Asymmetric 6-vertex model and classical Ruijsenaars-Schneider system of particles

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Abstract

We discuss the correspondence between models solved by Bethe ansatz and classical integrable systems of Calogero type. We illustrate the correspondence by the simplest example of the inhomogeneous asymmetric 6-vertex model parametrized by trigonometric (hyperbolic) functions.

1 Introduction

The correspondence between quantum or statistical models solved by Bethe ansatz and classical integrable many-body systems of Calogero type (the quantum-classical duality) was established in [1] for the case of XXX type models. See also [2, 11, 21, 19, 15, 16] for different aspects of this remarkable correspondence. It was extended to models of the XXZ type related to quantum affine algebras $\mathcal{U}_q(\hat{sl}_N)$ in [20, 6]. In this paper we illustrate the correspondence by the simplest example of the inhomogeneous asymmetric 6-vertex model parametrized by trigonometric (hyperbolic) functions which is related to the trigonometric (hyperbolic) Ruijsenaars-Schneider system of particles [18].

The asymmetric 6-vertex model can be thought of as the symmetric one in horizontal and vertical external fields [5, 17]. In a natural basis, the matrix of Boltzmann weights

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of the asymmetric 6-vertex model has the form
\[
R = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & c' & b' & 0 \\
0 & 0 & 0 & a'
\end{pmatrix}.
\]

The standard argument shows that the partition function with periodic boundary condition depends only on the product \(cc'\), so we can put \(c' = c\) from the very beginning without loss of generality.

In this paper we consider the integrable inhomogeneous 6-vertex model on \(L\) sites, with \(x_i (i = 1, \ldots, L)\) being inhomogeneity parameters at the sites of the lattice. As is known, the asymmetric model with the horizontal external field \(h (a/a' = b/b' = e^{2h})\) is equivalent to the symmetric one \((a' = a, b' = b)\) with twisted boundary conditions which preserve integrability. The twist matrix is \(g = \text{diag}(e^{Lh}, e^{-Lh})\). In fact the transfer matrices of the two models differ by a similarity transformation, so they have the same spectrum. The transfer matrix of the twisted model, \(T^{(h)}(x)\), as a function of the spectral parameter \(x\), has simple poles at the points \(x_i\). The residues at the poles, \(H_i = (\sinh \eta)^{-1} \text{res}_{x=x_i} T^{(h)}(x_i)\), where \(\eta\) is the anisotropy parameter, are commuting operators which can be simultaneously diagonalized. In the framework of the quantum-classical duality \[11\], their eigenvalues are to be identified with velocities, \(\dot{x}_i\), of the classical Ruijsenaars-Schneider particles while the inhomogeneity parameters are identified with their coordinates, \(x_i\), with the condition that the higher integrals of motion of the classical model take some prescribed values expressed through spectral invariants of the twist matrix. We will also show that eigenvalues of another distinguished set of commuting operators, \(G_i = T^{(h)}(x_i - \eta)\), should be identified with \(-\eta^{-1}e^{-\eta p_i}\), where \(p_i\) are momenta of the Ruijsenaars-Schneider particles.

We thus see that the different eigenstates of the transfer matrix correspond to intersection points of two Lagrangian submanifolds in the Ruijsenaars-Schneider phase space: one of them is the hyperplane \(x_i = \text{const}\) and the other one is the level set of classical Hamiltonians in involution.

### 2 The asymmetric inhomogeneous 6-vertex model

**The symmetric model.** We start with the well known symmetric 6-vertex model. The Boltzmann weights \(a' = a, b' = b, c' = c\) are given by the \(R\)-matrix
\[
R(x) = \begin{pmatrix}
\frac{\sinh(x+\eta)}{\sinh x} & 0 & 0 & 0 \\
0 & 1 & \frac{\sinh \eta}{\sinh x} & 0 \\
0 & \frac{\sinh \eta}{\sinh x} & 1 & 0 \\
0 & 0 & 0 & \frac{\sinh(x+\eta)}{\sinh x}
\end{pmatrix}.
\]
in the standard trigonometric (hyperbolic) parametrization [5], where \( x \) is the spectral parameter and \( \eta \) is the anisotropy parameter.

Let \( V_i \cong \mathbb{C}^2 \) be several copies of the linear space \( \mathbb{C}^2 \), then \( R_{ij}(x) \) is a linear operator on \( \bigotimes_i V_i \) which acts non-trivially in \( V_i \otimes V_j \). This \( R \)-matrix satisfies the Yang-Baxter equation

\[
R_{12}(x-x')R_{13}(x)R_{23}(x') = R_{23}(x')R_{13}(x)R_{12}(x-x'),
\]

where the both sides are operators in \( V_1 \otimes V_2 \otimes V_3 \). The anisotropy parameters are the same for the three \( R \)-matrices.

