ABOUT SOME FAMILY OF ELLIPTIC CURVES

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Abstract. We examine the moduli space $E \cong T^*$ of complex tori $T(\tau) \cong \mathbb{C}/L(\tau)$ where $L(\tau) = const \cdot \eta^2(\tau)L_\tau$. We find that the Dedekind eta function furnishes a bridge between the euclidean and hyperbolic structures on $T^*$ and the theta function for the lattice $E_8$. The former one allows us to rewrite the Lame equation for the Bers embedding of $T_{1,1}$ in a new form. We show that $L_0$ has natural decomposition into 8 sublattices (each equivalent to $L_0$), together with appropriate half-points and that this leads to local functions of the form $\vartheta_8(0, \tau_\alpha)$ for a local map $(U_\alpha, \tau_\alpha)$ and to a relation with $E_8$.

1. Introduction

We have shown in [1] that the natural algebraic structures associated to the punctured torus $T^* \cong H/\Gamma'$, (here $\Gamma'$ is the commutator group of the modular group $\Gamma = SL_2\mathbb{Z}$, $\Gamma' = [\Gamma, \Gamma]$) viewed as the Veech modular curve of complex tori, produce exactly the generating matrix for the binary error correcting Golay code $G_{24}$. This is a reason why in this paper we investigate the (on the other hand well known) punctured torus $T^*$ more carefully. We will find that the Dedekind eta function plays very important role. It furnishes not only a bridge between the hyperbolic and euclidean geometries on $T^*$ but it also connects (see the formula (5.5)) the doubly-periodic Weierstrass function $\wp(p(z_\alpha), L_0)$ on $T^*$ with the theta function for the lattice $E_8$, that is with $\Theta_{E_8}(\tau_\alpha) = \sum_{m=0}^{\infty} r_{E_8}(m) \eta_0^{m}$ (here $\tau_\alpha = \tau_\alpha(x')$, $z_\alpha = p(\tau_\alpha)$, $x' \in U_\alpha \subset T^*$ and $r_{E_8}(m)$ is the number of elements $\underline{w} \in E_8$ such that $\underline{v} \cdot \underline{w} = 2m$).

Since the Veech modular curve $T^*$ naturally carries the modular $J$-invariant we may view each of the objects $G_{24}$ and $E_8$ as a sort of a hidden structure associated to the Klein $J$-function that is encoded in the projection $J: T^* \to Y(1) = H/\Gamma$.

In [2] we have shown that, similarly to strong consequences coming from relations between $\Gamma'$ and the subgroups $\Gamma(2)$, $\Gamma(3)$, $\Gamma_c$ and $\Gamma_{ns}(3)$ of the modular group $\Gamma$ (and investigated in this note) the relations between $\Gamma = SL_2\mathbb{Z}$ and $\Gamma_0(p)$ (for the supersingular primes) introduce a hidden structure asociated to the $J$-function whose the full symmetry group $K$ must have the order that is devided by each of these primes $p$. Since the full automorphism group of $G_{24}$ (given by the Matieu group $M_{24}$) must be a subgroup of $K$, the conditions that $p||K$ together with the requirement that $K$ is a simple group implies that $K$ has to be the monster group $M$.

We will start with the family of lattices $L(\tau) = const \cdot \eta^2(\tau)L_\tau$ on $H$, where $L_\tau = [1, \tau]$ and we will show that the moduli space for complex tori $T(\tau) \cong \mathbb{C}/L(\tau)$
is an elliptic open curve $\mathcal{E} : t^2 = 4u^3 - 1$. Using the ramification scheme for appropriate natural projections we obtain that $u(\tau)$ and $t(\tau)$ coincide with the absolute invariants for $\Gamma_3^+$ (3) and for $\Gamma_c$ respectively as well as that the curve $\mathcal{E}$ is analytically isomorphic to $\mathbf{T}^* \cong H/\Gamma' \cong \mathbb{C} - L_0/L_0$. In section 3 we find relations $dz_\alpha = s\eta^4(\tau_\alpha)d\tau_\alpha$, (s is a global constant) between local coordinates on $\mathbf{T}^*$ and we investigate their consequences. We introduce some Hecke operators and we find some sort of hidden $E_8$-symmetry on $\mathbf{T}^*$.

2. Preliminaries

2.1. Curve $\mathcal{E}$. Each element $\tau$ of the upper half-plane $H$ determines a lattice $L_\tau = [1, \tau]$ and a complex torus $\mathbf{T}_\tau = \mathbb{C}/L_\tau$. However, we will consider, instead of the standard family $\{\mathbf{T}_\tau\}_{\tau \in \mathbb{H}}$ of compact complex tori, a family $\{\mathbf{T}(\tau) = \mathbb{C}/L(\tau)\}_{\tau \in \mathbb{H}}$ where $L(\tau) = \mu(\tau)L_\tau$, $\mu(\tau) = 2\pi^3/\eta^2(\tau)$ and $\eta(\tau)$ is the standard Dedekind eta function. Now, each torus $\mathbf{T}(\tau)$ is analytically isomorphic to the curve

\begin{equation}
E_{L(\tau)} : Y^2 = 4X^3 - g_2(L(\tau))X - g_3(L(\tau))
\end{equation}

and we will define a function $u(\tau)$ as given by $u(\tau) := \frac{1}{3\sqrt[3]{2}} g_2(L(\tau))$ and the function $t(\tau) := g_3(L(\tau))$. We have

\begin{equation}
g_2(L(\tau)) = \mu(\tau)^{-4} g_2(\tau) = \frac{3}{(2\pi)^3} \frac{g_2(\tau)}{\eta(\tau)^6}
\end{equation}

and

\begin{equation}
g_3(L(\tau)) = \mu(\tau)^{-6} g_3(\tau) = \frac{3\sqrt{2}}{2(2\pi)^6} \frac{g_3(\tau)}{\eta(\tau)^{12}}
\end{equation}

where $g_k(\tau) = g_k(L_\tau)$ for $k = 2, 3$ are the standard Eisenstein series. We see that the functions $u(\tau)$ and $t(\tau)$ satisfy the equation $4u^3 - t^2 - 1 = 0$ and hence determine an elliptic open curve

\begin{equation}
\mathcal{E} : t^2 = 4u^3 - 1
\end{equation}

Each point $(u(\tau), t(\tau)) \in \mathcal{E}$ corresponds to a curve

\begin{equation}
E_{u,t} : Y^2 = 4X^3 - 3\sqrt[3]{4}uX - t
\end{equation}
When point \( P = (u, t) \) of \( E \) has both coordinates different from zero then there exist exactly six distinct points \((\rho^k u, \pm t) \) \((k = 0, 1, 2, \rho = e^{\frac{2\pi i}{3}})\) on \( E \) which correspond to six isomorphic elliptic curves representing the same equivalent class of complex tori. When \( P = (u, 0) \) then we must have \( u = 4^{-\frac{1}{2}} \rho^k \) and points \((\rho^k 4^{-\frac{1}{2}}, 0) \) with \( k = 0, 1, 2 \) correspond to three isomorphic curves representing the equivalence class \([\mathbb{C}/\mathbb{Z}[\rho]]\). When \( P = (0, t) \) then we must have \( t = \pm i \) and both curves \( E_{0, \pm i} \) represent the equivalence class \([\mathbb{C}/\mathbb{Z}[\rho]]\) of tori. (Here the square bracket denotes the equivalence class of complex tori i.e. a point of the modular space \( H/\Gamma, \Gamma = SL_2\mathbb{Z}. \))

From the form of the equation (2.4) the elliptic curve \( E \) is itself analytically isomorphic to a complex torus that belongs to the class \([\mathbb{C}/\mathbb{Z}[\rho]]\). Since both functions \( u(\tau) \) and \( t(\tau) \) have the hyperbolic nature to find their realizations in terms of the Weierstrass functions \( \wp \) and \( \wp' \) (which belong to the flat geometry) we must consider the relationships between \( E \) and some modular curves of level 2 and of level 3 structures respectively.

