Relative velocities in bi-disperse turbulent aerosols: simulations and theory

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We perform direct numerical simulations of a bi-disperse suspension of heavy spherical particles in forced, homogeneous, and isotropic three-dimensional turbulence. We compute the joint distribution of relative particle distances and longitudinal relative velocities between particles of different sizes, and compare the results with recent theoretical predictions [Meibohm et al. Phys. Rev. E 96 (2017) 061102] for the shape of this distribution. We also compute the moments of relative velocities as a function of particle separation, and compare with the theoretical predictions. We observe good agreement.

I. INTRODUCTION

Here we are concerned with small but heavy particles moving in a turbulent flow. How frequently and at what speeds do such particles collide with each other in turbulence? This question plays a central role in attempting to understand collisions and coalescence of microscopic water droplets in turbulent clouds [1], and to understand the formation of planetesimals in proto-planetary disks [2–4]. The particles in these turbulent aerosols are small and collisions between them are few and far between, consequently fluctuations matter. To understand how the distribution of particle sizes changes as a function of time, it is therefore not sufficient to merely consider the average collision rate. To account for the fluctuations it is necessary to consider the joint distribution of particle separations and their relative velocities [5–7]. A mean-field like description based solely on the first moment of particle separations and their relative velocities [5–7]. A mean-field description based solely on the first moment of relative particle velocities neglects fluctuations and may therefore not be reliable.

Völk et al. [8–10] and others [11, 12] formulated inertial-range theories for relative velocities of particles, referring to particle separations in the inertial range of turbulence. A criticism of this approach is that the collisions between the particles happen deep inside the dissipation range when the particle sizes are much smaller than the Kolmogorov length, η. It has been observed in direct numerical simulations (DNS) that inertial-range theories for the moments of relative velocities [8–10] fail at small Stokes numbers [13] (the Stokes number is a dimensionless measure of the importance of particle inertia). The predictions of Ref. [12] for the far tail of the distribution of relative velocities between nearby identical particles assume large Stokes numbers and a well-developed inertial range. This is difficult to achieve in DNS, and therefore it remains to be determined under which circumstances the prediction may hold.

Gustavsson et al. [6, 14–16] developed a dissipation-range theory for the distribution of relative velocities of identical particles, when the collision radius – the sum of the particle radii – is in the dissipation range of turbulence. An asymptotic form of the distribution was obtained by matching two limiting cases and using that inertial particles of identical sizes distribute on a fractal attractor in phase space [6, 14]. The result is a non-Gaussian distribution, with power-law tails that reflect large fluctuations. The theory applies in the limit where the Stokes number is large enough for particles to detach from the streamlines of the flow. But since the theory [6, 14–16] neglects inertial-range fluctuations, it may require modifications at very large Stokes numbers where the particle separations explore the inertial range.

In the astrophysical literature, DNS results for the relative-velocity distribution were recently reported by Ishihara et al. [13], as well as by Pan and Padoan [17, 18]. These authors fit the distribution to stretched exponentials. This raises the question how universal the power-law tails predicted in Refs. [6, 14] are. For Stokes numbers of order unity, the power laws were clearly seen in DNS [19, 20].

The findings and open questions described above apply to identical particles. But to understand how the size distribution of particles in turbulent aerosols changes as a result of collisions and coalescences, the distribution for particles of different sizes (different Stokes numbers) is needed. Meibohm et al. [21] developed a dissipation-rate theory for the distribution of relative velocities of particles that have different Stokes numbers, by analyzing a statistical model in the white-noise limit. The predictions of Ref. [21] have not been tested in DNS yet.

To understand the distribution of relative velocities in turbulent aerosols is an important problem to study – both in theory and in simulations – because it is hard to obtain direct measurements of droplet velocities in clouds, and quite impossible as far as grain velocities in proto-planetary disks are concerned. There are two laboratory experiments [22, 23] that have measured the distribution of relative velocities of micron-sized particles in turbulence, and their mean and root-mean square values as functions of particle separations. Experimental limitations make it difficult to measure at which relative velocities particles actually collide in these experiments. For micron-sized particles this occurs at separations deep inside the dissipative range, at present outside the spatial...
resolution of the experiments.

It is therefore important to validate existing theories for collision velocities of particles in turbulence by comparison with results of DNS. This is the purpose of the present paper. It is organized as follows: in Section II we describe the model and details of the DNS. In Section III we summarize the key theoretical results of Refs. [14, 21]. In Section IV we present our DNS results for the relative velocities between particles with different Stokes numbers. We compare the DNS results for the joint probability distribution of relative velocities and separations with the theoretical predictions of Meibohm et al. [21]. The distribution is non-Gaussian. When the difference between the Stokes numbers is not too large, then the distribution exhibits power-law tails as predicted by theory. At small separations and relative velocities, the power law in relative velocities is cut off, it becomes a broad Gaussian (approximately uniform), verifying the new velocity scale \( V_c \) predicted by theory [21]. Also the distribution of separations becomes uniform for separations smaller than \( R_c \). This scale was predicted in Refs. [24, 25]. We show how the scales \( V_c \) and \( R_c \) are related. Finally, we have determined the root-mean-square (RMS) relative velocities of particles when one of the particle have very small Stokes number and compared it with newly developed theory. The results are in agreement with the predictions of the theory. We conclude in Section VI.

