REGULARIZATION OF RATIONAL GROUP ACTIONS

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Abstract. We give a modern proof of the Regularization Theorem of André Weil which says that for every rational action of an algebraic group $G$ on a variety $X$ there exist a variety $Y$ with a regular action of $G$ and a $G$-equivariant birational map $X \dashrightarrow Y$. Moreover, we show that a rational action of $G$ on an affine variety $X$ with the property that each $g$ from a dense subgroup of $G$ induces a regular automorphism of $X$, is a regular action.

The aim of this note is to give a modern proof of the following Regularization Theorem due to André Weil, see \cite{Wei55}. We will follow the approach in \cite{Zai95}. Our base field $k$ is algebraically closed. A variety is an algebraic $k$-variety, and an algebraic group is an algebraic $k$-group.

**Theorem 1.** Let $G$ be an algebraic group and $X$ a variety with a rational action of $G$. Then there exists a variety $Y$ with a regular action of $G$ and a birational $G$-equivariant morphism $\phi: X \dashrightarrow Y$.

We do not assume that $G$ is linear or connected, nor that $X$ is irreducible. This creates some complications in the arguments. The reader is advised to start with the case where $G$ is connected and $X$ irreducible in a first reading.

We cannot expect that the birational map $\phi$ in the theorem is a morphism. Take the standard Cremona involution $\sigma$ of $\mathbb{P}^2$, given by $(x : y : z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$, which collapses the coordinate lines to points. This cannot happen if $\sigma$ is a regular automorphism. However, removing these lines, we get $k^* \times k^*$ where $\sigma$ is a well-defined automorphism.

More generally, consider the rational action of $G := \text{PSL}_2 \times \text{PSL}_2$ on $\mathbb{P}^2$ induced by the birational isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Then neither an open set carries a regular $G$-action, nor $\mathbb{P}^2$ can be embedded into a variety $Y$ with a regular $G$-action.

As we will see in the proof below, one first constructs a suitable open set $U \subseteq X$ where the rational action of $G$ has very specific properties, and then one shows that $U$ can be equivariantly embedded into a variety $Y$ with a regular $G$-action.

### 1.1. Rational maps

We first have to define and explain the different notion used in the theorem above. We refer to \cite{Bla16} for additional material and more details.

Recall that a rational map $\phi: X \dashrightarrow Y$ between two varieties $X, Y$ is an equivalence class of pairs $(U, \phi_U)$ where $U \subseteq X$ is an open dense subset and $\phi_U: U \rightarrow Y$ a morphism. Two such pairs $(U, \phi_U)$ and $(V, \phi_V)$ are equivalent if $\phi_U|_{U \cap V} = \phi_V|_{U \cap V}$. We say that $\phi$ is defined in $x \in X$ if there is a $(U, \phi_U)$ representing $\phi$ such that $x \in U$. The set of all these points forms an open dense subset $\text{Dom}(\phi) \subseteq X$ called the domain of definition of $\phi$. We will shortly say that $\phi$ is defined in $x$ if $x \in \text{Dom}(\phi)$.

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For all $(U, φ_U)$ representing $φ: X \rightarrow Y$ the closure $\overline{φ_U(U)} \subseteq Y$ is the same closed subvariety of $Y$. We will call it the closed image of $φ$ and denote it by $φ(X)$.

The rational map $φ$ is called dominant if $φ(X) = Y$. It follows that the composition $ψ \circ φ$ of two rational maps $φ, ψ: X \rightarrow Y$ and $ψ: Y \rightarrow Z$ is a well-defined rational map $ψ \circ φ: X \rightarrow Z$ in case $φ$ is dominant.

A rational map $φ: X \rightarrow Y$ is called birational if it is dominant and admits an inverse $ψ: Y \rightarrow X$, $ψ \circ φ = \text{id}_X$. It then follows that $ψ$ is also dominant and that $φ \circ ψ = \text{id}_Y$. Clearly, $ψ$ is well-defined by $φ$, and we shortly write $ψ = φ^{-1}$. It is easy to see that $φ$ is birational if and only if there is a $(U, φ_U)$ representing $φ$ such that $φ_U: U \hookrightarrow Y$ is an open immersion with a dense image. The set of birational maps $φ: X \rightarrow Y$ is a group under composition which will be denoted by Bir($Y$).

A rational map $φ: X \rightarrow Y$ is called biregular in $x$ if there is an open neighborhood $U \subseteq \text{Dom}(φ)$ of $x$ such that $φ_U: U \hookrightarrow Y$ is an open immersion. It follows that the subset $X' := \{x \in X \mid φ$ is biregular in $x\}$ is open in $X$, and the induced morphism $φ: X' \hookrightarrow Y$ is an open immersion. This implies the following result.

