Survival and extreme statistics of work in steady-state heat engines

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We derive universal bounds for the finite-time survival probability of the stochastic work extracted in steady-state heat engines. We also find estimates for the time-dependent thresholds that the stochastic work does not surpass with a prescribed probability. At long times, the tightest thresholds are proportional to the large deviation functions of stochastic entropy production. Our results, which entail an extension of martingale theory for entropy production, are tested with numerical simulations of a stochastic photoelectric device.

The magnitude of extreme values in stochastic processes and how often these may occur is a topic of primary interest in statistical physics with multidisciplinary applications [1]. Extreme fluctuations, although generally associated to small probabilities, may have a critical impact in many physical systems across different scales, like seismic activity leading to earthquakes or Tsunami waves, prize fluctuations producing crashes in the stock market, or violent winds impacting the performance of wind turbine loads and the power grid [2]. In small systems, fluctuations play a prominent role, often driving systems far away from equilibrium [3]. As a consequence, they are of crucial importance for the performance and robustness of small microscopic motors, heat engines and refrigerators [4–14].

Recent work [15–19] has applied the theory of Martingales [20]—a mathematical framework widely used in quantitative finance—to unveil new universal statistical properties in nonequilibrium stochastic thermodynamics. A milestone of this theory is the fact that the stochastic entropy production \(\dot{S}_{\text{tot}}(\tau) = \dot{S}_{\text{tot}}(X_\tau)\) generated along single stationary trajectories \(X_\tau = \{X(s)\}_{s=\tau}^\tau\) is an exponential martingale [15, 23], i.e., \(e^{-\dot{S}_{\text{tot}}(\tau)/kB}[X_\tau] = e^{-\dot{S}_{\text{tot}}(t)/Kn}\), where \(kB\) denotes Boltzmann’s constant.

Here and further, we denote by \((\Omega(\tau)|X_\tau)\) the conditional average of a functional \(\Omega(\tau) \equiv \Omega(X_\tau)\) over trajectories \(X_\tau\) whose history up to time \(t < \tau\) is known. Applying Doob’s theorems [20] for martingale processes to stochastic entropy production, universal equalities and inequalities concerning first-passage and minima of entropy production have been recently derived [15, 18, 24–28] and tested experimentally in nanoelectronic [24, 28, 29] and granular systems [30]. However, it remains an open, active area of research to derive universal “survival” statistics (e.g. the probability to remain below or above a given threshold) of fluctuating physical quantities in steady-state nonequilibrium processes [31, 32]. In the context of stochastic thermodynamics, it is relevant to study extrme and survival statistics of the work extracted by microscopic heat engines, and the peaks in the consumption of chemical fuel driving a molecular machine. Two important questions are: (i) what is the survival probability for the work not to exceed/fall below a certain threshold value? (ii) what is the “optimal” threshold that guarantees a prescribed value of the survival probability for the work extracted by a stationary heat engine?

In this Letter, we provide insights about the above questions by first deriving universal bounds for the cumulative distribution of the finite-time maximum and minimum of stochastic entropy production. We then apply these results to bound the survival statistics of the work extracted by steady-state engines permanently coupled to two heat baths at cold \(T_c\) and hot \(T_h\) temperatures. Here, we focus on the survival probability for the stochastic work extracted \(W(t)\) to not surpass a threshold \(w \geq 0\) anytime within the interval \(t \in [0, \tau]\) (with \(\tau\) sufficiently large). We show that the finite-time survival probability of the work can exceed a given prescribed probability \(\alpha < 1\), i.e.,

\[
Q(w, \tau) = \Pr\left(\max_{0 \leq t \leq \tau} W(t) < w\right) \geq 1 - \alpha
\]

whenever the threshold satisfies

\[
w \geq w_+(\tau) = \eta k_B T_c \left[\frac{\ln \left(\min_{p \geq 1} \left(e^{pS_{\text{tot}}(\tau)/kB}\right)^{1/p}\right)}{\eta C - \eta}\right].
\]

Equivalently, we show that the right tail of the distribution of work maxima is bounded by \(\Pr(\max_{t \leq \tau} W(t) > w_+(\tau)) \leq \alpha\). In other words, we find that with reliability \(1 - \alpha\), the maxima (i.e. peaks) of the work that a load system powered by an engine are given by the right-hand side of Eq. (2). Analogously, we also obtain a family of lower bounds for the probability of the minimum of the work extracted to not fall below a threshold \(-w \leq 0\) in \(t \in [0, \tau]\). Namely, we find that

\[
Q(-w, \tau) = \Pr\left(\min_{t \in [0, \tau]} W(t) > -w\right) \geq 1 - \alpha,
\]

where the inequality is verified whenever the threshold obeys

\[
w \geq w_-(\tau) = \frac{\eta k_B T_c}{\eta C - \eta} \left[\frac{\ln \left(\min_{p \geq 1} \left(e^{-pS_{\text{tot}}(\tau)/kB}\right)^{1/p}\right)}{\alpha^{1/p}}\right].
\]
The optimal thresholds \( w_{\pm}(\tau) \), given by the right-hand side of Eqs. (2) and (4), depend on the engine’s efficiency \( \eta \) and its distance from Carnot efficiency \( \eta_C = 1 - T_c/T_h \), the temperature of the cold bath \( T_c \), and the entropy-production generating function \( e^{\pm S_{\text{tot}}(\tau)} \) at the final time \( \tau \). This implies that reaching Carnot efficiency \( \eta \to \eta_C \) at finite dissipation, \( e^{\pm S_{\text{tot}}(\tau)} > \alpha \) leads to diverging power extreme fluctuations, as suggested by Ref. [9]. Remarkably, the \( \tau \)-dependent optimal thresholds depend on the desired probability \( \alpha \) of absorption. Thus, they provide bounds for the extreme excursions of work for different lengths of the operation interval \( \tau \), as illustrated in Fig. 1 for the example of a photoelectric device.

