Parallel Type Decomposition Scheme for Quasi-Linear Abstract Hyperbolic Equation

Nana Dikhaminjia, Jemal Rogava and Mikheil Tsiklauri

Abstract

Cauchy problem for an abstract hyperbolic equation with the Lipschitz continuous operator is considered in the Hilbert space. The operator corresponding to the elliptic part of the equation is a sum of operators $A_1, A_2, \ldots, A_m$. Each addend is a self-adjoint and positive definite operator. A parallel type decomposition scheme for an approximate solution of the stated problem is constructed. The main idea of the scheme is that on each local interval classic difference problems are solved in parallel (independently from each other) respectively with the operators $A_1, A_2, \ldots, A_m$. The weighted average of the received solutions is announced as an approximate solution at the right end of the local interval. Convergence of the proposed scheme is proved and the approximate solution error is estimated, as well as the error of the difference analogue for the first-order derivative for the case when the initial problem data satisfy the natural sufficient conditions for solution existence.

Keywords and phrases: Decomposition scheme, Abstract hyperbolic equation, Operator splitting, Cauchy problem, Parallel algorithm.

MSC 2010: 65M12, 65M15, 65M55, 49M27.

1 Introduction

First works dedicated to construction and investigation of decomposition schemes were published in the fifties and sixties of the XX century (see G. A. Baker, T. A. Oliphant [1], G. Birkhoff, R. S. Varga [2], G. Birkhoff, R. S. Varga, D. Young [3], J. Douglas [4], J. Douglas, H. Rachford [5], E. G. Diakonov [6], D. G. Gordeziani [7], N. N. Ianenko [8], V. P. Il'in [9], A. N. Konovalov [10], G. I. Marchuk, N. N. Ianenko [11], G. I. Marchuk, U. M. Sultangazin [12], D. Peaceman, H. Rachford [13], A. A. Samarskii [14], [15], V. P. Il'in [16], G. I. Marchuk, U. M. Sultangazin [17], D. Peaceman, H. Rachford [18], A. A. Samarskii [19], [20]). It may be said that the works of these authors become a basis for further research on decomposition schemes.

Decomposition schemes in view of numerical calculation can be divided into two groups: schemes of sequential account (see for example G. I. Marchuk [21], A. A. Samarskii, P. N. Vabishchevich [22]) and schemes of parallel account (D. G. Gordeziani [23], [24], D. G. Gordeziani, H. V. Meladze [25], D. G. Gordeziani, A. A. Samarskii [26], A. M. Kuzyk, V. L. Makarov [27]).

In view of numerical calculations, with the development of parallel processing, obviously the parallel type decomposition schemes have clear advantage. The present work discusses the construction and investigation of parallel type decomposition scheme.

It is well-known that for error estimation of approximate solution of evolution problem, usually the solution is required to be of the higher order smooth than it is necessary following from natural conditions. In case of decomposition schemes this detail gains more importance. Demand on increasing the smoothness can be conditioned by the fact that the operators corresponding to the problems obtained by splitting are non-commutative. Therefore, it is important to build the decomposition schemes, whose numerical calculation and methodology of error estimate of approximate solution does not require sharp increase of the solution smoothness. These details are much more complicated for the second order evolution equation compared to the first order.
one. One of the reasons for this is that natural scheme for the first order evolution equation is two-layer, and for the second order - three-layer. For the most cases, in comparison with the two-layer schemes, investigation of the three-layer schemes are related with certain difficulties. At the first sight, this issue might be overcome: the second order evolution equation by introducing the additional unknown can be deduced to the first order system. However, in this case, if the operator in the initially given equation is self-adjoint, in the obtained system there will be non self-adjoint matrix operator, that significantly complicates solving of the corresponding discrete problem.

In the present work the decomposition scheme for the second order evolution equation is proposed that does not require to increase smoothness of the solution in view of numerical calculation. In addition, the methodology that we use for error estimate of the approximate solution makes it possible to find convergence order in the conditions with almost natural limitations.

For the investigation of the decomposition scheme we use polynomials of certain class, which we call two-variable polynomial. These polynomials are represented by means of second order classical Chebyshev polynomials.

We should note that several works are devoted to use of orthogonal polynomials in approximate solution schemes for differential equations: V. L. Makarov [21], A. G. Morris, T. S. Horner [25], V. A. Novikov, G. V. Demidov [26], V. A. Rastrenin [29]. In the work [21] many aspects of using orthogonal polynomials in the difference problems are presented quite widely.

Recent results related to the construction and investigation of decomposition schemes for evolution equations are obtained by the following authors: S. Blanes, F. Casas and M. Thalhammer [1], D. He, K. Pan and H. Hu [13], J. L. Padgett and Q. Sheng [27], J. Zhao, R. Zhan and Y. Xu [35]. We also note the work [5] in which a high-order accuracy decomposition scheme is considered for an abstract hyperbolic equation.

2 Statement of the problem and decomposition scheme

Let us consider the Cauchy problem for abstract hyperbolic equation in the Hilbert space $H$:

$$
\frac{d^2 u(t)}{dt^2} + A u(t) + M(u(t)) = f(t), \quad t \in [0, T], \quad (2.1)
$$

$$
u(0) = \varphi_0, \quad \frac{du(0)}{dt} = \varphi_1. \quad (2.2)
$$

where $A$ is a self-adjoint ($A$ does not depend on $t$), positive definite (generally unbounded) operator with the definition domain $D(A)$, which is everywhere dense in $H$, i.e. $\overline{D(A)} = H$, $A = A^*$ and

$$(Au, u) \geq \alpha \|u\|^2, \quad \forall u \in D(A), \quad \alpha = const > 0,
$$

where by $\|\|$ and $(\cdot, \cdot)$ are defined correspondingly the norm and scalar product in $H$; nonlinear operator $M(\cdot)$ satisfies Lipschitz condition,

$$
\|M(u) - M(v)\| \leq a\|u - v\|, \quad \forall u, v \in H,
$$

$a = const > 0$ ; $\varphi_0$ and $\varphi_1$ are given vectors from $H$; $u(t)$ is a continuous, twice continuously differentiable, searched function with values in $H$, and $f(t)$ is given continuous function with values in $H$.

Similar to the linear case, $u(t)$ vector function with values in $H$, defined on the interval $[0, T]$, is called a solution of the problem (2.1), (2.2) if it satisfies the following conditions: (a) $u(t)$ is twice continuously differentiable in the interval $[0, T]$; (b) $u(t) \in D(A)$ for any $t$ from $[0, T]$ , the function $A u(t)$ is continuous and $M(u(t))$ is continuous; (c) $u(t)$ satisfies equation (2.1) on the $[0, T]$ interval and the initial condition (2.2).

**Remark 2.1.** If $f(t)$ is continuously differentiable on $[0, T]$ (or $f(t) \in D(A^{1/2})$ for any $t$ from $[0, T]$ and the function $A^{1/2} f(t)$ is continuous), $\varphi_0 \in D(A)$ and $\varphi_1 \in D(A^{1/2})$, then there exists only solution $u(t)$ of the problem (2.1), (2.2) (without Lipschitz continuous operator)
that satisfies the condition: the function \( u'(t) \) gets the values from \( D(A^{1/2}) \) and \( A^{1/2}u'(t) \) is continuous on \([0, T]\) (see [19], Theorem 1.5, p. 301).

Let
\[
A = \sum_{j=1}^{m} A_j \quad \text{and} \quad A_j = A_j^* \geq \alpha_j I \quad \text{with} \quad \alpha_j = \text{const} > 0.
\]  

(2.3)

Then approximate solution of problem (2.1), (2.2) at the points \( t = t_{k+1} = (k+1)\tau \), \( k = 1, \ldots, n - 1 \), \( \tau = T/n \) \((n > 1)\) is defined by the following formula:
\[
v_{k+1} = \sum_{j=1}^{m} \eta_j y_{j,k+1}, \quad \sum_{j=1}^{m} \eta_j = 1, \quad 0 < \eta_j < 1,
\]
where \( y_{j,k+1} \) is a solution of the following difference problem:
\[
\eta_j y_{j,k+1} - \frac{2v_k + v_{k-1}}{\tau} + A_j y_{j,k+1} = \delta_{1,j} [f(t_k) - M(v_k)],
\]  

(2.4)

v_0 = \varphi_0, \quad v_1 = \varphi_0 + \tau \varphi_1,

(2.5)

where \( j = 1, \ldots, m \), \( \delta_{1,j} \) is a Kronecker symbol.

