The heat equation with order-respecting absorption and particle systems with topological interaction

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March 7, 2021

Abstract

A PDE formulation is proposed, referred to as a heat equation with order-respecting absorption, aimed at characterizing hydrodynamic limits of a class of particle systems on the line with topological interaction that have so far been described by free boundary problems. It consists of the heat equation with measure-valued injection and absorption terms, where the absorption measure respects the usual order on \( \mathbb{R} \) in the sense that, for all \( r \in \mathbb{R} \), it charges \((−\infty, r)\) only at times when the solution vanishes on \((r, \infty)\). The formulation is used to obtain new hydrodynamic limit results for two models. One is a variant of the main model studied by Carinci, De Masi, Giardinà and Presutti [5] where Brownian particles undergo injection according to a general injection measure, and removal that is restricted to the rightmost particle of the configuration. This partially addresses a conjecture of [5]. Next a Brownian particle system is considered where the \( Q \)-quantile member of the population is removed until extinction, where \( Q \) is a given \([0, 1]\)-valued continuous function of time. Here, unlike in earlier work on the subject, the removal mechanism acts on particles that are ‘at the boundary’ but are not rightmost or leftmost. Finally, further potential uses of order-respecting absorption are mentioned.

1 Introduction

Particle systems with topological interactions are models for particle dynamics on the real line where the so called boundary particles, namely the rightmost and leftmost ones, have a special role. Typically in these models, all particles undergo motion, branching or nonlocal duplication, whereas the boundary particles are in addition subject to a removal (or injection) mechanism. The study of their hydrodynamic limits and the characterization of these limits in terms of free boundary problems (FBP) has been the subject of interest in the literature [4, 5, 6, 7, 8, 10]. A prototype of this class of models is the following variant of the basic model studied by Carinci, De Masi, Giardinà and Presutti in the monograph [5]. It is a system of Brownian particles on the real line subject to injection of new particles and removal, where injection follows a given probability measure \( \pi \) and removal is constrained to the rightmost particle. Initially, \( N \) particles are distributed according to \( u_0(x)dx \), where \( u_0 \) is given. The cumulative number of injections (resp., removals) by time \( t \) is \( [NI_t] \) (resp., \( [NJ_t] \)) where \( I \) and \( J \) are given continuous nondecreasing functions starting at 0, and

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$1 + I - J > 0$. The particle configuration measure at time $t$ is denoted by $\xi^N_t$, and its rescaled version by $\bar{\xi}^N_t = N^{-1}\xi^N_t$. Key to a new viewpoint proposed in this paper is the fact that the removal mechanism is order-respecting: for every $r \in \mathbb{R}$, at times when a particle is removed from $(-\infty, r)$, no particles occupy $(r, \infty)$. This condition translates in a natural way to a condition about the associated heat equation, provided it is formulated with a measure-valued RHS. This equation, for the unknown $(u, \beta)$, takes the form

$$L u = \alpha - \beta,$$

where $L = \partial_t - \frac{1}{2} \partial_{xx}$, $\alpha$ and $\beta$ are measures on $\mathbb{R} \times \mathbb{R}_+$ governing injection and removal, respectively, and $\alpha$ is given by the problem data via $\alpha(dx, dt) = \pi(dx)dI_t$. It is considered with initial condition $u(\cdot, 0) = u_0$, a positivity condition of $u$, a cumulative removal condition $\beta(\mathbb{R} \times [0, t]) = J_t$, and the order-respecting absorption (ORA) condition

$$\int_{\mathbb{R}_+} u((r, \infty), t)\beta((-\infty, r) \times dt) = 0 \quad \text{for all} \quad r \in \mathbb{R}, \quad (1)$$

where one denotes $u((r, \infty), t) = \int_r^{\infty} u(y, t)dy$. Under some restrictions on $(u_0, I, J)$, it is shown in this paper that there exists a unique solution $(u, \beta)$, and that the limit of $\bar{\xi}^N_t$ exists and has a density given by $u(\cdot, t)$ (see §2 for the precise definition of a solution and statement of convergence). This model is referred to in this paper as removal at boundary (RAB).

This formulation is to be contrasted with the FBP approach that has been successfully used many times as a means of characterizing scaling limits of this and various related models (see [5] and references therein). In particular, the basic model of [5] (studied alongside several others) corresponds to the above description with $\pi = \delta_0$ (throughout, the Dirac measure at $x \in \mathbb{R}$ or $x \in \mathbb{R} \times \mathbb{R}_+$ is denoted by $\delta_x$), $I_t = J_t = jt$, $j > 0$ a constant, and where the Brownian particles live in $\mathbb{R}_+$, reflecting at 0. The FBP [5] consists of the heat equation $Lu = 0$ on a time varying domain $(0, X_t)$, with initial condition $u(\cdot, 0) = u_0$, boundary condition $-\frac{1}{2} \partial_x u(0, t) = j$, and the following conditions at the free boundary,

$$u(X_t, t) = 0, \quad -\frac{1}{2} \partial_x u(X_t, t) = j.$$

This problem is closely related to the Stefan problem, for which it is well known that local existence and uniqueness of classical solutions hold (for classical initial conditions) but global existence fails in general. Hence a relaxed notion of solutions is developed in [5], for which existence and uniqueness are established. It is also established that this solution describes the hydrodynamic limit of the particle configuration process.

In [5], constraining the particles to $\mathbb{R}_+$ and injecting at the leftmost site assures that injection always occurs to the left of the free boundary. It is conjectured in [5, §15.1] that when applying a general injection distribution, which may result in injection to the right of the free boundary, the hydrodynamic limit is ruled by a FBP (specifically, (15.1.1)–(15.1.3) in [5]). Such a condition is not important when it comes to the ORA formulation, which completely avoids the use of a free boundary. While our results are not concerned with a FBP, they address the conjecture in a weaker sense, as they show that for a general injection distribution, (and further, time varying rates and no mass conservation condition) the limit exists, and moreover is characterized by an alternative PDE, namely the heat equation with ORA.
Another model treated in this paper is referred to as removal at quantile (RAQ). Initially \( N \) particles are distributed on the line according to \( u_0(x)dx \). At rate \( N \), the \( Q \)-quantile member of the population is removed until no particles are left, where \( Q \) is a given continuous function of time with values in \([0,1]\). The motivation to study this model is that, according to a formal argument, it gives rise to a topological interaction of a somewhat different nature than models studied earlier. Namely, denote the mass to the left and right of the \( Q \)-quantile member by \( q_L(t) = (1-t)(1-Q(t)) \) and, resp., \( q_R(t) = (1-t)Q(t) \). Focus first on the case where \( Q \) is \( C^1 \) and takes values in \((0,1)\), and moreover, \( \lambda_L(t), \lambda_R(t) \in [0,1] \) for all \( t \), where \( \lambda_L(t) = -\frac{dq_L(t)}{dt}, \lambda_R(t) = -\frac{dq_R(t)}{dt} \). In this special case, the FBP formally associated with the model is given by the heat equation \( \mathcal{L}u = 0 \) on the time varying domain \((-\infty, X_t) \cup (Y_t, \infty)\), \( X_t < Y_t \), for \( t \in (0,1) \), coupled with the initial condition \( u(\cdot,0) = u_0 \) and the conditions at the free boundary

\[
\begin{align*}
u(X_t,t) &= u(Y_t,t) = 0, & -\frac{1}{2}\partial_x u(X_t,t) &= \lambda_L(t), & \frac{1}{2}\partial_x u(Y_t,t) &= \lambda_R(t), & t \in (0,1).
\end{align*}
\]

Thus the macroscopic configuration is supported on \((-\infty, X_t) \cup [Y_t, \infty)\) and absorption takes place at \( X_t \) and \( Y_t \). Unlike in earlier work \([4,5,6,7,8]\), according to this description, the removal mechanism acts on particles that are ‘at the boundary’ but are not rightmost or leftmost. If \( Q \) is allowed to vary in \([0,1]\) then one must deal with free boundaries taking values in \([-\infty, \infty]\), for \( X_t = -\infty \) (resp., \( Y_t = \infty \)) when \( Q(t) = 1 \) (resp., \( Q(t) = 0 \)). For the more general case where \( \lambda_L \) or \( \lambda_R \) are not restricted to \([0,1]\) (or when \( Q \) is merely continuous) we make no attempt to propose a FBP formulation, which seems to be even more challenging.

We propose an alternative formulation via a variant of the ORA condition. It does not rely on the above heuristic or on its implication that the macroscopic configuration is supported on two connected components. Rather, it expresses an absorption rule that again reflects the particle system’s removal mechanism. It consists of the heat equation \( \mathcal{L}u = -\beta \) with an initial condition \( u(\cdot,0) = u_0 \), a cumulative removal condition \( \beta(\mathbb{R} \times [0,t]) = t \land 1 \), and the conditions

\[
\begin{align*}
\int_{\mathbb{R}_+} [u((r, \infty), t) - q_R(t)]^+ \beta((-\infty, r) \times dt) &= 0, & r \in \mathbb{R}, \\
\int_{\mathbb{R}_+} [u((-\infty, r), t) - q_L(t)]^+ \beta((r, \infty) \times dt) &= 0, & r \in \mathbb{R}.
\end{align*}
\]

It is shown that the hydrodynamic limit exists and is characterized as the unique solution to this equation.

The first paper to study hydrodynamic limits for a model closely related to the aforementioned RAB was \([4]\), where particles perform random walks on \([0, N] \cap \mathbb{Z}\) rather than Brownian motion. In \([6]\), a one dimensional symmetric simple exclusion process with birth of the leftmost hole and death of the rightmost particle was considered, and convergence at the hydrodynamic scale was proved. A formal limit gave rise to a FBP that is close to that of \([5]\), but a rigorous connection was not proved because existence and uniqueness of solution to this FBP are not known. A related class of models is the Brunet-Derrida evolution-selection mechanisms, of which the \( N \)-branching Brownian motion is a special case. In this model, Brownian particles on the line undergo branching, and the leftmost particle is removed whenever a new one is born. The hydrodynamic limit was proved to exist in \([7]\) and the limit was identified as the local solution of a FBP, but the identification on an arbitrary time interval was conditional on existence of a global solution and on continuity of
the free boundary. These issues were then settled in [3]. A variant of this model, in which the branching is nonlocal, was studied in [8], where the hydrodynamic limit was proved to exist. Its characterization as the solution of a FBP was also proved conditionally on existence of a classical solution to the FBP. However, existence is not known in general.

The results of this paper demonstrate that ORA may provide a route to circumvent obstacles related to FBP (existence of classical solutions, regularity of the free boundary, etc.), and consequently may be applicable in cases where FBP formulations are too difficult to analyze. In the future, it is desired to justify this statement more broadly. For example, it is straightforward to formulate (as we do in §6) ORA for the N-BBM and its extensions alluded to above, and so it would be of interest to apply the approach in settings such as [8] where the FBP is challenging.

ORA is closely related to (indeed, inspired by) the Skorohod map. Particularly, the Skorohod map in measure space was introduced in [2] as a means of describing priority in queueing models, especially in settings of a continuum of priority classes, and was used to establish law-of-large-numbers limits of such systems. A relation between ORA and the Skorohod map can be made precise, and moreover, a solution to the heat equation with ORA can be seen as a fixed point of a related map (see §2). However, we were not able to base a proof of our main results on fixed point techniques.

The proof of uniqueness of solutions to the heat equation relies heavily on the method of barriers. We refer to [5] for the history of its use; it has been employed in particular in [4, 5, 6, 7, 8, 10]. Barriers are discrete approximate solutions that trap the true solutions from above and below in the sense of mass transport inequalities, and that have a unique separating element. The specific form of the barriers from [5] is used, up to simple adaptations, in the construction of the lower barriers for the RAB model. For the upper barriers of RAB and both lower and upper barriers of RAQ, additional ideas are necessary and the constructions differ considerably. Existence of solutions is not proved analytically, but follows as a consequence of the convergence of the probabilistic model. For convergence we deviate from the method employed in [5, 4, 7] in that we do not construct stochastic barriers, but instead our arguments are based on tightness of the rescaled processes.

Organization of this paper

§2.1 introduces the particle system with RAB and asserts that the hydrodynamic limit exists (Theorem 2.1). In §2.2, a general treatment from [1] of parabolic initial value problems involving measures is mentioned in the special case of the heat equation. The uniqueness results from this reference are crucially used in the sequel. Then, the heat equation with ORA is formulated (8), and a uniqueness result is stated (Theorem 2.2). The main result regarding this model, Theorem 2.3, is given in §2.3, stating that the scaling limit has a density and identifying this density as the unique solution of (8). Also, a comparison principle for solutions is stated (Theorem 2.4), according to which solutions depend on the problem data in such a way that an increase in the absorption rate (i.e., the derivative of $J$) causes a decrease of the solution $u$ w.r.t. the mass transport partial order. §3 gives the proof of Theorem 2.2, starting with a representation of $u$ in terms of $\alpha$ and $\beta$ as a mild solution (§3.1) followed by mass transport inequalities, some of which are borrowed from [5], stated in §3.2. Lower and upper barriers are constructed in §3.3 and §3.4, resp. The result is finally proved in §3.5 by showing that the upper and lower barriers cannot be separated by more than one element. In §4.1, Theorem 2.3 is proved, and as a corollary, Theorem 2.1. The proof
is based on a tightness argument (Lemma 4.2), followed by estimates on the empirical density, used in conjunction with a result due to Doob found in [9] on existence of a measurable density, to construct such a density for any subsequential limit of the rescaled configuration process $\hat{\xi}^N$ (Lemma 4.3). In Lemma 4.4 it is deduced from the particle system’s removal structure that these densities, $u$ (paired with $\beta$, also obtained by a scaling limits), satisfy the ORA condition. Based on these lemmas, the proof of Theorem 2.3 that appears at the end of the section shows that $(\hat{u}, \beta)$ constitute solutions to equation (8). The uniqueness of such solution established in Theorem 2.2 then implies the existence of limits. In §4.2 Theorem 2.4 is proved via a coupling argument for the particle system. §5 treats the RAQ model starting with §5.1 that construct the model. The heat equation (35) is presented and the main result regarding this model, Theorem 5.1, is stated in §5.2. §5.3 lists and proves further mass transport inequalities. Barriers are constructed in §5.4, and §5.5 completes the proof. In §6 further examples are provided of models that can be formulated via ORA, yet to be studied.

**Notation**

The ‘spacial’ and ‘temporal’ domains we work with are $\mathbb{R}$ and, respectively, $\mathbb{R}_+ = [0, \infty)$. For $X = \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}_+$ let $C_c^\infty(X)$ be the space of compactly supported smooth functions on $X$. Let $C_0(X)$ denote the space of continuous functions on $X$ vanishing at infinity, and endow it with the maximum norm. For $s \in \mathbb{R}$ and $p \in [0, \infty]$ denote by $W^s_p(X)$ the usual Sobolev space, so that $W^0_p(X) = L_p(X)$. Abbreviate $W^s_p(\mathbb{R})$ and $L_p(\mathbb{R})$ to $W^s_p$ and $L_p$, resp. Denote by $\mathcal{M}(X)$ the space of locally finite signed Borel measures on $X$ and endow it with the topology of weak convergence on compacts. Let $\mathcal{M}_+(X) \subset \mathcal{M}(X)$ denote the set of positive measures. Denote by $\mathcal{M}_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ the space of signed Borel measures that are finite on $\mathbb{R} \times [0, T]$ for any finite $T$. Use similarly the notation $L_{p, \text{loc}}(\mathbb{R}_+, L_p)$, etc.

For $u, v \in \mathcal{B}(\mathbb{R}, \mathbb{R})$ and $\xi \in \mathcal{M}_+(\mathbb{R})$, denote $\langle u, \xi \rangle = \int_R u \, d\xi$ and $\langle u, v \rangle = \int_R uv \, dx$, provided the integrals exist. For $\xi \in \mathcal{M}_+(\mathbb{R})$, and an interval, say $[a, b]$, $\xi[a, b]$ is used as shorthand for $\xi([a, b])$, and for $v \in L_1$, $v[a, b]$ is shorthand for $\int_a^b v \, dx$.

For $X$ a Polish space, let $C(\mathbb{R}_+, X)$ be the space of continuous paths, and $D(\mathbb{R}_+, X)$ the space of càdlàg paths. Endow the latter with the Skorohod $J_1$ topology.

Let $C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ denote the subset of $C(\mathbb{R}_+, \mathbb{R}_+)$ of nondecreasing functions that vanish at zero. For $I \in C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$, denote by $dI_t$ the corresponding Stieltjes measure on $\mathbb{R}_+$. Let $AC^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ denote the subset of $C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ of absolutely continuous functions.

