CURVATURE ESTIMATES OF SPACELIKE SURFACES IN DE SITTER SPACE.

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Abstract. Local estimates of the maximal curvatures of admissible spacelike hypersurfaces in de Sitter space for k-symmetric curvature functions are obtained. They depend on interior and boundary data.

1. Introduction

In this work we will consider solutions to fully nonlinear PDEs of the form

\[ F(A) = f(\lambda_1, \ldots, \lambda_n) = \psi, \quad \text{in } \Omega \subset S^n, \]

where \( A \) is the second fundamental form of a spacelike hypersurface in de Sitter space \( S^{n+1}_1 \). Furthermore \( f \) is a symmetric function of the eigenvalues of \( A \), and \( \psi \) is a function of the position vector and the tilt of the hypersurface to be defined below. We will assume that the hypersurface is the graph of a function over an open set of the sphere. More precisely, let \( \Omega \subset S^n \) be an open set and \( u : \Omega \rightarrow I \) a smooth function, where \( I = [R_1, R_2] \) is the real interval \( 0 < R_1 < R_2 \), such that the graph

\[ \Sigma = \text{graph}(u) = \{ Y = (u(\xi), \xi) \mid \xi \in \Omega \} \subset S^{n+1}_1 \]

is a spacelike hypersurface in de Sitter space \( S^{n+1}_1 \).

For \( 1 \leq k \leq n \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), let \( S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \), and define the normalised symmetric polynomial \( H_k(\lambda) = \binom{n}{k}^{-1} S_k \). In this paper we consider the case when \( f \) is the homogeneous function of degree one given by

\[ f(\lambda) = H_k^{1/k}(\lambda), \]

defined in an open convex cone \( \Gamma \) which is symmetric, with vertex at the origin and contains the positive cone \( \Gamma^+ = \{ \lambda \in \mathbb{R}^n \mid \lambda_i > 0, \forall i = 1, 2, \ldots, n \} \).

Since \( f \in C^2(\Gamma) \cap C^0(\Gamma) \), \( f_{\lambda_i} > 0 \) for all \( i = 1, 2, \ldots, n \), and \( f(\lambda) \) is concave in \( \Gamma \), it follows that \( F \) is elliptic and concave. A solution \( u \) will be called admissible if the principal curvatures \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of the spacelike hypersurface \( \Sigma \) given by (2) belong to the connected component of \( \Gamma_k \) containing \( \Gamma^+ \), where \( \Gamma_k := \{ \lambda \in \mathbb{R}^n \mid H_k(\lambda) > 0 \} \).

The existence of solutions of such equations has been studied in [4] by L. Caffarelli, L. Nirenberg and J. Spruck. In [5], they proved the existence of starshaped hypersurfaces in Euclidean space with prescribed \( k \)-symmetric curvature using the
a priori $C^{2,\alpha}$ estimate needed to carry out the continuity method. By the Evans-Krylov theorem it is sufficient obtain the apriori $C^0$, $C^1$ and $C^2$ estimates for admissible solutions, where the last one follows from an estimate of the maximal principal curvature of the hypersurface.

For various ambient Riemannian manifolds, curvature estimates for starshaped hypersurfaces with given $k$-symmetric curvature have also been proved. Namely for hypersurfaces in the sphere, the lower order and the curvature estimate are given in [2] by M. Barbosa, L. Herbert and V. Oliker. These were used for the existence result by Y. Li and V. Oliker in [11]. The curvature estimate and the existence result for hypersurfaces in the hyperbolic space was proved by Q. Jin and Y. Li in [10] using similar arguments of W. Sheng, J. Urbas and X. Wang in [12]. The lower order estimates for this case are also contained in [2] and used to complete the existence result. For spacelike hypersurfaces in Minkowski space and Lorentz manifolds various results have been proved by R. Bartnik and L. Simon [3], C. Gerhardt [6, 7, 8], Y. Huang [9] and the references provided in them.

We obtain similar curvature estimates as in [9] in de Sitter space. As in [9] we impose a growth assumption on the right hand side in terms of the tilt $\tau$ (see (19)). We introduce in Section 2 the geometric formulae of hypersurfaces in Lorentzian Manifolds, and provide explicit expressions for hypersurfaces in de Sitter space. In Section 3 we prove the following

Theorem 1. Let $\Omega \subset S^n$ be a domain in the round sphere, and let $u \in C^4(\Omega) \cap C^2(\Omega)$ an admissible solution of the boundary value problem

$$\begin{cases} F(A) = H_k^2(\lambda(A)) = \psi(Y,\tau) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases},$$

where $A$ is the second fundamental form of a spacelike surface $\Sigma$ in de Sitter space given by (10), $\psi \in C^\infty(\bar{\Omega})$, $\psi > 0$ and convex in $\tau$. Assume additionally that

$$\psi_\tau(X,\tau)\tau - \psi(X,\tau) \geq 0,$$

for all $X \in S^{n+1}_1$ and $\tau \in [1, \infty)$. Then

$$\sup_{\Omega} |A| \leq C,$$

where $C$ depends on $n$, $\|\varphi\|_{C^4(\Omega)}$, $\|\psi\|_{C^2(I \times \Omega \times [1,\infty))}$ and $\sup_{\partial\Omega} |A|$.

And finally in Section 4 we give an interior estimate when the growth condition is strict and the boundary data is spacelike and affine.

