Negation and Involution Adjunctions

Kosta Došen and Zoran Petrić

Mathematical Institute, SANU
Knez Mihailova 35, p.f. 367
11001 Belgrade, Serbia
email: {kosta, zpetric}@mi.sanu.ac.yu

Abstract
This note analyzes in terms of categorial proof theory some standard assumptions about negation in the absence of any other connective. It is shown that the assumptions for an involutive negation, like classical negation, make a kind of adjoint situation, which is named involutive adjunction. The notion of involutive adjunction amounts in a precise sense to adjunction where an endofunctor is adjoint to itself.

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Dedicated to Dov Gabbay on the occasion of his 60th birthday

1 Introduction

The goal of this note is to present a phenomenon of adjunction present in assumptions about an involutive negation connective, like classical negation. Proof-theoretical assumptions concerning such a negation make an adjoint situation that we call an involutive adjunction. The notion of involutive adjunction amounts, in a sense to be made precise, to adjunction where an endofunctor is adjoint to itself, which in [2] is called self-adjunction.

In a series of papers, which starts with [4] (see [5], [7] and [6]), Dov Gabbay has been working on characterizations of negation in terms of assumptions about a consequence relation. Sometimes, as in this note, Gabbay concentrates
on negation in the absence of any other connective. The context of the present note replaces Gabbay’s logical framework of a consequence relation by a consequence graph, as this is done in categorial proof theory. We do not have any more only a relation between premises and conclusions, but we have arrows between them, and there may be more than one such arrow. We are interested in equalities between these arrows. Often these equalities, which are proof-theoretically motivated, exemplify important notions of category theory. This note shows that with an involutive negation we fall on a particular notion of adjunction. This is yet another corroboration of Lawvere’s thesis that all logical constants are tied to adjoint situations (see [8]), and of Mac Lane’s slogan that adjunction arises everywhere (see [9], Preface).

2 Self-adjunctions

To fix notation and terminology, we will rely on the following definition of the notion of adjunction (cf. [9], Section IV.1, and [1], Section 4.1.3).

An adjunction is a sextuple \((A, B, F, G, \varphi, \gamma)\) where

\- \(A\) and \(B\) are categories,
\- \(F\) from \(B\) to \(A\) and \(G\) from \(A\) to \(B\) are functors,
\- \(\varphi\) is a natural transformation of \(A\) from the composite functor \(FG\) to the identity functor of \(A\), which means that the following equation holds in \(A\) for every arrow \(f : A_1 \rightarrow A_2\) of \(A\):
\[(\varphi \text{ nat})\quad f \circ \varphi_{A_1} = \varphi_{A_2} \circ FGf,\]
\- \(\gamma\) is a natural transformation of \(B\) from the identity functor of \(B\) to the composite functor \(GF\), which means that the following equation holds in \(B\) for every arrow \(g : B_1 \rightarrow B_2\) of \(B\):
\[(\gamma \text{ nat})\quad GFg \circ \gamma_{B_1} = \gamma_{B_2} \circ g,\]

the following triangular equations hold in \(A\) and \(B\) respectively:

\[(\varphi \gamma F)\quad \varphi_{FB} \circ F \gamma_B = 1_{FB},\]
\[(\varphi \gamma G)\quad G \varphi_A \circ \gamma_{GA} = 1_{GA}.\]

A self-adjunction is a quadruple \(\langle S, L, \varphi, \gamma \rangle\) where \(\langle S, S, L, L, \varphi, \gamma \rangle\) is an adjunction (this notion is taken over from [2], Section 10). So, in a self-adjunction, \(L\) is an endofunctor, and the equations \((\varphi \text{ nat})\) and \((\gamma \text{ nat})\) become
\[ f \circ \varphi_A = \varphi_{A_2} \circ LLf, \]
\[ LLf \circ \gamma_{A_1} = \gamma_{A_2} \circ f, \]
while the triangular equations become
\[ (\varphi \gamma L) \quad \varphi_LA \circ L\gamma_A = L\varphi_A \circ \gamma_LA = 1_{LA}. \]

A \textit{K-self-adjunction} is a self-adjunction that satisfies the additional equation
\[ (\varphi \gamma K) \quad L(\varphi_A \circ \gamma_A) = \varphi_LA \circ \gamma_LA, \]
and a \textit{J-self-adjunction} is a self-adjunction that satisfies the additional equation
\[ (\varphi \gamma J) \quad \varphi_A \circ \gamma_A = 1_A \]
(these notions are also from [2], Section 10). It is easy to see that every \textit{J-self-adjunction} is a \textit{K-self-adjunction} (the converse need not hold).

