Ultrasoft Effects in Heavy-Quarkonium Physics

BERND A. Kniehl*
Max-Planck-Institut für Physik (Werner-Heisenberg-Institut),
Föhringer Ring 6, 80805 Munich, Germany

ALEXANDER A. PENIN†
II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany

Abstract

In the framework of nonrelativistic QCD, we consider a new class of radiative corrections, which are generated at next-to-next-to-next-to-leading order through the chromoelectric dipole interaction of heavy quarkonium with ultrasoft virtual gluons. We provide analytical formulae from which the resulting shifts in the quarkonium energy levels and the wave functions at the origin may be calculated. We discuss the phenomenological implications for the top-antitop and Υ systems and point out some limitations of describing charmonium using a Coulomb potential.

PACS numbers: 12.38.Bx, 12.39.Jh, 14.40.Gx

*Permanent address: II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany.
†Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312, Russia.
1 Introduction

Recently, essential progress has been made in the theoretical investigation of the pair production of heavy quarks at threshold, and analytical results are now available up to next-to-next-to-leading order (NNLO). The NNLO corrections have turned out to be so sizeable that it appears to be indispensable to also gain control over the next-to-next-to-next-to-leading order (N³LO), both in regard of phenomenological applications and in order to understand the structure and the peculiarities of the threshold expansion. In this paper, we take the first step in this direction and investigate a particular class of N³LO corrections, namely those arising from the emission and absorption of virtual ultrasoft gluons by the heavy quarks. Such ultrasoft corrections are absent in lower orders and represent a genuinely new feature of the N³LO.

Specifically, we study the near-threshold behavior of the vacuum-polarization function $\Pi(q^2)$ of a heavy-quark vector current $j_\mu = \bar{q}\gamma_\mu q$,

$$
(q_\mu q_\nu - g_\mu_\nu q^2) \Pi(q^2) = i \int d^4xe^{iq_x} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle .
$$

(1)

Its imaginary part gives (up to a constant factor) the spectral density of $q\bar{q}$ production in $e^+e^-$ annihilation. Near threshold, the heavy quarks are nonrelativistic, so that one may consider the quark velocity $\beta$ (or inverse quark mass) as a small parameter. An expansion in $\beta$ may be performed directly in the Lagrangian of quantum chromodynamics (QCD) by using the framework of effective field theory. In the threshold problem, there are four different scales: (i) the hard scale (energy and momentum scale like $m_q$); (ii) the soft scale (energy and momentum scale like $\beta m_q$); (iii) the potential scale (energy scales like $\beta^2 m_q$, while momentum scales like $\beta m_q$); and (iv) the ultrasoft scale (energy and momentum scale like $\beta^2 m_q$). The ultrasoft scale is only relevant for gluons.

By integrating out the hard scale of QCD, one arrives at the effective theory of nonrelativistic QCD (NRQCD). If one also integrates out the soft scale and the potential gluons, one obtains the effective theory of potential NRQCD (pNRQCD), which contains potential quarks and ultrasoft gluons as active particles. The dynamics of the quarks is governed by the effective, nonrelativistic Schrödinger equation and by their interaction with the ultrasoft gluons. To get a regular perturbative expansion within pNRQCD, this interaction should be expanded in multipoles. The corrections from harder scales are contained in the Wilson coefficients, leading to an expansion in $\alpha_s$, as well as in the higher-dimensional operators of the nonrelativistic Hamiltonian, corresponding to an expansion in $1/m_q$.

This paper is organized as follows. In Section 2, we introduce the basic features of the pNRQCD formalism and derive formulas from which the leading ultrasoft corrections to the energy levels and the wave functions at the origin of heavy quarkonia may be evaluated. In Section 3, we present a numerical analysis and discuss phenomenological implications of our results. Section 4 contains our conclusions. In the Appendix, we explain how the “QCD Bethe logarithms” introduced in Section 2 may be reduced to one-dimensional integrals of elementary functions.
2 Ultrasoft corrections to energy levels and wave functions

In this section, we briefly recall the formalism of pNRQCD and calculate the leading ultrasoft corrections to the energy levels and the wave functions at the origin of heavy quarkonia. The basic quantity of pNRQCD is the nonrelativistic Green function, \( G^{s,o} \), of the Schrödinger equation,

\[
(H^{s,o} - E) G^{s,o}(x, y, E) = \delta^{(3)}(x - y),
\]

where \( H^{s,o} \) is the nonrelativistic Hamiltonian for the quark pair in the colour-singlet \( (s) \) [colour-octet \( (o) \)] state, defined by

\[
H^{s,o} = -\frac{\Delta_x}{m_q} + V^{s,o}(x) + \ldots,
\]

\[
V^{s,o}(x) = V^{s,o}_C(x) + \ldots,
\]

with \( \Delta_x = \partial_x^2 \) and \( x = |x| \). The ellipses stand for the higher-order terms in \( \alpha_s \) and \( 1/m_q \).

