Gravity as a gauge theory of translations

J. Martín–Martín * A. Tiemblo†

Abstract

The Poincaré group can be interpreted as the group of isometries of a minkowskian space. This point of view suggests to consider the group of isometries of a given space as the suitable group to construct a gauge theory of gravity. We extend these ideas to the case of maximally symmetric spaces to reach a realistic theory including the presence of a cosmological constant. Introducing the concept of “minimal tetrads” we deduce Einstein gravity in the vacuum as a gauge theory of translations.

1 Introduction

Universality is, without any doubt, the characteristic property of gravity, a force affecting in the same manner to all kinds of matter and energy. It was probably this universality the key idea leading Einstein to identify gravity with a very fundamental ingredient of reality. According to the contemporary mathematical achievements, he found in Geometry the final answer. Coherently with this basic assumption, our description of the gravitational field is given by some geometrical background perturbed by the presence of matter or energy.

As a matter of fact our phenomenological experience of the space–time nature is, roughly speaking, very close to an almost pseudoeuclidean geometry slightly modified by the presence of masses or energy. Here the term slightly alludes to the experimental evidence of the extreme weakness of gravity when compared to the other natural forces. These properties are on the basis of the so-called “weak field approximation” which constitutes an effective approach well adapted to many different problems. In any case the idea that gravity could be described by some expansion involving a characteristic parameter must be, in our opinion taken seriously into account. A part of this paper is precisely devoted to investigate the possible theoretical grounds of such an expansion.

*Departamento de Física Fundamental, Universidad de Salamanca, 37008 Salamanca, Spain, e-mail: chmm@usal.es
†Instituto de Física Fundamental, C/ Serrano 113 bis, CSIC, 28006, Madrid, Spain, e-mail: Tiemblo@imaff.cfmac.csic.es
In the absence of fermions the theory can be established in a purely geometrical framework. Nevertheless, a relevant question arises when we attempt to couple fermions with gravity, a program which implies the necessity to enlarge the framework with the introduction of an internal symmetry group suitable to include spinorial fields in the theory.

There is, in fact, a very rich literature [1] [2] [3] [4] [5] [6] [7] [8] [9] suggesting different extensions of the gauge principle to space-time groups, an attempt that exploits the properties of the Yang–Mills Theories in the hope to express gravity as mediated by gauge potentials only, as it happens with the other interactions.

Special mention is deserved to the occurrence in these kind of theories of the tetrad a typical structure which is neither a purely geometrical object nor an internal tensor, but both things at the same time, as deduced from the double behavior with respect to space-time coordinates and internal degrees of freedom. To explain tetrad’s properties several proposals have been considered and to this purpose Hehl’s Poincaré gauge theory as well as metric affine gravity [10] [11] [12] must be mentioned.

It must be stressed that tetrads can be considered as the footprint of the unavoidable presence of translations in a true gauge theory of gravity. Using Feynman’s words “gravity is that field which corresponds to a gauge invariance with respect to displacement transformations”. We claim that the cornerstone to include translations in such a gauge theory is given by non linear realizations of space-time symmetry groups containing translations [13] [14] [15] [16] [17] [18] [19]. On these grounds we are going to show that the formal structure of tetrads can be expressed in terms of well known gauge objects, connections and Goldstone–like bosons associated to the displacement transformations that, as we will see, can be locally chosen as coordinates that, in this way, can be interpreted as well as dynamical objects.

We devote the first section to a brief review of the local nonlinear realizations. The application to the Poincaré group is also included, obtaining the formal structure of the tetrads and discussing the possible dynamical interpretation of its different elements.

Being the quantum vacuum the basic ingredient of the physical reality, any realistic attempt to construct a dynamical description of the space–time nature, must include the role of the cosmological constant as an initial requirement. In this aim the extension of the gauge principle to the group of isometries of a space of constant curvature, appears as the natural generalization of the Poincaré group that can be considered as a limiting case when the curvature tends to zero.

The procedure provides a structure for the tetrads that allows us to identify the minimal structure satisfying all the requirements demanded for a tetrad. Curiously the remaining part is associated only to the translational connections that, in this manner, can be interpreted, following Feynman
ideas, as the true gravitational contributions.

On these assumptions a decomposition, analogous to the “weak field approach” is naturally obtained, it nevertheless appears not as a reasonable approximation but as an exact structure predicted by the non linear realization of the symmetry group.

2 Non linear tetrads

To construct a gauge theory of gravity, one is constrained to start from our phenomenological evidence of the space–time observable properties. Therefore, being Poincaré the group of isometries of the Minkowskian space, the presence of translations in the gauge description is a necessary consequence.

The non linear gauge realizations of space–time symmetry groups containing translations have been the object, in the past, of several papers, [20] [21] [22] [23] [24]. Nevertheless to facilitate the reading of this work we include here a brief review of the methods and main results.