**Lemma 1.** Let $φ: X \rightarrow Y$ be a birational map. Then the set
$$\text{Breg}(φ) := \{x \in X \mid φ$ is biregular in $x\}$$
is open and dense in $X$.

**Remark 1.** If $X$ is irreducible, a rational dominant map $φ: X \rightarrow Y$ defines a $k$-linear inclusion $φ^*: k(Y) \hookrightarrow k(X)$ of fields. Conversely, for every inclusion $α: k(Y) \hookrightarrow k(X)$ of fields there is a unique dominant rational map $φ: X \rightarrow Y$ such that $φ^* = α$. In particular, we have an isomorphism Bir($X$) $\cong$ Aut$_k(k(X))$ of groups, given by $φ$ $\mapsto$ $(φ^*)^{-1}$.

### 1.2. Rational group actions.

**Definition 1.** Let $X, Z$ be varieties. A map $φ: Z \rightarrow \text{Bir}(X)$ is called a morphism if there is an open dense set $U \subseteq Z \times X$ with the following properties:

- (i) The induced map $(z, x) \mapsto φ(z)(x): U \rightarrow X$ is a morphism of varieties.
- (ii) For every $z \in Z$ the open set $U_z := \{x \in X \mid (z, x) \in U\}$ is dense in $X$.
- (iii) For every $z \in Z$ the birational map $φ(z): X \rightarrow X$ is defined in $U_z$.

Equivalently, we have a rational map $Φ: Z \times X \rightarrow X$ such that, for every $z \in Z$,

- (i) the open subset $\text{Dom}(Φ) \cap (\{z\} \times X) \subseteq \{z\} \times X$ is dense, and
- (ii) the induced rational map $Φ_z: X \rightarrow X, x \mapsto Φ(z, x)$, is birational.

This definition allows to define the Zariski-topology on Bir($X$) in the following way.

**Definition 2.** A subset $S \subseteq \text{Bir}(X)$ is closed if for every morphism $ρ: Z \rightarrow \text{Bir}(X)$ the inverse image $ρ^{-1}(S) \subseteq Z$ is closed.

Now we can define rational group actions on varieties. Let $G$ be an algebraic group and let $X$ be a variety.

**Definition 3.** A rational action of $G$ on $X$ is a morphism $ρ: G \times X \rightarrow X$ such that the following holds:
(a) \( \text{Dom}(\rho) \cap \{(g) \times X\} \) is dense in \( \{g\} \times X \) for all \( g \in G \),
(b) the induced rational map \( \rho_g : X \dasharrow X, x \mapsto \rho(g, x) \), is birational,
(c) the map \( g \mapsto \rho_g \) is a homomorphism of groups.

If \( \rho \) is defined in \( (g, x) \) and \( \rho(g, x) = y \) we will say that \( g \cdot x \) is defined and \( g \cdot x = y \).

We will also use the birational map
\[
\tilde{\rho} : G \times X \dasharrow G \times X, \quad (g, x) \mapsto (g, \rho(g, x)),
\]
see section 1.5 below.

**Remark 2.** Note that if \( \rho : G \times X \dasharrow X \) is defined in \( (g, x) \), then \( \rho_g : X \dasharrow X \) is defined in \( x \), but the reverse implication does not hold. An example is the following.

Consider the regular action of the additive group \( G_a \) on the plane \( \mathbb{A}^2 = \mathbb{A}^2 \) by translation along the \( x \)-axis: \( s \cdot x := x + (s, 0) \) for \( s \in G_a \) and \( x \in \mathbb{A}^2 \). Let \( \beta : X \rightarrow \mathbb{A}^2 \) be the blow-up of \( \mathbb{A}^2 \) in the origin. Then we get a rational \( G_a \)-action on \( X \), \( \rho : G_a \times X \dasharrow X \). It is not difficult to see that \( \rho \) is defined in \( (e, x) \) if and only if \( \beta(x) \neq 0 \), i.e. \( x \) does not belong to the exceptional fiber, but clearly, \( \rho_e = \text{id} \) is defined everywhere.

If \( \phi : Z \rightarrow \text{Bir}(X) \) is a morphism such that \( \phi(Z) \subseteq \text{Aut}(X) \), the group of regular automorphisms, one might conjecture that the induced map \( Z \times X \rightarrow X \) is a morphism. I don’t know how to prove this, but maybe the following holds.

**Conjecture.** Let \( \rho : G \rightarrow \text{Bir}(X) \) be a rational action. If \( \rho(G) \subseteq \text{Aut}(X) \), then \( \rho \) is a regular action.

We can prove this under additional assumptions.