Theorem 2. Let $\Omega \subset S^n$ be a domain in the round sphere, and let $u \in C^4(\Omega) \cap C^2(\Omega)$ an admissible solution of the boundary value problem

$$\begin{cases} F(A) = H_k^2(A(A)) = \psi(Y,\tau) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases},$$

where $A$ is the second fundamental form of a spacelike surface $\Sigma$ in de Sitter space given by (10), $\psi \in C^\infty(\bar{\Omega})$, $\psi > 0$ and convex in $\tau$. Assume also that

$$\psi_\tau(X,\tau)\tau - \psi(X,\tau) > 0,$$

for all $X \in S^{n+1}_1$ and $\tau \in [1, \infty)$, and that the domain $\Omega$ is $C^2$, uniformly convex. If the boundary value $\varphi$ is spacelike and affine, namely $\varphi$ is the restriction of an affine function on ambient Minkowski space of $n + 2$ dimension. Then for any
where \( \Omega' \subset \Omega \), there is a constant \( C \) depending only on \( n, \Omega, \mathrm{dist}(\Omega', \partial \Omega) \), \( \|\varphi\|_{C^1(\Omega)} \) and \( \|\psi\|_{C^2(I \times \Omega \times [1, \infty))} \), such that

\[
\sup_{\Omega'} |A| \leq C.
\]

2. **Geometric formulae for hypersurfaces in de Sitter space**

We will recall some geometric formulae for hypersurfaces in Lorentzian manifolds and at the end we will apply them to the case of spacelike hypersurfaces in de Sitter space.

Let \( \{\partial_1, \ldots, \partial_n, N\} \) be a coordinate frame for a Lorentzian manifold \((M, \bar{g})\) and \( M \) a Lorentzian (not necessarily spacelike) hypersurface with induced metric \( g \) such that \( \{\partial_i\} \) span \( TM \), and let \( N \) be the unit normal field to \( M \) and put \( \epsilon = \bar{g}(N, N) \). When the induced metric is positive definite, then we say that \( M \) is a spacelike hypersurface. The metric \( g \) can be represented by the matrix \( g_{ij} = g(\partial_i, \partial_j) \) with inverse denoted by \( g^{ij} \).

The **Gauss formula** for \( X, Y \in T\Sigma \) reads

\[
D_X Y = \nabla_X Y + \epsilon h(X, Y)N,
\]

here \( D \) is the connection on \( M \), \( \nabla \) is the induced connection on \( M \) and the **second fundamental form** \( h \) is the normal projection of \( D \). In a coordinate basis we write

\[
h_{ij} = h(\partial_i, \partial_j).
\]

The **shape operator** is obtained by raising an index with the inverse of the metric

\[
h^i_j = g^{ik}h_{kj}.
\]

The **principal curvatures** of the hypersurface \( \Sigma \) are the eigenvalues of the symmetric matrix \( (h^i_j) \). The tangential projection of the covariant derivative of the normal vector field \( N \) on \( \Sigma \), \( \nabla_j N = (D_\partial_j N)\top \), is related to the second fundamental form by the **Weingarten equation**

\[
\nabla_j N = -h^i_j \partial_i = -g^{ik}h_{kj}\partial_i.
\]

The **curvature tensor** is defined for \( X, Y, Z \in T\Sigma \) as

\[
R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.
\]

The **Christoffel symbols** are given by

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),
\]

and the curvature tensor in terms of Christoffel symbols is

\[
R_{ijkl} = R^m_{ijkm} \partial_m = (\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^m_{js} \Gamma^s_{ik} - \Gamma^m_{sk} \Gamma^s_{jk}) \partial_m.
\]

Contracting with the metric

\[
R_{ijkl} = g(R(\partial_i, \partial_j)\partial_k, \partial_l) = g_{lm}R^m_{ijk}.
\]

We can also write the curvature tensor of the ambient manifold in terms of the curvature of the surface and the second fundamental form

\[
R_{ijk} = R^m_{ijk} \partial_m
\]

\[
= D_j(D_i \partial_k) - D_i(D_j \partial_k)
\]

\[
= (\nabla_j + D_j\top)(\nabla_i \partial_k + \epsilon h_{ik} N) - (\nabla_i + D_i\top)(\nabla_j \partial_k + \epsilon h_{jk} N)
\]

\[
= R_{ijk} + \epsilon h_{ik} \nabla_j N - \epsilon h_{jk} \nabla_i N + \epsilon D_j\top(hN)_{ik} - \epsilon D_i\top(hN)_{jk},
\]

\[
\Omega' \subset \subset \Omega.\]

\[\sup_{\Omega'} |A| \leq C.\]
where \( D^\perp(hN)_{jk} = D^\perp(h_{jk}N) - \Gamma^r_{ji}h_{rj}N - \Gamma^r_{kj}h_{rk}N. \)

From the last identity, when the ambient manifold is flat, we obtain the \textit{Codazzi equation} given by the identity

\[
\nabla_i h_{jk} = \nabla_j h_{ik}.
\]

Note that the first and second covariant derivatives of the second fundamental form are given by

\[
\begin{align*}
\nabla_i h_{ij} &= \partial_i h_{ij} - \Gamma^r_{ji}h_{ir} - \Gamma^r_{ij}h_{ri}, \\
\nabla_k \nabla_i h_{ij} &= \partial_k (\nabla_i h_{ij}) - \Gamma^r_{ki} \nabla_i h_{ij} - \Gamma^r_{ki} \nabla_i h_{ij} - \Gamma^r_{kj} \nabla_i h_{ir}.
\end{align*}
\]

The \textit{Gauss Equation} expressed in orthonormal coordinates, is given by

\[
R_{ijkl} = \epsilon (h_{ik} h_{jl} - h_{il} h_{jk}).
\]

When \( M \) is a hypersurface of a flat manifold \( \bar{R}_{kiij} = 0 \), the last equation simplifies to the identity

\[
R_{ijkl} = \epsilon (h_{ik} h_{jl} - h_{il} h_{jk}).
\]

Note that \( A \) is a bilinear symmetric tensor, and the following \textit{Ricci identity} holds

\[
\nabla_k \nabla_i A_{ij} - \nabla_i \nabla_k A_{ij} = R_{kljr} A_{ir} + R_{klir} A_{rj}.
\]

Let \( \mathbb{R}^{n+2}_1 = (\mathbb{R}^{n+2}, \bar{g}) \) be the Minkowski space with metric \( \bar{g} = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2 \) and covariant derivative \( \bar{D} \). Then \textit{de Sitter} space is defined as \( S^{n+1}_1 = \{ x \in \mathbb{R}^{n+2}_1 : \bar{g}(x, x) = 1 \} \) with the induced Lorentzian metric which we will denote by \( g \), and covariant derivative \( D \). Moreover, any point in \( S^{n+1}_1 \) can be written as \((r, \xi) \in \mathbb{R}^+ \times S^n\), with the induced metric

\[
g = -dr^2 + \cosh^2(r)\sigma,
\]

where \( \sigma \) is the round metric on \( S^n \), and later we will use \( \nabla \) to denote the covariant derivative for the metric \( \sigma \). The vector field \( \partial_r \) will be written separately from any other index notation \( \partial_{\alpha}, \partial_\beta, \ldots \), etc., the latter indices taking values form 1 to \( n \).