Another important property of the \( R \)-matrix (1) is its invariance under the diagonal Cartan subgroup of \( GL(2) \times GL(2) \) which means commutativity with \( g \otimes g \), where \( g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \) is any diagonal \( 2 \times 2 \) matrix. Set \( g_1 = g \otimes 1, g_2 = 1 \otimes g \), then it is easy to check that

\[
g_1 g_2 R_{12}(x) = R_{12}(x) g_1 g_2.
\]

The asymmetric model. Let us label by 0 the horizontal ("auxiliary") space and by \( i = 1, \ldots, L \) the vertical ("quantum") spaces. The matrices of the Boltzmann weights for the asymmetric model with the horizontal field \( h \) and the vertical field \( v \) are defined as

\[
R_{0i}^{h,v}(x) = e^{\frac{h}{2} \sigma_0^z} e^{\frac{v}{2} \sigma_i^z} R_{0i}(x) e^{\frac{h}{2} \sigma_0^z} e^{\frac{v}{2} \sigma_i^z}
= \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix}_0 \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}_i R_{0i}(x) \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix}_0 \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}_i.
\]

The explicit form of this "asymmetric" \( R \)-matrix in the trigonometric parametrization is

\[
R_{0i}^{h,v}(x) = \begin{pmatrix} e^{h+i \frac{\sinh(x+\eta)}{\sinh x}} & 0 & 0 \\ 0 & e^{h-v} \frac{\sinh \eta}{\sinh x} & 0 \\ 0 & \frac{\sinh \eta}{\sinh x} & e^{-h+v} \\ 0 & 0 & 0 \end{pmatrix}.
\]

The Yang-Baxter equation (2) combined with the Cartan subgroup invariance property (3) implies the following Yang-Baxter equation for the asymmetric \( R \)-matrices with the same parameter \( \eta \):

\[
R_{12}^{v',v}(x-x') R_{13}^{h,v}(x) R_{23}^{h,v'}(x') = R_{23}^{h,v'}(x') R_{13}^{h,v}(x) R_{12}^{v',v}(x-x').
\]

In [7], the existence of an \( R \)-matrix that intertwines \( R_{0i}^{h,v}(x) \) and \( R_{0i}^{h,v'}(x) \) was proved by a direct solution of the Yang-Baxter equation.
Below we consider the inhomogeneous asymmetric 6-vertex model with periodic boundary condition in the horizontal direction. The transfer matrix of the model is defined in the usual way [12]:

\[ T^{h,v}(x) = \text{tr}_0 \left( R_{01}^h(x - x_1) R_{02}^h(x - x_2) \ldots R_{0L}^h(x - x_L) \right). \] (7)

The inhomogeneity parameters \( x_i \) at the sites are shifts of the spectral parameter. We assume that they are in general position, i.e. \( x_i \neq x_j \) and \( x_i \neq x_j \pm \eta \) for any \( i \neq j \). It follows from the Yang-Baxter equation (6) that the transfer matrices with different \( x \)'s and \( v \)'s (but the same \( \eta, h \) and \( \{x_i\}_L \)) commute: \([T^{h,v}(x), T^{h,v'}(x')] = 0\). It is easy to see that the dependence of the transfer matrix on the vertical field \( v \) is very simple:

\[ T^{h,v}(x) = e^{vS^z} T^{h,0}(x), \] (8)

where

\[ S^z = \sum_{i=1}^{L} \sigma_i^z = M_1 - M_2 \] (9)

is the operator that counts the (conserved) difference between the total number of up (\( M_1 = \frac{1}{2} \sum_{i=1}^{L} (1 + \sigma_i^z) \)) and down (\( M_2 = \frac{1}{2} \sum_{i=1}^{L} (1 - \sigma_i^z) \)) looking arrows on vertical edges. Note that \( M_1 + M_2 = L \mathbf{1} \), where \( \mathbf{1} \) is the unity operator.

In fact the transfer matrix of the asymmetric model with periodic boundary condition is connected with the transfer matrix of the symmetric model with twisted boundary condition by a similarity transformation. Set (cf. [9])

\[ U = \mathbf{1} \otimes e^{h\sigma_z} \otimes e^{2h\sigma_z} \otimes \ldots \otimes e^{(L-1)h\sigma_z} = \exp \left( L \sum_{j=1}^{L} (j-1)h \sigma_j^z \right). \]

The Cartan subgroup invariance [3] implies the relation

\[ UT^{h,v}(x) U^{-1} = e^{vS^z} T^{(h)}(x), \]

where

\[ T^{(h)}(x) = \text{tr}_0 \left( R_{01}(x - x_1) R_{02}(x - x_2) \ldots R_{0L}(x - x_L) e^{Lh\sigma_z^z} \right) \] (10)

is the transfer matrix for the symmetric model with the boundary condition twisted by the diagonal group element \( g = e^{Lh\sigma_z^z} \).

Diagonal matrix elements of the \( R \)-matrix [11] are periodic under the shift \( x \rightarrow x + i\pi \) while the non-diagonal ones are anti-periodic. Therefore, the trace (10) enjoys the periodicity condition \( T^{(h)}(x + i\pi) = T^{(h)}(x) \). Since it has first order poles at the points \( x_i \), its pole expansion can be written as

\[ T^{(h)}(x) = C + \sinh \eta \sum_{k=1}^{L} H_k \coth(x - x_k), \] (11)

where

\[ C = \frac{1}{2} \left( T^{(h)}(\infty) + T^{(h)}(-\infty) \right), \quad H_k = (\sinh \eta)^{-1} \text{res}_{z=x_k} T^{(h)}(z) \]
are some commuting operators. They can be regarded as Hamiltonians of an integrable quantum spin chain with long range interaction. The limiting values of $T^{(h)}(x)$ as $x \to \pm \infty$ can be easily found:

$$T^{(h)}(\infty) = C + \sinh \eta \sum_k H_k = e^{Lh} e^{\eta M_1} + e^{-Lh} e^{-\eta M_2},$$

$$T^{(h)}(-\infty) = C - \sinh \eta \sum_k H_k = e^{Lh} e^{-\eta M_1} + e^{-Lh} e^{-\eta M_2}.$$