A \textit{J-self-adjunction} that satisfies
\[ (\gamma \varphi) \quad \gamma_A \circ \varphi_A = 1_{LLA} \]
is called a \textit{trivial} self-adjunction. Note that for trivial self-adjunctions it is superfluous to assume the equations (\gamma nat) and (\varphi \gamma G), or alternatively (\varphi nat) and (\varphi \gamma F); these equations can be derived from the remaining ones.

The \textit{free self-adjunction} \langle S, L, \varphi, \gamma \rangle generated by \{p\}, where we call \( p \) a letter, is defined as follows. The category \( S \) has as objects the formulae of the propositional language generated by \{p\} with a unary connective \( L \). We may identify the formulae \( p, Lp, LLp, \ldots \) of this language with the natural numbers 0, 1, 2, \ldots

The arrow terms of \( S \) are defined inductively out of the primitive arrow terms
\[ 1_A : A \to A, \quad \varphi_A : LLA \to A, \quad \gamma_A : A \to LLA, \]
for every object \( A \) of \( S \), with the help of the operations of composition \( \circ \) and the unary operation that assigns to the arrow term \( f : A \to B \) the arrow term \( Lf : LA \to LB \). On these arrow terms we impose the equations of self-adjunctions. In the set of these equations we have of course all the equations \( f = f \), and this set is closed under symmetry and transitivity of equality, and under the rules.
\[
\begin{align*}
\text{(cong \(\circ\))} & \quad f = f_1 \quad g = g_1 \\
& \quad f \circ g = f_1 \circ g_1 \\
\text{(cong \(L\))} & \quad f = g \\
& \quad Lf = Lg
\end{align*}
\]

We assume for \(f\) and \(g\) in (cong \(\circ\)) that they have composable types, such that \(f \circ g\) is defined; the same assumption is made for \(f_1\) and \(g_1\).

We define analogously the free \(K\)-self-adjunction, the free \(J\)-self-adjunction and the free trivial self-adjunction generated by \(\{p\}\), just by imposing additional equations.

### 3 Involutive adjunctions

Consider a category \(\mathcal{A}\) and a contravariant functor \(\neg\) from \(\mathcal{A}\) to \(\mathcal{A}\), which means that for \(f : A \to B\) in \(\mathcal{A}\) we have \(\neg f : \neg B \to \neg A\) in \(\mathcal{A}\), and the following equations are satisfied:

\[
\begin{align*}
\text{(\(\neg\1\))} & \quad \neg1_A = 1_{\neg A} \\
\text{(\(\neg(\circ\))} & \quad \neg(f \circ g) = \neg g \circ \neg f, \quad \text{for } f : A \to B \text{ and } g : C \to A.
\end{align*}
\]

The contravariant functor \(\neg\) may be conceived either as a functor from the category \(\mathcal{A}^{\text{op}}\) to \(\mathcal{A}\), which we denote by \(\neg\) too, or as a functor from \(\mathcal{A}\) to \(\mathcal{A}^{\text{op}}\), which we denote by \(\neg^{\text{op}}\).

Suppose that for every object \(A\) of \(\mathcal{A}\) we have an arrow \(n_A : \neg\neg A \to A\) of \(\mathcal{A}\). The arrow \(n_A\) becomes the arrow \(n_A^{\text{op}} : A \to \neg\neg A\) in \(\mathcal{A}^{\text{op}}\).

