A unified theory of gravity and electromagnetism: Classical and quantum aspects

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MS received 28 August 2018; revised 28 May 2019; accepted 3 June 2019; published online 23 August 2019

Abstract. A unified classical theory of gravity and electromagnetism with a torsion vector $\Gamma_\ell \neq 0$, proposed by S N Bose in 1952, is introduced. In this theory, the torsion vector acts as a magnetic current and it is shown that (i) the electromagnetism is invariant under continuous Heaviside–Larmor transformations and (ii) the electric and magnetic charges are topologically quantised, satisfying the Dirac quantisation condition, without implying any Dirac string provided $\Gamma_\ell$ is curl-less.

Keywords. Unified theory; topological charges.

PACS No. 04.20.Cv

1. Introduction

Einstein’s famous equation

$$R_{ik} - \frac{1}{2} g_{ik} R = \kappa T_{ik} \quad (1)$$

of general relativity (GR) has been extremely successful in explaining and predicting various weak-field phenomena such as the precession of the perihelion of mercury, the bending of light by stars, the Shapiro delay time and the frame-dragging precession of gyroscopes measured by the gravity probe B experiment [1]. In the strong field sector, the observation of gravitational waves originating in the merger of binary black holes [2] has also come as a reassurance of the general correctness of the theory.

The successful weak-field predictions are solutions of the field equations $R_{ik} = 0$, i.e. with $T_{ik} = 0$ ‘outside’ a spherically symmetric body having mass and angular momentum, such as the Schwarzschild and Kerr metrics [3]. Hence, $R_{ik} = 0$ are not necessarily ‘vacuum equations’ in the sense of a completely empty Universe. On the other hand, attempts to use solutions of eq. (1) with $T_{ik} \neq 0$ in Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmology have led to many intractable problems such as the hypothetical non-baryonic dark matter [4], dark energy and the cosmological constant problem [5], the horizon problem [6] and the flatness problem [7] which show no signs of going away.

Einstein himself was very unhappy with the role that the stress–energy tensor $T_{ik}$ played in GR. He had repeatedly emphasised that it was only a phenomenological representation of matter, to be regarded with caution. In 1936 he wrote [8]:

“[General Relativity] is sufficient – as far as we know – for the representation of the observed facts of celestial mechanics. But it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low-grade wood (right side of the equation). The phenomenological representation of matter is, in fact, only a crude substitute for a representation which would do justice to all known properties of matter”.

Also, in a letter to Michele Besso [9] he wrote:

“But it is questionable whether the equation $R_{ik} – (1/2) g_{ik} R = T_{ik}$ has any reality left within it in the face of quanta. I vigorously doubt it. In contrast, the left-hand side of the equation surely contains a deeper truth. If the equation $R_{ik} = 0$ really determines the behavior of the singularities, then a law describing this behavior would be justified far more deeply than the aforementioned equation, which is not unified and only phenomenologically justified”.

Dedicated to the memory of Prof. S N Bose on his 125th birth anniversary.
These quotes show that the stress–energy tensor was unsatisfactory to Einstein for two reasons. First, it is not geometrical in nature like the left side of eq. (1) and hence not unified with it, and secondly, it does not reflect the quantum nature of matter and radiation. It is merely phenomenological, a placeholder for a more satisfactory theory of matter. This is why later on Einstein preferred the quantum nature of matter and radiation. It is mere not hence not unified with it, and secondly, it does not reflect equations. The purpose of this paper is to consider one

rent as a purely geometric aspect of the nonlinear field

cannot determine the $\Gamma$-field completely but only

2. A maximally symmetric unified theory with $T_{ik} = 0$

Let $U_4$ be a smooth manifold with signature $(-, +, +, +)$ and endowed with a non-symmetric affine connection $\Gamma$ and a non-symmetric metric $g$. Let

$$R_{ik} = \Gamma^\alpha_{ik,\alpha} - \Gamma^\alpha_{ia,k} + \Gamma^f_{ik} \Gamma^\lambda_{f\xi} - \Gamma^f_{ik} \Gamma^\lambda_{\xi k}$$

be the non-symmetric curvature tensor. Let us also define

$$\Gamma^\lambda_{(ik)} = \frac{1}{2} (\Gamma^\lambda_{ik} + \Gamma^\lambda_{ki}),$$

$$\Gamma^\lambda_{[ik]} = \frac{1}{2} (\Gamma^\lambda_{ik} - \Gamma^\lambda_{ki}),$$

$$\Gamma_{l} = \frac{1}{2} (\Gamma^\lambda_{il} - \Gamma^\lambda_{li}).$$

$\Gamma^\lambda_{[ik]}$ is called the Cartan torsion tensor. Following Einstein [16], let us put $\Gamma_{l} = 0$ because it is not determined by any equation in the theory. Similarly, let