### 2 Removal at boundary: model and results

#### 2.1 Brownian particles with injection and boundary removal

A tuple $(u_0, \pi, I, J)$ is said to be an admissible data if $u_0 \in L_1(\mathbb{R}, \mathbb{R}_+) \cap L_\infty(\mathbb{R}, \mathbb{R}_+)$, $\|u_0\|_1 = 1$, $\pi$ is a Borel probability measure on $\mathbb{R}$, $I \in C^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \cap C^0(\mathbb{R}_+, \mathbb{R}_+)$ and $J \in AC^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \cap C^0(\mathbb{R}_+, \mathbb{R}_+)$ for some $\rho_0 > 1/2$, and

$$\varepsilon_0 := \inf_{t \in \mathbb{R}_+} (1 + I_t - J_t) > 0. \tag{2}$$
In addition, it is required that \( \sup_t I_t = \sup_t J_t = \infty \), but this condition is not an important one; it is imposed only to slightly simplify the description of the particle system. The roles played by these four ingredients are to determine, respectively, the initial configuration of particles, the distribution of injected particles, and the cumulative number of injected (resp., removed) particles. The precise details are as follows.

The model is indexed by \( N \), where \( N \) is the initial number of particles. Let an admissible data \((u_0, \pi, I, J)\) be given. The construction uses three mutually independent tuples \( \{x^i, 1 \leq i \leq N\} \), \( \{x^i, i \geq N + 1\} \) and \( \{\tilde{B}^i_t, t \in \mathbb{R}_+, i \in \mathbb{N}\} \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Here, \( x^i \), \( 1 \leq i \leq N \) are IID real-valued RVs distributed according to the probability measure \( u_0(x)dx \), representing the initial positions of those particles that are in the system at time 0. Also, \( x^i, i \geq N \) are IID real-valued RVs distributed according to \( \pi \), representing the injection positions of particles injected after time 0. Finally, \( \{\tilde{B}^i_t, t \geq 0\}, i \in \mathbb{N} \) are mutually independent standard Brownian motions (SBMs).

The \( N \) initial particles are labeled by \( \{1, \ldots, N\} \), and the remaining ones by \( \{N + 1, N + 2, \ldots\} \), in the order of injection. The injections occur at deterministic times determined by the function \( I \). The number of particles introduced (present initially or injected later) by time \( t \) is given by \( N^I_t := N + [NI_t] \). Thus for \( i \geq N + 1 \), the injection time of particle \( i \) is at \( \sigma^i = \sigma^i(N) = \inf\{t : N^I_t \geq i\} \). Set \( \sigma^i = 0 \) for \( i \leq N \). For \( i \in \mathbb{N} \), we refer to the process \( \{\tilde{B}^i_t, t \in [\sigma^i, \infty)\} \) as the potential trajectory associated with particle \( i \in \mathbb{N} \), where

\[
\tilde{B}^i_t = x^i + \tilde{B}^i_{t-\sigma^i}, \quad t \geq \sigma^i. \tag{3}
\]

Particle removal times are dictated similarly by the function \( J \). The number of removals by \( t \) is \( N^J_t = [NJ_t] \), and the \( i \)-th time of removal is \( \eta^i = \eta^i(N) = \inf\{t : N^J_t \geq i\} \), for \( i \in \mathbb{N} \).

The number of particles in the system at time \( t \) is \( N + [NI_t] - [NJ_t] \geq N\varepsilon_0 - 1 \). Thus (2) ensures that the number of particles is always positive, provided \( N\varepsilon_0 - 1 \geq 1 \), a condition assumed throughout.

The collection of potential trajectories and removal times are used to construct dynamics under which, at each time \( \eta_t \), the rightmost particle is removed. It is possible that an injection and a removal occur simultaneously; in this case the construction obeys the rule ‘inject and then remove’, and therefore it is possible for an injected particle to be removed immediately.

The details are as follows. For a finite nonempty subset \( C \) of \( \mathbb{N} \) and \( g : C \to \mathbb{R} \), denote \( \arg\max\{g(i) : i \in C\} = \min\{i \in C : g_i = \max_C g\} \). Let \( [a, b] = \emptyset \) if \( b \leq a \) and \( [a, b] = \emptyset \) if \( b < a \). A process \( A_t \), representing the population of particles alive at time \( t \), taking values in the set of finite nonempty subsets of \( \mathbb{N} \), is constructed recursively. On \([0, \eta^1)\), set \( A_t = \{i : i \leq N^I_t\} \). For \( j \geq 1 \), \( A_t \) is defined on \([\eta^j, \eta^{j+1})\) based on \( A_{[0, \eta^j)} \). Let

\[
C_j = A_{\eta^j} \cup \{k : \sigma_k = \eta^j\}.
\]

Thus \( C_j \) consists of those particles that are alive at \( \eta^j \) and, possibly one more particle, namely the one injected exactly at \( \eta^j \), if there is one. Next, the rightmost particle is removed, by setting

\[
A_{\eta^j} = C_j \setminus \{\arg\max\{\tilde{B}^i_{\eta^j} : i \in C_j\}\}.
\]

Finally, for \( t \in (\eta^j, \eta^{j+1}) \), new injections are accounted for by letting \( A_t = A_{\eta^j} \cup \{i : N^I_{\eta^j} < i \leq N^I_t\} \). This defines \( A_t \) for all \( t \in [0, \infty) \).
The removal time of particle \( i \in \mathbb{N} \), denoted \( \tau_i \), can now be deduced from the construction of \( A_t \). If \( i \in \mathbb{N} \) and \( i \in \bigcup_t A_t \), let \( \tau_i = \sup \{ t : i \in A_t \} \). Otherwise, \( i \) must be a particle injected and removed at the same time, in which case let \( \tau_i = \sigma_i \). The \textit{trajectory associated with particle} \( i \) is given by \( \{ B_t^i : t \in [\sigma_i, \tau_i] \} \). By construction, \( \tau_i \) is a stopping time on the filtration \( F_t = \sigma \{ B_s^i, s \in [\sigma_i, t], i \in \mathbb{N} \} \).

The configuration process, with sample paths in \( D(\mathbb{R}_+, \mathcal{M}_+ (\mathbb{R})) \) is defined as

\[
\xi^N_t(dx) = \sum_{i=1}^{N^I_t} \delta_{B^i_t}(dx)1_{\{t < \tau_i\}} = \sum_{i \in \mathbb{N}} \delta_{B^i_t}(dx)1_{\{\sigma^i \leq t < \tau^i\}}.
\]

The removal positions and times are recorded by an \( \mathcal{M}_{\text{loc}}(\mathbb{R} \times \mathbb{R}) \)-valued RV,

\[
\beta^N_t(dx, dt) = \sum_{i \in \mathbb{N}: \tau^i < \infty} \delta_{B^i_t}(dx, dt).
\]

The injection positions and times are encoded in the (deterministic) member of \( \mathcal{M}_{\text{loc}}(\mathbb{R} \times \mathbb{R}) \),

\[
\alpha^N_t(dx, dt) = \sum_{i \geq N+1} \delta_{(x_i, \sigma_i)}(dx, dt).
\]

It is also useful to consider two extended configuration process. One keeps track of the removed particles, viewed at their removal positions, and another merely ignores the removal mechanism. That is,

\[
\gamma^N_t = \xi^N_t + \int_{[0,t]} \beta^N(s, ds) = \sum_{i \leq N^I_t} \delta_{B^i_t \wedge \tau_i},
\]

\[
\zeta^N_t = \sum_{i=1}^{N^I_t} \delta_{B^i_t}.
\]

Denote normalized versions by \( \bar{\xi}^N = N^{-1} \xi^N \), \( \bar{\beta}^N = N^{-1} \beta^N \), etc.

Because the removal occurs at the rightmost particle, the construction satisfies, for every \( i \in \mathbb{N} \) with \( \tau_i < \infty \) and every \( r \in \mathbb{R} \),

\[
B_{\tau_i} < r \implies \xi_{\tau_i}(r, \infty) = 0.
\]

As a result,

\[
\int_{(-\infty, r) \times \mathbb{R}_+} (1_{[r, \infty)} \bar{\xi}^N_t, \bar{\beta}^N_t)(dx, dt) = 0, \quad r \in \mathbb{R}.
\]

View \( \bar{\xi}^N \) and \( \bar{\beta}^N \) as random elements taking values in \( D(\mathbb{R}_+, \mathcal{M}(\mathbb{R})) \) and \( \mathcal{M}_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+) \), respectively, and endow the product space with the product topology.

\textbf{Theorem 2.1.} There exists a pair \( (\xi, \beta) \in C(\mathbb{R}_+, \mathcal{M}_+ (\mathbb{R})) \times \mathcal{M}_{+, \text{loc}}(\mathbb{R} \times \mathbb{R}_+) \) such that, as \( N \to \infty \), \( (\bar{\xi}^N, \bar{\beta}^N) \to (\xi, \beta) \) in probability.

The proof of this result is given at the end of §4, and is based on uniqueness of solutions to the heat equation with an ORA condition, described next.
2.2 Heat equation with ORA

First, consider the heat equation with measure-valued RHS. Namely, for $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ and $u_0 \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ consider the problem

\[
\begin{align*}
\partial_t u - \frac{1}{2} \partial_{xx} u &= \mu \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \\
(\cdot, 0) &= u_0 \quad \text{on } \mathbb{R}.
\end{align*}
\]

(7)

Let $q \in (1, \infty)$. A (weak) $L_q$-solution of (7) is a function $u \in L_1,\text{loc}(\mathbb{R}^+,L_q)$ satisfying

\[
-\int_0^\infty \left( \partial_t \varphi + \frac{1}{2} \partial_{xx} \varphi, u \right) dt = (\varphi(\cdot,0),u_0) + \int_{\mathbb{R} \times \mathbb{R}^+} \varphi d\mu
\]

for all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$. Such problems were analyzed in far greater generality in [1]. In particular, by [1, Theorem 1 and Remark 1(b)], for $1 < q < \infty$, $0 \leq \sigma < \frac{q+1}{q}$ and $p \geq 1$ such that $\frac{2}{p} + \frac{1}{q} > 1 + \sigma$, this problem possesses a unique $L_q$-solution $u$. Moreover, $u \in L_{p,\text{loc}}(\mathbb{R}^+,W^\sigma_{q})$, and is independent of $q \in (1, \infty)$. Furthermore, $u \in L_{p,\text{loc}}(\mathbb{R}^+,C_0(\mathbb{R}) \cap C^\rho(\mathbb{R}))$, for $p \in [1,2)$ and $\rho \in [0,1)$.

We base on this result the following problem formulation. Let an admissible data $(u_0,\pi,I,J)$ be given. Denote by $\alpha \in \mathcal{M}_{+,\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ the measure $\alpha(dx,dt) = \pi(dx) dI(t)$. Recall our notation $v[y,z] = \int_y^z v \, dx$ when $v \in L_1(\mathbb{R},\mathbb{R})$. Consider the problem

\[
\begin{align*}
(i) \quad & \partial_t u - \frac{1}{2} \partial_{xx} u = \alpha - \beta \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \\
(ii) \quad & u(\cdot,0) = u_0 \quad \text{on } \mathbb{R}, \\
(iii) \quad & \int_{\mathbb{R}^+} u([r,\infty),t)\beta((-\infty,r) \times dt) = 0 \quad \text{for all } r \in \mathbb{R}, \\
(iv) \quad & \beta([\mathbb{R},[0,t]]) = J_t \quad \text{for all } t \in \mathbb{R}^+.
\end{align*}
\]

(8)

Define a (weak) solution to (8) to be a pair $(u,\beta) \in L_{1,\text{loc}}(\mathbb{R}^+,L_1) \cap L_{\infty,\text{loc}}(\mathbb{R}^+,L_\infty) \times \mathcal{M}_{+,\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ such that $u$ is a.e. nonnegative, conditions (8)(iii) and (8)(iv) hold, and moreover for any test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$,

\[
-\int_0^\infty \left( \partial_t \varphi + \frac{1}{2} \partial_{xx} \varphi, u \right) dt = (\varphi(\cdot,0),u_0) + \int_{\mathbb{R} \times \mathbb{R}^+} \varphi d\alpha - \int_{\mathbb{R} \times \mathbb{R}^+} \varphi d\beta.
\]

(9)

**Theorem 2.2.** Assume there exists a solution $(u, \beta)$ to (8). Then it is unique.

This result is proved in §3.

2.3 Main result

The main result regarding this model summarizes Theorems 2.1 and 2.2, and relates the limit $(\xi, \beta)$ from the former to the solution $(u, \beta)$ from the latter.
Theorem 2.3. There exists a unique solution \((u, \beta)\) to (8). Moreover, there exists a version of \(u\) such that letting \(\xi_t\) be defined by \(\xi_t(dx) = u(x,t)dx\) for every \(t \in \mathbb{R}_+\), one has \((\xi, \beta) \in C(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R})) \times \mathcal{M}_{+, \text{loc}}(\mathbb{R} \times \mathbb{R}_+)\) and \((\tilde{\xi}^N, \tilde{\beta}^N) \to (\xi, \beta)\) in probability, as \(N \to \infty\).

See Remark 3.3 for further properties of the solution following from [1]. The proof of Theorem 2.3 is based on §3–4 and is given at the end of §4.

Following is comparison principle for solutions of (8). It is largely a corollary of the main result. To state it we need the notion of a mass transport inequality, used extensively throughout.

On \(L_1(\mathbb{R}, \mathbb{R})\), the partial order \(u \leq v\) is defined as \(u|_{[r, \infty)} \leq v|_{[r, \infty)}\) for all \(r \in \mathbb{R}\). The same notation is used for an analogous partial order on the space of finite Borel measures on \(\mathbb{R}\), that is, \(\nu \leq \mu\) is defined as \(\nu|_{[r, \infty)} \leq \mu|_{[r, \infty)}\) for all \(r \in \mathbb{R}\).

Theorem 2.4. Let each of the tuples \((u_0, \pi, I, J)\) and \((\tilde{u}_0, \tilde{\pi}, \tilde{I}, \tilde{J})\) be admissible data and assume \(\tilde{u}_0 \leq u_0, \tilde{\pi} \leq \pi, I - \tilde{I} \in C^1((\mathbb{R}_+, \mathbb{R}_+)), \tilde{J} - J \in C^1((\mathbb{R}_+, \mathbb{R}_+))\). Then the corresponding solutions \((u, \beta), (\tilde{u}, \tilde{\beta})\) of (8) satisfy \(\tilde{u}(\cdot, t) \leq u(\cdot, t), t \in \mathbb{R}_+\).

The proof is provided in §4.2 and uses, besides Theorem 2.3, a coupling argument which shows a similar property for the configuration process.

Remark 2.1. The condition (8)(iii) is closely related to the Skorohod map \(\Phi : D(\mathbb{R}_+, \mathbb{R}) \to D(\mathbb{R}_+, \mathbb{R})\) given by \(\Phi(\varphi)(t) = - \inf_{s \in [0, t]}(\varphi(s) \land 0)\). It is shown that the latter can be used as an alternative to the former in the problem formulation. Further, it is shown that the component \(\beta\) of the solution can be seen as the fixed point of a transformation defined using \(\Phi\).

Let \((u, \beta)\) be a solution to (8). Denote
\[
\hat{\xi}_t(r) = u([r, \infty), t), \quad \hat{\beta}_t(r) = \beta((-\infty, r) \times [0, t]), \quad \hat{\gamma}_t(r) = u([r, \infty), t) + \beta([r, \infty) \times [0, t]) - J_t.
\]
(10) Then by (8)(iv),
\[
\tilde{\xi}_t(r) = \hat{\xi}_t(r) + \beta((-\infty, r) \times [0, t]) + \beta([r, \infty) \times [0, t]) - J_t
\]
\[
= \hat{\gamma}_t(r) + \hat{\beta}_t(r).
\]
One also has \(\tilde{\xi}_t(r) \geq 0\) and, by (8)(iii), for every \(r\),
\[
\int_{\mathbb{R}_+} \tilde{\xi}_t(r)d\hat{\beta}_t(r) = 0.
\]
Skorohod’s lemma then states that \(\tilde{\xi}\) and \(\hat{\beta}\) can be recovered from \(\hat{\gamma}\) via
\[
\hat{\beta}_t(r) = - \inf_{s \in [0, t]}(\hat{\gamma}_s(r) \land 0), \quad \hat{\xi}_t(r) = \hat{\gamma}_t(r) + \hat{\beta}_t(r).
\]
(11)

From this one may draw two conclusions. First, an alternative way to define a solution \((u, \beta)\) is to replace (8)(iii)–(iv) by the relation (11), where again \(\hat{\beta}\) and \(\hat{\gamma}\) are given by (10).

Second, let \(\Psi\) denote the solution map for the problem (7), so that \(u = \Phi(\mu; u_0)\). Then a solution \((u, \beta)\) of (8) satisfies \(u = \Phi(\alpha - \beta; u_0)\). As a result, the third part of (10) gives \(\hat{\gamma}\) as the image of \((\Phi(\alpha - \beta), \beta, J)\) under a map. Considering \((u_0, \alpha, J)\) as fixed, let this relation be written as \(\tilde{\gamma} = \hat{\Phi}(\beta)\) for a suitable \(\hat{\Phi}\). Next, let the relation expressed in the first part of (11) be written as \(\hat{\beta} = \Psi(\hat{\gamma})\). Since \(\hat{\beta}\) determines \(\beta\), one further has \(\beta = \tilde{\Psi}(\hat{\gamma})\). Thus \(\beta\) is a fixed point of \(\tilde{\Psi} \circ \hat{\Phi}\).