2.2. \( \Gamma', \Gamma_c \) and \( \Gamma(2). \). Let \( r_N \) denote the modulo \( N \) homomorphism \( r_N : SL_2\mathbb{Z} \to SL_2\mathbb{Z}/N \). The image \( r_2(\Gamma) = SL_2\mathbb{Z}/2 \cong S_3 \) whereas the image of \( \Gamma' = [\Gamma, \Gamma] \) is the normal subgroup of \( S_3 \) given by \( C_3 \cong \mathbb{Z}_3 \). Let \( \Gamma_c \) denote the subgroup \( r_2^{-1}(C_3) \) of \( \Gamma \). It has genus zero, it has only one cusp of width 2 and it has index 2 in \( \Gamma \). Moreover we may take \( \{I, T\} \) as a set of its coset representatives in \( \Gamma, T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \). Let \( T' \)
denote the punctured torus \( H/\Gamma' \) which is analytically isomorphic to \( \mathbb{C} - L_0/L_0 \) for some lattice \( L_0 = \text{const} \cdot L_0 \) and let \( X' \) be \( T' \cup \{\infty\} \cong H'/\Gamma' \) where \( H' \) denotes the extended half-plane \( H \cup \mathbb{Q} \cup \{\infty\} \). We have the following natural projections:

\[
X' \xrightarrow{\pi'} X_c \xrightarrow{\pi} X(1) \text{ with } X_c \cong H'/\Gamma_c, X(1) \cong H'/\Gamma \text{ with projections } \pi'_c \text{ of degree 3 and } \pi_c \text{ of degree 2.}
\]

The absolute invariant for \( \Gamma_c \) is given by \( J_c(\tau) = (J(\tau) - 1)^2 \), [3], and it is also \( \Gamma' \)-invariant (using (2.3) it may be identified with the function \( t(\tau) \)). The comparison of the ramification scheme for \( \pi'_c : X' \to X_c \) and for \( \wp' : X' \to \mathbb{C}P_1 \) implies that (after the identification of \( \mathbb{C}P_1 \) with \( J_c \)-plane \( X_c \)) the \( \Gamma' \)-automorphic function \( t(\tau) \) coincides with the lifting to \( H \) of the function \( \wp' \) on \( T' \). In other words we have shown that the following is true:

**Lemma 1.** Let \( p : H \to H/N \) be the natural projection corresponding to the group \( N = [\Gamma', \Gamma] \) with \( H/N \cong \mathbb{C} - L_0 \). The lifting of \( \wp(z, L_0) \) on \( \mathbb{C} - L_0 \) to \( H \) determined by \( p \) produces exactly the \( \Gamma' \)-automorphic function \( t(\tau) \).

At this point it is worth to notice that (since the modulo 2 homomorphism maps both groups \( \langle g \rangle \) and \( \langle a \rangle \) onto \( C_3 \) and since \( a \) is \( \Gamma(2) \)-equivalent to \( g^2 \) and \( a^2 \) is \( \Gamma(2) \)-equivalent to \( g \)) we may view the modular curve \( X_c \) (of \( C_3 \)-equivalent level two structures) as the quotient \( X(2)/C_3 \) (here \( g = ST, a = TS, S = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \)).

2.3. \( \Gamma', \Gamma(3) \) and \( \Gamma^+_n(3). \) A similar situation occurs when we pass to the modulo 3 homomorphism. The image \( r_3(\Gamma') \) is the normal subgroup of \( \Gamma/\Gamma(3) \cong SL_2(3) \) but now this subgroup is not an abelian one. It is isomorphic to the quaternion group \( Q_8 \) and we have

\[
(2.6) \quad 1 \longrightarrow Q_8 \longrightarrow SL_2(3) \longrightarrow \mathbb{Z}_3 \longrightarrow 1
\]

The subgroup \( r_3^{-1}(Q_8) \) of \( \Gamma \) is associated to the non-split Cartan subgroup of \( GL_2(3) \) and is usually denoted by \( \Gamma^+_n(3), [4] \). It has index 3 in \( \Gamma \) and we may take
the set \( \{I, T, T^2\} \) as a set of its coset representatives. The modular curve \( \mathbf{X}_c^+(3) \) of \( Q_8 \)-equivalent level 3 structures (in fact, since the normal subgroup \( N \) of \( Q_8 \) acts trivially, these structures are \( Q_8/N \) equivalent) has genus zero and only one cusp of width 3.

The absolute invariant for \( \Gamma'_{ns}(3) \) can be taken as \( J_n(\tau) = J(\tau)^3, [5] \), and hence this uniformizer of \( \mathbf{X}_{ns}^+(3) \) coincides with the \( \Gamma' \)-automorphic function \( \sqrt[3]{4u} \) introduced earlier. Taking into account the ramification scheme given by Pict.1

\[
\begin{array}{cccccccc}
X' & \infty' & \rho_1 & \rho_2 & i_1' & i_2' & x_1' & x_2' & x_3' \\
2 & 2 & 1 & 1 & 2 & 2 & x_1 & x_2 & x_3 \\
X_{ns}^+(3) & \infty_1 & \rho & 3 & 11 & x \\
3 & 3 & 3 & 1 & 1 & \\
X(1) & \infty & 0 & 1 & \\
\end{array}
\]

Pict.1

we obtain immediately:

**Lemma 2.** Let \( p \) be the natural projection \( H \to \mathbb{C} - L_0 \) introduced earlier. The lifting of the Weierstrass function \( \wp(z, L_0) \) on \( \mathbb{C} - L_0 \) to \( H \) determined by \( p \) produces exactly the \( \Gamma' \)-automorphic function \( u(\tau) = J_n(\tau) \).

Since functions \( u(\tau) \) and \( t(\tau) \) are liftings to \( H \) of the Weierstrass functions \( \wp \) and \( \wp' \) respectively we have the following

**Corollary 1.** An elliptic curve \( \mathcal{E} \): \( t^2 = 4u^3 - 1 \) that forms the moduli space of elliptic curves associated to the family of lattices \( \{L(\tau) = \mu(\tau) L_\tau\} \) with \( \tau \in H \) is analytically isomorphic to the punctured torus \( \mathbf{T}' = H/\Gamma' \cong \mathbb{C} - L_0/L_0 \) with isomorphism given by \( z \to (u(\tau), t(\tau), 1) \) for any \( \tau \) with the property that \( p(\tau) \in z + L_0 \).

Thus, the \( \Gamma' \)-automorphic functions \( u(\tau) = \frac{1}{3\sqrt[3]{4}} g_2(L(\tau)) \) and \( t(\tau) = g_3(L(\tau)) \) are objects of both: of the euclidean geometry (since \( u(\tau) = \wp(p(\tau), L_0) \) and \( t(\tau) = \wp'(p(\tau), L_0) \)) and of the hyperbolic geometry (as \( u(\tau) \) is the lifting to \( \Gamma' \) of a Hauptmodule \( J_{ns}(\tau) \) for \( \Gamma_{ns}^+(3) \) and \( t(\tau) \) is the lifting of a Hauptmodule \( J_c \) for \( \Gamma_c \)).

In other words we have the following commutative diagrams:

\[
\begin{array}{ccc}
\mathbf{X}' & \overset{\wp'}{\longrightarrow} & \mathbb{C}P_1 \\
\pi_c' \downarrow & & \downarrow \pi_n' \\
\mathbf{X}_c & \overset{J_c}{\longrightarrow} & \mathbb{C}P_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{X}' & \overset{\wp'^{\Gamma_0}}{\longrightarrow} & \mathbb{C}P_1 \\
\pi_n' \downarrow & & \downarrow \pi_n \\
\mathbf{X}_{ns}^+(3) & \overset{J_n}{\longrightarrow} & \mathbb{C}P_1 \\
\end{array}
\]

3. **A Matter of The Dedekind Eta Function**

3.1. **Hyperbolic and Euclidean.** We have already introduced a universal covering \( p \) which projects \( H \) onto the infinite punctured plane \( \mathbb{C} - L_0 \) with the deck group corresponding to a homomorphism of \( \Pi_1(\mathbb{C} - L_0) \to N \). \( N = [\Gamma', \Gamma] \). So, \( N\tau \Leftrightarrow z \in \mathbb{C} - L_0 \), \( \Gamma'\tau \Leftrightarrow z + L_0 \) and \( L_0 = c[1, \rho] \) for some constant \( c \). Let \( r \) be
the local inverse of \( p \), that is, \( \{ r, z \} = \frac{1}{2} p(z, L_0) \) (here \( \{ \} \) denotes the Schwarzian derivative). Now the \( \Gamma' \)-automorphic functions \( u \) and \( t \) can be locally viewed as

\[
u(r(z)) = \psi(z, L_0) \quad t(r(z)) = \psi'(z, L_0)
\]

Let \( \{(U_\alpha, \tau_\alpha)\}_\alpha \) be an atlas on \( \mathbb{T} \cong H/\Gamma' \) coming from the universal covering \( \pi' \): \( H \to \mathbb{T} \) i.e. for \( (u, t) = x' \in U_\alpha \cap U_\beta \), we have \( \tau_\beta(x') = \gamma \tau_\alpha(x') \) for some \( \gamma \in \Gamma' \).

