VARIATIONS OF THE RAMANUJAN POLYNOMIALS AND REMARKS ON $\zeta(2j+1)/\pi^{2j+1}$

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Abstract. We observe that five polynomial families have all of their zeros on the unit circle. We prove the statements explicitly for four of the polynomial families. The polynomials have coefficients which involve Bernoulli numbers, Euler numbers, and the odd values of the Riemann zeta function. These polynomials are closely related to the Ramanujan polynomials, which were recently introduced by Murty, Smyth and Wang [10]. Our proofs rely upon theorems of Schinzel [13], and Lakatos and Losonczi [9] and some generalizations.

1. Introduction

In a recent paper, Murty, Smyth and Wang considered the Ramanujan polynomials [10]. They were defined by Gun, Murty and Rath [7] using

$$R_{2k+1}(z) := \sum_{j=0}^{k-1} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2j},$$

(1)

where $B_j$ is the $j$th Bernoulli number. Among other fascinating results, Murty, Smyth and Wang showed that $R_{2k+1}(z)$ has all of its non-real zeros on the unit circle. The purpose of this paper is to study some variants of $R_{2k+1}(z)$, which also have many zeros on the unit circle.

Claim 1. Let $B_j$ denote the Bernoulli numbers, and let $E_j$ denote the Euler numbers. Suppose that $k \geq 2$. The following polynomials have all of their non-zero roots
on the unit circle:

\[
P_k(z) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} \left( \frac{2k}{2j} \right) z^{2j} + \zeta(2k-1) \left( z^{2k-1} + (-1)^k z \right), \\
Q_k(z) := (2^{2k} + 1) P_k(z) - 2^{2k} P_k(z/2) - P_k(2z), \\
Y_k(z) := \frac{\pi}{2^{2k}} (Q_k(i\sqrt{z}) + Q_k(-i\sqrt{z})) \\
W_k(z) := (2^{2k-1} + 2) P_k(z) - 2^{2k} P_k(z/2) - P_k(2z), \\
S_k(z) := \sum_{j=0}^{k} E_{2j} E_{2k-2j} \left( \frac{2k}{2j} \right) z^j.
\]

We will offer a general proof of Claim 1 for \( Q_k(z), Y_k(z), W_k(z), \) and \( S_k(z) \). It seems that \( P_k(z) \) is more difficult to handle. In Section 4 we offer several partial results concerning \( P_k(z) \).

An important secondary goal of this work, is to highlight a connection with the odd values of the Riemann zeta function. Recall that the Riemann zeta function is defined by

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

While it is a classical fact that \( \zeta(2j)/\pi^{2j} \) is rational when \( j \geq 1 \), very little is known about the arithmetic nature of \( \zeta(2j+1) \). The only theorems in this direction are celebrated irrationality results. For instance, Apéry showed that \( \zeta(3) \) is irrational [2, 11], Rivoal proved that infinitely many odd zeta values are irrational [12], and Zudilin established that at least one element of the set \{\( \zeta(5), \zeta(7), \zeta(9), \zeta(11) \)\} is also irrational [14].

Therefore, let us briefly consider the identity which gave birth to the Ramanujan polynomials [3, pg. 276]. The formula can be written as

\[
\frac{(2\pi)^{2k-1}}{2(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} \left( \frac{2k}{2j} \right) z^{2j} + \frac{\zeta(2k-1)}{2} \left( (-1)^k z + z^{2k-1} \right) \\
= -\sum_{n=1}^{\infty} \frac{z^{2k-1}}{n^{2k-1} e^{2\pi n z} - 1} + (-1)^{k+1} \sum_{n=1}^{\infty} \frac{1}{n^{2k-1} e^{2\pi n z - 1}}.
\]

This identity holds whenever \( z \not\in i\mathbb{Q} \). The restriction is necessary to ensure that both infinite series converge. Not surprisingly, this formula is also mentioned in works dealing with the irrationality of \( \zeta(3) \) [11]. Notice that Gun, Murty and Rath used (7) to express odd zeta values in terms of Eichler integrals [7]. Now consider the case when \( k = 2 \). A brief numerical calculation shows that the polynomial obtained from the left-hand side

\[
z^4 + 5z^2 + 1 - \frac{90\zeta(3)}{\pi^3} (z^3 + z) = 0,
\]

(8)
has all of its zeros on the unit circle. Notice that if we truncate the right-hand side of (7), then we can nearly obtain a closed form for $\zeta(3)$:

$$\zeta(3) \approx \left( \frac{z^4 + 5z^2 + 1}{z^3 + z} \right) \frac{\pi^3}{90},$$

(9)

where

$$0 = \frac{z}{e^{2\pi/z} - 1} + \frac{z^{-1}}{e^{2\pi z} - 1}.$$  

This approximation gives six decimal places of numerical accuracy. The accuracy can be increased by including higher order terms in the truncation. It would be extremely interesting if this idea could be used to say something about the irrationality of $\zeta(3)/\pi^3$. Unfortunately such a theorem is well beyond the scope of this paper. We will settle for the more modest goal of studying the polynomial families listed in Claim 1.

2. $S_k(z)$, $Y_k(z)$ and the theorems of Schinzel, Lakatos and Losonczi

In this section we prove that $S_k(z)$ and $Y_k(z)$ have all of their non-zero roots on the unit circle. Our proofs follow from applying the theorems of Schinzel [13], Lakatos and Losonczi [9], and Lakatos [8]. Lakatos proved that any reciprocal polynomial $\sum_{j=0}^{k} A_j z^j$, with real-valued coefficients, which satisfies

$$|A_k| \geq \sum_{j=0}^{k} |A_j - A_k|,$$

(10)

must have all of its zeros on the unit circle. If the inequality is strict then the polynomial has only simple zeros. Equation (10) is a very strong restriction. There have been a number of recent improvements to (10) with a similar flavor (see [13] and [9]). Schinzel proved that any self-inversive polynomial which satisfies

$$|A_k| \geq \inf_{c,d \in \mathbb{C}} \sum_{|d|=1}^{k} \sum_{j=0}^{k} |cA_j - d^{k-j}A_k|,$$

(11)

must have all of its zeros on the unit circle [13]. Self-inversive polynomials have complex-valued coefficients which satisfy $A_j = \epsilon A_{k-j}$, for some fixed $|\epsilon| = 1$. Notice that the class of self-inversive polynomials includes both reciprocal and anti-reciprocal polynomials. In Theorems 1 and 2 we apply Schinzel’s theorem with $d = 1$.