The propagation of the quark-antiquark pair in the singlet and octet states is described by the singlet and octet Green functions, respectively, which have the following spectral representations [12]:

\[
G^s(x, y, E) = \sum_{n=1}^{\infty} \frac{\psi_n^s(x)\psi_n(y)}{E_n - E} + \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{\psi^*_k(x)\psi^*_k(y)}{k^2/m_q - E},
\]

\[
G^o(x, y, E) = \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{\psi^o_k(x)\psi^o_k(y)}{k^2/m_q - E},
\]

where \( \psi_m \) and \( \psi^o_k \) are the wave functions of the \( q\bar{q} \) bound and continuum states. Note that a discrete part of the spectrum (bound states) only exists for the singlet Green function.

The nonrelativistic expansion in \( \alpha_s \) and \( \beta = \sqrt{1 - 4m_q^2/s} \), where \( s \) is the \( q\bar{q} \) centre-of-mass energy, provides us with the following representation of the heavy-quark vacuum-polarization function near threshold:

\[
\Pi(E) = \frac{N_c}{2m_q} G^s_C(0, 0, E) + \ldots,
\]
where \( E = \sqrt{s} - 2m_q \) is the \( q\bar{q} \) energy counted from the threshold and \( G_C^\alpha \) is the leading-order Coulomb Green function, which sums up the \( (\alpha_s/\beta)^n \) terms singular near the threshold. The ellipsis stands for the higher-order terms in \( \alpha_s \) and \( \beta \).

We are interested in the correction to the Green function induced by a virtual ultrasoft gluon. The corresponding diagram is shown in Fig. 1. The leading contribution is the one due to the chromoelectric dipole interaction \( g_s(|r_q - r_{\bar{q}}| \cdot E \) of the quark-antiquark pair with the ultrasoft gluon \[12\]. We wish to calculate the corresponding corrections to the energy levels and the wave functions at the origin of several low-lying resonances, which represent key parameters for the analysis of the threshold production of top- and bottom-quark pairs. Near the \( n \)th pole of the discrete spectrum, the correction to \( G_C^\alpha \) reads

\[
\Delta G(0, 0, E)|_{E \to E_n} = -\frac{|\psi_n(0)|^2}{(E_n - E)^2} J(E),
\]  

where

\[
J(E) = C_F g_s^2 \int_0^\infty \frac{d^3k}{(2\pi)^3} \langle r_i \rangle_{kn} \langle r_j \rangle_{nk} I^{ij}(E - k^2/m_q),
\]

with

\[
I^{ij}(p) = -i \int \frac{d^Dl}{(2\pi)^D} \frac{l_0^3(\delta^{ij} - l^il^j / l^2)}{l^2(p - l^0)},
\]

in \( D \) space-time dimensions. After integration over \( l_0 \), we recover the well-known non-relativistic perturbation theory expression. The pole in the nonrelativistic propagator is bypassed according to the standard prescription \( E \to E + i\epsilon \). The remaining integral over \( l \) is ultraviolet divergent for \( D = 4 \). To obtain a finite result, we use dimensional regularization with \( D = 4 - \epsilon \). The nonrelativistic perturbation theory of quantum electrodynamics (QED) in dimensional regularization is comprehensively described in Refs. \[13\],\[14\]. We thus obtain

\[
I^{ij}(p) = p^3 \frac{\delta^{ij}}{6\pi} \left( \frac{1}{\epsilon} + \ln \frac{\mu_F^2}{p^2} + \frac{5}{6} - \ln 2 \right),
\]

where \( 1/\epsilon = 1/\epsilon + [\ln(4\pi) - \gamma_E]/2 \). Note that this divergence is spurious. It arises in the process of scale separation due to the use of pNRQCD perturbation theory at short distances where it is inapplicable. In the total N\(^3\)LO result, the pole in \( 1/\epsilon \) is cancelled by the infrared poles coming from the hard- and soft-scale corrections. However, since these corrections are still unknown, we subtract the divergent part according to the \( \overline{\text{MS}} \) scheme. This means that the same scheme must be used for the calculation of the hard- and soft-scale corrections. As a consequence, the partial result for the ultrasoft contribution depends on the auxiliary “factorization scale” \( \mu_f \), which drops out in the total result.

