K-theory and index pairings for C*-algebras generated by q-normal operators

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Abstract

The paper presents a detailed description of the K-theory and K-homology of C*-algebras generated by q-normal operators including generators and the index pairing. The C*-algebras generated by q-normal operators can be viewed as a q-deformation of the quantum complex plane. In this sense, we find deformations of the classical Bott projections describing complex line bundles over the 2-sphere, but there are also simpler generators for the $K_0$-groups, for instance 1-dimensional Powers–Rieffel type projections and elementary projections belonging to the C*-algebra. The index pairing between these projections and generators of the even K-homology group is computed, and the result is used to express the $K_0$-classes of the quantized line bundles of any winding number in terms of the other projections.

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# Introduction

In this paper we give a complete description of the K-theory of all possible C*-algebras generated by one of the most prominent relations occurring in the theory of $q$-deformed spaces:

$$zz^* = q^2 z^* z, \quad q \in (0, 1).$$

(1)

The complex *-algebra with generators $z$ and $z^*$ subject to the relation (1) is known as the coordinate ring $\mathcal{O}(\mathbb{C}_q)$ of the quantum complex plane. On the analytic side, a densely defined closed linear operator on a Hilbert space satisfying (1) is called $q$-normal operator. In other words, any $q$-normal operator yields a Hilbert space representation of $\mathcal{O}(\mathbb{C}_q)$. These representations have been classified in [2] and [3]. The results therein include that non-zero $q$-normal operators are never bounded. As a consequence, $\mathcal{O}(\mathbb{C}_q)$ cannot be equipped with a $C^*$-norm, which is consistent with the idea that $\mathcal{O}(\mathbb{C}_q)$ should be viewed as the coordinate ring of a non-compact quantum space. To study non-compact quantum spaces in the $C^*$-algebra setting, one may apply Woronowicz’s theory of $C^*$-algebras generated by unbounded elements [17]. This has been done by the authors in [3]. It turned out that the $C^*$-algebra generated by a $q$-normal operator $z$ depends only on the spectrum of the self-adjoint operator $|z| = \sqrt{z^* z}$. Among all these $C^*$-algebras, there is a universal one, namely when the spectrum of $|z|$ coincides with the whole interval $[0, \infty)$. This algebra is viewed as the algebra of continuous functions vanishing at infinity on the quantum complex plane and is denoted by $C_0(\mathbb{C}_q)$.

A natural question is if the passage from the commutative $C^*$-algebra $C_0(\mathbb{C})$ to the $q$-deformed version $C_0(\mathbb{C}_q)$ preserves topological invariants. Preserving topological invariants gives another justification for calling $C_0(\mathbb{C}_q)$ the algebra of continuous functions vanishing at infinity on the quantum complex plane. Nevertheless, we are also interested in detecting quantum effects, i.e., situations where the computations of invariants differ from the classical case. In favorable situations, the quantum case even leads to a simplification.

To provide answers to these questions, we compute the K-theory for all $C^*$-algebras generated by $q$-normal operators. Our results in Theorem 1 show that $C_0(\mathbb{C}_q)$ does actually have the same K-groups as the commutative $C^*$-algebra $C_0(\mathbb{C})$. The analogy goes even further since the non-zero elements of $K_0(C_0(\mathbb{C}_q))$ can be described by $q$-deformed versions of the classical Bott projections describing complex line bundles of winding number $n \in \mathbb{Z}$ over the 2-sphere. However, as a quantum effect, the $K_0$-classes can also be given by 1-dimensional Power–Rieffel type projections in the $C^*$-algebra $C_0(\mathbb{C}_q)$. Note that $C_0(\mathbb{C})$ does not contain non-trivial projections since $\mathbb{C}$ is connected.
The situation changes if \( \text{spec}(|z|) \neq [0, \infty) \). Then the \( K_1 \)-group of the C*-algebra generated by the \( q \)-normal operator \( z \) is trivial as in the classical case but the \( K_0 \)-group depends on the number of gaps in the set \( \text{spec}(|z|) \cap [q, 1] \), and any of the groups \( \mathbb{Z}^n \), \( n > 1 \), as well as \( \oplus_{n \in \mathbb{N}} \mathbb{Z} \) can occur (Theorem 6). Moreover, the \( K_0 \)-groups are generated by elementary 1-dimensional projections belonging to the C*-algebra. If one wants to consider all C*-algebras generated by a \( q \)-normal operator as deformations of the complex plane, then quantization leads to a whole family of quantum spaces with different topological properties and also simplifies the description of generators of the \( K_0 \)-groups.

A practical method of determining \( K_0 \)-classes is by computing the index pairing with K-homology classes. We present for all C*-algebras generated by a \( q \)-normal operator a set of generators for the (even) K-homology group and show that it gives rise to a non-degenerate pairing with the \( K_0 \)-group. The index pairing between these generators and all the projections mentioned above is computed (Theorem 9 and Theorem 12), and the result is used to express the \( K_0 \)-class of the quantized line bundles of winding number \( n \in \mathbb{Z} \) in terms of the different projections (Corollary 13). Moreover, one can find generators of the even K-homology group that compute exactly the rank or the winding number of the quantized line bundles. Remarkably, for the elementary projections, the computation of the winding number boils down to its simplest form: the computation of a trace of a projection onto a finite dimensional subspace. Thus one can say that quantization leads to a significant simplification of the index computation.

The description of the K-groups is only the first step, albeit an essential one, of the larger program of understanding the noncommutative geometry of the quantum complex plane. A further step would be to find a Dirac operator satisfying the axioms of a spectral triple which might be a noncommutative analogue of the Dirac operator on \( \mathbb{R}^2 \) with the flat metric or of the Dirac operator on the Riemannian 2-sphere in local coordinates. In view of a possible \( q \)-deformed differential calculus associated to the commutation relation (1), it seems to be natural to look for a twisted spectral triple in the sense of Connes and Moscovici [5]. However, this admittedly more difficult problem is beyond the scope of the present paper.

## 2 C*-algebras generated by q-normal operators

In this section we collect the most important facts on C*-algebra generated by q-normal operators from [3] (see also [2]). Let \( q \in (0, 1) \) and \( z \) be a \( q \)-normal operator, that is, \( z \) is a densely defined closed linear operator on a Hilbert space \( \mathcal{H} \) such that (1) holds on \( \text{dom}(z^*z) = \text{dom}(zz^*) \). By [3, Corollary 2.2], the Hilbert space \( \mathcal{H} \) de-
composes into the direct sum \( \mathcal{H} = \ker(z) \oplus \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \), where, up to unitary equivalence, we may assume \( \mathcal{H}_n = \mathcal{H}_0 \). For \( h \in \mathcal{H}_0 \) and \( n \in \mathbb{Z} \), let \( h_n \) denote the vector in \( \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \) which has \( h \) in the \( n \)-th component and 0 elsewhere. Then the action of \( z \) on \( \mathcal{H} \) is determined by

\[
z = 0 \text{ on } \ker(z), \quad z h_n = q^n (Ah)_{n-1} \text{ on } \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n,
\]

(2)

where \( A \) is a self-adjoint operator \( A \) on \( \mathcal{H}_0 \) such that \( \text{spec}(A) \subset [q, 1] \) and \( q \) is not an eigenvalue. A non-zero representation of a \( q \)-normal operator \( z \) is irreducible if and only if \( \ker(z) = \{0\} \) and \( \mathcal{H}_0 = \mathbb{C} \). In this case, \( A \) can be viewed as a real number in \( (q, 1] \).

In [3, Section 3], it has been shown that the C*-algebra generated in the sense of Woronowicz [17] by a \( q \)-normal operator \( z \) depends only on the spectrum

\[
X := \text{spec}(|z|) = \{0\} \cup \cup_{n \in \mathbb{Z}} q^n \text{spec}(A) \subset [0, \infty).
\]

(3)

More than that, it can be described as a C*-subalgebra of the crossed product algebra \( C_0(X) \rtimes \mathbb{Z} \) without referring to the Hilbert space \( \mathcal{H} \). Here, for any \( q \)-invariant locally compact subset \( X \subset [0, \infty] \) (such as \( \text{spec}(|z|) \)), the \( \mathbb{Z} \)-action is given by the automorphism

\[
\alpha_q : C_0(X) \longrightarrow C_0(X), \quad \alpha_q(f)(x) := f(qx).
\]

(4)

Recall that the crossed product algebra \( C_0(X) \rtimes \mathbb{Z} \) can be described as enveloping C*-algebra generated by functions \( f \in C_0(X) \) and a unitary operator \( U \) subjected to the relation

\[
U^* f U = \alpha_q(f).
\]

By [3, Theorem 3.2], the C*-algebra generated by a non-zero \( q \)-normal operator \( z \) is isomorphic to

\[
C^*_0(z, z^*) := \| \cdot \|\text{-cls} \left\{ \sum_{\text{finite}} f_k U^k \in C_0(X) \rtimes \mathbb{Z} : k \in \mathbb{Z}, \ f_k(0) = 0 \text{ if } k \neq 0 \right\},
\]