$$\tilde{g}^{(ik)} = \frac{1}{2} \sqrt{-g} (g^{ik} + g^{ki}),$$

$$\tilde{g}^{[ik]} = \frac{1}{2} \sqrt{-g} (g^{ik} - g^{ki}).$$

To restrict the number of possible covariant terms in a non-symmetric theory, Einstein [16] imposed ‘transposition invariance’ and ‘$\lambda$-transformation invariance’ on the theory. Let $\tilde{\Gamma}^\lambda_{ik} = \Gamma^\lambda_{ik}$ and $\tilde{g}_{ik} = g_{ki}$. Then terms that are invariant under the simultaneous replacements of $\Gamma^\lambda_{ik}$ and $g_{ik}$ by $\tilde{\Gamma}^\lambda_{ik}$ and $\tilde{g}_{ik}$, respectively, are called transposition invariants. For example, the tensor $R_{ik}$ (2) is not transposition invariant because it is transposed to

$$\tilde{R}_{ik} = \Gamma^\alpha_{kl,\alpha} - \Gamma^\alpha_{ai,k} + \Gamma^f_{kl} \Gamma^\lambda_{f\xi} - \Gamma^f_{kl} \Gamma^\lambda_{\xi k}.$$

Next, define the transformations

$$\Gamma^i_{kl} = \Gamma^i_{kl} + \delta^i_k \lambda_j,$$

$$g^{ik} = \tilde{g}^{ik},$$

where $\lambda$ is an arbitrary function of the coordinates. Then $R_{ik}$ (eq. (2)) is $\lambda$-transformation invariant (or projective invariant). What this means is that a theory characterised by $R_{ik}$ cannot determine the $\Gamma$-field completely but only
up to an arbitrary function \( \lambda \). Hence, in such a theory, \( \Gamma \) and \( \Gamma' \) represent the same field. Further, this ‘\( \lambda \)-transformation’ produces a non-symmetric \( \Gamma' \) from a \( \Gamma \) that is symmetric or antisymmetric in the lower indices. Hence, the symmetry condition for \( \Gamma \) loses its objective significance. This sets the ground for a genuine unification of gravity and electromagnetism, the former determined by the symmetric part and the latter by the antisymmetric part of the action.

Let us write the simplest transposition invariant and \( \lambda \)-transformation invariant Lagrangian [17]:

\[
\mathcal{L} = \frac{1}{2} \sqrt{-g} (g^{ik} R_{ik} + \tilde{g}^{ik} \tilde{R}_{ik}),
\]

which can be expressed as (see Appendix A)

\[
\mathcal{L} = \tilde{g}^{(ik)} \left( R_{(ik)} - \Gamma^{\lambda}_{[i\xi]} \Gamma_{\lambda}^{\xi} \right) + \tilde{g}^{[ik]} \Gamma^{\lambda}_{[i\lambda];\lambda} := \tilde{g}^{(ik)} R'_{(ik)} + \tilde{g}^{[ik]} \Gamma^{\lambda}_{[i\lambda];\lambda}.
\]

The variation of the action \( \int \mathcal{L} d^4x \) holding \( \tilde{g}^{(ik)} \) and \( \tilde{g}^{[ik]} \) constant at once implies

\[
R'_{(ik)} = R_{(ik)} - \Gamma^{\lambda}_{[i\xi]} \Gamma_{\lambda}^{\xi} = 0,
\]

\[
\Gamma^{\lambda}_{[i\lambda];\lambda} = 0.
\]

It can also be shown (see Appendix A) using the variational principle that the equation connecting \( g \) and \( \Gamma \) in this non-symmetric theory is

\[
\tilde{g}^{ik} \Gamma^{\alpha}_{\lambda} \Gamma^{\rho}_{\lambda} \Gamma^{\tau}_{\lambda} + \tilde{g}^{[ik]} \Gamma^{\alpha}_{\lambda} \Gamma^{\rho}_{\lambda} \Gamma^{\tau}_{\lambda} = 0,
\]

where

\[
\Gamma^{\alpha}_{\lambda} = \Gamma^{\alpha}_{\rho\lambda},
\]

\[
\Gamma^{\alpha}_{\rho\lambda} = \Gamma^{\alpha}_{\lambda},
\]

and further that

\[
\tilde{g}^{[\alpha\lambda]} = 0.
\]

The last equation can be interpreted as Maxwell’s equations for electrodynamics by identifying \( \tilde{g}^{[ik]} \) with the dual electromagnetic field \( \tilde{F}^{ik} \). The electric current is given by

\[
j^i = \frac{1}{3!} \epsilon^{iv\lambda\rho} \left( \tilde{g}_{[v\lambda],\rho} + \tilde{g}_{[\lambda\rho],v} + \tilde{g}_{[\rho v],\lambda} \right).
\]

\[
= 0.
\]

The current vanishes because of the Bianchi identity in the first line. Hence, this theory describes the free electromagnetic fields with the same strength as gravity.