**Theorem 2.** Let \( \rho : G \rightarrow \text{Bir}(X) \) be a rational action where \( X \) is affine. Assume that there is a dense subgroup \( \Gamma \subseteq G \) such that \( \rho(\Gamma) \subseteq \text{Aut}(X) \). Then the \( G \)-action on \( X \) is regular.

The proof will be given in the last section 1.9.

**Definition 4.** Given rational \( G \)-actions \( \rho \) on \( X \) and \( \mu \) on \( Y \), a dominant rational map \( \phi : X \rightarrow Y \) is called \( G \)-equivariant if the following holds:

(Equi) For every \( (g, x) \in G \times X \) such that (1) \( \rho \) is defined in \( (g, x) \), (2) \( \phi \) is defined in \( x \) and in \( \rho(g, x) \), and (3) \( \mu \) is defined in \( (g, \phi(x)) \), we have \( \phi(\rho(g, x)) = \mu(g, \phi(x)) \).

Note that the set of \( (g, x) \in G \times X \) satisfying the assumptions of (Equi) is open and dense in \( G \times X \) and has the property that it meets all \( \{g\} \times X \) in a dense open set.

**Remark 3.** If \( G \) acts rationally on \( X \) and if \( X' \subseteq X \) is a nonempty open subset, then \( G \) acts rationally on \( X' \), and the inclusion \( X' \hookrightarrow X \) is \( G \)-equivariant. Moreover, if \( G \) acts rationally on \( X \) and if \( \phi : X \dasharrow Y \) is a birational map, then there is uniquely define rational action of \( G \) on \( Y \) such that \( \phi \) is \( G \)-equivariant.

Note that for a rational \( G \)-action \( \rho \) on \( X \) and an open dense set \( X' \subseteq X \) with induced rational \( G \)-action \( \rho' \) we have
\[
\text{Dom}(\rho') = \{(g, x) \in \text{Dom}(\rho) \mid x \in X' \text{ and } g \cdot x \in X'\},
\]
\[
\text{Breg}(\tilde{\rho}') = \{(g, x) \in \text{Breg}(\tilde{\rho}) \mid x \in X' \text{ and } g \cdot x \in X'\}.
\]
1.3. The case of a finite group $G$. Assume that $G$ is finite and acts rationally on an irreducible variety $X$. Then every $g \in G$ defines a birational map $g: X \rightarrow X$ and thus an automorphism $g^*$ of the field $\mathbb{k}(X)$ of rational functions on $X$. In this way we obtain a homomorphism $G \rightarrow \text{Aut}_\mathbb{k}(\mathbb{k}(X))$ given by $g \mapsto (g^*)^{-1}$.

By Remark 3 above we may assume that $X$ is affine. Hence $\mathbb{k}(X)$ is the field of fractions of the coordinate ring $\mathcal{O}(X)$. Since $G$ is finite we can find a finite-dimensional $\mathbb{k}$-linear subspace $V \subseteq \mathbb{k}(X)$ which is $G$-stable and contains a system of generators of $\mathcal{O}(X)$.

Denote by $R \subseteq \mathbb{k}(X)$ the subalgebra generated by $V$. By construction,

(a) $R$ is finitely generated and $G$-stable, and

(b) $R$ contains $\mathcal{O}(X)$.

In particular, the field of fractions of $R$ is $\mathbb{k}(X)$. If we denote by $Y$ the affine variety with coordinate ring $R$, we obtain a regular action of $G$ on $Y$ and a birational morphism $\psi: Y \rightarrow X$ induced by the inclusion $\mathcal{O}(X) \subseteq R$. Now the Regularization Theorem follows in this case with $\phi := \psi^{-1}: X \rightarrow Y$.

There is a different way to construct a “model” with a regular $G$-action, without assuming that $X$ is irreducible. In fact, there is always an open dense set $X_{\text{reg}} \subseteq X$ where the action is regular. It is defined in the following way (cf. Definition 5 below). For $g \in G$ denote by $X_g \subseteq X$ the open dense set where the rational map $\rho_g: x \mapsto g \cdot x$ is biregular. Then $X_{\text{reg}} := \bigcap_{g \in G} X_g$ is open and dense in $X$ and the rational $G$-action on $X_{\text{reg}}$ is regular. In fact, $\rho_g$ is biregular on $X_{\text{reg}}$, hence also biregular on $h \cdot X_{\text{reg}}$ for all $h \in G$ which implies that $h \cdot X_{\text{reg}} \subseteq X_{\text{reg}}$.

