Volume growth of quasihyperbolic balls

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Abstract. The purpose of this paper is to study the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain $G \subset \mathbb{R}^n$ in terms of their radii.

Bibliography: 23 titles.

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§ 1. Introduction

Since its introduction three decades ago, the quasihyperbolic metric has become a popular tool in many subfields of geometric function theory, for instance, in the study of quasiconformal maps of $\mathbb{R}^n$, $n \geq 2$, and of Banach spaces (see [5] and [19]), the analysis of metric spaces [8] and of hyperbolic type metrics [7]. A natural question is whether or not and to what extent do the results of hyperbolic geometry have counterparts in quasihyperbolic geometry. For instance in [11] it was noticed that some facts from hyperbolic trigonometry of the plane have counterparts in the quasihyperbolic setup.

The purpose of this paper is to study the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain $G \subset \mathbb{R}^n$ in terms of their radii. Below we give an explicit growth estimate for the case of domains with Ahlfors regular boundary.

For a compact set with empty interior $E \subset \mathbb{R}^n$ we consider the layer sets $E(s,t) = \{ z \in \mathbb{R}^n : s \leq d(z,E) \leq t \}$, where $0 < s < t$, and relate their volume to the metric size of $E$ via its (Hausdorff) dimension under the additional assumption that $E$ be Ahlfors $Q$-regular for some $0 < Q < n$. It is essentially equivalent to formulate this idea in terms of the number of Whitney cubes of the Whitney decomposition of $\mathbb{R}^n \setminus E$ that meet $E(s,t)$. This idea goes back to [15]. One of the key ideas of [15] was to estimate the metric size of $E$ from above in terms of the size of the layer sets. In Lemmas 1 and 2, we will refine some results from [15].

In § 5 we estimate the growth of the quasihyperbolic volume of balls from below in the case when $G = \mathbb{R}^n \setminus E$ is $\psi$-uniform and $E$ is Ahlfors regular. The main result of the paper is Theorem 2, which gives a lower bound for the volume growth.

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§ 2. Notation

We assume \( n \geq 2 \) and use the notation \( B^n(x, r) = \{ z \in \mathbb{R}^n : |x - z| < r \} \) for the Euclidean ball, and its boundary is the sphere \( S^{n-1}(x, r) = \partial B^n(x, r) \), where the centre \( x \) can be omitted if \( x = 0 \).

The quasihyperbolic distance between two points \( x \) and \( y \) in a proper subdomain \( G \) of Euclidean space \( \mathbb{R}^n, n \geq 2 \), is defined by

\[
k_G(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_{\alpha} |dz|,
\]

where \( d(z) = d(z, \partial G) \) is the (Euclidean) distance between the point \( z \in G \) and the boundary of \( G \) (denoted \( \partial G \)), and \( \Gamma_{xy} \) is the collection of all rectifiable curves in \( G \) joining \( x \) and \( y \). Note that the quasihyperbolic metric in a half-space is just the hyperbolic metric, and that for a ball the two are bilipschitz equivalent. Some basic properties of the quasihyperbolic metric can be found in [22]. In particular, Martin and Osgood [13], p. 38, showed that for \( x, y \in \mathbb{R}^n \setminus \{0\} \) and \( n \geq 2 \)

\[
k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\theta^2 + \log^2 \frac{|x|}{|y|}}, \tag{2.1}
\]

where \( \theta = \angle(x, 0, y) \in [0, \pi] \).

For the volume of the unit ball in \( \mathbb{R}^n \) we use the notation

\[
\Omega_n := m(B^n(0, 1)) = \frac{\pi^{n/2}}{\Gamma((n/2) + 1)},
\]

where \( m \) is the \( n \)-dimensional Lebesgue measure and \( \Gamma \) stands for the usual \( \Gamma \)-function; see [1], Ch. 6. The surface area of the unit sphere is

\[
\omega_{n-1} := m_{n-1}(S^{n-1}(0, 1)) = n\Omega_n.
\]

The quasihyperbolic volume of a Lebesgue measurable set \( A \subset G \) is defined by

\[
\text{vol}_{k_G}(A) = \int_A \frac{dm(z)}{d(z)^n}.
\]

We also use the notation \( \text{vol}_k(A) \) for \( \text{vol}_{k_G}(A) \) if the domain \( G \) is clear from the context.

