INHOMOGENEOUS SYSTEMS AND THEIR RECTIFICATION PROPERTIES

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Abstract

We explore the possibility of obtaining unidirectional current in a symmetric (periodic) potential system without the application of any obvious (apparent) externally applied bias. There are many physical models proposed to accomplish this nonequilibrium effect. In the present work we consider inhomogeneous systems so that the friction coefficient and/or temperature could vary in space. We find out a model with minimal conditions that the inhomogeneous system assisted by fluctuating forces must satisfy, in order to obtain unidirectional current. In the process we discuss about thermal and frictional ratchets that are of current interest. We argue that different models of frictional ratchets work under the same basic principle of alteration of relative stability of otherwise locally stable states in the presence of temperature inhomogeneity. We also discuss in detail the nature of currents in rocked frictional ratchets. In particular we analyse a novel phenomenon of multiple current reversals and the efficiency of the energy transduction in these systems.

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1 Introduction

The study of interplay of noise and nonlinear dynamics presents many challenges in systems under non equilibrium conditions. These systems exhibit wide variety of novel physical outcomes (simple to complex). A nonequilibrium system is one in which there is a net energy flow from external sources. Many different models employing stochastic processes in physics, chemistry, engineering and biological sciences have lead to discovery of several noise induced phenomena in systems far away from equilibrium. Prominent examples are noise induced phase transitions, stochastic resonance, resonant activation, noise induced stability of unstable states, noise induced unidirectional transport of particles in the absence of obvious bias (thermal ratchets or theory of molecular motors), noise induced ordering ranging from separation of different materials into heterogeneous final state to the formation of a rich variety of regular patterns, self organization, etc [1-8]. The list is not exhaustive by any means. In all these phenomena mentioned above noise plays an active role (active noise paradigm). Noise or fluctuations arise either because of the coupling of system to an external unknown system or from the thermal bath. The presence of noise can alter the behaviour of system in a fundamental way - a change that has nothing to do with the sensitive dependence of initial conditions as in the chaotic systems. In contrast to the general notion that noise is undesirable and destructive, in many nonequilibrium systems it plays a constructive and stabilizing role in the dynamics. The most of the phenomena mentioned above exist only in the presence of noise, i.e., they can not be observed in the absence of noise. Hence noise acts as a generator of order as opposed to generator of disorder. Even when the magnitude of noise is small, the probability of the macrostate depends on the details of the global kinetics and can not be determined by the macrostate alone. In other words the stability criteria which examine only the immediate vicinity of a locally stable state are inadequate to assess the relative stability of states in a nonequilibrium system. The kinetics of unstable intermediate state even as these are rarely populated can have a dramatic effect on the relative stability of states. This leads to the notion of local versus global stability criteria that will be discussed in the following sections.

In the present work we will be mainly concerned with the nature of directed motion induced by the random noise in inhomogeneous systems in the absence of bias. The noise induced active transport in a fluctuating environment arises from so-called ratchet mechanism [7-24]. Here, nonequilibrium fluctuations combined with spatial or temporal anisotropy conspire to generate systematic motion even in the absence of any net bias. It should be noted that in accordance with the second law of thermodynamics usable work cannot be extracted if only equilibrium fluctuations are present. In thermal equilibrium the principle of detailed balance prohibits net particle current in any system. In contrast, in a nonequilibrium situation,
where the detailed balance is lost, net current flow is possible, i.e., one can extract energy by rectifying fluctuations or at the expense of overall increased entropy. The problem of rectification at Brownian scale was posed much earlier by Smoluchowski\cite{25} and Feynman\cite{26}. A major motivation for these studies (stimulating interaction between biology and physics) comes from the plausible theoretical arguments for the motion of molecular kinesin\cite{1,7,8,11}. The kinesin molecule belongs to a class of proteins known as motor molecules. These molecules which include dyneins and myosin move along the structural filaments such as microtubules, microfilaments. The motor molecules are used for the transportation of organelles (cargo, chemicals) for intra cellular transport and muscle contraction or to power muscles. The energy source for these molecules comes from hydrolysis of ATP. In ATP energy is stored in the phosphate bonds, this energy is released when the bond is hydrolyzed and ADP is produced. The motor proteins use this energy to bring about unidirectional motion along the biopolymers. Here chemical energy is converted into mechanical energy. The fluctuations in the potentials experienced by the motors are believed to arise from the binding and dissociation of ATP, and the anisotropic periodic potential as representing the electrostatic potential along the long structural filament. In these systems it is known that there is no known gradient of chemical concentration or the temperature, to determine the direction of the movement. Moreover, it is also clear that the motion of these motors can be described as a over damped motion of a Brownian particles. At any time the velocity $v$ of the particle is proportional to the force $F$ on the particle. These particles experience random kicks from the surrounding medium and the average thermal energy of a particle is $k_B T$. This energy $k_B T$ is comparable to the other involved energy scales in the problem, such as the barrier heights. Hence Brownian motion plays an essential role in the action of motors. Thus, protein motors operate in a Brownian regime where inertia is negligible and thermal fluctuations are important.

With a primary motivation to explain the unidirectional motion of molecular motors the subject of noise induced transport has gone beyond the biological realm. In physics, area of thermal ratchets are being explored to investigate new methods for controlled devices of high resolution for particle separation\cite{27}. These devices are expected to be superior to existing methods such as electrophoretic method for particles of micrometer scale, like cells, latex spheres, DNA or proteins. For this the current reversal phenomenon is one of the most interesting aspect of the theory of Brownian ratchets, i.e., magnitude and the direction of the Brownian particles are very sensitive to their masses. New questions regarding the nature of heat engines (reversible and irreversible) at atomic or molecular scales, their energetics, and the efficiency of the energy conversion are being studied\cite{28}. In Brownian heat engines one would also like to understand the possible sources of irreversibility and whether the irreversibility can be suppressed such that the efficiency approaches that of a Carnot cycle.

The collection of large number of interacting Brownian particles exhibit cooperative ef-
flicts. Interactions can lead to dynamical phase transitions and instabilities that characterise the behaviour of motor collections. It turns out that cooperative motors can generate a directed force even if the system is symmetric\cite{29}. The direction of motion of a symmetric system is selected by spontaneous symmetry breaking. In symmetric case, the transition is isomorphous to a paramagnet-ferromagnet transition, in the asymmetric case to a liquid-vapor transition. In some special cases the coupled systems display amazing features when exposed to an externally applied force. The most prominent being a zero-bias negative conductance and anomalous hysteresis\cite{30}, which are completely new collective phenomena. In game theory based on the theory of Brownian ratchets, new area of paradoxical gambling games have emerged under the subject of Parronodo’s paradoxes\cite{31}. Here, two losing (with probability one) gambling games, when combined or played in a random sequence, can lead to a winning game with probability one! It has also been suggested that mobility-induced transport can be used as a concept of possibilities to think about social and financial systems that are so often out of equilibrium\cite{32}. Ratchet devices based on quantum processes have been proposed. Using quantum dots these devices have been investigated experimentally. Net currents due to quantum coherent motion of electrons have been observed even when the applied ac voltage is zero on average. Strikingly the direction of the current is reversed as a function of temperature, exhibiting quantum to classical cross over, the phenomenon is of fundamental importance in the foundations of quantum mechanics\cite{33-34}.

Several qualitatively different physical models for noise induced transport have been proposed, namely, rocking ratchets, flashing ratchets, diffusion ratchets, correlation ratchets, frictional ratchets, etc. For this we refer the readers to an excellent recent review by Reimann\cite{8}. Most of these studies are restricted to non-interacting particles. In this article we confine to the physics in inhomogeneous medium where Brownian particles experience the friction coefficient and/or temperature to be non uniform in space. We find out a model with minimal conditions that the inhomogeneous system assisted by fluctuating forces must satisfy in order to obtain unidirectional current. For this we develop a self consistent approach based on microscopic treatment\cite{24,35}. In the process we discuss about thermal and frictional ratchets that are of current interest. In these inhomogeneous systems we show that directed current can be obtained even in spatially periodic symmetric potentials. The inversion symmetry in these systems being broken dynamically by the space dependent frictional coefficient or temperature inhomogeneity. Several other conditions to obtain noise induced transport can be relaxed in regard to the nature of external noise and their statistics as compared to non frictional ratchets. We explain the possibility of obtaining current reversals even in the adiabatic regime of rocked frictional ratchets as a function of system parameters\cite{36,37}. In a non-adiabatic regime multiple current reversals can be observed\cite{38}. Following the treatment of stochastic energetics we discuss the efficiency of the energy transduction and give conditions
under which noise can facilitate the energy conversion.

Nature of system inhomogeneity plays important role in deciding the nonequilibrium and kinetic properties of the system. Most of the systems that one comes across in nature are inhomogeneous. These inhomogeneities could be structural, configurational, entropic, temperature non-uniformities, etc. Brownian motion in confined geometries or in porous media show space dependent friction[39]. Particles diffusing close to surface have a space dependent friction coefficient[39,40]. It is believed that molecular motor proteins move close along the periodic structure of microtubules and therefore experience a position dependent mobility[32]. Frictional inhomogeneities are common in super lattice structures and in semiconductor systems[20]. In Josephson junctions periodically varying frictional coefficient corresponds to the term representing interference between the quasiparticle tunneling and the Cooper pair tunneling[41]. Nonuniformity in temperature can have important consequences on the particle motion, for instance, the kinetics of growth of crystalline nuclei in the melt around its critical size. The latent heat generation being, in this example, responsible for the creation of nonuniform temperature field across the surface of the nucleus. One can have inhomogeneous temperature field because of nonuniform distribution of phonons and electrons(or of quasiparticles in general) with different characteristic temperatures in the solid[22]. Temperature inhomogeneities can also be induced by external pumping of noise in the system.

