A Fresh View of Cosmological Models Describing Very Early Universe: General Solution of the Dynamical Equations

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Abstract—The dynamics of any spherical cosmology with a scalar field ('scalaron') coupling to gravity is described by the nonlinear second-order differential equations for two metric functions and the scalaron depending on the ‘time’ parameter. The equations depend on the scalaron potential and on arbitrary gauge function that describes time parameterizations. This dynamical system can be integrated for flat, isotropic models with very special potentials. But, somewhat unexpectedly, replacing the independent variable $t$ by one of the metric functions allows us to completely integrate the general spherical theory in any gauge and with arbitrary potentials. In this approach, inflationary solutions can be easily identified, explicitly derived, and compared to the standard approximate expressions. This approach is also applicable to intrinsically anisotropic models with a massive vector field ('vecton') as well as to some non-inflationary models.

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1. INTRODUCTION

In this paper we extend the new method for constructing the exact solutions of FLRW cosmological models with a scalaron (inflaton), which was proposed in [1], to fairly general homogeneous non-isotropic cosmologies with scalaron or vecton. The next step will be to identify, classify, and study different classes of solutions in connection with inflation and other possible cosmological scenarios. The third step should be to study their interrelations in the frame of the two-dimensional theory of spherical gravity coupled to scalarons. This program had been attempted to partly implement in [2–6], in the context of the multi-Liouville and Toda-Liouville integrable models, where the three sectors of the solutions were derived and classified. The difficult problem that was not really touched in that papers is the derivation and study of the solutions in presence of small perturbations. Here we discuss only the first step—formulating the approach to constructing exact solutions of a rather general cosmological dynamics with arbitrary potentials $\mathcal{V}(\alpha) \equiv \mathcal{V}(\psi(\alpha))$.

We derive the cosmological models by the standard spherical reduction of the four-dimensional model of gravity coupled to scalaron $\psi$ and vecton $A$, considered in [8, 9]:

$$\mathcal{L} = \sqrt{-g} \left[ R[g] - \kappa \left( \frac{1}{2} F_{\alpha \beta} F^{\alpha \beta} + m^2 A_i A^i + g_{\mu \nu} \partial \psi \partial \psi + \mathcal{V}(\psi) \right) \right], \quad (1)$$

where $g_{\mu \nu}$ is the metric of the four-dimensional space-time, $F_{\alpha \beta}$ is the field strength of the vecton, $\kappa$ is the gravitational constant, $m^2$ is the mass parameter of the vecton, and $\mathcal{V}(\psi)$ is the scalaron potential.

The cosmological models are derived by a further dimensional reduction of the two-dimensional scalaron or vecton theories. With this aim, we apply a kind of a direct separating the $t$ and $r$ variables that does not require any group-theoretical considerations (see, e.g., discussion in [4, 5]). Thus we get nonlinear dynamical systems of functions that depend on $t$ or on $r$ and thus describe cosmological models or static states. By solving these systems we are finding some possible cosmologies (or static black holes) but this does not give us the real processes of how these emerge from the solutions of the two-dimensional dynamical system. This is a separate problem not discussed in this paper. Note that for different solutions, the stability with respect to small gravitational perturbations was studied in [12]. If the solution is stable we can regard it as a sort of attractor in the space of solutions. In this sense, Friedmann-type solutions must be attractors.

2. SPHERICAL NON-ISOTROPIC GRAVITY

For future considerations we first write the approximate four-dimensional model of gravity coupled to scalaron $\psi$ and vecton $A$, considered in [8, 9]:

$$\mathcal{L} = \sqrt{-g} \left[ R[g] - \kappa \left( \frac{1}{2} F_{\alpha \beta} F^{\alpha \beta} + m^2 A_i A^i + g_{\mu \nu} \partial \psi \partial \psi + \mathcal{V}(\psi) \right) \right]. \quad (1)$$

The author supposes that the reader is familiar with basics of cosmology including present status of inflationary models as presented in, e.g., [13–16]. In addition, this paper is not completely independent of [1] and some familiarity with it is also supposed (it also contains many references to related work).
where the arbitrary potential \( \nu(\psi) \) may include the cosmological term \( 2\Lambda \) and, possibly, the mass term for \( \psi \). To find the spherically reduced theory we write the metric in the form

\[
ds^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega^2(\theta, \phi) - e^{2\gamma} dt^2 + 2e^{2\delta} dr dt,
\]

where \( \alpha, \beta, \gamma, \delta \) depend on \( t, r \) and \( d\Omega^2(\theta, \phi) \) is the metric on the 2-dimensional sphere \( S^{(2)} \).