The equation sets (13)–(16) are the fundamental equations of the theory.

Note that the Ricci tensor \( R'_{(ik)} \) in the theory, which is flat, has an additional term compared to the GR Ricci tensor \( R_{(ik)} \). Clearly, the additional curvature has its origin in the torsion in the manifold \( U_4 \).

In this context, the scepticism expressed by Pauli concerning a unified theory of this kind in his classic book ‘Theory of relativity’ [18] is worth recalling:

“Whether the field equations of this theory, which are based on the formal postulates of \( \lambda \)-invariance and of transposition invariance without any obvious geometrical and physical meaning, can actually be connected with physics at all, is rather doubtful.”

We shall see in the next section how the connection to physics can be established by breaking one of the symmetries, namely \( \lambda \)-invariance.

3. A non-maximally unified theory with \( T_{ik} = 0, \Gamma_i \neq 0 \)

In this section we shall relax the Einstein condition \( \Gamma_i = 0 \) and consider a more general form of unified theory proposed by Bose [17] in 1952. Bose’s Lagrangian

\[
\mathcal{L} = \sqrt{-g} \left( [g^{ik} R_{ik} + \tilde{g}^{ik} \tilde{R}_{ik}] + a \tilde{g}^{(ik)} \Gamma_{i\lambda} + b \tilde{g}^{[ik]} \Gamma_{[i\lambda]} \right)
\]

has two additional terms to the maximally symmetric Lagrangian (10) with \( a \) and \( b \) being two arbitrary dimensionless parameters. Projective or \( \lambda \)-invariance would require \( a \) to vanish but not \( b \). Since \( a \neq 0 \) in Bose’s theory, it is only transposition invariant. Equations (13) and (14) are modified to

\[
R_{ik} = R_{(ik)} - Q_{[i\xi]}^{\lambda} \Gamma_{\lambda}^{\xi} + x \Gamma_i \Gamma_k
\]

\[
= R_{(ik)} - \Gamma_{[i\xi]}^{\lambda} \Gamma_{\lambda}^{\xi} + a \Gamma_i \Gamma_k = 0,
\]

\[
Q_{[i\xi]}^{\lambda} \lambda = y (\Gamma_i \Gamma_k - \Gamma_k \Gamma_i) = 0,
\]

where

\[
Q_{[i\xi]}^{\lambda} \lambda = \frac{1}{3} \delta_k^{\xi} \Gamma_k - \frac{1}{3} \delta_k^{\xi} \Gamma_i,
\]

\[
Q_{[i\xi]}^{\lambda} \lambda = Q_{[i\xi]}^{\lambda} \lambda - Q_{[i\xi]}^{\lambda} \lambda + Q_{[i\xi]}^{\lambda} \lambda + Q_{[i\xi]}^{\lambda} \lambda
\]

and \( x = a + (1/3), y = (1/6) - b \). Hence, the new Ricci tensor \( R_{ik} \) is also flat though the Universe has other fields than gravity, and is hence not empty. Note that one can also write eq. (19) in the form

\[
R_{(ik)} = \left( \Gamma_{[i\xi]}^{\lambda} \Gamma_{\lambda}^{\xi} - a \Gamma_i \Gamma_k \right) := \kappa \left( T_{ik} - \frac{1}{2} g_{(ik)} T \right),
\]
where \( \kappa = 8\pi G/c^4 \) is the Einstein constant, which shows that \( \kappa^{-1}(\Gamma_{ij}^\xi \Gamma_{\xi k}^{j,i} - a \Gamma_i \Gamma_k) \) can be interpreted as a traceless stress tensor. As will be seen presently, the torsion vector \( \Gamma_i \) is basically the source current of the dual electromagnetic field.

Equation (19) will lead to corrections to the Schwarzschild and Kerr metrics that are analogous to the Reissner–Nordström and Kerr–Newman metrics.

The variational principle is a little more complex because \( Q_{i\kappa}^\lambda = 0 \) and all the 24 components of \( Q_{i\kappa}^\lambda \) are not independent, and consequently one has to use a set of Lagrange multipliers \( k^i \) [17]. As shown in Appendix B the equations connecting the \( g \)'s and \( \Gamma \)'s in this case are of the form

\[
g^{i\kappa} \partial_{\kappa} \Gamma^i_{\alpha\lambda} + g^{\alpha k} \Gamma^i_{\alpha\lambda} = 3 g^{i\kappa} \Phi_\kappa, \tag{24}
\]

\[
\Phi_\kappa = \frac{1}{g(\lambda\beta)k^\beta}, \tag{25}
\]

where

\[
\Gamma_{\alpha\lambda} = \Gamma_{(\alpha\lambda)} + \frac{1}{\sqrt{-g}} \left( g_{\alpha\beta} k^\beta g^{\kappa\lambda} - g_{\alpha\kappa} k^\lambda g^{\beta\lambda} \right),
\]

\[
k^\beta = -\frac{1}{2} \left( \frac{x}{3y} \right) \sqrt{-g} g^{(\alpha\beta)} \Gamma_\alpha.
\]