Since the multiplier system of \( \eta^2(\tau) \) restricted to the subgroup \( \Gamma' \) of \( \Gamma \) is a trivial one, on any intersection \( U_\alpha \cap U_\beta \) we obtain

\[
L(\tau_\beta) = \mu(\tau_\beta)L_{\tau_\beta} = \mu(\tau_\alpha)L_{\tau_\alpha} = L(\tau_\alpha)
\]

This means that at each point \( x' \in \mathbb{T}, \ x' = (u, t) \), we have well define lattice \( L(x') = L(\tau_\alpha(x')) = L(\tau_\beta(x')) \) and hence we have an analytic isomorphism between \( \mathbb{C}/L(x') \) and \( E_{u, t}: Y^2 = 4X^3 - 3\sqrt{3u}X - t \).

Let us introduce another atlas \( \{(U_\alpha, z_\alpha)\}_\alpha \) on \( \mathbb{T} \) with holomorphic bijections \( z_\alpha: U_\alpha \to V_\alpha \subset \mathbb{C} - L_0 \) coming from the projection \( p': \mathbb{C} - L_0 \to \mathbb{T} \) and with the property that

\[
\tau_\alpha(p'(z_\alpha)) = r(z_\alpha) \quad z_\alpha(p'(\tau_\alpha)) = p(\tau_\alpha)
\]

(3.3)

(If necessary we may pass to some refinement of an open covering \( \{U_\alpha\} \) of \( \mathbb{T} \).)

Now, for each \( x' \in U_\alpha \cap U_\beta \) we have \( \tau_\beta(x') = \gamma \tau_\alpha(x') \) for some \( \gamma \in \Gamma' \) and \( z_\beta(x') = z_\alpha(x') + w \) for some \( w \in L_0 \).

Since

\[
u(\tau_\alpha) = \psi(p(\tau_\alpha), L_0) = \psi(p(\tau_\beta), L_0) = u(\tau_\beta)
\]

and analogously

\[
t(\tau_\alpha) = \psi'(z_\alpha, L_0) = \psi'(z_\beta, L_0) = t(\tau_\beta)
\]

the relation

\[
t(\tau_\alpha) = \psi'(z_\alpha, L_0) = \frac{d\psi(p(\tau_\alpha), L_0)}{dp(\tau_\alpha)} = \frac{d\psi(p(\tau_\alpha))}{d\tau_\alpha} \frac{d\tau_\alpha}{dz_\alpha} = \frac{d\psi(\tau_\alpha)}{dz_\alpha} \frac{dz_\alpha}{d\tau_\alpha}
\]

implies that

\[
\frac{du(\tau_\alpha)}{d\tau_\alpha} = t(\tau_\alpha) \frac{dz_\alpha}{d\tau_\alpha}
\]

(3.4)

Since we already know that

\[
u(\tau_\alpha) = \left(\frac{1}{4}J(\tau_\alpha)\right)^{\frac{1}{2}}, \quad t(\tau_\alpha) = (J(\tau_\alpha) - 1)^{\frac{1}{2}}
\]

we may use the well known formula [6]

\[
\eta^{24}(\tau) = \frac{1}{(48\pi^2)^3 (J(\tau))^4 (1 - J(\tau))^3}
\]

(3.5)

to find that on \( U_\alpha \subset \mathbb{T} \) we have

\[
\frac{dz_\alpha}{d\tau_\alpha} = s \eta^4(\tau_\alpha), \quad s = 2k\pi \frac{\sqrt{2}}{\sqrt{3}}
\]

and \( k \) is a global constant given by a 6-th root of \(-1\). Hence, on any intersection \( U_\alpha \cap U_\beta \) on \( \mathbb{T} \) we have

\[
dz_\alpha = s \eta^4(\tau_\alpha)d\tau_\alpha = s \eta^4(\tau_\beta)d\tau_\beta = dz_\beta
\]
as expected. We see that the Dedekind eta function provides the transition between the local flat coordinates $z_\alpha(x')$ and the hyperbolic $\tau_\alpha(x')$ coordinates on $T^*$. In other words it plays the role of a bridge between the euclidean geometry on $T^* \cong H/\Gamma'$ and its natural hyperbolic geometry. Moreover, from the formula (3.6), we obtain that

$$\wp(z_\alpha, L_0) dz_\alpha^2 = \frac{k}{3(2\pi)^3} g_2(\tau_\alpha) d\tau_\alpha^2$$

Let $q$ be the holomorphic quadratic differential on $T^*$ that is determined by the Eisenstein series $g_2(\tau)$ i.e. with respect to the atlas $\{(U_\alpha, \tau_\alpha)\}_\alpha$ it can be written as $q = (\frac{k}{3(2\pi)^3} g_2(\tau_\alpha) d\tau_\alpha^2)$. Now, with respect to the atlas $\{(U_\alpha, z_\alpha)\}_\alpha$, $q$ takes the form:

$$q = (\wp(z_\alpha, L_0) dz_\alpha^2)$$

Similarly, it is easy to check that

$$t(\tau_\alpha) = \frac{2\pi \sqrt{3}}{\sqrt{2}} \frac{du(\tau_\alpha)}{d\tau_\alpha} \eta^{-4}(\tau_\alpha)$$

and hence on each $U_\alpha$ we can write

$$g_3(\tau_\alpha) = \frac{(2\pi)^7}{3 \sqrt{2}} e^{-\frac{\pi^2}{3} \eta^8(\tau_\alpha)} \frac{du(\tau_\alpha)}{d\tau_\alpha}$$

Let $\xi$ denote the holomorphic differential on $T^*$ which is determined by $g_3(\tau)$. The above formulae allow us to write

$$\xi = (\wp'(z_\alpha, L_0) dz_\alpha^3) = (\frac{2}{(2\pi)^9} e^{\frac{\pi^2}{3}} g_3(\tau_\alpha) d\tau_\alpha^3)$$

In other words we have shown the following:

**Lemma 3.** The $\Gamma'$-automorphic forms on $H$ corresponding to the differentials $q = (\wp(z_\alpha, L_0) dz_\alpha^2)$ and $\xi = (\wp'(z_\alpha, L_0) dz_\alpha^3)$ on $T^*$ are exactly ones determined by the standard Eisenstein series $g_2(\tau)$ and $g_3(\tau)$ respectively. More precisely we have

$$\frac{k^2}{3(2\pi)^2} g_2(\tau) = \frac{k^2 \sqrt{3}}{3} (2\pi)^2 \eta^8(\tau) u(\tau)$$

and

$$\frac{2k^3}{(2\pi)^3} g_3(\tau) = \frac{k^3 \sqrt{2}}{3} (2\pi)^2 \eta^8(\tau) \frac{du(\tau)}{d\tau}$$

respectively.

From the relations (3.12) and (3.13) we obtain (after differentiating the first equation):

$$\frac{dg_2(\tau)}{d\tau} = \frac{8}{\eta(\tau)} g_2(\tau) + \frac{3k}{\pi} g_3(\tau)$$

We notice that when we choose the 6-th root $k$ of $-1$ as $k = -i$ then the latter formula is equivalent to the Serre derivative of the modular form $E_4(\tau) = \frac{3}{2\pi^2} g_2(\tau)$. 