Thus, to construct approximate solution \( v_{k+1} \) for problem (2.1), (2.2) at the point \( t_{k+1} \), it is necessary to solve \( m \) problems independent from each other. Therefore, scheme (2.4) can be called parallel type decomposition scheme. These kind schemes for the first time were discussed in the work by D. Gordeziani (see [9], [10]). Parallel type decomposition schemes also are considered in the works: D. Gordezianis, A. Samarski [12], D. G. Gordeziani, H. V. Meladze [11], A. M. Kuryk, V. L. Makarov [20]. Specifically the scheme (2.4) (without Lipschitz-continuous operator) is given in [31].

3 Representation of the approximate solution error by means of Chebyshev polynomial and the main theorem

Let the problem (2.1), (2.2) has a solution.

Then the equation (2.1) at the point \( t = t_{k+1} \) can be written as
\[
u(t_{k+1}) - 2u(t_k) + u(t_{k-1}) + Au(t_{k+1}) = g_k,
\]  

(3.1)

where
\[
g_k = \tilde{f}(t_k) + A \left[ u(t_{k+1}) - u(t_k) \right] + \tau^{-2} \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \left[ u''(t) - u''(t_k) \right] dt \]
\[+ \tau^{-2} \int_{t_{k-1}}^{t_k} (t - t_{k-1}) \left[ u''(t) - u''(t_k) \right] dt.
\]

(3.2)

and where \( \tilde{f}(t) = f(t) - M(u(t)) \).

From (3.1) it follows that
\[
u(t_{k+1}) - 2Lu(t_k) + Lu(t_{k-1}) = \tau^2 Lg_k,
\]

(3.2)

where \( k = 1, \ldots, n - 1 \),
\[
L = (I + \tau^2 A)^{-1}.
\]
we call two-variable Chebyshev polynomials. These polynomials are defined by the following
\[ S_j = (I + \tau^2 \eta_j^{-1} A_j)^{-1}. \]
If we multiply both sides of equality (3.3) on \( \eta_j \) and summarize, we get
\[ v_{k+1} - 2Sv_k + Sv_{k-1} = \tau^2 \psi_k, \tag{3.4} \]
where \( k = 1, \ldots, n - 1, \)
\[
S = \sum_{j=1}^{m} \eta_j S_j, \quad \psi_k = \sum_{j=1}^{m} \delta_{1,j} S_j [f(t_k) - M(v_k)] = S_1 [f(t_k) - M(v_k)].
\]
If we take (3.4) from (3.2), we get
\[ z_{k+1} - 2Sz_k + Sz_{k-1} = r_k, \tag{3.5} \]
where \( z_k = v_k - u(t_k), \)
\[
r_k = r_{0,k} + r_{1,k} - L \left( \tau^2 r_{2,k} + r_{3,k} \right) + r_{4,k}, \quad r_{0,k} = (S - L) u(t_k), \quad r_{1,k} = (S - L) [u(t_k) - u(t_{k-1})], \quad r_{2,k} = A [u(t_{k+1}) - u(t_k)], \quad r_{3,k} = \tau^2 \left[ \psi_k - L\tilde{f}(t_k) \right],
\]
\[
r_{3,k} = \int_{t_{k-1}}^{t_{k+1}} (t_{k+1} - t) \left[ u''(t) - u''(t_k) \right] dt + \int_{t_{k-1}}^{t_k} (t - t_{k-1}) \left[ u''(t) - u''(t_k) \right] dt.
\]
To present solution of equation (3.5) in explicit form, we need certain class polynomials, which
we call two-variable Chebyshev polynomials. These polynomials are defined by the following
recurrent relation:
\[
\tilde{U}_{k+1}(x, y) = x\tilde{U}_k(x, y) - y\tilde{U}_{k-1}(x, y), \quad k = 1, 2, \ldots, \tag{3.6}
\]
\[
\tilde{U}_1(x, y) = x, \quad \tilde{U}_0(x, y) = 1.
\]
Notice that the works \[33\], \[32\] are devoted to research of three-layer semi-discrete schemes
by means of Chebyshev polynomials.
We call \( \tilde{U}_k(x, y) \) two-variable Chebyshev polynomials as \( U_k(x) = \tilde{U}_k(2x, 1) \)
represents second order Chebyshev polynomials.
The following formula can be easily obtained
\[
\tilde{U}_k(x, y) = \sqrt{y^2} \tilde{U}_k(\xi, 1), \quad \xi = \frac{x}{\sqrt{y}}, \quad y > 0, \tag{3.7}
\]
that relates \( \tilde{U}_k(x, y) \) to \( \tilde{U}_k(x) \).
Now we can explicitly write solution of equation (3.5) by means of polynomials \( \tilde{U}_k(x, y) \).
Using induction we get
\[
z_{k+1} = \tilde{U}_k(2S, S) z_1 - S\tilde{U}_{k-1}(2S, S) z_0 + \sum_{i=1}^{k} \tilde{U}_{k-i}(2S, S) r_i. \tag{3.8}
\]
Obviously, as $S_j$ ($j=1,\ldots,m$) are self-adjoint, non-negative, bounded operators, then operator $S$ also will be self-adjoint non-negative and bounded. Hence, as known (see, e. g., [30], Chapter VI), there exists the only square root $S^{1/2}$. Taking this into account and using formula (3.7), we obtain

$$
\tilde{U}_k (2S, S) = B^k \tilde{U}_k (2B, I) = B^k U_k (B), \quad B = S^{1/2}.
$$

Due to this equality, (3.8) will be

$$
z_{k+1} = B^k U_k (B)z_1 - B^{k+1} U_{k-1} (B)z_0 + \sum_{i=1}^{k} B^{k-i} U_{k-1} (B)r_i. \tag{3.9}
$$

Formula (3.9) is a main relation, by using of which the following theorem is proved.

**Theorem 3.1.** If the problem (2.1), (2.2) has a solution, then the estimate is valid for approximate solution error:

$$
\|z_{k+1}\| \leq \exp (cT_{k-1}) \left( \gamma_0 \left\| \Delta z_0 \right\| + \gamma_1 \|z_0\| + \Theta_k (\tau) \right), \tag{3.10}
$$

where $z_k = v_k - u (t_k)$, $\Delta z_0 = z_1 - z_0$, $c = \nu^{-1/2} a$, $\nu = \min_{1 \leq j \leq m} (\alpha_j)$, $\gamma_0 = \nu^{-1/2} + c\tau^2$, $\gamma_1 = 1 + c\tau$.

$$
\Theta_k (\tau) = \tau^2 \sum_{i=1}^{k} \left[ c_1 \|Au(t_i)\| + c_3 \|\tilde{f}(t_i)\| \right] + \tau \sum_{i=1}^{k} \left[ c_1 J_i (t_{i-1}, A^{1/2} u) + c_2 J_i (t_{i+1}, A^{1/2} u) \right] + c_2 \sum_{i=1}^{k} \int_{t_{i-1}}^{t_{i+1}} \left[ J_i (t, A^{1/2} u) + J_i \left(t, A^{-1/2} \tilde{f}\right) \right] dt,
$$

and where

$$
J_i (t, u) = \|u(t) - u(t)\|, \quad \tilde{f}(t) = f(t) - M(u(t)),
$$

$$
c_1 = \sum_{j=1}^{m} \eta_j^{-3/2} \left( \eta_j^{-1} a_j + 1 \right), \quad a_j = \|A_j A^{-1}\| < \infty,
$$

$$
c_2 = m + c_0, \quad c_0 = \sum_{j=1}^{m} \eta_j^{-1/2} a_j, \quad c_3 = \eta_1^{-1/2} + m + c_0.
$$

**Result 2.2.** If the functions $f(t)$ and $A^{1/2} u(t)$ on $[0, T]$ satisfies Holder condition by the index $\lambda$ ($0 < \lambda \leq 1$), then

$$
\|u(t_k) - u_k\| \leq c\tau^\lambda, \quad c = const > 0.
$$

4 Auxiliary lemmas and remarks

**Remark 4.1.** As $A_j A^{-1}$ ($j = 1, \ldots, m$) are closed operators (it can be proved easily), therefore due to Closed Graph Theorem they are bounded, i.e. $a_j = \|A_j A^{-1}\| < \infty$.