The matrix element \( \langle r \rangle_{kn} \) is the one between the singlet Coulomb wave function of principal quantum number \( n \) and the octet Coulomb wave function of momentum \( k \). Since this matrix element is finite, we may use the wave functions of the Schrödinger equation in three dimensions, with \( \epsilon = 0 \). Writing

\[
E_n = E_n^C + \Delta E_n,
\]

\[
|\psi_n(0)|^2 = |\psi_n^C(0)|^2 \left( 1 + \Delta^2 \right),
\]

4
where the Coulomb values are

\[ E_n^C = -\frac{\lambda_s^2}{m_q n^2} , \]

\[ |\psi_n^C(0)|^2 = \frac{\lambda_s^3}{n^3} , \]  

(12)

with \( \lambda_s = \alpha_s C_F m_q / 2 \), we obtain the leading ultrasoft corrections as

\[ \Delta E_n = J(E)|_{E = E_n^C} , \]

\[ \Delta \psi_n^2(0) = \left. \frac{\partial J(E)}{\partial E} \right|_{E = E_n^C} . \]  

(13)

Inserting Eq. (8), we thus obtain

\[ \Delta E_n = -2 C_F \alpha_s \frac{\pi}{3} \int_0^\infty \frac{d^3 k}{(2\pi)^3} |\langle r_{kn} \rangle|^2 \left( E_n^C - k^2 / m_q \right)^3 \left( \ln \frac{E_1}{E_n^C - k^2 / m_q} + \ln \frac{\mu_f}{E_1} + \frac{5}{6} - \ln 2 \right) , \]

\[ \Delta \psi_n^2(0) = -2 C_F \alpha_s \pi \int_0^\infty \frac{d^3 k}{(2\pi)^3} |\langle r_{kn} \rangle|^2 \left( E_n^C - k^2 / m_q \right)^2 \left( \ln \frac{E_1}{E_n^C - k^2 / m_q} + \ln \frac{\mu_f}{E_1} + \frac{1}{2} - \ln 2 \right) . \]  

(14)

Except for the terms involving \( \ln [E_1 / (E_n^C - k^2 / m_q)] \), we may evaluate the right-hand sides of Eq. (14) analytically. In fact, making use of the completeness relation,

\[ \int_0^\infty \frac{d^3 k}{(2\pi)^3} |\langle r_{kn} \rangle|^2 (E_n^C - k^2 / m_q)^m = \langle r (E_n^C - \mathcal{H}^\omega)^m r \rangle_{nn} , \]  

(15)

this problem may be reduced to the calculation of the diagonal matrix elements of appropriate local operators,

\[ \langle r (E_n^C - \mathcal{H}^\omega)^3 r \rangle_{nn} = \left\langle \frac{1}{m_q} \left( V_s - V^\omega \right) (E_n^C - 3V_s + V^\omega) - 2 \frac{\partial^2 V^\omega}{m_q^2} \right\rangle_{nn} \]

\[ = E_n^C \left[ \frac{1}{4} C_A^3 + \frac{2}{n} C_A^2 C_F + \left( \frac{6}{n} - \frac{1}{n^2} \right) C_A^2 C_F + \frac{4}{n^2} C_F^3 \right] , \]

\[ \langle r (E_n^C - \mathcal{H}^\omega)^2 r \rangle_{nn} = \left\langle \frac{1}{m_q} \left( V_s - V^\omega \right)^2 + \frac{1}{m_q} (E_n^C - 2V_s + V^\omega) \right\rangle_{nn} \]

\[ = \left( \frac{1}{4} C_A^2 C_F + \frac{1}{n^2} C_A C_F^2 + \frac{1}{n^2} C_F^3 \right) . \]  

(16)

The \( C_F^3 \) contribution is purely Abelian and coincides with the QED result for the positronium bound state [13]. The “maximal non-Abelian” contribution proportional to \( C_A^3 \) in the local part of the energy shift may be read off from the analysis of the infrared properties of the non-Abelian static potential [15]. The \( C_A^3 / \epsilon \) term in the ultrasoft contribution to the energy shift cancels the corresponding infrared pole in the potential [16]. Note that there
is no such term in the wave-function correction. This may be understood by observing that, in contrast to the case of the energy shift, the wave function receives an additional infrared $C_A^3/\epsilon$ contribution from the hard matching coefficient of the heavy-quark vector current. This hard contribution must cancel the potential-related divergence in the correction to the wave function, but it does not affect the energy levels.