II. NUMERICAL METHOD

A. Particle dynamics

We describe the motion of a heavy particle in a turbulent flow by the Stokes model [20]:

\[
\frac{d}{dt} \mathbf{x} = \mathbf{v} , \quad \frac{d}{dt} \mathbf{v} = \frac{1}{\tau} \left[ \mathbf{u}(\mathbf{x}, t) - \mathbf{v} \right].
\]

Here \( \mathbf{x} \) and \( \mathbf{v} \) are the position and velocity of the particle, the characteristic response time of the particle is \( \tau \). The response time depends upon the particle size, \( a \). In the Stokes limit, \( \tau = (2 \rho_p / \rho) a^2 / \nu \). Here \( \rho_p \) and \( \rho \) are the mass densities of the particle and the fluid, and \( \nu \) is the kinematic viscosity. Finally \( \mathbf{u}(\mathbf{x}, t) \) is the flow velocity. This model assumes that the effect of gravitational acceleration is small compared to the acceleration due to the turbulent flow, that fluid-inertia corrections are small, and that particle-particle interactions can be neglected.

B. Direct numerical simulation of turbulence

The flow velocity \( \mathbf{u}(\mathbf{x}, t) \) is determined by solving the Navier–Stokes equation

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2a}
\]

\[
\rho \frac{D}{Dt} \mathbf{u} = -\nabla p + \mu \nabla \cdot \mathbf{S} + \mathbf{f}. \tag{2b}
\]

TABLE I. Parameters for our DNS runs with \( N^3 \) collocation points, \( N_p \) is the number of particles. Further, \( \nu \) is the kinematic viscosity, \( \epsilon \) in the mean rate of energy dissipation, \( \eta \equiv (\nu^2/\epsilon)^{1/4} \) and \( \tau_\eta = (\nu/\epsilon)^{1/4} \) are the Kolmogorov length and time scales. These numbers are quoted in dimensionless units (see text). The Reynolds number \( Re \) is based on the large length scale at which the fluid motion is forced, and \( T_{eddy} \) is large eddy turn-over time scale of the flow (see text).

| \( N \) | \( \nu \) | \( N_p \) | \( Re \) | \( \epsilon \) | \( \eta \) | \( \tau_\eta \) | \( T_{eddy} \) |
|---|---|---|---|---|---|---|---|
| 512 | 5.0 \times 10^{-4} | 10^{7} | 89 | 3.25 \times 10^{-3} | 1.4 \times 10^{-2} | 0.39 | 0.86 |

Here \( \frac{D}{Dt} \equiv \partial_t + \mathbf{u} \cdot \nabla \) is the Lagrangian derivative, \( p \) is the pressure of the fluid, and \( \rho \) is its density as mentioned above. The dynamic viscosity is denoted by \( \mu \equiv \rho \nu \), and \( \mathbf{S} \) is the second-rank tensor with components \( S_{ij} = \partial_i u_j + \partial_j u_i - \delta_{ij} (2/3) \partial_k u_k \) (Einstein summation convention). Here \( \partial_i u_j \) are the elements of the matrix \( \mathbf{A} \) of fluid-velocity gradients. We use the ideal gas equation of state with a constant speed of sound, \( c_s = 1 \).

Our simulations are performed in a three-dimensional periodic box with sides \( L_x = L_y = L_z = 2 \pi \). To solve Eqs. (3), we use the pencil code [27], which uses a sixth-order finite-difference scheme for space derivatives and a third-order Williamson–Runge-Kutta [28] scheme for time derivatives. The external forcing \( \mathbf{f} \), which is a white-in-time, Gaussian, stochastic process concentrated on a shell of wavenumber with radius \( k_f \) in Fourier space [29], is integrated by using the Euler–Marayuma scheme [30]. Under the action of the force the flow attains a statistically steady state where the average energy dissipation by viscous forces is balanced by the average energy injection by the external forcing \( \mathbf{f} \). The amplitude of the external forcing is chosen such that the Mach number, \( Ma \equiv u_{rms}/c_s \) is always less than 0.1, i.e., the flow is weakly compressible which has no important effect on our results; please see the discussion in Ref. [20], section II and Appendix A in Ref [24, 25] for further details. The same setup has been used before in studies of scaling and intermittency in fluid and magnetohydrodynamic turbulence [31, 32].

We introduce the particles into the simulation after the flow has reached a statistically steady state. Initially, the position of the heavy particles are random but statistically homogeneous with zero initial velocity. Then we simultaneously solve Eqs. (1) and (2). To this end we must interpolate the flow velocity to typically off-grid positions of the heavy inertial particles. We use a tri-linear method for interpolation.

We define the Reynolds number by \( Re \equiv u_{rms}/(\nu k_f) \), where \( u_{rms} \) is the root-mean-square velocity of the flow averaged over the whole domain and the kinematic viscosity \( \nu \equiv \mu/\rho \). The mean energy dissipation rate \( \epsilon \equiv 2\nu \Omega \) where the enstrophy \( \Omega \equiv \langle \omega^2 \rangle \), and \( \omega \equiv \nabla \times \mathbf{u} \) is the vorticity. The Kolmogorov length is defined as \( \eta \equiv (\nu^2/\epsilon)^{1/4} \), the characteristic time scale of dissipation is given by \( \tau_\eta = (\nu/\epsilon)^{1/4} \) and \( u_\eta \equiv \eta/\tau_\eta \) is the char-
acteristic velocity scale at the dissipation length scale. In what follows, unless otherwise stated, we use \( \eta \), \( \tau \), and \( u_0 \) to non-dimensionalize length, time, and velocity respectively. The large eddy turnover-time is given by \( T_{\text{eddy}} \equiv 1/(k \eta u_{\text{rms}}) \).