**Lemma 4.2.** For any $j$ ($j = 1, \ldots, m$) $D(A^{1/2}) \subset D(A_j^{1/2})$ and

$$
\|A_j^{1/2} u\| \leq \|A^{1/2} u\|, \quad \forall u \in D(A^{1/2}). \tag{4.1}
$$
Proof: According to condition (2.3) we have:

\[(A_j u, u) \leq (Au, u), \quad \forall u \in D(A) \subset D(A_j).\]

From here it follows

\[\left\| A_j^{1/2} u \right\| \leq \left\| A^{1/2} u \right\|, \quad \forall u \in D(A) \subset D(A_j).\]  \hspace{1cm} (4.2)

It is known that \(D(A)\) is a ball for \(A^{1/2}\) (see [17], p. 354). This means: for each \(u \in D(A^{1/2})\) there exists a sequence \(u_n \in D(A)\) such that \(u_n \to u\) and \(A_j^{1/2}u_n \to A^{1/2}u\). From here, according to (4.2) it follows that \(A_j^{1/2}u_n\) is a Cauchy Sequence. Obviously it will be convergent due to completeness of \(H\). Since \(A_j^{1/2}\) is closed (\(A_j\) is given as self-adjoint and positive definite), therefore \(u \in D(A_j^{1/2})\) and \(A_j^{1/2}u_n \to A^{1/2}u\). Thus \(D(A^{1/2}) \subset D(A_j^{1/2})\), and in addition, the inequality

\[\left\| A_j^{1/2}u_n \right\| \leq \left\| A^{1/2}u_n \right\|, \quad u_n \in D(A) \subset D(A_j)\]

gives (4.3).

Remark 4.3. If self-adjoint positive definite operators \(A\) and \(B\) are such that \(D(A) \subset D(B)\) and \(B \leq A\) (\((Bu, u) \leq (Au, u), \forall u \in D(A)\)), then \(A^{-1} \leq B^{-1}\).

Indeed, let’s \(u = B^{-1}f\) and \(v = A^{-1}f\), \(f \in H\). The relation is true (see the proof of Theorem VI.2.21 from [17]):

\[
(A^{-1}f, f)^2 = (v, Bu)^2 = (B^{1/2}v, B^{1/2}u)^2 \\
\leq \left\| B^{1/2}v \right\|^2 \left\| B^{1/2}u \right\|^2 = (Bv, u) \\
\leq (Av, v)(Bu, u) = (f, A^{-1}f)(f, B^{-1}f) \\
= (A^{-1}f, f)(B^{-1}f, f).
\]

After reduction we get

\[
(A^{-1}f, f) \leq (B^{-1}f, f),
\]
i. e. \(A^{-1} \leq B^{-1}\).

Remark 4.4 The formula is valid

\[S - L = \tau^2 \sum_{j=1}^{m} \eta_j^{-1} (I - S_j) (\eta_j^{-1} A_j A^{-1} - I) AL.\]  \hspace{1cm} (4.3)

Indeed, as

\[I - S = \sum_{j=1}^{m} \eta_j (I - S_j) = \tau^2 \sum_{j=1}^{m} A_j S_j,\]

therefore

\[S - L = [S (I + \tau^2 A) - I] L = [(S - I) + \tau^2 A S] L \\
= \tau^2 \sum_{j=1}^{m} (\eta_j S_j A - A_j S_j) L.\]

From here, taking the equality into account

\[
\eta_j S_j A - A_j S_j = \eta_j S_j A - S_j A_j = S_j (\eta_j A - A_j) \\
= (S_j - I) (\eta_j A - A_j) + (\eta_j A - A_j),
\]

we get (4.3).
We can consider the closeness of operators $S$ and $L$ also by the following formulas:

$$S_j = \tau^4 \eta_j^{-2} A_j^2 S_j - \tau^2 \eta_j^{-1} A_j + I,$$

$$S = \tau^4 \sum_{j=1}^{m} \eta_j^{-1} A_j^2 S_j - \tau^2 A + I,$$

$$L = \tau^4 A^2 L - \tau^2 A + I.$$

**Lemma 4.5.** The inequalities are valid:

$$\| (I - S)^{-1/2} A L f \| \leq \tau^{-1} \left( m \| A^{1/2} L f \| + c_0 \| A^{1/2} L^{1/2} f \| \right), \quad f \in H,$$

$$\| (I - S)^{-1/2} (S - L) f \| \leq \tau^2 c_1 \| A L f \|, \quad f \in H,$$

$$\| (I - S)^{-1/2} L A u \| \leq \tau^{-1} c_2 \| A^{1/2} u \|, \quad u \in D(A),$$

$$\| (I - S)^{-1/2} L f \| \leq \tau^{-1} c_2 \| A^{-1/2} f \|, \quad f \in H.$$  

**Proof:** Let us prove the inequality (4.4). As

$$(I - S) \geq \eta_j (I - S_j) = \tau^2 A_j S_j > 0,$$

therefore (see remark 4.3)

$$(I - S)^{-1} \leq \eta_j^{-1} (I - S_j)^{-1} = \tau^{-2} \left( I + \tau^2 \eta_j^{-1} A_j \right) A_j^{-1}
\leq \tau^{-2} A_j^{-1} + \eta_j^{-1} I \leq \left( \tau^{-1} A_j^{-s} + \eta_j^{-s} I \right)^2, \quad s = \frac{1}{2}.$$  

(4.8)

Thus we have

$$\| (I - S)^{-s} f \| \leq \| (\tau^{-1} A_j^{-s} + \eta_j^{-s} I) f \| \leq \tau^{-1} \| A_j^{-s} f \| + \eta_j^{-s} \| f \|, \quad f \in H.$$  

Using this inequality we get (below everywhere $s = 1/2$):

$$\| (I - S)^{-s} A L f \| = \left\| (I - S)^{-s} \sum_{j=1}^{m} A_j L f \right\| \leq \tau^{-1} \left( \sum_{j=1}^{m} \left( \| A_j^{-s} L f \| + \tau \eta_j^{-s} \| A_j L f \| \right) \right).$$  

(4.9)

According to lemma 4.2 we have

$$\| A_j^{-s} L f \| \leq \| A^s L f \|. \quad (4.10)$$

According to remark 4.1 we have

$$\| A_j L f \| \leq \| A_j A_j^{-1} \| \| A L f \| = a_j \| A L f \|. \quad (4.11)$$

From (4.9), taking into account (4.10) and (4.11), we obtain

$$\| (I - S)^{-s} A L f \| \leq m \tau^{-1} \| A^s L f \| + c_0 \| A L f \|.$$  

Hence, taking into account inequality

$$\| A L f \| = \| A^s L (A^s L f) \| \leq \| A^s L \| \| A^s L f \| \leq \tau^{-1} \| A^s L f \|$$

7
Let us prove inequality (4.5). From (4.8) we have
\[ \| (I - S)^{-1/2} h \| \leq \eta_j^{-1/2} \| (I - S_j)^{-1/2} h \|, \quad h \in H. \] (4.12)

If in (4.12) we substitute \( h = (I - S_j) f \), we get
\[ \| (I - S)^{-1/2} (I - S_j) f \| \leq \eta_j^{-1/2} \| (I - S_j)^{1/2} f \| \leq \eta_j^{-1/2} \| f \|. \] (4.13)

From this inequality and (4.3) it follows
\[ \| (I - S)^{-1/2} (I - S_j) f \| \leq \tau^{-1} \left( m \| A^{1/2} L u \| + c_0 \| A^{1/2} L^{1/2} u \| \right) \]
\[ \leq \tau^{-1} \left( m \| L A^{1/2} u \| + c_0 \| L^{1/2} A^{1/2} u \| \right) \]
\[ \leq \tau^{-1} c_2 \| A^{1/2} u \|. \]

Let us prove the inequality (4.6). Obviously, operators \( A \) and \( L \) are commutative on \( D(A) \). Then due to (4.4), the relation is valid:
\[ \| (I - S)^{-1/2} L A u \| = \| (I - S)^{-1/2} A L u \| \]
\[ \leq \tau^{-1} \left( m \| A^{1/2} L u \| + c_0 \| A^{1/2} L^{1/2} u \| \right) \]
\[ \leq \tau^{-1} \left( m \| L A^{1/2} u \| + c_0 \| L^{1/2} A^{1/2} u \| \right) \]
\[ \leq \tau^{-1} c_2 \| A^{1/2} u \|. \]

Obviously inequality (4.7) is a result of (4.6).