We would like to emphasize and show explicitly later that the variations in space dependent friction does not alter the equilibrium properties of the system, however, affects the dynamics of the system in a nontrivial way. The relative stability of the competing states is generally governed by the usual Boltzmann factor. In the presence of nonuniform temperature the relative stability of two states is sensitive to detailed kinetics all along the pathways (on the potential surface between the two states under comparison). We reason in the present work that system inhomogeneity may provide a clear and unifying framework to approach the problem of macroscopic motion under discussion. The existing popular models,[8-19] currently in the literature, mostly take the nonequilibrium fluctuations to be non-Gaussian white (colored) noise together with a ratchetlike periodic potential to aid unidirectional motion of an overdamped Brownian particle. The ratchetlike periodic system potentials \( V(q) \), obviously violate parity \( V(q) \neq V(-q) \). For such potentials one can readily calculate steady current flow \( J(F) \) of a Brownian particle in the presence of an external field \( F \). It turns out that \( J(F) \) is not an odd function of \( F \) and in general, \( J(F) \neq -J(-F) \) in the regime of nonlinear response. In other words, reversal of the external force may not lead to a reversed current of the same magnitude in sharp contrast to the case of a nonratchetlike (symmetric) periodic potential system where \( J(F) = -J(-F) \) follows. From this general observation, in a ratchetlike potential, it can be easily concluded that on application of a zero time averaged periodic field, say \( F=Asin\omega t \), one can obtain net unidirectional current. Thus ratchet may also be
described as a nonlinear rectifier. The currents exhibit maxima both as a function of the forcing amplitude as well as temperature. When the forcing amplitude is small the motion is determined by the thermal activation rates (Kramer’s time scales) or overcoming the barriers to the left and right. Jump rates being asymmetric, we obtain a finite average drift velocity for nonzero forcing term. The average velocity is quite small at low temperatures due to the Arrhenius prefactor. At intermediate temperature, the jumps are more frequent and at high temperatures, jumps in both direction are of equal likelihood and the magnitude of the net current falls. For this reason we observe a maxima in current at finite temperature. This is the basic physics behind some of the physical models (known as rocked ratchet) used to obtain current rectification in a periodic potential system. There are models, however, that do not use oscillating external fields. Instead, colored noise of zero average strength—dichotomous, Ornstein-Uhlenbeck, Kangaroo processes,...[11-13], is used to drive the Brownian particle to obtain macroscopic motion in a ratchetlike potential system. We would like to emphasize that in rocked ratchet it is possible to obtain unidirectional current even in the absence of thermal noise, i.e., in the deterministic limit when the potential is asymmetric and the amplitude of the rocking force can be such that tilt in one direction can destroy all energy barriers allowing the particle to slide down (running state). However, they still have barriers when the potential is tilted by the same amount in the opposite direction and hence preventing the particle motion in the opposite direction. In this deterministic limit, motion exhibit intriguing structures as the function of forcing amplitude, such as current quantization and phase locking behaviour[38]. Inclusion of inertial effect allows the possibility of having both regular and chaotic dynamics. This deterministically induced chaos can mimic the role of noise. The system exhibits multiple current reversals with respect to the forcing amplitude. However, all these effects are not robust in the presence of noise.

There are further interesting models where the potential barriers themselves are allowed to fluctuate (flashing ratchet), for instance, with finite time correlations between two states under the influence of a noise source[8,11]. An example being an overdamped Brownian particle subjected to a ratchetlike periodic potential, where the asymmetric saw-tooth potential (in which two edges of the teeth make different angles with the vertical) is switched on to its full strength for time $\tau_{on}$ during which the Brownian particle slides down the potential slope to the bottom of the potential minima (fixed point). At the end of $\tau_{on}$, the system is put in the other (off) state during which the potential is set equal to a constant (say $= 0$, simplest case) for an interval $\tau_{off}$ and the particle executes free diffusive motion. At the end of $\tau_{off}$ the system is put back in the on state for interval $\tau_{on}$. This process of flipping of states is repeated ad-infinitum. If $\tau_{off}$ is adjusted in such a way that by the end of $\tau_{off}$ the diffusive motion just takes the particle out of the (now nonexistent) potential minima in the steeper slope direction (smaller distance, say in right direction) of the saw-tooth potential but
fails to do so in the gentler slope direction (larger distance), the immediate next on interval will take the particle to the adjacent minimum in the steeper slope side of the saw-tooth potential. Repetition of such sequential flipping of states for a large number of times lead to a net unidirectional macroscopic current (in the right direction) of the Brownian particle. It should be noted that a symmetrical nonratchetlike potential (fluctuating between two states) would, instead, have yielded symmetrical excursions of the Brownian particle and, hence, no net unidirectional motion. In this mechanism to obtain net current of the Brownian particle the system is supplied with the required energy externally to flip the system between the two states keeping the interval $\tau_{on}$ and $\tau_{off}$ fixed (thus the system is in a nonequilibrium state). There is a lot of freedom to play around with the parameters $\tau_{on}$, $\tau_{off}$, the saw tooth potential, and the thermal noise strength. And a judicious tuning of these parameters could even result in the reversal of the current. The flipping process could, however, be effected also by a finite time-correlated fluctuating dichotomous force. It should be noted that the former flipping process of definite $\tau_{on}$ and $\tau_{off}$ have been practically exploited in the particle separation techniques, whereas the latter fluctuating flipping time process has some appeal to natural processes. The diversity of the models just does not end here. Unlike rocking ratchets (deterministic limit with high amplitude of rocking), thermal fluctuations are essential to get net unidirectional motion in all models of flashing ratchets. Interestingly in all these flashing ratchet models particle need not climb the barrier, it only has to slide down the potential slope. There have been attempts too to obtain macroscopic current with Gaussian white noise under nonratchetlike symmetric periodic potential field as well but subjected to temporally asymmetric periodic external fields [16-18]. We do not attempt here, however, to give a review of the models considered. We present, in the following, a framework to obtain macroscopic motion in an inhomogeneous system with space dependent friction coefficient and nonuniform temperature fields restricting ourselves to periodic potential systems.

In all the models just mentioned [8] (The list is not exhaustive.) the system was taken to be homogeneous as far as the question of diffusivity was concerned. However, in some of the works, earlier to the ones alluded to so far, the nonuniformity of the diffusion constant of the system was considered to yield macroscopic transport [20-21]. The diffusion coefficient could be space dependent or state dependent and so the system may dissipate energy during its time evolution differently at different space points. Unlike homogeneous systems, however, the physics of inhomogeneous systems has not been free from controversies [21,44], such as, whether the equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial q^2} D(q)P, \quad (1a)$$

or
\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial q} D(q) \frac{\partial P}{\partial q},
\]

should be the correct form of diffusion equation. Nevertheless, such controversies apart, Büttiker [20] and also van Kampen [21] have shown that one can expect macroscopic transport of a Brownian particle in a periodic potential field when the diffusion coefficient is also periodic with the same periodicity but shifted by a phase difference other than 0 and \( \pi \) with respect to the periodic potential field. It should be noted that the potential field is not required to be ratchetlike. The system is rendered nonequilibrium by diffusion coefficient inhomogeneity in the system and the "stationary state" of the system is no longer governed by the usual Boltzmann factor. In the theory, one needs to go beyond the phenomenological description.

In reference [35], a microscopic treatment is given for the derivation of the macroscopic equations of motion in an inhomogeneous medium (space dependent friction coefficient and spatially nonuniform temperature) starting from a microscopic Hamiltonian of the system in contact with (phonon) heat bath(s). Moreover, a proper overdamped limit of the Langevin equation of motion in such an inhomogeneous medium is derived. A correct form of the corresponding Fokker-Planck equation is obtained and it is explicitly shown that neither of the two forms of the diffusion equation mentioned above [eq.(1)] is correct. From this macroscopic equation of motion one obtains an expression for the average current which depends on the details of the potential field and the inhomogeneities of the system.

As mentioned earlier, the nonuniformity of diffusion coefficient can arise either because of the space dependence of the friction coefficient \( \eta(q) \) or that of the temperature \( T(q) \) or because of both[40.44]. We, however, discuss possibilities of macroscopic current flow as a result of various kind of inhomogeneities in a symmetric periodic potential system. The first case we consider is when \( \eta(q) \) and \( T(q) \) are space dependent. By taking \( \eta(q) \) and \( T(q) \) periodic one gets a tilt in the potential field throughout the sample as discussed phenomenologically in ref.[20], resulting in a macroscopic current because of thermal fluctuations. In this case the temperature inhomogeneity is crucial and one can obtain current even when the friction coefficient becomes uniform. Friction coefficient inhomogeneity alone, however, does not generate macroscopic current. In the second case, we consider a thermal particle in a system with space dependent friction coefficient but subjected to external white noise fluctuations, and in the third case a thermal particle subjected to external space dependent white noise is considered. We then discuss the case of a Brownian particle coupled to two thermal baths. It should be noted, however, that in all the cases that we have considered do not require the potential to be ratchetlike nor do we require the fluctuating forces to be correlated in time to obtain macroscopic current. Finally we discuss the noise induced currents, their efficiency
and the phenomenon of multiple current reversals in rocked frictional ratchets.

