ZARISKI DENSE ORBIT CONJECTURE ON BIRATIONAL AUTOMORPHISMS OF PROJECTIVE THREEFOLDS

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Abstract. Under the framework of dynamics on projective varieties by Kawamata, Nakayama and Zhang [22, 28, 29, 40], Hu and the author [19], we may reduce the Zariski dense orbit conjecture for automorphisms \( f \) on projective threefolds \( X \) with either trivial canonical divisor or negative Kodaira dimension to the following three cases: (i) weak Calabi-Yau threefolds and \( f \) is primitive (ii) rationally connected threefolds and (iii) uniruled threefolds admitting a special MRC fibration over an elliptic curve. And we prove the Zariski dense orbit conjecture is true for either (1) automorphisms of normal projective varieties \( X \) with the irregularity \( q(X) \geq \dim X - 1 \), or (2) automorphisms \( f \) of projective varieties \( X \) such that \( f^* D \equiv D \) for some big \( \mathbb{R} \)-divisors \( D \) and the periodic points of \( f \) are Zariski dense.

1. Introduction

1.1. Zariski dense orbit conjecture. Let \( X \) be a projective variety over an algebraically closed field \( k \) of characteristic zero and \( f : X \dasharrow X \) be a dominant rational self-map. Denote by \( k(X)^f \) the field of \( f \)-invariant rational functions on \( X \). Let \( X_f(k) \) be the set of \( x \in X(k) \) whose orbit \( O_f(x) \) is well-defined.

The following Zariski dense orbit conjecture was proposed by Medvedev and Scanlon [26, Conjecture 5.10], by Amerik, Bogomolov and Rovinsky [2] and strengthens a conjecture of Zhang [37].

**Conjecture 1.1.** Let \( X \) be a projective variety over an algebraically closed field \( k \) of characteristic zero and \( f : X \dasharrow X \) a dominant rational self-map. Then either \( k(X)^f \neq k \) or there is a point \( x \in X_f(k) \) whose orbit \( O_f(x) \) is Zariski dense in \( X(k) \).

**Remark 1.2.** The condition \( k(X)^f \neq k \) is the same as saying that there is nonconstant rational function \( \psi : X \to Y : = \mathbb{P}^1 \) such that \( \psi \circ f = \psi \). This implies the descent endomorphism \( f_Y \) of \( Y \) may be an identify map. So there is no any point \( x \in X_f(k) \) such that \( O_f(x) \) is Zariski dense. It is immediate to see that such a condition is absolutely...
necessary in order to hope for conclusion in Conjecture 1.1 to hold; the difficult in Conjecture 1.1 is prove that such a condition is indeed sufficient for the existence of a Zariski dense orbit.

1.2. Historical note. In [13, Theorem 5.1], Fakhruddin proved that $f$ has a Zariski dense forward orbit if $f$ is polarized and $k$ is uncountable. And then Amerik and Campana [1] proved Conjecture 1.1 when only assume that $k$ is uncountable. In [3, Corollary 9], Amerik proved the existence of non-preperiodic algebraic point when $f$ is of infinite order. In fact, Conjecture 1.1 is also true in positive characteristic, as long as the base field $k$ is uncountable (see [7, Corollary 6.1]). On the other hand, when the transcendence degree of $k$ over $\mathbb{F}_p$ is smaller than the dimension of $X$, there are counterexamples to the corresponding variant of Conjecture 1.1 in characteristic $p$ (as shown in [7, Example 6.2]).

In [2], Amerik, Bogomolv and Rovinsky proved Conjecture 1.1 under the assumption $k = \mathbb{Q}$ and $f$ has a fixed point $p$ which is smooth and such that the eigenvalues of $df|_p$ are nonzero and multiplicatively independent.