(5)

where \( X = \text{spec}(|z|) \) and \( \| \cdot \|\text{-cls} \) denotes the norm closure in \( C_0(X) \rtimes \mathbb{Z} \). The case \( \text{spec}(|z|) = [0, \infty) \) has the universal property that

\[
C_0([0, \infty)) \rtimes \mathbb{Z} \ni \sum_{\text{finite}} f_k U^k \longmapsto \sum_{\text{finite}} f_k|_X U^k \in C_0(X) \rtimes \mathbb{Z}
\]
yields a well-defined *-homomorphism of crossed product algebras which restricts to the corresponding C*-subalgebras defined in (5). This was one of the motivations in [3] to define the C*-algebra algebra of continuous functions vanishing at infinity on a quantum complex plane as the C*-algebra generated by a $q$-normal operator $z$ satisfying spec$(|z|) = [0, \infty)$, i.e.,

$$C_0(\mathbb{C}_q) := \| \cdot \|_{\text{cls}} \left\{ \sum_{\text{finite}} f_k U^k \in C_0([0, \infty)) \rtimes \mathbb{Z} : f_k(0) = 0 \text{ if } k \neq 0 \right\}. \quad (6)$$

Furthermore, its unitization

$$C(S^2_q) := C_0(\mathbb{C}_q) + \mathbb{C}1 \quad (7)$$

is viewed as the C*-algebra of continuous functions on a quantum 2-sphere obtained from a one-point compactification of the quantum complex plane. To distinguish $C_0(\mathbb{C}_q)$ from the C*-algebras $C_0^* (z, z^*)$ generated by $q$-normal operator $z$ such that spec$(|z|) \neq [0, \infty)$, we call the latter case generic.

By (2), $0 \in X := \text{spec}(|z|)$ for any $q$-normal operator $z$. As 0 is invariant under multiplication by $q$,

$$\text{ev}_0 : \left\{ \sum_{\text{finite}} f_k U^k : f_k \in C_0(X) \right\} \rightarrow \mathbb{C}, \quad \text{ev}_0 \left( \sum_{\text{finite}} f_k U^k \right) = \sum_{\text{finite}} f_k(0) \quad (8)$$

yields a well-defined *-homomorphism, in particular a so-called covariant representation, where one may set $\text{ev}_0(U) := 1$. Since the C*-norm of the enveloping C*-algebra is given by taking the supremum of the operator norms over all *-representations [16], the map (8) is norm decreasing and thus extends to a continuous *-homomorphism $\text{ev}_0 : C_0(X) \rtimes \mathbb{Z} \rightarrow \mathbb{C}$. From the definitions of $C_0(X) \rtimes \mathbb{Z}$, $C_0^* (z, z^*)$ and $\text{ev}_0$, we get the following exact sequence of C*-algebras:

$$0 \rightarrow C_0(X \setminus \{0\}) \rtimes \mathbb{Z} \overset{\iota}{\rightarrow} C_0^* (z, z^*) \overset{\text{ev}_0}{\rightarrow} \mathbb{C} \rightarrow 0. \quad (9)$$

For $C_0^* (z, z^*) \cong C_0(\mathbb{C}_q) \subset C_0([0, \infty)) \rtimes \mathbb{Z}$, this exact sequence becomes

$$0 \rightarrow C_0([0, \infty)) \rtimes \mathbb{Z} \overset{\iota}{\rightarrow} C_0(\mathbb{C}_q) \overset{\text{ev}_0}{\rightarrow} \mathbb{C} \rightarrow 0. \quad (10)$$

Note finally that we can identify the unit in $C_0^* (z, z^*) + \mathbb{C}1$ with the constant function $1 \in C(X \cup \{\infty\})$, $1(x) = 1$. Then

$$C_0^* (z, z^*) + \mathbb{C}1 = \| \cdot \|_{\text{cls}} \left\{ \sum_{\text{finite}} f_k U^k \in C(X \cup \{\infty\}) \rtimes \mathbb{Z} : k \in \mathbb{Z}, \ f_k(0) = f_k(\infty) = 0 \text{ if } k \neq 0 \right\}, \quad (11)$$
and the natural projection \( C^*_0(z, z^*) + \mathbb{C} 1 \rightarrow (C^*_0(z, z^*) + \mathbb{C} 1)/C^*_0(z, z^*) \cong \mathbb{C} \) can be written
\[
ev_\infty : C^*_0(z, z^*) + \mathbb{C} 1 \rightarrow \mathbb{C}, \quad \ev_\infty \left( \sum_{\text{finite}} f_k U^k \right) = f_0(\infty) . \tag{12}
\]
The formula (8) remains unchanged for the unitalization \( C^*_0(z, z^*) + \mathbb{C} \).

3 K-theory

3.1 K-theory of the quantum complex plane

There are several ways to compute the K-theory of \( C_0(\mathbb{C}_q) \). To keep the paper elementary, we will use the standard six-term exact sequence in K-theory for C*-algebra extensions. We could have used Exel’s generalized Pimsner–Voiculescu six-term exact sequence for generalized crossed product algebras defined by partial automorphisms [6] but the gain would be minor at the cost of introducing more terminology.

The C*-algebra extension (10) yields the following six-term exact sequence of K-theory:
\[
\begin{array}{ccc}
K_0(C_0((0, \infty)) \rtimes \mathbb{Z}) & \xrightarrow{i_*} & K_0(C_0(\mathbb{C}_q)) \xrightarrow{\ev_{0*}} K_0(\mathbb{C}) \\
\delta_{10} & & \delta_{01} \\
K_1(\mathbb{C}) & \leftarrow \xrightarrow{\ev_{0*}} & K_1(C_0(\mathbb{C}_q)) & \leftarrow i_1 & K_1(C_0((0, \infty)) \rtimes \mathbb{Z}).
\end{array}
\tag{13}
\]

To resolve (13), we need to know \( K_0(C_0((0, \infty)) \rtimes \mathbb{Z}) \) and \( K_1(C_0((0, \infty)) \rtimes \mathbb{Z}) \). These K-groups can easily obtained from the Pimsner–Voiculescu six-term exact sequence [1, Theorem 10.2.1] by noting that the automorphism \( \alpha_q \) is homotopic to \( \text{id} = \alpha_1 = \lim_{q \to 1} \alpha_q \) so that the induced group homomorphisms yield \( (\alpha_q)_* - \text{id} = 0 \). Another way of deducing these K-groups would be to consider a continuous field of C*-algebras (see e.g. [12]), or to use the fact that \( \mathbb{Z} \) acts freely and properly on \( (0, \infty) \) so that \( C_0((0, \infty)) \rtimes \mathbb{Z} \) is Morita equivalent to \( C(\mathbb{R}/\mathbb{Z}) \cong C(S^1) \) (see e.g. [16, Remark 4.16]). In any case, the outcome is
\[
K_0(C_0((0, \infty)) \rtimes \mathbb{Z}) = \mathbb{Z}, \quad K_1(C_0((0, \infty)) \rtimes \mathbb{Z}) = \mathbb{Z} . \tag{14}
\]
The isomorphism \( K_1(C_0((0, \infty)) \rtimes \mathbb{Z}) \cong K_1(C_0((0, \infty))) \) in the Pimsner–Voiculescu six-term exact sequence also shows that a generator of \( K_1(C_0((0, \infty)) \rtimes \mathbb{Z}) \) may be given by the \( K_1 \)-class of the unitary
\[
e^{-2\pi i h} \in C_0((0, \infty)) + \mathbb{C} \subset (C_0((0, \infty)) \rtimes \mathbb{Z}) + \mathbb{C}, \tag{15}
\]
where \( h : [0, \infty) \to \mathbb{R} \) denotes a continuous function such that \( h(0) = 1 \) and \( \lim_{t \to \infty} h(t) = 0 \). As \( h \in C_0(\mathbb{R}_+) \subset C_0(\mathbb{C}_q) \) is a self-adjoint lift of the trivial projection \( 1 \in \mathbb{C}_q \) under \( \text{ev}_0 \), we get \( \delta_0(1) = [e^{-2\pi i h}] \in K_1(C_0((0, \infty)) \rtimes \mathbb{Z}) \), see e.g. [15, p. 172]. Since \( \delta_0 \) maps a generator into a generator, it is an isomorphism and thus the adjacent homomorphisms \( \text{ev}_0^* \) and \( \iota^* \) in (13) are 0.