In section 2 we provide a derivation of the macroscopic equation of motion in an inhomogeneous system from a microscopic Hamiltonian of a Brownian particle interacting with a (phonon) heat bath. We, then, obtain proper Smoluchowski equation from the derived Langevin equation of motion following the prescription of Sancho et al [45]. We use, in Sec.3, this overdamped equation of motion in an inhomogeneous system with space dependent friction coefficient and nonuniform temperature field to obtain nonzero macroscopic current. In the same section we elaborate three other possible cases of inhomogeneous systems where macroscopic current could be possible. The section 4 is devoted to AC driven frictional ratchets and 5 for discussions.

2 Equation of motion in inhomogeneous systems

We consider an inhomogeneous system where the inhomogeneity could arise either because of the space dependence of friction coefficient, or the nonuniformity of the temperature field or because of the combined effect of both. The effect of the nonuniformity of temperature or temperature gradient, however, cannot be incorporated as a potential term in the Hamiltonian formalism in sharp contrast to, for instance, the amenability of incorporation of electric field gradient in the Hamiltonian of a charged particle. We, therefore, incorporate the effect of temperature inhomogeneity at the end directly into the equation of motion obtained from the microscopic Hamiltonian suited to take care of the space dependence of the friction coefficient.

2.1 Equation of motion in a space dependent friction field

We consider a (subsystem) Brownian particle, of mass \( M \), described by a coordinate \( Q \) and momentum \( P \) moving in a potential field \( V(Q) \) of the system and being in contact with a thermal (phonon) bath. The bath oscillators are described by coordinates \( q_\alpha \), momenta \( p_\alpha \) and mass \( m_\alpha \) with characteristic frequencies \( \omega_\alpha \). We consider the total Hamiltonian \[ H = \frac{P^2}{2M} + V(Q) + \sum_{\alpha} \left[ \frac{p^2_\alpha}{2m_\alpha} + \frac{m_\alpha \omega^2_\alpha}{2} \left( q_\alpha - \frac{\lambda_\alpha A(Q)}{m_\alpha \omega^2_\alpha} \right)^2 \right], \]

The interaction of the subsystem with the thermal bath is through the linear (in \( q \)) coupling term \( \lambda_\alpha q_\alpha A(Q) \). From (2) one obtains the following equations of motion.

\[
\dot{Q} = \frac{P}{M},
\]
\[ \dot{P} = -V'(Q) + \sum_{\alpha} \lambda_{\alpha} A'(Q) \left[ q_{\alpha} - \frac{\lambda_{\alpha} A(Q)}{m_{\alpha} \omega_{\alpha}^2} \right], \]  

(3b)

\[ \dot{q} = \frac{p_{\alpha}}{m_{\alpha}}, \]  

(3c)

\[ \dot{p}_{\alpha} = -m_{\alpha} \omega_{\alpha}^2 q_{\alpha} + \lambda_{\alpha} A(Q), \]  

(3d)

where \( A'(Q) \) is the derivative of \( A(Q) \) with respect to \( Q \). After solving (3c) and (3d) for \( q_{\alpha} \) by using the method of Laplace transform and substituting its value in (3b), we obtain

\[ \dot{Q} = \frac{P}{M}, \]  

(4a)

\[ \dot{P}(t) = - V'(Q) - \sum_{\alpha} \frac{\lambda_{\alpha}^2 A'(Q)}{m_{\alpha} \omega_{\alpha}^2} \int_{0}^{t} dx \cos \omega_{\alpha}(t - t') A'(Q) \frac{P(t')}{M} \]

\[ + A'(Q) \sum_{\alpha} \lambda_{\alpha} \left[ x_{\alpha}(0) \cos(\omega_{\alpha} t) + \frac{\dot{x}_{\alpha}(0)}{\omega_{\alpha}} \sin(\omega_{\alpha} t) \right] + A'(Q) \sum_{\alpha} \frac{A(Q_{0}) \lambda_{\alpha}^2}{m_{\alpha} \omega_{\alpha}^2} \cos(\omega_{\alpha} t). \]  

(4b)

Here \( Q_{0} \) is the initial value of the particle co-ordinate \( Q \) and \( x_{\alpha}(0) \) and \( \dot{x}_{\alpha}(0) \) are the initial co-ordinates and velocities, respectively, of the bath variables. The second term in the right hand side of equation (4b) depends on the momenta at all times previous to \( t \). At this stage Markovian limit is imposed so that

\[ g(t - t') = \sum_{\alpha} \frac{\lambda_{\alpha}^2}{m_{\alpha} \omega_{\alpha}^2} \cos \omega_{\alpha}(t - t') = 2\eta \delta(t - t'). \]  

(5)

The equation (5) follows readily from the well known Ohmic spectral density distribution for the bath oscillators, i.e.,

\[ \rho(\omega) = \frac{\pi}{2} \sum_{\alpha} \frac{\lambda_{\alpha}^2}{m_{\omega_{\alpha}}} \delta(\omega - \omega_{\alpha}) = \eta \omega e^{-\frac{\omega}{\omega_{c}}}. \]  

(6)

where \( \omega_{c} \) is an upper cutoff frequency set by the oscillator spectrum of the thermal bath. The Markovian approximation (5) has the effect of neglecting the transient terms involving the initial coordinate \( Q_{0} \), in the equation of motion, for long time behaviour [46]. In other words, the equation should well describe the motion of the Brownian particle in time scales \( t > \omega_{c}^{-1} \).

The equation of motion thus assumes the form

\[ \dot{Q} = \frac{P}{M}, \]  

(7a)
\[ \dot{P} = -V'(Q) - \frac{\eta}{M} [A'(Q)]^2 P + A'(Q) f(t), \quad (7b) \]

where
\[ f(t) = \sum_{\alpha} \lambda_{\alpha} \left[ q_{\alpha}(0) \cos(\omega_{\alpha} t) + \frac{\dot{q}_{\alpha}(0)}{\omega_{\alpha}} \sin(\omega_{\alpha} t) \right]. \quad (8) \]

The force \( f(t) \) is fluctuating in character because of the associated uncertainties in the initial conditions \( q_{\alpha}(0) \) and \( \dot{q}_{\alpha}(0) \) of the bath variables. However, as the thermal bath is characterised by its temperature \( T \), the equilibrium distribution \( P_{eq}(q_{\alpha}(0), \dot{q}_{\alpha}(0)) \) of bath variables is given by the Boltzmannian form
\[ P(q_{\alpha}(0), \dot{q}_{\alpha}(0)) = \frac{1}{Z} \prod_{\alpha} e^{-\frac{1}{k_B T} \left( m_{\alpha} \ddot{q}_{\alpha}(0)^2 + m_{\alpha} \omega_{\alpha}^2 q_{\alpha}(0)^2 \right)}, \quad (8a) \]

where \( Z \) is the partition function. Using equation (8a) and (6) one can easily compute the statistical properties of the fluctuating force \( f(t) \). It is Gaussian with
\[ \langle f(t) \rangle = 0, \quad (9a) \]

and
\[ \langle f(t) f(t') \rangle = k_B T g(t - t') = 2k_B T \eta \delta(t - t'). \quad (9b) \]

It should be noted that the effect of the interaction term \( \lambda_{\alpha} q_{\alpha} A(Q) \) in the Hamiltonian (2) is to introduce a friction term and a fluctuating term \( f(t) \) in the equation of motion (7b). Moreover, \( A'(Q) = \text{constant} \) corresponds to a uniform friction coefficient. We redefine, \( [A'(Q)]^2 \eta = \eta(Q) \) and \( \frac{f(t)}{\sqrt{\eta}} \to f(t) \), and put \( M = 1 \), in (7) to obtain,
\[ \dot{Q} = P, \quad (10a) \]
\[ \dot{P} = -V'(Q) - \eta(Q) P + \sqrt{k_B T \eta(Q)} f(t), \quad (10b) \]

with
\[ \langle f(t) \rangle = 0, \quad (11a) \]

and
\[ \langle f(t) f(t') \rangle = 2 \delta(t - t'). \quad (11b) \]

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From eqs. (9) it follows that the derived Langevin equation of motion (10b) of a Brownian particle, in a system with space dependent friction $\eta(Q)$ but at constant uniform temperature $T$, is internally consistent and obeys fluctuation-dissipation theorem. We now proceed to incorporate the effect of space dependence of temperature, in a thermally nonuniform system, into the Langevin equation of motion by assuming that the Brownian particle comes in contact with a continuous sequence of independent temperature baths as its coordinate $q$ changes in time. (For notational simplicity, we replace the coordinate $Q$ and momenta $P$ by the corresponding lower case letters $q$ and $p$, respectively, reserving $P$ for probability distribution.)