In the large operation limit, \( \tau \to \infty \), we can approximate \( \ln(e^{\pm S_{\text{tot}}(\tau)})^{1/p} \approx (\tau/p)\lambda_S(\pm p) \), where \( \lambda_S(k) \equiv \lim_{\tau \to \infty} (1/\tau) \ln(e^{kS_{\text{tot}}(\tau)}) \) is the scaled cumulant generating function of entropy production [33]. Notably, in this limit, the work optimal thresholds become linear with \( \tau \) and the optimization problem can be solved exactly:

\[
w_{\pm} (\tau) \simeq \frac{\eta k_B T_c}{\eta_C - \eta} \min_{p \geq 1} \left[ \lambda_S(\pm p) - \frac{\ln \alpha}{\tau p} \right]
= \frac{\eta k_B T_c}{\eta_C - \eta} \left[ \tau \lambda_S(\pm 1) - \frac{\ln \alpha}{\tau} \right], \tag{5}\]

where we have used the convexity of the scaled cumulant generating function and the fluctuation theorem \( \lambda_S(-p) = \lambda_S(p + 1) \) [21]. Note that because \( e^{-S_{\text{tot}}(\tau)/k_B} = 1 \) then \( \lambda_S(-1) = 0 \) and hence the optimal lower threshold saturates to a time-independent value \( w_- \approx -(\ln \alpha)\eta k_B T_c/(\eta_C - \eta) \). On the other hand, the optimal upper threshold reaches a linear saturating behaviour \( w_+(\tau) \approx \tau \lambda_S(1)\eta k_B T_c/(\eta_C - \eta) \) whose slope is independent on the significance level \( \alpha \). In practical situations, it might be approximated using the lower bounds to the generating function of entropy production [34–36].

The theoretical results in Eq. (1)–(4) are based on a family of universal bounds for the probability of entropy production maxima and minima which we derive and discuss in the following [Eqs. (8a)–(8b)], together with inequalities for the averages maximum and minimum of entropy production [Eqs. (10)]. In what follows, we consider a system described by a discrete variable \( X(t) \) which evolves in time following a Markovian, continuous-time nonequilibrium stationary process. The system is assumed to be in contact with one or several heat baths at temperatures \( T_k \) \((k \geq 1)\) such that every transition \( n \to m \) takes place at an average rate \( W_k(n,m) \) in contact with a single bath \( k \). Hence, the rates obey local detailed balance, i.e., \( W_k(n,m) = W_k(m,n)e^{\sigma_k(n,m)/k_B} \), with the \( \sigma_k(n,m) \) entropy change of the \( k \)-th bath in the \( n \to m \) transition. The stochastic entropy production associated with a single trajectory \( X_\tau \) is given by:

\[
S_{\text{tot}}(\tau) = k_B \ln \frac{P[X_\tau]}{P[X_\tau']} = \Delta S(\tau) - \sum_k \frac{Q_k(\tau)}{T_k}, \tag{6}\]

where \( P[X_\tau] \) is the probability of to observe the time-reversed path \( X_\tau' \equiv \{ X(\tau - t) \}_{t=0}^\tau \). Here, we have also introduced \( \Delta S(t) = k_B \ln(P_{st}[X(0)]/P_{st}[X(\tau)]) \) the stochastic system entropy change [22] and \( Q_k(\tau) \equiv -T_k \int_0^\tau \alpha_k[X(t),X(t + dt)] \) the stochastic heat absorbed from the \( k \)-th bath along the trajectory \( X_\tau \). From the definition (6) it follows that \( e^{-S_{\text{tot}}(t)/k_B} \) is a martingale. This implies that \( e^{\alpha S_{\text{tot}}(\tau)/k_B} \) is a submartingale process for any real \(|q| \geq 1\), that is

\[
\langle e^{qS_{\text{tot}}(\tau)/k_B} | X_\tau \rangle \geq e^{\alpha S_{\text{tot}}(t)/k_B}. \tag{7}\]