Below we use the Whitney decomposition of the complement of a closed set in \( \mathbb{R}^n \). If \( E \subset \mathbb{R}^n \) is nonempty and closed, then \( \mathbb{R}^n \setminus E \) can be represented ([18], p. 16) as a union of closed dyadic cubes \( Q^i_j \)

\[
\mathbb{R}^n \setminus E = \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^{N_i} Q^i_j, \tag{2.2}
\]

where the cubes \( Q^i_j \) have the following properties:

(i) each \( Q^i_j \) has faces parallel to the coordinate planes and edges of length \( 2^{-i} \);

(ii) the interiors of the cubes \( Q^i_j \) and \( Q^k_l \) are mutually disjoint provided \( l \neq j \) or \( i \neq k \);
of Whitney cubes.  

We set $\nu(E) = \mu(E)$. We refer to $\mu$ as the \textit{measure} for all $E \subset \mathbb{R}^n$. If $E$ is uniformly porous, then $E$ is a subset of some $Q$-regular set for some $p, r_0 > 0$. A set $E$ is uniformly porous if it is $Q$-regular for some $p, r_0 > 0$. We refer to $p, r_0$ and $n$ as the \textit{uniform porosity data}.

A set $E \subset \mathbb{R}^n$ is $Q$-\textit{regular} for $0 < Q < n$, if there is a (Borel regular, outer-) measure $\mu$ with $\text{spt}(\mu) = E$ and constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha r^Q \leq \mu(B^n(x, r)) \leq \beta r^Q$$

for all $x \in E$ and $r$, $0 < r < \text{diam}(E)$. Here $\text{spt}(\mu)$ denotes the smallest closed set with full $\mu$-measure. We refer to $n$, $\alpha$, $\beta$ and $\text{diam}(E)$ as the $Q$-\textit{regularity data}.

There is a close connection between the notions of uniform porosity and $Q$-regularity. Indeed, it is well known and easy to see (for example, [3], Lemma 3.12) that if $E$ is $Q$-regular for some $0 < Q < n$, then it is uniformly porous. Conversely, if $E$ is uniformly porous, then it is a subset of some $Q$-regular set for some $0 < Q < n$ (see [9], Theorem 5.3, for a more precise quantitative statement).

Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism with $\psi(0) = 0$. A domain $G \subset \mathbb{R}^n$ is said to be $\psi$-\textit{uniform} [23] if

$$k_G(x, y) \leq \psi \left( \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right)$$

for all $x, y \in G$. In particular, the domain $G$ is \textit{L-uniform} if $\psi(t) = L \log(1 + t)$ with $L > 1$. A domain is called \textit{uniform} if it is an $L$-uniform domain for some $L > 1$. These domains have been studied by many authors; see [6], [12], [14], [21] and [23].

§ 3. Basic examples for the quasihyperbolic volume

As mentioned in § 1, for a given domain $G$ and a fixed point $x \in G$, our goal is to find estimates for the asymptotic behaviour of $\text{vol}_k(B_k(x, r))$ when

$$B_k(x, r) = \{ z \in G : k_G(x, z) < r \}$$

is the quasihyperbolic ball.