\section*{2.2 Equation of motion in a space dependent friction and temperature field}

We consider each space point $q$ of the system to be in equilibrium with a thermal bath characterised by temperature $T(q)$. Also, it should be noted that one could take $\eta(q)$ to be constant piecewise along $q$, and in each piece of these $q$ segments eq.(10b) would correspond to an equation of motion with the constant friction coefficient but with the same statistical character of $f(t)$ (11a-11b) in all $q$ intervals. Let us discretise the system, for the sake of argument, into segments $\Delta q$ around $q$ and represent them by indices $i$. Let us further assume that each segment is connected to an independent thermal bath at temperature $T_i$ with corresponding random forces $f_i(t)$ so that the equation of motion (10b), in the segment $i$, will have the last term $\sqrt{k_B T_i}\eta(Q) f_i(t)$. As the two different segments $i$ and $j$ are each coupled to an independent temperature bath we have $\langle f_i(t) f_j(t') \rangle = 2\delta_{ij}\delta(t-t')$. Because $f(t)$ is $\delta$ correlated in time, as the particle evolves dynamically the fluctuation force $f_i(t)$ experienced by the Brownian particle while in the space segment $i$ at time $t$ will have no memory about the fluctuating force experienced by it at some previous time $t'$ while in the space segment $j \neq i$. The space-dependent index $i$ in $f_i(t)$, therefore, can be ignored and the equation of motion becomes local in time as well as in space. Therefore, in the continuum limit, the stochastic equations of motion of the Brownian particle, in an inhomogeneous medium with space dependent friction and nonuniform temperature, acquire the simple forms

\begin{align*}
\dot{q} &= p, \quad (12a) \\
\dot{p} &= -V'(q) - \eta(q)p + \sqrt{k_B T(q)\eta(q)} f(t), \quad (12b)
\end{align*}

with

$$\langle f(t) f(t') \rangle = 2\delta(t-t'). \quad (12c)$$
2.3 The Smoluchowski Equation

From eq.(12b) one can readily write down the Fokker-Planck equation or the Kramer’s equation for the full probability distribution $P(q,v,t)$. However, in most of the practical situations the marginal probability distribution $P(q,t)$ for the variable $q$ alone suffices to describe the motion of the Brownian particle. This probability distribution $P(q,t)$ can be obtained in the overdamped limit of the Langevin equation (12b) which is valid on time scales larger than the inverse friction $\eta^{-1}$. In other words in the overdamped case the fast variable, velocity $v$, is eliminated from the equation of motion. In the case of homogeneous systems one simply puts $\dot{p} = 0$ in eq.(12b) to obtain the overdamped Langevin equation. However, in case of inhomogeneous systems, the above method of adiabatic elimination of fast variables does not work, and leads to unphysical equilibrium distribution. The proper prescription for the elimination of fast variables has been given in Ref.[45] for systems with space dependent friction. The method retains all terms upto order $\eta^{-1}$ and the resulting overdamped Langevin equation yields physically valid equilibrium distribution. We, therefore, apply the same prescription to obtain the overdamped Langevin equation of motion in an inhomogeneous system with space dependent friction $\eta(q)$ and nonuniform temperature field $T(q)$. We obtain,

$$
\dot{q} = -\frac{V'(q)}{\eta(q)} - \frac{k_B}{2[\eta(q)]^2} [T(q)\eta'(q) + \eta(q)T'(q)] + \sqrt{\frac{k_BT(q)}{\eta(q)}} f(t), \quad (13)
$$

with

$$
\langle f(t)f(t') \rangle = 2\delta(t-t'). \quad (14)
$$

Using van Kampen Lemma[47] and the Novikov’s theorem[48] we obtain the corresponding Fokker-Planck equation as

$$
\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q} \left[ \frac{1}{\eta(q)} \left( \frac{\partial}{\partial q} k_BT(q)P(q,t) + V'(q)P(q,t) \right) \right]. \quad (15)
$$

Eq.(15) is the Smoluchowski equation for an overdamped Brownian particle moving in an inhomogeneous system with space dependent friction and nonuniform temperature. It should be noted that eq.(15) gives the correct form of diffusion equation instead of either of the two forms mentioned in eqs.(1). It is clear that the temperature and the friction coefficients influence the particle motion in a qualitatively different fashion and they cannot be plugged together to get effective diffusion coefficient to satisfy either of the forms of eq.(1). In the next section we discuss how the system inhomogeneity can help maintain a macroscopic unidirectional current.
3 Unidirectional currents in inhomogeneous systems (frictional ratchet)

We consider inhomogeneous systems where the inhomogeneity could be an internal property of the system or it could be imposed externally. As mentioned earlier we consider four cases where macroscopic motion can be obtained.

3.1 Rectification in an inhomogeneous system with space dependent friction and nonuniform temperature

When the system is bounded at \( q \to \pm \infty \), i.e., \( V \to \infty \) as \( q \to \pm \infty \), the system attains steady (stationary) state with zero probability current. In such a situation, we can calculate the steady state probability distribution \( P_s(q) \), from the Smoluchowski equation (15), by setting the probability current

\[
\frac{1}{\eta(q)} \left[ V'(q)P(q,t) + \frac{\partial}{\partial q} k_B T(q) P(q,t) \right] = 0
\]

(16)
equal to zero, as

\[
P_s(q) = Ne^{-\psi(q)},
\]

(17)
where

\[
\psi(q) = \int^q \left( \frac{V'(x) + kT'(x)}{k_B T(x)} \right) dx,
\]

(18)
and \( N \) is a normalization constant. It is very clear from the expression, eq.(18), for \( \psi(q) \) (effective or generalized potential) that the peaks of \( P_s(q) \) are determined not by the minima of \( V(x) \) alone but are determined as a combined effect with \( T(x) \). \( P_s(q) \) may even peak at positions which would be quite less likely to be populated in the stationary situations for uniform temperature, \( T(x) = T \) condition. In this respect, nonequilibrium situations appear strange. It is quite common in biological systems where, for example, otherwise less likely ion channels are, in some situations, found to be more active for ionic transport. Recently such nonequilibrium behaviour in biological systems have been theoretically attributed to the effect of nonequilibrium fluctuations and the process has been termed as kinetic focusing[49]. Moreover, it should be noted that the relative stability of two states of a system with nonuniform temperature field is not determined by the local function \( V(q) \) but by the entire pathway.
through a continuous sequence of intervening states between the two states under comparison. The temperature variation may modify the kinetics of these intervening states drastically and hence their contribution towards the relative stability will be substantial even when they are sparsely populated. For example, application of a localized heating at a point on the reaction coordinate lying between the lower energy minimum and potential energy barrier maximum can raise the relative population of the higher-lying energy minimum over that of a lower minimum given by usual Boltzmann factor. It should further be noted that \( \psi(q) \) is not determined by \( \eta(q) \) as it should be. Moreover, the functional form of \( \psi(q) \) is similar to \( \int q v(x) D(x) dx \), of course, in this case \( V'(q) \) has been augmented by a compensating force \( kT'(q) \).

\[ v(q) = \eta^{-1}(q)[V'(q) + kT'(q)] \]

is the drift velocity and \( D(q) = \eta^{-1}(q)k_BT(q) \) is the effective diffusion coefficient.

So far we have not assigned any functional form to \( V(q), T(q) \) and \( \eta(q) \). In ref.[20] it is shown that at least in one case the system can generate nonzero probability current, namely, when both \( V(q) \) and \( D(q) \) are periodic with same periodicity but having a phase difference other than 0 and \( \pi \). In our present problem if we assume \( V(q), T(q) \) and \( \eta(q) \) to be periodic functions with periodicity, say, \( 2\pi \) then the probability current is given by [20]

\[ J = \frac{1 - e^{-\delta}}{\int_0^{2\pi} dy e^{-\psi(y)} \int_0^{\psi(y)+2\pi} dx e^{\psi(x)}}, \]

where \( \delta = \psi(q) - \psi(q + 2\pi) \) determines the effective slope of a generalized potential \( \psi(q) \) and hence \( \delta \) being + or −ve determines the direction of current. It is obvious from the expression for \( \delta \) that the phase difference \( \phi \), between \( V(q) \) and \( T(q) \), alone determines the direction of current. It should further be noted that the amplitude of variation of \( \eta(q) \) does not determine the direction of current but does affect the magnitude of current. A periodic variation of \( \eta(q) \) and \( V(q) \) but uniform \( T(q) \) will yield no unidirectional current. For \( V(q) = V_0(1 - \alpha \cos(q - \phi)) \) and \( T(q) = T_0(1 - \alpha \cos(q - \phi)) \), with \( 0 < \alpha < 1 \) (for positive temperature) \( \delta \) turns out to be \( \frac{2\pi V_0 \sin\phi}{k_BT} \left[ \frac{1}{\sqrt{1 - \alpha^2}} - 1 \right] \) which is definitely nonzero for \( \phi \neq n\pi \), \( n = 0, \pm1, \pm2, \ldots \). Thus \( \phi \) alone determines the direction of nonzero current \( J \). In this case the periodic variation of temperature plays the crucial role and may yield current even when \( \eta(q) = \eta_0 = \text{constant} \). The dependence of current on \( \phi \) alone can readily be observed from the fact that the effective potential \( \psi(q) \) shows a tilt(or finite slope) in positive or in negative direction depending on the magnitude of \( \phi \). The transport arises because particles starting from the potential minima can climb the hot slope more easily than they can climb the cold slope, created by phase difference between the potential and temperature profile. Thus transition rates
from one value to another in the right direction is different from that in the left direction (\( \phi \) breaks the symmetry between left and right transition rates). We consider another simple case for which \( V(q) = V_0(1 - \cos(q)) \), \( T(q) = T_0/(1 - \alpha \cos(q - \phi)) \), with \( 0 \leq \alpha < 1 \) and space independent friction with magnitude \( \eta_0 \). For this simple case effective potential \( \psi(q) = V_0(1 - \cos(q) - (\alpha/4)[\cos(\phi) - \cos(2q - \phi)] - (\alpha/2)\sin(\phi)q)/T_0 \), where the last term clearly shows tilt responsible for the current. The magnitude of current is given by the eqn. (19) with \( \delta = V_0\alpha\sin(\phi)/(2T_0) \). Current is zero for \( \alpha = 0 \) and for \( \phi = n\pi \). Direction of current depends on \( \phi \). In fig.[4] we have plotted dimensionless current \( j \) versus \( T_0 \) the average value of the temperature field over the spatial period for various values of \( \alpha \). Here temperature is in a dimensionless form and is scaled with respect to barrier heights. Other parameters are given in the figure caption. Interestingly the current increases with \( T_0 \) starting from 0 at \( T_0 = 0 \) and saturates to a constant value at high \( T_0 \) limit. This saturation in the high temperature limit is specific to this model. In the inset we have shown the variation of current \( j \) with \( \phi \) for various values of \( T_0 \). It may be noted again that the direction of current is determined solely by \( \phi \). For further analysis and graphical presentation of the effective potential we refer to [20]. We now consider cases, where \( \eta(q) \) plays a decisive role.