whenever \( \tau \geq t \geq 0 \). Equation (7) follows from applying Jensen’s inequality for conditional expectations. In particular, for the martingale \( M(t) = e^{-S_{\text{tot}}(t)/k_B} \) and the convex function \( f(x) = x^{-q} \) \((|q| \geq 1)\), we have \( \langle f(M(\tau)) | X_\tau \rangle \geq M(t) \) for any \( \tau \geq t \geq 0 \). Equivalently, for the choice \( f(x) = -\ln x \) one gets \( \langle S_{\text{tot}}(\tau)/k_B \rangle \geq 2 S_{\text{tot}}(t) \) which generalizes the Second law \( \langle S_{\text{tot}}(\tau) \rangle \geq 0 \).
We now apply Doob’s maximal inequality [20, 37] 
\[ \Pr(\max_{t \leq \tau} e^{\eta S_{\text{tot}}(t)/kB} \geq s) \leq \left( e^{\eta S_{\text{tot}}(\tau)/kB} / s \right) \Pr(\text{positive submartingales } e^{\eta S_{\text{tot}}(\tau)/kB}) \] 
(recall that \( |q| \geq 1 \)),
which yields, changing variables, the following universal bounds for the cumulative distribution of entropy production maxima and minima—and thus for its survival probability
\[ \Pr(S_{\text{max}}(\tau) \geq s) \leq e^{-ps/kB} \langle e^{pS_{\text{tot}}(\tau)/kB} \rangle \]
and
\[ \Pr(S_{\text{min}}(\tau) \leq -s) \leq e^{-ps/kB} \langle e^{-pS_{\text{tot}}(\tau)/kB} \rangle, \]
where we used here the short-hand notation \( S_{\text{max}}(\tau) = \max_{t \leq \tau} S_{\text{tot}}(t) \) and \( S_{\text{min}}(\tau) = \min_{t \geq \tau} S_{\text{tot}}(t) \), and here \( s > 0 \) and \( p \geq 1 \). Note that, applying \( p = 1 \) in Eq. (8b), one retrieves the bound \( \Pr(S_{\text{min}}(\tau) \leq -s) \leq e^{-s/kB} \) for the entropy-production infimum statistics derived in Ref. [15]. As we will show below, the value of \( p \) at which the tightest bounds are obtained is strongly dependent on \( \tau \) and the statistics of \( S_{\text{tot}}(\tau) \). Moreover, Eqs. (8a)-(8b) provide information on the survival probability with two absorbing, symmetric barriers located at \( \pm s \). The probability that \( S_{\text{tot}} \) escapes the interval \([−s, s]\) before time \( t \) obeys \( \Pr(T_{\text{esc}} \leq t) \leq \Pr(\max_{t \leq \tau} S_{\text{tot}}(t) \geq s) + \Pr(\min_{t \geq \tau} S_{\text{tot}}(t) \leq -s) \), which using Eqs. (8a)-(8b) yields
\[ \Pr(T_{\text{esc}} \leq t) \leq 2e^{-ps/kB} \langle \cosh(pS_{\text{tot}}(\tau)/kB) \rangle. \]

Equation (9) reveals that the cumulative escape-time distribution from \([−s, s]\) is bounded by only the even moments of entropy production.

Another related result concerns the average finite-time running maximum and minimum of entropy production. To this aim, we make use of Doob’s \( L^p \) inequality,
\[ \langle \max_{t \in [0, \tau]} |Z(t)|^p \rangle^{1/p} \leq \frac{p}{p/(p - 1)} \langle |Z(\tau)|^p \rangle^{1/p}, \]
where \( p > 1 \). Applying this theorem to the submartingales \( e^{\pm S_{\text{tot}}(\tau)/kB} \) (see Supplemental Material [38] for details) we derive the following universal bounds:
\[ \langle S_{\text{max}}(\tau) \rangle \leq kB \ln \left( \frac{p}{p - 1} \langle e^{pS_{\text{tot}}(\tau)/kB} \rangle^{1/p} \right) \]
and
\[ \langle S_{\text{min}}(\tau) \rangle \geq -kB \ln \left( \frac{p}{p - 1} \langle e^{-pS_{\text{tot}}(\tau)/kB} \rangle^{1/p} \right), \]
where here \( p > 1 \). Notably, these inequalities generalise the infimum law (\( S_{\text{min}}(\tau) \)) \( \geq -kB \) derived in Ref. [15]. As we will show below, the optimal bounds to the maximum and minimum of entropy production are obtained for different values of \( p \) that depend crucially on \( \tau \).

Close to equilibrium \( S_{\text{tot}} \) becomes Gaussian and hence
\[ \langle e^{\pm S_{\text{tot}}(\tau)/kB} \rangle = 1 + (p^2 \pm p) \langle S_{\text{tot}}(\tau) \rangle / kB, \]
and all the bounds (10) only depend on the average dissipation.