Whether existence and uniqueness of a fixed point, thus of solutions to (8), can be deduced from abstract fixed point theorems is not obvious to us.
3 Removal at boundary: uniqueness via barriers

This section proves uniqueness of solutions to (8) by constructing barriers, which are shown to constitute upper and lower bounds to any solution, in the sense of mass transport inequalities. In §3.1 a representation of $u$ in terms of $\alpha$ and $\beta$ is given. In §3.2, some mass transport inequalities are provided, used in the sequel. The lower and upper barriers are constructed in §3.3 and §3.4. Finally, the proof of Theorem 2.2 appears in §3.5. Throughout this section, an admissible data $(u_0, \pi, I, J)$ is fixed.

3.1 Mild solutions

Let the heat kernel be denoted by $G_t(x, y) = (2\pi t)^{-1/2}e^{-(x-y)^2/2t}$. Denote by $G_t$ the operator $G_t u(y) = \int_\mathbb{R} G_t(x, y)u(x)dx$, for $u \in L_1(\mathbb{R}, \mathbb{R}^+)$.

For $\gamma \in M_{+\text{,loc}}(\mathbb{R} \times \mathbb{R}^+)$, denote

$$G * \gamma(y, t) = \int_{\mathbb{R} \times [0, t]} G_{t-s}(x, y)\gamma(dx, ds),$$

$$G * \gamma(y, t; \tau) = \int_{\mathbb{R} \times [\tau, t]} G_{t-s}(x, y)\gamma(dx, ds),$$

for $t \geq \tau$.

Lemma 3.1. Assume that a solution $(v, \beta)$ to (8) exists. Then $v$ has a version $u$ given by

$$u(y, t) = G_t u_0(y) + G * \alpha(y, t) - G * \beta(y, t).$$

(12)

Moreover, for $0 \leq \tau < t$,

$$u(y, t) = G_{t-\tau} u(\cdot, \tau)(y) + G * \alpha(y, t; \tau) - G * \beta(y, t; \tau).$$

(13)

Furthermore, $u \in L_{\infty,\text{loc}}(\mathbb{R}^+, L_1) \cap L_{\infty,\text{loc}}(\mathbb{R}^+, L_\infty)$.

Remark 3.1. If $(v, \beta)$ is a solution to (8) and $u = v$ a.e. then $(u, \beta)$ is also a solution; specifically, Condition (8)(iii) is inherited from the former to the latter. Indeed, for a given $r$, $v([r, \infty), t)$ and $u([r, \infty), t)$ may differ for $t$ in a null set w.r.t. the Lebesgue measure $dt$. However, this is a null set also w.r.t. the measure $\beta((\infty, r) \times dt) \ll \beta(\mathbb{R} \times dt) = dJ_t \ll dt$. The latter is a consequence of our assumption $J \in AC^\uparrow(\mathbb{R}^+, \mathbb{R}^+)$. 

Remark 3.2. Throughout what follows, by a solution to (8) we will always mean the version given by (12). In view of the above lemma there is no loss of generality in doing so.

Proof. Let $(v, \beta)$ be a solution, and define $u$ by (12). First we argue that the integrals in (12) are finite. By monotone convergence, it suffices that $\int_{\mathbb{R} \times [0, t-\epsilon]} G_{t-s}(x, y)\alpha(dx, ds)$ remains bounded as
\[ \varepsilon \downarrow 0. \] To show this, bound the above integral by
\[
\int_{[0,t-[\varepsilon]]} (t-s)^{-1/2} dI_s = I_{t-[\varepsilon]} \varepsilon^{-1/2} - \frac{1}{2} \int_{0}^{t-[\varepsilon]} I_s(t-s)^{-3/2} ds
\]
\[
= (I_{t-[\varepsilon]} - I_t) \varepsilon^{-1/2} + \frac{1}{2} \int_{0}^{t-[\varepsilon]} (I_t - I_s)(t-s)^{-3/2} ds + I_t t^{-1/2}
\]
\[
\leq c \int_{0}^{t} (t-s)^{\rho_0 - \frac{3}{2}} ds + I_t t^{-1/2} < \infty, \tag{14}
\]
where \( \rho_0 > 1/2 \) is used. A similar argument applies for the integral in (12) by appealing to (8)(iv) and the Hölder continuity of \( J \).

Next, given a test function \( \varphi \), identity (9) is satisfied by \((u, \beta)\) as follows by direct calculation that uses the fact that \( G_t(x, y) \) solves the heat equation and again the finiteness of \( \int_{[0,t]} (t-s)^{-1/2} dI_s \). We do not include this calculation because it is standard.

We now show that \( u \) is a member of \( L_{\infty, loc}(\mathbb{R}_+, L_1) \) (hence \( L_{1, loc}(\mathbb{R}_+, L_1) \)) and of \( L_{\infty, loc}(\mathbb{R}_+, L_\infty) \). We have \( |u(y, t)| \leq G_t u_0(y) + G * \alpha(y, t) + G * \beta(y, t) \). Hence
\[
\|u(\cdot, t)\|_1 \leq \|u_0\|_1 + \|\pi\|_1 I_t + J_t = 1 + I_t + J_t,
\]
\[
\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty + \int_{[0,t]} (t-s)^{-1/2}(dI_s + dJ_s).
\]
The above integral is locally bounded as the calculation (14) shows, upon using one more time the fact \( I, J \in C^{\rho_0} \). This shows the claim.

We can now appeal to the uniqueness result due to Amann, of weak \( L_q \)-solutions to (7), for \( q \in (1, \infty) \), where we take \( \mu = \alpha - \beta \) (thus \( \beta \) in this case is taken as data). Namely, as already mentioned, by [1, Theorem 1], this problem possesses a unique weak \( L_q \)-solution. Both \( u \) and \( v \) are in \( L_{1, loc}(\mathbb{R}_+, L_1) \cap L_{\infty, loc}(\mathbb{R}_+, L_\infty) \), and consequently in \( L_{1, loc}(\mathbb{R}_+, L_q) \) for all \( q \in (1, \infty) \). Since (9) is satisfied by both \( u \) and \( v \), they are both weak \( L_q \)-solutions to (7) and as a result, \( u = v \) a.e.

Finally, (13) follows from (12) by a direct calculation that we omit, upon using the identity \( \int G_{t-r}(x, y) G_{r}(y, z) dy = G_t(x, z) \).

### 3.2 Mass transport inequalities

We borrow from [5] the use of mass transport inequalities.

For \( \delta > 0 \), \( K^\delta \) acts on \( H_\delta = \{ u \in L_1(\mathbb{R}, \mathbb{R}_+) : \|u\|_1 > \delta \} \) by cutting mass of size \( \delta > 0 \) that is rightmost. That is, for \( u \in H_\delta \), let
\[
R_\delta(u) = \sup \{ x \in \mathbb{R} : u[x, \infty) \geq \delta \}
\]
and
\[
K^\delta u(x) = u(x)1_{(-\infty, R_\delta(u))}(x).
\]
When \( \delta = 0 \) set \( K^0 = \text{id} \), the identity map. Also, denote \( \tilde{K}^\delta = \text{id} - K^\delta \). We also use an operator that removes the mass of size \( \delta \) lying between \( R_{\Delta + \delta} \) and \( R_\Delta \). More precisely, given \( \Delta > 0 \) and \( \delta \geq 0 \), let \( \delta \) denote the pair \((\Delta, \delta)\). Then the operator \( K^\delta \) acts on \( H_{\Delta + \delta} \) as
\[
K^\delta u(x) = K^{\Delta, \delta} u(x) = u(x)1_{(-\infty, R_{\Delta + \delta}(u) \cup (R_{\Delta}(u), \infty))}(x).
\]
Also set $\hat{K}^\delta = \mathbf{i} d - K^\delta$.

**Lemma 3.2.** Let $\delta \geq 0$ and assume $u, v \in H_\delta$.

i. If $u \not\leq v$ then $K^\delta u \not\leq K^\delta v$ and $G^\delta u \not\leq G^\delta v$.

ii. If also $w \in L_1(\mathbb{R}, \mathbb{R}^+)$ is such that $\|w\|_1 \leq \delta$ and $u - w \in L_1(\mathbb{R}, \mathbb{R}^+)$ then $K^\delta u \not\leq u - w$.

Next let $\Delta > 0$ and assume $u, v \in H_{\Delta + \delta}$.

iii. If $u \not\leq v$ then $K^{\Delta, \delta} u \not\leq K^{\Delta, \delta} v$.

iv. If $0 < \Delta \leq \Delta$ then $K^{\Delta, \delta} u \not\leq K^{\Delta, \delta} u$.

**Proof.** i. These are well known properties. For monotonicity of $K^\delta$ see [5, Lemma 6.6]. As for $G^\delta$, the same lemma in [5] shows monotonicity of the Neumann Laplacian evolution operator on $\mathbb{R}_+$, but the same proof holds for the one on $\mathbb{R}$.

ii. We have $K^\delta = u - \hat{w}$ where $\hat{w} = u1_{(c, \infty)}$, $\int_{\infty}^\infty u = \delta$. Hence it suffices to show that $\int_r^\infty w \leq \int_r^\infty \hat{w}$ for all $r$. First consider $r \geq c$. In this case, noting that $w \leq u$ pointwise, we have $\int_r^\infty w \leq \int_r^\infty \hat{w}$.

Next, if $r < c$ we have $\int_r^\infty w \leq \delta$ but $\int_r^\infty \hat{w} = \delta$.

iii. We have $u(r, \infty) \leq v(r, \infty)$ for all $r$. We must show that $\hat{u}(r, \infty) \leq \hat{v}(r, \infty)$ for all $r$, where

\[
\hat{u} = u1_{(-\infty, a) \cup (b, \infty)}, \quad \text{where } u(a, b) = \delta, \quad u(b, \infty) = \Delta,
\]

\[
\hat{v} = v1_{(-\infty, a) \cup (b, \infty)}, \quad \text{where } v(a, b) = \delta, \quad v(b, \infty) = \Delta.
\]

To this end, note first that $b \leq \hat{b}$ and $a \leq \bar{a}$. We split into cases.

For $r \geq \bar{b}$ it is clear that $\hat{u}(r, \infty) \leq \hat{v}(r, \infty)$.

For $r \in [\bar{a}, \bar{b}]$, $\hat{v}(r, \infty) = \Delta$ but $\hat{u}(r, \infty) \leq \hat{u}(a, \infty) = \Delta$.

For $r \in [a, \bar{a}]$, $\hat{u}(r, \infty) \leq \hat{u}(a, \infty) = \Delta$, but $\hat{v}(r, \infty) \geq \hat{v}(a, \infty) = \Delta$.

For $r < a$, $\hat{u}(r, \infty) = u(r, \infty) - \delta$ and $\hat{v}(r, \infty) = v(r, \infty) - \delta$.

iv. Note that $K^{\Delta, \delta} u = K^\delta v + z$, $K^{\Delta, \delta} u = K^{\Delta, \delta} v + z$,

where

\[
v = K^{\Delta} u, \quad z = \hat{K}^{\Delta} u.
\]

Hence it suffices to prove $K^\delta v \not\leq K^{\Delta, \delta} v$. To this end, note that $K^{\Delta, \delta} v = v - w \in L_1(\mathbb{R}, \mathbb{R}^+)$, and $\|w\|_1 = \delta$. Therefore we can use part (ii) of the lemma, by which $K^\delta v \not\leq v - w$. This completes the proof.

### 3.3 Lower barriers

For $u \in L_1(\mathbb{R}, \mathbb{R}^+)$ and $0 \leq \tau < t$ let

\[
P^{\tau, \delta} u = u + G \ast \alpha(\cdot, t; \tau).
\]

Between time $(n - 1)\delta$ and $n\delta$, the amount of mass removed is given by $j_n(\delta) := J_{n\delta} - J_{(n-1)\delta}$.

Denote $K_{n, \delta} = K^{\gamma_n(\delta)}$. The injection of mass during the same time interval is according to $P_{n, \delta} := P^{(n-1)\delta, n\delta}$. The lower barriers are defined for each $\delta > 0$ and $n \in \mathbb{Z}_+$ as $u^{(\delta, \gamma)}_{0n} = u_n$ and

\[
u^{(\delta, \gamma)}_{n_0} = K_{n, \delta} P_{n, \delta} G_{\delta} u^{(\delta, \gamma)}_{n_0} - (n-1)\delta, \quad n \in \mathbb{N}.
Note that for the RHS to be well defined one must have
\[ P_{n,δ}Gδu_{(n-1)δ}^{δ-} ∈ H_{j_n(δ)}. \]  \( (15) \)

Our goal here is to show that the lower barriers form lower bounds on solutions to \((8)\).

**Proposition 3.1.** Fix \( δ > 0 \). Then \((15)\) holds for all \( n ∈ \mathbb{N} \), and consequently the lower barriers are well defined. Moreover, let \((u, β)\) be a solution to \((8)\). Then for \( n ∈ \mathbb{Z}_+ \),
\[ u_{nδ}^{δ-} ≤ u_{nδ}. \]

**Proof.** We have \( ∥u_0∥_1 = 1 \). Therefore for \( n = 1 \), the \( L_1 \) norm of the LHS of \((15)\) is \( 1 + J_δ ≥ J_δ \), where \((2)\) is used. This shows that \((15)\) holds for \( n = 1 \). Next, assume that \((15)\) holds for all \( n ∈ \{1, …, m\} \). Then \( u_1, …, u_m \) are well defined, and
\[ ∥u_{mδ}\|_1 = ∥u_{(m-1)δ}\|_1 + (I_{mδ} - I_{(m-1)δ}) - (J_{mδ} - J_{(m-1)δ}) \]  \( (16) \)

whence \( ∥u_{mδ}\|_1 = 1 + I_{mδ} - J_{mδ} \). As a result, the \( L_1 \) norm of the LHS of \((15)\) with \( n = m + 1 \) is
\[ 1 + I_{(m+1)δ} - J_{mδ} ≥ J_{m+1}(δ) \]  \( (17) \)
by \((2)\). This shows that \((15)\) holds for \( n = m + 1 \), which completes the argument for the first assertion.

Next, to prove the partial order, again arguing by induction, assume \( f ≼ g \) where \( f = u_{(n-1)δ}^{δ-} \) and \( g = u_{(n-1)δ} \). Write \( K, P, G \) and \( K_{n, δ}, P_{n, δ} \) and \( G_{n, δ} \), resp. We have \( u_{nδ}^{δ-} = KPGf \). Moreover, by Lemma 3.1, \( u_{nδ} = PGg - h \), where
\[ h(y) = \int_{\mathbb{R}×[(n-1)δ,nδ]} G_{nδ-s}(x, y)β(dx, ds). \]

Hence the proof will be complete provided we show that \( KPGf ≼ PGg - h \).

By Lemma 3.2(i), both operators \( G \) and \( K \) preserve \( ≼ \). It is trivial that this is also true for \( P \). Denote \( w = PGg \). Suppose one shows \( Kw ≼ w - h \). Then
\[ KPGf ≼ KPGg = Kw ≼ w - h, \]
and the proof would be complete.

It thus suffices to show \( Kw ≼ w - h \). Denote \( r_0 = R_{j_n(δ)}(w) \). Then \( \int_{r_0}^∞ w(x)dx = j_n(δ) \) and \( Kw = w - q \) where \( q(y) = w(y)1_{\{y>r_0\}} \). Next, the function \( w - h \) is nonnegative (as required by the definition of a solution) and \( \int h(y)dy = j_n(δ) \). Hence for \( r ≥ r_0 \),
\[
\begin{align*}
w(r, ∞) - h(r, ∞) &= w(r, ∞) - w(r, ∞) \\
&= 0 \\
&≤ w(r, ∞) - h(r, ∞).
\end{align*}
\]

For \( r < r_0 \),
\[
\begin{align*}
w(r, ∞) - q(r, ∞) &= w(r, ∞) - j_n(δ) \\
&≤ w(r, ∞) - h(r, ∞).
\end{align*}
\]
This shows \( w - q ≼ w - h \) and completes the proof. 