Inserting \( K_0(\mathbb{C}) = \mathbb{Z}, K_1(\mathbb{C}) = 0 \) and (14) into (13) yields \( K_0(C_0(\mathbb{C}_q)) \cong \mathbb{Z} \) and \( K_1(C_0(\mathbb{C}_q)) \cong 0 \). Adjoining furthermore a unit to \( C_0(\mathbb{C}_q) \), we have proven the following theorem:

**Theorem 1.** The \( K \)-groups of the \( C^* \)-algebras \( C_0(\mathbb{C}_q) \) and \( C(S^2_q) \) from Equations (6) and (7), respectively, are given by

\[
\begin{align*}
K_0(C_0(\mathbb{C}_q)) &\cong \mathbb{Z}, & K_0(C(S^2_q)) &\cong \mathbb{Z} \oplus \mathbb{Z}, \\
K_1(C_0(\mathbb{C}_q)) &\cong 0, & K_1(C(S^2_q)) &\cong 0.
\end{align*}
\]

**Remark 2.** In Corollary [11] below, we will show that generators for \( K_0(C_0(\mathbb{C}_q)) \) are given by the \( K_0 \)-classes \([P_{\pm 1}] - [1]\) with the Bott projections \( P_{\pm 1} \) defined in (29) and (30), and by the \( K_0 \)-class \([R_1]\) with the Powers–Rieffel type projection \( R_1 \) defined in (32). As a trivial consequence, each of these elements together with \([1]\) generate \( K_0(C(S^2_q)) \).

**Remark 3.** Note that the \( K \)-groups in Theorem 1 are isomorphic to the classical counterparts since \( K_0(C_0(\mathbb{C})) = \mathbb{Z}, K_1(C_0(\mathbb{C})) = 0, K_0(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z} \) and \( K_1(C(S^2)) = 0 \).

### 3.2 K-theory of \( C^* \)-algebras generated by generic \( q \)-normal operators

In this section, we describe the K-theory of the \( C^* \)-algebra \( C^*_q(z, z^*) \) defined in (5), where \( z \) is a \( q \)-normal operator such that \( X \subseteq \text{spec}(|z|) \neq [0, \infty) \). By replacing \( z \) by \( tz \), \( t > 0 \), we may assume that \( 1 \notin \text{spec}(|z|) \). From the \( q \)-invariance of \( \text{spec}(|z|) \) and the properties of the self-adjoint operator \( A \) described below (2), we conclude that \( Y := \text{spec}(|z|) \cap (q, 1) = \text{spec}(A) \) is a compact subset of \((q, 1)\) and \( X = \{0\} \cup \bigcup_{n \in \mathbb{Z}} q^n Y \), see (8).

Similar to the previous section, we consider the standard six-term exact sequence...
associated to the C*-algebra extension \((\mathfrak{F})\), i.e.,

\[
\begin{array}{ccc}
K_0(C_0(X\setminus\{0\}) \rtimes \mathbb{Z}) & \xrightarrow{\iota_*} & K_0(C_0^*(z, z^*)) \\
\delta_{10} & & \delta_{01} \\
K_1(\mathbb{C}) & \xleftarrow{\ev_0} & K_1(C_0^*(z, z^*)) & \xrightarrow{\lambda_*} & K_1(C_0(X\setminus\{0\}) \rtimes \mathbb{Z})
\end{array}
\]  \tag{18}

Now we need to know the K-groups of the crossed product algebra \(C_0(X\setminus\{0\}) \rtimes \mathbb{Z}\). Clearly, the \(\mathbb{Z}\)-action on the disjoint union \(X\setminus\{0\} = \bigcup_{n \in \mathbb{Z}} q^n Y\) is free and proper, and the quotient space \((X\setminus\{0\})/\mathbb{Z}\) can be identified with \(Y\). It follows from \([8\text{ Corollary 15}]\) that \(C_0(X\setminus\{0\}) \rtimes \mathbb{Z} \cong C(Y) \otimes K(\ell_2(\mathbb{Z}))\). By C*-stabilization, we have

\[
K_i(C_0(X\setminus\{0\}) \rtimes \mathbb{Z}) \cong K_i(C(Y) \otimes K(\ell_2(\mathbb{Z}))) \cong K_i(C(Y)), \quad i = 0, 1, \tag{19}
\]

where a set of generators of \(K_i(C_0(X\setminus\{0\}) \rtimes \mathbb{Z})\) is given by a set of generators of \(K_i(C(Y))\) under the embeddings \(C(Y) \subset C_0(X\setminus\{0\}) \subset C_0(X\setminus\{0\}) \rtimes \mathbb{Z}\). So we are reduced to determining the K-theory of \(C(Y)\).

The K-groups of \(C(Y)\) for arbitrary compact planar sets \(Y \subset \mathbb{C}\) are well known and a characterization of them can be found, e.g., in \([9\text{ Section 7.5}]\). For an explicit description of the generators and later reference, we will introduce some notation in the next remark and then state the result in a proposition.

**Remark 4.** Let \(Y \subset (q, 1)\) be a compact set and let \(s \in (q, 1)\) denote the maximum of \(Y\). Consider the family \(\{I_j : j \in J\}\) of connected components of \((q, s) \setminus Y\). These connected components are of course open intervals in \((q, 1)\). If \(Y\) has a finite number of connected components, say \(n \in \mathbb{N}\), then \(J\) has the same number of elements and we may choose \(J = \{1, \ldots, n\}\). If \(Y\) has an infinite (possibly uncountable) number of connected components, then \(J\) is countable infinite and we may take \(J = \mathbb{N}\). For each \(j \in J\), we choose a \(c_j \in I_j\) so that we arrive at the following situation:

\[
Y^c := (q, 1) \setminus Y = (s, 1) \cup \bigcup_{j \in J} I_j, \quad I_j \cap I_k = \emptyset \text{ if } j \neq k, \quad c_j \in I_j \subset (q, s). \quad \tag{20}
\]

Moreover, we will frequently use the indicator function of a subset \(A \subset \mathbb{R}\) given by \(\chi_A(t) := 1\) for \(t \in A\) and 0 otherwise. Note that \(\chi_{(x,y)}\) is a continuous projection for all \(x, y \in Y^c := (q, 1) \setminus Y\), i.e., \(\chi_{(x,y)} \in C(Y)\) and \((\chi_{(x,y)})^2 = \chi_{(x,y)} = (\chi_{(x,y)})^*\).

**Proposition 5.** Let \(Y \subset (q, 1)\) be a non-empty compact set. For all such sets,

\[
K_1(C(Y)) = 0. \tag{21}
\]


If $Y$ has $n \in \mathbb{N}$ connected components, then

$$K_0(C(Y)) \cong \mathbb{Z}^n.$$  \hfill (22)

If $Y$ has an infinite number of connected components, then

$$K_0(C(Y)) \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z} \quad \text{(infinite direct sum).}$$  \hfill (23)

For any choice of real numbers $c_j \in I_j$ as in Remark 4, the equivalence classes of the projections $\chi_{(c_j, 1)} \in C(Y)$, $j \in J$, generate freely $K_0(C(Y))$.

Proof. Equation (21) follows immediately from [9, Propositions 7.5.2 and 7.5.3] since $\mathbb{C} \setminus Y$ has no bounded connected component.

For a description of $K_0(C(Y))$ by a complete set of generators, we first identify projections $P \in \text{Mat}_k(C(Y))$, $k \in \mathbb{N}$, with projection-valued continuous functions $P : Y \to \text{Mat}_k(\mathbb{C})$. As in [9, Definition 7.5.1], let $\tilde{H}^0(Y, \mathbb{Z})$ denote the group of continuous, integer-valued functions on $Y$. By [9, Proposition 7.5.2], the map

$$h_0 : K_0(C(Y)) \to \tilde{H}^0(Y, \mathbb{Z}), \quad h_0([P])((t)) := \text{rank}(P(t)),$$  \hfill (24)

is an isomorphism. Since the continuous function $h_0([P])$ takes values in a discrete set, it is locally constant. By the compactness of $Y$, it can only have a finite number of jumps and each jump can only occur if the distance of neighboring points is greater than 0. Hence there exist $c_{j_1}, \ldots, c_{j_{k_0}} \in (q, 1) \setminus Y$ such that $c_k$ belongs to the connected component $I_{j_k}$ as described in (20); $c_{j_1} < \cdots < c_{j_{k_0}}$ and $h_0([P])$ is constant on $(c_{j_k}, c_{j_{k+1}}) \cap Y$ for $k = 1, \ldots, k_0$, where we set $c_{j_{k_0}+1} := 1$. Let $n_k \in \mathbb{N}_0$ such that $h_0([P])((t)) = n_k \in \mathbb{N}_0$ for all $t \in (c_{j_k}, c_{j_{k+1}}) \cap Y$. Then

$$h_0([P]) = \sum_{k=1}^{k_0} n_k \chi(c_{j_k}, c_{j_{k+1}}) = n_1 \chi(c_{j_1}, 1) + \sum_{k=2}^{k_0} (n_k - n_{k-1}) \chi(c_{j_k}, 1).$$  \hfill (25)

Define $[p] \in K_0(C(Y))$ by

$$[p] := \sum_{k=1}^{k_0} n_k \chi(c_{j_k}, c_{j_{k+1}}) = n_1 \chi(c_{j_1}, 1) + \sum_{k=2}^{k_0} (n_k - n_{k-1}) \chi(c_{j_k}, 1).$$  \hfill (26)

Since obviously $h_0([p]) = h_0([P])$, it follows from the injectivity of $h_0$ that $[p] = [P]$ in $K_0(C(Y))$.