Equation (16) is modified to (see Appendix B)

\[
g^{[i\alpha]} = 6k^i = 3\theta g^{(i)j} \Gamma_j, \quad \theta = -\frac{x}{3y}, \quad y \neq 0. \tag{27}
\]

Thus, \( \Gamma^i \) turns out to be the source of dual electromagnetic field unless \( x = 0 \), i.e. \( a = -1/3 \) and \( k^i = 0 \). If \( \Gamma_i = k_i = 0 \), one gets back the maximally unified theory. By multiplying \( \Gamma_i \) by the dimensional parameter \( \zeta = \sqrt{\mu_0/\kappa} \), where \( \mu_0 \) is free space permeability, we get

\[
\bar{F}_{i\alpha} = j_{im}, \tag{28}
\]

where \( j_{im} = 3\theta \zeta \Gamma^i \) is the magnetic source current which is automatically conserved because \( \bar{F}_{i\alpha} = -\bar{F}^{ai} \). Defining \( F_{ki} = \epsilon_{kli\alpha} \bar{F}_{i\alpha} \), we have

\[
\partial^k F_{ki} = \epsilon_{kli\alpha} \bar{F}_{i\alpha} = j_l, \tag{29}
\]

where \( j_l \) is the electric source current which is also automatically conserved. Hence, using the definitions \( \alpha G = Gm_p^2/hc \) and \( \alpha = e^2/hc \), we get \( \alpha/\alpha G = e^2/m_p^2 = 10^{-6} \) in Planck units in which \( G = c = \hbar = 4\pi \epsilon_0 = 1 \).

Equations (28) and (29) constitute the complete set of Maxwell equations in the presence of the two geometric source currents:

\[
\bar{\nabla} \times \bar{B} = \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} = \bar{j}, \quad \bar{\nabla} \cdot \bar{E} = \rho_e, \tag{30}
\]

\[
\bar{\nabla} \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = -\bar{j}_m, \quad \bar{\nabla} \cdot \bar{B} = \rho_m. \tag{31}
\]

The two Bianchi identities that must be satisfied are

\[
F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0, \tag{32}
\]

\[
\bar{F}_{\mu\nu,\lambda} + \bar{F}_{\nu\lambda,\mu} + \bar{F}_{\lambda\mu,\nu} = 0. \tag{33}
\]

These identities are consistent with the inhomogeneous Maxwell equations provided one makes the following identifications:

\[
F^{0i} = -(E^i - E'^i), \quad F^{ij} = e^{ijk}(B_k - B'_k), \tag{34}
\]

\[
\bar{F}^{0i} = (B^i - B'^i), \quad \bar{F}^{ij} = e^{ijk}(E_k - E'_k), \tag{35}
\]

where \( \bar{E}', \bar{B}' \) are auxiliary fields which also satisfy the inhomogeneous Maxwell equations:

\[
\bar{\nabla} \cdot \bar{E}' = \rho_e, \quad \bar{\nabla} \times \bar{E}' + \frac{\partial \bar{B}'}{\partial t} = -\bar{j}_m, \tag{36}
\]

\[
\bar{\nabla} \cdot \bar{B}' = \rho_m, \quad \bar{\nabla} \times \bar{B}' - \frac{1}{c^2} \frac{\partial \bar{E}'}{\partial t} = \bar{j}, \tag{37}
\]

so that \( F^{0i} \) and \( \bar{F}^{0i} \) satisfy the free Maxwell equations. One can define potentials \( A^\mu = (\phi/c, \bar{A}), B^\mu = (\phi/c, \bar{A}) \) and \( \bar{A}^\mu = (\Phi/c, \bar{A}) \), \( \bar{B}^\mu \) through the relations

\[
\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla} \phi, \tag{38}
\]

\[
\bar{E}' = -\frac{\partial \bar{A}'}{\partial t} - \bar{\nabla} \phi', \tag{39}
\]

\[
\bar{B} = \bar{\nabla} \times \bar{A} + \bar{B}_m, \quad \bar{B}_m = -\bar{\nabla} \phi, \tag{40}
\]

\[
\bar{B}' = \bar{\nabla} \times \bar{A}' + \bar{B}_m', \quad \bar{B}_m' = -\bar{\nabla} \phi' \tag{41}
\]

with the conditions

\[
\square \bar{A} = \nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\bar{j}, \tag{42}
\]

\[
\partial_\mu \bar{A}^\mu = \bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0, \tag{43}
\]

