\sqrt{\mu} \mathbf{H}$ [18], the time derivatives $\partial_t \mathbf{E}, \partial_t \mathbf{H}$ can also be expressed in the form

$$\partial_t \mathbf{F} = -iv\nabla \times \mathbf{F} - \mathbf{j}_f/\sqrt{\epsilon}, \quad v = c/\sqrt{\epsilon \mu}.$$

The related energy flow is described by the transport equation

$$\partial_t w + \nabla \cdot \mathbf{Y} + \mathbf{E} \cdot \mathbf{j}_f = 0,$$

where $w = (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})/2 = \mathbf{F} \cdot \mathbf{F}^*/2, (= T_{00})$, is the energy density, and $\mathbf{Y} = \epsilon \mathbf{E} \times \mathbf{H} = iv\mathbf{F} \times \mathbf{F}^*/2$, is the Poynting vector.

In $\mathbb{R}^n$, with the volume element $d^nx \equiv dx_1 \wedge \ldots \wedge dx_n$, and metric $ds^2 \equiv \sigma_1 dx_1^2 + \ldots + \sigma_n dx_n^2, \sigma_i = \pm 1$, if $d^nx = \alpha \wedge \beta$, where $\alpha = dx_{i_1} \wedge \ldots \wedge dx_{i_p}, p \leq n$, then $\star \alpha \equiv \sigma_{i_1} \ldots \sigma_{i_p} \beta$.

$^7t \equiv [\mathbf{p}, \mathbf{r}], t^2 = [1,-1], \text{defined by } \hat{\ast}(\omega_f; \ast \omega_f) \equiv (\star \omega_f; -\omega_f), \hat{P}_c(J_m; J_e) \equiv (J_e; -J_m)$.

$^8$A term $\mathbf{P}_0/\mathbf{M}_0$ can stand for an antenna-source of electric/magnetic-type Hertz vector waves. In [17], the intrinsic atomic electric and magnetic dipole moments are parts of a photon detection device.
The linearity in $\partial_t$ of (25) was a reason in [11] to interpret $F$ as quantum photon wave function, while the linearity of (26) has been used in [19] to find a classical Hamilton function for photons. Unlike $F$ and $w$, the potential $A|_{\nabla A=0}$ satisfies the inhomogeneous wave equation

$$\partial_t^2 A - v^2 \Delta A = c j_f/\epsilon - c \partial_t \nabla V,$$  

(27)

containing $\partial_t^2$. In the absence of sources ($j_f = 0, \partial_t \nabla V = 0$), this equation can be obtained by using the total field energy $E = \int d^3 x \; w$ as a Hamiltonian functional

$$H_F = \frac{1}{2} \int d^3 x \; [\mu v^2 \Pi^2 + \frac{1}{\mu} (\nabla \times A)^2],$$

(28)

of the canonical coordinates $A, \Pi = \dot{A}/\mu v^2$, $\{A_{\alpha}, \Pi_{j\nu}\} = \delta_{ij} \delta^3(\mathbf{x} - \mathbf{x}')$, $(\nabla \cdot A = 0$ is only a constraint on the solutions). Similarly to (21), the Fourier transform of $A_{\alpha}$ introduces the complex coordinates $A'_{\mathbf{k}}, A^*_{\mathbf{k}} = A'_{-\mathbf{k}}$, such that

$$A_{\alpha} = \frac{1}{(2\pi)^{3/2}} \int d^3 k \; e^{-ik\cdot x} A'_{\mathbf{k}}, \; \mathbf{k} = (k_1, k_2, k_3),$$