In what follows we use \( \eta \), \( \tau \), and \( u_0 \) to non-dimensionalize length, time, and velocity respectively. We define the Stokes number as \( \tau \equiv \tau / \eta \), where \( \tau \) is the particle response time. It is obtained by replacing the gas \([2, 3, 35]\), we must use a different expression for different value of the density ratio. Also, since the sizes responds to dust in proto-planetary disks one must use the background fluid that corresponds to water droplets from its Stokes number we have used typical values of particle inertia.

\begin{align*}
\text{In the Introduction, this parameter measures the importance of particle response time in Eq. (1). As mentioned in the}
\text{We define the Stokes number as } \text{St} \equiv \tau / \eta, \text{where } \tau \text{ is the particle}
\text{response time. It is obtained by replacing the gas } [2, 3, 35], \text{we must use a different expression for different value of the}
\text{density ratio. Also, since the sizes responds to dust in proto-planetary disks one must use the background fluid that}
\text{corresponds to water droplets from its Stokes number we have used typical values of particle inertia.}
\end{align*}

**III. THEORETICAL BACKGROUND**

In this Section we summarize the dissipation-range theory for the distribution of relative velocities between two particles with different Stokes numbers \([21]\). We denote the relative-particle velocity by \( \mathbf{V} = \mathbf{v}_2 - \mathbf{v}_1 \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are the individual particle velocities. The distance between the particles is denoted by \( R = |\mathbf{R}| \), where \( \mathbf{R} = \mathbf{x}_2 - \mathbf{x}_1 \) is the separation vector between the particle positions, and the longitudinal relative velocity is defined as \( V_R = \mathbf{V} \cdot \mathbf{R} / |\mathbf{R}| \). We denote the steady-state distribution of relative velocities and separations by \( \mathcal{P}(R, V_R) \).

The moments of the distribution are characterized by

\begin{align*}
\langle |V_R|^p \rangle_R &= \frac{m_p(R)}{m_0(R)}, m_p(R) = \int dV_R |V_R|^p \mathcal{P}(R, V_R). \quad (3)
\end{align*}

The factor \( m_0(R) \) is related to the pair correlation function by \( m_0(R) = g(R) R^{d-1} \) \([6]\).

**A. Distribution of relative velocities and separations**

Gustavsson and Mehlig \([6, 14, 15]\) developed a theory for the distribution of relative velocities of nearby identical particles. The theory takes into account particle inertia, and it rests on the observation that such particles form fractal spatial patterns in turbulence \([20]\), and that caustics can give rise to large relative velocities at small separations \([36, 38]\). The theory predicts that the distribution of relative velocities \( V_R \) at small separations \( R \) is a power law, reflecting fractal clustering in phase space. The power-law exponent is related to the phase-space correlation dimension \( D_2 \) \([6, 14, 21]\). The distribution determines the scaling of relative-velocity moments \([9]\) with separation \( R \) \([15]\). These predictions for identical particles should hold for turbulence as well as statistical-model flows. In the white-noise limit, the theory was derived from first principles in Refs. \([6, 14]\). For turbulent flows, the theoretical predictions were verified using DNS \([15, 20, 39]\) and using kinematic turbulence simulations \([15]\). See also Refs. \([40–42]\).

The correlation dimension \( D_2 \) is not universal. In the white-noise limit \( D_2 \) can be calculated in perturbation theory \([14, 22]\), but in general it must be determined numerically. As is well known, \( D_2 \) depends non-monotonically on \( \text{St} \) with a minimum at \( \text{St} \) of order unity \([14]\).

Particles with different Stokes numbers cluster on distinct fractal attractors, so that the distribution of separations between particles with different Stokes numbers is cut off at a small spatial scale, \( R_c \), that depends on the difference between the Stokes numbers \([21, 22]\). How are the relative velocities of nearby particles affected? In Ref. \([21]\) a statistical model for relative velocities between particles with different Stokes numbers was analyzed in the white-noise limit. It was shown that there is a new velocity scale \( V_c \), and that the distribution of \( V_R \) and \( R \) is a broad Gaussian below these scales \([21]\), in other words approximately uniform:

\begin{align*}
\mathcal{P}(R, V_R) = \mathcal{N} R^{d-1} \begin{cases} 
1 & \text{for } |V_R| < V_c \text{ and } R < V_c / z^*, \\
R^{\mu_0-d-1} & \text{for } R > V_c / z^* \text{ and } |V_R| < z^* R, \\
(V_R / z^*)^{\mu_0-d-1} & \text{for } |V_R| > V_c \text{ and } z^* R < |V_R|, \\
0 & \text{for } |V_R| > V_0.
\end{cases} \quad (4)
\end{align*}

In addition to the normalization \( \mathcal{N} \) there are four more parameters in Eq. \((4)\): the two velocity scales \( V_c \) and \( V_0 \),
the power-law exponent \( \mu_c \), and the parameter \( z^* \).

The last parameter, \( z^* \), defines the line \( V_R = z^* R \) in the \( R-V_R \) plane where known limiting behaviors of \( \mathcal{P}(R, V_R) \) in the dissipative range are matched to obtain the theoretical predictions for \( \mathcal{P}(R, V_R) \).

The exponent \( \mu_c \) is related to the phase-space correlation dimension \( D_2(\overline{St}) \) of the mono-disperse system with Stokes number \( St \)

\[
\mu_c = \min\{D_2(\overline{St}), d+1\},
\]

where \( d = 3 \) is the spatial dimension, and \( \overline{St} \) is the harmonic mean of the two Stokes numbers,

\[
\overline{St} = \frac{2St_1St_2}{St_1 + St_2}.
\]

The parameter \( D_2 \) can be calculated analytically in the white-noise limit \([21, 45, 46]\), but in turbulent flows it must be determined numerically.