We now apply our generic results to the example of Markovian steady-state heat engines that are in contact with two thermal baths. The baths, at inverse temperatures \( T_h = 1/T_h \) and \( T_c = 1/T_c \) and chemical potentials \( \mu_h \) and \( \mu_c \), exchange energy and particles with the engine. The transition rates are assumed to fulfill local detailed balance \( \Gamma_{i \rightarrow j} = \Gamma_{j \rightarrow i} e^{\sigma_{i,j}} \), with \( \sigma_{i,j} = -\beta_h [\Delta E(i,j) - \mu_h \Delta N(i,j)] \) and \( \sigma_{j,i} = -\beta_c [\Delta E(i,j) - \mu_c \Delta N(i,j)] \) the entropy change in the hot and cold baths respectively when a transition \( i \rightarrow j \) happens. Here, \( \Delta E(i,j) = E(j) - E(i) \) is the system change in energy, and \( \Delta N(i,j) \) is the change of particle number in any of the baths. The work extracted reads
\[ W(t) = -\int_0^t \sum_k \mu_k \Delta N_k(t) \nu(t + dt) \] 
Using the first law \( \Delta E(\tau) + W(\tau) = Q_h(\tau) + Q_c(\tau) \) inside Eq. (6), one finds the stochastic entropy production for this model \( S_{\text{tot}}(\tau) = -\beta_h [W(t) + \Delta F_c(\tau)] + (\beta_c - \beta_h) Q_h(\tau) \), where \( \Delta F_c(\tau) \equiv \Delta E(\tau) - kB T_c \Delta S(\tau) \) is a nonequilibrium free energy change. Note that in the stationary state, all non-extensive quantities such as energy and entropy changes vanish on average, and all time-extensive currents become on average proportional to each other. In particular, \( (W(\tau)) = \eta Q_h(\tau) \) with \( \eta \) the macroscopic efficiency of the heat engine. The stochastic heat absorbed from the hot reservoir can be then approximated as \( Q_h(\tau) \approx \eta^{-1} W(\tau) + \xi(\tau) \), where \( \xi(\tau) \) is a bounded, zero mean \( \langle \xi(\tau) \rangle = 0 \) term that is non-extensive in time. Under this approximation we get \( S_{\text{tot}}(\tau) = \beta_c \left( \frac{2e^{-\eta}}{\eta} \right) W(\tau) + \beta [\eta_c \xi(\tau) - \Delta F_c(\tau)] \) It follows that
\[ \max_{t \leq \tau} S_{\text{tot}}(t) \leq \beta_c \left( \frac{\eta c - \eta}{\eta} \right) \max W(t) + \kappa, \]
\[ \min_{t \leq \tau} S_{\text{tot}}(t) \geq \beta_c \left( \frac{\eta c - \eta}{\eta} \right) \min W(t) - \kappa, \]
where \( \kappa \geq \beta_c \max_{t \in [0, \tau]} [\eta_c \xi(\tau) - \Delta F_c(\tau)] \geq 0 \) is a stochastic, bounded, non-extensive quantity. Using the inequalities (11a)-(11b) in Eqs. (8a)-(8b), we obtain the following bounds for the finite-time survival probability of the work extracted
\[ Q(\tau, w, \tau) \geq 1 - e^{-pw} \left( \frac{\eta c - \eta}{\eta c} \right) \rho_p(\tau, w) \] 
\[ Q(-w, \tau, -w) \leq 1 - e^{-pw} \left( \frac{\eta c - \eta}{\eta c} \right) \rho_p(\tau, w), \]
Using Eqs. (12a)-(12b) we can estimate what is the work threshold value \( w_+ > 0 \) \((-w_- < 0\) that a steady-state heat engine does not surpass (fall below) in a time interval of duration \( \tau \) with a prescribed probability \( 1 - \alpha \).They are obtained by solving \( Q(\tau, w, \tau) \geq 1 - \alpha \) and \( Q(-w, -\tau, \tau) \geq 1 - \alpha \), which yield
\[ w_{\pm}(\tau) = \left( \frac{\eta c T_c}{\eta c - \eta} \right) \ln \left( \frac{e^{\pm pS_{\text{tot}}(\tau)/kB}}{\alpha^{1/p}} \right) \pm \kappa, \]
with \( p \geq 1 \). Note that the threshold values in Eq. (13) increase with time, since \( e^{\pm pS_{\text{tot}}(\tau)/kB} \geq (e^{-S_{\text{tot}}(\tau)})^{-p} = 1 \) due to the fluctuation theorem, and it increases monotonically with time due to the submartingale property (7). Thus, for \( \tau \) large \( e^{\pm pS_{\text{tot}}(\tau)} \gg \alpha^{\pm pN} \), the term \( \kappa \) can be safely neglected yielding the approximate optimal bounds in Eqs. (2) and (4).
We now illustrate our results for the paradigmatic case of a stochastic photoelectric device [5, 39, 40], see Fig. 1B for an illustration. Two dots at with energy levels $\epsilon_1$ (left) and $\epsilon_r$ (right) exchange electrons with two fermionic baths that are at the same temperature $T_c$, but different chemical potentials $\mu_l$ and $\mu_r$, respectively. A photonic reservoir temperature $T_h$ induces transitions between the two dots, while non-radiative recombination processes are neglected [39, 41]. Due to Coulomb repulsion, either zero or one electron (in one of the two dots) can be occupying simultaneously the two energy levels, i.e. there are three possible configurations of the system. The energies of the three states of the system are $\epsilon_0 = 0$, $\epsilon_1 > 0$, and $\epsilon_r > \epsilon_1$ see Fig. 1B for an illustration with the transition rates. The tunnelling rates from the leads $(i = l, r)$ are given by $\Gamma^{+}_{i} = \gamma_0 f_i(\epsilon_{i}), \Gamma^{-}_{i} = \gamma_0 f_i(1 - \epsilon_{i})$, with $\gamma_0 > 0$, and $f_i(x) = 1/(e^{(x-\mu)/T_i} + 1)$ the corresponding Fermi distribution. For the bosonic reservoir $\Gamma^{+}_c = \Gamma^{-}_c = N_\hbar$, $\Gamma^{\pm}_c = \Gamma_\hbar(\tilde{n} + 1)$ where $\tilde{n} = 1/(e^{(\epsilon_r-\epsilon_{i})/T_h} + 1)$ is the average number of photons in the bath. For simplicity we assume in the following $\gamma_0 = \Gamma_0$. It can be straightforwardly checked that $\Gamma^{-}_{i} = \Gamma^{+}_{i} e^{(-\mu_{i})/T_{i}}$ for $i = l, r$ and $\Gamma^{-}_{c} = \Gamma^{+}_{c} e^{(-\epsilon_r-\epsilon_{i})/T_h}$. The efficiency of the device is given by the ratio between the applied voltage between leads $V = (\mu_r - \mu_l)/q_e$, with $q_e$ the elementary charge, and the difference in energies of the dots, as $\eta = q_e V / (\epsilon_r - \epsilon_1)$.