\( \square \)
3.4 Upper barriers

The upper barriers are defined for $\delta = (\Delta, \delta) \in (0, \infty)^2$ and $n \in \mathbb{Z}_+$ as follows. Let

$$K_{n, \delta} = K_{\Delta, j_n(\delta)}.$$ 

Set $u_0^{(\delta, +)} = u_0$, and for $n \in \mathbb{N},$

$$u_{n\delta}^{(\delta, +)} = K_{n, \delta} P_{n, \delta} G_{\delta} u_{(n-1)\delta}^{(\delta, +)} + e_{n, \delta},$$

$$e_{n, \delta}(y) = e^{-\Delta^j/2} j_n(\delta) 1_{[\sigma_n, \sigma_n+1]}(y), \quad y \in \mathbb{R}_+,$$

$$\sigma_n = \sigma_n(\delta) = R_\Delta(P_{n, \delta} G_{\delta} u_{(n-1)\delta}^{(\delta, +)}).$$

Once again, for the definition to be valid, one must assure that for all $n \in \mathbb{N},$

$$P_{n, \delta} G_{\delta} u_{(n-1)\delta}^{(\delta, +)} \in H_{\Delta + j_n(\delta)}.$$

**Proposition 3.2.** For $\Delta \in (0, \varepsilon_0) \cup (\Delta, \delta) \in \mathbb{Z}_+$ and consequently the upper barriers are well defined. Moreover, let $(u, \beta)$ be a solution to (8). Then given $T \in (0, \infty)$ there exists $\Delta_0 \in (0, \varepsilon_0)$ such that for every $\Delta \in (0, \Delta_0)$ there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $n \in \mathbb{Z}_+$ one has

$$u_{n\delta} \preceq u_{n\delta}^{(\delta, +)},$$

provided $n\delta \leq T$.

Given $\gamma \in \mathcal{M}_{+, \text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ and $[t_1, t_2] \subset \mathbb{R}_+$, the infimum and the supremum of the support of the measure $\gamma(\cdot) = \gamma(\cdot \times [t_1, t_2])$ are denoted by $\rho_\pi(\gamma; [t_1, t_2])$ and, respectively, $\rho^+(\gamma; [t_1, t_2])$.

If we ignore for a moment the ‘error’ terms $e_{n, \delta}$, the recursion consists of iterates of $K_{n, \delta} P_{n, \delta} G_{\delta}$. The cutting operator removes mass at a point that leaves $\Delta$ units of mass to its right. Of course, the PDE (8) also has a removal mechanism, expressed by $\beta$ and constrained by conditions (8)(iii)–(iv). A central part of the argument is the removal of the two removal mechanisms. That is, if $\rho_\pi(\beta; [(n-1)\delta, n\delta])$, which expresses the lowest reach of $\beta$ during the interval $[(n-1)\delta, n\delta]$, can be bounded below then cutting a mass sufficiently to the left will result in removal mostly to the left of that occurring in the PDE model. For this to be useful one needs to show that removal at the point that leaves mass of size $\Delta$ to its right, for arbitrarily small $\Delta > 0$, is indeed “sufficiently to the left”. This requires $\delta$ to be much smaller than $\Delta$, and is made precise by including the aforementioned error terms.

**Lemma 3.3.** Given a solution $(u, \beta)$, $\delta > 0$ and $n \in \mathbb{N}$, denote $\rho_{n, \delta} = \rho_\pi(\beta; [(n-1)\delta, n\delta])$. Then

$$\rho_{n, \delta} \geq b(n, \delta, u_{(n-1)\delta}) := R_{3j_n(\delta)}(u_{(n-1)\delta}),$$

provided $u_{(n-1)\delta} \in H_{3j_n(\delta)}$ and $j_n(\delta) > 0$.

**Proof.** We consider only $n = 1$; the proof is similar for all $n$. Fix $\delta$ and a solution $(u, \beta)$. By contradiction, assume $\rho_{1, \delta} < b = b(1, \delta, u_0) = R_{3J_1}(u_0)$. (Note that $R_{3J_1}(u_0)$ is well defined and finite by the assumption $u_0 \in H_{3J_1}$.) Then there exists $x < b$ such that $\beta((-\infty, x] \times [0, \delta]) > 0$, hence $\beta((-\infty, b) \times [0, \delta]) > 0$. Using (8)(iii), $\int_{[0, \delta]} u([b, \infty), t) \beta((-\infty, b) \times dt) = 0$, we deduce that there exists $t \in [0, \delta]$ such that $u([b, \infty), t) = 0$. It is impossible that $t = 0$ since $u_0[b, \infty) = $
$3J_\delta > 0$. Fix such $t$. We now appeal to identity (12). Denoting the last term there by $\eta(y) = \int_{R \times [0,t]} G_{t-\delta}(x,y) \beta(dx,ds)$, clearly

$$\int_b^\infty \eta(y)dy \leq \int_{-\infty}^\infty \eta(y)dy = \beta([R \times [0,t]]) = J_t \leq J_\delta.$$ 

Since $u(\cdot, t)$ vanishes on $[b, \infty)$, we have for the first term in (12),

$$\int_b^\infty \int_R G_t(x,y)u_0(x)dxdy \leq \int_b^\infty \eta(y)dy \leq J_\delta.$$

Using $\int_x^\infty G_t(x,y)dy = 1/2$,

$$J_\delta \geq \int_{x \in [b,\infty)} \int_{y \in [b,\infty)} G_t(x,y)u_0(x)dxdy \geq \frac{1}{2} \int_b^\infty u_0(x)dx = \frac{3J_\delta}{2},$$

a contradiction due to the assumption $J_\delta = j_1(\delta) > 0$. This proves the claim. □

**Proof of Proposition 3.2.** First, to prove that when $\Delta < \varepsilon_0$ one has (19) for all $n$ (and therefore the upper barriers are well defined) we argue as in the proof of Proposition 3.1. In place of equality in (16) we write an inequality $\geq$, and in place of (17), we write $1 + I_{(m+1)\delta} - J_{m\delta} > \Delta + j_{m+1}(\delta)$, that again follows from (2). This shows the first claim.

We turn to the main assertion. This result is stated for $n\delta \in [0,T]$. Fix $T$. Assume that $\Delta < \varepsilon_0/2$. It follows from Lemma 3.1 and (2) that $\|u(\cdot,t)\|_1 \geq \varepsilon_0$. Since $J$ is continuous on $[0,T]$, we may and will assume also that $\delta$ is so small that $\Delta + 3j_n(\delta) < \varepsilon_0$ whenever $n\delta \in [0,T]$. As a result, the bound asserted in Lemma 3.3 is valid provided merely that $j_n(\delta) > 0$.

It follows from Lemma 3.1 that there exist constants $c_1$, $c_\infty$, depending only on $T$ and $I_{[0,T]}$, $J_{[0,T]}$ such that any solution $(u, \beta)$ (corresponding to the given admissible data) satisfies $\|u(\cdot,t)\|_1 \leq c_1$ and $\|u(\cdot,t)\|_\infty \leq c_\infty$, $t \in [0,T]$. In what follows we consider only $\Delta$ so small that $2c_\infty \Delta^2 < \Delta/6$ and, as already mentioned, $\Delta < \varepsilon_0/2$.

Arguing by induction, assume that $f \leq g$ where $f = u_{(n-1)\delta}$ and $g = u_{(n-1)\delta}^{(\delta,+)}$. Write $K$, $P$ and $G$ for $K_{n,\delta}$, $P_{n,\delta}$ and $G_{\delta}$, respectively. Then using Lemma 3.1, $u_{n\delta} = PGf - h$, where

$$h(y) = \int_{R \times [(n-1)\delta,n\delta]} G_{n\delta-s}(x,y) \beta(dx,ds),$$

whereas $u_{n\delta}^{(\delta,+)} = KP\bar{G}g + e_{n,\delta}$. Denote $w = PGf$. If one shows

$$w - h \leq Kw + e_{n,\delta}$$

(20)

then $u_{n\delta} = w - h \leq KP\bar{G}g + e_{n,\delta} \leq KP\bar{G}g + e_{n,\delta} = u_{n\delta}^{(\delta,+)}$, by the monotonicity of $G$ and of $K^{\Delta \delta}$ (Lemma 3.2(i) and (iii)), which completes the proof. It remains to show (20).

First, if $j_n(\delta) = 0$ then both sides of (20) coincide with $w$, because $h = 0$, $e_{n,\delta} = 0$ and $K = \text{id}$. 

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Next consider \( j_\delta > 0 \). Denote \( j = j_\delta \) and \( b = R_{3j_\delta}(f) = R_{3j}(f) \). The assumptions of Lemma 3.3 are thus satisfied, and the lemma asserts that \( \rho_{n, \delta} \geq b \). Write \( h = h_1 + h_2 \), where

\[
h_1(y) = h(y)1_{\{y < b - \Delta^2\}}, \quad h_2(y) = h(y)1_{\{y \geq b - \Delta^2\}}.
\]

Note that \( \|h\|_1 = j \). A straightforward calculation based on

\[
\int_{R \times [(n-1)\delta, n\delta]} G_{n\delta-s}(x, y)\beta(dx, ds) = \int_{[b, \infty) \times [(n-1)\delta, n\delta]} G_{n\delta-s}(x, y)\beta(dx, ds)
\]

(which follows from Lemma 3.3), the identity \( \beta(R_+ \times [n\delta, t]) = J_t - J_{(n-1)\delta} \), \( t \in [(n-1)\delta, n\delta] \) and the estimate \( G_t(0, [a, \infty)) \leq e^{-a^2/2t} \), \( a > 0 \), \( t < a^2 \), shows that the mass of \( h_1 \) is much smaller, specifically,

\[
\|h_1\|_1 \leq e^{-\Delta^2/2\delta} j,
\]

provided \( \delta < \Delta^4 \). Let \( q \in (0, 1] \) be defined by \( \|h_2\|_1 = qj \). Then \( q \geq 1 - e^{-\Delta^4/2\delta} \).

Our next step toward showing (20) is to show

\[
w - h \preceq K^{\Delta qj}w.
\]

By definition, \( h_2 \) is supported to the right of \( b - \Delta^2 \). On the other hand, \( K^{\Delta qj}w = w - \tilde{h} \), where \( \|\tilde{h}\|_1 = qj \) and \( \tilde{h} \) is supported to the left of \( R_\Delta(w) \). Thus if we can show

\[
R_\Delta(w) \leq R_{3j}(f) - \Delta^2,
\]

then using \( \|\tilde{h}\|_1 = \|h_2\|_1 \) we would have \( w - h_2 \preceq K^{\Delta qj}w \), hence

\[
w - h = w - h_1 - h_2 \preceq w - h_2 \preceq K^{\Delta qj}w,
\]

giving (22).

To show (23), note first, by boundedness \( \|f\|_\infty < c_\infty \), that for all \( \delta \) so small that \( 3j < \Delta/6 \) and recalling \( 2c_\infty\Delta^2 < \Delta/6 \),

\[
R_{\Delta/3}(f) \leq R_{3j}(f) - 2\Delta^2.
\]

Also, we argue that for all small \( \delta \),

\[
R_{2\Delta/3}(Gf) \leq \theta := R_{\Delta/3}(f) + \Delta^2.
\]

To show this we must to show \( (Gf)[\theta, \infty) \leq 2\Delta/3 \). Let \( f = f_1 + f_2 = \tilde{K}^{\Delta/3}f + K^{\Delta/3}f \). Because \( \|f_1\|_1 = \Delta/3 \), we have \( \|Gf_1\|_1 = \Delta/3 \), and

\[
(Gf)[\theta, \infty) = (Gf_1)[\theta, \infty) + (Gf_2)[\theta, \infty) \leq \Delta/3 + (GA)[\theta, \infty)
\]

where \( A = \|f_2\|_1\delta_x \) is an atom of mass equal to that of \( f_2 \) at the point \( x = R_{\Delta/3}(f) \). Recalling that \( \|f\|_1 < c_1 \), we have \( (GA)[\theta, \infty) \leq c_1G_\delta(0, [\Delta^2, \infty)) \), which of course is smaller than \( \Delta/3 \) if \( \delta \) is small. This shows (25).

For \( \tilde{\pi} := G * \alpha(\cdot, n\delta; (n-1)\delta) \) we have \( \|\tilde{\pi}\|_1 = I_{n\delta} - I_{(n-1)\delta} \). Hence for all small \( \delta \), \( \|\tilde{\pi}\|_1 < \Delta/3 \). As a result,

\[
R_\Delta(w) = R_\Delta(PGf) = R_\Delta(Gf + \tilde{\pi}) \leq R_{2\Delta/3}(Gf).
\]
Combining this with (24) and (25) gives (23). Hence (22) follows.

Finally, note that
\[ K^{\Delta,q}w = K^{\Delta,j}w + \widehat{K}^{\Delta,q}w. \]

The last term is a nonnegative function supported to the left of \( R_\Delta(w) \), hence \( \leq (1-q)j1_{[R_\Delta(w),R_\Delta(w)+1]} \).
Now, \( f \preceq g \) hence \( w = PGf \preceq PGg \). Hence \( R_\Delta(w) \leq R_\Delta(PGg) \), and
\[ \widehat{K}^{\Delta,q}w \preceq (1-q)j1_{[R_\Delta(PGg),R_\Delta(PGg)+1]}. \]

Combining this with (22),
\[ w - h \preceq K^{\Delta,q}w \preceq K^{\Delta,j}w + (1-q)j1_{[R_\Delta(PGg),R_\Delta(PGg)+1]}. \]
Recalling that by (21), \( 1 - q \leq e^{-\Delta^4/2\delta} \), it follows that the last term is \( \leq e_{n,\delta} \). This proves (20) and completes the proof.

3.5 Proof of uniqueness result

The last step is showing that the lower and upper barriers become close upon taking \( \delta \to 0 \) then \( \Delta \to 0 \).

Proposition 3.3. Given \( T \), let \( \Delta_0 = \Delta_0(T) \) and \( \delta_0 = \delta_0(T, \Delta_0) \) be as in Proposition 3.2. Then for \( \Delta \in (0, \Delta_0) \), \( \delta \in (0, \delta_0) \) and \( n \in \mathbb{Z}_+ \) \( n\delta \leq T \), one has
\[ u^{(\delta,+)\infty}_{n\delta} \simeq u^{(\delta,-)}_{n\delta} + \varepsilon_n, \]
where \( \varepsilon_n = \varepsilon_n(T, \tilde{\delta}) \in L_1(\mathbb{R}, \mathbb{R}_+) \) and
\[ \|\varepsilon_n\|_1 \leq \Delta + e^{-\Delta^4/2\delta} \sum_{i=1}^n j_i(\delta) = \Delta + e^{-\Delta^4/2\delta} J_{n\delta}. \]

Proof. By induction. Abbreviate \( u^{(\delta,-)} \), \( u^{(\delta,+)} \), \( e_{n,\delta} \), \( P_{n,\delta} \) and \( G_{\delta} \) to \( u^{(-)} \), \( u^{(+)} \), \( e_n \), \( P \) and \( G \), respectively. For \( n = 0 \) the claim holds because \( u^{(\pm)}_0 = u_0 \).

Next, consider \( n \geq 1 \). Assume \( u^{(\pm)}_{(n-1)\delta} \preceq u^{(-)}_{(n-1)\delta} + \varepsilon_{n-1} =: f_{n-1} \), and \( \varepsilon_{n-1} \) is nonnegative and \( \|\varepsilon_{n-1}\|_1 \leq \Delta_{n-1} := \Delta + e^{-\Delta^4/2\delta} \sum_{i=1}^{n-1} j_i(\delta) \). Then by definition of the upper barriers,
\[ u^{(+)}_{n\delta} = K^{\Delta,j_n(\delta)}PGu^{(+)}_{(n-1)\delta} + e_n. \]
Now, \( \Delta \leq \Delta_{n-1} \). Hence by Lemma 3.2(iv),
\[ u^{(+)}_{n\delta} \preceq K^{\Delta_{n-1},j_n(\delta)}PGu^{(+)}_{(n-1)\delta} + e_n \]
\[ \preceq K^{\Delta_{n-1},j_n(\delta)}PGf_{n-1} + e_n, \]
where monotonicity of \( K, P \) and \( G \) is used. We can rewrite the above as
\[ K^{j_n(\delta)}K^{\Delta_{n-1}}PGf_{n-1} + \widehat{K}^{\Delta_{n-1}}PGf_{n-1} + e_n. \]
Note that \( w := G\varepsilon_{n-1} \) is pointwise nonnegative, \( \|w\|_1 \leq \Delta_{n-1} \) and
\[
PGu_{(n-1)\delta}^{(-)} = PGf_{n-1} - w.
\]
Hence by Lemma 3.2(ii), \( K^{\Delta_{n-1}}PGf_{n-1} \ll PGu_{(n-1)\delta}^{(-)} \). By the monotonicity of \( K^{\delta n} \), denoting \( \varepsilon_n = K^{\Delta_{n-1}}PGf_{n-1} + \varepsilon_n \),
\[
u_{n\delta}^{(+)} = K^{\delta n}PGu_{(n-1)\delta}^{(-)} + \varepsilon_n = u_{n\delta}^{(-)} + \varepsilon_n.
\]
Now, \( \varepsilon_n \) is nonnegative. Moreover, \( \|\varepsilon_n\|_1 = \|K^{\Delta_{n-1}}PGf_{n-1}\|_1 + \|\varepsilon_n\|_1 \leq \Delta_{n-1} + e^{-\Delta^{4}/2\delta_{n}(\delta)} = \Delta_n \).
This completes the proof.

