Colour groups in tilings with singularities

Reinhard Lück
Weilstetter Weg 16, D-70567 Stuttgart, Germany
r.v.lueck@web.de

Abstract. The paper is focussed on colour symmetry of regular tilings with singularities. Colour indices of the tilings considered are not only determined by the symmetry of the tilings but additionally by the rotational symmetry of the singularities. The number of the derived colour indices is found to be limited for a tiling. Coloured examples are presented. The combination of phyllotactic analysis and colour symmetry is discussed. Periodic arrangements of singularities can be introduced to visualize special hyperbolic tilings.

1. Introduction
Colour symmetries have been discussed for Euclidean periodic and quasiperiodic [1, 2] as well as for hyperbolic tilings [3]. The advantage of this research is – beside the aesthetics of the results – a better understanding of atomic superstructures and of coincidence site lattices. Regular tilings with singularities were described by Grünbaum and Shephard [4] in the Section ‘Well-behaved tilings’ of their textbook. Such tilings are self-similar with a scaling factor like quasiperiodic patterns. There are three types of regular patterns in two dimensions and Euclidean space. These are composed of (i) 4 quadrangles sharing a vertex ($4^4$), (ii) 6 triangles sharing a vertex ($3^6$) and (iii) 3 hexagons sharing a vertex ($6^3$). Tilings of the type ($4^4$) and the tilings ($3^6$) and ($6^3$) will be treated separately. The singularities have a rotational symmetry different from the vertex symmetry; this is similar as for so-called Volterra dislocations or inclinations related to disclinations ([5] and references therein). The colour symmetry of regular patterns without singularities was described by Baake [1] and in other papers [2] and references therein. In the present case, the colour symmetry should be compatible with both sufficient rotational colour symmetry of the undistorted regular pattern and the rotational symmetry of the singularity. In addition, periodic arrangements of singularities will be introduced for the visualization of special hyperbolic tilings. The colour indices of possible colourings in the considered tilings are not only determined by the symmetry of the tiling – as described in references [1], [2] and [3] – but additionally by the rotational symmetry of the singularities. Examples including phyllotactic analysis will be presented and discussed.

2. Tilings of the type ($4^4$)
2.1. Five-fold singular symmetry
There are several tiles of quadrangular shape which form tilings with singularities. The Penrose kite [4] may be regarded as a very common one. As the Penrose kite is related to five-fold symmetry in quasiperiodic patterns, a pattern with five-fold singularity can be designed. The scaling factor of the
resulting self-similar tiling is the golden mean $\tau = (5^{1/2} + 1)/2$ as in quasiperiodic patterns. The five-colour (colour index $h=5$) decoration of the (undistorted) square lattice is well-known from the $\Sigma 5$ coincidence site lattice (CSL) [2]. The corresponding colouring is not a perfect one, since there is no colour mirror symmetry; it may be considered as semi-perfect. Moreover, an enantiomorphic pair of colourings exists. This colouring can be easily applied on a tiling with five-fold singularity. The colouring of a tiling composed of Penrose kites with a five-fold singularity is demonstrated in Fig. 1. The tiling $(4^5)$ can be decorated with two colours as it is well known from the chess board. This colouring can be transformed to the kite tiling with singularity. A combination of 5 and 2 colours results in 10 colours. A combination of the two different colourings with 5 colours results in a perfect colouring with 25 colours. No other colour indices than 1, 2, 5, 10, 25, and 50 exist for this pattern.

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{figure1.png}
\caption{Five-fold pattern composed of Penrose kites. $(4^5)$ tiling with five-fold singular centre. The scaling factor is $\tau$ with rotation (e.g. by $\pi$) and $\tau^2$ without rotation. Decoration is performed with five colours ($h=5$). There is no mirror symmetry for colours and therefore no perfect colouring. An enantiomorphic pair exists for 5 colours. 2, 25 and 50 colours are possible with perfect colouring. An enantiomorphic pair with 10 colours can be derived.}
\end{figure}

