PROOF OF A CONJECTURE BY GAZEAU ET AL. USING THE GOULD-HOPPER POLYNOMIALS

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Abstract. We prove the “strong conjecture” expressed in [1] about the coefficients of the Taylor expansion of the exponential of a polynomial. This implies the “weak conjecture” as a special case. The proof relies mainly about properties of the Gould-Hopper polynomials.

1. Introduction

In [1], the authors state the following conjecture:

Conjecture 1. If \( \{a_i, 2 \leq i \leq p\} \) are positive numbers, and with the notation \( x_n! = \prod_{k=1}^{n} x_k \), then the numbers \( x_i \) such that

\[
\exp \left( t + \sum_{i=2}^{p} \frac{a_i t^i}{i} \right) = \sum_{n \geq 0} \frac{t^n}{x_n!}
\]

satisfy the recurrence relation

\[
x_n = \frac{n + 1}{1 + \sum_{i=2}^{p} a_i x_{n-i-1}!}.
\]

In the following, we prove this conjecture using Gould-Hopper polynomials as defined in [2] and some integral representations of these polynomials as introduced in [3].

2. Preliminary Tools

In [4], Nieto and Truax consider the operator

\[
I_j = \exp \left[ \left( c \frac{d}{dx} \right)^j \right]
\]

where \( c \) is a constant and \( j \) an integer. They remark that, for any well-behaved function \( f \), \( I_1 \) acts as the translation operator

\[
I_1 f(x) = f(x + c),
\]

which can also be viewed as the probabilistic expectation

\[
I_1 f(x) = \mathbb{E} f(x + Z_1)
\]

where \( Z_1 \) is the deterministic variable equal to 1.

In the case \( j = 2 \), with \( Z_2 \) denoting a Gaussian random variable with variance 2, \( I_2 \) acts as the Gauss-Weierstrass transform

\[
I_2 f(x) = \mathbb{E} f(x + cZ_2).
\]

It was shown in [4] that this result can be extended to any integer value of \( j \) as follows:

Proposition 2. For any integer \( j \geq 1 \), there exists a complex-valued random variable \( Z_j \) such that the following representation

\[
I_j = \mathbb{E} f(x + cZ_j)
\]

holds.
The properties of the complex-valued random variable $Z_j$ were studied further in [3]. The only important property we need to know here is that

$$\mathbb{E} Z_j^k = \begin{cases} 0 & \text{if } k \neq 0 \mod j \\ \frac{(pj)!}{p!} & \text{if } k = pj, \ p \in \mathbb{N} \end{cases}$$

and that, as a consequence, its characteristic function

$$\mathbb{E} \exp (uZ_j) = \exp (u^j), \ u \geq 0,$$

since a straightforward computation gives

$$\mathbb{E} \exp (uZ_j) = \sum_{k=0}^{+\infty} \frac{u^k}{k!} \mathbb{E} Z_j^k = \sum_{p=0}^{+\infty} \frac{u^{pj} p!}{p!} = \exp (u^j).$$

**Definition 3.** The Gould-Hopper polynomials [2, p.58] are defined as

$$g_m^n (x, h) = \mathbb{E} \left( x + \sum_{i=2}^{p} h_i^i Z_i \right)^n$$

and can be naturally generalized as

$$g_n (x, h) = \mathbb{E} \left( x + \sum_{i=2}^{p} h_i^i Z_i \right)^n$$

for any vector $h = [h_2, \ldots, h_p]$ such that $\{h_i \geq 0, \ 2 \leq i \leq p\}$.

**Lemma 4.** The Gould-Hopper polynomials satisfy the following identity

$$g_n (x, h) = \exp \left( \sum_{i=2}^{p} h_i^i \frac{d^i}{dx^i} \right) x^n.$$

**Proof.** We have

$$\exp \left( \sum_{i=2}^{p} h_i^i \frac{d^i}{dx^i} \right) x^n = \prod_{i=2}^{p} \exp \left( h_i^i \frac{d^i}{dx^i} \right) x^n$$

and the result follows by applying successively (2.1), we deduce the result. □

**Lemma 5.** The generating function of the Gould-Hopper polynomials $g_n (x, h)$ is

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n (x, h) = \exp \left( xt + \sum_{i=2}^{p} h_i t^i \right).$$

**Proof.** From the definition (2.4), we deduce, with $Z_2, \ldots, Z_p$ as in Proposition 2

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} g_n (x, h) = \mathbb{E} \exp \left( t \left( x + \sum_{i=2}^{p} h_i^i Z_i \right) \right) = \exp (xt) \prod_{i=2}^{p} \mathbb{E} \exp \left( t h_i^i Z_i \right)$$

and the result follows from (2.3). □

From this lemma, we deduce that the factorial coefficients $x_n!$ satisfy

$$(x_n!)^{-1} = \frac{g_n (x, h)}{n!}.$$

In order to obtain a recurrence formula for the numbers $x_n$, we need the following recurrence relation on the Gould-Hopper polynomials.
Lemma 6. The Gould-Hopper polynomials \((2.6)\) satisfy the difference equation
\[
g_{n+1}(x, h) = xg_n(x, h) + \sum_{k=2}^{p} kh_k \frac{n!}{(n-k+1)!} g_{n+1-k}(x, h).
\]