Let $G$ be a Lie group having a subgroup $H$, we assume that the elements $C(\varphi)$ (cosets) of the quotient space $G/H$ can be characterized by a set of parameters say $\varphi$. Let us denote by $\psi$ an arbitrary linear representation of the subgroup $H$.

The non linear realization can be deduced from the action of a general element “$g$” of the whole group on the cosets representatives defined in the form:

$$gC(\varphi) = C(\varphi')h(\varphi, g)$$

(1)

where $h(\varphi, g) \in H$. It acts linearly on the representation space $\psi$ according to:

$$\Psi' = \rho[h(\varphi, g)]\Psi,$$

(2)

being $\rho$ a representation of the subgroup $H$. It can also be seen that the coset realization is the most general one which preserves linear the action of the subgroup $H$.

To construct a non linear local theory, the next step is to define suitable gauge connections, that can be obtained by substituting the ordinary Cartan 1–form $\omega = C^{-1}dC$ by a generalized expression of the form:

$$\Gamma = C^{-1}DC$$

(3)

where $D = d + \Omega$ is the covariant differential constructed with the $\Omega$ 1-form connection defined on the algebra of the whole group and having the canonical transformation law:

$$\Omega' = g\Omega g^{-1} + gdg^{-1}$$

(4)
The main point to be emphasized concerns the components of the non-linear connection $\Gamma$ that can be classified into two different categories:

- Those associated to the subgroup $H$, having the character and transformation properties of an ordinary gauge connection with respect to the action of $H$.
- Those associated to the coset sector, having the meaning of covariant differentials of the coset fields $\varphi$ and transforming as a true tensor with respect to the subgroup $H$.

The method can be applied to the Poincaré group \cite{25} by choosing $H = \text{Lorentz}$ and parametrizing the cosets in the form $C(\varphi) = e^{i\varphi P_i}$, where $P_i$ are the translational operators and $\varphi_i$ a set of continuous parameters.

Using (1) a simple calculation yields the variations of the cosets parameters $\varphi$ which reads:

$$\delta \varphi^i = \epsilon^i + \beta^i_j \varphi^j$$

where $\beta^i_j$ are the infinitesimal parameters of the Lorentz transformations and $\epsilon^i$ the corresponding ones for the translations.

The general non-linear connection (3) contains the ordinary Lorentz linear gauge connection $A_{ij}$ and a Lorentz tensorial object $e_i$ linked to the coset sector that can be identified with the tetrad, namely:

$$e_i = D\varphi_i + \Gamma(T)_i$$

where $D$ is the Lorentz covariant differential and $\Gamma(T)_i$ is a translational gauge connection transforming as

$$\delta \Gamma(T)_i = \beta^i_j \Gamma(T)_j - D\epsilon_i$$

According to the transformation properties (4) the fields $\varphi^i$ can be identified with the cartesian coordinates, a nice result as long as in this way they can be dynamically interpreted as Goldstone bosons with respect to the gauged translations. A second question arises from the anomalous dimensionality of the translational connection $\Gamma(T)$ in equation (6). Being a connection an object with the same formal dimensionality of a derivative, we introduce a constant characteristic length, say $\lambda$, to render it homogeneous with the ordinary Lorentz connection $A_{ij}$. Thus redefining $\Gamma(T)_i = \lambda \gamma_i$ we rewrite explicitly:

$$e(\lambda)_{\mu i} = \partial_\mu \varphi_i + A_{\mu ij} \varphi^j + \lambda \gamma_{\mu i}$$

The occurrence of a fundamental length in gravitational physics is an almost commonly accepted idea, it appears for instance at the Planck scale, lattices or string theories, \cite{26} \cite{27} \cite{28} \cite{29} \cite{30} \cite{31}. We claim that a natural
place to include a fundamental length is precisely the translational connection in equation (6), as we will see the smallness of $\lambda$ gives a coherent support to the treatment of gravity as a perturbation of a background metric.

As it has been pointed out, the essential property of a tetrad is given by its double character, transforming as an ordinary general vector with respect to coordinates transformations in the Greek indices and as a Lorentz vector in the Latin ones. It acts, in this way, as a link between both spaces. Nevertheless from equation (8) we realize that, being a covariant derivative of a Lorentz vector, $e(0)_{\mu i} = \partial_{\mu} \varphi^i + A_{\mu ij} \varphi^j$ is the minimal structure suitable by itself to accomplish this function, we outline that the distinction between $e(0)_{\mu i}$ and $e(\lambda)_{\mu i}$ depends only on the behavior with respect to translations.