As an example, we first consider very regular domains for which the correct asymptotic behaviour can be obtained by direct calculation.
3.1. The unit ball. We will find the quasihyperbolic volume of $B_k(0, r)$ in the unit ball $G = B^n = B^n(0, 1)$. Observe first that

$$B_k(0, t) = B^n(0, 1 - e^{-t}).$$

We need the following integral representation for the hypergeometric function: for $c > b > 0$,

$$F(a, b; c; x) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt,$$

where $B(b, c - b) = \Gamma(b)\Gamma(c - b)/\Gamma(c)$. By definition

$$\text{vol}_k(B^n(0, r)) = \int_{B^n(r)} \frac{dm(x)}{(1-|x|)^n} = \omega_{n-1} \int_0^r \frac{t^{n-1}}{(1-t)^n} dt = \frac{\omega_{n-1} r^n}{n(1-r)^{n-1}} F(1, 1; n+1; r),$$

(3.1)

where the last equality follows from

$$F(a, b; c; x) = (1-x)^{c-a-b} F(a, c-b; x)$$

(see [1], 15.3.3). Now $\text{vol}_k(B_k(0, r)) = \text{vol}_k(B^n(0, 1-e^{-r}))$ and hence for $0 < r < \infty$

$$\text{vol}_k(B_k(0, r)) = \frac{\omega_{n-1}(1-e^{-r})^n}{n(1-(1-e^{-r}))^{n-1}} F(1, 1; n+1; (1-e^{-r}))$$

$$\sim \frac{2^{1-n} \omega_{n-1}}{n-1} e^{(n-1)r} \text{ as } r \to \infty,$$

since by [1], 15.1.20,

$$F(1, 1; n+1; 1) = \frac{\Gamma(n+1)\Gamma(n-1)}{\Gamma(n)^2} = \frac{n}{n-1}.$$

Proposition 1. For $s > 0$, $\lambda > 1$, $G = B^n$ and $E = \mathbb{R}^n \setminus G$

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \to 1 - \lambda^{1-n}$$

as $s \to 0^+$.

Proof. Let $0 < s < 1/\lambda$. Then the claim is equivalent to

$$\frac{\text{vol}_k(B^n(1-s) \setminus B^n(1-\lambda s))}{\text{vol}_k(B^n(1-s))} \to 1 - \lambda^{1-n}$$

as $s \to 0^+$.
By (3.1)
\[
\frac{\text{vol}_k(B^n(1-s) \setminus B^n(1-\lambda s))}{\text{vol}_k(B^n(1-s))} = 1 - \frac{\lambda^{1-n}(1-\lambda s)^n F(1,1;n+1;1-\lambda s)}{(1-s)^n F(1,1;n+1;1-s)} \to 1 - \lambda^{1-n} \quad \text{as } s \to 0.
\]

**Remark 1.** For \( G = B^n \), let \( m_h(B^n(r)) \) be the hyperbolic volume of the Euclidean ball \( B^n(r) \). For a Lebesgue measurable set \( A \subset B^n \) its hyperbolic volume is defined by
\[
m_h(A) = \int_A \frac{2^n \, dm(x)}{(1-|x|^2)^n}.
\]
It follows that
\[
\left(\frac{2}{1+r}\right)^n \text{vol}_k(B^n(r)) = \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1+r)^n (1-t)^n} \, dt \leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t^2)^n} \, dt = m_h(B^n(r))
\]
and
\[
m_h(B^n(r)) \leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t^2)^n} \, dt = 2^n \, \text{vol}_k(B^n(r)).
\]
It is easy to see that
\[
m_h(B^n(r)) = \omega_{n-1} 2^n r^n \frac{1}{n} F\left(\frac{n}{2}, n; 1 + \frac{n}{2}; r^2\right).
\]
Proposition 1 is true for the hyperbolic metric with the same limiting value \( 1 - \lambda^{n-1} \).

### 3.2. Punctured space

We will consider the growth of quasihyperbolic volume in multiply connected domains. It can be shown (see [11], Lemma 4.56, Theorem 4.57 and Corollary 4.58) that in \( \mathbb{R}^2 \setminus \{0\} \) the quasihyperbolic area of \( B_k(x,r) \) is equal to \( \pi r^2 \) if \( r \leq \pi \), and
\[
2\pi \sqrt{r^2 - \pi^2} + 2\pi^2 \tan^{-1}\left(\frac{\pi}{\sqrt{r^2 - \pi^2}}\right) \quad \text{if } r > \pi.
\]
Therefore,
\[
2\pi \sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x,r)) \leq 4\pi r \quad \text{if } r > \pi.
\]
We will now find similar lower and upper bounds for the quasihyperbolic volume of \( B_k(x,r) \) in \( \mathbb{R}^n \setminus \{0\} \) for \( n > 2 \).