Therefore, we have the following sum rules:

$$C = e^{Lh} \cosh(\eta M_1) + e^{-Lh} \cosh(\eta M_2), \quad (12)$$

$$\sum_{k=1}^L H_k = e^{Lh} \frac{\sinh(\eta M_1)}{\sinh \eta} + e^{-Lh} \frac{\sinh(\eta M_2)}{\sinh \eta}. \quad (13)$$

For the needs of finding the partition function one is interested in the solution of the common spectral problem

$$\begin{cases} T^{(h)}(x) \Psi = T(x) \Psi \\ M_i \Psi = M_i \Psi \end{cases} \quad \text{or} \quad \begin{cases} H_i \Psi = H_i \Psi \\ M_i \Psi = M_i \Psi \end{cases}, \quad i = 1, \ldots, L \quad (14)$$

Another distinguished set of commuting “Hamiltonians” is

$$G_i = T^{(h)}(x_i - \eta). \quad (15)$$

It can be shown that

$$G_i H_i = \prod_{k \neq i}^L \frac{\sinh(x_i - x_k + \eta)}{\sinh(x_i - x_k)} 1. \quad (16)$$

**The Bethe ansatz solution.** The operators $T^{h,v}(x)$, $S^z$ can be diagonalized simultaneously for any $x$. Below we will work with the operator $T^{(h)}(x)$ or, equivalently, with the set of commuting “Hamiltonians” $H_k$ (they generalize Hamiltonians of the trigonometric Gaudin model). This problem is usually solved by the algebraic Bethe ansatz \[10, 12\]. In the sector where $S^z$ has eigenvalue $S^z = L - 2M_2 \geq 0$ the eigenvalues $T(x)$ of $T^{(h)}(x)$ are given by the formula

$$T(x) = e^{Lh} \prod_{k=1}^L \frac{\sinh(x - x_k + \eta)}{\sinh(x - x_k)} \prod_{\alpha=1}^{M_2} \frac{\sinh(x - u_\alpha - \eta)}{\sinh(x - u_\alpha)} + e^{-Lh} \prod_{\alpha=1}^{M_2} \frac{\sinh(x - u_\alpha + \eta)}{\sinh(x - u_\alpha)} \quad (17)$$

(recall that $M_1 + M_2 = L$). The Bethe roots $u_\alpha$ are to be found from the system of Bethe equations

$$e^{2Lh} \prod_{k=1}^L \frac{\sinh(u_\alpha - x_k + \eta)}{\sinh(u_\alpha - x_k)} = \prod_{\beta=1, \neq \alpha}^{M_2} \sinh(u_\alpha - u_\beta + \eta) \sinh(u_\alpha - u_\beta - \eta). \quad (18)$$

The corresponding eigenvalues of $H_j$ and $G_j$ are

$$H_j = e^{Lh} \prod_{k=1, \neq j}^L \frac{\sinh(x_j - x_k + \eta)}{\sinh(x_j - x_k)} \prod_{\alpha=1}^{M_2} \frac{\sinh(x_j - u_\alpha - \eta)}{\sinh(x_j - u_\alpha)}, \quad (19)$$

$$G_j = e^{-Lh} \prod_{\alpha=1}^{M_2} \frac{\sinh(x_j - u_\alpha)}{\sinh(x_j - u_\alpha - \eta)}. \quad (20)$$
3 The trigonometric Ruijsenaars-Schneider model

The Ruijsenaars-Schneider (RS) system of particles is the relativistic generalization of the Calogero-Moser-Sutherland model. Let \( p_i, x_i \) be canonical variables with the Poisson brackets \( \{ p_i, x_j \} = \delta_{ij} \). The trigonometric (hyperbolic) RS system of \( L \) particles is defined by the classical Hamiltonian

\[
H = \sum_{i=1}^{L} e^{\eta p_i} \prod_{k \neq i} \frac{\sinh(x_i - x_k + \eta)}{\sinh(x_i - x_k)},
\]

where the parameter \( \eta \) has the meaning of the inverse velocity of light. The velocities of particles are given by

\[
\dot{x}_i = \frac{\partial H}{\partial p_i} = \eta e^{\eta p_i} \prod_{k \neq i} \frac{\sinh(x_i - x_k + \eta)}{\sinh(x_i - x_k)}.
\]

The equations of motion \( \dot{p}_i = -\frac{\partial H}{\partial x_i} \) are

\[
\ddot{x}_j = -\sum_{k=1, \neq j}^{L} \frac{2\dot{x}_j \dot{x}_k \sinh^2 \eta \cosh(x_j - x_k)}{\sinh(x_j - x_k + \eta) \sinh(x_j - x_k) \sinh(x_j - x_k - \eta)}.
\]

Two interesting special cases of the trigonometric RS model are \( \eta = \pm \infty \) and \( \eta = i\pi/2 \). In the former case the equations of motion simplify to

\[
\eta = \pm \infty : \quad \ddot{x}_j = 2 \sum_{k=1, \neq j}^{L} \dot{x}_j \dot{x}_k \coth(x_j - x_k).
\]

In the latter case they have the form

\[
\eta = \frac{i\pi}{2} : \quad \ddot{x}_j = 4 \sum_{k=1, \neq j}^{L} \frac{\dot{x}_j \dot{x}_k}{\sinh 2(x_j - x_k)}.
\]