Now with the help of the general tetrad (8) we define, in the usual way, the metric tensor

$$g(\lambda)_{\mu \nu} = e(\lambda)^{\mu}_{\mu} e(\lambda)_{\nu \nu} = g(0)_{\mu \nu} + \lambda \gamma_{\mu \nu} + \lambda^2 \gamma_{\mu \rho} \gamma_{\nu \sigma} g(0)^{\rho \sigma}$$

where $g(0)_{\mu \nu} = e(0)_{\mu i} e(0)^{i} \nu$ and we have used $e(0)_{\mu i}$ to transform indices.

Equation (9) strongly resembles a weak field expansion, playing $g(0)_{\mu \nu}$ the role of a background metric, nevertheless it must be emphasized that equation (9) is an exact result deduced from the structure of the tetrads. Therefore to elucidate the properties, structure and dynamical nature of $g(0)_{\mu \nu}$ becomes a relevant and interesting question which is going to be investigated in what follows.

As we have pointed out the choice Poincaré as the internal group is based on our phenomenological knowledge of space–time which suggests us a configuration very close to a minkowskian space. Notwithstanding a pure pseudoeuclidean model excludes the presence of a cosmological constant, a contribution that describes essential properties of the quantum vacuum or, in other words, of the real physical space-time. Therefore being Minkowsky the simplest case of the spaces of constant curvature it seems natural to extend the study to the general case to include the possible occurrence of a cosmological constant. The relevance of this point will be evident in the next section.

A maximally four dimensinoal symmetric space, in fact, admits a maximal number of Killing vectors defined as:

$$\hat{p}_{i} = i \left\{ \partial_{i} + \frac{k}{4} (2x_{i} x_{j} - \delta_{i}^{j} r^{2}) \partial_{j} \right\}$$

$$L_{ij} = i (\delta_{i}^{k} x_{j} - \delta_{j}^{k} x_{i}) \partial_{k}$$

where $r^{2} = \eta_{ij} x^{i} x^{j}$ and being $k$ the curvature. They define a semisimple Lie algebra of the form:

$$[\hat{p}_{i}, \hat{p}_{j}] = i k L_{ij}$$

$$[L_{ij}, \hat{p}_{k}] = i \eta_{k[i} \hat{p}_{j]}$$
\[ [L_{ij}, L_{kl}] = -i \{ \eta_{i[k} L_{l]j} - \eta_{j[k} L_{l]i} \} \] (14)

which reduces to Poincaré when \( k \to 0 \).

To construct the local theory we follow the general scheme outlined for the Poincaré case. Therefore we maintain for the cosets the same definition taking \( C = e^{i \varphi^i \hat{p}_i} \), parametrizing the group element \( g \) and the elements of \( H = Lorentz \) respectively as:

\[ g = e^{i \varphi^i \hat{p}_i} e^{\frac{1}{2} \beta^{ij} L_{ij}} \approx I + i (\hat{p}_i + \frac{1}{2} \beta^{ij} L_{ij}) \] (15)

and

\[ h = e^{\frac{1}{2} u^{ij} L_{ij}} \] (16)

where \( e^i \) and \( \beta^{ij} \) are the corresponding infinitesimal parameters.

The non linear expression of the connections can be deduced from the general formula (3) which in this case reads out:

\[ \Gamma = C^{-1} D C = e^{-i \varphi^i \hat{p}_i} [d + i \lambda \gamma^i \hat{p}_i + \frac{i}{2} A^{ij} L_{ij}] e^{i \varphi^i \hat{p}_i} \] (17)

With the help of the commutation relations (12), (13), (14) and using Hausdorff–Campbell formulas to deal with exponentials we obtain, after a little calculation, the value of \( \Gamma \)

\[ \Gamma = i \hat{e}_i \hat{p}^i + \frac{i}{2} \hat{A}_{ij} L^{ij} \] (18)

with

\[ \hat{e}_i = N D \varphi_i + \frac{1 - N}{\mu^2} (\varphi^i D \varphi_j) \varphi_i + \lambda M \gamma_i + \lambda \frac{1 - M}{\mu^2} (\gamma^i \varphi_j) \varphi_i \] (19)

where \( D \) is the Lorentz covariant differential, \( \mu = \varphi_i \varphi^i \) being \( M \) and \( N \), respectively:

\[ M = 1 - \frac{k \mu^2}{2!} + \frac{(k \mu^2)^2}{4!} \cdots \sim \cos \sqrt{k \mu^2} \] (20)

\[ N = 1 - \frac{k \mu^2}{3!} + \frac{(k \mu^2)^2}{5!} \cdots \sim \frac{\operatorname{sen} \sqrt{k^2}}{\sqrt{k \mu^2}} \] (21)

and

\[ \hat{A}_{ij} = A_{ij} + \frac{1 - M}{\mu^2} \varphi_{[i} D \varphi_{j]} + \lambda N \varphi_{[i} \gamma_{j]} \] (22)

To use only non linear connections we introduce in (19) the value of \( A_{ij} \) as deduced from (22), so that we rewrite (19) in the form:

\[ \hat{e}_i = \frac{N}{M} \hat{D} \varphi_i + \frac{1}{\mu^2} (1 - \frac{N}{M}) (\varphi^i D \varphi_j) \varphi_i + \lambda \frac{1}{M} \gamma_i^* + \lambda \frac{\varphi_i \varphi_j}{\mu^2} \gamma_j \] (23)
where $\hat{D}$ means the Lorentz covariant differential in terms of $\hat{A}_{ij}$, and $\gamma_i^* = (\delta_{ij} - \varphi_i \varphi_j / \mu^2) \gamma^j$.