3.2. Some Hecke Operators. Let us introduce (see [7]) the operator $T_{(g),k}$ of weight $k \in \mathbb{Z}$ acting on the space of functions $f : H \to \mathbb{C}$ as follows

\begin{equation}
(T_{(g),k}f)(\tau) = \sum_{r=1}^{3} j_{\gamma}(r,\tau)f(\gamma^r \tau)
\end{equation}

where $j_{\gamma}(\tau) = c\tau + d$ for any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2\mathbb{R}$. Let $A_k(G)$ denote the space of $G$-automorphic forms of weight $k$ for a Fuchsian group $G$. Since we have

\begin{equation}
T_{(g),k} : A_k(\Gamma) \to A_k(\Gamma_c)
\end{equation}

and

\begin{equation}
T_{(g),k} : A_k(\Gamma(2)) \to A_k(\Gamma_c)
\end{equation}

we may find some relations between $k$-forms for $\Gamma(2)$ and $k$-forms for $\Gamma'$. Namely, these Hecke operators together with the projections $\pi_8'$ and $\pi_2^\gamma : X(2) \to X_c$ allow us to transform $\hat{\tau}$-differentials on $X'$ into $\hat{\tau}$-differentials on $X(2)$ and vice versa. Let us denote the composition of $(\pi_2^\gamma)^*$ and of $T_{(g),k}$ as the operator $\hat{H}_k$. Let us check what are the images of $\hat{H}_k$ produced by the $\Gamma'$-automorphic functions $u(\tau)$ and $t(\tau)$. Since $L(g\tau) = i\hat{\pi}^\tau L(\tau)$ and $L(g^2\tau) = -i\hat{\pi}^{-\tau} L(\tau)$ we have $g_2(L(g\tau))) = \rho^2g_2(L(\tau))$ and $g_2(L(g^2\tau)) = \rho g_2(L(\tau))$. Hence

\[
\hat{H}_0u(\tau) = 0 \quad \text{and} \quad \hat{H}_0t(\tau) = 3t(\tau)
\]

So, the Weierstrass function $\wp$ on $T'$ produces the zero function on $X(2)$ but the Weierstrass function $\wp'$ produces a multiple of the lifting of the absolute invariant $J_c$ from $X_c$ to $X(2)$. We already know that the regular quadratic differential $(dz_2^2)_{\alpha}$ on $T'$ corresponds to the $\Gamma'$-automorphic form $s^2\eta^8(\tau)$ on $H$. It occurs that the image under the operator $\hat{H}_4 = (\pi_2^\gamma)^* \circ T_{(g),4}$ (transforming $A_4(\Gamma')$ into $A_4(\Gamma(2))$)

\[
\eta^8(\tau)
\]

vanishes. Although $\hat{H}_0u(\tau) = 0$ and $\hat{H}_4\eta^8(\tau)=0$ the operator $\hat{H}_4$ acts on their product $u(\tau)\eta^8(\tau)$ by multiplication by 3. This is because the product is a $\Gamma$-automorphic form i.e. $u(\tau)\eta^8(\tau) \in A_4(\Gamma) \subset A_4(\Gamma')$. Generally we have

Lemma 4. For any $\varphi \in A_k(\Gamma)$ and for any $f \in A_0(\Gamma')$ we have $\hat{H}_k(f \varphi) = \varphi \hat{H}_0(f)$.

Proof. Simple. □

We have exactly the same properties when we replace $A_k(\Gamma')$ by $A_k(\Gamma(2))$ and the operator $\hat{H}_k$ by the the operator $\hat{H}_k$ defined as the composition $\pi_2^\gamma \circ T_{(g),k}$ and transforming $A_k(\Gamma(2))$ into $A_k(\Gamma(G))$. Since we have

\begin{equation}
T_{(g),4}\vartheta_3(\tau)^8 = \frac{3}{(2\pi)^4}g_2(\tau)
\end{equation}

the image by $\hat{H}_4$ of the differential on $X(2)$ determined by $\vartheta_3(\tau)^8 \in A_4(\Gamma(2))$ produces the differential

\begin{equation}
(\frac{1}{k^2} \frac{3}{2\pi} \varphi^2(z_\alpha, L_0)dz_\alpha^2)_{\alpha} = \frac{3}{(2\pi)^4}g_2(\tau_\alpha)dz_\alpha^2
\end{equation}

on $T'$. (Here $\vartheta_3(\tau) \equiv \vartheta_3(0,\tau)$ is the standard theta function on $H$.) However when we start with $g_2(\tau)$ as $\Gamma'$-automorphic form then, using the operatore $\hat{H}_4$ we will not
return to $\vartheta_3(\tau)^8 \in A_4(\Gamma(2))$. Instead of we obtain $\widehat{H}_4g_2(\tau) = 3g_2(\tau)$ as an element of $A_4(\Gamma(2))$.

3.3. Bers embedding. Since the differential $q$ given by (3.8) has a pole of order 2 at the puncture of $\mathbf{T}^*$ it is not integrable so, although it is holomorphic on $\mathbf{T}^*$, it does not correspond to any element of the Banach space $\mathcal{B}_2(L, \Gamma')$ (the space of all holomorphic Nehari-bounded forms on the lower half-plane $L$ of weight 4, [8]). This means that we cannot use $q$ to construct the Bers embedding $\mathcal{T}_{1,1} \to \mathcal{B}_2(L, \Gamma')$. However, we see from (3.6) that the holomorphic differential on $\mathbf{T}^*$

\begin{equation}
\varphi = (dz^2)_{\alpha} = (s^2 \eta^8(\tau) d\tau^2)_{\alpha}
\end{equation}

(3.20)

corresponds to $\Phi = \phi(\tau) d\tau^2$ with $\varphi(\tau) = s^2 \eta^8(\tau)$ and hence it corresponds to an element of $\mathcal{B}_2(L, \Gamma')$ which may be used to find a concrete Bers embedding. However now, the space $\mathcal{T}_{1,1}$ must have its origin at $\mathbf{T}^*$. This means that we must find the domain of complex numbers $b$ such that the Schwarzian differential equation

\begin{equation}
\{w, \tau\} = b\varphi(\tau)
\end{equation}

(3.21)

has a schlicht solution $w$ which has a quasiconformal extention $\hat{w}$ to all $\mathbb{C}$ compatible with $\Gamma'$. Since a schlicht solution $w$ of (3.21) can be written as the quotient $\frac{y_1}{y_2}$ of two linearly independent solutions of $y''(\tau) + \frac{1}{2} b\varphi(\tau) y(\tau) = 0$ to find the values of $b$ for which $\hat{w}\Gamma'\hat{w}^{-1}$ is a quasi-Fuchsian of signature $(1; 1)$ we should consider the linear differential equation

\begin{equation}
y''(\tau) + \frac{bs^2}{2} \eta^8(\tau) y(\tau) = 0 \quad \tau \in L
\end{equation}

(3.22)

Till now, to find a Bers embedding of $\mathcal{T}_{1,1}$ we take the the Teichmueller space $\mathcal{T}_{1,1}$ originating at the punctured torus, usually $\mathbb{C} - L_i/L_i$, and we are looking for the values $b \in \mathbb{C}$ for which the Lame equation

\begin{equation}
y'' + \frac{1}{2} \left( \frac{1}{2} \varphi(z, L_i) + b \right) y = 0
\end{equation}

(3.23)

has a purely parabolic monodromy group (which is the commutator subgroup of the quasi-Fuchsian group $\hat{w}\Gamma_1\hat{w}^{-1}$ of signature $(1; 1)$) Thus, the relation (3.6) allows us to consider the equation (3.22) instead of (3.23) and, since till now the equation (3.22) had not been investigated (to the author’s knowledge), there is a possibility that we obtain new, more transparent understanding of the domain of Bers embedding of the Teichmueller space $\mathcal{T}_{1,1}$ that originates at $\mathbf{T}^* = H/\Gamma'$ (instead of at $\mathbf{T}'$).

4. Fundamental Domains for $\Gamma'$

The standard quadrilateral fundamental domains $\mathcal{F}_i'$ for $\Gamma'$ and $\mathcal{F}(\Gamma(2))$ for $\Gamma(2)$ have the same underlying set $\mathcal{F}$, given by the quadrilateral $(-1, 0, 1, \infty)$, and hence we may choose the same set of their coset representatives in $\overline{\Gamma} = \text{PSL}_2\mathbb{Z}$. We may decompose the set $\mathcal{F}$ into copies of a fundamental region $F(\Gamma) = (i - 1, \rho, i, \infty)$.
or into copies of a fundamental region $F_{\Gamma} = (0, \rho + 1, \infty)$ of the modular group according to

$$\mathfrak{F} = \mathfrak{S}_1 F(\Gamma) = \mathfrak{S}_2 F_{\Gamma},$$

where $\mathfrak{S}_1 = \{I, g, g^2, T, Tg, Tg^2\}$ and $\mathfrak{S}_2 = \{I, a, a^2, S, Sa, Sa^2\}$ are two sets of coset representatives in $\tilde{\Gamma}$. When we start with the set $\mathfrak{F}$, to determine whether we have $\Gamma(2)$ or $\Gamma'$ quadrilateral domain, we have to use either geometric or algebraic considerations. Geometrically, we have different identifications on the border $\partial \mathfrak{F}$ given by the generators of $\Gamma' = \langle A, B \rangle$ and of $\Gamma(2) = \langle -1, T^2, U \rangle$ respectively.