In [5, Theorem 1.3], Bell, Ghioca and Tucker proved that if $f$ is an automorphism without nonconstant invariant rational function, then there exists a subvariety of codimension 2 whose orbit under $f$ is Zariski dense. In [8, Corollary 1.7], Bell, Ghioca, Reichstein and Satriano proved that the conjecture for all smooth minimal 3-folds of Kodaria dimension 0 with sufficiently large Picard number, contingent on certain conjectures in the minimal model program.

In [35, Theorem 1.1], Xie proved that Conjecture 1.1 for dominant polynomial endomorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$. In [36, Theorem 1.11], Xie proved that Conjecture 1.1 is proved for $f = (f_1, \cdots, f_n) : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$, where the $f_i$’s are endomorphisms of $\mathbb{P}^1$, in [36, Theorem 1.11]. See also [6, Theorem 14.3.4.2], where $f_i$’s are not post-critically finite; and Medvedev and Scanlon [26, Theorem 7.16] for endomorphism $f : \mathbb{A}^n \to \mathbb{A}^n$ with

$$f(x_1, \cdots, x_n) = (f_1(x_1), \cdots, f_n(x_n)), f_i(x_i) \in k[x_i].$$

If $X$ is a (semi-) abelian variety and $f$ is a dominant self-map, Conjecture 1.1 has been proved by Ghioca, Satriano and Scanlon in [15, Theorem 1.2] and [16, Theorem 1.1]. In [18, Theorem 1.4], Ghioca and Xie showed that if Conjecture 1.1 holds for the dynamical system $(X \times \mathbb{A}_k^k, \psi)$, then it also holds for the dynamical system $X \times \mathbb{A}_k^k \to X \times \mathbb{A}_k^k$ is given by $(x, y) \mapsto (\varphi(x), A(x)y)$ and $A \in \text{GL}_d(k(X)).$

When $X$ is an algebraic surface, and $f$ is a birational self-map, Conjecture 1.1 has been proved by Xie in [36, Corollary 3.27]. On the other hand, Xie in [36, Theorem 1.15] proved Conjecture 1.1 when $f$ is a surjective endomorphism of a smooth projective surface. And then Xie, Jia and Zhang [21, Theorem 1.9] proved Conjecture 1.1 for a surjective endomorphism of a projective surface.
1.3. Main results. It is known that there is no $f$-equivariant minimal model program for automorphism groups of projective varieties in general (cf. [19, Remark 1.3(1)]). This makes dynamics of automorphisms of projective varieties challenging. Therefore, it is reasonable to assume that $X$ is minimal, e.g. $K_X \sim 0$ when $\dim X = 3$ and $\kappa(X) = 0$.

It is known that we have the Beauville-Bogomolov-decomposition for minimal models with trivial canonical class due to Höring and Peternell [20, Theorem 1.5] as follows. Let $X$ be a normal projective variety at most klt singularities such that $K_X \equiv 0$. There exists a finite cover étale in codimension one $\pi: \tilde{X} \to X$ such that

$$\tilde{X} \cong A \times \prod Y_j \times \prod Z_k$$

where $A$ is an abelian variety, the $Y_j$ are singular Calabi-Yau varieties and the $Z_k$ are singular irreducible holomorphic symplectic varieties (see [14, Definition 1.3]). However, it is thus not clear to us that we can always lift the automorphisms of $X$ to some splitting cover $\tilde{X}$ (cf. [19, Remark 3.5]). Instead of utilizing their strong decomposition theorem, we use a weak version (cf. [19, Lemma 2.7]) due to Kawamata [22], and developed by Nakayama-Zhang [29]. For more details about automorphisms of projective varieties with trivial canonical divisor, we refer to [40, Theorem 1.1 and 2.4] and [19, Theorem 1.2]. For automorphisms on projective varieties with negative Kodaira dimension, we can use special MRC fibration due to Nakayama [28] which have the descent property. We refer to [19, Lemma 2.11] for more details about it.

We first quote the following result in [24, Theorem 1.3] which which will be used frequently in this paper.