**Theorem 1.** Suppose that $k \geq 1$. Then all of the zeros of

$$S_k(z) = \sum_{j=0}^{k} E_{2j} E_{2k-2j} \binom{2k}{2j} z^j$$

lie on the unit circle. Furthermore, all of the zeros are simple.
Proof. With (11) in mind, let us begin by setting

\[ A_j := E_{2j}E_{2k-2j}\binom{2k}{2j}. \]

The sign of \( E_{2n} \) alternates with respect to \( n \). This implies that all of the coefficients of \( S_k(z) \) have sign \((-1)^k\).

Our proof consists of three main steps. First we remove the absolute values from the sum (11). This is easily accomplished by showing that 

\[ (cA_j - A_k - 2j) > 0 \]

for \( c = \frac{\pi}{4(1 + 3^{-1-2k})} \). Next we evaluate \( \sum_{j=0}^{k} A_j \) explicitly, and finally we deduce the desired upper bound.

In order to remove the absolute value signs from (11), we need to demonstrate that 

\[ (cA_j - A_k - 2j) > 0. \]

If we take \( c = \frac{\pi}{4(1 + 3^{-1-2k})} \), then the previous inequality implies \((-1)^k A_k - c(-1)^k A_j > 0 \).

Let \( E_n(z) \) denote the classical Euler polynomials, and recall a standard convolution identity [6]:

\[ \sum_{j=0}^{n} \binom{n}{j} E_j(v)E_{n-j}(w) = 2(1 - w - v)E_n(v + w) + 2E_{n+1}(v + w). \]

Set \( v = w = \frac{1}{2} \), and then use \( E_n = 2^nE_n\left(\frac{1}{2}\right) \), to obtain an expression for \( S_k(1) \). We have

\[ |S_k(1)| = |2^{2k+1}E_{2k+1}(1)| = 2^{2k+1}(2^{2k+2} - 1)|B_{2k+2}|/k + 1. \]
The evaluation of $E_{2k+1}(1)$ follows from [1, pg. 805]. Thus
\[
\sum_{j=0}^{k} |cA_j - A_k| = (-1)^k \sum_{j=0}^{k} (A_k - cA_j) = (k+1)(-1)^k A_k - c(-1)^k S_k(1) = (k+1)|A_k| - c|S_k(1)| = (k+1)|A_k| - \frac{\pi 2^{2k-1} (2^{2k+2} - 1)|B_{2k+2}|}{(k+1)(1 + 3^{-1-2k})}.
\]
To finish the verification of (11), we need to show that the last expression is bounded from above by $|A_k|$. This is equivalent to showing that
\[
\frac{\pi 2^{2k-1} (2^{2k+2} - 1)|B_{2k+2}|}{(k+1)(1 + 3^{-1-2k})} \geq k|A_k| = k|E_{2k}|.
\]
(12)
We will resort to an inequality for Bernoulli numbers [1, pg. 805]:
\[
\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1 - 2^{-2n})}.
\]
(13)
Thus we find
\[
\frac{\pi 2^{2k-1} (2^{2k+2} - 1)|B_{2k+2}|}{(k+1)(1 + 3^{-1-2k})} > \frac{\pi 2^{2k-1} (2^{2k+2} - 1) 2(2k+2)!}{(k+1)(1 + 3^{-1-2k}) (2\pi)^{2k+2}} = \frac{2^{k+1} (1 - 2^{-2-2k}) (2k+1)!}{(1 + 3^{-1-2k}) \pi^{2k+1}}.
\]
(14)
On the other hand, we have already used the fact that Euler numbers are bounded from above by
\[
k|E_{2k}| < k \frac{4^{2k+1}(2k)!}{\pi^{2k+1}}.
\]
(15)
Substituting (14) and (15) into (12), reduces the inequality to
\[
\frac{2k+1}{2k} > \frac{1 + 3^{-1-2k}}{1 - 2^{-2-2k}}.
\]
This final inequality is easily verified with elementary calculus for $k \geq 1$. Since the inequality is strict, we conclude immediately that $S_k(z)$ has only simple zeros which all lie on the unit circle.

We have proved that $S_k(z)$ has all of its zeros on the unit circle. Perhaps it is interesting to note that $S_k(z)$ satisfies
\[
\frac{(\pi/2)^{2k+1}}{2(2k)!} S_k(-z^2) = z^{2k} \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)\text{sech}(\pi n/2z)}{n^{2k+1}}
\]
\[
+ (-1)^k \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)\text{sech}(\pi nz/2)}{n^{2k+1}},
\]
where $\chi_{-4}(n)$ is the non-principle character mod 4. This formula appears immediately after equation (7) in Ramanujan’s notebook [3, pg. 276]. As a result it is
easy to approximate the zeros of $S_k(z)$ by the zeros of exponential polynomials. It remains to be seen whether or not there are any interesting applications for this observation.

To illustrate this method a second time, we prove that the polynomial $Y_k(z)$ has all of its non-zero roots on the unit circle. Notice that $Y_k(z)$ is a close analogue of $S_k(z)$, except that it involves Bernoulli numbers rather than Euler numbers.

**Theorem 2.** Suppose that $k \geq 2$. The polynomial

$$Y_k(z) = \frac{\pi}{2^{2k}} (Q_k(i\sqrt{z}) + Q_k(-iz))$$

$$= \frac{\pi^{2k}}{(2k)!} \sum_{j=0}^{k} B_{2j}B_{2k-2j}(2^{2j} - 1)(2^{2k-2j} - 1) \binom{2k}{2j} z^j,$$

has all of its non-zero roots on the unit circle. Furthermore, all of the zeros are simple.

**Proof.** Observe that $Y_k(z)$ has degree $k - 1$, since the coefficients of $z^k$ and $z^0$ are identically zero. We prove that $Y_k(z)/z$ satisfies the hypothesis of Schinzel’s theorem [13]. If we eliminate the trivial factor of $z$, then we obtain a polynomial of the form

$$\frac{Y_k(z)}{z} = \sum_{j=0}^{k-2} A_j z^j,$$

where

$$A_j = \frac{(2\pi)^{2k}}{(2k)!} \binom{2k}{2j+2} (1 - 2^{-2-2j})(1 - 2^{2-2k+2j})B_{2j+2}B_{2k-2j-2}.$$

Notice that $Y_k(z)/z$ is reciprocal, since $A_{k-2-j} = A_j$. By elementary properties of Bernoulli numbers, the sign of $A_j$ is $(-1)^k$ for all $j$.

Schinzel’s theorem can be applied if the following inequality holds:

$$|A_{k-2}| \geq \sum_{j=0}^{k-2} |cA_j - A_{k-2}|,$$  \hspace{1cm} (16)

for some $c \in \mathbb{C}$. We prove that (16) holds when $c = \frac{\pi^2(1-2^{-2k})}{8(1-2^{2-2k})}$. Our proof follows the same three steps as in the case of $S_k(z)$.