(29)

and if $\partial_t^2 A = v^2 \Delta A, \nabla \cdot A = 0$, then $A'_{\mathbf{k}} = -\omega_k^2 A'_{\mathbf{k}}, \omega_k = v|\mathbf{k}|, \mathbf{k} \cdot A'_{\mathbf{k}} = 0$. The previous solutions (10) correspond in this case to photon wave modes, having "rotational" amplitudes $|A'_{\mathbf{k}}| = \text{constant}$, such as the linearly polarized plane waves $A^\pm_{\alpha p}(x_1, t) = u^0(x_1 \mp vt) e_{\mathbf{p} \sigma}$, where $u^0$ is the initial condition in (10), and $e_{\mathbf{p} \sigma} = (0, e_2, e_3)$ is a constant, real polarization vector. More specific solutions, with circular polarization, are $A^\pm_{\alpha p}(x_3, t) = A_\perp (\cos \phi_\perp, \pm \sin \phi_\perp, 0), \phi_\perp = k(x_3 \mp vt)$, related to the real part of the complex vector potential $U$ for $F$, (Appendix 1), by $A^\pm_{\alpha p} = \sqrt{\mu}(U_{\pm1} + U_{\pm1}^*)/2$, where

$$U_{\sigma}(x_3, t) = \sqrt{\frac{2}{\mu}} A_\perp e^{-i(k_3 - \sigma vt)} e_{\mathbf{p} \sigma}, \; e_{\mathbf{p} \sigma} = \frac{1}{\sqrt{2}} (1, i\sigma, 0), \; \sigma = \pm 1.$$  

(30)

Aside the coordinates $A'_{\mathbf{k}}$, one can define variables similar to (11),

$$\bar{\psi}_{\mathbf{k}}' = \sqrt{\frac{\omega_k}{2\mu v^2}} (A^*_{\mathbf{k}} - i \omega_k A'_{\mathbf{k}}) \equiv \sum_{\tau=1,2} \sqrt{\eta_{k\tau}} e^{i\varphi_{k\tau}} e_{k\tau},$$

(31)

where $e_{k\tau}$ are two real polarization vectors, $e_{k\tau} \cdot e_{k\tau'} = \delta_{\tau\tau'}$, orthogonal to $\mathbf{k}$, and $\eta_{k\tau}, \varphi_{k\tau}$ are the "oscillator"action-angle coordinates. Because $i\partial_t \bar{\psi}_{\mathbf{k}}' = \omega_k \bar{\psi}_{\mathbf{k}}'$, the associated "action field"

$$\bar{\psi}_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} \int d^3 k' \; e^{i\mathbf{k}\cdot \mathbf{x}} \sqrt{\frac{\omega_k}{2\mu v^2}} (A^*_{\mathbf{k}} + i \omega_k A'_{\mathbf{k}})$$

(32)

satisfies $i\partial_t \bar{\psi} = \hat{H}_f \bar{\psi}$, with

$$\hat{H}_{fxx'} = \frac{1}{(2\pi)^3} \int d^3 k \; \omega_k e^{i(k\cdot x - k'\cdot x')} \; \hat{H}_f^2 = -v^2 \Delta.$$  

(33)
Expressing the amplitudes $A'_k$ of (22) in terms of $\vec{\psi}$, one obtains

$$A_k = \frac{c}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (e^{ik\cdot x} \psi_k^* + e^{-ik\cdot x} \psi_k^*) . \quad (34)$$

If we "quantize" $A_k$ by replacing for $\sqrt{\eta_{kr} e^{ik\cdot r}}$ and $\sqrt{\eta_{kr} e^{-i k\cdot r}}$ in $\psi_k^*$, $\psi_k^*$ the operators $\sqrt{\eta \hat{a}_{kr}}$, respectively $\sqrt{\eta \hat{a}_{kr}^\dagger}$, with (scaled) commutator $[\hat{a}_{kr}, \hat{a}_{kr}^\dagger] = \delta_{rr} \delta^3(k - k')$, then in vacuum ($\epsilon = 1$), using the units $c = 1$, $h = 1$, (34) becomes the field operator of [1], p. 90. However, one can proceed also by considering (32), after a suitable normalization, as photon wave function.