**Proposition 2.** For \( x \in \mathbb{R}^n \setminus \{0\}, n > 2 \) and \( r > \pi \)
\[
2\omega_{n-1} \sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x,r)) \leq 2\omega_{n-1} r.
\]
Proof. Since $B_k(x, r)$ is invariant under inversion in $S^{n-1}(x)$, we have

$$\text{vol}_k(B_k(x, r)) = 2 \text{vol}_k(B_k(x, r) \cap B^n(|x|)).$$

By [22], (3.9), $B_k(x, r) \subset \mathbb{R}^n \setminus B^n(|x|e^{-r})$. Let $S = S^{n-1}(|x|e^{-\sqrt{n-1}r^2})$, then by (2.1), $\max_{y \in S} k_{\mathbb{R}^n \setminus \{0\}}(x, y) = r$ implies that

$$B^n(|x|) \setminus B^n(|x|e^{-\sqrt{n-1}r^2}) \subset B_k(x, r).$$

Hence we have

$$\text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-\sqrt{n-1}r^2})) \leq \text{vol}_k(B_k(x, r) \cap B^n(|x|)) \quad (3.2)$$

and

$$\text{vol}_k(B_k(x, r) \cap B^n(|x|)) \leq \text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-r})). \quad (3.3)$$

Next we find the quasihyperbolic volume of the annulus $E(a, b) = \{x \in \mathbb{R}^n : a < |x| < b\}$, $0 < a < b < \infty$. By definition

$$\text{vol}_k(E(a, b)) = \int_{E(a, b)} \frac{dm(x)}{|x|^n} = \omega_{n-1} \int_a^b \frac{t^{n-1}}{t^n} dt = \omega_{n-1} \log \frac{b}{a}. \quad (3.4)$$

Now the assertion follows from equations (3.2) and (3.3).

**Corollary 1.** For $s > 0$, $\lambda > 1$, $G = B^n \setminus \{0\}$ and $E = \mathbb{R}^n \setminus G$

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \to 1 - \lambda^{1-n}$$

as $s \to 0$.

**Proof.** Let $s < 1/\lambda$. Then $E(s, \lambda s) = E_1 \cup E_2$, where $E_1 = B^n(1-s) \setminus B^n(1-\lambda s)$ and $E_2 = B^n(\lambda s) \setminus B^n(s)$. Then by (3.1) and (3.4)

$$\text{vol}_{k_G}(E(s, \lambda s)) = \text{vol}_{k_{B^n}}(E_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E_2)$$

$$= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n + 1; 1 - s)$$

$$- \frac{\omega_{n-1}(1 - \lambda s)^n}{n(\lambda s)^{n-1}} F(1, 1; n + 1; 1 - \lambda s) + \omega_{n-1} \log \lambda$$

$$\sim \frac{\omega_{n-1}}{n - 1} (1 - \lambda^{1-n}) s^{1-n} \text{ as } s \to 0.$$ 

Similarly $E(s, \infty) = B^n(1-s) \setminus B^n(s) = E'_1 \cup E'_2$, where $E'_1 = B^n(1-s) \setminus B^n(1/2)$ and $E'_2 = B^n(1/2) \setminus B^n(s)$, and

$$\text{vol}_{k_G}(E(s, \infty)) = \text{vol}_{k_{B^n}}(E'_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E'_2)$$

$$= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n + 1; 1 - s)$$

$$- \frac{\omega_{n-1}}{2n} F\left(1, 1; n + 1; \frac{1}{2}\right) + \omega_{n-1} \log \frac{1}{2s}$$

$$\sim \frac{\omega_{n-1}}{n - 1} s^{1-n} + \omega_{n-1} \log \frac{1}{s} \text{ as } s \to 0.$$ 

Now we have

$$\lim_{s \to 0} \frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} = \lim_{s \to 0} \frac{(\omega_{n-1}/(n-1))(1 - \lambda^{1-n}) s^{1-n}}{(\omega_{n-1}/(n-1)) s^{1-n} + \omega_{n-1} \log(1/s)} = 1 - \lambda^{1-n}.$$
3.3. **Half-space.** The next two propositions concern the case of the half-space, in which case the quasihyperbolic volume coincides with the classical hyperbolic volume. The hyperbolic volume of hyperbolic simplexes in the upper half-space was considered by Milnor in [17].