Now consider the upper velocity scale \( V_0 \). It was assumed in deriving Eq. (4) that it suffices to consider separations in the dissipative range where the turbulent fluid velocities are spatially smooth. This range extends up to separations \( R \) somewhat larger than the Kolmogorov length \( \eta \). The theory mirrors the distribution of spatial separations for \( R < 1 \) to distributions in relative velocities, just as it does for identical particles. Therefore the upper cutoff for the \( V_R \) power laws is

\[
V_0 = z^*.
\]

How this parameter depends upon the Stokes number is not known in general. In a one-dimensional statistical model this parameter was calculated in the white-noise limit in Ref. [6].

In Eq. (7), the distribution was simply set to zero for \( V_R > V_0 \). This is an oversimplification, in particular for turbulence where the far tails of the \( V_R \)-distribution at small spatial separations result from particle pairs that have had separations in the inertial range in the past. For large Stokes numbers and when the inertial range is well developed it was argued in Ref. [12] that the tail of the conditional distribution \( \mathcal{P}(R=0, V_R) \) has the form \( \sim C_1/(\varepsilon \tau)^{1/2} \exp[-C_2|V_R|^{4/3}/(\varepsilon \tau)^{2/3}] \) for very large Stokes numbers. A statistical-model calculation with an inertial range yields the prefactors \( C_1 \) and \( C_2 \) in the white-noise limit, but they could have different parameter dependencies in turbulence \([47]\). At smaller \( R_c \), when the inertial range is not well developed, one may argue that the tail should be well approximated by a Gaussian with variance \( \propto u_{rms}^2 \). The RMS turbulent velocity is an estimate of the relative velocities of particles that move independently at large separations, of the order of the system size. In summary, the far tail of the relative-velocity distribution is not universal. Here we simply set

\[
V_0 = u_{rms} \quad \text{(8)}
\]

when we compare with our DNS data.

The fourth parameter in Eq. (4) is the scale \( V_c \). It depends upon the difference of the two Stokes numbers. We follow Ref. [21] and write

\[
\theta = \frac{|St_1 - St_2|}{St_1 + St_2}. \quad \text{(9)}
\]

The white-noise model predicts that \([21]\)

\[
V_c \propto \theta \quad \text{(10)}
\]

at small \( \theta \). In this case, the power-law tails of the distribution \( \mathcal{P} \) are expected to contribute to the relative velocity moments. According Eq. (4), the tails of the distribution beyond \( V_c \) are simply those of the mono-disperse system.

Eq. (4) implies that the distribution of separations becomes uniform in \( R \) for \( R < R_c \), as predicted in Refs. [24, 18]. Their spatial scale \( R_c \) is thus related to our velocity scale as follows:

\[
R_c \equiv V_c/z^*, \quad \text{(11)}
\]

and therefore \( R_c \propto \theta \) at small \( \theta \).

### B. Moments of relative velocities

Theoretical predictions for \( \langle |V_R|^p \rangle_R \) are obtained by integrating the distribution \( \mathcal{P} \), as determined by Eq. (4). We first quote the results when \( \theta \) is small, when the distribution exhibits a clear power law. This power law is cut off at small relative velocities at \( max(V_c, z^* R) = z^* \max(R_c, R) \), consequently the result for \( \langle |V_R|^p \rangle_R \) depends on whether \( R > R_c \) or not. When \( R > R_c \) we find

\[
m_p(R) = b_p R^{\mu_c + p - 1} + c_p R^{\mu_c + 1}, \quad \text{(12)}
\]

with

\[
b_p = \frac{\mathcal{N} (1 + d - \mu_c) z^{* p + 1}}{(p + 1)(\mu_c - d + p)}, \quad \text{(13)}
\]

\[
c_p = \frac{\mathcal{N} z^{* p + 1} (V_0)^{\mu_c - d + p}}{\mu_c - d + p},
\]

where \( \mathcal{N} \) is the normalization factor in Eq. (4). For large values of \( p \), the coefficients \( b_p \) and \( c_p \) are sensitive to the form of the distribution beyond the cutoff \( z^* \), which depends on the nature of the turbulent fluctuations. Also, the value of \( \mu_c = D_2(\overline{St}) \) is not universal, and neither is the parameter \( z^* \).

The \( R \)-dependence predicted by Eq. (4) is universal. It is equal to the scaling form of \( m_p(R) \) for identical particles \([13]\), as expected for small \( \theta \). But for particles with different Stokes numbers the coefficients \( b_p \) and \( c_p \) depend upon \( \theta \), although only through the global normalization constant \( \mathcal{N} \). The new scale \( V_c \) does not enter explicitly because \( R > R_c \).
Now consider $R < R_c$. Then the uniform part in Eq. (4) dominates the moments. At $R < R_c$, particles of two different sizes $a_1$ and $a_2$ move approximately independently from each other. In this case the moments take the form:

$$m_p(R) \sim c_p' R^{d-1},$$

with

$$c_p' = c_p - \frac{\mathcal{N} (1 + d - \mu_c) (V_c / z^*)^{\mu_c - d + p} z^{*p+1}}{(\mu_c - d + p)(p + 1)}.$$  (15)

For $p = 1, 2, 3, \ldots$ one finds that $c_p' < c_p$ for heavy particles in incompressible turbulence at not too large Stokes numbers [DNS show that $D_2 > d - 1$, and that $D_2 < d + 1$ for not too large Stokes numbers, see Eq. (3)]. The moments for larger $\theta$ are nevertheless usually larger than those for $\theta \to 0$, because the term $b_p R^{D_2 + p - 1}$ makes a large negative contribution unless $R$ is extremely small, and this term is absent in Eq. (14). In general, if $\mathcal{S}$ is small enough so that caustics are rare, then Eq. (13) can give a contribution for different particles that is much larger than for identical particles, leading to a significantly higher collision rate. The dependence on $R$ is of the same form as the caustic contribution in Eq. (3) in the limit $\theta \to 0$.