Then the two-dimensional reduction of four-dimensional theory (1) can easily be found (the prime denotes differentiations with respect to \( r \) and the dot—

with respect to \( t \)):

\[
\mathcal{L}^{(2)} = \mathcal{L}_{gr} + e^{2\beta} \leq 2^{\epsilon} (A_i - A_0)^2 + e^{2\alpha}(\psi^2 + m^2 A_0^2)
\]

\[
- e^{2\gamma}(\psi^2 + m^2 A_0^2) - e^{2\gamma} \nu(\psi)
\]

where \( \nu = \psi(t, r), A_i = A_i(t, r), A_0 = F_{0r}, \) and

\[
\mathcal{L}_{gr} \equiv e^{2\alpha}(2\beta^2 + 4\beta\gamma)
\]

\[
- e^{2\gamma}(2\beta^2 + 4\beta\alpha) + 2ke^{2\gamma},
\]

where we omit the total derivatives, which do not affect the equations of motion. Notice that we also omit the terms depending on \( \delta \) by taking the formal limit \( \delta \rightarrow -\infty \). The variation in this parameter gives the zero momentum constraint, which in this limit is simply

\[
-\beta' - \beta\beta' + \alpha\beta' + \beta\gamma' = \frac{1}{2}(\psi' \nu + A_0A_i).
\]

All other equations are obtained by varying the metric functions \( \alpha, \beta \).

To find all possible cosmological models we require \( \nu' = 0, A_i = 0 \) (\( A_i = A(t) \)) and make further reductions by separating the variables \( t \) and \( r \) in the metric

\[
\alpha = \alpha_0(t) + \alpha_i(t), \quad \beta = \beta_0(t) + \beta_i(t).
\]

\[
\gamma = \gamma_0(t) + \gamma_i(r).
\]

The momentum constraint severely restricts possible cosmological models that can be derived from the two-dimensional Lagrangian. It is not difficult to find that there are four types of cosmological and static solutions. Two main solutions are: \( \beta' = \gamma' = 0 \) ("general anisotropic" cosmology) and \( \alpha = \beta, \gamma' = 0 \) (general isotropic, or, FLRW-cosmology). For other solutions of (5) there emerge strong restrictions on \( \nu(\psi) \), requiring \( \nu = 0 \) or even \( \nu' = 0 \). The two-dimensional theory reduced by conditions \( \nu' = 0 \), (6), and (5) describe homogeneous cosmologies with scalaron \( \psi(t) \) if we require their three-dimensional curvature to be constant (notice unusual definition of \( K, k \)),

\[
K^{(3)} = e^{2\beta} R_{ij}
\]

\[
= 2ke^{2\beta_i} - 2e^{-2\alpha}(2\beta'' + 3\beta'^2 - 2\beta')(\alpha_i) = -6k.
\]

In addition, the three-dimensional subspace is isotropic if

\[
R^i = -2e^{-2\alpha}(2\beta'' + 3\beta'^2 - 2\beta')(\alpha_i) = R^i
\]

which is equivalent to the relation

\[
ke^{2\beta - 2\alpha} + \beta'' - \beta'\alpha_1 = 0.
\]

Taking into account that we can choose \( \alpha' = 0 \), the two conditions are equivalent to one:

\[
\beta'' - ke^{2\beta} = 0.
\]

These relations also directly follow from the conditions \( \alpha = \beta, \gamma' = 0 \) and from the requirement of separation in the equation of motion. We discussed these matters in more detail in papers quoted above. Here we only remind that the 'general anisotropic' cosmology (with \( \beta' = \gamma' = 0 \)) becomes isotropic if \( k = 0 \); then also \( k = 0 \) but nevertheless, for the general solution, \( \alpha = \beta \) as is demonstrated below. We will immediately see that, in this case, there exists the unique solution with \( \alpha = \beta \). The 4-dimensional solution then necessary satisfies \( \gamma' = 0 \) condition and therefore it is the true FLRW-type solutions.