Now consider the group homomorphism

$$\Phi : \bigoplus_{j \in J} \mathbb{Z} \to K_0(C(Y)), \quad \Phi((g_j)_{j \in J}) := \sum_{k=1}^{N} g_{n_k} \chi(c_{j_k}, 1),$$  \hfill (27)
where \( N \in \mathbb{N} \) and \( g_{n_1}, \ldots, g_{n_N} \) are the non-zero elements of \((g_j)_{j \in J} \in \bigoplus_{j \in J} \mathbb{Z} \setminus \{0\}\). From the representation (26) of any \( K_0 \)-element, one sees immediately that \( \Phi \) is surjective. The injectivity follows from the linear independence of the set of functions \( \{\chi(c_{n_j}, 1) : j \in J\} \) since \( h_0(\Phi((g_j)_{j \in J})) = \sum_{k=1}^N g_{n_k} \chi(c_{n_k}, 1) \) and \( \ker(h_0) = \{0\} \). Hence \( \Phi \) defines an isomorphism and \( \{[\chi(c_{j}, 1)] = \Phi((\delta_{ji})_{i \in I}) : j \in J\} \) yields a set of generators.

Returning to the computation of the \( K \)-groups of \( C_0^\ast(z, z^*) \), we can now insert the results of Proposition \( 5 \) and Equation (19) into the six-term exact sequence (18). Then the second line has only trivial groups in the corners, thus \( K_1(C_0^\ast(z, z^*)) \cong 0 \). The remaining short exact sequence is obviously split exact with a splitting homomorphism \( \sigma : K_0(\mathbb{C}) \rightarrow K_0(C_0^\ast(z, z^*)) \) given by sending the generator \([1] \in K_0(\mathbb{C})\) to class of the continuous projection \( \chi(0, q) \in C_0(Y) \subset C_0^\ast(z, z^*) \). Thus, to obtain \( K_0(C_0^\ast(z, z^*)) \), one only has to add one free generator to \( K_0(C_0(Y \setminus \{0\}) \times \mathbb{Z}) \cong K_0(C(Y)) \), for instance, we may take \([\chi(0, q)] = \sigma([1])\). This proves the following theorem.

**Theorem 6.** Let \( z \) be a \( q \)-normal operator such that \( X := \text{spec}(\|z\|) \neq [0, \infty) \) and assume without loss of generality that \( q, 1 \notin X \). If \((q, 1) \cap X \) has \( n \in \mathbb{N} \) connected components, then

\[
K_0(C_0^\ast(z, z^*)) \cong \mathbb{Z}^{n+1}, \quad K_1(C_0^\ast(z, z^*)) = 0.
\]

If \((q, 1) \cap X \) has an infinite number of connected components, then

\[
K_0(C_0^\ast(z, z^*)) \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z} \quad \text{(infinite direct sum)}, \quad K_1(C_0^\ast(z, z^*)) = 0.
\]

A set of generators for \( K_0(C_0^\ast(z, z^*)) \) is given by \([\chi(0, q)]\) and \([\chi(c_{j}, 1)]\), \( j \in J \), where \( J \) and \( c_j \in (q, 1) \) are defined as in Remark 4, for \( Y := (q, 1) \cap X \).

**Remark 7.** For all \( q \)-normal operators \( z \) such that \( X := \text{spec}(\|z\|) \neq [0, \infty) \), we see by Theorem 6 that \( K_0(C_0^\ast(z, z^*)) \) contains more copies of \( \mathbb{Z} \) than \( K_0(C_0(\mathbb{C})) \cong \mathbb{Z} \) since \((q, 1) \cap X \) has at least one connected component. This observation justifies the definition of \( C_0(\mathbb{C}) \) as the \( \mathcal{C}^\ast \)-algebra generated by a \( q \)-normal operator \( z \) such that \( \text{spec}(\|z\|) = [0, \infty) \) because only then the equalities \( K_0(C_0(\mathbb{C})) = K_0(C_0(\mathbb{C})) \) and \( K_1(C_0(\mathbb{C})) = K_1(C_0(\mathbb{C})) \) hold.

However, if one wants to consider the \( \mathcal{C}^\ast \)-algebras \( C_0^\ast(z, z^*) \) from Theorem 6 also as algebras of continuous functions vanishing at infinity on a quantum complex plane, then, by [3, Corollary 2.4], all the abelian groups \( \mathbb{Z}^n \), \( n \in \mathbb{N} \), as well as \( \bigoplus_{n \in \mathbb{N}} \mathbb{Z} \) can occur as a \( K_0 \)-group of a quantum complex plane.
3.3 Bott projections and Powers–Rieffel type projections

Our interest in the Bott projections lies in the observation that they can be viewed as representing noncommutative complex line bundles of any winding number. These projections are given by $2 \times 2$-matrices whose entries are rational functions in the generators $z$ and $z^*$. Taking advantage of the noncommutativity of the involved crossed product algebras, we can also find non-trivial 1-dimensional projections belonging to the C*-algebra. The defining formulas for the 1-dimensional projections are completely analogous to the so-called Powers-Rieffel projections for the irrational rotation C*-algebra (noncommutative torus) $C(S^1) \rtimes \mathbb{Z}$. The Bott projections and Powers–Rieffel type projections exist for all C*-algebras $C_0^*(z, z^*)$ generated by a $q$-normal operator $z$, but for $C_0(C_q)$ they take on an added significance because they are used to express all $K_0$-classes of $C_0(C_q)$. This will be shown in the next section by computing the index pairing. As can be seen in Theorem 6, there are much more elementary projections in $C_0^*(z, z^*)$ if $\text{spec}(|z|) \neq [0, \infty)$. The relations between the Bott projections, Powers–Rieffel type projections and the elementary projections will be revealed in the next section by computing the index pairing.

By classical Bott projections, we mean the following projections representing line bundles of winding number $\pm n \in \mathbb{Z}$ over the classical 2-sphere [7, Section 2.6]:

\[ p_n := \frac{1}{1 + z^n z^*} \begin{pmatrix} z^n & z^n \\ z^n & 1 \end{pmatrix} = \frac{1}{1 + z^n z^*} \begin{pmatrix} z^n \\ 1 \end{pmatrix} \begin{pmatrix} z^n & 1 \end{pmatrix}, \]
\[ p_{-n} := \frac{1}{1 + z^n z^*} \begin{pmatrix} z^n & z^n \\ z^n & 1 \end{pmatrix} = \frac{1}{1 + z^n z^*} \begin{pmatrix} z^n \\ 1 \end{pmatrix} \begin{pmatrix} z^n & 1 \end{pmatrix}, \quad n \in \mathbb{N}_0. \]

Setting $v_n := \frac{1}{\sqrt{1 + z^n z^*}} \begin{pmatrix} z^n & 1 \end{pmatrix}$ and $v_{-n} := \frac{1}{\sqrt{1 + z^n z^*}} \begin{pmatrix} z^n & 1 \end{pmatrix}$ for $n \in \mathbb{N}_0$, we can write $p_k = v_k^* v_k$ for all $k \in \mathbb{Z}$. From the simple observation that $v_k v_k^* = 1$, it follows that $v_k$ is a partial isometry and thus $p_k$ is a projection.

We apply the same ideas to define Bott projections in the non-commutative case, the only difference being the replacement of the unbounded continuous function $z \in C(\mathbb{C})$ by the unbounded $q$-normal operator $z : \text{dom}(z) \subset \mathcal{H} \rightarrow \mathcal{H}$. For $n \in \mathbb{N}_0$, let

\[ V_n := \frac{1}{\sqrt{1 + z^n z^*}} \begin{pmatrix} z^n & 1 \end{pmatrix}, \quad V_{-n} := \frac{1}{\sqrt{1 + z^n z^*}} \begin{pmatrix} z^n & 1 \end{pmatrix}, \quad P_{\pm n} := V_{\pm n} V_{\pm n}. \quad (28) \]

Clearly, $V_{\pm n} V_{\pm n}^* = 1$, hence $P_{\pm n}$ is a self-adjoint projection. Writing $z$ in its polar decomposition $z = U|z|$ and using $U f(|z|) U^* = f(q|z|) = \alpha_q(f)(|z|)$ for every Borel
function $f$ on $[0, \infty)$ [2 Proposition 1], one computes

$$P_n = \begin{pmatrix} \frac{q^{-n+1}|z|^{2n}}{1 + q^{-n+1}|z|^{2n}} & \frac{q^{-n}|z|^n}{1 + q^{-n}|z|^n} U_{2n} \\ \frac{q^{n+1}|z|^{2n}}{1 + q^n|z|^{2n}} & \frac{q^n|z|^{2n}}{1 + q^{n+1}|z|^{2n}} \end{pmatrix},$$  \hspace{1cm} (29)

$$P_{-n} = \begin{pmatrix} \frac{q^{n+1}|z|^{2n}}{1 + q^n|z|^{2n}} & \frac{q^{-n}|z|^n}{1 + q^{-n+1}|z|^{2n}} U_{2n} \\ \frac{q^{-n+1}|z|^{2n}}{1 + q^{-n}|z|^{2n}} & \frac{q^{-n}|z|^n}{1 + q^{-n+1}|z|^{2n}} \end{pmatrix}.$$  \hspace{1cm} (30)

By identifying the rational functions in $|z|$ with continuous functions on $\text{spec}(|z|)$, we can view $P_n$ and $P_{-n}$ as projections in $\text{Mat}_2(C_0^*(z, z^*) + \mathbb{C})$. Furthermore, they present $K_0$-classes $[P_n] - [1], [P_{-n}] - [1] \in K_0(C_0^*(z, z^*))$ and $[P_n], [P_{-n}] \in K_0(C_0^*(z, z^*) + \mathbb{C})$.