Finally consider larger values of $\theta$, large enough so that the power laws in Eq. (3) disappear. In a Gaussian white-noise model the distribution $\mathcal{P}(R, V_R)$ is Gaussian in this limit [21].

When one of the particles has a very small Stokes number, $St_2 \ll 1$ say, we can evaluate the coefficient $c_p'$ term in Eq. (14) in terms of single-particle observables. We now outline the calculation for $p = 2$. When $St_2 \ll 1$, we can expand the equation of motion up to leading order in $St_2$ to obtain the velocity of the second particle:

$$v_2 \approx u(x,t) - \mathbf{\Lambda} \cdot R - St^2 \frac{Du}{Dt} (x + R, t).$$

The relative velocity between two particles can then be written as

$$V(R) \approx v - u(x,t) + \mathbf{\Lambda} \cdot R + St^2 \frac{Du}{Dt} (x + R, t).$$

(17)

The first line of the right-hand-side of Eq. (17) is $St^1$ times the acceleration of a single particle; at small $|V_R|$ and $St_2$ this is the leading order contribution to the relative velocity. The distribution of the acceleration has been studied extensively and is known to have exponential tails [48, 50]. This information allows us to approximately relate the structure functions to single-particle averages, as shown below.

To calculate $\langle V^2 \rangle_R$ for $R$ much smaller than $R_c$, it is sufficient to consider one component of $V$, square Eq. (17), and then to take steady-state averages. Assuming that $R \ll 1$ we obtain:

$$\langle V^2 \rangle_R \approx \frac{1}{3} \left[ \langle u^2 \rangle - \langle v^2 \rangle \right] \left( 1 - 2 St_2 \right) - \frac{2}{3} St_2 \langle (u - v) \cdot \mathbf{\Lambda} \cdot (u - v) \rangle.$$  (18)

All averages on the r.h.s. of Eq. (18) are evaluated for a single particle with Stokes number $St_1$. The only $St_2$-dependence appears in the prefactors on the r.h.s. of Eq. (18). We remark that there is no $R$-dependence (since all averages are single-particle averages). This is the result of neglecting the gradient term $\mathbf{\Lambda} \cdot R$ in the equation for the particle separations. As explained in Section II.A of Ref. [21] this is allowed provided that $R < R_c$. But not that in Ref. [21] the white-noise limit was analyzed, while Eq. (18) applies to a turbulent flow.

IV. DNS RESULTS

A. Distribution of relative velocities and separations

Fig. 1 shows a comparison between the theory Eq. (4) and our DNS results for $\mathcal{P}(R, V_R)/R^2$ for different values of $\theta$. The first column of panels in this Figure shows contour plots of $\mathcal{P}(R, V_R)/R^2$. As predicted by the theory [41], there is a region in the $R-V_R$ plane where the distribution is a broad Gaussian. In a log-log plot this appears as an approximately uniform region where $\mathcal{P}/R^2$ is approximately constant. Outside this region, and for small values of $\theta$, the equidistant contour lines show that the distribution exhibits the power laws, as predicted by the theory.

To analyze the power laws in relative velocities in more detail, the second column of panels in Fig. 1 shows plots of $\mathcal{P}(R, V_R)/R^2$ as functions of $|V_R|$ for several different values of $R$. We can clearly distinguish the power-law from the broad Gaussian at small $|V_R|$, where $\mathcal{P}/R^2 \approx \text{const}$. Eq. (4) says that the cross over between these two behaviors occurs at $\min(V_c, z^* R)$. We estimate this cross-over scale by drawing two lines: a horizontal one at small $|V_R|$, and a power-law fit for larger $|V_R|$. The scale at which these two lines intersect is our estimate of the cross-over scale. For small values of $R$ the fits yield a velocity scale that is independent of $R$, this is $V_c$. For slightly larger values of $R$, the velocity scale is proportional to $R$, as predicted by theory, and the constant of proportionality defines the parameter $z^*$.

Dissipation-range theory [21] says that $V_c = c \theta$ for small $\theta$, but the theory does not determine the constant of proportionality $c$. This constant is system specific, as is the value of $z^*$. In the white-noise limit these parameters can be calculated analytically [4, 21], but not in general.

Therefore it is important to determine these constants by DNS. The results are shown in Fig. 2. Panel (a) shows that $z^*$ is essentially independent of $\theta$, while panel (b) demonstrates that $V_c$ is proportional to $\theta$ at small $\theta$, as
FIG. 1. (color online) DNS results for joint distribution $\mathcal{P}(R, V_R)$ of $R$ and $V_R$, divided by $R^2$. Parameters: $\tilde{\mu} = 2$ and $\theta = 0.005$ (top row), $\theta = 0.05$ (second row), and $\theta = 0.1$ (bottom row). First column: Contour plots of $\mathcal{P}(R, V_R)/R^2$ color coded according to $\log_{10}(\mathcal{P}(R, |V_R|)/R^2)$. The blue lines in the bottom left corner of these plots show the scales $R_c$ and $V_c$ (see text). The dashed lines show the theoretical matching condition $V_R = z^* R$ (see text). Second column: plots of $\mathcal{P}(R, V_R)/R^2$ as functions of $|V_R|$ for different values of $R$ as indicated in the panels. Also shown are fits (solid lines) to the theoretical power-law prediction $|V_R|^{\mu_c - 4}$, Eq. (4), to determine $\mu_c$ as a function of $\tilde{\mu}$. The crossover between the approximately uniform (broad Gaussian) part at small $|V_R|$ (and small $R = 0.03, 0.06$, horizontal solid lines) and the power-law at intermediate $R$ sets the scale $V_c$ (dashed vertical lines).
predicted by the theory. Fig. 2(b) also shows that the prefactor depends on $\text{St}$ as $\text{St}^{-1/2}$, at least for the parameters simulated. This follows from the fact that the DNS data for $V_c/\text{St}^{-1/2}$ collapse onto a single line. However, there is no theoretical explanation for this result, as far as we know.