Algebraically, the free generators $S$ and $g$ of $\tilde{\Gamma} = \langle S \rangle * \langle g \rangle$ determine distinct permutations of the copies of fundamental domains for $\tilde{\Gamma}$ depending whether their union forms $\mathfrak{F}(\Gamma(2))$ or $\mathfrak{F}'$. More precisely, following the Millington construction [9], both $S$ and $g$ determine permutations $\mu$ and $\sigma$ of a set of coset representatives. A permutation group $\Sigma = \langle \mu, \sigma | \mu^2 = \sigma^3 = I \rangle$ acts transitively on a set of cosets and the disjoint cycle decomposition of $\mu$, $\sigma$ and of their product $\mu \sigma$ provides the genus and inequivalent cusp widths for an appropriate subgroup of $\tilde{\Gamma}$.

For example, if we consider cosets represented by elements of $\mathfrak{S}_1$ and if we re-numerate its elements as $\{I, T, g^2, Tg^2, g, Tg\} \leftrightarrow \{0, 1, 2, 3, 4, 5\}$ respectively then, the permutation $\mu = (03)(14)(25) \in S_6$ for the subgroups $\Gamma(2)$ and $\Gamma'$ of $\tilde{\Gamma}$. However the motion $g$ produces the permutation $\sigma' = (04)(13)$(5) for $\Gamma'$ and hence $\mu \sigma' = (01, 2, 3, 4, 5)$. The corresponding permutation group $\Sigma(\Gamma') = \mu, \sigma' \rangle$ tells us that $\Gamma'$ has genus 1, no elliptic elements and the single cusp of width 6 (equal to the length of the cycle $\mu \sigma'$). For $\Gamma(2)$, $g = ST$ generates the permutation $\sigma = (042)(135)$. So the product $\mu \sigma = (01)(23)(45)$ and we have three inequivalent cusps of width equal to 2 each.

We notice that the cycle structures of the generators of $\Sigma(\Gamma')$ and of $\Sigma(\Gamma(2))$ are the same but the permutations given by the products of the generators introduce distinction in the properties of cusps for $\Gamma'$ and for $\Gamma(2)$ respectively. Of course we could take different enumeration of cosets and different decompositions of the fundamental region of a given subgroup of $\tilde{\Gamma}$. The permutation group obtained by using these new data will have generators (i.e. permutations representing $S$ and $g$) that are simultaneously conjugate in $S_6$ either to $\{\mu, \sigma'\}$ (in the case of $\Gamma'$) or to $\{\mu, \sigma\}$ (for $\Gamma(2)$).

Thus, when we start with the quadrilateral region $\mathfrak{F} = (-1, 0, 1, \infty)$ then we must perform some operations for the cusp of width 6 of $\Gamma'$ to be seen. However, the hexagonal fundamental domain $\mathfrak{S}'_6 = (\rho - 2, \rho - 1, \rho, \omega, \omega + 1, \omega + 2, \omega + 3, \infty)$ has the parabolic vertex of index 6 already. Moreover, if we choose the following fundamental standard domains: $R = (\rho, \omega, \infty)$ for $\tilde{\Gamma}$, $F(\Gamma_c) = T^{-2}R \cup T^{-1}R$ for $\Gamma_c$ and $F(\Gamma_{ns}^+(3)) = T^{-2}R \cup T^{-1}R \cup R$ for $\Gamma_{ns}^+(3)$ then we have immediately the relations between the appropriate sets given by

$$\mathfrak{S}'_6 = (I \cup T^3)F(\Gamma_{ns}^+(3)) = (I \cup T^2 \cup T^4)F(\Gamma_c).$$

These relations immediately describe the ramifications of $X' \to X_{ns}^+(3)$ and of $X' \to X_c$ at $\infty$ respectively.

When we work with the quadrilateral domain $\mathfrak{F}'$, then, in fact, we are dealing with the subgroups of $PSL_2\mathbb{Z}/\Gamma'$ that may be identified with the finite subgroups $\langle S \rangle$ and $\langle g \rangle$ of the modular group itself. But when we consider the hexagonal fundamental domain $\mathfrak{S}'_6$ then the more natural is to view the quotient $PSL_2\mathbb{Z}/\Gamma'$
as given by $\langle T \rangle \mod T^6$. Although we have that $T^3$ is equivalent to $S$ modulo $\Gamma'$ (more precisely $T^3 = S[S^{-1}, T][S^{-1}, T^{-1}]$) and we can write
\begin{equation}
\tilde{\Gamma}/\Gamma' \cong \langle S \rangle \times \langle g \rangle \cong \langle T^3 \rangle \times \langle T^2 \rangle \mod T^6
\end{equation}
we notice that the elements $S$ and $g$ have finite order in $\tilde{\Gamma}$ whereas both $T^3$ and $T^2$ are generators of infinite parabolic subgroups of $PSL_2 \mathbb{Z}$. So we are dealing with transparent differences between the nature of algebraic objects that may be associated to $\mathfrak{g}$ and $\mathfrak{g}'$ respectively and which are involved in the hidden structure of the Veech curve determined by the dynamical system of a billiard (in a $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{g}^*)$-triangle) and described by the error correcting Golay code $G_{24}$ in $[1]$. These differences become even deeper when we consider a non-unitary representation $\chi$ of $\Gamma$ in $\mathbb{C}^2 = \text{Span}\{J_n, J_{\infty}\}$. Since the $\Gamma'$-automorphic functions $u(\tau)$ and $t(\tau)$ are given by the liftings of $J_n$ and of $J_c$ from $\Gamma'_n(3)$ and from $\Gamma_c$ to $\Gamma'$ appropriately we may identify the underlying vector space $\mathbb{C}^2$ for $\chi$ with the linear span of the Weierstrass functions $\wp(p(\tau), L_0) \equiv u(\tau)$ and $\wp'(p(\tau), L_0) \equiv t(\tau)$ respectively. Since
\begin{align*}
u(-1) &= u(\tau) \quad u(\tau + 1) = \rho u(\tau) \\
t(-1) &= t(\tau) \quad t(\tau + 1) = -t(\tau)
\end{align*}
the transformation $S$ acts as identity. We have: $\chi(S) = I$, $\chi(T) = \begin{pmatrix} -1 & 0 \\ 0 & \rho \end{pmatrix}$ and $\chi(T^6) = I$. Thus to see $\chi$ as a representation of $\tilde{\Gamma}/\Gamma'$ on $\text{Span}\{\wp, \wp'\}$ we must take the set $\{T^k, k = 0 \ldots 5\}$ as a set of the cosets representatives of $\Gamma'$ in $\tilde{\Gamma}$. In other words, it is the cusp of $\Gamma'$ and its ramification indices over $X_c$ and over $X^+_{2,3}(3)$ respectively that are important here, and it is the hexagonal domain $\mathfrak{g}_6$ which immediately produces the relations (4.2).