The logarithmic contributions in Eq. (14) represent a pure “retardation effect” and cannot be reduced to local-operator contributions. It is convenient to introduce the “QCD Bethe logarithms”

$$
L_n^E = \frac{1}{C_F C_n^3} \int_0^\infty \frac{d^3k}{(2\pi)^3} |\langle r|k\rangle_n|^2 \left( E_n^C - k^2/m_q \right)^3 \ln \frac{E_n^C}{E_n^C - k^2/m_q},
$$

$$
L_n^\psi = \frac{1}{C_F} \int_0^\infty \frac{d^3k}{(2\pi)^3} |\langle r|k\rangle_n|^2 \left( E_n^C - k^2/m_q \right)^2 \ln \frac{E_n^C}{E_n^C - k^2/m_q}. \quad (17)
$$

In the Appendix, we explain how these QCD Bethe logarithms may be reduced to one-dimensional integrals of elementary functions. For $n = 1, 2, 3$, we obtain the following numerical values:

$$
L_1^E = -81.5379, \quad L_2^E = -37.6710, \quad L_3^E = -22.4818,
$$

$$
L_1^\psi = -5.7675, \quad L_2^\psi = 0.7340, \quad L_3^\psi = 2.2326. \quad (18)
$$

The final results for the ultrasoft corrections to the heavy-quarkonium energy levels and wave functions at the origin read

$$
\Delta E_n = -\frac{2\alpha_s^3}{3\pi} E_n \left\{ \left[ \frac{1}{4} C_A + \frac{2}{n} C_A^2 + \left( \frac{6}{n} - \frac{1}{n^2} \right) C_A C_F^2 + \frac{4}{n} C_F^3 \right] \ln \frac{\mu_f}{E_1^C} + \frac{5}{6} - \ln 2 \right\} + C_F^3 L_n^E,
$$

$$
\Delta \psi_n^2 = -\frac{2\alpha_s^3}{\pi} \left\{ \left[ \frac{1}{4} C_A C_F + \frac{1}{n^2} C_A C_F^2 + \frac{1}{n^2} C_F^3 \right] \ln \frac{\mu_f}{E_1^C} + \frac{1}{2} - \ln 2 \right\} + C_F^3 L_n^E \right\} . \quad (19)
$$

### 3 Phenomenological applications

We are now in a position to present a numerical analysis and to discuss the phenomenological implications of our results. As illustrative examples, we consider, in Figs. 2 and 3, the $b\bar{b}$ and $t\bar{t}$ ground states, with $n = 1$, respectively, and investigate the size and the dependence on the factorization scale $\mu_f$ of the ultrasoft corrections to the energy level $E_1$ and the square of the wave function at the origin $|\psi_1(0)|^2$. As input values for the pole masses and the strong coupling constant, we use $m_b = 4.8$ GeV, $m_t = 175$ GeV and $\alpha_s(M_Z) = 0.118$, respectively. Note that, in Eq. (19), one power of $\alpha_s$ refers to the ultrasoft gluon interaction and should be evaluated at the ultrasoft scale, $\alpha_s^2 m_q$, while the two residual powers of $\alpha_s$ originate from the Coulomb Green function and should be evaluated at the Coulomb scale, $\alpha_s m_q$. This leads us to solve the functional equations $\alpha_s(\alpha_s^2 m_q) = \alpha_s$ and $\alpha_s(\alpha_s m_q) = \alpha_s$ for the ultrasoft and Coulomb regimes, respectively.
In the top-quark case, we have $\alpha_s = 0.146$ at the Coulomb scale and $\alpha_s = 0.196$ at the ultrasoft scale. Both solutions are in the perturbative region. In the bottom-quark case, we obtain $\alpha_s = 0.34$ at the Coulomb scale and $\alpha_s = 0.47$ at the ultrasoft scale. The last value seems to be too large for a reliable perturbative calculation. Thus, to be on the safe side, we redefine the ultrasoft scale to be $3\alpha_s^2m_b$. This leads to $\alpha_s = 0.34$, which coincides with the value at the Coulomb scale. The $\mu_f$ dependence is cancelled in the total result by hard terms of the form $\ln(\mu_f/m_q)$ and by soft terms of the form $\ln(\mu_f/\alpha_s m_q)$. In order to minimize these unknown logarithmic contributions and to absorb the large logarithms into the known ultrasoft terms, we select $\mu_f$ from the interval $\alpha_s m_q < \mu_f < m_q$.