**Lemma 4.6.** For operator polynomials \( U_k(B) \) the following estimate is valid:
\[ \| B U_k(B) \| \leq (\tau \sqrt{\nu})^{-1}, \quad \nu = \min_{1 \leq j \leq m} (\alpha_j), \] (4.14)
\[ \left\| U_k(B) \left( I - B^2 \right)^{1/2} \right\| \leq 1, \] (4.15)
\[ \left\| U_k(B) - B U_{k-1}(B) \right\| \leq 1, \] (4.16)
\[ \left\| B U_k(B) - U_k(B) \right\| \leq 1. \] (4.17)

**Proof:** As it is well-known, for Chebyshev second-order polynomials the estimate is valid (see, e.g., [37]):
\[ |U_k(x)| \leq \frac{1}{\sqrt{1 - x^2}}, \quad x \in [-1, 1], \] (4.18)
that follows from the well-known formula
\[ U_k(x) = \sin \left( (k + 1) \arccos x \right) \frac{1}{\sqrt{1 - x^2}}, \quad x \in [-1, 1]. \]

From this formula, by means of simple calculations, we also obtain the estimate
\[ |U_k(x) - U_{k-1}(x)| \leq \sqrt{\frac{2}{1 + x}}, \quad x \in [-1, 1]. \] (4.19)
Indeed we have
\[ |U_k(x) - U_{k-1}(x)| = \frac{2}{\sqrt{1-x^2}} \left| \cos\left(k + \frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \right| \]
\[ \leq \frac{2}{\sqrt{1-x^2}} \sqrt{\frac{1 - \cos \theta}{2}} \]
\[ = \sqrt{\frac{2}{1+x}}, \quad x \in ]-1,1[, \]
where \( \theta = \arccos x \).

The following estimation is valid:
\[ |U_k(x) - xU_{k-1}(x)| \leq |\cos((k+1)\theta)| \leq 1. \quad (4.20) \]

Obviously, similarly we get the estimate
\[ |xU_k(x) - U_{k-1}(x)| \leq 1. \quad (4.21) \]

For the estimation of norm of polynomial operators \( U_k(B) \), we need to estimate the spectrum of operator \( B = S^{1/2} \), that obviously can be reduced to the estimate of spectrum of operator \( S \).

Let us first estimate spectrum of operator \( S_j = \left( I + \tau^2 \eta_j^{-1} A_j \right)^{-1} \). We obviously have
\[ \left( I + \tau^2 \eta_j^{-1} A_j \right) \geq \left( 1 + \tau^2 \nu^{-1} \right) I > 0 . \]

From here, due to remark 4.3, it follows
\[ 0 < S_j \leq \left( 1 + \tau^2 \eta_j^{-1} \alpha_j \right)^{-1} I < \left( 1 + \tau^2 \nu^{-1} \right)^{-1} I . \quad (4.22) \]

If we take into account representation of \( S \), then according to (4.22), we have
\[ 0 < S \leq \sum_{j=1}^{m} \eta_j \left( 1 + \tau^2 \nu \right)^{-1} I = \left( 1 + \tau^2 \nu \right)^{-1} I . \]

This means that
\[ \text{Sp}(S) \subset \{0, \left( 1 + \tau^2 \nu \right)^{-1} \} \]

From here it follows that (due to well-known theorem on spectral mapping)
\[ \text{Sp}(B) \subset \{0, \left( 1 + \tau^2 \nu \right)^{-1/2} \} . \quad (4.23) \]

Further we can easily show estimates of (4.14)-(4.17).

Let us show estimate (4.14). As is known, norm of operator-function, when the argument represents self-adjoint bounded operator, is equal to the \( C \)-norm of the corresponding scalar function on the spectrum (see, e. g., [29], Chapter VII). Due to this result we have
\[ \|BU_k(B)\| \leq \max_{x \in \text{Sp}(B)} |xU_k(x)| . \]

From here, by the inequality (4.18) and relation (4.23), we get
\[ \|BU_k(B)\| \leq \frac{x}{\sqrt{1-x^2}} \leq \frac{1}{\tau \sqrt{\nu}} . \]

Similarly is obtained (4.15). Indeed we have
\[ \left\| U_k(B) \left( I - B^2 \right)^{1/2} \right\| \leq \max_{x \in \text{Sp}(B)} \left| U_k(x) \sqrt{1-x^2} \right| \]
\[ \leq \max_{x \in \text{Sp}(B)} \left( \frac{1}{\sqrt{1-x^2}} \sqrt{1-x^2} \right) = 1 . \]

From the inequalities (4.20) and (4.21) respectively follows the estimates (4.16) and (4.17).
5 Error estimate of approximate solution

In this section we continue to prove Theorem 3.1, that concerns error estimate of approximate solution.

Let us rewrite formula (3.9) in the following form

\[ z_{k+1} = \tau B^k U_k(B) \frac{\Delta z_0}{\tau} + B^k (U_k(B) - BU_{k-1}(B)) z_0 + \sum_{i=1}^{k} B^{k-i} U_{k-i}(B) r_i , \] (5.1)

where \( \Delta z_0 = z_1 - z_0 \).

Let us note that \( \Delta z_0/\tau \) is an error of the difference analog of the first order derivative of approximate solution at the point \( t = 0 \),

\[ \frac{\Delta z_0}{\tau} = \frac{\Delta u_0}{\tau} - \frac{\Delta u(0)}{\tau}, \quad \Delta u(0) = u(\tau) - u(0) . \]

If we move to norms in (5.1), we obtain

\[ \| z_{k+1} \| \leq \tau \left\| B^{k-1} \right\| \| BU_k(B) \left\| \frac{\Delta z_0}{\tau} \right\| + \| B^k \| \| U_k(B) - BU_{k-1}(B) \| \| z_0 \| + \sum_{i=1}^{k} \left\| B^{k-i} \right\| \left\| U_{k-i}(B)(I - B^2)^{1/2} \right\| \left\| (I - B^2)^{-1/2} r_i \right\| . \]

From here, taking into account estimates (4.14), (4.15) and (4.16) (\( \| B \| \leq 1 \)), we have

\[ \| z_{k+1} \| \leq \nu^{-1/2} \left\| \frac{\Delta z_0}{\tau} \right\| + \| z_0 \| + \sum_{i=1}^{k} \left\| (I - B^2)^{-1/2} r_i \right\| . \] (5.2)

Obviously we have

\[ \left\| (I - B^2)^{-1/2} r_i \right\| = \left\| (I - S)^{-1/2} r_i \right\| \leq \lambda_{0,i} + \lambda_{1,i} + \tau^2 \lambda_{2,i} + \lambda_{3,i} + \lambda_{4,i} , \] (5.3)

where

\[ \lambda_{s,i} = \left\| (I - S)^{-1/2} r_{s,i} \right\| , \quad s = 0, 1, 4, \]
\[ \lambda_{s,i} = \left\| (I - S)^{-1/2} L r_{s,i} \right\| , \quad s = 2, 3 . \]