Using the same techniques the non linear transformations properties of the fields $\varphi_i$ can be deduced from the non linear action definition (1), namely:

$$\delta \varphi_i = \frac{1}{N} \left[ \varepsilon_i + (N - 1) \frac{(\varphi^j \varepsilon_j) \varphi_i}{\mu^2} \right] + u_{ij} \varphi^j$$  \hspace{1cm} (24)$$

with:

$$u_{ij} = \beta_{ij} + \frac{1 - M}{N \mu^2} \varphi_i \varepsilon_j$$  \hspace{1cm} (25)$$

It can be easily seen that for a vanishing $k$ we recover the Poincaré approach as given by (8). Of course the limit $K \to 0$ must be considered carefully, they are in fact different group structures with specific properties. The Cassimir operator, for instance have the expression $C = \frac{1}{6k} \Delta$ which depends on the inverse of $K$.

A redefinition of the form $\hat{\varphi}_i = \varphi_i N$ leads us finally to an alternative more compact expression for $\hat{e}_i$, namely:

$$\hat{e}_i = \frac{1}{M} (\hat{D} \hat{\varphi}_i + \lambda \gamma_i^*) + \lambda \frac{\hat{\varphi}_i \hat{\varphi}_j}{\mu^2} \gamma^j$$  \hspace{1cm} (26)$$

To conclude this section we explicit write $\hat{e}_i$ in its tensorial form as:

$$e(\lambda)_{\mu i} = e(0)_{\mu i} + \lambda \gamma_{\mu i} = e(0)_{\mu i} (\delta_{\mu}^\nu + \lambda \gamma_{\mu}^\nu)$$  \hspace{1cm} (27)$$

where

$$\gamma_{\mu i} = \frac{1}{M} \gamma_{\mu i}^* + \frac{\varphi_i \varphi_j}{\mu^2} \gamma^j_{\mu}$$  \hspace{1cm} (28)$$

and

$$e(0)_{\mu i} = \frac{1}{M} \hat{D}_{\mu} \hat{\varphi}_i$$  \hspace{1cm} (29)$$

In this manner the theory can be interpreted as depending on two different dynamical variables, a tetrad $e(0)_{\mu i}$ (minimal tetrad) playing the role to change coordinates between both spaces and a true tensor $\gamma_{\mu i}$ related to displacement transformations, an idea which is implicit in the expression (9) of the general metric tensor.

3 Minimal tetrads

Assuming that the structure of the non linear tetrads is given by equations (8) and (26), we are now interested in the behavior of the theory when considered as an expansion in $\lambda$. To start from the lower terms, we are going to call “minimal tetrads” the limit $\lambda \to 0$ of the previously mentioned equations (8) and (26) which are precisely the objects to be analyzed in this
section. However it is probably useful to briefly reassert previously the standard gauge lagrangian formalism of gravity to introduce some definitions and properties to be used in the following.

Once the non linear dynamical variables $\hat{e}_{\mu i}$ and $\hat{A}_{\mu ij}$ have been obtained, the ordinary Einstein equations, in terms of the metric tensor $g(\lambda)_{\mu \nu}$, can be deduced from gravitational gauge lagrangian build up with the Field–Strength Tensor $\hat{F}_{\mu \nu ij} = \partial_{\mu} \hat{A}_{\nu ij} + \hat{A}_{\mu ik} \hat{A}^{k}_{\nu j}$. Taking advantage however of the knowledge of its internal structure as given by equations (8), (23) or (26) we are ready to reach a deeper insight into the meaning and possibilities of the non linear approach.

Nevertheless being the cosmological constant an essential feature of the quantum vacuum we are going to include it as an ingredient of the theory in the empty space. The presence or not of a cosmological term has, as we will see, important consequences.