2.2. Six-fold and three-fold singular symmetries

Kites may be also arranged with other symmetries and/or other scaling factors. In an arrangement with six-fold symmetry eight (Fig. 2) or nine (Fig. 3) colours are possible. The tilings of Figs 2 and 3 are equivalent, but their scaling factors are different. The maximum colour index is 72. The kites may be distorted to asymmetric quadrangles. Figure 4 with nine colours is equivalent to Fig. 3; the pattern in Fig. 4 has the same linear scaling factor $3^{1/2}$. Possible colour indices are 1, 2, 4, 8, 9, 18, 36, and 72. Figure 5 is another example of an asymmetric tiling. The colour index is 5, no other colour indices than 1, 5, 9, and 45 are possible. It should be emphasized that there are many more rotational symmetries of singularities in $(4^5)$ patterns.

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{8 colours in a $(4^5)$ tiling with six-fold singular centre.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{figure3.png}
\caption{9 colours in a $(4^5)$ tiling with six-fold singular centre. Symmetric version.}
\end{figure}
3. Tilings with curved lines

Before discussing tilings composed of triangles and hexagons some modifications with curved boundaries should be introduced. From Figs 4 and 5 it may be concluded that spirals are essentials of such tilings. The spirals may be taken directly from the tilings with scaling properties. Another possibility is a conformal transformation of the regular tilings. Conformal transformation or conformal mapping is a standard mathematical procedure described e.g. in textbooks (compare the classical Books by Zeev Negari [6] or by Ludwig Bieberbach [7]) or mathematical handbooks [8]. Conformal transformation can be performed by a holomorphic function in the planar representation of complex numbers. Interest in the theory of conformal transformation arose in the second half of the 19th century; applications were developed in the 20th century. For the present purpose it is important that such conformal transformation results in logarithmic spirals. The presented examples are designed using logarithmic spirals. The property of the logarithmic spiral provides that all conformal symmetric (4^4) patterns are constructed with the same spiral. Other spirals (e.g. taken from the self-similar tilings) can be applied to change the scaling factor. We applied both methods, some results will be presented. Figure 6 is analogous to Fig. 1. Five colours are used to decorate. The numbers of left-hand and right-hand spirals are both 5. These numbers determine the maximum colour index. Figure 7 is a conformal

![Figure 4](image1.png)

**Figure 4.** 9 colours in a (4^4) tiling with six-fold singular centre. Asymmetric version.

![Figure 5](image2.png)

**Figure 5.** 5 colours in an asymmetric (4^4) tiling with a three-fold singular centre. Maximum colour index is 45.

![Figure 6](image3.png)

**Figure 6.** Conformal transformation of (4^4) tiling with a five-fold singular centre. Decoration with 5 colours.

![Figure 7](image4.png)

**Figure 7.** Decoration of an conformal (4^4) tiling with an eight-fold centre with 8 colours.
transformation of a \((4^4)\) pattern with an eight-fold centre. The numbers of spirals are 8 for both orientations. The figure is decorated with 8 colours. Possible colour indices for this tiling are 1, 2, 4, 8, 16, 32, 64, 128. If the numbers of the left-hand and right-hand spirals are respectively \(m\) and \(n\), the maximum colour index for \((4^4)\) tilings is

\[
h_{\text{max}} = m^2 + n^2
\]  

(1)

There are more colour indices if the colour group with maximum colour index allows subgroups. The colour indices are exactly the orders of the subgroups of the colour group of maximum colour index; there are no subgroups not corresponding to a colouring.

**Figure 8.** Four colours in a conformal \((3^6)\) tiling with twelve- singular centre.

**Figure 9.** 16 colours in a conformal \((6^3)\) tiling with twelve-fold singular centre.

### 4. Tilings of the types \((3^6)\) and \((6^3)\)

Tilings composed of triangles or hexagons may be designed by self-similar extension or by conformal transformation. Examples are presented in Figs 8 and 9. A \((3^6)\) tiling with a twelve-fold singular centre is decorated with 4 colours in Fig. 8. Figure 9 represents a \((6^3)\) conformal tiling with a twelve-fold singular centre which is decorated with 16 colours. In case of regular hexagons or vertices of equilateral triangles, the colour indices of undistorted patterns may be generalized by

\[
h = M^2 + 3N^2
\]  