Let \( u : S^n \to [0, \infty) \) be a smooth function and consider a spacelike hypersurface in \( S^{n+1}_1 \) given by the graph \( \Sigma = \{(u(\xi), \xi)\} \). The tangent space of the hypersurface at a point \( Y \in \Sigma \) is spanned by the tangent vectors \( Y_j = u_j \partial_r + \partial_j \), the covariant derivative \( \nabla \) corresponding to the induced metric on \( \Sigma \) which is given by

\[
G_{ij} = -u_i u_j + \cosh^2(u)\sigma_{ij}.
\]

Since the metric is positive definite, its inverse can be computed

\[
G^{ij} = \cosh^{-2}(u)\sigma^{ij} + \frac{\sigma^{\gamma\mu} u_{\gamma} \sigma^{\eta\nu} u_{\eta}}{\cosh^2(u) - \cosh^2(u)|\nabla u|^2},
\]

where \( \nabla u = \sigma^{ij} u_j \partial_i \) and \( |\nabla u| := \sigma^{ij} u_i u_j \). Note that for this to be well defined we need to have \( |\nabla u|^2 \neq \cosh^2(u) \), and this is the case when the surface is spacelike.

A unit normal vector to \( \Sigma \) at the point \( Y \) can be obtained by solving the equation \( g(Y_{\alpha}, \hat{n}) = 0 \), and then we get

\[
\hat{n} = -\frac{\cosh^2(u)\partial_r + \nabla u}{\sqrt{\epsilon ( - \cosh^2(u) + \cosh^2(u)|\nabla u|^2)}},
\]

and moreover, since \( \Sigma \) is spacelike, then the following inequality must hold

\[
|\nabla u| \leq \cosh(u),
\]
because the unit vector $\hat{n}$ normal to $\Sigma$ is time-like, that is $g(\hat{n}, \hat{n}) = -1$.

The second fundamental form is the projection of the second derivatives of the parameterisation $D_{Y_1} Y_2$ on the normal direction. Notice that from (5), and writing $\hat{\Gamma}$ for the Christoffel symbols of the metric $\sigma$, we have

$$D_{\partial_r} \sigma = 0; \quad D_{\partial_r} \partial_j = \tanh(r) \partial_j; \quad D_{\partial_r} \partial_j = \cosh(r) \sigma_{ij} \partial_r + \hat{\Gamma}_{ij}^k \partial_k,$$

and using these identities we compute

$$D_{Y_1} Y_j = D_{u_1 \partial_r + \partial_i} (u_j \partial_r + \partial_j) = u_j u_1 D_{\partial_r} \partial_r + u_i D_{\partial_\lambda} \partial_j + u_i D_{\partial_r} \partial_r + D_{\partial_\lambda} \partial_j.$$

Let $W^2 = \cosh^4(u) - \cosh^2(u) |\nabla u|^2$, then $A_{ij} = g(D_{Y_1} Y_j, \hat{n})$ is given explicitly by

$$(10) \quad A_{ij} = \frac{\cosh^2(u)}{W} \left( \nabla^2_{ij} u - 2 \frac{\sinh(u)}{\cosh(u)} u_i u_j + \sinh(u) \cosh(u) \sigma_{ij} \right).$$

Recalling that the Minkowski space is a flat Lorentzian manifold, and letting $\h$ denote the second fundamental form of de Sitter space $S_1^{n+1}$, when we apply the Gauss equation (7) to the surface as a submanifold of codimension two $\Sigma \subset S_1^{n+1} \subset \mathbb{R}^{n+1}$, we have

$$0 = \bar{R}_{ijkl} = \bar{R}_{ijkl} - \epsilon_1 (h_{ik} h_{jl} - h_{il} h_{jk}) = R_{ijkl} - \epsilon_2 (A_{ik} A_{jl} - A_{il} A_{jk}) - \epsilon_1 (h_{ik} h_{jl} - h_{il} h_{jk}).$$

The Gauss formula applied twice reads

$$(11) \quad D_{Y_1} Y_j = \nabla_{Y_1} Y_j - A_{ij} \hat{n} - (Y_i, Y_j) Y.$$

For any function $f : S_1^{n+1} \times \mathbb{R} \to \mathbb{R}$, the partial derivative on $S_1^{n+1}$ and $\Sigma$ are defined respectively as

$$(13) \quad D^\sigma f = \sigma^{\alpha \beta} \frac{\partial f}{\partial x_\alpha} \partial_\beta, \text{ and } \nabla^\sigma f = (D^\sigma f)^\top.$$

Finally let us remark that at a given point of $\Sigma$ we can use coordinates such that the second fundamental form $\{A_{ij}\}$ is diagonal, thus $\lambda_n = A_{ii}$ at the point, and through the paper we assume $\lambda_1 \geq \cdots \geq \lambda_n$, and we may also assume that $\lambda_1 \geq 1$. The fact that $A$ is diagonal at a point also implies that $F^{ij} := \frac{\partial F}{\partial x_{ij}}$ is also diagonal and we can also write $F^{ii} = f_i$.