Using (4.5), we get:

\[ \lambda_{0,i} = \left\| (I - S)^{-1/2} (S - L) u(t_i) \right\| \leq \tau^2 c_1 \| ALu(t_i) \| \leq \tau^2 c_1 \| Au(t_i) \| , \] (5.4)

\[ \lambda_{1,i} = \left\| (I - S)^{-1/2} (S - L) [u(t_i) - u(t_{i-1})] \right\| \leq \tau^2 c_1 \| A^{1/2} [u(t_i) - u(t_{i-1})] \| \leq \tau c_1 \| A^{1/2} \left\| A^{1/2} [u(t_i) - u(t_{i-1})] \right\| . \] (5.5)
Using (4.6) and (4.7), respectively we get:

\[
\tau^2 \lambda_{2,i} = \tau^2 \| (I - S)^{-1/2} LA [u(t_{i+1}) - u(t_i)] \|
\leq \tau c_2 \| A^{1/2} [u(t_{i+1}) - u(t_i)] \| ,
\]

(5.6)

\[
\lambda_{3,i} = \| (I - S)^{-1/2} Lr_{3,i} \| \leq \tau^{-1} c_2 \| A^{-1/2} r_{3,i} \|
\]

\[
\leq \tau^{-1} c_2 \int_{t_i}^{t_{i+1}} (t_{i+1} - t) A^{-1/2} [u''(t) - u''(t_i)] dt
\]

\[
+ \tau^{-1} c_2 \int_{t_{i-1}}^{t_i} (t - t_{i-1}) A^{-1/2} [u''(t) - u''(t_i)] dt
\]

\[
\leq c_2 \int_{t_{i-1}}^{t_{i+1}} \| A^{-1/2} [u''(t) - u''(t_i)] \| dt .
\]

From here, taking into account equation (2.1), we have

\[
\lambda_{3,i} \leq c_2 \int_{t_{i-1}}^{t_{i+1}} \| A^{1/2} [u(t) - u(t_i)] \| dt
\]

\[
+ c_2 \int_{t_{i-1}}^{t_{i+1}} \| A^{-1/2} \tilde{f}(t) - \tilde{f}(t_i) \| dt .
\]

(5.7)

Let us estimate \( \lambda_{4,i} \). Let’s rewrite \( r_{4,i} \) as

\[
r_{4,i} = \tau^2 \left[ \psi_i - L \tilde{f}(t_i) \right] = \tau^2 S_1 [f(t_i) - M(v_i)] - \tau^2 L \tilde{f}(t_i)
\]

\[
= \tau^2 S_1 [f(t_i) - M(u(t_i))] + \tau^2 S_1 [M(u(t_i)) - M(v_i)] - \tau^2 L \tilde{f}(t_i)
\]

\[
= \tau^2 S_1 [M(u(t_i)) - M(v_i)] + \tau^2 S_1 \tilde{f}(t_i) - \tau^2 L \tilde{f}(t_i)
\]

\[
= \tau^2 S_1 [M(u(t_i)) - M(v_i)] + \tau^2 (S_1 - I) \tilde{f}(t_i) + \tau^2 (I - L) \tilde{f}(t_i)
\]

\[
= \tau^2 S_1 [M(u(t_i)) - M(v_i)] + \tau^2 (S_1 - I) \tilde{f}(t_i) + \tau^4 A L \tilde{f}(t_i) .
\]

(5.8)

Hence we have

\[
\lambda_{4,i} = \| (I - S)^{-1/2} r_{4,i} \|
\leq \tau^2 \| (I - S)^{-1/2} S_1 [M(u(t_i)) - M(v_i)] \|
\]

\[
+ \tau^2 \| (I - S)^{-1/2} (I - S) \tilde{f}(t_i) \|
\]

\[
+ \tau^4 \| (I - S)^{-1/2} A L \tilde{f}(t_i) \| .
\]

(5.9)

If we use inequality (4.12), we get:

\[
\| (I - S)^{-1/2} S_1 h \| \leq \eta_{e}^{-1/2} \| (I - S_1)^{-1/2} S_1 h \|
\]

\[
= \eta_{e}^{-1/2} \| \tau^2 \eta_{e}^{-1} A S_1 \|^{-1/2} S_1 h \|
\]

\[
= \tau^{-1} \| A_i^{-1/2} S_1^{1/2} h \| \leq (\tau \sqrt{\eta})^{-1} \| h \| ,
\]

(5.10)

\[
\| (I - S)^{-1/2} (I - S_1) h \| \leq \eta_{e}^{-1/2} \| (I - S_1)^{1/2} h \| \leq \eta_{e}^{-1/2} \| h \| .
\]

(5.11)
Taking into account that $\|\tau A^{1/2} L^{1/2}\| \leq 1$, then from (4.4) we get

$$\tau^2 \| (I - S)^{-1/2} A L \| \leq \tau \left( m \| A^{1/2} L f \| + c_0 \| A^{1/2} L^{1/2} f \| \right) \leq (m + c_0) \| f \| .$$

(5.12)

From (5.9), taking into account estimates (5.10), (5.11) and (5.12), we get

$$\lambda_{k,i} \leq \tau \nu^{-1/2} \| M(u_{t,i}) - M(v_{t,i}) \| + \tau^2 c_3 \| f(t_{i}) \| ,$$

(5.13)

where $c_3 = \eta^{-1/2} + m + c_0$.

Since nonlinear operator $M$ satisfies Lipschitz condition, therefore from (5.13) we have

$$\lambda_{k,i} \leq \tau \nu^{-1/2} a \| u(t_{i}) - v_{t,i} \| + \tau^2 c_3 \| f(t_{i}) \| ,$$

(5.14)

where $a$ is a Lipschitz constant.

If in (5.3) we substitute (5.4), (5.5), (5.6), (5.7) and (5.14), we get

$$\| (I - B^2)^{-1/2} r_{i} \| \leq \tau \nu^{-1/2} a \| z_i \| + \tau^2 c_3 \| f(t_{i}) \| + \tau c_2 J_i(t_{i+1}, A^{1/2} u) + \tau c_1 J_i(t_{i} - 1, A^{1/2} u)$$

$$+ c_2 \int_{t_{i-1}}^{t_{i+1}} \left[ J_i(t, A^{1/2} u) + J_i(t, A^{-1/2} f) \right] dt ,$$

(5.15)

where

$$J_i(t, u) = \| u(t_{i}) - u(t) \| .$$

From (5.2), taking into account (5.15), we get

$$\delta_{k+1} \leq c \tau \delta_k + \lambda_k ,$$

(5.16)

where $\delta_k = \| z_k \| , c = \nu^{-1/2} a$.

$$\lambda_k = \nu^{-1/2} \left\| \frac{\Delta z_0}{\tau} \right\| + \| z_0 \| + \Theta_k(\tau) .$$

From (5.16), according to discrete analog of Gronwell’s lemma, we have

$$\delta_{k+1} \leq \exp(c \tau \delta_{k-1}) \left( c \tau \delta_1 + \lambda_k \right) .$$

Obviously from here, taking into account inequality

$$\delta_1 = \| z_1 \| \leq \tau \left\| \frac{\Delta z_0}{\tau} \right\| + \| z_0 \| ,$$

We get the estimate (3.10).