(2)

This equation holds also for the maximum number of colours in six-fold patterns with singularities. The meaning of \(M\) and \(N\) is determined by the number of spirals crossing perpendicularly edges (\(M\)) and vertices (\(N\)), respectively. More generally, for colouring hexagons three types of spirals (including straight lines) approaching the singular centre should be considered. If the numbers of these dual lines (i.e. they are crossing centres of hexagons and crossing perpendicularly hexagon edges) are \(l\), \(m\), and \(n\), the maximum colour index was found to be

\[
h_{\text{max}} = (l^2 + m^2 + n^2)/2
\]  

(3)

### 5. Phyllotactic analysis and colourings

One application of the discussion of tilings with singularities is the phyllotactic analysis of supposed or real botanical materials. Several papers are published on this subject in literature on aperiodics ([9], [10], [11] and references therein) but nothing is said about colours. A short test of several cones (e.g. pine-, scots-pine- or larch-cones) resulted in the combination of consecutive Fibonacci numbers for the spirals involved. For quadrangles, equation (1) should be applied; and the following numbers result for the maximum colour indices \(h_{\text{max}}\):
Numbers within parentheses are examples which were not found in the present analysis of cones. The maximum colour indices are again Fibonacci numbers. In the language of Kepler [12, 13], these are the Fibonacci numbers producing ‘males’, i.e. the product of consecutive numbers exceeds a square number by one (Kepler wrote ‘Einheit’). The skipped Fibonacci numbers produce ‘females’, where the product of consecutive numbers is less by one than a square number. 5, 13, 89, 233 ... are prime numbers; therefore, these groups do not have subgroups. Important subgroups of the colour indices 34 and 610 are formed by the colour index 2. For hexagons, however, applying equation (3) the maximum colour indices \( h_{\text{max}} \) are

\[(7, 19, 49, 129, 337, (883, ...)]

The tilings can be characterized by three consecutive Fibonacci numbers for \( l, m, \) and \( n: \)

\[(1, 2, 3; 2, 3, 5; 3, 5, 8; 5, 8, 13; 8, 13, 21; (13, 21, 34;...)]

7, 19, 337, 883... are prime numbers, the colour groups do not have subgroups. 49 as a square number implies an enantiomorphic pair of subgroups with colour index 7; 129 has subgroups with 3 and 43 as colour index. Pine-cones of the type 5, 8, 13 may be decorated with three colours.

6. Periodic arrangement of singularities and hyperbolic tilings

Coloured hyperbolic tilings were investigated recently [3], [14]. Colour indices were derived for several tilings; perfect and semi-perfect colourings were distinguished [3]. Examples with ten colours were published using the Poincaré model [14]. A disadvantage of the Poincaré model is that the tile size is increasingly shrinking with increasing distance from the centre. Some colour symmetry properties are therefore difficult to be detected. Searching for other possibilities of visualization we introduce a periodic arrangement of singularities. Such figures require the general symmetry of a periodic regular tiling. Another condition is a violation of Euler’s equation for Euclidian geometry. The determination of colour index requires the consideration of an additional condition. As a consequence, the number of colour indices is reduced. Figure 10 represents a \((4^\circ)\) tiling with a periodic arrangement of four-fold singularities (white areas), the pattern is decorated with 6 colours.

![Figure 10. Pattern \((4^\circ)\) with a periodic arrangement of four-fold singular points, decoration with 6 colours.](image)

![Figure 11. A tiling \((3^\circ)\) with a periodic arrangement of three-fold singular points. It is decorated with 8 colours.](image)
Figure 11 is a representation of the tiling \((3^8)\) with three-fold singular points. This tiling allows a decoration with 2, 4, 8, 16, or 32 colours. Figure 11 is decorated with 8 colours.