We say that \(\langle \mathcal{A}, \neg, n^{-}\rangle\) is an \(n^{-}\)-adjunction when

\[
\langle \mathcal{A}, \mathcal{A}^{\text{op}}, \neg, \neg^{\text{op}}, n^{-}, n^{-^{\text{op}}} \rangle
\]

is an adjunction. This means that in \(\mathcal{A}\) we have for every \(f : A_1 \to A_2\) the equation

\[
\text{(\(n^{-\text{nat}}\))} \quad f \circ n_{A_1}^{-} = n_{A_2}^{-} \circ \neg f,
\]

alternatively written \(f \circ n_{A_1}^{-} = n_{A_2}^{-} \circ \neg^{\text{op}} f\), which also delivers \((n^{-^{\text{op}}} \text{ nat})\) in \(\mathcal{A}^{\text{op}}\), and the equation
\[(n \to \text{triang}) \quad n_{\neg A} \circ n_{\neg A} = 1_{\neg A},\]

which delivers both the equation \((\varphi \gamma F)\), i.e. \((n \to n \to \neg \neg)\), in \(A\), and the equation \((\varphi \gamma G)\), i.e. \((n \to n \to \neg \neg \circ \neg \neg)\), in \(A^{op}\).

Suppose now that we have as before a category \(A\) and a contravariant functor \(\neg\) from \(A\) to \(A\), and that for every object \(A\) of \(A\) we have an arrow \(n_{\neg} : A \to \neg \neg A\) of \(A\). The arrow \(n_{\neg}\) becomes the arrow \(n_{\neg}^{op} : \neg \neg A \to A\) in \(A^{op}\).

We say that \(\langle A, \neg, n_{\neg} \rangle\) is an \(n_{\neg}\)-adjunction when

\[\langle A^{op}, A, \neg^{op}, \neg^{op}, n_{\neg}^{op}, n_{\neg}^{op} \rangle\]

is an adjunction. This means that in \(A\) we have for every \(f : A_1 \to A_2\) the equation

\[(n_{\neg} \text{ nat}) \quad \neg f \circ n_{\neg A_1} = n_{\neg A_2} \circ f,\]

which also delivers \((n_{\neg} \circ n_{\neg} \text{ nat})\) in \(A^{op}\), and the equation

\[(n_{\neg} \text{ triang}) \quad \neg n_{\neg A} \circ n_{\neg A} = 1_{\neg A},\]

which delivers both the equation \((\varphi \gamma F)\), i.e. \((n \to n \to \neg \neg \circ \neg \neg)\), in \(A^{op}\), and the equation \((\varphi \gamma G)\), i.e. \((n \to n \to \neg \neg \circ \neg \neg)\), in \(A\). Note that what we call \(n_{\neg}\)-adjunction is called self-adjunction in [11] (Section 3.1; cf. also [10], Section I.8), which should not be confused with our notion of self-adjunction in the preceding section.

We say that \(\langle A, \neg, n_{\neg}, n_{\neg} \rangle\) is an involutive adjunction when \(\langle A, \neg, n_{\neg} \rangle\) is an \(n_{\neg}\)-adjunction and \(\langle A, \neg, n_{\neg} \rangle\) is an \(n_{\neg}\)-adjunction.

A \(K\)-involutive adjunction is an involutive adjunction that satisfies the additional equation

\[(n_{\neg} \to K) \quad \neg (n_{\neg A} \circ n_{\neg A}) = n_{\neg A} \circ n_{\neg A},\]

and a \(J\)-involutive adjunction is an involutive adjunction that satisfies the additional equation

\[(n_{\neg} \to J) \quad n_{\neg A} \circ n_{\neg A} = 1_{\neg A}.\]

It is easy to see that every \(J\)-involutive adjunction is a \(K\)-involutive adjunction (the converse need not hold).

A \(J\)-involutive adjunction that satisfies

\[(n_{\neg} \to J) \quad n_{\neg A} \circ n_{\neg A} = 1_{\neg A} \]

is an \(n_{\neg}\)-adjunction.
is called a trivial involutive adjunction.

Note that for trivial involutive adjunctions it is superfluous to assume the equations \( (n^\leftarrow \text{nat}) \) and \( (n^\leftarrow \text{triang}) \), or alternatively \( (n^\rightarrow \text{nat}) \) and \( (n^\rightarrow \text{triang}) \); these equations can be derived from the remaining ones. In trivial involutive adjunctions we have the equations

\[
\begin{align*}
n^\leftarrow_{\neg A} & = \neg n^\rightarrow_A, \\
n^\rightarrow_{\neg A} & = \neg n^\leftarrow_A.
\end{align*}
\]

The free involutive adjunction \( \langle A, \neg, n^\rightarrow, n^\leftarrow \rangle \) generated by \( \{p\} \) is defined as follows. The category \( A \) has as objects the formulae of the propositional language generated by \( \{p\} \) with a unary connective \( \neg \). We may identify these formulae with the natural numbers.