\[
\square \bar{A}' = \nabla^2 \bar{A}' - \frac{1}{c^2} \frac{\partial^2 \bar{A}'}{\partial t^2} = -\bar{j}, \tag{44}
\]

\[
\partial_\mu \bar{A}^\mu = \bar{\nabla} \cdot \bar{A}' + \frac{1}{c^2} \frac{\partial \phi'}{\partial t} = 0, \tag{45}
\]

\[
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\rho_e, \tag{46}
\]

\[
\nabla^2 \phi' - \frac{1}{c^2} \frac{\partial^2 \phi'}{\partial t^2} = -\rho_e. \tag{47}
\]
\partial \tilde{B}_m / \partial t = \partial \tilde{B}_m' / \partial t = - \tilde{j}_m, \quad (45)

so that both sets of Maxwell equations (30), (31), (36) and (37) are satisfied. Thus, \( \tilde{A}' = \tilde{A} + \tilde{K} \) and \( \phi' = \phi + \kappa \), where \( \tilde{K} \) is a constant vector and \( \kappa \) a constant scalar.

The fields are invariant under the gauge transformations \( \tilde{A} \rightarrow \tilde{A} + \tilde{\nabla} \chi, \phi \rightarrow \phi - \partial \chi / \partial t \), where \( \chi \) is any twice differentiable function of \((\tilde{x}, t)\) that satisfies the equation \( \Box \chi = 0 \).

Notice that eq. (31) implies that the magnetic field \( \tilde{B} \) can no longer be just \( \tilde{\nabla} \times \tilde{A} \) as in standard electrodynamics – there must be an additional curl-free field \( \tilde{B}_m \) as in (40) satisfying condition (45). This is in accordance with the Helmholtz decomposition theorem which states that any sufficiently smooth vector field falling off to zero at infinity can be uniquely written as the sum of two vector fields, one divergence-free (solenoidal) and the other curl-free (irrotational). Consequently, the curl-free field \( \tilde{B}_m \) can be ignored only at the cost of introducing a singularity, as shown by Dirac [19]. Dirac’s famous solution is

\[ \tilde{A} = q_m (1 - \cos \theta) \hat{\phi}, \quad \theta \neq \pi \]

in spherical polar coordinates. This has a line of singularity along the negative \( z \)-axis characterised by \( \theta = \pi \). This is the Dirac string. Such a singular solution becomes unnecessary in the presence of the curl-free field \( \tilde{B}_m \). Using eq. (45) one can write

\[ \tilde{B}_m = - \int_{\tilde{x} = \text{const}} \tilde{j}_m \, dt + \tilde{B}_0, \]

\[ \tilde{B}_0 = \text{constant vector}, \quad (47) \]

so that on using the continuity equation for \( \tilde{j}_m \) one has

\[ \tilde{\nabla} \cdot \tilde{B} = \tilde{\nabla} \cdot \tilde{B}_m = \rho_m. \]

This solution is different from the ones proposed by Wu and Yang [20] based on fibre bundle theory and by Cabibbo and Ferrari [21] based on a double potential. It is interesting to note that in all these solutions a single potential describing the magnetic field of a monopole throughout all space (as used by Dirac) is replaced by two potentials.

The presence of \( \Gamma_i \) makes the electrodynamics invariant under continuous transformations:

\[ \tilde{E} \rightarrow \tilde{E}' = \tilde{E} \cos \theta - \tilde{B} \sin \theta, \]

\[ \tilde{B} \rightarrow \tilde{B}' = \tilde{E} \sin \theta + \tilde{B} \cos \theta, \]

where \( 0 \leq \theta \leq \pi / 2 \). Hence,

\[ \tilde{j}' = \tilde{j} \cos \theta - \tilde{j}_m \sin \theta, \quad (51) \]

\[ \tilde{j}_m' = \tilde{j} \sin \theta + \tilde{j}_m \cos \theta, \quad (52) \]

\[ \rho_e' = \rho_e \cos \theta - \rho_m \sin \theta, \quad (53) \]

\[ \rho_m' = \rho_e \sin \theta + \rho_m \cos \theta. \quad (54) \]

For \( \theta = \pi / 2 \) one has \( \tilde{E} \rightarrow - \tilde{B}, \tilde{B} \rightarrow \tilde{E}, (\rho_e, \tilde{j}) \rightarrow (-\rho_m, -\tilde{j}_m), (\rho_m, \tilde{j}_m) \rightarrow (\rho_e, \tilde{j}) \) [22,23]. This shows that there is complete equivalence and continuous freedom in the choice of electric and magnetic quantities.