We perform numerical simulations of the photoelectric device for two values of the voltage bias in the device (see insets), close and far from the equilibrium point, $V_\star = (\epsilon_r - \epsilon_{i}) q_C/q_e$, and considered large operation times $\tau = 230 \Gamma_0^{-1}$ such that $\kappa \sim 10^{-1}$ can be safely neglected in our analysis. The performance of our bounds for the survival probability of work maxima $Q(w, \tau)$ [Eq. (12a)] and minima $Q(-w, \tau)$ [Eq. (12b)] are respectively illustrated in Fig. 2A and Fig. 2B. These bounds become tight for large $w$, corresponding to the right (left) tail of the maximum (minimum) work probability distribution. This implies that the optimal work thresholds $w_+(\tau)$ and $w_-(\tau)$ in Eqs. (1)-(3) are tight for small values of the failure probability $\alpha$. We also observe that close to the equilibrium point, as $w$ increases large values of $p$ provide the tightest bounds to $Q(w, \tau)$ and $Q(-w, \tau)$ (black solid lines). Far from equilibrium however, $p = 1$ is accurate even for large $w$.

In Fig. 2C we illustrate our results for the upper and lower work thresholds in Eq. (13) as a function of the failure probability $\alpha$ for fixed $\tau$ and different values of $p$ (dashed lines) [42]. The black solid lines show the optimal thresholds in Eqs. (1)-(3) obtained by optimizing over the range of $p$ that we explored. As can be appreciated in Fig. 2(c), for large values of $\alpha$, the tighter thresholds are obtained for $p = 1$, while as we reduce $\alpha$ the thresholds corresponding to higher values of $p$ become the most accurate ones. As the system is pushed far away from the equilibrium point, the transition from lower to higher $p$-values becomes more abrupt and occurs at lower values of $\alpha$ [38]. We also mention that, in general, high $p$ values are important to determine the optimal thresholds only for short lengths of the operation interval, while in the limit of large $\tau$ the optimal thresholds are determined by the $p = 1$ case, leading respectively to a linear growth for $w_+(\tau)$ and a constant behavior in $w_-(\tau)$, as predicted by Eq. (5) as shown in Fig. 1.

Fig. 3 illustrates the generalized supremum and infimum laws for entropy production given by Eqs. (10). Large values of $p$ provide tighter upper and lower bounds for short times, where the minimum of the entropy production becomes relevant. On the other hand, small values of $p$ dominate as longer observation times are considered, where the maximum becomes the dominant. Moreover we observe that, for all the time interval lengths considered here, the optimal lower bounds to $\langle S_{\text{min}}(\tau) \rangle$ with $p \geq 2$ are tighter than the infimum law $\langle S_{\text{min}}(\tau) \rangle \geq -k_B$ (dashed-dotted line).

The above results can be generalized to any other integrated current proportional to the total entropy produc-
tion, such as the heat dissipated into the cold reservoir \( Q_{\text{diss}} = -Q_c \), providing thresholds analogous to Eq. (13) which limit the fluctuations in the events of extreme heat dissipation by the engine (see [38]). Moreover, while we focused on heat engines, analogous results can be obtained for refrigerators.

In summary, we derived universal nonequilibrium inequalities for the entropy production extrema statistics (running maximum and minimum), together with time-dependent threshold bounds for the extreme fluctuations of work and heat in generic nonequilibrium steady-state heat engines. Our results may be relevant in the design of suitable load systems connected to the engine, which need to both absorb and release the corresponding peaks of work. Similar implications follow for the design of accurate detectors monitoring its heat exchange [43].

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**Supplemental Material to: “Survival and extreme statistics of work in steady-state heat engines”**

In this Supplemental Material we provide extra details on the derivations for obtaining the universal bounds on the survival probability of entropy production and work reported in the main text and additional material from the numerical simulations of the photoelectric device used to illustrate our results. In particular in Sec. S1 we provide a detailed derivation of Eqs. (8a) and (8b) in the main text for the probability of maxima and minima of entropy production in the interval [0, τ]. In Sec. S2 we provide a proof of Eq. (10) of the main text for the generalized maximum and minimum laws. Section S3 provides details on how our theory can be also applied to the fluctuations of the dissipative heat, rather than work in generic steady-state heat engines. Finally, the last Sec. S4 is dedicated to provide extra details on the results obtained for the photoelectric device introduced in the main text.