To calculate the field equations we start from the gravitational gauge lagrangian density written in the form:

$$L = \hat{e} (\lambda)^{\mu i} \hat{e} (\lambda)_{\nu j} \hat{F}_{\mu \nu ij} + \hat{e} \Lambda$$

(30)

where $\hat{e}$ is the determinant, $\Lambda$ the cosmological constant and $\hat{e} (\lambda)^{\mu i}$ the formal inverse of $\hat{e} (\lambda)_{\mu i}$. To explore the dependence on $\lambda$ of the solutions we split the tetrad $\hat{e} (\lambda)_{\mu i}$ as:

$$\hat{e} (\lambda)_{\mu i} = \hat{e} (0)_{\mu i} + \lambda \hat{\gamma}_{\mu i}$$

(31)

where according to (26)

$$\hat{e} (0)_{\mu i} = \frac{1}{M} (\partial_{\mu} \hat{\varphi}_{i} + \hat{A}_{\mu ij} \hat{\varphi}_{j})$$

(32)

and $\hat{\gamma}_{\mu i}$ is given by (28)

To take into account (32) and (28) suitable Lagrange multipliers $\Delta^{\mu i}$ and $\Sigma^{\mu i}$ can be introduced. So that we add to (30) the following auxiliary term:

$$\Delta L = \hat{e} \Delta^{\mu i} \left[ \hat{e} (0)_{\mu i} - \frac{1}{M} (\partial_{\mu} \hat{\varphi}_{i} + \hat{A}_{\mu ij} \hat{\varphi}_{j}) \right] + \hat{e} \Sigma^{\mu i} \left[ \hat{\gamma}_{\mu i} - \left( \frac{1}{M} \hat{\varphi}_{i}^{*} \hat{\varphi}_{j} \right) \hat{\gamma}_{\mu j} \right]$$

(33)

Under all these assumptions the field equations can be written in the following form:

$$\hat{F}(\lambda)_{\mu i} - \frac{1}{2} \hat{e}(\lambda)_{\mu i} \hat{F}(\lambda) = \hat{e}(\lambda)_{\mu i} \Lambda$$

(34)

$$\partial_{\nu} M^{\nu \mu ij} - M^{\mu [i k} A_{j k]} = 0$$

(35)

with the addition of the conditions (28), (32) and being the Lagrange multipliers equal to zero. Here $\hat{F}_{\mu i} = \hat{e}(\lambda)^{\nu j} \hat{F}_{\mu \nu ij}$, $\hat{F} = \hat{e}(\lambda)^{\mu i} \hat{F}_{\mu i}$ and $M^{\nu \mu ij} = \hat{e}(\lambda)\hat{e}(\lambda)^{\mu i} \hat{e}(\lambda)^{\nu j}$.
The solution of (35) is highly simplified when $\hat{A}_{\mu ij}$ is redefined as follows:

$$\hat{A}_{\mu ij} = \hat{e}(\lambda)^{\alpha}_{i} D_{\mu} \hat{e}(\lambda)^{\alpha}_{j} + B_{\mu ij}, \quad (36)$$

where $D_{\mu}$ is the ordinary Christoffel covariant derivative acting on the index $\alpha$ of the tetrad $\hat{e}(\lambda)^{\alpha}_{j}$. Introducing the last in (35) we get immediately $B_{\mu ij} = 0$. So that finally the motion equations are given again by the conditions (32), (28) and the more familiar relations:

$$G(\lambda)^{\mu\nu} = \frac{1}{2} g(\lambda)^{\mu\nu} \Lambda \quad (37)$$

$$\hat{A}_{\mu ij} = \hat{e}(\lambda)^{\alpha}_{i} D_{\mu} \hat{e}(\lambda)^{\alpha}_{j} \Rightarrow B_{\mu ij} = 0 \quad (38)$$

where $G(\lambda)^{\mu\nu}$ is the Einstein tensor written in terms of the previously defined metric $g(\lambda)^{\mu\nu}$. Again for a vanishing $k$ the field equations becomes the corresponding ones to the Poincaré group.

We have employed Lagrange multipliers to reach a cleaner result, however it is interesting to notice that the formal variations with respect to $\varphi_{i}$ in the lagrangian (30) formally leads to a motion equation which becomes the covariant divergence of the Einstein tensor and therefore is satisfied identically. This result can be easily understood by considering that, as deduced from its transformation properties, the fields $\varphi_{i}$ are isomorphic to coordinates (the cartesian ones in the Poincaré case); therefore they can be eliminated from the theory by a simple choice of coordinates. A result that allows us, coherently with the non linear approach, to interpret them dynamically as Goldstone bosons of the theory. We remark that, due to the use of Lagrange multipliers, equation (32) is strictly, at this level, a pure definition useful to discuss the structure of the dynamical variables.