Proof of Theorem 2.2. Note first that once uniqueness is established for the \( u \) component of the solution \((u, \beta)\), uniqueness of the \( \beta \) component follows immediately from (9). To show the former, argue by contradiction and assume that \((u_i^{(i)}, \beta_i^{(i)})\), \( i = 1, 2 \) are two solutions where \( u_1 \) and \( u_2 \) are distinct. Then there exist \( t > 0 \) and \( r \in \mathbb{R} \) such that, say, \( u_1^t[r, \infty) < u_2^t[r, \infty) \). Fix such \( t \) and \( r \). Given \( n \in \mathbb{N} \) denote \( \delta_n = tn^{-1} \). Then by Propositions 3.1 and 3.2, for every small \( \Delta > 0 \) there exists \( n_0 \) such that for every \( n > n_0 \),
\[
u_t^{(\delta_n, -)}[r, \infty) \leq u_1^t[r, \infty) < u_2^t[r, \infty) \leq u_t^{(\Delta, \delta_n, +)}[r, \infty).
\]
By Proposition 3.3,
\[
u_t^{(\Delta, \delta_n, +)}[r, \infty) - u_t^{(\delta_n, -)}[r, \infty) \leq \|\varepsilon_n(t, \Delta, \delta_n)\|_1 \leq \Delta + e^{-\Delta^{4}/2\delta_{n}(\delta)}J_t.
\]
On taking \( n \to \infty \) and then \( \Delta \downarrow 0 \), the RHS converges to zero, a contradiction.

Remark 3.3. Here are further properties of the solution deduced from [1, Theorem 1 and Remark 1(b)]. For \( 1 < q < \infty, 0 \leq \sigma < \frac{q+1}{q} \) and \( p \geq 1 \) such that \( \frac{2}{p} + \frac{1}{q} > 1 + \sigma \), \( u \in L_{p, \text{loc}}(\mathbb{R}^+, W^\sigma_q) \), and for \( p \in [1, 2) \) and \( \rho \in [0, 1) \), \( u \in L_{p, \text{loc}}(\mathbb{R}^+, C^\rho_0(\mathbb{R}) \cap C^\rho(\mathbb{R})) \).

4 Removal at boundary: convergence

4.1 Proof of main result

In this section the convergence result is proved based on the uniqueness of solutions to the PDE. It proceeds in four steps. First, in Lemma 4.1, the rescaled processes are shown to satisfy equation (9) with a certain error term. Lemma 4.2 establishes tightness of these processes. Existence of measurable density for any subsequential limit of \( \hat{\xi}^N \) is argued in Lemma 4.3. Then, Lemma 4.4 shows that subsequential limits satisfy the ORA condition. The proof of Theorem 2.3 appears at the end of the section.

A first relation to the PDE is as follows.
Lemma 4.1. Fix $\varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^+)$ and $T$ so large that $\varphi(\cdot, t) = 0$ for all $t \geq T$. Then
\[-\int_0^\infty ((\partial_t + \frac{1}{2} \partial_{xx})\varphi(\cdot, t), \xi_t^N)dt = \langle \varphi(\cdot, 0), \xi_0^N \rangle + \int_{\mathbb{R} \times \mathbb{R}^+} \varphi d\alpha^N - \int_{\mathbb{R} \times \mathbb{R}^+} \varphi d\beta^N + \bar{M}_T^N.\]

Here, $\bar{M}_T^N$ is an $\{F_t\}$-martingale that starts at zero with quadratic variation bounded by
\[\langle \bar{M}_N^N \rangle_t \leq \|\partial_x \varphi\|_\infty^2 N^{-1} (1 + J_t)t.\] (26)

**Proof.** By the assumed property $\varphi(\cdot, T) = 0$ and (4), we have
\[0 = \int_{\mathbb{R}} \varphi(x, T) \xi_T^N (dx) = N^{-1} \sum_{i \leq N^T_t} \varphi(B^i_{T \wedge \tau^i}, T).\]

For any $t$ and $i \leq N^T_t$, we have $\sigma_i \leq t$, hence by Itô’s formula,
\[\varphi(B^i_{t \wedge \tau^i}, t) = \varphi(x^i, \sigma_i) + \int_{\sigma_i}^t \partial_t \varphi(B^i_{s \wedge \tau^i}, s)ds + \int_{\sigma_i}^t \partial_x \varphi(B^i_s, s)dB^i_s + \frac{1}{2} \int_{\sigma_i}^t \partial_{xx} \varphi(B^i_s, s)ds.\]

For $i \in \mathbb{N}$ denote $M^i_t = M^{N^T_t,i}_t = \int_{\sigma_i}^{(t \wedge \tau^i)} \partial_x \varphi(B^i_s, s)dB^i_s$. Thus for $i \leq N^T_t$,
\[\varphi(B^i_{T \wedge \tau^i}, T) = \varphi(x^i, \sigma_i) + \int_{\sigma_i}^{T \wedge \tau^i} (\partial_t + \frac{1}{2} \partial_{xx}) \varphi(B^i_s, s)ds - \varphi(B^i_{T \wedge \tau^i}, T \wedge \tau^i) + M^i_T.\]

Hence
\[0 = \int_{\mathbb{R}} \varphi(\cdot, 0) d\xi_0^N + \int_{\mathbb{R} \times [0, T]} \varphi d\alpha^N + \int_0^T (\partial_t + \frac{1}{2} \partial_{xx}) \varphi(x, t) \xi_t^N (dx) dt - \int_{\mathbb{R} \times [0, T]} \varphi d\beta^N + \bar{M}_T^N,\]

where $\bar{M}_T^N = N^{-1} \sum_{i \leq N^T_t} M^i_T$. The integration range $\mathbb{R} \times [0, T]$ can be replaced by $\mathbb{R} \times \mathbb{R}^+$ thanks to the property $\varphi(\cdot, t) = 0$, $t \geq T$. Finally, the quadratic variation bound (26) is straightforward. \(\square\)

Recall that $\mathcal{M}(\mathbb{R})$ is equipped with the weak convergence topology and $D(\mathbb{R}^+, \mathcal{M}(\mathbb{R}))$ with the corresponding $J_1$ topology. $\mathcal{M}_{loc}(\mathbb{R} \times \mathbb{R}^+)$ is equipped with topology of weak convergence on compacts. Together they are equipped with the product topology.

**Lemma 4.2.** The sequence of laws of $(\xi^N, \beta^N)$, $N \in \mathbb{N}$, is tight. For every subsequential limit $(\xi, \beta)$, one has $\mathbb{P}(\xi \in C(\mathbb{R}^+, \mathcal{M}_+)(\mathbb{R}))) = 1$.

**Proof.** Tightness may be argued separately for each component. For $\beta^N$, it will be shown that for every $T$, the sequence of law of $\beta^N|_{\mathbb{R} \times [0, T]}$ is tight with respect to the topology of weak convergence. This is sufficient. Fix $T < \infty$ and write $\beta^N$ for the restriction $\beta^N|_{\mathbb{R} \times [0, T]}$. Consider the following subset of $\mathcal{M}_+(\mathbb{R} \times [0, T])$. Given a constant $c$ and a collection of constants $k(\varepsilon), \varepsilon \in \mathcal{E} := \{n^{-1} : n \in \mathbb{N}\}$ satisfying $\lim_{\varepsilon \to 0} k(\varepsilon) = 0$, let
\[K(c, k) = \{\beta : \text{for every } \varepsilon \in \mathcal{E}, \beta(\mathbb{R} \times [0, T]) \leq c, \beta([-k(\varepsilon), k(\varepsilon)] \times [0, T]) \leq \varepsilon\}.\]
By Prohorov’s theorem this set has a compact closure, and by Portmanteau’s theorem it is closed. Hence $K(c, k)$ is compact.

Now, the total mass of $\bar{\beta}^N$ is deterministic, and given by $\bar{\beta}^N(\mathbb{R} \times [0, T]) = N^{-1}N_T^J \leq J_T$.

The position of each particle that has lived between time 0 and $T$ (which includes the position where it was removed) can be bounded in terms of its initial position and the maximal displacement of the driving BM. Hence denoting $y^i = \sup_{t \in [0, T]} |\tilde{B}^i_t|$, 

$$\bar{\beta}^N([-k, k]^c \times [0, T]) \leq N^{-1} \sum_{i \leq N_T^J} 1_{\{\sup_{t \in [s^i, t]} |B^i_t| \geq k\}} \leq N^{-1} \sum_{i \leq N_T^J} (1_{\{|x^i| \geq k/2\}} + 1_{\{y^i \geq k/2\}}) \to P(x^1 \geq k/2) + I_T P(z \geq k/2) + (1 + I_T) P(y^1 \geq k/2),$$

in probability, where we recall that $x^1$ is distributed according to $\pi$, let $z$ be distributed according to $u_0(x)dx$, and use the LLN. For every $\varepsilon$ let $k = k(\varepsilon)$ be so large that the RHS is less than $\varepsilon/2$.

Then setting $K = K(J_T, k(\cdot))$ one has $P(\bar{\beta}^N \in K) \to 0$ as $N \to \infty$. This shows tightness of the laws of $\bar{\beta}^N$.

Denote by $d_L(\cdot, \cdot)$ the Levy distance on $\mathcal{M}_+(\mathbb{R})$ and recall that it induces weak convergence. For $f : \mathbb{R}_+ \to \mathbb{R}$ denote $w(f, \delta, T) = \sup\{|f(s) - f(t)| : s, t \in [0, T], |s-t| \leq \delta\}$, and for $f : \mathbb{R}_+ \to \mathcal{M}_+(\mathbb{R})$ use the same notation for the modulus of continuity with respect to $d_L$. The argument for $\xi^N$ is based on showing (i) for every $T$ there exists a compact $K = K(T) \subset \mathcal{M}_+(\mathbb{R})$ such that $P(\xi^N \in K) \to 1$ as $N \to \infty$; and (ii) for every $T$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_N P(w(\xi^N, \delta, T) \geq \varepsilon) \leq \varepsilon.$$

Once these two properties are proved we can use [11] Corollary 3.7.4 (p. 129) by which $\bar{\xi}^N$ is a relatively compact sequence. Moreover, because we use $w$ rather than $w'$ (cf. (3.6.2) of [11] (p. 122)), this in fact establishes $C$-tightness, which proves the second statement. The argument for (i) follows by similar considerations to the one given above for $\bar{\beta}^N$, and is thus omitted.

It remains to show (ii). Recall that $A^N_t$ denotes the index set for the particles existing in the configuration at time $t$. Consider $s, t \in [0, T]$ such that $0 < t - s \leq \delta$. The cardinality of the symmetric different between $A^N_s$ and $A^N_t$ is given by

$$N^J_t - N^J_s + N^J_s - N^J_t \leq Nw_1(\delta), \tag{27}$$

where $w_1(\delta) = w(I, \delta, T) + w(J, \delta, T)$.

Now, $w(\tilde{B}^i, \delta, T)$ are IID and each converges to zero as $\delta \to 0$. Therefore given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_N P(\Omega^N) = 0,$$

where $\Omega^N = \Omega^N_{\varepsilon, \delta} = \left\{\frac{\#\{i \leq N_T^J : w(\tilde{B}^i, \delta, T) \geq \eta\}}{N_T^J} \geq \varepsilon\right\}$, and we have denoted $\eta = c\varepsilon$, $c = (2 + I_T)^{-1}$. Given a set $C \subset \mathbb{R}$ let $C^\eta$ denote the $\eta$-neighborhood. On the event $\Omega^N$, the following holds: except for at most $w_1(\delta)N$ particles (removed between $s$
and \(t\), and at most \(\eta N_I^t\) particles (whose displacement exceeds \(\eta\)), each particle \(i \in A^N_s\) travels less than \(\eta\) between \(s\) and \(t\). Hence, for any Borel set \(C\), on \(\Omega^N\),

\[
\xi^N_s(C) \leq \xi^N_t(C') + w_1(\delta)N + \eta N_I^t.
\]

Similarly,

\[
\xi^N_t(C) \leq \xi^N_s(C') + w_1(\delta)N + \eta N_I^t.
\]

By making \(\delta\) smaller if necessary, we may assume that \(w_1(\delta) < \eta\). Consequently, \(w_1(\delta)N + \eta N_I^t < \eta N + \eta N(1 + I_T) = \varepsilon\). We obtain that on \(\Omega^N\),

\[
d_L(\xi^N_s, \xi^N_t) < \varepsilon.
\]

This shows that

\[
\limsup_N P(w(\xi^N, \delta, T) \geq \varepsilon) = 0,
\]

and the proof is complete. \(\square\)

The next two lemmas address limit points \((\xi, \beta)\) of \((\tilde{\xi}^N, \tilde{\beta}^N)\). The first states that for every \(t\), \(\xi_t(dx)\) has a density.

**Lemma 4.3.** Let \((\xi, \beta)\) be a subsequential limit of \((\tilde{\xi}^N, \tilde{\beta}^N)\). Then, given \(T < \infty\),

\[
\sup_{t \in [0, T]} \sup_{a, b \in \mathbb{R}: a < b} (b - a)^{-1} \xi_t(a, b) \leq c \quad a.s.,
\]

where \(c < \infty\) is a deterministic constant that depends only on \(T\) and the data. Moreover, there exists a full measure event \(\Omega^1\) and a \(\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}\)-measurable function \(u(x, t, \omega)\) such that for every \((t, \omega) \in \mathbb{R}_+ \times \Omega^1\), \(u(\cdot, t, \omega)\) is a density of \(\xi_t(\cdot, \omega)\) with respect to the Lebesgue measure on \(\mathbb{R}\).

**Proof.** Fix \(T\). The first step is to show that there exists a constant \(c_1\) such that for every \(t \in [0, T]\) and every \(a < b\),

\[
\xi_t(a, b) \leq c_1(b - a) \quad a.s. \quad (29)
\]

Recall that \(\zeta^N\) denotes the particle configuration without removals. Note that by construction, \(\xi^N_t(B) \leq \zeta^N_t(B)\) for all \(t \in \mathbb{R}_+, B \in \mathcal{B}(\mathbb{R})\), \(\omega \in \Omega\). The argument toward (29) is based on showing, for \(t, a, b\) fixed,

\[
P(\zeta^N_t(a, b) > c_1(b - a)) \to 0. \quad (30)
\]

If the above is true, the same is true for \(\xi^N_t(a, b)\). Since along a subsequence one has \(\xi^N \Rightarrow \xi\) and the latter has continuous sample paths, one also has \(\xi_t^N \Rightarrow \xi_t\). As a result, \(P(\xi_t(a, b) > c_1(b - a)) \leq \limsup P(\xi^N_t(a, b) > c_1(b - a)) = 0\).

The convergence (30) is a standard result but we have not found a reference that covers precisely this setting. The proof is simple. We have by (5), \(\zeta^N_t(a, b) = \sum_{i=1}^{N^t_i} 1_{\{B_i \in (a, b)\}}\). For a first moment
We first prove a version of (31) for rational $r$. More precisely, consider the collection $\Sigma$ of all (deterministic) tuples $\sigma = (r, t_1, t_2, \eta, \delta) \in Q^5$, $0 \leq t_1 < t_2$, $\eta > 0$, $\delta > 0$. We show that for every
\[ \sigma \in \Sigma, \mathbb{P}(\Omega_\sigma) = 0 \text{ where} \]

\[ \Omega_\sigma = \{ \inf_{t \in [t_1, t_2]} \xi_t[r, \infty) > \eta, \beta((-\infty, r) \times (t_1, t_2)) > \delta \}. \]

Fix \( \sigma = (r, t_1, t_2, \eta, \delta) \). If \( \mathbb{P}(\Omega_\sigma) > 0 \) then by the weak convergence \((\tilde{\xi}^N, \tilde{\beta}^N) \Rightarrow (\xi, \beta)\) and the a.s. continuity of the limit \( \xi \), one must have for all large \( N \),

\[ \mathbb{P}\left( \inf_{t \in [t_1, t_2]} \tilde{\xi}_t^N[r, \infty) > \eta/2, \tilde{\beta}_t^N(-\infty, r) \times (t_1, t_2) > \delta/2 \right) > 0. \]

However, by (6), the above probability is zero for all \( N \). This shows \( \mathbb{P}(\Omega_\sigma) = 0 \). Consequently, \( \mathbb{P}(\cup \Omega_\sigma) = 0 \).