The functional 2-form $\Omega_F = \int d^3x \ dA \cdot \wedge W = i \int d^3x \ d\psi \cdot \wedge d\overline{\psi}^*$ indicates that $A_W = 2\pi \langle \psi | \psi \rangle = 2\pi \int d^3k (\eta_{k1} + \eta_{k2})$ represents the total phase-space area bounded by a wave-like orbit $\eta_{kr} = \text{constant}$, such that the quantization condition $A_W = N\hbar$, $N \in \mathbb{N}$, can be used to normalize $\psi$ to a given number of action quanta (photons). Moreover, a photon quasi-density $f_N$ on the phase-space $T^*\mathbb{R}^3$ can be obtained by the Wigner transform of $\overline{\psi}/\sqrt{\hbar}$,

$$f_N(x, p) = \frac{1}{(2\pi)^3\hbar} \int d^3k \ e^{-ik\cdot p} \, \frac{\psi^*}{\sqrt{\hbar}} \cdot \frac{\psi}{\sqrt{\hbar}} , \quad (35)$$

$$\int d^3x d^3p \ f_N(x, p) = N. \quad \text{According to the general identity} \ [10, 20]$$

$$\int d^3x d^3p \ f_1(x, p) f_2(x, p) \equiv \hbar^3 Tr(f_1 f_2) , \quad (36)$$

the total wave-field energy $E_W = \int d^3x \ w = \langle \psi | \hat{H}_f | \psi \rangle = \hbar^2 Tr(f_N \hat{H}_f)$ becomes in the phase-space representation $E_W = \int d^3x d^3p \ f_N(x, p) \epsilon_p$, where $\epsilon_p = \hbar \omega_p / c = v | p |$ is the photon energy. At a finite temperature $T$, the equilibrium energy density takes the form $\tilde{\omega}_T = \int d^3p \ f^T(x, p) \epsilon_p$, independent of $N$, where

$$f^T(x, p) = \frac{1}{\hbar^3 \epsilon_p / k_B T - 1}$$

is the Planck distribution function [11] (Appendix 2). The Wien’s displacement law in vacuum, written as $\lambda_m p_T = 1.26\hbar$, indicates that for a given "thermal momentum" $p_T = k_B T / c$, the maximum of the spectral energy density $U^T_\lambda$, ($\tilde{\omega}_T = \int d\lambda U^T_\lambda$), appears at a wavelength $\lambda_m$ close to the "dual" value $\hbar / p_T$. The number of thermal photons in a cube of volume $\lambda_m^3$ is $N_m = \lambda_m^3 \int d^3p \ f^T(x, p) = 0.48$, at any temperature.

4 Concluding remarks

The fundamental constituents of matter can be associated, by structural stability, with elementary adiabatic invariants, abstract entities encountered in the treatment of complex dynamical systems, including classical fields. The "wave quanta" are such invariants, as phase-space area elements $h = 2\pi \hbar = 4.1$ meV-ps, defining a partition of the
total action integral for the wave-field.

In the example of Section 2, (11) and (14) define the functional \( \psi_{[u],[v]}(x,t) \), normalized (using \( A_D = N\hbar \)) by summation over \( x \) at a fixed wave configuration \([u],[v]\), instead of integrating over all configurations at given \( x \). Therefore, if \( N \) is large, it should be regarded as density amplitude for a quasi-classical ensemble of \( N \) action quanta \( \hbar \). Nevertheless, it becomes closer to a true Schrödinger wave function, depending only on the "field coordinates" \( u \), on the subset of the "rotational" solutions \( u^\pm \), when it reduces to \( \psi_{[u]}^\pm(x,t) \). In particular, for \( N = 1 \), \( |\psi_{[u]}^\pm(x,t)|^2/\hbar \) can be interpreted as quantum probability density for the spatial localization of a phonon.