Fig. 2(c) shows the power-law exponents $\mu_c$. We extracted $\mu_c$ for different values of $\text{St}$ and for two different values of $\theta$ by fitting power laws to the DNS results for the distribution of relative velocities. Panel (c) shows the resulting exponents $\mu_c$ together with $D_2$ for the case $\text{St}_1 = \text{St}_2$ from Ref. 21. Up to the numerical accuracy in our DNS we find for $D_2 < 4$ that $\mu_c = D_2$, independent of $\theta$ for small values of $\theta$. The phase-space correlation dimension $D_2$ has a characteristic minimum at $\text{St}$ of order unity and monotonously approaches the spatial dimension $2d$ for large $\text{St}$ [see Fig. 2(c)].

In summary we observe good agreement between our DNS and the theory, Eq. (4), in particular for small $\theta$. As $\theta$ increases, the velocity scale $V_c$ grows so that the range of the power law between $V_c$ and $V_0$ becomes smaller. For large enough values of $\theta$, the power laws disappear. In this limit the distribution is a broad Gaussian, approximately uniform. In our log-log plots, $P/R^2$ is approximately constant in this region.

B. Moments of relative velocities

Fig. 3 summarizes our DNS results for the moments of relative velocities as a function of particle separation. Panel (a) shows DNS results for $m_0(R)/R^2$ as a function of $R$ (symbols), while panel (b) shows $m_2(R)/R^2$, also as a function of $R$. The parameters are given in the Figure caption. Also shown is the scaling of the smooth contribution predicted by Eq. (12) (solid line). Dashed vertical lines correspond to the scale $R_c = V_c/\gamma^*$. The parameters $V_c, \mu_c$, and $\gamma^*$ were determined separately, as described in Section IV A.

As predicted by Eq. (12), the moments scale as $R^{d-1}$ for $R < R_c$. For $R > R_c$ smooth contribution dominates for $m_0(R)$ for both values of $\text{St}$, whereas for higher order moment $m_2(R)$ smooth contribution dominates only for the smaller mean Stokes number. For larger mean Stokes number, the caustic contribution $c_p R^{d-1}$ swamps the smooth part for $R$ below $R_c$. In limit the relative-velocity moments $m_p(R)$ are dominated by the singular $R^{d-1}$-contribution provided that $p$ is large enough. While the $R$-dependence of this contribution is the same for identical particles and for particles with different Stokes numbers, the physical origin of this power law is slightly different in the two cases. For identical particles, the singular term is caused by caustics. For particles with different Stokes numbers, by contrast, the singular contribution is due to the uncorrelated motion between nearby $(R < R_c)$ particles with different Stokes numbers [21].

Fig. 3(c) shows DNS results for $(V_R^2)_{R=a_1+a_2}$ at the collision radius $R = a_1 + a_2$ for $\text{St}_2 \ll 1$ as a function of $\text{St}_1$ (red circles), that is for large values of $\theta$. Also shown is the theoretical expression, Eq. (13) (green squares). The averages on the r.h.s. of Eq. (13) are determined by DNS, by averaging along heavy-particle paths in the steady state. The agreement is good at small values of $\text{St}_1$, but we observe deviations at larger values of the Stokes number. It is possible that this is due to higher-$\text{St}$-terms neglected in (13). Plotting only the first term of Eq. (13) yields slightly different results, although the deviations are smaller than those between the full theory and the DNS results. We have checked that the gradient term $\nabla \cdot R$ in the equation of motion for the separation $R$ is negligible. For all data points shown, $\theta$ is large enough so that $a_1 + a_2$ is much less than $R_c$. In this range the DNS results do not depend upon $R$. This is the plateau region seen in Fig. 3(a).

V. DISCUSSION

Our results in agreement with the theory that the distribution of relative velocities is non-Gaussian when $\theta$ is small. For a fairly wide range of $\theta$ (up to $\theta \sim 0.1$), the distribution has power-law tails $\sim |V_R|^{\mu_c - 4}$ at small separations. The dissipation-range theory predicts that the exponent $\mu_c$ is determined by the phase-space correlation dimension $D_2(\text{St})$ for a mono-disperse system with Stokes number $\text{St}$ [Eq. (13)]. In our simulations, the numerical values of $\mu_c$ vary from approximately 2.4 to 3.5, and in this range there is good agreement between the theory and the numerical values of $\mu_c$ obtained from the DNS [31].

In the astrophysical literature, several papers have reported DNS results for the distribution of relative particle velocities [13, 17, 18]. These authors attempted to fit the distributions to stretched exponentials, of the form $\exp[-(|V_R|/\beta)\gamma]$ with fitting parameters $\beta$ and $\gamma$. The parameter $\gamma$ is usually quoted to be smaller than unity. This law is neither consistent with our power-law predictions, nor with the large-St prediction from Ref. [12]. We have reanalyzed the data in Fig. 12 of Ref. [13] for the two smallest Stokes numbers, and find clear power laws over one decade of $V_R/u_0$, with exponents $\mu_c - 4$ in good agreement with the dissipation-range theory (the values of $\mu_c$ were obtained from the plots of the pair correlation function in Fig. 8 of the same paper).