**S1. PROOF OF THE BOUNDS ON THE ENTROPY PRODUCTION EXTREMA STATISTICS**

Martingales are stochastic process $M(\tau)$ that are bounded, $|M(t)| < \infty$ for any finite time $t$, and that verify $\langle M(\tau) |_{\gamma(0,t)} \rangle = M(t)$ for $0 \leq t \leq \tau$. Similarly, submartingales $Z(t)$ are bounded stochastic process that verify $\langle Z(\tau) |_{\gamma(0,t)} \rangle \geq Z(t)$ [37]. A powerful property of non-negative martingale and submartingale processes is that they verify Doob’s submartingale inequality [20, 37]:

$$\lambda \Pr(\max_{t \in [0,\tau]} Z(t) \geq \lambda) \leq \langle Z(\tau) \rangle,$$

where $\lambda > 0$ is a positive constant. The above inequality puts a constraint on the right tail of the distribution of the maximum of $Z$ along the interval $[0, \tau]$ by its mean at the final time $\tau$, and the value of the threshold $\lambda$. Another related result is Doob’s $L^p$ inequality, which bounds the average of the maximum of $Z(t)$ in the interval over many trajectories $\langle \max_{t \in [0,\tau]} Z(t)^p \rangle^{1/p} \leq q \langle Z(\tau)^p \rangle^{1/p}$, where $p^{-1} + q^{-1} = 1$ and $p > 1$.

Let us consider now the finite-time maxima and minima of entropy production, $S_{\max}(\tau) \equiv \max_{t \in [0,\tau]} S_{tot}(t)$ and $S_{\min}(\tau) \equiv \min_{t \in [0,\tau]} S_{tot}(t)$, during the generic time interval $[0, \tau]$. We apply Doob’s maximal inequality [Eq. (S1)] to $Z(t) = e^{+pS_{tot}(t)}$ in Eq. (7) with $p \geq 1$. For the case $Z(t) = e^{+pS_{tot}(t)}$ we obtain the following family of exponential bounds for the statistics of entropy production maxima:

$$\Pr(S_{\max}(\tau) \geq S_+) \leq e^{-pS_+ \langle e^{pS_{tot}(\tau)} \rangle},$$

where $S_+ > 0$ is a positive threshold and we made the identification $\lambda = e^{pS_+}$ in Eq. (S1). Remarkably, the above equations imply that the probability of extreme (positive) entropy production events during the interval $[0, \tau]$, decreases exponentially with the threshold $S_+$ (and $p \geq 1$), multiplied by the moment generating function of the entropy production $\langle e^{pS_{tot}(\tau)} / k_B \rangle$ at the end of the interval. This set of bounds provide a tight envelope to the statistics of $S_{\max}(\tau)$, with higher (lower) values of $p$ providing narrow bounds for larger (smaller) values of the threshold $S_+$.

Analogously, the case $Z(t) = e^{-pS_{tot}(t)}$ give us a family of lower bounds for the probability that the finite-time minimum of entropy production, $S_{\min}(\tau)$, lies above some negative threshold $-S_-$ (with $S_- \geq 0$). To obtain this family of bounds we notice that $S_{\min}(\tau) = -\max_{t \in [0,\tau]} \ln Z(t) / p = -(1/p) \ln \max_{t \in [0,\tau]} Z(t)$ since the logarithm is monotonically increasing. Then by using Eq. (S1) and the above chain of equalities we obtain:

$$\Pr(S_{\min} \leq -S_-) \leq e^{-pS_- \langle e^{-pS_{tot}(\tau)} \rangle},$$

where again $p \geq 1$ and we identified $\lambda = e^{pS_+}$. This provides a family of exponential and time-dependent bounds to entropy production minima. For $p = 1$ we recover from Eq. (8b) and the integral fluctuation theorem $(e^{-S_{tot}(\tau)}) = 1$, the result in Ref. [15]. The later bound is saturated in the continuous (diffusive) limit when $t \to \infty$, as reported in Refs. [24, 25].

**S2. PROOF OF THE GENERALIZED SUPREMEUM AND INFIMUM LAWS**

The generalized maximum and minimum laws reported in Eq. (10) of the main text can be derived by using Doob’s $L^p$ inequality [20, 37]:

$$\langle \max_{t \in [0,\tau]} |Z(t)|^p \rangle^{1/p} \leq \left( \frac{p}{p - 1} \right) \langle |Z(\tau)|^p \rangle^{1/p},$$

which verifies Doob’s submartingale inequality. The later bound is saturated in the continuous (diffusive) limit when $t \to \infty$, as reported in Refs. [24, 25].
where $Z(t)$ is a submartingale and $p > 1$. Applying this inequality to the positive submartingales $Z(t) = e^{\pm S_{\text{tot}}(t)/kB}$ we obtain, respectively:

$$
\langle \max_{t \in [0,\tau]} e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p} \leq \left( \frac{p}{p-1} \right) \langle e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p}.
$$