5. DECOMPOSITION OF $L_0$

The projection $p : H \to H/N \cong \mathbb{C} - L_0$, $N = [\Gamma', \Gamma']$ corresponds to the abelianization of $\Gamma'$, $\Gamma'/N \cong \mathbb{Z}^2$. More precisely, let $\Gamma'$ be generated by $A = [S, T^{-1}] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and by $B = [S, T] = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Any element $\gamma \in \Gamma'$ has the abelianized form
\begin{equation}
\gamma = A^m B^n \mathfrak{n}, \quad (m, n) \in \mathbb{Z}^2, \quad \mathfrak{n} \in N
\end{equation}
We usually write $\gamma = mA + nB$, $[10]$, so that
\begin{align*}
p(n\tau) &= p(\tau) + z \in \mathbb{C} - L_0, \quad L_0 = [\omega_1, \omega_2] = c[1, \rho] \\
p(\gamma \tau) &= p(\tau) + m\omega_1 + n\omega_2 & \text{for} \quad \gamma = A^m B^n \mathfrak{n} \in \Gamma'
\end{align*}
Let the quaternion group $Q_8 = \langle \alpha, \beta | \alpha^4 = 1, \alpha^2 = \beta^2, \alpha \beta = \beta \alpha^{-1} \rangle$ be realized by the following matrices in $SL_2(3)$:
\begin{align*}
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \\
\beta &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \quad \beta \alpha = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad \alpha \beta = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\end{align*}
The group $\mathbb{Z}_3$ that occurs in (2.6) is generated by $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and acts on $Q_8$ by the automorphisms determined by $X \alpha X^{-1} = \beta$ and $X \beta X^{-1} = \alpha \beta$. Let $r'_3$ denote the restriction of the homomorphism $r_3$ to $\Gamma'$. It maps
\begin{align*}
A &\to \beta, \quad A^2 \to \beta^2, \quad A^3 \to \beta^3, \quad A^4 \to I \\
B &\to \beta \alpha, \quad AB \to \alpha^3, \quad A^2 B \to \alpha \beta, \quad A^3 B \to \alpha
\end{align*}
(5.3)

From now on we will use the following enumeration of the elements of $Q_8$:
\begin{align*}
\{I, \beta, \beta^2, \beta^3, \beta \alpha, \alpha^3, \alpha \beta, \alpha\} &\equiv \{q_1, q_2, \ldots, q_8\}
\end{align*}
(5.4)
respectively. Let $\sigma \in S_8$ be the permutation $\sigma = (13)(24)(57)(68)$ of $\{q_1, \ldots, q_8\}$ corresponding to the multiplication by $\alpha^2 = \beta^2 = (\alpha \beta)^2 = -I$.

**Lemma 5.** The homomorphism $r'_3 : \Gamma' \to Q_8$ induces a unique mapping $\kappa : \Gamma'/N \to Q_8 \times Q_8$ such that $\kappa(m, n) = (q_k, \sigma q_k)$ for some unique, appropriate $k \in \{1, \ldots, 8\}$.

**Proof.** Let $N = \{1, \alpha^2\}$ denote a normal subgroup of $Q_8$ and let $r'_{3,N}$ denote the restriction of $r'_3$ to the normal subgroup $N$ of $\Gamma'$. Let $K_N$ denote the kernel of the homomorphism $r'_{3,N}$, $K_N \triangleleft N$, so that we have $N = K_N \cup A$ as a set, with $A = r'_{3,N}^{-1}(\alpha^2)$. Since each coset $(m, n)$ of $N$ has the decomposition
\begin{align*}
(m, n) &\equiv A^m B^n N = A^m B^n K_N \cup A^m B^n A
\end{align*}
(5.5)
and all elements of the set $\{A^m B^n K_N\}$ are mapped onto some concrete $q_k$ whereas elements of $\{A^m B^n A\}$ are all mapped onto $\sigma q_k$, the lemma follows.

For each $k \in \{1, \ldots, 8\}$ we introduce the subset $A_k$ of $\Gamma'$ as the union
\begin{align*}
A_k = A_k \cup B_k
\end{align*}
with
\begin{align*}
\mathfrak{A}_k = \{A^m B^n | n \in K_N; r'_3(A^m B^n) = q_k\}
\end{align*}
(5.6)
and
\begin{align*}
\mathfrak{B}_k = \{A^m B^n | n \in A; \ r'_3(A^m B^n) = \sigma q_k\}
\end{align*}
(5.7).

**Lemma 6.** The decomposition $\Gamma' = \bigcup_{k=1}^{8} A_k$ determines a one-one correspondence between the set of elements of $Q_8$ and the set of elements of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

**Proof.** Let us write
\begin{align*}
\mathfrak{A}_k = \{A^{m_k} B^{n_k}\} K_N \quad \text{and} \quad \mathfrak{B}_k = \{A^{m'_k} B^{n'_k}\} A
\end{align*}
for appropriate pairs of integers $(m_k, n_k)$ and $(m'_k, n'_k)$ in $\mathbb{Z}^2$. Let $s_k$ and $t_k$ denote the smallest nonnegative integers such that $r'_3(A^{s_k} B^{t_k}) = q_k$. We see immediately that
\begin{align*}
\{(m_k, n_k)\} &= \{(4m + s_k, 4n + t_k), (4m + 2 + s_k, 4n + 2 + t_k), m, n \in \mathbb{Z}\}
\end{align*}
(5.9)
and
\begin{align*}
\{(m'_k, n'_k)\} &= \{(4m + 2 + s_k, 4n + t_k), (4m + s_k, 4n + 2 + t_k), m, n \in \mathbb{Z}\}
\end{align*}
(5.10)
Thus the set $A_k \subset \Gamma'$ is uniquely determined by the pair $(s_k, t_k) \in \mathbb{Z}_4 \times \mathbb{Z}_2$ and the lemma follows.
Lemma 7. The homomorphism $r'_3 : \Gamma' \to Q_8$ determines the decomposition of the lattice $L_0$ into 8 disjoint sublattices.

Proof. We have seen that we can decompose the set of all $N$-cosets in $\Gamma'$ into 8 subsets of cosets given by $\{(m_k, n_k) \in \mathbb{Z}^2 | A^{m_k}B^{n_k} \sim q_k\}$ i.e. produced by cosets representatives $A^{m_k}B^{n_k} \in A_k$. Now, to each such subset we may uniquely associate the subset $L_k \subset \mathbb{Z}^2$ of the form:

\begin{equation}
L_k = \{a_k + 4\mathbb{Z}^2\} \cup \{a_k + (2, 2) + 4\mathbb{Z}^2\}
\end{equation}

Using the correspondence $(m, n) \leftrightarrow m\omega_1 + n\omega_2$ as well as the symmetry properties of the lattice $L_0 = [\omega_1, \omega_2] = \omega_1[1, \rho]$ expressed by $[1, \rho] = [1, \omega]$ for $\omega = \rho + 1 = e^{\pi i/4}$ we obtain immediately that each subset $L_k \subset \mathbb{Z}^2$, $k \in \{1, \ldots, 8\}$ determines a unique sublattice $\tilde{L}_k$ of $L_0$ given by

\begin{equation}
\tilde{L}_k = \tilde{a}_k + \omega_1[4, 2\omega] \subset L_0, \quad \tilde{a}_k = s_k\omega_1 + t_k\omega_2
\end{equation}

\[\square\]

We notice that the $K$-multiple (for $K = 5/2h^2(0, \omega)$) of the lattice $[4, 2\omega]$ gives the primitive periods of the function sinus amplitudis $sn(2Kz)$. We will not pursue this direction here. Instead of we will look at $\tilde{L}_k$ as the sublattice $L_k = \tilde{a}_k + 4L_0$ of $L_0$ together with its halfpoints $\{h_k\} = \tilde{a}_k + 2(\omega_1, \omega_2) + 4L_0$. We see immediately that although the decomposition of $L_0$ into mutually disjoint subsets $\tilde{L}_k$, $k = \{1, \ldots, 8\}$ is uniquely determined by $q_k$'s, the realization of each $\tilde{L}_k$ as a sublattice of $L_0$ (more precisely a 4-dilate of $L_0$) together with appropriate half points is not a canonical one. Namely we can do this in three distinct ways. To analyse the situation let us introduce the lattices:

\begin{equation}
\Lambda_3 = [\omega^3_1, \omega^3_2] = 4[\omega_1, \omega_2]
\end{equation}

\begin{equation}
\Lambda_4 = [\omega^4_1, \omega^4_2] = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \circ \Lambda_3 = [\omega^3_2, -\omega^3_1 - \omega^3_2]
\end{equation}

\begin{equation}
\Lambda_2 = [\omega^2_1, \omega^2_2] = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \circ \Lambda_3 = [-\omega^3_1 - \omega^3_2, \omega^3_1]
\end{equation}

Equivalently we could write $\Lambda_4 = g \circ \Lambda_3$ and $\Lambda_2 = g^2 \circ \Lambda_3, g = ST$. Although all these three lattices $\Lambda_i$'s, $i = 2, 3, 4$ are equivalent, the realizations of each subsets $\tilde{L}_k$ by distinct $\Lambda_i$'s requires distinct half points of these lattices. Thus we have:

\begin{equation}
I. \quad \tilde{L}_k = \{\tilde{a}_k + \Lambda_3\} \cup \{\tilde{a}_k + \frac{\omega^3_1 + \omega^3_2}{2} + \Lambda_3\}
\end{equation}

or

\begin{equation}
II. \quad \bar{L}_k = \{\bar{a}_k + \Lambda_4\} \cup \{\bar{a}_k + \frac{\omega^3_2}{2} + \Lambda_4\}
\end{equation}

The explicit relations between $Q_8$ and $\mathbb{Z}_3 \times \mathbb{Z}_2$ are given in the table.