Now we will construct 1-dimensional projections in $C_0^*(z, z^*)$ similar to the Powers–Rieffel projections in the irrational rotation $C^*$-algebra [11]. To this end, choose a continuous function

$$\phi : [q, 1] \longrightarrow \mathbb{R} \hspace{0.5cm} \text{such that} \hspace{0.5cm} 0 \leq \phi \leq 1, \hspace{0.5cm} \phi(q) = 0, \hspace{0.5cm} \phi(1) = 1,$$

and define for $n \in \mathbb{N}$

$$h(t) := \begin{cases} \sqrt{\phi(t)}(1 - \phi(t)), & t \in [q, 1], \\ 0, & t \notin [q, 1], \end{cases} \hspace{1cm} f_n(t) := \begin{cases} \phi(t), & t \in [q, 1], \\ 1, & t \in (1, q^{n+1}), \\ 1 - \phi(q^n t), & t \in [q^{-n+1}, q^{-n}], \\ 0, & t \notin [q, q^{-n}]. \end{cases}$$  \hspace{1cm} (31)

With $U$ denoting the unitary element from $C_0(X) \rtimes \mathbb{Z}$ implementing the $\mathbb{Z}$-action, let

$$R_n := U^n h + f_n + h U_{2n}, \hspace{1cm} n \in \mathbb{N}. \hspace{1cm} (32)$$

Since $h, f_n \in C_0(X)$ and $h(0) = 0$, we have $R_n \in C_0^*(z, z^*)$. Obviously, $R_n^* = R_n$. Furthermore, direct computations show that $R_n^2 = R_n$. Hence $[R_n] \in K_0(C_0^*(z, z^*))$ defines a $K_0$-class, and one can write $-[R_n] = [1 - R_n] - [1]$ with the 1-dimensional projection $1 - R_n \in C_0^*(z, z^*) + \mathbb{C}$.

\section{K-homology and index pairings}

\subsection{Fredholm modules}

Even and odd Fredholm modules for a $C^*$-algebra $\mathcal{A}$ define equivalence classes in the $K$-homology groups $K^0(\mathcal{A})$ and $K^1(\mathcal{A})$, respectively. As we are interested in the
index pairing of Fredholm modules with K-theory, and as it has been shown that the $K_1$-groups of C*-algebras generated by a q-normal operator are trivial, we will only consider even Fredholm modules which pair with $K\pi$ index pairing of Fredholm modules with K-theory, and as it has been shown that the $K\pi$ bounded *-representations

Let $p$ be a Fredholm operator and the index map

$$\langle[(\pi_-, \pi_+)], [p]\rangle := \text{ind}(\pi_+(p)|_{\pi_-(p)\mathcal{H}^N})$$

(33)

defines a pairing between $K^0(\mathcal{A})$ and $K_0(\mathcal{A})$. Provided that $\pi_-(p) - \pi_+(p)$ is of trace class, the index pairing can be computed by the trace formula

$$\langle[(\pi_-, \pi_+)], [p]\rangle = \text{Tr}_\mathcal{H}\left(\text{Tr}_{\text{Mat}_N(\mathcal{A})}(\pi_-(p) - \pi_+(p))\right)$$

(34)

see e.g. [4] or [7].

For the index pairing (33) to be well-defined, one does not need to assume that $\pi_+$ and $\pi_-$ are unital representations. In particular, consider the *-homomorphisms $\text{ev}_\infty$, $\text{ev}_0 : C_0^*(z, z^*) + \mathbb{C} \rightarrow \mathbb{C}$, where $\text{ev}_\infty$ was defined in (12) and $\text{ev}_0$ denotes the extension of the homomorphism $\mathcal{A}$ to $C_0^*(z, z^*) + \mathbb{C}$. Setting $\mathcal{H}_- = \mathcal{H}_+ := \mathbb{C}$, the pairs $(\text{ev}_\infty, 0)$ and $(\text{ev}_0, 0)$ yield trivially Fredholm modules for $C_0^*(z, z^*) + \mathbb{C}$ and the trace formula (34) reads

$$\langle[(\text{ev}_\infty, 0)], [p]\rangle = \text{Tr}_{\text{Mat}_N(\mathbb{C})}(\text{ev}_\infty(p)), \quad \langle[(\text{ev}_0, 0)], [p]\rangle = \text{Tr}_{\text{Mat}_N(\mathbb{C})}(\text{ev}_0(p)).$$

(35)

Note that (35) computes the rank of the projections $\text{ev}_\infty(p)$, $\text{ev}_0(p) \in \text{Mat}_N(\mathbb{C})$. As $\mathbb{C} \cong (C_0^*(z, z^*) + \mathbb{C})/C_0^*(z, z^*)$ corresponds to evaluating functions at the classical point $\infty$, we can view the number $\text{Tr}_{\text{Mat}_N(\mathbb{C})}(\text{ev}_\infty(p))$ as the rank of the noncommutative vector bundle determined by $p \in \text{Mat}_N(C_0^*(z, z^*) + \mathbb{C})$ in the spirit of the Serre–Swan theorem.

Less trivially, the next proposition associates a Fredholm module to any irreducible *-representation from [2].

**Proposition 8.** For any q-normal operator $z$ and real number $y \in (q, 1] \cap \text{spec}(|z|)$, consider the Hilbert space representation $\pi_y : C_0^*(z, z^*) + \mathbb{C} \rightarrow \text{B}(\ell_2(\mathbb{Z}))$ given on an orthonormal basis of $\ell_2(\mathbb{Z})$ by

$$\pi_y(U)e_n = e_{n-1}, \quad \pi_y(f)e_n = f(q^n y)e_n, \quad f \in C(\text{spec}(|z|) \cup \{\infty\}).$$

(36)
Furthermore, let $\Pi_+$ and $\Pi_-$ denote the orthogonal projections onto the closed subspaces $\overline{\text{span}\{e_k : k > 0\}}$ and $\overline{\text{span}\{e_k : k \leq 0\}}$ of $\ell_2(\mathbb{Z})$, respectively, and define Hilbert space representations $\pi_0, \pi_\infty : C_0^q(z, z^*) + \mathbb{C} \to B(\ell_2(\mathbb{Z}))$ by

$$
\pi_0(a) := \text{ev}_0(a)\Pi_+, \quad \pi_\infty(a) := \text{ev}_\infty(a)\Pi_-, \quad a \in C_0^q(z, z^*) + \mathbb{C}, \quad (37)
$$

with the *-homomorphisms $\text{ev}_0, \text{ev}_\infty : C_0^q(z, z^*) + \mathbb{C} \to \mathbb{C}$ defined at the end of Section 3. Then the pair $(\pi_y, \pi_0 \oplus \pi_\infty)$ is an even Fredholm module for $C_0^q(z, z^*) + \mathbb{C}$.

Proof. Direct computation show that (36) and (37) define unital *-representations. By density and continuity, it suffices to show that $\pi_y(fU^n) - \pi_0(fU^n) - \pi_\infty(fU^n)$ is compact for the generators $fU^n \in C_0^q(z, z^*) + \mathbb{C}$, $n \in \mathbb{Z}$. First, let $n \neq 0$. Then $f(0) = f(\infty) = 0$ by (11) and therefore $\pi_0(fU^n) + \pi_\infty(fU^n) = 0$ by (37). The operator $\pi_y(f)$ is diagonal on the basis elements $e_k$ with eigenvalues $f(q^ky)$ converging to $f(0) = f(\infty) = 0$ as $k \to \pm \infty$. Therefore $\pi_y(f)$ is a compact operator and so is $\pi_y(fU^n) - \pi_0(fU^n) - \pi_\infty(fU^n) = \pi_y(f)\pi_y(U^n)$.