As we have pointed out the minimal structure having the formal properties of a tetrad is given by $e(0)_{\mu i}$. An important object as far as, according to (9), it can be used to provide us with a basis suitable to express gravity as an expansion in $\lambda$. Therefore the question is now to calculate the formal expression and consequences of the introduction of this minimal tetrad. We are going to consider however, in the first place, the Poincaré group, briefly reassuming the results obtained in a previous paper [25], at this purpose we take the limit $\lambda \rightarrow 0$ in equation (8) which, in this case, reads:

$$e(0)_{\mu i} = \partial_{\mu} \varphi_{i} + A_{\mu ij} \varphi^{j} \quad (39)$$

Taking $\lambda = 0$ in equation (36) and multiplying then by $\varphi^{j}$ one gets:

$$A_{\mu ij} \varphi^{j} = e(0)^{\alpha}_{i} D(0)_{\mu} e(0)_{\alpha j} \varphi^{j} + B_{\mu i} = e(0)^{\alpha}_{i} D(0)_{\mu} [e(0)_{\alpha j} \varphi^{j}] - \partial_{\mu} \varphi_{i} + B_{\mu i} \quad (40)$$

where $D(0)_{\mu}$ is the covariant Christoffel derivative with respect to the Greek index of the tetrad, constructed with the metric tensor $g(0)^{\mu\nu}$, and $B_{\mu i} = B_{\mu ij} \varphi^{j}$. When $k \rightarrow 0$ it follows from the definition (40)

$$e(0)_{\alpha j} \varphi^{j} = \partial_{\alpha} \sigma \quad (41)$$
with $\sigma = \frac{1}{2} \mu^2 = \frac{1}{2} \varphi_i \varphi^i$. Using (42) and (43) in (40) we obtain:

$$e(0)_{\mu i} = e(0)^{\alpha}_{\mu} D(0)_\mu D(0)_{\alpha} \sigma + B_{\mu i}. \quad (42)$$

Multiplying now by $e(0)_{\nu}^i$ we get finally the expression of the minimal metric tensor which reads:

$$g(0)_{\mu \nu} = D(0)_\mu D(0)_{\nu} \sigma + B_{\mu \nu}, \quad (43)$$

that for the vacuum solutions, $B_{\mu \nu} = 0$, reduces to:

$$g(0)_{\mu \nu} = D(0)_\mu D(0)_{\nu} \sigma \quad (44)$$

It can be easily seen that (46) is a particular case of the more general one:

$$D_\mu D_\nu \sigma = \frac{1}{d} g_{\mu \nu} \Box \sigma \quad (45)$$

being $d$ the dimensionality of the space. It must be emphasized that here the question is not to find the structure of $\sigma$, that we know "a priori", but to state that the existence of a scalar density $\sigma$ satisfying (45) implies, as an integrability condition, the maximally symmetric character of the space. Not surprisingly when equation (44) is concerned it can be shown that the space becomes directly Minkowskian. This is again a consequence of the role played by the fields $\varphi_i$ as Goldstone bosons of the theory with respect to the translations. In fact, choosing the $\varphi_i$ in (44) as coordinates we easily verify that the metric adopt the pseudoeuclidean value $\eta_{ij}$, valid only in the absence of the cosmological constant. To summarize briefly the Poincaré case puts in evidence that the non linear gauging of the group of isometries leads us to a minimal tetrad which generates precisely the metric of a pseudoeuclidean background space, a result that support the idea to link gravity to the translational connections. At the same time, from the limit $\lambda \to 0$ in equation (37), we realize that Poincaré group excludes (at expected) the presence of a cosmological constant.

As a consequence space–time physics can be described by two different dynamics, one related with the characteristic length $\lambda$ is of course the "true" gravitational interaction, while that associated with the properties of the quantum vacuum, relevant in the limit $\lambda \to 0$, is phenomenologically characterized by the cosmological constant $\Lambda$. The necessity therefore to include $\Lambda$ in a realistic scheme strongly suggests to study the group of isometries of a maximally symmetric space. Thus we recover the explicit form of equation (29) which becomes:

$$\hat{e}(0)_{\mu i} = \frac{1}{M} (\partial_\mu \hat{\varphi}_i + \hat{A}_{\mu ij} \hat{\varphi}^j) \quad (46)$$

following the same lines of the previous calculations we get:

$$\hat{e}(0)_{\mu i} = \frac{1}{M} \hat{e}(0)^{\alpha}_{\mu} D(0)_\mu \left[ \frac{1}{M} D(0)_{\alpha} \hat{\sigma} \right] \quad (47)$$
where \( \hat{\sigma} = \eta^{ij} \hat{\phi}_i \hat{\phi}_j \).

Now multiplying by \( \hat{e}(0)_\nu \) we obtain finally

\[
g(0)_{\mu \nu} = \frac{1}{M^2} \left[ D(0)_\mu D(0)_\nu \hat{\sigma} - \frac{\dot{M}'}{M} D(0)_\mu \hat{\sigma} D(0)_\nu \hat{\sigma} \right]
\]

(48)

being \( \dot{M}' \) the derivative of \( \dot{M} \) with respect to \( \hat{\sigma} \).