From now on we ignore the subtleties related to higher-dimensional solutions and only consider (one-dimensional) cosmological dynamical equations.

### 3. General Dynamical Equations

Now let us introduce cosmological reductions of theories (1). As was shown in [8], the approximate cosmological Lagrangian\(^5\) can be written in the form

\[
\mathcal{L}_c = e^{2\alpha}(\beta'^2 + \beta\beta' + \beta\gamma) + 2ke^{2\gamma}.
\]

\[
- e^{-2\gamma}(2\beta^2 + 4\beta\alpha - \nu - \nu') - 6ke^{2\gamma}.
\]

\(^3\) A detailed description of the reduction procedure was described in our earlier work, e.g., [4, 5].

\(^4\) We call the 'special' anisotropic the cosmology with \( \beta' = \gamma' = 0 \) and \( \alpha = 0 \), which is dual to FLRW. The flat isotropic cosmology is obtained from the general anisotropic one if in addition \( K = k = 0 \) and \( \alpha = \beta \).

\(^5\) In the cosmological models of [7–9] with the massive vector \( A_i \) there is no \( \nu \). In the small \( \Lambda_i^2 \) approximation there remain from its kinetic part the term \( A_i^2 \) and the constant potential \( \nu = 2\Lambda \).
To write the equations of motion in a more compact form, we introduce notation
\[ 3p = (\alpha + 2\beta), \quad 3\sigma = (\beta - \alpha), \]
\[ \alpha = \rho - 2\sigma; \quad 3A = e^{-2p+4\sigma}(A^2 + m^2 e^{2\gamma}A^2). \tag{12} \]
Then the exact Lagrangian for the vector-scalar cosmology is:
\[ \mathcal{L} = e^{3p-\gamma}(\psi^2 - 6\rho^2 + 6\sigma^2) \]
\[ - e^{3\gamma+\nu}(\psi) - 6ke^{2-2p+\gamma}e^{3\gamma+3A}. \tag{13} \]
Here \( e^\gamma \) is the Lagrangian multiplier, variations of which yield the energy constraint:
\[ \mathcal{H}_c = \psi^2 - 6\rho^2 + 6\sigma^2 + e^{2\gamma}(\psi) \]
\[ + 6ke^{2-2p+\sigma} + 3A = 0. \tag{14} \]
As in any gauge theory with one constraint of this type, we can choose one gauge fixing condition. The standard conditions are \( \gamma = 0, \gamma = \alpha, \gamma = 3\alpha \). The other equations are
\[ 4\dot{\rho} + 6\dot{\rho}^2 - 4\dot{\psi} \dot{\gamma} + 6\dot{\sigma}^2 + \dot{\psi}^2 \]
\[ - e^{2\gamma}(\psi) = 2ke^{2-2p+\sigma} - A, \tag{15} \]
\[ \dot{\sigma} + (3\dot{\rho} - \dot{\gamma})\dot{\sigma} = ke^{2-2p+\sigma} + A, \tag{16} \]
\[ \dot{\psi} + (3\dot{\rho} - \dot{\gamma})\dot{\psi} + e^{2\gamma}(\psi)/2 = 0; \tag{17} \]
\[ \dot{A} + (\dot{\rho} + 4\dot{\sigma} - \dot{\gamma})A + e^{2\gamma}m^2A = 0. \tag{18} \]
Equation (15) can be replaced by two equations obtained by subtracting from it Eq. (14),
\[ \dot{\rho} + (3\dot{\rho} - \dot{\gamma})\dot{\rho} - e^{2\gamma}(\psi)/2 \]
\[ = 2ke^{2-2p+\sigma} + (3A + A)/4, \tag{19} \]
or alternatively, by adding them together (note that
\[ 3A = 0 \text{ if } m^2 > 0, \]
\[ \dot{\rho} + 3\dot{\sigma}^2 - \dot{\rho} \dot{\gamma} + \dot{\psi}^2/2 \]
\[ = -ke^{2-2p+\sigma} - (3A + A)/4. \tag{20} \]