3. Proof of Theorem 1

We are now going to prove that if $u$ is an admissible solution of (1) then the curvature of the hypersurface is bounded, then the $C^2$ estimate of the solution will be a consequence of the equation of the second fundamental form (10) and lower order estimates. We will need the commutator formula for second order derivatives of the second fundamental form, given by Ricci’s identity (8), together with the Gauss equation of the surface as a codimension 2 spacelike submanifold of the Minkowski space. With this in account and together with equation (11) we obtain the following

$$(14) \quad R_{ijkl} = -(A_{ik} A_{jl} - A_{il} A_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}),$$
where we are using \( A_{ij} \) for the second fundamental form of the spacelike hypersurface in de Sitter space, and \( h_{ij} \) denotes the second fundamental form of de Sitter space in flat Minkowski space. Substituting in equation (8) we get

\[
\nabla_k \nabla_l A_{ij} = \nabla_l \nabla_k A_{ij} + \sum_r R_{lkjr} A_{ir} + \sum_R R_{kljr} A_{ri}
\]

Moreover, notice that by the Codazzi equation, the Ricci identity (8) and summing over \( r \) we get

\[
\nabla_i \nabla_j A_{kk} = \nabla_j \nabla_i A_{jj} = \nabla_k \nabla_k A_{jj} + F_{jj} A_{2jj} + \sum_j F_{jj} - \psi \left( n + \sum_j A_{jj}^2 \right).
\]

Let \( H = \sum_k A_{kk} \), we will use the identities above to compute \( F_{jj} \nabla_j \nabla_j H \) that will be used later. From (17) we have

\[
F_{jj} \nabla_j \nabla_j H = F_{jj} \nabla_k \nabla_k A_{jj} + H F_{jj} A_{jj}^2 - n F_{jj} A_{jj} - \psi \left( n + \sum_j A_{jj}^2 \right).
\]

Since \( H_{kk}^{1/k} \) is homogeneous of degree 1, it holds that \( F_{jj} A_{jj} = \psi \), and then

\[
F_{jj} \nabla_j \nabla_j H = \sum_k F_{jj} \nabla_k \nabla_k A_{jj} + H \left( F_{jj} A_{jj}^2 + \sum_j F_{jj} \right) - \psi \left( n + \sum_j A_{jj}^2 \right).
\]
Finally, the third identity is obtained using the previous equation for the hessian and we get

\[ F^{ij} \nabla_j \nabla_i H = - \sum_k F^{ij,k} \nabla_k A_j \nabla_k A_i + \sum_k \nabla_k \nabla_k \psi + H \left( F^{jj} A_{jj}^2 + \sum_j F^{jj} \right) - \psi \left( n + \sum_j A_{jj}^2 \right). \]

Now we consider the following parameterisation of the hypersurface

\[ Y = \sinh(u(\xi))E_1 + \cosh(u(\xi))\xi, \quad \xi \in \mathbb{S}^n, \]

where \( E_1 = (1, 0, ..., 0) \in \mathbb{R}^{n+1,1}. \) The tangent space to \( \Sigma \) is spanned by the vectors \( Y_i = u_i (\cosh(u)E_1 + \sinh(u)\xi) + \cosh(u)\xi_i = u_i \partial_r + \partial_i. \) We will write \( Y_i = \nabla_i \) and \( u_i = \partial_i u = \cosh(u)\xi_i u. \)

Note that

\[ \cosh(u)\partial_r = E_1 + \sinh(u)Y. \]

The tilt and the height functions are given respectively by

\[ \tau = \langle \hat{n}, E_1 \rangle = \frac{\cosh^2(u)}{\sqrt{\cosh^2(u) - |\nabla u|^2}}; \quad \eta = \langle Y, E_1 \rangle = -\sinh(u), \]

and

\[ \exp[\Phi(u, \xi)] = \frac{A_{11}}{g_{11}} \exp[\alpha(\tau) - \beta \eta]. \]

**Proposition 1.** For \( \tau \) and \( \eta \) defined as above, the following hold:

1. \( \nabla_i \eta = -\tau A_{ij} - \eta g_{ij}. \)
2. \( \nabla_j \tau = -g^{ik} A_{kj} \nabla_i \eta. \)
3. \( \nabla_j \nabla_i \tau = -g^{mn} \nabla_n A_{ij} \nabla_m \eta + \tau A_{mj} g^{mn} A_{ni} + A_{ij} \eta. \)

**Proof.** Using the Weingarten equation (4) we obtain the second identity

\[ \nabla_j \tau = \langle \nabla_j \hat{n}, E_1 \rangle = -A_{1j} Y_j, E_1 \]

\[ = -g^{ik} A_{kj} \langle Y_i, E_1 \rangle = -g^{ik} A_{kj} \nabla_i \langle Y, E_1 \rangle = -g^{ik} A_{kj} \nabla_i \eta. \]

The first of the identities follows using the Gauss formula applied twice (12)

\[ \nabla_i \nabla_j \eta = \langle E_1, \nabla_i \nabla_j Y \rangle = \langle E_1, -A_{ij} \hat{n} - g_{ij} Y \rangle = -\tau A_{ij} - \eta g_{ij}. \]

Finally, the third identity is obtained using the previous equation for the hessian of \( \eta \) as follows

\[ \nabla_j \nabla_i \eta = \nabla_j (-g^{mn} A_{nj} \nabla_m \eta) \]

\[ = -\nabla_j g^{mn} A_{nj} \nabla_m \eta - g^{mn} \nabla_j A_{nj} \nabla_m \eta - g^{mn} A_{nj} \nabla_m \eta \]

\[ = -g^{mn} \nabla_j A_{nj} \nabla_m \eta - g^{mn} A_{nj} \nabla_m \eta \]

\[ = -g^{mn} \nabla_n A_{ij} \nabla_m \eta + \tau A_{mj} g^{mn} A_{ni} + g^{mn} A_{ni} \eta g_{mj} \]

\[ = -g^{mn} \nabla_n A_{ij} \nabla_m \eta + \tau A_{mj} g^{mn} A_{ni} + A_{ij} \eta. \]

\[ \square \]
Proof of Theorem 1. We will estimate $|H|$ and since $H^2 = |A|^2 + 2S_2$, we will get the desired estimate by admissibility. Since $\psi = \psi(Y, \tau)$ we first note that

\begin{equation}
\nabla_k \nabla_l \psi = \nabla_k (\nabla^T_l \psi + \psi \nabla_l \tau)
\end{equation}

and also

\begin{equation}
\nabla_k \nabla^T_l \psi = D_k^L D_l^T \psi - (\nabla^T_l Y_k)(\psi) = -A_{kl} D_n^L \psi - g_{kl} D_A^L \psi.
\end{equation}