Next, let $n = 0$. Then $(\pi_y(f) - \pi_0(f) - \pi_\infty(f))e_k = (f(q^ky) - f(0))e_k$ if $k > 0$ and $(\pi_y(f) - \pi_0(f) - \pi_\infty(f))e_k = (f(q^ky) - f(\infty))e_k$ if $k \leq 0$. Again, $\{e_k : k \in \mathbb{Z}\}$ is a complete set of eigenvectors and the sequence of eigenvalues converges to 0 since $\lim_{k \to \infty} (f(q^ky) - f(0)) = f(0) - f(0) = 0$ and $\lim_{k \to -\infty} (f(q^ky) - f(\infty)) = f(\infty) - f(\infty) = 0$. Therefore $\pi_y(f) - \pi_0(f) - \pi_\infty(f)$ yields again a compact operator. □

Recall that the equivalence relation in K-homology is defined by operator homotopy. If $y_1, y_2 \in (q, 1] \cap \text{spec}(|z|)$ belong to the same connected component of $\text{spec}(|z|) \subset \mathbb{R}$, then $[0, 1] \ni t \to (\pi_{y_1 + t(y_2 - y_1)}, \pi_0 \oplus \pi_\infty)$ yields an operator homotopy between $(\pi_{y_1}, \pi_0 \oplus \pi_\infty)$ and $(\pi_{y_2}, \pi_0 \oplus \pi_\infty)$, hence these Fredholm modules define the same class in $K^0(C_0^q(z, z^*) + \mathbb{C})$. The non-degenerateness of the index pairing in Theorem 12 will show that, if $y_1, y_2 \in (q, 1] \cap \text{spec}(|z|)$ belong to different connected components, then the corresponding Fredholm modules yield different K-homology classes.

4.2 Index pairings for the quantum complex plane

The aim of this section is to compute the index pairing for the unital $C^*$-algebra $C(S^2_q) = C_0(\mathbb{C}_q) + \mathbb{C}$ viewed as the algebra of continuous functions on the one-point compactification of the quantum complex plane. A family of projections describing $K_0$-classes was given in Section 3.3. However, since the computations apply to any $q$-normal operator, we state the result for a general $C^*$-algebra $C_0^q(z, z^*) + \mathbb{C}$. Since, by Theorem 1, $K_0(C(S^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is torsion free, it follows from the universal coefficient
Theorem 9. Let $z$ be a $q$-normal operator and $n \in \mathbb{N}$. For $y \in (q, 1] \cap \text{spec}(|z|)$, consider the $K$-homology class $[\langle \pi_y, \pi_0 \oplus \pi_\infty \rangle]$ from Proposition 8 and let $[(\text{ev}_\infty, 0)]$ denote the $K$-homology class from Equation (35). Then the index pairing between these $K$-homology classes and the $K$-theory classes of $C^*_0(z, z^*) + \mathbb{C}$ defined by the Bott projections $P_{\pm n}$ from Equations (29) and (30), and the Powers–Rieffel projections $R_n$ from Equation (32), is given by

$$
\langle [(\text{ev}_\infty, 0)], [P_{\pm n}] \rangle = 1, \quad \langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [P_{\pm n}] \rangle = \pm n, \\
\langle [(\text{ev}_\infty, 0)], [R_n] \rangle = 0, \quad \langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n] \rangle = n.
$$

Moreover, $\langle [(\text{ev}_\infty, 0)], [1] \rangle = 1$ and $\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [1] \rangle = 0$.

Proof. By setting $|z| = 0$ and taking the limit $|z| \to \infty$, one sees that the application of the evaluation maps (8) and (12) to the projections from (29), (30) and (32) yields

$$
ev_0(P_{\pm n}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{ev}_\infty(P_{\pm n}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (38)
$$

$$
ev_0(R_n) = f_n(0) = 0, \quad \text{ev}_\infty(R_n) = f_n(\infty) = 0, \quad (39)
$$

and also $\text{ev}_0(1) = 1 = \text{ev}_\infty(1)$. In particular, by (35),

$$
\langle [(\text{ev}_\infty, 0)], [P_{\pm n}] \rangle = 1, \quad \langle [(\text{ev}_\infty, 0)], [R_n] \rangle = 0, \quad \langle [(\text{ev}_\infty, 0)], [1] \rangle = 1.
$$

Furthermore, $\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [1] \rangle = 0$ by (34) since $\pi_y(1) - (\pi_0 \oplus \pi_\infty)(1) = 1 - 1 = 0$.

We continue by computing the pairing $\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n] \rangle$. From (37) and (39), it follows that $(\pi_0 \oplus \pi_\infty)(R_n) = 0$. Therefore (34) reduces to

$$
\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n] \rangle = \text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(U^n h + f_n + hU^*n)). \quad (40)
$$

As $\pi_y(U^n h)e_k = h(q^k y)e_{k-n}$ acts as a weighted shift operator, the trace of $\pi_y(U^n h)$ vanishes, and so does the trace of its adjoint $\pi_y(hU^*n) = \pi_y((U^n h)^*)$. Therefore, computing the trace in (10) reduces to summing the matrix elements $\langle e_k, \pi_y(f_n)e_k \rangle$, $k \in \mathbb{Z}$. Applying (31) and (36), we get

$$
\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n] \rangle = \text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(f_n)) = \phi(y) + \left( \sum_{k=1}^{n-1} 1 \right) + 1 - \phi(y) = n.
$$
It remains to compute \( \langle [\pi_y, \pi_0 \oplus \pi_\infty], [P_{\pm n}] \rangle \). From (37) and (38), it follows that
\[
\text{Tr}_{\text{Mat}_2(C^*_0(\pi, \pi_0^*), + C)}(\pi_0 \oplus \pi_\infty)(P_{\pm n}) = \Pi_\pm + \Pi_\mp = 1.
\]
Thus, for \( [P_n] \) from (29), the trace formula (34) and the Hilbert space representation (36) give
\[
\langle [\pi_y, \pi_0 \oplus \pi_\infty], [P_n] \rangle = \text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(\frac{q^{-n(n-1)|z|^2_n}}{1 + q^{-n(n-1)|z|^2_n}} + \frac{1}{1 + q^n(n+1)|z|^2_n}), [P_n] - 1)
\]
(41)
\[
= \sum_{k \in \mathbb{Z}} \left( \frac{q^{-n(n-1)}(q^k y)^{2n}}{1 + q^{-n(n-1)}(q^k y)^{2n}} + \frac{1}{1 + q^n(n+1)(q^k y)^{2n}} - 1 \right).
\]
\[
= \sum_{k=0}^{\infty} \left( \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{-n^2+n+2nk y^{2n}}} + \frac{1}{1 + q^{n^2+n+2nk y^{2n}}} - 1 \right) + \sum_{k=1}^{\infty} \left( \frac{q^{-n^2+n-2nk y^{2n}}}{1 + q^{-n^2+n-2nk y^{2n}}} + \frac{1}{1 + q^{n^2+n-2nk y^{2n}}} \right)
\]
\[
= \sum_{k=0}^{\infty} \left( \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{-n^2+n+2nk y^{2n}}} + \frac{1}{1 + q^{n^2+n+2nk y^{2n}}} \right).
\]
Observe that
\[
\sum_{k=0}^{\infty} \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{-n^2+n+2nk y^{2n}}} = \sum_{k=0}^{n-1} \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{-n^2+n+2nk y^{2n}}} + \sum_{k=0}^{\infty} \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{n^2+n+2nk y^{2n}}},
\]
(42)
where the second sum was obtained by shifting the summation index from \( k \) to \( n+k \). Similarly,
\[
\sum_{k=1}^{\infty} \frac{1}{1 + q^{n^2+n-2nk y^{2n}}} = \sum_{k=0}^{n-1} \frac{1}{1 + q^{n^2+n-2nk y^{2n}}} + \sum_{k=1}^{\infty} \frac{1}{1 + q^{n^2+n-2nk y^{2n}}}
\]
(43)
by shifting the summation index from \( k \) to \( n-k \). Inserting (42) and (43) into (41) yields
\[
\langle [\pi_y, \pi_0 \oplus \pi_\infty], [P_n] \rangle = \sum_{k=0}^{n-1} \left( \frac{q^{-n^2+n+2nk y^{2n}}}{1 + q^{-n^2+n+2nk y^{2n}}} + \frac{1}{1 + q^{n^2+n+2nk y^{2n}}} \right) = \sum_{k=0}^{n-1} 1 = n.
\]
(44)
Analogously,
\[
\langle [\pi_y, \pi_0 \oplus \pi_\infty], [P_{-n}] \rangle = \text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(\frac{q^{n(n+1)}|z|^2_n}{1 + q^{n(n+1)}|z|^2_n}) + \frac{1}{1 + q^n(n+1)|z|^2_n}, [P_{-n}] - 1)
\]
\[
= \sum_{k \in \mathbb{Z}} \left( \frac{q^{n(n+1)}(q^k y)^{2n}}{1 + q^{n(n+1)}(q^k y)^{2n}} + \frac{1}{1 + q^n(n+1)(q^k y)^{2n}} - 1 \right).
\]
\[
= \sum_{k=0}^{\infty} \left( \frac{q^{n^2+n+2nk y^{2n}}}{1 + q^{n^2+n+2nk y^{2n}}} - \frac{1}{1 + q^{n^2+n+2nk y^{2n}}} \right) + \sum_{k=1}^{\infty} \left( \frac{q^{n^2+n-2nk y^{2n}}}{1 + q^{n^2+n-2nk y^{2n}}} + \frac{1}{1 + q^{n^2+n-2nk y^{2n}}} \right)
\]
\[
= -\langle [\pi_y, \pi_0 \oplus \pi_\infty], [P_n] \rangle = -n,
\]
where we used (41) and (44) in the last line. \( \Box \)
As announced in Remark 2, we will now give explicit generators for \( K_0(C_0(C_q)) \) and \( K_0(C(S^2_q)) \).

