Eisenstein series for Jacobi forms of lattice index

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Structure of the talk

1. Introduction to Jacobi forms
2. Jacobi–Eisenstein series
3. Fourier coefficients of Eisenstein series
1. Jacobi forms

- the arithmetic theory of scalar Jacobi forms was developed in 1985 by Eichler & Zagier
  - Eisenstein series
  - Taylor expansions
  - Hecke operators
  - relation with half-integral weight elliptic modular forms and with vector-valued modular forms (theta expansion)
  - relation with Siegel modular forms
- Jacobi forms and Algebraic Geometry (from the work of V. Gritsenko)
  - orthogonal modular forms can be obtained as lifts of Jacobi forms; the former determine Lorentzian Kac–Moody Lie (super) algebras of Borcherds type
  - Jacobi forms are solutions to the *mirror symmetry* problem for $K3$ surfaces
  - the *elliptic genus* of a Calabi–Yau manifold is a weak Jacobi form
  - and much more...
Notation

- \( e_c(x) = \exp \left( \frac{2\pi i x}{c} \right) \) and \( e(x) = e_1(x) \)
- the **weight** of a Jacobi form will be \( k \) in \( \mathbb{N} \) and the **index** \( \underline{L} = (L, \beta) \):
  - \( L \cong \mathbb{Z}^{\text{rk}(L)} \) is a **free, finite rank** \( \mathbb{Z} \)-module
  - \( \beta : L \times L \to \mathbb{Z} \) is a **symmetric, positive-definite, even** \( \mathbb{Z} \)-bilinear form
- set \( \beta(\lambda) := \frac{1}{2} \beta(\lambda, \lambda) \)
- the **dual lattice** of \( L \): \( L^\# := \{ t \in L \otimes \mathbb{Z} \mathbb{Q} : \beta(\lambda, t) \in \mathbb{Z} \text{ for all } \lambda \text{ in } L \} \)
  - the **determinant** of \( L \) is \( \det(L) := |L^\# / L| \)
- the integral **Jacobi group associated to** \( L \) is \( J_L := \Gamma \ltimes L^2 \)
- \( J_L^L \) acts on \( \text{Hol}(\mathfrak{h} \times (L \otimes \mathbb{C}) \to \mathbb{C}) \); given \( g = (A, (\lambda, \mu)) \) in \( J_L^L \), set

\[
\phi|_{k,L} g(\tau, z) := \phi \left( A\tau, \frac{z + \lambda \tau + \mu}{c\tau + d} \right) (c\tau + d)^{-k} \times e \left( \frac{-c\beta(z + \lambda \tau + \mu)}{c\tau + d} + \tau \beta(\lambda) + \beta(\lambda, z) \right)
\]
Definition

A function \( \phi \) in \( \text{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}) \) is called a Jacobi form of weight \( k \) and index \( L \) if:

1. \( \phi|_{k,L}(A, h) = \phi \), for all \((A, h)\) in \( J^L \);
2. \( \phi \) has a Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{D \in \mathbb{Q}_{\leq 0}, t \in L^\#} C(D, t) e \left( (\beta(t) - D)\tau + \beta(t, z) \right).
\]

for fixed \( k \) and \( L \), denote the \( \mathbb{C} \)-vector space of all such functions by \( J_{k,L} \)

**Jacobi cusp forms**: \( D < 0 \); denote the subspace of cusp forms of weight \( k \) and index \( L \) by \( S_{k,L} \)
2. Jacobi–Eisenstein series

Why the interest?