3.2 Rectification in an inhomogeneous system with space dependent friction in the presence of an external parametric white noise

Unlike the case considered in subsection 3.1, where the overdamped Brownian particle experiences a fixed (in time) local (nonuniform) temperature profile \( T(q) \) during its sojourn \( q(t) \) for all \( t \), we consider, in this subsection, a system with uniform temperature \( T(q) = T \) but a spatially varying \( \eta(q) \). The Langevin equation of motion is given by

\[
\dot{p} = -V'(q) - \eta(q)p + \sqrt{k_BT\eta(q)}f(t)
\]

and the corresponding overdamped equation is

\[
\dot{q} = -\frac{V'}{\eta(q)} - \frac{k_BT\eta'(q)}{2[\eta(q)]^2} + \sqrt{\frac{k_BT}{\eta(q)}}f(t),
\]

with

\[
\langle f(t) \rangle = 0,
\]

and

\[
\langle f(t)f(t') \rangle = 2\delta(t-t').
\]
We, now, subject the system to an external parametric additive white noise fluctuating force $\xi(t)$, so that the equation of motion becomes

$$\dot{q} = -\frac{V'(q)}{\eta(q)} - \frac{k_B T \eta'(q)}{2[\eta(q)]^2} + \sqrt{\frac{k_B T}{\eta(q)}} f(t) + \xi(t),$$

(22)

with

$$\langle \xi(t) \rangle = 0,$$

$$\langle \xi(t) \xi(t') \rangle = 2\Gamma \delta(t - t'),$$

where $\Gamma$ is the strength of the external white noise $\xi(t)$. We can immediately write down the corresponding Fokker-Planck (Smoluchowski) equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial q} \left[ \left\{ \frac{V'(q)}{\eta(q)} \right\} P + \left\{ \frac{k_B T}{\eta(q)} + \Gamma \right\} \frac{\partial P}{\partial q} \right].$$

(23)

For periodic functions $V(q)$ and $\eta(q)$, with periodicity $2\pi$, one obtains unidirectional current following earlier procedure using equation (23). The resulting expression for current $J$ takes the same functional form as given in eq.(19) where $\psi(q)$ is now given by

$$\psi(q) = \int_0^q dx \frac{V'(x)}{k_B T + \Gamma \eta(x)}$$

(24)

and the effective diffusion coefficient

$$D(q) = (k_B T + \Gamma \eta(q))/\eta(q),$$

(25)

with

$$\delta = \psi(q) - \psi(q + 2\pi).$$

For $V(q) = V_0(1 - \cos(q))$ and $\eta(q) = \eta_0(1 - \alpha \cos(q - \phi))$, $\delta$ turns out to be equal to

$$\frac{2\pi V_0 \sin \phi}{\alpha \eta_0 \sqrt{\left(k_B T + \eta_0\right)^2 - (\eta_0 \alpha)^2}} - 1.$$ \(\text{As earlier the direction of current is determined by the phase difference } \phi \text{ between the periodic functions } V(q) \text{ and } \eta(q). \)

It is important to notice that there is no way one could obtain macroscopic current in the absence of the external white noise $\xi(t)$. This case, however, is similar in essence to the previous case of nonuniform temperature. In the present situation the overdamped Brownian particle is subjected to an external parametric random noise. The noise being externally imposed, the system always absorbs energy (without the presence of corresponding loss factor)
Also, the overdamped particle moves slowly wherever the friction coefficient $\eta(q)$ is large and the possibility of absorption of energy from the external white noise at those elements $q$ of the system, therefore, is correspondingly large. Thus, the effective temperature $T(q)$ of the system is given by $k_B T + \Gamma \eta(q)$, which modulates as $\eta(q)$ varies and hence the macroscopic current results as in the case 3.1.

### 3.3 Rectification in a homogeneous system but subjected to an external parametric space dependent white noise

The overdamped Langevin equation is,

$$\dot{q} = -\frac{V''(q)}{\eta} + \sqrt{k_B T \eta} f(t),$$

with $\langle f(t) \rangle = 0$ and $\langle f(t) f(t') \rangle = 2 \delta(t - t')$. Eq.(26) obeys fluctuation-dissipation theorem and hence in the absence of any external bias potential there can be no net current irrespective of the form of the periodic potential $V(q)$. We now subject the system to an external multiplicative Gaussian white noise fluctuation. The corresponding overdamped Langevin equation is given by

$$\dot{q} = -\frac{V'(q)}{\eta} + \sqrt{k_B T \eta} f(t) + g(q) \xi(t),$$

where $g(q)$ is an arbitrary function of $q$ and $\xi(t)$ is a white noise with

$$\langle \xi(t) \rangle = 0,$$

and

$$\langle \xi(t) \xi(t') \rangle = 2\Gamma \delta(t - t').$$

The associated Fokker-Planck equation can be immediately written down as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial q} \left( \frac{V''(q)}{\eta} P \right) + \frac{k_B T}{\eta} \frac{\partial}{\partial q} g(q) \frac{\partial}{\partial q} g(q) P + \Gamma \frac{\partial}{\partial q} g(q) \frac{\partial}{\partial q} g(q) P$$

(28)

Now, if we assume $V(q)$ and $g(q)$ to be periodic functions with periodicity $2\pi$, the net unidirectional current can be obtained and is given by eq.(19), with

$$\psi(q) = \int^q dx \frac{V'(x) + \eta \Gamma g(x) g'(x)}{k_B T + \eta \Gamma g^2(x)},$$

(29)
and the effective diffusion coefficient,
\[ D(q) = \frac{k_B T + \eta \Gamma g^2(q)}{\eta}. \]

For the specific form of the periodic functions
\[ V(q) = V_0(1 - \cos q), \]
and
\[ g(q) = \sqrt{g_0(1 - \alpha \cos(q - \phi))}, \]
we obtain,
\[ \delta = \frac{2\pi V_0 \sin \phi}{\eta \Gamma g_0 \alpha} \left[ \frac{k_B T + \eta \Gamma g_0}{\sqrt{(k_B T + \eta \Gamma g_0)^2 - (\eta \Gamma g_0)^2}} - 1 \right]. \]
The phase \( \phi \) being +ve or -ve determines the sign of \( \delta \) and consequently direction of the current \( J \) (eq.(19)).

It should be noted that, as in case 3.2, the overdamped Brownian particle experiences an effective space dependent temperature \( T(q) = k_B T + \eta \Gamma g(q)^2 \). The first case (3.1) corresponds to a system which is intrinsically nonequilibrium and requires an internal mechanism such as the generation of latent heat at the interface in first order transitions to maintain the temperature profile \( T(q) \). The other two cases (3.2 and 3.3) are, however, supplied with energy externally via the externally applied white noise. And finally, we consider a case where the Brownian particle is subjected to two thermal baths.

### 3.4 Rectification in an inhomogeneous system under the action of two thermal (noise) baths

We now consider the situation in which the system is in contact with an additive thermal noise bath at temperature \( T \) and a multiplicative thermal noise bath at temperature \( T' \). The corresponding equation of motion of the Brownian particle can be derived from a microscopic Hamiltonian and is given by [22]
\[ M\ddot{q} = -V'(q) - \Gamma(q)\dot{q} + \xi_A(t) + \sqrt{f(q)}\xi_B(t), \]  
\[ (30) \]
\( \xi_A(t) \) and \( \xi_B(t) \) are two independent Gaussian white noise fluctuating forces with statistics,
\[ \langle \xi_A(t) \rangle = 0, \]  
\[ (31a) \]
\[ \langle \xi_A(t)\xi_A(t') \rangle = 2\Gamma_A k_B T \delta(t-t'), \quad (31b) \]

and

\[ \langle \xi_B(t) \rangle = 0, \quad (32a) \]
\[ \langle \xi_B(t)\xi_B(t') \rangle = 2\Gamma_B k_B T \delta(t-t'), \quad (32b) \]

where, \( T \) and \( T \) are temperatures of the two baths A and B, respectively. It should be noted that, \( \xi_A(t) \) and \( \xi_B(t) \) represent internal fluctuations and together satisfy the fluctuation-dissipation theorem \( \Gamma(q) = \Gamma_A + \Gamma_B f(q) \). The bath B is associated with a space dependent friction coefficient \( f(q) \). When the two temperatures \( T \) and \( T \) become equal the system will be in equilibrium and no net current can flow. By making \( T \) and \( T \) different the system is rendered nonequilibrium and one can extract energy at the expense of increased entropy.