### 3.1 Quantisation

Note that \( (\rho_m, \rho_e) \) are time components of the corresponding four currents which are determined in terms of continuous fields (see eqs (17) and (27)). Also, note that no condition has been imposed so far on \( \Gamma_i \). Instead of imposing the Einstein condition \( \Gamma_i = 0 \) if one imposes the weaker condition

\[ \Gamma_{i,k} - \Gamma_{k,i} = 0, \quad (55) \]

but \( b \neq 0 \), then like \( R_{ik} = 0, Q'_{[ik];\lambda} = 0 \), and one immediately gets a very interesting result, namely that \( \Gamma_i \), and hence the magnetic source current \( j_{mi} = \zeta \Gamma_i \) is an ‘irrotational’ or curl-less axial vector. Let \( S = \mathbb{R}^3 \setminus \{(0,0,z \leq 0) | z \in \mathbb{R} \} \) be the usual three-dimensional space with the negative \( z \)-axis, along which \( \tilde{\Gamma} \neq 0 \) removed. Then the curl-less vector \( \tilde{\Gamma} = - \tilde{\nabla} \Phi, \nabla^2 \Phi = 0 \) has vortex solutions \( \tilde{\Gamma} = \tilde{\varepsilon}_\phi / r \) where \( \tilde{\varepsilon}_\phi \) is a unit vector and the integral over a unit counterclockwise circular path \( C \) in the \( xy \) plane enclosing the origin is

\[ \frac{1}{2\pi} \oint_C \tilde{\varepsilon}_\phi \, d \phi = n, \quad n \in \mathbb{Z}, \quad (56) \]

where \( n \) is a winding number which can be interpreted as the number of magnetic charges enclosed by the unit circle. For \( n = 1 \) one has a single magnetic charge, i.e. a magnetic monopole carrying some charge \( g \). It is a straightforward consequence of condition (55). Because of the Larmor–Heaviside symmetry, there is also an electric monopole, i.e. a particle carrying electrical charge \( e \). Thus, particles emerge as topological charges. The product \( eg \) has the dimension of action and the fundamental unit of action in nature being the Planck constant \( h \), it follows that \( eg / h = \text{constant} \), which is essentially the topological basis of Dirac quantisation.
4. Concluding remarks

A remarkable feature of Bose’s unified theory with \( T_{ik} = 0, \Gamma_i \neq 0 \) is the occurrence of topological charges satisfying the Dirac quantisation condition without implying any Dirac string, provided \( \Gamma_j \) is curl-less. Thus, Einstein’s dream of deriving quanta from a unified geometric theory of continuous fields is at least partially realised. Magnetic monopoles are also predicted by other unified gauge theories [24,25], and hence the failure so far to detect them remains a deep puzzle in physics.

It must be emphasised that if \( a = b = 0 \) as in the theory with maximal symmetry, none of the above results can be derived.

Bose proposed his theory in 1952. He worked out many of the mathematical results given in this paper (eqs (18)–(27) and Appendix B). I have endeavoured to derive some new and interesting consequences of his theory, namely the two Bianchi identities, the presence of magnetic charges and currents without Dirac strings, the complete equivalence and freedom in the choice of electric and magnetic quantities in the theory, gauge invariance and, most strikingly, the topological quantisation of charge as a consequence of adding condition (55) on \( \Gamma_i \) not considered by Bose. This type of quantisation is not possible in a theory with \( \Gamma_i = 0 \). The presence of the magnetic current \( \xi \theta \Gamma_i \) also provides a natural classical theory of magnetism in the Universe originating in a non-zero torsion vector. Magnetism is ubiquitous in the Universe, and primordial magnetic fields are specially important for probing the physics of the early Universe [26].

Acknowledgements

The author wishes to acknowledge very helpful comments by one of the referees which have led to some revisions in the presentation, particularly relating to symmetry breaking and Dirac strings. He is deeply indebted to Prof. S N Bose for his suggestion to look at some of his old papers to see if they have any modern relevance. The present work is a result of that search. The author also is thankful to the National Academy of Sciences, India, for a grant.

Appendix A

The straightforward algebra gives

\[
\mathcal{L} = \frac{1}{2} \sqrt{-g} \left( g^{ik} R_{ik} + \tilde{g}^{ik} \tilde{R}_{ik} \right) = \left\{ \tilde{g}^{(ik)}(R_{ik} + \tilde{R}_{ki}) + \tilde{g}^{ik}[(R_{ik} - \tilde{R}_{ki})] \right\}
\]

Thus, \( H \) is free of the partial derivatives of \( \Gamma^\lambda_{(ik)} \) and \( \Gamma^\lambda_{[ik]} \), and the four-divergence term in the action integral is equal to a surface integral at infinity on which all arbitrary variations are taken to vanish.