In order to remove the absolute value signs from (16), we need to demonstrate that $(-1)^k(cA_j - A_{k-2}) > 0$. We demonstrate this by comparing an upper bound on $(-1)^kA_{k-2}$, with a lower bound on $(-1)^kA_j$. The lower bound on $|A_j|$ is a consequence of an inequality from [5]:

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}(1-2^{-2n})}.$$

In particular we find

$$(-1)^kA_j = |A_j| > 4.$$  \hspace{1cm} (17)
By (13), we find an upper bound for $|A_{k-2}|$:

$$(-1)^k A_{k-2} = |A_{k-2}| < \frac{\pi^2}{2 \cdot 1 - 2^3 - 2k} 2^{2 - 2k},$$

(18)

Comparing (18) and (17), allows us to easily conclude

$$(-1)^k (cA_j - A_{k-2}) > 0,$$

(19)

whenever $k > 2$.

Since we have proved (19), Schinzel’s sum immediately reduces to

$$\sum_{j=0}^{k-2} |cA_j - A_{k-2}| = -(k-1)(-1)^k A_{k-2} + (-1)^k c \sum_{j=0}^{k-2} A_j.$$  (20)

Now we simplify the remaining sum. Let $B_j(z)$ denote the usual Bernoulli polynomials. By standard evaluations of Bernoulli polynomials [1, pg. 805], we have

$$A_j = \frac{(2\pi)^{2k}}{4(2k)!} \left( \frac{2k}{2j + 2} \right) \left( B_{2j+2} \left( \frac{1}{2} \right) - B_{2j+2}(0) \right) \left( B_{2k-2j-2} \left( \frac{1}{2} \right) - B_{2k-2j-2}(0) \right).$$

Next we use a well known convolution identity for Bernoulli polynomials [6]:

$$\sum_{j=0}^{n} \binom{n}{j} B_j(v) B_{n-j}(w) = n(v + w - 1)B_{n-1}(v + w) - (n - 1)B_n(v + w).$$  (21)

Considering all of the cases where $(v, w) \in \{ \left( \frac{1}{2}, \frac{1}{2} \right), (0, \frac{1}{2}), \left( \frac{1}{2}, 0 \right), (0, 0) \}$, leads to

$$\sum_{j=0}^{k-2} A_j = \frac{(2\pi)^{2k}}{(2k)!} \frac{(2k-1)}{2} \left( B_{2k} \left( \frac{1}{2} \right) - B_{2k}(0) \right)$$

$$= -\frac{(2\pi)^{2k}}{(2k)!} (2k - 1)(1 - 2^{-2k})B_{2k}.$$  (22)

Substituting (22) into (16), leads to a closed form expression for the sum we are interested in:

$$\sum_{j=0}^{k-2} |cA_j - A_{k-2}| = -(k-1)(-1)^k A_{k-2} - (-1)^k c \frac{(2\pi)^{2k}}{(2k)!} (2k - 1)(1 - 2^{-2k})B_{2k}.$$  

The proof can be completed by showing that this last expression is bounded from above by $|A_{k-2}|$ or

$$\frac{\pi^2(1 - 2^{2-2k})(2\pi)^{2k}}{8(1 - 2^3 - 2k)(2k)!} (2k - 1)(1 - 2^{-2k})|B_{2k}| < k|A_{k-2}|.$$  

Employing

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n}(2\pi)^{2n}}{2(2n)!}$$

reduces the desired inequality to

$$\frac{1}{2} (2k - 1) (1 - 2^{-2k}) \zeta(2k) < k(1 - 2^{3-2k}) \zeta(2k - 2).$$
It is elementary to show that this inequality holds for \( k > 1 \). \( \square \)

3. Generalizing the criteria to other families

Conditions such as (11) appear to be too restrictive to apply to polynomial families such as \( P_k(z) \), \( W_k(z) \) and \( Q_k(z) \). In this section we prove that \( Q_k(z) \) and \( W_k(z) \) have all their roots on the unit circle, by extending the theorems used in the previous section. Let us briefly recall how to derive results such as (10) and (11). For a real-valued reciprocal polynomial \( V_k(z) = \sum_{j=0}^{k} A_j z^j \), the condition

\[
|A_k| > \sum_{j=0}^{k} |c A_j - A_k|,
\]

immediately implies that

\[
1 > \left| \frac{c}{A_k} V_k(z) - v_k(z) \right|,
\]

where \( v_k(z) = \frac{k+1}{k-1} \). Notice that if \( v_k(z) \) is expanded in a geometric series, then (25) can be derived from (24) as a simple consequence of the triangle inequality. Despite the fact that (25) does not imply (24), it turns out that (25) is easily strong enough to conclude that \( V_k(z) \) has all of its zeros on the unit circle. To demonstrate this, first restrict \( z \) to the unit circle, and write \( z = e^{i\theta} \) with \( \theta \in (0, 2\pi) \), and

\[
v_k(\theta) = z^{-(k+1)/2} V_k(z) = \frac{\sin \left( \frac{(k+1)\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)}.
\]

If \( j < 2k+2 \) is a positive odd integer, then it is easy to show that \( \tilde{v}_k \left( \frac{j\pi}{k+1} \right) \) has sign \((-1)^{(j-1)/2}\), and \( |\tilde{v}_k \left( \frac{j\pi}{k+1} \right)| \geq 1 \). This implies that \( \tilde{v}_k(\theta) \) has at least \( k+1 \) interlacing positive and negative values in the interval \((0, 2\pi)\), and it has absolute value \( \geq 1 \) at each of those points. By (25) we can write \( \frac{1}{A_k} z^{-(k+1)/2} V_k(z) = \tilde{v}_k(\theta) + ET \), where the error term has absolute value less than 1. It follows that \( \frac{1}{A_k} z^{-(k+1)/2} V_k(z) \) changes sign at least \( k \) times for \( \theta \in (0, 2\pi) \). By the intermediate value theorem we conclude that \( V_k(z) \) has at least \( k \) zeros on the unit circle. Since the polynomial has at most \( k \) zeros, all of its zeros must lie on the unit circle.

We can easily extend this idea by selecting a different \( v_k(z) \).\(^1\) This typically entails constructing \( v_k(z) \) to approximate specific polynomial families.

**Definition 1.** Let \( f(\theta) : (a, b) \to \mathbb{R} \) be a continuous function. We call \( f(\theta) \) a \( k \)th order alternating function on \((a, b)\), if it assumes alternating positive and negative (or negative and positive) values at points \( p_i \), where \( a < p_1 < \cdots < p_{k+1} < b \). We say that \( f(\theta) \) has oscillation distance \( d \), if \( |f(p_i)| > d \) for each \( i \in \{1, \ldots, k+1\} \).