The RS model is known to be integrable. It has the Lax representation \( \dot{L} = [A, L] \) with the Lax matrix

\[
L_{ij} = L_{ij}(\{x_k\}_L, \{\dot{x}_k\}_L) = \frac{\sinh \eta \dot{x}_i}{\sinh(x_i - x_j - \eta)}
\]

and the A-matrix

\[
A_{jk} = \left( \sum_{l \neq j} \dot{x}_l \coth(x_j - x_l) - \sum_{l} \dot{x}_l \coth(x_j - x_l + \eta) \right) \delta_{jk} + \frac{1 - \delta_{jk}}{\sinh(x_j - x_k)}
\]

The Lax representation implies that the time evolution of the Lax matrix is a similarity transformation: \( L(t) = U(t)LU^{-1}(t) \). In terms of momenta we have:

\[
L_{ij} = \frac{\eta \sinh \eta}{\sinh(x_i - x_j - \eta)} e^{\eta p_i} \prod_{k \neq i} \frac{\sinh(x_i - x_k + \eta)}{\sinh(x_i - x_k)}.
\]

\(^1\)The Lax matrix used in [6] is \( \tilde{L} = -L^t \), where \( t \) means transposition.
Note that $H = -\eta^{-1} \text{tr} L$. The integrals of motion in involution are given by

$$H_k = \text{tr} L^k, \quad H = -\eta^{-1} H_1. \quad (29)$$

The generating function of conserved quantities is the characteristic polynomial $Q(\lambda) = \det(\lambda I - L)$.

Let $X = \text{diag}(x_1, x_2, \ldots, x_L)$ be the diagonal matrix with the diagonal entries being coordinates of the particles. It is easy to check that the matrices $X, L$ satisfy the relation

$$e^{-\eta} e^X L e^{-X} - e^\eta e^{-X} L e^X = 2 \sinh \eta \dot{X} E,$$

where $E$ is the $L \times L$ matrix of rank 1 with all entries equal to 1.

The Lax matrix of the RS model admits a simple factorization:

$$L = \dot{X} C, \quad (30)$$

where $X = \text{diag}(x_1, x_2, \ldots, x_L)$ and $C$ is the trigonometric Cauchy matrix

$$C_{ij} = \frac{\sinh \eta}{\sinh(x_i - x_j - \eta)}. \quad (31)$$

It allows one to calculate the characteristic polynomial explicitly. We use the known fact that the coefficient in front of $\lambda^{L-k}$ in the polynomial $\det_{L \times L}(\lambda I + M)$ equals the sum of all diagonal $k \times k$ minors of the matrix $M$. All such minors can be found using decomposition $(30)$ and the explicit expression for the determinant

$$\det_{L \times L}(\lambda I - L) = \sum_{n=0}^{L} (-1)^n E_n \lambda^{L-n}, \quad (32)$$

where

$$E_n = (-1)^n \sum_{1 \leq i_1 < \ldots < i_n \leq L} x_{i_1} \ldots x_{i_n} \prod_{1 \leq \alpha < \beta \leq L} C(x_{i_\alpha} - x_{i_\beta}).$$

The integrals of motion $E_n$ are related to the integrals of motion $H_n$ by the Newton’s formula $\sum_{k=0}^{L} (-1)^k E_{L-k} H_k = 0$, where $H_0 = \text{tr} L^0 = L$.

In fact the Lax matrix admits another factorization which is non-trivial:

$$L = -\eta e^{\eta P} D_\eta (V^t)^{-1} S^{-1} V^t (D_\eta)^{-1}, \quad (33)$$

(see [3, 13, 11] for a similar representation in the rational case). Here $P, D, S$ are diagonal matrices $P = \text{diag}(p_1, p_2, \ldots, p_L)$,

$$(D_\eta)_{ij} = \delta_{ij} \prod_{k \neq i}^{L} \sinh(x_i - x_k + \eta) \quad (34)$$

$$(S)_{ij} = \delta_{ij} e^{-(2i-L-1)\eta} \quad (35)$$

and $V$ is the Vandermonde type matrix

$$V_{ij} = e^{(2j-L-1)x_i}. \quad (36)$$

Equation $(33)$ is the classical version of the factorized $L$-operator for the quantum trigonometric RS model [4].
4 The correspondence between the 6-vertex model and the RS model

Consider the Lax matrix [26] of the $L$-particle RS model, where the coordinates of particles, $x_i$, are identified with the inhomogeneity parameters and the inverse “velocity of light”, $\eta$, is identified with the anisotropy parameter. Let us also substitute $\dot{x}_i = -H_i$ and consider the matrix $L = L(\{x_i\}_L, \{-H_i\}_L)$:

$$
L = \begin{pmatrix}
H_1 & \sinh \eta H_1 & \sinh \eta H_1 & \cdots & \sinh \eta H_1 \\
\sinh \eta H_2 & \sinh(x_2-x_1+\eta) & \sinh \eta H_2 & \cdots & \sinh \eta H_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sinh \eta H_L & \sinh \eta H_L & \sinh \eta H_L & \cdots & H_L \\
\sinh(x_1-x_L+\eta) & \sinh(x_2-x_L+\eta) & \sinh(x_1-x_L+\eta) & & \\
\end{pmatrix}.
$$