Equation (50) can be alternatively written in a more compact expression of the form:

\[
g(0)_{\mu \nu} = \frac{\text{Const}}{M} D(0)_\mu D(0)_\nu \tilde{\sigma}
\]

(49)

where \( \tilde{\sigma} = F(\hat{\sigma}) \hat{\sigma} \) and \( F(\hat{\sigma}) \) satisfy the differential equation:

\[
F'(\sigma) \tilde{\sigma} + F(\sigma) = \text{Const} \dot{M}.
\]

(50)

Taking now the constant equal to one, the trace of equation (51) becomes:

\[
\dot{M} = \frac{1}{4} \Box(0) \tilde{\sigma}
\]

(51)

which substituted in (51) leads us to:

\[
D(0)_\mu D(0)_\nu \tilde{\sigma} = \frac{1}{4} g(0)_{\mu \nu} \Box(0) \tilde{\sigma},
\]

(52)

in which we recognize equation (45) which implies, as an integrability condition, that \( g(0)_{\mu \nu} \) describes the metric of a maximally symmetric space having, consequently, a constant curvature, thus compatible with the presence of a cosmological term in the theory. We remark that the presence of the factor \( M^{-1} \) in equation (51) is essential to obtain a condition like (45). The absence of this factor in the Poincaré case leads us to (44) which implies the pseudoeuclidean character of the space.

## 4 Summary and conclusions

As we have seen, equation (32) as well as its limit when \( k \) tends to zero are suitable structures to construct a gauge approach of the Lorentz Group contained in a larger theory including translations. To reach the integrability conditions of equations (44) and (45) we have used a redefinition of the gauge field \( \hat{A}_{\mu ij} \) of the form:

\[
\hat{A}_{\mu ij} = \hat{e}(\lambda)_i \alpha_j D_\mu \hat{e}(\lambda)_{\alpha_j} + B_{\mu ij},
\]

(53)

where the object \( B_{\mu ij} \) has the meaning of a torsion term, that in the presence of matter takes into account the coupling of the spin densities to the field \( \hat{A}_{\mu ij} \). In the absence of matter terms the field equations lead us to
the result \( B_{\mu ij} = 0 \). So that as long as (44) and (49) have been obtained in the absence of this contribution they are no longer only a consequence of the definition of \( \hat{e}(0)_{\mu i} \) but a result of the vacuum motions equations. Notwithstanding it must be emphasized the purely algebraic character of the motion equation deduced from \( \delta A_{\mu ij} \). It implies that, when only classical fields equations are concerned, \( A_{\mu ij} \) can be eliminated from the theory by substituting everywhere its value as given by (36). As a consequence, once the expression for the minimal metric tensor is fixed by integrability conditions, the only relevant dynamical variable to describe pure gravity is given by the translational connection \( \gamma_{\mu i} \) or, in other words, a gauge theory of translations.

It is not in the scope of this paper to consider the introduction of matter fields, limiting ourselves to the vacuum solutions in the presence of a cosmological constant. It is immediate to see that when we work with (53) the equation (49), for instance, should be substituted by the more general symmetric expression:

\[
g(0)_{\mu \nu} = \frac{1}{M} \delta(0)_{\mu} D(0)_{\nu} \hat{\sigma} + B_{\mu \nu},
\]

where \( B_{\mu \nu} = \frac{1}{2} B_{(\mu ij)} \hat{\phi}^j \hat{e}(0)^i_{\nu} \).

It is worth mentioning that in the standard local field theory in the presence of gravity \( B_{\mu ij} \) stands for the matter sources coupled linearly to \( A_{\mu ij} \), (i.e.) the spin densities. In a gauge field theory only fermions give rise to these kind of contributions. It is known, on the other hand, that the fermion spin densities \( B_{\mu ij} \) are completely antisymmetric when all indexes are reduced to have the same nature, therefore a symmetric part of \( B_{\mu \nu} \) should be absent in (54). In any case this is a question to be considered in detail in the general framework of a theory containing matter fields.

The “anomalous” dimensionality of the translational connection \( \Gamma(T)_i \) in equation (6) allows us to identify a natural source for the introduction in the theory of a characteristic length \( \lambda \) intimately related to the gravitational forces. We assume that \( \lambda \) accomplish a fundamental role giving support to a natural expansion for the gravitational interaction. Nevertheless being \( \lambda \) a dimensional constant, some comments are probably in order. As we have pointed out the idea of a characteristic length in gravitational physics is commonly accepted by many authors and implicit in recent developments. It can be easily understood by assuming that when the scale of distances involved in a process are very large with respect to \( \lambda \), the relevant terms are the lower ones, the results obtained when \( \lambda \to 0 \) widely support an interpretation of this kind. Going however a little forward one could consider \( \lambda \) as a natural constant probably related with the limit of validity of a geometrical description.

