Linked Cluster Expansions on non-trivial topologies

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Linked cluster expansions provide a useful tool both for analytical and numerical investigations of lattice field theories. The expansion parameter is the interaction strength fields at neighboured lattice sites are coupled. They result into convergent series for free energies, correlation functions and susceptibilities. The expansions have been generalized to field theories at finite temperature and to a finite volume. Detailed information on critical behaviour can be extracted from the high order behaviour of the susceptibility series. We outline some of the steps by which the 20th order is achieved.

1. SERIES EXPANSIONS

Realistic physical models evolve enormous complexity and in many instances force us to use approximations or expansions about solvable cases. Convergent expansions allow for precise quantitative statements within their range of convergence. Under general conditions they are suitable for an investigation of phase transitions and of critical phenomena. The price to be paid is that normally high orders have to be computed in order to reveal collective behaviour, and special techniques have to be developed and applied.

The linked cluster expansion (LCE) amounts to a convergent hopping parameter expansion applied to the free energy or connected correlation functions of lattice models \[1\]. To be specific, let us consider the \(O(N)\) symmetric scalar model defined on the hypercubic lattice \(\Lambda\) by the action

\[
S(\Phi, v) = \sum_{x \in \Lambda} V(\Phi_x) - \frac{1}{2} \sum_{x,y \in \Lambda} \sum_{a=1}^{N} \Phi_x^a v_{xy} \Phi_y^a
\]

with

\[
v_{xy} = \begin{cases} 
2 \kappa, & \text{x, y nearest neighbour,} \\
0, & \text{otherwise,}
\end{cases}
\]

and lattice site action \(V(\Phi_x)\) appropriately bounded from below. With the free energy \(W\) defined by

\[
e^{W(v)} = \int \prod_{x \in \Lambda} d^N \Phi_x \ e^{-S(\Phi, v)}
\]

we obtain the LCE as the Taylor expansion

\[
W(v) = \exp \left( \sum_{x,y} v_{xy} \frac{\partial}{\partial v_{xy}} \right) W(\hat{v}) \bigg|_{\hat{v}=0}
\]

and the obvious generalization to connected correlations and susceptibilities

\[
\chi_{2n} = \sum_{x_2, x_3, \ldots, x_{2n}} <\Phi_0^1 \Phi_{x_2}^1 \Phi_{x_3}^1 \cdots \Phi_{x_{2n}}^1>_{conn}.
\]

The rapidly increasing combinatorical complexity can be managed by the appropriate graphical device. The application of the LCE in field theory has been pioneered by Lüscher and Weisz \[2\]. Lüscher and Weisz studied a lattice \(\Phi^4\) theory close to its continuum limit in four dimensions. They succeeded in a 14th order computation of susceptibility series for arbitrary quartic couplings.

Normally, these expansions are applied to models on a lattice that is unbounded in all direction, and to the study of 2nd order transitions. There are recent new fields of application and generalization of LCE. A major topic are field theories at
finite temperature \(^{3}\). Here one dimension of the lattice is compactified to a torus of length given by the inverse temperature \(L_0\) in lattice units. If \(L_0\) is an even number, equivalence classes of graphs can be introduced that do not depend on \(L_0\) and hence are the same as for \(L_0 = \infty\). There are then mainly two adjustments necessary in order to generalize LCE to this case. First, embedding numbers change. The second point is more laborious. Comparing field theory models at zero and non-zero temperature, the shift of transition regions typically is rather small in bare parameters, whereas critical behaviour such as critical exponents in many instances undergo a considerable change. This different critical behaviour has to be resolved by the different high order behaviour of the susceptibility series. It amounts to an even higher order computation than on lattices unbounded in all directions. Graphs contributing to the expansion must be able to wind around the temperature torus sufficiently often in order to pick up sufficient finite temperature dependence.

We briefly mention two other recent generalizations of LCE. The first one is to apply LCE in a finite volume and to utilize finite size scaling analysis in order to study both 1st and 2nd order transitions, as well as tricritical behaviour \(^{4}\). This is the topic of the talk given by H. Meyer-Ortmanns, these proceedings. The second generalization applies to models that couple fields at lattice sites which are not nearest neighbours \(^{5}\).

2. ALGEBRA OF GRAPHS

In the range of orders considered, the approximated computational costs of increasing the LCE series by two orders are a factor of about 30-40. Fast algorithms have to be developed to achieve this computation. Some of the problems that one is faced with are briefly described in the following.

A first step is to restrict the classes of graphs to be considered and write correlation functions in terms of quantities that allow for expansions into those restricted graph classes, such as 1PI ones. Furthermore, 1PI graphs are composed of 1VI graphs \(S\) and renormalized moments \(Q\) (one external vertex only).