The correspondence between the 6-vertex model and the RS model consists in the fact that if the $H_k$’s are eigenvalues of the operators $H_k$, then the eigenvalues of the RS Lax matrix are

$$
e^{Lh-(M_1-1)\eta+2\eta j}, \quad j = 0, 1, \ldots, M_1 - 1, \quad (38)
$$

$$
e^{-Lh-(M_2-1)\eta+2\eta j}, \quad j = 0, 1, \ldots, M_2 - 1.
$$

In the terminology of the Bethe ansatz technique, they form “strings” of lengths $M_1, M_2$ centered at $e^\pm Lh$. We see that the spectrum of $L$ depends only on the horizontal external field $h$ (and on $M_1, M_2$). This allows one to say that the spectral problem for the 6-vertex transfer matrix is equivalent to the following inverse spectral problem for the RS Lax matrix: given $x_i$’s, to find $H_i$’s in such a way that the eigenvalues of the Lax matrix have the fixed prescribed values (38). Equivalently, one fixes the values of the RS integrals of motion to be

$$
\mathcal{H}_n = \text{tr} L^n = e^{Lhn} \frac{\sinh(M_1\eta n)}{\sinh(\eta n)} + e^{-Lhn} \frac{\sinh(M_2\eta n)}{\sinh(\eta n)}. \quad (39)
$$

According to (31), (32), we have (31), where

$$
\mathcal{E}_n = \sum_{1 \leq i_1 < \ldots < i_n \leq L} H_{i_1} \ldots H_{i_n} \prod_{1 \leq \alpha < \beta \leq n} \frac{\sinh^2(x_{i_\alpha}-x_{i_\beta})}{\sinh(x_{i_\alpha}-x_{i_\beta}+\eta) \sinh(x_{i_\alpha}-x_{i_\beta}-\eta)}. \quad (40)
$$

The spectrum of the operators $H_i$ can be found by solving the algebraic equations

$$
\mathcal{E}_n = e_n, \quad n = 1, \ldots, L, \quad (41)
$$

where $e_n$ are elementary symmetric functions, $e_n = \sum_{1 \leq i_1 < \ldots < i_n \leq L} \xi_{i_1} \ldots \xi_{i_n}$, of the variables...
\( \xi_i \) which are taken from the set
\[
\{ \xi_k \}_L = \left\{ e^{Lh-(M_1-1)\eta} e^{Lh-(M_1-3)\eta} \ldots e^{Lh+(M_1-1)\eta} \right\}_{M_1}
\]
\[
e^{-(Lh-(M_2-1)\eta)} e^{-(Lh-(M_2-3)\eta)} \ldots e^{-Lh+(M_2-1)\eta} \right\}_{M_2}
\]
of eigenvalues of the matrix \( L \). These equations have many solutions which correspond to different eigenstates.

It is interesting to note that eigenvalues of the commuting Hamiltonians \( \mathbf{G}_i \) are related to momenta of the RS particles. More precisely, it follows from (16) and (22) that as soon as we identify \( H_i = -\dot{x}_i \) we should also identify
\[
G_i = -\eta^{-1} e^{-\eta x_i}, \quad i = 1, \ldots, L.
\]

## 5 Proof of the correspondence

The proof of the correspondence is straightforward but rather involved. In particular, it employs the non-trivial factorization \( K \) of the Lax matrix.

First let us prove the following lemma [6].

**Lemma 1.** Let \( Q, \tilde{Q} \) be a pair of \( N \times N \) and \( M \times M \) matrices
\[
Q_{ij}(\{x_i\}_N, \{y_\alpha\}_M, g) = \frac{g \sinh \eta}{\sinh(x_j-x_i+\eta)} \prod_{k \neq i}^N \frac{\sinh(x_i-x_k+\eta)}{\sinh(x_i-x_k)} \prod_{\gamma=1}^M \frac{\sinh(x_i-y_\gamma)}{\sinh(x_i-y_\gamma+\eta)}
\]  
(43)
where \( i, j = 1, \ldots, N \) and
\[
\tilde{Q}_{\alpha\beta}(\{y_\gamma\}_M, \{x_i\}_N, g) = \frac{g \sinh \eta}{\sinh(y_\beta-y_\alpha+\eta)} \prod_{\gamma \neq \alpha}^M \frac{\sinh(y_\alpha-y_\gamma-\eta)}{\sinh(y_\alpha-y_\gamma)} \prod_{k=1}^N \frac{\sinh(y_\alpha-x_k)}{\sinh(y_\alpha-x_k-\eta)}
\]  
(44)
where \( \alpha, \beta = 1, \ldots, M \) (for definiteness, we assume that \( M \leq N \)). Then the following identity holds true:
\[
\det_{N \times N} (\lambda - Q(\{x_i\}_N, \{y_\alpha\}_M, g)) = \det_{(N-M) \times (N-M)} (\lambda - g S_{N-M}) \det_{M \times M} (\lambda - \tilde{Q}(\{y_\alpha\}_M, \{x_i\}_N, g))
\]  
(45)
Here we use the notation \( (S_K)_{ij} = \delta_{ij} e^{-(2i-K-1)\eta} \) \( (i, j = 1, \ldots, K) \) for the matrix of the form \( K \) of size \( K \times K \).
This means that the matrix \( Q \) (43) has \( N - M \) eigenvalues of the form \( g e^{-(2i-N+M-1)\eta} \), \( i = 1, \ldots, N - M \). In particular, at \( M = 0 \) we have
\[
\det_{N \times N} (\lambda - Q(\{x_i\}_N, \emptyset, g)) = \det_{N \times N} (\lambda - g S_N) = \prod_{i=0}^{N-1} (\lambda - g e^{-(2i-N+1)\eta}).
\]  
(46)
**Proof.** The both sides of (45) are rational functions of \( t_i = e^{2x_i} \). It is enough to prove that they have the same residues at the poles and the same values at infinity.
For the proof we need the factorization of the matrices $Q$, $\tilde{Q}$ which is similar to \((33)\):

$$Q\left(\{x_i\}_N,\{y_\alpha\}_M, g\right) = gW^{(N,M)}D_q(\{x_i\}_N)(V^t)^{-1}(\{x_i\}_N)S_N^{-1}V^t(\{x_i\}_N)D^{-1}_\eta(\{x_i\}_N),$$  \hfill (47)

$$\tilde{Q}\left(\{y_\alpha\}_M,\{x_i\}_N, g\right) = g\tilde{W}^{(N,M)}D_0^{-1}(\{y_\alpha\}_M)V(\{y_\alpha\}_M)S_MV^{-1}(\{y_\alpha\}_M)D_0(\{y_\alpha\}_M).$$  \hfill (48)