**Lemma 1.** Suppose that \( f(\theta) \) is a \( k \)th order alternating function on \((a, b)\), with oscillation distance \( d \). Let \( F(\theta) : (a, b) \to \mathbb{R} \) be a continuous function such that \(|F(\theta) - f(\theta)| < d \) for all \( \theta \). Then \( F(\theta) \) has at least \( k \) zeros.

\(^1\)This principle was inspired by a careful study of the proof in [10]
Proof. This lemma is essentially a restatement of the intermediate value theorem. The proof follows immediately from the method described in the previous discussion. \qed

3.1. The zeros of \( W_k(z) \) lie on the unit circle. The main result of this subsection is the following theorem:

**Theorem 3.** Suppose that \( k \geq 2 \). The polynomial

\[
W_k(z) = (2^{2k-1} + 2)P_k(z) - 2^k P_k(z/2) - P_k(2z)
\]

\[
= \frac{(2\pi)^{2k-1}}{(2k)!} 2^2 \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} (1 - 2^{1-2j})(1 - 2^{1-2k+2j}) \binom{2k}{2j} z^{2j}
\]

has all of its zeros on the unit circle. Furthermore, all of the zeros are simple.

In order to prove Theorem 3 we first need to establish that a certain trigonometric polynomial possesses the alternating property with oscillation distance 0.3.

**Lemma 2.** Suppose that \( k > 10 \). The function

\[
w_k(\theta) := 2 \cos(k\theta) + \frac{\pi^2}{3} \cos((k-2)\theta) + \frac{2}{(1-2^{1-2k})} \sin((k-3)\theta) \sin \theta
\]

is an alternating function of order \( 2k \) on \((-\pi, \pi)\), with oscillation distance 0.3.

**Proof.** We need to demonstrate that \(|w_k(\theta)| > .3\) for \(2k + 1\) values of \(\theta \in (-\pi, \pi)\). We must also show that the sign of \(w_k(\theta)\) alternates over successive points in this set. Since \(w_k(\theta)\) is even with respect to \(\theta\), and since \(w_k(0) > 3\), we only need to demonstrate that there are an additional \(k\) such points in \((0, \pi)\). Suppose that \(k > 10\), let \(\alpha\) be defined by

\[
\alpha = \frac{1}{\pi} \arccos \left( \frac{.3}{\frac{\pi^2}{3} - 2} \right) = .42\ldots, \tag{26}
\]

and let

\[
j_0 = [(k-1)\alpha] + 1. \tag{27}
\]

We claim that \(w_k(\theta)\) satisfies the necessary conditions on the following set of points:

\[
S = \left\{ \frac{\pi}{k-1}, \ldots, \frac{(j_0 - 1)\pi}{k-1} \right\} \cup \left\{ \frac{(j_0 - 1/2)\pi}{k-1}, \ldots, \frac{(k-j_0 - 1/2)\pi}{k-1} \right\}
\]

\[
\cup \left\{ \frac{(k-j_0)\pi}{k-1}, \ldots, \frac{(k-2)\pi}{k-1} \right\} \cup \left\{ \frac{(k-(1+\epsilon))\pi}{k-1} \right\},
\]

where \(\epsilon > 0\) is sufficiently small. First expand \(w_k(\theta)\) using trigonometric identities

\[
w_k(\theta) = \cos((k-1)\theta) \cos(\theta) \left( \frac{\pi^2}{3} + 2 - \frac{4}{1-2^{1-2k}} \right)
\]

\[
+ \sin((k-1)\theta) \left( \left( \frac{\pi^2}{3} - 2 - \frac{4}{1-2^{1-2k}} \right) \sin \theta + \frac{2}{1-2^{1-2k}} \csc \theta \right).
\]
Notice that \( S \) (essentially) arises from cases where either \( \cos((k-1)\theta) = 0 \), or \( \sin((k-1)\theta) = 0 \).

Begin by considering the cases where \( \theta = \frac{j\pi}{k-1} \) and \( j \in \{1, \ldots, k-2\} \). Then

\[
w_k \left( \frac{j\pi}{k-1} \right) = (-1)^j \cos \left( \frac{j\pi}{k-1} \right) \left( \frac{\pi^2}{3} + 2 - \frac{4}{1 - 2^{1-2k}} \right). \tag{28}
\]

In order to have \( |w_k \left( \frac{j\pi}{k-1} \right)| > 0.3 \), we need to restrict \( j \) so that

\[
\left| \cos \left( \frac{j\pi}{k-1} \right) \right| > \frac{0.3}{\frac{\pi^2}{3} + 2 - \frac{4}{1 - 2^{1-2k}}}. 
\]

In other words we must have

\[
\frac{j}{k-1} \notin (\alpha_k, 1 - \alpha_k),
\]

where

\[
\alpha_k = \frac{1}{\pi} \arccos \left( \frac{0.3}{\frac{\pi^2}{3} + 2 - \frac{4}{1 - 2^{1-2k}}} \right).
\]

Since \( k > 10 \) we have \( (k-1)(\alpha_k - \alpha) \ll 1 \). Therefore it is sufficient that

\[
\frac{j}{k-1} \notin (\alpha, 1 - \alpha),
\]

where \( \alpha \) is defined in (26). This implies that \( j \in \{1, 2, \ldots, j_0 - 1\} \cup \{k-j_0, \ldots, k-2\} \), with \( j_0 \) defined in (27). If \( j \in \{1, 2, \ldots, j_0 - 1\} \), then by (28) \( w_k \left( \frac{j\pi}{k-1} \right) \) has sign \((-1)^j\). If \( j \in \{k-j_0, \ldots, k-2\} \) then the cosine in (28) contributes an extra minus sign, and \( w_k \left( \frac{j\pi}{k-1} \right) \) has sign \((-1)^{j+1}\).

Now consider the case where \( \theta = \frac{(j-1/2)\pi}{(k-1)} \) and \( j \in \{j_0, \ldots, k-j_0\} \). By elementary properties of trigonometric functions, \( w_k(\theta) \) reduces to

\[
w_k \left( \frac{(j-1/2)\pi}{k-1} \right) = (-1)^{j+1} \left( \left( \frac{\pi^2}{3} - 2 - \frac{4}{1 - 2^{1-2k}} \right) \sin \left( \frac{(j-1/2)\pi}{k-1} \right) \right. 
\]

\[
\left. + \frac{2}{1 - 2^{1-2k}} \csc \left( \frac{(j-1/2)\pi}{k-1} \right) \right). \tag{29}
\]

In order to place a lower bound on this expression, first choose an interval \((\beta, 1 - \beta)\), which contains the set of rational numbers \( \left\{ \frac{j_0-1/2}{k-1}, \ldots, \frac{k-j_0-1/2}{k-1} \right\} \). This can be accomplished by selecting

\[
\beta = \begin{cases} 
\alpha & \text{if } j_0 > \alpha(k-1) + \frac{1}{2}, \\
\alpha - \frac{1}{2(k-1)} & \text{if } j_0 < \alpha(k-1) + \frac{1}{2}.
\end{cases}
\]