- they are an example of Jacobi forms

\[ \vartheta_{L,y}(\tau, z) = \sum_{t \in L \#_{t \equiv y \mod L}} e(\beta(t)\tau + \beta(t, z)) \]

- they should be perpendicular to Jacobi cusp forms with respect to a suitably defined Petersson scalar product and hence we obtain the following decomposition:

\[ J_{k,L} = S_{k,L} \oplus J_{Eis}^{k,L} \]

- we are interested in a theory of newforms with respect to Hecke operators (defined by Ajouz in 2015)
• the *isotropy set* of $L$ is $\text{Iso}(D_L) := \{ r \in L^# / L : \beta(r) \in \mathbb{Z} \}$

• define $J_L^\infty := \{ ( ( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} ), (0, \mu) ) : n \in \mathbb{Z}, \mu \in L \}$

**Definition**

For every $r$ in $\text{Iso}(D_L)$, let $g_L,r(\tau, z) := e(\beta(r)\tau + \beta(r, z))$ and define the Eisenstein series of weight $k$ and index $L$ associated to $r$ as

$$E_{k,L,r}(\tau, z) := \frac{1}{2} \sum_{\gamma \in J_L^\infty \setminus J_L} g_L,r|_{k,L} \gamma(\tau, z).$$

• it is absolutely and uniformly convergent on compact subsets of $\mathcal{H} \times (L \otimes \mathbb{C})$ for $k > \frac{rk(L)^2}{2} + 2$

• their *twists* by primitive Dirichlet characters modulo divisors of $N_r$ (order of $r$ in $L^#/L$) form a *basis of eigenforms* of $\text{Span}\{ E_{k,L,r} : r \in \text{Iso}(D_L) \}$

$$E_{k,L,r,\chi}(\tau, z) := \sum_{d \in (\mathbb{Z}/\mathbb{Z}N_r)^\times} \chi(d)E_{k,L,dr}(\tau, z)$$

with eigenvalues given by *twisted divisor sums*
Example (scalar case - Eichler & Zagier)

- consider \( L_m = (\mathbb{Z}, (x, y) \mapsto 2mxy) \), with \( m \) in \( \mathbb{N} \)
  - the dual is \( \frac{1}{2m} \mathbb{Z} \)
- then \( J_{k,L_m} \) is the space of scalar Jacobi forms \( J_{k,m} \)
- take \( L_m \) as the index, \( r = 0 \) and \( k \) even; then \( E_{k,L_m,0} \) has Fourier coefficients \( C\left(0, \frac{t}{2m}\right) = 1 \) for all \( \frac{t}{2m} \) in \( \mathbb{Z} \) and

\[
C\left(\frac{t^2}{4m} - n, \frac{t}{2m}\right) = \frac{(2\pi)^{k-\frac{1}{2}} i^k}{(2m)^{\frac{1}{2}} \Gamma(k - \frac{1}{2})} \left(n - \frac{t^2}{4m}\right)^{k-\frac{3}{2}} \sum_{c \geq 1} c^{-k} \times \sum_{\lambda,d(\mod c)} e_c(md^{-1} \lambda^2 - 2mr \lambda + nd).
\]

- when \( m = 1 \), we have \( C\left(\frac{t^2}{4} - n, \frac{t}{2}\right) = \frac{L_{t^2-4n}(2-k)}{\zeta(3-2k)} \)
- when \( m \) is square-free, we have

\[
C\left(\frac{t^2}{4m} - n, \frac{t}{2m}\right) = \frac{1}{\zeta(3 - 2k) \sigma_{k-1}(m)} \sum_{d|\(n,t,m\)} d^{k-1} \frac{L_{t^2-4nm}}{d^2} (2 - k)
\]

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Eisenstein series for Jacobi forms of lattice index
For $k > \frac{rk(L)}{2} + 2$ and for every $r$ in $\text{Iso}(D_L)$, the Eisenstein series $E_{k,L,r}$ is an element of $J_{k,L}$ and it is orthogonal to $S_{k,L}$ with respect to a suitably defined Petersson scalar product. It has the following Fourier expansion:

$$E_{k,L,r}(\tau, z) = \frac{1}{2} \left( \vartheta_{L,r}(\tau, z) + (-1)^k \vartheta_{L,-r}(\tau, z) \right)$$

$$+ \sum_{\substack{D \in \mathbb{Q}_{<0}, t \in \mathcal{L}\# \\beta(t) - D \in \mathbb{Z}}} C_{k,L,r}(D, t) e \left( (\beta(t) - D)\tau + \beta(t, z) \right),$$