The system, thus, acts as a Maxwell’s-demon type information engine which extracts work by rectifying internal fluctuations. In ref. [22] an expression for current is obtained in the overdamped limit. The overdamped limit of the Langevin equation is taken by setting the left hand side of eq.(30) equal to zero. This procedure of obtaining overdamped limit is not correct as explained in section 1. Following the procedure of ref.[45] the correct Fokker-Planck equation in the overdamped limit is given by [23]

\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial q} \left\{ \frac{V'(q)P}{\Gamma(q)} + TT_A \frac{\partial}{\partial q} \frac{\partial}{\partial q} \frac{\Gamma(q)}{P} + \sqrt{f(q)} \frac{\partial}{\partial q} \frac{\sqrt{f(q)}}{\Gamma(q)} P + \sqrt{f(q)}' \sqrt{f(q)} \right\}. \quad (33) \]

For periodic functions \( V(q) \) and \( f(q) \) with periodicity \( 2\pi \) the noise induced transport current \( J \) is given by eq.(19), where, now

\[ \psi(q) = \int_q^q \frac{V'(x)\Gamma(x)}{TT_A + T_B f(x)} + \frac{(T - T)}{\Gamma(x)} \frac{\Gamma_A f'(x)}{(TT_A + T_B f(x))} \right\} dx, \quad (34) \]

and \( D(q) = \frac{TT_A + T_B f(q)}{(f(q))^2} \) and \( \delta = \psi(q) - \psi(q + 2\pi) \).

As in earlier cases, taking specific periodic forms of \( V(q) = V_0(1 - \cos q) \) and \( f(q) = f_0(1 - \alpha \cos(q - \phi)) \), the exponent \( \delta \) in eq.(19) for current is obtained as

\[ \delta = \left(1 - \frac{T}{T} \right) \frac{2\pi V_0 \sin \phi}{TT_B f_0 \alpha} \left[ \frac{TT_A + T_B f_0}{\sqrt{(TT_A + T_B f_0)^2 - (T_B f_0 \alpha)^2}} - 1 \right] \quad (35) \]
It is clear from the expression for $\delta$ that, again as in earlier cases (3.1-3.3), the phase difference $\phi$ between $V(q)$ and $f(q)$ determines the direction of current $J$. It is to be noted that, the current will flow in one direction if $T > T$ and will flow in the opposite if $T < T$ for given $\phi$. Thus, the system acts like a Carnot engine which extracts work by making use of two thermal baths at different temperatures ($T \neq T$). Moreover $\delta$ vanishes when $f(q)$ becomes space independent constant $f_0$, i.e., when $\alpha = 0$, and the current $J$ becomes zero. It should be noted further that when the amplitudes of $f(q)$ and $f'(q)$ are small compared to the amplitude of $V(q)$, the problem turns out to be equivalent to a particle moving in a spatially varying temperature field, $T(q) = (T \Gamma_A + T \Gamma_B f(q))/\Gamma(q)$ and, as discussed in section 3.1, such a nonuniform temperature field yields net unidirectional current.

4 AC driven frictional ratchets

Having shown that the average unidirectional motion is possible in a periodic system subjected to an external white noise fluctuations, we now turn our attention to rectification affect in inhomogeneous systems driven by ac force as opposed to white noise fluctuations. These ratchets are called as rocked frictional ratchets. We study phenomena of currents, their reversals and efficiency of energy transduction in these systems. We obtain separately results both in adiabatic (low frequency limit) and in nonadiabatic limit of external driving. We show that in adiabatic limit, when the potential is symmetric efficiency can me maximised as the function of noise strength or temperature. However in the case of asymmetric potential, temperature may or may not facilitate the energy conversion. For this case current reversal can also be obtained in the adiabatic limit as the function of temperature and the amplitude of the periodic drive. In the nonadiabatic limit by properly tuning the parameter the system exhibits multiple current reversals, both as a function of thermal noise strength and as a function of rocking force. Current reversals also occur under deterministic conditions and exhibits intriguing structures. These results are due to mutual interplay between potential asymmetry, noise, driving frequency and inhomogeneous friction.

4.1 Current reversals and efficiency in the adiabatic regime

In this subsection we study the motion of an over-damped Brownian particle in a potential $V(q)$ subjected to a space dependent friction coefficient $\eta(q)$ and an external force field $F(t) = A \cos(\omega t + \theta)$ at temperature $T$. The motion is described by the Langevin equation [24,35-38]

$$\frac{dq}{dt} = -\frac{(V'(q) - F(t))}{\eta(q)} - k_B T \frac{\eta'(q)}{[\eta(q)]^2} + \sqrt{k_B T \eta(q)} \xi(t),$$

(36)
where \( \xi(t) \) is a randomly fluctuating Gaussian white noise with zero mean and correlation, i.e., \( \langle \xi(t)\xi(t') \rangle = 2\delta(t-t') \). We take \( V(q) = V_0(q) + qL \). For generality we take \( V_0(q) \) to be asymmetric periodic potential, \( V_0(q + 2n\pi) = V_0(q) = -\sin(q) + \frac{\mu}{4}\sin(2q) \), where \( \mu \) is the asymmetry parameter and \( n \) being any natural integer. \( L \) is a constant force (load) representing the slope of the washboard potential against which the work is done. Also, we take the friction coefficient \( \eta(q) \) to be periodic: \( \eta(q) = \eta_0(1 - \lambda \sin(q + \phi)) \), where \( \phi \) is the phase difference with respect to \( V_0(q) \). The equation of motion is equivalently given by the Fokker-Planck equation

\[
\frac{\partial P(q,t)}{\partial t} = \frac{1}{\eta(q)} \left[ k_BT \frac{\partial P(q,t)}{\partial q} + (V'(q) - F(t))P(q,t) \right].
\]

This equation can be solved for the probability current \( j \) when \( F(t) = A = \) constant, and is given by [51-53]

\[
j = \int_0^{2\pi} k_BT \left( 1 - \exp(-2\pi (A - L)/k_BT) \right) \exp(\frac{-V_0(q) + (A - L)q}{k_BT}) d\xi + \int_{\pi}^{\pi} \eta(x) \exp(\frac{V_0(q) - (A - L)q}{k_BT}) d\xi.
\]

It may be noted that even for \( L = 0 \), \( j(A) \) may not be equal to \(-j(-A)\) for \( \phi \neq 0, \pi \). This leads to the rectification of current (or unidirectional current) in the presence of an applied ac field \( F(t) \). We assume \( F(t) \) changes slowly enough, i.e., its frequency is smaller than any other frequency related to relaxation rate in the problem. For a field \( F(t) \) of a square wave amplitude \( A \), an average current over the period of oscillation is given by \( \langle j \rangle = \frac{1}{2} [j(A) + j(-A)] \). This particle current can even flow against the applied load \( L \) and thereby store energy in a useful form. In the quasi-static limit following the method of stochastic energetics it can be shown [54-56] that the input energy \( R \) (per unit time) and the work \( W \) (per unit time) that the ratchet system extracts from the external noise are given by \( R[F(t)] = \frac{1}{t_f - t_i} \int_{x(t_i)}^{x(t_f)} F(t) dx(t) \) and \( W = \frac{1}{t_f - t_i} \int_{x(t_i)}^{x(t_f)} dV[x(t)] \). As defined earlier \( V \) is the total potential including the load. The efficiency of the energy transformation is given by \( \eta = \frac{W}{R} \). For the square wave of amplitude \( A \) we get \( R = \frac{1}{2}A[\langle j(A) \rangle - \langle j(-A) \rangle] \) and \( W = \frac{1}{2}L[\langle j(A) \rangle + \langle j(-A) \rangle] \) respectively. Thus the efficiency (\( \eta \)) of the system to transform the external fluctuation to useful work is given by

\[
\eta = \frac{L[j(A) + j(-A)]}{A[j(A) - j(-A)]}.
\]

For details we refer to [54,55]. Before proceeding further we would like to emphasize that from eqn. (32), one can obtain the mobility of the particle in a periodic potential tilted by a constant external field moving in an inhomogeneous media. The mobility is given by current
divided by constant external field and is a nonlinear function of external field. Unlike the constant friction coefficient case, where the mobility monotonically increases and saturates to a constant value, here the mobility shows stochastic resonance (SR) and asymptotically saturates to the previous constant value. The occurrence of SR in the absence of weak periodic signal is indeed a new phenomena[51-52], which essentially reduces the constraints for the observability of SR[4,5]. Besides SR, the mobility also shows resonance like phenomena as a function of external applied force and noise induced stability of states (mobility decreases as a function of noise strength). The noise induced stability is yet another counterintuitive novel concept aided by the presence of noise[6,52]. This phenomena is also related to the delay of decay of unstable states by enhancing the strength of thermal noise.