1.4. A basic example. We now give an example which should help to understand the constructions and the proofs below. Let $X$ be a variety with a regular action of an algebraic group $G$. Choose an open dense subset $U \subseteq X$ and consider the rational $G$-action on $U$. Then $X := \bigcup_{g \in G} gU \subseteq X$ is open and dense in $X$ and carries a regular action of $G$.

The rational $G$-action $\rho$ on $U$ is rather special. First of all we see that $\rho$ is defined in $(g, u)$ if and only if $g \cdot u \in U$. This implies that $\rho$ is defined in $(g, u)$ if and only if $\rho_g$ is defined in $u$. Next we see that if $\rho$ is defined in $(g, u)$, then $\tilde{\rho}: G \times U \rightarrow G \times U$, $(g, x) \mapsto (g, \rho(g, x))$, is biregular in $(g, u)$. And finally, for any $x$ the (open) set of elements $g \in G$ such that $\tilde{\rho}$ is biregular in $(g, x)$ is dense in $G$.

A first and major step in the proof is to show (see section 1.5) that for every rational $G$-action on a variety $X$ there is an open dense subset $X_{\text{reg}} \subseteq X$ with the property that for every $x \in X_{\text{reg}}$ the rational map $\tilde{\rho}: G \times X_{\text{reg}} \rightarrow G \times X_{\text{reg}}$ is biregular in $(g, x)$ for all $g$ in a dense (open) set of $G$. Then, in a second step in section 1.6, we construct from $X_{\text{reg}}$ a variety $Y$ with a regular $G$-action together with an open $G$-equivariant embedding $X_{\text{reg}} \hookrightarrow Y$.

1.5. $G$-regular points and their properties. Let $X$ be a variety with a rational action $\rho: G \times X \rightarrow X$ of an algebraic group $G$. Define

$$\tilde{\rho}: G \times X \rightarrow G \times X, \quad (g, x) \mapsto (g, \rho(g, x)).$$

It is clear that $\text{Dom}(\tilde{\rho}) = \text{Dom}(\rho)$ and that $\tilde{\rho}$ is birational with inverse $\tilde{\rho}^{-1}(g, y) = (g, \rho(g^{-1}, y))$, i.e. $\tilde{\rho}^{-1} = \tau \circ \tilde{\rho} \circ \tau$ where $\tau: G \times X \rightarrow G \times X$ is the isomorphism $(g, x) \mapsto (g^{-1}, x)$.

The following definition is crucial.
Lemma 2. This implies the following result.

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Proof. (a) Let $G$ dense (open) set of

Lemma 4. Consider a rational

Proposition 1. In particular, $\text{pr}_Y(\Gamma(\phi)) = \phi(\text{Dom}(\phi))$.

The main proposition is the following.

Proposition 1. (a) $X_{\text{reg}}$ is open and dense in $X$.

(b) If $x \in X_{\text{reg}}$ and if $\tilde{\rho}$ is biregular in $(g, x)$, then $g \cdot x \in X_{\text{reg}}$.

Proof. (a) Let $G = G_0 \cup G_1 \cup \cdots \cup G_n$ be the decomposition into connected components. Then $D_i := \text{Breg}(\rho) \cap (G_i \times X)$ is open and dense for all $i$ (Lemma 1), and the same holds for the image $D_i \subseteq X$ under the projection onto $X$. Since $X_{\text{reg}} = \bigcap_i D_i$, the claim follows.

(b) If $\tilde{\rho}$ is biregular in $(g, x)$, then $\tilde{\rho}^{-1} = \tau \circ \tilde{\rho} \circ \tau$ is biregular in $(g, g \cdot x)$, hence $\tilde{\rho}$ is biregular in $\tau(g, g \cdot x) = (g^{-1}, g \cdot x)$. If $x$ is $G$-regular, then $\rho_h$ is biregular in $x$ for all $h$ from a dense open set $G' \subseteq G$. Now Lemma 2(b) implies that $\tilde{\rho}$ is biregular in $(h g^{-1}, g \cdot x)$ for all $h \in G'$, hence $g \cdot x \in X_{\text{reg}}$.

Note that for an open dense set $U \subseteq X$ a point $x \in U$ might be $G$-regular for the rational $G$-action on $X$, but not for the rational $G$-action on $U$. However, Proposition 1(b) implies the following result.

Corollary 1. For the rational $G$-action on $X_{\text{reg}}$ every point is $G$-regular.

This allows to reduce to the case of a rational $G$-action such every point is $G$-regular.

Lemma 3. Assume that $X = X_{\text{reg}}$. If $\rho_g$ is defined in $x$, then $\rho_g$ is biregular in $x$.