**Proposition 3.** For \( x \in \mathbb{H}^2 \) and \( r > 0 \) we have
\[
\text{vol}_k(B_k(x, r)) = 2\pi (\cosh r - 1).
\]

**Proof.** Choose \( x = te_2 = (0, t) \in \mathbb{H}^2 \), which implies that
\[
B_k(x, r) = B^2(e_2 t \cosh r, t \sinh r).
\]
Now
\[
\text{vol}_k(B_k(x, r)) = \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \sqrt{\frac{t^2 \sinh^2 r - (\cosh r - h)^2}{h^2}} \, dh = 2\pi (\cosh r - 1).
\]

**Proposition 4.** For \( x \in \mathbb{H}^3 \) and \( r > 0 \) we have
\[
\text{vol}_k(B_k(x, r)) = \pi (\sinh(2r) - 2r).
\]

**Proof.** Choose \( x = te_3 = (0, 0, t) \in \mathbb{H}^3 \), so that
\[
B_k(x, r) = B^3(e_3 t \cosh r, t \sinh r).
\]
Now
\[
\text{vol}_k(B_k(x, r)) = \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \frac{\pi (t^2 \sinh^2 r - (\cosh r - h)^2)}{h^3} \, dh = \pi (\sinh(2r) - 2r).
\]

§ 4. **The number of Whitney cubes**

Recall that \( N_i \) denotes the number of the \( i \)th-generation Whitney cubes of \( \mathbb{R}^n \setminus E \). We also define the number
\[
\bar{N}_i = N_{i+k_0} + \cdots + N_{i+k_1}, \quad (4.1)
\]
where \( k_0 \) is the smallest integer satisfying \( 8\sqrt{n} 2^{-k_0} \leq 1 \) and \( k_1 \) is the largest integer with \( 80\sqrt{n} 2^{-k_1} \geq 1 \).

**Lemma 1.** If \( E \subset \mathbb{R}^n \) is compact and uniformly porous, then there are constants \( 0 < c < C < \infty \) and \( i_0 \in \mathbb{N} \) (depending only on the uniform porosity data) such that for all \( i \geq i_0 \)
\[
c 2^m(E(2^{-i})) \leq \bar{N}_i \leq C 2^m(E(2^{-i})).
\]

**Proof.** Let \( 0 < \lambda < 1 < \Lambda < \infty \). Choose \( 0 < p < (1 + \lambda)/(2\Lambda) \) and \( r_0 > 0 \) such that \( E \) is \((2p, r_0)\)-uniformly porous. Given \( r > 0 \) such that \( R = (1 + \lambda)r/(2p) < r_0 \), let \( B_1, \ldots, B_N \) be a maximal collection of disjoint balls of radius \( R \) centred at \( E \). Then by elementary geometry
\[
N \Omega_n R^n \leq m(E(R)) \leq 3^n N \Omega_n R^n. \quad (4.2)
\]
For each $B_i$, we can find $y_i \in B_i/2$ with $d(y_i, E) \geq pR = (1 + \lambda)r/2$. Moreover (since we live in the Euclidean space), we can in fact find such a $y_i$ with $d(y_i, E) = (1 + \lambda)r/2$. Then $B(y_i, (1 - \lambda)r/2) \subset E(r) \setminus E(\lambda r)$ and combining this with (4.2),

$$m(E(r) \setminus E(\lambda r)) \geq N\Omega_n \left(\frac{1 - \lambda}{2}\right)^n r^n \geq \left(\frac{p(1 - \lambda)}{3(1 + \lambda)}\right)^n 3^n N\Omega_n R^n$$

(recall that $R \geq \Lambda r$).