The arrow terms of \( A \) are defined inductively out of the primitive arrow terms

\[
1_A: A \rightarrow A, \quad n^\rightarrow_A: \neg \neg A \rightarrow A, \quad n^\leftarrow_A: A \rightarrow \neg \neg A,
\]

for every object \( A \) of \( A \), with the help of the operations of composition \( \circ \) and the unary operation that assigns to the arrow term \( f: A \rightarrow B \) the arrow term \( \neg f: \neg B \rightarrow \neg A \). On these arrow terms we impose the equations of involutive adjunctions. In the set of these equations we have of course all the equations \( f = f \), and this set is closed under symmetry and transitivity of equality, under the rule \( (\text{cong } \neg) \), and also under the rule

\[
(\text{cong } \neg) \quad \frac{f = g}{\neg f = \neg g}.
\]

We define analogously the free \( K \)-involutive adjunction, the free \( J \)-involutive adjunction and the free trivial involutive adjunction generated by \( \{p\} \), just by imposing additional equations.

Note that the category of the free involutive adjunction generated by an arbitrary set having more than one letter would be the disjoint union of isomorphic copies of the category \( A \) of the free involutive adjunction generated by \( \{p\} \). An analogous remark applies to the category of the free self-adjunction generated by an arbitrary set having more than one member: it would be the disjoint union of isomorphic copies of the category \( S \) of the free self-adjunction generated by \( \{p\} \).
4 Self-adjunctions and involutive adjunctions

We are now going to prove that in the free self-adjunction \(\langle S, L, \varphi, \gamma \rangle\) and the free involutive adjunction \(\langle A, \neg, n^\rightarrow, n^\leftarrow \rangle\), both generated by \(\{p\}\), the categories \(S\) and \(A\) are isomorphic categories.

First, we define \(\neg, n^\rightarrow\) and \(n^\leftarrow\) in \(S\) in the following manner. On objects we have that \(\neg\) is \(L\), while for the arrow term \(f: A \to B\) of \(S\) we define the arrow term \(\neg f: \neg B \to \neg A\) of \(S\) inductively as follows:

\[
\begin{align*}
\neg 1_A & = L1_A = 1_{LA} = 1_{\neg A}, \\
\neg \varphi_A & = L\gamma_A, \\
\neg \gamma_A & = L\varphi_A, \\
\neg (f \circ g) & = -g \circ -f, \\
-Lf & = L\neg f.
\end{align*}
\]

That this defines an operation \(\neg\) on the arrows of \(S\) is shown by verifying that if \(f = g\) in \(S\), then \(\neg f = \neg g\) in \(S\); we verify, namely, that the equations of \(S\) are closed under the rule \((\text{cong } \neg)\) of the preceding section. This is done by a straightforward induction on the length of the derivation of \(f = g\) in \(S\). For that we use the fact that for every arrow term \(f\) of \(S\) the arrow term \(\neg f\) is equal in \(S\) to an arrow term of the form \(Lf'\).

Finally, we have

\[
n_{A}^\rightarrow = df \varphi_A, \quad n_{A}^\leftarrow = df \gamma_A.
\]

Next, we define \(L, \varphi\) and \(\gamma\) in \(A\) in the following manner. On objects we have that \(L\) is \(\neg\), while for the arrow term \(f: A \to B\) of \(A\) we define the arrow term \(Lf: LA \to LB\) of \(A\) inductively as follows:

\[
\begin{align*}
L1_A & = \neg 1_A = 1_{\neg A} = 1_{LA}, \\
Ln_{A}^\rightarrow & = -n_{A}^\rightarrow, \\
Ln_{A}^\leftarrow & = -n_{A}^\leftarrow, \\
L(f \circ g) & = Lf \circ Lg, \\
L\neg f & = -Lf.
\end{align*}
\]

That this defines an operation \(L\) on the arrows of \(A\) is shown by verifying that if \(f = g\) in \(A\), then \(Lf = Lg\) in \(A\); we verify, namely, that the equations of \(A\) are
closed under the rule \((\text{cong} L)\) of §2 above. This is done by a straightforward induction on the length of the derivation of \(f = g\) in \(A\). For that we use the fact that for every arrow term \(f\) of \(A\) the arrow term \(Lf\) is equal in \(A\) to an arrow term of the form \(\neg f'\).