Proof. Assume that $\rho_g$ is defined in $x \in X$. There is an open dense subset $G' \subseteq G$ such $\rho_h$ is biregular in $g \cdot x$ and $\rho_h \circ \rho_g$ is biregular in $x$ for all $h \in G'$. Since $\rho_h \circ \rho_g = \rho_{h g}$ we see that $\rho_g$ is biregular in $x$.

For a rational map $\phi: X \rightarrow Y$ the graph $\Gamma(\phi)$ is defined in the usual way:

$$\Gamma(\phi) := \{(x, y) \in X \times Y \mid \phi \text{ is defined in } x \text{ and } \phi(x) = y\}.$$

In particular, $\text{pr}_X(\Gamma(\phi)) = \text{Dom}(\phi)$ and $\text{pr}_Y(\Gamma(\phi)) = \phi(\text{Dom}(\phi))$.

The next lemma will play a central role in the construction of the regularization.

Lemma 4. Consider a rational $G$-action $\rho$ on a variety $X$ and assume that every point of $X$ is $G$-regular. Then, for every $g \in G$, the graph $\Gamma(\rho_g)$ is closed in $X \times X$. 

Proof. Let $\Gamma := \overline{\{(\rho) \}}$ be the closure of the graph of $\rho_0$ in $X \times X$. We have to show that for every $(x_0, y_0) \in \Gamma$ the rational map $\rho_0$ is defined in $x_0$, or, equivalently, that the morphism $\pi_1 := \text{pr}_1 |_{\Gamma}: \Gamma \rightarrow X$ induced by the first projection is biregular in $(x_0, y_0)$.

Choose $h \in G$ such that $\rho_{h_0}$ is biregular in $x_0$ and $\rho_h$ is biregular in $y_0$, and consider the induced birational map $\Phi := (\rho_{h_0} \times \rho_h): X \times X \dasharrow X \times X$. If $\Phi$ is defined in $(x, y) \in \Gamma(\rho_{h_0})$, $y := g \cdot x$, then $\Phi(x, y) = ((hg) \cdot x, (hg) \cdot x) \in \Delta(X)$ where $\Delta(X) := \{(x, x) \in X \times X \mid x \in X\}$ is the diagonal. It follows that $\Phi(\Gamma) \subseteq \Delta(X)$.

$$
\begin{array}{c}
\begin{array}{c}
X \times X \overset{\rho_{h_0} \times \rho_h}{\longrightarrow} X \times X \\
\downarrow \subseteq \\
\Gamma \\
\downarrow \phi \\
\pi_1 \\
\downarrow \text{pr}_1 \\
X \overset{\rho_{h_0}}{\longrightarrow} X
\end{array}
\end{array}
$$

Since $\Phi$ is biregular in $(x_0, y_0)$, we see that $\phi := \Phi |_{\Gamma}: \Gamma \dasharrow \Delta(X)$ is also biregular in $(x_0, y_0)$. By construction, we have $\rho_{h_0} \circ \pi_1 = \text{pr}_1 \circ \phi$. Since $\rho_{h_0} \circ \pi_1$ is biregular in $\phi(x_0, y_0)$ and $\phi$ is biregular in $(x_0, y_0)$ (and $\text{pr}_1 |_{\Delta(X)}$ is an isomorphism) it follows that $\pi_1$ is biregular in $(x_0, y_0)$, hence the claim.  

The last lemma is easy.

Lemma 5. Consider a rational action $\rho$ of $G$ on a variety $X$. Assume that there is a dense open set $U \subseteq X$ such that $\tilde{\rho}$ defines an open immersion $\tilde{\rho}: G \times U \hookrightarrow G \times X$. Then the open dense subset $Y := \bigcup g \cdot U \subseteq X$ carries a regular $G$-action.

Proof. It is clear that every $\rho_g$ induces an isomorphism $U \xrightarrow{\sim} g \cdot U$. This implies that $Y$ is stable under all $\rho_g$. It remains to see that the induced map $G \times Y \rightarrow Y$ is a morphism. By assumption, this is clear on $G \times U$, hence also on $G \times g \cdot U$ for all $g \in G$, and we are done. \hfill \Box

1.6. The construction of a regular model. In view of Corollary 1 our Theorem 1 will follow from the next result.

Theorem 3. Let $X$ be a variety with a rational action of $G$. Assume that every point of $X$ is $G$-regular. Then there is a variety $Y$ with a regular $G$-action and a $G$-equivariant open immersion $X \hookrightarrow Y$.