Notice that one of these situations must occur, because (27) guarantees that \( j_0 \in (\alpha(k-1), \alpha(k-1) + 1) \). We obtain the following lower bound from (29):

\[
\left| w_k \left( \frac{(j-1/2)\pi}{k-1} \right) \right| \geq \min_{\theta \in (\pi\beta, \pi(1-\beta))} \left| \left( \frac{\pi^2}{3} - 2 - \frac{4}{1 - 2^{1-2k}} \right) \sin \theta + \frac{2}{1 - 2^{1-2k}} \csc \theta \right|. 
\]
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The right-hand side is minimized at the end points of the interval $(\pi \beta, \pi(1 - \beta))$, so it follows that

$$\left| w_k \left( \frac{(j - 1/2)\pi}{k - 1} \right) \right| \geq \left| \left( \frac{\pi^2}{3} - 2 - \frac{4}{1 - 2^{1-2k}} \right) \sin \pi \beta + \frac{2}{1 - 2^{1-2k}} \csc \pi \beta \right|$$

Consider both choices of $\beta$, and recall the assumption that $k > 10$. A few easy calculations are sufficient to obtain

$$\left| w_k \left( \frac{(j - 1/2)\pi}{k - 1} \right) \right| > \begin{cases} 0.57 & \text{if } j_0 > \alpha(k - 1) + \frac{1}{2}, \\ 0.34 & \text{if } j_0 < \alpha(k - 1) + \frac{1}{2}, \end{cases}$$

for all values of $j \in \{j_0, \ldots, k - 2j_0\}$. It is easy to deduce from (29) that the sign of $w_k \left( \frac{(j - 1/2)\pi}{k - 1} \right)$ is precisely $(-1)^j$.

Finally consider the value of $w_k \left( \frac{(k - (1-\epsilon))\pi}{k - 1} \right)$. Notice that

$$w_k(\pi) = (-1)^k \left( \frac{\pi^2}{3} + 2 + \frac{2k - 6}{1 - 2^{1-2k}} \right).$$

Since $k > 10$ it follows easily that $|w_k(\pi)| > 19$, and $w_k(\pi)$ has sign $(-1)^k$. If $\epsilon$ is sufficiently small then $w_k \left( \frac{(k - (1-\epsilon))\pi}{k - 1} \right)$ also has sign $(-1)^k$, and absolute value much larger than .3.

To briefly summarize the sign values of $w_k(\theta)$, we have the following table:

| $\theta$     | $\text{Sign (}w_k(\theta)\text{)}$ |
|--------------|-----------------------------------|
| 0            | $(-1)^0$                          |
| $\pi$        | $(-1)^1$                          |
| $\frac{k-\pi}{2\pi}$ | $(-1)^2$                          |
| $\vdots$     | $\vdots$                          |
| $\frac{(j_0-1)\pi}{k-1}$  | $(-1)^{j_0-1}$                    |
| $\frac{(j_0-1/2)\pi}{k-1}$ | $(-1)^{j_0}$                      |
| $\vdots$     | $\vdots$                          |
| $\frac{(k-j_0-1/2)\pi}{k-1}$  | $(-1)^{k-j_0}$                    |
| $\frac{(k-j_0)\pi}{k-1}$  | $(-1)^{k-j_0+1}$                  |
| $\vdots$     | $\vdots$                          |
| $\frac{(k-2)\pi}{k-1}$  | $(-1)^{k-1}$                      |
| $\frac{(k-(1-\epsilon)\pi}{k-1}$ | $(-1)^k$                         |

This table shows that $w_k(\theta)$ changes sign at least $k$ times over the interval $(0, \pi)$. \qed

Now we use Lemma 2 to prove our main result.
Proof. (Theorem 3). Let us define $A_j$ using

$$W_k(iz) = \frac{(2\pi)^{2k-1}2^{2k}}{(2k)!} \sum_{j=0}^{k} B_{2j}B_{2k-2j}(1 - 2^{1-2j})(1 - 2^{1-2k+2j}) \binom{2k}{2j} z^{2j}$$

$$= \sum_{j=0}^{k} A_j z^{2j}.$$

By Lemma 1 it suffices to prove that $|z^{-k}W_k(iz) - w_k(z)| < 0.3$ where

$$w_k(z) = (z^k + z^{-k}) + \frac{\pi^2}{6} (z^{k-2} + z^{2-k}) + \frac{2}{(1 - 2^{1-2k})} \frac{z^{k-3} - z^{3-k}}{z - z^{-1}},$$

and $z = e^{i\theta}$. Thus we write

$$\left| \frac{z^{-k}W_k(iz)}{A_0} - w_k(z) \right| = \left| \sum_{j=0}^{k} \frac{A_j}{A_0} z^{2j-k} - (z^k + z^{-k}) - \frac{\pi^2}{6} (z^{k-2} + z^{2-k}) \right.$$}

$$- \frac{2}{(1 - 2^{1-2k})} \frac{z^{k-3} - z^{3-k}}{z - z^{-1}} \right|$$

$$\leq 2 \left| A_1 - \frac{\pi^2}{6} \right| + \sum_{j=2}^{k-2} \frac{A_j}{A_0} - \frac{2}{(1 - 2^{1-2k})},$$

where the second step makes use of a geometric series and the triangle inequality.

If we recall that $B_0 = -1/2$, and use both inequalities from (13), then we find

$$\frac{A_j}{A_0} = \frac{B_{2j}B_{2k-2j}(1 - 2^{1-2j})(1 - 2^{1-2k+2j}) \binom{2k}{2j}}{-B_{2k}(1 - 2^{1-2k})} < \frac{2}{(1 - 2^{1-2k})}.$$

Additionally we have, by (23),

$$\left| A_1 - \frac{\pi^2}{6} \right| = \left| \frac{\pi^2}{6} \zeta(2k - 2) - \frac{\pi^2}{6} \right|.$$

This second expression goes to zero as $k \to \infty$. For example, it is not hard to see that the absolute value is less than 0.01 for $k > 4$. 