It seems to be interesting to transform such figures with periodic arrangements of singularities into tilings with locally regular vertices. A conformal periodic tiling should require that the singularities are formed by closed lines of six-, three-, four-, or two-fold symmetry. A sketched pattern analogous to Fig. 11 is presented in Fig. 12 as an example. An attempt to transform the vertices to a locally regular eight-fold rotational symmetry similar as in the Poincaré model of \((3^8)\) is presented in Fig. 13. Both representations (Figs 12 and 13) are decorated with 16 colours. To visualize a decoration with 4 colours, the 16 colours are organized in four groups (bluish, greenish, yellowish/brownish and reddish) of four different tones. The comparison of the periodic arrangement of singularities and the Poincaré disc is very helpful to understand colour symmetry and colour orbits. Figure 12 demonstrates periodicity of colours, this corresponds to a symmetry operation difficult to detect in the Poincaré disc. However, not all hyperbolical colour symmetries can be transformed to a pattern with a periodic arrangement of singularities. A hyperbolic tiling as it is represented in the Poincaré model can not be completely transformed into a conformal mapping with such a periodic arrangement of singularities; in other words: there are (relatively small) areas in Fig. 13 not represented in Fig. 14.

7. Conclusion
Tilings with singularities can be designed by a self-similar procedure or by conformal transformation of a regular tiling. Only a limited number of colourings were found to be possible for regular tilings with a single singularity. The maximum colour index can be easily determined counting the number of differently oriented spirals approaching the singularity. The same procedure can be applied to colour phyllotactic arrangements. Some coloured hyperbolic tilings can be represented by a periodic arrangement of singularities.

8. Acknowledgement
I am grateful to the Bielefeld group, especially to Dirk Frettlöh, for a discussion of the present topic, to Professor Ellen Baake for a discussion on phyllotactic analysis, to David H. Warrington for a critical reading of the manuscript and to an unknown referee for some important proposals.
References
[1] Baake M 1997 Combinatorial aspects of colour symmetry J. Phys. A: Math. Gen. 30 2687-98
[2] Lück R 2008 Colour symmetry of 25 colours in quasiperiodic patterns Phil. Mag. 88 2049-58
[3] Frettlöh D 2008 Counting perfect colourings of plane regular tilings Z. Krist. 223 773-776
[4] Grünbaum B and Shephard C G 1987 Tiling and Patterns (Freeman, New York) 113-164
[5] Kleman M 2009 Interplay between grain boundaries and disclinations in condensed matter physics Int. J. Mat. Res. 100 1449-1455
[6] Nehari Z 1952 Conformal mapping McGraw-Hill, New York – Toronto – London, reprinted 1975 Dover Publication Inc. New York
[7] Bieberbach L 1956 Einführung in die konforme Abbildung 5th edition, Berlin; 1953 Conformal mapping, translation of the 4th edition which was originally published 1949, Chelsea
[8] Meyers Handbuch über die Mathematik 1972, H Meschkowski ed. 2nd edition Bibliographisches Institut, Mannheim – Wien - Zürich
[9] Rivier N Pieranski P and Rothen F 1994 Conformal crystals and their defects Aperiodic’94 (World Scientific, Singapore) Ed Chapuis G and Paciorek W 1211-25
[10] Rivier N and Goldar A 1997 Entropy of aperiodic crystal generated by spirals Aperiodic’97 (World Scientific, Singapore) Ed de Boissieu M Verger-Gaugry J-L and Currat R 131-135
[11] Jean R V 1992 Nomothetical modelling of spiral symmetry in biology Fivefold Symmetry (World Scientific, Singapore) Ed Hargittai I 505-528
[12] Brandmüller J 1992 Fivefold symmetry in mathematics, physics, chemistry, biology and beyond Fivefold Symmetry (World Scientific, Singapore) Ed Hargittai I 11- 31
[13] Kepler J 1619 Harmonices Mundi Libri Quinque (Five books on the harmony of the world) Joannis Plancus, Lincii Austria; translated and commented by M Caspar, Weltharmonie (Harmony of the world) R. Oldenbourg-Verlag, Munich 1984 (cited after [12])
[14] Lück R and Frettlöh D 2008 Ten colours in quasiperiodic and regular hyperbolic tilings Z. Krist. 223 782-784