From now on $X$ is a variety with a rational $G$-action $\rho$ such that $X_{\text{reg}} = X$. Let $S := \{g_0 := e, g_1, g_2, \ldots, g_m\} \subseteq G$ be a finite subset. These $g_i$’s will be carefully chosen in the proof of Theorem 2 below. Let $X^{(0)}, X^{(1)}, \ldots, X^{(m)}$ be copies of the variety $X$. On the disjoint union $X(S) := X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(m)}$ we define the following relations between elements $x_i, x'_i \in \Xi$:

(1) For any $i$: $x_i \sim x'_i$ $\iff$ $x_i = x'_i$;

(2) For $i \neq j$: $x_i \sim x_j$ $\iff$ $\rho_{g_j^{-1}} g_i$ is defined in $x_i$ and sends $x_i$ to $x_j$.

It is not difficult to see that this defines an equivalence relation. (For the symmetry one has to use Lemma 3.) Denote by $\tilde{X}(S) := X(S)/ \sim$ the set of equivalence classes endowed with the induced topology.
Lemma 6. The maps \( \iota_i : \Xi \to \hat{X}(S) \) are open immersions and endow \( \hat{X}(S) \) with the structure of a variety.

Proof. By definition of the equivalence relation and the quotient topology the natural maps \( \iota_i : \Xi \to \hat{X}(S) \) are injective and continuous. Denote the image by \( \hat{X}^{(i)} \).

We have to show that \( \hat{X}^{(i)} \) is open in \( \hat{X}(S) \), or, equivalently, that the inverse image of \( \hat{X}^{(i)} \) in \( X(S) \) is open. This is clear, because the inverse image in \( \Xi \) of the intersection \( \hat{X}^{(i)} \cap \hat{X}^{(j)} \) is the open set of points where \( \rho_{g_{ij}^{-1}} \) is defined.

It follows that \( \hat{X}(S) \) carries a unique structure of a prevariety such that the maps \( \iota_i : \Xi \to \hat{X}(S) \) are open immersions. It remains to see that the diagonal \( \Delta(\hat{X}(S)) \subseteq \hat{X}(S) \times \hat{X}(S) \) is closed. For this it suffices to show that \( \Delta_{ij} := \Delta(\hat{X}(S)) \cap (\hat{X}^{(i)} \times \hat{X}^{(j)}) \) is closed in \( \hat{X}^{(i)} \times \hat{X}^{(j)} \) for all \( i, j \). This follows from Lemma 4, because \( \Delta_{ij} \) is the image of \( \Gamma(\rho_{g_{ij}^{-1}}) \subseteq \Xi \times \hat{X}(S) \). In fact, for \( x_i \in \Xi \) and \( x_j \in \hat{X}^{(j)} \), we have \( (\bar{x}_i, \bar{x}_j) \in \Delta_{ij} \) if and only if \( x_i \sim x_j \). This means that \( \rho_{g_{ij}^{-1}} \) is defined in \( x_i \) and \( \rho_{g_{ij}^{-1}}(x_i) = x_j \), i.e. \( (x_i, x_j) \in \Gamma(\rho_{g_{ij}^{-1}}) \).

\( \square \)

Fixing the open immersion \( \iota_0 : X = X^{(0)} \to \hat{X}(S) \) we obtain a rational \( G \)-action \( \bar{\rho} = \bar{\rho}_S \) on \( \hat{X}(S) \) such that \( \iota_0 \) is \( G \)-equivariant (Remark 3). If we consider each \( \Xi \) as the variety \( \hat{X}(S) \) with the rational \( G \)-action \( \rho^{(i)}(g, x) := \rho(g, g_{ij}^{-1} x) \), then, by construction of \( \hat{X}(S) \), the open immersions \( \iota_i : \Xi \to \hat{X}(S) \) are all \( G \)-equivariant.

Lemma 7. For all \( i \), the rational map \( \bar{\rho}_{g_i} \) is defined on \( \hat{X}^{(0)} \) and defines an isomorphism \( \bar{\rho}_{g_i} : \hat{X}^{(0)} \cong \hat{X}^{(i)} \).

Proof. Consider the open immersion \( \tau_i := \iota_i \circ \iota_0^{-1} : \hat{X}^{(0)} \to \hat{X}(S) \) with image \( \hat{X}^{(i)} \).

We claim that \( \tau_i(\bar{x}) = g_i \cdot \bar{\bar{x}} \). It suffices to show that this holds on an open dense set of \( \hat{X}^{(0)} \). Let \( U \subseteq X \) be the open dense set where \( g_i \cdot x \) is defined. For \( x \in U \) and \( y := g_i \cdot x \in X \) we get, by definition, \( \iota_0(y) = \iota_i(x) \). On the other hand, \( \iota_0(y) = \iota_0(g_i \cdot x) = g_i \cdot \iota_0(x) \). Hence, \( g_i \cdot \iota_0(x) = \iota_i(x) \), and so \( \tau_i(\bar{x}) = g_i \cdot \bar{x} \) for all \( \bar{x} \in \iota_0(U) \).