Finally, we have

\[\varphi_A = df \ n_A^+, \quad \gamma_A = df \ n_A^- .\]

We verify easily by induction on the complexity of the arrow term \(f\) that both in \(S\) and in \(A\) we have the equation

\[(LL\neg) \quad LLf = \neg f .\]

Next we verify that the equations of involutive adjunctions hold for the defined \(\neg\), \(n^+\) and \(n^-\) in \(S\). This is done in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use \((LL\neg), (\varphi \text{ nat})\) and \((\gamma \text{ nat})\) to verify \((n^+ \text{ nat})\) and \((n^- \text{ nat})\), while the equations \((n^+ \text{ triang})\) and \((n^- \text{ triang})\) reduce to \((\varphi \gamma L)\). In the induction step, we rely on the closure of \(S\) under \((\text{cong } \neg)\), which we established above.

We verify also that the equations of self-adjunctions hold for the defined \(L\), \(\varphi\) and \(\gamma\) in \(A\). This is done again in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use \((LL\neg), (n^+ \text{ nat})\) and \((n^- \text{ nat})\) to verify \((\varphi \text{ nat})\) and \((\gamma \text{ nat})\), while the equations \((\varphi \gamma L)\) reduce to \((n^+ \text{ triang})\) and \((n^- \text{ triang})\). In the induction step, we rely on the closure of \(A\) under \((\text{cong } L)\), which we established above.

We have a functor \(F_A\) from \(S\) to \(A\) that maps the object of \(S\) corresponding to the natural number \(n\) to the object of \(A\) corresponding to \(n\), and that maps every arrow of \(S\) to the homonymous arrow in the defined \(S\) structure of \(A\). For example,

\[F_A \varphi LLp = \varphi_{\neg p} = n_{\neg p}^+ .\]

We define analogously a functor \(F_S\) from \(A\) to \(S\). That \(F_A\) and \(F_S\) are indeed functors follows from what we established above.

It is trivial that on objects we have that \(F_S F_A A\) is \(A\), and that \(F_A F_S B\) is \(B\). We show next by induction on the complexity of \(f\) that in \(S\) we have

\[F_S F_A f = f .\]
When $f$ is of the form $Lf'$, we make an auxiliary induction on the complexity of $f'$, in which we use $(LL\vdash\dashv)$. We show analogously that in $\mathcal{A}$ we have

$$F_{\mathcal{A}}F_{\mathcal{S}}g = g.$$ 

This concludes the proof that $\mathcal{S}$ and $\mathcal{A}$ are isomorphic categories.

We demonstrate analogously that the categories of, respectively,

- the free $\mathcal{K}$-self-adjunction and the free $\mathcal{K}$-involutive adjunction,
- the free $\mathcal{J}$-self-adjunction and the free $\mathcal{J}$-involutive adjunction,
- the free trivial self-adjunction and the free trivial involutive adjunction,

all generated by $\{p\}$, are isomorphic categories.

The interest of considering $\mathcal{K}$ and $\mathcal{J}$ versions of self-adjunctions and involutive adjunctions comes from connections with Temperley-Lieb algebras and the associated geometrical interpretation (see [2] and references therein). Roughly speaking, $\mathcal{K}$ is what we find in Temperley-Lieb algebras, where only the number of circles (which correspond to $\varphi_A \circ \gamma_A$ or $n_{\mathcal{A}}^A \vdash n_{\mathcal{A}}^A$) counts, while in $\mathcal{J}$ circles are disregarded.

The free trivial self-adjunction, and hence also the free trivial involutive adjunction, are preorders; namely, all arrows with the same source and target are equal. This follows from the results of [2] (unabridged version) or [3].

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