6 Error estimate for the difference analog of the first order derivative of approximate solution

In this section, on the basis of the results obtained in the previous section, we obtain the a priori estimates for error of the difference analog of the first order derivative of the decomposition scheme solution.
Theorem 6.1. If the problem (2.1), (2.2) has a solution and \( \varphi_0 \in D(A) \), then the estimate is valid:

\[
\left\| \frac{\Delta z_k}{\tau} \right\| \leq \left\| \frac{\Delta z_0}{\tau} \right\| + (\tau \rho_k)^{-1/2} \| z_0 \| + \tilde{\Theta}_k(\tau),
\]

where \( \Delta z_k = z_{k+1} - z_k \).

\[
\tilde{\Theta}_k(\tau) = \tau \sum_{i=1}^{k} \{ c_i J_i \left( t_{i-1}, u'' \right) + J_i \left( t_{i+1}, u'' \right) + ac_i J_i \left( t_{i-1}, u \right) \}
\]

\[
+ \tau \sum_{i=1}^{k} \left[ c_7 J_i \left( t_{i-1}, f \right) + J_i \left( t_{i-1}, f \right) \right]
\]

\[
+ \sum_{i=1}^{k} \left( \int_{t_{i-1}}^{t_i} J_i \left( t, u'' \right) dt + \tau a \| z_i \| \right)
\]

\[
+ \tau \left[ c_1 \| A \varphi_0 \| + c_6 \left( \| f(0) \| + \| M(\varphi_0) \| \right) \right],
\]

and where \( c_5 = c_4 + c_1 \), \( c_6 = \eta_1^{-1/2} + m + c_6 \), \( c_7 = c_4 + c_1 + c_6 \),

\[
c_4 = \sum_{j=1}^{m} \eta_j^{-1} \left( \eta_j^{-1} a_j + 1 \right), \quad J_i \left( t, u \right) = \| u(t_i) - u(t) \|.
\]

Proof: Due to formula (3.9) we have

\[
z_{k+1} - z_k = (R_k - R_{k-1}) z_1 - (R_{k-1} - R_{k-2}) B^2 z_0 + \Phi_k,
\]

where \( R_k = B^k U_k(B) \),

\[
\Phi_k = \sum_{i=1}^{k} R_{k-i} r_i - \sum_{i=1}^{k-1} R_{k-1-i} r_i.
\]

Let us rewrite (6.2) as

\[
\frac{\Delta z_k}{\tau} = \frac{(R_k - R_{k-1}) \Delta z_0}{\tau} + \tau^{-1} \left[ R_k + B^2 R_{k-2} - (I + B^2) R_{k-1} \right] z_0 + \tau^{-1} \Phi_k,
\]

where \( \Delta z_k = z_{k+1} - z_k \).

By the simple transformation we get

\[
R_k + B^2 R_{k-2} - (I + B^2) R_{k-1} = B^{k-1} \left[ B \left( U_k + U_{k-2} \right) - (I + B^2) U_{k-1} \right]
\]

\[
= -B^{k-1} \left( I - B^2 \right) U_{k-1}.
\]

From here, taking into account (4.15), we have

\[
\left\| R_k + B^2 R_{k-2} - (I + B^2) R_{k-1} \right\| \leq \| B^{k-1} \left( I - B^2 \right) \| \left\| U_{k-1} \right\| \leq \| B^{k-1} \left( I - B^2 \right)^{1/2} \| \leq \max_{0 \leq x \leq 1} \left[ x^{k-1} (1 - x^2)^{1/2} \right] \leq \frac{1}{\sqrt{k}}.
\]

Obviously, according to (4.17), we have

\[
\left\| R_k - R_{k-1} \right\| \leq \| B^{k-1} \left( BU_k(B) - U_{k-1}(B) \right) \| \leq \| B^{k-1} \| \| BU_k(B) - U_{k-1}(B) \| \leq 1.
\]
We can give to \( \Phi_k \) the following form

\[
\Phi_k = \sum_{i=1}^{k} R_{k-i} \left( r_i - r_{0,i} - \tau^2 \zeta_i \right) - \sum_{i=1}^{k-1} R_{k-1-i} \left( r_i - r_{0,i} - \tau^2 \zeta_i \right)
\]
\[
+ \sum_{i=1}^{k} R_{k-i} \left( r_{0,i} + \tau^2 \zeta_i \right) - \sum_{i=1}^{k-1} R_{k-1-i} \left( r_{0,i} + \tau^2 \zeta_i \right)
\]
\[
= \sum_{i=1}^{k} \left( R_{k-i} - R_{k-1-i} \right) \left( r_i - r_{0,i} - \tau^2 \zeta_i \right)
\]
\[
+ \sum_{i=1}^{k} R_{k-i} \left( r_{0,i} + \tau^2 \zeta_i \right) - \sum_{i=2}^{k} R_{k-i-1} \left( r_{0,i-1} + \tau^2 \zeta_{i-1} \right)
\]
\[
= \sum_{i=1}^{k} \left( R_{k-i} - R_{k-1-i} \right) \left( r_i - r_{0,i} - \tau^2 \zeta_i \right)
\]
\[
+ \sum_{i=1}^{k} R_{k-i} \left( (r_{0,i} - r_{0,i-1}) + \tau^2 (\zeta_i - \zeta_{i-1}) \right]
\]
\[
+ R_{k-1} \left( r_{0,0} + \tau^2 \zeta_0 \right) ,
\]
\[
(6.6)
\]

where \( R_{-1} = 0 \),
\[
\zeta_i = (S_i - I) \tilde{f}(t_i) + \tau^2 AL \tilde{f}(t_i) .
\]

Taking into account (6.5), we have
\[
\| (R_{k-i} - R_{k-1-i}) (r_i - r_{0,i} - \tau^2 \zeta_i) \|
\]
\[
\leq \left( \| r_{1,i} \| + \tau^2 \| Lr_{2,i} \| + \| Lr_{2,i} \| + \| r_{1,i} - \tau^2 \zeta_i \| \right) .
\]
\[
(6.7)
\]

By formula (4.3), we get
\[
\| r_{1,i} \|
\]
\[
= \| (S - L)[u(t_i) - u(t_{i-1})] \|
\]
\[
= \tau^2 \left( \sum_{j=1}^{m} \eta_j^{-1} (I - S_j) (\eta_j^{-1} A_j A^{-1} - I) AL [u(t_i) - u(t_{i-1})] \right)
\]
\[
\leq \tau^2 c_4 \| A \| [u(t_i) - u(t_{i-1})] ,
\]
\[
(6.8)
\]

Obviously, for \( Lr_{2,i} \) we have
\[
\| Lr_{2,i} \| \leq \| A \| [u(t_{i+1}) - u(t_i)] .
\]
\[
(6.9)
\]

From inequalities (6.8) and (6.9), with account of equation (2.1), we get
\[
\| r_{1,i} \| + \tau^2 \| Lr_{2,i} \|
\]
\[
\leq \tau^2 c_4 \left( \| u'' (t_i) - u'' (t_{i-1}) \| + \| f(t_i) - f(t_{i-1}) \| \right)
\]
\[
+ \tau^2 \left( \| u'' (t_{i+1}) - u'' (t_i) \| + \| f(t_{i+1}) - f(t_i) \| \right) .
\]
\[
(6.10)
\]

It is also obvious, that from (6.8) it follows
\[
\| r_{1,i} - \tau^2 \zeta_i \| \leq \tau^2 \| S_i [M (u(t_i)) - M (v_i)] \| \leq \alpha \tau^2 \| u(t_i) - v_i \| .
\]
\[
(6.11)
\]
From the representation of $r_{3,1}$ it follows that

$$
\| L r_{3,1} \| \leq \int_{t_i}^{t_{i+1}} (t_{i+1} - t) \left\| u^{(2)}(t) - u^{(2)}(t_i) \right\| dt \\
+ \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \left\| u^{(2)}(t) - u^{(2)}(t_i) \right\| dt \\
\leq \tau \int_{t_i}^{t_{i+1}} \left\| u^{(2)}(t) - u^{(2)}(t_i) \right\| dt .
$$

(6.12)

If we substitute inequalities (6.10), (6.11) and (6.12) in (6.7), we get

$$
\left\| \left( R_{k-1} - R_{k-1-i} \right) \left( \tau_i - r_{0,i} - \tau^2 \zeta_i \right) \right\| \\
\leq \tau^2 c_4 \left( \left\| u^{(2)}(t_i) - u^{(2)}(t_{i-1}) \right\| + \left\| f(t_i) - f(t_{i-1}) \right\| \\
+ \tau^2 \left( \left\| u^{(2)}(t_{i+1}) - u^{(2)}(t_i) \right\| + \left\| f(t_{i+1}) - f(t_i) \right\| \\
+ \tau \int_{t_i}^{t_{i+1}} \left\| u^{(2)}(t) - u^{(2)}(t_i) \right\| dt + a c_4 \tau^2 \left\| u(t_i) - v_i \right\| .
$$

(6.13)