(S5)

We now use that since the exponential function is a monotonically increasing function of $x$ we have $\max_{t \in [0,\tau]} e^{\pm p S_{\text{tot}}(t)/kB} = \exp(p \max_{t \in [0,\tau]} S_{\text{tot}}(t)/kB)$ and since $\max_{t \in [0,\tau]} S_{\text{tot}}(t)/kB = -\min_{t \in [0,\tau]} S_{\text{tot}}(t)/kB$, we obtain the two following inequalities:

$$
\langle \exp[p S_{\text{max}}(\tau)/kB] \rangle^{1/p} \leq \left( \frac{p}{p-1} \right) \langle e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p},
$$

(S6)

$$
\langle \exp[-p S_{\text{min}}(\tau)/kB] \rangle^{1/p} \leq \left( \frac{p}{p-1} \right) \langle e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p},
$$

(S7)

where we identified $S_{\text{max}}(\tau)$ and $S_{\text{min}}(\tau)$ respectively. Now we use in the left-hand-side of Eq. (S6) that the exponential function is convex, hence fulfilling Jensen’s inequality $\langle e^x \rangle \geq e^{\langle x \rangle}$, and in the left-hand-side of Eq. (S7) that the function $\phi(x) = x^p$ is also convex for $x > 0$ (we recall that $e^{-S_{\text{min}}(\tau)/kB}$ is non-negative) and hence Jensen’s inequality implies $|\langle x^p \rangle| \geq |\langle x \rangle|^p$. We then obtain the following lower bounds to the right hand sides of Eqs. (S6) and Eq. (S7) respectively:

$$
\exp[S_{\text{max}}(\tau)/kB] \leq \left( \frac{p}{p-1} \right) \langle e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p}.
$$

(S8)

$$
\exp[-S_{\text{min}}(\tau)/kB] \leq \left( \frac{p}{p-1} \right) \langle e^{\pm p S_{\text{tot}}(t)/kB} \rangle^{1/p}.
$$

(S9)

Finally by taking logarithms in both sides of the equations Eq. (S8) and (S9) and multiplying Eq. (S9) by $-1$ we immediately recover Eqs. (10) in the main text.

**S3. BOUNDS FOR THE DISSIPATED HEAT**

We can then use an analogous approximate relation to link the heat currents from the hot and cold baths as $Q_h(\tau) \simeq -Q_c(\tau)/(1-\eta) + \xi(\tau)$, where the stochastic quantity $\xi(\tau)$ is bounded and has zero mean, so that at the average level we recover $\langle Q_h(\tau) \rangle \simeq -Q_c(\tau)/(1-\eta)$. It is related to $\xi(\tau)$ in the main text as $\xi(\tau) = (\Delta E - \eta \xi(\tau))/(1-\eta)$. Using the above relation, the stochastic entropy production in Eq. (6) of the main text can be conveniently rewritten as $S_{\text{tot}}(\tau) = \Delta S(\tau) - Q_c(\tau)(\eta \xi - \eta)/(1-\eta) - \beta_\xi \xi(\tau)$. Then considering the maximum over the interval $[0,\tau]$ we obtain the following relations between the maximum and minimum of entropy production and the heat *dissipated* into the cold reservoir, $Q_{\text{diss}} \equiv -Q_c$:

$$
S_{\text{max}}(\tau) \leq \beta_\xi \left( \frac{\eta \xi - \eta}{1-\eta} \right) Q_{\text{diss}}^{\text{max}}(\tau) + \kappa',
$$

(S10)

$$
S_{\text{min}}(\tau) \geq \beta_\xi \left( \frac{\eta \xi - \eta}{1-\eta} \right) Q_{\text{diss}}^{\text{min}}(\tau) - \kappa',
$$

(S11)

with $Q_{\text{diss}}^{\text{max}}(\tau) = \max_{t \in [0,\tau]}[-Q_c(t)]$ the maximum of the heat dissipated into the cold reservoir during the interval $[0,\tau]$, and $Q_{\text{diss}}^{\text{min}}(\tau) = \min_{t \in [0,\tau]}[-Q_c(t)]$ the corresponding minimum (i.e. the maximum heat absorbed). Here $\kappa' \geq \max_{t \in [0,\tau]} |\Delta S(t) - \beta_\xi \xi(t)|$ bounding again the terms non-extensive in time. Following the same procedure as in the main text we obtain analogous relations for the extreme dissipation events dumping heat to the cold reservoir $Q_{\text{diss}} = -Q_c$. These can be expressed in terms of the survival probability for the heat dissipated, namely

$$
D(q,\tau) \equiv \Pr\left( \max_{0 \leq t \leq \tau} Q_{\text{diss}}(t) < q \right) \geq 1 - \alpha
$$

(S12)

$$
D(-q,\tau) \equiv \Pr\left( \min_{0 \leq t \leq \tau} Q_{\text{diss}}(t) > -q \right) \geq 1 - \alpha
$$

(S13)
which are respectively verified when \( q \geq \min_{\gamma \geq 1} q_{\pm}^{(p)} \) for the thresholds:

\[
q_+^{(p)}(\tau) = T_c \left( \frac{1 - \eta}{\eta_C - \eta} \right) \left[ \log \left( e^{pS_{\text{tot}}(\tau)}/\alpha^{1/p} \right) + \kappa' \right],
\]