Applying the above estimate with $r = 2^{-i-3}$, $\lambda = 1/2$ and $\Lambda = 8$ and using (2.5) implies that

$$m(E(2^{-i})) \leq cm(E(2^{-i-3}) \setminus E(2^{-i-4})) \leq c \sum_{j=k_0}^{k_1} N_{i+j} 2^{-(i+j)n} \leq cm(E(2^{-i})).$$

where $c < +\infty$ depends only on $n$ and $p$. Lemma 1 is proved.

We also need the following simple fact.

**Lemma 2.** If $E \subset \mathbb{R}^n$ is compact and $Q$-regular for some $0 < Q < n$, then there are constants $0 < c < C < \infty$ (depending only on the $Q$-regularity data) such that $cr^{n-Q} \leq m(E(r)) \leq Cr^{n-Q}$ for $0 < r < \text{diam}(E)$.

**Proof.** Let $r \in (0, \text{diam}(E))$. Let $B_1, \ldots, B_N$ be a maximal collection of disjoint balls, each of radius $r$ and centred at points in $E$. Then, as in (4.2),

$$\bigcup_{i=1}^{N} B_i \subset E(r) \subset \bigcup_{i=1}^{N} 3B_i$$

and in particular,

$$N\Omega_n r^n \leq m(E(r)) \leq N\Omega_n 3^n r^n. \quad (4.3)$$

Using the $Q$-regularity, we have

$$N\alpha r^Q \leq \sum_{i} \mu(B_i) \leq \mu(E) = m(E(r)) \leq \sum_{i} \mu(3B_i) \leq N\beta 3^Q r^Q. \quad (4.4)$$

The result follows if we combine (4.3) and (4.4).

Putting the two previous lemmas together, we obtain the following result.

**Corollary 2.** If $E \subset \mathbb{R}^n$ is compact and $Q$-regular for some $0 < Q < n$, then there are constants $0 < c < C < \infty$ and $i_0 \in \mathbb{N}$ (depending only on the $Q$-regularity data) such that for all $i \geq i_0$

$$c 2^{iQ} \leq \tilde{N}_i \leq C 2^{iQ}.$$

**Remark 2.** 1) The porosity and regularity assumptions in the above results are essential because without them we cannot guarantee that $m(E(2r) \setminus E(r))$ is comparable to $m(E(r))$ for small $r > 0$.

2) In [10] the authors considered the relations between the number of Whitney balls and Minkowski dimension in an Ahlfors regular space, which also can be reformulated equivalently in terms of the measures of the layer sets. They also showed a close connection between the ‘surface area’ of the layer sets and the number of Whitney balls.
§ 5. Growth of the quasihyperbolic volume

Our next goal is to prove growth estimates of the above type for a larger class of domains, the so-called $\varphi$-uniform domains. We first observe that the quasihyperbolic volume of any Whitney cube is essentially a constant.

**Lemma 3.** Let $E$ be a closed subset of $\mathbb{R}^n$ and let $G$ be a component of $\mathbb{R}^n \setminus E$. Then for the Whitney cubes of $\mathbb{R}^n \setminus E$ contained in $G$ we have

$$2^{-2n} n^{-n/2} \leq \text{vol}_k(Q^i_j) \leq n^{-n/2}.$$  

**Proof.** Since $m(Q^i_j) = 2^{-in}$, by (2.3) we have

$$\text{vol}_k(Q^i_j) \geq \frac{2^{(i-2)n}}{\sqrt{n}} 2^{-in} = 2^{-2n} n^{-n/2}$$
and

$$\text{vol}_k(Q^i_j) \leq \frac{2^{in}}{\sqrt{n}} 2^{-in} = n^{-n/2}.$$  

**Theorem 1.** Let $G$ be a proper subdomain of $\mathbb{R}^n$ with compact and $Q$-regular boundary for some $Q \in (0, n)$. For each $x \in G$ there exists a constant $C < \infty$, which depends only on the $Q$-regularity data and on $d(x)$, such that for all sufficiently large $r$ we have