Then, in an orthonormal frame such that $A$ is symmetric and proceeding as in [9], we have

\begin{equation}
\sum_k \nabla_k \nabla_k \psi = \sum_k \nabla_k \nabla^T_k \psi + 2 \sum_k \nabla_k \nabla \tau \nabla \nabla \tau + \psi \nabla \nabla \tau \nabla \nabla \tau + \psi \sum_k \nabla_k \nabla_k \tau.
\end{equation}

From the assumption that $\psi$ is convex in $\tau$ and its regularity, and Proposition 1 it follows

\begin{equation}
\sum_k \nabla_k \nabla_k \psi \geq \psi \tau \sum_k \nabla_k \nabla_k \tau + \psi \tau \sum_k (\nabla_k \tau)^2 - C_1 H - C_2
\end{equation}

Note that at the maximum of $H$ we have $\nabla H \geq 0$ and $\nabla_j \nabla_i H \leq 0$, then it follows

\begin{equation}
0 \geq F^{ij} \nabla_j \nabla_i H. \end{equation}

We continue from equation (18) and using the last inequality (23), the concavity of $F$, the fact that $H \geq 0$ and $\sum_j F^{jj} \geq 0$ we get

\begin{equation}
0 \geq \sum_k \nabla_k \nabla_k \psi + H \left( F^{jj} A_{jj}^2 + \sum_j F^{jj} \right) - \psi \left( n + \sum_j A_{jj}^2 \right)
\end{equation}

\begin{equation}
\geq \psi \tau \left( \sum_k \tau A_{kk}^2 + H \eta \right) - C_1 H - C_2 + H F^{jj} A_{jj}^2 - \psi \left( n + \sum_j A_{jj}^2 \right)
\end{equation}

\begin{equation}
\geq -C_2 - n \psi + (\psi \tau - C_1) H + F^{jj} A_{jj}^2 H + (\psi \tau - \psi) \sum_k A_{kk}^2.
\end{equation}

Since $(\psi \tau - \psi) \geq 0$, we can improve the last inequality by dropping the last term. Using the Newton-Maclaurin inequalities $H_{k+1} H_{k-1} \leq H_k^2$ one can show (see [13]) the following

\begin{equation}
F^{ij} A_{kl} A_{ij} \geq \frac{1}{n} S^1_k S_1,
\end{equation}

and from this it follows that

\begin{equation}
0 \geq -C_2 - n \psi + (\psi \tau - C_1) H + C_3 \psi H^2
\end{equation}

which implies $H$ is bounded, hence $A$ is bounded. 

\end{proof}

4. Proof of Theorem 2

Proof. Consider the function $\gamma = \varphi - u$, $\gamma > 0$ in $\Omega$. Let

\begin{equation}
\Phi(\xi) = \ln(A_{11}) + \alpha(\tau) + \beta \ln(\gamma),
\end{equation}

its first covariant derivative

\begin{equation}
\nabla_j \Phi = \frac{\nabla_j A_{11}}{A_{11}} + \alpha' \nabla_j \tau + \beta \frac{\nabla_j \gamma}{\gamma}.
\end{equation}
The second covariant derivative is
\[
\nabla_j \nabla_j \Phi = \frac{\nabla_j \nabla_j A_{11}}{A_{11}} - \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2 + \alpha'' (\nabla_j \tau)^2
\]
\[
+ \alpha' \nabla_j \nabla_j \tau + \beta \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Using the commutator formula (15), we can replace the first term in the right hand side of the last equation, and we also multiply the first derivatives of the equation, to get an expression for \( F_{jj} \nabla_j \nabla_j \Phi \). Here, as usual, the notation indicates a sum over the repeated \( j \) index. Thus we get
\[
F_{jj} \nabla_j \nabla_j \Phi = \frac{1}{A_{11}} \left( \sum_j F_{jj} \nabla_j \nabla_j A_{jj} + F_{jj} A_{11} \right) + \frac{1}{A_{11}} \left( \nabla_j A_{11} \right)^2 + \alpha'' F_{jj} (\nabla_j \tau)^2
\]
\[
+ \alpha' F_{jj} \nabla_j \nabla_j \tau + \beta F_{jj} \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta F_{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Note that in coordinates such that \( h_{ij} = -\delta_{ij} \), some terms in the brackets cancel. Now, using the identity \( F_{jj} A_{jj} = \psi \) from the homogeneity of (3), we can write
\[
F_{jj} \nabla_j \nabla_j \Phi = \frac{1}{A_{11}} \left( \sum_j F_{jj} \nabla_j \nabla_j A_{jj} + F_{jj} A_{jj} \right) - \left( A_{11} + \frac{1}{A_{11}} \right) \psi
\]
\[
+ \sum_j F_{jj} - F_{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2 + \alpha'' F_{jj} (\nabla_j \tau)^2
\]
\[
+ \alpha' F_{jj} \nabla_j \nabla_j \tau + \beta F_{jj} \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta F_{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Using equation (16) in the last equation we get
\[
(25) \quad F_{jj} \nabla_j \nabla_j \Phi = - \frac{1}{A_{11}} F_{jj,kl} \nabla_k A_{ij} \nabla_l A_{jk} + \frac{\nabla_i \nabla_j \psi}{A_{11}}
\]
\[
- \left( A_{11} + \frac{1}{A_{11}} \right) \psi + F_{jj} A_{11}^2 + \sum_j F_{jj}
\]
\[
- F_{jj} \left( \frac{\nabla_i A_{11}}{A_{11}} \right)^2 + \alpha'' F_{jj} (\nabla_j \tau)^2 + \alpha' F_{jj} \nabla_j \nabla_j \tau
\]
\[
+ \beta F_{jj} \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta F_{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Then as in [9], by Proposition 1.(3) and using \( \psi(Y, \tau) \) we have
\[
\nabla_i \nabla_j \psi \geq \psi \nabla_i \nabla_1 \tau - C_1 A_{11} - C_2
\]
\[
= \psi \left( - \sum_r \nabla_r A_{11} \nabla_r \eta + A_{11}^2 \tau + A_{11} \delta_{11} \right) - C_1 A_{11} - C_2.
\]
Then we have the following inequality:

\[
\frac{\nabla_1 \nabla_1 \psi}{A_{11}} \geq -\frac{\psi_r}{A_{11}} \sum_r \nabla_r A_{11} \nabla_r \eta + \psi_r A_{11} \nabla_r \tau + \psi_r \delta_{11} - C_1 - \frac{C_2}{A_{11}}.
\]