Therefore we can remove the absolute value signs from (31). We find that

\[ |z^{-kw_k(iz)}/A_0 - w_k(z)| \leq 2A_1/A_0 - \pi^2/3 + 2(k-3)/1 - 2^{1-2k} - \sum_{j=2}^{k-2} A_j/A_0 \]

\[ = 2 + 4A_1/A_0 - \pi^2/3 + 2(k-3)/1 - 2^{1-2k} - \sum_{j=0}^{k} A_j/A_0 \]

\[ = 2 + 2\pi^2 \zeta(2k-2)/\zeta(2k) - \pi^2/3 + 2(k-3)/1 - 2^{1-2k} - 2k-1/1 - 2^{1-2k} \]

\[ = 2 + 2\pi^2 \zeta(2k-2)/\zeta(2k) - \pi^2/3 - 5/1 - 2^{1-2k} \]

\[ \leq -3 + \left( 2 - 2^{1-2k} - 1 \right) \pi^2/3. \]

Notice that we evaluated \( \sum_j A_j \) using the same Bernoulli convolution identity (21). In addition, we have used the inequality

\[ \frac{1}{1 - 2^{-n}} < \zeta(n) < \frac{1}{1 - 2^{1-n}}, \]

which are easy to deduce from the Euler product formula and the Dirichlet eta function. As \( k \to \infty \) this final upper bound approaches a limit of \( \pi^2/3 - 3 \approx .289 \). It is easy to see that it becomes \( < 0.3 \) for \( k > 6 \).

In summary, we have proved the theorem for \( k > 10 \). The cases for \( k \leq 10 \) are easily checked with the numerical method outlined in section 4. \( \square \)

3.2. The zeros of \( Q_k(z) \) lie on the unit circle. Notice that the coefficients of \( Q_k(z) \) involve the odd values of the Riemann zeta function. The primary goal of this subsection is to prove the following theorem:

**Theorem 4.** Suppose that \( k \geq 2 \). The polynomial

\[ Q_k(z) = (2^{2k} + 1)P_k(z) - 2^{2k}P_k(z/2) - P_k(2z) \]

\[ = \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j}B_{2k-2j}(2^{2j} - 1)(2^{2k-2j} - 1) \binom{2k}{2j} z^{2j} \]

\[ + \zeta(2k-1)(2^{2k-1} - 1)((-1)^kz + z^{2k-1}), \]

has all of its non-zero roots on the unit circle. Furthermore, all of the zeros are simple.

As in the proof of Theorem 3, the first step is to construct a trigonometric polynomial which approximates \( Q_k(z) \). Notice that \( Q_k(z) \) has degree \( 2k - 1 \), and that it has a trivial zero at \( z = 0 \). Therefore we need to prove that it has \( 2k - 2 \) zeros on the unit circle.
Lemma 3. Suppose that $k > 5$. Then
\[ q_k(\theta) := 2 \cos((k - 2)\theta) + \frac{4}{\pi} \sin((k - 1)\theta) + \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \sin(\theta) \]
is an alternating function of order $2k - 2$ on $(-\pi, \pi)$, with oscillation distance $0.03$. 

Proof. We need to demonstrate that $|q_k(\theta)| > 0.03$ for $2k - 1$ values of $\theta \in (-\pi, \pi)$. We must also show that the sign of $q_k(\theta)$ alternates over successive points in this subset. The proof is similar to the proof of Lemma 2, so we will be brief. Let $\alpha$ be defined by
\[ \alpha = \frac{1}{\pi} \arccos \left( \frac{0.03}{2 - \frac{4\theta}{\pi}} \right) = .47 \ldots , \]and let
\[ j_0 = [(k - 1)\alpha] + 1. \]We claim that $|q_k(\theta)| > 0.03$ on the following set of $2k + 1$ points:
\[ S = \{0\} \cup \left\{ \pm \frac{\pi}{k - 1}, \ldots, \pm \frac{j_0 - 1}{k - 1}\pi \right\} \cup \left\{ \pm \frac{(j_0 - 1/2)\pi}{k - 1}, \ldots, \pm \frac{k - j_0 - 1/2}{k - 1}\pi \right\} \]
\[ \cup \left\{ \pm \frac{(k - j_0)\pi}{k - 1}, \ldots, \pm \frac{k - 2}{k - 1}\pi \right\} \cup \left\{ \pm \frac{(k - (1 - \epsilon))\pi}{k - 1}\right\} . \]

If we consider $S \setminus \{(\frac{j_0 - 1/2}{k - 1}\pi, \frac{k - j_0 - 1/2}{k - 1}\pi\}$, then we obtain a subset of $2k - 1$ points where the sign of $q_k(\theta)$ alternates over successive points.

In order to prove this claim, first expand $q_k(\theta)$ using trigonometric identities
\[ q_k(\theta) = 2 \cos((k - 1)\theta) \cos(\theta) \left( 1 - \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \right) \]
\[ + \sin((k - 1)\theta) \left( 2 \sin(\theta) - \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \right) + \frac{4}{\pi} + \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \csc(\theta) . \]

Now consider the cases where $\theta = \frac{j\pi}{k - 1}$, with $-(k - 2) \leq j \leq (k - 2)$ and $j \neq 0$. We have
\[ q_k \left( \frac{j\pi}{k - 1} \right) = 2(-1)^j \cos \left( \frac{j\pi}{k - 1} \right) \left( 1 - \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \right) . \]To ensure that $|q_k(\frac{j\pi}{k - 1})| > 0.03$, we need to restrict $j$ so that
\[ \left| \cos \left( \frac{j\pi}{k - 1} \right) \right| > \frac{0.03}{2 \left( 1 - \frac{8(1 - 2^3 - 2k)}{\pi^2(1 - 2^{2k})} \right)} . \]

By similar reasoning as in the proof of Lemma 2, it is sufficient that
\[ \frac{j}{k - 1} \notin \{-(1 - \alpha), -\alpha\} \cup \{\alpha, 1 - \alpha\}, \]
where $\alpha$ is defined in (32). This immediately implies that $j \in \{\pm 1, \ldots, \pm (j_0 - 1)\} \cup \{\pm (k - j_0), \ldots, \pm (k - 2)\}$, where $j_0$ is defined in (33). A careful inspection of (34) reveals that the function has sign $(-1)^j$ for $j \in \{\pm 1, \ldots, \pm (j_0 - 1)\}$, and sign $(-1)^{j+1}$ for $j \in \{\pm (k - j_0), \ldots, \pm (k - 2)\}$.
Next consider the cases where \( \theta = \pm \frac{(j-1/2)\pi}{(k-1)} \) and \( j \in \{j_0, \ldots, (k-j_0)\} \). We obtain

\[
q_k \left( \pm \frac{(j-1/2)\pi}{2(k-1)} \right) = (-1)^{j-1} \left( 2 \sin \left( \frac{(j-1/2)\pi}{k-1} \right) \left( 1 - \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \right) \right)
\]

\[
\pm \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \csc \left( \frac{(j-1/2)\pi}{k-1} \right) + \frac{4}{\pi} \right)
\]  