Next consider the event

\[ E^0 = \{ \text{that there exists } r \in \mathbb{R} \text{ such that } X(\xi, \beta, r > 0) \}. \]

Given any \( r \in \mathbb{R} \), consider \( r_n \downarrow r, r_n \in \mathbb{Q} \). Then \( \beta((-\infty, r), dt) \ll \beta((-\infty, r_n), dt) \) for every \( n \). Moreover, \( \xi_t[r_n, \infty) \rightarrow \xi_t[r, \infty) \) as \( n \rightarrow \infty \) uniformly for \( t \) in any compact set (as follows from the Lipschitz property \( x \rightarrow \xi_t[x, \infty) \) stated in Lemma 4.3). Consequently, if \( X(\xi, \beta, r) > 0 \) then there exists \( n \) such that \( X(\xi, \beta, r_n) > 0 \). Hence on \( E^0 \) there exists \( r \in \mathbb{Q} \) such that \( X(\xi, \beta, r) > 0 \). Next, the condition \( X(\xi, \beta, r) > 0 \) (with \( r \in \mathbb{Q} \)) implies that there exists \( \eta \in \mathbb{Q} \cap (0, \infty) \) such that \( \int_{A_\eta} a_t db_t > 0 \) where \( a_t = \xi_t[r, \infty) = \xi_t(r, \infty), b_t = \beta(-\infty, r) \times [0, t] \) and \( A_\eta = \{ t : a_t > 2\eta \} \). The trajectory \( t \mapsto a_t \) is continuous on \((0, \infty)\) (using the fact that \( \xi \in C(\mathbb{R}_+, \mathcal{M}(\mathbb{R})) \) and that for each \( t, \xi_t \) has no atoms). Hence \( A_\eta \) is a countable union of disjoint open intervals. For one of those intervals, say \((t_1, t_2)\), we have \( \int_{t_1, t_2} a_t db_t > 0 \). By taking a subset we may assume without loss that \( t_1, t_2 \in \mathbb{Q} \). Consequently, \( \xi_t(r, \infty) = a \geq 2\eta > \eta \) on the closed interval \([t_1, t_2] \), while \( b_{t_2} > b_{t_1} \). Moreover, notice that by construction, for every \( a, \tilde{\beta}_t^N(\mathbb{R} \times [0, s]) = N^{-1}[NJ_s] \). Hence by the continuity of \( J, \beta \) does not charge sets \( \mathbb{R} \times \{ t \} \). As a result, there exists \( \delta > 0 \) such that

\[ \beta((-\infty, r) \times (t_1, t_2)) > \delta. \]

This shows that \( \mathbb{P}(E^0) \leq \mathbb{P}(\cup_{\sigma \in \Sigma} \Omega_\sigma) = 0. \)

**Proof of Theorem 2.3.** Again consider a subsequential limit \((\xi, \beta) \) of \((\tilde{\xi}^N, \tilde{\beta}^N) \). Let \( u \) be the density from Lemma 4.3. We show that a.s., \((u, \beta) \) is a weak solution to (8).

First, to address (9), recall that \( \{ x^i, 1 \leq i \leq N \} \) are IID \( u_0(\mathbb{R}) \), hence \( \langle \varphi(\cdot, 0), \tilde{\xi}^N_0 \rangle \rightarrow (\varphi(\cdot, 0), u_0) \) in probability. Using this and the relation \( \xi_t(dx) = u(x, t) dx \) in Lemma 4.1 shows that \((u, \beta) \) satisfies (9) a.s.

That condition (8)(iii) is satisfied by \((u, \beta) \) follows from Lemma 4.4.

Condition (8)(iv) is an immediate consequence of the particle system construction where the removal count is given by \( N_t^I = [NJ_t] \).

To show that \((u, \beta) \) is a.s. a weak solution to (8), it remains to prove that \( u \in L_{1,loc}(\mathbb{R}_+, L_1) \cap L_{\infty,loc}(\mathbb{R}_+, L_\infty) \). The former condition is a consequence of \( \tilde{\xi}^N_t(\mathbb{R}) = N^{-1}(N_t^I - N_t^I) \leq 1 + I_t \) and the latter follows the Lipschitz bound (28). We have thus shown that \((u, \beta) \) is a.s. a weak solution to (8). By Theorem (2.2), it is in fact the unique weak solution to (8). In particular, the limit \((\xi, \beta) \) does not depend on the subsequence. In other words, there exists a unique weak solution \((u, \beta) \) to (8), and \((\tilde{\xi}^N, \tilde{\beta}^N) \Rightarrow (\xi, \beta) \) in probability, where, for every \( t, \xi_t(dx) = u(x, t) dx \).

**Proof of Theorem 2.1.** This result is contained in Theorem 2.3.
4.2 Comparison via coupling

In this section we prove the comparison principle, Theorem 2.4.

Proof of Theorem 2.4. The proof is based on a coupling. Because it is a straightforward coupling construction, we only give a sketch. Before we can apply coupling, we need two preparatory steps.

Step 1. A right-continuous nondecreasing function from $\mathbb{R}_{+}$ to $\mathbb{Z}_{+}$ starting at 0, for which every jump is of size 1, is said to be a simple step function. The construction from §2.1 uses $N^I_t = N + [NI_t]$ and $N^J_t = [NJ_t]$ as the injection and removal counts, but we claim that the main result still holds if these are replaced by any sequence $N^I_{t,N} = N + \hat{N}^I_{t,N}$ and $N^J_{t,N}$ such that $\hat{N}^I_{t,N}$ and $N^J_{t,N}$ are simple step functions, as long as $N^{-1}\hat{N}^I_{t,N} \to I$ and $N^{-1}N^J_{t,N} \to J$, as $N \to \infty$, uniformly on compacts. The only place in the proof that needs some attention is the tightness argument, Lemma 4.2, where the inequality (27) should be modified to add an $o(N)$ term on the RHS. The proof carries over with this change.

Step 2. By performing discretization (such as $x \mapsto [x]$) to $N(\tilde{J}_t - J_t)$ and $N^J_t$ it is seen that simple step functions $N^{\tilde{J},N}$ and $N^{J,N}$ can be constructed in such a way that the jumps of the former dominate those of the latter (every time the latter function jumps, so does the former), and moreover, their normalized versions converge uniformly on compacts to $\tilde{J}$ and $J$. (If a jump of size 2 occurs for $N^{\tilde{J},N}$ via the above discretization, as it may, one splits it to two nearby jumps).

Similarly $\hat{N}^I_{t,N}$ and $\hat{N}^I_{t,N}$ can be constructed in such a way that the jumps of the latter dominate those of the former, with convergence of their normalized versions to $\tilde{I}$ and $I$.

Using the domination of the jumps of $N^I_{t,N}$ over those of $N^I_{t,N}$, it follows by an elementary argument that a coupled pair of particle systems can be constructed in such a way that the counting measures $\tilde{\xi}^N$ and $\xi^N$ (that, we recall, account merely for the potential trajectories) satisfy a.s., for all $t$,

$$\tilde{\xi}^N_t \leq \xi^N_t.$$

To deduce a similar statement about the counting measures of the true trajectories, argue by induction on the times of removal. At times when a joint removal occurs, the rightmost particle is removed at both systems. Since by the induction assumption the counting measures are ordered as $\tilde{\xi}^N \leq \xi^N$ right before the removal, the removal of rightmost particle from both configurations preserves this order. At times of a jump of $N^{\tilde{J},N}$ alone, a particle is removed only from the configuration $\tilde{\xi}^N$, again keeping the order. As a consequence, $\tilde{\xi}^N_t \leq \xi^N_t$ holds a.s. at all times. The result is now a simple consequence of Theorem 2.3.

5 Removal at quantile

In this section the approach is applied to RAQ, a model that follows a far more general removal mechanism, where, for a given continuous function $Q$, the $Q(t)$-quantile member of the population is removed at time $t$. On the other hand, the model is simpler than the RAB model in two respects: the removal rate is constant, and there are no injections, and therefore the population dies out after $N$ removals.

Whereas the construction and analysis of the barriers, as well as some further details of the uniqueness proof are different, the convergence proof is almost identical to the one provided for RAB.
The particle system is constructed in §5.1. The heat equation with a (modified) ORA condition is described in §5.2, where the main result regarding this model is also stated. §5.3 provides several further mass transport inequalities used later. The barriers are constructed and analyzed in §5.4. Finally, in §5.5 the proof of uniqueness and convergence is provided.

5.1 Particle system with quantile removal

Simply put, the system consists of Brownian particles whose initial number is \( N \), subjected to removal defined in terms of a given \( Q : [0,1] \rightarrow [0,1] \). At a fixed rate \( N \), the particle that is the \( Q(t) \)-quantile member of the current population is removed, until there are no particles left. The precise details of the construction are as follows.

Let \( u_0 \in L_1(\mathbb{R},\mathbb{R}_+) \cap L_\infty(\mathbb{R},\mathbb{R}_+) \), \( \|u_0\|_1 = 1 \) be given, and let \( q(t) = (1-t)Q(t), \) \( 0 \leq t \leq 1 \). It turns out to be more natural to regard \( q \), rather that \( Q \), as data; note that \( Q(t) \) can be recovered from \( q(t) \) for all \( 0 \leq t < 1 \). Thus an admissible data for this problem is a pair \( (u_0,q) \), where \( u_0 \) is as above, and \( q : \mathbb{R}_+ \rightarrow [0,1] \) is a continuous function satisfying \( q(t) \leq 1-t \) for \( t \in [0,1) \) and vanishing on \( [1,\infty) \).

The initial number of particles is \( N \). Mutually independent BMs \( \{B^i_t, t \geq 0 \}, \) \( 1 \leq i \leq N \) are given, where \( B^i \) starts at \( x^i \) for \( 1 \leq i \leq N \), and \( x^i \) are IID, distributed according to the probability measure \( u_0(x)dx \).

The configuration is constructed recursively on intervals \( \text{INT}_k \) given by \( [t_k,t_{k+1}) \) where \( t_k = kN^{-1}, \) \( 0 \leq k \leq N \), while \( t_{N+1} = \infty \), where at each time \( t_k \), \( 1 \leq k \leq N \), one particle is removed. Given admissible data \( (u_0,q) \), let \( Q(t) = (1-t)^{-1}q(t) \) for \( 0 \leq t < 1 \) (it is not necessary to define \( Q(1) \) because at time 1– there is only one particle to remove). The collection of particles present in the system during \( \text{INT}_k \) is denoted by \( \text{POP}_k \); in particular, \( |\text{POP}_k| = N-k \). For \( t \in \text{INT}_0 \), the \( N \) particle positions are \( B^i_t \), and \( \text{POP}_0 = \{1,\ldots,N\} \). For \( 1 \leq k \leq N-1 \), let \( i_k \) be the \( Q(t_k) \)-quantile member of the particle configuration at \( t_k \). That is,

\[
i_k = \min\{j \in P : \#\{i \in P : b_i \geq b_j\} = \lfloor nQ(t_k) \rfloor\},
\]

where \( P = \text{POP}_{k-1} \), \( n = N-k+1 = |\text{POP}_{k-1}| \), and for \( i \in P \), \( b_i = B^i_{t_k} \). Here, the convention \( [0] = 1 \) is used. Set \( \text{POP}_k = \text{POP}_{k-1} \setminus \{i_k\} \). At the final removal time, \( t_N = 1 \), the last remaining particle is removed. Thus \( i_N \) is the unique member of \( \text{POP}_{N-1} \), and \( \text{POP}_N \) is the empty set. Finally, \( \tau^i \), the removal time of particle \( i \), is well defined for all \( 1 \leq i \leq N \) via the relation \( \tau^{i_k} = t_k \), for \( 1 \leq k \leq N \).

The configuration process and the removal measure are defined by

\[
\xi^N_t(dx) = \sum_{i=1}^{N} \delta_{B^i_t}(dx)1_{\{t<\tau^i\}}, \quad \beta^N_t(dx,dt) = \sum_{i \leq N} \delta_{(B^i_{\tau^i},\tau^i)}(dx,dt),
\]

and the two extended configuration processes by

\[
\gamma^N_t(dx) = \sum_{i \leq N} \delta_{B^i_{\tau^i}}(dx), \quad \zeta^N_t(dx) = \sum_{i \leq N} \delta_{B^i}(dx).
\]

The rescaled versions are denoted by \( \tilde{\xi}^N = N^{-1}\xi^N \), etc.
Fix $r \in \mathbb{R}$. If at time $t = t_k = kN^{-1}$ a particle is removed at a position within $(-\infty, r)$ then, again denoting $n = N-k+1$, one has $\xi_k^N(r, \infty) = \xi_k^N(r, \infty) \leq \lfloor nQ(t_k) \rfloor - 1 \leq nQ(t_k) \leq Nq(t_k) + 1$. Similarly if a particle is removed within $(r, \infty)$ then $\xi_k^N(\infty, r) \leq N(1 - t_k - q(t_k)) + 1$.

To express this in terms of $\bar{\xi}_N$ and $\bar{\beta}_N$ use the following notation. For $(\xi, \beta, c) \in D(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R})) \times \mathcal{M}_{+,\text{loc}}(\mathbb{R} \times \mathbb{R}_+) \times C(\mathbb{R}_+, [0, 1])$, let

$$I^+(\xi, \beta, c) = \int_{\mathbb{R}_+} [\xi_t(r, \infty) - c(t)]^+ \beta((-\infty, r) \times dt),$$

$$I^-(\xi, \beta, c) = \int_{\mathbb{R}_+} [\xi_t(\infty, r) - (1 - t - c(t))]^+ \beta((r, \infty) \times dt).$$

Then

$$I^+\left(\bar{\xi}_N^N, \bar{\beta}_N^N, q(\cdot) + \frac{1}{N}\right) = I^-\left(\bar{\xi}_N^N, \bar{\beta}_N^N, q(\cdot) - \frac{1}{N}\right) = 0, \quad r \in \mathbb{R}.\quad (34)$$

### 5.2 PDE and main result

With a slight abuse of notation, for $u \in L_{1,\text{loc}}(\mathbb{R}_+, L_1)$, let $I^\pm(u, \beta, c)$ be defined as in (32)–(33) with $u((r, \infty), t)$ in place of $\xi_t(r, \infty)$, etc. Consider the problem

$$\begin{cases}
(i) & \partial_t u - \frac{1}{2} \partial_{xx} u = -\beta & \text{in } \mathbb{R} \times \mathbb{R}_+, \\
(ii) & u(\cdot, 0) = u_0 & \text{on } \mathbb{R}, \\
(iii) & I^+(u, \beta, q) = 0 & \text{for all } r \in \mathbb{R}, \\
(iv) & \beta(\mathbb{R} \times [0, t]) = t \wedge 1 & \text{for all } t \in \mathbb{R}_+.
\end{cases}\quad (35)$$

A solution to (35) is a pair $(u, \beta) \in L_{1,\text{loc}}(\mathbb{R}_+, L_1) \cap L_{\infty,\text{loc}}(\mathbb{R}_+, L_\infty) \times \mathcal{M}_{+,\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ such that $u$ is a.e. nonnegative, conditions (35)(iii) and (35)(iv) hold, and for any $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+)$,

$$- \int_0^\infty (\partial_t \varphi + \frac{1}{2} \partial_{xx} \varphi, u) dt = (\varphi(\cdot, 0), u_0) - \int_{\mathbb{R} \times \mathbb{R}_+} \varphi d\beta.\quad (36)$$

**Theorem 5.1.** There exists a unique solution $(u, \beta)$ to (35). Moreover, if for every $t$ we set $\xi_t(dx) = u(x, t)dx$ then $(\xi, \beta) \in C(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R})) \times \mathcal{M}_{+,\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ and $(\bar{\xi}_N^N, \bar{\beta}_N^N) \to (\xi, \beta)$ in probability, as $N \to \infty$.

### 5.3 More mass transport inequalities

In this section it is convenient to work with an extension of $K^\Delta$, denoted by $L^\Delta$, to $\Delta \in \mathbb{R}$ (but still $\delta > 0$). Introduce, analogously to $R_\delta$, for $\delta > 0$ and $u \in H_\delta$,

$$\tilde{R}_\delta(u) = \inf\{x \in \mathbb{R} : u(-\infty, x] \geq \delta\},$$

and analogously to $K^\delta$,

$$\tilde{K}^\delta u(x) = u(x)1_{[\tilde{R}_\delta(u), \infty)}(x).$$
This operator cuts off the leftmost mass of size $\delta$. For $\Delta \in \mathbb{R}$, $\delta > 0$ and $u \in H_\delta$,

$$L^{\Delta,\delta} = \begin{cases} K_\delta^u & \Delta \leq 0, \\ K_{\Delta,\delta}^u & \Delta > 0, \Delta + \delta < \|u\|_1, \\ \tilde{K}_\delta^u & \Delta > 0, \Delta + \delta \geq \|u\|_1. \end{cases}$$

**Lemma 5.1.** Let $\Delta \in \mathbb{R}$, $\delta > 0$, $u, v \in H_\delta$ be given, where $u \preceq v$. Then

i. $\tilde{K}_\delta^u \preceq \tilde{K}_\delta^v$.

ii. $L^{\Delta,\delta} u \preceq L^{\Delta,\delta} v$.

iii. If in addition $w \in L_1(\mathbb{R}, \mathbb{R}_+)$ is such that $\|w\|_1 = \delta$, and $u - w \in L_1(\mathbb{R}, \mathbb{R}_+)$ then $u - w \preceq \tilde{K}_\delta^u$.

iv. If $-\infty < \Delta \leq \Delta < \infty$ then $L^{\Delta,\delta} u \preceq L^{\Delta,\delta} v$.