On the other hand, using the assumption that \( \varphi \) is affine then

\[
F^{jj} \nabla_j \nabla_j \gamma \geq -C.
\]

Also we are assuming control over \( |\nabla_j \gamma| \leq C \), and then

\[
F^{jj} \nabla_j \gamma \nabla_j \gamma \leq C \sum_j F^{jj},
\]

which will be used at the end.

If we now continue using inequalities (27) and (26) in (25) we obtain

\[
F^{jj} \nabla_j \nabla_j \Phi \geq -\frac{1}{A_{11}} F^{j;j,k} \nabla_1 A_{1j} \nabla_1 A_{kl} - \frac{\psi_r}{A_{11}} \sum_r \nabla_r A_{11} \nabla_r \eta
+ \psi_r A_{11} \tau + \psi_r \delta_{11} - C_1 - \frac{C_2}{A_{11}} + F^{jj} A^2_{jj}
- \left( A_{11} + \frac{1}{A_{11}} \right) \psi + \sum_j F^{jj} - F^{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2
+ \alpha'' F^{jj} \left( \nabla_j \tau \right)^2 + \alpha' F^{jj} \nabla_j \nabla_j \tau - \beta \frac{C}{\gamma} - \beta F^{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Using again Proposition 1-(3), we replace the term \( \alpha' F^{jj} \nabla_j \nabla_j \tau \) to get

\[
F^{jj} \nabla_j \nabla_j \Phi \geq -\frac{1}{A_{11}} F^{j;j,k} \nabla_1 A_{1j} \nabla_1 A_{kl} - \frac{\psi_r}{A_{11}} \sum_r \nabla_r A_{11} \nabla_r \eta
+ \psi_r A_{11} \tau + (\psi_r + \alpha' \psi) \delta_{11} - C_1 - \frac{C_2}{A_{11}} + \sum_j F^{jj}
+ (1 + \alpha' \tau) F^{jj} A^2_{jj} - \left( A_{11} + \frac{1}{A_{11}} \right) \psi - F^{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2
+ \alpha'' F^{jj} \left( \nabla_j \tau \right)^2 - \alpha' \sum_r \nabla_r \psi \nabla_r \eta - \beta \frac{C}{\gamma} - \beta F^{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\]

Now, at the maximum, we also have

\[
-\psi_r \sum_r \nabla_r A_{11} \nabla_r \eta = \psi_r \sum_r \left( \alpha' \nabla_r \tau + \beta \frac{\nabla_r \gamma}{\gamma} \right) \nabla_r \eta,
\]

and since \( \nabla_r \psi = \psi_r + \psi_r \nabla_r \tau \), we have that

\[
-\psi_r \sum_r \frac{\nabla_r A_{11}}{A_{11}} \nabla_r \eta - \alpha' \sum_r \nabla_r \psi \nabla_r \eta = \sum_r \left( \psi_r \beta \frac{\nabla_r \gamma}{\gamma} - \alpha' \psi_r \right) \nabla_r \eta \geq -\frac{C \beta}{\gamma} - C,
\]
then,

\begin{equation}
F^{jj} \nabla_j \nabla_j \Phi \geq -\frac{1}{A_{11}} F^{ij,kl} \nabla_i \nabla_k A_{ij} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C \\
+ (\psi_\tau - \psi) A_{11} + (\psi_\tau + \alpha' \psi) \delta_{11} \\
+ (1 + \alpha' \tau) F^{jj} A_{jj}^2 - \frac{\psi}{A_{11}} + \sum_j F^{jj} \\
- F^{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2 + \alpha'' F^{jj} (\nabla_j \tau)^2 - \beta F^{jj} \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\end{equation}

Case 1: There is a constant $\mu > 0$ such that $A_{nn} \leq -\mu A_{11}$.

In this case we will use the concavity of $F$ and drop the term with the second derivatives $F^{ij,kl}$ in the inequality (29). Note that the last equation implies that

\begin{equation}
F^{jj} A_{jj}^2 \geq \frac{\mu^2}{n} A_{11}^2 \sum_j F^{jj},
\end{equation}

and also

\begin{equation}
F^{nn} \geq \frac{1}{n} \sum_j F^{jj}.
\end{equation}

Note as well that

\begin{equation}
F^{jj} (\nabla_j \tau)^2 = F^{jj} A_{jj}^2 (\nabla_j \eta)^2 \leq C F^{jj} A_{jj}^2.
\end{equation}

At the maximum of $\Phi$ we have $\nabla_j \Phi = 0$ and from (24) we have

\begin{equation}
\left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2 = \left( \alpha' \nabla_j \tau + \beta \frac{\nabla_j \gamma}{\gamma} \right)^2,
\end{equation}

and moreover, for all $\epsilon > 0$ we have

\begin{equation}
\left( \alpha' \nabla_j \tau + \beta \frac{\nabla_j \gamma}{\gamma} \right)^2 < (1 + \epsilon) (\alpha')^2 (\nabla_j \tau)^2 + (1 + \epsilon^{-1}) \beta^2 \left( \frac{\nabla_j \gamma}{\gamma} \right)^2.
\end{equation}