(S14)

\[
q_-^{(p)}(\tau) = T_c \left( \frac{1 - \eta}{\eta_C - \eta} \right) \left[ \log \left( e^{pS_{\text{tot}}(\tau)}/\alpha^{1/p} \right) - \kappa' \right].
\]

(S15)

The thresholds depend again on the failure probability \( \alpha \), and guarantee a precise limit on events leading to an extreme dissipation on the cold reservoir that may lead to e.g. its overheating. We notice that \( q_+^{(p)}(\tau) \) above diverge if \( \eta \to \eta_C \) at finite power, and that the non-extensive term \( \kappa' \) can again be neglected when \( \tau \) is large. These results are in complete analogy to the work thresholds reported in the main text in Eq. (13).

S4. EXTRA DETAILS ON THE WORK THRESHOLDS FOR THE PHOTOELECTRIC DEVICE

In this section we provide extra details about our numerical results for the photoelectric device used in the main text to illustrate our universal bounds for the survival probability of work using the thresholds in Eq.(13) of the main text. In particular in the main text we mention that a feature of the work thresholds \( w_{\pm}(\tau) \) in Eq.(13) is that in far from equilibrium situations, where the entropy production is high, there is a more abrupt transition in the optimal thresholds from the bounds given by low values of \( p \) to high values.

This effect can be appreciated by comparing Fig. S1a and Fig. S1b where we show, respectively, the family of thresholds (13) for different values of \( p \) in close (\( \eta = 0.95\eta_C \)) and far-from-equilibrium situations (\( \eta = 0.54\eta_C \)) for long intervals \( \Gamma_{0}\tau = 230 \) as a function of the failure probability \( \alpha \). As it can be appreciated there, while in the former case all different values of \( p \) play an important role for some value of \( \alpha \), leading to a smooth curve, in the latter situation the thresholds for \( p = 1 \) dominate until very low values of \( \alpha \) where the thresholds quickly saturate to high values, \( p = 10 \) (notice that in Figs. S1b and S1c we do not include the more demanding case \( p = 100 \)), making intermediate values superfluous, and quickly bending the curve at \( \alpha \approx 10^{-4} \). This also implies that in far from equilibrium situations, computing the simpler case \( p = 1 \) gives a good approximation to the optimal thresholds even for moderate and low values of the failure probability up to \( \alpha \approx 10^{-4} \). Another feature that distinguish close from far-from-equilibrium situations is that while the upper threshold \( w_+(\tau) \) take similar values in both situations, substantial differences appear on the values of the lower threshold \( w_-(\tau) \), whose values can be multiplied by a factor 4 in the far-from-equilibrium case with respect to close to equilibrium. In other words, close to equilibrium the work thresholds are more symmetric and centered around zero, indicating that high fluctuations of either work extracted or performed by the load are similar. In contrast far away from equilibrium high fluctuations for work exerted on the system get suppressed more quickly than those for work extracted. Moreover the standard deviation of the distribution \( \Delta W(\tau) \) becomes notably smaller in this case.

FIG. S1. Work thresholds in Eq.(13) of the main text as a function of the failure probability \( \alpha \) for different values of \( p \) (dashed lines) see legend. The optimal thresholds in Eqs.(2) and (3) of the main text that guarantee bounded fluctuations are highlighted by the solid black lines. (a) Close to equilibrium conditions as given by \( \eta = 0.95\eta_C \) \( (q_{S}V = 1.95k_B T_c) \) and interval length \( \Gamma_{0}\tau = 230 \). (b) Far from equilibrium conditions, \( \eta = 0.54\eta_C \) \( (q_{S}V = 1.1k_B T_c) \) and interval length \( \Gamma_{0}\tau = 230 \). (c) Same far from equilibrium conditions and short time interval \( \Gamma_{0}\tau = 70 \). In the three cases we also show the average work \( \langle W(\tau) \rangle \) at the end of the interval \([0, \tau] \), together with its standard deviation \( \Delta W(\tau) \). Parameters of the simulation: \( T_h = 15T_c, \mu_i = 0.8k_B T_c, \epsilon_1 = 1.3k_B T_c, \epsilon_s = 3.5k_B T, N \sim 10^4 \) trajectories.
A similar trend as described above for the transition between work thresholds from low to high values of $p$, and on the asymmetry of upper and lower thresholds, can be also appreciated by comparing with Fig. S1c, which corresponds to the same far-from-equilibrium conditions than Fig. S1b ($\eta = 0.54 \eta_C$), but much smaller time-intervals, $\Gamma_0 \tau = 70$. We also see that for shorter lengths of the time-interval the transition from low to high values of $p$ in the optimal thresholds is displaced towards higher values of $\alpha$. This implies that the simplest case $p = 1$ give the optimal threshold (or a good approximation to it) only for values of $\alpha$ up to $10^{-3}$, in contrast to the $10^{-4}$ obtained for larger intervals. The later imply confidence intervals of 99.9% and 99.99% respectively for the work fluctuations to not surpass the thresholds.