The variations of \( H \) with respect to \( \Gamma^\lambda_{(ik)} \) and \( \Gamma^\lambda_{[ik]} \) give

\[
\tilde{g}^{ik} + \tilde{g}^{ia} \Gamma^k_{(i)} - \tilde{g}^{ak} \Gamma^i_{(i)} = \tilde{g}^{ik} \Gamma^\alpha_{(i)} - \tilde{g}^{ik} \Gamma^\alpha_{(i)}
\]

On adding (A3) and (A4), we get

\[
\tilde{g}^{ik} \Gamma^\alpha_{(i)} = 0.
\]
By contracting (A6) once with respect to \((k, \lambda)\), then with respect to \((i, \lambda)\), and subtracting the equations term by term, one gets eq. (16).

Appendix B

In this appendix we shall use the Greek symbols \(\mu, \nu\) instead of \(i, k\). Define

\[
s^{\mu\nu} = \frac{1}{2} \sqrt{-g} (g^{\mu\nu} + g^{\nu\mu}) = \frac{1}{2} \left(g^{\mu\nu} + \bar{g}^{\mu\nu}\right)
\]

\[
a^{\mu\nu} = \frac{1}{2} \sqrt{-g} (g^{\mu\nu} - g^{\nu\mu}) = \frac{1}{2} \left(g^{\mu\nu} - \bar{g}^{\mu\nu}\right)
\]

The equations of connection in the broken symmetric theory are obtained by writing

\[
\mathcal{L} = H + \frac{dx^\lambda}{dx^\mu} f_{\mu}(x)
\]

with

\[
X^\lambda = \lambda^{\mu\nu} \Gamma^{\lambda}_{(\mu\nu)} - \lambda^{\mu\nu} \Gamma^\nu_{(\mu\nu)} + a^{\mu\nu} Q_{\mu\nu}^\lambda + \frac{2}{3} \lambda^{\mu\nu} \Gamma^\nu_{\mu} + \Gamma^\lambda,
\]

\[
H = -s^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)} + s^{\mu\nu} \lambda^{\mu\nu} + \frac{2}{3} s^{\mu\nu} \lambda^{\mu\nu} \Gamma^\nu_{\mu} + s^{\mu\nu}(\Gamma^{\xi}_{(\mu\nu)} + \xi^{\xi}_{(\mu\nu)} - \xi^{\xi}_{(\mu\nu)})
\]

\[
+ s^{\mu\nu}(Q_{\mu\nu}^\lambda - \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)} - \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)})
\]

\[
+ s^{\mu\nu}(-Q_{\mu\nu}^\lambda + \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)} - \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)})
\]

\[
+ a^{\mu\nu}(\Gamma^{\xi}_{(\mu\nu)} + \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)} + \lambda^{\mu\nu} \lambda^{\lambda}_{(\mu\nu)})
\]

\[
- k^{\mu} \lambda^{\mu\nu} \lambda^\lambda_{\mu\nu},
\]

\[
Q_{\mu\nu}^\lambda = \gamma^{\mu\lambda}_{(\mu\nu)} + \frac{1}{3} \delta^{\lambda}_{\mu} \Gamma^\nu_{\mu} + \frac{1}{3} \delta^{\lambda}_{\nu} \Gamma^\mu_{\nu}.
\]

Note that

\[
\delta \int \frac{dx^\lambda}{dx^\mu} d^4x = \delta \int x_{\lambda} d\sigma^\lambda = 0.
\]

Hence, only variations of \(H\) will contribute. It is easy to see that variations of the function

\[
H - 2k^{\mu} Q_{\mu\lambda}^\lambda,
\]

where \(k^{\mu}\) is a four-vector Lagrange multiplier, with respect to \(\Gamma^\lambda_{(\mu\nu)}, Q_{\mu\nu}^\lambda\) and \(\Gamma^\mu_{\nu}\) give, respectively, the three equations

\[
s^{\mu\nu} + s^{\mu\alpha} \gamma^\nu_{(\mu\alpha)} + s^{\alpha\nu} \gamma^\mu_{(\alpha\nu)} - s^{\mu\nu} \gamma^\alpha_{(\lambda\alpha)}
\]

\[
= -a^{\mu\alpha} Q_{\lambda\alpha}^\nu + a^{\nu\alpha} Q_{\alpha\nu}^\mu.,
\]

\[
a^{\mu\nu} + a^{\mu\alpha} \gamma^\nu_{(\mu\alpha)} + a^{\nu\alpha} \gamma^\mu_{(\alpha\nu)} - a^{\mu\nu} \gamma^\alpha_{(\lambda\alpha)}
\]

\[
- k^{\mu} \delta^\nu_{\lambda} + k^{\nu} \delta^\mu_{\lambda} = -\left[s^{\mu\alpha} Q_{\lambda\alpha}^\nu + s^{\nu\alpha} Q_{\alpha\nu}^\mu\right]
\]

and

\[
y a^{\mu\nu} + x^\nu \gamma^\mu_{\nu} = 0.
\]

It follows from the first two equations that

\[
s^{\mu\alpha} + s^{\alpha\beta} \gamma_{(\alpha\beta)} + a^{\alpha\beta} Q_{\alpha\beta}^\mu = 0,
\]

\[
a^{\mu\nu} = 3k^{\mu}.
\]

Hence, it follows from eq. (B9) and the last equation that

\[
k^{\mu} = \theta s^{\mu\nu} \gamma_{\nu}, \quad \theta = -\frac{x}{3y},
\]

\[
k^{\mu} = 0.
\]

Equations (B11) and (B12) result in eq. (27) in §3.