**Theorem 1.3.** Let $f$ be a birational automorphism on a normal projective variety $X$ with non-negative Kodaira dimension. If $X$ admits a $f$-equivariant morphism $\pi: X \to \mathbb{P}^1$, then $X$ does no have any dense $f$-orbit. In particular, Conjecture 1.1 is true for $(X,f)$.

The notion of a primitive birational automorphism was introduced by Zhang [38] as follows.

**Definition 1.4.** A birational automorphism $f: X \to X$ is *imprimitive* if there exists a variety $B$ with $1 \leq \dim B < \dim X$, a birational map $g: B \dashrightarrow B$, and a dominant rational map $\pi: X \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$. The map $f$ is called *primitive* if it is not imprimitive.

Now we give our main result of Conjecture 1.1 for projective threefolds as follows.

**Theorem 1.5.** Let $X$ be a normal projective threefold $X$ and an abelian subgroup $G$ of $\text{Aut}(X)$. Then following statements hold.
(1) Suppose $K_X \sim 0$ and $f \in G$. Then to prove Conjecture 1.1, we may assume that $X$ is weak Calabi-Yau and $f$ is primitive.

(2) Suppose $\kappa(X) = -\infty$. Then to prove Conjecture 1.1, we may assume that $X$ is either rationally connected threefold or a uniruled threefold admitting a special MRC fibration over an elliptic curve.

Note that a birational automorphism $f$ on a minimal Calabi-Yau threefold $X$ of Picard number $\rho(X) \geq 2$ is primitive if the action $f^*|_{\text{NS}_Q(X)}$ is irreducible over $\mathbb{Q}$ (cf. [30, Corollary 1.3]). This motivates the following question.

**Question 1.6.** Let $f$ be a birational automorphism of a weak Calabi-Yau variety $X$ with $\rho(X) \geq 2$. Suppose that $f^*|_{\text{NS}_Q(X)}$ is irreducible over $\mathbb{Q}$. Then is Conjecture 1.1 true for $(X,f)$?

For Conjecture 1.1 on irregular projective varieties, we have the following theorem.

**Theorem 1.7.** (cf. Theorem 3.4) Let $f$ be an automorphism of a normal projective variety $X$. Then Conjecture 1.1 is true for $(X,f)$ if $q(X) \geq \dim X - 1$.

Notice that Xie proved in [34, Proposition 6.2] that an automorphism $f$ of a projective variety $X$ has finite order if $f^*|_{\text{NS}_R(X)} = \text{id}$ and the periodic points of $f$ are Zariski dense. We may extend his result after replacing the condition that $f^*|_{\text{NS}_R(X)} = \text{id}$ by $f^*D \equiv D$ for some big $\mathbb{R}$-divisor $D$ as follows.

**Theorem 1.8.** Let $f$ be an automorphism of a projective variety $X$. If the periodic points of $f$ are Zariski dense and $f^*D \equiv D$ for some big $\mathbb{R}$-divisor $D$. Then there is an integer $n > 0$ such that $f^n = \text{id}$. In particular, $f$ does not have any Zariski dense forward orbit.

Finally, using Theorem 1.8 we may give a result on Conjecture 1.1 on projective varieties $X$ with $\rho(X) = 1$ as follows.

**Proposition 1.9.** Let $f$ be a surjective endomorphism of a normal projective variety $X$ with at most klt singularities and $\rho(X) = 1$. Then the following statements hold.

(1) If $\delta_f = 1$ and the periodic points of $f$ are Zariski dense, then $f$ does not have any Zariski dense orbit.

(2) Suppose $\delta_f > 1$. Then to prove Conjecture 1.1, we may assume that $X$ is Fano.