Incidence matrices provide a convenient way to represent graphs algebraically. For a graph \(\Gamma\) with \(V\) vertices, the incidence matrix \(I_\Gamma\) is the symmetric \(V \times V\)-matrix obtained as follows.

- Enumerate the vertices in any order.
- \(I_\Gamma(i, i) = \text{number of external lines attached to the vertex } v_i\).
- For \(i \neq j\), \(I_\Gamma(i, j) = \text{number of common lines of the vertices } v_i \text{ and } v_j\).

As an example, the graph

\[
\begin{array}{cccccc}
& 2 & 1 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 & 2 & 1 & 2 \\
& 0 & 0 & 0 & 0 & 0
\end{array}
\]

is represented by the incidence matrix

\[
I_\Gamma = \begin{pmatrix}
2 & 1 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 & 2 & 1 & 2 \\
& 0 & 0 & 0 & 0
\end{pmatrix}
\]

This way of representing graphs is not unique because the vertices can be enumerated arbitrarily. Comparison of two graphs needs huge factorial numbers of simultaneous permutations of rows and columns. The actual construction of graphs is done by iteration on the number of lines. Equivalent graphs have to be kept only once, and the number of comparisons is a measure of the efficiency of the algorithms.

A "canonical" representation of a graph \(\Gamma\) that is both unique and efficient is the following. Introduce a (partial) order relation on the set of vertices and enumerate the vertices in accordance with it. The incidence matrix then is defined by

\[
I_\Gamma^{\text{can}}(i, j) = \max_{\pi \in \Pi'} I_\Gamma(\pi(i), \pi(j)),
\]

where \(\Pi'\) is the subset of permutations that exchange vertices only if they stay unordered relative to each other under the preorder. The maximum is with respect to any total order relation on symmetric matrices.

The preorder of vertices should be as complete as possible. If all "local" properties of vertices
are taken into account in an optimal way we obtain the order relation introduced by Lüscher and Weisz. They succeeded in a 14th order computation for 2- and 4-point functions. Even higher orders need to take into account more global properties. A good example is given by

\[
\begin{align*}
&\text{2} & &\text{3} & &\text{3} & &\text{3} & &\text{3} & &\text{3} \\
&\text{3} & &\text{3} & &\text{3} & &\text{3} & &\text{3} & &\text{3} & &\text{1} \\
\end{align*}
\]

with many 2-vertices staying unordered. Taking into account properties like the distance to vertices that have more than 2 lines attached, vertex ordering can be enhanced to give

\[
\begin{align*}
&\text{2} & &\text{4} & &\text{6} & &\text{7} & &\text{5} & &\text{3} \\
&\text{4} & &\text{6} & &\text{7} & &\text{5} & &\text{3} & &\text{3} & &\text{1} \\
\end{align*}
\]

There are \((3!)^5\) permutations left to build \(I^\text{an}_L\) instead of \((15!)\). Furthermore, order relations can be iteratively improved by taking into account order numbers of neighboured vertices, next to nearest neighbours and so one. Normally, one or two iterations are sufficient.

The table shows the number of graphs for some classes generated as yet. The index \(k = 2, 4, 6\) denotes the number of external lines attached, \(L\) is the order (number of internal lines).

| \(L\) | \(Q_2(L)\) | \(S_2(L)\) | \(S_4(L)\) | \(S_6(L)\) |
|-------|---------|---------|---------|---------|
| 0     | 1       | 1       | 1       | 1       |
| 1     | 0       | 0       | 0       | 0       |
| 2     | 1       | 0       | 1       | 1       |
| 3     | 0       | 1       | 1       | 2       |
| 4     | 4       | 0       | 4       | 6       |
| 5     | 0       | 2       | 4       | 11      |
| 6     | 15      | 3       | 20      | 46      |
| 7     | 0       | 8       | 27      | 91      |
| 8     | 79      | 9       | 117     | 349     |
| 9     | 0       | 40      | 214     | 837     |
| 10    | 439     | 68      | 815     | 3140    |
| 11    | 0       | 247     | 1830    | 8401    |
| 12    | 2877    | 470     | 6721    | 31187   |
| 13    | 0       | 1779    | 17028   | 90599   |
| 14    | 20507   | 3937    | 61653   | 336582  |
| 15    | 0       | 14801   | 170923  | 1042392 |
| 16    | 161459  | 35509   | 621191  | 3895341 |
| 17    | 0       | 135988  | 1834324 |         |
| 18    | 1376794 | 350614  | 5548427 |         |
| 19    | 0       | 1361878 | 20967387|         |
| 20    | 12693105| 3705467 |         |         |

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