According to estimate (4.15) we have

$$
\left\| R_{k-1} (r_{0,i} - r_{0,i-1}) \right\| \\
= \left\| R_{k-1} \left( I - B^2 \right)^{1/2} \left( I - B^2 \right)^{-1/2} \left( r_{0,i} - r_{0,i-1} \right) \right\| \\
\leq \left\| U_{k-1} \left( I - B^2 \right)^{1/2} \right\| \left\| \left( I - S \right)^{-1/2} \left( r_{0,i} - r_{0,i-1} \right) \right\| \\
\leq \left\| \left( I - S \right)^{-1/2} \left( r_{0,i} - r_{0,i-1} \right) \right\| .
$$

(6.14)

Similarly we get

$$
\left\| R_{k-1} (\zeta_i - \zeta_{i-1}) \right\| \\
\leq \left\| \left( I - S \right)^{-1/2} (\zeta_i - \zeta_{i-1}) \right\| \\
\leq \left\| \left( I - S \right)^{-1/2} (S_i - I) \left( \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right) \right\| \\
+ \tau^2 \left\| \left( I - S \right)^{-1/2} \left( \tilde{L} \left( \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right) \right) \right\| .
$$

(6.15)

If we substitute in (6.14) representation of $r_{0,i}$, $(r_{0,i} = (S - L) u(t_i))$ and take into account estimate (4.15), we get

$$
\left\| R_{k-1} (r_{0,i} - r_{0,i-1}) \right\| \leq \tau^2 c_4 \left\| A \left[ u(t_i) - u(t_{i-1}) \right] \right\| .
$$

(6.16)

Similarly we have

$$
\left\| R_{k-1} \varphi_0 \right\| \leq \tau^2 c_4 \left\| A u(0) \right\| = \tau^2 c_4 \left\| A \varphi_0 \right\| , \quad \varphi_0 \in D(A).
$$

(6.17)

From (6.16), taking into account equation (2.1), we have

$$
\left\| R_{k-1} (r_{0,i} - r_{0,i-1}) \right\| \\
\leq \tau^2 c_1 \left( \left\| u^{(2)}(t_i) - u^{(2)}(t_{i-1}) \right\| + \left\| f(t_i) - f(t_{i-1}) \right\| \right) .
$$

(6.18)

If in (6.15) we take into account estimates (4.13), (4.4) and $\tau \left\| A^{1/2} L^{1/2} h \right\| \leq h$, we get

$$
\left\| R_{k-1} (\zeta_i - \zeta_{i-1}) \right\| \\
\leq \eta_1^{1/2} \left\| \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right\| + \tau m \left\| A^{1/2} L \left( \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right) \right\| \\
+ \tau \left\| A^{1/2} L^{1/2} \left( \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right) \right\| \leq c_6 \left\| \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right\| \\
\leq c_6 \left( \left\| f(t_i) - f(t_{i-1}) \right\| + a \left\| u(t_i) - u(t_{i-1}) \right\| \right) .
$$

(6.19)
where \( c_5 = \eta_1^{-1/2} + m + c_0 \).

Analogously to (6.19), the inequality is true

\[
\| R_{k-1} \Omega_0 \| \leq c_5 \left( \| f(0) \| + \| M(\varphi_0) \| \right). \tag{6.20}
\]

Obviously, from inequalities (6.18) and (6.19) it follows

\[
\| R_{k-1} \left[ (r_{0,i} - r_{0,i-1}) + \tau^2 (\zeta_i - \zeta_{i-1}) \right] \|
\leq \tau^2 (c_1 \| u''(t_i) - u''(t_{i-1}) \| + ac_6 \| u(t_i) - u(t_{i-1}) \|)
+ \tau^2 (c_1 + c_6) \| f(t_i) - f(t_{i-1}) \|. \tag{6.21}
\]

If in (6.6) we move to norms and take into account inequalities (6.19), (6.21), (6.17) and (6.20), we get

\[
\| \Phi_k \| \leq \tau^2 c_4 \sum_{i=1}^{k} \left( \| u''(t_i) - u''(t_{i-1}) \| + \| f(t_i) - f(t_{i-1}) \| \right)
+ \tau^2 \sum_{i=1}^{k} \left( \| u''(t_{i+1}) - u''(t_i) \| + \| f(t_{i+1}) - f(t_i) \| \right)
+ \tau^2 \sum_{i=1}^{k} \left( \int_{t_{i-1}}^{t_{i+1}} \| u''(t) - u''(t_i) \| \, dt + a\tau^2 \| \zeta_i \| \right)
+ \tau^2 (c_1 + c_6) \sum_{i=1}^{k} \| f(t_i) - f(t_{i-1}) \|
+ \tau^2 c_1 \| A\varphi_0 \| + \tau^2 c_6 \left( \| f(0) \| + \| M(\varphi_0) \| \right). \tag{6.22}
\]

Or the same

\[
\| \Phi_k \| \leq \tau^2 \sum_{i=1}^{k} \left[ c_5 J_i \left( t_{i-1}, u'' \right) + J_i \left( t_{i+1}, u'' \right) + ac_6 J_i \left( t_{i-1}, u \right) \right]
+ \tau^2 \sum_{i=1}^{k} \left[ c_7 J_i \left( t_{i-1}, f \right) + J_i \left( t_{i+1}, f \right) \right]
+ \tau \sum_{i=1}^{k} \left( \int_{t_{i-1}}^{t_{i+1}} J_i (t, u'') \, dt + \tau a \| \zeta_i \| \right)
+ \tau^2 \left[ c_1 \| A\varphi_0 \| + c_6 \left( \| f(0) \| + \| M(\varphi_0) \| \right) \right], \tag{6.22}
\]

where \( c_\tau = c_4 + c_1 + c_6 \).

From (6.3), taking into account (6.4), (6.5) and (6.22), it follows (6.1).

**Result 2.** If functions \( f(t) \) and \( u''(t) \) on the interval \([0, T]\) satisfies Holder condition with the index \( \lambda \) \((0 < \lambda \leq 1)\), then

\[
\left\| u' (t_k) - \frac{u_{k+1} - u_k}{\tau} \right\| \leq c \tau^\lambda, \quad c = \text{const} > 0. \tag{6.23}
\]

With account of (6.1), estimate (6.23) follows from the following equality

\[
\frac{u' (t_k) - u_{k+1} - u_k}{\tau} = \tau^{-1} \int_{t_k}^{t_{k+1}} \left[ u' (t_k) - u' (t) \right] \, dt - \frac{z_{k+1} - z_k}{\tau}
\]
7 Numerical Results

The calculations were performed for the following problem.

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u = \sin(u) + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T),
\]

\[
u(x, y, 0) = \varphi_0(x, y), \quad u'(x, y, 0) = \varphi_1(x, y),
\]

\[
\Omega = (0, 1) \times (0, 1),
\]

\[
u_{\partial \Omega \times [0, T]} = 0.
\]

For this problem two different test case were calculated.

**Test 1.**

\[
u(x, y, t) = t^{7/2} \sin(2\pi x) \cos(2\pi x),
\]

\[(x, y, t) \in [0, 1] \times [0, 1] \times [0, 1],
\]

\[
\tau = h_x = h_y = 0.05.
\]

Fig[1] shows exact solution, approximate solution and error. This test example is interesting as fourth derivative of the solution is discontinuous. As we see from Fig. 1 the numerical algorithm was able to resolve this problem with good accuracy. Maximum error equals to 0.0046.

![Figure 1: Exact and approximate solutions of Test 1.](image)

**Test 2.**

\[
u(x, y, t) = t^{7/2} \sin(10\pi x) \cos(10\pi x),
\]

\[(x, y, t) \in [0, 1] \times [0, 1] \times [0, 1],
\]

\[
\tau = h_x = h_y = 0.01.
\]

Similarly, Fig[2] shows exact solution, approximate solution and error. This test example is interesting as fourth derivative of the solution is discontinuous and also has high oscillations, in the interval (0, 1) it changes the sign ten times. As we see from Fig. 2 the numerical algorithm was able to detect these oscillations and recovered original function with good accuracy. Maximum error for this case equals to 0.0074.
Figure 2: Exact and approximate solutions of Test 2.

On Fig. 3 there is given a dependence of the logarithm of the relative error of the approximated solution on the logarithm of number of division by special variable. Aim of this figure is to find the convergence rate of the method by means of the numerical experiment. If the method is second order of accuracy, then, the curve of the function (logarithm of the solution error) should approach to the line, the tangent of which equals two. On Fig. 3 it is clearly seen that, the curve approaches the line, the tangent of which equals to two, and this verifies the theoretical result proved in the article.