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$q_k$ & $q_1$ & $q_2$ & $q_3$ & $q_4$ & $q_5$ & $q_6$ & $q_7$ & $q_8$
\hline
$\alpha_k = (s_k, t_k)$ & (0,0) & (1,0) & (2,0) & (3,0) & (0,1) & (1,1) & (2,1) & (3,1)
\hline
\end{tabular}
We observe that the realizations \( I, II \) and \( III \) are associated to the pairs \( (Q_8, I), (Q_8, g) \) and to \( (Q_8, g^2) \) respectively.

6. Conclusions

We have seen that we needed both homomorphisms, modulo 2 and modulo 3, to find that the Weierstrass functions \( \wp \) and \( \wp' \) on \( p(H) \cong \mathbb{C} - L_0 \) have liftings to \( H \) given by the absolute invariants \( J_n(\tau) = (J(\tau))^n \) (for \( \Gamma_3(3) \)) and \( J_n(\tau) = (J(\tau) - 1)^n \) (for \( \Gamma_2(3) \)) respectively. Further, we have obtained that the homomorphism \( r_3^k \) determines the decomposition of \( \Gamma' \) into subsets \( A_k = r_3^{-1}(q_k) \), \( k = 1, \ldots, 8 \) (equivalently into the cosets of a normal subgroup \( \Gamma' \cap \Gamma(3) \) in \( \Gamma' \)).

Then we decomposed each \( A_k \subset \Gamma' \) as \( A_k = \mathcal{A}_k \cup \mathcal{B}_k \) according to (5.6) and (5.7). We have noticed that the set of pairs \( (m_k, n_k) \in \mathbb{Z}^2 \) with \( A^{m_k}B^{n_k} \in \mathcal{A}_k \) determines a sublattice \( \widetilde{L}_k \) of the lattice \( L_0 \cong \mathbb{Z}^2 \). Although the sublattice \( \widetilde{L}_k \) is not a dilate of \( L_0 \) we may view it as given by a lattice equivalent to \( L_0 \) together with the set of all of its appropriate half points. Such realization is not a canonical one and we have three ways, \( I, II \) and \( III \), to do this. In other words, we obtain the decomposition of \( L_0 \) into eight disjoint subsets, \( L_0 = \bigcup_{k=1}^8 \widetilde{L}_k \), each of which can be seen as

\[
\widetilde{L}_k = \{ \tilde{a}_k + \Lambda_1 \} \cup \{ \tilde{a}_k + h(l) + \Lambda_1 \} \quad k = 1, \ldots, 8
\]

for \( l = 2, 3, 4 \) (here \( \Lambda_l = g^l \circ \Lambda_3 \) and \( h(l) \) is an appropriate, depending on \( l \), half-point of \( \Lambda_1 \)). All lattices \( \Lambda_l \)'s are 4-dilates of \( L_0 \) and the essential differences between \( I, II \) and \( III \) lie in the different positions of half-points. These three realizations correspond to the elements of the group \( \langle g \rangle < SL_2 \mathbb{Z} \) involved in the formulæ (5.13) - (5.15). More precisely, although the three lattices \( \Lambda_l = [\omega_l', \omega_2] = g^l \circ \Lambda_3, l = 2, 3, 4 \) coincide (and are all 4-dilates of the lattice \( L_0 \)) the fact that \( g \notin \Gamma(2) \) implies that the half-points of the lattices \( g^l \circ \Lambda_3 \)'s are not preserved. Since \( r_3(\langle g \rangle) \cong SL_2(3)/Q_8 \) we may view the group \( Q_8 \circ SL_2(3) \) as producing the decomposition \( L_0 = \bigcup_{k=1}^8 \widetilde{L}_k \) and we may view the quotient \( SL_2(3)/Q_8 \) (which is associated to the symmetries of the lattice \( L_0 \) described by the cyclic group \( \langle g \rangle \) as responsible for the three realizations given by (5.16), (5.17) and (5.18) respectively. We see that:

- The realization \( I \) is associated to \( (\Lambda_1, \frac{\omega_3^2 + \omega_2^2}{2}) \) and involves the half points that correspond to the zeros of \( \vartheta_3(v, \rho) \)
- The realization \( II \) is associated to \( (\Lambda_4, \frac{\omega_2^4}{2}) \) and involves the half points that correspond to the zeros of \( \vartheta_4(v, \rho) \)
- The realization \( III \) is associated to \( (\Lambda_2, \frac{\omega_2^4}{2}) \) and involves the half points that correspond to the zeros of \( \vartheta_2(v, \rho) \)

Here \( v = \frac{\tau}{\omega_3}, L_0 = [\omega_1, \omega_2], z = p(\tau) \) for \( \tau \in H \) and we use exactly the same subindex \( l \) for a lattice \( \Lambda_l \) and for the corresponding even theta function. Moreover, the relations \( \Lambda_4 = g \circ \Lambda_3 \) and \( \Lambda_2 = g^2 \circ \Lambda_3 \) are parallel to the following relations respectively:

\[
\vartheta_2^g(0, \tau) = (j_g(\tau))^{-4} \vartheta_3^g(0, \tau) \quad \vartheta_2^g(0, \tau) = (j_g(\tau)^2)^{-4} \vartheta_3^g(0, g^2\tau)
\]

Now, the global section \( L(x') \) of lattices over \( T^* \) introduced earlier leads to the fiber space over \( T^* \) whose fiber at any point \( x' \in T^* \) is a complex torus \( T_{x'} \) given by
\( \mathbb{C}/L(x') \) and attached to \( x' \) at the origin. However although for each point \( x' \in \mathbb{T}^* \) the lattice \( L(x') \) is well defined it is not equipped with any concrete basis and hence its half points are determined only up to permutations. The situation will change when we restrict ourselves to a single map \( (U_\alpha, \tau_\alpha) \), that is to \( x' \in U_\alpha \subset \mathbb{T}^* \). Now we can write

\[
L(x') = L(\tau_\alpha) = \mu(\tau_\alpha)[1, \tau_\alpha] = [\omega_1^\alpha, \omega_2^\alpha]
\]

and the half points are given by \( h_1^\alpha = \frac{\omega_1^\alpha}{2}, h_2^\alpha = \frac{\omega_2^\alpha}{2} \) and by \( h_3^\alpha = \frac{\omega_1^\alpha + \omega_2^\alpha}{2} \) respectively. Since the decomposition of \( L_0 \) corresponds to the decomposition of \( \mathbb{Z}^2 \) given by (5.11) the realization \( I \) defines the decompositions of each \( L(\tau_\alpha(x')), x' \in U_\alpha \) onto eight sublattices together with their half points as follows:

\[
(6.2) \quad \widetilde{L}_k^\alpha = \{ \tilde{a}_k^\alpha + 4L(\tau_\alpha) \} \cup \{ \tilde{a}_k^\alpha + 4h_3^\alpha + 4L(\tau_\alpha) \}, \quad k = 1, \ldots, 8
\]

where \( \tilde{a}_k^\alpha = s_k \omega_1^\alpha + t_k \omega_2^\alpha \). Each \( \widetilde{L}_k^\alpha \) produces torus isomorphic to \( T(\tau_\alpha(x')) \) attached at the origin to \( x' \) together with well defined half-point on it (corresponding to the zero of \( \theta_3(z, \tau_\alpha) : U_\alpha \times \mathbb{C} \to \mathbb{C} \)). Since \( k = 1, \ldots, 8 \), we may consider (on a set \( U_\alpha \)) the field \( (4L(\tau_\alpha), 4h_3^\alpha + 4L(\tau_\alpha)) \) and hence we naturally obtain the function \( \theta_3^6(0, \tau_\alpha) \) (on \( U_\alpha \)). Similarly, starting with the realization \( II \) or \( III \) we arrive to the functions \( \theta_3^6(0, \tau_\alpha) \) or to \( \theta_3^6(0, \tau_\alpha) \) respectively. None of these functions \( \theta_3^6(0, \tau_\alpha) \), \( l = 2, 3, 4 \) can be (using the atlas \( \{ (U_\alpha, \tau_\alpha) \}_\alpha \) extended to the whole \( \mathbb{T}^* \) to define any meaningful object on it.