From Figs. 2a and 3a, we observe that, depending on $\mu_f$, the ultrasoft N$^3$LO corrections to the energy levels of the $b\bar{b}$ and $t\bar{t}$ systems may be as large as $-120$ MeV and $+120$ MeV, respectively. As we see from Figs. 2b and 3b, the ultrasoft N$^3$LO wave-function correction reaches $-40\%$ for the $b\bar{b}$ system and $-7\%$ for the $t\bar{t}$ system. The circumstance that the corrections are close to zero at some points of the interval $\alpha_s m_q < \mu_f < m_q$ justifies this choice of factorization scale. It is interesting to compare the ultrasoft N$^3$LO corrections presented in Figs. 2 and 3 with the corresponding NLO and NNLO corrections, which we extract from Ref. [3]. In the bottom-quark case, the NLO (NNLO) contributions to $\Delta E_1$ and $\Delta \psi^2_1$ approximately amount to $-100$ MeV ($-200$ MeV) and $+50\%$ ($+150\%$), respectively. In the top-quark case, the corresponding values are $-700$ MeV ($-700$ MeV) and $-15\%$ ($+15\%$), respectively. This comparison nicely illustrates the numerical significance of the new N$^3$LO corrections.

Interesting, relevant and timely applications of our results include the studies of top-quark pair production at threshold, of low-lying $\Upsilon$ resonances and of $\Upsilon$ sum rules. In the case of the 1S $\Upsilon$ resonance, where local duality is expected to work, it is interesting to compare the perturbative ultrasoft contribution, related to the scale $\alpha_s^2m_b \approx 1$ GeV, with the leading nonperturbative contribution of the gluon condensate, due to nonperturbative fluctuations at the scale $\Lambda_{QCD}$, which originates from a diagram of the type shown in Fig. 1 with a broken gluon propagator. In the case of the 1S energy level, the latter is given by

$$\Delta E_1 = \frac{117m_q}{1275\lambda_s^3} \left< \alpha_s G_{\mu\nu}^a G^{a\mu\nu}_\mu \right>. \quad (20)$$

Using $\left< \alpha_s G^2 \right> = 0.06$ GeV$^4$, we have $\Delta E_1 \approx 60$ MeV, which is comparable to the ultrasoft contribution.

In the case of the $J/\psi$ resonance, which is sometimes optimistically considered to be a Coulomb system, the ultrasoft scale is of the order of $\Lambda_{QCD}$ if one assumes $\alpha_s$ to freeze at 1 GeV. Thus, one has to accurately separate perturbative and nonperturbative contributions. Nevertheless, our results suggest that the $J/\psi$ resonance cannot be regarded as a Coulomb potential system. Indeed, the potential model can be destroyed either by the large contribution from the gluon condensate, which is nonpotential because the nonperturbative scale $\Lambda_{QCD}$ is below the potential scale $\alpha_s m_c \approx 700$ MeV, or by the large perturbative retardation contribution. The first one is proportional to $1/\alpha_s^4$, while the second one is proportional to $\alpha_s^5$. We observe that, due to their different dependence on $\alpha_s$, these two contributions cannot be simultaneously small for any low-scale evolution.
of the coupling constant. For $m_c \approx 1.4$ GeV and $\alpha_s \approx 0.5$, the typical size of nonpotential contributions to the $J/\psi$ energy levels is about 400 MeV, which is comparable to the inverse Bohr radius. This clearly indicates that the $J/\psi$ system is far from being Coulombic.