1.4. **Notation.** We denote by $\mathbb{N}$ the set of positive integers, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $X$ be a projective variety. Denote by $\text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X)$ the usual Néron-Severi group (in the sense of Cartier divisors). Let $\text{NS}_F(X) := \text{NS}(X) \otimes_{\mathbb{Z}} F$, where $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. The symbols $\sim$ (resp. $\sim_\mathbb{Q}, \sim_\mathbb{R}$) and $\equiv$ denote the linear equivalence (resp. $\mathbb{Q}$-linear equivalence, $\mathbb{R}$-linear equivalence) and the numerical equivalence on divisors.
We denote by $O_{\phi}(\alpha)$ the orbit of any point $\alpha \in X(k)$ under $\Phi$, i.e. the set of all $\Phi(\alpha)$ for $n \in \mathbb{N}_0$. We say that $\alpha$ is periodic if there exists $n \in \mathbb{N}$ such that $\Phi^n(\alpha) = \alpha$. We say that $\alpha$ is preperiodic if there exists $m \in \mathbb{N}_0$ such that $\Phi^m(\alpha)$ is periodic.

Let $G$ be a subgroup of the automorphism group $\text{Aut}(X)$ of $X$. We say that a rational map $\pi : X \dashrightarrow Y$ is $G$-equivariant or $f$-equivariant, if there is a group homomorphism $\rho : G \to \text{Aut}(Y)$ such that $\pi \circ f = \rho(f) \circ \pi$ for all $f \in G$. By taking the normalization which is $f$-equivariant, we may assume that $X$ is normal. Given a $G$-action on an algebraic variety $W$, i.e., there is a group homomorphism $G \to \text{Aut}(W)$, we denote by $G|W$ the image of $G$ in $\text{Aut}(W)$. The action of $G$ is faithful, if $G \to \text{Aut}(W)$ is injective.

A surjective endomorphism $f : X \to X$ of projective variety is said to be int-amplified if $f^*H - H = L$ for some ample Cartier divisors $H$ and $L$, or equivalently, if all eigenvalues of $f^*|_{\text{NS}_k(X)}$ are of modulus greater than 1 (cf. [27, Theorem 1.1]). A prime divisor $V$ on $X$ is said to be totally invariant under the endomorphism $f$, if $f^{-1}(V) = V$ set-theoretically. To simplify, we give a notation for the set of totally invariant prime divisors for the int-amplified endomorphism $f : X \to X$:

$$\text{TI}_f(X) = \{ V | V \text{ is a prime divisor on } X \text{ such that } f^{-1}(V) = V \}.$$ 

A normal projective variety $X$ is said to be $Q$-abelian if there is a finite surjective morphism $\pi : A \to X$ étale in codimension 1 with $A$ being an abelian variety.

Let $f : X \dashrightarrow X$ be a dominant rational map of a projective variety $X$ and $H$ an ample divisor on $X$. We define the first dynamical degree $\delta_f$ of $f$ as

$$\delta_f = \lim_{n \to \infty} ((f^n)^*H : H^{\dim X - 1})^{1/n},$$

where $H$ is a nef and big Cartier divisor of $X$. We say $f$ is of positive entropy if $\delta_f > 1$, otherwise $f$ is of zero entropy (and $\delta_f = 1$).

Let

$$h_X : X(\overline{\mathbb{Q}}) \to [0, \infty)$$

be a Weil height on $X$ associated with an ample divisor, and let $h_X^+ = \max \{ 1, h_X \}$. The arithmetic degree of $f$ at $P \in X_f(\overline{\mathbb{Q}})$ is the quantity

$$\alpha_f(P) = \lim_{n \to \infty} h_X^+(f^n(P))^{1/n},$$

if the limit exists.

The paper is organized as follows. In Section 2, we discuss basic facts on Conjecture 1.1. Theorem 1.7 is proved in Section 3. Theorem 1.5 is proved in Section 4. Theorem 1.8 and Proposition 1.9 are proved in Section 5.
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2. Preliminaries

Conjecture 1.1 is invariant under a generically finite dominant rational map.