The complex field \( F = \sqrt{c}E + i\sqrt{\mu}H \) is associated to the complex 2-form \( \omega_f = F \cdot dZ \),

\[
\tilde{\omega}_f = \frac{1}{\sqrt{\hbar}} \omega_f - \frac{i}{\sqrt{\epsilon}} (\ast \omega_f - \omega_{pol}) \quad , \quad dZ = \frac{1}{\sqrt{\epsilon \mu}} dS_0 + i dS
\]  

In the absence of free charges and currents \( d\omega_f = 0 \), and there exists a complex 1-form \( \theta_f = U_0 d\sigma_0 + U \cdot d\mathbf{x} \), (defined up to a gauge term), where \( U = (U_0, U) \) is a complex potential, such that \( \tilde{\omega}_f = -d\theta_f \). This equality reduces to \( F = \sqrt{c}E(\ast - \partial_0 U + \nabla U_0) = i\nabla \times U \), so that \( U \) should satisfy

\[
i(\partial_t U - c\nabla U_0) = v \nabla \times U \quad , \quad v = c/\sqrt{\epsilon \mu} \]  

By choosing the gauge such that \( U_0 = 0 \), one obtains \( i\partial_t U = v \nabla \times U \), (of the form (25) at \( j_f = 0 \)), which means that in the Fourier expansion

\[
U(x,t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{-ik \cdot x} U'_k(t) \]  

the vector \( U'_k \) performs a precession around \( k \), according to \( \dot{U}'_k = -vk \times U'_k \). For the complex vectors \( U'_{k\sigma} \) such that \( k \times U'_{k\sigma} = -i\sigma |k|U'_{k\sigma} \), where \( \sigma = \pm 1 \) is the helicity,
this precession reduces to a phase factor, and $U'_{k\sigma}(t) = e^{i\sigma\omega_k t}U'_{k\sigma}(0)$, $\omega_k = v|k|$, similar to $\tilde{\psi}_k(t) = e^{-i\omega_k t}\tilde{\psi}_k(0)$ in (43). Another interesting solution is

$$U(\rho, \varphi, z, t) = e^{ik(z-\nu t)}(\rho e_p + i\sqrt{\frac{3}{k}}e_z) , \quad \nabla \cdot U = 0 \ ,$$

where $(\rho, \varphi, z)$ are the cylindrical coordinates with unit tangent vectors $(e_\rho, e_\varphi, e_z)$, and $e_p = (e_\rho + ie_\varphi)/\sqrt{2}$ has definite helicity, $e_z \times e_p = -ie_p$. This result shows that the cylindrical symmetry, known for its special properties in the case of the gravitational waves [21], also provides peculiar electromagnetic waves, having both transversal and longitudinal components.

**Appendix 2: The quantum noise and thermal radiation**

An ensemble of free photons in a homogeneous, isotropic, non-dispersive medium, can be described by the phase-space distribution function (55),

$$f(x, p) = \frac{1}{(2\pi)^3 h} \int d^3k' e^{-ik'p} \tilde{\psi}_{x+k'}/x \cdot \tilde{\psi}_{x} = \frac{1}{(2\pi)^3 h^3} \int d^3k e^{ikx} \tilde{\psi}_x + \frac{k}{2} \tilde{\psi}_x^* - \frac{k}{2} \ ,$$

where $\tilde{\psi}_k'(t) = e^{-i\omega_k t}\tilde{\psi}_k'(0)$. Presuming that the main contribution to the integral arises from the subset $|k| < |p|/h$, then for $a = p/h + k/2$, $b = p/h - k/2$, we get $\omega_a - \omega_b \approx (a - b) \cdot \nabla_k \omega |_{k=p/h} = k \cdot v_p$, $v_p = v/|p|$, and

$$\partial_t f = -\nabla \cdot y \ , \quad y = v_p f \ .$$

If (21) contains $j_f \neq 0$, $\nabla \cdot j_f = 0$, then $\tilde{A}'_k = -\omega_2^2 A'_k + c j_f' / \epsilon$, and $\tilde{\psi}'_k$ of (31) satisfies $i\partial_t \tilde{\psi}'_k = \omega_k \tilde{\psi}'_k - j_f / \sqrt{2e\omega_k}$. In particular, when $j_f$ reduces to the (quasi) Ohmic current $j_\Omega = \sigma q E$ generated by the ensemble of photons, specified by