**Proof.** i. Although in the special case when $\|u\|_1 = \|v\|_1$, the monotonicity of $\tilde{K}_\delta^u$ follows from that of $K_\delta^u$ (via reflection $x \mapsto -x$), this is not the case in general. We must show $\tilde{K}_\delta^u(r, \infty) \leq \tilde{K}_\delta^v(r, \infty)$ for all $r$. Note first that one must have $\|u\|_1 \leq \|v\|_1$. Let $r_u = \tilde{R}_\delta(u)$ and $r_v = \tilde{R}_\delta(v)$.

Case 1: $r_u \leq r_v$. For $r \geq r_v$, this follows from $u \preceq v$. For $r \leq r_v$,

$$\tilde{K}_\delta^u(r, \infty) \leq u(r, \infty) \leq \|u\|_1 - \delta \leq \|v\|_1 - \delta = \tilde{K}_\delta^v(r, \infty).$$

Case 2: $r_u > r_v$. Again, the claim holds trivially when $r > r_u$. For $r \leq r_v$, $\tilde{K}_\delta^u(r, \infty) = \|u\|_1 - \delta \leq \|v\|_1 - \delta = \tilde{K}_\delta^v(r, \infty)$. For $r_v \leq r \leq r_u$,

$$\tilde{K}_\delta^u(r, \infty) \leq u(r, \infty) \leq v(r, \infty) = \tilde{K}_\delta^v(r, \infty).$$

ii. For $\Delta \leq 0$, the claim holds by the monotonicity of $K_\delta^u$ (Lemma 3.2(i)). Next assume $\Delta > 0$. If $\Delta + \delta < \|u\|_1 \lor \|v\|_1$, the claim follows from the monotonicity of $K_{\Delta,\delta}^u$ (Lemma 3.2(iii)). If $\Delta + \delta \geq \|u\|_1 \lor \|v\|_1$, the claim follows from the monotonicity of $\tilde{K}_\delta^u$ just proved. Because $\|u\|_1 \leq \|v\|_1$, the only remaining case is $\|u\|_1 \leq \Delta + \delta < \|v\|_1$. In this case we must show $\tilde{K}_\delta^u(r, \infty) \leq \tilde{K}_{\Delta,\delta}^u(r, \infty)$. Denote $r_u = \tilde{R}_\delta(u)$ and $r_v = \tilde{R}_\delta(v)$.

Case 1: $r_u \leq r_v$. For $r \geq r_v$, the claim clearly holds. For $r \leq r_v$,

$$\tilde{K}_\delta^u(r, \infty) \leq \|\tilde{K}_\delta^u\|_1 = \|u\|_1 - \delta \leq \Delta = \tilde{K}_{\Delta,\delta}^u(r_v, \infty) \leq \tilde{K}_{\Delta,\delta}^v(r_v, \infty).$$

Case 2: $r_u > r_v$. For $r \geq r_v$, the claim clearly holds. For $r \leq r_v$, again

$$\tilde{K}_\delta^u(r, \infty) = \|u\|_1 - \delta \leq \Delta = \tilde{K}_{\Delta,\delta}^u(r_v, \infty) \leq \tilde{K}_{\Delta,\delta}^v(r_v, \infty).$$

For $r_v < r < r_u$,

$$\tilde{K}_\delta^u(r, \infty) \leq u(r, \infty) \leq v(r, \infty) = \tilde{K}_{\Delta,\delta}^v(r, \infty).$$

iii. Because $\|u - w\|_1 = \|\tilde{K}_\delta^u\|_1$, this claim follows from Lemma 3.2(ii) by reflection.

iv. One can write $L^{\Delta,\delta} u$ as $K^{g(a),\delta}^u$ where $g(a) = (a \lor 0) \land (\|u\|_1 - \delta)$. Since $g$ is a nondecreasing function, the result follows from Lemma 3.2(iv).
v. If $\hat{\Delta} > 0$ and $\Delta < \|u\|_1 - \delta$ then both sides of the asserted inequality are equal to $K^{\Delta, \delta} u$. If $\hat{\Delta} > 0$ and $\Delta \geq \|u\|_1 - \delta$ then both sides of the inequality are equal to $\tilde{K}^{\delta} u$.

Finally, consider the case $\hat{\Delta} \leq 0$ and denote the RHS of the asserted inequality by $\nu$. Then $\nu = K^{\Delta - \hat{\Delta}, \delta} u$. Since by assumption $0 \leq \Delta - \hat{\Delta} \leq \|u\|_1 - \delta$, one has $\nu = L^{\Delta - \hat{\Delta}, \delta} u$. But $\Delta \leq \Delta - \hat{\Delta}$, hence $L^{\Delta, \delta} u \leq L^{\Delta - \hat{\Delta}, \delta} u = v$ by part (iv) above.

5.4 Barriers

Lemma 5.2. Assume that a solution $(v, \beta)$ to (35) exists. Then $v$ has a version $u$ given by

$$u(y, t) = G_t u_0(y) - G^* \beta(y, t). \quad (37)$$

Moreover, for $0 \leq \tau < t$,

$$u(y, t) = G_{t-\tau} u(\cdot, \tau)(y) - G^* \beta(y, t; \tau). \quad (38)$$

Furthermore, $u \in L^\infty_{\text{loc}} (\mathbb{R}^+, L^1) \cap L^\infty_{\text{loc}} (\mathbb{R}^+, L^\infty)$.

As was the case in §3, a solution to (35) will always mean the version given by (37), and this results in no loss of generality.

Proof. The proof of Lemma 3.1 does not rely on the ORA condition (8)(iii) in any way, and hence the result is a special case of Lemma 3.1.

The barriers are defined for parameters $0 < \delta < \Delta$. Denote again $\bar{\delta} = (\Delta, \delta)$. Let

$$q_n^{(-)} = \min_{(n-1)\delta \leq \delta} q, \quad q_n^{(+)} = \max_{(n-1)\delta, \delta} q$$

and

$$L_n^{(+)} = L_n^{q_n^{(+)} + \Delta}. \quad (39)$$

Then the upper barriers are defined by $u_{n\delta}^{(\delta, +)} = u_0$, and for $n \in \mathbb{N}$ such that $n\delta < 1 - \Delta$,

$$u_{n\delta}^{(\delta, +)} = L_n^{(+)\delta} G_{\delta} u_{(n-1)\delta} + e_n^{(+)\delta}$$

$$e_n^{(+)\delta}(y) = e^{ -\Delta^2 / 2 \delta^2} \delta_1_{[u_n^{(+)\delta}, \sigma_n^{(+)\delta} + 1]}(y), \quad y \in \mathbb{R},$$

$$\sigma_n^{(+)\delta} = \sigma_n^{(+)\delta} = R\Delta (G_\delta u_{(n-1)\delta}).$$

For the first and third lines to be well defined one must have $\|u_{(n-1)\delta}^{(\delta, +)}\|_1 > \delta$ and, respectively, $> \Delta$. Because every iteration cuts off a mass of size $\delta$ (and adds the mass of the error term), the condition $n\delta < 1 - \Delta$ does assure $\|u_{(n-1)\delta}^{(\delta, +)}\|_1 \geq 1 - (n-1)\delta > \Delta > \delta$, hence the upper barriers are well defined.

The lower barriers are constructed analogously (and are again well defined). Namely, let

$$L_{n, \delta}^{(-)} = L_{n, \delta}^{q_n^{(-)} - \Delta - \delta, \delta}. \quad (39)$$
Then $u_{0}^{(δ,-)} = u_{0}$, and for $n ∈ \mathbb{N}$, $nδ < 1 - \Delta$,

$$
u_{nδ}^{(δ,-)} = L_{nδ}^{(-)}G_{δ}u_{(n-1)δ} + e_{nδ}^{(-)}$$

$$e_{nδ}^{(-)}(y) = e^{-\Delta^{2}/2δ}f_{1}[u_{nδ}^{(-)} - \sigma_{nδ}^{(-)}](y), \quad y ∈ \mathbb{R}$$

$$\sigma_{nδ}^{(-)} = \sigma_{n}^{(-)}(\tilde{\delta}) = \tilde{R}_{\delta}(G_{δ}u_{(n-1)δ}).$$

(40)

Following [5], define, for $u, v ∈ L_{1}(\mathbb{R}, \mathbb{R}^{+})$ and $m > 0$, the relation $u ▪ v \mod m$ as

$$u(r, ∞) ≤ v(r, ∞) + m, \quad \text{for all } r ∈ \mathbb{R}$$

**Proposition 5.1.** For every $Δ > 0$ sufficiently small and every $0 < δ < δ_{0} = δ_{0}(Δ)$ and $n ∈ \mathbb{Z}_{+}$ such that $nδ < 1 - Δ$, for any solution $(u, β)$ to (35) one has

$$u_{nδ}^{(δ,-)} ≍ u_{nδ} \mod e^{-\Delta^{2}/2δ}, \quad \text{and} \quad u_{nδ} ≍ u_{nδ}^{(δ,+)}.$$

By Lemma 5.2, there is a constant $c_{∞}$ such that for any solution $(u, β)$, $∥u(·, t)∥_{1} ≤ 1$ and $∥u(·, t)∥_{∞} ≤ c_{∞}$, $t ∈ [0, 1]$.

**Lemma 5.3.** There exists $δ_{0} > 0$ such that for any solution $(u, β)$ and any $(n, δ) ∈ \mathbb{N} × (0, δ_{0})$ satisfying $nδ ≤ 1$,

$$ρ_{s}(β; [(n-1)δ, nδ]) ≥ R_{q_{n,δ}^{(+)} + c_{∞}δ^{1/2}}(u_{(n-1)δ}),$$

provided that $q_{n,δ}^{(+)} + c_{∞}δ^{1/2} ≤ ∥u_{(n-1)δ}∥_{1}$, and

$$ρ^{*}(β; [(n-1)δ, nδ]) ≤ R_{q_{n,δ}^{(-)} - c_{∞}δ^{1/2}}(u_{(n-1)δ}),$$

provided that $q_{n,δ}^{(-)} - c_{∞}δ^{1/2} ≥ 0$.

**Proof.** It suffices to prove the result for $n = 1$, and to consider only the lower bound. Fix $δ$ and a solution $(u, β)$. Denote $r = R_{q_{1,δ}^{(+)} + c_{∞}δ^{1/2}}(u_{0})$. First, in the special case where $q_{1,δ}^{(+)} + c_{∞}δ^{1/2} = ∥u_{0}∥_{1}$ and $u_{0}(−∞, x) > 0$ for all $x$, there is nothing to prove because $r = −∞$. In all other cases, $r$ is finite. Therefore it is assumed that what follows that $r$ is finite.

Arguing by contradiction, assume $ρ_{s}(β; [0, δ]) < r$. Then $β((−∞, r) × [0, δ]) > 0$. Using (35)(iii),

$$\int_{[0, δ]} [u((r, ∞), t) - q(t)]+ β((−∞, r) × dt) = 0,$$

it follows that there exists $t ∈ [0, δ]$ such that $u((r, ∞), t) ≤ q(t) ≤ q_{1,δ}^{(+)}$. It is impossible that $t = 0$ since $u_{0}(r, ∞) = q_{1,δ}^{(+)} + c_{∞}δ^{1/2}$. Fix such $t$. Denote the last term in (37) by $η(y) = \int_{\mathbb{R} × [0, t]} G_{1-s}(x, y)β(dx, ds)$. Then

$$\int_{r}^{∞} η(y)dy ≤ \int_{−∞}^{∞} η(y)dy = β(\mathbb{R} × [0, t]) = t ≤ δ.$$

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Since \( u((r, \infty), t) \leq q_1^{(+)} \), we have by (37),
\[
\int_r^\infty \int_\mathbb{R} G_t(x, y)u_0(x)dy \leq q_1^{(+)} + \int_r^\infty \eta(y)dy \leq q_1^{(+)} + \delta.
\]
Hence if \( A = \int_r^\infty G_t(x, (-\infty, r))u_0(x)dx \),
\[
q_1^{(+)} + \delta \geq \int_r^\infty G_t(x, (r, \infty))u_0(x)dx = -A + u_0(r, \infty)
\]
\[
= -A + q_1^{(+)} + c_\infty \delta^{1/2}.
\]
However,
\[
A \leq c_\infty \int_r^\infty G_t(x, (-\infty, r))dx
\]
\[
= c_\infty \int_0^\infty G_1(0, (x, \infty))dx = c_\infty (2\pi)^{-1/2}t^{1/2} < \frac{c_\infty}{2} \delta^{1/2}.
\]
This gives \( c_\infty \delta^{1/2} - \delta < \frac{1}{2} c_\infty \delta^{1/2} \), which is impossible when \( \delta \) is small.

\[\square\]

**Proof of Proposition 5.1.** There is an obvious symmetry in equation (35) w.r.t. the reflection \( x \mapsto -x \), thanks to which the lower bound follows from the upper bound. More precisely, let \( (u, \beta) \) be a solution to (35). Define \((\tilde{u}, \tilde{\beta})\) by composing \( (u, \beta) \) with the reflection \( x \mapsto -x \). Then \((\tilde{u}, \tilde{\beta})\) is a solution of (35) with data \((u_0(-\cdot), 1 - t - q(t))\) in place of \((u_0, q)\). Further, there is symmetry in the construction of the upper and lower barriers: If \( U^{(\delta, -)} \) are the lower barriers for the data \((u_0, q)\) and \( U^{(\delta, +)} \) are the upper barriers for the data \((u_0(-\cdot), 1 - t - q(t))\), then \( U^{(\delta, +)} = U^{(\delta, -)}(-\cdot) \). Assume now that the statement regarding the upper barriers is true. Apply the upper bound \( \tilde{u} \leq U^{(\delta, +)} \).

Then for all \( r \), suppressing the time variable,
\[
u(-\infty, r) \leq U^{(\delta, -)}(-\infty, r)
\]
hence
\[
u^{(\delta, -)}[r, \infty) \leq u[r, \infty] + \|u^{(\delta, -)}\|_1 - \|u\|_1.
\]
Now, directly from (40), \( \|u^{(\delta, -)}\|_1 = 1 - n\delta + ne^{-\Delta^4/2\delta} \). By Lemma 5.2 and (35)(iv), \( \|u_{n\delta}\|_1 = 1 - n\delta \). This shows that \( u^{(\delta, -)} \leq u_{n\delta} \mod e^{-\Delta^4/2\delta} \), which gives the lower bound. It thus suffices to prove the upper bound.

The proof strategy is similar to that of Lemma 3.2, but various details differ, hence we present the proof in full.

In what follows, the superscript ‘+’ is suppressed from all notation except the upper barriers themselves that we continue to denote as \( U^{(\delta, +)} \). The parameter \( \Delta > 0 \) is assumed to satisfy \( 2c_\infty \Delta^2 < \Delta/4 \).

We argue by induction. Assume that \( f \leq g \) where \( f = u_{(n-1)\delta} \) and \( g = u_{(n-1)\delta}^{(\delta, +)} \). Write \( L \) and \( G \) for \( L_{n, \delta}^{(+)} \) and \( G_{n, \delta} \), respectively. By Lemma 5.2, \( u_{n\delta} = Gf - h, h(y) = \int_{\mathbb{R} \times \{(n-1)\delta, n\delta\}} G_{n\delta - s}(x, y)\beta(dx, ds) \).
Moreover, \( u_{n\delta}^{(\delta, +)} = LG + e_{n, \delta} \). Denoting \( w = Gf \), consider the inequality
\[
w - h \leq Lw + e_{n, \delta}.
\]
(41)
If it holds then, using $f \preceq g$ and monotonicity of $G$ and $L$ (see Lemma 5.1(ii)), it will follow that $Gf - h \preceq LGg + e_{n, \delta}$, hence $u_{n, \delta} = Gf - h \preceq LGg + e_{n, \delta}$, and the proof will be complete. It thus remains to show (41). Let $q^* = e_{n, \delta}$.

First, if $q^* + \Delta + \delta \geq \|f\|_1 = \|w\|_1$ then by the way in which the operator $L$ is defined, $Lw = \tilde{K}^\delta w$. Hence by Lemma 5.1(iii), noting that $\|h\|_1 = \delta$, we have $w - h \preceq \tilde{K}^\delta w$. This shows (41).