First, a few general remarks on Eqs. (14)–(20). This system can be simplified by applying a gauge fixing condition while some subsystems can be solved in any gauge. Further, taking account of the fact that the sums of the first two terms in Eqs. (16)–(19) are essentially time derivatives of the momenta of the dynamical variables \( (\dot{\rho}, \dot{\psi}, \dot{\sigma}) \) and \( A \),
\[ (p_\rho, p_\psi, p_\sigma) = 2e^{3p-\gamma}(-6\rho, \psi, 6\sigma), \]
\[ p_A = 2e^{\gamma+4\sigma-\gamma}A. \tag{21} \]
we see that these definitions together with Eqs. (16)–(19) are the Hamiltonian equations with the canonical Hamiltonian (compare to pure scalaron model in [1]):
\[ \mathcal{H}_c = \frac{1}{24}(6p_\rho^2 + p_\sigma^2 - p_\rho^2 + 6p_\rho^2 e^{2p-4\sigma})e^{3p-\gamma} \]
\[ + \nu(\psi)e^{\gamma+\nu} + 6ke^{\gamma-2\nu} + m^2 A^2 e^{\gamma+p+\sigma}. \tag{22} \]

In [1] we exploited only the simplest gauge choices, which in the present general model look like \( \gamma + \epsilon \rho = 0 \). When \( A = 0, k = 0 \), the gauge \( \gamma - 3p = 0 \) is most useful (unlike the ‘standard’ gauge \( \gamma = 0 \)). More general gauges are \( g(\rho, \sigma; \gamma) = 0 \). Thus if we try to choose a gauge in which r.h.s. of (16) vanishes we must express \( A^2 \) in \( A \) in terms of \( p_A \). Then we find that this condition does not fix \( \gamma \) because all the terms are proportional to \( e^{2\gamma} \). Other interesting properties of these equations are: independence of the \( \psi \)-equation of \( k, \sigma, A \) and independence of the \( A \)-equation of \( k, \psi \). Note that \( \rho(t) = \xi(t) \) is the most important cosmological function, which in the isotropic limit coincides with the standard Hubble parameter (function) usually denoted by \( H(t) \). We keep the same notation for our generalized ‘Hubble function’ \( \rho(t) \). Equations (19), (20) give useful restrictions on \( \dot{H}(t) \):
\[ \dot{H}(t) \equiv \ddot{H}(t) \geq 0, \text{ if } \gamma = 3\rho, \nu \geq 0, \quad k \geq 0; \]
\[ \dot{H}(t) \leq 0, \text{ if } \gamma = 0, \quad k \geq 0. \tag{23} \]

By the way, inequality \( \dot{\rho} \geq 0 \) for \( \nu \geq 0, k \geq 0 \) keeps in any gauge. This may hint at idea of the gauge independent solution, our main result presented in next section.

Formally, there exist six essentially different special cases to consider. Let us denote them by the symbol \( \{S, V, C\} \), where: \( S = \psi \) or \( \rho \), \( V = A \) or \( 0 \), \( C = k \) or \( 0 \). In this paper, we only solve the general case \( \{\psi, 0; k\} \) with arbitrary \( k \). When \( k = 0 \), the solution coincides with the standard FLRW solution, if we assume that \( \sigma \equiv 0 \). When \( k \neq 0 \), we have not the FLRW cosmology because \( \sigma \neq 0 \) and, if even \( \sigma \to 0 \), the 4-dimensional solution does not coincide with FLRW, for which \( \beta \neq 0 \). The essentially different and much more difficult cases \( \{0, A; k\} \) and \( \{\psi, A; k\} \) will be considered in a separate publication. An important property of the general system of equations is independence of the \( \psi \)-equation of \( k, \sigma, A \) and independence of the \( A \)-equation of \( k, \psi \).