Here

$$V_{ij}(\{q_k\}_K) = e^{(2j-K-1)q_i}, \quad i, j = 1, \ldots, K,$$

$$\left(\mathcal{D}_\xi\right)_{ij}(\{q_k\}_K) = \delta_{ij} K \prod_{k \neq j} \sinh(q_i - q_k + \xi), \quad i, j = 1, \ldots, K,$$

$$W_{ij}^{(N,M)} = \delta_{ij} M \prod_{\gamma = 1}^M \frac{\sinh(y_\gamma - x_i)}{\sinh(y_\gamma - x_i - \eta)}, \quad i, j = 1, \ldots, N,$$

$$W_{\alpha\beta}^{(N,M)} = \delta_{\alpha\beta} N \prod_{k = 1}^N \frac{\sinh(y_\alpha - x_k)}{\sinh(y_\alpha - x_k - \eta)}, \quad \alpha, \beta = 1, \ldots, M.$$

Let us note that $\det W^{(N,M)} = \det \tilde{W}^{(N,M)}$. Hence the statement of the lemma acquires the form

$$\det_{N \times N} \left(\lambda(W^{N,M})^{-1} - Q_0(\{x_i\}_N, g)\right)$$

$$= \det_{(N-M) \times (N-M)} \left(\lambda I - gS_{N-M}\right) \det_{M \times M} \left(\lambda(W^{N,M})^{-1} - \tilde{Q}_0(\{y_\alpha\}_M, g)\right),$$

where $Q_0(\{x_i\}_N, g) = Q(\{x_i\}_N, \emptyset, g)$, $\tilde{Q}_0(\{y_\alpha\}_M, g) = \tilde{Q}(\{y_\alpha\}_M, \emptyset, g)$.

First let us prove that the left hand side of \((53)\) does not have poles at the points $x_i = x_k$ and $x_i = x_k + \eta$. For this we write

$$V_{ij}(\{x_k\}_N) = e^{(1-N)x_i}(e^{2x_i})^{i-1} = T_{ii}(\{x_k\}_N)V_{ij}(\{x_k\}_N),$$

where $V$ is the Vandermonde matrix of variables $t_i = e^{2x_i}$ ($V_{ij} = t_i^{j-1}$) and $T$ is the diagonal matrix ($T_{ii} = e^{(1-N)x_i}$). Then one can rewrite the left hand side of \((53)\) as

$$\det_{N \times N} \left(\lambda(W^{N,M})^{-1} - Q_0(\{x_i\}_N, g)\right)$$

$$= \det_{N \times N} \left(\lambda(W^{N,M})^{-1} - gD_q(V^t)^{-1}S_N^{-1}V^tD^{-1}_\eta\right)$$

$$= \det_{N \times N} \left(\lambda(W^{N,M})^{-1} - gT^{-1}(\tilde{V}^t)^{-1}S_N^{-1}\tilde{V}^tT\right)$$

$$= \det_{N \times N} \left(\lambda \tilde{V}^t(W^{N,M})^{-1}(\tilde{V}^t)^{-1} - gS_N^{-1}\right).$$

The inverse to the Vandermonde matrix is given by the explicit expression

$$(\tilde{V}^t)^{-1}_{kj} = \frac{1}{(j-1)!} \left[ \prod_{l \neq k}^N \frac{s - t_l}{t_k - t_l} \right]_{s=0}.$$

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Then the matrix element \((\tilde{\mathbf{V}} \mathbf{W}^{-1} \tilde{\mathbf{V}}t)^{-1})_{ij}\) has the form

\[
(\tilde{\mathbf{V}} \mathbf{W}^{-1} \tilde{\mathbf{V}}t)^{-1} = \sum_{k=1}^{N} \tilde{v}_{ik} \mathbf{W}^{-1}_{kk} (\tilde{\mathbf{V}}t)^{-1}_{kj}
\]

\[
= \sum_{k=1}^{N} t_{i}^{-1} \mathbf{W}^{-1}_{kk} \frac{1}{(j-1)!} \left. \partial^{j-1}_{s} \prod_{l \neq k}^{N} \frac{s-t_{l}}{t_{k}-t_{l}} \right|_{s=0} .
\]

The expression

\[
\sum_{k=1}^{N} t_{i}^{-1} \mathbf{W}^{-1}_{kk} \prod_{l \neq k}^{N} \frac{s-t_{l}}{t_{k}-t_{l}} = \sum_{k=1}^{N} (\tilde{\mathbf{V}} \mathbf{W}^{-1} \tilde{\mathbf{V}}t)^{-1}_{ik} s^{k}
\]

is the generating function of the matrix elements. We see that the poles at \(x_{a} = x_{b} + \eta\) are absent. The pole at \(x_{a} = x_{b}\) comes from the terms with \(k = a, b\). The residue at this point is given by the expression

\[
\prod_{m=1}^{N} (s-t_{m}) \left( t_{a}^{-1} \mathbf{W}^{-1}_{aa} \prod_{l \neq a,b}^{N} \frac{s-t_{l}}{t_{a}-t_{l}} - t_{b}^{-1} \mathbf{W}^{-1}_{bb} \prod_{l \neq a,b}^{N} \frac{s-t_{l}}{t_{b}-t_{l}} \right)
\]

which is zero at \(x_{a} = x_{b}\).