We shall first discuss the effect of inhomogeneity in a symmetric potential (i.e when $\mu = 0$) and under adiabatic conditions and then proceed to show in the next subsection how asymmetry and finite frequency drive brings about a qualitative change in the current characteristics. It is found, numerically, that the effect of $\lambda$, the amplitude of periodic modulation of $\eta(q)$, for instance on $< j >$ is more pronounced for larger value of $\lambda(0 \leq \lambda < 1)$. We choose $\lambda = 0.9$ throughout our work. Contrary to the homogeneous friction case where the net current goes to zero, the detailed analysis of the Eqn. (38) shows that the net current $< j >$ quickly saturates monotonically to a value equal to $-\frac{\lambda}{2} \sin \phi$ as a function of $A$ irrespective of the value of temperature $T$. All the interesting features are captured at smaller values of $A$ itself. We shall confine ourselves to $A < 1$, the threshold value at which the barrier of the potential $V_0(q) = -\sin q$ just disappears (in this regime the deterministic motion is in the locked state). In this case as mentioned earlier it is a phase lag $\phi$ creates spatial asymmetry responsible for currents, $\phi$ determines the direction of asymmetry of the ratchet. For $\phi > \pi$, we have a forward moving ratchet (current flowing in the positive direction) and for $\phi < \pi$ we have the opposite, in the presence of external quasi-static force $F(t)$. When the system is homogeneous ($\lambda = 0$), $< j > = 0$ for $L = 0$ and for all values of $A$ and $T$, because the potential $V(q)$ is symmetric. However, when $\lambda \neq 0, < j > \neq 0$ for all $\phi \neq 0, \pi$. Fig.(2) clearly illustrates this. In fig.(2) we have plotted $< j >$ (in dimensionless units) for various values of $\phi$ at $A = 0.5$ as a function of temperature $T$ (in dimensionless units). Henceforth all our variables like $< j >, A, T$ are in dimensionless units [53]. The inset shows the behaviour of $< j >$ with temperature for various $A$ when $\phi = 1.3\pi$. We have chosen $L = 0.02$ (corresponding to a positive mean slope of $V(q)$). One would expect, for $\lambda = 0, < j > < 0$ for all values of $A$ and $T \neq 0$ due to the presence of load or biasing field in the negative direction. Fig.(2) shows that $< j > > 0$ for all values of $\phi$ (the net current being up against the mean slope of $V(q)$). Though the result is counterintuitive, but it is still understandable. For $j(-A)$ the particle is expected to have acquired higher mean velocity before it hits the large $\eta(q)$ region and hence faces maximum resistive force and therefore though $j(-A)$ is negative it is small
in magnitude compared to \(j(A)(>0)\), where the particle faces just the reverse situation. The
average current \(<j>\) exhibits maximum as a function of temperature (SR like phenomena).
At large value of temperature, \(<j>\) becomes negative. In this regime of high temperature
the ratchet effect disappears and the current which is negative is in response to the load. The
peaking behaviour of \(<j>\) as a function of \(T\) is due to the synergetic effect of the thermal
fluctuations and the space dependent friction coefficient \(\eta(q)\). Of course, this result is obtained
in the quasi-static limit when the time scale of variation of \(F(t)\) is much larger compared to
any other relevant time scales involved in the system.

Fig.(3) shows the efficiency \(\eta\) of flow of current against the load (slope) \(L\). The parameter
values chosen for figs.(2 and 3) are same. \(\eta\) shows a maximum as a function of temperature.
This implies that thermal noise can facilitate energy conversion by ratchet system. In the
adiabatically rocked ratchet this can be seen only in the presence of space dependent friction.
It is to be noted that the temperature corresponding to maximum efficiency is not close
to the temperature at which the current \(<j>\) is maximum. From this we can conclude
that the condition of maximum current does not correspond to maximum efficiency of current
generation. This fact has been pointed out earlier but in a different system [55].

In fig.(4) we have plotted the average current \(<j>\) in the absence of load as function of
temperature for various values of \(A\) for given \(\phi=1.3\pi\). It shows that in the absence of load
\(L\), \(<j>\) never changes sign from positive to negative.

We now turn our attention to asymmetric potentials. It is possible to get a current reversal
as a function of temperature if, instead of a symmetric potential one considers an asymmetric
potential \(V_0(q) = -\sin q - \mu \sin 2q\) (where \(\mu\) lies between 0 and 1, describes an asymmetry
parameter) [57]. In fig.(5) we have plotted \(<j>\) in the absence of load as function of
temperature with asymmetry parameter \(\mu = 1\). In fig.(5) we notice that the current reverses
its direction as function of temperature or noise strength. Here too the current reversal
phenomena is solely due to the space dependent friction. In the large temperature regime
(compared to the barrier energies) the current due to space dependent friction dominates and
hence current does not depend on the asymmetry parameter \(\mu\). We have chosen \(\phi\) such that in
the high temperature limit current is negative. Even in the absence of spatial inhomogeneity
asymmetric potential exhibits current. In our case this current is in positive direction and
dominates in the low temperature regime even after the inclusion of inhomogeneities. Thus
going from the low temperature to the high temperature regime we have current reversal. Thus
current reversal can be obtained by properly tuning the system parameters. In homogeneous
systems (but asymmetric potential) current reversal is possible when the frequency of the
field swept \(F(t)\) is large[10,58] (nonadiabatic regime).

In the presence of spatial asymmetry efficiency exhibits a complex behaviour. Depending
on the system behaviour efficiency can monotonically decrease as a function of temperature
or it can show a peaking behaviour. Thus thermal fluctuations need not facilitate the energy conversion[36-37]. This is in contrast to the behaviour in the presence of symmetric potential, where the efficiency always exhibits a peaking behaviour as a function of the strength of thermal noise.

4.2 Current reversals in nonadiabatic regime

In this subsection we discuss the phenomena of current reversals in nonadiabatic regime in some detail. For finite frequency drive, the probability current density $J(x,t)$ is given by

$$J(x,t) = -\frac{1}{\eta(x)}[(V'(x) - F(t)) + k_B T \frac{\partial}{\partial x} P(x,t)],$$

where $F(t + \tau) = F(t)$ ($\tau$ being the period of the forcing term). Since the potential and the driving force have spatial and temporal periodicity respectively, therefore $J(x,t) = J(x + 1, t + \tau)$, [10,53]. The net current $j$ in the system is given by $j = \lim_{t \to \infty} \frac{1}{\tau} \int_{t}^{t+\tau} \int_{0}^{1} J(x,t) dx$. $j$ is independent of the initial phase $\theta$ of the driving force. We solve the FPE numerically by the method of finite difference and calculate the current $j$[38].

In the Fig. (6A), the average current $j$ is plotted as a function of temperature $T$ for different values of $\omega$. Here the asymmetry parameter $\mu = 1.0$ and $\lambda = 0.0$. In this case the current reverses its sign (only once) for frequencies sufficiently large as shown in the $\omega = 4.0, \omega = 5.0$ case. These frequencies corresponds to nonadiabatic regime [10] and are larger than intrawell relaxation frequency $\omega_0 = 3.18$. Current in the adiabatic limit can be obtained analytically[36-37]. In the absence of asymmetric potential and presence of space dependent friction ($\lambda = 0.9$), there is no current reversal irrespective of $\omega$ and phase shift $\phi$ as shown in Fig (6B). Hence asymmetric potential is essential for current reversal. However, the direction of the current depends on the phase lag $\phi$. Separately in both these cases absolute value of current exhibits a maxima as a function of $T$, reminiscent of SR. In the absence of space dependent friction fig. (6A), for frequencies higher than intrawell frequency $\omega_0$, the low temperature scenario is governed by interplay between potential asymmetry and $\omega$. Due to higher frequency the Brownian particles do not get enough time to cross the barriers. Most of the particles move about the potential minima. The probability of finding the Brownian particles near the minima increases with increasing frequency, consequently the probable number of particles near the potential barrier decreases. Since the distance from a potential minima to the basin of attraction of next minima is less from steeper side (which is at left side of the minima) than from the slanted side, hence in one period the particles get enough time to climb the potential barrier from the steeper side than from the slanted side, resulting in a negative current. On increasing the temperature, the particles get kick of larger
intensity and hence they easily cross the slanted barrier, resulting in a current reversal and positive current.

Now we study the combined effect of spatial asymmetry and frictional inhomogeneity in the nonadiabatic regime. In fig. (7A) we have plotted $j$ vs $T$ with $\mu = 1.0$, $\lambda = 0.1$ and $\phi = 0.2\pi$ for different values of $\omega$. In the presence of finite frequency drive there are two current reversals as shown in the figure (for the case of $\omega = 4.0$). This phenomena of twice current reversal with temperature $T$ is the foremost feature in our model, previously unseen in any overdamped system. In this case $\phi$ has been chosen in such a manner so that current in the high temperature limit (which is predominantly determined by frictional inhomogeneity than spatial asymmetry) is opposite to pure spatially asymmetric case ($\lambda = 0.0$). This is the cause of second current reversal as seen for the $\omega = 4.0$ case. The first current reversal being the effect of finite frequency driving and spatial asymmetry as discussed earlier. When the phase difference $\phi$ is such that the current due to space dependent friction alone is in the same direction as that of current due to potential asymmetry only, then we do not have current reversal in the adiabatic case as shown in Fig (7B) where $\phi = 1.2\pi$, though a single current reversal due to finite frequency drive may be present. In all the cases studied so far we have observed that the current reversal do not take place above a critical frequency $\omega_c$ of driving, which in turn depends sensitively on $\phi$ and other parameters in the problem.