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6 Here \( g = \psi, \rho, \sigma, A \) and \( p \) – the corresponding momenta. As \( g \) are independent of \( \gamma \), a simple and safe gauge condition is \( g(\rho; \gamma) = 0 \) with \( \partial_\gamma g \neq 0 \). The restriction on general \( g \) is \( \partial_\rho g + \partial_\gamma g \neq 0 \).
4. GAUGE INDEPENDENT SOLUTION OF SCALARON COSMOLOGY

Now we turn to the \{\psi, 0, k\} equations which we rewrite in the first-order form for the momentum-like variables \(\xi, \eta, \zeta\) treated as functions of \(\rho\) (using \(d/dt = \xi d/d\rho\)).

\[
(\rho, \psi, \sigma) = [\xi(\rho), \eta(\rho), \zeta(\rho)] = [\xi(\rho), \psi'(\rho), \sigma'(\rho)] = [\xi(\rho), \eta(\rho), \sigma(\rho)],
\]

where \(\chi(\rho) = \eta/\xi = \psi'(\rho)\) and \(\alpha(\rho) = \zeta/\xi = \sigma'(\rho)\) are gauge invariant functions, the integrals of which, \(\psi(\rho)\) and \(\sigma(\rho)\), give a ‘portrait’ of the scalaron cosmology. If we could derive the portrait, we would find all the characteristics of the cosmology. In paper [1] we thoroughly studied \(\chi(\rho)\) and its another incarnation \(\chi(\psi)\) as well as \(\psi(\rho)\) when \(\sigma = 0\) the \(\chi\) function can be derived if we take as the input either potential \(\bar{\psi}(\rho)\) or the generalized ‘Hubble function’ \(H(\rho) = \dot{\chi}(\rho)\). The \(\chi\)-equation with the \(v(\psi)\) input can be exactly solved for a mere handful of potentials. Actually, it was studied in various approximations, described in [1], and we here confine ourselves to equations with potentials depending on \(\rho\).

First, let us transform the dynamical equations into first-order form by generalizing approach of [1]. In \{\psi, 0, k\} case, the dynamical system consists of Eqs. (16), (17) plus (19) or (20). The constraint (14), being its integral, is automatically satisfied when the arbitrary integration constant is chosen zero. Then the only problem remaining is the presence of \(v(\psi)\) and of \(\psi(\rho)\) dependence when \(k \neq 0\). The formal substitution \(\bar{\psi}(\rho) = v(\psi(\rho))\) allows us to solve all equations in the \(\rho\)-version and, as soon as we derive \(\chi(\rho)\), we will be able to find the potential in the \(\psi\)-version, \(\bar{\psi}(\rho) = \bar{\psi}(\psi(\rho))\) for arbitrary \(\bar{\psi}(\rho)\). The obvious relation,

\[
v'(\psi) = \frac{d\psi}{d\eta} = \frac{\bar{\psi}(\rho)}{\xi(\rho)} = \bar{\psi}(\rho)/\chi(\rho),
\]

then solves the \(v(\psi)\) problem. Comparing the definitions (24), (25), and their discussion with the corresponding consideration in [1], it is not difficult to find that (16), (17), (19) with \(A_2 = 0, \ k = 0\) are linear differential equations for \(\xi^2, \eta^2, \zeta^2\) in any gauge \(\gamma(\rho)\) and can be exactly solved for any potential \(\bar{\psi}(\rho)\). The \(k\)-terms are proportional to \(\exp[-2\sigma(\rho)]\), but, in cosmological consideration, we may suppose that \(\sigma \ll \rho\) as will be argued below.