The present scheme depends on two different constants that govern the physics of the space-time. The first one, specially relevant when \( \lambda \to 0 \),
is the cosmological constant essential to determine the choice of the gauge group concerned. The second one is, of course, \( \lambda \) itself related to what we could call properly gravitational forces. There appears, in this way, open problems to be elucidated. From the point of view of the group theory for instance, it is evidently interesting to try to understand the meaning and behavior of displacement transformations when a characteristic length like \( \lambda \) is present, an old problem on the other hand admitting different approaches. Concerning dynamics the close and foreseeable relationship between \( \lambda \) and the gravitational constant is another obvious and relevant point. Both are questions to be considered on the grounds of a general scheme including matter terms, work on these aspects is actually in progress.

**Acknowledgements**

We acknowledge Prof. A. Fernández Rañada and J. Julve for useful discussions. One of us (J. Martín) acknowledge financial support under the projects FIS2006-05319 of the Spanish MEC and SA010CO5 of the Junta de Castilla y León.

**References**

[1] R. Utiyama, *Phys. Rev.* **101**, 1597 (1956)

[2] D. Sciama, *Rev. Mod Phys.* **36**, 463 (1964) and *Rev. Mod Phys.* **36**, 1103 (1964)

[3] T. W. B. Kibble, *J. Math. Phys.* **2**, 212 (1961)

[4] K. Hayashi & Shirafuji, *Prog. Theor. Phys.* **64**, 866 (1980)

[5] D. Ivanenko & G. Sardanashvily, *Phys. Rept.* **94**, 1 (1983)

[6] E. A. Lord, *Gen, Rel. Grav.* **19**, 983 (1987)

[7] E. A. Lord & P. Goswami, *J. Math. Phys.* **29**, 258 (1988)

[8] G. Sardanashvily, *Teor. Math. Phys.* **132**, 1163 (2002)

[9] A. Ashtekar, *Phys. Rev. Lett.* **57**, 2244 (1986)

[10] F. W. Hehl, G. D revenues, & P. von der Heyde, *Phys. Rev. D** **10**, 1066 (1974)

[11] F. W. Hehl, G. D Kerlick, P. von der Heyde & J. Nester, *Rev. Mod. Phys.* **48**, 393 (1976)
[12] F. W. Hehl, J. D Mc Crea, E. W. Mielke & Y. Neeman, Phys. Rept. 258, 1 (1995)

[13] S. R. Coleman, J. Wess & B. Zumino Phys. Rev. 5177, 2239 (1969)

[14] J. Callan, G. Curtis, S. R. Coleman, J. Wess & B. Zumino Phys. Rev. 177, 2247 (1969)

[15] A. Salam & J. Strathdee Phys. Rev. 184, 1750 (1969)

[16] C. J. Isham, A. Salam & J. Strathdee Annals Phys. 62, 98 (1971)

[17] A. B. Iborisov & V. Ogievetskii Teor. Math. Fiz. 21, 239 (1974)

[18] Y. M. Cho Phys. Rev. D 18, 2810 (1978)

[19] K. S. Stelle & P. C. West Phys. Rev. D 21, 1466 (1980)

[20] J. Julve, A. López Pinto, A. Tiemblo & R. Tresguerres Gen. Rel. Grav. 28, 759 (1996)

[21] J. Julve, A. López Pinto, A. Tiemblo & R. Tresguerres (1996) in New Frontiers in Gravitation, G. A. Sardanashvily & R. Santilli (eds.). Hadronics Press Inc. Palm Harbord, p. 115

[22] A. López Pinto, A. Tiemblo & R. Tresguerres Class. Q. Grav. 12, 1503 (1995)

[23] A. López Pinto, A. Tiemblo & R. Tresguerres Class. Q. Grav. 13, 2255 (1996) and Class. Q. Grav. 14, 549 (1997)

[24] R. Tresguerres Phys. Rev D, 66, 064025 (2002)

[25] J. Martín & A. Tiemblo Int. Journal Geom. Meth. Mod. Phys., 5N2, 253 (2008)

[26] H. H. Borzeskowski & H. J. Treder, The Meaning of Quantum Gravity, Reidel Dordrecht (1971)

[27] L. J. Garay Int. J. Mod. Phys. A 10, 145(1985)

[28] B. A. Berg & B. Krishnan Phys. Lett. B 318, 59 (1993)

[29] G. Feinberg, R. Friedberg, T. D. Lee & H. C. Ren Nucl. Phys. B 245, 342 (1984)

[30] M. Kato Phys. Lett. B 245, 43 (1990) ¡OJO!

[31] K. Konishi, G. Paffuti & P. Provero Phys. Lett. B 234, 276 (1999)