where

$$C_{k,L,r}(D, t) = \frac{(2\pi)^{k-\frac{rk(L)}{2}} \Gamma(k-\frac{rk(L)}{2})}{2 \det(L)^{1/2} \Gamma \left( k - \frac{rk(L)}{2} \right)} (-D)^{k-\frac{rk(L)}{2} - 1}$$

$$\times \sum_{c \geq 1} c^{-k} \left( H_{L,c}(r, D, t) + (-1)^k H_{L,c}(-r, D, t) \right)$$

and $H_{L,c}(r, D, t)$ is the lattice sum

$$\sum_{\lambda \in L/cL, d \in (\mathbb{Z}/c\mathbb{Z})^\times} e_c \left( \beta(\lambda + r)d^{-1} + (\beta(t) - D)d + \beta(t, \lambda + r) \right).$$
Sketch of proof

\[ E_{k,L,r}(\tau, z) = \frac{1}{2} \sum_{\gamma \in J_{L}^L \setminus J_{L}^\infty} g_{L,r \mid k,L} \gamma(\tau, z) \]  
\text{(Reminder)}

- Invariance under \(|k,L\) action of \(J^L\) follows by construction.
- The given Fourier expansion completes the modularity argument.
- For orthogonality to cusp forms, use unfolding technique:

\[ \langle \phi, E_{k,L,r} \rangle := \int_{\bar{\mathfrak{H}} J_{L}^L} \phi(\tau, z) \sum_{\gamma \in J_{L}^L \setminus J_{L}^\infty} g_{r \mid k,L} \gamma(\tau, z) \mathfrak{H}(\tau)^k e^{-\frac{4\pi \beta(\Im(z))}{\Im(\tau)}} dV_{L}(\tau, z) \]

\[ = \int_{\bar{\mathfrak{H}} J_{L}^\infty} \phi(\tau, z) g_{r}(\tau, z) \mathfrak{H}(\tau)^k e^{-\frac{4\pi \beta(\Im(z))}{\Im(\tau)}} dV_{L}(\tau, z) \]

- Choose a convenient fundamental domain.
- Insert Fourier expansion of \(\phi\) and use orthogonality relations for exponential function, which imply that the integral in \(\Re(z)\) vanishes.
3. Fourier coefficients of Eisenstein series

\[ E_{k,L,r}(\tau, z) = \frac{1}{2} \sum_{\gamma \in J^L_{\infty} \setminus J^L} g_{L,r}|k,L \gamma(\tau, z) \]

(Reminder)

- choose a convenient set of coset representatives for \( J^L_{\infty} \setminus J^L \)
- write each \( \gamma \) as \((A, (a\lambda, b\lambda))\) and separate contributions coming from terms with \( c = 0 \) and \( c \neq 0 \)
  - \( c = 0 \) gives the singular term
    \[
    \sum_{\substack{t \in L \# \\
    t \equiv r \mod L}} e(\beta(t)\tau) \left( e(\beta(t, z)) + (-1)^k e(\beta(-t, z)) \right)
    \]
  - terms with \( c < 0 \) give same contribution as those with \( c > 0 \), multiplied by \((-1)^k\) and with \( z \) replaced by \(-z\)
  - contribution coming from \( c > 0 \) is:
    \[
    \sum_{c > 0} c^{-k} \sum_{\lambda(c), d(c) \times} e_{c} (\beta(\lambda + r)d^{-1}) F \left( \tau + \frac{d}{c}, z - \frac{1}{c} \lambda \right),
    \]
where \( F(\tau, z) \) has period \( \mathbb{Z} \) in \( \tau \) and period \( L \) in \( z \).
For every \( r \) in \( \text{Iso}(D_L) \), the Eisenstein series \( E_{k,L,r} \) is an element of \( J_{k,L} \) and it is orthogonal to \( S_{k,L} \) with respect to a suitably defined Petersson scalar product. It has the following Fourier expansion:

\[
E_{k,L,r}(\tau, z) = \frac{1}{2} \left( \vartheta_{L,r}(\tau, z) + (-1)^k \vartheta_{L,-r}(\tau, z) \right) + \sum_{D \in \mathbb{Q}_{<0}, t \in L^\#} C_{k,L,r}(D, t) e \left( (\beta(t) - D)\tau + \beta(t, z) \right),
\]

where

\[
C_{k,L,r}(D, t) = \frac{(2\pi)^{k - \frac{rk(L)}{2}} i^k}{2 \det(L)^{\frac{1}{2}} \Gamma \left( k - \frac{rk(L)}{2} \right)} (-D)^{k - \frac{rk(L)}{2} - 1} \times \sum_{c \geq 1} c^{-k} \left( H_{L,c}(r, D, t) + (-1)^k H_{L,c}(-r, D, t) \right)
\]

and \( H_{L,c}(r, D, t) \) is the lattice sum

\[
\sum_{\lambda \in L/cL, d \in (\mathbb{Z}/c\mathbb{Z})^\times} e_c \left( \beta(\lambda + r)d^{-1} + (\beta(t) - D)d + \beta(t, \lambda + r) \right).
\]
• suppose that \( \text{rk}(L) \) is even

• define \( \Delta(L) := (-1)^{\frac{\text{rk}(L)}{2}} \det(L) \) and write \( \Delta(L) = \varpi^2 \), with \( \varpi \) the discriminant of \( \mathbb{Q}(\sqrt{\Delta(L)}) \); set \( \chi_d(\cdot) := (\frac{d}{\cdot}) \)

**Theorem (M., 2017)**

*When \( k \) is odd, \( E_0 \equiv 0 \). When \( k \) and \( \text{rk}(L) \) are even, \( E_0 \) has the Fourier expansion*

\[
E_0(\tau, z) = \vartheta_{L,0}(\tau, z) + \sum_{(D,t) \in \text{supp}(L)} C_0(D, t)e \left( (\beta(t) - D)\tau + \beta(t, z) \right)
\]

and

\[
C_0(D, t) = \frac{2(-1)^{\frac{\text{rk}(L)}{4}} (-D|\varpi|)^{k - \frac{\text{rk}(L)}{2} - 1}}{\text{fL} \left( 1 - k + \frac{\text{rk}(L)}{2} , \chi_\varpi \right)} \sum_{d \mid f} \frac{\mu(d)}{d^{k - \frac{\text{rk}(L)}{2}}} \chi_\varpi(d) \sigma_{1 - 2k + \text{rk}(L)} \left( \frac{f}{d} \right) \times \prod_{p \mid 2DN_t^2 \det(L)} \frac{L_p(k - 1)}{1 - \chi\Delta(L)(p)p^{\frac{\text{rk}(L)}{2} - k}}.
\]
Sketch of proof

- Define the representation numbers

\[ R(b) := \#\{ \lambda \in L/bL : \beta(\lambda + t) - D \equiv 0 \mod b \} \]

- Set \( r = 0 \) in the formula for \( C_r(D, t) \), introduce the Möbius function to remove the coprimality conditions in \( H_{L,c}(0, D, t) \) and obtain

\[
C_{k,L,0}(D, t) = \frac{(2\pi)^{k - \frac{rk(L)}{2}} i^k (-D)^{k - \frac{rk(L)}{2} - 1}}{2 \det(L)^{-\frac{1}{2}} \Gamma\left(k - \frac{rk(L)}{2}\right) \zeta(k - rk(L))} (1 + (-1)^k) \sum_{b \geq 1} \frac{R(b)}{b^{k-1}}
\]

- Reminder:

\[
singular-term(E_{k,L,0}) = \frac{1 + (-1)^k}{2} \vartheta_{L,0}
\]

- The \( R(b) \)'s are multiplicative functions of \( b \); the Dirichlet series \( L(s) := \sum_{b \geq 1} R(b)b^{-s} \) arises in Brunier & Kuss (2001)