4 Conclusions

We conclude with a few general remarks concerning the structure of higher-order corrections in pNRQCD. At NLO, the only source of corrections is the running of $\alpha_s$. At NNLO, higher-dimensional operators start to contribute. At $N^3$LO, retardation effects, which cannot be described by local operators, enter the game. The leading retardation effects, which are under consideration here, arise from the chromoelectric dipole interaction of heavy quarkonium with virtual ultrasoft gluons as depicted in Fig. 1. To our knowledge, these effects have not been studied elsewhere in the literature. We emphasize that they constitute a genuinely new feature, which is absent in NLO and NNLO. In particular, they are not contained in any of the popular renormalon-based mass definitions. On the other hand, they are expected to be the last source of unexpectedly large corrections. Thus, our result sets the scale of the $N^3$LO corrections.

In this paper, we took a first step towards the $N^3$LO analysis of the heavy-quark threshold dynamics. Our results entirely account for the ultrasoft-scale physics and should be complemented by the soft and hard contributions. The analysis of the last one includes the calculations of the three-loop hard matching coefficient and the three-loop potential. The corresponding two-loop results may be found in Refs. [17,18], respectively. Some $N^3$LO results on the effective Lagrangian were presented in Ref. [19].

Acknowledgements

B.A.K. thanks the Theory Group of the Werner-Heisenberg-Institut for the hospitality extended to him during a visit when this paper was finalized. A.A.P. gratefully acknowledges discussions with K. Melnikov. The work of A.A.P. is supported in part by the Volkswagen Foundation under Contract No. I/73611, by the Russian Academy of Sciences through Grant No. 37 and by and Russian Fund for Basic Research under Contract No. 97-02-17065.
References

[1] J.H. Kühn, A.A. Penin and A.A. Pivovarov, Nucl. Phys. B 534 (1998) 356.

[2] K. Melnikov and A. Yelkhovsky, Nucl. Phys. B 528 (1998) 59; Phys. Rev. D 59 (1999) 114009.

[3] A.A. Penin and A.A. Pivovarov, Phys. Lett. B 435 (1998) 413; Nucl. Phys. B 549 (1999) 217; B 550 (1999) 375; Preprint No. MZ-TH/98-61 and hep-ph/9904278 (December 1998).

[4] A.H. Hoang, Phys. Rev. D 59 (1999) 014039; Preprint No. CERN-TH/99-152 and hep-ph/9905550 (May 1999).

[5] A.H. Hoang and T. Teubner, Phys. Rev. D 58 (1998) 114023; Preprint No. CERN-TH/99-59, DESY 99-047 and hep-ph/9904468 (April 1999).

[6] M. Beneke, A. Signer and V.A. Smirnov, Phys. Lett. B 454 (1999) 137.

[7] T. Nagano, A. Ota and Y. Sumino, Preprint No. TU-562 and hep-ph/9903498 (March 1999).

[8] M. Beneke and A. Signer, Preprint No. CERN-TH/99-163, DTP/99/60 and hep-ph/9906475 (June 1999).

[9] M. Beneke and V.A. Smirnov, Nucl. Phys. B 522 (1998) 321.

[10] W.E. Caswell and G.P. Lepage, Phys. Lett. 167B (1986) 437; G.P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, Phys. Rev. D 46 (1992) 4052.

[11] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. 64 (1998) 428.

[12] M.B. Voloshin, Nucl. Phys. B 154 (1979) 365; M.A. Shifman and M.B. Voloshin, Yad. Fiz. 41 (1985) 187 [Sov. J. Nucl. Phys. 41 (1985) 120]; H. Leutwyler, Phys. Lett. 98B (1981) 447.

[13] A. Pineda and J. Soto, Phys. Lett. B 420 (1998) 391; Phys. Rev. D 59 (1999) 016005.

[14] A. Czarnecki, K. Melnikov and A. Yelkhovsky, Phys. Rev. A 59 (1999) 4316.

[15] T. Appelquist, M. Dine and I.J. Muzinich, Phys. Rev. D 17 (1978) 2074.

[16] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Preprint No. CERN-TH/99-61 and hep-ph/9903355 (March 1999); Preprint No. CERN-TH/99-199, HEPHY-PUB 716/99, UB-ECM-PF 99/06, UWThPh-199-34 and hep-ph/990724 (July 1999).
[17] A. Czarnecki and K. Melnikov, Phys. Rev. Lett. 80 (1998) 2531; M. Beneke, A. Signer and V.A. Smirnov, Phys. Rev. Lett. 80 (1998) 2535.