Proposition 2.1. (cf. [8, Lemma 4.1] and [36, Lemma 3.23]) Let $\pi : X \dashrightarrow Y$ be a dominant rational map of projective varieties over a number field $K$. Let $f : X \dashrightarrow X$ and $g : Y \dashrightarrow Y$ be dominant rational maps such that $g \circ \pi = \pi \circ f$. Suppose $\pi$ is generically finite. Then Conjecture 1.1 is true for $(X, f)$ if and only if Conjecture 1.1 is true for $(Y, g)$.

Proof. By taking the graph of $\pi$, it suffices for us to consider the case when $\pi$ is a generically finite surjective morphism. Let $Z(X, f)$ be the condition "$f : X \dashrightarrow X$ has Zariski dense orbit" and $R(X, f)$ the condition "$f$ preserves a nonconstant function".

Now we may assume that there exists a nonempty open subset $V$ of $Y(K)$ and a point $y \in V$ such that the orbit of $y$ is well-defined, contained in $\pi(X \setminus I(f)) \cap (Y \setminus I(g))$ and Zariski dense in $Y$. Then for every $x \in \pi^{-1}(y)$, the orbit $O_f(x)$ is Zariski dense in $X$. On the other hand, we may assume that there exists a nonempty open subset $U$ of $X(K)$ and a point $x \in U$ such that the orbit of $x$ is well-defined, contained in $\pi^{-1}(Y \setminus I(g))$ and Zariski dense in $X$. Then $\pi(x) \in \pi(U) \subseteq Y(K)$, the orbit $O_g(\pi(x))$ is Zariski dense in $Y$. Therefore, $Z(X, f)$ holds if and only if $Z(Y, g)$ holds.

Now we shall show $R(Y, g)$ implies that $R(X, f)$. We may assume that there is a nonconstant rational function $\varphi$ on $Y$ such that $\varphi \circ g = \varphi$. Then $\psi := \varphi \circ \pi$ is a nonconstant rational function on $X$ such that $\psi \circ f = \psi$. So Conjecture 1.1 is true for $(X, f)$ if Conjecture 1.1 is true for $(Y, g)$.

Now we shall show $R(X, f)$ implies that $R(Y, g)$. We may assume that there is a nonconstant rational function $\theta$ on $X$ such that $\theta \circ f = \theta$. We have $\theta \in K(X) \subseteq K(Y)$. Set $m := [K(X) : K(Y)]$. Then $K(X)$ is a $m$ dimensional $K(Y)$-vector space. Denote by

$$T^m + \sum_{i=1}^m (-1)^i P_i T^{m-i}$$

the characteristic polynomial of the $K(Y)$-linear operator

$$K(X) \to K(X) : h \mapsto \theta \cdot h.$$ 

We have $P_i \in K(Y)$ and $g^*(P_i) = P_i, i = 1, \cdots, m$. If $P_i \in K$ for $i = 1, \cdots, m$, then $\theta \in K$, which is a contradiction. It follows that there exists $i \in \{1, \cdots, m\}$ such that $P_i$
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is a nonconstant rational function on $X$. Therefore, $g$ preserves a nonconstant fibration. So Conjecture 1.1 is true for $(Y, g)$ if Conjecture 1.1 is true for $(X, f)$. □

Proposition 2.2. [8, Lemma 2.1] In order to prove Conjecture 1.1 for the dynamical system $(X, f)$, it is sufficient to prove Conjecture 1.1 for an iterate $(X, f^m)$, for some $m \in \mathbb{Z}_{>0}$. Moreover, let $G \leq \text{Aut}(X)$, Conjecture 1.1 is true for any $g \in G$ if and only if Conjecture 1.1 is true for any $g' \in G'$, where $G'$ is a finite-index subgroup of $G$.