- It converges for \( \Re(s) > \rk(L) \) and that it can be continued meromorphically to \( \Re(s) > \frac{\rk(L)}{2} + 1 \), with a simple pole at \( s = \rk(L) \)
• define $w_p = 1 + 2\text{ord}_p(2NtD)$ and the local Euler factor

$$L_p(s) := p^{-wp^s} R(p^{wp}) + \left(1 - p^{-(s-\text{rk}(L)+1)}\right)^{wp-1} \sum_{l=0}^{wp-1} p^{-ls} R(p^l)$$

• using results of Siegel (1935) on representation numbers of quadratic forms modulo prime powers, we obtain

$$L(s) = \frac{\zeta(s-\text{rk}(L)+1)}{L\left(s - \frac{\text{rk}(L)}{2} + 1, \chi\Delta(L)\right)} \prod_{p|2DN_t^2, \text{det}(L)} \frac{L_p(s)}{1 - \chi\Delta(L)(p)p^{-(s-\frac{\text{rk}(L)}{2}+1)}}$$

• functional equations, Gauss sums, the duplication formula and Euler’s reflection formula for the Gamma function, values of the Riemann zeta function at positive even integers complete the proof
we have obtained:

\[
C_0(D, t) = \frac{2(-1)^{\left\lceil \frac{rk(L)}{4} \right\rceil} (-D|D|)^k \frac{rk(L)}{2} - 1}{\sqrt{fL \left(1 - k + \frac{rk(L)}{2}, \chi_D\right)} \sum_{d|f} \frac{\mu(d)}{d^{k - \frac{rk(L)}{2}}} \chi_D(d) \sigma_{1 - 2k + rk(L)} \left(\frac{f}{d}\right) \times \prod_{p|2DN^2 \det(L)} \frac{L_p(k - 1)}{1 - \chi\Delta(L)(p)p^{\frac{rk(L)}{2} - k}}}
\]

**Corollary**

*The Fourier coefficients of $E_0$ are rational numbers.*

use results of Zagier (1981)

\[
L(-n, \chi) = -\frac{M^n}{n + 1} \sum_{l=1}^{M} \chi(l)B_{n+1} \left(\frac{l}{M}\right)
\]
can we say something about the ‘bad’ Euler factors?

Cowan, Katz and White (2017): set $\mathcal{D} := \prod_{p | N_t^2 D, \gcd(p, 2 \det(L)) = 1} p^{v_p(N_t^2 D)}$

\[
\prod_{p | N_t^2 D, p \nmid 2 \det(L)} \frac{L_p(k - 1)}{1 - \chi_L(p)p^{-(k - \frac{\rk(L)}{2})}} = \chi_L(\mathcal{D})\mathcal{D}^{-(k - \frac{\rk(L)}{2} - 1)} \sum_{d | \mathcal{D}} \chi_L(d)d^{k - \frac{\rk(L)}{2} - 1}
\]

the formulas in CKW were implemented in Sage by B. Williams:
https://math.berkeley.edu/~btw/local-L-functions.sagews

Example

If $L$ is unimodular ($L^\# = L$), then

\[
C_0(D, t) = \frac{\rk(L) - 2k}{B_{k - \frac{\rk(L)}{2}}^{k - \frac{\rk(L)}{2} - 1}}(D).
\]

This was also shown by Woitalla (2018).
Fourier coefficients of non-trivial Eisenstein series

- let $x \in \mathbb{L}^\# / L$ and define the following Schrödinger representation
  \[ \sigma_x : \mathbb{Z}^3 \to \text{Span}_\mathbb{C} \{ \vartheta_{L,y} : y \in \mathbb{L}^\# / L \} : \]
  \[ \sigma_x(\lambda, \mu, \nu)\vartheta_{L,y} := e(\mu \beta(x, y) + (\nu - \lambda \mu) \beta(x)) \vartheta_{L,y - \lambda x} \]
- note that $\mathbb{Z}^3$ has group law
  \[ (\lambda, \mu, t)(\lambda', \mu', t') = (\lambda + \lambda', \mu + \mu', t + t' + \lambda \mu' - \mu \lambda') \]
- every Jacobi form has a theta expansion:
  \[ \phi(\tau, z) = \sum_{y \in \mathbb{L}^\# / L} h_{\phi,y}(\tau)\vartheta_{L,y}(\tau, z) , \]
  where
  \[ h_{\phi,y}(\tau) = \sum_{D \in \mathbb{Q}} \sum_{(D,y) \in \text{supp}(L)} C(D, y)q^{-D} \]
Definition