We remark that the distribution of relative velocities in bidisperse suspensions was recently studied in Ref. [52]. This study did not report power-laws for the distribution of relative velocities. As our results show, possible reasons for the absence of power laws are, firstly, that the distributions were calculated at quite large separations (of the order of the Kolmogorov length, $R \sim \eta_K$). Secondly, the values of $\theta$ were quite large, too large to see power laws as our theory and DNS data demonstrate.
Pan and Padoan [17] did not plot the radial relative velocity \( V_R \) (that determines how particles approach each other), but instead the RMS relative velocity \( V_{\text{rms}} = \sqrt{V_1^2 + V_2^2 + V_3^2} \). The power law of the distribution of \( V_{\text{rms}} \) is a different power law \[14\]: \( |V_{\text{rms}}|^{\mu - 2d} \). We have compared this prediction with the data shown in Fig. 14 of Ref. [17]. There is a clear power law, with exponent \( \approx -3.7 \) for \( St = 1.55 \). Theory says that the exponent should equal \( D_2 - 6 \), but Ref. [17] does not give values for the fractal correlation dimension \( D_2 \). Estimating \( D_2 \) from our data at \( St = 1.55 \) (albeit at a different Reynolds number), we find \( D_2 - 6 \approx -3.4 \), in reasonable but not perfect agreement with the DNS results of Ref. [17].

Ishihara et al. state that their distribution approaches a Gaussian when \( \theta \) is not small. This is consistent with theory [21], predicting a broad Gaussian for the body of the distribution. In our log-log plots, Fig. 1, the broad Gaussian appears as a region where \( \mathcal{P}/R^2 \) is approximately constant. When \( \theta \) is large enough, this region extends out to \( V_0 \), approximately equal to the RMS turbulent velocity, \( u_{\text{rms}} \). The form of the far tails beyond \( V_0 \) is difficult to determine, because the tails describe rare events, and since there is no theoretical prediction apart from the law predicted in Ref. [12]. Yet this applies only at large Stokes numbers, and when there is a well-developed inertial range.

In both cases, when \( \theta \) is small and when it is large, the RMS relative velocity is determined by the upper cutoff, \( V_0 \). We have simply set \( V_0 = u_{\text{rms}} \) here, but this is a simplification. In general, the upper cutoff \( V_0 \) must also depend on particle inertia (Stokes number). We have neglected this dependence here. Taking \( V_0 = u_{\text{rms}} \) implies that the moments of particle relative velocities depend on the Reynolds number \( Re \) when determined by the upper cutoff \( V_0 \), since \( u_{\text{rms}} \propto Re^{1/4} \). With our present computational capabilities we cannot explore such a weak dependence on \( Re \); hence we have concentrated our efforts
Changing Re while keeping the system size $L$ (in their paper), obtaining a fairly strong dependence on relative particle velocities for different values of Re (Fig. 3 changes the Kolmogorov length $\eta$ and hence $R = r/\eta$ is different for different value of Re. Unless $R < R_c$ (whether this condition is satisfied or not is determined by the values of the Stokes numbers), the relative velocity statistics depends on $R$, as the dissipation-range theory shows. Thus evaluating the moments at $r = 10^{-3}L$ for changing $\eta$ may give rise to a spurious Re dependence. It would be of interest to test quantitatively whether the Re-dependence predicted by the dissipation-range theory is consistent with this explanation.

It is a strength of the dissipation-range theory summarized in Section IV that it predicts how the moments of relative velocities depend upon particle separation $R$. The microscopic dust grains in accretion disks are much smaller than the Kolmogorov length $\eta$, so that the collision radius $R = a_1 + a_2$ is well below $\eta$. Inertial-range theories [8–12] do not refer to scales below $\eta$. As a consequence they cannot describe collisions that occur deep in the dissipation range. In DNS it is also difficult to reach to such small scales, much smaller than $\eta$, simply because particles rarely come so close. But collisional aggregation in turbulent aerosols is fluctuation dominated when the systems are dilute, so that such rare events matter. Several recent works [13, 17, 53] give results for RMS relative velocities at fixed separations, usually of order $\eta$, irrespective of the size of the particles. The theory [12, 15] allows to extrapolate the DNS results to $R = a_1 + a_2$. Here the parameter $R_c = V_c/z^*$ plays an important role. If $R < R_c$ then the theory shows that the relative particle-velocity statistics is independent of the separation $R$.

A weakness of the dissipation-range theory is that it expresses the prefactors $b_p$ and $c_p$ in the $R$-dependence of the moments in terms of parameters $z^*$, $\mu_c$, $V_c$, and $V_0$ that must be determined separately, by DNS for example. The theory shows, however, that the prefactors are not universal. It would therefore be of great interest to find alternative ways of computing these prefactors. One possibility, although numerical, is to use the approach of Zaichik and collaborators [54, 55] and its refinements [56].

VI. SUMMARY AND CONCLUSIONS

Let us summarize the key findings here. We used direct numerical simulations of particle-laden, homogeneous and isotropic, forced turbulence to study the statistics of relative velocities and separations between particles with different Stokes numbers. We computed the joint distribution of particle separations and their relative velocities. We found that the shape of the distribution is in good agreement with the predictions of dissipation-range theory [21]. When the difference between the two Stokes numbers is small enough, then the distribution exhibits power laws, and the exponent is related to fractal patterns in phase space [26]. We found that the power laws are cut off at small relative velocities, at a new scale $V_0$. When the difference between the Stokes number is small, then we found that $V_0$ depends linearly on this difference, in agreement with the theoretical prediction [21].