$$\text{vol}_k(B_k(x, r)) \leq Ce^{Qr}.$$  

**Proof.** Let $E = \partial G$. We have

$$B_k(x, r) \subset \{ z \in G : e^{-r} d(x) < d(z) < e^r d(x) \} =: D$$
and

$$Q^i_j \subset E(2^{-i} \sqrt{n}, 5 \cdot 2^{-i} \sqrt{n}).$$

If $5 \cdot 2^{-i} \sqrt{n} < e^{-r} d(x)$ or $2^{-i} \sqrt{n} > e^r d(x)$, that is, if

$$i > \frac{r + \log(5\sqrt{n}/d(x))}{\log 2} =: K \quad \text{or} \quad i < -\frac{r - \log(\sqrt{n}/d(x))}{\log 2} =: K',$$
then $Q^i_j \cap D = \emptyset$, and thus

$$B_k(x, r) \subset \bigcup_{i=[K']}^{[K]} \bigcup_{j=1}^{N_i} Q^i_j.$$  

(5.1)

Note that the inequality $N_i \leq M (\text{diam}(E)) 2^{ni}$ always holds for all $i \in \mathbb{N}$ [15] and therefore $N_i \leq C (\text{diam}(E), i_0)$ for $i \leq i_0$, where $i_0$ is as in Lemma 1. Hence for all sufficiently large $r$, by (5.1) and Lemma 3, we have

$$\text{vol}_k(B_k(x, r)) \leq \sum_{i=[K']}^{[K]} \sum_{j=1}^{N_i} \text{vol}_k(Q^i_j) \leq n^{-n/2} \sum_{i=[K']}^{[K]} N_i.$$
Using the estimates for the number of Whitney cubes from Lemma 1 and for the measure of level sets from Lemma 2, we get
\[
\text{vol}_k(B_k(x, r)) \leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) + n^{-n/2} \sum_{i=i_0}^{[K]} c 2^{in}m(E(2^{-i}))
\leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) + n^{-n/2} \sum_{i=i_0}^{[K]} c 2^{in}c'(2^{-i})^{n-Q}
\leq n^{-n/2}(i_0 - [K'])C(\text{diam}(E), i_0) + \frac{cc'n^{-n/2}}{2Q-1} 2^{KQ} \leq Ce^{Qr},
\]
where the last inequality holds for sufficiently large \(r\) by the definition of \(K\) and \(K'\).

Theorem 1 is proved.

Lemma 4. Let \(\emptyset \neq E \subset \mathbb{R}^n\) be a compact set such that \(G = \mathbb{R}^n \setminus E\) is a \(\psi\)-uniform domain and let \(x \in G\). Then for all \(r > 0\),
\[
E\left(\frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)}, d(x)\right) \subset B_k(x, r).
\]

Proof. We only need to consider \(r \geq \psi((\text{diam}(E) + 2d(x))/d(x))\). Otherwise, this is trivial. For arbitrary
\[
y \in E\left(\frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)}, d(x)\right),
\]
we have
\[
d(y) < d(x) \quad \text{and} \quad \psi\left(\frac{\text{diam}(E) + 2d(x)}{d(y)}\right) < r. \tag{5.2}
\]
By (5.2) and the \(\psi\)-uniformity of \(G\) we get
\[
k_G(x, y) \leq \psi\left(\frac{|x - y|}{\min\{d(x), d(y)\}}\right) \leq \psi\left(\frac{\text{diam}(E) + d(x) + d(y)}{\min\{d(x), d(y)\}}\right)
\leq \psi\left(\frac{\text{diam}(E) + 2d(x)}{d(y)}\right) < r,
\]
which implies the desired inclusion.

Theorem 2. Let \(E \subset \mathbb{R}^n\) be a compact \(Q\)-regular set with \(0 < Q < n\) such that \(G = \mathbb{R}^n \setminus E\) is a \(\psi\)-uniform domain. Then for each \(x \in G\) there exist \(c > 0\) and \(r_1 > 0\) such that
\[
\text{vol}_k(B_k(x, r)) \geq c(\psi^{-1}(r))^Q
\]
for all \(r > r_1\), where \(c\) and \(r_1\) depend only on the \(Q\)-regularity data and \(d(x)\).