[18] M. Peter, Phys. Rev. Lett. 78 (1997) 602; Nucl. Phys. B 501 (1997) 471; Y. Schröder, Phys. Lett. B 447 (1999) 321.

[19] A. V. Manohar, Phys. Rev. D 56 (1997) 230; A. Pineda and J. Soto, Phys. Rev. D 58 (1998) 114011.
Appendix

Only the S-wave component of the (colour-singlet) Green function, with angular-momentum quantum number \( l = 0 \), contributes to its value at the origin and, therefore, to the leading ultrasoft corrections to the vacuum-polarization function. This means that only the \( l = 1 \) component of the colour-octet wave function contributes to the matrix element \( \langle r \rangle_{kn} \). The corresponding Coulomb wave functions for the attractive (singlet) and repulsive (octet) potentials read

\[
\psi_{n0}(r) = \frac{1}{\sqrt{4\pi}} R_{n0}(r), \\
\psi_{k1}^o(r) = e^{i(\pi/2-\delta_C^o)} \frac{3}{2k} R_{k1}(r) \frac{k \cdot r}{kr},
\]

where \( \delta_C^o \) is the \( l = 1 \) Coulomb phase and

\[
R_{n0}(r) = 2 \left( \frac{\lambda_s}{n} \right)^{3/2} e^{-\lambda_s r/n} F \left( 1 - n, 2, \frac{2\lambda_s r}{n} \right), \\
R_{k1}(r) = \sqrt{\frac{8\pi}{3}} k r \left( \frac{\nu^2 + 1}{\nu (\nu^2 - 1)} \right)^{1/2} e^{ikr} F(2 + i\nu, 4, -i2kr).
\]

Here, \( F \) is the confluent hypergeometric function, \( \nu = \lambda_s \rho_1 / k \) and

\[
\rho_n = \frac{1}{n} \left( \frac{C_A}{2C_F} - 1 \right) = \frac{1}{8n}.
\]

After some algebra, we obtain the QCD Bethe logarithms of Eq. (17) in terms of one-parameter integrals of elementary functions, as

\[
L_{n}^{E,\psi} = \int_0^\infty d\nu Y_n^{E,\psi}(\nu) X_n^2(\nu),
\]

where

\[
Y_n^E(\nu) = \frac{2^6 \rho_n^5 \nu (\nu^2 + 1) e^{4\nu \arctan \nu/\rho_n}}{n^2 (\nu^2 + \rho_n^2)^3 (e^{2\nu} - 1)} \ln \frac{n^2 \nu^2}{\nu^2 + \rho_n^2}, \\
Y_n^\psi(\nu) = \frac{\nu^2}{(\nu^2 + \rho_n^2)} Y_n^E(\nu),
\]

and

\[
X_1(\nu) = \rho_1 + 2, \\
X_2(\nu) = \frac{\nu^2 (2\rho_2^2 + 9 \rho_2 + 8) - \rho_2^2 (\rho_2 + 4)}{(\nu^2 + \rho_2^2)}, \\
X_3(\nu) = \frac{\nu^4 (8\rho_3^3 + 60 \rho_3^2 + 123 \rho_3 + 66) - 2\nu^2 \rho_3^2 (6\rho_3^2 + 41 \rho_3 + 54) + 3 \rho_3^4 (\rho_3 + 6)}{3 (\nu^2 + \rho_3^2)^2}.
\]

Since it is usually sufficient to consider \( n = 1, 2, 3 \) in practical applications and the expressions for \( X_n \) with \( n > 3 \) are somewhat cumbersome, we refrain to listing the latter.
**Figure captions**

**Fig. 1.** Feynman diagram giving rise to the ultrasoft contribution at N\(^3\)LO. The single and double lines stand for the singlet and octet Green functions, respectively, the wavy line represents the ultrasoft-gluon propagator in the Coulomb gauge, and the vertices correspond to the chromoelectric dipole interaction.

**Fig. 2.** Ultrasoft corrections to (a) the energy level \(E_1\) and (b) the square of the wave function at the origin \(|\psi_1(0)|^2\) as functions of the factorization scale \(\mu_f\) for the \(b\bar{b}\) ground state, with principal quantum number \(n = 1\).

**Fig. 3.** Same as in Fig. 2, but for \(t\bar{t}\).
Fig. 2a

Fig. 2b
Fig. 3a

Fig. 3b