In what follows we address the case $q^* + \Delta + \delta < \|f\|_1$. In this case $Lw = K^{q^* + \Delta} w$. Let $r_* = R_{q^* + c_\infty, \delta^{1/2}}(f)$, which is a finite number provided $\delta$ is small. By Lemma 5.3 $\rho_*([\beta; [(n-1)\delta, n\delta]) \geq r_*$. Let $h = h_1 + h_2$, where

\[
h_1(y) = h(y)1_{\{y < r_* - \Delta^2\}}, \quad h_2(y) = h(y)1_{\{y \geq r_* - \Delta^2\}}.
\]

Again, recall $\|h\|_1 = \delta$. Using

\[
\int_{R \times [(n-1)\delta, n\delta]} G_{n\delta - s}(x, y) \beta(dx, ds) = \int_{[r_*, \infty) \times [(n-1)\delta, n\delta]} G_{n\delta - s}(x, y) \beta(dx, ds)
\]

and $G_t(0, [a, \infty)) \leq e^{-a^2/2t}$ $a > 0$, $t < a^2$, it is easy to see that

\[
\|h_1\|_1 \leq e^{-\Delta^4/2\delta} \delta, \quad (42)
\]

provided $\delta < \Delta^4$. Let $p \in (0, 1]$ be defined by $\|h_2\|_1 = p\delta$. Then $p \geq 1 - e^{-\Delta^4/2\delta}$.

In order to show (41) it will now be argued that

\[
w - h \preceq K^{q^* + \Delta} p\delta w. \quad (43)
\]

Because $h_2$ is supported to the right of $r_* - \Delta^2$ and $K^{q^* + \Delta} p\delta w = w - \tilde{h}$, where $\|\tilde{h}\|_1 = p\delta$ and $\tilde{h}$ is supported to the left of $R_{q^* + \Delta}(w)$, if one has

\[
R_{q^* + \Delta}(w) \leq R_{q^* + c_\infty, \delta^{1/2}}(f) - \Delta^2, \quad (44)
\]

then one could deduce $w - h_2 \preceq K^{q^* + \Delta} p\delta w$, hence

\[
w - h = w - h_1 - h_2 \preceq w - h_2 \preceq K^{q^* + \Delta} p\delta w,
\]

giving (43). Toward showing (44), it follows from $\|f\|_\infty < c_\infty$ that for all $\delta$ small (specifically, $c_\infty \delta^{1/2} < \Delta/4$ and recalling $2c_\infty \Delta^2 < \Delta/4$),

\[
R_{q^* + \Delta/2}(f) \leq R_{q^* + c_\infty, \delta^{1/2}}(f) - 2\Delta^2. \quad (45)
\]

Moreover, for small $\delta$,

\[
R_{q^* + \Delta}(Gf) \leq \theta := R_{q^* + \Delta/2}(f) + \Delta^2, \quad (46)
\]

as we now show. This is shown by proving that $(Gf)[\theta, \infty] \preceq q^* + \Delta$. Let $f = f_1 + f_2 = \tilde{K}^{q^* + \Delta/2} f + K^{q^* + \Delta/2} f$. Then $\|f_1\|_1 = q^* + \Delta/2$, $\|Gf_1\|_1 = q^* + \Delta/2$, and

\[
(Gf)[\theta, \infty] = (Gf_1)[\theta, \infty] + (Gf_2)[\theta, \infty] \preceq q^* + \Delta/2 + (GA)[\theta, \infty]
\]
where $A = \|f_2\|_{\delta_\times}$ with $x = R_{q^* + \Delta/2}(f)$. Because $\|f\|_1 \leq 1$, $(GA)[\theta, \infty) \leq G_\delta(0, [\Delta^2, \infty)) < \Delta/2$ for small $\delta$. This shows (46).

The bounds (45) and (46) imply (44). This establishes (43).

Next,

$$K^{q^* + \Delta, p\delta} w = K^{q^* + \Delta, \delta} w + K^{q^* + \Delta + p\delta, (1 - p)\delta} w.$$  

The last term on the RHS is nonnegative and supported to the left of $R_\Delta(w)$, with total mass $(1 - p)\delta$, hence it is $\lesssim (1 - p)\delta_1[R_\Delta(w), R_\Delta(w) + 1]$. Moreover, since $w \lesssim Gg$, one has $R_\Delta(w) \leq R_\Delta(Gg)$, hence the same term is dominated by $(1 - p)\delta_1[R_\Delta(Gg), R_\Delta(Gg) + 1]$. Using this in (43),

$$w - h \approx K^{q^* + \Delta, p\delta} w \approx K^{q^* + \Delta, \delta} w + (1 - p)\delta_1[R_\Delta(Gg), R_\Delta(Gg) + 1].$$

Since $1 - p \leq e^{-A^4/2\delta}$, the last term is $\approx \epsilon_{n, \delta}$, and (41) follows.

The proof of the following result is similar to that of Proposition 3.3, but various details differ and so the proof is presented in full.

**Proposition 5.2.** For all $\Delta$ sufficiently small and $0 < \delta < \delta_1(\Delta)$ and all $n \in \mathbb{Z}_+$ such that $n\delta < 1 - 3\Delta - e^{-A^4/2\delta}$, one has

$$u_{n\delta}^{(\delta, +)} \approx u_{n\delta}^{(\delta, -)} + \epsilon_n,$$

where $\epsilon_n = \epsilon_n(\delta) \in L_1(\mathbb{R}, \mathbb{R}_+)$ and

$$\|\epsilon_n\|_1 \leq 3\Delta + e^{-A^4/2\delta}n.$$

**Proof.** Abbreviate $u^{(\delta, \pm)}$, $u^{(\delta, \pm)}_{n, \delta}$, $L^{(\pm)}_{n, \delta}$ and $G\delta$ to $u^{(\pm)}$, $u^{(\pm)}_n$, $L^{(\pm)}$ and $G$. For $n = 0$ one has $u^{(\pm)}_0 = u_0$.

Let $n \geq 1$. The induction assumption is $u_{n-1\delta}^{(\pm)} \approx u_{(n-1)\delta}^{(\pm)} + \epsilon_{n-1} =: f_{n-1}$, $\epsilon_{n-1}$ is nonnegative, $\|\epsilon_{n-1}\|_1 \leq \Delta_{n-1} := 3\Delta + e^{-A^4/2\delta}n$. Then $u_{n\delta}^{(\pm)} = L_n^{(\pm)}Gu_{(n-1)\delta}^{(\pm)} + e_n^{(\pm)}$. Hence

$$u_{n\delta}^{(\pm)} = L_n^{(\pm)} + \Delta, \delta)Gu_{(n-1)\delta}^{(\pm)} + e_n^{(\pm)}$$

$$\approx L_n^{(\pm)} + \Delta, \delta)Gf_{n-1} + e_n^{(\pm)}$$

where one uses the monotonicity of $L$ (Lemma 5.1(ii)) and $G$. Using Lemma 5.1(v),

$$u_{n\delta}^{(\pm)} \approx L_n^{(\pm)} + \Delta - \Delta_{n-1}, \delta)K^{\Delta_{n-1}}Gf_{n-1} + \hat{K}^{\Delta_{n-1}}Gf_{n-1} + e_n^{(\pm)}.$$  

Because $q$ is continuous on $[0, 1]$, one has $\max_{\delta < \delta_1} (q_{n, \delta}^{(\pm)} - q_{n, \delta}^{(\pm)}) < \Delta$ for all small $\delta$. Hence for $\delta$ small, $q_{n, \delta}^{(\pm)} + \Delta - \Delta_{n-1} \leq q_{n, \delta}^{(\pm)} - \Delta - \delta$. Thus by Lemma 5.1(iv), denoting $\epsilon_n = \hat{K}^{\Delta_{n-1}}Gf_{n-1} + e_n^{(\pm)}$ and $z = K^{\Delta_{n-1}}Gf_{n-1},$

$$u_{n\delta}^{(\pm)} \approx L_n^{(\pm)} + \Delta - \delta, \delta)Gz + e_n^{(\pm)} + \epsilon_n.$$  

Suppose $z \approx Gu_{(n-1)\delta}^{(\pm)}$. Then $u_{n\delta}^{(\pm)} \approx u_{n\delta}^{(\pm)} + \epsilon_n$, and $\|\epsilon_n\|_1 \leq \Delta_{n-1} + \|e_n^{(\pm)}\|_1 = \Delta_n$, as claimed.

Hence it remains to show $z \approx Gu_{(n-1)\delta}^{(\pm)}$. But $w := G\varepsilon_n - 1$ is pointwise nonnegative, $\|w\|_1 \leq \Delta_n - 1$ and

$$Gu_{(n-1)\delta}^{(\pm)} = Gf_{n-1} - w.$$  

Hence by Lemma 3.2(ii), $z \approx Gu_{(n-1)\delta}^{(\pm)}$, and the proof is complete. 

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5.5 Uniqueness and convergence

Proof of Theorem 5.1.

Step 1: The uniqueness result.

It will be shown that if there exists a solution \((u, \beta)\) to (35) then it is unique (the existence, as in the case of RAB, follows from the convergence result proved afterwards). As in the proof of Theorem 2.2, if uniqueness is established for the \(u\) component, uniqueness of the \(\beta\) component follows. To show uniqueness of \(u\), argue by contradiction assuming \((u^i, \beta^i), i = 1, 2\) are two solutions with distinct \(u^1\) and \(u^2\). Note that Lemma 5.2 and (35)(iv) dictate that \(u(\cdot, t) = 0\) for \(t \geq 1\). Hence there exist \(0 < t < 1\) and \(r \in \mathbb{R}\) such that, say, \(u^1_t[r, \infty) < u^2_t[r, \infty)\). For \(n \in \mathbb{N}\) let \(\delta_n = tn^{-1}\). Then by Proposition 5.1, for every small \(\Delta > 0\) and every \(n > n_0(\Delta)\),

\[
a(n, \Delta) := u^{(\delta_n, -)}_t[r, \infty) - e^{\Delta t / 2\delta_n} \leq u^1_t[r, \infty) < u^2_t[r, \infty) \leq b(n, \Delta) := u^{(\Delta, \delta_n, +)}_t[r, \infty),
\]

whereas by Proposition 5.2,

\[
b(n, \Delta) - a(n, \Delta) \leq ||\varepsilon_n(t, \Delta, \delta_n)||_1 + e^{-\Delta t / 2\delta_n} \leq 3\Delta + 2e^{-\Delta t / 2\delta_n}.
\]

A contradiction is obtained upon sending \(n \to \infty\) then \(\Delta \downarrow 0\). This establishes uniqueness.

Step 2: The convergence result.

Given step 1, the proof of existence of a solution and the convergence assertion follow very closely the steps of the proof of Theorem 2.3, comprising §4. These steps are Lemmas 4.1, 4.2, 4.3 and 4.4, and finally the proof of Theorem 2.3. The only part of §4 where the specifics of the removal mechanism are used is the proof of Lemma 4.4 (that is not to say that the properties of the function \(J\), governing the cumulative removal count, are not used; they are used for example in Lemma 4.2. However the removal form is not used). Thus the statements of Lemmas 4.1, 4.2 and 4.3 and their proofs cover the current model as a special case. The only part of §4 that needs to be modified is Lemma 4.4. A modified statement and proof are provided below in Lemma 5.4.

Likewise, the proof of Theorem 2.3 based on uniqueness and Lemmas 4.1–4.4 is valid here, with Lemma 5.4 in place of Lemma 4.4. This completes the proof of Theorem 5.1.

Lemma 5.4. Let \((\xi, \beta)\) be a subsequential limit of \((\bar{\xi}^N, \bar{\beta}^N)\). Then a.s., for all \(r \in \mathbb{R}\) one has \(\mathcal{I}_r^\pm(\xi, \beta, q) = 0\).

Proof. We provide a proof for \(\mathcal{I}_r^+\); the one for \(\mathcal{I}_r^-\) is similar. Recalling (32), we must show

\[
\mathcal{I}_r^+(\xi, \beta, q) = \int_{\mathbb{R}^+} [\xi_t(r, \infty) - q(t)]^+ \beta((-\infty, r) \times dt) = 0, \quad \text{for all } r \in \mathbb{R}.
\]  

Let \(\Sigma\) be the collection of tuples \(\sigma = (r, t_1, t_2, \eta, \delta) \in \mathbb{Q}^5, 0 \leq t_1 < t_2, \eta > 0, \delta > 0\). It is shown that for every \(\sigma \in \Sigma, \mathbb{P}(\Omega_\sigma) = 0\) where

\[
\Omega_\sigma = \{ \inf_{t \in [t_1, t_2]} [\xi_t(r, \infty) - q(t)] > \eta, \beta((-\infty, r) \times (t_1, t_2)) > \delta \}.
\]

Fix \(\sigma\). If \(\mathbb{P}(\Omega_\sigma) > 0\) then in view of the convergence \((\bar{\xi}^N, \bar{\beta}^N) \Rightarrow (\xi, \beta)\) and the a.s. continuity of the sample paths of the limit \(\xi\), one has for all large \(N\),

\[
\mathbb{P}\left( \inf_{t \in [t_1, t_2]} [\xi^N_t(r, \infty) - q(t)] > \eta/2, \bar{\beta}^N((-\infty, r) \times (t_1, t_2)) > \delta/2 \right) > 0.
\]
However, the above probability is zero for all $N$, in view of (34). This shows $\mathbb{P}(\Omega_\sigma) = 0$, and consequently, $\mathbb{P}(\cup\Omega_\sigma) = 0$.

The remainder of the proof is as that of Lemma 4.4 (with trivial adaptations).

6 Other models

Additional models formulated via ORA are presented with no rigorous claims or proofs. The goal is twofold: to propose alternatives to additional macroscopic models that have previously been formulated via FBP (items 1 and 2), and to demonstrate that the approach gives rise to new models that are quite far from FBP (items 3 and 4).

1. Removal on both extremes. This is a straightforward extension of the basic model where both the leftmost and the rightmost particles are removed according to prescribed functions $J^L$ and $J^R$. The equation (8) is then modified as $L u = \alpha - \beta^L - \beta^R$ with $u(\cdot,0) = u_0$, $\beta^L(\mathbb{R} \times [0,t]) = J^L_t$, $\beta^R(\mathbb{R} \times [0,t]) = J^R_t$, and two ORA conditions of opposite directions,

$$\int_{\mathbb{R}^+} u((\infty,r],t)\beta^L((r,\infty) \times dt) = 0, \quad \int_{\mathbb{R}^+} u([r,\infty),t)\beta^R((\infty,r) \times dt) = 0, \quad r \in \mathbb{R}.$$

A FBP for this model (without injection, until extinction) was instrumental in [12] for the study of a hydrodynamic limit of a polymer pinning model.

2. The $N$-BBM model. As mentioned in the introduction, hydrodynamic limits of this model were identified in [7] as a solution to a FBP analyzed in [3]. An alternative macroscopic model consists of the equation $L u = u - \beta$ coupled with an initial condition, a mass conservation condition $\int_{\mathbb{R}} u(x,t)dx = 1$, and the condition

$$\int_{\mathbb{R}^+} u((\infty,r],t)\beta((r,\infty) \times dt) = 0, \quad r \in \mathbb{R}.$$ 

It is straightforward to extend this equation to the case of nonlocal branching studied in [8].

3. Injection at quantile. Similarly to removal at quantile, one can formulate an equation with injection at a specified quantile of the population. For example, consider a system of $N$ Brownian particles on the line with rate-$N$ injection and removal, injection occurring at the location of the $a$-quantile member of the population and removal is restricted to the $b$-quantile member. Here, $a \in (0,1)$ and $b \in [0,1]$ are parameters. It is of interest to relate the hydrodynamic limit to an equation, for the unknown $(u, \alpha, \beta)$, given by $L u = \alpha - \beta$ coupled with an initial condition, the condition $\alpha(\mathbb{R} \times [0,t]) = \beta(\mathbb{R} \times [0,t]) = t$, and a variant of (35)(iii),

$$\mathcal{J}_r^\pm(u, \alpha, a) = \mathcal{J}_r^\pm(u, \beta, b) = 0, \quad r \in \mathbb{R},$$

where

$$\mathcal{J}_r^+(u, \gamma, c) = \int_{\mathbb{R}^+} [u((r,\infty),t) - c]^+ \gamma((\infty,r) \times dt),$$

$$\mathcal{J}_r^-(u, \gamma, c) = \int_{\mathbb{R}^+} [u((\infty,r),t) - (1-c)]^+ \gamma((r,\infty) \times dt).$$
4. Other total orders. Equation (8) could be considered with condition (8)(iii) replaced by
\[ \int_{x:x<r,t\in\mathbb{R}_+} \int_{y:y<y} u(y,t)\,dy \beta(dx,dt) = 0, \quad r \in \mathbb{R}, \]
for a given (measurable) total order \( \prec \) on \( \mathbb{R} \). This formulation is meaningful also for the heat equation in higher dimension, in which case \( \prec \) is a given total order of \( \mathbb{R}^d \). Of course, these equations are no longer alternatives to FBP, especially in higher dimension. A serious challenge in implementing barrier techniques, even in dimension 1, is that the monotonicity properties of the heat kernel extensively used in the proofs fail to hold w.r.t. other orders.

Acknowledgment The author would like to thank Kavita Ramanan for valuable discussions and comments. This research was supported in part by the Israel Science Foundation grant 1035/20.

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