**Corollary 10.** Each of the \( K_0 \)-classes \([R_1]\), \([P_1] - [1]\) and \([P_{-1}] - [1]\) generates \( K_0(C_0(C_q)) \cong \mathbb{Z} \), and a pair of generators for \( K_0(C(S^2_q)) \cong \mathbb{Z} + \mathbb{Z} \) is obtained by adding the trivial class \([1]\) \( \in K_0(C(S^2_q)) \).

**Proof.** First we consider \( K_0(C_0(C_q)) \cong \mathbb{Z} \). Since any multiple of a generator would yield a multiple of 1 in the index pairing with K-homology classes, it is sufficient to find an element in \( K_0(C_0(C_q)) \) such that the pairing with a K-homology class yields \( \pm 1 \).

By Theorem 9, \( \langle ([\pi_y, \pi_0 \oplus \pi_{\infty}], [R_1]) = 1 \) and \( \langle ([\pi_y, \pi_0 \oplus \pi_{\infty}], [P_{\pm 1}] - [1]) = \pm 1 \), therefore each of the three \( K_0 \)-classes \([R_1]\), \([P_1] - [1]\) and \([P_{-1}] - [1]\) freely generates \( K_0(C_0(C_q)) \). For a set of generators of \( K_0(C(S^2_q)) \), one only needs to add the trivial class \([1]\).

By analogy to the classical Bott projections, we may view the projective modules \( C(S^2_q)^2P_n \) as the continuous sections of a non-commutative complex line bundle over the quantum sphere \( S^2_q \) with winding number \( n \in \mathbb{Z} \). Then \( \langle ([\pi_y, \pi_0 \oplus \pi_{\infty}], [P_n]) \rangle \) computes the winding number and the pairing with \([\text{ev}_{\infty}, 0]\) detects the rank of a noncommutative vector bundle (in the classical point \( \infty \)). Moreover, an isomorphism \( \mathbb{Z} \cong K_0(C_0(C_q)) \) is given by \( n \mapsto [P_n] - [1] \).

As an application of the index pairing in Theorem 9 we will give an alternative description of non-commutative complex line bundles by the 1-dimensional projections \( R_n \), \( n \in \mathbb{N} \), without the need of specifying equivalence relations in \( K_0(C(S^2_q)) \).

**Corollary 11.** Let \( n \in \mathbb{N} \). Given the Bott projections \( P_{\pm n} \) from Equations (29) and (30), and the Powers–Rieffel projections \( R_n \) from Equation (32), the following equalities hold in \( K_0(C(S^2_q)) \):

\[
[P_n] = [1] + [R_n] = \begin{pmatrix} R_n & 0 \\ 0 & 1 \end{pmatrix}, \quad [P_{-n}] = [1 - R_n].
\]

**Proof.** Since the K-homology classes \([\text{ev}_{\infty}, 0]\) and \([([\pi_y, \pi_0 \oplus \pi_{\infty}]]) \) from Theorem 9 separate the generators of \( K_0(C(S^2_q)) \) from Corollary 10, it suffices to show that the index pairings coincide, which is straightforward.

Clearly, there are no 1-dimensional projections in \( C(S^2) \) since \( S^2 \) is connected, so the existence of the 1-dimensional projections \( R_n \) can be regarded as a quantum effect. Note moreover that the index pairing with the \( K_0 \)-classes \([R_n]\) reduces to the computation of very simple traces and is also much simpler than the computation of the index pairing with the \( K_0 \)-classes determined by the Bott projections. In a certain
sense, one can say that the quantization of $S^2$ leads to a significant simplification of the index pairing.

### 4.3 Index pairings, generic case

In this section, we compute the index pairings for the C*-algebra $C_0^*(z, z^*)$ generated by a $q$-normal operator such that $X := \text{spec}(|z|) \neq [0, \infty)$. As in Section 3.2 we assume that $1 \notin \text{spec}(|z|)$, and as in the previous section, we state the results for unitalization $C_0^*(z, z^*) + \mathbb{C}$ since the same results apply to the non-unital case after some minor modifications. The main difference to the previous section is that now the $K_0$-group is generated by simple projections of the type $\chi_A(|z|) \in C_0^*(z, z^*) + \mathbb{C}$, see Theorem 6, where one has to add the trivial generator $[1] \in K_0(C_0^*(z, z^*) + \mathbb{C})$.

As for any compact set of real numbers, the connected components of $Y$ are closed intervals $K_\gamma := [a_\gamma, b_\gamma]$. As customary, we identify a singleton $\{y\}$ with the closed interval $[y, y]$. Let $\{K_\gamma : \gamma \in \Gamma\}$ denote the set of the connected components of $Y$. For each $\gamma \in \Gamma$, choose a $y_\gamma \in K_\gamma$ and consider the Fredholm module

$$F_\gamma := (\pi_{y_\gamma}, \pi_0 \oplus \pi_\infty) \quad (45)$$

from Proposition 8. The next theorem shows that the index pairings with these $K$-homology classes together with the classes $[(\text{ev}_0, 0)]$ and $[(\text{ev}_\infty, 0)]$ from 35 determine uniquely any $K_0$-class of $C_0^*(z, z^*) + \mathbb{C}$.

**Theorem 12.** Let $\{K_\gamma : \gamma \in \Gamma\}$ and $F_\gamma$ be defined as above. The index pairing

$$\mathcal{K} := \mathbb{Z}[(\text{ev}_0, 0)] \oplus \bigoplus_{\gamma \in \Gamma} \mathbb{Z}[F_\gamma] \oplus \mathbb{Z}[(\text{ev}_\infty, 0)]$$

and $K_0(C_0^*(z, z^*) + \mathbb{C})$. The index pairing is determined by

$$\langle [(\text{ev}_\infty, 0)], [1] \rangle = 1, \quad \langle [(\text{ev}_\infty, 0)], [\chi_{(0,q)}] \rangle = 0, \quad \langle [(\text{ev}_0, 0)], [\chi_{(c,j,1)}] \rangle = 0, \quad (46)$$

$$\langle [(\text{ev}_0, 0)], [1] \rangle = 1, \quad \langle [(\text{ev}_0, 0)], [\chi_{(0,q)}] \rangle = 1, \quad \langle [(\text{ev}_0, 0)], [\chi_{(c,j,1)}] \rangle = 0, \quad (47)$$

$$\langle [F_\gamma], [1] \rangle = 0, \quad \langle [F_\gamma], [\chi_{(0,q)}] \rangle = 0, \quad (48)$$

$$\langle [F_\gamma], [\chi_{(c,j,1)}] \rangle = 1 \quad \text{if} \quad y_\gamma \in (c_j, 1), \quad \langle [F_\gamma], [\chi_{(c_j,1)}] \rangle = 0 \quad \text{if} \quad y_\gamma \notin (c_j, 1). \quad (49)$$

**Proof.** We first compute the index pairings for the generators of $K_0(C_0^*(z, z^*) + \mathbb{C})$. Equations 46 and 47 are a simple consequence of 35 by evaluating the 1-dimensional projections in $\infty$ and 0, respectively. From 37 and the just mentioned evaluation maps, it also follows that

$$(\pi_0 \oplus \pi_\infty)(1) = 1, \quad (\pi_0 \oplus \pi_\infty)(\chi_{(0,q)}) = \Pi_+, \quad (\pi_0 \oplus \pi_\infty)(\chi_{(c_j,1)}) = 0. \quad (50)$$
Moreover, \((46)\) yields \(\pi_{y_\gamma}(\chi_A)e_n = e_n\) if \(q^n y_\gamma \in A \subset [0, \infty)\) and \(\pi_{y_\gamma}(\chi_A)e_n = 0\) otherwise. Therefore \(\pi_{y_\gamma}(1) = 1\) and

\[
\pi_{y_\gamma}(\chi_{[0,q)}) = \Pi_+ \quad \pi_{y_\gamma}(\chi_{(c_j,1)}) = 0 \quad \text{if} \quad y_\gamma \notin (c_j, 1) \quad \pi_{y_\gamma}(\chi_{(c_j,1)}) = \Pi_{e_0} \quad \text{if} \quad y_\gamma \in (c_j, 1)
\]

(51)

where \(\Pi_{e_0}\) denotes the 1-dimensional orthogonal projection onto \(\text{span}\{e_0\}\). Combining (40) and (51) with (34) yields for \(F_{\gamma}\) from (45)

\[
\langle [F_{\gamma}], [1]\rangle = \text{Tr}_{\ell_2}(1 - 1) = 0, \quad \langle [F_{\gamma}], [\chi_{[0,q)}]\rangle = \text{Tr}_{\ell_2}(\Pi_+ - \Pi_+) = 0, \quad (52)
\]

\[
\langle [F_{\gamma}], [\chi_{(c_j,1)}]\rangle = \text{Tr}_{\ell_2}(0 - 0) = 0 \quad \text{if} \quad y_\gamma \notin (c_j, 1), \quad (53)
\]

\[
\langle [F_{\gamma}], [\chi_{(c_j,1)}]\rangle = \text{Tr}_{\ell_2}(\Pi_{e_0}) = 1 \quad \text{if} \quad y_\gamma \in (c_j, 1).
\]

(54)

This finishes the proof of (46)–(49).