Now select \( \beta \) so that \( \left\{ \frac{j_0-1/2}{k-1}, \ldots, \frac{k-j_0-1/2}{k-1} \right\} \subset (\beta, 1 - \beta) \). Following Lemma 2, this is accomplished by selecting

\[
\beta = \begin{cases} 
\alpha & \text{if } j_0 > \alpha(k-1) + \frac{1}{2}, \\
\alpha - \frac{1}{2(k-1)} & \text{if } j_0 < \alpha(k-1) + \frac{1}{2}.
\end{cases}
\]

Therefore we obtain

\[
\left| q_k \left( \pm \frac{(j-1/2)\pi}{2(k-1)} \right) \right| \geq \min_{\theta \in (\pi\beta, \pi(1-\beta))} \left| 2 \left( 1 - \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \right) \sin \theta + \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \csc \theta \pm \frac{4}{\pi} \right|
\]

\[
\geq \left| 2 \left( 1 - \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \right) \sin \pi \beta + \frac{8(1-2^3-2k)}{\pi^2(1-2^2-2k)} \csc \pi \beta \pm \frac{4}{\pi} \right|
\]

Checking both possible values of \( \beta \), and both possible signs of \( \pm \), leads to a lower bound which holds for \( k > 5 \):

\[
\left| q_k \left( \pm \frac{(2j-1)\pi}{2(k-1)} \right) \right| > 0.08...
\]

The final signs are summarized in the table below.

The only remaining cases are when \( j \in \{0\} \cup \{\pm \frac{(k-1-\epsilon)\pi}{k-1}\} \). These cases can be easily dispensed with by elementary properties of trigonometric functions.

To briefly summarize the sign values of \( q_k(\theta) \), we have the following table:

| \( \theta \) | \( \text{Sign} (q_k(\theta)) \) | \( \theta \) | \( \text{Sign} (q_k(\theta)) \) |
|---|---|---|---|
| \( \pm \frac{(j-1/\epsilon)\pi}{k-1} \) | \( (-1)^k \) | \( 0 \) | \( (-1)^0 \) |
| \( \pm \frac{(j-2)\pi}{k-1} \) | \( (-1)^{k-1} \) | \( \frac{\pi}{k-1} \) | \( (-1)^1 \) |
| \( \pm \frac{(j-j_0)\pi}{k-1} \) | \( (-1)^{j_0-1} \) | \( \frac{2\pi}{k-1} \) | \( (-1)^2 \) |
| \( \pm \frac{(j-j_0-1/2)\pi}{k-1} \) | \( (-1)^{j_0} \) | \( \frac{(j_0-1)\pi}{k-1} \) | \( (-1)^3 \) |
| \( \pm \frac{(j_0-1/2)\pi}{k-1} \) | \( (-1)^{j_0-1} \) | \( \frac{(j_0-1/2)\pi}{k-1} \) | \( (-1)^4 \) |
| \( \pm \frac{2\pi}{k-1} \) | \( (-1)^2 \) | \( \frac{(j_0)\pi}{k-1} \) | \( (-1)^5 \) |
| \( \pm \frac{\pi}{k-1} \) | \( (-1)^1 \) | \( \frac{(k-2)\pi}{k-1} \) | \( (-1)^6 \) |
| \( \pm \frac{(k-1-\epsilon)\pi}{k-1} \) | \( (-1)^k \) | \( \frac{(k-1-\epsilon)\pi}{k-1} \) | \( (-1)^7 \) |
Notice that there are precisely $2k + 1$ values of $\theta$ in this table. If we exclude the cases where $\theta \in \{(j_0 - 1/2)\pi, (k - j_0 - 1/2)\pi\}$, then the sine of $q_k(\theta)$ alternates over the remaining $2k - 1$ values of $\theta$. \hfill \Box

Next we use Lemma 3 to establish that $Q_k(z)$ has all of its non-zero roots on the unit circle for $k \geq 2$.

**Proof.** (Theorem 4). Let us define $A_j$ using

$$Q_k(iz) = \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} B_{2j} B_{2k-2j} (2^{2j} - 1) (2^{2k-2j} - 1) \binom{2k}{2j} z^{2j} + i (-1)^k \zeta (2k - 1) (2^{2k-1} - 1) (z - z^{2k-1})$$

In order to simplify the following analysis, we have intentionally defined $A_j$ to only involve the even coefficients of $Q_k(iz)$. Notice that $A_0 = A_k = 0$, and that

$$A_1 = \frac{(2\pi)^{2k-1}}{(2k)!} B_{2k-2} (2^{2k-2} - 1) \frac{k(2k - 1)}{2} = (-1)^k \zeta (2k - 2) (2^{2k-2} - 1).$$

Suppose that $k > 5$. Then by Lemma 1 it suffices to prove that $\left| \frac{z^{-k} Q_k(iz)}{A_1} - q_k(z) \right| < 0.03$ where

$$q_k(z) = (z^{k-2} + z^{2-k}) - \frac{2i}{\pi} \left( z^{k-1} - z^{1-k} \right) + \frac{8}{\pi^2} \frac{(1 - 2^{3-2k}) (z^{k-3} - z^{3-k})}{(1 - 2^{2-2k}) z - z^{-1}},$$  \hfill (36)  

and $z = e^{i\theta}$.

Therefore let us begin by writing

$$\left| \frac{z^{-k} Q_k(iz)}{A_1} - q_k(z) \right| = \sum_{j=0}^{k} \frac{A_j z^{2j-k}}{A_1} + (-1)^k \frac{\zeta (2k - 1) (2^{2k-1} - 1)}{A_1} (z^{1-k} - z^{k-1})$$

$$- (z^{-k} + z^{2-k}) + \frac{2i}{\pi} \left( z^{k-1} - z^{1-k} \right) - \frac{8}{\pi^2} \frac{(1 - 2^{3-2k}) (z^{k-3} - z^{3-k})}{(1 - 2^{2-2k}) z - z^{-1}}$$

$$\leq \sum_{j=2}^{k-2} \left( \frac{A_j}{A_1} - \frac{8}{\pi^2} \frac{(1 - 2^{3-2k})}{(1 - 2^{2-2k})} \right) + 2 \left| (-1)^k \frac{\zeta (2k - 1) (2^{2k-1} - 1)}{A_1} - \frac{2}{\pi} \right|.$$  

The second step follows from substituting a geometric series, and then applying the triangle inequality. We know from equation (19) (after noting the change in definition of $A_j$), that

$$\frac{A_j}{A_1} > \frac{8}{\pi^2} \frac{(1 - 2^{3-2k})}{(1 - 2^{2-2k})}.$$
In addition
\[ \left| (-1)^k \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1} - \frac{2}{\pi} \right| = \left| \frac{2\zeta(2k-1)(1-2^{1-2k})}{\pi \zeta(2k-2)(1-2^{2-2k})} - \frac{2}{\pi} \right|. \]