Proof. The moment representation \((2.5)\) yields
\[
g_{n+1}(x, h) = E \left( x + \sum_{i=2}^{p} h_i Z_i \right)^{n+1} = E \left( x + \sum_{i=2}^{p} h_i Z_i \right)^{n+1}
\]
\[= xE \left( x + \sum_{i=2}^{p} h_i Z_i \right)^{n} + \sum_{k=2}^{p} h_k E Z_k \left( x + \sum_{i=2}^{p} h_i Z_i \right)^{n}.
\]
The first term is identified as \(xg_n(x, h)\) and the second term is computed using the following lemma. □

Lemma 7. The random variables \(Z_j\) as defined in Proposition \((2)\) satisfy the following Stein identity
\[
E \left( Z_k f \left( x + \sum_{i=2}^{p} h_i Z_i \right) \right) = kh_k \frac{1}{k} E f^{(k-1)} \left( x + \sum_{i=2}^{p} h_i Z_i \right)
\]
for any smooth function \(f\).

Proof. The partial derivative
\[
\frac{\partial}{\partial h_k} E \left( f \left( x + \sum_{i=2}^{p} h_i Z_i \right) \right) = E \left( Z_k f' \left( x + \sum_{i=2}^{p} h_i Z_i \right) \right) \frac{1}{k} h_k - 1
\]
can also be computed from \((2.6)\) as
\[
\frac{\partial}{\partial h_k} \exp \left( \sum_{i=2}^{p} h_i \frac{d^i}{dx^i} \right) f(x) = \exp \left( \sum_{i=2}^{p} h_i \frac{d^i}{dx^i} \right) \frac{d^k}{dx^k} f(x)
\]
\[= \exp \left( \sum_{i=2}^{p} h_i \frac{d^i}{dx^i} \right) f^{(k)}(x) = E f^{(k)} \left( x + \sum_{i=2}^{p} h_i Z_i \right)
\]
so that
\[
E \left( Z_k f' \left( x + \sum_{i=2}^{p} h_i Z_i \right) \right) \frac{1}{k} h_k - 1 = E f^{(k)} \left( x + \sum_{i=2}^{p} h_i Z_i \right)
\]
which is the result after replacing \(f'\) by \(f\) in both sides. Using this result yields with \(f(x) = x^n\) yields the proof of Lemma \((6)\)

We can now prove the Conjecture \((1)\) as follows: by Lemma \((6)\) the quantities
\[
(x_n !)^{-1} = \frac{1}{n !} g_n(x, h)
\]
satisfy the recurrence
\[
(n + 1) (x_{n+1} !)^{-1} = x (x_n !)^{-1} + \sum_{k=2}^{p} kh_k (x_{n+1-k} !)^{-1}.
\]
Dividing both sides by \((x_n !)^{-1}\), and remarking that
\[
x_{n+1}^{-1} = \frac{(x_{n+1} !)^{-1}}{(x_n !)^{-1}},
\]
we deduce
\[
(n + 1) x_{n+1}^{-1} = x + \sum_{k=2}^{p} kh_k \frac{x_{n+1} !}{x_{n+1-k} !}.
\]
Choosing \(h_k = \frac{\alpha_k}{k}\) and \(x = 1\), we deduce the result.
We note that we have a proved slightly more general result than Conjecture \textsuperscript{11} namely the fact that the coefficients $x_n$ in the expression

$$\exp\left( xt + \sum_{i=2}^{p} \frac{a_i}{i!} t^i \right) = \sum_{n \geq 0} \frac{t^n}{x_n!}$$

satisfy the recurrence

$$x_{n+1} = \frac{n + 1}{x + \sum_{k=2}^{p} a_k \frac{x_{n+1-k}}{x_{n+1-k}!}}.$$

**References**

[1] H. Bergeron, E. M. F. Curado, J.-P. Gazeau, Ligia M. C. S. Rodrigues, Generating functions for generalized binomial distributions, arXiv:1203.3936v1 [math-ph]

[2] H. W. Gould and A. T. Hopper, Duke Math. J. 29-1, 51-63, 1962

[3] C. Vignat, A probabilistic approach to some results by Nieto and Truax, Journal of Mathematical Physics 51, 123505, 2010

[4] M. M. Nieto and D. R. Truax, Arbitrary-order Hermite generating functions for obtaining arbitrary-order coherent and squeezed states, Physics Letters A 208, 8-16, 1995

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