To show the non-degeneracy of the index pairing, let

\[
p := l[\chi_{[0,q)}] + \sum_{k=1}^{N} n_k[\chi_{(c_j,1)]} + m[1] \in K_0(C_0^*(z,z^*) + \mathbb{C}), \quad N \in \mathbb{N}, \quad l, n_k, m \in \mathbb{Z},
\]

and suppose that \(\langle F, p \rangle = 0\) for all \(F \in \mathcal{K}\). Pairing first with \([\text{ev}_\infty, 0]\) and then with \([\text{ev}_0, 0]\) gives \(m = 0\) and \(l = 0\) by (46) and (47). Thus \(p = \sum_{k=1}^{N} n_k[\chi_{(c_j,1)]}\).

Without loss of generality, we may assume that \(c_{j_1} > \ldots > c_{j_N}\). As \(c_{j_N}\) and \(c_{j_{N-1}}\) belong to different connected components of \((q, 1) \setminus Y\), there exists a \(\gamma_N \in \Gamma\) such that \(c_{j_{N-1}} > y_{\gamma_N} > c_{j_N}\). Then (53) and (51) yield \(0 = \langle [F_{\gamma_N}], [p]\rangle = n_N\). Continuing inductively, choosing in each step a \(\gamma_k \in \Gamma\) such that \(c_{j_{K-1}} > y_{\gamma_k} > c_{j_k}\), where we set \(c_{j_0} := 1\), it follows that \(n_N = \ldots = n_1 = 0\), hence \(p = 0\).

Finally, let

\[
F := l[\text{ev}_0, 0] + \sum_{k=1}^{N} n_k[F_{\gamma_k}] + m[\text{ev}_\infty, 0] \in \mathcal{K}, \quad N \in \mathbb{N}, \quad l, n_k, m \in \mathbb{Z},
\]

and suppose that \(\langle F, p \rangle = 0\) for all \(p \in K_0(C_0^*(z,z^*) + \mathbb{C})\). Similarly to the above, we may assume that \(y_{\gamma_1} > \ldots > y_{\gamma_N}\). As each \(y_{\gamma_k}\) belongs to a different connected component of \(Y\), there exist \(j_k \in J\) such that \(y_{\gamma_k} > c_{j_k} > y_{\gamma_{k+1}}\) for \(k = 1, \ldots, N\), where \(y_{\gamma_{N+1}} := q\). From (46), (47) and (49), we obtain \(0 = \langle F, [\chi_{(c_j,1)]}\rangle = n_1\). Continuing by induction on \(k = 2, \ldots, N\), and applying in each step the same argument, we conclude that \(n_2 = \ldots = n_N = 0\). Thus \(F = l[\text{ev}_0, 0] + m[\text{ev}_\infty, 0]\). Now (40) and (47) imply first \(0 = \langle F, [\chi_{[0,q)}]\rangle = l\) and then \(0 = \langle F, [1]\rangle = m[\langle [\text{ev}_\infty, 0], [1]\rangle = m\), therefore \(F = 0\).
As an application of the index pairing in Theorem 12 we will use elementary projections $\chi_A \in C_0^*(z, z^*) \oplus \mathbb{C}$ to give an alternative description of the $K_0$-classes of the non-commutative complex line bundles determined by the Bott projections and Powers–Rieffel type projections from Section 3.3.

**Corollary 13.** For $n \in \mathbb{N}$, let $P_{\pm n}$ denote the Bott projections defined in (29) and (30), and let $R_n$ denote the Powers–Rieffel type projections from (32). Then the following equalities hold in $K_0(C_0^*(z, z^*) \oplus \mathbb{C})$.

\[
[P_n] = [1] + [R_n] = [1] + n[\chi(q, 1)] = \begin{bmatrix} 1 & 0 \\ 0 & \chi(q^n, 1) \end{bmatrix},
\]
\[
[P_{-n}] = [1] - [R_n] = [1] - n[\chi(q, 1)] = [1 - \chi(q^n, 1)].
\]

**Proof.** As $1 \notin X$ and thus $\{q^k : k \in \mathbb{Z}\} \cap X = \emptyset$ by the $q$-invariance of $X = \text{spec}([z])$, we know that $\chi_{(q^n, q^k)}$ is a projection in $C_0(X) \subset C_0^*(z, z^*) \oplus \mathbb{C}$ for all $n, k \in \mathbb{Z}, n > k$.

We will prove (55) and (56) by showing that the index pairings with the $K$-homology classes $[(ev_0, 0)], [(ev_\infty, 0)]$ and $[F_\gamma], \gamma \in \Gamma$, coincide. Then, by the non-degeneracy statement of Theorem 12, Equations (55) and (56) yield identities in K-theory.

Applying first Theorem 9 and then Theorem 12 with $c_j$ replaced by $q$, one readily sees that

\[
\langle [(ev_\infty, 0)], [P_{\pm n}] \rangle = \langle [(ev_\infty, 0)], [1] \pm [R_n] \rangle = \langle [(ev_\infty, 0)], [1] \pm n[\chi(q, 1)] \rangle = 1,
\]
\[
\langle [(ev_0, 0)], [P_{\pm n}] \rangle = \langle [(ev_0, 0)], [1] \pm [R_n] \rangle = \langle [(ev_0, 0)], [1] \pm n[\chi(q, 1)] \rangle = 1,
\]
\[
\langle [F_\gamma], [P_{\pm n}] \rangle = \langle [F_\gamma], [1] \pm [R_n] \rangle = \langle [F_\gamma], [1] \pm n[\chi(q, 1)] \rangle = \pm n.
\]

As a consequence, $[P_{\pm n}] = [1] \pm [R_n] = [1] \pm n[\chi(q, 1)]$. By the obvious relations in K-theory, it now suffices to verify that $[\chi(q^n, 1)] = n[\chi(q, 1)]$. Clearly,

\[
\langle [(ev_p, 0)], [\chi(q^n, 1)] \rangle = 0 = \langle [(ev_p, 0)], n[\chi(q, 1)] \rangle, \quad p \in \{0, \infty\},
\]

since all evaluation maps give 0. This also shows that $(\pi_0 \oplus \pi_\infty)(\chi(q^n, 1)) = 0$ by (57). Furthermore, Equation (36) implies that $\pi_y(\chi(q^n, 1))$ is the orthogonal projection onto span$\{e_0, \ldots, e_{n-1}\}$ for all $y \in Y \subset (q, 1)$. Therefore the index pairing (34) yields

\[
\langle [F_\gamma], [\chi(q^n, 1)] \rangle = \text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(\chi(q^n, 1))) = n = n\text{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(\chi(q, 1))) = \langle [F_\gamma], n[\chi(q, 1)] \rangle,
\]

which completes the proof. \qed
Note that the projections $\chi(q^n, 1)$ are utterly elementary, i.e., continuous functions with values in $\{0, 1\}$. In particular, the computation of index pairing reduces to its simplest possible form, namely to the calculation of a trace of a finite-dimensional projection. Also, by unitary equivalence of $K_0$-classes, we obtain from (55) and (56) the following isomorphisms of finitely generated projective modules:

$$
(C_0^*(z, z^*) \oplus \mathbb{C})^2 P_{-n} \cong (C_0^*(z, z^*) \oplus \mathbb{C}) \chi(q^n, 1),
$$

$$
(C_0^*(z, z^*) \oplus \mathbb{C})^2 P_n \cong (C_0^*(z, z^*) \oplus \mathbb{C}) \oplus (C_0^*(z, z^*) \oplus \mathbb{C}) \chi(q^n, 1),
$$

where the right hand sides are considerably more simple.

The interest in the projections $P_{\pm n}$ arose from the observation that they can be regarded as deformations of the classical Bott projections representing complex line bundles of winding number $\pm n$ over the 2-sphere. Recall that we defined the $C^*$-algebra of the quantum 2-sphere as $C(S_q^2) := C_0^*(z, z^*) \oplus \mathbb{C}$, where $\text{spec}(|z|) = [0, \infty)$, because only in that case the deformation preserves the classical $K$-groups. However, if one wants to view any $q$-normal operator $z$ as a deformation of the complex plane, then the deformations satisfying $\text{spec}(|z|) \neq [0, \infty)$ lead to a substantial simplification of the description of complex line bundles and the index computation.

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