When $\theta$ is large, by contrast, theory predicts that the body of the distribution is broad Gaussian [21], in agreement with the DNS of [12, 53]. In a log-log plot [Fig. 4] this Gaussian appears as a region where $\mathcal{P}/R^2$ is roughly constant. The shape of the distribution beyond $V_0$ (here simply set to zero) is not known. There are indications [53] that the theory of Ref. [12] may work for the tails. But this could not be unequivocally shown, and it must be borne in mind that the prediction of Ref. [12] applies to large Stokes numbers in systems with a very well developed inertial range, so that the scale-dependent Stokes number at the largest scale is much less than unity. These questions remain for further studies.

Dissipation-range theory [6, 14, 16, 21] predicts how the relative-velocity fluctuations depend on particle separation. This power-law dependence of the relative-velocity moments upon particle separation is universal (but the prefactors of the power laws are not). The original inertial-range theories discussed above do not refer to particle separations in the dissipation range, and attempts to modify inertial-range theories to take into account dissipation-range dynamics [57, 58] were shown to fail (Fig. 5 in Ref. [13]), so that they cannot be used to model collision velocities of microscopic dust grains in circumstellar accretion disks, where collisions happen in the dissipation range. It is challenging to use DNS to determine collision rates and velocities of small grains deep in the dissipation range, because such encounters are infrequent, yet significant. Usually, DNS data on relative-particle velocities [13, 17, 53] are evaluated at fixed separations of order $\eta$, as discussed above. The theory described and tested here allows to extrapolate the DNS results to the relevant scales, often much smaller than the Kolmogorov length $\eta$.

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letters 97, 048501 (2006).
[39] M. Voß kuhle, A. Pumir, E. Lévéque, and M. Wilkinson, “Prevalence of the sling effect for enhancing collision rates in turbulent suspensions,” Arxiv (2013).
[40] J. Bec, L. Biferale, M. Cencini, A. Lanotte, and F. Toschi, “Intermittency in the velocity distribution of heavy particles in turbulence,” J. Fluid Mech. 646, 527–536 (2010).
[41] J. Bec, L. Biferale, M. Cencini, A. Lanotte, and F. Toschi, “Spatial and velocity statistics of inertial particles in turbulent flows,” Journal of Physics: Conference Series 333, 012003 (2011).
[42] J. P. L. C. Salazar and L. R. Collins, “Inertial particle relative velocity statistics in homogeneous isotropic turbulence,” JFM 696, 45–66 (2012).
[43] Martin James and Samriddhi Sankar Ray, “Enhanced droplet collision rates and impact velocities in turbulent flows: The effect of poly-dispersion and transient phases,” Scientific Reports 7, 12231 (2017).
[44] Jeremie Bec, Luca Biferale, Massimo Cencini, A Lanotte, Stefano Musacchio, and Federico Toschi, “Heavy particle concentration in turbulence at dissipative and inertial scales,” Physical review letters 98, 084502 (2007).
[45] M. Wilkinson, B. Mehlig, and K. Gustavsson, “Correlation dimension of inertial particles in random flows,” Europhys. Lett. 89 (2010), 50002.
[46] K. Gustavsson, B. Mehlig, and M. Wilkinson, “Analysis of the correlation dimension of inertial particles,” Phys. Fluids 27 (2015), 073305.
[47] K. Gustavsson and B. Mehlig, “Distribution of velocity gradients and rate of caustic formation in turbulent aerosols at finite Kubo numbers,” Phys. Rev. E 87 (2013), 023016.
[48] J Bec, Antonio Celani, M Cencini, and S Musacchio, “Clustering and collisions of heavy particles in random smooth flows,” Physics of Fluids (1994-present) 17, 073301 (2005).
[49] Jeremy Bec, Luca Biferale, Guido Boffetta, Antonio Celani, Massimo Cencini, Alessandra Lanotte, S Musacchio, and Federico Toschi, “Acceleration statistics of heavy particles in turbulence,” Journal of Fluid Mechanics 550, 349–358 (2006).
[50] Akshay Bhatnagar, Direct Numerical Simulations of Fluid Turbulence: (A) Statistical Properties of Tracer and Inertial Particles (B) Cauchy-Lagrange Studies of the Three-Dimensional Euler Equation, Ph.D. thesis, Dept. of Physics, Indian Institute of Science, Bangalore. (2016).
[51] This agreement should be understood in the following manner. The theory does not allow a calculation of $\mu_c$ from first principle, but it shows that $\mu_c = D_2(\text{St})$ for small $\theta$. This is indeed what we confirm from DNS.
[52] Rohit Dhariwal and Andrew D Bragg, “Small-scale dynamics of settling, bidisperse particles in turbulence,” Journal of Fluid Mechanics 839, 594–620 (2018).
[53] Liubin Pan, Paolo Padoan, and John Scalo, “Turbulence-induced relative velocity of dust particles. II. The bidisperse case,” The Astrophysical Journal 791, 48 (2014).
[54] Leonid I Zaichik, Olivier Simonin, and Vladimir M Alipchenkov, “Two statistical models for predicting collision rates of inertial particles in homogeneous isotropic turbulence,” Physics of Fluids 15, 2995 (2003).
[55] Leonid I Zaichik, Vladimir M Alipchenkov, and Emmanuil G Sinaiski, Particles in turbulent flows (John Wiley & Sons, 2008).
[56] Liubin Pan and Paolo Padoan, “Relative velocity of inertial particles in turbulent flows,” Journal of Fluid Mechanics 661, 73–107 (2010).
[57] CW Ormel and JN Cuzzi, “Closed-form expressions for particle relative velocities induced by turbulence,” Astronomy & Astrophysics 466, 413–420 (2007).
[58] Liubin Pan and Paolo Padoan, “Turbulence-induced Relative Velocity of Dust Particles V. Testing Previous Models,” The Astrophysical Journal 812, 10 (2015).