Proposition 2.3. [16, Lemma 3.1] Let $f$ be a self-map of a projective variety $X$ and an automorphism $g \in \text{Aut}(X)$. In order to prove Conjecture 1.1 for $(X, f)$, it is sufficient to prove Conjecture 1.1 for $(X, g^{-1} \circ f \circ g)$.

The following is a variant of [3, Corollary 9].

Proposition 2.4. Let $f$ be a surjective endomorphism of a projective variety $X$ over a number field $K$ with $\delta_f > 1$. Then $f$ has a point $P \in X(K)$ such that $\# \mathcal{O}_f(P) = \infty$ and $\delta_f = \alpha_f(P)$.

Proof. By [31, Theorem 1.1], if $\alpha_f(P) = \delta_f$, then $P$ is non-preperiodic. The existence of non-preperiodic point $P$ with $\delta_f = \alpha_f(P)$ is proved by [32, Theorem 1.8]. □

Proposition 2.5. Conjecture 1.1 is true for any surjective endomorphism $f$ of a $\mathbb{Q}$-abelian variety $X$.

Proof. There exists a finite surjective morphism $\pi : A \to X$ with $A$ being an abelian variety, such that $f$ lifts to a surjective endomorphism $f_A : A \to A$ by [12, Corollary 8.2]. Note that Conjecture 1.1 is true for $(A, f_A)$ by [15, Theorem 1.2]. Therefore, Conjecture 1.1 is true for $(X, f)$ by Proposition 2.1. □

The following theorem is due to Cantat [11], Bell, Moosa and Topaz [10].

Theorem 2.6. Let $f$ be a self-map of a projective variety $X$ defined over a field $F$ of characteristic zero. Then the following are equivalent:

1. There exists infinitely many totally invariant prime divisors.
2. There exists a nonconstant rational function $g$ such that $g \circ f = g$.

Proof. Apply [10, Theorem 1.1] to $Z = X, \phi_1 = f$ and $\phi_2 = \text{id}_X$. □

Remark 2.7. When $f$ is int-amplified, Zhong proved in [41, Theorem 1.1 (1)] that

$$\# \text{TI}_f(X) \leq \dim X + \rho(X).$$

Therefore, to show Conjecture 1.1 for the case that $f$ is int-amplified, it suffices to show that there is a point $P \in X(F)$ with a Zariski dense $f$-orbit by Theorem 2.6. This result
can be shown quite straightforward. Indeed, we assume that there is a rational fibration \( \pi: X \to Y := \mathbb{P}^1 \) such that \( \pi \circ f = \pi \). This implies that the descent endomorphism \( f|_Y \) of \( f \) may be an identity map. However, \( f|_Y \) should be int-amplified by [27, Lemma 3.4]. This gives a contradiction.

3. Proof of Theorem 1.7

Proposition 3.1. [8, Lemma 5.1] Let \( \pi: X \to Y \) be a dominant rational map of projective varieties over a number field \( K \). Let \( f: X \to X \) and \( g: Y \to Y \) be dominant rational maps such that \( g \circ \pi = \pi \circ f \). Suppose \( f \) is an automorphism, \( \dim X = \dim Y + 1 \) and Conjecture 1.1 is true for \((Y, g)\). Then Conjecture 1.1 is true for \((X, f)\).

Proposition 3.2. Let \( \pi: X \to Y \) be a surjective endomorphism of normal projective varieties with \( 3 = \dim X > \dim Y \geq 1 \). Let \( f: X \to X \) and \( g: Y \to Y \) be dominant rational maps such that \( g \circ \pi = \pi \circ f \). Suppose \( f \) is an automorphism. Then to show Conjecture 1.1 holds for \((X, f)\), we only to assume that \( Y \) is \( \mathbb{P}^1 \) or an elliptic curve. In particular, if \( \kappa(X) \geq 0 \), then \( Y \) is an elliptic curve.