The existence of the three pictures \( I, II \) and \( III \) of each \( \tilde{L}_k \), \( k = 1, \ldots, 8 \) comes from the symmetry properties of the lattice \( P(\Gamma(\infty)) = L_0 \). Since the group \( (g) \) is responsible for the existence of these three realizations we may naturally involve the Hecke operators \( T_{(g),k} \) introduced in the subsection (3.2). Thus, for \( l = 2, 3, 4 \) on each \( U_\alpha \subset \mathbb{T}^* \) we obtain

\[
T_{(g),4}\theta_3^g(0, \tau_\alpha) = \theta_3^g(0, \tau_\alpha) + \theta_3^g(0, \tau_\alpha) + \theta_3^g(0, \tau_\alpha)
\]

Since for \( x' \in U_\alpha \cap U_\beta \) we have \( \tau_\beta(x') = \gamma \tau_\alpha(x') \) for some \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \) and

\[
T_{(g),4}\theta_3^g(0, \tau_\beta) = (c \tau_\alpha + d)^4 T_{(g),4}\theta_3^g(0, \tau_\alpha)
\]

the family \( \{ T_{(g),4}\theta_3^g(0, \tau_\alpha) \}_\alpha \) forms well defined quadratic differential on \( \mathbb{T}^* \) which is exactly the same for each \( l = 2, 3, 4 \). The necessity of applying on \( U_\alpha \) the Hecke operator \( T_{(g),4} \) to \( \theta_3^g(0, \tau_\alpha) \) (or equivalently, the necessity of taking equally weighted sum \( \sum_{l=2}^4 \theta_3^g(0, \tau_\alpha) \) on \( U_\alpha \)) reflects the fact that each one of these three realizations is equally important. Thus, the explicit forms of the Thetanullverde \( \theta_1(0, \tau_\alpha) \), \( l = 2, 3, 4 \) on \( U_\alpha \) (which result from these all three realizations) provide

\[
(6.3) \quad T_{(g),4}\theta_3^g(0, \tau_\alpha) = \sum_{\omega \in \mathbb{Z}^8} q_\alpha^{2\omega^2} + \sum_{\omega \in \mathbb{F}^8} (-1)^{mu} q_\alpha^{2\omega^2} + \sum_{\omega \in \mathbb{Z}^8} q_\alpha^{(\omega - \omega')^2}
\]

and is further equal to the following expression

\[
\sum_{\omega \in \mathbb{Z}^8} q_\alpha^{2\omega^2} + \sum_{\omega \in \mathbb{F}^8} q_\alpha^{(\omega - \omega')^2} = 2\Theta_{E_\alpha}(\tau_\alpha)
\]

Here \( \omega_\alpha = e^{i\pi \tau_\alpha}, \quad \alpha = \{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \} \in \mathbb{Q}^8 \), and \( 1 = 2\omega \).

Another argument which leads to the sum of all \( \theta_3^g(0, \tau_\alpha) \), \( l = 2, 3, 4 \) comes from the subsection (3.2). Namely, for any holomorphic atlas on \( Y(2) \cong H/\Gamma(2) \) the transition functions preserve (pointwise) all half-points of each nonsingular fiber.
of the modular elliptic surface over \( X(2) \). Hence \( \vartheta_{3}^{8}(0, \tau) = \sum_{q \in \mathbb{Z}^{8}} q^{2} \) is a \( \Gamma(2) \)-
automorphic form of weight 4. Roughly speaking, the existence of a global section of half-points over the moduli space \( Y(2) \) allows us to consider only the first part of the right side of (6.3) which contains only the lattice \( \mathbb{Z}^{8} \). When we pass to the moduli space \( T^{*} \) of complex tori, it is no longer possible and (on each \( U_{\alpha} \)) we must also involve the remaining terms of the left side of (6.3), that is, we must consider the lattice \( E_{8} \) instead of merely \( \mathbb{Z}^{8} \) as for \( Y(2) \).

Summarizing, the occurrence of the \( E_{8} \)-symmetry related to the moduli space \( T^{*} \) can be seen as a consequence of the relation between \( \Gamma' \) and \( Q_{8} \) coming from the modulo 3 homomorphism and of the existence of the equally important three realizations of the decomposition of the lattice \( L_{0} \) into 8 mutually disjoint subsets.

Moreover, from local relations \( H_{4}(\vartheta_{3}^{8}(\tau) d\tau^{2}) \cong const \varphi(z) dz^{2} \) coming off the subsection (3.2) we obtain

\[
(6.4) \quad u(\tau_{\alpha}) = \varphi(p(\tau_{\alpha}), L_{0}) = \text{const} \cdot \frac{\Theta_{E_{8}}(\tau_{\alpha})}{\eta^{8}(\tau_{\alpha})} \quad \text{on} \quad U_{\alpha}
\]

and hence

\[
(6.5) \quad \varphi(p(\tau_{\alpha}), L_{0}) = \text{const} \cdot q_{\alpha}^{-4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varrho_{E_{8}}(m)p_{8}(n-m)q_{\alpha}^{n}
\]

So, we may view the Weierstrass function on the moduli curve \( T^{*} \) as a function which encodes the information about the decompositions of \( L_{0} \).

Let us notice the difference between the Jacobi and our approach. Although Jacobi forms involve both euclidean variable \( z \) and hyperelliptic variable \( \tau \) (in particular, the ratio of the Jacobi-Eisenstein forms of index 1 and weight 10 and 12 respectively gives a constant multiple of the Weierstrass \( \varphi \)-function for each \( L_{\tau}, \tau \in H \) ) in the Jacobi picture we must work with meromorphic functions on \( H \times \mathbb{C} \) satisfying some concrete conditions. In our approach we simply translate the hyperbolic objects for \( \Gamma' \), \( \Gamma_{n} \), \( \Gamma_{n}(3) \), etc. into the euclidean objects on \( \mathbb{C} - L_{0} \) and vice versa and this is a reason for the appearance of 8 sublattices of \( L_{0} \) together with their appropriate half-points and further the appearance of \( \Theta_{E_{8}}(q_{\alpha}) \) on \( U_{\alpha} \).

We have shown that the bridge between the hyperbolic structure of the universal covering space \( H \) of \( T^{*} \cong H/\Gamma' \) and the euclidean structure of \( \mathbb{C} - L_{0} \), \( (T^{*} \cong \mathbb{C} - L_{0}/L_{0}) \) is given by the function \( \eta^{8}(\tau) \). Now, rewriting the formula (6.4) as

\[
(6.6) \quad \varphi(p(\tau_{\alpha}), L_{0}) \eta^{8}(\tau_{\alpha}) = \text{const} \cdot \Theta_{E_{8}}(\tau_{\alpha})
\]

we may view \( \eta^{8}(\tau_{\alpha}) \) as a bridge between 2-periodic with respect to \( L_{0} \), function \( \varphi \) and the theta function of the lattice \( E_{8} \) (which may be produced by the decomposition of \( L_{0} \) into 8 sublattices together with appropriate half points in three distinct ways respectively).

Let us also notice that the function \( \eta^{8}(\tau) \) provides very strong interrelation between the groups \( \Gamma \) and \( \Gamma' \). It is expressed by the fact that the ring of modular forms for \( \Gamma \), that is the ring \( \mathcal{M}(\Gamma) = \mathbb{C}[g_{2}(\tau), g_{3}(\tau)] \) can be written as \( \mathcal{M}(\Gamma) = \mathbb{C}[\eta^{8}(\tau) u(\tau), \eta^{8}(\tau) u'(-\tau)] \) and hence can be given as:

\[
(6.7) \quad \mathcal{M}(\Gamma) = \mathbb{C}[\eta^{8}(\tau) \varphi(p(\tau), L_{0}), \eta^{8}(\tau) \varphi'(p(\tau), L_{0})]
\]

This tells us that although the generators \( g_{2}(\tau) \) and \( g_{3}(\tau) \) are algebraically independent they produce differentials \( \varphi(z) dz^{2} \) and \( \varphi'(z) dz^{3} \) respectively and hence we have some “elliptic” type of differential relation between them.
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