Multiple current reversals can also be seen when the amplitude ($A$) of the forcing term is varied in a suitable parameter regime of our system. In Fig. 8, the plot of $j$ versus $A$ is shown for different values of $\omega$, keeping $\lambda$, $\phi$ and $T$ fixed at $0.1$, $0.88\pi$ and $0.05$ respectively. For $\omega = 4.0$ curve, we can see as many as four current reversals. For very large value of $A$, the current asymptotically goes to a constant value $-\frac{A}{2}\sin(\phi)$, as was previously shown for the adiabatic case. It should be noted that in the same asymptotic regime current goes to zero in the absence of space dependent friction. As the asymptotic value of current depends on $\phi$, so we can choose it appropriately to make it positive or negative. In the present case, $\phi$ has been chosen such that the asymptotic current is negative which guarantees at least one current reversal irrespective of frequency. The oscillatory behaviour in the $j - A$ characteristics is the reminiscent of deterministic dynamics. The inset in Fig. 8 shows current reversal even for the deterministic case also. In addition it exhibits the interesting phenomena of current quantization and phase locking. This current quantization give rise to oscillatory behaviour due to broadening of the steps in presence of small thermal noise. On further increasing the temperature these oscillatory behaviour vanishes.

In Fig (8) we have plotted $j$ versus $A$ for $\phi = 1.2\pi$ for various values of $\omega$. There is no current reversal in the adiabatic regime. Here $\phi$ has been chosen such that, the asymptotic current is in the same direction as that due to the spatial asymmetry of the potential (i.e positive). Hence no multiple current reversal can be seen in the nonadiabatic regime even
though there is oscillatory behaviour. The only reversal at low value of \( A \) is that of finite frequency drive as discussed previously. The observation of multiple current reversals can be attributed to a cooperative interplay between the spatial asymmetry of the potential, the friction inhomogeneity and the finite frequency drive. Depending on the system parameters we may have multiple current reversal or no current reversal at all (see Fig. (8 and 9)). All the above results can be understood qualitatively.

5 Summary and Discussion

Transport in a nonequilibrium periodic system has become, in recent times, a field of very active research. We have just tried, in the beginning of this work, only to enumerate various working ideas to build a plausible model of thermal ratchet. The brief enumeration is, of course, not complete. The models are being gradually refined and simplified to be close either to the experimental reality or to invent techniques to be useful in practice. Transport in inhomogeneous nonequilibrium systems has attracted attention since long. We have presented in Sec.2 a microscopic approach to obtain macroscopic equation of motion in such systems.

With the help of these equations in Sec. 3.1 we have considered a medium with space dependent temperature and friction and have discussed the origin of unidirectional current in these systems. In the situations considered, in sections 3.1 and 3.2, the system is subjected to external white noise fluctuations violating the fluctuation-dissipation theorem. Also, these two cases, in a sense, are physically equivalent to having a spatially varying temperature field as considered in section 3.1. In section 3.4 we have considered an inhomogeneous system under the action two thermal baths. This system acts like a Carnot engine which extracts work by making use of two thermal baths being at different temperature. This model too in the limit of small friction field modulation amplitude corresponds to a spatially varying temperature field. These observations seem to suggest that the case of inhomogeneous systems with spatially varying temperature field provides a general paradigm to obtain macroscopic current and several variants considered to model fluctuation induced transport may fall in the same general class of problems as considered in section 3.1. Moreover in all our models to obtain unidirectional current we require external white noise fluctuations as opposed to various models for homogeneous systems where application of colored noise (correlated noise) is must. For example, earlier models of thermal ratchets driven by colored noise in a small correlation time expansion (or in the unified colored noise approximation for arbitrary time) become identical to a Brownian particle moving in an inhomogeneous medium with space dependent diffusion coefficient [59]. The interesting idea of relative stability of states, which affects current, in the presence of temperature nonuniformity which is central to our treatment, however, has not
received the attention it deserves in the area of nonequilibrium thermodynamics. We remark that we do not require the periodic potential field of the system to be ratchetlike nor do we require the fluctuating force to be a colored noise to obtain macroscopic current. Thus we have put the problem of macroscopic unidirectional motion in nonequilibrium systems on a more general footing.

In Sec. 4 we have discussed the current rectification and their reversals in ac driven frictional ratchet both in adiabatic and in nonadiabatic regime. Both spatially symmetric and asymmetric potential has been considered. We observe several novel and complex features arising due to the combined effect of asymmetry and inhomogeneous friction and finite frequency driving force. Currents in the low temperature regime is mostly influenced by the asymmetry of the potential. At higher temperatures it is controlled by the modulation parameter $\lambda$ of the friction coefficient. We find current reversal with temperature even when the forcing is adiabatic. In the presence of finite frequency, twice current reversal occurs. As function of amplitude of the forcing term we observe multiple current reversals. Current even reverses its sign in the adiabatic deterministic regime and exhibits intriguing feature like phase locking and current quantization. Current reversals in ratchets are very sensitive to the nature of potential and system parameters. Even the condition of current direction cannot be readily predicted a priori. Ratchets of different kind (and there are as many as discussed by Reimann [8]) require different conditions for current reversals, which have been worked out in some limiting cases [8]. However, all ratchets exhibit great sensitivity to the form of underlying potential and its derivatives over the entire period (see the introduction of [8]). For example, it has recently been shown [60] that by barrier subdivision of potential profile with an integer $n$ number of modulations (in a randomly rocked ratchet) can lead to $n-1$ current reversals as a function of temperature. However, there is no universal rule to obtain current reversal. We expect that our analysis should be applicable for the motion of particle in porous media and for molecular motors where space dependent friction can arise due to the confinement of particles.

We have briefly discussed the efficiency of energy conversion in these systems in the adiabatic limit. We have shown that efficiency for energy transduction can be maximised as a function of noise strength and temperature. This in itself is a interesting observation. However, in the case of asymmetric potential the temperature may or may not facilitate the energy conversion. Some of our recent studies reveal that in nonadiabatic limit, efficiency of the system may be small or large compared to the adiabatic regime. This indicates that by going away from the quasi-static limit(adiabatic limit) efficiency can be increased contrary to the behaviour for reversible engines. For heat engines at molecular levels(molecular motors) the generalisation of thermodynamic principles to nonequilibrium steady states is a subject of current interest[61]. In a recent work[32] it has been shown that in two state ratchet mod-
els (flashing ratchets) mobility can take over the role of the potential. Contrary to previous models even no microscopic forces (or potentials) are involved in the transport mechanism. We further expect that several novel cooperative effects and quantum effects discussed in the introduction for homogeneous systems may lead to a rich variety of phenomenon in inhomogeneous systems. System inhomogeneities may further enhance our understanding of non-Debye relaxation (approach to equilibrium) even in the absence of potential disorder. To conclude we expect that the further studies of dynamics of particles in inhomogeneous non equilibrium systems may lead to new concepts in thermodynamics and statistical physics.

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Figure 1: Current is plotted as function of temperature $k_B T_0$ for $\phi = \frac{\pi}{2}$ and $\alpha = 0.5$. The inset shows current as function of $\phi$ for $k_B T_0 = 2.0, k_B T_0 = 1.0, k_B T_0 = 0.5$ from top to bottom and $\alpha = 0.4$. 
Figure 2: The net current $<j>$ as a function of $T$ for various values of $\phi$ at $A = 0$. The inset shows the variation of $<j>$ with $T$ for different $\phi$ with $<f>$ for various values of $\phi$. The inset figure 2 shows the net current $<j>$ as a function of $T$ for different $\phi$ at $A = 0.2$. The inset represents $<j>$ versus $T$ for both the figures.
Figure 3: Efficiency \( \eta \) as a function of \( T \) for various values of \( \phi \) at \( A = 0.02 \). The inset shows the variation of \( \eta \) as a function of \( T \) for different values of \( A \) at \( \phi = 1.06 \pi \) and \( \phi = 1.3 \pi \).
Figure 4: Current $\langle j \rangle$ as a function of $T$ for various values of $F$ at $\phi = 1.3\pi$. $L = 0$ in this case.
Figure 5: Current in asymmetric potential for various values of $\phi$ (for different values of $F_0$ in the inset) at $\phi = 0.3\pi$, $\phi = 0.2\pi$, and $\phi = 0$ when load $L = 0.3$.
Figure 6: Mean current $j$ vs temperature $T$ for (A) $\lambda = 0, \mu = 1.0$ and (B) $\lambda = 0.1, \mu = 0$ and $\phi = 0.2\pi$. Note there is no current reversal when the potential is symmetric.

Figure 7: The mean current $j$ vs temperature $T$ for $\phi = 0.2\pi$, $A = 0.5$ and $\lambda = 0.1$. The driving frequencies are $\omega = 3.0, 4.0$ and 5.0. The right hand side figure shows current $j$ vs $T$ for $\phi = 1.2\pi$. 

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Figure 8: The mean current $j$ with amplitude $A$ of the forcing term for $\phi = 0.88\pi$, $T = 0.05$ and $\lambda = 0.1$ with $\omega = 3.0, 4.0, 5.0$. The inset shows the reversal of deterministic current vs the amplitude of the forcing for $\omega = 0.25$.

Figure 9: The mean current $j$ with amplitude $A$ of the forcing term for $\phi = 1.2\pi$, $T = 0.05$ and $\lambda = 0.1$ with $\omega = 3.0, 4.0, 5.0$. The inset shows the deterministic current vs the amplitude of the forcing for $\omega = 0.25$. 

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