$$j'_\Omega(k) = -\frac{\sigma q}{c} A'_k = -i\gamma \sqrt{\frac{e\omega_k}{2}} (\tilde{\psi}'_k - \tilde{\psi}'_{-k}) , \quad \gamma = \frac{\sigma q}{e} > 0 \ ,$$

one obtains

$$\partial_t (\tilde{\psi}'_a \cdot \tilde{\psi}'_b) = -i(\omega_a - \omega_b)\tilde{\psi}'_a \cdot \tilde{\psi}'_b - \gamma \tilde{\psi}'_a \cdot \tilde{\psi}'_{-b} + \frac{\sigma q}{2} \tilde{\psi}'_a \cdot \tilde{\psi}^*_{-b} + \frac{\sigma q}{2} \tilde{\psi}'_{-a} \cdot \tilde{\psi}^*_{b} \ .$$

The last two terms include the fast oscillating factors $e^{\pm(i\omega_a + \omega_b)t}$ and their contribution to $\partial_t f$ can be neglected, so that the (quasi) Ohmic environment changes (12) into

$$\partial_t f = -\nabla \cdot y - \gamma f \ .$$

This shows that the energy loss described by (20), $\partial_t w = -\nabla \cdot Y - \gamma w$, is due to the overall decrease of the photon number, while the energy $\epsilon_p = v|p|$ of each photon is a constant, by contrast to an ensemble of massive particles, where the number (of particles / degrees of freedom) is a constant.

Aside $j_\Omega$, at finite temperature $j_f$ has also a random part of zero mean (noise) $j_n$,
due to the phonon modes and thermal fluctuations. The effects of \( j_n \) can be described by extending (44) to a Fokker-Planck equation, but this requires several assumptions on the correlation function between \( j_n(t) \) and \( j_n(t') \). Because the Fourier transform of a random field can also be seen as a peculiar weighted average, particularly important is the choice of the defining representation, either as the coordinate \((x)\), or the momentum \((p = \hbar k)\) space. Let us presume that the correlation function is diagonal in the momentum ("quantum") representation, having the general form

\[
\langle \langle j^*_{nk}(t) \cdot j'_{nk'}(t') \rangle \rangle = 2\delta(t - t')\delta^3(k - k')Q_T(|k|)
\]

(45)

where \( \langle \langle ... \rangle \rangle \) denotes the average over the statistical ensemble of the free charges, and \( Q_T(|k|) \) still needs to be specified. In this case, (44) is extended by a source term \( \sim Q_T(|p|/\hbar) \), becoming

\[
\partial_t f + \nabla \cdot y = \gamma \left( \frac{1}{\hbar^3\sigma_q\epsilon_p} Q_T - f \right)
\]

(46)

At equilibrium, \( \partial_t f = 0, \nabla \cdot y = 0, \) and \( f^T(x, p) = Q_T/\hbar^3\sigma_q\epsilon_p \). The "classical" Rayleigh-Jeans formula (consistent with the usual radiation reaction force [22]), is provided by \( Q_T^{RJ} = 2\sigma_q k_B T \), and the "quantum" Wien distribution by \( Q_T^{W} = 2\sigma_q\epsilon_p e^{-\epsilon_p/k_B T} \), where \( k_B = 0.086 \) meV/K is the Boltzmann’s constant. The Planck distribution (37) can also be obtained by introducing the density-dependent correction factor \( (1 + \hbar^3 f/2) \) to the \( Q_T^{W} \) source term of (46), accounting for the stimulated emission in each polarization mode of an elementary cell. One should note though that for applications to realistic situations of interest now, involving large \((N \sim 10^{11})\), non-equilibrium ensembles of short wavelength photons [23], further corrections in both source, and absorption terms, are necessary.

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