Note that we will find below an $\alpha$ such that $(\alpha'' - (1 + \epsilon) (\alpha')^2) < 0$, so

\begin{equation}
(\alpha'' - (1 + \epsilon) (\alpha')^2) F^{jj} (\nabla_j \tau)^2 \geq C_1 (\alpha'' - (1 + \epsilon) (\alpha')^2) F^{jj} A_{jj}^2,
\end{equation}

then from (29),

\begin{equation}
F^{jj} \nabla_j \nabla_j \Phi \geq -2\beta \frac{C}{\gamma} - C - C_1 - \frac{C_2}{A_{11}} + (\psi_\tau - \psi) A_{11} \\
+ (\psi_\tau + \alpha' \psi) \delta_{11} + \left\{ (1 + \alpha' \tau) + C_1 (\alpha'' - (1 + \epsilon) (\alpha')^2) \right\} F^{jj} A_{jj}^2 \\
- \frac{\psi}{A_{11}} + \left\{ 1 - (\beta + (1 + \epsilon^{-1}) \beta^2) \frac{1}{\gamma^2} \right\} \sum_j F^{jj}.
\end{equation}

Now, in order to control the coefficients of $F^{jj} A_{jj}^2$, we solve the following ordinary equation

\begin{equation}
\alpha'' - (\alpha')^2 = 0,
\end{equation}

and we find solutions of the form

\begin{equation}
\alpha = -\ln(\tau + a),
\end{equation}
where $a > 0$ to be specified. Moreover, the first and second derivatives are

$$
\alpha' = -\frac{1}{\tau + a}, \quad \alpha'' = \frac{1}{(\tau + a)^2},
$$

and then it is clear that

$$
\alpha'' - (1 + \epsilon)(\alpha')^2 = -\frac{\epsilon}{(\tau + a)^2} \leq 0,
$$

from which we can also see that for $\epsilon = a^2/2C_1$ we have

$$(\alpha'\tau + 1) + C_1(\alpha'' - (1 + \epsilon)(\alpha')^2) = \frac{a}{\tau + a} - \frac{C_1\epsilon}{(\tau + a)^2}
= \frac{a(\tau + a)}{(\tau + a)^2} - \frac{C_1\epsilon}{(\tau + a)^2} > \frac{a^2}{2(\tau + a)^2} \geq C_3 > 0,$$

then form (34) we get

$$
0 \geq -2\beta C \frac{\gamma}{\gamma} - C - C_1 - \frac{C_2}{A_{11}}
+ (\psi\tau - \psi)A_{11} + (\psi\tau + \alpha'\psi)\delta_{11}
+ C_3F_{jj}A_{11}^2 - \frac{\psi}{A_{11}} + \left\{1 - (\beta + (1 + \epsilon^{-1})\beta^2) \frac{1}{\gamma^2}\right\} \sum_j F_{jj}.
$$

Note $A_{11} \geq \cdots \geq A_{nn}$ and this implies that

$$
\sum_j F_{jj} = \frac{1}{\psi^{k-1}}H_{k-1},
$$

from this it follows that

$$
\sum_j F_{jj} \geq C_4 > 0.
$$

Using the growth assumption $\psi\tau - \psi > 0$, the inequality (30), and choosing $\beta > 0$ such that $\{1 - (\beta + (1 + \epsilon^{-1})\beta^2) \frac{1}{\gamma^2}\} > 0$, we obtain

$$
0 \geq -2\beta C \frac{\gamma}{\gamma} - C - C_1 - \frac{C_2}{A_{11}} - \frac{\psi}{A_{11}} + \frac{\mu^2}{n}C_3A_{11}^2.
$$

Now we make use of the assumption $\lambda_1 \geq 1$ so that

$$
\frac{C(\beta)}{\mu} \geq \gamma A_{11}.
$$

Case 2: Looking back at inequality (29), the assumption for this case is the existence of $\mu > 0$ such that

$$
A_{nn} \geq -\mu A_{11},
$$

and in this case we will make use of the term with $F_{jj,kl}$. Note also that $A_{jj} \geq -\mu A_{11}$, for all $j = 1, 2, \ldots, n$ since $A_{11} \geq A_{22} \geq \cdots \geq A_{nn}.

Consider the following partition of the indices $\{1, 2, \ldots, n\}$,

$$
I = \{j \mid F_{jj} \leq 4F_{11}\}, \quad \text{and} \quad J = \{j \mid F_{jj} > 4F_{11}\}.
$$
Now, for \( j \in I \), at the maximum, equation (31) and inequality (32) hold for any \( \varepsilon > 0 \), namely

\[
\left( \alpha' \nabla_{j} \tau + \beta \frac{\nabla_{j} \gamma}{\gamma} \right)^{2} < (1 + \varepsilon)(\alpha')^{2}(\nabla_{j} \tau)^{2} + (1 + \varepsilon^{-1})\beta^{2} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2}, \quad j \in I.
\]

For \( j \in J \), at the maximum, since \( \nabla_{j} \Phi = 0 \) in equation (24), we have for any \( \varepsilon > 0 \) that

\[
\beta^{-1} \left( \alpha' \nabla_{j} \tau + \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2} \leq \frac{1 + \varepsilon}{\beta} (\alpha')^{2}(\nabla_{j} \tau)^{2} + \frac{1 + \varepsilon^{-1}}{\beta} \left( \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2}.
\]

From these two inequalities we can get

\[
\beta F^{jj} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2} + F^{jj} \left( \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2} \leq \frac{1 + \varepsilon}{\beta} (\alpha')^{2} \sum_{j \in J} F^{jj}(\nabla_{j} \tau)^{2} + \frac{1 + \varepsilon^{-1}}{\beta} \sum_{j \in J} \left( \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2}
\]

\[
+ \beta \sum_{j \in I} F^{jj} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2} + \sum_{j \in J} F^{jj}(\nabla_{j} \tau)^{2} + (1 + \varepsilon)(\alpha')^{2} \sum_{j \in I} F^{jj}(\nabla_{j} \tau)^{2}
\]

\[
+ (1 + \varepsilon^{-1})\beta^{2} \sum_{j \in I} F^{jj} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2} \leq 4n \{ \beta + (1 + \varepsilon^{-1})\beta^{2} \} F^{11} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2}
\]

\[
+ (1 + \varepsilon)(1 + \beta^{-1})(\alpha')^{2} F^{jj}(\nabla_{j} \tau)^{2} + \{ 1 + (1 + \varepsilon^{-1})\beta^{-1} \} \sum_{j \in J} F^{jj} \left( \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2}.
\]