Adding (B7) and (B8), we get

\[
\tilde{g}^{\mu\nu} + \tilde{g}^{\mu\alpha} (\gamma^\nu_{(\alpha\nu)} + Q_{\alpha\nu}^\mu) + \tilde{g}^{\alpha\nu} (\gamma^\mu_{(\alpha\nu)} + Q_{\alpha\nu}^\mu) - \tilde{g}^{\mu\nu} \gamma^\alpha_{(\lambda\alpha)} = k^{\mu} \delta^\nu_{\lambda} - k^{\nu} \delta^\mu_{\lambda},
\]

where \(\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}\). Multiplying (B14) by \(\tilde{g}_{\mu\nu}\) and using the results

\[
\tilde{g}^{\mu\nu} \gamma_{\nu\lambda} = \delta^\nu_{\lambda}, \quad \tilde{g}^{\mu\nu} \gamma_{\nu\lambda} = \delta^\mu_{\lambda}, \quad \gamma^\lambda_{\nu\lambda} = 0,
\]

we first observe that

\[
\gamma^\alpha_{(\lambda\alpha)} = \frac{|g|_{\cdot,\lambda}}{2\sqrt{-g}} + \frac{1}{2} (\tilde{g}_{\lambda\alpha} - \tilde{g}_{\alpha\lambda}) k^{\beta}
\]

\[
\equiv \frac{|g|_{\cdot,\lambda}}{2\sqrt{-g}} + \tilde{g}_{\cdot,\beta} k^{\beta}.
\]

Hence, dividing (B14) by \(\sqrt{-g}\), and also using (B16) and the results

\[
g^{\mu\alpha} g_{\beta\alpha} k^{\beta} = k^{\mu} \quad \text{and} \quad g^{\alpha\nu} g_{\alpha\nu} k^{\beta} = k^{\nu},
\]

we get

\[
g^{\mu\nu} g_{\lambda\nu} \gamma_{\mu\lambda} k^{\beta} = k^{\mu} \quad \text{and} \quad g^{\alpha\nu} g_{\lambda\alpha} k^{\beta} = k^{\nu},
\]

\[
ge^{\mu\nu} \gamma_{(\lambda\mu)\nu} + \gamma_{(\lambda\mu)\nu} + \frac{1}{2\sqrt{-g}} (g_{\lambda\beta} k^{\beta} \delta^\nu_{\nu} - g_{\beta\lambda} k^{\beta} \delta^\nu_{\nu}) + \frac{1}{2\sqrt{-g}} (g_{\beta\alpha} k^{\beta} \delta^\mu_{\nu} - g_{\alpha\beta} k^{\beta} \delta^\mu_{\nu})
\]

\[
= 3 g^{\mu\nu} g_{(\lambda\beta)} k^{\beta}.
\]

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Now, define the new affine coefficients
\[
\Gamma'^{\nu}_{\lambda\alpha} = \left( \Gamma^\nu_{(\lambda\alpha)} + Q^\nu_{\lambda\alpha} + \frac{1}{\sqrt{-g}} (g_{\lambda\beta}k^\beta \delta^\nu_\alpha - g_{\lambda\alpha}k^\beta \delta^\nu_\lambda) \right),
\]
\[
\Gamma'^{\mu}_{\alpha\lambda} = \left( \Gamma^\mu_{(\alpha\lambda)} + Q^\mu_{\alpha\lambda} + \frac{1}{\sqrt{-g}} (g_{\alpha\beta}k^\beta \delta^\mu_\lambda - g_{\alpha\lambda}k^\beta \delta^\mu_\alpha) \right)
\]
\[\text{B19}\]
\[\text{B20}\]
and
\[
\Phi_\lambda = \frac{1}{2} g_{[\lambda\beta]} k^\beta,
\]
\[\text{B21}\]
which is eq. (25) in §3. Then, eq. (B18) can be written in the form
\[
g_{\lambda\alpha} + g_{\mu\alpha} \Gamma'^{\nu}_{\lambda\alpha} + g_{\nu\alpha} \Gamma'^{\mu}_{\alpha\lambda} = 3g_{\lambda\nu} \Phi_\lambda.
\]
\[\text{B22}\]
This is eq. (24) in §3.

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