Rewriting the dynamical equations in terms of the positive functions,

\[
[x(\rho), y(\rho), z(\rho)] = \exp(6\rho - 2\gamma)[\xi^2(\rho), \eta^2(\rho), \zeta^2(\rho)]
\]

we see that (17) gives the only equation independent on the curvature term \(- k e^{-2\sigma}\):

\[
y'(\rho) + y(\rho) - 6y(\rho) = 0, \quad y \equiv e^{6\gamma}(\rho).
\]

Two other equations depend on the curvature term and the whole system is not closed:

\[
x'(\rho) - V(\rho) = 4k e^{-2\sigma},
\]

\[
z'(\rho) = 2k e^{-2\sigma}\sigma'(\rho).
\]

The additional equation that makes it closed is the definition of \(\psi(\rho)\) written as \((\sigma')^2 = z/x \equiv \omega^2(\rho)\). This will allow us to show that \(\sigma\) satisfy a closed equation.

We first find the formal solution of Eqs. (27), (28) plus constraint (14), which is

\[
6x(\rho) = y(\rho) + V(\rho) + 6z(\rho) + 6k e^{-2\sigma}.
\]

To this end, it is sufficient to integrate equations (27), (28a) and use (29):

\[
y = 6\left(C_x + \int V(\rho)\right) - V(\rho),
\]

\[
x = \left(C_x + \int V(\rho)\right) + 4k \int e^{-2\sigma(\rho)},
\]

where the integration (all integrals) is taken over the interval \(-\infty \leq \rho_0, \rho\), while

\[
z = x(\rho)\sigma^2(\rho) = C_x + k\left[4\int e^{-2\sigma(\rho)} - e^{-2\sigma(\rho)}\right]
\]

Eq. (31) would be nonlinear integro-differential equation for \(\sigma(\rho)\) if we knew \(x(\rho)\). Supposing \(\sigma \ll 1\) we can derive \(\sigma\) by some sort of iterations. The first approximation for \(x\), with \(\sigma = 0\), coincides with the result of [1], because \(z \equiv 0\) and therefore \(C_x = C_y\). The generalized function \(\chi^2\) in this case also coincides with the previous result. The new function, giving the gauge-invariant measure of anisotropy is given by \(\sigma^2 = z(x)/x(\sigma) = (\sigma')^2\). From (31) we see that, when \(k = 0\), there exists solution \(\sigma^2(\rho) = C_x \neq 0\). If \(V(\rho) > 0\) and grows with \(\rho\), there may exist non-isotropic solution that becomes (almost) isotropic for large \(\rho\). The isotropic solution \(z \equiv 0\) is, probably, the limiting or enveloping one for solutions with \(C_x \neq 0\).

**In summary**, there are four types of the solutions: (1) \(k = 0, \sigma' = 0, C_z = 0\) – isotropic solution coinciding with the FLRW cosmology; (2) \(k = 0, \sigma' \neq 0, C_z \neq 0\)
the simplest anisotropic solution, which for a wide class of the potentials $\mathcal{V}(\rho)$ becomes isotropic at large $\rho$, i.e. $\sigma^* \to 0$ for $\rho \to +\infty$; (3) $k \neq 0, \sigma^*=0, C_z=0$ — isotropic cosmology, not of FLRW type; (4) $k \neq 0, \sigma^* \neq 0$ — general anisotropic solution; under some restrictions on $\mathcal{V}$, it can also vanish at $\rho \to +\infty$.

We conclude that for large $\rho$ there exists a wide class of the potentials $\mathcal{V}(\rho)$ for which $\gamma(\rho) \equiv \gamma^*(\rho)$ is small. If we define the generalized conditions for inflation (inflation) by the two inequalities, $\omega^2 \leq 1$ (small anisotropy) and $\chi^2 \leq 1$ we find that the generalized second inflationary parameter $\hat{r}$ for $k = 0$ can be defined\textsuperscript{10} as follows:

$$
\hat{r}(\rho) = \frac{\psi^2 e^{-2\gamma}/\mathcal{V}(\psi)}{\mathcal{V}(\rho)} = \frac{\psi}{\mathcal{V}} = \chi^2 (6(1 - \omega^2) - \chi^2)^{-1}.
$$

(32)

The r.h.s. coincides with Eq. (75) of [1], if $\omega = 0$ and $\rho = \alpha$. Note that, for positive potential, this relation requires $0 < \chi^2 < 6(1 - \omega^2)$. Now we can mutatis mutandis repeat considerations of paper [1], Sec. 4.3, and derive the most important relation (rem., $k = 0$):

$$
\hat{r} + \frac{6C_y}{V(\rho)} = -1 + \frac{6}{V} \int V(\rho)
$$

(33)