Proof. When \( \dim Y = 2 \), then Conjecture 1.1 holds for \((Y, g)\) by [21, Theorem 1.9]. Then Conjecture 1.1 holds for \((X, f)\) by Proposition 3.1. Then we may assume that \( \dim Y = 1 \). If \( g(Y) \geq 2 \) (i.e., \( \kappa(Y) > 0 \)), then \( Y \) does no have any dense \( f \)-orbit by [25, Remark 1.3]. So Conjecture 1.1 is true for \((X, f)\). Therefore, \( Y \) is \( \mathbb{P}^1 \) or an elliptic curve. If \( \kappa(X) \geq 0 \), then we end the proof of Proposition 3.2 by Theorem 1.3.

Proposition 3.3. [25, Proposition 3.7] Let \( f \) be a surjective endomorphism of a normal projective variety \( X \). To prove Conjecture 1.1 is true for \((X, f)\), then we may assume the Albanese morphism \( \pi: X \to A \) is surjective.

Theorem 3.4. Let \( f \) be an automorphism of a normal projective variety \( X \). Then Conjecture 1.1 is true for \((X, f)\) if \( q(X) \geq \dim X - 1 \).

Proof. By Proposition 3.3, the Albanese morphism \( \pi: X \to A \) is surjective and \( \dim A = q(X) \). Notice \( f \) descents to \( f_A \) of \( A \) by the universal of the Albanese morphism. If \( \dim A = \dim X \), then Conjecture 1.1 is true for \((X, f)\) by Proposition 2.1 and Theorem 2.5. If \( \dim A = \dim X - 1 \), then Conjecture 1.1 is true for \((X, f)\) by Proposition 3.1 and Theorem 2.5.

4. Proof of Theorems 1.5

We first quote the definition of a weak Calabi-Yau variety in [19, Definition 2.4].

Definition 4.1. A normal projective variety \( X \) is called a weak Calabi-Yau variety, if
• $X$ has only canonical singularities;
• the canonical divisor $K_X \sim 0$; and
• the augmented irregularity $\widetilde{q}(X) = 0$.

Here, the augmented irregularity $\widetilde{q}(X)$ of $X$ is defined as the supremum of $q(Y)$ of all normal projective varieties $Y$ with finite surjective morphism $Y \to X$, étale in codimension one. Namely,

$$\widetilde{q}(X) = \sup \{ q(Y) : Y \to X \text{ is a finite surjective and étale in codimension one} \}.$$

Now we divide the proof of Theorem 1.5 into two parts as follows.

Proof of Theorem 1.5(1). It is known that the rank of $G \leq \text{Aut}(X)$ is less and equal than $\dim X - 1$ (cf. [39]). By [19, Theorems 1.1 and 1.2] and Propositions 2.1 and 2.2, we may assume that $X$ to be one of the following three cases:

(i) $X$ is a $Q$-abelian threefold.
(ii) $X$ is a product of $E \times S$, where $E$ is an elliptic curve and $S$ is a weak Calabi-Yau surface.
(iii) $X$ is a weak Calabi-Yau threefold.

The case (i) follows from Proposition 2.5. For the case (ii), by [19, Lemma 2.9], every automorphism of $f$ of $E \times S$ splits, i.e. there exists automorphisms $f_E$ and $f_S$ of $E$ and $S$ such that $f = f_E \times f_S$. By [21, Theorem 1.9], Conjecture 1.1 is true for $(S, f_S)$. Then Lemma 3.1 implies Conjecture 1.1 is true for $(X, f)$.

Now we deal with the case (iii). Assume that $X$ has a $f$-equivariant rational map $\pi : X \to Y$. We take $W$ as the normalization of the graph $\Gamma_\pi$ of $\pi$ which admits a natural faithful $G|_W$-action and then $q : W \to X$ is $G|_W$-equivariant. So we may assume $\pi$ is morphism by Proposition 2.1. By Proposition 3.2 we may assume that $Y$ is an elliptic curve. This completes the proof of Theorem 1.5 (1) as $X$ has trivial Albanese. □

The following special MRC fibration is due to Nakayama [28].