This limit tends to zero. A simple calculation shows that the quantity is less than
\[ 0.001 \] for \( k > 3 \).
Therefore we can remove the absolute value signs from the inequality. We are left
with
\[ \left| \frac{z^{-k}Q_z(i\pi)}{\pi} - q_k(z) \right| \]
\[ \leq \sum_{j=2}^{k-2} \frac{A_j}{A_1} - (k-3) \frac{8}{\pi^2} \frac{(1-2^{-2k})}{(1-2^{-2k})} + 4 \frac{\pi}{3} - 2(-1)^k \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1} \]
\[ = \sum_{j=1}^{k-1} \frac{A_j}{A_1} - 2 - (k-3) \frac{8}{\pi^2} \frac{(1-2^{-2k})}{(1-2^{-2k})} + 4 \frac{\pi}{3} - 4 \frac{\zeta(2k-1)(1-2^{1-2k})}{\pi \zeta(2k-2)(1-2^{2-2k})} \]
\[ = (2k-1) \frac{4}{\pi^2} \frac{\zeta(2k-2)(1-2^{-2k})}{\zeta(2k-2)} - 2 - (k-3) \frac{8}{\pi^2} \frac{(1-2^{-2k})}{(1-2^{-2k})} + 4 \frac{\pi}{3} - 4 \frac{\zeta(2k-1)(1-2^{1-2k})}{\pi \zeta(2k-2)(1-2^{2-2k})} \]
As usual, we have evaluated \( \sum_j A_j \) using (21). The limit of the upper bound is
\[ \frac{20}{\pi^2} - 2. \] It is easy to see that it becomes \( < 0.03 \) for \( k > 6 \). The cases for \( k < 6 \) are
easily proved with the numerical method described in Section 4.

We conclude this section by deriving a second approximation for \( \zeta(3)/\pi^3 \). If we
begin with the expression for \( Q \) in terms of \( P \), and then substitute a truncated
version of (7) for \( P \), we can obtain
\[ \zeta(3) \approx \frac{z}{\pi^3}, \quad \text{(37)} \]
where
\[ 0 = \frac{2z}{e^{4\pi z} - 1} + \frac{8z^3}{e^{\pi z} - 1} - \frac{17z}{e^{2\pi z} - 1} - \frac{17z^3}{e^{3\pi z} - 1} + \frac{8z}{e^{2\pi z} - 1} + \frac{2z^3}{e^{4\pi z} - 1}. \quad \text{(38)} \]
Selecting the zero given by \( z \approx .92 - .39i \), yields 4 decimal places of accuracy in
(37). Notice that this approximation is slightly worse than (9).

4. Partial results on \( P_k(z) \)
We have made a number of unsuccessful attempts to apply the theorems of
Schinzel, Lakatos and Losonczi, and their generalizations to the case of \( P_k(z) \). A
piece of evidence indicating that these methods may not work is given by the result

\[ \text{The most general result that we can prove is that } P_k(z) \text{ has at least } k-1 \text{ zeros in half of the unit circle, using the construction from [10].} \]
in [10] which shows that $R_{2k+1}(z)$ has four zeros that do not lie in the unit circle (by comparison $Y_k(z)$ has all of its roots on the unit circle).

We will briefly describe one instance were Lakatos’s condition (10) fails, because it leads to an interesting formula. Let us define the $A_j$’s as follows:

$$\sum_{j=0}^{4k} A_j z^j = |P_k(iz)|^2$$

$$= \left( \frac{2}{\pi} \sum_{j=0}^{k} \zeta(2j)\zeta(2k-2j)z^{2j} \right)^2 + \zeta^2(2k-1)\left(z^{2k-1} - z\right)^2. $$

Notice that $|P_k(iz)|^2$ has all of its zeros on the unit circle, if and only if $P_k(z)$ also has all of its zeros on the unit circle. Computational experiments helped us to make the following observation:

**Observation 1.** Suppose that $k \geq 2$, then

$$4k(k-1)|A_{4k}| = \sum_{j=0}^{4k} |A_{4k} - A_j|.$$  \hfill (39)

Formula (39) can be proved with the identities for Bernoulli numbers that we used in Theorem 2. This immediately rules out the possibility of applying (10) (condition (11) can also be ruled out by slightly different methods). It is curious to note that the right-hand side of (39) appears to involve odd zeta values, whereas the left-hand side does not. It turns out that when (39) is explicitly calculated, the odd zeta values drop out.

**Theorem 5.** Suppose that $2 \leq k < 1000$. Then all of the zeros of $P_k(z)$ lie on the unit circle. Furthermore, all of the zeros are simple.

While we have not been able to prove a general theorem concerning $P_k(z)$, we have been able to prove Theorem 5 for $k < 1000$. The proof uses a standard computational method based on the intermediate value theorem. Notice that the map

$$z + z^{-1} \rightarrow 2u,$$

sends the unit circle to the real interval $[-1, 1]$. Under this transformation, we also have

$$z^k + z^{-k} \rightarrow 2T_k(u),$$

where $T_k(u)$ is the usual Chebyshev Polynomial. If we write $(z^{2k} + (-1)^k)P_k(z) = z^{2k}(P_k(z) + P_k(1/z)) = 2z^{2k}P_k^*(u)$, then it follows that $P_k(z)$ has all of its zeros on the unit circle, if and only if

$$P_k^*(u) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} (-1)^j B_{2j} B_{2k-2j} \binom{2k}{2j} T_{2j}(u)$$

$$+ \zeta(2k-1) \left(T_{2k-1}(u) + (-1)^k T_1(u) \right),$$
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has all of its zeros in the interval $[-1, 1]$. It is easy to count real zeros of real-valued polynomials. The intermediate value theorem allows one to find zeros by detecting sign changes. Since $P_k^*(u)$ has degree $2k$, it is only necessary to detect $2k$ sign changes in $[-1, 1]$ (fewer sign changes are required if zeros lie at $u = \pm 1$). We have successfully carried out these calculations for $k < 1000$.

5. Conclusion

In conclusion, we have shown that $S_k(z)$, $Y_k(z)$, $W_k(z)$, and $Q_k(z)$ have all of their non-zero roots on the unit circle. These polynomials have a strong connection to the Ramanujan polynomials. We were disappointed that we were unable to deduce a similar theorem concerning $P_k(z)$, however we are hopeful that the approach outlined in Section 3 might eventually succeed in this case. An additional avenue might involve studying the zeros of a truncated version of the right-hand side of (7). Notice that the zeros of $P_k(z)$ are very well approximated by the zeros of

$$0 = \frac{z^{k-1}}{e^{2\pi/z} - 1} + (-1)^{k+1} \frac{z^{1-k}}{e^{2\pi z} - 1}.$$ 

Thus, it should be a worthwhile endeavor to study the zeros of these auxiliary functions.

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