This relation is the generalization of Eq. (78) in [1], the basic tool for inflationary perturbation theory generated by iterations of Eq. (82). It is reproduced if we take $C_y = C_z = 0$ and thus $\omega^2 = 0, C_x = 0$. For a wide class of the potentials $\mathcal{V}(\rho)$ the $C_y$-corrections to $\chi^2$ are exponentially small for large $\rho$. Moreover, supposing invariance of $\chi^2$ under scaling of the potential (see [1]), $C_y$ must be arbitrarily small or zero for the inflationary solution.

If $k \neq 0$, the $k$-term in (30) is of order $ke^{4\rho}$ for $\rho \to +\infty$, $\sigma(\rho) \to 0$. Using the expansion in (33) we can estimate the behavior of $\chi^2$ for large $\rho$ if $6C_y/V(\rho) \ll \bar{T}(\rho) < 1$, where $\bar{T}(\rho) = \mathcal{V}(\rho)/\mathcal{V}(\rho)$.\textsuperscript{11} Using the first terms of $y(\rho)$ and $x(\rho)$ we find

$$
\chi^2 = \frac{y}{x} = \left[\frac{36C_y}{V} - \bar{T} + o(\bar{T})\right]
$$

$$
\times \left[\frac{6C_y}{V} \left(1 - \frac{1}{6} \bar{T} + o(\bar{T})\right) + ke^{4\rho}(1 + O(\sigma))\right]^{-1}.
$$

(34)

This expression demonstrates that in inflationary solution the finite $k$ corrections may be larger than anisotropic ones. To see this, let us look once more on $\omega^2(\rho)$.

When $6C_y/V$ is negligible and $\omega^2 \ll 1$ we can use the results derived in [1]\textsuperscript{12}, but here we have to add some iteration scheme for deriving $\omega^2$, which we cannot discuss here. When $k = 0$ we however have the simple gauge-independent expression\textsuperscript{13}

$$
\omega^2 = \frac{C_z}{x(\rho)} = C_z \left[C_x + \int V(\rho)\right]^{-1}
$$

$$
= \frac{C_z}{V} \left[\frac{6C_x}{V} + 1 - \frac{1}{6} \bar{T}(\rho) + \ldots\right]^{-1}.
$$

(35)

As argued in [1], to get a pure inflationary solution we suppose that $C_y = 0$, but then, to take account of the anisotropic corrections we must take $C_x = C_z \neq 0$. In any case, the gauge invariant ‘anisotropy’ remains small and can be neglected in the scalaron inflationary models. The $k \neq 0$ correction may be more important.

5. REMARKS

The general solution in the scalaron cosmology depends on two arbitrary constants $C_y$ and $C_z \equiv C_x - C_y$. Formally, this solution is defined for $-\infty < \rho < +\infty$. However, in the interval $-\infty < \rho < \rho_0 \ll \rho_+$, where $\rho_0$ is some small positive number, the classical picture is invalid. If our classical dynamical system can be used for small and negative $\rho$, we may assume that there exist two intervals: ‘classical’ ($\rho_0, \rho_+$) and ‘quantum’ ($\rho_-, \rho_0$), in which a simple quantization approach can be applied (see, e.g., [17, 18]). As a first step one may try to quantize the classical inflationary system (with $C_y = C_z = 0$). Much more difficult is the vector theory, which is intrinsically anisotropic. Its analysis will be given elsewhere.

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\textsuperscript{12}The inflationary solution corresponds to $C_y = 0$, small $C_z = C_x, \omega^2$, and $\chi^2 < 6$.

\textsuperscript{13}The general solution of Eq. (16) with $k = A_x = 0$ in the gauge $\gamma = 3\rho$ is $\sigma = C(t - t_0)$. In any gauge, there exists the integral $\sigma = C \exp(1 - 3\rho)$ from which $\sigma(\rho)$ can be derived if we know $\xi(\rho)$.
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