Definition 4.2. [19, Definition 2.10] Given a projective variety $X$, a dominant rational map $\pi : X \dashrightarrow Z$ is called the special MRC fibration of $X$, if it satisfies the following conditions:

(1) The graph $\Gamma_\pi \subseteq X \times Z$ of $\pi$ is equidimensional over $Z$.
(2) The general fibers of $\Gamma_\pi \to Z$ are rationally connected.
(3) $Z$ is a non-uniruled normal projective variety.
(4) If $\pi' : X \dashrightarrow Z'$ is a dominant rational map satisfying (1)-(3), then there is a birational morphism $v : Z' \to Z$ such that $\pi = v \circ \pi'$. 


The existence and the uniqueness (up to isomorphism) of the special MRC fibration is proved in [28, Theorem 4.18]. One of the crucial advantages of the special MRC is the following descent property.

Theorem 4.3. [24, Theorem 4.3] Let $\pi : X \longrightarrow Z$ be the special MRC fibration, and $f \in \text{Aut}(X)$. Then there exists a birational morphism $p : W \rightarrow X$ and an automorphism $f_W \in \text{Aut}(W)$ and an equidimensional surjective morphism $q : W \rightarrow Z$ satisfying the following conditions:

1. $W$ is a normal projective variety.
2. A general fiber of $q$ is rationally connected.
3. $W$ admits $f_W$-equivariant fibration over $X$ and $Z$.

Now we resume the proof of Theorem 1.5.

Proof of Theorem 1.5(2). Consider the special MRC fibration $\pi : X \longrightarrow Z$. By Theorem 4.3 and Proposition 2.1, we may assume that $\pi : X \rightarrow Z$ is $f$-equivariant morphism and the general fiber of $q$ is rationally connected. If $\dim Y = 0$, then $X$ is rationally connected. Now assume that $\dim Y > 0$. Then the proof follows from Proposition 3.2 as $Y$ is non-uniruled. $\square$

5. Proof of Theorem 1.8 and Proposition 1.9

Proof of Theorem 1.8. Let $f^*D \equiv D$ for some big $\mathbb{R}$-divisor $D$. Then

$$f \in \text{Aut}_{[D]}(X) := \{ f \in \text{Aut}(X) | f^*[D] = [D] \}.$$ 

Therefore, there is an integer $n > 0$ such that $f^n \in \text{Aut}_0(X)$ by a Fujiki-Lieberman type theorem (cf. [23, Theorem 1.4]). So $f^n$ acts on $\text{NS}_{\mathbb{R}}(X)$ as the identity. This completes the proof of Theorem 1.8 by [34, Proposition 6.2]. $\square$

Now we give the proof of Proposition 1.9 as follows.

Proof of Proposition 1.9. It is well-known that $f^*D \equiv \delta_f D$ for some nef $\mathbb{R}$-divisor $D$. Then $D$ is ample as $\rho(X) = 1$. Therefore, $(f^*D)^{\dim X} = (\deg f) H^n = \delta_f^n H^n$. This yields that $\deg f = \delta_f^{\dim X}$ as $H^n > 0$. If $\delta_f = 1$, then $f^*D \equiv D$. Then the proof follows from Proposition 1.8. Now suppose $\delta_f > 1$. Then $f$ is int-amplified. If $K_X$ is pseudo-effective, then $X$ is $Q$-abelian by [27, Theorem 1.9]. Therefore, Conjecture 1.1 is true for $(X,f)$ by Proposition 2.5. If $K_X$ is not pseudo-effective (hence $-K_X$ is effective), then $-K_X$ is ample as every effective $\mathbb{R}$-divisor is ample when $\rho(X) = 1$. Therefore, $X$ is Fano. $\square$
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