\( \square \)

Proof of Theorem 3. (a) Since \( X_{\text{reg}} = X \), we see that for any \( x \in X \) there is a \( g \in G \) such that \( (g, x) \in D \), hence \( \bigcup gD = G \times X \) where \( G \) acts on \( G \times X \) by left-multiplication on \( G \). As a consequence, we have \( \bigcup gD = G \times X \) for a suitable finite subset \( S = \{g_0 = e, g_1, \ldots, g_m\} \subseteq G \). This set \( S \) will be used to construct \( \hat{X}(S) \).

(b) Let \( D^{(0)} \subseteq G \times \hat{X}^{(0)} \) be the image of \( D \), and consider the rational map \( \bar{\rho}_S : G \times \hat{X}^{(0)} \to G \times \hat{X}(S), (g, \bar{x}) \mapsto (g, \bar{\rho}(g, \bar{x})) \). We claim that \( \bar{\rho}_S \) is biregular. In fact, for any \( i \), the map \( (g, x) \mapsto (g, g \cdot x) \) is the composition of \( (g, x) \mapsto (g, g_i^{-1} g x) \) and \( (g, y) \mapsto (g, g_i y) \) where the first one is biregular on \( g_i D^{(0)} \) with image in \( G \times \hat{X}(S) \), and the second is biregular on \( G \times \hat{X}^{(0)} \), by Lemma 7. Now the claim follows, because \( G \times \hat{X}^{(0)} = \bigcup g_i D^{(0)} \), by (a).

(c) It follows from (b) that the rational action \( \bar{\rho} \) of \( G \) on \( \hat{X}(S) \) has the property that \( \bar{\rho}_S \) defines an open immersion \( G \times \hat{X}^{(0)} \to G \times \hat{X}(S) \). Now Theorem 3 follows from Lemma 5, setting \( Y := \hat{X}(S) \).

\( \square \)
1.7. Normal and smooth models. If $X$ is an irreducible $G$-variety, i.e., a variety with a regular action of $G$, then it is well-known that the normalization $\tilde{X}$ has a unique structure of a $G$-variety such that the normalization map $\eta: \tilde{X} \to X$ is $G$-equivariant. If $X$ is reducible, $X = \bigcup_i X_i$, we denote by $\tilde{X}$ the disjoint union of the normalizations of the irreducible components $X_i$, $\tilde{X} = \bigcup_i \tilde{X}_i$, and by $\eta: \tilde{X} \to X$ the obvious morphism which will be called the normalization of $X$. The proof of the following assertion is not difficult.

**Proposition 2.** Let $X$ be a $G$-variety and $\eta: \tilde{X} \to X$ its normalization. Then there is a unique regular $G$-action on $\tilde{X}$ such that $\eta$ is $G$-equivariant.

It is clear that for any $G$-variety $X$ the open set $X_{smooth}$ of smooth points is stable under $G$. Thus smooth models for a rational $G$-action always exist.

The next result, the equivariant resolution of singularities, can be found in KOLLÁR’s book [Kol07]. He shows in Theorem 3.36 that in characteristic zero there is a functorial resolution of singularities $BR(X): X' \to X$ which commutes with surjective smooth morphisms. This implies (see his Proposition 3.9.1) that every action of an algebraic group on $X$ lifts uniquely to an action on $X'$.

**Proposition 3.** Assume $\text{char } k = 0$, and let $X$ be a $G$-variety. Then there is a smooth $G$-variety $Y$ and a proper birational $G$-equivariant morphism $\phi: Y \to X$.

1.8. Projective models. The next results show that there are always smooth projective models for connected algebraic groups $G$. More precisely, we have the following propositions.

**Proposition 4.** Let $G$ be a connected algebraic group acting on a normal variety $X$. Then there exists an open cover of $X$ by quasi-projective $G$-stable varieties.

**Proposition 5.** Let $G$ be a connected algebraic group acting on a normal quasi-projective variety $X$. Then there exists a $G$-equivariant embedding into a projective $G$-variety.

**Outline of Proofs.** Both propositions are due to SUMIHIRO in case of a connected linear algebraic group $G$ [Sum74, Sum75]. They were generalized to a connected algebraic group $G$ by BRION in [Bri10, Theorem 1.1 and Theorem 1.2].

In this context let us mention the following equivariant Chow-Lemma. For a connected linear algebraic group $G$ it was proved by SUMIHIRO [Sum74] and later generalized to the non-connected case by REICHSTEIN-YOSSUSIN [RY02]. It implies that projective models always exist for linear algebraic groups $G$.