In a similar way, one can show that there are no poles at \(y_{a} = y_{\beta}\) and \(y_{a} = y_{\beta} + \eta\) in the right hand side of (53). This means that the both sides have poles only at the points \(x_{i} = y_{a}\).

The next step is induction in \(M\). At \(M = 0\) we have

\[
det_{N \times N} (\lambda - \mathbf{Q}) = det_{N \times N} (\lambda - g \mathbf{D}_{\eta}(\mathbf{V}t)^{-1} \mathbf{S}_{N}^{-1} \mathbf{V}t \mathbf{D}_{\eta}^{-1})
\]

\[
= \det_{N \times N} (\lambda - g \mathbf{S}_{N}^{-1}) = \det_{N \times N} (\lambda - g \mathbf{S}_{N})
\]

which agrees with the statement of the lemma (the second determinant of the 0\(\times\)0 matrix in (45) is set to be equal to 1). The assumption of the induction is that the statement of the lemma holds true at \(M - 1\) and for any \(N \geq M - 1\). Pass from \(M - 1\) to \(M\) and consider the residue at \(x_{i} = y_{a}\) in the left hand side of (53):

\[
res_{x_{i}=y_{a}} \det_{N \times N} (\lambda(\mathbf{W}^{N,M})^{-1} - \mathbf{Q}_{0}(\{x_{k}\}^{N}, g))
\]

\[
= \det_{(N-1) \times (N-1)} (\lambda(\mathbf{W}^{N-1,M})^{-1} - \mathbf{Q}_{0}^{ii}(\{x_{k}\}^{N}, g)) \times \lambda \sinh \eta \prod_{\gamma \neq \alpha}^{M} \frac{\sinh(x_{i} - y_{\gamma} + \eta)}{\sinh(x_{i} - y_{\gamma})}
\]

\[
\times \prod_{k=1, k \neq i}^{N} \frac{\sinh(x_{k} - y_{\alpha} + \eta)}{\sinh(x_{k} - y_{\alpha})} \prod_{\gamma = 1, \neq \alpha}^{M} \frac{\sinh(x_{i} - y_{\gamma} + \eta)}{\sinh(x_{i} - y_{\gamma})}.
\]

In the second line \(\mathbf{Q}_{0}^{ii}\) is the matrix \(\mathbf{Q}_{0}\) without its \(i\)-th row and \(i\)-th column. In a similar way, the residue in the right hand side of (53) is

\[
res_{x_{i}=y_{a}} \det_{M \times M} (\lambda(\tilde{\mathbf{W}}^{N,M})^{-1} - \tilde{\mathbf{Q}}_{0}(\{y_{\gamma}\}_{M}, g))
\]
\[
\det_{M-1 \times M-1} \left( \lambda (\tilde{W}^{N,M-1})^{-1} - Q_0^{(0)} \{ y_\gamma \}_M, g \right) \times \lambda \sinh \eta \prod_{k=1, \neq i}^N \frac{\sinh(x_k-y_\alpha + \eta)}{\sinh(x_k-y_\alpha)} \\
= \prod_{\gamma=1, \neq \alpha}^M \frac{\sinh(x_\gamma - y_\alpha + \eta)}{\sinh(x_\gamma - y_\gamma)} \times \prod_{k=1, \neq i}^N \frac{\sinh(x_k-y_\alpha + \eta)}{\sinh(x_k-y_\alpha)}. 
\]

We see that the multipliers near the determinants in the both sides are the same, so the equality of the residues in \([53]\) for \(N, M\) is reduced to \([53]\) for \(N-1, M-1\) which holds true according to the assumption of the induction. Therefore, the poles and the residues in all variables in the both sides of \([53]\) are the same.

We have thus proved that
\[
\det_{N \times N} \left( \lambda (W^{N,M})^{-1} - Q_0 \{ x_i \}_N, g \right) = \prod_{(N-M) \times (N-M)} \left( \lambda I - g S_{N-M} \right) \det_{M \times M} \left( \lambda (\tilde{W}^{N,M})^{-1} - Q_0 \{ y_\gamma \}_M, g \right) + C_{N,M}, 
\]
where \(C_{N,M}\) are some constants. They can be found from the limit \(y_\alpha \to \infty\). We have:
\[
\lim_{y_\alpha \to \infty} \det_{N \times N} \left( \lambda (W^{N,M})^{-1} - Q_0 \{ x_k \}_N, g \right) = \left( \lambda e^{-N \eta} - g e^{-(M-1)\eta} \right) \det_{(M-1) \times (M-1)} \left( \lambda (\tilde{W}^{N,M-1})^{-1} - Q_0 \{ y_\gamma \}_M, g \right), \\
\lim_{y_\alpha \to \infty} \det_{M \times M} \left( \lambda (\tilde{W}^{N,M})^{-1} - Q_0 \{ y_\gamma \}_M \right) = \left( \lambda e^{-N \eta} - g e^{-(M-1)\eta} \right) \det_{(N-M) \times (N-M)} \left( \lambda I - g S_{N-M} \right),
\]
Using the trivially checked identity
\[
\det_{(N-M+1) \times (N-M+1)} \left( \lambda e^{-\eta I} - g S_{N-M+1} \right) = \left( \lambda e^{(M-N-1)\eta} - g \right) \det_{(N-M) \times (N-M)} \left( \lambda I - g S_{N-M} \right),
\]
one can see that \(C_{N,M} = C_{N,M-1}\). But we know that \(C_{N,0} = 0\) for any \(N\). Therefore, \(C_{N,0} = 0\) for any \(N, M\) and the lemma is proved.