Using the last two estimates in (29) at the maximum we obtain

\[
0 \geq -\frac{1}{A_{11}} F^{ij, kl} \nabla_{1} A_{ij} \nabla_{1} A_{kl} - C_{1} - \frac{C_{2}}{A_{11}} - 2\beta \frac{C}{\gamma} - C
\]

\[
+ (\psi_{\tau} - \psi) A_{11} + (\psi_{\tau} + \alpha' \psi) \delta_{11}
\]

\[
+ (1 + \alpha' \tau) F^{jj} \frac{\nabla^{2}}{A_{11}} + \sum_{j} F^{jj}
\]

\[
- F^{jj} \left( \frac{\nabla_{j} A_{11}}{A_{11}} \right)^{2} + \alpha'' F^{jj}(\nabla_{j} \tau)^{2} - \beta F^{jj} \left( \frac{\nabla_{j} \gamma}{\gamma} \right)^{2}.
\]
Solving \( \alpha'' - (\alpha')^2 = 0 \) as in Case 1, we obtain (33), then

\[
0 \geq -\frac{1}{A_{11}} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C
\]
\[
+ (\psi \tau - \psi A_{11} + (\psi + \alpha' \psi) \delta_{11} - \frac{\psi}{A_{11}}
- 4n \{ \beta + (1 + \epsilon^{-1})\beta^2 \} F^{11} \left( \frac{\nabla_1 \gamma}{\gamma} \right)^2 + \sum_j F^{jj}
+ \{ (1 + \alpha' \tau) + C_1 (\alpha'' - (1 + \epsilon)(1 + \beta^{-1})(\alpha')^2) \} F^{jj} A_{11}^2
- \{ 1 + (1 + \epsilon^{-1})\beta^{-1} \} \sum_{j \in J} F^{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2,
\]

and moreover, for \( \epsilon = \epsilon(a) \), there is a \( C_0 > 0 \) such that the last term is improved by

\[
0 \geq -\frac{1}{A_{11}} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C
\]
\[
+ (\psi \tau - \psi A_{11} + (\psi + \alpha' \psi) \delta_{11} - \frac{\psi}{A_{11}}
- 4n \{ \beta + (1 + \epsilon^{-1})\beta^2 \} F^{11} \left( \frac{\nabla_1 \gamma}{\gamma} \right)^2 + \sum_j F^{jj}
+ \{ (1 + \alpha' \tau) + C_1 (\alpha'' - (1 + \epsilon)(1 + \beta^{-1})(\alpha')^2) \} F^{jj} A_{11}^2
- \{ 1 + C_0 \beta^{-1} \} \sum_{j \in J} F^{jj} \left( \frac{\nabla_j A_{11}}{A_{11}} \right)^2.
\]

It is also known (see for instance Lemma 2.20 and Lemma 2.21 in [1]) that for any symmetric matrix \( \eta_{ij} \) we have

\[
F^{ij,kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ij} \eta_{ji} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij},
\]

and whenever \( F \) is concave, then the second term of the right hand side of the equation is non-positive and it should be read as a limit when \( \lambda_i = \lambda_j \). Then, using this Lemma, the Codazzi equation (6) and since \( 1 \notin J \) we have the following inequality

\[
-\frac{1}{\lambda_1} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} \geq -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} | \nabla_1 A_{11} |^2
= -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} | \nabla_j A_{11} |^2.
\]
Then following from (35) we get

\[0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C + (\psi \tau - \psi) A_{11} + (\psi \tau + \alpha' \psi) \delta_{11}\]

\[+ C_3 F^{jj} A^2_{jj} - \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1 + \epsilon^{-1})\beta^2\} F^{11} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2\]

\[- (1 + C_0 \beta^{-1}) \sum_{j \in J} F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 - \frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} |\nabla_j A_{11}|^2.\]

Put \(\delta = C_0 \beta^{-1}\), and recall that since \(j \in J\) we have \(f_j > 4f_1\). If \(\lambda_j > 0\) then the equation

\[(37) \quad (1 - \delta) f_j \lambda_1 \geq 2f_1 \lambda_1 - (1 + \delta) f_j \lambda_j, \quad \text{for} \quad j \in J,\]

holds with \(\delta = \frac{1}{4}\). If \(\lambda_j \leq 0\), then since \(\lambda_n \geq -\mu \lambda_1\) and thus \(\lambda_j \geq -\mu \lambda_1\) for all \(j = 1, 2, \ldots, n\), then we have \(|\lambda_j| \leq \mu \lambda_1\). This implies that (37) is also satisfied if \(\delta = 1/4\) and \(\mu = 1/5\). Recall that this choices implies a value for \(\beta\) which depends on \(\sup_{\Omega} |\nabla u|\).

Equation (37) implies the inequality

\[\frac{2}{\lambda_1} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} \geq (1 + C_0 \beta^{-1}) \frac{f_j}{\lambda_1}, \quad j \in J,\]

for \(\beta\) sufficiently small, and then we can drop the last two terms in (36)

\[0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C + (\psi \tau - \psi) A_{11} + (\psi \tau + \alpha' \psi) \delta_{11}\]

\[+ C_3 F^{jj} A^2_{jj} - \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1 + \epsilon^{-1})\beta^2\} F^{11} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2\]

Now, recall from (28) we get

\[0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C + (\psi \tau - \psi) A_{11} + (\psi \tau + \alpha' \psi) \delta_{11}\]

\[+ C_3 F^{jj} A^2_{jj} - \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1 + \epsilon^{-1})\beta^2\} \frac{F^{11}}{\gamma^2},\]

which gives us at the end an estimate of the type

\[C_4 \lambda_1 + C_3 F^{11} \lambda_1^2 \leq C \left(1 + \frac{\frac{F^{11}}{\gamma^2}}{\gamma^2}\right),\]

which concludes the proof the theorem.
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