Figure 3: Dependence of logarithm of relative error on logarithm of number of divisions for Test 2.

References

[1] G. A. Baker and T. A. Oliphant. An implicit, numerical method for solving the two-dimensional heat equation. *Quart. Appl. Math.*, 17:361–373, 1959/60.

[2] G. Birkhoff and R. S. Varga. Implicit alternating direction methods. *Trans. Amer. Math. Soc.*, 92:13–24, 1959.
[3] G. Birkhoff, R. S. Varga, and D. Young. Alternating direction implicit methods. In *Advances in Computers, Vol. 3*, pages 189–273. Academic Press, New York, 1962.

[4] S. Blanes, F. Casas, and M. Thalhammer. Splitting and composition methods with embedded error estimators. *Appl. Numer. Math.*, 146:400–415, 2019.

[5] N. Dikhaminjia, J. Rogava, and M. Tsiklauri. Construction and investigation of a fourth order of accuracy decomposition scheme for nonhomogeneous multidimensional hyperbolic equation. *Numer. Funct. Anal. Optim.*, 35(3):275–293, 2014.

[6] E. G. D’jakonov. Difference schemes with splitting operator for higher-dimensional non-stationary problems. *Zh. Vychisl. Mat i Mat. Fiz.*, 2:549–568, 1962.

[7] J. Douglas. On the numerical integration of $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = \partial u/\partial t$ by implicit methods. *J. Soc. Indust. Appl. Math.*, 3:42–65, 1955.

[8] J. Douglas and H. H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. *Trans. Amer. Math. Soc.*, 82:421–439, 1956.

[9] D. G. Gordeziani. On application of local one dimensional method for solving parabolic type multidimensional problems of $2m$-degree. In *Proc. Acad. Sci. GSSR*, volume 3, pages 535–542, 1965.

[10] D. G. Gordeziani. A certain economical difference method for the solution of a multidimensional equation of hyperbolic type. *Gamoqeneb. Math. Inst. Sem. Mohsen. Anotacie.*, 1(4):11–14, 1971.

[11] D. G. Gordeziani and G. V. Meladze. The simulation of the third boundary value problem for multidimensional parabolic equations in an arbitrary domain by one-dimensional equations. *Zh. Vychisl. Mat i Mat. Fiz.*, 14:246–250, 271, 1974.

[12] D. G. Gordeziani and A. A. Samarskij. Certain thermoelastic problems of plates and shells and the summary approximation method. *Complex Anal. Appl., Collect. Artic., Steklov Math. Inst., Moscow*, pages 173–186, 1978.

[13] D. He, K. Pan, and H. Hu. A spatial fourth-order maximum principle preserving operator splitting scheme for the multi-dimensional fractional Allen-Cahn equation. *Appl. Numer. Math.*, 151:44–63, 2020.

[14] N. N. Ianenko. On Economic Implicit Schemes (Fractional steps method). *Dokl. Akad. Nauk SSSR*, 134(5):84–86, 1960.

[15] N. N. Ianenko. *The method of fractional steps for solving multidimensional problems in mathematical physics*. Novosibirsk: Izdat. “Nauka” - Sibirsk. Otdel. 196 pp., 1967.

[16] V. P. Il’in. On the splitting of difference parabolic and elliptic equations. *Sibirsk. Mat. Æ.*, 6:1425–1428, 1965.

[17] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[18] A. N. Konovalov. The method of fractional steps for solving the Cauchy problem for a multi-dimensional wave equation. *Dokl. Akad. Nauk SSSR*, 147:25–27, 1962.

[19] S. G. Krein. *Linear differential equations in Banach space*. Translations of Mathematical Monographs, Vol. 29. American Mathematical Society, Providence, R.I., 1971. Translated from the Russian by J. M. Danskin.
A. M. Kuzyk and V. L. Makarov. Estimation of the accuracy of the method of summary approximation of the solution of an abstract Cauchy problem. *Dokl. Akad. Nauk SSSR*, 275(2):297–301, 1984.

V. L. Makarov. On the difference schemes with exact and explicit spectrum. Doctoral Dissertation, Taras Shevchenko National University of Kyiv, 1976.

G. I. Marchuk. Metody rasschepleniya. Moskva: Nauka, 264 pp., 1988.

G. I. Marčuk and N. N. Janenko. Solution of a higher-dimensional kinetic equation by a splitting method. *Dokl. Akad. Nauk SSSR*, 157:1291–1292, 1964.

G. I. Marčuk and U. M. Sultangazin. On a proof of the splitting method for the equation of radiation transfer. *Zh. Vyčisl. Mat i Mat. Fiz.*, 5:852–863, 1965.

A. G. Morris and T. S. Horner. Chebyshev polynomials in the numerical solution of differential equations. *Math. Comp.*, 31(140):881–891, 1977.

V. A. Novikov and G. V. Demidov. A remark on a certain method of constructing schemes of high accuracy. *Čisl. Metody Mekh. Splošnoi Sredy*, 3(4):89–91, 1972.

J. L. Padgett and Q. Sheng. Convergence of an operator splitting scheme for abstract stochastic evolution equations. In *Advances in mathematical methods and high performance computing*, volume 41 of *Adv. Mech. Math.*, pages 163–179. Springer, Cham, 2019.

D. W. Peaceman and H. H. Rachford. The numerical solution of parabolic and elliptic differential equations. *J. Soc. Indust. Appl. Math.*, 3:28–41, 1955.

V. A. Rastrenin. The application of a certain difference method to abstract hyperbolic equations. *Differencial'nye Uravnenija*, 9:2222–2226, 2300, 1973.

M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

D. L. Rogava. An averaged semidiscrete scheme of summary approximation for an abstract hyperbolic equation. In *Current problems in mathematical physics, Vol. I (Russian) (Tbilisi, 1987)*, pages 338–348, 491–492. Tbilis. Gos. Univ., Tbilisi, 1987.

Dzh. L. Rogava. *Poludiskretnye skhemy dlya operatornykh differentsial'nykh uravnenii*. Izdatel’stvo “Tekhnicheskogo Universitets”, Tbilisi, 1995.

J. Rogava. The study of the stability of semidiscrete schemes by means of Chebyshev orthogonal polynomials. *GSSR Mecn. Akad. Moambe*, 83(3):545–548, 1976.

A. A. Samarskiĭ. An efficient difference method for solving a multidimensional parabolic equation in an arbitrary domain. *Zh. Vyčisl. Mat i Mat. Fiz.*, 2:787–811, 1962.

A. A. Samarskiĭ. Locally homogeneous difference schemes for higher-dimensional equations of hyperbolic type in an arbitrary region. *Zh. Vyčisl. Mat i Mat. Fiz.*, 4:638–648, 1964.

A. A. Samarskiĭ and P. N. Vabishchevich. *Additienye skhemy dlya zadach matematicheskoi fiziki*. “Nauka”, Moscow, 1999.

G. Szegő. *Orthogonal polynomials*, volume XXIII of *Colloq. Publ., Am. Math. Soc.* Providence, RI: Am. Math. Soc. (AMS), Fourth edition, 1975.

J. Zhao, R. Zhan, and Y. Xu. The analysis of operator splitting for the Gardner equation. *Appl. Numer. Math.*, 144:151–175, 2019.
Authors’ addresses:

Nana Dikhaminjia
School of Business, Technology and Education, Ilia State University (ISU), Kakutsa Cholokashvili Ave 3/5, Tbilisi 0162, Georgia.
E-mail: nana.dikhaminjia@iliauni.edu.ge

Jemal Rogava
Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University (TSU), Ilia Vekua Institute of Applied Mathematics (VIAM), 2 University St., Tbilisi 0186, Georgia.
E-mail: jemal.rogava@tsu.ge

Mikheil Tsiklauri
Missouri University of Science and Technology, Electromagnetic Compatibility Laboratory, 4000 Enterprise Drive, Rolla, MO 65409, USA.
E-mail: tsiklaurim@mst.edu mtsiklauri@gmail.com