**Theorem 1.** Let \(H_i\) be eigenvalues of the operators \(H_i\), then spectrum of the matrix \(L(\{ x_k \}_L, \{ \hat{x}_k = -H_k \}_L)\) is the following:
\[
\text{Spec} L(\{ x_k \}_L, \{ \hat{x}_k = -H_k \}_L) = \left\{ e^{Lh-(M_1-1)\eta}, e^{Lh-(M_1-2)\eta}, \ldots, e^{Lh+(M_1-1)\eta}, e^{Lh-(M_2-1)\eta}, e^{Lh-(M_2-2)\eta}, \ldots, e^{Lh+(M_2-1)\eta} \right\}.
\]

Obviously, the statement of the theorem is equivalent to
\[
\det_{L \times L} \left[ \lambda I - L(\{ x_k \}_L, \{ -H_k \}_L) \right] = \det_{M_1 \times M_1} \left[ \lambda I - e^{Lh} S_{M_1} \right] \det_{M_2 \times M_2} \left[ \lambda I - e^{-Lh} S_{M_2} \right]. 
\]

**Proof.** The value of \(H_k\) through the Bethe roots is given by equation \([19]\). Substituting \([19]\) into \(L(\{ x_k \}_L, \{ \hat{x}_k = -H_k \}_L)\), we see that
\[
L(\{ x_k \}_L, \{ \hat{x}_k = -H_k \}_L) = Q(\{ x_k - \eta \}_L, \{ u_\alpha \}_M, e^{Lh})
\]
where $Q$ is given by (43). Lemma 1 implies that
\[
\det_{L \times L} (\lambda - L) = \det_{(L-M_2) \times (L-M_2)} (\lambda - e^{Lh} S_{L-M_2}) \det_{M_2 \times M_2} (\lambda - \tilde{Q}(\{u_\alpha\}_{M_2}, \{x_i - \eta\}_L, e^{Lh}))
\] (56)
with
\[
\tilde{Q}_{\alpha\beta}(\{u_\alpha\}_{M_2}, \{x_i - \eta\}_L, e^{Lh}) = \frac{e^{Lh} \sinh \eta}{\sinh(u_\beta - u_\alpha + \eta)} \prod_{\gamma \neq \alpha}^{M_2} \frac{\sinh(u_\alpha - u_\gamma - \eta)}{\sinh(u_\alpha - u_\gamma)} \prod_{k=1}^{L} \frac{\sinh(u_\alpha - x_k + \eta)}{\sinh(u_\alpha - x_k)}
\]

Imposing the Bethe equations (18), we have:
\[
\tilde{Q}_{\alpha\beta}(\{u_\alpha\}_{M_2}, \{x_i - \eta\}_L, e^{Lh}) \bigg|_{BE} = \frac{e^{-Lh} \sinh \eta}{\sinh(u_\beta - u_\alpha + \eta)} \prod_{\gamma \neq \alpha}^{M_2} \frac{\sinh(u_\alpha - u_\gamma + \eta)}{\sinh(u_\alpha - u_\gamma)}
\]
\[
= Q_{\alpha\beta}(\{u_\alpha\}_{M_2}, \emptyset, e^{-Lh})
\]
The second determinant in (56) is then equal to
\[
\det_{M_2 \times M_2} (\lambda - \tilde{Q}(\{u_\alpha\}_{M_2}, \{x_i - \eta\}_L, e^{Lh})) = \det_{M_2 \times M_2} (\lambda - Q(\{u_\alpha\}_{M_2}, \emptyset, e^{-Lh})) = \det_{M_2 \times M_2} (\lambda - e^{-Lh} S_{M_2})
\]
(The second equality again follows from Lemma 1.) Combining this with (56), we get (55).

6 Conclusion

In this paper we have discussed the extension of the quantum-classical correspondence to the simplest model of the XXZ type. The main result is that the methods suggested in [11] for models parametrized by rational functions are successfully applicable to this case as well. The anisotropy parameter of the 6-vertex model determines the splitting of spectrum of the related classical Lax matrix.

The known examples suggest that the quantum-classical correspondence is specific for integrable models. However, its origin and range of generality still remain obscure. One of the open questions is the extension to models parametrized by elliptic functions (in particular, to the 8-vertex model).

At the same time, similar phenomena are encountered in somewhat different contexts. For example, the interpretation of the Planck constant as relativistic deformation parameter is used in construction of the relativistic Euler-Arnold tops [13] via quasi-classical description of the one-site spin chain. In some particular cases these tops are gauge equivalent to the Ruijsenaars-Schneider models, and the Planck constant entering the quantum $R$-matrix is identified with the relativistic parameter. Another example is the Matsuo-Cherednik approach [14] [8] to the quantum Calogero-Moser models. The spectral parameters of the classical $r$-matrices in the Knizhnik-Zamolodchikov connections play the role of particle’s coordinates in the Calogero-Moser models. Close topics are discussed in [15]. At last, let us point out one more interrelation between classical and quantum integrable systems similar to the quantum-classical correspondence. It was observed in [16] that the eigenstates of quantum Hitchin system correspond to intersection points of two Lagrangian submanifolds in the classical phase space given by the moduli of flat connections.
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