Proof. Let \(K\) be the largest integer such that
\[
2^{-(K+k_1)} \sqrt{n} \geq \frac{\text{diam}(E) + 2d(x)}{\psi^{-1}(r)}, \tag{5.3}
\]
and let
\[ r_1 = \psi \left( (\text{diam}(E) + 2d(x)) \max \left\{ \frac{10 \cdot 2^{k_1 - k_0}}{d(x)}, \frac{2^{i_0 + k_1}}{\sqrt{n}} \right\} \right), \]
where \( i_0 \) is as in Lemma 1, and \( k_1 \) and \( k_0 \) are as in (4.1). Then for all \( r > r_1 \) we have
\[ 2^{-K} < \frac{C}{\psi^{-1}(r)}, \quad C := \frac{2^{k_1 + 1}(\text{diam}(E) + 2d(x))}{\sqrt{n}}, \quad (5.4) \]
and
\[ 5\sqrt{n} 2^{-(K+k_0)} \leq d(x). \quad (5.5) \]
For \( r > r_1 \), it follows from (2.5), (5.3) and (5.5) and Lemma 4 that for \( K + k_0 \leq m \leq K + k_1 \) all the \( m \)-th-generation Whitney cubes belong to \( B_k(x, r) \). Together with Corollary 2 and Lemma 3, for large values of \( r \) this yields the required estimate
\[ \text{vol}_k(B_k(x, r)) \geq c_0 \tilde{N}_K \geq c_1 2^{KQ} \geq c(\psi^{-1}(r))^Q, \]
where the last inequality follows from (5.4), and the constant \( c > 0 \) only depends on the \( Q \)-regularity data and \( d(x) \). Theorem 2 is proved.

By combining Theorem 1, Theorem 2 and the definition of a uniform domain we obtain the following corollary.

**Corollary 3.** Let \( E \subset \mathbb{R}^n \) be a compact \( Q \)-regular set with \( 0 < Q < n \) such that \( G = \mathbb{R}^n \setminus E \) is a uniform domain with the uniformity constant \( L > 1 \). Then for each \( x \in G \) and sufficiently large \( r > 0 \),
\[ ce^{Qr/L} \leq \text{vol}_k(B_k(x, r)) \leq Ce^{Qr}, \]
where \( C < \infty \) depends only on the \( Q \)-regularity data and \( L \), and \( c > 0 \) depends only on the \( Q \)-regularity data, \( L \) and \( d(x) \).

**Example 1.** Let \( E \subset \mathbb{R}^n \) be a self-similar set whose complement is a uniform domain. For instance, \( E \) can be a Cantor set on a hyperplane, the \((1/4)\)-Cantor set in the plane, or more generally any self-similar set satisfying the strong separation condition (see, for example, [4] for the definitions). Then \( E \) is \( Q \)-regular in its dimension and Corollary 3 can be applied.

**Remark 3.** Although we stated Theorem 2 for unbounded domains, essentially the same proof works for all \( \psi \)-uniform domains \( G \subset \mathbb{R}^n \) whose boundary is \( Q \)-regular and uniformly porous in \( G \). The last assumption means that the porosity holes in the definition of the uniform porosity lie completely inside \( G \). For instance, in \( \mathbb{R}^2 \), \( \partial G \) can be the Von-Koch snowflake curve. More generally, the boundary could be the union of a finite number of such \( Q_i \)-regular pieces, and in this case we could apply the lower bound with the exponent \( r \min\{Q_i\}/L \) and the upper bound with the exponent \( r \max\{Q_i\} \).

It is not known what the best possible lower bound in Corollary 3 is. An interesting open problem is whether the exponent \( Qr/L \) could be replaced by \( cr \) for some uniform constant \( c > 0 \) independent of \( L \).
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