**Proposition 6** ([Sum74, Theorem 2], [RY02, Proposition 2]). Let $G$ be a linear algebraic group. For every $G$-variety $X$ there exists a quasi-projective $G$-variety $Y$ and a proper birational $G$-equivariant morphism $Y \to X$ which is an isomorphism on a $G$-stable open dense subset $U \subseteq Y$.

1.9. Proof of Theorem 2. We start with a rational action $\rho: G \to \text{Bir}(X)$ of an algebraic group $G$ on a variety $X$, and we assume that there is a dense subgroup $\Gamma \subseteq G$ such that $\rho(\Gamma) \subseteq \text{Aut}(X)$.

(a) We first claim that the rational $G$-action on the open dense set $X_{reg} \subseteq X$ is regular. For every $x \in X_{reg}$ there is a $g \in \Gamma$ such that $\tilde{\rho}$ is biregular in $(g, x)$. Since, by assumption, the $\rho_h$ are biregular on $X$ for all $h \in \Gamma$ it follows from Lemma 2(b)
that $\bar{\rho}$ is biregular in $(g', x)$ for any $g' \in \Gamma$. Moreover, by Proposition 1(b), we have $g' \cdot x \in X_{\text{reg}}$, hence $X_{\text{reg}}$ is stable under $\Gamma$.

By Theorem 3 we have a $G$-equivariant open immersion $X_{\text{reg}} \hookrightarrow Y$ where $Y$ is a variety with a regular $G$-action. Since the complement $C := Y \setminus X_{\text{reg}}$ is closed and $\Gamma$-stable we see that $C$ is stable under $\bar{\Gamma} = G$, hence the claim.

(b) From (a) we see that the rational map $\rho: G \times X \rightarrow X$ has the following properties:

(i) There is a dense open set $X_{\text{reg}} \subseteq X$ such that $\rho$ is regular on $G \times X_{\text{reg}}$.

(ii) For every $g \in \Gamma$ the rational map $\rho_g: X \rightarrow X$, $x \mapsto \rho(g, x)$, is a regular isomorphism.

Now the following lemma implies that $\rho$ is a regular action in case $X$ is affine, proving Theorem 2.

Lemma 8. Let $X, Y, Z$ be varieties and let $\phi: X \times Y \rightarrow Z$ be a rational map where $Z$ is affine. Assume the following:

(a) There is an open dense set $U \subseteq Y$ such that $\phi$ is defined on $X \times U$;

(b) There is a dense set $X' \subseteq X$ such that the induced maps $\phi_x: \{x\} \times Y \rightarrow Z$ are morphisms for all $x \in X'$.

Then $\phi$ is a regular morphism.

Proof. We can assume that $Z = A^1$, so that $\phi = F$ is a rational function on $X \times Y$. We can also assume that $X, Y$ are affine and that $U = Y_f$ with a non-zero divisor $f \in \mathcal{O}(Y)$. This implies that $f^kF \in \mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$ for some $k \geq 0$. Write $f^kF = \sum_{i=1}^n h_i \otimes f_i$ with $k$-linearly independent $h_1, \ldots, h_n \in \mathcal{O}(X)$. Setting $F_x(y) := f(x, y)$ for $x \in X$, the assumption implies that $F_x = \sum_{i=1}^n h_i(x) \frac{f_i}{f}$ is a regular function on $Y$ for all $x \in X'$.

We claim that there exist $x_1, \ldots, x_n \in X'$ such that the $n \times n$-matrix $(h_i(x_j))_{i,j=1}^n$ is invertible. This implies that the rational functions $\frac{f_i}{f}$ are $k$-linear combinations of the $F_{x_i} = f(x_i, y) \in \mathcal{O}(Y)$. Hence they are regular, and thus $F$ is regular. The lemma follows.

It remains to prove the claim. Assume that we have found $x_1, \ldots, x_m \in X'$ ($m < n$) such that the $m \times m$-matrix $(h_i(x_j))_{i,j=1}^m$ is invertible. Then there are uniquely defined $\lambda_1, \ldots, \lambda_m \in k$ such that $h_{m+1}(x_i) = \sum_{j=1}^m \lambda_j h_j(x_i)$ for $i = 1, \ldots, m$. Since $h_1, \ldots, h_m, h_{m+1}$ are linearly independent, it follows that $h_{m+1} \neq \sum_{j=1}^m \lambda_j h_j$. This implies that there exists $x_{m+1} \in X'$ such that $h_{m+1}(x_{m+1}) \neq \sum_{j=1}^m \lambda_j h_j(x_{m+1})$, and so the matrix $(h_i(x_j))_{i,j=1}^{m+1}$ is invertible. Now the claim follows by induction. □

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