Define the averaging operator at $x$ in the following way:

$$\text{Av}_x \phi(\tau, z) := \frac{1}{N_x^2} \sum_{(\lambda, \mu) \in (\mathbb{Z}/\mathbb{Z}N_x^2)^2} \sigma_x^*(\lambda, \mu, 0) \phi(\tau, z).$$

- defined for vector-valued modular forms by Williams (2018)
- $\sigma_x$ is unitary and $\text{Av}_x$ maps $J_{k,L}$ to $J_{k,L}$

Proposition (M., 2017)

Suppose that $k$ is even and let $r$ be an element of $\text{ISO}(D_L)$. Then:

$$\sum_{\lambda \in \mathbb{Z}/\mathbb{Z}N_r} E_{k,L,\lambda r}(\tau, z) = \sum_{t \in L/\#L} \left( \sum_{\lambda \in \mathbb{Z}/\mathbb{Z}N_r} h_{E_{k,L,0}, t + \lambda r}(\tau) \right) \vartheta_{L,t}(\tau, z).$$
Sketch of proof

- for any $r$ in $L^\# / L$,

$$Av_r E_{k,L,0}(\tau, z) = \sum_{\lambda \in \mathbb{Z}/ZN_r^2} E_{k,L,\lambda r}(\tau, z)$$

- insert definition of $Av_r$ and theta expansion of $E_{k,L,0}$ on left-hand side and expand:

$$Av_r E_{k,L,0}(\tau, z) = N_r \sum_{t \in L^\# / L} h_{E_{k,L,0},t}(\tau) \sum_{\lambda \in \mathbb{Z}/ZN_r} \vartheta_{L,t-\lambda r}$$

- if $\beta(r) \in \mathbb{Z}$, then trivially

$$\sum_{\lambda \in \mathbb{Z}/ZN_r^2} E_{k,L,\lambda r}(\tau, z) = N_r \sum_{\lambda \in \mathbb{Z}/ZN_r} E_{k,L,\lambda r}(\tau, z),$$
• with this formula, we can compute Fourier coefficients of $E_{k,L,r}$ for any $r$ of order 2, 3, 4 or 6

Example

**Suppose that $r$ in has order 2. Then:**

$$E_{k,L,0}(\tau, z) + E_{k,L,r}(\tau, z) = \sum_{\substack{t \in \mathbb{L}^\# / \mathbb{L} \\ \beta(r,t) \in \mathbb{Z}}} (hE_{k,L,0,t} + hE_{k,L,0,t+r}) (\tau) \vartheta_{L,t}(\tau, z)$$

and hence

$$C_{k,L,r}(D, t) = \begin{cases} C_{k,L,0}(D, t + r), & \text{if } \beta(r, t) \in \mathbb{Z} \\ -C_{k,L,0}(D, t), & \text{otherwise} \end{cases}$$
Poincaré series:

\[ C_{k,L,F,r}(D,t) := \delta_L(F,r,D,t) + (-1)^k \delta_L(F,-r,D,t) + \frac{2\pi i^k}{\det(L)^{1/2}} \]

\[ \times \left( \frac{D}{F} \right)^{k - \frac{\text{rk}(L)}{4} - \frac{1}{2}} \sum_{c \geq 1} J_{k - \frac{\text{rk}(L)}{2} - 1} \left( \frac{4\pi (DF)^{1/2}}{c} \right) c^{-\frac{\text{rk}(L)}{2}} - 1 \]

\[ \times \left( H_{L,c}(F,r,D,t) + (-1)^k H_{L,c}(F,-r,D,t) \right) \]

Schwagenscheidt (2018): Fourier coefficients of \( E_{D_L,r,\chi} \)

Ran and Skoruppa (in progress): Fourier coefficients of \( E_{k,L,r,\chi} \)

Thank you!